Strong Novikov conjecture for low degree cohomology and exotic group C*-algebras

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Abstract

We strengthen a result of Hanke–Schick about the strong Novikov conjecture for low degree cohomology by showing that their non-vanishing result for the maximal group C*-algebra even holds for the reduced group C*-algebra. To achieve this we provide a Fell absorption principle for certain exotic crossed product functors.

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1 Introduction

Recall the following result of Hanke and Schick [HS08]. Let $G$ be a discrete group and denote by $\Lambda^\ast(G) \subset H^\ast(BG; \mathbb{Q})$ the subring of the singular cohomology theory $H^\ast(BG; \mathbb{Q})$ with rational coefficients generated by $H^\leq 2(BG; \mathbb{Q})$, the rational cohomology classes of degree at most two. Further, let $\text{ch}: K_\ast(BG) \to H_\ast(BG; \mathbb{Q})$ be the homological Chern character from the $K$-homology to the homology of the classifying space $BG$ of $G$.

Theorem A ([HS08]). Let $h \in K_\ast(BG)$ such that there is $c \in \Lambda^\ast(G)$ with $\langle c, \text{ch}(h) \rangle \neq 0$.

Then $h$ is not mapped to zero under the assembly map $K_\ast(BG) \to K_\ast(C^\ast_{\text{max}}G) \otimes \mathbb{R}$.

Since the rational injectivity of the assembly map is known as the strong Novikov conjecture, their result states that the strong Novikov conjecture is true for those $K$-homology classes which can be detected by low degree cohomology classes.

Note that the strong Novikov conjecture firstly implies the classical Novikov conjecture about homotopy invariance of higher signatures, and secondly provides obstructions (the higher $\hat{A}$-genera) to the existence of positive scalar curvature metrics on manifolds. The Novikov conjecture for low degree cohomology classes was proven with different methods by Connes, Gromov and Moscovici [CGM93] and by Mathai [Mat03].

If $G$ is discrete and torsion free, the Baum–Connes conjecture states that the analytic assembly map $K_\ast(BG) \to K_\ast(C^\ast_rG)$ is an isomorphism. Note importantly, that on the right hand side we use the reduced group C*-algebra. Hence the Baum–Connes conjecture, which is already known for many groups, predicts that in the above theorem of Hanke and Schick we should actually be able to put the reduced, instead of the maximal, group C*-algebra on the right hand side. This idea is supported by the results in [AAS18], where it is shown that the image of the analytic assembly map is related to the $\tau$-part of the $K$-theory of the reduced group C*-algebra. On the other hand, the $\tau$-part of the $K$-theory for the reduced and the maximal group C*-algebras are canonically identified.

The goal of this paper is to confirm this fact. Let $G$ be a finitely presented group and, as before, denote by $\Lambda^\ast(G) \subset H^\ast(BG; \mathbb{Q})$ the subring generated by $H^\leq 2(BG; \mathbb{Q})$, the cohomology classes of degree at most two.

Theorem B. Let $h \in K_\ast(BG)$ such that there is $c \in \Lambda^\ast(G)$ with $\langle c, \text{ch}(h) \rangle \neq 0$.

Then $h$ is not mapped to zero under the assembly map $K_\ast(BG) \to K_\ast(C^\ast_rG) \otimes \mathbb{R}$.

In order to prove Theorem B we will use the machinery of exotic crossed products, and especially the recent result that the group C*-algebra corresponding to the minimal exact crossed product is actually the reduced group C*-algebra [BEW18b].

We have written the proof of Theorem B in such a way that it may be read without first reading Section 2 on exotic crossed products. Everything one needs to know about

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1Hanke and Schick first prove their theorem for finitely presented groups, and then use that every discrete group is a filtered colimit of finitely presented groups to generalize their result to all discrete groups. But in our situation, since the $K$-theory of the reduced group C*-algebra is not known to be functorial for arbitrary group homomorphisms, we can not carry out the last step generalizing to all discrete groups.
the kind of exotic crossed products we are using in the proof of Theorem \[ \text{II} \] is summarized in Properties \[ 3.3 \].

The main technical result that we will prove in Section \[ 2 \] is a Fell absorption principle for exotic crossed products (Lemma \[ 2.10 \]). A consequence of it, which we will exploit in our proof of Theorem \[ \text{II} \], is the following:

**Theorem C.** Let \( - \rtimes \mu \) be a correspondence crossed product functor.

Then the coproduct \( \Delta : \mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G, \sum a_g g \mapsto \sum a_g (g \otimes g) \) extends continuously to a \(*\)-homomorphism

\[
\Delta : C^*_\mu G \to C^*_\text{max} G \otimes \mu C^*_\mu G.
\]

If we denote by \( - \rtimes \epsilon \) the minimal exact correspondence crossed product functor \cite[Cor. 8.8]{BEW18a}, then the above theorem combined with the fact that the associated group \( C^*-\text{algebra} C^*_\epsilon G := \mathbb{C} \rtimes \epsilon G \) coincides with the reduced group \( C^*-\text{algebra} C^*_r G \), we get the \(*\)-homomorphism \( \Delta : C^*_r G \to C^*_\text{max} G \otimes \epsilon C^*_r G \). Note that up to now it was only known that the coproduct integrates to a continuous map \( C^*_r G \to C^*_\text{max} G \otimes_C C^*_r G \). The fact that we are able to lift it to a map where we use an exact tensor product is a crucial ingredient in our proof of Theorem \[ \text{II} \].

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**2 Exotic crossed products**

In this section we describe and prove some properties of exotic crossed products. Even though, in applications to the Novikov conjecture only discrete groups are involved, we present these facts in full generality. Let \( G \) be a locally compact group with a fixed (left invariant) Haar measure (which we simply write as \( ds \) in integrals) and let \( \Delta \) the associated modular function on \( G \).

- We let \( \text{C}^*\text{-Alg}(G) \) (resp. \( \text{CP}(G) \)) denote the category of \( G \)-\( \text{C}^* \)-algebras with \( G \)-equivariant \(*\)-homomorphisms (resp., \( G \)-equivariant completely positive maps) as morphisms.

- For \( \text{C}^*\text{-Alg}(\{e\}) \) (resp. \( \text{CP}(\{e\}) \)), where \( e \) denotes the trivial group, we also write \( \text{C}^*\text{-Alg} \) (resp. \( \text{CP} \)).

- Further, let \( \text{*-Alg} \) denote the category of involutive algebras with \(*\)-homomorphisms as morphisms.
Definition 2.1 (Crossed products).

1. By \( -\rtimes_{\text{alg}} G : \text{C}^*\text{-Alg}(G) \to \text{*-Alg} \) we denote the functor mapping a \( G \)-C*-algebra \( A \) with action \( \alpha \) to

\[
A \rtimes_{\text{alg}} G := C_c(G, A)
\]

as a vector space equipped with the product

\[
(f \ast g)(t) := \int f(s)\alpha_s(g(s^{-1}t))ds
\]

for \( f, g \in A \rtimes_{\text{alg}} G \) and \( t \in G \), and the involution

\[
f^*(s) := \Delta(s^{-1})\alpha_s((f(s^{-1}))^*)
\]

for \( f \in A \rtimes_{\text{alg}} G \) and \( s \in G \), and mapping a \( G \)-equivariant *-homomorphism \( \varphi : A \to B \) to the *-homomorphism

\[
\varphi \rtimes_{\text{alg}} G : A \rtimes_{\text{alg}} G \to B \rtimes_{\text{alg}} G, f \mapsto \varphi \circ f.
\]

2. By \( -\rtimes_{\text{max}} G : \text{C}^*\text{-Alg}(G) \to \text{C}^*\text{-Alg} \) we denote the maximal crossed product functor, which comes with a natural assignment \( (\kappa_A : A \rtimes_{\text{alg}} G \to A \rtimes_{\text{max}} G)_{A \in \text{C}^*\text{-Alg}(G)} \) consisting of injective *-homomorphisms with dense image.

3. Finally, let \( -\rtimes_{\text{r}} G : \text{C}^*\text{-Alg}(G) \to \text{C}^*\text{-Alg} \) be the reduced crossed product functor. There is a natural transformation \( \Lambda : -\rtimes_{\text{max}} G \to -\rtimes_{\text{r}} G \) between the functors \( -\rtimes_{\text{max}} G \) and \( -\rtimes_{\text{r}} G \) consisting of surjective *-homomorphisms such that \( \Lambda \circ \kappa \) consists of injective *-homomorphisms (with dense image). ♦

A crossed product functor \( -\rtimes_{\mu} G \) is a functor

\[
-\rtimes_{\mu} G : \text{C}^*\text{-Alg}(G) \to \text{C}^*\text{-Alg}
\]

together with natural transformations

\[
q : -\rtimes_{\text{max}} G \to -\rtimes_{\mu} G \quad \text{and} \quad s : -\rtimes_{\mu} G \to -\rtimes_{\text{r}} G
\]

consisting of surjective *-homomorphisms such that \( s \circ q = \Lambda \). In particular, we have that \( q \circ \kappa : -\rtimes_{\text{alg}} G \to -\rtimes_{\mu} G \) is a natural assignment consisting of injective *-homomorphisms with dense image. Therefore we can consider \( A \rtimes_{\text{alg}} G \) as a *-subalgebra of \( A \rtimes_{\mu} G \) for all \( G \)-C*-algebras \( A \).

We write \( \mathcal{M}(A) \) for the multiplier algebra of a C*-algebra \( A \). Let \( \varphi : A \to \mathcal{M}(B) \) be a nondegenerate *-homomorphism between a C*-algebra \( A \) and the multiplier algebra \( \mathcal{M}(B) \) of a C*-algebra \( B \). This means that \( \varphi(A)B \) is dense in \( B \). In this case there is a unique extension of \( \varphi \) to a homomorphism on \( \mathcal{M}(A) \); we denote it by \( \overline{\varphi} : \mathcal{M}(A) \to \mathcal{M}(B) \).
If $-\rtimes_{\mu} G$ is a crossed product functor, then we typically write $C^*_\mu G$ for $C \rtimes_{\mu} G$. Furthermore, let $A$ be a $G$-$C^*$-algebra and let $(A \rtimes_{\text{max}} G, \iota_A, \iota_G)$ be the maximal crossed product of $A$ together with the universal covariant representation $\iota_A: A \to \mathcal{M}(A \rtimes_{\text{max}} G)$ and $\iota_G: G \to \mathcal{M}(A \rtimes_{\text{max}} G)$. Then

$$
\iota_{A,\mu} := \overline{\iota_A} \circ \iota_A : A \to \mathcal{M}(A \rtimes_{\mu} G)
$$

$$
\iota_{G,\mu} := \overline{\iota_A} \circ \iota_G : G \to \mathcal{M}(A \rtimes_{\mu} G)
$$

is a covariant representation of $A$.

