A FLOW OF CONFORMALLY BALANCED METRICS WITH KÄHLER FIXED POINTS

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Abstract

While the Anomaly flow was originally motivated by string theory, its zero slope case is potentially of considerable interest in non-Kähler geometry, as it is a flow of conformally balanced metrics whose stationary points are precisely Kähler metrics. We establish its convergence on Kähler manifolds for suitable initial data. We also discuss its relation to some current problems in complex geometry.

1 Introduction

The main purpose of this paper is to study a geometric flow which is of potential interest from several viewpoints, including mathematical physics, non-Kähler complex geometry, and the theory of non-linear partial differential equations. Let $X$ be a compact $n$-dimensional complex manifold, which admits a non-vanishing holomorphic $(n,0)$-form $\Omega$. The flow which we shall consider is the flow $t \rightarrow \omega(t)$ of Hermitian metrics defined by

$$\partial_t(\|\Omega\|_{\omega}^{n-1}) = i\partial\bar{\partial}\omega^{n-2}$$

with an initial data $\omega_0$ which is conformally balanced, in the sense that

$$d(\|\Omega\|_{\omega_0}^{n-1}) = 0.$$  

Here $\|\Omega\|_\omega$ is the norm of $\Omega$, defined by $\|\Omega\|_{\omega_0}^2 = i^{n^2}n! \Omega \wedge \bar{\Omega} \omega_0^{-n}$.

The flow (1.1) is a generalization to arbitrary dimension, with the slope parameter $\alpha'$ set to 0, of the Anomaly flow introduced in [56] for $n = 3$. This flow provides a systematic approach for solving the Hull-Strominger system [38, 39, 65] for supersymmetric compactifications of the heterotic string. Thus it is highly desirable to gain a better understanding for it, and the case $\alpha' = 0$ provides already an important and non-trivial special case. Another defining feature of the flow (1.1) is that it preserves the conformally balanced condition (1.2), and its stationary points are astheno-Kähler metrics (see §2.2 for definition). This implies that its stationary points are Kähler [24, 53], so the convergence of the flow is closely related to a well-known question in non-Kähler geometry, namely when is a conformally balanced manifold actually Kähler. Also closely related is another

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fundamental question in non-Kähler geometry and algebraic-geometric stability conditions [16, 47, 77, 78], namely when does a positive \((p, p)\) cohomology class admit as representative the \(p\)-th power of a Kähler form. Finally, on Kähler manifolds the flow (1.1) can be viewed as a complex version of the inverse Gauss curvature flow studied extensively in convex geometry, see for example [2, 5, 12, 13, 25, 75], and is in itself quite interesting as a fully non-linear partial differential equation. More details on all these motivations will be provided in Section §2 below.

In [59] another generalization of the Anomaly flow to arbitrary dimension \(n\) was proposed which agrees with the flow (1.1) in dimension \(n = 3\). For that generalization, when \(\alpha' = 0\) for all dimensions \(n\), the flow was shown to continue to exist, as long as the curvature and torsion remain bounded and the norm of \(\Omega\) does not tend to 0. But this is all which is known at the present time. In this paper, we shall prove the following theorems:

**Theorem 1** Let \(X\) be an \(n\)-dimensional compact complex manifold with a nowhere vanishing \((n, 0)\) holomorphic form \(\Omega\), \(n \geq 3\). Let the initial data \(\omega_0\) be a Hermitian metric which is conformally balanced, i.e. the condition (1.2) holds. Then the flow is parabolic, and admits in particular a unique solution on some time interval \([0, T)\) with \(T > 0\).

Clearly the flow (1.1) can only converge if the manifold \(X\) is Kähler. Conversely, if the manifold \(X\) is Kähler, we can prove:

**Theorem 2** Consider the flow (1.1) on a compact \(n\)-fold \(X\) equipped with a holomorphic \((n, 0)\)-form \(\Omega\) as in the previous theorem, \(n \geq 3\). Assume that the initial data \(\omega(0)\) satisfies

\[
\|\Omega\|_{\omega(0)}^{n-1} = \hat{\chi}^{n-1},
\]

where \(\hat{\chi}\) is a Kähler metric. Then the flow (1.1) exists for all time \(t > 0\), and as \(t \to \infty\), the solution \(\omega(t)\) converges smoothly to a metric \(\omega_\infty\) satisfying

\[
\omega_\infty = \|\Omega\|_{\chi_\infty}^{-2/(n-2)} \chi_\infty,
\]

where \(\chi_\infty\) is the unique Kähler Ricci-flat metric in the cohomology class \([\hat{\chi}]\), and \(\|\Omega\|_{\chi_\infty}\) is a constant given by

\[
\|\Omega\|_{\chi_\infty}^2 = \frac{n!}{|\hat{\chi}|^n} \left( \int_X i^n \Omega \wedge \bar{\Omega} \right).
\]

In particular, \(\omega_\infty\) is Kähler and Ricci-flat.

As a consequence we obtain another proof of the classical theorem of Yau [79] on the existence of Ricci-flat Kähler metrics. A first parabolic proof of Yau’s theorem was the one by Cao [10] using the Kähler-Ricci flow. More recent parabolic proofs using inverse Monge-Ampère flow were obtained in [11, 14, 17]. But, as stressed in [14] and also discussed in Section §2.2 below, alternative approaches are interesting not just for alternative proofs
in themselves, but also for the singularities that they would develop when no stationary point exists.

Theorem 2 will be reduced to the following theorem on a flow of complex Monge-Ampère type which may be of independent interest:

**Theorem 3**  Let $X$ be a compact Kähler manifold with Kähler metric $\hat{\chi}$. Let $f \in C^\infty(X, \mathbb{R})$ be a given function. Consider the flow

$$\partial_t \varphi = e^{-f} \frac{(\hat{\chi} + i\partial\bar{\partial}\varphi)^n}{\hat{\chi}^n}, \quad \varphi(x, 0) = 0,$$

where $\varphi(t)$ is subject to the plurisubharmonicity condition

$$\hat{\chi} + i\partial\bar{\partial}\varphi > 0.$$  

Then the flow $\varphi(t)$ exists for all $t > 0$, and the averages $\bar{\varphi}(t) = \varphi(t) - \frac{1}{V} \int_X \varphi \hat{\chi}^n$, with $V = \int_X \hat{\chi}^n$, converge in $C^\infty$.

The flow (1.6) shares with the Kähler-Ricci flow and the inverse Monge-Ampère flow ($MA^{-1}$-flow) introduced by Collins-Hisamoto-Takahashi [14] the fact that the Monge-Ampère determinant $(\hat{\chi} + i\partial\bar{\partial}\varphi)^n/\hat{\chi}^n$ appears in the right hand side. However, it appears here as a first power, and not as a log or as an inverse power, which makes it not concave. In particular, it does not fall within the scope of standard parabolic PDE methods such as those developed in [63].

The paper is organized as follows. In §2, we provide some background and motivations for the flow (1.1). In §3, we compute the evolution equation for the metric and prove the short time existence as claimed in Theorem 1. In §4, the Anomaly flow is reduced to the study of a complex Monge-Ampère flow for initial data satisfying (1.3). We establish all the necessary estimates for the long-time behavior of the flows. In §5, we prove the convergence of the flow which completes the proof of Theorems 2 and 3. In §6, we provide some further remarks about the Anomaly flow and its possible connection to some interesting problems in complex geometry. The appendix contains some calculations used in §3.

## 2 Motivations for the flow

We provide now some details on the three different contexts which make the flow (1.1) of particular interest.

### 2.1 The Hull-Strominger system

Let $X$ be a compact 3-fold equipped with a nowhere vanishing holomorphic $(3, 0)$-form $\Omega$. Let $t \to \Phi(t)$ be a flow of $(2, 2)$-forms with $d\Phi = 0$ for any $t$. Then the following flow was
\[ \partial_t (\| \Omega \| \omega^2) = i \partial \bar{\partial} \omega - \frac{\alpha'}{4} (\text{Tr} (Rm \wedge Rm) - \Phi(t)) \]  

(2.1)

with initial data \( \omega_0 \) required to satisfy the condition \( d(\| \Omega \| \omega_0^2) = 0 \). Here \( \alpha' \) is a physical parameter called the slope, and \( Rm \) is the Riemann curvature of the metric \( \omega \), viewed as a \((1,1)\)-form valued in the bundle of endomorphisms of \( T^{1,0}(X) \). One case of particular interest is when \( X \) is also equipped with a holomorphic vector bundle \( E \to X \) with \( c_1(E) = 0 \), and \( \Phi(t) = \text{Tr}(F \wedge F) \) where \( F \) is the curvature of a Hermitian metric \( H(t) \) on \( E \) which itself evolves simultaneously under the Donaldson heat flow

\[ H^{-1} \partial_t H = -(\Lambda_{\omega} iF), \quad H(0) = H_0. \]  

(2.2)

Here \( H_0 \) is a given initial metric, \( F \) is viewed as a \((1,1)\)-form valued in the bundle of endomorphisms of \( E \), and \( (\Lambda_{\omega} iF)^{\alpha \beta} = g^{jk} F_{kj}^{\alpha \beta} \) is the usual Hodge contraction. The stationary points of the simultaneous flows (2.1) and (2.2) are precisely the solutions of a system of equations proposed independently by C. Hull [38, 39] and A. Strominger [65] for supersymmetric compactifications of the heterotic string to \( M^{3,1} \times X \), where \( M^{3,1} \) is Minkowski space-time. The Hull-Strominger system of equations is a generalization of the more specific case considered earlier by Candelas, Horowitz, Strominger and Witten [9], where \( E \) was set to \( T^{1,0}(X) \). If we consider only the stationary points of the system, we can set \( H = \omega \), and consequently \( \Phi = \text{Tr}(Rm \wedge Rm) \). In this case the term \( \text{Tr}(Rm \wedge Rm) - \Phi(t) \) vanishes trivially, and it was shown in [9] that the stationary points of both (2.1) and (2.2) are given by the same condition of \( \omega \) being a Ricci-flat Kähler metric.

A major reason for considering the flow (2.1), among all the flows which have the same stationary points, is that it preserves the condition that the form \( \| \Omega \| \omega \omega^2 \) be closed. In fact, the \((2,2)\) de Rham class of \( \| \Omega \| \omega(t)^2 \) can be easily read off from the flow. This is however only a weak substitute for the analogous statement for the \((1,1)\) Kähler class of \( \omega \) in the Kähler-Ricci flow, and this accounts for many new difficulties that one encounters in the study of (2.1).

The flow (2.1) can be expressed more explicitly in terms of a flow for the \((1,1)\)-form \( \omega(t) \) itself [59]. Perhaps surprisingly, this formulation shows that (2.1) can be viewed as a generalization of the Ricci flow with a triple complication, namely the metrics are not Kähler (or Levi-Civita), the norms \( \| \Omega \| \omega \) also occur, and so do quadratic expressions in the curvature tensor.

The flow (2.1) was shown in [60] to produce a simpler and unified way of recovering the solutions of the Hull-Strominger system on Calabi-Eckmann-Goldstein-Prokushkin fibrations [8, 36] originally found by Fu and Yau [31, 32]. It also inspired directly the solutions of the Fu-Yau Hessian equations found recently in [62], which extended our earlier work [55, 58]. However, it is still largely unexplored, and its long-term behavior is already
known to exhibit in general a rather intricate behavior, such as a particular sensitivity to the initial data [20, 60, 61].

Because of all these complications, it is necessary to examine the flow (2.1) in some simpler settings. The above choice of $\Phi$ in [9] has the effect of eliminating the terms in the right hand side of (2.1) which are quadratic in the curvature tensors. This effect is also achieved simply by considering the case $\alpha' = 0$. The flow (2.1) can then be considered on its own. And while the dimension 3 for $X$ is required for the interpretation of the Hull-Strominger system as a compactification of the heterotic string, from the pure mathematical standpoint, we can consider all dimensions $n \geq 3$, and the flow we propose in (1.1) is the natural generalization of the Anomaly flow (2.1) with $\alpha' = 0$.

### 2.2 Generalizations of the Kähler condition

The flow (1.1) also fits into the broad question of when a compact complex manifold $X$ may admit a Kähler metric or a weaker substitute. We review some of the relevant notions.

