GIRARD TYPE THEOREMS FOR DE SITTER TRIANGLES WITH NON-NUL EDGE

BAKI KARLIGA AND UMUT TOKESER

Abstract. Girard’s Theorem subjects to the area depending interior angles of a spherical triangle. In this paper, we introduce to its analogues for proper de Sitter triangles with non-null edges.

1. Introduction

If something exerts a force on a particle, then this phenomenon is called gravity. A free moving or falling particle follows a geodesic in a space-time. Thus, the geometry of space-time is modified by the sources of gravity. At far away, the sources of gravity is called dark energy reveals de Sitter rather than Minkowski space-time. Thus, de Sitter space is a suitable model for the universe as it consistently explains its structure.

A space (space-time) is called flat or curved if it has zero or not zero curvature. A geodesic in a flat space (space-time) is always straight lines while in a curved space (space-time) is always curved line. Although a geodesic of curved space has only space-like causal direction, a geodesic of curved space-time has one of three different causal directions which are space-like, time-like and light-like. While it is only hyperbola (great circle) in hyperbolic (spherical) curved space, a geodesic of de Sitter curved space-time is one of three curves which are ellipse, hyperbola and straight line. Thus, geodesics in de Sitter space-time is quite different and rich from the spherical and hyperbolic curved space. In view of triangular shapes, de Sitter space is richer and more universal than Euclidean, Minkowskian, spherical and hyperbolic spaces.

For a de Sitter triangle, the plane spanned by two tangent vectors at a vertex is called angle plane of that vertex. If the angle plane of a vertex is space-like, light-like or time-like, then angle is called space-like, light-like or time-like. The plane containing an edge of a de Sitter triangle is called edge

2010 Mathematics Subject Classification. Primary 51B20, 51M25, 97G30; Secondary 53A35, 83C80, 51P05.

Key words and phrases. Girard’s Theorem, triangle, de Sitter triangle.
plane of that edge. If an edge plane is space-like, light-like or time-like, then edge is called space-like, light-like, or time-like.

In space-time geometry, triangles can be classified according to causal type of its angles and edges [2]. In de Sitter space, the triangle classification according to causal type of edges is given by Asmus [3]. He showed that there are ten different triangles and only four of them (spatiolateral, tempolateral, chorosceles and chronosceles) have a polar triangle.

The complex valued pseudoangle, and the complex valued area depending on interior pseudo-angles of a de Sitter triangle is given in [4]. Peiro, in [5, Theorem 1.1], gave the relationship the dihedral angle with the angle between normals of two edge planes.

The area depending on interior angles $\alpha, \beta, \gamma$ of a triangle on unit sphere is given by Girard’s Theorem as $(\alpha + \beta + \gamma) - \pi$ [1]. Hyperbolic analogue of Girard’s Theorem, known Lambert’s Theorem, gives the area depending on interior angles $\alpha, \beta, \gamma$ of a triangle on unit hyperbolic plane as $\pi - (\alpha + \beta + \gamma)$ [6]. The complex valued analogue of Girard’s theorem for de Sitter triangles with non-null edge is introduced by Dzan[4].

By introducing the relationship with complex valued pseudo-angle and angle in $\mathbb{R}^3_1$, we give Girard’s theorems subjects to area depending interior angles of a contractible spatiolateral, tempolateral, chrosceles and chronosceles de Sitter triangles.

2. Geodesics and Triangles in de Sitter Space $S^2_1$

If $w = q - \langle p, q \rangle p$ and $W = sp\{p, q\}$ for $p, q \in S^2_1$, then by [7] and [8], one can easily prove the following results.

Theorem 1.

1. $|\langle p, q \rangle| < 1 \iff w$ is space-like
2. $|\langle p, q \rangle| > 1 \iff w$ is time-like
3. $|\langle p, q \rangle| = 1 \iff w$ is null.

Theorem 2.

1. $|\langle p, q \rangle| < 1 \iff W$ space-like
2. $|\langle p, q \rangle| > 1 \iff W$ time-like
3. $|\langle p, q \rangle| = 1 \iff W$ null.

