CONFORMAL INVARIENCE IN CLASSICAL FIELD THEORY

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ABSTRACT

A geometric generalization of the first-order Lagrangian formalism is used to analyse a conformal field theory for an arbitrary primary field. We require that the global conformal transformations are Noetherian symmetries and we prove that the action functional can be taken strictly invariant with respect to these transformations. In other words, there does not exists a ”Chern-Simons” type Lagrangian for a conformally invariant Lagrangian theory.

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1. Introduction

Conformal field theories continue to be a domain of active research. (See for instance [1] and references quoted there). For various reasons all investigations are done in the framework of quantum field theory. The investigation of these theories in the framework of classical field theory seems to be absent from the literature. This can be explained, as pointed out in [1] p. 190, because the local conformal transformations cannot be defined everywhere in the complex plane; the analiticity forces them to diverge somewhere. So these type of transformations cannot be interpreted as invariance transformations for a classical Lagrangian. However one can consider only the global conformal transformations (also called the homografic transformations) which can be defined as bijective applications of the completed complex plane $\mathbb{C} \cup \{\infty\}$. We think that such an analysis is not without interest.

In particular, we are adressing the following question. Let us consider a first-order Lagrangian theory for a primary field. (This hypotesis is made for simplicity, but in principle one can consider more than one primary field). We impose the condition that the global conformal transformations are Noetherian symmetries i.e. according to the usual definitions (see e.g. [2], p. 16) the action functional is invariant up to a trivial action ($\equiv$ an action giving trivial Euler-Lagrange equations of motions). We will be able to prove that one can redefine the action functional such that it will be strictly invariant with respect to these transformations. So, there are no Lagrangians of the “Chern-Simons” type for a (global) conformal field theory.

The main technical tool will be the geometric Lagrangian formalism developped in [3]-[7] (see also [8]) for classical field theory starting from original ideas of Poincaré, Cartan and Lichnerowicz. The same technique was applied by the same author for studying gauge theories, gravitation theory, etc. (see ref. [8] and references quoted there).

In Section 2 we present the general framework (more details can be found in [8]). In Section 3 we particularize the results of Section 2 for a conformal field theory. In Section 4 we derive the most general Lagrangian theory (in the sense of Section 2) compatible with conformal invariance.
2. General Theory

2.1 Let \( S \) be a differentiable manifold of dimension \( n + N \). The first order Lagrangian formalism is based on an auxiliary object, namely the bundle of 1-jets of \( n \)-dimensional submanifolds of \( S \), denoted by \( J^1_n(S) \). This differentiable manifold is, by definition:

\[
J^1_n(S) \equiv \bigcup_{p \in S} J^1_n(S)_p
\]

where \( J^1_n(S)_p \) is the manifold of \( n \)-dimensional linear subspaces of the tangent space \( T_p(S) \) at \( S \) in the point \( p \in S \). This manifold is naturally fibered over \( S \) and we denote by \( \pi \) the canonical projection. Let us construct charts on \( J^1_n(S) \) adapted to this fibered structure.

We first choose a local coordinate system \((x^\mu, \psi^A)\) on the open set \( U \subseteq S \); here \( \mu = 1, \ldots, n \) and \( A = 1, \ldots, N \). Then on the open set \( V \subseteq \pi^{-1}(U) \), we shall choose the local coordinate system \((x^\mu, \psi^A, \chi^A_{\mu})\), defined as follows: if \((x^\mu, \psi^A)\) are the coordinates of \( p \in U \), then the \( n \)-dimensional plane in \( T_p(S) \) corresponding to \((x^\mu, \psi^A, \chi^A_{\mu})\) is spanned by the tangent vectors:

\[
\frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} + \chi^A_{\mu} \frac{\partial}{\partial \psi^A}.
\]

We will systematically use the summation convention over the dummy indices.

By an evolution space we mean any (open) subbundle \( E \) of \( J^1_n(S) \).

2.2 Let us define for a given evolution space \( E \):

\[
\Lambda_{LS} \equiv \{ \sigma \in \wedge^{n+1}(J^1_n(S)) | i_{Z_1} i_{Z_2} \sigma = 0, \forall Z_i, \text{ s. t. } \pi_* Z_i = 0, i = 1, 2 \}. \tag{2.2}
\]

It is clear that any \( \sigma \in \Lambda_{LS} \) can be written in the local coordinates from above as follows:

\[
\sigma = \varepsilon_{\mu_1, \ldots, \mu_n} \sum_{k=0}^{n} \frac{1}{k!} C^k_{\mu_0, \ldots, \mu_k} \sigma_{A_0, \ldots, A_k} d\chi_{\mu_0} A_0 \wedge \cdots \wedge d\chi_{\mu_k} A_k + \cdots \wedge d\chi_{\mu_n}.
\]

\[
\varepsilon_{\mu_1, \ldots, \mu_n} \sum_{k=0}^{n} \frac{1}{(k + 1)!} C^k_{\mu_1, \ldots, \mu_k} \delta\psi^A_{A_0} \wedge \cdots \wedge \delta\psi^A_{A_k} \wedge \delta\psi^A_{A_{k+1}} \wedge \cdots \wedge \delta\psi^A_{A_n}. \tag{2.3}
\]

Here \( \varepsilon_{\mu_1, \ldots, \mu_n} \) is the signature of the permutation \((1, \ldots, n) \mapsto (\mu_1, \ldots, \mu_n)\), \( \delta\psi^A \) is by definition:

\[
\delta\psi^A \equiv d\psi^A - \chi^A_{\mu} dx^\mu. \tag{2.4}
\]
We can suppose that the functions $\sigma_{A_0,\ldots,A_k}^{\mu_0,\ldots,\mu_k}$ are completely antisymmetric in the indices $\mu_1, \ldots, \mu_k$ and also in the indices $A_1, \ldots, A_k$, and the functions $\tau_{A_0,\ldots,A_k}^{\mu_1,\ldots,\mu_k}$ are completely antisymmetric in the indices $\mu_1, \ldots, \mu_k$ and also in the indices $A_0, \ldots, A_k$.

