Coding for Interactive Communication with Small Memory and Applications to Robust Circuits

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Abstract

Classically, coding theory has been concerned with the problem of transmitting a single message in a format which is robust to noise. Recently, researchers have turned their attention to designing coding schemes to make two-way conversations robust to noise. That is, given an interactive communication protocol Π, an interactive coding scheme converts Π into another communication protocol Π′ such that, even if errors are introduced during the execution of Π′, the parties are able to determine what the outcome of running Π would be in a noise-free setting.

We consider the problem of designing interactive coding schemes which allow the parties to simulate the original protocol using little memory. Specifically, given any communication protocol Π we construct robust simulating protocols which tolerate a constant noise rate and require the parties to use only $O(\log d \log s)$ memory, where $d$ is the depth of Π and $s$ is a measure of the size of Π. Prior to this work, all known coding schemes required the parties to use at least $Ω(d)$ memory, as the parties were required to remember the transcript of the conversation thus far. Moreover, our coding scheme achieves a communication rate of $1 - O(\sqrt{ε})$ over oblivious channels and $1 - O(\sqrt{ε \log \log \frac{1}{ε}})$ over adaptive adversarial channels, matching the conjecturally optimal rates. Lastly, we point to connections between fault-tolerant circuits and coding for interactive communication with small memory.

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1 Introduction

1.1 Interactive Coding Theory

In the well-studied field of coding theory, which dates back to the seminal work of Shannon [Sha48], researchers attempt to understand the fundamental limits on the transfer of information imposed by unreliable communication channels. Until recently, all work has focused on the one-way model of communication, where one party (henceforth referred to as Alice) wishes to send a single message to a second party (Bob). While classical coding schemes have found innumerable applications in technology and the field of theoretical computer science more broadly, modern distributed systems have motivated the development of radically new coding schemes. Computational tasks are now regularly accomplished by multiple processors working in parallel, interleaving brief computations with brief communications. If these communications are subjected to noise, the computation could be derailed.

Motivated by this scenario, Schulman [Sch96] initiated the study of the following model of interactive communication over a noisy channel. There are two parties who wish to carry out a conversation. The additional wrinkle: the channel through which the parties communicate is now unreliable. Therefore, the goal is to transform the original protocol into a new protocol which is still guaranteed to correctly determine the outcome of the original protocol (or, say, succeed with high probability), even if some errors are introduced into the conversation. The new protocol is referred to as a robust simulation of the original protocol. In the literature, errors may be random or adversarial; and, if the errors are adversarial, they may be selected by an oblivious, computationally bounded, or fully adaptive adversary.

In this work, we consider the problem of designing robust simulations of interactive protocols in which the parties use limited memory. Suppose we wish to robustly simulate a depth $d$ protocol of size $s$, where the size of a protocol is a notion we introduce in Section 2.2.3 which corresponds, loosely speaking, to the memory the players require to run the protocol. In all prior work, the parties are required to remember at least $\Omega(d)$ bits, as they need to remember the entire transcript of the conversation thus far. Unfortunately, as complicated computational tasks are more regularly being solved by large distributed networks, storing the entire history of the transcript can be impractical. It is therefore beneficial to have noise-resilient communication protocols which are more space-efficient. We would also like the depth of the simulating protocol to be not too much greater than the depth of the original protocol. Fortunately, the communication rate of our robust simulations match the best known communication rates of any robust simulation (where communication rate denotes the ratio of the depth of the original protocol to the depth of the simulating protocol).

Next, we mention that the design of interactive protocols has recently been used to construct fault-tolerant formulas (that is, circuits whose underlying graph structure is a tree). In order to ensure that the noise-resilient formulas are not too much larger than the original formula, the communication protocol must also be reasonably “compact” which translates to the requirement that the parties use a small amount of memory. This observation is what originally motivated us to study this problem. We discuss this direction more fully in Sections 1.2 and 7.

Our Result Our main result is a space-efficient interactive coding scheme of high communication rate.
Theorem 1.1 (Informal Statement; cf. Theorems 6.4 and 6.5). An interactive protocol of length \(d\) and of size \(s\) can be robustly simulated over a fully adaptive adversarial channel in which the parties use only \(O(\log s \log d)\) memory. Moreover, the communication rate matches the communication rate of the most efficient known protocol.

We remark that \(\Omega(\log s)\) memory is necessary, as otherwise the parties cannot even remember where they are in the original protocol, and will therefore not know when they are finished. It is an interesting open problem to determine if there could be a protocol using only \(O(\log s)\) memory. We think that this is probably impossible.

Prior Work The problem of communicating over a noisy channel was originally formulated by Schulman [Sch96], and he provided a constant rate scheme tolerating up to a \(1/240\)-fraction of bit flips. Much later, Braverman and Rao [BR11] revived the field by introducing a scheme for simulating any communication protocol assuming the channel flips at most \(1/4 - \epsilon\) transmissions. Moreover, they show that this is the best possible for non-adaptive protocols. Later, Ghaffari, Haeupler and Sudan [GHS14] showed that a \(2/7 - \epsilon\) error fraction can be tolerated if one allows adaptivity. In adaptive protocols, the order of speaking is not fixed ahead of time, so the parties determine who should speak next given their input and the transcript of the communication thus far.

In the small-error regime, if \(\epsilon\) denotes the error rate, Haeupler [Hae14] provides a coding scheme with information rate \(1 - O(\sqrt{\epsilon})\) against random, oblivious and computationally bounded adversaries; and information rate \(1 - O(\sqrt{\epsilon \log \log(1/\epsilon)})\) against adaptive adversaries. Our algorithms are largely inspired by a comment given at the end of this paper, although some nontrivial additional ideas are required in the analysis to make this approach work.

To the best of our knowledge, the problem of interactive coding with limited memory has not appeared in the literature prior to this work. However, the problem of space-bounded communication complexity was considered by Brody et al. [BCP+13]. Our model for a space-bounded communication protocol is influenced by their model.

1.2 Robust Circuits

We now briefly introduce the notion of robust circuits, which we discuss more fully in Section 7. The problem of designing fault-tolerant circuits can be traced back to von Neumann [VN56]. In the von Neumann model, each gate independently has its output flipped with some probability \(p < 1/2\). A natural limitation of the von Neumann model is that no circuit can be guaranteed to compute the correct output with probability better than \(1 - p\), as with probability \(p\) the output of the final gate could be flipped. This motivated Kleitman et al. [KLM94] to consider the model of short-circuit errors. A gate \(g\) is said to be short-circuited if the value it computes is fixed to be one of its inputs. In our work, short-circuit errors are chosen adversarially and may even depend on the input to the circuit.

In [KLR12], the authors show how to efficiently construct formulas which function correctly even if at most a \(\frac{1}{6} - \epsilon\) fraction of the gates along each path from an input gate to an output gate are short-circuited. Moreover, the size of the noise-resilient formula is polynomially bounded by the size of the formula it
simulates. They achieve this via a reduction to interactive coding theory. We extend their methodology to handle general circuits, at the cost of placing stricter requirements on the interactive protocol.

1.3 Organization

We begin with some preliminaries on interactive coding theory in Section 2. In Section 3 we provide some high-level intuition for our protocol, before introducing some notation and proving some technical lemmas in Section 4. In Section 5 we describe a robust simulation over a feedback channel, which acts as a warm-up for the robust simulation for general channels we provide in Section 6. In Section 7 we describe how interactive coding schemes of a specific type could be used to obtain robust circuits. Finally, Appendix A contains the observation that robust formulas naturally yield robust protocols, but that this direction fails for general circuits.

2 Preliminaries

2.1 Communication Protocols

A communication protocol Π consists of two parties/players named Alice and Bob who communicate by sending symbols from an alphabet Σ over a channel. The communication proceeds round-by-round, and in each round, one party communicates while the other party listens. The transcript of the protocol is the concatenation of all the symbols sent across the channel, together with a label indicating who sent each symbol. We let d denote the length of the protocol (d stands for “depth”). We will consider communication protocols with private randomness; that is, each party is provided with an infinite random string at the start of the protocol, and the strings Alice and Bob see are sampled independently.

In order to speak of “memory-bounded” interactive protocols, we introduce the concept of a player’s knowledge. A player’s knowledge may consist of (a) its input; (b) any randomness to which it has access; and (c) the transcript of the execution of the protocol thus far. We will often consider protocols in which a party’s knowledge is less than this, i.e., in which its knowledge consists of some data that is computable from these data and requires less memory to store. The decision to speak or listen and, if speaking, the symbol that a party sends across the channel is required to be a function of the party’s knowledge. We think of a player’s knowledge as being encoded in a bit-string.

More precisely, the players’ actions in an m-memory-bounded communication protocol are defined by a pair of transition functions

\[ T_A : \Sigma \times \{0,1\}^m \times \mathcal{X} \to \Sigma \times \{0,1\}^m \times (\mathcal{Z} \sqcup \{\perp\}) , \]

\[ T_B : \Sigma \times \{0,1\}^m \times \mathcal{Y} \to \Sigma \times \{0,1\}^m \times (\mathcal{Z} \sqcup \{\perp\}) . \]

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1We use the terms party and player interchangeably.
2It is more standard in the interactive coding theory literature to use n for the length of the protocol. However, we reserve n to denote the input length of a circuit.
\[ T_A(\sigma, \alpha, x) = (\sigma', \alpha, h) \] means that Alice, upon receiving symbol \( \sigma \) with memory contents \( \alpha \) and input \( x \), sends \( \sigma' \) to Bob and updates her memory contents to \( \alpha' \). If \( h \neq \perp \), then Alice halts and outputs \( h \). \( T_B \) is understood similarly for Bob.

We also think of communication protocols as a pebble game that is played on a rooted directed acyclic graph (DAG). Each non-leaf node is owned by one of the two players, and each leaf node is given a label. Initially, the pebble is placed on the root. If a player owns the node on which the pebble is currently placed, it is her turn to communicate. Given a node \( v \) owned by Alice, \( v(x) \in \Sigma \) denotes the symbol she sends across the channel when the pebble is on \( v \) and she has input \( x \), and we define \( v(y) \) similarly for nodes owned by Bob. Each symbol in the alphabet defines one of the edges leaving the current node. When a symbol is sent across the channel, the pebble is moved across the corresponding edge. The protocol terminates when the pebble arrives at a leaf. We refer to this leaf as the correct leaf and denote its label by \( \ell(x, y) \). Throughout, \( s \) will denote the size of the DAG and \( d \) its depth.

Note that if the protocol’s associated DAG (henceforth referred to as protocol DAG) has \( s \) nodes, then the players may simulate the protocol with memory-bound \( m = \lceil \log_2 s \rceil \): they are merely required to remember the current location of the pebble, which can be done with \( m \) bits.

Remark. In the communication complexity literature, one typically restricts attention to protocols whose underlying graph structure is a tree. However, since we are concerned with space-bounded interactive protocols, it is desirable to first “compress” the communication protocol into its smallest possible representation, which will be a directed acyclic graph.

\[ 2.2 \quad \text{Interactive Coding Theory} \]

\[ 2.2.1 \quad \text{Channel Models} \]

An interactive coding scheme is again a communication protocol \( \Pi \), where now the parties converse through a noisy channel, denoted \( \mathcal{C} \). That is, if a party intends to send a symbol \( \sigma \), the other party receives the symbol \( \mathcal{C}(\sigma) \), which may be different from \( \sigma \). We consider various models of channels in this work.

In the fully adaptive channel model with error rate \( \varepsilon > 0 \), the number of symbol corruptions is at most \( \varepsilon d \) in a length \( d \) protocol, where a symbol corruption is an error where a sent symbol is replaced by an arbitrary symbol. The adversary therefore knows \( d \) and its “budget” \( \varepsilon d \) of corruptions it may introduce. The adversary may choose corruptions based on its internal state, its own randomness, and the observed communication history. Note that its decisions may not be based on any inputs given to Alice or Bob, nor on their private randomness, unless it is able to infer these data from the communication history.

There are many relaxations of this model which are regularly considered in the literature. If all of the adversary’s decisions may be computed in probabilistic polynomial time in \( d \), then the adversary is computationally bounded. If the adversary is not allowed to base its decisions on the observed communication history, and therefore must choose which rounds it will corrupt before the protocol starts, the adversary is deemed oblivious. As a special case, if the adversary chooses to corrupt each round with probability \( \varepsilon \), it is deemed random.

\[ 3 \quad \text{That is, a node with out-degree } 0. \]
Lastly, if a channel has feedback, that means that the party who transmits a symbol learns what symbol was received at the other end of the channel. We introduce feedback channels for two reasons: first, we provide a description of a space-bounded interactive coding scheme over a feedback channel as a “warm-up” to our main result; and secondly, they are the channel of interest when we attempt to use interactive coding theory to design robust circuits.

### 2.2.2 Adaptivity in Interactive Protocols

In early work on interactive coding theory, communication protocols were always non-adaptive in the sense that which party speaks in a given round is determined solely by the round number. In an adaptive protocol, which party speaks in a given round is determined not only by the round number, but also by the players’ observed communication history and perhaps their randomness. Adaptivity has already proved to be a useful concept: for example, adaptivity allows the communication rate of the protocols in [Hae14] to exceed the rate limit given in [KR13]. However, one must be careful in its definition to prevent “silence” from giving the parties a free, uncorruptable symbol they can send across the channel. Thus, we stipulate the following. In each round, the parties must choose to either listen or communicate, but not both. If both parties try to speak in the same round, neither party will receive a symbol. If both parties are listening, the adversary is given a “free” corruption in the sense that he may give both parties an arbitrary symbol, and these corruptions do not count towards his budget of $\varepsilon d$ corruptions. Our main result is an adaptive protocol, although the simpler protocol for feedback channels is not.

### 2.2.3 Robust Simulations

In the theory of coding for interactive communication, one is given a protocol which accomplishes a certain task in the noise-free setting, and one describes a robust simulation of the given protocol which still accomplishes the task, even if the players must communicate via a noisy channel. If the robust simulation involves randomness, we require that the parties accomplish the task in a noisy setting, except with some (hopefully small) failure probability. We crystallize this in the following definition, where we make use of the DAG formulation of a protocol introduced in Section 2.1.

**Definition 2.1 (Robust Simulation).** A protocol $\Pi'$ is said to robustly simulate a deterministic protocol $\Pi$ over a channel $\mathcal{C}$ if, given any inputs $(x, y)$ to $\Pi$, both parties can uniquely decode $\ell(x, y)$, the leaf that the parties would have arrived at in a noise-free execution of $\Pi$ on inputs $(x, y)$. If the protocols $\Pi'$ or $\Pi$ are randomized, $\Pi'$ is said to robustly simulate $\Pi$ with failure probability $p$ over a channel $\mathcal{C}$ if both parties can uniquely decode $\ell(x, y)$, except with probability at most $p$.

In general, the robust simulation $\Pi'$ will use a larger number of rounds than $\Pi$, typically of the form $\alpha d$ for some $\alpha = \alpha(\varepsilon) > 1$. It is also possible that $\Pi'$ will use an alphabet $\Sigma'$ that is different from the alphabet $\Sigma$ used by $\Pi$. The ratio $\frac{\log |\Sigma|}{\alpha \log |\Sigma'|}$ then gives the communication rate of the protocol, which is the ratio of the amount of information communicated in $\Pi$ to the amount communicated in $\Pi'$.

We think of a robust simulation $\Pi'$ as having oracle access to the protocol $\Pi$. This means that the parties may query an oracle $\mathcal{O}$ whose behavior is defined as follows. Let $v$ be a node in $\Pi$’s DAG and
\( \sigma \in \Sigma \) a symbol. \( \mathcal{O}(v) \) outputs either the symbol in \( \Sigma \) that the player should send when the pebble is on node \( v \); or \( \perp \) if the player does not own node \( v \) or if \( v \) is a leaf. \( \mathcal{O}(v, \sigma) \) outputs a node \( v' \), where \( v' \) is the child of \( v \) corresponding to \( \sigma \); or \( \perp \) if \( v \) is a leaf.