\section*{2.1 Properties of correspondence crossed product functors}

\begin{lemma}
Let $-\rtimes_{\mu} G$ be a crossed product functor. Then $-\rtimes_{\mu} G$ preserves direct sums of $C^*$-algebras. To be more precise, if $\{A_i: i \in I\}$ is any collection of $G$-$C^*$-algebras, then there is a canonical isomorphism

$$
\left( \bigoplus_{i \in I} A_i \right) \rtimes_{\mu} G \cong \bigoplus_{i \in I} (A_i \rtimes_{\mu} G).
$$

\end{lemma}

\begin{proof}
Let $A$ be the direct sum $\bigoplus_i A_i$. This is the universal $G$-$C^*$-algebra generated by orthogonal copies of $A_i$ as ideals. The $G$-equivariant inclusion $\phi_i: A_i \to A$ then lifts to a *-homomorphism $\phi_i \rtimes_{\mu} G: A_i \rtimes_{\mu} G \to A \rtimes_{\mu} G$ and the images of these maps are mutually orthogonal, so we get a well-defined *-homomorphism $\phi: \bigoplus_i (A_i \rtimes_{\mu} G) \to A \rtimes_{\mu} G$, which is clearly also surjective. It is also injective because the canonical $G$-equivariant projections $\psi_i: A \to A_i$ by functoriality yield *-homomorphisms $\psi_i \rtimes_{\mu} G: A \rtimes_{\mu} G \to A_i \rtimes_{\mu} G$ that therefore give a *-homomorphism $\psi: A \rtimes_{\mu} G \to \prod_i (A_i \rtimes_{\mu} G)$ such that $\psi \circ \phi$ equals the canonical embedding $\bigoplus_i (A_i \rtimes_{\mu} G) \to \prod_i (A_i \rtimes_{\mu} G)$.
\end{proof}

We are going to use a class of crossed products which is well behaved with respect to completely positive maps. These are proven in \cite{BEW18a} to be exactly the correspondence crossed products, i.e. the crossed products which are functorial for correspondences defined as bimodules in the sense of Kasparov. Indeed they allow for the construction of a descent morphism in equivariant $KK$-theory. Among several equivalent definitions (cf. \cite[Thm. 4.9]{BEW18a}) we recall the one related to completely positive maps.

\begin{definition}
A crossed product functor $-\rtimes_{\mu} G$ has the cp-map property (or equivalent, is a correspondence crossed product functor), if $-\rtimes_{\mu} G$ extends to a functor

$$
-\rtimes_{\mu} G: \text{CP}(G) \to \text{CP}
$$

in the following sense:

For all $G$-$C^*$-algebras $A$ and $B$ and every $G$-equivariant completely positive map $\varphi: A \to B$, there is a completely positive map $\varphi \rtimes_{\mu} G: A \rtimes_{\mu} G \to B \rtimes_{\mu} G$ determined by $(\varphi \rtimes_{\mu} G)(f) = \varphi \circ f$ for all $f \in A \rtimes_{\text{alg}} G$.
\end{definition}

The next lemma is a direct combination of Theorem 4.9 and Lemma 3.3 in \cite{BEW18a} (taking into account the implication “hereditary subalgebra property $\Rightarrow$ ideal property”).
Lemma 2.4. Every correspondence crossed product functor $- \rtimes_{\mu} G$ is functorial for generalised homomorphisms. In other words, for all $G$-C*-algebras $A$ and $B$ and every $G$-equivariant *-homomorphism $\phi : A \to \mathcal{M}(B)$, there exists a *-homomorphism

$$\phi \rtimes_{\mu} G : A \rtimes_{\mu} G \to \mathcal{M}(B \rtimes_{\mu} G)$$

given by

$$(\phi \rtimes_{\mu} G)(f)g = (\phi \circ f) \ast g$$

for all $f \in A \rtimes_{\text{alg}} G$ and $g \in B \rtimes_{\text{alg}} G$.

Remark 2.5. Note that for a crossed product functor $- \rtimes_{\mu} G$ which is functorial for generalised homomorphisms, the unitary group representation $\iota_{G,\mu} : G \to \mathcal{M}(A \rtimes_{\mu} G)$ integrates to a *-homomorphism $\iota_{G,\mu} : C^*_{\mu} G \to \mathcal{M}(A \rtimes_{\mu} G)$ for all $G$-C*-algebras $A$.

To see this, consider the $G$-equivariant unital *-homomorphism $m : C \to \mathcal{M}(A)$ (which is just the scalar multiplication with the unit $1_A$); since $- \rtimes_{\mu} G$ is functorial for generalised homomorphisms, $m$ induces a *-homomorphism $m \rtimes_{\mu} G : C^*_{\mu} G \to \mathcal{M}(A \rtimes_{\mu} G)$, which is the integrated form of $\iota_{G,\mu}$.

Furthermore, a consequence of Lemma 2.4 is the following:

Proposition 2.6. Let $G$ be a discrete group, $A$ a $G$-C*-algebra and $(A \rtimes_{\text{max}} G, \iota_A, \iota_G)$ the maximal crossed product together with the universal covariant representation

$$\iota_A : A \to \mathcal{M}(A \rtimes_{\text{max}} G) \quad \text{and} \quad \iota_G : G \to \mathcal{M}(A \rtimes_{\text{max}} G).$$

Then $(A \rtimes_{\text{max}} G, \text{Ad}(\iota_G))$ is a $G$-C*-algebra and $\iota_A$ is a $G$-equivariant *-homomorphism $A \to A \rtimes_{\text{max}} G$, where the $G$-action on $A \rtimes_{\text{max}} G$ is now given by $\text{Ad}(\iota_G)$.

Further, if $- \rtimes_{\mu} G$ is a correspondence crossed product functor, then the induced *-homomorphism

$$\iota_A \rtimes_{\mu} G : A \rtimes_{\mu} G \to (A \rtimes_{\text{max}} G) \rtimes_{\mu, \text{Ad}(\iota_G)} G$$

is injective.

Proof. Notice that, since $G$ is a discrete group, the image of $\iota_A$ is contained in $A \rtimes_{\text{alg}} G \subseteq A \rtimes_{\text{max}} G$; indeed, it is given by $\iota_A(a) = a\delta_e$ for all $a \in A$ where $e \in G$ is the identity element.

Let $E : A \rtimes_{\text{max}} G \to A$ be the canonical conditional expectation; this is the continuous extension of the evaluation $A \rtimes_{\text{alg}} G \to A$ at the identity element $e \in G$. It is a $G$-equivariant completely positive map. Since $E \circ \iota_A = \text{id}_A$, we obtain by the cp-map property

$$\text{id}_{A \rtimes_{\mu} G} = (E \rtimes_{\mu} G) \circ (\iota_A \rtimes_{\mu} G),$$

which proves the injectivity of $\iota_A \rtimes_{\mu} G$.}

Let $(A, \alpha)$ be a $G$-C*-algebra with a $G$-action $\alpha$. Recall that the action $\alpha$ is called unitarily implemented if there exists a strictly continuous unitary group representation $u : G \to \mathcal{M}(A)$ such that

$$\alpha_s = \text{Ad}(u_s)$$

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for all $s \in G$. In the language of \cite{BEW18a} Sec. 5] this means that $\alpha$ is exterior equivalent to the trivial action.

The next proposition follows from \cite{BEW18a} Lem. 5.2 and \cite{BEW18a} Thm. 4.9.

**Proposition 2.7.** Let $(A, \alpha)$ be a $G$-$C^*$-algebra with a unitarily implemented action $\alpha$, and denote by $u: G \to \mathcal{M}(A)$ the corresponding unitary group representation.

If $- \rtimes_{\mu} G$ is a correspondence crossed product functor, then

$$
\varphi: A \rtimes_{\text{alg}, \alpha} G \to A \rtimes_{\text{alg}, \text{id}} G, \quad f \mapsto (G \to A, s \mapsto f(s)u_s)
$$

extends to a $*$-isomorphism $\varphi: A \rtimes_{\mu, \alpha} G \to A \rtimes_{\mu, \text{id}} G$.

### 2.2 Fell absorption principle for crossed product functors

Let $- \rtimes_{\mu} G$ be a correspondence crossed product functor. First of all, we define a $C^*$-tensor product

$$
- \otimes_{\mu} C^*_\mu G: C^*-\text{Alg} \to C^*-\text{Alg}
$$

with the group $C^*$-algebra $C^*_\mu G$. For this purpose, we let $A$ be a $C^*$-algebra. Then $A$ can be considered as a $G$-$C^*$-algebra with $G$ acting trivially on $A$. We set

$$
A \otimes_{\mu} C^*_\mu G := A \rtimes_{\mu, \text{id}} G.
$$

The following proposition proves that this is indeed a $C^*$-tensor product of $A$ and $C^*_\mu G$ and justifies our notation.

We first recall some notation. Let $A$ be a $C^*$-algebra and let $(A \rtimes_{\text{max}} G, \iota_A, \iota_G)$ be the maximal crossed product together with the universal covariant representation

$$
\iota_A: A \to \mathcal{M}(A \rtimes_{\text{max}} G) \quad \text{and} \quad \iota_G: G \to \mathcal{M}(A \rtimes_{\text{max}} G).
$$

Let $q_A: A \rtimes_{\text{max}} G \to A \rtimes_{\mu} G$ be the natural quotient map and $\iota_{A, \mu} := q_A \circ \iota_A$. Finally, let $\iota_{G, \mu}: C^*_\mu G \to \mathcal{M}(A \rtimes_{\mu} G)$ be the canonical $*$-homomorphism described in Remark 2.5.

**Proposition 2.8.** Let $G$ act trivially on $A$. Then the $*$-homomorphisms $\iota_{A, \mu}$ and $\iota_{G, \mu}$ commute and we have

$$
\iota_{A, \mu}(a) \cdot \iota_{G, \mu}(f) \in A \rtimes_{\mu} G
$$

for every $a \in A$ and $f \in \mathbb{C} \rtimes_{\text{alg}} G$. The induced $*$-homomorphism

$$
\iota_{A, \mu} \otimes \iota_{G, \mu}: A \otimes C^*_\mu G \to A \rtimes_{\mu} G
$$

is injective and has dense range. In particular, $A \rtimes_{\mu} G$ is a tensor product for $A$ and $C^*_\mu G$. 

