Preconditioned space–time boundary element methods for the one-dimensional heat equation

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1 Introduction

Space–time discretization methods, see, e.g., [8], became very popular in recent years, due to their ability to drive adaptivity in space and time simultaneously, and to use parallel iterative solution strategies for time–dependent problems. But the solution of the global linear system requires the use of some efficient preconditioner.

In this note we describe a space–time boundary element discretization of the heat equation and an efficient and robust preconditioning strategy which is based on the use of boundary integral operators of opposite orders, but which requires a suitable stability condition for the boundary element spaces used for the discretization. We demonstrate the method for the simple spatially one-dimensional case. However, the presented results, particularly the stability analysis of the boundary element spaces, can be used to extend the method to the two- and three-dimensional problem [2].

Let \( \Omega = (a, b) \subset \mathbb{R} \), \( \Gamma := \partial \Omega = \{a, b\} \) and \( T > 0 \). As a model problem we consider the Dirichlet boundary value problem for the heat equation,

\[
\alpha \partial_t u - \Delta x u = 0 \text{ in } Q := \Omega \times (0, T), \quad u = g \text{ on } \Sigma := \Gamma \times (0, T), \quad u = u_0 \text{ in } \Omega
\]

with the heat capacity constant \( \alpha > 0 \), the given initial datum \( u_0 \), and the boundary datum \( g \). The solution of (1) can be expressed by using the representation formula for the heat equation (1), i.e. for \( (x,t) \in Q \) we have

\[
u(x,t) = \int_\Omega U^*(x-y,t)u_0(y)dy + \frac{1}{\alpha} \int_\Sigma U^*(x-y,t-s) \frac{\partial}{\partial n_y} u(y,s)ds, \]

(2)

\[
- \frac{1}{\alpha} \int_\Sigma \frac{\partial}{\partial n_y} U^*(x-y,t-s)g(y,s)ds.
\]
where \( U^* \) denotes the fundamental solution of the heat equation given by

\[
U^*(x-y,t-s) = \begin{cases} \\
\left( \frac{\alpha}{4\pi(t-s)} \right)^{1/2} \exp \left( -\frac{\alpha|y|^2}{4(t-s)} \right), & s < t, \\
0, & \text{else}
\end{cases}
\]

Hence it suffices to determine the yet unknown Cauchy datum \( \partial_n u_\Sigma \) to compute the solution of (1). It is well known [5] that for \( u_0 \in L^2(\Omega) \) and \( g \in H^{1/2,1/4}(\Sigma) \) the problem (1) has a unique solution \( u \in H^{1/2}(Q, \alpha \partial_t - \Delta_x) \) with the anisotropic Sobolev space

\[
H^{1/2}(Q, \alpha \partial_t - \Delta_x) := \left\{ u \in H^{1/2}(Q) : (\alpha \partial_t - \Delta_x)u \in L^2(Q) \right\}.
\]

In the one-dimensional case the spatial component of the space–time boundary \( \Sigma \) collapses to the points \( \{a, b\} \) and therefore we can identify the anisotropic Sobolev spaces \( H^{s,\beta}(\Sigma) \) with \( H^s(\Sigma) \). The unknown density \( w := \partial_n u_\Sigma \in H^{-1/4}(\Sigma) \) can be found by applying the interior Dirichlet trace operator \( \gamma_0^{\text{int}} : H^{1/2}(Q) \to H^{1/4}(\Sigma) \) to the representation formula (2).

\[
g(x,t) = (M_0u_0)(x,t) + (Vw)(x,t) + \left( \frac{1}{2} I - K \right)g(x,t) \quad \text{for (x,t) \in \Sigma}.
\]

The initial potential \( M_0 : L^2(\Omega) \to H^{1/4}(\Sigma) \), the single layer boundary integral operator \( V : H^{-1/4}(\Sigma) \to H^{1/4}(\Sigma) \), and the double layer boundary integral operator \( \frac{1}{2} I - K : H^{1/4}(\Sigma) \to H^{1/4}(\Sigma) \) are obtained by composition of the potentials in (2) with the Dirichlet trace operator \( \gamma_0^{\text{int}} \), see, e.g., [1, 5]. In fact, we have to solve the variational formulation to find \( w \in H^{-1/4}(\Sigma) \) such that

