Introduction to Quantum Group Theory

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Abstract
This is a short, self-contained expository survey, focused on algebraic and analytic aspects of quantum groups. Topics covered include the definition of “quantum group,” the Yang-Baxter equation, quantized universal enveloping algebras, representations of braid groups, the KZ equations and the Kohno-Drinfeld theorem, and finally compact quantum groups and the analogue of Haar measure for compact quantum groups.

1 Introduction

There are two main approaches to quantum group theory, the purely algebraic approach, and the $C^*$-algebra approach, which uses deep connections with functional analysis. A $C^*$-algebra is a Banach algebra satisfying $\|a^* a\| = \|a\|^2$ for all elements $a$. Equivalently, a $C^*$-algebra is a norm closed $*$-subalgebra of $B(\mathcal{H})$, where $\mathcal{H}$ denotes a Hilbert space. The simple example of quantum matrix groups fits naturally into both frameworks. In this paper we endeavor to

- Define Hopf algebras, and their quasitriangular structures, which is the starting point of the algebraic approach.
- Discuss the Yang-Baxter equation and quantum matrix algebras, including an algebro-geometric view of the moduli of solutions.
- Give the construction of the quantized universal enveloping algebra in detail, both for the simplest nontrivial example ($\mathfrak{sl}_2$) and the general finite-dimensional Lie algebra $\mathfrak{g}$.
- Discuss the connection of quantum groups with representations of braid groups.
- Discuss an application of these techniques to understanding recent results of Drinfeld and Kohno regarding the Knizhnik-Zamolodchikov Equations.
- Discuss compact quantum groups, Haar measure on these groups, and their representation theory. This can be viewed in light of Connes’ formulation of noncommutative geometry.
2 Hopf Algebras and the Universal R-matrix

2.1 Notation

We begin by introducing the following *leg numbering notation*, which is due to Sweedler. If \( A \) is any unital algebra, and \( R \in A \otimes A \), then in general \( R \) admits a representation in the form \( R = \sum \alpha_i \otimes \beta_i \) (finite sum) for \( \alpha_i, \beta_i \) elements of \( A \). There are exactly three ways of extending \( R \) to an element of \( A \otimes A \otimes A \) by tensoring with \( 1 \in A \). The leg numbering notation is a way of distinguishing between them:

\[
R_{12} := \sum_i \alpha_i \otimes \beta_i \otimes 1 \\
R_{13} := \sum_i \alpha_i \otimes 1 \otimes \beta_i \\
R_{23} := \sum_i 1 \otimes \alpha_i \otimes \beta_i
\]

This notation is useful, in particular, for specifying the defining relations of the universal R-matrix, which we will do. More generally, we will write \( R_{ij} \) for the extension of \( R \) to \( A \otimes A \otimes A \) which "is" \( R \) in the \( i \)th and the \( j \)th component.

In this paper, \( \tau \) will always denote the flip operator which acts linearly on the second tensor power of a module by \( \tau(a \otimes b) = b \otimes a \).

2.2 Hopf Algebras

We first define coalgebras and bialgebras. The starting point is to write \( \mu : A \otimes A \to A \) for the multiplication map of an associative algebra over a field \( k \), and to note that a unit for the algebra \( A \) certainly defines a map \( k \to A \) which we can call \( \eta \). Then, the associativity of the multiplication and the nice multiplicative property of the unit can be written as follows:

\[
\begin{align*}
\text{associativity:} & \\
A \otimes A \otimes A & \xrightarrow{\mu \otimes \text{id}} A \otimes A \\
A \otimes A & \xrightarrow{\text{id} \otimes \mu} A \otimes A \\
A \otimes A & \xrightarrow{\mu \otimes \text{id}} A \otimes A \\
A & \xrightarrow{\mu} A \\
\text{unit:} & \\
A \otimes A & \xrightarrow{\eta \otimes \text{id}} A \\
A \otimes A & \xrightarrow{\text{id} \otimes \eta} A \\
A \otimes A & \xrightarrow{\mu} A \\
A & \xrightarrow{\mu} A
\end{align*}
\]

We thus take the definition of an *algebra* to be a vector space with the additional structure of a pair of linear maps \((\mu, \eta)\) making (1) a commutative diagram.

