Series-Like Iterative Functional Equation for PM Functions

M. Suresh Kumar\textsuperscript{1} and V. Murugan\textsuperscript{2}
\textsuperscript{1}Department of Mathematics, The Gandhigram Rural Institute, Gandhigram - 624 302, Dindigul, India
\textsuperscript{2}Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal - 575 025, Mangalore, India
E-mail: sureshkumarmusu2009@gmail.com, murugan@nitk.edu.in

Abstract. Given a non-empty subset $X$ of the real line and a self map $G$ on $X$, the functional equation representing $G$ as an infinite linear combination of iterations of a self map $g$ on $X$ is known as the series-like functional equation. The solutions of the series-like functional equation have been studied only for the class of continuous strictly monotone functions. In this paper, we prove the existence of solutions of series-like functional equations for the class of continuous non-monotone functions using characteristic interval.

Keywords: Iterative root, Fort, Characteristic Interval, PM functions.

1. Introduction
Given a set $X \subseteq \mathbb{R}$, a function $G : X \rightarrow X$ and a sequence of real numbers $\{\lambda_n\}$, the functional equation of the form
\begin{equation}
\sum_{n=1}^{\infty} \lambda_n g^n(t) = G(t) \quad \text{for all } t \in X
\end{equation}
is called a series-like iterated functional equation. Any function $f : X \rightarrow X$ satisfies (1) is called a solution to the series-like iterated functional equation (1). If $\lambda_n = 0$ for all $n > m$ for some $m \in \mathbb{N}$ then the functional equation (1) reduces to
\begin{equation}
\sum_{n=1}^{m} \lambda_n g^n(t) = G(t) \quad \text{for all } t \in X.
\end{equation}
The functional equation (2) is known as polynomial-like iterative functional equation \cite{12}. If $\lambda_1 = 1$ and $\lambda_n = 0$ for all $n \geq 2$ then the functional equation (1) further reduces to
\begin{equation}
g^n(t) = G(t) \quad \text{for all } t \in X.
\end{equation}
The functional equation (3) is known as an iterative functional equation \cite{14}. The equation (3) is one of the simplest form of iterative functional equation, and it is examined in the classical works of Babbage \cite{1}. The solution of (3) has been studied extensively for the class of continuous monotone functions in \cite{2} and \cite{4}. More precisely, for a given strictly increasing continuous function $G$ on an interval $I$ the equation (3) has solution for all $n \in \mathbb{N}$,
and for a strictly decreasing continuous onto function on an interval $I$ the equation (3) has solution for all odd $n \in \mathbb{N}$ and has no solution for all even $n \in \mathbb{N}$ (see Theorem 11.2.2 and 11.2.4 [5]). Zhang [14], introduced the concept of height and characteristic interval for a continuous non-monotone functions having finitely many non-monotone points, known as Piecewise Monotone (PM) functions, and studied the solutions of (3). Further developments and some open problems on the existence of solutions of (3) for PM functions and non-PM functions can be found in [7],[3],[6],[8].

The polynomial-like iterative functional equation (2), which is the generalization of (3), whose continuous and differentiable solutions has been studied in [11], [12], [13] for the class of continuous strictly monotone functions. The existence of solutions of (2) for the class of PM functions has been studied in [15] by using the method of characteristic interval.

The existence of continuous solutions of series-like iterative functional equation (1) has been discussed in [9],[10], but only for the class of strictly monotone functions. The existence of solutions of series-like iterative functional equation (1) for the class of PM functions remains unsolved. In this paper, we prove the existence of solutions of series-like iterative functional equation (1) for the class of PM functions by using the method of characteristic interval. At first, we prove the existence of solution of (1) on the characteristic interval and then extend that solution to the whole domain of the PM function. We also provide an example to illustrate our main theorem.

2. Preliminaries

Throughout this paper, we fix the domain of all functions to be the closed and bounded interval $I = [a, b]$, and all functions are assumed to be continuous on $I$. We say that a point $t \in I$ is a fort of the function $g$, if $g$ is non-monotone in any neighborhood of $t$. Further, $g$ is said to be piecewise monotone (PM) if it has finite number of forts. We denote the set of all continuous functions from $I$ into itself by $C(I)$ and the set of all PM functions from $I$ into itself by $PM(I)$.

