A NATURAL MIN-MAX CONSTRUCTION FOR
GINZBURG-LANDAU FUNCTIONALS

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Abstract. We use min-max techniques to produce a family of nontrivial solutions \( u_\epsilon : M^n \to \mathbb{R}^2 \) of the Ginzburg-Landau equation

\[
\Delta u_\epsilon + \frac{1}{\epsilon^2}(1 - |u_\epsilon|^2)u_\epsilon = 0
\]
on a given compact Riemannian manifold \( M^n \), whose energy grows like \( |\log \epsilon| \) as \( \epsilon \to 0 \). Building on the analysis of [5], we show that when the degree one cohomology \( H^1_{dR}(M) = 0 \), the energy of these solutions concentrates on a nontrivial stationary, rectifiable \((n-2)\)-varifold \( V \).

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## 1. Introduction

Let \((M, g)\) be a compact Riemannian manifold of dimension \( n \geq 2 \). Given a complex-valued map \( u : M \to \mathbb{R}^2 \), we define for \( \epsilon > 0 \) the Ginzburg-Landau functionals

\[
E_\epsilon(u) := \int_M e_\epsilon(u) = \int_M \frac{1}{2} |du|^2 + \frac{1}{\epsilon^2}W(u).
\]

Here, \( W : \mathbb{R}^2 \to \mathbb{R} \) is a smooth, bounded potential satisfying

\[
W(z) = \frac{1}{4}(1 - |z|^2)^2 \text{ for } |z| < 2,
\]

\[
W(z) \geq 2 \text{ for } |z| \geq 2,
\]

and

\[
\sup_{z \in \mathbb{R}^2} |DW(z)| < \infty.
\]
Critical points $u_\epsilon : M \to \mathbb{R}^2$ of the energy $E_\epsilon$ solve the Ginzburg-Landau equation

\begin{equation}
\Delta u_\epsilon = \frac{1}{\epsilon^2} D W(u_\epsilon).
\end{equation}

Clearly, the global minimizers of $E_\epsilon$ are just the constant maps taking values in the unit circle. On a bounded domain $\Omega \subset \mathbb{R}^n$, we can find more interesting solutions of (1.5) by minimizing $E_\epsilon(u)$ among maps with fixed Dirichlet data

\begin{equation}
u|\partial \Omega = h_\epsilon.
\end{equation}

When $\Omega \subset \mathbb{R}^2$ is a simply-connected planar domain, and $h_\epsilon$ is a fixed map $h : \partial \Omega \to S^1$ of degree $d$, the asymptotic behavior of these minimizers $u_\epsilon$ as $\epsilon \to 0$ was characterized by the work of Bethuel-Brezis-Hélein [4] and Struwe [20]. Namely, they showed that (along some subsequence $\epsilon_j \to 0$), there exist $d$ points $a_1, \ldots, a_d \in \Omega$ such that

\begin{equation}
\lim_{\epsilon \to 0} e_{\epsilon}(u_\epsilon) \pi \cdot \log \epsilon \delta_{a_j} \text{ in } (C^0)^*,
\end{equation}

while

\begin{equation}
\epsilon \to u \text{ in } C^\infty_{\text{loc}}(\Omega \setminus \{a_1, \ldots, a_d\}),
\end{equation}

where $u : \Omega \to S^1$ is a weakly harmonic map with singularities at $\{a_1, \ldots, a_d\}$ ([4], [20]). In particular, these results establish the variational theory of $E_\epsilon$ as a natural means for producing singular harmonic maps to $S^1$ in situations where finite-energy solutions aren’t available.

For solutions in higher dimensions, a still richer structure emerges, with connections to geometric measure theory. (We assume here some familiarity with the basic definitions and results of geometric measure theory, as found in [8] and [19]; see especially [19] for the theory of varifolds.) For domains $\Omega \subset \mathbb{R}^n$, $n \geq 3$, Lin and Rivière studied minimizers of $E_\epsilon$ under boundary conditions $h_\epsilon : \partial \Omega \to D^2$ that approximate a map $\partial \Omega \to S^1$ with singularity along a fixed $n-3$-dimensional submanifold $S \subset \partial \Omega$ [14]. In a striking extension of the two-dimensional results, they showed that (along a subsequence) the measures

\begin{equation}
\mu_\epsilon := \frac{e_{\epsilon}(u_\epsilon)}{\pi \log \epsilon} dx
\end{equation}

converge to the weight measure $\mu_T$ of an integral $(n-2)$-current $T \in I_{n-2}(\Omega)$ solving the Plateau problem

\begin{equation}
\partial T = S, \quad M(T) \leq M(T + \partial W) \text{ for all } W \in I_{n-1}(\Omega),
\end{equation}

while, away from $\text{spt}(T)$, $u_\epsilon$ again converges to a harmonic map $u : \Omega \setminus \text{spt}(T) \to S^1$ [14]. The proof of this statement doesn’t rely on the existence
of a solution to (1.9), so these results yield a new existence proof for the codimension-two Plateau problem via Ginzburg-Landau functionals [14].

In [5], Bethuel, Brezis, and Orlandi employed ideas from [3] and [14] to produce similar results for non-minimizing solutions $u_\epsilon$ of (1.5) with boundary data $h_\epsilon$ similar to that used in [14]. For such solutions, they showed that the normalized energy measures $\mu_\epsilon$ concentrate on a stationary, rectifiable varifold of codimension two, away from which the maps $u_\epsilon$ converge smoothly to a harmonic map to $S^1$. In particular, their results give us reason to hope that the variational theory of the Ginzburg-Landau functional could be used to produce nontrivial critical points of the $(n-2)$-area functional.

In this paper, we introduce a natural min-max procedure for the Ginzburg-Landau energies to produce solutions on an arbitrary compact manifold whose energy concentration measures $\mu_\epsilon$ have mass bounded above and below:

**Theorem 1.1.** On any compact Riemannian manifold $(M^n, g)$, there exists a family of nontrivial solutions $u_\epsilon : M \to \mathbb{R}^2$ of the Ginzburg-Landau equations (1.5) satisfying energy bounds of the form

$$c |\log \epsilon| \leq E_\epsilon(u_\epsilon) \leq C|\log \epsilon|$$

for some positive constants $C = C(M)$, $c = c(M)$.

Moreover, when $M$ has vanishing degree one cohomology $H^1_{dR}(M) = 0$, we show that the analysis of [5] can be extended to arbitrary global solutions of (1.5) satisfying bounds of the form (1.10), to conclude that

**Theorem 1.2.** If $H^1_{dR}(M) = 0$, then there exists a subsequence $\epsilon_j \to 0$ and a nontrivial stationary, rectifiable $(n-2)$-varifold $V$ on $M$ such that

$$\mu_{\epsilon_j} := \frac{e_{\epsilon_j}(u_{\epsilon_j})}{|\log \epsilon_j|} dv_g \to \|V\|.$$

**Remark 1.3.** When $H^1_{dR}(M) \neq 0$, it is no longer true that bounds of the form (1.10) yield compactness results for the solutions $u_\epsilon$, or $(n-2)$-rectifiability of the energy concentration measure. For instance, consider $M = S^1 \times N$ endowed with the product metric, let $\epsilon_k = e^{-k^2}$, and let $u_{\epsilon_k} : M \to \mathbb{C}$ be given by $u_{\epsilon_k}(z,y) = (1 - k^2 \epsilon_k^2)^{1/2} z^k$. These $u_{\epsilon_k}$ then solve $(GL)_{\epsilon_k}$ with energy bounds of the form (1.10), but the energy concentration measures $\mu_{\epsilon_k}$ converge to a multiple of the volume measure $dv_g$ as $k \to \infty$. We nonetheless expect that the conclusion of Theorem 1.2 will hold for our min-max solutions when $H^1_{dR}(M) \neq 0$, but such a result will necessarily rely in a nontrivial way on the min-max construction.