We can now dualize this construction, so that a *coalgebra* is a vector space \( C \) with the additional structure of a pair of linear maps \((\Delta, \epsilon)\) making (2) a commutative diagram.
A bialgebra structure on a vector space $A = C$ is a quadruple of objects $(\mu, \eta, \triangle, \epsilon)$ which satisfy all of the commutative diagrams (1)-(2) as well as the following compatibility equations:

$$\triangle(hg) = \triangle(h)\triangle(g), \quad \triangle(1) = 1 \otimes 1, \quad \epsilon(hg) = \epsilon(h)\epsilon(g), \quad \epsilon(1) = 1,$$

for all $g, h \in A$.

We define the convolution product of two linear maps $f, g : A \rightarrow A$ by the formula $f \ast g = \mu(f \otimes g)\triangle$. Not surprisingly, $\ast$ is associative and has neutral element $\eta \circ \epsilon$. In this situation, a bialgebra $(A, \mu, \eta, \triangle, \epsilon)$ is said to be a Hopf Algebra if the identity map $\text{id} : A \rightarrow A$ is invertible for the convolution product, and its inverse is the antipode $S$. The defining property of this antipode can also be represented as a commutative diagram:

**Definition 1** A quasitriangular Hopf algebra, also called a braided Hopf algebra, consists of the data of a Hopf algebra, $(A, \mu, \eta, \triangle, \epsilon, S)$ together with an invertible element $R \in A \otimes A$ satisfying the following two conditions:

1. $R\triangle(x)R^{-1} = \triangle'(x)$ for all $x \in A$.
2. $(\triangle \otimes \text{id})(R) = R_{13}R_{23}$ and $(\text{id} \otimes \triangle)(R) = R_{13}R_{12}$.

where $\triangle' = \tau \circ \triangle$ is the opposite comultiplication. In this situation, $R$ is called the universal R-matrix.

### 3 The Yang-Baxter Equation

The Yang-Baxter equation (3) is simple to write down, but it is more complicated to understand the motivation for its study; hence, this deeper kind of understanding will form the subject matter of this section.

**Lemma 1** From axioms (i)-(ii) in Definition 1 it follows that $R$ satisfies the Yang-Baxter Equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$  (3)
Proof of Lemma 1. Compute \((\text{id} \otimes \tau \circ \triangle) R\) in two ways, using the second part of (ii), or by first using (i) and then the second part of (ii). \(\square\).

Example 1 If \(G\) is an algebraic group or monoid then we define \(\mathcal{O}(G)\) to be the bialgebra of polynomial functions on \(G\). Of particular interest is \(G = M(n) = n \times n\) matrices over a field. There is a “standard” quantization of \(\mathcal{O}(M(n))\) for which the quantizing Yang-Baxter matrix is given by

\[
R = \exp(t\gamma_q) \exp(t\beta) \exp(t\gamma_q)
\]

where \(\gamma_q = \sum_{i<j} e_{ij} \wedge e_{ji}\), and \(\beta\) is a generic element of \(M(n) \otimes M(n)\).

3.1 Quantum Matrices and the Yang-Baxter Variety

Specializing to the matrix algebra \(M_n = M_n(k)\), we fix an element \(R \in M_n \otimes M_n\) satisfying (3), (an “R-matrix” from now on) and we first define the bialgebra \(A(R)\) to be generated by unit element 1, together with \(n^2\) indeterminates \(t = \{t^i_j\}\) with relations \(R t_1 t_2 = t_2 t_1 R\) and coalgebra structure:

\[
\Delta t = t \otimes t, \quad \epsilon t = \text{id} \tag{4}
\]

We resist the temptation to write everything using index notation on the tensors, but as an example, note that the first of equations (4) is shorthand for \(\Delta t^i_j = t^i_a \otimes t^a_j\), in which the sum over \(a\) is understood according to the summation convention. The bialgebra \(A(R)\) is called the algebra of quantum matrices.

