On semi-equivalence of generically-finite polynomial mappings

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Abstract Let \( f, g : X \to Y \) be continuous mappings. We say that \( f \) is topologically equivalent to \( g \) if there exist homeomorphisms \( \Phi : X \to X \) and \( \Psi : Y \to Y \) such that \( \Psi \circ f \circ \Phi = g \). Moreover, we say that \( f \) is topologically semi-equivalent to \( g \) if there exist open, dense subsets \( U, V \subset X \) and homeomorphisms \( \Phi : U \to V \) and \( \Psi : Y \to Y \) such that \( \Psi \circ f \circ \Phi|_U = g|_U \). Let \( X, Y \) be smooth irreducible affine complex varieties. We show that every algebraic family \( F : M \times X \ni (m, x) \mapsto F(m, x) = f_m(x) \in Y \) of polynomial mappings contains only a finite number of topologically non-equivalent proper mappings and only a finite number of topologically non-semi-equivalent generically-finite mappings.

In particular there are only a finite number of classes of topologically non-equivalent proper polynomial mappings \( f : \mathbb{C}^n \to \mathbb{C}^m \) of a bounded (algebraic) degree. The same is true for a number of classes of topologically non-semi-equivalent generically-finite polynomial mappings \( f : \mathbb{C}^n \to \mathbb{C}^m \) of a bounded (algebraic) degree.

Mathematics Subject Classification 14 D 99 · 14 R 99 · 51 M 99

1 Introduction

Let \( f, g : X \to Y \) be continuous mappings. We say that \( f \) is topologically equivalent to \( g \) if there exist homeomorphisms \( \Phi : X \to X \) and \( \Psi : Y \to Y \) such that \( \Psi \circ f \circ \Phi = g \). Moreover, we say that \( f \) is topologically semi-equivalent to \( g \) if there exist open, dense subsets \( U, V \subset X \) and homeomorphisms \( \Phi : U \to V \) and \( \Psi : Y \to Y \) such that \( \Psi \circ f \circ \Phi|_U = g|_U \).

In the case \( X = \mathbb{C}^n \) and \( Y = \mathbb{C} \) René Thom stated a Conjecture that there are only finitely many topological types of polynomials \( f : X \to Y \) of bounded degree. This Conjecture was confirmed by Fukuda [2]. Also a more general problem was considered: how many
topological types are there in the family $P(n, m, k)$ of polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ of degree bounded by $k$? Aoki and Noguchi [1] showed that there are only a finite number of topologically non-equivalent mappings in the family $P(2, 2, k)$. Finally Nakai [8] showed that each family $P(n, m, k)$, where $n, m, k > 3$, contains infinitely many different topological types even if we consider only generically-finite mappings. Hence the General Thom Conjecture is not true even for generically-finite mappings. However, we show in this paper that there are only a finite number of classes of topologically semi-equivalent generically-finite polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ of a bounded (algebraic) degree.

As a by product of our considerations we give a simple proof of the following interesting fact: for every $n, m$ and $k$ there are only a finite number of topological types of proper polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ of (algebraic) degree bounded by $k$. Hence we can say that Thom Conjecture is true for proper polynomial mappings. We show also that if $n \leq m$ and $\Omega_n(d_1, \ldots, d_m)$ denotes the family of all polynomial mappings $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ of a multi-degree bounded by $(d_1, \ldots, d_m)$, then any two general member of this family are topologically equivalent.

In fact we prove more: if $X, Y$ are smooth affine irreducible varieties, then every algebraic family $\mathcal{F}$ of polynomial mappings from $X$ to $Y$ contains only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) mappings. Moreover, if a family $\mathcal{F}$ is irreducible, then two generic members of $\mathcal{F}$ are in the same equivalence class.

Let us recall here, that a mapping $f : X \to Y$ is generically finite, if for general $x \in X$ the set $f^{-1}(f(x))$ is finite. Our proof goes as follows. Let $M$ be a smooth affine irreducible variety and let $\mathcal{F}$ be a family of polynomial mappings induced be a regular mapping $F : M \times X \to Y$, i.e., $\mathcal{F} := \{ f_m : X \ni x \mapsto F(m, x) \in Y, m \in M \}$.

