Abel rings and super-strongly clean rings

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Abstract In this note, we first show that a ring $R$ is Abel if and only if the $2 \times 2$ upper triangular matrix ring $(\begin{array}{cc} R & R \\ 0 & R \end{array})$ over $R$ is quasi-normal. Next, we give the notion of super-strongly clean ring (that is, an Abel clean ring), which is inbetween uniquely clean rings and strongly clean rings. Some characterizations of super-strongly clean rings are given.

Keywords Abel rings · quasi-normal rings · clean rings · super-strongly clean rings · strongly exchange rings

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All rings considered in this paper are associative rings with identity. Let $R$ be a ring, write $E(R)$, $U(R)$, $J(R)$ and $Z(R)$ to denote the set of all idempotents, the set of units, the Jacobson radical and the center of $R$, respectively.

A ring $R$ is called Abel if $E(R) \subseteq Z(R)$. The study of Abel rings seems to originate from [6]. In the next fifty-five years many scholars have studied Abel rings such as [1], [4], [5], [7], [9], [10] and [11]. A ring $R$ is called quasi-normal if $eR(1-e)Re = 0$, for each $e \in E(R)$. Clearly, Abel rings are quasi-normal, but the converse is not true in general by [10].

According to [7], a ring $R$ is called exchange if for each $a \in R$ there exists $e \in E(R)$ such that $e = ab$ and $1 - e = (1 - a)c$, for some $b, c \in R$ and $R$ is said to be:

1. clean if, for each $a \in R$, $a = u + e$ for some $u \in U(R)$ and $e \in E(R)$;
2. uniquely clean if the representation of $a$ in (1) is unique;

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strongly clean if $ue = eu$ holds in the representation of $a$ mentioned above (1). In [7], it is shown that clean rings are exchange, but the converse is not true unless $R$ is Abel.

In this note we introduce two new members of the clean family, that is super-strongly clean rings and superclean rings, the relations among these rings are discussed.

1 Some characterizations of Abel rings

**Theorem 1.1** A ring $R$ is Abel if and only if $S_2(R) = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ is quasi-normal.

**Proof.** First, we assume that $R$ is Abel and $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in E(S_2(R))$. Then

$$\begin{align*}
a^2 &= a, \\
c^2 &= c, \\
b &= ab + bc. \tag{1.3}
\end{align*}$$

Now, for any $B = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}, C = \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \in S_2(R)$, one has

$$AB(1 - A)CA = \begin{pmatrix} ax(1-a)ua & ax(1-a)ub + ax(1-a)vc \\
0 & -axbwc + ay(1-c)wc + bz(1-c)wc \\
& cz(1-c)wc \end{pmatrix}. \tag{1.4}$$

Since $R$ is Abel, (1.1), (1.2) and (1.3) imply $a, c \in Z(R)$. Hence

$$ax(1-a)ua = ax(1-a)ub = ax(1-a)vc = 0,$$

$$cz(1-c)wc = ag(1-c)wc = bz(1-c)wc = 0.$$ 

By (1.3), one gets

$$axbwc = ax(ab + bc)wc = axabwc + axbcwc = axbwc + axbwc \tag{1.5}$$

this gives

$$axbwc = 0.$$ 

Thus $AB(1 - A)CA = 0$ and so $S_2(R)$ is quasi-normal.

Conversely, assume that $S_2(R)$ is quasi-normal and $e \in E(R)$. Then $\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} \in E(S_2(R))$, so for each $x \in R$, one has

$$\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -e \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} = 0$$

that is, $\begin{pmatrix} 0 & ex(1-e) \\ 0 & 0 \end{pmatrix} = 0$. Thus $ex(1-e) = 0$, for each $x \in R$, it follows that $eR(1-e) = 0$, for each $e \in E(R)$. Using $1-e$ instead of $e$, one obtains that $(1-e)Re = 0$, this gives $ae = eae = ea$, for each $a \in R$. Hence $R$ is Abel. \quad \square
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[10, Theorem 2.9] implies that a ring \( R \) is quasi-normal if and only if \( T_2(R) = \{ \left( \begin{array}{cc} a & b \\ 0 & a \end{array} \right) | a, b \in R \} \) is quasi-normal. Hence, by Theorem 1.1, we have the following corollary.

**Corollary 1.2** A ring \( R \) is Abel if and only if

\[
TW_4(R) = \left\{ \left( \begin{array}{ccc} a_1 & a_2 & a_4 \\ 0 & a_3 & a_6 \\ 0 & 0 & a_1 \\ 0 & 0 & 0 \end{array} \right) | a_1, a_2, a_3, a_4, a_5, a_6 \in R \right\}
\]

is quasi-normal.

