Matching LTB and FRW spacetimes through a null hypersurface

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Abstract

Matching of a LTB metric representing dust matter to a background FRW universe across a null hypersurface is studied. In general, an unrestricted matching is possible only if the background FRW is flat or open. There is in general no gravitational impulsive wave present on the null hypersurface which is shear-free and expanding. Special cases of the vanishing pressure or energy density on the hypersurface is discussed. In the case of vanishing energy momentum tensor of the null hypersurface, i.e. in the case of a null boundary, it turns out that all possible definitions of the Hubble parameter on the null hypersurface, being those of LTB or that of FRW, are equivalent, and that a flat FRW can only be joined smoothly to a flat LTB.

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1 Introduction

All our observation in cosmology is along our past null-cone. What if we implement the local inhomogeneities we observe in a globally homogeneous universe in accordance with the cosmological principle \[1\]? Assume now a model universe in which the background outside our past light cone is smoothed out to a Friedmann-Robertson-Walker (FRW) universe, but on and within the light cone in our vicinity the spacetime is inhomogeneous \[2\]. As a simple case we assume the inhomogeneous part be given by a Lemaître-Tolman-Bondi (LTB) solution to the Einstein equations. Hence, the separating surface of the two spacetime will become a null surface for the LTB observers. Now, the question is if such a matching is exactly doable within the general relativity assuming Einstein dynamics? We therefore formulate the general problem of gluing a LTB metric to a FRW one along a null hypersurface. Similar problem for the case of a timelike hypersurface has already been studied \[3\]. The null hypersurface is however more delicate as we will see in this paper, at the same time it has more cosmological applications.

For this purpose we use Barrabès-Israel (BI) null shell formalism \[4\] to investigate the matching and find the junction conditions. Section 2 is devoted to the formulation of the problem. The junction conditions are then explicitly written in section 3. Section 4 is devoted to the discussion of the impulsive gravitational wave on the null hypersurface and the Penrose classification of the hypersurface for different cases. Different applications and special cases are then discussed in section 5. A conclusion follows then in section 6.

Conventions. Natural geometrized units, in which \(G = c = 1\) are used throughout the paper. The null hypersurface is denoted by \(\Sigma\). The symbol \(|\Sigma|\) means "evaluated on the null hypersurface". We use square brackets \([F]\) to denote the jump of any quantity \(F\) across \(\Sigma\). Latin indices range over the intrinsic coordinates of \(\Sigma\) denoted by \(\xi^a\), and Greek indices over the coordinates of the 4-manifolds.

2 Formulation of the gluing LTB to FRW

Consider an inhomogeneous metric containing dust matter represented by a LTB cosmological model glued to a FRW background universe along a null hypersurface. We choose the LTB metric to be written in the synchronous comoving coordinates in the form \[5\]

\[
ds^2 = -dt_\perp^2 + \frac{R'^2}{1 + E(r_-)}dr_\perp^2 + R^2(t_-, r_-)(d\theta^2 + \sin^2 \theta d\varphi^2),
\]

where the overdot and prime denote partial differentiation with respect to \(t_\perp\) and \(r_\perp\), respectively, and \(E(r_-)\) is an arbitrary real function such that \(E(r_-) > -1\), and we take \(R' > 0\) and \(R > 0\) to avoid shell crossing and shell focusing, respectively, of the dust matter during their radial motion. Then the corresponding Einstein field equations turn out to be

\[
\dot{R}^2(t_-, r_-) = E(r_-) + \frac{2M(r_-)}{R},
\]

\[
4\pi \rho_L(t_-, r_-) = \frac{M'(r_-)}{R^2 R'},
\]

\[2\]
where $\rho_L$ is the energy density and $M$ is another arbitrary function. We take a FRW background universe described by the following metric

$$ds^2 = -dt^2 + \frac{a^2(t_+)}{1 - kr^2_+}dr^2_+ + r^2_+a^2(t_+)(d\theta^2 + \sin^2\theta d\varphi^2),$$

(4)

where $k = +1, 0, -1$ distinguishes the closed, spatially flat, and the open cosmological models, respectively. To glue the LTB inhomogeneous patch and FRW background universe along the null hypersurface $\Sigma$ we need to have

$$r_- = r_-(t_-), \quad \frac{dr_-}{dt_-} = \frac{\sqrt{1 + E}}{R'(t_-, r_-)}|_{\Sigma},$$

(5)

in the minus coordinates, and

$$r_+ = r_+(t_+), \quad \frac{dr_+}{dt_+} = \frac{\sqrt{1 - kr^2_+}}{a(t_+)}|_{\Sigma},$$

