OPTIMAL REGULARITY FOR THE OBSTACLE PROBLEM FOR THE
p-LAPLACIAN

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ABSTRACT. In this paper we discuss the obstacle problem for the p-Laplace operator. We prove optimal growth results for the solution. Of particular interest is the point-wise regularity of the solution at free boundary points. The most surprising result we prove is the one for the p-obstacle problem: Find the smallest $u$ such that
$$\text{div}(\nabla u)^{p-2} \nabla u) \leq 0, \quad u \geq \phi, \quad \text{in } B_1,$$
with $\phi \in C^{1,1}(B_1)$ and given boundary datum on $\partial B_1$. We prove that the solution is uniformly $C^{1,1}$ at free boundary points. Similar results are obtained in the case of an inhomogeneity belonging to $L^\infty$. When applied to the corresponding parabolic problem, these results imply that any solution which is Lipschitz in time is $C^{1,\frac{p-1}{p}}$ in the spatial variables.

1. INTRODUCTION

1.1. Problem formulation. In this paper we consider the optimal regularity of minimizers of the constrained $p$-Dirichlet energy
$$\int_{B_1} \frac{\nabla v}{p}^p + f v \, dx, \quad v \in \mathbb{K} := \{ w \in W^{1,p}(B_1) : w \geq \phi, \ w = g \text{ on } \partial B_1 \},$$
where $B_1 \subset \mathbb{R}^n$ ($n \geq 2$) is the unit ball, and $\phi$ and $g$ are given functions (in appropriate spaces). This is equivalent to finding the smallest function $u$ such that
$$\Delta_p u \leq f, \quad u \geq \phi,$$
given the boundary conditions on $\partial B_1$. Here, and in the sequel, $\Delta_p u = \text{div}(\nabla u)^{p-2} \nabla u)$ is the $p$-Laplace operator and $1 < p < \infty$.

Of particular interest is the set $\Omega = \{ u > \phi \} \cap B_1$ and the free boundary $\Gamma = \partial \{ u > \phi \} \cap B_1$. To better understand the free boundary, $\Gamma$, it is important to first understand the point-wise regularity of the solution $u$. In the homogeneous case $f \equiv 0$, we prove, the rather “unexpected” result, that the point-wise regularity of $u$ at a free boundary point is the same as the regularity of the obstacle $\phi$, at least up to $C^{1,1}$. That is, if $\phi \in C^{1,1}$ then $u$ leaves $\phi$ in a quadratic fashion. This surprising result implies, in turn, that the presence of the obstacle actually improves the regularity of the solution to the solution to the $p$-harmonic obstacle problem, at free boundary points.
In the more general and inhomogeneous case we prove that if $\phi \in C^{1,\beta}$ and if $f \in L^\infty(B_1)$ then $u$ leaves $\phi$ in $r^{1+\alpha}$-fashion, where
\[
\alpha = \min\left(\frac{1}{p-1}, \beta\right).
\]

In the second part of the paper we apply the aforementioned results to the $p$-parabolic obstacle problem. The $p$-parabolic obstacle problem amounts to finding the smallest function $u$, defined on $B_1 \times (0, T)$, with given boundary and initial data, such that
\[
\begin{cases}
\Delta_p u - \frac{\partial u}{\partial t} \leq 0, \\
u \geq \phi.
\end{cases}
\]
Under the assumption that $\frac{\partial u}{\partial t} \in L^\infty$, we obtain the optimal growth in the spatial variable of order $1 + \alpha$, where
\[
\alpha = \min\left(\frac{1}{p-1}, \beta\right).
\]
In addition, we show that if the initial datum satisfies $|\Delta_p g| \leq C$ and the obstacle and the spatial boundary datum are Lipschitz in time, then so is the solution.

