Kaluza-Klein theory for teleparallel gravity

Chao-Qiang Geng,\textsuperscript{1,2,3,*} Chang Lai,\textsuperscript{1†} Ling-Wei Luo,\textsuperscript{2‡} and Huan-Hsin Tseng\textsuperscript{2§}

\textsuperscript{1}Chongqing University of Posts \& Telecommunications, Chongqing, 400065, China
\textsuperscript{2}Department of Physics, National Tsing Hua University, Hsinchu 300, Taiwan
\textsuperscript{3}Physics Division, National Center for Theoretical Sciences, Hsinchu 300, Taiwan

Abstract

We study teleparallel gravity in the original Kaluza-Klein (KK) scenario. Our calculation of the KK reduction of teleparallel gravity indicates that the 5-dimensional torsion scalar $^{(5)}T$ generates the non-Brans-Dicke type effective Lagrangian in 4-dimension due to an additional coupling between the derivative of the scalar field and torsion, but the result is equivalent to that in general relativity. We also discuss the cosmological behavior in the FLRW universe based on the effective teleparallel gravity.

* Electronic address: geng@phys.nthu.edu.tw
† Electronic address: laichang@cqupt.edu.cn
‡ Electronic address: d9622508@oz.nthu.edu.tw
§ Electronic address: d943335@oz.nthu.edu.tw
I. INTRODUCTION

The extra dimension theory was originally proposed to unify the electromagnetism and gravity theory into one theory via 5-dimension with gauge invariant suggested by Kaluza [1]. Klein then realized the Kaluza’s cylindrical condition as the zero mode of harmonic expansion fields with a compactification of the 5th-dimension into a small circle $S^1$ [2]. In the Kaluza-Klein (KK) theory, particles are described by series of the mass spectrum, called as the KK towels. The contributions from the extra dimension give rise to the different mass scales by the KK towels. In the low energy scale, the KK dimensional reduction leads to the effective theory with gravity interacting with electromagnetic and scalar fields in 4-dimension [3].

An alternative gravity theory with absolutely parallelism in Weitzenb"ock geometry called teleparallel equivalent to general relativity (TEGR) was first considered by Einstein [4]. Recently, several generalizations related to teleparallel gravity (teleparallelism) have been presented in the literature, such as teleparallel dark energy [5] and $f(T)$ [6, 7] models.

Teleparallelism in the KK scenario has been discussed in Refs. [8–12]. In this article, we compute the KK reduction of TEGR at the low energy in the absence of the electromagnetic field. The reduction generates an additional non-minimal coupled term for the effective action, which matches neither with the result in Ref. [11] nor with the Brans-Dicke type theory. We also show that there exists an Einstein frame for the non-minimal teleparallel gravity by including the additional coupling. Finally, we study the flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology and discuss the solution in the KK scenario.

II. KALUZA-KLEIN REDUCTION FOR TELEPARALLEL GRAVITY

A. Brief Review of Teleparallel Gravity

In the teleparallel theory, one introduces an orthonormal frame $\vartheta^i = e^i_\mu \, dx^\mu$ with a given metric $g_{\mu\nu} = \eta_{ij} \, e^i_\mu \, e^j_\nu$, where $\eta_{ij} = \text{diag}(+1, -1, -1, -1)$ is Minkowski metric, $\mu, \nu, \rho, \ldots = 0, 1, 2, 3$ label coordinate frame indices, and $i, j, k, \ldots = \hat{0}, \hat{1}, \hat{2}, \hat{3}$ denote orthonormal frame indices. As a result, we have $e_i = e^\mu_i \partial_\mu$ and $\vartheta^i = e^i_\mu \, dx^\mu$ as the fields in the orthonormal frame, also called vielbein fields denoted by the coefficients $e^i_\mu$. The duality of coframes and frames leads to the relations $e^i_\mu \, e^\mu_j = \delta^i_j$ and $e^i_\nu \, e^\nu_i = \delta^\nu_i$. One can define the absolute parallelism, i.e., the curvature-free condition, via the covariant derivative of vielbein fields of...
\[ \nabla_\nu e^i_\mu = \partial_\nu e^i_\mu - e^i_\rho \Gamma^\rho_{\mu \nu} = 0 \]

