SUBGROUPS OF CATEGORICALLY CLOSED SEMIGROUPS

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Abstract. Let $C$ be a class of topological semigroups. A semigroup $X$ is called (1) $C$-closed if $X$ is closed in every topological semigroup $Y \in C$ containing $X$ as a discrete subsemigroup, (2) ideally $C$-closed if for any ideal $I$ in $X$ the quotient semigroup $X/I$ is $C$-closed; (3) absolutely $C$-closed if for any homomorphism $h : X \to Y$ to a topological semigroup $Y \in C$, the image $h[X]$ is closed in $Y$, (4) injectively $C$-closed (resp. $C$-discrete) if for any injective homomorphism $h : X \to Y$ to a topological semigroup $Y \in C$, the image $h[X]$ is closed (resp. discrete) in $Y$. Let $T_2S$ be the class of Tychonoff zero-dimensional topological semigroups. We prove the following results: (i) for any ideally $T_2S$-closed semigroup $X$, every subgroup of the center $Z(X) = \{ z \in X : \forall x \in X \ (xz = z) \}$ is bounded; (ii) for any $T_2S$-closed semigroup $X$, every subgroup of the ideal center $IZ(X) = \{ z \in Z(X) : zX \subseteq Z(X) \}$ is bounded; (iii) for any $T_2S$-discrete or injectively $T_2S$-closed semigroup $X$, every subgroup of $Z(X)$ is finite, (iv) for any viable idempotent $e$ in an ideally (and absolutely) $T_2S$-closed semigroup $X$, the maximal subgroup $H_e$ is ideally (and absolutely) $T_2S$-closed and has bounded (and finite) center $Z(H_e)$.

1. Introduction and Main Results

In many cases, completeness properties of various objects of General Topology or Topological Algebra can be characterized externally as closedness in ambient objects. For example, a metric space $X$ is complete if and only if $X$ is closed in any metric space containing $X$ as a subspace. A uniform space $X$ is complete if and only if $X$ is closed in any uniform space containing $X$ as a uniform subspace. A topological group $G$ is Raïkov complete if and only if it is closed in any topological group containing $G$ as a subgroup.

On the other hand, for topological semigroups there are no reasonable notions of (inner) completeness. Nonetheless we can define many completeness properties of semigroups via their closedness in ambient topological semigroups.

A topological semigroup is a topological space $X$ endowed with a continuous associative binary operation $X \times X \to X$, $(x, y) \mapsto xy$.

Definition. Let $C$ be a class of topological semigroups. A topological semigroup $X$ is called

- $C$-closed if for any isomorphic topological embedding $h : X \to Y$ to a topological semigroup $Y \in C$ the image $h[X]$ is closed in $Y$;
- injectively $C$-closed if for any injective continuous homomorphism $h : X \to Y$ to a topological semigroup $Y \in C$ the image $h[X]$ is closed in $Y$;
- absolutely $C$-closed if for any continuous homomorphism $h : X \to Y$ to a topological semigroup $Y \in C$ the image $h[X]$ is closed in $Y$.

For any topological semigroup we have the implications:

absolutely $C$-closed \(\Rightarrow\) injectively $C$-closed \(\Rightarrow\) $C$-closed.

Definition. A semigroup $X$ is defined to be (injectively, absolutely) $C$-closed if so is $X$ endowed with the discrete topology.

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We will be interested in the (absolute, injective) \( C \)-closedness for the classes:

- \( \mathcal{T}_1 \mathcal{S} \) of topological semigroups satisfying the separation axiom \( T_1 \);
- \( \mathcal{T}_2 \mathcal{S} \) of Hausdorff topological semigroups;
- \( \mathcal{T}_z \mathcal{S} \) of Tychonoff zero-dimensional topological semigroups.

A topological space satisfies the separation axiom \( T_1 \) if all its finite subsets are closed. A topological space is zero-dimensional if it has a base of the topology consisting of clopen (= closed-and-open) sets. It is well-known (and easy to see) that every zero-dimensional \( T_1 \) topological space is Tychonoff.

Since \( \mathcal{T}_z \mathcal{S} \subseteq \mathcal{T}_2 \mathcal{S} \subseteq \mathcal{T}_1 \mathcal{S} \), for every semigroup we have the implications:

\[
\begin{align*}
\text{absolutely } & \mathcal{T}_1 \mathcal{S}\text{-closed} \quad \Rightarrow \quad \text{absolutely } \mathcal{T}_2 \mathcal{S}\text{-closed} \quad \Rightarrow \quad \text{absolutely } \mathcal{T}_z \mathcal{S}\text{-closed} \\
\text{injectively } & \mathcal{T}_1 \mathcal{S}\text{-closed} \quad \Rightarrow \quad \text{injectively } \mathcal{T}_2 \mathcal{S}\text{-closed} \quad \Rightarrow \quad \text{injectively } \mathcal{T}_z \mathcal{S}\text{-closed} \\
\mathcal{T}_1 \mathcal{S}\text{-closed} \quad \Rightarrow \quad \mathcal{T}_2 \mathcal{S}\text{-closed} \quad \Rightarrow \quad \mathcal{T}_z \mathcal{S}\text{-closed}.
\end{align*}
\]

\( C \)-Closed topological groups for various classes \( C \) were investigated by many authors [1, 2, 3, 13, 27, 38, 48]. In particular, the closedness of commutative topological groups in the class of Hausdorff topological semigroups was investigated in [44, 58]; \( C \)-closed topological semilattices were investigated in [5, 6, 40, 41, 53, 54]. Some notions of completeness in Category Theory were investigated in [19, 20, 31, 32, 35, 36, 47]. In particular, closure operators in different categories were studied in [14, 16, 17, 18, 23, 24, 37, 56, 57].

In some cases the injective \( C \)-closedness can be reduced to the \( C \)-closedness and \( C \)-discreteness.

Definition. Let \( C \) be a class of topological semigroups. A semigroup \( X \) is called

- \( C \)-discrete (or else \( C \)-nontopologizable) if for any injective homomorphism \( h : X \to Y \) to a topological semigroup \( Y \in C \) the image \( h[X] \) is a discrete subspace of \( Y \);
- \( C \)-topologizable if \( X \) is not \( C \)-discrete.

The study of topologizable and nontopologizable semigroups is a classical topic in Topological Algebra that traces its history back to Markov’s problem [49] of topologizability of infinite groups, which was resolved in [52, 42] and [50] by constructing examples of nontopologizable infinite groups. The topologizability of semigroups was investigated in [9, 10, 21, 22, 25, 26, 28, 29, 34, 45, 46, 55].

The following theorem was proved by the authors in [8, Proposition 3.2].

Theorem 1.1 (Banakh–Bardyla). Let \( C \) be a class of topological semigroups (such that \( C = \mathcal{T}_1 \mathcal{S} \)). A semigroup \( X \) is injectively \( C \)-closed if (and only if) \( X \) is \( C \)-closed and \( C \)-discrete.

This paper is a continuation of the papers [7, 8, 11] providing inner characterizations of various closedness properties of semigroups. In order to formulate such inner characterizations, let us recall some properties of semigroups.

A semigroup \( X \) is called

- chain-finite if any infinite set \( I \subseteq X \) contains elements \( x, y \in I \) such that \( xy \notin \{x, y\} \);
- singular if there exists an infinite set \( A \subseteq X \) such that \( AA \) is a singleton;
- periodic if for every \( x \in X \) there exists \( n \in \mathbb{N} \) such that \( x^n \) is an idempotent;
- bounded if there exists \( n \in \mathbb{N} \) such that for every \( x \in X \) the \( n \)-th power \( x^n \) is an idempotent;
- group-finite if every subgroup of \( X \) is finite;
- group-bounded if every subgroup of \( X \) is bounded.

The following theorem (proved in [7]) characterizes \( C \)-closed commutative semigroups.
**Theorem 1.2** (Banakh–Bardyla). Let $C$ be a class of topological semigroups such that $T_2S \subseteq C \subseteq T_1S$. A commutative semigroup $X$ is $C$-closed if and only if $X$ is chain-finite, nonsingular, periodic, and group-bounded.

