Entangling two defects via a surrounding crystal

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We theoretically show how two impurity defects in a crystalline structure can be entangled through coupling with the crystal. We demonstrate this with a harmonic chain of trapped ions in which two ions of a different species are embedded. Entanglement is found for sufficiently cold chains and for a certain class of initial, separable states of the defects. It results from the interplay between localized modes which involve the defects and the interposed ions, it is independent of the chain size, and decays slowly with the distance between the impurities. These dynamics can be observed in systems exhibiting spatial order, viable realizations are optical lattices, optomechanical systems, or cavity arrays in circuit QED.

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Entanglement is a quantum mechanical property with no classical counterpart. One of its peculiar features are the correlations between the measurement outcomes for two entangled objects, even when they are at large distances [1], which make entanglement a precious resource for quantum communication protocols and quantum metrology [2]. On the other hand, its quantum mechanical nature makes it fragile against external perturbations [3].

The issue of identifying open-system dynamics in which entanglement is robust recently motivated a series of studies [4, 5], which showed that a bath can mediate entanglement between two objects interacting with it [6–11]. These findings lead to the question of how bath-mediated entanglement scales with the distance between the objects. When the bath is composed by oscillators forming a linear chain, entanglement between any pair of oscillators decays quickly on a length scale of the order of the interparticle distance [12, 13]. A further study, based on a phenomenological model, suggested that the cutoff wavelength, which in a linear chain coincides with the interparticle distance, is the characteristic length over which bath-mediated entanglement vanishes [14].

These predictions are however restricted to specific settings. In Ref. [13], for instance, it was argued that Markovian models can not capture all dynamical processes which can lead to entanglement between distant objects. Moreover, in Ref. [12], a microscopic model was proposed, that allowed two defect oscillators embedded in the chain to become entangled by the chain itself instead of thermalizing at the chain’s temperature [15].

This is possible since the impurities break the discrete translational symmetry of the lattice, giving rise to a set of normal modes involving the defects and the ions between them. Such modes are essential for the generation of entanglement at steady state. We remark that a linear chain of oscillators in a thermal state and coupled to a single impurity is a realization of Rubin’s model, it leads to thermalization of the impurity and is an example of Brownian motion in a solid-state environment [17]. The addition of a second impurity introduces a new symmetry modifying this behaviour. In the context of recent studies on non-Markovianity [15, 18–21] one can say that the second defect modifies the reservoir making it non-Markovian [16].

The findings of Ref. [16] imply that two impurity defects can be entangled by the surrounding bulk, when the latter exhibits some sort of spatial order. In addition, entanglement could be found between two defects at any distance, independent of the size of the bulk. The latter statement thus goes beyond existing studies, which analyze entanglement generation between two systems at the edges of a chain [22, 23]. It further addresses the question, whether it is possible to generate entanglement between two macroscopically-distant objects through interaction with a large common bulk. Understanding this issue is relevant at the fundamental level, for example in the context of recent studies of thermalization in quantum systems [24, 25]. Moreover, it is important for studies on biological systems [21], and for implementing protocols based on dissipative dynamics in quantum networks [26].
In this Letter we analyse the dependence of bath-mediated entanglement on the distance by considering the specific physical system of two ions in a linear Paul trap, which are embedded in a linear chain of ions of different species, as shown in Fig. 1. Let \( N \) and \( Q \) be the number of ions and their charge. The ions are aligned along the \( z \) axis inside a linear Paul trap with position (canonically-conjugated momentum) \( \mathbf{p}_j = (p_{jx}, p_{jy}, p_{jz}) \), where \( j = 1, \ldots, N \) labels the ions. The impurity defects are two ions of different species far away from the chain edges, whose mass \( M \) is larger than the mass \( m \) of the other ions composing the chain. The trap potential for the ion \( j \) reads \( V_{\text{trap}}(\mathbf{r}_j) = \left( U_{0} |z_j|^2 + U_{\perp}(x_j^2 + y_j^2)/2 \right) \), where \( U_0 \) is determined by the static quadrupole potential and \( U_{\perp} = (U_0/m_j - U_0)/2 \) by the radio-frequency potential creating the transverse confinement. 