\[
\langle Vw, \tau \rangle_\Sigma = \langle \left( \frac{1}{2} I + K \right)g, \tau \rangle_\Sigma - \langle M_0u_0, \tau \rangle_\Sigma \quad \text{for all } \tau \in H^{-1/4}(\Sigma), \tag{3}
\]

where \( \langle \cdot, \cdot \rangle_\Sigma \) denotes the duality pairing on \( H^{1/4}(\Sigma) \times H^{-1/4}(\Sigma) \). The single layer boundary integral operator \( V \) is bounded and elliptic, i.e. there exists a constant \( c_1^V > 0 \) such that

\[
\langle Vw, w \rangle_\Sigma \geq c_1^V \|w\|_{H^{-1/4}(\Sigma)}^2 \quad \text{for all } w \in H^{-1/4}(\Sigma).
\]

Thus, the variational formulation (3) is uniquely solvable. When applying the Neumann trace operator \( \gamma_0^{\text{int}} : H^{1/2}(Q, \alpha \partial_t - \Delta_x) \to H^{-1/4}(\Sigma) \) to the representation formula (2) we obtain the second boundary integral equation

\[
w(x,t) = (M_1u_0)(x,t) + \left( \frac{1}{2} I + K' \right)w(x,t) + (Dg)(x,t) \quad \text{for (x,t) \in \Sigma}
\]
We consider an arbitrary decomposition of the space–time boundary \( \Sigma \). Note that there is no time-stepping scheme involved.

with the hypersingular boundary integral operator \( D : H^{1/4}(\Sigma) \rightarrow H^{-1/4}(\Sigma) \), and with the adjoint double layer boundary integral operator \( K' : H^{-1/4}(\Sigma) \rightarrow H^{1/4}(\Sigma) \).

Moreover, \( M_1 : L^2(\Omega) \rightarrow H^{-1/4}(\Sigma) \).

### 2 Boundary element methods

For the Galerkin boundary element discretization of the variational formulation \([3]\) we consider a family \( \{\Sigma_N\}_{N \in \mathbb{N}} \) of arbitrary decompositions of the space–time boundary \( \Sigma \) into boundary elements \( \sigma_\ell \), i.e. we have

\[
\Sigma_N = \bigcup_{\ell = 1}^N \sigma_\ell.
\]

In the one-dimensional case the boundary elements \( \sigma_\ell \) are line segments in temporal direction with fixed spatial coordinate \( x_\ell \in \{a, b\} \) as shown in Fig. 1. Let \((x_\ell, t_1)\) and \((x_\ell, t_2)\) be the nodes of the boundary element \( \sigma_\ell \). The local mesh size is then given as \( h_\ell := |t_2 - t_1| \) while \( h := \max_{\ell = 1, \ldots, N} h_\ell \) is the global mesh size.

For the approximation of the unknown Cauchy datum \( w = \gamma_{\text{int}} u \in H^{-1/4}(\Sigma) \) we consider the space \( S_0^h(\Sigma) := \text{span} \{ \phi^0_\ell \}_{\ell = 1}^N \) of piecewise constant basis functions \( \phi^0_\ell \), which is defined with respect to the decomposition \( \Sigma_N \). The Galerkin-Bubnov variational formulation of \([3]\) is to find \( w_h \in S_0^h(\Sigma) \) such that

\[
(V w_h, \tau_h)_\Sigma = \langle (1/2 I + K)g, \tau_h \rangle_\Sigma - \langle M_0 u_0, \tau_h \rangle_\Sigma \quad \text{for all } \tau_h \in S_0^h(\Sigma).
\]

This is equivalent to the system of linear equations \( V_h w = f \) where

\[
V_h[\ell, k] = (V \phi^0_\ell, \phi^0_k)_\Sigma, \quad f[\ell] = \langle (1/2 I + K)g, \phi^0_\ell \rangle_\Sigma - \langle M_0 u_0, \phi^0_\ell \rangle_\Sigma, \quad k, \ell = 1, \ldots, N.
\]

Due to the ellipticity of the single layer operator \( V \) the matrix \( V_h \) is positive definite and therefore the variational formulation \([4]\) is uniquely solvable as well. Moreover,
when assuming \( w \in H^t(\Sigma) \) for some \( t \in [0, 1] \), there holds the error estimate

\[
\| w - w_h \|_{H^{-1/4}(\Sigma)} \leq ch^{1/4 + t} |w|_{H^t(\Sigma)}.
\]

Using standard arguments we also conclude the error estimate

\[
\| w - w_h \|_{L^2(\Sigma)} \leq ch^t |w|_{H^t(\Sigma)}
\]

which implies linear convergence of the \( L^2(\Sigma) \)–error of the Galerkin approximation \( w_h \) if \( w \in H^1(\Sigma) \) is satisfied.