1.2. Known results. The non-degenerate elliptic obstacle problem, $p = 2$, is very well studied and the regularity properties of the solution are well known. It was proved by Frehse in [13] and Kinderlehrer in [16] (in two dimensions) that $u$ is $C^{1,1}$, provided the same is true for the obstacle. Later in [7] it was proved that the free boundary, except at cusp-like points, is a $C^\infty$ hypersurface. This result was sharpened even further in two dimensions by Monneau in [26]. A related but somewhat different problem was studied in [14] and [19]. See also [27] and [22] for regularity results relating to the $p$-harmonic obstacle problem. In [15], the corresponding result of Theorem 1 is proved, when $C^{1,\beta}$ is replaced by $C^{0,\alpha}$ for some small $\alpha \in (0, 1)$, for a more general class of quasilinear operators. The resemblance of the proofs is striking.

For the parabolic obstacle problem, there is a series of papers [4], [5], [3] and [2], where optimal regularity as well as the regularity of the free boundary is proved in the case when $p = 2$, for variable coefficients and variable right-hand side. In the papers [20] and [21], the right-hand side is allowed to be merely in $L^p$. In [28], the elliptic part of the operator is allowed to be fully nonlinear. A slightly more general free boundary problem of parabolic type is studied in [3], [12], [11] and [1].

In the $p$-parabolic case we refer the reader to the literature: [9], [17], [23], [24] and [18]. One of the authors studied a quite similar problem in [29].

1.3. Main idea. Roughly speaking, the main idea for the elliptic problem is the following: When the gradient is large then the equation is non-degenerate and classical estimates apply. But on the other hand, when the gradient is small we can rescale and obtain uniform bounds using the weak Harnack inequality (which applies to supersolutions).
1.4. Acknowledgement. We thank Peter Lindqvist for several encouraging and useful comments, after reading the manuscript at an early stage. The second author is supported by the Swedish Research Council, grant no. 2012-3124. The third author is partially supported by the Swedish Research Council.

2. The elliptic problem

In this part we treat the elliptic problem. Given an open, bounded set $\Omega$ and some boundary datum given by the restriction of $g \in W^{1,p}(\Omega)$ to $\partial \Omega$, we say that $u$ is a solution of the $p$-obstacle problem in $\Omega$ with obstacle $\phi \in C^{1,\beta}$, $g \geq \phi$, and with inhomogeneity $f \in L^\infty(B_1)$, if $u$ minimizes

$$\int_\Omega \frac{\lvert \nabla u \rvert^p}{p} + fu \, dx$$

subject to $u \geq \phi$ in $\Omega$ and $u = g$ on $\partial \Omega$.

The main result of this paper is the optimal growth at free boundary points.

**Theorem 1.** Let $p \in (1, \infty)$, $\beta \in (0, 1]$ and $u$ be a solution to the $p$-obstacle problem in $B_1$ with obstacle $\phi \in C^{1,\beta}(B_1)$ and $f \in L^\infty(B_1)$. Suppose further

$$\lVert \phi \rVert_{C^{1,\beta}(B_1)} \leq N, \quad \lVert f \rVert_{L^\infty(B_1)} \leq L.$$

Then for any point $y \in \Gamma \cap B_{1/2}$ and for $r < 1/2$

$$\sup_{x \in B_r(y)} \lvert u(x) - u(y) - (x - y) \cdot \nabla u(y) \rvert \leq C(N^{p-1} + L)^{\frac{1}{p-1}}r^{1+\alpha},$$

where $C = C(\beta, p)$ and

$$\alpha = \min \left( \frac{1}{p-1}, \beta \right),$$

and $\alpha = \beta$ if $f \equiv 0$. In particular,

$$\sup_{x \in B_r(y)} \lvert u(x) - \phi(x) \rvert \leq (C + 1)(N^{p-1} + L)^{\frac{1}{p-1}}r^{1+\alpha}.$$

**Proof.** We give the proof in the case $f \not\equiv 0$. The only difference in proving the case $f \equiv 0$ would be the scaling of order $\beta$. By simply considering the normalized function

$$\frac{u}{(N^{p-1} + L)^{\frac{1}{p-1}}}$$

we can assume that $u$ solves the $p$-obstacle problem with obstacle $\phi$ satisfying $\lVert \phi \rVert_{C^{1,\beta}(B_1)} \leq 1/2$ and with $\lVert f \rVert_{L^\infty(B_1)} \leq 1$. Then at any free boundary point $y$, we have $\lvert \nabla u(y) \rvert = \lvert \nabla \phi(y) \rvert \leq 1/2$. The proof is now divided into different cases. The correctly scaled estimate is then obtained in the end by multiplying the constant with $(N^{p-1} + L)^{\frac{1}{p-1}}$. 