Proof. For \( a \in A \) and \( f \in \mathbb{C} \times_{\text{alg}} G \) we have \( \iota_{A,\mu}(a) \cdot \iota_{G,\mu}(f) = (a \otimes f : G \to A, s \mapsto f(s)a) \), which is contained in \( A \times_{\text{alg}} G \subseteq A \times_\mu G \), and it is clear that the *-homomorphisms \( \iota_{A,\mu} \) and \( \iota_{G,\mu} \) commute. We also obtain \( \iota_{A,\mu}(a) \cdot \iota_{G,\mu}(x) \in A \times_\mu G \) for all \( a \in A \) and \( x \in C^*_\mu G \). It is also quickly verified that \( \text{span}\{ \iota_{A,\mu}(a) \cdot \iota_{G,\mu}(f) : a \in A, f \in \mathbb{C} \times_{\text{alg}} G \} \) is a dense subspace of \( A \times_\mu G \).

Therefore it remains to prove that \( \iota_{A,\mu} \otimes \iota_{G,\mu} \) is injective. For this purpose, assume that \( x \in A \otimes C^*_\mu G \) is an element in the kernel of \( \iota_{A,\mu} \otimes \iota_{G,\mu} \). Then there are elements \( a_1, \ldots, a_n \in A \) and \( b_1, \ldots, b_n \in C^*_\mu(G) \) such that \( x = \sum_i a_i \otimes b_i \) and \( \{ b_i : 1 \leq i \leq n \} \) is linearly independent. Since \( - \times_\mu G \) has the ep-map property, every state \( \varphi \in \mathcal{S}(A) \) on \( A \) induces a completely positive map \( \varphi \times_\mu G : A \times_\mu G \to C^*_\mu G \) given by

\[
(\varphi \times_\mu G)(f)(t) = \varphi(f(t))
\]

for \( f \in A \times_{\text{alg}} G \) and \( t \in G \). We obtain

\[
(\varphi \times_\mu G)(\iota_{A,\mu} \otimes \iota_{G,\mu}(x)) = \sum_i \varphi(a_i)b_i = 0
\]

for all states \( \varphi \in \mathcal{S}(A) \) on \( A \). This implies \( \varphi(a_i) = 0 \) for all \( \varphi \in \mathcal{S}(A) \) and \( 1 \leq i \leq n \) and finally \( a_i = 0 \) for all \( 1 \leq i \leq n \). This completes the proof.  

\[\square\]

Remark 2.9. In the following we will identify \( A \otimes C^*_\mu(G) \) with a *-subalgebra of \( A \times_\mu G \) (via \( \iota_{A,\mu} \otimes \iota_{G,\mu} \)) whenever \( A \) is a C*-algebra (with trivial \( G \)-action).

Note that the tensor product \( - \otimes_\mu C^*_\mu G \) is just the restriction of the original crossed product functor \( - \times_\mu G \) to C* -algebras with the trivial \( G \)-action and hence inherits properties like (generalised) functoriality, exactness, etc.  

Let \( - \times_\mu G \) be a correspondence crossed product functor and let \( A \) be a \( G \)-C*-algebra. Let \( (A \times_{\max} G, \iota_A, \iota_G) \) be the maximal crossed product of \( A \) and \( u_{G,\mu} : G \to \mathcal{M}(C^*_\mu G) \) the canonical unitary representation on \( C^*_\mu G \).

Lemma 2.10. The *-homomorphisms

- \( \iota_A \otimes_\mu 1 : A \to \mathcal{M}((A \times_{\max} G) \otimes_\mu C^*_\mu G) \) induced by

\[
(\iota_A \otimes_\mu 1)(a)(b \otimes c) := \iota_A(a)b \otimes c
\]

for \( a \in A \), \( b \in A \times_{\max} G \) and \( c \in C^*_\mu G \), and

- \( \iota_G \otimes_\mu u_{G,\mu} : G \to \mathcal{M}((A \times_{\max} G) \otimes_\mu C^*_\mu G) \) induced by

\[
(\iota_G \otimes_\mu u_{G,\mu})(s)(b \otimes c) = \iota_G(s)b \otimes u_{G,\mu}(s)c
\]

for all \( s \in G \), \( b \in A \times_{\max} G \) and \( c \in C^*_\mu G \)

exist and define a nondegenerate covariant representation of \( A \). The integrated form

\[
(\iota_A \otimes_\mu 1) \times_{\max} (\iota_G \otimes_\mu u_{G,G}) : A \times_{\max} G \to \mathcal{M}((A \times_{\max} G) \otimes_\mu C^*_\mu G)
\]

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factors through $q_A: A\rtimes_{\text{max}} G \to A\rtimes_{\mu} G$, i.e., there is a unique *-homomorphism

$$(t_A \otimes_{\mu} 1) \rtimes_{\mu} (t_G \otimes_{\mu} u_{G,G}): A\rtimes_{\mu} G \to \mathcal{M}((A\rtimes_{\text{max}} G) \otimes_{\mu} C_{\mu}^* G)$$

such that $(t_A \otimes_{\mu} 1) \rtimes_{\text{max}} (t_G \otimes_{\mu} u_{G,G}) = (t_G \otimes_{\mu} (t_G \otimes_{\mu} u_{G,G})) \circ q_A$. If $G$ is discrete, then $(t_A \otimes_{\mu} 1) \rtimes_{\mu} (t_G \otimes_{\mu} u_{G,G})$ is injective.

**Proof.** Let us first show existence of $t_A \otimes_{\mu} 1$ and $t_G \otimes_{\mu} u_{G,G}$. It is known [RW98, Prop. B.21] that there are *-monomorphisms

$$j_{A\rtimes_{\text{max}} G}: \mathcal{M}(A\rtimes_{\text{max}} G) \longrightarrow \mathcal{M}((A\rtimes_{\text{max}} G) \otimes_{\mu} C_{\mu}^* G)$$

$$j_{C_{\mu}^* G}: \mathcal{M}(C_{\mu}^* G) \longrightarrow \mathcal{M}((A\rtimes_{\text{max}} G) \otimes_{\mu} C_{\mu}^* G)$$

which satisfy

$$j_{A\rtimes_{\text{max}} G}(m)(a \otimes b) = ma \otimes b$$

$$j_{C_{\mu}^* G}(n)(a \otimes b) = a \otimes nb$$

$$j_{A\rtimes_{\text{max}} G}(a) \cdot j_{C_{\mu}^* G}(b) = a \otimes b$$

for all $m \in \mathcal{M}(A\rtimes_{\text{max}} G)$, $n \in \mathcal{M}(C_{\mu}^* G)$, $a \in A\rtimes_{\text{max}} G$ and $b \in C_{\mu}^* G$. Note that by the first two equations in the previous display the images of $j_{A\rtimes_{\text{max}} G}$ and $j_{C_{\mu}^* G}$ commute with each other. We now set

$$t_A \otimes_{\mu} 1 := j_{A\rtimes_{\text{max}} G} \circ t_A \quad \text{and} \quad t_G \otimes_{\mu} u_{G,G} = (j_{A\rtimes_{\text{max}} G} \circ t_G) \cdot (j_{C_{\mu}^* G} \circ u_{G,G}).$$

One immediately sees that $(t_A \otimes_{\mu} 1, t_G \otimes_{\mu} u_{G,G})$ is of the form claimed by the lemma. Starting from the formulas and, since the universal covariant representation $(t_A, t_G)$ is involved, it follows immediately that $(t_A \otimes_{\mu} 1, t_G \otimes_{\mu} u_{G,G})$ is a nondegenerate covariant representation of $A$. It integrates to a nondegenerate *-representation

$$(t_A \otimes_{\mu} 1) \rtimes_{\text{max}} (t_G \otimes_{\mu} u_{G,G}): A\rtimes_{\text{max}} G \to \mathcal{M}((A\rtimes_{\text{max}} G) \otimes_{\mu} C_{\mu}^* G).$$

It remains to show that $(t_A \otimes_{\mu} 1) \rtimes_{\text{max}} (t_G \otimes_{\mu} u_{G,G})$ factors through $q_A: A\rtimes_{\text{max}} G \to A\rtimes_{\mu} G$.

To this end, we note that $(A\rtimes_{\text{max}} G, Ad(t_G))$ defines a $G$-$C^*$-algebra whose action is unitarily implemented. Because $- \rtimes_{\mu} G$ is a correspondence crossed product functor by assumption, by Proposition 2.7 the $C^*$-algebra $(A\rtimes_{\text{max}} G)\rtimes_{\mu, Ad(t_G)} G$ is *-isomorphic to

$$(A\rtimes_{\text{max}} G) \rtimes_{\mu, \text{id}} G = (A\rtimes_{\text{max}} G) \otimes_{\mu} C_{\mu}^* G.$$

The *-isomorphism $\varphi: (A\rtimes_{\text{max}} G)\rtimes_{\mu, Ad(t_G)} G \to (A\rtimes_{\text{max}} G) \otimes_{\mu} C_{\mu}^* G$ is given by

$$\varphi(f) := (G \to A\rtimes_{\mu} G, s \mapsto f(s)t_G(s))$$

for $f \in (A\rtimes_{\text{max}} G)\rtimes_{\text{alg}, Ad(t_G)} G$. Furthermore, $t_A: A \to \mathcal{M}(A\rtimes_{\text{max}} G)$ is a $G$-equivariant generalised *-homomorphism, where $Ad(t_G)$ is the action used on $A\rtimes_{\text{max}} G$, and therefore induces a *-homomorphism

$$t_A \rtimes_{\mu} G: A\rtimes_{\mu} G \to \mathcal{M}((A\rtimes_{\text{max}} G) \rtimes_{\mu, Ad(t_G)} G).$$
We now define
\[ \pi := \varphi \circ (\iota_A \rtimes_{\mu} G) : A \rtimes_{\mu} G \to \mathcal{M}((A \rtimes_{\max} G) \otimes_{\mu} C_{\mu}^* G) \].

We have that \( \pi \) and \( (\iota_A \otimes_{\mu} 1) \rtimes_{\max} (\iota_G \otimes_{\mu} u_{G,\mu}) \) coincide on the dense subspace \( A \rtimes_{\alg} G \). Hence \( (\iota_A \otimes_{\mu} 1) \rtimes_{\max} (\iota_G \otimes_{\mu} u_{G,\mu}) \) factors through \( q_A : A \rtimes_{\max} G \to A \rtimes_{\mu} G \).