Remark. In prior work on interactive coding theory, it is typically required that at the end of the protocol over the noisy channel, both players know what the transcript of the conversation would have been in a noise-free setting. However, recall that we are interested in interactive coding schemes in which the players use limited memory, and the entire transcript is in general too large to fit in memory. Hence, we relax this requirement: the parties are now simply required to know the leaf of the protocol DAG that they would have agreed upon in the noise-free setting. In most prior work, the protocol DAG is in fact a tree, and therefore there is a one-to-one correspondence between paths (i.e., transcripts) and leaves. This is thus the natural generalization for protocols whose underlying graph structure is a DAG.

3 High-Level Intuition

We first give an informal discussion of our protocols. The general outline of each protocol is inspired by the “rewind-if-error” framework used in many interactive coding schemes. In these protocols, the idea is to communicate according to the original protocol in rounds, and then, at the end of some number of rounds, communicate certain check-bits to ascertain if there has been an error. In this way, the parties can determine if and how much they should rewind the computation.

Two problems immediately present themselves if one hopes to implement this procedure with low memory. First of all, since we cannot recall the entire transcript, we will have to carefully determine which previous states we should remember if we would like to backtrack later. We refer to the previous states that the parties remember as “meeting points”. Secondly, even once the parties have stored judiciously chosen lists of meeting points in case they need to backtrack, it is inevitable that the parties’ meeting points could be distinct. As such, one party may wish to backtrack to a previous step that the other party may not even have stored in memory.

In Section 5, since the interactive coding scheme we define is only promised to work over a feedback channel, it is much easier for us to ensure that the parties’ available meeting points remain consistent. Hence, we are able to focus more clearly upon the problem of determining which meeting points to store. To obtain our general result, we combine these ideas with the approach of Haeupler in [Hae14], which provides a general framework for ensuring that the parties remain synchronized.

In the next section, we will define the meeting points that the parties will store in the execution of our protocols. They will be defined so as to be “geometrically spread out”, with a greater density of available meeting points closer to the parties current location in the protocol. This allows the parties to quickly correct recent errors, and, by appropriately aggregating “votes” for meeting points, we can prevent the adversary from forcing the parties to backtrack large amounts while investing only a small number of corruptions.

However, it is inevitable that the players will have to correct an error and, due to the paucity of available meeting points, will be forced to backtrack very far and thereby undo much of the simulation which was correct. We need to therefore argue that the adversary must have invested many errors to
force the players to do this. We demonstrate this by introducing a parameter $L^-$ that quantifies how far the players have traversed since the last time they had been correctly simulating the protocol. We show in Lemmas 5.2 and 6.6 that the amount of “correct computation” the adversary can force the players to undo is on the order of $L^-$, indicating that the adversary had indeed introduced many corruptions. This constitutes a main technical contribution of our work.

4 Meeting Points and Preliminary Lemmas

Notation All logarithms are taken to the base 2. The expression $a|b$ asserts the integer $a$ divides the integer $b$ without remainder. We introduce the notation $\lfloor a \rfloor$ to denote the integer obtained by rounding $a$ down to the nearest multiple of $b$. For example, $\lfloor 5 \rfloor_2 = 4$ and $\lfloor 8 \rfloor_3 = 6$. Lastly, for integers $a \leq b$, $\{a, b\} = \{a, a + 1, \ldots, b - 1, b\}$ denotes the discrete, closed, interval with endpoints $a$ and $b$.

Meeting Points As alluded to in the previous section, our interactive coding schemes will require the parties to maintain a set of “meeting points” that they can backtrack to should the need arise. In this section, for an integers $a$ and $j$, we define the scale $j$ meeting point (of $a$) as the integer $\lfloor a \rfloor_{2j} - 2^j$. In Sections 5 and 6, the scale $j$ meeting point will be a node of the DAG, where the depth or “iteration-depth” (a notion we introduce in Section 6) of the node will play the role of $a$, and the scale $j$ meeting point will be the node that was passed in the simulation at “iteration-depth” $\lfloor a \rfloor_{2j} - 2^j$. However, to keep this discussion general, we ignore these implementation details.

We begin by proving a few lemmas regarding meeting points. For a given integer $a$, let

$$M_a = \{b \geq 0 : \exists j \geq 0 \text{ s.t. } b = \lfloor a \rfloor_{2j} - 2^j\}.$$  

If $j \geq \lceil \log(a) \rceil$, then $\lfloor a \rfloor_{2j} = 1$, so $\lfloor a \rfloor_{2j} - 2^j < 0$. Hence, $|M_a| = O(\log a)$.

Lemma 4.1. Let $a$ be a positive integer, and let $j$ be the non-negative integer such that $2^j|a$ but $2^{j+1}|a$.

(a) For all $j' \in [0, j]$ and any $a' \in [a + 2^j, a + 2^{j+1} - 1]$, we have $a = [a']_{2^{j'}} - 2^j$. Moreover, for all $j'' \neq j'$ we have $a \neq [a']_{2^{j''}} - 2^j$.

(b) For any $a' \geq a + 2^{j+1}$, for any $j' \geq 0$, $a \neq [a']_{2^{j'}} - 2^j$.

Proof. Write $a = k2^j$ with $k$ odd. We first prove part (a). Write $a' = a + \Delta$ for some $\Delta \in [2^{j'}, 2^{j'+1} - 1]$. Note that $[a']_{2^{j'}} = a + 2^{j'}$. Since

$$a + 2^{j'} = k2^{j'} + 2^{j'} = 2^{j'}(k2^{j'} - 2^{j'} + 1),$$

$a + 2^{j'}$ is a multiple of $2^{j'}$. However, the next multiple larger is

$$2^{j'}(k2^{j'} - 2^{j'} + 2) = k2^{j'} + 2^{j'+1} = a + 2^{j'+1} \notin [a + 2^{j'}, a + 2^{j'+1} - 1],$$

so it’s strictly larger than $a'$. Hence, $\lfloor a' \rfloor_{2^{j'}} - 2^{j'} = (a + 2^{j'}) - 2^{j'} = a$. 

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Now, suppose \( j'' \neq j' \). If \( j'' < j' \), \( [a]_{2j''} \geq [a']_{2j'} = a + 2^{j'} \), so
\[
[a']_{2j''} - 2^{j''} \geq a + 2^{j'} - 2^{j''} > a.
\]
If \( j'' > j' \), \( [a']_{2j''} \leq [a']_{2j'} = a + 2^{j'} \), as any multiple of \( 2^{j''} \) is also a multiple of \( 2^{j'} \). Hence,
\[
[a']_{2j''} - 2^{j''} \leq a + 2^{j'} - 2^{j''} < a.
\]
We now prove part (b). Write \( a' = a + \Delta \) for some \( \Delta \geq 2^{j+1} \). For \( j' \leq j \) we have
\[
[a']_{2j'} - 2^{j'} \geq a + 2^{j+1} - 2^{j'} > a.
\]
Finally, for \( j' > j \) we cannot have \( a = [a']_{2j'} - 2^{j'} \) as \( 2^{j'} ([a']_{2j'} - 2^{j'}) \), but the assumption on \( j \) tells us \( 2^{j'} | a \).

**Lemma 4.2.** For any positive integer \( a \) and any \( c \in [1, \lfloor a/3 \rfloor] \), there is a \( j \) such that \( [a]_{2j} - 2^j \in [a - 3c, a - c] \).

**Proof.** First of all, note that \( [a]_{2^0} - 2^0 = a - 1 \geq a - 3 \geq a - 3c \) and
\[
\left\lfloor \frac{a}{2^{\left\lfloor \log a \right\rfloor - 1}} \right\rfloor \leq 2^{\left\lfloor \log a \right\rfloor - 1} - 1 \leq \frac{a}{2} \leq \frac{2a}{3} = a - a/3 \leq x - c.
\]
Hence, if the lemma is false, there must exist an \( i \) such that
\[
[a]_{2i} - 2^i < a - 3c < a - c < [a]_{2i-1} - 2^{i-1}.
\]
On the one hand, since \( [a]_{2i-1} - 2^{i-1} > a - c, a - 2^{i-1} > a - c, \) so \( c > 2^{i-1} \). However,
\[
2c = (a - c) - (a - 3c)
\]
\[
< ([a]_{2i-1} - 2^{i-1}) - ([a]_{2i} - 2^i)
\]
\[
= ([a]_{2i-1} - [a]_{2i}) + 2^{i-1}
\]
\[
\leq 2^{i-1},
\]
so \( c < 2^{i-2} \). Contradiction.

**5 Simpler Case: Feedback Channel**

In this section, we describe a robust simulation \( \Pi' \) of a protocol \( \Pi \) which uses \( O(\log d \log s) \) memory when the players communicate via a feedback channel. For simplicity, we also assume that the protocol \( \Pi \) we wish to simulate is alternating, i.e., that Alice speaks in the odd rounds while Bob speaks in the even rounds. Note that this at most doubles the depth of \( \Pi \).
Informally, $\Pi'$ operates as follows. The protocol proceeds in iterations which each consist of 2 rounds: in the first round, Alice communicates and Bob listens, while in the second, Bob communicates and Alice listens. The players communicate as if they were running the protocol $\Pi$ normally by (implicitly) moving a pebble along the protocol DAG for $\Pi$, until a player learns that a transmission they sent was corrupted (which they learn from the feedback). Once a player realizes that a corruption has occurred and thereby caused the players to diverge from the correct computational path, they will send a special rewind symbol, which we denote by $\rho$. Once sufficiently many rewind symbols have been sent across the channel, the players will backtrack to an earlier point in the computation. In order to determine which point the players should backtrack to, the players maintain a meeting point set, which we describe next.

As the players progress down the protocol DAG of $\Pi$, they will maintain the set $M$ of stored meeting points that they will be able to backtrack to if a channel corruption causes them to traverse down an incorrect path of the DAG. This is done as follows. For a node $u$ that the pebble was placed on during the execution of the protocol, if $P$ denotes the path traced from the root to this node by the pebble, we let $\text{depth}(u) = |P|$ denote the number of edges in $P$ and \( \text{iteration-depth}(u) = \lfloor \text{depth}(u)/2 \rfloor \). Intuitively, as the protocol proceeds in iterations of length 2, it is easiest to reason about the amount of progress which is made in chunks of size 2. When the pebble is on a node at iteration-depth $a \leq d/2$, the set $M$ will be contained in the set

\[
\{(u, b) : \text{iteration-depth}(u) = b \text{ and } \exists j \geq 0 \text{ s.t. } b = \lfloor a/2^j \rfloor - 2^j\}.
\]

Since there are at most $O(\log a) = O(\log d)$ nodes in $M$ and only $O(\log s)$ bits are required to specify a node, $O(\log d \log s)$ bits are sufficient for the parties to maintain their meeting point sets.

Now, in order to determine which points the players should backtrack to, the players will compute the number of sent rewind symbols (which can be determined with a counter using $O(\log d)$ bits). For two nodes $u$ and $v$ on the simulated path $P$, define their iteration-distance to be $|\text{iteration-depth}(u) - \text{iteration-depth}(v)|$. As soon as the number of sent symbols exceeds the iteration-distance from the current node to the nearest point they have saved in the meeting point set $M$, the players roll back their computation to this meeting point and subtract from the rewind counter the distance they have backtracked. We show that, so long as the adversary is not permitted to corrupt too many transmissions, this strategy robustly simulates the original protocol.

**Confirmation Steps** As a final technical detail, we will be required to add “confirmation steps” to the end of the protocol, wherein the parties send each other a fixed symbol. Intuitively, these steps allow the parties to affirm that they have correctly simulated the protocol. The confirmation steps ensure that the parties never run out of steps to simulate, and furthermore ensure that the adversary cannot let the parties correctly simulate $\Pi$ and then force them to backtrack, leading them to end at an intermediate node of $\Pi$. As we only ever add $O(d)$ confirmation steps per leaf of the protocol DAG, the increase in the size of the DAG is at most by a multiplicative factor of $d$, and since we are aiming for memory bounds on the order of $\log s \log d$, this contribution is negligible.

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4For this reason, if the original protocol used the alphabet $\Sigma$, the simulating protocol $\Pi'$ uses the alphabet $\Sigma' = \Sigma \cup \{\rho\}$. 

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Algorithm 1 Space-bounded robust simulation over a feedback channel with error rate $\varepsilon$, Alice's perspective

1. $O \leftarrow$ Alice’s oracle for alternating protocol of depth $d$ + final confirmation steps
2. $cnt \leftarrow 0$ \(\triangleright\) Counts the rewind symbols
3. $error \leftarrow false$ \(\triangleright\) Indicates whether an error has occurred from the player’s perspective
4. $M \leftarrow \{(root, 0)\}$, $v \leftarrow root.$ \(\triangleright\) root denotes the root of the DAG, $v$ denotes the current node
5. $d_{total} \leftarrow \lceil \frac{1}{2-24\varepsilon}d \rceil$

6. for $d_{total}$ iterations do
\hspace{1em} \(\triangleright\) Each iteration consists of one round of Alice speaking followed by one round of Bob speaking.
\hspace{1em} Thus, from Bob’s perspective, he first listens, then communicates.

7. $\sigma \leftarrow O(v)$ \(\triangleright\) First, Alice communicates
8. if $error = false$ then
9. \hspace{1em} send $\sigma$
10. \hspace{1em} Let $\sigma_t$ denote transmitted symbol
11. \hspace{1em} if $\sigma \neq \sigma_t$ and $\sigma_t \neq \rho$ then $error \leftarrow true$
12. \hspace{1em} else send $\rho$

13. let $\sigma_r$ denote the received symbol \(\triangleright\) Next, Alice listens

14. if $\sigma_t = \rho$ or $\sigma_r = \rho$ then
15. \hspace{1em} if $\sigma_t = \rho$ and $\sigma_r = \rho$ then $cnt \leftarrow cnt + 2$
16. \hspace{1em} else $cnt \leftarrow cnt + 1$
17. else
18. \hspace{1em} temp_node $\leftarrow O(v, \sigma_t)$
19. \hspace{1em} $v \leftarrow O(temp\_node, \sigma_r)$

20. Let $D$ denote iteration-distance to nearest meeting point
21. while $D \leq cnt$ do
22. \hspace{1em} backtrack to closest meeting point $(u, b) \in M$ and remove it from $M$
23. \hspace{1em} update $v \leftarrow u$
24. \hspace{1em} $D \leftarrow$ iteration-distance to closest meeting point \(\triangleright\) Update the set of meeting points.
25. $M \leftarrow M \cup \{(v, \text{iteration-depth}(v))\}$ \(\triangleright\) Add the current node to the set $M$.
26. Remove all pairs $(u, c) \in M$ for which there is no $j$ for which $c = [\text{iteration-depth}(v)]_{2j} - 2^j$. 

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Theorem 5.1. For any error rate $\varepsilon \in \left(0, \frac{1}{12}\right)$, Algorithm 1 is a robust simulation of a protocol $\Pi$ using $O(\log d \log s)$ space, where the protocol DAG of $\Pi$ has size $s$ and depth $d$.

Proof. That the protocol uses $O(\log d \log s)$ space follows from earlier observations. We thus only prove that it robustly simulates $\Pi'$. First a couple definitions: the correct path refers to the path in $\Pi$'s DAG that the players would traverse in a noise-free execution of the protocol, while the simulated path refers to the path in $\Pi$’s DAG that has been traced out by $\Pi'$ in its simulation of $\Pi$ at a given point in its execution.