Let $(X, J)$ be a compact complex manifold of complex dimension $n$ equipped with a Hermitian metric $g$, and denote its Kähler form by $\omega = i \sum g_{kj} dz^j \wedge d\bar{z}^k$. Since the Kähler form of a Hermitian metric determines the metric (as $J$ is fixed), by an abuse of terminology we will not distinguish between the two notions.

If $d\omega = 0$, then $g$ is called a Kähler metric, and $X$ is called a Kähler manifold. For each $2 \leq k \leq n - 1$, it is natural to consider a weaker condition of the form

$$d\omega^k = 0.$$  \hspace{1cm} (2.1)

The case $k = n - 1$ was introduced by Michelsohn [54] who called such a metric balanced, and the manifold $X$ a balanced manifold. By an observation due to Gray and Hervella [37] conditions such as $d\omega^k = 0$ for some $2 \leq k \leq n - 2$ actually imply that $\omega$ is Kähler, so the case $k = n - 1$ is the only non-trivial generalization of the Kähler property. Michelsohn found an intrinsic characterization of compact manifolds with balanced metrics by means of positive currents. Using such a characterization, Alessandrini and Bassanelli proved that the existence of balanced metrics is preserved under birational transformations in [1]. Remarkably, as we just saw in the previous section §2.1, balanced metrics also arise in string theory since the torsion constraint equation $d(\|\Omega\|_2 \omega^2) = 0$ in dimension 3 of the Hull-Strominger system just means that the metric $\|\Omega\|_2 \omega$ is balanced, see [49]. The existence of balanced metrics on compact Hermitian manifolds has been studied extensively (see e.g. [18, 19, 21, 22, 26, 27, 28, 54, 71, 76] and references therein).

On the other hand, another natural generalization of the Kähler condition is

$$i\partial \bar{\partial} \omega^\ell = 0,$$  \hspace{1cm} (2.2)

for some $\ell$ with $1 \leq \ell \leq n - 1$. If $\ell = n - 1$, then $\omega$ is called a Gauduchon metric. In [34], Gauduchon proved that there always exists a unique Gauduchon metric, up to a constant
conformal factor, in the conformal class of a given Hermitian metric. Gauduchon manifolds also provide a natural setting for an extension [48, 51] of the Donaldson-Uhlenbeck-Yau theory of Hermitian-Yang-Mills metrics on stable bundles over Kähler manifolds.

If $\ell = n - 2$, then $\omega$ is called an astheno-Kähler metric. This notion was introduced by Jost and Yau in [41] to establish the existence of Hermitian harmonic maps, and it turns out to be particularly interesting for many analytic arguments to be useful. For example, Tosatti and Weinkove proved in [72] the Calabi-Yau theorems for Gauduchon and strongly Gauduchon metrics on the class of compact astheno-Kähler manifolds. Recently, the existence such metrics on compact complex manifolds has been studied widely, see for example [23, 24, 46, 52, 53].

An interesting question raised in [69] asks whether a compact complex non-Kähler manifold can admit both an astheno-Kähler metric and a balanced metric. Very recently, such examples were constructed in [23, 46]. In fact, the astheno-Kähler metric of Latorre-Ugarte [46] is $k$-th Gauduchon [29], meaning that $i\partial \bar{\partial} \omega^k \wedge \omega^{n-k-1} = 0$, for every $1 \leq k \leq n - 1$. However, a single metric on a compact manifold cannot be both balanced and astheno-Kähler, unless it is Kähler. This statement can be viewed as a generalization to arbitrary dimensions of the arguments of [9]. It was proved in Matsuo-Takahashi [53], and extended in Fino-Tomassini [24] to the case of metrics which are both conformally balanced and astheno-Kähler. For easy reference, we state these results as a lemma, and include an alternative proof in this paper (see Corollary 1 in §3 for (ii) and (e) in §6 for (i)):

**Lemma 1** [24, 53] Let $X$ be an $n$-dimensional compact complex manifold with Hermitian metric $\omega$. Then $\omega$ is Kähler if one of the following conditions is satisfied

(i) $\omega$ satisfies both $d\omega^{n-1} = 0$ and $i\partial \bar{\partial} \omega^{n-2} = 0$,

(ii) $X$ admits a nowhere vanishing holomorphic $(n, 0)$-form $\Omega$ and $\omega$ satisfies both $d(||\Omega||_\omega \omega^{n-1}) = 0$ and $i\partial \bar{\partial} \omega^{n-2} = 0$. In this case, $\omega$ is also Ricci-flat.

We return now to the flow (1.1). Since the right hand side of the flow is both $d$ and $i\partial \bar{\partial}$ exact, it follows that if $||\Omega||_\omega \omega^{n-1}$ is $d$-closed or $i\partial \bar{\partial}$-closed, then $||\Omega||_\omega \omega^{n-1}$ remains $d$-closed or $i\partial \bar{\partial}$-closed along the flow, and the de Rham cohomology class $[||\Omega||_\omega \omega^{n-1}]$ or the Bott-Chern cohomology class $[||\Omega||_\omega \omega^{n-1}]_{BC}$ is preserved.

Moreover, thanks to (ii) in Lemma 1, we see that if the Anomaly flow (1.1) converges, the limit metric must be a Kähler Ricci-flat metric. Therefore, the flow provides a deformation path in the space of conformally balanced metrics to a Kähler metric. If no Kähler metric exists on $X$, then the flow cannot converge, and either its singularities in finite-time or long-term behavior should provide an analytic measure of the absence of Kähler metrics.
2.3 Parabolic fully non-linear equations on Hermitian manifolds

The theory of parabolic fully non-linear equations on Hermitian manifolds has been developed extensively over the years, and has resulted in some powerful and general results. As an example, let \((X, \omega)\) be a compact Hermitian manifold equipped with a smooth closed form \(\hat{\chi}\), and consider the equation

\[
\partial_t u = F(A(i\partial\bar{\partial}u)) - \psi(z), \quad u(0) = u_0.
\]  

(2.3)

Here \(i\partial\bar{\partial}u\) is the complex Hessian of \(u\), and \(A(i\partial\bar{\partial}u)\) is the set of eigenvalues of the form \(\hat{\chi} + i\partial\bar{\partial}u\) with respect to the metric \(\omega\). The function \(F(\lambda)\) is a given function defined on a given cone \(\Gamma\), and \(A(i\partial\bar{\partial}u)\) is required to be in \(\Gamma\) for all time.

The function \(F\) is required to satisfy many conditions, including conditions amounting to the parabolicity of the flow (2.3). But an additional condition, originating from the earliest works of Caffarelli-Nirenberg-Spruck [7] and Krylov [43] on the theory of fully non-linear equations, is that \(F\) be a concave function of \(\lambda \in \Gamma\). This concavity condition, together with the existence of subsolutions and more technical hypotheses, is now known to lead to some very general theorems which can apply to a wide variety of geometric flows that have been studied in the literature (see e.g. [63] and references therein).

The Anomaly flow on Calabi-Eckmann-Goldstein-Prokushkin fibrations studied in [60] provides however a natural geometric example of a flow that does not satisfy the concavity condition, but which is nevertheless well-behaved. This suggests the existence of a useful theory going beyond concave equations, the formulation of which would require the treatment of many more examples. As we shall see, the flow (1.1), in its realization (1.6), provides another instructive example. Note that it corresponds to \(F = e^{f(z)} \prod_{j=1}^{n} \lambda_j\), unlike the Kähler-Ricci flow, which corresponds to \(F = \log \prod_{j=1}^{n} \lambda_j\) which is concave.

3 Short-time existence and proof of Theorem 1

To prove Theorem 1, we need a lemma which will allow us to compute relevant quantities with the Hodge star operator. For a \((p, q)\)-form \(\Theta\) on a manifold \(X\), its components \(\Theta_{k_1...k_qj_1...j_p}\) are defined by

\[
\Theta = \frac{1}{p!q!} \sum \Theta_{k_1...k_qj_1...j_p} dz^{j_p} \wedge \cdots \wedge dz^{j_1} \wedge d\bar{z}^{k_q} \wedge \cdots \wedge d\bar{z}^{k_1}.
\]  

(3.1)

For a \((p, p)\)-form \(\Theta\), we define

\[
\text{Tr} \Theta = \langle \Theta, \omega^p \rangle = i^{-p} \prod_{\ell=1}^{p} g^{k_\ell j_\ell} \Theta_{j_1k_1...j_pk_p}.
\]  

(3.2)
Lemma 2 Let \( \alpha \in \Omega^{1,1}(X, \mathbb{R}), \Phi \in \Omega^{2,2}(X, \mathbb{R}) \) and \( \Psi \in \Omega^{3,3}(X, \mathbb{R}) \) on a complex manifold \( X \) of dimension \( n \geq 3 \). Let \( \omega = ig_{kj}dz^j \wedge dz^k \) be a Hermitian metric and \( * \) its associated Hodge star operator. Then

\[
*(\alpha \wedge \omega^{n-2}) = -(n-2)! \alpha + (n-2)! (\text{Tr} \alpha \omega), \quad (3.3)
\]

\[
(*\Phi \wedge \omega^{n-3})_{kj} = i(n-3)! g^{s\bar{r}} \Phi_{\bar{r}s k j} + i(n-3)! \text{Tr} \Phi g_{kj}, \quad (3.4)
\]

and if \( n \geq 4 \), then

\[
(*\Psi \wedge \omega^{n-4})_{kj} = \frac{(n-4)!}{2} g^{q\bar{p}} g^{s\bar{r}} \Psi_{\bar{p}s \bar{q} k j} + i(n-4)! \text{Tr} \Psi g_{kj}. \quad (3.5)
\]

Proof: We will use the general formula for the Hodge star operator on \((n-1, n-1)\) forms, as given in (Lemma 3, [59]). For any \((n-1, n-1)\) form \( \Theta \), there holds

\[
(*\Theta)_{jk} = \frac{1}{(n-1)! (n^2 - 6n + 6)} \left\{ 6 i^{-(n-2)} \prod_{p=1}^{n-2} g^{k_p \bar{j}_p} \Theta_{j k_1 \bar{k}_1 \cdots j_{n-2} \bar{k}_{n-2}} + (n-6) \text{Tr} \Theta \right\} i g_{jk}.
\]

In fact, the proof of (3.3) can also be found in [59, 40], but we provide details here for completeness. First, we notice that

\[
(\alpha \wedge \omega^{n-2})_{kj} \bar{k}_{j_1 \cdots j_{n-2}} = i^{n-2} \alpha \{k_j g_{k_1 j_1} \cdots g_{k_{n-2} j_{n-2}}\}, \quad (3.6)
\]

\[
(\Phi \wedge \omega^{n-3})_{kj} \bar{k}_{j_1 \cdots j_{n-3}} = i^{n-3} \Phi \{k_j g_{k_1 j_1} \cdots g_{k_{n-3} j_{n-3}}\}, \quad (3.7)
\]

\[
(\Psi \wedge \omega^{n-4})_{kj} \bar{k}_{j_1 \cdots j_{n-4}} = i^{n-4} \Psi \{k_j g_{k_1 j_1} \cdots g_{k_{n-4} j_{n-4}}\}. \quad (3.8)
\]

Here we use the notation \( \{\} \) to mean antisymmetrization in both the barred and unbarred indices. Using the formula for the Hodge star on \((n-1, n-1)\) forms exhibited above, we deduce that we must have

\[
(*\alpha \wedge \omega^{n-2})_{kj} = a_n \alpha \bar{k}_j + ib_n (\text{Tr} \alpha) g_{kj}, \quad (3.9)
\]

\[
(*\Phi \wedge \omega^{n-3})_{kj} = i c_n g^{s\bar{r}} \Phi_{\bar{r}s k j} + id_n (\text{Tr} \Phi) g_{kj}, \quad (3.10)
\]

\[
(*\Psi \wedge \omega^{n-4})_{kj} = e_n g^{q\bar{p}} g^{s\bar{r}} \Psi_{\bar{p}s \bar{q} k j} + if_n (\text{Tr} \Psi) g_{kj}, \quad (3.11)
\]

for some coefficients \( a_n, b_n, c_n, d_n, e_n, f_n \) to be determined. If \( \text{Tr} \alpha = 0 \), a direct computation shows

\[
*(\alpha \wedge \omega^{n-2}) = -(n-2)! \alpha. \quad (3.12)
\]

Therefore \( a_n = -(n-2)! \). Taking \( \alpha = \omega \) and using \( *\omega^{n-1} = (n-1)! \omega \), we deduce the relation

\[
(n-1)! = a_n + nb_n. \quad (3.13)
\]

It follows that \( b_n = (n-2)! \) and we have established (3.3).
Next, we test $\Phi = \alpha \wedge \omega$, which has components
\[ (\alpha \wedge \omega)_{rsjk} = i(\alpha_{skj}g_{rs} - \alpha_{ksj}g_{r} - \alpha_{rjs}g_{ks} + \alpha_{rsj}g_{k}). \] (3.14)
Thus
\[ g^{\bf{e}}(\alpha \wedge \omega)_{rsjk} = (n - 2)i\alpha_{skj} - (\text{Tr } \alpha)g_{jk}. \] (3.15)
and
\[ \text{Tr} (\alpha \wedge \omega) = 2(n - 1)(\text{Tr } \alpha). \] (3.16)
Equating (3.9) with (3.10), we therefore we have the relation
\[-(n - 2)!\alpha_{skj} + i(n - 2)(\text{Tr } \alpha)g_{jk} = i^2c_n(n - 2)\alpha_{skj} - ic_n(\text{Tr } \alpha)g_{jk} + 2id_n(n - 1)(\text{Tr } \alpha)g_{jk}. \]
Taking $\alpha$ with $\text{Tr } \alpha = 0$, we see that $c_n = (n - 3)$. Solving for $d_n$ gives $d_n = \frac{(n-3)!}{2}$. This establishes (3.4).