It is interesting to consider the moduli space of all invertible elements \(R \in M_n \otimes M_n\) satisfying (3). In this situation, it makes sense to quotient by the equivalence of different normalizations. We define \(YB_n\) to be the resulting moduli space. It has naturally the topology of an algebraic variety, since an R-matrix can be interpreted as an element of the common zero loci of a collection of polynomials over \(k\). In this interpretation, the construction of the quantum matrix algebra \(A(R)\) which was discussed just above forms a kind of bundle over this variety, but the variety \(YB_n\) itself is far from being a smooth manifold. In general it contains singular disconnected points, lines, planes, and jumps in dimension (even when \(n\) is held fixed)!

3.2 Quantization of Poisson brackets

Drinfeld studied the quantization of Poisson brackets on the commutative algebra \(C^\infty(G)\) where \(G\) is a Lie group. Let \(X_i\) denote a basis of \(\mathfrak{g} = \text{the Lie algebra of } G\). Let \(\partial_i\) be the right-invariant vector field on \(G\) corresponding to \(X_i\), and let \(\partial'_i\) be the corresponding left-invariant vector field. Let \(R \in \bigwedge^2(\mathfrak{g})\) be given by \(R = r^{ij} X_i \otimes X_j\).

The Poisson brackets originally studied by Sklyanin, and more carefully by Drinfeld, are then given by equations of the form

\[
\{f, g\} = r^{ij} (\partial_i f \partial_j g - \partial'_i f \partial'_j g), \quad f, g \in C^\infty(G)
\]
4 \( \mathfrak{su}_2 \) and \( \mathfrak{sl}_2 \) Quantum Groups

In a course on finite-dimensional complex Lie algebras, it often makes sense to first study detailed properties of the Lie algebra \( \mathfrak{sl}_2 \), or its real form \( \mathfrak{su}_2 \), because the root space decomposition shows that a semisimple Lie algebra \( g \) can be viewed as containing a number of copies of \( \mathfrak{sl}_2 \) as subalgebras, and the properties of these subalgebras are crucial in deriving important properties of \( g \) and its representations. It turns out that something similar is true for the quantum groups obtained from semisimple complex Lie algebras, the Quantized Enveloping Algebras, and so we adopt a similar progression of ideas.

The classical Lie algebra we study, \( \mathfrak{sl}_2 \), is rank one and has generators and relations

\[
[H, X_\pm] = \pm 2X_\pm, \quad [X_+, X_-] = H
\]

Let \( q \) be a nonzero parameter. We define \( U_q(\mathfrak{sl}_2) \) as the noncommutative algebra generated by 1 and \( X^\pm, q^{H/2}, q^{-H/2} \), with the relations \( q^{\pm H} q^{\mp H} = 1 \) suggested by the notation, together with the nontrivial relations

\[
q^{H/2} X_\pm q^{-H/2} = q^{\pm 1} X_\pm, \quad [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}
\]

The following equations give this algebra the structure of a Hopf algebra:

\[
\Delta q^{H/2} = q^{H/2} \otimes q^{H/2}, \quad \Delta X_\pm = X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm
\]

(5)

\[
e q^{H/2} = 1, \quad e X_\pm = 0, \quad S X_\pm = -q^{\pm 1} X_\pm, \quad S q^{H/2} = q^{-H/2}
\]

(6)

If we work over the ring \( \mathbb{C}[[q]] \) of formal power series in the indeterminate \( q \), then the Hopf algebra defined by eqns. (5)-(6) is quasitriangular, with the following explicit universal R-matrix:

\[
R = q^{H/2} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]!} (q^{H/2} X_+ \otimes q^{-H/2} X_-)^n q^{-n(n-1)/2}
\]

where we have made use of the notation \([n] = q^n - q^{-n}\) and \([n]! = [n][n-1] \ldots [1]\). This completely specifies the algebra structure and its quantum R-matrix. In principle, one at this point should verify all of the Hopf algebra axioms. We will not do this here, but refer the reader to the literature [4].

