Vector and tensor perturbations in Horava-Lifshitz cosmology

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We study cosmological vector and tensor perturbations in Horava-Lifshitz gravity, adopting the most general Sotiriou-Visser-Weinfurtner generalization without the detailed balance but with projectability condition. After deriving the general formulas in a flat FRW background, we find that the vector perturbations are identical to those given in general relativity. This is true also in the non-flat cases. For the tensor perturbations, high order derivatives of the curvatures produce effectively an anisotropic stress, which could have significant effects on the high-frequency modes of gravitational waves, while for the low-frequency modes, the efforts are negligible. The power spectrum is scale-invariant in the UV regime, because of the particular dispersion relations. But, due to lower-order corrections, it will eventually reduce to that given in GR in the IR limit. Applying the general formulas to the de Sitter and power-law backgrounds, we calculate the power spectrum and index, using the uniform approximations, and obtain their analytical expressions in both cases.

I. INTRODUCTION

The background dynamics and the generation and evolution of perturbations during a period of inflation in the early universe, may deviate from the standard results if general relativity (GR) acquires significant ultra-violet (UV) corrections from a quantum gravity theory. Horava recently proposed such a theory [1], motivated by the Lifshitz theory in solid state physics [2]. Horava-Lifshitz (HL) theory has the interesting feature that it is non-relativistic in the UV regime, i.e., Lorentz invariance is broken. The effective speed of light in the theory diverges in the UV regime, which could potentially resolve the horizon problem without invoking inflation. Furthermore, scale-invariant super-horizon curvature perturbations could be produced without inflation [3, 10].

Originally, Horava assumed two conditions – detailed balance and projectability (though he also considered the case where the detailed balance condition is softly broken) [1]. Later, it was found that breaking the projectability condition is problematic [11] and gives rise to an inconsistent theory [12]. With detailed balance, on the other hand, the scalar field is not UV stable [13], and the theory requires a non-zero negative cosmological constant and breaks parity in the purely gravitational sector [14] (see also [1]).

To resolve these problems, various modifications have been proposed [15, 16]. In particular, the Sotiriou-Visser-Weinfurtner (SVW) generalization is the most general setup of the HL theory with the projectability condition and without detailed balance [14]. The preferred time that breaks Lorentz invariance leads to a reduced set of diffeomorphisms, and as a result, a spin-0 mode of the graviton appears. This mode is potentially dangerous and may be not stable, and cause strong coupling problems that could prevent the recovery of GR in the IR limit [11, 17, 18]. To address these issues and apply the theory to cosmology, linear perturbations of the Friedmann-Robertson-Walker (FRW) model with arbitrary spatial curvature in the SVW setup were studied, and shown explicitly that the spin-0 scalar mode of the graviton is stable in both the IR and the UV regimes, provided that $0 \leq \xi \leq 2/3$, where $\xi$ is a dynamical coupling parameter [9]. However, this stability condition has the unwanted consequence that the scalar mode is a ghost.

To tackle this problem, one may consider the theory in the range $\xi \leq 0$, so that the ghost problem is avoided. But, within this range, the spin-0 mode becomes unstable, since now the sound speed $c_s^2 = \xi/(2 - 3\xi)$ is non-positive. However in the limit that the sound speed becomes small, one should undertake a non-linear analysis to determine whether the strong self-coupling of the scalar mode decouples, as in the Vainshtein mechanism in massive gravity [19]. Taking these non-linear effects into account, Mukohyama recently showed that the continuous limit, $\xi \rightarrow 0$, of GR indeed exists for spherically symmetric, static, vacuum configurations [20]. In this paper, we assume that the strong-coupling problem in the cosmological background can also be addressed via this mechanism [21] or some other approach.

In addition, it could be quite possible that the legitimate background in the HL theory is not Minkowski. In particular, recent observations show that our universe is currently de Sitter-like [22]. Therefore, an alternative is to consider the de Sitter space as the background. Along this direction of thinking, the stability of the de Sitter spacetime in the SVW setup was studied recently, and shown that, in contrast to the Minkowski, the de Sitter space is stable [23].

With the above in mind, in [10] we studied perturbations of a scalar field cosmology. After deriving the generalized Klein-Gordon equation, which is sixth-order in spatial derivatives, we investigated scalar field perturbations coupled to gravity in a flat Friedmann-Robertson-Walker (FRW) background. In the sub-horizon regime, we found that in general the metric and scalar field modes have independent oscillations with different frequencies and phases. On super-horizon scales, the perturbations...
become adiabatic during slow-roll inflation driven by a single field, and the comoving curvature perturbation is constant.

In this paper we shall generalize our previous studies \(^3\) for scalar perturbations to vector and tensor perturbations in the SVW form of HL gravity \(^1\). We will investigate how standard results for linear perturbations in GR are modified and how it may still be possible to recover some standard results in the long-wavelength or low-energy limit. We will not consider the non-linear perturbations and consider only the linear evolution of perturbations in a flat background. But, for vector perturbations, our results also hold for the non-flat cases.

In Sec. II we briefly review the HL cosmology in the SVW setup for a flat background, while in Sec. III we present the general expressions for vector perturbations, and show explicitly that they are the same as those given in GR. We argue that this is also true even the background is not flat. In Sec. IV, we study the tensor perturbations and present the general formulas, from which we find that high order derivatives of curvatures act as an anisotropic stress, which could produce significant efforts on the high-frequency modes of gravitational waves. In Sec. V, as applications of our general formulas for tensor perturbations, we study the power spectrum and index in both the de Sitter and the power-law backgrounds, by using the uniform approximations, proposed recently by Habib et al. \(^{24}\). When there is only one turning point, we obtain the analytical expressions for the power spectrum and index in both cases. We conclude in Sec. VI.

