GEOMETRICAL STRUCTURES ON THE COTANGENT BUNDLE

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Abstract. In this paper we study the geometrical structures on the cotangent bundle using the notions of adapted tangent structure and regular vector fields. We prove that the dynamical covariant derivative on $T^*M$ fix a nonlinear connection for a given $J$-regular vector field. Using the Legendre transformation induced by a regular Hamiltonian, we show that a semi-Hamiltonian vector field on $T^*M$ corresponds to a semispray on $TM$ if and only if the nonlinear connection on $TM$ is just the canonical nonlinear connection induced by the regular Lagrangian.

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Key words: adapted tangent structure, regular vector field, dynamical covariant derivative, Jacobi endomorphism, Hamiltonian vector field.

1. Introduction

It is well known that many geometric structures on the cotangent bundle $T^*M$ of a differentiable manifold $M$ can be studied using the same methods as in the case of tangent bundle $TM$. The tangent bundle has a naturally defined integrable tangent structure and together with a semispray (second order differential equation vector field) induce a nonlinear connection [5, 9]. However, in the case of the cotangent bundle we do not have a canonical tangent structure or something similar to a semispray, but there are several dual objects, as the adapted almost tangent structure and regular vector fields [19]. Also, we have some canonical geometric objects: Liouville-Hamilton vector field, Liouville 1-form and symplectic structure [1, 17, 26]. The existence of a regular Hamiltonian on $T^*M$ or a regular Lagrangian on $TM$ permit us to transfer many geometrical structures between these spaces using the Legendre transformation [11, 17, 20]. In the paper [21] the dynamical covariant derivative on the cotangent bundle is introduced as a tensor derivation and the metric nonlinear connections are studied. In this paper we investigate more geometrical structures on the cotangent bundle and show how the dynamical covariant derivative induced by an adapted tangent structure and a regular vector field fix the nonlinear connection.

The paper is organized as follows. In the first section we present the preliminary results on the cotangent bundle (see for instance [17, 18, 19, 26]). Also, we study the tension and the strong torsion of the nonlinear connection and investigate the homogeneous case. Using the Frölicher-Nijenhuis bracket
we study the Jacobi endomorphism on the cotangent bundle. In the second section, using the notions of $J$-regular vector field and an arbitrary nonlinear connection we introduce the dynamical covariant derivative on the cotangent bundle and prove that the condition of compatibility with the adapted tangent structure, that is $\nabla J = 0$, determines the nonlinear connection. In the third section we investigate the nonlinear connection induced by a $J$-regular vector field. We introduce the almost complex structure using the Frölicher-Nijenhuis bracket and prove that its dynamical covariant derivative vanishes. These subjects can be found in the case of tangent bundle in a lot of papers (see for instance [3, 6, 10, 12, 13, 14, 22, 23, 24]).

In the last part, these geometrical structures are exemplified in the case of Hamiltonian space. Finally, using the Legendre transformation induced by a regular Hamiltonian, we prove that a semispray on $TM$ corresponds to a semi-Hamiltonian vector field on $T^*M$ if and only if the nonlinear connection on $TM$ determined by semispray is just the canonical nonlinear connection induced by the regular Lagrangian.

2. Preliminary structures on the cotangent bundle

If $M$ be a differentiable, $n$-dimensional manifold and $(T^*M, \tau, M)$ the cotangent bundle then the local coordinates on $\tau^{-1}(U)$ are denoted $(x^i, p_i)$, $(i = 1, n)$. The natural basis on $T^*M$ is denoted $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i})$ and we consider the following geometric objects

$$C^* = p_i \frac{\partial}{\partial p_i}, \quad \theta = p_i dx^i, \quad \omega = d\theta = dp_i \wedge dx^i,$$

where $(dx^i, dp_i)$ is the dual natural basis. The following properties hold:

1° $C^*$ is a vertical vector field, globally defined on $T^*M$, called the Liouville-Hamilton vector field.

2° The 1-form $\theta$ is globally defined on $T^*M$ and is called the Liouville 1-form.

3° $\omega$ is a symplectic structure, called canonical.

The Poisson bracket $\{\cdot, \cdot\}$ on $T^*M$, is defined by

$$\{f, g\} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial x^i}, \quad \forall f, g \in \mathcal{F}(T^*M).$$

In the following by a $d$-tensor field we mean a tensor field on $T^*M$ whose components, under a change of coordinates on $T^*M$, behave like the components of a tensor field on $M$.

We recall that if $L$ and $K$ are $(1, 1)$-type tensor field, Frölicher-Nijenhuis bracket $[L, K]$ is the vector valued 2-form [8]

$$[L, K](X, Y) = [LX, KY] + [KX, LY] + (LK + KL)[X, Y] - L[X, KY] - K[X, LY] - L[KX, Y] - K[LX, Y].$$
and the Nijenhuis tensor of $L$ is given by
\[ N_{L}(X, Y) = \frac{1}{2} [L, L] = [LX, LY] + L^2[X, Y] - L[X, LY] - L[LX, Y]. \]

For a vector field in $\mathcal{X}(M)$ and a $(1,1)$-type tensor field $L$ on $M$ the Frölicher-Nijenhuis bracket $[X, L] = \mathcal{L}_{X}L$ is the $(1,1)$-type tensor field on $M$ given by
\[ \mathcal{L}_{X}L = \mathcal{L}_{X} \circ L - L \circ \mathcal{L}_{X}, \]
where $\mathcal{L}_{X}$ is the usual Lie derivative.

