Cloning of spin coherent states

Rafal Demkowicz-Dobrzański, 1, 2 Marek Kuś, 1, 3 and Krzysztof Wódkiewicz 2

1 Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotników 32/44, 02-668 Warszawa, Poland
2 Institute of Theoretical Physics, Warsaw University, Warszawa, Poland
3 Faculty of Mathematics and Sciences, Cardinal Stefan Wyszyński University, Warszawa, Poland

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We consider optimal cloning of the spin coherent states in Hilbert spaces of different dimensionality \( d \). We give explicit form of optimal cloning transformation for spin coherent states in the three-dimensional space, analytical results for the fidelity of the optimal cloning in \( d = 3 \) and \( d = 4 \) as well as numerical results for higher dimensions. In the low-dimensional case we construct the corresponding completely positive maps and exhibit their structure with the help of Jamiołkowski isomorphism. This allows us to formulate some conjectures about the form of optimal coherent cloning CP maps in arbitrary dimension.

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I. INTRODUCTION

The no-cloning theorem \([1, 2, 3]\) claims that a universal and faithful cloning machine that would clone perfectly an arbitrary input quantum state, is incompatible with quantum mechanics. It is possible, however, to find imperfect cloning machines that would copy quantum states with some loss of quality. The interesting question is what are the optimal cloning machines allowed by quantum mechanics that would clone quantum states as good as possible?

Recently, many papers have appeared describing different optimal universal cloning machines designed to clone arbitrary states from a Hilbert space of dimension \( d \). Optimal schemes for cloning of pure states, giving the highest possible fidelity, were found for \( 1 \rightarrow 2 \) (two copies of the original input state are produced) and more generally \( N \rightarrow M \) (\( M \) copies of \( N \) identical input states) universal cloning in arbitrary dimension \([4, 5, 6, 7, 8]\).

One can consider also nonuniversal cloning machines designed specially for cloning certain subsets of states from a given Hilbert space. In this approach one attempts to optimize fidelity for cloning states from a subset without taking care how well other states (outside the subset) are cloned. Usually one also imposes a condition that all states from the chosen subset are cloned equally well.

The case when one wants to find the best possible machine for cloning of two given nonorthogonal states was discussed in \([3]\). Cloning of coherent states in an infinite dimensional space was considered in \([10, 11]\), where transformation for cloning of all coherent states with the same fidelity \( F = 2/3 \) was proposed. A proof that such a cloning procedure gives the highest possible fidelity for coherent states was later given in \([12]\).

The proof, however, was not a straightforward one and limited to the case of so called gaussian cloners. By a straightforward proof we mean a standard optimization procedure, where one starts with the most general unitary transformation on an input state, a blank state and an ancilla, adds unitary constraints, and then tries to optimize fidelity for cloning of the desired group of states. This seems to be very difficult to do in the case of optimizing cloning of coherent states, as these are states taken from infinite dimensional Hilbert space.

Instead of the optimization scheme, the authors of \([12]\) make use of limitations on so called joint measurement \([13]\). The idea of joint measurement is to perform some kind of a measurement on a quantum system that would give us simultaneously some information about two noncommuting observables (for example position and momentum). One way of performing joint measurement can be realized with the help of quantum cloning \([14]\). In order to measure simultaneously position and momentum of a particle, one can first clone its quantum state and then perform a position measurement on one of the clones and a momentum measurement on the other. As cloning must not be a way to circumvent fundamental limitations on accuracy of a joint measurement, authors arrive at the conclusion that the maximal attainable fidelity for cloning of coherent states must satisfy: \( F \leq 2/3 \).

In this paper we shall consider spin coherent states in a finite dimensional space and we shall look for the optimal cloning transformation for them. In the limit of \( d \rightarrow \infty \) the coherent states we use tend to the harmonic oscillator coherent states. Therefore for higher dimensions we shall be able to compare our results on maximal possible fidelity with those based on joint measurement arguments.

In section IV we discuss the same problem of cloning of spin coherent states using different approach. We analyze completely positive (CP) maps corresponding to the universal and the coherent cloning transformation in the case of \( d = 2, d = 3 \). In doing so, we follow the general approach, proposed by D’Ariano and Presti \([15]\) to the problem of nonuniversal covariant cloning. We exhibit the structure of cloning CP maps using Jamiołkowski isomorphism.
This section is intended to give a better insight into the form of the optimal coherent cloning transformation in low dimensions, which we believe may be helpful in finding an analytical solution for the problem of cloning of spin coherent states in an arbitrary dimension $d$.

II. COHERENT STATES

In the infinite dimensional space of the harmonic oscillator states we can construct the creation and annihilation operators, $a^\dagger, a$, obeying the boson commutation relation $[a, a^\dagger] = 1$. The coherent states of such a system (harmonic oscillator coherent states) are eigenvectors of the annihilation operator $a|\alpha\rangle = \alpha|\alpha\rangle$ and can be obtained as displacements of the ground state $|0\rangle$:

$$|\alpha\rangle = D(\alpha)|0\rangle, \quad D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a).$$

(1)

One of important properties of such coherent states is that they satisfy the lower bound on product of the dispersions of the position and momentum operators (or the quadrature operators) required by the Heisenberg uncertainty principle.