It should be noted that tensor perturbations were studied previously by several authors in the framework of the HL theory. In particular, Takahashi and Soda studied the efforts of primordial gravitational waves due to the parity violation \(^{25}\), while Koh studied the power spectrum and index of gravitational waves with the Corley-Jacobson dispersion relations \(^{20}\). Yamamoto, Kobayashi and Nakamura, on the other hand, studied the problems for both scalar and tensor perturbations, using the uniform approximations \(^{7}\). Gong, Koh and Sasaki investigated vector and tensor perturbations in a different setup (In particular, the actions used by these authors violate the parity) with a scalar field as the only source \(^{27}\), and found that the vector perturbations have zero-degree of freedom, as that in GR.

II. THE FLAT FRW BACKGROUND IN THE SVW SETUP

The SVW generalization \(^{14}\) of HL theory coupled with matter fields has been reviewed in our previous work \(^3\) \([\text{10}]\), and in this paper we shall directly adopt the notations and conventions given there without further explanations. For detail, we refer readers to \(^3\) \([\text{10}]\).

The flat homogeneous and isotropic universe is described by the metric, \(ds^2 = a^2(\eta) (-dt^2 + \delta_{ij}dx^i dx^j)\). For this metric, \(K_{ij} = -a^2 \delta_{ij}\), where \(\mathcal{H} = a'/a\) and a prime denotes derivative with respect to \(\eta\). Then, it can be shown that the Hamiltonian constraint,

\[
\int d^3x \sqrt{g} (\mathcal{L}_K + \mathcal{L}_V) = 8\pi G \int d^3x \sqrt{g} J^t, \tag{2.1}
\]

yields the (generalized) Friedmann equation,

\[
\left(1 - \frac{3}{2} \xi\right) \frac{H^2}{a^2} = \frac{8\pi G}{3} \bar{\rho} + \Lambda, \tag{2.2}
\]

while the dynamical equations,

\[
\frac{1}{N\sqrt{g}} (\sqrt{g} \pi^{ij})' = -2 (K^2)^{ij} + 2 (1 - \xi) KK^{ij} + \frac{1}{N} \nabla_k \left[ N^k \pi^{ij} - 2 \pi^{ki} (N^j) \right] + \frac{1}{2} \mathcal{L}_K g^{ij} + F^{ij} + 8\pi G T^{ij}, \tag{2.3}
\]

give rise to

\[
\left(1 - \frac{3}{2} \xi\right) \frac{2H' + H^2}{a^2} = -8\pi G \bar{p} + \Lambda, \tag{2.4}
\]

where a prime denotes derivative with respect to \(\eta\),

\[
\bar{J}^t = -2\bar{\rho}, \quad \bar{J}^i = 0, \quad \bar{\pi}_{ij} = a^2 \bar{p} \delta_{ij}, \tag{2.5}
\]

and \(\bar{\rho}\) and \(\bar{p}\) are the total density and pressure. Similar to that in GR, the momentum constraint,

\[
\nabla_j \pi^{ij} = 8\pi G J^j, \tag{2.6}
\]

is satisfied identically, while the conservation law of energy

\[
\int d^3x \sqrt{g} \left[ g_{kl} T^{kl} - \frac{1}{\sqrt{g}} (\sqrt{g} J^t)' + \frac{2N_k}{\sqrt{g}} (\sqrt{g} J^k) \right] = 0, \tag{2.7}
\]

yields

\[
\bar{\rho}' + 3H (\bar{\rho} + \bar{p}) = 0. \tag{2.8}
\]

For the FRW background, the conservation law of momentum

\[
\nabla^k T_{ik} - \frac{1}{N \sqrt{g}} (\sqrt{g} J_i)' - \frac{N_i}{N} \nabla_k J^k - \frac{J^k}{N} (\nabla_k N_i - \nabla_i N_k) = 0, \tag{2.9}
\]

is satisfied identically.

It should be noted that Eq. (2.8) can be also obtained directly from Eqs. (2.2) and (2.4). In addition, replacing \(G\) and \(\Lambda\) by \(G/(1 - 3\xi/2)\) and \(\Lambda/(1 - 3\xi/2)\), Eqs. (2.2) and (2.4) becomes identical to those given in GR.

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1 In \(^3\), scalar perturbations in a flat FRW background were studied without fixing the gauge. This allowed the authors to study properties of the gauge-invariant quantities easily.
III. COSMOLOGICAL VECTOR PERTURBATIONS