We now test $\Psi = \Phi \wedge \omega$. Its components can be worked out to be
\[ (\Phi \wedge \omega)_{rsjpqkj} = i\left\{ \Phi_{rsjpqk} + \Phi_{rjpqs}g_{kj} + \Phi_{rjqps}g_{kj} + \Phi_{ksrjq}g_{kj} + \Phi_{kjrs}g_{pq} ight. \\
+ \Phi_{kpjqk}g_{pq} + \Phi_{pskjg_{pq}} + \Phi_{pjqkg_{pq}} + \Phi_{pqkjg_{pq}} \} \] (3.17)
Then
\[ g^{\bf{e}}(\Phi \wedge \omega)_{rsjpqkj} = 2i(n - 3)g^{\bf{e}}\Phi_{kjrs} - i\text{Tr } \Phi g_{kj}, \] (3.18)
and
\[ \text{Tr} (\Phi \wedge \omega) = 3(n - 2)(\text{Tr } \Phi). \] (3.19)
Equating (3.10) and (3.11) with $\Psi = \Phi \wedge \omega$ gives the relation
\[ i(n - 3)g^{\bf{e}}\Phi_{rsjk} + i\frac{(n - 3)!}{2}(\text{Tr } \Phi)g_{jk} \\
= 2i(n - 3)e_n g^{\bf{e}}\Phi_{kjrs} - ie_n(\text{Tr } \Phi)g_{jk} + 3i(n - 2)f_n(\text{Tr } \Phi)g_{jk}. \] (3.20)
If we choose $\Phi$ with $\text{Tr } \Phi = 0$, we see that $e_n = \frac{(n-4)!}{2}$. Substituting the value of $e_n$ into (3.20), we deduce $f_n = \frac{(n-4)!}{6}$. Q.E.D.

Before stating the full evolution of the components of the metric in the Anomaly flow (1.1), we establish notation which will be used subsequently. The torsion of $\omega(t)$ is given by $T = i\partial \omega$ and $\bar{T} = -i\partial \omega$, with components
\[ T = \frac{1}{2}T_{kj\ell}dz^\ell \wedge dz^j \wedge d\bar{z}^k, \quad \bar{T} = \frac{1}{2}\bar{T}_{kj\bar{\ell}}dz^\ell \wedge d\bar{z}^j \wedge dz^k, \] (3.21)
given by
\[ T_{kj\ell} = \partial_jg_{\ell k} - \partial_\ell g_{kj}, \quad \bar{T}_{kj\bar{\ell}} = \partial_jg_{\bar{k}\ell} - \partial_{\bar{k}}g_{j\ell}. \] (3.22)
The torsion 1-form $\tau$ is given by
\begin{equation}
\tau = T_idz^i, \quad T_k = g^{jk}T_{kjl}.
\tag{3.23}
\end{equation}

We may take the norms of $T$ and $\tau$ by setting
\begin{equation}
|T|^2 = g^{mk}g^{jn}g^{lp}T_{kjl}T_{mnp}, \quad |\tau|^2 = g^{jk}T_jT_k.
\tag{3.24}
\end{equation}

The curvature tensor of the Chern connection $\nabla$ of $\omega$ is given by
\begin{equation}
R_{kj}^p = -\partial_k(g^{pl}\partial_jg_{lj}), \quad Rm = R_{kj}^p d^j \land d^k \in \Omega^{1,1}(X) \otimes \text{End}(T^{1,0}(X)).
\tag{3.25}
\end{equation}

For a general Hermitian metric $\omega$, there are 4 notions of Ricci curvature.
\begin{equation}
R_{kj} = R_{kj}^p p, \quad \tilde{R}_{kj} = R_p^{pj}k, \quad R'_{kj} = R_{kj}^p p, \quad R''_{kj} = R_{kj}^{pj}.\tag{3.26}
\end{equation}

Conformally balanced metrics have additional structure, and their curvatures satisfy (see Lemma 5 in [59])
\begin{equation}
R'_{kj} = R''_{kj} = \frac{1}{2}R_{kj}, \quad \tilde{R}_{kj} = \frac{1}{2}R_{kj} + \nabla^m T_{kjm}.
\tag{3.27}
\end{equation}

We can now state the formula for the evolution of the metric tensor along the Anomaly flow (1.1). The structure of the torsion terms can be compared to other geometric flows studied in different settings e.g. [4, 6, 42, 50, 66, 67, 73]. The appearance of quadratic torsion terms proportional to $g_{kj}$ when $n \geq 4$ is reminiscent of evolution of the metric in the Laplacian flow for closed $G_2$ structures [6, 42, 50].

The difference in expressions when $n = 3$ and $n \geq 4$ is due to the appearance of $(n-2)(n-3)i\partial\omega \land \partial\omega \land \omega^{n-3}$ when expanding $i\partial\bar{\partial}\omega^{n-2}$ if $n \geq 4$. In deriving (3.29), we cancelled coefficients of the form $\frac{(n-3)}{2}$, making it invalid to substitute $n = 3$ in this expression.

**Theorem 4** Start the Anomaly flow $\frac{d}{dt}(\|\Omega\|\omega^{n-1}) = i\partial\bar{\partial}(\omega^{n-2})$ with initial metric $\omega_0$ satisfying $d(\|\Omega\|\omega_0^{n-1}) = 0$. If $n = 3$, the evolution of the metric is given by
\begin{equation}
\partial_t g_{kj} = \frac{1}{2\|\Omega\|_\omega} \left[ -\tilde{R}_{kj} + g^m_{kl}g^{sp}T_{rmj}\bar{T}_{sjh} \right].
\tag{3.28}
\end{equation}

If $n \geq 4$, the evolution of the metric is given by
\begin{equation}
\partial_t g_{kj} = \frac{1}{(n-1)\|\Omega\|_\omega} \left[ -\tilde{R}_{kj} + \frac{1}{2(n-2)}(|T|^2 - 2|\tau|^2) g_{kj} + \frac{1}{2}g^{sp}g^{qr}T_{kqs}\bar{T}_{jpr} + g^{sr}(T_{kjr}\bar{T}_r + T_{s}T_{jkr}) + T_{j}T_{k} \right].
\tag{3.29}
\end{equation}
As a immediate consequence, we see that the flow has a short-time solution. Indeed, by definition \( \bar{R}_{kj} = -g^{pq} \partial_q \partial_p g_{kj} + g^{pq} g^{rs} \partial_q g_{ks} \partial_p g_{sj} \). The leading term of the linearization is the different operator

\[
\delta g_{kj} \rightarrow -g^{pq} \partial_p \partial_q \delta g_{kj}
\]  

(3.30)

which is elliptic. Hence the flow is strictly parabolic. Theorem 1 is proved.

**Proof:** The evolution equation when \( n = 3 \) was already given in [59], so we assume that \( n \geq 4 \). Since \( \| \Omega \|_\omega^2 = \Omega \bar{\Omega} (\det g)^{-1} \), we have

\[
\frac{d}{dt} \| \Omega \|_\omega = -\frac{1}{2} \| \Omega \|_\omega \text{Tr} \dot{\omega}.
\]  

(3.31)

The Anomaly flow (1.1) can be written as

\[
-\frac{1}{2} \| \Omega \|_\omega (\text{Tr} \dot{\omega}) \omega^{n-1} + (n-1) \| \Omega \|_\omega \dot{\omega} \wedge \omega^{n-2}
= (n-2)i\partial \bar{\partial} \omega \wedge \omega^{n-3} + i(n-2)(n-3) \text{T} \wedge \bar{T} \wedge \omega^{n-4}.
\]  

(3.32)

Wedging both sides by \( \omega \) gives the following equation of top forms

\[
-\frac{1}{2} \| \Omega \|_\omega (\text{Tr} \dot{\omega}) \omega^n + (n-1) \| \Omega \|_\omega \dot{\omega} \wedge \omega^{n-1}
= (n-2)i\partial \bar{\partial} \omega \wedge \omega^{n-2} + i(n-2)(n-3) \text{T} \wedge \bar{T} \wedge \omega^{n-3}.
\]  

(3.33)

Using the identities (A.3), (A.5), (A.7) for contracting forms, we obtain

\[
-\frac{1}{2} \| \Omega \|_\omega (\text{Tr} \dot{\omega}) + \frac{(n-1)}{n} \| \Omega \|_\omega (\text{Tr} \dot{\omega})
= -\frac{(n-2)}{2n(n-1)} g^{jk} g^{\ell m} (i \partial \bar{\partial} \omega)_{kj \ell m} - \frac{(n-3)}{6n(n-1)} g^{jk} g^{q\bar{p}} (T \wedge \bar{T})_{kj pq \bar{q} \bar{r}}.
\]  

(3.34)

Therefore

\[
(\text{Tr} \dot{\omega}) = \frac{1}{(n-1) \| \Omega \|_\omega} \text{Tr} (i \partial \bar{\partial} \omega) + \frac{i(n-3)}{3(n-1)(n-2) \| \Omega \|_\omega} \frac{1}{\text{Tr} (T \wedge \bar{T})}.
\]  

(3.35)

We now apply the Hodge star operator with respect to \( \omega \) to (3.32)

\[
-\frac{1}{2} \| \Omega \|_\omega (\text{Tr} \dot{\omega}) \star \omega^{n-1} + (n-1) \| \Omega \|_\omega \star (\dot{\omega} \wedge \omega^{n-2})
= (n-2) \star (i \partial \bar{\partial} \omega \wedge \omega^{n-3}) + i(n-2)(n-3) \star (T \wedge \bar{T} \wedge \omega^{n-4}).
\]  

(3.36)

Substituting Lemma 2,

\[
-\frac{(n-1)}{2} \| \Omega \|_\omega (\text{Tr} \dot{\omega}) g_{kj} + (n-1) \| \Omega \|_\omega \{ -\partial_t g_{kj} + (\text{Tr} \dot{\omega}) g_{kj} \}
= g^{sp} (i \partial \bar{\partial} \omega)_{p \bar{q} \bar{r} kj} + \frac{1}{2} (\text{Tr} i \partial \bar{\partial} \omega) g_{kj} + \frac{1}{2} g^{qp} g^{sr} (T \wedge \bar{T})_{\bar{r} \bar{q} \bar{p} \bar{q} \bar{r} \bar{p} \bar{q} \bar{r} kj} + \frac{i}{6} \text{Tr} (T \wedge \bar{T}) g_{kj}.
\]  

11
Substituting (3.35), we see that the \((\text{Tr} \, i\partial \bar{\partial} \omega)\) terms cancel and we are left with
\[
\partial_t g_{kj} = -\frac{1}{(n-1)\|\Omega\|_\omega} g^{s\bar{r}}(i\partial \bar{\partial} \omega)_{s\bar{r}kj} - \frac{1}{2(n-1)\|\Omega\|_\omega} g^{q\bar{p}}g^{s\bar{r}}(T \land \bar{T})_{s\bar{r}pqkj} \\
- \frac{i}{6(n-1)(n-2)\|\Omega\|_\omega} \text{Tr}(T \land \bar{T}) g_{kj}.
\]
(3.38)

Next, we compute the components of the terms of the right-hand side. By substituting the relation of Ricci curvatures for conformally balanced metrics (3.27) into identity (A.19) for components of \(i\partial \bar{\partial} \omega\) from the appendix, we obtain
\[
g^{s\bar{r}}(i\partial \bar{\partial} \omega)_{s\bar{r}kj} = \tilde{R}_{kj} - g^{m\bar{l}}g^{s\bar{r}}T_{\bar{r}mi}\bar{T}_{\bar{s}\bar{k}j}.
\]
(3.39)

Substituting this identity together with (A.13) and (A.14) into (3.38) gives equation (3.29) for the evolution of the metric. Q.E.D.

