KOROVKIN-TYPE RESULTS ON CONVERGENCE OF SEQUENCES OF 
POSITIVE LINEAR MAPS ON FUNCTION SPACES 

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Abstract. In this paper we deal with the convergence of sequences of positive linear maps to a 
(not assumed to be linear) isometry on spaces of continuous functions. We obtain generalizations 
of known Korovkin-type results and provide several illustrative examples. 

1. Introduction

One of the most impressive results in approximation theory is, without doubt, Korovkin’s theorem 
on convergence of positive linear operators on a space of continuous functions. More explicitly, 
Korovkin’s theorem (often called Korovkin’s first theorem) states that if a sequence \{T_n\} of positive 
linear maps on \(C_R[0,1]\) converges to the identity operator on the quadratic polynomials, then \(T_nf\) 
converges to \(f\) for all \(f \in C_R[0,1]\) \([8]\). This result arose from a generalization of the well-known 
proof of Weierstrass’s approximation theorem given by S. Bernstein. Its strength and simplicity 
have produced, as it is clearly imaginable, a wide range of applications and generalizations. One of 
them deals with substituting the identity operator by other operators and the closed interval \([0,1]\) 
by other spaces. Others center on finding subsets of function spaces, known as Korovkin sets or test 
functions, which guarantee that the convergence of a sequence of positive linear maps holds on the 
whole space provided it holds on them. For more details and other aspects of this topic, we refer to 
the monographs \([2, 6]\), the recent survey paper by Altomare \([1]\), and the references therein.

Let \(X\) and \(Y\) be compact Hausdorff spaces, \(M\) be a unital subspace of \(C(X)\), and \(S\) be a function 
space included in \(M\). In \([7]\), the authors studied the convergence of a sequence of unital linear 
contractions towards a fixed linear isometry. Indeed, they proved that, under certain assumptions, 
if each \(T_n : M \rightarrow C(Y)\) (\(n \in \mathbb{N}\)) is a unital linear contraction and \(T_\infty : M \rightarrow C(Y)\) is a 
linear isometry such that \(\{T_nf\}\) converges to \(T_\infty f\) for all \(f \in S\), then \(\{T_nf\}\) converges to \(T_\infty f\) 
for all \(f \in M\), not only pointwise but also uniformly. In this paper we deal with the convergence 
of sequences of (not necessarily contractions) positive linear maps to a (not assumed to be linear) 
isometry on spaces of continuous functions by combining ideas given in \([7]\) and in the original proof

2010 Mathematics Subject Classification. Primary 41A36; Secondary 46E15.

Key words and phrases: Function space, Korovkin’s theorem, Choquet boundary, positive linear map.

J.J. Font is supported by Spanish Government grant MTM2016-77143-P (AEI/FEDER, UE) and Generalitat 
Valenciana (Projecte GV/2018/110).
of Korovkin’s theorem. In particular, we obtain proper generalizations of [5, Theorems 3.1 and 4.1] and of several classical Korovkin-type results, and provide several illustrative examples.

2. Preliminaries

For any compact Hausdorff space $X$, let $C(X)$ denote the space of continuous real or complex-valued functions on $X$, equipped with the uniform norm $\| \cdot \|$. Note that we write $C_\mathbb{R}(X)$ instead of $C(X)$ when we want to consider only real-valued case. A unital subspace $S$ of $C(X)$ is called a function space on $X$ if $S$ separates the points of $X$ in the sense that for each $x, x' \in X$ with $x \neq x'$ there exists a function $f \in S$ such that $f(x) \neq f(x')$.

Let $S$ be a subspace of $C(X)$, which we always assume to be linear. We denote by $B_S^*$ the closed unit ball of the dual space of $(S, \| \cdot \|)$. A nonempty subset $E$ of $X$ is called a boundary for $S$ if each function in $S$ attains its maximum modulus within $E$. The Choquet boundary $Ch(S)$ of $S$ is the non-empty set of all points $x \in X$ for which $\delta_x$, the evaluation functional at $x$, is an extreme point of the closed unit ball $B_{S^*}$. Namely, we have $\text{ext}(B_{S^*}) = T Ch(S) = \{ \alpha x : \alpha \in T \text{ and } x \in Ch(S) \}$, where $T = \{ z \in \mathbb{C} : |z| = 1 \}$. It is known that $Ch(S)$ is a boundary for $S$. In particular, one can obtain the following remark immediately:

**Remark 2.1.** If for each $x \in X$ there is a function $h \in S$ such that $h(x) = 1$ and $|h(y)| < 1$ for any $y \neq x$, then $Ch(S) = X$. For example, as in Korovkin’s original theorem, if we assume $X = [0, 1]$ and $S = \text{Span}\{1, x, x^2\}$, then $h(x) := 1 - (x - a)^2$, $a \in [0, 1]$, yields $Ch(S) = [0, 1]$.

