Solvable Lie algebras are not that hypo

Diego Conti, Marisa Fernández and José A. Santisteban

November 2, 2010

Abstract

We study a type of left-invariant structure on Lie groups, or equivalently on Lie algebras. We introduce obstructions to the existence of a hypo structure, namely the 5-dimensional geometry of hypersurfaces in manifolds with holonomy SU(3). The choice of a splitting $g^* = V_1 \oplus V_2$, and the vanishing of certain associated cohomology groups, determine a first obstruction. We also construct necessary conditions for the existence of a hypo structure with a fixed almost-contact form. For non-unimodular Lie algebras, we derive an obstruction to the existence of a hypo structure, with no choice involved. We apply these methods to classify solvable Lie algebras that admit a hypo structure.

MSC classification: Primary 53C25; Secondary 53C15, 17B30, 53D15

Introduction

In [5], Salamon and the first author of the present work introduced hypo structures, namely the SU(2)-structures induced naturally on orientable hypersurfaces of Calabi-Yau manifolds of (real) dimension 6. They are defined as follows. An SU(2)-structure on a five-manifold is an almost-contact metric structure with additionally a reduction from the structure group SO(4) to SU(2); such a structure is entirely determined by the choice of differential forms $(\alpha, \omega_1, \omega_2, \omega_3)$, where $\alpha$ is the almost-contact 1-form and the $\omega_i$ are pointwise a distinguished orthonormal basis of $\Lambda^2_+ (\ker \alpha)$, which implies that the quadruplet $(\alpha, \omega_1, \omega_2, \omega_3)$ satisfies certain relations (see Section 1 Proposition 3). Since SU(2) is the stabilizer of a point under the action of SU(3) on $\mathbb{R}^6$, hypersurfaces in manifolds with holonomy contained in SU(3) or, equivalently, with an integrable SU(3)-structure, inherit a natural SU(2)-structure.

In fact, if $M$ is a Riemannian 6-manifold with holonomy contained in SU(3), then $M$ has a Hermitian structure, with Kähler form $F$, and a complex volume form $\Psi = \Psi_+ + i \Psi_-$, satisfying $dF = 0 = d\Psi$. Therefore, if $N \subset M$ is an orientable hypersurface, and $U$ is the unit normal vector field, the SU(3)-structure induces an SU(2)-structure $(\alpha, \omega_1, \omega_2, \omega_3)$ on $N$ defined by

$$\alpha = -U \cdot F, \quad \omega_1 = f^* F, \quad \omega_2 = U \cdot \Psi_-, \quad \omega_3 = -U \cdot \Psi_+,$$
where $f: N \to M$ is the inclusion.

The integrability condition on the ambient manifold immediately gives
\[ d\omega_1 = 0, \quad d(\omega_2 \wedge \alpha) = 0, \quad d(\omega_3 \wedge \alpha) = 0. \]

An SU(2)-structure satisfying this condition is called a hypo structure. Such a structure can also be characterized in terms of generalized Killing spinors, or by the condition that the intrinsic torsion is a symmetric tensor, which turns out to coincide with the second fundamental form of the hypersurface. In this sense, hypo geometry is the five-dimensional analogue of half-flat geometry in dimension six (see [3, 10]). Indeed, much as in the half-flat case, any real-analytic hypo manifold can be immersed isometrically in a Riemannian manifold with holonomy contained in SU(3), so as to invert the construction outlined above, and the immersion can be determined explicitly by solving certain evolution equations ([5]).

In order to construct examples of hypo structures, a natural place to look is left-invariant structures on 5-dimensional Lie groups. In the analogous half-flat case, this was the approach of [4, 2, 6, 7], focusing on the nilpotent case, and more recently of [12], considering products of three-dimensional Lie groups. In five-dimensions, only 9 isomorphism classes of nilpotent Lie groups exist, of which exactly six admit a hypo structure [5]. If one considers solvable Lie groups, things become more complicated. By Mubarakzyanov’s classification [11], there are 66 families of solvable Lie algebras of dimension 5, some of which depend on parameters; we refer to the comprehensive list of [1]. It was shown in [9] that precisely 35 out of these 66 families admit an invariant contact structure, at least generically (i.e., for generic values of the parameters). Moreover, without using Mubarakzyanov’s classification, it was proved in [8] that only 5 of the 66 admit a hypo-contact structure, namely a hypo structure $(\alpha, \omega_i)$ such that the underlying almost-contact metric structure is a contact metric structure.

In this paper we introduce some obstructions to the existence of a hypo structure on a Lie algebra, and use them to classify solvable Lie algebras with a hypo structure. The first obstruction follows a construction of [7]. One considers a splitting $\mathfrak{g}^* = V_1 \oplus V_2$, where $V_1$ has dimension two. This determines a doubly graded vector space $\Lambda^* \mathfrak{g}^* = \bigoplus \Lambda^{p,q}$, which is made into a double complex if
\[ d(\Lambda^{p,q}) \subset \Lambda^{p+2,q-1} \oplus \Lambda^{p+1,q}. \]

The double complex has an associated spectral sequence that collapses at the second step. If $H^{0,3} = H^{3,0} = 0$ and $H^{0,2} = H^{2,0} = 0$ are zero, relative to some choice of the splitting, then no hypo structure exists (see Proposition 3). In fact, the key property is
\[ Z^k \subset \Lambda^{2,k-2} \oplus \Lambda^{1,k-1}, \quad k = 2, 3, \]
where $Z^k$ denotes the space of closed $k$-forms; this condition does not require [1], whose main relevance is in giving a cohomological interpretation. This obstruction applies to 27 indecomposable Lie algebras and 10 decomposable Lie
algebras, at least generically, where decomposable means isomorphic to a direct sum of ideals.

A second set of obstructions comes from the fact that if \((\alpha, \omega_i)\) is a hypo structure on a Lie algebra \(g\), then the forms \(\omega_2 \wedge \alpha, \omega_3 \wedge \alpha\) lie in the space

\[
V = \{\gamma \in \Lambda^3 \mathfrak{g}^* \mid \gamma \wedge \alpha = 0, d\gamma = 0\}.
\]

If, for some \(\beta \in \mathfrak{g}^*\), either the space \(V \wedge \beta \subset \Lambda^4 \mathfrak{g}^*\) has dimension one or

\[
\dim(V \wedge \beta) = 2, \quad Z^2 \wedge \alpha \wedge \beta \subset V \wedge \beta, \quad (2)
\]

then necessarily \(\alpha\) and \(\beta\) are linearly dependent. This is an obstruction to the existence of a hypo structure with a fixed \(\alpha\) (see Proposition 4), but it can be combined with other arguments to prove that no hypo structure exists on a Lie algebra.

Indeed, we show that if a non-unimodular Lie algebra \(g\) has a hypo structure \((\alpha, \omega_i)\), then the 1-form \(\beta \in \mathfrak{g}^*\) defined by \(\beta(X) = \text{tr} \text{ad}(X)\) is orthogonal to \(\alpha\); this gives a canonical choice for \(\beta\) in (2). Explicitly, in Proposition 6, we prove that there is no hypo structure if either \(Z^3 \wedge \beta\) has dimension less than two, or

\[
\dim(Z^3 \wedge \beta) = 2, \quad \alpha \wedge \beta \wedge Z^2 \subset Z^3 \wedge \beta
\]

for any \(\alpha \in \mathfrak{g}^*\) such that \(\alpha \wedge \beta \wedge Z^3 = 0\). This obstruction applies to 6 indecomposable families and 12 decomposable families.

On the other hand, even for unimodular Lie algebras, the structure of the space of closed 3-forms may give restrictions on \(\alpha\) (see Lemma 10), which together with (2) enable one to show that certain Lie algebras have no structure. This obstruction accounts for 6 indecomposable families.

Finally, for 2 indecomposable families and one decomposable Lie algebra, we use the trivial fact that the space \((Z^3)^2 \wedge \alpha\) is non-zero, as it contains \((\omega_1)^2 \wedge \alpha \neq 0\).

Having obtained the classification, we can ask how often a solvable Lie algebra is hypo. We know from [5] that the answer is 6 times out of 9 for nilpotent Lie algebras. In fact, we obtain a shorter proof of this result, namely that the nilpotent Lie algebras denoted here by \(D_3, A_{5,3}, A_{4,1} \oplus \mathbb{R}\) have no hypo structure.

In the solvable case, the question is somewhat ambiguous, because the Lie algebras come in families. With reference to Mubarakzyanov’s list, it turns out that, given a family with more than one element, the subset of Lie algebras that have a hypo structure is always a proper subset, but not always discrete. This suggests recasting the question in the following form: how many families in Mubarakzyanov’s list of solvable Lie algebras contain at least one hypo Lie algebra? The answer is 21 out of 66, so the ratio is considerably less than in the nilpotent case.

If we further distinguish according to whether a family of Lie algebras is decomposable and whether it is generically contact, we obtain the following
For instance, the top-left entry states that of the 24 families in Mubarakzyanov’s list which are indecomposable and have a contact structure for generic choice of the parameters, precisely 7 have a hypo structure for some choice of the parameters.

