Hydrodynamical Models of Accretion Disks in
SU UMa Systems

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Abstract
We numerically test the mode-coupling model (Lubow 1991a) of tidal instability in SU UMa systems. So far, all numerical models confirming it have been based on SPH codes and isothermal equation of state. In our paper we present Eulerians models, using both isothermal approximation and the full energy equation. We also investigate influence of different ways of mass transfer. While isothermal models behave similarly to SPH simulations, the behaviour of models with full energy equation is quite different, and the mode-coupling model is not confirmed in this case.

1 Introduction
SU UMa stars form a subclass of dwarf novae, which, in turn, are a subclass of cataclysmic variables (CVs). Like all CVs, the SU UMa stars are semidetached binary systems consisting of a white dwarf (the primary) and a low-mass, main-sequence star (the secondary). The secondary fills its Roche lobe, and a stream of gas flows from its surface toward the primary through the inner Lagrangian point. Because of excessive angular momentum, the stream is deflected from its original direction, and an accretion disk is formed around the primary. The disk may be subject to a thermal instability (Smak 1999), resulting in episodes of enhanced accretion rate. To a distant observer, such an episode is visible as a temporary brightening of the star, commonly referred to as an outburst.

As opposed to ordinary dwarf novae, the SU UMa stars exhibit a clearly bimodal distribution of outbursts. Normal outbursts have an amplitude of $\sim 3$ mag, and last from one to four days. Superoutbursts are by $\sim 1$ mag stronger, and last for up to several weeks. The recurrence time of normal outbursts (days to weeks) is not constant, and it varies substantially from one system to another. The superoutbursts repeat more regularly, and their recurrence time is much longer (months to years). In the extreme case of WZ Sge stars, superoutbursts are nearly exclusively observed, with a recurrence time of up to several tens of years. In superoutbursts, the light curve of a SU UMa system is modulated with
a period a few per cent longer than the orbital period. Those modulations are referred to as superhumps. The superhump signal is known to originate from extended source(s) in the outer disk (Warner 1995).

Superoutbursts are thought to be driven by a combination of thermal and tidal instability. During normal outbursts, the disk grows in size as it diffuses under the influence of increased viscosity. In systems with mass ratios $q \lesssim 0.25$, it eventually reaches up to the location of the 3:1 resonance, at which the orbital frequency of the disk gas is three times larger than the orbital frequency of the binary (we define $q$ as the ratio $M_2/M_1$, where $M_1$ and $M_2$ are primary’s and secondary’s mass, respectively). Subsequently, the tidal instability sets in, the disk becomes eccentric, and, seen in the inertial frame, it performs a slow, prograde precession. The tidal influence of the secondary on (and the viscous dissipation in) the outer disk, is largest when the bulk of the disk passes the secondary. The superhump period is then the beat period between the precession period and the orbital period of the binary (Osaki 1996).

Disk precession and superhump phenomenon have been subject to rather intense theoretical investigations, largely based on numerical simulations. According to Lubow (1991a,b), the eccentricity builds up due to nonlinear interaction of waves, in which the $m=3$ component of the tidal field is a key factor. Heemskerk (1994) performed simulations using only that component, and he found that the disk became eccentric, but it precessed retrogradely. Moreover, with the full tidal potential, the accretion disk was kept away from the location of the resonance, and no significant eccentricity was produced. Heemskerk’s results are the only ones obtained with an Eulerian (fluid) code. All remaining models presented in the literature have been based on Lagrangian (particle) codes. A detailed review of those calculations can be found in Murray (1998). While a qualitative agreement with observations of superhump systems was reached in several aspects, the models did not entirely agree with the analytical theory of the tidal instability. In particular, the measured eccentricity growth rates were much smaller than the predicted ones. The models themselves were often based on an extremely simple physical scenario (fully isothermal disk; gas from the secondary uniformly "raining" onto a circular orbit within the disk). In some of them the tidal instability was initiated by an arbitrary increase of viscosity in the disk by a factor of 10. Finally, even in the models which properly followed the stream of gas between the inner Lagrangian point up to its collision with the edge of the disk, the resolution in the collision region was too poor to resolve strong shock waves responsible for the hot spot phenomenon.

In the present paper, we obtain Eulerian models of disks in SU Uma systems in order to isolate the influence of various approximations on the outcome of the simulations. The models can be directly compared to Lagrangian models of Murray (1996, 1998). We perform both isothermal simulations, and simulations in which the full energy equation with a realistic cooling term is solved. We also compare "rainfall-type" mass transfer models with those based on realistic modelling of the stream from the secondary. The physical assumptions on which our models are based are described in Sect. 2 together with numerical methods employed to solve the equations of hydrodynamics. The models are presented in
Sect. 3, and the results are discussed in Sect. 5.

