Data analysis of continuous gravitational wave: Fourier transform – I

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ABSTRACT
We present the Fourier Transform of a continuous gravitational wave. We have analysed the data set for a 1-d observation time and our analysis is applicable for an arbitrary location of detector and source. We have taken into account the effects arising due to the rotational as well as orbital motions of the Earth.

Key words: gravitational waves – methods: data analysis – pulsars: general.

1 INTRODUCTION
The first generation of long-baseline laser interferometers and ultracryogenic bar detectors will start collecting data very soon. The network of detectors is expected not only to confirm the existence of gravitational waves (GW) but will also provide the information about the structure and the dynamics of its source. At the present stage, the data analysis depends largely upon the study of the expected characteristic of its potential sources and the waveforms. The majority of the experimental searches is focused on the detection of burst and chirp signals. However, the interest in the data analysis for continuous gravitational wave (CGW) signals is growing. A prime example of a source of this type is a spinning neutron star. Many research groups around the globe are working extensively on the data analysis for spinning neutron stars (Jaranowski et al. 1998; Jaranowski & Królak 1999, 2000; Brady et al. 1998; Brady & Creighton 2000; Królak 1999).

The detection of GW signals in the noisy output of the detectors has its own problems, not the least of which is the sheer volume of data analysis. Bar detectors have essentially the same problems as interferometers in reference to CGW sources. Each detector produces a single data stream that may contain many kinds of signals. Detectors do not point, but rather sweep their broad quadrupolar beam pattern across the sky as the Earth moves. Therefore, possibilities of data analysis algorithms need to accommodate signals from any arbitrary location of its source.

In this and the subsequent paper (Srivastava & Sahay 2002), we present analysis of a Fourier transform (FT) of the output data of a ground-based laser interferometer. The output data has broad-band noise and the signal is to be extracted out of it. For this, one has to enhance signal-to-noise ratio (S/N). This is achieved by analysing noise and the signal is to be extracted out of it. For this, one has to analyse the data set for a 1-d observation time (\(\sqrt{\Omega_1}\)). However, in data for a long observation-time data, as S/N is directly proportional to the square root of observation time (\(\sqrt{\int t}\)). In addition to this, there is amplitude modulation (AM). As we will see in Srivastava & Sahay (2002), the amplitude of the detector output consists of simple harmonic terms with frequencies \(w_{\text{rot}}\) and \(2w_{\text{rot}}\) where, \(w_{\text{rot}}\) stands for angular rotational frequency of the Earth. Accordingly, the AM results in splitting of the FT into frequencies \(\pm w_{\text{rot}}\) and \(\pm 2w_{\text{rot}}\).

In the next section we present the noise-free response of the laser interferometric detector and obtain the explicit beam pattern functions. In Section 3 we discuss the Doppler effect and obtain Fourier transform (FT) of the FM signal for arbitrary source and detector locations taking into account the Earth’s rotational motion about its axis and its revolution around the Sun. In Section 4 the FT of the Doppler modulated complete response of the detector is obtained. Section 5 is the conclusion of the paper.

2 THE NOISE-FREE RESPONSE OF DETECTOR: BEAM PATTERN AND AMPLITUDE MODULATION
Let a plane GW fall on a laser interferometer and produce changes in the arms of the detector. In order to express these changes quantitatively we have to specify the wave and the detector. Let \(XYZ\) and \(xyz\) represent respective frames characterizing the wave and the detector. We assume the direction of propagation of the wave to be the X-axis and the vertical at the place of the detector to be the z-axis. The difference of the changes \(\delta l\) in the arm-lengths of the detector may be given via

\[
R(t) = \frac{\delta l}{l_o} = -\sin 2\Omega \left[ (A_x^X A_y^X - A_y^X A_x^X) h_x + (A_x^X A_y^X + A_y^X A_x^X) h_x \right]
\]