Let us recall that a congruence on a semigroup $X$ is an equivalence relation $\approx$ on $X$ such that for any elements $x \approx y$ of $X$ and any $a \in X$ we have $ax \approx ay$ and $xa \approx ya$. For any congruence $\approx$ on a semigroup $X$, the quotient set $X/\approx$ has a unique semigroup structure such that the quotient map $X \to X/\approx$ is a semigroup homomorphism. The semigroup $X/\approx$ is called the quotient semigroup of $X$ by the congruence $\approx$.

A subset $I$ of a semigroup $X$ is called an ideal in $X$ if $IX \cup XI \subseteq I$. Every ideal $I \subseteq X$ determines the congruence $(I \times I) \cup \{(x, y) \in X \times X : x = y\}$ on $X \times X$. The quotient semigroup of $X$ by this congruence is denoted by $X/I$ and called the quotient semigroup of $X$ by the ideal $I$. If $I = \emptyset$, then the quotient semigroup $X/\emptyset$ can be identified with the semigroup $X$.

Theorem 1.2 implies that each subsemigroup of a $C$-closed commutative semigroup is $C$-closed. On the other hand, quotient semigroups of $C$-closed commutative semigroups are not necessarily $C$-closed, see Example 1.8 in [7]. This motivates us to introduce the following notions:

**Definition.** A semigroup $X$ is called

- **projectively $C$-closed** if for any congruence $\approx$ on $X$ the quotient semigroup $X/\approx$ is $C$-closed;
- **ideally $C$-closed** if for any ideal $I \subseteq X$ the quotient semigroup $X/I$ is $C$-closed.

It is easy to see that for every semigroup the following implications hold:

$$\text{absolutely } C \text{-closed } \Rightarrow \text{projectively } C \text{-closed } \Rightarrow \text{ideally } C \text{-closed } \Rightarrow C \text{-closed}.$$ 

For a semigroup $X$, let $E(X) \overset{\text{def}}{=} \{x \in X : xx = x\}$ be the set of idempotents of $X$.

For an idempotent $e$ of a semigroup $X$, let $H_e$ be the maximal subgroup of $X$ that contains $e$. The union $H(X) = \bigcup_{e \in E(X)} H_e$ of all subgroups of $X$ is called the Clifford part of $S$. A semigroup $X$ is called

- **Clifford** if $X = H(X)$;
- **Clifford+finite** if $X \setminus H(X)$ is finite.

Ideally and projectively $C$-closed commutative semigroups were characterized in [7] as follows.

**Theorem 1.3** (Banakh–Bardyla). Let $C$ be a class of topological semigroups such that $T_2S \subseteq C \subseteq T_1S$. For a commutative semigroup $X$ the following conditions are equivalent:

1. $X$ is projectively $C$-closed;
2. $X$ is ideally $C$-closed;
3. the semigroup $X$ is chain-finite, group-bounded and Clifford+finite.

For a semigroup $X$ let

$$Z(X) \overset{\text{def}}{=} \{z \in X : \forall x \in X \ (xz = zx)\}$$

be the center of $X$, and

$$IZ(X) \overset{\text{def}}{=} \{z \in Z(X) : zX \subseteq Z(X)\}$$

be the ideal center of $X$. Observe that the ideal center is the largest ideal in $X$, which is contained in $Z(X)$.

The following theorem is proved in Lemmas 5.1, 5.3, 5.4 of [7].

**Theorem 1.4** (Banakh–Bardyla). If a semigroup $X$ is $T_2S$-closed, then its center $Z(X)$ is chain-finite, periodic and nonsingular.

Theorems 1.2 and 1.3 suggest the following problem.

**Problem 1.5.** Is the center $Z(X)$ of any $T_2S$-closed semigroup $T_2S$-closed?
By Theorems 1.2 and 1.4 Problem 1.5 is equivalent to

**Problem 1.6.** Let $X$ be a $T_2S$-closed semigroup. Is any subgroup of $Z(X)$ bounded?

In this paper we shall give partial affirmative answers to Problems 1.5 and 1.6. To formulate them, we need to recall some information on viable idempotents and viable semigroups.

Following Putcha and Weissglass [51] we define a semigroup $X$ to be *viable* if for any $x, y \in X$ with $\{xy, yx\} \subseteq E(X)$ we have $xy = yx$. This notion can be localized using the notion of a viable idempotent.

An idempotent $e$ in a semigroup $X$ is defined to be *viable* if the set

$$H_e \overset{\text{def}}{=} \{x \in X : xe = ex \in H_e\}$$

is a *coideal* in $X$ in the sense that $X \setminus H_e$ is an ideal in $X$. By $VE(X)$ we denote the set of viable idempotents of a semigroup $X$.

By Theorem 3.2 of [4], a semigroup $X$ is viable if and only if each idempotent of $X$ is viable if and only if for every $x, y \in X$ with $xy = e \in E(X)$ we have $xe = ex$ and $ye = ey$. This characterization implies that every semigroup $X$ with $E(X) \subseteq Z(X)$ is viable. In particular, every commutative semigroup is viable.

The main result of this paper is the following theorem describing properties of subgroups of categorically closed semigroups and providing partial affirmative answers to Problem 1.6.

**Theorem 1.7.** Let $X$ be a semigroup and $i \in \{1, 2, 3\}$.

1. If $X$ is $T_2S$-discrete, then $Z(X)$ is group-finite.
2. If $X$ is injectively $T_2S$-closed, then $Z(X)$ is group-finite.
3. If $X$ is ideally $T_2S$-closed, then $Z(X)$ is group-bounded.
4. If $X$ is $T_2S$-closed, then $IZ(X)$ is group-bounded.
5. If $X$ is ideally $T_2S$-closed, then for every viable idempotent $e \in VE(X)$ the maximal subgroup $H_e$ of $X$ is projectively $T_2S$-closed and has bounded center $Z(H_e)$.
6. If $X$ is absolutely $T_2S$-closed, then for every viable idempotent $e \in VE(X)$ the maximal subgroup $H_e$ of $X$ is absolutely $T_2S$-closed and has finite center $Z(H_e)$.
7. If $X$ is ideally $T_2S$-closed, then the set $Z(X) \cap \sqrt{VE(X)} \setminus H(X)$ is finite.

In the last statement of Theorem 1.7, $\sqrt{VE(X)}$ stands for the set $\{x \in X : \exists n \in \mathbb{N} \ x^n \in VE(X)\}$. Theorem 1.7 combined with Theorems 1.2 and 1.4 implies the following two partial affirmative answers to Problem 1.5.

**Theorem 1.8.** If a semigroup $X$ is $T_2S$-closed, then its ideal center $IZ(X)$ is $T_1S$-closed.

**Theorem 1.9.** If a semigroup $X$ is $T_2S$-discrete or injectively $T_2S$-closed or ideally $T_2S$-closed, then its center $Z(X)$ is $T_1S$-closed.

Theorems 1.3, 1.7(3) and 1.9 motivate the following “ideal” version of Problem 1.5.

**Problem 1.10.** Is the center of an ideally $T_2S$-closed semigroup ideally $T_2S$-closed?

By Theorems 1.7 and 1.8 Problem 1.10 is equivalent to

**Problem 1.11.** Is the center of an ideally $T_2S$-closed semigroup Clifford+finite?

Theorems 1.3, 1.7(7) imply the following proposition which gives a partial affirmative answer to Problems 1.10 and 1.11.

**Proposition 1.12.** If a semigroup $X$ is ideally $T_2S$-closed, then any subsemigroup $S \subseteq Z(X) \cap \sqrt{VE(X)}$ of $X$ is projectively $T_1S$-closed and Clifford+finite.
A semigroup $X$ is called $Z$-viable if $E(X) \cap Z(X) \subseteq VE(X)$, i.e., each central idempotent of $X$ is viable. It is clear that each viable semigroup is $Z$-viable.

Proposition 1.12 combined with Theorem 1.4 and Lemma 2.5 implies the following two partial answers to Problem 1.10.