Let us recall that if $f : X \to Z$ is a generically finite polynomial mapping of affine varieties, then the set $\{ z \in Z : z \in \text{Sing}(Z) \text{ or } \# f^{-1}(z) \neq \mu(f) \}$, where $\mu(f)$ is the topological degree of $f$. The set $B(f)$ is always closed in $Z$. We show that there exists a Zariski open, dense subset $U$ of $M$ such that

1. for every $m \in U$ we have $\mu(f_m) = \mu(\mathcal{F})$, where we treat $f_m$ as a mapping $f_m : X \to Z_m := \overline{f_m(X)}$,
2. for every $m_1, m_2 \in U$ the pairs $(f_{m_1}(X), B(f_{m_1}))$ and $(f_{m_2}(X), B(f_{m_2}))$ are equivalent via a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \to Y$ such that $\Psi(f_{m_1}(X)) = f_{m_2}(X)$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

In particular the group $G = \pi_1(\overline{f_m(X)} \backslash B(f_m))$ does not depend on $m \in U$. Using elementary facts from the theory of topological coverings, we show that the number of topological semi-types (types) of generically-finite (proper) mappings in the family $\mathcal{F}_U$ is bounded by the number of subgroups of $G$ of index $\mu(\mathcal{F})$, hence it is finite. Then we conclude the proof by induction. Finally, the case of arbitrary $M$ can be easily reduced to the smooth, irreducible, affine case.

Remark 1.1 In this paper we use the term “polynomial mapping” for every regular mapping $f : X \to Y$ of affine varieties.

2 Bifurcation set

Let $X, Z$ be affine irreducible varieties of the same dimension and assume that $X$ is smooth. Let $f : X \to Z$ be a dominant polynomial mapping. It is well known that there is a Zariski open non-empty subset $U$ of $Z$ such that for every $x_1, x_2 \in U$ the fibers $f^{-1}(x_1), f^{-1}(x_2)$...
have the same number \( \mu(f) \) of points. We say that \( \mu(f) \) is the topological degree of \( f \). Recall the following (see [5,6]).

**Definition 2.1** Let \( X, Z \) be as above and let \( f : X \to Z \) be a dominant polynomial mapping. We say that \( f \) is **finite at a point** \( z \in Z \) if there exists an open neighborhood \( U \) of \( z \) such that the mapping \( f_{|\mu^{-1}(U)} : \mu^{-1}(U) \to U \) is proper.

It is well-known that the set \( S_f \) of points at which the mapping \( f \) is not finite is either empty or it is a hypersurface (see [5,6]). We say that \( S_f \) is the set of **non-properness** of \( f \).

**Definition 2.2** Let \( X \) be a smooth affine \( n \)-dimensional variety and let \( Z \) be an affine variety of the same dimension. Let \( f : X \to Z \) be a generically finite dominant polynomial mapping of geometric degree \( \mu(f) \). The bifurcation set of \( f \) is

\[
B(f) = \{ z \in Z : z \in \text{Sing}(Z) \text{ or } \#f^{-1}(z) \neq \mu(f) \}.
\]

**Remark 2.3** The same definition makes sense for those continuous mapping \( f : X \to Z \), for which we can define the topological degree \( \mu(f) \) and singularities of \( Z \). In particular if \( Z_1, Z_2 \) are affine algebraic varieties, \( f : X \to Z_1 \) is a dominant polynomial mapping and \( \Phi : Z_1 \to Z_2 \) is a homeomorphism which preserves singularities, then we can define \( B(\Phi \circ f) \) as \( \Phi(B(f)) \). Moreover, the mapping \( \Phi \circ f \) behaves topologically as an analytic covering. We will use this facts in the proof of Theorem 3.5.

We have the following theorem (see also [7]).

**Theorem 2.4** Let \( X, Z \) be affine irreducible complex varieties of the same dimension and suppose \( X \) is smooth. Let \( f : X \to Z \) be a polynomial dominant mapping. Then the set \( B(f) \) is closed and \( B(f) = K_0 \cup S_f \cup \text{Sing}(Z) \).

**Proof** Let us note that outside the set \( S_f \cup \text{Sing}(Z) \) the mapping \( f \) is a (ramified) analytic covering of degree \( \mu(f) \). By Lemma 2.5 below, if \( z \notin \text{Sing}(Z) \) we have \( \#f^{-1}(z) \leq \mu(f) \). Moreover, since \( f \) is an analytic covering outside \( S_f \cup \text{Sing}(Z) \) we see that for \( y \notin S_f \cup \text{Sing}(Z) \) the fiber \( f^{-1}(y) \) has exactly \( \mu(f) \) points counted with multiplicity. Take \( X_0 := X \setminus f^{-1}(\text{Sing}(Z) \cup S_f) \). If \( z \in K_0(f_{|X_0}) \), the set of critical values of \( f_{|X_0} \), then \( \#f^{-1}(z) < \mu(f) \).

Now let \( z \in S_f \setminus \text{Sing}(Z) \). There are two possibilities:

(a) \( \#f^{-1}(z) = \infty \).

(b) \( \#f^{-1}(z) < \infty \).