As a consequence of the above calculation, we obtain a proof for case (ii) in Lemma 1.

**Corollary 1** Stationary points of the Anomaly flow \(\frac{d}{dt}(\|\Omega\|_\omega \omega^{n-1}) = i\partial \bar{\partial}(\omega^{n-2})\) with initial metric \(\omega_0\) satisfying \(d(\|\Omega\|_\omega \omega_0^{n-1}) = 0\) are Ricci-flat Kähler metrics.

**Proof:** The case \(n = 3\) was done in [59], so we assume \(n \geq 4\). Setting (3.29) to zero and taking the trace gives
\[
0 = -\tilde{R} + \frac{n}{2(n-2)}(|T|^2 - 2|\tau|^2) - \frac{1}{2}|T|^2 + 3|\tau|^2.
\]
(3.40)

By the definition of Ricci curvatures (3.26),
\[
\tilde{R} = g^{j\bar{k}}R_{p\bar{k}j} = -g^{j\bar{k}}\partial_j \partial_{\bar{k}} \log \det g_{\ell m},
\]
(3.41)

hence \(\tilde{R} = \Delta \log \|\Omega\|_\omega^2\). Simplifying, we obtain
\[
(n-2)\Delta \log \|\Omega\|_\omega^2 = |T|^2 + 2(n-3)|\tau|^2.
\]
(3.42)

By the maximum principle, we conclude that \(\log \|\Omega\|_\omega^2\) is constant and \(|T|^2 = 0\). It follows that \(\omega\) is Kähler with zero Ricci curvature. Q.E.D.

### 4 A complex Monge-Ampère flow

#### 4.1 The Anomaly flow and a scalar Monge-Ampère flow

Let \(X\) be a compact Kähler manifold with Kähler metric \(\hat{\chi} = i\hat{\chi}_{kj} dz^j \land d\bar{z}^k\). Given a potential function \(\varphi : X \to \mathbb{R}\), we will use the notation
\[
F[\varphi] = (\hat{\chi} + i\partial \bar{\partial} \varphi)^n.
\]
(4.1)
Let \( f \in C^\infty(X, \mathbb{R}) \) be a given function. Consider the Monge-Ampère flow of potentials
\[
\partial_t \varphi = e^{-f} F[\varphi], \quad \varphi(x, 0) = 0,
\]
subject to the plurisubharmonic condition
\[
\chi = \hat{\chi} + i \partial \bar{\partial} \varphi > 0.
\]

Assume further that \( X \) admits a nowhere holomorphic \((n, 0)\)-form \( \Omega \) and consider the flow (1.1). We claim that, with the function \( f \) chosen by
\[
e^{-f} = \frac{1}{(n-1)} \| \Omega \|_{\hat{\chi}}^{-2},
\]
the Monge-Ampère flow (4.2) is just the Anomaly flow (1.1) with initial data
\[
\| \Omega \|_{\omega_0}^{n-1} = \hat{\chi}^{n-1}.
\]
Indeed, let \( t \to \varphi(t) \) be the solution of the Monge-Ampère flow (4.2) and set
\[
\chi(t) = \hat{\chi} + i \partial \bar{\partial} \varphi(t) > 0, \quad \| \Omega \|_{\omega(t)}^{n-1} = \chi(t)^{n-1}.
\]
Then
\[
\partial_t(\| \Omega \|_{\omega}^{n-1}) = \partial_t \chi^{n-1} = (n-1) i \partial \bar{\partial} (\partial_t \varphi) \wedge \chi^{n-2}.
\]
Next, the above relation between \( \omega(t) \) and \( \chi(t) \) can be inverted: taking the determinants of both sides gives \( \| \Omega \|_{\omega}^{1/(n-1)} = \| \Omega \|_{\chi}^{2/(n-2)} \), and thus the relation can also be written as
\[
\omega^{n-2} = \| \Omega \|_{\chi}^2 \chi^{n-2}.
\]
It follows that
\[
i \partial \bar{\partial} \omega^{n-2} = i \partial \bar{\partial} (\| \Omega \|_{\chi}^{-2} \chi^{n-2}) = i \partial \bar{\partial} (\| \Omega \|_{\chi}^{-2}) \wedge \chi^{n-2}.
\]
It suffices then to show that
\[
(n-1) \partial_t \varphi = \| \Omega \|_{\chi}^{-2}
\]
since this would imply that \( \omega \) is a solution of the Anomaly flow (1.1), which is uniquely determined by the parabolicity of the flow established in Theorem 1. But by the definition of \( \varphi \), we have
\[
(n-1) \partial_t \varphi = (n-1) e^{-f} F[\varphi] = \frac{\det \hat{\chi}}{\Omega \Omega} \cdot \frac{\det (\hat{\chi} + i \partial \bar{\partial} \varphi)}{\det \hat{\chi}} = \frac{1}{\| \Omega \|_{\chi}^2}.
\]
This completes the proof of our claim.
4.2 Evolution identities

In this section, we compute the evolution of various basic quantities along the flow (1.6). Let us first establish notations.

We will denote $\chi_{kj} = \hat{\chi}_{kj} + \phi_{kj}$, and use upper indices $\chi_{kj}^i$ to denote the inverse matrix of $\chi_{kj}$. We will sometimes write $\dot{\phi}$ for $\partial_t \phi$. Subscripts on a function will denote spacial partial derivatives and $\hat{\nabla}$ and $\Delta_{\hat{\chi}} = \hat{\chi}^{pq} \hat{\nabla}_p \hat{\nabla}_q$ will denote covariant derivatives with respect to the background Kähler metric $\hat{\chi}$. We use the Chern connection defined as usual by $\hat{\nabla}_k V^p = \partial_k V^p$, $\hat{\nabla}_k V^p = \hat{\chi}^{pq} \partial_k (\hat{\chi}_{qi} V^i)$ for any section $V$ of $T^{1,0} X$, and the curvature convention

$$[\hat{\nabla}_j, \hat{\nabla}_k]W_i = -\hat{R}_{kj}^{\quad pq} W_p,$$

for any section $W$ of $(T^{1,0} X)^*$. We introduce the linearized operator

$$L = e^{-f} F \chi_{kj}^i \hat{\nabla}_j \hat{\nabla}_k.$$

We will also use the relative endomorphism $h_{ij} = \hat{\chi}_{ij} \chi_{kj}$, which satisfies

$$\text{Tr } h = n + \Delta_{\hat{\chi}} \varphi, \quad \text{Tr } h^{-1} = \chi^{pq} \hat{\chi}_{pq}.$$

4.2.1 Evolution of the potential

We first compute

$$(\partial_t - L) \varphi = e^{-f} F - e^{-f} F \chi_{kj}^i \varphi_{jk}$$

$$= e^{-f} F - e^{-f} F \chi_{kj}^i (\chi_{jk} - \hat{\chi}_{jk}).$$

Therefore

$$(\partial_t - L) \varphi = -(n - 1) e^{-f} F + e^{-f} F \text{Tr } h^{-1}.$$

4.2.2 Evolution of the determinant

Next, differentiating the evolution equation in time gives

$$\partial_t (e^{-f} F) = L (e^{-f} F).$$

Expanding, we obtain

$$(\partial_t - L) F = e^f L (e^{-f} F) - 2 F e^{-f} \text{Re} \{\chi^{jk} f_j F_k\}.$$
4.2.3 Evolution of the cohomological representative

Differentiating equation (1.6) once gives

\[ \partial_t \partial_\rho \varphi = e^{-f} F^{k\bar{j}} \hat{\nabla}_q \varphi_{jk} - e^{-f} F \partial_\rho f. \]  

(4.19)

We introduce the notation

\[ F^{k\bar{j},r\bar{s}} = F^{r\bar{s}} \chi^{k\bar{j}} - F^{k\bar{j}} \chi^{r\bar{s}}, \]  

(4.20)

so that

\[ \hat{\nabla}_q (F \chi^{k\bar{j}}) = F^{k\bar{j},r\bar{s}} \hat{\nabla}_q \varphi_{sr}. \]  

(4.21)

We note that unlike when \( F \) is concave, the quantity \(-F^{k\bar{j},r\bar{s}} M_{jk} \overline{M_{rs}}\) for a Hermitian tensor \( M_{ab} \) does not have a sign. Differentiating the flow twice, which amounts to differentiating (4.19) once more gives

\[ \partial_t \varphi_{pq} = e^{-f} F^{k\bar{j}} \hat{\nabla}_q \hat{\nabla}_p \varphi_{jk} + e^{-f} F^{k\bar{j},r\bar{s}} \hat{\nabla}_q \hat{\nabla}_p \varphi_{sr} - e^{-f} F \partial_q \partial_p f + e^{-f} F \partial_q f \partial_p f. \]

(4.22)

Commuting derivatives

\[ e^{-f} F^{k\bar{j}} \hat{\nabla}_q \hat{\nabla}_p \varphi_{jk} = e^{-f} F^{k\bar{j}} \hat{\nabla}_k \hat{\nabla}_j \varphi_{pq} + e^{-f} F \chi^{k\bar{j}} \hat{R}_{j\bar{k}p} \partial_{\bar{a}} \varphi_{aq} + e^{-f} F \chi^{k\bar{j}} \hat{R}_{j\bar{k}p} \partial_{\bar{a}} \varphi_{a\bar{k}}. \]

(4.23)

Thus we obtain the evolution of \( \chi_{\bar{p}q} = \hat{\chi}_{\bar{p}q} + \varphi_{\bar{p}q} \)

\[ (\partial_t - L) \chi_{\bar{p}q} = e^{-f} F^{k\bar{j}} \hat{R}_{j\bar{k}p} \partial_{\bar{a}} \varphi_{aq} + e^{-f} F \chi^{k\bar{j}} \hat{R}_{j\bar{k}p} \partial_{\bar{a}} \varphi_{a\bar{k}} + e^{-f} F^{k\bar{j},r\bar{s}} \hat{\nabla}_q \chi_{jk} \hat{\nabla}_p \chi_{sr} - e^{-f} F \partial_q \hat{\nabla}_p \chi_{jk} \hat{f}_p \\
- e^{-f} F \hat{f}_{pq} + e^{-f} F \hat{f}_p \hat{f}_q. \]

(4.24)

Since \( L \) involves covariant derivatives with respect to the background \( \hat{\chi} \), we may take the trace with respect to \( \hat{\chi} \) and derive

\[ (\partial_t - L) \text{Tr} h = e^{-f} F^{k\bar{j}} \hat{R}_{j\bar{k}p} \partial_{\bar{a}} \varphi_{ap} + e^{-f} F \chi^{k\bar{j}} \hat{R}_{j\bar{k}p} \partial_{a\bar{k}} \varphi \]

\[ + e^{-f} F^{k\bar{j},r\bar{s}} \chi^{q\bar{p}} \hat{\nabla}_q \hat{\nabla}_r \chi_{jk} \hat{\nabla}_p \chi_{sr} - 2e^{-f} \text{Re} \{ \chi^{q\bar{p}} F_{qfp} \} - e^{-f} F \hat{\chi}^{q\bar{p}} \hat{f}_{qfp} - e^{-f} F \hat{\chi}^{q\bar{p}} \hat{f}_p \hat{f}_q. \]