**Theorem 1.13.** If a $Z$-viable semigroup $X$ is ideally $T_2S$-closed, then its center $Z(X)$ is projectively $T_1S$-closed.

**Theorem 1.14.** If a semigroup $X$ is ideally $T_2S$-closed, then its ideal center $IZ(X)$ is projectively $T_1S$-closed.

The statements 1–3, 4, 5–6, and 7 of Theorem 1.7 are proved in Sections 3, 4, 5, 6, respectively.

## 2. Preliminaries

We denote by $\omega$ the set of finite ordinals, by $\mathbb{N}$ the set of positive integers, and by $\mathbb{C}$ the set of complex numbers. The family of all finite subsets of a set $X$ is denoted by $[X]^{<\omega}$.

Let

$$\mathbb{T} \overset{\text{def}}{=} \{ z \in \mathbb{C} : |z| = 1 \}$$

be the compact topological group endowed with the operation of multiplication of complex numbers. The following lemma is a classical result of Baer [33, 21.1].

**Lemma 2.1** (Baer). For any distinct elements $x, y$ of a commutative group $X$ there exists a homomorphism $h : X \to \mathbb{T}$ such that $h(x) \neq h(y)$.

The following lemma is proved in Claim 7.1 of [3].

**Lemma 2.2.** For any unbounded commutative group $X$ there exists a homomorphism $h : X \to \mathbb{T}$ whose image $h[X]$ is infinite.

For a semigroup $X$, its

- **0-extension** is the semigroup $X^0 = \{0\} \cup X$ where $0 \notin X$ is any element such that $0x = 0 = x0$ for all $x \in X^0$;
- **1-extension** is the semigroup $X^1 = \{1\} \cup X$ where $1 \notin X$ is any element such that $1x = x = x1$ for all $x \in X^1$.

If $X$ is a topological semigroup, then we shall assume that $X^0$ and $X^1$ are topological semigroups containing $X$ as a clopen subsemigroup.

For any semigroup $X$, the set $E(X)$ is endowed with the natural partial order $\leq$ defined by $x \leq y$ iff $xy = yx = x$. For two idempotents $x, y \in E(X)$ we write $x < y$ if $x \leq y$ and $x \neq y$.

For an element $a$ of a semigroup $X$, the set

$$H_a \overset{\text{def}}{=} \{ x \in X : (xX^1 = aX^1) \wedge (X^1x = X^1a) \}$$

is called the $\mathcal{H}$-class of $a$. By Corollary 2.2.6 [13], for every idempotent $e \in E(X)$ its $\mathcal{H}$-class $H_e$ coincides with the maximal subgroup of $X$, containing the idempotent $e$.

**Lemma 2.3.** Let $X$ be a semigroup, $e, f \in E(X)$, $x \in H_e$ and $y \in H_f$. If $xy = yx$, then $xy \in H_{ef} = H_{fe}$ and $(xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1}$.

**Proof.** Observe that $ey = x^{-1}xy = x^{-1}yx = x^{-1}yxe = x^{-1}xye = e = eyxx^{-1} = eyx^{-1} = xyx^{-1} = yyx^{-1} = ye$. By analogy we can prove that $xf = fx$. Next, observe that $ef = eyy^{-1} = ye^{-1} = fye^{-1} = fey^{-1} = fef = y^{-1}yef = y^{-1}yf = y^{-1}ey = y^{-1}ye = fe$. 

Then for the idempotent \( u = ef = fe \) we have \( xyX^1 = xfX^1 = fxX^1 = feX^1 = uX^1 \) and \( X^1 xy = X^1 ey = X^1 ye = X^1 fe = X^1 u \), which means that \( xy \in H_u \subseteq H(X) \). Observe that
\[
x^{-1} f = x^{-1} ef = x^{-1} fe = x^{-1} fx^{-1} = x^{-1} f x^{-1} = ef x^{-1} = fe x^{-1} = f x^{-1}.
\]
By analogy we can prove that \( y^{-1} e = ey^{-1} \). Then \( x^{-1} y^{-1} X^1 = x^{-1} f X^1 = f x^{-1} X^1 = fe X^1 = u X^1 \) and \( X^1 x^{-1} y^{-1} = X^1 ey^{-1} = X^1 ye^{-1} = X^1 fe = X^1 u \), which means that \( x^{-1} y^{-1} \in H_u \). By analogy we can prove that \( y^{-1} x^{-1} \in H_u \). It follows from \( xy y^{-1} x^{-1} = x f x^{-1} = f x x^{-1} = f e = e u = u \) that \( y^{-1} x^{-1} = (x y)^{-1} \). Also \( x y x^{-1} y^{-1} = y x x^{-1} y^{-1} = y e y^{-1} = e y y^{-1} = e f = u \) implies that \( x^{-1} y^{-1} = (x y)^{-1} = y^{-1} x^{-1} \). □

For a subset \( A \) of a semigroup \( X \), let
\[
\sqrt[n]{A} = \bigcup_{n \in \mathbb{N}} \sqrt[A]{\mathcal{A}} \quad \text{where} \quad \sqrt[A]{\mathcal{A}} = \{ x \in X : x^n \in A \}.
\]
For an element \( a \in X \), the set \( \sqrt[a]{\mathcal{A}} \) will be denoted by \( \sqrt[A]{a} \).

The following lemma is proved in [7, Lemma 3.1].

**Lemma 2.4.** For any idempotent \( e \) of a semigroup we have \( (\sqrt[1]{e} \cdot H_e) \cup (H_e \cdot \sqrt[1]{e}) \subseteq H_e \).

**Lemma 2.5.** Let \( X \) be a semigroup. Every idempotent \( e \in IZ(X) \) is viable.

**Proof.** Since \( e \in Z(X) \), the map \( h : X \to eX, h : x \mapsto ex \), is a homomorphism. The semigroup \( eX \subseteq Z(X) \) is commutative and hence is viable. Then the set \( e X \setminus \frac{eX}{e} = \{ x \in eX : ex \notin H_e \} \) is an ideal in \( eX \) and its preimage \( h^{-1}[e X \setminus \frac{eX}{e}] = \{ x \in X : ex \notin H_e \} = X \setminus \frac{eX}{e} \) is an ideal in \( X \). Therefore, \( \frac{eX}{e} \) is a coideal in \( X \) and the idempotent \( e \) is viable. □

**Remark 2.6.** The inclusion \( E(Z) \cap IZ(X) \subseteq VE(X) \) proved in Lemma 2.5 cannot be improved to the inclusion \( E(X) \cap Z(X) \subseteq VE(X) \): by [12] and [15] there exist infinite congruence-free monoids. In each congruence-free monoid \( X \neq \{1\} \) the idempotent \( 1 \) is central but not viable.

### 3. Proof of Theorem 1.7(1–3)

First we prove a useful lemma on topologizations of semigroups with the help of uniformities. We refer the reader to [30, §8] for the theory of uniform spaces. A topology \( \tau \) on a group \( G \) is called a **group topology** on \( G \) if \((G, \tau)\) is a Hausdorff topological group.

**Lemma 3.1.** Let \( X \) be a semigroup and \( H \) be a subgroup of the center \( Z(X) \). For any group topology \( \tau_H \) on \( H \) there exists a uniformity \( \mathcal{U} \) on \( X \) such that the completion \( \overline{X} \) of the uniform space \((X, \mathcal{U})\) has a unique structure of a topological semigroup containing \( X \) as a subsemigroup and \((H, \tau_H)\) as a topological subgroup.

**Proof.** Let \( \mathcal{U} \) be the uniformity on \( X \) generated by the base consisting of the entourages \( W_U = \{(x, y) \in X \times X : x = y\} \cup \bigcup_{x \in X} (U x \times U x) \)
where \( U = U^{-1} \in \tau_H \) is a neighborhood of the idempotent \( e \) of the group \( H \).

