Boosted perturbations at the end of inflation

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We study the effect on the primordial cosmological perturbations of a sharp transition from inflationary to a radiation and matter dominated epoch respectively. We assume that the perturbations are generated by the vacuum fluctuations of a scalar field slowly rolling down its potential, and that the transition into the subsequent epoch takes place much faster than a Hubble time. The behaviour of the superhorizon perturbations corresponding to cosmological scales in this case is well known. However, it is not clear how perturbations on scales of and smaller than the Hubble horizon scale at the end of inflation may evolve through such a transition. We derive the evolution equation for the gravitational potential $\Psi$, which allows us to study the evolution of the perturbations on all scales under these circumstances. We show that for a certain range of scales inside the horizon at the end of inflation, the amplitude of the perturbations are enhanced relative to the superhorizon scales. This enhancement may lead to the overproduction of Primordial Black Holes (PBHs), and therefore constrain the dynamics of the transitions that take place at the end of inflation.

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I. INTRODUCTION

Inflation has become the main paradigm to understand the presence of small inhomogeneities during the early universe. The observed Cosmic Microwave Background (CMB) and Large Scale Structure (LSS) spectrum of primordial perturbations, reveals that simple inflationary models are consistent with the analysis of the current data. Nevertheless, CMB and LSS observations only probe primordial perturbations on scales that leave the horizon long before the end of inflation, and possible cosmological implications of the perturbations on subhorizon scales have not been studied much.

In this paper, we investigate the behavior of curvature perturbations on scales of and smaller than the horizon size at the end of inflation. We assume standard slow-roll inflation and that the perturbations are generated from the vacuum fluctuations of the inflaton field. It is expected that the universe enters either a stage of radiation-domination or matter-domination right after inflation, depending on the efficiency of preheating or reheating, and the transition takes place within a timescale shorter than the Hubble time. We therefore model this transition as a change of the equation of state from $w \approx -1$ to $w = 1/3$ or to $w = 0$ within a time scale $\Delta t \ll H_e^{-1}$ where $H_e$ is the Hubble parameter at the end of inflation.

On large scales where the wavelengths are much larger than $\Delta t$, we may regard the transition as instantaneous, occurring on a given spacelike hypersurface. Then we can match the perturbations between inflation and the succeeding epoch by requiring that the intrinsic and extrinsic curvatures are continuous on the hypersurface at which we do the matching. It is natural to assume that the end of inflation occurs when the inflaton field reaches a critical value, $\phi_c(x, \tau) = \text{constant}$. This implies that the transition occurs on a comoving surface on which $\phi$ is spatially homogeneous. Then one can show that we have the following junction conditions,

$$[R_k]^+_{\tau} = 0, \quad \Psi_k^+ = 0,$$

where $R$ is the curvature perturbation on comoving hypersurfaces, and $\Psi$ is the generalized gravitational potential. The junction conditions give the initial conditions for the following radiation- or matter-dominated stage. On superhorizon scales, the potential $\Psi$ then reaches a constant value, $\Psi \sim R$, a few Hubble times after the end of inflation.

What we are interested in here is perturbations on scales of or smaller than the horizon scale. On subhorizons scales, but still larger than the scale $\Delta t$, the resulting perturbations are the same as in the case of superhorizon perturbations if the universe is matter-dominated after the transition. On the other hand, the result is not so simple if the universe is radiation-dominated after the transition. In Refs. [4, 5], we investigated this case, and argued there that the relative large amplitude of the potential $\Psi$ can be achieved on subhorizon scales and may lead to the overproduction of PBHs at the end of inflation. The constraint on the amplitude of the curvature perturbation is then tighter than the current bound from the PBHs formed throughout the radiation epoch (for the constraints on the "standard" formation of PBHs see [6–16]).
In this paper we investigate the effect of the transition on curvature perturbations on all subhorizon scales, including those on sufficiently small scales on which the transition cannot be regarded instantaneous any more. The paper is organized as follows. In Sec. (II) we derive the governing equation for the evolution of the potential $\Psi$ for an arbitrary equation of state and velocity of sound. We review the generation of the perturbations during slow-roll inflation in Sec. (III). We study the evolution of $\Psi$ through a transition into radiation and matter domination in Secs. (IV) and (V) before concluding. In Appendices A and B we present analytical approximations that we have employed to reproduce and understand certain features that we have found in our numerical estimations.

Throughout the paper, we use the Planck units $8\pi G = M_P^{-2} = 1$.

II. GAUGE INVARIANT COSMOLOGICAL PERTURBATIONS

In this section, we introduce the governing equations for the scalar perturbations about the homogeneous and isotropic background that we will use throughout this paper.

The background space-time geometry for a spatially flat universe is given by the FLRW metric,
\[ ds^2 = a^2(\tau) \left( -d\tau^2 + dx^2 + dy^2 + dz^2 \right) , \]  
where $\tau$ is the conformal time defined in terms of the cosmic time $t$ as $d\tau = dt/a$, and $a(\tau)$ is the scale factor. Relative to the above coordinates the energy-momentum tensor of a perfect fluid at rest is
\[ T^0_0 = -\rho , \]  
\[ T^0_i = P \delta^i_0 , \]  
where $\rho$ and $P$ are the energy density and pressure of the fluid.

The governing equations for the system are determined by the Einstein Field equations and the energy momentum conservation. That is
\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = T^{\mu\nu} , \]  
\[ T^{\mu\nu} ; \nu = 0 . \]  
In the present case they give the following set of equations:
\[ 3H^2 = \rho a^2 , \]  
\[ \rho' + 3H (\rho + P) = 0 , \]  
where a prime (') denotes the differentiation with respect to $\tau$, and the conformal Hubble parameter $H$ is defined by $H = a'/a$, hence is related to the physical Hubble parameter $H$ by $H = aH$. A combination of the above two equations gives a second order differential equation for $a(\tau)$,
\[ H' = -\frac{1}{6} (\rho + 3P) a^2 . \]  

Ignoring the vector and tensor perturbations, the perturbed metric can be written to first order as
\[ ds^2 = a^2(\tau) \left[ - (1 + 2A) d\tau^2 - 2B_i d\tau dx^i + [(1 + 2D) \delta_{ij} + 2E_{ij}] dx^i dx^j \right] , \]  
where a comma denotes the partial differentiation with respect to the spatial coordinates, and $A$, $B$, $D$, and $E$ are the functions that describe the scalar perturbations. The scalar part of the perturbations in the components of the energy momentum tensor for a perfect fluid are
\[ \delta T^0_0 = -\delta \rho , \]  
\[ \delta T^0_i = (\rho + P) (B_i + v_i) , \]  
\[ \delta T^i_j = \delta P \delta^i_j , \]  
where we have decomposed the fluid 3-velocity $v^i$ into a divergence free vector and the gradient of a scalar, $v^i_x$, to extract the scalar part of the perturbations as usual.
It is convenient to express the governing equations in terms of quantities which are invariant under infinitesimal coordinate transformations. Bardeen [17] gives the following two gauge invariant quantities for the metric,

$$\Phi = A + \frac{1}{a} [a (B - E')]' ,$$

$$\Psi = D + H (B - E') ,$$

and for the matter variables,

$$\delta C = \delta + 3 \left( \frac{\rho + P}{\rho} \right) H (v - B) ,$$

$$\bar{v} = v - E' .$$

The functions \( \Psi \) and \( \Phi \) represent the lapse function perturbation and the curvature perturbation in the longitudinal gauge respectively. For the matter variables \( \delta C \) is the density contrast \( \delta \rho / \rho \) on the comoving slice and \( \bar{v} \) is the velocity of matter on the longitudinal gauge relative to the \( x^i = \text{constant} \) observers [18].