If \( G \) is discrete, then Proposition \( \ref{prop:2.6} \) states that \( \iota_A \rtimes_{\mu} G \) is injective and hence \( \pi \) is injective. This implies that \( (\iota_A \otimes_{\mu} 1) \rtimes_{\mu} (\iota_G \otimes_{\mu} u_{G,\mu}) \) is injective.

\[ \Box \]

### 2.3 Exact correspondence crossed product functors

**Definition 2.11.** Let \( - \rtimes_{\mu} G \) be a crossed product functor. It is called exact if for every \( G \)-\( C^* \)-algebra \( A \) and every \( G \)-invariant ideal \( I \) of \( A \) the sequence
\[ 0 \to I \rtimes_{\mu} G \to A \rtimes_{\mu} G \to (A/I) \rtimes_{\mu} G \to 0 \]
is exact.

**Corollary 2.12.** Let \( - \rtimes_{\mu} G \) be a correspondence crossed product functor for a discrete group \( G \). Then it is exact if and only if it is exact for trivial actions, that is, if the functor \( A \mapsto A \otimes_{\mu} C_{\mu}^* G \) is exact on the category of \( C^* \)-algebras.

**Proof.** The forward direction is trivial.

For the converse we use the embedding \( A \rtimes_{\mu} G \hookrightarrow (A \rtimes_{\max} G) \otimes_{\mu} C_{\mu}^* G \) provided by Lemma \( \ref{lemma:2.10} \). Then the proof is exactly the same as the one for the fact that a group is exact if and only if its reduced group \( C^* \)-algebra is exact. For convenience we sketch the proof here. Given a \( G \)-invariant ideal \( I \subseteq A \), the exactness of \( - \rtimes_{\max} G \) yields a short exact sequence \( I \rtimes_{\max} G \hookrightarrow A \rtimes_{\max} G \to (A/I) \rtimes_{\max} G \). Using the maps provided by Lemma \( \ref{lemma:2.10} \) for \( I, A \) and \( A/I \), we get a commutative diagram
\[
\begin{array}{c}
(I \rtimes_{\max} G) \otimes_{\mu} C_{\mu}^* G \\
\downarrow \quad \downarrow \quad \quad \downarrow \quad \downarrow \quad \quad \downarrow \\
I \rtimes_{\mu} G & \quad A \rtimes_{\mu} G & \quad (A/I) \rtimes_{\mu} G
\end{array}
\]

where the vertical arrows are the injective homomorphisms provided by Lemma \( \ref{lemma:2.10} \) and the top line is exact by assumption.

The following argument now yields the exactness of the bottom line at \( A \rtimes_{\mu} G \): let \( x \in A \rtimes_{\mu} G \) with \( q(x) = 0 \). Then \( \delta(x) \in \ker(\tilde{q}) = J := (I \rtimes_{\max} G) \otimes_{\mu} C_{\mu}^* G \). Let \( (e_i) \) be an approximate unit for \( I \rtimes_{\mu} G \). Since \( \delta \) is nondegenerate (by Lemma \( \ref{lemma:2.10} \)), \( \delta(e_i) \) is an approximate unit for \( J \). Since \( \delta(x) \in J \) we have \( \delta(xe_i) = \delta(x)\delta(e_i) \to \delta(x) \). But since \( \delta \) is isometric, this is equivalent to \( xe_i \to x \). Therefore \( x \in I \rtimes_{\mu} G \).

The injectivity of \( i \) follows from the ideal property of \( \rtimes_{\mu} \), and surjectivity of \( q \) holds, since it is quickly seen that it must have dense image. \[ \Box \]
Exact correspondence crossed product functors exist in abundance, as the next lemma shows. In many cases it is possible to start with a group C*-algebra and to construct then the corresponding crossed product functor.

**Lemma 2.13.** Let $G$ be a second-countable locally compact group, and let $C^*_r G$ be a group C*-algebra such that the dual space $(C^*_r G)^*$ of $C^*_r G$ is (isomorphic to) an ideal in the Fourier–Stieltjes algebra of $G$ [KLQ13].

Then there exists an exact correspondence crossed product functor $- \rtimes G$ such that $\mathbb{C} \rtimes G = C^*_r G$.

**Proof.** Let $- \rtimes_{KLQ} G$ be the KLQ-functor satisfying $\mathbb{C} \rtimes_{KLQ} G = C^*_r G$, see [BEW17, Prop. 3.9]. There exists a minimal exact correspondence functor $- \rtimes G$ that dominates $- \rtimes_{KLQ} G$. So it remains to show that $\mathbb{C} \rtimes G = \mathbb{C} \rtimes_{KLQ} G$.

For this purpose, we let $- \rtimes_{\mathcal{E}(E)} G$ be the minimal exact crossed product functor that dominates $- \rtimes_{KLQ} G$. Then by [BEW18b, Prop. 2.6] we have $\mathbb{C} \rtimes_{\mathcal{E}(E)} G = \mathbb{C} \rtimes_{KLQ} G$. Let $- \rtimes_{\text{sep}, \mathcal{E}(E)} G$ be given by

$$A \rtimes_{\text{sep}, \mathcal{E}(E)} G = \lim_{i \in I_A} A_i \rtimes_{\mathcal{E}(E)} G$$

for $G$-C*-algebras $A$, where $I_A$ is the directed set of all separable sub-$G$-C*-algebras of $A$ ordered by inclusion. This defines in a canonical sense an exact crossed product functor that dominates $- \rtimes_{\mathcal{E}(E)} G$ (see [BEW18a, Lem. 8.11]).

Note that, by [BEW18b, Prop. 4.4], the functor $- \rtimes_{\mathcal{E}(E)} G$ is Morita compatible. Hence by [BEW18a, Prop. 8.10] the functor $- \rtimes_{\mathcal{E}(E)} G$ has the full projection property on the category of separable $G$-C*-algebras. Therefore, by [BEW18a, Lem. 4.11], $- \rtimes_{\mathcal{E}(E)} G$ has the projection property on the category of separable $G$-C*-algebras. Hence $- \rtimes_{\text{sep}, \mathcal{E}(E)} G$ has the projection property (see [BEW18a, Lem. 8.11] and therefore $- \rtimes_{\text{sep}, \mathcal{E}(E)} G$ is a correspondence functor. Since $- \rtimes_{\text{sep}, \mathcal{E}(E)} G$ is exact as well, $- \rtimes_{\text{sep}, \mathcal{E}(E)} G$ dominates $- \rtimes G$. This completes the proof.

**Corollary 2.14.** Let $G$ be a countable, discrete group. Then there exists an exact correspondence crossed product functor $- \rtimes G$ such that $C^*_r G \cong C^*_r G$, where the latter denotes the reduced group C*-algebra.

**Proof.** We apply Lemma 2.13 to the reduced group C*-algebra to get an exact correspondence crossed product functor $- \rtimes G$ with $C^*_r G \cong C^*_r G$.\hfill\Box

**Remark 2.15.** Note that we do not need Lemma 2.13 to get an exact correspondence crossed product functor $- \rtimes G$ with $C^*_r G \cong C^*_r G$. That such a functor exists was already proven in [BEW18b].\hfill\blacktriangleleft

For the proof of Theorem 3.4 we need to extend an exact correspondence crossed product functor $- \rtimes G$ to an exact correspondence crossed product functor $- \rtimes (G \times \mathbb{Z})$ such that $C^*_r (G \times \mathbb{Z}) \cong C^*_r G \otimes C^* \mathbb{Z}$. This is the content of the next lemma.

**Lemma 2.16.** Let $- \rtimes G$ be a crossed product functor. Then there is a crossed product functor $- \rtimes G \times \mathbb{Z}$ such that $C^*_r (G \times \mathbb{Z}) \cong C^*_r G \otimes C^* \mathbb{Z}$. Further,
1. if $\kappa_{\mu} G$ is exact, then $\kappa_{\bar{\mu}} (G \times \mathbb{Z})$ is exact, and

2. if $\kappa_{\mu} G$ is a correspondence functor, then $\kappa_{\bar{\mu}} (G \times \mathbb{Z})$ also is one.

**Proof.** First of all, note that the functor $-\kappa_{\mu} G$ induces a functor from $C^*\text{-Alg}(G \times \mathbb{Z})$ to $C^*\text{-Alg}(\mathbb{Z})$, which we again denote by $-\kappa_{\mu} G$, in the following way:

For a $(G \times \mathbb{Z})$-C*-algebra $A$ with action $\alpha: G \times \mathbb{Z} \to \text{Aut}(A)$ we write $\alpha_G = \alpha_{G \times \{0\}}$ and $\alpha_\mathbb{Z} = \alpha_{\{e\} \times \mathbb{Z}}$. Then $\alpha_\mathbb{Z}(n)$ is a $G$-equivariant *-homomorphism on $(A, \alpha_G)$ for all $n \in \mathbb{Z}$. Hence it induces a $\mathbb{Z}$-homomorphism $\alpha_\mathbb{Z}(n) \kappa_{\mu} G: A \kappa_{\mu, \alpha_G} G \to A \kappa_{\mu, \alpha_G} G$ for all $n \in \mathbb{Z}$. Since $-\kappa_{\mu} G$ is a functor, $\alpha_\mathbb{Z} \kappa_{\mu} G: \mathbb{Z} \to \text{Aut}(A \kappa_{\mu, \alpha_G} G)$, $n \mapsto \alpha_\mathbb{Z}(n) \kappa_{\mu} G$ defines a group homomorphism. Finally, $(A \kappa_{\mu, \alpha_G} G, \alpha_\mathbb{Z} \kappa_{\mu} G)$ defines a $\mathbb{Z}$-C*-algebra.

For a $(G \times \mathbb{Z})$-equivariant *-homomorphism $\varphi: A \to B$ between $(G \times \mathbb{Z})$-C*-algebras $A$ and $B$ with actions $\alpha$ and $\beta$ we obtain a $\mathbb{Z}$-equivariant *-homomorphism

$$
\varphi \kappa_{\mu} G: (A \kappa_{\mu, \alpha_G} G, \alpha_\mathbb{Z} \kappa_{\mu} G) \to (B \kappa_{\mu, \beta_G} G, \beta_\mathbb{Z} \kappa_{\mu} G).
$$

Hence we define $-\kappa_{\bar{\mu}} (G \times \mathbb{Z})$ to be $(-\kappa_{\mu} G) \circ (-\kappa_{\mu} G): C^*\text{-Alg}(G \times \mathbb{Z}) \to C^*\text{-Alg}$. By construction, $C^*_\bar{\mu}(G \times \mathbb{Z}) = (C \kappa_{\mu} G) \times \mathbb{Z} \cong C^*_\mu G \otimes C^* \mathbb{Z}$.