In the sequel, unless otherwise stated, it is assumed that $X$ and $Y$ are compact Hausdorff spaces, $M$ is a self-conjugate subspace of $C(X)$ in the sense that $\tilde{f} \in M$ whenever $f \in M$, and $S$ is a function space included in $M$.

A linear map $T : M \rightarrow C(Y)$ is called positive if $T f \geq 0$ holds for all $f \geq 0$.

Let $f, f_1, f_2, \ldots \in C(X)$ and $X_0 \subseteq X$. If $\{f_n\}$ converges pointwise to $f$ on $X_0$, we write $f_n \rightharpoonup f$ on $X_0$. Also, we omit $X_0$ when $X_0 = X$.

Given $f, g \in C(X)$, we shall write $f \otimes 1 + 1 \otimes g$ to denote the function in $C(X \times X)$ such that $(f \otimes 1 + 1 \otimes g)(x, x') := f(x) + g(x')$. Furthermore, if $T, T' : S \subseteq C(X) \rightarrow C(Y)$, then we set $(T \otimes T') (f \otimes 1 + 1 \otimes g)(y) := Tf(y) + T1(y)T'g(y)$ for all $f, g \in S$ and $y \in Y$.

Finally let us state the following lemma which is used in the proofs of our results.

**Lemma 2.2.** [5, Theorem 2.2.6] Let $S$ be a function space on $X$ and $x_0 \in X$. Then $x_0 \in Ch(S)$ if and only if for any $\alpha, \beta \in (0, \infty)$ with $\alpha < \beta$ and any open neighborhood $U$ of $x_0$, there is a function $f \in S$ such that $\text{Ref} \leq 0$ on $X$, $\text{Ref} < -\beta$ on $U^c$ and $\text{Ref}(x_0) > -\alpha$. 

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3. RESULTS

**Theorem 3.1.** Suppose that \{T_n\} is a sequence of positive linear maps from \(M\) into \(C(Y)\), and \(T_\infty\) is an isometry from \(M\) onto a subspace \(T_\infty(M)\) of \(C(Y)\).

(a) If \(T_nf \to T_\infty f\) for all \(f \in S\), then \(T_nf \to T_\infty f\) on \(Ch(T_\infty(S))\) for all \(f \in M\).

(b) Let \(N := \text{Span} \bigcup_{1 \leq n \leq \infty} T_n(M)\). If, in part (a), \(Ch(N) \subseteq Ch(T_\infty(S))\) and the set \({T_n1 : n \in \mathbb{N}\}\) is bounded, then \(T_nf \to T_\infty f\) for all \(f \in M\).

**Proof.** We will base the proof of (a) through the following steps.

**Step 1.** For each triple of distinct points \(x, x', z \in Ch(M)\), there exists a function \(h \in M\) such that \(|h(x)| \neq |h(x')|\) and \(h(z) = 0\).

Since \(M\) is a self-conjugate function space we can find a real-valued function \(f \in M\) such that \(f(x) = 1\) and \(f(x') = 0\). Now we consider the following cases based on the value of \(f\) at \(z\):

- \(f(z) = 1\). Clearly, \(h = 1 - f\) is the desired function.
- \(f(z) \neq 1, \frac{1}{2}\). Take \(h = f - f(z)\).
- \(f(z) = \frac{1}{2}\). In this case we choose a non-negative function \(g \in M\) with \(g(x), g(x') > 3\) and \(g(z) < \frac{1}{2}\), by Lemma 2.2. If \(g(x') - g(x) = 2\), then \(h = g - g(z)\) is the desired function. Otherwise, we can see that \(h = 2f + g - g(z) - 1\) satisfy the requested properties.

**Step 2.** \(T_\infty\) is a linear isometry.

Note that \(T_\infty 0 = \lim T_n 0 = 0\). Then according to the Mazur-Ulam theorem [10], \(T_\infty\) is a real-linear isometry. Hence now we only need to consider the complex case. Let us point out that \(T_\infty 1 = \lim T_n 1 \geq 0\). Taking into account Step 1, from [9] Theorem 2.3 it follows that \(T_\infty 1 = 1\) and there exist a (possibly empty) clopen subset \(K\) of \(Ch(T_\infty(M))\), and a continuous surjective map \(\phi : Ch(T_\infty(M)) \to Ch(M)\) such that for all \(f \in M\),

\[
T_\infty f = \begin{cases} 
  f \circ \phi & \text{on } K, \\
  \overline{f \circ \phi} & \text{on } Ch(T_\infty(M)) \setminus K.
\end{cases}
\]

But \(T_\infty i = \lim T_n i = i \lim T_n 1 = iT_\infty 1 = i\), which implies that \(K = Ch(T_\infty(M))\). Hence taking into account that \(Ch(T_\infty(M))\) is a boundary for \(T_\infty(M)\), we deduce that \(T_\infty\) is a linear isometry.