4.1 Real Forms

The classical Lie algebra \( \mathfrak{sl}_2 \) has two inequivalent real forms: \( \mathfrak{su}_2 \) and \( \mathfrak{su}(1,1) \cong \mathfrak{sl}(2, \mathbb{R}) \). In this section, we describe the corresponding theorem in the case of quantum groups, without giving proofs.

In the classical case, one can define a real-form of a complex simple Lie algebra \( g \) to be an anti-linear anti-involution \( * \) on \( g \). The different real subalgebras which complexify to \( g \) are then identified with the \( * \)-invariant subspaces. The \( * \)-structure
on \( g \) corresponds to a Hopf-\( * \) structure on the universal enveloping algebra \( U(g) \), so viewing real forms of classical Lie algebras in terms of \( * \) operations abstracts directly to the case of quantum groups. Thus we define a real form of a Hopf algebra over \( \mathbb{C} \) to be a specification of a Hopf \( * \) structure. In particular, this now gives a well-defined meaning to a unitary representation, the latter being defined as a representation which is \( * \)-equivariant.

Given these definitions, one finds a direct analogy: just as is the case for classical \( \mathfrak{sl}_2 \), there are two inequivalent real forms of the quantum group \( U_q(\mathfrak{sl}_2) \).

5 The Quantized Enveloping Algebra \( U_q(g) \)

Let \( g \) be a finite dimensional complex semisimple Lie algebra \( g \) of rank \( N \), with Cartan matrix \((a_{ij})\). In this situation, the matrix \( a_{ij} \) is symmetrizable, which means that there are relatively prime positive integers \( d_1, \ldots, d_N \) with the property that the matrix \((d_i a_{ij})\) is symmetric. Let \( \{\alpha_1, \ldots, \alpha_N\} \) be the simple roots, and \( Q \) the root lattice. Let \( (\ , \ ) \) be the positive definite symmetric bilinear form on the root space defined by the equations

\[
(\alpha_i, \alpha_j) = d_ia_{ij}
\]

which can be realized by dualizing the Killing form \( K \). We choose Cartan-Chevalley generators \( H_i, X_{\pm i} \) according to the following (completely classical) prescription:

\[
\alpha_i(H_j) = K(d_i H_i, H_j) = a_{ji}, \quad K(X_{+i}, X_{-j}) = d_i^{-1}\delta_{ij}
\]

so that we have relations

\[
[H_i, H_j] = 0, \quad [H_i, X_{\pm j}] = \pm a_{ij} X_{\pm j}, \quad [X_{+i}, X_{-j}] = \delta_{ij} H_i
\]

To deform this into the quantized enveloping algebra, we can either associate to this root system a set of generators \( \{q_i^{\pm H_i/2}, X_i, X_{-i}\} \) of a Hopf algebra over \( \mathbb{C} \) (as we have done for \( \mathfrak{sl}_2 \) previously in Section 4) or we can define \( U_q(g) \) over the formal power series ring \( \mathbb{C}[[t]] \) with \( q = e^{t/2}, q_i = q_i^{d_i} \).