It is remarkable that the following relation has an intrinsic global meaning:

$$\sum_{i,j=1}^{k} (-1)^{i+j} \sigma_{A_i,A_1,\ldots,\hat{A}_i,\ldots,A_k}^{\mu_j,\mu_1,\ldots,\mu_k} = 0. \quad (2.5)$$

for $k = 1, \ldots, n$.

This can be verified directly by computing the transformation law for the functions $\sigma$ with respect to a change of charts on $E$ induced by a change of charts on $\pi(E) \subseteq S$.

More abstractly [6], one can prove this as follows. One defines first the local operator $K$ on $\Lambda_{LS}$ by:

$$K\sigma \equiv \sum_{\delta x^p} \delta x^p \frac{\partial}{\partial \chi^A_p} \left( \delta \psi^A \wedge \sigma \right). \quad (2.6)$$

and proves that $K$ is in fact globally defined. Then one can show that (2.5) is the local expression of the global relation:

$$K\sigma = 0. \quad (2.7)$$

We say that $\sigma \in \Lambda_{LS}$ is a Lagrange-Souriau form on $E$ if it verifies (2.7) (or locally (2.5)) and is also closed:

$$d\sigma = 0. \quad (2.8)$$

In practical computations we will need the local form of (2.8). By some work one arrives at the following relations:

$$\frac{\partial \sigma_{A_0,\ldots,A_k}^{\mu_0,\ldots,\mu_k}}{\partial \chi^{A_{k+1}}_{\mu_{k+1}}} - \sigma_{A_0,\ldots,A_{k+1}}^{\mu_0,\ldots,\mu_{k+1}} - (A_0 \leftrightarrow A_{k+1}, \mu_0 \leftrightarrow \mu_{k+1}) = 0. \quad (2.9)$$

$$\frac{\delta \tau_{B_0,\ldots,B_k}^{\mu_1,\ldots,\mu_k}}{\delta x^p} - \sum_{i=0}^{k} (-1)^i \left( \frac{\partial \tau_{B_0,\ldots,B_i}^{\mu_1,\ldots,\mu_k}}{\partial \psi^A_i} + \frac{\partial \tau_{A_0,\ldots,A_{k+1}}^{\mu_1,\ldots,\mu_k}}{\partial \chi^{B \mu_{k+1}}} \right) - \tau_{B_0,\ldots,B_k}^{\mu_1,\ldots,\mu_k} = 0. \quad (2.10)$$

$$\frac{\delta \tau_{A_0,\ldots,A_k}^{\mu_1,\ldots,\mu_{k+1}}}{\delta x^{\mu_0}} + \sum_{i=0}^{k+1} (-1)^i \frac{\partial \tau_{A_0,\ldots,\hat{A}_i,\ldots,A_{k+1}}^{\mu_1,\ldots,\mu_k}}{\partial \psi^{A_i}} = 0. \quad (2.11)$$

for $k = 0, \ldots, n$. Here a hat means as usual omission.

We will call (2.5) and (2.9)-(2.11) the structure equations.
A Lagrangian system over $S$ is a couple $(E, \sigma)$ where $E \subseteq J^1_n(S)$ is some evolution space over $S$ and $\sigma$ is a Lagrange-Souriau form on $E$.

2.3 The purpose of the Lagrangian formalism is to describe *evolutions* i.e. immersions $\Psi : M \rightarrow S$, where $M$ is some $n$-dimensional manifold, usually interpreted as the space-time manifold of the system.

Let us note that frequently, one supposes that $S$ is fibered over $M$, but we do not need this additional restriction in developing the general formalism. Let us denote by $\hat{\Psi} : M \rightarrow J^1_n(S)$ the natural lift of $\Psi$. If $(E, \sigma)$ is a Lagrangian system over $S$, we say that $\Psi : M \rightarrow S$ verifies the *Euler-Lagrange equations* if:

$$\dot{\Psi}^* i_Z \sigma = 0.$$  \hfill (2.12)

for any vector field $Z$ on $E$.

2.4 By a *symmetry* of the Euler-Lagrange equations we understand a map $\phi \in Diff(S)$ such that if $\Psi : M \rightarrow S$ is a solution of these equations, then $\phi \circ \Psi$ is a solution of these equations also.

It is easy to see that if $\phi \in Diff(S)$ is such that $\dot{\phi}$ leaves $E$ invariant and:

$$\dot{\phi}^* \sigma = \sigma.$$  \hfill (2.13)

then it is a symmetry of the Euler-Lagrange equations (2.12). We call the symmetries of this type *Noetherian symmetries* for $(E, \sigma)$.

If a group $G$ act on $S$: $G \ni g \mapsto \phi_g \in Diff(S)$ then we say that $G$ is a *group of Noetherian symmetries* for $(E, \sigma)$ if for any $g \in G$, $\phi_g$ is a Noetherian symmetry. In particular we have:

$$(\dot{\phi}_g)^* \sigma = \sigma.$$  \hfill (2.14)

It is considered of physical interest to solve the following classification problem: given the manifold $S$ with an action of some group $G$ on $S$, find all Lagrangian systems $(E, \sigma)$ where $E \subseteq J^1_n(S)$ is on open subset and $G$ is a group of Noetherian symmetries for $(E, \sigma)$. This goal will be achieved by solving simultaneously (2.7), (2.8) and (2.14) in local coordinates and then investigating the possibility of globalizing the result.

