Strong* convergence of quantum channels

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Abstract

In arXiv:1712.03219 the existence of a strongly (pointwise) converging sequence of quantum channels that can not be represented as a reduction of a sequence of unitary channels strongly converging to a unitary channel is shown. In this work we give a simple characterization of sequences of quantum channels that have the above representation. The corresponding convergence is called the strong* convergence, since it is related to the convergence of selective Stinespring isometries of quantum channels in the strong* operator topology.

Some properties of the strong* convergence of quantum channels are considered. It is shown that for Bosonic Gaussian channels the strong* convergence coincides with the strong convergence.

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1 Introduction

The Stinespring theorem implies that any quantum channel \( \Phi \) from a system \( A \) to a system \( B \) can be represented as

\[
\Phi(\rho) = \text{Tr}_E V_\Phi \rho V_\Phi^* ,
\]

where \( V_\Phi \) is an isometrical embedding of the input Hilbert space \( \mathcal{H}_A \) into the tensor product of the output Hilbert space \( \mathcal{H}_B \) and some Hilbert space \( \mathcal{H}_E \) typically called environment \([8, 21]\). By using the Stinespring representation \((1)\) for any quantum channel \( \Phi \) from \( A \) to \( B \) one can find such quantum systems \( D \) and \( E' \) that

\[
\Phi(\rho) = \text{Tr}_{E'} U_\Phi \rho \otimes \sigma_0 U^*_\Phi,
\]

where \( \sigma_0 \) is a pure state in \( \mathcal{S}(\mathcal{H}_D) \) and \( U_\Phi \) is an unitary operator from \( \mathcal{H}_{AD} \) onto \( \mathcal{H}_{BE'} \) \([8]\). In the case \( A = B \) one can take \( D = E' = E \) in \((2)\) \([8, 12, 21]\).

Representation \((2)\) called unitary dilation of a quantum channel \( \Phi \) allows to consider this channel as a reduction of some unitary (reversible) channel between larger quantum systems \([8, 12, 21]\).

In study of quantum channels and their information characteristics it is necessary to consider topology (convergence) on the set of all quantum channels between given quantum systems. In finite-dimensions the uniform convergence generated by the diamond-norm metric between quantum channels is widely used \([1, 15], [21, \text{Ch.9}]\). But this convergence is too strong for analysis of real variations of infinite-dimensional channels \([18, 23]\). In this case it is natural to use the strong convergence of quantum channels generated by the family of seminorms \( \Phi \mapsto \|\Phi(\rho)\|_1, \rho \in \mathcal{S}(\mathcal{H}_A) \) \([11, 22]\). The strong convergence of a sequence \( \{\Phi_n\} \) of channels to a channel \( \Phi_0 \) means that

\[
\lim_{n \to \infty} \Phi_n(\rho) = \Phi_0(\rho) \text{ for all } \rho \in \mathcal{S}(\mathcal{H}_A).
\]

It is easy to see that a sequence of unitary channels \( \rho \mapsto U_n \rho U_n^* \) strongly converges to a unitary channel \( \rho \mapsto U_0 \rho U_0^* \) if and only if the sequence \( \{U_n\} \) converges to the operator \( U_0 \) in the strong operator topology.\(^1\) A characterization of the strong convergence of arbitrary quantum channels is presented in \([17, \text{Theorem 1}]\), it states that a sequence \( \{\Phi_n\} \) of quantum channels strongly

\(^1\)The strong operator topology on the set of unitary operators coincides with the weak, \( \sigma \)-weak and \( \sigma \)-strong operator topologies \([3]\).
converges to a quantum channel \( \Phi_0 \) if and only if there is a quantum system \( E \) and a sequence \( \{ V_{\Phi_n} \} \) of isometries from \( \mathcal{H}_A \) into \( \mathcal{H}_{BE} \) converging to an isometry \( V_{\Phi_0} \) in the strong operator topology such that (1) holds for all \( n \).

The nontrivial part of this characterization can be treated as continuity of the multi-valued map \( \Phi \mapsto V_{\Phi} \) (where \( V_{\Phi} \) is the isometry from representation (1)) w.r.t. the strong convergence topology on the set of channels and the strong operator topology on the set of isometries. It turns out that this continuity does not imply continuity of the multi-valued map \( \Phi \mapsto U_{\Phi} \) (where \( U_{\Phi} \) is the unitary from representation (2)) w.r.t. these topologies. This means the existence of a sequence \( \{ \Phi_n \} \) of quantum channels strongly converging to a quantum channel \( \Phi_0 \) that can not be represented in form (2) with some sequence \( \{ U_{\Phi_n} \} \) of unitaries converging to a unitary operator \( U_{\Phi_0} \) in the strong operator topology [17].

The above discontinuity of the unitary dilation is a specific feature of the strong convergence: by using the arguments from the proof of Theorem 1 in [14] one can show that any sequence \( \{ \Phi_n \} \) of quantum channels converging to a quantum channel \( \Phi_0 \) w.r.t. the diamond norm can be represented in form (2) with some sequence \( \{ U_{\Phi_n} \} \) of unitaries converging to a unitary operator \( U_{\Phi_0} \) in the operator norm topology [17, Proposition 4].

The discontinuity of the unitary dilation with respect to the strong convergence of quantum channels means the existence of a strongly converging sequence of channels that has no physical sense within the standard interpretation of a channel as a reduced unitary evolution of some larger system. Mathematically, this means that the strong convergence of quantum channels is too weak for describing physical perturbations of quantum channels.

The aim of this note is to describe all sequences of quantum channels that can be represented as a reduction of a sequence of unitary channels strongly converging to a unitary channel. We call the corresponding convergence of quantum channels the strong\(^*\) convergence, since it is related to the convergence of selective Stinespring isometries of quantum channels in the strong\(^*\) operator topology [3].