For the matter perturbations of scalar type, there is a third gauge invariant quantity often called the entropy perturbation, which can be constructed in terms of the matter variables only:

$$\delta P = \delta P - \frac{P'}{\rho} \delta \rho .$$

The entropy perturbation measures the difference between uniform density and pressure hypersurfaces. For an equation of state in which the pressure is a function of the energy density alone (the so-called barotropic equation of state), the uniform density and pressure hypersurfaces coincide and the pressure perturbation is called adiabatic. For a more general equation of state this is no longer the case and we get a non-zero entropy perturbation. However, there is one important exception in which the perturbation is purely adiabatic although the equation of state is not barotropic. That is the case of a single real scalar field. In this case, \( P = K - V(\phi) \) and \( \rho = K + V(\phi) \), where \( K = \dot{\phi}^2 / 2 \).

Then, it is known (or easily seen from the equations below) that one obtains a closed single second order differential equation for the scalar perturbation if one defines the sound velocity as the ratio between \( \delta P \) and \( \delta \rho \) on the comoving hypersurfaces (denoted by the suffix \( C \))

$$\delta P_C = c_s^2 \delta \rho_C .$$

In the present case, \( c_s = 1 \) since \( \delta P_C = \delta \rho_C = \delta K \). Note that, since \( \delta p = c_s^2 \delta \rho \) in any gauge for a barotropic fluid, the definition of the sound velocity \( c_s \) by Eq. (20) is valid both for a scalar field and for a barotropic fluid.

Now we can express the perturbed equations for Eqs. (6) and (7) in terms of the gauge invariant variables above as

$$\nabla^2 \Psi = - \frac{\rho a^2}{2} \delta C ,$$

$$\Phi + \Psi = 0 ,$$

$$\left( \Psi a \right)' = - \frac{1}{2} (\rho + P) a^2 (av) ,$$

$$\frac{1}{a} (av)' = - \Psi + \frac{1}{(\rho + P)} \delta P_C .$$

Then setting \( \delta P_C = c_s^2 \delta \rho_C \) in the last equation, we can reduce the equations above to the following second order differential equation for \( \Psi \),

$$a^2 (\rho + P) \left[ \frac{f_k'}{a^2 (\rho + P)} \right]' + \left[ k^2 c_s^2 - \frac{1}{2} (\rho + P) a^2 \right] f_k = 0 ,$$

where \( f_k = a \Psi_k \). The equation above determines the evolution of the potential \( \Psi \) for a given equation of state and velocity of sound \( c_s(\tau) \). In the following sections we use Eq. (25) to calculate the evolution of the perturbations generated during slow-roll inflation through a rapid transition into radiation domination and matter domination epochs.
III. THE GENERATION OF PERTURBATIONS DURING INFLATION

We assume the universe is dominated by a minimally coupled real scalar field $\phi$ during inflation. As mentioned in the previous section, for this universe, the energy density and pressure of the homogeneous scalar field $\phi \equiv \phi(\tau)$ are given by

$$\rho = K + V(\phi) , \quad P = K - V(\phi) ; \quad K \equiv \frac{1}{2} \dot{\phi}^2 = \frac{1}{2a^2} \phi^2. \quad (26)$$

The density and pressure perturbations are then related by

$$\delta P = \delta \rho - 2V,\phi \delta \phi , \quad (27)$$

which leads to $\delta P_C = \delta \rho_C$ on comoving hypersurfaces and hence $c_s = 1$. During this stage, the evolution equation for $f_k$, Eq. (25), may be written as

$$f''_k - 2 \phi'' \phi' f'_k + [k^2 + H' - H^2] f_k = 0. \quad (28)$$

Introducing a new variable $u$ given by

$$u_k = f_k \exp \left(-\int^\tau \frac{\phi''}{\phi'} d\tau \right) = 2k f_k \phi'^{-1} = \frac{2k}{\phi} \Psi_k , \quad (29)$$

the differential equation for $f$ can be expressed in the form,

$$u''_k + W^2(k, \tau) u_k = 0, \quad (30)$$

where $W^2(k, \tau)$ is given by

$$W^2(k, \tau) = k^2 c_s^2 + m^2_{\text{eff}} a^2 ; \quad m^2_{\text{eff}} a^2 = H' - H^2 - \left(\frac{\phi''}{\phi'}\right)^2 + \left(\frac{\phi'''}{\phi'}\right)' . \quad (31)$$

The coefficient $2k$ in the definition of $u_k$ is chosen so that the function $a q_k$ becomes a properly normalized mode function for an effective scalar field when quantized [22]. To be more specific, the field $Q \equiv a q$ becomes a conformally coupled scalar field with its effective mass-squared $m^2_{\text{eff}}$ defined in the above equation.

We can find approximate analytical solutions for $u_k$ as follows. We first introduce the following set of slow-roll parameters

$$\epsilon_H = \frac{\dot{\phi}^2}{2H^2} = -\frac{a}{H^2} \left(\frac{H}{a}\right)' , \quad (32)$$

$$\eta_H = -\frac{\dot{\phi}}{H \phi} = -\frac{a}{H \phi'} \left(\frac{\phi'}{a}\right)' . \quad (33)$$

Then we rewrite the function $W^2(k, \tau)$ as

$$W^2(k, \tau) = k^2 c_s^2 + m^2_{\text{eff}} a^2 = H' - H^2 - \left(\frac{\phi''}{\phi'}\right)^2 + \left(\frac{\phi'''}{\phi'}\right)' . \quad (34)$$

It is well known that the differential equations for the evolution of cosmological perturbations have analytical solutions if $\epsilon_H$ and $\eta_H$ are constant [20, 21]. Typically the slow-roll parameters are small, at least for the range of field values at which perturbations on scales constrained by observation are generated. Their time derivatives then, which are given by

$$\frac{1}{H} \dot{\epsilon}_H = 2 \epsilon_H (\epsilon_H - \eta_H) , \quad (35)$$

$$\frac{1}{H} \dot{\eta}_H = \eta_H \left(\frac{\dot{\phi}}{H \phi} + \epsilon_H + \eta_H \right) , \quad (36)$$

and
are small, and $\epsilon_H$ and $\eta_H$ can be approximately taken as constant, as least for a certain number of $e$-folds. In that case the conformal time is given by

$$\tau = \int \frac{da}{a^2 H} = - \frac{1}{aH} + \int \frac{\epsilon_H da}{a^2 H} = - \frac{1}{aH} \frac{1}{1 - \epsilon_H}$$

(37)

where for the time being we have set the integrating constant to zero. Writing down explicitly the time dependence in Eq. (30), it becomes a differential Bessel equation,

$$u_k'' + \left[ k^2 - \frac{(\nu^2 - 1/4)}{\tau^2} \right] u_k = 0,$$

(38)

where the order $\nu$ is approximately given by

$$\nu \simeq 1/2 + 2\epsilon_H - \eta_H.$$  

(39)

In the limit $k \gg aH$, we assume that the fluctuations of the field are in the Minkowski vacuum,

$$u_k \to \frac{1}{(2k)^{1/2}} e^{-ik\tau}.$$  

(40)

Then the solution for $u_k$ can be expressed in terms of the Hankel function of the first kind $H^{(1)}_\nu(x)$, and apart from an irrelevant constant phase factor, $\Psi_k$ can be written as

$$\Psi_k = \frac{\sqrt{\pi}}{2k^{3/2}} \frac{H^2}{\dot{\phi}} \epsilon_H (-k\tau)^{1/2} H^{(1)}_\nu(-k\tau).$$

(41)

On scales well outside the horizon, $k \ll aH$, the amplitude of $\Psi$ takes the following asymptotic limit,

$$\Psi_k(\tau) \simeq 2^{\nu-1} \Gamma(\nu) \frac{\Gamma(1/2)}{\sqrt{2\pi}} \epsilon_H (-k\tau)^{1/2-\nu} \left( \frac{H^2}{\phi} \right)_{k=aH}. $$

(42)

Evaluating the expression above at a given time, for example at the end of inflation, it can be seen that the scale dependence on superhorizon scales is given by the deviation of $\nu$ from the value $1/2$. Nevertheless, the scale dependence then is commonly given by the small variation of the amplitude, $(H^2/\dot{\phi})|_{k=aH}$, when perturbations on different scales leave the horizon during inflation. In this case one can write

$$\Psi_k(\tau) \simeq 2^{\nu-1} \frac{\Gamma(\nu)}{\sqrt{2\pi}} \frac{\Gamma(1/2)}{\epsilon_H (1 - \epsilon_H)^{1/2-\nu}} \left( \frac{H^2}{\phi} \right)_{k=aH}. $$

(43)

For scales that remain inside the horizon during the inflationary epoch, $\Psi$ does not display significant scale dependence even if the slow-roll parameters are relatively large. For $k\tau > 1$, $\Psi$ converges very quickly to

$$\Psi_k(\tau) \simeq \frac{1}{\sqrt{2kH}} \frac{H^2}{\phi} e^{-ik\tau},$$

(44)

and only perturbations on scales for which $k\tau \simeq 1$, may vary respect to the scale invariant case.