2 Physical assumptions and numerical methods

We simulate the flow of gas in the orbital plane of a binary consisting of two stars in circular orbits around the center of mass. We use spherical coordinates and a corotating reference frame centered on the primary, with the $z$-axis perpendicular to the orbital plane. Assuming that the ratio $H/r$ of the disk is constant, and the latitudinal velocity component is negligible, we can write the continuity equation and the equations describing conservation of radial and angular momentum in the following form:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]

\[
\frac{\partial \rho v_r}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}_r) = \frac{\rho v_\phi^2}{r} \frac{\partial p}{\partial r} + \rho f_r + F_{r}^{\text{visc}}
\]

\[
\frac{\partial j}{\partial t} + \nabla \cdot (j \vec{v}) = -\frac{\partial p}{\partial \phi} + \rho f_\phi + r F_{\phi}^{\text{visc}}
\]

where for any variable $a$

\[
\nabla \cdot (a \vec{v}) = \frac{1}{r^2} \frac{\partial r^2 a v_r}{\partial r} + \frac{1}{r} \frac{\partial a v_\phi}{\partial \phi}
\]

In these equations $j = \rho v_\phi$ is the angular momentum density measured in corotating frame, and $\vec{f}$ is the external force (gravitational and inertial) acting on unit volume. Viscous forces $F_{r}^{\text{visc}}$ are given by standard formulae (see eg. Landau and Lifshitz 1982). We assumed that kinematic viscosity coefficient and bulk to shear coefficients ratio are constant throughout the disk.

The models are either isothermal or radiative. In the first case we use an isothermal equation of state

\[
p = c_s^2 \rho,
\]

where $c_s$ is the isothermal sound speed, assumed to be constant in space and time. In the second case the equation of state of an ideal gas with the ratio of specific heats $\gamma$ equal to $5/3$ is used, and the energy equation

\[
\frac{\partial E}{\partial t} + \nabla \cdot (E \vec{v}) = -p \nabla \cdot \vec{v} + Q_{\text{visc}}^{\text{visc}} - Q_{\text{rad}}^{\text{rad}}
\]

(1)

is additionally solved, with $E$ standing for the internal energy density, $Q_{\text{visc}}^{\text{visc}}$ - for heat generated by viscous forces and $Q_{\text{rad}}^{\text{rad}}$ - for heat radiated away. Cooling processes are described in the same way as in Różycka and Spruit (1993).

The equations are solved with the help of an explicit Eulerian code described by Różycka (1985) and Różycka and Spruit (1993). The inner edge of the grid is located at $r_{in} = 0.1d$ from the primary (where $d$ is the distance between the components of the system), and the outer one ($R_{out}$) - at the inner Lagrangian
point $L_1$. For all models the same mass ratio $q = 3/17$ is assumed, resulting in $r_{L_1} = 0.736d$. The spacing of the grid is logarithmic in $r$ and uniform in the azimuthal coordinate $\phi$, and there are 50 and 60 grid points in $r$ and $\phi$, respectively.

The gas can freely flow into the computational domain or out of it through the outer boundary of the grid. At the inner grid boundary only free outflow is allowed for, and no inflow can occur. Due to the explicit character of the numerical scheme, the length of the time step is limited by the Courant condition. In the present simulations the Courant factor is 0.3.

3 Initial setup and results of simulations

In the present paper we repeat and refine two simulations performed by Murray (1996), also referred to as models 4 and 5 in a later discussion by Murray (1998). As mentioned in the Introduction, Murray uses a Lagrangian SPH code, whereas we work with an Eulerian, grid based one. Additionally, we employ a more realistic physical scenario, taking into account the effects of cooling, and using an ideal gas equation of state instead of an isothermal one. Following Murray we adopt $q = 3/17$, a kinematic viscosity coefficient of $2.5 \times 10^{-4} d^2 \Omega^* $ (where $\Omega^*$ is the orbital frequency of the binary), and a sound velocity of $0.02 d \Omega_*^{-1}$ for the isothermal cases.

As an initial condition we chose isothermal, Keplerian disk with density varying as $r^{-1.5}$. The disk extends from the inner boundary of the grid up to $r = 0.35d$. The rest of the grid is originally filled with a rarefied ambient medium at Keplerian rotation around the primary. The density of the ambient medium is 1% of the disk density at $r = 0.35$, and its pressure matches that of the disk.

