Characterizations of graph classes via convex geometries: A survey

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Abstract

Graph convexity has been used as an important tool to better understand the structure of classes of graphs. Many studies are devoted to determine if a graph equipped with a convexity is a convex geometry. In this work we survey results on characterizations of well-known classes of graphs via convex geometries. We also give some contributions to this subject.

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1. Introduction

A convexity on a nonempty set $V$ is a family $\mathcal{C}$ of subsets of $V$ (called convex sets) such that: (a) $\emptyset$ and $V$ are convex sets; (b) the intersection of convex sets is a convex set. If $V = V(G)$ for some graph $G$ then $\mathcal{C}$ is a graph convexity of $G$. The structure of a convexity is based on the way convex sets are defined. In the context of graph convexities, there are many examples in the literature where convex sets are defined over a path system, typically according to the following rule: fix a family $\mathcal{P}_G$ of paths (or, more generally, walks) of a graph $G$, and say that a set $S \subseteq V(G)$ is convex if no vertex $x \notin S$ lies in a member of $\mathcal{P}_G$ that starts and ends at two vertices of $S$. The most natural example is the so-called geodesic convexity \cite{20}, for which $\mathcal{P}_G$ is the family of all the shortest paths of $G$. Other important examples are: the monophonic convexity \cite{11,14}, the $m^3$-convexity \cite{13}, the $k$-convexity \cite{21}, the toll convexity \cite{2}, and the weakly toll convexity \cite{22}, defined over induced paths, induced paths of length at least three, induced paths of length at most $k$, tolled walks, and weakly toll walks, respectively. Other rules to define convex sets, not based on path systems, have also been investigated; an important example comes from the so-called Steiner convexity, introduced in \cite{6}. In this case, a set $S \subseteq V(G)$ is Steiner convex if, for every $S' \subseteq S$, the vertices of any Steiner tree with terminal set $S'$ belong to $S$.

Graph convexities have been studied in many contexts. A strong direction of research has focused on determining convexity invariants, such as the hull number, the interval number, and the convexity number, among others. A major reference work by Pelayo \cite{20} gives an extensive overview on convexity invariants, applied to the case of the geodesic convexity.

Other studies are devoted to determine if a graph convexity is a convex geometry. This line of research is closely linked to the original idea of defining a combinatorial abstraction of convex sets in Geometry. The convex sets in the plane are those subsets $S$ such that no point outside $S$ lies on a line segment with endpoints in $S$. The convex hull of a set of points $A$ in the plane is the smallest convex set that contains $A$. A point $x$ of a convex set $S$ is an extreme point of $S$ if $S \setminus \{x\}$ is also a convex set, that is, there is no line segment with endpoints in $S \setminus \{x\}$ containing $x$. For instance, an $n$-polygon (considered as a closed region) is clearly a convex set in the plane, and its extreme points are precisely its $n$ vertices\footnote{Here, the term vertices is used in the geometric sense.}. In addition, the polygon
is the convex hull of its $n$ extreme points. Another example is a circle (closed circular region); it is clearly a convex set in the plane, and all the points lying on the circle border are its extreme points. Again, the circle is the convex hull of such points.

The above concepts can be transferred to the combinatorial field in a natural way. We refer the reader to [17]. Let $G$ be a graph and let $\mathcal{C}$ be a convexity of $G$. Given a set $S \subseteq V(G)$, the smallest set $H \in \mathcal{C}$ containing $S$ is called the convex hull of $S$. A vertex $x$ of a convex set $S$ is an extreme vertex of $S$ if $S \setminus \{x\}$ is also convex. The convexity $\mathcal{C}$ is a convex geometry if it satisfies the Minkowski-Krein-Milman property [27]: Every convex set is the convex hull of its extreme vertices. The main question dealt with in this survey is: by fixing a rule $r$ to define the convex sets (e.g., a rule based on some path system), determine the class of graphs whose $r$-convexities are convex geometries. For instance, by fixing induced paths, we can obtain the following characterization: a graph $G$ is chordal if and only if the monophonic convexity of $G$ is a convex geometry [17]. Ptolemaic graphs, interval graphs, proper interval graphs, weak polarizable graphs, and 3-fan-free chordal graphs can also be characterized in this way by considering, respectively, the geodesic convexity [17], the toll convexity [2], the weakly toll convexity [22], the $m^3$-convexity [13], and the Steiner convexity [6]. In [21], a characterization of graphs with $l^k$-convexities that are convex geometries are studied. Section 3 discusses in detail most of these characterizations.