A special case, in which the slow-roll parameters defined above are constant and we get an exact analytical solution for $\Psi_k$, is power law inflation. In power law inflation $a \propto t^q$, and the slow-roll parameters give

$$\epsilon_H = \eta_H = \frac{1}{q}.$$  

(45)

Then the solution for $\Psi$ is given by Eq. (44) with

$$\nu = \frac{1}{2} \frac{(q + 1)}{(q - 1)}.$$  

(46)
For large $q$, when the slow-roll conditions hold strongly, $\Psi_k$ on superhorizon scales becomes \footnote{The relation in Eq. (47) holds barring an exact de Sitter background, which corresponds to $\dot{\phi} = 0$. Note that we are considering the quantity $(H^2/\dot{\phi})$ small to be consistent with observation.}

$$\Psi_k (\tau) = \frac{1}{\sqrt{2k^3} q} \left( \frac{H^2}{\dot{\phi}} \right)_{k=aH} \approx 0.$$ \hspace{1cm} (47)

On cosmological scales, which leave the horizon long before the end of inflation, it is common to describe the small inhomogeneities observed in CMB temperature fluctuations by the curvature perturbation on comoving hypersurfaces, $R$. The curvature perturbation $R$ is related to the Bardeen potential $\Psi$ by

$$R = \frac{2}{3} \frac{H^{-1} \Psi' + \Psi}{(1+w)} - \Psi,$$ \hspace{1cm} (48)

where $w = P/\rho$ and $(1 + w) = 2\epsilon_H/3$ during inflation.

For power law inflation, the derivative of $\Psi$ can be expressed as

$$\mathcal{H}^{-1} \Psi_k'(\tau) = \frac{\sqrt{\pi}}{2k^{3/2}} \frac{H^2}{\dot{\phi}} \epsilon_H (-k\tau)^{1/2} \left[ (-k\tau) (1 - \epsilon_H) H^{(1)}_{\nu+1} (-k\tau) - (1 + \epsilon_H) H^{(1)}_{\nu} (-k\tau) \right].$$ \hspace{1cm} (49)

Then we find the following exact expression for $R$:

$$R_k (\tau) = -\frac{\sqrt{\pi}}{2k^{3/2}} \frac{H^2}{\dot{\phi}} (1 - \epsilon_H) (-k\tau)^{3/2} H^{(1)}_{\nu+1} (-k\tau).$$ \hspace{1cm} (50)

On superhorizon scales, taking the asymptotic limit of the Hankel function, $(-k\tau) \rightarrow 0$, the curvature perturbation can be expressed as

$$R_k = \frac{2^{\nu-1/2}}{\sqrt{2k^3}} \frac{\Gamma (\nu + 1)}{\Gamma (3/2)} \left( \nu + 1/2 \right)^{-\nu-1/2} \left( \frac{H^2}{\dot{\phi}} \right)_{k=aH},$$ \hspace{1cm} (51)

in agreement with \footnote{The relation in Eq. (47) holds barring an exact de Sitter background, which corresponds to $\dot{\phi} = 0$. Note that we are considering the quantity $(H^2/\dot{\phi})$ small to be consistent with observation.}. In the limit $q \rightarrow \infty$, we get

$$R_k = \frac{1}{\sqrt{2k^3}} \frac{H^2}{\dot{\phi}}_{k=aH},$$ \hspace{1cm} (52)

which is the standard result for a slowly varying potential \footnote{The relation in Eq. (47) holds barring an exact de Sitter background, which corresponds to $\dot{\phi} = 0$. Note that we are considering the quantity $(H^2/\dot{\phi})$ small to be consistent with observation.}. For $\epsilon_H$ not too small this result is modified by the factor in Eq. (51).

In view of the expressions derived above for $R$ and $\Psi$, it is clear that for small values of the slow-roll parameters the amplitude of $\Psi$ is much smaller than $R$ on superhorizon scales. On scales well inside the horizon, $k > aH$, the curvature perturbation $R$ increases with the comoving scale as $k/aH$, while $\Psi$ is scale independent so the amplitude of $\Psi$ is further suppressed relative to $R$. It is natural then to neglect $\Psi$ respect to $R$ during inflation if the slow-roll parameters are not too close to 1.

As we stressed above, Eqs. (41) and (50) are exact solutions for the perturbed quantities $R$ and $\Psi$ generated during power law inflation. They provide approximate solutions for slow-roll inflation if we consider a suitable power law for the scale factor of the homogeneous background during a sufficiently short lapse of time. The approximation will be valid then for certain number of $e$-folds, depending on the variation of the slow-roll parameters. Here we are interested in the behaviour of the perturbations at the end of inflation, and therefore the power law solutions at that time may differ considerably from those at the time of horizon crossing for scales which are well outside the horizon at the end of inflation. Nevertheless, since we focus on the scales smaller than the horizon, this problem does not affect our analysis.

In the following sections, we give approximate analytical expressions for $\Psi$ for a transition from inflation to radiation and matter domination alongside of numerical calculations. For the numerical estimation it is convenient to normalize the solutions of $\Psi$ with respect to the value of $R$ at horizon crossing at the end of inflation. Thus we define this value, $A_R$, as

$$A_R \equiv \sqrt{2k^3} R_{k=H_e} = r (q) \left( \frac{H^2}{\dot{\phi}} \right)_{k=H_e},$$ \hspace{1cm} (53)
where \( r(q) \) is just the factor in Eq. (51). That is
\[
  r(q) = 2^{\nu - 1/2} \frac{\Gamma(\nu + 1)}{\Gamma(3/2)} (\nu + 1/2)^{-\nu - 1/2}.
\]

IV. THE TRANSITION INTO RADIATION DOMINATION

In this section, we study the behavior of the potential \( \Psi \) during a sharp transition into radiation domination from a slow-roll inflationary epoch. The evolution of the potential \( \Psi \) from Eq. (25) is determined by
\[
  f_k'' - \left[ \frac{(\rho + P)'}{(\rho + P)} + 2\mathcal{H} \right] f_k' + \left[ k^2 c_s^2 (\tau) - \frac{1}{2} (\rho + P) a^2 \right] f_k = 0.
\]

It is necessary to do a constant time shift of the conformal time to match the scale factor between inflation and the radiation dominated stage. If the transition into radiation domination occurs much more rapidly than a Hubble time, then the scale factor at the end of inflation is approximately given by \( a_e \simeq 1/H_e \tau_e \), and we have to replace in the formulae for the perturbations during inflation \( \tau \) by \( \tau - \tau_R \), where \( \tau_R \) is the constant conformal time shift. Neglecting \( \epsilon_H \) in Eq. (37), we get \( \tau_R = 2 \tau_e \).

One can in principle solve Eq. (55) numerically for a given equation of state and velocity of sound. Here we are interested in the qualitative behaviour of \( \Psi \) for a sharp transition into radiation domination, so we assume that the equation of state changes abruptly to reach its radiation domination value in some small fraction of a Hubble time.