where \(l_o\) is the normal length of the arms of the detector and 2\(\Omega\) expresses the angle between them (Schutz & Tinto 1987). The matrix \((A_x^X)\) represents the transformation expressing the rotations to
bring the wave frame \((X, Y, Z)\) to the detector frame \((x, y, z)\). The direction of the source may be expressed in any of the coordinates employed in spherical astronomy. However, it is convenient to define it in Solar system barycentre (SSB) frame, \((X', Y', Z')\). This SSB frame is nothing but astronomer’s ecliptic coordinate system. Let \(\theta\) and \(\phi\) denote the celestial colatitude and celestial longitude of the source. These coordinates are related to the right ascension, \(\alpha\) and declination, \(\delta\) of the source via

\[
\begin{align*}
\cos \theta &= \sin \delta \cos \epsilon - \cos \delta \sin \epsilon \sin \alpha \\
\sin \theta \cos \phi &= \cos \delta \cos \alpha \\
\sin \theta \sin \phi &= \sin \delta \sin \epsilon + \cos \delta \cos \epsilon \sin \alpha ,
\end{align*}
\]

(2)

where \(\epsilon\) represents obliquity of the ecliptic. We choose the \(x\)-axis as the bisector of the angle between the arms of the detector. At this stage the orientation of the detector in the horizontal plane is arbitrary. It is assigned with the help of the angle \(\gamma\) that the \(x\)-axis makes with the local meridian. The location of the detector on the Earth is characterized by the angles \(\alpha\) (colatitude) and \(\beta\) (the local sidereal time), expressed in radians. The transformation matrix \(\mathbf{A}_K\) may be expressed as

\[
\mathbf{A} = \mathbf{DCB}
\]

(3)

where \(\mathbf{B}\) is rotation required to bring \(XYZ\) to \(X'Y'Z'\), \(\mathbf{C}\) is the rotation required to bring \(X'Y'Z'\) to \(x'y'z'\) and \(\mathbf{D}\) is the rotation required to bring \(x'y'z'\) to \(xyz\). Here \(x'y'z'\) represents the frame associated with the Earth. The Euler angles defining the corresponding rotation matrices are given via

\[
\begin{align*}
\mathbf{B} &: (\theta, \phi, \psi) \\
\mathbf{C} &: (0, \epsilon, 0) \\
\mathbf{D} &: (\alpha, \beta + \pi/2, \gamma - \pi/2)
\end{align*}
\]

(4)

where \(\psi\) is a measure of the polarization of the wave (Goldstein 1980).

Let us write equation (1) as

\[
R(t) = \frac{\partial}{\partial \omega} = -\sin 2\Omega [F_h h_x + F_s h_x]
\]

(5)

The functions \(F_h\) and \(F_s\) involve the angles \(\theta, \phi, \psi, \epsilon, \alpha, \beta, \gamma\) and express the effect of the interaction of the wave and the detector. These are called antenna or beam patterns. The explicit functional dependences of \(F_h\) and \(F_s\) are given by Jotania & Dhurandhar (1994). It is pointed out that there are few minor errors in these expressions and these are later corrected by the authors in Jotania (1994). These functions appear complicated but may be written in simpler form by introducing the following abbreviations.

\[
\begin{align*}
A &= 2XY \cos \epsilon - \sin^2 \epsilon \sin \alpha \sin 2\gamma \\
B &= 2XY \sin^2 \epsilon - \cos^2 \epsilon \sin^2 \alpha \sin 2\gamma \\
C &= \cos \epsilon \sin \gamma + \cos \epsilon \sin \gamma \\
D &= -\sin \epsilon (U \sin \gamma - V \sin \alpha \cos \gamma), \\
E &= -2XY \cos \epsilon \sin \epsilon - \cos \epsilon \sin \epsilon \sin 2\alpha \sin 2\gamma \\
&+ 2 \epsilon (X \sin \gamma - Y \sin \alpha \cos \gamma).
\end{align*}
\]

(8)

After straightforward substitutions one obtains

\[
F_h(t) = \frac{1}{2}[(2(L^2 - M^2)UV + (N^2 - P^2)A + (Q^2 - R^2)B) \\
+ (LN - MP)C + (LQ + MR)D + (NQ + PR)E],
\]

(9)

\[
F_s(t) = 2LMUV + NPA - \frac{1}{2}B \sin \epsilon \epsilon \sin 2\phi + (LP + MN)C + (MQ - LR)D + (PQ - NR)E.
\]