To obtain the Weitzenböck connection \[ \Gamma^\rho_{\mu \nu} = e^i_\partial_\nu e^i_\mu \] in Weitzenböck geometry. The torsion tensor is defined by the anti-symmetric part of the Weitzenböck connection \[ T^\rho_{\mu \nu} \equiv \Gamma^\rho_{\nu \mu} - \Gamma^\rho_{\mu \nu} = e^i_\partial_\nu e^i_\mu - \partial_\nu e^i_\mu = -T^\rho_{\nu \mu} \] and the contorsion \[ K^\rho_{\mu \rho} \equiv (-1/2)(T^\rho_{\nu \mu} - T^\rho_{\mu \nu} - T^\rho_{\nu \mu}) = -K^\rho_{\mu \rho}. \]

The torsion scalar is constructed as \[ T = \frac{1}{4} T^\rho_{\mu \nu} T^\rho_{\nu \mu} + \frac{1}{2} T^\rho_{\mu \nu} T^\nu_{\rho \mu} T^\sigma_{\nu \rho} \tag{2.1a} \]

The torsion scalar can also be rewritten as \[ T = \frac{1}{4} T^\rho_{\mu \nu} S^\rho_{\nu \mu}, \] where \[ S^\rho_{\nu \mu} = \frac{1}{2} T^\rho_{\mu \nu} S^\nu_{\rho \mu} - \delta^\nu_\rho T^\sigma_{\nu \sigma} = -S^\nu_{\rho \mu} \] is called the superpotential. In general relativity, the Hilbert Lagrangian is the curvature scalar \[ R. \]

In analogy, the teleparallel gravitational action of TEGR is given by the torsion scalar

\[ S_{\text{TEGR}} = \frac{1}{2\kappa} \int d^4 x e T, \tag{2.2} \]

where \( \kappa = 8\pi G \) is the gravitational coupling and \( e \equiv \det(e^i_\mu). \)

In the 5-dimensional teleparallel gravity, the metric is given by \( \bar{g}_{MN} = \bar{\eta}_{IJ} e^I_M e^J_N \), where \( \bar{\eta}_{IJ} = \text{diag}(+1, -1, -1, -1, \varepsilon) \) with \( \varepsilon = \pm 1 \), \( M, N, O = 0, 1, 2, 3, 5 \) and \( I, J, K = \hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{5} \).

It is convenient to calculate torsion by the Cartan structure equation \( T^I = d\theta^I. \) The non-vanishing components of the torsion 2-form in the orthonormal frame are \( \bar{T}^k_{ij}, \bar{T}^k_{5j} \) and \( \bar{T}^5_{i5}. \) The 5-dimensional torsion scalar \( (5)T \) can be given in a similar way as Eq. (2.1b) in 5-dimension, decomposed as \[ (5)T = \bar{T} + \frac{1}{2} \left( \bar{T}^k_{ij} \bar{T}^k_{5j} + \bar{T}^k_{5j} \bar{T}^j_{k5} \right) + 2 \bar{T}^k_{ij} \bar{T}^5_{i5} - \bar{T}^5_{j5} \bar{T}^k_{k5}, \tag{2.3} \]

where \( \bar{T} \) is the induced 4-dimensional torsion scalar.

### B. Kaluza-Klein Theory

In the original Kaluza-Klein theory, the 5-dimensional spacetime bulk \( V = M \times S^1 \) is a product space of a hypersurface \( M \) and a circle \( S^1 \) with small radius \( r \) representing the compactified extra dimension. Because of the embedding, we can always write the 5-dimensional metric \( \bar{g} \) in the coordinate system as

\[ \bar{g}_{MN} = \begin{pmatrix} g_{\mu \nu}(x^\mu, y) & 0 \\ 0 & \varepsilon \phi^2(x^\mu, y) \end{pmatrix}, \tag{2.4} \]
where \( y = x^5 \) and \( \epsilon = -1 \). In the orthonormal frame, the vielbein fields are \( e_i^\mu \) and \( \epsilon_5^\mu = \phi(x^\mu, y) \). Consequently, the 5-dimensional torsion scalar in the coordinate frame is

\[
(5) T = \bar{T} + \frac{1}{2} (\bar{T}_\rho^\sigma \bar{T}_\nu^\rho \bar{T}_\nu^\sigma + \bar{T}_\rho^\nu \bar{T}_\nu^\rho \bar{T}_\rho^\sigma) + 2 \bar{T}_\sigma^\mu \bar{T}_\mu^5 - \bar{T}_\nu^\rho \bar{T}_\rho^5 \bar{T}_\sigma^\sigma ,
\]

