Index Theorems for One-dimensional Chirally Symmetric Quantum Walks with Asymptotically Periodic Parameters

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Abstract

We focus on index theory for chirally symmetric discrete-time quantum walks on the one-dimensional integer lattice. Such a discrete-time quantum walk model can be characterised as a pair of a unitary self-adjoint operator $\Gamma$ and a unitary time-evolution operator $U$, satisfying the chiral symmetry condition $U^* = \Gamma U$. The significance of this index theory lies in the fact that the index we assign to the pair $(\Gamma, U)$ gives a lower bound for the number of symmetry protected edge-states associated with the time-evolution $U$. The symmetry protection of edge-states is one of the important features of the bulk-edge correspondence. The purpose of the present paper is to revisit the well-known bulk-edge correspondence for the split-step quantum walk on the one-dimensional integer lattice. The existing mathematics literature makes use of a fundamental assumption, known as the 2-phase condition, but we completely replace it by the so-called asymptotically periodic assumption in this article. This generalisation heavily relies on analysis of some topological invariants associated with Toeplitz operators.

Keywords: Strictly local operator, Symmetry protection, Topological invariants, Index theory, Split-step quantum walk

1. Introduction

The major mathematical theme of the present article can be broadly described as index theory for unitary operators. More specifically, we focus on an abstract unitary
operator $U$ on a Hilbert space $H$, satisfying the following **chiral symmetry condition**:

$$U^* = \Gamma U \Gamma,$$

(1)

where $\Gamma$ is a fixed unitary self-adjoint operator on $H$. With the canonical decomposition $H = \ker(\Gamma - 1) \oplus \ker(\Gamma + 1)$ of the underlying Hilbert space in mind, we can naturally assign certain well-defined indices $\text{ind}_+ (\Gamma, U), \text{ind}_- (\Gamma, U)$ to the given pair $(\Gamma, U)$, in such a way that the following estimate holds true;

$$|\text{ind}_\pm (\Gamma, U)| \leq \dim \ker(U \pm 1),$$

(2)

where we assume that the essential spectrum of $U$, denoted by $\sigma_{\text{ess}}(U)$, does not contain $\pm 1$ (see §2.1 for the definition of $\sigma_{\text{ess}}(U)$). We typically impose the assumption of $\pm 1 \notin \sigma_{\text{ess}}(U)$ to ensure that the two indices on the left hand side of (2) can be viewed as well-defined Fredholm indices. As can be easily seen from (2), if $\text{ind}_\pm (\Gamma, U)$ is non-zero, then the eigenspace $\ker(U \mp 1)$ contains some non-trivial eigenstates. This implication, known as the **protection of eigenstates by chiral symmetry**, is one of the important features of the bulk-edge correspondence. A brief summary of the index theory mentioned so far can be found in §3.1.

The symmetry protection of eigenstates naturally arises in the context of so-called (discrete-time) **quantum walks** [Gud88, ADZ93, Mey96, ABN+01]. Quantum walk models are typically characterised by their associated unitary time-evolution operators $U$, and we may consider various symmetry types in a sense analogous to (1). Index theory for quantum walks on the integer lattice $\mathbb{Z}$ with various symmetry types has been a particularly active subject of recent mathematical studies of quantum walks [CGS+16, CGG+18, CGS+18]. In particular, the chiral symmetry condition (1) alone has attracted tremendous attention [Suz19, ST19a, Mat20, Tan21, CGWW21], and it is also the main subject of the present article. **Suzuki’s split-step quantum walk** [FFS17, FFS18, FFS19, Tan21, NOW21] can be viewed as a prominent example of a one-dimensional chirally symmetric quantum walk. This model is characterised by the following two operators defined on the Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^2) = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ of $\mathbb{C}^2$-valued square-summable sequences:

$$ \Gamma_{\text{suz}} := \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} \begin{pmatrix} p & \sqrt{1-p^2} \\ \sqrt{1-p^2} & -p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix}, $$

(3)

$$ U_{\text{suz}} := \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} \begin{pmatrix} p & \sqrt{1-p^2} \\ \sqrt{1-p^2} & -p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} a & b^* \\ b & -a \end{pmatrix}, $$

(4)

where $L$ is the bilateral left-shift operator on $\ell^2(\mathbb{Z})$ (see (20) for definition), and where $p = (p(x))_{x \in \mathbb{Z}}$ and $a = (a(x))_{x \in \mathbb{Z}}$ are real-valued sequences assuming values in the closed interval $[-1, 1]$. Such bounded sequences are identified with the corresponding bounded
multiplication operators on $\ell^2(\mathbb{Z})$ throughout this paper. It is shown in [NOW21, Corollary 4.4] that the model introduced above is equivalent to Kitagawa’s split-step quantum walk [KRBD10, KBF+12, Kit12].

It follows from a direct computation that if we set $(\Gamma, U) := (\Gamma_{suz}, U_{suz})$, then we obtain the chiral symmetry condition (1). To give a complete classification of the associated indices $\text{ind}_{\pm}(\Gamma, U)$, let us assume the existence of the following limits for each $\star = -\infty, +\infty$:

$$p(\star) := \lim_{x \to \star} p(x), \quad a(\star) := \lim_{x \to \star} a(x).$$

(5)

It is shown in [MST21, Theorem 1.1](i) that under (5), we have $\pm 1 \notin \sigma_{\text{ess}}(U)$ if and only if $p(\star) \neq \pm a(\star)$ for each $\star = -\infty, +\infty$. Moreover, in this case

$$\text{ind}_{\pm}(\Gamma, U) = \begin{cases} +1, & p(-\infty) \mp a(-\infty) < 0 < p(+\infty) \mp a(+\infty), \\ -1, & p(+\infty) \mp a(+\infty) < 0 < p(-\infty) \mp a(-\infty), \\ 0, & \text{otherwise}. \end{cases}$$

(6)

Note first that the index formula (6) is robust in the sense it depends only on the asymptotic values (5). In particular, if $|\text{ind}_{\pm}(\Gamma, U)| = 1$, then it follows from (2) that the eigenspace $\ker(U \mp 1)$ contains at least one non-trivial eigenstate. It is also shown in [MST21, Theorem 1.1](ii) that such symmetry protected eigenstates exhibit exponential decay in a certain well-defined sense.

This begs the following natural question. The existence of the two-sided limits (5) is a fundamental assumption in [MST21, Theorem 1.1], but is there some meaningful way to generalise this result? The ultimate purpose of the present article is to show that such a generalisation is actually possible, and we do so by replacing (5) with the so-called asymptotically periodic assumption (see §2 for definition).

The present article is organised as follows. The evolution operator of Suzuki’s split-step quantum walk given by (4) is an explicit example of so-called strictly local operators. In §2 we develop an elementary operator-algebraic method to classify some topological invariants associated with strictly local operators satisfying the asymptotically periodic assumption. In fact, this result is a generalisation of [Tan21, Theorem A]. In §3 we give a generalisation of [MST21, Theorem 1.1] as a direct application of §2. The paper concludes with several concluding remarks in §4.

On a final note, the present article focuses on some topological invariants associated with quantum walks, which only make sense in infinite dimensions. The novelty of our approach lies in the fact we can fully classify these topological invariants in the language of linear algebra. As we shall see in this paper, some crucially important arguments can
be eventually simplified to analysis of $n \times n$ matrices of the form:

$$
\begin{pmatrix}
\alpha_0 & \beta_0 & 0 & \cdots & 0 & \gamma_0 \\
\gamma_1 & \alpha_1 & \beta_1 & \cdots & 0 & 0 \\
0 & \gamma_2 & \alpha_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \alpha_{n-2} & \beta_{n-2} \\
\beta_{n-1} & 0 & 0 & \cdots & \gamma_{n-1} & \alpha_{n-1}
\end{pmatrix}
$$

(7)

2. Strictly local operators with asymptotically periodic parameters

We start with the following main result of [Tan21]:

**Theorem 2.1 ([Tan21, Theorem A]).** Let $k_0 \in \mathbb{N}$, and let $A_{-k_0}, \ldots, A_{k_0}$ be $n \times n$ matrices-valued sequences on $\mathbb{Z}$ admitting the following limits for $-k_0 \leq k \leq k_0$:

$$
A_k(L) := \lim_{x \to -\infty} A_k(x), \quad A_k(R) := \lim_{x \to +\infty} A_k(x).
$$

(8)

Let

$$
A := \sum_{k=-k_0}^{k_0} A_k \begin{pmatrix} L^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & L^k \end{pmatrix},
$$

(9)

$$
\hat{A}(\tilde{z}, z) := \sum_{k=-k_0}^{k_0} A_k(\tilde{z}) \begin{pmatrix} z^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z^k \end{pmatrix}, \quad z \in \mathbb{T}, \quad \tilde{z} = L, R.
$$

(10)

where $L$ is the bilateral left-shift operator on $\ell^2(\mathbb{Z})$, and where each $A_k$ in (9) is viewed as the bounded multiplication operator on $\ell^2(\mathbb{Z}, \mathbb{C}^n)$. Then the following assertions hold true:

(i) We have that $A$ is Fredholm if and only if $\mathbb{T} \ni z \mapsto \det \hat{A}(\tilde{z}, z) \in \mathbb{C}$ is nowhere vanishing on $\mathbb{T}$ for each $\tilde{z} = L, R$. In this case, the Fredholm index of $A$ is given by

$$
\text{ind } (A) = \text{wn } \left( \det \hat{A}(R, \cdot) \right) - \text{wn } \left( \det \hat{A}(L, \cdot) \right),
$$

(11)

where \( \text{wn } \left( \det \hat{A}(\tilde{z}, \cdot) \right) \) denotes the winding number of the continuous function $\mathbb{T} \ni z \mapsto \det \hat{A}(\tilde{z}, z) \in \mathbb{C}$ with respect to the origin for each $\tilde{z} = L, R$.

(ii) The essential spectrum of $A$ is given by

$$
\sigma_{\text{ess}}(A) = \bigcup_{z \in \mathbb{T}} \sigma \left( \hat{A}(R, z) \right) \cup \bigcup_{z \in \mathbb{T}} \sigma \left( \hat{A}(L, z) \right).
$$

(12)
Any operator $A$ of the form (9) is referred to as an $n$-dimensional \textit{strictly local operator} on the integer lattice $\mathbb{Z}$ throughout this paper. The purpose of the current section is to generalise the existing formulas (11) to (12), by replacing the assumption (8) with the so-called \textit{asymptotically periodic assumption}. More precisely, we assume that there exist natural numbers $n_L, n_R$ with the property that the following limits exist for $-k_0 \leq k \leq k_0$;

\begin{align*}
A_k(L, m) &:= \lim_{x \to -\infty} A_k(n_L \cdot x + m), \quad m \in \{0, \ldots, n_L - 1\}, \quad (13) \\
A_k(R, m) &:= \lim_{x \to +\infty} A_k(n_R \cdot x + m), \quad m \in \{0, \ldots, n_R - 1\}. \quad (14)
\end{align*}

In other words, the doubly-infinite sequences $A_{-k}, \ldots, A_k$ are \textit{asymptotically $(n_L, n_R)$-periodic}. Let us consider the following explicit example;

\textbf{Remark 2.2.} Let $A_k = (A_k(x))_{x \in \mathbb{Z}}$ be asymptotically $(3,2)$-periodic. That is, we assume the existence of the following $3 + 2 = 5$ limits:

\begin{align*}
A_k(L, 0) &= \lim_{x \to -\infty} A_k(3x + 0), & A_k(R, 0) &= \lim_{x \to +\infty} A_k(2x + 0), \\
A_k(L, 1) &= \lim_{x \to -\infty} A_k(3x + 1), & A_k(R, 1) &= \lim_{x \to +\infty} A_k(2x + 1), \\
A_k(L, 2) &= \lim_{x \to -\infty} A_k(3x + 2),
\end{align*}

On the other hand, one can rearrange $A_k(x)$ according to the following table;

\begin{center}
\begin{array}{ccccccc}
A_k(L, 0) & \leftarrow & A_k(-9) & A_k(-6) & A_k(-3) & \cdots \\
A_k(L, 1) & \leftarrow & A_k(-8) & A_k(-5) & A_k(-2) & \cdots \\
A_k(L, 2) & \leftarrow & A_k(-7) & A_k(-4) & A_k(-1) & \cdots \\
\cdots & \cdots & A_k(0) & A_k(2) & A_k(4) & \rightarrow & A_k(R, 0) \\
\cdots & \cdots & A_k(1) & A_k(3) & A_k(5) & \rightarrow & A_k(R, 1)
\end{array}
\end{center}

The first three rows show that $A_k(3x + 0), A_k(3x + 1), A_k(3x + 2)$ have well-defined limits as $x \to -\infty$, whereas the last two rows show that $A_k(2x + 0), A_k(2x + 1)$ have well-defined limits as $x \to +\infty$.