(6)

in the plus coordinates. Now, the requirement of the continuity of the induced metric on $\Sigma$ yields the following matching conditions:

$$t_+ = t_- = t, \quad r_+a^\Sigma = R, \quad \frac{dr_+}{dr_-} = \frac{R'\sqrt{1 - kr^2_+}}{a\sqrt{1 + E}}|_{\Sigma}.$$  

(7)

For further applications, we note that the differentiation of $r_+a = R$ on $\Sigma$ leads to

$$\sqrt{1 - kr^2_+} + RH^\Sigma = \sqrt{1 + E} + \dot{R},$$

(8)

where $H = \frac{1}{a}\frac{da}{dt_+}$ is the Hubble parameter of the FRW background. Taking $\xi^a = (t, \theta, \varphi)$ with $a = 1, 2, 3$ as the intrinsic coordinates on $\Sigma$ we must calculate the tangent basis vectors $e_a = \partial/\partial \xi^a$ on both sides of $\Sigma$. Having written $x^\mu_\pm$ in terms of $x^\mu_-$ by Eq. (7), we get

$$e^\mu_\mu|_- = \left(1, \frac{\sqrt{1 + E}}{R'}, 0, 0 \right)|_{\Sigma}, \quad e^\mu_\theta|_- = \delta^\mu_\theta, \quad e^\mu_\varphi|_- = \delta^\mu_\varphi,$$

(9)

$$e^\mu_\mu|_+ = \left(1, \frac{\sqrt{1 - kr^2_+}}{a}, 0, 0 \right)|_{\Sigma}, \quad e^\mu_\theta|_+ = \delta^\mu_\theta, \quad e^\mu_\varphi|_+ = \delta^\mu_\varphi.$$  

(10)

Let $t$ be a parameter on the null geodesic generators of $\Sigma$, we choose the tangent-normal vector $n^\mu$ to coincide with the tangent basis vector associated with the parameter $t$, so that $n^\mu = e^\mu_t$. We may then complete the basis by a transverse null vector $N^\mu$ uniquely defined by the four conditions $n_\mu N^\mu = -1, N^\mu e^\mu_A = 0 (A = \theta, \varphi)$, and $N_\mu N^\mu = 0$. We find

$$N_\mu|_- = \frac{1}{2} \left(-1, \frac{-R'}{\sqrt{1 + E}}, 0, 0 \right)|_{\Sigma},$$

(11)

$$N_\mu|_+ = \frac{1}{2} \left(-1, \frac{-a}{\sqrt{1 - kr^2_+}}, 0, 0 \right)|_{\Sigma}.$$  

(12)
Furthermore, the induced metric on \( \Sigma \) given by \( g_{\alpha\beta} = g_{\mu\nu}e^\mu_a e^\nu_b \big|_\pm \) is computed to be \( g_{\alpha\beta} = \text{diag}(0, R^2, R^2 \sin^2 \theta) \), which is the same on both sides of the hypersurface. Defining a pseudo-inverse of the induced metric \( g_{\alpha\beta} \) on \( \Sigma \) as \( g^{ac} g_{bc} = \delta^a_b + n^a \nabla_\mu N_\nu \), with \( n^a = \delta^a_t \), one gets \( g^{\alpha\beta} = \text{diag}(0, R^2, R^2 \sin^2 \theta) \). The final junction condition is formulated in terms of the jump in the extrinsic curvature. Using the definition \( K_{\alpha\beta} = e^\mu_a e^\nu_b \nabla_\mu N_\nu \), we may therefore compute the transverse extrinsic curvature tensor \([4]\) on both sides of \( \Sigma \). Its non-vanishing components on the minus side are found as

\[
K_{\theta\theta} \big|_\Sigma = \sin \theta \gamma_{\phi\phi} \big|_\Sigma = \frac{R}{2} (R - \sqrt{1 + E}) |_{\Sigma},
\]

\[
K_{tt} \big|_\Sigma = H R' |_{\Sigma},
\]

where we have defined \( H R' = \frac{R'}{R} \) as one of the possible definitions of the Hubble function for a LTB metric while the other possibility is \( H = \frac{R'}{R} \). Although these definitions coincide in the case of FRW, they are different for LTB. The corresponding non-vanishing components on the plus side are

\[
K_{\theta\theta} \big|_+ = \sin \theta \gamma_{\phi\phi} \big|_\Sigma = \frac{R}{2} (RH - \sqrt{1 - kr^2}) |_{\Sigma},
\]

\[
K_{tt} \big|_+ = H.
\]

The jump in the transverse extrinsic curvature across the null hypersurface, given by \( \gamma_{ab} = 2[K_{ab}] \), has the following non-zero components:

\[
\gamma_{\theta\theta} = \sin \theta \gamma_{\phi\phi} = 2R(\sqrt{1 + E} - \sqrt{1 - kr^2}) |_{\Sigma},
\]

\[
\gamma_{tt} = 2(H - H R') |_{\Sigma},
\]

where we have used Eq. (8).

