ON DEGREE OF APPROXIMATION BY PRODUCT MEANS OF CONJUGATE SERIES OF A FOURIER SERIES

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Abstract: In this paper a theorem on degree of approximation of a function $f \in \text{Lip}(\alpha , r)$ by product summability $(E, q)(\bar{N}, p_n)$ of conjugate series of Fourier series associated with $f$ has been established.

1. Introduction

Let $\Sigma a_n$be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that

$$\sum_{v=0}^{n} p_v \to \infty, \quad \text{as} \quad n \to \infty, \quad (p_{-i} = p_{-i} = 0, i \geq 0).$$

(1.1)

The sequence –to–sequence transformation

$$t_n = \frac{1}{p_n} \sum_{v=0}^{n} p_v s_v$$

(1.2)

defines the sequence $\{t_n\}$ of the $(\bar{N}, p_n)$ -mean of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$. If

$$t_n \to s, \text{ as } n \to \infty$$

(1.3)

then the series $\Sigma a_n$ is said to be $(\bar{N}, p_n)$ summable to $s$.

The conditions for regularity of $(\bar{N}, p_n)$-summability are easily seen to be[1]

$$\begin{align*}
(i) & \quad P_n \to \infty, \text{as } n \to \infty. \\
(ii) & \quad \sum_{i=0}^{n} p_i \leq C|P_n|, \text{as } n \to \infty.
\end{align*}$$

(1.4)

The sequence –to–sequence transformation, [1]

$$T_n = \frac{1}{(1+q)^n} \sum_{v=0}^{n} q^{n-v} s_v$$

(1.5)

defines the sequence $\{T_n\}$ of the $(E, q)$ mean of the sequence $\{s_n\}$. If

$$T_n \to s, \text{ as } n \to \infty,$$

(1.6)

then the series $\Sigma a_n$ is said to be $(E, q)$ summable to $s$.
Clearly $(E, q)$ method is regular. Further, the $(E, q)$ transform of the $(N, p_n)$ transform of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{-k} t_k$$

$$= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\ell=0}^{k} p_\ell s_\ell \right\}$$ (1.7)

If

$$\tau_n \to s, \quad \text{as} \quad n \to \infty,$$ (1.8)

then $\sum a_n$ is said to be $(E, q)(N, p_n)$-summable to $s$.

Let $f(t)$ be a periodic function with period $2\pi$ and $L$-integrable over $(-\pi, \pi)$. The Fourier series associated with $f$ at any point $x$ is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) \equiv \sum_{n=0}^{\infty} A_n(x)$$ (1.9)

and the conjugate series of the Fourier series (1.9) is

$$\sum_{n=1}^{\infty} \left( b_n \cos nx - a_n \sin nx \right) = \sum_{n=1}^{\infty} B_n(x)$$ (1.10)

Let $s_n(f; x)$ be the $n$-th partial sum of (1.10). The $L^\infty$-norm of a function $f: \mathbb{R} \to \mathbb{R}$ is defined by

$$\|f\|_\infty = \sup \{ f(x) : x \in \mathbb{R} \}$$ (1.11)

and the $L^\nu$-norm is defined by

$$\|f\|_\nu = \left( \int_0^{2\pi} |f(x)|^\nu dx \right)^{\frac{1}{\nu}}, \quad \nu \geq 1$$ (1.12)

The degree of approximation of a function $f: \mathbb{R} \to \mathbb{R}$ by a trigonometric polynomial $P_n(x)$ of degree $n$ under norm $\| \cdot \|$ is defined by [5].

$$\|P_n - f\|_\infty = \sup \{|P_n(x) - f(x)| : x \in \mathbb{R}\}$$ (1.13)

and the degree of approximation $E_n(f)$ of a function $f \in L_\nu$ is given by

$$E_n(f) = \min_{P_n} \|P_n - f\|_\nu$$ (1.14)

A function $f$ is said to satisfy Lipschitz condition (here after we write $f \in Lipa$) if

$$|f(x + t) - f(x)| = O\left(|t|^\alpha\right), 0 < \alpha \leq 1$$ (1.15)

and $f(x) \in Lip(a, r)$, for $0 \leq x \leq 2\pi$, if

$$\left( \int_0^{2\pi} |f(x + t) - f(x)|^r dx \right)^{\frac{1}{r}} = O\left(|t|^\alpha\right), 0 < \alpha \leq 1, r \geq 1, t > 0$$ (1.16)
We use the following notation throughout this paper:
\[
\psi(t) = \frac{1}{2} \{ f(x + t) - f(x - t) \},
\]
(1.17)

and
\[
\overline{K_n}(t) = \frac{1}{\pi (1 + q)^v} \sum_{k=0}^{n} \frac{(n)}{k} q^{n-k} \left\{ \frac{1}{p_k} \sum_{\nu=0}^{k} p_\nu \frac{\cos t/2 - \cos (\nu + 1/2) t}{\sin \frac{t}{2}} \right\}
\]
(1.18)

Further, the method \((E, q)(N, p_n)\) is assumed to be regular.