For every neighborhood \( U = U^{-1} \in \tau_H \) of \( e \) and every \( x \in X \), consider the ball \( B(x; W_U) = \{ y \in X : (x, y) \in W_U \} \) of radius \( W_U \). We claim that \( B(x; W_U) = \{ x \} \) for any \( x \in X \) such that \( x \neq ex \). Indeed, in the opposite case, we could choose an element \( y \in B(x; W_U) \setminus \{ x \} \) and find \( z \in X \) such that \( (x, y) \in U x \times U x \).

Then \( x = uz \) for some \( u \in U \subseteq H \) and hence \( ex = eu z = uz = x \), which contradicts our assumption.

Fix any \( x \in X \) satisfying \( x = ex \). Let us show that \( B(x; W_U) = U U^{-1} x \) for any symmetric open neighborhood \( U \in \tau_H \) of \( e \). For any \( y \in U U^{-1} x \) we can find \( u, v \in U \) such that \( y = uv^{-1} x \) and conclude
that for $z \overset{\text{def}}{=} v^{-1}x$ we have $x = ex = vv^{-1}x = vz$ and hence $(x, y) = (vz, uz) \in Uz \times Uz \subseteq W_U$, which implies that $y \in B(x; W_U)$. On the other hand, for every $y \in B(x; W_U) \setminus \{x\}$ there exists $z \in X$ such that $(x, y) \in Uz \times Uz$. It follows that there exists $p \in U$ such that $x = pz$. Then $p^{-1}x = ez$, witnessing that $ez \in U^{-1}x$. Hence $y \in Uz = Uez \subseteq UU^{-1}x$. Thus, $B(x; W_U) = UU^{-1}x$.

Let $\overline{X}$ be the completion of the uniform space $(X, \mathcal{U})$. Let us check that the semigroup operation $\cdot : X \times X \to X$, $(x, y) \mapsto xy$, is uniformly continuous with respect to the product uniformity on $X \times X$. Indeed, given any neighborhood $U = U^{-1} \in \tau_H$ of $e$ we can find a neighborhood $V = V^{-1} \in \tau_H$ of $e$ such that $VV \subseteq U$. Let us show that

$$B(x; W_U) \cdot B(y; W_U) \subseteq B(xy; W_U) \subseteq B(xy; W_U)$$

for every $x, y \in X$. For this fix any $t \in B(x; W_U)$ and $p \in B(y; W_U)$. Assume that $t \neq x$ and $p \neq y$. Then there exist $z_1, z_2 \in X$ and $v_1, v_2, v_3, v_4 \in V$ such that $x = v_1z_1$, $t = v_2z_1$, $y = v_3z_2$ and $p = v_4z_2$. Since the group $H$ is contained in the center of $X$ we get that $xy = (v_1v_3)(z_1z_2)$ and $tp = (v_2v_4)(z_1z_2)$. Since $v_1v_3 \in VV$ and $v_2v_4 \in VV$ we obtain that $tp \in B(xy; W_U) \subseteq B(xy; W_U)$, by the choice of $V$.

The three other cases: $t = x$ and $p \neq y$; $t \neq x$ and $p = y$; $t = x$ and $p = y$ can be treated similarly.

By [30] 8.3.10], the uniformly continuous map $\cdot : X \times X \to X$ can be extended to a uniformly continuous map $\overline{\cdot} : \overline{X} \times \overline{X} \to \overline{X}$. The density of $X$ in $\overline{X}$ implies that the binary operation $\overline{\cdot}$ is associative, witnessing that $\overline{X}$ is a topological semigroup. The definition of the uniformity $\mathcal{U}$ ensures that the topology induced by $\mathcal{U}$ on $H$ coincides with the topology $\tau_H$. 

The following lemma implies Theorem 1.7(1–3).

**Lemma 3.2.** Let $X$ be a semigroup.

1. If $X$ is injectively $T_2S$-closed or $T_2S$-discrete, then $Z(X)$ is group-finite.
2. If $X$ is ideally $T_2S$-closed, then $Z(X)$ is group-bounded.

**Proof.** If $Z(X)$ is not group-finite (and not group-bounded), then $Z(X)$ contains a countable infinite subgroup $H$ (which is not bounded).

Let $\overline{H}$ be the set of all homomorphisms from $H$ to the group $\mathbb{T} \overset{\text{def}}{=} \{z \in \mathbb{C} : |z| = 1\}$. By Lemma 2.1 the diagonal homomorphism $\delta : H \to \mathbb{T}^H$, $\delta : z \mapsto (x(z))_{x \in H}$, is injective. Let $K$ be the closure of the subgroup $\delta[H]$ in the compact topological group $\mathbb{T}^H$.

Let $\tau_H$ be the unique topology on $H$ such that the injective homomorphism $\delta : H \to K$ is a topological embedding. Let $\mathcal{U}$ be the uniformity on $X$ generated by the base consisting of the entourages

$$W_U = \{(x, y) \in X \times X : x = y\} \cup \bigcup_{z \in X} (Uz \times Uz)$$

where $U = U^{-1} \in \tau_H$ is a neighborhood of the idempotent $e$ of the group $H$. In the proof of Lemma 3.1 we have shown that for every $x \in X$ the ball $B(x; W_U) \overset{\text{def}}{=} \{y \in X : (x, y) \in W_U\}$ of radius $W_U$ equals $\{x\}$ if $x \neq ex$ and $B(x; W_U) = UU^{-1}x$ if $x = xe$. Let $\tau_X$ be the topology on $X$ generated by the uniformity $\mathcal{U}$. The topology $\tau_X$ consists of all subsets $V \subseteq X$ such that for every $x \in V$ there exists a neighborhood $U = U^{-1} \in \tau_H$ of $e$ such that $B(x; W_U) \subseteq V$.

By Lemma 3.1 the completion $\overline{X}$ of the uniform space $(X, \mathcal{U})$ carries the structure of a topological semigroup containing $X$ as a dense subsemigroup and $(H, \tau_H)$ as a topological subgroup. Since $H$ is a clopen subgroup of $(X, \tau_X)$, the closure $\overline{H}$ of $H$ in $\overline{X}$ is topologically isomorphic to $K$ and hence is a compact topological group.

Observe that for every $h \in H$ the closed subset $Z_h \overset{\text{def}}{=} \{x \in \overline{X} : xh = hx\}$ of $\overline{X}$ contains $X$ and hence $Z_h = \overline{X}$. Then for every $x \in \overline{X}$ the closed subset $Z_x \overset{\text{def}}{=} \{y \in \overline{X} : xy = yx\}$ of $\overline{X}$ contains $H$ and hence contains the closure of $H$ in $\overline{X}$. Therefore, $\overline{H} \subseteq Z(\overline{X})$. 

By Theorem 1.4, the semigroup $\mathfrak{S}[H]$ is not discrete and not closed in the compact topological group $K$ and hence $H$ is not periodic and not closed in $X$. Then there exists an element $a \in \mathfrak{S} \setminus H \subseteq X \setminus X$. Let $Y$ be the subsemigroup of $X$, generated by the set $X \cup \{a\}$. Since $a \in \mathfrak{S} \subseteq Z(X)$, the semigroup $Y$ coincides with the set $\{xa^n : x \in X, n \geq 0\}$, where we assume that $xa^n = x$ if $n = 0$.

Observe that the space $Y$ is Tychonoff being a subspace of the uniform (and thus Tychonoff) space $X$. For every $x \in X$ with $x \neq xe$, the singleton $\{x\}$ is clopen in $(X, \tau_X)$ and remains clopen in $X$. On the other hand, for every $x = xe \in X$ the subspace $xH$ is clopen in $X$ and $x\mathfrak{S}$ is clopen in $X$. Since the set $Y \cap x\mathfrak{S} = \{xya^n : y \in H, n \geq 0\}$ is countable, the Tychonoff space $Y$ is locally countable and hence zero-dimensional. Since the subgroup $H$ of $X$ is not discrete, the semigroup $X$ is $T_2S$-topologizable. Since the identity homomorphism $X \to Y \in T_2S$ is injective and has non-closed image in $Y$, the semigroup $X$ is not injectively $T_2S$-closed. These arguments complete the proof of statement (1).

