Systematic study of pasta nuclei in neutron stars with families of the empirical nuclear equations of state

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Abstract. The structures of pasta nuclei in the crust of a neutron star are investigated systematically by using a family of the equations of state (EOSs) of nuclear matter within the framework of the Thomas-Fermi theory. This EOS family is given as a function of the incompressibility $K_0$ of symmetric nuclear matter and the parameter $L$ that characterizes the density dependence of the symmetry energy. The remaining saturation parameters are chosen so that the masses and radii of stable nuclei calculated within the framework are consistent with the empirical values. The structure of a neutron-rich nucleus is determined by the local pressure balance of asymmetric nuclear matter, which is mainly controlled by $L$ and $K_0$. With this EOS family, we calculate the structure of pasta nuclei in the crust of a neutron star, and examine its ($K_0, L$) dependence. It is found that the existence of pasta nuclei is mainly determined by $L$. For sufficiently small $L$, as the matter density increases, the shape of the nuclei changes from sphere to cylinder, slab, cylindrical hole and spherical hole successively. For larger $L$, this sequence of the shape change may end before reaching spherical hole. All the pasta shape can exist for $L \leq 70$ MeV, while no pasta shape appears for $L \geq 110$ MeV. The density width of each pasta phase, the energy and volume fraction at the onset of each pasta phase have a clear correlation with $L$ while the onset density also has an appreciable $K_0$ dependence.

1. Introduction
The existence of pasta nuclei in the crust of neutron stars has been studied by various authors including us with various empirical interactions \cite{1, 2, 3}. So far it has been shown that the existence depends on the interactions adopted in the calculations \cite{3}. It is now of even greater importance to study the elastic and superfluid properties of matter in the pasta phases because quasi-periodic oscillations in the afterglow of giant flares observed from soft-gamma repeaters suggests the possible presence of pasta nuclei in the deepest region of the crust \cite{4}.

The energy per nucleon of nuclear matter at density $n$ close to the saturation density $n_0$ of symmetric nuclear matter with neutron excess $\alpha$ close to zero can be expanded as

$$w(n, \alpha) = w_0 + \frac{K_0}{2} u^2 + \frac{Q_0}{6} u^3 + \cdots + \left[ S_0 + Lu + \frac{K_{sym}}{2} u^2 + \frac{Q_{sym}}{6} u^3 + \cdots \right] \alpha^2 + \cdots \quad (1)$$
Here, \( u = (n - n_0)/(3n_0) \), and the coefficients \( w_0 \) and \( K_0 \) are the saturation energy and incompressibility of symmetric nuclear matter. The energy per nucleon of neutron matter can also be expanded as

\[
w(n, \alpha = 1) = w_{n0} + L_{n0}u + \frac{K_{n0}}{2}u^2 + \frac{Q_{n0}}{6}u^3 + \ldots
\]

(2)

The structure of a neutron-rich nucleus is determined from the local pressure equilibrium. The lowest order pressure parameters are the incompressibility \( K_0 \) of symmetric nuclear matter and the parameter \( L \) that characterizes the density dependence of the symmetry energy. Two of the present authors (OI) have already developed a family of the equations of state (OI EOSF), which were constructed in such a way to reproduce an average behavior of masses and radii of \( \beta \)-stable laboratory nuclei [8], and thereby shown that the existence of pasta nuclei has to depend mainly on \( L \) and that some nuclear shapes other than sphere could appear if \( L \leq 100 \) MeV [9]. Due to insufficient numerical accuracy, however, it was difficult to predict which nuclear shape could appear as a function of \( L \).

In this study, we remedy over-fitting of the OI EOSF parameters and examine the \((K_0, L)\) dependence of the other saturation parameters and correlations among them. Then, we examine how the pasta nuclear structures in neutron stars depend on the uncertain saturation parameters \((K_0, L)\) using the improved OI EOS family.

2. Macroscopic nuclear model

We begin with a macroscopic nuclear model, which is based on the nuclear energy density functional and constructed in such a way as to reproduce empirical masses and radii of stable nuclei. The energy of an atomic nucleus (a unit cell of matter in the neutron-star crust) is composed of the nuclear energy \( W_N \), the Coulomb energy, and the electron rest mass energy (the relativistic electron kinetic energy).