Theorem 3. Let $p, q \in S^2_1 \overrightarrow{pq}$ is light-like if and only if $\langle p, q \rangle = 1$. 
Theorem 4.

(1) $W$ is time-like and $\langle p, q \rangle > 1$ if and only if $p$ and $q$ is on the same part of hyperbola $W \cap S^n_1$.

(2) $W$ is time-like and $\langle p, q \rangle < -1$ if and only if $p$ and $q$ is on the different part of the hyperbola $W \cap S^n_1$.

(3) $W$ is space-like if and only if $|\langle p, q \rangle| < 1$.

(4) $W$ is null if and only if $|\langle p, q \rangle| = 1$.

Theorem 5. Let $l$ be geodesic segment bounded by $p$ and $q$, then

(1) $l$ is hyperbola part if and only if $\langle p, q \rangle > 1$.

(2) $l$ is ellipse part if and only if $|\langle p, q \rangle| < 1$.

(3) $l$ is null line segment if and only if $\langle p, q \rangle = 1$.

(4) $l$ is impossible line segment if and only if $\langle p, q \rangle < -1$.

A generalized de Sitter triangle $\Omega$ can be seen as follows in Asmus [3]:

(1) If $\Omega \subset H^2$, $\Omega$ is called hyperbolic triangle

(2) If $\Omega \subset (-H^2)$, $\Omega$ is called antipodal hyperbolic triangle

(3) If $\Omega \subset S^n_1$, $\Omega$ is called proper de Sitter triangle

(4) Otherwise, $\Omega$ is called strange triangle

(5) If at least one edge is empty, then, $\Omega$ is called impossible triangle.

Let $i, j$ and $k$ be the number of spacelike, timelike and lightlike edges of a de Sitter triangle $\Delta_k$.

$3\Delta_0, 1\Delta_1, 1\Delta_0, 0\Delta_1, 0\Delta_2, 1\Delta_0$ are the proper de Sitter triangles with null edges, and are called Lucilateral, Multiple, Photosceles with space-like base, Photosceles with time-like, Bimetrical Chronosceles, Bimetrical Chorosceles Triangle, respectively (see Figure 1).

$0\Delta_0, 0\Delta_3, 2\Delta_1$ and $1\Delta_2$ are the proper de Sitter triangles with non-null edges, and are called Spatiolateral, Tempolateral, Chorosceles, Chronosceles Triangle, respectively (see Figure 2).
Figure 1. (a) lucilateral (b) photosceles with space-like base (c) bimetrical chronosceles (d) photosceles with time-like base (e) bimetrical chorosceles (f) multiple triangle

Figure 2. (a) contractible spatiolateral (b) non-contractible spatiolateral (c) tempolateral (d) chorosceles (e) chronosceles
2.1. Pseudo-angle and Angle in Lorentz Space $\mathbb{R}^3_1$.

By [4], we have the following definition

**Definition 1.** The pseudo-norm $\|u\|_p$ of $u \in \mathbb{R}^3_1$ is defined by the complex number

$$\|u\|_p = \sqrt{\langle u, u \rangle} \in \mathbb{R}^+ \cup \{0\} \cup \mathbb{R}^+i,$$  

where $i = \sqrt{-1}$.

Then, we have

$$\|u\|_p = \begin{cases} 0, & u \text{ null} \\ \sqrt{|\langle u, u \rangle|}, & u \text{ space-like} \\ i \sqrt{|\langle u, u \rangle|}, & u \text{ time-like} \end{cases}$$

**Definition 2.** Let $u, v$ be unit non-null vectors in $\mathbb{R}^3_1$, then the complex number $\phi(u, v)$ satisfying $\cos \phi = \frac{\langle u, v \rangle}{\|u\|_p \|v\|_p}$ is called pseudo-angle between $u$ and $v$ ([4] and [9]).

Let $u, v$ be unit non-null vectors, and let $U$ be the subspace $sp\{u, v\}$ of $\mathbb{R}_1^3$. Then we have the following definitions.