**Remark 1.4.** As in [5], after establishing a positivity result for the $(n-2)$-density of the limiting measure, the concentration of energy on an $(n-2)$-rectifiable varifold in Theorem 1.2 follows from Ambrosio and Soner’s blow-up argument in [3]. In particular, it does not follow from our analysis that $V$ has integer density $H^{n-2}$-a.e. For applications to geometric measure
theory, it would be very interesting to extend the integrality results of [14] for minimizers to the min-max solutions constructed here.

These results are inspired in large part by Guaraco’s min-max program for the elliptic Allen-Cahn equation—the scalar analog of [15]. Building on results of Hutchinson-Tonegawa [12] and Tonegawa-Wickramasekera [21], it was shown in [11] that real-valued solutions of (1.5) arising from a natural mountain-pass construction exhibit energy blow-up on a stationary, integral \((n-1)\)-varifold, with singular set of Hausdorff dimension \(\leq n-8\). In particular, the analysis in [11] recovers the major results of the Almgren-Pitts min-max construction of minimal hypersurfaces [1], [17], while replacing a number of the original geometric measure theory arguments with (often simpler) pde methods.

The conclusions of [11] are particularly intriguing in light of recent applications of the min-max theory of minimal hypersurfaces to some long-standing problems in geometry, such as Marques and Neves’s resolution of the Willmore Conjecture [16], or their proof that manifolds of positive Ricci curvature contain infinitely many minimal hypersurfaces [15]. As a natural regularization of the Almgren-Pitts theory, the Allen-Cahn min-max serves as a bridge between these kinds of results and questions in semilinear pde. In [9], for example, Gaspar and Guaraco draw on this relationship by adapting the arguments in [15] to the Allen-Cahn setting, obtaining a number of new results about the solution space of semilinear pdes of this type.

Our results here suggest that the min-max theory of Ginzburg-Landau functionals may provide a similar regularization for the Almgren-Pitts min-max in codimension two. To this end, it would be desirable to extend our results by removing the cohomological constraint in Theorem 1.2 and investigating the integrality of the energy-concentration varifold. Da Rong Cheng has informed us that he has independently obtained the result of Theorem 1.2.

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2. The Min-Max Procedure

Our basic method for constructing critical points of \(E_\epsilon\) is a natural extension to codimension two of Guaraco’s mountain pass construction in [11]. Namely, we employ a simple two-parameter min-max procedure (following the presentation in [10]) to obtain nontrivial critical points of \(E_\epsilon\) on \(M\).

Let \(\varphi\) be a \(C^1\) functional on a Banach space \(X\), and suppose \(X\) splits into a sum

\[ X = Y \oplus Z, \]
where \( \dim(Y) = k < \infty \). Denote by \( B_Y \) the closed unit \( k \)-ball

\[
B_Y := \{ u \in Y \mid \|u\| \leq 1 \},
\]

and let

\[
S_Y := \{ u \in Y \mid \|u\| = 1 \}
\]

be its boundary \((k-1)\)-sphere. Let \( \Gamma \) be the collection of maps

\[
(2.1) \quad \Gamma := \{ F \in C^0(B_Y, X) \mid F|_{S_Y} = Id|_{S_Y} \},
\]

and \( c \) the associated min-max constant

\[
(2.2) \quad c := \inf_{F \in \Gamma} \max_{y \in B_Y} \varphi(F(y)).
\]

For any family \( F \in \Gamma \), given a projection \( P_Y : X \to Y \), we can apply elementary degree theory to the map

\[
P_Y \circ F : B_Y \to Y
\]

to conclude that \( P_Y \circ F \) must vanish somewhere, so that \( F(y) \in Z \) for some \( y \in B_Y \). If we have also an estimate of the form

\[
(2.3) \quad \inf \varphi(Z) > \sup \varphi(S_Y),
\]

it then follows from general versions of the min-max theorem (e.g., Theorem 3.2 in [10]) that

**Theorem 2.1.** For any sequence \( F_j \in \Gamma \) such that

\[
(2.4) \quad \lim_{j \to \infty} \sup_{y \in B_Y} \varphi(F_j(y)) = c,
\]

there exists a sequence \( u_j \in X \) such that

\[
(2.5) \quad \lim_{j \to \infty} \varphi(u_j) = c,
\]

\[
(2.6) \quad \lim_{j \to \infty} \|d\varphi(u_j)\| = 0,
\]

and

\[
(2.7) \quad \lim_{j \to \infty} \text{dist}(u_j, F_j(B_Y)) = 0.
\]

It’s not difficult to see how the Ginzburg-Landau energy \( E_\epsilon \) fits into this framework. By our assumptions on the structure of \( W \), \( E_\epsilon \) is a \( C^1 \) functional on the Sobolev space \( H^1(M, \mathbb{R}^2) \), with derivative \( E'_\epsilon \) given by

\[
\langle E'_\epsilon(u), v \rangle = \int_M \langle du, dv \rangle + \epsilon^{-2} \langle DW(u), v \rangle.
\]

If we consider the natural splitting

\[
H^1(M, \mathbb{R}^2) = \mathbb{R}^2 \oplus Z
\]

of \( H^1(M, \mathbb{R}^2) \) into the constant maps (identified with \( \mathbb{R}^2 \)) and the orthogonal complement

\[
Z := \{ u \in H^1(M, \mathbb{R}^2) \mid \int_M u = 0 \in \mathbb{R}^2 \},
\]
then we note that the unit circle $S^1 \subset \mathbb{R}^2$ in the $\mathbb{R}^2$ factor is precisely the subset of $H^1(M, \mathbb{R}^2)$ on which $E_\epsilon$ vanishes. Thus, to apply Theorem 2.1 to obtain a nice min-max sequence for $E_\epsilon$, it is enough to establish an estimate of the form (2.3): namely, we need to show that 

$$\inf_{u \in Z} E_\epsilon(u) > 0. \tag{2.8}$$

Such an estimate is easy to obtain: The Poincaré inequality furnishes us with a constant $\lambda_1(M) > 0$ such that

$$\int_M |u|^2 \geq \lambda_1(M) \int_M |\nabla u|^2$$

for all $u \in Z$; hence, for any $u \in Z$, we find that

$$E_\epsilon(u) = \int_M |\nabla u|^2 + \frac{W(u)}{\epsilon^2} \geq \int_M \frac{\lambda_1}{2} |u|^2 + \frac{W(u)}{\epsilon^2} \geq \int_{\{|u| \geq 1/2\}} \frac{\lambda_1}{2} |u|^2 + \int_{\{|u| < 1/2\}} \frac{W(u)}{\epsilon^2} \geq \min \left\{ \frac{\lambda_1}{8}, \frac{W(1/2)}{\epsilon^2} \right\} \cdot \frac{1}{2} \text{vol}(M) > 0. \tag{2.9}$$

Thus, (2.8) holds, and we are indeed in a position to apply the min-max theorem 2.1. That is, letting $D \subset \mathbb{R}^2$ denote the closed unit disk, and setting

$$\Gamma(M) := \{ F \in C^0(D, H^1(M, \mathbb{R}^2)) \mid F(y) \equiv y \text{ for } y \in S^1 \}, \tag{2.9}$$

and

$$c_\epsilon(M) := \inf_{F \in \Gamma(M)} \max_{y \in D} E_\epsilon(F(y)), \tag{2.10}$$

we can extract from any minimizing sequence of families

$$F_j \in \Gamma(M), \quad \lim_{j \to \infty} \max_{y \in D} E_\epsilon(F_j(y)) = c_\epsilon \tag{2.11}$$

a min-max sequence $u_j$ satisfying (2.5)-(2.7).