2. Known Theorem

Dealing with the degree of approximation by the product \((E, q)(C, 1)\) -mean of Fourier series, Nigam et al [3] proved the following theorem:

**Theorem 2.1.** If a function \(f\) is \(2\pi\) - periodic and belonging to class \(Lipa\), then its degree of approximation by \((E, q)(C, 1)\) summability mean on its Fourier series \(\sum_{n=0}^{\infty} A_n(t)\) is given by
\[
\| E_n^q C_n - f \|_\infty = O\left( \frac{1}{(n+1)^{\alpha}} \right), 0 < \alpha < 1
\]
where \(E_n^q C_n\) represents the \((E, q)\) transform of \((C, 1)\) transform of \(s_n(f; x)\).

Recently, Misra et al [2] proved the following theorem using \((E, q)(N, p_n)\) mean of conjugate series of the Fourier series:

**Theorem 2.2.** If \(f\) be \(2\pi\) - Periodic function of class \(Lipa\), then degree of approximation by the product \((E, q)(N, p_n)\) summability means of the conjugate series (1.10) of Fourier series (1.9) is given
\[
\| \tau_n - f \|_\infty = O\left( \frac{1}{(n+1)^{\alpha}} \right), 0 < \alpha < 1
\]
where \(\tau_n\) is as defined in (1.7).  

3. Main theorem

In this paper, we have proved a theorem on degree of approximation by the product mean \((E, q)(N, p)\) of conjugate series of Fourier series of a function of class \(Lip(a, r)\). We prove:

**Theorem 3.1.** If \(f\) be \(2\pi\) - Periodic function of class \(Lipa\), then degree of approximation by the product \((E, q)(N, p_n)\) summability means of the conjugate series of Fourier series (1.10) is given
\[
\| \tau_n - f \|_\infty = O\left( \frac{1}{(n+1)^{\alpha - 1/r}} \right), 0 < \alpha < 1
\]
by
\[
\| \tau_n - f \|_\infty = O\left( \frac{1}{n+1} \right), 0 < \alpha < 1
\]
where \(\tau\) is as defined in (1.7).

4. Required Lemmas

We require the following Lemmas to prove the theorem.

**Lemma 4.1**
\[
| K_n(t) | = O(n) \quad 0 \leq t \leq \frac{1}{n+1}
\]
Proof.

For $0 \leq t \leq \frac{1}{n+1}$ we have $\sin nt \leq n \sin t$ then

$$\left| K_n(t) \right| = \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} \cos \frac{t}{2} - \cos \left( \nu + \frac{1}{2} \right) \frac{t}{2} \sin \frac{t}{2} \right\} \right|$$

$$\leq \frac{1}{\pi (1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} \left( \frac{\cos \frac{t}{2} - \cos \nu \frac{t}{2} \cos \frac{t}{2} + \sin \nu \frac{t}{2} \sin \frac{t}{2}}{\sin \frac{t}{2}} \right) \right\}$$

$$\leq \frac{1}{\pi (1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} \left( 2 \sin \nu \frac{t}{2} \sin \frac{t}{2} + \nu \sin \frac{t}{2} \right) \right\}$$

$$\leq \frac{1}{\pi (1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} (O(\nu) + O(\nu)) \right\}$$

$$\leq \frac{1}{\pi (1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} O(k) \right\}$$

$$= O(n)$$

This proves the lemma.

Lemma 4.2

$$\left| K_n(t) \right| = O\left( \frac{1}{t} \right), \text{ for } \frac{1}{n+1} \leq t \leq \pi$$

Proof.

For $\frac{1}{n+1} \leq t \leq \pi$, by Jordan’s lemma, we have $\sin \left( \frac{t}{2} \right) \geq \frac{t}{\pi}$

Then

$$\left| K_n(t) \right| = \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} \cos \frac{t}{2} - \cos \left( \nu + \frac{1}{2} \right) \frac{t}{2} \sin \frac{t}{2} \right\} \right|$$

$$= \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} \left( \frac{\cos \frac{t}{2} - \cos \nu \frac{t}{2} \cos \frac{t}{2} + \sin \nu \frac{t}{2} \sin \frac{t}{2}}{\sin \frac{t}{2}} \right) \right\} \right|$$
This proves the lemma.