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\(^2\)It seems reasonable to assume that all physical perturbations of a unitary channel \( \rho \mapsto U\rho U^* \) is properly described by continuous deformations of the unitary \( U \) in the strong operator topology (coinciding in this case with the weak, \( \sigma \)-weak and \( \sigma \)-strong operator topologies [3]). I would be grateful for any comments concerning this question.
2 Preliminaries

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$ with the operator norm $\| \cdot \|$ and $\mathfrak{T}(\mathcal{H})$ the Banach space of all trace-class operators on $\mathcal{H}$ with the trace norm $\| \cdot \|_1$. Let $\mathfrak{S}(\mathcal{H})$ be the set of quantum states (positive operators in $\mathfrak{T}(\mathcal{H})$ with unit trace) [8, 21].

A quantum channel $\Phi$ from a system $A$ to a system $B$ is a completely positive trace preserving linear map from $\mathfrak{T}(\mathcal{H}_A)$ into $\mathfrak{T}(\mathcal{H}_B)$ [8, 21]. For any quantum channel $\Phi : A \to B$ the Stinespring theorem (cf. [19]) implies existence of a Hilbert space $\mathcal{H}_E$ and of an isometry $V_\Phi : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that representation (1) holds. The quantum channel

$$\mathfrak{T}(\mathcal{H}_A) \ni \rho \mapsto \tilde{\Phi}(\rho) = \text{Tr}_B V_\Phi \rho V_\Phi^* \in \mathfrak{T}(\mathcal{H}_E)$$

is called complementary to the channel $\Phi$ [8, Ch.6]. The complementary channel is uniquely defined up to isometrical equivalence, i.e. if $\tilde{\Phi}' : A \to E'$ is the channel defined by formula (1) via some other Stinespring isometry $V_{\tilde{\Phi}'} : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_{E'}$ then there exists a partial isometry $W : \mathcal{H}_E \to \mathcal{H}_{E'}$ such that $\tilde{\Phi}'(\rho) = W \tilde{\Phi}(\rho) W^*$ and $\tilde{\Phi}(\rho) = W^* \tilde{\Phi}'(\rho) W$ for all $\rho \in \mathfrak{S}(\mathcal{H}_A)$ [9].

The strong convergence of quantum channels is generated by the family of seminorms $\Phi \mapsto \| \Phi(\rho) \|_1$, $\rho \in \mathfrak{S}(\mathcal{H}_A)$ [11]. The strong convergence of a sequence $\{ \Phi_n \}$ of channels to a channel $\Phi_0$ means that

$$\lim_{n \to \infty} \Phi_n(\rho) = \Phi_0(\rho) \quad \text{for all} \quad \rho \in \mathfrak{S}(\mathcal{H}_A).$$

(5)

This convergence is more relevant for analysis of infinite-dimensional quantum channels than the diamond-norm convergence [18, 23]. Equivalent definitions of the strong convergence and its properties are described in [22].

If $\Phi$ is a linear bounded map from $\mathfrak{T}(\mathcal{H}_A)$ to $\mathfrak{T}(\mathcal{H}_B)$ then the map $\Phi^* : \mathfrak{B}(\mathcal{H}_B) \to \mathfrak{B}(\mathcal{H}_A)$ defined by the relation

$$\text{Tr}(\Phi(\rho)B) = \text{Tr}(\Phi^*(B)\rho) \quad \text{for all} \quad B \in \mathfrak{B}(\mathcal{H}_B)$$

(6)

is called dual to the map $\Phi$ [3, 16]. If $\Phi$ is a channel acting on quantum states, i.e. a channel in the Schrodinger picture, then $\Phi^*$ is a channel acting on quantum observables, i.e. a channel in the Heisenberg picture [8, 21].

The result in [7] implies that the trace-norm convergence in (5) is equivalent to the convergence of the sequence $\{ \Phi_n(\rho) \}$ to the state $\Phi_0(\rho)$ in the
weak operator topology. So, by noting that the set \( \mathcal{S}(\mathcal{H}_A) \) in (5) can be replaced by its subset consisting of pure states it is easy to show that the strong convergence of a sequence \( \{\Phi_n\} \) of quantum channels to a channel \( \Phi_0 \) means, in the Heisenberg picture, that

\[
\text{w- lim}_{n \to \infty} \Phi_n^*(B) = \Phi_0^*(B) \quad \text{for all } B \in \mathfrak{B}(\mathcal{H}_B),
\]

where \( \text{w-lim} \) denotes the limit in the weak operator topology in \( \mathfrak{B}(\mathcal{H}_A) \).

3 On sequences of quantum channels having strongly converging unitary dilations

The following theorem gives several criteria for existence of a strongly converging unitary dilation for a sequence of quantum channels.

**Theorem 1.** Let \( \{\Phi_n\}_{n \geq 0} \) be a sequence of quantum channels from \( A \) to \( B \). The following properties (i)-(v) are equivalent:

(i) there exist quantum systems \( D \) and \( E \), a sequence \( \{U_n\}_{n \geq 0} \) of unitary operators from \( \mathcal{H}_AD \) onto \( \mathcal{H}_BE \) and a sequence \( \{\sigma_n\} \) of states in \( \mathcal{S}(\mathcal{H}_D) \) converging to a state \( \sigma_0 \) such that \( \Phi_n(\rho) = \text{Tr}_E U_n \rho \otimes \sigma_n U_n^* \) for all \( n \geq 0 \) and \( \text{s-lim}_{n \to \infty} U_n = U_0 \);

(ii) there exist quantum systems \( D \) and \( E \), a sequence \( \{U_n\}_{n \geq 0} \) of unitary operators from \( \mathcal{H}_AD \) onto \( \mathcal{H}_BE \) and a pure state \( \sigma_0 \) in \( \mathcal{S}(\mathcal{H}_D) \) such that \( \Phi_n(\rho) = \text{Tr}_E U_n \rho \otimes \sigma_0 U_n^* \) for all \( n \geq 0 \) and \( \text{s-lim}_{n \to \infty} U_n = U_0 \);

