Relation of nonlinear oscillator design based on phase reduction method and fractional derivative

Junichi Hongu | Daisuke Iba

1Mechanical and Physical Engineering, Tottori University, Tottori, Japan
2Mechanical Engineering, Kyoto Institute of Technology, Kyoto, Japan

Correspondence
Junichi Hongu, Mechanical and Physical Engineering, Tottori University, 4-101, Koyama-cho minami, Tottori-shi, Tottori, Japan.
Email: hongu@tottori-u.ac.jp

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Summary
For wide application of the synchronization properties of nonlinear oscillators in control and signal processing engineering field, a common design method for nonlinear oscillator models is required. The output response of a nonlinear oscillator excited by a periodic signal implicitly includes information on both the instantaneous phase and the amplitude of the input signal. The design of a nonlinear oscillator model that can synchronize with an arbitrary cyclic phenomenon can be enabled by explicitly expressing the dynamics of both phase and amplitude with the phase reduction method and organizing them by the number of events in one period. Assuming the cyclic phenomenon to be a single periodic signal, the properties of the generalized nonlinear oscillator model depend on the phase resolution of the signal, which is associated with the phase-shift of fractional calculus. Thus, this study verifies the validity of fractional calculus through the input-output characteristic of the generalized nonlinear oscillator model. Numerical simulation using a fractional differentiator with backward-difference demonstrated that increasing the phase resolution of a single periodic signal improves the estimation accuracy of the phase and amplitude of the signal by employing a generalized nonlinear oscillator model.

KEYWORDS
fractional derivative, nonlinear oscillator, phase reduction method, phase resolution

1 | INTRODUCTION

1.1 Nonlinear oscillator and its application

Nonlinear oscillators (limit cycle oscillators) and their synchronization phenomena have long attracted the attention of mathematicians, physicists, and engineers. The oldest record of oscillator synchronization can be found in Huygens’ letter. Subsequently, many self-excited oscillatory phenomena have been found, the dynamics of which have been expressed by mathematical models, including mechanical to chemical oscillators, the Van der Pol oscillator as an electronic oscillator based on the vacuum-tube oscillator, the Duffing oscillator as a mechanical oscillator based on the hardening spring, the Belousov–Zhabotinsky reaction (BZ reaction) as a chemical oscillator caused by the chemical reaction between bromine and acid, and the FitzHugh–Nagumo model as a neural oscillator based on the neural firing. The common
FIGURE 1  Synchronization property. (A) When the oscillator has no external input, it has a periodic solution with a natural frequency; when excited by an external input, the oscillator locks the phase of the external input. (B) In general, the frequency domain that the oscillator can synchronize with the external input is expanded in proportion to increasing the external input amplitude. Moreover, the synchronous regimes of arbitrary order $n : m$ can be observed.

FIGURE 2  Types of application of the nonlinear oscillator. Schematic illustration of (A) open-loop, (B) closed-loop, and (C) mutual-synchronization types

characteristics among these oscillator models are that they are classified as nonlinear dynamics; their mathematical expressions are written by a nonlinear autonomous system of differential equations (Equation 1); furthermore, they provide one periodic solution when the oscillator models have no external input.

$$\dot{x} = f(x) \neq Ax. \quad (1)$$

The most notable feature of the nonlinear oscillator is its synchronization property (entrainment$^7$), whereby the oscillator can change its natural frequency and synchronize with a periodic signal (Figure 1A). This entrainment occurs within a certain frequency range called the synchronization region (Arnold tongue$^8$; Figure 1B). In the physics field that involves nonlinear dynamics, synchronization phenomena among coupled oscillator groups have been actively researched using the Kuramoto model$^9$ (Equation 2). The Kuramoto model shown in Equation (2) is a simple model for calculating mutual-synchronization among nonlinear oscillators (Figure 2C). Here, $\omega_i$ is the natural frequency of each oscillator, $N$ is the number of the oscillators, and $K$ is the coupling coefficient between oscillators. Moreover, a detailed review of the Kuramoto model has been described by Acebrón et al.$^{10}$

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1 \cdots N. \quad (2)$$

At the same time, in control and signal processing engineering, there has been an advancement in the practical applications of the synchronization property. Accordingly, the studies surrounding the nonlinear oscillator are cross-sectoral issues and are being approached from both theoretical and application perspectives.
In control and signal processing engineering, the applications of nonlinear oscillators can be categorized into three types: (a) open-loop type, (b) closed-loop type, and (c) mutual-synchronization type, shown in Figure 2.

The open-loop type passively uses the synchronization property of an oscillator (or an oscillator group). A typical example is the phase-lock loop (PLL), which is inserted into a radio receiver to track the radio frequency. Another example is the injection locking of a laser array to obtain a high-intensity laser beam. Injection locking use self-sustained oscillators that are influenced by a joint external force. In recent years, in the vibration analysis field, the application of injection locking of oscillators to time-frequency analysis, such as wavelet transform, has been reported.

The closed-loop type has an interconnection between an oscillator (or oscillator group) and a control object. A typical example is a locomotive robot that uses a central pattern generator (CPG). The CPG is a biological neural circuit in the spinal cord composed of coupled neural oscillators that produce rhythmic output and cyclic locomotion, such as walking, swimming, and flying. The cyclic motion generated by the closed-loop between the CPG and musculoskeletal system is called global entrainment. Cyclic locomotion in global entrainment is highly stable against external perturbations; therefore, a locomotive robot using CPG can move robustly to withstand disturbances. The mechanism of global entrainment has also attracted the attention of the researchers who have theoretically studied control. For example, Iwasaki et al. analytically explained the orbital stability of global entrainment using the harmonic-balance method.

The mutual-synchronization type uses the mutual coupling among the oscillators, also called the oscillator network or collective synchronization. A typical example is a vibration transport machine with the self-synchronization of unbalanced rotors. Suppose one unbalanced rotor implies one nonlinear oscillator, this vibration transport machine utilizes the property of the mutual synchronization between two oscillators. Meanwhile, collective synchronization such as the synchronous flashing of fireflies, is frequently observed in nature; therefore, the phenomenon of mutual synchronization is one special topic in nonlinear dynamics in physics. A considerable number of studies have dealt with collective synchronization in the physics field. However, this aspect is beyond the scope of the present study.

As described, nonlinear oscillators are an integral part of control and signal processing engineering, wherein either physical or the virtual nonlinear oscillators widely employed. In physical nonlinear oscillators, the oscillator is mounted as an electric circuit and a mechanical structure, while in the virtual nonlinear oscillators, the oscillator is mounted as a mathematical model. In particular, in applications that use virtual nonlinear oscillators, the closed-loop type attempts to construct a flexible and robust control system based on a foundation that is completely different from the classic and modern control theory.

