Radiative energy loss of relativistic charged particles in absorptive media

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Abstract

We determine the energy loss spectrum per time-interval of a relativistic charge traversing a dispersive medium. Polarization and absorption effects in the medium are modelled via a complex index of refraction. We find that the spectrum amplitude becomes exponentially damped due to absorption mechanisms. Taking explicitly the effect of multiple scatterings on the charge trajectory into account, we confirm results obtained in a previous work.

Keywords: radiative energy loss, LPM effect, damping effect

1. Introduction

The matter formed in high-energy heavy-ion collisions is an opaque QCD medium \cite{1,2,3,4,5}, in which relativistic partons seem to suffer from a substantial medium-induced energy loss. Based on perturbative QCD calculations, this loss is commonly understood as dominated by the radiation of gluons off the energetic parton, cf. \cite{6,7} for recent reviews. The properties of a medium can, however, significantly influence the probability for the formation of radiation and, in consequence, the amount of radiative energy loss \cite{8,9,10}. In \cite{11}, the impact of absorption and polarization effects in an infinite dispersive medium on the energy lost by a relativistic point-charge per unit length (or time) was investigated in linear response theory. These considerations are strictly valid only for electric charges in electro-magnetic plasmas, but can be understood as an Abelian approximation for the dynamics of a colour charge in a strongly interacting medium. The aim of the present work is to provide some more explanatory details for the results derived in \cite{11}. Thereby, we follow closely the reasoning of Landau and Pomeranchuk in their pioneering work \cite{12}, in particular, when including the effect of multiple scatterings the charge encounters during the radiation formation process. Throughout this work, natural units are used, i.e. $\hbar = c = 1$.

2. Derivation of the differential mechanical work

In line with the study in \cite{13}, we determine the energy lost by a charge traversing an absorptive and polarizable medium from the negative of the work that has to be performed for moving against the electric field the charge induces in the medium. The work $W$ is conveniently evaluated in the mixed spatial coordinate $\vec{r}$ and frequency $\omega$ representation via

$$W = 2 \text{Re} \left( \int d^3 \vec{r}' \int_0^{\infty} d\omega \hat{E}(\vec{r}', \omega) \hat{j}(\vec{r}', \omega) \right).$$

Here, $\hat{j}(\vec{r}', \omega)$ is the Fourier-transform of the real and time-dependent classical current $\hat{j}(\vec{r}', t) = q \delta(t) \delta^3(\vec{r}' - \vec{r}(t))$ for a point-charge $q$ produced in the remote past and moving with velocity $\vec{v}(t)$, while $\hat{E}(\vec{r}', \omega)$ is the corresponding total electric field inside the medium as determined from Maxwell’s equations for linear dispersive media.
Considering in the following an isotropic and homogeneous medium, cf. [14], the constitutive expression for the electric field induced by the above current reads in Fourier-space

\[ \frac{1}{\omega \mu(\omega)} \left[ \hat{k} \hat{E}_x(\omega) - k^2 \hat{E}_x(\omega) \right] + \omega \epsilon(\omega) \hat{E}_x(\omega) = - \frac{iq}{(2\pi)^2} \int_{-\infty}^{\infty} dt' \hat{\nu}(t') e^{i\omega t' \hat{k} \cdot \hat{r}(t')} . \]  

(2)

For simplicity, the permittivity \( \epsilon(\omega) \) and permeability \( \mu(\omega) \) of the medium are assumed to exhibit no momentum \( \hat{k} \)-dependence. Decomposing \( \hat{E}_x(\omega) \) as well as \( \hat{\nu}(t) \) into components parallel and orthogonal to \( \hat{k} \), Eq. (2) results in

\[ \hat{E}_x(\omega) = \frac{iq}{(2\pi)^2} \int dt' \frac{e^{i\omega t' \hat{k} \cdot \hat{r}(t')}}{\omega \epsilon(\omega)} \left\{ \frac{(\hat{k} \cdot \hat{v}(t'))^2}{(\omega^2 n(\omega)^2 - k^2)} - \frac{\hat{v}(t') \omega^2 n(\omega)^2}{(\omega^2 n(\omega)^2 - k^2)} \right\} \]  

for the total electric field, where \( n(\omega)^2 = \epsilon(\omega) \mu(\omega) \) denotes the \( \omega \)-dependent complex medium index of refraction squared.