(iii) there exist a quantum system \( E \) and a sequence \( \{V_n\}_{n \geq 0} \) of isometries from \( \mathcal{H}_A \) into \( \mathcal{H}_BE \) such that \( \Phi_n(\rho) = \text{Tr}_E V_n \rho V_n^* \) for all \( n \geq 0 \), \( \text{s-lim}_{n \to \infty} V_n = V_0 \) and \( \text{s-lim}_{n \to \infty} V_n^* = V_0^* \);

(iv) there exist a set of sequences \( \{A_i^n\}_{n \geq 0}, i \in I \), of operators from \( \mathcal{H}_A \) to \( \mathcal{H}_B \) such that \( \Phi_n(\rho) = \sum_{i \in I} A_i^n \rho [A_i^n]^* \) for all \( n \geq 0 \), \( \text{s-lim}_{n \to \infty} A_i^n = A_i^0 \) and \( \text{s-lim}_{n \to \infty} [A_i^n]^* = [A_i^0]^* \) for each \( i \in I \);

(v) \( \text{s-lim}_{n \to \infty} \Phi_n^*(B) = \Phi_0^*(B) \) for all \( B \in \mathfrak{B}(\mathcal{H}_B) \).

\(^3\text{s-lim}_{n \to \infty} X_n = X_0 \) denotes the strong convergence of a sequence \( \{X_n\} \) to operator \( X_0 \).

\(^4\)\( \Phi^* \) is the dual map to the channel \( \Phi \) defined by relation (6).
The equivalent properties (i)-(v) imply the strong convergence of the sequence \( \{ \Phi_n \} \) to the channel \( \Phi_0 \) (but the converse implication is not valid).

**Remark 1.** Property (v) in Theorem 1 can be replaced by the following property, which is more easily verified in some cases:

(v') the sequence \( \{ \Phi_n \} \) strongly converges to the channel \( \Phi_0 \) and

\[
\text{s-lim}_{n \to \infty} \Phi_n^*(B) = \Phi_0^*(B) \quad \text{for all } B \in \mathcal{B}_0,
\]

where \( \mathcal{B}_0 \) is a dense subset of \( \mathcal{B}(\mathcal{H}_B) \) in the strong operator topology.

This follows from the proof of the implication (v) \( \Rightarrow \) (iii) in Theorem 1.

**Proof of Theorem 1.** (i) \( \Rightarrow \) (iii). Since for any converging sequence of states in \( \mathcal{S}(\mathcal{H}_D) \) there is a converging sequence of purifications in \( \mathcal{S}(\mathcal{H}_D) \), where \( R \) is some system, and \( s\text{-lim}_{n \to \infty} U_n = U_0 \) implies \( s\text{-lim}_{n \to \infty} U_n \otimes I_R = U_0 \otimes I_R \), we may assume that the sequence \( \{ \sigma_n \} \) consists of pure states. Let \( \{ \tau_n \} \) be a sequence of unit vectors in \( \mathcal{H}_D \) converging to a unit vector \( \tau_0 \) such that \( \sigma_n = |\tau_n\rangle \langle \tau_n| \) for all \( n \geq 0 \). For each \( n \) let \( V_n : \mathcal{H}_A \to \mathcal{H}_{BE} \) and \( P_n : \mathcal{H}_A \to \mathcal{H}_{AD} \) be the operators defined by settings \( V_n|\varphi\rangle = U_n|\varphi \otimes \tau_n\rangle \) and \( P_n|\varphi\rangle = |\varphi \otimes \tau_n\rangle \) for any \( \varphi \in \mathcal{H}_A \). Then \( \Phi_n(\rho) = \text{Tr}_E V_n \rho V_n^* \) for all \( n \geq 0 \) and \( s\text{-lim}_{n \to \infty} V_n = V_0 \).

Since

\[
V_n^*|\psi\rangle \otimes |\tau_n\rangle = P_n V_n^* U_n |\psi\rangle = [P_n V_n^* U_n] U_n^* |\psi\rangle
\]

for any vector \( \psi \) in \( \mathcal{H}_{BE} \), to show that \( s\text{-lim}_{n \to \infty} V_n^* = V_0^* \) it suffices to note that \( s\text{-lim}_{n \to \infty} U_n^* = U_0^* \) and that the operator \( P_n V_n^* U_n \) is the orthogonal projector on the subspace \( \mathcal{H}_B \otimes \{ c\tau_n \} \) of \( \mathcal{H}_{AD} \) for each \( n \geq 0 \).

(iii) \( \Rightarrow \) (iv). Let \( \{ \tau_i \}_{i \in I} \) be a basic in \( \mathcal{H}_E \). For given \( n \) and \( i \) let \( A_n^i \) be the operator from \( \mathcal{H}_A \) to \( \mathcal{H}_B \) such that \( \langle \psi | A_n^i | \varphi \rangle = \langle \psi \otimes \tau_i | V_n | \varphi \rangle \) for any \( \varphi \in \mathcal{H}_A \) and \( \psi \in \mathcal{H}_B \). Then \( \Phi_n(\rho) = \sum_{i \in I} A_n^i \rho [A_n^i]^* \) for all \( n \geq 0 \). By noting that \( V_n|\varphi\rangle = \sum_{i \in I} A_n^i |\varphi \otimes \tau_i\rangle \) and \( V_n^* |\varphi \otimes \tau_i\rangle = [A_n^i]^* |\varphi\rangle \) for any \( i \) and \( \varphi \in \mathcal{H}_A \) it is easy to show that \( s\text{-lim}_{n \to \infty} A_n^i = A^0 \) and \( s\text{-lim}_{n \to \infty} [A_n^i]^* = [A^0]^* \) for each \( i \in I \).

