A possible hypercomputational quantum algorithm
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ABSTRACT
The term ‘hypermachine’ denotes any data processing device (theoretical or that can be implemented) capable of carrying out tasks that cannot be performed by a Turing machine. We present a possible quantum algorithm for a classically non-computable decision problem, Hilbert’s tenth problem; more specifically, we present a possible hypercomputation model based on quantum computation. Our algorithm is inspired by the one proposed by Tien D. Kieu, but we have selected the infinite square well instead of the (one-dimensional) simple harmonic oscillator as the underlying physical system. Our model exploits the quantum adiabatic process and the characteristics of the representation of the dynamical Lie algebra $\mathfrak{su}(1,1)$ associated to the infinite square well.

Keywords: Hypercomputation, computability, adiabatic quantum computation, infinite square well, dynamical Lie algebra $\mathfrak{su}(1,1)$, Hilbert’s tenth problem

1. INTRODUCTION

Once upon on time, back in the golden age of the recursive function theory, computability was an absolute. Richard Sylvan and Jack Copeland.

The hypercomputers, according to Copeland and Proudfoot, compute functions or numbers, or more generally, solve problems or carry out tasks, that cannot be computed or solved by a Turing machine (TM). The hypercomputation theory rejects the idea of an absolute computability (i.e. Turing computability), detached from logical, mathematical, physical or biological theories. Notwithstanding, the up coming of an academic community that works the concept of hypercomputation, and notwithstanding the proliferation of theoretical hypercomputation models, the possibility of real construction of a hypermachine is controversial and is still under analysis.

At first glance, we could think that the possibility of a hypercomputation model would be a refutation of the widely accepted Church-Turing thesis, which is usually interpreted as the identification of the naturally calculable functions with the TM-computable functions; but actually the existence of hypercomputation models refute the thesis M, which identifies the functions calculable by a machine with the TM-computable functions. In other words, the existence of the current proposed hypercomputation models coexists with acceptance of the Church-Turing thesis.

The existence of supported proposals of hypercomputation in quantum mechanics is ample, nevertheless this is not the case for those based on quantum computation. Considering the fact that quantum computation extends beyond its “standard” model (i.e. quantum Turing machines or quantum circuits) and it includes proposals such as continuous, adiabatic, and/or holonomic quantum computation, among others; this article presents the construction of a hypercomputation model from quantum computation. Our model is based on the one proposed...
by Tien D. Kieu,\textsuperscript{8–10} but we have selected the infinite square well (ISW) instead of the (one-dimensional) simple harmonic oscillator (SHO) as the underlying physical system. Our model exploits the quantum adiabatic process, and due to this change, our model is supported by some characteristics of the dynamical Lie algebra $\mathfrak{su}(1, 1)$, associated to the ISW; instead of the dynamical Lie algebra Weyl-Heisenberg $\mathfrak{g}_{\text{W-H}}$, associated to the SHO.

2. KIEU’S ALGORITHM

A Diophantine equation is of the following form

$$D(x_1, \ldots, x_k) = 0,$$

(1)

where $D$ is a polynomial with integer coefficients. In present terminology, Hilbert’s tenth problem may be paraphrased as: Given a Diophantine equation of type (1), we should build a procedure to determine whether or not this equation has a solution in non-negative integers $\mathbb{N}$. From the concluding results obtained by Matiyasevich, Davis, Robinson, and Putnam, we know that, in the general case, this problem is algorithmically unsolvable or more precisely, it is TM incomputable.\textsuperscript{11}

The hypercomputability of Kieu’s algorithm is due to the fact that this algorithm solves Hilbert’s tenth problem. From its preliminary version,\textsuperscript{12} Kieu has presented different refinements to his algorithm\textsuperscript{8–10, 13–16}. In addition, with basis in his algorithm, Kieu has reformulated Hilbert’s tenth problem in strictly mathematical terms, indicating a possible way of solution through the mathematical analysis and the theory of infinite-dimensional operators.\textsuperscript{17}

