An analogy between optical turbulence and activator-inhibitor dynamics

F. Spineanu and M. Vlad
National Institute of Laser, Plasma and Radiation Physics
Magurele, Bucharest 077125, Romania

Abstract

The propagation of laser beams through media with cubic non-linear polarization is part of a wide range of practical applications. The processes that are involved are at the limit of extreme (quasi-singular) concentration of intensity and the transversal modulational instability, the saturation and defocusing effect of the plasma generated through avalanche and multi-photon (MPI) ionization are competing leading to a complicated pattern of intensity in the transversal plane. This regime has been named “optical turbulence” and it has been studied in experiments and numerical simulations. Led by the similarity of the portraits we have investigated the possibility that the mechanism that underlies the creation of the complex pattern of the intensity field is the manifestation of the dynamics activator-inhibitor. In a previous work we have considered a unique connection, the complex Landau-Ginzburg equation, a common ground for the nonlinear Schrodinger equation (optical propagation) and reaction-diffusion systems (activator-inhibitor). The present work is a continuation of this investigation. We start from the exact integrability of the elementary self-focusing propagation (gas Chaplygin with anomalous polytropic) and show that the analytical model for the intensity can be extended on physical basis to include the potential barrier separating two states of equilibria and the drive due to competing Kerr and MPI nonlinearities. We underline the variational structure and calculate the width of a branch of the cluster of high intensity (when it is saturated at a finite value). Our result is smaller but satisfactorily in the range of the experimental observations.
1 Introduction

This work is an extension of our previous work on the possible parallel between the optical turbulence and the Labyrinth instability acting in a system with a dynamics of the type activator-inhibitor [1]. We recall that optical turbulence is one of the regimes of propagation in a medium with cubic Kerr nonlinearity of a pulse produced by a laser at powers much higher than the threshold for self-focalization. The multiple filamentation, saturation through generation of plasma followed by re-location and coalescence of zones of high intensity lead to a complicated distribution of intensity in the transversal plane. The basic mechanism for the apparently random distribution is similar to a competition of two fields in a reaction-diffusion system. One is auto-catalitic and the other acts to limit the expansion of the first.
Previously we have argued that a connection can be established between the analytical structure underlying the optical turbulence and the one of the labyrinth instability. The connection is provided by the complex Landau-Ginzburg equation for which exist mappings to the Nonlinear Schrodinger Equation and respectively to activator-inhibitor equations.

In the present work we start from the description of the self-focusing as an exactly integrable “Chaplygin gas with anomalous polytropic exponent” (or: “drop-on-ceil” [2]). We extend this pure self-focusing scheme by adding analytical terms which are manifestation of natural physical processes:

- the diffusion
- the difference in potential energy between the two extrema at equilibrium: \( I = I_{\text{max}} \) and \( I = 0 \);
- the competition between Kerr nonlinearity and the defocusing property of the plasma

Therefore we must note from the beginning that the theory is constructed on the basis of analytical implementation of properties that are identified in a physical analysis of the two real systems.

We show (Appendix A) that a modification of the exactly integrable “drop-on-ceil” instability exhibits the expected effect of increasing structuring in the transversal plane.

We study the possible stabilization of the width of a stripe belonging to the cluster of high intensity. For the range of parameters that permit stabilization, we can provide an approximative value. Compared with experimental observation, our analytical result is smaller, but the sources of improvement of the analytical approach are sufficiently rich to allow extensions.

1.1 The basic analytical model of the propagation with self-focusing

We start from the basic elements of the propagation of a high intensity laser pulse in a cubic nonlinear medium. Consider the equation for the amplitude of the electric field \( A(z, x, y) \) of a laser beam \((k_0, \omega_0)\) in a medium with Kerr nonlinearity \( \varepsilon_2 > 0 \),

\[
2ik_0 \frac{\partial A}{\partial z} + \Delta_{\perp} A + k_0^2 \frac{\varepsilon_2}{\varepsilon_0} |A|^2 A = 0
\]  

(1)
and take a new factorization, in which it is introduced the \textit{eikonal}

\[ A(r, z) = a(r, z) \exp[i k_0 S(r, z)] \] (2)

where \( S(r, z) \equiv \text{eikonal with unit } [S] = \text{length} \). The resulting equations are \((3), (4), (2)\), assuming axial symmetry in the transversal plane \( \text{i.e. only retaining the radial coordinate } r \)

\[
\frac{\partial a}{\partial z} + v \frac{\partial a}{\partial r} + \frac{a}{2r} \frac{\partial}{\partial r} (rv) = 0 \] (3)

\[
2 \frac{\partial S}{\partial z} + v^2 = \frac{\varepsilon_2}{\varepsilon_0} a^2 + \frac{1}{k_0} \Delta \frac{a}{a} \]

where the “velocity” is

\[ v = \frac{\partial S}{\partial r} \] (4)

nondimensional. The \textit{velocity} is the derivative of the eikonal to the radial coordinate. It actually is like a \textit{wavenumber} for a propagation in the transversal direction to the \textit{z} axis. It will govern the pattern formation in the transversal plane. The last term can be neglected in the limit \( \lambda \to 0 \). Then, adopting the new variable

\[ I \equiv a^2 \] (5)

we have

\[
\frac{\partial I}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (rvI) = 0 \] (6)

\[
\frac{\partial v}{\partial z} + v \frac{\partial v}{\partial r} = \frac{\varepsilon_2}{2\varepsilon_0} \frac{\partial I}{\partial r} \]

These equations are of type “drop-on-ceil instability” and belong to the class describing a \textit{gas Chaplygin with anomalous politropic exponent}. They can only be solved approximately. To advance the analytical description it is necessary to restrict to a single spatial coordinate in the transversal plane, which renders the system exactly integrable

\[
\frac{\partial I}{\partial z} + \frac{\partial}{\partial x} (vI) = 0 \] (7)

\[
\frac{\partial v}{\partial z} + v \frac{\partial v}{\partial x} = c_0 \frac{\partial}{\partial x} \left( \frac{I}{I_0} \right) \]

Here

\[ c_0^2 = \frac{\varepsilon_2}{2\varepsilon_0} \]

and \( I_0 = a_0^2 \) is the intensity at the entrance in the medium. These equations are solved in Appendix A using the hodograph transformation, as described in [2].
1.2 The optical turbulence

To investigate the possible validity of the parallel between optical turbulence and the activator-inhibitor dynamics we will not employ a detailed description of the random multiple filamentation pattern of intensity. We must retain that there are regions of high intensity and complementary regions of low intensity. Their spatial pattern is an intricate distribution of stripes (branches of a plane graph) as connected components of a cluster. Further we will mention that inside the regions of the cluster of high intensity there are spots of even higher intensity, where new filaments are initiated. This is because the intensity is still higher than the threshold for self-focusing. In such a spot it is generated plasma and the effect of the electrons of the plasma is to defocus locally the beam and to saturate the increase of the intensity. This is seen as a relocation of the high intensity from the region of concentration. We then recognize the basic dynamics of an activator with auto-catalitic evolution (the intensity) and a competing inhibitor (the plasma).

The sequence of physical processes is as follows: (1) The high intensity produced at self-focalization generates plasma; (2) Plasma acts as a negative lens; (3) Plasma pushes away the high intensity spots while it expands and de-localizes them. (This has experimental support: in a symmetric geometry\[5\] the axial region of the high-intensity pulse is moved symmetrically towards larger radii and a ring is formed. No substantial loss of energy occurs at these events. Then the ring collapses again on the axis.)

This is the physical picture that we will have to implement in an analytical description.

2 Expanding around the strict self-focusing dynamics

We will draw a parallel between the optical turbulence and the dynamics of an activator-inhibitor system. With only the Kerr nonlinearity retained, the equation for the intensity

$$\frac{\partial I}{\partial z} + \frac{\partial}{\partial x} (vI) = 0$$

(9)

is an equation of conservation where the effect of advection is produced by the transversal variation of the eikonal. The focusing effect creates in the transversal plane regions where the intensity \(I\) is high while in the complementary zone \(I\) is relatively low (see Ettoumi et al.\[6\]). As suggested by the
approach in the case of reaction-diffusion systems, we will simplify the representation of the intensity field by restricting it to only two values: $I = I_{\text{max}}$ and respectively $I = 0$, uniformly distributed inside mutually excluded zones [7]. These zones are stripes with meandering shapes in plane, each creating a connected cluster and separated by sharp interfaces (as in Fig.1 of Ref.[6]) from the complementary set. The evolution of the system from one state to another is constrained. This means that in a point $x$, through only successive steps consisting of focusing, plasma generation by ionization, defocusing and relocation of high-$I$ regions there can be transition from one state to another. This particularity is very often encountered (including to reaction-diffusion systems) and is represented schematically as a potential with two equilibrium states separated by a barrier

$$F[I] \sim I^2 (I - I_{\text{max}})^2$$

To solve Eq.(9) we must find $v(z,x)$, i.e. find from Eq.(6) the characteristics of the quasi-Lagrangian flow of $I$. However we would like to include at least a schematic description of the complex processes mentioned above: focalization, plasma generation and defocusing with re-location. Then we return to the Eulerian point of view by assuming that changes of $I$ from $I = 0$ to $I = I_{\text{max}}$ result from the competition between the potential energy $F$ and the external nonlinear drive, i.e. the Kerr focalization and the coupling with the plasma density. The flux

$$\Gamma_I = vI = -D \frac{\partial I}{\partial x}$$

ensure that the profiles are smooth. The external nonlinear drive arises from the difference between the Kerr-induced focalization and the defocusing effect of the density $\rho$ of electrons of the plasma, at the current value of the intensity $I$.

The structure of alternating stripes of high intensity and zones of low intensity (from where the high intensity has been pushed away and relocated) appears in experiments and in numerical simulations of multiple filamentation and optical turbulence [8], [9]. We are interested in the dynamics of a $x$-interval, a section of a stripe of high intensity $I$ bounded (to the left and right) by zones of low intensity. The high $I$ is necessarily associated with presence of electron plasma $\rho$. In activator-inhibitor dynamics the fronts of the activator ($I$) are sharp while the profiles of $\rho$ (inhibitor) are expected to be smooth and diffuse. We want to see if a stripe of high-$I$ is stabilized to a finite width limited by the left and right fronts.