The cosmological vector perturbations of the metric are given by [3]
\[
\delta g_{ij} = a^2(\eta)(F_{ij} + F_{j,i}),
\]
\[
\delta N^i = -S^i,
\]
where \(F_{ij} = \partial F_{i}/\partial x^j\) and
\[
S_i = 0,
\]
\[
F_i = 0,
\]
with \(S_i \equiv \delta^{ik} S_{j,k}\). The corresponding matter perturbations are given by
\[
\delta J^i = \frac{1}{a^4} q^i, \quad \delta J^0 = 0, \quad \delta \tau^{ij} = \frac{2}{a^2} \left( \Pi^{(i,j)} - \bar{\rho} F^{(i,j)} \right),
\]
where \(f^{(ij)} \equiv (f^{ij} + f^{ji})/2\) and
\[
q_i = 0 = \Pi^i.
\]
Note the slight difference between the definition of \(\delta \tau^{ij}\) used here and the one introduced in [3]. The vectors \(S^i, F^i, q^i\) and \(\Pi^i\) are in general functions of \(\eta\) and \(x^i\), and all their indices are lowered by \(\delta_{ij}\), for example, \(S_i \equiv \delta_{ik} S^k\) and so on. With the quasi-longitudinal gauge, one can set \(F_i = 0\). However, to have our results as much applicable as possible, we shall leave this possibility open, and consider the case with any \(F_i\) and \(S_i\). Then, we find that
\[
K_{ij} = -a \left( \mathcal{H} \delta_{ij} + F_{(i,j)} + 2 \mathcal{H} F_{(i,j)} + S_{(i,j)} \right), \quad R_{ij} = 0,
\]
\[
\mathcal{L}_K = -\frac{3(2 - 3\xi)}{a^2} \mathcal{H}^2, \quad \mathcal{L}_V = 2\Lambda,
\]
\[
F^{ij} = \sum_{s=0}^{8} g_s \zeta^s \left( F_s \right)^{ij} = -\frac{\Lambda}{a^4} \left( \delta^{ij} - 2 F^{(i,j)} \right).
\]
Hence, the Hamiltonian constraint (3.1) yields the generalized Friedmann equation (2.2), while the momentum constraint (2.6) gives
\[
\partial^2 \left( F_i' + S_i \right) = 16\pi G a q_i,
\]
where \(\partial^2 \equiv \delta^{ij} \partial_i \partial_j\). It is interesting to note that the quantity,
\[
\Phi_i \equiv F_i' + S_i,
\]
is gauge-invariant [3]. The dynamical equation (2.3), on the other hand, yields Eq. (2.4) to zeroth order, while to first order, it gives
\[
\left( F_{(i,j)} + S_{(i,j)} \right) + 2\mathcal{H} \left( F_{(i,j)}' + S_{(i,j)} \right) = 16\pi G a^2 \Pi_{(i,j)},
\]
(3.8)
The conservation law of energy (2.7) does not give new constraint, rather than Eq. (2.8), while the conservation of momentum yields,
\[
q_i' + 3\mathcal{H} q_i = a \partial^2 \Pi_i.
\]
(3.9)
However, this equation is not independent, and can be obtained from Eqs. (3.6) and (3.8).

It is remarkable that neither the parameter \(\xi\) nor high order derivatives are involved in Eqs. (3.6) - (3.9). The reasons are the following: All places that depend on \(\xi\) are through the term \((1 - \xi) K\), as one can see from the definitions of \(\mathcal{L}_K, \pi^{ij}\) and the dynamical equations Eq. (2.3). However, to first order, from Eq. (3.5) we find that \(K = -3\mathcal{H}/a\). Thus, the linear perturbations of \(K\) vanish identically. Then, all equations of linear perturbations do not depend explicitly on \(\xi\). On the other hand, all coupling constants \(g_2, g_3, ..., g_8\) are proportional to high order derivatives through the 3-dimensional Ricci tensor \(R_{ij}\) and its derivatives. Since \(R_{ij}\) also vanishes identically even when \(F_i \neq 0\), these high order derivatives have no contributions to the linearized equations. As a result, Eqs. (3.6) - (3.9) do not depend on \(\xi, g_2, ..., g_8\), and are identical to those given in GR [28], by noticing
\[
q_i = -a \delta q_i = a \left( \pi^0 - \bar{\rho} \right) (S_i - v_i),
\]
(3.10)
where \(q_i\) and \(v_i\) are quantities introduced in [28].

The above conclusion can be further generalized to the non-flat case. To see this, the simplest way is to work with the gauge \(F_i = 0\) [3], so the gauge-invariant vector defined by Eq. (3.7) reduces exactly to \(S_i\). In this gauge, since \(\delta R_{ij} = 0\), one can see that the high order derivatives of curvature have no contributions. In addition, to first order we also have \(\delta K = 0\). Then, as argued above, the parameter \(\xi\) will not appear in the linearized equations. Therefore, the resulting linearized equations for \(S_i\) do not depend explicitly on \(\xi, g_2, ..., g_8\) even when the spatial curvature of the FRW universe is different from zero, and must be identical to those given in GR [28]. Hence, the results obtained in GR can be easily generalized to the HL theory. For example, for a scalar field, both \(q_i\) and \(\Pi_i\) vanish identically in the HL theory [10]. Then, similar to that in GR, the vector perturbations are zero in all of space and shall remain so, if no sources of vorticity are introduced.