Case 1: When $|\nabla u(y)| < r^\alpha < (1/2)^\alpha$: When $|\nabla u(y)| < r^\alpha$ it follows from the triangle inequality that
\[
\sup_{x \in B_r(y)} |u(x) - u(y)| \leq C r^{1+\alpha}
\]
implies
\[
\sup_{x \in B_r(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| \leq (C + 1) r^{1+\alpha}.
\]
It is therefore enough to prove
\[
\sup_{B_{r/2}(y)} |u(x) - u(y)| \leq C r^{1+\alpha},
\]
for some constant $C = C(\beta, p)$. To this end we define the rescaled functions
\[
\tilde{\phi}(x) = \frac{\phi(rx + y) - \phi(y)}{r^{1+\alpha}}
\]
and
\[
\tilde{u}(x) = \frac{u(rx + y) - u(y)}{r^{1+\alpha}}.
\]
We may estimate the $L^\infty$ norm of $\tilde{\phi}$ according to
\[
\|\tilde{\phi}\|_{L^\infty(B_1)} = \left\| \frac{\phi(rx + y) - \phi(y)}{r^{1+\alpha}} \right\|_{L^\infty(B_1)} \leq \left\| \frac{\phi(rx + y) - \phi(y) - r \nabla \phi(y) \cdot (x - y)}{r^{1+\alpha}} \right\|_{L^\infty(B_1)} + \left\| \frac{\nabla \phi(y) \cdot (x - y)}{r^{\alpha}} \right\|_{L^\infty(B_1)} \leq 3/2,
\]
where we used Proposition 9 in the appendix to conclude that $|\nabla \phi(y)| = |\nabla u(y)| \leq r^\alpha$.

We note that $\tilde{u}$ minimizes
\[
\int_{B_1} \frac{|
abla \tilde{u}|^p}{p} + \tilde{f} \tilde{u} \, dx
\]
with $\tilde{\phi}$ as obstacle and where
\[
\tilde{f}(x) = r^{1-\alpha(p-1)} f(rx),
\]
so that $\|\tilde{f}\|_{L^\infty(B_1)} \leq 1$. Thus,
\[
\Delta_p \tilde{u} \leq \tilde{f}, \quad \text{in } B_1.
\]
The weak Harnack inequality, see for instance Theorem 3.13 in [25], applied to the non-negative function $\tilde{u} + 3/2 \geq -\|\tilde{\phi}\|_{L^\infty} + 3/2 \geq 0$ implies
\[
\|\tilde{u} + 3/2\|_{L^s(B_{3/2})} \leq C_1(p) \inf_{B_1} (\tilde{u} + 3/2) \leq C_1(p) \left( \tilde{\phi}(0) + 3/2 \right) \leq 4C_1(p) = C_2(p).
\]
for some $s > 1$. Now, let
\[
v = \max(\tilde{u} + 3/2, \sup_{B_1} \tilde{\phi} + 3/2).
\]
Then $\Delta_p v \geq \hat{f}$ and thus from the sup-estimate for subsolutions (cf. Corollary 3.10 in [25]) we can conclude together with the estimate above that

$$\sup_{B_r} v \leq C_3(p)\|v\|_{L^s(B_{3r})} \leq C_3(p)C_2(p).$$

This implies, upon relabeling the constants, that

(2) $\sup_{B_r} \tilde{u} \leq C_3(p).$

Since moreover $\tilde{u} \geq \hat{\phi} \geq -3/2$, $\tilde{u}$ is uniformly bounded in $L^\infty(B_{1/2})$, which implies the desired estimate, for $r < 1/4$. In order to obtain the estimate for $r \in (1/4, 1/2)$ one just needs to increase the constant by $2^{1+\alpha}$.