Since the construction involving the \( q_i^{\pm H_i/2} \) as generators was illustrated for the \( \mathfrak{sl}_2 \) case, we will now illustrate the general case using the second approach, based on formal power series. Accordingly, we define the relations of \( U_q(g) \) as

\[
[H_i, H_j] = 0, \quad [H_i, X_{\pm j}] = \pm a_{ij} X_{\pm j}, \quad [X_{+i}, X_{-j}] = \delta_{ij} \frac{q_i^{H_i} - q_i^{-H_i}}{q_i - q_i^{-1}}
\]

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} X_{\pm i}^{1-a_{ij}-k} X_{\pm j} X_{\pm i}^k = 0, \quad \text{for all } i \neq j
\]

where the expressions \( \binom{n}{m}_{q_i} \) are called \( q \)-binomial coefficients and has the same definition as the usual binomial coefficient, except that the factorial function is replaced...
by the quantum factorial \([n]_q! = [n]_q[n-1]_q \cdots [1]_q\), where \([n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}\). Finally, the coproduct, counit, and antipode are defined by the equations

\[
\Delta H_i = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta X_{\pm i} = X_{\pm i} \otimes q_i^{H_i/2} + q_i^{-H_i/2} \otimes X_{\pm i},
\]

(9)

\[
\epsilon(H_i) = \epsilon(X_{\pm i}) = 0, \quad S H_i = -H_i, \quad S X_{\pm i} = -q_i^{\pm 1} X_{\pm i}
\]

(10)

We denote the algebra defined by the generators and relations (7)-(10) by \(U_q(\mathfrak{g})\). As in the case of the quantized \(\mathfrak{sl}_2\), we omit the proof of these relations and merely state one of its most fundamental properties:

**Proposition 2** The algebra \(U_q(\mathfrak{g})\) defined by (7)-(10) is a Hopf algebra. The action of the antipode is \(-1\) times the conjugation by \(\rho^\vee\), where \(\rho\) is \(\frac{1}{2}\) times the sum of the positive roots, and \(\vee\) denotes dualization with respect to the Killing form.

### 6 Quantum Groups and Braid Groups

**Definition 2** The braid group on \(n\) strands \((n \geq 3)\) is the group \(B_n\) generated by \(n - 1\) elements \(\sigma_1, \ldots, \sigma_{n-1}\) with the relations

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad |i - j| > 1
\]

and

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for} \quad 1 \leq i, j \leq n - 1
\]

A trivial case is \(n = 2\), in which case \(B_2 = \mathbb{Z}\) is free on one generator.

Proposition 3 gives a connection between the Yang-Baxter equation and linear representations of the braid group.

**Proposition 3** Let \(V\) be a vector space and \(c \in GL(V)\) a linear automorphism satisfying the Yang-Baxter equation:

\[
(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c)
\]

Define, for \(1 \leq i \leq n - 1\), linear automorphisms \(c_i\) of \(V^\otimes n\) by:

\[
\begin{align*}
c_1 & = c \otimes \text{id}^{\otimes(n-2)}, \\
c_i & = \text{id}^{\otimes(i-1)} \otimes c \otimes \text{id}^{\otimes(n-i-1)}, \quad \text{if} \quad 1 < i < n - 1 \\
c_{n-1} & = \text{id}^{\otimes(n-2)} \otimes c
\end{align*}
\]

Then there is a unique homomorphism from \(B_n\) to the group of linear automorphisms of \(V^\otimes n\) sending \(\sigma_i\) to \(c_i\).

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1 quantum factorials \([n]_q!\) of the form utilised here were defined by Heine in 1846!
Thus in any quasitriangular (or “braided”) Hopf algebra $H$, the universal $R$-matrix allows to define braid group representations commuting with the adjoint action of $H$ in the tensor powers of any $H$-module. Here, the adjoint action of $H$ is an action of $H$ on itself, defined by the equation

$$\text{ad}(x)(y) = \sum x(1)yS(x(2))$$

in which the subscripts are a notation which is quite common: we avoid giving separate names to the homogeneous elements which make up a general element in a tensor power, and simultaneously take away the index labeling the sum, even though there is still a sum. Thus, if $R \in V \otimes 2$ is a non-homogeneous element in a second tensor power, then the same notation would be $R = \sum_{i=1}^n \alpha_i \otimes \beta_i$. We will follow convention and instead write $R = \sum R(1) \otimes R(2)$ with the understanding that the symbols $R(i)$ have no independent meaning.

