On the Impossibility of Global Convergence in Multi-Loss Optimization

Alistair Letcher
aletcher.github.io

Abstract

Under mild regularity conditions, gradient-based methods converge globally to a critical point in the single-loss setting. This is known to break down for vanilla gradient descent when moving to multi-loss optimization, but can we hope to build some algorithm with global guarantees? We negatively resolve this open problem by proving that any reasonable algorithm will exhibit limit cycles or diverge to infinite losses in some differentiable game, even in two-player games with zero-sum interactions. A reasonable algorithm is simply one which avoids strict maxima, an exceedingly weak assumption since converging to maxima would be the opposite of minimization. This impossibility theorem holds even if we impose existence of a strict minimum and no other critical points. The proof is constructive, enabling us to display explicit limit cycles for existing gradient-based methods. Nonetheless, it remains an open question whether cycles arise in high-dimensional games of interest to ML practitioners, such as GANs or multi-agent RL.

1 Introduction

Problem Setting

As multi-agent architectures proliferate in machine learning, it is becoming increasingly important to understand the dynamics of gradient-based methods when optimizing multiple interacting goals, otherwise known as differentiable games. This framework encompasses GANs [11], intrinsic curiosity [24], imaginative agents [25], synthetic gradients [12], hierarchical reinforcement learning [31, 29] and multi-agent RL in general [6]. The interactions between learning agents make for vastly more complex mechanics: naively applying gradient descent on each loss is known to diverge even in simple bilinear games.

Related Work

A large number of methods have been proposed recently to tackle the question of local convergence, or global convergence in the convex setting: adaptations of single-loss algorithms such as Extragradient (EG) [2] and Optimistic Mirror Descent (OMD) [7], Consensus Optimization (CO) for GAN training [20], Competitive Gradient Descent (CGD) based on solving a bilinear approximation of the loss functions [26], Local Symplectic Surgery (LSS) for finding local Nash equilibria [19], Symplectic Gradient Adjustment (SGA) based on a novel decomposition of game mechanics [4, 16], and opponent-shaping algorithms including Learning with Opponent-Learning Awareness (LOLA) [10] and its convergent counterpart, Stable Opponent Shaping (SOS) [17].

Each has shown promising theoretical implications and empirical results, but none offers insight into global convergence in the non-convex setting, which includes the vast majority of machine learning applications. One of the main roadblocks compared with single-loss optimization has been noted by Schaefer and Anandkumar [26]: “a convergence proof in the nonconvex case analogue to Lee et al. [14] is still out of reach in the competitive setting. A major obstacle to this end is the identification of a suitable measure of progress (which is given by the function value in the single agent setting), since norms of gradients can not be expected to decay monotonously for competitive dynamics in non-convex-concave games.”
The question has come up in a number of other papers including [19]: “It is important to emphasize that our analysis has been limited to neighborhoods of equilibria; the proposed algorithm can converge in principle to limit cycles at other locations of the space. These are hard to rule out completely.”

Global convergence has been established in a few special cases: potential and Hamiltonian games [4], two-player zero-sum games satisfying the two-sided Polyak-Łojasiewicz condition [32], and zero-sum linear quadratic games [33]. These are significant contributions with several applications of interest, but do not include any of the architectures mentioned above. Finally, Balduzzi et al. [5] show that GD dynamics are bounded under a ‘negative sentiment’ assumption in smooth markets, which do include GANs – but this implies nothing about convergence, as we will show.

On the other hand, failure of global convergence has been shown by Palaiopanos et al. [21] for the Multiplicative Weights Update method, and by Vlatakis-Gkaragkounis et al. [30], Bailey et al. [3] for simultaneous gradient descent (SGD) and alternating gradient descent (AGD). Limit cycles have been analysed there in great detail, with interesting connections to Poincaré recurrence, but nothing is claimed about other optimization methods.

Contribution This paper adresses the more general question of whether global convergence can be guaranteed for any learning algorithm. We show that the failure of SGD and AGD described by Vlatakis-Gkaragkounis et al. [30] is no accident, but rather inherent to any reasonable algorithm, namely, one that avoids strict maxima. This naively weak assumption turns out to be fundamentally incompatible with global convergence, even in the more tractable class of smooth markets defined by pairwise zero-sum interaction terms [5].

We construct a two-player market where any reasonable algorithm enters a limit cycle or diverges to infinite loss for both players. We plot these cycles explicitly for the panel of existing gradient-based methods mentioned above. Finally, we show that this behaviour persists even in the smaller class of two-player zero-sum games.

One might hope that global convergence could at least be guaranteed in games without strict maxima, or even stronger, games with a strict minimum (also called stable fixed point) and no other critical points. Unfortunately, our impossibility theorem extends also to this setting, despite the initial intuition that little could be claimed about reasonable algorithms in the absence of maxima.

This paper brings to an end the pursuit of global convergence in general settings, even in two-player games with zero-sum interactions. Nonetheless, it may be that cycles do not arise in high-dimensional games of interest, including GANs. It is essential in future work to focus in on these specific architectures, and prove or disprove global convergence there.

2 Background

2.1 Single losses: global convergence of gradient descent

Given a continuously differentiable function \( f : \mathbb{R}^d \to \mathbb{R} \), let

\[
\theta_{k+1} = \theta_k - \alpha \nabla f(\theta_k)
\]

be the iterates of gradient descent with learning rate \( \alpha \), initialised at \( \theta_0 \). Under regularity conditions, gradient descent converges globally to critical points:

**Proposition 1.** Assume \( f \in C^2 \) has compact sublevel sets and is either analytic or has isolated critical points. For any \( \theta_0 \in \mathbb{R}^d \), define \( U_0 = \{ f(\theta) \leq f(\theta_0) \} \) and let \( L < \infty \) be a Lipschitz constant for \( \nabla f \) in \( U_0 \). Then for any \( 0 < \alpha < 2/L \) we have \( \lim_k \theta_k = \bar{\theta} \) for some critical point \( \bar{\theta} \).

The requirements for convergence are relatively mild:

1. \( f \) has compact sublevel sets iff \( f \) is coercive, \( \lim_{\|\theta\| \to \infty} f(\theta) = \infty \), which mostly holds in machine learning since \( f \) is a loss function.
2. \( f \) has isolated critical points if it is a Morse function (nondegenerate Hessian at critical points), which holds for almost all \( C^2 \) functions. More precisely, Morse functions form an open, dense subset of all functions \( f \in C^2(\mathbb{R}^d, \mathbb{R}) \) in the Whitney \( C^2 \)-topology.
3. Lipschitz continuity of \( \nabla f \) is not assumed, which would fail even for cubic polynomials.
The goal of this paper is to prove that similar (even weaker) guarantees cannot be obtained in the multi-loss setting – not only for GD, but for any reasonable algorithm.

2.2 Differentiable games

Following Balduzzi et al. [4], we frame the problem of multi-loss optimization as a differentiable game among cooperating and competing agents; these may simply be different internal components of a single system, like the generator and discriminator in GANs.

Definition 1. A differentiable game is a set of $n$ agents with parameters $\theta = (\theta^1, \ldots, \theta^n) \in \mathbb{R}^d$ and twice continuously differentiable losses $L^i : \mathbb{R}^d \to \mathbb{R}$, where $\theta^i \in \mathbb{R}^d$ for each $i$ and $\sum_i d_i = d$.

In practice, losses need only be differentiable almost-everywhere. For instance, neural networks with rectified linear units (ReLU) would not be differentiable everywhere.

We do not assume that each $L^i$ is convex as a function of $\theta^i$ alone, for any fixed opponent parameters $\theta^{-i}$, nor do we restrict $\theta$ to the probability simplex. If $n = 1$, the ‘game’ is simply to minimise a given loss function. We write $\nabla_i L^k = \nabla_{\theta^i} L^k$ and $\nabla_{ij} L^k = \nabla_{\theta^i \theta^j} L^k$ for any $i, j, k$, and define the simultaneous gradient of the game as the concatenation of each player’s gradient,

$$\xi = (\nabla_1 L^1, \ldots, \nabla_n L^n)^T \in \mathbb{R}^d.$$  

The $i$th component of $\xi$ is the direction of greatest increase in $L^i$ with respect to $\theta^i$. If each agent minimises their loss independently from others under GD with learning rate $\alpha$, the parameter update for all agents is given by $\theta \leftarrow \theta - \alpha \xi(\theta)$. We call this simultaneous gradient descent (SGD), or GD for short. We call $\theta$ a critical point if $\xi(\bar{\theta}) = 0$. These are usually called fixed points, since they are fixed points of simultaneous GD, but critical points may not coincide with fixed points of other optimization algorithms. Now introduce the ‘Hessian’ of the game as the block matrix

$$H = \nabla \xi = \begin{pmatrix} \nabla_{11} L^1 & \ldots & \nabla_{1n} L^1 \\ \vdots & \ddots & \vdots \\ \nabla_{n1} L^n & \ldots & \nabla_{nn} L^n \end{pmatrix} \in \mathbb{R}^{d \times d}.$$  

This can equivalently be viewed as the Jacobian of the vector field $\xi$. Importantly, note that $H$ is not symmetric in general unless $n = 1$, in which case we recover the usual Hessian $H = \nabla^2 L$. However, $H$ can be decomposed into symmetric and anti-symmetric components as $H = S + A$ by Balduzzi et al. [4]. A second useful decomposition has appeared recently in [17] and [26]: $H = H_d + H_o$ where $H_d$ and $H_o$ are the matrices of diagonal and off-diagonal blocks; formally, $H_d = \bigoplus_i \nabla_{ii} L^i$. 

A solution concept for differentiable games, analogous to the single-loss case, is defined as follows.

Definition 2. A critical point $\bar{\theta}$ is a (strict, local) minimum if $H(\bar{\theta}) \succ 0$.

These were named stable fixed points by Balduzzi et al. [4], but the term is usually reserved in dynamical systems to the larger class defined by Hessian eigenvalues with positive real parts, which is implied but not equivalent to $H \succ 0$ for non-symmetric matrices.

