Spectral investigation of the noise influencing multi-qubit states

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Characterizing and understanding noise affecting quantum states has immense benefits in spectroscopy as well as in realizing quantum devices. Transverse relaxation times under a set of dynamical decoupling (DD) sequences with varying inter-pulse delays were earlier used for obtaining the noise spectral densities of single-qubit coherences. In this work, using a pair of homonuclear spins and NMR techniques, we experimentally characterize noise in certain decoherence-free subspaces. We also explore the noise of similar states in a heteronuclear spin-pair. Further using a 10-qubit system, we investigate noise profiles of various multi-qubit coherences and study the scaling of noise with respect to the coherence order. Finally, using the experimentally obtained noise spectrum of the 10-qubit NOON state, we predict the performance of Uhrig DD sequence and verify it experimentally.

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I. INTRODUCTION

The inevitable presence of local or global electromagnetic noise may cause loss of quantum coherences of spin systems or induce redistribution of spin populations. This phenomena which is often described in terms of decoherence or depolarization appears in NMR as a net relaxation of transverse or longitudinal magnetization. Combating decoherence is of utmost importance in spectroscopy and in realizing quantum devices such as quantum information processors (QIP). Passive techniques like decoherence-free subspaces (DFS) [1, 2] as well as active techniques like dynamical decoupling (DD) [3] and quantum error correcting codes [4, 5] have been developed to overcome decoherence. While the passive techniques rely on exploiting the symmetries in the interaction Hamiltonian, the active techniques focus on systematic modulation of the quantum states to suppress decoherence. In the following we discuss the noise in various types of quantum coherences including DFS, single-quantum, as well as multiple-quantum coherences.

An example of DFS is the singlet subspace in a two-qubit system [2]. In NMR, an excess population in the singlet state, over the uniformly distributed triplet states, is termed as a singlet order. It has been shown that, such an order, under favorable circumstances, has much longer life-times than the usual longitudinal relaxation time scales, and is therefore known as a long-lived singlet state (LLS) [6]. Similarly, the coherence between the singlet state and the zero-quantum triplet state also has longer life-times than the usual transverse relaxation time scales, and is therefore termed as a long-lived coherence (LLC) [7]. On the other hand, several other single- and multiple-quantum coherences lack the symmetry properties and are therefore prone to stronger decoherence [8].

In this work we attempt to extract the noise spectra acting on various quantum coherences of NMR spin-systems. Learning about noise-spectrum not only provides insights into the physical process of noise in quantum systems, but also assists in optimizing DFS conditions as well as in designing better controls for active suppression of noise. Quantum noise spectroscopy (QNS), a tool to characterize the environmental noise, was independently proposed by Yuge et al [9] and Álvarez and Suter [10].

The paper is organized as follows. In the following section we describe the theoretical formalism of QNS. In section III, we apply QNS and experimentally extract the noise spectra of some interesting quantum coherences. Finally we conclude in section IV.

II. THEORY AND METHODS

Here we review the theoretical aspects of characterizing the noise using a single two-level quantum system (qubit) as a probe. We consider the qubit to be coupled to a bath via a purely dephasing interaction. Assuming the system Hamiltonian $\mathcal{H}_S = \omega_0 \sigma_z / 2$ and the bath Hamiltonian $\mathcal{H}_B$, the joint-evolution is described by the Hamiltonian

$$\mathcal{H} = \mathcal{H}_S + \mathcal{H}_{SB} + \mathcal{H}_B.$$  

Here $\mathcal{H}_{SB} = j_{SB} \sigma_z B / 2$ describes the system-bath interaction with $B$ being the bath operator and $j_{SB}$ being the system-bath coupling strength. In the interaction picture of the bath Hamiltonian, the bath operator

$$B'(t) = e^{-i\mathcal{H}_B t} B e^{i\mathcal{H}_B t}$$

becomes time-dependent. After tracing-out the bath variables, the interaction Hamiltonian reduces to