Braid groups are important in modern representation theory, as is illustrated by our discussion of the Drinfeld-Kohno theorem (Theorem 4) in Section 7.

7 Application: the Knizhnik-Zamolodchikov Equations

Let $g$ be a complex semisimple Lie algebra as usual, and consider $\phi(z) = \phi(z_1, \ldots, z_n)$ to be a function of $n$ complex variables with values in the $n$-fold tensor product of $g$-modules

$$M := M_1 \otimes M_2 \otimes \cdots \otimes M_n$$

Fix an invariant symmetric inner product on $g$, which can be identified with a certain tensor $\Omega \in g \otimes g$. For $i \neq j$, we let $\Omega_{ij}$ denote the endomorphism of $M$ induced by letting the inner product $\Omega$ act on the $M_i$ and $M_j$ factors, keeping all other factors fixed.

Physicists discovered [6] that the correlation functions of Wess-Zumino-Witten models (two-dimensional conformal field theory) satisfy the following system of differential equations:

$$\frac{\partial \phi(z)}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i}^{\Omega_{ij}} \frac{\phi(z)}{z_i - z_j}$$

These are the famous Knizhnik-Zamolodchikov equations, after which this section is titled.

The important Kohno-Drinfeld theorem states roughly that the monodromy of the solutions of the KZ equations (11) can be expressed in terms of the R-matrix of the quantized universal enveloping algebra $U_q(g)$. The quantization parameter $q$ in the quantum group is connected with the complex constant $\kappa$ which appears in the KZ equations by the relation:

$$q = \exp(2\pi i/\kappa)$$
7.1 The precise statement of the Kohno-Drinfeld theorem

The following discussion of the Kohno-Drinfeld theorem is adapted from [3], in which a new (short!) proof of the theorem is given, using certain canonical categories obtained from the study of Lie bialgebras.

The category $\mathcal{O}$ for $\mathfrak{g}$ is defined to be the category of $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-representations, whose weights belong to a union of finitely many cones $\lambda - \sum_i \mathbb{Z}_+ \alpha_i$, $\lambda \in \mathfrak{h}^*$, and the weight subspaces are finite dimensional. Define also the category $\mathcal{O}[\hbar]$ of deformation representations of $\mathfrak{g}$, i.e. representations of $\mathfrak{g}$ on topologically free $k[[\hbar]]$ modules with similar assumptions on the weights ($\lambda \in \mathfrak{h}^*[\hbar]$).

In a similar way one defines the category $\mathcal{O}_\hbar$ for the algebra $\mathcal{U}$: it is the category of $\mathcal{U}$-modules which are topologically free over $k[[\hbar]]$ and satisfy the same conditions as in the classical case.

**Theorem 4 (Drinfeld-Kohno)** Let $k = \mathbb{C}$. Let $V \in \mathcal{O}_\hbar$, and $V_q = F(V)$ be its image in $\mathcal{O}_\hbar$. Let $F(z_1,\ldots,z_n)$ be a function of complex variables $z_1,\ldots,z_n$ with values in $V^\otimes n[\lambda][\hbar]$ (the weight subspace of weight $\lambda$) and consider the system of Knizhnik-Zamolodchikov differential equations:

$$\frac{\partial F}{\partial z_i} = \frac{\hbar}{2\pi i} \sum_{j \neq i} \Omega_{ij} F_{z_i - z_j}.$$

Then the monodromy representation of the braid group $B_n$ for this equation is isomorphic to the representation of $B_n$ on $V_q^\otimes n[\lambda]$ defined by the formula

$$b_i \rightarrow \sigma_i R_{ii+1},$$

where $b_i$ are generators of the braid group and $\sigma_i$ are the permutation of the $i$-th and $(i+1)$-th components.