For any PM function $g$, if we denote the number of forts by $N(g)$ then it is easy to observe that $\{N(g^n)\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive integers. We say a positive integer $k$ as height of $g$, if $k$ is the least positive integer such that $N(g^k) = N(g^{k+1})$. If there is no such positive integer then we say the height of $g$ is infinity [14]. Height of the function $g$ is denoted by $H(g)$.

**Definition 2.1.** [3] The characteristic interval of $g$ is the smallest closed interval containing the range of $g$ whose endpoints are either forts of $g$ or the endpoints of $I$ and it is denoted by $Ch_g$.

We observe that, for any PM function $g$, $H(g) \leq 1$ if and only if $g$ is strictly monotone in $Ch_g$. A detailed study of functions having different height can be found in [3, 6, 7, 14]. Let $a', b' \in I$ such that $a' < b'$ and $m, M$ be positive real numbers. We define a class of functions as follows:

$$
\mathcal{F}(I) = \{ g \in C(I) \mid g(a) = a, g(b) = b \}
$$

$$
\mathcal{F}(I, m, M) = \{ g \in \mathcal{F}(I) \mid m(t-s) \leq g(t) - g(s) \leq M(t-s) \forall t, s \in I \text{ with } t > s \}
$$

$$
\mathcal{S}([a', b'], m, M) = \{ g \in PM(I) \mid Ch_g = [a', b'], g|_{Ch_g} \in \mathcal{F}(I, m, M) \}.
$$

We observe that the set $\mathcal{F}([a', b'], m, M)$ is a convex, compact subset of $C(I)$ (see Proposition 2.2 of [10]). Also, as any function $g \in \mathcal{S}([a', b'], m, M)$ is strictly increasing in $Ch_g$, the set $\mathcal{S}([a', b'], m, M)$ consists of a collection of PM functions of height less than or equal to one whose characteristic interval is $[a', b']$.

Our main theorem proves the existence of solutions of series-like functional equation (1) for any function $G \in \mathcal{S}([a', b'], m, M)$ with some specific values of $m, M$. At first we prove the
existence of solution of (1) in the characteristic interval of $G$ and then we extend that solution to the whole interval. The following lemma will lead to our main result.

**Lemma 2.2.** Let $G \in PM(I)$ with $H(G) \leq 1$, $Ch_G = [a', b']$ and $G_0 = G_{|[a',b']}$. Let $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of non-negative numbers and $\alpha_n \leq \lambda_n \leq \beta_n$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$.

Suppose (i) $0 < m < 1$, $M > 1$ (ii) $\sum_{n=1}^{\infty} \beta_n M^n < \infty$ (iii) $K_0 = \sum_{n=1}^{\infty} \alpha_n m^{n-1} > 0$.

Then for any $G_0 \in \mathcal{F}([a', b'], K_0, M)$, there exists a function $g_0 \in \mathcal{F}([a', b'], m, M)$ such that

$$\sum_{n=1}^{\infty} \lambda_n g_0^n(t) = G_0(t) \text{ for all } t \in [a', b'],$$

where $K_1 = \sum_{n=1}^{\infty} \beta_n M^{n-1}$.

Proof of the Lemma 2.2 follows from Corollary 3.5 in [10]. For each $G \in S([a', b'], m, M)$, Lemma 2.2 guarantees the solution of the series-like iterative functional equation (1) on the characteristic interval of $G$. Hence, to study existence of solutions of equation (1) for PM functions, in addition to the hypothesis of Lemma 2.2, it is enough to discuss the extension of such solution to the whole interval.

3. Extension of Solutions from the Characteristic Interval

In this section we prove our main theorem of extending solutions of (1) from the characteristic interval of $G$ to the whole interval $I$. The following lemma will be useful in extending the above solution to the whole interval $I$.

**Lemma 3.1.** Let $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of non-negative numbers and $\alpha_n \leq \lambda_n \leq \beta_n$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$. Suppose $0 < m < 1$, $M > 1$ and

(i) $\sum_{n=1}^{\infty} \beta_n M^n < \infty$ (ii) $K_0 = \sum_{n=1}^{\infty} \alpha_n m^{n-1} > 0$.

Then for each $g \in \mathcal{F}([a', b'], K_0, M)$, the function $L_g : [a', b'] \rightarrow [a', b']$ defined by

$$L_g(t) = \sum_{n=1}^{\infty} \lambda_n g^{n-1}(t) \text{ for all } t \in [a', b']$$

is invertible and $L_g \in \mathcal{F}([a', b'], K_0, M_1)$. Further, $L_g^{-1} \in \mathcal{F}([a', b'], \frac{1}{K_1}, \frac{1}{K_1})$.

