Comparison of Different Boost Transformations for the Calculation of Form Factors in Relativistic Quantum Mechanics

Lukas Theußl¹, A. Amghar², B. Desplanques³, S. Noguera¹

¹ Departamento de Física Teórica, Universidad de Valencia, E-46100 Burjasot (Valencia), Spain
² Faculté des Sciences, Université de Boumerdes, 35000 Boumerdes, Algeria
³ Institut des Sciences Nucléaires (UMR CNRS/IN2P3–UJF), F-38026 Grenoble Cedex, France

Abstract. The effect of different boost expressions, pertinent to the instant, front and point forms of relativistic quantum mechanics, is considered for the calculation of the ground-state form factor of a two-body system in simple scalar models. Results with a Galilean boost as well as an explicitly covariant calculation based on the Bethe-Salpeter approach are given for comparison. It is found that the present so-called point-form calculations of form factors strongly deviate from all the other ones. This suggests that the formalism which underlies them requires further elaboration. A proposition in this sense is made.

1 Introduction

Among the forms of relativistic quantum mechanics originally proposed by Dirac [1], the point form is probably the least known and also the one least used. This may be due to the non-linear constraints on coordinates when quantization is performed on a hyperboloid. On the other hand, the dynamics is particularly appealing in this form, because the interaction is solely contained in the momentum operators, i.e., only four generators of the Poincaré group are dynamic, while, in particular, rotations and boosts are kinematic. Interest in the point form has been revived recently by the work of Klink [2] and its applications to the form factors of the pion [3], the deuteron [4] and the nucleon [5]. In the deuteron case, there is no significant improvement with respect to a non-relativistic calculation. In the nucleon case, the agreement with experimental data is at first sight spectacular, especially at the lowest $Q^2$ range considered by the authors.
However, analysing the calculations in detail, some questions arise. The well-known vector-meson dominance mechanism, which explains to a large part the nucleon form factors at low $Q^2$ via the coupling of the photon to the nucleon by $\rho$ and $\omega$ exchanges, is not accounted for at all, and nothing indicates that this might be achieved in a hidden way by incorporated relativistic effects. Furthermore, no intrinsic quark form factors (which could account for $\rho$ and $\omega$ exchanges implicitly) are employed, while they are necessary for the construction of the quark-quark interaction.

From these works, one may deduce an expression for the mean squared radius that scales like $1/M^2$, where $M$ is the total mass of the bound system. This would have the surprising consequence that the root mean squared radius increases with decreasing $M$, and, in particular, diverges in the limit $M \to 0$. This is contrary to physical intuition, where a smaller total mass, usually brought about by increasing the interaction, leads to a more compact system. Note that the limit $M \to 0$ is not a completely academic one since it applies to the pion. A simple scaling argument would lead to a squared radius of the pion of $3/m_\pi^2 \approx 6 \text{ fm}^2$ (!).

The success of the calculation in ref. [5] is attributed by the authors to relativistic boost effects. Curiously, taking into account the Lorentz-contraction effect by a simple replacement of the argument of the form factor, one gets an effect that increases the form factor for a given $Q^2$. This is the opposite of what was found in ref. [5].

In a series of recent papers, motivated by the above mentioned observations, the reliability of the employed point-form implementation has been tested on some simple scalar systems, which are academic, but offer the advantage of eliminating uncertainties, like spin effects and intrinsic form factors of the constituents, while allowing partly analytic results. These works include a two-body system composed of scalar particles exchanging a zero-mass boson [6] and a system corresponding to a zero-range interaction [7]. A systematic comparison with predictions of the other forms of dynamics (instant and front form), as well as with a non-relativistic and an explicitly covariant calculation based on the Bethe-Salpeter approach has been presented in ref. [8], where also the sensitivity of the results on the mass operator was investigated.

We shall only outline in this contribution the procedure that was followed in ref. [8], concentrating on the construction of the mass operator, where another questionable feature of recent point form applications arises. An example of numerical results for form factors will confirm the unusual results obtained in the more realistic calculations of refs. [4, 5].