In the regions of high intensity new spots of focalization are initiated with the tendency of formation of high concentration and further filamentation.
They are visible for example in Fig.5 of Ref.[10]. Since such a spot produces plasma with defocusing and re-location effect, one concludes that these are the positions where the modification of the interface takes place. The two factors: activator ($I$) and inhibitor ($\rho$) are always connected and $\rho$ follows $I$.

The result is that behind their permanent competition there remain zones with low values of both $I$ and $\rho$.

As discussed above, this complex process manifests itself as a barrier that makes the two equilibria states to be separated and not easily mutually accessible. It is represented by the potential $F$ with the two equilibria states and the barrier between them. We now must postulate that the two states of equilibrium have different potential energy, one of the states being favored: the mix of high intensity trying to focus but saturated through the effect of $\rho$ has higher potential energy than the empty regions which only remain behind such events. The difference is measured as [7]

$$ F[I] = f \frac{1}{4} \left(\bar{I}^2 - 1\right) + \left(r - \frac{1}{2}\right) \left(\frac{1}{2} \bar{I}^2 - \frac{1}{3} \bar{I}^3 - \frac{1}{12}\right) $$

(12)

with $\bar{I} \equiv \frac{I}{I_{\text{max}}}$ and $f$ is a dimensional factor. The drive produced on the variable $I$ is

$$ \frac{\delta F[I]}{\delta I} = f \frac{1}{I_{\text{max}}^3} I (I - rI_{\text{max}}) (I - I_{\text{max}}) $$

(13)

The difference between the potential energy of the two equilibrium states is

$$ \Delta F = F[\bar{I} = 1] - F[\bar{I} = 0] $$

(14)

$$ = f \frac{1}{6I_{\text{max}}^3} \left(r - \frac{1}{2}\right) $$

for $0 < r < 1$. Now regarding the source of local dynamics, we note that the change from one state to another can be done when there is no compensation between Kerr focusing and plasma defocusing. The terms arise from the subtraction: $\sim$ (Kerr focusing) $-$ (plasma defocusing), as in the original extended NSEq [11], [12]

$$ 2ik_0 \frac{\partial E}{\partial z} \sim \frac{2k_0 \omega_0}{c} n_2 |E|^2 E - k_0 \omega_0 \sigma \tau_0 \rho E $$

(15)

Then the coupling $C$ that acts like a drive is the difference, after factoring out $k_0$, can be written

$$ C \equiv \alpha I^2 - \alpha' I \rho $$

(16)

where

$$ \alpha \equiv \frac{2 \omega_0}{c} n_2 \quad \text{and} \quad \alpha' \equiv \omega_0 \sigma \tau_0 $$

(17)

This coupling is no more linear as it was in classical activator-inhibitor models [13], like FitzHugh-Nagumo.
3 The dynamics of the stripes of intensity

The basic analytical structure of the self-focusing instability is captured by the drop-on-ceil instability, Eq. (7). As discussed before this structure is now extended by adding the terms representing the potential energy cost of moving between the two distinct equilibria and by the drive resulting from the competition of the focusing and defocusing effects. We propose the equation

\[ \frac{\partial I}{\partial z} = D \frac{\partial^2 I}{\partial x^2} - \frac{\delta F}{\delta I} + \alpha I^2 - \alpha' I \]  

(18)

after replacing the flux \( \Gamma = -D \frac{\partial I}{\partial x} \). The coordinate \( x \) is measured across the section of connex stripes. The Eq. (18) can be derived from the functional

\[ W_I = \int dx \left[ \frac{1}{2} \left( \frac{\partial I}{\partial x} \right)^2 + F[I] - \frac{\alpha I^3}{3} \right] + \alpha' \frac{1}{2} \int dx \rho(x) I^2(x) \]  

(19)

4 The equation for the electron plasma density

The equation for \( \rho \) is \[11\], \[12\], \[8\]

\[ \frac{\partial \rho}{\partial t} = d \frac{\partial^2 \rho}{\partial x^2} - a \rho^2 + b I^K \]  

(20)

The first term in the RHS is the divergence of the local flux of density, i.e. the accumulation or depletion of density, the second is the decrease of the density through recombination and the source of density is the last term (note that we have neglected the avalanche ionization \( \sim \rho I \), which may be justified in the case of short time of pulse). The last term is the Multi-Photon Ionization (MPI) rate.

We will investigate the state where stripes of constant \( I = I_{\text{max}} \) alternate with stripes of low intensity, \( I = 0 \). Then we consider that the intensity has no spatial variation and the equation of \( \rho \) can be solved with constant and uniform \( I^K \).

\[ I^K = \text{const} \]  

(21)

The parameter \( d \equiv \delta^2/\tau \equiv \text{diffusion coefficient of electrons} \left( \frac{m^2}{s} \right) \) is estimated in the Appendix \[13\]. We choose

\[ \delta \sim 10^{-6} \ (m) \]  

(22)
which is a reasonable choice in the range of possible lengths of the electron mean free path. Using \( \tau \sim 1 \times 10^{-13} \) (s) \([9]\) we obtain

\[
d \sim \frac{10^{-12}}{10^{-13}} = 10 \left( \frac{m^2}{s} \right)
\]

(23)

Other parameters are \( a = 5 \times 10^{-13} \left( \frac{m^3}{s} \right) \) and

\[
\beta^{(K=7)} = 6.5 \times 10^{-104} \left( \frac{m^{11}}{W^6} \right)
\]

(24)

leading to

\[
b \equiv \frac{\beta^{(7)}}{K \hbar \omega_0} = 3.6 \times 10^{-86} \left( \frac{m^{11}}{J} \right)
\]

(25)

and

\[
E^{phys} = 9.15 \times 10^7 \left( \frac{V}{m} \right)
\]

(26)

In terms of intensity we have

\[
I \equiv \left| \tilde{E}_0 \right|^2 \left( \frac{W}{m^2} \right) \text{ alternatively } I = \left( |\sqrt{\epsilon_0}E^{phys}|^2 \right)
\]

where \( \tilde{E} = 5 \times 10^6 \left( \frac{W^{1/2}}{m} \right) \) such that, calculated below, we have for \( K = 7 \)

\[
bI^K = \frac{\beta^{(K=7)}}{K \hbar \omega_0} \left| \tilde{E}_0 \right|^2 = \frac{\beta^{(K=7)}}{K \hbar \omega_0} |\sqrt{\epsilon_0}E_0|^{2K} \sim 7.7 \times 10^7 \left( \frac{1}{m^3s} \right)
\]

(27)

It is interesting to estimate the density \( \rho \) that results if the only process were recombination \( \partial \rho / \partial t = |a\rho^2| \). Taking the time duration of the pulse \( \delta t = 80 \) (fs) we have the estimation \( 1/\rho = a \times \delta t \) or \( \rho \sim 2 \times 10^{25} \) (part/m^3). On the other hand one expects that the plasma density is approximately 1% of the air density. Then for various estimations we take

\[
\rho \sim 10^{23} \left( m^{-3} \right)
\]

(28)

Further, the equation can be integrated once

\[
d \frac{1}{2} \left( \frac{\partial \rho}{\partial x} \right)^2 = a \frac{1}{3} \rho^3 - b\rho I^K + C
\]

(29)

where

\[
[C] = \frac{1}{m^8 s}
\]

(30)
If the spot is symmetric the density created by $I^K$ has a maximum at the center of the spot and

$$\frac{\partial \rho}{\partial x} = 0 \text{ for } x = 0$$

$$C = \rho(0) bI^K - \frac{1}{3} a [\rho(0)]^3$$

(31)

We will use the notation $\rho_0 \equiv \rho(0)$. Replacing in the right hand side

$$\frac{d\rho}{\left[\left\{\frac{2b}{a} (\rho^3 - \rho_0^3) - \frac{2b}{a} I^K (\rho - \rho_0)\right\}\right]^{1/2}} = \pm dx$$

(32)

We recall that we look for a regime of fast inhibitor \[7\]. This setting of the problem assumes that there is no time variation of the density, in the sense that the formation of plasma is instantaneous under the effect of $I^K$. Only spatial variation of the electron density is considered. Then from a reference value of $\rho$, denoted $\rho(0)$ at $x = 0$ all other $\rho$’s are smaller $\rho - \rho_0 < 0$. Using the notation

$$\rho - \rho_0 = -\varepsilon < 0$$

(33)

the denominator becomes $\frac{2b}{a} [-\varepsilon^3 + s\varepsilon^2 + t\varepsilon]$ where

$$s \equiv 3\rho_0 > 0 \quad \quad \quad \quad \left[\text{m}^{-3}\right]$$

$$t \equiv \frac{3b}{a} I^K - 3\rho_0^2 > 0 \quad \quad \quad \left[\text{m}^{-6}\right]$$

(34)

and the integral

$$\int \frac{-d\varepsilon}{[-\varepsilon^3 + s\varepsilon^2 + t\varepsilon]^{1/2}} = \pm \sqrt{\frac{2a}{3d}} dx$$

(35)

**Digression on the magnitudes of the parameters $s$ and $t$** We want to underline a particularity of the problem connected with the estimation of the orders of magnitude of the terms involved in these equations. This problem will be found under different manifestations several times below.

Estimation of the magnitude of the parameters $s$ and $t$,

$$s \sim 3 \times 10^{23} \left(\text{m}^{-3}\right)$$

(36)

$$t = \frac{3b}{a} I^K - 3\rho_0^2 \sim 10^{21} \left(\frac{1}{\text{m}^6}\right) - 3 \times 10^{46} \left(\frac{1}{\text{m}^6}\right)$$

(37)

At the first sight $t$ is negative, $t < 0$ for $bI^K \sim 10^8 \left(\frac{W}{\text{m}^2}\right)$, where $I$ was taken $\sim 10^{15} \left(\frac{W}{\text{m}^2}\right)$. This is the uniform distribution in the cross section of
the beam and does not reflect the focusing effects, which can lead to locally quasi-singular concentrations of $I$. We must take into account that the first term can be much higher than it is here and this is precisely the situation that is interesting for us. It will be much larger when $bI^K$ will be multiplied by a coefficient "FACTOR". For the following calculations we take

$$t > 0$$  \hspace{1cm} (38)

which corresponds to the situation that the MPI is still higher than the recombination.