On the cotangent bundle $T^*M$ there exists the integrable vertical distribution $V_{u}T^{*}M$, $u \in T^{*}M$ generated locally by the basis $\{ \partial/\partial p_{i} \}_{i=1}^{n}$. A nonlinear connection $\mathcal{N}$ is a horizontal distribution $H_{u}T^{*}M$ which is supplementary to the vertical distribution, that is $T_{u}T^{*}M = V_{u}T^{*}M \oplus H_{u}T^{*}M$. If $\mathcal{N}$ is a nonlinear connection then on the every domain of the local chart $\tau^{-1}(U)$, the adapted basis of the horizontal distribution $HT^{*}M$ is
\[ \frac{\delta}{\delta x^{i}} = \frac{\partial}{\partial x^{i}} + \mathcal{N}_{ij} \frac{\partial}{\partial p_{j}}, \]
where $\mathcal{N}_{ij}(x, p)$ are the coefficients of the nonlinear connection. The dual adapted basis is
\[ \delta_{p_{i}} = dp_{i} - \mathcal{N}_{ij} dx^{j}. \]

The system of vector fields $\left( \frac{\delta}{\delta x^{i}}, \frac{\partial}{\partial p_{i}} \right)$ defines the local Berwald basis on $T^{*}M$. A nonlinear connection induces the horizontal and vertical projectors given by
\[ h = \frac{1}{2}(Id + \mathcal{N}), \quad v = \frac{1}{2}(Id - \mathcal{N}), \]
and locally
\[ h = \frac{\delta}{\delta x^{i}} \otimes dx^{i}, \quad v = \frac{\partial}{\partial p_{i}} \otimes \delta p_{i}. \]

The following properties hold
\[ h^2 = h, \quad v^2 = v, \quad hv = vh = 0, \quad h + v = Id, \quad h - v = \mathcal{N}. \]

We consider the nonlinear connection $\mathcal{N}$ and denote $\tau_{ij} = \frac{1}{2}(\mathcal{N}_{ij} - \mathcal{N}_{ji})$. The nonlinear connection $\mathcal{N}$ on $T^{*}M$ is called symmetric if \[ \omega(hX, hY) = 0, \quad X, Y \in \mathcal{X}(T^{*}M), \]
where $h$ is the horizontal projector induced by nonlinear connection. Locally, we obtain that the nonlinear connection is symmetric if and only if $\tau_{ij} = 0$, that is $\mathcal{N}_{ij} = \mathcal{N}_{ji}$.

The following equations hold
\[ \frac{\delta}{\delta x^{i}} \frac{\partial}{\partial p_{k}}, \quad \left[ \frac{\delta}{\delta x^{i}}, \frac{\partial}{\partial p_{j}} \right] = - \frac{\partial \mathcal{N}_{ir}}{\partial p_{j}} \frac{\partial}{\partial p_{r}}, \quad \left[ \frac{\partial}{\partial p_{i}}, \frac{\partial}{\partial p_{j}} \right] = 0 \]
\[ R_{ijk} = \frac{\delta \mathcal{N}_{jk}}{\delta x^{i}} - \frac{\delta \mathcal{N}_{ik}}{\delta x^{j}}. \]
The curvature of the nonlinear connection $\mathcal{N}$ on $T^*M$ is given by $\Omega = -\frac{1}{2} [h, h]$ where $h$ is the horizontal projector and $\frac{1}{2} [h, h]$ is the Nijenhuis tensor of $h$. In local coordinates we obtain

$$\Omega = -\frac{1}{2} R_{ijk} dx^i \wedge dx^j \otimes \frac{\partial}{\partial p_k},$$

where $R_{ijk}$ is given by (2) and is called the curvature $d$-tensor of the nonlinear connection $\mathcal{N}$. The curvature of a nonlinear connection is an obstruction to the integrability of the horizontal distribution. Using (1), it results that the horizontal distribution is integrable if and only if the curvature vanishes. Also, from the Jacobi identity we obtain

$$[h, \Omega] = 0.$$

**Definition 1.** The tension of the nonlinear connection $\mathcal{N}$ is given by

$$t = \frac{1}{2} L_{C^*} \mathcal{N}$$

In local coordinates we obtain

$$t = (p_k \frac{\partial N_{ij}}{\partial p_k} - N_{ij}) dx^i \otimes \frac{\partial}{\partial p_j}$$

It results that the tension of the nonlinear connection vanishes if and only if $\mathcal{N}$ is homogeneous of degree one with respect to $p$.

**Definition 2.** An almost tangent structure on $T^*M$ is a morphism $\mathcal{J} : \mathcal{X}(T^*M) \rightarrow \mathcal{X}(T^*M)$ of rank $n$ such that $\mathcal{J}^2 = 0$. The almost tangent structure is called adapted if $[19]$

$$\text{Im} \mathcal{J} = \text{Ker} \mathcal{J} = V T^*M.$$  

The following properties hold

(3) \quad $\mathcal{J} h = \mathcal{J}$, \quad $h \mathcal{J} = 0$, \quad $\mathcal{J} v = \mathcal{J}$, \quad $v \mathcal{J} = \mathcal{J}$.  