(10)

The compactification obtained here arises due to the fact that above introduced abbreviations find places in the transformation matrices as follows:

\[
\begin{pmatrix}
L & N & Q \\
M & P & R
\end{pmatrix}
\]

\[
\begin{pmatrix}
\sin \theta \sin \psi & \sin \theta \cos \psi & \cos \theta
\end{pmatrix}
\]

(11)

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \epsilon & \sin \epsilon \\
0 & -\sin \epsilon & \cos \epsilon
\end{pmatrix}
\]

(12)

\[
\begin{pmatrix}
U & V & \sin \alpha \cos \beta \\
X & Y & \sin \alpha \sin \beta \\
-\sin \alpha \cos \gamma & \sin \alpha \sin \gamma & \cos \alpha
\end{pmatrix}
\]

(13)

After algebraic manipulation equations (9) and (10) may be expressed as

\[
F_h(t) = F_{h1} \cos \beta + F_{h2} \sin \beta + F_{h3} \cos \beta + F_{h4} \sin \beta + F_{h5};
\]

(14)

\[
F_s(t) = F_{s1} \cos \beta + F_{s2} \sin \beta + F_{s3} \cos \beta + F_{s4} \sin \beta + F_{s5};
\]

(15)

where \(F_{\text{h}i}\) and \(F_{\text{s}i}\) \((i = 1, 2, 3, 4, 5)\) are time independent expressions given via

\[
\begin{align*}
F_{h1} &= -2G \cos \alpha \cos 2\gamma + \frac{\sin 2\epsilon}{2} \cos 2\alpha + 1, \\
F_{h2} &= H \cos \alpha \cos 2\gamma - F \sin 2\gamma \cos 2\alpha, \\
F_{h3} &= I \sin \alpha \cos 2\gamma + J \sin 2\alpha \cos 2\gamma, \\
F_{h4} &= K \sin \alpha \cos 2\gamma - \frac{1}{2} \sin 2\alpha \sin 2\gamma, \\
F_{h5} &= \frac{3 \sin^2 \epsilon \sin 2\gamma}{2} [H + L^2 - M^2].
\end{align*}
\]

(16)
\[ G = \frac{1}{2}[(LQ + MR) \sin \epsilon - (LN - MP) \cos \epsilon], \]
\[ H = \frac{1}{2}[(N^2 - P^2) \cos^2 \epsilon - (L^2 - M^2) + (Q^2 - R^2) \sin^2 \epsilon - (NQ + PR) \sin 2\epsilon], \]
\[ I = \frac{1}{2}[(Q^2 - R^2) \sin 2\epsilon - (N^2 - P^2) \sin 2\epsilon - 2(NQ + PR) \cos 2\epsilon], \]
\[ J = \frac{1}{2}[(LN - MP) \sin \epsilon + (LQ + MR) \cos \epsilon] \quad (17) \]

Let us note that \( F_i \times \) is related to \( F_i + \) via
\[ F_i(\theta, \phi, \alpha, \beta, \gamma, \epsilon) = F_i\left(\theta, \phi - \frac{\pi}{4}, \psi, \alpha, \beta, \gamma, \epsilon\right) ; \]
\[ i = 1, 2, 3, 4, 5. \quad (18) \]

This symmetry is representative of the quadrupolar nature of the detector and the wave. The detectors at different orientations will record different amplitudes in their responses. The explicit beam pattern functions may be computed easily for any instant of time. Due to the symmetries involved in \( F_i \) and \( F_s \) it is sufficient to evaluate either of the beam patterns.

The amplitude modulation of the received signal is a direct consequence of the non-uniformity of the sensitivity pattern. As remarked earlier, they are a fairly complicated function of their arguments. Equations (14) and (15) reveal that the monochromatic signal frequency will split, due to AM, into five frequencies. This results in the distribution of energy in various frequencies and consequent reduction of the amplitude of the signal. The periodicity of the beam pattern \( F_s \) and \( F_s \) with a period equal to 1 sidereal day is due to the diurnal motion of the Earth.