Proof. Proof of Lemma 3.1 follows from Lemma 3.2 in [10].

Suppose that $G \in PM(I)$ with $H(G) \leq 1$. Then $G_{|[a',b']}$ is strictly monotone, where $Ch_G = [a', b']$. Therefore, by using the hypothesis of Lemma 2.2, any $G_{|[a',b']} \in \mathcal{F}([a', b'], K_0, M)$ the series-like functional equation (1) has a solution in $\mathcal{F}([a', b'], m, M)$. By extending this solution, we prove that, any function $G \in S([a', b'], m, M)$ and for any non-negative sequence $\{\lambda_n\}$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$, the functional equation (1) has a solution.
Theorem 3.2. Let \{\lambda_n\}, \{\alpha_n\} and \{\beta_n\} are sequence of non-negative numbers and \(\alpha_n \leq \lambda_n \leq \beta_n\) such that \(\sum_{n=1}^{\infty} \lambda_n = 1\). Suppose \(0 < m < 1\), \(M > 1\) and

\[(i) \sum_{n=1}^{\infty} \beta_n M^n < \infty, \quad (ii) K_0 = \sum_{n=1}^{\infty} \alpha_n m^{n-1} > 0, \quad (iii) G(I) \subseteq G([a', b']).\]

Then for any \(G \in \mathcal{S}([a', b'], K_0 M, M)\) the series-like functional equation (1) has a solution \(g \in \mathcal{S}([a', b'], m, M)\), where \(K_1 = \sum_{n=1}^{\infty} \beta_n M^{n-1}\).

Proof. Put \(G_0 = G|_{[a', b']}\). Then \(G|_{[a', b']} \in \mathcal{F}([a', b'], K_1 m, K_0 M)\). Therefore, by Lemma 2.2, there exists \(g_0 \in \mathcal{F}([a', b'], m, M)\) such that

\[
\sum_{n=1}^{\infty} \lambda_n g_0^n(t) = G_0(t) \text{ for all } t \in [a', b'].
\]

We now extend this \(g_0\) to the whole interval \(I\). For this, define \(L_{g_0} : [a', b'] \to [a', b']\) by

\[
L_{g_0}(t) = \sum_{n=1}^{\infty} \lambda_n g_0^{n-1}(t) \text{ for all } t \in [a', b'].
\]

Then, by Lemma 3.1, \(L_{g_0} \in \mathcal{F}([a', b'], K_0, K_1)\) and \(L_{g_0}\) is invertible. Moreover \(L_{g_0}^{-1} \in \mathcal{F}([a', b'], \frac{1}{K_0}, \frac{1}{K_1})\). Now, we define \(g : I \to I\) as follows:

\[
g(t) := \begin{cases} L_{g_0}^{-1} \circ G(t) & \text{if } a \leq t < a' \\ g_0(t) & \text{if } a' \leq t \leq b' \\ L_{g_0}^{-1} \circ G(t) & \text{if } b' \leq t \leq b. \end{cases}
\]

The function \(g\) is well defined by condition (iii). Consequently, for each \(t \in [a', b']\) we have

\[
\sum_{n=1}^{\infty} \lambda_n g^n(t) = \sum_{n=1}^{\infty} \lambda_n g_0^n(t) = G_0(t) = G(t).
\]

On other hand, if \(t \in I \setminus [a', b']\) then

\[
\sum_{n=1}^{\infty} \lambda_n g^n(t) = \lambda_1 g(t) + \lambda_2 g^2(t) + \cdots + \lambda_n g^n(t) + \cdots = \lambda_1 L_{g_0}^{-1}(G(t)) + \lambda_2 g_0(L_{g_0}^{-1}(G(t))) + \cdots + \lambda_n g_0^{n-1}(L_{g_0}^{-1}(G(t))) + \cdots = \sum_{n=1}^{\infty} \lambda_n g_0^{n-1}(L_{g_0}^{-1}(G(t))) = L_{g_0}(L_{g_0}^{-1}(G(t))) = G(t).
\]