However, to the best of the authors' knowledge, a common design method for the nonlinear oscillator model has not been established. Although several nonlinear oscillator models exist, there is no common design method for nonlinear oscillator models that can synchronize various cyclic phenomena. The cyclic phenomena include all control objects with cyclic motions and all periodic signals, including scalar and vector signals. By formulating a common design method for the nonlinear oscillator model, the application field of the nonlinear oscillator can be extended. Particularly, the applications of the oscillator on the closed-loop system can meet the requirements of machines and mechanisms in the mechanical engineering field that have cyclic motions other than locomotive robots.

### 1.2 Phase reduction method

A powerful analysis method for understanding the synchronization phenomenon of a nonlinear oscillator is the phase reduction method. The phase-reduction method involves mapping from the periodic solution of the nonlinear oscillator model with n dimensions (Equation 1) to two scalar values: phase and amplitude. Two differential equations can be derived using the chain rule: phase equation and amplitude equation (Equation 3). Significantly, a well-known phase equation is the Kuramoto model (Equation 2). The foundation of the phase reduction method was established by Kuramoto and Winfree; subsequently, Kuramoto’s disciples have made significant contributions to theoretically expand the phase reduction method. In addition, to investigate the phase-noise problem, phase-amplitude analyses of the oscillator based on Floquet’s theory have been extensively performed in the electrical engineering.

\[
\begin{align*}
\dot{\theta} &= \omega_0 \quad \text{Phase equation} \\
\dot{\rho} &= -\kappa \rho \quad \text{Amplitude equation}
\end{align*}
\]
FIGURE 3  Input-output of generalized nonlinear oscillator model. The input is a periodic signal implicitly including the phase $\theta$ and the amplitude $A$. In this case, the outputs of the generalized nonlinear oscillator model $\theta$, $\rho$ are their estimated values.

As the phase-reduction method can be used for the analysis of an arbitrary mathematical model of a nonlinear oscillator, this method is a unified theory for nonlinear oscillators. In other words, if a nonlinear oscillator design method based on the phase reduction method is established, a widely usable nonlinear oscillator against various cyclic phenomena can be obtained.

1.3  Our current work

Hence, we attempted to establish a common design method for a nonlinear oscillator based on the phase-reduction method as follows.

Step 1. With the phase reduction method, differential equations expressing the dynamics of nonlinear oscillators can be transformed into phase and amplitude equations; next, they are expanded into the case where the nonlinear oscillator has a periodic signal.

Step 2. To obtain the expanded phase and amplitude equations that can synchronize with the various real control objects and real signals, the two differential equations are rewritten as two difference equations discretized by the number of events and their intervals in the cyclic phenomenon.

The advantage of this method is that the information necessary for the oscillator design is only the number of events and their intervals in one cycle; thus, our method enables us to design a nonlinear oscillator model against a sequence of phenomena whose start and end points coincide. Moreover, when the cyclic phenomenon is a single periodic signal, this generalized oscillator model also acts as a common-dimensional observer to estimate the instantaneous phase and amplitude of the signal (Figure 3).

1.4  Fractional calculus

In an attempt to establish a common design method for the nonlinear oscillator model, the relationship between the nonlinear oscillator and fractional calculus has gradually become evident. Fractional calculus is a mathematical analysis that studies the real-number powers of differential and integral operation. Historically, the $1/2$-derivative of a viscous fluid equation has long been known. In recent years, fractional-order PID controllers has been actively researched. The fractional-order proportional-integral-derivative (PID) controller is an extension of the classical PID controller.
Fractional-order controllers are less sensitive to changes in the parameters of a controlled system and controller; they can easily attain the property of iso-damping.\textsuperscript{46}

Assuming that the cyclic phenomenon is a single periodic signal, the number of events in a cycle indicates the phase resolution of the signal (Figure 4); moreover, the increased number of events further improves the convergence of the synchronization property of the generalized nonlinear oscillator. Meanwhile, one feature of fractional calculus in the signal processing process is an increase in the phase resolution of a signal. This implies that fractional calculus is useful in improving the convergence of the synchronization of the nonlinear oscillator, $\theta \to \vartheta$, $\rho \to A$ shown in Figure 4. Nonlinear oscillator models with the fractional calculus have been recently proposed and analyzed;\textsuperscript{47-50} however, to the best of the authors’ knowledge, no study has clarified the relationship between a nonlinear oscillator and fractional calculus.

1.5 | Purpose

Based on this background, the purpose of this study is to clarify the relation between the nonlinear oscillator and fractional calculus by investigating the input-output characteristic of the generalized nonlinear oscillator model, and to verify that the increment of the phase resolution of a signal is directly linked to the improvement of the estimation accuracy of the phase and amplitude of the signal using the generalized nonlinear oscillator model.

This paper is structured as follows: First, an explanation of the common design method for the nonlinear oscillator model is presented. Next, the input-output characteristics of the generalized nonlinear oscillator model are discussed. We reuse the content described in Hongu and Iba\textsuperscript{40} here. Finally, it is verified that utilizing fractional calculus enables us to improve the estimation accuracy of the phase and amplitude of the signal using the generalized nonlinear oscillator model.

2 | PHASE REDUCTION METHOD

The phase-reduction method\textsuperscript{9,29} is a perturbation method for the periodic solution of a nonlinear oscillator model. With the phase reduction method, the characteristics of the nonlinear oscillator model can be presented by two equations: the phase and the amplitude equations. This section introduces the basic concept of the phase-reduction method.

2.1 | Floquet theory

First, we express a nonlinear oscillator model with a periodic solution as a nonlinear autonomous system of differential equations, where $\dot{x}$ means the time derivative $\frac{dx}{dt}$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{phase_resolution}
\caption{Phase-resolution. This figure describes the concept of the phase resolution in the polar coordinates system with the input and outputs shown in Figure 3.}
\end{figure}
\[
\begin{align*}
\dot{x} &= f(x) \\
\mathbf{x} &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T \in \mathbb{R}^n \quad (4) \\
f(x) &= \begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \end{bmatrix}^T : \mathbb{R}^n \to \mathbb{R}^n.
\end{align*}
\]

By providing a periodic solution to Equation (4) is \( p \) and the infinitesimal change from the orbit is \( z \), the first approximation of the Taylor series of Equation (4) is

\[
\dot{x} = f(p + z) \approx f(p) + \frac{\partial f}{\partial \mathbf{x}} z. \quad (5)
\]

Here, \( \frac{\partial f}{\partial \mathbf{x}} \) denotes the Jacobian matrix. With \( A = \frac{\partial f}{\partial \mathbf{x}} \), the dynamics of \( z \) can be expressed as

\[
\dot{z} = Az. \quad (6)
\]

**Theorem 1** (Floquet theorem\textsuperscript{51}). If \( A \) is a periodic matrix, then a fundamental matrix solution \( \Phi = P \exp(-Bt) \) exists for \( \dot{\Phi} = A\Phi \), where \( \Phi, P, B \) are \( n \times n \) matrices.

