LOCAL MIRROR SYMMETRY FOR ONE-LEGGED
TOPOLOGICAL VERTEX

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Abstract. We prove the Bouchard-Mariño Conjecture for the
framed one-legged topological vertex by deriving the Eynar-Orantin
type recursion relations from the cut-and-join equation satisfied by
the relevant triple Hodge integrals. This establishes a version of
local mirror symmetry for the local $\mathbb{C}^3$ geometry with one $D$-brane.

1. Introduction

Local mirror symmetry relates the A-theory on an open toric Calabi-
Yau threefold with the B-theory on its mirror manifold. Mathemati-
cally in the local A-theory one studies the local Gromov-Witten in-
vants. In a series of work culminating in [1], a formalism called
the topological vertex based on duality with Chern-Simons theory has
been developed in the physics literature. A mathematical theory of the
topological vertex [16] has been developed, based on a series of earlier
papers [20, 14, 15]. The B-theory in genus 0 was originally studied by
the theory of variation of Hodge structures and Frobenius manifolds. In
higher genera, they have been studied from various points of view, in-
cluding holomorphic anomaly equation and $tt^*$-geometry [2]. Recently
a new formalism for the local B-theory on the mirror of toric Calabi-
Yau threefolds has been proposed in [17, 4], inspired by the recursion
procedure of [10] discovered first in the context of matrix models.

The new formalism of the B-theory makes it possible to verify local
mirror symmetry in arbitrary genera, and this has been done in many
cases in [17] and [4]. The simplest case is the one-legged framed topo-
logical vertex, Bouchard and Mariño [5] made a conjecture about it
based on the proposed new formalism of the B-theory in [17, 4]. In this
paper we will present a proof of this conjecture.

Mathematically the framed topological vertex involves some special
Hodge integrals associated with partitions of positive integers. The
one-legged case appear naturally in the formal localization calculations
of some open Gromov-Witten invariants by Katz and Liu [12]. A closed
formula for them was conjectured by Mariño and Vafa [18], and proved
mathematically by Liu-Liu-Zhou [14] and Okounkov-Pandharipande
We will refer to such Hodge integrals as triple Hodge integrals of KLMV type.

It is well-known that a certain limit of the Mariño-Vafa formula is the ELSV formula that relates linear Hodge integrals to Hurwitz numbers. Based on this, Bouchard and Mariño [5] made a corresponding conjecture for Hurwitz numbers and linear Hodge integrals. This conjecture has been proved by Borot-Eynard-Safnuk-Mulase [3] and Eynard-Safnuk-Mulase [9] by two different methods. In this paper we will follow [9] to prove the Bouchard-Mariño Conjecture for triple Hodge integrals of KLMV type. The following is our main result:

**Theorem 1.** For triple Hodge integrals of KLMV type, the Bouchard-Mariño recursion can be derived from the cut-and-join equation.

The starting point of our proof, as in [9], is the cut-and-join equation as suggested in [5]. This equation was originally studied for Hurwitz numbers from a combinatorial point of view [11], and later by a symplectic geometric point of view [13]. Inspired by the ELSV formula, it was proposed by the author in 2002 (first draft written in November 2002) that cut-and-join equation may be used as a tool to study Hodge integrals, in particular, a proof of the Mariño-Vafa formula by establishing the cut-and-join equation geometrically via localization on relative moduli spaces was proposed. In collaborations with Kefeng Liu and Melissa Liu [14] this proposal was carried out. Furthermore, it was generalized in [15] to obtain the two-partition Hodge integral formula conjectured in [21], and in [16] to obtain the three-partition Hodge integral formula in the mathematical theory of the topological vertex developed jointly with Jun Li. Our proof is a slight simplification of the method in [9] because we do not use the Laplace transform. After the completion of this work, there appears a paper [7] which gets the same result by the method of [9]. We believe the cut-and-join equation may play an important role in establishing the local mirror symmetry by relating the mathematical computations in the local A-theory with the new formalism of the local B-theory. We will present the corresponding results for the framed topological vertex in the two-legged and three-legged cases in a forthcoming work [22].

2. **Bouchard-Mariño Conjecture for Triple Hodge Integrals of KLMV Type**

In this section we state and reformulate Bouchard-Mariño Conjecture for triple Hodge integrals of KLMV type.
2.1. **Triple Hodge integrals of KLMV type.** For a partition \( \mu = (\mu_1, \ldots, \mu_{l(\mu)}) \) of \( d > 0 \), consider triple Hodge integrals of the form:

\[
W_{g,\mu}(a) = (-1)^{g + l(\mu)} \frac{|\text{Aut}(\mu)|^{l(\mu) - 1}}{\prod_{i=1}^{l(\mu)} (\mu_i - 1)!} \cdot \int_{\mathcal{M}_{g,l(\mu)}} \frac{\Lambda^\vee_g(1)\Lambda^\vee_g(-a - 1)\Lambda^\vee_g(a)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i^a)} \prod_{i=1}^{l(\mu)} \frac{1}{1 - \mu_i^a},
\]

where \( \Lambda^\vee_g(a) = \sum_{i=0}^{g} (-1)^i a^{g-i} \lambda_i \).

Note these are defined for \( 2g - 2 + l(\mu) > 0 \), so that \( \mathcal{M}_{g,n} \) makes sense. It is useful to extend the definition to the exceptional cases \( (g, l(\mu)) = (0, 1) \) and \( (g, l(\mu)) = (0, 2) \) by the following conventions:

\[
\begin{align*}
1. & \quad \int_{\mathcal{M}_{0,1}} \frac{\Lambda^\vee_0(1)\Lambda^\vee_0(a)\Lambda^\vee_0(-1 - a)}{1 - m^a} = m^{-2}, \\
2. & \quad \int_{\mathcal{M}_{0,2}} \frac{\Lambda^\vee_0(1)\Lambda^\vee_0(a)\Lambda^\vee_0(-1 - a)}{(1 - m^a)(1 - m^b)} = \frac{1}{m_1 + m_2}.
\end{align*}
\]

Write

\[
\langle \tau_{b_1} \cdots \tau_{b_n} T_g(a) \rangle_g = \int_{\mathcal{M}_{g,n}} \prod_{i=1}^{n} \psi_i^{b_i} \cdot \Lambda^\vee_g(1)\Lambda^\vee_g(a)\Lambda^\vee_g(-1 - a).
\]

For \( n \geq 1 \), following Bouchard and Mariño [5], define

\[
(3) \quad W_g(x_1, \ldots, x_n; a) = \sum_{l(\mu) = n} z_\mu W_{g,\mu}(a) m_\mu(x_1, \ldots, x_n) \prod_{i=1}^{n} \frac{dx_i}{x_i},
\]

where

\[
(4) \quad z_\mu = |\text{Aut}(\mu)| \prod_{i=1}^{n} \mu_i,
\]

\[
(5) \quad m_\mu(x_1, \ldots, x_n) = \frac{1}{|\text{Aut}(\mu)|} \sum_{\sigma \in S_n} x_{\sigma(i)}^{\mu_i}.
\]