The concept of coherent states is not restricted to the infinite dimensional space. In a finite dimensional space one can introduce different kinds of coherent states $|\alpha\rangle$. In this paper we shall concentrate on so called spin coherent states (SU(2) coherent states), which we define below [13].

Let us consider the Hilbert space of spin states with the total spin $j$, the space in question has the dimension $d = 2j + 1$. By $|m\rangle$, $m = -j, -j + 1, \ldots, j - 1, j$ we denote the basis consisting of the eigenvectors of $J_z$ operator. Spin coherent states are defined as rotations of the "ground" state $|−j\rangle$ by unitary operators from the irreducible SU(2) representation in the $2j + 1$ dimensional space:

$$|\theta, \phi\rangle = R_{\theta, \phi}|−j\rangle, \quad R_{\theta, \phi} = e^{-i\theta(J_z \sin \phi - J_y \cos \phi)}.$$

(2)

The operator $R_{\theta, \phi}$ corresponds to a rotation by the angle $\theta$ around the axis $\vec{n} = [\sin \phi, -\cos \phi, 0]$. For $j = 1/2$ the dimension of the space is $d = 2$ (qubit). In this case spin coherent states are actually all the pure states in the space (every pure state can be described by a direction on the Bloch sphere). In higher dimensions, however, spin coherent states constitute only a subset in the set of all states of a given Hilbert space.

Their similarity to the harmonic oscillator coherent states lies in the fact that they are constructed as rotations of the "ground" state just like the harmonic oscillator coherent states were constructed as displacements of the ground state. For us the most important feature of spin coherent states is that they approach harmonic oscillator coherent states when dimension of the space tends to infinity [19].

One can decompose a spin coherent state $|\theta, \phi\rangle$ in the $J_z$-eigenvectors basis:

$$\langle m|\theta, \phi\rangle = \left(\frac{2j}{j + m}\right)^{1/2} \sin^{j+m}(\theta/2) \cos^{-m}(\theta/2)e^{-i(j+m)\phi}.$$

(3)

III. OPTIMAL CLONING OF SPIN COHERENT STATES

The most general $1 \rightarrow 2$ cloning transformation which clones pure states from a $d$-dimensional Hilbert space $\mathcal{H}_1$, is a unitary transformation $U$ acting on Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_a$. The $d$-dimensional spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ correspond to the input system and a blank one which after cloning will carry cloned states, while $\mathcal{H}_a$ is the Hilbert space of ancilla states.

We shall denote a basis in the space $\mathcal{H}_1$ (or $\mathcal{H}_2$) by $|n\rangle$, $n = 0, \ldots, d - 1$, where $|0\rangle$ corresponds to the ground state $|−j\rangle$. Coherent states in this space are thus rotations of the state $|0\rangle$. We shall refer to the states $|n\rangle$ as the number states.

The cloning machine will act on the state $|\psi⟩_1|0⟩_2|\psi_0⟩$, where $|\psi⟩_1 \in \mathcal{H}_1$ is a state to be copied, while the blank system and the ancilla are always prepared in the same initial states denoted here by $|0⟩_2|\psi_0⟩$. The final state will be $U|\psi⟩_1|0⟩_2|\psi_0⟩$. After tracing out the output density matrix with respect to the ancilla states and the states of one of the clones, one obtains reduced density matrix for the remaining clone. In this paper only symmetric cloning is considered, so the two reduced density matrices for the two clones are the same and are denoted by $\rho^{\text{out}}$.

Looking for the best possible machine for cloning of spin coherent states from space $\mathcal{H}_1$ means maximizing the average fidelity for cloning of these states:

$$F = \int d\Omega(\theta, \phi)|\rho^{\text{out}}(\theta, \phi)|\theta, \phi⟩, \quad$$

(4)
where \(d\Omega = d\phi d\theta \sin \theta/4\pi\), and the \(\rho_{\theta,\phi}^\text{out}\) is the density matrix of the clones after cloning the input spin coherent state \(|\theta,\phi\rangle\).

In order to find the maximal attainable fidelity for cloning of spin coherent states and corresponding cloning transformation we shall generalize the method used by Gisin and Masar in [5].

Firstly we assume that the final state is symmetric with respect to the exchange of the two clones. Secondly we assume that our cloning machine clones all coherent states with the same fidelity. We may make these assumptions, because they do not lower possible attainable fidelity, as was explained in [5].