$$\mathcal{H}'_{S(B)} = j_{SB} B'(t) \sigma_z / 2,$$

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the filter-function becomes time independent and reduces to a delta-comb
\[ |F(\omega, n\tau_f)|^2 = \sum_{k=-\infty}^{\infty} \frac{n\sqrt{2\pi}}{\tau_f} \delta(\omega - \omega_k)|F(\omega, \tau_f)|^2 \]  
(6)
where \(
\omega_k = 2\pi k/\tau_f
\) and \(k \in [-\infty, \infty]\) is the Fourier index of \(f(t)\) [17]. The exponential decay factor now becomes time independent, i.e., \(R = 1/T_2\). Hence for a long time point \(t = n\tau_f\)
\[
\frac{1}{T_2} = \frac{2\pi}{\tau_f} \sum_{k=0}^{\infty} S(\omega_k)|F(\omega_k, \tau_f)|^2
\]
(7)
where \(A_k = \frac{2\pi}{\tau_f} |F(\omega_k, \tau_f)|^2\) [10].

In the case of a free-evolution without any DD sequence, the modulation function \(f(t)\) becomes constant and therefore, the filter-function \(F(\omega, \tau_f)\) is a sinc-function centered at \(\omega = 0\), and the decay rate \(1/T_2\) depends only on \(S(0)\).

For the CPMG sequence [18, 19] with uniformly distributed \(\pi\) pulses at an interval \(2\tau\), \(f(t)\) switches between +1 and −1 with a period \(\tau_f = 4\tau\). The schematic diagrams of \(f(t)\) and the corresponding filter functions \(|F(\omega, \tau_f)|^2\) for a set of \(\tau\) values are shown in Fig. 1. In this case, \(A_k = \frac{4}{\pi^2 k^2}\) for odd \(k\) and \(A_k = 0\) otherwise. Hence
\[
\frac{1}{T_2} = \frac{4\pi^2}{\pi^2} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} S(\omega_{2l+1}).
\]
(8)
Thus the decay rate \(1/T_2\) for a given \(\tau\) is determined by the harmonics at \(\omega_{2l+1} = \pi(2l+1)/2\tau\), as illustrated in Fig. 1. Hence from the experimentally measured \(T_2\) values for \(\tau \in [\tau_{\text{min}}, \tau_{\text{max}}]\), one can extract the spectral density points \(S(\omega_{2l+1})\) in the range \(\omega \in [\pi/2\tau_{\text{max}}, \pi/2\tau_{\text{min}}]\) by inverting the above equation. In the following we discuss two ways of extracting the noise spectrum \(S(\omega)\) from Eq. 8.

An approximate way is to truncate the series in Eq. 8 to the zeroth order term so that,
\[
S(\pi/2\tau) \approx \frac{\pi^2}{4T_2^2}.
\]
(9)
This method is suitable for spectral densities with sharp cut-offs at low-frequencies [20]. Otherwise, ignoring higher order terms may introduce an error up to about 10%.

On the other hand, we can account for the zeroth as well as many higher order terms of spectral density by using a suitable model function for the spectral density. Random isotropic rotations of liquid molecules usually lead to exponential autocorrelation function and therefore the corresponding spectral density is Lorentzian [12].
Multiple relaxation sources may lead to multi-Lorentzian spectral density as observed in the experiments described in the next section. Our phenomenological model thus consists of a linear combination of Lorentzians

\[ S_L(\omega) = \sum_{j=1}^{L} \frac{\lambda_j}{(\omega - \omega_j)^2 + \lambda_j^2}. \]  

The parameters \( \omega_j \) (center-frequency) and \( \lambda_j \) (line-width) can be determined by numerically maximizing the overlap between the experimental \( T_2 \) values and those calculated using the model function \( S_L(\omega) \). Another benefit of obtaining the functional form of spectral density is that it allows one to evaluate the performance of various DD sequences at arbitrary inter-pulse spacing, as illustrated in section III C.

Although, noise filtering techniques for multi-qubit states are being developed recently [21, 22], in this work we use a single probe qubit to capture effective noise influencing multiqubit states.