Convex geometries are equivalent, by complementation, to antimatroids (see [25]). An antimatroid consists of a finite family $\mathcal{F}$ of finite sets (called feasible sets) such that: (a) $\mathcal{F}$ is closed under unions; (b) if $S$ is a nonempty feasible set, then there is $x \in S$ for which $S \setminus \{x\}$ is also a feasible set (that is, $\mathcal{F}$ is an accessible set system). If $U$ is the union of the sets in $\mathcal{F}$ then the family of complementary sets $\mathcal{C} = \{U \setminus S \mid S \in \mathcal{F}\}$ is a convex geometry. It is not difficult to see that $\mathcal{C}$ is a convexity; in addition, it satisfies the anti-exchange property [15, 25], which is complementary to the accessibility property of antimatroids (property (b) above). In Section 2, we present in more detail the anti-exchange property, which will be useful to derive some results in Section 4.

This survey is organized as follows. Section 2 provides all the necessary background. In Section 3, we review the main results in the literature on characterizations of graph classes via convex geometries; chordal, Ptolemaic, strongly chordal, interval, and proper interval graphs, among other classes, are dealt with. In Section 4 we present some contributions; namely, we show
that forests, forests of stars, cographs, bipartite graphs, and planar graphs can be well characterized via convex geometries. Our concluding remarks are presented in Section 5.

2. Preliminaries

All the graphs in this work are connected, unless otherwise stated. Let \( \mathcal{P} \) be a function (called path system) that maps each connected graph \( G \) to a collection \( \mathcal{P}_G \) of walks of \( G \). For instance, \( \mathcal{P} \) can map \( G \) to its shortest paths, induced paths, tolled walks etc. The members of \( \mathcal{P}_G \) are generically called \( p \)-walks of \( G \). All the path systems dealt with in this paper satisfy the following property: For every \( G \), all the walks of \( G \) formed by a single vertex or two adjacent vertices are \( p \)-walks.

The interval \( I(u, v) \) of \( u, v \in V(G) \) is the set of all vertices that lie in a \( p \)-walk from \( u \) to \( v \). For \( S \subseteq V(G) \), we define \( I(S) = \cup_{u,v \in S} I(u,v) \). A set \( S \subseteq V(G) \) is convex if \( I(S) = S \). Let \( \mathcal{C} = \{ S \subseteq V(G) : I(S) = S \} \). Clearly, \( \mathcal{C} \) is a convexity of \( G \), defined over the collection \( \mathcal{P}_G \) of \( p \)-walks.

A vertex \( x \) of a convex set \( S \subseteq V(G) \) is an extreme vertex of \( S \) if \( S \setminus \{ x \} \) is also a convex set. This definition implies that: (a) no \( p \)-walk in \( G[S] \) contains \( x \) as an internal vertex; (b) \( x \notin I(u,v) \) for any pair of distinct vertices \( u, v \in S \). The set of extreme vertices of \( S \) is denoted by \( \text{ext}(S) \). The convex hull of a set \( S \subseteq V(G) \), denoted by \( H(S) \), is the smallest set of vertices of \( G \) that contains \( S \) and is convex. It can be shown that \( H(S) \) can be iteratively constructed by successive applications of the interval operator, as follows: starting with a set \( S \subseteq V(G) \), define \( S_0 = S \) and \( S_i = I(S_{i-1}) \), \( i \geq 1 \). Then there exists \( k \geq 1 \) such that \( S_k = S_{k-1} = H(S) \).

The concept of interval is the combinatorial analog of a line segment between two points in the plane. Just as the geometric convex hull of a set of points in the plane can be built using line segments, the convex hull of a set \( S \) of vertices in a graph uses the notion of interval for the iterative construction described above.

We say that a convexity \( \mathcal{C} \) of a graph \( G \) is a convex geometry (or is geometric) if it satisfies the following property: for every \( S \in \mathcal{C} \), it holds that \( S = H(\text{ext}(S)) \). This is called the Minkowski-Krein-Milman property [27]. Alternatively, the convexity \( \mathcal{C} \) is a convex geometry if the following axiom is satisfied:

**Anti-exchange property** [15, 25]

If \( y, z \notin H(S) \) and \( z \in H(S \cup \{ y \}) \) then \( y \notin H(S \cup \{ z \}) \).
The anti-exchange property is a combinatorial abstraction of the usual convex hull operator in the plane: for two points \( y \) and \( z \) not in the convex hull of \( S \), if \( z \) is in the convex hull of \( S \cup \{y\} \) then \( y \) is outside the convex hull of \( S \cup \{z\} \). See Figure 1.

![Figure 1: Anti-exchange property applied to \( S = \{a, b, c, d\} \).](image)

3. An overview of graph classes and convex geometries

In this section we briefly review the main existing results on convex geometries associated with path systems. Chordal graphs play an important role in this section. For basic facts on chordal graphs, see [20].

3.1. Monophonic convexity

A path (cycle) in a graph \( G \) is induced if there is no edge of \( G \) linking two nonconsecutive vertices of it. Let \( P^m \) be the path system that maps each graph \( G \) to its induced paths. The convexity defined over the induced paths of \( G \) is called the monophonic convexity of \( G \).