In case (b) we can assume that \( f^{-1}(z) \neq \emptyset \). Let \( U \) be an affine neighborhood of \( z \) disjoint from \( \text{Sing}(Z) \) over which the mapping \( f \) has finite fibers. Let \( V := f^{-1}(U) \). By the Zariski Main Theorem in the version given by Grothendieck (see [3]), there exists a normal variety \( \overline{V} \) and a finite mapping \( \overline{f} : \overline{V} \to U \) such that

1. \( V \subset \overline{V} \),
2. \( \overline{f}_{|V} = f \).

Since \( z \in \overline{f}(\overline{V} \setminus V) \), it follows from Lemma 2.5 below that \( \#f^{-1}(z) < \mu(f) \). Consequently, if \( z \in S_f \), we have \( \#f^{-1}(z) < \mu(f) \). Finally, we have \( B(f) = K_0(f_{|X_0}) \cup S_f \cup \text{Sing}(Z) \). However, the set \( K_0(f_{|X_0}) \) is closed in \( Z \setminus (S_f \cup \text{Sing}(Z)) \). Hence \( B(f) \) is closed in \( Z \).

**Lemma 2.5** Let \( X, Z \) be affine normal varieties of the same dimension. Let \( f : X \to Z \) be a finite mapping. Then for every \( z \in Z \) we have \( \#f^{-1}(z) \leq \mu(f) \).
\begin{proof}
Let \( \# f^{-1}(z) = \{x_1, \ldots, x_r\} \). We can choose a function \( h \in \mathbb{C}[X] \) which separates all \( x_i \) (in particular we can take as \( h \) the equation of a general hyperplane section). Since \( f \) is finite, the minimal equation of \( h \) over the field \( \mathbb{C}(Z) \) is of the form:

\[ T^s + a_1(f)T^{s-1} + \cdots + a_s(f) \in f^*\mathbb{C}[Z][T], \]

where \( s \leq \mu(f) \). If we substitute \( f = z \) into this equation we get the desired result. \( \square \)

\section{3 Main result}

We start with the following:

\begin{lemma}
Let \( f : X^k \rightarrow Y^l \) be a dominant polynomial mapping of affine irreducible varieties. There exists a Zariski open non-empty subset \( U \subset Y \) such that for any \( y \in U \) we have \( \text{Sing}(f^{-1}(y)) = f^{-1}(y) \cap \text{Sing}(X) \).
\end{lemma}

\begin{proof}
We can assume that \( Y \) is smooth. Since there exists a mapping \( \pi : Y^l \rightarrow \mathbb{C}^l \) which is generically etale, we can assume that \( Y = \mathbb{C}^l \). Let us recall that if \( Z \) is an algebraic variety, then a point \( z \in Z \) is smooth if and only if the local ring \( \mathcal{O}_z(Z) \) is regular, or equivalently \( \dim \mathcal{C} \ m/m^2 = \dim Z \), where \( m \) denotes the maximal ideal of \( \mathcal{O}_z(Z) \).

Let \( y = (y_1, \ldots, y_l) \in \mathbb{C}^l \) be a sufficiently generic point. Then by Sard’s Theorem the fiber \( Z = f^{-1}(y) \) is smooth outside \( \text{Sing}(X) \) and \( \dim Z = \dim X - l = k - l \). Note that the generic (scheme-theoretic) fiber \( F \) of \( f \) is reduced. Indeed, this fiber \( F = \text{Spec}(\mathbb{C}(Y) \otimes_{\mathbb{C}[Y]} \mathbb{C}[X]) \) is the spectrum of a localization of \( \mathbb{C}[X] \) and so a domain. Since we are in characteristic zero, the reduced \( \mathbb{C}(Y) \)-algebra \( \mathbb{C}(Y) \otimes_{\mathbb{C}[Y]} \mathbb{C}[X] \) is necessarily geometrically reduced (i.e. stays reduced after extending to an algebraic closure of \( \mathbb{C}(Y) \)). Since the property of fibres being geometrically reduced is open on the base, i.e. on \( Y \), thus the fibers over an open subset of \( Y \) will be reduced. Consequently, there is a Zariski open, non-empty subset \( U \subset Y \) such that for \( y \in U \) the fiber \( f^{-1}(y) \) is reduced. Hence we can assume that \( Z \) is reduced. It is enough to show that every point \( z \in Z \cap \text{Sing}(X) \) is singular on \( Z \).

