THE QUASI-POISSON GOLDMAN FORMULA

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Abstract. We prove a quasi-Poisson bracket formula for the space of representations of the fundamental groupoid of a surface with boundary, which generalizes Goldman’s Poisson bracket formula. We also deduce a similar formula for quasi-Poisson cross-sections.

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Introduction

Let $\Sigma$ be a closed oriented surface and $G$ a Lie group with a fixed invariant scalar product on its Lie algebra. Ignoring a singular part, the moduli space $X_G(\Sigma)$ of flat connections on principal $G$-bundles over $\Sigma$, well known to be identified with $\text{Hom}(\pi_1(\Sigma), G)/G$, is a symplectic manifold [3, 5]. If $\Sigma$ is a bordered surface, i.e., has non-empty boundary, then the symplectic structure generalizes to a Poisson structure.

For $\Sigma$ closed, Goldman [6] provided a nice Poisson bracket formula for certain functions on $X_G(\Sigma)$. When $G = \text{GL}_n\mathbb{R}$, the formula leads to the so-called Goldman algebra of loops on a surface. The same result for $\partial \Sigma \neq \emptyset$, although widely believed to be true, was given a proof first in [8].

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On the other hand, when $\partial \Sigma \neq \emptyset$, the quasi-Poisson theory of moduli spaces \cite{Givental:1994,Givental:1995} provides the following finite-dimensional construction of the Poisson structure on $X_G(\Sigma)$. We assume that $\partial \Sigma$ has $b$ components and choose a marked point $p_i$ on each of them. Consider the space of fundamental groupoid representations 

$$M_G(\Sigma) := \text{Hom}(\pi_1(\Sigma; p_1, \cdots, p_b), G),$$

which has a natural $G^b$-action and the quotient is $M_G(\Sigma)/G^b = X_G(\Sigma)$. Then the space of functions on $M_G(\Sigma)$, denoted by $\mathcal{O}_{M_G(\Sigma)}$, is equipped with a canonical quasi-Poisson $G^b$-bracket $\{\cdot, \cdot\}_{M_G(\Sigma)}$ whose restriction to invariant functions $\mathcal{O}_{M_G(\Sigma)}^G = \mathcal{O}_{X_G(\Sigma)}$ gives the Poisson structure of $X_G(\Sigma)$.

Our main result, Theorem 2.5, is a quasi-Poisson bracket formula for functions on $M_G(\Sigma)$, which generalizes Goldman’s formula. When $G = \text{GL}_n \mathbb{R}$, the formula leads to a quasi-Poisson version of the Goldman algebra, which was first discovered by Massuyeau and Turaev \cite{MasaTuraev:2008} using a different approach.

As a corollary, when $G$ is compact, we deduce a quasi-Poisson bracket formula (Theorem 3.2) for functions on the cross-section

$$L = \bigcap_{i=1}^b \mu_i^{-1}(U) \subset M_G(\Sigma),$$

where $\mu_i : M_G(\Sigma) \to G$ is the holonomy of the $i$-th component of $\partial \Sigma$ (with reversed orientation), and $U \subset G$ is a certain cross-section of the conjugation $G$-action on itself, so that $L$ is a Hamiltonian quasi-Poisson $H^b$-manifold for a subgroup $H \subset G$, see §3.1.

The present work was motivated by a relationship between $L$ and some particular rational functions on $X_G(\Sigma)$ in the case $G = \text{SL}_n \mathbb{R}$ \cite{Giroux:2013,Giroux:2014}. This aspect will appear in the author’s thesis \cite{Nie:2015} and a forthcoming paper.

As this paper was being prepared for release, Li-Bland and Ševera independently released the paper \cite{LiBlandSevera:2015}, whose results overlap with ours.

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1. QUASI-POISSON THEORY

In this preliminary section we recall a version of the quasi-Poisson theory \cite{Givental:1994,Givental:1995}. The presentation here can be viewed as a simplified version of a more general framework \cite{LiBlandSevera:2015}.  

1.1. Fundamental groupoids of surfaces. Let $\Sigma$ be a compact oriented surface such that $\partial \Sigma$ has $b \geq 1$ boundary components and a marked point $p_i$ is chosen on each of them. But a path on $\Sigma$ we always mean an oriented smooth curve whose starting and ending points belong to $\{p_1, \cdots, p_b\}$. Let

$$\pi_1(\Sigma) := \pi_1(\Sigma; p_1, \cdots, p_b),$$

denote the fundamental groupoid of $\Sigma$, i.e., the set of (end-points-fixing) homotopy classes of paths, equipped with the obvious partial multiplication.
Fix a Lie group $G$. A representation of $\pi_1(\Sigma)$ in $G$ is by definition a map $\pi_1(\Sigma) \to G$ preserving multiplications. Let the space of all representations be denoted by

$$M_G(\Sigma) := \text{Hom}(\pi_1(\Sigma), G).$$

If $\alpha$ is a path, the holonomy along $\alpha$ is the map

$$\text{Hol}_\alpha : M_G(\Sigma) \to G, \ m \mapsto m(\alpha).$$

Let $\beta_i$ denote the $i$-th boundary loop (with induced orientation). The $i$-th reversed boundary holonomy

$$\mu_i := \text{Hol}_{\beta_i^{-1}}$$

will play a special role later on.

There is a natural $G^b$-action on $M_G(\Sigma)$ given by

$$(g_1, \cdots, g_b).m(\alpha) = g_im(\alpha)g_i^{-1}$$

for any $m \in M_G(\Sigma)$ and path $\alpha$ going from $p_i$ to $p_j$. In other words, if $\alpha$ starts and ends both at $p_i$ (resp. only starts/ends at $p_i$), then $\text{Hol}_\alpha$ is equivariant with respect to the $i$-th $G$-action on $M_G(\Sigma)$ and the $G$-action on itself by conjugation (resp. left/right multiplication).

It is easy to see that the map

$$M_G(\Sigma) \to \text{Hom}(\pi_1(\Sigma; p_1), G)$$

induced by the injection $\pi_1(\Sigma; p_1) \to \pi_1(\Sigma)$ is a principle $G^{b-1}$-bundle, hence the quotient $M_G(\Sigma)/G^b$ is identified with $X_G(\Sigma) = \text{Hom}(\pi_1(\Sigma; p_1), G)/G$.

If $\Sigma = \Sigma_{g,b}$ is the connected surface with genus $g$ and $b$ boundary components, a set of $2(b-1) + 2g$ generators of $\pi_1(\Sigma)$ without relations can be constructed as follows. Take a path $\alpha_i$ from $p_1$ to $p_i$ for each $2 \leq i \leq b$, and two loops $\gamma_j, \delta_j$ based at $p_1$ for each $1 \leq j \leq g$, such that by cutting the surface along these paths we obtain a polygon whose edges are successively

$$\beta_1, \alpha_2, \beta_2^{-1}, \cdots, \alpha_b, \beta_b^{-1}, \gamma_1, \delta_1^{-1}, \gamma_1^{-1}, \delta_1, \cdots, \gamma_g, \delta_g, \gamma_g^{-1}, \delta_g^{-1},$$

see the second picture of Figure 1.1. Then $\{\alpha_i, \beta_i, \gamma_j, \delta_j\}$ form the required generators.

As a consequence, put

$$u_i = \text{Hol}_{\alpha_i}, \ v_i = \text{Hol}_{\beta_i}, \ a_j = \text{Hol}_{\gamma_j}, \ b_j = \text{Hol}_{\delta_j},$$

then $(u_i, v_i, a_j, b_j)$ form a $G$-valued coordinates system of $M_G(\Sigma)$, which identifies $M_G(\Sigma)$ with $G^{2(b-1)+2g}$. Clearly, $\mu_i = u_i^{-1}$ ($2 \leq i \leq b$) and

$$\mu_1 = u_2v_2u_2^{-1} \cdots u_bv_bu_b^{-1}[a_1, b_1] \cdots [a_g, b_g],$$

where $[a, b] = aba^{-1}b^{-1}$ is the commutator.

Here are two simplest examples

**Example 1.1** ($\Sigma_{0,2}$, the cylinder). The coordinates system $(u, v) = (u_2, v_2)$ identifies $M_G(\Sigma_{0,2})$ with $G^2$, on which the $G^2$ action reads

$$(u, v)^{(g_2, g_2)} = (g_1u, g_2v, g_2^{-1}),$$

and we have

$$\mu_1(u, v) = uv^{-1}, \mu_2(u, v) = v^{-1}.$$
Example 1.2 (Σ₁₁, the one-holed torus). The coordinates system \((a, b) = (a₁, b₁)\) identifies \(M_G(Σ₁₁)\) with \(G²\). The canonical \(G\)-action is the conjugation action on both factors. Put \(μ = \text{Hol}_β^{-1}\) where \(β\) is the boundary loop, then \(μ(a, b) = (a, b)\).

1.2. Splitting. By a splitting arc on \(Σ\) we mean an embedded segment \(∆\) which joins some marked point, say, \(p₁\), with another point on the same boundary component \(β₁\).

Splitting \(Σ\) along \(∆\) and splitting \(p₁\) into two marked points \(p₁^L\) and \(p₁^R\), we get a surface \(Σ_∆\) with \(b + 1\) boundary components, each of them still having a marked point. See the following picture.

Remark 1.3. The two particular marked points \(p₁^L, p₁^R \in ∂Σ_∆\) have a left-right distinction. Indeed, identifying a neighborhood \(U\) of \(p₁\) in \(Σ\) with the upper half-plan in an orientation-preserving manner, we assume that after splitting \(U\) into two, \(p₁^L\) is on the left half and \(p₁^R\) on the right.

The gluing map \(Σ_∆ → Σ\) induces a map between fundamental groupoids, and hence a map between representation spaces

\[ R_∆ : M_G(Σ) → M_G(Σ_∆) \]

By a suitable adaptation of the Van Kampen theorem, one can prove that \(R_∆\) is bijective. Identifying \(M_G(Σ)\) and \(M_G(Σ_∆)\) via \(R_∆\), it is easy to see that

- The \(G^b\)-action on \(M_G(Σ)\) is induced by the \(G^{b + 1}\)-action on \(M_G(Σ_∆)\) and the diagonal embedding of the first factor

\[ G^b → G^{b + 1}, \quad (g₁, g₂, \cdots, gₙ) ↦ (g₁, g₁, g₂, \cdots, gₙ) \]

- Consider the maps \(μ_1 : M_G(Σ) → G\) and

\[ μ₁^L := \text{Hol}_{(β₁^L)}^{-1}, \quad μ₁^R := \text{Hol}_{(β₁^R)}^{-1} : M_G(Σ_∆) → G, \]

where \(β₁^L\) (resp. \(β₁^R\)) is the boundary loop of \(Σ_∆\) at \(p₁^L\) (resp. \(p₁^R\)), then

\[ μ₁ = μ₁^L \cdot μ₁^R. \]

Here and below, the product of two maps \(M → G\) is pointwise defined using the product in \(G\).

