Evidence for the Unbinding of the $\phi^4$ Kink’s Shape Mode

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Abstract

The $\phi^4$ double-well theory admits a kink solution, whose rich phenomenology is strongly affected by the existence of a single bound excitation called the shape mode. We find that the leading quantum correction to the energy needed to excite the shape mode is $-0.115567\lambda/m$ in terms of the coupling $\lambda/4$ and the meson mass $m$ evaluated at the minimum of the potential. On the other hand, the correction to the continuum threshold is $-0.433\lambda/m$. A naive extrapolation to finite coupling then suggests that the shape mode melts into the continuum at the modest coupling of $\lambda/4 \sim 0.106m^2$, where the $Z_2$ symmetry is still broken.

1 Introduction

The $\phi^4$ double-well field theory admits a kink solution. This kink appears in a number of physical systems, from polyacetylene [1] to crystals [2] to graphene [3]. Besides the translation mode, it enjoys a single bound excitation, called the shape mode. The shape mode is believed to be responsible for most of the kink’s distinctive phenomenology [4] (but see Ref. [5]), such as the resonance phenomenon [6,7] (but see Ref. [8]), wobbling kink multiple scattering [9,10] and spectral walls [11,12]. While the shape mode deserves and has received significant attention in the literature, quantum corrections to the shape mode have not yet been considered.

In this note we report the one-loop quantum correction to the energies needed to excite both the kink’s shape mode and also the first continuum excitation. We find that the energy needed to excite a shape mode decreases linearly in the coupling, but that the energy needed to excite a continuum mode decreases more rapidly. If this behavior is extrapolated to intermediate couplings, then the shape mode energy will exceed that of a continuum meson and so the shape mode is expected to become unbound, dissolving into the continuum at $\lambda/m^2 \sim 0.422$. This occurs at smaller coupling than other expected features of the model, such as the restoration of the $\phi \rightarrow -\phi$ symmetry at $\lambda/m^2 \sim 1.2(1)$ found in Ref. [13], or
the point at which the meson mass exceeds twice \cite{14, 15} or \( \pi \) times \cite{16} the semiclassical \cite{17, 18} kink mass and one expects mesons to drop out of the spectrum.

2 Review of the \( \phi^4 \) Kink

The \( \phi^4 \) double-well quantum field theory is defined by the Hamiltonian

\[
H = \int dx \left[ \frac{1}{2} : \pi(x) \pi(x) : a + \frac{1}{2} : \partial_x \phi(x) \partial_x \phi(x) : a + : \frac{\phi^2}{4} \left( \sqrt{\lambda} \phi(x) - m \sqrt{2} \right)^2 : a \right] \tag{2.1}
\]

where \( : a \) is normal ordering with respect to the operators that create plane wave excitations of the \( \phi \) field about a classical vacuum. It admits a classical kink solution

\[
\phi(x, t) = f(x) = \frac{m}{\sqrt{2\lambda}} \left( 1 + \tanh \left( \frac{mx}{2} \right) \right) \tag{2.2}
\]

whose mass is classically

\[
Q_0 = \frac{m^3}{3\lambda}. \tag{2.3}
\]

Small, orthonormal perturbations about the kink solution with frequency \( \omega_k \) will all be called normal modes. They include a zero-mode proportional to \( f' \) as well as continuum modes \( g_k(x) \) and a shape mode \( g_S(x) \)

\[
g_k(x) = \frac{e^{-ikx}}{\omega_k \sqrt{m^2 + 4k^2}} \left[ 2k^2 - m^2 \right.
\]

\[
+ (3/2)m^2 \text{sech}^2 \left( \frac{mx}{2} \right) - 3imk \tanh \left( \frac{mx}{2} \right) \bigg] \tag{2.4}
\]

\[
g_S(x) = -i \frac{3m}{2} \tanh \left( \frac{mx}{2} \right) \text{sech} \left( \frac{mx}{2} \right).
\]

At tree level the energy required to excite a normal mode is its frequency \cite{17}

\[
\omega_k = \sqrt{m^2 + k^2}, \quad \omega_S = \frac{\sqrt{3}}{2} m. \tag{2.5}
\]
Figure 1: The first two diagrams give $\mu(1)(S)$. The next two are equal and sum to $\mu(2)(S)$. The last diagram is $\mu(3)(S)$. $\mu(4)(S)$ is found by replacing one $k$ in the first two diagrams with a zero mode. A loop factor $I(x)$ arises for each loop at a single vertex.