2.5 Now we make the connection with the usual Lagrangian formalism. We can consider that the open set $V \subseteq \pi^{-1}(U)$ is simply connected by choosing it small enough.
The first task is to exhibit somehow a Lagrangian. To this purpose, we use the structure equations (2.9)-(2.11). Using induction (from $k = n$ to $k = 0$) and applying repeatedly the Poincaré lemma, one shows rather easily that $\sigma::$ and $\tau::$ can be written in the following form:

$$\sigma_{A_0,\ldots,A_k}^{\mu_0,\ldots,\mu_k} = \frac{\partial L_{A_1,\ldots,A_k}^{\mu_1,\ldots,\mu_k}}{\partial x^{A_0}} - L_{A_0,\ldots,A_k}^{\mu_0,\ldots,\mu_k}. \quad (2.15)$$

$$\tau_{A_0,\ldots,A_k}^{\mu_1,\ldots,\mu_k} = \sum_{i=0}^{k} (-1)^i \frac{\partial L_{A_1,\ldots,A_k}^{\mu_1,\ldots,\mu_k}}{\partial \psi^{A_i}} - \frac{\delta L_{A_0,\ldots,A_k}^{\mu_0,\ldots,\mu_k}}{\delta x^{\mu_0}}. \quad (2.16)$$

where $L_{A_1,\ldots,A_k}^{\mu_1,\ldots,\mu_k}$ are (real) functions defined on $V$ and completely antisymmetric in the indices $\mu_1,\ldots,\mu_n$ and also in the indices $A_1,\ldots,A_n$. A more abstract way to prove this is given in [6]. From (2.8) one has in $V$:

$$\sigma = d\theta. \quad (2.17)$$

for some $n$-form $\theta$. Then one can show that by eventually redefining $\theta$: $\theta \to \theta + df$ one can exhibit it in the form:

$$\theta = \varepsilon_{\mu_1,\ldots,\mu_n} \sum_{k=0}^{n} \frac{1}{k!} C^k_n L_{A_1,\ldots,A_k}^{\mu_1,\ldots,\mu_k} \delta \psi^{A_1} \wedge \ldots \wedge \delta \psi^{A_k} \wedge dx^{\mu_{k+1}} \wedge \ldots \wedge dx^{\mu_n}. \quad (2.18)$$

Then (2.15) and (2.16) follow from (2.3), (2.17) and (2.18).

Finally, using the structure equation (2.5) one gets a recurrence relation for the functions $L_{A_1,\ldots,A_k}^{\mu_1,\ldots,\mu_k}$ and easily shows that:

$$L_{A_1,\ldots,A_k}^{\mu_1,\ldots,\mu_k} = \frac{1}{k!} \sum_{\sigma \in P_k} (-1)^{|\sigma|} \frac{\partial L}{\partial x^{A_1_{\mu_1(1)}} \ldots \partial x^{A_k_{\mu_k(1)}}}. \quad (2.19)$$

($P_k$ is the permutation group of the numbers $1,\ldots,k$) and $|\sigma|$ is the signature of $\sigma$). $L$ is called a local Lagrangian. If $\sigma$ is of the form (2.3) with the coefficients $\sigma::$ and $\tau::$ given by (2.16), (2.17) and (2.20) then we denote it by $\sigma_L$. The expressions (2.18)-(2.19) are exactly the ones appearing in [3]-[5]. So, we can conclude that the framework above generalizes the scheme from [3]-[5] in the following sense: the central object is now the Lagrange-Souriau form $\sigma$, not the Poincaré-Cartan form $\theta$. The connection between them is only local (see (2.17)) and $\sigma$ can be globally defined.

Now one can easily show that the local form of the Euler-Lagrange equations (2.12) coincides with the usual one. Namely, one chooses convenient local coordinates ($x^\mu, \psi^A$)
on an open set \( U \subseteq S \), such that the evolution \( \Psi : M \to S \) will be locally given by \( x^\mu \mapsto (x^\mu, \Psi^A(x)) \). Then \( \dot{\Psi} : M \to J^1(A)(S) \) is given by \( \dot{\Psi} \) by \( x^\mu \mapsto (x^\mu, \Psi^A(x), \frac{\partial \Psi^A}{\partial x^\mu}(x)) \) and the equations (2.12) have the local expression:

\[
\frac{\partial L}{\partial \psi^A} \circ \dot{\Psi} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial x^A} \circ \dot{\Psi} \right) = 0.
\] (2.20)

i.e. the usual Euler-Lagrange equations if one takes \( \sigma = \sigma_L \).

We also note the following result: The Euler-Lagrange equations (2.12) are trivial \( \text{iff} \) \( \sigma = 0 \). One can prove this fact elementary. Indeed if \( \sigma = 0 \), then it is clear that the Euler-Lagrange equations (2.12) are trivial. Conversely, suppose that (2.12) are identities. Substituting (2.3) into (2.12) one obtains that \( \sigma_{\mu \mu, 1}^A \equiv 0 \) and \( \tau_{A, 0} \equiv 0 \). Now one uses the structure equations (2.5) + (2.7)-(2.11) to prove by induction that \( \sigma_{\ldots, 0} \equiv 0 \) and \( \tau_{\ldots, 0} \equiv 0 \), i.e. \( \sigma \equiv 0 \). For another proof see [7].

Let us suppose now for the moment that \( \sigma \) is exact i.e. verifies (2.17) on the whole \( E \). Then one can define the action functional by the formula:

\[
\mathcal{A}(\Psi) \equiv \int_M \dot{\Psi}^* \theta.
\] (2.21)

One can show that \( \Psi \) is a solution of the Euler-Lagrange equations \( \text{iff} \) it is an extremal of the action functional.

We say that the functional \( \mathcal{A} \) is a trivial action if it gives trivial Euler-Lagrange equations. Then the usual definition of Noetherian symmetries is encoded in the relation:

\[
\mathcal{A}(\phi \circ \Psi) = \mathcal{A}(\Psi) + \text{a trivial action}
\] (2.22)

(see [2]).

Now one can establish that in this case (2.22) is equivalent to the definition (2.13) given in the general case. To prove this assertion one proceeds as follows. First, one notes that in this case one can write \( \theta \) as \( \theta_L \) for some Lagrangian \( L \), and (2.21) becomes the usual expression:

\[
\mathcal{A}(\Psi) = \int_M L \circ \dot{\Psi}.
\] (2.23)

Next, one plugs this expression into (2.22) and obtains (taking into account that \( \Psi \) is arbitrary):

\[
J_\phi L \circ \dot{\phi} = L + L_0.
\] (2.24)
Here $J_\phi$ follows from the (eventual) volume change induced by $\phi$ and $L_0$ is a trivial Lagrangian i.e. a Lagrangian giving trivial Euler-Lagrange equations of motions.