Given any family $F \in \Gamma(M)$, we can apply the nearest-point retraction $\Phi : \mathbb{R}^2 \to D$ to obtain a new family $\tilde{F} := \Phi \circ F \in \Gamma$; it is clear that $\text{Lip}(\Phi) = 1$ and $W \circ \Phi \leq W$, and therefore

$$E_\epsilon(\tilde{F}(y)) \leq E_\epsilon(F(y)) \text{ for each } y \in D.$$

In particular, starting from any minimizing sequence of families $F_j$ as in (2.11), we can apply $\Phi$ to obtain a new minimizing sequence $\tilde{F}_j$ satisfying

$$\|\tilde{F}_j(y)\|_{\infty} \leq 1. \tag{2.12}$$
If $u_j \in H^1(M, \mathbb{R}^2)$ is a min-max sequence satisfying (2.5)-(2.7) with respect to $\tilde{F}_j$, then the bound (2.12), together with (2.7), implies that $u_j$ is bounded in $L^2$, and since
\[
\lim_{j \to \infty} \frac{1}{2} \int_M |du_j|^2 \leq \lim_{j} E_\epsilon(u_j) = c_\epsilon,
\]
it follows that $u_j$ is bounded in the full $H^1$ norm.

It is a simple and well known fact (see, e.g., [11], [13]) that functionals of Ginzburg-Landau type satisfy the Palais-Smale condition along bounded sequences: that is, if
\[
\sup_j \|u_j\|_{H^1} < \infty \text{ and } \lim_{j \to \infty} \|E'_\epsilon(u_j)\| = 0,
\]
then $u_j$ contains a strongly convergent subsequence $u_j \to u$, whose limit necessarily satisfies
\[
E'_\epsilon(u) = 0 \text{ and } E_\epsilon(u) = \lim_{j \to \infty} E_\epsilon(u_j).
\]

Applying this fact to the min-max sequence of the previous paragraph, we obtain our basic existence result:

**Proposition 2.2.** For any $\epsilon > 0$, there exists a critical point $u_\epsilon \in H^1(M, \mathbb{R}^2)$ of $E_\epsilon$ such that
\[
(2.13) \quad E_\epsilon(u_\epsilon) = c_\epsilon(M) > 0,
\]
and
\[
(2.14) \quad \|u_\epsilon\|_\infty \leq 1.
\]

To prove Theorem 1.1 it remains to establish the energy estimates
\[
0 < \liminf_{\epsilon \to 0} \frac{c_\epsilon(M)}{\log \epsilon} \leq \limsup_{\epsilon \to 0} \frac{c_\epsilon(M)}{\log \epsilon} < \infty.
\]

3. Lower Bounds on the Energies

Since we’ve shown that the min-max constants $c_\epsilon(M)$ are positive critical values of the energy $E_\epsilon$, one obvious way to obtain lower bounds for $c_\epsilon(M)$ is to find lower bounds for the energy of arbitrary nontrivial solutions of (1.5). Simple examples show, however, that such estimates will not in general yield lower bounds of the desired form.

Consider, for instance, $M = S^1 \times N$ endowed with the product metric, and let $p : M \to S^1$ be the obvious projection. For each $\epsilon \in (0, 1)$, it’s easy to check that the maps
\[
p_\epsilon = (1 - \epsilon^2)^{\frac{1}{4}} \cdot p
\]
satisfy (1.5), while their energies $E_\epsilon(p_\epsilon)$ stay uniformly bounded as $\epsilon \to 0$. (As an aside, we note that the maps $p_\epsilon$ also satisfy $\int_M p_\epsilon = 0$, so, in contrast to the situation for the Allen-Cahn min-max [11], we can’t hope to establish the desired energy blow-up by proving lower bounds for $E_\epsilon$ over maps of zero average.)
The problem in the example above comes from the existence of a nontrivial harmonic map \( M \to S^1 \). Recall that (modulo rotation) smooth harmonic maps to \( S^1 \) are in one-to-one correspondence with harmonic one-forms representing integer cohomology classes in \( H^1_{dR}(M) \). In particular, when \( H^1_{dR}(M) = 0 \) there are no nontrivial harmonic maps \( M \to S^1 \), and in this case, we find the following:

**Lemma 3.1.** If \( H^1_{dR}(M) = 0 \), then for any family \( u_\epsilon \) of nontrivial solutions of (1.5), we have the lower energy bound

\[
(3.1) \quad \liminf_{\epsilon \to 0} \frac{E_\epsilon(u_\epsilon)}{|\log \epsilon|} > 0.
\]

**Proof.** To begin, we show that any nontrivial solution \( u_\epsilon \) must vanish somewhere. To see this, suppose \( u_\epsilon \) solves (1.5), and that

\[
(3.2) \quad |u_\epsilon(x)| > 0 \quad \text{for all} \quad x \in M.
\]

Let \( ju_\epsilon \) denote the pull-back of the one-form \( r^2 d\theta \in \Omega^1(\mathbb{R}^2) \) by \( u_\epsilon \)--i.e.,

\[
(3.3) \quad ju_\epsilon := u^1_\epsilon du^2_\epsilon - u^2_\epsilon du^1_\epsilon.
\]

Computing the divergence of \( ju_\epsilon \) and applying (1.5), we arrive at

\[
(3.4) \quad d^* ju_\epsilon = 0,
\]

a fundamental fact for solutions of (1.5). That is, for any \( \psi \in C^\infty(M) \), we have

\[
(3.5) \quad \int_M \langle ju_\epsilon, d\psi \rangle = 0.
\]

In light of (3.2), consider the smooth map

\[
\phi := \frac{u_\epsilon}{|u_\epsilon|} : M \to S^1,
\]

and observe that the pullback \( \phi^* d\theta \) is a closed one-form; hence, by our assumption on the cohomology of \( M \), there exists some \( \psi \in C^\infty(M) \) such that

\[
(3.6) \quad \phi^* d\theta = d\psi.
\]

On the other hand, we also note that

\[
\phi^* d\theta = \phi^*(r^2 d\theta) = |u_\epsilon|^{-2} ju_\epsilon,
\]

so that applying (3.5) to (3.6) yields

\[
\int_M |u_\epsilon|^2 |d\psi|^2 = 0.
\]

Thus, \( |d\phi| = |\phi^* d\theta| = 0 \), so that \( \phi \equiv \beta \) for some constant \( \beta \in S^1 \).

It then follows from (1.5) that

\[
\Delta(1 - |u_\epsilon|) = \epsilon^{-2}(1 - |u_\epsilon|^2)|u_\epsilon|.
\]
Multiplying both sides by \((1 - |u_\epsilon|)\) and integrating yields

\[
0 \geq - \int_M |d(1 - |u_\epsilon|)|^2 = \epsilon^{-2} \int_M (1 - |u_\epsilon|)^2 |u_\epsilon|(1 + |u_\epsilon|) \geq 0,
\]

and we immediately conclude that \(|u_\epsilon| \equiv 1\); hence, \(u_\epsilon \equiv \beta \in S^1\) is a trivial solution.