If we count Lie algebras with a hypo structure rather than families, we obtain the following table:

|         | generically contact | non-contact | all  |
|---------|---------------------|-------------|------|
| indecomposable | 7/24                | 9/15        | 16/39|
| decomposable   | 1/11                | 4/16        | 5/27 |
| all            | 8/35                | 13/31       | 21/66|

Thus, there are exactly ten solvable Lie algebras that have both a hypo and a contact structure, for half of which the structures can be chosen to be compatible [8].

Finally, we point out that there are only five non-unimodular hypo Lie algebras, contained in three families, all of them indecomposable and contact.

1 A first obstruction

In this section we introduce an obstruction to the existence of a hypo structure on a 5-dimensional Lie algebra. This obstruction is given in terms of the cohomology groups of a certain double complex associated to any n-dimensional Lie algebra.

Let \( \mathfrak{g} \) be an \( n \)-dimensional Lie algebra, and denote by \( d \) the Chevalley-Eilenberg differential on the dual \( \mathfrak{g}^* \). A coherent splitting of \( \mathfrak{g} \) is a splitting \( \mathfrak{g}^* = V_1 \oplus V_2 \), where \( V_1 \) and \( V_2 \) are vector spaces, \( \dim V_1 = r \geq 2 \) and

\[
d(V_1) \subset \Lambda^2 V_1, \quad d(V_2) \subset \Lambda^2 V_1 + V_1 \wedge V_2.
\]

Let \( \Lambda^{p,q} \) be the natural image of \( \Lambda^p V_1 \otimes \Lambda^q V_2 \) in \( \Lambda^{p+q} \mathfrak{g}^* \), with the convention that \( \Lambda^{p,q} = 0 \) whenever \( p \) or \( q \) is negative. A coherent splitting determines a double complex \( (\Lambda^{*,*}, \delta_1, \delta_2) \), \( \delta_1, \delta_2 \) being the operators:

\[
d_1 : \Lambda^{p,q} \longrightarrow \Lambda^{p+1,q}, \quad d_2 : \Lambda^{p,q} \longrightarrow \Lambda^{p+2,q-1}, \quad d = \delta_1 + \delta_2.
\]

They satisfy

\[
d_1^2 = 0 = d_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1.
\]

For any choice of coherent splitting on \( \mathfrak{g} \), we can define the cohomology groups \( H^{p,q}(\mathfrak{g}, V_1) \) as follows (see also [7]). For each \( k \geq 0 \) we define a filtration

\[
\Lambda^{r,k-r} \subset \Lambda^{r,k-r} + \Lambda^{r-1,k-r+1} + \Lambda^{r-2,k-r+2} \subset \cdots \subset \Lambda^{r,k-r} + \Lambda^{r-1,k-r+1} + \cdots + \Lambda^0, k = 1.
\]

4
Notice that in (3), the space $\Lambda_{p,k}^p$ is zero if $p > k$ or $p > r$. We denote by $Z^k \subset \Lambda^k$ the space of closed invariant $k$-forms. Taking the intersection with $Z^k$, the filtration (3) determines the filtration

$$Z_r^k \subset Z_{r-1}^k \subset Z_{r-2}^k \subset \cdots \subset Z_0^k = Z^k,$$

and taking the quotient by the $d$-exact forms, we obtain yet another filtration

$$H_r^k \subset H_{r-1}^k \subset H_{r-2}^k \subset \cdots \subset H_0^k = H^k.$$

We can now define the cohomology groups

$$H^{p,q}(g, V_1) = H^{p+q}_p \oplus H^{p+q}_{p+1} \oplus \cdots \oplus H^{p+q}_0 = H^{p,q}.$$

The notation is justified by the fact that whilst the spaces $\Lambda^{p,q}$ depend on both $V_1$ and $V_2$, the filtration (3), and therefore the cohomology groups, depend only on $V_1$. We define

$$h^{p,q}(g, V_1) = \dim H^{p,q}(g, V_1).$$

We can think of a coherent splitting as defined by a decomposable form which spans $\Lambda^r V_1$.

**Lemma 1.** Let $g$ be a Lie algebra of dimension $n$, and let $\phi$ be a decomposable $r$-form. Then $\phi$ defines a coherent splitting $g^* = V_1 \oplus V_2$, with $\dim V_1 = r$, if and only if

- $d\alpha \wedge \phi = 0$ for all $\alpha \in g^*$;
- $d\phi = 0$;
- $\mathcal{L}_X \phi$ is a multiple of $\phi$ for all $X$ in $g$, where $\mathcal{L}$ denotes the Lie derivative.

**Proof.** Given a coherent splitting with $\Lambda^r V_1$ generated by $\phi$, we have

$$d\phi \in \Lambda^{r+1,0} = \{0\} \quad \text{and} \quad d\alpha \wedge \phi \in \Lambda^{r+2,0} + \Lambda^{r+1,1} = \{0\}, \quad \text{for } \alpha \in \Lambda^1.$$

Also, since $\phi$ is closed, $\mathcal{L}_X \phi = d(X \cdot \phi) \in \Lambda^{r,0}$ (where $X \cdot$ denotes the contraction by $X$) which is spanned by $\phi$.

To prove the converse, let $\phi = \alpha^1 \wedge \cdots \wedge \alpha^r$, and complete $\alpha^1, \ldots, \alpha^r$ to a basis $\alpha^1, \ldots, \alpha^r, \beta^1, \ldots, \beta^{n-r}$. The first condition implies that the image of $d: \Lambda^1 \to \Lambda^2$ is contained in $\Lambda^{2,0} + \Lambda^{1,1}$. All we need to check in order to have a coherent splitting is that $d\alpha^j$ has type $(2,0)$. Suppose otherwise. Then

$$(d\alpha^j)^{1,1} = a_{jh}^i \beta_h \wedge \alpha_j.$$

Now since $\phi$ is closed, $d(X \cdot \phi) = \mathcal{L}_X \phi$ is a multiple of $\phi$ by hypothesis. So we have that

$$0 = (d(\alpha^1 \wedge \cdots \wedge \alpha^r))^{r-1,1} = \sum_{j \neq i,h} (-1)^{j+i-1} a_{jh}^i \beta_h \wedge \alpha^1 \wedge \cdots \wedge \alpha^j \wedge \cdots \wedge \alpha^r + \sum_{j \neq i,h} a_{jh}^i \beta_h \wedge \alpha^1 \wedge \cdots \wedge \alpha^i \wedge \cdots \wedge \alpha^r.$$
Hence \( a_{ih}^j = 0 \) for all \( i \neq j \) and \( h \), and \( \sum_{j \neq i} a_{jh}^j = 0 \) for all \( i, h \). Therefore \( a_{jh}^i = 0 \) for all \( i, j, h \).

We introduce the following notation. Let \( D_j \) be the annihilator of the kernel of \( d: \Lambda^j g^* \rightarrow \Lambda^{j+1} g^* \). In other words, if \( v^1, \ldots, v^n \) is a basis of the vector space \( \Lambda^{j+1} g^* \) dual to \( \Lambda^j g^* \), then \( D_j \) is spanned by the \( v_j \circ d \). Likewise, for any \( \phi \in \Lambda^k g^* \), let \( L^\phi_j \) be the annihilator of the kernel of the map

\[
\Lambda^j g^* \rightarrow \Lambda^{j+k} g^*, \quad \alpha \mapsto \alpha \wedge \phi.
\]

We can then give a specialized version of the lemma that accounts for the vanishing of certain cohomology groups. In the five-dimensional case we get:

**Proposition 2.** Let \( g \) be a 5-dimensional Lie algebra. Then \( g \) has a coherent splitting, with \( \dim V_1 = 2 \) and \( H^{0,2} = 0 = H^{0,3} \), if and only if there exists a nonzero 2-form \( \phi \) such that

- \( \phi \wedge \phi = 0 \);
- \( d\phi = 0 \);
- \( L_X \phi \) is a multiple of \( \phi \) for all \( X \) in \( g \);
- \( L^\phi_2 \subset D_2, L^\phi_3 \subset D_3 \).

**Proof.** Given a coherent splitting, it is clear that exact \( k \)-forms have no component in \( \Lambda^{0,k} \). Moreover, the condition \( H^{0,k} = 0 \) is equivalent to \( Z^k \) being contained in \( \Lambda^{k,0} + \cdots + \Lambda^{1,k-1} \). Thus, \( L^\phi_k \subset D_k \) if and only if \( H^{0,k} = 0 \).

Conversely, a 2-form \( \phi \) such that \( \phi \wedge \phi = 0 \) is decomposable, and therefore determines a splitting. If \( \phi \) is as in the hypothesis, the splitting is coherent because \( L^\phi_2 \subset D_2 \) implies that closed 2-forms, and in particular exact 2-forms, have no component in \( \Lambda^{0,2} \); therefore, \( d\alpha \wedge \phi = 0 \) for all \( \alpha \in g^* \) and Lemma 1 applies.