For a vertex \( v \in V(G) \), let \( N[v] \) denote the closed neighborhood of \( v \) in \( G \). If \( N[v] \) is a clique then \( v \) is said to be a simplicial vertex. It is easy to see that \( v \) is an extreme vertex of a (monophonically) convex set \( S \subseteq V(G) \) if and only if \( v \) is a simplicial vertex of \( G[S] \), the subgraph of \( G \) induced by \( S \).

Suppose that \( G \) contains an induced cycle \( C \) with at least four vertices. Clearly, \( S = H(V(C)) \) is convex and contains no simplicial vertices, i.e., \( \text{ext}(S) = \emptyset \). Thus, the Minkowski-Krein-Milman property fails in this case. In other words, if the monophonic convexity of \( G \) is a convex geometry then \( G \) is chordal. On the other hand, if \( G \) is chordal then every nonsimplicial vertex of \( G \) lies on an induced path between two simplicial vertices [17]. As chordality is an hereditary property, this implies that every convex subset of vertices of \( G \) is indeed the convex hull of its extreme vertices. Hence:

**Theorem 1.** [17] A graph \( G \) is chordal if and only if the monophonic convexity of \( G \) is a convex geometry.
3.2. Geodesic convexity

A path $P$ between two vertices $u$ and $v$ in a graph $G$ is a shortest path if the length of $P$ is equal to the distance between $u$ and $v$ in $G$. Let $\mathcal{P}^g$ be the path system that maps each graph $G$ to its shortest paths. The convexity defined over the shortest paths of $G$ is the geodesic convexity of $G$.

As in the previous subsection, it is easy to see that $v$ is an extreme vertex of a (geodesically) convex set $S \subseteq V(G)$ if and only if $v$ is a simplicial vertex of $G[S]$. In addition, if the geodesic convexity of $G$ is a convex geometry then $G$ must be clearly chordal. However, it is not true that, for every chordal graph $G$, the geodesic convexity of $G$ is a convex geometry. For instance, consider the graph $G'$ (called gem) with $V(G') = \{a, b, c, d, e\}$ and $E(G') = \{ab, bc, cd, de, ae, be, ce, de\}$. Assume that $G'$ is an induced subgraph of a chordal graph $G$. Clearly, the extreme vertices of $Y = H(V(G'))$ are in $\{a, d\}$. However, $H(\{a, d\}) \neq Y$.

A graph $G$ is Ptolemaic if it is chordal and contains no gem as an induced subgraph. The arguments in the previous paragraph tell us that if the geodesic convexity of $G$ is a convex geometry then $G$ is a Ptolemaic graph. Conversely, if $G$ is a Ptolemaic graph then it can be proved that every induced path of $G$ is a shortest path \cite{23}, i.e., the monophonic and the geodesic convexities of $G$ coincide in this case. Therefore, in a Ptolemaic graph, every nonsimplicial vertex of $G$ lies on a shortest path between two simplicial vertices. Consequently, every convex subset of vertices of $G$ is indeed the convex hull of its extreme vertices. Hence:

**Theorem 2.** \cite{17} A graph $G$ is Ptolemaic if and only the geodesic convexity of $G$ is a convex geometry.

3.3. Strong convexity

We say that a path $P = u_0u_1 \ldots u_n$ is even-chorded if it has no odd chord (an edge that connects two vertices that are an odd distance $d > 1$ apart from each other in a path or cycle) and, in addition, neither $u_0$ nor $u_n$ lies in a chord of $P$. Let $\mathcal{P}^s$ be the path system that maps a graph $G$ to its even-chorded paths. The convexity defined over such paths is called the strong convexity of $G$, and the associated convex sets are called strong convex sets.

An even cycle is a cycle with an even number of vertices. A graph is strongly chordal \cite{16} if it is chordal and every of its even cycles with at least six vertices has an odd chord. A vertex of a graph is simple if the neighborhoods of its neighbors form a nested family of sets. Note that a simple vertex
is simplicial, but not conversely. In [16], the following characterization of strongly chordal graphs is given:

**Theorem 3.** [16] A graph $G$ is a strongly chordal graph if and only if every induced subgraph of $G$ contains a simple vertex.

Suppose that the strong convexity of $G$ is a convex geometry. Since induced paths are even-chorded, an extreme vertex $v$ of an strong convex set $S$ must be simplicial in $G[S]$. This implies that $G$ is chordal, and, in this case, we can prove the following lemma:

**Lemma 4.** [17] Let $G$ be a chordal graph. Then a vertex $v \in V(G)$ is an extreme vertex of an strong convex set $S$ if and only if $v$ is a simple vertex in $G[S]$.