The original two models of Murray differ only in details of mass transfer. In the first case the gas from the secondary enters the computational domain as a narrow stream originating at $L_1$; in the second case the gas "rains" uniformly onto the orbit corresponding to the circularization radius. In the following, they are referred to as stream and rain models, respectively (note that, by the definition of the circularization radius, if the mass transfer rate is the same in both cases then the rate of angular momentum transfer is also the same). Below we describe four models: radiative stream, radiative rain, isothermal stream, and isothermal rain. Hereafter, they are referred to as Models (or Cases) 1 - 4, correspondingly.

All models were followed for 100 orbits of the binary. In order to check if the balance between the matter added to the disk and accreted onto the white dwarf is established in the course of evolution, we monitored the disk mass as a function of time (Fig.1). For models 1, 2 and 3 the mass of the disk grows rapidly in the beginning, and then it reaches an equilibrium value. That value is almost two times greater in Case 4 (isothermal rain) than in Case 2 (radiative rain). In Case 3, after the phase of initial growth, the mass of the disk begins to oscillate with a period of $\sim 45P$ (where $P$ is the orbital period of the binary).
The next step was to check, whether the disks in our models became eccentric. For natural numbers \( k, l \) we can calculate

\[
S_{k,l} = \left| \int_{r_{in}}^{r_{out}} \rho_{k,l} r^2 dr \right|
\]

where \( \rho_{k,l} \) are defined by equation

\[
\rho = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \rho_{k,l} \exp[i(k\theta - l\Omega_\star t)]
\]

and the term \( r^2 \) results from the assumption of constant ratio \( H/r \) (in practice, the lower integration limit can be equal to \( r_{in} \), whereas the upper one has to be placed at \( r = 0.6 \) in order to avoid highly asymmetric contribution from the stream). If the mass of the disk is constant, then an appropriate measure of the eccentricity is \( S_{1,0} \). If, however, the mass is varying in the course of simulation, \( S'_{1,0} \) should be used instead, where \( S'_{1,0} \equiv S_{1,0}/S_{0,0} \) (note that \( S_{0,0} \) is proportional to the mass of gas in the integration domain). Fig. 2 shows \( S'_{1,0} \) as a function of time. It can be immediately seen that \( S'_{1,0} \) is much greater in isothermal models than in radiative ones. In other words, isothermal disks more willingly become elliptic. Secondly, only Model 4 (i.e. isothermal rain) exhibits a phase of exponential growth of \( S'_{1,0} \) which was predicted by Lubow's
theory. Even in that case, however, another prediction of the theory is not met, namely, the behavior of the (2,3) mode. The strength of that mode should be proportional to the time-derivative of the strength of (1,0) mode, whereas in our simulations it is roughly proportional to the strength of (1,0) mode (see Fig. 3).

Following Murray (1996, 1998), we calculated the rate of viscous dissipation in the disk as a function of time. The results of Fourier analysis of that function for Models 3 and 4 in range of $60 - 100P$ are shown in Fig. 4 and 5. In Case 4 (isothermal rain) the rate of viscous dissipation reaches a stationary value, whereas in Case 3 (isothermal stream) it exhibits large amplitude oscillations (by a factor of 2) with a period of $\sim 45P$. Note that the maxima of the dissipation rate coincide with the maxima of the disk mass. In both models much more rapid oscillations of the dissipation rate are also seen, whose dominant frequency is a little lower than $1P^{-1}$ (all remaining peaks in power spectra can be interpreted as its higher harmonics). The corresponding periods are equal to 1.11P in Model 3., and 1.08P in Model 4. In Model 3 the high-frequency oscillations are visible only within dissipation rate maxima (on the ascending branch shortly before peak, and on the entire descending branch). In Model 4 they are excited when the mass of the disk approaches its maximum, and they persist till the end of simulation. A detailed examination of high-frequency oscillation reveals their similarity to those found by Murray (1998) in his Models 4 and 5 (see his Fig. 6). Because they are visible both in model with and without
stream, those oscillations cannot be excited by the interaction of the stream with the advancing and retreating edge of the disc, and they most probably must be related to the tidal influence of the secondary. We found no similar oscillations in our radiative models.