In particular, strict minima are (differential) Nash equilibria, as defined by Mazumdar et al. [19], since diagonal blocks must also be positive definite: $\nabla_{ii} L^i(\bar{\theta}) \succ 0$.

This paper is concerned with iterative optimization algorithms whose iterates $\theta_k$ are obtained by initialising parameters $\theta_0$ (usually following some distribution) and applying a continuous function to the previous iterates, namely, $\theta_{k+1} = F(\theta_k, \ldots, \theta_0)$ for some continuous function $F$. This evidently holds for all gradient-based methods; in fact, most of them are only functions of the current iterate $\theta_k$, so that $\theta_k = F^k(\theta_0)$ where $F^k$ denotes the $k$-times composition of $F$ with itself.

A large number of gradient-based methods have been proposed to address the pitfalls of simultaneous gradient descent, which is known to diverge even in simple bilinear games. Each of them is defined formally in Appendix A, along with experiment hyperparameters.

For single-player games, the goal of such algorithms is for $\theta_k$ to converge to a local (perhaps global) minimum as $k \to \infty$. The goal is less clear for differentiable games, but the goal is generally to reach a minimum or a Nash equilibrium. In the case of specific applications like GANs, the goal might be to reach parameters that produce realistic images, which is challenging to define formally.
2.3 Strict maxima and reasonable algorithms

Just as game theorists do not all agree on the 'best' or most appropriate solution concept, from Nash equilibria to Pareto optimality, it is not a priori clear what critical points should always be avoided. In single-player games, saddle points are the obvious candidate. We usually restrict our attention to strict saddles, defined by $\min \lambda(H) < 0$ (the presence of a negative eigenvalue in the Hessian), since they are second-order tractable and provably avoided by gradient-based methods [15]. Contained in strict saddles are strict maxima, defined by $\max \lambda(H) < 0$ (only negative eigenvalues).

For differentiable games, this is less obvious. We define and discuss possible generalisations in Appendix C, and simply point out here that a rich set of related concepts arise for interacting agents:

**Proposition 2.** Write $\lambda(A) = \text{Re}(\text{Spec}(A))$ for real parts of the eigenvalues of a matrix $A$. We have the following implications, and none of them are equivalences.

$$
\begin{align*}
\max \lambda(H) < 0 & \quad \iff \quad \min \lambda(H) < 0 \\
\max \lambda(S) < 0 & \quad \iff \quad \min \lambda(S) < 0 \\
\max \lambda(H_d) < 0 & \quad \iff \quad \min \lambda(H_d) < 0
\end{align*}
$$

The top row is dynamics-based, governed by the collective Hessian, while the bottom row has a game-theoretical flavor whereby $H_d = \bigoplus \nabla_i L^i$ decomposes into agentwise Hessians. The left and right triangles collide respectively to maxima and saddles for single losses, since $H = S = H_d = \nabla^2 L$.

Among these classes of potentially undesirable points, the weakest possible requirement is that algorithms avoid critical points $\theta$ with $\max \lambda(S(\theta)) < 0$, the smallest such class. Since $S$ is symmetric, this is equivalent to $S(\theta) \prec 0$, which is equivalent to $H(\theta) \prec 0$.

It is exceedingly reasonable to ask that optimization algorithms avoid these points almost surely. Firstly, they are are equivalent to strict maxima in the single-loss setting: converging there would be the opposite of our goal. Secondly, they are unstable with respect to GD dynamics since $H \prec 0$. Finally, the bottom-left implication $\max \lambda(H_d) < 0$ is equivalent to $\nabla_i L^i \prec 0$ for all $i$: they are strict maxima of each player’s individual loss function, so they can all improve their losses by moving anywhere away from them. Failing to avoid them would go against the very idea of optimization.

This class was named unstable fixed points in [17, 16, 5], but the term is usually reserved in dynamical systems for the larger class defined by $\min \lambda(H) < 0$. We prefer the more indicative name strict maxima, as justified by analogy with single losses above.

**Definition 3.** A critical point $\theta$ is a (strict, local) maximum if $H(\theta) \prec 0$.

The second reasonable condition to impose is that algorithms converge only to critical points. If not, $\nabla_i L^i(\theta) \neq 0$ for some $i$, implying that agent $i$ can strictly improve its losses by moving away from $\theta$, simply by following the negative gradient direction. Gradient-based methods should be expected to satisfy this, since there is no reason for an agent to stop improving if its gradient is non-zero.

**Definition 4 (Reason).** An algorithm is reasonable if it converges only to critical points and avoids strict maxima almost surely, for suitable hyperparameters.

Formally: $F(\theta) = \theta$ implies $\xi(\theta) = 0$, and for any strict maximum $\theta$ and bounded region $U$, $\mu(\{\theta_0 \in U \mid \lim_k \theta_k = \theta\}) = 0$ for suitable hyperparameters ($\mu$ denotes Lebesgue measure).

We could ask that reasonable algorithms avoid all unstable critical points, a.k.a. strict saddles defined by $\min \lambda(H) < 0$, but this is more than we need to prove the failure of global convergence.

Reason is not equivalent to rationality or self-interest. Reason is much weaker, imposing only that agents are well-behaved regarding strict maxima even if their individual behaviour is not self-interested. For instance, SGA agents do not behave out of self-interest [4].
3 Global Convergence in Differentiable Games

3.1 Any reasonable algorithm fails to converge globally

Our main contribution is to show that we cannot provide weak global guarantees for any reasonable algorithm. First recall that global convergence should not be expected in all games, since there may be a divergent direction with minimal loss (imagine minimising $L = x$ or $L = e^x$). It should however be asked that algorithms have bounded iterates in coercive games defined by

$$\lim_{\|\theta]\to\infty} L^i(\theta) = \infty$$

for all $i$. Indeed, unbounded iterates in coercive games would lead to infinite losses for all agents, the worst possible outcome. Once bounded iterates are guaranteed, recall from Proposition 1 that convergence to a critical point should hold if the Hessian is nondegenerate for all critical points, which implies isolated critical points. This condition can also be replaced by analyticity of the loss. In the spirit of weakest assumptions, we ask for convergence when both conditions hold.

**Definition 5 (Globality).** An algorithm is global if, in a coercive, analytic and nondegenerate game, for fixed $\theta_0$, iterates $\theta_k$ are bounded and converge for suitable hyperparameters.

Hyperparameters may depend on the given game and the initial point $\theta_0$. Note that GD is global for single-player differentiable games by Proposition 1, and reasonable by [14, Theorem 4]. We will extend this result to potential games in Section 3.7.

Unfortunately, reason and globality are fundamentally at odds as soon as we move to more general games. One might argue that expecting global guarantees in all differentiable games is excessive, since every continuous dynamical system arises as simultaneous GD on the loss functions of a differentiable game [5, Lemma 1]. For this reason, Balduzzi et al. [5] have introduced a vastly more tractable class of games called (smooth) markets, defined by playerwise zero-sum interactions.

**Definition 6.** A (smooth) market is a differentiable game where interactions between players are pairwise zero-sum, namely, $L^i(\theta) = L^i(\theta^i) + \sum_{j\neq i} g_{ij}(\theta^i, \theta^j)$ with $g_{ij}(\theta^i, \theta^j) = 0$ for all $i, j$.

This generalises zero-sum games while remaining amenable to optimization and aggregation, meaning that “we can draw conclusions about the gradient-based dynamics of the collective by summing over properties of its members” [5]. Moreover, this class captures a large number of applications including GANs and related architectures, intrinsic curiosity modules, adversarial training, task-suites and population self-play. One would modestly hope for some algorithm to converge globally in markets. Our main contribution is to prove that even this is too much to ask.

**Theorem 1.** There are no reasonable global algorithms, even for two-player markets. [There is a coercive, nondegenerate, analytic two-player market $G$ such that any reasonable algorithm produces bounded but non-convergent iterates, or divergent iterates with infinite losses for all players.]

**Proof.** Consider the analytic market $G$ given by

$$L^1(x, y) = x^6/6 - x^2/2 + xy + \frac{1}{4} \left( \frac{y^4}{1+x^2} - \frac{x^4}{1+y^2} \right)$$

$$L^2(x, y) = y^6/6 - y^2/2 - xy - \frac{1}{4} \left( \frac{y^4}{1+x^2} - \frac{x^4}{1+y^2} \right).$$

We prove in the appendix that $G$ is coercive, nondegenerate, and has a unique critical point at the origin, which is a strict maximum. A reasonable algorithm will almost surely converge nowhere, so iterates are either bounded but non-convergent or divergent with infinite losses.

**Remark.** One might wonder why this approach cannot be used to disprove global convergence for single losses. One reason is that coercive losses, unlike games, always have a global minimum: we cannot construct a coercive loss with no critical points other than strict maxima.
The game $G$ has four possible outcomes, each in some way unsatisfactory:

1. Iterates are unbounded, and all players diverge to infinite loss. [Not global]
2. Iterates are bounded and converge to the strict maximum. [Not reasonable]
3. Iterates are bounded and converge to a non-critical point. [Not reasonable]
4. Iterates are bounded but do not converge. [Not global]

The first three points are highly objectionable, as already discussed. The fourth is, in practice, much like the third, since we must eventually terminate the iteration. The resulting parameters will be an arbitrary point on the limit cycle, not a critical point. Even if we let agents update parameters endlessly while they play a game or solve a task, they will have oscillatory behaviour and fail to produce consistent outcomes (e.g. when generating an image or playing Starcraft).

The hope for machine learning is that limit cycles do not arise in high-dimensional applications we care about, such as GANs or intrinsic curiosity. This may well be the case, but proving or disproving global convergence in these specific settings is beyond the scope of this paper.

### 3.2 Implication: existing methods fail to converge globally

We expect all existing gradient-based methods to be reasonable – except LOLA, which may converge to non-critical points, but this was pointed out and resolved by Letcher et al. [17] with SOS.