We are now in a position to state the following generalisation of Theorem 2.1;

\textbf{Theorem A.} Let $k_0 \in \mathbb{N}$, and let $A_{-k_0}, \ldots, A_{k_0}$ be finitely many $n \times n$ matrices-valued sequences on $\mathbb{Z}$ admitting the following representations:

\begin{equation}
A_k(x) = \begin{pmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{pmatrix}, \quad x \in \mathbb{Z}, \quad -k_0 \leq k \leq k_0. \quad (15)
\end{equation}
We assume that there exist \( n_L, n_R \in \mathbb{N} \) with the property that the following limits exist for \( 1 \leq i, j \leq n \) and for \(-k_0 \leq k \leq k_0\):

\[
\begin{align*}
a^k_{ij}(L, m) &:= \lim_{x \to -\infty} a^k_{ij}(n_L \cdot x + m), \quad m \in \{0, \ldots, n_L - 1\}, \\
a^k_{ij}(R, m) &:= \lim_{x \to +\infty} a^k_{ij}(n_R \cdot x + m), \quad m \in \{0, \ldots, n_R - 1\}.
\end{align*}
\]

For each \( \sharp = L, R \) and each \( z \in \mathbb{T} \), let \( \hat{A}(\sharp, z) = (\hat{A}_{ij}(\sharp, z))_{ij} \) be the square matrix of dimension \( n \times n \) defined by the following block-matrix representation:

\[
\hat{A}(\sharp, z) := \begin{pmatrix} \hat{A}_{11}(\sharp, z) & \cdots & \hat{A}_{1n}(\sharp, z) \\ \vdots & \ddots & \vdots \\ \hat{A}_{n1}(\sharp, z) & \cdots & \hat{A}_{nn}(\sharp, z) \end{pmatrix}.
\]

\[
\hat{A}_{ij}(\sharp, z) := \sum_{k=-k_0}^{k_0} \begin{pmatrix} a^k_{ij}(\sharp, 0) & 0 & \cdots & 0 \\ 0 & a^k_{ij}(\sharp, 1) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & a^k_{ij}(\sharp, n_z - 1) & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ z \end{pmatrix},
\]

where \( 1 \) denotes the identity matrix of dimension \( n_z - 1 \). If \( A \) is a strictly local operator of the form (10), then the following the following assertions hold true:

(i) We have that \( A \) is Fredholm if and only if \( \mathbb{T} \ni z \mapsto \det(\hat{A}(\sharp, z)) \in \mathbb{C} \) is nowhere vanishing on \( \mathbb{T} \) for each \( \sharp = L, R \). In this case, the Fredholm index of \( A \) is given by (11).

(ii) The essential spectrum of \( A \) is given by (12).

In general, the Fredholm index and essential spectrum are meaningful only in infinite dimensions. Note, however, that Theorem A allows us to fully classify these two topological invariants for a strictly local operator in the language of linear algebra. In terms of practical applications, Theorem A can be applied to the time-evolution operator of a discrete-time quantum walk defined on the integer lattice \( \mathbb{Z} \), provided that it is an operator of the form (9) satisfying the asymptotically periodic assumptions (16) to (17).

Remark 2.3. Theorem 2.1 is a special case of Theorem A. Indeed, with the notation introduced in Theorem A if \( n_z = 1 \) for each \( \sharp = L, R \), then (16) to (17) become:

\[
\begin{align*}
a^k_{ij}(L, 0) &:= \lim_{x \to -\infty} a^k_{ij}(x), \\
a^k_{ij}(R, 0) &:= \lim_{x \to +\infty} a^k_{ij}(x).
\end{align*}
\]

In this case, we show that (18) is given by (11). We define the two matrices \( A_k(L), A_k(R) \) by (5) for each \( k \):

\[
A_k(\sharp) := \begin{pmatrix} a^k_{11}(\sharp, 0) & \cdots & a^k_{1n}(\sharp, 0) \\ \vdots & \ddots & \vdots \\ a^k_{n1}(\sharp, 0) & \cdots & a^k_{nn}(\sharp, 0) \end{pmatrix}, \quad \sharp = L, R.
\]
We have \( \hat{A}_{ij}(\hat{z}, z) = \sum_{k=-k_0}^{k_0} a^k_{ij}(\hat{z}, 0) z^k \), and so

\[
\hat{A}(\hat{z}, z) = \begin{pmatrix}
\sum_{k=-k_0}^{k_0} a^k_{11}(\hat{z}, 0) z^k & \cdots & \sum_{k=-k_0}^{k_0} a^k_{1n}(\hat{z}, 0) z^k \\
\vdots & \ddots & \vdots \\
\sum_{k=-k_0}^{k_0} a^k_{n1}(\hat{z}, 0) z^k & \cdots & \sum_{k=-k_0}^{k_0} a^k_{nn}(\hat{z}, 0) z^k
\end{pmatrix},
\]

which is consistent with (10).

2.1. Preliminaries

By operators we shall always mean everywhere-defined bounded linear operators between Banach spaces throughout this paper. An operator \( A \) on a Hilbert space \( \mathcal{H} \) is said to be Fredholm, if \( \ker A, \ker A^* \) are finite-dimensional and if \( A \) has a closed range. Given such \( A \), we define the Fredholm index of \( A \) by \( \text{ind}(A) := \dim \ker A - \dim \ker A^* \). It is well-known that the Fredholm index is invariant under compact perturbations. That is, given an operator \( A \) on \( \mathcal{H} \) and a compact operator \( K \) on \( \mathcal{H} \), we have that \( A \) is Fredholm if and only if so is \( A + K \), and in this case \( \text{ind}(A) = \text{ind}(A + K) \). The (Fredholm) essential spectrum of an operator \( A \) on \( \mathcal{H} \) is defined as the set \( \sigma_{\text{ess}}(A) \) of all \( \lambda \in \mathbb{C} \), such that \( A - \lambda \) fails to be Fredholm. Note that \( \sigma_{\text{ess}}(A) \) is also stable under compact perturbations.

The Hilbert space of all square-summable \( \mathbb{C} \)-valued sequences \( \Psi = (\Psi(x))_{x \in \mathbb{Z}} \) is denoted by the shorthand \( \ell^2(\mathbb{Z}) := \ell^2(\mathbb{Z}, \mathbb{C}) \). We have a natural orthogonal decomposition \( \ell^2(\mathbb{Z}) = \ell^2_1(\mathbb{Z}) \oplus \ell^2_2(\mathbb{Z}) \), where

\[
\ell^2_1(\mathbb{Z}) := \{ \Psi \in \ell^2(\mathbb{Z}) \mid \Psi(x) = 0 \ \forall \ x \geq 0 \}, \\
\ell^2_2(\mathbb{Z}) := \{ \Psi \in \ell^2(\mathbb{Z}) \mid \Psi(x) = 0 \ \forall \ x < 0 \}.
\]

The orthogonal projections of \( \ell^2(\mathbb{Z}) \) onto the above subspaces shall be denoted by \( P_L \) and \( P_R = 1 - P_L \) respectively. For each \( \hat{z} = L, R \), the orthogonal projection \( P_{\hat{z}} \) can be written as \( P_{\hat{z}} = \iota_\hat{z} \iota_{\hat{z}}^* \), where \( \iota_\hat{z} : \ell^2_\hat{z}(\mathbb{Z}) \hookrightarrow \ell^2(\mathbb{Z}) \) is the inclusion mapping. The left-shift operator \( L \) on \( \ell^2(\mathbb{Z}) \) is defined by

\[
L\Psi := \Psi(\cdot + 1), \quad \Psi \in \ell^2(\mathbb{Z}).
\]

For each \( m \in \mathbb{N} \) any operator \( X \) on \( \ell^2(\mathbb{Z}, \mathbb{C}^m) := \bigoplus_{j=1}^m \ell^2(\mathbb{Z}) \) admits the following unique block-operator matrix representation;

\[
X = \begin{pmatrix}
X_{11} & \cdots & X_{1m} \\
\vdots & \ddots & \vdots \\
X_{m1} & \cdots & X_{mm}
\end{pmatrix},
\]

where each \( X_{ij} \) is an operator on \( \ell^2(\mathbb{Z}) \). We shall agree to use the shorthand \( X = (X_{ij}) \) to mean that (21) holds true. With this representation of \( X \) in mind, for each \( \hat{z} = L, R \),
we define the following compression on $\ell^2(Z, C) := \bigoplus_{j=1}^m \ell^2(Z)$:

$$X_j := 
\begin{pmatrix}
\ell^2_1 X_1 & \cdots & \ell^2_1 X_m \\
\vdots & \ddots & \vdots \\
\ell^2_1 X_m & \cdots & \ell^2_1 X_m
\end{pmatrix},
$$

(22)

For each $m \in \mathbb{N}$ the operator $\tau_m : \bigoplus_{j=1}^m \ell^2(Z) \to \ell^2(Z)$ is defined as the inverse of the following unitary operator

$$\ell^2(Z) \ni \psi \mapsto \begin{pmatrix}
\psi(m) \\
\vdots \\
\psi(m \cdot m - 1)
\end{pmatrix} \in \bigoplus_{j=1}^m \ell^2(Z).
$$

(23)

In particular, $\tau_1$ is the identity operator on $\ell^2(Z)$. Similarly, for each $\tau = L, R$ and each $m \in \mathbb{N}$ we define the operator $\tau_{\tau,m} : \bigoplus_{j=1}^m \ell^2_1(Z) \to \ell^2_1(Z)$ by

$$\tau_{\tau,m} := \ell^2_1 t_m \left( \bigoplus_{j=1}^m t_1 \right).$$

It is easy to see that $\tau_{\tau,m}$ is a unitary operator, since its inverse $\tau^*_{\tau,m} = \left( \bigoplus_{j=1}^m t_1^* \right)^\dagger \tau_{\tau,m} t_1$ is given explicitly by the following formula;

$$\ell^2(Z) \ni \psi \mapsto \begin{pmatrix}
\psi(m) \\
\vdots \\
\psi(m \cdot m - 1)
\end{pmatrix} \in \bigoplus_{j=1}^m \ell^2(Z),
$$

(24)

where $\psi(m), \ldots, \psi(m \cdot m - 1) \in \ell^2(Z)$.

**Lemma 2.4.** If $a = (a(x))_{x \in Z}$ is a bounded $\mathbb{C}$-valued sequence, identified with the associated multiplication operator on $\ell^2(Z)$, then for each $n \in \mathbb{N}$ we have

$$\tau^*_n a \tau_n = \begin{pmatrix}
a(n) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & a(n \cdot +n - 1)
\end{pmatrix},$$

(25)

$$\tau^*_n L \tau_n = \begin{pmatrix}
1 \\
\vdots \\
0
\end{pmatrix},$$

(26)

where $1$ is the identity operator on $\bigoplus_{j=1}^{n-1} \ell^2(Z)$.8
Proof. For each \( \psi \in \ell^2(\mathbb{Z}) \) we have

\[
\tau_n^* a\psi = \begin{pmatrix} a(n\cdot)\psi(n\cdot) \\ \vdots \\ a(n \cdot + n - 1)\psi(n \cdot + n - 1) \end{pmatrix} = \begin{pmatrix} a(n\cdot) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & a(n \cdot + n - 1) \end{pmatrix} \tau_n^* \psi, 
\]

\[
\tau_n^* L\psi = \begin{pmatrix} \psi(n \cdot + 1) \\ \vdots \\ \psi(n \cdot + n) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ L & 0 & 0 & \ldots & 0 \end{pmatrix} \tau_n^* \psi.
\]

The claim follows. \( \square \)

**Corollary 2.5.** For each \( \sharp = L, R \) and each \( n \in \mathbb{N} \), we have \( \tau_n^* P_n^* \tau_n = \bigoplus_{j=1}^n P_k \).