Assume that \( z \in Z \cap \text{Sing}(X) \) is smooth on \( Z \). Let \( f : X \rightarrow \mathbb{C}^l \) be given as \( f = (f_1, \ldots, f_i) \), where \( f_i \in \mathbb{C}[X] \). Then \( \mathcal{O}_z(Z) = \mathcal{O}_z(X)/(f_1 - y_1, \ldots, f_i - y_i) \). In particular if \( m' \) denotes the maximal ideal of \( \mathcal{O}_z(Z) \) and \( m \) denotes the maximal ideal of \( \mathcal{O}_z(X) \) then \( m' = m/(f_1 - y_1, \ldots, f_i - y_i) \). Let \( \alpha_i \) denote the class of the polynomial \( f_i - y_i \) in \( m/m^2 \). Let us note that

\[ m'/m^2 = m/(m^2 + (\alpha_1, \ldots, \alpha_l)). \]

Since the point \( z \) is smooth on \( Z \) we have \( \dim \mathcal{C} \ m'/m^2 = \dim Z = \dim X - l \). Take a basis \( \beta_1, \ldots, \beta_{k-l} \) of the space \( m'/m^2 \) and let \( \overline{\beta_i} \in m/m^2 \) correspond to \( \beta_i \) under the correspondence (1). Note that the vectors \( \overline{\beta_1}, \ldots, \overline{\beta_{k-l}}, \alpha_1, \ldots, \alpha_l \) generate the space \( m/m^2 \). This means that \( \dim \mathcal{C} \ m/m^2 \leq k - l + l = k = \dim X \). Hence the point \( z \) is smooth on \( X \), a contradiction. \( \square \)

We have:

\begin{lemma}
Let \( X, Y \) be smooth complex irreducible algebraic varieties and \( f : X \rightarrow Y \) a regular dominant mapping. Let \( N \subset W \subset X \) be closed subvarieties of \( X \). Then there exists a non-empty Zariski open subset \( U \subset Y \) such that for every \( y_1, y_2 \in U \) the triples \( (f^{-1}(y_1), W \cap f^{-1}(y_1), N \cap f^{-1}(y_1)) \) and \( (f^{-1}(y_2), W \cap f^{-1}(y_2), N \cap f^{-1}(y_2)) \) are homeomorphic.
\end{lemma}
Lemma 3.4 Let $X$ be an algebraic completion of $X$ and let $\overline{Y}$ be a smooth algebraic completion of $Y$. Take $X' := \text{graph}(f) \subset X \times \overline{Y}$ and let $X_2$ be a desingularization of $X'$.

We can assume that $X \subset X_2$. We have an induced mapping $\overline{f} : X_2 \to \overline{Y}$ such that $\overline{f}|_X = f$. Let $Z = X_2 \setminus X$. Denote by $\overline{N}, \overline{W}$ the closures of $N$ and $W$ in $X_2$. Let $\mathcal{R} = \{\overline{N} \cap Z, \overline{W} \cap Z, \overline{N}, \overline{W}, Z\}$, a collection of algebraic subvarieties of $X_2$. There is a Whitney stratification $S$ of $X_2$ which is compatible with $\mathcal{R}$.

For any smooth strata $S_i \in S$ let $B_i$ be the set of critical values of the mapping $\overline{f}|_{S_i}$ and denote $B = \bigcup B_i$. Take $X_3 = X_2 \setminus \overline{f}^{-1}(B)$. The restriction of the stratification $S$ to $X_3$ gives a Whitney stratification which is compatible with the family $\mathcal{R}' := \mathcal{R} \cap X_3$. We have a proper mapping $f' := \overline{f}|_{X_3} : X_3 \to \overline{Y} \setminus B$ which is a submersion on each stratum. By the Thom first isotopy theorem there is a trivialization of $f'$ which preserves the strata. It is an easy observation that this trivialization gives a trivialization of the mapping $f : X \setminus f^{-1}(B) \to Y \setminus B := U$. In particular the fibers $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are homeomorphic via a stratum preserving homeomorphism. This means that the triples $(f^{-1}(y_1), W \cap f^{-1}(y_1), N \cap f^{-1}(y_1))$ and $(f^{-1}(y_2), W \cap f^{-1}(y_2), N \cap f^{-1}(y_2))$ are homeomorphic.

We also need the following:

Definition 3.3 Let $X, Y$ be smooth affine varieties. By a family of regular mappings $\mathcal{F}_M(X, Y, F) := \mathcal{F}$ we mean a regular mapping $F : M \times X \to Y$, where $M$ is an algebraic variety. The members of a family $\mathcal{F}$ are the mappings $f_m : X \ni x \to F(m, x) \in Y$. Let $G : M \times X \ni (m, x) \mapsto (m, F(m, x)) \in Z = G(M \times X) \subset M \times Y$.

If $G$ is generically finite, then by the topological degree $\mu(\mathcal{F})$ we mean the number $\mu(G)$. Otherwise we put $\mu(\mathcal{F}) = 0$.