**Proposition 1.3** A ring \( R \) is Abel if and only if \( T_2(R) \) is Abel.

**Proof.** Let \( R \) be an Abel ring and \( A = \left( \begin{array}{cc} a & b \\ 0 & a \end{array} \right) \in E(T_2(R)) \). Then

\[a^2 = a\]  

(1.6)

and

\[b = ab + ba.\]  

(1.7)

Since \( R \) is Abel, (1.6) implies \( a \in Z(R) \). Hence, by (1.7), one gets \( b = ab + ba \) and \( ab = a^2b + a^2b = ab + ab \), this gives \( ab = 0 \) and so \( b = 0 \).

Now, for any \( B = \left( \begin{array}{cc} x & y \\ 0 & x \end{array} \right) \in T_2(R) \), one has

\[AB = \left( \begin{array}{cc} ax & ay \\ 0 & ax \end{array} \right) = \left( \begin{array}{cc} xa & ya \\ 0 & xa \end{array} \right) = \left( \begin{array}{cc} x & y \\ 0 & x \end{array} \right) \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right) = BA.
\]

Thus \( T_2(R) \) is Abel. The converse is clear. \( \square \)

**Corollary 1.4** A ring \( R \) is Abel if and only if

\[
W_4(R) = \left\{ \left( \begin{array}{ccc} a_1 & a_2 & a_4 \\ 0 & a_1 & a_4 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{array} \right) | a_1, a_2, a_3, a_4, a_5, a_6 \in R \right\}
\]

is quasi-normal.

**Proof.** It follows from Corollary 1.4 and the fact that \( TW_4(R) \cong W_4(R) \). \( \square \)
2 Super-strongly clean rings

Let $R$ be a ring and $a \in R$. Recall that $a$ is said to be:

1. exchange if there exists $e \in E(R)$ such that $e \in aR$ and $1 - e \in (1-a)R$;
2. clean if $a = u + e$, for some $u \in U(R)$ and $e \in E(R)$;
3. uniquely clean if the representation of (2) is unique;
4. strongly clean if $a$ has a representation as (2) such that $ea = e$;
5. strongly exchange if there exists $e \in E(R)$ such that $e = ab = ba$ and $1 - e = (1-a)c = c(1-a)$, for some $b, c \in R$.

A ring $R$ has the property $P$ if all elements of $R$ have it, where $P$ refers to exchange, clean, uniquely clean, strongly clean and strongly exchange.

An element $a$ of $R$ is called super-strongly clean if $a$ is clean and $ea = e$ whenever $a = u + e$, for any $u \in U(R)$ and $e \in E(R)$. A ring $R$ is called super-strongly clean if every element of $R$ is super-strongly clean. Clearly, if $R$ is a clean Abel ring, then $R$ is super-strongly clean.

**Lemma 2.1** The following conditions are equivalent for a ring $R$:

(i) $R$ is Abel;
(ii) Every idempotent element of $R$ is super-strongly clean.

**Proof.** For $e \in E(R)$, one has that $e = (2e - 1) + (1 - e)$, hence $e$ is a clean. Thus (i) $\implies$ (ii) is trivial.

(ii) $\implies$ (i) Let $e \in E(R)$ and $a \in R$. Write $u = eae - ea + 1 - 2e$ and $g = ea - eae + e$, then $ue^2 = 1$, $g \in E(R)$ and $1 - e = u + g$. By (ii), one has $ug = gu$, this implies $ea = eae$, thus $e(1-e) = 0$, for each $a \in R$ and so $R$ is Abel. $\square$

By Lemma 2.1, we have the following theorem.

**Theorem 2.2** $R$ is a super strongly clean ring if and only if $R$ is an Abel clean ring.

By Theorem 2.2 and [8, Lemma 4], we have the following corollary.

**Corollary 2.3** Uniquely clean rings are super-strongly clean.

The following example illustrates that the converse of Corollary 2.3 is not true in general.

**Example 2.1** Let $R = \mathbb{Z}_5$. Then $U(R) = \{[1],[2],[3],[4]\}$ and $E(R) = \{[0],[1]\}$. Since $[0] = [4] + [1]; [1] = [1] + [0]; [2] = [1] + [1]; [3] = [2] + [1]; [4] = [3] + [1] = [4] + [0]$, $R$ is a clean ring but not uniquely clean. Since $R$ is a commutative ring, by Theorem 2.2, $R$ is super-strongly clean.