Assuming $t > 0$ MPI higher than recombination We make an approximation

$$\frac{d\varepsilon}{\sqrt{\varepsilon (s \varepsilon + t)}} = \mp \sqrt{\frac{2a}{3d}} dx$$  \hspace{1cm} (39)

by ignoring the high order $\varepsilon^3$. Neglecting $\varepsilon^3$ is equivalent to neglecting the highest effect of recombination.

$$\frac{d\varepsilon}{\sqrt{\varepsilon (s \varepsilon + t)}} = \frac{1}{\sqrt{s}} \ln \left( \sqrt{s (s \varepsilon^2 + \varepsilon t)} + 2s \varepsilon + t \right) \text{ for } \Delta < 0 \text{ and } 2s \varepsilon + t > \sqrt{-\Delta} = t$$  \hspace{1cm} (40)

(Gradshteyn Ryzhik 2.261). The equation

$$\frac{d\varepsilon}{\sqrt{\varepsilon (s \varepsilon + t)}} = \mp \sqrt{\frac{2a}{3d}} dx$$  \hspace{1cm} (41)

for $\varepsilon \equiv \rho_0 - \rho(x) \geq 0$ is now integrated

$$\sqrt{s^2 \varepsilon^2 + \frac{s}{l^2} \varepsilon + 2 \frac{s}{l} \varepsilon + 1} = \exp \left[ \mp \sqrt{3} \rho_0 \sqrt{\frac{2a}{3d}} (x - x_0) \right]$$  \hspace{1cm} (42)

where $x_0$ corresponds to the position where $\varepsilon = 0$, which is the same where the derivative of $\rho(x)$ is zero. Let

$$y \equiv \frac{s}{l} \varepsilon$$  \hspace{1cm} (43)

NOTE regarding the magnitude and sign for the new variable $y$. The magnitude is

$$|y| = \left| \frac{3 \times 10^{23}}{10^{21} \left( \frac{1}{m^2} \right) - 3 \times 10^{46} \left( \frac{1}{m^6} \right) \times 10^{23}} \right| \sim 1$$  \hspace{1cm} (44)
As results from $I \sim 10^{15} \left(\frac{W}{m^2}\right)$ the first term in the expression of $t$ is much smaller than the second

$$t = \frac{3b}{a} I^K - 3\rho_0^2 \sim 10^{21} \left(\frac{1}{m^6}\right) - 3 \times 10^{46} \left(\frac{1}{m^6}\right)$$  \hspace{1cm} (45)

and this would mean $y < 0$. This has been discussed above. It is the situation where we use the whole intensity of the beam without taking into account the focalization that is the origin of the formation of stripes. Certainly we cannot assume that the focalization is quasi-singular, with locally extremely high value for $I$ but we still must assume that the formation of plasma ($\text{MPI} \sim bI^K$) is possible and the recombination and diffusion just shape the profile.

Then

$$t > 0 \text{ and } t \sim 10^{46} \left(\frac{1}{m^6}\right)$$  \hspace{1cm} (46)

It follows that

$$y > 0$$  \hspace{1cm} (47)

We introduce the notation

$$h \equiv \exp \left[ \mp \sqrt{3\rho(0)} \sqrt{\frac{2a}{3d}} (x - x_0) \right]$$  \hspace{1cm} (48)

and make few estimations. Since

$$\frac{2a}{3d} \sim 0.6 \times \frac{5 \times 10^{-13} \left(\frac{m^3}{s}\right)}{10 \left(\frac{m^2}{s}\right)} \approx 3 \times 10^{-14} (m)$$  \hspace{1cm} (49)

for $\rho(0) \sim 10^{23} (m^{-3})$. The combination at the exponent

$$\sqrt{3\rho(0)} \sqrt{\frac{2a}{3d}} \approx 10^5 \left(\frac{1}{m}\right)$$  \hspace{1cm} (50)

We find that $h$ verifies the necessary constraint $h \ll 1$. Introducing the notation

$$\sqrt{3\rho(0)} \sqrt{\frac{2a}{3d}} \equiv \frac{1}{\xi} \left(\frac{1}{m}\right)$$  \hspace{1cm} (51)

with units $[\xi] = m$ we have

$$h = \exp \left(-\frac{x - x_0}{\xi}\right)$$  \hspace{1cm} (52)
The equation becomes

$$\sqrt{y^2 + y + 2y + 1} = h$$  \hfill (53)

Returning to $\varepsilon$ we have

$$\varepsilon = \frac{3\frac{b}{a} I K - 3\rho_0^2}{3\rho_0} \left[ 0.1 \pm 0.6\sqrt{1 - h^2} \right]$$ \hfill (54)

$$\rho_0 - \rho(x) = \left[ \frac{b I K}{a \rho_0} - \rho_0 \right] \left[ 0.1 \pm 0.6\sqrt{1 - h^2} \right] > 0$$ \hfill (55)

Note

$$p \equiv \frac{b I K}{a \rho_0} - \rho_0$$ \hfill (56)

we have

$$\rho(x) = \rho_0 - p \frac{1}{10} (5 + 3h^2)$$ \hfill (57)

We argue that the sign $+$ must be chosen. This is because we want that the overall term $-p \frac{1}{10} (5 + 3h^2)$ to remain negative since this reflects our choice of regime: fast generation of plasma through ionization followed by diffusion and recombination still under a source coming from MPI. If instead we had chosen $0.5 \mp 0.3h^2 = \frac{1}{10} (5 - 3h^2)$ the term $-p \frac{1}{10} (5 - 3h^2)$ were less negative.

**NOTE on the magnitude of the parameter $p$**  The notation used above introduces

$$p \equiv \frac{b I K}{a \rho_0} - \rho_0$$ \hfill (58)

As explained, the strong focalization that leads to plasma formation means that the assumption $b I K \sim 10^8 \left( \frac{1}{m^3} \right)$ is an underestimation. The MPI term should generically be multiplied with a $FACTOR$ that represents the amplification in a spot that initiate a filament. Then

$$p \sim \text{FACTOR} \times 0.2 \times 10^{-2} \left( \frac{1}{m^3} \right) - 10^{23} \left( \frac{1}{m^3} \right)$$ \hfill (59)

For example, for an increase in the amplitude of electric field $E$ with a factor of 100, the amplification of the MPI term is $\text{FACTOR} = \left( 10^2 \right)^K = 10^{28}$ for $K = 7$ leading to

$$p \sim 0.2 \times 10^{-2} \times 10^{28} - 10^{23} \left( \frac{1}{m^3} \right)$$ \hfill (60)
The parameter $p$ must be considered positive and with a magnitude similar to the one of the two competing components, $p \sim 10^{23} \text{ (m}^{-3}\text{)}$.

Finally we return to our equation

$$\frac{\partial I}{\partial z} = D \frac{\partial^2 I}{\partial x^2} - \frac{\delta F}{\delta I} + \alpha I^2 - \alpha' \rho I$$  \hspace{1cm} (61)$$

where we replace

$$\rho = \rho_0 - \left( \frac{b I^K}{a \rho_0} - \rho_0 \right) \frac{1}{10} (5 + 3h^2)$$  \hspace{1cm} (62)$$

Since we have assumed that the density that we study $\rho(x)$ is smaller (due to depletion by diffusion and recombination) than the density created at the maximum of the focalization of $I$, which is the maximum $\rho_0$,

$$\rho(x) - \rho_0 < 0$$  \hspace{1cm} (63)$$

$p$ must be positive such that the substraction to be correct

$$\rho(x) = \rho_0 - p \frac{1}{10} (5 + 3h^2) < \rho_0$$  \hspace{1cm} (64)$$

It is convenient to separate the expression of the density

$$\rho(x) = -IKw_1(x) + w_2(x)$$  \hspace{1cm} (65)$$

$$w_1(x) \equiv \frac{1}{10\rho_0 a} (5 + 3h^2)$$  \hspace{1cm} (66)$$

$$w_2(x) \equiv \rho_0 + \rho_0 \frac{1}{10} (5 + 3h^2)$$

The equation for $I$ becomes

$$\frac{\partial I}{\partial z} = D \frac{\partial^2 I}{\partial x^2} - \frac{\delta F}{\delta I} + \alpha I^2 - \alpha' \rho I \left[ -IKw_1(x) + w_2(x) \right]$$  \hspace{1cm} (67)$$

5 The stabilization of the stripe

We start from the differential equations for the activator field (the intensity $I$).

The equation

$$\frac{\partial I}{\partial z} = D \frac{\partial^2 I}{\partial x^2} - \frac{\delta F}{\delta I} + \alpha I^2 - \alpha' \rho I$$  \hspace{1cm} (68)$$
It can be derived from
\[ \mathcal{W}_I = \int dx \left[ \frac{1}{2} D \left( \frac{\partial I}{\partial x} \right)^2 + F[I] - \frac{\alpha I^3}{3} \right] + \alpha' \frac{1}{2} \int dx \rho(x) I^2(x) \]  

(69)

And, the equation for the density \( \rho \) is
\[ \frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2} - a \rho^2 + b I^K \]  

(70)

with the Energy functional
\[ \mathcal{W}_\rho = \int dx \left[ \frac{1}{2} d \left( \frac{\partial \rho}{\partial x} \right)^2 + a \rho^3 \right] + b \int dx \rho I^K \]  

(71)

We follow the work by Goldstein \[7\] to study the evolution of a stripe \( I = I_{\text{max}} \) between regions (also stripes) of \( I = 0 \).