(iv) \( \Rightarrow \) (iii). Let \( V_n|\varphi\rangle = \sum_{i \in I} A_n^i |\varphi \otimes \tau_i\rangle \) for any \( \varphi \in \mathcal{H}_A \), where \( \{ \tau_i \}_{i \in I} \) is a basic in an appropriate Hilbert space \( \mathcal{H}_E \). Then \( \Phi_n(\rho) = \text{Tr}_E V_n \rho V_n^* \) for all \( n \geq 0 \). Since \( s\text{-lim}_{n \to \infty} A_n^i = A^0 \) for all \( i \in I \), the sequence \( \{ V_n|\varphi\rangle \} \) weakly converges to the vector \( V_0|\varphi\rangle \). The norm convergence of this sequence follows

\footnote{See the proof of Proposition 2 in Section 5.}
from the fact that all the operators $V_n$ are isometries. Since $s\cdot \lim_{n \to \infty} [A_n^*]^* = [A_0]^*$ and $V_n^* |\varphi \otimes \tau_i\rangle = [A_i^*]^* |\varphi\rangle$ for all $i \in I$ and $n \geq 0$, the sequence $\{V_n^*\}$ strongly converges to the operator $V_0^*$.

(iii) $\Rightarrow$ (ii). Let $C$ and $D$ be any infinite-dimensional quantum systems and $\sigma_0 = |\tau_0\rangle \langle \tau_0|$, where $\tau_0$ is any unit vector in $\mathcal{H}_D$. If we identify the space $\mathcal{H}_A$ with the subspace $\mathcal{H}_A \otimes \{c\tau_0\}$ of $\mathcal{H}_{AD}$, then $\{V_n\}$ is a sequence of partial isometries from $\mathcal{H}_{AD}$ to $\mathcal{H}_{BEC} \cong \mathcal{H}_{AD}$ strongly converging to the partial isometry $V_0$ such that $V_n^* V_n = V_0^* V_0$, $\dim \ker V_n V_n^* = \dim \ker V_n^* V_n = +\infty$ for all $n \geq 0$ and $s\cdot \lim_{n \to \infty} V_n^* = V_0^*$. So, the existence of the sequence $\{U_n\}$ with the required properties (with the system $EC$ in the role of $E$) follows from Proposition 8 in the Appendix.

The implication (i) $\Rightarrow$ (v) is stated in Proposition 3 in [17]. Note that (v) directly follows from (iii), since $\Phi_n^*(B) = V_n^* B \otimes I_E V_n$ for any $B \in \mathfrak{B}(\mathcal{H}_B)$ and all $n \geq 0$. The implication (ii) $\Rightarrow$ (i) is trivial.

(v) $\Rightarrow$ (iii). Note that (v) imply the strong convergence of the sequence $\{\Phi_n\}$ to the channel $\Phi_0$, since this convergence is equivalent to (7). Assume first that the channel $\Phi_0$ has infinite Choi rank and $\Phi_0(\rho) = \text{Tr}_E V_0 \rho V_0^*$ is the minimal Stinespring representation of the channel $\Phi_0$ [8, Ch.6]. It means that the set

$$\{[B \otimes I_E] V_0 |\varphi\rangle | B \in \mathfrak{B}(\mathcal{H}_B), \varphi \in \mathcal{H}_A\} \quad (8)$$

is dense in $\mathcal{H}_{BE}$. By Lemma 1 below there is a sequence $\{V_n\}$ of isometries from $\mathcal{H}_A$ into $\mathcal{H}_{BE}$ strongly converging to $V_0$ such that $\Phi_n(\rho) = \text{Tr}_E V_n \rho V_n^*$ for all $n$. Since $\Phi_n^*(B) = V_n^* B \otimes I_E V_n$ for any $B \in \mathfrak{B}(\mathcal{H}_B)$, by using the condition $s\cdot \lim_{n \to \infty} V_n = V_0$ it is easy to show that $\lim_{n \to \infty} V_n^* |\psi\rangle = V_0^* |\psi\rangle$ for any vector $\psi$ from the set $[8]$. If the channel $\Phi_0$ has finite Choi rank then take any channel $\Psi : A \to C$ with infinite Choi rank and consider the sequence of channels

$$\tilde{\Phi}_n(\rho) = \frac{1}{2} \Phi_n(\rho) + \frac{1}{2} \Psi(\rho)$$

from $\mathfrak{S}(\mathcal{H}_A)$ into $\mathfrak{S}(\mathcal{H}_B \oplus \mathcal{H}_C)$ strongly converging to the channel $\tilde{\Phi}_0(\rho) = \frac{1}{2} \Phi_0(\rho) + \frac{1}{2} \Psi(\rho)$ with infinite Choi rank. It is easy to see that (v) implies

$$s\cdot \lim_{n \to \infty} \tilde{\Phi}_n^*(B) = \tilde{\Phi}_0^*(B) \quad \forall B \in \mathfrak{B}(\mathcal{H}_B \oplus \mathcal{H}_C).$$

By the above part of the proof property (iii) holds for the sequence $\{\tilde{\Phi}_n\}$, i.e. there exist a quantum system $E'$ and a sequence $\{\tilde{V}_n\}_{n \geq 0}$ of isometries from $\mathcal{H}_A$ into $(\mathcal{H}_B \oplus \mathcal{H}_C) \otimes \mathcal{H}_{E'}$ such that $\tilde{\Phi}_n(\rho) = \text{Tr}_{E'} \tilde{V}_n \rho \tilde{V}_n^*$ for all
n ≥ 0, $s$-lim $\tilde{V}_n = \tilde{V}_0$ and $s$-lim $\tilde{V}_n^* = \tilde{V}_0^*$. Let $P_B$ be the projector on the subspace $H_B$ of $H_B \oplus H_C$. Then

$$\Phi_n(\rho) = 2P_B\tilde{\Phi}_n(\rho)P_B = 2\text{Tr}_{E'}[P_B \otimes I_{E'}][\tilde{V}_n\rho\tilde{V}_n^*][P_B \otimes I_{E'}] \quad \forall n \geq 0.$$ 

Hence $V_n = \sqrt{2}[P_B \otimes I_{E'}]\tilde{V}_n$ is a Stinespring isometry for the channel $\Phi_n$ for all $n \geq 0$. Since $s$-lim $V_n = V_0$ and $s$-lim $V_n^* = V_0^*$, property (iii) holds for the sequence $\{\Phi_n\}$.

