**Strong $Z_c^+(3900) \rightarrow J/\psi \pi^+; \eta_c \rho^+$ decays in QCD**

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The widths of the strong decays $Z_c^+(3900) \rightarrow J/\psi \pi^+$ and $Z_c^+(3900) \rightarrow \eta_c \rho^+$ are calculated. To this end, the mass and decay constant of the exotic $Z_c^+(3900)$ state are computed by means of a two-point sum rule. The obtained results are then used to calculate the strong couplings $g_{Z_c^+,J/\psi}$ and $g_{Z_c^+,\eta_c \rho}$ employing QCD sum rules on the light-cone supplied by a technique of the soft-meson approximation. We compare our predictions on the mass and decay widths with available experimental data and other theoretical results.

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## I. INTRODUCTION

The exotic hadron states, i.e. ones that can not be included into the quark-antiquark and three-quark bound schemes of the standard spectroscopy already attracted interests of physicists [1, 2]. The quantitative investigations of such states are connected with the invention of the QCD sum rule method [3], which was employed for analysis of glueballs, hybrid [4], [5], [6], and exotic four-quark $u\bar{u}u\bar{s}d$ mesons [7], [8] and six-quark systems [8]. But because of problems of old experiments, stemmed mainly from difficulties in detecting heavy resonances, the existence of the exotic states was not then certainly established.

The situation changed dramatically during the last decade when the Belle, BaBar, LHCb and BES collaborations began copiously to yield experimental data providing an information on the masses, decay width and quantum numbers $J^{PC}$ of new exotic states. Starting from the discovery of the charmonium-like resonance $X(3872)$ by Belle [9], confirmed later by other experiments [10], studying of new $XYZ$ family of mesons became one of the interesting and rapidly growing branches of the high energy physics (see, the reviews [11–17] and references therein).

There were attempts to describe the new charmonium-like resonances as excitations of the ordinary $c\bar{c}$ charmonium: In order to compute the charmonium spectrum, various quark-antiquark potentials were used and their mass and radiative transitions to other charmonium states were studied [18]. It should be noted that some of new resonances allow interpretation as the excited $c\bar{c}$ states. But the bulk of the collected experimental data can not be included into this scheme, and hence for their explanation new-unconventional quark configurations are required. To this end various quark-gluon models were suggested. They differ from each other in elements of the substructure, and in mechanisms of the strong interactions between these elements that form bound states.

One of the most employed in this context models is the four-quark or the tetraquark model of the new resonances, already used to analyze the light-quark exotic mesons [2, 7]. In the renewed tetraquark model, the hadronic bound state is formed by two heavy and two light quarks $Q\bar{Q}q\bar{q}$. These quarks may group into compact tetraquark state, where all quarks have overlapping wave functions [19]. Their strong interaction can be studied by the quark potential models that include not only the 2-body potentials of pairwise quark interactions, but also the 3-body and 4-body potentials. In other model four quarks cluster into the colored diquark $Qq$ and antidiquark $\bar{Q}\bar{q}$, which emerge as the elements of the substructure. Diquark-antidiquarks are organized in such a way that reproduce quantum numbers of the corresponding exotic states [20]. In this picture the bound state forms due to not only the quark-antiquark potentials, but also owing to the diquark-antidiquark interactions. Alternatively, in the meson molecule picture, the quarks appear as color-singlet $Qq$ and $\bar{Q}\bar{q}$ mesons. Finally, in the hadro-quarkonium model suggested in Ref. [21], the four-quarks create the bound system consisting of colorless $Q\bar{Q}$ and $q\bar{q}$ pairs of the heavy and light quarks, respectively. In the molecule and hadro-quarkonium models the strong interactions are mediated by the meson exchange.

Another possibility to describe the four-quark state is the Born-Oppenheimer tetraquark structure proposed recently in Ref. [22]. In this approach the heavy quarks $Q$ and $\bar{Q}$ are considered as being embedded in the configuration of gluon and light-quark fields, which are not a flavor singlet, but have isospin 1. The exotic mesons can also be considered in the framework of the traditional hybrid models, as particles consisting of the heavy quarks and a gluon $Q\bar{Q}q$. The gluonic excitation in this approach is treated as a constituent-particle with definite quantum numbers. It should be noted, however that none of these models firmly succeeded in analysis of the variety of the available experimental data: in order to describe features of the observed exotic states one should involve different models.

The $XYZ$ meson masses and decay widths were calculated, and their quantum numbers $J^{PC}$ analyzed using all of the aforementioned models and various theoretical methods. Theoretical approaches within QCD include
the lattice simulations to explore the exotic and excited charmonium spectroscopy [23, 25], the calculations based on the different quark potential models [11, 14], and QCD sum rule method (see, for review Ref. [14, 15]). The evaluation of the masses, decay constants and widths of numerous exotic states and comparison of the obtained results with the accumulated experimental data yield valuable information on the quark-gluon structure of new states and mechanisms of the strong interactions between their building elements. Despite remaining problems, one can state that now important parts of the whole picture of exotic multi-quark states are clearer than in the beginning of the decade.

The $Z_c^\pm$ states discovered by BESIII in the process $e^+e^- \rightarrow \pi^+\pi^- J/\psi$ [26], were observed by the Belle collaboration [27] as well. Their existence were also confirmed in Ref. [28] on the basis of the CLEO-c data analysis. The $Z_c^\pm \rightarrow J/\psi\pi^\pm$ decays demonstrate that $Z_c^\pm$ are tetraquark states with constituents $ccud$ and $cd\bar{u}$. Observation of the neutral partner of $Z_c^-$ in the process $e^+e^- \rightarrow \pi^0 Z_c^-$ was reported in Ref. [29]. Theoretical investigations of the $Z_c$ states encompass different models and approaches (see, Refs. [30–34] and references therein).

In this work we evaluate the widths of the strong decays $Z_c^+ (3900) \rightarrow J/\psi\pi^+$ and $Z_c^+ (3900) \rightarrow \eta\rho^+$ of $Z_c^+ (3900)$ (in what follows denoted as $Z_c$) as well as the couplings of the strong $Z_c J/\psi\pi$ and $Z_c \eta\rho$ vertices allowing us to find the required decay widths. For calculation of the mass and decay constant we employ QCD two-point sum rule, whereas in the case of the strong couplings apply methods of QCD light-cone sum rule (LCSR) supplemented by the soft-meson approximation [35, 37]. The latter is necessary because $Z_c$ state contains the four valence quarks, as a result, the light-cone expansion of the correlation functions inevitably reduces to the short-distance expansion in terms of local matrix elements. In the context of LCSR approach this corresponds to the vanishing meson momentum. In the present work we adopt the zero-momentum limit for the mesons referring to the approach itself as the soft-meson approximation. This approximation is rather simple and, as we shall see, leads to nice agreement with the experimental data [24, 27]. Within the sum rule method $Z_c$ state was studied previously in Refs. [30–32]. Thus, in order to calculate strong couplings and decay widths of $Z_c$ state in Ref. [31] QCD three-point sum rule method was employed.