Now it is a matter of computation to show that (2.24) is equivalent to the following relation:

$$\dot{\phi}^* \theta_L = \theta_L + \theta_{L_0}. \quad (2.25)$$

Applying the exterior derivative one gets:

$$\dot{\phi}^* \sigma_L = \sigma_L + \sigma_{L_0}. \quad (2.26)$$

But, because the Euler-Lagrange equations for $L_0$ are trivial, we have $\sigma_{L_0} = 0$ according to a remark above, so we get (2.13) ($\sigma = \sigma_L$ in our case). Conversely, if one has (2.13) then applying Poincaré lemma one obtains (2.25) with $\theta_{L_0}$ an exact $n$-form. But in this case $\sigma_{L_0} = 0$ and the same remark above implies that $L_0$ is a trivial Lagrangian. The proof is finished.

The equivalence between (2.13) and (2.22) shows clearly why (2.13) is the most suitable definition of Noetherian symmetries: it makes sense also in the case when $\sigma$ is not exact (and (2.22) makes no sense).

We close this subsection with a comment about the necessity of working with the rather complicated expression (2.3). It is more or less obvious from what has been said above that the proof of the equivalence between (2.13) and (2.22) relies heavily on the specific structure of $\sigma$ given by (2.3). Moreover one can show [5] that if one considers only a truncated $\sigma$ (for instance one takes in (2.18) the sum from 0 to $p < n$) then one can save the implication $\sigma = \sigma_L \Rightarrow (2.20)$ if $p \geq 1$ but, in general, only the implication $\sigma = \sigma_L \Rightarrow (2.22)$ stays true. In other words, one can consider in (2.3) only the terms $k = 0, 1, 2$ and still has a geometrical way of expressing the Euler-Lagrange equations (2.12), but the set of transformations $\phi$ satisfying (2.13) is, in general, strictly smaller that the whole group of Noetherian symmetries. Only when one considers the whole sum these two sets are identical.

2.6 An important particular case of the general framework presented in the Section 2.2 is the following one. We say that a Lagrangian system $(E, \sigma)$ is of the Chern-Simons type if:

$$\sigma \in \Lambda_{CS} \equiv \{ \sigma \in \Lambda^{n+1}(E) | i_Z \sigma = 0, \forall Z, \ s. \ t. \ \pi_* Z = 0 \}. \quad (2.27)$$
(compare with (2.2)). The terminology is justified by the following fact \([10]\): for a gauge theory such a \(\sigma\) follows from a Chern-Simons Lagrangian.

It is clear that in the local coordinates used up till now, such a \(\sigma\) is of the form (2.3) where one makes: \(\sigma_{\cdots} \to 0\).

One can prove that in this case the (local) Lagrangian can be chosen a polynomial in the variables \(\chi^A_{\mu}\) of maximal degree \(n\).

### 3. Conformal Field Theories

3.1 We consider only the simplest case of a primary field. In the general framework of Section 2, we take \(S = R^2 \times R^2\) with coordinates \((x^1, x^2, \psi^1, \psi^2)\). As it is well known, it is more convenient to use as independent variables the following complex combinations:

\[
  z^\mu \equiv x^1 + \mu i x^2. \tag{3.1}
\]

\[
  \psi^\mu \equiv \psi^1 + \mu i \psi^2. \tag{3.2}
\]

for \(\mu = +, -\). As we have said in the Introduction, we will consider that in fact \(z^\mu\) takes values in the completed complex plane \(\mathbb{C} \cup \{\infty\}\). We take for the evolution space the manifold \(E \equiv J^1_2(S)\) with coordinates \((z^\mu, \psi^\nu, \chi^\nu_{\mu})\) We particularize now the expression (2.3) of \(\sigma\):

\[
  \sigma = 2\varepsilon_{\mu_1, \mu_2} \sigma_{\nu_0, \nu_1}^{\mu_0, \mu_1} d\chi_{\mu_0}^{\nu_0} \wedge \delta \psi^{\nu_1} \wedge \wedge d\psi^{\mu_2} + \varepsilon_{\mu_1, \mu_2} \sigma_{\nu_0, \nu_1}^{\mu_0, \mu_1, \mu_2} d\chi_{\mu_0}^{\nu_0} \wedge \delta \psi^{\nu_1} \wedge \delta \psi^{\mu_2} + \varepsilon_{\mu_1, \mu_2} \tau_{\nu_0}^{\nu_1} \delta \psi^{\nu_0} \wedge d\mu_1 \wedge d\mu_2 + \varepsilon_{\mu_1, \mu_2} \tau_{\nu_0, \nu_1}^{\mu_1} \delta \psi^{\nu_0} \wedge \delta \psi^{\nu_1} \wedge d\mu_2. \tag{3.3}
\]

where we have taken into account the fact that (2.5) for \(k = 1\) gives \(\sigma_{\nu_0}^{\mu_0} = 0\). Here:

\[
  \delta \psi^{\nu} \equiv d\psi^{\nu} - \chi^\nu_{\mu} dz^\mu. \tag{3.4}
\]

We list now the structure equations. From (2.5) we have:

\[
  \sigma_{\nu_0, \nu_1}^{\mu_0, \mu_1} - \sigma_{\nu_0, \nu_1}^{\mu_1, \mu_0} - \sigma_{\nu_0, \nu_1}^{\mu_0, \mu_1} + \sigma_{\nu_1, \nu_0}^{\mu_1, \mu_0} = 0. \tag{3.5}
\]

From (2.9) we have:

\[
  \sigma_{\nu_0, \nu_1}^{\mu_0, \mu_1} = \sigma_{\nu_1, \nu_0}^{\mu_1, \mu_0}. \tag{3.6}
\]
\[
\frac{\partial \sigma_{\nu_0,\mu_1}}{\partial \chi_{\mu_2}} \mu_0, \mu_1, \nu_0, \nu_1, \nu_2, \mu_2, \nu_2 - (\mu_0 \nu_0 \leftrightarrow \mu_2 \nu_2) = 0. \tag{3.7}
\]

\[
\frac{\partial \sigma_{\nu_0,\mu_1,\mu_2}}{\partial \chi_{\mu_3}} \mu_0, \mu_1, \mu_2, \nu_0, \nu_1, \nu_2, \mu_2, \nu_2 - (\mu_0 \nu_0 \leftrightarrow \mu_3 \nu_3) = 0. \tag{3.8}
\]