2 Mass operator

The starting point of every calculation is a dynamical equation (a mass operator), from which a wave function may be determined that is subsequently used for the calculation of observables. The important constraint to be fulfilled by such an equation is its covariance, i.e., the resulting spectrum should
be invariant with respect to Lorentz boosts. We choose an equation with a quadratic form of the energy- and the phase-space factors, which facilitates the calculations:

\[
\left( E_p^2 - (e_{p_1} + e_{p_2})^2 \right) \Phi(p_1, p_2) = \int \int \frac{dp_1'}{(2\pi)^3} \frac{dp_2'}{(2\pi)^3} \sqrt{2(e_{p_1} + e_{p_2})} \sqrt{2(e_{p_1}' + e_{p_2}')} V_{\text{int}}(p_1, p_2, p_1', p_2') \frac{\sqrt{2(e_{p_1} + e_{p_2})}}{\sqrt{2e_{p_1}' \sqrt{2e_{p_2}'}}} \Phi(p_1', p_2'),
\]

where \( E_p = \sqrt{M^2 + P^2} \), \( e_p = \sqrt{m^2 + p^2} \). The quantities \( M \) and \( P \) represent the total mass and the total momentum of the system under consideration.

One can easily determine the constraints that ensure the invariance of the mass spectrum of this equation. The same constraints also provide a direct way to relate a wave function calculated in a moving frame (with a finite momentum \( P \)), to the rest frame wave function. Provided \( V_{\text{int}}(p_1, p_2, p_1', p_2') \) is appropriately chosen, the above equation can always be transformed into the center of mass by a simple change of variables:

\[
(M^2 - 4e_k^2) \phi_0(k) = \int \frac{dk'}{(2\pi)^3} \frac{1}{\sqrt{e_k}} V_{\text{int}}(k, k') \frac{1}{\sqrt{e_{k'}}} \phi_0(k').
\]

This equation is of the form \( M^2 = M_0^2 + V \), i.e., \( M^2 \) gives an invariant mass operator whose solutions can be used in any form of relativistic quantum mechanics. It is just the relation between the set of particle momenta, \( p_1, p_2 \), to the set of 'internal' momenta, \( k, P \), that will be different from one form to the other. In ref. [8] this transformation has been explicitly demonstrated for the instant form of dynamics. It is given by an expression that is equivalent to the one obtained from the Bakamjian - Thomas construction [9]:

\[
\begin{align*}
p_1 &= k - P \frac{k \cdot P}{P^2} + P \frac{k \cdot P}{P^2} \sqrt{4e_k^2 + P^2} + e_k \frac{P}{2e_k}, \\
e_{p_1} &= e_k \frac{\sqrt{4e_k^2 + P^2}}{2e_k} + k \cdot \frac{P}{2e_k},
\end{align*}
\]

(3)

together with similar expressions for particle 2, where the change, \( k \rightarrow -k \) has to be made. In the instant form, the momenta of the constituent particles are related to the total momentum \( P \) by the equality \( p_1 + p_2 = p_1' + p_2' = P \), consistently with the property that the momentum has a kinematical character in this form. When the dynamics is described on a surface different from the instant-form one, \( t = \tau \), other relations between momenta are obtained. This is easiest to see by integrating plane waves over the hyper-surface under consideration. For a hyper-plane, \( \lambda \cdot x = \tau \) with \( \lambda^2 = 1 \), for instance, one obtains:

\[
\int d^4x \delta(\lambda \cdot x) e^{i(p_1 + p_2 - P) \cdot x} = \frac{1}{(2\pi)^3} \frac{1}{\lambda^0} \delta \left( p_1 + p_2 - P - \frac{\lambda}{\lambda^0}(e_1 + e_2 - E_P) \right),
\]

(4)
while for a hyperboloïd \((x^2 = \tau)\), one gets for the special case \(\tau = 0\):

\[
\int d^4x \delta(x \cdot x) e^{i(p_1 + p_2 - p) \cdot x} = \frac{4\pi}{(p_1 + p_2 - P)^2 - (e_1 + e_2 - E_P)^2 + i\epsilon}.
\] (5)