### 3 Doppler Shift and Frequency Modulation

The frequency of a monochromatic signal will be Doppler-shifted due to the translatory motion of the detector acquired from the motions of the Earth. Let us consider a CGW signal of constant frequency \( f_o \). The frequency \( f' \) received at the instant \( t \) by the detector is given by
\[ f'(t) = f_o \gamma_o \left(1 + \frac{v \cdot n}{c}(t)\right) ; \quad \gamma_o = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \quad (19) \]

where \( n \) is the unit vector from the antenna to the source, \( v \) is the relative velocity of the source and the antenna, and \( c \) is the velocity of light. The unit vector \( n \) from the antenna to the source, in view of the fact that the distance of the source is very large compared to the average distance of the centre of the SSB frame and the detector, may be taken parallel to the unit vector drawn from the centre of the SSB frame to the source. Hence,
\[ n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad (20) \]

As \( v \) keeps on changing continuously in both amplitude and direction, \( f' \) is a continuous function of \( t \). Furthermore, as \( v \ll c \), we take \( \gamma_o = 1 \).

The radius vector \( r(t) \) in the SSB frame is given by (Jotania et al. 1996)

![Figure 1. FT of a FM signal frequency, \( f_o = 80 \) Hz, from a source located at \((\pi/36, \pi)\) with a resolution of \(1.16 \times 10^{-5} \) Hz.](https://example.com/figure1.png)
Figure 2. FT of a FM signal frequency, $f_0 = 80$ Hz, from a source located at $(\pi/36, \pi)$ with a resolution of $10^{-6}$ Hz.

Figure 3. FT of a FM signal frequency, $f_0 = 80$ Hz, from a source located at $(\pi/36, \pi)$ with a resolution of $10^{-7}$ Hz.
where $R_\alpha$, $R_\infty$ and $w_{\text{obs}}$ represent respectively the Earth’s radius, the average distance between the centre of the Earth from the origin of SSB frame and the orbital angular velocity of the Earth. Here $t$ represents the time in s elapsed from the instant the Sun is at the vernal equinox and $\beta_0$ is the local sidereal time at that instant. The Doppler shift is now given via

$$\frac{f’ - f_0}{f_0} = \frac{v \cdot n}{c}(t) = \frac{\dot{r} \cdot n}{c}$$

$$= \frac{R_{\infty}w_{\text{obs}}}{c} \sin \theta \sin(\phi - w_{\text{obs}}t)$$

$$+ \frac{R_\alpha w_{\text{rot}}}{c} \sin \alpha \sin(\phi - w_{\text{obs}}t)$$

$$+ \frac{R_\alpha w_{\text{rot}}}{c} \sin \alpha \sin(\theta \cos \beta \cos \epsilon \sin \phi - \cos \phi \sin \beta)$$

$$+ \cos \beta \sin \epsilon \cos \theta$$

(23)

The phase $\Phi(t)$ of the received signal is given by

$$\Phi(t) = 2\pi \int_0^t f’(t') \, dt'$$

$$= 2\pi f_0 \int_0^t \left[ 1 + \frac{v \cdot n}{c}(t') \right] \, dt'.$$

(24)

Here we assume the initial phase of the wave to be zero. After straightforward calculation, we obtain

$$\Phi(t) = 2\pi f_0 \left\{ t + \frac{R_\alpha}{c} \sin \theta \cos \phi’$$

$$+ \frac{R_\alpha}{c} \sin \alpha \sin(\phi - \phi’ \cos \beta \cos \epsilon \sin \phi + \cos \phi \cos \beta)$$

$$+ \sin \beta \sin \epsilon \cos \theta - \frac{R_\alpha}{c} \sin \theta \cos \phi$$

$$- \sin \beta_0 \sin \epsilon \cos \theta \right\}$$

$$= 2\pi f_0 t + \frac{\alpha(t_0 - t)}{c} + \mathcal{P} \sin(w_{\text{rot}}t)$$

$$+ \mathcal{Q} \cos(w_{\text{rot}}t) - \mathcal{R} = \mathcal{Q},$$

(25)

where

$$\mathcal{P} = 2\pi f_0 R_\alpha \frac{R_\alpha}{c} \sin \alpha \sin(\theta - \epsilon \sin \phi)$$