From (2.10) we have:

\[
\frac{\delta \sigma_{\xi,\rho}}{\delta z_{\rho}} + \frac{\partial \tau_{\nu_0}}{\partial \chi_{\xi}} - \tau_{\nu_0,\xi} = 0. \tag{3.9}
\]

\[
\frac{\delta \sigma_{\xi,\mu_1,\mu_2}}{\delta \psi_{\rho_0}} - \frac{\partial \sigma_{\xi,\mu_1,\mu_2}}{\partial \psi_{\nu_1}} + \frac{\partial \sigma_{\xi,\mu_1,\mu_2}}{\partial \psi_{\nu_1}} = 0. \tag{3.10}
\]

\[
\frac{\partial \sigma_{\xi,\mu_1,\mu_2}}{\partial \psi_{\rho_0}} - \frac{\partial \sigma_{\xi,\mu_1,\mu_2}}{\partial \psi_{\nu_1}} + \frac{\partial \sigma_{\xi,\mu_1,\mu_2}}{\partial \psi_{\nu_1}} = 0. \tag{3.11}
\]

Finally, from (2.11) we have:

\[
\frac{\delta \tau_{\nu_0}}{\delta z_{\rho}} + \frac{\partial \tau_{\nu_1}}{\partial \psi_{\rho_0}} - \frac{\partial \tau_{\nu_0}}{\partial \psi_{\nu_2}} = 0. \tag{3.12}
\]

\[
\frac{\partial \tau_{\nu_1}}{\partial \psi_{\rho_0}} - \frac{\partial \tau_{\nu_0,\nu_1}}{\partial \psi_{\rho_0}} + \frac{\partial \tau_{\nu_0,\nu_1}}{\partial \psi_{\nu_2}} = 0. \tag{3.13}
\]

Here:

\[
\frac{\delta}{\delta z_{\mu}} \equiv \frac{\partial}{\partial z_{\mu}} + \chi_{\nu_{\mu}} \frac{\partial}{\partial \psi_{\nu}}. \tag{3.14}
\]

3.2 We now impose the invariance with respect to global conformal transformations. Let \( f^+ \) and \( f^- \) be homograic functions of the variables \( z^+ \) and \( z^- \) respectively. For consistency we must require that:

\[
(f^+(z^+))^* = f^-(z^-). \tag{3.15}
\]

We denote by \( f \) the map:

\[
f(z^+, z^-) \equiv (f^+(z^+), f^-(z^-)). \tag{3.16}
\]

Then we consider the following transformation on \( S \):

\[
\phi_f (z^\mu, \psi^\nu) = \left( f^\mu (z^\mu), \prod_p \left[ \hat{f}^p (z^\rho) \right] m^\nu_p \psi^\nu \right). \tag{3.17}
\]

Here \( m_+ \) and \( m_- \) are two real numbers characteristic of the primary field \( \psi \) (the conformal weights). By \( \hat{f} \) we denote the first derivative of \( f \).
We say that the Lagrangian system \((E, \sigma)\) is \textit{globally conformal invariant} if for any \(f^\mu\) we have:

\[
(\dot{\phi}_f)^* \sigma = \sigma. \tag{3.18}
\]

Let us make the connection with the usual definition of conformal invariance. We suppose for the moment that \(\sigma = d\theta\) and we define the action functional by (2.22).

If \(f\) is as above then the action of the corresponding conformal transformation on the set of immersions \(\Psi : M \to S\) (for a 2-dimensional manifold \(M\)) is:

\[
\Phi_f(z^\mu, \Psi^\nu(\cdot)) = \left(f^\mu(z^\mu), \left[\frac{\partial f^+}{\partial z^+}(\cdot)\right] m^\nu, \left[\frac{\partial f^-}{\partial z^-}(\cdot)\right] m^-\nu \Psi^\nu \circ f^{-1}(\cdot) \right). \tag{3.19}
\]

(see e.g. [1]).

Then (3.18) is equivalent to the usual definition of conformal invariance:

\[
\mathcal{A}(\Phi_f \Psi) = \mathcal{A}(\Psi) + a \text{ trivial action}. \tag{3.20}
\]

3.3 According to the strategy outlined in Section 2 we work with the more general definition (3.18). It is convenient to use this relation in an infinitesimal form. Namely, one takes:

\[
f^\mu(z^\mu) = z^\mu + \theta^\mu(z^\mu). \tag{3.21}
\]

with \(\theta^\mu\) infinitesimally small, computes the variation of \(\sigma\) with respect to \(\dot{\phi}_f\), and takes into account the fact that \(\theta^+\) and \(\theta^-\) are of the following form (see [1] p. 190):

\[
\theta^\mu(z^\mu) = a_{-1}^\mu z^\mu + a_0^\mu z^\mu + a_1^\mu (z^\mu)^2 \quad (\mu = \pm) \tag{3.22}
\]

with \(a_{-1}, a_0\) and \(a_1\) arbitrary complex numbers.