5.1 The variational equations

5.1.1 Variational equation for the intensity

The equation for \( I \) can be written in variational form. We separate the non-coupled parts in the functionals
\[ \mathcal{W}_I = \mathcal{E}_I + \mathcal{F}_I \]  

(72)

\[ \mathcal{E}_I = \int dx \left[ \frac{1}{2} D \left( \frac{\partial I}{\partial x} \right)^2 + F[I] - \frac{\alpha I^3}{3} \right] \]  

(73)

and the coupled part
\[ \mathcal{F}_I = \alpha' \int dx \frac{1}{2} \rho(x) I^2(x) \]  

(74)

and calculate first for \( I \). After an integration by parts
\[ \mathcal{E}_I[I] = \int dx \left[ -\frac{1}{2} D I \left( \frac{\partial^2 I}{\partial x^2} \right) + F[I] - \frac{\alpha I^3}{3} \right] \]  

(75)

By functional integration of \( \mathcal{E}_I[I] \) to \( I(x) \) we get a \( \delta(x - x') \) factor which will be integrated over \( x' \) and selects precisely the terms calculated at \( x \), i.e. the equation. The integration of product of identical functions like \( (\partial I/\partial x) \) will occur twice
\[ \frac{\delta \mathcal{E}_I}{\delta I} = -D \left( \frac{\partial^2 I}{\partial x^2} \right) + \frac{\delta F}{\delta I} - \alpha I^2 \]  

(76)
To this equation we add the result of functional variation of the coupling term
\[
\frac{\delta F_I}{\delta I} = \alpha' \rho (x) I (x) \tag{77}
\]
\[
\frac{\delta F_I}{\delta \rho} = \alpha I^2 (x) \tag{78}
\]

The equation for the variable \( I \) is
\[
\frac{\partial I}{\partial z} = D \frac{\partial^2 I}{\partial x^2} - \frac{\delta F}{\delta I} + \alpha I^2 - \alpha' \rho I \tag{79}
\]
can now be written
\[
\frac{\partial I}{\partial z} = \frac{\delta E_I}{\delta I} - \frac{\delta F_I}{\delta I} \tag{80}
\]

5.1.2 Variational equation for the density \( \rho \)

In an analogous calculation we separate in the energy functional the coupling term
\[
\mathcal{W}_\rho = \mathcal{E}_\rho + \mathcal{F}_\rho \tag{81}
\]

\[
\mathcal{E}_\rho = \int dx \left[ \frac{1}{2} \frac{d}{dx} \left( \frac{\partial \rho}{\partial x} \right)^2 + a \rho^3 \right] \tag{82}
\]

\[
\mathcal{F}_\rho = b \int dx \frac{1}{2} \rho^2 (x) I^K (x) \tag{83}
\]

Preparing for functional variation
\[
\mathcal{E}_\rho [\rho] = \int dx \left[ - \frac{1}{2} \frac{d}{dx} \rho \left( \frac{\partial^2 \rho}{\partial x^2} \right) + a \rho^3 \right] \tag{84}
\]

\[
\frac{\delta \mathcal{E}_\rho}{\delta \rho} = - d \left( \frac{\partial^2 \rho}{\partial x^2} \right) + a \rho^2 \tag{85}
\]

\[
\frac{\delta \mathcal{F}_\rho}{\delta I} = b K \rho (x) I^{K-1} (x) \tag{86}
\]

\[
\frac{\delta \mathcal{F}_\rho}{\delta \rho} = b I^K (x) \tag{87}
\]

The equation of motion
\[
\frac{\partial \rho}{\partial t} = d \frac{\partial^2 \rho}{\partial x^2} - a \rho^2 + b I^K \tag{88}
\]
is written as
\[
\frac{\partial \rho}{\partial t} = - \frac{\delta \mathcal{E}_\rho}{\delta \rho} + \frac{\delta \mathcal{F}_\rho}{\delta \rho} \tag{89}
\]
5.2 Is-there a gradient flow?

An important factor in the formation of a labyrinth pattern for an activator-inhibitor system is reduction of the dynamics to the gradient flow [13], [7]. We would like to check that the same structure exists for the two fields \((I, \rho)\).

We take infinitely fast inhibitor

\[
\frac{\partial \rho}{\partial t} = - \frac{\delta E}{\delta \rho} + \frac{\delta F}{\delta \rho} = 0
\]

and calculate \(\frac{\partial}{\partial z} (E_I + F_I)\). We use Eqs.(72) - (74)

\[
\frac{\partial E_I}{\partial z} = \frac{\partial E_I}{\partial I} \frac{\partial I}{\partial z} + \frac{\partial E_I}{\partial \rho} \frac{\partial \rho}{\partial z} = \int dx \left\{ -D \left( \frac{\partial^2 I}{\partial x^2} \right) + \frac{\delta F}{\delta I} + \alpha I^2 - \alpha' \rho I \right\}^2
\]

In the first square paranthesis we add and substract what is missing for the expression inside to become \(-\frac{\partial I}{\partial z}\) which means the second square paranthesis with negative sign

\[
\frac{\partial E_I}{\partial z} = \frac{\partial E_I}{\partial I} \frac{\partial I}{\partial z} = \int dx \left\{ -D \left( \frac{\partial^2 I}{\partial x^2} \right) + \frac{\delta F}{\delta I} + \alpha I^2 - \alpha' \rho I \right\}^2
\]

For the second part we have

\[
\frac{\partial F_I}{\partial z} = \frac{\partial F_I}{\partial I} \frac{\partial I}{\partial z} + \frac{\partial F_I}{\partial \rho} \frac{\partial \rho}{\partial z} = \int dx \left[ \alpha' \rho (x) I (x) \right] \left( D \frac{\partial^2 I}{\partial x^2} - \frac{\delta F}{\delta I} + \alpha I^2 - \alpha' \rho I \right]
\]

Adding the two expressions we obtain

\[
\frac{\partial}{\partial z} (E_I + F_I) = - \int dx \left( D \frac{\partial^2 I}{\partial x^2} - \frac{\delta F}{\delta I} + \alpha I^2 - \alpha' \rho I \right)^2 = - \int dx \left( \frac{\partial I}{\partial z} \right)^2 < 0
\]

and this confirms that we have a gradient flow.

The fact that the evolution of the intensity \(I\) is a gradient flow supports the idea that the optical turbulence and the activator-inhibitor have the same mathematical nature.
5.3 The energy of a stripe

We consider a stripe belonging to the cluster of high intensity, of time-dependent width $2Q$, $x \in [-Q,+Q]$. The axis of the stripe is considered a line and does not intervene in the calculation below. The energy functional for $I$ will be used to calculate the energy of the stripe on unit length along the axis

$$\mathcal{W}_I = \int dx \left[ \frac{1}{2} D \left( \frac{\partial I}{\partial x} \right)^2 + F [I] - \frac{\alpha I^3}{3} \right] + \frac{\alpha'}{2} \int dx \rho(x) I(x)^2$$  \hspace{1cm} (95)

According to the method developed by Goldstein \[7\], we must evaluate the contributions.

1. The “line tension” arises from the gradient at the front (interface)

$$\gamma \sim \int_{\text{front}} dx D \left( \frac{\partial I}{\partial x} \right)^2$$  \hspace{1cm} (96)

2. The pressure $\Pi$ is the density of the energy contained in the stripe relative to the “empty” regions around

$$\Pi = \frac{1}{2Q} \int_{-Q}^{Q} dx \{ F[I = I_{\text{max}}] - F[I = 0] \} = \Delta F$$  \hspace{1cm} (97)

3. For the third term we have to introduce the expression of $\rho(x)$ that we have calculated.

We remember that the stripe is defined by $I = I_{\text{max}} = \text{const}$ on a spatial region of length $2Q$ bounded by zones “empty” of intensity, $I = 0$.

$$I = I_{\text{max}} [\Theta(x + Q) - \Theta(x - Q)]$$  \hspace{1cm} (98)

that when we integrate over the stripe where $I = I_{\text{max}} = \text{const}$ we have

$$\int_{-Q}^{Q} dx \left[ -\frac{\alpha I_{\text{max}}^3}{3} \right] = -\frac{\alpha I_{\text{max}}^3}{3} 2Q$$  \hspace{1cm} (99)

and will contribute to variational terms.

The last term in the expression of $\mathcal{W}_I$ comes from the coupling with $\rho(x)$

$$\int_{-Q}^{Q} dx \left[ \frac{\alpha'}{2} \rho(x) I^2(x) \right] = \frac{\alpha'}{2} I_{\text{max}}^2 \int_{-Q}^{Q} dx \rho(x)$$  \hspace{1cm} (100)

where $\rho(x)$ is given in terms of $h(x)$.
The total energy

\[ \Delta E (Q) \approx 2\gamma + 2Q\Delta F - \alpha \frac{I_{\max}^3}{3} 2Q + \frac{\alpha'}{2} I_{\max}^2 \int_{-Q}^{Q} dx \rho(x) \]  

(101)

is the Lagrangian density for time-independent state

\[ \mathcal{L} [I] = -\Delta E (Q) \]  

(102)

The expression of \( \mathcal{L} \) must be employed in the Euler Lagrange variational equation. However there is an additional term that should be considered, \emph{i.e.} a dissipative term \( \mathcal{R}[\partial I/\partial z] \)

\[ \mathcal{R} [\partial I/\partial z] = \int_{-\infty}^{\infty} dx \frac{1}{2} \left( \frac{\partial I}{\partial z} \right)^2 \]  

(103)

and the Euler-Lagrange equation is

\[ \frac{d}{dz} \frac{\delta \mathcal{L}}{\delta (\partial I/\partial z)} - \frac{\delta \mathcal{L}}{\delta I} = - \frac{\delta \mathcal{R}}{\delta (\partial I/\partial z)} \]  

(104)

The functional that is considered dissipative, \( \mathcal{R} \), will be calculated replacing

\[ \frac{\partial I}{\partial z} = \frac{\partial I}{\partial x} \frac{\partial x}{\partial z} \]  

(105)

and taking into account that there is the boundary condition which is time dependent, \emph{i.e.} \( x(t) = Q(t) \).

\[ \mathcal{R} \left[ \frac{\partial I}{\partial z} \right] = \left( \frac{\partial Q}{\partial z} \right)^2 \int_{\text{front}} dx \left( \frac{\partial I}{\partial x} \right)^2 \]  

(106)

The integral involves the “line tension” and is replaced by

\[ \mathcal{R} \left[ \frac{\partial I}{\partial z} \right] \sim \left( \frac{\partial Q}{\partial z} \right)^2 \frac{\gamma}{D} \]  

(107)

and

\[ \frac{\delta \mathcal{R}}{\delta (\partial I/\partial z)} = \frac{\delta}{\delta (\partial I/\partial z)} \left\{ \left( \frac{\partial Q}{\partial z} \right)^2 \frac{\gamma}{D} \right\} \]  

\[ = \frac{\partial Q}{\partial z} \frac{2\gamma}{D} \]  

(108)
the variational equation becomes

\[ \mathcal{L} [I] = -\Delta E (Q) \]

\[ = - \left[ 2\gamma + 2Q \Delta F - \alpha \frac{I_{\text{max}}^3}{3} 2Q + \frac{\alpha'}{2} I_{\text{max}}^2 \int_{-Q}^{Q} d\rho (x) \right] \]

or

\[ \frac{2\gamma}{D} \frac{\partial Q}{\partial z} = - \frac{\partial (\Delta E)}{\partial Q} \]