We note the dependence of the transverse potential strength on the mass of the ion: this implies that the defect’s transverse motion is a localized oscillation in the chain for sufficiently large ratios \( M/m \). The ions are at a sufficiently low temperature \( T \) such that they perform harmonic vibrations about their respective equilibrium positions \( \mathbf{r}_j^{(0)} = (0, 0, z_j^{(0)}) \), at which the trap force and the Coulomb repulsion mutually balance. In this limit the dynamics is governed by the quadratic Hamiltonian \( H_0 = H_{\parallel} + H_{\perp}^{(x)} + H_{\perp}^{(y)} \), with

\[
H_{\parallel} = \sum_{j=1}^{N} \left( \frac{p_{jz}^2}{2m_j} + \frac{1}{2} U_0 |q_j|^2 + \frac{1}{4} \sum_{\ell \neq j} K_{j,\ell}(q_j - q_\ell)^2 \right), \\
H_{\perp}^{(x)} = \sum_{j=1}^{N} \left( \frac{p_{jx}^2}{2m_j} + \frac{1}{2} U_{\perp} x_j^2 - \frac{1}{8} \sum_{\ell \neq j} K_{j,\ell}(x_j - x_\ell)^2 \right),
\]

and \( H_{\perp}^{(y)} = H_{\perp}^{(x \rightarrow y)} \). Here, \( q_j = z_j - z_j^{(0)} \), while \( K_{j,\ell} = 2Q^2/|z_j^{(0)} - z_\ell^{(0)}|^3 \) is the coupling due to the Coulomb repulsion. We label the defect ions by \( j_1, j_2 \), with \( 1 \ll j_1 < j_2 \ll N \) and mutual distance \( d \gg j_2 - j_1 \ll N \).

We first analyze the spectrum of Hamiltonian \( H_0 \). For simplicity, we assume equally-spaced axial equilibrium positions with interparticle distance \( a = z_j^{(0)} - z_{j+1}^{(0)} \), which can be found in the central region of long ion chains and in anharmonic potentials. Figure 2(a) displays the spectra of the axial and transverse modes for \( M \approx 2.87 m_\text{e} \), which corresponds to In\(^{2+} \) ions embedded in a Ca\(^{+} \) chain. Two degenerate normal mode frequencies for each transverse spectrum are separated from the respective transverse branch by a gap: If this gap is sufficiently large, these modes approximately coincide with the defects transverse vibrations. In this limit they are strongly coupled with each other via the Coulomb interaction, and weakly coupled to the rest of the chain.

Their dynamics are governed by a beam-splitter type of interaction: The two defects can become entangled by the unitary evolution after preparing each transverse mode in a squeezed state. We verify this behaviour assuming that initially the chain is in a thermal state at temperature \( T \) while the transverse modes are prepared in identical squeezed vacuum states along the \( x \) direction with variances \( \Delta x_{jz}^2 = x_0^2 e^{2s}/2 \) and \( \Delta p_{jz}^2 = p_0^2 e^{-2s}/2 \). Here \( x_0 = \sqrt{\hbar/(M \Omega_{\perp})} \) is the size of the ground state of the defect oscillator with frequency \( \Omega_{\perp} = \sqrt{U_{\perp}(M)/M} \), \( m_\text{e} = \hbar/x_0 \), and \( s \) is the real-valued squeezing parameter. The defects’ state remains Gaussian under the evolution given by the quadratic Hamiltonian, Eq. (2), and is hence described by the first moments and the covariance matrix. Entanglement between the defect modes is quantified by means of the logarithmic negativity, \( E_N \), obtained from the symplectic eigenvalues of the partially transposed covariance matrix \( \tilde{\Sigma} \). The grey curves in Fig. 2(b) display \( E_N \) as a function of the time \( t \) for a fixed distance \( d = 7a \); \( E_N \) grows with \( t \) and increases with \( s \). The defects become entangled also when they are initially not squeezed \((s = 0)\), which is due to the fact that the normal modes of the isolated frequencies include some small displacements of the other ions of the chain. Numerical simulations show that \( E_N \) at a given time \( t \) decreases with the strength of the direct Coulomb coupling, which scales with the distance as \( 1/d^2 \).