III. EXPERIMENTS AND RESULTS

In this section, we describe the experimental noise spectroscopy of certain interesting multi-qubit coherences.

A. LLS and LLC

We used the two phenyl \(^1\)H nuclei of 2,3,6-trichlorophenol dissolved in dimethyl sulfoxide-D6. The experiments were carried out at 300 K in two different magnetic fields corresponding to Larmor frequencies \( \nu_0 = 400 \text{ MHz} \) as well as \( \nu_0 = 600 \text{ MHz} \). The chemical shift difference \( \Delta \nu \times 10^6/\nu_0 = 0.21 \text{ ppm} \) and the scalar coupling constant \( J = 8 \text{ Hz} \). Under weak-coupling approximation, the NMR Hamiltonian is

\[ H = \pi \Delta \nu I_z - \pi \Delta \nu S_z + \pi J 2I_zS_z, \]

where \( I_z \) and \( S_z \) are the spin operators.

The natural choice for expressing LLS and LLC is the singlet triplet basis, formed by the eigenvectors of the isotropic interaction Hamiltonian \( \mathbf{I} \cdot \mathbf{S} \), i.e.,

\[ |T_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \]
\[ |T_+\rangle = |00\rangle, \]
\[ |T_-\rangle = |11\rangle, \]
\[ |S_0\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \]

where \( \{|00\rangle, |01\rangle, |10\rangle, |11\rangle \} \) form the Zeeman eigenbasis.

In particular, we focus on the following coherences:

\[ \rho_{LLS} = |S_0\rangle\langle S_0| - |T_0\rangle\langle T_0| \]
\[ \rho_{LLC} = |S_0\rangle\langle T_0| + |T_0\rangle\langle S_0| \]
\[ \rho_{SL} = |T_+\rangle\langle T_+| - |T_-\rangle\langle T_-|. \]

In the above, \( \rho_{LLS}, \rho_{LLC}, \) and \( \rho_{SL} \) are realized by preparing the states \(-\mathbf{I} \cdot \mathbf{S}, I_x - S_x \), and \( I_x + S_x \) respectively, and applying a strong spin-lock along the \( x \) axis [6, 7]. Here we have considered \( \rho_{SL} \) for the sake of comparison with the other long-lived states. The pulse sequences corresponding to these states are shown in Fig. 2.

We use the multi-Lorentzian model function described in Sec. II to extract the noise spectrum. The best fit was achieved with a minimum of three Lorentzian functions (i.e., \( L = 3 \)) as described in Eq. 10. We scan over a range of spectral frequencies \( \omega = \pi/2\tau \) by varying the duration \( 2\tau \) between the \( \pi \) pulses, and measure the corresponding \( T_2 \) values. WALTZ-16 spin-lock of 2 kHz amplitude was applied along the \( x \) axis during the delays between the \( \pi \) pulses. The experimental \( T_2 \) values for all the three states and for \( \tau \) values ranging from 2 ms to 2 s are displayed in Fig. 3(a). The uncertainties in the noise spectrum (represented by the width of the bands) are estimated by several iterations of maximizations also considering the standard deviations in \( T_2 \) values.

As expected, \( \rho_{LLS} \) has the lowest noise in the whole-frequency range indicating long-lifetimes. On the other hand, \( \rho_{SL} \) has the highest noise indicating a relatively short-lived state. The long-lived coherence \( \rho_{LLC} \) has an intermediate noise-profile. Owing to the hardware limitations, the highest frequency sampled by the experi-