### 3 Junction conditions

The surface energy-momentum tensor of the lightlike shell having the null hypersurface \( \Sigma \) as its history is directly related to the jump in the transverse extrinsic curvature. It can be expressed in the intrinsic coordinates \( \xi^a \) as follows \([6]\)

\[
S^{ab} = f n^a n^b + p g^{ab} + j^a n^b + j^b n^a,
\]

where

\[
f = -\frac{1}{16\pi} g^{ab} \gamma_{ab}
\]

represents the surface energy density,

\[
p = -\frac{1}{16\pi} \gamma_{ab} n^a n^b
\]

displays the isotropic surface pressure, and

\[
j^a = -\frac{1}{16\pi} g^{ac} \gamma_{cd} n^d
\]
represents the surface current of the lightlike shell. All these surface quantities are measured by a family of freely-moving observers crossing the null hypersurface. Using the jumps in the extrinsic curvature obtained above, we notice first that the surface current term given by (22) vanishes identically. From Eqs. (20) and (21) the energy density and pressure are then calculated as

\[ 16\pi f = -\frac{2}{R^2}|_{\Sigma} \gamma_{\theta\theta} = \frac{4}{R}(\sqrt{1 - kr_{+}^2} - \sqrt{1 + E})|_{\Sigma}, \]  \tag{23} 

\[ 16\pi p = -\gamma_{tt} = 2(HR' - H)|_{\Sigma}. \]  \tag{24} 

Assuming the positivity of the surface energy density of the shell given by (23) we see that the matching of an LTB metric along a null hypersurface to the FRW metric is possible only if

\[ E(r-) \leq -kr_{+}^2. \]  \tag{25} 

Taking into account that there is always \(-1 < E\), the following cases are possible:

\begin{align*}
  k &= 0 \quad \Rightarrow \quad E = 0 \quad \text{and} \quad -1 < E < 0, \quad \tag{26} \\
  k &= +1 \quad \Rightarrow \quad -1 < E \leq -r_{+}^2, \quad \tag{27} \\
  k &= -1 \quad \Rightarrow \quad E = 0, \quad -1 < E < 0 \quad \text{and} \quad 0 < E \leq r_{+}^2. \quad \tag{28}
\end{align*}

Note that all these constraints are valid on the light cone, i.e. both \(r_-\) and \(r_+\) may, in general, take different values between zero and infinity, except for the FRW case with \(k = +1\), where we must have \(0 < r_+ < 1\). Therefore, cases bounded by \(-1 < E \leq -r_{+}^2\) and \(0 < E < r_{+}^2\) may not be valid globally. We will call such cases restricted matching. Therefore, to have an unrestricted matching, the background FRW has to be flat or open. In each case the LTB may then be flat or closed. It should be noted that for the case of a marginally bound (flat) inhomogeneous patch \((E = 0)\) glued to a flat FRW background \((k = 0)\), using Eq. (23), we find that \(f = 0\) so that the null hypersurface is the history of a lightlike shell with the only possibility of admitting a surface pressure. From (24) we also see that the LTB Hubble function \(HR'\) may be bigger or smaller than the FRW Hubble parameter \(H\) on the hypersurface \(\Sigma\), depending on the surface pressure on \(\Sigma\) being negative or positive.

4 The case of gravitational impulsive wave

Gluing of manifolds along null hypersurfaces is a tricky action. Although similar expression to (19) in the case of time-like or space-like hypersurfaces includes all the information about the junction, this is not so for the null case. In general, there is a part of \(\gamma_{ab}\), denoted by \(\hat{\gamma}_{ab}\), which does not contribute to the expression (19) for the intrinsic energy-momentum tensor \(S^{ab}\) of the null shell, and is interpreted as an impulsive gravitational wave propagating independently of the null shell. The expression for \(\hat{\gamma}_{ab}\) is given by [7]

\[ \hat{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}g^{cd}\gamma_{cd}g_{ab} + 2n^{c}\gamma_{c(a}N_{b)} + \gamma_{cd}n^{c}n^{d}N_{a}N_{b}, \]  \tag{29} 

5
where \( N_a = N^{\mu} e^a_\mu = (-1, 0, 0) \). In our case, however, it can easily be shown that all components of \( \hat{\gamma}_{ab} \) vanish identically. This is a hint that there is no gravitational impulsive wave across the null hypersurface of the junction between LTB and FRW.