Kieu’s algorithm is constructed beginning with the SHO characteristics. For the SHO with Hamiltonian

$$H = (P^2 + X^2)/2 = a\dagger a + 1/2,$$

(2)

the occupation-number states $|n\rangle$, the action of the creation $a\dagger$ and annihilation $a$ operators on occupation-number states, their commutation relations $[a, a\dagger], [a, a],$ and $[a\dagger, a\dagger]$, the occupation-number operator $N$ and the coherent states $|\alpha\rangle$ are given by

$$\{|n\rangle \mid n \in \mathbb{N}\},$$

(3)

$$a|0\rangle = 0, \quad a|n\rangle = \sqrt{n}|n-1\rangle, \quad a\dagger|n\rangle = \sqrt{n+1}|n+1\rangle,$$

(4)

$$[a, a\dagger] = 1, \quad [a, a] = [a\dagger, a\dagger] = 0,$$

(5)

$$N = a\dagger a,$$

(6)

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad \alpha \in \mathbb{C},$$

(7)

With the basis for the SHO, Kieu’s algorithm\textsuperscript{8} is shown in Table (1).

Kieu has indicated the following characteristics of his algorithm\textsuperscript{8–10}:

1. The algorithm is probabilistic such as are quantum algorithms in general.

2. The infinite-dimension Hamiltonian (2) acts on some Fock space whose orthonormal basis is (3), and satisfies that $|n\rangle = \frac{(a\dagger)^n}{\sqrt{n!}} |0\rangle$.

3. The codification of (1) is made by the occupation-number operator (6) which is diagonal on the basis (3), and whose eigenvalues are the non-negative integers.

\textsuperscript{1}In the Discussion List FOM - Foundations of Mathematics (www.cs.nyu.edu/pipermail/fom/) some characteristics of this algorithm have been analyzed and discussed.
Table 1. Kieu’s hypercomputational quantum algorithm.

Given a Diophantine equation with \( k \) unknowns of type (1), Kieu provides the following quantum algorithm to decide whether this equation has any non-negative integer solution or not:

1. Construct a physical process in which a system initially starts with a direct product of \( k \) coherent states \( \left| \psi(0) \right> \), and in which the system is subject to a time-dependent Hamiltonian \( H_A(t) \) over the time interval \([0, T] \), for some time \( T \), with the initial Hamiltonian \( H_I \) and the final Hamiltonian \( H_D \), given by

\[
| \psi(0) \rangle = \bigotimes_{i=1}^{k} | \alpha_i \rangle ,
\]

(8)

\[
H_A(t) = (1 - t/T)H_I + (t/T)H_D ,
\]

(9)

\[
H_I = \sum_{i=1}^{k} (a_i^\dagger - \alpha_i^* \alpha_i) (a_i - \alpha_i) ,
\]

(10)

\[
H_D = (D(N_1, \ldots, N_k))^2 .
\]

(11)

2. Measure through the time-dependent Schrödinger equation

\[
i \partial_t | \psi(t) \rangle = H_A(t) | \psi(t) \rangle , \quad \text{for } t \in [0, T]
\]

the maximum probability to find the system in a particular occupation-number state at the chosen time \( T \),

\[
P_{\text{max}}(T) = \max_{(n_1, \ldots, n_k) \in \mathbb{N}^k} \left| \langle \psi(T) | n_1, \ldots, n_k \rangle \right|^2 ,
\]

\[
= | \langle \psi(T) | \{ n \}_0 \rangle |^2 ,
\]

where \(| \{ n \}_0 \rangle \) (which is a direct product of \( k \) particular occupation-number state, \( \bigotimes_{i=1}^{k} | n_i \rangle_0 \)) provides that maximal probability among all other direct products of \( k \) occupation-number states.

3. If \( P_{\text{max}}(T) \leq 1/2 \), increase \( T \) and repeat all the steps above.

4. If

\[
P_{\text{max}}(T) > 1/2
\]

(12)

then \(| \{ n \}_0 \rangle \) is the ground state of \( H_D \) (assuming no degeneracy) and we can terminate the algorithm and deduce a conclusion from the fact that

\[
H_D | \{ n \}_0 \rangle = 0 \text{ iff the equation (1) has a non-negative integer solution.}
\]

4. The application of the adiabatic theorem to obtain from the ground state (8) associated with the eigenvalue zero of (10), the ground state \(| \{ n \}_0 \rangle \) of (11).