### 3 Preconditioning strategies

Since the boundary element discretization is done with respect to the whole space–time boundary \( \Sigma \) we need to have an efficient iterative solution technique. In fact, the linear system \( V_h w = f \) with the positive definite but nonsymmetric matrix \( V_h \) can be solved by using a preconditioned GMRES method. Here we will apply a preconditioning technique based on boundary integral operators of opposite order \([10]\), also known as operator or Calderon preconditioning \([3]\). Since the single layer integral operator \( V : H^{-1/4}(\Sigma) \rightarrow H^{1/4}(\Sigma) \) and the hypersingular integral operator \( D : H^{1/4}(\Sigma) \rightarrow H^{-1/4}(\Sigma) \) are both elliptic, the operator \( DV : H^{-1/4}(\Sigma) \rightarrow H^{1/4}(\Sigma) \) behaves like the identity. Hence we can use the Galerkin discretization of \( D \) as a preconditioner for \( V_h \). But for the Galerkin discretization \( D_h \) of the hypersingular integral operator \( D : H^{1/4}(\Sigma) \rightarrow H^{-1/4}(\Sigma) \) we need to use a conforming ansatz space \( Y_h = \text{span} \{ \psi_i \}_{i=1}^w \subset H^{1/4}(\Sigma) \) while the discretization of the single layer integral operator \( V \) is done with respect to \( S^0_h(\Sigma) \). Since the boundary element space \( S^0_h(\Sigma) \) of piecewise constant basis functions \( \phi_k^0 \) also satisfies \( S^0_h(\Sigma) \subset H^{1/4}(\Sigma) \) we can choose \( Y_h = S^0_h(\Sigma) \). The inverse hypersingular operator \( D^{-1} \) is spectrally equivalent to the single layer operator \( V \), therefore the approximation of the preconditioning operator corresponds to a mixed approximation scheme, and hence we need to assume a discrete stability condition to be satisfied.

**Theorem 1** \([3][10]\). Assume the discrete stability condition

\[
\sup_{0 \neq \psi \in Y_h} \frac{\langle \tau_h, \psi \rangle_{L^2(\Sigma)}}{\| \psi \|_{H^{1/4}(\Sigma)}} \geq c_4 \| \tau_h \|_{H^{-1/4}(\Sigma)} \quad \text{for all } \tau_h \in S^0_h(\Sigma).
\]

Then there exists a constant \( c_k > 1 \) such that

\[
\kappa \left( M^{-1}_h D_h M^{-1}_h \right) \leq c_k
\]

where, for \( k, \ell = 1, \ldots, N, \)

\[
V_h[\ell, k] = \langle V \phi_k^0, \phi_\ell^0 \rangle_{\Sigma}, \quad D_h[\ell, k] = \langle D \psi_\ell, \psi_k \rangle_{\Sigma}, \quad M_h[\ell, k] = \langle \phi_k^0, \psi_\ell \rangle_{L^2(\Sigma)}.
\]
Thus we can use $C_V^{-1} = M_h^{-1}D_hM_h^{-T}$ as a preconditioner for $V_h$. Since $M_h$ is sparse and spectrally equivalent to a diagonal matrix, the inverse $M_h^{-1}$ can be computed efficiently. It remains to define, for given $S_h^0(\Sigma)$, a suitable boundary element space $Y_h$ such that the stability condition (5) is satisfied. In what follows we will discuss a possible choice.