Then we have

\[
(6) \quad W_g(x_1, \ldots, x_n; a) = (-1)^{g+n}(a(a + 1))^{n-1} \cdot \sum_{b_1=0}^{3g-3+n} \langle \tau_{b_1} \cdots \tau_{b_n} T_g(a) \rangle_g \prod_{i=1}^{n} d\phi_{b_i}(x_i, a),
\]
for $2g - 2 + n > 0$, where

$$\phi_b(x; a) = \sum_{m=1}^{\infty} \frac{\prod_{j=1}^{m-1}(\mu a + j)}{(m - 1)!} m^b x^m.$$  

2.2. **Ramification point on the framed mirror curve.** Eynard-Orantin formalism [10] recursively defines a sequence of differentials on a plane algebraic curve. The relevant curve in this case is the framed mirror curve as suggested in [4]:

$$x = y^a - y^{a+1}.$$  

Near $(x, y) = (0, 1)$, one can invert the above equation to get:

$$y(x) = 1 - \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{n-2}(na + j)}{n!} x^n.$$  

Indeed, set

$$y = 1 - u,$$

then by applying Lagrange inversion to the equation:

$$x = u(1 - u)^a$$

one can get [8]:

$$u(x) = \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{n-2}(na + j)}{n!} x^n.$$  

**Remark 2.1.** After a “mirror transformation” of the form

$$x \mapsto -(-1)^a x, \quad y \mapsto -y,$$

equation (8) transforms into the equation:

$$x + y^a + y^{a+1} = 0.$$  

This is equation (4.3) for the framed mirror curve in [4].

The $x$-projection from this curve to $\mathbb{C}$ is ramified. We have

$$\frac{dx}{dy} = ay^{a-1} - (a + 1)y^a, \quad \frac{d^2 x}{dy^2} = a(a - 1)y^{a-2} - a(a + 1)y^{a-1},$$

therefore $\frac{dx}{dy} = 0$ if and only if $y = \frac{a}{a+1}$; furthermore, $\frac{d^2 x}{dy^2}\bigg|_{y=\frac{a}{a+1}} \neq 0$.

Hence the $x$-projection from this curve to $\mathbb{C}$ is two-to-one near

$$(x, y) = \left(\frac{a^a}{(a + 1)^{a+1}}, \frac{a}{a + 1}\right).$$
There are two points \( q \) and \( \tilde{q} \) on the curve near the ramification point such that \( x(q) = x(\tilde{q}) \). Write

\[
y(q) = \frac{a}{a+1} + z, \quad y(\tilde{q}) = \frac{a}{a+1} + P(z),
\]

where \( P(z) = -z + o(z) \). Then from

\[
y(q)^a - y(q)^{a+1} = y(\tilde{q})^a - y(\tilde{q})^{a+1},
\]

one can find [5]:

\[
P(z) = - z - \frac{2(a^2 - 1)}{3a}z^2 - 4\frac{(a^2 - 1)^2}{9a^2}z^3 - \frac{2(a + 1)^3(22a^3 - 57a^2 + 57a - 22)}{135a^3}z^4 + \ldots.
\]

2.3. Bouchard-Mariño Conjecture. By abuse of notations, we will write \( W(x_1, \ldots, x_n; a)dx_1 \cdots dx_n \) as \( W(y_1, \ldots, y_n; a) \). It was conjectured in [5], by making the proposal in [17] explicit in this situation, that the differentials \( W(y_1, \ldots, y_n; a) \) can be computed recursively by the Eynard-Orantin formalism as follows. The initial values are

\[
W_0(y; a) = \ln y(x; a) \frac{dx}{x},
\]

\[
W_0(y_1, y_2; a) = \frac{dy_1 dy_2}{(y_1 - y_2)^2} - \frac{dx_1 dx_2}{(x_1 - x_2)^2},
\]

and the recursion is given by:

\[
W_g(y_{[n]}; a) = \text{Res}_{z=0} \frac{dE_z(y_1)}{\omega(z)}
\]

\[
\left( W_{g-1} \left( \frac{a}{a+1} + z, \frac{a}{a+1} + P(z), y_{[n]}; a \right) + \sum_{q_1 + q_2 = q} W_{g_1} \left( \frac{a}{a+1} + z, y_{A}; a \right) W_{g_2} \left( \frac{a}{a+1} + P(z), y_{B}; a \right) \right),
\]

where

\[
\omega(z) = \left( \ln y(z) - \ln y(P(z)) \right) \frac{dx(z)}{x(z)},
\]

\[
dE_z(y_1) = \frac{1}{2} \left( \frac{1}{y(z_1) - y(z)} - \frac{1}{y(z_1) - y(P(z))} \right) dy_1.
\]

Note the differentials \( \omega_z \) and \( dE_z(y_1) \) are closely related to \( W_0(y; a) \) and \( W_0(y_1, y_2; a) \) respectively. On the right-hand side of (21), only exceptional cases \( W_0(\frac{a}{a+1} + z, y_i; a) \) and \( W_0(\frac{a}{a+1} + P(z), y_i; a) \) appear.
I.e., $W_0\left(\frac{a}{a+1} + z; a\right)$ and $W_0\left(\frac{a}{a+1} + P(z); a\right)$ are understood as 0 in (21).

Here and below we use the following notations:

\begin{align*}
  (24) \quad [n] &= \{1, \ldots, n\}, \\
  (25) \quad [n]_i &= \{1, \ldots, n\} - \{i\}, \\
  (26) \quad [n]_{ij} &= \{1, \ldots, n\} - \{i, j\}.
\end{align*}

We let $x_{[n]}$ stand for $x_1, \ldots, x_n$.

We now make the recursion relations (21) more explicit. Write $\psi_b(y; a) = \phi_b(x; a)$. From the definitions we have

\[ W_g(y_{[n]}; a) = (-1)^{g+n}(a(a+1))^{n-1} \sum \prod_{i=1}^{n} \tau_{b_i} \cdot T_g(a) \prod_{i=1}^{n} \psi_{b_i+1}(y_i; a) \frac{dx_i}{x_i}. \]

It follows that

\[
W_{g-1}\left(\frac{a}{a+1} + z, \frac{a}{a+1} + P(z); y_{[n]}\right) = (-1)^{g+n} \sum \prod_{i=2}^{n} \tau_{b_i} \cdot T_{g-1}(a) \prod_{i=1}^{n} \psi_{b_i+1}(y_i; a) \frac{dx_i}{x_i} \cdot \frac{dx(z)}{x(z)} \cdot \frac{dx(P(z))}{x(P(z))},
\]

\[
W_{g_1}\left(\frac{a}{a+1} + z, y_A\right) = (-1)^{g_1+|A|+1} \sum \prod_{i\in A} \tau_{b_i} \cdot T_{g_1}(a) \prod_{i\in A} \psi_{b_i+1}(y_i; a) \frac{dx_i}{x_i},
\]

\[
W_{g_2}\left(\frac{a}{a+1} + P(z), y_B\right) = (-1)^{g_2+|B|+1} \sum \prod_{i\in B} \tau_{b_i} \cdot T_{g_2}(a) \prod_{i\in B} \psi_{b_i+1}(y_i; a) \frac{dx_i}{x_i}.
\]