**Lemma 2.4** If $R$ is an Abel exchange ring, then $R$ is strongly exchange.

**Proof.** Let $a \in R$. Since $R$ is an exchange ring, there exists $e \in E(R)$ such that $e = ax$ and $1 - e = (1-a)y$, for some $x, y \in R$. Let $b = xe$ and $c = y(1-e)$, then $e = ab$ and $1 - e = (1-a)c$. Clearly, $b = be = bab$. Write $g = ba$, then $g = g^2$ and $g = ba = (be)a$. Since $R$ is Abel, $g = bae = ge$. Since $e = ee = a(ba)b = a(ba)b = gab = gab = ge$, $g = e$, that is $e = ab = ba$. Similarly, one can show that $1 - e = (1-a)c = c(1-a)$. Thus $R$ is strongly exchange. $\square$
By Lemma 2.4, Theorem 2.2, [3] and [7] one has the following corollary.

**Corollary 2.5** Super-strongly clean rings are strongly clean.

Recall that a ring $R$ is $\pi$-regular (strongly $\pi$-regular [2]) if, for each $a \in R$, there exists $n = n(a) \geq 1$ such that $a^n \in a^nRa^n$ ($a^n \in a^{n+1}R$).

The following example illustrates that the converse of Corollary 2.5 is not true in general.

**Example 2.2** Let $F$ be a field and $R = \left( \begin{array}{cc} F & F \\ 0 & F \end{array} \right)$. Let $0 \neq A = \left( \begin{array}{cc} a & b \\ 0 & c \end{array} \right) \in R$.

Case 1. If $a = c = 0$, then $A^2 = 0 \in A^3R$.

Case 2. If $a \neq 0$ and $c = 0$, then $A = A^2B$, where

$$B = \left( \begin{array}{cc} a^{-1} & a^{-2}b - a^{-1}b \\ 0 & 1 \end{array} \right).$$

Case 3. If $a \neq 0$ and $c \neq 0$, then $A \notin U(R)$ and $A = A^2B$ where $B = \left( \begin{array}{cc} 1 & 1 \\ 0 & c^{-1} \end{array} \right)$.

Thus $R$ is a strongly $\pi$-regular ring. By [2], $R$ is strongly clean. Since $R$ is not Abel, by Theorem 2.2, $R$ is not super-strongly clean.

By Theorem 2.2 and [7], one has the following corollary.

**Corollary 2.6** $R$ is a super-strongly clean ring if and only if $R$ is an Abel exchange ring.

A clean ring $R$ is called superclean if $R$ is also a quasi-normal ring. Clearly, super-strongly clean rings are superclean.

**Lemma 2.7** $R$ is a clean ring if and only if $S_2(R)$ is a clean ring.

**Proof.** First assume that $R$ is clean and $A = \left( \begin{array}{cc} a & b \\ 0 & c \end{array} \right) \in S_2(R)$. Since $R$ is clean, $a = u+f, c = v+g$ for some $u, v \in U(R)$ and $f, g \in E(R)$. Clearly, $A = \left( \begin{array}{cc} u & b \\ 0 & v \end{array} \right) + \left( \begin{array}{cc} f & 0 \\ 0 & g \end{array} \right)$,

where $\left( \begin{array}{cc} u & b \\ 0 & v \end{array} \right) \in U(S_2(R))$ and $\left( \begin{array}{cc} f & 0 \\ 0 & g \end{array} \right) \in E(S_2(R))$. Thus $S_2(R)$ is clean.

Next assume that $S_2(R)$ is clean and $a \in R$. Then

$$\left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} u & w \\ 0 & v \end{array} \right) + \left( \begin{array}{cc} f & h \\ 0 & g \end{array} \right),$$

where $\left( \begin{array}{cc} u & w \\ 0 & v \end{array} \right) \in U(S_2(R))$ and $\left( \begin{array}{cc} f & h \\ 0 & g \end{array} \right) \in E(S_2(R))$. By computing, one has $a = u + f$, where $u \in U(R)$ and $f \in E(R)$. Hence $R$ is clean. □

By Theorem 1.1, Theorem 2.2 and Lemma 2.7, we have the following theorem.
**Theorem 2.8**  
R is super-strongly clean if and only if $S_2(R)$ is superclean.

**Remark 2.1**  
Clearly, for any ring $R$, $S_2(R)$ is not Abel, hence Theorem 2.2 and Theorem 2.8 imply that superclean rings need not be super-strongly clean. Therefore superclean rings are proper generalization of super-strongly clean rings.

A natural question is that: *Is any superclean ring also strongly clean?*

**Proposition 2.9**  
Let $F$ be a field. Then $R = \begin{pmatrix} F & F & F \\ F & F & 0 \\ 0 & 0 & F \end{pmatrix}$ is strongly $\pi$-regular and so $R$ is strongly clean.