IV. COSMOLOGICAL TENSOR PERTURBATIONS

The cosmological tensor perturbations of the metric are given by [3]
\[
\delta g_{ij} = a^2(\eta)H_{ij}(\eta, x^k), \quad \delta N^i = 0 = \delta N,
\]
(4.1)
with the constraints
\[
H_i^0 = 0 = H_{ij}^j,\]
(4.2)
while the corresponding matter perturbations are given by
\[
\delta \tau^{ij} = \frac{1}{a^2} \left( (\Pi^{TT})^{ij} - \hat{p} H^{ij} \right), \quad \delta J^i = 0 = \delta J^j,
\] (4.3)
where
\[
\Pi^{(TT)} = 0 = \Pi^{(TT)}_{ij}, (4.4)
\]
and \(\Pi^{(TT)ij} = \Pi^{(TT)ij}(\eta, x^k)\). Note the difference between \(\Pi^{(TT)ij}\) used here and \(\Pi^j\) defined in [9]. All indices of \(H_{ij}\) and \(\Pi^{(TT)}_{ij}\) will be raised by \(\delta^{ij}\). Then, we find that to first-order the extrinsic curvature and Ricci tensors are given by
\[
K_{ij} = -\alpha H \delta_{ij} - \frac{a}{2} \left( H'_{ij} + 2 \dot{H} H_{ij} \right),
\]
\[
R_{ij} = -\frac{1}{2} \partial^2 H_{ij}. \quad (4.5)
\]
Because of the constraints (4.2), it can be shown that in the present case the first-order perturbations of \(\mathcal{L}_K\) and \(\mathcal{L}_V\) vanishes identically,
\[
\mathcal{L}_K = -\frac{3(2 - 3\xi)}{a^2} \mathcal{H}^2 + \mathcal{O}(H^2),
\]
\[
\mathcal{L}_V = 2\Lambda + \mathcal{O}(H^2). \quad (4.6)
\]
As a result, to zeroth-order the Hamiltonian constraint (2.1) yields the Friedmann equation, while to first-order it is satisfied identically. On the other hand, from the expression,
\[
\pi^{ij} = -\frac{2 - 3\xi}{a^3} \mathcal{H} \delta^{ij} + \frac{1}{2a^3} \left[ H''_{ij} + 2(2 - 3\xi) \mathcal{H} H^{ij} \right], \quad (4.7)
\]
we find that the momentum constraint (2.6) is also satisfied identically for tensor perturbations, where \(\delta J = 0\). This is also true for the momentum conservation (2.6), while the energy conservation (2.7) yields Eq. (2.8) to zero-th order, and is identically satisfied to first-order. To first-order we also find that
\[
F^{ij} = -\frac{1}{\sqrt{g}} \frac{\delta (\sqrt{g} \mathcal{L}_V)}{\delta g_{ij}} = \sum_{s=0}^8 g_s \zeta^s (F_s)^{ij} = -\frac{\Lambda}{a^2} \delta^{ij} + \frac{1}{2a^2} \left[ 2\Lambda + \frac{1}{a^2} \partial^2 - \frac{g_8}{\zeta^2 a^2} \partial^4 - \frac{g_8}{\zeta^4 a^4} \partial^6 \right] H'_{ij}. \quad (4.8)
\]
Then, the dynamical equations (2.3) yield,
\[
H''_{ij} + 2 \mathcal{H} H'_{ij} - \partial^2 H_{ij} = 16\pi G a^3 \Pi_{ij}^{(TT)} - \frac{1}{\zeta^2 a^2} \left( g_3 + \frac{g_8}{\zeta^2 a^2} \right) \partial^4 H_{ij}. \quad (4.9)
\]
Note that in writing the above equation, we had used Eq. (2.3). In addition, the Newtonian constant \(G\) is not modified by the factor \(1/(1 - 3\xi/2)\), as it was for scalar perturbations [9]. When \(g_3 = g_8 = 0\), it reduces exactly to that given in GR [28]. When they are different from zero, it shows clearly that these high order derivatives serve as anisotropic sources to produce primordial gravitational waves. However, since \(\zeta^2 = M_p^2/2\), they are highly suppressed in the IR regime.

During inflation we can neglect \(\Pi_{ij}^{(TT)} = 0\), since the inflaton has no anisotropic stress. Then, the effective gravitational stress,
\[
\Pi_{ij}^{HL} = -\frac{1}{16\pi G c^2 a^4} \left( g_3 + \frac{g_8}{\zeta^2 a^2} \right) \partial^4 H_{ij}, \quad (4.10)
\]
affects only the high-frequency modes, which could be very interesting, as they provide a mechanism to produce the initial seeds of gravitational waves even during the epoch of inflation.

Introducing two eigenmodes \(e_+^{(\eta, x)}(x)\) of the spatial Laplacian, \((\partial^2 + k^2/a^2) e_+^{(\eta, x)}(x) = 0\) with comoving wavenumber \(k\), we can decompose \(H_{ij}\) and \(\Pi_{ij}^{(TT)}\) into two independent components:
\[
H_{ij}(\eta, x) = H_{ij}(\eta, x) e_+^{(\eta, x)}(x), \quad (4.11)
\]
where \(e_+^{(\eta, x)}\) denote two possible polarization states of gravitational waves, + and ×. Then, Eq. (4.10) reduces to
\[
w''_k + \left( \omega_k^2 - a''/a \right) w_k = 16\pi G a^3 \Pi_{ij}^{(TT)}, \quad (4.12)
\]
where \(w_k = (w_+^k, w_\times^k)\) etc., and
\[
w_+^{(\eta, x)} = a H_+^{(\eta, x)}, \quad \omega_+^2 = k^2 + \frac{g_3 k_4^4}{\zeta^2 a^2} - \frac{g_8 k_6^6}{\zeta^4 a^4}. \quad (4.13)
\]
Note that in writing Eq. (4.12) we dropped the subindices + and × from \(w\) and \(\Pi^{(TT)}\). In the UV regime, \(\omega_+^2 \simeq -g_8 k_6^6/(\zeta^4 a^4)\), and to have stable modes we must assume
\[
g_8 < 0. \quad (4.14)
\]
Then, the primordial gravitational wave spectra are scale-invariant [3, 28]. In the IR regime, \(\omega_+^2 \simeq k^2\), and the \(H = \text{Const}\) mode on large scales is regained. Since the intermediate \(k^4\) part is not scale-invariant, there may be a peak in the spectra. In addition, both states + and × satisfy the same equation, in contrast to the case with detailed balance condition [28], circular polarization cannot be generalized when \(\Pi_{ij}^{(TT)} = 0\) in the current setup. This is because the SVW generalization preserves parity [13], while the HL theory with detailed balance condition does not [28].