(2.5)

where \( \bar{T} = T \) is a pure 4-dimensional object due to \( \bar{T}_\rho^\mu \bar{T}_\nu^\rho = T_\rho^\mu \bar{T}_\nu^\rho \). Note that as there only exists the KK zero mode in the effective low-energy theory based on the cylindrical condition, all fields have no dependence on the 5th-dimensional component, i.e., \( g_{\mu \nu,5} = 0 \), which is referred to as KK ansatz [3]. Within the ansatz, the metric is reduced to

\[
\bar{g}_{MN} = \begin{pmatrix} g_{\mu \nu}(x^\mu) & 0 \\ 0 & -\phi^2(x^\mu) \end{pmatrix} .
\]

(2.6)

The residual components of the non-vanishing torsion tensor are \( T_\rho^\mu \) and \( \bar{T}_5^\mu = \partial_\mu \phi / \phi \). The 5-dimensional torsion scalar is

\[
(5) T = T + 2 T_\sigma^\mu \bar{T}_5^\mu ,
\]

(2.7)

which can be used to construct the 5-dimensional action. The extra dimension \( y = r \theta \) is the ground state of \( \bar{g}_{55} \) with the radius \( r \). The invariant volume element is \( ^{(5)}e \, d^5 x = e \phi \, r \, d\theta \, d^4 x \) by the dimensional reduction with \( ^{(5)}e = \det(e_M^I) \). As a result, we obtain the 4-dimensional effective action

\[
S_{\text{eff}} = \frac{1}{2 \kappa_4} \int d^4 x \, e \left( \phi T + 2 T_\mu \partial_\mu \phi \right) ,
\]

(2.8)

where the effective coupling \( \kappa_4 := \kappa_5 / 2 \pi r = 8 \pi G_5 / 2 \pi r \) and \( T_\mu := \bar{T}_\sigma^\chi \bar{T}_\chi^\mu \) is the torsion trace vector. Hence, we have the KK reduction procedure in teleparallel gravity which replaces the 5-dimensional torsion scalar \( (5) T \) instead of \( T + 2 \phi^{-1} T_\mu \partial_\mu \phi \).

For \( F(T) \) gravity in the KK theory, where \( F(T) \) is an arbitrary function of \( T \), the Lagrangian density \( ^{(5)}e \, F(\bar{T}(T)) \) becomes \( 2 \pi e \phi \, F(T + 2 \phi^{-1} T_\mu \partial_\mu \phi) \). The effective action for \( F(T) \) gravity is

\[
S_{\text{eff}} = \frac{1}{2 \kappa_4} \int d^4 x \, e \phi \, F \left( T + \frac{2}{\phi} T_\mu \partial_\mu \phi \right) .
\]

(2.9)

By taking \( F(\bar{T}(T)) \) to be linear in the torsion scalar with a cosmological constant, i.e., \( F(\bar{T}(T)) = (5) T - 2 \Lambda_5 \), the action can be reduced to Eq. [2.8] with a varying 4-dimensional cosmological term \( \Lambda_4(x^\mu) = \Lambda_5 \phi(x^\mu) \).