In \(\mathcal{H}_1 \otimes \mathcal{H}_2\) space there are \(S = d(d-1)/2\) symmetric states. We shall denote them by \(|s\rangle_{12}, s = 0 \ldots S - 1\). The most general cloning transformation can thus be described as:

\[
U|n\rangle_1|0\rangle_2|A_0\rangle = |s\rangle_{12}|R_{ns}\rangle, \tag{5}
\]

where \(|R_{ns}\rangle\) are unnormalized states of the ancilla system and the summation convention is used. Unitarity of transformation imposes a set of constraints:

\[
(R_{n's}|R_{ns}) = \delta_{n'n}. \tag{6}
\]

Every coherent state can be decomposed in the basis of number states \(|\theta,\phi\rangle = O_n|n\rangle\). According to Eq. (8):

\[
O_n = \langle n|\theta,\phi\rangle = \left(\frac{d - 1}{n}\right)^{1/2} \sin^n(\theta/2) \cos^{d-n-1}(\theta/2) e^{-in\phi}, \tag{7}
\]

where \(d = 2j + 1, n = j + m\). Action of the cloning transformation \(U\) on such a coherent state gives:

\[
U|\theta,\phi\rangle_1|0\rangle_2|0\rangle_0 = O_n|s\rangle_{12}|R_{ns}\rangle. \tag{8}
\]

Calculating the fidelity from equation (8) one arrives at the formula:

\[
\mathcal{F} = \langle R_{n's'}|R_{ns}\rangle \langle k|l|s\rangle \langle s'|k'|l\rangle \int d\Omega O^*_n O_{k'} O^*_k O_n = \langle R_{n's'}|R_{ns}\rangle A_{n's'n's}. \tag{9}
\]

As explained in [5], due to the rotational symmetry of the cloning machine, it is enough to consider only one constraint from the set (6), namely \(\langle R_{ns}|R_{ns}\rangle = d\). Incorporating this constraint by means of the Lagrange multiplier \(\lambda\) we have to extremize the following expression:

\[
\mathcal{F} = \langle R_{n's'}|R_{ns}\rangle A_{n's'n's} - \lambda(\langle R_{n's'}|R_{ns}\rangle \delta_{n's} \delta_{n'n} - d). \tag{10}
\]

Denoting by \(\{|a\}\) an orthonormal basis in the ancilla system, we can insert the identity operator \(I = \sum_a |a\rangle \langle a|\) between \(\langle R_{n's'}|\) and \(|R_{ns}\rangle\). Then, varying over \(\langle R_{n'a'}|a\rangle\), we obtain an eigenvalue problem:

\[
(A_{n's'n's} - \lambda \delta_{n's} \delta_{n'n})|a\rangle |R_{ns}\rangle = 0. \tag{11}
\]

Multiplying on the left by \(\langle R_{n's'}|a\rangle\) and summing over \(n',s',\) and \(a\) one concludes that the fidelity is \(\mathcal{F} = d\lambda\). Thus looking for the best fidelity is equivalent to looking for the biggest eigenvalue of (11). From the eigenvectors corresponding to this eigenvalue one can infer the form of the unitary transformation which attains this fidelity.

In the case of the universal cloning machine in \(d\) dimensions it was proven [5] that the optimal fidelity for \(1 \rightarrow 2\) universal cloning reads:

\[
\mathcal{F}^\text{universal} = \frac{d + 3}{2d + 2}. \tag{12}
\]

As we attempt to clone only coherent states from a given space, fidelity of such cloning should be higher \(\mathcal{F}^\text{coherent} \geq \mathcal{F}^\text{universal}\) (equality holds only for \(d = 2\)). We obtained analytical solutions for the maximal fidelity of cloning of spin coherent states for \(d = 3\) and \(d = 4\):

\[
\mathcal{F}^\text{coherent}_3 = \frac{11 + \sqrt{21}}{20}, \quad \mathcal{F}^\text{coherent}_4 = \frac{79 + \sqrt{997}}{140}. \tag{13}
\]

We have also calculated the numerical values of fidelities for higher dimensions \(d \leq 16\) (The fact that we stopped our calculations at \(d = 16\), is merely because of the rapid growth of computation time with the increase of \(d\). (This
is because the number of elements of matrix $A$, we had to find the biggest eigenvalue of, was increasing as $d^6$.

Comparison between results for the maximal fidelities of the coherent and the universal cloning is shown in Figure 1.

The dotted line marks the value of the fidelity equal to $2/3$. Fidelity of universal cloning machines tends to $1$ as the dimension tends to infinity, and falls below $2/3$ as soon as for $d = 6$. Our numerical results show that with the increase of the dimension fidelity of the coherent cloning falls significantly slower than that of the universal cloning. In our calculations it has not fallen below $2/0.699$. These numerical results are useful in giving limitations for the maximal possible fidelity for cloning of harmonic oscillator coherent states, as for higher dimension spin coherent states tend to harmonic oscillator ones.

Calculation of fidelities for higher dimension gives a better bound on this fidelity. From these results we cannot conclude, however, whether the fidelity $F^\text{coherent}_\infty$ will actually reach the value $2/3$ or will it saturate somewhere above.

We have recently learned that using different approach and with the help of some numerics, Navez and Cerf have found that the optimal fidelity for cloning of harmonic oscillator coherent states is $0.6825$, which indeed falls into the range [2/3, 0.699].

