Continuous and Discrete Homotopy Operators with Applications in Integrability Testing

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In honor of Ryan Sayers (1982-2003)

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Abstract

We introduce calculus-based formulas for the continuous Euler and homotopy operators. The 1D continuous homotopy operator automates integration by parts on the jet space. Its 3D generalization allows one to invert the total divergence operator. As a practical application, we show how the operators can be used to symbolically compute local conservation laws of nonlinear systems of partial differential equations in multi-dimensions.

By analogy to the continuous case, we also present concrete formulas for the discrete Euler and homotopy operators. Essentially, the discrete homotopy operator carries out summation by parts. We use it to algorithmically invert the forward difference operator. We apply the discrete operator to compute fluxes of differential-difference equations in (1+1) dimensions.

Our calculus-based approach allows for a straightforward implementation of the operators in major computer algebra systems, such as Mathematica and Maple. The symbolic algorithms for integration and summation by parts are illustrated with elementary examples. The algorithms to compute conservation laws are illustrated with nonlinear PDEs and their discretizations arising in fluid dynamics and mathematical physics.

1 Introduction

This chapter focuses on symbolic methods to compute polynomial conservation laws of partial differential equations (PDEs) in multi-dimensions and differential-difference equations (DDEs) (semi-discrete lattices). For the latter we treat only (1+1) dimensional systems where time is continuous and the spacial variable has been discretized.

There are several strategies to compute conservation laws of PDEs. Some methods use a generating function [2], which requires the knowledge of key pieces of the Inverse Scattering Transform [1]. Other methods use Noether’s theorem to get conservation laws from variational symmetries. More algorithmic methods, some of which circumvent the existence of a variational principle [5, 6, 24, 33], require the solution of a determining system of ODEs or PDEs. Despite their power, only a few of these methods have been implement in computer algebra systems (CAS), such as Mathematica, Maple, and REDUCE. See [15, 33] for reviews.

We advocate a more direct approach by building the candidate density as a linear combination (with constant coefficients) of terms that are uniform in rank (with respect to the scaling symmetry of the PDE). Although restricted to polynomial densities and fluxes, our method is entirely algorithmic and can be implemented in most CAS. We refer the reader to [15, 16] for details about an implementation in Mathematica, which can be downloaded from [18]. An implementation in Maple is also available [11].

Our earlier algorithm [15, 16] worked only for nonlinear PDEs in one spacial variable. In this chapter we present an algorithm that works for systems of PDEs in multi-dimensions that appear in fluid mechanics, elasticity, gas dynamics, general relativity, (magneto-)hydrodynamics, etc. The new algorithm produces densities in which all divergences and equivalent terms have been removed. An additional advantage of our methods to compute densities and fluxes is that they can be applied to nonlinear DDEs [16, 19, 20].

During the development of our methods we came across tools from the calculus of variations and differential geometry that deserve attention in their own right. These tools are the
variational derivative, the higher Euler operators, and the homotopy operator.

To set the stage, we address a few issues arising in multivariate calculus:

(i) To determine whether or not a vector field $\mathbf{F}$ is conservative, i.e. $\mathbf{F} = \nabla f$ for some scalar field $f$, one must verify that $\mathbf{F}$ is irrotational, that is $\nabla \times \mathbf{F} = \mathbf{0}$. The field $f$ can be computed via standard integrations [29, p. 518, 522].

(ii) To test if $\mathbf{F}$ is the curl of some vector field $\mathbf{G}$, one must check that $\mathbf{F}$ is incompressible or divergence free, i.e. $\nabla \cdot \mathbf{F} = 0$. The components of $\mathbf{G}$ result from solving a coupled system of first-order PDEs [29, p. 526].

(iii) To verify whether or not a scalar field $f$ is the divergence of some vector function $\mathbf{F}$, no theorem from vector calculus comes to the rescue. Furthermore, the computation of $\mathbf{F}$ such that $f = \nabla \cdot \mathbf{F}$ is a nontrivial matter. In single variable calculus, it boils down to computing the primitive $F = \int f \, dx$.

In multivariate calculus, all scalar fields $f$, including the components $F_i$ of vector fields $\mathbf{F} = (F_1, F_2, F_3)$, are functions of the independent variables $(x, y, z)$. In differential geometry one addresses the above issues in much greater generality. There, the functions $f$ and $F_i$ can depend on arbitrary functions $u(x, y, z), v(x, y, z)$, etc. and their mixed derivatives (up to a fixed order) with respect to the independent variables $(x, y, z)$. Such functions are called differential functions [30]. As one might expect, carrying out the gradient-, curl-, or divergence-test requires advanced algebraic machinery. For example, to test whether or not $f = \nabla \cdot \mathbf{F}$ requires the use of the variational derivative (Euler operator) in 3D. The actual computation of $\mathbf{F}$ requires integration by parts. That is where the homotopy operator and the variational complex come into play.

In 1D problems the continuous total homotopy operator\(^1\) reduces the problem of symbolic integration by parts to an integration with respect to a single variable. In 2D and 3D, the homotopy operator allows one to invert the total divergence operator and, again, reduce the problem to a single integration. At the moment, no major CAS have reliable routines for integrating expressions involving unknown functions and their derivatives. As far as we know, no CAS offer a function to test if a differential function is a divergence. Routines to symbolically invert the total divergence are certainly lacking.

The continuous homotopy operator is a universal, yet little known, tool that can be applied to many problems in which integration by parts of arbitrary functions plays a significant role. We refer the reader to [30, p. 374] for a history of the homotopy operator in the context of inverse problems of the calculus of variations.

A major motivation for writing this chapter is to demystify the homotopy operators. Therefore, we purposely avoid differential forms and abstract concepts such as the variational bicomplex. Instead, we present down-to-earth calculus formulas for the homotopy operators which makes them readily implementable in major CAS.

By analogy with the continuous case, we also present formulas for the discrete versions of the Euler and homotopy operators. The discrete homotopy operator is a powerful tool to invert the forward difference operator, whatever the application is. It circumvents the necessary summation (by parts) by applying a set of variational derivatives followed by a one-dimensional integration with respect to an auxiliary parameter. We use the homotopy operator to compute conserved fluxes of DDEs. Numerous examples of such DDEs are given.

\(^1\)Hence forth, homotopy operator instead of total homotopy operator.
Beyond DDEs, the discrete homotopy operator has proven to be useful in the study of difference equations [22, 27]. To our knowledge, CAS offer no tools to invert the forward difference operator. Once fully implemented, our discrete homotopy operator will overcome the shortcomings.

As shown in [22, 27], the parallelism between the continuous and discrete cases can be made rigorous and both theories can be formulated in terms of variational bicomplexes. To make our work accessible to as wide an audience as possible, we do not explicitly use the abstract framework. Aficionados of de Rham complexes may consult [7, 8, 9, 25] and [22, 27, 28]. The latter papers cover the discrete variational bicomplexes.

2 Examples of Nonlinear PDEs

We consider nonlinear systems of evolution equations in (3 + 1) dimensions,

\[ u_t = G(u, u_x, u_y, u_z, u_{2x}, u_{2y}, u_{2z}, u_{xy}, u_{xz}, u_{yz}, \ldots), \]  

where \( x = (x, y, z) \) are space variables and \( t \) is time. The vector \( u(x, y, z, t) \) has \( N \) components \( u_i \). In the examples we denote the components of \( u \) by \( u, v, w, \) etc. Subscripts refer to partial derivatives. For brevity, we use \( u_{2x} \) instead of \( u_{xx} \), etc. and write \( G(u^{(n)}) \) to indicate that the differential function \( G \) depends on derivatives up to order \( n \) of \( u \) with respect to \( x, y, \) and \( z \). For simplicity, we assume that \( G \) does not explicitly depend on \( x \) and \( t \). No restrictions are imposed on the number of components, the order, and the degree of nonlinearity of the variables in \( G \).

We will predominantly work with polynomial systems, although systems involving one transcendental nonlinearity can also be handled. If parameters are present in (1), they will be denoted by lower-case Greek letters.

**Example 1:** The coupled Korteweg-de Vries (cKdV) equations [1],

\[ u_t - 6\beta uu_x + 6vv_x - \beta u_{3x} = 0, \quad v_t + 3uv_x + v_{3x} = 0, \quad \beta \neq 0, \]  

where \( \beta \) is a nonzero parameter, describes interactions of two waves with different dispersion relations. System (2) is known in the literature as the Hirota-Satsuma system. It is completely integrable [1, 21] when \( \beta = \frac{1}{2} \).

**Example 2** The sine-Gordon (SG) equation [10, 26], \( u_{2t} - u_{2x} = \sin u \), can be written as a system of evolution equations,

\[ u_t = v, \quad v_t = u_{2x} + \sin u. \]

This system occurs in numerous areas of mathematics and physics, ranging from surfaces with constant mean curvature to superconductivity.

**Example 3:** The (2+1)-dimensional shallow water wave (SWW) equations [12],

\[
\begin{align*}
    u_t + (u \cdot \nabla)u + 2\Omega \times u &= -\nabla(h\theta) + \frac{1}{2}h\nabla\theta, \\
    \theta_t + u \cdot (\nabla \theta) &= 0, \\
    h_t + \nabla \cdot (hu) &= 0,
\end{align*}
\]  

3
describe waves in the ocean using layered models. Vectors \( \mathbf{u} = u(x, y, t)\mathbf{i} + v(x, y, t)\mathbf{j} \) and \( \mathbf{\Omega} = \mathbf{\Omega}k \) are the fluid and angular velocities, respectively. \( \mathbf{i}, \mathbf{j}, \text{ and } \mathbf{k} \) are unit vectors along the \( x, y, \text{ and } z \)-axes. \( \theta(x, y, t) \) is the horizontally varying potential temperature field, and \( h(x, y, t) \) is the layer depth. The dot (\( \cdot \)) stands for Euclidean inner product and \( \nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} \) is the gradient operator. System (4) is written in components as

\[
\begin{align*}
  u_t + uu_x + vu_y - 2\Omega v + \frac{1}{2}h\theta_x + \theta h_x &= 0, \\
v_t + uv_x + vv_y + 2\Omega u + \frac{1}{2}h\theta_y + \theta h_y &= 0, \\
\theta_t + u\theta_x + v\theta_y &= 0, \\
h_t + uh_x + uh_z + hv_y + vh_y &= 0.
\end{align*}
\]

(5)

3 Key Definitions–Continuous Case

**Definition:** System (1) is said to be *dilation invariant* if it is invariant under a scaling (dilation) symmetry.

**Example:** The cKdV system (2) is invariant under the scaling symmetry

\( (x, t, u, v) \rightarrow (\lambda^{-1}x, \lambda^{-3}t, \lambda^2u, \lambda^2v) \),

(6)

where \( \lambda \) is an arbitrary scaling parameter.

**Definition:** We define the weight, \( W \), of a variable as the exponent of \( \lambda \) that multiplies the variable.

**Example:** We will always replace \( x \) by \( \lambda^{-1}x \). Thus, \( W(x) = -1 \) or \( W(\partial/\partial x) = 1 \). From (6), we have \( W(\partial/\partial t) = 3 \) and \( W(u) = W(v) = 2 \) for the cKdV equations.

**Definition:** The rank of a monomial is defined as the total weight of the monomial. An expression (or equation) is *uniform in rank* if its monomial terms have equal rank.

**Example:** Coincidentally, all monomials in both equations of (2) have rank 5. Thus, (2) is uniform in rank. The ranks of the equations in (1) may differ from each other. Conversely, requiring uniformity in rank for each equation in (1) allows one to compute the weights of the variables (and thus the scaling symmetry) with linear algebra.

**Example:** For the cKdV equations (2), one has

\[
\begin{align*}
  W(u) + W(\partial/\partial t) &= 2W(u) + 1 = 2W(v) + 1 = W(u) + 3, \\
  W(v) + W(\partial/\partial t) &= W(u) + W(v) + 1 = W(v) + 3,
\end{align*}
\]

(7)

which yields \( W(u) = W(v) = 2, W(\partial/\partial t) = 3 \), which leads to (6).

Dilation symmetries, which are special Lie-point symmetries, are common to many nonlinear PDEs. However, non-uniform PDEs can be made uniform by extending the set of dependent variables with auxiliary parameters with appropriate weights. Upon completion of the computations one can set these parameters to one.

**Example:** The sine-Gordon equation (3) is not uniform in rank unless we replace it by

\[
\begin{align*}
u_t &= v, \\
v_t &= u_{2x} + \alpha \sin u, \quad \alpha \in \mathbb{R}.
\end{align*}
\]

(8)

Using the Maclaurin series for the \( \sin \) function, uniformity in rank requires

\[
\begin{align*}
  W(u) + W(\partial/\partial t) &= W(v), \\
  W(v) + W(\partial/\partial t) &= W(u) + 2 = W(\alpha) + W(u) = W(\alpha) + 3W(u) = W(\alpha) + 5W(u) = \cdots.
\end{align*}
\]

(9)
This forces us to set $W(u) = 0$. Then, $W(\alpha) = 2$. By allowing the parameter $\alpha$ to scale, (8) becomes scaling invariant under the symmetry

$$(x, t, u, v, \alpha) \rightarrow (\lambda^{-1}x, \lambda^{-1}t, \lambda^0u, \lambda^1v, \lambda^2\alpha),$$  

(10)
corresponding to $W(\partial/\partial x) = W(\partial/\partial t) = 1, W(u) = 0, W(v) = 1, W(\alpha) = 2$. The first and second equations in (8) are uniform of ranks 1 and 2, respectively.