(109)

(110)

It results

\[ \frac{\gamma}{D} \frac{\partial Q}{\partial z} = - \left[ \Delta F - \alpha \frac{I_{\text{max}}^3}{3} + \frac{\alpha'}{2} I_{\text{max}}^2 \frac{1}{2} \frac{\partial}{\partial Q} \int_{-Q}^{Q} d\rho (x) \right] \]

(111)

To advance we have to examine the last term. It has been derived above, Eq. [65], the following expression for the density of plasma electrons determined by: the intensity \( I \), the recombination and diffusion

\[ \rho (x) = -I^K w_1 (x) + w_2 (x) \]

(112)

Now we make more explicit the last term

\[ \frac{\alpha'}{2} I_{\text{max}}^2 \frac{1}{2} \frac{\partial}{\partial Q} \int_{-Q}^{Q} d\rho (x) \]

\[ = \frac{\alpha'}{2} I_{\text{max}}^2 \frac{1}{2} \frac{\partial}{\partial Q} \left[ \int_{-Q}^{Q} dxw_2 (x) - I^K \int_{-Q}^{Q} dxw_1 (x) \right] \]

(113)

We have

\[ \int_{-Q}^{Q} dxw_2 (x) = 2Q \rho_0 3 \frac{3}{2} + \rho_0 \frac{3}{10} \xi \exp \left( \frac{2x_0}{\xi} \right) \sinh \left( \frac{Q}{\xi/2} \right) \]

(114)

and

\[ \int_{-Q}^{Q} dxw_1 (x) = \frac{b}{a} Q + \frac{3}{10} \xi \exp \left( \frac{2x_0}{\xi} \right) \sinh \left( \frac{Q}{\xi/2} \right) \]

(115)

Replacing

\[ \frac{\partial}{\partial Q} \int_{-Q}^{Q} d\rho (x) = -I^K \frac{b}{a} \rho_0 + 3 \rho_0 \]

\[ + \frac{3}{5} \left[ -I^K \frac{b}{a} \rho_0 + \rho_0 \right] \exp \left( \frac{2x_0}{\xi} \right) \cosh \left( \frac{Q}{\xi/2} \right) \]

(116)
we introduce the notation

\[ q \equiv -I^K b \frac{\partial}{\partial p} + 3 \rho_0 \]

\[ = -p + 2 \rho_0 \]  

and the result is represented as

\[ \frac{\partial}{\partial Q} \int_{-Q}^{Q} dx \rho(x) = q - p \frac{3}{5} \exp \left( \frac{2x_0}{\xi} \right) \cosh \left( \frac{Q}{\xi/2} \right) \]  

We can now write the functional

\[ \frac{\gamma}{D} \frac{\partial Q}{\partial z} = -\Delta F + \alpha I_{\text{max}}^3 - \frac{\alpha}{4} I_{\text{max}}^2 q + \frac{3}{4} \rho \frac{\exp \left( \frac{2x_0}{\xi} \right) \cosh \left( \frac{Q}{\xi/2} \right)}{\alpha} \]  

A stationary state for the stripe exists when \( \partial Q/\partial z = 0 \), which has the approximative form

\[ 6 \left( \frac{\Delta F}{I_{\text{max}}^2 \alpha p} - \frac{\alpha}{\alpha' p} + \frac{q}{p} \right) \exp \left( \frac{2x_0}{\xi} \right) = \cosh \left( \frac{Q}{\xi/2} \right) \]  

We introduce the notation

\[ r \equiv 6 \left( \frac{\Delta F}{p I_{\text{max}}^2 \alpha'} - \frac{\alpha}{\alpha' p} I_{\text{max}} - 1 + \frac{2 \rho_0}{p} \right) \]  

and for a stabilization of the stripe width we need \( r > 1 \). For an evaluation we use the magnitudes chosen above and adopt a hypothesis on the difference between the potential energies of the two basic states

\[ \frac{\Delta F}{p I_{\text{max}}^2 \alpha'} \sim 1 \]  

We conclude that the terms in \( r \) can lead to a negative value which means that there is no stabilization of the stripes.

If however the concentration of beam energy renders \( I_{\text{max}} \) higher by orders of magnitude compared with the uniformly distributed input \( I \) then \( r \) can be positive and of order few units. In this case, adopting \( x_0 = 0 \), we solve \( u^2 - 2ru + 1 = 0 \) and find \( u = \exp \left( \frac{2Q}{\xi} \right) \). Then \( 2Q \sim \xi \ln r \) leads to a rough estimation

\[ Q \gtrsim \xi \sim 10^{-5} \ (m) \]
where we used the estimation
\[ \xi \equiv \left[ \sqrt{3} \rho_0 \sqrt{\frac{2a}{3d}} \right]^{-1} \]
\[ \sim 10^{-5} \text{ (m)} \] (124)

The result is smaller than the width that can be retrieved from the pictures obtained experimentally by Ettoumi et al. [6] where one can infer an average width \( \sim 10^{-4} \text{ (m)} \).

We can improve the analytical framework with the purpose of a better description of the balance between numbers of very high magnitude (\( \sim 10^{23} \)) that are substracted in the competition between Kerr and plasma non-linearities. We will need new technical methods and some numerical work in parallel.

6 Conclusion

The previous work [1] has advanced a hypothesis that there is a common mathematical structure underlying the optical turbulence and the gradient flow of some nonlinear reaction diffusion system. The common ground is the activator-inhibitor dynamics where two fields, one auto-catalitic and the other acting to limit and inhibit the expansion of the first, compete and generate a complicated pattern. The distribution of the intensity of the laser pulse is mainly the result of self-focusing (Kerr) nonlinearity and defocusing effect of the plasma created by ionization. The basic model of self-focalization is exactly integrable and we argue that starting from here one can construct a mathematical model that incorporates the known physical processes of beam propagation in a way that makes transparent the analogy with the activator-inhibitor dynamics. The constructed model yields the analytical form Eq.(18) which, together with the equation for the density \( \rho(x) \) indeed shows the dynamics of activator-inhibitor type.

We show that it has the structure of gradient flow and we study the possible regimes consisting of suppression or, alternatively, saturation to a finite width of the stripe belonging to the cluster of high intensity.

As explained in the previous work, there is a practical utility in revealing this parallel between optical turbulence and the activator-inhibitor dynamics. The latter has been thoroughly investigated and many aspects can now be mapped on the corresponding behavior of the intensity in the transversal plane of a laser beam: formation of spots of high intensity, possibly with crystal spatial distribution, etc.
A numerical study devoted to this analogue mathematical behavior may be useful.

Acknowledgment This work has been supported in part by the Contract 4N/2016 of the Project PN 16 47 01 01 of the Romanian Ministry of Education and Scientific Research.

Appendices

A Appendix A. The hodograph transformation

We adopt the standard treatment of Trubnikov and Zhdanov \cite{2} of the nonlinear self-focusing. See also Appendix A of Ref.\cite{14}.

The equations are

\[
\frac{\partial I}{\partial z} + \frac{\partial}{\partial x} (vI) = 0 \tag{A.1}
\]

\[
\frac{\partial v}{\partial z} + v \frac{\partial v}{\partial x} = c_0 \frac{\partial}{\partial x} \left( \frac{I}{I_0} \right)
\]

where

\[
v = \frac{\partial S}{\partial x} \tag{A.2}
\]

\[=\text{transversal derivative of the eikonal}\]

\[
c_0^2 = \frac{\varepsilon_2}{2 \varepsilon_0} I_0 \tag{A.3}
\]

\[
I = a^2 \tag{A.4}
\]

\[
I_0 = a_0^2 \tag{A.5}
\]

\[=\text{intensity at the entrance in the medium}\]

and

\[
A(z, x) = a(z, x) \exp [ikS(z, x)] \tag{A.6}
\]

The variables are

\[(x, z) \rightarrow (I, v) \tag{A.7}\]
Now we apply the hodograph transformation to express \((x, z)\) in terms of
\((I, v)\) following closely the original treatment \[15\]

\[
\frac{dz}{dz} = 1 = \frac{\partial z}{\partial I} \frac{\partial I}{\partial z} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial z} \quad (A.8)
\]

and

\[
\frac{dz}{dx} = 0 = \frac{\partial z}{\partial I} \frac{\partial I}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \quad (A.9)
\]

This is a linear system with four equations and four unknowns. The first
equation from the first group and the first equation from the second group
are solved using the Jacobian

\[
\det \begin{pmatrix}
\frac{\partial z}{\partial I} & \frac{\partial z}{\partial v} \\
\frac{\partial z}{\partial x} & \frac{\partial z}{\partial x}
\end{pmatrix}
= \frac{\partial z}{\partial I} \frac{\partial x}{\partial v} - \frac{\partial z}{\partial v} \frac{\partial x}{\partial I} \quad (A.10)
\]

Then

\[
\frac{\partial I}{\partial z} = -\frac{1}{J} \frac{\partial x}{\partial v} \quad (A.11)
\]

\[
\frac{\partial v}{\partial z} = -\frac{1}{J} \left( -\frac{\partial x}{\partial I} \right) \quad (A.12)
\]

Now we repeat for: the second equation from the first group and the
second equation from the second group

\[
0 = \frac{\partial z}{\partial I} \frac{\partial I}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \quad (A.13)
\]

\[
1 = \frac{\partial x}{\partial I} \frac{\partial I}{\partial x} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial x}
\]

The result

\[
\frac{\partial I}{\partial x} = -\frac{1}{J} \left( -\frac{\partial z}{\partial v} \right) \quad (A.14)
\]

\[
\frac{\partial v}{\partial x} = -\frac{1}{J \frac{\partial I}{\partial I}}
\]
The result is
\[ \frac{\partial v}{\partial z} = \frac{1}{J} \frac{\partial x}{\partial I} \quad (A.15) \]
\[ \frac{\partial I}{\partial z} = -\frac{1}{J} \frac{\partial x}{\partial v} \]
\[ \frac{\partial v}{\partial x} = -\frac{1}{J} \frac{\partial z}{\partial I} \]
\[ \frac{\partial I}{\partial x} = \frac{1}{J} \frac{\partial z}{\partial v} \]
\[ \frac{\partial x}{\partial I} = \frac{1}{J} \frac{\partial z}{\partial v} \]