**Proof.** For each \( \sharp = L, R \), we can identify \( P_k \) with the multiplication operator \( \delta_k \). It follows from (25) that

\[
\tau_n^* \delta_k \tau_n = \begin{pmatrix} \delta_k(n\cdot) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \delta_k(n \cdot + n - 1) \end{pmatrix},
\]

where \( \delta_k(n\cdot) = \cdots = \delta_k(n \cdot + n - 1) = \delta_k \). Therefore, \( \tau_n^* P_k^* \tau_n = \bigoplus_{j=1}^n P_k \). \( \square \)

In fact, the special case of Theorem A where \( n_L = n_R \) can be easily proved by making use of Lemma 2.4. As for the general case \( n_L \neq n_R \), we require the following non-trivial fact:

**Lemma 2.6.** For each \( \sharp = L, R \) and each \( m \in \mathbb{N} \), we have

\[
\left( \bigoplus_{j=1}^n \tau_{j,m}^* \right) A_k \left( \bigoplus_{j=1}^n \tau_{j,m} \right) = \left( \bigoplus_{j=1}^n \tau_{j,m}^* \right) A \left( \bigoplus_{j=1}^n \tau_{j,m} \right)_{\sharp}.
\] (27)

More explicitly, the \( m \times n \)-dimensional strictly local operator \( \left( \bigoplus_{j=1}^n \tau_{j,m}^* \right) A \left( \bigoplus_{j=1}^n \tau_{j,m} \right) \) coincides with the block-operator matrix \( B(m) \) defined by the following formulas:

\[
B(m) := \begin{pmatrix} B_{11}(m) & \cdots & B_{1n}(m) \\ \vdots & \ddots & \vdots \\ B_{n1}(m) & \cdots & B_{nn}(m) \end{pmatrix},
\] (28)

\[
B_{ij}(m) := \sum_{k=-k_0}^{k_0} \begin{pmatrix} a_{ij}^*(m) & 0 & \ldots & 0 \\ 0 & a_{ij}^*(m+1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & a_{ij}^*(m+n-1) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}_k,
\] (29)

where \( 1 \) denotes the identity operator of dimension \( m - 1 \).
The formula (27) shows that ♯-compression and τₘ-unitary transforms can be interchanged.

**Proof.** Note that $A$ can be expressed as a block-operator matrix form $A = (A_{ij})$ according to (22), where

$$A_{ij} := k_0 \sum_{k=-k_0}^{k_0} a_{ij}^k L^k.$$

Note that the left-hand side of (27) becomes;

$$\left( \bigoplus_{j=1}^{n} \tau_{\sharp,j} \right) A \left( \bigoplus_{j=1}^{n} \tau_{\sharp,j} \right) = \begin{pmatrix} \tau_{\sharp,m} (\tau_{\sharp}^{*} A_{11} \tau_{\sharp}) \tau_{\sharp,m} & \cdots & \tau_{\sharp,m} (\tau_{\sharp}^{*} A_{1n} \tau_{\sharp}) \tau_{\sharp,m} \\ \vdots & \ddots & \vdots \\ \tau_{\sharp,m} (\tau_{\sharp}^{*} A_{n1} \tau_{\sharp}) \tau_{\sharp,m} & \cdots & \tau_{\sharp,m} (\tau_{\sharp}^{*} A_{nn} \tau_{\sharp}) \tau_{\sharp,m} \end{pmatrix} \bigoplus_{j=1}^{n} \ell_{\sharp,j} (\mathbb{Z}, C^m).$$

Note that for each $i, j$ we obtain

$$\tau_{\sharp,j}^* (\tau_{\sharp}^* A_{ij} \tau_{\sharp}) \tau_{\sharp,j} = \begin{pmatrix} \bigoplus_{j=1}^{m} \tau_{\sharp,j}^* \tau_{\sharp,j}^* (\tau_{\sharp}^* A_{ij} \tau_{\sharp}) \tau_{\sharp,j}^* \tau_{\sharp,j}^* \end{pmatrix} \left( \bigoplus_{j=1}^{m} \tau_{\sharp,j} \right)$$

where the second last equality follows from Corollary 2.5 and the last equality follows from $\tau_{\sharp,j}^* \tau_{\sharp,j}^* = 1$. Therefore, (27) holds true. Now,

$$\tau_{\sharp,j}^* A_{ij} \tau_{\sharp,j} = \sum_{k=-k_0}^{k_0} \tau_{\sharp,j}^* a_{ij}^k L^k \tau_{\sharp,j} = \sum_{k=-k_0}^{k_0} \tau_{\sharp,j}^* a_{ij}^k \tau_{\sharp,j} (\tau_{\sharp,j}^* L \tau_{\sharp,j})^k.$$

The claim follows from Lemma 2.4. \qed
2.2. Proof of the main theorem

Lemma 2.7. For each $\sharp = L, R$ let

$$A(\sharp) := \begin{pmatrix} A_{11}(\sharp) & \cdots & A_{1n}(\sharp) \\ \vdots & \ddots & \vdots \\ A_{1n}(\sharp) & \cdots & A_{nn}(\sharp) \end{pmatrix},$$

$$A_{ij}(\sharp) := \sum_{k=-k_0}^{k_0} \begin{pmatrix} a_{ij}^k(\sharp, 0) & 0 & \cdots & 0 \\ 0 & a_{ij}^k(\sharp, 1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{ij}^k(\sharp, n_\sharp - 1) \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ L \ 0 \ \cdots \ \cdots \ 0 \end{pmatrix}^k,$$

where $1$ denotes the identity operator of dimension $n_\sharp - 1$. Then the following operators are compact:

$$\left( \bigoplus_{j=1}^{n} \tau_{n_\sharp, n_\sharp}^* \right) A_2 \left( \bigoplus_{j=1}^{n} \tau_{n_\sharp, n_\sharp} \right) - A(\sharp)_2 = (B(n_\sharp) - A(\sharp))_2,$$

where $B(n_\sharp) - A(\sharp) = (B_{ij}(n_\sharp) - A_{ij}(\sharp))_{i,j}$. Since $\bigoplus_{j=1}^{n} \tau_{n_\sharp}^* = \bigoplus_{j=1}^{n} \tau_{n_\sharp}^* \bigoplus_{j=1}^{n} P_2$, it remains to show that $C_{ij}(\sharp) := \left( \bigoplus_{j=1}^{n} P_2 \right) (B_{ij}(n_\sharp) - A_{ij}(\sharp))$ is compact. We obtain

$$C_{ij}(\sharp) = \sum_{k=-k_0}^{k_0} \begin{pmatrix} \delta_{ij}(a_{ij}(\sharp, 0) - a_{ij}^k(n_\sharp - 1)) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_{ij}(a_{ij}(n_\sharp - 1) - a_{ij}^k(\sharp, n_\sharp - 1)) \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ L \ 0 \ \cdots \ \cdots \ 0 \end{pmatrix}^k,$$

where the last equality follows from Lemma 2.6. Note that each $\delta_x a_{ij}^k(n_\sharp \cdot + j)$ has the following 2-sided limits:

$$\lim_{x \to -\infty} \delta_x a_{ij}^k(n_\sharp x + j) = \begin{cases} a_{ij}^k(-\infty, j), & \sharp = L, \\ 0, & \sharp = R, \end{cases}$$

$$\lim_{x \to +\infty} \delta_x a_{ij}^k(n_\sharp x + j) = \begin{cases} 0, & \sharp = L, \\ a_{ij}^k(+\infty, j), & \sharp = R. \end{cases}$$

It follows that $\delta_x(a_{ij}^k(\sharp, j) - a_{ij}^k(n_\sharp x + j)) \to 0$ as $x \to \pm \infty$. The claim follows. \(\square\)
Proof of Theorem A. Note that \( A - A_L \oplus A_R \) is finite rank by [Tan21, Corollary 2.2]. Since the Fredholm index and essential spectrum are invariant under compact perturbations, it suffices to consider \( A' := A_L \oplus A_R \) from here on. It follows from Lemma 2.7 that the following difference is compact:

\[
\left( \bigoplus_{j=1}^{n} \tau_{z, n_j}^* \right) A_L \left( \bigoplus_{j=1}^{n} \tau_{z, n_j} \right) - A(z)_I.
\]

(i) Since the Fredholmness is invariant under unitary transforms and compact perturbations, we have that \( A' \) is Fredholm if and only if \( A(L)_L, A(R)_R \) are Fredholm. In this case,

\[
\text{ind } A' = \text{ind } A(L)_L + \text{ind } A(R)_R.
\]

On the other hand, it follows from [Tan21, Theorem 2.4](i) that for each \( \sharp = L, R \) the operator \( A(\sharp)_I \) is Fredholm if and only if \( T \ni z \mapsto \det \hat{A}(\sharp, z) \in \mathbb{C} \) is nowhere vanishing on \( T \). In this case, the Fredholm index of \( A(\sharp)_I \) is given by

\[
\text{ind } A(\sharp)_I = \begin{cases} \text{wn} \left( \det \hat{A}(+\infty, \cdot) \right), & \sharp = R, \\ -\text{wn} \left( \det \hat{A}(-\infty, \cdot) \right), & \sharp = L. \end{cases}
\]

The claim follows.

(ii) Since the essential spectrum is invariant under unitary transforms and compact perturbations, we have

\[
\sigma_{\text{ess}}(A') = \sigma_{\text{ess}}(A(L)_L) \cup \sigma_{\text{ess}}(A(R)_R) = \bigcup_{z \in T} \sigma \left( \hat{A}(R, z) \right) \cup \bigcup_{z \in T} \sigma \left( \hat{A}(L, z) \right),
\]

where the last equality follows from (11).

3. Applications of Theorem A

The purpose of the current section to generalise the existing index formula (6).

3.1. Two main theorems

We first give a brief description of the existing index theory for chirally symmetric unitary operators here (see, for example, [MST21] or [Tan21, §3.1]). Let \( \mathcal{H} \) be a Hilbert space, and let \((\Gamma, U)\) be a pair of a unitary self-adjoint operator \( \Gamma : \mathcal{H} \to \mathcal{H} \) and a unitary operator \( U : \mathcal{H} \to \mathcal{H} \), satisfying the chiral symmetry condition (1). It can then be shown that the real part \( R := (U + U^*)/2 \) and imaginary part \( Q := (U - U^*)/2 \) of \( U \) admit the following block-operator matrix representations:

\[
R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}_{\ker(\Gamma-1) \oplus \ker(\Gamma+1)}, \quad Q = \begin{pmatrix} 0 & Q_2 \\ Q_1 & 0 \end{pmatrix}_{\ker(\Gamma-1) \oplus \ker(\Gamma+1)}.
\]
where the first equality follows from the commutation relation \([\Gamma, R] := \Gamma R - R \Gamma = 0\), whereas the second equality follows from the anti-commutation relation \(\{\Gamma, Q\} := \Gamma Q + Q \Gamma = 0\) (see \cite{Suz19}, Lemma 2.2 for details). Since \(R, Q\) are self-adjoint, we have \(R^* j = R j\) for each \(j = 1, 2\), and \(Q^2 = Q^*_1\). It follows that the unitary operator \(U = R + iQ\) admits the following representation;

\[
U = \begin{pmatrix} R_1 & iQ_2 \\ iQ_1 & R_2 \end{pmatrix}_{\ker(\Gamma - 1) \oplus \ker(\Gamma + 1)}.
\] (31)

With (31) in mind, we introduce the following formal indices:

\[
\text{ind}^\pm(\Gamma, U) := \dim \ker(\Gamma R \mp 1) - \dim \ker(\Gamma R \mp 1),
\] (32)

\[
\text{ind}(\Gamma, U) := \dim \ker Q_1 - \dim \ker Q_2.
\] (33)

If \(\pm 1 \notin \sigma_{\text{ess}}(U)\), then \(\text{ind}^\pm(\Gamma, U)\) is a well-defined integer, and (2) holds true. We prove the following index formula in this section;

**Theorem B.** Let \((\Gamma, U) = (\Gamma_{\text{suz}}, U_{\text{suz}})\) be defined by (3) to (4). Suppose that there exist \(n_{-\infty}, n_{+\infty} \in \mathbb{N}\) with the property that the following limits exist for each \(\star = -\infty, +\infty\);

\[
\zeta(\star, m) := \lim_{x \to \star} \zeta(n_\star \cdot x + m), \quad \zeta \in \{p, a\}, \ m \in \{0, \ldots, n_\star - 1\}.
\] (34)

(i) Then \(\pm 1 \notin \sigma_{\text{ess}}(U)\) if and only if for each \(\star = -\infty, +\infty\)

\[
\prod_{m=0}^{n_\star - 1} (1 + p(\star, m))(1 \mp a(\star, m)) \neq \prod_{m=0}^{n_\star - 1} (1 - p(\star, m))(1 \pm a(\star, m)).
\] (35)

(ii) Let us impose the following condition;

\[
\prod_{m=0}^{n_\star - 1} (1 + p(\star, m))(1 \mp a(\star, m)) + \prod_{m=0}^{n_\star - 1} (1 - p(\star, m))(1 \pm a(\star, m)) > 0.
\] (36)

For each \(\star = -\infty, +\infty\) let \(p(\star), a(\star) \in [-1, 1]\) be uniquely defined through the following formula:

\[
\prod_{m=0}^{n_\star - 1} (1 + \zeta(\star, m)) \bigg/ \prod_{m=0}^{n_\star - 1} (1 - \zeta(\star, m)) = \left( \frac{1 + \zeta(\star)}{1 - \zeta(\star)} \right)^{n_\star}, \quad \zeta = p, a, \quad \star = -\infty, +\infty.
\] (37)

Then \(\pm 1 \notin \sigma_{\text{ess}}(U)\) if and only if \(p(\pm \infty) \neq \pm a(\pm \infty)\). Moreover, in this case, we have the following formula;

\[
\text{ind}^\pm(\Gamma, U) = \frac{\text{sgn}(p(+\infty) \mp a(+\infty)) - \text{sgn}(p(-\infty) \mp a(-\infty))}{2} \in \{-1, 0, 1\}. \quad (38)
\]
We shall make use of the following arithmetic convention for each \( r \in (0, \infty] \):
\[
r + \infty = \infty + r = \infty, \quad r \cdot \infty = \infty \cdot r = \infty, \quad 0^{-1} = \infty, \quad \infty^{-1} = 0,
\]
where \( 0 \cdot \infty, \infty \cdot 0 \) are left undefined throughout this paper. With this convention in mind, we have the homeomorphism \([0, \infty] \ni s \mapsto s^{-1} \in [0, \infty] \), where the extended half-line \([0, \infty]\) is viewed as a metric space in the obvious way.