**Definition**: System (1) is called **multi-uniform** in rank if it admits more than one dilation symmetry (which is not the result of adding auxiliary parameters with weights).

**Example**: Uniformity in rank for the SWW equations (5) requires, after some algebra, that

$$W(\partial/\partial t) = W(\partial/\partial x) = 1, W(u) = W(v) = W(\Omega) - 1,$$

$$W(\theta) = 2W(\Omega) - W(h) - 2,$$

(11)

where $W(h)$ and $W(\Omega)$ remain free. The SWW system is thus multi-uniform. The symmetry

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda^0\theta, \lambda^1h, \lambda^2\Omega),$$  

(12)

which is most useful for our computations later on, corresponds to $W(\partial/\partial x) = W(\partial/\partial y) = 1, W(\partial/\partial t) = 2, W(u) = W(v) = 1, W(\theta) = 1, W(h) = 1, \text{ and } W(\Omega) = 2$. A second symmetry,

$$(x, y, t, u, v, \theta, h, \Omega) \rightarrow (\lambda^{-1}x, \lambda^{-1}y, \lambda^{-2}t, \lambda u, \lambda v, \lambda^2\theta, \lambda^0h, \lambda^2\Omega),$$  

(13)

matches $W(\partial/\partial x) = W(\partial/\partial y) = 1, W(\partial/\partial t) = 2, W(u) = W(v) = 1, W(\theta) = 2, W(h) = 0, W(\Omega) = 2$.

## 4 Conserved Densities and Fluxes of Nonlinear PDEs

**Definition**: A scalar differential function $\rho(u^{(n)})$ is a conserved density if there exists a vector differential function $J(u^{(m)})$, called the associated flux, such that

$$D_t \rho + \text{Div} J = 0$$  

(14)
is satisfied on the solutions of (1). Equation (14) is called a local\(^2\) conservation law [30], and Div is called the total divergence \(^3\). Clearly, $\text{Div} J = (D_x, D_y, D_z) \cdot (J_1, J_2, J_3) = D_xJ_1 + D_yJ_2 + D_zJ_3$. In the case of one spacial variable ($x$), (14) reduces to

$$D_t \rho + D_x J = 0,$$  

(15)

where both density $\rho$ and flux $J$ are scalar differential functions. In the 1D case,

$$D_t \rho(u^{(n)}) = \frac{\partial \rho}{\partial t} + \sum_{k=0}^{n} \frac{\partial \rho}{\partial u_k} D_x^k u_t,$$  

(16)

where $u^{(n)}$ is the highest order term present in $\rho$. Upon replacement of $u_t, u_{tx}, \text{ etc. from } u_t = G$, one gets

$$D_t \rho = \frac{\partial \rho}{\partial t} + \rho(u)'[G],$$  

(17)

\(^2\)We only compute densities and fluxes free of integral terms.

\(^3\)Gradient, curl, and divergence are in rectangular coordinates.
where $\rho(u)'[G]$ is the Fréchet derivative of $\rho$ in the direction of $G$. Similarly,

$$D_x J(u^{(m)}) = \frac{\partial J}{\partial x} + \sum_{k=0}^{m} \frac{\partial J}{\partial u_{kx}} u_{(k+1)x}. \quad (18)$$

The generalization of (16) and (18) to multiple dependent variable is straightforward. For example, taking $\mathbf{u} = (u, v)$,

$$D_t \rho(u^{(m_1)}, v^{(n_2)}) = \frac{\partial \rho}{\partial t} + \sum_{k=0}^{m_1} \frac{\partial \rho}{\partial u_{kx}} D_x^2 u_t + \sum_{k=0}^{n_2} \frac{\partial \rho}{\partial v_{kx}} D_x^2 v_t, \quad (19)$$

$$D_x J(u^{(m_1)}, v^{(n_2)}) = \frac{\partial J}{\partial x} + \sum_{k=0}^{m_1} \frac{\partial J}{\partial u_{kx}} u_{(k+1)x} + \sum_{k=0}^{n_2} \frac{\partial J}{\partial v_{kx}} v_{(k+1)x}. \quad (20)$$

We will ignore densities and fluxes that explicitly depend on $x$ and $t$. If $G$ is polynomial then most, but not all, densities and fluxes are also polynomial.

**Example:** The first four density-flux pairs for the cKdV equations (2) are

$$\rho^{(1)} = u, \quad J^{(1)} = -3\beta u^2 + 3v^2 - \beta u_{2x}, \quad (any \ \beta) \quad (21)$$

$$\rho^{(2)} = u^2 - 2v^2, \quad J^{(2)} = -4\beta u^3 + \beta u_x^2 - 2\beta uu_{2x} + 2v^2 - 4v v_{2x}, \quad (any \ \beta) \quad (22)$$

$$\rho^{(3)} = uv, \quad J^{(3)} = 3u^2 v + 2u^3 - u_x v_x + u_{2x} v + uv_{2x}, \quad (\beta = -1) \quad (23)$$

and

$$\rho^{(4)} = (1 + \beta) u^3 - 3uv^2 - \frac{1}{2}(1 + \beta) u_x^2 + 3v_x^2, \quad (24)$$

$$J^{(4)} = -\frac{9}{2} \beta (1 + \beta) u^4 + 9\beta u^2 v^2 - \frac{9}{2} v^4 + 6\beta (1 + \beta) uu_x^2 - 3\beta (1 + \beta) u^2 u_{2x}$$

$$+ 3\beta v^2 u_{2x} - \frac{1}{2} \beta (1 + \beta) u_{2x}^2 + \beta (1 + \beta) u_x v_{3x} - 6\beta vu_x v_x + 12uv^2_x$$

$$-6uvv_{2x} - 3v_{2x}^2 + 6v_{3x} v_x \quad (\beta \neq -1). \quad (25)$$

The above densities are uniform in ranks 2, 4 and 6. Both $\rho^{(2)}$ and $\rho^{(3)}$ are of rank 4. The corresponding fluxes are also uniform in rank with ranks 4, 6, and 8. In [15], we listed a few densities of rank $\geq 8$, which only exist when $\beta = \frac{1}{2}$.

In general, if in (15) rank $\rho = R$ then rank $J = R + W(\partial/\partial t) - 1$. All the terms in (14) are also uniform in rank. This comes as no surprise since the conservation law (14) holds on solutions of (1), hence it “inherits” the dilation symmetry of (1).

**Example:** The first few densities [3, 13] for the sine-Gordon equation (8) are

$$\rho^{(1)} = 2\alpha \cos u + v^2 + u_x^2, \quad J^{(1)} = -2u_x v, \quad (26)$$

$$\rho^{(2)} = 2u_x v, \quad J^{(2)} = 2\alpha \cos u - v^2 - u_x^2, \quad (27)$$

$$\rho^{(3)} = 6\alpha vv_x \cos u + v^3 u_x + vu_x^3 - 8v_x u_{2x}, \quad (28)$$

$$\rho^{(4)} = 2\alpha^2 \cos^2 u - 2\alpha^2 \sin^2 u + 4\alpha v^2 \cos u + 20\alpha u_x^2 \cos u + v^4 + 6v^2 u_x^2 + u_x^4$$

$$-16v_x^2 - 16u_{2x}^2. \quad (29)$$

$J^{(3)}$ and $J^{(4)}$ are not shown due to length. Again, all densities and fluxes are uniform in rank (before $\alpha$ is set equal to 1).
Example: The first few conserved densities and fluxes for the SWW equations (5) are
\[ \rho^{(1)} = h, \quad J^{(1)} = \left( \begin{array}{c} uh \\ vh \end{array} \right), \quad \rho^{(2)} = h\theta, \quad J^{(2)} = \left( \begin{array}{c} uh\theta \\ vh\theta \end{array} \right), \quad \rho^{(3)} = h\theta^2, \quad J^{(3)} = \left( \begin{array}{c} uh\theta^2 \\ vh\theta^2 \end{array} \right), \quad \rho^{(4)} = (u^2 + v^2)h + h^2\theta, \quad J^{(4)} = \left( \begin{array}{c} u^3h + uv^2h + 2uh^2\theta \\ v^3h + u^2vh + 2vh^2\theta \end{array} \right), \quad \rho^{(5)} = v_x\theta - u_y\theta + 2\Omega\theta, \]
(30)
\[ J^{(5)} = \frac{1}{6} \left( \begin{array}{c} 12\Omega u\theta - 4uw_y\theta + 6uv_x\theta + 2vv_y\theta + u^2\theta_y + v^2\theta_y - h\theta\theta_y + h\theta^2 \\ 12\Omega v\theta + 4uv_x\theta - 6vu_y\theta - 2uu_x\theta - u^2\theta_x - v^2\theta_x + h\theta\theta_x - h\theta^2 \end{array} \right). \]
(31)

All densities and fluxes are multi-uniform in rank, which will substantially simplify the computation of the densities. Under either of the two scales, (12) or (13), \( \text{rank}(J) = \text{rank}(\rho) + 1 \).

With the exception of \( \rho^{(2)} \) and \( J^{(2)} \), the ranks of the densities under scales (12) and (13) differ by one. The same holds for the fluxes.

5 Tools from the Calculus of Variations

In this section we introduce the variational derivative (Euler operator), the higher Euler operators from the calculus of variations, and the homotopy operator from homological algebra. These tools will be applied to the computation of densities and fluxes in Section 7.

5.1 Continuous Variational Derivative (Euler Operator)

Definition: A scalar differential function \( f \) is a divergence if and only if there exists a vector differential function \( F \) such that \( f = \text{Div} F \). In 1D, we say that a differential function \( f \) is exact \(^4\) if and only if there exists a scalar differential function \( F \) such that \( f = D_x F \). Obviously, \( F = D_x^{-1}(f) = f f \, dx \) is then the primitive (or integral) of \( f \).

Example: Consider
\[ f = 3 u_x v_x^2 \sin u - u_x^3 \sin u - 6 v v_x \cos u + 2 u_x u_{2x} \cos u + 8 v_x v_{2x}, \]
which we encountered [3] while computing conservation laws for (8). \( f \) is exact. Indeed, upon integration by parts (by hand), one gets
\[ F = 4 v_x^2 + u_x^2 \cos u - 3 v_x^2 \cos u. \]
(34)

Most CAS, including Mathematica, Maple\(^5\) and Reduce, fail this elementary integration!

Example: Consider
\[ f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x. \]
(35)

It is easy to verify that \( f = \text{Div} F \) with
\[ F = (uv_y - u_x v_y, -uv_x + u_x v_y). \]
(36)

\(^4\)We do not use integrable to avoid confusion with complete integrability from soliton theory.

\(^5\)Version 9.5 of Maple can integrate such expressions as a result of our interactions with the developers.
As far as we know, the leading CAS have no tools to compute $F$. Three questions arise:

(i) Under what conditions for $f$ does a closed form for $F$ exist?
(ii) If $f$ is a divergence, what is it the divergence of?
(iii) Avoiding integration by parts, how can one design a fast algorithm to compute $F$?

To answer these questions we use the following tools from the calculus of variations: the variational derivative (Euler operator), its generalizations (higher Euler operators), and the homotopy operator.

**Definition:** The variational derivative (Euler operator), $\mathcal{L}_{u(x)}^{(0)}$, is defined [30, p. 246] by

$$\mathcal{L}_{u(x)}^{(0)} = \sum_{J} (-D)_{J} \frac{\partial}{\partial u_{J}},$$

where the sum is over all the unordered multi-indices $J$ [30, p. 95]. For example, in the 2D case the multi-indices corresponding to second-order derivatives can be identified with $\{2x, 2y, 2z, xy, xz, yz\}$. Obviously, $(-D)_{2x} = D_{x}^{2}$, $(-D)_{xy} = D_{x}D_{y}$, etc. For notational details see [30, p. 95, p. 108, p. 246].