5. The finiteness of the run time \( T \) to obtain with a high probability the ground state \(| \{ n \}(T) \rangle_0 \) of \( H_D \). Although the value of \( T \) is not calculable a priori, its finiteness is guaranteed because the ground state of \( H_A(sT) \) for \( 0 < s < 1 \) is non-degenerate, and this ground state never crosses with any other state during the adiabatic regime.

6. The halting criterion (12) is established by the maximum peak of density of probability associated with
the initial state \((8)\), where for any \(n\) and \(\alpha\)

\[
|\langle\alpha|n\rangle|^2 = e^{\alpha|\alpha|2n} \frac{n!}{n!} < 1/2
\]

In addition, it is necessary to verify that the probability of any excited state cannot be greater than \(1/2\) at anytime.

### 3. HYPERCOMPUTATIONAL QUANTUM ALGORITHM À LA KIEU

Although the physical referent used by Kieu for the construction of his algorithm was the SHO, parting from the description on Table (1), it can be observed that its constitutive elements, come forth from the dynamical algebra Weyl-Heisenberg \(g_{W-H}\) associated with the SHO. The dynamical algebra \(g_{W-H}\) satisfies the commutation relations (5). Beginning with its generating elements \(a, a^\dagger\) and 1, it is possible to factor the Hamiltonian from the SHO as pointed out by (2) and it is possible to construct the occupation-number operator (6). The coherent states (7) correspond to the eigenstates of annihilation operator \(a\), in other words, 

\[
a|\alpha\rangle = \alpha|\alpha\rangle.
\]

Furthermore, the algebra \(g_{W-H}\) has an infinite-dimensional irreducible representation wherein the action of its generators on basis (3) is given by (4). This representation makes possible that the eigenvalues of operator (6) are the non-negative integers and its eigenstates are the basis (3), in other words \(N|n\rangle = n|n\rangle\).

To carry out the construction of our algorithm à la Kieu, we have selected as physical referent the ISW, which has an associated dynamical algebra different to the one used by Kieu. For a particle with mass \(m\) trapped inside the infinite well \(0 < x < \pi l\), the Hamiltonian operator \(H^{\text{ISW}}\) and the energy levels \(E_n^{\text{ISW}}\) are

\[
H^{\text{ISW}} = i\hbar^2 \frac{d^2}{2m dx^2} - \frac{\hbar^2}{2ml^2}, \quad E_n^{\text{ISW}} = \frac{\hbar^2}{2ml^2} n(n+2),
\]

where the action of \(H^{\text{ISW}}\) on basis (3) is

\[
H^{\text{ISW}} |n\rangle = E_n^{\text{ISW}} |n\rangle.
\]

Due to the spectral structure of the ISW, the dynamical algebra associated with it, is the Lie algebra \(\mathfrak{su}(1,1)\). This is a three-dimensional algebra that satisfies the commutation relations

\[
[K_-, K_+] = K_3, \quad [K_-, K_3] = 2K_-, \quad [K_+, K_3] = -2K_+,
\]

where operators \(K_+\), \(K_-\) and \(K_3\) are called creation, annihilation and Cartan operators, respectively. The algebra \(\mathfrak{su}(1,1)\) admits an infinite-dimensional irreducible representation where actions of \(K_+, K_-\) and \(K_3\) on basis (3) are

\[
K_+ |n\rangle = \sqrt{(n+1)(n+3)} |n+1\rangle, \quad K_- |0\rangle = 0, \quad K_3 |n\rangle = (2n+3) |n\rangle.
\]