It is interesting to consider a conformal transformation \( \tilde{g}_{\mu \nu} = \Omega^2(x^\mu) \, g_{\mu \nu} \) for the effective action in Eq. [2.8]. The transformations of the vierbien and torsion tensor are \( \tilde{e}_\mu^i = \Omega e_{\mu}^i \).
and $\tilde{T}^{i}_{\mu\nu} = \Omega \tilde{\Gamma}^{i}_{\mu\nu} + e^{i}_{\mu} \partial_{\mu} \Omega - e^{i}_{\mu} \partial_{\nu} \Omega$, while the torsion scalar and the torsion trace vector are expressed as

\begin{align}
T &= \Omega^{2} \tilde{T} - 4 \Omega \tilde{g}^{\mu\nu} \tilde{\Gamma}_{\mu} \partial_{\nu} \Omega - 6 \tilde{g}^{\mu\nu} \partial_{\mu} \Omega \partial_{\nu} \Omega, \quad (2.10a) \\
T_{\mu} &= \tilde{T}_{\mu} + 3 \Omega^{-1} \partial_{\mu} \Omega, \quad (2.10b)
\end{align}

respectively. By choosing $\phi = \Omega^{2}$, the action reads

\begin{equation}
S_{\text{eff}} = \int d^{4}x \frac{1}{2\kappa_{4}} \tilde{T} + \frac{1}{2} \tilde{g}^{\mu\nu} \partial_{\mu} \psi \partial_{\nu} \psi, \quad (2.11)
\end{equation}

where $\psi = (6/\kappa_{4})^{1/2} \ln \Omega$ is a dilaton field. As seen from Eq. (2.11), there exists an \textit{Einstein frame} for the non-minimal coupled effective Lagrangian in Eq. (2.8) in teleparallel gravity.

By using Eq. (2.8) and the curvature-torsion relation

\begin{equation}
R_{(e)} = T - 2 \tilde{\nabla}_{\mu} T_{\mu}, \quad (2.12)
\end{equation}

where $R_{(e)}$ and $\tilde{\nabla}_{\mu}$ are the Riemann curvature constructed by the \textit{Levi-Civita connection} in terms of the vielbein and the covariant derivative with respect to the Levi-Civita connection, respectively, we obtain the action

\begin{equation}
-\frac{1}{2\kappa_{4}} \int d^{4}x \left( \phi R_{(e)} - 2 \tilde{\nabla}_{\mu}(\phi T_{\mu}) \right). \quad (2.13)
\end{equation}

It is interesting to note that in the 5-dimensional general relativity, the effective action is $(-1/2\kappa_{4}) \int d^{4}x \sqrt{-g} \phi \tilde{R}$ which is a specific case of the Brans-Dicke theory \cite{3, 14, 15}. It is clear that the form of the effective action in Eq. (2.8) is not a scalar-tensor-like due to the additional non-minimal coupling $2 T_{\mu} \partial_{\mu} \phi$, which is different from the reduction action in general relativity although it is still \textit{equivalent} to the Lagrangian $\phi \tilde{R}$ up to the total derivative term as shown in Eq. (2.13). We also note that our action in Eq. (2.9) is clearly different from Eq. (5) in Ref. \cite{11}, in which the function variable is $T + \phi^{-2} \partial^{\mu} \phi \partial_{\mu} \phi$ without coupling between $T$ and $\phi$ instead of $T + 2 \phi^{-1} T_{\mu} \partial_{\mu} \phi$.

\section{Equations of Motion}

By varying Eq. (2.8) with respect to $\epsilon^{i}_{\mu}$, the equation of motion of teleparallel gravity is

