Store-Forward and its Implications for Proportional Scheduling

Neil Walton
Switch Networks:

(BP) BackPressure – Classical switch policy

(PS) Proportional Scheduler – New switch policy

(SF) Store-Forward – A Related CTMC

One story:

PS is structurally simpler than BP

But... this talk:

SF is similar to PS and has lots of nice properties.

We study SF to postulate why PS is good:

Equilibrium, Delay, Product-Form, Lyapunov fns.
Switch Networks
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Communication networks are multihop
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Two Stabilizing Policies:
1. BackPressure
2. Proportional Scheduler
Switch Networks: BackPressure

Two Steps:
1. Determine weights

\[ w_j(Q) = \max_{k \in j} \{Q_k - Q_{n(k)}, 0\} \]

2. Schedule to maximize weights

\[ \text{Max} \sum_{j \in J} w_j(Q) \sigma_j \text{ over } \sigma \varepsilon < S > \]
Switch Networks: Proportional Scheduler

Two Steps:

1. Schedule to maximize

\[
\text{Max } \sum_{j} Q_j \log \bar{\sigma}_j \quad \text{over } \quad \bar{\sigma}_\varepsilon < S >
\]

\[
\text{e.g. } \bar{\sigma}_1 = \frac{4}{7}, \quad \bar{\sigma}_0 = \bar{\sigma}_2 = \frac{3}{7}.
\]

2. Serve queues with FIFO discipline
The Store-Forward Network is CTMC with feasible service rates

\[ \sigma_j^{SF}(Q) = \frac{\Phi(Q - e_j)}{\Phi(Q)} \]

where \( \Phi(Q) \) is a positive function that looks more complicated than it really is.
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One Story

| Diameter | BackPressure |
|----------|--------------|
| D=2      | 3            |
| D=4      | 6            |
| D=6      | 12           |
| ...      | ...          |
| D        | $3 \times 2^{(D/2-1)}$ |
### One Story

| Diameter | BackPressure | Proportional Scheduler |
|----------|--------------|------------------------|
| D=2      | 3            | 3                      |
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One Story

Compared with BackPressure, the Proportional Scheduler:

• requires far less information
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• ... (see W’14 Sigmetrics)
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This talk

Comparing BP & PS is not the main point...

• SF & PS are asymptotically equivalent:

\[ \sigma^{SF}(Q) \approx \sigma^{PS}(Q), \quad \text{for } Q \text{ large} \]

• SF has good properties. Thus, so should PF.
This talk

PS should inherit lots of properties known for SF...

- Equilibrium Distribution
- Delay
- Product form characterization
- Large deviations
- Lyapunov function
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FIFO Store-Forward
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We give results for Store-Forward.

For each there’s a conjecture for Proportional Scheduler.

Each result is difficult (or impossible) to establish for BackPressure.
Results on Store-Forward: Equilibrium

Theorem 1: A FIFO routed Store-Forward Network is positive recurrent when \((a_j : j \in \mathcal{J}) \in < \mathcal{S} >^\circ\) and has an equilibrium distribution of the form

\[
\pi(Q, \Gamma) = \Phi(Q) \prod_{j \in \mathcal{J}} \prod_{r : j \in r} \left(a_r \Gamma_{jr}(Q_j)\right).
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Results on SF: Delay

Theorem 2: For a stationary Store-Forward network, the delay on a route $r$ given by $D_r$ has expectation

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\mathbb{E}[D_r] = \sum_{j \in r} \sum_{l : j \in l} \frac{A_{lj}}{1 - a_l},
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Results on SF: Product-Form

Theorem 3: Consider a stationary Store-Forward Network on scheduling set

\[ < S > = \left\{ s \in \mathbb{R}^J_+: \sum_{j \in J} A_{lj} s_j \leq 1, l \in \mathcal{L} \right\}. \] (12)

If there are two queues \( j \) and \( j' \) such that there is no shared resource pool, i.e. \( \nexists \ l \in \mathcal{L} \) such that \( A_{lj} > 0 \) and \( A_{lj'} > 0 \), then the queues are statistically independent.
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Grey nodes are independent
i.e. Just need different multipliers, we don’t need product space.

Refs. Kang Kelly Lee Williams ’09, Shah, W., Zhong ’14
Results on SF: Product-Form

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Results on SF: Product-Form

Left-hand nodes are independent of right-hand nodes

i.e. Similar to a Gibbs Random Field / .
Results on SF: Large Deviations

We take a sequence of states \((Q^c, \Gamma^c)\) where

\[
\frac{Q^c}{c} \xrightarrow{c \to \infty} Q \quad \text{and} \quad \frac{\Gamma^c(cq)}{c} \xrightarrow{c \to \infty} \Gamma(q),
\]

Finally, we analyse the large deviations behaviour

\[
\lim_{c \to \infty} \frac{1}{c} \log P^c = - \max_{\sigma \in \langle S \rangle} \sum_{j \in J} Q_j \log \sigma_j - \sum_j \sum_r \int_0^{Q_j} \log \left( \frac{\Gamma_{jr}}{a_r} \right) d\Gamma_{jr}
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\]
Differentiating this Lyapunov Fn:

**Proposition:** For a FIFO Network under PS, for

\[
H(t) = \max_{\sigma \in \langle S \rangle} \sum_{j \in J} Q_j(t) \log \sigma_j + \sum_{j \in J} \sum_{r \in R} \int_0^{Q_j(t)} \log \left( \frac{\Gamma_{jr}}{a_r} \right) d\Gamma_{jr}
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then

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H'(t) = - \sum_{r \in R} D'_r(t) \log \frac{D'_r(t)}{A'_r(t)} - \sum_{j \in J} A'_j(t) \log \frac{A'_j(t)}{D'_j(t)} < 0
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Thank you for listening!

Some References:

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