The nuclear energy \( W_N \) is written as the integral of the local energy density,

\[
W_N = \int d^3r \left[ \epsilon_0(n_n(r), n_p(r)) + F_0 \nabla n(r)^2 + m_n n_n(r) + m_p n_p(r) \right].
\]

(3)

Here, \( n_n \) (\( n_p \)) is the local neutron (proton) density, \( n = n_n + n_p \) is the total nucleon density and \( m_n \) (\( m_p \)) is the neutron (proton) rest mass. The symbol \( \epsilon_0 = w(n, \alpha) \times n \) denotes the energy density of uniform nuclear matter. The surface energy stems not only from the local energy density \( \epsilon_0(n_n, n_p) \) but also from the gradient term in Eq. (3), in which the coefficient \( F_0 \) represents the finite range effects of the nuclear interactions.

The homogeneous energy density \( \epsilon_0 \) is written as the sum of the free kinetic energy, and the potential energy density, which is the weighted sum of the symmetric matter part \( v_s(n) \) and the neutron matter part \( v_n(n)\):

\[
\epsilon_0(n_n, n_p) = \frac{3}{5} (3\pi^2)^{2/3} \left( \frac{\hbar^2}{2m_n} n_n^{5/3} + \frac{\hbar^2}{2m_p} n_p^{5/3} \right) + (1 - \alpha^2)v_s(n) + \alpha^2 v_n(n).
\]

(4)

Here, the potential energy densities are parametrized as

\[
v_s(n) = a_1 n^2 + \frac{a_2 n^3}{1 + a_3 n}, \quad v_n(n) = b_1 n^2 + \frac{b_2 n^3}{1 + b_3 n}.
\]

(5)

The coefficients \( a_1 \) (\( b_1 \)) and \( a_2 \) (\( b_2 \)) are the two- and three-body energy coefficients for symmetric (neutron) matter, respectively. The coefficients \( a_3 \) (\( b_3 \)) and \( b_3 \) control the many-body \((N \geq 4)\) energies and work to soften the EOS at high densities [10]. It is hard to constrain the
$b_3$ value for neutron matter from stable nuclei. We set $b_3 = 1.58632$ fm$^3$ which is chosen to fit the neutron matter EOS by Friedman and Pandharipande [11] so as to give a reasonable value to the many body energy for neutron matter [2, 8].

In the present model, the values of the six parameters ($a_1, a_2, a_3, b_1, b_2$, and $F_0$) are fitted to reproduce the smoothed behavior of empirical masses and radii of stable nuclei [2]. The same data let us know the saturation properties of the nuclear energy and density, which in turn lead to a liquid drop picture of nuclei. In fact, the values of the four coefficients in the liquid drop mass formula (volume, surface, symmetry and Coulomb) are well constrained from those data. By comparison, in the present model, we have two extra degrees of freedom in potential parameters. In our systematic studies that started from [8], we choose $K_0$ and $L$ as the two extra degrees of freedom. Specifically, for a given set of $(K_0, L)$, the six parameter values are fitted to the empirical nuclear masses and radii [8]. In this way, all the interaction parameters are determined as functions of $(K_0, L)$. Therefore, we can calculate any physical quantity as a function of $(K_0, L)$.

In the present study, special attention is paid to avoidance of overfitting of the parameter values so that the parameters have smooth $(K_0, L)$ dependences. Figure 1 shows the values of the six interaction parameters ($a_1, a_2, a_3, b_1, b_2$, and $F_0$) as functions of $(K_0, L)$. It is interesting to note that the potential parameters ($a_1, a_2, a_3$) of symmetric matter depend mainly on $K_0$ and the potential parameters ($b_1, b_2$) of neutron matter depend mainly on $L$. The gradient energy parameter $F_0$ depends on both $K_0$ and $L$ appreciably.