Example 1.4. There is a splitting arc \(∆\) on \(Σ = Σ₁₁\) such that the standard generators \(γ, δ\) of \(π₁(Σ₁₁)\) becomes standard generators of \(Σ_∆ = Σ₀₂\), see the first picture of Figure 1.1. With the coordinates of Example 1.1 and 1.2, we have

\[ R_∆ : M_G(Σ₁₁) \xrightarrow{R_∆} M_G(Σ₀₂) \]

\[(a, b) \mapsto (u, v) = (a, b).\]
Moreover, we have $\mu = \mu_1 \cdot \mu_2$.

**Example 1.5.** Let $\Sigma = \Sigma_{g,b}$ and take paths $\alpha_i$ and $\delta_j, \gamma_j$ as in §1.1. Splitting $\Sigma$ successively along the $b-2 + g$ splitting arcs shown in the second picture of Figure 1.1, we get a surface $\hat{\Sigma}$ which is the disjoint union of $b-1$ copies of $\Sigma_{0,2}$ and $g$ copies of $\Sigma_{1,1}$, and a bijection between the representation spaces

$$M_G(\Sigma_{g,b}) \sim \rightarrow M_G(\hat{\Sigma}) = M_G(\Sigma_{0,2})^{b-1} \times M_G(\Sigma_{1,1})^g.$$ 

### 1.3. Quasi-Poisson manifolds.

**Definition 1.6.** Let $G$ be a Lie group with an invariant scalar product $(\cdot \mid \cdot)$ on the Lie algebra $\mathfrak{g}$. Let $M$ be a manifold with a $G$-action denoted by $\rho$. A $G$-invariant bivector field $P \in \Gamma(\bigwedge^2 TM)$ is called a quasi-Poisson $G$-tensor and $(M, P)$ called a quasi-Poisson $G$-manifold if $P$ satisfies

$$[P, P] = \rho \phi.$$ 

Here $[P, P] \in \Gamma(\bigwedge^3 TM)$ is the Schouten bracket, $\phi \in (\bigwedge^3 \mathfrak{g})^G$ is a canonical invariant trivector associated with $(\cdot \mid \cdot)$, which has the following expression in terms of an orthonormal basis $(e_i)$ of $\mathfrak{g}$:

$$\phi = \frac{1}{12} \sum_{i,j,k} (e_i \mid [e_j, e_k]) e_i \wedge e_j \wedge e_k,$$

and we have extended the Lie algebra homomorphism

$$\mathfrak{g} \to \Gamma(TM), \quad x \mapsto \rho_x,$$

where $\rho_x(m) := \left. \frac{d}{dt} \right|_{t=0} \rho_{\exp(-tx)}(m)$

to a map $\bigwedge^\bullet \mathfrak{g} \to \Gamma(\bigwedge^\bullet TM)$ preserving Schouten brackets.

The *quasi-Poisson bracket* $\{\cdot, \cdot\}$ associated to $P$ is defined as the anti-symmetric bilinear form on $\mathcal{O}_M$ given by

$$\{f, g\} = P(df, dg).$$

**Definition 1.7.** Let $(M, P)$ be a quasi-Poisson $G$-manifold, where the $G$-action is denoted by $\rho$. An equivariant map $\mu : M \to G$ (where $G$ acts on itself by conjugation) is called a *group-valued* moment map, and $(M, P, \mu)$ called a *Hamiltonian quasi-Poisson manifold*, if $\mu$ satisfies

$$\mu^* \theta(P^!(df)) = -\frac{1}{2}(1 + \text{Ad}^{-1}_\mu) \chi_f$$

in $\mathcal{O}_M$. 

---

**Figure 1.1.** Splitting $\Sigma_{1,1}$ and $\Sigma_{g,b}$
for any function $f \in \mathcal{O}_M$. Here $\theta \in \Omega^1(G, \mathfrak{g})$ is the left invariant Maurer-Cartan 1-form, $P^\theta : T^*M \to TM$ is defined by $P^\theta(df) = P(df, \cdot)$ and the variation map $\chi_f : M \to \mathfrak{g}$ of $f$ is defined by

$$
(\chi_f(m) \mid x) = \rho_x(f)(m), \quad \text{for any } m \in M, x \in \mathfrak{g}.
$$

**Remark 1.8.** A quasi-Poisson $G$-manifold $(M, P)$ is said to be **non-degenerate** if at any point $m \in M$ we have $T_m M = P^\theta_m(T^*M) + \rho_m(m)$. Any Hamiltonian quasi-Poisson manifold is foliated by non-degenerate ones.

In the present paper we mainly work with the canonical quasi-Poisson structure on $M_G(\Sigma)$, which is known to be non-degenerate.

The following fusion construction provides a way to get new quasi-Poisson manifolds from old ones:

**Definition 1.9.** Let $(M, P)$ be a quasi-Poisson $G \times G \times H$-manifold, where the $G \times G$-action is denoted by $\rho$. Let $(e_i)$ be an orthonormal basis of $\mathfrak{g}$ and put

$$
\psi = \frac{1}{2} \sum_i (e_i, 0) \wedge (0, e_i) \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g}).
$$

Then the bivector field

$$
P' = P - \rho \psi
$$

is a quasi-Poisson $G \times H$-tensor, where $G \times H$ acts on $M$ via the diagonal embedding

$$
G \times H \hookrightarrow G \times G \times H, \quad (g, h) \mapsto (g, g, h).
$$

The quasi-Poisson $G \times H$-manifold $(M, P')$ is called the **fusion** of $(M, P)$.

Furthermore, if $\mu = (\mu_1, \mu_2, \nu) : M \to G \times G \times H$ is a moment map for $P$, then $\mu' = (\mu_1 \cdot \mu_2, \nu) : M \to G \times H$ is a moment map for $P'$. The Hamiltonian quasi-Poisson $G \times H$-manifold $(M, P', \mu')$ is also called the fusion of $(M, P, \mu)$.

In particular, let $(M_i, P_i, (\mu_i, \nu_i))$ be a Hamiltonian quasi-Poisson $G \times H_i$-manifold ($i = 1, 2$). Clearly, $(M_1 \times M_2, P_1 + P_2, (\mu_1, \mu_2, \nu_1, \nu_2))$ is a Hamiltonian quasi-Poisson $G \times G \times H_1 \times H_2$-manifold. The fusion $(M_1 \times M_2, P', (\mu_1 \cdot \mu_2, \nu_1, \nu_2))$ is called the **fusion product** of $M_1$ and $M_2$, denoted by $M_1 \oplus M_2$.

**Example 1.10.** Let $G \times G$ acts on $G$ by $\rho_{(g, h)}(a) = gah^{-1}$. Then the trivial bivector field $P = 0$ on $G$ is a quasi-Poisson $G \times G$-tensor. It does not admit moment maps.

**Example 1.11.** Applying fusion to Example 1.10, we get a quasi-Poisson $G$-tensor on $G$ with respect to the conjugation action:

$$
P_G = \frac{1}{2} \sum_i e_i^R \wedge e_i^L.
$$

Here $e_i^L$ (resp. $e_i^R$) is the left (resp. right) invariant vector field on $G$ generated by $e_i \in \mathfrak{g} = T_e G$. It can be shown that the identity $G \to G$ is a moment map for $P_G$.

**Example 1.12.** The product of two copies of Example 1.10 is $M = G \times G$ with the $G^4$-action $\rho_{(g_1, g_2, g_3, g_4)}(a, b) = (g_1 ag_2^{-1}, g_3 bg_4^{-1})$. Applying fusions with respect to the first and last factor of $G^4$, then with respect to the second and third factor,
we get the following quasi-Poisson $G \times G$-tensor on $G \times G$ (where the action is $ho_{(g,h)}(a,b) = (gah^{-1},hbg^{-1})$):

$$P = \frac{1}{2} \sum_{i} (e_{i}^{1L} \wedge e_{i}^{2R} + e_{i}^{1R} \wedge e_{i}^{2L}).$$

Here $e_{i}^{1L}$ (resp. $e_{i}^{2L}$) is the left invariant vector field on the first (resp. second) factor generated by $e_{i}$.

It can be shown that

$$\mu : G \times G \to G \times G, (a,b) \mapsto (ab,a^{-1}b^{-1})$$

is a moment map in this case. This Hamiltonian quasi-Poisson is called the double of $G$, denoted by $D(G)$.

Applying fusion again to $D(G)$, we get a quasi-Poisson $G \times G$-tensor on $G \times G$ (where $G$-acts by conjugation on both factors), which has moment map $(a,b) \mapsto aba^{-1}b^{-1}$. This Hamiltonian quasi-Poisson manifold is called the fused double.

1.4. The Quasi-Poisson structure on $M_{G}^{b}$. A main result in quasi-Poisson theory is that there is a canonical quasi-Poisson $G^{b}$-tensor on $M_{G}^{b}$, whose reduction gives the standard Poisson structure on $X_{G}^{b}$.

First let us consider the simplest case $\Sigma = \Sigma_{0,2}$. As a $G^{2}$-manifold, $M_{G}^{b}(\Sigma_{0,2})$ is identified with the double $D(G)$ through

$$M_{G}^{b}(\Sigma_{0,2}) \rightarrow D(G)$$

$$(u,v) \mapsto (a,b) = (u,vu^{-1}).$$

Thus $M_{G}^{b}(\Sigma_{0,2})$ is a Hamiltonian quasi-Poisson $G^{2}$-manifold, with moment map

$$(ab,a^{-1}b^{-1}) = (uvu^{-1},v^{-1}) = (\mu_{1},\mu_{2}).$$

Next, under the splitting map $R_{\Delta} : M_{G}^{b}(\Sigma_{1,1}) \rightarrow M_{G}^{b}(\Sigma_{0,2}) \cong D(G)$ of Example 1.4, the $G$-action on $M_{G}^{b}(\Sigma_{1,1})$ coincides with the action of diagonal subgroup of $G^{2}$ on $D(G)$, hence we can also endow $M_{G}^{b}(\Sigma_{1,1})$ with the Hamiltonian quasi-Poisson structure of the fused double. The moment map $\mu_{1} \cdot \mu_{2}$ coincides with

$$\mu : M_{G}^{b}(\Sigma_{1,1}) \rightarrow G.$$

Similarly, in view of Example 1.5, for any $\Sigma = \Sigma_{g,b}$ we can endow $M_{G}^{b}(\Sigma)$ with the Hamiltonian quasi-Poisson $G^{b}$-manifold structure of the fusion product

$$M_{G}^{b}(\Sigma) = M_{G}^{b}(\Sigma_{0,2}) \oplus \cdots \oplus M_{G}^{b}(\Sigma_{0,2}) \oplus M_{G}^{b}(\Sigma_{1,1}) \oplus \cdots \oplus M_{G}^{b}(\Sigma_{1,1}),$$

and the moment map is

$$(\mu_{1},\cdots,\mu_{b}) : M_{G}^{b}(\Sigma) \rightarrow G^{b}.$$
Briefly speaking, the theorem comes from the fact that the standard Poisson structure on $X_G(\Sigma)$ arises as a reduction of the infinite-dimensional symplectic manifold $N_G(\Sigma)$ of flat connections, while the quasi-Poisson structure on $M_G(\Sigma)$ is a partial reduction. However, the first statement of the theorem can be shown in a straightforward way, see [9].