We use a potential function argument. Let $P$ denote the path that is the intersection of the simulated path and the correct path. Then, if $v$ denotes the current node, we let

$$\ell^+ = \frac{|P|}{2} \quad \text{and} \quad \ell^- = \text{iteration-depth}(v) - \ell^+.$$

Let $R$ denote the value currently stored by the $\text{cnt}$ variable (which is always the same for both players). Finally, let $L^-$ denote the maximum value that $\ell^-$ has obtained since the last point in the computation at which $\ell^-$ had equaled 0. Let

$$\Phi = \begin{cases} 
\ell^+ + R - \ell^- - 10L^- & \text{if } \ell^- > 0 \\
\ell^+ - R & \text{if } \ell^- = 0
\end{cases}$$

Before analyzing the potential function, we first prove the following lemma. For a node not on the correct path, we refer to the error point as the iteration-depth of the node of minimal depth after which the correct path and the simulated path diverge.

Lemma 5.2. Suppose that an error has occurred and that the players backjump from a node deeper than the error point to a node shallower than the error point. Let $D$ denote the iteration-distance from the error point to the nearest meeting point prior to it that was saved, which is the point that the players backjump to. Then $D < 2L^-$. 

Intuitively, this lemma states that if the adversary forces the players to backtrack and undo much of the “correct simulation” of $\Pi$, then it must be that he has introduced many corruptions to force the $L^-$ parameter to become so large.

Proof of Lemma 5.2. Let $c$ denote the iteration-depth of the deepest node reached since the last time $\ell^-$ was equal to 0. Let $b$ denote the iteration-depth of the error point. Let $a$ denote the iteration-depth of the deepest node in the players’ meeting point set at depth at most the depth of the error point. Then $D = b - a$ and $L^- = c - b$. Next, fix $j$ for which

$$|c|_{2j} - 2^j \leq b < |c|_{2j-1} - 2^{j-1},$$

and let $a' = |c|_{2j} - 2^j$. We make the following claim:

Claim 5.1. When the parties backjumped to correct $b$, the node at iteration-depth $a'$ was an available meeting point. Thus, $a' \leq a$. 

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Proof of Claim 5.1. Let \( t \) denote the time at which the backjump occurred. Assume for sake of contradiction that \( a' \) was not available as a meeting point.

Observe that, as the communication between the parties progresses, an iteration-depth may be added, then removed, multiple times. Thus, we may consider the time \( t' \) when \( a' \) was last removed. Suppose this happened when the parties were at an iteration-depth \( c' \). By Lemma 4.1, we have \( c' = a' + 2^{j'+1} \), where \( j' \geq j \) is such that \( 2^{j'}|a' \) but \( 2^{j'+1} \not|a' \). In particular, this implies \( c' > c \). By the definition of \( c \), this must mean that we reached \( c' \) before the error at \( b \) occurred. That is, if \( t_b \) denotes the time at which the error at iteration-depth \( b \) occurred, \( t' < t_b \).

We also observe that, in the interval \([a', b]\), there are no meeting points available at time \( t' \). Indeed, write \( a' = k2^{j'} \) with \( k \) odd, so we have \( c' = a' + 2^{j'+1} = (k+2)2^{j'} \). Then the scale \( j' \) meeting point is clearly \((k+1)2^{j'} \). Using that \( c < (k+2)2^{j} \) we have \( \lfloor c \rfloor_{2^{-j}} - 2^{j-1} \leq (2k+3)2^{j-1} - 2^{j-1} = (2k+2)2^{j-1} = (k+1)2^{j} \), and so \( b < (k+1)2^{j} \), whence \( (k+1)2^{j'} \geq (k+1)2^{j} > b \).

The scale \( j'+1 \) meeting point is

\[
\lfloor (k+2)2^{j'} \rfloor_{2^{j'+1}} - 2^{j'+1} = (k+1)2^{j'} - 2^{j'+1} = (k-1)2^{j'} < a' .
\]

Thus, there can be no meeting points available in the range \([a', b]\) at time \( t' \). Note that this will continue to be true for any time after \( t' \) before the parties backjump to a point above \( b \).

Since an error occurred at iteration-depth \( b \) at time \( t_b > t' \), we may let \( \tilde{t} \) denote the first time the parties backjumped from a node below \( b \) to a node above \( b \) with \( t' < \tilde{t} < t_b \). Then we must have jumped above \( a' \), as there are no meeting points available in the range \([a', b]\). But then, as the pebble moves from the point to which the players backjumped to the iteration-depth \( b \) (where the error occurred), the meeting point must have been reintroduced at iteration-depth \( a' \). This contradicts the definition of \( t' \), and so the claim is proved.

The proof of the lemma now follows from some simple arithmetic. We must show \( b - a < 2(c - b) \), or, equivalently, \( 3b < 2c + a \). Write \( c = a + \Delta \), so \( \Delta \in [2^j, 2^{j+1} - 1] \).

If \( \Delta \in [2^j, 2^j + 2^{j-1} - 1] \), then

\[
\lfloor c \rfloor_{2^{-j}} - 2^{j-1} = a + 2^j - 2^{j-1} = a + 2^{j-1} ;
\]

therefore,

\[
3b < 3(a + 2^{j-1}) = 2(a + 2^j) + (a - 2^{j-1}) \leq 2c + a .
\]

If \( \Delta \in [2^j + 2^{j-1}, 2^{j+1} - 1] \), then

\[
\lfloor c \rfloor_{2^{-j}} - 2^{j-1} = a + 2^j + 2^{j-1} - 2^{j-1} = a + 2^j ;
\]

therefore,

\[
3b < 3(b + 2^j) = (2a + 2^j + 2^{j-1}) + a \leq 2c + a .
\]

\( \boxdot \)
We now show that (a) in any error-free iteration of the protocol that proceeds without error, the potential \( \Phi \) increases by at least 1, while (b) in any iteration of the protocol in which a transmission is corrupted, the potential \( \Phi \) decreases by at most 11. We use \( \Phi, \ell^+, \ell^-, R, L^- \) to denote the values before the iteration and \( \Phi', \ell'^+, \ell'^-, R', L'^- \) to denote the values after the iteration. For any variable, \( \Delta \) denotes the change in that variable in a given iteration, e.g., \( \Delta \Phi = \Phi' - \Phi \).

Suppose we have just experienced an error-free iteration, so we want to show \( \Delta \Phi \geq 1 \). We consider multiple cases.

- If \( \ell^- = 0 \), then both parties send a new symbol which progresses the pebble correctly, so the pebble moves two steps correctly. If the players do not backjump after this, \( \ell^+ \) increases by 1 and \( R \) remains unchanged, so \( \Delta \Phi = 1 \). If the players backjump after this, note that they can only backjump by iteration-distance 1 (as otherwise they would have backjumped more in the previous round), after which 1 is subtracted from the \( \text{cnt} \) variable. So \( \Delta \ell^+ = 0 \) while \( \Delta R = -1 \), which again yields \( \Delta \Phi = 1 \).

- Suppose \( \ell^- > 0 \). Since one of the parties will be aware that an error occurred, that party will transmit \( \rho \). Thus, \( R \) increases by at least 1. There are a few cases that we now distinguish:
  - If the parties do not backtrack after this round, then \( R \) increases by at least 1 and all other quantities remain unchanged, so \( \Delta \Phi \geq 1 \).
  - Suppose the parties backtracking, \( \ell'^- > 0 \). Let \( \ell \) denote the iteration-distance of the jump. Then both \( \ell^- \) and \( R \) decrease by \( \ell \), although recall that \( R \) also increased by at least 1. Since \( \ell^+ \) and \( L^- \) are unchanged, we conclude \( \Delta \Phi \geq \ell - (\ell - 1) = 1 \).
  - Suppose the parties backtracking we now have \( \ell'^- = 0 \), and let \( \Delta \ell^- = \ell \). Note that the definition of \( \Phi \) goes from the top definition to the bottom definition. Let \( D \) denote the difference between the iteration-depth of the point that we backjumped to and the iteration-depth of the error point, so then \( \Delta \ell^+ = -D \). Lastly, note that prior to this iteration we had \( \ell + D - 2 \leq R < \ell + D \), as otherwise we would have backjumped to this meeting point in the previous iteration, and we also require that after this iteration (in which at most 2 rewind symbols may be sent), the rewind counter is at least \( \ell + D \). Thus, the contribution of \( R \) to \( \Delta \Phi \) is at least \(- (\ell + D + 2) \) (where we recall that \( R \) contributes positively in the top definition of \( \Phi \) and negatively in the bottom).

Assume first \( D = 0 \), so then \( \ell \geq 1 \). Then \( \Delta \Phi \) is at least \( \ell + 10 \ell - (\ell + 2) = 10 \ell - 2 \geq 0 \).

Assume now \( D \geq 1 \). By Lemma 5.2, \( L^- > \frac{1}{2} D \). Hence,

\[
\Delta \Phi \geq -D + \ell + 10(0.5D) - (\ell + D + 2) = 3D - 2 \geq 1 .
\]

We now consider an iteration in which an error occurred. We wish to show that \( \Delta \Phi \) is lower bounded by some (negative) constant. We again distinguish several cases.

- Suppose that prior to this round, we had \( \ell^- = 0 \).
  - If both of the symbols sent by the parties are not rewind symbols, then the pebble moves two steps. If the erroneous transmissions cause the parties to diverge from the correct path (so the
\[
\Delta \Phi \geq -(1 + 10) = -11.
\]

\section{Generalization to Channels Without Feedback}

We now turn our attention to the general model of an adversarial channel without feedback. Our protocol is very much inspired by [Hae14], and in particular a comment made at the end of Section 7. However, since the parties will be storing only a small number of meeting points, in the analysis we will need to make use of the \( L^- \) parameter introduced in the previous section.

\subsection{Hash Functions and Robust Randomness Exchange}

We begin by defining the two hash function families that our protocols will use. The first hash function is constructed from the \( \epsilon \)-biased probability spaces of [NN93] which have the following guarantee. We remark that we only use it in Algorithm 4.

\begin{lemma} (from [NN93]). \label{lem:hash}
For any \( n \), any alphabet \( \Sigma \), and any probability \( p \in (0,1) \), there exists \( s = \Theta(\log(n \log |\Sigma|)) + \log \frac{1}{p} \), \( o = \Theta(\log \frac{1}{p}) \), and a function \( h \) which, given an \( s \)-bit uniformly random seed \( S \) maps any string over \( \Sigma \) of length at most \( n \) into an \( o \)-bit output, such that the collision probability of any two \( n \)-symbol strings over \( \Sigma \) is at most \( p \).
\end{lemma}

This next hash family uses inner products which lend them to a simple analysis if the parties are forced to use \( \delta \)-biased random strings rather than uniformly random strings. This hash family is used in both Algorithms 3 and 4.
**Definition 6.2 (Inner Product Hash Function).** For an input length $L$ and an output length $o$, we define the inner product hash function $h_S(\cdot)$ as follows: For a given binary seed $S$ of length at least $2oL$ and a binary input string $X$ of length $l \leq L$, $h_S(X)$ outputs the $o$ inner products $(\langle \tilde{X}, S[i \cdot 2L + 1, i \cdot 2L + \tilde{l}] \rangle)_{i=0}^{o-1}$, where $\tilde{X} = (X, |X|)$ denotes the string $X$ concatenated with its length and $\tilde{l} = |\tilde{X}| \leq 2L$ denotes the length of $\tilde{X}$.

One can easily see that this hash family has collision probability exponentially small in its output length (assuming a sufficiently large uniformly random seed is used). Moreover, by linearity, replacing the seed by a $\delta$-biased seed does not significantly affect the collision probability. More precisely, we have the following guarantee:

**Lemma 6.3.** [Lemma 6.3 in [Hae14]] Consider $n$ pairs of binary strings $(X_1, Y_1), \ldots, (X_n, Y_n)$ where each string is of length at most $L$, and suppose $h$ is the inner product hash function for input length $L$ and any output length $o$. Suppose furthermore that $S$ is a random seed string of length at least $n \cdot 2oL$ which is sampled independently of the $X$ and $Y$ inputs and is cut into $n$ strings $S_1, \ldots, S_n$. Then the output distribution $(x_1, \ldots, x_n) = (h_{S_1}(X_1) - h_{S_1}(Y_1), \ldots, h_{S_n}(X_n) - h_{S_n}(Y_n))$ for a $S$ sampled from a $\delta$-biased distribution is $\delta$-statistically close to the output distribution for a uniformly sampled $S$, in which case the $x_i$ are independent and equal to the 0 string with probability $2^{-o}$.

### 6.2 Exchanging Randomness with an Adaptive Adversary

One of the main challenges posed by an adaptive adversary, as opposed to an oblivious adversary, is that the parties cannot trivially synchronize their private randomness by sending it across the channel. Indeed, if that occurs at, say, the start of the protocol, then the adversary will simply read the randomness the players are using, and then can use his corruptions to create hash collisions, etc. In [Hae14], this problem was overcome by having the players send a large amount of randomness across the channel protected by an error-correcting code (ECC), which the players then stretch (deterministically) to an even longer low-bias pseudorandom string. Given the quality of the error-correcting code, the adversary does not have a sufficiently large budget of errors to corrupt the random string, and also the large amount of (pseudo)randomness at the players’ disposal allows one to argue that, except with exponentially small probability, the number of hash collisions is on the order of the number of corruptions the adversary may introduce (i.e., $\Theta(d\varepsilon)$).

However, this strategy will not work for our purposes, as the players will be forced to store a long random string, violating our desired memory bound. Thus, we will be forced to break up the protocols into blocks. Each block will have roughly $\log d$ iteration, and prior to this iteration, the parties exchange roughly $\log d$ bits of randomness with an ECC, which is then stretched into a low-bias pseudorandom string. The protocol for this exchange is given in Algorithm 2. If the Robust Randomness Exchange protocol is successful at the start of a block, then we say that the parties have uncorrupted randomness.

### 6.3 Main Theorem

As stated previously, our algorithms are very much akin to the algorithms given in [Hae14]. The algorithm proceeds in iterations of $r$ rounds, where $r$ is chosen to be $\Theta(1/\sqrt{\varepsilon})$ in the case of a non-adaptive adversary,
Algorithm 2 Randomness Exchange($\ell, \delta$)

1: $\ell' \leftarrow \Theta(\log(1/\delta) + \log \ell)$
2: $C : \{0, 1\}^{\ell'} \rightarrow \{0, 1\}^{\Theta(\ell + I_{\text{total}})}$ is an error correcting code with distance $4I_{\text{total}}$

3: if Alice then
4: $R' \leftarrow$ uniform bit string of length $\ell'$.
5: Send $C(R')$ to Bob.
6: else if Bob then
7: Receive $C'$ from Alice.
8: $R' \leftarrow$ decoding of $C'$.
9: $R \leftarrow \delta$-biased pseudorandom string of length $\ell$ derived from $R'$.

and $\Theta\left(\sqrt{\log \log \frac{1}{\varepsilon}}\right)$ in the case of an adaptive adversary. In each iteration, the parties simulate the original protocol by moving a pebble along the protocol DAG for $r$ steps, assuming that they are convinced that they are synchronized and on the correct path (where the correct path is defined as before, i.e., the path that would be traced out by the pebble in a noise-free execution of the protocol). To determine if they are synchronized and on the correct path, the parties exchange hashes. If they are convinced that something has gone wrong, they will send “dummy symbols” (e.g., they will send a fixed $\sigma_0 \in \Sigma$ for $r$ steps).