To prove that $-\kappa_{\bar{\mu}} (G \times \mathbb{Z})$ is a crossed product functor, let $q: -\kappa_{\max} G \to -\kappa_{\mu} G$ and $s: -\kappa_{\mu} G \to -\kappa_{r} G$ be the natural transformations. They induce natural transformations

$$
\bar{q}: (-\kappa_{\max} G) \circ (-\kappa_{\max} G) \to (-\kappa_{\max} G) \circ (\kappa_{\mu} G)
$$

and

$$
\bar{s}: (-\kappa_{\max} G) \circ (-\kappa_{\mu} G) \to (-\kappa_{\max} G) \circ (-\kappa_{r} G)
$$

consisting of quotient maps. Furthermore, there exist natural isomorphisms between $-\kappa_{\max} (G \times \mathbb{Z})$ and $(\kappa_{\max} G) \circ (-\kappa_{\max} G)$ and between $-\kappa_{r} (G \times \mathbb{Z})$ and $(\kappa_{\max} G) \circ (-\kappa_{r} G)$ ([Wi07, Prop. 3.11]). Hence $-\kappa_{\bar{\mu}} (G \times \mathbb{Z})$ is a crossed product functor.

Note that if $-\kappa_{\mu} G$ is exact, then the new functor from $C^*\text{-Alg}(G \times \mathbb{Z})$ to $C^*\text{-Alg}(\mathbb{Z})$ is again exact. Then $-\kappa_{\bar{\mu}} (G \times \mathbb{Z})$ is, as a composition of exact functors, again exact.

Therefore, it remains to show that $\kappa_{\bar{\mu}} (G \times \mathbb{Z})$ is a correspondence crossed product functor if $-\kappa_{\mu} G$ is one. Let $A$ be a $(G \times \mathbb{Z})$-C*-algebra with action $\alpha$ and $p \in \mathcal{M}(A)$ be a $(G \times \mathbb{Z})$-invariant projection. Then $p$ is $G$-invariant and since $-\kappa_{\mu} G$ is a correspondence crossed product functor, the inclusion $pA \to A$ induces an injective *-homomorphisms $pA \kappa_{\mu, \alpha_G} G \to A \kappa_{\mu, \alpha_G} G$ (see [BEW18a, Thm. 4.9]). Because $-\kappa_{\mu} G$ maps injective $\mathbb{Z}$-equivariant *-homomorphisms to injective *-homomorphisms, we finally have that $pA \to A$ induces an injective *-homomorphism $pA \kappa_{\bar{\mu}} (G \times \mathbb{Z}) \to A \kappa_{\bar{\mu}} (G \times \mathbb{Z})$. Hence $-\kappa_{\bar{\mu}} (G \times \mathbb{Z})$ satisfies the projection property, which implies by [BEW18a, Thm. 4.9] that $-\kappa_{\bar{\mu}} (G \times \mathbb{Z})$ is a correspondence crossed product functor.

### 3 Application to the strong Novikov conjecture

In this section we will revisit the proof of Theorem A in Section 3.1 and then modify it in Section 3.2 such that we conclude the stronger statement claimed in Theorem B in the introduction. Commutativity of the big diagram occurring in the proof of Theorem B is proven in the separate Section 3.3.
3.1 The original proof of Hanke and Schick

We will start with a more general setup than the one of Theorem A in the introduction: we will first follow the exposition given in Hanke [Han11] about elements of infinite $K$-area.

Recall that there exists a natural pairing $K^*(X) \otimes K_*(X) \to \mathbb{Z}$ between the $K$-theory and $K$-homology of a compact Hausdorff space $X$. It can be described $KK$-theoretically as the Kasparov product $KK_* (\mathbb{C}, C(X)) \otimes KK_* (C(X), \mathbb{C}) \to KK_* (\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ under the identifications $K^*(X) \cong KK_{-*} (C(X), \mathbb{C})$ and $K_*(X) \cong KK_{*} (C(X), \mathbb{C})$. Concrete formulas for this pairing may be found in, e.g., [HR00, Sec. 8.7].

**Definition 3.1** (Hanke [Han11], Defn. 3.5). Let $M$ be a closed, smooth manifold and let us consider a $K$-homology class $h \in K_0 (M)$.

We say that $h$ has infinite $K$-area, if there exists a Riemannian metric on $M$ and a number $l > 0$ so that the following holds: For each $\varepsilon > 0$ there is a unital $C^*$-algebra $A_\varepsilon$ and a finitely generated Hilbert $A_\varepsilon$-module bundle $E_\varepsilon \to M$ which carries a holonomy representation which is $\varepsilon$-close to the identity at scale $l$ and satisfies

$$\langle [E_\varepsilon], h \rangle \neq 0 \in K_0 (A_\varepsilon) \otimes \mathbb{Q} \quad (3.1)$$

where $[E_\varepsilon] \in KK (\mathbb{C}, C(M) \otimes A_\varepsilon)$ is the element represented by $E_\varepsilon \to M$.

Let us recall what the condition at scale $l$ means: it precisely requires that parallel transport $P_\varepsilon$ along a smooth curve $c$ of length $\leq l$ is $\varepsilon \cdot \text{length}(c)$ close to the identity in operator norm.

Let $\alpha : K_* (M) \to K_* (C^*_\text{max} \pi_1 (M))$ be the higher index map. We briefly recall now how it is constructed. Let $\mathcal{MF} := \tilde{M} \times_{\pi_1 (M)} C^*_\text{max} \pi_1 (M)$ be the Mishchenko–Fomenko bundle. It is a bundle of finitely generated Hilbert $C^*_\text{max} \pi_1 (M)$-modules and hence defines a class $[\mathcal{MF}] \in K_0 (C(M) \otimes C^*_\text{max} \pi_1 (M))$. Then $\alpha := \langle [\mathcal{MF}], - \rangle$ is the index pairing with this class. If $M$ is a closed aspherical manifold, then the strong Novikov conjecture predicts it to be rationally injective.

**Theorem 3.2** (Hanke [Han11] Thm. 3.9). Let $M$ be a closed connected smooth manifold and let $h \in K_0 (M)$ be of infinite $K$-area.

Then we have

$$\alpha (h) \neq 0 \in K_0 (C^*_\text{max} \pi_1 (M)) \otimes \mathbb{R}.$$

**Proof.** We will give a sketch of Hanke’s proof since we will need the details of it later. We will provide the definitions of the appearing objects as we go along.

Hanke constructs a commutative diagram

$$\begin{array}{ccc}
K_0 (M) & \xrightarrow{[\mathcal{MF}, -]} & K_0 (C^*_\text{max} \pi_1 (M)) \xrightarrow{\phi_*} K_0 (Q) \\
\downarrow \alpha & & \downarrow = \\
K_0 (M) & \xrightarrow{[V], -} & K_0 (A) \xrightarrow{\psi_*} K_0 (Q)
\end{array} \quad (3.2)$$

$^2$The pairing used here is just the Kasparov product

$$KK(\mathbb{C}, C(M) \otimes A_\varepsilon) \otimes KK(C(M), \mathbb{C}) \to KK(\mathbb{C}, A_\varepsilon) \cong K^0 (A_\varepsilon).$$
and shows that $h$ is sent to something non-zero in the lower-right corner. The top-left arrow is the map $\alpha$ represented as twisting by the Mishchenko–Fomenko bundle $MF$ as explained before. The steps in Hanke’s proof are the following:

1. There are certain unital C*-algebras $A_{1/k}$ for each $k \in \mathbb{N}$ whose concrete form is of no interest for us until the proof of Corollary 3.9 (where we will give more information on their construction). Further, there are certain finitely generated Hilbert $A_{1/k}$-bundles $E_{1/k}$ over $M$.

In the above Diagram 3.2 we have that $A = \prod_{k=1}^{\infty} A_{1/k}$ (norm bounded sequences) and that $V \to M$ is a finitely generated Hilbert $A$-module bundle with transition functions in diagonal form. Then the $k$-component of $V$ is isomorphic to $E_{1/k}$ as a Hilbert $A_{1/k}$-module bundle. This bundle $V$ is provided by [Han11, Prop. 3.12].

The possibility of this construction is ensured by the fact that all the holonomy representations of the component bundles $E_{1/k}$ are uniformly close to the identity (at the same scale). Hence we can just take the product of the transition functions. Indeed, if this kind of Lipschitz uniformity in the transition functions is not ensured, such a construction cannot be performed as an example in loc. cit. shows.

Also an important point in the construction is the fact that we can assume the fiber of $E_{1/k}$ to be isomorphic to $q_k A_{1/k}$ with $q_k$ a projection in $A_{1/k}$ (this is obtained up to tensoring with matrices). It follows that the typical fiber of $V$ is just $qA$ with $q = (q_k)_k \in A$.

2. Let $A'$ be the closed ideal $\bigoplus A_{1/k}$ in $A$ and $Q$ be the quotient C*-algebra $A/A'$ with quotient map $\psi: A \to Q$. We define $\psi_k: A \to A_{1/k}$ as the projection onto the $k$-th component.

We get a bundle of finitely generated Hilbert $Q$-modules with fiber $\psi(q)Q$, namely $W := V \otimes Q$. Thanks to the crucial property on the holonomy representations of the component bundles one proves that $W$ is flat and is associated with a unitary representation

$$\phi: \pi_1(M) \longrightarrow \text{Hom}_Q(\psi(q)Q, \psi(q)Q) = \psi(q)Q \psi(q) \hookrightarrow Q,$$

see [Han11] Prop. 3.13. This is the holonomy of $W$ and the morphism $\phi$ in (3.2).

3. The composition $K_0(M) \xrightarrow{\langle [V], - \rangle} K_0(A) \xrightarrow{(\psi_k)_k} K_0(A_{1/k})$ sends the element $h$ to $\langle [E_{1/k}], h \rangle \in K_0(A_{1/k})$ which is rationally non-zero by assumption on $h$. Therefore, under the map

$$\chi: K_0(A) \to \prod_{k \in \mathbb{N}} K_0(A_{1/k}), \quad z \mapsto ((\psi_k)_*z)_{k=1,2,...} \tag{3.3}$$

the element $z := \langle [V], h \rangle$ is sent to a sequence all of whose components are non-zero.
4. We consider the short exact sequence $0 \rightarrow A' \rightarrow A \rightarrow Q \rightarrow 0$ which provides the long exact sequence

$$
\cdots \xrightarrow{\partial} K_0(A') \rightarrow K_0(A) \rightarrow K_0(Q) \xrightarrow{\partial} \cdots
$$

Assuming that $\psi_*(z) = 0 \in K_0(Q)$ we get a lift of $z$ to $K_0(A')$. Because $K$-theory commutes with direct sums, we have $K_0(A') \cong \bigoplus_{k=1}^{\infty} K_0(A_{1/k})$. But this implies that the sequence $\chi(z) = ((\psi_k)_*z)_{k=1,2,\ldots}$ is non-zero for only finitely many $k \in \mathbb{N}$, which is a contradiction. Hence $\psi_*(z) \neq 0$, and from the above diagram we therefore get that $\alpha(h) \neq 0$ in $K_0(C^*_\text{max}\pi(M))$.