Because $x(P(z)) = x(z)$, we actually have:

\[ \frac{dx(P(z))}{x(P(z))} = \frac{dx(z)}{x(z)}. \]

For the exceptional terms:

\[
W_0\left(\frac{a}{a+1} + z, y_i\right) = \frac{dx(z)}{x(z)} \left( \frac{x(z)dy_i}{\frac{dx_i}{dz}(\frac{a}{a+1} + z - y_i)^2} - \frac{x(z)dx_i}{(x(z) - x_i)^2} \right).
\]
Therefore, (21) can be rewritten as the following equation:

\[ \sum_{b_1 \geq 0} \left( \prod_{i=1}^{n} \tau_{b_1} \cdot T_{g}(a) \right) \cdot \prod_{i=1}^{n} \psi_{b_i+1}(y_i; a) \cdot \frac{\partial x_i}{\partial y_i} \]

\[ = \text{Res}_{z=0} \left( \frac{1}{2} \left( \frac{1}{z_1 - z} - \frac{1}{z_1 - P(z)} \right) \cdot \frac{dx(z)}{x(z)} \right) \]

\[ \left( a(a + 1) \sum_{b, c \geq 0} \langle \tau_b \tau_c \prod_{i=2}^{n} \tau_{b_i} \cdot T_{g-1}(a) \rangle_{g-1} \cdot \psi_{b+1}(y(z); a) \cdot \prod_{i \in [n_1]} \psi_{b_i+1}(y_i; a) \right) \]

\[ - \sum_{g_1 + g_2 = g \atop A \prod B = [n_1]} \langle \tau_b \prod_{i \in A} \tau_{b_i} \cdot T_{g_1}(a) \rangle_{g_1} \cdot \langle \tau_c \prod_{i \in B} \tau_{b_i} \cdot T_{g_2}(a) \rangle_{g_2} \cdot \psi_{b+1}(y(z); a) \cdot \prod_{i \in [n_1]} \psi_{b_i+1}(y_i; a) \]

\[ - \frac{1}{a(a + 1)} \sum_{i=2}^{n} \left( \frac{\partial x(z)}{\partial x_i} \left( \frac{1}{a + 1} + z - y_i \right)^2 \right) - \frac{x(z)x_i}{(x(z) - x_i)^2} \]

\[ \cdot \sum_{b \geq 0 \atop j \in [n_1]} \langle \tau_b \prod_{i \in [n_1]} \tau_{b_i} \cdot T_{g}(a) \rangle_{g} \cdot \psi_{b+1}(y(z); a) \cdot \prod_{j \in [n_1]} \psi_{b_j+1}(y_j; a) \]

\[ - \frac{1}{a(a + 1)} \sum_{i=2}^{n} \left( \frac{\partial x(z)}{\partial x_i} \left( \frac{1}{a + 1} + P(z) - y_i \right)^2 \right) - \frac{x(z)x_i}{(x(z) - x_i)^2} \]

\[ \cdot \sum_{b \geq 0 \atop j \in [n_1]} \langle \tau_b \prod_{i \in [n_1]} \tau_{b_i} \cdot T_{g}(a) \rangle_{g} \cdot \psi_{b+1}(y(z); a) \cdot \prod_{j \in [n_1]} \psi_{b_j+1}(y_j; a) \].

2.4. Change to \( t \)-coordinates. Note

\[ \frac{1}{2} \left( \frac{1}{z_1 - z} - \frac{1}{z_1 - P(z)} \right) = \sum_{k=1}^{\infty} \frac{z^k}{z_1^{k+1}} - \frac{P(z)^k}{z_1^{k+1}} = \sum_{m \geq 1} a_m(z_1) z^m, \]

where \( a_m(z_1) \in z_1^{-2} \mathbb{C}[z_1^{-1}] \). Therefore, each residue term on the right-hand side of (28) lies in \( z_1^{-2} \mathbb{C}[z_1^{-1}] \). This suggests to define

\[ t = \frac{1}{(a + 1)z}. \]
It is not hard to check that
\[(30) \quad t(x; a) = 1 + (a + 1) \sum_{n=1}^{\infty} \frac{\prod_{a+1}^{n-1}(na + a)}{(n - 1)!} x^n = 1 + (a + 1)\phi_0(x; a),\]
Now we have:
\[(31) \quad x = \frac{a^a}{(a + 1)^{a+1}}(1 - \frac{1}{t})(1 + \frac{1}{at})^a.\]
For positive integer \(a\), this defines \(x\) as a meromorphic function in \(t\) with a pole at \(t = 0\).

It is clear that for \(b \geq 0\),
\[(32) \quad \phi_b(x; a) = (x \frac{\partial}{\partial x})^b t(a; x) - \frac{1}{a+1}.\]

By easy calculations,
\[(33) \quad x \frac{\partial}{\partial x} = \frac{u(1-u)}{1-(a+1)u} \frac{\partial}{\partial u} = \frac{t}{a+1}(t-1)(at+1) \frac{\partial}{\partial t}.\]
Hence under the change of variables \(x \mapsto t\), \(\phi_b(x; a)\) becomes
\[(34) \quad \hat{\xi}_b(t; a) = D^b_t \frac{t-1}{a+1}\]
for \(b \geq 0\), where
\[(35) \quad D_t = \frac{1}{a+1} t(t-1)(at+1) \frac{\partial}{\partial t}.\]

It follows that \(\hat{\xi}_b(t; a)\) is a polynomial of degree \(2b+1\) in \(t\), for example,
\[(36) \quad \hat{\xi}_0(t; a) = \frac{t-1}{a+1},\]
\[(37) \quad \hat{\xi}_1(t; a) = \frac{1}{a+1} t(t-1)(at+1).\]

Actually for \(b \geq 0\), \(\hat{\xi}_{b+1}(t; a) \in t\mathbb{C}[t]\), therefore, \(\psi_{b+1}(y(z); a)\) lies in \(z^{-1}\mathbb{Z}[z^{-1}]\). By (33), we also have
\[\psi_{b+1}(y; a) \frac{dx}{x} = D_{t+1} t \cdot \frac{(a+1)dt}{t(t-1)(at+1)} = -\frac{1}{a+1} \frac{\partial}{\partial t} D^b_t \cdot \frac{dz}{z^2} \in \frac{dz}{z^2} \mathbb{C}[z^{-1}].\]
This means the left-hand side of (28) also lies in \(z^{-2}\mathbb{C}[z^{-1}]\).