Therefore, using the matrix transform \( y = P^{-1}z \), Equation (6) becomes

\[
\dot{y} = By. \quad (7)
\]

Given that the particular-periodic solutions of Equation (7) are the time derivatives \( \dot{p}, \ddot{p}, \ldots, p^{(n-1)} \) and a general-periodic solution of Equation (7) is \( q \), we have

\[
P = \begin{bmatrix} \dot{p} & \ddot{p} & \cdots & p^{(n-1)} & q \end{bmatrix}. \quad (8)
\]

In this case, the matrix \( B \) can be expressed as

\[
B = \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \lambda \end{bmatrix}. \quad (9)
\]

Here, \( \lambda \) is the Floquet factor. Considering the convergence condition \( t \to \infty \Rightarrow y \to 0 \), we find that the sufficient condition for asymptotic stability of the limit cycle is \( \lambda < 0 \).

With \( \lambda = -\kappa \), substituting \( z = qy \) into Equation (6) yields the essential relational expression, or

\[
\dot{q} = \kappa q + Aq. \quad (10)
\]

### 2.2 Phase reduction

Next, the two scalar fields \((\theta, \rho)\) are defined as

**Definition 1.**

\[
\mathbf{x}(\theta, \rho) := p(\theta) + \rho q(\theta). \quad (11)
\]

We then map the state vector of the nonlinear oscillator model onto the scalar fields (Figure 5). Here, \( \theta \) is the phase that increases linearly from 0 to \( 2\pi \) in the interval \( T \) of one cycle of the limit cycle \((\dot{\theta} = \frac{2\pi}{T} = \omega_0)\), and \( \rho \) is the scale parameter multiplied \( q(\theta) \) (not the Euclidean distance).

In this case, the gradients of the scalar fields \( \text{grad}_x \theta \) and \( \text{grad}_x \rho \) are defined as
Phase reduction method. In this figure, the black closed curve is the limit cycle of a nonlinear oscillator in the phase space; furthermore, by Definition 1 (Equation 11), the blue vector $\mathbf{p}(\theta)$ and the green vector $\rho \mathbf{q}(\theta)$ form the orange vector $\mathbf{x}(\theta, \rho)$ as the state vector of the nonlinear oscillator on an isochrone.

\[
\begin{align*}
\text{grad}_x \theta \cdot \frac{\partial x}{\partial \theta} &= I_{nnn}, \\
\text{grad}_x \rho \cdot \frac{\partial x}{\partial \rho} &= I_{nnn}.
\end{align*}
\] (12)

Here, $\text{grad}_x \theta \perp \text{grad}_x \rho$.

In scalar fields, by substituting Equation (13) into Equation (4) yields Equation (14).

\[
\begin{align*}
\dot{x} (\theta, \rho) &= \dot{p} (\theta) + \rho \dot{q} (\theta) + \rho \dot{q} (\theta) \\
\mathbf{f} (\mathbf{x} (\theta, \rho)) &\simeq \left\{ \frac{\partial \mathbf{p} (\theta)}{\partial \theta} + \rho \frac{\partial \mathbf{q} (\theta)}{\partial \theta} \right\} \dot{\theta} + \rho \mathbf{q} (\theta) \dot{\rho}, \\
\dot{p} (\theta) + \rho \dot{q} (\theta) + \rho \dot{q} (\theta) &\simeq \left\{ \frac{\partial \mathbf{p} (\theta)}{\partial \theta} + \rho \frac{\partial \mathbf{q} (\theta)}{\partial \theta} \right\} \dot{\theta} + \rho \mathbf{q} (\theta) \dot{\rho}.
\end{align*}
\] (13)

(14)

Assuming $\rho$ is extremely small, multiplying the left-right side of Equation (14) by $\text{grad}_x \theta$ yields the phase equation; likewise, multiplying the left-right side of Equation (14) by $\text{grad}_x \rho$ yields the amplitude equation. Using the relational expression in Equation (10), these equations are expressed as

\[
\begin{align*}
\dot{\theta} &\simeq \omega_0, \\
\dot{\rho} &\simeq -\kappa \rho,
\end{align*}
\] (15)

where $\omega_0$ is the natural frequency of the nonlinear oscillator.

2.3 External perturbation

Considering an external perturbation $\epsilon$ for the nonlinear oscillator model, we have

\[
\dot{\mathbf{x}} = \mathbf{f} (\mathbf{x}) + \epsilon.
\] (16)
Assuming that the perturbation $\varepsilon$ is very small, the effects of the perturbation on the scalar fields $(\theta, \rho)$ can be approximated as the phase and amplitude responses of the limit cycle. Hence,

$$\begin{align*}
\dot{\theta} &\approx \omega_0 + \nabla_x \theta \cdot \varepsilon \\
\dot{\rho} &\approx -\kappa \rho + \nabla_x \rho \cdot \varepsilon.
\end{align*}$$

(17)

From these operations, the response of the nonlinear oscillator model to external perturbation can be considered separately as the phase and amplitude responses.

This is the basic concept of the phase-reduction method. By replacing the perturbation with a periodic input, Equation (17) can be expanded and the synchronization property of the oscillator is discussed.

3 | DESIGN PREPARATION

Designing a nonlinear oscillator model based on the phase-reduction method involves creating both phase and amplitude equations. Thus, in this study, we consider the forms of the phase and amplitude equations that can be easily designed.

3.1 | Periodic input

Considering a periodic input $r$ in steady-state$^1$ for the nonlinear oscillator model, we have

$$\dot{x} = f(x) + r(t).$$

(18)

Next, two scalar fields $(\bar{\theta}, \bar{\rho})$ governed by the external input are newly defined as

**Definition 2.**

$$x(\bar{\theta}, \bar{\rho}) := p(\bar{\theta}) + \bar{\rho}q(\bar{\theta}).$$

(19)

We then map the state vector of the nonlinear oscillator model on the new scalar fields. Hence,

$$\begin{align*}
\dot{\bar{\theta}} &\approx \omega \\
\dot{\bar{\rho}} &\approx -\kappa \bar{\rho}.
\end{align*}$$

(20)

Here, $\omega$ is the frequency of the external input. Assuming that the external input is extremely small, substituting Equation (21) into Equation (18) yields Equation (22).

$$\begin{align*}
x(\bar{\theta}, \bar{\rho}) &= p(\bar{\theta}) + \bar{\rho}q(\bar{\theta}) \\
f(x) + r(t) &\approx f(p) + r(t)
\end{align*}$$

(21)

$$r(t) \approx \bar{\rho}q(\bar{\theta}).$$

(22)

Because the external input is in the steady state$^1$ ($\dot{\bar{\rho}} = 0$), we have

$$r(t) \approx \bar{\rho}q(\bar{\theta}).$$

(23)

By replacing $\varepsilon$ in Equation (17) by $r$, we obtain
\[
\begin{align*}
\dot{\theta} & \simeq \omega_0 + \overset{x=\mathrm{p}(\theta)}{\mathrm{grad}_x} \theta \cdot \bar{\rho} \mathbf{q} (\bar{\theta}), \\
\dot{\rho} & \simeq -\kappa \rho + \overset{x=\mathrm{p}(\theta)}{\mathrm{grad}_x} \rho \cdot \bar{\rho} \mathbf{q} (\bar{\theta}).
\end{align*}
\] (24)

\*\*1 In this case, the steady-state implies that the frequency- and amplitude-fluctuations of the periodic input are relatively small.