With applications in mind, we give explicit formulas for the variational derivatives in 1D, 2D, and 3D. For scalar component $u$ they are

$$\mathcal{L}_{u(x)}^{(0)} = \sum_{k=0}^{\infty} (-D)_{x}^{k} \frac{\partial}{\partial u_{xx}^{k}} = \frac{\partial}{\partial u} - D_{x} \frac{\partial}{\partial u_{x}} + D_{x}^{2} \frac{\partial}{\partial u_{xx}} - D_{x}^{3} \frac{\partial}{\partial u_{xxx}} + \cdots,$$  

$$\mathcal{L}_{u(x,y)}^{(0)} = \sum_{k_{x}=0}^{\infty} \sum_{k_{y}=0}^{\infty} (-D)_{x}^{k_{x}} (-D)_{y}^{k_{y}} \frac{\partial}{\partial u_{xx}^{k_{x}} y^{k_{y}}}$$

$$= \frac{\partial}{\partial u} - D_{x} \frac{\partial}{\partial u_{x}} - D_{y} \frac{\partial}{\partial u_{y}} + D_{x}^{2} \frac{\partial}{\partial u_{xx}} + D_{x} D_{y} \frac{\partial}{\partial u_{xy}} + D_{y}^{2} \frac{\partial}{\partial u_{yy}} - D_{x}^{3} \frac{\partial}{\partial u_{xxx}} - \cdots,$$  

and

$$\mathcal{L}_{u(x,y,z)}^{(0,0,0)} = \sum_{k_{x}=0}^{\infty} \sum_{k_{y}=0}^{\infty} \sum_{k_{z}=0}^{\infty} (-D)_{x}^{k_{x}} (-D)_{y}^{k_{y}} (-D)_{z}^{k_{z}} \frac{\partial}{\partial u_{xx}^{k_{x}} yy^{k_{y}} zz^{k_{z}}}$$

$$= \frac{\partial}{\partial u} - D_{x} \frac{\partial}{\partial u_{x}} - D_{y} \frac{\partial}{\partial u_{y}} - D_{z} \frac{\partial}{\partial u_{z}} + D_{x}^{2} \frac{\partial}{\partial u_{xx}} + D_{y}^{2} \frac{\partial}{\partial u_{yy}} + D_{z}^{2} \frac{\partial}{\partial u_{zz}}$$

$$+ D_{x} D_{y} \frac{\partial}{\partial u_{xy}} + D_{x} D_{z} \frac{\partial}{\partial u_{xz}} + D_{y} D_{z} \frac{\partial}{\partial u_{yz}} - D_{x}^{3} \frac{\partial}{\partial u_{xxx}} - \cdots.$$  

Note that $u_{x x x y y y}$ stands for $u_{xx x y y y}$ where $x$ is repeated $k_{x}$ times and $y$ is repeated $k_{y}$ times. Similar formulas hold for components $v, w,$ etc.

The first question is then answered by the following theorem [30, p. 248].

**Theorem:** A necessary and sufficient condition for a function $f$ to be a divergence, i.e. there exists a $F$ so that $f = \text{Div } F$, is that $\mathcal{L}_{u(x)}^{(0)}(f) \equiv 0$. In other words, the Euler operator annihilates divergences, just as the divergence annihilates curls, and the curl annihilates gradients.
If, for example, $u = (u, v)$ then both $L_{u(x)}^{(0)}(f)$ and $L_{v(x)}^{(0)}(f)$ must vanish identically. For the 1D case, the theorem says that a differential function $f$ is exact, i.e. there exists a $F$ so that $f = D_x F$, if and only if $L_{u(x)}^{(0)}(f) \equiv 0$.

**Example:** To test the exactness of $f$ in (33) which involves just one independent variable $x$, we apply the zeroth Euler operator (37) to $f$ for each component of $u = (u, v)$ separately. For component $u$ (of order 2), one computes

$$
L_{u(x)}^{(0)}(f) = \frac{\partial f}{\partial u} - D_x \frac{\partial f}{\partial u_x} + D_x^2 \frac{\partial f}{\partial u_{2x}} = 3u_x v^2 \cos u - u^3 \cos u + 6v v_x \sin u - 2u_x u_{2x} \sin u
$$

$$
- D_x [3v^2 \sin u - 3u_x^2 \sin u + 2u_{2x} \cos u] + D_x^2 [2u_x \cos u]
$$

$$
= 3u_x v^2 \cos u - u^3 \cos u + 6v v_x \sin u - 2u_x u_{2x} \sin u
$$

$$
- [3u_x v^2 \cos u + 6v v_x \sin u - 3u_x^3 \cos u - 6u u_{2x} \sin u
$$

$$
- 2u_x u_{2x} \sin u + 2u_{3x} \cos u]
$$

$$
+ [-2u_{3x} \cos u - 6u_x u_{2x} \sin u + 2u_{3x} \cos u]
$$

$$
\equiv 0.
$$

(41)

Similarly, for component $v$ (also of order 2) one readily verifies that $L_{v(x)}^{(0)}(f) \equiv 0$.

**Example:** As an example in 2D, one can readily verify that $f = u_x v_y - u_{2x} v_y - u_y v_x + u_{xy} v_x$ from (35) is a divergence. Applying (39) to $f$ for each component of $u = (u, v)$ gives $L_{u(x,y)}^{(0)}(f) \equiv 0$ and $L_{v(x,y)}^{(0)}(f) \equiv 0$.

### 5.2 Continuous Higher Euler Operators

To compute $F = \text{Div}^{-1}(f)$ or, in the 1D case $F = D_x^{-1}(f) = \int f \, dx$, we need higher-order versions of the variational derivative, called *higher Euler operators*. The general formulas are given in [30, p. 367]. With applications in mind, we restrict ourselves to the 1D, 2D, and 3D cases.

**Definition:** The *higher Euler operators* in 1D (with variable $x$) are

$$
L_{u(x)}^{(i)} = \sum_{k=0}^{\infty} \binom{k}{i} (-D_x)^{k-i} \frac{\partial}{\partial u_{kx}},
$$

(42)

where $\binom{k}{i}$ is the binomial coefficient. Note that the higher Euler operator for $i = 0$ matches the variational derivative in (38). The explicit formulas for the first three higher Euler operators (for component $u$ and variable $x$) are

$$
L_{u(x)}^{(1)} = \frac{\partial}{\partial u_x} - 2D_x \frac{\partial}{\partial u_{2x}} + 3D_x^2 \frac{\partial}{\partial u_{3x}} - 4D_x^3 \frac{\partial}{\partial u_{4x}} + \cdots,
$$

(43)

$$
L_{u(x)}^{(2)} = \frac{\partial}{\partial u_{2x}} - 3D_x \frac{\partial}{\partial u_{3x}} + 6D_x^2 \frac{\partial}{\partial u_{4x}} - 10D_x^3 \frac{\partial}{\partial u_{5x}} + \cdots,
$$

(44)

$$
L_{u(x)}^{(3)} = \frac{\partial}{\partial u_{3x}} - 4D_x \frac{\partial}{\partial u_{4x}} + 10D_x^2 \frac{\partial}{\partial u_{5x}} - 20D_x^3 \frac{\partial}{\partial u_{6x}} + \cdots.
$$

(45)
**Definition:** The higher Euler operators in 2D (with variables \( x, y \)) are given by

\[
\mathcal{L}^{(i_x,i_y)}_{u(x,y)} = \sum_{k_x=i_x}^{\infty} \sum_{k_y=i_y}^{\infty} \left( \frac{k_x}{i_x} \right) \left( \frac{k_y}{i_y} \right) \left( -D_x \right)^{k_x-i_x} \left( -D_y \right)^{k_y-i_y} \frac{\partial}{\partial u_{k_xk_y}} . \quad (46)
\]

Note that the higher Euler operator for \( i_x=i_y=0 \) matches the variational derivative in (39). The first higher Euler operators (for component \( u \) and variables \( x \) and \( y \)) are

\[
\begin{align*}
\mathcal{L}^{(1,0)}_{u(x,y)} &= \frac{\partial}{\partial u_x} - 2D_x \frac{\partial}{\partial u_{2x}} - D_y \frac{\partial}{\partial u_{xy}} + 3D_x^2 \frac{\partial}{\partial u_{3x}} + 2D_y D_y \frac{\partial}{\partial u_{xy}} + D_y^2 \frac{\partial}{\partial u_{2xy}} - \cdots , \\
\mathcal{L}^{(0,1)}_{u(x,y)} &= \frac{\partial}{\partial u_y} - 2D_y \frac{\partial}{\partial u_{2y}} - D_x \frac{\partial}{\partial u_{yx}} + 3D_y^2 \frac{\partial}{\partial u_{3y}} + 2D_y D_x \frac{\partial}{\partial u_{yx}} + D_x^2 \frac{\partial}{\partial u_{2yx}} - \cdots , \\
\mathcal{L}^{(1,1)}_{u(x,y)} &= \frac{\partial}{\partial u_{xy}} - 2D_x \frac{\partial}{\partial u_{2xy}} - 2D_y \frac{\partial}{\partial u_{4xy}} + 3D_x^2 \frac{\partial}{\partial u_{3xy}} + 4D_y D_y \frac{\partial}{\partial u_{2xy}} + \cdots , \\
\mathcal{L}^{(2,1)}_{u(x,y)} &= \frac{\partial}{\partial u_{3y}} - 3D_x \frac{\partial}{\partial u_{3xy}} - 2D_y \frac{\partial}{\partial u_{2xy}} + 3D_x^2 \frac{\partial}{\partial u_{3xy}} + 6D_y \frac{\partial}{\partial u_{4xy}} + 3D_y^2 \frac{\partial}{\partial u_{2xy}} - \cdots .
\end{align*}
\]

**Definition:** The higher Euler operators in 3D (with variables \( x, y, z \)) are

\[
\mathcal{L}^{(i_x,i_y,i_z)}_{u(x,y,z)} = \sum_{k_x=i_x}^{\infty} \sum_{k_y=i_y}^{\infty} \sum_{k_z=i_z}^{\infty} \left( \frac{k_x}{i_x} \right) \left( \frac{k_y}{i_y} \right) \left( \frac{k_z}{i_z} \right) \left( -D_x \right)^{k_x-i_x} \left( -D_y \right)^{k_y-i_y} \left( -D_z \right)^{k_z-i_z} \frac{\partial}{\partial u_{k_xk_yk_z}} . \quad (51)
\]

The higher Euler operator for \( i_x=i_y=i_z=0 \) matches the variational derivative given in (40).

### 5.3 Continuous Homotopy Operator

We now discuss the homotopy operator which will allow us to reduce the computation of \( F = \text{Div}^{-1}(f) \) (or in the 1D case, \( F = \text{Div}^{-1}(f) = \int f \, dx \)) to a single integral with respect to an auxiliary variable denoted by \( \lambda \) (not to be confused with \( \lambda \) in Section 3). Hence, the homotopy operator circumvents integration by parts and reduces the inversion of the total divergence operator, \( \text{Div} \), to a problem of single-variable calculus. The homotopy operator is given in explicit form, which makes it easier to implement in CAS. To keep matters transparent, we present the formulas of the homotopy operator in 1D, 2D, and 3D.

**Definition:** The homotopy operator in 1D (with variable \( x \)) [30, p. 372] is

\[
\mathcal{H}_{u(x)}(f) = \int_0^1 \sum_{j=1}^{N} I_{u_j}(f)[\lambda u] \frac{d\lambda}{\lambda} , \quad (52)
\]

where the integrand \( I_{u_j}(f) \) is given by

\[
I_{u_j}(f) = \sum_{i=0}^{\infty} D_x^i \left( u_j \mathcal{L}^{(i+1)}_{u_j(x)}(f) \right) . \quad (53)
\]

The integrand involves the 1D higher Euler operators in (42). In (52), \( N \) is the number of dependent variables and \( I_{u_j}(f)[\lambda u] \) means that in \( I_{u_j}(f) \) one replaces \( u(x) \rightarrow \lambda u(x) \), \( u_x(x) \rightarrow \lambda u_x(x) \), etc.
Given an exact function \( f \), the question how to compute \( F = D_x^{-1}(f) = \int f \, dx \) is then answered by the following theorem [30, p. 372].

**Theorem:** For an exact function \( f \), one has \( F = \mathcal{H}_{u(x)}(f) \).

Thus, in the 1D case, applying the homotopy operator (52) allows one to bypass integration by parts. A clever argument why the homotopy operator actually works is given in [6, p. 582]. As an experiment, one can start from a function \( \tilde{f} \), compute \( f = D_x \tilde{F} \), subsequently compute \( F = \mathcal{H}_{u(x)}(f) \), and finally verify that \( F - \tilde{F} \) is a constant.