It is the time to replace these expressions in the Chaplygin equations for self-focusing
\[ \frac{\partial I}{\partial z} + \frac{\partial }{\partial x} (vI) = 0 \quad (A.16) \]
\[ \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial x} = c_0^2 \frac{\partial }{\partial x} \left( \frac{I}{I_0} \right) \]

where we carry out the derivations
\[ \frac{\partial I}{\partial z} + \frac{\partial v}{\partial x} I + v \frac{\partial I}{\partial x} = 0 \quad (A.17) \]
\[ \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial x} = c_0^2 \frac{\partial I}{\partial x} \frac{1}{I_0} \]

and replace in the first equation
\[ -\frac{1}{J} \frac{\partial x}{\partial v} -\frac{1}{J} \frac{\partial z}{\partial I} I + v \left( \frac{1}{J} \frac{\partial z}{\partial v} \right) = 0 \quad (A.18) \]
\[ \frac{\partial x}{\partial v} + \frac{\partial z}{\partial I} I - v \frac{\partial z}{\partial v} = 0 \quad (A.19) \]
\[ \frac{\partial x}{\partial v} = v \frac{\partial z}{\partial v} - I \frac{\partial z}{\partial I} \]

Now we replace in the second equation
\[ \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial x} = c_0^2 \frac{\partial I}{\partial x} \frac{1}{I_0} \quad (A.20) \]

it is
\[ \frac{1}{J} \frac{\partial x}{\partial I} + v \left( \frac{1}{J} \frac{\partial z}{\partial I} \right) = \frac{c_0^2}{I_0} \frac{1}{J} \frac{\partial z}{\partial v} \quad (A.21) \]
\[ \frac{\partial x}{\partial I} = \frac{c_0^2}{I_0} \frac{\partial z}{\partial v} + v \frac{\partial z}{\partial I} \]
We must take care of the mixed derivatives

\[ \frac{\partial^2 x}{\partial I \partial v} = \frac{\partial^2 x}{\partial v \partial I} \]  

(A.22)

\[ \frac{\partial}{\partial I} \left[ v \frac{\partial z}{\partial v} - I \frac{\partial z}{\partial I} \right] = \frac{\partial}{\partial v} \left[ \frac{c_0^2}{I_0} \frac{\partial z}{\partial v} + v \frac{\partial z}{\partial I} \right] \]  

(A.23)

From this

\[ \frac{\partial v}{\partial I} \frac{\partial z}{\partial v} + v \frac{\partial^2 z}{\partial I^2} - \frac{\partial z}{\partial I} - I \frac{\partial^2 z}{\partial I^2} = \frac{c_0^2}{I_0} \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial I} + v \frac{\partial^2 z}{\partial v \partial I} \]  

(A.24)

We note that the second term from the LHS is reduced with the last term of the RHS and that the first term in the LHS is identically zero since \( v \) and \( I \) are independent variables of the second set, just like \((z, x)\).

\[ \frac{c_0^2}{I_0} \frac{\partial^2 z}{\partial v^2} + I \frac{\partial^2 z}{\partial I^2} + 2 \frac{\partial z}{\partial I} = 0 \]  

(A.25)

\[ \frac{1}{I} \frac{\partial}{\partial I} \left( I^2 \frac{\partial z}{\partial I} \right) + \frac{c_0^2}{I_0} \frac{\partial^2 z}{\partial v^2} = 0 \]  

(A.26)

We make the substitution

\[ r = I^2 \]  

(A.27)

\[ s = \frac{1}{2} \frac{v}{\sqrt{c_0^2/I_0}} \]  

We calculate

\[ I = r^2 \]  

(A.28)

\[ \frac{\partial}{\partial r} = \frac{\partial}{\partial I} \frac{\partial I}{\partial r} = 2r \frac{\partial}{\partial I} \]  

\[ \frac{\partial}{\partial I} = \frac{1}{2r} \frac{\partial}{\partial r} \]  

\[ s = \frac{1}{v} \frac{1}{2 \sqrt{c_0^2/I_0}} \]  

(A.29)

\[ v = \alpha s \text{ where } \alpha \equiv 2 \sqrt{c_0^2/I_0} \]  

\[ \frac{\partial}{\partial s} = \frac{\partial v}{\partial s} \frac{\partial}{\partial v} = \alpha \frac{\partial}{\partial v} \]  

\[ \frac{\partial}{\partial v} = \frac{1}{\alpha} \frac{\partial}{\partial s} \]
\[
\frac{1}{I} \frac{\partial}{\partial I} \left( I^2 \frac{\partial z}{\partial I} \right) + \frac{c_0^2}{I_0} \frac{\partial^2 z}{\partial v^2} = 0 \quad (A.30)
\]
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^4 \frac{\partial z}{\partial r} \right) + \frac{\alpha}{\alpha} \frac{\partial}{\partial \alpha} \left( \frac{1}{\alpha} \frac{\partial z}{\partial \alpha} \right) = 0
\]
\[
\frac{1}{4r^3} \frac{\partial}{\partial r} \left( r^3 \frac{\partial z}{\partial r} \right) + \frac{1}{4} \frac{\partial^2 z}{\partial s^2} = 0
\]

We can return to our problem. The equations
\[
\frac{\partial I}{\partial z} + \frac{\partial}{\partial x} (vI) = 0 \quad (A.31)
\]
\[
\frac{\partial v}{\partial z} + v \frac{\partial v}{\partial x} = c_0^2 \frac{\partial}{\partial x} P \left[ \frac{I}{I_0} \right]
\]
where until now
\[
P \left[ \frac{I}{I_0} \right] = \frac{I}{I_0} \quad (A.32)
\]
and from now-on
\[
P \left[ \frac{I}{I_0} \right] = \frac{I}{I_0} - \beta \left( \frac{I}{I_0} \right)^K \quad (A.33)
\]
and
\[
\frac{\partial}{\partial x} P \left[ \frac{I}{I_0} \right] = \delta P \left[ \frac{I}{I_0} \right] \frac{\partial}{\partial x} \frac{I}{I_0} \quad (A.34)
\]
We introduce the notation
\[
\frac{\delta P \left[ \frac{I}{I_0} \right]}{\delta \left( \frac{I}{I_0} \right)} \equiv G \left[ \frac{I}{I_0} \right] \quad (A.35)
\]
The operations are the same as above. The first equation leads to
\[
\frac{\partial x}{\partial v} = v \frac{\partial z}{\partial v} - I \frac{\partial z}{\partial I} \quad (A.36)
\]
The second equation leads to
\[
\frac{\partial x}{\partial I} = G \left[ \frac{I}{I_0} \right] \frac{c_0^2}{I_0} \frac{\partial z}{\partial v} + v \frac{\partial z}{\partial I} \quad (A.37)
\]
and impose the equality of the mixed derivatives
\[
\frac{\partial}{\partial I} \left( v \frac{\partial z}{\partial v} - I \frac{\partial z}{\partial I} \right) = \frac{\partial}{\partial v} \left( G \left[ \frac{I}{I_0} \right] \frac{c_0^2}{I_0} \frac{\partial z}{\partial v} + v \frac{\partial z}{\partial I} \right) \quad (A.38)
\]
\[
v \frac{\partial^2 z}{\partial I \partial v} - \frac{\partial z}{\partial I} - I \frac{\partial^2 z}{\partial I^2} = G \left[ \frac{I}{I_0} \right] \frac{c_0^2}{I_0} \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial I} + v \frac{\partial^2 z}{\partial I \partial v} \quad (A.39)
\]
We reduce the terms and obtain
\[ I \frac{\partial^2 z}{\partial I^2} + 2 \frac{\partial z}{\partial I} + G[I] \frac{c_0^2}{I_0} \frac{\partial^2 z}{I_0 \partial v^2} = 0 \]  
(A.40)

As before the terms with derivatives to \( I \) are grouped to give
\[ \frac{1}{I} \frac{\partial}{\partial I} \left( I^2 \frac{\partial z}{\partial I} \right) + G[I] \frac{c_0^2}{I_0} \frac{\partial^2 z}{I_0 \partial v^2} = 0 \]  
(A.41)

In the first attempt we proceed in an analogous manner as above. We make the substitution
\[ r = I^2 \]  
(A.42)
\[ s = \frac{1}{2} \frac{v}{\sqrt{c_0^2/I_0}} \]

We calculate
\[ I = r^2 \]  
(A.43)
\[ \frac{\partial}{\partial r} = \frac{\partial}{\partial I} \frac{\partial I}{\partial r} = 2r \frac{\partial}{\partial I} \]
\[ \frac{\partial}{\partial I} = \frac{1}{2r} \frac{\partial}{\partial r} \]

\[ s = \frac{1}{2} \frac{1}{\sqrt{c_0^2/I_0}} \]  
(A.44)
\[ v = \alpha s \quad \text{where} \quad \alpha \equiv 2 \sqrt{c_0^2/I_0} \]
\[ \frac{\partial}{\partial s} = \frac{\partial v}{\partial s} \frac{\partial}{\partial v} = \alpha \frac{\partial}{\partial v} \]
\[ \frac{\partial}{\partial v} = \frac{1}{\alpha} \frac{\partial}{\partial s} \]

This is replaced in the equation
\[ \frac{1}{I} \frac{\partial}{\partial I} \left( I^2 \frac{\partial z}{\partial I} \right) + G[I] \frac{c_0^2}{I_0} \frac{\partial^2 z}{I_0 \partial v^2} = 0 \]  
(A.45)
\[ \frac{1}{r^2} \frac{1}{2r} \frac{\partial}{\partial r} \left( r^4 \frac{1}{2r} \frac{\partial z}{\partial v} \right) + G[I] \frac{\alpha^2}{4} \frac{1}{\alpha} \frac{\partial}{\partial s} \left( \frac{1}{\alpha} \frac{\partial}{\partial s} \right) = 0 \]
\[ \frac{1}{r^3} \frac{\partial}{\partial r} \left( r^3 \frac{\partial z}{\partial r} \right) + G[I] \frac{\partial^2 z}{\partial s^2} = 0 \]
The final form is
\[ \frac{\partial^2 z}{\partial r^2} + \frac{3}{r} \frac{\partial z}{\partial r} + G[I] \frac{\partial^2 z}{\partial s^2} = 0 \] (A.46)

Here we must redefine \( G \) as
\[ G[I] \rightarrow G[\sqrt{r}] \] (A.47)