With basis in the algebra \(\mathfrak{su}(1,1)\), the Hamiltonian (13) is rewritten as

\[
H^{\text{ISW}} = \frac{\hbar^2}{2ml^2} K_+ K_-,
\]

and a new number operator \(N^{\text{ISW}}\) is given by

\[
N^{\text{ISW}} = (1/2)(K_3 - 3),
\]

where \(N^{\text{ISW}} |n\rangle = n |n\rangle\), in other words, the operator \(N^{\text{ISW}}\) presents the same spectral characteristics as the occupation-number operator (6).
Due to the dynamical algebra associated, the Barut-Girardello coherent states $|z\rangle$, $z \in \mathbb{C}$, for the ISW are eigenstates of annihilation operator $K_{-}$

$$|z\rangle = \frac{|z|}{\sqrt{I_2(2|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!(n+2)!}} |n\rangle,$$

(17)

where $I_v(x)$ is the modified Bessel function of the first kind.

With the presented elements, Kieu’s algorithm is rewritten in the following way. Instead of replacing each one of the variables of (1) by (6) to construct (11), these can be replaced by (16) to obtain

$$H_{ISW}^D = (D(N_1^{ISW}, \ldots, N_k^{ISW}))^2 .$$

(18)

Due to (18), it is necessary to construct a new initial Hamiltonian $H_{ISW}^I$ from creation and annihilation (14) operators of $\mathfrak{su}(1,1)$

$$H_{ISW}^I = \sum_{i=1}^{k} (K_{+,i} - z_i^*) (K_{-,i} - z_i) ,$$

(19)

which has associated with the eigenvalue zero, the ground state $|\psi_{ISW}(0)\rangle$, constructed from the coherent states (17)

$$|\psi_{ISW}(0)\rangle = \bigotimes_{i=1}^{k} |z_i\rangle .$$

(20)

Finally, from (18) and (19) the Hamiltonian (9) takes the form

$$H_{ISW}^A(t) = \left(1 - \frac{t}{T}\right) H_{ISW}^I + \left(\frac{t}{T}\right) H_{ISW}^D .$$

(21)

Some observations with respect to the new algorithm:

1. The infinite-dimension Hamiltonian (15) acts on some Fock space too, whose orthonormal basis is (3), and satisfies that $|n\rangle = \frac{(K_{+})^n}{\sqrt{n!(n+2)!/2}} |0\rangle$.

2. The Hamiltonians (9) and (21) are unbounded operators, therefore it is necessary to use a version of the adiabatic theorem for unbounded operators,\textsuperscript{20} for both algorithms. However, this fact is not very significant because the algorithm operates more in the infrared part of the spectrum than in the ultraviolet part.

3. The conditions required for the finiteness of the run time $T$ are satisfied by (21) adapting Kieu’s arguments for algebra $\mathfrak{g}_W$ to algebra $\mathfrak{su}(1,1)$.

4. In order to satisfy the halting criterion (12), it is necessary to select $z_i$ values such that $|z_i| > 1.6$ for the construction of (20), since according to the density of probability associated to the states (17) for any $n$ and $|z| > 1.6$

$$|\langle z | n \rangle|^2 = \frac{|z|^2}{2I_2(2|z|)} < 1/2 ,$$

In the simulations of our algorithm,\textsuperscript{21} this argument has been numerically confirmed.
5. Unlike the algebra \( g_{W-H} \), the codification of (1) in the algebra \( su(1,1) \) could be directly carried out with the diagonal operator \( K_3 \). Apparently, due to (14) this codification change, changes the problem to be resolved, given that instead of establishing if (1) has, or doesn’t have non-negative integers solutions, the only thing that could be established is if (1) has or doesn’t have solutions of the form \( 2n + 3 \), with \( n \in \mathbb{N} \). Nevertheless, this problem is equivalent to Hilbert’s tenth problem parting from the construction of an infinite system of Diophantine equations.¹¹

6. Unlike the algebra \( g_{W-H} \), the algebra \( su(1,1) \) admits different types of coherent states (Barut-Girardello, Klauder, Perelomov, etc.).¹⁹ The advantage of a type of coherent state over another one would be in computational complexity issues, rather than in computational power issues.