It is also interesting to note that, in contrast to scalar perturbations [9], Eq. (4.10) does not contain the coupling
constant $\xi$ explicitly, with the same reason as for vector perturbations as explained following Eq. (33). This is also true for non-flat FRW models, because the kinetic part is independent of the spatial curvature. Then, tensor perturbations in all FRW models does not have the ghost problem for any given coupling constant $\xi$, including the range $0 \leq \xi \leq 2/3$, in which ghosts were found in the scalar sector of perturbations [4, 17].

V. TENSOR PERTURBATIONS IN SOME SPECIFIC BACKGROUNDS

In this section, we consider tensor perturbations of Eq. (4.12) with the assumption that the efforts of $\Pi^{(TT)}$ are negligible, so Eq. (4.12) reduces to

$$w_k'' + \left(\omega_k^2 - \frac{a''}{a}\right) w_k = 0. \quad (5.1)$$

The above equations usually has the following asymptotic solutions,

$$w_k = \begin{cases} \frac{w_0}{\sqrt{2\omega_k}} e^{-i\omega_k \eta}, & k\eta \to -\infty, \\ \frac{A_k}{\omega_k}, & k\eta \to 0, \end{cases} \quad (5.2)$$

where $w_0^0$ and $A_k$ are constants. Then, in the superhorizon region ($k\eta \approx 0$), the power spectrum is given by

$$P_T(k)_{k\eta \to -0} = \frac{k^3}{2\pi^2} \left| \frac{w_k}{A_k} \right|^2 \approx \frac{k^3}{2\pi^2} |A_k|^2. \quad (5.3)$$

Therefore, to find $P_T(k)$ now simply reduces to find the constant $A_k$ by connecting the two asymptotic solutions. This can be done by a matching process in the intermediate region [cf. Fig. 1]. As pointed out by several authors [30, 31], such a process lacks error control and is not systematically improvable.

Recently, Habib et al. [24] advocated another method - the so-called uniform approximation [32]. The latter provides a single approximate solution for the whole range $k$, so it does not employ the intermediate matching process and the approximation procedure can be systematically improved and possesses an error control function. To show how this method works, we first write Eq. (5.1) in the form [7, 24],

$$w_k'' = \left[ g(k, \eta) + q(\eta) \right] w_k, \quad (5.4)$$

where

$$g(k, \eta) = \frac{a''(\eta)}{a(\eta)} + \frac{1}{4\eta^2} - \omega_k^2(k, \eta),$$

$$q(\eta) = -\frac{1}{4\eta^2}. \quad (5.5)$$

Note that the specific choice of the function $q(\eta)$ is to guarantee the convergence of the approximation [32]. Then, the single approximate solution can be written as

$$w_k = \left( \frac{y(k, \eta)}{g(k, \eta)} \right)^{1/4} \left[ a_k Ai(y) + b_k Bi(y) \right], \quad (5.6)$$

where $Ai(y)$ and $Bi(y)$ are Airy functions, and

$$y(k, \eta) = \begin{cases} y_+(k, \eta), & \eta > \tilde{\eta}, \\ y_-(k, \eta), & \eta < \tilde{\eta}, \end{cases} \quad (5.7)$$

with

$$y_{\pm}(k, \eta) = \mp \left\{ \frac{3}{2} \int_{\tilde{\eta}(k)}^{\eta} \sqrt{\pm g(k, \eta')} d\eta' \right\}^{2/3}, \quad (5.8)$$

and $\tilde{\eta}$ is the turning point, defined by $g(k, \tilde{\eta}) = 0$ [cf. Fig. 1]. The integration of $y_{\pm}$ is taken on the left of the turning point $\tilde{\eta}$, while the one of $y_+$ is taken on the right of $\tilde{\eta}$. In order to fix the coefficients $a_k$ and $b_k$, $w_k$ is required to reduce to its asymptotic form (5.2) as $k\eta \to -\infty$. In this limit, for well-behavior $\omega_T$, the function $y_{\pm}(k, \eta)$ is very large and negative. So, one can use the asymptotic forms of the Airy functions [33],

$$Ai(-x) \simeq \frac{1}{(\pi x^2)^{1/4}} \sin\left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right),$$

$$Bi(-x) \simeq \frac{1}{(\pi x^2)^{1/4}} \cos\left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right), \quad (5.9)$$

for $x \gg 1$ with $|\text{arg}(x)| < 2\pi/3$. Then, choosing

$$a_k = -i \sqrt{\frac{\pi}{2}} e^{i\pi/4}, \quad b_k = \sqrt{\frac{\pi}{2}} e^{i\pi/4}, \quad (5.10)$$

we find that

$$w_k = \frac{1}{\sqrt{2\omega_T}} \exp \left\{ -i \int_{\eta}^{\tilde{\eta}(k)} \omega_T(k, \eta') d\eta' \right\}, \quad (5.11)$$

FIG. 1: The three different regions: (a) the sub-horizon region ($k\eta \to -\infty$), denoted by Sub-H; (b) the intermediate region ($\eta \approx \bar{\eta}$), denoted by $I$; and (c) the super-horizon region ($k\eta \to 0^-$), denoted by Super-H. For well-defined $\omega_T$, the function $g(\eta)$ is positive for $\eta > \bar{\eta}$, and negative for $\eta < \bar{\eta}$, where $\bar{\eta}$ is the turning point, and given by the negative root of $g(\bar{\eta}) = 0$.