Eq. (4) would apply to an instant form of dynamics when \(\lambda = 0\), and we recover correctly \(p_1 + p_2 = P\) in this case. However, in front form (Eq. (4) with \(\lambda/\lambda^0\) equal to a unit vector \(n\)) and in point form, Eq. (5), the relations are more complicated. In particular, in neither of these forms we may set the sum of the particle momenta \((p_1 + p_2)\) equal to zero in the rest frame of the system \((P = 0)\). This contrasts with the prescription used in recent point form applications, where a relation of the form

\[ p_1 + p_2 = \frac{2e_k}{M} P \] (6)

is applied, i.e., \(p_1 + p_2 = 0\) in the center of mass, which can never be obtained from a relation of the form of eq. (5). The relation between particle and 'internal' momenta obtained in that way rather resembles the one in instant form given by Eq. (4), with factors \(2e_k\) replaced by the total mass \(M\). This suggests, like Eq. (6), that we might expect troubles in the limit \(M \to 0\) in this case.

### 3 Results for elastic form factors

Expressions for elastic charge form factors have been given in ref. [8] in the different forms of relativistic quantum mechanics and for different choices of the two-body interaction model. We reproduce in Table 1 results obtained for a zero-range interaction (corresponding to an infinite-mass exchange boson). This model has the advantage that some calculations may be carried out analytically [7], which is useful for checking certain properties like the exact logarithmic dependence of the high \(Q^2\) behavior of the form factors. The other extreme of a zero mass exchange boson has been discussed in ref. [6].

Inspecting the results, two features are noticed immediately:

- The form factors in point form depart significantly from all the others, especially for high \(Q^2\) and in the limit of small \(M\).
- The root mean squared radius of the bound state (proportional to the slope of the form factor at the origin) diverges in the limit \(M \to 0\).

These features are qualitatively the same as the ones found in recent works, where the so-called point form formalism was applied to the calculation of the nucleon form factor [5]. In particular, because of the wrong power law behavior at high \(Q^2\), the form factors in point form miss the Born amplitude, contrary to all the other approaches. This is a severe shortcoming and one has to wonder about the reasons for these peculiarities.

One possible explanation would be that two-body currents play a much more dominant role in the point form than in all the other approaches. This
Table 1. Elastic vector- and scalar form factors, $F_1(Q^2)$ and $F_0(Q^2)$, for a system bound by the exchange of an infinite-mass boson (zero-range interaction) for two values of the total mass $M$. The wave function used in the instant form (I.F.), front form (F.F.) and point form (P.F.) cases is issued from Eq. (2). B.S. gives the results for a Bethe-Salpeter calculation, while Gal. corresponds to a calculation employing a Galileian boost. Asymptotic behaviors for $F_1(Q^2)$ are $Q^{-2} (\log Q)^2$, $Q^{-2} (\log Q)^2$, $Q^{-4}$, $Q^{-2} (\log Q)^2$, and $Q^{-1}$ for I.F., F.F., P.F., B.S. and Gal., respectively.

| $Q^2/m^2$ | 0.01 | 0.1 | 1.0 | 10.0 | 100.0 |
|-----------|------|-----|-----|------|-------|
| $M = 1.6 m$ |
| $F_1$ | I.F. | 0.999 | 0.990 | 0.917 | 0.594 | 0.208 |
| $F_0$ | I.F. | 1.325 | 1.309 | 1.176 | 0.658 | 0.187 |
| $F_1$ | F.F. | 0.999 | 0.989 | 0.908 | 0.566 | 0.191 |
| $F_0$ | F.F. | 1.325 | 1.309 | 1.176 | 0.659 | 0.187 |
| $F_1$ | P.F. | 0.999 | 0.986 | 0.871 | 0.353 | 0.207-01 |
| $F_0$ | P.F. | 0.998 | 0.976 | 0.800 | 0.236 | 0.108-01 |
| $F_1$ | B.S. | 0.999 | 0.989 | 0.908 | 0.566 | 0.191 |
| $F_0$ | B.S. | 1.325 | 1.309 | 1.176 | 0.659 | 0.187 |
| $F_1$ | Gal. | 0.999 | 0.994 | 0.947 | 0.699 | 0.320 |