**Remark.** In the proof of Proposition 2 we can suppose that \( g \) has a coherent splitting, with \( \dim V_1 = r \geq 2 \), and conclude that \( H^{0,k} = 0 \) is equivalent to \( L^\phi_k \subset D_k \), since this works for any dimension \( n \) of \( g \) and for all values of \( r \). However, we need \( r = 2 \) to have that a 2-form \( \phi \) is decomposable if and only if \( \phi \wedge \phi = 0 \).

From now on, given a 5-dimensional Lie algebra \( g \) whose dual is spanned by \( \{ e^1, \ldots, e^5 \} \), we will write \( e^{ij} = e^i \wedge e^j \), \( e^{ijk} = e^i \wedge e^j \wedge e^k \), and so forth.

The relevance of the above proposition comes from **hyp geometry**. First we recall some facts about SU(2)-structures on a 5-manifold. (For more details, we refer to [5].) Let \( N \) be a 5-manifold and let \( L(N) \) be the principal bundle of linear frames on \( N \). An **SU(2)-structure** on \( N \) is an SU(2)-reduction of \( L(N) \). We have the following (see [5, Proposition 1]):

\[\text{Lemma 1:}\]
Proposition 3. SU(2)-structures on a 5-manifold $N$ are in one-to-one correspondence with quadruplets $(\alpha, \omega_1, \omega_2, \omega_3)$, where $\alpha$ is a 1-form and $\omega_i$ are 2-forms on $N$ satisfying at each point $\omega_i \wedge \omega_j = \delta_{ij} v$, $v \wedge \alpha \neq 0$ for some 4-form $v$, and

$$i_X \omega_1 = i_Y \omega_2 \Rightarrow \omega_3(X,Y) \geq 0,$$

where $i_X$ denotes the contraction by $X$.

Moreover, we need recall the following definition.

Definition 4. An SU(2)-structure $(\alpha, \omega_1, \omega_2, \omega_3)$ on a 5-manifold $N$ is said to be hypo if

$$d(\omega_2 \wedge \alpha) = d(\omega_3 \wedge \alpha) = d\omega_1 = 0. \quad (4)$$

Therefore, to a choice of a coframe $f^1, \ldots, f^5$ on a Lie algebra $g$, we associate an SU(2)-structure given by

$$\alpha = f^5, \quad \omega_1 = f^{12} + f^{34}, \quad \omega_2 = f^{13} + f^{42}, \quad \omega_3 = f^{14} + f^{23}, \quad (5)$$

and it is called hypo if $\omega_1$, $\omega_2 \wedge \alpha$, $\omega_3 \wedge \alpha$ are closed.

Definition 5. Let $f^1, \ldots, f^5$ be a coframe on a Lie algebra $g$ such that the quadruplet $(\alpha, \omega_1, \omega_2, \omega_3)$ given by (5) defines a hypo structure on $g$. Then, the coframe $f^1, \ldots, f^5$ is said to be adapted to the hypo structure $(\alpha, \omega_1, \omega_2, \omega_3)$.

Proposition 6. If $g$ has dimension 5, and there exists a coherent splitting $g^* = V_1 \oplus V_2$ with $\dim V_1 = 2$ and $h^{0,3}(g, V_1) = 0 = h^{0,2}(g, V_1)$, then there is no hypo structure.

Proof. Let $(\alpha, \omega_i)$ be a hypo structure, and let $\phi$ be a generator of $\Lambda^{2,0}$. We know that the forms $\omega_1$, $\alpha \wedge \omega_2$, $\alpha \wedge \omega_3$ are closed. Moreover, because $h^{0,3}(g, V_1) = 0 = h^{0,2}(g, V_1)$, we have

$$\phi \wedge \omega_1 = 0, \quad \phi \wedge (\alpha \wedge \omega_i) = 0.$$

If we decompose the space of two-forms on $\mathbb{R}^5$ according to

$$\Lambda^2 \mathbb{R}^5 = \alpha \wedge \Lambda^1 \mathbb{R}^4 \oplus \Lambda_2^2 \mathbb{R}^4 \oplus \Lambda_2^2 \mathbb{R}^4,$$

we find that $\phi$ must lie in $\Lambda_2^2 \mathbb{R}^4$. Since $\phi^2 = 0$, this implies $\phi = 0$. $\Box$

Remark. Strictly speaking Proposition 6 does not use the fact that the splitting is coherent, but only the conditions $Z^3 \wedge \phi = 0$, $Z^2 \wedge \phi = 0$; or, in the language of Proposition 2, the inclusions $L_3^3 \subseteq D_2$, $L_2^3 \subseteq D_3$. Indeed it is sometimes the case that a splitting with this property exists without being coherent. Consider for instance the Lie algebra

$$g = (e^{13}, e^{34}, -e^{24}, 0, 0)$$
(by this notation we mean that the dual $\mathfrak{g}^*$ has a fixed basis $e^1, \ldots, e^5$, such that
\[ de^1 = e^{13}, \quad de^2 = e^{34}, \quad de^3 = -e^{24}, \quad de^4 = 0 = de^5. \]
Then $e^{24}$ defines a splitting with $Z^2 \wedge \phi = Z^2 \wedge \phi = 0$, and yet this is not coherent. On the other hand, a different obstruction applies to this case (see Proposition 9 below).

2 Other obstructions

When looking at 5-dimensional solvable Lie algebras, the coherent splitting obstruction, shown in Proposition 3, is sometimes not sufficient to determine whether a hypo structure exists. In this section we describe two different obstructions that can be used in these cases.

For every 1-form $\gamma$, let $L_\gamma : \Lambda^j \to \Lambda^{j+1}$ be the map given by $L_\gamma(\eta) = \gamma \wedge \eta$.

Proposition 7. Let $\alpha, \beta$ be linearly independent one-forms on a Lie algebra $\mathfrak{g}$, and set $V = \ker L_\alpha \cap Z^3$. Suppose that either
\begin{itemize}
  \item $\dim L_\beta(V) < 2$; or
  \item $\dim L_\beta(V) = 2$ and $L_\alpha(L_\beta(Z^2)) \subset L_\beta(V)$.
\end{itemize}
Then there is no hypo structure on $\mathfrak{g}$ of the form $(\alpha, \omega_i)$ (in the sense that its almost-contact form is $\alpha$ itself).

Proof. Suppose for a contradiction that a hypo structure $(\alpha, \omega_i)$ exists, and let $f^1, \ldots, f^5$ be an adapted coframe. Up to rescaling the metric and up to SU(2) action, we can assume that $\beta = f^1 + af^5$, with $a$ a constant. Then $\omega_2 \wedge \alpha, \omega_3 \wedge \alpha$ lie in $V$ and
\[ L_\beta(V) \ni L_\beta(\omega_2 \wedge \alpha) = f^{1425}, \quad L_\beta(V) \ni L_\beta(\omega_3 \wedge \alpha) = f^{1235}. \]
So $\dim L_\beta(V) \geq 2$, and if equality holds then $L_\beta(V)$ is spanned by $f^{1425}, f^{1235}$. But then
\[ L_\alpha(L_\beta(Z^2)) \ni \alpha \wedge \beta \wedge \omega_1 = f^{5134}, \]
which cannot lie in $L_\beta(V)$.

For non-unimodular Lie algebras, it turns out that we have a canonical choice for $\beta$:

Lemma 8. Let $\mathfrak{g}$ be a non-unimodular Lie algebra and let $\beta \in \mathfrak{g}^*$ be the form corresponding to the linear map $\mathfrak{g} \to \mathbb{R}$, $X \to \text{tr} \text{ad} X$. If $\mathfrak{g}$ has a hypo structure $(\alpha, \omega_i)$, then $\alpha$ and $\beta$ are orthogonal with respect to the underlying metric.
Proof. In an adapted coframe \( f^1, \ldots, f^5 \), with dual frame \( f_1, \ldots, f_5 \), \( \alpha = f^5 \), we have
\[
df^{1234} = \sum_k f^k([f_k, f_5]) f^{12345} = -\beta(f_5) f^{12345}.
\]
However, since \( \omega_1 \) is closed, the left-hand side is zero and so \( \beta(f_5) = 0 \).

Thus, in the non-unimodular case Proposition 7 gives a fairly straightforward criterion:

**Proposition 9.** Let \( \mathfrak{g} \) be a non-unimodular Lie algebra, and let \( \beta \in \mathfrak{g}^* \) be the form corresponding to the linear map \( \mathfrak{g} \to \mathbb{R}, X \to \text{tr} \ ad X \). Suppose that either
- \( \dim L_\beta(Z^3) < 2 \); or
- \( \dim L_\beta(Z^3) = 2 \) and for every \( \alpha \in \mathfrak{g}^* \) such that \( L_\alpha(L_\beta(Z^3)) = 0 \),
  \( L_\alpha(L_\beta(Z^2)) \subset L_\beta(Z^3) \).

Then \( \mathfrak{g} \) has no hypo structure.

