Some results in the exceptional set

of

Twin Prime Problem

Goldtwe Anihe

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Abstract

In the paper, there are new found methods to determine the range of every exceptional element in exceptional set, we can solve Twin primes problem and Goldbach Conjecture problem basically.

Key Word

exceptional set Twin primes problem Goldbach Conjecture problem

Address: Gongnong Sicun 103--301
Shanghai, P. R. of China

Email: goldtwe@hotmail.com

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1 §1

As the writer of "Sieve Methods" (i.e.[1]) indicates, for Twin primes problem and Goldbach Conjecture problem, they belong to the same problem.

We can expand all conclusions (e.g. the result in [2], these results before the 1980s.) of Goldbach Conjecture problem to solve the problem of the twin prime, and vice-versa.

The problem of exceptional set exists in Goldbach Conjecture problem (cf.[2]) after the three primes is proven. Therefore, the problem of exceptional set also exists in Twin primes problem, and their distribution of the exceptional element are same.

We have the expression of exceptional set in Goldbach Conjecture problem as follows.

Let $x$ be a larger positive integer.

\[
\text{non-Goldbach number := } \{ n \neq p_1 + p_2, 2 < p_1 \leq n, 2 < p_2 \leq n, 2|n\}
\]

\[
E_g(x) := \{ a : a = \text{non-Goldbach number } n, 2 < n \leq x \}
\]

and for the number of all element in $E_g(x)$ set, we express it as $E_g(x)$.

In $(x/2, x]$, for all even number $n$ except $E_g(x)$ exceptional values, (cf.[2,3]) we have

\[
D_g(n) = 2C(n) \frac{n}{\ln n} + O\left(\frac{x(\ln \ln x)^3}{\ln^4 x}\right)
\]

(1)

where

\[
C(n) = \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|n} \frac{p-1}{p-2}
\]

and $D_g(n)$, cf.(10)

\[
E_g(x) \ll \frac{x}{\ln^4 x}
\]

(2)
where, \( A \) be an any give positive, and the contained constant depending on \( A \).

Here, In Twin primes problem,

\[
C(n) = \prod_{p > 2} \left(1 - \frac{1}{(p - 1)^2}\right)
\]

For the exceptional set of Twin primes problem, to the sieve function eye, we have

\[
A_g := \{ a : a = n - p, 2 < p \leq n, 2|n\}
\]

where \( A_g \) which is in the sieve function \( S_g(A_g, P, z) \) of Goldbach Conjecture problem be a set.

\[
A_t := \{ a : a = p + 2, 2 < p \leq n, 2|n\}
\]

where \( A_t \) which is in the sieve function \( S_t(A_t, P, z) \) of Twin primes problem be a set.

As for \( E_g(x) \), an even number \( n \) is an exceptional element when all \( n - p \) are not the prime in \( A_g \).

To \( E_t(x) \) of Twin primes problem, it is the same with \( E_g(x) \) of Goldbach Conjecture problem. i.e. An even number \( n \) is an exceptional element when all \( p + 2 \) are not the prime in \( A_t \). This conclusion developed from the \( E_g(x) \) of Goldbach Conjecture problem. Their distribution of the exceptional element are same. On this, we have

\[
D_t(n) = 2 \prod_{p > 2} \left(1 - \frac{1}{(p - 1)^2}\right) \frac{n}{\ln^2 n} + O\left(\frac{x(\ln \ln x)^3}{\ln^3 x}\right)
\]

where \( D_t(n) \), cf. (11).

\[
E_t(x) \ll \frac{x}{\ln^4 x}
\]

where, \( A \) be an any give positive, and the contained constant depending on \( A \).