$$- \sin \beta_0 \sin \theta \cos \phi,$$

$$\mathcal{Q} = 2\pi f_0 R_\alpha \frac{R_\alpha}{c} \sin \alpha \sin(\theta \cos \epsilon \sin \phi + \cos \theta \sin \epsilon)$$

$$+ \cos \beta_0 \sin \theta \cos \phi,$$

$$\mathcal{N} = \mathcal{P}^2 + \mathcal{Q}^2,$$

$$\mathcal{Z} = 2\pi f_0 \frac{R_\alpha}{c} \sin \phi,$$

$$\mathcal{R} = \mathcal{Z} \cos \phi.$$  

(26)

**Figure 4.** Power spectrum of the complete response of a Doppler modulated signal frequency, $f_0 = 80$ Hz, from a source located at ($\pi/36$, $\pi$) with a resolution of $10^{-1}$ Hz.
\[ \delta = \tan^{-1} \frac{\phi}{\xi}, \]
\[ \phi' = w_{\text{ob}} t - \phi, \]
\[ \xi_{\text{ob}} = w_{\text{ob}} t = a \xi_{\text{rot}}; \quad a = w_{\text{ob}} / w_{\text{rot}} \approx 1/365.26, \]
\[ \xi_{\text{rot}} = w_{\text{rot}} t. \]

The two polarization states of the signal can be taken as
\[ h_+ (t) = h_{\nu_+} \cos [\Phi (t)] \]
\[ h_- (t) = h_{\nu_-} \sin [\Phi (t)], \]
where \( h_{\nu_+}, h_{\nu_-} \) are the time-independent amplitude of \( h_+ (t) \) and \( h_- (t) \), respectively.

To understand the nature of the FM let us consider the function
\[ h(t) = \cos [\Phi (t)] \]
and analyse it for 1-d observation data. The FT is given via
\[ \langle \hat{h} (f) \rangle_d = \int_0^T \cos [\Phi (t)] e^{-i 2 \pi f t} \; dt; \]
\[ T = 1 \text{ sidereal day} = 86164 \text{ s}. \]

This splits into two terms as
\[ \langle \hat{h} (f) \rangle_d = \langle h_+ \rangle + \langle h_- \rangle; \]
\[ \langle \hat{h} (f) \rangle_d = \frac{1}{2 w_{\text{rot}}} \int_0^{2 \pi} e^{i \xi} \left[ \sum_{l=1}^{\infty} J_l (\xi) \cos (l - \delta) \right] d \xi, \]
where \( J \) stands for the Bessel function of the first kind. After performing the integration, we get
\[ \langle \hat{h} (f) \rangle_d \approx \frac{\nu}{2 w_{\text{rot}}} \sum_{l=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{i A B (C - i D)}; \]

\[ I_{\nu_+} = \frac{1}{2 w_{\text{rot}}} \int_0^{2 \pi} e^{i \xi} \left[ \sum_{l=1}^{\infty} J_l (\xi) \cos (l - \delta) \right] d \xi, \]
\[ I_{\nu_-} = \frac{1}{2 w_{\text{rot}}} \int_0^{2 \pi} e^{i \xi} \left[ \sum_{l=1}^{\infty} J_l (\xi) \cos (l - \delta) \right] d \xi, \]
\[ v_+ = f_0 + f; \quad \xi = \xi_{\text{rot}} = w_{\text{rot}} t. \]

Numerical result shows that \( I_{\nu_+} \) oscillates very fast and contributes very little to \( \langle \hat{h} (f) \rangle_d \). Hence, hereafter, we drop \( I_{\nu_+} \) from equation \( 32 \) and write \( v \) in place of \( v_- \). Using the identity
\[ e^{i \xi} \cos \theta = J_0 (\xi) + 2 \sum_{l=1}^{\infty} l J_l (\xi) \cos l \theta \]
we obtain
\[ \langle \hat{h} (f) \rangle_d \approx \frac{1}{2 w_{\text{rot}}} e^{i (R - Q)} \int_0^{2 \pi} e^{i \xi} \left[ J_0 (\xi) + 2 \sum_{l=1}^{\infty} J_l (\xi) \cos k (a \xi - \phi) \right] \]
\[ \times \sum_{m=-\infty}^{\infty} J_m (\xi) e^{i m (\xi - \delta)} d \xi, \]

\[ \langle \hat{h} (f) \rangle_d \approx \frac{\nu}{2 w_{\text{rot}}} \sum_{l=\delta}^{\infty} \sum_{m=-\infty}^{\infty} e^{i A B (C - i D)}; \]