The negative work in differential form for positive \( \omega \) following from Eq. (3) together with \( \hat{j}^{+}(\hat{r}', \omega) \) reads then

\[ - \frac{dW}{d\omega} = \text{Re} \left\{ \frac{i q^2}{8 \pi^3} \int dt \int dt' \int d^3 k \frac{e^{i\omega t' \hat{k} \cdot \hat{r}(t')} - e^{i\omega t \hat{k} \cdot \hat{r}(t)}}{\omega \epsilon(\omega)} \left\{ \frac{(\hat{k} \cdot \hat{v}(t'))^2}{(k^2 - \omega^2 n(\omega)^2)} - \frac{\hat{v}(t') \omega^2 n(\omega)^2}{(k^2 - \omega^2 n(\omega)^2)} \right\} \right\} , \]  

(4)

where \( \Delta \hat{r} = \hat{r}(t) - \hat{r}(t') \). The three-momentum integral in Eq. (4) can be evaluated analytically. In order to show this, we concentrate for the moment on the integral related to the first term in the parentheses of Eq. (4) and may rewrite, cf. [12],

\[ \int d^3 k \frac{(\hat{k} \cdot \hat{v}(t))(\hat{k} \cdot \hat{v}(t'))}{(k^2 - \omega^2 n(\omega)^2)} e^{i\Delta \hat{k} \cdot \hat{r}} = - \frac{2\pi}{i} \left( \nabla_{\Delta \hat{r}} \hat{v}(t))(\nabla_{\Delta \hat{r}} \hat{v}(t')) \right) \frac{1}{\Delta r} \int_{-\infty}^{\infty} dk \frac{k e^{i\Delta \hat{k} \cdot \hat{r}}}{(k^2 - \omega^2 n(\omega)^2)} . \]  

(5)

The remaining \( k \)-integral is calculated by contour integration, where the contour has to be closed in the upper half complex momentum-plane because \( \Delta r \) is positive semidefinite. Depending, in general, on the sign of the non-vanishing imaginary part of \( n(\omega) = n_r(\omega) + i n_i(\omega) \), only one of the two simple poles in \( k(\omega) = \text{sgn}(n_r(\omega)) \omega n(\omega) \) will be located within the closed contour. This results in

\[ \int_{-\infty}^{\infty} dk \frac{k e^{i\Delta \hat{k} \cdot \hat{r}}}{(k^2 - \omega^2 n(\omega)^2)} = i \pi \text{Re} \left[ \text{sgn}(n_r(\omega)) \omega n(\omega) \right] \cdot \exp[-i\Delta \hat{r} \cdot \hat{r}(t')] . \]  

(6)

The \( \hat{k} \)-integral related to the second term in the parentheses of Eq. (4) can be evaluated in a similar fashion. This leads to our main result for the negative differential work for positive \( \omega \) reading [11]

\[ - \frac{dW}{d\omega} = -\text{Re} \left\{ \frac{i q^2}{4 \pi^3 \omega \epsilon(\omega)} \int dt \int dt' e^{-i\omega(t-t')} \left\{ \omega^2 n(\omega)^2 \hat{v}(t) \hat{v}(t') + (\nabla_{\Delta \hat{r}} \hat{v}(t))(\nabla_{\Delta \hat{r}} \hat{v}(t')) \right\} e^{i \text{sgn}(n_r(\omega)) \omega n(\omega) |\Delta r|} \frac{1}{\Delta r} \right\} . \]  

(7)

Irrespective of the sign of \( n_r(\omega) \), the amplitude in Eq. (7) is exponentially damped which implies a potential reduction of the negative differential work as compared to the case with a real valued medium index of refraction. In the limit \( n_i(\omega) = 0 \) and \( n_r(\omega) = 1 \), Eq. (7) renders into the expression for the radiation intensity discussed in [12] and, thus, represents a radiative energy loss spectrum. In an absorptive medium, \( n_i(\omega) > 0 \), and in case \( n_r(\omega) > 0 \) one can show that, considering for simplicity \( \mu(\omega) = 1 \), Eq. (7) also describes a loss of energy, which is the physical situation assumed in our study.