2 §2

We quote the increment to the sieve function of Twin primes problem. And the fundamental properties of sieve function as follows.

\[
(i) S(A, P, z_1) \leq S(A, P, z_2) \quad , \quad (z_1 \geq z_2)
\]
(ii) $0 \leq S(A, P, z) \leq |A|$ \hspace{1cm} (6)

(iii) $S(A + \Delta A, P, z) = S(A, P, z) + S(\Delta A, P, z)$ \hspace{1cm} (7)

Here, $|A|$ be a number of all element in set $A$. $\Delta A$ be the non-empty subset of set $(A + \Delta A)$, and we also call it after an increment of set $A$. Its number is $|\Delta A|$. To $A$ and $\Delta A$, their cap is an empty set $\phi$, i.e.

$$A \cap \Delta A = \phi$$

Then

$$S(A + \Delta A, P, z_1) = S(A, P, z_1) + S(\Delta A, P, z_1)$$

$$< S(A, P, z_2) + |\Delta A|$$ \hspace{1cm} (8)

Where $z_1 > z_2$.

So, for three expressions in (8), if we know two expressions of they, then another of they has the upper or lower bound.

When we apply (8), (3), (4) and Drawer principle to the exceptional set of Twin primes problem, for $2(E_t(x) + 1)$ natural numbers in a closed interval of the natural number, \{2|M, [M, M + 2E_t(x)]], (x/2 < M, M + 2E_t(x) \leq x]\}, if an even number which is one of two endpoints is an exceptional element, then the non-exceptional element which is an even number in this closed interval must exists. And the difference between this two even numbers less than or equal to $2E_t(x)$. Besides, for §3, the exceptional element is the right endpoint. On this

$$|\Delta A| \leq 2E_t(x)$$

and $S(A, P, z_1)$ or $S(A, P, z_2)$ which one of they in (8) be the $D_t(n)$ of (3). At this very moment, $A > 4$ in (4), we have

$$\frac{|\Delta A|}{D_t(n)} \rightarrow 0 \hspace{1cm} , \hspace{1cm} x \rightarrow \infty$$

So, we must know that an exceptional element of Twin primes problem which is the another of they has the upper or lower bound. As for an exceptional element of Goldbach Conjecture problem, by the §1, it also has the upper or lower bound.
When even number \( n_1 > n_2 \), if the expressions of \( D_t(n_1) - D_t(n_2) \) as (15), (16) and (17), then we can determine the range of every exceptional element in exceptional set of Twin primes problem. So, for \( \alpha \) and \( \beta \) sequence of number as follows.

\[
\begin{array}{ccccccc}
  i & 1 & 2 & 3 & \ldots & n-1 & n \\
  \alpha & 1 & 2 & 3 & \ldots & n-1 & n, 2|n \\
  \beta_g & n-1 & n-2 & n-3 & \ldots & 1 \\
  \beta_t & 3 & 4 & 5 & \ldots & n+1 & n+2 \\
\end{array}
\]

First, by Eratosthenes’ sieve, we have all primes in \( \alpha \) and \( \beta \) sequence of number. Second, we map the process that sieve \( \beta \) sequence of number into \( \alpha \) sequence of number. The manner of mapping be \( n_{\alpha i} + n_{\beta_j} = n \) or \( n_{\beta_j} - n_{\alpha i} = 2 \). And we also sieve out these number which was mapped in a sequence of number. Then, we have (10) and (11). Let \( P \) is the prime set in \([2, n]\), its number \(|P|\) is \( \pi(n) \). i.e.

\[ P := \{p : 1 < p \leq n\} \quad , \quad |P| = \pi(n) \]

For \( \alpha \) sequence of number, we have the prime theorem as follows.

\[ \pi(n) = n + \sum_{d|\prod_{p \in P} p} \mu(d)\left[\frac{n}{d}\right] + O(\pi(\sqrt{n})) \quad (9) \]

After the mapping, for Goldbach Conjecture problem

\[ D_g := \{a \in P : a = n - p, 2 < p \leq n, 2|n\} \]

\[ D_g(n) = |D_g| \]

(cf.\( P \) and \(|P|\)) i.e.

\[ D_g(n) = \pi(n) + \sum_{d|\prod_{p \in P} p} \mu(d)\pi(n, d, n \mod d) + O(\pi(\sqrt{n})) \quad (10) \]

