A YANG-BAXTER EQUATION FOR METAPLECTIC ICE

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ABSTRACT. We will give new applications of quantum groups to the study of spherical Whittaker functions on the metaplectic $n$-fold cover of $GL(r, F)$, where $F$ is a nonarchimedean local field. Earlier Brubaker, Bump, Friedberg, Chinta and Gunnells had shown that these Whittaker functions can be identified with the partition functions of statistical mechanical systems. They postulated that a Yang-Baxter equation underlies the properties of these Whittaker functions. We confirm this, and identify the corresponding Yang-Baxter equation with that of the quantum affine Lie superalgebra $U_{\sqrt{v}}(\hat{gl}(1|n))$, modified by Drinfeld twisting to introduce Gauss sums. (The deformation parameter $v$ is specialized to the inverse of the residue field cardinality.)

For principal series representations of metaplectic groups, the Whittaker models are not unique. The scattering matrix for the standard intertwining operators is vector valued. For a simple reflection, it was computed by Kazhdan and Patterson, who applied it to generalized theta series. We will show that the scattering matrix on the space of Whittaker functions for a simple reflection coincides with the twisted $R$-matrix of the quantum group $U_{\sqrt{v}}(\hat{gl}(n))$. This is a piece of the twisted $R$-matrix for $U_{\sqrt{v}}(\hat{gl}(1|n))$, mentioned above.

1. Introduction

The formula of Casselman and Shalika [11] expresses values of the spherical Whittaker function for a principal series representation of a reductive algebraic group over a $p$-adic field in terms of the characters of irreducible finite-dimensional representations of the Langlands dual group. Their proof relies on knowing the effect of the intertwining integrals on the normalized Whittaker functional. Since the Whittaker functional is unique, the intertwining integral just multiplies it by a constant, which they computed.

In contrast with this algebraic case, Whittaker models of principal series representations of metaplectic groups are generally not unique. The effect of the intertwining operators on the Whittaker models was computed by Kazhdan and Patterson [23]. Specifically, they computed the scattering matrix of the intertwining operator corresponding to a simple reflection on the finite-dimensional vector space of Whittaker functionals for the $n$-fold metaplectic cover of $GL(r, F)$, where $F$ is a $p$-adic field. Some terms in this matrix are simple rational functions of the Langlands parameters, while others involve $n$-th order Gauss sums. Though complicated in appearance, this scattering matrix was a key ingredient in their study of generalized theta series, and also in the later development of a metaplectic Casselman-Shalika formula by Chinta and Offen [13] and McNamara [33].

One of the two main results of this paper is that this scattering matrix computed by Kazhdan and Patterson is the $R$-matrix of a quantum group, quantum affine $gl(n)$, modified by Drinfeld twisting to introduce Gauss sums. This appears to be a new connection between the representation theory of $p$-adic groups and quantum groups, which should allow one to use
techniques from the theory of quantum groups to study Whittaker functions on metaplectic groups.

Although we can now prove this directly, we were led to this result by studying lattice models whose partition function give values of Whittaker functions on the metaplectic cover of $GL(r, F)$. The existence of a solution to the Yang-Baxter equation that makes these models solvable was predicted in [7]. Such a solution has important applications in number theory: it gives easy proofs (in the style of Kuperberg’s proof of the alternating sign matrix conjecture) of several facts about Weyl group multiple Dirichlet series [9]. The other main result of this paper is the discovery of a solution of the Yang-Baxter equation which makes the models mentioned above solvable. Moreover, we relate this solution to the quantum affine $\mathfrak{gl}(1|n)$. The relation between the two main results follows from the inclusion of (quantum affine) $\mathfrak{gl}(n)$ into $\mathfrak{gl}(1|n)$.

Let us now explain our results in more detail. We will exhibit solvable lattice models whose partition functions are values of metaplectic Whittaker functions. This is a modification of a family of models proposed earlier in Brubaker, Bump, Chinta, Friedberg and Gunnells [7]. Let $\tilde{G}$ denote an $n$-fold metaplectic cover of $G := GL(r, F)$ where the non-archimedean local field $F$ contains the $2n$-th roots of unity. If $\lambda$ is a partition of length $\leq r$, a system $S_\lambda$ was exhibited in [7] whose partition function equaled the value of one particular spherical Whittaker function at

$$s \left( \begin{array}{c} p^{\lambda_1} \\ \vdots \\ p^{\lambda_r} \end{array} \right),$$

where $s : GL(r, F) \to \tilde{GL}(F)$ is a standard section.

The systems proposed in [7] were generalizations of the six-vertex model. The six-vertex model with field-free boundary conditions was solved by Lieb [27], Sutherland [37] and Baxter [2] and were motivating examples that led to the discovery of quantum groups. (See [26, 20, 14].) In Baxter’s work, the solvability of the models is dictated by the Yang-Baxter equation where the relevant quantum group is $U_q(\hat{sl}_2)$. In the special case $n = 1$ (so when we are working with non-metaplectic $GL(r, F)$), the systems proposed in [7] coincide with those discussed in Brubaker, Bump and Friedberg [8, 9] and there is a Yang-Baxter equation available. However even in this case these models differ from those considered by Lieb, Sutherland and Baxter since although they are true six-vertex models, they are not field-free. Based on the results of this paper, we now understand that the relevant quantum group for the lattice models in [8, 9] is $U_q(\hat{gl}(1|1))$, as we will make clear in subsequent sections.

It was explained in [7] that a Yang-Baxter equation for metaplectic ice would give new proofs of two important results in the theory of metaplectic Whittaker functions. The first is a set of local functional equations corresponding to the permutation of the Langlands-Satake parameters. The second is an equivalence of two explicit formulas for the Whittaker function, leading to analytic continuation and functional equations for associated Weyl group multiple Dirichlet series. The proof of this latter statement occupies the majority of [9].

However, no Yang-Baxter equation for the metaplectic ice in [7] could be found. In this paper we will make a small but crucial modification of the Boltzmann weights for the model in [7]. This change does not affect the partition function, but it makes possible a Yang-Baxter
equation. This is Theorem 3 in Section 3. The solutions to the Yang-Baxter equation may be encoded in a matrix commonly referred to as an \( R \)-matrix.

We further prove that the resulting \( R \)-matrix has two important properties:

1. It is a Drinfeld twist of the \( R \)-matrix obtained from the defining representation of quantum affine \( \widehat{\mathfrak{gl}}(1\mid n) \), a Lie superalgebra.
2. It contains the \( R \)-matrix of a Drinfeld twist of \( \widehat{\mathfrak{gl}}(n) \), which, as we have already explained, we will identify with the scattering matrix of the intertwining operators on the Whittaker models.

Consider the quantized enveloping algebra of the untwisted affine Lie algebra \( \widehat{\mathfrak{gl}}(n) \), i.e. the central extension of the loop algebra of \( \mathfrak{gl}(n) \). We denote the quantized enveloping algebra as \( U_{\sqrt{q}}(\widehat{\mathfrak{gl}}(n)) \) instead of the usual \( U_q \) because in our application the deformation parameter \( v \) will be \( q^{-1} \), where \( q \) is the cardinality of the residue field of \( F \). If \( V \) and \( W \) are vector spaces, let \( \tau = \tau_{V,W} \) denote the flip operator \( V \otimes W \to W \otimes V \). The Hopf algebra \( U_{\sqrt{q}}(\widehat{\mathfrak{gl}}(n)) \) is almost quasitriangular, so given any two modules \( V \) and \( W \), there is an \( R \)-matrix \( R_{V,W} \in \text{End}(V \otimes W) \) such that \( \tau R_{V,W} : V \otimes W \to W \otimes V \) is a module homomorphism (though it will not always be isomorphism). The \( R \)-matrices for \( U_{\sqrt{q}}(\widehat{\mathfrak{gl}}(n)) \) acting on the standard module were found by Jimbo [21] (see also Frenkel and Reshetikhin [16], Remark 4.1.); they satisfy a parametrized Yang-Baxter equation as is expected.

The quantum group \( U_{\sqrt{q}}(\widehat{\mathfrak{gl}}(n)) \) has an \( n \)-dimensional evaluation module \( V_+(z) \) for every complex parameter value \( z \). We will label a basis of the module \( v_{+a}(z) \) where \( a \) runs through the integers modulo \( n \). The parameter \( +a \) will be called a decorated spin (to be supplemented later by another one, denoted \( -0 \)), generalizing the so-called spin \( \pm \) in the literature of the six-vertex model. We may think of the decoration \( a \) (mod \( n \)) as roughly corresponding to the sheets of the metaplectic cover \( \tilde{G} \to \text{GL}(r) \) of degree \( n \).

The matrix \( R_{z_1,z_2} \in \text{End}(V_+(z_1) \otimes V_+(z_2)) \) may be defined by specifying the values \( R_{\alpha,\beta}^\gamma(z_1,z_2) \) for decorated spins \( \alpha, \beta, \gamma \) and \( \delta \) such that
\[
R_{z_1,z_2}(v_\alpha(z_1) \otimes v_\beta(z_2)) = \sum_{\gamma,\delta} R_{\alpha,\beta}^\gamma(z_1,z_2)v_\gamma(z_1) \otimes v_\delta(z_2).
\]

These are given by the following table:

| \( \alpha, \beta, \gamma, \delta \) | \( +a, +a, +a, +a \) (\( 0 \leq a \leq n \)) | \( +b, +a, +b, +a \) (\( 0 \leq a, b \leq n, a \neq b \)) | \( +b, +a, +a, +b \) (\( 0 \leq a, b \leq n, a \neq b \)) |
|---|---|---|---|
| \( R_{\alpha,\beta}^\gamma(z_1,z_2) \) | \( \frac{-v^{+(z_1/z_2) - n}}{1-v^{(z_1/z_2) - n}} \) | \( g(a-b)^{-n}(z_1/z_2)^{-n} \) | \( \frac{1}{1-v^{(z_1/z_2) - n}} \)
| \( \frac{1}{1-v(z_1/z_2) - n} \) a > b, \( \frac{1-v^{(z_1/z_2)} - n}{1-v(z_1/z_2) - n} \) a < b. |

Here \( g(a-b) \) is an \( n \)-th order Gauss sum. These are not present in the out-of-the-box \( U_{\sqrt{q}}(\widehat{\mathfrak{gl}}(n)) \) \( R \)-matrix but may be introduced by Drinfeld twisting that will be discussed in Section 6 and in Section 4 of [5]. This procedure does not affect the validity of the Yang-Baxter equations, but is better for comparison with the \( R \)-matrix for the partition functions of metaplectic ice giving rise to Whittaker functions.
To obtain the full $R$-matrix used in the Yang-Baxter equation for metaplectic ice, we must enlarge the set of decorated spins $+a$ to include one more, labelled $-0$. Then the $n$-dimensional vector space $V_+(z)$ is enlarged to an $n+1$ “super” vector space $V_{\pm}(z)$. In Section 3, we present an $R$-matrix that gives a solution of the Yang-Baxter equation for the metaplectic ice model. In Section 6, we show that the solution of the Yang-Baxter equation is equivalent to the $R$-matrix corresponding to the defining representation of the quantum affine Lie superalgebra $\hat{U}_{\sqrt{\tau}((\mathfrak{gl}(n|1))}$ modified by a Drinfeld twist.

Finally, we explain the connection between the $R$-matrix of Theorem 3 and the structure constants alluded to in item (2) above. The local functional equations for metaplectic Whittaker functions mentioned earlier may be understood as arising from intertwining operators. Let $\hat{T}$ be the diagonal torus in $GL(r, \mathbb{C})$, the Langlands dual group of $G$. Each diagonal matrix

$$z = \begin{pmatrix} z_1 & \cdots & z_r \end{pmatrix} \in \hat{T}(\mathbb{C})$$

indexes a principal series representation $\pi_z$ of $\hat{G}$. Let $\mathcal{W}^z$ be the finite-dimensional vector space of spherical Whittaker functions for $\pi_z$. If $n = 1$, $\mathcal{W}^z$ is one-dimensional, but not in general since if $n > 1$ the representation $\pi_z$ does not have unique Whittaker models. If $s_i$ is a simple reflection in the Weyl group $W$, then let $\mathcal{A}_{s_i}$ denote the standard intertwining integral $\mathcal{A}_{s_i} : \pi_z \rightarrow \pi_{s_i z}$ (see [13] for the precise definition). This induces a map $\mathcal{W}^z \rightarrow \mathcal{W}^{s_i z}$. If $n > 1$ then $\mathcal{A}_{s_i}$ has an interesting scattering matrix on the Whittaker model that was computed by Kazhdan and Patterson (Lemma I.3.3 of [23]). This calculation underlies their work on generalized theta series, and was used by Chinta and Offen [13] and generalized by McNamara [33] to study the analog of the Casselman-Shalika formula for the spherical Whittaker functions.