**Definition 3.** The angle $\theta$ between vectors $u$ and $v$ in $\mathbb{R}^3_1$ is given by

$$\theta = \begin{cases} \arccos(\langle u, v \rangle), & U \text{ is space-like} \\ \arccosh(-\langle u, v \rangle), & U \text{ is time-like and } \langle u, u \rangle \langle v, v \rangle = 1 \text{ and } \langle u, v \rangle < -1 \\ \arccos(\langle u, v \rangle), & U \text{ is time-like and } \langle u, v \rangle = 1 \text{ and } \langle u, v \rangle > 1 \\ \arcsinh(\langle u, v \rangle), & U \text{ is time-like and } \langle u, v \rangle = -1 \\ \end{cases}$$

By Definition 2 and Definition 3, we give $\phi$ depend on $\theta$.

**Definition 4.**

1. If $u, v$ are unit space-like vectors and $\theta > 0$, then

$$\phi = \begin{cases} \pi - i\theta, & \langle u, v \rangle < -1 \\ i\theta, & \langle u, v \rangle > 1 \\ \theta \in [0, \pi], & \langle u, v \rangle \in [-1, 1] \end{cases}$$

2. If $u, v$ are unit unit time-like vectors and $\theta > 0$, then

$$\phi = \begin{cases} -i\theta, & u \text{ and } v \text{ are same time cone} \\ \pi + i\theta, & u \text{ and } v \text{ different time cone} \end{cases}$$

If $u$ unit space-like, $v$ unit time-like and $\theta \in \mathbb{R}$ then

$$\phi = \frac{\pi}{2} + i\theta$$
3. Girard Type Theorems for Proper de Sitter Triangle with Non-null Edges

Let $\triangle$ be a proper de Sitter triangle with non null edge, and let $p_1, p_2, p_3$ be vertices of $\triangle$. Let $V^k_j, V^l_j$ and $u_j$ be unit tangent vectors at vertex $p_j$ pointing in the direction vertices $p_k, p_l$ and the unit outer normal to the edge plane opposite to vertex $p_j$, $j = 1, 2, 3$. Then one can see that

$$\langle V^k_j, V^l_j \rangle = \langle u_k, u_l \rangle, \quad k \neq j \neq l, \quad j, k, l = 1, 2, 3. \quad (3.1)$$

By [3, Remark 2.9], we have $V^k_j$ is time-like (space-like) if and only if $u_k$ is space-like (time-like).

**Theorem 6.** Let $\triangle$ be a triangle with vertices $p_1, p_2, p_3$ in $S^2_1$, and let $\phi_{kl}$ be pseudo-angle between unit tangent vectors $V^k_j$ and $V^l_j$ at vertex $p_j$ pointing in the direction vertices $p_k, p_l$. Then the area of $\triangle$ is

$$\nabla = (\phi_{12} + \phi_{13} + \phi_{23} - \pi) \in \mathbb{R}^+ i.$$

**Proof.** See [4, Theorem 5]. $\square$

3.1. Girard’s Theorem for $3\triangle_0$. $3\triangle_0$ is called contractible if the sum of lengths of edges is less then $2\pi$, non-contractible if greater then $2\pi$. The edges of non-contractible triangle are satisfy triangle inequality while contractible one are not [3].

A non-contractible and contractible triangle has a polar triangle being **hyperbolic and strange triangle with one time like and the other two edges are impossible**. Thus a contractible triangle has one and only one vertex at which the unit outer normals to the edge planes are in same time cone, but non-contractible spatiolateral triangle has three vertices at which the unit outer normals to the edge planes are in same time cone.

One can obtain a non-contractible spatiolateral triangle $\Omega$ from a contractible spatiolateral de Sitter triangle $\Omega$ by taking the antipodal of vertex at which the unit outer normals to the edge planes are in same time cone.