**Step 3.** For each \(f \in M\), \(T_nf \to T_\infty f\) on \(Ch(T_\infty(S))\).

By [7] Lemma 2.5] (or [3] Corollary 3.2], there is a continuous surjection \(\varphi : Ch(T_\infty(S)) \to Ch(S)\) such that

\[
T_\infty f(y) = f(\varphi(y)) \quad (f \in S, y \in Ch(T_\infty(S))).
\]
Let $f \in M$ and $\epsilon > 0$. Then we can define a function in $C(X \times X)$ as $F := f \otimes 1 - 1 \otimes f$. Clearly, $F = 0$ on the subset $\Delta_X = \{(x, x) : x \in X\}$ of $X \times X$. Then there is an open neighborhood $U$ of $\Delta_X$ with $|F| < \epsilon$ on $U$.

Let $y' \in Ch(T_\infty(S))$ and $x' = \varphi(y')$. Choose an open neighborhood $V_{x'}$ of $x'$ such that $V_{x'} \times V_{x'} \subseteq U$. By Lemma 2.2, we find a function $f_{y'} \in S$ such that

$$Ref_{y'} \geq 0 \text{ on } X, \quad Ref_{y'} \geq 1 \text{ on } V_{x'}, \quad Ref_{y'}(x') < \epsilon.$$

Put $F_{y'} = f_{y'} \otimes 1 + 1 \otimes f_{y'}$. It is clear that $ReF_{y'} \geq 0$ on $X \times X$ and $ReF_{y'} \geq 1$ on $U^c$. Hence we have

$$ReF \leq \|F\| \leq \|F\|ReF_{y'} \text{ on } U^c,$$

which yields $|ReF| \leq 1 \otimes \epsilon + \|F\|ReF_{y'}$ on $X \times X$. In other words,

$$-(1 \otimes \epsilon + \|F\|ReF_{y'}) \leq ReF \leq 1 \otimes \epsilon + \|F\|ReF_{y'} \text{ on } X \times X.$$

Hence for each $y \in X$ we get

$$-\epsilon - 2\|F\|ReF_{y'} - \|F\|ReF_{y'}(y) + Ref(y) \leq ReF - \|F\|Ref_{y'} \leq \epsilon + \|F\|Ref_{y'}(y) + Ref(y).$$

Since $\{T_n\}$ is a sequence of linear positive maps, it follows that

$$-2\|F\|T_n(ReF_{y'}) + (-\epsilon - \|F\|Ref_{y'}(y) + Ref(y))T_n1 \leq T_n(Ref) - \|F\|T_n(Ref_{y'}) \leq$$

$$T_n1(\epsilon + \|F\|Ref_{y'}(y) + Ref(y))$$

for each $y \in X$. Now, from the representation of $T_\infty$ on $M$ (Step 2), we deduce that

$$-2\|F\|T_n(ReF_{y'})(z) + T_\infty(-\epsilon - \|F\|Ref_{y'} + Ref)(z)T_n1(z) \leq T_n(Ref)(z) - \|F\|T_n(Ref_{y'})(z) \leq$$

$$T_n1(z)T_\infty(\epsilon + \|F\|Ref_{y'} + Ref)(z')$$

for any $z \in Y$ and $z' \in Ch(T_\infty(M))$. Thus, again since $T_\infty1 = 1$, $T_\infty$ is a positive linear map and also $Ch(T_\infty(M))$ is a boundary for $T_\infty(M)$, it is observed that the above relation holds for all $z, z' \in Y$. Therefore, especially we get

$$-\|F\|T_n(ReF_{y'}) - T_n1T_\infty(\epsilon + \|F\|Ref_{y'}) \leq T_n(Ref) - T_n1T_\infty(Ref) \leq T_n1T_\infty(\epsilon + \|F\|Ref_{y'}) + \|F\|T_n(Ref_{y'})$$

on $Y$. Rewriting the above inequality adopted to our notation in Section 2 we have

$$-(T_n \otimes T_n1T_\infty)(1 \otimes \epsilon + \|F\|ReF_{y'}) \leq (T_n \otimes T_n1T_\infty)(Ref) \leq (T_n \otimes T_n1T_\infty)(1 \otimes \epsilon + \|F\|ReF_{y'}),$$