(4.25)

Writing \( \chi_{\bar{k}k} = \hat{\chi}_{\bar{k}k} + \varphi_{\bar{k}k} \) and \( \hat{R}^{p \bar{a}}_{q \bar{a}} = \hat{R} \),

\[ (\partial_t - L) \text{Tr} h = e^{-f} F^{k\bar{j}} \hat{R}_{j\bar{k}p} \chi_{ap} + e^{-f} F \hat{R} + e^{-f} F^{k\bar{j},r\bar{s}} \chi^{q\bar{p}} \hat{\nabla}_q \chi_{jk} \hat{\nabla}_r \chi_{sr} - 2e^{-f} \text{Re} \{ \chi^{q\bar{p}} F_{qfp} \} - e^{-f} F \hat{\chi}^{q\bar{p}} \hat{f}_{qfp} - e^{-f} F \hat{\chi}^{q\bar{p}} \hat{f}_p \hat{f}_q. \]

(4.26)
4.3 Estimate of the speed

Differentiating the equation in time, we obtain

\[(\partial_t - L)\dot{\varphi} = 0.\]  \hspace{1cm} (4.27)

By the maximum principle, we have the bound

\[\inf_X \dot{\varphi}(0) \leq \dot{\varphi} \leq \sup_X \dot{\varphi}(0).\]  \hspace{1cm} (4.28)

Since \(\dot{\varphi}(0) = e^{-f}\), it follows that \(e^{-f}F = \dot{\varphi} \geq \inf_X e^{-f}\). As a consequence, we obtain

**Lemma 3** Let \(\varphi\) be a solution to (1.6) on \(X \times [0,T]\) satisfying the positivity condition (1.7). There exists a constants \(C > 0\) and \(\delta > 0\) depending only on \((X,\hat{\chi})\) and \(f\) such that

\[|\partial_t \varphi| \leq C, \; \delta \leq e^{-f}F[\varphi] \leq C, \; (\sup_X \varphi - \inf_X \varphi)(t) \leq C.\]  \hspace{1cm} (4.29)

**Proof:** It only remains to establish the oscillation estimate. This follows from applying Yau’s \(C^0\) estimate \([79]\) to the equation

\[(\hat{\chi} + i\partial\bar{\partial}\varphi)^n = (\partial_t \varphi) e^{f(\hat{\chi})^n}\]  \hspace{1cm} (4.30)

at each fixed time. Q.E.D.

4.4 Second order estimate

**Lemma 4** Let \(\varphi\) be a solution to (1.6) on \(X \times [0,T]\) satisfying the positivity condition (1.7). There exists a constants \(C > 0\) and \(A > 0\) depending only on \(X, \hat{\chi}\) and \(f\) such that

\[\Delta \hat{\chi} \varphi(x,t) \leq Ce^{A(\hat{\varphi}(x,t) - \inf_{X \times [0,T]} \hat{\varphi})},\]  \hspace{1cm} (4.31)

where

\[\hat{\varphi}(x,t) = \varphi - \frac{1}{V} \int_X \varphi \hat{\chi}^n, \; V = \int_X \hat{\chi}^n.\]  \hspace{1cm} (4.32)

We will apply the maximum principle to the following test function defined on \(X \times [0,T]\),

\[G(x,t) = \log \text{Tr}\ h - A\hat{\varphi} + \frac{B}{2}F^2,\]  \hspace{1cm} (4.33)

where \(A, B > 0\) are constants to be determined. Before preceding, we note the extra term \(\frac{B}{2}F^2\), which does not appear in the standard test function used in the study of the complex Monge-Ampère type equations (see for example, \([57, 79]\) or the Kähler-Ricci flow \([10]\). Indeed, this is the main innovation of this test function, and we use it to overcome the difficult terms caused by the lack of concavity of the Monge-Ampère flow, as well as the cross terms involving the conformal factor \(e^{-f}\).
We compute the evolution of the test function.

\[
(\partial_t - L)G = \frac{1}{\text{Tr} h} (\partial_t - L) \text{Tr} h + \frac{e^{-f} F}{\left(\text{Tr} h\right)^2} \chi^{jk} (\partial_j \text{Tr} h)(\partial_k \text{Tr} h) - A(\partial_t - L) \varphi \\
+ \frac{A}{V} \int_X \partial_t \varphi \hat{\chi}^n + BF(\partial_t - L)F - Be^{-f} F \hat{\chi}^{jk} F_j F_k. \tag{4.34}
\]

Substituting (4.16), (4.18) and (4.26)

\[
(\partial_t - L)G = \frac{1}{\text{Tr} h} \left\{ -e^{-f} F \chi^{kj} R_{jk} \hat{\rho} \chi_{ap} + e^{-f} F \hat{\rho} - e^{-f} F \hat{\chi}^{jk} f_{kj} + e^{-f} F \hat{\chi}^{jk} f_j f_k \\
+ e^{-f} F^{kj} \chi^{r_1 s_1} \hat{\nabla} q \chi_{r_1} \hat{\nabla} p \chi_{s_1} - 2e^{-f} \text{Re}\{\hat{\chi}^{jk} f_j f_k\} \right\} \\
+ A(n - 1)e^{-f} F - Ae^{-f} F \text{Tr} h^{-1} + \frac{A}{V} \int_X e^{-f} F \hat{\chi}^n \\
+ BF^3 \chi^{jk} (e^{-f})_{kj} - 2BF^2 e^{-f} \text{Re}\{\chi^{jk} f_j f_k\} - Be^{-f} F \hat{\chi}^{jk} F_j F_k. \tag{4.35}
\]

Using Lemma 3, we estimate

\[
\frac{1}{\text{Tr} h} \left\{ -e^{-f} F \chi^{kj} R_{jk} \hat{\rho} \chi_{ap} + e^{-f} F \hat{\rho} - e^{-f} F \hat{\chi}^{jk} f_{kj} + e^{-f} F \hat{\chi}^{jk} f_j f_k \right\} \\
\leq \frac{C}{\text{Tr} h} \left\{ (\text{Tr} h^{-1})(\text{Tr} h) + 1 \right\} \leq CT h^{-1}, \tag{4.36}
\]

where the constant $C$ depends on $e^{-f}$ and the curvature of the background metric $\hat{\chi}$. Next, by (4.20) we have

\[
e^{-f} F^{kj} \chi^{r_1 s_1} \hat{\nabla} q \chi_{r_1} \hat{\nabla} p \chi_{s_1} = e^{-f} F \chi^{r_1 s_1} \chi^{kj} \hat{\nabla} q \chi_{r_1} \hat{\nabla} p \chi_{s_1} - e^{-f} F \chi^{r_1} \chi^{k_1} \chi^{s_1} \hat{\nabla} q \chi_{r_1} \hat{\nabla} p \chi_{s_1} \\
= \frac{e^{-f} F}{\chi^{pq}} F_p F_q - e^{-f} F \chi^{r_1} \chi^{k_1} \hat{\nabla} q \chi_{r_1} \hat{\nabla} p \chi_{s_1}. \tag{4.37}
\]

The term involving $\hat{\chi}^{pq} F_p F_q$ is the new bad term compared to standard arguments, and it is the reason for the addition of $\frac{B^2}{2} F^2$ to the test function $G$. The main inequality becomes

\[
(\partial_t - L)G \leq \frac{1}{\text{Tr} h} \left\{ \frac{e^{-f} F}{\chi^{jk}} F_j F_k - 2e^{-f} \text{Re}\{\hat{\chi}^{jk} F_j F_k\} \right\} \\
+ \frac{e^{-f} F}{\text{Tr} h} \left\{ \frac{1}{\text{Tr} h} \chi^{jk}(\partial_j \text{Tr} h)(\partial_k \text{Tr} h) - \hat{\chi}^{pq} \chi^{r_1 s_1} \hat{\nabla} p \chi_{r_1} \hat{\nabla} p \chi_{s_1} \right\} \\
- 2BF^2 e^{-f} \text{Re}\{\chi^{jk} f_j f_k\} - Be^{-f} F \hat{\chi}^{jk} F_j F_k \\
+ \{C(1 + B) - Ae^{-f} F \} \text{Tr} h^{-1} + C(A + B + 1). \tag{4.38}
\]

By the inequality of Yau and Aubin [79, 3],

\[
\frac{1}{\text{Tr} h} \chi^{jk} (\partial_j \text{Tr} h)(\partial_k \text{Tr} h) - \hat{\chi}^{pq} \chi^{r_1 s_1} \hat{\nabla} p \chi_{r_1} \hat{\nabla} p \chi_{s_1} \leq 0. \tag{4.39}
\]
Next, we estimate
\[
\frac{1}{\text{Tr } h} \left\{ \frac{e^{-f}}{F} \hat{\chi}^{jk} F_j F_k - 2 e^{-f} \text{Re} \{ \hat{\chi}^{jk} F_j f_k \} \right\} \leq \frac{1}{\text{Tr } h} \left\{ \frac{2 e^{-f}}{F} \hat{\chi}^{jk} F_j F_k + e^{-f} F \hat{\chi}^{jk} f_j f_k \right\} \leq 2 \frac{e^{-f}}{F} \hat{\chi}^{jk} F_j F_k + C \text{Tr } h^{-1},
\]
and
\[-2BF^2 e^{-f} \text{Re} \{ \hat{\chi}^{jk} f_j F_k \} \leq \hat{\chi}^{jk} F_j F_k + CB^2 \text{Tr } h^{-1}.
\]
Applying these estimates in the main inequality yields
\[
(\partial_t - L)G \leq (1 + 2 e^{-f} F^{-1} - Be^{-f} F) \hat{\chi}^{jk} F_j F_k + \{C(1 + B + B^2) - Ae^{-f} F\} \text{Tr } h^{-1} + C(A + B + 1).
\]
By Lemma 3, we have the uniform lower bound $e^{-f} F \geq \delta > 0$. Thus may choose $B \gg 1$ such that $(1 + 2 e^{-f} F^{-1} - Be^{-f} F) \leq 0$. Next, we may choose $A \gg B \gg 1$ such that $(C(1 + B + B^2) - Ae^{-f} F) \leq -1$. Then if $G$ attains a maximum on $X \times [0, T]$ at a point $(x_0, t_0)$ with $t_0 > 0$, by the maximum principle we have the inequality
\[
0 \leq (\partial_t - L)G \leq - \text{Tr } h^{-1} + C(A + B + 1).
\]
It follows that the eigenvalues of $h$ are bounded below at $(x_0, t_0)$. The product of the eigenvalues of $h$ is given by $F$, which is uniformly bounded along the flow. Thus the eigenvalues of $h$ are bounded above at $(x, t_0)$, and so $\text{Tr } h \leq C$ at $(x_0, t_0)$. Therefore
\[
G(x, t) \leq G(x_0, t_0) \leq C - A \inf_{X \times [0, T]} \hat{\varphi}.
\]
If $G$ attains a maximum at $t_0 = 0$, we have already have $G(x, t) \leq C$. Therefore
\[
\log \text{Tr } h \leq C + A \left( \hat{\varphi} - \inf_{X \times [0, T]} \hat{\varphi} \right)
\]
along the flow. Q.E.D.

**Corollary 2** Let $\varphi$ be a solution to (1.6) on $X \times [0, T]$ satisfying the positivity condition (1.7). There exists a constant $C > 0$ depending only on $X$, $\hat{\chi}$ and $f$ such that
\[
C^{-1} \hat{\chi}_{kj}(z) \leq \chi_{kj}(z, t) \leq C \hat{\chi}_{kj}(z).
\]

**Proof:** We first use Lemma 4 to obtain an upper bound on $\Delta \hat{\chi} \varphi$ independent of time. Indeed, the oscillation of $(\sup_X \hat{\varphi} - \inf_X \hat{\varphi})(t)$ is bounded at each fixed time by Lemma 3. Since $\int \hat{\varphi} \hat{\chi}^n = 0$, we must have that $(\sup_X \hat{\varphi})(t) \geq 0$ and $(\inf_X \hat{\varphi})(t) \leq 0$, hence $\hat{\varphi}$ is uniformly bounded along the flow. By Lemma 4,
\[
\text{Tr } h \leq C.
\]
This gives the upper bound of $\chi$. For the lower bound, we use that the determinant $F[\varphi] = \chi^n/\hat{\chi}^n$ is uniformly bounded below by Lemma 3. Q.E.D.
4.5 Higher order estimates

In this section, we will follow the main line of the argument given in Tsai [75] to prove the higher order estimates. We note that we cannot directly apply a standard theorem to our equation $\partial_t \varphi = e^{-f}F[\varphi]$, $F[\varphi] = \det(\chi_{kj} + \varphi_{kj})/\det \chi_{kj}$, since $F[\varphi]$ is not concave in the second derivatives of $\varphi$. However, $\log F[\varphi]$ is concave as a function of $\varphi_{kj}$, and to introduce the logarithm we split the argument into several steps to treat space and time separately. First, we fix some notation.