We still use meeting points as in Section 5, although we slightly redefine iteration-depth to account for the fact that iterations now consist of $r$ rounds. For a node $u$ whose depth is a multiple of $r$, the iteration-depth of $u$ is defined to be $\text{depth}(u)/r$, and we denote it $\text{iteration-depth}(u)$. Then, if $a$ denotes the iteration-depth of the current node, the meeting point set $\mathcal{M}$ will be a subset of

$$\mathcal{M} \subseteq \{(u, b) : \exists j \geq 0 \text{ s.t. } b = \lfloor a \rfloor 2^j - j \text{ and } \text{iteration-depth}(u) = b\}.$$ 

We remark that maintaining this set requires at most $O(\log d \log s)$ bits to store: $\log s$ bits to describe $u$ and $\log d$ bits for $b$. When we write “update $\mathcal{M}$”, we mean that the players should adjust their meeting point sets so as to remember only those meeting points which are at some scale $j \geq 0$.

Next, every block will consist of $I_{\text{total}}$ iterations. At the start of every block the players exchange randomness using Algorithm 2. We will choose $I_{\text{total}} = \Theta(\log d)$, as then it will suffice for the players to store $O(\log d \log s)$ bits of randomness ($O(\log s)$ bits are needed to indicate the vertex of the DAG, and we need this much randomness for the hash families from Definition 6.2), which fits within their budget.

We are now ready to state our main theorems (cf. Theorem 1.1).

**Theorem 6.4.** Let $\Pi$ be a depth $d$ protocol of size $s$ over the alphabet $\Sigma$. Then Algorithm 3 is a $O(\log s \log d)$ space-bounded randomized coding scheme which robustly simulates $\Pi$ with failure probability $2^{-\Theta(d\varepsilon)}$ over any oblivious adversarial error channel with alphabet $\Sigma$ and error rate $\varepsilon$. The simulation uses $d(1 + \Theta(\sqrt{\varepsilon}))$ steps and therefore achieves a communication rate of $1 - \Theta(\sqrt{\varepsilon})$. 

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Theorem 6.5. Let $\Pi$ be a depth $d$ protocol of size $s$ over the alphabet $\Sigma$. Then Algorithm [4] is a $O(\log s \log d)$ space-bounded randomized coding scheme which robustly simulates $\Pi$ with failure probability $2^{-\Theta(d\varepsilon)}$ over any fully adaptive adversarial error channel with alphabet $\Sigma$ and error rate $\varepsilon$. The simulation uses $d(1 + \Theta(\sqrt{\varepsilon \log \log \frac{1}{\varepsilon}}))$ steps and therefore achieves a communication rate of $1 - \Theta(\sqrt{\varepsilon \log \log \frac{1}{\varepsilon}})$.

Potential Much of our analysis follows the analysis given in [Hae14]; in particular, its proof of Theorems 7.1 and 7.2. We first define quantities that the potential is a function of. The potential will depend on the variables $k, v, E$ of both parties. We use the subscript $A$ (respectively, $B$) to refer to the value of the variable for Alice (respectively, Bob). Where appropriate, we also use the subscript $AB$ to denote the sum of the variables, e.g., $k_{AB} = k_B + k_B$.

We use $\ell^+$ and $\ell^-$ to denote the number of iterations of agreement between the paths traced out by Alice and Bob. That is, if $P_A$ and $P_B$ denote the paths traced out by Alice and Bob, respectively:

$$
\ell^+ = \min\left\{ \ell : \exists v_A \in P_A, v_B \in P_B \text{ s.t. } \text{depth}(v_A) = \text{depth}(v_B) = \ell \right\};
$$

$$
\ell^- = \frac{|P_A| + |P_B|}{r} - 2\ell^+.
$$

We also use the variable $L^-$ to again denote the maximum value that $\ell^-$ has attained since the last time $\ell^-$ was 0.

Lastly, for sake of analysis (these quantities are not known to either player) we define the two variables $BVC_A$ and $BVC_B$ which are a “bad vote count” for Alice and Bob. The variables count the contribution of hash collisions and corruptions to $v_1$ and $v_2$. In any iteration in which the $v_1$ of either party increases even though $MP_1$ is either unavailable or does not match either $MP_1$ or $MP_2$ of the either party, we increase both $BVC_A$ and $BVC_B$. Analogously, if $v_2$ increases for either party although $MP_2$ is either unavailable or not equal to either meeting point of the other player, we increment both $BVC$ values. Conversely, if a meeting point is available and matches either of the other player’s meeting points but the corresponding vote does not increase (say, due to a corruption), we call this an uncounted vote and increment both players’ $BVC$ values. Finally, during every reset status (Lines 26, 29 or 32), we reset the $BVC$ of this player to 0.

We define, for constants $1 < C_1 < \cdots < C_6$, the potential $\Phi$ as follows:

$$
\Phi = \begin{cases} 
\ell^+ - C_3\ell^- - C_2L^- + C_1k_{AB} - C_5E_{AB} - 2C_6BVC_{AB} & \text{if } k_A = k_B \\
\ell^+ - C_3\ell^- - C_2L^- - .9C_4k_{AB} + C_4E_{AB} - C_6BVC_{AB} & \text{if } k_A \neq k_B 
\end{cases}
$$

We remark that this potential function is obtained by subtracting $C_2L^-$ from the potential function used in the proof of Theorems 7.1 and 7.2 of [Hae14].

Generalization of Lemma 5.2 First, we need to argue that any decrease in $\ell^+$ due to a large backjump is compensated by a large decrease in $L^-$. This is the analogous result to Lemma 5.2.
Algorithm 3 Space-bounded robust simulation for oblivious adversarial channels with oracle $O$

1: $\Pi \leftarrow d$-round protocol to be simulated $+$ $\Theta(\sqrt{\varepsilon d})$ final confirmation steps
2: $h \leftarrow$ inner product hash family from Definition 6.2 with output length $o = \Theta(1)$ and input length $t = \Theta(\log s)$

3: Initialize parameters: $r_c \leftarrow \Theta(1)$, $r \leftarrow \lceil \sqrt{\varepsilon d} \rceil$; $I_{total} \leftarrow \lceil \log d \rceil$; $B_{total} \leftarrow \lceil d/I_{total} \rceil + \Theta(d\varepsilon)$

4: $M \leftarrow \{(\text{root}, 0)\}; v \leftarrow \text{root}$

5: unavailable $\leftarrow$ reserved string, e.g., the all-1s string

6: Reset Status: $k, E, v_1, v_2 \leftarrow 0$

7: for $B_{total}$ blocks do $\triangleright$ Verification Phase

8: $R \leftarrow \text{RandomnessExchange}(\ell = tI_{total}, \delta = 2^{-\Theta(I_{total})})$ $\triangleright$ Requires $\Theta(I_{total})$ rounds.

9: for $I_{total}$ iterations do

10: $k \leftarrow k + 1$

11: $\text{MP1} \leftarrow$ scale $k + 1$ meeting point if it’s available, else unavailable.

12: $\text{MP2} \leftarrow$ scale $k$ meeting point if it’s available, else unavailable.

13: $S \leftarrow t$ new random bits from $R$

14: $(H_k, H_v, H_{\text{MP1}}, H_{\text{MP2}}) \leftarrow (h_S(k), h_S(v), h_S(\text{MP1}), h_S(\text{MP2}))$

15: Send $(H_k, H_v, H_{\text{MP1}}, H_{\text{MP2}})$

16: Receive $(H'_k, H'_v, H'_{\text{MP1}}, H'_{\text{MP2}})$

17: if $H_k \neq H'_k$ then $E \leftarrow E + 1$

18: else

19: if $H_{\text{MP1}} \in \{H'_{\text{MP1}}, H'_{\text{MP2}}\}$ and MP1 unavailable then $v_1 \leftarrow v_1 + 1$

20: else if $H_{\text{MP2}} \in \{H'_{\text{MP1}}, H'_{\text{MP2}}\}$ and MP2 unavailable then $v_2 \leftarrow v_2 + 1$

21: if $k = 1$ and $H_v = H'_v$ and $E = 0$ then $\triangleright$ Computation Phase

22: Continue computation, using $O$, for $r$ steps

23: Reset Status: $k, E, v_1, v_2 \leftarrow 0$

24: else

25: Do $r$ dummy communications

26: if $2E \geq k$ then Reset Status: $k, E, v_1, v_2 \leftarrow 0$ $\triangleright$ Transition Phase

27: else if $k = \bar{k}$ and $v_1 \geq 0.4 \cdot \bar{k}$ and MP1 $\neq$ not_available then

28: Backjump to MP1

29: Reset Status: $k, E, v_1, v_2 \leftarrow 0$

30: else if $k = \bar{k}$ and $v_2 \geq 0.4 \cdot \bar{k}$ and MP2 $\neq$ not_available then

31: Backjump to MP2

32: Reset Status: $k, E, v_1, v_2 \leftarrow 0$

33: else if $k = \bar{k}$ then $v_1, v_2 \leftarrow 0$

34: Update $M$ (if necessary)

35: Output the current node
Lemma 6.6. Suppose that an error has occurred at iteration-depth $b$: that is, $b = \ell^+ + \ell^- > 0$. Let $a$ denote the iteration-depth of the deepest node both parties have in their meeting point sets that is above $b$. Then $a - b < 2L^-$. 

Proof. Let $c$ denote the iteration-depth of the deepest node reached by a player since the last time $\ell^- = 0$. We claim that $a \geq |c|_{2^j} - 2^j$, where $j$ is such that $|c|_{2^j} - 2^j \leq b < |c|_{2^{j - 1}} - 2^{j - 1}$. 

To see this, let $a' = |c|_{2^j} - 2^j$. Let $t$ denote the current iteration, and assume by way of contradiction that $a'$ is not available as a meeting point to one of the players. Without loss of generality, suppose this player is Alice.

Observe that, as the simulation between the parties progresses, a node may be added as a meeting point, and removed, multiple times. Thus, we may consider the last time that Alice went from storing the node at iteration-depth $a'$ to removing this node (the case for Bob is analogous). Suppose Alice was at

Algorithm 4 Space-bounded robust simulation for fully adversarial channels with oracle $O$

1: $\Pi \leftarrow d$-round protocol to be simulated $+$ $\Theta(\sqrt{\varepsilon}d)$ confirmation steps
2: $h_1 \leftarrow$ inner product hash family from Definition 6.2 with output length $o_1 = \Theta(\log^{1/\varepsilon}s)$ and input length $t_1 = \Theta(\log s)$
3: $h_2 \leftarrow$ hash family from Lemma 6.1 with collision probability $p_2 = 0.1$, output length $o_2 = \Theta(1)$, and seed length $t_2 = \Theta(\log \log 1/\varepsilon)$
4: Initialize parameters: $r_c \leftarrow \Theta(\log\log 1/\varepsilon)$; $r \leftarrow \lceil \sqrt{\frac{d}{\varepsilon}} \rceil$; $I_{total} \leftarrow \lceil \log d \rceil$; $B_{total} \leftarrow \lceil d/I_{total} \rceil + \Theta(d\varepsilon)$
5: $M \leftarrow \{(\text{root},0)\}$; $v \leftarrow \text{root}$ $\triangleright$ root denotes the root of the DAG; $v$ denotes the current node
6: unavailable $\leftarrow$ reserved string, say, the all 1s string.
7: Reset Status: $k, E, v_1, v_2 \leftarrow 0$
8: for $B_{total}$ blocks do
9: $R \leftarrow \text{RandomnessExchange}(\ell = t_1 I_{total}, \delta = 2^{-\Theta(I_{total})})$ $\triangleright$ Requires $\Theta(I_{total})$ rounds
10: for $I_{total}$ iterations do
11: $k \leftarrow k + 1$;
12: $\text{MP1} \leftarrow$ scale $k + 1$ meeting point if it's available, else unavailable.
13: $\text{MP2} \leftarrow$ scale $k$ meeting point if it's available, else unavailable.
14: $S_1 \leftarrow s_1$ new preshared random bits from $R$; $S_2 \leftarrow s_2$ fresh random bits
15: $h(\cdot) := h_{2, S_2}(h_{1, S_1}(\cdot))$
16: Send $(S_2, h(k), h(v), h(\text{MP1}), h(\text{MP2}))$; Receive $(S'_2, H'_k, H'_v, H'_{\text{MP1}}, H'_{\text{MP2}})$
17: $h'(\cdot) := h_{2, S'_2}(h_{1, S_1}(\cdot))$
18: $(H_k, H_v, H_{\text{MP1}}, H_{\text{MP2}}) = (h'(k), h'(v, h'(\text{MP1}), h'(\text{MP2}))$
19: Remaining code as in lines 16 to 34 in Algorithm 3.
iteration-depth $c'$, so $c' = a' + 2^j' + 1$, where $j' \geq j$ is such that $2^j'|a'$ but $2^{j'+1} \not\mid a'$. In particular, observe that $c' > c$. By the definition of $c$, this must mean that Alice reached $c'$ before the error at iteration-depth $b$ occurred. That is, if $t_b$ denotes the time at which the error occurred, $t' < t_b$.

Also, at time $t'$, no meeting points were available to Alice in the interval $[a', b]$. Indeed, write $a' = k2^j$ with $k$ odd, so we have $c' = a' + 2^j' + 1 = (k+2)2^j$. Then the scale $j'$ meeting point is $(k+1)2^j$. Using that $c < (k+2)2^j$ we have $[c]_{2j-1} - 2^j-1 \leq (2k+3)2^j-1 - 2^j-1 = (2k+2)2^j-1 = (k+1)2^j$, and so $b < (k+1)2^j$,

whence

$$ (k+1)2^j' \geq (k+1)2^j > b . $$

The scale $j' + 1$ meeting point is

$$ [(k+2)2^j]_{2^j'+1} - 2^j'+1 = (k+1)2^j - 2^j'+1 = (k-1)2^j' < a' . $$

Thus, there can be no meeting points available to Alice in the range $[a', b]$ at time $t'$. Since an error occurred at iteration-depth $b$ at time $t_b > t'$, we may let $\tilde{t}$ denote the first time Alice backjumped from a node below $b$ to a node above $b$, so then $t' < \tilde{t} < t_b$. Then Alice must have backjumped above $a'$ as there are no meeting points available to her in the range $[a', b]$. But then, the meeting point $a'$ must have been reintroduced. Since we assumed that $a'$ was not available to Alice at time $t$, it must have been removed at some time after $t'$, contradicting the definition of $t'$.

The proof of the lemma now follows from some simple arithmetic. We must show $b - a < 2(c - b)$, or, equivalently, $3b < 2c + a$. Write $c = a + \Delta$ with $\Delta \in [2^j, 2^{j+1} - 1]$.

If $\Delta \in [2^j, 2^j + 2^{j-1} - 1]$, then

$$ [c]_{2j-1} - 2^{j-1} = a + 2^j - 2^{j-1} = a + 2^{j-1} ; $$

therefore,

$$ 3b < 3(a + 2^{j-1}) = 2(a + 2^j) + (a - 2^{j-1}) \leq 2c + a . $$

If $\Delta \in [2^j + 2^{j-1}, 2^{j+1} - 1]$, then

$$ [c]_{2j-1} - 2^{j-1} = a + 2^j + 2^j - 2^{j-1} = a + 2^j ; $$

therefore,

$$ 3b < 3(b + 2^j) = (2a + 2^j + 2^{j-1}) + a \leq 2c + a . $$

\[ \Box \]

### Analysis of the potential function

**Lemma 6.7.** In every computation and verification phase the potential decreases at most by a fixed constant, regardless of the number of errors and hash collisions. Furthermore, in the absence of an error or hash collision, and assuming the players have uncorrupted randomness, the potential increases by at least one.
Proof. All quantities on which the potential depend change at most by a constant during any computation and verification phase. The maximum potential change is therefore at most a constant.