To prove statement (2), assuming that the subgroup $H$ is not bounded, we will show that the semigroup $X$ is not ideally $T_2S$-closed. To derive a contradiction, assume that $X$ is ideally $T_2S$-closed. By Theorem 1.4, the semigroup $Z(X)$ is periodic and so is the group $H \subseteq Z(X)$. By Lemma 2.2 there exists a homomorphism $h : H \to \mathbb{T}$ whose image $h[H]$ is infinite and hence dense in $\mathbb{T}$. Consider the projection $pr_h : K \to \mathbb{T}$, $pr_h : y \mapsto y(h)$, of $K$ onto the $h$-th factor and observe that $pr_h[K]$ is a compact subgroup of $\mathbb{T}$ that contains the dense subgroup $h[H]$ of $\mathbb{T}$. The compactness of $K$ ensures that $pr_h[K] = \mathbb{T}$ and hence $K$ contains an element of infinite order. Since $K$ is topologically isomorphic to the closure $\overline{H}$ of $H$ in $X$, we can find an element $a \in \overline{H} \setminus H \subseteq X \setminus X$ of infinite order.

Then the monogenic subsemigroup $a^n \overset{\text{def}}{=} \{a^n : n \in \mathbb{N}\}$ of $\mathfrak{S}$ is disjoint with the periodic group $H$.

In the semigroup $X$ consider the ideal $I \overset{\text{def}}{=} \{x \in X : e \notin x\mathfrak{S}\}$. We claim that the ideal $I$ is clopen in $X$. To see that $I$ is closed in $X$, observe that for every $x \notin I$ we have $e = pxq$ for some $p, q \in X^{-1}$. Then for every $h \in \mathfrak{S} \subseteq Z(X)$ we have $e = ee = pxq = pxe = pxh^{-1}q \in \mathfrak{S}(xh)X^{-1} \subseteq X^{-1}(xh)X^{-1}$ and hence $xh \notin I$. Then $x\mathfrak{S}$ is a neighborhood of $x$ in $X$ that misses the set $I$ and witnesses that the ideal $I$ is closed in $X$. On the other hand, for every $x \in I$ we have $x\mathfrak{S} \subseteq I$ as $I$ is an ideal in $X$. Since $x\mathfrak{S}$ is a neighborhood of $x$ in $X$, the ideal $I$ is open in $X$.

In the subsemigroup $Y = \{xa^n : x \in X, n \geq 0\}$ of the topological semigroup $X$, consider the set $J \overset{\text{def}}{=} (Y \setminus X) \cup (I \cap Y)$. We claim that $J$ is an ideal in $Y$. In the opposite case we can find elements $j \in J$ and $y \in Y$ such that $jy \notin J$ or $yj \notin J$. First assume that $jy \notin J$. It follows from $j \in J = (Y \setminus X) \cup (I \cap Y)$ and $jy \notin J$ that $jy \notin I$ and $j \notin I$, as $I$ is an ideal. Then $j \in Y \setminus X$ but $jy \in X$. It follows from $j \in Y \setminus X$ that $j = xa^n$ for some $x \in X$ and $n \in \mathbb{N}$. Also the definition of the semigroup $Y \ni y$ ensures that $y = xya^m$ for some $x \in X$ and $m \geq 0$ (where we assume that $xya^m = x$ if $m = 0$).

It follows from $jy \notin I$ that $e = \tilde{p}\tilde{q}y\tilde{q}$ for some $\tilde{p}, \tilde{q} \in X^{-1}$. Since the set $\overline{H}$ is a clopen neighborhood of $e$ in $X$, there are neighborhoods $O_{\tilde{p}}, O_{\tilde{q}}$ of the elements $\tilde{p}, \tilde{q}$ in the topological semigroup $X$ such that $O_{\tilde{p}}\tilde{q}O_{\tilde{q}} \subseteq \overline{H}$. Choose any $p \in X \cap O_{\tilde{p}}$ and $q \in X \cap O_{\tilde{q}}$. Since $jy \in X$, we get $pjyq \in X \cap O_{\tilde{p}}\tilde{q}O_{\tilde{q}} \subseteq X \cap \overline{H} = H$. On the other hand, $pjyq = pxa^nxya^mq = pxx_yaq^{n+m}$ and hence $pxyq = (p)x_ya^{-(n+m)} \in X \cap \overline{H} = H$ and finally $pjyq = (p)x_yaq^{n+m} \notin H$ because $n + m \in \mathbb{N}$ and $a^{n+m} \cap H = \emptyset$. By analogy we can derive a contradiction assuming that $yj \notin J$. These contradictions show that $J$ is an ideal in $Y$.

Let $\tau_Y$ be the topology of the semigroup $Y$ inherited from the topological semigroup $X$. Define a stronger topology $\tau_Y'$ on $Y$ as follows: A subset $U \subseteq Y$ is open in $(Y, \tau_Y')$ if an only if for every $y \in J \cap U$ there exists an open neighborhood $V \in \tau_Y$ of $y$ such that $V \subseteq U$. The definition of $\tau_Y'$ implies that $Y \setminus J = X \setminus I$ is an open discrete subspace of $(Y, \tau_Y')$. Taking into account that $J$ is an ideal in $Y$, it can be shown that $(Y, \tau_Y')$ is a zero-dimensional topological semigroup containing $X$ as a non-closed
subsemigroup. Endow the quotient semigroup \( Y/(Y \cap I) \) with the strongest topology \( \tau \) in which the quotient homomorphism \( q : (Y, \tau_Y) \to Y/(Y \cap I) \) is continuous. Taking into account that the ideal \( I \cap Y \) is clopen in the zero-dimensional topological semigroup \((Y, \tau_Y)\), we conclude that \((Y/(Y \cap I), \tau)\) is a zero-dimensional topological semigroup containing the quotient semigroup \(X/(X \cap I)\) as a discrete non-closed subsemigroup. Hence the semigroup \(X\) is not ideally \( T_2 \)-closed. \( \square \)

4. Proof of Theorem 1.7 (4)

Given a \( T_2 \)-closed semigroup \( X \), we will prove that its ideal center \( IZ(X) \) is group-bounded.

By Theorem 1.4, the semigroup \( Z(X) \) is chain-finite, periodic and nonsingular. To derive a contradiction, assume that some subgroup of the ideal center \( IZ(X) \) is unbounded. Observe that for every idempotent \( e \in IZ(X) \), the maximal subgroup \( H_e = H_{ee} \) of \( X \) is contained in \( IZ(X) \). Therefore, \( H_e \) is a maximal subgroup of the semigroup \( IZ(X) \). Since the semigroup \( Z(X) \) is chain-finite, the partially ordered set \( E(X) \cap Z(X) \) is well-founded, i.e., each nonempty subset of \( E(X) \cap Z(X) \) contains a minimal element. Using this fact we can find an idempotent \( e \in IZ(X) \) such that the maximal subgroup \( H_e \) is unbounded but for any idempotent \( f \in E(IZ(X)) \) with \( f < e \) the maximal subgroup \( H_f \) is bounded.

By Lemma 2.4 the idempotent \( e \) is viable and hence the set \( X \setminus H_e \) is an ideal in \( X \).

Claim 4.1. For any \( a \in X \setminus H_e \), the set \( G_a = \{ x \in H_e : ax = ae \} \) is a subgroup of \( H_e \) such that the quotient group \( H_e / G_a \) is bounded.

Proof. Observe that for any \( x, y \in G_a \), \( axy = aey = aye = ace = ae = ae \). Hence \( G_a \) is a subsemigroup of \( H_e \). Since the group \( H_e \subseteq Z(X) \) is periodic, the subsemigroup \( G_a \) of \( H_e \) is a subgroup of \( H_e \). It remains to prove that the quotient group \( H_e / G_a \) is bounded. Since \( e \in IZ(X) \), the element \( ae \) belongs to the ideal center \( IZ(X) \). Since \( Z(X) \) is periodic, there exists \( n \in \mathbb{N} \) such that \( f^n = (ae)^n = an \) is an idempotent of the semigroup \( IZ(X) \). It is clear that \( ef = fe = f \) and hence \( f \leq e \). Assuming that \( f = e \), we conclude that \( ae \in \sqrt{f} = e \sqrt{e} \) and hence \( ae = aee \in e \sqrt{e} \subseteq H_e \) by Lemma 2.4. But the inclusion \( ae \in H_e \) contradicts \( a \not\in H_e \). This contradiction shows that \( f < e \). By the minimality of \( e \), the maximal group \( H_f \) is bounded. So, there exists \( p \in \mathbb{N} \) such that \( y^p = f \) for all \( y \in H_f \).