If we choose $Y_h = S_h^0(\Sigma)$ for the discretization of the hypersingular operator $D$, then $M_h$ becomes diagonal and is therefore easily invertible. In order to prove the stability condition (5) we need to establish the $H^{1/4}(\Sigma)$–stability of the $L^2(\Sigma)$–projection $Q_h^0 : L^2(\Sigma) \to S_h^0(\Sigma) \subset L^2(\Sigma)$ which is defined as

$$\langle Q_h^0 v, \tau_h \rangle_{L^2(\Sigma)} = \langle v, \tau_h \rangle_{L^2(\Sigma)} \quad \text{for all } \tau_h \in S_h^0(\Sigma).$$

Following [7], and when assuming local quasi-uniformity of the boundary element mesh $\Sigma_N$ we are able to establish the stability of $Q_h^0 : H^{1/4}(\Sigma) \to H^{1/4}(\Sigma)$, see [2] for a more detailed discussion: For $\ell = 1, \ldots, N$ we define $I(\ell)$ to be the index set of the boundary element $\Sigma_\ell$ and all its adjacent elements. We assume the boundary element mesh $\Sigma_N$ to be locally quasi-uniform, i.e. there exists a constant $c_L \geq 1$ such that

$$\frac{1}{c_L} \leq \frac{h_\ell}{h_k} \leq c_L \quad \text{for all } k \in I(\ell) \text{ and } \ell = 1, \ldots, N.$$ 

In this case the operator $Q_h^0 : H^{1/4}(\Sigma) \to H^{1/4}(\Sigma)$ is bounded, i.e. there exists a constant $c_S^0 > 0$ such that

$$\|Q_h^0 v\|_{H^{1/4}(\Sigma)} \leq c_S^0 \|v\|_{H^{1/4}(\Sigma)} \quad \text{for all } v \in H^{1/4}(\Sigma). \ (6)$$

By using the stability estimate (6) we can conclude

$$\frac{1}{c_S^0} \|\tau_h\|_{H^{-1/4}(\Sigma)} \leq \sup_{0 \neq v_h \in S_h^0(\Sigma)} \frac{\langle \tau_h, v_h \rangle_{L^2(\Sigma)}}{\|v_h\|_{H^{1/4}(\Sigma)}} \quad \text{for all } \tau_h \in S_h^0(\Sigma).$$

Hence the stability condition (5) holds and we can use $C_V^{-1} = M_h^{-1}D_hM_h^{-T}$ as a preconditioner for $V_h$.

### 4 Numerical results

For the numerical experiments we choose $\Omega = (0, 1)$, $T = 1$, and we consider the model problem (1) with homogeneous Dirichlet conditions $g = 0$, and some given initial datum $u_0$ satisfying the compatibility conditions $u_0(0) = u_0(1) = 0$. The Galerkin boundary element discretization of the variational formulation (3) is done by piecewise constant basis functions. The resulting system of linear equations $V_h w = f$ is solved by using the GMRES method. As a preconditioner we use the discretization $C_V^{-1} = M_h^{-1}D_hM_h^{-T}$ of the hypersingular operator $D$ with piecewise constant basis functions.
Uniform refinement

The first example corresponds to the initial datum $u_0(x) = \sin 2\pi x$ and a globally uniform boundary element mesh of mesh size $h = 2^{-L}$. Table 1 shows the $L^2(\Sigma)$-error $\|w - w_h\|_{L^2(\Sigma)}$ and the estimated order of convergence (eoc), which is linear as expected. Moreover, the condition numbers of the stiffness matrix $V_h$ and of the preconditioned matrix $C^{-1}V_h$ as well as the number of iterations to reach a relative accuracy of $10^{-8}$ are given which confirm the theoretical estimates.

Table 1 Error, condition and iteration numbers in the case of uniform refinement

| L  | N  | $\|w - w_h\|_{L^2(\Sigma)}$ | eoc | $\kappa(V_h)$ | It. | $\kappa(C^{-1}V_h)$ | It. |
|----|----|-----------------------------|-----|----------------|-----|---------------------|-----|
| 0  | 2  | 2.249                       | -   | 1.001          | 1   | 1.002               | 1   |
| 1  | 4  | 1.311                       | 0.778 | 2.808     | 2   | 1.279               | 2   |
| 2  | 8  | 0.658                       | 0.996 | 4.905     | 4   | 1.422               | 4   |
| 3  | 16 | 0.324                       | 1.021 | 7.548     | 8   | 1.486               | 8   |
| 4  | 32 | 0.160                       | 1.017 | 11.140   | 16  | 1.541               | 14  |
| 5  | 64 | 0.079                       | 1.010 | 16.724   | 31  | 1.563               | 13  |
| 6  | 128| 0.040                       | 1.006 | 13.470   | 41  | 1.590               | 13  |
| 7  | 256| 0.020                       | 1.003 | 22.053   | 50  | 1.615               | 12  |
| 8  | 512| 0.010                       | 1.001 | 32.043   | 59  | 1.636               | 12  |
| 9  | 1024| 0.005                      | 1.001 | 60.957   | 70  | 1.777               | 11  |
| 10 | 2048| 0.002                      | 1.000 | 88.488   | 82  | 1.762               | 11  |
| 11 | 4096| 0.001                      | 1.000 | 128.957 | 96  | 1.765               | 10  |