We will give below the result of this elementary but tedious computation. It is convenient to define the following differential operators on \(E\):

\[
D_\rho \equiv \sum_\nu m_{\nu\rho} \psi^\nu \frac{\partial}{\partial \psi^\nu} + \sum_{\mu, \nu} (m_{\nu\rho} - \delta_{\mu\rho}) \chi^\nu \mu \frac{\partial}{\partial \chi^\nu}_\mu. \tag{3.23}
\]

and:

\[
D'_\rho \equiv \sum_\nu m_{\nu\rho} \psi^\nu \frac{\partial}{\partial \chi^\nu}_\rho. \tag{3.24}
\]
Then (3.18) is infinitesimally equivalent to the following set of relations:

$$\frac{\partial \sigma^{\mu_0, \mu_1}_{\nu_0, \nu_1}}{\partial \rho} = 0.$$  \hspace{1cm} (3.25)

$$D_{\rho} \sigma^{\mu_0, \mu_1}_{\nu_0, \nu_1} = 0.$$  \hspace{1cm} (3.26)

$$D'_{\rho} \sigma^{\mu_0, \mu_1}_{\nu_0, \nu_1} = 0.$$  \hspace{1cm} (3.27)

$$\frac{\partial \sigma^{\mu_0, \mu_1, \mu_2}_{\nu_0, \nu_1, \nu_2}}{\partial \rho} = 0.$$  \hspace{1cm} (3.28)

$$D_{\rho} \sigma^{\mu_0, \mu_1, \mu_2}_{\nu_0, \nu_1, \nu_2} + (m_{\nu_0 \rho} + m_{\nu_1 \rho} + m_{\nu_2 \rho} - \delta_{\rho}^{\mu_0}) \sigma^{\mu_0, \mu_1, \mu_2}_{\nu_0, \nu_1, \nu_2} = 0.$$  \hspace{1cm} (3.29)

$$D'_{\rho} \sigma^{\mu_0, \mu_1, \mu_2}_{\nu_0, \nu_1, \nu_2} = 0.$$  \hspace{1cm} (3.30)

$$\frac{\partial \tau_{\nu_0}}{\partial \rho} = 0.$$  \hspace{1cm} (3.31)

$$D_{\rho} \tau_{\nu_0} + (m_{\nu_0 \rho} + 1) \tau_{\nu_0} = 0.$$  \hspace{1cm} (3.32)

$$D'_{\rho} \tau_{\nu_0} + \cdots = 0.$$  \hspace{1cm} (3.33)

$$\frac{\partial \tau^{\mu_1}_{\nu_0, \nu_1}}{\partial \rho} = 0.$$  \hspace{1cm} (3.34)

$$D_{\rho} \tau^{\mu_1}_{\nu_0, \nu_1} + (m_{\nu_0 \rho} + m_{\nu_1 \rho} + \delta_{\rho}^{\mu_1}) \tau^{\mu_1}_{\nu_0, \nu_1} = 0.$$  \hspace{1cm} (3.35)

$$D'_{\rho} \tau^{\mu_1}_{\nu_0, \nu_1} + \cdots = 0.$$  \hspace{1cm} (3.36)

In (3.33) and (3.36) we mean by $\cdots$ some algebraic expressions in $\sigma_{\cdots}$ which will not be needed.

In the following section we will analyse separately the cases $(m_+)^2 + (m_-)^2 \neq 0$ and $m_+ = m_- = 0$ and we will prove that $\sigma$ can be exhibited in the form $\sigma_L = d\theta_L$ such that the corresponding action functional is strictly invariant with respect to $\dot{\phi}_f$ defined by (3.19) above. The idea is to first analyse the functions $\sigma_{\cdots}$ and then to eliminate them completely from the game and ending up with a Chern-Simons Lagrangian theory. Of course, the result anticipated above means that this last contribution will be in fact trivial.

4. The main theorem
a) \((m_+)^2 + (m_-)^2 \neq 0\)

4.1 In this case we use instead of the variables \(\chi^\nu_{\mu}\) the new coordinates:

\[
X_\nu \equiv \sum_\nu \mu m_{-\nu\nu} \psi^-\nu \chi^\mu_{\nu}.
\] (4.1)

and:

\[
Y_\nu \equiv \sum_\nu m_{\mu\nu} \psi^-\mu \chi^\nu_{\mu}.
\] (4.2)

The Jacobian \(D(X,Y)/D(\chi)\) is non-singular outside the hypersurface \(\psi^+\psi^- = 0\).

Then it easily follows that (3.27) and (3.30) means that \(\sigma^\nu_{\mu\nu}\) do not depend on the variables \(Y_\nu\). So, taking into account the independence of \(z^\rho\) (see (3.25) and (3.28)) it follows that \(\sigma^\nu_{\mu\nu}\) are functions only of \(\psi\) and \(X_\nu\), i.e.

\[
\sigma^\mu_{\nu,\nu_1} = s^\mu_{\nu,\nu_1} \circ \Phi.
\] (4.3)

and:

\[
\sigma^\mu_{\nu,\nu_1,\nu_2} = s^\mu_{\nu,\nu_1,\nu_2} \circ \Phi.
\] (4.4)

where:

\[
\Phi(z,\psi,\chi) \equiv (\psi, m_-\psi^-\chi^+ + m_+\psi^+\chi^- + m_+\psi^-\chi^+ - m_-\psi^+\chi^-).
\] (4.5)

4.2 Next one rephrases (3.26) and (3.29) in terms of \(s^\nu_{\mu\nu}\). If we define:

\[
\hat{D}_\rho \equiv \sum_\nu m_{\nu\rho} \psi^\nu \frac{\partial}{\partial \psi^\nu} + \sum_\nu (m_+ + m_- - \delta_{\nu\rho}) X_\nu \frac{\partial}{\partial X^\nu}.
\] (4.6)

then we get respectively:

\[
\hat{D}_\rho s^\mu_{\nu_1,\nu_1} + (m_{\nu_1\rho} + m_{\nu_1\rho} - \delta_{\nu_1\rho} + \delta_{\nu_1\rho}) s^\mu_{\nu_1,\nu_1} = 0.
\] (4.7)

and:

\[
\hat{D}_\rho s^\mu_{\nu_1,\nu_1,\nu_2} + (m_{\nu_1\rho} + m_{\nu_1\rho} + m_{\nu_2\rho} - \delta_{\nu_2\rho}) s^\mu_{\nu_1,\nu_1,\nu_2} = 0.
\] (4.8)

