Notes on Naimark’s dilation theorem

Roberto Beneduci
Departimento di Fisica, Universitá della Calabria and Istituto Nazionale di Fisica Nucleare
via P. Bucci, cubo 30/B, 87036 Rende, Italy
E-mail: roberto.beneduci@unical.it

Abstract. After a short review of the main properties of unsharp observables we provide
examples of commutative unsharp observables obtained as the projection of sharp observables.
In particular, we consider a family of PVMs defined on the tensor product Hilbert space $\mathcal{H} \otimes \mathcal{K}$
which include the one introduced by Ozawa in his modification of the von Neumann position
measurement model and show that their projections onto $\mathcal{H}$ are commutative POVMs. Then, we
make some observations on the concept of Naimark dilation and its connections with the integral
representation of commutative POVMs. Finally, we introduce a partial order relation in the set
of Naimark dilations, show that there is a minimal element and that it is unique. Minimality is
a consequence of the partial ordered structure and coincides with the usual definition of minimal
Naimark dilation. In passing, we use Naimark’s theorem in order to prove some well known
properties of positive operator valued measures.

Dedication
The present notes are dedicated to the memory of my friend Paul Busch. In 2010 we started a
scientific collaboration which soon turned into a deep connection. He was a wise and precious
friend. Paul’s death leaves an unbridgeable loss for the community. Some of the observations I
present here are answers to questions that were raised during our conversations.

1. Introduction
Positive operator valued measures are fruitfully used in quantum mechanics in order to generalize
the concept of quantum observable which is otherwise represented by a self-adjoint operator.
Such a generalization is rooted in the statistical character of quantum physics and opened
new perspectives [1–9]. Just to recall some examples, it allowed a phase space representation
of quantum mechanics where the distribution functions describing the states of the system
are positive-definite [4, 7, 8, 10, 11], it made possible a rigorous description of the Heisenberg
error-disturbance relations [9, 12, 13], it made clear that commutativity is not equivalent to
compatibility [2, 7], and it made it possible to define a photon localization observable [4–6, 14–19].
As a relevant physical example that illustrates the problem of joint measurability in the framework
of POVMs, one can consider the case of the position and momentum operators, $Q, P$, in the
Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. Although they are incompatible (they do not commute) they can
be smeared to two commutative POVMs, $F^Q, F^P$ which are non-commuting but compatible,
i.e., they are the marginals of a joint POVM [2, 7]. In turn, $F^Q$ and $F^P$ provide examples of
commutative POVMs obtained as the randomization of a spectral measure. That is a general
feature of commutative POVMs, i.e., they are random versions of spectral measures [20–24]. To be more precise, they can be represented as the integral of a Markov kernel with respect to a spectral measure [22–24]. There are other integral representations of POVMs [25, 26]. In the commutative case, all of them are equivalent [23–27]. A different kind of connection between POVMs and projection valued measures (PVMs) is established by the Naimark dilation theorem which ensures that every POVM is the projection of a PVM in an extended Hilbert space [28–31]. From the physical viewpoint, the Naimark theorem ensures the existence of a measurement for any observable [7].

In the present note we make some observations on Naimark’s dilation theorem. For example, it is interesting to ask under what conditions a POVM obtained as the projection of a PVM is commutative. As an example, we analyze a family of PVMs which includes the one introduced by Ozawa [32] in his modification of the von Neumann position measurement model, and show that their projection is a commutative POVM. Then, we review some of the main results on the connection between the Naimark dilation theorem and integral representation of commutative POVMs, discuss the relevance of the Naimark theorem to the compatibility problem and derive some well known properties of POVMs by means of the Naimark theorem. Finally we analyze the concept of minimality of the Naimark dilation in the framework of partially ordered sets (posets) where minimality is a consequence of the poset structure and prove the existence of a unique minimal Naimark dilation.

In what follows, we denote by \(\mathcal{B}(X)\) the Borel \(\sigma\)-algebra of a topological space \(X\) and by \(\mathcal{L}_s(H)\) the space of all bounded self-adjoint linear operators acting in a Hilbert space \(H\) with scalar product \(\langle \cdot, \cdot \rangle\). The subspace of positive operators is denoted by \(\mathcal{L}_s^+(H)\).

**Definition 1.1.** A Positive Operator Valued Measure (for short, POVM) is a map \(F : \mathcal{B}(X) \to \mathcal{L}_s^+(H)\) such that:

\[
F\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} F(\Delta_n),
\]

where \(\{\Delta_n\}\) is a countable family of disjoint sets in \(\mathcal{B}(X)\) and the series converges in the weak operator topology. It is said to be normalized if

\[
F(X) = 1,
\]

where \(1\) is the identity operator.

**Definition 1.2.** A POVM is said to be commutative if

\[
[F(\Delta_1), F(\Delta_2)] = 0, \quad \forall \Delta_1, \Delta_2 \in \mathcal{B}(X).
\]

**Definition 1.3.** A POVM is said to be orthogonal if \(\Delta_1 \cap \Delta_2 = \emptyset\) implies

\[
F(\Delta_1)F(\Delta_2) = 0,
\]

where \(0\) is the null operator.

**Definition 1.4.** A Projection Valued Measure (for short, PVM) is an orthogonal, normalized POVM. A spectral measure is a real PVM \((X = \mathbb{R})\).

Let \(E\) be a PVM. By equation (2),

\[
0 = E(\Delta)E(X - \Delta) = E(\Delta)[1 - E(\Delta)] = E(\Delta) - E(\Delta)^2.
\]

We can then restate Definition 1.4 as follows.
Definition 1.5. A PVM $E$ is a POVM such that $E(\Delta)$ is a projection operator for each $\Delta \in B(X)$.

In quantum mechanics, non-orthogonal normalized POVMs are also called generalised or unsharp observables while PVMs are called standard or sharp observables.

Definition 1.6. The von Neumann algebra (or $W^*$-algebra) $A^W(F)$ generated by the POVM $F$ is the von Neumann algebra generated by the set $\{F(\Delta)\}_{\Delta \in B(X)}$. It is the smallest $*$-algebra of bounded operators closed in the weak operator topology and generated by $\{F(\Delta)\}_{\Delta \in B(X)}$.