5. The last step entails checking that the above arguments go through if we tensor everything with $\mathbb{R}$. This is straightforward. □

3.2 Incorporating Fell’s absorption principle

The proof of Theorem 3.4 works with any exact correspondence crossed product functor. The properties of such functors, that we will need in the proof, are the following ones:

Properties 3.3. Fix a discrete group $G$. Let $- \rtimes \mu G$ be an exact correspondence crossed product functor.

Then it has the following properties:

1. The corresponding group C*-algebra $C^*_\mu G := C \rtimes \mu G$ is a completion of $\mathbb{C}G$ and the identity on $\mathbb{C}G$ extends to surjective *-homomorphisms $C^*_\text{max}G \rightarrow C^*_\mu G \rightarrow C^*_\text{r}G$.

   This property holds by definition of crossed product functors.

2. For any C*-algebra $A$ with the trivial $G$-action, the crossed product $A \rtimes \mu G$ is a C*-completion of the algebraic tensor product $A \otimes C^*_\mu G$ and the identity map on $A \otimes C^*_\mu G$ extends to surjective *-homomorphisms

   $$A \otimes_{\max} C^*_\mu G \rightarrow A \rtimes \mu G \rightarrow A \otimes_{\min} C^*_\mu G.$$

   Because of this we denote $A \rtimes \mu G =: A \otimes_{\mu} C^*_\mu G$.

   This property is exactly Proposition 2.8.

3. The functor $- \otimes_{\mu} C^*_\mu G$ is exact, i.e., for every exact sequence $0 \rightarrow I \rightarrow A \rightarrow Q \rightarrow 0$ of C*-algebras with the trivial $G$-action we get an exact sequence

   $$0 \rightarrow I \otimes_{\mu} C^*_\mu G \rightarrow A \otimes_{\mu} C^*_\mu G \rightarrow Q \otimes_{\mu} C^*_\mu G \rightarrow 0.$$

   This is true because we just assume $- \rtimes \mu G$ to be exact. Note that it is actually equivalent to requiring that $- \rtimes \mu G$ is exact by Corollary 2.12.
4. The coproduct $\Delta : C G \to C G \otimes C G$, $\sum a_g g \mapsto \sum a_g (g \otimes g)$ extends continuously to a $^*$-homomorphism

$$\Delta : C_{\mu G}^* \to C_{\max G}^* \otimes_{\mu} C_{\mu G}^*.$$ 

This follows from Lemma 2.10. Note that because $G$ is a discrete group, $C_{\max G}^*$ and $C_{\mu G}^*$ are unital C*-algebras. Hence $C_{\max G}^* \otimes_{\mu} C_{\mu G}^*$ is also unital and so its multiplier algebra coincides with it.

Note that this property in particular requires that the coproduct extends to a map $C_{\mu G}^* \to C_{\max G}^* \otimes_{\min} C_{\mu G}^*$. And this is equivalent to require that the dual space $(C_{\mu G}^*)'$ is (isomorphic to) an ideal in the Fourier-Stieltjes algebra $B(G) = (C_{\max G}^*)'$, see [KLQ13, Cor. 3.13].

5. The canonical map

$$\bigoplus_{k \in \mathbb{N}} \left( A_k \otimes_{\mu} C_{\mu G}^* \right) \to \left( \bigoplus_{k \in \mathbb{N}} A_k \right) \otimes_{\mu} C_{\mu G}^*$$

is an isomorphism.

This property is exactly Lemma 2.2. ♦

We can now generalize the result of Hanke–Schick [HS08]. Let $G$ be a finitely presented group and denote by $\Lambda^*(G) \subset H^*(BG; \mathbb{Q})$ the subring generated by $H^{\leq 2}(BG; \mathbb{Q})$. Recall further the homological Chern character $\ch : K_*(BG) \to H_*(BG; \mathbb{Q})$; a definition of it using the Baum–Douglas model for $K$-homology may be found in [BD82, §11]. Note that we are also using the Baum–Douglas model for $K$-homology in the proof below.

**Theorem 3.4.** Let $h \in K_*(BG)$ be such that there is $c \in \Lambda^*(G)$ with $\langle c, \ch(h) \rangle \neq 0$, and let $- \rtimes_{\mu} G$ be an exact correspondence crossed product functor. Then $h$ is not mapped to zero under the assembly map $K_*(BG) \to K_*(C_{\mu G}^*) \otimes \mathbb{R}$.

**Remark 3.5.** By Corollary 2.14 we know that there is an exact correspondence crossed product functor whose group $C^*$-algebra is the reduced one. Using such a crossed product functor, Theorem 3.4 becomes Theorem 19 from the introduction.

Of course, Theorem 3.4 for the reduced group $C^*$-algebra subsumes the corresponding statements for any other group $C^*$-algebra due to Property 3.3.1. ♦

**Remark 3.6.** Let us explain the suspension trick used in the odd-dimensional case of the proof of Theorem 3.4. Let us denote by $z \in K_1(S^1)$ the generator. We have exterior products

$$- \boxtimes z : K_*(BG) \to K_{*+1}(B(G \times \mathbb{Z})),$$

$$- \boxtimes \alpha(z) : K_*(C_{\mu G}^*) \to K_{*+1}(C_{\mu G}^*(G \times \mathbb{Z})),$$

where we have used Lemma 2.16 to define the exterior product for the group $C^*$-algebras, and both exterior products are injective by the respective Kunneth theorems.
If \( h \in K_1(BG) \) satisfies the assumption of the Theorem 3.4, then \( h \otimes z \) also satisfies it (since \( c \otimes \varphi_z \) is an element of \( \Lambda^*(G \times \mathbb{Z}) \) if \( c \in \Lambda^*(G) \), where \( \varphi_z \in H^1(S^1) \) denotes the generator).

Therefore, if we can show that \( h \otimes z \) is not mapped to zero under the assembly map \( K_0(B(G \times \mathbb{Z})) \rightarrow K_0(C_\mu^*(G \times \mathbb{Z})) \otimes \mathbb{R} \), then \( h \) won’t be mapped to zero under the assembly map \( K_1(BG) \rightarrow K_1(C_\mu^*(G)) \otimes \mathbb{R} \).

Proof of Theorem 3.4. The proof is an adaptation of the proof of [Han11, Thm. 4.1], which is itself an elaboration on the proof given in [HS08].

By the suspension argument explained in Remark 3.6 we can assume that \( h \in K_0(BG) \).

For simplicity we also assume that \( c \in H^2(BG; \mathbb{Q}) \). The general case \( c \in \Lambda^*(G) \) reduces to the former case as described at the end of Section 2 of [HS08].

Using the Baum–Douglas model for \( K \)-homology, \( h \) is represented in terms of a finite-dimensional Hermitian vector bundle \( S \rightarrow M \) over a closed, connected, smooth manifold \( M \) and a continuous map \( f : M \rightarrow BG \) that can be assumed to induce isomorphism on fundamental groups. The infinite bundle construction we summarised in the proof of Theorem 3.2 is performed starting from a suitable deformation of the connection on the Hermitian line bundle classified by \( f^*(c) \).

It is important to note that the non-triviality of all the pairings \( \langle [E_\varepsilon], h \rangle \), see (3.1), is witnessed by a very particular trace in the present situation: every algebra \( A_\varepsilon \) comes with a natural trace \( \tau_\varepsilon : A_\varepsilon \rightarrow \mathbb{C} \). Note that these algebras \( A_\varepsilon \) and bundles \( E_\varepsilon \) are provided by the construction of Hanke and Schick, see Point (1) of the proof of Theorem 3.2. The induced map in \( K \)-theory maps to \( \mathbb{R} \) such that for every \( \varepsilon \) we have \( \tau_\varepsilon ([E_\varepsilon], h) \neq 0 \), which is again a property of their particular construction. We will recall more details further below in Section 3.3.

We look at the following diagram, which is a modified and expanded version of the Diagram (3.2). We will explain the arrows occuring in it further below.

\[
K_0(M) \xrightarrow{\langle [M]^\varepsilon, \cdot \rangle} K_0(C_\mu^*G) \xrightarrow{\Delta^*} K_0(C_{\text{max}}^*G \otimes_\mu C_\mu^*G) \xrightarrow{(\phi \otimes \text{id})^*} K_0(Q \otimes_\mu C_\mu^*G) \quad (3.4)
\]

\[
K_0(M) \xrightarrow{\langle [V], \cdot \rangle} K_0(A) \xrightarrow{\psi^*} K_0(Q)
\]

\[
K_0(A \otimes_\mu C_\mu^*G) \xrightarrow{(\psi \otimes \text{id})^*} K_0(Q \otimes_\mu C_\mu^*G)
\]

\[
\prod K_0(A_{1/k} \otimes_\mu C_\mu^*G)
\]

\[
\prod \mathbb{R} \rightarrow \prod \mathbb{R} / \oplus_{\text{alg}} \mathbb{R}
\]

The first arrow in the top line is the composition of the map \( \alpha : K_0(M) \rightarrow K_0(C_{\text{max}}^*G) \)

\[\text{This is the reason why we restrict in this argument to finitely presented groups } G.\]
from the Diagram \((3.2)\) with the map induced from the \(*\)-homomorphism \(C^*_{\max} G \to C^*_{\mu} G\) given by Property \(3.3.1\). Note that \(\alpha_{\mu}\) factors as the composition of \(K_0(M) \to K_0(BG)\) with the higher index map \(K_0(BG) \to K_0(C^*_{\mu} G)\), where the first of these maps is induced by a choice of classifying map \(M \to BG\).

The second arrow in the top line is induced by the coproduct whose existence is guaranteed by Property \(3.3.4\). The bottom vertical map on the left

\[
P(\tau_{1/k} \otimes \tau_e): \prod_0 K_0(A_{1/k} \otimes_{C^*_{\mu} G}) \to \prod_0 \mathbb{R}
\]

is induced from the maps \(A_{1/k} \otimes_{C^*_{\mu} G} \to A_{1/k} \otimes_{C^*_{\mu} G} (\text{which exist by Property } 3.3.2)\) composed with the tensor products \(\tau_{1/k} \otimes \tau_e\) of the given traces \(\tau_{1/k}\) on the algebras \(A_{1/k}\) with the canonical trace \(\tau_e\) on \(C^*_{\mu} G\). By \(\bigoplus_{\text{alg}} \mathbb{R}\) in the lower right corner we mean the algebraic direct sum, i.e., sequences with only finitely many non-zero entries.