Thus, if \(u_\epsilon\) is a nontrivial solution of (1.5) on \(M\) with \(H^1_{dR}(M) = 0\), there must be some point \(x_\epsilon \in M\) such that \(u_\epsilon(x_\epsilon) = 0\). Now we appeal to one of the central analytical lemmas of [5] (see also [14])—the so-called \(\eta\)-ellipticity theorem—to see that the existence of such a zero necessarily produces the desired energy blow up. Though the \(\eta\)-ellipticity theorem is originally stated for the Euclidean setting in [5], the arguments are purely local, and can be applied to small balls on compact manifolds to yield the following

**Theorem 3.2.** (Theorem 2 of [5]) There exist positive constants \(\epsilon_0(M), \delta_0(M), \eta_0(M) > 0\) such that if \(u_\epsilon\) solves (1.5) on a geodesic ball \(B_r(x)\), where \(\epsilon < \epsilon_0\), \(r \leq \delta_0\) and

\[
\int_{B_r(x)} e_\epsilon(u_\epsilon) \leq r^{n-2}\epsilon_0 \log(\epsilon/r),
\]

then

\[
|u_\epsilon|^2(x) \geq \frac{7}{8}.
\]

Applying this at the zeros \(x_\epsilon\) of our nontrivial solutions \(u_\epsilon\), with \(r = \delta_0(M)\), we see that for all \(\epsilon\) sufficiently small, we must have

\[
E_\epsilon(u_\epsilon) \geq \int_{B_{\delta_0}(x_\epsilon)} e_\epsilon(u_\epsilon) > \delta_0^{n-2}\eta_0(|\log \epsilon| - |\log \delta_0|),
\]

from which (3.1) follows. \(\square\)

Applying the preceding lemma to the nontrivial solutions of Proposition 2.2 we immediately obtain

**Lemma 3.3.** If \(M^n\) is a Riemannian manifold with \(H^1_{dR}(M) = 0\), then the min-max constants \(c_\epsilon(M)\) defined by (2.10) satisfy the lower bound of (1.10): namely,

\[
\liminf_{\epsilon \to 0} \frac{c_\epsilon(M)}{|\log \epsilon|} > 0.
\]

Next, we observe that these lower bounds can be extended to arbitrary manifolds by way of a simple trick, which can easily be applied to a wide range of min-max constructions. The resulting energy estimates are somewhat crude, but sufficient to establish the desired energy blow-up.

\[1\]Our choice of the constant \(\frac{7}{8}\) here was of course somewhat arbitrary; we could replace it with any constant in \((0, 1)\), changing \(\eta_0\) accordingly.
Let \((M^n, g)\) once again be an arbitrary compact manifold, and recall the definition of \(\Gamma(M)\):

\[
\Gamma(M) := \{ F \in C^0(D, H^1(M, \mathbb{R}^2)) \mid F(y) \equiv y \text{ for } y \in S^1 \}.
\]

Given a domain \(\Omega \subset M\) and a family \(F \in \Gamma(M)\), it’s clear that the family \(F|_{\Omega} \in C^0(D, H^1(\Omega, \mathbb{R}^2))\) given by restriction

\[
y \mapsto F(y)|_{\Omega}
\]

lies in \(\Gamma(\Omega)\), and trivially satisfies the bound

\[
E_\epsilon(F) \geq E_\epsilon(F|_{\Omega}).
\]

As a consequence, we obtain the simple estimate

\[
\tag{3.11} c_\epsilon(M) \geq \inf_{F \in \Gamma(\Omega)} \max_{y \in D} E_\epsilon(F(y))
\]

for any subdomain \(\Omega \subset M\).

Now, let \(B^n \subset M\) be an embedding of the closed \(n\)-ball into \(M\) (e.g., as a closed geodesic ball), and consider the map

\[
R : H^1(B^n, \mathbb{R}^2) \to H^1(S^n, \mathbb{R}^2)
\]

given by identifying \(B^n\) with a closed hemisphere and reflecting. That is, for \(u \in H^1(B^n, \mathbb{R}^2)\), define

\[
Ru(x_0, \ldots, x_n) := u \circ f(|x_0|, x_1, \ldots, x_n),
\]

where \(f : S^n_+ \to B^n\) is a diffeomorphism with the closed hemisphere

\[
S^n_+ = \{(x_0, \ldots, x_n) \in S^n \mid x_0 \geq 0 \}.
\]

It’s then straightforward to check that \(R\) is a bounded (hence continuous) linear map, and in particular,

\[
\tag{3.12} R \circ F \in \Gamma(S^n) \text{ for any } F \in \Gamma(B^n).
\]

Moreover, since reflection across the equator simply doubles the energy \(E_\epsilon\) of a map in \(H^1(S^n_+, \mathbb{R}^2)\), and \(f : S^n_+ \to B^n\) is necessarily bi-Lipschitz, we have an estimate of the form

\[
\tag{3.13} C^{-1} E_\epsilon(Ru) \leq E_\epsilon(u) \leq CE_\epsilon(Ru) \text{ for every } u \in H^1(M, \mathbb{R}^2),
\]

for some constant \(C\) depending on our choice of \(f\).

Applying \((3.11)\) to a fixed choice of closed ball \(B^n \subset M\), and fixing a choice of \(f : S^n_+ \to B^n\), we conclude from \((3.12)\) that

\[
\tag{3.14} c_\epsilon(M) \geq C^{-1} c_\epsilon(S^n, g_{\text{standard}})
\]

for some finite, positive constant \(C\) independent of \(\epsilon\). Finally, we note that since \(H^1_{\text{Riem}}(S^n) = 0\), we can apply Lemma 3.3 to \((S^n, g_{\text{standard}})\), and combining \((3.10)\) with \((3.14)\), we arrive at the desired lower bound:
Proposition 3.4. On any compact manifold \((M^n, g)\), the min-max constants \(c_\epsilon(M)\) satisfy
\[
\liminf_{\epsilon \to 0} \frac{c_\epsilon(M)}{|\log \epsilon|} > 0.
\]

4. Upper Bounds on the Energies

To find suitable upper bounds for the energies \(c_\epsilon(M)\), we just need to produce families \(F_\epsilon \in \Gamma(M)\) consisting of maps that behave roughly like model solutions of (1.5).

Given \(\epsilon > 0\), consider the map \(v_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2\) defined by
\[
v_\epsilon(z) = \begin{cases} 
\frac{z}{|z|} & \text{for } |z| > \epsilon, \\
\frac{z}{\epsilon} & \text{for } |z| \leq \epsilon.
\end{cases}
\]

Letting \(\pi_z^\perp : \mathbb{R}^2 \to \mathbb{R}^2\) denote orthogonal projection onto \([\mathbb{R}z]^\perp\), we then have
\[
dv_\epsilon(z) = \begin{cases} 
\pi_z^\perp |z| & \text{for } |z| > \epsilon \\
\frac{1}{\epsilon} Id & \text{for } |z| \leq \epsilon,
\end{cases}
\]
and, in particular,
\[
e_\epsilon(v_\epsilon) = \frac{1}{2} |dv_\epsilon|^2(z) + \frac{W(v_\epsilon(z))}{\epsilon^2} \leq \frac{1}{2|z|^2} \text{ for } |z| > \epsilon, \text{ and } \leq \frac{9}{4\epsilon^2} \text{ for } |z| \leq \epsilon.
\]

A quick computation then reveals an energy bound of the form
\[
E_\epsilon(v_\epsilon, D_R) = \int_{\{|z| \leq R\}} e_\epsilon(v_\epsilon) \leq \pi \log(R/\epsilon) + C
\]
for the restriction of \(v_\epsilon\) to the disk \(D_R\) of radius \(R\) about the origin.

Let \(\Omega \subset \mathbb{R}^2\) be a bounded domain, and consider the family of maps
\[
D \ni y \mapsto v_{y,\epsilon} \in Lip(\Omega, \mathbb{R}^2)
\]
geven by the translates
\[
v_{y,\epsilon}(z) = v_\epsilon(z + \frac{y}{1 - |y|}) \text{ for } |y| < 1,
\]
and
\[
v_{y,\epsilon}(z) = y \text{ for } y \in \partial D.
\]
Since \(\Omega\) is bounded, it follows from (4.6) and (1.2) that \(y \mapsto v_{y,\epsilon}\) is a continuous family in \(Lip(\Omega, \mathbb{R}^2)\), and thus, by (4.6), a member of \(\Gamma(\Omega)\). In light of the energy estimate (4.4), this family seems like a promising starting point for constructing well-behaved families on an arbitrary manifold.