Later we will sometimes identify the mapping $f_m$ with the mapping $G(m, \cdot) = (m, f_m) : X \to m \times Y$. The following lemma is important:

Lemma 3.4 Let $X, Y$ be smooth affine complex varieties. Let $M$ be a smooth affine irreducible variety and let $\mathcal{F}$ be the family induced by a mapping $F : M \times X \to Y$, i.e., $\mathcal{F} = \{f_m : X \ni x \mapsto F(m, x) \in Y, m \in M\}$. Assume that $\mu(\mathcal{F}) > 0$. Take $Z = G(M \times X)$ and put $Z_m = (m \times Y) \cap Z$.

Then

1. There is an open non-empty subset $U_1 \subset M$ such that for every $m \in U_1$ we have $\mu(f_m) = \mu(\mathcal{F})$;
2. There is a non-empty open subset $U_2 \subset U_1$ such that for every $m \in U_2$ we have $f_m(X) = Z_m := (m \times Y) \cap Z$ and $B(f_m) = B(G)_m := (m \times Y) \cap B(G)$;
3. There is a non-empty open subset $U_3 \subset U_2$ such that for every $m_1, m_2 \in U_3$ the pairs $(f_{m_1}(X), B(f_{m_1}))$ and $(f_{m_2}(X), B(f_{m_2}))$ are equivalent by means of a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \to Y$ such that $\Psi(f_{m_1}(X)) = f_{m_2}(X)$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

Proof (1) Take $G : M \times X \ni (m, x) \mapsto (m, F(m, x)) \in Z$. The mapping $G : M \times X \ni (m, x) \mapsto (m, F(m, x)) \in Z$ has a constant number of points in the fibers outside the bifurcation set $B(G) \subset Z$. Take $U = Z \setminus B(G)$. By Theorem 2.4 the set $U$ is open. Let $\pi : Z \ni (m, y) \mapsto m \in M$ be the projection. We show that the constructible set $\pi(U)$ is dense in $M$. Indeed, assume that $\pi(U) = N$ is a proper subset of $M$. Since $U$ is dense in $Z$, we have $\pi(Z) \subset N$, i.e., $Z \subset N \times Y$. This is a contradiction. In particular the set $\pi(U)$ is dense in $M$ and it contains a Zariski open, non-empty subset $U_1 \subset M$. Of course $\mu(f_m) = \mu(\mathcal{F})$ for $m \in U_1$. 

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Consider the projection $\pi : Z \ni (m, y) \mapsto m \in M$. As we know from (1), the mapping $\pi$ is dominant. By a well known result, after shrinking $U_1$ we can assume that every fiber $Z_m$ of $\pi$ $(m \in U_2 \subset U_1)$ is of pure dimension $d = \dim Z - \dim M = \dim X$. However, $Z_m = f_m(X) \cup B(G)_m$. Generically the dimension of $B(G)_m$ is less than $d$. Hence if we possibly shrink $U_2$, we get $Z_m = f_m(X)$ for $m \in U_2$. Moreover, by Lemma 3.1 (after shrinking $U_2$ if necessary), we can assume that $\text{Sing}(Z_m) = \text{Sing}(Z)_m := (m \times Y) \cap \text{Sing}(Z)$. Now it is easy to see that $B(f_m) = B(G)_m$.

(3) We have $f_m(X) = Z_m$ and $B(f_m) = B(G)_m$ for $m \in U_2$. Now apply Lemma 3.2 with $X = U_2 \times Y$, $W = (U_2 \times Y) \cap Z$, $N = (U_2 \times Y) \cap B(G)$ and $f : U_1 \times Y \ni (m, y) \mapsto m \in U_1$.

Now we are ready to prove our main result:

**Theorem 3.5** Let $X, Y$ be smooth affine irreducible varieties. Every algebraic family $\mathcal{F}$ of polynomial mappings from $X$ to $Y$ contains only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) mappings.

**Proof** The proof is by induction on $\dim M$. We can assume that $M$ is affine, irreducible and smooth. Indeed, $M$ can be covered by a finite number of affine subsets $M_i$, and we can consider the families $\mathcal{F}|_{M_i}$ separately. For the same reason we can assume that $M$ is irreducible. Finally $\dim M \setminus \text{Reg}(M) < \dim M$ and we can use induction to reduce the general case to the smooth one.