In what follows, we shall always refer to normalized POVMs and we shall use the term “measurable” for the Borel measurable functions. For any unit vector $\psi \in H$, the map $\langle F(\cdot)\psi, \psi \rangle : B(X) \to [0, 1], \quad \Delta \mapsto \langle F(\Delta)\psi, \psi \rangle,$ is a measure. In the following, we shall use the symbol $d\langle F_x\psi, \psi \rangle$ to mean integration with respect to $\langle F(\cdot)\psi, \psi \rangle$.

For any real, bounded and measurable function $f$ and for any POVM $F$, there is a unique [33] bounded self-adjoint operator $B \in L_s(H)$ such that $\langle B\psi, \psi \rangle = \int f(x)d\langle F_x\psi, \psi \rangle$, for each $\psi \in H$. (3)

If equation (3) is satisfied, we write $B = \int f(x)dF_x$ or $B = \int f(x)F(dx)$ equivalently.

Definition 1.7. The spectrum $\sigma(F)$ of a POVM $F$ is the closed set $\{x \in X : F(\Delta) \neq 0, \forall \Delta \text{ open}, x \in \Delta\}$.

By the spectral Theorem [34], there is a one-to-one correspondence between spectral measures, $E$, and self-adjoint operators, $B$, the correspondence being given by $B = \int \lambda dE^B_\lambda$.

Notice that the spectrum of $E^B$ coincides with the spectrum of the corresponding self-adjoint operator $B$. Moreover, in this case a functional calculus can be developed. Indeed, if $f : \mathbb{R} \to \mathbb{R}$ is a measurable real-valued function, we can define the self-adjoint operator [34] $f(B) = \int f(\lambda)dE^B_\lambda.$ (4)

If $f$ is bounded, then $f(B)$ is bounded [34]. In particular,

$$E[i] := \int t^i dE_t = \left( \int t dE_t \right)^i = A^i$$ (5)

and $A = \int t dE_t$ is the generator of the von Neumann algebra generated by $E$. We point out that if $F$ is not projection valued, equations (5) and (4) do not hold [35] and, in order to recover the generator of $A^W(F)$, we need all the moments of $F$. In particular, in the case of a real commutative POVM $F$ with bounded spectrum and such that $F(\Delta)$ is discrete for any $\Delta$, we have [36,37]

$$A = \sum_{i=0}^{\infty} \alpha_i F[i], \quad \alpha_i \geq 0, \quad \sum_{i=0}^{\infty} \alpha_i < \infty,$$

where $A$ is a generator of the von Neumann algebra $A^W(F)$.

In the following we assume $X$ to be Hausdorff, locally compact and second countable.
2. Naimark’s dilation theorem
A procedure to derive a POVM is to start from a PVM in a Hilbert space \( H \) and to project to a subspace \( H_0 \subset H \), i.e., \( F(\Delta) = PE(\Delta)P \) where \( PH = H_0 \). In this context it is worth analyzing the properties of the POVM \( F \) as for example its commutativity. Here we provide a family of PVMs whose projection onto a subspace gives a commutative POVM. In particular, we consider a PVM of the kind

\[
\bar{Q}(\tau) = cQ(0) \otimes I + dI \otimes \bar{Q}(0),
\]

where \( \bar{Q}(0) \) is a self-adjoint operator in the Hilbert space \( K \), \( Q(0) \) is a self-adjoint operator in the Hilbert space \( H \otimes K \) and \( c \) and \( b \) are functions depending on \( \tau \). We suppose \( H \) and \( K \) are infinite-dimensional. An observable of this kind has been derived by Ozawa [32] as the solution of the Heisenberg equation in his model for the measurement of position. In the Ozawa model, \( \bar{Q}(\tau) \) denotes the meter observable at time \( \tau \) and \( Q(0) \) the position observable of the system at time \( t = 0 \). The outcome of the measurement is obtained by measuring the meter observable \( \bar{Q} \) at time \( \Delta t^1 \). Analogous equations can be derived for the momentum. Now we show that the projection of \( \bar{Q}(\tau) \) onto the subspace \( H \subset H \otimes K \) is a commutative POVM. First we recall that the tensor product \( H \otimes K \) can be represented as the direct sum\(^2\)

\[
H \otimes K = H \oplus H \oplus \cdots
\]

Note also that \( H \) is isomorphic to \( H \otimes \phi \) where \( \phi \) is a unit vector in \( K \). Now, let \( \{\psi_i\}_i \) and \( \{\phi_j\}_j \) be bases for \( H \) and \( K \) respectively. Then

\[
H \otimes K = \bigoplus_j (H \otimes \phi_j)
\]

so that \( H \) is embedded in \( H \otimes K \) as

\[
(\mathcal{H} \otimes \phi_1) \oplus 0 \oplus 0 \cdots
\]

Let \( P \) be the projection operator from \( \mathcal{H} \otimes K \) onto \( \mathcal{H} \) defined by \( P(\xi_1 \otimes \phi_1, \ldots, \xi_i \otimes \phi_i, \ldots) = \xi_1 \otimes \phi_1 \) where \( \xi_i \in \mathcal{H}, i \in \mathbb{N} \). We have

\[
P(\psi \otimes \phi_j) = (\psi \otimes \phi_j)\delta_{ij}.
\]

Let \( E^{Q(0)} \) and \( E^{\bar{Q}(0)} \) denote respectively the spectral measures of \( Q(0) \) and \( \bar{Q}(0) \) in equation (6). Then, the spectral measure of \( \bar{Q}(\tau) \) is \( E^{Q(0)} \otimes E^{\bar{Q}(0)} \). Indeed, we have

\[
cQ(0) \otimes I + dI \otimes \bar{Q}(0) = \int \int (c \lambda + d \beta) \, dE^{Q(0)}(\lambda) \otimes dE^{\bar{Q}(0)}(\beta).
\]

What about the projection of \( E^{Q(0)} \otimes E^{\bar{Q}(0)} \) onto \( \mathcal{H} \)? For any \( \psi_i \in \mathcal{H} \), we have

\[
P(E^{Q(0)}(\Delta_1) \otimes E^{\bar{Q}(0)}(\Delta_2))P(\psi_i \otimes \phi_1) = P(E^{Q(0)}(\Delta_1) \otimes E^{\bar{Q}(0)}(\Delta_2))(\psi_i \otimes \phi_1)
\]
\[
= P(E^{Q(0)}(\Delta_1)\psi_i \otimes E^{\bar{Q}(0)}(\Delta_2)\phi_1)
\]
\[
= (E^{Q(0)}(\Delta_1)\psi_i \otimes \alpha_1\phi_1),
\]

\(^1\) \( \tau \) is chosen such that \( 0 \leq \tau \leq \Delta t \) with \( K \Delta t = 1 \) where \( K \) is the coupling constant which is supposed to be very large, \( K \gg 1 \).