7. The recent equivalence between the adiabatic computation and the “standard” quantum computation²² does not generate any contradiction with the hypercomputational characteristics of our algorithm (or Kieu’s algorithm), given that our algorithm is one of adiabatic quantum computation over infinite-dimensional spaces, and the demonstration of equivalence indicated is for the finite-dimensional case.

8. The Lie algebra \( su(1,1) \) is the dynamical algebra associated with different physical referents such as the infinite cylindrical wells, the Pöschl-Teller potentials, quantum optics systems with \( SU(1,1) \) symmetries, among others. Therefore, in principle, it is possible to select one of these referents as an underlying physical system of our hypercomputational quantum algorithm.²³

4. **NON-(TURING MACHINE) HYPERCOMPUTATION**

Within the context of hypercomputation, a hypermachine also capable of simulating a universal Turing machine is called a super-TM, otherwise it is called a non-TM.²⁴ The possible universality of our hypercomputation model based on the ISW would be established by its capability to generate a set of quantum gates, such that any unitary transformation \( U(2^n) \), that is, any quantum gate that operates upon \( n \)-qubits, can be approximated with sufficient exactness by a quantum circuit that is only made of a finite number of gates of this set.

For example⁴, for the controlled-NOT gate \( (CNOT^4) \), the transformation carried out upon the 2-qubit canonical basis \( \{|00\rangle^4, |01\rangle^4, |10\rangle^4, |11\rangle^4\} \), is given by

\[
|x,y\rangle^4 \xrightarrow{CNOT^4} |x,x \oplus y\rangle^4.
\]  

For the ISW with the Hamiltonian independent from the time (13), the states of the system evolve according to the Schrödinger equation solution of stationary states²⁵

\[
|n^\infty(t)\rangle = U^\infty(t) |n^\infty(0)\rangle = e^{-\frac{i}{\hbar}H_{ISW} t} |n^\infty(0)\rangle,
\]  

where \( U^\infty(t) \) is the unitary evolution operator and \( |n^\infty(0)\rangle = \sum_n c_n |n\rangle^\infty \). The \( U^\infty(t) \) matrix elements are

\[
U_{np}^\infty(t) = \exp \left(-i \frac{\hbar(n+2)t}{2ma^2} \right) \delta_{np},
\]  

where \( \delta_{np} \) is the Kronecker delta.

From different choices of the \( t \) parameter in (24), and different qubits coding, it is possible to find the evolutions corresponding to different quantum gates. In order to build the \( CNOT^\infty \) gate, based on the normalized

¹¹Henceforth, we will use the convention of a superindex over the operators (quantum gates) and over the states (qubits). This superindex will denote the dimension of Hilbert space upon which the operators act or upon which the states are defined.
Therefore, it is not possible to obtain the universality of our hypercomputation model parting from compound universal sets via gates that act upon a different number of qubits.

It is quite surprising that a quantum computation model over such a simple physical system different from the one selected by Kieu opens the possibility of obtaining new hypercomputation models supported by the quantum computation.

Kieu’s algorithm does not depend on any intrinsic peculiarity of the simple harmonic oscillator nor on its dynamical algebra $g_{WH}$. It is possible to consider that in principle any finite-dimensional dynamical algebra that admits an infinite-dimensional irreducible representation, and that it admits the formation of attainable coherent states in a known quantum system, is a good candidate to establish an algorithm à la Kieu.

## 5. CONCLUSIONS

It is quite surprising that a quantum computation model over such a simple physical system like the infinite square well has hypercomputational characteristics as it was shown. The success obtained by choosing an underlying physical system different from the one selected by Kieu opens the possibility of obtaining new hypercomputation models supported by the quantum computation.

Kieu’s algorithm does not depend on any intrinsic peculiarity of the simple harmonic oscillator nor on its dynamical algebra $g_{WH}$. It is possible to consider that in principle any finite-dimensional dynamical algebra that admits an infinite-dimensional irreducible representation, and that it admits the formation of attainable coherent states in a known quantum system, is a good candidate to establish an algorithm à la Kieu.

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