**Figure 5.** Power spectrum of a Doppler modulated signal at frequencies \( f + 2 f_{\text{rot}} \) of signal frequency, \( f_0 = 80 \text{ Hz} \), from a source located at \( (\tau/36), \tau \) with a resolution of \( 10^{-7} \text{ Hz} \).
The FT of the two polarization states of the wave can now be written as

\[
[\tilde{h}_{s}(f)]_d = h_{os}[\tilde{h}(f)]_d \doteq \frac{\nu h_{os}}{2w_{rot}} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{iA B[C - iD]}; \quad (40)
\]

\[
[\tilde{h}_{s}(f)]_d = -i h_{os}[\tilde{h}(f)]_d \doteq \frac{\nu h_{os}}{2w_{rot}} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{iA B[D - iC]}. \quad (41)
\]

The FT of the FM signal contains the double-series Bessel functions. The Bessel functions have contributions due to the rotational as well as the orbital motion of the Earth. It is remarked that Jotania et al. (1996) have analysed FT of FM signal for 1-d observation time. They have taken specific detector as well as source location. They have also neglected the orbital motion. Our analysis generalizes their results. We may now compute \([\tilde{h}(f)]_d\) and may plot its behaviour. To achieve this we have made use of Mathematica software. We know that the value of Bessel function decreases rapidly as its order exceed the argument. Accordingly, one computes in practice only a finite number of terms in the above infinite series. Fig. 1 represents such a plot for

\[
\begin{align*}
 f_o &= 80 \text{ Hz}, & h_{os} &= 1 \\
 \alpha &= \pi/3, & \beta_o &= \pi/4, & \gamma &= 2\pi/5, \\
 \theta &= \pi/36, & \phi &= \pi, & \psi &= \pi/6.
\end{align*}
\]

with a resolution equal to \(1/T_o = 1.16 \times 10^{-5} \text{ Hz}\). For the present case we found it sufficient to evaluate the infinite series for \(k = -21\,900 \, \text{to} \, +21\,900\) and \(m = -10 \, \text{to} \, +10\). Figs 2 and 3 represent the plot of the FT at resolution \(10^{-6} \text{ Hz}\) and \(10^{-7} \text{ Hz}\). A careful look at these plots reveals that the resolution of Fig. 1 does not represent the details of the dominant peaks around \(f_o\), and Fig. 3 does not give any new behaviour as compared to Fig. 2. Hence, we may say that a resolution of about \(10^{-6} \text{ Hz}\) is required to understand the correct behaviour of the FT for 1-d observation data. In this instance let us recall that the data analysis for fast Fourier transform (FFT) limits the resolution to \(1/T_o\). However, the detector output may provide us higher resolution. Thus the semi-analytical analysis presented here may provide more information as compared to FFT.

![Figure 6](https://example.com/image)

**Figure 6.** Power spectrum of a Doppler modulated signal at frequencies \(f - 2f_{rot}\) of signal frequency, \(f_o = 80 \text{ Hz}\), from a source located at \((\pi/36, \pi)\) with a resolution of \(10^{-7} \text{ Hz}\).
4 FOURIER TRANSFORM OF THE COMPLETE RESPONSE

The complete response $R(t)$, in view of equations (5), (14), (15), (28) and (29) may be written as

$$ R(t) = R_i(t) + R_s(t); \quad (43) $$

$$ R_i(t) = h_{\nu_i} \left[ F_{1+} \cos \beta + F_{2+} \sin \beta \right] \cos(\Phi(t)) $$
$$ + F_{3+} \cos \beta + F_{4+} \sin \beta \right] \sin(\Phi(t)) \quad (44) $$

$$ R_s(t) = h_{\nu_s} \left[ F_{1+} \cos \beta + F_{2+} \sin \beta \right] \cos(\Phi(t)) $$
$$ + F_{3+} \cos \beta + F_{4+} \sin \beta \right] \sin(\Phi(t)) \quad (45) $$