The dashed arrows will be constructed at the end of this proof, and the commutativity of the diagram will be shown in Corollary \(3.9\) below.

Given all the above, the claim of this theorem follows quickly: the left path from \(K_0(M)\) to \(\prod_0 \mathbb{R}\), applied to the element \(h\), results in a sequence all of whose entries are non-zero due to the assumptions of this theorem (compare to Step 3 of the proof of Theorem \(3.2\)). Hence it stays non-zero if we map it further to \(\prod_0 \mathbb{R}/ \bigoplus_{\text{alg}} \mathbb{R}\). By commutativity of the diagram this means that \(\alpha(h) \in K_0(C^*_{\mu} G)\) is non-zero. Tensoring the diagram with \(\mathbb{R}\) we still obtain the same conclusion. Hence \(\alpha(h) \neq 0 \in K_0(C^*_{\mu} G) \otimes \mathbb{R}\).

It remains to construct the dashed arrows in the Diagram \((3.4)\), i.e., the map

\[
K_0(Q \otimes_{C^*_{\mu} G}) \to \prod_0 \mathbb{R}/ \bigoplus_{\text{alg}} \mathbb{R}.
\]

An element \(x \in K_0(Q \otimes_{C^*_{\mu} G})\) is represented by a difference \(x = [p] - [q]\) of projections \(p\) and \(q\) in matrices over \(Q \otimes_{C^*_{\mu} G}\). We will now explain how to evaluate a single projection like \(p\) to something in \(\prod_0 \mathbb{R}/ \bigoplus_{\text{alg}} \mathbb{R}\).

By Property \(3.3.3\) we have an exact sequence

\[
0 \to A' \otimes_{C^*_{\mu} G} \to A \otimes_{C^*_{\mu} G} \to Q \otimes_{C^*_{\mu} G} \to 0,
\]

and we will show:

- for any small \(\varepsilon\), say \(\varepsilon < 1/8\), we find a \(K \in \mathbb{N}\) and a ”lift” \(p'_{>K} \in \prod_{>K} (A_{1/k} \otimes_{C^*_{\mu} G})\) which is an \(\varepsilon\)-projection and can be evaluated suitably by \(\prod_{>K} (\tau_{1/k} \otimes \tau_e)\). The result is in \(\prod_0 \mathbb{R}/ \bigoplus_{\text{alg}} \mathbb{R}\), and

- this is independent on the choices.

We give the details in the following.

From the exactness of \((3.5)\) we can lift \(p\) to a self-adjoint matrix \(\tilde{p}\) over \(A \otimes_{C^*_{\mu} G}\) which is a projection modulo matrices over \(A' \otimes_{C^*_{\mu} G}\). We can apply now the map \(\prod(\psi_k \otimes \text{id})\) to map \(\tilde{p}\) to a self-adjoint matrix \(p'\) over \(\prod(A_{1/k} \otimes_{C^*_{\mu} G})\). Because of the

\[\text{Note that there is no concrete reason here to tensor with } \mathbb{R}. \text{ We could instead also tensor with } \mathbb{Q}.
\]
Property 3.3.5 we have $\prod (\psi_k \otimes \text{id}) (A' \otimes \mu C_\mu^* G) \subseteq \bigoplus_k (A_{1/k} \otimes \mu C_\mu^* G)$, thus $p'$ will be a projection modulo matrices over $\bigoplus_k (A_{1/k} \otimes \mu C_\mu^* G)$. Hence, fixing an $\varepsilon < 1/8$, there will be $K \in \mathbb{N}$, such that $p'_{>K}$ is an $\varepsilon$-projection in $\prod_{>K} (A_{1/k} \otimes \mu C_\mu^* G)$. So, if $\text{proj}(-)$ denotes the characteristic function of the interval $[1 - 1/4, 1 + 1/4]$, we get an honest projection $\text{proj}(p'_{>K})$ in $\prod_{>K} (A_{1/k} \otimes \mu C_\mu^* G)$ which can be evaluated by $\prod_{>K} (\tau_{1/k} \otimes \tau_e)$ to a value in $\prod \mathbb{R} \big/ \bigoplus_{\text{alg}} \mathbb{R}$. It remains to show that this is well-defined. If we have a second lift $p''_{>K'}$, then $\text{proj}(p'_{>K})$ and $\text{proj}(p''_{>K'})$ will be $(1/2 + 1/10)$-close in $\prod_{>K''} (A_{1/k} \otimes \mu C_\mu^* G)$ for some large $K''$ and hence they will be unitarily equivalent.

5.3 Commutativity of the main diagram

Let us explain why commutativity of Diagram (3.4) is non-trivial. We denote by

- $\mathbb{C}G$ the complex group ring of $G$,
- $\Delta: \mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G$ the coproduct $\sum a_g g \mapsto \sum a_g (g \otimes g)$,
- $\iota: \mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G$ the inclusion $\sum a_g g \mapsto \sum a_g (g \otimes e)$, where $e$ is the identity element of the group $G$,
- $\tau: \mathbb{C}G \to \mathbb{C}$ any trace on $\mathbb{C}G$, and
- $\tau_e: \mathbb{C}G \to \mathbb{C}$ the canonical trace $\sum a_g g \mapsto a_e$ on $\mathbb{C}G$.

Now we consider the diagram

\[
\begin{array}{ccc}
\mathbb{C}G & \xrightarrow{\Delta} & \mathbb{C}G \otimes \mathbb{C}G \\
\downarrow{\iota} & & \downarrow{\tau \otimes \tau_e} \\
\mathbb{C}G \otimes \mathbb{C}G & \xrightarrow{\tau \otimes \tau_e} & \mathbb{C}
\end{array}
\] (3.6)

We have

$$( (\tau \otimes \tau_e) \circ \Delta ) \left( \sum a_g g \right) = a_e \cdot \tau(e)$$

and for the other composition in the diagram we have

$$( (\tau \otimes \tau_e) \circ \iota ) \left( \sum a_g g \right) = \sum a_g \tau(g) .$$

Hence Diagram (3.6) only commutes in the case that $\tau$ is a multiple of $\tau_e$.

Now Diagram (3.6) is similar to Diagram (3.4) in the sense that the composition of the top arrows of Diagram (3.4) with the dashed arrow is similar to the composition of the top and right vertical arrow in Diagram (3.6), and the composition of the horizontal arrows in the second row of Diagram (3.4) with the dashed arrow is similar in flavour to the composition of the left vertical and lower arrow in Diagram (3.6). So we expect commutativity of Diagram (3.4) only if the traces $\tau_{1/k}$ on the algebras $A_{1/k}$ used in the

\[\text{It is a general fact about C*-algebras that projections} \ P, Q \ \text{with} \ \|P - Q\| < 1 \ \text{are unitarily equivalent.}\]
definition of the dashed arrows in Diagram (3.4) are of similar kind as the canonical evaluation trace on the identity element in the group $C^*$-algebras.

In order to show that the traces $\tau_{1/k}$ on the algebras $A_{1/k}$ have the needed property, let us describe them more precisely. Hanke and Schick [HS08] construct on the associated Hilbert bundle $\widetilde{M} \times_G \ell^2(G)$, where $G$ acts diagonally (by the right regular representation on $\ell^2(G)$ and deck transformations on $\widetilde{M}$), a family $(\nabla^{1/k})_k$ of connections with curvature going to zero with $k \to \infty$. This is the reason for the Hilbert $Q$-module $W$ being flat (see Point 2 of the proof of Theorem 3.2).

More precisely, on $\widetilde{M}$ we have a connection form $\eta \in \Omega^1(\widetilde{M}, i\mathbb{R})$ and a natural left action $\varphi \mapsto g \cdot \varphi$ of $G$ on the space of forms with values $\text{End}(\ell^2(G))$. Using this action we can define a natural family of invariant connection forms $(\omega_{1/k})_k$. These are the forms that restrict to $1/k (g \cdot \eta)$ on the sub-bundle $\widetilde{M} \times Cg$. By $G$-invariance a corresponding family of connections on $\widetilde{M} \times_G \ell^2(G)$ is well defined.

Let us choose a reference point $p \in M$ and a point $\tilde{p} \in \widetilde{M}$ in its fiber. In this way the fiber of $\widetilde{M} \times_G \ell^2(G)$ at $p$ is identified with $\ell^2(G)$. For every $k$, the algebra $A_{1/k}$ is defined by the norm closure inside $\mathfrak{B}(\ell^2(G))$ of all the operators which arise by parallel translation for $\nabla^{1/k}$ along piecewise smooth loops. All these algebras are then naturally represented on the same Hilbert space and the traces are the vector states

$$\tau_{1/k}(\vartheta) := \langle \vartheta(e), e \rangle \quad \text{for } \vartheta \in A_{1/k}.$$  

Here $e \in \ell^2(G)$ is the characteristic function of the identity element $e$ of $G$ and $\langle -, - \rangle$ the inner product of $\ell^2(G)$. The construction has the following property which will be important for us later and says that the traces are akin to the trace $\tau_e$ on $C_r^*G$.

**Property 3.7.** Let $\phi_\gamma \in A_{1/k}$ be the parallel translation map associated to a loop $\gamma$ which represents an element in $\pi_1(M, p)$. Then $\tau_{1/k}(\phi_\gamma) = 0$ if $\gamma$ is not trivial in $\pi_1(M, p)$.

**Proof.** This is also mentioned in the beginning of the proof of [HS08, Lem. 2.2]. Let $\gamma$ a loop based on $p$; to compute $\tau_{1/k}(\phi_\gamma)$ we lift $\gamma$ to a path $\tilde{\gamma}$ in $\widetilde{M}$ and we compute the parallel translation along $\tilde{\gamma}$ with respect to the connection associated to $\omega_{1/k}$. We call this operator of parallel translation on the covering $\phi_\tilde{\gamma}$. Now assume the class of $\gamma$ in the fundamental group is $g \neq e$, then $\tilde{\gamma}$ is not a loop and its endpoint is exactly $\tilde{p} \cdot g$.

Since the connection forms preserve the sub-bundles $Cg$ the vector $\phi_{\tilde{p} \cdot g} e$ is represented in $\widetilde{M} \times \ell^2(G)$ by the couple $(\tilde{p} \cdot g, \lambda e)$ for some number $\lambda$. The corresponding inner product computing the trace is $\lambda \langle e, g^{-1} \rangle = 0$.

Before proving that Diagram (3.4) commutes we discuss some basic facts about almost projections and $K$-theory.