The last assertion of the theorem follows from the equivalent definition of the strong convergence and Corollary 3 in [17]. □

**Lemma 1.** Let $\{\Phi_n\}$ be a sequence of quantum channels from $A$ to $B$ strongly converging to a channel $\Phi_0$ with infinite Choi rank. For any given Stinespring isometry $V_0 : H_A \rightarrow H_{BE}$ of the channel $\Phi_0$ there is a sequence $\{V_n\}$ of isometries from $H_A$ into $H_{BE}$ strongly converging to $V_0$ such that $\Phi_n(\rho) = \text{Tr}_E V_n^*V_n^* \rho$ for all $n$.

**Proof.** Since $H_E$ is an infinite-dimensional Hilbert space, for each $n$ there exists an isometry $V_n : H_A \rightarrow H_{BE}$ such that $\Phi_n(\rho) = \text{Tr}_E V_n^*V_n^* \rho$. By the proofs of Lemma 1 and Theorem 1 in [17] (based on the results from [14]) there is a sequence $\{C_n\}$ of contractions in $\mathcal{B}(H_E)$ such that the sequence of isometries

$$\tilde{V}_n = [I_B \otimes C_n]V_n \oplus [I_B \otimes \sqrt{I_E - C_n^*C_n}]V_n$$

from $H_A$ into $H_B \otimes (H_E \oplus H_E)$ strongly converges to the isometry $V_0 \oplus 0$. By simple continuity arguments we may assume that all the operators $C_n$ are non-degenerate. Let $U_n$ be the isometry from the polar decomposition of $C_n$. Since the sequences $[I_B \otimes C_n]V_n$ and $[I_B \otimes \sqrt{I_E - C_n^*C_n}]V_n$ strongly converge to the isometry $V_0$ and to the zero operator correspondingly, it is easy to show that the sequence $W_n = (I_B \otimes U_n)V_n$ strongly converges to the isometry $V_0$. It is clear that $\Phi_n(\rho) = \text{Tr}_E W_n^*W_n^* \rho$ for all $n$. □

According to the operator theory terminology a sequence $\{T_n\}$ of operators from $H_A$ into $H_B$ is called **strongly** converging to an operator $T_0$ if $s$-$\lim_{n \rightarrow \infty} T_n = T_0$ and $s$-$\lim_{n \rightarrow \infty} T_n^* = T_0^*$ [3]. So, Theorem 1 states, in particular, that the existence of strongly converging sequence of unitary dilations is equivalent to the existence of strongly** converging sequence of Stinespring isometries. This motivates the following

**Definition 1.** A sequence $\{\Phi_n\}$ of quantum channels is called **strongly** converging to a channel $\Phi_0$ if the equivalent properties (i)-(v) in Theorem 1 hold.
Corollary 3 in [17] implies that the strong* convergence of quantum channels is stronger than the strong convergence. The difference between these types of convergence is best seen on the Heisenberg picture: as mentioned at the end of Section 2 the strong convergence of a sequence \( \{\Phi_n\} \) to a channel \( \Phi \) can be defined as
\[
\lim_{n \to \infty} \Phi_n^*(B) = \Phi_0^*(B) \quad \text{for all } B \in \mathcal{B}(\mathcal{H}_B),
\]
(the limit in the weak operator topology) while the strong* convergence of this sequence means that
\[
\lim_{n \to \infty} \Phi_n^*(B) = \Phi_0^*(B) \quad \text{for all } B \in \mathcal{B}(\mathcal{H}_B).
\]
(9)

The below example shows, in particular, that the strong* convergence is substantially weaker than the uniform (diamond norm) convergence.

**Example.** Let \( \Phi \) be an arbitrary quantum channel from \( A \) to \( B \) and \( \{P_n\} \) a sequence of finite rank projectors in \( \mathcal{B}(\mathcal{H}_B) \) strongly converging to the unit operator \( I_{\mathcal{H}_B} \). Let
\[
\Phi_n(\rho) = P_n\Phi(\rho)P_n + [\text{Tr}(I_{\mathcal{H}_B} - P_n)\Phi(\rho)]\sigma
\]
for all \( n \), where \( \sigma \) is a given state in \( \mathcal{S}(\mathcal{H}_B) \). It is clear that the sequence \( \{\Phi_n\} \) strongly converges to the channel \( \Phi \), but it does not converge uniformly to \( \Phi \) in general (it suffices to consider the case \( A = B \), \( \Phi = \text{Id}_{\mathcal{H}_A} \)). Sequences of this type are used in [11] for approximation of infinite-dimensional quantum channels by channels with finite-dimensional output system. Since the map \( B \mapsto \Phi^*(B) \) is continuous w.r.t. the strong operator topology, we have
\[
\lim_{n \to \infty} \Phi_n^*(B) = \lim_{n \to \infty} \Phi_n^*(P_nBP_n + [I_{\mathcal{H}_B} - P_n]\text{Tr}B\sigma) = \Phi^*(B) \quad \forall B \in \mathcal{B}(\mathcal{H}_B).
\]
So, the sequence \( \{\Phi_n\} \) strongly* converges to the channel \( \Phi \).

By using (9) as the simplest definition of the strong* convergence it is easy to show that this convergence is preserved under basic manipulations with quantum channels.

**Corollary 1.** Let \( \{\Phi_n\} \) and \( \{\Psi_n\} \) be sequences of quantum channels from \( A \) to \( B \) and from \( C \) to \( D \) correspondingly that strongly* converge to quantum channels \( \Phi_0 \) and \( \Psi_0 \).
A) The sequence \{\Phi_n \otimes \Psi_n\} of channels from AC to BD strongly* converges to the channel \Phi_0 \otimes \Psi_0.

B) If B = C then the sequence \{\Psi_n \circ \Phi_n\} of channels from A to D strongly* converges to the channel \Psi_0 \circ \Phi_0.

By using property (iii) in Theorem 1 as a criterion of the strong* convergence it is easy to prove the following observation which shows the continuity of the complementary operation \( \Phi \mapsto \tilde{\Phi} \) (defined in (1)) with respect to the strong* convergence of quantum channels.