Here, for simplicity, we have taken the angle between the arms of the detector equal to $\pi/2$, i.e. $\Omega = \pi/4$. Now the FT of the complete response may be expressed as

$$ \tilde{R}(f) = \tilde{R}_i(f) + \tilde{R}_s(f) \quad (46) $$

Substituting $\beta$ as given by equation (22), one obtains

$$ \tilde{R}_i(t) = \frac{h_{\nu_i}}{2} \left[ e^{-i2\omega_i t} (F_{1+} + iF_{2+}) e^{-i\Delta \omega_{rot} t} \right. $$
$$ + e^{i2\omega_i t} (F_{1+} - iF_{2+}) e^{i\Delta \omega_{rot} t} $$
$$ + e^{-i\omega_s t} (F_{3+} + iF_{4+}) e^{-i\Delta \omega_{rot} t} $$
$$ + e^{i\omega_s t} (F_{3+} - iF_{4+}) e^{i\Delta \omega_{rot} t} \left. \right] \cos(\Phi(t)) \quad (47) $$

and a similar expression for $\tilde{R}_s(t)$. Now it is straightforward to obtain the expressions for $\tilde{R}_i(f)$ and $\tilde{R}_s(f)$ as

$$ \tilde{R}_i(f)_{\nu_i} = \frac{h_{\nu_i}}{2} \left\{ e^{-i2\omega_i t} (F_{1+} + iF_{2+}) \left[ \hat{h}(f + 2f_{rot}) \right]_d \right. $$
$$ + e^{i2\omega_i t} (F_{1+} - iF_{2+}) \left[ \hat{h}(f - 2f_{rot}) \right]_d $$
$$ + e^{-i\omega_s t} (F_{3+} + iF_{4+}) \left[ \hat{h}(f + f_{rot}) \right]_d $$
$$ + e^{i\omega_s t} (F_{3+} - iF_{4+}) \left[ \hat{h}(f - f_{rot}) \right]_d \left. \right\} \quad (48) $$

$$ \tilde{R}_s(f)_{\nu_s} = \frac{h_{\nu_s}}{2} \left\{ e^{-i2\omega_i t} (F_{2+} - iF_{1+}) \left[ \hat{h}(f + 2f_{rot}) \right]_d \right. $$
$$ - e^{i2\omega_i t} (F_{2+} + iF_{1+}) \left[ \hat{h}(f - 2f_{rot}) \right]_d $$
$$ + e^{-i\omega_s t} (F_{4+} - iF_{3+}) \left[ \hat{h}(f + f_{rot}) \right]_d $$
$$ - e^{i\omega_s t} (F_{4+} + iF_{3+}) \left[ \hat{h}(f - f_{rot}) \right]_d \left. \right\} \quad (49) $$

Collecting our results the FT of the complete response of the detector for 1-d time-integration will be

$$ \tilde{R}(f)_{\nu_{\nu_i}} = \frac{1}{2} \left\{ e^{-i2\omega_i t} \left[ \hat{h}(f + 2f_{rot}) \right]_d h_{\nu_i} (F_{1+} + iF_{2+}) \right. $$
$$ + h_{\nu_i} (F_{2+} - iF_{1+}) \right\} + e^{i2\omega_i t} \left[ \hat{h}(f - 2f_{rot}) \right]_d h_{\nu_i} (F_{2+} + iF_{1+}) $$
$$ \times \left[ h_{\nu_s} (F_{3+} + iF_{4+}) \right] - h_{\nu_s} (F_{3+} - iF_{4+}) \right\} $$
$$ + e^{-i\omega_s t} \left[ \hat{h}(f + f_{rot}) \right]_d h_{\nu_i} (F_{3+} + iF_{4+}) $$
$$ - e^{i\omega_s t} \left[ \hat{h}(f - f_{rot}) \right]_d h_{\nu_i} (F_{3+} - iF_{4+}) $$
$$ \times \left[ h_{\nu_s} (F_{4+} - iF_{3+}) \right] + e^{i\omega_s t} \left[ \hat{h}(f + f_{rot}) \right]_d h_{\nu_i} (F_{4+} + iF_{3+}) $$
$$ \times \left[ h_{\nu_s} (F_{3+} + iF_{4+}) \right] - h_{\nu_s} (F_{3+} - iF_{4+}) \right\} $$
$$ + 2 \left[ \hat{h}(f) \right]_d h_{\nu_i} (F_{5+} + iF_{5+}) + h_{\nu_i} (F_{5+} - iF_{5+}) \right\} \quad (50) $$