Now we consider the case that no error or hash collision happened. In this case, the $BVC_{AB}$ value does not change. Furthermore, computation only proceeds if $v_A = v_B$ so $\ell^+$ can only increase and $\ell^-, L^-$ are 0 before and after. Lastly, both $k_A$ and $k_B$ increase by one and if they are not equal $E_A$ and $E_B$ increase by one, too. In the first case the increase of $k_{AB}$ leads to a total potential increase of $2C_2 > 1$. In the latter case the increase of $k_{AB}$ and $E_{AB}$ leads to a total potential change of $2(-0.9C_4 + C_4)$ which is at least one for sufficiently large $C_4$. The potential therefore increases by at least one in the computation and update phase when no error or hash collision happens, assuming the players’ randomness is not corrupted. □

Lemma 6.8. In every iteration the potential decreases by at most a fixed constant, regardless of errors and hash collisions. Furthermore, in the absence of an error or hash collision and assuming the players have uncorrupted randomness the potential increases by at least 1.

Proof. Given Lemma 6.7 it suffices to show that a transition phase never decreases the potential. We show exactly this, except for one case, in which the potential decreases by a small constant. In case of the iteration being error and hash collision free this constant is shown to be less than the increase of the potential.

If a transition occurs at line 27, we call it an error transition. If a transition occurs at line 29 or 32, we call it a meeting point transition. As before, we use $'$ to denote the value of a variable after the round, and use $\Delta$ to denote the change in the variable.

We proceed by case analysis, considering which transition(s) occurred and whether or not the parties’ $k$ parameters were synchronized.

- Suppose $k_A \neq k_B$ and exactly one error meeting point transition occurred. Then the disagreement in the $k$ parameter remains, that is, $k'_A \neq k'_B$. Assume without loss of generality that Alice does the transition. Since $BVC_{AB}$ never increases in an error and collision free iteration, i.e., $\Delta BVC_{AB} \leq 0$, any change in $BVC_{AB}$ can only increase $\Phi$. Moreover, since error counts increase by at most 1 per iteration and after any iteration $2E < k$ holds we have $E_A \leq k_A/2 + 0.5$. Note that all the remaining terms measured in the potential change by at most $k_A$. We therefore obtain a potential change of at least $(0.9C_4 - 0.5C_4 - C_3 - C_2 - 1)k_A - 0.5C_4$. If $k_A \geq 2$ then this is at least 1 for sufficiently large $C_4$. If $k_A = 1$ then this quantity could be negative. However, in this case Alice reset her status in the preceding iteration and is in the same position except possibly a one block shorter transcript at the end of this iteration while Bob increased $E_B$ and $k_B$. This leads to an overall potential change of at least $0.1C_4 - C_3 - C_2 - 1$ which is at least 1 for sufficiently large $C_4$.

- Suppose $k_A \neq k_B$ and any two transitions occurred. Then $k'_A = k'_B = BVC'_A = BVC'_B = 0$ which makes the potential exactly $\ell^+ - C_3\ell^- - C_2L^-$ after the transition. The contribution from $BVC_{AB}$ can thus only increase the potential so it suffices to show that the contributions of $\Delta k_{AB}$ and $\Delta E_{AB}$ are positive. As in the previous case $E_A \leq k_A/2 + 0.5$ and $E_B \leq k_B/2 + 0.5$ and thus $E_{AB} \leq 0.5k_{AB} + 1$. We therefore have a potential increase of at least $0.9C_4 k_{AB} - C_4 E_{AB} \geq C_4(0.9k_{AB} - (0.5k_{AB} + 1)) = C_4(0.4k_{AB} - 1) \geq 1$, where the last inequality holds for $C_4$ sufficiently large since $k_{AB} \geq 3$. 

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• Suppose \( k_A = k_B \) and at least one error transition occurred. Then the potential change is dominated by the reduction or re-weighting of the \( BVC_{AB} \) count or by the reduction in the \( E \) variable(s) of the transitioning party(ies). In particular, the \( BVC \) of a party does not increase and is never weighted higher before the transition than after. Any \( BVC \) change therefore only affects the potential positively. The \( E \) count of the party with the error transition on the other hand is at least \( 0.5k \) which leads to a potential change of at least \( C_5 \cdot 0.5k \). All other quantities influencing the potential are changing at most by \( 3k \) while being associated with smaller constants. This guarantees an overall potential change of at least one for a sufficiently large \( C_6 \).

• If \( k_A = k_B \) and one meeting point transition occurred alone then both parties had the same \( k \) count for the last \( k \) rounds and either both or none of the parties should have transitioned. This guarantees that the \( BVC \)-count of the transitioning party is at least \( k/4 \) which guarantees that the overall potential change is dominated by the \( C_6k/4 \) increase due to the new \( BVC \) count.

• Finally, suppose \( k_A = k_B \) and two meeting point transitions occurred. If \( \ell^- \neq 0 \) then both parties should not have transitioned, so this implies that there must have been many bad votes: \( BVC_{AB} \geq 0.4\ell^- \). Now, if \( \ell^- = 0 \) but \( k_A = k_B \geq 4L^- \) then applying Lemma 6.6 tells us that since there was an available meeting point at most \( 2L^- + \ell^- \leq 3L^- \) round-depths shallower, we should have transitioned to this meeting point during the previous \( L^- \) rounds. So again we find \( BVC_{AB} \geq 0.4L^- \geq 0.4\ell^- \). Thus, the change in the \( BVC \) count dominates any other potential changes, i.e., for sufficiently large \( C_6 \), \( \Delta \Phi \geq 1 \). Finally, if \( \ell^- = 0 \) and \( k_A = k_B \leq 4L^- \) then the total potential change is at least \( C_2L^- - (C_1 + 1)k_{AB} \) which is at least 1 assuming \( C_2 \) is sufficiently large compared to \( C_1 \).

This completes the proof of the lemma.

Analysis of the Hash Collisions

Our next goal is to argue that there will not be too many hash collisions. In fact, the number of hash collisions will (with high probability) be on the order of the number of corruptions the adversary may introduce, i.e., \( \Theta(d\varepsilon) \).

Our first step is to bound the number of blocks for which the adversary can corrupt the randomness.

**Lemma 6.9.** The number of blocks for which the players have distinct (pseudo)randomness strings is at most \( O(d\varepsilon/I_{total}) \). Thus there can be at most \( O(d\varepsilon) \) iterations in which the parties have corrupted randomness.

**Proof.** Since the error correcting code we use has distance \( \Theta(I_{total}) \), this is the number of errors the adversary must introduce to corrupt the players’ shared randomness. Since he has a total budget of \( \varepsilon d \) errors, he can corrupt at most \( \varepsilon d/\Theta(I_{total}) = O(d\varepsilon/I_{total}) \) many blocks.

We now consider the hash collisions which arise, even if the players have uncorrupted randomness. In order to bound the number of hash collisions, we first prove that the potential cannot grow too quickly.

**Lemma 6.10.** After \( t \) iterations, it holds that \( \Phi(t) \leq t + 20C_1d\varepsilon \).

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Proof. Since \( \ell^+ \) increases by at most 1 in each iteration, we have \( \ell^+ \leq T \). Furthermore, it always holds that \( 2E < k \) which implies \(-0.9C_4k_{AB} + C_4E_{AB} \leq -4C_4k_{AB} \leq 0 \). Hence, if \( k_A \neq k_B \), then we see that \( \Phi(t) \leq t \).

So suppose now that \( k_A = k_B \) and that \( \Phi(t) = t + x \) with \( x \geq 1 \). By inspecting \( \Phi \), we see that we need \( k_{AB} \) larger than \( L^- \); however for this to be the case, we argue that there must have been many corruptions introduced by the adversary. More precisely, if \( \Phi(t) = t + x \) then it must be that \( x \leq C_1k_{AB} - C_2L^- \) which implies \( x \leq C_1(k_{AB} - 2L^-) \) assuming \( C_2 \geq 2C_1 \). However, the only way for \( k_{AB} \) to be larger than \( 2L^- \) is if in the last \( k_{AB} - 2L^- \) rounds, at least 10\% of the votes were corrupted to appear non-matching. Since these errors cannot be caused by a hash collision, i.e., they must be introduced by the adversary, we see that \( .1(k_{AB} - 2L^-) \leq 2d\varepsilon \). This shows \( x \leq C_1(k_{AB} - 2L^-) \leq 20C_1d\varepsilon \), which completes the proof.

This allows us to show that the number of errors in Algorithm 3 is small with high probability.

Lemma 6.11. For any protocol \( \Pi \) and any oblivious adversary, the number of iterations with uncorrupted randomness suffering a hash collision in Algorithm 3 is at most \( O(d\varepsilon) \), with probability \( 1 - 2^{-\Theta(d\varepsilon)} \).

Proof. We call an iteration dangerous if prior to the iteration either \( \ell^- > 0 \) or \( k_{AB} > 0 \). Note that hash collisions may only occur during dangerous iterations (hence their name). Let \( D \) be the number of dangerous iterations and \( H \) the number of hash collisions. The idea is the following: we would like to show that if \( H \) is significantly larger than \( d\varepsilon \) then the fraction of dangerous iterations with a hash collision, \( \frac{H}{D} \), is significantly larger than the expected fraction of hash collisions which is \( p = 2^{-o} \), where the output length \( o \) will be chosen to be a sufficiently large constant. Recall that in Algorithm 3 we have set \( \log \frac{1}{2} = \Theta(I_{total}) = \Theta(\log d) \). By Lemma 6.3 in \[\text{HaeL1}\], by choosing the constant in the \( \Theta(\cdot) \) sufficiently large all \( I_{total} = \Theta(\log d) \) hashing steps in each block are \( \delta \)-statistically close to being fully independent. Hence, within every block with uncorrupted randomness, the hash collisions are statistically close to being distributed as Bernoulli(\( p \)) random variables.

Let \( B \) denote the number of blocks with uncorrupted randomness. We fix a particular distribution of the \( D \) dangerous iterations to the blocks and the \( H \) hash collisions to the \( D \) dangerous iterations. We show that this particular arrangement of hash collisions and dangerous iterations occurs with probability \( 2^{-\Theta(d\varepsilon)} \), assuming \( H, D \) and the ratio \( \frac{H}{D} \) are sufficiently large.

By Lemma 6.8 we have a constant \( C^- \) such that in every iteration with an error the potential decreases by at most \( C^- \), while in every error-free iteration the potential increases by at least 1. The total potential change which occurs in dangerous iterations is therefore at least \( (D - H - 2d\varepsilon) - C(H - 4d\varepsilon) \) while the potential accumulated during non-dangerous iterations is at least \( R_{total} - C(2d\varepsilon) \), so the total potential is at least \( R_{total} + (D - H + 2d\varepsilon) - C(H + 4d\varepsilon) \). However, Lemma 6.10 tells us that the total potential is at most \( R_{total} + 20C_1d\varepsilon \), and so \( (D - H) - CH = O(d\varepsilon) \) from whence it follows that \( D \leq \frac{C + 1}{H}H + O(d\varepsilon) \). From this, we conclude that if \( D = \Theta(d\varepsilon) \) is sufficiently large then \( H \) must be at least \( \frac{1}{2(C+1)}D \). However, if we choose \( o \) large enough so that \( p = 2^{-o} < \frac{1}{4(C+1)} \), the probability that we have this many hash collisions is \( 2^{-\Theta(d\varepsilon)} \).

We may now take a union bound over all possible distributions of (a) dangerous rounds amongst the blocks (of which there are at most \( \binom{B+D}{B} = 2^o(d) \)) and (b) distributions of hash collisions amongst
the dangerous rounds (of which there are at most \((D + H)\frac{1}{H} \leq 2^{O(d\varepsilon)}\)) and, by choosing the output length \(o\) sufficiently small, the probability that \(H\) and \(D\) both exceed \(O(d\varepsilon)\) is at most \(2^{-\Theta(d\varepsilon)}\), as claimed. Therefore \(H = O(d\varepsilon)\) with probability \(1 - 2^{-\Theta(d\varepsilon)}\).

Next, to deal with the case of fully adversarial channels, we show that the hash function \(h_1\) in Algorithm 3 also causes at most \(O(d\varepsilon)\) hash collisions.

**Lemma 6.12.** For any protocol \(\Pi\) and any fully adversarial channel the number of iterations in Algorithm 3 with a hash collision from \(h_1\) is at most \(O(d\varepsilon)\) with probability \(1 - \varepsilon^{\Theta(d\varepsilon)}\).

*Proof.* First, fix any oblivious adversarial strategy, and repeat the proof of Lemma 3.11, replacing \(h\) by \(h_1\) and \(o\) by \(o_1 = \Theta(\log \frac{1}{\varepsilon})\). The argument still applies, and shows that the probability of having \(\Theta(d\varepsilon)\) hash collisions in \(\Theta(d\varepsilon)\) dangerous iterations is at most \(\varepsilon^{\Theta(d\varepsilon)}\). Since the number of oblivious strategies which choose \(2d\varepsilon\) rounds to corrupt out of at most \(2d\varepsilon\) is \((\frac{2d}{2d\varepsilon})^{2d\varepsilon} < \left(\frac{4}{\pi}\right)^{2d\varepsilon}\), we may take a union bound over all possible adaptive adversaries. This extends the proof to handle adversarial channels.

From here, we turn to understanding the number of hash collisions caused by \(h_2\). This function only maps the \(o_1 = \Theta(\log \frac{1}{\varepsilon})\) long output of \(h_1\) into a \(o_2 = \Theta(1)\) length output, using seeds of length \(\Theta(\log \log \frac{1}{\varepsilon})\).

We remark that here the seeds are truly random, i.e., the seeds are not biased.

**Lemma 6.13.** For any protocol \(\Pi\) and any fully adversarial channel the number of iterations in Algorithm 4 with a hash collision from \(h_2\) is at most \(O(d\varepsilon)\) with probability \(1 - 2^{-\Theta(d\varepsilon)}\).

*Proof.* This time, we call an iteration dangerous if the hash values from both parties after applying \(h_1\) do not agree. Clearly, hash collisions from \(h_2\) can only occur during dangerous iterations. Let \(D\) denote the number of dangerous iterations and \(H\) the number of iterations with a hash collision from \(h_2\).

By Lemma 6.9 and Lemma 6.12 the number of errors or \(h_1\) hash collisions (which may be due to bad randomness) is at most \(O(d\varepsilon)\) with probability \(1 - \varepsilon^{-\Theta(d\varepsilon)}\). Repeating the argument of Lemma 6.11 shows that \(D \leq (C + 1)H + \Theta(d\varepsilon)\), so if \(H = \Theta(d\varepsilon)\) is sufficiently large then the number of rounds with a hash collision from \(h_2\) is at least \(\frac{1}{2(C + 1)}d\). However, just as in Lemma 6.11 if we choose \(o_2 = \Theta(1)\) large enough so that \(p_2 = 2^{-o_2} < \frac{1}{4(C + 1)}\), this occurs with probability \(2^{-\Theta(d\varepsilon)}\). Indeed, since the random \(h_2\) seeds are sampled afresh and independently at the beginning of each round the hash collisions are dominated by i.i.d. Bernoulli\((p_2)\) variables, and an application of the Chernoff bound tells us that the probability of a sum of Bernoulli\((p_2)\) deviating by a multiplicative constant from its expectation is at most \(2^{-\Theta(d\varepsilon)}\).

Given the \(O(d\varepsilon)\) bounds on the total number of hash collisions in both of the protocols, we may prove this section’s main theorems.