### 3.2 Phase-equation

For the phase-equation, using \( Y_\theta (\theta) = \overset{x=\mathrm{p}(\theta)}{\mathrm{grad}_x} \theta \) and \( \varphi = \bar{\theta} - \theta \), we obtain

\[
\dot{\varphi} \simeq \omega - \omega_0 - \bar{\rho} Y_\theta (\bar{\theta} - \varphi) \cdot \mathbf{q} (\bar{\theta}).
\] (25)

Assuming that the phase response is relatively faster than the change in \( \varphi \) in one cycle and averaging the fast component on the right side of this equation\*\*2, we have

\[
\dot{\varphi} \simeq \omega - \omega_0 - \bar{\rho} \int_0^{2\pi} Y_\theta (\bar{\theta} - \varphi) \cdot \mathbf{q} (\bar{\theta}) d\bar{\theta}
= \omega - \omega_0 - \bar{\rho} \Gamma (\varphi).
\] (26)

Here, \( \Gamma (\varphi) \) is called the phase coupling function. If \( \Gamma (\varphi) \) involves a stable point \( \dot{\varphi} d\varphi < 0 \), the phase locking occurs, that is, the nonlinear oscillator model can synchronize with the input. Additionally, when we choose the approximation \( \bar{\rho} \Gamma (\varphi) \approx K \sin \varphi \), we obtain the Kuramoto model.\*\*3 In contrast, we choose the approximation \( \bar{\rho} \Gamma (\varphi) \approx \alpha \varphi \) to obtain a mathematically tractable phase-equation. Assuming that the frequency detuning is very small\*\*3 \( (\omega - \omega_0 \ll \bar{\rho}) \), we have a simple phase-equation, or

\[
\dot{\varphi} \simeq -\alpha \varphi.
\] (27)

Here, the range of \( \varphi \) is limited to \(-\pi \leq \varphi < \pi \).

### 3.3 Amplitude-equation

Similarly, for the amplitude equation, using \( X_\theta (\rho) = \overset{x=\mathrm{p}(\theta)}{\mathrm{grad}_x} \rho \) and \( e = \bar{\rho} - \rho \), we have

\[
\dot{e} \simeq \kappa \bar{\rho} - \kappa \rho - \bar{\rho} X_\theta (\bar{\rho} - e) \cdot \mathbf{q} (\bar{\theta}).
\] (28)

Assuming that the phase difference is very small\*\*3 \( (\bar{\theta} - \theta \approx 0) \), using the relationship \( X_\theta (\bar{\rho} - e) \approx X_\theta (\bar{\rho} - e) \perp \mathbf{q} (\bar{\theta}) \), we have

\[
\dot{e} \simeq -\kappa e.
\] (29)

The phase equation considering a periodic input such as the Kuramoto model\*\*9 is well known, whereas the amplitude equation considering a periodic input is not well known; however, we consider both the phase and amplitude equations\*\*34,35 to provide the generalized nonlinear oscillator model. This is a unique feature of this study.

\*\*2 The error of the averaging method is bounded and expressed as \( \varphi = \varphi_{av} + O (\bar{\rho}) \) where \( \bar{\rho} \ll 1 \); thus, this method yields a good approximation for the qualitative behavior of a general nonlinear oscillator.
Although the frequency detuning and the phase difference do not become zero; however, on these assumptions, Equation (27) and Equation (29) capture the essence of synchronization of a general nonlinear oscillator in the sense that it independently operates its phase-response and amplitude-response in the neighborhood area of its limit cycle.

### 3.4 Design objective

Finally, based on the phase reduction method, we obtain the mathematically tractable phase equation and amplitude equation, or

\[
\begin{align*}
\dot{\phi} &= -\alpha \phi \\
\dot{e} &= -\kappa e
\end{align*}
\]  

or

\[
\begin{align*}
\dot{\theta} &= \omega_0 + \alpha \varphi \\
\dot{\rho} &= \kappa e
\end{align*}
\]  

This equation states that \( t \to \infty \Rightarrow \varphi \to 0, e \to 0 \) if \( a > 0, \kappa > 0 \). In this study, we used Equation (30) or Equation (31) as the design objective, and expand them against the various cyclic phenomena; then, we will establish a common design method for the nonlinear oscillator model.

### 4 NONLINEAR OSCILLATOR MODEL DESIGN

To establish a common design method for a nonlinear oscillator model that can synchronize with various cyclic phenomena, we consider the number of events \( N \) occurring in one cycle of the phenomenon. In this section, we discretize Equation (30) or Equation (31) by the number of events and obtain a generalized nonlinear oscillator model; then show its design process.

#### 4.1 Phase equation

##### 4.1.1 Event and phase

The virtual nonlinear oscillator used in the applications described in Section 1 should synchronize with the various periodic signals and cyclic motions. Phase locking requires the continuous phase information for these cyclic phenomena; unfortunately, we can only observe discrete phase information through the observable events. That is, the events explicitly contain phase information. Thus, we mapped an event to a phase and designed a generalized nonlinear oscillator model (Figure 6).

##### 4.1.2 Discretization

Given that the number of events is \( N \) and the number of cycles is \( L \), we discretize the phase equation in Equation (30). We then obtain the recursion formula as follows: where \( \xi_i \) denotes the interval between events.

\[
\begin{align*}
\varphi_{L+1/N} &= \varphi_L - \alpha \varphi_L \xi_1 \omega_0^{-1} \\
\varphi_{L+2/N} &= \varphi_{L+1/N} - \alpha \varphi_{L+1/N} \xi_2 \omega_0^{-1} \\
&\vdots \\
\varphi_{L+1} &= \varphi_{L+(N-1)/N} - \alpha \varphi_{L+(N-1)/N} \xi_N \omega_0^{-1}
\end{align*}
\]  

(32)
Events and phases of a cyclic phenomenon. The schematic illustrations of events and phases of a cyclic phenomenon are shown to the left and right. Here, the begin and end of the phenomenon correspond \( \theta = 0 \) and \( \theta = 2\pi \), respectively, and the event \( i \) corresponds \( \theta = \theta_i \).

\[
\varphi_{L+1} = \prod_{i=1}^{N} (1 - \omega_0^{-1} a_i \xi_i) \varphi_L. \tag{33}
\]

We replacing \( a_i \xi_i \) with \( a_i' \xi_i \) where \( a' = a_i \xi_i = \text{const.} \), we have

\[
\varphi_{L+1} = (1 - \omega_0^{-1} a')^N \varphi_L. \tag{34}
\]

In this case, the convergence condition of the recursion is \( 0 < a' < 2\omega_0 \) regardless of whether \( N \) is even or odd.