**Example:** Using (33), we show how the homotopy operator (52) is applied. For a system with \( N = 2 \) components, \( (u_1, u_2) = (u, v) \), the homotopy operator formulas are

\[
\mathcal{H}_{u(x)}(f) = \int_0^1 (I_u(f)[\lambda u] + I_v(f)[\lambda u]) \frac{d\lambda}{\lambda}, \tag{54}
\]

with

\[
I_u(f) = \sum_{i=0}^{\infty} D_x^i \left( u \mathcal{L}^{(i+1)}_{u(x)}(f) \right) \quad \text{and} \quad I_v(f) = \sum_{i=0}^{\infty} D_x^i \left( v \mathcal{L}^{(i+1)}_{v(x)}(f) \right). \tag{55}
\]

These sums have only finitely many non-zero terms. For example, the sum in \( I_u(f) \) terminates at \( p - 1 \) where \( p \) is the order of \( u \). Take, for example, \( f = 3 u_x v^2 \sin u - u_x^3 \sin u - 6 v v_x \cos u + 2 u_x u_{2x} \cos u + 8 v_x v_{2x} \). First, we compute

\[
I_u(f) = u \mathcal{L}^{(1)}_{u(x)}(f) + D_x \left( u \mathcal{L}^{(2)}_{u(x)}(f) \right)
= u \frac{\partial f}{\partial u_x} - 2 u D_x \left( \frac{\partial f}{\partial u_{2x}} \right) + D_x \left( u \frac{\partial f}{\partial u_{2x}} \right)
= 3 u v^2 \sin u - u u_x^2 \sin u + 2 u_x^2 \cos u. \tag{56}
\]

Next,

\[
I_v(f) = v \mathcal{L}^{(1)}_{v(x)}(f) + D_x \left( v \mathcal{L}^{(2)}_{v(x)}(f) \right)
= v \frac{\partial f}{\partial v_x} - 2 v D_x \left( \frac{\partial f}{\partial v_{2x}} \right) + D_x \left( v \frac{\partial f}{\partial v_{2x}} \right)
= -6 v^2 \cos u + 8 v_x^2. \tag{57}
\]

Formula (54) reduces to an integral with respect to \( \lambda \):

\[
F = \mathcal{H}_{u(x)}(f) = \int_0^1 (I_u(f)[\lambda u] + I_v(f)[\lambda u]) \frac{d\lambda}{\lambda}
= \int_0^1 \left( 3 \lambda^2 u v^2 \sin(\lambda u) - \lambda^2 u u_x^2 \sin(\lambda u) + 2 \lambda u_x^2 \cos(\lambda u) - 6 \lambda v^2 \cos(\lambda u) + 8 \lambda v_x^2 \right) d\lambda
= 4 v_x^2 + u_x^2 \cos u - 3 v^2 \cos u. \tag{58}
\]

We now turn to the problem of inverting the Div operator using the homotopy operator.

**Definition:** We define the homotopy operator in 2D (variables \( x, y \)) through its two components \( (\mathcal{H}_{u(x,y)}^{(x)}(f), \mathcal{H}_{u(x,y)}^{(y)}(f)) \). The \( x \)-component of the operator is given by

\[
\mathcal{H}_{u(x,y)}^{(x)}(f) = \int_0^1 \sum_{j=1}^N I_{u_j}^{(x)}(f)[\lambda u] \frac{d\lambda}{\lambda}, \tag{59}
\]

\[
11
\]
with $I_{u_j}^{(x)}(f)$ given by

$$I_{u_j}^{(x)}(f) = \sum_{i_x=0}^{\infty} \sum_{i_y=0}^{\infty} \left( \frac{1+i_x}{1+i_x+i_y} \right) D_x^i D_y^j (u_j \mathcal{L}_{u_j(x,y)}^{(i_x,i_y)}(f)).$$  \hspace{1cm} (60)

Analogously, the $y$-component is given by

$$\mathcal{H}_{u(x,y)}^{(y)}(f) = \int_0^1 \sum_{j=1}^N I_{u_j}^{(y)}(f)[\lambda u] \frac{d\lambda}{\lambda},$$  \hspace{1cm} (61)

with

$$I_{u_j}^{(y)}(f) = \sum_{i_x=0}^{\infty} \sum_{i_y=0}^{\infty} \left( \frac{1+i_y}{1+i_x+i_y} \right) D_x^i D_y^j (u_j \mathcal{L}_{u_j(x,y)}^{(i_x,1+i_y)}(f)).$$  \hspace{1cm} (62)

These integrands involve the 2D higher Euler operators in (46).

After verification that $f$ is a divergence, the question how to compute $F = (F_1, F_2) = \text{Div}^{-1}(f)$ is then answered by the following theorem.

**Theorem:** If $F$ is a divergence, then $F = (F_1, F_2) = \text{Div}^{-1}(f) = (\mathcal{H}_{u(x,y)}^{(x)}(f), \mathcal{H}_{u(x,y)}^{(y)}(f))$.

The superscript $(x)$ in $\mathcal{H}^{(x)}(f)$ reminds us that we are computing the $x$-component of $F$. As a test, one can start from any vector $\mathbf{F}$ and compute $f = \text{Div} \mathbf{F}$. Next, compute $F = (F_1, F_2) = (\mathcal{H}_{u(x,y)}^{(x)}(f), \mathcal{H}_{u(x,y)}^{(y)}(f))$ and, finally, verify that $\mathbf{F} - F$ is divergence free.

**Example:** Using (35), we show how the application of the 2D homotopy operator leads to (36), up to a divergence free vector. Consider $f = u_x v_y - u_2 v_y - u_y v_x + u_y v_x$, which is easily verified to be a divergence. In order to determine $\text{Div}^{-1}(f)$, we calculate

$$I_u^{(x)}(f) = u \mathcal{L}_{u(x,y)}^{(1,0)}(f) + D_x \left( u \mathcal{L}_{u(x,y)}^{(2,0)}(f) \right) + \frac{1}{2} D_y \left( u \mathcal{L}_{u(x,y)}^{(1,1)}(f) \right)$$

$$= u \left( \frac{\partial f}{\partial u_x} - 2D_x \frac{\partial f}{\partial u_{2x}} - D_y \frac{\partial f}{\partial u_{xy}} \right) + D_x \left( u \frac{\partial f}{\partial u_{2x}} \right) + \frac{1}{2} D_y \left( u \frac{\partial f}{\partial u_{xy}} \right)$$

$$= u v_y + \frac{1}{2} u_y v_x - u_x v_y + \frac{1}{2} u v_{xy}. \hspace{1cm} (63)$$

Similarly,

$$I_v^{(x)}(f) = v \mathcal{L}_{v(x,y)}^{(1,0)}(f) = v \frac{\partial f}{\partial v_x} = -u_y v + u_x v. \hspace{1cm} (64)$$

Hence,

$$F_1 = \mathcal{H}_{u(x,y)}^{(x)}(f) = \int_0^1 \left( I_u^{(x)}(f)[\lambda u] + I_v^{(x)}(f)[\lambda u] \right) \frac{d\lambda}{\lambda}$$

$$= \int_0^1 \lambda \left( u v_y + \frac{1}{2} u_y v_x - u_x v_y + \frac{1}{2} u v_{xy} - u_y v + u_x v \right) \, d\lambda$$

$$= \frac{1}{2} u v_y + \frac{1}{4} u_y v_x - \frac{1}{2} u_x v_y + \frac{1}{4} u v_{xy} - \frac{1}{2} u_y v + \frac{1}{2} u_x v. \hspace{1cm} (65)$$
Without showing the details, one computes in an analogous fashion
\[
F_2 = \mathcal{H}^{(y)}_{u(x,y)}(f) = \int_0^1 \left( I_u^{(y)}(f)[\lambda u] + I_v^{(y)}(f)[\lambda u] \right) \frac{d\lambda}{\lambda} \\
= \int_0^1 \left( -uv_x - \frac{1}{2}uv_{2x} + \frac{1}{2}u_xv_x + \lambda (u_xv - u_{2x}v) \right) d\lambda \\
= -\frac{1}{2}uv_x - \frac{1}{4}uv_{2x} + \frac{1}{4}u_xv_x + \frac{1}{2}u_xv - \frac{1}{2}u_{2x}v.
\]

One can readily verify that the resulting vector
\[
\mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}uv_y + \frac{1}{4}u_yv_x - \frac{1}{2}u_xv_y + \frac{1}{4}uv_{xy} - \frac{1}{2}uv_y + \frac{1}{4}u_{xy}v \\ -\frac{1}{2}uv_x - \frac{1}{4}uv_{2x} + \frac{1}{4}u_xv_x + \frac{1}{2}u_xv - \frac{1}{2}u_{2x}v \end{pmatrix}
\]

differs from \( \tilde{\mathbf{F}} = (uv_y - u_xv_y, -uv_x + u_xv_x) \) by a divergence free vector.

The generalization of the homotopy operator to 3D is straightforward.

**Definition:** The homotopy operator in 3D (with variables \( x, y, z \)) is \( (\mathcal{H}_{u(x,y,z)}^{(x)}(f), \mathcal{H}_{u(x,y,z)}^{(y)}(f), \mathcal{H}_{u(x,y,z)}^{(z)}(f)) \). By analogy with (59), the \( x \)-component is
\[
\mathcal{H}_{u(x,y,z)}^{(x)}(f) = \int_0^1 \sum_{j=1}^N I_{u_j}^{(x)}(f)[\lambda u] \frac{d\lambda}{\lambda},
\]
with,
\[
I_{u_j}^{(x)}(f) = \sum_{i_x=0}^\infty \sum_{i_y=0}^\infty \sum_{i_z=0}^\infty \left( \frac{1}{1 + i_x + i_y + i_z} \right) D_x^{i_x}D_y^{i_y}D_z^{i_z} \left( u_j \mathcal{L}_{u_j}^{(1+i_x,i_y,i_z)}(f) \right).
\]
The \( y \) and \( z \) operators are defined analogously. The integrands involve the 3D higher Euler operators in (51). By analogy with the 2D case the following theorem holds.

**Theorem:** Given a divergence \( f \) one has \( \mathbf{F} = \text{Div}^{-1}(f) = (\mathcal{H}_{u(x,y,z)}^{(x)}(f), \mathcal{H}_{u(x,y,z)}^{(y)}(f), \mathcal{H}_{u(x,y,z)}^{(z)}(f)) \).

### 6 Removing Divergences and Equivalent Terms

We present an algorithm to remove divergences and equivalent terms in order to make our computation of the densities simpler.

**Definition:** Two scalar differential functions, \( f^{(1)} \) and \( f^{(2)} \), are equivalent if and only if they differ by the divergence of some vector \( \mathbf{V} \), i.e. \( f^{(1)} \sim f^{(2)} \) if and only if \( f^{(1)} - f^{(2)} = \text{Div} \mathbf{V} \).

Obviously, if a scalar expression is equivalent to zero, then it is a divergence.

**Example:** Functions \( f^{(1)} = uu_{2x} \) and \( f^{(2)} = -u_x^2 \) are equivalent because \( f^{(1)} - f^{(2)} = uv \) and \( u_x^2 = D_x(uxx) \). Using (38), note that \( v_1 = \mathcal{L}_{u(x)}^{(0)}(uu_{2x}) = 2u_{2x} \) and \( v_2 = \mathcal{L}_{u(x)}^{(0)}(-u_x^2) = 2u_{2x} \) are equal. Also, \( f = u_{4x} = D_x(u_{3x}) \) is a divergence and, as expected, \( v_3 = \mathcal{L}_{u(x)}^{(0)}(u_{4x}) = 0 \).

**Example:** In the 2D case, \( f^{(1)} = (u_x - u_{2x})v_y \) and \( f^{(2)} = (u_y - uxy)v_x \) are equivalent since \( f^{(1)} - f^{(2)} = u_xv_y - u_{2x}v_y - uyv_x + u_{xy}v_x = \text{Div}(uv_y - u_xv_y, -uv_x + u_xv_x) \). Using (39), note that \( v_1 = \mathcal{L}_{u(x,y)}^{(0)}(f^{(1)}) = v_2 = \mathcal{L}_{u(x,y)}^{(0)}(f^{(2)}) = (-v_{xy} - v_{xxy}, -u_{xy} - u_{xxy}) \).

To remove divergences and equivalent terms we use the following algorithm.
Algorithm: Remove-Divergences-And-Equivalent-Terms(\(\mathcal{R}\))

/* Given is list \(\mathcal{R}\) of monomial differential functions */
/* Initialize two new lists \(S, B\) */
\(\mathcal{S} \leftarrow \emptyset\)
\(\mathcal{B} \leftarrow \emptyset\)
/* Find first member of \(S\) */
for each term \(t_i \in \mathcal{R}\)
\(\text{do } \mathcal{v}_i \leftarrow L_u(0)(t_i)\)
\(\text{if } \mathcal{v}_i \neq 0\)
\(\text{then } \mathcal{S} \leftarrow \{t_i\}\)
\(\mathcal{B} \leftarrow \{\mathcal{v}_i\}\)
\(\text{break}\)
\(\text{else discard } t_i \text{ and } \mathcal{v}_i\)
/* Find remaining members of \(S\) */
for each term \(t_j \in \mathcal{R} \setminus \{t_1, t_2, \ldots, t_i\}\)
\(\text{do } \mathcal{v}_j \leftarrow L_u(0)(t_j)\)
\(\text{if } \mathcal{v}_j \neq 0\)
\(\text{then if } \mathcal{v}_j \not\in \text{Span}(\mathcal{B})\)
\(\text{then } \mathcal{S} \leftarrow \mathcal{S} \cup \{t_j\}\)
\(\mathcal{B} \leftarrow \mathcal{B} \cup \{\mathcal{v}_j\}\)
\(\text{else discard } t_j \text{ and } \mathcal{v}_j\)

return \(\mathcal{S}\)
/* List \(\mathcal{S}\) is free of divergences and equivalent terms */

Example: Using the above algorithm, we remove divergences and equivalent terms in \(\mathcal{R} = \{u^3, u^2v, uv^2, v^3, u^2, u_xv_x, v_x^2, u_{2x}v, u_{2x}v, uv_{2x}, u_{4x}, v_{4x}\}\). Since \(\mathcal{v}_1 = L_u(0)(u^3) = (3u^2, 0) \neq (0, 0)\) we have \(\mathcal{S} = \{t_1\} = \{u^3\}\) and \(\mathcal{B} = \{\mathcal{v}_1\} = \{(3u^2, 0)\}\). The first for loop is halted and the second for loop starts. Next, \(\mathcal{v}_2 = L_u(0)(u^2v) = (2uv, u^2) \neq (0, 0)\). We verify that \(\mathcal{v}_1\) and \(\mathcal{v}_2\) are independent and update the sets resulting in \(\mathcal{S} = \{t_1, t_2\} = \{u^3, u^2v\}\) and \(\mathcal{B} = \{\mathcal{v}_1, \mathcal{v}_2\} = \{(3u^2, 0), (2uv, u^2)\}\).