Let $3\triangle_0$ be a spatiolateral triangle with vertex set $\{p_1, p_2, p_3\}$, and let $V^k_j$ and $V^l_j$ be unit tangent vectors at vertex $p_j$ pointing in the direction vertex $p_k$ and $p_l$. Denote by $u_j$ the unit outer normal to the edge plane opposite to vertex $p_j$, $j = 1, 2, 3$. Then $V^k_j, V^l_j$ are space-like, and $u_k, u_l$ are time-like vectors. By equation $(3.1)$,

$$\cos \phi_{23} = \langle V^1_1, V^3_1 \rangle; \quad \cos \phi_{13} = \langle V^1_2, V^3_2 \rangle; \quad \cos \phi_{12} = \langle V^1_3, V^2_3 \rangle \quad (3.2)$$
where $\phi_{kl}$ is the pseudo-angle at vertex $p_j$ of $^0_{3\Delta_0}$.

**Theorem 7.** Let $^0_{3\Delta_0}$ be contractible spatiolateral de Sitter triangle with the measure of interior angles $\theta_{23}, \theta_{12}, \theta_{13}$ and let $\theta_{23}$ be interior angle at vertex $p_1$ which the unit outer normals to the edge planes are in same time cone. Then the area $V$ of $^0_{3\Delta_0}$ is

$$V = -\theta_{23} + \theta_{12} + \theta_{13}.$$ 

**Proof.** By [3, Corollary 4.10], contractible spatiolateral triangle has a strange polar triangle whose two vertices in the same time cone other one in different time cone. Then by equation (3.1),

$$\langle V_1^2, V_1^1 \rangle < -1, \quad \langle V_2^1, V_2^3 \rangle > 1, \quad \langle V_3^1, V_3^3 \rangle > 1.$$

By Definition 4, we have

$$\phi_{23} = \pi - i\theta_{23}, \quad \phi_{13} = i\theta_{13}, \quad \phi_{12} = i\theta_{12}.$$

By Theorem 6, we obtain

$$\nabla = i(-\theta_{23} + \theta_{13} + \theta_{12}), \quad (3.3)$$

which is completes the proof. \qed

By equation (3.1) and Definition 3, we see that

$$\theta_{23} = \arccos h(-\langle V_1^2, V_1^3 \rangle), \quad \theta_{13} = \arccos h(\langle V_2^1, V_2^3 \rangle), \quad \theta_{12} = \arccos h(\langle V_3^1, V_3^2 \rangle).$$

By Theorem 7, the following corollary has been proved.

**Corollary 1.** If $u_j$ is the unit outer normal to the edge plane opposite to vertex $p_j$ of contractible spatiolateral triangle $^0_{3\Delta_0}$, then

$$V = -\arccos h(-\langle V_1^2, V_1^3 \rangle)) + \arccos h(\langle V_2^1, V_2^3 \rangle) + \arccos h(\langle V_3^1, V_3^2 \rangle).$$

**Remark 1.** By definition of non-contractible spatiolateral triangle, one can easily see that it is not restrict an area on de Sitter plane.

### 3.2. Girard’s Theorem for $^0_{\Delta_3}$

Let $^0_{\Delta_3}$ be a tempolateral de Sitter triangle with vertices $p_1, p_2, p_3$.

Then by [3, Lemma 3.3], $^0_{\Delta_3}$ has one and only one vertex at which the time-like unit tangent vectors are in different time cone. No loss of generality we choose that vertex $p_1$. By [3, Theorem 5.9 ], the interior angle $\theta_{23}$ at vertex $p_1$ is greater then the sum of other two interior angle $\theta_{12}, \theta_{13}$ of $^0_{\Delta_3}$. That is

$$\theta_{23} - \theta_{12} - \theta_{13} > 0.$$
Let $V^j_k$ be unit tangent vector at vertex $p_j$ pointing in the direction vertex $p_k$. Then by equation (3.1), we have

$$
\langle V^2_1, V^3_1 \rangle > 1 , \langle V^2_1, V^3_2 \rangle < -1 , \langle V^2_3, V^3_2 \rangle < -1.
$$

(3.4)

By Definition 4, we have

$$
\phi_{23} = \pi + i\theta_{23} , \ \phi_{13} = -i\theta_{13} , \ \phi_{12} = -i\theta_{12}.
$$

(3.5)

By Theorem 6, we have

$$
\nabla = i(\theta_{23} - \theta_{12} - \theta_{13}) \in \mathbb{R}^+i.
$$

Therefore, we have been proved the following theorem.

