The Laplace transform and polynomial approximation in $L^2$

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1 Introduction

This short note gives a sufficient condition for having the class of polynomials dense in the space of square integrable functions with respect to a finite measure dominated by the Lebesgue measure in the real line, here denoted by $L^2$. It is shown that if the Laplace transform of the measure in play is bounded in a neighbourhood of the origin, then the moments of all order are finite and the class of polynomials is dense in $L^2$. The existence of the moments of all orders is well known for the case where the measure is concentrated in the positive real line (see Feller, 1966), but the result concerning the polynomial approximation is original, even thought the proof is relatively simple. This tool is essential for constructing semiparametric extensions of classic parametric models.

A review on the Laplace transform theory is given in section 2. The main result is proved in section 3 and an alternative stronger condition easier to be verified not involving the calculation of the Laplace transform is given in section 4.

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2 Basic properties of the Laplace Transform

In this section we review the basic properties of the Laplace transform. Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a function such that for some $s \in \mathbb{R}$ the integral

$$M(s; f) = \int_{\mathbb{R}} e^{sx} f(x) \lambda(dx)$$

(1)
converges. Here $\lambda$ is a $\sigma$- finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The function $M(\cdot; f) : \mathbb{R} \rightarrow [0, \infty]$ such that for each $s \in \mathbb{R}$, $M(s; f)$ is given by (1) is said to be the Laplace transform of $f$.

We now study some properties of the functions with finite Laplace transform in a neighborhood of zero.

**Proposition 1** Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a continuous function such that for some $\delta > 0$ and for all $s \in (-\delta, \delta)$

$$M(s; f) < \infty$$

(2)

Then $f$ possesses finite moments of all orders, i.e. for all $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$,

$$\int_{\mathbb{R}} x^n f(x) \lambda(dx) \in \mathbb{R}.$$

**Proof:** Since for all $s \in (-\delta, \delta)$, $M(s; f) < \infty$, $e^{sx} \leq e^{sx} + e^{-sx}$ and using the series version of the monotone convergence theorem (see Billingsley 1986 page 214 theorem 16.6

\footnote{The referred theorem states: "If $f_n \geq 0$, then $\int \sum_n f_n d\lambda = \sum_n \int f_n d\lambda$."}) we have

$$\begin{align*}
\infty &> \int_{\mathbb{R}} e^{\delta x} f(x) \lambda(dx) + \int_{\mathbb{R}} e^{-\delta x} f(x) \lambda(dx) \\
&\geq \int_{\mathbb{R}} e^{\delta x} f(x) \lambda(dx) \\
&= \int_{\mathbb{R}} \left\{ \sum_{k=0}^{\infty} \frac{|\delta x|^k}{k!} \right\} f(x) \lambda(dx) \\
&= \sum_{k=0}^{\infty} \left\{ \int_{\mathbb{R}} \frac{|\delta x|^k}{k!} f(x) \lambda(dx) \right\},
\end{align*}$$

and we conclude that the moments of all orders of $f$ are in $\mathbb{R}$. \qed
The notion of Laplace transform can be extended to functions with range equal to the whole real line in the following way. Given a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) we define the positive and the negative part of \( f \) respectively by

\[
  f^+ (\cdot) = f(\cdot) \chi_{[0,\infty)} \{ f(\cdot) \} \quad \text{and} \quad f^- (\cdot) = -f(\cdot) \chi_{(-\infty,0]} \{ f(\cdot) \} .
\]

Here \( \chi_A (\cdot) \) is the indicator function of the set \( A \). We have clearly the decomposition

\[
  f (\cdot) = f^+ (\cdot) - f^- (\cdot) .
\]

We define the \textit{Laplace transform} of a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) as the function \( M(\cdot; f) : \mathbb{R} \rightarrow [-\infty, \infty] \) given by

\[
  M(\cdot; f) = M(\cdot; f^+) - M(\cdot; f^-) ,
\]

provided that at least one of the terms of the right side of (3) is finite (otherwise the Laplace transform of \( f \) is not defined).