By Theorem 3 and Lemma 4, $G$ is a strongly chordal graph. Conversely, suppose that $G$ is a strongly chordal graph. In [17], the following result is proved:

**Lemma 5.** [17] In a strongly chordal graph, every nonsimple vertex lies in an even-chorded path between simple vertices.

By Lemma 5, $V(G)$ is indeed the convex hull of its extreme vertices. But since strongly chordal graphs are hereditary, we have:

**Theorem 6.** [17] A graph $G$ is strongly chordal if and only if the strong convexity of $G$ is a convex geometry.

3.4. $m^3$-convexity

Let $\mathcal{P}^{\geq 3}$ be the path system that maps a graph $G$ to its induced paths of length at least three. The convexity defined over such paths is called the $m^3$-convexity of $G$, and the associated convex sets are called $m^3$-convex sets. Note that an $m^3$-convex set does not necessarily induce a connected subgraph.

Let $P_n$ denote the induced path with $n$ vertices. An alternative definition of “simplicial” is: $v$ is a simplicial vertex iff it is not a midpoint of an induced $P_3$. In [24], this concept is relaxed as follows: a vertex is semisimplicial if it is not an internal vertex of an induced $P_4$, and nonsemisimplicial otherwise. Clearly, a vertex $v$ is an extreme vertex of an $m^3$-convex set $S \subseteq V(G)$ if and only if $v$ is semisimplipcial in $G[S]$.
A hole is an induced cycle with at least five vertices. The house is the graph with vertices $a, b, c, d, e$ and edges $ab, bc, cd, ad, ae, be$. The domino is the graph with vertices $a, b, c, d, e, f$ and edges $ab, bc, cd, ad, ce, ef, df$. Finally, the $A$ is the domino minus the edge $ef$. A graph $G$ is weakly polarizable if $G$ contains no hole, house, domino, or $A$ as an induced subgraph. Weakly polarizable graphs are also called HHDA-free graphs. Using arguments similar to those presented in the preceding subsections, it can be shown if the $m^3$-convexity of $G$ is a convex geometry then $G$ is a weakly polarizable graph. Suppose the $m^3$-convexity of $G$ is a convex geometry and $G$ contains, for instance, a house $G'$ as an induced subgraph. Then the only possible semisimplicial vertex in the subgraph of $G$ induced by $Y = H(V(G'))$ is vertex $e$, i.e., $ext(Y) \subseteq \{e\}$. This implies $H(ext(Y)) \neq Y$, a contradiction. Analogously, $G$ cannot contain a hole, a domino, or an $A$ as an induced subgraph. Thus, $G$ is weakly polarizable. Conversely, if $G$ is weakly polarizable, it holds that every nonsemisimplicial vertex lies on an induced path of length at least three between semisimplicial vertices $[13]$. We then have the following theorem:

**Theorem 7.** $[13]$ A graph $G$ is weakly polarizable if and only if the $m^3$-convexity of $G$ is a convex geometry.

### 3.5. Toll convexity

A walk $u_0u_1 \ldots u_{k-1}u_k$ is a tolled walk if $u_0u_i \in E(G)$ implies $i = 1$ and $u_ju_k \in E(G)$ implies $j = k - 1$. In $[1]$, tolled walks were used to characterize interval graphs. A graph $G$ is an interval graph if its vertices can be associated with intervals on the real line such that two vertices are adjacent if and only if the associated intervals intersect.

Let $P'$ be the path system that maps a graph $G$ to its tolled walks. The convexity defined over such walks is called the toll convexity of $G$, and the associated convex sets are called toll convex sets.

If $I$ is an interval on the real line, let $R(I)$ and $L(I)$ be, respectively, the right and left endpoints of $I$. Given a family of intervals (or interval model) $\mathcal{I} = \{I_v\}_{v \in V(G)}$, associated with an interval graph $G$, we say that $I_a$ is an end interval if $L(I_a) = \min \bigcup I_v$ or $R(I_a) = \max \bigcup I_v$, i.e., $I_a$ appears as the first or the last interval in $\mathcal{I}$. A vertex $a \in V(G)$ is an end vertex if there exists some interval model of $G$ where $a$ is associated with an end interval. In addition, $a$ is an end simplicial vertex if $a$ is an end vertex and is simplicial. In $[2]$, two facts on extreme vertices in toll convex sets are given.
Lemma 8. If $v$ is an extreme vertex of a toll convex set $S$ of a graph $G$ then $v$ is simplicial in $G[S]$.

Lemma 9. A vertex $v$ of an interval graph $G$ is an extreme vertex of a toll convex set $S \subseteq V(G)$ if and only if $v$ is an end simplicial vertex of $G[S]$.