**Theorem 8.** Let $\triangle_3(p_1,p_2,p_3)$ be a tempolateral de Sitter triangle with the greater angle at vertex $p_1$. Then the area $V$ of $\triangle_3(p_1,p_2,p_3)$ is

$$
V = \theta_{23} - \theta_{12} - \theta_{13}.
$$

By Definition 3 and equation (3.4), we have

$$
\theta_{23} = \arccos h(\langle V^2_1, V^3_1 \rangle), \ \theta_{13} = \arccos h(-\langle V^2_1, V^3_2 \rangle), \ \theta_{12} = \arccos h(-\langle V^2_3, V^3_2 \rangle).
$$

Now we have the following result.

**Corollary 2.** Let $\triangle_3(p_1,p_2,p_3)$ be a tempolateral de Sitter triangle with the greater angle at vertex $p_1$. Then the area $V$ of $\triangle_3(p_1,p_2,p_3)$ is given by

$$
V = \arccos h(\langle V^2_1, V^3_1 \rangle) - \arccos h(-\langle V^2_1, V^3_2 \rangle) - \arccos h(-\langle V^2_3, V^3_2 \rangle).
$$

### 3.3. Girard’s Theorem for $\triangle_1$.

Let $\triangle_1$ be chorosceles de Sitter triangle with vertices $p_1, p_2, p_3$ and let $V^k_i$ and $V^l_i$ be tangent vectors at vertex $p_i$ pointing in the direction vertex $p_k$ and $p_l$. Denote by $u_i$ the unit outer normal to the edge plane opposite to vertex $p_i, i = 1, 2, 3$. Without no loss of generality, we choose the time-like edge opposite to vertex $p_1$. Then by Theorem 5, we have

$$
|\langle p_1, p_2 \rangle| < 1 , \ |\langle p_1, p_3 \rangle| < 1 , \ |\langle p_2, p_3 \rangle| > 1
$$

(3.6)

Since the polar triangle of $\triangle_1$ is strange triangle, we have

$$
\langle V^2_1, V^3_1 \rangle = \langle u_2, u_3 \rangle > 1
$$

or in another way, $u_2$ and $u_3$ are in different time cone. Thus the polar triangle of $\triangle_1$ has vertices as follows $u_1 \in S^2_1$ and $u_2, u_3 \in H^2 \cup (-H^2)$. 
Theorem 9. Let $\triangle_1$ be horosphere de Sitter triangle with interior angles $\theta_{23}, \theta_{13}$ and $\theta_{12}$. Then the area $V$ of $\triangle_1$ is given by

$$V = \theta_{23} + \theta_{12} + \theta_{13}.$$ 

Proof. By $u_2, u_3$ are time-like vectors in different time cone and Definition 4, we see that

$$\phi_{23} = i\theta_{23}, \quad \phi_{13} = \frac{\pi}{2} + i\theta_{13}, \quad \phi_{12} = \frac{\pi}{2} + i\theta_{12}$$

By Theorem 6, we obtain

$$\nabla = i(\theta_{23} + \theta_{12} + \theta_{13}).$$

This completes the proof. \qed

By Definition 3, we have

$$\theta_{23} = \text{arccosh}(\langle V_2^1, V_1^3 \rangle), \quad \theta_{13} = \text{arcsinh}(\langle V_2^1, V_2^3 \rangle), \quad \theta_{12} = \text{arcsinh}(\langle V_3^1, V_3^2 \rangle).$$

By using these equations in Theorem 9, we obtain the following corollary.