4.3 We pursue with the analysis of the functions \(s^\nu_{\mu\nu}\) translating everything in terms of a Lagrangian \(L_0\) depending only on the variables \(\psi\) and \(X\). To this purpose we start with the structure equations (3.8). Inserting (4.4) in this relation and applying Frobenius
theorem it is clear that one can find a system of functions $L_{\nu_1 \nu_2}^{\mu_1 \mu_2}$ depending on $\psi$ and $X$, with antisymmetry with respect to the transposition of the indices $\mu$ and also with respect to the transposition of the indices $\nu$ and such that:

$$\sigma_{\nu_0, \nu_1, \nu_2}^{\mu_0, \mu_1, \mu_2} = \frac{\partial}{\partial X^{\nu_0 \mu_0}} (L_{\nu_1 \nu_2}^{\mu_1 \mu_2} \circ \Phi). \quad (4.9)$$

Inserting (4.4) and (4.9) into (4.8) we get an equation of the same type for $L_{\nu_1 \nu_2}^{\mu_1 \mu_2}$:

$$\hat{D}_\rho L_{\nu_1 \nu_2}^{\mu_1 \mu_2} + (m_{\nu_1 \rho} + m_{\nu_2 \rho}) L_{\nu_1 \nu_2}^{\mu_1 \mu_2} = \cdots. \quad (4.10)$$

where by $\cdots$ we mean some $\psi$-dependent functions.

We iterate the procedure. Inserting (4.9) into (3.7) we get as before that $\sigma_{\nu_0, \nu_1}^{\mu_0, \mu_1}$ are of the form:

$$\sigma_{\nu_0, \nu_1}^{\mu_0, \mu_1} = \frac{\partial}{\partial X^{\nu_0 \mu_0}} (L_{\nu_1}^{\mu_1} \circ \Phi) - L_{\nu_0 \nu_1}^{\mu_0 \mu_1}. \quad (4.11)$$

with $L_{\nu_1}^{\mu_1}$ depending only on $\psi$ and $X$. Also we use (4.3) and (4.11) into (4.7) and end up with:

$$\hat{D}_\rho L_{\nu_1}^{\mu_1} + (m_{\nu_1 \rho} + \delta_{\mu_1 - \rho}) L_{\nu_1}^{\mu_1} = \cdots. \quad (4.12)$$

where by $\cdots$ we mean some expression depending only on $\psi$ and $X$ which is polynomial in $X$ of maximal degree 1.

Finally, we insert (4.11) into (3.6) and find out that there exists a $(\psi, X)$-dependent function $L_0$ such that:

$$L_{\nu}^{\mu} = \frac{\partial}{\partial X^{\nu \mu}} (L_0 \circ \Phi). \quad (4.13)$$

The derivation of (4.9), (4.11) and (4.13) is a process of the same type as the process leading to the equations (2.15).

If we insert (4.13) into (4.12) we can prove that in fact the right hand side of (4.12) does not depend on $X$ and we have:

$$\hat{D}_\rho L_0 + L_0 = F_{\rho}^{\mu} X_{\mu} + F_{\rho}. \quad (4.14)$$

for some $\psi$-dependent functions $F_{\rho}^{\mu}$ and $F_{\rho}$.

Now the structure equation (3.5) expresses $L_{\nu_1 \nu_2}^{\mu_1 \mu_2}$ also in terms of $L_0$:

$$L_{\nu_1 \nu_2}^{\mu_1 \mu_2} \circ \Phi = \frac{1}{2} \frac{\partial^2}{\partial X^{\nu_1 \mu_1} \partial X^{\nu_2 \mu_2}} (L_0 \circ \Phi) - (\mu_1 \leftrightarrow \mu_2). \quad (4.15)$$
Again (4.13) and (4.15) should be compared with (2.19).

4.4 We try to eliminate completely the functions $\sigma_{i\ldots i}$. As a byproduct we will show that $L_0$ (which is not unique) can be chosen such that in (4.14) the right hand side is in fact $X$-independent.

To this purpose we define:

$$ L \equiv L_0 \circ \Phi. \quad (4.16a) $$

$$ \sigma_{CS} \equiv \sigma - \sigma_L. \quad (4.16b) $$

It is clear that $\sigma_{CS}$ is of the Chern-Simons type and also that (3.18) is equivalent to:

$$ (\dot{\phi}_f)^* \sigma_L - \sigma_L + (\dot{\phi}_f)^* \sigma_{CS} - \sigma_{CS} = 0. \quad (4.17) $$

It is better to compute separately the first two terms and the last two terms in (4.17), using of course an infinitesimal transformation. For the first two terms it is convenient to compute first the variation of $\theta_L$ and then to use $\sigma = d\theta_L$. For the last two terms in (4.17) we make $\sigma_{i\ldots i} \to 0$ and $\tau_{i\ldots i} \to (\tau_{CS})_{i\ldots i}$ in the computations of Section 2. If we denote:

$$ P_{\rho} \equiv F^\rho_{\mu} X_\mu \circ \Phi + F_\rho. \quad (4.18) $$

then it is easy to prove that (4.17) gives:

$$ \frac{\partial (\tau_{CS})_{\nu_0}}{\partial z^\rho} = 0. \quad (4.19) $$

$$ D_\rho (\tau_{CS})_{\nu_0} + (m_{\nu_0 \rho} + 1) (\tau_{CS})_{\nu_0} + \frac{\partial P_{\rho}}{\partial \psi^\nu_0} = 0. \quad (4.20) $$

$$ D'_\rho (\tau_{CS})_{\nu_0} + \frac{\partial P_{\rho}}{\partial \chi^\nu_0} = 0. \quad (4.21) $$

and:

$$ \frac{\partial (\tau_{CS})^{\mu_1}_{\nu_0,\nu_1}}{\partial z^\rho} = 0. \quad (4.22) $$

$$ D_\rho (\tau_{CS})^{\mu_1}_{\nu_0,\nu_1} + (m_{\nu_0 \rho} + m_{\nu_1 \rho} + \delta^{\mu_1}_{-\rho}) (\tau_{CS})^{\mu_1}_{\nu_0,\nu_1} - \left( \frac{\partial^2 P_{\rho}}{\partial \chi^\nu_0 \partial \psi^{\nu_1}} - (\nu_0 \leftrightarrow \nu_1) \right) = 0. \quad (4.23) $$