equivalently,

$$|(T_n \otimes T_n1T_\infty)(Ref)| \leq (T_n \otimes T_n1T_\infty)(1 \otimes \epsilon + \|F\|ReF_{y'}).$$
Consequently, from the fact that each $T_n$ is a positive linear map and the representation of $T_\infty$, it follows that

$$|\text{Re}(T_n \otimes T_n 1T_\infty)(F)| = |\text{Re}T_n f - \text{Re}(T_n 1T_\infty f)|$$

$$= |T_n(\text{Re}f) - T_n 1T_\infty(\text{Re}f)|$$

$$= |(T_n \otimes T_n 1T_\infty)(\text{Re}F)|$$

$$\leq (T_n \otimes T_n 1T_\infty)(1 \otimes \epsilon + \|F\| \text{Re}F')$$

$$= (T_n \otimes T_n 1T_\infty)(1 \otimes \epsilon) + (T_n \otimes T_n 1T_\infty)(\|F\| \text{Re}F')$$

$$= \epsilon T_n 1 + \|F\|(T_n(\text{Re}F') + T_n 1T_\infty(\text{Re}F'))$$

$$= \epsilon T_n 1 + \|F\|(\text{Re}T_n f' + T_n 1\text{Re}T_\infty f')$$

$$\leq \epsilon T_n 1 + \|F\|(\|T_n f' - T_\infty f'\| + T_n 1\text{Re}T_\infty f' + \text{Re}T_\infty f')$$

which is to say,

$$|\text{Re}(T_n \otimes T_n 1T_\infty)(F)| \leq \epsilon T_n 1 + \|F\|(\|T_n f' - T_\infty f'\| + T_n 1\text{Re}T_\infty f' + \text{Re}T_\infty f').$$

Thus, from the latter inequality, the representation of $T_\infty$ and for any sufficiently large integer $n$, we get

$$|\text{Re}T_n f(y') - \text{Re}T_\infty f(y')| \leq |\text{Re}T_n f(y') - T_n 1(y')\text{Re}T_\infty f(y')| + |T_n 1(y')\text{Re}T_\infty f(y') - \text{Re}T_\infty f(y')|$$

$$\leq \epsilon T_n 1(y') + \|F\|(\|T_n f' - T_\infty f'\| + T_n 1(y')\text{Re}f'(x') + \text{Re}f'(x')) + \|T_n 1(y') - 1\|$$

$$\leq 2\epsilon + \|F\|\epsilon + \|f\|\epsilon$$

$$= (2 + 4\|F\| + \|f\|)\epsilon.$$

Hence $\text{Re}T_n f \longrightarrow \text{Re}T_\infty f$ on $Ch(T_\infty(S))$. By replacing $f$ by $-if$, we see that $\text{Im}T_n f \longrightarrow \text{Im}T_\infty f$ on $Ch(T_\infty(S))$. Therefore, $T_n f \longrightarrow T_\infty f$ on $Ch(T_\infty(S))$, which completes the proof of part (a).

(b) We first claim that $\|T_n\| \leq \sqrt{2}\|T_n 1\|$, where $\|T_n\|$ is the operator norm of $T_n$ (for each $n \in \mathbb{N}$). To see this, assume that $g \in M$ is real-valued and has supremum norm at most 1. Then $-1 \leq g \leq 1$ and thus, $-T_n 1 \leq T_n g \leq T_n 1$, which implies that $\|T_n g\| \leq \|T_n 1\|$. In the real case, this shows that $T_n$ is continuous and the claim holds. In the complex case, from this argument and the fact that $M$ is self-conjugate, it easily follows that $\|T_n\| \leq \sqrt{2}\|T_n 1\|$. Let $f \in M$. Taking into account the above claim and the boundedness of $\{T_n 1 : n \in \mathbb{N}\}$, we deduce that the set $\{T_n f : n \in \mathbb{N}\}$ is bounded. Now one can follow the last part of the proof of [7].
Theorem 3.3] to conclude that $T_n f \to T_{\infty} f$ on $Y$ and we include it for completeness. Assume that $\sim$ is the equivalence relation on $Y$ defined by

$$y \sim y' \iff g(y) = g(y') \quad \forall g \in N.$$  

The quotient space of $Y$ by $\sim$ is denoted by $Y/\sim$, and $\hat{g}$ will stand for the image of $y \in Y$ under the canonical map $\hat{\cdot}$ from $Y$ onto $Y/\sim$. Moreover, we define $\hat{g}(\hat{y}) = g(y)$ for all $g$ in $N$ and $y$ in $\hat{Y} = \{ \hat{y} : y \in Y \}$. It is apparent that $\hat{N} = \{ \hat{g} : g \in N \}$ is a function space on the compact space $\hat{Y}$.