Locally, an adapted almost tangent structure has the form

(4) \quad $\mathcal{J} = t_{ij} dx^i \otimes \frac{\partial}{\partial p_j}$,

where $t_{ij}(x, p)$ is a $d$-tensor field of rank $n$. The adapted almost tangent structure $\mathcal{J}$ is integrable if and only if

(5) \quad $\frac{\partial t_{ij}}{\partial p_k} = \frac{\partial t_{kj}}{\partial p_i}$,

where $t_{ij} t^{jk} = \delta_i^k$. The adapted almost tangent structure $\mathcal{J}$ is called symmetric if

$$\omega(\mathcal{J} X, Y) = \omega(\mathcal{J} Y, X).$$

Locally, this relation is equivalent with the symmetry of the tensor $t_{ij}(x, p)$. 

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Example 1. If \( g \) is a pseudo-Riemannian metric on vertical subbundle \( VT^*M \), then there exists a unique adapted almost tangent structure \( \mathcal{J} \) on \( T^*M \) such that
\[
\langle \mathcal{J}X, \mathcal{J}Y \rangle = -\omega(X, Y), \quad X, Y \in \mathfrak{X}(T^*M),
\]
and we say that \( \mathcal{J} \) is induced by the metric \( g \). Locally, if we consider
\[
g(x, p) = g^{ij} dp_i \otimes dp_j,
\]
then (6) implies that \( t_{ij} = g^{ij} \).

Definition 3. An adapted almost tangent structure \( \mathcal{J} \) is called homogeneous if
\[
\mathcal{L}_{C^*} \mathcal{J} = -\mathcal{J}.
\]

Proposition 1. The adapted almost tangent structure is homogeneous if and only if the local components \( t_{ij}(x, p) \) are 0-homogeneous with respect to \( p \).

Proof. By direct computation in local coordinates we obtain
\[
\mathcal{L}_{C^*} \mathcal{J} = \left( p_k \frac{\partial t_{ij}}{\partial p_k} - t_{ij} \right) dx^i \otimes \frac{\partial}{\partial p_j}
\]
and the equation (7) leads to \( p_k \frac{\partial t_{ij}}{\partial p_k} = 0 \), which end the proof. \( \square \)

Definition 4. The torsion of a nonlinear connection \( \mathcal{N} \) on \( T^*M \) is defined by \( T = [\mathcal{J}, h] \), where \( h \) is the horizontal projector and \( [\mathcal{J}, h] \) is the Frolicher-Nijenhuis bracket
\[
[\mathcal{J}, h](X, Y) = [\mathcal{J}X, hY] + [hX, \mathcal{J}Y] + \mathcal{J}[X, Y] - \mathcal{J}[X, hY] - \mathcal{J}[hX, Y] - h[X, \mathcal{J}Y] - h[\mathcal{J}X, Y].
\]

Locally, we consider
\[
T = \frac{1}{2} \mathcal{T}_{ijk}(dx^i \wedge dx^j) \otimes \frac{\partial}{\partial p_k},
\]
and by straightforward computation, it results
\[
\mathcal{T}_{ijk} = t_{ik} \frac{\partial N_{jk}}{\partial p_s} - t_{js} \frac{\partial N_{ik}}{\partial p_s} + \frac{\delta t_{jk}}{\delta x^j} \frac{\delta t_{ik}}{\delta x^j}.
\]

Proposition 2. The following equations hold
\[
[\mathcal{J}, \Omega] = [h, \mathcal{T}], \quad [C^*, \Omega] = [h, \mathcal{T}].
\]

Proof. We apply the Jacobi identity for the Frolicher-Nijenhuis bracket
\[
[\mathcal{J}, [h, h]] + [h, [h, \mathcal{J}]] + [h, [\mathcal{J}, h]] = 0,
\]
\[
-2[\mathcal{J}, \Omega] = -2[h, [h, \mathcal{J}]],
\]
which yields \( [\mathcal{J}, \Omega] = [h, \mathcal{T}] \). Also
\[
[C^*, [h, h]] + [h, [h, C^*]] + [h, [C^*, h]] = 0,
\]
\[
[C^*, [h, h]] = -2[h, [C^*, h]],
\]
and it results \([C^*, \Omega] = [h, t]\), because \(t = \frac{1}{2} \mathcal{L}_{C^*} N = [C^*, h]\).

Let \(\mathcal{J}\) be the adapted almost tangent structure on \(T^*M\). From [19] we set:

**Definition 5.** A vector field \(\rho \in \mathcal{X}(T^*M)\) is called \(\mathcal{J}\)-regular if it satisfies the equation

\[
\mathcal{J}[\rho, \mathcal{J}X] = -\mathcal{J}X, \quad \forall X \in \mathcal{X}(T^*M).
\]

Locally, a vector field on \(T^*M\) given in local coordinates by

\[
\rho = \xi^i(x, p) \frac{\partial}{\partial x^i} + \chi_i(x, p) \frac{\partial}{\partial p_i},
\]

is \(\mathcal{J}\)-regular if and only if

\[
t^{ij} = \frac{\partial \xi^j}{\partial p_i},
\]

where \(t^{ij}t^{jk} = \delta^i_k\). We have to remark that, if the equation \(\mathcal{J}[\rho, \mathcal{J}Y] = -\mathcal{J}Y\) is satisfied for any \(Y \in \mathcal{X}(T^*M)\), with the condition \(\text{rank} \left[ \frac{\partial \xi^j}{\partial p_i} \right] = n\), then \(\mathcal{J}\) is an integrable structure.