In Section 9 we will prove that the scattering matrix of the intertwining integrals on the spherical Whittaker functions is essentially $\tau R_{z_i, z_{i+1}}$, where $R_{z_i, z_{i+1}}$ is the $R$-matrix for a Drinfeld twist of $U_{\sqrt{\tau}((\mathfrak{gl}(n))}$.

**Theorem 1.** There is an isomorphism $\theta_z$ of the space $\mathcal{W}^z$ of spherical Whittaker functions to the vector space $V_+(z_1) \otimes \cdots \otimes V_+(z_r)$, which takes the vectors $v_{+a_1}(z_1) \otimes \cdots \otimes v_{+a_r}(z_r)$ into the basis of $\mathcal{W}^z$ given in [23, 13, 33] (see Section 8). Then the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{W}^z & \xrightarrow{\theta_z} & V_+(z_1) \otimes \cdots \otimes V_+(z_i) \otimes V_+(z_{i+1}) \otimes \cdots \otimes V_+(z_r) \\
\downarrow \mathcal{A}_{s_i} & & \downarrow I_{V_+(z_1) \otimes \cdots \otimes \tau R_{z_i, z_{i+1}} \otimes \cdots \otimes I_{V_+(z_r)}} \\
\mathcal{W}^{s_i z} & \xrightarrow{\theta_{s_i z}} & V_+(z_1) \otimes \cdots \otimes V_+(z_i) \otimes V_+(z_{i+1}) \otimes \cdots \otimes V_+(z_r)
\end{array}$$

where $\mathcal{A}_{s_i}$ denotes the map obtained by $W^z \mapsto W^z \circ \mathcal{A}_{s_i}$ for any spherical Whittaker function $W^z$ with normalized intertwining operator $\mathcal{A}_{s_i}$ defined in [14].

This offers a new and seemingly fundamental connection between the representation theory of quantum groups and $p$-adic metaplectic groups. We next describe two applications of the results in this paper which were both written after the present paper was circulated (in an earlier draft), and which continue this work.
The paper by Brubaker, Buciumas, Bump and Friedberg [5] was written after the first draft of this one was already posted to the arxiv, and it depends on it. In it we give a very general method of constructing representations of the affine Hecke algebra and show that examples of such representations can come either from the theory of Whittaker functionals on metaplectic $p$-adic groups or from certain Schur-Weyl dualities for quantum affine algebras. Theorem 1 in the present paper is used to prove the two representations mentioned are in fact the same. The paper also contains a more formal discussion of the Drinfeld twisting, an important supplement to the brief treatment we give below in Section 6.

The paper by Brubaker, Buciumas, Bump and Gray [6] was also written after this one. It uses the Yang-Baxter equations from this paper, and supplementary ones from Gray [19], to reprove the main result of [9], which may be expressed as the equality of the partition functions of two different ice models. One of the two ice models is described below in Section 2. The other one is similar but has different weights. The equality of the two partition functions is reminiscent of dualities for “physical phenomena,” similar for example to the Kramers-Wannier duality that relates the partition functions of the low-temperature and high temperature Ising models.

Future questions include the generalization of the above results to other Cartan types and relations between them and other literature such as Weissman [38]. It seems particularly important to understand the relation between our work and the geometric Langlands program initiated by Gaitsgory in [17], and more specifically the relation to the work of Lysenko [28] and Gaitsgory and Lysenko [18].

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2. The partition function

In the language of statistical mechanics, the partition function of a model is a generating function, that carries complete information about the system. It is a sum over a collection of states of the model. Each such state is then weighted by a Boltzmann weight. We first define the states of our model, and then the associated weights.

Consider a two-dimensional rectangular grid with $r$ rows and $N$ columns, where $N$ is any sufficiently large number. Every edge will be assigned a spin, which has value $+$ or $-$. The spins along the boundary edges will be fixed as part of the data specifying the system; the spins on the interior edges will be allowed to vary. Thus with the spins on the boundary fixed, a state of the system will be an assignment of spins to the interior edges.

We will use the following boundary conditions. We place $+$ spins on the left and bottom boundary edges, and $-$ on the right edge. As for the top edge, we proceed as follows. The columns will be numbered $N - 1, N - 2, \ldots, 2, 1, 0$ in decreasing order. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of length $\leq r$, and let $\rho = (r - 1, \ldots, 3, 2, 1, 0)$, so that $\lambda + \rho$ is a strict partition of length $r$. The spins along the left and bottom boundary edges of the grid will be $+$, while the spins along the right edge of the grid will all be $-$. The spins along the top edge will be a mixture of $+$ and $-$, and the number of $-$ signs appearing will be equal to $r$, the number of rows. For example, see Figure 1 for an example of a state.
Define the charge at each horizontal edge in the configuration to be the number of + spins at or to the right of the edge, along the same row. We also will speak of the charge at a vertex, defined to be the charge on the edge to the right of the vertex. The charges are labeled in Figure 1 as decorations above each vertex.

The state will be called admissible if the four spins on adjacent edges of any vertex are in one of the six configurations in Figure 2. It will be called \( n \)-admissible if it is admissible and if furthermore every horizontal edge with a \( - \) spin has charge \( \equiv 0 \) modulo \( n \).

An example of an admissible state is shown in Figure 1. (The appearance of labels \( z_i \) on the vertices in the figure will be explained momentarily.) The illustrated state is \( n \)-admissible only if \( n = 1 \) or \( 3 \), since it has a horizontal \( - \) edge with charge 3.

Now we explain how to attach a weight to each admissible state. This Boltzmann weight of a state is obtained as a product of weights attached to each vertex in the model. The weight attached to any vertex makes use of a pair of functions \( h \) and \( g \) on the integers satisfying the following properties.

Let \( n \) be a fixed positive integer and \( v \) a fixed parameter. Let \( g(a) \) be a function of the integer \( a \) which is periodic modulo \( n \), and such that \( g(0) = -v \), while \( g(a)g(n-a) = v \) if \( n \) does not divide \( a \). Let

\[
h(a) = \begin{cases} 
1 - v & \text{if } n|a, \\
0 & \text{otherwise}.
\end{cases}
\]

Remark. In [7] and [9], the functions \( g \) and \( h \) are defined using \( n \)-th order Gauss sums, with \( v = q^{-1} \), and shown to satisfy the above properties. We will use this specific choice later in Theorem 2 and in Section 9 to connect the partition function to metaplectic Whittaker functions. However, only the above properties are required for their study by statistical mechanical methods in subsequent sections.
To define the Boltzmann weight at a vertex, choose \( r \) nonzero complex numbers \( z_1, \ldots, z_r \), one for each row, as indicated in Figure 1. Then given a vertex in the \( i \)-th row, its Boltzmann weight is given in Figure 2. Note that this weight depends on the spins and the charges on adjacent edges, and the row \( i \) in which it appears. Then the Boltzmann weight \( B_{z_1, \ldots, z_r}(s) \) of the state \( s \) is the product of the Boltzmann weights over all vertices in the grid. We often omit the \( n \) or the \( z_1, \ldots, z_r \) in the notation for \( B \), as the weights may be stated uniformly for all such choices.

Thus in the example state in Figure 1 the vertex in the first row and column 5 is of type \( b_1 \) with charge 4, so the Boltzmann weight at this vertex is \( g(4) \). There are two \( c_2 \) vertices in columns 3 and 0, a \( c_1 \) vertex in column 2, and \( a_1 \) vertices in columns 4 and 1. The \( c_1 \) vertex and one of the \( c_2 \) vertices has charge 2, so the state is \( n \)-admissible only if \( n = 1 \) or 2. Assuming this, the \( g(4) \) from the \( b_1 \) vertex evaluates to \(-v\) and the \( c_1 \) vertex evaluates to \((1 - v)z_1\), while the remaining vertices in the row have weight 1. Thus the total contribution of this row is \((-v)(1 - v)z_1\). The second row has a vertex \( c_2 \) (with charge 0) the remaining vertices are all \( b_2 \) or \( a_2 \), so this row contributes \( z_2^2 \). The last row has a \( c_2 \) vertex (with charge 0) and two \( b_2 \) vertices. The remaining three vertices in the row are of type \( a_1 \) with Boltzmann weight 1. The Boltzmann weight of this state is \((-v)(1 - v)z_1 z_2^2 z_3^2 \) if \( n = 1 \) or 2, and 0 otherwise.

Note that, although the Boltzmann weights depend only on the charge \( a \) modulo \( n \), we take charge to be an integer, not an integer modulo \( n \). This will be important mainly in Section 5.

**Proposition 1.** Suppose that \( s \) is an admissible state such that the Boltzmann weight \( B^{(n)}(s) \neq 0 \). Then the state is \( n \)-admissible.

**Proof.** We must show that, under this assumption, the charge on every edge with spin \( - \) is a multiple of \( n \). Suppose not and consider the right-most edge in any row where this condition fails. Then the charge at the vertex \( v \) to the right of this edge is not a multiple of \( n \). We claim that the edge to the right of \( v \) has spin \( + \). We know that \( v \) is not the rightmost vertex in its row since its charge is nonzero. So if the edge to the right of \( v \) has spin \( - \), then the vertex to the right of \( v \) has the same charge as \( v \), contradicting our assumption that \( v \) is the rightmost counterexample in its row.
Since the edge to the left of \( v \) is \(-\) and the edge to the right is \(+\), consulting Figure 2 we see that the only admissible configuration of spins at the vertex \( v \) is of type \( c_1 \), so the Boltzmann weight at \( v \) is \( h(a)z_i = 0 \) because \( n \nmid a \). This contradicts our assumption that \( B^{(n)}(s) \neq 0 \).

Let \( \mathcal{S}^{(n)}_{\lambda} \) denote the set of \( n \)-admissible states with fixed boundary spins determined by a partition \( \lambda \). The sum \( Z(\mathcal{S}^{(n)}_{\lambda}) \) of its Boltzmann weights is called the partition function. (We could just as easily sum over all admissible states, since the Boltzmann weights of the states that are not \( n \)-admissible vanish.)

**Theorem 2** (Theorem 4 of [7]). Let \( \lambda \) be a partition with \( r \) parts and let \( n \) be a fixed positive integer. Then the partition function \( Z(\mathcal{S}^{(n)}_{\lambda}) \) is (up to normalization) a value of a \( p \)-adic spherical Whittaker function on the metaplectic \( n \)-fold cover of \( GL(r,F) \), where \( F \) is a nonarchimedean local field with residue field of cardinality \( q \equiv 1 \mod (2n) \).

In the above result, the Boltzmann weights used in \( Z(\mathcal{S}_{\lambda}) \) have parameter \( v = q^{-1} \) where \( q \) is the cardinality of the residue field of \( F \). In Proposition 7 of Section 9 we give a more precise and more refined version of this result.

### 3. The Yang-Baxter equation

The partition functions described in Section 2 differ from those of the classical six-vertex model in a crucial way: the Boltzmann weights depend on a global statistic, the charge. If we wish to use statistical mechanical techniques like the Yang-Baxter equation, we need the weight at any vertex to be local, that is, depending only on nearest-neighbor interactions. We achieve this by a slight change in point of view, introducing decorated spins for the horizontal edges. Given a fixed positive integer \( n \), a decorated spin is an ordered pair \((\sigma,a)\) where the spin \( \sigma \) is \(+\) or \(-\) and the decoration \( a \) is an integer mod \( n \). Moreover if \( \sigma = -\), we will only consider \( a = 0 \mod n \). In figures we will sometimes draw the spin \( \sigma \) in a circle and write the decoration \( a \) next to it. In text we will denote \((\sigma,a)\) as \( \sigma a \). The key point is that the decoration is now viewed as part of the data attached to a horizontal edge. Now there are \( n + 1 \) possible decorated spins for horizontal edges, rather than just the spins \(+\) and \(-\); we have left the six-vertex model.

Given a state of the system, each horizontal edge receives a decoration \( a \) depending on its spin \( \sigma \) and the charge \( b \) of the edge to its right. If \( \sigma = +\), then \( a = b + 1 \) and if \( \sigma = -\), then \( a = b \). If we set the initial decorations of the right-hand boundary edges (which all have spin \(-\)) to be \( 0 \), then this rule clearly recovers the charge (mod \( n \)) of the previous section. Thus the Boltzmann weights in Figure 2 may be interpreted as purely local; in the figure, we have indicated the decoration by writing it over the spin. We are justified in requiring the decoration at an edge to be \( 0 \) modulo \( n \) (without affecting the resulting partition functions) by Proposition 1.