In Fig. 6 and 7 we present sequences of disk density maps for Models 1 and 3 (radiative stream and isothermal stream), covering the time-span from $99P$ to $100P$ (at that time the rapid oscillations of viscous dissipation rate are fully developed in Model 3). In the radiative case the disk is almost circular. However, in the isothermal case the disk has a clearly nonaxisymmetric shape, which, at least in some frames, does not significantly differ from the elliptical one. The ellipse is precessing with a period a little longer than $1P$. In one precession period the maximum viscous dissipation occurs when the major axis of the ellipse is perpendicular to the line connecting the components of the binary ($t = 99.65P$ in Fig. 7) (the same results were obtained by Murray). In both cases a nonaxisymmetric feature in the form of two spiral arms is also seen, which in Case 1 remains stationary in the corotating frame. We interpret it as the ($2, 2$) mode, excited and maintained by the tidal interaction of the secondary.

Fig. 8 shows the density map of Model 3 at $T = 80P$ (i.e. between the large-scale maxima of the dissipation rate, when no short-period oscillations are visible). At that moment the disk is nearly axisymmetric, and only the ($2, 2$) mode has a significant amplitude. Thus, we conclude that the overall behaviour
Figure 4: Viscous dissipation rate (program units) in Model 3 as a function of time (top), and its power spectrum in time range of $60 - 100P$ (bottom). Dissipation rate in this time range.

Figure 5: Viscous dissipation rate (program units) in Model 4 as a function of time (top), and its power spectrum in time range of $60 - 100P$ (bottom).
Figure 6: Density maps for Model 1. The snapshots are taken at $t = 99.9P$, $99.2P$, $99.4P$, $99.6P$, $99.8P$ and $100P$. The white dwarf and the $L_1$ point are at the center, and in the middle of the lower edge of each frame.
Figure 7: Density maps for Model 3. The snapshots are taken at $t = 99.1P$, $99.2P$, $99.4P$, $99.6P$, $99.8P$ and $100P$. The white dwarf and the $L_1$ point are at the center, and in the middle of the lower edge of each frame.
Figure 8: Density map for model 3 at the time $t = 80P$. The white dwarf and the $L_1$ point are at the center, and in the middle of the lower edge of the frame.
of Model 3 is in agreement with the tidal instability model of superhumps.

4 Discussion

In the present paper we reported a series of 4 simulations of disk in Cataclysmic Variables of SU UMa type. The least realistic model was obtained with an isothermal equation of state, and a "rainfall-type" approximation for mass transfer from the secondary. The most realistic simulation involved a full energy equation, and a stream of matter originating at the inner Lagrangian point.

We found that only the isothermal disks exhibit a clear tendency toward elliptic distortions accompanied by precession. Within the framework of Lubow's (1992) theory, one could try to explain this result by assuming, that the strength of the tidally excited (2,2) mode is significantly larger in radiative models than in the isothermal ones. This is because, according to the theory, the (2,2) oscillations tend to keep the disk gas away from the 3:1 resonance responsible for the growth of the tidal instability. Such an assumption, however, is not confirmed by the analysis of our results: the (2,2) mode appears to be equally strong in all models.

The phase of an exponential growth of the (1,0) mode, foreseen by the theory, was found only in the least realistic model. However, contrary to theoretical predictions, even in that case the time derivative of the strength of (1,0) mode was not proportional to the strength of the (2,3) mode.

Oscillations with a period slightly longer than $P$, which may be tentatively associated with superhumps, were observed in isothermal models only. The period of oscillations found in radiative models was about 3 times shorter.

The main results of this work may be summarized in three points:

1. As foreseen by the tidal instability theory, isothermal develop an appreciable eccentricity, and begin to precess. The precession period is the same as the period of rapid fluctuations in viscous dissipation rate, and it is slightly longer than the orbital period of the binary. The behaviour of the (1,0) mode, however, is not consistent with the theory.

2. In radiative models an elliptic, precessing disk does not develop. The dominant oscillations in the viscous dissipation curve have periods of $\sim 1/3P$.

3. In all models the two-armed (2,2) mode, excited and maintained by tidal forces of the secondary, is very clearly seen.

Our conclusion is that the mechanism of superhump phenomenon is not yet entirely understood, and further research on this subject is desirable. Różyczka & Spruit (1993) simulated a viscosity-free disk with radiative losses, in which angular momentum was transported by spiral shocks. They found an eruptive instability, qualitatively similar to outbursts of Dwarf Novae. During the eruption, the radius of the disk increased substantially. We suggest that low-viscosity disks prone to this type of instability may develop eccentricity and exhibit superhump-like oscillations during outbursts. This issue is presently under investigation.
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