In the group \( H_e \) consider the subgroup \( G = \{ x^p : x \in H_e \} \). By Lemma 2.3 \( fH_e \subseteq H_f \). For every \( x \in H_e \) we have \( fxe = (fx)^p = f \) by the choice of \( p \). Then \( fG = \{ f \} \) and \( yG = (yf)G = y(fG) = yf = y \) for every \( y \in H_f \). Since \( a^ke = (ae)^k \in H_f \), we have \( a^pG = a^{pn}(eG) = (a^pG)eG = \{ a^ke \} \) and hence \( G \subseteq G_a \) is a subgroup of \( H_e : a^x = a^{x^p} \).

Let \( k \leq n \) be the smallest number such that the subgroup \( G \cap G_{ak} \) has finite index in \( G \).

If \( k > 1 \), then the subgroup \( G \cap G_{ak-1} \) has infinite index in \( G \), by the minimality of \( k \). Since the group \( G \cap G_{ak} \) has finite index in \( G \), the subgroup \( G \cap G_{ak-1} \) has infinite index in the group \( G \cap G_{ak} \).

So, we can find an infinite set \( J \subseteq G \cap G_{ak} \) such that \( x(G \cap G_{ak-1}) \cap y(G \cap G_{ak-1}) = \emptyset \) for any distinct elements \( x, y \in J \). Observe that for any distinct elements \( x, y \in J \) we have \( a^kx = a^ke = a^ky \) and \( a^{k-1}x = a^{k-1}y \) (assuming that \( a^{k-1}x = a^{k-1}y \), we obtain that \( a^{k-1}e = a^{k-1}x^k\) and hence \( yx^{-1} \in G \cap G_{ak-1} \) which contradicts the choice of the set \( J \)). Then the set \( A = a^{k-1}J \subseteq IZ(X) \) is infinite. We claim that \( AA \) is a singleton. Indeed, for any \( x, y \in J \) we have:

\[
a^{k-1}xa^{k-1}y = a^{k-1}xa^{k-2}y = a^{k}ea^{k-2}y = a^{k-2}ea^{k}y = e^{k-2}ea^{k}e = a^{2k-2}e.
\]

Therefore, \( AA = \{ a^{2k-2}e \} \). But the existence of such set \( A = IZ(X) \subseteq Z(X) \) contradicts the nonsingularity of the semigroup \( Z(X) \). This contradiction shows that \( k = 1 \) and hence the subgroup \( G \cap G_a \) has finite index in \( G \). Then the quotient group \( G/(G \cap G_a) \) is finite and bounded. Since the quotient group \( H_e / G \) is bounded, the quotient group \( H_e / (G \cap G_a) \) is bounded and so is the quotient group \( H_e / G_a \). \( \square \)
Let
$$\sqrt{1} = \{z \in \mathbb{C} : \exists n \in \mathbb{N} \ (z^n = 1)\}$$
be the quasi-cyclic group, considered as a dense subgroup of the compact topological group $T = \{z \in \mathbb{C} : |z| = 1\}$. Denote by $\hat{H}_e$ the set of all homomorphisms from $H_e$ to $T$. Since $H_e \subseteq \mathcal{I}Z(X) \subseteq Z(X)$ and the semigroup $Z(X)$ is periodic, the group $H_e$ is periodic and hence $\varphi[H_e] \subseteq \sqrt{1}$ for any homomorphism $\varphi \in \hat{H}_e$. By Lemma 2.1, the diagonal homomorphism $\delta : H_e \to T^{\hat{H}_e}$, $\delta : x \mapsto (\varphi(x))_{\varphi \in \hat{H}_e}$, is injective. Identify the group $H_e$ with its image $\delta[H_e] \subseteq \sqrt{1}^{\hat{H}_e}$ in the compact topological group $T^{\hat{H}_e}$ and let $\overline{H}_e$ be the closure of $H_e$ in $T^{\hat{H}_e}$.

By Lemma 2.2, there exists a homomorphism $h : H_e \to T$ with infinite image $h[H_e]$. The subgroup $h[H_e]$, being infinite, is dense in $T$. The homomorphism $h$ admits a continuous extension $\tilde{h} : \overline{H}_e \to T$, $\tilde{h} : (z_{\varphi})_{\varphi \in \hat{H}_e} \mapsto z_h$. The compactness of $\overline{H}_e$ and density of $h[H_e] = \overline{h[H_e]}$ in $T$ imply that $\tilde{h}(\overline{H}_e) = T$.

By Claim 4.1, for every $a \in X \setminus \frac{H_e}{e}$ the quotient group $H_e/G_a$ is bounded. So, we can find a number $n_a \in \mathbb{N}$ such that $x^{n_a} \in G_a$ for all $x \in H_e$. Moreover, for any non-empty finite set $F \subseteq X \setminus \frac{H_e}{e}$ and the number $n_F = \prod_{a \in F} n_a \in \mathbb{N}$, the intersection $G_F = \bigcap_{a \in F} G_a$ contains the $n_F$-th power $x^{n_F}$ of any element $x \in H_e$.

Then for every $y \in h[H_e] \subseteq \sqrt{1}$, we get $y^{n_F} \in h[G_F]$, which implies that the subgroup $h[G_F]$ is dense in $T$. Let $\overline{G}_F$ be the closure of $G_F$ in the compact topological group $\overline{H}_e$. The density of the subgroup $h[G_F]$ in $T$ implies that $\overline{h[G_F]} = \overline{h[G_F]} = T$.

By the compactness, $\overline{h[\bigcap_{F \subseteq [X]^{<\omega}} G_F]} = \bigcap_{F \subseteq [X]^{<\omega}} \overline{h[G_F]} = T$. So, we can fix an element $s \in \bigcap_{F \subseteq [X]^{<\omega}} G_F \subseteq \overline{H}_e$ whose image $\tilde{h}(s) \in T$ has infinite order in the group $T$. Then $s$ also has infinite order and its orbit $s^\mathbb{N}$ is disjoint with the periodic group $H_e$.

Consider the subsemigroup $S \subseteq \overline{H}_e$ generated by $H_e \cup \{s\}$. Observe that
$$S = \overline{H}_e \cup \{gs^n : g \in H_e, \ n \in \mathbb{N}\} \subseteq \bigcup_{\varphi \in \widehat{H}_e} \mathbb{Q}_\varphi$$
where $\mathbb{Q}_\varphi$ is the countable subgroup of $T$ generated by the set $\sqrt{1} \setminus \{\varphi(s)\}$.

It is clear that the subspace topology $\overline{\tau}$ on $S$, inherited from the topological group $\prod_{\varphi \in \hat{H}_e} \mathbb{Q}_\varphi$ is Tychonoff and zero-dimensional. Then the topology $\overline{\tau}$ on $S$ generated by the base
$$\{U \cap a\overline{G}_F : U \in \overline{\tau}, \ a \in \overline{H}_e, \ F \in [X \setminus \frac{H_e}{e}]^{<\omega}\}$$
is zero-dimensional, too. It is easy to see that $(S, \overline{\tau})$ is a topological semigroup and $S$ belongs to the closure of $H_e$ in the topology $\overline{\tau}$. Finally, endow $S$ with the topology $\tau = \{U \cup D : U \in \overline{\tau}, \ D \subseteq H_e\}$. The topology $\tau$ is well-known in General Topology as the Michael modification of the topology $\overline{\tau}$ (see [30] 5.1.22). Since the (group) topology $\overline{\tau}$ is zero-dimensional, so is its Michael modification $\tau$ (see [30] 5.1.22). Using the fact that $S \setminus H_e$ is an ideal in $S$, it can be shown that $(S, \tau)$ is a zero-dimensional topological semigroup, containing $H_e$ as a dense discrete subgroup. From now on we consider $S$ as a topological semigroup, endowed with the topology $\tau$.