As \( \Delta \hat{r} \rightarrow -\Delta \hat{r} \) under variable interchange between \( t \) and \( t' \), only the part of the integrand in Eq. (7) which is symmetric in \( t \) and \( t' \) gives rise to a non-vanishing contribution. Then, the time integrations in Eq. (7) may be rearranged in such a way that, together with an overall factor of 2, only \( t \geq t' \) has to be considered in the \( t \)-integral. Furthermore, in order to stay in line with the original work of Landau and Pomeranchuk [12], one may neglect those terms in Eq. (7) which are related to derivatives of \( 1/\Delta r \). Correspondingly, Eq. (7) reduces to

\[ - \frac{dW}{d\omega} = -\text{Re} \left\{ \frac{i q^2 \omega n(\omega)^2}{2 \pi^3 \omega \epsilon(\omega)} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt \cos[\omega(t-t')] \left\{ \hat{v}(t) \hat{v}(t') - \frac{\hat{v}(t') \Delta \hat{r} \cdot \hat{v}(t) + \hat{v}(t) \Delta \hat{r} \cdot \hat{v}(t')}{(\Delta r)^2} \right\} e^{i \text{sgn}(n_r(\omega)) \omega n(\omega) |\Delta r|} \right\} . \]  

(8)
Such an approximation might at first appear reasonable for large \( \Delta r \gg 1 \). It is worthwhile noting, however, that taking into account the terms that are omitted by such a procedure is, in principle, of equivalent importance. This can easily be seen, for instance, by realizing that in \([12]\) the correct limiting Bethe-Heitler radiation spectrum can only be obtained if one assumes a mean squared transverse momentum transfer that is twice as large as the one for a single Rutherford-scattering. This implies, as was already discussed in \([15]\), that the terms omitted in \([12]\) give exactly the same contribution as those that were taken into account in \([12]\).

3. Result for a specific charge trajectory

The important quantity entering Eqs. \([7]\) and \([8]\) is \( \Delta \vec{r} \), which is determined for a given \( \vec{v}(t) \) via \( \Delta \vec{r} = \int_{t'}^t \vec{v}(\tau) d\tau \). In case of a time-independent velocity this results in \( \Delta \vec{r} = \vec{v} \cdot (t - t') \), such that the differential work in Eq. \((8)\) exactly vanishes. For a time-dependent \( \vec{v}(t) \), one may rewrite the time-integrals appearing in Eq. \((8)\) as

\[
\mathcal{J} = \int_{-\infty}^{\infty} dt' \int_{t'}^{\infty} dt \cos[\omega(t-t')] \left\{ \langle \vec{v}(t) \rangle \vec{v}(t') - \frac{\langle \vec{v}(t) \rangle \Delta \vec{r}(\vec{v}(t') \Delta \vec{r})}{(\Delta r)^2} \right\} \frac{e^{i \text{sgn}(\omega) \omega \Delta r}}{\Delta r}
\]

by substituting \( t = t' + \tau \), where now \( \Delta \vec{r} = \int_{t'}^{t} \vec{v}(\tau + \tau) d\tau \).

In order to quantify \( \mathcal{J} \) in Eq. \((9)\), one may assume that any given initial velocity \( \vec{v}(t') \) is changed by multiple scatterings in the medium according to \( \vec{v}(t' + \tau) = \vec{v}(t') \cos \theta(\tau) + [\vec{v}(t')] \hat{\vec{e}}_n \sin \theta(\tau) \) \([12]\). Here, \( \hat{\vec{e}}_n \) is a unit vector perpendicular to \( \vec{v}(t') \) and \( \theta(\tau) \) is the angle relative to \( \vec{v}(t') \) that the charge accumulates through these scatterings within the time-interval between \( t' \) and \( t' + \tau \geq t' \). For this specific form of \( \vec{v}(t) \), one finds \( \vec{v}(t') \vec{v}(t' + \tau) = \vec{v}(t')^2 \cos \theta(\tau) = \vec{v}(t')^2(1 - \theta(\tau)^2)/2 \) in the approximation of small deflection angles within the time-duration \( \bar{t} \) after \( t' \). Likewise, one finds \( \Delta \vec{r} = \vec{v}(t')(\bar{t} - I_2/2) + \vec{v}(t') \hat{\vec{e}}_n \hat{\vec{e}}_1 \) for small deflection angles, where \( I_2 = \int_0^{\bar{t}} \theta(\tau)^2 d\tau \) and \( I_1 = \int_0^{\bar{t}} \theta(\tau) d\tau \).