During inflation (1 + \( w \)) is approximately given by
\[
  \frac{1}{2} \frac{w''}{1 + w} - \mathcal{H} \frac{w'}{1 + w},
\]

neglecting all the terms proportional to \( \mathcal{H}^2 \). The terms proportional to the derivatives of \( w \) are sharply peaked functions, with an approximate amplitude of order \( \mathcal{H}/\delta^2 \) during the transition when the equation of state varies. In Fig. 1, we show the time dependence of \( V_{\text{eff}}(\tau) \) for an equation of state such that during the transition (1 + \( w \)) it behaves like
\[
  (1 + w) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{\mathcal{H}_e (\tau - \tau_e)}{\delta} \right) \right]
\]

where we have ignored the small value proportional to \( \epsilon_H \) it has during inflation. We plot \( V_{\text{eff}} \) for \( \delta = 0.01 \) and two different values of \( q \). The height and width of the peaks shown in the figure depend on \( \delta \) as well as \( \epsilon_H \), which is
evident from the expression of $V_{\text{eff}}$ in Eq. (60). As we shall see, the variation of $V_{\text{eff}}$ with $\epsilon_H$ is responsible for the peculiar scale dependence of the spectrum of $\Psi_k$ on subhorizon scales. On superhorizon scales, one gets the usual red tilt on the spectrum for power law inflation.

Once the equation of state reaches its radiation domination constant value, the derivatives of $w$ vanish and $W^2(k, \tau)$ becomes

$$W^2(k, \tau) = \left( k^2 c_s^2 - \frac{2}{\tau^2} \right),$$

(62)

which gives a Bessel differential equation of order $\nu = 3/2$ for $u$ when $c_s$ is constant. Note that the solutions for half integer values of $\nu$ are just spherical Bessel functions.

For scales $k \gg \mathcal{H}/\delta$ then, the gradient term in $W^2(k, \tau)$ dominates. Assuming that $c_s^2(\tau)$ drops down from 1 to 1/3 during the transition, an approximate estimate for $\Psi_k$ on these small scales, $k \gg a\mathcal{H}/\delta$, throughout the transition and radiation domination epoch, is given by the WKB solution,

$$\Psi_k(\tau) \approx \frac{1}{(2k)^{3/2}} \frac{(\rho + p)^{1/2}}{\sqrt{c_s(\tau)}} M_p^{-\frac{1}{2}} e^{-ikf c_s(\tau) d\tau},$$

(63)

where we have recovered the Plank mass, $M_p = (8\pi G)^{-1/2}$, so that the dimensionless nature of $\sqrt{2k^3}\Psi_k$ is explicit. Hereafter, we call this non-dimensional amplitude as the amplitude of $\Psi$ at a comoving scale $k$.

In the opposite regime, $k \ll \mathcal{H}/\delta$, the variation of the equation of state from the slow-roll regime is responsible of the growth of the perturbed quantity $\Psi$. We neglect the terms proportional to $f_k$ in Eq. (25), and therefore it becomes

$$f''_k - \frac{(\rho + p)'}{(\rho + p)} + 2Hf'_k = 0,$$

(64)

which gives the following conserved quantity,

$$\frac{f_k'}{a^2(\rho + p)} = \text{constant}.$$  

(65)

Integrating the equation above, we find that $\Psi$ does not change substantially during the transition if the transition is much faster than a Hubble time. In fact, the relative variation of the potential $\Psi$ during the transition is roughly

$$\left( \frac{\Delta \Psi}{\Psi} \right) \sim \frac{\delta}{\epsilon_H}.$$  

(66)

Therefore taking $\Psi$ to be constant during the transition on these scales appears to be a good approximation in the limit $\delta \lesssim \epsilon_H$.

Neglecting the small variation of $\Psi$ and the scale factor during the transition, Eq. (65) implies that the curvature perturbation on comoving hypersurfaces, $\mathcal{R}$, remains constant during the transition. Thus for a rapid transition from slow-roll into radiation domination, there is a certain range of scales inside horizon for which the transition may be taken as instantaneous. Using the junction conditions in Eqs. (1) and (2), we get the result found in [4],

$$\Psi_k(\tau) = 2\mathcal{R}_k(\tau_c) \left( \frac{(x - x_c) \cos(x - x_c) - (1 + xx_c) \sin(x - x_c)}{x^3} \right),$$

(67)

where $x = c_s k \tau$ and $x_c = c_s k \tau_c$, with the velocity of sound given by $c_s = 1/\sqrt{3}$. On superhorizon scales, $x_c \to 0$ and $x \ll 1$, $\Psi$ reaches the constant value,

$$\Psi_k \approx -\frac{2}{3} \mathcal{R}_k(\tau_c),$$

(68)

whereas the modes inside the horizon at the end of inflation undergo damped oscillations.

We can check the qualitative behaviour of the perturbed potential $\Psi$, outlined in the preceding paragraphs, by solving numerically Eq. (65) for a definite equation of state which behaves like a step function with a given width. Using an error function for the variation of $(\rho + p)$ during the transition, with a typical width $\Delta \tau \sim \delta/\mathcal{H}_c$, we have
found that the numerical solutions agree very well with our analytical expectations in both regimes. In Fig. 1, we show the evolution of $\Psi$ for two different values of $q$ in power law inflation with comoving wavenumber $k \ll H_e/\delta$. We take values of $\delta$ such that $\delta \gtrsim 1/q$. We plot the modes with $\alpha = 0.01, 0.1, 1$ and $5$, where $\alpha = k/a_s H_e = k/H_e$. Thus the former two correspond to superhorizon perturbations at the end of inflation, while the latter two to subhorizon perturbations. The superhorizon modes reach the value $\Psi_{k}^2 = 2H_k(\tau_e)/3$ a few Hubble times after the end of inflation. For large values of $q$ the scale dependence of these modes is small. As $q$ decreases the deviation from a scale invariant spectrum in the superhorizon modes becomes more prominent. On subhorizon scales, $k > H_e$, $\Psi$ oscillates with decaying amplitude once the radiation dominated epoch begins. The relevant quantity to estimate the abundance of subhorizon PBHs formed at the end of inflation is the first maximum of the oscillations, $\Psi_M(k)$ [4, 5]. $\Psi_M(k)$ increases with $k$, until the comoving scale approaches $k \sim H_e/\delta$. $\Psi_M(k)$ then decreases and converges to the approximate value given in Eq. (63), $\Psi_M \sim H_e/M_p$, as shown in Fig. 2.

The numerical value of $\Psi_M(k)$ on subhorizon scales for two different values of $q$ is shown in the left panel of Fig. 2. The value of $\tau$ for the first maximum is approximately given by $\tau_M \approx \tau_e + \pi H_e/2kc_s$. It is convenient to express $\Psi_M(k)$ in $H_e/M_p$ units, because in the limit $k \gg H_e/\delta$ it approximately drops down to that value. For a given transition time, $\Delta t \sim \delta/H_e$, the amplitude of $\Psi_M$ increases as $q$ gets larger on an interval about the comoving scale $k \sim H_e/\delta$, where the resulting spectrum has a broad resonance. As we show in Appendix B, the resonance can be interpreted as the effect of $V_{eff}$ in Eq. (59) acting as a potential barrier for the incoming wave function. The higher the potential barrier is the larger is the maximum amplitude of the resonance. The width of the resonance is related to the transition time only, and it is of the order $H_e/\delta$. As $k$ becomes much larger than $H_e/\delta$, $\psi_M$ tends to $H_e/M_p$, independently of the values of $q$ and $\delta$.