We make the substitution that combines the coordinate \( r \) with the unknown function \( z \). [The coordinate \( r \) is a measure of the intensity \( I \).]
\[ \psi \equiv rz \] (A.48)

and replace the variable \( t \) by \( \psi \)

\[ z = \frac{\psi}{r} \] (A.49)
\[ \frac{\partial z}{\partial r} = -\frac{1}{r^2} \psi + \frac{1}{r} \frac{\partial \psi}{\partial r} \]
\[ \frac{\partial^2 z}{\partial r^2} = \frac{2}{r^3} \psi - \frac{1}{r^2} \frac{\partial \psi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} \]

and we have
\[ \frac{\partial^2 z}{\partial r^2} = \frac{\partial^2}{\partial r^2} \left( \frac{\psi}{r} \right) \] (A.50)
\[ = \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} - \frac{2}{r^2} \frac{\partial \psi}{\partial r} + \frac{2}{r^3} \psi \]
\[ \frac{3}{r} \frac{\partial z}{\partial r} = 3 \frac{\partial}{\partial r} \left( \frac{\psi}{r} \right) \] (A.51)
\[ = \frac{3}{r} \left( -\frac{1}{r^2} \psi + \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \]

and
\[ G[I] \frac{\partial^2 z}{\partial s^2} = G[I] \frac{\partial^2}{\partial s^2} \left( \frac{\psi}{r} \right) \text{ remember } r \text{ and } s \text{ are independent(A.52)} \]
\[ = G[I] \frac{1}{r} \frac{\partial^2 \psi}{\partial s^2} \]
\[
\frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} - \frac{2}{r^2} \frac{\partial \psi}{\partial r} + \frac{2}{r^3} \psi \\
+ \frac{3}{r} \left( - \frac{1}{r^2} \psi + \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \\
+ G[r] \frac{1}{r} \frac{\partial^2 \psi}{\partial s^2} = 0
\] (A.53)

\[
\frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \psi}{\partial r} - \frac{1}{r^3} \psi + G[r] \frac{1}{r} \frac{\partial^2 \psi}{\partial s^2} = 0
\] (A.54)

For comparison that will allow identification of the operator we mention

\[
\Delta f (r, \varphi, s) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial s^2}
\] (A.55)

We recognize the first two terms, containing the derivations to \( r \). Then the term \(-\frac{1}{r^2} \psi\) can be attributed to the operator of derivation with respect to the azimuthal variable \( \varphi \)

\[
\frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2} \rightarrow -\frac{1}{r^2} \psi
\] (A.56)

if \( f \sim \psi (r, s) \cos \varphi \) (A.57)

Then our equation is

\[
\Delta_{(r, \varphi)} \Psi + G[r] \frac{\partial^2 \Psi}{\partial s^2} = 0
\] (A.58)

where

\[
\Psi = \psi \cos \varphi
\] (A.59)

and

\[
\psi = rz
\] (A.60)

Now we comment on the result of this derivation. We remember that the variable \( s \) comes from \( v = \frac{\partial s}{\partial x} \) which is the derivative of the eikonal to the transversal coordinate \( x \).

If we introduce a harmonic variation on the \( s \) coordinate

\[
\Psi = \Xi (r, \varphi) \exp (ik_s s)
\] (A.61)
we get a Helmholtz equation
\[\Delta_{(r,\varphi)} \Xi (r, \varphi) \exp (ik_s s) + G [r] \left( - \kappa_s^2 \right) \Xi (r, \varphi) \exp (ik_s s) = 0 \tag{A.62}\]
\[\left( \Delta_{(r,\varphi)} - \kappa_s^2 G [r] \right) \Xi (r, \varphi) = 0 \tag{A.63}\]
where
\[\Xi (r, \varphi) = \psi \cos \varphi \tag{A.64}\]

Here
\[\psi = rz \tag{A.65}\]
\[= (\text{measure of the beam intensity, } I^2) \times (\text{distance } z \text{ on axis})\]

and
\[\varphi = \text{fictitious azimuthal angle} \tag{A.66}\]
in a cylindrical space where the radial coordinate is \(r = I^2\), the vertical coordinate is \(s \sim v \sim \frac{\partial S}{\partial x}\).

We note that instead of \(\kappa_s^2\) we now have \(G [r] \kappa_s^2\) (A.68), where \(G\) decreases when the intensity increases. This means that the effective wavenumber on the “vertical” coordinate \(s\) becomes smaller when \(I\) increases. The rate of variation of \(\Psi\) (which means \(\psi = rz\)) along the direction \(s\) becomes slower, with longer wavelengths along \(s \sim \frac{\partial S}{\partial x}\).

In the absence of \(G\) (i.e. in the usual situation of self-focusing) the two quantities \(\psi = rz\) and \(\partial S/\partial x\) evolve in a similar way: \(z\) increases approaching the focalization point, \(z \rightarrow z^*\). Simultaneously the intensity \(I \sim \sqrt{r}\) increases hence \(\psi \sim I^2 z\) increases. The same is true for the derivative of the eikonal since the field becomes more sharply concentrated on the transversal coordinate \(x\). Hence \(\partial S/\partial x\) also increases when the beam approaches focalization.

The explicit functional form of this correlated variation of the two quantities \(\psi\) and \(\partial S/\partial x\) is difficult to be derived. However we can see that by inserting \(G [r]\), which decreases when the beam approaches focalization, it is affected the relative rate of variation: \(\psi\) will be slowed down along \(s \sim \partial S/\partial x\) since \((\kappa_s^2)^{\text{eff}} = G [r] \kappa_s^2\) decreases as \(G\). This is equivalent to slowing down the process of increase of \(I\) as approaching the focalization. The concentration of the energy of the beam is slowed down. This is the manifestation of the well-known physical process: increase of the density of electrons weakens the focusing effect of the Kerr nonlinearity and the focalization saturates.
B Appendix B. Estimation of the physical parameters

B.1 Estimation of the diffusion coefficient

The distance travelled by an electron between two collisions is

\[ \delta \sim v_{th,e} \times \tau \]  

(B.1)

where the thermal velocity must correspond to few electron-volts since the electrons are just after being created with \( E_g = 11 \text{ (eV)} \) and then heated. We take

\[ E^{elect} \sim 1 \text{ eV} \]  

(B.2)

and the thermal velocity

\[ v_{th,e} = 4.19 \times 10^5 \sqrt{T_e (eV)} \left( \frac{m}{s} \right) \]  

(B.3)

and the time of collisions

\[ \tau = 10^{-13} \text{ (s)} \]  

(B.4)

Then the distance between two collisions

\[ \delta \sim v_{th,e} \times \tau \]  

\[ = 4 \times 10^5 \times 10^{-13} \]  

is of the order of \( 10^{-7} \).

On the other hand we have an alternative estimation

\[ \tau = \text{collision time} = \nu_{ee}^{-1} \]  

(B.6)

and

\[ \nu_{ee} = 2.91 \times 10^{-6} \ln \Lambda \times \frac{n_e (cm^{-3})}{[T_e (eV)]^{3/2}} \text{ (s)} \]  

(B.7)

Take

\[ n_e \sim 10^{23} \text{ (m}^{-3}) = 10^{17} \text{ (cm}^{-3}) \]  

(B.8)

\[ T_e \sim 1 \text{ (eV)} \]

\[ \ln \Lambda = 25 \]

It results

\[ \nu_{ee} = 3 \times 10^{-6} \times 25 \times \frac{10^{17}}{[1]^{3/2}} \]  

\[ = 75 \times 10^{11} \text{ (s}^{-1}) \]  

(B.9)
An order of magnitude is

\[ \nu_{ee} \sim 10^{13} \ (s^{-1}) \quad (B.10) \]

this is compatible with

\[ \tau \sim \nu_{ee}^{-1} \]
\[ \tau = 10^{-13} \ (s) \quad (B.11) \]

which is compatible with Ref.[9].

If we use as input the frequency of collisions \( \tau \) and calculate the temperature of the electron plasma

\[ \nu_{ee} = \tau^{-1} = 10^{13} \ (s^{-1}) \quad (B.12) \]

\[ T_e^{3/2} = \frac{2.91 \times 10^{-6} \times \ln \Lambda \times n_e}{\nu_{ee}} \quad (B.13) \]
\[ = \frac{3 \times 10^{-6} \times 25 \times 10^{17}}{10^{13}} = 75 \times 10^{-2} \]

it results

\[ T_e = (75 \times 10^{-2})^{2/3} \approx 0.8 \ (eV) \quad (B.14) \]

compatible with our assumption.

We can estimate the energy that can go to the plasma of electrons.

\[ P_{in} \sim 10^9 \ (W) \quad (B.15) \]

For this we introduce a parameter fraction that represents the amount from the total energy that goes to the electron plasma. The energy is

\[ W_{elect-plasma} = fraction \times P_{in} \times \Delta t \quad (B.16) \]
\[ = 1 \times 10^9 \times 100 \ (fs) \]
\[ = 10^9 \times 10^{-13} \]
\[ = 10^{-4} \ (J) \]

This energy is distributed on a number of particles \( N \)

\[ N = \rho \times Vol \quad (B.17) \]
\[ = 10^{23} \ (m^{-3}) \times a^3 \]
where
\[ a \sim 1 \text{ (mm)} = 10^{-3} \]  
\[ N = 10^{23} \times 10^{-9} = 10^{14} \text{ (particles)} \]

The amount of energy for each particle (electron) is
\[ \delta W_{\text{elect-plasma}} = \frac{W_{\text{elect-plasma}}}{N} \]
\[ = \frac{10^{-4} \text{ (J)}}{10^{14} \text{ (electrons)}} \]
\[ = 10^{-18} \text{ (J)} \]

This energy corresponds to
\[ T_{e} \sim \frac{\delta W_{\text{elect-plasma}}}{(eV)} = \frac{10^{-18} \text{ (J)}}{1.6 \times 10^{-19} \text{ (J/eV)}} \]
\[ \sim 10 \text{ (eV)} \]

we have
\[ \frac{m_e v_{th,e}^2}{2} = \delta W_{\text{elect-plasma}} \]
\[ v_{th,e}^2 = \frac{2 \delta W_{\text{elect-plasma}}}{m_e} = \frac{2 \times 10^{-18} \text{ (J)}}{9.1 \times 10^{-31} \text{ (kg)}} \]
\[ = 0.2 \times 10^{13} \left( \frac{J}{kg} \right) \]
\[ v_{th,e} = 1.4 \times 10^6 \left( \frac{m}{s} \right) \]