We use the term decoration (rather than “charge” again) to emphasize this change to the local point of view and because the Boltzmann weights of Figure 2 depend only on the congruence class of charge mod \( n \). Thus there are only finitely many admissible configurations of decorated spins at a given vertex.

Now that the weights at any vertex may be viewed as local, we are ready to present our solution to the Yang-Baxter equation. Three vertices with different Boltzmann weights will
appear in the Yang-Baxter equation (1) below. These will be labeled $z_1$, $z_2$ and $R_{z_1,z_2}$. Here $z_1$ and $z_2$ are nonzero complex numbers used in the Boltzmann weight of the associated vertex, which double as labels for the vertices. At the vertices labeled $z_1$ and $z_2$ we will use the Boltzmann weights already described in Figure 2.

In Figure 3 we describe the Boltzmann weights at the vertices labeled $R_{z_1,z_2}$. The Boltzmann weights depend only on the residue classes modulo $n$ of the integers $a, b, c, \cdots$ that appear in these formulas, but in some cases depend on a particular choice of representatives for residue classes. These choices are indicated in the description below the figure.

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
$a$ & $a$ & $z_2^n - v z_1^n$ & $0$\\
$+$ & $+$ & $g(a-b)(z_1^n - z_2^n)$ & $0$\\
& $+$ & $b$ & $z_1^n - z_2^n$ \\
& & $0$ & $(1-v)z_1^n z_2^n$ (*) \\
$+$ & $+$ & $v(z_1^n - z_2^n)$ & $0$\\
& & $a$ & $(1-v)z_1^n z_2^n$ (**)
\hline
\end{tabular}
\end{center}

\textbf{Figure 3.} Boltzmann weights for the $R$-vertex $R_{z_1,z_2}$. It is assumed that $b$ is not equal to $a$. (*) Here $c \equiv a - b \mod n$ with $0 \leq c < n$. (***) Here we choose the representative of $a$ modulo $n$ with $1 \leq a \leq n$, so if $a \equiv 0 \mod n$, $n-a$ means 0, not $n$.

\textbf{Theorem 3.} The partition functions of the following two systems are equal. That is, if we fix the charges $\sigma, \tau, \beta, \rho, \alpha$ and $\mu$ and the decorations $a, b, c, d$, and sum over the possible values of the inner edge (decorated) spins, we obtain the same result in both cases.

(1)

\begin{center}
\begin{tabular}{|c|c|}
\hline
$\beta$ & $\beta$\\
$\tau$ & $\tau$\\
$a$ & $b$\\
$\sigma$ & $\sigma$\\
$\eta$ & $\eta$\\
$z_1$ & $z_2$\\
$\theta$ & $\theta$\\
$\mu$ & $\mu$\\
$\delta$ & $\delta$\\
$\rho$ & $\rho$\\
\hline
\end{tabular}
\end{center}

\textbf{Proof.} In every admissible configuration there are an even number of + spins on the six boundary edges. Therefore there are 32 possible boundary conditions and we will consider these 32 cases separately. Each case breaks into subcases depending on the decorations at $\sigma, \tau, \theta$ and $\rho$. We exhaustively considered all cases where there is a nonzero contribution. To give the reader a feeling for the possibilities, we will do one case in detail, but in the interest of keeping the publication concise, the remaining cases may be found in [4].
In our enumeration of cases, the case we will consider carefully is Case 10. Assume that \((\sigma, \tau, \beta, \theta, \rho, \alpha) = (+, +, -, +, -, +)\).

**Case 10a**: With \(k \neq 0\), suppose that the (decorated) spins on the six boundary edges are as follows:

\[
\begin{array}{cccccc}
\sigma & \tau & \beta & \theta & \rho & \alpha \\
k+1 & 1 & + & k & 0 & + \\
+ & + & - & + & - & +
\end{array}
\]

On each side there is one \(n\)-admissible state:

|    | \(e\) | \(f\) | \(\gamma\) | weight                     |    | \(g\) | \(h\) | \(\delta\) | weight                     |
|----|------|------|------|----------------|----|------|------|------|----------------|
| k+1 | 1    |      |      | \((z^n_1 - z^n_2) g(k) g(-k)\) |    | k    | 0    |      | \((z^n_1 - z^n_2) v\)   |
| +   | +    |      |      |                |    | +    |      |      |                |

Thus the configurations are as follows:

Since \(g(k) g(-k) = v\), (1) is satisfied in this case.

**Case 10b**: With \(k \neq 0\),

|    | \(a\) | \(b\) | \(\beta\) | \(c\) | \(d\) | \(\alpha\) |
|----|------|------|------|------|------|------|
|    | \(\sigma\) | \(\tau\) | \(\theta\) | \(\rho\) | \(\alpha\) |
| 1  |      |      |      |      |      |      |
| k+1 | 1    |      |      |      |      |      |
| +   |      | +    |      |      |      |      |

|    | \(e\) | \(f\) | \(\gamma\) | weight                     |    | \(g\) | \(h\) | \(\delta\) | weight                     |
|----|------|------|------|----------------|----|------|------|------|----------------|
| k+1 | 1    |      |      | \((1 - v) z^n_1 z^{n-k}_2 g(k)\) |    | 0    |      |      | \((1 - v) z^n_1 z^{n-k}_2 g(k)\) |
| +   |      | +    |      |                |    | -    |      |      |                |

**Case 10c**:
This completes the proof of Case 10. See the appendix in [4] for the other cases. □

4. Partition function identities with weights $R_{z_1,z_2}$

If $\alpha = (\pm, c)$ is a decorated spin, let us say that $\alpha$ is visible if either the spin is $+$, or if the decoration $c \equiv 0$ modulo $n$. It is a consequence of Proposition 1 that only $n$-admissible states have nonzero Boltzmann weights, and in any $n$-admissible state, only vertices with visible spins on all adjacent edges produce nonzero Boltzmann weights as given in Figure 2. Furthermore the nonzero Boltzmann weights of type $R_{z_1,z_2}$ given in Figure 3 occur only at vertices with visible spins on all adjacent edges.

**Theorem 4.** Let $z_1$, $z_2$ and $z_3$ be given. Then for every choice of visible decorated boundary spins $\alpha, \beta, \gamma, \delta, \epsilon, \phi$, the partition functions of the following two systems are equal:

This is through the matrix $\tau R_{z_1,z_2}$ where $\tau(x \otimes y) = y \otimes x$. **Theorem 4** implies that the morphism $R_{\tilde{\sigma}}$ is well defined, depending only on the choice of braid. Our goal is to embed this category in a category of modules of a quantum group, which will be a twist of the quantized enveloping algebra of $\widehat{\mathfrak{gl}}(1|n)$.
Theorem 5. Let $\alpha, \beta, \gamma, \delta$ be visible decorated spins. Then the partition function

\[
R_{z_2, z_1} R_{z_1, z_2}
\]

equals

\[
\begin{cases}
(z^n_1 - vz^n_2)(z^n_2 - vz^n_1) & \text{if } \alpha = \gamma, \beta = \delta \\
0 & \text{otherwise.}
\end{cases}
\]

As we mentioned in the introduction, we will show in Section 9 that Theorem 4 is related to the intertwining integrals for principal series representations of the metaplectic group, which were calculated in Kazhdan and Patterson [23]. From this point of view, Theorem 5 is related to Theorem I.2.6 of [23].

Because of the last result, it is almost true that if we modified the $R$-matrix $R_{z_1, z_2}$ by dividing by $z^n_1 - vz^n_2$, the associated quantum (super) group (which will be identified in Section 6) would be triangular in the sense of Drinfeld [14]. However because this factor $z^n_1 - vz^n_2$ can be zero, this is not quite true, and the braided category of modules is also not triangular.

5. Proofs of Theorems 4 and 5

The goal of this section will be to prove the theorems in Section 4. Some of the facts we uncover are of interest beyond this goal. Although the statement of these two theorems is independent of the systems $\mathfrak{S}_\lambda$ in Section 2, those systems will appear in the proof.

In this section, we view the Boltzmann weights in Figures 2 and 3 as depending on the charge, an integer, so that $a$ and $b$ appearing in those figures are integer valued rather than integers mod $n$. Let us consider some implications of this alternative point of view. In Figure 2 we recorded Boltzmann weights which were 0 unless the adjacent charges were of the form $(a, a)$ or $(a + 1, a)$ for some $a$ mod $n$. Now wherever $a + 1$ appears, it must be interpreted as the integer $a + 1$. If we replaced it by $a' + 1$ where $a' \equiv a$ modulo $n$, the Boltzmann weight would be interpreted as zero unless $a' = a$. In particular, we may change the charge $a$ adjacent to a vertex to $a'$ without changing the Boltzmann weight, but we must change it both adjacent edges – to the left and right of the vertex. A similar situation occurs with Figure 3; if we change $a$ to $a'$ where $a' \equiv a$ modulo $n$, we must change it in both places where it occurs.

The use of integer-valued charge leads to the following paradox: If $a \equiv b$ modulo $n$ but $a \neq b$, observe that the integer-valued pair of charges $(a, b)$ is no longer allowable in the top-left pattern in Figure 3. Instead it is associated with the second and the third patterns in the top row of the table. At first this is disconcerting because it seems to suggest a new proof of the Yang-Baxter equation is required. However, this is not a problem because the contributions of the second and third patterns together give the same result as the first pattern in this case. Indeed in this case, $g(a - b) = -v$ and so

\[
(3) \quad z^n_2 - vz^n_1 = [g(0)(z^n_1 - z^n_2)] + [(1 - v)z^n_2].
\]
Let nonzero complex numbers $z_1, \ldots, z_r$ be given. Regard $z = (z_1, \ldots, z_r)$ as coordinates in a complex torus $T$, and $\Lambda = X^*(T) \cong \mathbb{Z}^r$ will be the $GL_r$ weight lattice. Elements of the weight lattice will be called \textit{weights}, not to be confused with Boltzmann weights. We will denote $z^\mu = \prod z_i^{\mu_i}$ if $\mu = (\mu_1, \ldots, \mu_r) \in \Lambda$. We fix an integer $N$, large enough so that $N \geq \mu_i$ whenever $\mu = (\mu_1, \ldots, \mu_r)$ is a weight that occurs in our considerations.

In Section 2, the partition function $Z(\mathcal{G}_\lambda)$ was a sum over all admissible states with boundary spins corresponding to the choice of $\lambda$. If we view the data on any edge as both a spin and an integer charge, then the partition function $Z(\mathcal{G}_\lambda)$ can be written as a sum of much smaller partition functions having both fixed boundary spins \textit{and} fixed boundary charges. To explain this point, let $a = (a_1, \ldots, a_r)$ where $a_1, \ldots, a_r$ are positive integers. Consider the system $\mathcal{G}_\lambda$ with the same grid and the same Boltzmann weights as in Section 2, but at each horizontal boundary edge, we attach a spin and a charge as follows. On the horizontal edges along the right-hand boundary, use the decorated spins $-\varepsilon$. On the left-hand boundary, we use $+a_1, \ldots, +a_r$. Let $\mathcal{G}_{\lambda, a}$ be the set of states whose left and right boundary conditions match these, and whose top and bottom edges have the same boundary conditions (using undecorated spins) as $\mathcal{G}_\lambda$. Then the following identity of partition functions is immediate:

$$Z(\mathcal{G}_\lambda) = \sum_a Z(\mathcal{G}_{\lambda, a}),$$

where $Z(\mathcal{G}_{\lambda, a})$ is the partition function of the restricted system $\mathcal{G}_{\lambda, a}$. Let $N$ be the width of the grid, that is, the number of vertices in each row. Since the only admissible verticals augment charge by at most one moving from right to left, then $N$ is an upper bound for each possible choice of $a_i$ such that $\mathcal{G}_{\lambda, a}$ is nonempty. In particular, the above sum is finite.

**Lemma 1.** Let $a = (a_1, \ldots, a_r)$, and let $\mu = (N - a_1, \ldots, N - a_r)$. Then the Boltzmann weight of every state in $\mathcal{G}_{\lambda, a}$ equals $z^\mu$ times a polynomial in $v$.

**Proof.** Consulting Figure 2 we see that the power of $z_i$ equals the number of vertices in the $i$-th row whose configuration of adjacent spins is of type $a_2, b_2$ or $c_1$. These are exactly the cases where the vertex to the left of the vertex is $-$, that is, precisely those that do not contribute to the charge at the left edge of the row. Since this charge is $a_i$, the power of $z_i$ that occurs is $N - a_i = \mu_i$. \qed

Thus we will call $\mu$ the \textit{weight} of the state. Let $W$ be the Weyl group of $\Lambda$, which is the symmetric group $S_r$.