2. The quasi-Poisson bracket on $M_G(\Sigma)$

The aim of this section is to prove the main result of the paper, Theorem 2.5, and deduce some simple corollaries. First we need some more notations.

2.1. Notations.

**Definition 2.1.** Let $G$ be a Lie group with an invariant scalar product $(\cdot | \cdot)$ on its Lie algebra $\mathfrak{g}$. For any $\Phi \in \mathcal{O}_G$, we define maps $\Phi^\wedge, \Phi^\vee : G \to \mathfrak{g}$ by

$$x^R(\Phi)(g) = (\Phi^\wedge(g) | x), \quad x^L(\Phi)(g) = (\Phi^\vee(g) | x),$$

for any $x \in \mathfrak{g}$ and $g \in G$.

We will often make use of the following characterizing property of $\Phi^\vee$ and $\Phi^\wedge$: let $\theta \in \Omega^1(G, \mathfrak{g})$ (resp. $\bar{\theta} \in \Omega^1(G, \mathfrak{g})$) denote the left (resp. right) invariant Maurer-Cartan 1-form, then for any manifold $M$ and map $u : M \to G$, we have

$$d(\Phi(u)) = (\Phi^\wedge(u) | u^*\bar{\theta}) = (\Phi^\vee(u) | u^*\theta).$$

Here and below, $\Phi^I(u)$ means the composition of $u$ with $\Phi$ and $\Phi^I (I = \wedge, \vee)$, respectively.

**Example 2.2** (Matrix entry functions). Put $G = \text{GL}_n\mathbb{R}$ and equip the Lie algebra $\mathfrak{g} = \text{gl}_n\mathbb{R}$ with the Ad-invariant scalar product $(x | y) = \text{Tr}(xy)$. Let $\Phi_{ij} \in \mathcal{O}_G$ $(1 \leq i, j \leq n)$ be the matrix entry function given by $\Phi_{ij}(g) = g_{ij}$ (the $(i, j)$-entry of $g$).

Let $E_{ij}$ be the $n \times n$ matrix whose $(i, j)$-entry is 1 and all other entries vanish. The definition of $\Phi_{ij}^\vee$ yields

$$(\Phi_{ij}^\vee(g) | x) = d\Phi_{ij}(x^L(g)) = d\Phi_{ij}(gx) = (gx)_{ij} = \text{Tr}(E_{ji}gx) = (E_{ji}g | x),$$

from which we get

$$\Phi_{ij}^\vee(g) = E_{ji}g,$$

and similarly

$$\Phi_{ij}^\wedge(g) = gE_{ji}.$$

Here are some elementary properties of $\Phi^\vee$ and $\Phi^\wedge$.

**Lemma 2.3.**

(i) $\Phi^\vee$ and $\Phi^\wedge$ are related by

$$\text{Ad}_g \Phi^\vee(g) = \Phi^\wedge(g), \quad \forall g \in G.$$

(ii) Let $\hat{\Phi}$ be the function on $G$ given by $\hat{\Phi}(g) = \Phi(g^{-1})$, then for any $g \in G$ we have

$$\hat{\Phi}^\vee(g) = -\Phi^\wedge(g^{-1}), \quad \hat{\Phi}^\wedge(g) = -\Phi^\vee(g^{-1}).$$

(iii) Fix $a, b \in G$ and let $\tilde{\Phi}$ be the function on $G$ given by $\tilde{\Phi}(g) = \Phi(agh)$, then for any $g \in G$ we have

$$\tilde{\Phi}^\vee(g) = \text{Ad}_g \Phi^\vee(agh), \quad \tilde{\Phi}^\wedge(g) = \text{Ad}_{g^{-1}} \Phi^\wedge(agh).$$
(iv) Let \( M \) be a \( G \)-manifold and \( u : M \to G \) be a map. We denote respectively by \( L, R \) and \( \text{Conj} \) the left, right and conjugation \( G \)-action on itself, then

\[
\chi_{\Phi(u)} = \begin{cases} 
-\Phi^L(u) & \text{if } u \text{ is } L\text{-equivariant}, \\
\Phi^R(u) & \text{if } u \text{ is } R\text{-equivariant}, \\
-\Phi^L(u) + \Phi^R(u) & \text{if } u \text{ is Conj-equivariant}.
\end{cases}
\]

(Recall that for \( f \in \mathcal{O}_M \) the variation map \( \chi_f : M \to \mathfrak{g} \) is defined by (1.3).)

We also need some notations concerning paths on \( \Sigma \) (see §1.1 for assumptions on \( \Sigma \)).

Given \( \Phi \in \mathcal{O}_G \) and a path \( \alpha \), we denote

\[
\Phi_\alpha := \Phi(\text{Hol}_\alpha) \in \mathcal{O}_{M_G(\Sigma)}, \quad \Phi_\alpha^I := \Phi(I)(\text{Hol}_\alpha) : M_G(\Sigma) \to \mathfrak{g} \quad (I = \wedge, \vee).
\]

The starting direction (resp. ending direction) of \( \alpha \), denoted by \( \alpha^\wedge \) (resp. \( \alpha^\vee \)) is the tangent vector at the starting (resp. ending) point of \( \alpha \) up to positive scaling.

If \( p \in \partial \Sigma \) is a marked point, we use the notation “\( \alpha^I \vdash p^\wedge \) (where \( I = \vee, \wedge \)) to indicate that “\( \alpha^I \) lies on \( p^\wedge \), i.e., \( \alpha \) starts from \( p \) if \( I = \wedge \) and \( \alpha \) ends at \( p \) if \( I = \vee \).

Two paths \( \alpha \) and \( \beta \) are said to be in general position if their interior intersection points are transversal double points (called crossings), and \( \alpha^\wedge, \alpha^\vee, \beta^\wedge, \beta^\vee \) do not coincide with each other.

Let \( \alpha \# \beta := (\alpha \cap \beta) \setminus \partial \Sigma \) denote the set of crossings. For any \( q \in \alpha \# \beta \), we let \( \varepsilon_q(\alpha, \beta) = \pm 1 \) be the oriented intersection number of \( \alpha \) and \( \beta \) at \( q \), and let \( \alpha \ast_q \beta \) denote the path which starts at \( \alpha^\wedge \), runs along \( \alpha \) before \( q \), then switches to \( \beta \) at \( q \), runs along \( \beta \) and ends at \( \beta^\vee \).

For any two symbols \( I, J = \wedge, \vee \), we also let \( \varepsilon(\alpha^I, \beta^J) = 0, \pm \frac{1}{2} \) denote the “oriented intersection number” of \( \alpha^I \) and \( \beta^J \); namely, define \( \varepsilon(\alpha^I, \beta^J) = 0 \) if \( \alpha^I \) and \( \beta^J \) do not lie on the same marked point; otherwise both \( \alpha^I \) and \( \beta^J \) are in \( T_p \Sigma \) for a marked point \( p \in \partial \Sigma \), and we define \( \varepsilon(\alpha^I, \beta^J) = \frac{1}{2} \) if the frame \( (\alpha^I, \beta^J) \) is compatible with the orientation of \( \Sigma \).

Finally, we define the algebraic intersection number of \( \alpha \) and \( \beta \) as

\[
i(\alpha, \beta) := \sum_{I,J=\wedge,\vee} \varepsilon(\alpha^I, \beta^J) + \sum_{q \in \alpha \# \beta} \varepsilon_q(\alpha, \beta).
\]

This generalizes the usual notion of algebraic intersection number for closed curves. It can be shown that the new notion is invariant under (endpoints-fixing) homotopy.

Let us remark that Lemma 2.3 (iv) has the following formulation when \( M = M_G(\Sigma) \) and \( u = \text{Hol}_\alpha \), which will be used a few times below. We denote the variation map of \( f \in \mathcal{O}_{M_G(\Sigma)} \) with respect to the \( G \)-action associated to a marked point \( p \in \{p_1, \ldots, p_b\} \) by \( \chi_f^p : M_G(\Sigma) \to \mathfrak{g} \).

**Lemma 2.4.** Put \( \varepsilon_{1J} = 1 \) if \( I = J \) and \( \varepsilon_{IJ} = -1 \) if \( I \neq J \). Let \( \Phi \in \mathcal{O}_G \), and \( \alpha \) be an path on \( \Sigma \). Then for any marked point \( p \in \partial \Sigma \) we have

\[
\chi_{\phi_\alpha}^p = \sum_{I: \alpha^I + p} \varepsilon_{1I} \Phi_\alpha^I.
\]

### 2.2. The quasi-Poisson bracket formula.

**Theorem 2.5.** For any \( \Phi, \Psi \in \mathcal{O}_G \) and paths \( \alpha, \beta \) in general position. The quasi-Poisson bracket of the functions \( \Phi_\alpha \) and \( \Psi_\beta \) on \( M_G(\Sigma) \) is

\[
\{ \Phi_\alpha, \Psi_\beta \}_{M_G(\Sigma)} = \sum_{I, J = \wedge, \vee} \varepsilon(\alpha^I, \beta^J)(\Phi_\alpha^I | \Psi_\beta^J) + \sum_{q \in \alpha \# \beta} \varepsilon_q(\alpha, \beta)B^q_{\Phi_\alpha, \Psi_\beta}. 
\]
where \( B^q_{\Phi,\alpha,\Psi,\beta} \in \mathcal{O}_{MG}(\Sigma) \) is defined by

\[
B^q_{\Phi,\alpha,\Psi,\beta} = (\Phi^\wedge | \text{Ad}_{\text{Hol}_{\alpha^q \beta}} \Psi^\vee).
\]

A proof will be given in the next subsection. We shall first give some remarks and easy consequences of the theorem.