This article is organized in the following way. In section IV we calculate the mass and decay constant of the $Z_c$ state within two-point QCD sum rule approach. Section V is devoted to calculation of the strong $Z_c J/\psi\pi$ and $Z_c \eta\rho$ vertices, where the sum rules for the couplings $g_{Z_c J/\psi\pi}$ and $g_{Z_c \eta\rho}$ are derived. Here we also calculate the widths of the decay channels under consideration. Numerical computations of the mass, decay constant, strong couplings, and decay widths are performed in Section VI. The obtained results are compared with the available experimental data, as well as with existing theoretical calculations. This Section contains also our conclusions. The explicit expression of the spectral density $\rho^{QCD}(s)$ necessary for computation of the mass and decay constant of $Z_c$ state is moved to Appendix A.

II. THE MASS AND DECAY CONSTANT OF THE $Z_c$ STATE

In order to calculate the mass and decay constant of the $Z_c^+$ state in the framework of QCD sum rules, we start from the two-point correlation function

$$\Pi_{\mu\nu}(q) = i \int d^4 x e^{i q \cdot x} \langle 0 | T \{ J_{\mu}^{Z_c^+}(x) J_{\nu}^{Z_c^+ \dagger}(0) \} | 0 \rangle, \quad (1)$$

where the interpolating current with required quantum numbers $J^{PC} = 1^{--}$ is given by the following expression

$$J_{\mu}^{Z_c^+}(x) = \frac{i \bar{c} \gamma_{\mu} c}{\sqrt{2}} \{ [u_a^T(x) C \gamma_5 c_b(x)] [\bar{d}_d(x) \gamma_\nu C \tau_5^T(x)] - [u_a^T(x) C \gamma_\nu c_b(x)] [\bar{d}_d(x) \gamma_5 C \tau_5^T(x)] \}. \quad (2)$$

Here we have introduced the short-hand notations $\epsilon = \epsilon_{abc}$ and $\bar{\epsilon} = \epsilon_{dec}$. In Eq. (2) $a, b, c, d, e$ are color indexes and $C$ is the charge conjugation matrix.

In order to derive QCD sum rule expression we first calculate the correlation function in terms of the physical degrees of freedom. Performing integral over $x$ in Eq. (1), we get

$$\Pi^{\text{Phys}}_{\mu\nu}(q) = \frac{\langle 0 | J_{\mu}^{Z_c^+}(x) | Z_c(q) \rangle \langle Z_c(q) | J_{\nu}^{Z_c^+ \dagger}(0) \rangle}{m_{Z_c}^2 - q^2} + \ldots$$

where $m_{Z_c}$ is the mass of the $Z_c$ state, and dots stand for contributions of the higher resonances and continuum states. We define the decay constant $f_{Z_c}$ through the matrix element

$$\langle 0 | J_{\mu}^{Z_c^+}(x) | Z_c(q) \rangle = f_{Z_c} \epsilon_{\mu\nu} \epsilon, \quad (3)$$

with $\epsilon_{\mu\nu}$ being the polarization vector of $Z_c$ state. Then in terms of $m_{Z_c}$ and $f_{Z_c}$, the correlation function can be written in the following form

$$\Pi^{\text{Phys}}_{\mu\nu}(q) = \frac{m_{Z_c}^2 f_{Z_c}^2}{m_{Z_c}^2 - q^2} \left( -g_{\mu\nu} + \frac{q_{\mu} q_{\nu}}{m_{Z_c}^2} \right) + \ldots \quad (4)$$

The Borel transformation applied to Eq. (4) yields

$$\mathcal{B}_q \Pi^{\text{Phys}}_{\mu\nu}(q) = m_{Z_c}^2 f_{Z_c}^2 e^{-m_{Z_c}^2/M^2} \left( -g_{\mu\nu} + \frac{q_{\mu} q_{\nu}}{m_{Z_c}^2} \right) + \ldots \quad (5)$$

The same function in QCD side, $\Pi^{QCD}_{\mu\nu}(q)$, has to be determined employing of the quark-gluon degrees of freedom. To this end, we contract the heavy and light quark
The correlation function $\Pi$ decomposition over the Lorentz structures in both $\Pi$ constant can be derived after choosing the same structure with

$$\Pi^{QCD}(q) = \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \left\{ \begin{array}{l} \text{Tr} \left[ \gamma_\lambda S_{\alpha'}(x) \right] \\ \times \gamma_\mu S_{\beta'}(x) \left[ \text{Tr} \left[ \gamma_\mu \gamma_5 S_{\alpha'}(x) \right] - \text{Tr} \left[ \gamma_\mu S_{\alpha'}(x) \gamma_\lambda S_{\beta'}(x) \right] \right] \\ - \text{Tr} \left[ \gamma_\mu \gamma_5 S_{\alpha'}(x) \right] \text{Tr} \left[ \gamma_\mu S_{\alpha'}(x) \right] \right\} \right\} \), (6)

where

$$S_{\alpha}(q) = CS_{\alpha}(q)C.$$ (7)

Here the heavy quark propagator $S_{\alpha}(q)$ is given by the expression (8)

$$S_{\alpha}(q) = \frac{1}{2m_{\alpha}} = e^{-kx} \left[ \begin{array}{l} \delta_{ij} (k + m_{\alpha}) \\ - \frac{g G_{ij}}{4} \sigma_{\alpha\beta} (k + m_{\alpha}) + (k + m_{\alpha}) \sigma_{\alpha\beta} \\ + \frac{g^2}{12} C_{\alpha\beta} G_{ij} \delta_{ij} k^2 + m_{\alpha} k \\ \frac{1}{(k^2 - m_{\alpha}^2)^2} + \ldots \right].$$

In Eq. (7) the short-hand notation

$$G_{ij}^{\alpha\beta} \equiv G_{ij}^A \delta_{ij} \alpha = 1, 2 \ldots 8,$$

is used, where $i, j$ are color indexes, and $t^A = \lambda^A/2$ with $\lambda^A$ being the standard Gell-Mann matrices. The first term in Eq. (7) is the free (perturbative) massive quark propagator, next ones are nonperturbative gluon corrections. In the nonperturbative terms the gluon field strength tensor $G_{ij}^{\alpha\beta} \equiv G_{ij}^{\alpha\beta}(0)$ is fixed at $x = 0$.