**Proposition 2** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( \delta > 0 \) be such that for all \( s \in [-\delta, \delta] \), \( M(s; f) \in \mathbb{R} \). Then, for all \( n \in \mathbb{N} \) and all \( s \in (-\delta/2, \delta/2) \), we have

\[
  M[s; (\cdot)^n f (\cdot)] \in \mathbb{R} .
\]

**Proof:** Assume without loss of generality that the function \( f \) is nonnegative. Take an arbitrary \( s \in [-\delta/2, \delta/2] \) and \( n \in \mathbb{N} \). By hypothesis, \( f \) has finite Laplace transform in a neighborhood of zero; then, from proposition 1, \( f \) has finite moments of all orders, in particular

\[
  \int_{\mathbb{R}} x^{2n} f(x) \lambda(dx) \in \mathbb{R} .
\]

Using the Cauchy-Schwartz inequality we obtain

\[
  |M[s; (\cdot)^n f (\cdot)]| = | < e^{(\cdot)s} , (\cdot)^n f (\cdot) >_{\lambda} | \\
  = | < e^{(\cdot)s} f^{1/2} (\cdot) , (\cdot)^n f^{1/2} (\cdot) >_{\lambda} | \\
  \quad \text{(Cauchy-Schwartz inequality)} \\
  \leq \left| e^{(\cdot)s} f^{1/2} (\cdot) \right| \left| (\cdot)^n f^{1/2} (\cdot) \right| \\
  = \left\{ \int_{\mathbb{R}} e^{2sx} f(x) \lambda(dx) \right\}^{1/2} \left\{ \int_{\mathbb{R}} x^{2n} f(x) \lambda(dx) \right\}^{1/2} < \infty
\]

\[\Box\]
3 Polynomial approximation in $L^2$

In this section we give a sufficient condition for having the class of polynomials dense in $L^2(a)$. Here $a$ is a density with respect to the $\sigma$-finite measure $\lambda$ of a positive finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $L^2(a)$ is endowed with the usual inner product and norm denoted by $\langle \cdot, \cdot \rangle_a$ and $\| \cdot \|_a$ respectively. The conditions we give will ensure that the measure $a$ possesses all moments finite, i.e. for all $k \in \mathbb{N}$,

$$\int_{\mathbb{R}} x^k a(x) \lambda(dx) \in \mathbb{R}.$$  

In that case we can define the sequence of polynomials $\{e_i(\cdot)\}_{i \in \mathbb{N}_0} \subseteq L^2(a)$ as the result of a Gram-Schmidt orthonormalization process applied to the sequence $\{1, (\cdot), (\cdot)^2, \ldots\}$. The following theorem gives a sufficient condition for $\{e_i(\cdot)\}$ to be a complete sequence in $L^2(a)$, which implies that the polynomials are dense in $L^2(a)$.

**Theorem 1** Let $a : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that

$$\forall x \in \mathbb{R}, a(x) > 0; \quad (4)$$

$$\exists \delta > 0 \text{ such that } \forall s \in [-\delta, \delta], M(s; a) = \int_{\mathbb{R}} e^{sx} a(x) \lambda(dx) < \infty. \quad (5)$$

Then the orthonormal sequence $\{e_i(\cdot)\}_{i \in \mathbb{N}_0}$ is complete in $L^2(a)$.

**Proof:** First of all we observe that condition (5) implies that the measure determined by $a$ possesses finite moments of all orders (see proposition 1).

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function in $L^2(a)$ such that for all $k \in \mathbb{N}_0$,

$$\int_{\mathbb{R}} x^k f(x) a(x) \lambda(dx) = 0. \quad (6)$$

We prove that $f(\cdot) = 0$ a.e. which implies the theorem (see Luenberg, 1969, Lemma 1, page 61).