**Remark 2.6.**

(i) Let us show that the right-hand side of the above formula is anti-symmetric when \((\Phi, \alpha)\) and \((\Psi, \beta)\) are exchanged.

The first sum is anti-symmetric because \(\varepsilon(\alpha^I, \beta^J) = -\varepsilon(\beta^J, \alpha^I)\). For the second sum, since \(\varepsilon_q(\alpha, \beta) = -\varepsilon_q(\beta, \alpha)\), it is sufficient to show

\[
B^q_{\Phi,\alpha,\Psi,\beta} = B^q_{\Psi,\beta,\Phi,\alpha}.
\]

This is proved by the following computation, using the Ad-invariance of \((\cdot | \cdot)\), Lemma 2.3 and the observation that the path \(\beta(\alpha^q \beta)^{-1} \alpha\) is homotopic to \(\beta^* \alpha\):

\[
B^q_{\Phi,\alpha,\Psi,\beta} = (\Phi^\wedge | \text{Ad}_{\text{Hol}_{\alpha^q \beta}} \Psi^\vee) = (\text{Ad}_{\text{Hol}_{\alpha^q \beta}} \Phi^\wedge | \text{Ad}_{\text{Hol}_{\alpha^q \beta}} \text{Ad}_{\text{Hol}_{\beta}}^{-1} \Psi^\vee)
= (\text{Ad}_{\text{Hol}_{\beta^q \alpha^*}} \Phi^\wedge | \Psi^\vee) = B^q_{\Psi,\beta,\Phi,\alpha}.
\]

(ii) By similar computations as above, one can obtain the following more general expressions of \(B^q_{\Phi,\alpha,\Psi,\beta}\), which will be used in §2.3 when we prove the theorem.

For any fixed \(I\) and \(J\), we define a path

\[
\gamma = (\alpha^I \wedge) * q (\beta^J \vee),
\]

where \(\varepsilon_I \wedge, \varepsilon_J \vee = \pm 1\) are defined in Lemma 2.4. Then we have

\[
B^q_{\Phi,\alpha,\Psi,\beta} = (\Phi^I | \text{Ad}_{\text{Hol}_{\gamma}} \Psi^J).
\]

(iii) Given \(\Phi, \Psi \in \mathcal{O}_G\), \(\Phi_\alpha\) and \(\Psi_\beta\) only depend on the homotopy classes of \(\alpha\) and \(\beta\), so the right-hand side of (2.1) should be invariant under homotopy. This fact will be established in the next subsection as a step in the proof of the theorem.

A simple consequence of Theorem 2.5 is the following

**Corollary 2.7.** If \(\alpha\) is a simple path (i.e. has no self-intersection), then

\[
\{\Phi_\alpha, \Psi_\alpha\} = 0 \quad \forall \Phi, \Psi \in \mathcal{O}_G.
\]

**Proof.** Let \(\beta\) be a path homotopic to \(\alpha\) such that \(\alpha \# \beta = \emptyset\) and that the relative configuration of \(\alpha\) and \(\beta\) at endpoints, in the case where \(\alpha\) is closed or non-closed, are as shown in the following local pictures respectively.
Put \( a = \text{Hol}_\alpha = \text{Hol}_\beta \). In the non-closed case we have \( \varepsilon(\alpha^\vee, \beta^\vee) = -\frac{1}{2}, \varepsilon(\alpha^\wedge, \beta^\wedge) = -\frac{1}{2} \) and \( \varepsilon(\alpha^\wedge, \beta^\vee) = \varepsilon(\alpha^\vee, \beta^\wedge) = 0 \). So Theorem 2.5 gives

\[
\{ \Phi_\alpha, \Psi_\beta \} = \{ \Phi_\alpha, \Psi_\beta \} = \frac{1}{2} (\Phi^\vee(a) \mid \Psi^\vee(a)) - \frac{1}{2} (\Phi^\wedge(a) \mid \Psi^\wedge(a)),
\]

which vanishes because \( \Phi^\wedge(a) = \text{Ad}_a \Phi^\vee(a), \Psi^\wedge(a) = \text{Ad}_a \Psi^\vee(a) \) and the scalar product \( (\cdot \mid \cdot) \) is Ad-invariant. In the closed case the argument is similar. \( \square \)

We also get the following particular case of Theorem 2.5 by straightforward computations using Example 2.2. Here \( i(\alpha, \beta) \) is the algebraic intersection number defined in the previous subsection.

**Corollary 2.8.** Assume \( G = \text{GL}_n \mathbb{R} \) and the invariant scalar product is \( (x \mid y) = \text{Tr}(xy) \). For a path \( \alpha \) on \( \Sigma \), we set \( \alpha_{ij} := \Phi_{ij} \circ \text{Hol}_\alpha \), where \( \Phi_{ij}(g) = q_{ij} \) is the \((i,j)\)-matrix entry function. Let \( \alpha \) and \( \beta \) be paths in general position. Then

\[
(2.2) \quad \{ \alpha_{ij}, \beta_{kl} \}_{M_G(\Sigma)} = \sum_{q \in \alpha \parallel \beta} \varepsilon_q(\alpha, \beta)(\alpha \ast_q \beta)_{il}(\beta \ast_q \alpha)_{kj} + \varepsilon(\alpha^\wedge, \beta^\vee)\alpha_{kj} \delta_{il} + \varepsilon(\alpha^\vee, \beta^\wedge)\alpha_{il} \delta_{kj} + \delta_{ij} \varepsilon(\alpha^\vee, \beta^\wedge)(\alpha^\wedge)_l.
\]

Here \( \delta_{ij} = 0, 1 \) is the Kronecker delta.

**Remark 2.9.** Corollary 2.8 implies that all the \( \alpha_{ij} \)'s generates a quasi-Poisson subalgebra \( \mathcal{A}_n(\Sigma) \subset \mathcal{O}_{M_G(\Sigma)} \). \( \mathcal{A}_n(\Sigma) \) can be described intrinsically as the commutative algebra over \( \mathbb{R} \) generated by all symbols of the form \( \alpha_{ij} \) (where \( 1 \leq i, j \leq n \) and \( \alpha \) is a non-trivial element in \( \pi_1(\Sigma) \)) with relations \( \sum_{j=1}^n \alpha_{ij} \beta_{jk} = (\alpha \beta)_{ik} \). This algebra first appears in [10] (where \( b = 1 \)).

More consequences of the theorem can be found in the author’s thesis [11]. For instance, if \( \Phi, \Psi \) are invariant by conjugation and \( \alpha, \beta \) are closed, then Theorem 2.5 recovers Goldman’s formula [6], hence gives a new proof of the latter. The flow generated by the vector field \( P(d\Phi_\alpha) \) (where \( P \) is the quasi-Poisson tensor on \( M_G(\Sigma) \)) can also be explicitly determined when \( \alpha \) is simple.

### 2.3. Proof of Theorem 2.5

We shall use the following terminology in the course of proof. Let \( \alpha \) and \( \beta \) be paths in general position on \( \Sigma \). We say that the theorem is true for the triple \((\Sigma, \alpha, \beta)\) if formula (2.1) holds for any choice of \( \Phi, \Psi \in \mathcal{O}_G \).

The strategy of proof is to successively reduce to simpler cases. Let us begin with some easy reductions.

**Lemma 2.10.** If the theorem is true for a triple \((\Sigma, \alpha, \beta)\), then it is true for the following triples as well:

(i) \((\Sigma, \beta, \alpha)\);
(ii) \((\Sigma, \alpha, \beta)\), where \( \Sigma \) denotes \( \Sigma \) with the opposite orientation\(^1\);
(iii) \((\Sigma, \alpha^{-1}, \beta)\) and \((\Sigma, \alpha^{-1}, \beta^{-1})\).

**Proof.** For (i) this is essentially Remark 2.6 (i).

As for (ii), this is because on one hand we have \( \{ \Phi_\alpha, \Psi_\beta \}_{M_G(\Sigma)} = -\{ \Phi_\alpha, \Psi_\beta \}_{M_G(\Sigma)} \) because the quasi-Poisson tensors on \( \Sigma \) and \( \Sigma \) are opposite; and on the other hand, under the reversed orientation, \( \varepsilon(\alpha^t, \beta^t) \) and \( \varepsilon_q(\alpha, \beta) \) are \(-1\) times of the old ones.

---

\(^1\) M_G(\Sigma) and M_G(\Sigma) are the same as \(\mathcal{O}_b\)-manifolds, but with opposite quasi-Poisson tensors and moment maps.
For the triples in (iii), in view of (i), it is sufficient to prove for \((\Sigma, \alpha, \beta^{-1})\). Put \(\hat{\Psi}(g) := \Psi(g^{-1})\). By hypothesis we have
\[
\{\Phi_\alpha, \Psi_{\beta^{-1}}\}_{MC(\Sigma)} = \{\Phi_\alpha, \hat{\Psi}_\beta\}_{MC(\Sigma)}
\]
\[
= \sum_{I, J} \varepsilon(\alpha^I, \beta^J)(\Phi_\alpha^I | \hat{\Psi}_\beta^J) + \sum_{q \in \alpha \# \beta} \varepsilon_q(\alpha, \beta) B^q_{\Phi, \alpha, \Psi, \beta}.
\]

Using Lemma 2.3 (ii), we get
\[
\text{Ad}_{\text{Hol}_{\alpha, q, \beta}} \hat{\Psi}_\beta^\prime = -\text{Ad}_{\text{Hol}_{\alpha, q, \beta}^{-1}} \Psi_\beta^\prime(\text{Hol}_{\beta^{-1}}) = -\text{Ad}_{\text{Hol}_{\alpha, q, \beta}^{-1}} \Psi_\beta^\prime,
\]
which implies \(B^q_{\Phi, \alpha, \Psi, \beta} = -B^q_{\Phi, \alpha, \Psi, \beta^{-1}}\), hence
\[
\varepsilon_q(\alpha, \beta) B^q_{\Phi, \alpha, \Psi, \beta} = \varepsilon_q(\alpha, \beta^{-1}) B^q_{\Phi, \alpha, \Psi, \beta^{-1}}.
\]

Similarly, for any \(I, J = \land, \lor\) we have
\[
\varepsilon(\alpha^I, \beta^J)(\Phi_\alpha^I | \hat{\Psi}_\beta^J) = \varepsilon(\alpha^I, (\beta^{-1})^J)(\Phi_\alpha^I | \Psi_{\beta^{-1}}^J),
\]
where \(J'\) denotes the symbol opposite to \(J\). Therefore, we have the required equality
\[
\{\Phi_\alpha, \Psi_{\beta^{-1}}\}_{MC(\Sigma)} = \sum_{I, J} \varepsilon(\alpha^I, (\beta^{-1})^J)(\Phi_\alpha^I | \Psi_{\beta^{-1}}^J) + \sum_{q \in \alpha \# \beta^{-1}} \varepsilon_q(\alpha, \beta^{-1}) B^q_{\Phi, \alpha, \Psi, \beta^{-1}}.
\]

The main ingredients of our proof of Theorem 2.5 are the following reductions.