**Proposition 2.** Suppose that $\mu = (\mu_1, \ldots, \mu_r)$ is a weight with $0 \leq \mu_i \leq N$, and suppose that the $\mu_i$ are all distinct. Let $a_i = N - \mu_i$. Then there exists a dominant weight $\lambda$ such that for every permutation $w \in W$ the system $\mathcal{G}_{\lambda, w(a)}$ consists of a single element. The Boltzmann weight of this state is $z^{w\lambda}$ times a nonzero constant depending only on $w$.

**Proof.** Since $\mu$ has distinct entries, there exists a dominant weight $\lambda$ such that the $\mu_i$ are the elements of $\lambda + \rho$, permuted. We will show that $\mathcal{G}_\lambda$ has a unique state with weight $w(\mu)$ for any $w \in W$.

We will make use of the bijection between states of $\mathcal{G}_\lambda$ and strict Gelfand-Tsetlin patterns with top row $\lambda$. (A Gelfand-Tsetlin pattern is called \textit{strict} if its rows are strictly decreasing.) This bijection is explained in [7], [8], and [9] (Proposition 19.5). Briefly, the elements of the $i$-th row of the pattern consists of the columns in which there are $+$ spins in the vertical
edges above the vertices in the $i$-th row of the grid. Thus the state illustrated in Figure 1 corresponds to the Gelfand-Tsetlin pattern

$$
\begin{bmatrix}
5 & 3 & 0 \\
5 & 2 & 0 \\
2 & 0 & 0 \\
\end{bmatrix}.
$$

The weight of the state can be read off from the pattern: the exponents of $z_1, z_2, \cdots$ are the differences between consecutive row sums. This is Lemma 11 in [8]. In the example above, the row sums are 8, 7, and 2, and so the weight is $(8 - 7, 7 - 2, 2) = (1, 5, 2)$, and indeed, the Boltzmann weight is $h(2) g(4) z_1 z_2 z_3$.

A pattern is called stable if the corresponding pattern involves no vertices in configuration $c_1$. By Proposition 1, any $c_2$ configuration that occurs has charge 0. Consequently, every vertex has weight 1, or $z_i$ or $g(a)$ for some $a$. In terms of the Gelfand-Tsetlin pattern, every entry (except of course those of the top row) equals one of the two above it, since otherwise (it is easy to see) it would produce an configuration of type $c_1$. This means that each row of the pattern (except the first row, which is $\lambda + \rho$) is obtained from the row above it by deleting a single entry, and so the differences of the row sums are the entries of the top row in some order. There are thus $r!$ stable patterns, and their weights are permutations of $\lambda + \rho$, that is, permutations of $\mu$. It is clear from the Gelfand-Tsetlin description that these are the extremal weights that can occur; in other words, every weight lies in the convex hull of the polytope spanned by the weights of the stable patterns, that is, by the $w(\mu)$ with $w \in W$. Moreover, only the stable pattern can produce such a weight, so the weights $w(\mu)$ are each produced by a unique state. It follows that if $\mathbf{a} = (N - \mu_1, \cdots, N - \mu_r)$, then $\mathcal{S}_{\lambda, \mathbf{a}}$ consists of a single state.

The Boltzmann weight of a stable state is nonzero, since (due to the fact that $c_1$ vertices do not appear and $c_2$ vertices only appear with charge 0) it equals $z^{w(\mu)}$ times a product of $g(a)$, which do not vanish.

We will need a generalization of this statement. To this end, we slightly generalize the class of boundary conditions. We still require + signs along the bottom vertical spins, and − signs along the right-hand edge, but now, in addition to allowing both + and − spins along the top edge, we allow both + and − spins along the left edge. We will call such a system a mixed system.

**Proposition 3.** Let $\alpha_1, \cdots, \alpha_r$ be visible spins. Assume that the spins among these with $\alpha_i = +a_i$ have distinct charges $a_i$, and that the spins $-a_i$ have $a_i = 0$. Then there exists a mixed system that has a unique state with left edge spins $\alpha_i$ (in order).

**Proof.** First we discard the $\alpha_i$ with spin −, and build a system as in Proposition 2. Then we modify this by inserting, corresponding to each $\alpha_i$ with spin −, another row with (as boundary conditions) − on both the left and right edge. This system will have a unique solution with left edge spins $\alpha_i$. The last step is to add back rows with − spins to the left and right at the locations where the $\alpha_i$ with spin − were discarded.

Let us illustrate this with an example. Suppose that $r = 3$, $N = 5$ and

$$
\begin{align*}
\alpha_1 &= +3, \\
\alpha_2 &= -0, \\
\alpha_3 &= +1.
\end{align*}
$$
First we discard $\alpha_2$. With $\mu = (N - a_1, N - a_3) = (2, 4)$ we must have $\lambda + \rho = (4, 2)$. This tells us the boundary condition of a 2-rowed system according to the recipe in Proposition 2. We construct the system with top row spins $-, +, -, +$ (in order) and left row spins $+, +$. In accordance with Proposition 2 this system has a unique state with left charges 2, 4 corresponding to $\alpha_1$ and $\alpha_3$. This state is

![Diagram of a 2-rowed system with top row spins $-, +, -, +$ and left row spins $+, +$.]

Now we insert a row of $-$ spins:

![Diagram of the system with an additional row of $-$ spins.]

to obtain the desired mixed system.

Let us prove Theorem 4. We will make two assumptions that we will remove later.

- We will assume that the $\alpha$, $\beta$ and $\gamma$ appearing in (2) are decorated $+$ spins. Thus we write $\gamma = +a_1$, $\beta = +a_2$ and $\alpha = +a_3$ for some integer charges $a_i$.
- We will also assume that $a_1$, $a_2$ and $a_3$ are distinct.

Note that both partition functions in (2) vanish unless $\delta$, $\epsilon$, and $\phi$ are a permutation of the $\alpha$, $\beta$, and $\gamma$, and we will assume this.

Consider the system $\mathcal{S}_\lambda$ described in Proposition 2. The dependence of this system on $z = (z_1, z_2, z_3)$ will be important and so we emphasize this by writing $\mathcal{S}_\lambda(z)$. Begin by attaching the configurations in Theorem 4 to the left of the three-rowed system $\mathcal{S}_\lambda(z)$ as
We are now regarding $\alpha$, $\beta$ and $\gamma$ as fixed boundary spins, part of the boundary conditions. However $\phi$, $\epsilon$ and $\delta$ are interior spins and must be summed over, though as we have noted, only a permutation of $\alpha$, $\beta$ and $\gamma$ can produce a nonzero weight.

By Proposition 2 if

$$\delta = +a_{w(1)}, \quad \epsilon = +a_{w(2)}, \quad \phi = +a_{w(3)},$$

then $\mathcal{S}_{\lambda,w(a)}$ consists of a single state whose Boltzmann weight is $g(w)z^{w(\mu)}$ for some nonzero value $g(w)$ made from Gauss sums. Let $B(w)$ and $C(w)$ be the partition functions of the two systems in Theorem 1 when $w$ is the permutation of $\gamma$, $\beta$, $\alpha$ that produces $\phi$, $\epsilon$, $\delta$. We have proved that the composite system described above has partition function

$$\sum_{w \in W} B(w) g(w) z^{w(\mu)}.$$

On the other hand, using the Yang-Baxter equation of Theorem 3, (4) equals the partition function of

Here $w_0 z = (z_3, z_2, z_1)$. Although the spins of interior edges are summed over all possibilities, we have taken the liberty of filling in the spins of six interior edges that are forced to be $-0$ because, according to the Boltzmann weights in Figure 3 each $R_{z_i,z_j}$ has only one admissible
state with right edge spins zero. We may fill in the values of these states and we have proved that
\[ \sum_{w \in W} B(w) g(w) z^{w(\mu)} = (z_1^n - vz_2^n)(z_1^n - vz_3^n)(z_2^n - vz_3^n) Z(\mathcal{G}_\lambda(w_0 z)). \]

Identical arguments apply when we attach the second configuration in Theorem 4 to the same system \( \mathcal{G}_\lambda(z) \) producing the same evaluation, and therefore
\[ \sum_{w \in W} B(w) g(w) z^{w(\mu)} = \sum_{w \in W} C(w) g(w) z^{w(\mu)}. \]

We need to deduce from this that each individual term agrees, namely \( B(w) = C(w) \), which is the content of Theorem 4. This requires a new idea. Even though we have repeatedly insisted for the purpose of this proof that the charges are integers, not integers modulo \( n \), we may still make use of the periodicity of the Boltzmann weights. Let \( K \) be some large positive integer that is a multiple of \( n \). Recalling that \( \gamma = +a_1, \beta = +a_2 \) and \( \delta = +a_3 \), let \( \gamma' = +a_1', \beta' = +a_2', \alpha' = +a_3' \), where
\[ (a_1', a_2', a_3') = (a_1, a_2, a_3) + (2K, K, 0). \]
Moreover, recalling that \( \delta, \epsilon, \phi \) are permutations of \( \gamma, \beta, \alpha \) (since otherwise Theorem 4 is trivially true), let \( \delta', \epsilon', \phi' \) be the corresponding permutation of \( \alpha', \beta', \gamma' \). By the charge-periodicity of the Boltzmann weights for \( R_{z_1, z_2} \), and because \( \alpha \equiv \alpha' \) modulo \( n \), etc. the partition function of the first configuration in Theorem 4 equals the partition function of
\[ R_{z_1, z_3} \]
\[ R_{z_2, z_3} \]

Before proceeding, we may now remove one of the two assumptions that we imposed on the boundary spins. We will continue to assume that \( \alpha, \beta \) and \( \gamma \) are decorated + spins, but we may drop the assumption that the \( a_i \) are distinct. Because we will be applying (5) to the last configuration, we will need the \( a_i' \) to be distinct, but this will certainly be true if \( K \) is sufficiently large. Let us suppose for the sake of explanation that \( a_1 = a_2 \), and that \( \beta = \gamma = \delta = \epsilon \) while \( a_3 \) is distinct. Using (3) we see that the first partition function in Theorem 4 equals the sum of the partition functions (6) plus another similar one with \( \beta' \) and \( \gamma' \) interchanged. Since the same is true of the second partition function in Theorem 4 this is sufficient.

Let \( A_0 \) be the \( \mathbb{C} \)-algebra generated by \( v \) and the \( g(a) \). Given an expression
\[ \xi = \sum_{\mu \in \Lambda} c(\mu) z^\mu, \quad c(\mu) \in A_0, \]
we define \( \text{supp}(\xi) = \{ \mu \in \Lambda | c(\mu) \neq 0 \} \). We will use the following trivial observation to separate out the individual terms in (5). Namely, if \( \xi + \eta = \xi' + \eta' \) and \( \text{supp}(\xi) \cup \text{supp}(\xi') \) is disjoint from \( \text{supp}(\eta) \cup \text{supp}(\eta') \) then \( \xi = \xi' \) and \( \eta = \eta' \). Now, consider what happens to
the supports of the individual terms on both sides of (5) as we increase $K$. (This requires increasing $N$, but that is harmless.) As we vary $K$, the support of $B(w)$ and $C(w)$ are unchanged. However as the vector $\mu = (N - a_1 - 2K, N - a_1 - K)$ becomes increasingly antidominant the weights $w(\mu)$ move apart. Hence if $K$ is large enough, the individual terms on both sides have disjoint support, and we conclude that they are individually equal. Recalling that $g(w) \neq 0$, this proves that $B(w) = C(w)$ for all $w$.

Theorem 4 is now proved except that we must remove the assumption that $\alpha$, $\beta$ and $\gamma$ are decorated + spins. This can be accomplished by a similar argument making use of the systems in Proposition 3. We leave the details to the reader. Theorem 5 is simpler and may be proved in the same way.

6. Metaplectic Ice and Supersymmetry

Perk and Schultz [35] found new solutions of the Yang-Baxter equation. Meanwhile graded (supersymmetric) Yang-Baxter equations were introduced by Bazhanov and Shadrikov [3]. It was found by Yamane [39] that the Perk-Schultz equations were related to the $R$-matrix of the quantized enveloping algebra of the $\mathfrak{gl}(m|n)$ Lie superalgebra in the standard representation. The quantized enveloping algebra of the corresponding affine Lie superalgebra was considered by [41]. A convenient reference for us is Kojima [24]. See also Zhang [40].