\(^2\) We recall that the direct sum of a countable family of Hilbert spaces \( \{(H_i, \langle \cdot, \cdot \rangle_i)\}_{i \in \mathbb{N}} \) is the space \( \bigoplus_{i=1}^{\infty} H_i = \{(\psi^1, \ldots, \psi^i, \ldots) | \psi^i \in H_i; \sum_{i=1}^{\infty} ||\psi^i||^2 < \infty \} \) with the inner product \( \langle (\psi^1, \ldots, \psi^i, \ldots), (\phi^1, \ldots, \phi^i, \ldots) \rangle = \sum_{i=1}^{\infty} \langle \psi^i, \phi^i \rangle_i. \)
where $\alpha_1$ comes from the expansion $E^{\bar{Q}(0)}(\Delta_1)\phi_1 = \sum_i \alpha_i \phi_i$. Moreover, $\forall \psi_i \in \mathcal{H}$,

$$P(E^{\bar{Q}(0)}(\Delta_1') \otimes E^{\bar{Q}(0)}(\Delta_2'))P(E^{\bar{Q}(0)}(\Delta_1)\psi_i \otimes \alpha_1 \phi_1) =$$
$$= P(E^{\bar{Q}(0)}(\Delta_1')E^{\bar{Q}(0)}(\Delta_1)\psi_i \otimes E^{\bar{Q}(0)}(\Delta_2')\alpha_1 \phi_1)$$
$$= (E^{\bar{Q}(0)}(\Delta_1')E^{\bar{Q}(0)}(\Delta_1)\psi_i \otimes \alpha'_1 \alpha_1 \phi_1),$$

where $\alpha'_1$ comes from the expansion $E^{\bar{Q}(0)}(\Delta_2')\phi_1 = \sum_i \alpha'_i \phi_i$. We can now calculate

$$P(E^{\bar{Q}(0)}(\Delta_1') \otimes E^{\bar{Q}(0)}(\Delta_2'))P(\psi_i \otimes \phi_1) = P(E^{\bar{Q}(0)}(\Delta_1') \otimes E^{\bar{Q}(0)}(\Delta_2'))(\psi_i \otimes \phi_1)$$
$$= (E^{\bar{Q}(0)}(\Delta_1')\psi_i \otimes \alpha'_1 \phi_1)$$

and

$$P(E^{\bar{Q}(0)}(\Delta_1) \otimes E^{\bar{Q}(0)}(\Delta_2))P((E^{\bar{Q}(0)}(\Delta_1')\psi_i \otimes \alpha'_1 \phi_1))$$
$$= P(E^{\bar{Q}(0)}(\Delta_1)E^{\bar{Q}(0)}(\Delta_1')\psi_i \otimes E^{\bar{Q}(0)}(\Delta_2')\alpha'_1 \phi_1)$$
$$= (E^{\bar{Q}(0)}(\Delta_1')E^{\bar{Q}(0)}(\Delta_1)\psi_i \otimes \alpha'_1 \alpha_1 \phi_1),$$

which show that the POVM $P(E^{\bar{Q}(0)}(\Delta_1') \otimes E^{\bar{Q}(0)}(\Delta_2'))P$ is commutative. Thus $P(E^{\bar{Q}(0)}(\Delta_1') \otimes E^{\bar{Q}(0)}(\Delta_2'))P$ should be the randomization of a sharp position observable and it would be interesting to analyze the properties of the Markov kernel that realizes the randomization. Moreover, the sharp version should be equivalent to the projection of the pointer observable $cQ(0) \otimes I + dI \otimes \bar{Q}(0)$ (see Theorem 2.7 below). Those are problems deserving further work. Anyway, we have a relevant physical example (perhaps a family of physical examples) where the projection of a pointer observable is a commutative POVM.

An important result concerning the connections between PVMs and POVMs is due to Naimark. He proved that not only the projection of a PVM provides a POVM but that every non-orthogonal POVM is the projection of a PVM.

**Theorem 2.1** (Naimark [3,8,28–31]). Let $F$ be a POVM. Then, there exist an extended Hilbert space $\mathcal{H}^+$ and a PVM $E^+$ on $\mathcal{H}^+$ such that

$$F(\Delta)\psi = PE^+(\Delta)\psi, \quad \forall \psi \in \mathcal{H},$$

where $P$ is the operator of projection onto $\mathcal{H}$. The dilation $E^+: \mathcal{H}^+ \rightarrow \mathcal{H}^+$ can be chosen such that the extended Hilbert space $\mathcal{H}^+$ coincides with the closure of $\text{span}\{E^+(\Delta)\psi : \psi \in \mathcal{H}, \Delta \in \mathcal{B}(X)\}$. In this case the dilation is said to be minimal and is unique up to unitary transformations.

Therefore, up to isomorphisms, there is a unique minimal Naimark dilation of $F$. In subsection 2.3 we reformulate the concept of minimality of the Naimark dilation in the language of partially ordered sets (posets) where the existence of a minimal dilation can be proved by Zorn’s lemma. In this framework, the definition of minimal dilation (see Theorem 2.1) becomes a consequence of the poset structure.