Let $Y = S \sqcup (X \setminus H_e)$ be the topological sum of the topological space $S$ and the discrete topological space $X \setminus H_e$. It is clear that $Y$ contains $X$ as a proper dense discrete subspace.

It remains to extend the semigroup operation of $X$ to a continuous commutative semigroup operation on $Y$. In fact, for any $a \in X, \ b \in H_e$ and $n \in \mathbb{N}$ we should define the product $a(bs^n)$. By the periodicity of the semigroup $Z(X)$, there is a number $p \in \mathbb{N}$ such that $f := (ae)^p$ is an idempotent. If $fe < e$, then we put $a(bs^n) = ab$. If $fe = e$, then $f = (ae)^p = (ae)^p e = fe = e$ and hence $ae$ belongs to the semigroup $\sqrt{1}^{\overline{H}_e}$. By Lemma 2.3, $ae = ae \in \sqrt{1}^{\overline{H}_e} \subseteq H_e$. So, we can put $a(bs^n) = (ae)bs^n$.

The choice of $s \in \bigcap_{F \subseteq [T]^{<\omega}} G_F$ guarantees that the extended binary operation is continuous. Now the density of $X$ in $Y$ implies that the extended operation is commutative and associative. Since
$Y \in TzS$, the semigroup $X$ is not $TzS$-closed, which is a desired contradiction completing the proof of Theorem 4.

5. PROOF OF THEOREM 5, 6

In this section we prove two lemmas implying statements (5) and (6) of Theorem 6.

Lemma 5.1. Let $i \in \{1, 2, z\}$. If $X$ is an ideally $TzS$-closed semigroup, then for every viable idempotent $e \in VE(X)$ the maximal subgroup $H_e$ of $X$ is projectively $TzS$-closed and has bounded center.

Proof. Assume that $X$ is an ideally $TzS$-closed semigroup and $e \in VE(X)$. To prove that the subgroup $H_e$ is projectively $TzS$-closed, take any homomorphism $h : H_e \to Y$ to a topological semigroup $Y \in TzS$ such that $h[H_e]$ is a discrete subgroup in $Y$. We need to prove that $h[H_e]$ is closed in $Y$. Replacing the topological semigroup $Y$ by the closure $\overline{h[H_e]}$ of $h[H_e]$ in $Y$, we can assume that $Y = \overline{h[H_e]}$.

Let us show that the complement $Y \setminus h[H_e]$ is an ideal in $Y$. In the opposite case we can find elements $y \in Y \setminus h[H_e]$ and $x \in Y$ such that $xy$ or $yx$ does not belong to $Y \setminus h[H_e]$. If $xy \notin Y \setminus h[H_e]$, then $xy \in h[H_e]$. Since $h[H_e]$ is a discrete subgroup of $Y$, there exists a neighborhood $O_{xy}$ of $xy$ in $Y$ such that $O_{xy} \cap h[H_e] = \{xy\}$. By the continuity of the semigroup operation in $Y$, there exist neighborhoods $O_x$ and $O_y$ of $x$ and $y$ in $Y$ such that $O_xO_y \subseteq O_{xy}$. Since $h[H_e]$ is dense in $Y$, we can choose an element $z \in O_x \cap h[H_e]$ and conclude that $z(O_y \cap h[H_e]) \subseteq h[H_e] \cap O_{xy} = \{xy\}$ and hence $O_y \cap h[H_e] \subseteq \{z^{-1}xy\}$, which is not possible as $y \in Y \setminus h[H_e]$ is an accumulation point of the set $h[H_e]$ in $Y$. By analogy we can derive a contradiction assuming that $yx \notin Y \setminus h[H_e]$.

Let $Y^0 = Y \cup \{0\}$ be the 0-extension of $Y$ by an external isolated zero. Since the idempotent $e$ is viable the set $I \eqdef X \setminus \frac{H_e}{e}$ is an ideal in $X$. So we can consider the quotient semigroup $X/I$, and the homomorphism $h_e : X/I \to Y^0$, defined by

$$h_e(x) = \begin{cases} h(xe) & \text{if } x \in \frac{H_e}{e} \\ 0 & \text{otherwise.} \end{cases}$$

The definition of the set $\frac{H_e}{e} = \{x \in X : xe = ex \in H_e\}$ guarantees that the homomorphism $h_e$ is well-defined.

Now consider the semigroup $Z = (X/I) \cup (Y \setminus h[H_e])$ in which $Y \setminus h[H_e]$ and $X/I$ are subsemigroups and for any $x \in X/I$ and $y \in Y \setminus h[H_e]$ the products $xy$ and $yx$ are defined as $h_e(x)y$ and $yh_e(x)$, respectively. Endow the semigroup $Z$ with the topology $\tau$ consisting of the sets $W \subseteq Z$ such that for every $y \in W \cap (Y \setminus h[H_e])$ there exists a neighborhood $O_y$ of $y$ in $Y$ such that $(O_y \setminus h[H_e]) \cup [h_e^{-1}O_y] \subseteq W$. It can be shown that $Z$ is a topological semigroup in the class $TzS$ containing $X/I$ as a discrete subsemigroup. Since the semigroup $X$ is ideally $TzS$-closed, the semigroup $X/I$ is closed in $Z$, which implies that the set $h[H_e]$ is closed in $Y$.

Therefore, the maximal group $H_e$ is projectively $TzS$-closed. By Lemma 3.2(2), the center $Z(H_e)$ of the group $H_e$ is bounded. □

We say that a class $C$ of topological semigroups is closed under 0-extensions if for any topological semigroup $Y \in C$ its 0-extension $Y^0 = Y \cup \{0\}$ belongs to the class $C$. It is clear that for every $i \in \{1, 2, z\}$ the class $TzS$ is closed under 0-extensions.

Lemma 5.2. Let $C$ be a class of topological semigroups such that $TzS \subseteq C \subseteq TzS$ and $C$ is closed under 0-extensions. If a semigroup $X$ is absolutely $C$-closed, then for every viable idempotent $e \in VE(X)$, the maximal subgroup $H_e$ of $X$ is absolutely $C$-closed and has finite center $Z(H_e)$.
Proof. Since the idempotent \( e \) is viable, the set \( I_e \defeq X \setminus H_e \) is an ideal in \( X \). By the definition of the set \( H_e \defeq \{ x \in X : xe = ex \in H_e \} \), the map \( h_e : X \to H_e^0 \),

\[
h_e(x) = \begin{cases} xe & \text{if } x \in H_e; \\ 0 & \text{otherwise}; \end{cases}
\]

is a well-defined homomorphism. If \( h_e[X] = H_e \), then the absolute \( C \)-closedness of \( X \) implies the absolute \( C \)-closedness of the maximal subgroup \( H_e \) and we are done. If \( h_e[X] = H_e^0 \), then the absolute \( C \)-closedness of \( X \) implies the absolute \( C \)-closedness of the semigroup \( H_e^0 \). To prove that the group \( H_e \) is absolutely \( C \)-closed, take any homomorphism \( f : H_e \to Y \) to a topological semigroup \( Y \in C \). Extend the homomorphism \( f \) to the homomorphism \( f^0 : H_e^0 \to Y^0 \) such that \( f(0) = 0 \). By our assumption, the class \( C \) is closed under 0-extensions and hence contains the topological semigroup \( Y^0 \). Since the semigroup \( H_e^0 \) is absolutely \( C \)-closed, the image \( f^0[H_e^0] \) is closed in \( Y^0 \). Since the set \( Y \) is closed in \( Y^0 \), the image \( f[H_e] = Y \cap f^0[H_e^0] \) is closed in \( Y \), witnessing that the group \( H_e \) is absolutely \( C \)-closed. By Lemma 6.2, the center \( Z(H_e) \) of the group \( H_e \) is finite. \( \square \)

6. Proof of Theorem 1.7(7)

In this section we prove three lemmas implying the statement 7 of Theorem 1.7.