Assume that $M$ is smooth and affine. If $\mu(\mathcal{F}) = 0$, then $\mathcal{F}$ does not contain any generically-finite mapping. Hence we can assume that $\mu(\mathcal{F}) = k > 0$. By Lemma 3.4 there is a non-empty open subset $U \subset M$ such that for every $m_1, m_2 \in U$ we have

1. $\mu(f_{m_1}) = \mu(f_{m_2}) = k$,
2. The pairs $(f_{m_1}(X), B(f_{m_1}))$ and $(f_{m_2}(X), B(f_{m_2}))$ are equivalent by means of a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \rightarrow Y$ such that $\Psi(f_{m_1}(X)) = f_{m_2}(X)$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

Fix a pair $Q = f_{m_0}(X), B = B(f_{m_0})$ for some $m_0 \in U_3$. For $m \in U_3$ the mapping $f_m : X \rightarrow Y$ is topologically equivalent to the continuous mapping $f'_m = \Psi_m \circ f_m$ with $f'_m(X) = Q$ and $B(f'_m) = B$ (Lemma 3.4). Every mapping $f'_m$ induces a topological covering $f^1_m : X \setminus f^{-1}_m(B) = P^1_m \rightarrow R = Q \setminus B$. Take a point $a \in R$ and let $a f'_m \in f'_m^{-1}(a)$. We have an induced homomorphism

$$f_\ast : \pi_1(P^1_m, a f'_m) \rightarrow \pi_1(R, a).$$

Denote $H_f = f_\ast(\pi_1(P_f, a f))$ and $G = \pi_1(R, a)$. Hence $[G : H_f] = k$. It is well known that the fundamental group of a smooth algebraic variety is finitely generated. In particular the group $G := \pi_1(Q \setminus B, a)$ is finitely generated. Let us recall the following result of M. Hall (see [4]):

**Lemma 3.6** Let $G$ be a finitely generated group and let $k$ be a natural number. Then there are only a finite number of subgroups $H \subset G$ such that $[G : H] = k$.

By Lemma 3.6 there are only a finite number of subgroups $H_1, \ldots, H_r \subset G$ with index $k$. Choose generically-finite (proper) mappings $f_i = f'_m = \Psi_i \circ f_m : X \rightarrow Y$ such that $H_{f_i} = H_i$ (of course only if such a mapping $f'_i$ does exist). We show that every generically-finite (proper) mapping $f'_m (m \in U)$ is semi-equivalent (equivalent) to one of mappings $f_i$. 

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Indeed, let $H_{f_i} = H_{f_i}$ (here $f_i' = \Psi_m \circ f_m$). We show that $f_m' := f$ is equivalent to $f_i$. Let us consider two coverings $f : (P_f, a_f) \to (R, a)$ and $f_i : (P_{f_i}, a_{f_i}) \to (R, a)$. Since $f_n(\pi_1(P_f, a_f)) = f_n(\pi_1(P_{f_i}, a_{f_i}))$ we can lift the covering $f$ to a homeomorphism $\phi : P_f \to P_{f_i}$ such that following diagram commutes:

\[
\begin{tikzpicture}
  \node (A) at (0,0) {$(P_f, a_f)$};
  \node (B) at (4,4) {$(P_{f_i}, a_{f_i})$};
  \node (C) at (4,0) {$(R, a)$};
  \draw[->] (A) to node[above] {$f$} (B);
  \draw[->] (A) to node[below] {$f_i$} (C);
  \draw[->] (B) to node[above] {$\phi$} (C);
\end{tikzpicture}
\]

Hence for generically-finite mappings we have

\[
(\Psi_{\iota})^{-1} \circ \Psi_m \circ f_m \circ \phi^{-1}|_U = f_m|_U,
\]

where $V = X \setminus f^{-1}_m(B(f_m))$ and $U = X \setminus f^{-1}_m(B(f_{m_i}))$. Hence $f_m$ is semi-equivalent to $f_{m_i}$.

In the case of proper mappings we show additionally that the mapping $\phi$ can be extended to a continuous mapping $\Phi$ on the whole of $X$. Indeed, take a point $x \in f^{-1}(B)$ and let $y = f(x)$. The set $f_i^{-1}(y) = \{b_1, ..., b_s\}$ is finite. Take small open disjoint neighborhoods $W_i(r)$ of $b_i$, such that $W_i(r)$ shrinks to $b_i$ as $r$ tends to 0. We can choose an open neighborhood $V(r)$ of $y$ so small that $f_i^{-1}(V(r)) \subset \bigcup_{j=1}^s W_i(r)$. Now take a small connected neighborhood $P_s(r)$ of $x$ such that $f(P_s(r)) \subset V(r)$. The set $P_s(r) \setminus f^{-1}(B)$ is still connected and it is transformed by $\phi$ into one particular set $W_0(r)$. We take $\Phi(x) = b_{i_0}$. It is easy to see that the mapping $\Phi$ so defined is a continuous extension of $\phi$. In fact $\phi(P_s(r) \setminus f^{-1}(B))$ shrinks to $b_{i_0}$ if $r$ goes to 0. Moreover, we still have $f = f_i \circ \Phi$.