**Proof.** Suppose \( \mathfrak{g} \) has a hypo structure \((\alpha, \omega_i)\). By Lemma 8 we know that \( \alpha \) and \( \beta \) are linearly independent. Consider the space \( V = \ker L_\alpha \cap Z^3 \). If \( L_\beta(Z^3) \) has dimension two, then \( L_\beta(V) \subseteq L_\beta(Z^3) \) may only have dimension two if equality holds, implying \( L_\alpha(L_\beta(Z^3)) = 0 \). Then the statement follows from Proposition 7.

In order to apply Proposition 7 effectively, one needs information on what the 1-form \( \alpha \) can be. The condition of Lemma 8 is often useful but not always sufficient, since in practice it only tells us that \( \alpha \) and \( \beta \) are linearly independent; moreover, it does not apply to unimodular Lie algebras (for which \( \beta \) is zero). The following result gives useful restrictions on the 1-form \( \alpha \); it is labeled a lemma because we view it as an aid toward the application of either Proposition 7 or Proposition 9.

**Lemma 10.** Suppose \( \mathfrak{g} \) has a hypo structure \((\alpha, \omega_i)\). If \( \dim(X \lrcorner Z^3) \land \gamma < 2 \), where \( X \in \mathfrak{g} \) and \( \gamma \in \mathfrak{g}^* \), then \( \alpha(X) = 0 \).

**Proof.** Suppose otherwise; fix an adapted coframe \( f^1, \ldots, f^5 \) with dual frame \( f_1, \ldots, f_5 \). Then \( X = af_1 + f_5 \) up to a multiple and \( SU(2) \) action. Therefore \( X \lrcorner Z^3 \) contains
\[
X \lrcorner (f^{135} + f^{425}) = f^{13} + f^{42} + af^{35},
\]
\[
X \lrcorner (f^{145} + f^{235}) = f^{14} + f^{23} + af^{45}.
\]
Now by hypothesis some linear combination
\[
\delta = \lambda(f^{13} + f^{42} + af^{35}) + \mu(f^{14} + f^{23} + af^{45})
\]
gives zero on wedging with \( \gamma \). But
\[
(\lambda(f^{13} + f^{42} + af^{35}) + \mu(f^{14} + f^{23} + af^{45}))^2 = 2(\lambda^2 + \mu^2) f^{1234} + \text{other terms},
\]
which is nonzero. By non-degeneracy \( \delta \land \gamma \neq 0 \), which is absurd. 

9
Remark. Regardless of Proposition 7, this lemma may have more immediate applications. Indeed, a hypo structure \((\alpha, \omega_i)\) always satisfies
\[
0 \neq (\omega_1)^2 \land \alpha \in (\mathbb{Z}^2)^2 \land \alpha.
\] (6)

3 Diatta’s algebras

Let us recall firstly that a contact form \(\eta\) on a five-dimensional Lie algebra \(g\) is a 1-form on \(g\) (that is, \(\eta \in g^*\)) such that
\[
\eta \land (d\eta)^2 \neq 0.
\]

The existence of a hypo structure on \(g\) is independent of the existence of a contact form. In fact, in this section we will consider indecomposable solvable Lie algebras of dimension 5 having a contact form \(\eta\), and we will see that many of those Lie algebras do not admit a hypo structure. Notice that we are not requiring that the almost-contact 1-form \(\alpha\) associated to the hypo structure coincide with the contact form \(\eta\).

In [9], Diatta gives a list of 24 (families of) indecomposable five-dimensional solvable Lie algebras \(D_1, \ldots, D_{24}\) that admit a left-invariant contact 1-form. They correspond to the algebras \(A_{5,k}\) of [1] under
\[
D_k \rightarrow \begin{cases} 
    A_{5,k+3} & k = 1, 2, 3 \\
    A_{5,k+15} & 4 \leq k \leq 24
\end{cases}.
\]

We shall use the notation \(D_k(p_1, \ldots, p_n)\) to denote special instances of a family for assigned values of the parameters. Notice that Diatta’s list, as well as the one in [1] from which it was extracted, contains conditions on the parameters. We shall ignore these conditions to keep things simpler. This has two consequences: first, the same Lie algebra may appear more than once, and second, a Lie algebra \(D_k(p_1, \ldots, p_n)\) may not have a contact structure for some choice of the parameters \(p_1, \ldots, p_n\). However, these “degenerate” cases turn out to never have a hypo structure.

Proposition 11. The indecomposable solvable Lie algebras that have both a
contact structure and a hypo structure are the following:

\[ D_1 = (e^{24} + e^{35}, 0, 0, 0, 0) \]
\[ D_2 = (e^{34} + e^{25}, e^{35}, 0, 0, 0) \]
\[ D_4(-1/2, -3/2) = \left( -\frac{1}{2} e^{15} - e^{23}, -e^{25}, \frac{1}{2} e^{35}, \frac{3}{2} e^{45}, 0 \right) \]
\[ D_4(1, -3) = (-e^{23} - 2e^{15}, -e^{25}, -e^{35}, 3e^{45}, 0) \]
\[ D_4(-2, 3) = (e^{15} - e^{23}, -e^{25}, 2e^{35}, -3e^{45}, 0) \]
\[ D_{15}(-1) = (-e^{15} - e^{24}, e^{34}, e^{35}, -e^{45}, 0) \]
\[ D_{18}(-1, -1) = (-e^{14}, -e^{25}, e^{34} + e^{35}, 0, 0) \]
\[ D_{20}(-2, 0) = (2e^{14}, -e^{24} - e^{35}, e^{25} - e^{34}, 0, 0) \]
\[ D_{22} = (e^{23} + 2e^{14}, e^{24} + e^{35}, e^{34} - e^{25}, 0, 0) \]

A Lie algebra \( D_k(p_1, \ldots, p_n) \) is hypo if and only if it belongs to this list.

**Proof.** First, we produce a hypo structure for each Lie algebra appearing in the statement.

The Lie algebras \( D_4(1, -3), D_{15}(-1), D_{18}(-1, -1) \) and \( D_{22} \) appear in [8] and have hypo contact structures given by the coframes

\[ e^5, \frac{1}{5}(e^1 - e^4), \frac{1}{2} e^3, \frac{1}{5} e^2, -\frac{1}{5} e^1 - \frac{2}{15} e^4 \]
\[ D_4(1, -3). \]

\[ \frac{1}{2}(-e^1 + e^3), e^5, \sqrt{2} e^4, \sqrt{2} e^2, -e^1 - e^3 \]
\[ D_{15}(-1). \]

\[ -\frac{1}{2\sqrt{3}}(e^4 + 2e^5), \frac{1}{2\sqrt{3}}(e^2 - e^3), \frac{1}{3} e^1 - \frac{1}{6} e^2 - \frac{1}{6} e^3, \frac{1}{2} e^4, \frac{1}{3}(e^1 + e^2 + e^3) \]
\[ D_{18}(-1, -1). \]

\[ e^4, e^1, \sqrt{2} e^3, \sqrt{2} e^2, \frac{1}{3}(e^5 - 3e^1) \]
\[ D_{22}. \]

The Lie algebra \( D_1 \) is nilpotent and has a well-known hypo-contact structure:

\[ e^2, e^4, e^3, e^5, e^1 \]
\[ D_1. \]

The Lie algebra \( D_2 \) is also nilpotent and equivalent to \((0, 0, 0, 12, 13 + 24)\), hence hypo by [5]; a hypo structure is given by the coframe

\[ e^2, e^4, e^5, -e^1, -e^1 + e^3 \]
\[ D_2. \]

Hypo structures on the three remaining Lie algebras are new. They are defined by

\[ e^1, e^3, e^2, e^5, e^4 \]
\[ -e^3, 2e^3, -2e^1, 2e^2, -e^4 \sqrt{2} \]
\[ D_{4}(1/2, -3/2). \]
\[ D_{4}(-2, 3). \]
and, for $D_{20}(-2,0)$, by

$$3e^2 - 3\sqrt{3}e^3 - \sqrt{3}e^4 + 2\sqrt{3}e^5 - 2\sqrt{3}e^2 - \sqrt{3}e^3, 9e^1 + 3e^3 + 3e^4,$$

$$- 2\sqrt{3}e^4 - \sqrt{3}e^2, -5e^1 - 2e^2 - 4e^3 - e^4 + 2e^5.$$

It is straightforward to check that all these Lie algebras have a contact form.