We will work locally on a ball $B_r(0)$ and cylinder $Q = B_r \times (T_0, T)$. For functions $v : B_r \to \mathbb{R}$ and $u : Q \to \mathbb{R}$, we define

$$
\|v\|_{C^\alpha(B_r)} = \|v\|_{L^\infty(B_r)} + \sup_{x \not= y \in B_r} \frac{|v(x) - v(y)|}{|x - y|^\alpha},
$$

and

$$
\|u\|_{C^\alpha,\alpha/2(Q)} = \|u\|_{L^\infty(Q)} + \sup_{(x,t) \not= (y,s) \in Q} \frac{|u(x,t) - u(y,s)|}{(|x - y| + |t - s|^{1/2})^\alpha}.
$$

The main estimate of this section is the following.

**Lemma 5** Let $\varphi$ be a solution to (1.6) on $X \times [0, \epsilon)$ satisfying the positivity condition (1.7). Let $B_1$ be a coordinate chart on $X$ such that $B_1 \subset \mathbb{R}^n$ is a unit ball. Then there exists $0 < \alpha < 1$ and $C > 0$, depending on $\hat{\chi}$, $f$, and $\epsilon$, such that on $Q = B_{1/2} \times [\frac{\epsilon}{2}, \epsilon)$,

$$
\|\partial_t \varphi\|_{C^\alpha,\alpha/2(Q)} + \|\varphi_{kj}\|_{C^\alpha,\alpha/2(Q)} \leq C.
$$

We first notice that the speed function of the flow satisfies the equation

$$
\partial_t(e^{-f}F) = L(e^{-f}F) = e^{-f}F\chi^{jk}(e^{-f}F)_{kj}.
$$

By Lemma 3 and Corollary 2, we see that this is a uniformly parabolic linear equation with bounded coefficients on $X \times [0, \epsilon)$. It follows from the Krylov-Safanov Harnack inequality [45] that

$$
\|e^{-f}F\|_{C^\alpha,\alpha/2(Q)} \leq C,
$$

for some $\alpha \in (0, 1)$. This implies that

$$
\|F\|_{C^\alpha,\alpha/2(Q)} \leq C, \quad \|\partial_t \varphi\|_{C^\alpha,\alpha/2(Q)} \leq C.
$$

For each fixed $t \in [\frac{\epsilon}{2}, \epsilon)$, by covering the compact manifold $X$ we can control the Hölder norm of $F(\cdot, t)$ on all of $X$, and we have the space norm estimate

$$
\|F(\cdot, t)\|_{C^\alpha(B_1)} \leq C,
$$

where $C$ is a constant independent of $t \in [\frac{\epsilon}{2}, \epsilon)$. Recall that $F[\varphi] = \det(\chi_{kj} + \varphi_{kj})/\det \chi_{kj}$, so after possibly taking a smaller $0 < \alpha < 1$, we may apply the estimates in Tosatti-Weinkove-Wang-Yang [74] to establish

$$
\|\varphi_{kj}(\cdot, t)\|_{C^\alpha(B_1)} \leq C
$$

(4.55)
where \( C \) is independent of \( t \in [\frac{\epsilon}{2}, \epsilon) \). Therefore, in view of a standard lemma [44] in parabolic Hölder spaces which allows us to treat time and space separately, it remains to show that

\[
\sup_{s, t \in [\frac{\epsilon}{2}, \epsilon], s \neq t} \frac{|\varphi_{kj}(z, s) - \varphi_{kj}(z, t)|}{|s - t|^{\frac{\alpha}{2}}} \leq C, \tag{4.56}
\]

for some \( C \) independent of \( z \in B_{1/2} \). Following Tsai [75], for \( 0 < h < \frac{\epsilon}{2} \) we consider the function

\[
U_{\lambda, h}(z, t) = \lambda \varphi(z, t) + (1 - \lambda) \varphi(z, t + h) \text{ with } 0 \leq \lambda \leq 1, \tag{4.57}
\]

defined on \( B_1 \times [\frac{\epsilon}{2}, \epsilon - h) \). Compute

\[
\log F(z, t) - \log F(z, t + h) = \int_0^1 \frac{d}{d\lambda} \log \det \left( \partial_{\lambda} \partial_k U_{\lambda, h}(z, t) \right) d\lambda \tag{4.58}
\]

where \( \chi^{jk}_{\lambda, h}(z, t) \) is the inverse of \( (\partial_{\lambda} \partial_k U_{\lambda, h}) > 0 \). Denote

\[
a_{h}^{jk}(z, t) = \int_0^1 \chi^{jk}_{\lambda, h}(z, t) d\lambda. \tag{4.59}
\]

It follows from Corollary 2 and the space norm estimate (4.55) that \( a_{h}^{jk} \) is uniform elliptic and satisfies

\[
\|a_{h}^{jk}(\cdot, t)\|_{C^\alpha(B_1)} \leq C \tag{4.60}
\]

for some constant \( C \) independent of \( h \) and \( t \in [\frac{\epsilon}{2}, \epsilon - h) \). Denote

\[
\varphi_h(z, t) = \frac{\varphi(z, t) - \varphi(z, t + h)}{|h|^{\frac{\alpha}{2}}} \tag{4.61}
\]

with \((z, t) \in B_1 \times [\frac{\epsilon}{2}, \epsilon - h)\). Then, \( \varphi_h \) satisfies the equation

\[
a_{h}^{jk}(z, t) \partial_j \partial_k \varphi_h(z, t) = g_h(z, t) \tag{4.62}
\]

with

\[
g_h(z, t) = \frac{\log F(z, t) - \log F(z, t + h)}{|h|^{\frac{\alpha}{2}}}. \tag{4.63}
\]

As we discussed above, at a fixed time we have that \( a_{h}^{jk}(\cdot, t) \) is uniformly elliptic and Hölder continuous, with constants independent of time \( t \) and parameter \( h \). We need the following lemma to estimate the Hölder norm of \( g_h \).
Lemma 6 The function \( g_h(z, t) \) satisfies
\[
\| g_h(\cdot, t) \|_{C^{4}(B_1)} \leq C, \tag{4.64}
\]
where \( C \) is independent of \( h \) and \( t \in [\frac{\epsilon}{2}, \epsilon - h) \).

Given this lemma, we can apply the elliptic Schauder estimate to equation (4.62) at a fixed time \( t \) and obtain
\[
\| \varphi_h(\cdot, t) \|_{C^2(B_{1/2})} \leq C \left( \| g_h(\cdot, t) \|_{C^{4}(B_1)} + \| \varphi_h(\cdot, t) \|_{L^\infty(B_1)} \right), \tag{4.65}
\]
with \( C \) independent of \( t \) and \( h \). By the estimate of the speed function in Lemma 3, we have \( \| \varphi_h(\cdot, t) \|_{L^\infty(B_1)} \leq C \) with \( C \) independent of \( t \) and \( h \). This together with Lemma 6 implies
\[
\sup_{z \in B_{1/2}} \frac{|\varphi_{kj}(z, t) - \varphi_{kj}(z, t + h)|}{|h|^{\frac{4}{4}}} \leq C, \tag{4.66}
\]
for all \( h > 0 \) small, where \( C \) is independent of \( h \) and \( t \in [\frac{\epsilon}{2}, \epsilon - h) \). Hence, we complete the proof of Lemma 5.

Proof of Lemma 6: Since \( F[\varphi] \) is uniformly bounded away from zero along the flow, we know that by (4.53) that
\[
\| \log F \|_{C^{\alpha/2}(B_1 \times [\frac{\epsilon}{2}, \epsilon])} \leq C.\tag{4.67}
\]
This implies that
\[
\frac{1}{|h|^{\frac{4}{4}}} | \log F(x, t) - \log F(x, t + h) - \log F(y, t) + \log F(y, t + h) | \leq C |x - y|^{\frac{4}{4}}. \tag{4.68}
\]
Indeed, if \( |h| \leq |x - y| \), then
\[
\frac{1}{|h|^{\frac{4}{4}}} \left\{ | \log F(x, t) - \log F(x, t + h) | + | \log F(y, t) - \log F(y, t + h) | \right\}
\leq C |h|^{\frac{4}{4}} \leq C |x - y|^{\frac{4}{4}}. \tag{4.69}
\]
On the other hand, if \( |h| \geq |x - y| \), then
\[
\frac{1}{|h|^{\frac{4}{4}}} \left\{ | \log F(x, t) - \log F(y, t) | + | \log F(x, t + h) - \log F(y, t + h) | \right\}
\leq C \frac{|x - y|^{\frac{4}{4}}}{|h|^{\frac{4}{4}}} |x - y|^{\frac{4}{4}} \leq C |x - y|^{\frac{4}{4}}. \tag{4.70}
\]
Combining these two cases and using the triangle inequality proves (4.68). Thus for any \( t \in [\frac{\epsilon}{2}, \epsilon - h) \) there holds
\[
\left\| \frac{\log F(\cdot, t) - \log F(\cdot, t + h)}{|h|^{\frac{4}{4}}} \right\|_{C^{\alpha/4}(B_1)} \leq C, \tag{4.71}
\]
where \( C \) is independent of \( t \) and \( h \). This completes the proof of the lemma. Q.E.D.
5 Proof of Theorem 2 and Theorem 3

In this section, we prove the long time existence and convergence of the complex Monge-Ampère flow, which completes the proof for Theorem 3. Then, from the discussion in §4.1, we also obtain the proof of Theorem 2.

5.1 Long-time existence

In this section, we show that the solution \( \varphi \) and its normalization \( \tilde{\varphi} \) are smooth and exist for all time.

Since the flow is parabolic, a solution exists on a maximal time interval \([0, T)\) with \( T > 0 \). Differentiating the equation, we obtain

\[
\partial_t \partial_p \varphi = e^{-f} F_{k^j} \hat{\nabla}_k \hat{\nabla}_j \partial_p \varphi - e^{-f} F \partial_p \varphi. \tag{5.1}
\]

By Corollary 2 and Lemma 5, this is a uniformly parabolic equation for \( \partial_p \varphi \) with Hölder continuous coefficients. By parabolic Schauder estimates (e.g. [44]), we obtain the \( C^{2+\alpha, 1+\alpha/2} \) estimate for \( \partial_p \varphi \). A bootstrapping argument gives estimates on all derivatives of \( \varphi \).

If \( T < \infty \), then our estimates allow us to take a subsequential limit and then extend the flow using the short-time existence theorem. It follows that a smooth solution exists on \([0, \infty)\). We already noted in the proof of Corollary 2 that \( \tilde{\varphi} \) is uniformly bounded, and the above argument shows that \( \tilde{\varphi} \) and all its derivatives are bounded along the flow.

5.2 Dilaton functional

We now return to the Anomaly flow \( \partial_t(\|\Omega\|_{\omega}^{n-1}) = i\partial\bar{\partial}(\omega^{n-2}) \) with ansatz \( \|\Omega\|_{\omega}^{n-1} = \chi^{n-1} \). Recall that this ansatz is preserved and the conformally balanced metric \( \omega(t) \) is given by the expression

\[
\omega = \|\Omega\|_{\chi}^{-2/(n-2)} \chi, \quad \chi(t) = \hat{\chi} + i\partial\bar{\partial} \varphi(t), \tag{5.2}
\]

where \( \varphi \) solves the Monge-Ampère flow (1.6). By the previous section, the Anomaly flow for the Hermitian metric \( \omega(t) \) exists for all time. Let

\[
M(\omega) = \int_X \|\Omega\|_{\omega}^n \tag{5.3}
\]

denote the dilaton functional. This functional was introduced by Garcia-Fernandez, Rubio, Shahbazi and Tipler [33] to formulate a variational principle for the Hull-Strominger system on holomorphic Courant algebroids. Here we compute its evolution along the Anomaly flow.

