Determinant for the Coulomb potential

on the three–sphere

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The functional determinant for a Coulomb potential (or mass squared) on a three–sphere is computed numerically.
1. Introduction.

In order to elucidate the stability, or not, of the Hartle–Hawking wave functional, $\Psi_{HH}[\phi]$, Anninos et al. [1] have calculated, on the three–sphere, the probability $|\Psi_{HH}|^2$, on the basis of the conjectured dS/CFT correspondence, for uniform, and non–uniform, profiles $\phi$, which can be thought of as the mass (squared) in a free anticommuting scalar boundary CFT, the further details of which I do not need. Maldacena’s correspondence says that $\Psi_{HH}$ is, up to a factor, the determinant of the ‘propagating’ operator of the CFT.

Anninos et al calculate this quantity in the special case that the profile is a function of only the angular ‘radial’ coordinate ($\chi$) on the three–sphere (so retaining an SO(3) symmetry). The technical evaluation of the determinant then uses, after a conformal transformation, the Dunne–Kirsten higher dimensional, $\mathbb{R}^n$, extension of the Gel’fand–Yaglom formula.

The particular profiles selected in [1] are the radial harmonics (i.e. SU(2) group characters) and any combination of these, thus giving a general central function. In this brief note, I draw attention to such a function for which the spectrum can be calculated exactly. This is the Coulomb potential (or Kepler problem) and I enlarge on this in the next section.

2. The Coulomb potential.

The relevant equation which determines the eigenvalues from which the determinant is constructed is essentially Schrödinger’s equation on $S^3$, solved, for the Coulomb case, originally by Schrödinger, [2]. I will use the formulation of Infeld, [3]. He is concerned with an atomic interpretation which accounts for his choice of constants. Since this is not relevant here, I write down the eigenvalue equation in the form,

$$(-\Delta_2 + 1 - \alpha^2 + V)\phi = \lambda\phi$$

where the Coulomb potential is

$$V(\chi) = 2\beta \cot \chi.$$  \hspace{1cm} (1)

$\beta$ is a constant giving the strength of the potential and I have offset the eigenvalue by a constant for convenience.

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2 See also Stevenson, [4], for a conventional solution.
3 I am therefore setting Infeld’s constants to be $2\mu = 1$, $Ze^2 = 2\beta$, $R = 1$ and $\hbar = 1$. 
Although the calculation of $\lambda$ is interesting it is now relatively standard and I just give the answers, [3] eqn.7.11. The eigenvalues are,

$$\lambda = n^2 - \alpha^2 - \frac{\beta^2}{n^2}, \quad n = 1, 2, \ldots$$

(Infeld has $l = n - 1$.) There is no continuum and the degeneracy of the $n$th level is $n^2$.  

These are all the facts one requires to proceed to the calculation of the determinant.

For present purposes, I do not need the flexibility provided by the $\alpha^2$ and so set it to zero.

### 3. The determinant

I take an unsophisticated approach to the evaluation of the determinant, which I denote by $D(\beta)$, constructed from the eigenvalues (2). In fact I compute the ratio $D(\beta)/D(0)$. Although $D(0)$ can be calculated separately, and is known, so allowing $D(\beta)$ to be found, I will not make this trivial adjustment.

Then, by basic definition,

$$\text{Re} \log \left( \frac{D(\beta)}{D(0)} \right) = \sum_{n=1}^{\infty} n^2 \log \left| 1 - \frac{\beta^2}{n^4} \right|,$$

which is reasonably amenable to direct calculation. The result is plotted in Figure 1, which shows a typical behaviour, cf [1].

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4 A question I have not been able to resolve, unrelated to the present calculation, is to find a ‘momentum space’ explanation of the $n^2$ degeneracy analogous to that in the flat space Coulomb problem. The difficulty here is that momentum space is discrete. Clearly, as the radius of the sphere increases, the SO(4) symmetry should emerge. See Higgs [5].
4. Discussion

Expanding the Coulomb potential in characters, I get,

\[ V(\chi) = \frac{1}{2\pi} \sum_{l=2,4,...} \frac{l \sin l\chi}{l^2 - 1 \sin \chi} , \]

which is orthogonal to the uniform mode, as can be seen more immediately by integrating \( V \) over the sphere to give zero. Figure 1 confirms the conclusion in [1] that, for such a radial deformation, the wavefunction(al) \( \sim D(\beta) \), is normalisable.

The exact solution on the hyperbolic sphere, \( H^3 \), is also known. There is then a continuum.

References.
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