*Proof of Theorem 6.4 and Theorem 6.5.* That the protocol uses \(O(\log d \log s)\) space follows from earlier observations. We first show that the protocols correctly simulate \(\Pi\). Due to Lemma 6.9, Lemma 6.11, Lemma 6.12 and Lemma 6.13 in both Algorithm 3 and Algorithm 4 there are at most \(O(d\varepsilon)\) rounds with bad randomness, a hash collision or an error. Moreover, using Lemma 6.8 the total potential drop in these iterations is \(O(d\varepsilon)\), while the potential increases by at least 1 in the remaining \(B_{total}R_{total} - O(d\varepsilon)\)
iterations. Thus, for $B_{total} = \lceil d/I_{total} \rceil + \Theta(d\varepsilon)$ sufficiently large, we have $\Phi(R_{total}B_{total}) \geq \lceil d/r \rceil + 20C_1d\varepsilon$. By Lemma 6.10 this implies $\ell^+ \geq \lceil d/r \rceil$, which implies that the parties have traced out the correct path and will therefore output the correct leaf.

Within every block, the total round complexity is $\Theta(I_{total}) + I_{total}(r + r_c) = \Theta(I_{total}) + rI_{total}(1 + \frac{r_c}{r})$. Hence, recalling $B_{total}I_{total} = \lceil d/r \rceil + \Theta(d\varepsilon)$, the round complexity of the entire protocol is

$$B_{total} (\Theta(I_{total}) + I_{total}r (1 + \frac{r_c}{r})) = \Theta(d/r) + ((d/r) + \Theta(d\varepsilon))r (1 + \frac{r_c}{r})$$

$$= \Theta(d/r) + d(1 + \Theta(r\varepsilon))(1 + \frac{r_c}{r}) = d \left(1 + \Theta \left(\frac{r_c}{r} \right)\right).$$

Since we set $r = \lceil \sqrt{\frac{r_c}{\varepsilon}} \rceil$, the round complexity is $d(1 + \Theta(\sqrt{r_c\varepsilon}))$. In Algorithm 3 we set $r_c = \Theta(1)$ leading to a round complexity of $d(1 + \Theta(\sqrt{\varepsilon}))$, while in Algorithm 4 $r_c = \Theta(\log \log \frac{1}{\varepsilon})$ so the round complexity is $d(1 + \Theta(\sqrt{\varepsilon} \log \log \frac{1}{\varepsilon}))$. Thus the communication rate of Algorithm 3 is $1 - \Theta(\sqrt{\varepsilon})$ while the communication rate of Algorithm 4 is $1 - \Theta(\sqrt{\varepsilon} \log \log \frac{1}{\varepsilon})$. This completes the proof.

7 Applications to Robust Circuit Size

We mentioned in the introduction that Kalai et al. showed how to use interactive coding theory to design robust circuits [KLR12]. Unfortunately, their approach only applies to formulas. A natural question is therefore to determine if a similar approach could be used to design robust circuits, where we insist that the robust circuit be not too much larger than the original circuit. This implies, in particular, that we cannot take the robust simulation of a formula computing the same function as the original, as the formula may have size exponential in the original circuit. We show that interactive protocols of a very specific kind do indeed yield constructions of robust circuits. Unfortunately, the condition that we place upon the interactive communication protocol appears fairly unnatural; in particular, none of the protocols we have given satisfy the criterion. Informally, we require that how a party updates their memory contents after sending a symbol only depends on the received symbol, and not the sent symbol or their privately held input. We further develop the connection between interactive coding theory and robust circuits in Appendix A where we “reverse” the transformation given in [KLR12].

We begin by providing some preliminaries on robust circuits.

7.1 Robust Circuits

A circuit $C$ consists of a directed acyclic graph (DAG) with $n$ input nodes (nodes with fan-in 0) labeled $z_1, \ldots, z_n$ and one output node (a node with fan-out 0). All non-input nodes are labeled by one of $\land, \lor$ or $\lnot$, and we refer to such nodes as gates. The gates labeled $\land$ or $\lor$ have fan-in at least 2 while the gates labeled $\lnot$ have fan-in 1. The size of $C$ is the number of nodes in the underlying graph, while the depth of $C$ is the length of the longest path in $C$. If the underlying undirected graph happens to be a tree, the circuit is called a formula.

The output of $C$ on an input $x \in \{0, 1\}^n$, denoted $C(x)$, is computed in the natural way. Formally, on input $x$ each node $v$ is assigned a value, denoted $v(x)$, which is computed recursively as follows: $v(x) = x_i$ if
v is the ith input node, and otherwise v(x) is the value obtained by applying the logical operation labeling v to the values of the nodes incident to v. Then, C(x) is simply the value of the output node.

In this work, we will always assume that all negations are directly applied to the input nodes (which is no loss of generality due to De Morgan’s laws). We therefore think of circuits as having 2n input nodes labeled z₁, ..., zₙ, ẑ₁, ..., ẑₙ. Moreover, we will assume that our circuits are connected. For this to be no loss of generality, we will allow circuits that have less than 2n input nodes.

Given a circuit C, a short-circuit error replaces an ∧ or an ∨ gate by a dictatorship of one of the inputs. That is, it replaces the gate by some function g such that, if x = (x₁, ..., x_k) denotes the values of the gates incident to the original gate, g(x) = xᵢ for some i ∈ [k].

Let Vᵦ be the set of gates in C labeled by ∧ and let Vᵥ the set of gates labeled by ∨. For simplicity, we assume that each ∧ and ∨ gate has the same fan-in k – one can always add “dummy inputs” to make this so. An error string is some e = (eᵦ, eᵥ) ∈ ([k] ∪ {*})Vᵦ × ([k] ∪ {*})Vᵥ. Intuitively, given a gate gᵢ, if eᵢ = * then the gate behaves normally, while if eᵢ = i for some i ∈ [k] then the gate is replaced by a dictatorship of the ith input. If eᵢ ≠ *, we say that e has short-circuited the gate gᵢ.

Formally, let g be a gate, x an input, and e = (eᵦ, eᵥ) an error string. Let g₁, ..., gₖ denote the nodes incident to g. The value of the gate g on the input x with error-string e is defined to be

\[ g^e(x) = \begin{cases} g^e_i(x) & \text{if } e_i = i, \\ g^e_1(x) \land \cdots \land g^e_k(x) & \text{if } e_i = * \text{ and } g \text{ is an } \land \text{ gate,} \\ g^e_1(x) \lor \cdots \lor g^e_k(x) & \text{if } e_i = * \text{ and } g \text{ is an } \lor \text{ gate.} \end{cases} \]

The value of the ith input node zᵢ is defined to be xᵢ. We reserve e* = (eᵦ*, eᵥ*) to refer to the all-* error string. Given a circuit C and an error string e, we let Cₑ denote the circuit obtained by replacing each gate g in C by gₑ.

For kᵦ, kᵥ ≥ 0, we call e = (eᵦ, eᵥ) a (kᵦ, kᵥ) error string if, on every directed path P = g₁, ..., gₘ from a gate g₁ adjacent to an input node to the output gate gₘ,

\[ |\{i : g_i \in Vᵦ, e_i \neq *\}| \leq kᵦ \quad \text{and} \quad |\{i : g_i \in Vᵥ, e_i \neq *\}| \leq kᵥ. \]

That is, on every input-output gate path, at most kᵦ ∧-gates are corrupted and at most kᵥ ∨-gates are corrupted.

We now define what it means for a circuit to robustly compute a function.

**Definition 7.1** (Robust Circuit Computation). Fix a subset E ⊆ ([k] ∪ {*})Vᵦ × ([k] ∪ {*})Vᵥ. We say that a circuit C robustly simulates a function f : {0, 1}ⁿ → {0, 1} with respect to E if for any x ∈ {0, 1}ⁿ and any e ∈ E, Cₑ(x) = f(x).

When we say that a circuit robustly simulates another circuit, we mean that the circuit robustly computes the function computed by the other circuit.

For example, let ε > 0 and suppose that E denotes the set of all (εd, εd)-error strings, and d is the depth of C. Then, to say that a circuit robustly simulates a function with respect to E means that the circuit computes the function correctly even if, on any path from the output gate to an input node, a
fraction of at most $\varepsilon$ gates are short-circuited. In this case, we say that $C$ robustly simulates $f$ against an $\varepsilon$ fraction of errors. Also, for positive $k_A, k_B$, we say we will say that $C'$ $(k_A, k_B)$-robustly simulates $C$ if $C'$ robustly simulates $C$ with respect to the set of all $(k_A, k_B)$ error strings.

Remark. Given a circuit $C$, we often identify it with the DAG defining it. However, at certain points, we will want to explicitly study the graph theoretic nature of the DAG which underlies it, particularly when we want to transfer between circuits and communication protocols. Thus, when convenient, we will use a different symbol to refer to the underlying DAG, typically $G$.

7.2 Robust Simulation of Protocols, Take 2

In this section we provide a new definition for a robust simulation of a protocol which is applicable to robust circuits. In the remainder of this section, we think of a communication protocol as a pebble game played on a rooted DAG. As before, the pebble is placed on nodes which are owned by Alice or Bob, and is moved along edges according the players’ private inputs. When an error occurs, the pebble is moved from a node to a child different than the one the player intended.

The crucial ingredient that we add at this point is the requirement that the DAG, in a sense, “completely defines” the robust simulation. To formalize this, let $V_A$ denote the set of vertices owned by Alice and let $V_B$ the set of vertices owned by Bob. An error pattern is an element of the set $([k] \cup \{\ast\})^{V_A} \times ([k] \cup \{\ast\})^{V_B}$, where $[k]$ denotes the fan-out of the nodes in the DAG (which corresponds to the alphabet size of the protocol) and $\ast$ is some new symbol. Given an error pattern $e$, we typically write $e = (e_A, e_B)$ where $e_A$ denotes $e$ restricted to $V_A$ and $e_B$ denotes $e$ restricted to $V_B$. The error pattern $e$ defines an error as follows: when the pebble is placed on node $v$ and $v$’s children are enumerated $v_1, \ldots, v_k$, if the player owning $v$ wishes to move the pebble to $v_i$, the pebble moves to

$$
\begin{cases}
  v_j & \text{if } e_v = j \in [k] \\
  v_i & \text{if } e_v = \ast
\end{cases}
$$

Thus, if $e_v \in [k]$, we speak of the transmission as being corrupted, while if $e_v = \ast$ the transmission is unaffected. We reserve $e^\ast = (e_A^\ast, e_B^\ast)$ to refer to the all-\ast error string. For a protocol $\Pi$, we let $\Pi^e(x, y)$ denote the output of the protocol when Alice receives input $x$, Bob receives input $y$, and the error string $e$ is applied to the transmissions as above.

For $k_A, k_B \geq 0$, we call $e = (e_A, e_B)$ a $(k_A, k_B)$ error string if, on every directed path $P = v_1, \ldots, v_m$ from the root to a leaf,

$$
|\{i : g_i \in V_A, e_{g_i} \neq \ast\}| \leq k_A \quad \text{and} \quad |\{i : g_i \in V_B, e_{g_i} \neq \ast\}| \leq k_B .
$$

That is, on every root to leaf path, at most $k_A$ of Alice’s transmissions are corrupted and at most $k_B$ of Bob’s transmissions are corrupted.

We now provide a new definition of what it means for a protocol to be a robust simulation. When we refer to a robust simulation in this section and in the appendix, it is to this notion that we are referring.
Definition 7.2 (Robust Simulation of a Protocol, Take 2). Fix a subset \( E \subseteq ([k] \cup \{\ast\})^{V_A} \times ([k] \cup \{\ast\})^{V_B} \). We say that a protocol \( \Pi' \) robustly simulates a protocol \( \Pi \) with respect to \( E \) if for any inputs \((x, y)\) and error strings \( e \in E \), \( \Pi'(x, y) = \Pi(x, y) \).

For example, let \( \varepsilon > 0 \) and suppose that \( E \) denotes the set of all \((\varepsilon d, \varepsilon d)\)-error strings, where \( d \) is the depth of \( \Pi' \). Then, to say that \( \Pi' \) robustly simulates the protocol \( \Pi \) with respect to \( E \) means that, even if an adversary corrupts at most \( \varepsilon d \) of Alice's and Bob's transmissions, we still correctly simulate \( \Pi \). Also, for positive \( k_A, k_B \), we say we will say that \( \Pi' \) \((k_A, k_B)\) robustly simulates \( \Pi \) if \( \Pi' \) robustly simulates \( \Pi \) with respect to the set of all \((k_A, k_B)\) error strings.

Remark. Just as with circuits, given a protocol \( \Pi \), we often identify it with the DAG defining it. However, when we want to directly study the graph-theoretic structure of the DAG underlying \( \Pi \) so as to transfer between communication protocols and circuits, we will use a different symbol to refer to the DAG, typically \( G \).

Discussion of this definition of robust simulation Say we have an \( m \)-memory-bounded protocol \( \Pi' \) which is a robust simulation of the protocol \( \Pi \), where we think of the protocols as being defined by transition functions (cf. Section 2). A natural approach for obtaining a protocol DAG \( G' \) from \( \Pi' \) is the following. The vertex set is \( \{0, 1\}^m \times \{0, 1\}^m \), i.e., the set of all possible pairs of configurations for the 2 players. Let \( v \) denote a vertex (corresponding to a particular configuration), and suppose that it is Alice's turn to speak. Then, for each symbol \( i \in [k] \), we put an edge from \( v \) to \( u \) labeled by \( i \) if, when Bob receives symbol \( i \), the parties' update their memory contents to those corresponding to \( u \).

While this construction might appear promising, there is a subtle issue. If Bob receives symbol \( i \) but Alice intended to send symbol \( j \neq i \), Alice may wish to update her memory contents differently than if she had intended to send the symbol \( i \). For example, in Algorithm 1 she assigns the parameter \texttt{error} the value \texttt{true}. However, by the previous construction, Alice must update her memory contents the same way, regardless of the symbol she intended to send.

One may then wonder if the robust simulations from Section 6 would be sufficient for our purposes. However, the transition function in that protocol does depend on the input to the players. However, when we think of a robust simulation on a DAG, observe that where the pebble moves from a given node is only permitted to depend on the received symbol.

Thus, in order to be able to apply the robust Karchmer-Wigderson transformation Lemma \ref{lem:karchmer-wigderson} from the next section, we need to start with what we term an input-oblivious robust simulation.

Definition 7.3. [Oblivious Robust Simulation] A robust simulation \( \Pi' \) of a communication protocol \( \Pi \) is input-oblivious (or just oblivious) if the transition function does not depend on the input and, in the case of a feedback channel, does not depend on the sent symbol but only the received symbol.

7.3 Karchmer-Wigderson Transformation

The Karchmer-Wigderson game \cite{KW90} translates the problem of proving (depth) lower bounds on circuits into a communication complexity lower bound. Let \( f : \{0, 1\}^n \to \{0, 1\} \) be a (non-constant) Boolean function. In the Karchmer-Wigderson game (KW-game), Alice is given \( x \in f^{-1}(0) \) and Bob is given
$y \in f^{-1}(1)$. Note that, in particular, $x \neq y$, so $\exists i \in [n]$ such that $x_i \neq y_i$. The objective of the game is for Alice and Bob to agree upon an index $i$ such that $x_i \neq y_i$. If we restrict our attention to circuits in which all $\land$ and $\lor$ gates have fan-in 2, then we have the following theorem.

**Theorem 7.4.** The minimum depth of a circuit computing $f$ is equal to the minimum number of bits Alice and Bob need to communicate to solve the KW-game for $f$.

Implicit in [KW90] is the following result.

**Theorem 7.5.** Suppose we have a circuit $C$ computing the function $f$. Then there is a protocol DAG for the KW-game for $f$ obtained by replacing every $\land$-gate with a node where Alice speaks, every $\lor$-gate with a node where Bob speaks, and every input gate with a leaf.