The light-quark propagator employed in our work reads

$$S_{q}(x) = e^{-kx} \left[ \begin{array}{l} \delta_{ij} (k + m_{\alpha}) \\ - \frac{g G_{ij}}{4} \sigma_{\alpha\beta} (k + m_{\alpha}) + (k + m_{\alpha}) \sigma_{\alpha\beta} \\ + \frac{g^2}{12} C_{\alpha\beta} G_{ij} \delta_{ij} k^2 + m_{\alpha} k \\ \frac{1}{(k^2 - m_{\alpha}^2)^2} + \ldots \right].$$ (8)

The correlation function $\Pi^{QCD}(q)$ has also the following decomposition over the Lorentz structures

$$\Pi^{QCD}(q) = \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \left\{ \begin{array}{l} \text{Tr} \left[ \gamma_\lambda S_{\alpha'}(x) \right] \\ \times \gamma_\mu S_{\beta'}(x) \left[ \text{Tr} \left[ \gamma_\mu \gamma_5 S_{\alpha'}(x) \right] - \text{Tr} \left[ \gamma_\mu S_{\alpha'}(x) \gamma_\lambda S_{\beta'}(x) \right] \right] \\ - \text{Tr} \left[ \gamma_\mu \gamma_5 S_{\alpha'}(x) \right] \text{Tr} \left[ \gamma_\mu S_{\alpha'}(x) \right] \right\} \right\} \), (6)

with the term $\sim q_\mu q_\nu$ and invariant function $\Pi^{QCD}(q^2)$, which can be represented as the dispersion integral

$$\Pi^{QCD}(q^2) = \int_{4m_c^2}^\infty \frac{\rho^{QCD}(s)}{s - q^2 + \ldots},$$ (10)

where $\rho^{QCD}(s)$ is the corresponding spectral density.

The QCD sum rule calculations requires utilization of some consecutive operations: we recall only the main steps in the computational scheme used in the present work to derive the spectral density $\rho^{QCD}(s)$. Thus, having employed the transformation

$$\frac{1}{(x^2)^n} = \int \frac{d^D t}{(2\pi)^D} e^{-i^t x (t - 1)^n + 2 D - 2 n D^2 / \pi} t^{2-n} \frac{\Gamma(2 - n)}{\Gamma(n)},$$ (11)

we first replace, where it is necessary, $x_\mu$ by $-i \partial / \partial q_\mu$, and calculate the $x$ integral. As a result, we get the delta function with a combination of the momenta in its argument. This Dirac delta is used to remove one of the momentum integrals. The remaining integrations over $t$ and over the momentum require invoking the Feynman parametrization and performing rearrangements of denominators in obtained expressions. Then we carry out integration over $t$ and perform the last integral over $k$ by means of the formulas

$$\int d^4 k \frac{1}{(k^2 + L)^n} = \frac{i n \pi^2 (-1)^n \Gamma(n - 2)}{\Gamma(n)} (-L)^n - \frac{m_{\alpha}^2}{2}$$ (12)

and

$$\int d^4 k \frac{k_{\mu} k_{\nu}}{(k^2 - 2 A k q + A^2 - B m_{\alpha}^2)^n} = \frac{i n \pi^2 (-1)^n + \Gamma(n - 2)}{\Gamma(n)} \left[ \frac{q_{\mu} + A^2 (\alpha - 3)}{2} - \frac{q_{\nu} + A^2 (\alpha - 3)}{2} q_{\mu} q_{\nu} \right].$$ (13)

In Eqs. (12) and (13) we use the notations

$$A = 2r(w + z - 1), \ B = r(w + z)(w - 1),$$

and

$$L = \frac{1}{w^2 + (w + z)(z - 1)}.$$ (14)

By applying the replacement

$$\Gamma \left( \frac{D}{2} - n \right) \left( -1 \right)^n \frac{1}{(n - 2)!} (-L)^{n-2} \ln(-L),$$ (15)

in the obtained expression, we get the imaginary part of the correlation function. The remaining integrals over
the Feynman parameters $w$ and $z$ in some simple cases can be carried out explicitly, or kept in their original form supplemented as a factor by the Heaviside function $\theta (L)$.

The results of our calculations of the spectral density $\rho_{QCD} (s)$ performed within this scheme are collected in Appendix A.

Applying the Borel transformation on the variable $q^2$ to the invariant amplitude $\Pi_{QCD} (q^2)$, equating the obtained expression with the relevant part of $B_q \Pi_{\mu \nu}^{\text{phys}} (q)$, and subtracting the continuum contribution, we finally obtain the required sum rule. Thus, the mass of the $Z_c$ state can be evaluated from the sum rule
\begin{equation}
 m_{Z_c}^2 = \frac{\int_{4m_c^2}^{s_0} ds \rho_{QCD} (s) e^{-s/M^2}}{\int_{4m_c^2}^{s_0} ds \rho (s) e^{-s/M^2}},
\end{equation}
whereas to extract the numerical value of the decay constant $f_{Z_c}$ we employ the formula
\begin{equation}
 f_{Z_c}^2 e^{-m_{Z_c}^2/M^2} = \int_{4m_c^2}^{s_0} ds \rho_{QCD} (s) e^{-s/M^2}.
\end{equation}

The last two expressions are required sum rules to evaluate the $Z_c$ state’s mass and decay constant, respectively.

## III. THE STRONG VERTICES $Z_c J/\psi \pi$ AND $Z_c \eta \rho$

This section is devoted to the calculation of the widths of the $Z_c \to J/\psi \pi$ and $Z_c \to \eta \rho$ decays. To this end we calculate the strong couplings $g_{Z_c J/\psi \pi}$ and $g_{Z_c \eta \rho}$ using methods of the QCD sum rules on the light-cone in conjunction with the soft-meson approximation.