For the sake of curiosity we can try to fit to our numerical results, a rational function (with three free parameters $\alpha, \beta, \gamma$) of the form $F^\text{fit}_d = \frac{\alpha d + \beta}{d + \gamma}$ (we know, however, that the exact formula for the optimal fidelity of cloning of spin coherent states is not a rational function, as we have found analytical solutions for $d = 3$ and $d = 4$ [13]). This fitting gives the asymptotic behaviour $F^\text{fit}_\infty = 0.6812$, which may be seen as an independent indication that optimal fidelity for cloning of harmonic oscillator coherent states is indeed above $2/3$.

The lowest dimension for which universal cloning and coherent cloning differ from each other is $d = 3$. It is interesting to compare the explicit form of the universal and the coherent cloning transformations in this case. Both transformations require that the ancilla system is at least three dimensional. We denote the three orthogonal ancilla states as $|A_0\rangle, |A_1\rangle, |A_2\rangle$. Optimal universal cloning transformation reads:

\[
\begin{align*}
|0\rangle|0\rangle|A_0\rangle &\rightarrow \frac{1}{\sqrt{2}}|0, 0\rangle|A_0\rangle + \frac{1}{2}|0, 1\rangle|A_1\rangle + \frac{1}{2}|0, 2\rangle|A_2\rangle \\
|1\rangle|0\rangle|A_0\rangle &\rightarrow \frac{1}{\sqrt{2}}|1, 1\rangle|A_1\rangle + \frac{1}{2}|0, 1\rangle|A_0\rangle + \frac{1}{2}|1, 2\rangle|A_2\rangle \\
|2\rangle|0\rangle|A_0\rangle &\rightarrow \frac{1}{\sqrt{2}}|2, 2\rangle|A_2\rangle + \frac{1}{2}|0, 2\rangle|A_0\rangle + \frac{1}{2}|1, 2\rangle|A_1\rangle,
\end{align*}
\]

where $|i, i\rangle$ denotes $|i\rangle|i\rangle$ state while by $|i, j\rangle$ (for $i \neq j$) we mean the symmetrized state $1/\sqrt{2}(|i\rangle|j\rangle + |j\rangle|i\rangle)$. The

![FIG. 1: Comparison between fidelity of coherent and universal cloning machines.](image-url)
optimal coherent cloning transformation in this case is given by:

\[
\begin{align*}
|0\rangle\langle 0|A_0 & \rightarrow \alpha|0,0\rangle|A_0\rangle + \frac{1}{\sqrt{2}}\beta|0,1\rangle|A_1\rangle + (\delta|0,2\rangle|A_2\rangle) \\
|1\rangle\langle 1|A_0 & \rightarrow (\gamma|1,1\rangle + \sqrt{2}\beta|0,2\rangle)|A_1\rangle + \frac{1}{\sqrt{2}}\alpha|0,1\rangle|A_0\rangle + \frac{1}{\sqrt{2}}\alpha|1,2\rangle|A_2\rangle \\
|2\rangle\langle 2|A_0 & \rightarrow \alpha|2,2\rangle|A_2\rangle + \frac{1}{\sqrt{2}}\alpha|1,2\rangle|A_1\rangle + (\delta|0,2\rangle|A_0\rangle),
\end{align*}
\]

with the coefficients:

\[
\begin{align*}
\alpha &= \sqrt{\frac{1}{11}(63 + 13\sqrt{21})} \approx 0.764 \\
\beta &= \sqrt{\frac{1}{2} - \frac{1}{5\sqrt{21}}} \approx 0.159 \\
\gamma &= \sqrt{\frac{1}{70}(21 + \sqrt{21})} \approx 0.604 \\
\delta &= \sqrt{\frac{1}{20} - \frac{21}{20\sqrt{21}}} \approx 0.315.
\end{align*}
\]

Note that while in the case of the universal cloning none of basis states was distinguished (all were cloned in the similar manner), in the case of the coherent cloning the state \(|1\rangle\) is cloned differently than two other basis states. The reason for this is that the state \(|1\rangle\) is not a spin coherent state while states \(|0\rangle, |2\rangle\) are.

We can compare the reduced density matrices for the two clones in the case of the universal and the coherent cloning. With the state \(|0\rangle\) at the input reduced density matrices after cloning with the universal and the coherent cloning machines are respectively given by:

\[
\begin{align*}
\rho_{\text{out}}^{\text{universal}} &= 0.750|0\rangle\langle 0| + 0.125|1\rangle\langle 1| + 0.125|2\rangle\langle 2| \\
\rho_{\text{out}}^{\text{coherent}} &= (\frac{9}{16}\alpha^2 + \delta^2)|0\rangle\langle 0| + (\frac{1}{4}\alpha^2 + \beta^2)|1\rangle\langle 1| + \delta^2|2\rangle\langle 2| \approx \\
&\approx 0.779|0\rangle\langle 0| + 0.171|1\rangle\langle 1| + 0.049|2\rangle\langle 2|.
\end{align*}
\]

After the universal cloning the resulting reduced density matrix (for cloning in any dimension) is a mixture of the initial density matrix \(|\psi\rangle\langle \psi|\) and the identity matrix of the appropriate dimension. That is why \(\rho_{\text{out}}^{\text{universal}}\) has equal contributions from \(|1\rangle\langle 1|\) and \(|2\rangle\langle 2|\) and a dominant term with \(|0\rangle\langle 0|\).