By the same manner, for Twin primes problem

\[ D_t := \{a \in P : a = p + 2, 2 < p \leq n, 2|n\} \]
$$D_t(n) = |D_t|$$

(cf. $P$ and $|P|$) i.e.

$$D_t(n) = \pi(n) + \sum_{d \mid \prod_{p \in P} p} \mu(d) \pi(n, d, d - 2) + O(\pi(\sqrt{n}))$$  \hfill (11)

It may be seen, when $n \to \infty$, we have the following

$$D_g(n) \longrightarrow \pi(n) + \sum_{d \mid \prod_{p \in P} p} \mu(d) \pi(n, d, n \pmod{d})$$ \hfill (12)

$$D_t(n) \longrightarrow \pi(n) + \sum_{d \mid \prod_{p \in P} p} \mu(d) \pi(n, d, d - 2)$$ \hfill (13)

where $d - 2$ of the second term does not change with $n$, and this second term be the element number of a union of set. i.e.

$$- \sum_{d \mid \prod_{p \in P} p} \mu(d) \pi(n, d, d - 2) = \left| \bigcup_{2 < p \leq \sqrt{n}} \{ a \in P : a \equiv p - 2 \pmod{p} \} \right|$$ \hfill (14)

(cf. $P$ and $|P|$).

So, when even number $n_1 > n_2$, Let

$$P_1 := \{ p : 2 < p \leq n_1 \}$$

$$P_2 := \{ p : 2 < p \leq n_2 \}$$

$$\Delta P := \{ p : n_2 < p \leq n_1 \}$$

Then

$$D_t(n_1) - D_t(n_2) \longrightarrow \pi(n_1) - \pi(n_2) -$$
$$- \left| \bigcup_{2 < p \leq \sqrt{n_1}} \{ a \in P_1 : a \equiv p - 2 \pmod{p} \} \right| +$$
$$+ \left| \bigcup_{2 < p \leq \sqrt{n_2}} \{ a \in P_2 : a \equiv p - 2 \pmod{p} \} \right|$$ \hfill (15)

(cf. (13))
Suppose,

\[\text{set1} := \{a \in P_2 : a \equiv p - 2 \pmod{p}, \sqrt{n_2} < p \leq \sqrt{n_1}\}\]

\[\text{set2} := \{a \in \Delta P : a \notin \Delta P \cap \{b \in \Delta P : b \equiv p - 2 \pmod{p}, 2 < p \leq \sqrt{n_1}\}\}\]

Then

\[D_t(n_1) - D_t(n_2) \rightarrow -|\text{set1}| + |\text{set2}|\]

\[< |\text{set1}| + |\text{set2}| \quad (16)\]

(cf. \(P\) and \(|P|\)), where

\[\text{set1} \in \bigcup_{2 < p \leq \sqrt{n_1}} \{a \in P_1 : a \equiv p - 2 \pmod{p}\}\]

Now we evaluate the value of |set1| and |set2|.

\[0 \leq \text{set2} < |\Delta P| < n_1 - n_2 \quad (cf.(6), P \text{ and } |P|)\]

\[0 \leq \text{set1} < \frac{n_2}{p_{\text{min}}} (\sqrt{n_1} - \sqrt{n_2}) < [\sqrt{n_1n_2} - n_2] < (n_1 - n_2)\]

where \(p_{\text{min}} \rightarrow \sqrt{n_2}\), cf. \(\sqrt{n_2} < p \leq \sqrt{n_1}\)

So,

\[0 \leq D_t(n_1) - D_t(n_2) < O(n_1 - n_2) \quad (17)\]

When \(A > 4\) in (4), every exceptional element in Twin primes problem are evaluated as the ending part of \(\S 2\). Thus we solve Twin primes problem basically.

By the same manner, every exceptional element in Goldbach Conjecture problem are also evaluated. Thus we also solve Goldbach Conjecture problem basically.

(end)

4 References

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