**Corollary 2.** If \{\Phi_n\} is a sequence of quantum channels from A to B strongly* converging to a channel \Phi_0 then there exists a sequence \{\Psi_n\} of quantum channels from A to some system E strongly* converging to a channel \Psi_0 from A to E such that \( \Psi_n = \tilde{\Phi}_n \) for all \( n \geq 0 \).

### 4 Representation of converging channels via a partial trace channel

The Stinespring representation (1) of any quantum channel \( \Phi_0 \) from A to B means that \( \Phi_0(\rho) = \Theta(V_0\rho V_0^*) \), where \( \Theta(\rho) = \text{Tr}_{E\rho} \) is the partial trace channel from BE to B and \( V_0 \) is an isometrical embedding of \( \mathcal{H}_A \) into \( \mathcal{H}_{BE} \). Assume that \{\( W_n \)\} is a sequence of partial isometries on \( \mathcal{H}_{BE} \) such that \( W_n^* W_n = P_0 \) for all \( n \), where \( P_0 \) is the projector on the subspace \( V_0(\mathcal{H}_A) \). Let

\[
\Phi_n(\rho) = \Theta(W_n V_0 \rho V_0^* W_n^*), \quad \rho \in \mathcal{S}(\mathcal{H}_A), \quad (10)
\]

be a quantum channel from A to B for each n. It is easy to show that

- if the sequence \{\( W_n \)\} converges to the projector \( P_0 \) in the operator norm then the sequence \{\( \Phi_n \)\} converges to the channel \( \Phi_0 \) in the diamond norm;

- if the sequence \{\( W_n \)\} strongly converges to the projector \( P_0 \) then the sequence \{\( \Phi_n \)\} strongly converges to the channel \( \Phi_0 \);

- if the sequence \{\( W_n \)\} strongly* converges\(^7\) to the projector \( P_0 \) then the sequence \{\( \Phi_n \)\} strongly* converges to the channel \( \Phi_0 \) (see Def.1).

\(^7\)It means that \( s\inf\lim_{n \to \infty} W_n = P_0 \) and \( s\inf\lim_{n \to \infty} W_n^* = P_0 \).
The results in [14],[17] and Theorem 1 in Section 3 imply

**Proposition 1.** Any sequence \( \{\Phi_n\} \) of channels converging in the diamond norm (correspondingly, strongly converging, strongly* converging) to a channel \( \Phi_0 \) can be represented in the form (10) with some isometrical embedding \( V_0 \) of \( \mathcal{H}_A \) into \( \mathcal{H}_{BE} \) and some sequence \( \{W_n\} \) of partial isometries converging to the projector \( P_0 \) w.r.t the operator norm topology (correspondingly, the strong operator topology, the strong* operator topology).

Usefulness of the representation (10) of strongly converging sequences of quantum channels is illustrated by the proof of Theorem 2 in [17].

### 5 Convergence of Bosonic Gaussian channels

In this section we show that the strong* and strong convergences coincide on the class Bosonic Gaussian channels playing a central role in continuous variable quantum information theory [8, 20].

Let \( \mathcal{H}_X (X = A,B,...) \) be the space of irreducible representation of the Canonical Commutation Relations (CCR)

\[
W_X(z)W_X(z') = e^{\left[-\frac{i}{2}z^\top \Delta_X z\right]}W_X(z' + z), \quad z,z' \in Z_X,
\]

with a symplectic space \( (Z_X, \Delta_X) \) and the Weyl operators \( W_X(z) \) [8, Ch.12]. Denote by \( s_X \) the number of modes of the system \( X \), i.e. \( 2s_X = \dim Z_X \).

A state \( \rho \) in \( \mathfrak{S}(\mathcal{H}_X) \) is called Gaussian if it has Gaussian characteristic function

\[
\phi_\rho(z) \doteq \text{Tr} W_X(z)\rho = e^{\left[i mz - \frac{1}{2} z^\top \sigma z\right]}, \quad (11)
\]

where \( m \) is a \( 2s_X \)-dimensional real row and \( \sigma \) is a \( (2s_X) \times (2s_X) \) real symmetric matrix satisfying the inequality

\[
\sigma \geq \pm i \Delta_X. \quad (12)
\]

The row \( m \) consists of the mean values of the canonical observables at the state \( \rho \), while \( \sigma \) is the covariance matrix of these observables [8, Ch.12].

A Bosonic Gaussian channel \( \Phi : \mathfrak{F}(\mathcal{H}_A) \rightarrow \mathfrak{F}(\mathcal{H}_B) \) is defined via the action of its dual \( \Phi^* : \mathfrak{B}(\mathcal{H}_B) \rightarrow \mathfrak{B}(\mathcal{H}_A) \) on the Weyl operators:

\[
\Phi^*(W_B(z)) = W_A(Kz)e^{\left[i tz - \frac{1}{2} z^\top \alpha z\right]}, \quad z \in Z_B, \quad (13)
\]
where $K$ is a linear operator $Z_B \rightarrow Z_A$, $\ell$ is a $2s_B$-dimensional real row and $\alpha$ is a $(2s_B) \times (2s_B)$ real symmetric matrix satisfying the inequality

$$
\alpha \geq \pm \frac{i}{2} \left[ \Delta_B - K^\top \Delta_A K \right].
$$

If $\rho$ is a state in $\mathcal{S}(\mathcal{H}_A)$ with the characteristic function $\phi_\rho(z) = \text{Tr} W_A(z) \rho$ and $\Phi$ is a Gaussian channel defined in (13) then the state $\Phi(\rho)$ has the characteristic function

$$
\phi_{\Phi(\rho)}(z) = \text{Tr} W_B(z) \Phi(\rho) = \text{Tr} \Phi^\ast(W_B(z))\rho = \phi_{\rho}(Kz)e^{[iz-\frac{1}{2}z^\top \alpha z]}.
$$

In particular, if $\rho$ is a Gaussian state with the characteristic function (11) then $\Phi(\rho)$ is a Gaussian state with the characteristic function

$$
\phi_{\Phi(\rho)}(z) = e^{[i(mK+\ell)z-\frac{1}{2}z^\top(\alpha+K^\top \sigma K)z]}.
$$

**Proposition 2.** For Bosonic Gaussian channels the strong* convergence is equivalent to the strong convergence and weaker than the uniform (diamond norm) convergence.