We now demonstrate that entanglement between the defects can be generated and substantially enhanced by coupling with the axial phonons of the chain. For this purpose we consider an interaction which couples the defects’ transverse and axial displacements. The dynamics are now given by Hamiltonian \( H = H_0 + H_{\ell}(t) \), where

\[
H_{\ell}(t) = \frac{\gamma(t)}{2} \left[ (x_{j_1} - q_{j_1})^2 + (x_{j_2} - q_{j_2})^2 \right],
\]

with \( \gamma(t) = \gamma \Theta(t) \) being an effective coupling strength and \( \Theta(t) \) the Heaviside function. This coupling could be realized, for instance, using a standing-wave laser in the \( x \) - \( z \) plane illuminating the defects in the Lamb-Dicke regime and with nodes at their equilibrium positions. For this geometry the motion along \( y \) is decoupled and will be ignored from now on. At \( t > 0 \) a displacement of each defect along \( x \) excites a wave packet of axial phononic modes.

We first note that these dynamics lead to thermalization of a single defect, but are significantly modified when both defects are coupled. In order to understand why, let us consider that the defects are symmetrically placed with respect to the trap center. By performing a change of coordinates, \( q_j^\pm = (q_{N-j} \pm q_j)/\sqrt{2} \) and \( X_\pm = (x_{j_1} \pm x_{j_2})/\sqrt{2} \), it becomes evident that the defects’ transverse center-of-mass (COM) displacement, \( X_+ \), couples to the axial displacements \( q_j^+ \), and the same holds for \( X_- \) and \( q_j^- \). The action of the axial vibrations

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on the dynamics of the defects’ collective variable can be characterized in terms of the spectral densities \( J^{\pm}(\omega) \)

\[
J^{\pm}(\omega) = \pi \sum_{k=1}^{N/2} \frac{(\gamma_{k}^{\pm})^{2}}{2m\omega_{k}^{\pm}} \delta(\omega - \omega_{k}^{\pm}),
\]

where \( \gamma_{k}^{\pm} \) is the coupling strength of the coordinate \( X_{\pm} \) to the \( k \)-th normal mode at eigenfrequency \( \omega_{k}^{\pm} \) of the chain \( q_{k}^{\pm} \). The inset of Fig. 2(b) displays the spectral densities \( J_{\pm}(\omega) \) for \( d = 7a \). The appearance of frequency values \( \omega_{k}^{+} (\omega_{k}^{-}) \) at which the spectral density \( J_{+} (J_{-}) \) vanishes, is a signature of “dark” normal modes, namely, localized excitations which involve the collective variable \( X^{+} (X^{-}) \) and which are decoupled from the rest of the chain. These zeroes equal the number of ions between the two defects, and each frequency \( \omega_{k}^{+} (\omega_{k}^{-}) \) corresponds to a decoherence-free subspace for \( X_{+} (X_{-}) \) when the defect frequency fulfills \( \Omega_{\pm} \simeq \omega_{k}^{+} (\Omega_{\pm} \simeq \omega_{k}^{-}) \) \[14\]. Entanglement between the defects can thus be generated for an initial state in which both impurities are in a squeezed state while the rest of the chain is in a thermal state \[42\]. In fact, taking for instance \( \Omega_{+} = \omega_{-}^{+} \), the dynamics leads to thermalization of the defects’ COM motion, \( X_{z} \), and destroys all initial correlations between the defects’ COM and relative motion, while the relative coordinate preserves part of the initial squeezing. Sufficiently large squeezing \( s \) and low temperatures \( T \) lead to two-mode squeezing of the defects’ transverse motion \[10, 44\], and hence to entanglement \[45\].

The logarithmic negativity is determined after numerically evaluating the defects’ dynamics, starting from the formal solution of the coupled Heisenberg equations of transverse, axial, and defect modes of the Hamiltonian \( H \) \[44\]. The solid curves in Fig. 2(b) display \( E_{N} \) in a chain of 50 ions, and for times over which finite-size effects are negligible. Entanglement builds up after a transient time and reaches values which are an order of magnitude larger than the values found in the absence of the coupling laser. It increases with the strength of the initial squeezing \( s \), and we checked that it also increases when decreasing the temperature of the chain, which is consistent with the results of previous works \[10, 16, 44\].

We note that, while cooling a large chain to ultralow temperatures is a challenging task, the basic requirement for observing the dynamics predicted here is that the axial mode of the chain resonant with the defect frequency is prepared in the ground state. In a chain of trapped ions this could be realized, for example, by cooling the motion of the defect ions while the laser coupling transverse and axial displacement is switched on \[46\]. Noise will not significantly affect the predicted dynamics, provided that the dark mode is cold and protected from external noise sources \[47\].