One can also find explicit expressions for \(\hat{\xi}_{-1}(t; a)\) and \(\hat{\xi}_{-2}(t; a)\). Note
\[(38) \quad x \frac{\partial}{\partial x} \phi_{-1}(x; a) = \phi_0(x; a) = \frac{t-1}{a+1},\]
therefore,
\[(39) \quad \frac{1}{a+1} t(t-1)(at+1) \frac{\partial}{\partial t} \hat{\xi}_{-1}(t; a) = \frac{t-1}{a+1}.\]
Integrating once, one gets:

\[
\hat{\xi}_{-1}(t; a) = \ln t - \ln(t + \frac{1}{a}) - \ln \frac{a}{a + 1},
\]

where we have used

\[
t(x; a)|_{x=0} = 1, \quad \phi_{-1}(x; a)|_{x=0} = 0.
\]

One can rewrite (40) as

\[
\hat{\xi}_{-1}(t; a) = -\ln(1 - u) = -\ln y.
\]

Similarly, one can integrate

\[
\frac{1}{a + 1}t(t - 1)(at + 1) \frac{\partial}{\partial t} \hat{\xi}_{-2}(t; a) = \hat{\xi}_{-1}(t; a)
\]

to get an expression of \(\hat{\xi}_{-2}(t; a)\) in \(t\).

2.5. Residue calculations. Now we have known that every term in (28) is an element in \(z_{-2}C[z_1^{-1}]\).

Proposition 2.1. The residue

\[
\text{Res}_{z=0} \frac{\frac{1}{z_1 - z} - \frac{1}{z_1 - P(z)}}{\ln y(z) - \ln y(P(z))} \cdot \psi_{b+1}(y(z); a)\psi_{c+1}(y(P(z)); a) \frac{dx(z)}{x(z)}
\]

is the principal part in \(z_1\) of

\[
\phi_{b+1}(z_1; a)\psi_{c+1}(P(z_1); a) + \psi_{b+1}(P(z_1); a)\psi_{c+1}(z_1; a) \frac{\partial x_1}{\partial z_1}.
\]

Proof. Because \(\phi_{b+1}(y(z); a)\) is a polynomial in \(t = \frac{1}{(a+1)z}\),

\[
f(z)dz := \frac{\frac{1}{z_1 - z} - \frac{1}{z_1 - P(z)}}{\ln y(z) - \ln y(P(z))} \cdot \psi_{b+1}(y(z); a)\psi_{c+1}(y(P(z)); a) \frac{dx(z)}{x(z)}
\]

is a meromorphic form with a pole at \(z = 0\), a simple pole at \(z = z_1\) and a simple pole at \(z = P(z_1)\). Here we have assumed \(|z_1|\) very small. By Cauchy’s residue theorem, for sufficiently small \(\epsilon > 0\),

\[
\frac{1}{2\pi i} \int_{|z| = \epsilon} f(z)dz = \text{Res}_{z=0} f(z)dz + \text{Res}_{z=z_1} f(z)dz + \text{Res}_{z=P(z_1)} f(z)dz.
\]

The LHS is a function \(r_1(z_1)\) analytic in \(z_1\). At the simple poles we have:

\[
\text{Res}_{z=z_1} f(z)dz = -\frac{\psi_{b+1}(y(z_1); a)\psi_{c+1}(y(P(z_1)); a)}{\ln y(z_1) - \ln y(P(z_1))} \cdot \frac{\partial x_1}{\partial z_1}.
\]
and

$$\text{Res}_{z=P(z_1)} f(z)dz = \phi_{b+1}(y(P(z_1)); a) \phi_{c+1}(y(z_1); a) \frac{\frac{\partial x}{\partial z}(P(z_1))}{\ln y(P(z_1)) - \ln y(z_1)} \cdot \frac{\frac{\partial y}{\partial z}(P(z_1))}{x_1 \cdot P'(P(z_1))}. $$

Using $P(P(z)) = z$ we get $P'(P(z)) \cdot P'(z) = 1$ and using $x(P(z)) = x(z)$ we get $\frac{\partial x}{\partial z}(P(z)) = \frac{\partial x}{\partial z}(P(z)) \cdot P'(z) = x'(z)$. Combining these two identities, one gets:

$$\frac{\partial x}{\partial z}(P(z)) = x'(z).$$

\[\square\]

In the same fashion one can prove the following:

**Proposition 2.2.** The residue

$$\text{Res}_{z=0} \left( \frac{1}{z_1-z} - \frac{1}{z_1-P(z)} \right) \cdot \frac{dx(z)}{x(z)} \cdot \left( \frac{x(z)x_i \frac{\partial y_i}{\partial z}}{(y_1-y_i)^2 - (x_1-x_i)^2} - \frac{x(z) x_i}{(x(z) - x_i)^2} \right) \cdot \phi_{b+1}(y(P(z_1)); a)$$

is the principal part in $z_1$ of

$$\frac{1}{\ln y(z_1) - \ln y(P(z_1))} \cdot \frac{\frac{\partial x_1}{\partial z_1}}{x_1} \cdot \left[ \frac{x_1 x_i y_i}{(y_1-y_i)^2 - (x_1-x_i)^2} \cdot \phi_{b+1}(y(P(z_1)); a) \right. $$

$$+ \left. \frac{\partial y_i}{\partial z_1} (P(z_1); a) x_i \frac{\partial y_i}{\partial x_1} \frac{x_1 x_i}{(x_1-x_i)^2} \cdot \phi_{b+1}(y(z_1); a) \right],$$

and the residue

$$\text{Res}_{z=0} \left( \frac{1}{z_1-z} - \frac{1}{z_1-P(z)} \right) \cdot \frac{dx(z)}{x(z)} \cdot \left( \frac{x(z)dy_i}{\frac{\partial y_i}{\partial z} (a+1 + P(z) - y_i)^2} - \frac{x(z)dx_i}{(x(z) - x_i)^2} \right) \cdot \phi_{b+1}(y(z); a)$$
is the principal part in $z_1$ of

\begin{equation}
\frac{1}{\ln y(z_1) - \ln y(P(z_1))} \cdot \frac{\partial x_1}{\partial z_1} \cdot \left[ \left( x_1 \frac{\partial y_1}{\partial x_1} x_i \frac{\partial y_i}{\partial x_i} - \frac{x_1 x_i}{(y_1 - y_i)^2} \right) \phi_{b+1}(y(P(z_1)); a) \right.
\end{equation}

\[ + \left( x_i \frac{\partial y_i}{\partial x_i} (P(z_1); a) dy_i - \frac{x_1 x_i}{(y(P(z_1); a) - y_i)^2} \right) \phi_{b+1}(y(z_1); a) \left] \right].