By [4, Section V] and [12, Section 4], for any $y \in Y$, there exists a positive measure $\mu$ on the $\sigma$-ring of subsets of $B_N^\ast$ generated by $ext(B_{N^\ast})$ and the Baire subsets of $B_{N^\ast}$ which represents $\hat{y}$ and $\mu(B_{N^\ast}) = 1$. From part (a), it is clear that $\hat{T_n f} \to \hat{T_{\infty} f}$ on $Ch(T_{\infty}(S))$. Hence, since $ext(B_{N^\ast}) = \mathcal{T} Ch(N) \subseteq \mathcal{T} Ch(T_{\infty}(S))$ and the set $\{ T_n f : n \in \mathbb{N} \}$ is bounded, from the Lebesgue’s dominated convergence theorem we get

$$T_n f(y) = \hat{T_n f}(\hat{y}) = \int_{B_N^\ast} \hat{T_n f} d\mu = \hat{T_{\infty} f}(\hat{y}) = T_{\infty} f(y).$$

Therefore, $T_n f \to T_{\infty} f$, as desired. \hfill \Box

Let us recall here the famous Arzela-Ascoli theorem, which will be used in the proof of the next result.

**Theorem (Arzela-Ascoli).** Given a subset $A$ of $C(X)$, the following statements are equivalent:

1. $A$ is a compact subset of $(C(X), || \cdot ||)$.
2. $A$ is closed, bounded, and equicontinuous in the sense that for each $x \in X$ and $\epsilon > 0$, there exists a neighborhood $V$ of $x$ such that $|f(y) - f(x)| < \epsilon$ for all $f \in A$ and $y \in V$.

**Theorem 3.2.** Let $\{ T_n \}$ be a sequence of positive linear maps from $M$ into $C(Y)$, and $T_{\infty}$ be an isometry from $M$ onto a subspace $T_{\infty}(M)$ of $C(Y)$.

(a) If $\{ T_n f \}$ converges uniformly to $T_{\infty} f$ for all $f \in S$, then $\{ T_n f \}$ converges uniformly to $T_{\infty} f$ on each compact subset of $Ch(T_{\infty}(S))$ for all $f \in M$.

(b) If, furthermore, either $Ch(T_{\infty}(S))$ or $Ch(N)$ is compact and $Ch(N) \subseteq Ch(T_{\infty}(S))$, then $\{ T_n f \}$ converges uniformly to $T_{\infty} f$ for any $f \in M$, where $N$ is as in Theorem 3.1.

**Proof.** (a) As in the proof of Theorem 3.1, there is a continuous surjection $\varphi : Ch(T_{\infty}(S)) \to Ch(S)$ such that for all $f \in S$,

$$T_{\infty} f(y) = f(\varphi(y)) \ (f \in S, y \in Ch(T_{\infty}(S))).$$

Suppose that $K$ is a compact subset of $Ch(T_{\infty}(S))$. Let $f \in M$, $y' \in K$ and $\epsilon > 0$. Put $F = f \otimes 1 - 1 \otimes f$ and $x' = \varphi(y')$. As before, we choose an open neighborhood $V_{x'}$ of $x'$ and a function
\[ f_{y'} \in S \text{ such that } \text{Re} f_{y'} \geq 0 \text{ on } X, \text{Re} f_{y'} \geq 1 \text{ on } V_{x'}^\epsilon \text{ and } \text{Re} f_{y'}(x') < \epsilon, \text{ and we also have} \]
\[
|\text{Re}T_n f - \text{Re}T_{\infty} f| \leq \epsilon T_n 1 + \|F\|(|T_n f_{y'} - T_{\infty} f_{y'}| + \text{Re}T_{\infty} f_{y'} + T_n 1 \text{Re}T_{\infty} f_{y'}) + |\text{Re}T_{\infty} f|T_n 1 - 1|,
\]
on \(Y\). Now, we prove the following claim.