In particular, Figure 1 confirms that existing methods all fail to converge globally in $G$ (even LOLA, which does not have any other fixed points in this particular game). The behaviour of (time-varying) LSS is rather unique, oscillating around a non-critical point instead of the origin. This is an artifact of the time-varying term: LSS would in fact converge to a non-critical point without it. The behaviour of SGA also differs slightly from the others, which is explained by the presence of a non-continuous parameter $\lambda$ jumping between $\pm 1$ according to an alignment criterion.

It is worth noting that existing methods also have non-convergent, bounded iterates in the zero-sum game given by $L^1 = xy - x^2/2 + y^2/2 + x^4/4 - y^4/4 = -L^2$. This game also has a strict maximum at the origin, and no other critical points, so one might wish to extend Theorem 1 to zero-sum games. However, zero-sum games cannot be coercive since $L^1 \to \infty$ implies $L^2 \to -\infty$. We do not want to impose globality in non-coercive games, since this would fail even for GD on single losses. However, we may extend our theorem by weakening the definition of coercivity to

$$\lim_{\|\theta^i\| \to \infty} L^i(\theta^i, \theta^{-1}) = \infty$$

for all $i$ and fixed $\theta^{-1}$. This is up to debate. In any case, limit cycles arise for existing methods:

**Proposition 3.** There is an analytic, nondegenerate zero-sum game such that existing gradient-based methods have bounded but non-convergent iterates for suitable hyperparameters.
3.3 Implication: there are no suitable measures of progress

A crucial step in proving global convergence of GD on single losses is showing that the set of accumulation points is a subset of critical points, using the function value as a ‘measure of progress’. The fact that this fails for differentiable games, as shown by Figure 1, implies that there can be no suitable measure of progress. We formalise this below, bringing to an end the question of Schaefer and Anandkumar [26] quoted in the introduction.

**Definition 7.** A measure of progress for an algorithm given by \( \theta_{k+1} = F(\theta_k) \) is a continuous map \( M : \mathbb{R}^d \rightarrow \mathbb{R} \), bounded below, such that \( M(F(\theta)) \leq M(\theta) \) and \( M(F(\theta)) = M(\theta) \) iff \( F(\theta) = \theta \).

Measures of progress are very similar to descent functions, as defined by Luenberger and Ye [18], and somewhat akin to Lyapunov functions. The result below is analogous to the Global Convergence Theorem of Luenberger and Ye [18].

**Proposition 4.** Assume a measure of progress \( M \) exists for some reasonable algorithm \( F \). Then the set of accumulation points of \( \theta_k \) is a subset of critical points.

The function value \( f \) is a measure of progress for single-loss GD under the usual regularity conditions, while the gradient norm \( \|\xi\| \) is a measure of progress for GD in strictly convex differentiable games:

\[
\|\xi(\theta - \alpha\xi)\|^2 = \|\xi\|^2 - \alpha\xi^T H^T \xi + o(\alpha) \leq \|\xi\|^2
\]

for appropriate \( \alpha \). Unfortunately, games like \( G \) prevent the existence of such measures in general.

**Corollary 1.** There are no measures of progress for reasonable algorithms, even in two-player markets, provided iterates are bounded in coercive games for suitable hyperparameters.

Assuming the algorithm to be reasonable is necessary: any map is a measure of progress for the unreasonable algorithm \( F(\theta) = \theta \).

3.4 What if there are strict minima?

One might wonder if it is purely the absence of strict minima that causes non-convergence, since strict minima are locally attracting under gradient dynamics. Can we guarantee global convergence if we impose existence of a minimum, and more, the absence of any other critical points?

Unfortunately we cannot. Assuming parameters are initialised following any continuous measure \( \nu \) on \( \mathbb{R}^d \), any reasonable algorithm will produce limit cycles or diverge to infinite loss in some two-player market with arbitrarily high probability.

**Theorem 2.** There are no reasonable global algorithms, even for two-player markets with a strict minimum and no other critical points. [There is a coercive, nondegenerate, almost-everywhere analytic two-player market with a unique and minimum critical point such that any reasonable algorithm produces bounded but non-convergent iterates, or divergent iterates with infinite losses for all players, with arbitrarily high probability.]

**Proof.** The proof idea is to modify \( G \) by deforming a small region around the maximum, turning it into a minimum while leaving the dynamics unchanged outside of this region; see appendix.

Before concluding, we show that gradient descent is reasonable for differentiable games, as should be expected. On the contrary, we prove that descending on the gradient norm is global for differentiable games – but cannot be reasonable, by Theorem 1. The final section reaffirms the existence of reasonable and global algorithms for potential and Hamiltonian games, as already stated by Balduzzi et al. [4], though the practical impact of these results is limited.

3.5 Gradient descent is reasonable

We prove in the appendix that GD avoids strict saddles almost surely, a simple adaptation of the single-player case proven by Lee et al. [14]. This was shown by Daskalakis and Panageas [8] for zero-sum games, but the proof trivially extends to all games. In particular, GD avoids strict maxima.

**Proposition 5.** GD is reasonable for differentiable games.
3.6 Gradient-norm descent is global

On the opposite side of the spectrum, consider descending on the gradient norm \( \mathcal{H} = \| \xi \|^2 / 2 \). The iteration is named Gradient-Norm Descent (GND), given by

\[
F(\theta) = \theta - \alpha \nabla \mathcal{H}(\theta) = \theta - \alpha H^\top \xi(\theta)
\]

This is agnostic to the stability of critical points, and may therefore converge to strict maxima. However, descending on this single function is guaranteed to converge as a corollary to Proposition 1, provided \( \mathcal{H} \) is coercive.

**Proposition 6.** GND is global for all differentiable games with coercive gradient norm.

The assumption on gradient norm coercivity is necessary: for instance, GND will diverge on the coercive loss \( L = (1 + x^2)^{1/4} \) for any \( x_0^2 \geq 2 \) and any learning rate.

3.7 Special case: potential and Hamiltonian games

Though reason and globality are fundamentally at odds for classes of games including two-player markets, global guarantees exist for potential and Hamiltonian games under further conditions. However, one should keep in mind that these are extremely small classes which fail to capture most ML applications of interest.

The following results were already stated (for the first) and proven (for the second) by Balduzzi et al. [4]; we include and prove them in the appendix for completeness. We also prove that Hamiltonian games have constant Hessian, a minor but novel claim to the best of our knowledge.

Recall the decomposition of the Hessian into symmetric and antisymmetric components, \( H = S + A \). Balduzzi et al. [4] define a game to be (exact) potential if \( A \equiv 0 \), and Hamiltonian if \( S \equiv 0 \).

**Proposition 7.** GD is reasonable and global for potential games with coercive potential.

**Proposition 8.** GND is reasonable and global for Hamiltonian games with everywhere-invertible Hessian and coercive gradient norm.

The assumption on everywhere-invertibility is far from trivial. For instance, there are no Hamiltonian games with invertible Hessian in odd dimension \( d \):

\[
\det(H) = \det(H^\top) = \det(-H) = (-1)^d \det(H) = -\det(H)
\]

It follows that fixed points of GND in odd-dimensional Hamiltonian games may not be critical points. This is far from reasonable, so the result is to be taken lightly.

4 Conclusion

We have proven that global convergence is too much to hope for in multi-loss optimization. Any reasonable algorithm can exhibit limit cycles or diverge to infinite losses, even in simple two-player markets. This arises because coercive games, unlike losses, may have no critical points other than strict maxima. However, this is not the only point of failure: even in markets with a strict minimum and no other critical points, reasonable algorithms will almost surely fail to converge.

Limit cycles are not necessarily bad; they may even have game-theoretic significance [23]. This paper nonetheless shows that some games, even in the class of two-player markets, have no satisfactory outcome. Players should neither escape to infinite losses, nor converge to strict maxima or non-critical points, so cycling may be the lesser evil. The community is accustomed to optimization problems whose solutions are single points, but cycles may have to be accepted as solutions in themselves.

The hope for machine learning practitioners is that local minima with large regions of attraction prevent limit cycles from arising in applications of interest, including GANs. Proving or disproving this is an interesting and important avenue for further research, with real implications on what to expect when agents learn while interacting with others. Cycles may for instance be unacceptable in self-driving cars, where oscillatory predictions may have life-threatening implications.
Broader impact

Researchers in optimization and game theory will benefit from this work through increased knowledge that global convergence of gradient-based methods does not transfer to the multi-loss setting. In particular, the richer dynamics of games (even two-player markets) prevent the existence of a measure of progress if convexity is not assumed. We hope this will help researchers avoid a fruitless quest for a general proof of global convergence, and instead focus on more specific classes of games for which guarantees may exist. This has been shown for potential and Hamiltonian games, zero-sum linear quadratic games, and two-player zero-sum games satisfying the two-sided Polyak-Łojasiewicz condition, but it would be interesting to answer this question for other classes of interest.

Machine learning practitioners will benefit from this work by becoming aware that gradient-based methods may fail to converge when optimizing interacting losses simultaneously, not because of inappropriate hyperparameters, but for fundamental dynamical reasons. This applies most particularly to widespread architectures such as GANs or multi-agent RL, but extends to other multi-agent settings including intrinsic curiosity modules, imaginative agents, synthetic agents and task-suites. However, as stated throughout the paper, it is not clear that limit cycles arise in high-dimensional applications of interest. We hope the community will push for a better understanding of global convergence in these specific settings – whether to prove or disprove it.

On the ethical side, this paper brings to light the possibility that learning agents interacting in the real world may have oscillatory behaviour. This could have serious implications in cases such as self-driving cars. Our work is a first step in pursuing a better understanding of global dynamics: despite being concerned with ruling out the general setting above all, and markets more specifically, the constructive nature of our proofs indicates why global convergence can fail and what assumptions may be required for it to be guaranteed.

The authors do not believe that anyone could be put at a disadvantage from this research.

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We first prove a lemma and state a standard optimization result.

We can now prove Proposition 1, which avoids requiring Lipschitz continuity by proving that iterates

Accompanying code for all experiments can be found at https://github.com/aletcher/

The proof is a straightforward adaptation of [22, Lemma 7], without requiring

In all experiments we use a learning rate $\lambda = 0.02$ for all algorithms, $\gamma = 0.1$ for CO and $a = b = 0.5$ for SOS. The $\lambda$ parameter for SGA is obtained by the alignment criterion introduced in the original paper; similarly for $p$ in SOS and $\lambda$ in LSS. The vector $u$ is set to the vector of ones for LSS.