It follows from Theorem B(i) that the assumption (36) is a necessary condition for \( \pm 1 \in \sigma_{\text{ess}}(U) \). Note that the assumption (36) ensures that the left-hand side of (37) is a well-defined number in \([0, \infty]\), since the problematic case \( 0/0 \) never occurs. Note also that \( \zeta(*) \) can be indeed uniquely defined through (37), since we have another homeomorphism \( \Lambda : [-1, 1] \to [0, \infty] \) defined by
\[
\Lambda(s) := \frac{1 + s}{1 - s}, \quad s \in [-1, 1].
\]
The function \( s \mapsto \Lambda(s) \) increases from \( \Lambda(-1) = 0 \) to \( \Lambda(+1) = \infty \) as in the following figure;

![Figure 1: This figure represents the graph of \( t = \Lambda(s) \).](image)

For each \( s, s' \in [-1, 1] \), we have \( \Lambda(-s) = \Lambda(s)^{-1} \). Furthermore, if \( ss' \neq -1 \), then the product \( \Lambda(s)\Lambda(s') \) is a well-defined extended non-negative real number, and the following two assertions hold true:
\[
\Lambda(s)\Lambda(s') = \Lambda \left( \frac{s + s'}{1 + ss'} \right), \quad (39)
\]
\[
\Lambda(s)\Lambda(s') \leq 1 \text{ if and only if } s + s' \leq 0, \quad (40)
\]
where \( 1 + ss' > 0 \) in (39), and where the notation \( \leq \) in (40) simultaneously denotes the three binary relations \( >, =, < \).

**Theorem C.** Let \( (\Gamma, U) = (\Gamma_{\text{aux}}, U_{\text{aux}}) \) be defined by (3) to (4). Suppose that there exist \( n_{-\infty}, n_{+\infty} \in \mathbb{N} \) with the property that the limits (34) exist for each \( * = -\infty, +\infty \). Let
Suppose that exist for each operator, and that there exist where we let

\[
\begin{bmatrix}
\frac{r_0(\pm, 0)}{r_1(\pm, 0)} + \frac{r_1(\pm, 0)}{r_0(\pm, 1)} z^s
\end{bmatrix}
\]

\(n_s = 1,\)

\[
2 \tilde{R}_{\pm}(x, z) :=
\begin{cases}
\frac{r_0(\pm, 0)}{r_1(\pm, 0)} + \frac{r_1(\pm, 0)}{r_0(\pm, 1)} z^s, & n_s = 2,
\frac{r_0(\pm, 0)}{r_1(\pm, 0)} + \frac{r_1(\pm, 0)}{r_0(\pm, 1)} z^s,
\end{cases}
\]

\(n_s \geq 3,\)

\[
\begin{align*}
&\frac{r_0(\pm, m)}{r_1(\pm, m)} := p(\pm, m) \pm 1) a(\pm, m + 1), \\
&\frac{r_1(\pm, m)}{r_0(\pm, 1)} := \sqrt{(1 \mp p(\pm, m))(1 \pm p(\pm, m + 1))(1 - a(\pm, m + 1)^2)},
\end{align*}
\]

where we let \(p(\pm, 0) := p(0, 0)\) and \(a(\pm, n_s) := a(\pm, 0)\). Then

\[
\sigma_{ess}(U) = \sigma(-\infty) \cup \sigma(\pm \infty),
\]

\[
\sigma(\pm) := \left\{ z \in \mathbb{T} \mid \Re z \in \sigma(\tilde{R}_-(x, z)) \cup \sigma(\tilde{R}_+(x, z)) \right\}, \quad \pm = -\infty, +\infty.
\]

3.2. Proof of Theorem B

We shall make use of the following remark in what follows;

**Remark 3.1.** Suppose that \(A = \alpha_{-1} L^{-1} + a_0 + a_1 L\) is a one-dimensional strictly local operator, and that there exist \(n_{-\infty}, n_{+\infty} \in \mathbb{N}\) with the property that the following limits exist for each \(\pm = -\infty, +\infty;\)

\[
\alpha_j(\pm, m) := \lim_{x \to \pm \infty} \alpha_j(n_s \cdot x + m), \quad j = -1, 0, 1, \quad m \in \{0, \ldots, n_s - 1\}.
\]

We introduce the following matrices according to [13] to [19] for each \(\pm = -\infty, +\infty;\)

\[
\hat{A}(x, z) := \sum_{j=-1,0,1} \left( \begin{array}{ccc}
\alpha_j(\pm, 0) & 0 & 0 \\
0 & \alpha_j(\pm, 1) & 0 \\
0 & 0 & \alpha_j(\pm, n_s - 1)
\end{array} \right) \left( \begin{array}{c}
1 \\
0 \\
0
\end{array} \right), \quad z \in \mathbb{T},
\]

where \(1\) denotes the identity matrix of dimension \(n_s - 1\). Note that each \(\hat{A}(x, z)\) admits the following explicit representation;

\[
\hat{A}(x, z) =
\begin{cases}
\frac{\alpha_0(\pm, 0)}{\alpha_0(\pm, 1)} z^s + \frac{\alpha_1(\pm, 0)}{\alpha_0(\pm, 1)} z, & n_s = 1,
\frac{\alpha_0(\pm, 0)}{\alpha_0(\pm, 1)} z^s + \frac{\alpha_1(\pm, 0)}{\alpha_0(\pm, 1)} z^s,
\end{cases}
\]

\(n_s = 2,\)

\[
\hat{A}(x, z) =
\begin{cases}
\frac{\alpha_0(\pm, 0)}{\alpha_0(\pm, 1)} z^s + \frac{\alpha_1(\pm, 0)}{\alpha_0(\pm, 1)} z^s, & n_s = 3,
\frac{\alpha_0(\pm, 0)}{\alpha_0(\pm, 1)} z^s + \frac{\alpha_1(\pm, 0)}{\alpha_0(\pm, 1)} z^s,
\end{cases}
\]

\(n_s \geq 3.\)
Lemma 3.2. If \((\Gamma, U) = (\Gamma_{suz}, U_{suz})\) is defined by \((49)\) to \((49)\), then there exist two unitary operators \(\epsilon, \eta\) on \(\ell^2(Z, \mathbb{C}^2)\), such that the following four decompositions hold true:

\[
\epsilon^* \Gamma \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon^* U_{\text{suz}} \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\eta^*)^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\eta^*) \quad \text{for each} \quad \epsilon, \eta.
\]

(48)

\[
\eta^* \Gamma^* \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta^* U_{\text{suz}} \eta = (\eta^*)^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\eta^*)^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(49)

More explicitly, if we let \(\zeta_k := \sqrt{1 + \zeta}\) for each \(\zeta = p, a\), then the unitary operators \(\epsilon, \eta\) are given respectively by

\[
\epsilon := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} \begin{pmatrix} p_+ & -p_- \\ p_- & p_+ \end{pmatrix}, \quad \eta := \frac{1}{\sqrt{2}} \begin{pmatrix} a_+ & -a_- \\ a_- & a_+ \end{pmatrix}.
\]

(50)

Proof. Note first that we have the following unitary diagonalisation for each \(\zeta = p, a\) and each \(x \in Z\) (see, for example, [Tan21, Example 3.1]):

\[
\left( \begin{array}{cc} \zeta_k(x) & -\zeta_k(x) \\ \zeta_k(x) & \zeta_k(x) \end{array} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_k(x) & -\zeta_k(x) \\ \zeta_k(x) & \zeta_k(x) \end{array} \right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

This result motivates us to introduce the unitary operators \(\epsilon, \eta\) defined by \((50)\). Indeed,

\[
\epsilon^* \Gamma \epsilon = \left( \begin{array}{cc} p_+ & -p_- \\ p_- & p_+ \end{array} \right) \left( \begin{array}{cc} p & \sqrt{1 - \zeta_k(x)^2} \\ \sqrt{1 - \zeta_k(x)^2} & -p \end{array} \right) \left( \begin{array}{cc} p_+ & -p_- \\ p_- & p_+ \end{array} \right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
\eta^* \Gamma^* \eta = \left( \begin{array}{cc} a_+ & -a_- \\ a_- & a_+ \end{array} \right) \left( \begin{array}{cc} a & \sqrt{1 - \zeta_k(x)^2} \\ \sqrt{1 - \zeta_k(x)^2} & -a \end{array} \right) \left( \begin{array}{cc} a_+ & -a_- \\ a_- & a_+ \end{array} \right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

On the other hand,

\[
\epsilon^* U_{\text{suz}} \epsilon = (\epsilon^* \epsilon) (\epsilon^* \Gamma \epsilon) = (\epsilon^* \epsilon) (\epsilon^* \epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\eta^*)^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\eta^*) \quad \text{for each} \quad \epsilon, \eta.
\]

\[
\eta^* U_{\text{suz}} \eta = (\eta^*)^* (\eta^* \epsilon) (\eta^* \epsilon) (\eta^* \eta) = (\eta^*)^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\eta^*)^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

With the notation introduced in Lemma 3.2, it is easy to see that the operator \(F := \eta^* \epsilon\) is given explicitly by

\[
F = \frac{1}{2} \begin{pmatrix} p_+ a_+ + a_- L^* p_- & -p_- a_+ + a_- L^* p_+ \\ -p_+ a_- + a_+ L^* p_- & -p_- a_+ + a_+ L^* p_+ \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}.
\]

(51)
It follows from [CGWW21, Lemma 3.2] that $U \mp 1$ is Fredholm if and only if $F_{1, \pm}, F_{2, \pm}$ are Fredholm. In this case, we have
\[
\text{ind}_\pm(T; U) = \text{ind}_\pm(e^* T e, e^* U e) = \text{ind} F_{1, \pm} = -\text{ind} F_{2, \pm},
\]
where the first equality follows from the unitary invariance of the indices ind$_\pm$, and where the last two equalities follow from [CGWW21, Lemma 3.2]. Therefore, it remains to compute the Fredholm index of the strictly local operators $F_{1, \pm} = \mp p_+ a_\mp + p_- (-1) a_\pm L^*$ with the aid of the following lemma;

**Lemma 3.3.** Let $\alpha, \beta \in \mathbb{R}$, and let $f(z) := \alpha + \beta z^*$ for each $z \in \mathbb{T}$. Then $f$ is nowhere vanishing if and only if $|\alpha| \neq |\beta|$. In this case, we have
\[
\text{wn}(f) = \begin{cases} -1, & |\alpha| < |\beta|, \\ 0, & |\alpha| > |\beta|. \end{cases}
\]

**Proof.** On one hand, if $\beta = 0$, then the constant function $f = \alpha$ is nowhere vanishing with $\text{wn}(f) = 0$ if and only if $|\alpha| \neq 0$. On the other hand, if $\beta \neq 0$, then the image of $f$ is the circle centred at $\alpha$ with the non-zero radius $|\beta|$. Note that the intersection of the circle and the real line $\mathbb{R}$ is the two-point set $\{\alpha - |\beta|, \alpha + |\beta|\}$. It follows that $f$ is nowhere vanishing if and only if $|\alpha| \neq |\beta|$, where the winding number of $f$ is either $-1$ or $0$. We have $\text{wn}(f) = -1$ if and only if $\alpha - |\beta| < 0 < \alpha + |\beta|$ if and only if $|\alpha| < |\beta|$. Similarly, $\text{wn}(f) = 0$ if and only if $|\alpha| > |\beta|$. \qed