**Claim:** The set \(\{T_n f : n \in \mathbb{N}\}\) is equicontinuous at \(y'\).

Since \(\{T_n f_{y'}\}\) and \(\{T_n 1\}\) converge uniformly to \(T_{\infty} f_{y'}\) and 1, respectively, there is an integer \(n_0\) such that for each \(n \geq n_0\), \(\|T_n f_{y'} - T_{\infty} f_{y'}\| < \epsilon\) and \(\|T_n 1 - 1\| < \epsilon\). On the other hand, \(\text{Re}T_{\infty} f_{y'}(y') < \epsilon\) and so, from the continuity of \(\text{Re}T_{\infty} f_{y'}\) and \(T_{\infty} f\), we can choose a neighborhood \(W_{y'}\) of \(y'\) so that the inequalities \(\text{Re}T_{\infty} f_{y'} < \epsilon\) and \(|T_{\infty} f - T_{\infty} f(y')| < \epsilon\) hold on \(W_{y'}\). Hence, letting \(\eta = \sup_{i \in \mathbb{N}} \|T_i 1\|\), for each \(y \in W_{y'}\) and \(n \geq n_0\) we get
\[
|\text{Re}T_n f(y) - \text{Re}T_n f(y')| \leq |\text{Re}T_n f(y) - \text{Re}T_{\infty} f(y)| + |\text{Re}T_n f(y') - \text{Re}T_{\infty} f(y')| + \]
\[
\eta T_{\infty} f_{y'}(y) + \|f\||T_n 1(y) - 1| + \eta \epsilon + \|F\|(|T_n f_{y'}(y') - T_{\infty} f_{y'}(y')| + \]
\[
\text{Re}T_{\infty} f_{y'}(y) + \eta T_{\infty} f_{y'}(y') + \|f\||T_n 1(y') - 1| + |\text{Re}T_{\infty} f(y) - \text{Re}T_{\infty} f(y')| \]
\[
\leq \eta \epsilon + \|F\|(\epsilon + \epsilon + \eta \epsilon) + \|f\|\epsilon + \|f\|(\epsilon + \epsilon + \eta \epsilon) + \|f\|\epsilon + \epsilon \]
\[
= \epsilon (2\eta + 2\|F\| + 4\|F\| 2\eta \|F\|) + \epsilon.
\]

Now, from the continuity of \(T_1 f, ..., T_{n_0} f\), it follows that the set \(\{\text{Re}T_n f : n \in \mathbb{N}\}\) is equicontinuous at \(y'\). Similarly, the set \(\{\text{Im}T_n f : n \in \mathbb{N}\}\) is equicontinuous at \(y'\), and, as a consequence, \(\{T_n f : n \in \mathbb{N}\}\) is equicontinuous at \(y'\), as claimed.

Moreover, as observed in the proof of Theorem \[3.1(\text{b})\], \(\{T_n f : n \in \mathbb{N}\}\) is bounded. Therefore, from the Arzela-Ascoli theorem and Theorem \[3.1(\text{a})\], it follows that each subsequence \(\{T_n f\}\) has a uniformly convergent sequence to \(T_{\infty} f\) on \(K\). This argument shows that \(\{T_n f\}\) converges uniformly to \(T_{\infty} f\) on the compact set \(K\).

(b) When either \(Ch(T_{\infty}(S))\) or \(Ch(N)\) is compact, then, from the above discussion, we deduce that \(\{T_n f\}\) converges uniformly to \(T_{\infty} f\) on \(Ch(N)\). Next, since \(Ch(N)\) is a boundary for \(N\), it is immediately seen that \(\{T_n f\}\) converges uniformly to \(T_{\infty} f\) on \(Y\). \(\Box\)

**Remark 3.3.** We would like to remark that the sequential version of Korovkin’s theorem does not yield its net version (see [13]). However, it can be easily checked that our techniques hold true when we replace the sequence \(\{T_n\}\) by a net of positive linear maps.

In the following corollary, we obtain the main results of [7], namely, [7, Theorem 3.3] and [7, Theorem 4.1] as consequences of Theorems 3.4 and 3.2.
**Corollary 3.4.** Let $M$ be a subspace of $C(X)$, $S \subseteq M$ be a function space, $\{T_n\}$ be a sequence of unital linear contractions from $M$ into $C(Y)$, $T_\infty$ be a linear isometry from $M$ into $C(Y)$, and $Ch(N) \subseteq Ch(T_\infty(S))$, where $N := \text{Span} \bigcup_{1 \leq n \leq \infty} T_n(M)$.

(a) If $T_nf \to T_\infty f$ for all $f \in S$, then $T_nf \to T_\infty f$ for all $f \in M$.

(b) If $\{T_nf\}$ converges uniformly to $T_\infty f$ for all $f \in S$, then $\{T_nf\}$ converges uniformly to $T_\infty f$ on each compact subset of $Ch(T_\infty(S))$ for any $f \in M$. If, furthermore, $Ch(T_\infty(S))$ or $Ch(N)$ is compact, then $\{T_nf\}$ converges uniformly to $T_\infty f$ for all $f \in M$.