By these results, we see that the Bouchard-Mariño recursion is equivalent to:

\begin{equation}
\sum_{b_1, \ldots, b_n \geq 0} \sum_{i=1}^{n} \langle \prod_{b_i \in A} T_{\bar{g}i}(a) \cdot \prod_{b_i \in B} \phi_{b_i+1}(y_i; a) \rangle \cdot \frac{\partial x_1}{\partial y_1} \cdot \frac{1}{x_1}
\end{equation}

\[ = \frac{1}{2 \ln y(z_1) - \ln y(P(z_1))} \cdot \frac{1}{x_1}
\end{equation}

\[ \sum_{b_1, \ldots, b_n \geq 0} \left( a(a + 1) \sum_{b, c \geq 0} \langle \tau_b \tau_c \prod_{i=2}^{n} T_{\bar{g}i-1}(a) \rangle \prod_{i=2}^{n} \phi_{b_i+1}(y_i; a) \right.
\end{equation}

\[ \left. \cdot \left[ \phi_{b+1}(y(z_1); a) \phi_{c+1}(y(P(z_1); a) + \phi_{b+1}(y(P(z_1); a) \phi_{c+1}(y(z_1); a) \right] \right)
\end{equation}

\[ - \sum_{A \cup B = \{n\}} \sum_{i \in A} \langle \tau_b \prod_{b \in A} T_{\bar{g}b}(a) \rangle \sum_{b, c \geq 0} \langle \tau_c \prod_{b \in B} T_{\bar{g}b}(a) \rangle \prod_{i=2}^{n} \phi_{b_i+1}(y_i; a)
\end{equation}

\[ \cdot \left[ \phi_{b+1}(y(z); a) \phi_{c+1}(y(P(z); a) + \phi_{b+1}(y(P(z); a) \phi_{c+1}(y(z); a) \right]
\end{equation}

\[ - 2 \sum_{i=2}^{n} \sum_{c \geq 0} \langle \tau_c \prod_{j \in [n]_i} T_{\bar{g}j}(a) \rangle \prod_{j \in [n]_i} \phi_{b_j+1}(y_j; a)
\end{equation}

\[ \frac{1}{a(a + 1)} \left[ \left( x_1 \frac{\partial y_1}{\partial x_1} x_i \frac{\partial y_i}{\partial x_i} - \frac{x_1 x_i}{(y_1 - y_i)^2} \right) \phi_{b+1}(y(P(z_1)); a) \right.
\end{equation}

\[ + \left( x_i \frac{\partial y_i}{\partial x_i} (P(z_1); a) x_i \frac{\partial y_i}{\partial x_i} - \frac{x_1 x_i}{(y(P(z_1); a) - y_i)^2} \right) \phi_{b+1}(y(z_1); a) \left] \right),
\end{equation}

modulo a term analytic in $z_1$.

2.6. Local Airy curve coordinate. Following \cite{9}, introduce a new coordinate $w$ by

\begin{equation}
x = \frac{a^a}{(a + 1)^{a+1}} e^{-w},
\end{equation}
so that the ramification point has \( w = 0 \). By (31),

\[
e^{-w} = (1 - \frac{1}{t})(1 + \frac{1}{at})^a.
\]

Therefore,

\[
w = -\ln(1 - \frac{1}{t}) - a \ln(1 + \frac{1}{at}) = \sum_{n=2}^{\infty} \frac{1}{n} \left( \frac{1}{t^n} + \frac{(-1)^n}{a^n-1} \right)
\]

for \(|t| > 1\). The leading term is \( \frac{a+1}{2at^2} \). Let

\[
v = \sqrt{\frac{a+1}{a}t} \left( 1 + \frac{1}{3}(1 - \frac{1}{a})\frac{1}{t} + \frac{1}{36}(7 - \frac{5}{a} + \frac{7}{a^2})\frac{1}{t^2}
\]

\[
+ \frac{1}{540}(73 - \frac{48}{a} + \frac{48}{a^2} - \frac{73}{a^3})\frac{1}{t^3}
\]

\[
+ \frac{1}{12960}(1331 - \frac{842}{a} + \frac{1036}{a^2} - \frac{842}{a^3} + \frac{1331}{a^4})\frac{1}{t^4} + \cdots
\]

that solves \( w = \frac{1}{2}v^2 \). This expresses \( v \) as analytic function in \( \frac{1}{t} \) for \( \frac{1}{t} \) and \( v \) near 0, therefore, by taking the inverse function, \( \frac{1}{t} \) is analytic in \( v \):

\[
\frac{1}{t} = \hat{v}(1 - \frac{1}{3}(1 - \frac{1}{a})\hat{v} + \frac{1}{36}(1 - \frac{11}{a} + \frac{1}{a^2})\hat{v}^2
\]

\[
+ \frac{1}{270}(1 + \frac{24}{a} - \frac{24}{a^2} + \frac{1}{a^3})\hat{v}^3
\]

\[
+ \frac{1}{4320}(1 - \frac{22}{a} + \frac{267}{a^2} - \frac{22}{a^3} + \frac{1}{a^4})\hat{v}^4 + \cdots),
\]

where \( \hat{v} = \sqrt{\frac{a+1}{a^2}}v \), hence

\[
t = \frac{1}{\hat{v}} \left( 1 + \frac{1}{3}(1 - \frac{1}{a})\hat{v} + \frac{1}{12}(1 + \frac{1}{a} + \frac{1}{a^2})\hat{v}^2
\]

\[
+ \frac{1}{135}(2 + \frac{3}{a} - \frac{3}{a^2} - \frac{2}{a^3})\hat{v}^3
\]

\[
+ \frac{1}{864}(1 + \frac{3}{a} + \frac{3}{a^2} + \frac{2}{a^3} + \frac{1}{a^4})\hat{v}^4 + \cdots)
\]

is meromorphic in \( v \) with a simple pole at \( v = 0 \).

The involution \( P \) in the \( z \)-coordinate (and the involution \( p \) in the \( t \)-coordinate) becomes simply

\[
v(p(t)) = -v(t).
\]

In other words,

\[
p(t) = t(-v; a).
\]
Because of (29) and (53),
\[
z = \frac{1}{a+1} t = \frac{1}{a+1} \hat{v}(1 - \frac{1}{3} (1 - \frac{1}{a}) \hat{v} + \frac{1}{36} (1 - \frac{11}{a} + \frac{1}{a^2}) \hat{v}^2
\]
(57)
\[
+ \frac{1}{270} (1 + \frac{24}{a} - \frac{24}{a^2} - \frac{1}{a^3}) \hat{v}^3
\]
\[
+ \frac{1}{4320} (1 - \frac{22}{a} + \frac{267}{a^2} - \frac{22}{a^3} + \frac{1}{a^4}) \hat{v}^4 + \cdots
\]
is analytic in \(v\) near \(v = 0\), and so a function analytic in \(z\) is also analytic in \(v\). Furthermore, because
\[
\frac{dz}{dv} = \sqrt{\frac{a}{(a+1)^3}} \neq 0,
\]
we have
\[
(59) \quad \text{Res}_{z=0} f(z) = \text{Res}_{v=0} f(z(v)).
\]

2.7. **The odd part of \(\xi_b(v; a)\).** For \(b \geq 0\),
\[
\xi_b(v; a) = (-\frac{1}{v \partial v})^b t - 1
\]
is a meromorphic function in \(v\) with a pole of order \(2b + 1\) at \(v = 0\). Furthermore, if we write
\[
t = t_o + t_e,
\]
where \(t_o\) and \(t_e\) are odd and even functions in \(v\) respectively:
\[
(60) \quad \begin{align*}
t_o & = \frac{1}{\hat{v}} + \frac{1}{12} (1 + \frac{1}{a} + \frac{1}{a^2}) \hat{v} + \frac{1}{864} (1 + \frac{1}{a} + \frac{1}{a^2})^2 \hat{v}^3 + \cdots, \\
t_e & = \frac{1}{3} (1 - \frac{1}{a}) + \frac{1}{135} (2 + \frac{3}{a} - \frac{3}{a^2} - \frac{2}{a^3}) \hat{v}^2 + \cdots,
\end{align*}
\]
then under the repeated action of \(\frac{1}{v \partial v}\), \(t_e\) will only contribute non-negative powers of \(v\). It follows that the odd part of \(\hat{\xi}_b\) is completely determined by \(t_o\). In particular for \(b > 0\) the principal part of \(\hat{\xi}_b\) is completely determined by \(t_o\) and \(\hat{\xi}_{-b}\) is analytic in \(v\).