Remark. The converse of Theorem 7.5, i.e., that a protocol DAG of a certain size and depth yields a circuit of the same size and depth, is also true. However, as we do not need the converse, we do not show it here.

An important corollary for our purposes is the following.

**Corollary 7.6.** Suppose we have a circuit $C$ computing the function $f$ of size $s$ and depth $d$. Then there is a protocol DAG for the KW-game for $f$ of size $s$ and depth $d$.

We prove Theorem 7.5 for completeness.

Proof. We proceed by induction on $d$. If $d = 0$, then the output gate of $C$ is an input node, either $z_i$ or $\bar{z}_i$, and $C$ consists of only this node. Thus, $i$ is always a solution to the KW-game for $f$, so Alice and Bob both output $i$. The protocol DAG can be taken to consist of a single leaf.

Suppose the theorem is true for circuits of depth $d - 1$. Consider the output gate of $C$: without loss of generality, assume it is an $\land$-gate. Then, we can write $C = C_1 \land C_2$ where the maximum depth of $C_1$ and $C_2$ is $d - 1$. Let $f_j$ denote the function computed by $C_j$ for $j = 1, 2$. Thus, $f = f_1 \land f_2$. By induction, we obtain protocol DAG $G_j$ for the KW-game for $f_j$ obtained as described in the theorem for $j = 1, 2$.

Construct the protocol $G$ for the KW-game for $f$ by adding a node where Alice speaks incident to the roots of the protocol DAGs $G_1$ and $G_2$. $G$ is therefore of the correct form, so it remains to show that $G$ solves the KW-game for $f$.

Let $x \in f^{-1}(0)$ denote Alice’s input and $y \in f^{-1}(1)$ denote Bob’s input. Since $f(x) = 0$, for some $j^* \in \{1, 2\}$, $f_{j^*}(x) = 0$. Since $f(y) = 1$, $f_1(y) = f_2(y) = 1$. Thus, $x \in f_{j^*}^{-1}(0)$ and $y \in f_{j^*}^{-1}(1)$, i.e. $(x, y)$ is a valid input pair for the KW-game for $f_{j^*}$. Thus, the protocol described by the DAG $G_{j^*}$ solves KW-game on this input pair, i.e., it will output an index $i$ for which $x_i \neq y_i$. Thus, on input $x$, Alice will determine which of $f_1(x)$ and $f_2(x)$ is 0, and move the pebble to the root node of the corresponding DAG $G_j$.

### 7.4 Robust Karchmer-Wigderson Transformation

We are now in a position to state our version of the robust Karchmer-Wigderson Transformation, which is very much in the spirit of Lemma 8 in [KLR12].
Lemma 7.7 (Robust Karchmer-Wigderson Transformation). Let $f : \{0,1\}^n \to \{0,1\}$ and let $\Pi$ be a protocol with associated DAG $G$. Let $V_A$ denote the nodes owned by Alice and $V_B$ the nodes owned by Bob. Let $T \subseteq f^{-1}(0) \times ([k] \cup \{\ast\})^{V_A}$ and $U \subseteq f^{-1}(1) \times ([k] \cup \{\ast\})^{V_B}$ be non-empty sets such that $\Pi$ solves the KW game on all inputs in $T \times U$. Assume moreover that every node that is a descendant of the root of $G$ can be reached using an input in $T \times U$. Then, there is a circuit $C$ obtained by replacing every node where Alice speaks with a $\land$ gate, every node where Bob speaks with a $\lor$ gate, and every leaf with a literal such that for every $(x,e^A) \in T$ and $(y,e^B) \in U$, $C(e^A,e^B^*) (x) = 0$ and $C(e^A^*,e^B^*) (y) = 1$.

Remark. Note that if $g$ is a gate, $x$ is an input and $(e^A,e^B)$ is any error pattern, then $g(x,e^A,e^B^*) = 0 \Rightarrow g(x,e^A,e^B) = 0$ and $g(x,e^A^*,e^B) = 1 \Rightarrow g(x,e^A,e^B) = 1$. Indeed, short-circuiting an $\lor$ gate can only change the value of the gate from 1 to 0, and since all negations are applied at the top level this can only change the value of the circuit from 1 to 0. The case of $\land$ gates is analogous. Hence, Lemma 7.7 actually shows that for any $(x,e^A) \in T$, $(y,e^B) \in U$ and any error strings $e^A^*,e^B^*$, we have $C(x,e^A^*) = 0$ and $C(y,e^A^*,e^B) = 1$.

Remark. For $k_A,k_B \geq 0$, let $E_{A,k_A}$ denote the set of all error strings corrupting at most $k_A$ of Alice's transmissions, and define $E_{B,k_B}$ similarly. When we apply Lemma 7.7 we will typically assume that $T = f^{-1}(0) \times E_{A,k_A}$ and $U = f^{-1}(1) \times E_{B,k_B}$. That is, $T \times U$ is the set of all inputs to the Karchmer-Wigderson game and all $(k_A,k_B)$-error strings.

Proof. The proof proceeds by induction on the depth of the communication protocol. If $\Pi$ has depth 0, there must exist an index $i$ for which $x_i \neq y_i$ for all $(x,e^A,y,e^B) \in T \times U$. Since $T \times U$ is a rectangle, this implies that $x_i = 0$ and $y_i = 1$ or $x_i = 1$ and $y_i = 0$ for all such $x$ and $y$. The literal $z_i$ or $\overline{z_i}$ satisfies the hypotheses in this case. (Note that the error strings do not come into play in this argument.)

For the inductive step, suppose (without loss of generality) that the root of $G$ is in $V_A$. Let $T_1,\ldots,T_k$ be the partition of $T$ induced by the first symbol sent across the channel. That is, $T_i$ is the set of pairs $(x,e^A)$ such that an $i$ is sent across the channel in the first round on this input. Since we have assumed that every vertex is reachable, all of these sets are non-empty. Let $G_i$ for $i \in [k]$ denote the subgraph induced by the nodes that the pebble can reach on an input in $T_i \times U$, excluding the root of $G$. The induction hypothesis implies that there are circuits $C_1,\ldots,C_k$ satisfying the lemma for $G_1,\ldots,G_k$ and $T_1 \times U,\ldots,T_k \times U$. Moreover, given the manner in which the circuit $C_i$ is obtained from $G_i$, $C_1,\ldots,C_k$ can be pasted together with an extra output $\land$ gate at the root of $G$, obtaining a circuit $C$ of the same size as $G$. That is, if a node $v$ appears in multiple $G_i$'s, then that node is transformed into the same type of gate (i.e., either $\land$ or $\lor$) in each $C_i$ which we may consistently label $g$, and if the pebble can move to the same child in $G_i$ and $G_j$, then $g$ will be connected to the same inputs in $C_i$ and $C_j$.

Let $(x,e^A),(y,e^B)$ be any elements of $T \times U$. Suppose that $j \in [k]$ is such that $(x,e^A) \in T_j$. The induction hypothesis implies that $C_j(e^A,e^B^*) (x) = 0$. Now, if $r$ denotes the root of $G$, observe that $e^A_i \in \{j,\ast\}$. If $e^A_i = j$, then $C_j(e^A,e^B^*) (x) = C_j(e^A^*,e^B^*) (x) = 0$, as required. Else, $C_r(e^A,e^B^*) (x) = C_1(e^A,e^B^*) (x) \land \cdots \land C_k(e^A,e^B^*) (x)$, which is still equal to 0 since $C_j(e^A,e^B^*) (x) = 0$. Finally, by induction we have $C_1(e^A^*,e^B^*) (y) = \cdots = C_k(e^A^*,e^B^*) (y) = 1$, so $C(e^A^*,e^B^*) (y) = C_1(e^A^*,e^B^*) (y) \land \cdots \land C_k(e^A^*,e^B^*) (y) = 1$. □
7.5 Putting it all Together: How to Obtain a Robust Circuit from an Oblivious Robust Simulation

We conclude this section by explaining how one can use an oblivious robust simulation of a communication protocol to obtain a construction of a robust circuit.

**Theorem 7.8.** Suppose that every alternating communication protocol with associated DAG of size $s$ and depth $d$ has an oblivious robust simulation over a feedback channel with error rate $\varepsilon > 0$ of size $f(s)$ and depth $g(d)$. Then, given any circuit $C$ of size $s$ and depth $d$, there is a circuit $C'$ with size $\leq f(2s)$ and depth $\leq g(2d)$ which robustly simulates $C$ against a $\frac{\varepsilon}{2}$ fraction of adversarially chosen short-circuit errors along any input to output gate path.

**Proof.** Given a circuit $C$ computing a boolean function $f$, we first apply Corollary 7.6 to obtain a protocol $\Pi$ for the KW-game for $f$ with protocol DAG $G$ of size $s$ and depth $d$. Next, we force the protocol to be alternating, which increases the depth and size by a factor of $\leq 2$. After this, we apply the assumption of the theorem, obtaining an oblivious robust simulation $\Pi'$ over a feedback channel with error rate $\varepsilon$ of size $\leq f(2s)$ and depth $\leq g(2d)$. Let $D \leq g(2d)$ denote the depth of $\Pi'$. Since $\Pi'$ can tolerate $\varepsilon D$ adversarially chosen errors, it can tolerate $\frac{\varepsilon}{2} D$ adversarially chosen errors when Alice speaks and $\frac{\varepsilon}{2} D$ adversarially chosen errors when Bob speaks. Using the error string notation, this implies that it is robust to any $(\frac{\varepsilon}{2} D, \frac{\varepsilon}{2} D)$-error string. We then apply Lemma 7.7 to obtain a circuit $C'$ from $\Pi'$ with size $\leq f(2s)$ and depth $\leq g(2d)$ which is robust to any $(\frac{\varepsilon}{2} D, \frac{\varepsilon}{2} D)$ error string. This circuit can therefore tolerate a $\frac{\varepsilon}{2}$ fraction of adversarially chosen short-circuit errors along any input to output gate path.

**Conclusions** Suppose that our protocols from Sections 5 and 6 were oblivious in the sense of Definition 7.3. Then, by appealing to Theorem 7.8 and using the construction of the protocol DAG outlined in Section 7.2, we obtain the following: given any circuit $C$ of size $s$ and depth $d$, for some constant $\varepsilon > 0$, there exists a circuit $C'$ of size $2^{O(\log d \log s)}$ and depth $O(d)$ that robustly simulates $C$ against an $\varepsilon$ fraction of errors. Thus, we would obtain a quasipolynomial blow-up in the size of the circuit. If we were to appeal to [KLR12], then just converting the circuit into an equivalent formula already results in an exponential blow-up.

**References**

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3We implicitly assume here that $f$ and $g$ are non-decreasing functions.
A Reverse Robust Karchmer-Wigderson Transformation

In this section, we show that robust formulas yield robust protocols for a KW-game in a natural way. This provides the converse to Lemma 7.7 for formulas. Unfortunately, we show in Appendix A.2 that one cannot in general obtain a robust protocol from a robust circuit by providing a simple robust circuit that cannot be the protocol DAG of any communication protocol tolerating even 1 corruption from the channel.

A.1 Proof of the Reverse Transformation for Formulas

We begin by describing the intuition of the protocol; see Algorithm 5 for a formal description. Just as in the original (non-robust) KW-transformation [KW90], Alice will speak at $\land$-gates, Bob at $\lor$-gates, and when the pebble arrives at an input gate the parties will output $i$ such that the input gate is labeled by
Lemma A.1. Suppose we have a formula $C'$ which $(k_A,k_B)$-robustly simulates a circuit $C$ computing a boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$. Then there is a protocol $\Pi'$ which $(k_A,k_B)$-robustly simulates $\Pi$, where $\Pi$ is the KW-game for $f$, obtained as follows. The underlying DAG of $\Pi'$ is the same as that of $C'$. 

We now formally state and prove the converse to Lemma 7.7 for formulas.
Every $\land$-gate is replaced by a node where Alice speaks, every $\lor$-gate is replaced by a node where Bob speaks, and every input node $z_i$ or $\bar{z}_i$ is replaced by a leaf where the parties output $i$. Moreover, if each gate has fan-in $\leq k$, then the protocol can be run over the alphabet $[k]$.

**Remark.** For simplicity, we assume that each gate has the same fan-in $k$ (this just makes the notation a little cleaner).

**Proof.** Let $x \in f^{-1}(0)$ denote Alice’s input, $y \in f^{-1}(1)$ Bob’s. We begin by proving the following claims:

**Claim A.1.** Suppose the pebble arrives at an $\land$-gate $g$. Let $t_A$ denote the number of Alice’s transmissions that have been corrupted thus far, and let $C_1, \ldots, C_k$ denote the subformulas incident to $g$. Then:

(i) Given any error strings $e^1, \ldots, e^k$ short-circuiting at most $k_A - t_A - 1$ $\land$-gates on $C_1, \ldots, C_k$, respectively, $C_j^e(x) = 0$ for all $j = 1, \ldots, k$.

(ii) There is a $j^* \in [k]$ such that given any error strings $e'$ short-circuiting at most $k_A - t_A$ $\land$-gates on $C_j^e$, $C_j^{e'}(x) = 0$.

We have a completely analogous claim for $\lor$-gates.

**Claim A.2.** Suppose the pebble arrives at an $\lor$-gate $g$. Let $t_B$ denote the number of Bob’s transmissions that have been corrupted thus far, and let $C_1, \ldots, C_k$ denote the subformulas incident to $g$. Then:

- Given any error strings $e^1, \ldots, e^k$ short-circuiting at most $k_B - t_B - 1$ $\lor$-gates on $C_1, \ldots, C_k$, respectively, $C_j^e(y) = 1$ for all $j = 1, \ldots, k$.

- There is a $j^* \in [k]$ such that given any error strings $e'$ short-circuiting at most $k_B - t_B$ $\lor$-gates on $C_j^e$, $C_j^{e'}(y) = 1$.

We will just prove Claim A.1, the proof of Claim A.2 is completely analogous.

**Proof of Claim A.1.** Suppose that $g$ is a shallowest gate contradicting the claim (that is, all shallower gates contradict neither (i) nor (ii)). Let $P = g_1, \ldots, g_r$ denote the path from $g =: g_0$ to $g_r$ taken by the pebble, where $g_r$ is defined in one of 2 ways. If every one of Alice’s transmissions have been corrupted thus far, $g_r$ is the output gate. Else, $g_r$ is an input gate to $g'$, where $g'$ is the deepest $\land$-gate that the pebble reached at which Alice spoke and was not corrupted. Note in the latter case that Alice intended the pebble to move from gate $g'$ to gate $g_r$. Let $C$ denote the subformula of $C'$ with output gate $g_r$ and let $\bar{t}_A$ denote the number of times that Alice had been corrupted prior to the pebble arriving at gate $g_r$. 

Suppose first that $g$ contradicts (i). Without loss of generality, we can then find a $(k_A - t_A - 1, 0)$ error string $e^1$ such that $C^e_1(x) = 1$. We construct an error string $\bar{e} \in (([k] \cup \{\ast\})^C$ as follows: $\bar{e}|_{C_1} = e^1$, $\bar{e}(g) = 1$, and for each $\land$-gate $g_i$ on the path $P$, $\bar{e}(g_i) = j_i$, where $j_i \in [k]$ is the symbol that the channel forced Alice to send when she spoke at the gate $g_i$. Note first that there are at most $k_A - \bar{t}_A$ $\land$-gates which are corrupted by $\bar{e}$, i.e., $\bar{e}$ is a $(k_A - \bar{t}_A, 0)$-error string on $C$. However, we claim that $C^\bar{e}(x) = 1$, i.e., $g_r^\bar{e}(x) = 1$. First, $g_0^\bar{e}(x) = C^{e^1}(x) = 1$. Then, assuming $g_i^\bar{e}(x) = 1$, we show $g_{i+1}^\bar{e}(x) = 1$: if $g_{i+1}$ is
an $\lor$-gate this is true since $g_i$ is an input to $g_{i+1}$, while if $g_i$ is an $\land$ gate, $g_i$ is the $j$th input to $g_{i+1}$ so $g_{i+1}^e(x) = g_i^e(x) = 1$ by the definition of $\tilde{e}$. So $g_r^e(x) = 1$.