**Case 2:** When $|\nabla u(y)| \geq r^\alpha$, $r < 1/2$: From Case 1 we know that

(3) $\sup_{B_{r}(y)} u(x) \leq C(p)r_y^{1+\alpha}$

where $r_y^\alpha = |\nabla u(y)|$. Let

$$\tilde{\phi}(x) = \frac{\hat{\phi}(r_y x + y) - \hat{\phi}(y)}{r_y^{1+\alpha}}.$$  

Then

(4) $|\nabla \tilde{\phi}(0)| = |\nabla \tilde{u}(0)| = 1.$

Define also

$$\tilde{u}(x) = \frac{u(r_y x + y) - u(y)}{r_y^{1+\alpha}}$$

and

$$\tilde{f}(x) = r_y^{1-\alpha(p-1)}f(r_y x).$$

Then $\tilde{u}$ solves the $p$-obstacle problem in $B_1$ with $\tilde{\phi}$ as an obstacle and with inhomogeneity $\tilde{f}$, satisfying $\|\tilde{f}\|_{L^n(B_1)} \leq 1$. Moreover, from the assumption $\|\hat{\phi}\|_{C^{1,\beta}} \leq 1/2$,

$$\|\tilde{\phi}\|_{C^{1,\beta}(B_{1/2})} \leq C_4,$$

and by (3) $\tilde{u}$ is uniformly bounded in $L^\infty(B_{1/2})$. From Proposition 9 it follows that

$$\|\tilde{u}\|_{C^{1,\tau}(B_{1/2})} \leq C_5, \quad \tau = \tau(p), \quad C_5 = C_5(p).$$

This together with (4) implies that we can find $r_0 = r_0(p)$ so that $|\nabla \tilde{u}| \geq 1/2$ in $B_{r_0}$. Hence, $\tilde{u}$ is a uniformly bounded solution to the obstacle problem for a uniformly elliptic

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1Observe that $\Delta_p v = \Delta_p (\max(\tilde{u} - \sup \hat{\phi}, 0) + 3/2 + \sup \hat{\phi}) = \hat{f}$ and $\Delta_p (\tilde{u} - \sup \hat{\phi}) = \hat{f}$ in the set $\{\tilde{u} - \sup \hat{\phi} > 0\}$ and it is zero outside this set. By taking a test function of the form $\phi \eta((\tilde{u} - \sup \hat{\phi})^+)$, with $\phi \in C_0^\infty(B_1)$ and $\eta$ a linear approximation of the identity, it follows that $\Delta_p v \geq \hat{f}$. 


operator with $C^r$-coefficients in $B_{r_0}$, with a $C^{1,\beta}$ regular obstacle and uniformly bounded inhomogeneity. From Proposition 9 and Proposition 10
\[ \| \tilde{u} \|_{C^{1,\alpha}(B_{r_0})} \leq C(p, \beta). \]
Scaling back we obtain
\[ \| u \|_{C^{1,\alpha}(B_{r_0\cdot r}(y))} \leq C(p, \beta), \]
which in particular implies
\[ \sup_{B_{r}(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| \leq C r^{1+\alpha}, \]
for $r < r_0 r_y = r_0 |\nabla u(y)|^{\frac{1}{\alpha}}$.