Define for each $k \in \mathbb{N}_0$, $t \in [-\delta/2, \delta/2]$ and $x \in \mathbb{R}$,

$$f_k(x) = (xt)^k f(x) a(x).$$

We will use a series version of the dominated convergence theorem applied to $\{f_k\}$. In the following we find a Lebesgue integrable function dominating uniformly (i.e. for all
We prove that the function $g$ inequality (see Luenberg, 1969, lemma 1, page 47) we obtain

$$|f(x)|a(x) \leq |f(x)|a(x) \sum_{k=0}^{\infty} \frac{|xt|^k}{k!}$$

Then the second term in the right side of (8) is in $L^2$ where the function $g$ is given, for all $x \in \mathbb{R}$, by

$$g(x) = \left\{ |f(x)|\sqrt{a(x)} \right\} \left\{ \sqrt{a(x)}(e^{xt} + e^{-xt}) \right\}.$$

We prove that the function $g$ is Lebesgue integrable. For, note that

$$\left\| f(\cdot)\sqrt{a(\cdot)} \right\|_{L^2(\lambda)}^2 = \int_{\mathbb{R}} |f(x)|^2 a(x) \lambda(dx) = \|f(\cdot)\|^2_a < \infty.$$ 

Then the first term in the right side of (8) is in $L^2(\lambda)$. On the other hand,

$$\left\| \sqrt{a(\cdot)}e^{(-x)t} \right\|_{L^2(\lambda)}^2 = \int_{\mathbb{R}} e^{2tx} a(x) \lambda(dx) = M(2t; a) < \infty$$

and

$$\left\| \sqrt{a(\cdot)}e^{-(-x)t} \right\|_{L^2(\lambda)}^2 = \int_{\mathbb{R}} e^{-2tx} a(x) \lambda(dx) = M(-2t; a) < \infty.$$ 

Then the second term in the right side of (8) is in $L^2(\lambda)$. Using the Cauchy-Schwartz inequality (see Luenberg, 1969, lemma 1, page 47) we obtain

$$\left| \int_{\mathbb{R}} g(x) \lambda(dx) \right| = \left| \langle f(\cdot)\sqrt{a(\cdot)}, \sqrt{a(\cdot)}(e^{(\cdot)t} + e^{-(\cdot)t}) \rangle \lambda \right|$$

$$\leq \left\| f(\cdot)\sqrt{a(\cdot)} \right\|_{L^2(\lambda)} \left\| \sqrt{a(\cdot)}(e^{(\cdot)t} + e^{-(\cdot)t}) \right\|_{L^2(\lambda)} < \infty.$$
Since (7) holds for each \( n \in \mathbb{N} \), \( x \in \mathbb{R} \), \( t \in [-\delta/2, \delta/2] \) and \( g \) is Lebesgue integrable we can use the series version of the dominated convergence theorem (see Billingsley, 1986, theorem 16.7 page 214) to obtain
\[
\int_{\mathbb{R}} e^{xt} f(x) a(x) \lambda(dx) = \int_{\mathbb{R}} \left\{ \sum_{k=0}^{\infty} \frac{(xt)^k}{k!} f(x) a(x) \right\} \lambda(dx)
\]
(from the series dominated convergence theorem)
\[
= \sum_{k=0}^{\infty} \left\{ \int_{\mathbb{R}} \frac{(xt)^k}{k!} f(x) a(x) \lambda(dx) \right\} = 0.
\]
We conclude that for all \( t \in [-\delta/2, \delta/2] \),
\[
M[t; f(\cdot)a(\cdot)] = 0. \tag{9}
\]
We show that (9) implies that \( f(\cdot) = 0 \) a.e. For,
\[
\|f(\cdot)\|^2_a = |< f(\cdot), 1>_a |
\]
\[
= |< \sqrt{f(\cdot)} \sqrt{f(\cdot)} e^{(\cdot)\delta/4} e^{(-\cdot)\delta/4}>_a |
\]
\[
= |< \sqrt{f(\cdot)} e^{(\cdot)\delta/4}, \sqrt{f(\cdot)} e^{(-\cdot)\delta/4}>_a |
\]
(from the Cauchy-Schwartz inequality)
\[
\leq \left\| \sqrt{f(\cdot)} e^{(\cdot)\delta/4} \right\|_a \left\| \sqrt{f(\cdot)} e^{(-\cdot)\delta/4} \right\|_a
\]
\[
= \left\{ \int_{\mathbb{R}} f(x) e^{\delta/2x} a(x) \lambda(dx) \right\}^{1/2} \left\{ \int_{\mathbb{R}} f(x) e^{-\delta/2x} a(x) \lambda(dx) \right\}^{1/2}
\]
\[
= \{M[\delta/2, f(\cdot)a(\cdot)]\}^{1/2} \{M[-\delta/2, f(\cdot)a(\cdot)]\}^{1/2}
\]
\[
= (\text{from (9)}) = 0.
\]
\[\Box\]