**Proof.** Let $\{\Phi_n\}$ be a sequence of Gaussian channels strongly converging to a Gaussian channel $\Phi_0$. By using Lemma 2 below and the coincidence of the weak operator topology with the strong operator topology on the set of unitary operators it is easy to show that $s\text{-}\lim_{n\to\infty} \Phi_n^\ast(W_B(z)) = \Phi_0^\ast(W_B(z))$ for all $z \in Z_B$. Since the linear hull of the family $\{W_B(z)\}_{z \in Z_B}$ is dense in $\mathcal{B}(\mathcal{H}_B)$ in the strong operator topology, the sequence $\{\Phi_n\}$ strongly* converges to the channel $\Phi_0$ by Remark 1 after Theorem 1.

To show that the strong* convergence is weaker than the uniform (diamond norm) convergence one can consider the single-mode Bosonic quantum limited attenuator defined by its action on the family $\{|\eta\rangle\langle\eta|\}_{\eta \in C}$ of coherent states as follows

$$
\Phi_k(|\eta\rangle\langle\eta|) = |k\eta\rangle\langle k\eta|.
$$

It is proved in [23] that $\|\Phi_k - \Phi_{k'}\|_o = 2$ for all $k \neq k'$. It is also mentioned in [23] that the channel $\Phi_{k'}$ strongly (and hence strongly*) converges to the channel $\Phi_k$ as $k' \rightarrow k$. Other examples showing the nonequivalence of the strong* and uniform convergences of Gaussian channels can be found in [22].
Lemma 2. Let $\Phi_n$, $n \geq 0$, be Gaussian channels between given Bosonic systems $A$ and $B$ defined by relation (13) with parameters $K_n, \ell_n, \alpha_n$. The sequence $\{\Phi_n\}$ strongly converges to the channel $\Phi_0$ if and only if

$$\lim_{n \to \infty} K_n = K_0, \quad \lim_{n \to \infty} \ell_n = \ell_0 \quad \text{and} \quad \lim_{n \to \infty} \alpha_n = \alpha_0,$$

where the limits in any topology on sets of finite matrices (rows).

Proof. Let $\rho$ be any state in $\mathcal{S}(\mathcal{H}_A)$. It follows from (14) that condition (16) implies pointwise convergence of the sequence $\{\phi_{\Phi_n}(\rho)\}$ to the function $\phi_{\Phi_0}(\rho)$. By the quantum version of Levy’s continuity theorem (see [6]) this implies convergence of the sequence $\{\Phi_n(\rho)\}$ to the state $\Phi_0(\rho)$.

Assume that the sequence $\{\Phi_n\}$ strongly converges to the channel $\Phi_0$. Then for any Gaussian state $\rho$ the sequence $\{\phi_{\Phi_n}(\rho)\}$ pointwise converges to the function $\phi_{\Phi_0}(\rho)$. By expression (15) this means that

$$\lim_{n \to \infty} e^{i(mK_n + \ell_n)z - \frac{1}{2} z^\top (\alpha_n + K_n^\top \sigma K_n)z} = e^{i(mK_0 + \ell_0)z - \frac{1}{2} z^\top (\alpha_0 + K_0^\top \sigma K_0)z}$$

for all $z \in \mathbb{Z}_B$. Since this relation holds for any mean row $m$ and covariance matrix $\sigma$ satisfying (12), it is easy to show the validity of condition (16). □

Proposition 2 states that any sequence of Gaussian channels strongly converging to a Gaussian channel can be represented as a reduction of a sequence of unitary channels strongly converging to a unitary channel.

At the same time, it is well known that any Gaussian channel has a Gaussian unitary dilation, i.e. it can be represented as a reduction of a Gaussian unitary channel between composite Bosonic systems [4, 5][8, Ch.12]. So, it seems reasonable to assume that Proposition 2 can be strengthened as follows

Conjecture. Any sequence of Gaussian channels strongly converging to a Gaussian channel can be represented as a reduction of a sequence of Gaussian unitary channels strongly converging to a Gaussian unitary channel.

This conjecture seems natural from the physical point of view but the explicit forms of Gaussian unitary dilations of a single Gaussian channel presented in [4, 5] show that its direct proof requires serious technical efforts.

---

8Gaussian unitary channel is a channel $\rho \mapsto U_T \rho U_T^*$, where $U_T$ is the canonical unitary corresponding to a symplectic trasformation $T$ [8, Ch.12].
Appendix

Below we present results concerning possibility to dilate a strongly converging sequence of partial isometries to strongly converging sequence of unitary operators.

**Proposition 3.** Let \( \{V_n\} \) be a sequence of partial isometries on a separable Hilbert space \( \mathcal{H} \) strongly converges to a partial isometries \( V_0 \) such that \( V_n^*V_n = V_0^*V_0 = P \) and \( \dim \ker P = \dim \ker Q_n \leq +\infty \), where \( Q_n = V_nV_n^* \), for all \( n \geq 0 \). The following properties are equivalent:

(i) there exists a sequence \( \{U_n\} \) of unitaries on \( \mathcal{H} \) strongly converging to a unitary operator \( U_0 \) such that \( U_nP = V_n \) for all \( n \geq 0 \);

(ii) the sequence \( \{Q_n\} \) strongly converges to the operator \( Q_0 \);

(iii) the sequence \( \{V_n^*\} \) strongly converges to the operator \( V_0^* \).

**Remark 2.** A sequence \( \{V_n\} \) of partial isometries satisfying the assumptions of Proposition 3 for which the properties (i)-(iii) do not hold can be found in the proof of Corollary 3 in [17].