Now, let \(M\) be a compact manifold, and let \(f : M \to \mathbb{R}^2\) be a Lipschitz map. By the preceding discussion, it’s clear that
\[
D \ni y \mapsto F_y := v_{y,\epsilon} \circ f
\]
defines a valid family in \(\Gamma(M)\); thus, we can estimate the min-max constants \(c_\epsilon\) from above by making a reasonable choice of \(f \in Lip(M, \mathbb{R}^2)\).
With $f$ and $F_y$ as above, setting $w := \frac{-y}{1-|y|}$, it follows from (4.3) that
\begin{equation}
E_\epsilon(F_y) \leq \int_{f^{-1}(C \setminus D_\epsilon(w))} \frac{1}{2} \operatorname{Lip}(f)^2 \frac{1}{2|f(x) - w|^2} + \frac{9}{4\epsilon^2} |f^{-1}(D_\epsilon(w))|,
\end{equation}
\[(4.8)\]

Suppose now that the Jacobian $|Jf| = |df^1 \wedge df^2|$ and the level sets $f^{-1}(\{z\})$ of $f$ satisfy estimates of the form
\begin{equation}
|Jf(x)| \geq C^{-1} \text{ a.e. } x \in M
\end{equation}
\[(4.9)\]

and
\begin{equation}
\sup_{z \in \mathbb{C}} \mathcal{H}^{n-2}(f^{-1}(\{z\})) \leq C
\end{equation}
\[(4.10)\]

for some finite, positive constants $C$. Then the coarea formula for Lipschitz maps (as stated in, e.g., [7]), together with (4.8), yields
\[
E_\epsilon(F_y) \leq C \operatorname{Lip}(f)^2 \int_{f(M) \setminus D_\epsilon(w)} \frac{1}{|z - w|^2} \cdot \mathcal{H}^{n-2}(f^{-1}(\{z\}))dz
+ \frac{C}{\epsilon^2} \int_{D_\epsilon(w)} \mathcal{H}^{n-2}(f^{-1}(\{z\}))dz
\leq C^2 \int_{f(M) \setminus D_\epsilon(w)} \frac{1}{|z - w|^2} + C^2.
\]

Finally, since the image $f(M)$ is a bounded subset of $\mathbb{R}^2$, we arrive at an estimate
\begin{equation}
E_\epsilon(F_y) \leq C_1 |\log \epsilon| + C_2,
\end{equation}
\[(4.11)\]

where $C_1$ and $C_2$ are constants depending only on $f$. Summarizing, we've proved the following:

**Lemma 4.1.** Given a Lipschitz map $f : M \to \mathbb{R}^2$ satisfying estimates of the form (4.9) and (4.10), the families $F^\epsilon \in \Gamma(M)$ defined by
\[F^\epsilon(y) := v_{y,\epsilon} \circ f\]

satisfy
\begin{equation}
\limsup_{\epsilon \to 0} \frac{1}{|\log \epsilon|} \max_{y \in D} E_\epsilon(F^\epsilon(y)) < \infty.
\end{equation}
\[(4.12)\]

Our goal now is to construct $f \in \text{Lip}(M, \mathbb{R}^2)$ satisfying (4.9) and (4.10). We do this via triangulation. Let $\Phi : M \to |\mathcal{K}|$ be a bi-Lipschitz map from $M$ to the underlying space of a finite simplicial complex $\mathcal{K}$ in some $\mathbb{R}^L$ (see, e.g., [22] for the classical construction). For each $k$-simplex $\Delta \in \mathcal{K}$, denote by $V(\Delta)$ the $k$-plane through the origin of $\mathbb{R}^L$ parallel to $\Delta$. Since $\mathcal{K}$ is finite, we can choose a generic 2-plane $\Pi \subset \mathbb{R}^L$ such that the restriction
\[
p|_{V(\Delta)} : V(\Delta) \to \Pi
\]
of the orthogonal projection
\[p : \mathbb{R}^L \to \Pi\]
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has rank 2 for every $\Delta \in \mathcal{K}$ of dimension $\geq 2$ and rank 1 when $\dim \Delta = 1$. Now identify $\Pi$ with $\mathbb{R}^2$, and set

$$f := p \circ \Phi.$$  

Since $\Phi$ is bi-Lipschitz, $\exists c > 0$ such that, for a.e. $x \in M$, the pullback

$$\Phi^*: \bigwedge^2 T^{*}_{\Phi(p)}|\mathcal{K}| \to \bigwedge^2 T^{*}_p M$$

satisfies

$$|\Phi^*(\zeta)| \geq c|\zeta| \text{ for every } \zeta \in \bigwedge^2 T^{*}_{\Phi(p)}|\mathcal{K}|.$$  

Furthermore, almost every $x \in M$ lies in the preimage of the interior $\Delta^0$ of some $n$-dimensional simplex $\Delta \in \mathcal{K}$. At such a point $x$, the differential $df$ of (5.13) is given by

$$p|V(\Delta) \circ d\Phi,$$

and since $p|V(\Delta)$ has full rank by our choice of $\Pi$, it follows that

$$|Jf(x)| = |d\Phi^*(p|V(\Delta)(e^1 \wedge e^2))| \geq c|p|V(\Delta)(e^1 \wedge e^2)| \geq C^{-1}$$

for some finite positive constant $C$. Thus, our chosen $f$ satisfies (4.19), and it remains to check (4.10).

This is similarly straightforward. For each $\Delta \in \mathcal{K}$ and $z \in \Pi$, our constraints on the rank of $p|V(\Delta)$ imply that $p^{-1}\{\{z\}\} \cap \Delta$ is given by the intersection of $\Delta$ with a translate of some subspace of $V(\Delta)$ of dimension $\leq n - 2$. Consequently, we have simple bounds of the form

$$\mathcal{H}^{n-2}(p^{-1}\{\{z\}\} \cap \Delta) \leq c_n \cdot diam(\Delta)^{n-2},$$

and thus, letting $N$ denote the number of simplices in $\mathcal{K}$,

$$\sup_{z \in \Pi} \mathcal{H}^{n-2}(p^{-1}\{\{z\}\} \cap |\mathcal{K}|) \leq Nc_n diam(\Delta)^{n-2} < \infty.$$  

It then follows that

$$\mathcal{H}^{n-2}(f^{-1}(\{z\})) \leq Lip(\Phi^{-1})^{n-2} \mathcal{H}^{n-2}(p^{-1}\{\{z\}\} \cap |\mathcal{K}|) \leq C,$$

for each $z \in \Pi$, so that (4.10) holds as well.

Thus, $f$ satisfies the hypotheses of Lemma 4.4, so we have families $F^\epsilon \in \Gamma(M)$ satisfying the estimate (4.12), and consequently,

$$\limsup_{\epsilon \to 0} \frac{c_\epsilon(M)}{\log \epsilon} = \limsup_{\epsilon \to 0} \frac{1}{\log \epsilon} \inf_{F \in \Gamma(M)} \max_{y \in D} E_\epsilon(F(y)) < \infty.$$  

Proposition 3.4 and (4.18) then combine to give us the estimate (1.10), completing the proof of Theorem 1.1.
5. The Energy Concentration Varifold when \( H^1_{dR}(M) = 0 \)

For \( \epsilon \in (0, 1) \), let \( u_\epsilon \) be a solution of (1.5) as constructed in Theorem 1.1. Our goal in this section is to show that when \( H^1_{dR}(M) = 0 \), along some subsequence \( \epsilon_j \to 0 \), the energy concentration measures

\[
\mu_\epsilon := \frac{e_\epsilon(u_\epsilon)}{|\log \epsilon|} dv_g
\]

concentrate on a stationary, rectifiable \((n - 2)\)-varifold. The key analytical lemma we’ll need to establish this is the density estimate

**Lemma 5.1.** If \( H^1_{dR}(M) = 0 \), then for any limiting measure \( \mu = \lim_{j \to \infty} \mu_{\epsilon_j} \), the \((n - 2)\)-density

\[
\Theta^{n-2}(\mu, x) := \lim_{r \to 0} \frac{\mu(B_r(x))}{\omega_{n-2}r^{n-2}} > 0
\]

for every \( x \in \text{spt}(\mu) \).