**Proposition 2.11.**

(i) Let \(\alpha'\) and \(\beta'\) be two paths in general position which are endpoints-fixing-homotopic to \(\alpha\) and \(\beta\) respectively. Then the right-hand side of (2.1) gives the same function when \(\alpha\) and \(\beta\) are replace by \(\alpha'\) and \(\beta'\), respectively. In particular, if the theorem is true for the triple \((\Sigma, \alpha, \beta)\), then it is also true for \((\Sigma, \alpha', \beta')\).

(ii) Let \(\alpha_1, \ldots, \alpha_r\) and \(\alpha\) be paths such that \(\alpha\) and each \(\alpha_i\) are in general position with \(\beta\), and \(\alpha\) is endpoints-fixing-homotopic to the composition \(\alpha_1 \cdots \alpha_r\), where we assume that each \(\alpha_i\) ends at the point where \(\alpha_{i+1}\) starts from. If the theorem is true for each of the triples \((\Sigma, \alpha_1, \beta), \ldots, (\Sigma, \alpha_r, \beta)\), then it is also true for \((\Sigma, \alpha, \beta)\).

(iii) Let \(\Delta\) be a splitting arc in \(\Sigma\). Let \(\tilde{\alpha}, \tilde{\beta}\) be paths in general position on \(\Sigma_\Delta\), and \(\alpha, \beta\) be their images in \(\Sigma\). If the theorem is true for the triple \((\Sigma_\Delta, \tilde{\alpha}, \tilde{\beta})\), then it is also true for \((\Sigma, \alpha, \beta)\).

**Proof of Proposition 2.11 (i).** A straightforward generalization of results in [6] §5 to surfaces with boundary shows that there exists a sequence of pairs of paths in general position
\[
(\alpha, \beta) = (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r) = (\alpha', \beta')
\]
such that \((\alpha_{i+1}, \beta_{i+1})\) is obtained from \((\alpha_i, \beta_i)\) by applying one of the following moves \((\omega_1)-(\omega_4)\), as shown in Figure 2.1-2.4.

To apply a move \((\omega_i)\) to \((\alpha, \beta)\), first we find an open set \(U\) in \(\Sigma\) which is diffeomorphic to a disk (for \(j = 1, 2, 3\)) or a half disk (for \(j = 4\), such that there are subintervals of \(\alpha\) and \(\beta\) in \(U\) as shown in the pictures, then we replace these intervals by the new ones shown in the pictures.
We only need to prove that if \((\alpha', \beta')\) is obtained from \((\alpha, \beta)\) by one of the above moves, then replacing \(\alpha\) and \(\beta\) on the right-hand side of formula (2.1) by \(\alpha'\) and \(\beta'\) gives the same result.

Clearly, the move \((\omega_1)\) does not give rise to any change of the right-hand side of the formula. The next two moves \((\omega_2)\) and \((\omega_3)\) do not change the starting/ending directions \(\alpha^I, \beta^J (I, J = \land, \lor)\), thus do not change the contribution from endpoints. Regarding the crossings, it can be shown (see p.293 of [6] for details) that \((\omega_2)\) turns two points \(p, q \in \alpha' \# \beta'\) into new ones \(p', q' \in \alpha' \# \beta'\), such that \(\alpha *_p \beta\) and \(\alpha *_q \beta\) are homotopic to \(\alpha' *_{p'} \beta'\) and \(\alpha' *_{q'} \beta'\) respectively, and moreover \(\varepsilon_p(\alpha, \beta) = \varepsilon_{p'}(\alpha', \beta')\), \(\varepsilon_q(\alpha, \beta) = \varepsilon_{q'}(\alpha', \beta')\); whereas the move \((\omega_3)\) creates two new interior intersection points \(p, q \in \alpha' \# \beta'\) such that \(\alpha' *_p \beta'\) and \(\alpha' *_q \beta'\) are homotopic to each other and \(\varepsilon_p(\alpha', \beta') = -\varepsilon_q(\alpha', \beta')\). As a result, the formula remains unchanged under these two moves.

Finally, let us see what happens when applying the move \((\omega_4)\) to \((\alpha, \beta)\) near a marked point \(p\). For two specific indices \(I, J\) such that \(\alpha^I, \beta^J \vdash p\), the move reverses the relative position of \(\alpha^I\) and \(\beta^J\), and creates a new interior intersection
point \( q \in \alpha' \# \beta' \). The only change of the right-hand side of formula (2.1) is that the term \( \varepsilon(\alpha',\beta')(\Phi^I_\alpha \mid \Psi^J_\beta) \) becomes

\[
\varepsilon(\alpha'',\beta'')(\Phi^I_\alpha \mid \Psi^J_\beta) + \varepsilon_q(\alpha',\beta')B^q_{\Phi,\alpha',\Psi,\beta'}.
\]

We need to identify this with the original term. Observe that

\[
\varepsilon(\alpha',\beta') = -\varepsilon(\alpha'',\beta'') = \frac{1}{2}\varepsilon_q(\alpha',\beta'),
\]

so it is sufficient to show

\[
B^q_{\Phi,\alpha',\Psi,\beta'} = (\Phi^I_\alpha \mid \Psi^J_\beta).
\]

This follows from the expression of \( B^q_{\Phi,\alpha',\Psi,\beta'} \) given in Remark 2.6 (ii): put \( \gamma := (\alpha'^{\#} \gamma) *_q (\beta'^{\#} \gamma) \), then we have

\[
B^q_{\Phi,\alpha',\Psi,\beta'} = (\Phi^I_\alpha \mid \Ad_{\Hol}(\Psi^J_\beta)).
\]

But it is easy to see that \( \gamma \) is the path which starts from \( p \), runs along \( \alpha \) until \( q \), and then comes back to \( p \) along \( \beta \), hence is homotopically trivial. \( \square \)

**Proof of Proposition 2.11 (ii).** It is sufficient to treat the \( r = 2 \) case, from which the general case follows by recurrence. For brevity, we put

\[
\begin{align*}
u_1 &= \Hol_{\alpha_1}, & u_2 &= \Hol_{\alpha_2}, & u &= \Hol_\alpha = u_1u_2, & v &= \Hol_\beta.\end{align*}
\]

We fix a point \( m_0 \in M_G(\Sigma) \), define \( u^{(1)}, u^{(2)} : M_G(\Sigma) \to G \) by

\[
\begin{align*}
u^{(1)}(m) &= u_1(m)u_2(m_0), & u^{(2)}(m) &= u_1(m_0)u_2(m) & \forall m \in M_G(\Sigma),
\end{align*}
\]

and define \( \Phi_1, \Phi_2 \in \mathcal{O}_G \) by

\[
\Phi_1(g) = \Phi(gu_2(m_0)), \quad \Phi_2(g) = \Phi(u_1(m_0)g) & \forall g \in G.
\]

Then the derivation of \( \Phi_\alpha = \Phi(u) = \Phi(u_1u_2) \in \mathcal{O}_{M_G(\Sigma)} \) at the point \( m_0 \) is the sum of the derivations of \( (\Phi_1)_{\alpha_1} = \Phi_1(u_1) \) and \( (\Phi_2)_{\alpha_2} = \Phi_2(u_2) \). As a result, we get

\[
\begin{align*}(2.3) \quad \{ \Phi_\alpha, \Psi_\beta \}(m_0) &= \{(\Phi_1)_{\alpha_1}, \Psi_\beta \}(m_0) + \{(\Phi_2)_{\alpha_2}, \Psi_\beta \}(m_0) \\
&= \sum_{I,J} \varepsilon(\alpha_I',\beta_J')(\Phi^I_1(u_1(m_0)) \mid \Psi^J_\beta(v(m_0))) + \sum_{q_1 \in \alpha_1 \# \beta} \varepsilon_{q_1}(\alpha_1,\beta)B^q_{\Phi,\alpha_1,\Psi,\beta}(m_0) \\
&\quad + \sum_{I,J} \varepsilon(\alpha_I',\beta_J')(\Phi^I_2(u_2(m_0)) \mid \Psi^J_\beta(v(m_0))) + \sum_{q_2 \in \alpha_2 \# \beta} \varepsilon_{q_2}(\alpha_2,\beta)B^q_{\Phi,\alpha_2,\Psi,\beta}(m_0).
\end{align*}
\]

The second equality is because of the hypothesis that the theorem is true for the triples \( (\Sigma, \alpha_1, \beta) \) and \( (\Sigma, \alpha_2, \beta) \).

Our goal is to show that the right-hand side of (2.3) coincides with the right-hand side of formula (2.1) evaluated at \( m_0 \) for some \( \alpha \) homotopic to \( \alpha_1\alpha_2 \).

Let us choose \( \alpha \) in the following way. First we modify \( \alpha_1 \) and \( \alpha_2 \) by homotopy such that all starting/ending directions of the three paths \( \alpha_1, \alpha_2, \beta \) are distinct. Let \( p \) be the ending point of \( \alpha_1 \). Then we get \( \alpha \) by smoothing the “corner” of \( \alpha_1\alpha_2 \) within a small neighborhood \( U \) of \( p \), see Figure 2.5.

The smoothing does not change the crossings of \( \alpha_1\alpha_2 \) with \( \beta \) outside \( U \), whereas it creates up to two crossings \( q_J \ (J = \land, \lor) \) in \( U \), depending on the relative position of \( \alpha_1', \alpha_2', \beta' \). There are two cases:

(a) If \( \beta' \) does not lie on \( p \), or if \( \beta' \cap p \) and the directions \( \alpha_1', \alpha_2' \) are on the same side of \( \beta' \) (i.e., both on the left or both on the right), then the smoothing creates no crossing near \( \beta' \).
(b) If \( \beta^J \vdash p \) and the directions \( \alpha_1^\vee, \alpha_2^\vee \) are on the two sides of \( \beta^J \) respectively, then the smoothing creates a crossing \( q_J \in U \).

![Diagram](Image)

**Figure 2.5.** A typical local picture: here \( \beta^\vee \) belongs to the case (a), and \( \beta^< \) to the case (b).