3 Energy Required to Excite a Shape Mode

The one-loop energy required to excite the shape mode is [19]

$$\mu(S) = \sum_{i=1}^{4} \mu^{(i)}(S), \quad \mu^{(1)}(S) = \int_{+}^{d^2 k / (2\pi)^2} \frac{(\omega_{k_1} + \omega_{k_2}) |V_{k_1 k_2 S}|^2}{8\omega_S \omega_{k_1} \omega_{k_2} (\omega_S^2 - (\omega_{k_1} + \omega_{k_2})^2)}$$

$$\mu^{(2)}(S) = - \int_{+}^{d^2 k / 2\pi} \frac{V_{S k S} V_{T-k}}{4\omega S \omega_k^2}, \quad \mu^{(3)}(S) = \frac{V_{T-S-S} S}{4\omega_S}$$

$$\mu^{(4)}(S) = \frac{1}{4Q_0} \int_{+}^{d^2 k / 2\pi} \left( \frac{\omega_S}{\omega_k} + \frac{\omega_k}{\omega_S} \right) \Delta_{-S-k} \Delta_{S_k}. \quad (3.1)$$

The corresponding diagrams, defined in Ref. [19], are drawn in Fig. 1. Here we have defined $\int_{+}^{d^2 k / (2\pi)}$ to be an integral over all real values of $k$ plus the same expression with $k$ replaced by the shape mode. The matrix $\Delta$ is defined by

$$\Delta_{S_k} = \int dx g_S(x) g'_k(x) \quad (3.2)$$

where it is understood that $g_{-S} = g_S^*$. Let $V^{(n)}(x)$ be the $n$th functional derivative of $H$ with respect to $\phi(x)$, evaluated at $\phi(x) = f(x)$

$$V^{(3)}(x) = 3\sqrt{2} \lambda \tanh \left( \frac{m x}{2} \right), \quad V^{(4)}(x) = 6\lambda. \quad (3.3)$$

Then we have also defined the symbol

$$V_{k_1 \ldots k_n} = \int dx V^{(n)}(x) \prod_{i=1}^{n} g_k(x), \quad V_{T k_1 \ldots k_n} = \int dx V^{(2+n)}(x) I(x) \prod_{i=1}^{n} g_k(x) \quad (3.4)$$

where the loop function $I(x)$ is

$$I(x) = \frac{|g_S(x)|^2}{2\omega_S} + \int \frac{dk}{2\pi} \left( \frac{|g_k(x)|^2}{2\omega_k} - 1 \right) \quad (3.5)$$

$$= \frac{1}{4\sqrt{3}} \sech^2 \left( \frac{m x}{2} \right) \tanh^2 \left( \frac{m x}{2} \right) - \frac{3}{8\pi} \sech^4 \left( \frac{m x}{2} \right).$$
Most of the relevant \( x \) integrals were already evaluated analytically in Ref. [18]

\[
\Delta_{Sk} = \frac{i\pi \sqrt{3} (3m^2 + 4k^2) \sqrt{m^2 + 4k^2}}{m^{3/2} \omega_k} \text{sech} \left( \frac{\pi k}{m} \right) \tag{3.6}
\]

\[
V_{SSS} = i\frac{9\sqrt{3} \lambda}{32\sqrt{2}} m^{3/2}, \quad V_{kSS} = i\frac{3\sqrt{3} \lambda \omega_k (m^2 - 2k^2)}{\sqrt{2}} \frac{m^3}{m^2 \sqrt{m^2 + 4k^2}} \text{csch} \left( \frac{\pi k}{m} \right) \tag{3.7}
\]

\[
V_{k1k2S} = -i\frac{3\sqrt{3} \lambda}{2\sqrt{2}} \frac{(\frac{17}{16} m^4 - (k_1^2 - k_2^2)^2) (m^2 + 4k_1^2 + 4k_2^2)}{m^{3/2} \omega_{k1} \omega_{k2} \sqrt{m^2 + 4k_1^2} \sqrt{m^2 + 4k_2^2}} \text{sech} \left( \frac{\pi (k_1 + k_2)}{m} \right)
\]

including some terms with a loop factor \( I(x) \)

\[
V_{I_k} = i\frac{\sqrt{3} \lambda}{\sqrt{6} m^4 \sqrt{m^2 + 4k^2}} \left[ \pi (-m^2 + 2k^2) + 3\sqrt{3} \omega_k^2 \right] \text{csch} \left( \frac{\pi k}{m} \right) \tag{3.8}
\]

\[
V_{I_S} = i\frac{3\sqrt{3} \lambda}{64\sqrt{3}} (3\sqrt{3} - 2\pi).
\]

We will also need

\[
V_{I_{S-S}} = \left( \frac{3\sqrt{3} \pi - 18}{35\pi} \right) \lambda.
\]

Substituting these expressions into (3.1) one can immediately evaluate the third and fourth terms

\[
\mu^{(3)}(S) = \left( \frac{3\pi - 6\sqrt{3}}{70\pi} \right) \frac{\lambda}{m} \sim -0.00439962 \frac{\lambda}{m} \tag{3.9}
\]

\[
\mu^{(4)}(S) = \frac{3\sqrt{3} \pi^2 \lambda}{2^{11} m^7} \int \frac{dk}{2\pi} \frac{(7m^2 + 4k^2)(3m^2 + 4k^2)(m^2 + 4k^2)}{(m^2 + k^2)^{3/2}} \text{sech}^2 \left( \frac{\pi k}{m} \right)
\]

\[
\sim 0.27419991 \frac{\lambda}{m}.
\]