2.8. **Bouchard-Mariño Conjecture in \(v\)-coordinates.** Note
\[
(63) \quad \frac{\partial x_1}{\partial \nu_1} x_1 = \frac{\partial x_1}{\partial \nu_1} x_1 = -v_1 \frac{1}{\partial \nu_1},
\]
Now by (53) one can see that (48) is equivalent to

\[
\sum_{b_1,\ldots,b_n \geq 0} \langle \prod_{i=1}^{n} \tau_{b_i} \cdot T_g(a) \rangle_g \cdot \prod_{i=1}^{n} \xi_{b_i+1}(v_i, a)
\]

\[
= \frac{1}{2} - \xi_{-1}(v_1; a) + \xi_{-1}(-v_1; a) \left( a(a+1) \sum_{b,c \geq 0} \langle \tau_{b} \tau_{c} \prod_{i=1}^{n} \tau_{b_i} \cdot T_{g-1}(a) \rangle_{g-1} \right)
\]

\[
\cdot \left[ \xi_{b+1}(v_1; a) \xi_{c+1}(-v_1; a) + \xi_{b+1}(-v_1; a) \xi_{c+1}(v_1; a) \right] \prod_{i=2}^{n} \xi_{b_i+1}(v_i, a)
\]

\[
- \sum_{g_1+g_2=g, A \prod B = [n]_1}^{\text{stable}} \sum_{i \in A} \langle \tau_{b_i} \prod_{j \in A} \tau_{b_j} \cdot T_{g_1}(a) \rangle_{g_1} \cdot \sum_{b,c \geq 0} \langle \tau_{c} \prod_{i \in B} \tau_{b_i} \cdot T_{g_2}(a) \rangle_{g_2}
\]

\[
\cdot \left[ \xi_{b+1}(v_1; a) \xi_{c+1}(-v_1; a) + \xi_{b+1}(-v_1; a) \xi_{c+1}(v_1; a) \right] \prod_{i=2}^{n} \xi_{b_i+1}(v_i, a)
\]

\[
- 2 \sum_{i=2}^{n} \sum_{c \geq 0} \langle \tau_{c} \prod_{j \in [n]_1 \setminus i} \tau_{b_j} \cdot T_{g}(a) \rangle_g \cdot \prod_{j \in [n]_1 \setminus i} \xi_{b_j+1}(v_j; a)
\]

\[
\left[ \left( \frac{x_1 x_i \frac{\partial y}{\partial x}(z_1; a) \frac{\partial y_i}{\partial x_i}}{(y_1 - y_i)^2} - \frac{x_1 x_i}{(x_1 - x_i)^2} \right) \xi_{b+1}(-v_1; a) \right.
\]

\[
+ \left( \frac{x_1 x_i \frac{\partial y}{\partial x}(P(z_1; a) \frac{\partial y_i}{\partial x_i})}{(y(P(z_1; a) - y_i)^2} - \frac{x_1 x_i}{(x_1 - x_i)^2} \right) \xi_{b+1}(v_1; a) \right]
\]

modulo a term with at most a simple pole at 0 in \(v_1\). This is because in determining the LHS from the RHS, only terms with degrees \(-3\) and lower in \(v_1\) are involved. For \(f(v_1, \ldots, v_n)\) write

\[
f(v_1, \ldots, v_n)^0 = \frac{1}{2} (f(v_1, \ldots, v_n) - f(-v_1, \ldots, v_n)).
\]

Noticing

\[
\left( \frac{x_1 x_i}{(x_1 - x_i)^2} \right)^0 = 0,
\]
we get from (64):

\[
\sum_{b_1, \ldots, b_n \geq 0} \left[ \prod_{i=1}^{n} \tau_{b_i} \cdot T_g(a) \right] (a) \cdot \xi_{b_i+1}(v_i, a)
\]

\[
= \frac{1}{2} \xi_{o-1}(v_1; a) \left( a(a+1) \sum_{b, c \geq 0} \langle \tau_b \tau_c \prod_{i=2}^{n} \tau_{b_i} \cdot T_g(a) \rangle \right)_{g-1}
\]

\[
\cdot \xi_{o-b+1}(v_1; a) \cdot \xi_{o-c+1}(v_1; a) \cdot \prod_{i=2}^{n} \xi_{b_i+1}(v_i, a)
\]

\[
- \sum_{g_1+g_2=g, A \cap B = [n]_1} \sum_{i \in A} \sum_{j \in B} \langle \tau_b \cdot T_{g_1}(a) \rangle_{g_1} \cdot \langle \tau_c \cdot T_{g_2}(a) \rangle_{g_2}
\]

\[
\cdot \xi_{b+1}(v_1; a) \xi_{c+1}(v_1; a) \cdot \prod_{i=2}^{n} \xi_{b_i+1}(v_i, a)
\]

\[
+ \frac{2}{a(a+1)} \sum_{i=2}^{n} \sum_{c \geq 0} \langle \tau_c \prod_{j \in [n]_i} \tau_{b_j} \cdot T_g(a) \rangle_{g} \cdot \prod_{j \in [n]_i} \xi_{b_j+1}(v_j; a)
\]

\[
\cdot \left( \frac{x_1 x_1 \frac{\partial}{\partial z_1}(z_1; a) \frac{\partial}{\partial z_1}}{(y_1 - y_i)^2} \right)^o \xi_{b+1}(v; a),
\]

modulo a term with at most a simple pole at 0 in \( v_1 \). This is an equivalent reformulation of the Bouchard-Mariño recursion that we will establish in the next section.

3. Proof of Bouchard-Mariño Conjecture

In this section we derive (65) from the cut-and-join equation.