Conclusion: In both Case 1 and Case 2 we concluded that
\[ \sup_{B_{r}(y)} |u(x) - u(y) - x \cdot \nabla u(y)| \leq C r^{1+\alpha}, \]
for all $r < 1/2$ such that $r \leq r_0 |\nabla u(y)|^{\frac{1}{\alpha}}$ (Case 2) and when $r > |\nabla u(y)|^{\frac{1}{\alpha}}$ (Case 1). Therefore we only need to fill the gap when
\[ r_0 |\nabla u(y)|^{\frac{1}{\alpha}} < r < |\nabla u(y)|^{\frac{1}{\alpha}}. \]
Assume that $r$ is in the interval specified in (5). Then
\[ \sup_{B_{r}(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| \leq \sup_{B_{r}(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)|. \]
Hence,
\[ \sup_{B_{r}(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| \leq \sup_{B_{r}(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| \leq C r_0^{1+\alpha} + C r_{y}^{1+\alpha} + r^{1+\alpha}. \]
We thus have the estimate for all $r < 1/2$. To obtain the estimate for the original $u$ (not rescaled by a factor $(N^{p-1} + L)^{\frac{1}{r-1}}$) one just needs to multiply the constant $C$ with $(N^{p-1} + L)^{\frac{1}{r-1}}$.

The last estimate follows from
\[ \sup_{x \in B_{r}(y)} |u(x) - \phi(x)| = \sup_{x \in B_{r}(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y) + \phi(y) + (x - y) \cdot \nabla \phi(y) - \phi(x)| \leq (C + 1) r^{1+\alpha}. \]
3. NON-DEGENERACY AND POROSITY OF THE FREE BOUNDARY IN THE HOMOGENEOUS CASE

In this section we treat the homogeneous case in more detail, assuming also that \( p > 2 \).

We prove by standard arguments that the difference \( u - \phi \) cannot decay faster than quadratic around free boundary points. This combined with the optimal quadratic growth implies, by a standard argument, that the free boundary \( \Gamma \) is porous. We recall that \( \Gamma \cap B_{1/2} \) is said to be porous if there exists a \( \delta > 0 \) such that for every \( y \in \Gamma \cap B_{1/2} \) and \( r \in (0, 1/4) \)

\[
\frac{|\Gamma \cap B_r(y)|}{|B_r(y)|} \leq 1 - \delta.
\]

Since this directly implies that \( \Gamma \) has no Lebesgue points it follows that the free boundary has measure zero. The notion of porosity was introduced in [10]; See also the survey [30].

**Proposition 2.** Let \( p \in (2, \infty) \) and let \( u \) be a solution to the \( p \)-obstacle problem in \( B_1 \) with obstacle \( \phi \in C^2(B_1) \) with \( f \equiv 0 \). Suppose further that \( \Delta_p \phi < 0 \). Then there is a constant \( \varepsilon = \varepsilon(\sup \Delta_p \phi) \) such that for any \( x^0 \in \Gamma \) and \( r < \text{dist}(x^0, \partial B_1) \) there holds

\[
\sup_{\partial B_r(x^0) \cap \{u > \phi\}} (u - \phi) \geq \varepsilon r^2.
\]

**Proof.** The proof is standard. Take \( y \in \{u > \phi\} \). Let \( v(x) = \phi(x) + \varepsilon|x - y|^2 \), where \( \varepsilon \) is chosen small enough such that \( \Delta_p v < 0 \). This is indeed possible since \( \Delta_p v \) is continuous with respect to \( \varepsilon \). Pick \( r < \text{dist}(x^0, \partial B_1) \). Then \( \Delta_p u = 0 \geq \Delta_p v \) in \( \{u > \phi\} \cap B_r(x^0) \).

Moreover, \( u(y) \geq \phi(y) = v(y) \). From the comparison principle it follows that there is \( z_y \in \partial (\{u > \phi\} \cap B_r(x^0)) \) such that \( u(z_y) \geq v(z_y) \). Since \( u < v \) on \( B_r(x^0) \cap \partial \{u > \phi\} \) there must be \( z_y \in \{u > \phi\} \cap \partial B_r(x^0) \) such that \( u(z_y) \geq v(z_y) \). The result follows by continuity and by letting \( y \to x^0 \).

**Remark 3.** The only place where we need to impose the condition \( p > 2 \) is in the proof above. The proof requires that

\[
\Delta_p (\phi(x) + \varepsilon|x - y|^2)
\]

is continuous with respect to \( \varepsilon \). This is not necessarily true when \( p < 2 \). However, we believe that the result holds true even in the case \( p < 2 \), and that the assumption is merely an artifact of the proof.