It remains to show that the remaining $D_k(p_1, \ldots, p_n)$ do not have a hypo structure; to that effect, we apply the results of Sections [1, 2]. Looking at the list and applying Proposition [2], we see that the algebras in the list that admit a coherent splitting with

$$H^0.2 = H^0.3 = 0$$

are precisely the following ($\phi$ denotes a generator of $\Lambda^2.0$ in each case):

- $D_4 = (-1 + p)e^{15} - e^{23} - e^{25} - pe^{35} - qe^{45}, 0), \phi = e^{25}$ if all of $p + q$, $2p + 1, 2p + 1 + q$ are non-zero, or $\phi = e^{35}$ if all of $p + q + 2, p + 2, 1 + q$ are non-zero;
- $D_5 = (-e^{15}(1 + p) - e^{23} - e^{45}, -e^{25} - pe^{35} - e^{45}(1 + p), 0), \phi = e^{25}$ if both of $1 + 2p, 2 + 3p$ are non-zero, or $\phi = e^{35}$ otherwise;
- $D_6 = (2e^{15} + e^{23}, e^{25}, e^{45} + e^{35}, 0), \phi = e^{25};$
- $D_7 = (e^{23}, 0, e^{25}, e^{45}, 0), \phi = e^{25};$
- $D_8 = (-2e^{15} - e^{23}, -e^{25}, -e^{35}, -pe^{45}, 0), \phi = e^{25};$
- $D_9 = (2e^{15} + e^{23} + e^{45}, e^{25}, e^{35} + e^{25}, 2e^{45}, 0), \phi = e^{25};$
- $D_{12} = (e^{45} + e^{15} + e^{23}, 0, e^{35}, e^{35} + e^{45}, 0), \phi = e^{25};$
- $D_{13} = (-e^{15}(1 + p) - e^{23}, -e^{25} - pe^{35} - e^{45}, 0), \phi = e^{25}$ if both of $p + 2, p + 3$ are non-zero, or $\phi = pe^{25} + (1 - p)e^{35}$ otherwise;
- $D_{14} = (e^{15} + e^{23}, e^{25}, 0, e^{45}, 0), \phi = e^{25};$
- $D_{15} = (-e^{24} - e^{15}(2 + p), -e^{25}(1 + p) - e^{34}, -pe^{35} - e^{45}, 0), \phi = e^{45}$ if both of $1 + 2p, 1 + p$ are non-zero;
- $D_{16} = (e^{24} + 3e^{15}, e^{34} + 2e^{25}, e^{35} + e^{45}, e^{45}, 0), \phi = e^{45};$
- $D_{17} = (-e^{15} - e^{24} - pe^{35}, -e^{25} - e^{34}, e^{35} 0, 0), \phi = e^{45};$
- $D_{18} = (-e^{14}, -e^{25}, -pe^{34} - qe^{35}, 0, 0), \phi = e^{45}$ if $(p, q)$ is not $(0, -1), (-1, 0)$ or $(1, -1);$
- $D_{19} = (-e^{15} - pe^{14}, -e^{35} - e^{24}, -e^{34}, 0, 0), \phi = e^{45};$
- $D_{20} = (-qe^{15} - pe^{14}, -e^{24} - e^{35}, e^{25} - e^{34}, 0, 0), \phi = e^{45}$ if $(p, q)$ is not $(-2, 0);$
• \(D_{23} = (e^{14}, e^{25}, e^{45}, 0, 0), \phi = e^{45}\);
• \(D_{24} = (e^{14} + e^{25}, e^{24} - e^{15}, e^{45}, 0, 0), \phi = e^{45}\).

Other non-unimodular Lie algebras are ruled out by Proposition 9. They are listed below; here and throughout the paper, the 1-form \(\beta\) is given up to multiple.

• \(D_4(1, -1) = (-e^{23} - 2e^{15}, -e^{25}, -e^{35}, e^{45}, 0), \beta = e^5\)
• \(D_4(0, -1) = (-e^{15} - e^{23}, -e^{25}, 0, e^{45}, 0), \beta = e^5\)
• \(D_{15}(-1/2) = (-e^{24} - \frac{3}{2}e^{15}, -\frac{1}{2}e^{25} - e^{34}, \frac{1}{2}e^{35}, -e^{45}, 0), \beta = e^5\).

To address the remaining Lie algebras, we apply either Proposition 7 or Equation 6.

• \(D_3 = (e^{25} + e^{34}, e^{35}, e^{45}, 0, 0)\). This Lie algebra is nilpotent and isomorphic to \((0, 0, 12, 13, 23 + 14)\), therefore not hypo by [5]; however, we can prove it directly using the methods of Section 2. We compute

\[
Z^2 = \text{Span}\{-e^{14} + e^{23}, e^{15} + e^{24}, e^{25}, e^{34}, e^{35}, e^{45}\}
\]
\[
Z^3 = \text{Span}\{-e^{125} + e^{134}, e^{135}, e^{145}, e^{234}, e^{235}, e^{245}, e^{345}\}
\]

Then the spaces \((Z^i \land e^5, i = 1, 2)\) are one-dimensional, so by Lemma 10 \(\alpha\) is a linear combination of \(e^3, e^4, e^5\). In particular, if \(\alpha\) is linearly independent of \(e^5\), then setting \(\beta = e^5\) in Proposition 7 we see that

\[
L_\beta(V) \subset \text{Span}\{e^{1345}, e^{2345}\}
\]

contains \(Z^2 \land \alpha \land \beta\). Otherwise, we may set \(\beta = e^4\) and obtain the same result.

• \(D_4(-2, 2) = (e^{15} - e^{23}, -e^{25}, 2e^{35}, -2e^{45}, 0)\). We compute

\[
Z^2 = \text{Span}\{e^{12}, -e^{15} + e^{23}, e^{15}, e^{25}, e^{34}, e^{35}, e^{45}\}
\]
\[
Z^3 = \text{Span}\{e^{125}, e^{135}, e^{235}, e^{234}, e^{245}, e^{345}\}
\]

Therefore \(Z^3 \land e^5\) is one-dimensional, hence by Lemma 10 \(\alpha(e_i) = 0, i = 1, 2, 3, 4\), i.e. \(\alpha = e^5\) up to a multiple. Now \(Z^2 \land \alpha\) is spanned by \(e^{125}, e^{235}, e^{345}\). Setting \(\beta = e^5\) in Proposition 7 gives a contradiction, as \(L_\beta(V)\) is spanned by \(e^{1235}, e^{2345}\) and \(Z^2 \land \alpha \land \beta\) is spanned by \(e^{1235}\).

• \(D_4(-1/2, -1) = (-\frac{1}{2}e^{15} - e^{23}, -e^{25}, \frac{1}{2}e^{35}, e^{45}, 0)\) is similar to \(D_4(-2, 2)\) in that

\[
Z^2 = \text{Span}\left\{e^{13}, \frac{1}{2}e^{15} + e^{23}, e^{24}, e^{25}, e^{34}, e^{35}, e^{45}\right\},
\]
\[
Z^3 = \text{Span}\left\{e^{125}, e^{135}, e^{234}, \frac{1}{2}e^{145}, e^{235}, e^{245}, e^{345}\right\}.
\]

The same argument applies, except that now \(Z^2 \land \alpha \land \beta\) is spanned by \(e^{2345}\).

13
• $D_{10} = (-2pe^{15} - e^{23}, -pe^{25} + e^{35}, -pe^{35} - e^{25}, -qe^{45}, 0)$. A basis of $Z^3$
is given by
\[ e^{345}, e^{235}, e^{135}, e^{125}, e^{245}, (2p + q)e^{145} - e^{234}, \]
plus $e^{123}$ if $p = 0$, whereas
\[ Z^2 = \text{Span} \left\{ e^{35}, e^{45}, 2e^{15} + e^{23}, e^{25} \right\}. \]
Now $\beta = (4p + q)e^5$, and if $4p + q \neq 0$ then Proposition 9 applies; in
general, we have that $e_1 \wedge Z^3$ and $e_4 \wedge Z^3$ wedged with $e^5$ are at most one-
dimensional, hence by Lemma 10 $\alpha$ lies in the span of $e^2, e^3, e^5$. But then
$(Z^2)^2 \wedge \alpha = 0$, which is a contradiction.

• $D_{11} = (-2e^{15}p - e^{23} - e^45, -e^{25}p + e^{35}, -e^{35}p - e^{25}, -2e^{45}p, 0)$. A basis
of $Z^2$ is given by
\[ 2pe^{15} + e^{23}, e^{35}, e^{45}, e^{25}, \]
whereas a basis of $Z^3$ is given by
\[ e^{345}, e^{235}, e^{135}, e^{125}, e^{245}, -4pe^{145} + e^{234}, \]
plus $e^{145} - e^{123}$ if $p = 0$. Thus $(e_1 \wedge Z^3) \wedge e^5, (e_4 \wedge Z^3) \wedge e^5$ have dimension
one and we see that $\alpha$ is in $\text{Span} \left\{ e^2, e^3, e^5 \right\}$, whence $\alpha \wedge (Z^2)^2 = 0$, a
contradiction.