In order to characterize the graphs with toll convexities that are convex geometries, we need to resort to a well-known characterization of interval graphs. Three vertices of a graph form an asteroidal triple if between any pair of them there exists a path that avoids the neighborhood of the third vertex.

Theorem 10. A graph $G$ is an interval graph if and only if $G$ is chordal and contains no asteroidal triple.

Suppose that the toll convexity of $G$ is a convex geometry. Let us show first that $G$ is chordal. By the assumption, $V(G)$ (and every of its toll convex subsets) is the convex hull of its extreme vertices. Let $v$ be an extreme vertex of $V(G)$. By definition of extreme vertex, $V(G) \setminus \{v\}$ is toll convex in $G$. Thus, the toll convexity of $G - v$ is a convex geometry (note that any subset of $V(G) \setminus \{v\}$ is toll convex in $G - v$ iff it is toll convex in $G$). Using induction, $G - v$ is chordal. Since by Lemma 8 $v$ is a simplicial vertex of $G$, it follows that $G$ is chordal as well (recall that a graph is chordal iff there is an ordering $v_1, \ldots, v_n$ of its vertices such that $v_i$ is simplicial in $G[\{v_i, v_{i+1}, \ldots, v_n\}]$, $1 \leq i \leq n$).

Now, let us show that $G$ contains no asteroidal triple. Suppose by contradiction that vertices $a, b, c$ form an asteroidal triple in $G$. Consider the walk $W$ from $a$ to $c$ formed by the concatenation of induced paths $P_{ab}$, from $a$ to $b$, and $P_{bc}$, from $b$ to $c$, such that $P_{ab}$ avoids $N[c]$ and $P_{bc}$ avoids $N[a]$. Note that $a$ is adjacent only to one vertex of $P_{ab}$ and to no vertices of $P_{bc}$. Likewise, $c$ is adjacent only to one vertex of $P_{bc}$ and to no vertices of $P_{ab}$. Thus, $W$ is a tolled walk from $a$ to $c$ passing through $b$. Let $Y = H(\{a, b, c\})$. Since $V(W) \subseteq Y$, $b$ is not an extreme vertex of $Y$. Analogously, $a$ and $c$ are not extreme vertices of $Y$. Since no other vertices of $Y$ are extreme vertices, this implies that $Y$ has no extreme vertices, a contradiction.

The precedent paragraphs tell us that if the toll convexity of $G$ is a convex geometry then, by Theorem 10 $G$ is an interval graph. To prove the converse, we use the following useful lemma:
Lemma 11. Let $G$ be an interval graph. Then every vertex that is not an end simplicial vertex lies in a tolled walk between two end simplicial vertices of $G$.

Using Lemma 9, Lemma 11 and the fact that interval graphs are hereditary, we have:

Theorem 12. A graph $G$ is an interval graph if and only if the toll convexity of $G$ is a convex geometry.

3.6. Weakly toll convexity

A walk $u_0u_1 \ldots u_{k-1}u_k$ is a weakly toll walk if $u_0u_i \in E(G)$ implies $u_i = u_1$ and $u_ju_k \in E(G)$ implies $u_j = u_{k-1}$. The concept of weakly toll walk is a relaxation of the concept of toled walk (in the sense that every tolled walk is a weakly toll walk). In the previous subsection, tolled walks have been used to characterize interval graphs via convex geometries. Similarly, weakly toll walks are used as a tool to characterize proper interval graphs. A proper interval graph is an interval graph that admits an interval model in which no interval properly contains another, or, equivalently, an interval model in which all the intervals have the same length. Roberts [30] proved that proper interval graphs are exactly the interval graphs containing no $K_{1,3}$ as an induced subgraph. The graph $K_{1,3}$ consists of four vertices $a, b, c, d$ and three edges $ab, ac, ad$.

Let $\mathcal{P}^w$ be the path system that maps a graph $G$ to its weakly toll walks. The convexity defined over such walks is called the weakly toll convexity of $G$, and the associated convex sets are called weakly toll convex sets.

Lemmas 8 and 9 have similar versions for weakly toll convex sets:

Lemma 13. If $v$ is an extreme vertex of a weakly toll convex set $S$ of a graph $G$ then $v$ is simplicial in $G[S]$.

Lemma 14. A vertex $v$ of a proper interval graph $G$ is an extreme vertex of a weakly toll convex set $S \subseteq V(G)$ if and only if $v$ is an end simplicial vertex of $G[S]$.

Using arguments similar to those used in the previous section, one can prove that if the weakly toll convexity of $G$ is a convex geometry then $G$ is chordal and cannot contain asteroidal triples and induced subgraphs isomorphic to $K_{1,3}$. Hence, using the characterization of proper interval graphs in [30], $G$ is a proper interval graph. Conversely, assume that $G$ is a proper
interval graph. Then, it is an interval graph. This means that, by Lemma 11, every vertex of $G$ that is not an end simplicial vertex lies in a tolled walk between two end simplicial vertices. Since every tolled walk is a weakly tolled walk and, by Lemma 14, end simplicial vertices of a proper interval graph are extreme vertices, we have:

**Lemma 15.** Let $G$ be a proper interval graph. Then every vertex that is not an end simplicial vertex lies in a weakly tolled walk between two end simplicial vertices of $G$.