### A. THE $Z_c J/\psi$ VERTEX

We start our analysis from the vertex $Z_c J/\psi$ aiming to calculate $g_{Z_c J/\psi \pi}$: we consider the correlation function
\begin{equation}
 \Pi_{\mu \nu} (p, q) = i \int d^4x e^{i p x} \langle \pi (q) | T \{ J_{\mu}^{J/\psi} (x) J_{\nu}^{J/\psi \dagger} (0) \} \rangle | 0 \rangle,
\end{equation}
where
\begin{equation}
 J_{\mu}^{J/\psi} (x) = \tau_{\mu} (x) \gamma_\mu c (x),
\end{equation}
and $J_{Z_c}^{J/\psi} (x)$ is defined by Eq. (4). Here $p$, $q$ and $p' = p + q$ are the momenta of $J/\psi$, $\pi$ and $Z_c$, respectively. A sample diagram describing the process $Z_c \to J/\psi \pi$ is depicted in Fig. 1.

To derive the sum rules for the coupling, we calculate $\Pi_{\mu \nu} (p, q)$ in terms of the physical degrees of freedom. Then it is not difficult to obtain
\begin{equation}
 \Pi_{\mu \nu}^{\text{phys}} (p, q) = \frac{\langle 0 | J_{\mu}^{J/\psi} | J/\psi (p) \rangle \langle J/\psi (p) \pi (q) | Z_c (p') \rangle}{p'^2 - m_{Z_c}^2} \times \frac{Z_c (p') | Z_c^{J/\psi} \dagger | 0 \rangle}{p'^2 - m_{Z_c}^2} + \ldots
\end{equation}

where the dots denote contribution of the higher resonances and continuum states.

We introduce the matrix elements
\begin{equation*}
 \langle 0 | J_{\mu}^{J/\psi} | J/\psi (p) \rangle = f_{J/\psi} m_{J/\psi} \epsilon_{\mu},
\end{equation*}
\begin{equation*}
 \langle Z_c (p') | J_{\pi}^{J/\psi \dagger} | 0 \rangle = f_{Z_c} m_{Z_c} \epsilon_{\pi}^\dagger,
\end{equation*}
\begin{equation*}
 \langle J/\psi (p) \pi (q) | Z_c (p') \rangle = \langle (p \cdot p') (\epsilon^* \cdot \epsilon') - (p \cdot \epsilon') (p' \cdot \epsilon^*) \rangle g_{Z_c J/\psi \pi},
\end{equation*}
\begin{equation}
 = \Pi_{\pi}^{\text{phys}} (p^2, (p + q)^2) g_{\mu \nu} + \Pi_{\mu \nu} (p^2, (p + q)^2) p'_\mu p'_\nu.
\end{equation}

For calculation of the strong coupling under consideration we choose to work with the structure $\sim g_{\mu \nu}$. Then, for the corresponding invariant function, we get
\begin{equation}
 \Pi_{\pi}^{\text{phys}} (p^2, (p + q)^2) = f_{J/\psi} f_{Z_c} m_m z m_{J/\psi} g_{Z_c J/\psi \pi}
 \times \frac{m_{Z_c}^2 + m_{J/\psi}^2}{2} + \Pi_{(RS:C)} (p^2, (p + q)^2),
\end{equation}
where $\Pi_{(RS:C)} (p^2, (p + q)^2)$ is the contribution arising from the higher resonances and continuum states, that
can be written down as the double dispersion integral:

\[
\Pi^{(RS:C)}(p^2, (p + q)^2) = \int \int \frac{\rho^{(s_1, s_2)}(s_1, s_2) ds_1 ds_2}{(s_1 - p^2)(s_2 - p'^2)} + \int \frac{\rho^{(s_1)}(s_1)}{(s_1 - p^2)} + \int \frac{\rho^{(s_2)}(s_2)}{(s_2 - p'^2)}. \tag{24}
\]

This formula contains also single dispersion integrals that are necessary to make the whole expression finite: As we shall see below, they play an important role in the soft-meson approximation adopted in the present work.

In the standard LCSR approach in order to get sum rules for the strong couplings \[37, 39, 40\] one applies to Eq. (23) double Borel transformation in variables \(p^2\) and \(p'^2\) that vanishes the single dispersion integrals leaving in the physical side of the LCSR only contributions of the ground state and the double spectral integral. In other words, effects of higher resonances and continuum states from the theoretical side of QCD reveals its interesting features. As is seen from Eqs. (22) and (18), the tetraquark state contains four quarks at the same space-time location, therefore contractions of the and quark fields given at \(x = 0\) with the relevant fields at \(x\) from the \(J/\psi\) meson yield expressions where the remaining light quarks are sandwiched between the pion and vacuum states forming local matrix elements. In other words, we encounter with the situation when dependence of the correlation function on the meson distribution amplitudes disappears and integrals over the meson DAs reduce to overall normalization factors. Within framework of LCSR method such situation is possible in the kinematical limit \(q \to 0\), when the light-cone expansion reduces to the short-distant one. As a result, instead of the expansion in terms of DAs one gets expansion over the local matrix elements. \[37\]. In this limit \(p' = p\) and relevant invariant amplitudes in the correlation function depend only on one variable \(p^2\). Here we adopt this approach, and following Ref. \[37\] refer to the limit \(q \to 0\) as the soft-meson approximation bearing in mind that it actually implies calculation of the correlation function with the equal initial and final momenta \(p' = p\), and dealing with the obtained double pole terms.

The soft-meson approximation considerably simplifies the QCD side of the sum rules, but leads to more complicated expression for its hadronic representation. In the soft \(p' \to p\) limit, as it has been just emphasized above, the ground state contribution depends only on the variable \(p^2\). With some accuracy, it can be written in the form

\[
\Pi^{\text{phys}}(p^2) = \frac{f_{J/\psi} f_{Z_c} m_{Z_c} m_{J/\psi}}{(p^2 - m^2)^2} m^2, \tag{25}
\]

where \(m^2 = (m_{Z_c}^2 + m_{J/\psi}^2)/2\). The Borel transformation in the variable \(p^2\) applied to this correlation function yields

\[
\Pi^{\text{phys}}(M^2) = \frac{1}{M^2} \left\{ \frac{f_{J/\psi} f_{Z_c} m_{Z_c} m_{J/\psi}}{M^2} g_{Z_c, J/\psi} \right\} e^{-m^2/M^2}. \tag{26}
\]

But because within the soft-meson approximation we employ the one-variable Borel transformation, now the single dispersion integrals also contribute to the hadronic part of the sum rules. Non-vanishing contributions correspond to transitions from the exited states in the \(J/\psi\) channel with \(m^* > m_{J/\psi}\) to the ground state \(J/\psi\) (the similar arguments are valid for the \(Z_c\) channel, as well). They are not suppressed relative to the ground state contribution even after the Borel transformation \[36, 37\]. Hence, taking into account all unsuppressed contributions to \(\Pi^{\text{phys}}(M^2)\), denoted below as \(A\), the hadronic part of the sum rules can be schematically written in the form \[37\]

\[
\Pi^{\text{phys}}(M^2) \simeq \frac{1}{M^2} \left\{ \frac{f_{J/\psi} f_{Z_c} m_{Z_c} m_{J/\psi}}{M^2} g_{Z_c, J/\psi} + A M^2 \right\} e^{-m^2/M^2}. \tag{27}
\]

It is evident, that the terms \(\sim A\) emerge here as an undesired contamination and make extraction of the strong coupling problematic. In order to remove them from the hadronic part it is convenient to follow a prescription suggested in Ref. \[38\] and act by the operator

\[
\left( 1 - M^2 \frac{d}{dM^2} \right) M^2 e^{-m^2/M^2}, \tag{28}
\]

to both sides of the sum rules.