• $D_{18}(-1, 0) = (-e^{14}, -e^{25}, e^{34}, 0, 0)$. We compute
\[ Z^2 = \text{Span} \left\{ e^{13}, e^{14}, e^{25}, e^{34}, e^{45} \right\}, \]
\[ Z^3 = \text{Span} \left\{ e^{125} + e^{124}, e^{134}, e^{135}, e^{145}, -e^{234} + e^{235}, e^{245}, e^{345} \right\}, \]
and $\beta = e^5$. Moreover the spaces
\[ (e_1 \wedge Z^3) \wedge (e^4 + e^5), \quad (e_2 \wedge Z^3) \wedge (e^4 + e^5), \quad (e_3 \wedge Z^3) \wedge (e^4 - e^5) \]
are one-dimensional, so by Lemma 10 and Lemma 5 we get $\alpha = e^4 + ae^5$, for some constant $a$. Then setting $\beta = e^5$ in Proposition 7 we see that
$L_\beta(V)$ is at most two-dimensional, and it contains $e^{145}$. Since $Z^2 \wedge \alpha \wedge \beta$
is spanned by $e^{1345}$, there is no hypo structure.

• $D_{18}(0, -1) = (-e^{14}, -e^{25}, e^{35}, 0, 0)$ is really isomorphic to $D_{18}(-1, 0)$, as
one can check by considering the coframe $(e^2, e^3, e^5, e^4)$, so it has no
hypo structure.

• $D_{21} = (e^{23} + e^{14}, e^{24} - e^{25}, e^{35}, 0, 0)$. Then $\beta = e^4$ and
\[ Z^2 = \text{Span} \left\{ e^{14} + e^{23}, e^{25} - e^{24}, e^{35}, e^{45} \right\}, \]
\[ Z^3 = \text{Span} \left\{ e^{125} - 2e^{124}, e^{135} + e^{134}, e^{234}, e^{235} + e^{145}, e^{245}, e^{345} \right\}. \]
Therefore, the spaces
\[(e_1 \wedge Z^3) \land (e^4 + e^5), \quad (e_2 \wedge Z^3) \land (e^5 - 2e^4), \quad (e_3 \wedge Z^3) \land (e^4 + e^5)\]
are one-dimensional, so by Lemma \[10\] and Lemma \[8\] we get \(\alpha = ae^4 + e^5\), for some constant \(a\). Then setting \(\beta = e^4\) in Proposition \[7\] we see that \(L_\beta(V)\) is at most two-dimensional, and it contains \(e^{2345}\). Since \(Z^2 \land \alpha \land \beta\) is spanned by \(e^{2345}\), there is no hypo structure.

\[\square\]

4 Indecomposable Lie algebras without contact form

We now pass on to indecomposable solvable Lie algebras that do not have a contact structure.

**Proposition 12.** The indecomposable solvable Lie algebras which have a hypo structure but not a contact structure are those given in Table \[1\], all of them unimodular.

Observe that \(A_{5,1}\) and \(A_{5,2}\) are nilpotent, and so appear in \[5\].

**Proof.** It is straightforward to verify that the co.frames given in the table define indeed hypo structures. To show that no other Lie algebras of the specified type have a hypo structure, we use the classification in \[1\].

- \(A_{5,3} = (e^{25}, e^{45}, e^{24}, 0, 0)\) is nilpotent and known not to have a hypo structure \[5\]. It also has a coherent splitting \(\phi = e^{45}\) with \(H^{0,2} = H^{0,3} = 0\).

- \(A_{5,7} = (e^{15}, pe^{25}, qe^{35}, re^{45}, 0)\) where \(p, q, r \neq 0\) is not hypo unless, up to permutation of the parameters, \(r = -1\) and \(p + q = 0\). Indeed, suppose first that \(p, q, r \neq -1\). Then if \(p + q \neq 0, -1\) we find a coherent splitting \(\phi = e^{45}\) with \(H^{0,2} = H^{0,3} = 0\). Since we can act by an automorphism to permute \(p, q, r\), the same happens if \(p + r \neq 0, -1\) or \(q + r \neq 0, -1\). Thus, still assuming \(p, q, r \neq -1\), we are left with the cases \((-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\) and \((-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\). In the former case, \(\phi = e^{15}\) defines a coherent splitting with \(H^{0,2} = H^{0,3} = 0\). In the latter case, the Lie algebra is non-unimodular with \(\beta = e^{5}\), and \(L_\beta(Z^3)\) has dimension one.

Now, if \(r = -1\) and \(p + q \neq 0\), the Lie algebra is non-unimodular with \(\beta = e^{5}\), and \(Z^3\) is spanned by \(e^{345}, e^{145}, e^{235}, e^{125}, e^{245}, e^{135}\) plus

\[e^{234}\] if \(p + q - 1 = 0\),

\[e^{123}\] if \(p + q + 1 = 0\).

Therefore, \(L_\beta(Z^3)\) is at most one-dimensional.
\begin{itemize}
  \item $A_{5,8} = (e^{25}, 0, e^{35}, pe^{45}, 0)$ has a coherent splitting given by $\phi = e^{25}$ with $H^{0,2} = H^{0,3} = 0$ if $p \neq -1$.
  \item $A_{5,9} = (e^{15} + e^{25}, e^{25}, pe^{35}, qe^{45}, 0)$ has a coherent splitting with $H^{0,2} = H^{0,3} = 0$. If $p + q \neq 0$ we can take $\phi = e^{25}$; if $p = -q$ but $p \neq 1, 2$ then $\phi = e^{35}$, and otherwise we can take $\phi = e^{45}$.
  \item $A_{5,10} = (e^{25}, e^{35}, 0, e^{45}, 0)$ has a coherent splitting given by $\phi = e^{35}$ with $H^{0,2} = H^{0,3} = 0$.
  \item $A_{5,11} = (e^{15} + e^{25}, e^{35} + e^{25}, e^{35}, -pe^{45}, 0)$ has a coherent splitting given by $\phi = e^{35}$ with $H^{0,2} = H^{0,3} = 0$.
  \item $A_{5,12} = (e^{15} + e^{25}, e^{25} + e^{45}, e^{35} + e^{45}, e^{45}, 0)$ has a coherent splitting given by $\phi = e^{45}$ with $H^{0,2} = H^{0,3} = 0$.
  \item $A_{5,13} = (e^{15}, pe^{25}, qe^{35} + re^{45}, qe^{45} - re^{35}, 0)$, where we assume $r \neq 0$ (as $A_{5,13}(p, q, 0)$ is isomorphic to $A_{5,7}(p, q, q)$), has a coherent splitting given by $\phi = e^{25}$ with $H^{0,2} = H^{0,3} = 0$ if $q \neq 0, -\frac{1}{2}$. If $q = -1/2, p \neq 1$ then the same holds of $\phi = e^{15}$. The only cases left out are $(p, q) = (1, -\frac{1}{2})$ and $q = 0$. In general, a basis of $Z^3$ is given by
    
    \begin{align*}
    e^{145}, e^{245}, e^{135}, e^{345}, e^{125}, e^{235}
    \end{align*}
  
  plus $e^{34}$ if $1 + 2q$ is zero, plus $e^{234}$ if $p + 2q$ is zero. A basis of $Z^2$ is given by
    
    \begin{align*}
    e^{45}, e^{25}, e^{15}, e^{35}
    \end{align*}
  
  plus $e^{34}$ if $q$ is zero, plus $e^{12}$ if $p = -1$. Thus, if $(p, q) = (1, -\frac{1}{2})$, then $(Z^2)^2 = 0$ contradicting \[ \text{for any } \alpha. \] On the other hand, if $q = 0$, then $\alpha$ cannot be independent of $\beta = e^5$, as $\dim L_\beta(Z^3) < 2$. But then \[ \text{is only satisfied if } (p, q) = (-1, 0), \] in which case we already know that a hypo structure exists.
  \item $A_{5,14} = (e^{25}, 0, e^{45} + pe^{35}, -e^{35} + pe^{45}, 0)$ has a coherent splitting given by $\phi = e^{25}$ with $H^{0,2} = H^{0,3} = 0$ if $p \neq 0$.
  \item $A_{5,15} = (e^{15} + e^{25}, e^{25}, e^{45} + pe^{35}, pe^{45}, 0)$ has a coherent splitting given by $\phi = e^{45}$ with $H^{0,2} = H^{0,3} = 0$ if $p \neq -1$.
  \item $A_{5,16} = (e^{25} + e^{15}, e^{25}, pe^{35} + qe^{45}, pe^{45} - qe^{35}, 0)$ has a coherent splitting given by $\phi = e^{25}$ if $p \neq 0$. If $p = 0$, then $\beta = e^5$ and $L_\beta(Z^3) = 0$.
  \item $A_{5,17} = (pe^{15} + e^{25}, pe^{25} - e^{15}, re^{45} + qe^{35} - re^{35} + qe^{45}, 0), r \neq 0$. Then $Z^3 \wedge e^5 = 0$. So if $p + q \neq 0$, then $\beta = e^5$ and Proposition \[ \text{applies.} \] Otherwise, the same argument together with Proposition \[ \text{shows that necessarily } \alpha = e^5. \] Now if $p + q = 0$ but $p \neq 0$ and $r \neq \pm 1$, then $Z^2 = e^5 \wedge \Lambda^1$, so by \[ \text{no hypo structure exists.} \]
  \item $A_{5,18} = (pe^{15} + e^{35} + e^{25}, pe^{25} + e^{45} - e^{15}, pe^{35} + e^{45}, -pe^{45} - e^{35}, 0)$. Then $\beta = pe^5$, and $L_\psi(Z^3) = 0$. So if $p \neq 0$ we obtain an obstruction.
\end{itemize}
Table 1: Nondecomposable, non-contact hypo Lie algebras