**Theorem 16.** A graph $G$ is a proper interval graph if and only if the weakly tolled convexity of $G$ is a convex geometry.

### 3.7. $l^k$-convexity

Let $\mathcal{P}^{\leq k}$ be the path system that maps a graph $G$ to its induced paths of length at most $k$. The convexity defined over such paths is called the $l^k$-convexity of $G$, and the associated convex sets are called $l^k$-convex sets. The $l^2$-convexity is also called $P_3^+$-convexity.

Suppose that the $l^2$-convexity of $G$ is a convex geometry. It is easy to see that the extreme vertices of the $l^2$-convex set $V(G)$ are the simplicial vertices of $G$. Thus, if $G$ contains an induced cycle $C$ with at least four vertices then $H(V(C))$ does not have simplicial (extreme) vertices, a contradiction. This means that $G$ is chordal. Now, we prove that $G$ is also a cograph (a graph that contains no $P_4$ as an induced subgraph – see [9] for details). Let $P = x_0, \ldots, x_p$ be a maximum induced path of $G$, between two simplicial vertices of $G$. Such a path exists because $G$ is chordal [10]. Assume $p \geq 3$. The set $H(V(P))$ is an $l^2$-convex set of $G$ whose extreme vertices are $x_0$ and $x_p$. If there exists $z \in H(V(P)) \setminus V(P)$ that lies in an induced path of length two between $x_0$ and $x_p$, $z$ must be adjacent to every $x_i$; otherwise, $P$ plus the edges $x_0z$ and $zx_p$ forms an induced cycle of $G$ with at least four vertices. Since $N[x_0]$ and $N[x_p]$ are cliques in $H(V(P))$,

$$H(V(P)) = (N[x_0] \cap N[x_p]) \cup \{x_0, x_p\}.$$ 

This implies $H(V(P)) \neq H(\{x_0, x_p\})$, a contradiction. Thus $G$ is a chordal cograph.

Conversely, let $G$ be a chordal cograph, and let $S \subseteq V(G)$ be an $l^2$-convex set of $G$. Since $G[S]$ is chordal, every vertex in $S$ lies in an induced path between two simplicial vertices of $G[S]$. Such a path must have length two,
otherwise $G$ contains an induced path of length at least four as an induced subgraph. Hence,

**Theorem 17.** \cite{1, 21} A graph $G$ is a chordal cograph if and only if the $l^2$-convexity of $G$ is a convex geometry.

We now analyze graphs with $l^3$-convexities that are convex geometries. A vertex $u$ is a *universal vertex* in a graph $G$ if $u$ is adjacent to every other vertex of $G$. An $n$-*gem* is a graph $G_n$ such that: (i) $V(G_n) = \{x_0, \ldots, x_n, u_n\} (n \geq 4)$; (ii) $x_0, \ldots, x_n$ is an induced path; (iii) $u_n$ is a universal vertex. We say that $G_n$ is solved if there exists in $G$ a $P_4$ connecting $x_0$ and $x_n$ that avoids $u_n$. In \cite{21}, graphs with $l^3$-convexities that are convex geometries are characterized as follows:

**Theorem 18.** \cite{21} Let $G$ be a graph. The $l^3$-convexity of $G$ is a convex geometry if and only if the following conditions hold:

1. $G$ is chordal;
2. $\text{diam}(G) \leq 3$;
3. every induced $n$-gem ($n \geq 4$) contained in $G$ is solved.

The proof of the above theorem contains many details and will be omitted. The “only if” part can be generalized to prove the following result:

**Theorem 19.** \cite{21} Let $G$ be a graph and let $k \geq 2$. If the $l^k$-convexity of $G$ is a convex geometry then $G$ is chordal and $\text{diam}(G) \leq k$.

Despite the above result, a complete characterization of graphs with $l^k$-convexities that are convex geometries, for arbitrary $k$, remains still as an open question.

A surprising fact is that the class of graphs characterized in Theorem 18 is not hereditary. The following example is elucidative:

**Example 20.** Let $G$ be the graph depicted in Figure 2. Note that the $l^3$-convexity of $G$ is a convex geometry. However, for $x \in \{2, 5\}$, the $l^3$-convexity of $G - x$ is not a convex geometry, since:

- $V(G - x)$ is trivially an $l^3$-convex set of $G - x$,
- $\text{ext}(V(G - x)) = \{1, 7\}$, and
• \( H(\{1, 7\}) = \{1, 7\} \neq V(G - x) \).