The distance traversed in a time \( \tau = 10^{-13} \text{ (s)} \) is
\[ \delta = v_{th,e} \times \tau = 10^6 \times 10^{-13} = 10^{-7} \text{ (m)} \]

Exactly the same result as above.

\[ \delta \sim 10^{-6} \text{ (m)} \text{ rather arbitrary} \]
\[ \tau \sim 1 \times 10^{-13} \text{ (s)} \text{ according to Mlejnek} \]
\[ d \sim \frac{10^{-12}}{10^{-13}} = 10 \left( \frac{m^2}{s} \right) \]
Possibly the range of the diffusion coefficient would be

\[ d \in [0.1, 10] \left( \frac{m^2}{s} \right) \]  \hspace{1cm} (B.26)

We choose

\[ d = 10 \left( \frac{m^2}{s} \right) \]  \hspace{1cm} (B.27)

### B.2 Estimation of the effect of focusing and defocusing terms

We will use

\[ n_2 = 3.2 \times 10^{-19} \left( \frac{cm^2}{W} \right) \]  \hspace{1cm} (Ref. [16]) \hspace{1cm} (B.28)

\[ = 3.2 \times 10^{-23} \left( \frac{m^2}{W} \right) \]

\[ \sigma \sim 5.1 \times 10^{-24} \left( m^2 \right) \]  \hspace{1cm} (Ref. [19]) \hspace{1cm} (B.29)

\[ \tau_0 \sim 3.5 \times 10^{-13} \left( s \right) \]  \hspace{1cm} (Ref. [9]) \hspace{1cm} (B.30)

\[ \rho = 10^{23} \left( m^{-3} \right) \]  \hspace{1cm} (B.31)

\[ I_0 \sim 10^{15} \left( \frac{W}{m^2} \right) \ldots 10^{17} \left( \frac{W}{m^2} \right) \]  \hspace{1cm} (B.32)

This is intensity on the whole area. In spots where self-focalization takes place, it can be orders of magnitude higher.

\[ \lambda = 775 \left( nm \right) \]  \hspace{1cm} (B.33)

From the last data

\[ k_0 = \frac{2\pi}{\lambda} = \frac{2\pi}{775 \times 10^{-9} \left( m \right)} = \frac{2\pi}{0.775 \times 10^{-6} \left( m \right)} \]

\[ \sim 8 \times 10^6 \left( m^{-1} \right) \]  \hspace{1cm} (B.34)

\[ \frac{\omega_0}{k_0} = c \]  \hspace{1cm} (B.35)

\[ \omega_0 = k_0c = 8 \times 10^6 \left( \frac{1}{m} \right) \times 3 \times 10^8 \left( \frac{m}{s} \right) = 24 \times 10^{14} \left( s^{-1} \right) \]
Then
\[
\frac{2}{c} n_2 = 2 \times \frac{1}{3 \times 10^8 \left( \frac{2 m}{s} \right)} \times 10^{-23} \left( \frac{m^2}{W} \right) \tag{B.36}
\]
\[
= 0.6 \times 10^{-31} \left( \frac{m^2 s^2}{J} \right)
\]
from where
\[
\alpha \equiv \frac{2 \omega_0}{c} n_2 = (\omega_0) \times \frac{2}{c} n_2 \tag{B.37}
\]
\[
\sim 24 \times 10^{14} \left( \frac{1}{s} \right) \times 0.6 \times 10^{-31} \left( \frac{m^2 s^2}{J} \right)
\]
\[
= 1.44 \times 10^{-16} \left( \frac{m}{W} \right)
\]
The constant in the defocusing term
\[
\sigma \tau_0 \sim 5 \times 10^{-24} \left( \frac{m^2}{s} \right) \times 3.5 \times 10^{-13} (s) \tag{B.38}
\]
\[
= 1.75 \times 10^{-36} \left( \frac{m^2 s}{m^2} \right)
\]
from where
\[
\alpha' \equiv \omega_0 \sigma \tau_0 \tag{B.39}
\]
\[
\sim 24 \times 10^{14} \left( \frac{1}{s} \right) \times 1.75 \times 10^{-36} \left( \frac{m^2 s}{m^2} \right)
\]
\[
= 4.2 \times 10^{-21} \left( \frac{m^2}{m^2} \right)
\]
Now we can estimate the two terms that compete
\[
\alpha I^2 - \alpha' \rho I \tag{B.40}
\]
factorizing a $I$ we have
\[
\alpha I - \alpha' \rho \tag{B.41}
\]
\[
\sim 1.44 \times 10^{-16} \left( \frac{m}{W} \right) \times 10^{15} \left( \frac{W}{m^2} \right) - 4.2 \times 10^{-21} \left( \frac{m^2}{m^2} \right) \times 10^{23} \left( \frac{1}{m^3} \right)
\]
\[
= 0.144 \left( \frac{1}{m} \right) - 420 \left( \frac{1}{m} \right)
\]
If instead of $I \sim 10^{15} \left( \frac{W}{m^2} \right)$ we would have taken
\[
I \sim 10^{17} \left( \frac{W}{m^2} \right) \tag{B.42}
\]
Then
\[ \alpha I - \alpha' \rho \sim 14 - 420 \]
and the two terms were closer, with still huge dominance of the second term, which represents defocusing due to plasma, over the focusing term due to Kerr nonlinearity.

However in the spots of focalization, which develop spontaneously in a strip of high \( I \), the local intensity is higher. Then the focalization overcomes the defocusing action of the electrons.

*It looks that we must work at the limit of balance of the focusing and defocusing, with a certain dominance of the Kerr-induced focusing, since we want to study the displacement of the front and motion of the interface associated with the relocation of the high-\( I \) zone.*

We conclude after using the usual values of the parameters [16], [9]
\[ \alpha \sim 1.44 \times 10^{-16} \left( \frac{m}{W} \right) \] \hspace{1cm} (B.44)
\[ \alpha' \sim 4.2 \times 10^{-21} \left( \frac{m^2}{W} \right) \]
and may be used with
\[ I \sim 10^{17} \left( \frac{W}{m^2} \right) \] \hspace{1cm} (or higher) \hspace{1cm} (B.45)
\[ \rho \sim 10^{23} \left( \frac{1}{m^3} \right) \]

For recombination
\[ a = 5 \times 10^{-13} \left( \frac{m^3}{s} \right) \] \hspace{1cm} (B.46)
and for MPI \( \beta^{(K=7)} = 6.5 \times 10^{-104} \left( \frac{m^{11}}{W} \right) \) we have
\[ b \equiv \frac{\beta^{(K=7)}}{K \hbar \omega_0} = 3.6 \times 10^{-86} \left( \frac{m^{11}}{J} \right) \]
\hspace{2cm} (B.47)

Taking \( E^{\text{phys}} = 9.15 \times 10^7 \left( \frac{V}{m} \right) \) we obtain \( I \equiv \left| \tilde{E}_0 \right|^2 \left( \frac{W}{m^2} \right) \), alternatively \( I = \left( |\sqrt{\epsilon_0 E^{\text{phys}}}|^2 \right) \) such that, calculated below, we have for \( K = 7 \)
\[ b I^K \sim 7.7 \times 10^7 \left( \frac{1}{m^3 s} \right) \] \hspace{1cm} (B.48)

These are the values of the parameters that are used in the main text.
References

[1] F. Spineanu and M. Vlad. The filamentation of the laser beam as a labyrinth instability (http://arxiv.org/pdf/1506.04245.pdf). arxiv, 2015.

[2] B.A. Trubnikov and S.K. Zhdanov. Unstable quasi-gaseous media. Physics Reports, 155(3):137 – 230, 1987.

[3] V.I. Talanov. Self-focusing of wave beams in nonlinear media. JETP Letters, 2:138–141, 1965.

[4] A.B. Schvartsburg. Self-constriction of a wave packet in a non-linear medium. Physics Letters A, 48(4):257 – 259, 1974.

[5] S. Tzortzakis, L. Bergé, A. Couairon, M. Franco, B. Prade, and A. Mysyrowicz. Breakup and fusion of self-guided femtosecond light pulses in air. Phys. Rev. Lett., 86:5470–5473, Jun 2001.

[6] W. Ettoumi, J. Kasparian, and J.-P. Wolf. Laser filamentation as a new phase transition universality class. Phys. Rev. Lett., 114:063903, Feb 2015.

[7] Raymond E. Goldstein, David J. Muraki, and Dean M. Petrich. Interface proliferation and the growth of labyrinths in a reaction-diffusion system. Phys. Rev. E, 53:3933–3957, Apr 1996.

[8] M. Mlejnek, M. Kolesik, J. V. Moloney, and E. M. Wright. Optically turbulent femtosecond light guide in air. Phys. Rev. Lett., 83:2938–2941, Oct 1999.

[9] M. Mlejnek, E. M. Wright, and J. V. Moloney. Dynamic spatial replenishment of femtosecond pulses propagating in air. Opt. Lett., 23(5):382–384, Mar 1998.

[10] G. Méchain, C.DAmico, Y.-B. André, S. Tzortzakis, M. Franco, B. Prade, A. Mysyrowicz, A. Couairon, E. Salmon, and R. Sauerbrey. Range of plasma filaments created in air by a multi-terawatt femtosecond laser. Optics Communications, 247(13):171 – 180, 2005.

[11] A. Couairon and A. Mysyrowicz. Femtosecond filamentation in transparent media. Physics Reports, 441(24):47 – 189, 2007.

[12] Luc Bergé. Wave collapse in physics: principles and applications to light and plasma waves. Physics Reports, 303(56):259 – 370, 1998.
[13] Rashimi C. Desai and Raymond Kapral. *Dynamics of self-organized and self-assembled structures*. Cambridge University Press, 2009.

[14] F. Spineanu and M. Vlad. A model for the reversal of the toroidal rotation in tokamak. *Nuclear Fusion*, 52:114019, 2012.

[15] B.A. Trubnikov, S.K. Zhdanov, and S.M. Zverev. *Hydrodynamics of unstable media*. CRC Press, 1996.

[16] Stefan Skupin, Ulf Peschel, Christoph Etrich, Lutz Leine, Dirk Michaelis, and Falk Lederer. Intense pulses in air: breakup of rotational symmetry. *Opt. Lett.*, 27(20):1812–1814, Oct 2002.