**Proposition 3.** If \(\mathcal{J}\) is a homogeneous adapted tangent structure then a vector field \(\rho\) is \(\mathcal{J}\)-regular if and only if

\[
\mathcal{J}\rho = C^*.
\]

**Proof.** Suppose that \(\rho\) is \(\mathcal{J}\)-regular, then \(t^{ij} = \frac{\partial \xi^j}{\partial p_i}\) and \(t_{ij}\) is 0-homogeneous, hence \(\xi^j\) is 1-homogeneous with respect to \(p\), therefore \(p_i \frac{\partial \xi^j}{\partial p_i} = \xi^j\) that is \(p_i t^{ij} = \xi^j\) which yields \(\mathcal{J}\rho = C^*\). Conversely, if \(\mathcal{J}\rho = C^*\) then \(\xi^j = p_i t^{ij}\) implies

\[
\frac{\partial \xi^j}{\partial p_k} = \frac{\partial t^{ij}}{\partial p_k} p_i + t^{kj} = \frac{\partial t^{kj}}{\partial p_i} p_i + t^{kj} = t^{kj},
\]

so \(\rho\) is \(\mathcal{J}\)-regular. \(\Box\)

**Definition 6.** The strong torsion of the nonlinear connection has the form

\[
T = i_\rho T - t.
\]

where \(i_\rho\) is the interior product with respect to the \(\mathcal{J}\)-regular vector field.

In local coordinates we obtain that

\[
T = \left( \xi^i T_{ijk} + N_{jk} - p_s \frac{\partial N_{jk}}{\partial p_s} \right) dx^j \otimes \frac{\partial}{\partial p_k},
\]

with \(T_{ijk}\) from (8). Using the relations (11) and (12) we obtain:

**Proposition 4.** The strong torsion of a nonlinear connection vanishes if the weak torsion and the tension vanish. Conversely is true if the nonlinear connection is homogeneous of degree one.
Definition 7. For a $\mathcal{J}$-regular vector field $\rho$ and an arbitrary nonlinear connection, consider the vertically valued $(1,1)$-type tensor field on $T^*M \setminus \{0\}$ given by

$$\Phi = v \circ L_\rho h,$$

which will be called the Jacobi endomorphism.

It results that

$$\Phi = v \circ L_\rho h = -v \circ L_\rho v = v \circ (L_\rho \circ h - h \circ L_\rho) = v \circ L_\rho \circ h.$$

In local coordinates we obtain

$$L_\rho \frac{\delta}{\delta x^i} = -\frac{\delta \xi^j}{\delta x^i} + R_{jk} \frac{\partial}{\partial p_j}, \quad L_\rho \frac{\partial}{\partial p_j} = -t_{ji} \frac{\delta}{\delta x^i} + (t_{ji} N_{ik} - \frac{\partial \chi_k}{\partial p_j}) \frac{\partial}{\partial p_k}.$$

Locally, the Jacobi endomorphism has the form

$$\Phi = R_{ij} dx^i \otimes \frac{\partial}{\partial p_j}, \quad R_{ij} = \frac{\delta \xi^i}{\delta x^j} N_{ik} - \frac{\delta \chi_k}{\delta x^j} + \rho(N_{ij}).$$

Proposition 5. The following result holds

$$\Phi = i_\rho \Omega + v \circ L_{v\rho} h.$$  

**Proof.** Indeed, $\Phi(X) = v[\rho, hX] = v[h\rho, hX] + v[v\rho, hX]$ and $\Omega(\rho, X) = v[h\rho, hX]$, which yields $\Phi(X) = \Omega(\rho, X) + v[v\rho, hX]$. □

Proposition 6. If $\rho$ is a horizontal vector field then

$$\Phi = i_\rho \Omega.$$  

**Proof.** We have $\rho = h\rho$ and $v\rho = 0$ which yields $\Phi = i_\rho \Omega$ and locally it results

$$\rho = \xi^i \frac{\delta}{\delta x^i}, \quad \chi_i = \xi^k N_{ki}, \quad R_{ij} = R_{kij} \xi^k,$$

which show us the relation between the Jacobi endomorphism given by (13) and curvature tensor from (2).

A vector field $\rho$ on $T^*M$ is called a Hamiltonian vector field if it is $\mathcal{J}$-regular and

$$L_\rho \omega = 0,$$

where $\omega$ is the canonical symplectic structure. In local coordinates for $\rho = \xi^i \frac{\partial}{\partial x^i} + \chi_i \frac{\partial}{\partial p^i}$ this condition is equivalent with [19]

$$\begin{align*}
(a) \quad & \frac{\partial \xi^i}{\partial p_i} = \frac{\partial \xi^j}{\partial p_j}, \\
(b) \quad & \frac{\partial \chi_i}{\partial p_j} = -\frac{\partial \xi^j}{\partial x^i}, \\
(c) \quad & \frac{\partial \chi_i}{\partial x^j} = \frac{\partial \chi_j}{\partial x^i}.
\end{align*}$$

A vector field $X \in \mathcal{X}(T^*M)$ is a semi-Hamiltonian vector field if it is $\mathcal{J}$-regular and satisfies the equation

$$i_\nu (L_X \omega) = 0, \quad \forall \nu \in \Gamma(VT^*M),$$

where $i_\nu$ is the interior product. By direct computation, it results that in the case of semi-Hamiltonian vector field, only the conditions (14) $a$ and $b$ are satisfied.
3. Dynamical covariant derivative on cotangent bundle

The notion of dynamical covariant derivative was introduced for the first time in the case of tangent bundle by Cariñena and Martinez [4] as a derivation of degree 0 along the tangent bundle projection (see also [3, 6, 14, 15, 25]). An extensive study and discussion about the dynamical covariant derivative which is associated to a second order vector field (semispray) on $TM$ can be found in [23]. Using [21] we set:

**Definition 8.** A map $\nabla : \mathcal{T}(T^*M \setminus \{0\}) \to \mathcal{T}(T^*M \setminus \{0\})$ is said to be a tensor derivation on $T^*M \setminus \{0\}$ if the following conditions are satisfied:

1) $\nabla$ is $\mathbb{R}$-linear,

2) $\nabla$ is type preserving, i.e. $\nabla(T^r_s(T^*M \setminus \{0\}) \subset T^r_s(T^*M \setminus \{0\})$, for each $(r, s) \in \mathbb{N} \times \mathbb{N}$,

3) $\nabla$ obeys the Leibnitz rule $\nabla(T \otimes S) = \nabla T \otimes S + T \otimes \nabla S$,

4) $\nabla$ commutes with any contractions.