**Proof.** In the context of real-valued function spaces, since every linear map $T$ with $\|T\| = T(1) = 1$ is positive ([13]), the result follows immediately from Theorems 3.1 and 3.2. Now let us consider the complex case. We note that

$$M + \overline{M} = \{f + \overline{g} : f, g \in M\}$$

is a self-conjugate subspace of $C(X)$. According to [7, Lemma 2.5] (or [3, Corollary 3.2]), there is a continuous surjection $\varphi : Ch(T_\infty(M)) \to Ch(M)$ such that

$$T_\infty f(y) = f(\varphi(y)) \ (f \in M, y \in Ch(T_\infty(M))).$$

Since $Ch(T_\infty(M) + \overline{T_\infty(M)}) = Ch(T_\infty(M))$ and $Ch(M + \overline{M}) = Ch(M)$ ([7, Lemma 2.3]) are boundaries, $T_\infty$ can be extended to a linear isometry $\tilde{T}_\infty : M + \overline{M} \to C(Y)$ such that

$$\tilde{T}_\infty(f + \overline{g})(y) = f(\varphi(y)) + \overline{g(\varphi(y))} \ (f, g \in M, y \in Ch(T_\infty(M))).$$

Moreover, by [7, Lemma 3.2], each $T_n$ can be extended to a positive linear map $\tilde{T}_n$ from $\overline{M} + M$ into $C(Y)$. Now, we get the result from Theorems 3.1 and 3.2. □

4. Examples

In this section we provide several examples which show how our results can be applied.

**Example 4.1.** Let $k \in \mathbb{N} \cup \{0, \infty\}$ and $C^{(k)}(I)$ denote the space of $k$-times continuously differentiable functions on the interval $I = [0, 1]$ which is a self-conjugate space. Suppose that $\{T_n\}$ is a sequence of positive linear maps from $C^{(k)}(I)$ into $C(I)$ satisfying

$$T_n 1 \to 1, \ T_n x \to x, T_n x^2 \to x^2.$$

For each $a \in I$, the function $h(x) = 1 - (x - a)^2$ belongs to the function space $S = \text{Span}\{1, x, x^2\}$. Since $h(a) = 1$ and $|h(y)| < 1$ for any $y \neq a$, we infer $Ch(S) = I$, by Remark 2.1. Now from Theorem 3.1 we conclude that $T_nf \to f$ for all $f \in C^{(k)}(I)$. Meantime, by Theorem 3.2, the same result holds true for "uniformly convergence" instead of "pointwise convergence", which can be also obtained from Korovkin’s first theorem.
Example 4.2. Let $\Omega$ be a non-empty open subset of $\mathbb{R}^p$ and $K$ be a compact subset of $\Omega$. The term *multi-index* denotes an ordered $p$-tuple $\alpha = (\alpha_1, ..., \alpha_p)$ of nonnegative integers $\alpha_i$. For each multi-index $\alpha$, consider the differential operator

$$D^\alpha = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} ... \left(\frac{\partial}{\partial z_p}\right)^{\alpha_p},$$

if $\alpha \neq 0$, and $D^\alpha f = f$ if $\alpha = 0$. A function $f$ on $\Omega$ is said to belong to $C^\infty(\Omega)$ if $D^\alpha f \in C(\Omega)$ for all multi-index $\alpha$. By $D_K$ we denote the space $\{f|_K : f \in C^\infty(\Omega)\}$. Since $D_K$ may be considered as a function space on $K$, from our results we deduce the following.

If $\{T_n : D_K \rightarrow C(K) : n \in \mathbb{N}\}$ is a sequence of positive linear maps such that $T_n1 \rightarrow 1$, $T_n(P_k) \rightarrow P_k$, $T_n(\sum_{k=1}^{p^2} P_k) \rightarrow \sum_{k=1}^{p^2} P_k$, where $P_k$ is the projection

$$P_k(x) = x_k \text{ for } x = (x_1, ..., x_p),$$

then $T_nf \rightarrow f$ for all $f \in D_K$. A similar result holds true for "uniformly convergence" instead of "pointwise convergence".

Let us remark that for any $a = (a_1, ..., a_p) \in K$, the function

$$h(x) = b_1 - (P_1(x) - a_1)^2 + ... + b_k - (P_p(x) - a_p)^2 \quad (x = (x_1, ..., x_p) \in \Omega),$$

where $b_i = \max\{|P_i(x) - a_i| : x \in K\}$, $i = 1, ..., p$, implies that $a$ belongs to the Choquet boundary of $S = \text{Span}\{1, P_1, ..., P_p, P_1^2, ..., P_p^2\}$ by Remark 2.1.