### 4.1.3 Difference equation to differential equation

Using the zero-order hold function and delta function, the recursion formula becomes

\[
\dot{x} = -a' \text{hold} \left[ \sum_{i=1}^{N} \xi_i^{-1} \varphi_i \cdot \delta(t - t_i) \right] \{ t_i \in \mathbb{R} | \theta(t_i) = \theta_i \}. \tag{35}
\]

Here,

\[
\text{hold} (x(t)) := \begin{cases} x(t) \cdots x \neq 0 \\ x(t - dt) \cdots x = 0 \end{cases} \tag{36}
\]

### 4.1.4 Event detection function

To map an event to a phase, we arrange Equation (35) as following:

First, because \( \varphi_i = \theta_i - \theta \), the vector function, called the phase-sensitivity function, is defined as

\[
Y = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_N \end{bmatrix}^T = \begin{bmatrix} \xi_1^{-1}w(\theta_1 - \theta) & \xi_2^{-1}w(\theta_2 - \theta) & \cdots & \xi_N^{-1}w(\theta_N - \theta) \end{bmatrix}^T. \tag{37}
\]

Here,

\[
w(x) := \text{wrap} \ x \ to \ -\pi \leq x < \pi. \tag{38}
\]
FIGURE 7  Event detection function. In this figure, the gray planes mean the isochrones of each event \( \theta = \theta_i \); furthermore, the scalar function \( \Lambda_i(x) \) detects the event when the state of phenomenon \( x \) passes the isochrone.

Next, the event-detection function is defined as

\[
\mathbf{H} = \begin{bmatrix} H_1 & H_2 & \cdots & H_N \end{bmatrix}^\text{tr} = \begin{bmatrix} h_1(r) & h_2(r) & \cdots & h_N(r) \end{bmatrix}^\text{tr}.
\]  

(39)

Here,

\[
h_i(x) := \begin{cases} 1 & \Lambda_i(x) \uparrow 0 \\ 0 & \Lambda_i(x) \neq 0 \end{cases}
\]

(40)

\[
\Lambda_i(x) := \begin{cases} > 0 & \cdots \theta(x) > \theta_i \\ 0 & \cdots \theta(x) = \theta_i \{ \Lambda_i(x) : \mathbb{R}^m \to \mathbb{R} \} \\ < 0 & \cdots \theta(x) < \theta_i \end{cases}
\]

(41)

The vector signal \( r \) is the observable state of a cyclic phenomenon, and \( \Lambda_i(x) \) is a scalar function that determines the passage of an isochrone as positive or negative (Figure 7).

Finally, Equation (35) becomes the simple expression, or

\[
\dot{\varphi} = -\alpha' \text{hold} \left[ \mathbf{Y}(\theta) \cdot \mathbf{H}(r) \right] = -\alpha' \varphi'
\]

(42)

or

\[
\dot{\theta} = \omega_0 + \alpha' \text{hold} \left[ \mathbf{Y}(\theta) \cdot \mathbf{H}(r) \right] = \omega_0 + \alpha' \varphi'.
\]

(43)

If we consider a larger number of events, this equation approaches the design objective: \( \lim_{N \to \infty} \alpha' \varphi' = \alpha \varphi \).

4.1.5  Synchronization region

Because phase locking occurs in a frequency range called the synchronization region, this range must be known. Given that the frequency of a cyclic phenomenon is \( \omega \), we average the fast component on the right side of Equation (43).
**Figure 8** Block diagram of phase-equation. This system has the state vector of a cyclic phenomenon \( r \) as an input and the phase of the oscillator \( \theta \) as an output. Here, the black signal-lines and elements represent the forward signals and elements, while the blue signal-lines and elements represent the feedback signals and elements related to \( \theta \).

\[
\dot{\theta} = \omega_0 + \frac{1}{2\pi} \int_0^{2\pi} \alpha' \phi' d\theta = \omega_0 + \frac{\alpha'N}{2\pi} \phi. \tag{44}
\]

With \( \dot{\phi} = \omega - \phi \), we have

\[
\dot{\phi} = \omega - \omega_0 - \frac{\alpha'N}{2\pi} \phi. \tag{45}
\]

The stable condition \((\dot{\phi}d\phi < 0)\) can be written as

\[
|\omega - \omega_0| < \frac{\alpha'N}{2}. \tag{46}
\]

Subsequently, by replacing \( \omega_0 \) with \( \omega \) in Equation (34), we have

\[
\varphi_{L+1} = \left(1 - \omega^{-1} \alpha'\right)^N \varphi_L. \tag{47}
\]

The convergence condition of this equation can be written as:

\[
\frac{\alpha'}{2} < \omega. \tag{48}
\]

Thus, the following equation defines the one-by-one synchronization region of the designed oscillator model.

\[
-\frac{\alpha'N}{2} < \omega - \omega_0 < \frac{\alpha'N}{2} \cap \frac{\alpha'}{2} < \omega. \tag{49}
\]

### 4.1.6 Block diagram

A block diagram of the phase equation is shown in Figure 8, which can be easily simulated by MATLAB/Simulink.

### 4.1.7 Frequency tracking

Considering the integration term of the phase difference, we can add the functionality of frequency tracking; furthermore, we have
\[ \dot{\varphi} = -\alpha \varphi - \beta \int \varphi \, dt. \] (50)

The time derivative of this equation is

\[ \ddot{\varphi} = -\alpha \dot{\varphi} - \beta \varphi. \] (51)

Thus, \( t \to \infty \Rightarrow \varphi \to 0, \varphi \to 0 \) if \( \alpha > 0, \beta > 0 \).