For a $J$-regular vector field $\rho$ and an arbitrary nonlinear connection we consider the map $\nabla : \mathcal{X}(T^*M \setminus \{0\}) \to \mathcal{X}(T^*M \setminus \{0\})$ given by

$$ (15) \quad \nabla = h \circ L_\rho \circ h + v \circ L_\rho \circ v, $$

which is called the dynamical covariant derivative. By setting $\nabla f = \rho(f)$, for $f \in C^\infty(T^*M \setminus \{0\})$ using the Leibnitz rule and the requirement that $\nabla$ commutes with any contraction, we can extend the action of $\nabla$ to arbitrary tensor fields on $T^*M \setminus \{0\}$. For a 1-form on $T^*M \setminus \{0\}$ the dynamical covariant derivative is given by

$$(\nabla \varphi)(X) = \rho(\varphi)(X) - \varphi(\nabla X).$$

For a $(1, 1)$-type tensor field $T$ on $T^*M \setminus \{0\}$ the dynamical covariant derivative has the form

$$ \nabla T = \nabla \circ T - T \circ \nabla. $$

By direct computation we obtain $\nabla h = \nabla v = 0$ and

$$ (16) \quad \nabla \frac{\delta}{\delta x^j} = -\frac{\delta \xi^i}{\delta x^j} \frac{\delta}{\delta x^i}, \quad \nabla dx^j = \frac{\delta \xi^j}{\delta x^i} dx^i. $$

$$ (17) \quad \nabla \frac{\partial}{\partial p_j} = \left( t^{ik} N_{ik} - \frac{\partial x_k}{\partial p_j} \right) \frac{\partial}{\partial p_k}, \quad \nabla \delta p_j = -\left( t^{ki} N_{ij} - \frac{\partial x_j}{\partial p_k} \right) \delta p_k, $$

**Proposition 7.** The following results hold:

$$ (18) \quad h \circ L_\rho \circ J = -h, \quad J \circ L_\rho \circ v = -v, $$

$$ (19) \quad \nabla J = L_\rho J + h - v, \quad \nabla J = \left( \rho(t_{ij}) + t_{kj} \frac{\partial \xi^k}{\partial x^i} - t_{ik} \frac{\partial x_j}{\partial p_k} + 2 N_{jk} \right) dx^i \otimes \frac{\partial}{\partial p_j}. $$
Proof. From (9) we get
\[ J[\rho, JX] = -JX \Rightarrow J([\rho, JX] + X) = 0 \Rightarrow [\rho, JX] + X \in V^*T^*M \]

\[ h([\rho, JX] + X) = 0 \Rightarrow h[\rho, JX] = -hX \Leftrightarrow h \circ L_\rho \circ J = -h. \]
Also, in \( J[\rho, JX] + JX = 0 \) considering \( JX = vZ \) it results \( J[\rho, vZ] = -vZ \Leftrightarrow J \circ L_\rho \circ v = -v \).

In the following we have
\[ \nabla \circ J = h \circ L_\rho \circ h \circ J + v \circ L_\rho \circ v \circ J = v \circ L_\rho \circ J = (Id - h) \circ L_\rho \circ J = L_\rho \circ J - h \circ L_\rho \circ J = L_\rho \circ J + h. \]

But, on the other hand
\[ J \circ \nabla = J \circ L_\rho \circ h = J \circ L_\rho \circ (Id - v) = J \circ L_\rho - J \circ L_\rho \circ v = J \circ L_\rho + v. \]
and it results
\[ \nabla \circ J - J \circ \nabla = L_\rho \circ J + h - J \circ L_\rho - v \Rightarrow \nabla J = L_\rho J + h - v. \]