\begin{equation}
\frac{1}{2} \epsilon^{\mu}_{i} \left( \phi T + 2 T_{\sigma} \partial_{\sigma} \phi \right) - \epsilon^{\mu}_{i} \left( \tilde{\Gamma}^{j}_{i\rho\sigma} S_{j}^{\mu\rho} \right) - \epsilon^{\nu}_{i} \left( \partial_{\sigma} \phi \tilde{T}_{\mu}^{\mu,\sigma} + \partial_{\nu} \phi \tilde{T}_{\mu}^{\mu} + \partial_{\mu} \phi \tilde{T}_{\nu}^{\mu} \right) \\
+ \frac{1}{\epsilon} \partial_{\mu} \left( \epsilon S_{i}^{\mu\nu} + \epsilon^{i}_{\mu} \partial^{\nu} \phi - \epsilon^{\nu}_{i} \partial^{\mu} \phi \right) = \kappa_{4} \Theta^{\mu}_{i}, \quad (3.1)
\end{equation}
with \( \Theta^\mu_i = (-1/e)(\delta L_m/\delta e^\mu_i) \), where we have assumed the energy-momentum tensor as a perfect fluid with \( \Theta^\mu_\nu = \text{diag}(\rho, -P, -P, -P) \). For the flat FLRW universe, we have \( g_{\mu\nu} = \text{diag}(1, -a^2(t), -a^2(t), -a^2(t)) \) and \( e^i_\mu = \text{diag}(1, a(t), a(t), a(t)) \). In such coordinates, the non-vanishing torsion component is \( T^\alpha_0_\alpha = \dot{a}/a \) without summation, where \( \alpha, \beta, \gamma, \ldots = 1, 2, 3 \) are in the coordinate frame and \( a, b, c, \ldots = \hat{1}, \hat{2}, \hat{3} \) in the orthonormal frame. The only torsion trace vector of \( T^\mu_\mu \) is the \( \mu = 0 \) component, given by \( T^0_0 = -3(\dot{a}/a) \). From the identity \( K^\rho_\mu_\rho = T^\rho_\rho_\mu \), we have \( K^\alpha_0_\alpha = -\dot{a}/a, K^0_\alpha_\alpha = a \dot{a}, \) and \( S^\alpha_0_\alpha = -2(\dot{a}/a) \). The Friedmann equations are given by

\[
3 \phi H^2 + 3 H \dot{\phi} = \kappa_4 \rho, \tag{3.2a}
\]

\[
3 \phi H^2 + 2 \dot{\phi} H + 2 \phi \dot{H} + \ddot{\phi} = -\kappa_4 P, \tag{3.2b}
\]

where \( H = \dot{a}/a \) is the Hubble parameter. It can be checked that Eq. (3.2) can be reduced to the Friedmann equations of TEGR by taking \( \phi = 1 \).

The equation of motion of the scalar field \( \phi \) is

\[
T - 2 \partial_\mu T^\mu - 2T^\mu \Gamma^\nu_\nu_\mu = 0, \tag{3.3}
\]

where \( \Gamma^\nu_\nu_\mu = e^\nu_i \partial_\mu e^\mu_i \) with the non-vanishing component being \( e^\nu_a \partial_\mu e^\mu_a = 3(\dot{a}/a) \) in the coordinate. Subsequently, we obtain \( a \ddot{a} + \dot{a}^2 = 0 \). The equation of motion is independent of the scalar field and the scale factor can be solved directly. By taking the solution to be proportional to \( t^m \), we find that \( m = 0 \) and \( 1/2 \), leading to

\[
a(t) = a_s + b \sqrt{t}, \tag{3.4}
\]

where \( a_s (> 0) \) and \( b \) are constants of \( \mathcal{O}(1) \). By substituting the solution in Eq. (3.4) back to \( a \ddot{a} + \dot{a}^2 = 0 \), we find that the condition \( a_s \cdot b = 0 \) must hold, which results in two cases: (i) \( a_s \neq 0, b = 0 \) and (ii) \( a_s = 0, b \neq 0 \).

For (i), we obtain \( a = a_s \), which corresponds to a static universe, where \( a_s \) is the scale factor of the valid energy scale for the effective teleparallel gravity in Eq. (2.8). For (ii), we get \( a = b \sqrt{t} \). In this case, \( t \) has to be larger than \( t_{\text{cut}} \) which represents the cut-off energy scale for the low-energy effective teleparallel gravity. Consequently, \( \ddot{a} = -b/(4t^{2/3}) \) and \( H = 1/(2t) \). For \( b < 0 (> 0) \), the universe is accelerated (decelerated) expanding.

Comparing to the effective Lagrangian in general relativity, the equation of motion of \( \phi \) is \( \ddot{R}(\epsilon) = 0 \), which also leads to the same solution for the scale factor. We conclude that teleparallel gravity and general relativity are equivalent in the KK scenario.
Finally, we remark that the KK reduction generates an effective low-energy theory that is improper to be applied to the inflationary stage. At the high energy scale, it has to consider the KK modes of gravitational and scalar fields, i.e., there exist massive gravitational and scalar fields. Clearly, the situation is much more complicated than the one discussed in Ref. \[11\].