**Proof.** Since all the partial isometries have the same initial space, the sequence \( \{W_n = V_nV_0^*\} \) consists of partial isometries and strongly converges to the projector \( Q_0 = V_0V_0^* \). Note that \( W_nW_n^* = Q_n \) and \( W_n^*W_n = Q_0 \) for all \( n \). So, the assertion of the proposition follows from Lemma 3 below. \( \square \)

**Lemma 3.** Let \( \{S_n = \{\varphi_i^n\}_{i \in I}\}_{n \geq 0} \) be a sequence of orthonormal systems of vectors in a separable Hilbert space \( \mathcal{H} \) such that \( \dim S_n^\perp = \dim S_0^\perp \leq +\infty \) for all \( n \). Let \( P_n = \sum_{i \in I} |\varphi_i^n\rangle \langle \varphi_i^n| \) be the projector on the subspace \( \mathcal{H}_n \) generated by \( S_n \) and \( W_n = \sum_{i \in I} |\varphi_i^n\rangle \langle \varphi_i^0| \) a partial isometry. Assume that \( \lim_{n \to \infty} \varphi_i^n = \varphi_i^0 \) for each \( i \in I \). The following properties are equivalent:

(i) for each \( n \geq 0 \) there is an orthonormal basis \( S_n^e = \{\psi_i^n\}_{i \in I} \cup \{\psi_j^n\}_{j \in J} \) in \( \mathcal{H} \) obtained by extension of the system \( S_n \) such that \( \lim_{n \to \infty} \psi_j^n = \psi_j \) for each \( j \in J \);

(ii) the sequence \( \{P_n\} \) strongly converges to the operator \( P_0 \);

(iii) the sequence \( \{W_n^*\} \) strongly converges to the operator \( P_0^* \);

9I am sure that these results can be found in the literature. So, I would be grateful for any references concerning this question.
Proof. (i) ⇒ (iii). It follows from (i) that

$$U_n = \sum_{i \in I} |\varphi_i^n\rangle\langle\varphi_i^0| + \sum_{j \in J} |\psi_j^n\rangle\langle\psi_j^0|$$

is an unitary operator strongly converging to the unit operator $I_H$ as $n \to \infty$. Then the unitary operator $U_n^*$ strongly converges to the unit operator as well, i.e.

$$\sum_{i \in I} |\varphi_i^0\rangle\langle\varphi_i^0| \ominus \sum_{j \in J} |\psi_j^0\rangle\langle\psi_j^0| \to \sum_{i \in I} |\varphi_i^0\rangle\langle\varphi_i^0| \ominus \sum_{j \in J} |\psi_j^0\rangle\langle\psi_j^0|$$

as $n \to \infty$ for any vector $\theta$ in $H$. Hence $W_n^*$ strongly converges to $P_0$.

(iii) ⇒ (ii). Since $W_n$ strongly converges to $P_0$ by the assumption, it follows from (iii) that $P_n = W_nW_n^*$ strongly converges to $P_0$.

(ii) ⇒ (i). Let $S_0^0 = \{\varphi_i^0\}_{i \in I} \cup \{\psi_j^0\}_{j \in J}$ be an orthonormal basis (o.n.b. in what follows) in $H$ obtained by extension of the system $S_0$. Sequentially applying Lemma 4 below one can construct, for any natural $m$ and $n$, an orthonormal system $\{\alpha_1^n, \ldots, \alpha_m^n\}$ in $S_n^\perp$ in such a way that $\lim_{n \to \infty} \alpha_j^n = \psi_j^0$ for all $j = 1, m$. This gives the required sequence of o.n.b. $S_n^0 = S_n \cup \{\psi_j^n\}_{j \in J}$ in the case $\dim S_0^\perp = +\infty$. If $\dim S_0^\perp = +\infty$ this sequence can be constructed as follows:

$$\psi_1^1 = \alpha_1^1 \text{ and } \{\psi_j^1\}_{j > 1} \text{ is any o.n.b. in } [\{\alpha_1^1\} \cup S_1]^\perp,$$

$$\psi_2^1 = \alpha_2^1, \psi_2^2 = \alpha_2^2 \text{ and } \{\psi_j^2\}_{j > 2} \text{ is any o.n.b. in } [\{\alpha_1^1, \alpha_2^2\} \cup S_2]^\perp,$$

$$\vdots$$

$$\psi_1^n = \alpha_1^n \ldots, \psi_j^n = \alpha_j^n \text{ and } \{\psi_j^n\}_{j > n} \text{ is any o.n.b. in } [\{\alpha_1^n, \ldots, \alpha_m^n\} \cup S_n]^\perp,$$

$$\vdots$$

**Remark 3.** A sequence $\{S_n\}$ of orthonormal systems satisfying the assumptions of Lemma 3 for which properties (i)-(iii) of this lemma do not hold can be easily constructed: let $\{\tau_i\}$ be a countable orthonormal system of vectors, $\varphi_i^n = \tau_i$ for all $i \neq n$ and $\varphi_n^n = \psi$, where $\psi$ is any unit vector in $\{\tau_i\}^\perp$.

**Lemma 4.** Let the assumptions of Lemma 3 hold and $\psi_0$ be any unit vector in $S_0^\perp$. If the sequence $\{P_n\}$ strongly converges to the operator $P_0$ then there is a sequence $\{\psi_n\}$ of unit vectors converging to the unit vector $\psi_0$ such that $\psi_n \in S_n^\perp$ for all $n$.

**Proof.** Let $\bar{P}_n = I_H - P_n$ and $|\psi_n\rangle = \bar{P}_n|\psi_0\rangle / \|\bar{P}_n|\psi_0\rangle\|$ if $\|\bar{P}_n|\psi_0\rangle\| \neq 0$ and $|\psi_n\rangle$ be any vector in $S_n^\perp$ otherwise. Since the sequence $\{\bar{P}_n\}$ strongly	

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converges to the operator $\bar{P}_0$ and $\bar{P}_0|\psi_0\rangle = |\psi_0\rangle$ the sequence $\{|\psi_n\rangle\}_{n}$ has the required properties. □

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