In a similar way the mapping $\Lambda$ determined by $\phi^{-1}$ is continuous. It is easy to see that $\Lambda \circ \Phi = \Phi \circ \Lambda = identity$, hence $\Phi$ is a homeomorphism. Consequently, the mapping $f_i \circ \Phi = \Psi_{\iota} \circ f_m \circ \Phi$ is equal to $f = \Psi_m \circ f_m$. Finally, we get

\[
(\Psi_{\iota})^{-1} \circ \Psi_m \circ f_m \circ \Phi^{-1} = f_m|_U.
\]

This means that the family $F_{|U}$ contains only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) mappings. In fact, the number of topological semi-types (types) of generically-finite (proper) mappings in $F_{|U}$ is bounded by the number of subgroups of $G$ of index $\mu(F)$.

Let $T = M \setminus U$. Hence $\dim T < \dim M$. By the induction the family $F_{|T}$ also contains only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) mappings. Consequently so does $F$.

\begin{corollary}
There is only a finite number of topologically non-semi-equivalent (non-equivalent) generically-finite (proper) polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$ of a bounded algebraic degree.
\end{corollary}
4 Families of proper mappings

In this section we slightly extend our previous result in the case of irreducible families of proper (or generically-finite) mappings. First we prove a following lemma:

**Lemma 4.1** Let $Y = \mathbb{R}^n$ and let $Z \subset Y$ be a linear subspace of $Y$. Fix $\epsilon > 0$ and take $\eta < \epsilon$. Let $B(0, \eta)$ be a ball of radius $\eta$. Let $\gamma : I \ni t \mapsto \gamma(t) \in B(0, \eta) \cap Z$ be a smooth path. Then there exists a continuous family of homeomorphisms $\Phi_t : Y \to Y$, $t \in [0, 1]$ such that

1. $\Phi_t(\gamma(t)) = \gamma(0)$ and $\Phi_t(z) = z$ for $\|z\| \geq \epsilon$.
2. $\Phi_0 = \text{identity}$.
3. $\Phi_t(Z) = Z$.

**Proof** Let $v_t = \gamma(0) - \gamma(t) \in T\mathbb{R}^n$. We construct a family of diffeomorphisms $\Phi_t$, which are interpolation between translation $x \mapsto x + v_t$ and identity.

Let $\sigma : Y \to [0, 1]$ be a differentiable function such that $\sigma = 1$ on $B(0, \eta)$ and $\sigma = 0$ outside $B(0, \epsilon)$. Define a vector field $V(x) = \sigma(x)v_t$. Integrating this vector field we get desired diffeomorphisms $\Phi_t$, for any $t$. \hfill $\square$

**Corollary 4.2** Let $Y$ be a smooth manifold and $Z$ be a smooth submanifold. For every point $a \in Z$ and every open neighborhood $V_a$ of the point $a$, there is an open connected neighborhood $U_a$ of the point $a$, such that:

(a) $\overline{U_a} \subset V_a$,
(b) if $\gamma : I \ni t \mapsto \gamma(t) \in U_a \cap Z$ is a smooth path, then there is a continuous family of diffeomorphism $\psi_t : Y \to Y$, $t \in [0, 1]$ such that

1. $\psi_t(\gamma(t)) = \gamma(0)$,
2. $\psi_t(x) = x$ for $x \notin V_a$ and $\psi_0 = \text{identity}$,
3. $\psi_t(Z) = Z$.

Now we are in a position to prove:

**Theorem 4.3** Let $X, Y$ be smooth affine irreducible varieties. Let $\mathcal{F} : M \times X \to Y$ be an algebraic family of proper polynomial mappings from $X$ to $Y$. Assume that $M$ is an irreducible variety. Then there exists a Zariski open dense subset $U \subset M$ such that for every $m, m' \in U$ mappings $f_m$ and $f_{m'}$ are topologically equivalent.

**Proof** We follow the proof of Theorem 3.5 and we use here the same notation. By Lemma 3.4 there is a non-empty open subset $U \subset M$ such that for every $m_1, m_2 \in U$ we have

1. $\mu(f_{m_1}) = \mu(f_{m_2}) = k$,
2. The pairs $(f_{m_1}(X), B(f_{m_1}))$ and $(f_{m_2}(X), B(f_{m_2}))$ are equivalent by means of a homeomorphism, i.e., there is a homeomorphism $\Psi : Y \to Y$ such that $\Psi(f_{m_1}(X)) = f_{m_2}(X)$ and $\Psi(B(f_{m_1})) = B(f_{m_2})$.

Fix a pair $(Q = \overline{f_{m_0}(X)}, B = B(f_{m_0}))$ for some $m_0 \in U$. For $m \in U$ the mappings $f_m$ and $f_{m_0}$ can be connected by a continuous path $f_t$, $f_0 = f_{m_0}$, $f_1 = f_m$. Moreover we have also a continuous family of homeomorphisms $\Psi_t : Y \to Y$ such that $\Psi_t(f_i(X)) = f_0(X)$ and $\Psi(B(f_i)) = B(f_0)$. It is enough to prove that mappings $F_t = \Psi_t \circ f_t$ are locally (in the sense of parameter $t$) equivalent.