The second term can be decomposed into the contributions \( \mu^{(20)} \) and \( \mu^{(21)} \) in which the dummy index is the shape mode or a continuum mode respectively

\[
\mu^{(20)}(S) = -\frac{V_{SSS} V_{I_{S-S}}}{4\omega_S^3} = \frac{9\pi}{2^{11}} (3\sqrt{3} - 2\pi) \frac{\lambda}{m} \sim -0.01500739 \frac{\lambda}{m} \tag{3.10}
\]

\[
\mu^{(21)}(S) = -\int \frac{dk}{2\pi} \frac{V_{kSS} V_{I_{k-S}}}{4\omega_S \omega_k^3}
\]

\[
= \frac{\pi \lambda}{4m^8} \int \frac{dk}{2\pi} \frac{k^4 (m^2 - 2k^2) (3\sqrt{3} - \pi) m^2 + (3\sqrt{3} + 2\pi) k^2 \text{csch}^2 \left( \frac{\pi k}{m} \right)}{m^2 + 4k^2}
\]

\[
\sim -0.00339318 \frac{\lambda}{m}.
\]
Finally we may decompose the first contribution into terms $\mu^I$ with $I$ dummy indices running over continuous modes $k$

$$\mu^{(10)}(S) = \frac{(2\omega_S)|V_{SSS}|^2}{8\omega_S^3(\omega_S^2 - (2\omega_S)^2)} = -\frac{|V_{SSS}|^2}{12\omega_S^3} = -\frac{9\pi^2}{2^{11}S} \lambda m \sim -0.17348914 \frac{\lambda}{m} \quad (3.11)$$

$$\mu^{(11)}(S) = 2\int \frac{dk}{2\pi} \frac{(\omega_k + \omega_S)|V_{kSS}|^2}{8\omega_S^2\omega_k(\omega_S^2 - (\omega_S + \omega_k)^2)}$$

$$= \frac{3\pi^2\lambda}{m^3} \int \frac{dk}{2\pi} \frac{(2\omega_k + \sqrt{3}m) \omega_k k^4}{(m^2 - (\sqrt{3}m + 2\omega_k)^2)^2 + 4k^2) \cosh^2\left(\frac{\pi k}{m}\right)}$$

$$\sim -0.01112149 \frac{\lambda}{m}$$

$$\mu^{(12)}(S) = \int \frac{d^2k}{(2\pi)^2} \frac{(\omega_{k_1} + \omega_{k_2})|V_{k_1k_2S}|^2}{8\omega_{k_1}\omega_{k_2}(\omega_{k_1}^2 - (\omega_{k_1} + \omega_{k_2})^2)}$$

$$= \frac{9\sqrt{3}\pi^2\lambda}{8m^4} \int \frac{d^2k}{(2\pi)^2} \frac{(\omega_{k_1} + \omega_{k_2})}{\omega_{k_1}\omega_{k_2} \cosh^2\left(\frac{\pi (k_1 + k_2)}{m}\right)}$$

$$\times \frac{\left(\frac{15}{16}m^4 - (k_1^2 - k_2^2)^2\right) (m^2 + 4k_1^2 + 4k_2^2) + 8m^2k_1^2k_2^2}{(m^2 + 4k_1^2)(m^2 + 4k_2^2)(3m^2 - 4(\omega_{k_1} + \omega_{k_2})^2)} \sim -0.1823560(2) \frac{\lambda}{m}.$$
vacuum and the normal modes with plane waves and dividing by a normalization factor of $2\pi\delta(0)$. Any method yields

$$\mu(0) = \frac{9m\lambda}{2} \int dp \frac{1}{2\pi \omega_p(m^2 - 4\omega_p^2)} = -\frac{\sqrt{3}}{4} \frac{\lambda}{m} \sim -0.433013\frac{\lambda}{m}. \quad (4.1)$$

Including the leading order and one-loop contributions, the energy required to excite a continuum mode is then

$$m - \frac{\sqrt{3}}{4} \frac{\lambda}{m} \quad (4.2)$$

and that required to excite a shape mode is

$$\frac{\sqrt{3}}{2} m - 0.1155669(2) \frac{\lambda}{m}. \quad (4.3)$$

These are equal when $\lambda/m^2 \sim 0.422$, at which point one expects the shape mode to delocalize into oblivion, drastically affecting the phenomenology. As the coupling constant is $\lambda/4$, this corresponds to a rather small value of the coupling and so it seems reasonable to speculate that the higher order corrections only displace but do not avoid this conclusion.

We have presented evidence that, at reasonably low but finite coupling, the $\phi^4$ kink loses its only bound excitation that is not a zero mode. It is not clear from our calculation whether such behavior is generic for kinks, or indeed for solitons. Recently, in Ref. [20] (see also [21, 22]), a number of new examples of such bound excitations have been found, and an efficient algorithm was described for generating them. Our method can only be applied to stable kinks, as we require Hamiltonian eigenstates, but using the stability criterion in Refs. [23, 24] one may easily filter examples. Our formula (3.1) can then be applied to study the evolution of the bound modes.

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