3.1. The cut-and-join equation for triple Hodge integrals of KLMV type. For a partition \( \mu = (\mu_1, \ldots, \mu_{l(\mu)}) \) of \( d > 0 \), consider the triple Hodge integral:

\[
\mathcal{C}_{g, \mu}(a) = \left( -\frac{\sqrt{\frac{1}{|\text{Aut}(\mu)|}}}{(a(a+1))^{l(\mu)}-1} \prod_{i=1}^{l(\mu)} \frac{\mu_i a + a}{\mu_i - 1} \right)
\]

\[
\cdot \int_{\mathcal{M}_{g, \mu}(a)} \frac{\Lambda^g(1) \Lambda^g(-a-1) \Lambda^g(a)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)}
\]

and their generating series

\[
\mathcal{C}_\mu(\lambda; a) = \sum_{g \geq 0} \lambda^{2g-2+l(\mu)} \mathcal{C}_{g, \mu}(a)
\]
and

\[ C(\lambda; a; p) = \sum_{|\mu| \geq 1} C_\mu(\lambda; a)p_\mu, \]

where \( p_\mu = \prod_{i=1}^{(\mu)} p_{\mu_i} \). They satisfy the following cut-and-join equation \cite{20, 14}:

\[ \frac{\partial C}{\partial a} = \frac{\sqrt{-1}}{2} \lambda \sum_{i,j \geq 1} \left( ij p_{i+j} \frac{\partial^2 C}{\partial p_i \partial p_j} + ij p_{i+j} \frac{\partial C}{\partial p_i} \frac{\partial C}{\partial p_j} + (i + j)p_ip_j \frac{\partial C}{\partial p_{i+j}} \right). \]

3.2. Symmetrization. One can also define

\[ C_g(p; a) = \sum_{\mu} C_{g,\mu}(a)p_\mu. \]

Because \( C_g(a; p) \) is a formal power series in \( p_1, p_2, \ldots, p_n, \ldots \), for each \( n \), one can obtain from it a formal power series \( \Phi_{g,n}(x_1, \ldots, x_n; a) \) by applying the following linear symmetrization operator \cite{11, 6}:

\[ p_\mu \mapsto \left( \sqrt{-1} \right)^{(n+|\mu|)} \delta_{(\mu),n} \sum_{\sigma \in \mathcal{S}_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n}. \]

From the definition, we have for \( 2g - 2 + n > 0 \),

\[ \Phi_{g,n}(a; x_1, \ldots, x_n) = -(a(a + 1))^{n-1} \sum_{b_1, \ldots, b_n \geq 0} \langle \tau_{b_1} \cdots \tau_{b_n} T_g(a) \rangle_g \prod_{i=1}^{n} \phi_{b_i}(x_i; a), \]

where

\[ \langle \tau_{b_1} \cdots \tau_{b_n} T_g(a) \rangle_g = \int_{M_{g,n}} \prod_{i=1}^{n} \psi_i^{b_i} \cdot \Lambda_{g}^{\nu}(1)\Lambda_{g}^{\nu}(a)\Lambda_{g}^{\nu}(-1 - a), \]

\[ \phi_{b}(x; a) = \frac{1}{a} \sum_{m \geq 1} \prod_{j=0}^{m-1} \frac{(ma + j)}{m!} x^m. \]

We have two exceptional cases. For \((g, n) = (0, 1)\),

\[ \Phi_{0,1}(x_1; a) = -\sum_{m=1}^{\infty} \frac{\prod_{j=1}^{m-1} (ma + j)}{(m-1)!} m^{-2} x_1^m = -\phi_{-2}(x_1; a); \]

for \((g, n) = (0, 2)\),

\[ \Phi_{0,2}(x_1, x_2; a) = -a(a + 1) \sum_{m_1, m_2 \geq 1} \prod_{i=1}^{2} \frac{\prod_{j=1}^{m_i-1} (m_i a + j)}{(m_i - 1)!} \frac{x_1^m x_2^{m_2}}{m_1 + m_2}. \]
By the analysis in [11, 6], the symmetrized cut-and-join equation of triple Hodge integrals of KLMV type is

\[
\frac{\partial}{\partial a} \Phi_{g,n}(x[n]; a) = -\frac{1}{2} \sum_{i=1}^{n} \frac{z_1}{z_1} \frac{\partial}{\partial z_1} \Phi_{g-1,n+1}(z_1, z_2, x[n]; a)|_{z_1, z_2 = x_i} \\
- \frac{1}{2} \sum_{i=1}^{n} \sum_{\text{stable}} x_i \frac{\partial}{\partial x_i} \Phi_{g_1,|A|+1}(x_i, x_A; a) \\
\cdot x_i \frac{\partial}{\partial x_i} \Phi_{g_2,|B|+1}(x_i, x_B; a) \\
- \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \Phi_{0,1}(x_i; a) \cdot x_i \frac{\partial}{\partial x_i} \Phi_{g,n}(x[n]; a) \\
- \sum_{i=1}^{n} \sum_{j \in [n]} (x_i \frac{\partial}{\partial x_i} \Phi_{0,2}(x_i, x_j; a) - \frac{x_j}{x_i - x_j}) \cdot x_i \frac{\partial}{\partial x_i} \Phi_{g,n-1}(x[n]; a).
\]

(68)

It is clear that

\[
W_g(y_1, \ldots, y_n; a) = (-1)^{g+n-1} \prod_{i=1}^{g} \frac{\partial}{\partial x_i} \Phi_{g,n}(x, x[n]; a) \cdot \prod_{i=1}^{n} dx_i.
\]

(69)

So it is natural to derive Bouchard-Mariño recursion from the symmetrized cut-and-join equation.

3.3. The initial values. By (69), we have

\[
x \frac{\partial}{\partial x} \Phi_{0,1}(x; a) = -\phi_{-1}(x; a) = -\xi_{-1}(v; a) = -\ln y.
\]

(70)

From this one then verifies (19). We also need to find similar expressions for \(\Phi_{0,2}(x_1, x_2; a)\). By (67),

\[
(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}) \Phi_{0,2}(x_1, x_2; a) = -a(a+1) \frac{t_1 - 1}{a+1} \cdot \frac{t_2 - 1}{a+1} \\
= -a(a+1) \frac{y_1 - 1}{(a+1)y_1 - a} \cdot \frac{y_2 - 1}{(a+1)y_2 - a}.
\]

(71)

Note if \(A(x_1, x_2) = \sum_{m_1, m_2 \geq 1} A_{m_1, m_2} x_1^{m_1} x_2^{m_2}\) satisfies

\[
(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}) A(x_1, x_2) = \sum_{m_1, m_2 \geq 1} B_{m_1, m_2} x_1^{m_1} x_2^{m_2},
\]

(72)
then $A(x_1, x_2)$ is uniquely determined by

$$A_{m_1, m_2} = \frac{1}{m_1 + m_2} B_{m_1, m_2}.$$  

Therefore, one can verify that:

$$\Phi_{0, 2}(x_1, x_2; a) = -\ln(y_2 - y_1) + \ln\left(\frac{1 - y_1}{x_1}\right) + \ln\left(\frac{1 - y_2}{x_2}\right).$$

Indeed, differentiating this equation one gets:

$$x_1 \frac{\partial}{\partial x_1} \Phi_{0, 2}(x_1, x_2; a) = -\left(\frac{1}{y_1 - y_2} - \frac{1}{y_1 - 1}\right) \frac{y_1^2 - y_1}{(a + 1)y_1 - a} + \frac{x_2}{x_1 - x_2}.$$  

One can use (33) to get

$$x_1 \frac{\partial}{\partial x_1} \Phi_{0, 2}(x_1, x_2; a) = -\left(\frac{1}{y_1 - y_2} - \frac{1}{y_1 - 1}\right) \frac{y_1^2 - y_1}{(a + 1)y_1 - a} + \frac{x_2}{x_1 - x_2}.$$  

One gets

$$x_2 \frac{\partial}{\partial x_2} \Phi_{0, 2}(x_1, x_2; a)$$

by switching 1 and 2, then (71) can be checked.