![FIG. 2. Pulse sequences used to measure the noise spectrum of (a) \( \rho_{LLS} \), (b) \( \rho_{LLC} \), and (c) \( \rho_{SL} \). Here \( \tau_1 = 1/(4J) \), \( \tau_2 = 1/(4J) + 1/(2\Delta\nu) \), \( \tau_3 = 1/(4\Delta\nu) \), and \( n \) is the number of times the loop is repeated. (d) Structure of 2,3,6-trichlorophenol. The CPMG DD sequence with spin-lock along x-axis is shown in the inset (DDSL).](image-url)
A heteronuclear singlet-state, though easy to prepare, is no longer an eigenstate of the interaction Hamiltonian. A pulse sequence to measure their noise spectrum is shown in Fig. 5a. It begins with a $\theta = \cos^{-1}(1/4)$ pulse on $^1$H spin followed by a pulsed-field-gradient to equalize the polarizations and prepare the state $I_z + S_z$. The following RF pulses and delays convert it to $-I_zS_x - I_yS_y \equiv |S_0\rangle\langle S_0| - |T\rangle\langle T|$. A CPMG DD sequence with a variable $\tau$ delay followed by a final $90^\circ_y$ on $^1$H is then used to measure the noise spectrum. The results are shown in Fig. 6. For comparison, we have also included the noise spectra of single-spin states $I_x$ and $S_x$. Here $^1$H spin has longer $T_2$ values and accordingly lower noise profile compared to $^{13}$C. Unlike in the homonuclear case, the heteronuclear singlet has the shortest $T_2$ values and a heteronuclear spin-pair, such as $^1$H-$^{13}$C in $^{13}$C Chloroform (dissolved in CDCl$_3$; see Fig. 5b), the singlet subspace is not a DFS, because a strong magnetic field breaks the symmetry between two spins and a spin-lock to restore the symmetry is not practical. Therefore,

![Diagram](https://via.placeholder.com/150)

**FIG. 3.** (a) Experimental decay constants (dots) of 2,3,6-trichlorophenol (averaged for both protons) in 400 MHz spectrometer for a range of $\tau$ values and for different states as indicated. The solid lines correspond to decay constants obtained from the best fit by the 3-Lorentzian model as described in Eq. 10. (b) The corresponding noise spectral density bands. The dashed line at 125 Hz corresponds to the maximum harmonics sampled with $\tau = 2$ ms.

**FIG. 4.** (a) The experimental decay constants (dots) at various $\tau$ delays for the singlet state $\rho_{LLS}$ at two different magnetic fields, i.e., 400 MHz and 600 MHz as indicated. The solid lines correspond to decay constants obtained from the best fit by the 3-Lorentzian model as described in Eq. 10. (b) The corresponding noise spectra.

**FIG. 5.** (a) Pulse sequence to measure noise spectrum in a heteronuclear spin system. (b) Molecular structure of Chloroform. Here singlet state is prepared on $^1$H and $^{13}$C spins with a coupling constant $J_{CH} = 209$ Hz between them. The CPMG DD sequence is shown in the inset.

B. Heteronuclear spin-pair

In a heteronuclear spin pair, such as $^1$H-$^{13}$C in $^{13}$C Chloroform (dissolved in CDCl$_3$; see Fig. 5b), the singlet subspace is not a DFS, because a strong magnetic field breaks the symmetry between two spins and a spin-lock to restore the symmetry is not practical. Therefore,
FIG. 6. (a) The experimental decay constants (dots) of $^{13}$C-Chloroform at various $\tau$ delays for single spin states $\rho_H = I_x$, $\rho_C = S_x$, and the singlet state $\rho_S$ at 500 MHz. The solid lines correspond to decay constants obtained from the best fit by the 3-Lorentzian model as described in Eq. 10. (b) The corresponding noise spectra.

therefore highest noise profile. Therefore a heteronuclear singlet is not an LLS at high fields [25].