The following example includes the complex Korovkin theorem.

Example 4.3. If $\{T_n : C(\mathbb{T}) \rightarrow C(\mathbb{T}) : n \in \mathbb{N}\}$ is a sequence of positive linear maps such that $T_n1 \rightarrow 1$ and $T_nz \rightarrow z$, then $T_nf \rightarrow f$ for all $f \in C(\mathbb{T})$. Notice that here if $z_0 \in \mathbb{T}$, then the function $h(z) = \frac{z^{2} - z_{0}^{2}}{2}$ works for Remark 2.1 ($S = \text{Span}\{1, z\}$).

Let $D$ be the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$ and $\{T_n : C(D) \rightarrow C(D) : n \in \mathbb{N}\}$ be a sequence of positive linear maps such that $T_n1 \rightarrow 1$, $T_nz \rightarrow z$, $T_n|z|^2 \rightarrow |z|^2$, then $T_nf \rightarrow f$ for all $f \in C(D)$.

It should be noted that since $T_n$ is positive, it is easily seen that $T_n\bar{z} = \bar{T_n}z$, which yields $T_n\bar{z} \rightarrow \bar{z}$, hence for each $z_0 \in D$, the function $h(z) = 1 - \frac{|z - z_0|^2}{4} = 1 - \frac{|z|^2 - \bar{z}_0z + \bar{z}_0z_0}{4}$, which belongs to $S = \text{Span}\{1, z, \bar{z}, |z|^2\}$, is the appropriate function for Remark 2.1.

The two above results holds true for "uniformly convergence" instead of "pointwise convergence".

Remark 4.4. From our theorems, one can obtain the Korovkin-type results of [11] and [15] (with respect to both "uniformly convergence" and "pointwise convergence"), which are generalizations of Korovkin’s second theorem on convergence of a sequence of positive linear maps for the space of real-valued continuous $2\pi$-periodic functions on $\mathbb{R}$. 

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REFERENCES

[1] F. Altomare, *Korovkin-type theorems and approximation by positive linear operators*, Surv. Approx. Theory. 5 (2010), 92-164.

[2] F. Altomare and M. Campiti, *Korovkin-Type Approximation Theory and its Applications*, de Gruyter Studies in Mathematics, 17, Walter de Gruyter Co., Berlin, 1994.

[3] J. Araujo and J. J. Font, *Linear isometries between subspaces of continuous functions*, Trans. Amer. Math. Soc. 349 (1997), 413-428.

[4] E. Bishop and K. de Leeuw, *The representation of linear functionals by measures on sets of extreme points*, Ann. Inst. Fourier (Grenoble) 9 (1959), 305-331.

[5] A. Browder, *Introduction to Function Algebras*, W. A. Benjamin, New York-Amsterdam, 1969.

[6] K. Donner, *Extension of Positive Operators and Korovkin Theorems*, Lecture Notes in Math. 904, Springer-Verlag, Berlin, 1982.

[7] T. Hachiro and T. Okayasu, *Some theorems of Korovkin type*, Studia Math. 155 (2003), no. 2, 131-143.

[8] P. P. Korovkin, *On convergence of linear positive operators in the space of continuous functions*, Doklady Akad. Nauk SSSR (N.S.) 90 (1953), 961-964 (in Russian).

[9] H. Koshimizu, T. Miura, H. Takagi and S. E. Takahasi, *Real-linear isometries between subspaces of continuous functions*, J. Math. Anal. Appl. 413 (2014), 229-241.

[10] S. Mazur and S. Ulam, *Sur les transformations isométriques d’espaces vectoriels normés*, C. R. Math. Acad. Sci. Paris 194 (1932), 946-948.

[11] E. N. Morozov, *Convergence of a sequence of positive linear operators in the space of continuous $2\pi$-periodic functions of two variables*, Kalinin. Gos. Ped. Inst. Uchen. Zap. 26 (1958), 129-142 (in Russian).

[12] R. R. Phelps, *Lectures on Choquet’s Theorem*, 2nd ed., Lecture Notes in Math. 1757, Springer, Berlin, 2001.

[13] R. R. Phelps, *The range of $Tf$ for certain linear operators $T$*, Proc. Amer. Math. Soc. 16 (1965), 381-382.

[14] E. Scheffold, *Über die punktweise konvergenz von operatoren in $C(X)$*, Rev. Acad. Ci. Zaragoza 28 (1973), 5-12.

[15] V. I. Volkov, *Conditions for convergence of a sequence of positive linear operators in the space of continuous functions of two variables*, Kalinin. Gos. Ped. Inst. Uchen. Zap. 26 (1958), 27-40 (in Russian).

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