In the right panel of Fig. 2, we show numerical values for the peak of the resonance, which here we denote it as $\Psi_{RES}$, as a function of $q$ for different values of the transition time. The solid lines represent the fitting curve,

$$\sqrt{2k^3} \Psi_{RES}(q, \delta) = \sqrt{q} \left[ A + B\sqrt{\delta} + C\delta \right], \tag{69}$$

where $A = 2.47$, $B = -4.74$ and $C = -1.58$ in $H_e/M_p$ units. The estimated value of peak of the resonance above is in agreement with the result found in [4], where the case of an instantaneous transition into radiation domination was analyzed. Calculating the maximum of Eq. (69), we get [4]

$$\sqrt{2k^3} \Psi_{RES} \approx 2\sqrt{3} \left( 1 - \epsilon_H \right) \left( \frac{H^2}{\phi_c} \right). \tag{70}$$

This is the same value we get in Eq. (69) if we take the limit $\delta \rightarrow 0$ and express $\Psi_M$ in terms of $A_\Gamma$ for $\epsilon_H$ small, noting that during inflation,

$$\frac{H^2}{\phi} = \frac{1}{\sqrt{2}} \left( \frac{H}{M_p} \right) \epsilon_H^{-1/2}. \tag{71}$$

We can therefore have a rough idea of the shape of the wide resonance if $q$ and $\delta$ are given. The height of the resonance is of the order of the curvature perturbation at the end of inflation, $A_\Gamma$, while the width is approximately given by $1/\delta$. It is interesting to note as well that while on superhorizon scales a smaller value of $q$ tends to enhance the tilt of the spectrum, it suppresses the amplitude of the resonance and in general of all the modes well inside the horizon about the comoving scale $k \sim H_e/\delta$.

V. THE TRANSITION INTO MATTER DOMINATION

If an epoch of matter domination succeeds the inflationary epoch, then the equation of state is $P \simeq 0$ and the differential equation for the perturbations during this period is

$$u_k'' + \left( c_s^2 k^2 - \frac{6}{\tau^2} \right) u_k = 0, \tag{72}$$

where the conformal time during the matter domination period is $\tau = 2/H$, corresponding to $a \propto \tau^2$, and the velocity of sound $c_s$ is practically zero. In the limit, $c_s k \rightarrow 0$, the two independent solutions for $u_k$ behave like a power law, $u_k \propto \tau^3$ and $\tau^{-2}$. Then the solution for $\Psi_k$ ($\propto \rho^{1/2} u_k$) is given by

$$\Psi_k(\tau) \simeq C_k + D_k \tau^{-5}, \tag{73}$$
FIG. 1: The figure on the left shows $V_{\text{eff}}$ through a transition into radiation domination as a function of $\tau$ in $H_{E}^{-1}$ units for two different values of $q$ in power law inflation. The transition time corresponds to $\delta = 0.01$. The figure on the right shows the evolution of $\sqrt{2k^3|\Psi_{k}|}$ in $A_{R}$ units for 4 different modes with $\alpha \equiv k/a_{E}H_{E} = 5, 1, 0.1, 0.01$, and the higher frequency modes corresponding to larger wavenumbers $k$. The parameters are the same as the left figure.

FIG. 2: On the left hand side, we show the scale dependence of the amplitude of $\Psi_{M}$ in $H_{E}/M_{P}$ units, defined as $\sqrt{2k^3|\Psi_{M}(k)|}$, for a transition into radiation domination. The two curves are evaluated for $\delta = 0.01$. The comoving wavenumber $k$ is in $H_{E}^{-1}$ units. The dashed line represents the maximum of the analytic expression in Eq. (67). On the right panel, we show the value of the maximum for the resonance, $\Psi_{RES}$, as a function of $q$, for $\delta = 0.05, 0.01$ and $0.001$. A larger $\delta$ results in a smaller value of $\Psi_{RES}$. The solid lines represent the interpolation of the numerical data.
where the integration constants depend on the comoving scale $k$. Thus as the universe enters a stage of matter domination, perturbations on scales for which $c_s k \tau$ is small reach certain constant values a few Hubble times after the end of inflation.

As in the preceding section, we consider that the equation of state changes very sharply from the inflationary value at the end of inflation, determined by the slow-roll parameter $\epsilon_H$ in the case of power law inflation, to the matter domination value $P = 0$. The behaviour of the perturbed quantities $\Psi$ and $\mathcal{R}$ during this transition is similar to the case of the transition into radiation domination for $k \ll \mathcal{H}/\delta$. On these scales, we can neglect the gradient terms in Eq. (25) and therefore the perturbed quantities $\Psi$ and $\mathcal{R}$ do not grow appreciably during the transition if $\delta \lesssim \epsilon_H$. However in the opposite regime, $k \gg \mathcal{H}/\delta$, if we assume that the velocity of sound $c_s(\tau)$ drops down to a negligible value during the transition, the potential $\Psi$ oscillates with a decreasing frequency, given by $W(k, \tau)$ in Eq. (59), until $W(k, \tau)$ reaches zero to eventually become negative. At this point, $\Psi$ starts to grow and eventually settle down to a constant value during matter domination.

Assuming that $\Psi$ and $\mathcal{R}$ do not vary during the transition, which is valid for $k \ll \mathcal{H}/\delta$ and $\delta \lesssim \epsilon_H$, and taking the slow-roll limit $\Psi \ll \mathcal{R}$ during inflation, we can determine the integrating constants $C_k$ and $D_k$. Setting the scale factor at the end of inflation to $a_e \approx 2/\tau_e \mathcal{H}_e$, the potential $\Psi$ is given by

$$\Psi_k(\tau) \approx -\frac{3}{5} \mathcal{R}_k(\tau_e) \left[ 1 - \left( \frac{\tau_e}{\tau} \right)^5 \right].$$

(74)

Thus, a few Hubble times after the end of inflation $\Psi$ reaches the constant value $\Psi_k \approx 3\mathcal{R}_k(\tau_e)/5$ on these scales. This is the standard result for superhorizon scales during matter domination. Here we have shown that this result will hold for $k \ll \mathcal{H}/\delta$ as long as $\delta \lesssim \epsilon_H$. For perturbations on scales $k \gtrsim \mathcal{H}/\delta$, $\Psi$ and $\mathcal{R}$ vary during the transition because the wavelength of the perturbations is the order or smaller than the transition time, and therefore the estimates above are no longer valid. Nevertheless, $\Psi$ also reaches a constant value during matter domination which in this case is scale dependent.

Approximate analytical solutions are harder to obtain than in the case of a transition into radiation domination for $k \gg \delta/\mathcal{H}$ because the gradient term vanishes during matter domination. Neglecting the derivatives of the equation of state, the evolution equation for $u_k$ can be written as

$$u_k'' + \left[k^2 c_s^2(\tau) - \frac{3}{2} \mathcal{H}^2 (1 + w)\right] u_k = 0,$$

(75)

where both $c_s^2(\tau)$ and $(1 + w)$ behave like step like functions with the same width determined by the parameter $\delta$. We have tried 1st order WKB solutions by using Airy functions to match the solutions between inflation and matter domination. Despite this solution gives the same scale dependence as our numerical results, that is $\Psi_k \sim (k/\mathcal{H})^{1/2}$, the amplitude is about a factor of 4 smaller. This is probably due to the fact that the WKB approximation fails marginally for large $\tau$ if $c_s^2 = 0$. An asymptotic matching using higher order WKB solutions would reduce the error. In Appendix A we show the details of the first order WKB approximation. The solution with the appropriate boundary conditions has the form,

$$\Psi_k(\eta) \approx 4.4 \left(\tau, \mathcal{H}_e\right)^{-3/2} \alpha^{1/2} \left(\frac{\mathcal{H}_e}{M_p}\right),$$

(76)

where $\alpha = k/\mathcal{H}_e$, and $\tau_*$ is the turning point of the differential equation. The turning point marks the time at which the frequency of the differential equation changes sign. In this case it is given by $W(k, \tau_e) = 0$ with

$$W^2(k, \tau) = k^2 c_s^2(\tau) - \frac{3}{2} \mathcal{H}^2 (1 + w).$$

(77)

Numerical solutions of the differential equation in Eq. (25) are shown in Figs. 3 and 4. We have modeled the time variation of $P = P(\rho)$ and $c_s(\tau)$ as in the previous section by an error function with a transition time interval of $\Delta \tau \sim \delta/\mathcal{H}$ until they settle down to the values corresponding to matter domination. The numerical results for the perturbed quantity $\Psi$ are in good agreement with our expectations based on the qualitative analysis. In the paragraphs below we describe our numerical results summarized in Fig. 3 and 4.