C. Large quantum coherences

Consider an $N$-spin star-topology system wherein a central spin (denoted by $M$) is uniformly coupled to $N-1$ magnetically equivalent spins (denoted by $A$). Such a system allows a convenient way to prepare many large quantum coherences. The method involves applying a Hadamard gate (denoted by $H$) on the central spin followed by a CNOT gate as described in Fig. 7. In thermal equilibrium, the central spin will have an excess $|0\rangle_M$ population while the surrounding spins have a Boltzmann distribution over all the states $|N-1,0\rangle_A$ to $|0,N-1\rangle_A$, wherein the first and second numbers denote the numbers of spins in $|0\rangle$ and $|1\rangle$ states respectively. The effect of Hadamard and CNOT gates can now be described as

$$\begin{align*}
|0\rangle_M \sum_{k=0}^{N-1} |N-1-k,k\rangle_A \xrightarrow{H} & \rho_H \\
\frac{|0\rangle_M + |1\rangle_M}{\sqrt{2}} \sum_{k=0}^{N-1} |N-1-k,k\rangle_A \xrightarrow{\text{CNOT}} & \rho_C
\end{align*}$$

The last sum represents a collection of coherences with quantum numbers $N, N-2, \cdots, 0$ for even $N$ and $N, N-2, \cdots, 1$ for odd $N$. Such coherences are often referred to as $|\text{MSSM}\rangle$ (many-some, some-many) states [8]. A special MSSM state is the $N$-quantum $|\text{NOON}\rangle$ state

$$|\text{NOON}\rangle = (|000..0\rangle + |111..1\rangle)/\sqrt{2}. \quad (14)$$

The MSSM states can be individually studied by selective filtering of their signals using a pair of pulsed-field-gradients (PFG) (see Fig. 7). If $\gamma_A$ and $\gamma_M$ denote the respective gyromagnetic ratios of $A$ and $M$ spins, we can express the dephasing caused by the first PFG by,

$$\phi_1(k) \propto \gamma_M + (N-2k-1)\gamma_A \quad (15)$$

where the term in the right hand side is known as the lopsidedness of the MSSM state and

$$\frac{\gamma_M - (N-1)\gamma_A}{\gamma_A} \leq \lambda(k) \leq \frac{\gamma_M + (N-1)\gamma_A}{\gamma_A}. \quad (16)$$

Each MSSM state is converted back into an observable single-quantum $M$ spin coherence by the application of a second CNOT

$$\begin{align*}
\frac{1}{\sqrt{2}} \sum_{k=0}^{N-1} |0\rangle_M |N-1-k,k\rangle_A + e^{i\phi_1(k)} |1\rangle_M |k, N-1-k\rangle_A \xrightarrow{\text{CNOT}} & \frac{1}{\sqrt{2}} \sum_{k=0}^{N-1} |0\rangle_M |N-1-k,k\rangle_A + |1\rangle_M |k, N-1-k\rangle_A.
\end{align*}$$

Selection of the signal from a desired MSSM state with a particular $\lambda(k)$ value is achieved with the help of a second PFG which introduces a phase $\phi_2(k) = -\phi_1(k)$. The noise spectroscopy of the MSSM states can be studied by inserting the DD sequence just before the second CNOT (see Fig. 7).
Experiments were carried out in a Bruker 500 MHz spectrometer at 300 K. Trimethylphosphite (see Fig. 8) dissolved in DMSO was used as a 10-spin star-topology system including a central $^{31}$P spin ($M$ spin) and the nine surrounding $^1$H spins ($A$ spins). The scalar spin-spin coupling $J_{PH}$ was about 11 Hz. The signals from various MSSM states (obtained with the pulse-sequence shown in Fig. 7) along with a reference spectrum are shown in Fig. 8.

Results and discussions: The results of the noise spectroscopy of various MSSM states are shown in Fig. 9. As expected, the spectral density profiles appear to go higher with the magnitude of the lopsidedness, and accordingly the NOON state has the highest noise profile.

It is interesting to study the scaling of the low-frequency noise ($\approx S(0)$) versus the lopsidedness. The inset of Fig. 9 shows the experimental values of low-frequency noise (at lowest frequencies sampled) and a fit with a shifted parabola $c_2 l^2 + c_0$. The best fit was found at $c_2 = 0.06\pm0.01$ and $c_0 = 3.37\pm0.34$. A quadratic scaling of noise with lopsidedness is obvious from the inset in Fig. 9.