**Proof of Theorem 2** For each $\zeta = p, a$, and each $* = \pm \infty$, let
\[
\zeta_\pm(\ast, m) := \sqrt{1 \pm \zeta(\ast, m)}, \quad m \in \{0, \ldots, n_* - 1\}.
\]
If we let $f_{0, \pm} := \mp p_+ a_\mp$ and $f_{-1, \pm} := p_- (-1) a_\pm$, then $F_{1, \pm} = f_{-1, \pm} L^* + f_{0, \pm} + 0L$. For each $z \in \mathbb{T}$ we introduce the following matrix according to [17]:
\[
2 F_{1, \pm}(\ast, z) := \begin{cases} f_{0, \pm}(\ast, 0) + f_{-1, \pm}(\ast, 0) z^*, & n_* = 1, \\ \begin{pmatrix} f_{0, \pm}(\ast, 0) & f_{-1, \pm}(\ast, 0) z^* \\ f_{-1, \pm}(\ast, 1) & f_{0, \pm}(\ast, 1) \\ 0 & f_{-1, \pm}(\ast, 2) f_{0, \pm}(\ast, 2) \\ 0 & 0 & \ddots & \ddots \\ f_{0, \pm}(\ast, n_* - 2) & 0 & \ddots & \ddots \\ 0 & 0 & 0 & f_{0, \pm}(\ast, n_* - 1) f_{-1, \pm}(\ast, n_* - 1) \\ 0 & 0 & 0 & 0 & f_{0, \pm}(\ast, n_* - 1) f_{-1, \pm}(\ast, n_* - 1) \end{pmatrix}, & n_* \geq 3. \end{cases}
\]

(i) Note first that the following equality holds true, if $n_* = 1, 2$;
\[
\det(2 F_{1, \pm}(\ast, z)) = \prod_{m=0}^{n_* - 1} f_{0, \pm}(\ast, m) + (-1)^{n_* + 1} \left( \prod_{m=0}^{n_* - 1} f_{-1, \pm}(\ast, m) \right) z^*. \tag{52}
\]
In fact, the co-factor expansion easily allows us to prove that (52) also holds true for any \( n_* \geq 3 \), since the determinant of any triangular matrix is the product of its diagonal entries. It follows from (52) that

\[
\det(2F_{1,\pm}(\star,z)) = \prod_{m=0}^{n_*-1} p_+(\star,m) a_+(\star,m) + \left( -1 \right)^{n_*+1} \prod_{m=0}^{n_*-1} p_-(\star,m) a_{\pm}(\star,m) \]

since \( p'_\cdot := p_-(\cdot - 1) \) satisfies

\[
\prod_{m=0}^{n_*-1} p'_\cdot(\star,m) = p_-(\star, n_* - 1) p_-(\star, 0) \ldots p_-(\star, n_* - 2) = \prod_{m=0}^{n_*-1} p_-(\star,m).
\]

It follows from Lemma 33 that \( z \mapsto \det F_{1,\pm}(\star,z) \) is nowhere vanishing if and only if (55) holds true for each \( \star = -\infty, +\infty \), since

\[
\prod_{m=0}^{n_*-1} \mp p_\star(\star,m) a_\pm(\star,m) = (\mp 1)^{n_*} \left( \prod_{m=0}^{n_*-1} (1 + p(\star,m))(1 \mp a(\star,m)) \right)^{1/2},
\]

\[
(-1)^{n_*+1} \prod_{m=0}^{n_*-1} p_-(\star,m) a_{\pm}(\star,m) = (-1)^{n_*+1} \left( \prod_{m=0}^{n_*-1} (1 - p(\star,m))(1 \pm a(\star,m)) \right)^{1/2}.
\]

Moreover, if (55) holds true, then \( w_\pm(\star) := \text{wn}(\det F_{1,\pm}(\star,\cdot)) \) is given by

\[
w_\pm(\star) = \begin{cases} 
-1, & \prod_{m=0}^{n_*-1} (1 + p(\star,m))(1 \mp a(\star,m)) < \prod_{m=0}^{n_*-1} (1 - p(\star,m))(1 \pm a(\star,m)), \\
0, & \prod_{m=0}^{n_*-1} (1 + p(\star,m))(1 \mp a(\star,m)) \geq \prod_{m=0}^{n_*-1} (1 - p(\star,m))(1 \pm a(\star,m)).
\end{cases}
\]

(ii) Suppose that (56) holds true. Note that

\[
\Lambda(-\zeta(\star))^{n_*} = (\Lambda(\zeta(\star)))^{n_*^{-1}} = \left( \frac{\prod_{m=0}^{n_*-1} (1 + \zeta(\star,m))}{\prod_{m=0}^{n_*-1} (1 - \zeta(\star,m))} \right)^{-1} \equiv \frac{\prod_{m=0}^{n_*-1} (1 - \zeta(\star,m))}{\prod_{m=0}^{n_*-1} (1 + \zeta(\star,m))},
\]

We consider

\[
\prod_{m=0}^{n_*-1} (1 + p(\star,m)) \prod_{m=0}^{n_*-1} (1 \mp a(\star,m)) \leq \prod_{m=0}^{n_*-1} (1 - p(\star,m)) \prod_{m=0}^{n_*-1} (1 \pm a(\star,m)),
\]

where the notation \( \equiv \) simultaneously denotes the three binary relations \(<, =, >\). On one hand, if \( \prod_{m=0}^{n_*-1} (1 - p(\star,m))(1 \pm a(\star,m)) > 0 \), then (54) is equivalent to

\[
\Lambda(p(\star))^{n_*} \Lambda(\mp a(\star))^{n_*} \equiv 1 \iff \Lambda(p(\star))\Lambda(\mp a(\star)) \equiv 1 \iff p(\star) \mp a(\star) \not\equiv 0,
\]

where the first equivalence follows from the fact that \([0, \infty] \ni s \mapsto s^{n_*} \in [0, \infty] \) is an increasing function. On the other hand, if \( \prod_{m=0}^{n_*-1} (1 + p(\star,m))(1 \mp a(\star,m)) \neq 0 \), then (54) is equivalent to

\[
1 \equiv \Lambda(-p(\star))^{n_*} \Lambda(\pm a(\star))^{n_*} \iff 0 \leq -p(\star) \pm a(\star) \iff p(\star) \mp a(\star) \leq 0.
\]
It follows that \( \mathcal{P} \) is equivalent to \( p(*) \mp a(*) \leq 0 \). It follows from (i) that \( \pm 1 \notin \sigma_{\text{ess}}(U) \) if and only if \( p(*) \mp a(*) \neq 0 \). In this case, \( \mathcal{P} \) becomes

\[
\begin{aligned}
w_{\pm}(*) &= \begin{cases}
-1, & p(*) \mp a(*) < 0, \\
0, & p(*) \mp a(*) > 0,
\end{cases} \\
&= \frac{\text{sgn}(p(*) \mp a(*)) - 1}{2}.
\end{aligned}
\]

We get

\[
\text{ind}_{\pm}(\Gamma, U) = w_{\pm}(+\infty) - w_{\pm}(-\infty) = \frac{\text{sgn}(p(+\infty) \mp a(+\infty)) - \text{sgn}(p(-\infty) \mp a(-\infty))}{2}.
\]

The claim follows.

3.3. Proof of Theorem C

Note that the evolution operator \( U_{\text{suz}} \) of Suzuki’s split-step quantum walk satisfying the asymptotically periodic assumption \( \mathcal{P} \) is a 2-dimensional strictly local operator. In theory, it is possible to compute \( \sigma_{\text{ess}}(U_{\text{suz}}) \) by making use of Theorem [A](ii), but we shall end up with spectral analysis of \( 2n_a \times 2n_a \) matrices according to [18]. In order to reduce the complexity of computations, we may only focus on the real part of \( U_{\text{suz}} \) as the following two lemmas suggest:

**Lemma 3.4.** Let \( (\Gamma, U) \) be any chiral pair on an abstract Hilbert space \( \mathcal{H} \), and let \( R \) be the real part of \( U \). Then

\[
\sigma_{\text{ess}}(U) = \{ z \in \mathbb{T} \mid \Re z \in \sigma_{\text{ess}}(R) \}.
\]

**Proof.** We shall make use of \( \sigma_{\text{ess}}(R) = \{ \Re z \mid z \in \sigma_{\text{ess}}(U) \} \), a simple proof of which can be found in [Tan20], Lemma 3.6. It suffices to prove \( \{ z \in \mathbb{T} \mid \Re z \in \sigma_{\text{ess}}(R) \} \subseteq \sigma_{\text{ess}}(U) \), since the reverse inclusion is obvious. If \( z \in \mathbb{T} \) satisfies \( \Re z \in \sigma_{\text{ess}}(R) \), then there exists \( z_0 \in \sigma_{\text{ess}}(U) \), such that \( \Re z = \Re z_0 \). That is, either \( z = z_0 \) or \( z = \bar{z}_0 \), where \( z_0^* \in \sigma_{\text{ess}}(U) \) by the chiral symmetry condition [1]. We get \( z \in \sigma_{\text{ess}}(U) \) in either case. The claim follows.

**Lemma 3.5.** With the notation introduced in Lemma 3.3, let \( R, Q \) be the real and imaginary parts of \( U_{\text{suz}} \). Then the unitary operator \( \epsilon \) gives the following decomposition:

\[
\epsilon^* R \epsilon = \begin{pmatrix} R_{e_1} & 0 \\ 0 & R_{e_2} \end{pmatrix}, \quad \epsilon^* Q \epsilon = \begin{pmatrix} 0 & Q_{a_0}^* \\ Q_{a_0} & 0 \end{pmatrix},
\]

where the three operators \( R_{e_1}, R_{e_2}, Q_{a_0} \) are defined respectively by

\[
2R_{e_1} := p_- Lp_+ \sqrt{1 - a^2} + p_+ \sqrt{1 - a^2} L^* p_- + (1 + p)a - (1 - p)a(\cdot + 1),
\]

\[
2R_{e_2} := p_+ Lp_- \sqrt{1 - a^2} + p_- \sqrt{1 - a^2} L^* p_+ - (1 - p)a + (1 + p)a(\cdot + 1),
\]

\[
-2iQ_{a_0} := p_+ Lp_+ \sqrt{1 - a^2} - p_- \sqrt{1 - a^2} L^* p_- - \sqrt{1 - p^2}(a + a(\cdot + 1)).
\]
In fact, the formula (51) can be found [Tan21, Lemma 3.2]. It is possible to prove (57) to (58) by an analogous argument as [Tan21, Remark 3.3(ii)] suggests, but we shall make use of the half-step operator decomposition (51) in the below proof.

**Proof.** Recall that \( F := \eta^* \epsilon \) is given by (51). It follows from the second equality in (48)

\[
\epsilon^* U_{suz} \epsilon = \begin{pmatrix} F_{1,-}^* F_{1,+} - F_{1,+}^* F_{1,-} & -(F_{2,-}^* F_{1,+} - F_{2,+}^* F_{1,-})^* \\ F_{2,-}^* F_{1,+} - F_{2,+}^* F_{1,-} & F_{2,+}^* F_{2,-} - F_{2,-}^* F_{2,+} \end{pmatrix},
\]

Since \( \epsilon^* R \epsilon, \epsilon^* Q \epsilon \) are the real and imaginary parts of \( \epsilon^* U_{suz} \epsilon \) respectively, we get

\[
\epsilon^* R \epsilon = \begin{pmatrix} F_{1,-}^* F_{1,+} - F_{1,+}^* F_{1,-} & 0 \\ 0 & F_{2,+}^* F_{2,-} - F_{2,-}^* F_{2,+} \end{pmatrix},
\]

\[
\epsilon^* Q \epsilon = \begin{pmatrix} 0 & i(F_{2,-}^* F_{1,+} - F_{2,+}^* F_{1,-})^* \\ -i(F_{2,-}^* F_{1,+} - F_{2,+}^* F_{1,-}) & 0 \end{pmatrix},
\]

where \( 2F_{1,\pm} = \mp p_\pm a_\pm + a_\pm L^* p_\pm \) and \( 2F_{2,\pm} = \mp p_\pm a_\pm + a_\pm L^* p_\pm \). The claim follows from the following direct computations;

\[
4(F_{1,-}^* F_{1,+} - F_{1,+}^* F_{1,-}) = 4R_{\epsilon_1},
\]

\[
4(F_{2,-}^* F_{2,+} - F_{2,+}^* F_{2,-}) = 4R_{\epsilon_2},
\]

\[
4(F_{2,-}^* F_{1,+} - F_{2,+}^* F_{1,-}) = 4iQ_{\epsilon_0}.
\]

\[\square\]

**Remark 3.6.** It immediately follows from (55) to (56) that

\[
\sigma_{\text{ess}}(U) = \{ z \in \mathbb{T} \mid \text{Re} \ z \in \sigma_{\text{ess}}(R_{\epsilon_1}) \cup \sigma_{\text{ess}}(R_{\epsilon_2}) \},
\]

where each \( R_{\epsilon_i} \) is a one-dimensional strictly local operator, unlike the evolution operator \( U \) itself. We are now in a position to apply the argument outlined in Remark 3.1 to \( R_{\epsilon_i} \).