For large values of $q$, the superhorizon modes do not show substantial scale dependence and $\Psi$ grows and converge to an approximate value of $\Psi \sim 3A_R/5$. As $q$ decreases the scale dependence of these modes gets larger increasing the red tilt of the spectrum, as shown in Fig. 4. The modes corresponding to scales in the range $\mathcal{H}_e \lesssim k \ll \mathcal{H}/\delta$, reach the approximate value,

$$\sqrt{2k^3}|\Psi_k| \approx \frac{3\alpha}{5} \frac{H_e^2}{\mathcal{H}_e} = \frac{3\alpha}{5} q^{1/2} \left(\frac{\mathcal{H}_e}{M_p}\right),$$

(78)
taking the asymptotic limit \( k \tau_0 \gg 1 \) of \( R_k \) in Eq. \([50]\). On these range of scales \( \Psi_k \) increases as the slow-roll parameter \( \epsilon_H = 1/q \) gets smaller. (See the right panels of Fig. 3 and Fig. 4)

On the opposite regime, \( k \gg H_c/\delta \), the amplitude of \( \Psi_k \) grows with scale as \( a^{1/2} \). As \( k \) decreases there are a series of small oscillations in the spectrum followed by a resonance approximately located at \( k \sim H_c/\delta \). The resonance is as in the radiation domination case the result of the effective potential \( V_{\text{eff}} \), on the modes with wavenumbers about \( k \sim H_c/\delta \). The maximum amplitude of the resonance depends on the height and width of the effective potential \( V_{\text{eff}} \). These are determined by the parameters \( q \) (or \( \epsilon_H \)) and \( \delta \). For a fixed value of \( \delta \), the potential barrier described by \( V_{\text{eff}} \) defined in Eq. \([60]\) behaves as \( V_{\text{eff}} \propto 1/(1+w)^2 \), increasing its height as \( \epsilon_H \) becomes smaller. As the height of the potential barrier increases, the effect on the a fixed mode \( k \) of the field \( u \) that propagates across it, is to increase its amplitude (as it can be observed in the left panel of Fig. 4). We have described in more detail the effect of a simple squared potential on the amplitude of \( \Psi_k \) in Appendix B. Despite its simplicity, the model reproduces qualitatively the features observed in the spectrum of \( \Psi_k \) shown in Fig. 4.

Numerical results for the maximum of the resonance in the spectrum of \( \Psi_k \) are shown in the right panel of Fig. 4. A good fitting curve in the regime shown in the figure is

\[
\Psi_{\text{RES}}(q,\delta) = A \frac{q^{1/2}}{\delta} \left( \frac{H_c}{M_p} \right) = A\sqrt{2} \left( \frac{H_c^2}{\phi_c^2} \right),
\]

(79)

where \( A = 0.195 \). The dependence of the amplitude of the resonance with \( q \) is the same as in the radiation domination case. However, in this case \( \Psi_{\text{RES}} \) diverges in the limit of \( \delta \) going to zero. This result is not surprising because if we consider an instantaneous transition and use the junction conditions \([1]\) and \([2]\), then \( \Psi_k \) is given by Eq. \([78]\) and it would diverge in the ultraviolet limit.

In reality, the sound velocity will not be exactly zero in the matter-dominated stage. In the case of a free massive scalar field, the perturbation will not grow on very small scales where the wavenumber exceeds the geometrical mean of the Hubble parameter and the mass, \( k/a > (k/a)_e \sim \sqrt{\frac{a}{M}} \) \([24]\). Furthermore, if there is a \( \lambda \phi^4 \) self-interaction, the critical scale is modified drastically to \( (k/a)_e \sim \lambda^{1/4}m^2/M_p \) \([24]\). In any case, these scales provide a natural ultraviolet cut-off and regulate the ultraviolet divergence.

To conclude this section we would like to remark the similar and different features of the solutions in this case compared to the transition into radiation domination we studied in the previous section. As in the radiation domination case, the potential \( \Psi \) does not depend on the slow-roll parameters in the limit \( k \gg H_c/\delta \). It is only in the opposite regime, when \( \mathcal{R} \) does not vary considerably during the transition, that its value at horizon crossing at the end of inflation is relevant for the perturbed quantity \( \Psi \). The scale dependence of the solutions for \( k \lesssim \delta/H_c \) is similar as well, with large values of \( q \) giving a nearly scale invariant spectrum while increasing the amplitude of the resonance. In contrast, although the height of the resonance is proportional to \( A_{\mathcal{R}} \), it depends strongly on \( \delta \).

VI. DISCUSSION AND CONCLUSIONS

We have studied the behaviour of the primordial perturbations through a sharp transition from inflation into the radiation and matter dominated epochs. We have found that for transitions that occur much faster than a Hubble time, there is a range of scales inside the horizon for which the amplitude of the primordial perturbations is enhanced, relative to the amplitude of perturbations that exit the horizon a few e-folds before the end of inflation, if slow-roll conditions hold during the final stages of inflation. For a Gaussian distribution of the perturbations, this relatively large amplitude increase the probability of strong perturbations on these small scales, that may lead to a significant production of PBHs.

For a transition from inflation into a radiation dominated universe, the maximum amplitude of the peculiar gravitational potential is about \( \Psi \sim 2\sqrt[3]{3}H_c^2/\phi_c \). Assuming a Gaussian distribution for the primordial perturbations, one can estimate the PBH abundance produced at the end of inflation, using for example Press-Schechter approach. (See however \([25]\) for the effect of non-Gaussian perturbations.) Given that the observed amplitude of the curvature perturbation on cosmological scales is roughly given by \( \mathcal{R} \sim 10^{-5} \), it turns out that unless the spectrum of \( \mathcal{R} \) is considerably larger at the end of inflation, the amount of PBH production would not have any cosmological significance \([5]\). However, a much larger amplitude on scales \( k \sim H_c \) is still compatible with current observational data even within the slow-roll paradigm \([12]\). Furthermore, PBHs may be overproduced as well with a more complicated scale dependence if slow-roll conditions do not hold throughout the whole period of inflation.

In the case of a transition into a matter dominated universe the production of PBHs could be more dramatic even with a very flat spectrum of \( \mathcal{R} \) on all scales. The maximum amplitude for \( \Psi \) now depends on delta as \( \Psi \sim 1/\delta \). Therefore if the velocity of sound does vanish on a very small scale, such that \( c_s = 0 \) for \( k \sim H_c/\delta \), there may be a significant production of small mass PBHs even if the transition time is not very rapid.
FIG. 3: The figures show the time evolution of $\sqrt{2k}|\Psi_k|$ in $A_R$ units through a transition into matter domination for 3 different modes. On the left graph, we show the superhorizon modes $\alpha \equiv k/H_e = 0.2, 0.1$ and 0.01 for perturbations generated during power law inflation. The smaller wavenumbers result in larger amplitudes as a consequence of the red tilt of the spectrum. On the right graph, the corresponding modes are $\alpha = 1, 5$ and 10. In this case the larger wavenumbers result in larger amplitudes. The transition time corresponds to $\delta = 0.01$. 

FIG. 4: The figure on the left shows the matter domination spectrum of $\Psi_k \sqrt{2k}|\Psi_k|$, in $H_e/M_p$ units for two different values of $q$. The transition time is fixed at $\delta = 0.01$. The dashed lines represent our analytical approximations onto the left and right of the resonance, $\Psi_k \propto \alpha$ and $\sqrt{\alpha}$, respectively. On the right, the peak value at the resonance, $\Psi_{\text{RES}}$, is plotted as a function of $q$ for $\delta = 0.05, 0.01$ and 0.001. The smaller values of $\delta$ result in larger amplitudes for the resonance.
Finally we would like to stress that a detailed analysis of specific models, in which the transition from inflation occurs very rapidly, may reveal the overproduction of PBHs, placing constraints on the spectrum of the primordial perturbations on scales that are too small for conventional observations.

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Appendix A: Matter perturbations on scales $k \gg H_e/\delta$.