For the last relation, we have
\[
\nabla J = \nabla \left( t_{ij}dx^i \otimes \frac{\partial}{\partial p_j} \right) = \nabla t_{ij}dx^i \otimes \frac{\partial}{\partial p_j} + t_{ij}dx^i \otimes \nabla \frac{\partial}{\partial p_j} \\
= (\rho(t_{ij})dx^i + t_{ij}\nabla dx^i) \otimes \frac{\partial}{\partial p_j} + t_{ij}dx^i \otimes \nabla \frac{\partial}{\partial p_j} \\
= \rho(t_{ij})dx^i \otimes \frac{\partial}{\partial p_j} + t_{ij}\frac{\delta \xi^i}{\delta x^k}dx^k \otimes \frac{\partial}{\partial p_j} + t_{ij}dx^i \otimes \left( t_{sj}N_{sk} - \frac{\partial \chi_k}{\partial p_j} \right) \frac{\partial}{\partial p_k} \\
= \left( \rho(t_{ij}) + t_{kj}\frac{\delta \xi^k}{\delta x^i} + N_{ij} - t_{ik}\frac{\partial \chi_j}{\partial p_k} \right) dx^i \otimes \frac{\partial}{\partial p_j} \\
= \left( \rho(t_{ij}) + t_{kj}\left( \frac{\delta \xi^k}{\delta x^i} + N_{is}\frac{\partial \xi^k}{\partial p_s} \right) + N_{ij} - t_{ik}\frac{\partial \chi_j}{\partial p_k} \right) dx^i \otimes \frac{\partial}{\partial p_j} \\
= \left( \rho(t_{ij}) + t_{kj}\frac{\delta \xi^k}{\delta x^i} + t_{kj}t_{sk}N_{is} + N_{ij} - t_{ik}\frac{\partial \chi_j}{\partial p_k} \right) dx^i \otimes \frac{\partial}{\partial p_j} \\
= \left( \rho(t_{ij}) + t_{kj}\frac{\delta \xi^k}{\delta x^i} - t_{ik}\frac{\partial \chi_j}{\partial p_k} + 2N_{ij} \right) dx^i \otimes \frac{\partial}{\partial p_j}. \]

\( \square \)

The equations (19) lead to the following result:

**Theorem 1.** For a \( J \)-regular vector field \( \rho \), an arbitrary nonlinear connection \( N \) and \( \nabla \) the dynamical covariant derivative, the following conditions are equivalent:

i) \( \nabla J = 0 \),

ii) \( L_\rho J + h - v = 0 \),

iii) \( N_{ij} = \frac{1}{2} \left( t_{ik}\frac{\partial \chi_j}{\partial p_k} - t_{kj}\frac{\partial \xi^k}{\delta x^i} - \rho(t_{ij}) \right) \).
3.1. The nonlinear connection induced by a $\mathcal{J}$-regular vector field.

Let $\mathcal{J}$ be an adapted tangent structure and $\rho$ a $\mathcal{J}$-regular vector field on $T^*M$. Then

\begin{equation}
N = -\mathcal{L}_\rho \mathcal{J},
\end{equation}

is a nonlinear connection on $T^*M$. In local coordinates the coefficients of the above nonlinear connection are given by \[19\]

\begin{equation}
N_{ij} = \frac{1}{2} \left( t_{ik} \frac{\partial X_j}{\partial p_k} - t_{kj} \frac{\partial X_i}{\partial p_k} - \rho(t_{ij}) \right).
\end{equation}

**Definition 9.** The almost complex structure has the form

\[ F = h \circ \mathcal{L}_\rho h - \mathcal{J}. \]

**Proposition 8.** The following results hold:

\[ F^2 = -\text{Id}, \quad F \circ \mathcal{J} = h, \quad \mathcal{J} \circ F = v, \quad v \circ F = F \circ h = -\mathcal{J}, \]

\[ h \circ F = F \circ v = F + J, \quad N \circ F = F + 2J, \quad F = \mathcal{L}_\rho h - F - J. \]

**Proof.** It results that $F = h \circ \mathcal{L}_\rho h - h \circ \mathcal{L}_\rho - J = h \circ \mathcal{L}_\rho \circ (h - \text{Id}) - J = -h \circ \mathcal{L}_\rho \circ v - J$, $F^2 = \mathcal{L}_\rho \circ v - J$, $F \circ \mathcal{J} = h \circ \mathcal{J} \circ \mathcal{L}_\rho \circ v = -h - v = -\text{Id}$. In the same way, using (18) the other relations can be proved. □

In local coordinates we have

\[ F = t^{ij} \frac{\delta}{\delta x^i} \otimes \delta p_j - t_{ij} \frac{\partial}{\partial p_i} \otimes \delta x^j. \]

For a $\mathcal{J}$-regular vector field $\rho$, we consider the $\mathbb{R}$-linear map $\nabla_0 : \mathcal{X}(T^*M \setminus \{0\}) \to \mathcal{X}(T^*M \setminus \{0\})$ given by

\[ \nabla_0 X = h[\rho, hX] + v[\rho, vX], \quad \forall X \in \mathcal{X}(T^*M \setminus \{0\}). \]

It results that

\[ \nabla_0(fX) = \rho(f)X + f \nabla_0 X, \quad \forall f \in C^\infty(T^*M \setminus \{0\}), \quad X \in \mathcal{X}(T^*M \setminus \{0\}) \]

Any tensor derivation on $T^*M \setminus \{0\}$ is completely determined by its actions on smooth functions and vector fields on $T^*M \setminus \{0\}$ (see [25] generalized Willmore’s theorem). Therefore there exists a unique tensor derivation $\nabla$ on $T^*M \setminus \{0\}$ such that

\[ \nabla |_{C^\infty(T^*M \setminus \{0\})} = \rho, \quad \nabla |_{\mathcal{X}(T^*M \setminus \{0\})} = \nabla_0. \]

We will call the tensor derivation $\nabla$, the *dynamical covariant derivative* induced by the $\mathcal{J}$-regular vector field $\rho$.