The absence of gravitational waves having the null hypersurface \( \Sigma \) as history can explicitly be seen in the following way. Let us first construct a null tetrad frame on \( \Sigma \). Consider a congruence of timelike geodesics with continuous 4-velocity \( u^\alpha \) across \( \Sigma \) so that \( [u_a u^a] = [u_\alpha e^\alpha_a] = 0 \). On the null hypersurface \( \Sigma \) we have the normal \( n^\mu \), which is tangential to \( \Sigma \), and the timelike vector field \( u^\mu \) tangent to the matter world lines in the LTB manifold and FRW background crossing the null hypersurface such that \( u^\alpha n_\alpha = -s < 0 \). It is then advantageous to introduce on \( \Sigma \) a transverse null vector field \( l^\mu \) defined by

\[
l^\mu = \frac{1}{2} \left( 1, -\frac{\sqrt{1+E}}{R'}, 0, 0 \right) |_{\Sigma}.
\]

Using this null tetrad, the Newman-Penrose component of the singular part of the Weyl tensor of Petrov type N characterizing an impulsive gravitational wave with history \( \Sigma \) is calculated as

\[
\tilde{\Psi}_4 = \frac{1}{2} \gamma_{ab} \bar{m}^a \bar{m}^b,
\]

where we have used Eqs. (17) and (32). This shows explicitly that there is no impulsive gravitational wave present and the null hypersurface \( \Sigma \) is just the history of a lightlike shell of matter being characterized in general by the surface energy density and isotropic pressure given by Eqs. (23), and (24), respectively. In this case the induced geometry on \( \Sigma \), inherited from the embedding spacetimes, is of type I according to a classification introduced by Penrose [8, 9].

The expansion \( \theta \) and complex shear \( \sigma \) of the geodesic generators of the null hypersurface \( \Sigma \) can now be defined by the following relations using the null tetrad [8]:

\[
\theta = m^\mu \bar{m}^\nu \nabla_\nu n_\mu = \frac{1}{R} (RH + \sqrt{1-kr^2_{+}}) |_{\Sigma},
\]

\[
\sigma = m^\mu m^\nu \nabla_\nu n_\mu = 0.
\]
The LTB inhomogeneous manifold is, therefore, joined to the background FRW universe through a shear-free, expanding null hypersurface $\Sigma$.

We have, therefore, shown that the matching of a LTB metric to an arbitrary FRW metric is possible given the constraint junction (25) is satisfied. Although the hypersurface of junction is in general supporting energy and pressure, there is no impulsive gravitational wave with the history $\Sigma$, which turns out to be shear-free but expanding.

5 Application to Special Cases

We have already seen that not all cases of matching a LTB to FRW along a null hypersurface is possible. Now, we would like to check different interesting cosmological cases which are possible, depending on the different forms of the energy momentum tensor on the hypersurface.

i) Pressureless hypersurface

Let us here consider the case that there is no surface isotropic pressure. Then it follows from (24) that there is no jump in the Hubble function across $\Sigma$: $H = H_R'$. In this case the energy density $f$ is the only non-vanishing surface quantity due to the presence of a lightlike shell of matter with the history $\Sigma$. Now, we have $\gamma_a = \gamma_{ab} n^b = \gamma_{\mu} = 0$. Therefore, the induced geometry on $\Sigma$ is of type III, according to the Penrose classification of induced geometries on $\Sigma$ [8, 9]. We may now write down the junction equation relating the properties of $\Sigma$ to those of outside medium described by the energy-momentum tensor $T_{\mu\nu}^\pm$ [8]:

$$8\pi \left[T_{\mu\nu} n^\mu n^\nu\right]_{\Sigma} = \theta \gamma^\dagger,$$

where $\gamma^\dagger = \gamma_a n^a = 0$ since the induced geometry on $\Sigma$ is of type III. Then by noting that LTB and FRW are perfect fluid spacetimes we see that Eq. (36) is reduced to the form

$$\rho_L \Sigma = \rho_b + p_b,$$

where $\rho_L$ and $\rho_b$ are the energy densities in LTB and FRW spacetimes respectively, and $p_b$ is the fluid pressure in FRW universe. Particularly, in the case of the pressure-free FRW universe we see that having a pressureless shell on the null hypersurface of junction requires that the jump in the energy density across $\Sigma$ vanish.