Now we compute the right-hand side of (2.3). By Lemma 2.3 (iii), we have the following identities of functions on \( G \),

\[
\Phi_1^\wedge(g) = \Phi^\wedge(gu_2(m_0)), \quad \Phi_1^\vee(g) = \text{Ad}_{u_2(m_0)}\Phi^\vee(gu_2(m_0)),
\]

and

\[
\Phi_2^\wedge(g) = \text{Ad}^{-1}_{u_1(m_0)}\Phi^\wedge(u_1(m_0)g), \quad \Phi_2^\vee(g) = \Phi^\vee(u_1(m_0)g).
\]

It follows that

\[
\Phi_1^\wedge(u_1(m_0)) = \Phi^\wedge(u(m_0)), \quad \Phi_1^\vee(u_1(m_0)) = \text{Ad}_{u_2(m_0)}\Phi^\vee(u(m_0)),
\]

\[
\Phi_2^\wedge(u_2(m_0)) = \text{Ad}^{-1}_{u_1(m_0)}\Phi^\wedge(u(m_0)), \quad \Phi_2^\vee(u_2(m_0)) = \Phi^\vee(u(m_0)).
\]

Using (2.4), we find that certain terms in (2.3) coincide with those in (2.1). Namely,

\[
\varepsilon(\alpha_1^\wedge, \beta^J)(\Phi_1^\wedge(u_1(m_0)) | \Psi^J(v(m_0))) = \varepsilon(\alpha^\wedge, \beta^J)(\Phi^\wedge(u(m_0)) | \Psi^J(v(m_0))),
\]

\[
\varepsilon(\alpha_2^\wedge, \beta^J)(\Phi_2^\wedge(u_2(m_0)) | \Psi^J(v(m_0))) = \varepsilon(\alpha^\vee, \beta^J)(\Phi^\vee(u(m_0)) | \Psi^J(v(m_0))),
\]

and

\[
\varepsilon_{q_i}(\alpha^\vee, \beta^J)B_{\Phi,\alpha^\vee,\beta^J}(m_0) = \varepsilon_{q_i}(\alpha^\wedge, \beta^J)B_{\Phi,\alpha^\wedge,\beta^J}(m_0), \quad \forall q_i \in \alpha_i \# \beta
\]

for \( i = 1, 2 \).

We see that each term in (2.1), except for those coming from the possible new crossings \( q_\ell \) and \( q_\ell' \), coincides with a term in (2.3). It remains to be shown that

\[
\varepsilon(\alpha_1^\vee, \beta^J)(\Phi_1^\wedge(u_1(m_0)) | \Psi^J(v(m_0))) + \varepsilon(\alpha_2^\vee, \beta^J)(\Phi_2^\wedge(u_2(m_0)) | \Psi^J(v(m_0)))
\]

\[
= \begin{cases} 
0 & \text{in the case (a),} \\
\varepsilon_{q_J}(\alpha, \beta)B_{\Phi,\alpha,\beta}(m_0) & \text{in the case (b).}
\end{cases}
\]

In Case (a), \( \varepsilon(\alpha_1^\vee, \beta^J) \) and \( \varepsilon(\alpha_2^\vee, \beta^J) \) are opposite, and it follows from (2.4) and Lemma 2.3 (iii) that

\[
(2.5) \quad \Phi_1^\wedge(u_1(m_0)) = \Phi_2^\wedge(u_2(m_0)).
\]

Thus we get the required equality.

In Case (b), it is easy to see that

\[
\varepsilon(\alpha_1^\vee, \beta^J) = \varepsilon(\alpha_2^\wedge, \beta^J) = \frac{1}{2}\varepsilon_{q_J}(\alpha, \beta).
\]

Using (2.5) and (2.4) again, we get

\[
\varepsilon(\alpha_1^\vee, \beta^J)(\Phi_1^\wedge(u_1(m_0)) | \Psi^J(v(m_0))) + \varepsilon(\alpha_2^\wedge, \beta^J)(\Phi_2^\wedge(u_2(m_0)) | \Psi^J(v(m_0)))
\]

\[
= \varepsilon_{q_J}(\alpha, \beta)(\Phi_1^\wedge(u(m_0)) | \text{Ad}^{-1}_{u_2(m_0)}\Psi^J(v(m_0))).
\]
On the other hand, by Remark 2.6 (ii),
\[ B^J_{\Phi_\alpha,\Psi_\beta} = (\Phi^J_\alpha(u) \mid \text{Ad}_\alpha, \Psi^J(v)), \]
where \( \gamma := (\alpha^{-1})_\ast_q \beta \) if \( J = \vee \), and \( \gamma := (\alpha^{-1})_\ast_q (\beta^{-1}) \) if \( J = \wedge \). Noting that \( \gamma \) is homotopic to \( \alpha_2 \), we get \( \text{Hol}_\gamma = u_2 \). This concludes the proof of the required equality.

Proof of Proposition 2.11 (iii). Suppose that the splitting arc \( \Delta \) issues from a marked point \( p \in \partial \Sigma \), and \( p \) split into two marked point \( p_1, p_2 \in \partial \Sigma \), where \( p_1 \) is on the left and \( p_2 \) on the right (see §1.2). For \( i = 1, 2 \), we let \( \mu_i : M_G(\Sigma) \to G \) denote the reversed boundary holonomy at \( p_i \), and let \( \chi^i \) denote the variation map (see (1.3) for the definition) of \( f \in \mathcal{O}_{M_G(\Sigma)} \) with respect to the \( G \)-action on \( M_G(\Sigma) \) associated to \( p_i \).

We consider \( M := M_G(\Sigma) \cong M_G(\Sigma) \) as the same manifold via the splitting homeomorphism \( R_\Delta \). If \( \tilde{\alpha} \) is an path on \( \Sigma \) and \( \alpha \) is its image on \( \Sigma \), then \( \text{Hol}_\alpha : M_G(\Sigma) \to G \) and \( \text{Hol}_\alpha : M_G(\Sigma) \to G \) are same map, in particular \( \Phi_\alpha = \Phi_\alpha \in \mathcal{O}_M \).

It follows from Theorem 1.13 and the definition of fusion (1.4) that
\[ \{ \Phi_\alpha, \Psi_\beta \}_{M_G(\Sigma)} = \{ \Phi_\alpha, \Psi_\beta \}_{M_G(\Sigma)} - \frac{1}{2}(\chi^{(1)}_\Phi \mid \chi^{(2)}_\Psi) + \frac{1}{2}(\chi^{(1)}_\Psi \mid \chi^{(2)}_\Phi). \]
The theorem is true for \((\Sigma, \alpha, \beta)\) by hypothesis, hence
\[ \{ \Phi_\alpha, \Psi_\beta \}_{M_G(\Sigma)} = \sum_{I,J} \varepsilon(\tilde{\alpha}^I, \tilde{\beta}^J)(\Phi^I_\alpha \mid \Psi^J_\beta) + \sum_{q \in \tilde{\alpha}^I \tilde{\beta}^J} \varepsilon_q(\tilde{\alpha}, \tilde{\beta})B^J_{\Phi_\alpha, \Psi_\beta} \]
\[ = \sum_{I,J} \varepsilon(\alpha^I, \beta^J)(\Phi^I_\alpha \mid \Psi^J_\beta) + \sum_{q \in \alpha^I \beta^J} \varepsilon_q(\alpha, \beta)B^J_{\Phi_\alpha, \Psi_\beta}. \]
To prove that the theorem is true for \((\Sigma, \alpha, \beta)\), we need to show
\[ \sum_{I,J} \varepsilon(\tilde{\alpha}^I, \tilde{\beta}^J)(\Phi^I_\alpha \mid \Psi^J_\beta) - \frac{1}{2}(\chi^{(1)}_\Phi \mid \chi^{(2)}_\Psi) + \frac{1}{2}(\chi^{(1)}_\Psi \mid \chi^{(2)}_\Phi) \]
\[ = \sum_{I,J} \varepsilon(\alpha^I, \beta^J)(\Phi^I_\alpha \mid \Psi^J_\beta). \]

By Lemma 2.4, for \( i = 1, 2 \) we have
\[ \chi^{(i)}_{\Phi_\alpha} = \sum_{I, \tilde{\alpha}^I \vdash p_i} \varepsilon_{I \vee \tilde{\alpha}^I}, \quad \chi^{(i)}_{\Psi_\beta} = \sum_{J, \tilde{\beta}^J \vdash p_i} \varepsilon_{J \vee \tilde{\beta}^J}. \]
Therefore, the left-hand side of (2.6) equals
\[ \left( \sum_{I,J} \varepsilon(\tilde{\alpha}^I, \tilde{\beta}^J) - \sum_{\tilde{\alpha}^I \vdash p_1, \tilde{\beta}^J \vdash p_2} \frac{\varepsilon_{I,J}}{2} + \sum_{\tilde{\alpha}^I \vdash p_2, \tilde{\beta}^J \vdash p_1} \frac{\varepsilon_{I,J}}{2} \right) (\Phi^I_\alpha \mid \Psi^J_\beta). \]
Consider both sides of (2.6) as a sum of four terms, corresponding to the four choices of \((I, J)\). Comparing the local pictures of \((\Sigma, \alpha, \beta)\) and \((\Sigma, \alpha, \beta)\) near \( p \), we see that for any \((I, J)\),
- If \( \tilde{\alpha}^I \vdash p_1 \) and \( \tilde{\beta}^J \vdash p_2 \), then \( \varepsilon(\tilde{\alpha}^I, \tilde{\beta}^J) = 0 \), \( -\frac{\varepsilon_{I,J}}{2} = \varepsilon(\alpha^I, \beta^J) \).
• If $\tilde{\alpha} \parallel p_2$ and $\tilde{\beta} \parallel p_1$, then
  $$\varepsilon(\tilde{\alpha}, \tilde{\beta}) = 0, \quad \frac{\varepsilon_{IJ}}{2} = \varepsilon(\alpha^I, \beta^J);$$

• If it is not the above two cases, then $\varepsilon(\tilde{\alpha}, \tilde{\beta}) = \varepsilon(\alpha^I, \beta^J)$.

This implies the required equality (2.6).

Using the above proposition, we will essentially reduce Theorem 2.5 to the following three simplest situations, which can be verified directly.

**Proposition 2.12.**  
(i) If $\Sigma = \Sigma_1 \cup \Sigma_2$ is a disjoint union, and $\alpha, \beta$ are contained in $\Sigma_1$, $\Sigma_2$, respectively, then the theorem is true for $(\Sigma, \alpha, \beta)$.