In the case of a commutative POVM $F$, Naimark’s theorem can be used to define a self-adjoint operator $A^+$ in $\mathcal{H}^+$ whose projection onto $\mathcal{H}$ gives the sharp reconstruction of $F$ [36–41]. Let us first recall the integral representation of commutative POVMs. In the following $\Lambda$ denotes a measurable space.
Definition 2.2. A Markov kernel is a map $\mu : \Lambda \times \mathcal{B}(X) \to [0, 1]$ such that,

1. $\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(X)$,
2. $\mu(\cdot)(\lambda)$ is a probability measure for each $\lambda \in \Lambda$.

Definition 2.3. A Feller Markov kernel is a Markov kernel $\mu(\cdot, \cdot) : \Lambda \times \mathcal{B}(X) \to [0, 1]$ such that the function

$$G(\lambda) = \int_X f(x) \mu_{dx}(\lambda), \quad \lambda \in \Lambda$$

is continuous and bounded whenever $f$ is continuous and bounded.

Theorem 2.4 ([24]). A POVM $F : \mathcal{B}(X) \to \mathcal{L}^+_+(\mathcal{H})$ is commutative if and only if there exists a bounded self-adjoint operator $A = \int \lambda dE_\lambda$ with spectrum $\sigma(A) \subset [0, 1]$, a subset $\Gamma \subset \sigma(A)$, $E(\Gamma) = 1$, a ring $\mathcal{R}$ which generates $\mathcal{B}(X)$ and a Feller Markov Kernel $\mu : \Gamma \times \mathcal{B}(X) \to [0, 1]$ such that

1. $F(\Delta) = \int_\Gamma \mu_\Delta(\lambda) dE_\lambda, \quad \Delta \in \mathcal{B}(X)$.
2. $A^W(F) = A^W(A)$.
3. $\mu$ separates the points in $\Gamma$.
4. $\mu_\Delta$ is continuous for each $\Delta \in \mathcal{R}$.
5. $A$ is called the sharp version of $F$ and is unique up to bijections.

An equivalence relation $\sim_a$ such that $A \sim_a F$ whenever $A$ is discrete can be introduced. That can be interpreted as the equivalence of $A$ and $F$ with respect to their informational content [40].

Definition 2.5. Let $A$ and $B$ be self-adjoint operators. Whenever there exists a one-to-one measurable function $f$ such that $A = f(B)$, we say that $A$ is equivalent to $B$ and write $A \leftrightarrow B$.

Theorem 2.6 ([38]). Let $F : \mathcal{B}(\mathbb{R}) \to \mathcal{L}^+_+(\mathcal{H})$ be a real POVM. Let $f$ be a bounded measurable real valued function. Let $A^+$ be the self-adjoint operator corresponding to the Naimark dilation $E^+$ of $F$. Then,

$$F(f) := \int f(t) dF_t = Pf(A^+)P.$$

In the case $f$ is unbounded, the domain of definition of the operators must be taken into account [9,35].

Theorem 2.7 ([36,37]). Let $F : \mathcal{B}(\mathbb{R}) \to \mathcal{L}^+_+(\mathcal{H})$ be a commutative POVM such that the operators in the range of $F$ are discrete.\(^3\) Let $A$ be the sharp version of $F$ and $A^+ = \int \lambda dE^+_\lambda$ the Naimark operator corresponding to the Naimark dilation $E^+$. Then, there are two bounded, one-to-one functions $f$ and $h$ such that

$$h(A) = \int f(t) dF_t = P^+ f(A^+)P.$$

Theorem 2.7 establishes that $h(A)$ is the projection of $f(A^+)$ with $h$ and $f$ one-to-one. According to Definition 2.5 we can say that a correspondence between $A$ and $A^+$ is established. We denote such a correspondence by $A \leftrightarrow PrA^+$.

\(^3\) $F(\Delta)$ is discrete if it has a complete set of eigenvectors.
2.1. Conditions for the joint measurability
In the present section we recall the definition and some of the main theorems on the joint measurability of two POVMs. Then, we show how Naimark’s dilation theorem can be used in order to give necessary and sufficient conditions for joint measurability.

**Definition 2.8.** Two POVMs $F_1 : \mathcal{B}(X_1) \to \mathcal{L}_+^+(\mathcal{H})$, $F_2 : \mathcal{B}(X_2) \to \mathcal{L}_+^+(\mathcal{H})$ are compatible (or jointly measurable) if they are marginals of a joint POVM $F : \mathcal{B}(X_1 \times X_2) \to \mathcal{L}_+^+(\mathcal{H})$.

We recall that the symbol $\mathcal{B}(X_1 \times X_2)$ denotes the product $\sigma$-algebra generated by the family of sets $\{\Delta_1 \times \Delta_2 : \Delta_1 \in \mathcal{B}(X_1), \Delta_2 \in \mathcal{B}(X_2)\}$.

Two POVMs $F_1$ and $F_2$ commute if $[F_1(\Delta_1), F_2(\Delta_2)] = 0$, for each $\Delta_1 \in \mathcal{B}(X_1)$ and $\Delta_2 \in \mathcal{B}(X_2)$. In the following, the commutativity of two POVMs $F_1$ and $F_2$ is denoted by the symbol $[F_1, F_2] = 0$.

As the following theorem shows, in the case of two POVMs, commutativity implies compatibility but the converse is not true, i.e., commutativity is not a necessary condition for the compatibility. That is one of the main advantage in using POVMs in order to represent quantum observables.

**Theorem 2.9 ([42]).** Two commuting POVMs are compatible.

Now, we use Naimark’s dilation theorem in order to characterize the compatibility of two POVMs. The proof is provided for completeness.

**Theorem 2.10 ([43]).** Two POVMs $F_1 : \mathcal{B}(X_1) \to \mathcal{L}_+^+(\mathcal{H})$ and $F_2 : \mathcal{B}(X_2) \to \mathcal{L}_+^+(\mathcal{H})$ are compatible if and only if there are two Naimark dilations $E_1^+ : \mathcal{B}(X_1) \to \mathcal{L}_+^+(\mathcal{H})$ and $E_2^+ : \mathcal{B}(X_2) \to \mathcal{L}_+^+(\mathcal{H})$ such that $[E_1^+, E_2^+] = 0$.