3. Saturation parameters and correlations among them
The values of the saturation parameters in Eqs. (1) and (2) can be calculated from Eq. (4). The improved EOS family exhibits better and clearer correlations among the interaction parameters and saturation parameters than the previous version of OI EOSF. As shown in Fig. 2, for symmetric nuclear matter, the saturation density $n_0$ correlates with $K_0$, while the saturation energy $w_0$ does with the gradient energy coefficient $F_0$. The symmetry energy coefficient $S_0$ and the incompressibility parameter $K_{n0}$ of neutron matter have a clear correlation with $L$. It is noted that the $S_0 - L$ and $K_{n0} - L$ correlations are consistent with those obtained from phenomenological interactions by Tews et al. [12]. Therefore, the improved version of the OI EOSF is expected to reasonably cover uncertainties of the phenomenological interactions of contemporary use.

4. Existence of pasta nuclei
Using the improved OI EOSF, it is confirmed that the existence of pasta nuclei is mainly determined by $L$. For sufficiently small $L$, the stable nuclear shape changes from sphere (sp), cylinder (cy), slab (sl), cylindrical hole (cyh) and spherical hole (sph) successively with increasing density. For larger $L$, this sequence of the shape change may end before reaching sph. Figure 3 (left) summarizes the last nuclear shape as a function of $L$. All the pasta shape can exist for $L < 70$ MeV, while no pasta shape appears for $L > 110$ MeV.

As shown in Fig. 3 (right), the volume fraction of the nucleus (proton cluster) at the onset of each pasta shape is consistent with the prediction by the compressible-liquid-drop model, which is independent of the nuclear interaction [5, 6, 7]. This suggests that the volume fraction is a good measure of the nuclear shape as in the pasta-like structure in diblock copolymer systems [13].

Figures 4 and 5 display the density and energy at the onset of each pasta shape, which are dependent mainly on $L$. While these absolute values also have an appreciable $K_0$ dependence, the density width of each pasta phase and the relative energy to uniform matter show a much clearer dependence on $L$. 
5. Summary and outlook

We obtained the improved values of the interaction and saturation parameters over the previous version of the OI EOSF: the results are essentially smooth functions of \((K_0, L)\). With this numerically improved OI EOSF, we examined correlations among the saturation parameters and the properties of pasta nuclei quantitatively as functions of \((K_0, L)\). From the viewpoint of the \(S_0 - L\) and \(K_{n0} - L\) correlations, the improved OI EOSF seems to reasonably cover the contemporary phenomenological neutron matter EOSs. The resultant properties of pasta nuclei are mainly dominated by \(L\). This seems natural because the matter considered here is very close to neutron matter.

The neutron-star matter EOS that can be obtained from the improved OI EOSF is consistent with the one constrained from GW170817 via Bayesian analysis [14] up to about \(n = 2n_0\) but too soft at higher densities, as can be seen from Fig. 6 showing two extreme cases of EOS E \((K_0 = 320 \text{ MeV}, L = 43 \text{ MeV})\) and Bp \((K_0 = 280 \text{ MeV}, L = 98 \text{ MeV})\) in comparison with the EOS constrained from GW170817. This behavior persists even if we set \(b_3 = 0\) to make the neutron matter EOS stiffest (see EOSs Bp and E with \(b_3 = 0\) shown by dashed lines in Fig. 6). This is partly because the dense symmetric matter EOS adopted here is too soft: the neutron-star matter EOS can become soft with increase of the proton fraction even if the neutron matter EOS is reasonably stiff.
Figure 2. The correlations of among the saturation parameters. The shaded areas are calculated with fitting formulae in Ref. [12], which enclose 68.3% of their accepted interactions.

Figure 3. The last nuclear shape before uniform matter appears (left) and the volume fraction of the nucleus at the onset of each shape (right), which are plotted as functions of $L$. In the right panel, the liquid drop prediction [7] is shown by dashed lines.

Figure 4. The onset density (left) and the density width of each pasta phase (right) as functions of $L$. 
Figure 5. The energy (left) and the energy difference with respect to uniform matter (right) at the onset of each pasta phase as functions of $L$.

We are now finalizing an OI EOS and nuclear table that can be obtained from the improved OI EOSF and preparing for its publication. We are also developing still more improved versions of the OI EOSF by modifying the high density neutron matter EOS (OI-H EOSF), the high density symmetric matter EOS (OI-Q EOSF) and the range of the interactions (OI-G EOSF), and eventually examining their effects to pasta nuclei.

Figure 6. The neutron-star matter EOS obtained in the present analysis (red and blue lines), which is compared with the one constrained from GW170817 in Ref. [14].

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