$$ D'_\rho (\tau_{CS})^{\mu_1}_{\nu_0,\nu_1} = 0. \quad (4.24) $$

(compare with (3.31)-(3.35)).
We still have the structure equations for \((\tau_{CS})_{\nu_0}\). These equations can be obtained from (3.9), (3.10), (3.12) and (3.13) making \(\sigma \rightarrow 0\) and \(\tau \rightarrow (\tau_{CS})_{\nu_0}\); we get:

\[
\frac{\partial (\tau_{CS})_{\nu_0}}{\partial \chi^\omega} - (\tau_{CS})_{\nu_0, \omega} = 0. 
\]

\[
\frac{\partial (\tau_{CS})_{\mu_1, \nu_0}}{\partial \chi^\omega} = 0. 
\]

\[
\frac{\partial (\tau_{CS})_{\nu_1}}{\partial \psi^{\nu_0}} - \frac{\partial (\tau_{CS})_{\nu_0}}{\partial \psi^{\nu_1}} = 0. 
\]

\[
\frac{\partial (\tau_{CS})_{\mu_1, \nu_1}}{\partial \psi^{\nu_0}} - \frac{\partial (\tau_{CS})_{\mu_1, \nu_2}}{\partial \psi^{\nu_1}} + \frac{\partial (\tau_{CS})_{\mu_1, \nu_1}}{\partial \psi^{\nu_2}} = 0. 
\]

It is rather easy to analyse (4.19)-(4.28). From (4.22) and (4.26) it follows that \((\tau_{CS})_{\mu_1, \nu_1}\) is only a \(\psi\)-dependent function. Then (4.19) and (4.25) give:

\[
(\tau_{CS})_{\nu_0}(\psi, \chi) = t_{\nu_0}(\psi) + (\tau_{CS})_{\mu_1, \nu_1}(\psi)\chi^{\nu_1}_{\mu_1}. 
\]

Substituting this expression into (4.24) it follows that in fact:

\[
(\tau_{CS})_{\mu_1, \nu_1} = 0. 
\]

Then (4.29) tells us that \((\tau_{CS})_{\nu_0}\) is only a \(\psi\)-dependent function. Inserting this information into (4.21) we get:

\[
\frac{\partial P^\rho}{\partial \chi^{\nu_0}} = 0 
\]

so, taking into account (4.18) it follows that in fact:

\[
F^\mu_\rho = 0 
\]

This relation gives the following form of (4.14):

\[
\hat{D}_\rho L_0 + L_0 = F^\rho_\mu. 
\]

(We recall that \(F^\rho_\mu\) is \(\psi\)-independent).

Because \((\tau_{CS})_{\nu_0}\) is only a \(\psi\)-dependent function, the relation (4.27) gives us:

\[
(\tau_{CS})_{\nu_0} = \frac{\partial l}{\partial \psi^{\nu_0}}. 
\]
for some \( \psi \)-dependent function \( l \). Now, it is quite easy to see that if we redefine: \( L_0 \rightarrow L_0 - l \), then we have:

\[
\sigma = \sigma_L.
\]  

(4.33)

and (4.31) stays true if we redefine conveniently the functions \( F_\rho \).

But now the invariance conditions are equivalent to:

\[
\frac{\partial F_\rho}{\partial \psi^{\nu_0}} = 0.
\]  

(4.34)

(see (4.20)) i.e. \( F_\rho \) are some constants. From (4.31) it follows immediately that in this case we must have \( F_+ = F_- \equiv F \).

So, by redefining \( L_0 \rightarrow L_0 - F \), we do not affect (4.33), but instead of (4.31) we have:

\[
\hat{D}_\rho L_0 + L_0 = 0.
\]  

(4.35)

4.5 The equations (4.33) and (4.35) provides us with the most general solution in the case a).

In fact, the analysis can be pushed a little bit further if we note that (4.35) is the infinitesimal form of:

\[
\lambda_+ \lambda_- L_0(\lambda_+^{m_+} \psi^+, \lambda_-^{m_-} \psi^-, \lambda_+^{m_+ + m_- - 1} \lambda_-^{m_+ + m_- - 1} X^+, \lambda_+^{m_+ + m_- - 1} \lambda_-^{m_+ + m_- - 1} X^-) = L_0(\psi, X).
\]  

(4.36)

for any \( \lambda_+, \lambda_- \in R_+ \). (This in turn is equivalent to the invariance of the corresponding action with respect to \( \Phi_f \) defined by (3.19)).

If we take \( \lambda_\mu = X_\mu \) for \( \mu = +, - \) then (4.36) tells us that \( L_0 \) is of the following form:

\[
L_0(\psi, X) = X^+ X^- l(X^{m_+} \psi^+, X^{m_-} \psi^-, (X^+ X^-)^{m_+ + m_-}).
\]  

(4.37)

b) \( m_+ = m_- = 0 \)

In the same way as at a) we can prove that \( \sigma \) can be exhibited in the form (4.33) where \( L \) verifies:

\[
\sum \chi^\nu \mu \frac{\partial L}{\partial \chi^\nu \mu} + L = 0.
\]  

(4.38)

which is the infinitesimal form of:

\[
\lambda_+ \lambda_- L(\psi, \lambda_+ \chi^\nu +, \lambda_- \chi^\nu -) = L(\psi, \chi^\nu +, \chi^\nu -).
\]  

(4.39)
for any $\lambda_\mu \in R_+$. (Again this expresses the invariance of the corresponding action with respect to $\Phi_f$).

So, we have:

**Theorem** Any conformally invariant first-order Lagrangian theory for a primary field is of the form $(E, d\theta L)$ where $L$ is such that the corresponding action functional is strictly invariant with respect to $\Phi_f$:

$$A(\Phi_f \Psi) = A(\Psi).$$

(4.40)

In particular, if $m_+ m_- \neq 0$ then $L$ is determined by $L = L_0 \circ \Phi$ with $L_0$ of the form (4.37) and if $m_+ = m_- = 0$ then $L$ is determined by the homogeneity property (4.39).

**Remark** It is quite plausible that a result of the same type stays true for a more general case of a Lagrangian theory depending on two or more primary fields.

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