To see why Lemma (5.1) is sufficient to establish Theorem 1.2, we recall some of the results of Ambrosio and Soner in [3]. Given an integer \( 0 \leq m \leq n \), denote by

\[ A^0_m(M) \subset \text{End}(TM) \]

the compact (fiber-)subbundle of \( \text{End}(TM) \) given by

\[
A^0_m(M) := \{ S \in \text{End}(TM) \mid S = S^*, -nId \leq S \leq Id, \text{tr}(S) \geq m \}.
\]

In the language of [3], a **generalized** \( m \)-varifold is a nonnegative Radon measure on the fiber bundle \( A^0_m(M) \). Note that the Grassmannian bundle \( G_m(TM) \) is naturally included in \( A^0_m(M) \) by identifying \( m \)-dimensional subspaces with the associated orthogonal projections, and thus every standard \( m \)-varifold (in the sense of [2],[19]) also defines a generalized \( m \)-varifold.

As with standard \( m \)-varifolds, one can define the first variation \( \delta V \) of a generalized \( m \)-varifold \( V \) as follows [3]: given a smooth vector field \( X \) on \( M \), we set

\[
\delta V(X) := \int_{A^0_m(M)} \langle S, \nabla X \rangle dV(S),
\]

and we call \( V \) **stationary** if \( \delta V = 0 \). Naturally, one also defines the mass measure \( \| V \| \) on \( M \) as the pushforward of \( V \) by the projection \( \pi : A^0_m(M) \to M \) [3]. Given a sequence \( V_j \) of generalized \( m \)-varifolds converging in \( C^0(A^0_m)^* \) to a generalized \( m \)-varifold \( V \), it follows from the standard properties of nonnegative Radon measures on compact spaces that \( \| V_j \| \to \| V \| \) in \( C^0(M)^* \); moreover, if \( \delta V_j = 0 \), it follows immediately from the definition of \( (C^0)^* \) convergence that \( \delta V = 0 \) as well [3].

---

2In the definition of generalized varifold in [3], the trace inequality \( \text{tr}(S) \geq m \) is replaced by equality, but the upper bound on the trace plays no role in their analysis.
Aside from standard varifolds, the most important (and motivating) examples of generalized varifolds come from the stress-energy tensors associated with solutions of pdes satisfying an inner variation equation. For solutions of (1.5), one considers the tensor (viewed as a symmetric endomorphism)
\[(5.3) \quad T_\epsilon(u_\epsilon) := e_\epsilon(u)Id - du^*du;\]
it follows from (1.5) that
\[\text{div}(T_\epsilon) = 0,\]
and thus, for any smooth vector field \(X\) on \(M\),
\[(5.4) \quad \int_M \langle T_\epsilon(u_\epsilon), \nabla X \rangle = 0.\]
In particular, writing
\[(5.5) \quad P_\epsilon := \text{Id} - e_\epsilon(u_\epsilon)^{-1}du^*du,\]
it follows that the measure
\[(5.6) \quad V_\epsilon := \delta P_\epsilon \times \mu_\epsilon\]
defines a stationary generalized \((n-2)\)-varifold with weight measure \(\mu_\epsilon\).

For solutions, like those of Theorem 1.1 satisfying an energy bound \(\mu_\epsilon(M) \leq C\), we can extract a subsequence \(\epsilon_j \to 0\) such that
\[V_{\epsilon_j} \to V\]
as generalized varifolds, so that \(V\) is again a stationary generalized \((n-2)\)-varifold with weight measure
\[\|V\| = \mu := \lim_{\epsilon_j \to 0} \mu_{\epsilon_j}.\]

We now recall the key measure-theoretic result of [3]:

**Proposition 5.2.** (Theorem 3.8 of [3]) If \(V\) is a generalized \(m\)-varifold for which \(\delta V \in [C^0(M,TM)]^*\) and \(\Theta^m(\|V\|, x) > 0\) at \(\|V\|\)-a.e. \(x \in \text{spt}(V)\), then there is a rectifiable \(m\)-varifold \(\tilde{V}\) with \(\|\tilde{V}\| = \|V\|\) and \(\delta \tilde{V} = \delta V\).

**Remark 5.3.** If the given \(V\) is a (standard) varifold, this is simply Allard’s rectifiability theorem (see [2], Section 5). The key observation of [3] is that positive density and bounded first variation force the fiber-wise center of mass of \(V\) to have the structure of orthogonal projection onto an \(m\)-dimensional subspace. In particular, if \(V\) has the structure of a Dirac mass in each fiber of \(A_m\) (as in (5.6)), the induced varifold \(\tilde{V} = V\).

**Remark 5.4.** In our statements of Lemma 5.1 and Proposition 5.2, we have implicitly used the fact that generalized \(m\)-varifolds \(V\) with bounded first variation satisfy a monotonicity property identical to that of standard \(m\)-varifolds, to ensure that the density \(\Theta^m(\|V\|, x)\) is well-defined without the decorations \(\ast\) or \(\ast\). (The monotonicity follows from the standard computations of, e.g., Section 5 of [2], or Section 40 of [19].)
From Proposition 5.2 and the preceding discussion, it is now clear that Lemma 5.1 will be sufficient to establish the conclusion of Theorem 1.2; the remainder of this section will be devoted to establishing this positive density condition.

To this end, we need a better understanding of the structure of the limiting measure $\mu$. By the local estimates of (6, Theorem 1.1 and Proposition 1.3), the upper bound in (1.10) immediately gives us the uniform bound

$$\int_M |d|u_\epsilon|^2 + \frac{W(u_\epsilon)}{\epsilon^2} \leq C,$$

where $C$ is independent of $\epsilon$. Defining $j u_\epsilon$ as before (3.3), we observe that

$$|u_\epsilon|^2 |du_\epsilon|^2 = |j u_\epsilon|^2 + |u_\epsilon|^2 |d|u_\epsilon|^2,$$

so it suffices to understand the contributions of $|j u_\epsilon|^2$ and $(1 - |u_\epsilon|^2) |du_\epsilon|^2$ to the energy concentration.

Next, note that the equation (1.5) on $(M, g)$ is equivalent to

$$\Delta g_\epsilon u_\epsilon = DW(u_\epsilon) = -(1 - |u_\epsilon|^2) u_\epsilon$$

in the dilated metric $g_\epsilon = \epsilon^{-2} g$. It follows that

$$\Delta g_\epsilon \frac{1}{2}(1 - |u_\epsilon|^2) = (1 - |u_\epsilon|^2) |u_\epsilon|^2 - |du_\epsilon|^2_{g_\epsilon},$$

and, via the Bochner formula,

$$\Delta g_\epsilon \frac{1}{2} |du_\epsilon|^2_{g_\epsilon} = \frac{1}{2} |d|u_\epsilon|^2_{g_\epsilon} - (1 - |u_\epsilon|^2) |du_\epsilon|^2_{g_\epsilon} + \langle Ric_{g_\epsilon}, du_\epsilon^* du_\epsilon \rangle_{g_\epsilon} + |Hess(u_\epsilon)|_{g_\epsilon}^2.$$

Thus, setting

$$w := \frac{1}{2} |du_\epsilon|^2_{g_\epsilon} - \frac{b}{2}(1 - |u_\epsilon|^2),$$

for some constant $b$, we find that

$$\Delta g_\epsilon w \geq (b - 1) |du_\epsilon|^2_{g_\epsilon} + 2 |u_\epsilon|^2 w + \epsilon^4 \langle Ric_{g_\epsilon}, du_\epsilon^* du_\epsilon \rangle_g$$

$$\geq (b - 1 - \epsilon^2 |Ric_{g_\epsilon}|_{g}) |du_\epsilon|^2_{g_\epsilon} + 2 |u_\epsilon|^2 w.$$ 