| Name      | Structure constants                                                                 | Hypo structure     |
|-----------|-------------------------------------------------------------------------------------|--------------------|
| \(A_{5,1}\) | \((e^{45}, e^{35}, 0, 0, 0)\)                                                      | \(e^1, e^2, e^4, e^5\) |
| \(A_{5,2}\) | \((e^{45}, e^{35}, e^{45}, 0, 0)\)                                                | \(e^1, e^2, e^3, e^5, e^5\) |
| \(A_{5,7}(p, -p, -1)\) | \((e^{15}, pe^{25}, -pe^{35}, -e^{45}, 0)\)                                      | \(e^1, e^2, e^3, e^4, e^5\) |
| \(A_{5,8}(-1)\) | \((e^{25}, 0, e^{35}, -e^{45}, 0)\)                                               | \(e^1, e^2, e^3, e^4, e^5\) |
| \(A_{5,13}(-1, 0, r)\) | \((e^{15}, -e^{25}, re^{45}, -re^{35}, 0)\)                                       | \(e^1, e^2, e^3, e^4, e^5\) |
| \(A_{5,14}(0)\) | \((e^{25}, 0, e^{45}, -e^{35}, 0)\)                                               | \(e^1, e^2, e^3, e^4, e^5\) |
| \(A_{5,15}(-1)\) | \((e^{15} + e^{25}, e^{25}, e^{45} - e^{35}, -e^{45}, 0)\)                       | \(e^1, e^2, e^3, e^4, e^5\) |
| \(A_{5,17}(0, 0, r)\) | \((e^{25} - e^{15}, re^{45} - e^{35}r, 0)\)                                      | \(e^1, e^2, e^3, e^4, e^5\) |
| \(A_{5,17}(p, -p, 1)\) | \((e^{25} + pe^{15} - e^{15} + pe^{25}, e^{45} - pe^{35}, -e^{35} - pe^{45}, 0)\) | \(e^1, e^2, e^3, e^4, e^5\) |
| \(A_{5,17}(p, -p, -1)\) | \((e^{25} + pe^{15}, -e^{15} + pe^{25}, -e^{45} - pe^{35}, e^{35} - pe^{45}, 0)\) | \(e^1, e^2, e^3, e^4, e^5\) |
| \(A_{5,18}(0)\) | \((e^{45} + e^{25}, -e^{15} + e^{45}, e^{45}, -e^{35}, 0)\)                      | \(e^1, e^2, e^3, e^4, e^5\) |

5 Decomposable contact Lie algebras

By [9], there are two types of decomposable 5-dimensional Lie algebras with an invariant contact form. First, the Lie algebras \((0, e^{12}) \oplus g_3\), where \((0, e^{12})\) is the Lie algebra of affine transformations of \(\mathbb{R}\), and \(g_3\) is any Lie algebra of dimension three other than \((0, e^{12}, e^{13})\) or \((0, 0, 0)\). Second, the Lie algebras of the form \(g_4 \oplus \mathbb{R}\), where \(g_4\) is a four-dimensional Lie algebra carrying an exact symplectic form. In this section we show that only one of these families admits a hypo structure, and it belongs to the first type.

**Proposition 13.** If \(g_3\) is a solvable Lie algebra of dimension three, then \((0, e^{12}) \oplus g_3\) has a hypo structure if and only if \(g_3 = A_{3,8} = (e^{23}, -e^{13}, 0, 0)\).

**Proof.** First, observe that \(A_{3,8} \oplus (0, e^{12}) = (e^{23}, -e^{13}, 0, 0, e^{45})\) has a hypo structure given by the coframe \(e^1, e^2, e^4, e^3, e^5\).

To prove uniqueness, we resort once again to the list in [1]. There are nine families of solvable Lie algebras of dimension three, of which the following five have a coherent splitting with \(H^{0,2} = H^{0,3} = 0\):

\[
\begin{align*}
A_{3,2} \oplus (0, e^{12}) &= (0, e^{12}, 0, 0, e^{45}), & \phi &= e^{14} \\
A_{3,4} \oplus (0, e^{12}) &= (e^{23} + e^{13}, e^{23}, 0, 0, e^{45}), & \phi &= e^{14} \\
A_{3,5} \oplus (0, e^{12}) &= (e^{13}, e^{13}, 0, 0, e^{45}), & \phi &= e^{14} \\
A_{3,7} \oplus (0, e^{12}) &= (e^{13}, qe^{23}, 0, 0, e^{45}), & 0 < |q| < 1, & \phi &= e^{14} \\
A_{3,9} \oplus (0, e^{12}) &= (qe^{13} + e^{23}, qe^{23} - e^{13}, 0, 0, e^{45}), & q > 0, & \phi &= e^{14}.
\end{align*}
\]

Therefore, by Proposition [2] there is no hypo structure on \(A_{3,i}\), for \(i = 2, 4, 5, 7, 9\).

For \(A_{3,1} \oplus (0, e^{12}) = (0, 0, 0, 0, e^{45})\), we find that \(\beta = e^4\) and \(L_\beta(Z^3)\) is spanned by \(e^{1234}\), so Proposition [3] applies.
For \( A_{3,3} \oplus (0, e^{12}) = (e^{23}, 0, 0, e^{45}) \), we have that \( \beta = -e^4 \) and 
\[
L_\beta(Z^3) = \text{Span}\{-e^{1234}, e^{2345}\}.
\]
So, by Proposition 9 if \((\alpha, \omega_i)\) is a hypo structure then
\[
\alpha \in \text{Span}\{e^2, e^3, e^4\},
\]
implies that \( Z^2 \wedge \alpha \wedge \beta \) is contained in \( L_\beta(Z^3) \), which is absurd.

Finally, the Lie algebra \( A_{3,6} \oplus (0, e^{12}) = (e^{13}, -e^{23}, 0, 0, e^{45}) \) satisfies
\[
Z^2 = \text{Span}\{e^{12}, e^{13}, e^{23}, e^{34}, e^{45}\},
\]
\[
Z^3 = \text{Span}\{e^{123}, e^{124}, e^{145} - e^{135}, e^{234}, e^{245} + e^{235}, e^{345}\}.
\]
So \( \alpha \) lies in the span of \( e^3, e^4 \) by Lemma 10. Moreover \( \beta = e^4 \), thus \( \alpha \) has the form \( e^3 + ae^4 \). Defining \( V \) as in Proposition 7, we see that \( e^4 \wedge V \) is contained in the span of \( e^{1234} \) and \( e^{2345} \). Since \( Z_2 \wedge e^{34} = e^{1234} \), there is no hypo structure. \( \square \)

Remark. Notice that Proposition 13 does not apply to contact Lie algebras alone, but also to the non-contact Lie algebras \( A_{3,1} \oplus (0, e^{12}) \) and \( A_{3,5} \oplus (0, e^{12}) \). Decomposable contact Lie algebras of the type \( g_4 \oplus \mathbb{R} \) are not unimodular, because the volume form is exact, and so it makes sense to apply Proposition 9. This turns out to be sufficient in order to show that no hypo structure exists on these Lie algebras.

**Proposition 14.** If \( g_4 \) is a 4-dimensional solvable Lie algebra with an exact symplectic form, then \( g_4 \oplus \mathbb{R} \) has no hypo structure.

**Proof.** Observe that \( g_4 \) is necessarily indecomposable, because it admits an exact symplectic form. From the list in [1], \( g_4 \) must belong to one of four families, to each of which we apply Proposition 9:

- The Lie algebra \( A_{4,7} \oplus \mathbb{R} = (e^{23} + 2e^{14}, e^{24} + e^{34}, e^{34}, 0, 0) \) has \( \beta = e^4 \), and \( L_\beta(Z^3) \) has dimension one.
- The Lie algebra \( A_{4,8} \oplus \mathbb{R} = (e^{23} + (1 + q)e^{14}, e^{24}, qe^{34}, 0, 0) \), \(-1 < q \leq 1\) has \( \beta = e^4 \), and \( L_\beta(Z^3) \) has dimension one except if \( q = -1/2 \). In this case, it is spanned by \( e^{1345}, e^{2345} \), and
  \[
  Z^2 = \text{Span}\left\{e^{13}, e^{23} + \frac{1}{2}e^{14}, e^{24}, e^{34}, e^{45}\right\},
  \]
  so by Proposition 9 there is no hypo structure.
- The Lie algebra \( A_{4,9} \oplus \mathbb{R} = (e^{23} + 2qe^{14}, qe^{24} + e^{34}, -e^{24} + qe^{34}, 0, 0) \), with \( q > 0 \). Then \( \beta = e^4 \), and \( L_\beta(Z^3) \) is one-dimensional.
- The Lie algebra \( A_{4,10} \oplus \mathbb{R} = (e^{13} + e^{24}, e^{23} - e^{14}, 0, 0, 0) \) has \( \beta = e^3 \), and 
  \[
  L_\beta(Z^3) = \text{Span}\{e^{2345}, e^{1345}\}.
  \]
  So \( L_\alpha \) kills \( L_\beta(Z^3) \) if and only if \( \alpha \) lies in the span of \( e^3, e^4, e^5 \), in which case \( \alpha \wedge \beta \wedge Z^2 \) is contained in \( L_\beta(Z^3) \).
Thus, in neither case is there a hypo structure.