We are now ready to give the proof of porosity.

**Corollary 4.** Under the assumptions in Proposition 2, the free boundary is porous. In particular it has Lebesgue measure zero.

**Proof.** At a closer look at Theorem 1, we see that in fact we prove

\[
|u(x) - \phi(x)| \leq CN(\text{dist}(x, \Gamma))^2
\]
for $x \in B_{\frac{1}{2}}$, given that $\phi$ is $C^{1,1}$. Now, pick $x^0 \in \Gamma$. By Proposition 2 for $r$ small enough, there is $y \in \partial B_r(x^0)$ such that

$$u(y) - \phi(y) \geq \varepsilon r^2.$$  

Combining the above estimates we arrive at

$$\varepsilon r^2 \leq CN(\text{dist}(y, \Gamma))^2,$$

so that

$$\text{dist}(y, \Gamma) > \left(\frac{\varepsilon}{CN}\right)^{\frac{1}{2}} r := \delta r.$$  

Hence,

$$\Gamma \cap B_{\delta r}(y) = \emptyset.$$  

Hence, we can find a point $z \in B_{\delta r}(y)$ so that

$$B_{\frac{1}{2}r}(z) \subset B_{\delta r}(y) \cap B_r(x^0),$$

which implies

$$\frac{|\Gamma \cap B_r(x^0)|}{|B_r(x^0)|} \leq 1 - \left(\frac{\delta}{2}\right)^n,$$  

which means exactly that $\Gamma$ is porous. From Lebesgue’s density theorem it follows that $\Gamma$ has zero Lebesgue density. □

4. APPLICATION TO THE PARABOLIC PROBLEM

In this part we mention an application of the previous results to the parabolic problem introduced earlier. Throughout this section, $q = \frac{p}{p-1}$. We introduce the notation

$$Q_r^- (x, t) = B_r(x, t) \times (-r^q + t, t], \quad \partial_p Q_r^- (x, t) = \partial B_r(x, t) \times (-r^q + t, t] \cup B_r(x, t) \times \{t\},$$

with the simplification $Q_r^- = Q_r^- (0, 0)$. Given boundary datum $g$ on $\partial_p Q_r^-$, we say that $u$ is a solution of the $p$-parabolic obstacle problem in $Q_r^-$ with obstacle $\phi$ if $u$ satisfies

$$\begin{align*}
\max(\Delta_p u - u_t, u - \phi) &= 0 \text{ in } Q_r^- , \\
u &= g \text{ on } \partial_p Q_r^-.
\end{align*}$$

As a simple corollary of Theorem 1 we obtain that if a solution is Lipschitz in time, then it has the optimal growth of order $q$ in the spatial variables, at free boundary points. We also give an example of assumptions under which the solution is Lipschitz in time. The main result of this section is stated below:

**Theorem 5.** Let $p \in (1, \infty)$ and let $u$ be a solution to the $p$-parabolic obstacle problem in $Q_1^-$ with obstacle $\phi \in C^2(Q_1^-)$. Suppose further that

$$|u_t| \leq L, \quad \|\phi\|_{C^2(Q_1^-)} \leq N.$$  

Then for any point $(y, s) \in \Gamma \cap Q_{1/2}^-$ and for $r < 1/4$

$$\sup_{(x, t) \in Q_r^- (y, s)} |u(x, t) - u(y, s) - (x - y) \cdot \nabla u(y, s)| \leq Cr^q, \quad q = \frac{p}{p-1}.$$
where $C = C(p, L, N)$.

The assumption that the solution is Lipschitz in time in Theorem 5 is rather unsatisfactory since we do not know if it is true in general even for solutions of the equation $u_t = \Delta_p u$, without the presence of an obstacle. However, if the initial datum $g$ satisfies $|\Delta_p g| \leq C$ and the obstacle and the boundary datum are Lipschitz in time, then so is the solution, as is shown below. In the following lemma we will, for notational convenience, assume that the solution is defined in $Q^+_1 \cap \Gamma$ instead of $Q^+_1$.