We will explain how to relate Theorem 4 to the $\mathfrak{gl}(1|n)$ $R$-matrix. The relationship is not an entirely simple one, since we will have to perform manipulations on the Perk-Schultz $R$-matrix in order to make the comparison. These manipulations preserve the Yang-Baxter equation, but (among other things) they introduce the Gauss sum $g(a)$ and in our view this is a nontrivial change. For example, the partition functions on Section 2 would be fundamentally changed without the $g(a)$. They could not be used as $p$-parts of multiple Dirichlet series ([7, 9]). Therefore, although we are able to relate the Yang-Baxter equation in Theorem 4 to the Perk-Schultz equation, the relationship is more subtle than one might guess.

The quantized enveloping algebra of $\mathfrak{gl}(1|n)$ (or its affinization) has a universal $R$-matrix which produces an $R$-matrix in $\text{End}(V_1 \otimes V_2)$ where $V_1$ and $V_2$ are any pair of modules. Taking three such modules, there is then a graded Yang-Baxter equation in $\text{End}(V_1 \otimes V_2 \otimes V_3)$ which we wish to compare with that in Theorem 4. We will be interested in the standard module, which is a graded $(1|n)$-dimensional vector space. Using the corresponding affine Lie superalgebra produces one copy of this module for every value of a parameter $z \in \mathbb{C}^\times$, which may be identified with the parameter in Figure 2. A basis of the graded $(1|n)$-dimensional vector space can be taken to be the decorated edge spins $-a$ for the even part, and $+a$ with $a$ modulo $n$ for the odd part. As noted by Kojima [24], we may change some signs in the graded Yang-Baxter equation to produce an ungraded Yang-Baxter equation. We will return to this point at the very end of the discussion.

Referring to [24] for notation, we will take the decorated spin $-0$ to have graded degree 0, and the spins $+a$, where $a$ is an integer modulo $n$, to have degree 1. Thus we are concerned with $\hat{\mathfrak{gl}}(1|n)$. For the sake of comparing our results to Kojima’s, the parameter $q$ in this section will be Kojima’s $q$, which will equal $\sqrt{v}$; it is not the same as $q$ in the other sections of this paper.
In Figure 4 we have the Boltzmann weights from Figure 3 divided by \( z_1^n \), compared with the corresponding coefficients from (2.4)-(2.7) of [24] which we have multiplied by the constant \( 1 - q^2 z \).

We will give two ways of modifying the \( R \)-matrix to obtain another \( R \)-matrix that is also a solution of the Yang-Baxter equation. One method only affects the weights in cases III, VII and VIII. The other only affects the weights in cases II, V and VI. After these changes, we will be able to match the Kojima Boltzmann weights up to sign, with \( z = z_2/z_1 \) and \( q^2 = v \). (Then we will have to discuss the sign.)

| I. | This paper | Kojima |
|----|------------|--------|
| \( + \) \( a \) \( + \) \( a \) | \((z_2/z_1)^n - v\) | \( z - q^2\) |
| \( + \) \( a \) \( + \) \( a \) | \( v(1 - (z_2/z_1)^n)\) | \( q(1 - z)\) |

| II. | This paper | Kojima |
|----|------------|--------|
| \( + \) \( a \) \( + \) \( b \) \( + \) \( a \) | \( g(a - b)(1 - (z_2/z_1)^n)\) | \( q(1 - z)\) |
| \( + \) \( b \) \( + \) \( a \) | \( (1 - (z_2/z_1)^n)\) | \( q(1 - z)\) |

| III. | This paper | Kojima |
|----|------------|--------|
| \( + \) \( a \) \( + \) \( a \) \( + \) \( b \) \( + \) \( b \) | \( (1 - v)(z_2/z_1)^{n-a+b}\) if \( a > b\), \( (q^2 - 1)\) if \( a < b\) | \( (1 - v)(z_2/z_1)^{n-a}\) \( z(1 - q^2)\) |
| \( + \) \( b \) \( + \) \( b \) | \( (1 - v)(z_2/z_1)^{-a+b}\) | \( z(1 - q^2)\) |

| IV. | This paper | Kojima |
|----|------------|--------|
| \( 0 \) \( 0 \) \( 0 \) \( 0 \) | \( 1 - v(z_2/z_1)^n\) | \( q^2 z - 1\) |
| \( 0 \) \( 0 \) \( 0 \) | \( (1 - v)(z_2/z_1)^a\) | \( 1 - q^2\) |

**Figure 4.** Left: The \( R \)-matrix from Figure 3 divided by \( z_1^n \). Right: the Boltzmann weights of Kojima’s \( \hat{\mathfrak{gl}}(1|n) \) \( R \)-matrix multiplied by \( 1 - q^2 z \). Just taking the first three cases (and discarding any case with a decorated spin \( -0 \)) gives the \( \hat{\mathfrak{gl}}(n) \) \( R \)-matrix.

For each nonzero complex number \( z \), let \( V(z) \) be an \( n + 1 \) dimensional vector space with basis \( v_\alpha = v_\alpha(z) \), where \( \alpha \) runs through \( n + 1 \) “decorated spins.” These are the ordered pairs \( +a \) with \( 0 \leq a < n \) and \( -0 \).

Previously we interpreted the vertex \( R_{z_1,z_2} \) as a vertex in a graph with certain Boltzmann weights attached to it. We now reinterpret it as an endomorphism of a vector space, as usual in the application of quantum groups to soluble lattice models. If \( \alpha, \beta, \gamma, \delta \) are decorated spins, let \( R_{\alpha,\beta}(z_1, z_2) \) be the Boltzmann weight of vertex \( R_{z_1,z_2} \) with the decorated spins
\(\alpha, \beta, \gamma, \delta\) arranged as follows:

\[
\begin{array}{ccc}
\alpha & \gamma & \beta \\
\beta & \gamma & \delta \\
\alpha & \beta & \delta
\end{array}
\]

We assemble these into an endomorphism \(R_{z_1,z_2}\) of \(V(z_1) \otimes V(z_2)\) as follows:

\[
R_{z_1,z_2}(v_\alpha \otimes v_\beta) = \sum_{\gamma,\delta} R_{\alpha,\beta}^{\gamma,\delta}(z_1, z_2) \, v_\gamma \otimes v_\delta.
\]

**Lemma 2.** We have

\[
(R_{z_2,z_3})_{23}(R_{z_1,z_3})_{13}(R_{z_1,z_2})_{12} = (R_{z_1,z_2})_{12}(R_{z_1,z_3})_{13}(R_{z_2,z_3})_{23}
\]

as endomorphisms of \(V(z_1) \otimes V(z_2) \otimes V(z_3)\).

Here the notation is (as usual in quantum group theory) that \(R_{ij}\) where \(1 \leq i < j \leq 3\) means a matrix \(R\) acting on the \(i, j\) components in \(V(z_1) \otimes V(z_2) \otimes V(z_3)\) with the identity acting on the third component.

**Proof.** Applying the left-hand side of (7) to \(v_\alpha \otimes v_\beta \otimes v_\gamma\) and extracting the coefficient of \(v_\delta \otimes v_\epsilon \otimes v_\phi\); the coefficient is found to be

\[
\sum_{\mu,\nu,\sigma} R_{\alpha,\beta}^{\mu,\sigma}(z_1, z_2) R_{\delta,\nu}^{\gamma,\sigma}(z_1, z_3) R_{\alpha,\beta}^{\epsilon,\delta}(z_2, z_3),
\]

which is the partition of the first system in Theorem 4. The same calculation applied to the right hand side of (7) gives the partition function of the second system in Theorem 4. So they are equal. \(\square\)

We will now describe two operations that one may perform on the Boltzmann weights that do not affect the validity of the Yang-Baxter equation.

**Change of basis.** We may change basis in \(V(z)\). Let \(f(\alpha, z)\) be a function of a decorated spin \(\alpha\) and a complex number \(z\). Let \(u_\alpha = f(\alpha, z)v_\alpha\) for \(v_\alpha \in V(z)\). Then

\[
R_{z_1,z_2}(u_\alpha \otimes u_\beta) = \sum_{\gamma,\delta} \hat{R}_{\alpha,\beta}^{\gamma,\delta}(z_1, z_2) \, u_\gamma \otimes u_\delta
\]

where

\[
\hat{R}_{\alpha,\beta}^{\gamma,\delta} = \frac{f(\alpha, z_1)f(\beta, z_2)}{f(\gamma, z_1)f(\delta, z_2)} \, R_{\alpha,\beta}^{\gamma,\delta}.
\]

Note that replacing \(R\) by \(\hat{R}\) only affects the weights in cases III, VII, and VIII in Figure 4.

Let us translate this into the language of Boltzmann weights. At the moment we are only concerned with Theorem 4. Later in Section 9 we will apply this technique to the first Yang-Baxter equation in Theorem 3. Thus we note the affect on the weights for both types of vertices. Taking the Boltzmann weights from Figures 3 and 2 with \(z_i\) and \(z_j\) as in those figures, the weights of

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
\alpha & \beta & \delta
\end{array}
\]

and

\[
\begin{array}{ccc}
\alpha & \beta \\
\alpha & \beta
\end{array}
\]

are given by

\[
\hat{R}_{\alpha,\beta}^{\gamma,\delta}(z_1, z_2),
\]

where

\[
\hat{R}_{\alpha,\beta}^{\gamma,\delta} = \frac{f(\alpha, z_1)f(\beta, z_2)}{f(\gamma, z_1)f(\delta, z_2)} \, R_{\alpha,\beta}^{\gamma,\delta}.
\]
will respectively be multiplied by

\[
\frac{f(\alpha, z_i) f(\beta, z_j)}{f(\gamma, z_i) f(\delta, z_j)}, \quad \frac{f(\alpha, z_i)}{f(\beta, z_i)}
\]

The first statement is a paraphrase of (8), and the second is checked the same way.

Returning to the comparison with Kojima’s weights, we take

\[
f(\alpha, z) = \begin{cases} 
z^a, & \alpha = +a, \\ 1, & \alpha = -0
\end{cases}
\]

This puts our $R$-matrix into agreement with Kojima in cases III, VII, and VIII but has no effect on the other cases.

**Twisting.** The modification in the previous subsection did not fundamentally change the $R$-matrix. We simply made a change of basis in the vector space on which it acts. In this section we will consider a more fundamental change of the $R$-matrix which does not affect the validity of the Yang-Baxter equation. This procedure is closely related to Drinfeld [15] twisting, a procedure which modifies the comultiplication of a Hopf algebra. It also changes the $R$-matrix when the Hopf algebra is quasitriangular. See Majid [29], Theorem 2.3.4 and Chari and Pressley [12] Section 4.2.E. In Reshetikhin [36] section 3, Drinfeld twisting is used to obtain multiparameter deformations of $U_q(\mathfrak{g}(n))$.

In this paper we will use a simple procedure to introduce Gauss sums into the $R$-matrix. In [5] (which is a sequel to this paper) we show that this procedure is equivalent to Drinfeld twisting when working with the quantum affine algebra $U_q(\mathfrak{gl}(n))$. We can give a simple explanation of this by making use of a feature of the Boltzmann weights in Figure 3. That is, if we have a nonzero weight for the vertex of form

\[
\begin{array}{c}
+ \\
\pm \\
\pm \\
\pm \\
\end{array}
\]

then either $a = c$ and $b = d$ or $a = d$ and $b = c$.

Now let us consider a modification of the Boltzmann weights in case II. We will multiply this weight by a function $\phi(a, b)$ of the decorations $a, b$ that has the following properties. First, it is independent of $z_1$ and $z_2$. Second, $\phi(a, b) \phi(b, a) = 1$.

**Proposition 4.** If $\tilde{R}$ is the $R$-matrix with this modification of the weights in case II, then $\tilde{R}$ also satisfies the same Yang-Baxter equation that $R$ does (Theorem 4).

**Proof.** From the Boltzmann weights in Figure 4 we see that the decorated spins of the two edges to the right of the vertex will have the same decorations as the two edges to the left of the vertex, in some order. Therefore, referring to the proof of Theorem 4 if either partition function is nonzero, the decorated spins $\delta, \epsilon$ and $\phi$ must be the same as $\alpha, \beta$ and $\gamma$ in some order. From this ordering, we may infer the number of case II vertices, and (with an exception to be explained below) it will be the same for both partition functions. That is, if $+a$ and $+b$ occur on the left in the opposite order that they do on the right, then a case II crossing must occur somewhere on a vertex between the four edges. And this will be true on both sides of the equation, so multiplying the case II Boltzmann weight by $\phi(a, b)$ will have the same effect on both sides of the equation.
The exception is that if two weights appear in the same order on the left and right, there may be two case II vertices or none between them. Thus suppose that \( \alpha = \phi = +a \) and \( \beta = \epsilon = +b \). Then in the first partition function in Theorem 5, we may have \( R_{z_1, z_2} \) and \( R_{z_2, z_3} \) either both in case II or both in case III. However if they are both in case II, the factor that we have to multiply is \( \phi(a, b) \phi(b, a) \), which equals 1 by assumption. \( \square \)

We may use this method of twisting in order to remove the \( g(a - b) \) in case II, and replace them by \( q \), since \( g(a - b) g(b - a) = v = q^2 \). We may also adjust the weights in cases V and VI so that in both cases the coefficient agrees with Kojima’s weights.