In case of a coherent cloning the state \(|0\rangle\) (a coherent state) is cloned better than by the universal machine (cf. higher coefficient in front of \(|0\rangle\langle 0|\)). Additionally, other basis states \(|n\rangle\langle n| (n \neq 0)\) contribute with different coefficients, the bigger is \(n\) the lower is the coefficient.

It is in accordance with what we know about cloning of the harmonic oscillator coherent states. When one clones a vacuum state through the optimal gaussian cloning machine for coherent states, reduced density matrix of clones describes then a thermal state with mean number of photons equal to 1/2. It is a mixture of number states \(|n\rangle\langle n|\) with the coefficients decreasing when \(n\) increases.

\section{IV. CP MAP AND POSITIVE OPERATOR APPROACH}

In this section we do not present any new cloning transformations, nor new results for optimal cloning fidelities. Instead, we analyze the structure of completely positive (CP) maps corresponding to the universal and the coherent cloning transformations obtained in previous section. This section is intended to give a better insight into the form of the optimal coherent cloning transformation in low dimensions, which we believe may be helpful in finding an analytical solution for the problem of cloning of spin coherent states in an arbitrary dimension \(d\).

The essential elements in the cloning process are the input state and the two output clones. Therefore instead of considering cloning process as a unitary operation \(U\) on the full space \(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_d\), one can consider a corresponding complete positive (CP), trace preserving map \(\mathcal{E} : \mathcal{L}(\mathcal{H}_1) \rightarrow \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)\), where \(\mathcal{L}(\mathcal{H})\) denotes a space of linear bounded operators on the Hilbert space \(\mathcal{H}\). The relation between the CP map \(\mathcal{E}\) and the unitary transformation \(U\) is the following:

\[
\mathcal{E}(\rho) = \text{Tr}_a(U\rho \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|U^\dagger),
\]
where $\rho = |\psi\rangle\langle\psi|$ is the state to be copied and the trace is calculated with respect to the ancilla states.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. For any CP map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$, one can define a positive operator $P_\varepsilon \in \mathcal{L}(\mathcal{K} \otimes \mathcal{H})$:

$$P_\varepsilon = \mathcal{E} \otimes \mathcal{I}(\|\|)(\|\|),$$

(19)

where $\|\| = \sum_n |n\rangle \otimes |n\rangle$ is an unnormalized state in $\mathcal{H} \otimes \mathcal{H}$ (maximally entangled state), $|n\rangle$ are base vectors in $\mathcal{H}$, and $\mathcal{I}$ is the identity map on $\mathcal{L}(\mathcal{H})$. This definition gives one to one correspondence between CP maps and positive operators known as the Jamiołkowski isomorphism [17]. As we consider trace preserving CP maps, we must impose an additional condition on $P_\varepsilon$:

$$\text{Tr}_\mathcal{K}(P_\varepsilon) = \mathbb{I} \in \mathcal{L}(\mathcal{H}).$$

(20)

Cloning of spin coherent states is a special case of a nonuniversal covariant cloning i.e. a cloning scheme covariant under a proper subgroup (all the states generated by this subgroup will be cloned in the same way) of the unitary group in d dimensions $U(d)$. In the case of the cloning of spin coherent states this subgroup is the $SU(2)$ group. General approach to the problem of optimizing nonuniversal covariant cloning using positive operator $P_\varepsilon$ picture was presented in [15]. We shall now look at our previous results for cloning of spin coherent states from this point of view.

We are interested in the covariant cloning with respect to $SU(2)$ group in d dimensional space. In terms of CP map this means that if $R_d$ is an irreducible representation of $SU(2)$ in d dimensional space then the condition

$$\mathcal{E}(R_d(g)\rho R_d(g)^\dagger) = R_d(g)^{\otimes 2}\mathcal{E}(\rho)R_d(g)^{\dagger^{\otimes 2}},$$

(21)

must be satisfied for any $g \in SU(2)$ and $\rho \in \mathcal{L}(\mathcal{H}_1)$.

The covariance condition for corresponding positive operator $P_\varepsilon \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_1)$ reads [15]:

$$[P_\varepsilon, R_d(g)^{\otimes 2} \otimes R_d^\dagger(g)] = 0,$$

(22)

where $R_d^\dagger$ denotes the complex conjugate representation. One can decompose the tensor product of representations into a direct sum of irreducible representation:

$$R_d^{\otimes 2} \otimes R_d^\dagger = \oplus_i R_i^\dagger R_i^\dagger.$$

(23)

The full Hilbert space is decomposed into invariant subspaces $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_1 = \oplus_i \mathcal{M}_i$. The representation $R_i^\dagger$ acts in the subspace $\mathcal{M}_i$.