1. **First step of the proof.** Let $C_t \subset X$ denotes the preimage by $F_t$ of the set $B$ (in fact $C_t = f_t^{-1}(B(f_t))$) and put $X_t = X \setminus C_t$. Put $Q' := Q \setminus B$. Assume that for all mappings $F_t$ there is a point $a \in (X \setminus \bigcup t \in I C_t)$ such that for all $t \in I$ we have $F_t(a) = b$. 

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We have an induced homomorphism $G_{t*} : \pi_1(X_t, a) \to \pi_1(Q', b)$. We show that the subgroup $F_{t*}(\pi_1(X_t, a)) \subset \pi_1(Q', b)$ does not depend on $t$.

Indeed let $\gamma_1, \ldots, \gamma_t$ be generators of the group $\pi_1(X_{t_0}, a)$. Let $U_t$ be an open relatively compact neighborhoods of $\gamma_t$ such that $\overline{U}_t \cap C_{t_0} = \emptyset$. For sufficiently small number $\epsilon > 0$ and $t \in (t_0 - \epsilon, t_0 + \epsilon)$ we have $\overline{U}_t \cap C_t = \emptyset$. Let $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Note that the loop $F_t(\gamma_t)$ is homotopic with the loop $F_{t_0}(\gamma_t)$. In particular the group $F_{t_0*}(\pi_1(X_{t_0}, a))$ is contained in the group $F_{t*}(\pi_1(X_t, a))$. Since they have the same (finite!) index in $\pi_1(Y', b)$ they are equal. This means that the subgroup $G_{t*}(\pi_1(X_t, a)) \subset \pi_1(Y', b)$ is locally constant, hence it is constant.

Let us consider two coverings $F_1 : (X_t, a) \to (Q', b)$ and $F_0 : (X_0, a) \to (Q', b)$. Since $F_{t*}\pi_1(X_t, a) = F_{0*}\pi_1(X_0, a)$ we can lift the covering $F_t$ to a homeomorphism $\phi_t : X_t \to X_0$. As before we can extend the homeomorphism $\phi_t$ to the homeomorphism $\Phi_t : X \to X$, such that $F_0 \circ \Phi_t = F_t$.  

(2) The general case. Now we can prove Theorem 4.3. Since in general there is no a point $a \in (X \setminus \bigcup_{t \in I} C_t)$ such that for all $t \in I$ we have $F_t(a) = b$, we have to modify our construction.

First we prove that for every $t_0 \in I$ there exists $\epsilon > 0$ and a family of homeomorphisms $\Phi_t : X \to X$, $t \in (t_0 - \epsilon, t_0 + \epsilon)$ such that $F_t = F_{t_0} \circ \Phi_t$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Take a point $a \in X_{t_0}$ and choose $\epsilon > 0$ so small that $a \in X_t$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Put $y(t) \ni t \mapsto F_t(a) \in Y'$. We can take $\epsilon$ so small that the hypothesis of Corollary 4.2 is satisfied. Applying Corollary 4.2 with $Y' = Y \setminus B$ and $Z = Q \setminus B$ we have a continuous family of diffeomorphisms $\psi_t : Y \to Y$ which preserves $Q$ and $B$, $t \in (t_0 - \epsilon, t_0 + \epsilon)$ such that $\psi_t(F_t(a)) = F_0(a)$. Take $G_t = \psi_t \circ F_t$. Arguing as in the first part of our proof all $G_t$ are topologically equivalent for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Hence also all $F_t$ are topologically equivalent for $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Since $F_t$ are locally topologically equivalent, they are topologically equivalent for every $t \in I$.

Corollary 4.4 Let $n \leq m$ and let $\Omega_n(d_1, \ldots, d_m)$ denotes the family of all polynomial mappings $F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ of a multi-degree bounded by $(d_1, \ldots, d_m)$. Then any two general members of this family are topologically equivalent.

Proof Indeed, it is enough to note that a generic mapping $f \in \Omega_n(d_1, \ldots, d_m)$ is proper. 

Using the same method we can prove:

Theorem 4.5 Let $X, Y$ be smooth affine irreducible varieties. Let $F : M \times X \to Y$ be an algebraic family of generically-finite polynomial mappings from $X$ to $Y$. Assume that $M$ is an irreducible variety. Then there exists a Zariski open dense subset $U \subset M$ such that for every $m, m' \in U$ the mappings $f_m$ and $f_{m'}$ are topologically semi-equivalent.

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