IV. CONCLUSIONS

We have examined the KK reduction of telaparallel gravity. Our result in Eq. (2.8) has shown that there is a coupled term between the derivative of the scalar field and torsion trace vector, which implies that the KK reduction procedure can not be applied in telaparallelism to obtain the Brans-Dicke type theory. The effective Lagrangian is different from general relativity although it is equivalent to $\phi \tilde{R}$ up to the total derivative term. The property of the additional coupling leads to an Einstein frame by the conformal transformation for the non-minimal coupled teleparallel gravity, which is different from the result in the literature \[11\]. In cosmology, we have obtained the equation of motion in the FLRW universe and found that the accelerated expansion of the universe can be achieved by the effective teleparallel gravity, which is the same as the effective KK scenario in general relativity.

ACKNOWLEDGMENTS

We thank Professor K. Bamba for communications and discussions. The work was supported in part by National Center for Theoretical Sciences, National Science Council (NSC-101-2112-M-007-006-MY3) and National Tsing Hua University (103N2724E1).

[1] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 1921, 966 (1921).
[2] O. Klein, Z. Phys. 37, 895 (1926) [Surveys High Energ. Phys. 5, 241 (1986)].
[3] J. M. Overduin and P. S. Wesson, Phys. Rept. 283, 303 (1997).
[4] A. Einstein, Sitzungsber. Preuss. Akad. Wiss., 1928(XVII), 217221, (1928).
[5] C. Q. Geng, C. C. Lee, E. N. Saridakis and Y. P. Wu, Phys. Lett. B 704, 384 (2011); C. Q. Geng, C. C. Lee and E. N. Saridakis, JCAP 1201, 002 (2012); J. A. Gu, C. C. Lee and
C. Q. Geng, Phys. Lett. B 718, 722 (2013); C. Q. Geng, J. A. Gu and C. C. Lee, Phys. Rev. D 88, 024030 (2013); J. T. Li, Y. P. Wu and C. Q. Geng, Phys. Rev. D 89, 044040 (2014).

[6] R. Ferraro and F. Fiorini, Phys. Rev. D 75, 084031 (2007).

[7] E. V. Linder, Phys. Rev. D 81, 127301 (2010) [Erratum-ibid. D 82, 109902 (2010)]; K. Bamba, C. Q. Geng and C. C. Lee, arXiv:1008.4036 [astro-ph.CO]; P. Wu and H. W. Yu, Eur. Phys. J. C 71, 1552 (2011); K. Bamba, C. Q. Geng, C. C. Lee and L. W. Luo, JCAP 1101, 021 (2011); S. H. Chen, J. B. Dent, S. Dutta and E. N. Saridakis, Phys. Rev. D 83, 023508 (2011); Y. F. Cai, S. H. Chen, J. B. Dent, S. Dutta and E. N. Saridakis, Class. Quant. Grav. 28, 215011 (2011).

[8] V. C. de Andrade, L. C. T. Guillen and J. G. Pereira, Phys. Rev. D 61, 084031 (2000).

[9] A. L. Barbosa, L. C. T. Guillen and J. G. Pereira, Phys. Rev. D 66, 064028 (2002).

[10] F. Fiorini, P. A. Gonzalez and Y. Vasquez, arXiv:1304.1912 [gr-qc].

[11] K. Bamba, S. ’i. Nojiri and S. D. Odintsov, Phys. Lett. B 725, 368 (2013).

[12] C. Q. Geng, L. W. Luo, H. H. Tseng, Class. Quant. Grav. 31, 185004 (2014) [arXiv:1403.3161 [hep-th]].

[13] A. Einstein, Sitzungsber. Preuss. Akad. Wiss., 1929(X), 156159, (1929).

[14] Y. Fujii and K. Maeda, “The scalar-tensor theory of gravitation,” Cambridge, USA: Univ. Pr. (2003) 240 p

[15] S. M. Carroll, “Spacetime and geometry: An introduction to general relativity,” San Francisco, USA: Addison-Wesley (2004) 513 p