Accompanying code for all experiments can be found at https://github.com/aletcher/impossibility-global-convergence.

### B Proofs

#### B.1 Proof of Proposition 1

We first prove a lemma and state a standard optimization result.

**Lemma 0.** Let $G \in C^1(U, \mathbb{R}^d)$ for some open set $U$. If $G$ is $L$-Lipschitz then $\sup_{\theta \in U} \| \nabla G(\theta) \| \leq L$.

The proof is a straightforward adaptation of [22, Lemma 7], without requiring $U$ to be convex.

**Proof.** Fix any $\theta \in U$ and $\epsilon > 0$. Since $U$ is open, the ball $B_r(\theta)$ of radius $r$ centred at $\theta$ is contained in $U$ for some $r > 0$. By Taylor expansion, for any unit vector $\theta'$,

$$
\|G(\theta + r\theta') - G(\theta)\| \geq r \|\nabla G(\theta)\theta'\| - o(r) \geq r \|\nabla G(\theta)\theta'\| - cr
$$

for $r$ sufficiently small. Since $G$ is $L$-Lipschitz, we obtain

$$
r \|\nabla G(\theta)\theta'\| \leq \|G(\theta + r\theta') - G(\theta)\| + r\epsilon \leq r(L + \epsilon).
$$

Since $\epsilon$ was arbitrary, $\|\nabla G(\theta)\theta'\| \leq L$ for any unit $\theta'$. By definition of the norm, we obtain

$$
\|\nabla G(\theta)\| = \sup_{\|\theta'\| = 1} \|\nabla G(\theta)\theta'\| \leq L
$$

for all $\theta \in U$ and hence $\sup_{\theta \in U} \|\nabla G(\theta)\| \leq L$. \(\square\)

**Proposition ([13, Prop. 12.4.4] and [1, Th. 4.1]).** Assume $f$ has $L$-Lipschitz gradient and is either analytic or has isolated critical points. Then for any $0 < \alpha < 2/L$ and $\theta_0 \in \mathbb{R}^d$ we have

$$
\lim_k \|\theta_k\| = \infty \quad \text{or} \quad \lim_k \theta_k = \hat{\theta}
$$

for some critical point $\hat{\theta}$. If $f$ moreover has compact sublevel sets then the latter holds, $\lim_k \theta_k = \hat{\theta}$.

We can now prove Proposition 1, which avoids requiring Lipschitz continuity by proving that iterates are contained in the sublevel set given by $\theta_0$ for appropriate learning rate $\alpha$.

**Proposition 1.** Assume $f \in C^2$ has compact sublevel sets and is either analytic or has isolated critical points. For any $\theta_0 \in \mathbb{R}^d$, define $U_0 = \{ f(\theta) \leq f(\theta_0) \}$ and let $L < \infty$ be a Lipschitz constant for $\nabla f$ in $U_0$. Then for any $0 < \alpha < 2/L$ we have $\lim_k \theta_k = \hat{\theta}$ for some critical point $\hat{\theta}$.
Proof. Note that $\nabla f \in C^1$, so $f$ has $L$-Lipschitz gradient inside any compact set $U$ for some finite $L$, and $\sup_{\theta \in U} \|\nabla^2 f(\theta)\| \leq L$ by Lemma 0. Now define $U_\alpha = \{\theta - t\alpha \nabla f(\theta) \mid t \in [0, 1], \theta \in U_0\}$ and the continuous function $L(\alpha) = \sup_{\theta \in U_\alpha} \|\nabla^2 f(\theta)\|$. Notice that $U_0 \subset U_\alpha$ for all $\alpha$. We prove that $\alpha L(\alpha) < 2$ implies $U_\alpha = U_0$ and in particular, $L(\alpha) = L(0)$. By Taylor expansion,  
$$f(\theta - t\alpha \nabla f) = f(\theta) - \alpha \|\nabla f(\theta)\|^2 + \frac{t^2 \alpha^2}{2} \nabla f(\theta)^T \nabla^2 f(\theta - t' \alpha \nabla f) f(\theta)$$
for some $t' \in [0, t] \subset [0, 1]$. Since $\theta - t' \alpha \nabla f \in U_\alpha$, it follows that

$$f(\theta - t\alpha \nabla f) \leq f(\theta) - \alpha \|\nabla f(\theta)\|^2 (1 - \alpha L(\alpha)/2) \leq f(\theta)$$

for all $\alpha L(\alpha) < 2$. In particular, $\theta - t\alpha \nabla f \in U_0$ and hence $U_\alpha = U_0$. We conclude that $\alpha L(\alpha) < 2$ implies $L(\alpha) = L(0)$, implying in turn $\alpha L(0) < 2$. We now claim the converse, namely that $\alpha L(0) < 2$ implies $\alpha L(\alpha) < 2$. For contradiction, assume otherwise that there exists $\alpha L(0) < 2$ with $\alpha L'(\alpha') \geq 2$. Since $\alpha L(\alpha)$ is continuous and $0L(0) = 0 < 2$, there exists $\bar{\alpha} \leq \alpha'$ such that $\bar{\alpha} L(0) < 2$ and $\bar{\alpha} L(\bar{\alpha}) = 2$. This is in contradiction with continuity:

$$2 = \bar{\alpha} L(\bar{\alpha}) = \lim_{\alpha \to \bar{\alpha}} \alpha L(\alpha) = \lim_{\alpha \to \bar{\alpha}} \alpha L(0) = \bar{\alpha} L(0).$$

Finally we conclude that $U_\alpha = U_0$ for all $\alpha L(0) < 2$, and in particular, for all $\alpha L < 2$. Finally, $\theta_k \in U_\alpha$ implies $\theta_{k+1} \in U_\alpha = U_0$ and hence $\theta_k \in U_0$ by induction. The result now follows by applying the previous proposition to $f|_{U_0}$. 

\[ \Box \]

B.2 Proof of Proposition 2

Proposition 2. Write $\lambda(A) = \text{Re}(\text{Spec}(A))$ for real parts of the eigenvalues of a matrix $A$. We have the following implications, and none of them are equivalences.

\[
\begin{align*}
\text{max } \lambda(H) &< 0 \quad \iff \quad \text{min } \lambda(H) < 0 \\
\text{max } \lambda(S) &< 0 \quad \iff \quad \text{min } \lambda(S) < 0 \\
\text{max } \lambda(H_d) &< 0 \quad \iff \quad \text{min } \lambda(H_d) < 0
\end{align*}
\]

Proof. We begin with the leftmost implications. If $\text{max } \lambda(S) < 0$ then $S < 0$ by symmetry of $S$, implying both $H < 0$ since $u^T Hu = u^T Su$ for all $u \in \mathbb{R}^d$, and negative definite diagonal blocks $\nabla^2 L(u) \leq 0$; finally $H_d < 0$. In particular this implies $\text{max } \lambda(H) < 0$ and $\text{max } \lambda(H_d) < 0$ since real parts of eigenvalues of a negative definite matrix are negative.

The rightmost implications follow as above by contraposition: if $\text{min } \lambda(S) \geq 0$ then $S \geq 0$, which implies $H \geq 0$ and $H_d \geq 0$ and hence $\text{min } \lambda(H) \geq 0$, $\text{min } \lambda(H_d) \geq 0$.

The top and bottom implications are trivial.

The diagonal implications hold by a trace argument:

$$\sum_i \lambda_i(H) = \text{Tr}(H) = \text{Tr}(H_d) = \sum_i \lambda_i(H_d),$$

hence $\text{max } \lambda(H) < 0$ implies the LHS is negative and thus $\sum_i \lambda_i(H_d) < 0$. It follows that $\lambda_i(H_d) < 0$ for some $i$ and finally $\text{min } \lambda(H_d) < 0$. The other diagonal holds identically.

We now prove that no implication is an equivalence. For the leftmost implications,

$$H = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

has $\text{max } \lambda(H_d) = -1 < 0$ while $\text{max } \lambda(S) = 3 > 0$, and

$$H = \begin{pmatrix} 2 & 4 \\ -4 & -4 \end{pmatrix}$$
has \( \max \lambda(H) = -1 < 0 \) while \( \max \lambda(S) = 2 > 0 \). This also proves the diagonal implications: the first matrix has \( \min \lambda(H_d) = -1 < 0 \) but \( \max \lambda(H) = 3 > 0 \), and the second matrix has \( \min \lambda(H) = -1 < 0 \) but \( \max \lambda(H_d) = 2 > 0 \).

For the rightmost implications, swap the sign of the diagonal elements for the two matrices above.