Using the relationship \(-\alpha \varphi \equiv -\alpha' \varphi'\), Equation (50) can be rewritten as

\[ \dot{\varphi} = -\alpha' \varphi' - \beta' \int \varphi' \, dt. \] (52)

Arranging Equation (52) with the number of events \( N \) and the number of cycles \( L \) yields the recursion formula, or

\[ \varphi_{L+1} = (I - \omega_0^{-1} \alpha')^N \varphi_L. \] (53)

Here,

\[
\begin{align*}
\varphi &= \begin{bmatrix} \varphi' \\ \varphi \end{bmatrix} \\
\alpha' &= \begin{bmatrix} \alpha' & \beta' \\ 1 & 0 \end{bmatrix}.
\end{align*}
\] (54)

In this case, the convergence condition \( \varphi_L \to 0 \) is \( 0 < \beta' \leq \frac{\alpha'^2}{4} \).

When we add the functionality of frequency tracking, the synchronization region becomes \( \frac{\omega'}{2} < \omega < \infty \); however, the nonlinear oscillator has a hysteresis property (frequency discrimination\(^{11}\)) outside the original synchronization region shown in Equation (49). Thus, we use the wrap function to consider this or

\[ \dot{\theta} = \omega_0 + \alpha' \varphi' + W \left( \beta' \int \varphi' \, dt \right). \] (55)

Here,

\[ W(x) := \text{wrap } x \text{ to } \omega_{\min} - \omega_0 \leq x < \omega_{\max} - \omega_0. \] (56)

The maximum-minimum frequency range \( \omega_{\min} < \omega < \omega_{\max} \) can be calculated using Equation (49).

By adding the frequency tracking functionality, the design parameter is increased by one: \( 0 < \beta' \leq \frac{\alpha'^2}{4} \). Assuming that the frequency change of the cyclic phenomenon is relatively slow\(^{4}\), we can design parameter \( \beta' \) as \( \beta' \ll \alpha' \) in a practical manner.

\(^{4}\)In this case, we assume that the cyclic phenomenon is in steady-state.

### 4.2 Amplitude-equation

#### 4.2.1 Concept of amplitude

For the nonlinear oscillator model, its amplitude \( \rho \) is an abstract scalar value. In other words, the phase \( \theta \) is uniquely determined for a cyclic phenomenon, whereas the amplitude \( \rho \) is not. Therefore, we define the amplitude as the instantaneous amplitude of a signal \( r \in \mathbb{R} \) in this study.
4.2.2 Formulation of amplitude-equation

When signal $r$ is assumed to be

$$r \approx A \sin \theta,$$  \hspace{1cm} (57)

we have

$$e = A - \rho.$$  \hspace{1cm} (58)

However, the signal implicitly involves instantaneous amplitude $A$. Thus, we define the following equation instead of Equation (58).

$$e' = |r| - \rho |\sin \theta|$$  \hspace{1cm} (59)

Therefore, the amplitude equation can be written as:

$$\dot{e} = -\kappa' e'$$  \hspace{1cm} (60)

or

$$\dot{\rho} = \kappa' e'.$$  \hspace{1cm} (61)

Assuming that the amplitude response is relatively faster than the amplitude changes of the signal, and averaging the fast component on the right side of this equation, we have

$$\dot{e} \approx -\kappa' \left( A \frac{1}{2\pi} \int_0^{2\pi} |\sin \theta| \, d\theta - \rho \frac{1}{2\pi} \int_0^{2\pi} |\sin \theta| \, d\theta \right).$$  \hspace{1cm} (62)

Assuming that the convergence speed of phase $\theta \to \theta$ is sufficiently faster than the convergence speed of amplitude $\rho \to A$, we have

$$\dot{e} \approx -\frac{2\kappa'}{\pi} (A - \rho).$$  \hspace{1cm} (63)

As the convergence speed of phase increases with a larger number of events; with a larger number of events, this equation approaches the design objective: $\lim_{N \to \infty} \kappa' e' = \kappa e$.

In addition, we can also define multiple amplitudes using the following equation because the amplitude is not uniquely determined:

$$\begin{cases} \dot{\rho}_i = \kappa' e'_i, \\ e'_i = |r_i| - \rho_i |\sin (\theta - \theta_i)| \end{cases}$$  \hspace{1cm} (64)

\hspace{0.5cm} *The error of the averaging method is bounded and expressed as $e = e_{av} + O(\kappa')$ where $\theta \approx \bar{\theta}$; thus, this method yields a good approximation for the qualitative behavior of Equation (60).

4.3 Summary

Finally, the nonlinear oscillator model that we generalized using the number of events of a cyclic phenomenon is expressed as
FIGURE 9  Block diagram of generalized nonlinear oscillator model. This system has the state vector of a cyclic phenomenon $r$ as an input and the phase and amplitude of the oscillator $\theta$, $\rho$ as outputs. Here, the black signal-lines and elements represent the forward signals and elements, while the blue signal-lines and elements represent the feedback signals and elements related to $\theta$.

\[
\begin{align*}
\dot{\theta} &= \omega_0 + \alpha' \varphi' + W (\beta' \int \varphi' dt), \\
\dot{\rho} &= \kappa' \varrho'
\end{align*}
\]  

(65)

where

\[
0 < \alpha' < 2\omega_0, 0 < \beta' \leq \frac{\alpha'^2}{4}, \kappa' > 0.
\]

(66)

The design procedure is as follows.

1. Phase equation
   i. Count the number of observable events in one cycle: $N$.
   ii. Determine whether the phases correspond to each event and their intervals: $\theta_i$, $\xi_i$.
   iii. Choose the observable state: $r$.
   iv. Define the event detection functions: $\Lambda_i$.
   v. Tune the natural frequency of the oscillator model nearly equaling the frequency of a cyclic phenomenon: $\omega_0 \approx \omega$.
   vi. Design the parameters considering the synchronization region and the convergence speed: $\alpha'$, $\beta'$.

2. Amplitude equation (optional)
   i. Choose the observable state: $r \in r$.
   ii. Design the parameters in consideration of the convergence speed: $\kappa'$.

A full-block diagram of the generalized nonlinear oscillator model is shown in Figure 9, which can also be easily simulated by MATLAB/Simulink.

As mentioned above, we attempt to build a common design method for a nonlinear oscillator model that can synchronize various cyclic phenomena based on the phase reduction method. Our concept is to discretize the design objective (Equation 30 or Equation 31) by the number of events of the phenomenon.

5 | SYNCHRONIZATION PROPERTY WITH SINGLE PERIODIC SIGNAL

This section briefly presents the input-output characteristic as an open-loop system of the generalized nonlinear oscillator model when the cyclic phenomenon is a single periodic signal.
FIGURE 10 Events and phases of a single periodic signal. The left part shows the schematic of events of the single periodic signal, and the right part shows the phases corresponding to them on a unit circle, where the number of events is $N = 2$.

| Number of events | $N = 2$ |
|------------------|---------|
| Phases of events | $\theta_1 = 0, \theta_2 = \pi$ |
| Intervals of events | $\xi_1 = \xi_2 = \pi$ |
| Observable state (phase equation) | $r = \begin{bmatrix} p & -p \end{bmatrix}$ |
| Event detection function | $\Lambda_1 (r) := p, \Lambda_2 (r) := -p$ |
| Observable state (amplitude equation) | $r = p$ |

5.1 Single periodic signal

Assuming that the cyclic phenomenon is a single periodic signal, we have

$$p = A \sin (\omega t + \delta).$$  \hspace{1cm} (67)

The events and phases involved in this phenomenon can be described as in Figure 10. From Figure 10, we immediately observe that the number of events is $N = 2$ by its zero-crossing timing. The parameters for the structural design of the oscillator are summarized in Table 1.