**Lemma 7** Let \( \omega(t) \) be a solution to the Anomaly flow (1.1) with \( n \geq 3 \). Then the dilaton functional evolves by

\[
\frac{d}{dt} M(\omega(t)) = \frac{1}{2} \frac{1}{(n-1)(n-2)} \int_X \{ |T|^2 - 2|\tau|^2 \} \omega^n. \tag{5.4}
\]
Proof: Wedging the equation \( \frac{d}{dt}(\|\Omega\|_{\omega}^{n-1}) = i\partial \bar{\partial}(\omega^{n-2}) \) with \( \omega \) gives

\[
\left( \frac{d}{dt}\|\Omega\|_{\omega} \right) \omega^n + (n-1)\|\Omega\|_{\omega}^{n-1} \wedge \dot{\omega} = i\partial \bar{\partial}\omega^{n-2} \wedge \omega. \tag{5.5}
\]

By definition \( \|\Omega\|_{\omega}^2 = \Omega \bar{\Omega}(\det g)^{-1} \), and we have

\[
\left( \frac{d}{dt}\|\Omega\|_{\omega} \right) \omega^n = -\frac{n}{2}\|\Omega\|_{\omega}^{n-1} \wedge \dot{\omega}. \tag{5.6}
\]

Therefore

\[
\frac{(n-2)}{2}\|\Omega\|_{\omega}^{n-1} \wedge \dot{\omega} = i\partial \bar{\partial}\omega^{n-2} \wedge \omega. \tag{5.7}
\]

It follows that

\[
\frac{d}{dt}(\|\Omega\|\omega^n) = \frac{d}{dt}(\|\Omega\|_{\omega}^{n-1}) \wedge \omega + \|\Omega\|_{\omega}^{n-1} \wedge \dot{\omega} = \left( \frac{n}{n-2} \right) i\partial \bar{\partial}\omega^{n-2} \wedge \omega. \tag{5.8}
\]

We now use Stokes theorem to obtain

\[
\frac{d}{dt}M(t) = \int_X \frac{d}{dt}(\|\Omega\|\omega^n) = -\frac{n}{n-2} \int_X i\partial\omega^{n-2} \wedge \bar{\partial}\omega. \tag{5.9}
\]

Hence

\[
\frac{d}{dt}M(t) = -n \int_X i\partial\omega \wedge \bar{\partial}\omega \wedge \omega^{n-3}. \tag{5.10}
\]

Applying identity (A.15) from the appendix with \( T = i\partial\omega \) and \( \bar{T} = -i\bar{\partial}\omega \), we may rewrite the integral as

\[
-n \int_X i\partial\omega \wedge \bar{\partial}\omega \wedge \omega^{n-3} = \frac{1}{2(n-1)(n-2)} \int_X \{|T|^2 - 2|\tau|^2\} \omega^n. \tag{5.11}
\]

This completes the proof of the lemma. Q.E.D.

Lemma 8 Let \( \omega(t) \) be a solution to the Anomaly flow (1.1) with \( n \geq 3 \). Further assume that the solution \( \omega(t) \) is conformally Kähler. Then the dilaton functional evolves by

\[
\frac{d}{dt}M(\omega(t)) = -\frac{1}{2(n-1)} \int_X |T|^2 \omega^n. \tag{5.12}
\]

In particular, the dilaton functional is monotone decreasing along the flow.

Proof: Fix a time \( t \), and let \( \omega(t) = e^{\psi} \omega \) with \( d\dot{\omega} = 0 \). The components of the torsion of \( \omega = ig_{kj}dz^j \wedge dz^k \) are given by

\[
T_{kj} = \partial_j \psi g_{k\ell} - \partial_k \psi g_{j\ell}, \quad T_\ell = -(n-1)\partial_\ell \psi. \tag{5.13}
\]

Computing the norms of these torsion tensors gives

\[
|T|^2 = 2(n-1)|\nabla \psi|^2, \quad |\tau|^2 = (n-1)^2|\nabla \psi|^2. \tag{5.14}
\]

Therefore \( 2|\tau|^2 = (n-1)|T|^2 \) in this case, and the identity follows from Lemma 7. Q.E.D.
Lemma 9 Let \( \omega(t) \) be a solution on \([0, \infty)\) to the Anomaly flow (1.1) with \( n \geq 3 \) with initial data satisfying

\[
\| \Omega \|_{\omega(0)} \omega(0)^{n-1} = \hat{\chi}^{n-1}, \tag{5.14}
\]

where \( \hat{\chi} \) is a Kähler metric. Then \( M(\omega(t)) \) is monotonically decreasing and furthermore \( \frac{d}{dt} M(\omega(t)) \to 0 \) as \( t \to \infty \).

Proof: By the formula for the ansatz (4.8), we have \( \omega = \| \Omega \|_{\hat{\chi}}^{-2/(n-2)} \chi \), and the metric is conformally Kähler \( \omega(t) = e^{\psi(t)} \chi(t) \) with

\[
\psi = \frac{1}{(n-2)} \log \| \Omega \|_{\chi}^{-2}. \tag{5.15}
\]

Furthermore

\[
|T|^2 = 2(n-1)|\nabla \psi|^2 = \frac{2(n-1)}{(n-2)^2} \| \Omega \|_{\chi}^{-2/2} \| \nabla \log \| \Omega \|_{\chi}^{-2} \|_{\chi}^2. \tag{5.16}
\]

Also,

\[
\omega^n = \| \Omega \|_{\chi}^{-2/n/(n-2)} \chi^n = \| \Omega \|_{\chi}^{-2/n/(n-2)} \| \Omega \|_{\chi}^{-2} \| \chi^n. \tag{5.17}
\]

We may apply Lemma 8 and obtain that \( M(t) \) is monotonically decreasing along the flow. Recall that \( (n-1) \dot{\varphi} = \| \Omega \|_{\chi}^{-2} \) gives the flow of the potential. Then

\[
\frac{d}{dt} M(t) = -\frac{1}{(n-2)^2} \int_X \left( \| \Omega \|_{\chi}^{-2} \right)^{2n-3} \| \nabla \log \| \Omega \|_{\chi}^{-2} \|_{\chi}^2 \| \Omega \|_{\chi}^2 \chi^n. \tag{5.18}
\]

We compute

\[
\frac{d^2}{dt^2} M(t) = -\frac{1}{(n-2)^2} \left\{ \frac{1}{n-2} \int_X \varphi \frac{(n-3)}{n-2} \partial_{\ell} \varphi | \nabla \varphi |_{\chi}^2 \| \Omega \|_{\chi}^2 \chi^n
- \int_X \varphi \frac{1}{n-2} \chi^{ij} \partial_j \varphi \partial_k \varphi \varphi_{ij} \| \Omega \|_{\chi}^2 \chi^n
+ \int_X \varphi \frac{1}{n-2} \chi^{jk} \left( \partial_j \partial_k \varphi \partial_{\ell} \varphi + \partial_j \varphi \partial_k \partial_{\ell} \varphi \right) \| \Omega \|_{\chi}^2 \chi^n \right\}. \tag{5.19}
\]

By our estimates, \( \varphi \) and \( \chi \) are uniformly bounded above and away from zero, and all space-time derivatives of \( \varphi \) are bounded. \( M(t) \) is uniformly bounded, monotone decreasing, and \( \frac{d^2}{dt^2} M(t) \) is uniformly bounded. It follows that \( \frac{d}{dt} M(t) \to 0 \) as \( t \to 0 \). Q.E.D.
5.3 Convergence

It remains only to show the convergence of the Anomaly flow (1.1), which we shall do, using the dilaton functional. Suppose there exists a sequence of times $t_j \to \infty$ such that $\omega(t_j)$ does not converge to $\omega_\infty$ as given in the theorem. By our estimates and the Arzela-Ascoli theorem, upon taking a subsequence we have that $\omega(t_j) \to \omega' \infty$ smoothly. Since $\frac{d}{dt} M(t) \to 0$, we conclude by Lemma 8 that

$$\int_X |T(\omega')|^2(\omega')^n = 0 \quad (5.20)$$

and hence $\omega' \infty$ is Kähler. By the ansatz (4.8),

$$\omega' \infty = \|\Omega\|_{\chi' \infty}^{-2/(n-2)} \chi' \infty, \quad (5.21)$$

and so $\|\Omega\|_{\chi' \infty}$ is constant. It follows that $\chi' \infty = \hat{\chi} + i \partial \bar{\partial} \varphi_\infty$ is the unique Kähler Ricci-flat metric in $[\hat{\chi}]$. Since

$$\int_X \|\Omega\|^2_{\chi' \infty} \frac{(\chi' \infty)^n}{n!} = \int_X i^n \Omega \wedge \bar{\Omega}, \quad (5.22)$$

the constant $\|\Omega\|_{\chi' \infty}$ is identified. Thus $\omega' \infty = \omega_\infty$ given in the theorem, and we have smooth convergence. Q.E.D.

We chose the argument using the dilaton functional in the belief that it will be useful in future studies of the Anomaly flow. For those readers who are only interested in the scalar equation (1.6), there are alternate ways to establish convergence of $\bar{\varphi}$. For example, the functional $\int_X e^{-f} F^2 \bar{\chi}^n$ is also monotone decreasing. Alternatively, convergence can be obtained by using the Li-Yau Harnack inequality as in [10, 35, 63]. In this case, we would use the Li-Yau Harnack estimate for the heat equation $u_t = g^{j k} u_{j k}$ on a Hermitian manifold $(M, g)$ proved by Gill [35], and apply it to the differentiated equation $\partial_t \bar{\varphi} = e^{-f} F \chi^{j k} \bar{\varphi}_{j k}$.

6 Further Developments

We conclude with some observations and open questions.

(a) The convergence theorems established in this paper for the flow (1.1) should be viewed as only the first step in a fuller theory yet to be developed. For example, we do not know at this moment whether the flow (1.1) will converge if the initial data is only known to satisfy $\|\Omega\|_{\omega_0} \omega_0^{n-1} \in [\hat{\chi}^{n-1}]$. We expect that it will not, unless $\|\Omega\|_{\omega_0} \omega_0^{n-1} = (\chi')^{n-1}$ for some Kähler form $\chi'$, in which case $[\chi'] = [\hat{\chi}]$. If so, whether the flow (1.1) converges with initial data $\|\Omega\|_{\omega_0} \omega_0^{n-1}$ may serve as a criterion for whether $\|\Omega\|_{\omega_0} \omega_0^{n-1}$ is the $(n-1)$-th power of a Kähler form.

In general, we actually expect the failure of convergence of the flow to provide important geometric information. As just stated above, this failure may be caused by the choice of
initial data, even within the \((n-1,n-1)\)-cohomology class \([\hat{\chi}^{n-1}]\) with \(\hat{\chi}\) Kähler. More important, it may be caused by the class \([||\Omega||_{\omega_0}\omega_0^{n-1}]\) not containing \(\hat{\chi}^{n-1}\) for any Kähler form \(\chi\), and in particular by the manifold \(X\) not being Kähler. In all these situations, we expect the formation of singularities of the flow and/or long-time behavior to reflect the non-Kähler setting. We shall return to these issues elsewhere.

(b) The existence of an initial metric \(\omega_0\) satisfying the condition (4.5) is equivalent to the existence of a Kähler metric \(\hat{\chi}\). Indeed, if \(\hat{\chi}\) is a Kähler metric, we can set \(\omega_0 = ||\Omega||_{\hat{\chi}}^{-\frac{2}{n-2}}\hat{\chi}\) to obtain a metric satisfying (4.5).