This shows that the class of graphs with \( l^3 \)-convexities that are convex geometries is not hereditary. As far as the authors know, this is the only practical example of a nonhereditary class of graphs with convex geometries of a certain type.

![Figure 2: The \( l^3 \)-convexity of \( G \) is a convex geometry.](image)

4. Other convexities

In this section we present some contributions to the study of convex geometries. Subsections 4.3 and 4.4 provide examples of convex geometries not associated with path systems.

4.1. \( P_3 \) convexity

Consider the path system that maps a graph \( G \) to its paths of length two (not necessarily induced). The convexity defined over such paths is called the \( P_3 \) convexity of \( G \) [7], and the associated convex sets are called \( P_3 \) convex sets. Note that \( S \) is a \( P_3 \) convex set if and only if every \( x \not\in S \) has at most one neighbor in \( S \). Thus, the determination of the convex hull of a set in the \( P_3 \) convexity may be viewed as an infection process where noninfected vertices having two or more infected neighbors become infected (for more details, see [7] and references therein).

**Theorem 21.** The \( P_3 \) convexity of \( G \) is a convex geometry if and only if \( G \) is a forest of stars.

**Proof.** First, suppose that the \( P_3 \) convexity of \( G \) is a convex geometry. If there are vertices \( a, b, c \in V(G) \) inducing a \( K_3 \), then \( a \in H(\{b, c\}) \) and \( b \in H(\{a, c\}) \), which contradicts the anti-exchange property, since \( \{c\} \) is
a convex set. Therefore, $G$ is $K_3$-free and the endpoints of any edge of $G$ form a convex set. We also have that $G$ is $C_4$-free, because if there is an induced $C_4 = abcd$ in $G$ then $S = \{c, d\}$ is a convex set, $a \in H(S \cup \{b\})$, and $b \in H(S \cup \{a\})$, which contradicts the anti-exchange property. Now, suppose that $G$ contains an induced $C_5 = abcdea$ in $G$. Since $\{c, d\}$ is a convex set and $G$ is $C_4$-free, we have $H(\{c, e\}) = \{c, d, e\}$, because if there is $v \in H(\{c, e\}) \setminus \{c, d, e\}$ then $\{c, d, e, v\}$ would be an induced $C_4$. Since $a \in H(S \cup \{b\})$ and $b \in H(S \cup \{a\})$, the anti-exchange property does not hold. Therefore, $G$ is $C_5$-free. Next, suppose that $G$ contains an induced $P_4 = abed$. We have that $S = \{a, d\}$ is a convex set, because if there is a $P_3 = aed$ in $G$ then we would have that $be, ce \notin E(G)$ (since $G$ is $K_3$-free) and that $abcde$ is an induced $C_5$, which is not possible. But then the anti-exchange property does not hold because $b \in H(\{a\} \cup S)$ and $c \in H(\{b\} \cup S)$. Thus, $G$ is $P_4$-free.

Since $G$ is $(K_3, C_4, P_4)$-free, we can say that $G$ is a cograph having no triangles or induced cycles with 4 vertices. We know that every nontrivial connected cograph $G$ is the join of two graphs $G_1$ and $G_2$. Since $G$ is $K_3$-free, we have that the vertices of $G_i$ for every $i \in \{1, 2\}$ induce an independent set; and since $G$ is $C_4$-free, we have that $|V(G_i)| = 1$ for at least one $i \in \{1, 2\}$. Therefore, every connected component of $G$ is a star.

Conversely, consider that $G$ is a forest of stars. Since a star has at most one vertex with degree at least 2, $G$ satisfies the anti-exchange property. Thus the $P_3$ convexity of $G$ is a convex geometry.

4.2. Triangle path convexity

We say that a path $P$ in graph $G$ is a triangle path if every two vertices of $P$ with distance greater than 2 in $P$ are non-adjacent in $G$. A set $S \subseteq V(G)$ is tp-convex if for any $x, y \in S$, every vertex belonging to some triangle path between $x$ and $y$ belongs to $S$. The convexity associated with the tp-convex sets of $G$ is the triangle path convexity of $G$.

Theorem 22. The triangle path convexity of $G$ is a convex geometry if and only if $G$ is a forest.

Proof. First, suppose that the triangle path convexity of $G$ is a convex geometry. If there are vertices $a, b, c \in V(G)$ inducing a $K_3$ then $a \in H(\{b, c\})$.

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2The join of two graphs $G_1$ and $G_2$ is the graph $G$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{xy \mid x \in V(G_1), y \in V(G_2)\}$.