According to Redfield theory of relaxation, the transverse relaxation is a result of two processes - adiabatic and nonadiabatic [12, 27]. The energy conserving adiabatic part arises by longitudinal noise and leads to dephasing. The nonadiabatic part is due to the transverse noise and can induce transitions. Tang et al had observed that the completely correlated longitudinal noise results in relaxation rates that vary quadratically with the coherence order [28]. In our system, the coherence order is characterized by lopsidedness. Thus the quadratic dependence of spectral density with lopsidedness points out that the noise is predominantly correlated, i.e., noise affects all the spins identically. The background part in the scaling ($c_0$) is due remaining contributions including the nonadiabatic relaxation and the self-relaxation of the probe qubit ($^{31}$P).

It can be noted that similar studies of scaling of decoherence were earlier reported in a solid state NMR system by Krojanski et al [29].

An immediate application of extracting the noise spectrum is in evaluating the performances of various types DD sequences and selecting the optimum sequence for preserving quantum coherences. Uhrig dynamical decoupling (UDD) [30], for example involves, a nonuniform distribution of $\pi$ pulses placed at time instants

$$t_j = \tau_c \sin^2 \left( \frac{\pi j}{2N_\pi + 2} \right),$$

(17)

where $N_\pi$ is the total number of $\pi$ pulses in one period ($\tau_c$), also known as the order of the UDD sequence (denoted UDD-$N_\pi$). It can be easily seen that UDD-1 and UDD-2 are identical to a CPMG sequence.

![FIG. 8. The spectral lines corresponding to various MSSM states with varying lopsidedness $l$. Each spectral line is individually normalized. The reference spectrum with all the lines is shown at the front. The structure of Trimethylphosphite is also shown at the top-left corner.](image)

![FIG. 9. Trimethylphosphite noise-spectra for various MSSM states with different lopsidedness $l$. The dashed lines parallel to $l$-axis represent the maximum frequency (250 Hz) sampled in experiments. The inset shows the scaling of low-frequency spectral density values with $l$.](image)

![FIG. 10. Decay rates versus UDD-3 cycle duration $\tau_c$ calculated using Eq. 7 for the experimental noise spectrum of 10-qubit NOON state. The dots correspond to experimental results.](image)
Having the functional form of the noise spectral density we can now predict the relative decay rates of a quantum state under a given DD sequence. As an example, the band in Fig. 10 shows the the predicted decay rates of the NOON state (spectral density shown in Fig. 9) under UDD-3 sequence for a range of $\tau_c$ values. The corresponding experimental decay rates are shown by dots. The reasonable agreement between experimental and predicted values of decay rates demonstrates the benefit of extracting the spectral distribution of noise. Similar results were obtained in the case of other MSSM states. It should be noted that imperfections in the $\pi$-pulses such as finite duration, sensitivity to RF inhomogeneity over the sample volume, and calibration errors may introduce additional uncertainties in the noise-spectrum estimation and may affect DD performance as well.

IV. CONCLUSIONS

While we are entering the era of quantum devices, noise remains a hurdle in storing quantum superpositions. Exploiting decoherence-free-subspaces (DFS) is one of the convenient ways to preserve quantum coherences. DFS is already being used for storing hyper-polarization [31], studying slow molecular dynamics [32], characterizing molecular diffusion [33, 34], precise measurements of coupling constants [35], as well as in fault-tolerant quantum computing [36]. However the noises influencing such special quantum coherences have not been hitherto characterized experimentally. In this work we have experimentally characterized and compared noise spectral densities of various multi-qubit coherences.

We found that the noise spectrum of the long-lived singlet state (LLS) under spin-lock of a homonuclear spin-pair had the lowest profile indicating the strong protection offered by the symmetry in DFS resulting in long-livedness of the state. The long-lived-coherence (LLC) between singlet and the zero-quantum triplet had a higher noise profile, but still lower than the normal uncorrelated (single-spin) coherence. We have also measured the extent of noise in LLS under different field strengths and as expected, we found a higher noise with a stronger field, although the overall spectral features remained similar. On the other hand, the uncorrelated spins showed lower noise content compared to singlet states in a heteronuclear spin system, indicating an asymmetry in the system. Further, we have also explored the noise profiles of various higher-order coherences in a 10-spin system, and found a predominantly quadratic scaling of noise with respect to coherence order. Finally, using the noise spectrum of the NOON state we predicted its decay rates under a 3rd order Uhrig dynamical decoupling sequence and verified the same with experiments.