In fact, any conformally balanced initial metric which is conformally Kähler satisfies (4.5). Let \(\omega_0\) be an initial conformally balanced metric such that \(\omega_0 = e^\psi\hat{\chi}\) where \(\psi: X \to \mathbb{R}\) is a smooth function and \(\hat{\chi}\) is a Kähler metric. Substituting \(\omega_0 = e^\psi\hat{\chi}\), we obtain
\[
||\Omega||_{\omega_0}\omega_0^{n-1} = (e^{(\frac{n}{2}-1)\psi}||\Omega||_{\hat{\chi}})^{\hat{\chi}^{n-1}}. \tag{6.1}
\]
Since \(d(||\Omega||_{\omega_0}\omega_0^{n-1}) = 0\), we conclude that \(e^{(\frac{n}{2}-1)\psi}||\Omega||_{\hat{\chi}} = C\) where \(C > 0\) is a constant. It follows that \(||\Omega||_{\omega_0}\omega_0^{n-1} = C\hat{\chi}^{n-1}\). After replacing \(\hat{\chi}\) by \(C^{1/(n-1)}\hat{\chi}\), we see that the ansatz (4.5) is satisfied.

In particular, we have shown that the Anomaly flow (1.1) preserves the conformally Kähler condition.

(c) Given an initial metric \(\omega_0\) satisfying \(d(||\Omega||_{\omega_0}\omega_0^{n-1}) = 0\), the Anomaly flow (1.1) preserves the balanced class of the initial metric.
\[
\frac{d}{dt}[||\Omega||_{\omega(t)}\omega(t)^{n-1}] = [i\partial\bar{\partial}\omega^{n-2}] = 0. \tag{6.2}
\]
Here we take cohomology classes in Bott-Chern cohomology
\[
H^{n-1,n-1}_{BC}(X, \mathbb{R}) = \left\{\text{closed real } (n-1,n-1)\text{ forms} \right\}/\left\{i\partial\bar{\partial}\beta : \beta \in \Omega^{n-2,n-2}(X, \mathbb{R})\right\}. \tag{6.3}
\]
Thus
\[
||\Omega||_{\omega(t)}\omega(t)^{n-1} \in [||\Omega||_{\omega_0}\omega_0^{n-1}], \tag{6.4}
\]
where
\[
[||\Omega||_{\omega_0}\omega_0^{n-1}] \in H^{n-1,n-1}_{BC}(X, \mathbb{R}) \tag{6.5}
\]
is the balanced class of \(\omega_0\). Since stationary points of the Anomaly flow are Kähler metrics, the Anomaly flow could potentially be used to study the relation between the balanced cone and Kähler cone on a Kähler Calabi-Yau manifold \((X, \Omega)\). The interaction between these two cones was explored by J.-X. Fu and J. Xiao [30], and they raised the question of detecting when a balanced class contains a Kähler metric (or more generally, a limit of Kähler metrics). Examples are given in [30], [70] of positive balanced classes on a Kähler manifold which do not contain a Kähler metric. As discussed above in (a), the formation
of singularities of the Anomaly flow may be related to the properties of the initial balanced class.

(d) Similar questions to (a) have been brought to our attention in informal discussions with T. Collins, in connection with a conjecture of Lejmi-Székelyhidi [47]. We briefly describe here one special case of the Lejmi-Székelyhidi conjecture. Let $\omega$ and $\alpha$ be two Kähler metrics on $X$. It is conjectured that if
\[ \int_X (\omega^n - n\omega \wedge \alpha^{n-1}) \geq 0, \int_D (\omega^{n-1} - \alpha^{n-1}) > 0, \] (6.6)
for every irreducible divisor $D$, then there exists a Kähler metric $\omega' \in [\omega]$ satisfying the positivity condition
\[ \omega^{n-1} - \alpha^{n-1} > 0. \] (6.7)
It was proved by Xiao [78] that one can find a balanced Hermitian metric $\tilde{\omega}$ such that $[\tilde{\omega}^{n-1}] = [\omega^{n-1}]$ and $\tilde{\omega}^{n-1} - \alpha^{n-1} > 0$, but it remains to find a Kähler metric in the balanced class $[\omega^{n-1}]$ with the desired positivity. The positivity condition (6.7) is of interest, as it corresponds to subsolutions to the fully nonlinear PDE
\[ n\alpha^{n-1} \wedge (\omega + i\partial\bar{\partial}u) = (\omega + i\partial\bar{\partial}u)^n. \] (6.8)
The existence of such subsolutions provides the existence of a genuine solution, as established in [64, 17] (see also [68, 15, 63] for extensions and generalizations).

(e) There is another flow superficially similar to (1.1), but which can be considered for any compact complex manifold $X$, and not just manifolds which admit a nowhere holomorphic $(n,0)$-form $\Omega$,
\[ \partial_t \omega^{n-1} = i\partial\bar{\partial} \omega^{n-2}, \] (6.9)
with initial data $\omega_0$ satisfying $d\omega_0^{n-1} = 0$. A similar computation to the one in the proof of Theorem 4, using Lemma 4 in [59], shows that the flow (6.9) can be expressed as
\[ \partial_t g_{kj} = -\frac{1}{(n-1)} \nabla^m T_{kjm} + \frac{1}{2(n-1)} \left\{ -g^{q\bar{p}} g^{s\bar{r}} T_{kqs} T_{j\bar{p}\bar{r}} + \frac{|T|^2}{n-1} g_{kj} \right\} \] (6.10)
for $n \geq 4$. From this formula, it is easy to deduce case (i) in Lemma 1, namely that the stationary points of the flow (6.9) are Kähler metrics: it suffices to set (6.10) to zero and to take the trace in order to obtain $|T|^2 = 0$. However, the flow (6.9) may be hard to use, because it is not parabolic, and its stationary set may be too large, as it contains all Kähler metrics.

(f) The flow (1.1) can be viewed as a Kähler analogue of the inverse Gauss curvature flow. Indeed, if we consider a flow of a strictly closed convex hypersurface $M_t$ in $\mathbb{R}^n$ by the inverse of its Gauss curvature, then it can be expressed [75] as
\[ \partial_t u = \frac{\det(ug_{ij} + \nabla_i \nabla_j u)}{\det g_{ij}}, \quad u(x,0) = u_0(x) > 0, \] (6.11)
where \( u \) is the support function \( u : S^n \times [0, T) \) defined by \( u(N, t) = \langle P, N \rangle \) where \( P \in M_t \) is the point on \( M_t \) with normal \( N \), and \( (S^n, g_{ij}) \) is the standard sphere. The right hand side of this flow exhibits the determinant of the Hessian of the unknown \( u \), just as the right hand side of the equation (1.6).

A Appendix

In this Appendix, we group together our conventions for differential forms and several identities needed for the proof of Theorem 1 and Theorem 4.

A.1 Components of a differential form

Let \( \varphi \) be a \((p,q)\)-form on the manifold \( X \). We define its components \( \varphi_{k_1 \ldots k_q j_1 \ldots j_p} \) by

\[
\varphi = \frac{1}{p!q!} \sum \varphi_{k_1 \ldots k_q j_1 \ldots j_p} dz^j \wedge \cdots \wedge dz^q \wedge dz^{k_1} \wedge \cdots \wedge dz^{k_q}.
\]  

(A.1)

A.2 Contraction identities

We note a few basic identities for contracting differential forms of degree \((p, p)\). Let \( \omega = ig_{kj} dz^j \wedge d\bar{z}^k \) be a Hermitian metric. For a \((1, 1)\) form \( \alpha \)

\[
\alpha = \alpha_{pq} dz^q \wedge d\bar{z}^p,
\]  

(A.2)

we have

\[
\alpha \wedge \frac{\omega^{n-1}}{(n-1)!} = -i(g^{jk}\alpha_{kj}) \frac{\omega^n}{n!}.
\]  

(A.3)

Next, for a \((2, 2)\) form \( \Phi \) with components

\[
\Phi = \frac{1}{4} \Phi_{pqrs} dz^p \wedge d\bar{z}^q \wedge dz^r \wedge d\bar{z}^s,
\]  

(A.4)

we have

\[
\Phi \wedge \frac{\omega^{n-2}}{(n-2)!} = -\frac{1}{2} \left\{ g^{jk} g^{lm} \Phi_{kjm\ell} \right\} \frac{\omega^n}{n!}.
\]  

(A.5)

For a \((3, 3)\) form \( \Psi \) with components

\[
\Psi = \frac{1}{36} \Psi_{kjpqrs} dz^k \wedge d\bar{z}^j \wedge dz^p \wedge d\bar{z}^q \wedge d\bar{z}^r \wedge dz^s,
\]  

(A.6)

we have

\[
\Psi \wedge \frac{\omega^{n-3}}{(n-3)!} = \frac{i}{6} \left\{ g^{jk} g^{pq} g^{sr} \Psi_{kjpqrs} \right\} \frac{\omega^n}{n!}.
\]  

(A.7)
A.3 Computing $T \wedge \bar{T}$

Next, let $T$ be $(2, 1)$ form.

$$T = \frac{1}{2} T_{ksj} dz^j \wedge dz^s \wedge dz^k. \quad (A.8)$$

Then

$$T \wedge \bar{T} = \frac{1}{4} T_{ksj} \bar{T}_{qp\bar{r}} dz^j \wedge dz^k \wedge dz^s \wedge dz^p \wedge dz^q \wedge dz^p. \quad (A.9)$$

Antisymmetrizing

$$T \wedge \bar{T} = \frac{1}{(3!)^2} (T \wedge \bar{T})_{pqrskj} dz^j \wedge dz^k \wedge dz^s \wedge dz^r \wedge dz^q \wedge dz^p. \quad (A.10)$$

where

$$(T \wedge \bar{T})_{pqrskj} = \left\{ T_{ksj} \bar{T}_{qp\bar{r}} + T_{rsj} \bar{T}_{qp\bar{r}} + T_{qsj} \bar{T}_{qp\bar{r}} + T_{kqs} \bar{T}_{j\bar{p}k} + T_{kqs} \bar{T}_{j\bar{p}k} \right\}. \quad (A.11)$$

Let

$$\tau = T_q dz^q, \quad T_q = g^{jk} T_{kj}, \quad |\tau|^2 = g^{pq} T_q T_p, \quad |T|^2 = g^{pq} g^{rs} g^{jk} T_{pqj} T_{qrk}. \quad (A.12)$$

Then

$$g^{pq} g^{rs} (T \wedge \bar{T})_{pqrskj} = g^{pq} g^{rs} (2 T_{pqj} \bar{T}_{qrk} + T_{kqs} \bar{T}_{j\bar{p}k}) - 2 g^{rs} (T_{kjs} \bar{T}_p + T_{s} \bar{T}_{j\bar{p}}) - 2 T_{j} \bar{T}_{k} \quad (A.13)$$

and

$$g^{jk} g^{pq} g^{rs} (T \wedge \bar{T})_{pqrskj} = 3 \{|T|^2 - 2|\tau|^2\}. \quad (A.14)$$

Applying formula (A.7), we obtain

$$T \wedge \bar{T} \wedge \omega^{n-3} = \frac{i}{2} \frac{1}{n(n-1)(n-2)} \{|T|^2 - 2|\tau|^2\} \omega^n. \quad (A.15)$$

A.4 Computing $i\partial \bar{\partial} \omega$

We have

$$i\partial \bar{\partial} \omega = \frac{1}{2^2} (i\partial \bar{\partial} \omega)_{kj\bar{m}} dz^m \wedge dz^j \wedge dz^k, \quad (A.16)$$

with

$$(i\partial \bar{\partial} \omega)_{kj\bar{m}} = \partial_k \partial_j g^{km} - \partial_k \partial_m g_{kj} - \partial_k \partial_j g_{\bar{m}l} + \partial_k \partial_m g_{\bar{m}l}. \quad (A.17)$$

Using the definition of curvature,

$$(i\partial \bar{\partial} \omega)_{kj\bar{m}} = R_{kj\bar{m}} - R_{km\bar{e}} + R_{\bar{e}m\bar{k}} - R_{\bar{e}km} - g^{sr} T_{\bar{r}mj} \bar{T}_{s\bar{k}}. \quad (A.18)$$

Therefore

$$g^{jk} (i\partial \bar{\partial} \omega)_{kj\bar{m}} = \bar{R}_{\bar{e}m} - R_{\bar{e}m} + R_{\bar{e}m} - R_{\bar{e}m} - g^{jk} g^{sr} T_{\bar{r}mj} \bar{T}_{s\bar{k}}. \quad (A.19)$$
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