**Proof.** Suppose $F_1$ and $F_2$ are compatible. Then, there is a POVM $F$ of which $F_1$ and $F_2$ are the marginals; i.e., $F_1(\Delta_1) = F(\Delta_1 \times X_2)$, $F_2(\Delta_2) = F(X_1 \times \Delta_2)$. Let $E^+$ be a Naimark dilation of $F$ and consider the POVMs $E_1^+(\Delta_1) = E^+(\Delta_1 \times X_2)$ and $E_2^+(\Delta_2) = E^+(X_1 \times \Delta_2)$. We have, $P E_1^+(\Delta_1) P = P E^+(\Delta_1 \times X_2) P = F(\Delta_1 \times X_2) = F_1(\Delta_1)$ and $P E_2^+(\Delta_2) P = P E^+(X_1 \times \Delta_2) P = F(X_1 \times \Delta_2) = F_2(\Delta_2)$. Moreover, $E_1^+$ and $E_2^+$ commute since they are the marginals of the POVM $E^+$.

Conversely, suppose there are two Naimark dilations $E_1^+$ and $E_2^+$ such that $[E_1^+, E_2^+] = 0$. Thanks to the commutativity $[E_1^+, E_2^+] = 0$, there is a joint POVM $E^+$; i.e., $E_1^+(\Delta_1) = E^+(\Delta_1 \times X_2)$, $E_2^+(\Delta_2) = E^+(X_1 \times \Delta_2)$. We have,

\[
\begin{align*}
F_1(\Delta_1) &= P E_1^+(\Delta_1) P = P E^+(\Delta_1 \times X_2) P = F(\Delta_1 \times X_2); \\
F_2(\Delta_2) &= P E_2^+(\Delta_2) P = P E^+(X_1 \times \Delta_2) P = F(X_1 \times \Delta_2),
\end{align*}
\]

where $F := P E^+ P$. Therefore, $F$ is a joint POVM for $F_1$ and $F_2$. \(\square\)

Note that if $F_1$ and $F_2$ are POVMs, theorems 2.10 and 2.9 imply that $F_1$ and $F_2$ are compatible if and only if they commute. Indeed, if $P E_1^+ P$ and $P E_2^+ P$ are projections then $[P, E_i^+] = 0$, $i = 1, 2$. Hence $[F_1, F_2] = 0$. Moreover [35], two POVMs are compatible if and only if they are the marginals of a POVM.

Theorem 2.10 is illustrated in the following diagram.

\[ E_1^+ \xrightarrow{\text{c}} E_2^+ \]
\[ \xrightarrow{P} \]
\[ F_1 \xleftarrow{\text{c}} F_2 \]

where the arrow $\xrightarrow{\text{c}}$ denotes compatibility, $\xrightarrow{P}$ denotes the relationship between a POVM and its dilation as expressed by Naimark’s theorem.
2.2. On some properties of POVMs
In the present subsection, we use the Naimark dilation theorem and the results in section 2.1 to prove some well known theorems about POVMs and compatibility of POVMs.

**Theorem 2.11.** Let $F : \mathcal{B}(X) \to \mathcal{L}_+^+(\mathcal{H})$ be a POVM and suppose there is $\Delta \in \mathcal{B}(X)$ such that $F(\Delta)^2 = F(\Delta)$. Then, $[F(\Delta), F(\Delta')] = 0$, $\forall \Delta' \in \mathcal{B}(X)$.

**Proof.** Let $E^+\Delta$ be a Naimark dilation of $F$. Then $PE^+\Delta|\mathcal{H} = F(\Delta)$. Since by hypothesis $F(\Delta)$ is a projection, it must be $[P, E^+\Delta] = 0$. Then, for any $\psi \in \mathcal{H}$ and $\Delta' \in \mathcal{B}(X)$,

$$F(\Delta)F(\Delta')\psi = PE^+\Delta PE^+\Delta'\psi = PE^+\Delta E^+\Delta'\psi = PE^+\Delta E^+\Delta P\psi = PE^+\Delta E^+\Delta\psi = F(\Delta)F(\Delta')\psi.$$

\[\Box\]

**Theorem 2.12.** Let $E : \mathcal{B}(X_1) \to \mathcal{L}_+^+(\mathcal{H})$ and $F : \mathcal{B}(X_2) \to \mathcal{L}_+^+(\mathcal{H})$ be a PVM and a POVM respectively. They are compatible if and only if they commute.

**Proof.** If they commute they are compatible by Theorem 2.9. If they are compatible there are commuting dilations $E^+ : \mathcal{B}(X_1) \to \mathcal{L}_+^+(\mathcal{H}^+)$ and $F^+ : \mathcal{B}(X_2) \to \mathcal{L}_+^+(\mathcal{H}^+)$ of $E$ and $F$ respectively. Since for any $\Delta_1 \in \mathcal{B}(X_1)$, $E(\Delta_1)$ is a projection, it must be $[P, E^+\Delta_1] = 0$. Therefore, for any $\psi \in \mathcal{H}$,

$$F(\Delta_2)E(\Delta_1)\psi = P\tilde{E}(\Delta_2)PE^+(\Delta_1)\psi = P\tilde{E}(\Delta_2)E^+(\Delta_1)P\psi = PE^+(\Delta_1)\tilde{E}^+(\Delta_2)\psi = PE^+(\Delta_1)\tilde{E}^+(\Delta_2)P\psi = E(\Delta_1)F(\Delta_2)\psi.$$

\[\Box\]

2.3. Minimal dilation
In the present subsection we analyze the problem of the existence of a minimal Naimark dilation in the framework of partially ordered sets (posets). We introduce a preorder in the set of all Naimark dilations, then we consider the partial order it induces and prove the existence of a minimal element as well as its uniqueness. In this approach the definition of the minimal Naimark dilation becomes a consequence of the partial order structure. First we reformulate the definition of Naimark’s dilation in a language which is more appropriate to the partial order sets framework.