Proceeding in a similar fashion, since the first seven terms are indeed independent, we have \(\mathcal{S} = \{t_1, t_2, \ldots, t_7\}\) and \(\mathcal{B} = \{\mathcal{v}_1, \mathcal{v}_2, \ldots, \mathcal{v}_7\} = \{(3u^2, 0), (2uv, u^2), \ldots, (0, -2v_{2x})\}\).

For \(t_8 = uu_{2x}\) we compute \(\mathcal{v}_8 = L_u(0)(uu_{2x}) = (2u_{2x}, 0)\) and verify that \(\mathcal{v}_8 = -\mathcal{v}_5\). So, \(\mathcal{v}_8 \in \text{Span}(\mathcal{B})\) and \(t_8\) and \(\mathcal{v}_8\) are discarded (i.e. not added to the respective sets). For similar reasons, \(t_9, t_{10}\) and \(t_{11}\) as well as \(\mathcal{v}_9, \mathcal{v}_{10}\), and \(\mathcal{v}_{11}\) are discarded. The terms \(t_{12} = u_{4x}\) and \(t_{13} = v_{4x}\) are discarded because \(\mathcal{v}_{12} = \mathcal{v}_{13} = (0, 0)\). So, \(\mathcal{R}\) is replaced by \(\mathcal{S} = \{u^3, u^2v, uv^2, v^3, u^2, u_xv_x, v_x^2\}\) which is free of divergences and equivalent terms.

7 Application: Conservation Laws of Nonlinear PDEs

As an application of the Euler and homotopy operators we show how to compute conserved densities and fluxes for the three PDEs in Section 2. The first PDE illustrates the 1D case
In (21) through (24) we gave the first four density-flux pairs. As an example, we will compute 7.1 Conservation Laws for the Coupled KdV Equations

\[
\{ \rho, J \} = \{ \rho, \text{Div} J \} = 0 \text{ of systems of nonlinear PDEs, we use a direct approach. First, we build the candidate density } \rho \text{ as a linear combination (with constant coefficients } c_i \text{) of terms which are uniform in rank (with respect to the scaling symmetry of the PDE). It is of paramount importance that the candidate density is free of divergences and equivalent terms. If such terms were present, their coefficients could not be determined because such terms can be moved into the flux, } J. \text{ To construct the shortest density, we will use the algorithm of Section 6.}

Second, we evaluate } D_t \rho \text{ on solutions of the PDE, thus removing all time derivatives from the problem. The resulting expression (called } E \text{) must be a divergence (of the as yet unknown flux). Thus, we set } L_{u(x)}^{(0)}(E) = 0. \text{ Setting the coefficients of like terms to zero leads to a linear system for the undetermined coefficients } c_i. \text{ In the most difficult case, such systems are parameterized by the constant parameters appearing in the given PDE. If so, a careful analysis of the eliminant (and solution branching) must be carried out. For each branch, the solution of the linear system is substituted into } \rho \text{ and } E. \text{ Third, since } E = \text{Div} J \text{ we use the homotopy operator } H_{u(x)} \text{ to compute } J = \text{Div}^{-1}(E). \text{ The computations are carried out with our Mathematica packages [18].}

7.1 Conservation Laws for the Coupled KdV Equations

In (21) through (24) we gave the first four density-flux pairs. As an example, we will compute density } \rho^{(4)} \text{ and associated flux } J^{(4)}.

Recall that the weights for the cKdV equations are } W(\partial/\partial x) = 1 \text{ and } W(u) = W(v) = 2. \text{ The parameter } \beta \text{ has no weight. Hence, } \rho^{(4)} \text{ has rank 6. The algorithm has three steps:}

Step 1: Construct the form of the density

Start from } \mathcal{V} = \{ u, v \}, \text{ i.e. the list of dependent variables with weight. Construct } \mathcal{M} = \{ u^3, v^3, u^2v, uv^2, u^2, u, v, u, v, 1 \}, \text{ which contains all monomials of selected rank 6 or less (without derivatives). Next, for each monomial in } \mathcal{M}, \text{ introduce the correct number of } x\text{-derivatives so that each term has rank 6. For example,}

\[
\begin{align*}
\frac{\partial^2 u^2}{\partial x^2} &= 2u_x^2 + 2uu_{2x}, & \frac{\partial^2 v^2}{\partial x^2} &= 2v_x^2 + 2vv_{2x}, & \frac{\partial^2 (uv)}{\partial x^2} &= u_{2x}v + 2u_xv_x + uv_{2x}, \\
\frac{\partial^4 u}{\partial x^4} &= u_{4x}, & \frac{\partial^4 v}{\partial x^4} &= v_{4x}, & \frac{\partial^6 1}{\partial x^6} &= 0.
\end{align*}
\]

(70)

Ignore the highest-order terms (typically the last terms) in each of the right hand sides of (70). Augment } \mathcal{M} \text{ with the remaining terms, after stripping off numerical factors, to get } \mathcal{R} = \{ u^3, u^2v, uv^2, v^3, u_xv_x, u_xv, v_x, u_{2x}v \}, \text{ where the 8 terms are listed by increasing order.}

Note that keeping all terms in (70) would have resulted in the list } \mathcal{R} \text{ (with 13 terms) given in the example at the end of Section 6. As shown, the algorithm would reduce } \mathcal{R} \text{ to 7 terms.}

Use the algorithm of Section 6, to replace } \mathcal{R} \text{ by } \mathcal{S} = \{ u^3, u^2v, uv^2, v^3, u_xv_x, u_xv, v_x^2 \}. \text{ Linearly combine the terms in } \mathcal{S} \text{ with constant coefficients to get the shortest candidate density:}

\[
\rho = c_1 u^3 + c_2 u^2v + c_3 uv^2 + c_4 v^3 + c_5 u_x^2 + c_6 u_xv_x + c_7 v_x^2.
\]

(71)
Step 2: Determine the constants $c_i$

Compute
\[
E = D_t \rho = \frac{\partial \rho}{\partial t} + \rho'(u)[F] = \frac{\partial \rho}{\partial u} u_t + \frac{\partial \rho}{\partial u_x} u_{tx} + \frac{\partial \rho}{\partial v} v_t + \frac{\partial \rho}{\partial v_x} v_{tx}
\]
\[
= (3c_1 u^2 + 2c_2 uv + c_3 v^2)u_t + (2c_5 u_x + c_6 v_x)u_{tx} + (c_2 u^2 + 2c_3 uv + 3c_4 v^2)v_t
\]
\[
+ (c_6 u_x + 2c_7 v_x)v_{tx}.
\]
(72)

Replace $u_t, v_t, u_{tx}$ and $v_{tx}$ from (2) to obtain
\[
E = (3c_1 u^2 + 2c_2 uv + c_3 v^2)(6\beta u u_x - 6v v_x + \beta u_3 x) + (2c_5 u_x + c_6 v_x)(6\beta u u_x - 6v v_x + \beta u_3 x)
\]
\[
-(c_2 u^2 + 2c_3 uv + 3c_4 v^2)(3v v_x + v_3 x) - (c_6 u_x + 2c_7 v_x)(3v v_x + v_3 x).
\]
(73)

Since $E = D_t \rho = -D_x J$, the expression $E$ must be exact. Therefore, apply the variational derivative (38) and require that $\mathcal{L}^{(0)}_u(E) \equiv 0$ and $\mathcal{L}^{(0)}_v(E) \equiv 0$. Group like terms and set their coefficients equal to zero to obtain the following (parameterized) linear system for the unknown coefficients $c_i$ through $c_7$:

\[
(3 + 4\beta)c_2 = 0, \quad 3c_1 + (1 + \beta)c_3 = 0, \quad 4c_2 + 3c_4 = 0, \quad (1 + \beta)c_3 - 6c_5 = 0,
\]
\[
\beta(c_1 + 2c_5) = 0, \quad \beta c_2 - c_6 = 0, \quad (1 + \beta)c_6 = 0, \quad c_4 + c_6 = 0,
\]
\[
2(1 + \beta)c_2 - 3(1 + 2\beta)c_6 = 0, \quad 2c_2 - (1 + 6\beta)c_6 = 0, \quad \beta c_3 - 6c_5 - c_7 = 0, \quad c_3 + c_7 = 0.
\]
(74)

Investigate the eliminant of the system. In this example, there exists a solution for any $\beta \neq -1$. Set $c_1 = 1$ and obtain
\[
c_1 = 1, \quad c_2 = c_4 = c_6 = 0, \quad c_3 = -\frac{3}{1 + \beta}, \quad c_5 = -\frac{1}{2}, \quad c_7 = \frac{3}{1 + \beta}.
\]
(75)

Substitute the solution into (71) and multiply by $1 + \beta$ to get
\[
\rho = (1 + \beta)u^3 - 3uv^2 - \frac{1}{2}(1 + \beta)u_x^2 + 3v_x^2,
\]
(76)

which is $\rho^{(4)}$ in (24).

Step 3: Compute the flux $J$

Compute the flux corresponding to $\rho$ in (76). Substitute (75) into (73), reverse the sign and multiply by $1 + \beta$, to get
\[
E = 18\beta(1 + \beta)u^3 u_x - 18\beta u^2 v v_x - 18\beta u u_x v^2 + 18v^3 v_x - 6\beta(1 + \beta)u_x^3 - 6\beta(1 + \beta)u u_x u_{2x}
\]
\[
+ 3\beta(1 + \beta)u^2 u_{3x} - 3\beta v^2 u_{3x} - 6v v_x v_{4x} - \beta(1 + \beta)u_x u_{4x} + 6uv v_{3x} + 6(\beta - 2)u_{x}^2 v_x
\]
\[
+ 6(1 + \beta)u_x v v_{2x} - 18v v v_{2x}.
\]
(77)

Apply (52) and (53) to (77) to obtain
\[
J = -\frac{9}{2} \beta(1 + \beta)u^4 + 9\beta u^2 v^2 - \frac{9}{2} v^4 + 6\beta(1 + \beta)u u_x^2 - 3\beta(1 + \beta)u_{x}^2 u_{2x}
\]
\[
+ 3\beta v^2 u_{2x} - \frac{9}{2} \beta(1 + \beta)u_{x}^2 + \beta(1 + \beta)u_x u_{3x} - 6\beta v v_x v_x
\]
\[
+ 12u_{x}^2 v - 6uv v_{2x} - 3v_{2x}^2 + 6v_{x} v_{3x},
\]
(78)

which is $J^{(4)}$ in (25).

The cKdV equations (2) are completely integrable if $\beta = \frac{1}{2}$ and admit conserved densities at every even rank.
7.2 Conservation Laws for the sine-Gordon Equation

Recall that the weights for the sine-Gordon equation (3) are $W(\frac{\partial}{\partial y}) = 1, W(u) = 0, W(v) = 1$, and $W(\alpha) = 2$. The first few (of infinitely many) densities and fluxes were given in (26) through (29). We show how to compute densities $\rho^{(1)}$ and $\rho^{(2)}$, both of rank 2, and their associated fluxes $J^{(1)}$ and $J^{(2)}$.

In contrast to the previous example, the candidate density will no longer have constant undetermined coefficients $c_i$ but functional coefficients $h_i(u)$ which depend on the transcendental variable $u$ with weight zero [3]. To avoid having to solve PDEs, we tacitly assume that there is only one dependent variable with weight zero.

**Step 1: Construct the form of the density**

Augment the list of dependent variables with $\alpha$ (with non-zero weight) and replace $u$ by $u_x$ (since $W(u) = 0$). Hence, $V = \{\alpha, u_x, v\}$. Compute $R = \{\alpha, v^2, v^2, u_{2x}, u_x v, u_x^2\}$ and remove divergences and equivalent terms to get $S = \{\alpha, v^2, u_{2x}, u_x v\}$. The candidate density

$$\rho = \alpha h_1(u) + h_2(u)v^2 + h_3(u)u_x^2 + h_4(u)u_x v,$$

(79)

with undetermined functional coefficients $h_i(u)$.