Figure 11 shows the input-output of the designed nonlinear oscillator model when the cyclic phenomenon is a single periodic signal. From Figure 11, it can be observed that, although there is the frequency detuning between the signal and the oscillator, phase-locking occurs; furthermore, the oscillator's amplitude has a value close to the input's amplitude. This implies that the designed nonlinear oscillator model can synchronize with a single periodic signal. In addition, the theoretical synchronization region calculated using Equation (49) is $0.5 < \omega < 2$ when $N = 2, \omega_0 = 1, \alpha' = 1$, which is in agreement with the simulation result. The synchronization region is determined using a chirp signal.

5.2 Nonlinear oscillator design problem and observer design problem

Furthermore, supposing that the cyclic phenomenon is a single periodic signal, the outputs of the phase and the amplitude equations of the generalized nonlinear oscillator model correspond to the instantaneous phase and instantaneous amplitude of the signal. In other words, the design problem for the generalized nonlinear oscillator model equals the observer design problem for the instantaneous phase and amplitude of the signal.

Assuming that a real signal and its dynamics are expressed as Equation (68) and Equation (69), we define the dynamics of the virtual signal as Equation (70).

$$R(t) = A(t) \exp (j\Theta(t)), \hspace{1cm} (68)$$

$$\begin{cases} \dot{\Theta} = \Omega \\ \dot{A} = 0 \end{cases} \hspace{1cm} (69)$$
FIGURE 11  Synchronization property with a single periodic signal. (a) The input of the designed oscillator $p$ is a single periodic signal. (b) The outputs of the designed oscillator $\theta, \rho$ are the estimated values of the input's instantaneous phase and amplitude. (c) The vertical axis is the frequency ratio of the input and output, and the horizontal axis is the input's frequency. In this case, the flat domain in this graph represents the synchronization region of the designed oscillator.

\[
\begin{align*}
\dot{\hat{\Theta}} &= \Omega \\
\dot{\hat{A}} &= 0
\end{align*}
\]

(70)

With $\Phi = \Theta - \hat{\Theta}$ and $E = A - \hat{A}$, the terms for these differences are added to Equation (70) yielding

\[
\begin{align*}
\dot{\Phi} &= -\mathcal{A}\Phi \\
\dot{E} &= -\mathcal{K}E
\end{align*}
\]

(71)

or

\[
\begin{align*}
\dot{\hat{\Theta}} &= \Omega + \mathcal{A}\Phi \\
\dot{\hat{A}} &= \mathcal{K}E
\end{align*}
\]

(72)

where, $\mathcal{A}$ and $\mathcal{K}$ are design parameters that determine the convergence of the observer.

This equation states that $t \to \infty \Rightarrow \Phi \to 0, E \to 0$ if $\mathcal{A} > 0, \mathcal{K} > 0$. The range of $\Phi$ is limited by $-\pi \leq \Phi < \pi$.

The statements of Equation (30) and Equation (71) are introduced using two different but equivalent methods: the phase reduction method and the common-dimensional observer design method. Hence, the design problem of the non-linear oscillator model with the phase coupling function $\Gamma(\phi) = \alpha \phi$ and Floquet factor $\lambda = -\kappa$ is the same as the observer design problem for the instantaneous phase and amplitude of the single periodic signal, or
5.3 Pseudo-property of iso-damping and flat-gain

As the faculties of the generalized nonlinear oscillator model and the observer are equal, we can generate a new signal from the output of the nonlinear oscillator model. Additionally, we can independently add an arbitrary phase-shift $\psi$ and an arbitrary gain $\sigma$ to the new signal. This mathematical operation can achieve the pseudo-property of iso-damping and flat-gain (Figure 12).

Given that $p$ and $q$ are the input signal and the newly generated signal, respectively, the transfer function can be approximately expressed as Equation (75).

$$
\begin{align*}
    p &= A \exp(j\theta) \\
    q &= \sigma \rho \exp(j\theta) \exp(j\psi) \approx \sigma A \exp(j\theta) \exp(j\psi),
\end{align*}
$$

To verify this pseudo-property, we simulated the frequency response from $p$ to $q$ using a chirp signal, where $\psi = 0, \sigma = 1$. Next, we obtain the phase-plot and the magnitude-plot shown in Figure 13 by considering the following equation.

$$
\begin{align*}
    \angle G(s) &\approx \theta - \theta \\
    |G(s)| &\approx \frac{\sigma}{A}
\end{align*}
$$

From Figure 13, we find that the frequency response from $p$ to $q$ has the pseudo-property of both iso-damping and flat-gain in the oscillator’s synchronization region.
As presented above, assuming that the cyclic phenomenon is a single periodic signal, the generalized nonlinear oscillator model has the role as a common dimensional observer to attain the pseudo-property of iso-damping and flat-gain. More design examples and demonstrations are described in Hongu and Iba.\cite{Hongu2016}

6 | RELATION OF GENERALIZED NONLINEAR OSCILLATOR MODEL AND FRACTIONAL DERIVATIVE

As the number of events of a signal indicates its phase resolution, which is deeply associated with the faculty of phase-shift present in the fractional calculus, we can expand the generalized nonlinear oscillator model to the fractional-order signal. This section describes a technique to increase the number of events using a fractional derivative.

6.1 | Fractional derivative

In this study, we use the Grünwald-Letnikov derivative\cite{Laskar2002} which is an expression of fractional calculus, or

\[
D^\gamma f(t) := \lim_{h \to 0} \left\{ \frac{1}{h^\gamma} \sum_{0 \leq k < \infty} (-1)^k \frac{\Gamma(\gamma + 1)}{\Gamma(k + 1)\Gamma(\gamma - k + 1)} f(t - kh) \right\} \cdots \gamma \in \mathbb{R}.
\] (77)

Here, $\Gamma(\cdot)$ is the gamma function.