ii) Vanishing the surface energy-momentum tensor: boundary layer

Looking now for a special case of great cosmological interest, i.e. smooth matching of an inhomogeneous patch to a homogeneous FRW background, let in addition to the surface pressure $p$, the surface energy density $f$ vanish too. From (25) this requires that $E(r_-) \Sigma = -kr_R^2$. Then in this case there would be neither a surface energy-momentum tensor on $\Sigma$ nor the impulsive gravitational wave. We therefore conclude that the null hypersurface $\Sigma$ is just a smooth boundary, like our past like cone in observational cosmology. On the other hand, since $\gamma_{\theta\theta} = 0$ and
using Eqs. (17) and (3), we find that

\[ H_\Sigma = H_R = H_{R'}, \]  

(38)

which is an interesting result which has been verified numerically \[2\]. It says that for the case of a smooth matching of a LTB metric to a FRW background, the LTB Hubble functions on \( \Sigma \) related to the metric functions \( R \) or \( R' \) are identical to the corresponding FRW Hubble function. We may therefore define a Hubble parameter without any ambiguity. These results are valid for any open or flat model, and in the restricted matchings, also for closed ones.

\section*{iii) Junction of the flat-flat case along a null boundary hypersurface}

A more closed look at the relation (23) shows that a flat or open FRW (LTB) can only be matched smoothly to a flat or open LTB (FRW), respectively. Therefore, if one is going to model the actual local inhomogeneities of the universe using a LTB part joined to a flat FRW along the past light cone, then it must also be flat, otherwise there will be a null matter shell with the history \( \Sigma \).

The relation between \( r_- \) and \( r_+ \) is an interesting question giving an insight in the LTB metric, and best studied in the flat-flat case. From (7) we infer for the flat-flat case with \( E(r) = 0 = k \)

\[ \frac{dr_+}{dr_-} = \frac{R'}{a}|_{\Sigma}. \]  

(39)

Now, let us see if one may have \( r_+ = r_- \) on the null hypersurface. For this to be the case one must have \( R'(r_-, t) \mid_\Sigma = a(t) \). We know, however, that for the LTB solution in the flat case the metric function is given by

\[ R(r_-, t) = \left( \frac{4\pi}{3} \rho_c \right)^{1/3} r_- (t - t_n(r_-))^{2/3}, \]  

(40)

where \( \rho_c \) is just a constant and the \( t_n \) is the so-called bang time \[1, 2\]. Using this, from the above condition, we obtain \( t_n |_{\Sigma} = 0 \). Therefore, the derivative of the bang time must be zero all along the null hypersurface, i.e. for all \( r_- \). We then conclude that the bang time must be a constant. This is equivalent to reducing LTB to a FRW metric. Hence, for a flat LTB to be glued to a flat FRW the comoving coordinate \( r \) can not be the same on different sides of the hypersurface. This may also be stated in the following way: although on the null hypersurface the relation \( R(r_-, t) \mid_\Sigma = a(t) r_+ \) is valid, we always have \( R'|_{\Sigma} \neq a(t) \), except for the case of LTB being reduced to FRW. This has explicitly been shown for some special cases in \[2\]. Therefore, for the junction of LTB to FRW, we always have \( r_- \neq r_+ \) on the null hypersurface. Obviously, this fact being proved for the flat-flat case, is also true in general.

\section*{6 Conclusion}

We have examined the matching of a LTB inhomogeneous solution to a background FRW universe across a null hypersurface. In general, the null hypersurface is the history of a lightlike
matter shell having surface energy density and isotropic pressure but no impulsive gravitational wave. We have shown that the assumption of positivity of the surface energy density of the null shell imposes the criterion \(^{[25]}\) for gluing a LTB metric to the FRW universe, with the result that an unrestricted matching is only possible if the background FRW is open or flat.
If the surface pressure on the null hypersurface vanishes, then the jump of the energy density across the null hypersurface is related to the fluid pressure of the background FRW universe. The case of vanishing both surface energy density and pressure on the null hypersurface leads to the interesting result that the FRW Hubble parameter and both definitions of the LTB Hubble parameters are equivalent. Therefore, in the case of a surface boundary junction a Hubble parameter may be defined without ambiguity. It has also been shown that a flat FRW can only be joined to a flat LTB along a null boundary.

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