**Definition 2.13.** Let $F : \mathcal{B}(X) \to \mathcal{L}_+^+(\mathcal{H})$ be a POVM on the Hilbert space $\mathcal{H}$. A Naimark dilation is a couple $(E^+, \mathcal{H}^+)$, where $\mathcal{H}^+$ is a Hilbert space containing $\mathcal{H}$ and $E^+$ is a PVM, such that

$$F(\Delta)\psi = PE^+(\Delta)\psi, \quad \forall \psi \in \mathcal{H} \text{ and } \forall \Delta \in \mathcal{B}(X),$$

where $P$ is the operator of projection onto $\mathcal{H}$.

Let $\mathcal{N}_d(F)$ be the set of all the Naimark dilations of a POVM $F$. We can introduce a preorder in $\mathcal{N}_d(F)$.

**Definition 2.14.** Let $F$ be a POVM on the Hilbert space $\mathcal{H}$. Let $(E^+, \mathcal{H}^+), (E^-, \mathcal{H}^-) \in \mathcal{N}_d(F)$. We say that $(E^-, \mathcal{H}^-)$ extends $(E^+, \mathcal{H}^+)$, and write $(E^-, \mathcal{H}^-) \sqsubseteq (E^+, \mathcal{H}^+)$ if there exists a unitary transformation $U : \mathcal{H}^+ \to U\mathcal{H}^+ \subset \mathcal{H}^-, \mathcal{H} \subset U\mathcal{H}^+$, such that $UE^+U^\dagger = E^-|_{U\mathcal{H}^+}$. 

8
Now we use the preorder to establish an equivalence relation between the elements of \( N_d(F) \).

**Definition 2.15.** We say that \( (E^+, H^+) \), \( (E^-, H^-) \) \( \in N_d(F) \) are equivalent and write \( (E^+, H^+) \sim (E^-, H^-) \) if \( (E^-, H^-) \subseteq (E^+, H^+) \) and \( (E^+, H^+) \subseteq (E^-, H^-) \). The set \( N_d(F) \) modulo the equivalence relation is denoted \( N_d(F)^\sim \).

Now the relation \( \subseteq \) induces a partial order in the set \( (N_d(F))^\sim \). Indeed, let \([a] \) denote the equivalence class of \( a \in N_d(F) \). Let \([a], [b] \in N_d(F)^\sim \). We say that \([a] \subseteq [b] \) if \( a \subseteq b \). It is easy to see that the \( (N_d(F)^\sim , \subseteq) \) is a poset.

**Proposition 2.16.** \( (E^+, H^+) \sim (E^-, H^-) \) if and only if there is a unitary transformation \( U : H^+ \rightarrow H^- \) such that \( UE^+U^\dagger = E^- \).

**Proof.** Since \( (E^-, H^-) \subseteq (E^+, H^+) \), there is an isometry \( U_1 : H^+ \rightarrow U_1(H^+) \subseteq H^- \) such that \( U_1E^+U_1^\dagger = E^- \mid_{U_1H^+} \). Since \( (E^+, H^+) \subseteq (E^-, H^-) \), there is an isometry \( U_2 : H^- \rightarrow U_2(H^-) \subseteq H^+ \) such that \( U_2E^-U_2^\dagger = E^+ \mid_{U_2H^-} \). Since the map \( U_2U_1 : H^+ \rightarrow H^+ \) preserves the inner product, it must be unitary (surjective and isometric). Hence \( U_1 \) and \( U_2 \) are unitary as well. In particular \( U_1(H^+) = H^+ \) and \( U_1E^+U_1^\dagger = E^- \). It is easy to prove the converse. \( \square \)

Now, we want to prove that \( N_d(F)^\sim \) has a \( \subseteq \)-maximal element.

**Theorem 2.17.** Let \( F \) be a POVM on the Hilbert space \( H \). Then, there is a Naimark dilation \( (E^+, H^+) \in N_d(F)^\sim \) which is \( \subseteq \)-maximal.

**Proof.** We prove that every chain in \( N_d(F)^\sim \) has an upper bound. Then, an application of Zorn’s lemma will prove the thesis. Let \( \{ (E^\alpha, H^\alpha) \}_{\alpha \in A} \) be a chain in \( N_d(F)^\sim \) (i.e., it is a non-empty subset \( N_d(F)^\sim \) such that for all \( [a], [b] \in N_d(F)^\sim \), either \([a] \subseteq [b] \) or \([b] \subseteq [a] \). Let \( \pi \in A \) and set \( \Pi := \{ \alpha \in A : (E^\alpha, H^\alpha) \subseteq (E^\pi, H^\pi) \} \). By definition of the order relation, for each \( \alpha \in \Pi \), \( H^\alpha \) can be embedded isometrically into \( H^\pi \), i.e., there exists a unitary isomorphism \( U^\alpha : H^\alpha \rightarrow U^\alpha H^\alpha \subseteq H^\pi \) such that \( U^\alpha E^\alpha(U^\alpha)^\dagger = E^\pi \mid_{U^\alpha H^\alpha} \). Now, we claim that \( (\overline{E}, \overline{H}) \) with \( \overline{H} := \bigcap \overline{U^\alpha H^\alpha} \supseteq H \) and \( \overline{E} = E^\pi \mid_{\overline{H}} \) is an upper bound of the collection \( \{ (E^\alpha, H^\alpha) \}_{\alpha \in \Pi} \).

First we suppose that \( \overline{E}(\overline{H}) \subset \overline{H} \). Then, \( \overline{H} \supseteq U^\alpha H^\alpha \), for any \( \alpha \in \Pi \). Hence, \( E^\pi \psi \in U^\alpha H^\alpha \), for any \( \alpha \in \Pi \) and \( \overline{E} \psi = E^\pi \psi \subseteq \bigcap \overline{U^\alpha H^\alpha} = \overline{H} \).