The top and bottom implications are trivially not equivalences:

\[
H = H_d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

has \( \min \lambda(H) = \min \lambda(H_d) = -1 < 0 \) but \( \max \lambda(H) = \max \lambda(H_d) = 1 > 0 \).

\[
\begin{align*}
\text{B.3 Proof of Theorem 1} \\
\text{Theorem 1.} & \quad \text{There are no reasonable global algorithms, even for two-player markets. [There is a coercive, nondegenerate, analytic two-player market } G \text{ such that any reasonable algorithm produces bounded but non-convergent iterates, or divergent iterates with infinite losses for all players.]}
\end{align*}
\]

\[
\text{Proof.} \quad \text{Consider the analytic market } G \text{ given by}
\]

\[
L^1(x, y) = x^6/6 - x^2/2 + xy + \frac{1}{4} \left( \frac{y^4}{1 + x^2} - \frac{x^4}{1 + y^2} \right)
\]

\[
L^2(x, y) = y^6/6 - y^2/2 - xy - \frac{1}{4} \left( \frac{y^4}{1 + x^2} - \frac{x^4}{1 + y^2} \right)
\]

with simultaneous gradient

\[
\xi = \begin{pmatrix}
x^5 - x + y - \frac{y^4 x}{2(1 + x^2)^2} - \frac{x^3}{1 + y^2} \\
y^5 - y - x - \frac{x^4 y}{2(1 + y^2)^2} - \frac{y^3}{1 + x^2}
\end{pmatrix}.
\]

One may painfully prove that the only real solution to \( \xi = 0 \) is the origin ‘by hand’, but we may as well take advantage of computer algebra systems to find the exact number of real roots using the resultant matrix and Sturm’s theorem. Singular [9] is one such free and open-source system for polynomial computations. First convert the equations into polynomials:

\[
\begin{align*}
2(1 + x^2)^2(1 + y^2)(x^5 - x + y) - y^4 x(1 + y^2) - 2 x^3 (1 + x^2)^2 &= 0 \\
2(1 + y^2)^2(1 + x^2)(y^5 - y - x) - x^4 y(1 + x^2) - 2 y^3 (1 + y^2)^2 &= 0.
\end{align*}
\]

We compute the resultant matrix determinant of the system with respect to \( y \), a univariate polynomial \( P \) in \( x \) whose zeros are guaranteed to contain all solutions in \( x \) of the initial system. We then use the Sturm sequence of \( P \) to find its exact number of real roots. This is implemented with the following Singular code, whose output is 1.\(^1\)

\[
\begin{align*}
\text{LIB "solve.lib"; LIB "rootsur.lib";} \\
\text{ring r = (0,x),(y),dp;} \\
poly p1 = 2*(1+x^2)^2*(1+y^2)*(x^5-x+y)-y^4*x*(1+y^2)-2*x^3*(1+x^2)^2; \\
poly p2 = 2*(1+y^2)^2*(1+x^2)*(y^5-y-x)-x^4*y*(1+x^2)-2*y^3*(1+y^2)^2; \\
\text{ideal i = p1,p2;} \\
poly f = \text{det(mp_res_mat(i));} \\
\text{ring s = 0,(x,y),dp;} \\
poly f = \text{imap(r, f);} \\
\text{nrroots(f);} \\
\end{align*}
\]

We know that \( \bar{x} = 0 \) is a real solution, so \( \bar{x} \) must be the unique critical point. The Hessian at \( \bar{x} \) is

\[
H(\bar{x}) = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix},
\]

\(^1\)Alternatively, we can use Singular’s solve procedure on the ideal i, which outputs a list of 89 solutions, only one being real. This relies on Gröbner bases, but we are not familiar with the procedure’s inner workings.
which is negative definite since $S(\bar{x}) = -I \prec 0$, so $\bar{x}$ is a nondegenerate strict maximum. We now prove coercivity of $G$, which follows by noticing that the dominant terms are $x^6/6$ and $y^4/(1 + x^2)$.

Formally, first note that $\frac{x^2}{1+y^2} \leq x^4$, hence

$$L^1 \geq x^6/6 - x^4/4 - x^2/2 + xy + \frac{1}{4} \left( \frac{y^4}{1+x^2} \right).$$

Now $xy \geq -|xy| \geq -(2x^2 + y^2/8)$ by Young’s inequality, hence

$$L^1 \geq x^6/6 - x^4/4 - 5x^2/2 - y^2/8 + \frac{1}{4} \left( \frac{y^4}{1+x^2} \right).$$

For any sequence $\|x\| \to \infty$, either $|x| \to \infty$ or $|x|$ is bounded above by some $k \in \mathbb{R}$ and $|y| \to \infty$. In the latter case, we have

$$\lim_{\|x\| \to \infty} L^1 \geq \lim_{|y| \to \infty} -k^4/4 - 5k^2/2 - y^2/8 + \frac{y^4}{4(1+k^2)} = \infty$$

since the leading term $y^4$ is of even degree and has positive coefficient, so we are done. Otherwise, for $|x| \to \infty$, we pursue the previous inequality to obtain

$$L^1 \geq x^6/6 - x^4/4 - 5x^2/2 + \frac{y^2}{8} \left( \frac{2y^2}{1+x^2} - 1 \right).$$

Now notice that $y^2 \geq x^2 \geq 1$ implies

$$L^1 \geq x^6/6 - x^4/4 - 5x^2/2 + \frac{x^2}{8} \left( \frac{x^2 - 1}{1+x^2} \right) \geq x^6/6 - x^4/4 - 5x^2/2 - x^2/8.$$

On the other hand, $x^2 \geq y^2$ also implies

$$L^1 \geq x^6/6 - x^4/4 - 5x^2/2 - x^2/8$$

by discarding the first (positive) term in the brackets. Both cases lead to the same inequality and hence, for any sequence with $|x| \to \infty$,

$$\lim_{\|x\| \to \infty} L^1 \geq \lim_{|x| \to \infty} x^6/6 - x^4/4 - 5x^2/2 - x^2/8 = \infty$$

since the leading term $x^6$ has even degree and positive coefficient. Hence $L^1$ is coercive, and the same argument holds for $L^2$ by swapping $x$ and $y$. A reasonable algorithm almost surely converges nowhere since the unique critical point is a strict minimum, so iterates are either bounded but non-convergent or divergent with infinite loss for both players. \qed

B.4 Non-convergence of existing algorithms in a zero-sum game

**Proposition 3.** There is an analytic, nondegenerate zero-sum game such that existing gradient-based methods have bounded but non-convergent iterates for suitable hyperparameters.

**Proof.** Consider the zero-sum analytic game $G'$ given by

$$L^1 = xy - x^2/2 + y^2/2 + x^4/4 - y^4/4 = -L^2$$

with simultaneous gradient

$$\xi = \left( \begin{array}{c} y - x + x^3 \\ -x - y + y^3 \end{array} \right).$$

The only solution to $\xi = 0$ is the origin since $\xi_1 = 0$ implies $y = x(1 - x^2)$ and so

$$0 = \xi_2 = x \left( -2 + x^2 + x^2(1 - x^2) \right).$$

The polynomial inside the brackets has no real roots, so $x = 0$ and hence $\xi_1 = 0 = y$. The game is nondegenerate since the origin has invertible Hessian

$$H = \left( \begin{array}{cc} -1 & 1 \\ -1 & -1 \end{array} \right).$$

The game is weakly coercive since $L^1(x, y) \to \infty$ for any fixed $\bar{y}$ by domination of the $x^4$ term; similarly for $L^2(x, y)$ by domination of the $y^4$ term. We show that existing methods have bounded but non-convergent iterates in this game in Figure 2. \qed

14
Theorem 2. The only critical point of $G$ is the strict maximum $\bar{\theta}$. The sequence $M(\theta_k)$ is monotonically decreasing and bounded below, hence convergent. In particular, $\lim_{k \to \infty} M(\theta_{k+1}) = \lim_{k \to \infty} M(\theta_k)$. By continuity of $M$ and $F$, we have

$$M(F(\bar{\theta})) = \lim_{k \to \infty} M(F(\theta_k)) = \lim_{k \to \infty} M(\theta_k) = M(\bar{\theta})$$

and hence $F(\bar{\theta}) = \bar{\theta}$. Since $F$ is reasonable, $\bar{\theta}$ must be a critical point.

Corollary 1. There are no measures of progress for reasonable algorithms, even in two-player markets, provided iterates are bounded in $G$ for suitable hyperparameters.

Proof. The only critical point of $G$ is the strict maximum $\bar{\theta}$. Since $G$ is coercive, $\theta_k$ is a bounded sequence and must have at least one accumulation point, by Bolzano-Weierstrass. If a measure of progress exists, any accumulation point must be $\bar{\theta}$ by the previous proposition. A sequence with exactly one accumulation point is convergent, hence $\theta_k \to 0$. This is in contradiction with the algorithm being reasonable. $\square$

B.5 Proof of Proposition 4 and Corollary 1

Proposition 4. Assume a measure of progress $M$ exists for some reasonable algorithm $F$. Then the set of accumulation points of $\theta_k$ is a subset of critical points.

Proof. We follow the proof of Lange [13, Prop. 12.4.2]. Consider any accumulation point $\bar{\theta} = \lim_{m \to \infty} \theta_{k_m}$. The sequence $M(\theta_k)$ is monotonically decreasing and bounded below, hence convergent. In particular, $\lim_{m} M(\theta_{k_{m+1}}) = \lim_{m} M(\theta_{k_{m}})$. By continuity of $M$ and $F$, we have

$$M(F(\bar{\theta})) = \lim_{m} M(F(\theta_{k_{m}})) = \lim_{m} M(\theta_{k_{m}}) = M(\bar{\theta})$$

and hence $F(\bar{\theta}) = \bar{\theta}$. Since $F$ is reasonable, $\bar{\theta}$ must be a critical point. $\square$

Corollary 1. There are no measures of progress for reasonable algorithms, even in two-player markets, provided iterates are bounded in $G$ for suitable hyperparameters.

Proof. The only critical point of $G$ is the strict maximum $\bar{\theta}$. Since $G$ is coercive, $\theta_k$ is a bounded sequence and must have at least one accumulation point, by Bolzano-Weierstrass. If a measure of progress exists, any accumulation point must be $\bar{\theta}$ by the previous proposition. A sequence with exactly one accumulation point is convergent, hence $\theta_k \to 0$. This is in contradiction with the algorithm being reasonable. $\square$

B.6 Proof of Theorem 2

Parameters are initialised following any continuous measure $\nu$ on $\mathbb{R}^d$.

Theorem 2. There are no reasonable global algorithms, even for two-player markets with a strict minimum and no other critical points. There is a coercive, nondegenerate, almost-everywhere analytic two-player market with a unique and minimum critical point such that any reasonable algorithm produces bounded but non-convergent iterates, or divergent iterates with infinite losses for all players, with arbitrarily high probability.