Using the two methods available to us, we see that we can adjust the Boltzmann weights to agree with Kojima’s, up to sign. We must now discuss the sign. We have agreement for all signs except case IV. As Kojima notes (below his equation (2.12)) his \( R \)-matrix, being supersymmetric, satisfies a graded Yang-Baxter equation. As he points out, an ungraded Yang-Baxter equation may be obtained by changing the sign when all edges are odd-graded. For us, this would mean changing the sign in cases I, II and III. However it works equally well to change the sign in the case where all edges are even-graded, that is, in case IV.

In conclusion, the supersymmetric Yang-Baxter equation in Kojima [24] is equivalent to our Theorem 4.

7. Metaplectic Preliminaries

In these last sections of the paper, we present a more refined version of Theorem 2, an identity between Whittaker functions on metaplectic covers of \( \text{GL}(r) \) and the partition functions of Section 2. Moreover we connect intertwining operators for metaplectic principal series representations to the Boltzmann weights in the \( R \)-matrix given in Figure 3. Before getting to these, we need to introduce the metaplectic group over a local field and its representation theory. In describing this construction, we rely on the prior work and notation of McNamara in [32] and [33], building on the foundational papers of Matsumoto [30] and Kazhdan-Patterson [23]. The reader should refer to these works for complete details; we say just enough here to make the presentation self-contained.

Let \( F \) be a non-archimedean local field with ring of integers \( \mathfrak{o}_F \) and a choice of local uniformizer \( \varpi \). Let \( q \) be the cardinality of the residue field \( \mathfrak{o}_F / \varpi \mathfrak{o}_F \). Let \( n \) be a fixed positive integer. We assume that \( q \equiv 1 \mod 2^n \) so that \( F \) contains the \( 2^n \)-th roots of unity. Let \( \mu_n \) denote the group of \( n \)-th roots of unity in \( F \) and fix an embedding \( \epsilon : \mu_n \to \mathbb{C}^\times \).

Let \( G := \text{GL}(r, F) \) with maximal split torus \( T(F) \). We begin by constructing a metaplectic \( n \)-fold cover of \( G \), denoted \( \tilde{G}^{(n)} \) or just \( \tilde{G} \) when the degree of the cover is understood. Recall that \( \tilde{G} \) is constructed as a central extension of \( G \) by \( \mu_n \):

\[
1 \to \mu_n \to \tilde{G} \to G \to 1.
\]

Thus as a set, \( \tilde{G} \cong G \times \mu_n \), but the multiplication in \( \tilde{G} \) is dictated by a choice of cocycle in \( H^2(G, \mu_n) \). One may construct the cocycle explicitly, as in Kubota [25], Matsumoto [30], Kazhdan and Patterson [23] and Banks-Levi-Sepanski [1], or realize the central extension as coming from an extension of \( K_2(F) \) constructed by Brylinski-Deligne [10]. For the applications at hand, we need only a few facts about the multiplication on \( \tilde{T} := p^{-1}(T) \), the inverse image of a maximal split torus \( T := T(F) \) in \( G \), and the splitting properties of some familiar subgroups.
Our cocycle $\sigma \in H^2(G, \mu_n)$ is chosen so that its restriction to $T(F) \times T(F) \to \mu_n$ is given on any $x, y$ in $T(F)$ explicitly by

$$\sigma(x, y) = \sigma\left(\begin{pmatrix} x_1 & \cdots & x_r \\ \cdots & \cdots & \cdots \\ y_1 & \cdots & y_r \end{pmatrix}, \begin{pmatrix} y_1 & \cdots & y_r \\ \cdots & \cdots & \cdots \\ x_1 & \cdots & x_r \end{pmatrix}\right) = (\det(x), \det(y))_{2n} \prod_{i>j} (x_i, y_j)^{-1},$$

where $(\cdot, \cdot) : F^\times \times F^\times \to \mu_n$ is the $n$-th power Hilbert symbol and $(\cdot, \cdot)_{2n}$ is the $2n$-th power Hilbert symbol. In this formula the Hilbert symbol $(a, b)$ is the unramified $n$-th norm residue symbol. It is defined, for $a, b \in F^\times$ to be the image of the unit

$$\left((-1)^{v(a)v(b)} \frac{a^{v(b)}}{b^{v(a)}}\right)^{\frac{n-1}{2}},$$

under a fixed $n$-th order character of $(\mathfrak{o}/\mathfrak{o}^\sigma)^\times$. General properties of the Hilbert symbol may be found in [33] noting that the symbol there is the inverse of ours; one property we use frequently is that $(x, x) = 1$ for any element $x \in F^\times$, since $F$ contains the $2n$-th roots of unity.

The cocycle $\sigma$ in (10) is the inverse of the one appearing on p. 39 of [23]. A short computation shows that the commutator of any pair of elements $\tilde{x}, \tilde{y}$ in $\hat{T}$ projecting to $x$ and $y$, respectively, in $T(F)$ is

$$[\tilde{x}, \tilde{y}] = \prod_{i=1}^r (x_i, y_i).$$

In particular if $x, y \in F^\times$ and $\lambda, \mu$ are elements of $X_s(T)$, let $\lambda(\tilde{x}), \mu(\tilde{y}) \in \hat{T}$ map to $x^\lambda$ and $y^\mu$, respectively, under the projection $p$ to $T(F)$. Then according to (11),

$$[\lambda(\tilde{x}), \mu(\tilde{y})] = (x, y)^{(\lambda, \mu)},$$

where $(\cdot, \cdot)$ denotes the usual dot product on $X_s(T) \simeq \mathbb{Z}^r$.

In order to make use of results in [33], we must connect this explicit construction to the one used there. In [33] the construction of $\hat{G}$ is obtained by first constructing the extension of $G(F)$ by $K_2(F)$ using a $W$-invariant quadratic form $Q$, and then using a push forward from $K_2(F)$ to the residue field, containing $\mu_n$. The calculation in (12) implies that the bilinear form $B(\lambda, \mu) := Q(\lambda + \mu) - Q(\lambda) - Q(\mu)$ for our extension, as described in Equation (2.1) of [33], is given by the dot product.

Finally, we record that the cocycle splits over any unipotent subgroup and over the maximal compact subgroup $K = G(\mathfrak{o})$ has a splitting in $\hat{G}$. The splitting over the maximal unipotent is clear from the description of the cocycle in [23] and the splitting over $K$ is their Proposition 0.1.2.

Next we recall the construction of unramified principal series on $\hat{G}$. Let $H := C_{\hat{T}}(\tilde{T}(\mathfrak{o}))$ denote the centralizer of $\hat{T} \cap K$ in $\hat{T}$. The subgroup $H$ is abelian. It consists of elements in $\hat{T}$ whose projection to the torus $t=(t_1, \ldots, t_r) \in T \simeq (F^\times)^r$ has $\text{ord}_{\mathfrak{o}_v}(t_j) \equiv 0 \ (n)$ for $j = 1, \ldots, r$. Thus we may identify $\hat{T}/\mu_n\hat{T}(\mathfrak{o})$ and $H/\mu_n\hat{T}(\mathfrak{o})$ with lattices $\Lambda$ and $n\Lambda$, respectively. In particular $\Lambda$ is isomorphic to the cocharacter lattice $X_s(T)$ of $T$. The map $\lambda \mapsto \varpi^\Lambda$ induces an isomorphism from $X_s(T)$ to $T/T(\mathfrak{o})$. Let $s : G \to \hat{G}$ denote the standard
section. By abuse of notation we will also denote by \( \varpi^\lambda \) the image of \( \varpi^\lambda \) under \( s \). Thus coset representatives for \( \bar{T}/H \) may be taken of the form \( \varpi^\lambda \) with \( \lambda \in X_*(T) \).

The principal series is constructed starting from a genuine character \( \chi \) on \( H \), trivial on \( \bar{T} \cap K \); we call these characters “unramified.” First we induce the character \( \chi \) up to \( \bar{T} \), and denote the resulting representation by \( i(\chi) = \text{Ind}_H^{\bar{T}}(\chi) \). Then we perform normalized parabolic induction to obtain a representation of \( \bar{G} \). This is done by first inflating the representation from \( \bar{T} \) to \( \bar{B} \), the inverse image of the standard Borel subgroup \( B \supset T \) in \( G \) and then inducing to obtain \( I(\chi) := \text{Ind}_B^{\bar{G}}(i(\chi)) \). Explicitly \( I(\chi) \) is the space of locally constant functions \( f : \bar{G} \to i(\chi) \) such that

\[
f(bg) = \delta^{1/2} \chi(b) f(g) \quad \text{for all } g \in \bar{G}, b \in \bar{B},
\]

where \( \delta \) denotes the modular quasicharacter. Thus \( I(\chi) \) is a \( \bar{G} \)-module under the action of right translation. Let \( \phi_K \) denote any of the \( i(\chi) \)-valued functions in the one dimensional space of \( K \)-fixed vectors in \( I(\chi) \); our results will be independent of this choice.

8. Intertwining operators and Whittaker functions

For any positive root \( \alpha \in \Phi^+ \), let \( U_\alpha \) denote the one-parameter unipotent subgroup corresponding to the embedding \( \iota_\alpha : SL_2 \to G \). To any element \( w \in W \), the Weyl group, we may define the unipotent subgroup \( U_w \) by

\[
U_w := \prod_{\alpha \in \Phi^+, w(\alpha) \in \Phi^-} U_\alpha.
\]

Then define the intertwining operator \( A_w : I(\chi) \to I(^w\chi) \) by

\[
A_w(f)(g) := \int_{U_w} f(w^{-1}ug) \, du
\]

whenever the above integral is absolutely convergent, and by the usual meromorphic continuation in general. Recall that on any spherical vector \( \phi_K \),

\[
A_w \phi_K^w = c_w(\chi) \phi_K^{\chi w}
\]

where for any simple reflections \( s = s_\alpha \) and any \( w \) such that the length function \( \ell(s_\alpha w) = \ell(w) + 1 \),

\[
c_s(\chi) = \frac{1 - q^{-1} z^{-n_\alpha}}{1 - z^{-n_\alpha}}, \quad \text{and} \quad c_{sw}(\chi) = c_s(\chi^w) c_w(\chi).
\]

Let \( \bar{A}_w \) denote the normalized intertwiner:

\[
\bar{A}_w := c_w(\chi)^{-1} A_w.
\]

Recall that, to any unramified character \( \psi \) on \( U \), the unipotent radical of \( B \), a Whittaker functional on a representation \( \pi, V \) of \( \bar{G} \) is a linear functional \( W^\pi \) for which

\[
W^\pi(\pi(u)v) = \psi(u)W^\pi(v) \quad \text{for all } u \in U \text{ and } v \in V.
\]

As stated in Section 6 of [33], the dimension of the space of Whittaker functionals for the principal series \( I(\chi) \) is equal to the cardinality of \( \bar{T}/H \). Let \( W^\chi \) denote the \( i(\chi) \)-valued Whittaker functional on \( I(\chi) \) defined by

\[
W^\chi(\phi) := \int_{\bar{T}/H} \phi(uw_0) \psi(u) \, du : I(\chi) \to i(\chi).
\]
Then there is an isomorphism between the linear dual $i(\chi)^*$ and Whittaker functionals to $\mathbb{C}$ on $I(\chi)$ given by

$$\mathcal{L} \mapsto \mathcal{L}(W^\chi), \quad \text{for } \mathcal{L} \text{ in } i(\chi)^*. \tag{15}$$

Let us describe a particularly nice basis of $i(\chi)^*$ for the computation of the spherical function under the Whittaker functional. Let $v_0 := \phi_K(1)$, an element of $i(\chi)$. Let $\pi_\chi$ denote the representation of $\tilde{T}$ on $i(\chi)$. Then to any set of coset representatives $b$ for $\tilde{T}/H$, the set $\{\pi_\chi(b)v_0\}$ is a basis for the representation on $i(\chi)$. Let $\{\mathcal{L}_b^{(\chi)}\}$ denote the corresponding dual basis of $i(\chi)^*$. More explicitly, given any weight $\mu$, we write $\mu = \beta + \gamma$ where $\beta$ is one of the chosen coset representatives in $\Lambda/n\Lambda$ and $\gamma \in n\Lambda$, the lattice for our isotropic subgroup. Then writing our $\tilde{T}/H$ coset representative $b = \varpi^\nu$, we define

$$\mathcal{L}_b^{(\chi)}(\pi_\chi(\varpi^\mu)\phi_K(1)) = \mathcal{L}_b^{(\chi)}(\pi_\chi(\varpi^{\beta+\gamma})\phi_K(1)) = \chi(\varpi^\gamma)\mathcal{L}_b^{(\chi)}(\pi_\chi(\varpi^\beta)\phi_K(1))$$

$$= \chi(\varpi^\gamma) \cdot \begin{cases} 1 & \beta = \nu, \\ 0 & \text{else.} \end{cases} \tag{16}$$

Thus we obtain a corresponding basis of the space of Whittaker functionals on $I(\chi)$, denoted $W_b^\chi$, according to the isomorphism (15).