One can also take $x_2 \frac{\partial}{\partial x_2}$ on both sides of (75) to get:

$$x_2 \frac{\partial}{\partial x_2} \Phi_{0, 2}(x_1, x_2; a) = -\left(\frac{1}{y_1 - y_2} - \frac{1}{y_1 - 1}\right) \frac{y_1^2 - y_1}{(a + 1)y_1 - a} + \frac{x_2}{x_1 - x_2}.$$  

By this one verifies (20).

3.4. Symmetrized cut-and-join equation in the $v$-coordinates.
Recall for $2g - 2 + n > 0$,

$$\Phi_{g,n}(x_{[n]}; a) = -(a(a + 1))^{n-1} \sum_{b_1 \geq 0} \langle \prod_{i=1}^{n} \tau_{b_i} \cdot T_g(a) \rangle \prod_{i=1}^{n} \xi_{b_i}(v_i; a).$$

It is straightforward to check that

$$x \frac{\partial}{\partial x} = -\frac{\partial}{\partial w} = -\frac{1}{v} \frac{\partial}{\partial v}.$$  

Therefore,

$$x_j \frac{\partial}{\partial x_j} \Phi_{g,n}(x_{[n]}; a) = -\frac{1}{v_j} \frac{\partial}{\partial v_j} \Phi_{g,n}(x_{[n]}; a)$$

$$= (a(a + 1))^{n-1} \sum_{b_1, \ldots, b_n \geq 0} \langle \prod_{i=1}^{n} \tau_{b_i} \cdot T_g(a) \rangle \prod_{i=1}^{n} \xi_{b_i + \delta_{ij}}(v_i; a).$$
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By (75),

\[
 x_1 \frac{\partial}{\partial x_1} \Phi_{0,2}(x_1, x_2; a) - \frac{x_2}{x_1 - x_2} = \left( \frac{1}{y_1 - 1} - \frac{1}{y_1 - y_2} \right) x_1 \frac{\partial y_1}{\partial x_1}.
\]

Now the symmetrized cut-and-join equation (68) can be written as:

\[
- \frac{\partial}{\partial a} \left( (a(a + 1))^{n-1} \sum_{b_i \geq 0} \langle \prod_{i=1}^{n} \tau_{b_i} \cdot T_g(a) \rangle_{g} \prod_{i=1}^{n} \xi_{b_i}(v_i; a) \right)
\]
\[
= \frac{1}{2} \sum_{i=1}^{n} (a(a + 1))^n \sum_{b,c,b_i \geq 0} \langle \tau_b \prod_{j \in [n]} \tau_{b_j} \cdot T_{g_1(a)} \rangle_{g_1} \xi_{b_1+1}(v_i) \prod_{j \in [n]} \xi_{b_j}(v_j; a)
\]
\[
- \frac{1}{2} \sum_{i=1}^{n} \sum_{\text{stable}} (a(a + 1))^{\lvert A \rvert} \langle \tau_c \prod_{j \in A} \tau_{b_j} \cdot T_{g_1(a)} \rangle_{g_1} \xi_{b_1+1}(v) \prod_{j \in A} \xi_{b_j}(v_j; a)
\]
\[
\cdot (a(a + 1))^{\lvert B \rvert} \langle \tau_c \prod_{j \in B} \tau_{b_j} \cdot T_{g_2(a)} \rangle_{g_2} \xi_{c+1}(v) \prod_{j \in B} \xi_{b_j}(v_j; a)
\]
\[
- \sum_{i=1}^{n} \xi_{-1}(v_i; a) \cdot (a(a + 1))^{n-1} \sum_{b_j \geq 0} \langle \prod_{j=1}^{n} \tau_{b_j} \cdot T_g(a) \rangle_{g} \prod_{j=1}^{n} \xi_{b_j+\delta_j(v_j; a)}
\]
\[
+ \sum_{i=1}^{n} \sum_{j \in [n]} \left( \frac{1}{y_i - 1} - \frac{1}{y_i - y_j} \right) x_i \frac{\partial y_i}{\partial x_i}
\]
\[
\cdot (a(a + 1))^{n-2} \sum_{b_1, \ldots, b_n \geq 0} \langle \prod_{k \in [n]} \tau_{b_k} \cdot T_g(a) \rangle_{g} \xi_{b_1+1}(v_i; a) \prod_{k \in [n]} \xi_{b_k}(v_k; a).
\]

We regard both sides of this equation as meromorphic functions in $v_1$, take the principal parts and then take only the even powers in $v_1$. The
left-hand side has no contribution, so we get:

\[ \xi_{-1}(v_1; a) \cdot \sum_{b_j \geq 0} \langle \prod_{j=1}^{n} \tau_{b_j} \cdot T_g(a) \rangle \xi_{b_1+1}(v_1; a) \prod_{j=2}^{n} \xi_{b_j}(v_j; a) \]

\[ = \frac{1}{2} a(a+1) \sum_{b,c,b_i \geq 0} \langle \tau_b \tau_c \prod_{j \in [n]} \tau_{b_j} \cdot T_{g_1}(a) \rangle \xi_{c+1}(v_1) \prod_{j \in [n]} \xi_{b_j}(v_j; a) \]

\[ - \frac{1}{2} \sum_{\text{stable}} \langle \tau_b \prod_{j \in A} \tau_{b_j} \cdot T_{g_1}(a) \rangle \langle \tau_c \prod_{j \in B} \tau_{b_j} \cdot T_{g_2}(a) \rangle \xi_{b+1}(v_1) \xi_{c+1}(v_1) \prod_{j \in [n]} \xi_{b_j}(v_j; a) \]

\[ + \frac{1}{a(a+1)} \sum_{j \in [n]} \left[ \left( \frac{1}{y_1 - 1} - \frac{1}{y_1 - y_j} \right) x_1 \frac{\partial}{\partial x_1} \right] \cdot \prod_{b_1, \ldots, b_j, b_i \geq 0} \langle \prod_{k \in [n]} \tau_{b_k} \cdot T_{g}(a) \rangle \xi_{b_1+1}(v_1; a) \prod_{k \in [n]} \xi_{b_k}(v_k; a) \]

modulo terms analytic in \( v_1 \). One gets (65) by taking \( \prod_{i=2}^{n} x_i \frac{\partial}{\partial x_i} \) on both sides then dividing both sides by \( \xi_{-1}(v; a) \) of this equation. This completes the proof of Theorem 1.

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