8 Compact Quantum Groups

The treatment of Compact Quantum Groups in this paper owes much to [4].

**Definition 3** Let $G = (A, \Phi)$ where $A$ is a separable unital $C^*$-algebra and $\Phi : A \rightarrow A \otimes A$ is a unital *-algebra homomorphism. We say that $G$ is a compact quantum group if

1. The following diagram commutes:

$$\begin{array}{ccc}
A & \xrightarrow{\Phi} & A \otimes A \\
\Phi \downarrow & & \downarrow \Phi \otimes \text{id} \\
A \otimes A & \xrightarrow{\text{id} \otimes \Phi} & A \otimes A \otimes A
\end{array}$$

2. The sets $\{(b \otimes I)\Phi(c) : b,c \in A\}$ and $\{(I \otimes b)\Phi(c) : b,c \in A\}$ are linearly dense subsets of $A \otimes A$. 

9
Compact quantum matrix groups can be recovered from the above formalism in the following way. If \((A, u)\) is a compact quantum matrix group, and \(\Phi\) is the corresponding comultiplication, then \((A, \Phi)\) satisfies the axioms given above.

As one might expect, the simplest case to understand is the case in which the algebra \(A\) defining the quantum group is commutative. In this situation, it follows from Gelfand-Naimark theory that \(\exists\) some compact space \(\Lambda\) such that \(A = C(\Lambda)\). The comultiplication map \(\Phi\) can also be understood within the context of Gelfand-Naimark theory. In this case the implication is that there exists a continuous map \(\Lambda \times \Lambda \rightarrow \Lambda\) denoted by \((\lambda_1, \lambda_2) \mapsto \lambda_1 \cdot \lambda_2\) such that for any \(a \in C(\Lambda)\) and for any \(\lambda_1, \lambda_2\), we have

\[(\Phi a)(\lambda_1, \lambda_2) = a(\lambda_1 \cdot \lambda_2)\]

Moreover, condition (1) of Definition[3] means that the “multiplication” \(\cdot\) is associative, and the two density conditions in (2) of Def. [3] imply the two-sided cancellation law for ordinary group multiplication. It follows that \(\Lambda\) in this situation is a topological group. This justifies the notion that a compact quantum group is a generalization of the classical idea of a compact group to an underlying “noncommutative space,” and as such, fits into Alain Connes’ formulation of non-commutative geometry.

9 Haar Measure on Quantum Groups

Let \(G = (A, \Phi)\) be a compact quantum group. The goal of this section will be to describe the generalization of Haar measure to this case. First it is necessary to define the convolution product. In the following definitions, \(\xi, \xi'\) are continuous linear functionals on \(A\), and \(a \in A\). One can convolve an element of \(A\) with a functional to produce an element of \(A\), and one can convolve two functionals to produce another functional. The precise statements are:

\[
\begin{align*}
\xi \ast a & := (\text{id} \otimes \xi)\Phi(a) \\
a \ast \xi & := (\xi \otimes \text{id})\Phi(a) \\
\xi' \ast \xi & := (\xi' \otimes \xi)\Phi
\end{align*}
\]

By definition, a state \(\xi\) on a \(C^*\)-algebra \(A\) is a positive normalized linear functional on \(A\). This terminology comes from the relationship which exists between quantum mechanics and \(C^*\)-algebras, provided by the Gelfand-Naimark-Segal (GNS) construction. Briefly, from the data \((A, \xi)\), the GNS construction gives a Hilbert space \(\mathcal{H}\), a unit cyclic vector \(\psi \in \mathcal{H}\), and a \(\ast\)-representation \(\pi_\xi : A \rightarrow \mathcal{B}(\mathcal{H})\) such that

\[\xi(a) = \langle \pi_\xi(a)\psi, \psi \rangle \text{ for all } a \in A\]

The following fundamental result establishes the analogue of Haar measure for compact quantum groups.