5. Proof of theorem 3.1

Using Riemann–Lebesgue theorem, we have for the n-th partial sum $s_n (f ; x)$ of the conjugate Fourier series (1.10) of $f (x)$, following Titchmarsh [4]

$$\overline{s_n (f ; x) - f (x)} = \frac{2}{\pi} \int_0^\pi \psi(t) \overline{K_n} \, dt$$

the $(N, pn)$ transform of $s_n (f ; x)$ using (1.2) is given by

$$t_n - f(x) = \frac{2}{\pi P_n} \int_0^\pi \psi(t) \sum_{k=0}^n \sum_{k=0}^P \left( \frac{\cos \frac{t}{2} - \sin \left( \frac{n+1}{2} \right) \frac{t}{2}}{2 \sin \left( \frac{t}{2} \right)} \right) \, dt$$

denoting the $(E, q)(N, pn)$ transform of $S_n (f ; x)$ by $\tau_n$, we have

$$\|\tau_n - f\| = \frac{2}{\pi (1 + q)^n} \int_0^\pi \psi(t) \sum_{k=0}^n \sum_{k=0}^P \left( \frac{\cos \frac{t}{2} - \sin \left( \frac{\nu + 1}{2} \right) \frac{t}{2}}{2 \sin \left( \frac{t}{2} \right)} \right) \, dt$$

$$= \int_0^\pi \psi(t) \overline{K_n (t)} \, dt$$

$$= I_1 + I_2 \, , \text{ say}$$

(5.1)
Now

\[ |I_1| = \frac{2}{\pi (1 + q)^r} \left| \int_0^{\pi} \psi(t) \left( \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left( \frac{1}{P_k} \sum_{\nu=0}^{k} \frac{\cos \frac{t}{2} - \cos \left( \nu + \frac{1}{2} \right)}{2 \sin \frac{t}{2}} \right) \right) dt \right| \]

\[ \leq \left| \int_0^{\pi} \psi(t) \overline{K}(t) dt \right| \]

\[ = \left( \int_0^{\pi} (\psi(t))^r dt \right)^{\frac{1}{r}} \left( \int_0^{\pi} (\overline{K}(t))^s dt \right)^{\frac{1}{s}}, \text{ using Holder’s inequality} \]

\[ \leq O \left( \frac{1}{(n+1)^{\alpha}} \right) \left( \int_0^{\pi} n^z dt \right)^{\frac{1}{s}}, \text{ using Lemma 4.1} \]

\[ = O \left( \frac{n}{(n+1)^{\alpha z}} \right) \]

\[ = O \left( \frac{1}{(n+1)^{\alpha z - 1}} \right) \]

\[ = O \left( \frac{1}{(n+1)^{\alpha z - 1}} \right) = O \left( \frac{1}{(n+1)^{\alpha z - 1}} \right) \quad (5.2) \]

Next

\[ |I_2| \leq \left( \int_{\frac{1}{n+1}}^{\pi} \left( \psi \left( \frac{t}{n+1} \right) \right)^r dt \right)^{\frac{1}{r}} \left( \int_{\frac{1}{n+1}}^{\pi} \left( \overline{K} \left( \frac{t}{n+1} \right) \right)^s dt \right)^{\frac{1}{s}}, \text{ using Holder’s inequality} \]

\[ \leq O \left( \frac{1}{(n+1)^{\alpha}} \right) \left( \int_{\frac{1}{n+1}}^{\pi} \left( \frac{1}{t} \right)^s dt \right)^{\frac{1}{s}}, \text{ using Lemma 4.2} \]

\[ = O \left( \frac{1}{(n+1)^{\alpha}} \right) \left( \frac{1}{n+1} \right)^{\frac{1-s}{s}} \]

\[ = O \left( \frac{1}{(n+1)^{\alpha z - 1}} \right) \quad (5.3) \]
Then from (5.2) and (5.3), we have

\[
|r_n - f(x)| = O\left(\frac{1}{(n+1)^{\frac{1}{r}}}ight), \text{ for } 0 < \alpha < 1, \ r \geq 1.
\]

Hence,

\[
\|r_n - f(x)\|_x = \sup_{-\pi < x < \pi} |r_n - f(x)| = O\left(\frac{1}{(n+1)^{\frac{1}{r}}}ight), \ 0 < \alpha < 1, \ r \geq 1
\]

This completes the proof of the theorem.

References

[1] Hardy, G.H: Divergent series, First edition, Oxford University press 70(19).

[2] Misra, U.K., Misra, M., Padhy, B.P. and Buxi, S.K: On degree of approximation by product means of conjugate series of Fourier series”, International Jour. of Math. Sciences, and Engineering Applications, ISSN 0973 – 9424, Vol.6 No.122 (Jan. 2012), pp 363 – 370.

[3] Nigam, H.K and Ajay Sharma: On degree of Approximation by product means, Ultra Scientist of Physical Sciences, Vol.22 (3) M, 889-894, (2010).

[4] Titchmarsh, E.C: The theory of functions, oxford university press, p.p402-403(1939).

[5] Zggmund , A : Trigonometric Series , second Edition ,Vol.I , Cambridge University press , Cambridge , (1959).