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and $b \in H\{a,c\}$, which contradicts the anti-exchange property, since $\{c\}$ is a convex set. Therefore, $G$ is $K_3$-free and the endpoints of any edge of $G$ form a convex set. We also have that $G$ is $C_k$-free for $k \geq 4$, because if there is an induced $C_k = v_1 \ldots v_kv_1$ in $G$ ($k \geq 4$) then $S = \{v_2,v_3\}$ is a convex set, $v_1 \in H(S \cup \{v_4\})$, and $v_4 \in H(S \cup \{v_1\})$, which contradicts the anti-exchange property. Hence, $G$ is acyclic.

Conversely, suppose that $G$ is a forest. As $G$ is $K_3$-free, the triangle path and the monophonic convexities of $G$ are identical. Since $G$ is a forest, $G$ is a chordal graph, which means, by Theorem 1, that both the monophonic and triangle path convexities of $G$ are convex geometries.

4.3. $\mathcal{F}$-free convexities

Let $H$ be a nontrivial graph. Given a graph $G$, we say that $S \subseteq V(G)$ is $H$-free convex if for every $S' \subseteq S$ with $|S'| = |V(H)| - 1$ and $x \in V(G)$, if the subgraph of $G$ induced by $S' \cup \{x\}$ is isomorphic to $H$ then $x \in S$. Given a family $\mathcal{F}$ of nontrivial graphs, we say that $S \subseteq V(G)$ is $\mathcal{F}$-free convex if $S$ is $H$-free convex for every $H \in \mathcal{F}$. The convexity associated with the $\mathcal{F}$-free convex sets of $G$ is the $\mathcal{F}$-free convexity of $G$.

**Theorem 23.** The $\mathcal{F}$-free convexity of $G$ is a convex geometry if and only if $G$ is $\mathcal{F}$-free.

*Proof.* First, suppose that $G$ is $\mathcal{F}$-free. By definition, every subset of $V(G)$ is $\mathcal{F}$-free convex, which means that the anti-exchange property is valid. Therefore, the $\mathcal{F}$-free convexity of $G$ is a convex geometry.

Conversely, suppose that the $\mathcal{F}$-free convexity of $G$ is a convex geometry and $G$ contains a graph of $\mathcal{F}$ as an induced subgraph. Let $S \subseteq V(G)$ with minimum size such that $G[S]$ is isomorphic to a graph $H \in \mathcal{F}$. Let $x,y$ be distinct vertices of $S$ and write $S' = S \setminus \{x,y\}$. By the minimality of $S$, we have that $S'$ is $\mathcal{F}$-free convex. Note also that $x \in H(S' \cup \{y\})$ and $y \in H(S \cup \{x\})$, which means that the anti-exchange property does not hold. This is a contradiction. Therefore, $G$ is $\mathcal{F}$-free. □

**Corollary 24.** Let $\mathcal{F}$ be the family of odd cycles. Then the $\mathcal{F}$-free convexity of $G$ is a convex geometry if and only if $G$ is bipartite.

**Corollary 25.** Let $\mathcal{F}$ be the family of graphs that can be obtained from $K_5$ or $K_{3,3}$ by subdivision of edges. Then the $\mathcal{F}$-free convexity of $G$ is a convex geometry if and only if $G$ is planar.
4.4. $P_4^+$-convexity

Recall that a cograph is a $P_4$-free graph. Given a graph $G$, we say that $S \subseteq V(G)$ is $P_4^+$-convex if, for every induced $P_4 = abcd$ in $G$, if $a, b, d \in S$ then $c \in S$. The convexity associated with the $P_4^+$-convex sets of $G$ is the $P_4^+$-convexity of $G$.

**Corollary 26.** The $P_4^+$-convexity of $G$ is a convex geometry if and only if $G$ is a cograph.

**Proof.** Follows from Theorem 23.

5. Conclusions

In this survey we presented characterizations of several classes of graphs via convex geometries. The characterizations follow the following structure: A graph $G$ is in class $\mathcal{G}$ if and only if the $C$-convexity of $G$ is a convex geometry. The table below summarizes the results.

| $\mathcal{G}$                  | $\mathcal{C}$                        |
|--------------------------------|--------------------------------------|
| chordal                        | monophonlic                          |
| Ptolemaic                      | geodesic                              |
| strongly chordal               | strong                               |
| weakly polarizable interval     | $m^3$                                |
| proper interval                | toll                                 |
| chordal cograph                | weakly toll                           |
| (Theorem 18)                   | $l^2$                                |
| forest of stars                | $P_3$                                |
| forests                        | triangle path                        |
| $\mathcal{F}$-free             | $\mathcal{F}$-free convexity         |
| bipartite                      | (Corollary 24)                       |
| planar                         | (Corollary 25)                       |
| cograph                        | $P_4^+$                              |

An interesting question is to increase the table below by including other important classes, such as distance-hereditary graphs and split graphs. Another line of research is to investigate convex geometries defined on digraphs.
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