Assuming random kicks from the medium constituents, stochastic averaging over the deflection angles yields, for instance, \( \langle J_2 \rangle = \hat{q} \hat{q}^2 / E^2 \), where the quantity \( \hat{q} \) is defined here as \( \hat{q} = \langle \theta(\tau)^2 E^2 / (2\tau) \rangle \), i.e. one half of the mean of the transverse momentum transfer squared per unit time\([3]\) and \( E \) is the energy of the charge. As a consequence, one obtains \( \langle \Delta \vec{r} \rangle \approx \vec{v}(t')^2 \sqrt{1 - \hat{q}(3E^2)} \). Omitting terms of order \( O(\theta^2) \) and approximating the exact average of \( \mathcal{J} \) by taking the mean of each individual term entering Eq. \((9)\), we find

\[
\langle \mathcal{J} \rangle \approx -\hat{q} / 3E \int_{-\infty}^{\infty} dt' \vec{v}(t') \int_0^{\infty} d\omega \cos[\omega \bar{t}] e^{i \text{sgn}(\omega) \omega \Delta r},
\]

where \( \langle \Delta r \rangle \approx \vec{v}(t')(\bar{t} - \hat{q}(3E^2)) \). Inserting \( \langle \mathcal{J} \rangle \) from Eq. \((10)\) into Eq. \((8)\) finally results in the energy loss spectrum per time-interval presented in \([11]\). We note that the expression obtained in Eq. \((10)\) is a direct consequence of both the particular form of \( \langle \mathcal{J} \rangle \) and the assumption of small deflection angles. The latter gives rise to the condition \( \hat{q}(3E^2) \ll 1 \) which implies that, for a given \( E, \bar{t} \) in the integral in Eq. \((9)\) may not be considered too large or, otherwise, \( \langle \Delta r \rangle \) becomes unphysical. This condition is, however, naturally satisfied as the relevant contributions to the \( \bar{t} \)-integral in Eq. \((9)\) stem from the region, in which the oscillating functions vary only slowly, i.e. for \( \bar{t} \) smaller than or of the order of the formation time, cf. \([11][16]\).

4. Characterizing the absorptive medium by a complex index of refraction

Absorption and polarization effects in the dispersive medium can effectively be taken into account via a complex index of refraction \([14]\). Assuming that the radiated quanta obey a dispersion relation that is influenced by the medium, a suitable ansatz for a momentum-independent \( n(\omega)^2 \) is

\[
n(\omega)^2 = 1 - \frac{m^2}{\omega^2} + 2i \frac{\Gamma}{\omega},
\]

\[1\]We note that in Ref. \([11]\) the same quantity \( \hat{q} \) was incorrectly identified with the mean of the transverse momentum transfer squared per unit time. With the common definition of \( \hat{q} = \langle \theta(\tau)^2 E^2 / (2\tau) \rangle \), the expression for \( -dW/da \) found in \([11]\) for the specific charge trajectory considered here and in \([11]\) is modified to some extent, see \([19]\).
In this way, one attributes both an effective mass \( m \) and, as result of damping mechanisms, a finite width \( \Gamma \) to the quanta. In general, \( m \) and \( \Gamma \) can be free parameters of a Lorentz-type spectral function for these excitations \( \Gamma \).

A consequence of the special ansatz Eq. (11) is that radiation cannot be emitted for \( \omega < m \). This follows directly from the dispersion relation as determined by \( \text{Re}(\omega^2n(\omega)^2 - k^2) = 0 \). In addition, the latter implies also that the emitted quanta are time-like excitations with \( \omega \), for which \( n_i(\omega) \) does not vanish. In [13, 19], in contrast, the dielectric functions for a strongly interacting medium were determined from the leading-order hard-thermal-loop gluon self-energy. In this case, emitted gluons are also time-like quanta with \( \omega \), however, for which \( n_i(\omega) = 0 \).

5. Conclusions

In this work, we presented a more detailed derivation of the results obtained in [11] for the energy loss spectrum per time-interval of a relativistic charge traversing an infinite dispersive medium. Absorption mechanisms were found to result in an exponential damping of the spectrum amplitude. Our classical study, which takes as in [12] multiple scatterings of the charge in the medium into account, is restricted to the regime of small \( \omega \ll E \) and is strictly valid only for describing electro-magnetic phenomena. Missing the important non-Abelian effect of gluon rescatterings, our approach may, nevertheless, be viewed as an Abelian approximation for the dynamics of colour charges in the absorptive quark-gluon plasma.

Our main result for the negative differential work given in Eq. (7) was obtained by assuming that the charge was created in the remote past and that the complex index of refraction was \( \vec{k} \)-independent. Such simplifications are, however, unrealistic for the situation encountered in high-energy heavy-ion collisions. In particular, the interference between the medium induced radiation and the initial bremsstrahlung due to the production process of the charge within the medium may be of importance. Consequently, the considerations presented here will necessarily have to be improved before firm quantitative conclusions can be drawn. We leave these issues to be addressed in a forthcoming publication.

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