Now we need to calculate the correlation function in terms of the quark-gluon degrees of freedom and find the QCD side of the sum rules. Having contracted heavy quarks fields we get

\[
\Pi^{\text{QCD}}(p, q) = \int d^4 x e^{i p x} \frac{\bar{\gamma}_5 S_c(x) \gamma_\mu}{\sqrt{2}} \left[ \bar{S}_c(-x) \gamma_\nu + \bar{\nu}_5 \bar{S}_c(x) \gamma_\mu \bar{\nu}_5 S_c(-x) \gamma_5 \right]_{\alpha \beta} \times \langle \pi(q) | \bar{\nu}_5(0) \gamma_\nu \gamma_\mu \nu_5(0) | 0 \rangle, \tag{29}
\]

where \(\alpha\) and \(\beta\) are the spinor indices.

To proceed we use the expansion

\[
\bar{\nu}_5 d_\beta^\mu \rightarrow \frac{1}{4} \Gamma^j \alpha \beta \left( \pi^i \Gamma^j d^i \right), \tag{30}
\]

where \(\Gamma^j\) is the full set of Dirac matrices

\[
\Gamma^j = 1, \gamma_5, \gamma_\lambda, i \gamma_5 \gamma_\lambda, \sigma_{\lambda \rho} / \sqrt{2},
\]
Within the LCSR method we have also to use the light-cone expansion for the $c$-quark propagator

\begin{equation}
\langle 0| T\{c(x)\bar{c}(0)\}|0\rangle = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - m_c^2} \left[ \frac{k + m_c}{2(k^2 - m_c^2)} G^{\mu\nu}(vx) \gamma_\mu \gamma_\nu \right] + \ldots
\end{equation}

But because the light quark fields in the local matrix elements are fixed at the point $x = 0$, in the second piece of the propagator [Eq. (31)] we expand the gluon field strength tensor at $x = 0$ keeping only the leading order term that is equivalent to usage the first two terms from $S_\pi^G(x)$ (see, Eq. (7)) in the calculations.

In order to determine the required local matrix elements we consider first the perturbative components of the heavy-quark propagators. To this end, it is convenient to take sums over the color indexes. Using the overall color factor $\epsilon_abc\epsilon_{dec}G^{\mu\nu}$, color factors of the propagators, as well as the projector onto a color-singlet state $\delta^{ad}/3$, it is easy to demonstrate that for the perturbative contribution, the replacement

\begin{equation}
\frac{1}{4} \Gamma^{j}_{\beta\alpha} (\pi^\alpha \Gamma^j d^d) \to \frac{1}{4} \Gamma^{j}_{\beta\alpha} (\pi^\alpha \Gamma^j G^{abd} d).
\end{equation}

is legitimate. For nonperturbative contributions, forming as a product of the perturbative part of one of the propagators with the term $\sim G$ from the other one, we find, for example,

$$
\epsilon_abc\epsilon_{dec}G^{\mu\nu}\frac{1}{4} \Gamma^{j}_{\beta\alpha} (\pi^\alpha \Gamma^j d^d) \to -\frac{1}{4} \Gamma^{j}_{\beta\alpha} (\pi^\alpha G^{abd} d).$$

This rule allows us to insert quark matrix elements the gluon field strength tensor $G$ that effectively leads to three-particle components and corresponding matrix elements of the pion. We neglect the terms $\sim GG$ appearing from the product of one-quark components of the heavy propagators. Having finished a color summation one can calculate the traces over spiner indexes.

The spectral density $\rho^{QCD}_c(s)$ has been found employing the approach outlined in the Section[11]. Calculations demonstrate that the pion local matrix element that, in the soft-meson limit, contributes to the both structures of $\text{Im} \Pi^{QCD}_{\mu\nu}(p, q = 0)$ is

\begin{equation}
\langle 0| \bar{d}(0)i\gamma_\mu u(0)|\pi(q)\rangle = f_\pi \mu_\pi,
\end{equation}

where

\begin{equation}
\mu_\pi = \frac{m_\pi^2}{m_u + m_d} = -\frac{2(\gamma_0)}{f_\pi^2}.
\end{equation}

The second equality in Eq. (34) is the relation between $m_\pi$, $f_\pi$, the quark masses and the quark condensate $\langle \bar{q}q \rangle$ arising from the partial conservation of axial vector current (PCAC).

Choosing the structure $\sim g_{\mu\nu}$ for our analysis it is straightforward to derive the corresponding spectral density

\begin{equation}
\rho^{QCD}_c(s) = \frac{f_\pi \mu_\pi(s + 2m_c^2)\sqrt{s(s - 4m_c^2)}}{12\sqrt{2}\pi^2 s}.
\end{equation}

The continuum contribution can be subtracted in a standard manner after $\rho^b_c(s) \to \rho^{QCD}_c(s)$ replacement. The final sum rule to evaluate the strong coupling reads

\begin{equation}
g_{Z_c,J/\psi} = \frac{2}{f_{J}/\psi} \int_{m_c^2}^{\infty} dM^2 \left[ 1 - M^2 \frac{d}{dM^2} \right] M^2 \int_{4m_c^2}^{\infty} ds e^{s(2m_c^2 + m_{J/\psi}^2 - 2s)/2M^2} \rho^{QCD}_c(s).
\end{equation}

The width of the decay $Z_c \to J/\psi\pi$ can be found applying the standard methods and definitions for the strong coupling alongside with other matrix elements [Eq. (21)] and parameters of the $Z_c$ state. Our calculations give

\begin{equation}
\Gamma(Z_c \to J/\psi\pi) = g_{Z_c,J/\psi}^2 \frac{m_{J/\psi}}{24\pi} \lambda(m_{Z_c}, m_{J/\psi}, m_\pi) \times \left[ 3 + \frac{2\lambda^2(m_{Z_c}, m_{J/\psi}, m_\pi)}{m_{J/\psi}^2} \right].
\end{equation}

where

\begin{equation}
\lambda(a, b, c) = \sqrt{a^2 + b^2 + c^2 - 2(a^2b^2 + a^2c^2 + b^2c^2)}.
\end{equation}

The final expressions (36) and (37) will be used for numerical analysis of the decay channel $Z_c \to J/\psi\pi$.