where $Ai(y)$ and $Bi(y)$ are Airy functions, and

$$y(k, \eta) = \begin{cases} y_+(k, \eta), & \eta > \bar{\eta}, \\ y_-(k, \eta), & \eta < \bar{\eta}, \end{cases} \quad (5.7)$$

with

$$y_{\pm}(k, \eta) = \mp \left\{ \frac{3}{2} \int_{\bar{\eta}(k)}^{\eta} \sqrt{\pm g(k, \eta')} d\eta' \right\}^{2/3}, \quad (5.8)$$

and $\bar{\eta}$ is the turning point, defined by $g(k, \bar{\eta}) = 0$ [cf. Fig. 1]. The integration of $y_{\pm}$ is taken on the left of the turning point $\bar{\eta}$, while the one of $y_+$ is taken on the right of $\bar{\eta}$. In order to fix the coefficients $a_k$ and $b_k$, $w_k$ is required to reduce to its asymptotic form (5.2) as $k\eta \to -\infty$. In this limit, for well-behavior $\omega_T$, the function $y_{\pm}(k, \eta)$ is very large and negative. So, one can use the asymptotic forms of the Airy functions [33],

$$Ai(-x) \simeq \frac{1}{(\pi x^2)^{1/4}} \sin\left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right),$$

$$Bi(-x) \simeq \frac{1}{(\pi x^2)^{1/4}} \cos\left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right), \quad (5.9)$$

for $x \gg 1$ with $|\text{arg}(x)| < 2\pi/3$. Then, choosing

$$a_k = -i \sqrt{\frac{\pi}{2}} e^{i\pi/4}, \quad b_k = \sqrt{\frac{\pi}{2}} e^{i\pi/4}, \quad (5.10)$$

we find that

$$w_k = \frac{1}{\sqrt{2\omega_T}} \exp \left\{ -i \int_{\eta}^{\bar{\eta}(k)} \omega_T(k, \eta') d\eta' \right\}, \quad (5.11)$$
as $\eta \to -\infty$, which is exactly the required adiabatic form in the sub-horizon region ($-k\eta \gg 1$). It can be shown that the Wronskian normalization condition,

$$w_k w''_k - w'_k w'_k = i,$$

(5.12)
is satisfied for the choice of Eq. (5.10).

To find the asymptotic behavior of the solution (5.6) as $k\eta \to 0^-$, we first notice that Eq. (5.10),

$$Ai(x) \simeq \frac{1}{2(x^2/4)\exp(-\frac{1}{2}x^3/2)}, \qquad (\text{arg}(x) < \pi),$$

$$Bi(x) \simeq \frac{1}{(x^2/4)\exp(\frac{1}{2}x^3/2)}, \qquad (\text{arg}(x) < \pi/3),$$

(5.13)
as $x \to \infty$. Then, Eq. (5.6) has the asymptotics,

$$w_k \simeq \frac{e^{i\pi/4}}{[4g(k, \eta)]^{1/4}} \exp[D(k, \eta)],$$

(5.14)
as $k\eta \to 0^-$, where

$$D(k, \eta) \equiv \int_{\eta(k)}^{\eta} \sqrt{g(k, \eta')} \, d\eta'.$$

(5.15)

Hence, we find that

$$P_T(k) = \lim_{k\eta \to 0^-} \frac{k^3}{4\pi^2 a^2 \sqrt{g(k, \eta)}} \exp\left\{2D(k, \eta)\right\},$$

(5.16)

and the spectrum index of the quantum fluctuations is given by

$$n_T \equiv \frac{d \ln P_T}{d \ln k} \bigg|_{k\eta \to 0^-} = 3 + \lim_{k\eta \to 0^-} 2k \frac{dD(k, \eta)}{dk}. $$

(5.17)

Note that in writing down the above expression, we had assumed that

$$\lim_{k\eta \to 0^-} \frac{k g_{kk}(k, \eta)}{g(k, \eta)} = 0.$$

(5.18)

It is interesting to note that both $P_T(k)$ and $n_T$ are uniquely determined by the function $D(k, \eta)$. It is also important to note that the above formulas are valid for any $\omega_T(k, \eta)$, as long as $g(k, \eta) = 0$ has only one negative root. When $g(k, \eta)$ has multiple zeros for $\eta < 0$, different treatment is needed. In this paper, we shall consider only the case where $g(k, \eta) = 0$ has only one negative root, as shown in Fig. 1. In particular, we shall consider two explicit backgrounds: the de Sitter spacetime, and the spacetime with power-law expansion.

\section{The de Sitter Background}

In the de Sitter background, we have $a(\eta) = -1/(H\eta)$, and

$$\omega^2 = k^2 + \frac{g_3 H^2 k^4}{\zeta^2} - \frac{g_8 H^4 k^6}{\zeta^4}.$$ 

(5.19)

Before studying the above equation, it is interesting to compare it with the one obtained in [25] with detailed balance condition, for which the tensor perturbations can be also cast in the form of Eq. (4.13) but now with a different $\omega^2_T$, given by

$$\omega^2_{T,TS} = c^2 k^2 \left[1 + \beta(ck\eta)^2 \left(1 + c\kappa^4 \gamma k\eta\right)^2\right],$$

(5.20)

where $c$ is the “emergent speed” of light, and

$$c^2 = \frac{\kappa^4 \mu^2 \Lambda_w}{16(1-3\lambda)}, \quad \beta = \frac{(1-3\lambda)H^2}{c^2 \Lambda_w}, \quad \gamma = \frac{2H}{c \mu w^2},$$

(5.21)

and $\lambda = 1 - \xi$. The constants $\mu$, $w$ and $\Lambda_w$ are the free parameters in the model, and $\epsilon^A = \pm 1$. When $\epsilon^A = 1$, it is called the right-handed mode, and when $\epsilon^A = -1$ it is called the left-handed mode. As mentioned above, the difference between left- and right- handed is exactly due to the violation of the parity.