| $M = 0.1 m$ |
| $F_1$ | I.F. | 0.999 | 0.996 | 0.963 | 0.759 | 0.343 |
| $F_0$ | I.F. | 1.498 | 1.487 | 1.389 | 0.920 | 0.320 |
| $F_1$ | F.F. | 0.999 | 0.995 | 0.954 | 0.723 | 0.315 |
| $F_0$ | F.F. | 1.498 | 1.487 | 1.389 | 0.920 | 0.320 |
| $F_1$ | P.F. | 0.839 | 0.222 | 0.82-02 | 0.141-03 | 0.20-05 |
| $F_0$ | P.F. | 0.699 | 0.130 | 0.42-02 | 0.071-03 | 0.10-05 |
| $F_1$ | B.S. | 0.999 | 0.995 | 0.954 | 0.723 | 0.315 |
| $F_0$ | B.S. | 1.498 | 1.487 | 1.389 | 0.920 | 0.320 |
| $F_1$ | Gal. | 0.999 | 0.998 | 0.980 | 0.846 | 0.475 |

possibility is currently investigated [10]. In view of the unitary equivalence of all relativistic quantum mechanics approaches, and because of the huge effect, one might wonder whether this is the most efficient way to proceed.

An alternative to this approach was sketched in ref. [7]. It is based on the observation that, in practice, recent applications of the point-form [4, 5, 6]
rely on employing wave functions issued from a mass operator whose solutions can also be identified with instant-form ones in the center of mass system. As noticed by Sokolov [11], this point-form approach is not identical to the one proposed by Dirac, where quantization is performed on a hyperboloid, $x \cdot x = \tau$.

When the system at rest described on the hyper-plane, $\lambda_0 \cdot x = \tau$, is kinematically boosted to get initial and final states with four-momenta, $P^\mu_i$ and $P^\mu_f$, these ones appear as described (quantized) on different surfaces, $\lambda_i \cdot x = \tau$ and $\lambda_f \cdot x = \tau$, where $\lambda_i^\mu \propto P_i^\mu$ with $\lambda_i^2 = 1$. This feature results from the identification of point- and instant-form wave functions in the center of mass. It does not correspond to the usual description of a process which, generally, relies on the same definition of the surface at all steps.

What is understood as point form in recent works is usually defined without any reference to surfaces at all. This is not really in the spirit of Dirac, but it is true that one does not need surfaces in order to construct a set of generators that satisfies the Poincaré algebra. However, the problem pointed out here concerns less the construction of the generators of the Poincaré algebra in terms of the total momentum $P$ and the internal variable $k$, but rather the relation of this set of variables to the physical ones. A point form approach in the Dirac sense, with quantization performed on a hyperboloid, would at least resolve the issues of relations between variables, that were raised in this paper.

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References

1. P. A. Dirac: Rev. Mod. Phys. 21, 392 (1949).
2. W. H. Klink: Phys. Rev. C58, 3587 (1998).
3. T. W. Allen, W. H. Klink: Phys. Rev. C58, 3670 (1998).
4. T. W. Allen, W. H. Klink, W. N. Polyzou: Phys. Rev. C63, 034002 (2001).
5. R. F. Wagenbrunn, S. Boffi, W. Klink, W. Plessas, M. Radici: Phys. Lett. B511, 33 (2001).
6. B. Desplanques, L. Theufl: Eur. Phys. J. A13, 461 (2002).
7. B. Desplanques, L. Theufl, S. Noguera: Phys. Rev. C65, 038202 (2002).
8. A. Amghar, B. Desplanques, L. Theufl: arXiv:nucl-th/0202046 to appear in Nucl. Phys. A.
9. B. Bakamjian, L.H. Thomas: Phys. Rev. 92, 1300 (1953).
10. L. Theufl, B. Desplanques: in preparation.
11. S. N. Sokolov: Theor. Math. Phys. 62, 140 (1985).