Remark. In the proof of Proposition 14 we have left out the cases $A_{4,8}(-1)$ and $A_{4,9}(0)$ because they do not have any exact symplectic form. They have no hypo structure either. Indeed, $A_{4,8}(-1)$ has a coherent splitting $\phi = e^{24}$ with $H^{0,2} = H^{0,3} = 0$. For $A_{4,9}(0)$, we apply Lemma 10 to show that $\alpha$ has no component along $e^1$, contradicting (6).

6 Decomposable non-contact Lie algebras

Decomposable Lie algebras of dimension five may either be of the form $g_3 \oplus h_2$, where we are allowing the factors themselves to be decomposable, or $g_4 \oplus \mathbb{R}$. In the former case, by Proposition 13 we can assume $h_2 = \mathbb{R}^2$. Without resorting to Mubarakzyanov’s classification, we can characterize which of these Lie algebras have a hypo structure.

Proposition 15. Let $g_3$ be a Lie algebra of dimension 3. Then $g = g_3 \oplus \mathbb{R}^2$ admits a hypo structure if and only if $g_3$ is unimodular.

Proof. Let $e^1, \ldots, e^5$ be a coframe reflecting the splitting $g = g_3 \oplus \mathbb{R}^2$, so that $e^1, e^2, e^3$ is a basis of $g_3^* \subset g^*$ and $e^4, e^5$ a basis of $(\mathbb{R}^2)^*$.

If $g_3$ is unimodular, then the coframe $e^1, e^2, e^4, e^5$ determines a hypo structure by (3), because $e^{12}, e^{13}, e^{23}$ are closed.

If $g_3$ is not unimodular, we can assume that $e^3 = \beta$ as defined in Lemma 8. Then $e^3$ is closed and $de^{12} \neq 0$. Moreover $de^i \wedge e^3 = 0, i = 1, 2$. This is because if $e_1, e_2, e_3$ is a basis of $g_3$ dual to $e^1, e^2, e^3$, then

$$0 = \text{tr}(\text{ad} e_2) = e^1([e_2, e_1]) + e^3([e_2, e_3]) = e^1([e_2, e_1]).$$

Thus $Z^3 = (e^3 \wedge \Lambda^2) \oplus W$, where

$$W \subset \Lambda^3(\text{Span} \{e^1, e^2, e^4, e^5\}).$$

Since $de^{12} \neq 0$, $W \subset \text{Span} \{e^{145}, e^{245}\}$; so $L_\beta(Z_4)$ has the same dimension as $\ker d \cap \text{Span} \{e^1, e^2\}$, which is at most one since $g_3$ is not abelian. By Proposition 9 this concludes the proof.

For the other case, we must refer to Mubarakzyanov’s classification.

Proposition 16. If $g_4$ is an indecomposable solvable Lie algebra of dimension four, then $g_4 \oplus \mathbb{R}$ has no hypo structure.

Proof. By (11), there are 10 families $A_{4,1}, \ldots, A_{4,10}$ of solvable Lie algebras of dimension four. The families $A_{4,7}$ through $A_{4,10}$ have no hypo structure by
Proposition 14 and the subsequent remark. The following families have a coherent splitting with $H^{0,2} = H^{0,3} = 0$:

\[ A_{4,1} \oplus \mathbb{R} = (e^{24}, e^{34}, 0, 0, 0), \quad \phi = e^{34} \]
\[ A_{4,2} \oplus \mathbb{R} = (q e^{14}, e^{24} + e^{34}, 0, 0), \quad q \neq 0, \quad \phi = e^{34} \]
\[ A_{4,3} \oplus \mathbb{R} = (e^{14}, e^{34}, 0, 0, 0), \quad \phi = e^{34} \]
\[ A_{4,4} \oplus \mathbb{R} = (e^{14} + e^{24} + e^{34}, 0, 0), \quad \phi = e^{45} \]
\[ A_{4,5} \oplus \mathbb{R} = (e^{14}, q e^{24}, p e^{34}, 0, 0), \quad p, q \neq 0, \quad \phi = \begin{cases} e^{14}, & (p, q) = (-1, -1) \\ e^{34}, & q \neq -1 \\ e^{24}, & p \neq -1 \end{cases} \]

Notice that $A_{4,1} \oplus \mathbb{R}$ is nilpotent and isomorphic to $(0, 0, 0, e^{12}, e^{14})$.

The remaining family is

\[ A_{4,6} \oplus \mathbb{R} = (q e^{14}, e^{34} + p e^{24}, -e^{24} + p e^{34}, 0, 0), \quad q \neq 0, \quad p \geq 0. \]

If $p > 0$, $\phi = e^{14}$ defines a coherent splitting with $H^{0,2} = H^{0,3} = 0$. If $p = 0$, then $\beta = e^{4}$, and

\[ Z^{3} = \text{Span} \{ e^{235}, e^{145}, e^{245}, e^{134}, e^{234}, e^{124}, e^{345} \}. \]

Therefore $L_{\beta}(Z^{3})$ has dimension one, and Proposition 9 applies.

Applying the classification of three-dimensional Lie algebras, we finally obtain:

**Theorem 17.** A solvable Lie algebra of dimension five has a hypo structure if and only if it appears in the list of Proposition 14 it appears in Table 1 or it is one of the following:

\[ (0, 0, 0, 0, 0), \quad (e^{23}, 0, 0, 0, 0), \quad (e^{23}, -e^{13}, 0, 0, 0), \quad (e^{13}, -e^{23}, 0, 0, 0), \quad (e^{23}, -e^{13}, 0, 0, e^{45}). \]

**References**

[1] P. Basarab-Horwath, V. Lahno and R. Zhdanov, *The structure of Lie algebras and the classification problem for partial differential equations*, Acta Appl. Math. (1) 69 (2001), 43–94.

[2] S. Chiossi and A. Fino, *Conformally parallel $G_{2}$ structures on a class of solvmanifolds*, Math. Z. (4) 252 (2006), 825–848.

[3] S. Chiossi and S. Salamon, *The intrinsic torsion of SU(3) and $G_{2}$ structures*, In: Proceedings of the “International Conference held to honour the 60th Birthday of A.M. Naveira”, Valencia (Spain), July 8-14, 2001 (O. Gil-Medrano, V. Miquel, eds.), World Scientific, Singapore, 2002, pp. 115–133.
[4] S. Chiossi and A. Swann, *G_2*-structures with torsion from half-integrable nilmanifolds*, J. Geom. Phys. **54** (2005), 262–285.

[5] D. Conti and S. Salamon, *Generalized Killing spinors in dimension 5*, Trans. Amer. Math. Soc. (11) **359** (2007), 5319–5343.

[6] D. Conti and A. Tomassini, *Special symplectic six-manifolds* Q. J. Math. (3) **58** (2007), 297–311.

[7] D. Conti, *Half-flat nilmanifolds*, Math. Ann. (2010), in press, available at arxiv.org:0903.1175v3 [math.DG].

[8] L. C. de Andrés, M. Fernández, A. Fino and L. Ugarte, *Contact 5-manifolds with SU(2)-structure*, Q. J. Math. (4) **60** (2009), 429–459.

[9] A. Diatta, *Left invariant contact structures on Lie groups*, Differ. Geom. Appl. (5) **26** (2008), 544–552.

[10] N. Hitchin, *Stable forms and special metrics*. In: “Global Differential Geometry: The Mathematical Legacy of Alfred Gray”, Proceedings of International Congress on Differential Geometry, Bilbao (Spain), September 18-23, 2000 (M. Fernández, J.A. Wolf, eds.), Contemporary Mathematics vol. **288**, pp. 70–89, Amer. Math. Soc., Providence, RI, 2001.

[11] G. M. Mubarakzyanov, *The classification of the real structure of five-dimensional Lie algebras*, Izv. Vys. Ucheb. Zaved. Matematika (3) **34** (1963), 99–105.

[12] F. Schulte-Hengesbach, *Half-flat structures on products of three-dimensional Lie groups*, J. Geom. Phys. **60** (2010), 1726-1740.

Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, Via Cozzi 53, 20125 Milano, Italy.

diego.conti@unimib.it

Universidad del País Vasco, Facultad de Ciencia y Tecnología, Departamento de Matemáticas, Apartado 644, 48080 Bilbao, Spain.

marisa.fernandez@ehu.es

joseba.santisteban@ehu.es