We believe that such studies are useful for understanding the physics of noise affecting quantum systems as well as to design ways to suppress decoherence. A better understanding of noise and their suppression will be crucial not only for the physical realization of quantum devices but also for general spectroscopic applications.

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REFERENCES

[1] Daniel A Lidar, Isaac L Chuang, and K Birgitta Whaley. Decoherence-free subspaces for quantum computation. Physical Review Letters, 81(12):2594, 1998.
[2] Daniel A. Lidar and K. Birgitta Whaley. Irreversible Quantum Dynamics, chapter Decoherence-Free Subspaces and Subsystems. 2003.
[3] Lorenza Viola, Emanuel Knill, and Seth Lloyd. Dynamical decoupling of open quantum systems. Physical Review Letters, 82(12):2417, 1999.
[4] David G Cory, MD Price, W Maas, E Knill, Raymond Laflamme, Wojciech H Zurek, Timothy F Havel, and SS Somaroo. Experimental quantum error correction. Physical Review Letters, 81(10):2152, 1998.
[5] John Preskill. Reliable quantum computers. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 454(1969):385–410, 1998.
[6] Marina Carvavaeta and Malcolm H Levitt. Long-lived nuclear spin states in high-field solution nmr. Journal of the American Chemical Society, 126(20):6228–6229, 2004.
[7] Riddhiman Sarkar, Puneet Ahuja, Paul R Vasos, and Geoffrey Bodenhausen. Long-lived coherences for homogeneous line narrowing in spectroscopy. Physical Review Letters, 104(5):053001, 2010.
[8] Jonathan A Jones, Steven D Karlen, Joseph Fitzsimons, Arzhang Ardavan, Simon C Benjamin, G Andrew D Briggs, and John JL Morton. Magnetic field sensing beyond the standard quantum limit using 10-spin noon states. Science, 324(5931):1166–1168, 2009.
[9] Tatsuro Yuge, Susumu Sasaki, and Yoshiro Hirayama. Measurement of the noise spectrum using a multiple-pulse sequence. Physical review letters, 107(17):170504, 2011.
[10] Gonzalo A Álvarez and Dieter Suter. Measuring the spectrum of colored noise by dynamical decoupling. Physical review letters, 107(23):230501, 2011.
[11] Leigh M. Norris, Gerardo A. Paz-Silva, and Lorenza Viola. Qubit noise spectroscopy for non-gaussian dephasing environments. Phys. Rev. Lett., 116:150503, Apr 2016.
[12] Anatole Abragam. *The principles of nuclear magnetism*. Number 32. Oxford university press, 1961.

[13] M J Biercuk, A C Doherty, and H Uys. Dynamical decoupling sequence construction as a filter-design problem. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 44(15):154002, 2011.

[14] A. G. Kofman and G. Kurizki. Universal dynamical control of quantum mechanical decay: Modulation of the coupling to the continuum. *Phys. Rev. Lett.*, 87:270405, Dec 2001.

[15] A. G. Kofman and G. Kurizki. Unified theory of dynamically suppressed qubit decoherence in thermal baths. *Phys. Rev. Lett.*, 93:130406, Sep 2004.

[16] Lukasz Cywiński, Roman M. Lutchyn, Cody P. Nave, and S. Das Sarma. How to enhance dephasing time in superconducting qubits. *Phys. Rev. B*, 77:174509, May 2008.

[17] Ashok Ajoy, Gonzalo A. Álvarez, and Dieter Suter. Optimal pulse spacing for dynamical decoupling in the presence of a purely dephasing spin bath. *Phys. Rev. A*, 83:032303, Mar 2011.