Recall that for a subset \( A \) of a semigroup \( X \) we denote the set \( \{ x \in X : \exists n \in \mathbb{N} \, x^n \in A \} \).

Lemma 6.1. If \( X \) is an ideally \( T_2 \)-S-closed semigroup, then the set

\[
B = \{ e \in VE(X) : (\sqrt{H_e} \cap Z(X)) \setminus H_e \neq \emptyset \}
\]

is finite.

Proof. To derive a contradiction, assume that the set \( B \) is infinite. By Theorem 1.4 the semigroup \( Z(X) \) is chain-finite, periodic and nonsingular. Given any idempotent \( e \in B \), choose an element \( x \in \sqrt{H_e} \cap Z(X) \). By the periodicity of \( Z(X) \), \( x^n = e \) for some \( n \in \mathbb{N} \). It follows from \( x \in Z(X) \) that \( e \in Z(X) \). Therefore, \( B \subseteq VE(X) \cap Z(X) \). Since the semigroup \( Z(X) \) is chain-finite, we can apply the Ramsey Theorem \([29]\) Theorem 5] and find an infinite subset \( C \subseteq B \) such that \( xy = xy \notin \{ x, y \} \) for any distinct elements \( x, y \in C \).

For every \( e \in E(X) \) let \( \sqrt[e]{e} = \{ x \in E(X) : x < e \} \). By the definition of a viable idempotent, for every \( e \in VE(X) \) the set \( X \setminus H_e \) is an ideal in \( X \). Then for every \( e \in C \) the intersection

\[
I_e \defeq \bigcap_{c \in C \setminus \sqrt[e]{e}} (X \setminus H_e)
\]

is an ideal in \( X \). It follows from \((H_e \cdot H_e) \cup (H_e \cdot H_e) \subseteq H_e \) that the union \( J_e \defeq I_e \cup H_e \) is an ideal in \( X \) and hence the union

\[
J \defeq \bigcup_{e \in C} J_e
\]

is an ideal in \( X \).

Claim 6.2. For any \( e \in C \) the set \( Z(X) \cap H_e \setminus H_e \) contains an element \( a_e \) such that \( a_e^2 \in H_e \).

Proof. Since \( e \in C \subseteq B \), there exists \( x \in (\sqrt{H_e} \cap Z(X)) \setminus H_e \). If \( x^2 \in H_e \), then put \( a_e = x \). Otherwise, by the periodicity of \( Z(X) \), there exists \( n \in \mathbb{N} \) such that \( x^n \in H_e \). Let \( m \in \mathbb{N} \) be the smallest number such that \( x^m \in H_e \). Note that \( 2 < m \leq n \) and \( x^{m-1} \in (\sqrt{H_e} \cap Z(X)) \setminus H_e \). Let \( a_e = x^{m-1} \) and observe that

\[
a_e^2 = x^{2m-2} = x^m x^{m-2} \in H_e \sqrt{H_e} \subseteq H_e,
\]

by Lemma 2.4. Also \( a_e e \in \sqrt{H_e} H_e \subseteq H_e \) and hence \( a_e \in \frac{H_e}{H_e} \setminus H_e \). \( \square \)
Consider the set \( A \defeq \{ a_e : e \in C \} \).

**Claim 6.3.** \( A \subseteq X \setminus J \).

**Proof.** Assuming that \( A \not\subseteq X \setminus J \), we can find two idempotents \( e, c \in C \) such that \( a_e \in J_c \). We claim that \( e \neq c \). In the opposite case, \( a_e \notin H_e \) implies \( a_e \in J_c \setminus H_e = J_c \setminus J_e \subseteq I_c \subseteq X \setminus \frac{H_e}{e} \), which contradicts the choice of \( a_e \) in Claim 6.2. This contradiction shows that \( a \neq c \).

Since \( a_e \in J_c \), we have \( a_e^2 \in J_c \) and \( e = a_e^2(a_e^2)^{-1} \in J_c = I_c \cup H_c \). If follows from \( e \neq c \) that \( e \in J_c \setminus H_c \subseteq I_c \). Since \( ce = ec \notin \{ e, c \} \), we obtain that \( e \in C \setminus \downarrow c \) and hence \( e \in I_c \subseteq X \setminus \frac{H_e}{e} \), which contradicts the obvious inclusion \( e \in \frac{H_e}{e} \). This contradiction completes the proof of \( A \subseteq X \setminus J \). \( \square \)

**Claim 6.4.** For any distinct idempotents \( e, c \in C \) we have \( a_e a_c \in I_e \cap I_c \) and \( a_e \neq a_c \).

**Proof.** Assuming that \( a_e a_c \notin I_e \), we can find an idempotent \( z \in C \setminus \downarrow e \) such that \( a_e a_c \in \frac{H_e}{z} \). Since \( X \setminus \frac{H_e}{z} \) is an ideal, \( a_e a_c \in \frac{H_e}{z} \) implies \( a_e, a_c \in \frac{H_e}{z} \) and hence \( za_e, za_c \in H_z \). Since the semigroup \( X \) is periodic, there exists \( n \in \mathbb{N} \) such that \( a^n_e = e \) and \( a^n_c = c \). Then \( ze = za^n_e = (za^n_e)^n \in H_z \) and \( zc = zc^n \in H_z \). Taking into account that \( ze = z = ze \) and hence \( z \leq c \) and \( z \leq e \). The choice of the set \( C \) ensures that \( e = z = e \), which contradicts the choice of \( e \neq c \). This contradiction shows that \( a_e a_c \in I_e \). By analogy we can show that \( a_e a_c \notin I_e \). Assuming that \( a_e = a_c \), we obtain that \( a_e a_c = a^n_e \in H_e \subseteq \frac{H_e}{e} \), which contradicts \( a_e a_c \in I_e \subseteq X \setminus \frac{H_e}{e} \). \( \square \)

Claims 6.3 and 6.4 imply that \( A \) is an infinite subset of \( X \setminus J \) such that

\[ AA \subseteq J \cup \{ a^2_e : e \in C \} \subseteq J \cup \bigcup_{e \in C} H_e = J. \]

Since \( X \) is ideally \( T_2 \)-closed, the quotient semigroup \( X/J \) is \( T_2 \)-closed. However, the set \( A \subseteq Z(X) \setminus J \) is a witness to the singularity of \( Z(X/J) \) which contradicts Theorem 1.7. The obtained contradiction implies that the set \( B \) is finite. \( \square \)

The following lemma has been proved in [7, Lemma 7.5].

**Lemma 6.5.** Let \( X \) be an ideally \( T_2 \)-closed semigroup such that for some \( e \in E(X) \cap Z(X) \) the semigroup \( H_e \cap Z(X) \) is bounded. Then the set \( (\sqrt{\mathcal{P}_e} \cap Z(X)) \setminus H_e \) is finite.

The next lemma proves the last statement of Theorem 1.7.

**Lemma 6.6.** If \( X \) is an ideally \( T_2 \)-closed semigroup, then the set \( Z(X) \cap \sqrt{\mathcal{V}E(X)} \setminus H(X) \) is finite.

**Proof.** By Theorems 1.4 and 1.7(3), the semigroup \( Z(X) \) is periodic and group-bounded. By Lemma 6.1, the set \( B = \{ e \in \mathcal{V}E(X) : Z(X) \cap \sqrt{\mathcal{P}_e} \setminus H_e \neq \emptyset \} \) is finite. By Lemma 6.5, for every \( e \in B \) the set \( Z(X) \cap \sqrt{\mathcal{P}_e} \setminus H_e \) is finite. Now we see that the set

\[ Z(X) \cap \sqrt{\mathcal{V}E(X)} \setminus H(X) \subseteq \bigcup_{e \in B} Z(X) \cap \sqrt{\mathcal{P}_e} \setminus H_e \]

is finite. \( \square \)

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