**Step 2: Determine the functions $h_i(u)$**

Compute

$$E = D_t \rho = \frac{\partial \rho}{\partial t} + \rho'(u)[F] = \frac{\partial \rho}{\partial u} u_t + \frac{\partial \rho}{\partial u_x} u_{xt} + \frac{\partial \rho}{\partial v} v_t$$

$$= (\alpha h_1' + v^2 h_2' + u_x^2 h_3' + u_x v h_4') v + (2u_x h_3 + v h_4) v_x + (2v h_2 + u_x h_4)(\alpha \sin(u) + u_x v).$$

(80)

where $h_i'$ means $\frac{dh_i}{du}$. Since $E = D_t \rho = -D_x J$, the expression $E$ must be exact. Therefore, require that $L^{(0)}_{u(x)}(E) \equiv 0$ and $L^{(0)}_{v(x)}(E) \equiv 0$. Set the coefficients of like terms equal to zero to get a mixed linear system of algebraic and ODEs:

$$h_2(u) - h_3(u) = 0, \quad h_2'(u) = 0, \quad h_3'(u) = 0, \quad h_4'(u) = 0, \quad h_2''(u) = 0,$$

(81)

$$h_4''(u) = 0, \quad 2h_2''(u) - h_3''(u) = 0, \quad 2h_2''(u) - h_3''(u) = 0,$$

(82)

$$h_1''(u) + 2h_2(u) \sin u = 0, \quad h_1''(u) + 2h_2(u) \sin u + 2h_2(u) \cos u = 0.$$  

(83)

Solve the system [3] and substitute the solution

$$h_1(u) = 2c_1 \cos u + c_3, \quad h_2(u) = h_3(u) = c_1, \quad h_4(u) = c_2,$$

(84)

(with arbitrary constants $c_i$) into (79) to obtain

$$\rho = c_1(2\alpha \cos u + v^2 + u_x^2) + c_2 u_x v + c_3 \alpha.$$

(85)

**Step 3: Compute the flux $J$**

Compute the flux corresponding to $\rho$ in (85). Substitute (84) into (80), to get

$$E = c_1(2u_{2x}v + 2u_x v_x) + c_2(v v_x + u_x u_{2x} + \alpha u_x \sin u).$$

(86)
Since \( E = D_t \rho = -D_x J \), one must integrate \( f = -E \). Applying (55) yields \( I_u(f) = -2c_1u_x v - c_2 (u_x^2 + \alpha u \sin u) \) and \( I_v(f) = -2c_1u_x v - c_2 v^2 \). Use formula (54) to obtain

\[
J = \mathcal{H}_{u(x)}(f) = \int_0^1 (I_u(f) | \lambda u) + I_v(f) | \lambda u) \frac{d\lambda}{\lambda} \\
= -\int_0^1 (4c_1 \lambda u_x v + c_2 (\lambda u_x^2 + \alpha u \sin(\lambda u) + \lambda v^2)) \, d\lambda \\
= -c_1(2u_x v) - c_2 \left( \frac{1}{2} v^2 + \frac{1}{2} u_x^2 - \alpha \cos u \right). \\
\tag{87}
\]

Finally, split density (85) and flux (87) into independent pieces (for \( c_1 \) and \( c_2 \)):

\[
\begin{align*}
\rho^{(1)} &= 2\alpha \cos u + v^2 + u_x^2 \quad \text{and} \quad J^{(1)} = -2u_x v, \\
\rho^{(2)} &= u_x v \quad \text{and} \quad J^{(2)} = -\frac{1}{2} v^2 - \frac{1}{2} u_x^2 + \alpha \cos u.
\end{align*}
\tag{88}
\tag{89}
\]

For \( E \) in (86), \( J \) in (87) can easily be computed by hand [3]. However, the computation of fluxes corresponding to densities of ranks \( \geq 2 \) is cumbersome and requires integration with the homotopy operator.

### 7.3 Conservation Laws for the Shallow Water Wave Equations

In contrast to the previous two examples, as far as we know, (5) is not completely integrable. One cannot expect infinitely many conserved densities and fluxes (of different ranks).

The first few densities and fluxes were given in (30). We show how to compute densities \( \rho^{(1)}, \rho^{(3)}, \rho^{(4)}, \) and \( \rho^{(5)} \), which are of rank 3 under the following (choice for the) weights

\[
W(\partial/\partial x) = W(\partial/\partial y) = 1, W(u) = W(v) = 1, W(\theta) = 1, W(h) = 1, W(\Omega) = 2. \tag{90}
\]

We will also compute the associated fluxes \( J^{(1)}, J^{(3)}, J^{(4)}, \) and \( J^{(5)} \).

The fact that (5) is multi-uniform is advantageous. Indeed, one can use the invariance of (5) under one scale to construct the terms of \( \rho \), and, subsequently, use additional scale(s) to split \( \rho \) into smaller densities. This “divide and conquer” strategy drastically reduces the complexity of the computations.

**Step 1: Construct the form of the density**

Start from \( \mathcal{V} = \{ u, v, \theta, h, \Omega \} \), i.e. the list of variables and parameters with weights. Use (90) to construct \( \mathcal{M} = \{ \Omega u, \Omega v, \ldots, u^3, v^3, \ldots, u^2 v, uv^2, \ldots, u^2 v, v^2, uv, \ldots, u, v, \theta, h \} \), which has 38 monomials of rank 3 or less (without derivatives).

The terms of rank 3 in \( \mathcal{M} \) are left alone. To adjust the rank, differentiate each monomial of rank 2 in \( \mathcal{M} \) with respect to \( x \) ignoring the highest-order term. For example, in \( \frac{d u^2}{dx} = 2u u_x \), the term can be ignored since it is a total derivative. The terms \( u_x v \) and \( -u v_x \) are equivalent since \( \frac{d(u v)}{dx} = u_x v + u v_x \). Keep \( u_x v \). Likewise, differentiate each monomial of rank 2 in \( \mathcal{M} \) with respect to \( y \) and ignore the highest-order term.

Produce the remaining terms for rank 3 by differentiating the monomials of rank 1 in \( \mathcal{M} \) with respect to \( x \) twice, or \( y \) twice, or once with respect to \( x \) and \( y \). Again ignore the highest-order terms. Augment the set \( \mathcal{M} \) with the derivative terms of rank 3 to get \( \mathcal{R} = \{ \Omega u, \Omega v, \ldots, u^2 v, u_x v, u_x \theta, u_x h, \ldots, u_y v, u_y \theta, \ldots, \theta_y h \} \) which has 36 terms.
Instead of applying the algorithm of Section 6 to \( \mathcal{R} \), use the “divide and conquer” strategy to split \( \mathcal{R} \) into sublists of terms of equal rank under the (general) weights

\[
W(\partial/\partial t) = W(\Omega), \quad W(\partial/\partial y) = W(\partial/\partial x) = 1, \quad W(u) = W(v) = W(\Omega) - 1, \quad W(\theta) = 2W(\Omega) - W(h) - 2, \quad (91)
\]

where \( W(\Omega) \) and \( W(h) \) are arbitrary. Use (91), to compute the rank of each monomial in \( \mathcal{R} \) and gather terms of like rank in separate lists.

Apply the algorithm from Section 6 to each \( \mathcal{R}_i \) to get the list \( \mathcal{S}_i \). Coincidentally, in this example \( \mathcal{R}_i = \mathcal{S}_i \) for all \( i \). Linearly combine the monomials in each list \( \mathcal{S}_i \) with coefficients to get the shortest candidate densities \( \rho_i \). In Table 1, we list the 10 candidate densities and the final densities with their ranks.

| i  | Rank                        | Candidate \( \rho_i \) | Final \( \rho_i \) | Final \( J_i \) |
|----|-----------------------------|-------------------------|-------------------|----------------|
| 1  | \( 6W(\Omega) - 3W(h) - 6 \) | \( c_1 \theta^3 \)     | 0                 | 0              |
| 2  | \( 3W(h) \)                 | \( c_1 \theta^3 \)     | 0                 | 0              |
| 3  | \( 5W(\Omega) - 2W(h) - 5 \) | \( c_1 u \theta^2 + c_2 v \theta^2 \) | 0         | 0              |
| 4  | \( W(\Omega) + 2W(h) - 1 \) | \( c_1 u h^2 + c_2 v h^2 \) | 0                 | 0              |
| 5  | \( 4W(\Omega) - W(h) - 4 \) | \( c_1 u^2 \theta + c_2 u v \theta + c_3 v^2 \theta + c_4 \theta^2 h \) | \( \theta^2 h \) | \( \begin{pmatrix} uh^2 \\
v \theta^2 \end{pmatrix} \) |
| 6  | \( 2W(\Omega) + W(h) - 2 \) | \( c_1 u^2 h + c_2 u h \theta + c_3 v^2 h + c_4 \theta h^2 \) | \( u^2 h + u^2 h + \theta h^2 \) | \( \begin{pmatrix} v^3 h + u^2 v h + 2u^2 h^2 \theta \\
v^3 h + u^2 v h \end{pmatrix} \) |
| 7  | \( 3W(\Omega) - W(h) - 2 \) | \( c_1 \Omega \theta + c_2 u_0 \theta + c_3 v_0 \theta + c_4 u_0 \theta + c_5 v_0 \theta \) | \( 2\Omega \theta - u_0 \theta + v_0 \theta \) | \( \Omega \) |
| 8  | \( W(\Omega) + W(h) \)      | \( c_1 \Omega h + c_2 u_0 h + c_3 v_0 h + c_4 u_0 h + c_5 v_0 h \) | \( \Omega h \) | \( \Omega \) |
| 9  | \( 2W(\Omega) - 1 \)        | \( c_1 \Omega u + c_2 \Omega v + c_3 u_0 v + c_4 \theta \ h + c_5 u_0 v + c_6 \theta \ h \) | 0         | 0              |
| 10 | \( 3W(\Omega) - 3 \)       | \( c_1 u^2 + c_2 u^2 v + c_3 u^2 v + c_4 \theta h + c_5 u^2 v + c_6 \theta v h \) | 0         | 0              |

Table 1: Candidate densities for the SWW equations.

**Step 2: Determine the constants** \( c_i \)

For each of the densities \( \rho_i \) in Table 1 compute \( E_i = D_\tau \rho_i \) and use (5) to remove all time derivatives. For example, proceeding with \( \rho_7 \),

\[
E_7 = \rho_7'(u)[F] = \frac{\partial \rho_7}{\partial u_x} u_{tx} + \frac{\partial \rho_7}{\partial u_y} u_{ty} + \frac{\partial \rho_7}{\partial v_x} v_{tx} + \frac{\partial \rho_7}{\partial v_y} v_{ty} + \frac{\partial \rho_7}{\partial \theta} \theta_t
\]

\[
= -c_4 \theta(uu_x + vv_y - 2\Omega v + \frac{1}{2}h \theta_x + \theta h_x) - c_7 \theta(uu_x + vv_y - 2\Omega v + \frac{1}{2}h \theta_x + \theta h_x) \]

\[
= -c_5 \theta(uu_x + vv_y - 2\Omega v + \frac{1}{2}h \theta_x + \theta h_x) - c_7 \theta(uu_x + vv_y - 2\Omega v + \frac{1}{2}h \theta_x + \theta h_x)
\]

\[
= -(c_1 \Omega + c_2 u_0 + c_3 v_0 + c_4 u_0 \theta + c_5 v_0 \theta)(u \theta_x + v \theta_y) . \quad (92)
\]

Require that \( L_{u(x,y)}^{(0,0)}(E_7) = L_{v(x,y)}^{(0,0)}(E_7) = L_{\theta(x,y)}^{(0,0)}(E_7) = L_{h(x,y)}^{(0,0)}(E_7) = 0 \), where, for example, \( L_{u(x,y)}^{(0,0)} \) is given in (39). Gather like terms. Equate their coefficients to zero to obtain

\[
c_1 + 2c_2 = 0, \quad c_3 = c_4 = 0, \quad c_1 - 2c_5 = 0, \quad c_2 + c_5 = 0. \quad (93)
\]
Set $c_1 = 2$. Substitute the solution

$$c_1 = 2, \ c_2 = -1, \ c_3 = c_4 = 0, \ c_5 = 1.$$  \hfill (94)

into $\rho_7$ to obtain $\rho_7 = 2\Omega\theta - u_y\theta + v_x\theta$, which corresponds to $\rho^{(5)}$ in (30).

Proceed in a similar way with the remaining nine candidate densities to obtain the results given in the third column of Table 1.