[18] H. Y. Carr and E. M. Purcell. Effects of diffusion on free precession in nuclear magnetic resonance experiments. *Phys. Rev.*, 94:630–638, May 1954.

[19] Saul Meiboom and David Gill. Modified spin-echo method for measuring nuclear relaxation times. *Review of scientific instruments*, 29(8):688–691, 1958.

[20] Swathi S Hegde and TS Mahesh. Engineered decoherence: Characterization and suppression. *Physical Review A*, 89(6):062317, 2014.

[21] Zhi-Kun Su and Shao-Ji Jiang. Filter-design perspective applied to dynamical decoupling of a multi-qubit system. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 45(2):025502, 2012.

[22] Gerardo A Paz-Silva, Seung-Woo Lee, Todd J Green, and Lorenza Viola. Dynamical decoupling sequences for multi-qubit dephasing suppression and long-time quantum memory. *New Journal of Physics*, 18(7):073020, 2016.

[23] Giuseppe Pileio and Malcolm H Levitt. Theory of long-lived nuclear spin states in solution nuclear magnetic resonance. ii. singlet spin locking. *The Journal of chemical physics*, 130(21):214501, 2009.

[24] Giuseppe Pileio and Malcolm H Levitt. Theory of long-lived nuclear spin states in solution nuclear magnetic resonance. ii. singlet spin locking. *The Journal of chemical physics*, 130(21):214501, 2009.

[25] Malcolm H Levitt. Singlet nuclear magnetic resonance. *Annual review of physical chemistry*, 63:89–105, 2012.

[26] John Cavanagh, Wayne J Fairbrother, Arthur G Palmer III, and Nicholas J Skelton. *Protein NMR spectroscopy: principles and practice*. Academic Press, 1995.

[27] A.G. Redfield. The theory of relaxation processes. *Advances in Magnetic and Optical Resonance*, 1:1–32, 1965.

[28] J Tang and A Pines. Multiple quantum nmr and relaxation of an oriented ch3 group. *The Journal of Chemical Physics*, 72(5):3290–3297, 1980.

[29] Hans Georg Krojanski and Dieter Suter. Scaling of decoherence in wide nmr quantum registers. *Physical review letters*, 93(9):090501, 2004.

[30] Götz S. Uhrig. Keeping a quantum bit alive by optimized π-pulse sequences. *Phys. Rev. Lett.*, 98:100504, Mar 2007.

[31] Puneet Ahuja, Riddhiman Sarkar, Sami Jannin, Paul R Vasos, and Geoffrey Bodenhausen. Proton hyperpolarisation preserved in long-lived states. *Chemical Communications*, 46(43):8192–8194, 2010.

[32] Riddhiman Sarkar, Paul R Vasos, and Geoffrey Bodenhausen. Singlet-state exchange nmr spectroscopy for the study of very slow dynamic processes. *Journal of the American Chemical Society*, 129(2):328–334, 2007.

[33] Riddhiman Sarkar, Puneet Ahuja, Paul R Vasos, and Geoffrey Bodenhausen. Measurement of slow diffusion coefficients of molecules with arbitrary scalar couplings via long-lived spin states. *ChemPhysChem*, 9(16):2414–2419, 2008.

[34] Simone Cavadini, Jens Dittmer, Sasa Antonijevic, and Geoffrey Bodenhausen. Slow diffusion by singlet state nmr spectroscopy. *Journal of the American Chemical Society*, 127(45):15744–15748, 2005.

[35] Aurélien Bornet, Sami Jannin, JA Konter, Patrick Haupte, Ben Van Den Brandt, and Geoffrey Bodenhausen. Ultra high-resolution nmr: sustained induction decays of long-lived coherences. *Journal of the American Chemical Society*, 133(39):15644–15649, 2011.

[36] Dave Bacon, Julia Kempe, Daniel A Lidar, and KB Whaley. Universal fault-tolerant quantum computation on decoherence-free subspaces. *Physical Review Letters*, 85(8):1758, 2000.