We show that this yields a contradiction. If $g_r$ is the output gate of the circuit $C'$, then $C'^{\tilde{e}}(x) = 1 \neq 0 = C(x)$, and since $\tilde{e}$ is a $(k_A - \tilde{t}_A, 0)$-error string, this contradicts the assumption that $C'$ $(k_A, k_B)$-robustly simulates the circuit $C$. Else, observe that by the minimality of the depth of $g$, it must be that the claim holds for $g'$. Hence, there is a $j \in [k]$ such that for any $(k_A - t_A, 0)$-error string $e', C_j'^e(x) = 0$, where $C_j'$ are the input subformulas to $g'$, and by the definition of $\Pi'$ Alice should have sent such a symbol $j'$ across the channel. Since Alice was not corrupted when she spoke, if $g_j$ is the $j$th input to $g'$ she intended to send $j'$ across the channel. But $\tilde{e}$ is a $(k_A - \tilde{t}_A, 0)$-error pattern such that $C_{j'}^{\tilde{e}}(x) = 1$. This means that Alice should have sent a different symbol across the channel: contradiction.

We now suppose that (ii) of the claim is violated. Then for all $j \in [k]$ there is a $(k_A - t_A)$-error string $e^j$ such that $C_j^{e^j}(x) = 1$. We again define a $(k_A - \tilde{t}_A, 0)$-error string $\tilde{e} \in ([k] \cup \{\ast\})^C$ as follows: we set $\tilde{e}|_{C_j} = e^j$ for $j = 1, \ldots, k$ (note that since the formulas $C_j$ are disjoint, this is a valid definition); we set $\tilde{e}(g) = \ast$; and, just as before, if $g_i$ is an $\land$-gate on the path $P$ and Alice was forced to send symbol $j_i \in [k]$, we set $\tilde{e}(g_i) = j_i$. This again yields a $(k_A - \tilde{t}_A, 0)$-error string. Note first that since $C_j^{\tilde{e}|_{C_j}}(x) = C_j^{e^j}(x)$ for all $j \in [k]$, $\tilde{g}_0^e(x) = C_1^{e^1}(x) \land \cdots \land C_k^{e^k}(x) = 1$. Then, just as in the previous argument, one can show that $\tilde{g}_r^e(x) = 1$, which again yields a contradiction.

It remains to show that Claim A.1 and Claim A.2 combine to yield the lemma. Suppose that after running protocol $\Pi'$ the pebble arrived at a leaf labeled by $z_i$ or $\bar{z}_i$. We show that the formula defined by the leaf (i.e., either $z_i$ or $\bar{z}_i$) leaf evaluates to 0 on input $x$ and evaluates to 1 on input $y$, so $x_i \neq y_i$.

Suppose that prior to this the pebble was on an $\land$-gate $g_0$ and that the leaf was (W.L.O.G.) the 1st input to $g$, so we refer to the subformula incident to $g_0$ consisting of just this input gate as $C_1$. If Alice intentionally sent the symbol 1 across the channel, then this input gate must have evaluated to 0 under $x$, as if $t_A$ is the number of times that Alice has been corrupted thus far, for any $(k_A - t_A, 0)$ error string $e'$, $C_1^{e'}(x) = 0$ by (i). Since $e'$ cannot not affect $C_1$, this just means that $C_1(x) = 0$, i.e., that the leaf evaluates to 0. Even if Alice was corrupted into sending a 1 across the channel, then by (ii), for any $(k_A - t_A - 1, 0)$ error string $e'$, $C_1^{e'}(x) = 0$, and again since error strings cannot affect $C_1$ we conclude that $C_1(x) = 0$, i.e., the leaf evaluates to 0.

It remains to show that the leaf evaluates to 1 on input $y$. Let $g'$ denote the last $\lor$-gate that the pebble was on, and let $g_r, \ldots, g_0$ denote the $\land$ gates on which the pebble moved from $g'$ to $g_0$, i.e., $g_r$ is an input to $g'$. For $i = 1, \ldots, r - 1$. Let $t_B$ denote the number of times that Bob had been corrupted prior to speaking at the gate $g'$. Let $\tilde{C}$ denote the subformula starting at gate $g'$, and let $\tilde{C}_1, \ldots, \tilde{C}_k$ denote the input subformulas, and assume W.L.O.G. that $\tilde{C}_1$ is the subformula starting at gate $g_r$.

If Bob intended to send 1 across the channel, then for any $(0, k_B - t_B)$ error string $e'$, $\tilde{C}_1^{e'}(y) = 1$, so a fortiori $\tilde{C}_1(y) = 1$ (a $(0, k_B - t_B)$-error string can only switch the output of a circuit from 1 to 0). If Bob did not intend to send 1 across the channel, we still know that for any $(0, k_B - t_B - 1)$-error string $e'$, $\tilde{C}_1^{e'}(y) = 1$, so again a fortiori $\tilde{C}_1(y) = 1$. Hence, $g_r(y) = 1$, which implies that all its inputs evaluate to 1.

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6The case of $\lor$-gates is analogous.
Figure 1: A circuit robustly computing the AND of its inputs. There is no communication protocol with Alice speaking at $\land$-gates and Bob speaking at $\lor$-gates robust to 1 error such that the pebble is guaranteed to arrive at a correct input gate.

on input $y$: in particular, $g_{r-1}(y) = 1$ evaluates to 1 on input $y$. Since $g_{r-1}(y) = 1$, all its inputs evaluate to 1 on input $y$, so $g_{r-2}(y) = 1$, etc., all the way down to $g_0(y) = 1$. Since all of $g_0$’s inputs must evaluate to 1 on input $y$, the leaf evaluates to 1 on input $y$, as desired. 

A.2 Counter-Example for Circuits

Unfortunately, Lemma A.1 does not apply to general circuits. Consider the circuit $C$ in Appendix A.2, which we claim $(1,0)$-robustly computes the AND of its inputs.

It is clear that $C(x_1, x_2) = x_1 \land x_2$, i.e., $C$ does compute the $\land$ function. To see that it $(1,0)$-robustly simulates the $\land$-function, it suffices to observe that it computes 0 on inputs 01 and 10 even if 1 $\land$-gate is short-circuited, which may be readily verified by inspection.

However, there is no protocol $\Pi$ with underlying DAG given by this circuit in which Alice speaks at $\land$-gates, Bob speaks at $\lor$-gates such that if one of Alice’s transmissions are corrupted we are guaranteed to arrive at a literal $z_i$ such that $x_i \neq y_i$. Suppose Alice gets input 10 and Bob gets 11. The adversary will let Alice send whatever symbol she likes from the output gate. Her next transmission, however, will be corrupted so that the pebble always arrives at the $z_1$ input. But $x_1 = y_1$, so Alice and Bob have not succeeded, i.e., $\Pi'$ does not $(1,0)$-robustly compute the AND of 2 literals.

Constant Separation Between Size of Circuit and Size of Protocol

In general, there is a circuit which $(k,0)$-robustly computes $\text{AND}(x_1, x_2)$ of size $1 + 2 + \cdots + (k + 1) = \frac{(k+2)(k+1)}{2}$. Conversely, we show that there is no $(k,0)$-robust protocol which successfully solves the KW game for $\text{AND}$ requires at least
This shows a constant multiplicative gap between the size of a robust circuit and the size of any robust protocol for the KW game.

Here is a construction of a \((k, 0)\)-robust circuit for AND: Construction. We will refer to the circuit by \(C\).

- Each gate is \(\land\), except inputs \(z_1\) and \(z_2\).
- The circuit consists of \(k + 1\) layers \(\ell_1, \ldots, \ell_{k+1}\), where each \(\ell_i\) is a path of length \(i\) directed from \(g_{i,1}, \ldots, g_{i,i}\).
- For all \(i \in [k + 1]\), \(j \in [i]\), \(j' \in [i - 1]\), there is an arc \(g_{i,j} \rightarrow g_{(i-1),j'}\).
- For all \(j \in [k + 1]\), there are arcs \(z_1 \rightarrow g_{k+1,j}\) and \(z_2 \rightarrow g_{k+1,j}\).

Claim A.3. Suppose at most \(k\) gates are corrupted by an error string \(e\) on any path from an input gate to the root. Then \(C^e(10) = C^e(01) = 0\).

Remark. Since there is a path connecting all the gates in \(C\), every \((k, 0)\) error string corrupts at most \(k\) gates. Also, since the circuit has no negations, it is clear that \(C^e(00) = 0\) and \(C^e(11) = 1\) for any error strings \(e\). Thus, proving the claim shows that \(C\) is a \((k, 0)\)-robust simulation of the AND-gate.

Proof. We make two observations. Let \(x \in \{10, 01\}\).

Observation 1. Suppose that there is a layer \(i\) such that

- \(e\) corrupts no gate in \(\ell_i\);
- there is a \(j_0\) such that \(g^e_{(i+1),j_0}(x) = 0\) or \(i = k + 1\).

Then each \(g_{i,j}\), \(j \in [i]\), evaluates to 0. So for all \(i' \leq i\) and \(j' \in [i']\), \(g^e_{i',j'}(x) = 0\), as there will be no inputs to any of these gates carrying a 1. In particular, \(C^e(x) = g^e_{1,1}(x) = 0\).

Observation 2. Suppose that layer \(i\) is such that

- there is a \(j \in [i]\) such that \(e\) does not corrupt \(g_{i,j}\);
- there is a \(j_0\) such that \(g^e_{(i+1),j_0}(x) = 0\) or \(i = k + 1\).

Then \(g^e_{i,j}(x) = 0\).

Combining the 2 observations, we conclude that if \(C^e(x) = 1\), it must be that there exists a maximal layer \(i\) such that each gate on layer \(i\) is corrupted and for each layer \(i' > i\), at least one gate is corrupted. Indeed, if there is no layer \(i\) such that each gate on that layer is corrupted, then Observation 2 tells us that on each layer, at least one gate evaluates to 0 on error string \(e\): in particular, this is true for layer 1, i.e., \(C^e(x) = g^e_{1,1}(x) = 0\). So there is a maximal \(i\) such that \(e\) corrupts each gate on layer \(i\). By Observation 3, we are not including input nodes/leafs.
if there is a layer $i' > i$ such that $e$ corrupts no gate on layer $i'$, then, since the maximality of $i$ tells us that either $i' = k + 1$ or there is a gate on layer $i' + 1$ evaluating to 0, we conclude that $C^e(x) = 0$.

But, such an error string $e$ corrupts at least $(k + 1 - i) + i = k + 1$ gates, and they all lie on an input-to-output gate path.

Conversely, one can show that every protocol which $(k, 0)$-robustly computes the KW-game for AND has size at least $(k + 1)^2$.

**Claim A.4.** Suppose $V$ is the set of non-leaf nodes of a protocol $\Pi$ which is a $(k, 0)$-robust simulation of the KW-game for AND. Then $|V| \geq (k + 1)^2$.

**Proof.** We may assume that $\Pi$ is chosen with $|V|$ as small as possible. First of all, observe that in such a protocol $\Pi$, Alice owns each node in $V$, i.e., $V_A = V$ and $V_B = \emptyset$. For suppose Bob spoke at some node $v$. Since Bob is only ever given input 00, the function defined by his input at node $v$ must be constant (as it has a domain of cardinality 1). Thus, Bob always sends the pebble to the same child, say $v'$. The protocol obtained by contracting the edge $(v, v')$ then behaves identically to $\Pi$, and has one fewer node, contradicting the minimality of $|V|$. Moreover, at each node $v$ owned by Alice (that is, every node $v$), if $v^x$ denotes the child Alice sends the pebble to on input $x$, then $v^{01} \neq v^{10}$. Indeed, since $\Pi$ is simply chosen to minimize $|V|$, it is no loss of generality to assume that Alice behaves identically on inputs 01 and 00. Then, if $v^{01} = v^{10}$, we could contract the edge $(v, v^{01})$ and the resulting protocol would still succeed but have one fewer node, again contradicting the minimality of $|V|$.

Let $A = (\{0, \ldots, k\})^2$, so $|A| = (k + 1)^2$. We claim that $|A| \leq |V|$. To do this, given any $v \in V$, label $v$ by $(a, b)$ if

- $a$ is the smallest integer such that there is an error string $e$ such that, when Alice is given input 01, the pebble arrives at node $v$ and $e$ has corrupted $a$ of Alice’s earlier transmissions;
- $b$ is defined similarly with respect to input 10.

By definition, no node can receive two labels. To prove $|A| \leq |V|$, it suffices to show that for all $(a, b) \in A$, there is a node $v$ such that $v$ is labeled $(a, b)$. We prove this by induction on $a + b$. For the base case, it is clear that the root node receives label $(0, 0)$.

For the induction step, given $(a, b)$, we know that for all pairs $(a', b') \in A$ with $a' + b' < a + b$, there is a node labeled by $(a', b')$. Choose $(a', b') \in A$ labeling a deepest node $v'$ subject to the condition that $a' \leq a$, $b' \leq b$ and $a' + b' < a + b$.

First, we claim that $(a', b') = (a - 1, b)$ or $(a', b') = (a, b - 1)$. For suppose that $a' + b' \leq a + b - 2$, and suppose without loss of generality that $a' < a$. Let $v''$ denote the child of $v'$ that Alice sends the pebble to on input 10. Then there is an error string $e^{01}$ which, on input 01, corrupts $a' + 1$ of Alice’s transmissions prior to the pebble arriving at node $v''$, and there is an error string $e^{10}$ which, on input 10, corrupts $b'$ of Alice’s transmissions prior to the pebble arriving at node $v''$. Thus, if $(a'', b'')$ is the label $v''$, $a'' \leq a' + 1$ and $b'' \leq b'$, so $a'' + b'' \leq a + b$ and $a'' \leq a$, $b'' \leq b$. But this contradicts the maximality of the depth of $v'$.

Therefore assume without loss of generality that $v'$ is labeled by $(a - 1, b)$. Let $v^{01}$ denote the child of $v'$ that Alice sends the pebble to on input 01, and $v^{10}$ the child she sends the pebble to on input 10.
Suppose \( v^{10} \) receives label \((\tilde{a}, \tilde{b})\). Note that there is an error string which, on input 01, corrupts \( a \) of Alice’s transmissions prior to the pebble arriving at \( v^{10} \); and there is an error string which, on input 10, corrupts \( b \) of Alice’s transmissions prior to the pebble arriving at \( v^{01} \). Thus, \( \tilde{a} \leq a \) and \( \tilde{b} \leq b \). Since \( v^{10} \) is deeper than \( v' \), the maximality of the depth of \( v' \) implies that we cannot have \( \tilde{a} + \tilde{b} < a + b \), so we conclude \( \tilde{a} = a \) and \( \tilde{b} = b \).

To complete the induction step, it remains to show \( v^{10} \) is not a leaf node. It cannot be a leaf labeled by 1, as there is an error string corrupting \( a \leq k \) of Alice’s transmissions on input 01 causing the pebble to arrive at this leaf. It also cannot be a leaf labeled by 2, as there is an error string corrupting \( b \leq k \) of Alice’s transmissions on input 10 causing the pebble to arrive at this leaf. So \( v^{10} \in V \), as desired. \( \Box \)