We wish to investigate the functions $W_b^\chi(\pi(\varpi^\lambda)\phi_K)$ for dominant weights $\lambda$. Indeed, the Whittaker functions $g \mapsto W_b^\chi(\pi(g)\phi_K)$ are determined by their values on $T(F)/T(\mathfrak{o})$, and it is easily shown that these Whittaker functions vanish off of dominant weights. It is perhaps most natural to view $W_b^\chi(\pi(\varpi^\lambda)\phi_K)$ as a function of the Langlands parameters $z = (z_1, \ldots, z_r)$ defining the character $\chi =: \chi_z$ on the subgroup $H$. Thinking in this way,

$$z \mapsto W_b^\chi(\pi(\varpi^\lambda)\phi_K)$$

takes values in $\mathcal{O}(\tilde{T}) \simeq \text{Frac}(\mathbb{C}[\Lambda])$, the field of fractions of the group algebra of the cocharacter lattice. Similarly, the functionals $\mathcal{L}_b$ in $i(\chi_z)^*$ may be viewed as functions of $z$ taking values in $\mathcal{O}(\tilde{T})$. We highlight that, from this point of view, the functional $\mathcal{L}_b$ (and hence the corresponding Whittaker functional $W_b$) takes values on the subfield $\text{Frac}(\mathbb{C}[\Lambda])$, where $\Lambda$ is the lattice corresponding to $H/\mu_0\tilde{T}(\mathfrak{o})$.

One approach to studying the resulting Whittaker function in (17), going back to Casselman and Shalika in the linear group case and used by Kazhdan and Patterson for metaplectic covers of $\text{GL}(r)$, is to exploit the fact that $W_{\nu'}^\chi \circ \overline{A}_w$ is an $i(\chi)$ valued Whittaker functional for $I(\chi)$. Composing with the functional $\mathcal{L}_a$ for some coset representative $a \in \tilde{T}/H$, we obtain an expansion:

$$W_a^\chi \circ \overline{A}_w = \sum_{b \in \tilde{T}/H} \tau_{a,b} W_b^\chi \tag{18}$$

for some rational functions $\tau_{a,b} := \tau_{\tilde{w},b}^{(w)}(z)$ in $\text{Frac}(\mathbb{C}[\Lambda])$. It suffices to understand these structure constants on simple reflections $w = s_\alpha$. These were computed for metaplectic covers of $\text{GL}(r)$ by Kazhdan and Patterson, and we discuss their explicit form in the following result. A nice alternate proof of this result is given as Theorem 13.1 in [33], and our notation more closely parallels this latter source. We take our coset representatives $a, b$ for $\tilde{T}/H$ of the form $\varpi^\mu$ with $\mu \in X_s(T)$ and use the more succinct notation $\tau_{\mu,\nu}(z)$ for their associated structure constants.
Proposition 5 (Kazhdan-Patterson, [23], Lemma I.3.3). The structure constants \( \tau_{\nu,\mu} := \tau_{(w)} \) for coset representatives \((a,b) = (w^\nu, w^\mu)\) appearing in (18) with \( w = s_\alpha, \) a simple reflection, can be broken into two pieces:

\[
\tau_{\nu,\mu} = \tau_{1,\nu,\mu} + \tau_{2,\nu,\mu}
\]

where \( \tau_1 \) vanishes unless \( \nu \sim \mu \pmod{\Lambda} \) and \( \tau_2 \) vanishes unless \( \nu \sim s(\mu) + \alpha \pmod{\Lambda} \). Moreover:

(19) \[
\tau_{1,\nu,\mu} = (1 - q^{-1}) \frac{z^{-n\lfloor \langle \alpha, \mu \rangle \rangle_n}}{1 - q^{-1}z^{-n\alpha}}
\]

where \( \lfloor x \rfloor \) denotes the smallest integer at least \( x \). And

(20) \[
\tau_{2,s(\mu)+\alpha,\mu} = g(\langle \alpha, \mu - \rho \rangle) \frac{1 - z^{-n\alpha}}{1 - q^{-1}z^{-n\alpha}}.
\]

Remark. In order to translate from the notation of [33], we choose \( Q(\alpha) = 1 \) on simple roots \( \alpha \) which then implies \( n_\alpha := n / \gcd(n, Q(\alpha)) = n \). Our cocycle has been chosen so that \( B(\alpha, \mu) = \langle \alpha, \mu \rangle \). Moreover, our \( n \)-th order Gauss sum \( g \) is denoted \( g \) in [33]. Finally, make the substitution \( x_\alpha \mapsto z^{-\alpha} \) in Theorem 13.1 of [33] to obtain the above formulas.

Remark. We caution the reader that \( \tau_1 \) is not independent of the choice of representative \( a = w^\nu \) in the quotient \( \Lambda / n \Lambda \). The function \( \tau_2 \) does satisfy this property, since replacing \( \nu \) by \( \nu + \gamma \) with \( \gamma \in \Lambda \) increases the inner product by a multiple of \( n = n_\alpha \) which leaves the evaluation of the Gauss sum \( g \) fixed (as it depends only on the congruence class of the argument mod \( n \)). Note however, that the formula for \( \tau_2 \) is only applicable if both \( \mu \) and its “dot” action \( s \cdot \mu = s(\mu) + \alpha \) are coset representatives.

9. INTERTWINING INTEGRALS AS R-MATRICES

Now specialize to the case \( G = \text{GL}(r) \). In this section, we prove relations between the spherical Whittaker functions \( W_{\chi}^\lambda(\pi(\varpi^\lambda) \phi_K) \) of the previous section and certain partition functions of metaplectic ice with fixed boundary charges. Then we show that the identity (18) is likewise expressible in terms of braided ice models, where the structure constants \( \tau_{a,b} \) are given by corresponding \( R \)-vertices matching those in Figure 3 up to a simple normalization. We continue to assume that the set of coset representatives \( b \in \tilde{T}/H \) are of the form \( b = \varpi^\nu \) with \( \nu \in X_\alpha(T) \), considered as elements of \( \tilde{G} \) under the split section discussed above.

In order to compare Whittaker functions and metaplectic ice, we use a modified system of Boltzmann weights from those in Figures 2 and 3. The weights we need now are given in Figures 5 and 6. We remind the reader that in Section 2, we stressed that it suffices to consider only the decoration 0 associated to a \( - \) spin, and the table below reflects this assumption. So we omit other \( -a \) decorated spins with \( a \neq 0 \).

**Proposition 6.** The Yang-Baxter equation is satisfied with the weights in Figures 3 and 6.

**Proof.** To obtain these from the weights in Figure 2 and 3, we make use of the change of basis method described in Section 6. We take the function \( f(\alpha, z) \) to equal

\[
\begin{cases}
  z^a & \text{if } \alpha = +a, \ 0 \leq a < n, \\
  1 & \text{if } \alpha = -0.
\end{cases}
\]
In Figure 5 we further divide each weight by \( z_i \), and in Figure 6 we divide by \( z_1^n - vz_2^n \). These multiplications apply to all weights, so we may do this at our convenience without affecting the Yang-Baxter equation.

We only have to check that this procedure gives the advertised weights. Let us just check this for the \( a_1 \) case in Figure 5. The corresponding weight in Figure 2 is 1. If \( a \neq n - 1 \), then according to (9), we multiply by \( z_i^{a+1} \) and divide by \( z_i^a \). However we are additionally dividing by \( z_i \), so we obtain 1, which is consistent with Figure 5 since \( \delta(a + 1) = 0 \). On the other hand, if \( a = n - 1 \), then \( a + 1 \) modulo \( n \) is zero, so we multiply by 1 and divide by \( z_i^n = z_i^{n-1} \). Additionally dividing by \( n \), we obtain \( z_i^{-n} \), which again equals \( z_i^{-n\delta(a+1)} \). We leave the remaining cases to the reader. □

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
a_1 & a_2 & b_1 & b_2 & c_1 & c_2 \\
\hline
\begin{array}{c}
\overset{a_1 \to a}{\text{a} + 1} \\
\text{1} \\
\text{a} \\
\text{z}^{-n\delta(a+1)}
\end{array} & \begin{array}{c}
\overset{(a+1)z^{-n\delta(a+1)}}{0} \\
\text{0} \\
\text{0} \\
\text{1}
\end{array} & \begin{array}{c}
\overset{b_1 \to a}{\text{a} + 1} \\
\text{0} \\
\text{a} \\
\text{g(a)z}^{-n\delta(a+1)}
\end{array} & \begin{array}{c}
\overset{b_2 \to a}{\text{a} + 1} \\
\text{0} \\
\text{0} \\
\text{1 - v}
\end{array} & \begin{array}{c}
\overset{c_1 \to a}{\text{c} + 1} \\
\text{0} \\
\text{0} \\
\text{z}^{-n\delta(1)}
\end{array} & \begin{array}{c}
\overset{c_2 \to a}{\text{c} + 1} \\
\text{0} \\
\text{0} \\
\text{0}
\end{array} \\
\hline
\end{array}
\]

**Figure 5.** Modified Boltzmann weights

Let \( e_i \) denote the weight \((0, \ldots, 0, 1, 0, \ldots, 0)\) with a 1 in the \( i \)-th position under the isomorphism \( X_*(T) \cong \mathbb{Z}^r \). Because we want to present very precise equalities between Whittaker functions and partition functions, we must choose specific representatives for the space of Whittaker functionals indexed by cosets in \( \Lambda/n\Lambda \). We choose a basis for the space of Whittaker functionals consisting of \( \pi_a \) with \( a = \pi^\nu \) and \( \nu \in \Lambda \) such that we may write

\[
\nu - \rho = c_1 e_1 + \cdots + c_r e_r, \quad \text{with } c_i \in \{0, \ldots, n - 1\} \text{ for all } i.
\]

**Proposition 7.** Fix a dominant weight \( \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \) and any coset representative \( a = \pi^\nu \) with \( \nu - \rho = c_1 e_1 + \cdots + c_r e_r \) such that each \( c_i \in \{0, \ldots, n - 1\} \). Then using Boltzmann weights in Figure 2

\[
\pi_\lambda(z) \overset{(21)}{=} z^{\nu - \rho} \pi_\lambda(e^\lambda \phi_K) = z_1^{\lambda_1 + r - c_1} \cdots z_r^{\lambda_r + r - c_r} Z_\lambda(z; c_1, \ldots, c_r).
\]

**Proof.** The result follows by refining theorems from the papers [7] and [31]. Indeed if we sum over all possible left-charge configurations \( c = (c_1, \ldots, c_r) \) with \( c_i \in \mathbb{Z}/n\mathbb{Z} \), then

\[
\sum_{c = (c_1, \ldots, c_r) \in \{0, n\}^r} z_1^{\lambda_1 + r - c_1} \cdots z_r^{\lambda_r + r - c_r} Z_\lambda(z; c_1, \ldots, c_r)
\]

is equal to the partition function called \( Z(\mathfrak{g}^\Gamma) := Z(\mathfrak{g}^\Gamma_\lambda) \) in Section 3.4 of [7]. (Note the partition function in [7] is made using the Boltzmann weights from Figure 2 which are normalized slightly differently than the modified weights above and account for the monomial in \( z \) in \( \langle 22 \rangle \).) In Theorem 4 of [7], we explain that

\[
Z(\mathfrak{g}^\Gamma) = \pi_\lambda = z^{\nu - \rho} W^\circ(\pi^\lambda) \quad \text{where} \quad W^\circ(\pi^\lambda) := \mathcal{L}(W^\lambda(\pi_\lambda(e^\lambda \phi_K)))
\]

and \( \mathcal{L}^\circ : i(\chi) \to \mathbb{C} \) is the linear functional defined in Proposition 5.2 of [31]. More explicitly, \( \mathcal{L}^\circ \) is defined as the unique functional \( \mathcal{L} \) such that, taking coset representatives of the form
$\varpi^\nu$ for $\tilde{T}/H$, where $H$ is the maximal abelian subgroup of $\tilde{T}$:
\[ \mathcal{L}(\pi_{\chi}(\varpi^\nu)\phi_K) = \chi(\varpi^\nu). \]

Here we have extended the definition of $\chi := \chi_{\varpi}$ on $H$ to elements of the form $\varpi^\nu$ by
\[ \chi(\varpi^\nu) = z^\nu. \]

(Choosing the functional compatible with the choice of spherical vector $\phi_K$ in this way gives a result independent of the choice of spherical vector in the one-dimensional space of them.)