**Proof.** Let $0 \neq A = \begin{pmatrix} a & b & c \\ 0 & d & s \\ 0 & 0 & t \end{pmatrix} \in R$. We shall divide the following several cases to prove:

- **Case 1.** If $a = d = t = 0$, then $A^3 = 0$ and so $A^3 = A^4$.
- **Case 2.** If $a = d = 0$ and $t \neq 0$, then $A^2 = \begin{pmatrix} 0 & 0 & bs + ct \\ 0 & 0 & st \\ 0 & 0 & t^2 \end{pmatrix}$ and $A^3 = \begin{pmatrix} 0 & 0 & bst + ct^2 \\ 0 & 0 & st^2 \\ 0 & 0 & t^3 \end{pmatrix}$, hence $A^2 = A^3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}$.
- **Case 3.** If $a = t = 0$ and $d \neq 0$, then $A^2 = A^3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & d^{-1} & 0 \\ 0 & 0 & d^{-1} \end{pmatrix}$.
- **Case 4.** If $a = 0$ and $dt \neq 0$, then $A = A^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & d^{-1} & -d^{-1}st^{-1} \\ 0 & 0 & t^{-1} \end{pmatrix}$.
- **Case 5.** If $a \neq 0$ and $d = t = 0$, then $A^2 = A^3 \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}$.
- **Case 6.** If $ad \neq 0$ and $t = 0$, then
  \[
  A = A^2 \begin{pmatrix} a^{-1} & -a^{-1}bd^{-1} & -a^{-1}bd^{-1}s + a^{-2}c - a^{-2}bs \\ 0 & d^{-1} & d^{-2}s \\ 0 & 0 & 0 \end{pmatrix}.
  \]
- **Case 7.** If $at \neq 0$ and $d = 0$, then
  \[
  A = A^2 \begin{pmatrix} a^{-1} & -a^{-2}b - a^{-1}ct^{-1} - a^{-2}bst^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}.
  \]
- **Case 8.** If $adt \neq 0$, then $A \in U(R)$ and $A = A^2A^{-1}$.

Thus $R$ is strongly $\pi$-regular. $\Box$
By [10, P1858], one knows that the ring appeared in Proposition 2.9 is not quasi-normal, hence it is not superclean. Thus there exists a strongly clean ring which is not superclean.

It is easy to show that a ring $R$ is clean if and only if $T_2(R)$ is clean. Hence, by [10, Theorem 2.9], we have the following proposition.

**Proposition 2.10** $R$ is a superclean ring if and only if $T_2(R)$ is a superclean ring.

According to [10, Proposition 4.1], a quasi-normal ring $R$ is clean if and only if $R$ is exchange. Hence one has the following proposition.

**Proposition 2.11** $R$ is a superclean ring if and only if $R$ is a quasi-normal exchange ring.

**Proposition 2.12** If $R$ is a superclean ring then $R/J(R)$ is super-strongly clean.

**Proof.** Since $R$ is superclean, $R$ is clean and quasi-normal, this implies idempotents can be lifted modulo $J(R)$. Let $\bar{a} \in E(R)$ where $\bar{R} = R/J(R)$. Then, there exists $e \in E(R)$ such that $e - a \in J(R)$. Since $R$ is quasi-normal, $eR(1 - e)Re = 0$, this implies $aR(1 - a)Ra = 0$. Since $R$ is semiprime, $aR(1 - a) = 0$, this gives $\bar{R}$ is Abel. Hence $R/J(R) = \bar{R}$ is super-strongly clean because $\bar{R}$ is clean. ☐

The following example illustrates the converse of Proposition 2.12 is not true in general.

**Example 2.3** Let $F$ be a field and $R = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}$. Clearly, $J(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $R/J(R)$ is a super-strongly clean ring. Since $R$ is not quasi-normal, $R$ is not superclean by Proposition 2.12.

Finally, we give a characterization of local rings.

**Proposition 2.13** $R$ is a local ring if and only if $R$ is a clean ring and $R/J(R)$ has no nonzero zero divisors.

**Proof.** The necessity is clear.

The sufficiency: let $e \in E(R)$, then in $\bar{R} = R/J(R)$, $e(1 - \bar{e}) = \bar{0}$, by hypothesis, $\bar{e} = 0$ or $1 - \bar{e} = 0$, this gives $e \in J(R)$ or $1 - e \in J(R)$, so $e = 0$ or $1 - e = 0$. Hence $E(R) = \{0, 1\}$. Now, let $a \in R$. If $a \notin J(R)$, then there exists $b \in R$ such that $1 - ab \notin U(R)$. Since $R$ is clean, $1 - ab = u + e$ for some $u \in U(R)$ and $e \in E(R)$. Clearly $e = 1$, so $ab = -u$. Since $0 \neq -bu^{-1}a \in E(R)$, $-bu^{-1}a = 1$, this implies $a \in U(R)$. Thus $R$ is local. ☐

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