**Lemma 6.** Let $p \in (1, \infty)$. Assume that $u$ is a solution to the $p$-parabolic obstacle problem in $Q^+_1 \cap \Gamma$ with obstacle $\phi$ and spatial boundary datum $f$ and initial datum $g$. Suppose further that $|f| \leq N$, $|\phi_t| \leq N$ in $Q^+_1$, $|\Delta_p g| \leq N$, for some constant $N > 0$. Then $|u_t| \leq N$.

**Proof.** The function $g(x) + Nt$ is a supersolution of the equation. Moreover, $g(x) + N t \geq u(x, t)$ on $\partial B_1$ and for $t = 0$. In addition, since $g(x) \geq \phi(x, 0)$ and $|\phi_t| \leq N$, also $g(x) + N t \geq \phi(x, 0) + N t \geq \phi(x, t)$. Hence, $g(x) + N t \geq u(x, t)$ in $Q^+_1$. Now, let $w(x, t) = u(x, t + h)$ for some $h > 0$. Then $w(x, t) = f(x, t + h) \leq f(x, t) + Nh$ for $x \in \partial B_1$ and $w(x, 0) = u(x, h) \leq g(x) + Nh$. Moreover, $w(x, t) \geq \phi(x, t + h)$, where $\phi(x, t + h) \leq \phi(x, t) + Nh$. Hence $w$ is less than or equal to the solution of the $p$-parabolic obstacle problem in $Q^+_1$ with spatial boundary datum $f(x, t) + Nh$, initial datum $g(x) + Nh$ and obstacle $\phi(x, t) + Nh$. This solution is of course given by $u(x, t) + Nh$ so that we obtain $u(x, t + h) = w(x, t) \leq u(x, t) + Nh$. By similar arguments, $u(x, t + h) \geq u(x, t) - Nh$. □

The corollary below is immediate.

**Corollary 7.** Let $p \in (1, \infty)$. Assume the hypotheses of Lemma 6 and that $\phi \in C^2(Q^-_1)$. Then for any point $(y, s) \in \Gamma \cap Q^-_{1/2}$ and for $r < 1/4$

$$
\sup_{(x, t) \in Q^-_{r}(y, s)} |u(x, t) - u(y, s) - (x - y) \cdot \nabla u(y, s)| \leq Cr^q,
$$

where $C = C(p, N, \|\phi\|_{C^2(Q^-_1)})$.

**Remark 8.** It seems plausible that without any boundedness condition on $D_t u$, one should be able to deduce an optimal growth for $u$, of order $q = p/(p - 1)$, from the free boundary. An argument, used by one of the authors (see [29]), and based on scaling and blow-up technique was done for the Stefan problem (with the assumption $D_t u \geq 0$). Although we could not reverify the inequality (2.15) therein, and what follows, of the main result in [29], we still believe that the statement should be true even under weaker assumptions (with no bound on $D_t u$).

In the scaling argument of [29], one ends up with a global solution $u_0$ for $t < 0$. The solution also behaves like $R^q$ for large $R$. Meanwhile we also have $u_0 \geq 0$, and $u_0(0, 0) = 0$. With the extra condition $D_t u_0 \geq 0$, and hence $u_0(0, t) = 0$ for $t < 0$. It is also not hard to conclude that $u_0$ is a solution to the $p$-parabolic equation; indeed any solution to the
homogeneous obstacle problem with zero obstacle is a solution to the equation itself. One can use these properties along with intrinsic Harnack’s inequality to conclude a behavior of type \(c (|x|^p / - t)^{1/(p-2)}\) for \(t < 0\). The question, that we were not able to answer, is whether such a non-trivial solution exists? The answer we conjecture is that no, there is no such non-trivial solution.