Adaptive refinement

For the second example we consider the initial datum $u_0(x) = 5e^{-10x} \sin \pi x$ which motivates the use of a locally quasi-uniform boundary element mesh resulting from some adaptive refinement strategy. The numerical results as given in Table 2 again confirm the theoretical findings, in particular the robustness of the proposed preconditioning strategy in the case of an adaptive refinement which is not the case when using none or only diagonal preconditioning $C_V = \text{diag}V_h$.

5 Conclusions and outlook

In this note we have described a space–time boundary element discretization of the spatially one-dimensional heat equation and an efficient and robust preconditioning strategy which is based on the use of boundary integral operators of opposite orders, but which requires a suitable stability condition for the boundary element spaces used for the discretization. In the particular case of the spatially one-dimensional
Table 2 Error, condition and iteration numbers in the case of adaptive refinement

| L  | N  | \( \|w - w_h\|_{L_2(\Sigma)} \) | \( \kappa(V_h) \) | It. | \( \kappa(C^{-1}V_h) \) It. | \( \kappa(C^{-1}V_h) \) It. |
|----|----|------------------------------|----------------|-----|---------------------|---------------------|
| 0  | 2  | 1.886                        | 1.00           | 2   | 1.001               | 2                   |
| 1  | 3  | 1.637                        | 3.97           | 3   | 2.553               | 3                   |
| 2  | 5  | 1.272                        | 12.23          | 5   | 4.055               | 4                   |
| 3  | 7  | 0.914                        | 34.21          | 7   | 3.611               | 6                   |
| 4  | 9  | 0.615                        | 92.08          | 9   | 3.164               | 8                   |
| 5  | 11 | 0.401                        | 118.59         | 11  | 2.945               | 10                  |
| 6  | 13 | 0.267                        | 338.26         | 13  | 2.803               | 12                  |
| 7  | 20 | 0.166                        | 621.77         | 20  | 3.524               | 18                  |
| 8  | 31 | 0.101                        | 1608.08        | 31  | 4.457               | 27                  |
| 9  | 47 | 0.063                        | 2344.90        | 47  | 5.779               | 32.54               |
| 10 | 74 | 0.039                        | 6141.47        | 74  | 8.348               | 37                  |
| 11 | 114| 0.024                       | 8409.92        | 114 | 10.950              | 42                  |
| 12 | 177| 0.015                       | 23007.60       | 173 | 14.324              | 47                  |
| 13 | 278| 0.010                       | 27528.30       | 200 | 21.094              | 53                  |

When using the discontinuous boundary element space \( S^0_h(\Sigma) \) for the approximation of the unknown flux we need to choose an appropriate boundary element space \( Y_h \) to ensure the stability condition (5). A possible approach is the use of a dual mesh using piecewise constant basis functions for the approximation of \( V \), and piecewise linear and continuous basis functions for the approximation of \( D \), see Fig. 2 for the situation in 1D. For a more detailed analysis of the proposed preconditioning strategy and suitable choices of stable boundary element spaces we refer to [2].

An efficient solution of local Dirichlet boundary value problems is an important tool when considering domain decomposition methods for the heat equation, see e.g. [9] in the case of the Laplace equation. Moreover, the preconditioning strategy of using operators of opposite order can also be used when considering related Schur complement systems on the skeleton, as they also appear in tearing and interconnecting domain decomposition methods, see, e.g., [4]. This also covers the coupling...
Fig. 2. Sample dual mesh. The piecewise linear and continuous functions $\phi_i^1$ are used for the discretization of $D$. The piecewise constant basis functions $\tilde{\phi}_i^0$ are used for the discretization of $V$ of space–time finite and boundary element methods. Related results on the stability and error analysis as well as on efficient solution strategies for space–time domain decomposition methods will be published elsewhere.

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