**Step 3: Compute the flux $J$**

Compute the flux corresponding to all $\rho_i \neq 0$ in Table 1. For example, continuing with $\rho_7$, substitute (94) into (92) to get

$$E_7 = -\theta(u_x v_x + uv_{2x} + v_x v_y + vv_{xy} + 2\Omega u_x + \frac{1}{2} \theta_y h_y - u_x u_y - uu_{xy} - u_y v_y - u_2 v
+ 2\Omega v_y - \frac{1}{2} \theta_y h_x) - (2\Omega u_x + 2\Omega v_y - uu_y \theta_x - u_y v \theta_y + uv_x \theta_x + vv_x \theta_y).$$  \hfill (95)

Apply the 2D homotopy operator in (59)-(61) to $E_7 = -\text{Div} J_7$. So, compute

$$I^{(x)}_u(E_7) = uL^{(1,0)}_{u(x,y)}(E_7) + D_x \left( uL^{(2,0)}_{u(x,y)}(E_7) \right) + \frac{1}{2} D_y \left( uL^{(1,1)}_{u(x,y)}(E_7) \right)
= u \left( \frac{\partial E_7}{\partial u_x} - 2D_x \left( \frac{\partial E_7}{\partial u_{2x}} \right) - D_y \left( \frac{\partial E_7}{\partial u_{xy}} \right) \right) + D_x \left( u \frac{\partial E_7}{\partial u_{2x}} \right) + \frac{1}{2} D_y \left( u \frac{\partial E_7}{\partial u_{xy}} \right)
= -uv_x \theta - 2\Omega u \theta - \frac{1}{2} u^2 \theta_y + uu_y \theta. \hfill (96)$$

Similarly, compute

$$I^{(x)}_v(E_7) = -vv_y \theta - \frac{1}{2} u^2 \theta_y - uv_x \theta, \hfill (97)$$

$$I^{(x)}_\theta(E_7) = \frac{1}{2} \theta^2 h_y - 2\Omega u \theta + uu_y \theta - uv_x \theta, \hfill (98)$$

$$I^{(x)}_h(E_7) = \frac{1}{2} \theta y h. \hfill (99)$$

Next, compute

$$J^{(x)}_7(u) = -H^{(x)}_{u(x,y)}(E_7)
= -\int_0^1 \left( I^{(x)}_u(E_7)|\lambda u| + I^{(x)}_v(E_7)|\lambda u| + I^{(x)}_\theta(E_7)|\lambda u| + I^{(x)}_h(E_7)|\lambda u| \right) \frac{d\lambda}{\lambda}
= \int_0^1 \left( 4\Lambda \Omega u \theta + \lambda^2 \left( 3uv_x \theta + \frac{1}{2} u^2 \theta_y - 2uv_y \theta + vv_y \theta + \frac{1}{2} v^2 \theta_y + \frac{1}{2} \theta^2 h_y - \frac{1}{2} \theta y h \right) \right) d\lambda
= 2\Omega u \theta - \frac{2}{3} uu_y \theta + uv_x \theta + \frac{1}{3} vv_y \theta + \frac{1}{6} u^2 \theta_y + \frac{1}{6} v^2 \theta_y - \frac{1}{6} h \theta \theta_y + \frac{1}{6} h_y \theta^2. \hfill (100)$$

In analogous fashion, compute

$$J^{(y)}_7(u) = -H^{(y)}_{u(x,y)}(E_7)
= 2\Omega \theta + \frac{2}{3} vv_x \theta - uv_y \theta - \frac{1}{3} uu_x \theta - \frac{1}{6} u^2 \theta_x - \frac{1}{6} v^2 \theta_x + \frac{1}{6} h \theta \theta_x - \frac{1}{6} h_x \theta^2. \hfill (101)$$
Hence,

\[ J_T = \frac{1}{6} \left( 12\Omega u\theta - 4uu_y\theta + 6uv_x\theta + 2vv_y\theta + u^2\theta_y + v^2\theta_y - h\theta \theta_x + h_y \theta^2 \right), \]

which matches \( J^{(5)} \) in (32).

Proceed in a similar way with the remaining nonzero densities to obtain the fluxes given in the last column of Table 1.

System (5) has conserved densities of the form

\[ \rho = hf(\theta) \quad \text{and} \quad \rho = (v_x - u_y + 2\Omega)g(\theta), \]

where \( f \) and \( g \) are arbitrary functions. Our algorithm can only find polynomial \( f \) and \( g \). A comprehensive study of all conservation laws of (5) is beyond the scope of this chapter.

8 Examples of Nonlinear DDEs

We consider nonlinear systems of DDEs of the form

\[ \dot{u}_n = G(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots), \]

where \( u_n \) and \( G \) are vector-valued functions with \( N \) components. The integer \( n \) corresponds to discretization in space\(^6\); the dot denotes differentiation with respect to the continuous time \( t \). For simplicity, we write \( G(u_n) \), although \( G \) depends on \( u_n \) and a finite number of its forward and backward shifts. We assume that \( G \) is polynomial with constant coefficients. No restrictions are imposed on the forward or backward shifts or the degree of nonlinearity in \( G \). In the examples we denote the components of \( u_n \) by \( u_n, v_n, \) etc. If present, parameters are denoted by lower-case Greek letters. We use the following two DDEs to illustrate the theorems and algorithms.

**Example 4:** The Kac-van Moerbeke (KvM) lattice [23],

\[ \dot{u}_n = u_n(u_{n+1} - u_{n-1}), \]

arises in the study of Langmuir oscillations in plasmas, population dynamics, etc.

**Example 5:** The Toda lattice [32] in polynomial form [16],

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}), \]

models vibrations of masses in a lattice with an exponential interaction force.

9 Dilation Invariance and Uniformity in Rank for DDEs

The definitions for the discrete case are analogous to the continuous case. For brevity, we use Example 5 to illustrate the definitions and concepts.

\(^6\)We only consider DDEs with one discrete variable.
As shown in Table 3, the Toda lattice (106) is invariant under the scaling symmetry
\[(t, u_n, v_n) \rightarrow (\lambda^{-1}t, \lambda u_n, \lambda^2 v_n).\]  

**Definition:** The *weight* \(W\) of a variable equals the exponent of the scaling parameter \(\lambda\). [16, 17]. Weights of dependent variables are nonnegative and rational. We tacitly assume that weights are independent of \(n\). For example, \(W(u_{n-1}) = W(u_n) = W(u_{n+1})\), etc.

**Example:** Since \(t\) is replaced by \(\frac{t}{\lambda}\) we have \(W(\frac{t}{\lambda}) = W(D) = 1\). From (107) we have \(W(u_n) = 1\) and \(W(v_n) = 2\).

**Definition:** The *rank* of a monomial equals the total weight of the monomial. An expression (or equation) is uniform in rank if all its monomial terms have equal rank.

**Example:** The three terms in the first equation in (106) have rank 2; all terms in the second equation have rank 3. Each equation is uniform in rank.

Conversely, requiring uniformity in rank in (106) yields \(W(u_n) + 1 = W(v_n)\), and \(W(v_n) + 1 = W(u_n) + W(v_n)\). Hence, \(W(u_n) = 1\), \(W(v_n) = 2\). So, the scaling symmetry can be computed with linear algebra.

Many integrable nonlinear DDEs are scaling invariant. If not, they can be made so by extending the set of dependent variables with parameters with weights.

### 10 Conserved Densities and Fluxes of Nonlinear DDEs

By analogy with \(D_x\) and \(D_x^{-1}\), we define the following operators acting on monomials \(m_n\) in \(u_n, v_n\), etc.

**Definition:** \(D\) is the *up-shift operator* (also known as the forward- or right-shift operator) \(D m_n = m_{n+1}\). Its inverse, \(D^{-1}\), is the *down-shift operator* (or backward- or left-shift operator), \(D^{-1} m_n = m_{n-1}\). The identity operator is denoted by \(I\), \(I m_n = m_n\) and \(\Delta = D - I\), is the forward difference operator. So, \(\Delta m_n = (D - I) m_n = m_{n+1} - m_n\).

**Definition:** A *conservation law* of (104),
\[D_t \rho_n + \Delta J_n = 0,\]  

which holds on solutions of (104), links a *conserved density* \(\rho_n\) to a *flux* \(J_n\). Densities and fluxes depend on \(u_n, v_n\) as well as forward and backward shifts of \(u_n\).

**Definition:** Compositions of \(D\) and \(D^{-1}\) define an *equivalence relation* \(\equiv\) on monomials. All shifted monomials are equivalent.

**Example:** For example, \(u_{n-1} v_{n+1} \equiv u_n v_{n+2} \equiv u_{n+1} v_{n+3} \equiv u_{n+2} v_{n+4}\). Factors in a monomial in \(u_n\) and its shifts are ordered by \(u_{n+j} < u_{n+k}\) if \(j < k\).

**Definition:** The *main representative* of an equivalence class is the monomial with \(u_n\) in the first position [16, 17].

**Example:** The main representative in class \(\{\cdots, u_{n-2} u_n, u_{n-1} u_{n+1}, u_n u_{n+2}, \cdots\}\) is \(u_n u_{n+2}\) (not \(u_{n-2} u_n\)).

For monomials involving \(u_n, v_n, w_n\), etc. and their shifts, we lexicographically order the variables, i.e. \(u_n < v_n < w_n\), etc. For example, \(u_n v_{n+2}\) (not \(u_{n-2} v_n\)) is the main representative of \(\{\cdots, u_{n-2} v_n, u_{n-1} v_{n+1}, u_n v_{n+2}, u_{n+1} v_{n+3}, \cdots\}\).
To stress the analogy between PDEs and DDEs, we put the defining equations next to each other in Table 2.

| Evolution Equation | Continuous Case (PDE) | Semi-discrete Case (DDE) |
|---------------------|------------------------|--------------------------|
| $u_t = G(u, u_x, u_y, \ldots, u_{2x}, \ldots)$ | $u_t = G(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots)$ |
| Conservation Law | $\nabla \cdot J = 0$ | $\rho_n + \Delta J_n = 0$ |

Table 2: Defining equations for conservation laws for PDEs and DDEs.

Table 3 shows the KvM and Toda lattices with their scaling invariance (and weights) and a few conserved densities. Notice that the conservation law “inherits” the scaling symmetry of the DDE. Indeed, observe that all $\rho_n$ in Table 3 are uniform in rank, be it of different ranks.

Table 3: Examples of nonlinear DDEs with weights and densities.

### 11 Discrete Euler and Homotopy Operators

#### 11.1 Discrete Variational Derivative (Euler Operator)

Given is a scalar function $f_n$ in discrete variables $u_n, v_n, \ldots$ and their forward and backward shifts. The goal is to find the scalar function $F_n$ so that $f_n = \Delta F_n = F_{n+1} - F_n$. We illustrate the computations with the following example:

$$f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n.$$  \hspace{1cm} (109)

By hand, one readily computes

$$F_n = v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1}. \hspace{1cm} (110)$$

Below we will address the questions:

(i) Under what conditions for $f_n$ does $F_n$ exist in closed form?

(ii) How can one compute $F_n = \Delta^{-1}(f_n)$?

(iii) Can one compute $F_n = \Delta^{-1}(f_n)$ in an analogous way as in the continuous case?

Expression $f_n$ is called exact if it is a total difference, i.e. there exists a $F_n$ so that $f_n = \Delta F_n$. With respect to the existence of $F_n$ in closed form, the following exactness criterion is well-known and frequently used [4, 20].

**Theorem**: A necessary and sufficient condition for a function $f_n$, with positive shifts, to be exact is that $\mathcal{L}^{(0)}_{u_n}(f_n) \equiv 0.$
\( \mathcal{L}_u^{(0)} \) is the variational derivative (discrete Euler operator of order zero) \([4]\) defined by

\[
\mathcal{L}_u^{(0)} = \frac{\partial}{\partial u_n} \left( \sum_{k=0}^{\infty} D^{-k} \right) = \frac{\partial}{\partial u_n} \left( I + D^{-1} + D^{-2} + D^{-3} + \cdots \right). \tag{111}
\]

A proof of the theorem is given in e.g. \([20]\). In practice, the series in (111) terminates at the highest shift occurring in the expression the operator is applied to. To verify that an expression \( E(u_{n-q}, \ldots, u_n, \ldots, u_{n+p}) \) involving negative shifts is a total difference, one must first remove the negative shifts by replacing \( E_n \) by \( \tilde{E}_n = D^q E_n \).

**Example:** We return to (109), \( f_n = -u_n u_{n+1} v_n - v_n^2 + u_{n+1} u_{n+2} v_{n+1} + v_{n+1}^2 + u_{n+3} v_{n+2} - u_{n+1} v_n \). We first test if \( f_n \) is exact (i.e., the total difference of some \( F_n \) to be computed later). We apply the discrete zeroth Euler operator to \( f_n \) for each component of \( \mathbf{u}_n = (u_n, v_n) \) separately. For component \( u_n \) (with maximum shift 3) one readily verifies that

\[
\mathcal{L}^{(0)}_{u_n}(f_n) = \frac{\partial}{\partial u_n} \left( I + D^{-1} + D^{-2} + D^{-3} \right) (f_n) \equiv 0. \tag{112}
\]

Similarly, for component \( v_n \) (with maximum shift 2) one checks that \( \mathcal{L}^{(0)}_{v_n}(f_n) \equiv 0 \).

### 11.2 Discrete Higher Euler and Homotopy Operators

To compute \( F_n \), we need higher-order versions of the discrete variational derivative. They are called discrete higher Euler operators \( \mathcal{L}^{(i)}_{u_n} \) in analogy with the continuous case \([30]\).