Proof. We modify the construction from Theorem 1 by deforming a small region around the maximum to replace it with a minimum. This leads to identical gradient dynamics provided we initialise outside of the region, which can be made arbitrarily small. First let $0 < \sigma < 0.1$ and define

$$f_\sigma(x, y) = \begin{cases} 
(x^2 + y^2 - \sigma^2)/2 & \text{if } \|x\| \geq \sigma \\
(y^2 - 3x^2)(x^2 + y^2 - \sigma^2)/(2\sigma^2) & \text{otherwise}.
\end{cases}$$

Note that $f_\sigma$ is continuous since

$$\lim_{\|x\| \to \sigma^+} f_\sigma(x, y) = 0 = \lim_{\|x\| \to \sigma^-} f_\sigma(x).$$

Figure 2: Existing algorithms exhibit limit cycles in $G$ with learning rate $\alpha = 0.02$. Algorithms may diverge to infinite losses for larger $\alpha$, and CO may converge to the strict maximum for large $\gamma$. 
Now consider the two-player market $G_\sigma$ given by

$$L^1(x, y) = x^6/6 - x^2 + f_\sigma(x, y) + xy + \frac{1}{4} \left( \frac{y^4}{1 + x^2} - \frac{x^4}{1 + y^2} \right)$$

$$L^2(x, y) = y^6/6 - f_\sigma(x, y) - xy - \frac{1}{4} \left( \frac{y^4}{1 + x^2} - \frac{x^4}{1 + y^2} \right).$$

The resulting losses are continuous but not differentiable; however, they are analytic (in particular smooth) almost everywhere, namely, for all $x$ not on the circle of radius $\sigma$. This is sufficient for the purposes of gradient-based optimization, noting that games such as GANs also fail to be everywhere-differentiable in the presence of rectified linear units.

We claim that $G_\sigma$ has a single critical point at the origin $\bar{x} = 0$, for any $0 < \sigma < 0.1$. First note that

$$\xi_{G_\sigma} = \xi_{G_0} = \left( x^5 - x + y - \frac{y^4 x}{2(1 + x^2)} - \frac{x^3}{1 + y^2} \right)
\left( y^5 - y - x - \frac{x^4 y}{2(1 + y^2)} - \frac{y^3}{1 + x^2} \right) = \xi_{G_0}$$

for all $||x|| \geq \sigma$, where $G$ is the game from Theorem 1. It was proved there that the only real solution to $\xi = 0$ is the origin, which does not satisfy $||x|| \geq \sigma$. Any critical point must therefore satisfy $||x|| < \sigma$, for which

$$\xi = \xi_{G_\sigma} = \left( x^5 + x + y - 2x(3x^2 + y^2)/\sigma^2 - \frac{y^4 x}{2(1 + x^2)} - \frac{x^3}{1 + y^2} \right)
\left( y^5 + y - x - 2y (y^2 - x^2)/\sigma^2 - \frac{x^4 y}{2(1 + y^2)} - \frac{y^3}{1 + x^2} \right).$$

First note that $\bar{x} = 0$ is a critical point; we prove that there are no others. The continuous parameter $\sigma$ prevents us from using a formal verification system as in Theorem 1, so we must work ‘by hand’. Warning: the proof is a long inelegant string of case-by-case inequalities.

Assume for contradiction that $\xi = 0$ with $x \neq 0$. First note that $||x|| < \sigma$ implies $|x|, |y| < \sigma$, and $x = 0$ or $y = 0$ implies $x = y = 0$ using $\xi_1 = 0$ or $\xi_2 = 0$ respectively. We can therefore assume $0 < |x|, |y| < \sigma$. We deal with $x > 0$, the opposite case following almost identically.

1. We begin with the case $\sigma/2 \leq x < \sigma$. First notice that

$$x + y - 2x(3x^2 + y^2)/\sigma^2 = x(1 - 6x^2/\sigma^2) + y(1 - 2xy/\sigma^2) \leq x(1 - 3/2) + y(1 - y/\sigma)$$

and the rightmost term attains its maximum value for $y = \sigma/2$, hence

$$x + y - 2x(3x^2 + y^2)/\sigma^2 \leq -x/2 + \sigma/4 \leq 0.$$

This implies

$$\xi_1 \leq x^5 - \frac{y^4 x}{2(1 + x^2)} - \frac{x^3}{1 + y^2} < x^5 - \frac{x^3}{1 + y^2} < x^3 \left( 1 - y^2 - \frac{1}{1 + y^2} \right) = \frac{-x^3 y^4}{1 + y^2} < 0$$

using $x^2 + y^2 < 1$, which is a contradiction to $\xi = 0$.

2. We proceed with the case $x < \sigma/2$ and $|y| \leq \sigma/2$. First, $y < 0$ implies the contradiction

$$\xi_2 < y - 2y^3/\sigma^2 - \frac{x^4 y}{2(1 + y^2)} < y^3/1 + x^2 < y/2 - y \left( \frac{y^4}{2} + \frac{\sigma^4}{2} \right) < y \left( \frac{1}{2} - \frac{1}{25} - \frac{1}{25} \right) < 0,$$

so we can assume $y > 0$. In particular we have $(1 - 2y(y + x)/\sigma^2) > 0$. If $y \leq x$ we also obtain

$$\xi_2 < y^5 + (y - x) (1 - 2y(y + x)/\sigma^2) - \frac{y^3}{1 + x^2} < y^3 \left( y^2 - \frac{1}{1 + x^2} \right) = \frac{-y^3 x^4}{1 + x^2} < 0,$$

so we can assume $x < y$. There are again two cases to distinguish. If $x < \sigma/2 - 2b\sigma^2$ with $b = 0.08$,

$$x(1 - 6x^2/\sigma^2) + y(1 - 2xy/\sigma^2) > x(1 - 3(1/2 - \sigma b)) + x(1 - (1/2 - \sigma b)) > 4\sigma bx$$

and $x(1 - 6x^2/\sigma^2) + y(1 - 2xy/\sigma^2) > x(1 - 3(1/2 - \sigma b)) + x(1 - (1/2 - \sigma b)) > 4\sigma bx$
We claim that the rightmost term is negative. Indeed, the quantity inside the brackets has derivative
\[ y \frac{d}{dx} \left( \frac{y^4}{2} \right) = \frac{4y^3}{2} > 0. \]
We can therefore assume
\[ \frac{y^3}{2} > 0. \]
which implies the contradiction
\[ \xi_1 > 4\sigma y - \frac{y^4}{2(1 + x^2)^2} > -\frac{x^3}{1 + y^2} > -\sigma x \left( 4b - \frac{\sigma^4}{2} - \frac{\sigma^2}{2} \right) > 0. \]
Finally assume \( x \geq \sigma/2 - b\sigma^2 \). Then we have
\[ (y - x)(1 - 2y(x + y)/\sigma^2) < b\sigma^2(1 + 4x^2/\sigma^2) < b\sigma^2(1 - (1 - 2\sigma b)^2) = 4\sigma^3 b^2(1 - \sigma b) < 4\sigma^3 b^2 \]
and obtain
\[ \xi_2 < y^5 + 4\sigma^3 y^2 - \frac{y^3}{1 + x^2} < \sigma^3 \left( \frac{y^2}{2} + 4b^2 - \frac{(1/2 - \sigma b)^3}{1 + \sigma^2/4} \right). \]
We claim that the rightmost term is negative. Indeed, the quantity inside the brackets has derivative
\[ \sigma/2^4 + \frac{(1/2 - \sigma b)^2}{(1 + \sigma^2/4)^2} (3b(1 + \sigma^2/4) + \sigma(1/2 - \sigma b)/2) > 0 \]
and so its supremum across \( \sigma \in [0, 0.1] \) must be attained at \( \sigma = 0.1 \). We obtain the contradiction
\[ \xi_2 < \sigma^3 \left( 0.01/2^5 + 4b^2 - \frac{(1/2 - \sigma b)^3}{1 + 0.01/4} \right) \approx -0.09 < 0. \]
as required.

3. Finally, consider the case \( x < \sigma/2 \) and \( |y| > \sigma/2 \). First, \( y < 0 \) implies the contradiction
\[ \xi_1 < x + y - 2x(3x^2 + y^2)/\sigma^2 < -2x(3x^2 + y^2) < 0 \]
so we can assume \( y > 0 \). Now assume \( y < \sigma - x(1 + \sigma^2) \). Then
\[ x(1 - 6x^2/\sigma^2) + y(1 - 2xy/\sigma^2) > -x/2 + y(1 - y/\sigma) > -x/2 + x(1 + \sigma^2) > x(1/2 + \sigma^2), \]
which yields the contradiction
\[ \xi_1 > x \left( \frac{1}{2} + \sigma^2 - \frac{y^4}{2(1 + x^2)^2} - \frac{x^3}{1 + y^2} \right) > x \left( 1/2 + \sigma^2 - \sigma^4 - \sigma^2/4 \right) > x(1/2 - 1/4) > 0. \]
We can therefore assume \( y \geq \sigma - x(1 + \sigma^2) \). We have
\[ (y - x)(1 - 2y(y + x)/\sigma^2) < (y - x)(1 - (y + x)/\sigma) \leq (y - x)(1 - (1 - \sigma x)) < \sigma x(y - x) \]
which attains its maximum in \( x \) at \( x = y/2 \), hence
\[ \xi_2 < y^5 - \frac{y^3}{1 + x^2} + \frac{\sigma y^2}{4} < \frac{\sigma y^2}{4} \left( 4\sigma^2 - \frac{2}{1 + \sigma^2} + 4 \right). \]
Finally we obtain the contradiction
\[ \xi_2 < \frac{\sigma y^2}{4} \left( \frac{5\sigma^2 + 4\sigma^4 - 1}{1 + \sigma^2} \right) < 0 \]
for all \( \sigma < 0.1 \). All cases lead to contradictions, so we conclude that \( \bar{x} \) is the only critical point, with positive definite Hessian
\[ H(\bar{x}) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} > 0, \]
hence \( \bar{x} \) is a strict minimum. Now notice that \( \mathcal{G}_0 \) has the same dominant terms as \( \mathcal{G} \) from Theorem 1, so coercivity of \( \mathcal{G}_0 \) follows from the same argument. Since \( \mathcal{G}_{\sigma} \) is identical to \( \mathcal{G}_0 \) outside the \( \sigma \)-ball \( B_{\sigma} = \{(x, y) \in \mathbb{R}^2 \mid \|x\| < \sigma \} \), coercivity of \( \mathcal{G}_0 \) implies coercivity of \( \mathcal{G}_{\sigma} \) for any \( \sigma \).