### B. THE $Z_c\eta_c\rho$ VERTEX

The coupling $g_{Z_c,\eta_c\rho}$ can be calculated utilizing the correlation function

\begin{equation}
\Pi_\nu(p, q) = i \int d^4xe^{ipx}\langle p(q)|T\{J_\nu^c(x)J_{\nu}^{c\dagger}(0)\}|0\rangle,
\end{equation}

where the current $J_\nu^c(x)$ is defined as

$$
J_\nu^c(x) = \bar{c}_i(x)i\gamma_\nu c_i(x).
$$

In order to find the hadronic representation of the correlation function we define the matrix element:

\begin{equation}
\langle 0| J^{c\nu}|\eta_c(p)\rangle = \frac{f_{\eta_c} m_{\eta_c}^2}{2m_c},
\end{equation}

with $m_{\eta_c}$ and $f_{\eta_c}$ being the $\eta_c$ meson’s mass and decay constant. The vertex $Z_c\eta_c\rho$ is defined as in Eq. (21)

\begin{equation}
\langle \eta_c(p)\rho(p')|Z_c(p')\rangle = [(q \cdot p')(\epsilon^* \cdot \epsilon')
\end{equation}

\begin{equation}
- (q \cdot \epsilon')(p' \cdot \epsilon)] g_{Z_c,\eta_c\rho}.
\end{equation}
but with \( q \) and \( \varepsilon \) being now the momentum and polarization vector of the \( \rho \)-meson, respectively.

Then the calculation of the hadronic representation \( \Pi^{\mathrm{phys}}_\nu(p, q) \) is straightforward and yields

\[
\Pi^{\mathrm{phys}}_\nu(p, q) = \frac{\langle 0| J^{\mu}_{\rho}(p) \gamma_{\rho}(q) Z_{c}(p') \rangle}{p^2 - m_{\eta_c}^2} \times \frac{Z_c(p')|J^{\mu}_{\rho}|0}{p'^2 - m_{\eta_c}^2} + \ldots \quad (41)
\]

Employing the corresponding matrix elements we find for the ground state contribution

\[
\Pi^{\mathrm{phys}}_\nu(p, q) = \frac{f_{\eta_c} f_{Z_c} m_{\eta_c} m_{Z_c}^2}{2m_{c} (p^2 - m_{Z_c}^2)} (p^2 - m_{\eta_c}^2)
\times \left( \frac{m_{\eta_c}^2 - m_{Z_c}^2}{2} c_{\nu} + p' \cdot e^* q_{\nu} \right) + \ldots \quad (42)
\]

In the soft-meson limit only the structure \( \sim e^*_{\nu} \) survives. The relevant invariant amplitude is given by the formula

\[
\Pi^{\mathrm{phys}}_\rho(p^2) = \frac{f_{\eta_c} f_{Z_c} m_{\eta_c} m_{Z_c}^2}{4m_{c}} (p^2 - m_{\eta_c}^2) (m_{\eta_c}^2 - m_{Z_c}^2) + \ldots \quad (43)
\]

where \( m^2 = (m_{\eta_c}^2 + m_{Z_c}^2)/2 \). The Borel transformation of \( \Pi^{\mathrm{phys}}_{\rho}(p^2) \) yields

\[
\Pi^{\mathrm{phys}}_{\rho}(M^2) = \frac{f_{\eta_c} f_{Z_c} m_{\eta_c} m_{Z_c}^2}{4m_{c}} (m_{\eta_c}^2 - m_{Z_c}^2) \times \frac{1}{M^2} e^{-(m_{\eta_c}^2 + m_{Z_c}^2)/2M^2} + \ldots \quad (44)
\]

Computation of the vertex \( Z_c \eta_c \rho \) in terms of the quark-gluon degrees of freedom is the next step to get the required sum rule. For the correlation function \( \Pi^{\mathrm{QCD}}_{\rho}(p, q) \) we obtain :

\[
\Pi^{\mathrm{QCD}}_{\rho}(p, q) = -i \int d^4 x e^{ipx} \frac{\varepsilon^*_{\rho} \varepsilon_{\gamma}}{\sqrt{2}} \left[ \gamma_5 \tilde{S}^\mu_c(x) \gamma_5 \right]
\times \tilde{S}^\mu_c(-x) \gamma_\nu + \gamma_\nu \tilde{S}^\mu_c(x) \gamma_5 \tilde{S}^\nu_c(-x) \gamma_5 \right]_{\alpha \beta}
\times \langle \rho(q) \gamma_\mu d_{\alpha}(0) | d_{\beta}(0) | 0 \rangle. \quad (45)
\]

In the soft-meson limit only the matrix elements

\[
\langle 0| \gamma_\mu d(0) | p, \lambda \rangle = \epsilon_{\mu}^{(\lambda)} f_{p} m_{p},
\langle 0| \gamma_\mu g G_{\mu\nu} \gamma_5 d(0) | p, \lambda \rangle = f_{p} m_{p}^{3} \epsilon_{\mu}^{(\lambda)} \zeta_{4} \quad (46)
\]

contribute to \( \text{Im} \Pi^{\mathrm{QCD}}_{\rho}(p, q = 0) \). The last equality in Eq. (46) is the matrix element of the twist-4 operator [41]: numerical value of the parameter \( \zeta_{4} \) was evaluated within QCD sum rule at the renormalization scale \( \mu = 1 \) GeV in Ref. [6] and was found to be equal to \( \zeta_{4} = 0.15 \pm 0.10 \).