\[\text{2}\]The theorem states: "If \( \sum f_n \) converges almost everywhere and \( |\sum k=1^n f_k| \leq g \) almost everywhere, where \( g \) is integrable, then \( \sum f_n \) and the \( f_n \) are integrable, and \( \int \sum f_n d\lambda = \sum \int f_n d\lambda \)."
4 Functions with exponentially decaying tails

The following proposition gives a sufficient condition for having the Laplace transform defined in a neighborhood of zero, which is easy to verify.

**Proposition 3** Let \( f : \mathbb{R} \rightarrow [0, \infty) \) be a continuous function such that for some \( \delta > 0 \) and for all \( s \in [-\delta, \delta] \)

\[
\lim_{x \to +\infty} e^{sx} f(x) = \lim_{x \to -\infty} e^{sx} f(x) = 0.
\]

Then we have:

i) For all \( s \in (-\delta, \delta) \) the Laplace transform of \( f \), \( M(s; f) \), is finite.

ii) For all \( k \in \mathbb{N} \),

\[
\lim_{x \to +\infty} x^k f(x) = \lim_{x \to -\infty} x^k f(x) = 0
\]

**Proof:**

i) Take \( s \in (-\delta, \delta) \). Condition (10) implies that there exists \( L \in \mathbb{R}_+ \) such that for all \( x \in \mathbb{R} \setminus [-L, L] \), \( e^{\delta x} f(x) < 1 \) and \( e^{-\delta x} f(x) < 1 \). We have then

\[
M(s; f) = \int_{\mathbb{R}} e^{sx} f(x) \lambda(dx)
= \int_{[-L,L]} e^{sx} f(x) \lambda(dx) + \int_{[L,\infty)} e^{sx} f(x) \lambda(dx) + \int_{(-\infty,-L]} e^{sx} f(x) \lambda(dx)
= \int_{[-L,L]} e^{sx} f(x) \lambda(dx) + \int_{[L,\infty)} e^{(s-\delta)x} e^{\delta x} f(x) \lambda(dx) + \int_{(-\infty,-L]} e^{(\delta-s)x} e^{-\delta x} f(x) \lambda(dx)
\leq \int_{[-L,L]} e^{sx} f(x) \lambda(dx) + \int_{[L,\infty)} e^{(s-\delta)x} \lambda(dx) + \int_{(-\infty,-L]} e^{(\delta-s)x} \lambda(dx) < \infty.
\]

ii) For each \( k \in \mathbb{N} \)

\[
\lim_{x \to +\infty} x^k f(x) = \lim_{x \to +\infty} \{e^{-\delta x} x^k\} \{e^{\delta x} f(x)\} = 0
\]

and

\[
\lim_{x \to -\infty} x^k f(x) = \lim_{x \to -\infty} \{e^{\delta x} x^k\} \{e^{-\delta x} f(x)\} = 0
\]
References

[1] Billingsley, P. (1986). *Probability and Measure*. Second edition. John Wiley and Sons. New York.

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[3] Luenberg, D.G. (1969). *Optimization by Vector Space Methods*. John Wiley and Sons. New York.