The Laplace transform of the fractional derivative is defined as follows:

\[
\mathcal{L}\{D^\gamma f(t)\} := s^\gamma \mathcal{L}\{f(t)\}.
\] (78)

In this case, to improve the sensitivity of the oscillator, we use the fractional derivative and not the fractional integral; hence, $0 < \gamma < 2$. Moreover, for the simulations in the following sections, we used the backward-difference approximation to mount the fractional differentiator.
FIGURE 14  Events and phases of a fractional-order signal. The left part shows the schematic of events of the fractional-order signal, and the right part shows the phases corresponding to them on a unit circle, where the number of events is $N = 2$.

| Table 2  | Oscillator’s structural design for a fractional-order signal |
|----------------|----------------------------------------------------------|
| Number of events | $N = 2$ |
| Phases of events | $\theta_1 = (1 - 0.5 \gamma) \pi, \theta_2 = (2 - 0.5 \gamma) \pi$ |
| Intervals of events | $\xi_1 = \xi_2 = \pi$ |
| Observable state (phase equation) | $r = \begin{bmatrix} -s^\gamma & s \end{bmatrix}^T p$ |
| Event detection function | $\Lambda_1(r) := -s^\gamma p, \Lambda_2(r) := s^\gamma p$ |
| Observable state (amplitude equation) | $r = p$ |

FIGURE 15  Synchronization property with a fractional-order signal. (A) The blue line is the fractional-order signal $s^{1/2} p$ that is the input of the designed oscillator, and the gray line is its zero-order signal $p$ as a reference. (B) The outputs of the designed oscillator $\theta, \rho$ are the estimated values of the input’s instantaneous phase and amplitude. (C) This plot compares the zero-order signal $p$ and the generated signal $q$ and represents that the oscillator can synchronize with the fractional-order signal well.
### 6.2 Fractional-order signal

We can also design a nonlinear oscillator model for a fractional-order signal in the same manner as the zero-order signal described in Section 5.1.

Assuming the cyclic phenomenon is a fractional-order signal, we posit Figure 14. From Figure 14, we find that the number of events is \( N = 2 \) by its zero-crossing timing. The parameters for the structural design of the oscillator are summarized in Table 2.

Figure 15 shows the input-output of the designed nonlinear oscillator model when the cyclic phenomenon is a fractional-order signal. Here, we provide a periodic signal with amplitude changes, \( p = A(t) \sin (\omega t) \), which is input from 15 s. Figure 15, although a frequency detuning exists between the input and the oscillator, phase-locking occurs; furthermore, the oscillator amplitude values are close to the input amplitude. This implies that the designed nonlinear oscillator model can synchronize with the fractional-order signal.

### 6.3 Set of fractional-order signals

We can increase the number of signal events by using multiple fractional differentiators, as shown in Figure 16. The parameters for the structural design of the oscillator are summarized in Table 3. where, \( M \) is the number of fractional differentiators and \( 0 < \gamma_k < 2 & \gamma_k > \gamma_k+1 \cdots (k = 1, 2 \cdots M) \).

![Diagram](image)

**Figure 16** Block diagram of generalized nonlinear oscillator model with fractional differentiators. This system has the single periodic signal \( p \) as an main input and the phase and amplitude of the oscillator \( \theta, \rho \) as outputs. Using fractional differentiators increases the number of signal events and can improve the synchronization property. Here, the black signal-lines and elements represent the forward signals and elements, while the blue signal-lines and elements represent the feedback signals and elements related to \( \theta \).
Table 3 Oscillator’s structural design for a set of fractional-order signals

| Number of events | $N = 2M + 2$ |
|------------------|-------------|
| Phases of events | $\theta_i = \begin{cases} 0 & \cdots i = 1 \\ (1 - 0.5\gamma_k)\pi & \cdots 2 \leq i \leq N/2 \\ \pi & \cdots i = N/2 + 1 \\ (2 - 0.5\gamma_k)\pi & \cdots N/2 + 2 \leq i \leq N \end{cases}$ |
| Intervals of events | $\delta_i = \begin{cases} \theta_{i+1} - \theta_i & \cdots 1 \leq i \leq N + 1 \\ \theta_i + 2\pi - \theta_i & \cdots i = N \end{cases}$ |
| Observable state (phase equation) | $r = \begin{bmatrix} 1 & -s^{\gamma_1} & -s^{2\gamma_1} & \cdots & -s^{M\gamma_1} & -1 & s^{\gamma_1} & s^{2\gamma_1} & \cdots & s^{M\gamma_1} \end{bmatrix}^T p$ |
| Event detection function | $\Lambda_i(r) = \begin{cases} p & \cdots i = 1 \\ -s^{\gamma_1}p & \cdots 2 \leq i \leq N/2 \\ -p & \cdots i = N/2 + 1 \\ s^{\gamma_1}p & \cdots N/2 + 2 \leq i \leq N \end{cases}$ |
| Observable state (amplitude equation) | $r = p$ |

**Figure 17** Synchronization property with a set of fractional-order signals. In (A–D), the top and the middle part show the input and outputs of the oscillator, and the bottom part compares the zero-order signal $p$ and the generated signal $q$, as shown in Figure 15.

### 6.4 Synchronization property of generalized nonlinear oscillator model with fractional derivatives

Figures 17–18 show the synchronization properties of the generalized nonlinear oscillator model with fractional derivatives and their phase and amplitude errors. In this simulation, we used the same input and oscillator parameters as those shown in Figure 15, and change the number of the fractional differentiators, $M = 0 – 3$. 
From Figures 17 and 18, we find that the generalized nonlinear oscillator model can synchronize with the set of fractional-order signals; furthermore, using the multiple fractional differentiators enables us to improve the signal’s phase resolution and the oscillator’s synchronization property. Thus, the fractional derivatives improve the oscillator’s estimation accuracy and convergence of the simultaneous phase and amplitude as a common-dimensional observer: $\theta \rightarrow \varphi, \rho \rightarrow A$. This implies that the pseudo-property of iso-damping and flat-gain utilizing the oscillator’s output can approach the ideal transfer function using fractional calculus; hence, the nonlinear oscillator and the fractional calculus are mutually related via the phase resolution.

However, these simulation results include errors coursed by the approximation calculation of the fractional calculus. Thus, this technique requires high-precision and high-speed calculations of the fractional differentiator.

**7 CONCLUSIONS**

This study explicitly rediscovered the effectiveness of fractional calculus by studying the input-output characteristic as an open-loop system of the generalized nonlinear oscillator model. As a result, we found that combining a nonlinear oscillator and a fractional calculus is essential to obtain the ideal frequency transmission characteristic. This finding might be useful in the applications of the open-loop and closed-loop types, as shown in Figure 2.

Furthermore, researchers who study the application of nonlinear oscillators and the fractional-order controller face the same issue inherent to “phase resolution”. Their cooperation is essential in developing next-generation control and signal processing techniques.

In future research, we propose exploring the use of a fractional differentiator with greater precision and calculation speed in our method.

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**ORCID**

Junichi Hongu [https://orcid.org/0000-0001-5025-3207](https://orcid.org/0000-0001-5025-3207)
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