The spectral density \( \rho^{\mathrm{QCD}}_{p}(s) \) can be derived in accordance with the prescription described above. As a result we get:

\[
\rho^{\mathrm{QCD}}_{p}(s) = f_{p} m_{p} \sqrt{s - m_{p}^2} \left( 1 + \frac{\zeta_{4} m_{p}^2}{s} \right). \quad (47)
\]

### IV. NUMERICAL COMPUTATIONS AND CONCLUSIONS

The QCD sum rules expressions for the mass and decay constant of the \( Z_c \) state, as well as ones for the couplings \( g_{Z_c J/\psi \pi} \) and \( g_{Z_c \eta_c \rho} \) contain numerous parameters that should be fixed in accordance with the standard prescriptions. Thus, for numerical computation of the \( Z_c \) state’s mass and decay constant we need values of the quark and gluon condensates. Apart from that QCD sum rules for the couplings contain masses and decay constants of the heavy (\( J/\psi, \eta_c \)) and light (\( \pi, \rho \)) mesons. The values of all these parameters are collected in Table 1.

| Parameters | Values         |
|------------|----------------|
| \( m_c \)  | (1.275 ± 0.025) GeV |
| \( m_{J/\psi} \) | (3096.92 ± 0.01) MeV |
| \( m_{\eta_c} \) | (2983.6 ± 0.6) MeV |
| \( m_{\pi} \)  | (139.57018 ± 0.00035) MeV |
| \( m_{\rho} \)  | (775.26 ± 0.25) MeV |
| \( f_{J/\psi} \) | 0.405 GeV |
| \( f_{\eta_c} \)  | 0.35 GeV |
| \( f_{\pi} \)  | 131.5 MeV |
| \( f_{\rho} \)  | 157 MeV |
| \( \langle \bar{q} q \rangle \) | (-0.24 ± 0.01) GeV³ |
| \( \langle \bar{c} c \rangle \) | (0.012 ± 0.004) GeV⁴ |
| \( m_0^2 \)  | (0.8 ± 0.1) GeV³ |

**FIG. 2:** The mass \( m_{Z_c} \) as a function of the Borel parameter \( M^2 \) for different values of \( s_0 \).

Applying the operator from Eq. (28) and equating the physical representation for the invariant amplitude with its QCD expression we, finally, obtain the sum rule to evaluate the strong coupling \( g_{Z_c \eta_c \rho} \). The width of the decay \( Z_c \to \eta_c \rho \) is given by Eq. (47) with replacements \( m_{c} \to m_{\eta_c} \) and \( m_{J/\psi} \to m_{\rho} \).
Sum rules calculations require fixing of the threshold parameter \(s_0\) and a region within of which it may be varied. For \(s_0\) we employ

\[(3.9 + 0.3)^2 \text{ GeV}^2 \leq s_0 \leq (3.9 + 0.5)^2 \text{ GeV}^2. \quad (48)\]

We also find the range \(2 \text{ GeV}^2 < M^2 < 5 \text{ GeV}^2\) as a reliable one for varying the Borel parameter, where the effects of the higher resonances and continuum states, and contributions of the higher dimensional condensates meet all requirements of QCD sum rules calculations. Additionally, in this interval the dependence of the mass and decay constant on \(M^2\) is stable, and we may expect that the sum rules give the correct results. By varying the parameters \(M^2\) and \(s_0\) within the allowed ranges, as well as taking into account ambiguities arising from other input parameters we estimate uncertainties of the whole calculations. The results for the mass \(m_{Z_c}\) and decay constant \(f_{Z_c}\) are drawn as the functions of the Borel parameter in Figs. 2 and 3 respectively. The sensitivity of the obtained predictions to the choice of \(s_0\) are also seen in these figures, where three different values for \(s_0\) are utilized. Our result for \(m_{Z_c}\) together with the prediction of Ref. [32] are written down in Table II. As it has been emphasized above, the exotic tetraquark \(Z_c\) state was observed by BESIII and Belle collaborations [26, 27], which measured its mass and width. The experimental data on the mass of \(Z_c\) state are also shown in Table II. As is seen, experimental results for \(m_{Z_c}\) are rather precise (\(\sim 0.15\% - 0.20\%\)), whereas theoretical prediction made in the present work suffers from errors \(\sim 5\% - 10\%\), which are inherent to sum rules calculations, and may be considered as acceptable in the case under consideration. Our finding for \(m_{Z_c}\) is consistent with the data, despite large theoretical uncertainties originating mainly from the choice of the parameters \(s_0\) and \(M^2\). It also agrees with the theoretical prediction of Ref. [32].

For the decay constant we get:

\[f_{Z_c} = (0.46 \pm 0.03) \times 10^{-2} \text{ GeV}^4. \quad (49)\]

\[
\begin{array}{|c|c|}
\hline
\text{Source} & m_{Z_c} \text{ (MeV)} \\
\hline
\text{BESIII} & 3899 \pm 6 \\
\text{Belle} & 3895 \pm 8 \\
\text{Present Work} & 3900 \pm 210 \\
Z. Wang, T. Huang & 3910^{+110}_{-50} \\
\hline
\end{array}
\]

TABLE II: Experimental data and theoretical predictions for the mass of \(Z_c\) state.

The same quantity was also calculated in Ref. [32], where the following prediction was made:

\[\lambda_{Z_c} = 2.20^{+0.36}_{-0.29} \times 10^{-2} \text{ GeV}^5. \quad (50)\]

Taking into account differences in definitions of the decay constants, and re-scaling \(f_{Z_c}\) by the factor \(m_{Z_c}\) in our case we get:

\[m_{Z_c} f_{Z_c} = (1.79 \pm 0.12) \times 10^{-2} \text{ GeV}^5. \quad (51)\]

The discrepancy between two result may be attributed to the more complicated phenomenological part in QCD sum rules calculations employed in Ref. [32].

The values for the mass and decay constant, as well as corresponding sum rules formulas are used to evaluate the strong couplings \(g_{Z_c \to J/\psi \pi}\) and \(g_{Z_c \to \eta \rho}\), and width of the decays \(Z_c \to J/\psi \pi\) and \(Z_c \to \eta \rho\). In the evaluation of the couplings we employ the same range for \(s_0\) and \(M^2\) as in calculations of the mass and decay constant. For \(g_{Z_c \to J/\psi \pi}\) we find:

\[g_{Z_c \to J/\psi \pi} = (0.39 \pm 0.06) \text{ GeV}^{-1}. \quad (52)\]

The width of the decay \(Z_c \to J/\psi \pi\) can be obtained by means of Eq. (57)

\[\Gamma(Z_c \to J/\psi \pi) = (41.9 \pm 9.4) \text{ MeV}. \quad (53)\]

For the coupling \(g_{Z_c \to \eta \rho}\) and width of the decay \(Z_c \to \eta \rho\) we get

\[g_{Z_c \to \eta \rho} = (1.39 \pm 0.15) \text{ GeV}^{-1}, \quad (54)\]

and

\[\Gamma(Z_c \to \eta \rho) = (23.8 \pm 4.9) \text{ MeV}. \quad (55)\]