![Figure 7](https://example.com/figure7.png)

Figure 7. Power spectrum of a Doppler modulated signal at frequencies $f + f_{rot}$ of signal frequency, $f_o = 80$ Hz, from a source located at $(\pi/36, \pi)$ with a resolution of $10^{-7}$ Hz.
This shows that, due to AM, every Doppler-modulated FM signal will split into four additional lines at $f \pm 2f_{\text{rot}}$ and $f \pm f_{\text{rot}}$, where $f_{\text{rot}}$ is the rotational frequency of the Earth ($f_{\text{rot}} \approx 1.16 \times 10^{-5}$ Hz).

We have plotted in Fig. 4 the power spectrum of the noise free complete response of the signal for its various parameters as given by (42). The contribution in the power spectrum of the modulation at frequencies $f + 2f_{\text{rot}}$, $f - 2f_{\text{rot}}$, $f + f_{\text{rot}}$ and $f - f_{\text{rot}}$ are represented respectively in Figs 5, 6, 7, 8 and 9. It is observed that the most power will be at $f + 2f_{\text{rot}}$ and the least will be at $f - f_{\text{rot}}$.

5 CONCLUSION

In this paper we have considered the effect of the Earth’s motion on the response of the detector through FT analysis. It can be easily inferred from equations (50) and (40, 41) that the splitting of frequencies (i) in AM arises explicitly due to rotational motion and (ii) in FM arises due to rotational as well as orbital motion of the Earth. In view of the fact that the data output at the detector is available in discrete form, the analytical FT is not very convenient and one normally employs the popular FFT. However, FFT has resolution limited to $1/T_o$. Further, it is important to understand for how much time one can ignore the frequency shift arising due to Doppler effect. In fact, Schutz (1991) has demonstrated that these effects due to rotational motion are important after the time given by

$$T_{\text{max}} = \left( \frac{2r}{\omega_{\text{rot}} f_o R_c} \right)^{1/2} \approx 70 \left( \frac{f_o}{1 \text{ kHz}} \right)^{-1/2} \text{ min.} \quad (45)$$

This means that for GW signal for a frequency of 80 Hz, one has to take into account these effects after data time $\approx 4$ h. The analytical FT studied in this paper leads to following inferences:

(i) FFT for 1-d observation data will not provide sufficient resolution as to represent the correct picture of the frequency splitting;
(ii) the adequate resolution required for 1-d observation is $10^{-6}$ Hz;
(iii) the frequency split due to FM for frequency $f_o = 80$ Hz and source at $(\theta, \phi) = (\pi/36, \pi)$ is $\approx 2 \times 10^{-4}$ Hz and due to AM is $\approx 4.64 \times 10^{-5}$ Hz;
(iv) the drop in amplitude due to FM alone is about 56 per cent;
(v) the drop in amplitude due to AM alone is about 18 per cent;
(vi) the drop in amplitude for the complete response is about 74 per cent;
(vii) the maximum power due to AM is associated with $f_o + 2f_{\text{rot}}$.

It is remarked that the drop of the amplitude in complete response is severe both due to AM and FM as the relevant frequency range lies in the same region.

We have presented the FT analysis assuming the phase of the GW to be zero at that instant $t = 0$. However, one may relax this condition and may obtain the results easily by taking into consideration the effects of change of the time origin.
Figure 9. Power spectrum of a Doppler modulated signal at frequencies $f$ of signal frequency, $f_o = 80$ Hz, from a source located at $(\pi/36, \pi)$ with a resolution of $10^{-7}$ Hz.

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