Corollary 3. The area $V$ of horosphere de Sitter triangle $\triangle_1$ is given by

$$V = \text{arccosh}(\langle V_2^1, V_1^3 \rangle + \text{arcsinh}(\langle V_2^1, V_2^3 \rangle) + \text{arcsinh}(\langle V_3^1, V_3^2 \rangle)).$$

3.4. Girard’s Theorem for $\triangle_2$.

Let $\triangle_2$ be horosphere de Sitter triangle with vertices $p_1, p_2, p_3$ and let $V_j^k$ and $V_j^l$ be unit tangent vectors at vertex $p_j$ pointing in the direction vertex $p_k$ and $p_l$. Denote by $u_j$ the unit outer normal to the edge plane opposite to vertex $p_j$, $j = 1, 2, 3$. Without no loss of generality, we choose the space-like edge opposite to vertex $p_1$. Then by Theorem 4, we have

$$\langle p_2, p_3 \rangle < 1, \quad \langle p_1, p_2 \rangle > 1, \quad \langle p_1, p_3 \rangle > 1.$$

The polar triangle of $\triangle_2$ is strange triangle with space-like edge bounded by $u_2, u_3 \in S_1^2$. Therefore we have

$$\langle V_1^2, V_1^3 \rangle = \langle u_2, u_3 \rangle < -1.$$

By Definition 4, we have

$$\phi_{23} = -i\theta_{23}, \quad \phi_{12} = \frac{\pi}{2} + i\theta_{12}, \quad \phi_{13} = \frac{\pi}{2} + i\theta_{13}.$$

By Theorem 6, we obtain

$$\nabla = i(-\theta_{23} + \theta_{12} + \theta_{13}).$$
So, we have been proved the following theorem.

**Theorem 10.** Let \( \triangle_{-2}^{0} \) be choronosceles de Sitter triangle with interior angles \( \theta_{23}, \theta_{13} \) and \( \theta_{12} \). Then the area \( V \) of \( \triangle_{-2}^{0} \) is given by

\[
V = -\theta_{23} + \theta_{12} + \theta_{13}.
\]

By Definition 3, we have

\[
\theta_{23} = \arccosh(-\langle V_{1}^{2}, V_{1}^{3} \rangle), \quad \theta_{13} = \arcsinh(\langle V_{2}^{1}, V_{2}^{3} \rangle), \quad \theta_{12} = \arcsinh(\langle V_{3}^{1}, V_{3}^{2} \rangle).
\]

Then, we obtain the following corollary.

**Corollary 4.** The area \( V \) of choronosceles de Sitter triangle \( \triangle_{-2}^{0} \) is given by

\[
V = -\arccosh(-\langle V_{1}^{2}, V_{1}^{3} \rangle) + \arcsinh(\langle V_{2}^{1}, V_{2}^{3} \rangle) + \arcsinh(\langle V_{3}^{1}, V_{3}^{2} \rangle).
\]

**References**

[1] Weets, J.R., The Shape of Space 2nd ed., Marcel Decker, Inc.New York, Basel, 2002.
[2] Birman, G.S. and Nomizu, K., Trigonometry in Lorentzian Geometry, Amer. Math. Monthly, 91(9), 543-549, 1984.
[3] Asmus, I., Duality Between Hyperbolic and de Sitter Geometry, J. Geom., 96(1-2),11-40, 2009.
[4] Dzan, J.J., Trigonometric Laws on Lorentzian sphere \( S_{1}^{2} \), J. Geom., 24(1),6-13, 1985.
[5] Suárez-Peiró, E., A Schläfli Differential Formula for Simplices in Semi-Riemannian Hyperquadrics, Pacific J. Math., 194(1), 229-255, 2000.
[6] Ratcliffe, J.G., Foundations of Hyperbolic Manifolds, Graduate texts in Mathematics, 149, Springer, NewYork, 2006.
[7] López, R., Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space, Int. Electron. J. Geom., 7(1)), 44-17, 2014.
[8] O’Neill, B. Semi-Riemannian Geometry, Pure and Applied Mathematics,103, Academic Press, Inc., NewYork, 1983.
[9] Enzyklopädie der Elementer Mathematik, Vol.V(Geometrie), Berlin,1979.