The decay width of the channels \(Z_c \to J/\psi \pi\) and \(Z_c \to \eta \rho\) were also analyzed in Ref. [31]. The authors considered the \(Z_c\) state as the isospin 1 partner of the \(X(3872)\), and therefore within SU(2) symmetry the masses of two particles were accepted being exactly equal to each other. In other words, here independent computation of the mass \(m_{Z_c}\) was not carried out. The couplings \(g_{Z_c \to J/\psi \pi}\) and \(g_{Z_c \to \eta \rho}\) were extracted from the three-point correlation function using standard methods of QCD sum rules. They obtained the value \(\Gamma(Z_c \to J/\psi \pi) = 29.1 \pm 8.2 \text{ MeV}\) for the width in \(J/\psi \pi\).
channel, which is lower than our prediction. At the same
time, the decay width
\[
\Gamma(Z_c \to \eta_c \rho) = 27.5 \pm 8.5 \text{ MeV},
\]
is in accord with our result [see, Eq. (55)].

Considering the transitions \(Z_c \to J/\psi \pi\) and \(Z_c \to \eta_c \rho\)
as dominant channels we obtain for the total width of \(Z_c\) approximately
\[
\Gamma_{Z_c} = 65.7 \pm 10.6 \text{ MeV},
\]
which is in accord with our result [see, Eq. (55)].

The ratio of width in \(\eta_c \rho\) channel to that of \(J/\psi \pi\) is equal to 0.56 \pm 0.24 which
is in agreement with the prediction of [34], obtained using a type II tetraquark model, i.e. within the color-spin
Hamiltonian approach by neglecting the spin-spin interaction outside the diquarks.

In conclusion, we have employed QCD two-point sum
rule to calculate the mass and decay constant of the ex-
otic \(Z_c\) state. The obtained results have been used as input
information for studying the strong vertices \(Z_c J/\psi \pi\)
and \(Z_c \eta_c \rho\), and evaluating of the couplings \(g_{Z_c J/\psi \pi}\) and
\(g_{Z_c \eta_c \rho}\). To this end, we have utilized the methods of
QCD sum rules on the light-cone and soft-meson approx-
imation. Finally, the widths of the decays channels \(Z_c \to J/\psi \pi\) and \(Z_c \to \eta_c \rho\) have been found. The the-
etorical framework used here is rather simple and allows one to calculate the relevant quantities in terms a few
local matrix elements of the pion and \(\rho\) meson. Its ac-
ccuracy was checked by explicit calculations in Ref. [37],
where the strong \(D^+D\pi\) and \(B^+B\pi\) vertices were explored in
the full LCSR approach, and its soft \(q \to 0\) limit. It was
demonstrated that both computational schemes lead to
almost identical predictions for the strong couplings
\(g_{D^+D\pi}\) and \(g_{B^+B\pi}\). The soft-meson(pion) approximation was also successfully employed to investigate some other
processes [42, 43].

Our results for the mass and total width of \(Z_c\) are consistent with available experimental data as well as other
theoretical predictions. The ratio of width in \(\eta_c \rho\) channel to
that of \(J/\psi \pi\) is also in agreement with the prediction of [34], obtained via a different approach. The observed
discrepancy between our result for \(f_{Z_c}\) and that of Ref. [32]
can be connected with accuracy and features of the used approaches. Further experimental measurements
and theoretical computations on parameters of the ex-
otic states may help us to improve schemes and methods
for their investigations.

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Appendix: A

In this appendix we have collected the results of our
calculations of the spectral density
\[
\rho^{\text{QCD}}(s) = \rho^{\text{pert}}(s) + \sum_{k=3}^{5} \rho_k(s), \tag{A.1}
\]
necessary for evaluation of the \(Z_c\) meson mass \(m_{Z_c}\) and
its decay constant \(f_{Z_c}\) from the QCD sum rule. In Eq.
(A.1) by \(\rho_k(s)\) we denote the nonperturbative contribu-
tions to \(\rho^{\text{QCD}}(s)\). Neglecting the terms proportional to
light quark masses, the explicit expressions for \(\rho^{\text{pert}}(s)\)
and \(\rho_k(s)\) are presented below as the integrals over the
Feynman parameters \(z\) and \(w\):

\[
\rho^{\text{pert}}(s) = \frac{1}{384\pi^6} \int_0^1 dz \int_0^{1-z} dw \frac{p^s}{(w-1)} \left( wz^2 \left[ swz(w+z-1) - m_c^2 \frac{w+z}{r} \right]^2 \right. \\
\left. \times \left( m_c^2 \frac{w+z}{r} (4w(w-1) + 3z(w-1) + 3z^2) - swz(w+z-1)(3z^2 + (w-1)(7w+3z)) \right) \right) \theta(L), \tag{A.2}
\]

\[
\rho_3(s) = \frac{1}{16\pi^4} \int_0^1 dz \int_0^{1-z} dw m_c e^5 w z (w+z-1) \left[ (\bar{u}u) w + (d\bar{d}) z \right] \left( m_c^2 \frac{w+z}{r} - swz(w+z-1) \right) \theta(L), \tag{A.3}
\]
\[ \rho_4(s) = -\frac{1}{36864\pi^4} \left\{ \frac{G F}{\pi} \int_0^1 dz \int_0^{1-z} dw \frac{w z^2 e^6 (w + z - 1)^2}{(w - 1)} \right\} \times (5w + 3z) + 2m^2 r_x \{ 48z^4 (z - 1)^3 + 64w^6 (z^3 - 1) + w z (z - 1) [-15 + z (z - 1)] \\
\times (-135 + 16z (z - 1)^2 (15z - 4)) \} + w^5 \left\{ -61 + 16z (-7 + z^2 (19z - 20)) \right\} + w^4 \{ 329 + 8z (-43 + 2z (3z - 1)) \\
\times (6 - 22z + 13z^2) \} + w^3 \left\{ -15 + 339z + z^2 (-405 + 32z (z - 1)^2 (17z - 10)) \right\} + w^2 \left\{ -219 + 795z \right. \\
\left. + 2z^3 (-249 - 344z + 16z^2 (63 - 66z + 23z^2)) \right\} \theta (L) \] 
\[ \rho_5(s) = \frac{m c}{16\pi^4} \int_0^1 dz \int_0^{1-z} dw r^5 w^2 z^2 (w + z - 1)^2 m^2_0 \left\{ \langle u (w) \rangle w + \langle d (\bar{w}) \rangle \bar{z} \right\} \theta (L), \]
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