(ii) Let $\nu$ be a loop which is homotopic to the reversed boundary loop of $\Sigma$ at a marked point $p$, and $\beta$ be any path in general position with $\nu$. Then the theorem is true for $(\Sigma, \nu, \beta)$.

(iii) Let $\alpha$ be an paths on $\Sigma_{0,2}$ joining the the two marked points. Then
\[
\{\Phi_\alpha, \Psi_\beta\}_{M_G(\Sigma_{0,2})} = 0, \quad \forall \Phi, \Psi \in O_G.
\]

As a result, let $\alpha'$ be a path in general position with $\alpha$ and homotopic to $\alpha$, then the theorem is true for $(\Sigma_{0,2}, \alpha, \alpha')$.

**Proof.** (i) is clear. We proceed to show (ii).

Because of Proposition 2.11 (i), we can modify $\nu$ by homotopy. Let us bring $\nu$ to a small neighborhood of the boundary, such that $\nu$ and $\beta$ have no crossings, and any $\beta^J \parallel p$ $(J = \wedge, \vee)$ lies between $\nu^\wedge$ and $\nu^\vee$. Put $\delta_p(\beta^J) = 1$ if $\beta^J \parallel p$, and otherwise $\delta_p(\beta^J) = 0$. Then it is easy to see that the $\varepsilon(\nu^I, \beta^J)$'s are given by
\[
(2.7) \quad \varepsilon(\nu^I, \beta^J) = \delta_p(\beta^\wedge), \quad \varepsilon(\nu^I, \beta^\vee) = -\delta_p(\beta^\vee), \quad \text{for } I = \wedge, \vee.
\]

Now we compute the quasi-Poisson bracket. Since $\mu := \text{Hol}_\nu$ is a component of the moment map $M_G(\Sigma) \to G^\text{h}$, (1.2) implies that
\[
(2.8) \quad \{\Phi_\nu, \Psi_\beta\}_{M_G(\Sigma)} = -d\Phi_\nu (P^\text{d}(d\Psi_\beta)) = -(\Phi_\nu^\wedge (\mu) \mid \nu^\wedge \theta(P^\text{d}(d\Psi_\beta)))
\]
\[
= \frac{1}{2} (\Phi_\nu^\wedge (\mu) \mid (1 + \text{Ad}_\mu^{-1}) \chi_{\Psi_\beta}^p) = \frac{1}{2} (\Phi_\nu^\vee (\mu) + \Phi_\nu^\wedge (\mu) \mid \chi_{\Psi_\beta}^p),
\]
where $\chi_{\Psi_\beta}^p$ denotes the variation function of $f \in O_{M_G(\Sigma)}$ with respect to the $G$-action associated to $p$. By Lemma 2.4, we have
\[
\chi_{\Psi_\beta}^p = -\delta_p(\beta^\wedge) \Psi_\beta^\wedge + \delta_p(\beta^\vee) \Psi_\beta^\vee.
\]

Inserting this into (2.8) and using (2.7), we conclude that
\[
\{\Phi_\alpha, \Psi_\beta\}_{M_G(\Sigma)} = \sum_{I,J} \varepsilon(\nu^I, \beta^J) (\Phi_I(\mu) \mid \Psi_J),
\]
which agrees with formula (2.1) because $\nu$ and $\beta$ have no crossings.

To prove (iii), let $p_1$ and $p_2$ be the starting and ending point of $\alpha$, respectively, and $\beta$ be the boundary loop at $p_2$. Set $a := \text{Hol}_\alpha$ and $b := \text{Hol}_{\beta^{-1}}$.

\[
(a, b) : M_G(\Sigma_{0,2}) \xrightarrow{\sim} G \times G = D(G)
\]
identifies $M_G(\Sigma_{0,2})$ with the double $D(G)$, on which the canonical quasi-Poisson bivector field $P$ is given in Example 1.12.

To prove the first assertion, it is sufficient to show that
\[
a^* \theta(p^\text{d}(d\Phi_\alpha)) = 0.
\]
This is done by straightforward computation. In fact, using \( d\Phi_\alpha = d(\Phi \circ a) = \Phi'(a) | a^*\theta \) and the expression of \( P \), we get
\[
P^\sharp (d\Phi_\alpha) = \frac{1}{2} \sum_i (\Phi'(a_i^\sharp L)) e_i^2 + \frac{1}{2} \sum_i (\Phi'(a_i^\sharp R)) e_i^2,
\]
whence \( P^\sharp (d\Phi_\alpha) \) is tangent to the second factor of \( D(G) \), and the required property follows.

Finally, we have already seen in the proof of Corollary 2.7 that the theorem gives zero when applied to \((\Sigma_0, \alpha, \alpha')\). Thus the theorem is true for \((\Sigma_0, \alpha, \alpha')\).

□

We are now in position to prove the theorem.

Proof of Theorem 2.5. Since any path on \( \Sigma \) is homotopic to the composition of a number of simple paths (this follows, e.g., from the presentation of the fundamental groupoid \( \pi_1(\Sigma) \) given in §1.1), applying Proposition 2.11 (i), it is sufficient to prove that the theorem is true for any \((\Sigma, \alpha, \beta)\) where \( \alpha \) and \( \beta \) are simple paths in general position.

We shall now do a further reduction to the case where \( \alpha \) and \( \beta \) have no crossings. If \( \alpha \# \beta \) is non-empty, suppose that it consists of points \( q_1, \ldots, q_r \) ordered by the orientation of \( \alpha \). We shall decompose \( \alpha \) into paths \( \alpha_0, \ldots, \alpha_r \) in the following way, such that each \( \alpha_i \) is simple and has no crossing with \( \beta \). Then the required reduction follows by applying Proposition 2.11 (i) again. First, \( \alpha_0 \) is obtained from the path \( \alpha \ast q_1 \beta \) by smoothing its corner at \( q_1 \) and homotoping it away from \( \beta \). Then we obtain \( \alpha_i \) \( (1 \leq i \leq r-1) \) from \( \beta^{-1} \ast p, \alpha \ast p_{i+1} \beta \), and \( \alpha_r \) from \( (\beta^{-1}) \ast p, \alpha \) in a similar way. See Figure 2.6.

![Figure 2.6. Decomposing \( \alpha \) into paths which do not cross \( \beta \).](image)

Therefore, we only need to prove the theorem for any triple \((\Sigma, \alpha, \beta)\) where \( \alpha \) and \( \beta \) are simple paths on \( \Sigma \) in general position and without crossings. By Lemma 2.10, we could exchange the roles of \( \alpha \) and \( \beta \), reverse the orientation of \( \Sigma \), or replace \( \alpha \) and/or \( \beta \) by their inverse. Performing these modifications if necessary, we can always bring the relative configuration of \( \alpha \) and \( \beta \) into the following three situations.

(a) The path \( \alpha \) is a simple loop issuing from a marked point \( p \). Furthermore, we assume that \( \alpha^\wedge \) is on the left of \( \alpha^\vee \), and also on the left of any \( \beta^J \vdash p \) (where \( J = \wedge, \vee \)).
(b) Both \( \alpha \) and \( \beta \) are embedded segments, and they share at most one endpoint. Furthermore, we assume that their common endpoint \( p \), if exists, is the starting point of both \( \alpha \) and \( \beta \), and \( \alpha^\wedge \) is on the left of \( \beta^\wedge \).
(c) Both \( \alpha \) and \( \beta \) are embedded segments, and they both go from a marked point \( p_1 \) to another \( p_2 \).
In case (a), we can find a splitting arc $\Delta$ issuing from $p$ and homotopic to $\alpha$, such that $\Delta$ is disjoint from $\alpha$ and $\beta$ except at $p$ (see Figure 2.7). Thus $\alpha$ and $\beta$ lifts to paths $\tilde{\alpha}$ and $\tilde{\beta}$ on the split surface $\Sigma_\Delta$. By Proposition 2.11 (iii), it is sufficient to prove the theorem for the triple $(\Sigma_\Delta, \tilde{\alpha}, \tilde{\beta})$. But $\tilde{\alpha}$ is homotopic to the reversed boundary loop of $\Sigma_\Delta$ at the left split marked point, so we conclude by applying Proposition 2.12 (ii).

![Figure 2.7. Typical situations in case (a) (left) and case (b) (right).](image)

In case (b), the argument is similar. Let $p$ be the starting point of $\alpha$, and $\nu$ the reversed boundary loop at the ending point of $\alpha$. We can find a splitting arc $\Delta$ issuing from $p$ and homotopic to $\alpha\nu\alpha^{-1}$, such that $\Delta$ is disjoint from $\alpha$ and $\beta$ except at $p$ (see Figure 2.7). The split surface $\Sigma_\Delta$ is the disjoint union of an annulus $\Sigma_{0,2}$ and a surface $\Sigma'$, whose number of boundary components is one less than $\Sigma$. The lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of $\alpha$ and $\beta$ are contained in $\Sigma_{0,2}$ and $\Sigma'$, respectively. This time we conclude by applying Proposition 2.12 (i).

Finally, in case (c), we decompose $\beta$ into $(\beta\alpha^{-1})\alpha$. We homotope the loop $\beta\alpha^{-1}$ into an loop $\gamma$ which has no crossings with $\alpha$, and also homotope $\alpha$ into an path $\alpha'$ in general position with $\alpha$. By Proposition 2.11 (ii) and the already proved case (b), it is sufficient to prove the theorem for the triple $(\Sigma, \alpha, \alpha')$. But this follows from the same splitting as in case (b) and Proposition 2.12 (iii). Thus the proof of Theorem 2.5 is complete. $\square$

3. QUASI-POISSON BRACKETS ON CROSS-SECTIONS

3.1. QUASI-POISSON CROSS-SECTIONS. In this subsection we recall the quasi-Poisson cross-section theorem from [1].

Assume that $G$ is compact and $(\cdot | \cdot)$ is a positive-definite invariant scalar product on $g$. Given $g \in G$, let $H$ be the stabilizer of $g$ with respect to the conjugation action of $G$ on itself, and $\mathfrak{h}$ be its Lie algebra. Then $g$ has a neighborhood $U$ in $H$ such that $U$ is a cross-section for the conjugation action, in the sense that the map $G \times_H U \to G.U, (g,h) \mapsto ghg^{-1}$ is a diffeomorphism onto its image.