In this Appendix, we find an approximate analytical solution for the potential $\Psi$ during the matter domination epoch on scales such that $k \gg H_e/\delta$. As explained in Sec. (V), we assume that the velocity of sound $c_s$ drops down from a value of 1 during inflation to zero at the end of inflation much faster than a Hubble time. In this case the gradient terms vanish and the simple oscillatory WKB solution we found for the radiation domination case is not valid. However, we can still find an approximate asymptotic solution to first order in the WKB expansion that reproduces the scale dependence of the spectrum of $\Psi$ accurately.

Neglecting the time derivatives in the equation of state, the evolution equation for the field $u_k$ is given by

$$u''_k + \left[ k^2 c_s^2(\tau) - \frac{3}{2} H^2 (1+w(\tau)) \right] u_k = 0,$$

(A1)

where $u_k = 2k(\rho + p)^{-1/2} \Psi_k$. It is convenient, to introduce an adimensional time parameter $\eta = \tau H_e$, where $H_e \equiv H(\tau_e)$. Then the equation of motion may be written as

$$\ddot{u} = Q(\eta) u,$$

(A2)

where $Q(\eta)$ is defined as

$$Q(\eta) \equiv \frac{3}{2} H^2 (1+w) - \alpha^2 c_s^2(\eta),$$

(A3)

and $\alpha = k/a_e H_e$ and $\dot{\mathcal{H}} = \mathcal{H}/H_e$.

This equation has a turning point $Q(\eta_*) = 0$ at $\eta_*$. For $\eta < \eta_*$, $Q < 0$ and the solutions are oscillatory. For $\eta > \eta_*$, $Q > 0$ and the two independent solutions may be given by growing and decaying modes. Then during inflation and matter domination, the 1st order WKB solutions are given by

$$u_{\text{inf}}^{\pm}(\eta) \sim |Q(\eta)|^{-1/4} \exp \left[ \pm i \int_{\eta}^{\eta_*} \sqrt{|Q(\eta)|} d\eta \right],$$

(A4)

$$u_{\text{mat}}^{\pm}(\eta) \sim Q(\eta)^{-1/4} \exp \left[ \pm \int_{\eta_*}^{\eta} \sqrt{Q(\eta)} d\eta \right].$$

(A5)

To match asymptotically the solutions above, we can expand $Q(\eta)$ to first order about the turning point and use the large argument expansions for the Airy functions. However, it is simpler to apply the following Liouville transformation,

$$Y = \left( \frac{d\xi}{d\eta} \right)^{1/2} u,$$

(A6)

$$\xi = \left( \frac{d\eta}{d\xi} \right)^2 Q(\eta),$$

(A7)

where $\xi$ and $\eta$ are analytic functions of each other at the turning point. Then we get the differential equation for $Y$,

$$\frac{d^2 Y}{d\xi^2} = [\xi + \vartheta(\xi)] Y,$$

(A8)
where the function $\vartheta(\xi)$ above is

$$\vartheta(\xi) = \left( \frac{d\xi}{d\eta} \right)^{-1/2} \frac{d^2}{d\xi^2} \left[ \left( \frac{d\xi}{d\eta} \right)^{1/2} \right].$$  \hfill (A9)

The differential equation for $Y$ becomes an Airy differential equation if we neglect $\vartheta$. An approximate solution for $u$ then is given by

$$u = \left( \frac{\xi}{Q} \right)^{1/4} \left[ C_1 Ai(\xi) + C_2 Bi(\xi) \right],$$  \hfill (A10)

which we use below to match the first order solutions across the turning point.

During inflation $\xi < 0$, and then we have

$$2^{3/2} \xi^{3/2} \approx \int_{\eta_*}^{-\eta} \alpha c_s(\eta) d(-\eta) \simeq k\tau,$$  \hfill (A11)

where $\tau \ll \tau_*$ and we have taken $c_s(\tau) = 1$ during inflation. In the models we consider here the velocity of sound during inflation always has that value, so from now on we just consider this case. In the limit $\eta \ll \eta_*$, the asymptotic expressions for the Airy functions in this limit are

$$Ai(-z) \sim \pi^{-1/2} z^{-1/4} \sin \left( \frac{2}{3} z^{3/2} + \frac{\pi}{4} \right),$$  \hfill (A12)

$$Bi(-z) \sim \pi^{-1/2} z^{-1/4} \cos \left( \frac{2}{3} z^{3/2} + \frac{\pi}{4} \right).$$  \hfill (A13)

Taking the initial Bunch-Davies vacuum value for the field $u_k$,

$$u_k \rightarrow \left( \frac{1}{2k} \right)^{1/2} e^{-ik\tau},$$  \hfill (A14)

one gets the following values for the integrating constants,

$$C_1 \sim -\left( \frac{1}{2k} \right)^{1/2} \alpha^{1/2} \sqrt{\pi} e^{i\pi/4},$$  \hfill (A15)

$$C_2 \sim -C_1.$$  \hfill (A16)

On the other hand, during the matter domination period $\xi > 0$, and the asymptotic behaviour of the Airy functions in this regime is

$$Ai(z) \sim 2^{-1/2} \pi^{-1/2} z^{-1/4} e^{-\frac{2}{3} z^{3/2}},$$  \hfill (A17)

$$Bi(z) \sim \pi^{-1/2} z^{-1/4} e^{\frac{2}{3} z^{3/2}}.$$  \hfill (A18)

Taking $c_s(\tau)$ and $w(\tau)$ to be zero for $\eta \gg \eta_*$, $Q^{-1/4}$ is approximately given by

$$Q^{-1/4}(\eta) \simeq \frac{1}{\sqrt{2}} \left( \frac{2}{3} \right)^{1/4} \eta^{1/2},$$  \hfill (A19)

and therefore the solution for $\Psi$ during the matter domination epoch, neglecting the decaying mode, is approximately given by

$$\Psi_k = \frac{(\rho + p)^{1/2}}{\sqrt{2k^3 M_p^2}} u_k \sim \frac{4.4}{\sqrt{2k^3}} \times \alpha^{1/2} \eta^{-5/2} \exp \left( \frac{2}{3} \xi^{3/2} \right) \left( \frac{H_c}{M_p} \right),$$ \hfill (A20)

where we have taken $\tau_c = 2/H_c$ and

$$\frac{(\rho + p)^{1/2}}{M_p^2} \simeq 8\sqrt{3} \left( \frac{H_c}{M_p} \right) \eta^{-3}.$$  \hfill (A21)
During the matter dominated epoch, the exponential term in Eq. (A20) is
\[ \frac{2}{3} \xi^{3/2} = \int_{\eta_s}^{\eta} \frac{3}{2} H^2 (1 + w(\tau)) - \alpha^2 \tau^2(\tau) d\eta. \]
By integrating by parts expression Eq. (A22), we get
\[ \frac{2}{3} \xi^{3/2} = \sqrt{6} \left[ \ln(\eta) - \int_{\eta_s}^{\eta} \ln(\eta) g^{-1/3} \frac{dg}{d\eta} d\eta \right], \]
where \( g \) is
\[ g(\eta) = \left[ (1 + w) - \frac{2k^2 \tau^2}{3H^2} \right]. \]
Neglecting the variation of the logarithmic term in the integrand, \( \xi^{3/2} \) is approximately given by
\[ \frac{2}{3} \xi^{3/2} \sim \sqrt{6} \ln \left( \frac{\eta}{\eta_s} \right), \]
and therefore the time variation of \( \Psi \) is negligible,
\[ \sqrt{2k^3} \Psi_k \sim 4.43 \times 10^{-0.45} \eta_{\ast}^{-0.15} \eta_{\ast}^{-0.15}, \]
if \( \eta \) is not too large.
This approximate solution is the first order solution in the WKB expansion. The scale dependence of \( \Psi \) on this regime is \( \alpha^{1/2} = (k/H_c)^{1/2} \), the same that we have found on our numerical estimations. The solution is not time independent, although the time dependence is very weak. On the other hand, the approximate value of the amplitude in Eq. (A22) results in a smaller value than the one we have obtained numerically. These two differences in the analytic result, suggest that a higher order WKB solution would correct this deficiency. In fact, for the 1st order WKB solution to be valid uniformly for all \( \tau \) it is necessary for \( Q(\tau) \) to decrease much more rapidly than \( 1/\tau^2 \) when \( \tau \) goes to infinity. During matter domination \( Q \sim 1/\tau^2 \), and therefore we believe that a second or higher order WKB asymptotic matching would yield a more accurate result. Here we are mostly interested in reproducing analytically the scale dependence of the spectrum, and when we compare the analytic with the numeric results in Fig. 4, we just regulate the lower limit of the integrand in Eq. (A22) to obtain a reasonable estimate. We believe that this approach is justified given that a relative small error in \( \xi \) could in principle result in a significant error in \( \Psi \propto \exp(2\xi^{3/2}/3). \)

Appendix B: The effect of a potential barrier on the potential \( \Psi \): the broad resonance.