Now, we prove that \( (E^\alpha, H^\alpha) \subseteq (\overline{E}, \overline{H}) \) for each \( \alpha \in \Pi \). The map \( \overline{U}^\alpha := (U^\alpha)^\dagger : H^\alpha \rightarrow (U^\alpha)^\dagger(\overline{H}) \subseteq H^\alpha \) is unitary with inverse \( (\overline{U}^\alpha)^\dagger : \overline{H} \rightarrow \overline{H} \) such that \( (\overline{U}^\alpha)^\dagger = U^\alpha_{|\overline{H}} \). Moreover, \( \overline{U}^\alpha E(U^\alpha)^\dagger = E^\alpha_{|\overline{H}} \). Indeed, \( (E^\pi, H^\pi) \subseteq (E^\alpha, H^\alpha) \) implies

\[
(U^\alpha)E^\alpha(U^\alpha)^\dagger = E^\pi_{|U^\alpha H^\alpha}
\]

and then,

\[
E^\alpha = (U^\alpha)^\dagger E^\pi_{|U^\alpha H^\alpha} U^\alpha = (U^\alpha)^\dagger E^\pi U^\alpha.
\]

Therefore,

\[
E^\pi_{|\overline{H}} = (U^\alpha)^\dagger E^\pi U^\alpha_{|\overline{H}} = (U^\alpha)^\dagger E^\pi U^\alpha_{|\overline{H}} = (U^\alpha)^\dagger E(U^\alpha)^\dagger = \overline{U}^\alpha E(U^\alpha)^\dagger
\]

so that \( (E^\alpha, H^\alpha) \subseteq (\overline{E}, \overline{H}) \), for each \( \alpha \in \Pi \). Thus \( (\overline{E}, \overline{H}) \) is an upper bound for the chain. By the Zorn lemma, there is a \( \subseteq \)-maximal element \( (E^+, H^+) \) in \( N_d(F)^\sim \).

\( \square \)

Now, we can introduce a partial order \( \subseteq^* \) in \( N_d(F)^\sim \) which is the dual of \( \subseteq \) and such that \( (E^+, H^+) \) is \( \subseteq^* \)-minimal.
Definition 2.18. We say that \((E^\sim, H^\sim) \sqsubseteq (E^+, H^+)\) if \((E^+, H^+) \sqsubseteq (E^\sim, H^\sim)\).

Corollary 2.19. There is a \(\sqsubseteq^*\)-minimal element in \(N_d(F)^{\sim}\).

Proof. Since \((E^+, H^+)\) is \(\sqsubseteq\)-maximal in \(N_d(F)^{\sim}\), it is \(\sqsubseteq^*\)-minimal in \(N_d(F)^{\sim}\). \(\Box\)

The minimal Naimark dilation with respect to the relation \(\sqsubseteq^*\) is exactly the minimal dilation that is usually introduced by definition.

Proposition 2.20. Let \((E^+, H^+)\) be a \(\sqsubseteq^*\)-minimal Naimark dilation. Then,

\[ H^+ = \hat{H} = \text{span}\{E^+(\Delta)\psi | \Delta \in B(X), \psi \in H\}. \]

Proof. First we prove that \((E^+_1, \hat{H}_1)\) is a Naimark dilation. Indeed, \(H \subset \hat{H}\) and for any

\[ \psi^+ \in \{E^+(\Delta)\psi | \Delta \in B(X), \psi \in H\}, \]

we have

\[ E^+_1(\Delta)\psi^+ = E^+(\Delta)[E^+(\Delta')\psi] \]

\[ = E^+(\Delta \cap \Delta')\psi \in \{E^+(\Delta)\psi | \Delta \in B(X), \psi \in H\}. \]

Then, by linearity and continuity \(E^+_1(\Delta)(\hat{H}) \subset \hat{H}\) and then

\[ F(\Delta)\psi = PE^+(\Delta)\psi = PE^+_1(\Delta)\psi. \]

Moreover, \((E^+_1, \hat{H}) \sqsubseteq^* (E^+, H^+)\) which, by the minimality of \((E^+, H^+)\), implies \((E^+_1, \hat{H}) = (E^+, H^+)\). \(\Box\)

Corollary 2.21. A \(\sqsubseteq^*\)-minimal Naimark dilation is minimal in the sense of Theorem 2.1.

Theorem 2.22. A \(\sqsubseteq^*\)-minimal Naimark dilation is unique.

Proof. Let \((E^+_1, H^+_1)\) and \((E^+_2, H^+_2)\) be two minimal Naimark dilations of \(F\). By proposition 2.20,

\[ H^+_1 = \text{span}\{E^+_1(\Delta)\psi | \Delta \in B(X), \psi \in H\}, \]

\[ H^+_2 = \text{span}\{E^+_2(\Delta)\psi | \Delta \in B(X), \psi \in H\}. \]

By proposition 2.16, it is sufficient to prove that there is a unitary map \(U : H^+_1 \to H^+_2\) such that \(UE^+_1U^\dagger = E^+_2\). Let us prove that the linear map

\[ U : E^+_1(\Delta)\psi \mapsto E^+_2(\Delta)\psi \]

is an isomorphism. It is sufficient to show that it is surjective and isometric. First we verify that \(U\) preserves the norm, for any choice of \(\Delta \in B(X)\) and \(\psi \in H\). We have

\[ \|E^+_1(\Delta)\psi\|^2 = \langle E^+_1(\Delta)\psi, E^+_1(\Delta)\psi \rangle = \langle E^+_1(\Delta)\psi, \psi \rangle \]

\[ = \langle F(\Delta)\psi, \psi \rangle \]

\[ = \langle E^+_2(\Delta)\psi, \psi \rangle = \langle E^+_2(\Delta)\psi, E^+_2(\Delta)\psi \rangle = \|E^+_2(\Delta)\psi\|^2. \]

By the linearity of the inner product and the continuity of the norm, \(U\) can be extended to a linear isometry from \(H^+_1\) onto \(H^+_2\). It is onto by definition.
It remains to be proven that $UE_1^+(\Delta')U^\dagger = E_2^+(\Delta')$ for every $\Delta' \in B(X)$. We have

$$[UE_1^+(\Delta')U^\dagger](E_2^+(\Delta)\psi) = UE_1^+(\Delta')E_2^+(\Delta)\psi$$

$$= UE_1^+(\Delta' \cap \Delta)\psi$$

$$= E_2^+(\Delta' \cap \Delta)\psi$$

$$= [E_2^+(\Delta')](E_2^+(\Delta)\psi),$$

which, by the linearity and the continuity of the inner product, can be extended to $\mathcal{H}_2^+$. □

As a consequence of proposition 2.20 the $\sqcap^*$-minimality coincides with the minimality in the sense of Naimark (see Theorem 2.1). That puts the concept of minimality of the Naimark dilation in its natural framework: the theory of posets.