Let $T \subset G$ a maximal torus, $t$ be its Lie algebra, and $\mathfrak{A} \subset t$ be a (closed) Weyl alcove. Without loss of generality we can assume $g \in \exp(\mathfrak{A})$. A standard choice of $U$ in this case is as follows. Let $\sigma$ be the open face of $\mathfrak{A}$ such that $g \in \exp(\mathfrak{A})$. The stabilizer of any element in $\sigma$ is the same subgroup $H = G_\sigma \subset G$. Let $V_\sigma$ be the union of all open faces $\tau$ of $\mathfrak{A}$ such that $\tau \supset \sigma$. Then we can take $U = G_\sigma \cdot \exp(V_\sigma)$. In particular, we can take $U = \exp(\mathfrak{A}^c)$ if $g \in \exp(\mathfrak{A}^c)$.

Notice that with the above choice of $U$, the stabilizer of any $h \in U$ is contained in $G_\sigma = H$. It follows that $(\text{Ad}_h - 1)|_{h^\perp}$ is invertible.
Let \((M, P, \mu)\) be a Hamiltonian quasi-Poisson \(G\)-manifold. By equivariance of \(\mu\), \(L = \mu^{-1}(U)\) is a \(H\)-invariant smooth submanifold of \(M\) and is a cross-section of the \(G\)-action in the sense that \(G \times H L \to \mu^{-1}(G U)\), \((g, m) \mapsto g \cdot m\) is a diffeomorphism. It follows that there is a splitting
\[ TM|_L = TL \oplus (L \times h^\perp). \]
Here we identify \((m, x) \in L \times h^\perp\) with \(\rho_x(m) \in TM|_L\).

**Theorem 3.1** (Cross-Section Theorem). (i) There is a decomposition
\[ P|_L = P_L + P_L^\perp, \]
where \(P_L \in \Gamma(\bigwedge^2 TL)\) and \(P_L^\perp : L \to \bigwedge^2 h^\perp\). Furthermore, \(P_L^\perp(m) \in \bigwedge^2 h^\perp \cong \bigwedge^2 (h^\perp)^*\) (here we identify \(h\) with \(h^*\) via \((\cdot \mid \cdot)_{h^*}\)) is the following skew-symmetric bilinear form on \(h^\perp:\)
\[ (x, y) \mapsto -\frac{1}{2}\left(\frac{\text{Ad}_\mu(m) + 1}{\text{Ad}_\mu(m) - 1}\right)x \mid y. \]
(ii) \((L, P_L, \mu|_L)\) is a Hamiltonian quasi-Poisson \(H\)-manifold.

We will consider cross-sections of the quasi-Poisson \(G^b\)-manifold \(M_G(\Sigma)\). From now on we put
\[ L = \bigcap_{i=1}^b \mu_i^{-1}(U) \subset M_G(\Sigma). \]
Then \(L\) is a smooth \(H^b\)-invariant submanifold of \(M_G(\Sigma)\). The above theorem implies that there is a bivector field \(P_L\) on \(L\) so that \((L, P_L, (\mu_1|_L, \cdots, \mu_b|_L))\) is a Hamiltonian quasi-Poisson \(H^b\)-manifold.

### 3.2. The quasi-Poisson bracket formula for cross-sections.

We shall deduce from Theorem 2.5 and the Cross-Section Theorem a formula for quasi-Poisson brackets of functions of the form \(\Phi_{\alpha}|_L\) on \(L\). This will involve an invertible linear map \(\Theta_h : g \to g\) depending on a parameter \(h \in U\) defined by
\[ \Theta_h = \text{Pr}_h + \frac{2}{1 - \text{Ad}_h} \text{Pr}_h^\perp, \]
where \(\text{Pr}_h\) and \(\text{Pr}_h^\perp\) are projections of \(g\) onto \(h\) and \(h^\perp\).

It is easy to see that the transpose of \(\Theta_h\) is
\[ \Theta_h^\top = \text{Pr}_h + \frac{2}{1 - \text{Ad}_h^\top} \text{Pr}_h^\perp, \]
and we have the identity
\[ \Theta_h + \Theta_h^\top = 2. \]

**Theorem 3.2.** Let \(\{\cdot , \cdot\}_L\) be the quasi-Poisson bracket defined by the quasi-Poisson structure of \(L\), then
\[ \{\Phi_{\alpha} , \Psi_{\beta}\}_L = \sum_{I, J = \lambda, \vee} \varepsilon(\alpha^I, \beta^J) A^I_J(\Phi_{\alpha}, \Psi_{\beta}) + \sum_{q \in \alpha \neq \beta} \varepsilon_q(\alpha, \beta) B^q_{\Phi_{\alpha}, \Psi_{\beta}}, \]
where \(\Phi_{\alpha}, \Psi_{\beta}\) and \(B^q_{\Phi_{\alpha}, \Psi_{\beta}}\) are the same as in Theorem 2.5, but restricted to \(L\) here. \(A^I_J(\Phi_{\alpha}, \Psi_{\beta}) \in \mathcal{O}_L\) is defined as follows. If \(\alpha^I\) and \(\beta^J\) are at the same marked...
point \( p_i \) then we define

\[
A_{\Phi,\alpha,\Psi,\beta}^{ij} = \begin{cases} 
(\Theta \mu_i \Phi^I_\alpha \mid \Psi^J_\beta) & \text{if } \alpha^I \text{ is on the left of } \beta^J \text{ at } p_i, \\
(\Phi^I_\alpha \mid \Theta \mu_i \Psi^J_\beta) & \text{if } \alpha^I \text{ is on the right of } \beta^J \text{ at } p_i;
\end{cases}
\]

otherwise we set \( A_{\Phi,\alpha,\Psi,\beta}^{ij} = 0 \).

By “\( \alpha^I \) is on the left of \( \beta^J \) at \( p_i \)” is meant the following local picture, where we identify a neighborhood of \( p_i \) with the upper-half plane in an orientation-preserving manner.

\[
\begin{array}{c}
\alpha \\
\mid \mid \\
\beta \\
\mid \mid \\
p_i
\end{array}
\]

**Proof.** By definition of the quasi-Poisson tensor on \( L \),

\[
\{\Phi_\alpha, \Psi_\beta\}_L = P_L(d\Phi_\alpha, d\Psi_\beta) = P|_L(d\Phi_\alpha|_b, d\Psi_\beta|_b) - P^+_L(d\Phi_\alpha|_b^+, d\Psi_\beta|_b^+)
\]

\[
= \{\Phi_\alpha, \Psi_\beta\}_{MG(\Sigma)}|_L - P^+_L(d\Phi_\alpha|_b^+, d\Psi_\beta|_b^+)
\]

Let \( \chi^{(i)}_{\Phi}, \chi^{(i)}_{\Psi} : MG(\Sigma) \to g \) denote the variation maps (see (1.3) for the definition) of \( \Phi_\alpha \) and \( \Psi_\beta \) with respect to the action of the \( i \)-th component of \( G^b \) on \( MG(\Sigma) \). Using the expression of \( P^+_L \) given in Theorem 3.1 and Lemma 2.4, we get

\[
-P^+_L(d\Phi_\alpha|_b^+, d\Psi_\beta|_b^+) = \frac{1}{2} \sum_{i=1}^b \left( \frac{\Ad \mu_i + 1}{\Ad \mu_i - 1} \right) Pr_{h^+}(\chi^{(i)}_{\Phi} | Pr_{h^+}(\chi^{(i)}_{\Psi}))
\]

\[
= \frac{1}{2} \sum_{i=1}^b \sum_{I,J} \varepsilon_{I,J} \left( \frac{\Ad \mu_i + 1}{\Ad \mu_i - 1} \right) Pr_{h^+}(\Phi^I_\alpha | \Psi^J_\beta),
\]

where for any fixed \( i \) the summation “\( \sum_{I,J} \)” runs over symbols \( I, J = \land, \lor \) such that both \( \alpha^I \) and \( \beta^J \) lie on \( p_i \).

Inserting the the above equality and Theorem 2.5 into (3.2), we get

\[
\{\Phi_\alpha, \Psi_\beta\}_L = \sum_{q \in \alpha \neq \beta} \varepsilon_q(\alpha, \beta) B^q_{\Phi,\alpha,\Psi,\beta}
\]

\[
+ \sum_{i=1}^b \sum_{I,J} \left( \varepsilon_i(\alpha^I, \beta^J) + \frac{1}{2} \varepsilon_{I,J} \left( \frac{\Ad \mu_i + 1}{\Ad \mu_i - 1} \right) Pr_{h^+} \right) \Phi^I_\alpha | \Psi^J_\beta,
\]

where \( \varepsilon_i(\alpha^I, \beta^J) = 0, \pm \frac{1}{2} \) is the oriented intersection number of \( \alpha^I \) and \( \beta^J \) at \( p_i \). Namely, \( \varepsilon_i(\alpha^I, \beta^J) = \varepsilon(\alpha^I, \beta^J) \) if both \( \alpha^I \) and \( \beta^J \) lie on \( p_i \) and \( \varepsilon_i(\alpha^I, \beta^J) = 0 \) otherwise.
It is easy to see that if \( \alpha^I \) and \( \beta^J \) both lie on \( p_i \) and \( \alpha^I \) is on the left of \( \beta^J \), then 
\[
\varepsilon_i(\alpha^I, \beta^J) = -\frac{1}{2} \varepsilon_{IJ}.
\]
This implies
\[
\varepsilon_i(\alpha^I, \beta^J) + \frac{1}{2} \varepsilon_{IJ} \left( \frac{\operatorname{Ad}_{\mu_i} + 1}{\operatorname{Ad}_{\mu_i} - 1} \right) \operatorname{Pr}_{\mathbb{h}^\perp} = \varepsilon_i(\alpha^I, \beta^J) \left( \operatorname{Pr}_{\mathbb{h}} + \operatorname{Pr}_{\mathbb{h}^\perp} \right) - \frac{1}{2} \varepsilon_{IJ} \left( \frac{\operatorname{Ad}_{\mu_i} + 1}{\operatorname{Ad}_{\mu_i} - 1} \right) \operatorname{Pr}_{\mathbb{h}^\perp}
\]
\[
= \varepsilon_i(\alpha^I, \beta^J) \Theta_{\mu_i}.
\]
Similarly, if \( \alpha^I \) is on the right of \( \beta^J \), then we have 
\[
\varepsilon_i(\alpha^I, \beta^J) = \frac{1}{2} \varepsilon_{IJ}
\]
and
\[
\varepsilon_i(\alpha^I, \beta^J) + \frac{1}{2} \varepsilon_{IJ} \left( \frac{\operatorname{Ad}_{\mu_i} + 1}{\operatorname{Ad}_{\mu_i} - 1} \right) \operatorname{Pr}_{\mathbb{h}^\perp} = \varepsilon_i(\alpha^I, \beta^J) \Theta_{\mu_i}.
\]
Inserting these into (3.3), we get the required equality (3.1).

\[ \square \]

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