In this Appendix, we explain qualitatively the origin of the broad resonance and small oscillations that we have obtained in our numerical calculations for the spectrum of \( \Psi \). In the two transitions from inflation studied here, the broad resonance is approximately located at the comoving scale \( k \sim H_c/\delta \) and it is followed by a series of small oscillations with an amplitude that is strongly suppressed as \( k \) increases (see the left panels of Figs. 3 and 4). About those scales, the derivatives of the equation of state are not negligible, and the function \( q'(k, \tau) \) has a complicated form that makes difficult to find approximate analytical solutions. For a step like equation of state with a width \( \delta/H_c \), these terms typically result in a sharply peaked function with an approximate amplitude of \( H_c^2/\delta^2 \). This suggests that, on scales about \( k \sim H_c/\delta \), the terms proportional to \( u' \) in \( q'(k, \tau) \) act as a potential barrier for the wave equation \( u_k \) that propagates from the inflation to the radiation and matter domination epochs.
In order to reproduce the broad resonance in the spectrum then, we adopt a square potential acting as a potential barrier, for which solutions can be easily found. The model is very crude, but as we will see it captures the fundamental phenomenon occurring on such scales. We restrict this analysis to the transition from inflation to matter domination. This case is easier to analyze because the modes of the potential reach a constant value. The radiation domination modes after the transition from inflation oscillate with a decaying amplitude. The quantity of interest then is the maximum amplitude during the oscillations, which is more cumbersome to calculate. Nevertheless, the relative enhancement of the modes is caused essentially by the same phenomenon as in the matter domination case.
We are interested in the evolution of the field \( u_k \) after the transition into radiation or matter domination on scales such that \( k \gtrsim H_c/\delta \). We assume that the velocity of sound, remains constant while the wave crosses through
the potential barrier. This is a good approximation for the radiation domination case because \(c_s\) does not vary considerably during the transition. In the matter domination case, we can consider the idealized situation in which \(c_s\) drops down to zero once the wave function has crossed the potential barrier. To find an approximation solution for the perturbations during this epoch then, we can take the resulting solution as the initial condition for (Eq. A1).

With the simplifying assumptions considered above in mind, we now study the behaviour of the fluctuations in the field \(u_k\) for a square potential. The evolution of the field then is determined by the differential equation,

\[
 u''_k + \left( \alpha^2 - V(\tau) \right) u_k = 0 ,
\]

where \(\alpha \equiv k/H_e\), and \(V(\tau)\) is given by

\[
 V = \begin{cases} 
 0, & \tau < -a, \\
 v^2, & -a < \tau < 0, \\
 0, & \tau > 0 .
\end{cases}
\]

Taking the initial value of the field \(u_k\) corresponding to the Minkowski vacuum,

\[
 u_k = \frac{1}{\sqrt{2k}} e^{-ik\tau} ,
\]

the solution for \(\tau > 0\) is

\[
 u_k = \frac{1}{\sqrt{2k}} \left( A e^{-ik\tau} + B e^{ik\tau} \right) ,
\]

where \(q = \sqrt{k^2 - v^2}\), and the constants \(A\) and \(B\) are given by

\[
 A = \frac{(k + q)^2 - (k - q)^2 e^{-2iqa}}{4kq e^{-i(q-k)a}} ,
\]

\[
 B = \frac{v^2 (1 - e^{-2iqa})}{4kq e^{-i(q-k)a}} .
\]

As in the previous section, an approximate asymptotic solution valid for throughout the transition and the matter domination epoch is

\[
 u = \left( \frac{\xi}{\Omega} \right)^{1/4} \left[ D_1 Ai(\xi) + D_2 Bi(\xi) \right] ,
\]

where \(D_1\) and \(D_2\) are integrating constants. Taking the asymptotic expressions of the Airy functions for \(\tau \ll \tau_*\) in Eqs. (A17) and (A18), we find that the initial conditions in Eq. (B4) are satisfied if

\[
 D_1 \sim \frac{1}{\sqrt{2k}} \pi^{1/2} \alpha^{1/2} (B - A) ,
\]

\[
 D_2 \sim \frac{1}{\sqrt{2k}} \pi^{1/2} \alpha^{1/2} (A + B) .
\]

The asymptotic solution for \(\tau \gg 0\) during matter domination for the potential \(\Psi_k\) is approximately given by

\[
 \Psi_k = \frac{(\rho + p)^{1/2}}{2k} u_k \sim \frac{1}{\sqrt{2k^3}} 4\eta_* |F| \alpha^{1/2} \left( \frac{H_e}{\sqrt{M_p}} \right) ,
\]

where \(\alpha = k/H_e\), \(\eta_* = \tau_* H_e\) denotes the time at the turning point (see Appendix A), and \(F \equiv A + B\). The function \(|F|\) which modulates the solution \(\Psi \propto \alpha^{1/2}\) we have found in Appendix A can be written as

\[
 |F|^2 = \frac{\alpha^2 - v^2 \cos^2 \left( l/\alpha^2 - v^2 \right)}{\alpha^2 - v^2} ,
\]

for \(\alpha > v\), which is the region of the comoving scale that we are interested in reproducing.
FIG. 5: The figure on the left we compare the matter domination spectrum of \( \Psi_k \) we have obtained numerically with the analytical approximation in Appendices A and B. The numerical solution corresponds to \( \delta = 0.01 \) and \( q = 100 \). On the right of the resonance, we have found a reasonable fit with the analytical approximation for \( v = 118 \), \( l = 0.025 \) and \( \eta_* = 0.2 \). The dashed line represents our analytical approximation onto the left of the resonance, \( \Psi \propto \alpha \).

In the limit \( k \gg H_e \) we get the trivial result \( |F| \approx 1 \), which corresponds to \( A \approx 1 \) and \( B \approx 0 \). On this regime, the variation in the potential does not affect the propagation of the field \( u_k \), and \( \Psi_k \propto \alpha^{1/2} \). As the comoving scale \( k \) decreases \( \Psi_k \) decreases until \( k \) approaches the scale close to the height of the potential \( \alpha = v \). About that scale, the amplitude of \( \Psi_k \) increases achieving its maximum amplitude at scale about \( \alpha \sim v \). For \( \alpha \lesssim v \), the amplitude of the perturbations decrease again. The overall effect then is that we find a broad resonance on the spectrum of \( \Psi_k \) approximately located at the scale \( \alpha \sim v \).

In Fig. 5 we compare the result of our numerical calculation of the spectrum of \( \Psi_k \) with the analytical approximation above in Eq. (B9). The numerical result is for \( \delta = 0.01 \) and \( q = 100 \). To the left of the resonance, we have plotted the analytical approximation for \( k \lesssim H_e \) in Eq. (78) which is a good approximation on that scales. To the right of the resonance, on scales such that \( k > \sim H_e/\delta \), we show the analytical approximation above in Eq. (B9) for \( v = 118 \) and \( \delta = 0.02 \). To regulate the integral in Eqs. (A22) and (A23) we have used the lower limit of integration \( \eta_* = 0.2 \), which gives the right result for the scales shown in Fig. 5. For these values of the parameters \( v \) and \( \delta \), we see that the square potential reproduces the resonance we have found in our numerical calculations sufficiently accurately given the crude analytical approximation we have done.

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