**Proof of Theorem 4.** Note that the two operators \( R_+ := R_{\epsilon_1} \) and \( R_- := R_{\epsilon_2} \), defined respectively by (57) to (58), are operators of the form \( 2R_{\pm} = r_{1,\pm} \pm (-1) L^{-1} + r_{0,\pm} + r_{1,\pm} L \). The formula (57) motivates us to define \( \hat{R}_{\pm}(\star, z) \) by (51). It follows from Theorem 3.3(ii) that

\[
\sigma_{\text{ess}}(R_{\pm}) = \bigcup_{\star = -\infty, +\infty} \left( \bigcup_{z \in \mathbb{T}} \sigma \left( \hat{R}_{\pm}(\star, z) \right) \right).
\]

(61)

We get

\[
\sigma_{\text{ess}}(U) = \{ z \in \mathbb{T} \mid \text{Re} \ z \in \sigma_{\text{ess}}(R_-) \cup \sigma_{\text{ess}}(R_+) \} = \bigcup_{\star = -\infty, +\infty} \sigma(\star),
\]

where the first equality follows from (60), and where the last equality follows from (61).

\[\square\]
4. Discussion

4.1. The essential spectrum of the one-dimensional split-step quantum walk

It is shown in Theorem B that $\sigma_{\text{ess}}(U) = \sigma(-\infty) \cup \sigma(+\infty)$, where for each $\star = -\infty, +\infty$ the subset $\sigma(\star)$ of $\mathbb{T}$ is defined by (15). The purpose of this subsection is to give a further classification of $\sigma(\star)$ by restricting attention to $n_\star = 1$ and $n_\star = 2$. To do so, we introduce the following notation for simplicity:

$$q := \sqrt{1-p^2}, \quad b := \sqrt{1-a^2}.$$ Given a fixed real number $r_0$ and a compact interval $[r_1, r_2]$ in $\mathbb{R}$, we let

$$r_0 + [r_1, r_2] := [r_0 + r_1, r_0 + r_2], \quad r_0 - [r_1, r_2] := [r_0 - r_2, r_0 - r_1].$$

4.1.1. The asymptotically 1-periodic case

We focus on the case $n_\star = 1$ first. The following proposition can be found in [Tan21], but we give an alternative derivation via Theorem C.

**Proposition 4.1** ([Tan21, Theorem B][i]). With the notation introduced in Theorem C in mind, if $n_\star = 1$, then we have $\sigma(\star) = \sigma(\hat{R}_+(*, z)) = \sigma(\hat{R}_-(\star, z))$ for each $z \in \mathbb{T}$. More precisely,

$$\sigma(\star) = \{z \in \mathbb{T} \mid \Re z \in I(\star)\},$$

where the closed subinterval $I(\star)$ of $[-1, 1]$ is defined by

$$I(\star) := p(\star, 0)a(\star, 0) + [-q(\star, 0)b(\star, 0), q(\star, 0)b(\star, 0)].$$

Moreover, $\pm 1 \notin \sigma(\star)$ if and only if $p(\star) \neq \pm a(\star)$.

**Proof.** It follows from (11) that if $n_\star = 1$, then

$$2\hat{R}_{\pm}(\star, e^{it}) = r_{0, \pm}(\star, 0) + 2r_{1, \pm}(\star, 0)\cos(t),$$

where

$$r_{0, \pm}(\star, 0) = (p(\star, 0) \pm 1)a(\star, 0) + (p(\star, 0) \mp 1)a(\star, 0) = 2p(\star, 0)a(\star, 0),$$

$$r_{1, \pm}(\star, 0) = \sqrt{(1 + p(\star, 0))(1 \mp p(\star, 0))(1 - a(\star, 0)^2)} = q(\star, 0)b(\star, 0).$$

It follows that $\hat{R}_+(\star, e^{it}) = \hat{R}_-(\star, e^{it}) = \hat{R}_0(\star, e^{it})$ for each $t \in [0, 2\pi]$, and we get (62). It is easy to see that $I(\star) = [p(\star, 0)a(\star, 0) - q(\star, 0)b(\star, 0), p(\star, 0)a(\star, 0) + q(\star, 0)b(\star, 0)]$ is a subset of $[-1, 1]$;

$$|p(\star, 0)a(\star, 0)| + q(\star, 0)b(\star, 0) \leq \frac{p(\star, 0)^2 + a(\star, 0)^2}{2} + \frac{(1 - p(\star, 0)^2) + (1 - a(\star, 0)^2)}{2} \leq 1.$$ It remains to show that $\pm 1 \notin \sigma(\star)$ is equivalent to $p(\star) \neq \pm a(\star)$, but we defer the proof until Remark 4.3. \qed
4.1.2. The asymptotically 2-periodic case

Next, we focus on the case \( n_\ast = 2 \).

**Theorem 4.2.** With the notation introduced in Theorem \( \mathbb{C} \) in mind, if \( n_\ast = 2 \), then we have \( \sigma(\ast) = \bigcup_{z \in \mathbb{T}} \sigma \left( \hat{R}_+(\ast, z) \right) = \bigcup_{z \in \mathbb{T}} \sigma \left( \hat{R}_-(\ast, z) \right) \) for each \( z \in \mathbb{T} \). More precisely,

\[
\sigma(\ast) = \left\{ z \in \mathbb{T} \mid \Re z \in I_1(\ast) \cup I_2(\ast) \right\},
\]

where each closed subinterval \( I_\ast(\ast) \) of \([-1, 1] \) is defined by

\[
I_\ast(\ast) := d(\ast) + (-1)^j \left[ \sqrt{d(\ast)^2 + d_1(\ast)}, \sqrt{d(\ast)^2 + d_2(\ast)} \right],
\]

\[
d(\ast) := \frac{4}{2} \left( p(\ast, 0) + p(\ast, 1) \right) \left( a(\ast, 0) + a(\ast, 1) \right),
\]

\[
d_j(\ast) := \frac{2 - (1 + p(\ast, 0)p(\ast, 1))(1 + a(\ast, 0)a(\ast, 1)) + (-1)^j \prod_{m=0,1} q(\ast, m)b(\ast, m)}{2}.
\]

Furthermore, we have the following assertions:

(i) The set \( I_1(\ast) \) given by (65) is a well-defined closed interval in the sense that \( 0 \leq d(\ast)^2 + d_1(\ast) \leq d(\ast)^2 + d_2(\ast) \). Moreover, \( I_1(\ast) \) lies to the left of \( I_2(\ast) \).

(ii) We have \( \pm 1 \notin \sigma(\ast) \) if and only if

\[
\prod_{m=0,1} (1 + p(\ast, m))(1 \mp a(\ast, m)) \neq \prod_{m=0,1} (1 - p(\ast, m))(1 \pm a(\ast, m))
\]

(iii) If \( \prod_{m=0,1} (1 + p(\ast, m))(1 \mp a(\ast, m)) + \prod_{m=0,1} (1 - p(\ast, m))(1 \pm a(\ast, m)) > 0 \), then we uniquely define \( p(\ast), a(\ast) \in [-1, 1] \) through

\[
\left( \frac{\prod_{m=0}^{n_\ast-1} (1 + \zeta(\ast, m))}{\prod_{m=0}^{n_\ast-1} (1 - \zeta(\ast, m))} \right) = \left( \frac{1 + \zeta(\ast)}{1 - \zeta(\ast)} \right)^2, \quad \zeta = p, a.
\]

Then \( \pm 1 \notin \sigma(\ast) \) if and only if \( p(\ast) \neq \pm a(\ast) \).

(iv) The sets \( I_1(\ast), I_2(\ast) \) are singleton sets if and only if \( \{ p(\ast, 0), p(\ast, 1), a(\ast, 0), a(\ast, 1) \} \) contains either \(-1\) or \(+1\). In this case, each \( I_j(\ast) \) is given explicitly by

\[
I_j(\ast) = \left\{ d(\ast) + (-1)^j \sqrt{d(\ast)^2 + \frac{2 - (1 + p(\ast, 0)p(\ast, 1))(1 + a(\ast, 0)a(\ast, 1))}{2}} \right\}.
\]

We show first that Proposition \( \mathbb{D} \) is a special case of Theorem \( \mathbb{E} \).

**Remark 4.3.** With the notation introduced in Theorem \( \mathbb{C} \) in mind, let \( n_\ast = 2 \). If \( p(\ast, 0) = p(\ast, 1) \) and if \( a(\ast, 0) = a(\ast, 1) \), then

\[
d(\ast) = \frac{4}{2} \left( p(\ast, 0) + p(\ast, 1) \right) \left( a(\ast, 0) + a(\ast, 1) \right) = p(\ast, 0)a(\ast, 0),
\]

\[
d_j(\ast) = \frac{2 - (1 + p(\ast, 0)^2)(1 + a(\ast, 0)^2) + (-1)^j(1 - p(\ast, 0)^2)(1 - a(\ast, 0)^2)}{2}.
\]
It follows that \( d(\star)^2 + d_1(\star) = 0 \), and that \( d(\star)^2 + d_2(\star) = (1 - p(\star, 0)^2)(1 - a(\star, 0)^2) \).

\[
I_1(\star) = [p(\star, 0)a(\star, 0) - q(\star, 0)b(\star, 0), p(\star, 0)a(\star, 0)],
\]

\[
I_2(\star) = [p(\star, 0)a(\star, 0), p(\star, 0)a(\star, 0) + q(\star, 0)b(\star, 0)].
\]

Therefore, \( I_1(\star) \cup I_2(\star) = [p(\star, 0)a(\star, 0) - q(\star, 0)b(\star, 0), p(\star, 0)a(\star, 0) + q(\star, 0)b(\star, 0)] \) coincides with \( I(\star) \) given by (63). It follows from Theorem 14.2(ii),(iii) that \( \pm 1 \notin \sigma(\star) \) if and only if \( p(\star) \neq \pm a(\star) \).

We prove Theorem 14.2 with the aid of the following lemma;

**Lemma 4.4.** Given \( \alpha_1, \alpha_2 \in \mathbb{R} \) and \( \beta_1, \beta_2 \geq 0 \), let us consider the one-parameter family \( \{R(z)\}_{z \in \mathbb{T}} \) of \( 2 \times 2 \) Hermitian matrices defined by the following formula;

\[
R(z) := \frac{1}{2} \begin{pmatrix}
\alpha_1 & \beta_1 + \beta_2 z^* \\
\beta_1 + \beta_2 z & \alpha_2
\end{pmatrix}, \quad z \in \mathbb{T}.
\]

(70)

For each \( j = 1, 2 \), let

\[
I_j := \frac{\alpha_1 + \alpha_2}{4} + (-1)^j \left[ \frac{\sqrt{(\alpha_1 - \alpha_2)^2 + 4(\beta_1 - \beta_2)^2}}{4}, \frac{\sqrt{(\alpha_1 - \alpha_2)^2 + 4(\beta_1 + \beta_2)^2}}{4} \right].
\]

Then the following assertions hold true:

(i) We have \( \bigcup_{z \in \mathbb{T}} \sigma(R(z)) = I_1 \cup I_2 \), where \( I_1 \) lies to the left of \( I_2 \).

(ii) The set \( I_1 \cup I_2 \) is connected if and only if \( \alpha_1 = \alpha_2 \) and \( \beta_1 = \beta_2 \). In this case, we have \( I_1 \cup I_2 = [\alpha_1/2 - \beta_1, \alpha_1/2 + \beta_1] \).

Note that \( I_1, I_2 \) are well-defined, since \( (\beta_1 - \beta_2)^2 \leq (\beta_1 + \beta_2)^2 \) follows from \( (\beta_1 \pm \beta_2)^2 = \beta_1^2 \pm 2\beta_1\beta_2 + \beta_2^2 \).