The commutation condition [22] imposes a certain block structure on the operator $P_\varepsilon$:

$$P_\varepsilon = \sum_{ij} c_{ij} \mathbb{I}_j^i,$$

(24)

where $c_{ij} = 0$ if representations $R^i$ and $R^j$ are inequivalent, and $\mathbb{I}_j^i$ denotes isomorphism between spaces $\mathcal{M}_i$ and $\mathcal{M}_j$ (which exists if $R^i$ and $R^j$ are equivalent). In order to have a positive operator $P_\varepsilon$ each block must be a positive matrix.

As we are interested in the symmetric cloning, in addition to the covariance with respect to the $SU(2)$ group we have to impose covariance with respect to permutation of the two output clones. This is equivalent to commutativity of the operator $P_\varepsilon$ and $S_2 \otimes \mathbb{I}$, where $S_2$ denotes the permutation of two clones (the commutation in the space $\mathcal{H}_1 \otimes \mathcal{H}_2$). As a consequence $c_{ij}$ coefficients are zero even if representations $R^i$ and $R^j$ are equivalent, but states from the corresponding spaces $\mathcal{M}^i$, $\mathcal{M}^j$ have different symmetry (i.e. they are symmetric or antisymmetric) with respect to exchange of clones. From now on we shall not distinguish between spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and both will be denoted by $\mathcal{H}$.

In order to find explicitly the decomposition [23] of the tensor product of three d-dimensional representations of $SU(2)$ group one has to follow the scheme of adding angular momenta of three particles. We shall first add angular momenta of the particles 1 and 2 using Clebsch-Gordan coefficients and then add the third particle in the same manner [20]. We choose this order because we want the states from invariant subspaces to have definite symmetry with respect to the exchange of particles 1 and 2 (in our case they correspond to the two clones).

The situation here is a little bit different, however, from the case of adding angular momenta of three particles, as the third representation in tensor product [23] is a complex conjugate one. Fortunately in the case of representations of $SU(2)$ a complex conjugate representation $R^*$ is always equivalent to $R$ and the only thing we have to modify when performing the decomposition is to make a substitution $|n\rangle \rightarrow (-1)^n|d - n - 1\rangle$ for the basis states in the third space [20].
A. \( d=2 \)

As an illustration we shall first consider spin coherent cloning in the case \( d=2 \) (universal cloning of qubit). The decomposition (23) reads now:

\[
R_2 \otimes R_2 \otimes R_2^\ast = R_2 \oplus R_2 \oplus R_4.
\]  

(25)

In the language of angular momentum this means that adding three particles with spin \( 1/2 \) we have two invariant spaces corresponding to spin \( 1/2 \) and one space corresponding to spin \( 3/2 \). The Hilbert space can be decomposed into three invariant subspaces \( \mathcal{H}^{\otimes 2+1} = \oplus_{i=1}^3 \mathcal{M}^i \). This decomposition is shown in Table I.

TABLE I: Invariant subspaces in \( \mathcal{H}^{\otimes 2+1} \), \( \dim(\mathcal{H}) = 2 \). \( S,A \) denote the symmetry of states (Symmetric or Antisymmetric) with respect to exchange of clones (first two spaces in the tensor product). \( d \) is the dimension of an invariant subspace.

| space \( \mathcal{M} \) | \( d \) | basis | \( S \) | \( A \) |
|----------------------|-----|-------|------|-----|
| \( \mathcal{M}^1 \)   | 2   | \( -\frac{1}{\sqrt{2}}|0,1,1\rangle + \frac{1}{\sqrt{2}}|1,0,1\rangle \) | \( A \) | \( S \) |
| \( \mathcal{M}^2 \)   | 2   | \( \frac{1}{\sqrt{3}}|0,0,0\rangle + \frac{1}{\sqrt{3}}|0,1,1\rangle + \frac{1}{\sqrt{3}}|1,0,0\rangle \) | \( S \) | \( S \) |
| \( \mathcal{M}^3 \)   | 4   | \( -\frac{1}{\sqrt{3}}|0,0,0\rangle + \frac{1}{\sqrt{3}}|0,1,1\rangle + \frac{1}{\sqrt{3}}|1,0,1\rangle \) | \( S \) | \( S \) |

According to what was said before, in this case the operator \( P_\mathcal{E} \), which is now a \( 8 \times 8 \) matrix, must be diagonal, as there are no equivalent representations acting in spaces with the same symmetry of states, i.e.

\[
P_\mathcal{E} = a\mathbb{1}^1_2 + b\mathbb{1}^2_2 + c\mathbb{1}^3_2,
\]

(26)

where \( a,b,c \geq 0 \).

The known transformation for the optimal cloning of a qubit [4], gives \( P_\mathcal{E} \) of the above form with \( a = 0, b = 1, \) and \( c = 0 \). (One could also use above formalism to find such a transformation, if it was not known, by maximizing fidelity with respect to \( a, b, \) and \( c \) with constraints imposed by the trace preserving condition (20)). In other words the operator \( P_\mathcal{E} \) has two non-zero eigenvalues both equal 1, and the corresponding eigenvectors belong to the two dimensional invariant subspace \( \mathcal{M}^2 \), symmetric with respect to permutations of the two clones.