Writing $A(M, g) := \max_{x \in M} |Ric_{g}|_g$, it follows that for $b > 1 + A \epsilon^2$, $w$ must be negative at its maximum, and we therefore conclude that

$$|du_\epsilon|^2_{g_\epsilon} \leq (1 + A \epsilon^2)(1 - |u_\epsilon|^2)$$

everywhere. Scaling back, we arrive at the gradient estimate

$$|du_\epsilon|^2 \leq \frac{1}{\epsilon^2} + A(1 - |u_\epsilon|^2).$$

From (5.10) and (5.7), we obtain the estimate

$$\int_M (1 - |u_\epsilon|^2) |du_\epsilon|^2 \leq C' \int_M \frac{W(u_\epsilon)}{\epsilon^2} \leq C'',$$
so that the only nontrivial contribution to the energy blow-up must come from $ju_{\epsilon}$. That is, given a convergent subsequence $\mu_{\epsilon_j} \to \mu$, combining (5.7), (5.8), and (5.11), we conclude that

$$\mu = \lim_{j \to \infty} \frac{1}{2} \frac{|ju_{\epsilon_j}|^2}{|\log \epsilon_j|} dv_g. \tag{5.12}$$

Now, following the arguments of [5], choose a smooth function $f : [0, 1] \to [1, 2]$ satisfying

$$f(t) = \frac{1}{t} \text{ for } t \geq \frac{3}{4}, \quad f(t) = 1 \text{ for } t \leq \frac{1}{2}, \quad \text{and } |f'| \leq 2. \tag{5.13}$$

Defining the one-forms

$$\gamma_{\epsilon} := f(|u_{\epsilon}|^2) ju_{\epsilon}, \tag{5.14}$$

it follows from the choice of $f$ that

$$||\gamma_{\epsilon}||^2 - |ju_{\epsilon}|^2| \leq C(1 - |u_{\epsilon}|^2)|ju_{\epsilon}|^2 \leq C(1 - |u_{\epsilon}|^2)|du_{\epsilon}|^2,$$

and thus, in light of (5.12) and (5.11), we have

$$\mu = \lim_{j \to \infty} \frac{1}{2} \frac{|\gamma_{\epsilon_j}|^2 dv_g}{|\log \epsilon_j|}. \tag{5.15}$$

As in [5], our estimates for $\gamma_{\epsilon_j}$ will come from estimates on the components of its Hodge decomposition; naturally, this is the point in our analysis where the constraint $H^1_{dR}(M) = 0$ becomes crucial. (We also assume throughout that $M$ is orientable, but this is ultimately of no analytic significance, as we can always pass to a double cover.) Choosing $\theta_{\epsilon} \in C^\infty(M)$ and $\xi_{\epsilon} \in \Omega^2(M)$ such that

$$\Delta \theta_{\epsilon} = \text{div}(\gamma_{\epsilon}) \tag{5.16}$$

and

$$\Delta_H \xi_{\epsilon} = d\gamma_{\epsilon} \tag{5.17}$$

(where $\Delta_H = dd^* + d^*d$ is the usual Hodge Laplacian), we use the fact that $H^1_{dR}(M) = 0$ to conclude that

$$\gamma_{\epsilon} = d\theta_{\epsilon} + d^*\xi_{\epsilon}. \tag{5.18}$$

Remark 5.5. Without the assumption that $H^1_{dR}(M) = 0$, we could still carry out the analysis of this section provided we had some a priori control on the harmonic part of $\gamma_{\epsilon}$. However, without such control, we allow for solutions like those discussed in Remark 1.3, for which the harmonic part of $\gamma_{\epsilon}$ dominates the energy blow-up.

We show next that the exact part $d\theta_{\epsilon}$ of $\gamma_{\epsilon}$ contributes negligibly to $\mu$. Since $\text{div}(ju_{\epsilon}) = 0$, the defining equation (5.16) for $d\theta_{\epsilon}$ and (5.14) yield

$$\Delta \theta_{\epsilon} = \text{div}(\gamma_{\epsilon}) = f'(|u_{\epsilon}|^2) \langle d|u_{\epsilon}|^2, ju_{\epsilon} \rangle.$$
Multiplying by $\theta_\epsilon$ and integrating, we see that the $L^2$ norm of $d\theta_\epsilon$ is given by

\begin{equation}
\int_M |d\theta_\epsilon|^2 = -\int_M \theta_\epsilon f'(|u_\epsilon|^2) \langle|u_\epsilon|^2, ju_\epsilon\rangle.
\end{equation}

(5.19)

Applying the coarea formula to the $|u_\epsilon|^2$ terms, we recast this as

\begin{align*}
\int_M |d\theta_\epsilon|^2 &= -\int_0^1 f'(t) \left( \int_{\partial\{u_\epsilon^2 < t\}} \langle \theta_\epsilon ju_\epsilon, \nu \rangle \right) dt \\
&= -\int_0^1 f'(t) \left( \int_{\{u_\epsilon^2 < t\}} \operatorname{div}(\theta_\epsilon ju_\epsilon) \right) dt \\
&= -\int_0^1 f'(t) \left( \int_{\{u_\epsilon^2 < t\}} \langle d\theta_\epsilon, ju_\epsilon \rangle \right) dt \\
&\leq 2 \int_0^1 \left( \int_{\{u_\epsilon^2 < t\}} |d\theta_\epsilon||ju_\epsilon| \right) dt \\
&\leq \int_0^1 \left( \frac{1}{2} \int_{\{u_\epsilon^2 < t\}} |d\theta_\epsilon|^2 + 2|ju_\epsilon|^2 \right) dt,
\end{align*}

from which it follows that

\begin{equation}
\int_M |d\theta_\epsilon|^2 \leq 4 \int_0^1 \left( \int_{\{u_\epsilon^2 < t\}} |ju_\epsilon|^2 \right) dt.
\end{equation}

(5.20)

Now, by (5.10), we know that

\begin{equation}
|ju_\epsilon|^2 \leq |du_\epsilon|^2 \leq \frac{C}{\epsilon^2},
\end{equation}

(5.21)

while it follows from (5.7) that

\begin{equation}
|\{u_\epsilon^2 \leq t\}| \leq \frac{4\epsilon^2}{(1-t)^2} \int_M \frac{W(u_\epsilon)}{\epsilon^2} \leq \frac{C\epsilon^2}{(1-t)^2}.
\end{equation}

(5.22)

Splitting the right-hand side of (5.20) into integrals over $t \in [0, 1-|\log \epsilon|^{-1/2}]$ and $t \in [1-|\log \epsilon|^{-1/2}, 1]$ (taking now $\epsilon < \frac{1}{e}$), we apply (5.21), (5.22), and (1.10) to estimate

\begin{align*}
\int_M |d\theta_\epsilon|^2 &\leq 4 \int_0^{1-|\log \epsilon|^{-1/2}} \left( \int_{\{u_\epsilon^2 < t\}} |ju_\epsilon|^2 \right) dt + \int_{1-|\log \epsilon|^{-1/2}}^1 \left( \int_{\{u_\epsilon^2 < t\}} |ju_\epsilon|^2 \right) dt \\
&\leq 4 \int_0^{1-|\log \epsilon|^{-1/2}} \frac{C}{\epsilon^2} \cdot \frac{C\epsilon^2}{(1-t)^2} dt + |\log \epsilon|^{1/2} \mu_\epsilon(M)
\end{align*}

\begin{align*}
&\leq C' |\log \epsilon|^{1/2}.
\end{align*}
It then follows that, for any \( U \subset M \),
\[
\frac{1}{|\log \epsilon|} \int_U (|d\theta_\epsilon|^2 + 2|d\theta_\epsilon, d^*\xi_\epsilon|) \leq \frac{C}{|\log \epsilon|^{1/2}} + \frac{1}{|\log \epsilon|} \int_U |\log \epsilon|^{1/4}|d\theta_\epsilon|^2
\]
\[
+ \frac{1}{|\log \epsilon|} \int_U |\log \epsilon|^{-1/4}|d^*\xi_\epsilon|^2
\]
\[
\leq \frac{C}{|\log \epsilon|^{1/4}} \quad \rightarrow \quad 0 \quad \text{as} \quad \epsilon \rightarrow 0,
\]
and consequently, we obtain our final reduction
\[
\mu = \lim_{j \rightarrow \infty} \frac{1}{2} \frac{|d^*\xi_j|^2}{|\log \epsilon_j|}.
\]