Let $A$ be a unital $C^*$-algebra and keep fixed a small $\varepsilon > 0$ for the entire following discussion. Let us explain how self-adjoint almost idempotents over $A$ define $K$-theory classes (see [XYT14, Sec. 2.2]). An almost idempotent is an element $p \in A$ which satisfies

---

6The self-adjointness is actually not strictly necessary for this. We have incorporated it since in our situation it can always be arranged.
\[ \|p^2 - p\| < \varepsilon. \] Let \( p \) be a self-adjoint almost idempotent in \( M_n(A) \); then there are disjoint open sets \( U, V \subset \mathbb{C} \) separating 0 and 1 in its spectrum:

\[ \text{spec}(p) \subset U \cup V, \quad 0 \in U, \quad 1 \in V. \]

Choose a real-valued function \( h \) on \( \mathbb{C} \) with \( h(\xi) = 0 \) on \( U \) and \( h(\xi) = 1 \) on \( V \). Then the holomorphic functional calculus, integrating the function \( h \) over a contour \( \mathcal{C} \) surrounding \( \text{spec}(z) \) inside \( U \cup V \), produces an honest projection and we define the \( K \)-theory class of \( p \) to be

\[ [p] := \left[ \frac{1}{2\pi i} \int_{\mathcal{C}} h(\xi)(\xi - p)^{-1}d\xi \right]. \]

Notice that we don’t actually need the holomorphic functional calculus because we are discussing the construction using selfadjoints. However, since this this works just well for quasi-idempotents, we continue using it. Denote by \( V_e(A) \) the space of self-adjoint almost idempotents of matrices over \( A \); by considering formal differences in the procedure before, we have constructed a surjection \( V_e(A) \times V_e(A) \to K_0(A) \). Every \( \ast \)-homomorphism \( f: A \to B \) of (unital) \( C^* \)-algebras \( A \) and \( B \) is contractive, therefore it restricts to a map \( V_e(A) \to V_e(B) \) which induces a map \( f_*: K_0(A) \to K_0(B) \).

Let now \( T: A \to Z \) be a positive tracial map with \( Z \) a commutative \( C^* \)-algebra; it induces a map \( T_*: K_0(A) \to Z \). Note that \( T_* \) maps to the selfadjoint elements in \( Z \). Let us compute \( T_*([p]) \) for the class of a self-adjoint almost idempotent. Continuity implies

\[ T_*([p]) = \frac{1}{2\pi i} \int_{\mathcal{C}} h(\xi)T((\xi - p)^{-1})d\xi. \]

To keep track of these classes which are defined by self-adjoint almost idempotents it can be useful to introduce the notation

\[ \left[ \int y \right] = \left[ \frac{1}{2\pi i} \int_{\mathcal{C}} h(\xi)(\xi - y)^{-1}d\xi \right]. \]

In this way, under a morphism \( F \) we have \( F_*[\int y] = [\int F(y)] \).

With this in mind we reconstruct the dashed map \( \tau_*: K_0(Q \otimes \mu C^*_\mu G) \to \prod \mathbb{R}/\bigoplus_{alg} \mathbb{R} \) using self-adjoint almost idempotents. Since general classes are formal differences, we explain as before the construction for a single self-adjoint almost idempotent element \( p \in M_n(Q \otimes \mu C^*_\mu G) \) with \( \|p^2 - p\| < \varepsilon \). Let \( p' \in M_n(A \otimes \mu C^*_\mu G) \) be any lift in the sequence \([3.5]\). By considering \( (p' + p')^2/2 \) we can assume that \( p' \) is self-adjoint. By definition of the quotient norm

\[ \inf_{y \in M_n(A \otimes \mu C^*_\mu G)} \|p'^2 - p' - y\| = \|p^2 - p\| < \varepsilon. \]

We can thus find a \( y_0 \in M_n(A \otimes \mu C^*_\mu G) \) with \( \|p'^2 - p' - y_0\| < \varepsilon \); without loss of generality we may assume that \( y_0 \) is self-adjoint (by passing to \( (y_0 + y_0^*)/2 \)). We continue to denote by \( \psi_k \otimes \text{id} \) the extension of the morphism to matrix algebras. Of course, we also exchange the tensor product with \( M_n \) and \( \prod \). Then we look at the image

\[ (\prod_k \psi_k \otimes \text{id})(p') = (q_k)_k \in \prod_k M_n(A_{1/k} \otimes \mu C^*_\mu G). \]

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Recall that $A'$ is the direct sum of the $A_{1/k}$, and let be $K$ such that $\| (\psi_k \otimes \text{id}) y_0 \| < \varepsilon$ for $k > K$. It follows that

$$q > K := (q_k)_{> K} \in \prod_{k > K} M_n(A_{1/k} \otimes_\mu C^*_\mu G)$$

is a self-adjoint almost idempotent with $\| q^2_{> K} - q_{> K} \| < 2\varepsilon$. Extend as customary the map $\prod_k \tau_k \otimes \tau_e$ to matrices over $\prod_k A_{1/k} \otimes_\mu C^*_\mu G$ by tensoring with the matrix trace. We continue to denote it with the same notation.

Now it remains to apply the functional calculus to define

$$\tau_*([p]) := \left( \prod_{k > K} \tau_k \otimes \tau_e \right)_* \left[ \int q_{> K} \right] \text{ mod } \oplus_{\text{alg}} \mathbb{R}.$$ 

This procedure define a map $\tau_* : K_0(Q \otimes_\mu C^*_\mu G) \to \prod \mathbb{R} / \oplus_{\text{alg}} \mathbb{R}$ which coincides with the dashed arrow constructed in the proof of Theorem 3.4.

**Proposition 3.8.** We have a commutative diagram

$$\begin{array}{cccc}
K_0(C^*_{\text{max}} G) & \xrightarrow{(\phi_{\text{max}}^*)} & K_0(C^*_{\text{max}} G) \otimes_\mu C^*_\mu G & \xrightarrow{(\phi \otimes \text{id})_*} & K_0(Q \otimes_\mu C^*_\mu G) \\
\Delta_* & \downarrow & & \tau_* & \rightarrow \rightarrow \rightarrow \rightarrow \\
K_0(C^*_\mu G) & \xrightarrow{\Delta_*} & K_0(C^*_{\text{max}} G) \otimes_\mu C^*_\mu G & \xrightarrow{(\phi \otimes \text{id})_*} & K_0(Q \otimes_\mu C^*_\mu G)
\end{array}$$

**Proof.** Start with a class $[x] \in K_0(C^*_{\text{max}} G)$ represented by a true projection over $C^*_{\text{max}} G$. By density we can find a self-adjoint $y \in M_n(\mathbb{C} G)$ which approximates $x$. We write it as a finite sum

$$y = \sum a_g g, \quad a_g \in M_n(\mathbb{C}).$$

Then $y$ is a self-adjoint almost idempotent. It follows that $y$ represents $[x]$ in the sense of the discussion before. Let us move $[x]$ to $K_0(Q \otimes_\mu C^*_\mu G)$ according to the up route and the down route in the above diagram. We find two elements:

$$y_1 = \left[ \int \sum_g a_g \phi(g) \otimes e \right] \quad \text{and} \quad y_2 = \left[ \int \sum_g a_g \phi(g) \otimes g \right].$$

We have to show that applying $\tau_*$ we get the same result. Remembering that $A$ is the product of all the holonomy algebras along loops, for these two elements we have a class of preferred lifts in $A \otimes_\mu C^*_\mu G$. Indeed for any element $g \in G$ we choose a smooth loop $\gamma(g)$ representing $g$ in the fundamental group. Denote with $\text{Hol}(\gamma(g)) = (\text{Hol}^{1/k}(\gamma(g)))_k \in A$ the collection of the parallel translations along this loop with respect to $\nabla^{1/k}$, then we have two lifts of $y_1$ and $y_2$. These are

$$z_1 = \sum a_g \text{Hol}(\gamma(g)) \otimes e, \quad \text{and} \quad z_2 = \sum a_g \text{Hol}(\gamma(g)) \otimes g$$

in $M_n \otimes (A \otimes_\mu C^*_\mu G) \cong M_n(A \otimes_\mu C^*_\mu G)$. We can assume that $z_i$ with $i = 1, 2$ are:
• selfadjoint (for if not, just take \((z_i + z_i^*)/2\) — they lift in the same way, because \(y_1\) and \(y_2\) are selfadjoint),

• almost idempotent (because if not, we can cut away a finite number of components of the \(((\phi_g^{1/k})_k\) as in the discussion before).

Now given the integral formula for \(\tau_\star\) and the use of the functional calculus the proof will be complete if we manage to show that for any polynomial with real coefficients \(f\) we have

\[
(\prod_k \tau_k \otimes \tau_e)(\prod_k \psi_k \otimes \text{id}) f(z_1) = (\prod_k \tau_k \otimes \tau_e)(\prod_k \psi_k \otimes \text{id}) f(z_2).
\]

Let us check it for any power: we have

\[
z_1^m = \sum_{(g_1, \ldots, g_m) \in G^m} a_{g_1} \cdots a_{g_m} \operatorname{Hol}(\gamma(g_1) \cdots \gamma(g_m)) \otimes e
\]

and

\[
z_2^m = \sum_{(g_1, \ldots, g_m) \in G^m} a_{g_1} \cdots a_{g_m} \operatorname{Hol}(\gamma(g_1) \cdots \gamma(g_m)) \otimes g_1 \cdots g_m.
\]

We apply our maps \(\prod \tau_k \otimes \tau_e\) (recall that these maps have been extended to matrices) and then look at the \(k\)th-component of the result:

\[
\left(\left(\prod_k \tau_k \otimes \tau_e\right)(\prod_k \psi_k \otimes \text{id}) z_1^m\right)_k = \sum_{(g_1, \ldots, g_m) \in G^m} \text{tr}_{M^n}(a_{g_1} \cdots a_{g_m}) \tau_k \operatorname{Hol}^{1/k}(\gamma(g_1) \cdots \gamma(g_k)) \quad (3.7)
\]

The element \(\gamma(g_1) \cdots \gamma(g_m)\) represents the product \(g_1 \cdots g_m \in \pi_1(M,p)\); it follows by Property \(3.7\) that the sum in \((3.7)\) is just performed on the elements \((g_1, \ldots, g_m)\) such that \(g_1 \cdots g_m = e\). This is to say that

\[
(\prod_k \tau_k \otimes \tau_e)(\prod_k \psi_k \otimes \text{id}) z_1^m = (\prod_k \tau_k \otimes \tau_e)(\prod_k \psi_k \otimes \text{id}) z_2^m
\]

finishing this proof. \(\square\)

From the above proposition we can conclude the sought corollary:

**Corollary 3.9.** Diagram \((3.4)\) commutes.
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