Recall from Lemma 1 that the partition function $Z_\lambda(z; \lambda_1 + r - c_1, \ldots, \lambda_1 + r - c_r)$ made from the weights in Figure 2 is the unique summand in (22) taking values in $\mathbb{C}[\Lambda]$ that are restricted to the the coset $z^\nu \mathbb{C}[n\Lambda]$. Thus using the modified weights of Figure 5, each summand
\[ z_1^{\lambda_1 + r - c_1} \cdots z_r^{\lambda_1 + r - c_r} Z_\lambda(z; c_1, \ldots, c_r) \]
appearing in (22) gives precisely the contribution to $W^\varphi(\varpi^\lambda)$ taking values on the coset $z_1^{c_1} z_2^{c_2} \cdots z_r^{c_r} \mathbb{C}[n\Lambda]$.

To finish, it remains to relate this functional $\mathcal{L}^\varphi$ to the functionals $\mathcal{L}_a$ as defined in (16) with $\mathcal{L}_a \circ \mathcal{W} =: W_a$ defined above. From the definition of $\mathcal{L}_a$ with $a = \varpi^\nu$, we see
\[ \mathcal{L}^\varphi = \sum_{a \in \tilde{T}/H} \chi(a) \mathcal{L}_a = \sum_{a \in \tilde{T}/H} z^\nu \mathcal{L}_a, \]
so we immediately conclude that
\[ (24) \quad W^\varphi(\varpi^\lambda) = \sum_{a = \varpi^\nu \in \tilde{T}/H} z^\nu W_a(\pi_{\chi}(\varpi^\lambda)\phi_K). \]

Since each of the functionals $\mathcal{L}_a$ take values on $\mathbb{C}[n\Lambda]$, the summand $z^\nu W_a(\pi_{\chi}(\varpi^\lambda)\phi_K)$ is the unique summand in (24) taking values on the coset $z^\nu \mathbb{C}[n\Lambda]$. Combining this with (23) gives the result.

Now that we have connected the Whittaker models $W_b$ to sums of partition functions on metaplectic ice with fixed boundary charge, we make a connection between braided ice (using the $R$-vertices for metaplectic ice discussed above) and the identity of Whittaker functionals in (18). It says that, with the right choice of normalization on the $R$-matrix, the structure constants $\tau_{a,b}^{(s)}$ are given by a subset of the $R$-vertices. More precisely, the Boltzmann weights associated to $R$-vertices needed for this matching are given in the following figure. They are obtained by dividing each of the entries in Figure 3 by the value of the $a_2$ $R$-vertex. We refer to the resulting system of weights as $\bar{R}$, which have been expressed in terms of $z^{-a}$ in the table. (For the application to metaplectic Whittaker functionals, set $v = q^{-1}$ as usual.)

In the table, we’re assuming that charges depicted as $a$ and $b$ are in distinct residue classes mod $n$. In the case that $a \equiv b \mod n$, the reader may easily check that the weight of the first entry in the top row above is equal to the sum of the weights of the second and third entries in the top row, analogous to the identity (3) earlier.

**Proposition 8.** Let $\mu \in X_*(T) \simeq \mathbb{C}[\Lambda]$ with $\mu - \rho = c_1 e_1 + \cdots + c_r e_r$ for some integers $c_i \in [0,n)$. Let $\tau_{\nu,\mu}(z) := \tau_{\nu,\mu}^{(s)}(z)$ as in Proposition 3. Let $w_t$ be the weights for $\bar{R}$ in Figure 6.
\begin{align*}
\tau^1_{\mu,\mu}(z) &= \text{wt} \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a}
\end{array}
\end{array}
\end{array} \right) \quad \text{and} \quad \tau^2_{s(\mu)+\alpha_i,\mu}(z) = \text{wt} \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a}
\end{array}
\end{array} \right) \\text{b}
\end{array} \right) \\
\text{If } a \equiv b \mod n, \text{ then} \\
\tau^1_{\mu,\mu}(z) + \tau^2_{s(\mu)+\alpha_i,\mu}(z) &= \text{wt} \left( \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{a}
\end{array}
\end{array} \right) \\text{a} \\text{a} \\text{b}
\end{array} \right)
\end{align*}

\textbf{Proof.} Suppose that } a \not\equiv b \mod n. \text{ Then we must compare the weights in the above table with the results of Proposition}[5]\text{. Recall from}[19]\text{ that with } \alpha = \alpha_i
\begin{align*}
\tau^1_{\mu,\mu} &= (1 - q^{-1}) \frac{z^{-n[\langle \alpha, \mu \rangle]_{n}}}{1 - q^{-1}z^{-n}} = \frac{1 - q^{-1}}{1 - q^{-1}z^{-n}} \begin{cases}
\frac{z^{-n\alpha}}{1 - q^{-1}z^{-n\alpha}} & \text{if } c_i - c_{i+1} > 0 \\
1 & \text{if } c_i - c_{i+1} \leq 0.
\end{cases}
\end{align*}

Since } c_i - c_{i+1} \equiv a - b, \text{ the above matches the weight of the } \hat{R}\text{-vertex in the statement of the lemma according to Figure}[3]\text{. From}[20],
\begin{align*}
\tau^2_{s(\mu)+\alpha,\mu} &= g(\langle \alpha, \mu - \rho \rangle) \frac{1 - z^{-n\alpha}}{1 - q^{-1}z^{-n}} = g(a - b) \frac{1 - z^{-n\alpha}}{1 - q^{-1}z^{-n}}
\end{align*}

since } \langle \alpha, \mu - \rho \rangle = c_i - c_{i+1} \equiv a - b \mod n.
If instead \( a \equiv b \mod n \), then the result follows from the identity of \( \hat{R} \) vertices analogous to (3):

\[
\text{wt} \left( \begin{array}{ccc}
& a & \\
& + & + \\
a & + & \\
\end{array} \right) = \text{wt} \left( \begin{array}{ccc}
& a & b \\
& + & + \\
b & + & a \\
\end{array} \right) + \text{wt} \left( \begin{array}{ccc}
& a & a \\
& + & + \\
a & + & b \\
\end{array} \right)_{a=b} + \text{wt} \left( \begin{array}{ccc}
& a & a \\
& + & + \\
a & + & b \\
\end{array} \right)_{a=b+0}.
\]

By \( a = b + 0 \) we mean that we take the value in the third case of Figure 6 when \( a > b \). So remembering that \( g(0) = -v \) equation (25) means

\[
-\frac{v + z^{-n_\alpha}}{1 - vz^{-n_\alpha}} = (-v) \frac{1 - z^{-n_\alpha}}{1 - vz^{-n_\alpha}} + \frac{(1 - v)z^{-n_\alpha}}{1 - vz^{-n_\alpha}}.
\]

\[\square\]

**Proposition 9.** Let \( \omega^\nu \) be a representative in \( \hat{T}/H \) with \( \nu - \rho = c_1 e_1 + \cdots + c_r e_r \) with \( c_i \in [0, n) \). Then for a simple reflection \( s_i \), the fundamental scattering matrix identity

\[
W_{\omega^\nu} \circ A_{s_i}(\pi(\omega^\lambda)\phi_K) = \tau_{\nu,\nu}^{1} W_{\omega^\nu}(\pi(\omega^\lambda)\phi_K) + \tau_{\nu,s_i}^{2} W_{s_i,\omega^\nu}(\pi(\omega^\lambda)\phi_K)
\]

is equivalent to the following identity of partition functions.

\[
(27) \quad Z_\lambda \left( \begin{array}{ccc}
c_{i-1} & + & z_{i-1} \\
c_i & + & z_{i+1} \\
c_{i+1} & + & z_i \\
c_{i+2} & + & z_{i+2} \\
\vdots & & \\
\end{array} \right) = Z_\lambda \left( \begin{array}{ccc}
c_{i-1} & + & z_{i-1} \\
c_i & + & z_i \\
c_{i+1} & + & z_{i+1} \\
c_{i+2} & + & z_{i+2} \\
\vdots & & \\
\end{array} \right)
\]

\[= \text{wt} \left( \begin{array}{ccc}
c_i & + & c_i \\
c_{i+1} & + & c_{i+1} \\
\end{array} \right) Z_\lambda(z; c) + \text{wt} \left( \begin{array}{ccc}
c_i & + & c_i \\
c_{i+1} & + & c_{i+1} \\
\end{array} \right) Z_\lambda(z; s_i(c)) \]

where the Boltzmann weights are given by Figures 5 and 6 and the weights \( \text{wt} \) are understood to be the right-hand side of (25) in the special case \( c_i = c_{i+1} \).

The identity (27) is just a consequence of the Yang-Baxter equation for our choice of Boltzmann weights. Indeed, by repeated use of the Yang-Baxter equation, we may push the \( R \)-vertex appearing in the partition function above rightward until it reaches the right-hand side. On the right-hand side its only admissible configuration is an \( R \)-vertex of type \( a_2 \) with modified Boltzmann weight equal to 1 according to Figure 3. Thus it matches the partition
function on the left-hand side of (27). Note that the spectral parameters $z_i$ and $z_{i+1}$ are reversed in the two models, as a consequence of these Yang-Baxter moves.

**Proof.** We apply the previous two Propositions to show that each of the Whittaker functions and structure constants $\tau$ appearing in (26) is equal to the corresponding partition function or Boltzmann weight for metaplectic ice in (27).

For the left-hand side of (26), recall that the normalized intertwiner $\hat{A}_{s_i}$ takes the spherical vector $\phi_K$ in $I(\chi)$ to the spherical vector $\phi_K$ in $I(s_i\chi)$ and commutes with the action by right-translation by $\pi(\omega \lambda)$. Thus,

$$W_{\omega \lambda}^{s_i \chi} \circ \hat{A}_{s_i}(\pi(\omega \lambda)\phi_K^X) = W_{\omega \lambda}^{s_i \chi}(\pi(\omega \lambda)\phi_K^X)$$

and this Whittaker function is given by the partition function on the left-hand side above, according to Proposition 7 (up to a power of $z$ that will cancel with a matching power of $z$ in the identity of partition functions).

The coefficients $\tau^1$ and $\tau^2$ appearing on the right-hand side of (26) match the weights of the two $R$-vertices appearing on the right-hand side of (27) by Proposition 8. The matching of Whittaker functions and partition functions on the respective right-hand sides (up to a constant in $z$) is again given by Proposition 7. To apply the Proposition, we must compare the decorations $c$ defining the left-hand boundary of $Z_{s_i}$’s and the representatives $\nu$ and $s_i \cdot \nu$ in the Whittaker functions. Note that we are using the “dot” action defined by

$$s_i \cdot \nu = s_i(\nu) + \alpha_i = s_i(\nu - \rho) + \rho.$$

Thus

$$s_i \cdot (\nu - \rho) = s_i(\nu - \rho) = s_i(c_1 e_1 + \cdots + c_r e_r),$$

where $s_i$ acts by swapping $e_i$ and $e_{i+1}$, reversing the roles of the decorations $e_i$ and $e_{i+1}$ in the partition function. \hfill \square

**Proof of Theorem 1.** For any choice of $z = (z_1, \ldots, z_r)$, define the map $\theta_z$ in the statement of the Theorem according to Proposition 7 – each $W_b^\chi$ is mapped to the collection of left-edge boundary decorations whose partition function matches $W_b$. That is, write $a = \omega \nu$ with $\nu - \rho = a_1 e_1 + \cdots + a_r e_r$, so $\theta_z$ maps $W_b^\chi(\phi_K)$ to $v_{+a_1}(z_1) \otimes \cdots \otimes v_{+a_r}(z_r)$. Then the commutativity of the diagram is immediate from Proposition 9 \square

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