Now, consider a point \( x \in M \) at which the density \( \Theta^{n-2}(\mu, x) = 0 \); that is, suppose
\[
\lim_{r \rightarrow 0} \frac{\mu(B_r(x))}{r^{n-2}} = 0.
\]
To establish Lemma 5.1, we need to show that \( x \notin \text{spt}(\mu) \); i.e., we need to find some ball \( B_r(x) \) about \( x \) for which
\[
\mu(B_r(x)) = \lim_{j \rightarrow \infty} \frac{1}{2} \frac{|d^*\xi_j|^2}{|\log \epsilon_j|} = 0.
\]

We begin by arguing as in Section VII of [5]. Let \( \delta_0(M), \eta_0(M) > 0 \) be as in the \( \eta \)-ellipticity Theorem 3.2. Let \( \delta_0(B_r) \) by (5.24), we can select \( R \in (0, \delta_0) \) such that
\[
\frac{1}{2} \eta_0 \geq \frac{\mu(B_{2R}(x))}{R^{n-2}} = \lim_{\epsilon \rightarrow 0} \frac{R^{2-n}}{|\log \epsilon|} \int_{B_{2R}(x)} e_\epsilon(u_\epsilon) = \lim_{\epsilon \rightarrow 0} \frac{R^{2-n}}{|\log (\frac{R}{\epsilon})|} \int_{B_{2R}(x)} e_\epsilon(u_\epsilon).
\]
Applying Theorem 3.2 at each point in \( B_R(x) \) for \( \epsilon \) sufficiently small, we conclude that
\[
|u_\epsilon|^2(y) \geq \frac{7}{8} \quad \text{for every} \quad y \in B_R(x),
\]
and observe that, by (5.13) and the definition of \( \gamma_\epsilon \),
\[
\gamma_\epsilon = \frac{1}{|u_\epsilon|^2} j u_\epsilon = j(u_\epsilon/|u_\epsilon|) \quad \text{on} \quad B_R(x).
\]
In particular, it follows that
\[
d^*\xi_\epsilon = d\gamma_\epsilon = 0 \quad \text{on} \quad B_R(x),
\]
and defining \( \varphi_\epsilon \in C^\infty(B_R(x)) \) by
\[
\Delta \varphi_\epsilon = 0, \quad \frac{\partial \varphi_\epsilon}{\partial \nu} = d^*\xi_\epsilon(\nu) \quad \text{on} \quad \partial B_R(x), \quad \text{and} \quad \int_{B_R(x)} \varphi_\epsilon = 0,
\]
we have
\[
d\varphi_\epsilon = d^*\xi_\epsilon \quad \text{on} \quad B_R(x).
\]
Since $\Delta \varphi_\epsilon = 0$ and $\int_{B_R(x)} \varphi_\epsilon = 0$, it follows from standard elliptic estimates that

$$\int_{B_{R/2}} |d^* \xi_\epsilon|^2 = \int_{B_{R/2}} |d\varphi_\epsilon|^2 \leq C_p \int_{B_R} |d\varphi_\epsilon|^p = C_p \int_{B_R} |d^* \xi_\epsilon|^p$$

for any $p \in (1, \infty)$. We now recall one of the central observations of [5]:

**Claim 5.6.** For each $1 \leq p < \frac{n}{n-1}$, there exists $C_p$ (independent of $\epsilon$) such that

$$\int_M \|d^* \xi_\epsilon\|^p \leq C_p.$$  

Once the claim is established, we'll obtain from (5.30) a uniform bound

$$\int_{B_{R/2}(x)} |d^* \xi_\epsilon|^2 \leq C$$

independent of $\epsilon$, and as a result,

$$\mu(B_{R/2}(x)) \leq \lim_{j \to \infty} \frac{1}{\log \epsilon_j} \int_{B_{R/2}(x)} \frac{1}{2} |d^* \xi_{\epsilon_j}|^2 = 0,$$

which is precisely what we needed to complete the proof of Lemma 5.1.

To prove the claim, we follow closely the arguments in Section VI and the Appendix of [5]. Fix $p \in \left[1, \frac{n}{n-1}\right)$, and denote by $q = \frac{p}{p-1} > n$ its Hölder conjugate. The $L^p$ norm of $d^* \xi_\epsilon$ is then given by

$$\|d^* \xi_\epsilon\|_{L^p(M)} = \sup \left\{ \int_M \langle d^* \xi_\epsilon, \beta \rangle \mid \beta \in \Omega^1(M), \|\beta\|_{L^q} = 1 \right\}.$$  

Since $d^* \xi_\epsilon$ is co-exact, for any $\beta \in \Omega^1(M)$, we have that

$$\int_M \langle d^* \xi_\epsilon, \beta \rangle = \int_M \langle d^* \xi_\epsilon, d^* d\alpha \rangle = \int_M \langle dd^* \xi_\epsilon, d\alpha \rangle,$$

where $\alpha := \Delta_H^{-1} \beta$ is the unique solution of $\Delta_H \alpha = \beta$. (If we allowed $H^1_{dR}(M) \neq 0$, we would of course have to carry this out on the orthogonal complement of the harmonic one-forms.) Now, it follows from the $L^q$ regularity theory of the Hodge Laplacian (see [18] for a careful treatment) that

$$\|d\alpha\|_{W^{1,q}} \leq C_q \|\beta\|_{L^q},$$

and since $q > n$, the Sobolev inequality yields

$$\|d\alpha\|_{L^\infty} \leq C_q \|\beta\|_{L^q}.$$  

Recalling that

$$dd^* \xi_\epsilon = d\gamma_\epsilon,$$

we can combine (5.33)-(5.35) to obtain the bound

$$\|d^* \xi_\epsilon\|_{L^p(M)} \leq C_p \|d\gamma_\epsilon\|_{L^1}.$$
All that remains is to bound the $L^1$ norm of $d\gamma_\epsilon$ as in [5], and this is straightforward. As we noted earlier, it follows from the definition of $f$ that

$$\gamma_\epsilon = j\left(\frac{u_\epsilon}{|u_\epsilon|}\right)$$

is closed on $\{|u_\epsilon|^2 \geq \frac{3}{4}\}$, so that

$$spt(d\gamma_\epsilon) \subset \{|u_\epsilon|^2 \leq \frac{3}{4}\},$$

and by (5.22),

$$(5.37)\quad |spt(d\gamma_\epsilon)| \leq C \epsilon^2.$$ 

Next, noting that

$$|d\gamma_\epsilon| = |f'(|u_\epsilon|^2)d|u_\epsilon|^2 \wedge j u_\epsilon + f(|u_\epsilon|^2)du_\epsilon| \leq C|du_\epsilon^1 \wedge du_\epsilon^2|,$$

we conclude from (5.10) that

$$(5.38)\quad |d\gamma_\epsilon| \leq \frac{C}{\epsilon^2}$$

pointwise. Combining (5.37) and (5.38), we obtain the desired $L^1$ bound

$$(5.39)\quad \|d\gamma_\epsilon\|_{L^1} \leq C,$$

which together with (5.36) completes the proof of the claim. The conclusion of Lemma 5.1 and, consequently, Theorem 1.2 then follow.

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