3. Conclusions
In the present note we reviewed some results about Naimark’s dilation theorem. First we focused on its relationships with the integral characterization of a commutative POVM $F$. Here the sharp version $A$ of $F$ can be thought of as a kind of ‘mirror image’ of the Naimark operator $A^+$ in the extended Hilbert space $\mathcal{H}^+$. Next, we remarked on the role of Naimark’s theorem in characterizing the joint measurability of generalized observables and showed that when we move from the Hilbert space $\mathcal{H}$ to its extension $\mathcal{H}^+$, the compatibility of two POVMs (which is in general different from their commutativity) becomes the commutativity of their Naimark dilations. The Naimark dilation theorem establishes a connection between a given POVM and a PVM (its Naimark dilation). The connection can be used to prove some of the properties of POVMs (subsection 2.2). Finally we provided a characterization of minimality of the Naimark dilation in the framework of partially ordered sets. Here minimality is connected to the order structure and it can be shown that minimal Naimark dilations always exist and coincide with the ones defined by the Naimark theorem.

4. Acknowledgement
The present work has been realized in the framework of the activities of the INDAM (Istituto Nazionale di Alta Matematica).

References
[1] Ludwig G 1983 Foundations of Quantum Mechanics I (New York: Springer)
[2] Davies E B 1986 Quantum Mechanics of Open Systems (London: Academic Press)
[3] Holevo A S 1982 Probabilistic and Statistical Aspects of Quantum Theory (Amsterdam: North Holland)
[4] Prugovečki E 1984 Stochastic Quantum Mechanics and Quantum Spacetime (Dordrecht: Reidel)
[5] Ali S T and Emch G G 1974 J. Math. Phys. 15 176–82
[6] Ali S T 1985 Riv. del Nuovo Cim. 8 1–127
[7] Busch P, Grabowski M and Lahti P 1995 Operational Quantum Physics (Berlin: Springer)
[8] Schroeck Jr F E 1996 Quantum Mechanics in Phase Space (Dordrecht: Kluwer)
[9] Busch P, Lahti P, Pellonpää J-P and Ylinen K 2016 Quantum Measurement (Berlin: Springer)
[10] Guz W 1984 23 157–84
[11] Stulpe W 1997 Classical Representations of Quantum Mechanics Related to Statistically Complete Observables (Berlin: Wissenschaft und Technik) Preprint quant-ph/0610122
[12] Busch P, Lahti P, and Werner R F 2013 Phys. Rev. Lett. 111 160405
[13] Busch P, Lahti P and Werner R F 2014 Rev. Mod. Phys. 86 1261–81
[14] Kraus K 1977 The Uncertainty Principle and Foundations of Quantum Mechanics ed W C Price and S S Chissick (London: Wiley) pp 293–320
[15] Hegerfeldt G C 1974 Phys. Rev. D 10 3320–1
[16] Jauch J M and Piron C 1967 Helv. Phys. Acta 40 559–70
[17] Castrigiano D P I 1981 *Lett. Math. Phys.* 5 303–9
[18] Brooke J A and Schroock Jr F E 1996 *J. Math. Phys.* 37 5958–86
[19] Beneduci R and Schroock F 2019 *Found. Phys.* 49 561–76
[20] Beneduci R and Nisticó G 2003 *J. Math. Phys.* 44 5461–73
[21] Cattaneo G and Nisticó G 2000 *J. Math. Phys.* 41 4365–78
[22] Beneduci R 2006 *J. Math. Phys.* 47 062104
[23] Jenčová A and Pulmannová S 2009 *Found. Phys.* 39 613–24
[24] Beneduci R 2016 *J. Math. Anal. and App.* 442 50–71
[25] Holevo A S 1972 *Trans. Mosc. Math. Soc.* 26 133–47
[26] Ali S T 1982 *Differential Geometric Methods in Mathematical Physics* (Lecture Notes in Mathematics vol 905) ed H D Doebner et al (Berlin: Springer) pp 207–28
[27] Beneduci R 2018 *Russ. J. Math. Phys.* 25 158–82
[28] Naimark M A 1940 *Izv. Akad. Nauk SSSR Ser. Mat.* 4 277–318
[29] Riesz F and Nagy B S 1990 *Functional Analysis* (New York: Dover)
[30] Akhiezer N I and Glazman I M 1963 *Theory of Linear Operators in Hilbert Space* (New York: Ungar)
[31] Paulsen V 2020 *Completely Bounded Maps and Operator Algebras* (Cambridge: Cambridge University Press)
[32] Ozawa M 2014 *J. Phys.: Conf. Series* 504 1–13
[33] Berberian S K 1966 *Notes on Spectral Theory* (Princeton: Van Nostrand)
[34] Reed M and Simon B 1980 *Methods of Modern Mathematical Physics* (New York: Academic Press)
[35] Lahti P, Pelloppää J-P and Ylinen K 1999 *J. Math. Phys.* 40 2181–9
[36] Beneduci R 2010 *Bull. Lond. Math. Soc.* 42 441–51
[37] Beneduci R 2010 *Lin. Alg. App.* 43 1224–39
[38] Beneduci R 2006 *Int. J. Geom. Meth. Mod. Phys.* 3 1559–71
[39] Beneduci R 2007 *J. Math. Phys.* 48 022102
[40] Beneduci R 2010 *Int. J. Theor. Phys.* 49 3030–8
[41] Beneduci R 2011 *Int. J. Theor. Phys.* 50 3724–36
[42] Lahti P 2003 *Int. J. Theor. Phys.* 42 893–906
[43] Beneduci R 2017 *Rep. Math. Phys.* 79 197–214