To express the action of $\nabla$, we note that for $\rho = \xi^i \frac{\partial}{\partial x^i} + \chi^i \frac{\partial}{\partial p_i}$ it results

\begin{equation}
\left[ \rho, \frac{\partial}{\partial p_j} \right] = -t^{ij} \frac{\delta}{\delta x^i} + \left( t^{ij} \delta_{ik} - t_{ik} \frac{\partial}{\partial p_j} \right) \frac{\partial}{\partial p_k}.
\end{equation}
3.1.1. Application to Hamiltonian case

The action of the dynamical covariant derivative on the Berwald basis has the form

\[ \nabla \frac{\partial}{\partial p_j} = v \left[ \rho, \frac{\partial}{\partial p_j} \right] = \left( \delta_i^j \nabla J - \frac{\partial \chi_k}{\partial p_j} \right) \frac{\partial}{\partial p_k} \]

**Proposition 9.** The dynamical covariant derivative is given by

\[ \nabla = \mathcal{L}_\rho + F + J - \Phi. \]

**Proof.** Using (15) we obtain

\[
\nabla = h \circ \mathcal{L}_\rho \circ h + v \circ \mathcal{L}_\rho \circ v = h \circ (\mathcal{L}_\rho h + h \circ \mathcal{L}_\rho) + v \circ (\mathcal{L}_\rho v + v \circ \mathcal{L}_\rho) = h \circ \mathcal{L}_\rho h + v \circ \mathcal{L}_\rho v + (h + v) \circ \mathcal{L}_\rho = \mathcal{L}_\rho + F + J - \Phi.
\]

**Proposition 10.** The following properties hold:

\[ \nabla J = 0, \quad \nabla F = 0. \]

**Proof.** From (19) it results \( \nabla J = 0 \). From \( F = -h \circ \mathcal{L}_\rho \circ v - J \) and \( \nabla F = \nabla \circ F - F \circ \nabla \) we obtain

\[
\nabla F = (h \circ \mathcal{L}_\rho \circ h + v \circ \mathcal{L}_\rho \circ v) \circ (h \circ \mathcal{L}_\rho \circ v) = -h \circ \mathcal{L}_\rho \circ h \circ \mathcal{L}_\rho \circ v + h \circ \mathcal{L}_\rho \circ v \circ \mathcal{L}_\rho \circ v = h \circ \mathcal{L}_\rho \circ (v \circ \mathcal{L}_\rho \circ v) = h \circ \mathcal{L}_\rho \circ \mathcal{L}_\rho \circ (J \circ \mathcal{L}_\rho \circ v) - (h \circ \mathcal{L}_\rho \circ J) \circ \mathcal{L}_\rho \circ v = -h \circ \mathcal{L}_\rho \circ \mathcal{L}_\rho \circ v + h \circ \mathcal{L}_\rho \circ \mathcal{L}_\rho \circ v = 0.
\]

using relations (18).

We have to remark that if \( \rho \) is a horizontal vector field then \( \nabla \rho = 0 \). Indeed \( \rho = h \rho, \quad v \rho = 0 \) and it results \( \nabla \rho = h \circ \mathcal{L}_\rho \circ h \rho + v \circ \mathcal{L}_\rho \circ v \rho = h \circ \mathcal{L}_\rho \circ \rho = 0 \).

3.1.1. Application to Hamiltonian case. A Hamilton space \([17]\) is a pair \((M, H)\) where \( M \) is a differentiable, \( n \)-dimensional manifolds and \( H \) is a function on \( T^* M \) with the properties:

1° \( H : (x, p) \in T^* M \to H(x, p) \in \mathbb{R} \) is differentiable on \( T^* M \) and continue on the null section of the projection \( \tau : T^* M \to M \)

2° The Hessian of \( H \) with respect to \( p_1 \) is nondegenerate

\[ g^{ij} = \frac{\partial^2 H}{\partial p_i \partial p_j}, \quad \text{rank} \| g^{ij}(x, p) \| = n \text{ on } \hat{T}^* M = T^* M \setminus \{0\} \]
3° $d$-tensor field $g^{ij}(x, p)$ has constant signature on $\tilde{T}^*M$.

The Hamiltonian $H$ on $T^*M$ induces a pseudo-Riemannian metric $g_{ij}$ with $g_{ij}g^{jk} = \delta^k_i$ and $g^{jk}$ given by (24) on $VT^*M$. It induces a unique adapted tangent structure, denoted

$$\mathcal{J}_H = g_{ij}dx^i \otimes \frac{\partial}{\partial p_i},$$

such that (5) is satisfied. A $\mathcal{J}$-regular vector field induced by the regular Hamiltonian $H$ is given by

$$\rho_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} + \chi_i \frac{\partial}{\partial p_i}.$$

There exists a unique Hamiltonian vector field $\rho_H \in \mathcal{X}(T^*M)$ such that

$$i_{\rho_H}\omega = -dH,$$

given by

$$\rho_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^j} \frac{\partial}{\partial p_i}.$$

Indeed, $\mathcal{L}_{\rho_H}\omega = (i_{\rho_H} \circ d + d \circ i_{\rho_H})\omega = i_{\rho_H} \circ d\omega + d \circ i_{\rho_H}\omega = -d^2H = 0$ and $\rho_H$ is a Hamiltonian vector field. The nonlinear connection

$$\mathcal{N} = -\mathcal{L}_{\rho_H}\mathcal{J}_H,$$

has the coefficients given by [17, 20]

$$N_{ij} = \frac{1}{2} \left( \{g_{ij}, H\} - \left( g_{ik} \frac{\partial^2 H}{\partial p_k \partial x^j} + g_{jk} \frac{\partial^2 H}{\partial p_k \partial x^i} \right) \right),$$

and is called the canonical nonlinear connection of the Hamilton space $(M, H)$ which is a metric nonlinear connection [21].