**Proof.** We shall identify the unit-circle \( \mathbb{T} \) with \([0, 1]\) through \([0, 1] \ni t \mapsto e^{it} \in \mathbb{T} \).

(i) We have

\[
2 \cdot \text{tr} R(t) = \alpha_1 + \alpha_2,
\]

\[
4 \cdot \text{det} R(t) = \alpha_1 \alpha_2 - |\beta_1 + \beta_2 e^{it}|^2 = \alpha_1 \alpha_2 - (\beta_1^2 + \beta_2^2) - 2\beta_1 \beta_2 \cos t.
\]

The eigenvalues of \( R(t) \) are given by

\[
\lambda_j(t) = \frac{\alpha_1 + \alpha_2 + (-1)^j \sqrt{(\alpha_1 - \alpha_2)^2 + 4(\beta_1^2 + 2\beta_1 \beta_2 \cos t + \beta_2^2)}}{4},
\]

where \( j = 1, 2 \). We get

\[
\bigcup_{t \in [0, \pi]} \sigma(R(t)) = \bigcup_{t \in [0, \pi]} \{\lambda_1(t)\} \cup \bigcup_{t \in [0, \pi]} \{\lambda_2(t)\}.
\]
The range of \([0,2\pi] \ni t \mapsto \sqrt{(\alpha_1 - \alpha_2)^2 + 4(\beta_1^2 + 2\beta_1\beta_2 \cos t + \beta_2^2)} \in \mathbb{R}\) is
\[
\left[\sqrt{(\alpha_1 - \alpha_2)^2 + 4(\beta_1^2 + \beta_2^2)}, \sqrt{(\alpha_1 - \alpha_2)^2 + 4(\beta_1 + \beta_2)^2}\right],
\]
where \((\beta_1 - \beta_2)^2 \leq (\beta_1 + \beta_2)^2\). Therefore, \(\bigcup_{t \in [0,2\pi]} \{\lambda_j(t)\} = I_j\) for each \(j = 1, 2\). Note also that \(I_1\) is located to the left of \(I_2\):
\[
\Delta(I_1, I_2) := \min I_2 - \max I_1 = \sqrt{(\alpha_1 - \alpha_2)^2 + 4(\beta_1 - \beta_2)^2} \geq 0.
\]
(ii) The gap \(\Delta(I_1, I_2)\) becomes 0 if and only if \(\alpha_1 - \alpha_2 = 0 = \beta_1 - \beta_2\). In this case,
\[
I_1 \cup I_2 = [\alpha_1/2 - \beta_1, \alpha_1/2 + \beta_1].
\]

**Proof of Theorem 4.2.** Since \(n_* = 2\), it follows from (70) that
\[
\hat{R}_\pm(\ast, z) = \frac{1}{2} \left( \begin{array}{cc}
 r_{0,\pm}(\ast, 0) & r_{1,\pm}(\ast, 0) + r_{1,\pm}(\ast, 1)z^\ast \\
r_{0,\pm}(\ast, 1) & r_{0,\pm}(\ast, 1)
\end{array} \right) .
\] (71)
Let us first prove that \(\bigcup_{z \in \mathbb{R}} \sigma(\hat{R}_\pm(\ast, z))\) does not depend on the choice of \(\pm\) in order to show (70). For each \(\zeta = p, q, a, b\), and each \(m = 0, 1\), we write \(\zeta_m := \zeta(\ast, m)\) for simplicity from here on. With this convention in mind, we let
\[
\begin{align*}
 r_{0,\pm}(\ast, 0) &= (p_0 \pm 1)a_0 + (p_0 \mp 1)a_1 =: \alpha_{1,\pm}, \\
r_{0,\pm}(\ast, 1) &= (p_1 \pm 1)a_1 + (p_1 \mp 1)a_0 =: \alpha_{2,\pm}, \\
r_{1,\pm}(\ast, 0) &= \sqrt{(1 + p_0)(1 \mp p_1)b_1} =: \beta_{1,\pm}, \\
r_{1,\pm}(\ast, 1) &= \sqrt{(1 + p_0)(1 \mp p_1)b_0} =: \beta_{2,\pm},
\end{align*}
\]
where \(\alpha_{1,\pm}, \alpha_{2,\pm} \in \mathbb{R}\), and where \(\beta_{1,\pm}, \beta_{2,\pm} \geq 0\). It follows that (70) is a special case of (71). We shall make use of the following equalities in order to apply Lemma 4.3(i) to \(\hat{R}_\pm(\ast, z)\):
\[
\begin{align*}
\alpha_{1,\pm} + \alpha_{2,\pm} &= (p_0 + p_1)(a_0 + a_1), \\
(\alpha_{1,\pm} - \alpha_{2,\pm})^2 &= (p_0 - p_1)^2(a_0 + a_1)^2 + 4(a_0 - a_1)^2 \pm 4(p_0 - p_1)(a_0^2 - a_1^2), \\
(\beta_{1,\pm} + (-1)^j\beta_{2,\pm})^2 &= (1 - p_0p_1)(2 - a_0^2 + a_1^2) + 2(-1)^jqaqb_1b_1 + (p_0 - p_1)(a_0^2 - a_1^2),
\end{align*}
\]
where \(j = 1, 2\), and where we use \(\beta_{1,\pm}^2 + \beta_{2,\pm}^2 = (1 - p_0p_1)(2 - a_0^2 - a_1^2) + (p_0 - p_1)(a_0^2 - a_1^2)\) in the last equality. It follows that \(d'_j := (\alpha_{1,\pm} - \alpha_{2,\pm})^2 + 4(\beta_{1,\pm} + (-1)^j\beta_{2,\pm})^2 \geq 0\) does not depend on the choice of \(\pm\) for each \(j = 1, 2\). Moreover,
\[
\begin{align*}
d'_j &= (p_0 - p_1)^2(a_0 + a_1)^2 + 4(a_0 - a_1)^2 + 4(1 - p_0p_1)(2 - a_0^2 - a_1^2) + 8(-1)^jqaqb_1b_1 \\
&= (p_0 + p_1)^2(a_0 + a_1)^2 + 8(2 - (1 + p_0p_1)(1 + a_0a_1) + (-1)^jqaqb_1b_1) \\
&= 16(d(\ast)^2 + d_j(\ast)),
\end{align*}
\]

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where the second equality follows from \((p_0 - p_1)^2 = (p_0 + p_1)^2 - 4p_0p_1\). If we define \(I_j(*)\) according to (65), then it follows from Lemma 4.4(i) that

\[
\bigcup_{z \in \mathbb{T}} \sigma\left(\hat{R}_+(\ast, z)\right) = \bigcup_{z \in \mathbb{T}} \sigma\left(\hat{R}_-(\ast, z)\right) = I_1(*) \cup I_2(*).
\]

Note that (64) follows from Theorem C.

(i) It is obvious that \(0 \leq d(*)^2 + d_1(*) \leq d(*)^2 + d_2(*)\), and that \(I_1(*)\) lies to the left of \(I_2(*)\). It remains to show \(I_1(*) \cup I_2(*) \subseteq [-1, 1]\). Let

\[
d_{\pm}(*) := d(*) \pm \sqrt{d(*)^2 + d_2(*)},
\]

where \(d_-(*)\) (resp. \(d_+(*)\)) is the minimum (resp. maximum) of \(I_1(*) \cup I_2(*)\). Let the notation \(\leq\) simultaneously denote \(\leq\) and \(=\). We are required to prove that the following equivalent conditions hold true with \(|d(*)| \leq 1\) in mind:

\[
\pm d_{\pm}(*) \leq 1 \text{ if and only if } 0 \leq (1 + p_0p_1)(1 + a_0a_1) \mp (p_0 + p_1)(a_0 + a_1) - q_0q_1b_0b_1, \quad (72)
\]

where \((1 + p_0p_1)(1 + a_0a_1) \mp (p_0 + p_1)(a_0 + a_1) \geq 0\). Indeed,

\[
(1 + p_0p_1)(1 + a_0a_1) \mp (p_0 + p_1)(a_0 + a_1) = \begin{cases} 0, & p_0p_1 = -1 \text{ or } a_0a_1 = -1; \\ (1 + p_0p_1)(1 + a_0a_1) \left(1 \mp \frac{p_0 + p_1}{1+ p_0p_1 a_0 + a_1} \right), & \text{otherwise.} \end{cases}
\]

It remains to prove

\[
0 \leq ((1 + p_0p_1)(1 + a_0a_1) \mp (p_0 + p_1)(a_0 + a_1))^2 - (1 - p_0^2)(1 - p_1^2)(1 - a_0^2)(1 - a_1^2). \quad (73)
\]

Let

\[
s_{\pm} := ((1 + p_0p_1)(1 + a_0a_1) \mp (p_0 + p_1)(a_0 + a_1))^2,
\]

\[
s_{\pm}' := ((1 + p_0p_1)(a_0 + a_1) \mp (1 + a_0a_1)(p_0 + p_1))^2.
\]

We show that the right hand side of (73) is \(s_{\pm}' \geq 0\):

\[
s_{\pm} - s_{\pm}' = (1 + p_0p_1)^2((1 + a_0a_1)^2 - (a_0 + a_1)^2) - (p_0 + p_1)^2((1 + a_0a_1)^2 - (a_0 + a_1)^2)
\]

\[
= ((1 + p_0p_1)^2 - (p_0 + p_1)^2)((1 + a_0a_1)^2 - (a_0 + a_1)^2)
\]

\[
= (1 - p_0^2)(1 - p_1^2)(1 - a_0^2)(1 - a_1^2).
\]

It follows that (73) holds true. We get \(-1 \leq d_-(*) \leq d_+(*) \leq 1\) by (72), and so \(I_1 \cup I_2 \subseteq [-1, 1]\).

(ii) It follows from (i) that

\[
d_{\pm}(*) = \pm 1 \text{ if and only if } s_{\pm}' = 0. \quad (74)
\]
It follows from a direct computation that
\[
\prod_{m=0,1} (1+p_m)(1\mp a_m) - \prod_{m=0,1} (1-p_m)(1\pm a_m) = \mp 2((1+p_0p_1)(a_0+a_1)\mp (1+a_0a_1)(p_0+p_1)).
\]
Therefore, \(\pm 1 \notin \sigma(*)\) if and only if \(\prod_{m=0,1} (1+p_m)(1\mp a_m) \neq \prod_{m=0,1} (1-p_m)(1\pm a_m)\).

(iii) If \(\prod_{m=0,1} (1+p_m)(1\mp a_m) + \prod_{m=0,1} (1-p_m)(1\pm a_m) > 0\), then we define \(p(*)\), \(a(*)\) \(\in [-1,1]\) through (69). Note that (69) is equivalent to \(p(*) \mp a(*) \neq 0\) as in the proof of Theorem 4.2. The claim follows from (ii).

(iv) Note that \(I_j(*)\) given by (69) is a singleton set if and only if \(d_1(*) = d_2(*)\) if and only if \(\prod_{m=0,1} q(*,m)b(*,m) = 0\). The claim follows.

\[\square\]

4.1.3. The general case

It is desirable to give a complete classification of \(\sigma(*)\) in full generality. The special cases \(n_* = 1, 2\) we have considered in this subsection are intended as motivating examples for this general approach. It is worth noting that the proof of Theorem 4.2 is already far from obvious. The general case \(n_* \geq 3\) naturally leads to spectral analysis of Hermitian matrices of the form (7), but it is not known to the authors whether or not there is a general standard method for this.

4.2. Exponential decay

Note that (68) can also be written as (6). The following result is also one of the main theorems of the present article;

**Theorem 4.5.** Let \((\Gamma, U) = (\Gamma_{\text{aux}}, U_{\text{aux}})\) be defined by (33) to (3). Suppose that there exist \(n_{-\infty}, n_{+\infty} \in \mathbb{N}\) with the property that the limits of the form (67) exist for each \(* \equiv \pm\infty\), and that
\[
\sup_{x \in \mathbb{Z}} |\zeta(x)| < 1, \quad \zeta \in \{p, a\}, n_0 \in \{0, \ldots, n_* - 1\}.
\]
Let the four numbers \(p(\pm\infty), a(\pm\infty) \in (-1,1)\) be uniquely defined through (77), and let \(p(\pm\infty) \neq \pm a(\pm\infty)\). Then the following assertions hold true:

(i) We have \(\dim \ker(U \mp 1) = |\text{ind}_\pm(\Gamma, U)|\), where \(\text{ind}_\pm(\Gamma, U)\) is given by (4).