We now demonstrate that the created entanglement is not a finite-size effect, and is indeed independent of the chain size. Fig. 2(c) displays the logarithmic negativity...
as a function of time for various chain sizes and a mutual
distance of $d = 17a$ between the defects. Here the COM
motion is decoupled by tuning its frequency to a value at
which the corresponding spectral density vanishes (see in-
set). The curves are displayed for times that are shorter
than the revival time and have been rescaled by the size-
dependent frequency $\omega_\parallel = \omega_{\text{rad}} \sqrt{\log N/N}$, where $\omega_{\text{rad}}$ is a
constant and the size-dependent factor warrants that the
interparticle distance at the chain center is independent
of $N$ (therefore keeping constant the cutoff frequency)
[34]. One observes that the logarithmic negativity os-
cillates about a (quasi) stationary state, whose value is
independent of the chain size. For increasing chain sizes
the revival time, and thus the time window over which
this entanglement is found, correspondingly increases.

Let us finally show that the entanglement generated
by the interaction with the chain slowly decays with the
mutual distance. Figure 3 displays the mean value of the
logarithmic negativity, $E_N$, averaged over the time in
which it regularly oscillates, as a function of the mu-
tual distance when the COM is decoupled. The differ-
ent points correspond to different zeroes of the spectral den-
sity: optimal entanglement is thus achieved by suitably
tuning the frequency of the impurity, so that it matches
the frequency of an optimal dark normal mode. Steady-
state entanglement is found on time scales $t_0$ of the order
of $d/c_s$, where $c_s \simeq \omega_\parallel a$ is the sound velocity [34]. For
the parameters and the chains here considered $t_0 \sim 100\mu s$.
At large $d$, $E_N$ decays linearly with the distance, which is
confirmed by a systematic analysis performed on a model
with nearest-neighbor interactions [48].

The resulting entanglement can be measured by ex-
tending the method developed in Ref. [49] to two
Gaussian modes. The procedure consists of coupling
the transverse oscillation of each defect with an an-
cillary qubit, such as an electronic transition of the
defect ion. This can be done using the Hamiltonian
$H^{(\text{int})}_{j:j'=1,2} = \hbar g_j(t)\sigma_j^x(e^{-i\Omega t} - 1)\sigma_j^x + a_j^\dagger a_j e^{i\Delta t}$, where $a_j = (x_j/x_0 + ip_j/p_0)/\sqrt{2}$ annihilates a phonon
of the defect oscillator at frequency $\Omega_j$, $g_j$ denotes
the Rabi coupling with the internal transition, and $\sigma_j^{x,y,z}$
are the Pauli operators. The expectation value $\langle \mathcal{T} \rangle = \langle \sigma_j^x \otimes \sigma_j^x - \sigma_j^y \otimes \sigma_j^y + i\sigma_j^z \otimes \sigma_j^y + i\sigma_j^y \otimes \sigma_j^x \rangle$ gives
the characteristic function $\chi(\beta_1, \beta_2)$ of a two-mode sys-
tem, where $\beta_1,2 = 2i \int_0^\infty dt_j g_{j,j'}(t') e^{i\Delta_j t_j - s t'}$. The recon-
struction of the oscillators state and thus of its entan-
glement properties is granted by properly designed coupling
profiles. Note that it is sufficient to probe the character-
istic function of Gaussian states close to the phase-space
origin since they are characterised by their first and sec-
ond moments.

Our study sheds light on the role of the reservoir in
establishing quantum correlations. It further identifies
the basic requirements for entangling two distant ob-
jects which interact with a bulk exhibiting long-range order.
Here, non-local, decoherence-free subspaces ap-
pear which are associated with normal modes involving
the defects vibrations. In principle, it is thus possible to
entangle arbitrarily distant objects via a common bath.
An intrinsic limitation to such dynamics is the fact that
the required time linearly scales up with the distance,
while the zeroes of the dark normal modes become denser
and denser. A further challenge is the realization of arbi-
tary large regular lattices. Tuning the defects’ frequency
is the key for entanglement generation in other physi-

cal platforms. These dynamics can also be observed, for
instance, in dipolar quantum gases in the Mott insula-
tor phase in an optical lattice, where the optical traps
can be engineered in order to tune the frequency of the
transverse oscillations of different species [50, 51]. Other
examples are cavity arrays in circuit QED [52] and opt-
omechanical systems as in [53], where the frequencies

can be controlled by means of proper designs and the use
of refractive media.

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