In the SVW setup, the theory is explicitly parity-preserving, so the right- and left-handed modes satisfy the same equation, and $\omega^2_T$ does not depend on $\epsilon^A$, and is given by Eq. (5.19). Then, from Eq. (5.19) we find that

$$g(k, \eta) = k^2 \left[9 \frac{4}{z^2} - \left(1 + \tilde{g}_3 z^2 + \tilde{g}_8 z^4\right)\right],$$

(5.22)

where

$$z = k\eta, \quad \tilde{g}_3 = \frac{g_3 H^2}{\zeta^2}, \quad \tilde{g}_8 = \frac{|g_8| H^4}{\zeta^4} > 0.$$ 

(5.23)

It can be shown that the function $g(k, \eta)$ defined above satisfies the condition (5.18).

Let first consider the case $g_3 = g_8 = 0$, for which the turning point is at

$$\bar{\eta} = -\frac{3}{2k},$$

(5.24)

and Eq. (5.19) yields

$$D(k, \eta) = -\frac{9}{4} - k^2 \eta^2 + \frac{3}{2} \ln \left(\frac{(1-k\eta)}{\sqrt{\frac{1}{4} + \frac{9}{2} - k^2 \eta^2}}\right).$$

(5.25)

Then, from Eqs. (5.16) and (5.17) we obtain

$$P_T(k) = \frac{9 H^2}{2\pi^2 e^2}, \quad n_T = 0,$$

(5.26)

which are the well-known results obtained in GR [34].

When $g_3 g_8 \neq 0$, it can be shown that $g(k, \eta) = 0$ can have at most three different turning points, depending on the signs of $g_3$, as shown in Fig. 2. In particular, when $g_3 \geq 0$, $g(k, \eta)$ has only one turning point. When $g_3 < 0$, $g(k, \eta)$ can have one, two or three turning points, depending on the ratio of $g_3/g_8$. In this paper, we consider only
When \( g_3 = g_8 = 0 \), we have \( \bar{n} = -3/(2k) \), and the above expression yields \( n_T = 0 \), which is exactly the result given by Eq. \( (5.20) \). When \( g_3 \) and \( g_8 \) are different from zero, the last two terms in the right-hand side represent corrections from the high order derivatives of the curvature, which are suppressed by the Planck scale \( \zeta^2 = M_{pl}^2/2 \).

Inserting Eq. \( (5.32) \) into Eq. \( (5.10) \), on the other hand, we find that

\[
\Pr(k) = \frac{4H^2\zeta^3}{3\pi^2 e^3} \exp \left\{ 2H^2\zeta^4 \left( g_3\zeta^2 + |g_8|\zeta^2H^2 \right) \right\},
\]

which is suppressed exponentially by the Planck scale.

### B. The Power-Law Background

When \( a(t) \propto t^{1+n} \) or \( a(\eta) = -(H\eta)^{-(1+1/n)} \), we find that

\[
g(k, \eta) = k^2 \left\{ \frac{\beta^2}{x^2} - \left[ 1 + \tilde{g}_3x^{2(1+1/n)} + \tilde{g}_8x^{4(1+1/n)} \right] \right\},
\]

where \( x \equiv -H\eta \), and

\[
\beta = \frac{(2 + 3n)H}{2nk}, \quad \tilde{g}_3 \equiv \frac{g_3k^2}{\zeta^2}, \quad \tilde{g}_8 \equiv \frac{|g_8|k^4}{\zeta^4}.
\]

Unlike that in GR, now inflation can be realized when \( n > -2/3 \). The de Sitter universe corresponds to \( n = \infty \).

When \( g_3 = g_8 = 0 \), we find that the zero of \( g(k, \eta) \) is at \(-k\bar{\eta} = \beta\), and Eq. \( (5.15) \) yields,

\[
D(k, \eta) = -\frac{k}{H}\sqrt{\beta^2 - x^2} - \frac{3n + 2}{2n} \ln \left( \frac{x}{\sqrt{\beta^2 - x^2} + \beta} \right).
\]

Then, Eqs. \( (5.16) \) and \( (5.17) \) give

\[
P_T(k) = \frac{1}{2\pi^2 e^{3+2/n}} \left[ \frac{(3n + 2)H}{n} \right]^{2(1+1/n)} k^{-2/n},
\]

\[
N_T = -\frac{2}{n}.
\]

As \( n \to \infty \), the above expressions reduce to the ones given by Eq. \( (5.20) \).