In Table 4, we have put the continuous and discrete higher Euler operators side by side. Note that the discrete higher Euler operator for \( i = 0 \) is the discrete variational derivative. The first three higher Euler operators for component \( u_n \) from Table 4 are

\[
\mathcal{L}^{(1)}_{u_n} = \frac{\partial}{\partial u_n} \left( D^{-1} + 2D^{-2} + 3D^{-3} + 4D^{-4} + \cdots \right), \tag{113}
\]

\[
\mathcal{L}^{(2)}_{u_n} = \frac{\partial}{\partial u_n} \left( D^{-2} + 3D^{-3} + 6D^{-4} + 10D^{-5} + \cdots \right). \tag{114}
\]

\[
\mathcal{L}^{(3)}_{u_n} = \frac{\partial}{\partial u_n} \left( D^{-3} + 4D^{-4} + 10D^{-5} + 20D^{-6} + \cdots \right). \tag{115}
\]

Similar formulae hold for \( \mathcal{L}^{(i)}_{v_n} \).

| Integrand Operator | Continuous Case | Discrete Case |
|-------------------|-----------------|---------------|
| \( I_{u_j}(f) = \sum_{i=0}^{\infty} D^i \left( u_j \mathcal{L}^{(i+1)}_{u_i}(f) \right) \) | \( I_{u_j,n}(f) = \sum_{i=0}^{\infty} \Delta^i \left( u_{j,n} \mathcal{L}^{(i+1)}_{u_{j,n}}(f) \right) \) |}

Table 4: Continuous and discrete Euler and homotopy operators in 1D side by side.
Also in Table 4, we put the formulae for the *discrete homotopy operator* $H_{u_n}$ and the continuous homotopy operator side by side. The integrand $I_{u_j,n}(f)$ of the homotopy operator involves the discrete higher Euler operators. As is the continuous case, $N$ is the number of dependent variables and $I_{u_j,n}(f)\lambda u_n$ means that after $I_{u_j,n}(f)$ is applied one replaces $u_n$ by $\lambda u_n$, $u_{n+1}$ by $\lambda u_{n+1}$, etc. To compute $F_n$, one can use the following theorem [19, 22, 27].

**Theorem:** Given an exact function $f_n$, one can compute $F_n = \Delta^{-1}(f_n)$ from $F_n = H_{u_n}(f_n)$.

Thus, the homotopy operator reduces the inversion of $\Delta$ (and summation by parts) to a set of differentiations and shifts followed by a single integral with respect to an auxiliary parameter $\lambda$. We present a simplified version [19] of the homotopy operator given in [22, 27], where the problem is dealt with in greater generality and where the proofs are given in context of variational complexes.

For a system with components, $(u_{1,n}, u_{2,n}) = (u_n, v_n)$, the discrete homotopy operator from Table 4 is

$$H_{u_n}(f) = \int_0^1 (I_{u_n}(f)[\lambda u_n] + I_{v_n}(f)[\lambda u_n]) \frac{d\lambda}{\lambda},$$  \hspace{1cm} (116)

with

$$I_{u_n}(f) = \sum_{i=0}^{\infty} \Delta^i \left( u_n L^{(i+1)}_{u_n}(f) \right) \quad \text{and} \quad I_{v_n}(f) = \sum_{i=0}^{\infty} \Delta^i \left( v_n L^{(i+1)}_{v_n}(f) \right).$$  \hspace{1cm} (117)

**Example:** We return to (109). Using (117),

$$I_{u_n}(f_n) = u_n L^{(1)}_{u_n}(f_n) + \Delta \left( u_n L^{(2)}_{u_n}(f_n) \right) + \Delta^2 \left( u_n L^{(3)}_{u_n}(f_n) \right)
\quad \quad = u_n \frac{\partial}{\partial u_n} \left( D^{-1} + 2D^{-2} + 3D^{-3} \right)(f_n) + \Delta \left( u_n \frac{\partial}{\partial u_n} \left( D^{-2} + 3D^{-3} \right)(f_n) \right)
\quad \quad + \Delta^2 \left( u_n \frac{\partial}{\partial u_n} D^{-3}(f_n) \right)
\quad \quad = 2u_nu_{n+1}v_n + u_{n+1}v_n + u_{n+2}v_{n+1},$$  \hspace{1cm} (118)

and

$$I_{v_n}(f_n) = v_n L^{(3)}_{v_n}(f_n) + \Delta \left( v_n L^{(2)}_{v_n}(f_n) \right)
\quad \quad = v_n \frac{\partial}{\partial v_n} \left( D^{-1} + 2D^{-2} \right)(f_n) + \Delta \left( v_n \frac{\partial}{\partial v_n} D^{-2}(f_n) \right)
\quad \quad = u_n u_{n+1} v_n + 2v_n^2 + u_{n+1}v_n + u_{n+2}v_{n+1}.$$  \hspace{1cm} (119)

The homotopy operator (116) thus leads to an integral with respect to $\lambda$ :

$$F_n = \int_0^1 (I_{u_n}(f_n)[\lambda u_n] + I_{v_n}(f_n)[\lambda u_n]) \frac{d\lambda}{\lambda}
\quad \quad = \int_0^1 \left( 2\lambda v_n^2 + 3\lambda^2 u_n u_{n+1} v_n + 2\lambda u_{n+1} v_n + 2\lambda u_{n+2}v_{n+1} \right) d\lambda
\quad \quad = v_n^2 + u_n u_{n+1} v_n + u_{n+1} v_n + u_{n+2} v_{n+1},$$  \hspace{1cm} (120)

which agrees with (110), previously computed by hand.
12 Application: Conservation Laws of Nonlinear DDEs

In [14, 20], different algorithms are presented to compute fluxes of nonlinear DDEs. In this section we show how to compute fluxes with the discrete homotopy operator. For clarity, we compute a conservation law for Example 5 in Section 8. The computations are carried out with our Mathematica packages [18]. The completely integrable Toda lattice (106) has infinitely many conserved densities and fluxes. As an example, we compute density \( \rho_n^{(3)} \) (of rank 3) and corresponding flux \( J_n^{(3)} \) (of rank 4). In this example, \( G = (G_1, G_2) = (v_{n-1} - v_n, v_n(u_n - u_{n+1})). \) Assuming that the weights \( W(u_n) = 1 \) and \( W(v_n) = 2 \) are computed and the rank of the density is selected (say, \( R = 3 \)), our algorithm works as follows:

**Step 1: Construct the form of the density**

Start from \( V = \{u_n, v_n\} \), i.e. the list of dependent variables with weight. List all monomials in \( u_n \) and \( v_n \) of rank 3 or less: \( M = \{u_n^3, u_n^2, u_n v_n, u_n, v_n\} \).

Next, for each monomial in \( M \), introduce the correct number of \( t \)-derivatives so that each term has rank 3. Using (106), compute

\[
\begin{align*}
\frac{d^0 u_n^3}{dt^0} &= u_n^3, \\
\frac{d^0 u_n v_n}{dt^0} &= u_n v_n, \\
\frac{d u_n^2}{dt} &= 2u_n \dot{u}_n = 2u_n v_{n-1} - 2u_n v_n, \\
\frac{d v_n}{dt} &= \dot{v}_n = u_n v_n - u_{n+1} v_n, \\
\frac{d^2 u_n}{dt^2} &= \frac{d (v_{n-1} - v_n)}{dt} = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n.
\end{align*}
\]

(122)

Augment \( M \) with the terms from the right hand sides of (122) to get \( \mathcal{R} = \{u_n^3, u_n v_{n-1}, u_n v_n, u_{n-1} v_{n-1}, u_{n+1} v_n\} \).

Identify members belonging to the same equivalence classes and replace them by their main representatives. For example, \( u_n v_{n-1} \equiv u_{n+1} v_n \), so the latter is replaced by \( u_n v_{n-1} \). Hence, replace \( \mathcal{R} \) by \( \mathcal{S} = \{u_n^3, u_n v_{n-1}, u_n v_n\} \), which has the building blocks of the density. Linearly combine the monomials in \( \mathcal{S} \) with coefficients \( c_i \) to get the candidate density:

\[
\rho_n = c_1 u_n^3 + c_2 u_n v_{n-1} + c_3 u_n v_n.
\]

(123)

**Step 2: Determine the coefficients**

Require that (108) holds. Compute \( D_t \rho_n \). Use (106) to remove \( \dot{u}_n \) and \( \dot{v}_n \) and their shifts. Thus,

\[
E_n = D_t \rho_n = (3c_1 - c_2) u_n^2 v_{n-1} + (c_3 - 3c_1) u_n^2 v_n + (c_3 - c_2) v_{n-1} v_n + c_2 u_{n-1} u_n v_{n-1} + c_2 v_{n-1}^2 - c_3 u_n u_{n+1} v_n - c_3 v_n^2.
\]

(124)

To remove the negative shift \( n - 1 \), compute \( \tilde{E}_n = D E_n \). Apply \( \mathcal{L}^{(0)}_{u_n} \) to \( \tilde{E}_n \), yielding

\[
\mathcal{L}^{(0)}_{u_n}(\tilde{E}_n) = \frac{\partial}{\partial u_n}(I + D^{-1} + D^{-2})(\tilde{E}_n)
= 2(3c_1 - c_2) u_n v_{n-1} + 2(c_3 - 3c_1) u_n v_n + (c_2 - c_3) u_{n-1} v_{n-1}
+ (c_2 - c_3) u_n v_{n+1}.
\]

(125)
Next, apply $\mathcal{L}^{(0)}_{v_n}$ to $\tilde{E}_n$, yielding
\[
\mathcal{L}^{(0)}_{v_n}(\tilde{E}_n) = \frac{\partial}{\partial v_n} (I + D^{-1})(\tilde{E}_n)
= (3c_1 - c_2)u_{n+1}^2 + (c_3 - c_2)v_{n+1} + (c_2 - c_3)u_nv_n + 2(c_2 - c_3)v_n
+ (c_3 - 3c_1)u_n^2 + (c_3 - c_2)v_{n-1}.
\]
(126)

Both (125) and (126) must vanish identically. Solve the linear system
\[
3c_1 - c_2 = 0, \quad c_3 - 3c_1 = 0, \quad c_2 - c_3 = 0.
\]
(127)

Set $c_1 = \frac{1}{3}$ and substitute the solution $c_1 = \frac{1}{3}, c_2 = c_3 = 1$, into (123)
\[
\rho_n = \frac{1}{3}u_n^3 + u_n(v_{n-1} + v_n).
\]
(128)

**Step 3: Compute the flux**

In view of (108), one must compute $J_n = -\Delta^{-1}(E_n)$. Substitute $c_1 = \frac{1}{3}, c_2 = c_3 = 1$ into (124). Then, $\tilde{E}_n = DE_n = u_nu_{n+1}v_n + v_n^2 - u_{n+1}u_{n+2}v_{n+1} - v_{n+1}^2$.

Apply (117) to $-\tilde{E}_n$ to obtain
\[
I_{u_n}(-\tilde{E}_n) = 2u_nu_{n+1}v_n, \quad I_{v_n}(-\tilde{E}_n) = u_nu_{n+1}v_n + 2v_n^2.
\]
(129)

Application of homotopy operator (116) yields
\[
\tilde{J}_n = \int_0^1 (I_{u_n}(-\tilde{E}_n)[\lambda u_n] + I_{v_n}(-\tilde{E}_n)[\lambda v_n]) \frac{d\lambda}{\lambda}
= \int_0^1 (3\lambda^2u_nu_{n+1}v_n + 2\lambda v_n^2) d\lambda
= u_nu_{n+1}v_n + v_n^2.
\]
(130)

After a backward shift, $J_n = D^{-1}(\tilde{J}_n)$, we obtain $J_n$. With (128), the final result is then
\[
\rho_n = \frac{1}{3}u_n^3 + u_n(v_{n-1} + v_n), \quad J_n = u_{n-1}u_nv_{n-1} + v_{n-1}^2.
\]
(131)

The above density corresponds to $\rho^{(3)}_n$ in Table 3.

### 13 Conclusion

Based on the concept of scaling invariance and using tools of the calculus of variations, we presented algorithms to symbolically compute conservation laws of nonlinear polynomial and transcendental systems of PDEs in multi-spacial dimensions and DDEs in one discrete variable. We covered the symbolic computation of densities and fluxes.

The continuous homotopy operator turned out to be a powerful, algorithmic tool to compute fluxes explicitly. Indeed, the homotopy operator handles integration by parts in multi-variables which allowed us to invert the total divergence operator. Likewise, the discrete
homotopy operator handles summation by parts and inverts the forward difference operator. In both cases, the problem is reduced to an explicit integral from 1D calculus.

Homotopy operators have a wide range of applications in the study of PDEs, DDEs, fully discretized lattices, and beyond. We extracted the continuous and discrete Euler and homotopy operators from pure mathematics, introduced them into applied mathematics, and therefore make them readily applicable to computational problems. We purposely avoided differential forms [30] and abstract concepts from differential geometry and homological algebra.

Our down-to-earth approach might appeal to scientists who prefer not to juggle exterior products and Lie derivatives. Our calculus-based formulas for the Euler and homotopy operators can be readily implemented in major CAS.

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