Fix any reasonable algorithm \( \mathcal{A} \), any continuous measure \( \nu \) on \( \mathbb{R}^d \) and any \( \epsilon > 0 \). We abuse notation somewhat and write \( F_{\sigma}^k(x_0) \) for the \( k \)th iterate of \( \mathcal{A} \) in \( \mathcal{G}_{\sigma} \) with initial parameters \( x_0 \). We claim that for any bounded region \( \bar{U} \), there exists \( \sigma > 0 \) satisfying
\[ P_{\nu} \left( x_0 \in U \text{ and } \lim_{k} F_{\sigma}^k(x_0) = \bar{x} \right) < \epsilon. \]
We prove more: GD avoids strict saddles in all differentiable games. The proof is essentially identical to the analogous result for single losses, as found in [22, Th. 3], except that we consider bounded iterates or divergent iterates with infinite loss with probability $1 - \epsilon$, proving the theorem.

First, by continuity of $\nu$ with respect to Lebesgue measure, we can choose $\sigma' > 0$ such that $P_{\nu}(x_0 \in B_{\sigma'}) < \epsilon/2$, since $\mu(B_{\sigma'}) \to 0$ as $\sigma \to 0$.

Now let $\bar{U}$ be the closure of $U$ and define $D = \bar{U} \cap \{|x| \geq \sigma'\}$. Note that $D$ is compact since $\bar{U}$ is compact and closed subsets of a compact set are compact. $A$ is reasonable, $D$ is bounded and $\bar{x}$ is a strict maximum in $g_0$, so there are hyperparameters such that the stable set

$$Z = \{x_0 \in D \mid \lim_{k} F^k_0(x_0) = \bar{x}\}$$

has zero measure. By the regularity theorem for Lebesgue measure, there exist open sets which contain $Z$ with arbitrary small measure. By continuity of $\mu$, we can therefore find $Z'$ open such that

$$P_{\nu}(x_0 \in Z') < \epsilon/2$$

with $Z \subset Z' \subset D$. Now for any $x_0 \in D \setminus Z'$, we know that $\lim_{k} F^k_0(x_0) \neq 0$ and in particular, $\inf_{k \in \mathbb{N}} \{\|F^k_0(x_0)\|\} > 0$. We can thus define the continuous function $h : D \setminus Z' \to \mathbb{R}^+$ given by

$$h(x_0) = \inf_{k \in \mathbb{N}} \{\|F^k_0(x_0)\|\}.$$ 

Since $Z'$ is open and $D$ is compact, $D \setminus Z'$ is compact and hence $h(D \setminus Z')$ is compact in $\mathbb{R}^+$. In particular, there exists $\delta > 0$ such that $h(x_0) \geq \delta$ and thus $\|F^k_0(x_0)\| \geq \delta$ for all $k \in \mathbb{N}$ and $x_0 \in D \setminus Z'$. Now let $\sigma = \min\{\sigma', \delta\}$ and notice that

$$x_0 \in D \setminus Z' \implies \inf_{k} \|F^k_0(x_0)\| \geq \delta \geq \sigma \implies \inf_{k} \|F^k_0(x_0)\| \geq \sigma.$$ 

The last implication holds since $g_0$ and $g_0$ are indistinguishable in $\{|x| \geq \sigma\}$, so the algorithm must have identical iterates $F^k_0(x_0) = F^k_\sigma(x_0)$ for all $k$. It follows in particular that $\lim_{k} F^k_\sigma(x_0) = \bar{x}$ implies $x_0 \notin D \setminus Z'$, and so $x_0 \in Z'$ or $x_0 \notin D$. Finally we obtain

$$P_{\nu}\left(x_0 \in U \text{ and } \lim_{k} F^k_\sigma(x_0) = \bar{x}\right) = P_{\nu}\left(x_0 \in U \cap Z' \text{ or } x_0 \in U \setminus D\right) \leq P_{\nu}\left(x_0 \in U \cap Z'\right) + P_{\nu}\left(x_0 \in U \setminus D\right) \leq P_{\nu}\left(x_0 \in Z'\right) + P_{\nu}\left(x_0 \in B_{\sigma'}\right) < \epsilon/2 + \epsilon/2 = \epsilon$$

as required. \(\square\)

### B.7 Proof of Proposition 5

We first state the Stable Manifold Theorem from dynamical systems theory, adapted from [27, p. 65].

**Theorem (Stable Manifold Theorem).** Let $\bar{\theta}$ be a fixed point for the $C^1$ local diffeomorphism $F : U \to \mathbb{R}^d$, where $U$ is a neighbourhood of $\bar{\theta}$ in $\mathbb{R}^d$. Let $E_a \oplus E_u$ be the generalised eigenspaces of $\nabla F(\bar{\theta})$ corresponding to eigenvalues with $|\lambda| \leq 1$ and $|\lambda| > 1$ respectively. Then there exists a local stable center manifold $W$ with tangent space $E_a$ at $\bar{\theta}$ and a neighbourhood $B$ of $\bar{\theta}$ such that $F(W) \cap B \subset W$ and $\cap_{n=0}^{\infty} F^{-n}(B) \subset W$.

In particular, if $\nabla F(\bar{\theta})$ has at least one eigenvalue $|\lambda| > 1$ then $E^u$ has dimension at least 1. Since $W$ has tangent space $E^s$ at $\bar{x}$, with codimension at least one, it follows that $W$ has measure zero in $\mathbb{R}^d$.

**Proposition 5.** GD is reasonable for differentiable games.

We prove more: GD avoids strict saddles in all differentiable games. The proof is essentially identical to the analogous result for single losses, as found in [22, Th. 3], except that we consider bounded neighbourhoods to avoid requiring global Lipschitz continuity of $\xi$; extra care is required.

**Proof.** Fixed points of GD coincide exactly with critical points, since $F(\theta) = \theta - \alpha \xi(\theta) = \theta$ iff $\xi(\bar{\theta}) = 0$. Let $\bar{\theta}$ be a strict saddle and take any bounded neighbourhood $U \ni \bar{\theta}$. Since $\xi \in C^1$ and $U$ is bounded, $\xi_U : U \to \mathbb{R}^d$ is $L$-Lipschitz for some finite $L$. We prove that the stable set

$$Z = \{\theta \in U \mid \lim_{k} F^k(\theta) = \bar{\theta}\}$$
has Lebesgue measure zero for any $0 < \alpha < 1/L$. First, we claim that $F_U : U \to \mathbb{R}^d$ is a $C^1$ local diffeomorphism, and a diffeomorphism onto its image. By Lemma 0, the eigenvalues of $H$ in $U$ satisfy $|\lambda| \leq \|H\| = \|\nabla \xi\| \leq L$, hence $\nabla F_U = I - \alpha H$ has eigenvalues $1 - \alpha \lambda \geq 1 - \alpha |\lambda| \geq 1 - \alpha L > 0$. It follows that $\nabla F_U$ is invertible everywhere, so $F_U$ is a local diffeomorphism by the Inverse Function Theorem [28, Th. 2.11]. To prove that $F_U : U \to F(U)$ is a diffeomorphism, it is sufficient to show injectivity of $F_U$. Assume for contradiction that $F_U(\theta) = F_U(\theta')$ with $\theta \neq \theta'$. Then by definition,
\[\theta - \theta' = \alpha(\xi_U(\theta') - \xi_U(\theta))\]
and so
\[\|\theta - \theta'\| = \alpha \|\xi_U(\theta') - \xi_U(\theta)\| \leq \alpha L \|\theta - \theta'\| < \|\theta - \theta'\|,
\]
a contradiction. We conclude that $F_U$ is a diffeomorphism onto its image.

We can now invoke the Stable Manifold Theorem to obtain a neighbourhood $B$ of $\bar{\theta}$ and a stable center manifold $W$ such that $\cap_{n=0}^\infty F_U^{-n}(B) \subset W$. Since $\bar{\theta}$ is a strict saddle, $H(\bar{\theta})$ has a negative eigenvalue $-\alpha + ib$ with $a > 0$, so $\nabla F(\bar{\theta}) = I - \alpha H(\bar{\theta})$ has an eigenvalue $\lambda = 1 + \alpha a - i\alpha b$ with $|\lambda| \geq |1 + \alpha a| > 1$. This implies that the eigenspace $E^s$ of eigenvalues with $|\lambda| \leq 1$ has codimension at least 1. Since $W$ has tangent space $E^s$ at $\bar{\theta}$, we conclude that $W$ has Lebesgue measure zero. Now notice that $\theta \in Z$ implies the existence of $K \in \mathbb{N}$ such that $F_U^K(\theta) \in B$ for all $k \geq K$, hence
\[F_U^K(\theta) \in \cap_{n=0}^\infty F_U^{-n}(B) \subset W\]
and thus
\[\theta \in \cup_{k=0}^\infty F_U^{-k}(W).
\]
This implies $Z \subset \cup_{k=0}^\infty F_U^{-k}(W)$. Now $F_U$ is a diffeomorphism onto its image, with continuously differentiable inverse $F_U^{-1}$, hence locally Lipschitz, thus preserving measure zero sets. By induction, $\mu(F_U^{-k}(W)) = 0$ for all $k$, and countable unions of measure zero sets have zero measure, so finally
\[\mu(Z) \leq \mu \left( \cup_{k=0}^\infty F_U^{-k}(W) \right) = 0\]
as required. \hfill \Box

### B.8 Proof of Proposition 6

**Proposition 6.** GND is global for all differentiable games with coercive gradient norm.

**Proof.** In an analytic game $G$, $\mathcal{H} = \|\xi\|^2 / 2$ must also be analytic, in particular $C^2$. By coercivity and Proposition 1, GD on $\mathcal{H}$ has convergent iterates $\theta_k$ for any $\theta_0$ and suitable $\alpha > 0$. \hfill \Box

### B.9 Proof of Proposition 7

**Proposition 7.** GD is reasonable and global for potential games with coercive potential.

**Proof.** GD is reasonable for all differentiable games by Proposition 5.

It was shown in [4] that a game is potential iff there exists a so-called potential function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that $\nabla_i L^i = \nabla_i \phi$ for all $i$. It follows that $\xi = \nabla \phi$, so simultaneous GD corresponds to GD on the single loss $\phi$. To prove globality, assume the game is analytic. Then $\nabla \phi = \xi$ is analytic, and so must be the potential $\phi$. [A roundabout but curious way to see this is that $\Delta \phi$ is analytic, where $\Delta$ is the Laplace operator, hence $\phi$ is analytic by the elliptic regularity theorem.]