It is noteworthy that the function

\[ U = c_p (|x|^p / - t)^{1/(p-2)}, \quad t < 0, \]

and with \(c_p = (p-2) \left( \frac{p}{p-2} \right)^{p-1} \left( \frac{p}{p-2} + n \right)^{1/p} \) is a solution to the equation. Another example is that of Barenblatt

\[ B_p = t^{-n/p} \left( c - \frac{p-2}{p} \lambda^{1/(p-1)} \left( \frac{|x|}{t^{1/p}} \right)^{p/(p-1)} \right)^{(p-1)/(p-2)}. \]

Here one can see two types of behavior for the Barenblatt solution, one is when the solution touches zero-obstacle, where we see an order of \((p-1)/(p-2)\) and the next is along the \(t\)-axis, with order \(p/(p-1)\).

Another example with order \(p/(p-1)\) can be given by

\[ u = c_p (x_+)^{p/(p-1)} + t, \quad c_p = (p/(p-1))^{p-1}, \quad 1 < p < \infty, \]

where the obstacle is \(\phi = t\).

Yet another example with zero obstacle is

\[ u = A (1 - x + ct)^{\frac{p-2}{p-1}}, \quad A = c_p^{1/(p-1)} \left( \frac{p-2}{p-1} \right)^{\frac{p-1}{p-2}}. \]

In the examples above we see three different behaviors

\[ r^{p-1}, \quad r^{\frac{p-2}{p-1}}, \quad r^{\frac{p}{p-2}}. \]

It is tantalizing to find out how many different growth rates that can be found for solutions, and what are the largest and smallest rates.

5. Appendix

In this section, we recall some well known facts. The proposition below states that if the obstacle is in \(C^{1,\beta}(B_1)\) and the inhomogeneity is in \(L^\infty(B_1)\), then any bounded solution to the \(p\)-obstacle problem with is locally in \(C^{1,\alpha}\) for some \(\alpha\), see [27] and [22].

**Proposition 9.** Let \(p \in (1, \infty)\) and \(u\) solve the \(p\)-obstacle problem in \(B_1\) with \(\phi \in C^{1,\beta}(B_1)\) as obstacle and \(f \in L^\infty(B_1)\) as inhomogeneity. Then there is \(\alpha(p, \|u\|_{L^\infty(B_1)}, \|\phi\|_{C^{1,\beta}(B_1)}, \|f\|_{L^\infty(B_1)})\) such that

\[ \|u\|_{C^{1,\alpha}(B_{\frac{1}{2}})} \leq C(p, \|u\|_{L^\infty(B_1)}, \|\phi\|_{C^{1,\beta}(B_1)}, \|f\|_{L^\infty(B_1)}). \]
It is also well known that the solution to the obstacle problem for a uniformly elliptic operator with $C^\alpha$ coefficients leaves the obstacle in a $r^{1+\gamma}$-fashion if the obstacle is $C^{1,\gamma}$-regular. This follows for instance from Corollary 2.6 in [8].

**Proposition 10.** Let $u$ be a solution of the following obstacle problem: The smallest $u$ such that
\[
\begin{cases}
\text{div}(A \nabla u) \leq f \\
u \geq \phi
\end{cases}
in B_1
\]
where $\phi \in C^{1,\gamma}(B_1)$ with $\gamma \in (0, 1)$, $A \in C^\alpha(B_1)$, $f \in L^\infty(B_1)$ and
\[
\lambda|\xi|^2 \leq A\xi \cdot \xi \leq \Lambda|\xi|^2, \quad 0 < \lambda < \Lambda
\]
for all $\xi \in \mathbb{R}^n$. Then for $y \in \partial\{u > \phi\}$ and $r < 1/2$
\[
\sup_{B_r(y)} |u(x) - u(y) - (x - y) \cdot \nabla \phi(y)| \leq Cr^{1+\gamma},
\]
where $C = C(p, \|u\|_{L^\infty(B_1)}, \|\phi\|_{C^{1,\gamma}(B_1)}, \|A\|_{C^\alpha(B_1)}, \|f\|_{L^\infty(B_1)}, \Lambda, \lambda, \gamma)$. In the homogeneous case $f \equiv 0$ we are allowed to choose $\gamma = 1$.

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