When \( g_3 \) and \( g_8 \) are different from zero and \( \epsilon = k^2/\zeta^2 \ll 1 \), the zero of \( g(k, \eta) = 0 \) is well approximated by \( x = \beta \). Then, \( g(k, \eta) \) can be written as \( \tilde{g} \),

\[
g(k, \eta) = \frac{k^2}{x^2} \left( \beta^2 - x^{2(1+\nu)} \right),
\]

where

\[
\nu(k, x) = \frac{1}{2} \frac{d\ln \tilde{g}(x)}{d\ln x} = \frac{1 + n}{n\tilde{g}(x)} \left( \epsilon g_3 + 2|g_8|\epsilon^2x^{2(1+1/n)} \right) x^{2(1+1/n)},
\]

\[
\tilde{g}(x) = 1 + \epsilon g_3x^{2(1+1/n)} + |g_8|\epsilon^2x^{4(1+1/n)}.
\]
Thus, we find that
\[
D(k, \eta) = \frac{-k^2}{(1 + \nu)H} \left[ \frac{\sqrt{\beta^2 - x^{2(1+\nu)}}}{\beta} + (1 + \tilde{\nu}) \ln \frac{x}{\beta} \right],
\]
where
\[
\tilde{\nu} \equiv \nu(k, \beta) = \frac{1 + n}{n\zeta^4} \left\{ g_3 \xi^2 k^2 \beta^2 (1+1/n) + (2|g_s| - g_3^2) k^4 \beta^3 (1+1/n) \right\}.
\]
From the above expressions we obtain
\[
P_T(k) \simeq \frac{k^{-2/n}}{2\pi^2} \left[ \frac{(3n + 2)H}{n} \right]^{2(1+1/n)} \times \exp \left\{ - \frac{(3n + 2)}{n(1 + \tilde{\nu})} \right\},
\]
\[
n_T \simeq -\frac{2}{n} \frac{2(1 + n)(2 + 3n)(1 - \ln 2)}{n^3 (1 + \nu)^2 \zeta^2 k^{4/n}} \left\{ g_3 k^{2/n} + \frac{2(2|g_s| - g_3^2)}{\zeta^2} \left[ \frac{(3n + 2)H}{2n} \right]^{2(1+1/n)} \right\} \times \left[ \frac{(3n + 2)H}{2n} \right]^{2(1+1/n)}.
\]
When \( \epsilon = k^2/\zeta^2 \gg 1 \), on the other hand, the zero of \( g(k, \eta) = 0 \) is well approximated by
\[
\bar{x} = -H\bar{\eta} = \left[ \frac{(3n + 2)H^2 \zeta^2}{2n k^3 \sqrt{|g_s|}} \right]^{\frac{1}{2(1+1/n)}}.
\]
Then, \( g(k, \eta) \) can be written as
\[
g(k, \eta) = \frac{k^6}{x^3 \xi^4} \left( \tilde{\beta}^2 - x^{2(1+\nu)} \right),
\]
but now with
\[
\tilde{\beta} = \frac{(3n + 2)\zeta^2 H}{2nk^3},
\]
\[
\nu(k, x) = \frac{1 + n}{ng(x)} \left( 2|g_s|x^{2(1+1/n)} + \frac{g_3}{\epsilon} \right) x^{2(1+1/n)},
\]
\[
\bar{g}(x) \equiv |g_s| x^{4(1+1/n)} + \frac{g_3}{\epsilon} x^{2(1+1/n)} + \frac{1}{\epsilon^2}.
\]
Then, we find that
\[
D(k, \eta) = -\frac{3n + 2}{2n(1 + \tilde{\nu})} \left[ \frac{\sqrt{\tilde{\beta}^2 - x^{2(1+\nu)}}}{\tilde{\beta}} + (1 + \tilde{\nu}) \ln \frac{x}{\tilde{\beta}} \right] + \ln \left( \frac{\tilde{\beta}}{\tilde{\beta} + \sqrt{(\tilde{\beta}^2 - x^{2(1+\nu)}}} \right),
\]
where \( \tilde{\nu} \equiv \nu(k, \bar{x}) \), from which we obtain
\[
P_T(k) \simeq \frac{\zeta^2}{2\pi^2} \left[ \frac{(3n + 2)\zeta^2 H}{nk^3} \right]^{2(1+1/n)} \times \exp \left\{ - \frac{(3n + 2)}{n(1 + \tilde{\nu})} \right\},
\]
\[
n_T \simeq -\frac{6(1 + n)}{n} + \frac{4(1 - \ln 2)(1 + n)\zeta^2}{n^2 (1 + \nu)^2 g_s^2} \left[ \frac{(3n + 2)\zeta^2 H}{(3n + 2)\zeta^2 H} \right]^{2(1+1/n)}
\]
\[
\times \left\{ \frac{\zeta^2 (g_3^2 - 2|g_s|)}{(3n + 2)\zeta^2 H} \right\} \left( \frac{2n \sqrt{|g_s|}}{(3n + 2)\zeta^2 H} \right)^{2(1+1/n)} - g_3 |g_s| \right\} k^\frac{4}{n+2}. (5.47)
\]
\[
\text{VI. CONCLUSIONS}
\]
We have studied cosmological vector and tensor perturbations in the most general SVW setup of the HL theory with the projectability condition but without the detailed balance. For the vector perturbations, we have showed explicitly that the resulted expressions are identical to those given in GR. Thus, all the results obtained in GR regarding to the vector perturbations also hold here in the HL theory.

For the tensor perturbations, we found that, among other things, the high order derivatives of curvatures produces an effective stress, which could produce high-frequency gravitational waves even when the matter anisotropic stress vanishes. These terms have negligible efforts on the low-frequency modes of gravitational waves. The dispersion relations contain three different terms, proportional to, respectively, \( k^2 \), \( k^4 \) and \( k^6 \). As a result, in the UV regime the power spectrum is scalar-invariant, while in the IR limit it eventually reduces to that given in GR.

Applying our general formulas for tensor perturbations to the background of de Sitter as well as the power-law expansions, we were able to calculate the corresponding power spectra and indices analytically, using the uniform approximations proposed recently by Habib et al.\[24\].

In this paper, we assumed that the strong-coupling problem\[11,17,18\] in cosmological backgrounds can also be addressed via the Vainshtein mechanism\[14,21\] or some other approach. We wish to come back to this issue soon.

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