It now follows by assumption of coercivity and Proposition 1 that for any fixed $\theta_0$, GD on $\phi$ has convergent iterates $\theta_k$ to some critical point $\bar{\theta}$ of $\phi$ for appropriate $\alpha$. Finally, $\bar{\theta}$ is a critical point since $\xi(\bar{\theta}) = \nabla \phi(\bar{\theta}) = 0$. Hence GD is global in potential games with coercive potential. \hfill \Box

### B.10 Proof of Proposition 8

**Proposition 8.** GND is reasonable and global for Hamiltonian games with everywhere-invertible Hessian and coercive gradient norm.
Proof. GND is global for all differentiable games with coercive gradient norm by Proposition 6.

Any algorithm trivially avoids strict maxima in Hamiltonian games: $S \equiv 0$ implies there are none. Fixed points of GND are critical points since $H^T(\bar{\theta})\xi(\bar{\theta}) = 0$ implies $\xi(\bar{\theta}) = 0$ by invertibility. 

We can superficially weaken the invertibility assumption by proving that Hamiltonian games have constant Hessian; everywhere-invertibility becomes equivalent to invertibility at some point.

Proposition. Hamiltonian games have constant Hessian.

Proof. First denote $\partial_n f = \partial f / \partial \theta_n$, where $\theta_n$ is the $n$th coordinate of $\theta^i$. Since $S \equiv 0$, we have $\nabla_{ij} L^i = -\nabla_{ij} L^j$ for all $i, j$. This is equivalent to $\partial_n \partial^i_{mn} L^i = -\partial_n \partial^i_{lm} L^j$ for all $i, j, n, m$. Now for any agent $k$ and coordinate $l$, using Schwarz’s theorem to cycle through agents,

$$\partial^k_n \partial^i_{ln} \partial^j_{ml} L^i = -\partial^k_n \partial^i_{ln} \partial^j_{ml} L^j = -\partial^i_{ln} \partial^k_{ml} L^j = -\partial^i_{ln} \partial^k_{ml} L^j = -\partial^i_{ml} \partial^k_{ln} L^j = -\partial^k_{ml} \partial^i_{ln} L^j.$$

We obtain $\partial^k_n (\partial^i_{ln} \partial^j_{ml} L^i) = 0$ for all $k, l$ and thus $\partial^i_{ln} \partial^j_{ml} L^i$ is constant for all $i, j, n, m$. This implies that $\nabla_{ij} L^i$ is constant for all $i, j$, and so must be the Hessian as given by $H_{ij} = \nabla_{ij} L^i$.

The careful reader may notice that we assumed $\nabla_{ij} L^i$ to be twice continuously differentiable to invoke Schwarz’s theorem, while we only know $\nabla_{ij} L^i \in C^1$. It turns out that $\nabla_{ij} L^i \in C^2$ necessarily holds in Hamiltonian games: first notice $\nabla_{ij} L^i = 0$ implies $\nabla_{ij} L^i$ constant in $i$; now $\nabla_{ij} (\nabla_{ij} L^i) = -\nabla_{ij} L^j$ is constant in $j$ since $\nabla_{ij} L^j$ is constant in $j$, hence $\nabla_{ij} L^i = \nabla_{ij} L^j \theta^j + F$ for some $F$ constant in $i, j$. Now the LHS $\nabla_{ij} L^i$ is continuously differentiable with respect to $\theta_k$ and so is the RHS term

$$F = \nabla_{ij} L^i - \nabla_{ij} L^i \theta^j = \nabla_{ij} L^i |\theta^j = 0$$

since $F$ is independent of $\theta^j$. Hence the middle term $\nabla_{ij} L^i$ must also be continuously differentiable for any $j$. We conclude that $\nabla_{ij} L^i$ exists and is continuous for all $i, j, k$ as required.

However, the property of having some nondegenerate point is no generic property in the class of Hamiltonian games, despite being weaker than the Morse property for single functions. For instance, there are no Hamiltonian games with invertible Hessian in odd dimension $d$:

$$\det(H) = \det(H^T) = \det(-H) = (-1)^d \det(H) = -\det(H).$$

As stated in the main text, fixed points of GND in odd-dimensional Hamiltonian games may therefore not be critical points. This is far from reasonable.

C Solution concepts in differentiable games

As highlighted by Proposition 2, standard optimization concepts (minima, maxima, saddles) generalise in different ways to differentiable games, giving rise to a rich set of concepts. Let us consider here the case of strict saddles, which are defined for single losses as critical points $\bar{\theta}$ whose Hessian $H(\bar{\theta})$ has at least one negative eigenvalue, namely, $\min \lambda(H) < 0$. We consider three possible generalisations below, corresponding to the right-hand triangle in Proposition 2. Recall that $\lambda(A) = \Re(\text{Spec}(A))$.

Definition 8. Let $\bar{\theta}$ be a critical point.

1. $\bar{\theta}$ is called a strict saddle if $\min \lambda(H(\bar{\theta})) < 0$.
2. $\bar{\theta}$ is called a strict Nash saddle if $\min \lambda(H_d(\bar{\theta})) < 0$.
3. $\bar{\theta}$ is called a strict symmetric saddle if $\min \lambda(S(\bar{\theta})) < 0$.

This already-rich set of concepts does not arise in single-player games since all three concepts collide: $H = H_d = S = \nabla^2 L$. The terminology is justified as follows:

1. Strict saddles are the most immediate generalisation, and correspond to those points almost surely avoided by GD, as shown in Proposition 5. In this sense, they are the ‘natural’ or ‘simultaneous’ generalisation of single-player strict saddles.
2. Since $H_d = \bigoplus_i \nabla_i L^i$, strict Nash saddles are also defined by $\nabla_i L^i(\tilde{\theta})$ having a negative eigenvalue for some $i$. In other words, player $i$ can decrease its loss locally around $\tilde{\theta}$ to second order, at odds with the concept of Nash equilibrium.

3. Strict symmetric saddles refer to the symmetric part $S$ of the Hessian. The authors do not believe this concept to be particularly intuitive or appropriate, but define it here for the sake of completeness. However, the class is a superset of the other two, see Proposition 2.

The first two classes are appealing in their own respects, but neither is a subset of the other.

**Proposition 9.** In a given game, let $\mathcal{X}, \mathcal{Y}$ be the set of strict saddles and Nash saddles respectively. Then there exist games such that $\mathcal{X} \not\subseteq \mathcal{Y}$ and $\mathcal{Y} \not\subseteq \mathcal{X}$.

**Proof.** For $\mathcal{X} \not\subseteq \mathcal{Y}$, consider the game

$$L_1 = x^2/2 + 2xy \quad \text{and} \quad L_2 = y^2/2 + 2xy$$

with

$$H = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$ 

There is a unique fixed point at the origin, where $H$ has a negative eigenvalue $-1$, while $H_d = I$ has positive eigenvalues $1$.

For $\mathcal{Y} \not\subseteq \mathcal{X}$, consider the game

$$L_1 = -x^2 + 4xy \quad \text{and} \quad L_2 = 2y^2 + 4xy.$$ 

with

$$H = \begin{pmatrix} -2 & 4 \\ -4 & 4 \end{pmatrix}.$$ 

There is a unique fixed point at the origin, where $H$ has positive eigenvalues $1 \pm i\sqrt{7}$, while $H_d$ has a negative eigenvalue $-2$.}

Let us dissect the two examples above to see how the concepts differ. In the first game, the origin is a strict saddle, but also a strict and global Nash equilibrium. This is rather surprising, and some of the community may rule out strict saddles as inappropriate for this reason. However, note crucially that both players can achieve arbitrarily low losses by following the anti-diagonal direction $y = -x \to \pm \infty$ instead of staying put at the origin. In this sense, the origin is to be avoided, despite being a Nash equilibrium, both justifying the ‘saddle’ terminology and challenging the validity of Nash equilibria as a solution concept in the context of optimization.

In the second game, the origin is a strict Nash saddle, but not a strict saddle. In fact, the origin is a stable equilibrium with respect to GD dynamics, and all recent gradient-based methods converge there (except for LSS, which is aimed precisely at converging to Nash equilibria). Again, this may be counterintuitive to game theorists since the first player always has a local incentive to move away from the origin, so converging there seems irrational. However, it is not advantageous to move away in the long run, since the second player will always respond by moving in the opposite direction and lead to a positive loss, at which point the first player will have to cross the origin again. In this sense, the origin is ‘stable’ and perhaps more reasonable than perpetual cycling around it.

It is clear now that both concepts have their raison d’être. The authors personally prefer strict saddles in the context of optimization, for the reasons discussed above, but it is intelligible to prefer one or the other in different settings.

In the same vein, maxima generalise in different ways to games. Recall that strict maxima are defined by $\max \lambda(H) < 0$ for single losses, or equivalently, $H \prec 0$. This again may be generalised to $\max \lambda(H) < 0$, $\max \lambda(H_d) < 0$ or $\max \lambda(S) < 0$. This final concept is equivalent to $S \prec 0$, which is equivalent to $H \prec 0$. In this sense it is perhaps the ‘closest’ to the single-loss case, thus our naming them ‘strict maxima’. It is also the smallest, being contained in the other two by Proposition 2.

Imposing that reasonable algorithms avoid strict maxima is therefore the weakest possible requirement of this kind, hoping to further convince the community of the impossibility of global convergence.