B. \( d=3 \)

In the case \( d=3 \), the tensor product (23) decomposes in the following manner:

\[
R_3 \otimes R_3 \otimes R_3^\ast = R_1 \oplus R_3 \oplus R_3 \oplus R_4 \oplus R_5 \oplus R_5 \oplus R_7.
\]

(27)

We have seven invariant subspaces \( \mathcal{H}^{\otimes 2+1} = \oplus_{i=1}^7 \mathcal{M}^i \) described in Table II. One can see that in case of \( d=3 \) the form of \( P_\mathcal{E} \) can be more complicated. It needs not to be diagonal as we have two equivalent representations in the subspaces \( \mathcal{M}^3 \) and \( \mathcal{M}^4 \), which are both symmetric under exchange of the clones. The most general \( P_\mathcal{E} \) satisfying the covariance condition will consist of one block for \( \mathcal{M}^3 \) and \( \mathcal{M}^4 \) subspaces and will be diagonal in the remaining part of the space.

Using this picture we can now look at the coherent and the universal cloning transformations for \( d=3 \) analyzed earlier. Interestingly, the \( 27 \times 27 \) matrices \( P_\mathcal{E} \) for both universal and coherent cloning have only three non-zero eigenvalues, all equal to 1. The difference between the two cloning transformations is in the eigenvectors of the corresponding \( P_\mathcal{E} \) operators. In both cases, however, all eigenvectors are confined to the subspace \( \mathcal{M}^3 \oplus \mathcal{M}^4 \). It means that the only non-zero elements of \( P_\mathcal{E} \) are in the block part considered above.

Let us denote by \( |\phi^i_k\rangle \) the \( n \)-th basis state in \( \mathcal{M}^i \), as ordered in Table II. In both cases of the universal and the coherent cloning the three eigenvectors corresponding to the eigenvalue 1 are given by:

\[
|v_k\rangle = a|\phi^3_k\rangle + b|\phi^4_k\rangle, \quad k = 1, 2, 3,
\]

(28)
TABLE II: Invariant subspaces in $\mathcal{H}^{d+1}$, dim($\mathcal{H}$) = 3. $\mathcal{S}$, $\mathcal{A}$ denote the symmetry of states (Symmetric or Antisymmetric) with respect to exchange of clones. $d$ is the dimension of an invariant subspace.

| $\mathcal{M}$ | $d$ | basis | $\mathcal{S}$ | $\mathcal{A}$ |
|--------------|-----|-------|---------------|--------------|
| $\mathcal{M}^1$ | 1   | $\frac{1}{\sqrt{2}} (-|0, 1, 0\rangle - |0, 2, 1\rangle + |1, 0, 0\rangle - |1, 2, 2\rangle + |2, 0, 1\rangle + |2, 1, 2\rangle)$ | A |
| $\mathcal{M}^2$ | 3   | $\frac{1}{\sqrt{2}} (-|0, 1, 1\rangle - |0, 2, 2\rangle + |1, 0, 1\rangle + |2, 0, 2\rangle)$ $\frac{1}{\sqrt{2}} (|0, 1, 0\rangle - |1, 0, 0\rangle - |1, 2, 2\rangle + |2, 1, 2\rangle)$ $\frac{1}{\sqrt{2}} (|0, 2, 0\rangle - |1, 2, 1\rangle - |2, 0, 0\rangle - |2, 1, 1\rangle)$ | A |
| $\mathcal{M}^3$ | 3   | $\frac{1}{\sqrt{2}} (|0, 2, 2\rangle - |1, 1, 2\rangle + |2, 0, 2\rangle)$ $\frac{1}{\sqrt{2}} (-|0, 2, 2\rangle - |1, 1, 2\rangle - |2, 0, 2\rangle)$ $\frac{1}{\sqrt{2}} (|0, 2, 0\rangle - |1, 1, 0\rangle + |2, 0, 0\rangle)$ | S |
| $\mathcal{M}^4$ | 3   | $\frac{1}{\sqrt{2}} (6|0, 0, 0\rangle + 3|0, 1, 1\rangle + |0, 2, 2\rangle + 3|1, 0, 1\rangle + 2|1, 1, 2\rangle + 2|2, 0, 1\rangle)$ $\frac{1}{\sqrt{2}} (3|0, 1, 0\rangle + 2|0, 2, 2\rangle + |1, 1, 1\rangle)$ $\frac{1}{\sqrt{2}} (|2, 0, 2\rangle + |2, 1, 1\rangle)$ | S |
| $\mathcal{M}^5$ | 5   | $\frac{1}{\sqrt{2}} (-|0, 1, 1\rangle + |1, 0, 2\rangle)$ $\frac{1}{\sqrt{2}} (|0, 1, 1\rangle - |0, 2, 2\rangle - |1, 0, 1\rangle + |2, 0, 2\rangle)$ $\frac{1}{\sqrt{2}} (|0, 2, 0\rangle + |1, 2, 1\rangle + |2, 0, 0\rangle + |2, 1, 1\rangle)$ | S |
| $\mathcal{M}^6$ | 5   | $\frac{1}{\sqrt{2}} (|2, 0, 0\rangle + |0, 1, 1\rangle + |0, 2, 2\rangle + |1, 1, 1\rangle + |1, 2, 2\rangle + |2, 0, 2\rangle)$ $\frac{1}{\sqrt{2}} (-|0, 1, 0\rangle - |1, 2, 2\rangle)$ $\frac{1}{\sqrt{2}} (-|0, 2, 0\rangle - |1, 2, 1\rangle - |2, 0, 0\rangle - |2, 1, 1\rangle + |2, 2, 2\rangle)$ | S |
| $\mathcal{M}^7$ | 7   | $\frac{1}{\sqrt{2}} (-|0, 1, 1\rangle + |0, 1, 2\rangle)$ $\frac{1}{\sqrt{2}} (-|0, 0, 0\rangle - |0, 2, 2\rangle - |0, 1, 1\rangle + |0, 2, 2\rangle - |1, 1, 1\rangle + |2, 1, 2\rangle + |2, 0, 2\rangle)$ $\frac{1}{\sqrt{2}} (|0, 2, 0\rangle + |1, 1, 0\rangle - |2, 0, 0\rangle - |2, 0, 0\rangle - |2, 1, 1\rangle + |2, 2, 2\rangle)$ | S |