Hence \(g\) satisfies the functional equation (1) for all \(t \in I\). To prove \(g\) is continuous on \(I\), by definition of \(g\), it is suffices to prove \(g\) is continuous at the points \(a'\) and \(b'\). If \(\{t_n\}\) be a sequence in \([a, a']\) such that \(t_n \to a'\) as \(n \to \infty\) then

\[
\lim_{n \to \infty} g(t_n) = \lim_{n \to \infty} L_{g_0}^{-1}(G(t_n)) = L_{g_0}^{-1}(G(a')) = L_{g_0}^{-1}(a') = g(a').
\]
Therefore $g$ is continuous at the point $a'$. In a similar manner, we can show that $g$ is continuous at the point $b'$. We now prove that $g \in S([a', b'], m, M)$. To do so, it suffices to show that the characteristic interval of $g$ is $[a', b']$. First, we observe that the only forts of $g$ are the forts of $G$, in particular, the points $a'$ and $b'$ are forts of $g$. We also observe, from the existence of $g_0$, $L_{g_0}^{-1}$ and condition (iii), that the range of $g$ is contained in $[a', b']$. Further, by the choice of $g_0$, the function $g$ has no forts in $[a', b']$. This shows that the interval $[a', b']$ is the smallest closed interval that contains the range of $g$ whose endpoints are the forts of $g$. Therefore $Ch_g = [a', b']$ and the proof is completed.

The following example illustrates our main theorem.

**Example 1.** Let $G : [0, 1] \to [0, 1]$ be the function defined by

\[
G(t) = \begin{cases} 
    t & \text{if } t \in [0, \frac{1}{2}] \\
    1 - t & \text{if } t \in \left(\frac{1}{2}, 1\right].
\end{cases}
\]

Clearly $G \in PM([0, 1])$ with $H(G) = 1$ and $N(G) = 1$. Now, consider the series-like iterative functional equation (1) with $\lambda_1 = \frac{3}{4}$ and $\lambda_n = \frac{1}{2^{n+1}}$ for all $n \geq 2$. Choose $\alpha_n = \beta_n = \lambda_n$ for all $n \in \mathbb{N}$ and $m = \frac{1}{4}$, $M = 2$. Note that

\[
\sum_{n=1}^{\infty} \beta_n M^n = \frac{4}{3} + 2 \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} < \infty,
\]

and

\[
K_0 = \sum_{n=1}^{\infty} \alpha_n m^{n-1} = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{4^{2n-2}} = \frac{11}{15},
\]

also

\[
K_1 = \sum_{n=1}^{\infty} \beta_n M^{n-1} = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{2^{n-2}} = \frac{8}{3}.
\]

Clearly $G \in S([0, \frac{1}{2}], \frac{3}{4}, \frac{22}{3})$. Therefore, by Theorem 3.2, the series-like functional equation (1) has a solution $g \in S([0, \frac{1}{2}], \frac{1}{4}, 2)$.

Further, we can construct the solution of the series like functional equation (1), if we know the solution in the characteristic interval of $G$. It is easy to verify that the function $g_0 : Ch_{g_0} = [0, \frac{1}{2}] \to Ch_{g_0}$ defined by $g_0(t) = t$ for all $t \in Ch_{g_0}$ satisfies the series like functional equation (1) on $Ch_{g_0}$ and $g_0 \in \mathcal{F}([0, \frac{1}{2}], \frac{1}{4}, 2)$. Also, an easy calculation shows that, $L_{g_0} : Ch_{g_0} \to Ch_{g_0}$ is the function defined by $L_{g_0}(t) = t$ for all $t \in [0, \frac{1}{2}]$. Therefore the function $g : [0, 1] \to [0, 1]$, constructed by using Theorem 3.2, given below

\[
g(t) = \begin{cases} 
    t & \text{if } t \in [0, \frac{1}{2}) \\
    1 - t & \text{if } t \in [\frac{1}{2}, 1].
\end{cases}
\]

satisfies the series like functional equation (1) on $[0, 1]$ and $g \in S([0, \frac{1}{4}], \frac{1}{4}, 2)$.

**4. Conclusions**

The existence of solutions of the series-like functional equation (3) has been addressed in this note. More precisely, the existence of solutions of (3) has been investigated on the characteristic interval of a PM function whose height is less than two. Further, we extended that solution from the characteristic interval to the whole domain of a PM function. Note that our extension theorem (Theorem 3.2) will work not only for PM functions with height less than two but all PM functions, provided (3) has a solution on the characteristic interval. The existence of solutions of (3) on the characteristic interval for PM functions with height greater than one is still an open unsolved problem.
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