**Theorem 5** Let \(G = (A, \Phi)\) be a compact quantum group. Then there exists a unique state \(h\) on \(A\) satisfying

\[a \ast h = h \ast a = h(a)I \text{ for all } a \in A\]  \hspace{1cm} (12)

10
To relate this to the classical theory of Haar measure on a compact group, one must notice that if \( A = C(\Lambda) \) is the algebra of continuous, complex-valued functions on a compact topological group \( \Lambda \), then the Haar measure \( \mu \) on \( \Lambda \) is certainly a positive measure on the Borel \( \sigma \)-field of measurable sets in \( \Lambda \). However, it can be equivalently described as a functional on \( A \) given by \( \mu(f) := \int_\Lambda f \, d\mu \). Convolutions of measures with test functions can be defined as in distribution theory, and it is straightforward to show that (12) holds for the Haar measure (state) \( \mu \).

10 Representations of Compact Quantum Groups

Before we can continue, it is necessary to introduce some preliminary notation and terminology from operator algebra theory. Let \( \mathfrak{A} \) be a \( C^* \)-algebra and let \( A, B \) be bounded linear operators on \( \mathfrak{A} \). We say that \( B \) is the adjoint of \( A \), if \( A(x^*y) = x^*B(y) \) for all \( x, y \in \mathfrak{A} \). In this situation we denote \( B \) by \( A^* \). The multiplier algebra \( M(\mathfrak{A}) \) is defined to be the subalgebra of the bounded operators \( B(\mathfrak{A}) \), defined by

\[
M(\mathfrak{A}) = \{ A \in B(\mathfrak{A}) \mid A^* \text{ exists} \}
\]

**Definition 4** Let \( G = (A, \Phi) \) be a compact quantum group and \( \mathcal{H} \) a Hilbert space. Let \( K \) denote the algebra of bounded, compact operators on \( \mathcal{H} \). A (strongly continuous) unitary representation of \( G \) on \( \mathcal{H} \) is a unitary element \( v \in M(K \otimes A) \) such that

\[
(id \otimes \Phi)v = v_{12}v_{13}
\]

One should think of \( v \) “acting on” \( \mathcal{H} \) in the following sense. As an element of the multiplier algebra \( M(K \otimes A) \), \( v \) is a bounded operator on \( K \otimes A \) possessing an adjoint with respect to the \( * \)-product on \( K \otimes A \) induced by the tensor product construction. Thus, using \( v \), every element \( a \in A \) “acts on” the compact operators \( K \) via the pairing \((a, k) \rightarrow v(k \otimes a)\).

If the Hilbert space \( \mathcal{H} \) is just \( \mathbb{C}^N \), then the multiplier algebra of \( K \otimes A \) is \( M_N(\mathbb{C}) \). A finite-dimensional representation of a compact quantum group is therefore by definition, a unitary matrix (with matrix elements from the algebra \( A \))

\[
v = (v_{k\ell})_{k,\ell=1,\ldots,N}
\]

satisfying the condition that

\[
\Phi(v_{k\ell}) = \sum_r v_{kr}v_{r\ell} \text{ for all } k, \ell = 1, \ldots, N
\]

In other words, matrix multiplication is the quantum “group law” on \( G \), determined by the comultiplication. Equation (13) is the analogue (with appropriate arrows reversed) of the classical notion that for a matrix representation of a group \( \rho : G \rightarrow GL(V) \), the matrix elements of products are given by the formula \( \rho(gh)_{k\ell} = \sum_r \rho(g)_{kr}\rho(h)_{r\ell} \).

It is a remarkable result that even at this level of generality, one obtains a complete reducibility result for these representations.

**Theorem 6** Let \( v \) be a unitary representation of a compact quantum group \( G = (A, \Phi) \) on any Hilbert space \( \mathcal{H} \). Then \( v \) decomposes as a direct sum of finite-dimensional irreducible representations.
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