with the coefficients:

| Basis | $a$ | $b$ |
|-------|-----|-----|
| universal | $\sqrt{\frac{1}{\sqrt{2}}}$ | $\sqrt{\frac{5}{\sqrt{2}}}$ |
| coherent | $\sqrt{\frac{1}{\sqrt{2}} - \frac{13}{6\sqrt{21}}}$ | $\sqrt{\frac{1}{\sqrt{2}} + \frac{13}{6\sqrt{21}}}$ |

Looking at these examples one may try to formulate some conjectures concerning the operator $P_E$ for higher dimensions $d$. It seems that it should have $d$ eigenvalues equal to 1 and all other $d^2 - d$ of them vanishing. Moreover, eigenvectors corresponding to the non-zero eigenvalues should be spanned by vectors from $d$-dimensional invariant subspaces, symmetric with respect to exchange of clones.

V. SUMMARY

We have analyzed the problem of cloning spin coherent states using the method proposed by Gisin and Massar. We found an explicit optimal cloning transformation for spin coherent states in the case when dimension of a Hilbert space is $d = 3$. Maximal attainable fidelities for higher dimensions ($d \leq 16$) were found. These results allowed us to make some comments regarding the problem of cloning harmonic oscillator coherent states. In particular our numerical results for fidelities show that that maximal attainable fidelity for cloning of harmonic oscillator coherent
states is in the range $\frac{2}{3} \leq \mathcal{F} \leq 0.699$. Additionally we analyzed more closely the cloning transformation for $d = 2$ and $d = 3$ with the help of Jamiołkowski isomorphism between CP maps and positive operators [15]. Doing so we followed the general approach towards nonuniversal covariant cloning, proposed by D'Ariano and Presti [15]. Finally we formulated some conjectures concerning the structure of CP maps corresponding to optimal spin coherent cloning transformations in arbitrary dimension $d$.

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[1] W. K. Wootters and W. Zurek, Nature 299, 802 (1982).
[2] H. P. Yuen, Phys. Lett. A 113, 405 (1985).
[3] H. Barnum et al., Phys. Rev. Lett. 76, 2818 (1995).
[4] V. Buzek and M. Hillery, Phys. Rev. A 54, 1844 (1996).
[5] N. Gisin and S. Massar, Phys. Rev. Lett. 79, 2153 (1997).
[6] D. Bruß et al., Phys. Rev. Lett. 81, 2598 (1997).
[7] R. F. Werner, Phys. Rev. A 58, 1827 (1998).
[8] H. Fan, K. Matsumoto, and M. Wadati, Phys. Rev. A 64, 064301 (2001).
[9] D. Bruß et al., Phys. Rev. A 57, 2368 (1997).
[10] N. J. Cerf, A. Ipe, and X. Rottenberg, Phys. Rev Lett 85, 1754 (2000).
[11] S. Braunstein et al., Phys. Rev. Lett. 86, 4938 (2000).
[12] N. J. Cerf and S. Iblisdir, Phys. Rev. A 62, 040301 (2000).
[13] E. Arthurs and K. J, Bell Syst. Tech. J. 44, 725 (1965).
[14] G. M. D’Ariano, C. Macchiavello, and M. F. Sacchi, J. Opt. B 3, 44 (2001).
[15] G. M. D’Ariano and P. LoPresti, Phys. Rev. A 64, 042308 (2001).
[16] P. Navez and N. Cerf, 2nd Workshop on Continuous-Variable Quantum Information Processing, Aix-en-Provence, France, April 2003.
[17] A. Jamiołkowski, Rep. Math. Phys 3, 275 (1972).
[18] A. Perelomov, Generalized Coherent States and Their Applications (Springer-Verlag, 1987).
[19] F. Arechi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A 6, 2211 (1972).
[20] J. Normand, A Lie group: Rotations in quantum mechanics (North-Holland, 1980).