Let us consider the regular Hamiltonian $H(x, p)$ on $T^*M$ which induces a local diffeomorphism $\Psi : T^*M \to TM$ given by

$$x^i, \quad y^i = \xi^i(x, p) = \frac{\partial H}{\partial p_i},$$

and $\Psi^{-1}$ has the following components

$$x^i, \quad p_i = \zeta_i(x, y) = \frac{\partial L}{\partial y^i},$$

where

$$L(x, y) = \zeta_i y^i - H(x, p),$$

is the Legendre transformation. From the condition for $\Psi^{-1}$ to be the inverse of $\Psi$ we obtain [19]

$$\frac{\partial \zeta_i}{\partial y^j} \circ \Psi = g_{ij}, \quad \frac{\partial \zeta_i}{\partial x^j} \circ \Psi = -g_{ik} \frac{\partial \xi^k}{\partial x^j}, \quad g^{ij} = \frac{\partial \xi^j}{\partial p_i}, \quad \frac{\partial^2 H}{\partial p_i \partial p_j}, \quad g_{ij}g^{jk} = \delta^k_i.$$

$$\Psi_* \left( \frac{\partial}{\partial p_i} \right) = (g^{jk} \circ \Psi^{-1}) \frac{\partial}{\partial y^k}, \quad \Psi_* \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} + \left( \frac{\partial \xi^k}{\partial x^i} \circ \Psi^{-1} \right) \frac{\partial}{\partial y^k},$$
\[ (30) \quad \Psi^{-1}_* \left( \frac{\partial}{\partial y^i} \right) = g_{ki} \frac{\partial}{\partial p_k}, \quad \Psi^{-1}_* \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} - g_{kh} \frac{\partial \xi^h}{\partial x^i} \frac{\partial}{\partial p_k}, \]

where \( \Psi_* \) is the tangent application of \( \Psi \).

We know \([5, 9]\) that if \( S = y^i \frac{\partial}{\partial x^i} + S^i \frac{\partial}{\partial y^i} \) is a semispray and \( J = dx^i \otimes \frac{\partial}{\partial y^i} \) is the canonical tangent structure on \( TM \), then the automorphism

\[ N = -\mathcal{L}_S J, \]

is a nonlinear connection on \( TM \) with the coefficients given by

\[ (31) \quad N^i_j(x, y) = -\frac{1}{2} \frac{\partial S^i}{\partial y^j}. \]

For a regular Lagrangian \( L \) on \( TM \) there exists the canonical nonlinear connection with the coefficients \( (31) \) where

\[ (32) \quad S^i = g^{ij} \left( \frac{\partial L}{\partial x^j} - \frac{\partial^2 L}{\partial x^k \partial y^j} y^k \right), \]

which is a metric nonlinear connection \([2]\). The next result shows the relation between the canonical nonlinear connection induces by the regular Lagrangian on \( TM \) and the semi-Hamiltonian vector field on \( T^*M \), via Legendre transformation. Let us consider a semispray \( S \) on \( TM \) and \( \Psi^{-1} \) the local diffeomorphism given by \( (28) \).

**Theorem 2.** The vector field \( \rho = \Psi^{-1}_* S \) is a semi-Hamiltonian \( J \)-regular vector field on \( T^*M \) if and only if the nonlinear connection on \( TM \) induces by semispray is the canonical nonlinear connection induced by the regular Lagrangian.

**Proof.** We consider a semispray \( S = y^i \frac{\partial}{\partial x^i} + S^i \frac{\partial}{\partial y^i} \) on \( TM \) and from \( (30) \) it results

\[ \Psi^{-1}_* S = \xi^i \frac{\partial}{\partial q^i} + \left( -\xi^i g_{kj} \frac{\partial \xi^j}{\partial x^i} + S^i g_{ik} \right) \frac{\partial}{\partial p_k}. \]

This together with the conditions \((14) \ b)\) and \((30)\) lead to

\[ g^{kj} \left( \frac{\partial \xi_i}{\partial x^k} - \frac{\partial \xi_k}{\partial x^i} + \frac{\partial S^i}{\partial y^k} g_{li} + \xi^l \frac{\partial g_{ik}}{\partial x^l} + S^l \frac{\partial g_{ik}}{\partial y^l} \right) = 0, \]

and using \((31)\) we obtain

\[ \frac{\partial \xi_i}{\partial x^k} - \frac{\partial \xi_k}{\partial x^i} = S(g_{ik}) - 2N^l_k g_{li}, \]

and it results, considering the symmetric part (interchanging \( i \) with \( k \) and totalizing)

\[ (33) \quad S(g_{ik}) - N^l_k g_{li} - N^l_i g_{lk} = 0 \iff \nabla g \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^k} \right) = 0, \]

which means that \( N \) on \( TM \) is a metric nonlinear connection, and the antisymmetric part

\[ \frac{\partial \xi_i}{\partial x^k} - \frac{\partial \xi_k}{\partial x^i} = N^l_i g_{lk} - N^l_k g_{li}, \]
where \( g_{il} = g_{il} \circ \Psi^{-1} \) by abuse, which is equivalent with

\[
N^l_{ik} - N^l_{ik} + \frac{\partial^2 L}{\partial x^k \partial y^i} - \frac{\partial^2 L}{\partial x^i \partial y^k} = 0 \iff \omega_L(hX, hY) = 0,
\]

where \( \omega_L \) is the symplectic structure induced by the regular Lagrangian on \( TM \).

From (33) and (34), using the Theorem 3.1 from [2] it results that \( N \) is the canonical nonlinear connection induced by the regular Lagrangian on \( TM \).

\[\Box\]

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