(ii) If \((-1)^j(p(-\infty) \mp a(-\infty)) < 0 < (-1)^j(p(+\infty) \mp a(+\infty))\) for some \(j = 1, 2\), then for any non-zero vector \(\Psi_\pm \in \ker(U \mp 1)\) there exists a unique non-zero vector \(\psi_\pm \in \ker\left(L + \sqrt{\Lambda((-1)^j p)\Lambda((-1)^j a)}\psi_\pm\right)\), such that
\[
\Psi_\pm = \left(\pm(-1)^j \sqrt{\Lambda((-1)^j a)}\psi_\pm\right).
\]
Moreover, the eigenstate \(\Psi_\pm\) characterised by (76) exhibits exponential decay in the sense that there exist positive constants \(c_\pm^1, c_\pm^2, \kappa_\pm^1, \kappa_\pm^2, x_\pm\), such that
\[
k_\pm^1 e^{-c_\pm^1|x|} \leq \|\Psi_\pm(x)\|^2 \leq k_\pm^2 e^{-c_\pm^2|x|}, \quad |x| \geq x_\pm.
\]
Remark 4.6.

(i) Note that (75) implies

\[ 1 > \sup_{x \in \mathbb{Z}} |\zeta(x)| \geq \limsup_{x \to \infty} |\zeta(\pm x)|. \]

It follows that \(|\zeta(\star, n_0)| < 1\) for each \(\zeta \in \{p, a\}\) and each \(n_0 \in \{0, \ldots, n_\star - 1\}\).

(ii) It is shown in the proof of Theorem 4.5 below that the four positive constants \(c_\downarrow^\pm, c_\uparrow^\pm, \kappa_\downarrow^\pm, \kappa_\uparrow^\pm\) in (77) can be expressed in terms of \(p, a\) (see (92) to (93) for details).

We introduce the following lemma in order to prove Theorem 4.5.

Lemma 4.7. Let \((\alpha(x))_{x=0}^\infty\) be a sequence of positive numbers, and let us assume that there exists a natural number \(n_0 \in \mathbb{N}\) such that the following limits exist in \((0, \infty)\):

\[ \alpha(+\infty, m) := \lim_{x \to \infty} \alpha(n_0 x + m), \quad m \in \{0, \ldots, n_0 - 1\}. \]  

(78)

Then \(\left(\prod_{m=0}^{n_0-1} \alpha(m)\right)^{1/x} \to \left(\prod_{m=0}^{n_0-1} \alpha(+\infty, m)\right)^{1/n_0}\) as \(x \to \infty\).

Note that the special case \(n_0 = 1\) is nothing but the well-known result that the geometric mean of a convergent positive sequence converges to its limit. We shall make use of this result in what follows.

Proof. Let \(m_0 \in \{1, \ldots, n_0\}\) be fixed. If let \(\beta(x) = \left(\prod_{m=0}^{n_0-1} \alpha(m)\right)^{1/x}\) for each \(x \in \mathbb{N}\), then

\[ \beta(n_0 x + m_0) = \left(\prod_{m=0}^{n_0 x - 1} \alpha(m) \prod_{m=0}^{n_0 - 1} \alpha(n_0 x + m)\right)^{1/x} = \left(\prod_{m=0}^{n_0 x - 1} \alpha(m) \prod_{m=0}^{n_0 - 1} \alpha(n_0 x + m)\right)^{1/x} = \prod_{m=0}^{n_0 - 1} \alpha(n_0 x + m), \]

where \(\left(\prod_{m=0}^{n_0 - 1} \alpha(n_0 x + m)\right)_{x \in \mathbb{N}}\) converges to the positive number \(\prod_{m=0}^{n_0 - 1} \alpha(+\infty, m)\). Moreover, we get as \(x \to \infty\)

\[ \log \left(\prod_{m=0}^{n_0 - 1} \alpha(n_0 x + m)\right)^{n_0 x + m_0} = \frac{\log \prod_{m=0}^{n_0 - 1} \alpha(n_0 x + m)}{n_0 x + m_0} \to 0, \]

where the last step follows from the fact that \(\left(\log \prod_{m=0}^{n_0 - 1} \alpha(n_0 x + m)\right)_{x \in \mathbb{N}}\) is a bounded sequence. It follows from the continuity of the exponential function that

\[ \left(\prod_{m=0}^{n_0 - 1} \alpha(n_0 x + m)\right)^{n_0 x + m_0} \to e^0 = 1. \]  

(79)
On the other hand, it follows that as $x \to \infty$

$$
\prod_{m=0}^{n_0-1} \prod_{x_m=0}^{x-1} \alpha(x_m n_0 + m)^{\frac{1}{x}} \to \prod_{m=0}^{n_0-1} \alpha(+\infty, m)^{\frac{1}{x_0}}.
$$

(80)

It follows from (79) to (80) as $x \to \infty$ we have

$$
\beta(n_0 x + m_0) \frac{1}{x_0 + m_0} = \left( \prod_{m=0}^{n_0-1} \prod_{x_m=0}^{x-1} \alpha(x_m n_0 + m)^{1/x} \right)^{\frac{1}{x_0 + m_0}} \to \left( \prod_{m=0}^{n_0-1} \alpha(+\infty, m) \right)^{\frac{1}{x_0}}.
$$

(81)

(82)

(i) Note first that the two non-negative numbers $\Delta_{1,\pm}$ and $\Delta_{2,\pm}$ cannot be simultaneously finite, since $|\delta_{1,\pm}(y)\delta_{2,\pm}(y)|^2 = 1$ for each $y \in \mathbb{Z}$. With this result in mind, [MST21, Theorem 3.1](i) gives the following classification;

$$
\text{ind}_{\pm}(I, U) = \dim \ker(U \mp 1),
$$

(83)

$$
\text{ind}_{\pm}(I, U) = \begin{cases} 
+1, & \Delta_{1,\pm} < \infty, \\
-1, & \Delta_{2,\pm} < \infty, \\
0, & \Delta_{1,\pm} = \Delta_{2,\pm} = \infty.
\end{cases}
$$

(84)

We are required to show that (83) agrees with (6) by making use of the root test. Since the function $\Lambda$ is continuous, for each $\zeta = -p, +p, -a, +a$ and each $\star = \pm \infty$, the following numbers belong to $(0, \infty)$;

$$
\Lambda(\zeta(\star, y)) = \lim_{x \to \star} \Lambda(\zeta(n_\star \cdot x + y)), \quad y \in \{0, \ldots, n_\star - 1\},
$$

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where \(|\zeta(\ast, n_0)| < 1\). It follows from Lemma 4.7 that as \(x \to \infty\)
\[
\lim_{x \to \infty} \left( \prod_{j=0}^{x-1} \Lambda(\zeta(-y-1)) \right)^{\frac{1}{x}} = \left( \prod_{j=0}^{n_x-1} \Lambda(\zeta(-\infty, y)) \right)^{\frac{1}{n_x}}.
\]
\[
\lim_{x \to \infty} \left( \prod_{j=0}^{x-1} \Lambda(\zeta(y)) \right)^{\frac{1}{x}} = \left( \prod_{j=0}^{n_x-1} \Lambda(\zeta(\infty, y)) \right)^{\frac{1}{n_x}}.
\]

Since \(|\delta_{j, \pm}(y)|^2 = (\Lambda(p(y))\Lambda(\pm a(y)))^{(-1)^j}\) for each \(y \in \mathbb{Z}\), we have
\[
\lim_{x \to \infty} \left( \prod_{j=0}^{x-1} \delta_{j, \pm}(-y-1)|^{-2} \right)^{\frac{1}{x}} = (\Lambda(p(-\infty))\Lambda(\mp a(-\infty)))^{(-1)^j+1},
\]
\[
\lim_{x \to \infty} \left( \prod_{j=0}^{x-1} |\delta_{j, \pm}(y)|^2 \right)^{\frac{1}{x}} = (\Lambda(p(+\infty))\Lambda(\mp a(+\infty)))^{(-1)^j},
\]
where \(\Lambda(p(\ast))\Lambda(\mp a(\ast)) \neq 1\) for each \(\ast = \pm \infty\), since we assume \(p(\ast) \mp a(\ast) \neq 0\). That is, the root test is applicable to each of the two infinite series on the right hand side of (82), and we obtain the following equivalence for each \(j = 1, 2\);
\[
\Delta_{j, \pm} < \infty \text{ if and only if } (-1)^j(p(+\infty) \mp a(+\infty)) < 0 < (-1)^j(p(-\infty) \mp a(-\infty)). \tag{85}
\]
It is now easy to see that (84) becomes (6).

(ii) Let \(\Delta_{j, \pm} < \infty\) for some \(j = 1, 2\) throughout. It follows from [MST21, Theorem 3.1(ii)] that we have the following linear isomorphism;
\[
\ker(L - \delta_{j, \pm}) \in \psi \longmapsto \left( \mp(-1)^j \sqrt{\Lambda(\mp(-1)^j a)} \psi \right) \in \ker(U \mp 1), \tag{86}
\]
where \(\dim \ker(U \mp 1) = 1\). In other words, for any non-zero vector \(\Psi_{\pm} \in \ker(U \mp 1)\) there exists a unique non-zero vector \(\psi_{\pm} \in \ker \left(L + \sqrt{\Lambda((-1)^j p)}\Lambda(\mp(-1)^j a)\right)\), such that \(\Psi_{\pm}\) is given explicitly by (76). Finally, we introduce the following positive constants to show that \(\Psi_{\pm}\) exhibits exponential decay.
\[
\hat{\delta}_{j, \pm} := \min \left\{ (\Lambda(p(-\infty))\Lambda(\mp a(-\infty)))^{(-1)^j+1}, (\Lambda(p(+\infty))\Lambda(\mp a(+\infty)))^{(-1)^j} \right\}, \tag{87}
\]
\[
\hat{\Delta}_{j, \pm} := \max \left\{ (\Lambda(p(-\infty))\Lambda(\mp a(-\infty)))^{(-1)^j+1}, (\Lambda(p(+\infty))\Lambda(\mp a(+\infty)))^{(-1)^j} \right\}, \tag{88}
\]
\[
\Lambda_{j, \pm} := \inf_{x \in \mathbb{Z}} \Lambda(\mp(-1)^j a(x)) + 1, \tag{89}
\]
\[
\Lambda_{j, \pm} := \sup_{x \in \mathbb{Z}} \Lambda(\mp(-1)^j a(x)) + 1. \tag{90}
\]
Note that 0 < \(\hat{\delta}_{j, \pm} \leq \hat{\delta}_{j, \pm} < 1\), where the first inequality follows from Remark 4.6(i), and where the last inequality follows from (83) with \(\Delta_{j, \pm} < \infty\). Let \(c > 0\) be small enough,
so that $0 < \delta_{j,\pm}^\dagger - \epsilon < \delta_{j,\pm}^\dagger + \epsilon < 1$ holds true. It then follows from \cite{MST21}, Theorem 3.1(iii) that there exists $x_\pm \in \mathbb{N}$, such that

\[
\Lambda_{j,\pm}^\dagger \left( \delta_{j,\pm}^\dagger - \epsilon \right) |x| \leq \frac{\|\Psi(x)\|^2}{|\psi(0)|^2} \leq \Lambda_{j,\pm}^\dagger \left( \delta_{j,\pm}^\dagger + \epsilon \right) |x|, \quad |x| \geq x_\pm.
\]

We obtain (77), if we let

\[
\kappa_{j,\pm}^\dagger := |\psi(0)|^2 \Lambda_{j,\pm}^\dagger, \quad c_{j,\pm}^\dagger := - \log \left( \delta_{j,\pm}^\dagger - \epsilon \right),
\]

\[
\kappa_{j,\pm}^\uparrow := |\psi(0)|^2 \Lambda_{j,\pm}^\uparrow, \quad c_{j,\pm}^\uparrow := - \log \left( \delta_{j,\pm}^\uparrow + \epsilon \right).
\]

**Remark 4.8.** The proof of Theorem 4.5 above gives yet another derivation of the index formula (6) via (84). This latter derivation relies only on elementary analysis of first-order difference equations inspired by \cite{FFS18}, while the former derivation outlined in §3.2 makes extensive use of Toeplitz operators. Note, however, that despite its simplicity the latter method alone is insufficient to justify where the technical assumption $p(\pm\infty) \neq \pm a(\pm\infty)$ comes from. It is precisely the language of Toeplitz operators that allows us to establish the non-trivial equivalence of this assumption and the essential gap condition $\pm 1 \notin \sigma_{\text{ess}}(U)$ (see Theorem B(ii) for details).

$$\square$$

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