A Sharp Decay Estimate for Degenerate Oscillatory Integral Operators Using Broad-Narrow Method

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Abstract
We use broad-narrow method to establish the sharp $L^4$ decay estimate for a class of degenerate oscillatory integral operators in $(2 + 1)$ dimensions. Especially, the model phase function is

$$x^2t + yt^2,$$

a cubic homogeneous polynomial which is degenerate in the sense of Tang (Forum Math 18:427–444, 2006).

Keywords Degenerate oscillatory integrals · Sharp decay · Broad-narrow method

Mathematics Subject Classification 42B20 · 47G10

1 Introduction

It is always a longstanding problem to determine the decay rate for integrals with integrands containing oscillatory elements. For the scalar oscillatory integral

$$I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda S(x)} \psi(x) dx,$$

we are interested in determining the optimal decay rate whenever the phase function satisfies certain assumptions. The stationary phase method tells that if the critical points of $S(x)$ are nondegenerate, then we have the sharp decay estimate

$$|I(\lambda)| \leq C \lambda^{-\frac{d}{2}}.$$
It should be pointed out that the implicit constant $C$ depends on the support function $\psi$ and the phase function $S$. It strongly demonstrates the "local" characteristic contrast to the "global" van der Corput-type estimates. Readers may refer to [3, 10] for more comprehensive background. It is natural that we can further consider the oscillatory integrals with degenerate phases. If the phase function is real-analytic, Arnold posed the hypothesis that the sharp decay rate is determined by the Newton distance of the phase. This was confirmed by [30] in which the phase functions are assumed to satisfy certain nonsingular conditions.

As was generalized to the operator setting, we call the operators of the form

$$T_\lambda f(x) := \int_{\mathbb{R}^d} e^{i\lambda S(x,y)} \psi(x, y) f(y) dy, \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}^d,$$

as $(d_X + d_Y)$-dimensional operators. If the phase function $S$ is nondegenerate in the sense that the Hessian does not vanish on the support of $\psi$, then Hörmander’s lemma [18] gives the optimal decay estimate whenever $d_X = d_Y = d$. The model phase is $S(x, y) = x \cdot y$ which corresponds to Fourier transform. In fact, there is another geometric understanding for Hörmander’s lemma. Based on the phase function, we can write the canonical relation

$$C_S := \{(x, \xi; y, \eta)\} = \{(x, \nabla_x S; y, -\nabla_y S)\} \subset T^*\mathbb{R}^d \times T^*\mathbb{R}^d,$$

and view this as a Lagrangian submanifold when endowed with a symplectic form $dx \wedge d\xi - dy \wedge d\eta$. Define the left and right projection mapping, respectively, as

$$\pi_L : C_S \longrightarrow (x, \xi), \quad \pi_R : C_S \longrightarrow (y, \eta).$$

In the language of geometry, Hörmander’s lemma says that if both $\pi_L$ and $\pi_R$ are local diffeomorphisms, then $\|T_\lambda\|_{L^2 \rightarrow L^2}$ has the optimal decay estimate $\lambda^{-d/2}$. It is known that the singularity types of the mappings $\pi_L$ or $\pi_R$ influence the decay rates of $T_\lambda$. For further researches in this direction, see [6, 7, 13–16, 21].

However, a frustrating fact is that even the phase function is a simple homogeneous polynomial, the singularities of the corresponding mappings $\pi_L$ and $\pi_R$ are complicated. One way out of the geometric constraints is to focus on the algebraic and analytic properties, this leads to the thorough understanding of $(1 + 1)$-dimensional operators in the works on $L^2$ mapping properties [11, 22–24, 26] and $L^p$ mapping properties [25, 27, 28, 31, 32].

In the $(2 + 1)$-dimensional case, Tang [29] obtained the (nearly) sharp $L^2$ decay estimates for operators with homogeneous polynomial phases satisfying an algebraic nondegeneracy condition. [12] extended them to higher dimensional cases. However, the general higher dimensional cases are little understood. When $d_X \neq d_Y$, researches about $T_\lambda$ are largely motivated by Fourier restriction (or extension) phenomenon initially raised by Stein in 1960s. For instance, if the underlying geometric object is codimension-1 sphere or paraboloid, it can be generalized to Hörmander-type oscillatory integral operators, and progress can be found in [4, 5, 18, 19]; if the underlying
geometric objects are space curves, for instance, see [8], the generalization to corresponding oscillatory integral operators can be found in [1, 2]. For further research concerning more degenerate phases, we refer to [17].

Since the tools of restriction estimates are fruitful, then we aim to explore the possibility of applying some of these tools to establish sharp $L^2$ decay estimates for degenerate oscillatory integral operators.

Here, we consider the following $(2+1)$—dimensional operators

$$T_\lambda^\psi f(x, y) = \int_{\mathbb{R}} e^{i\lambda(x^2 t + y t^2)} \psi(x, y, t) f(t) dt.$$

Although the phase function is a simple cubic homogeneous polynomial, it does not satisfy the Carleson-Sjölin condition and is also outside scope of [29]. We use the broad-narrow method of Bourgain–Guth [5] and the classical bilinear method dating back to Fefferman [9] to give the following theorem.

**Theorem 1.1** For the operator $T_\lambda^\psi$, we have the sharp $L^4$ decay estimate

$$\|T_\lambda^\psi f\|_{L^4(\mathbb{R}^2)} \lesssim C_\psi \lambda^{-3/8} \|f\|_{L^4(\mathbb{R})}. \quad (1.1)$$

In fact, by simple linear transformation, we can generalize this theorem to more general cases.

**Corollary 1.2** Consider the following operators

$$\tilde{T}_\lambda^\psi f(x, y) = \int_{\mathbb{R}} e^{i\lambda[(ax+by)t^2+(cx+dy)t]2} \psi(x, y, t) f(t) dt,$$

if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is nonsingular, we also have the sharp $L^4$ decay estimate

$$\|\tilde{T}_\lambda^\psi f\|_{L^4(\mathbb{R}^2)} \lesssim C_{\psi, a, b, c, d} \lambda^{-3/8} \|f\|_{L^4(\mathbb{R})}.$$

By Hölder’s inequality, we know that both $T_\lambda^\psi$ and $\tilde{T}_\lambda^\psi$ map $L^p$ into $L^p$ with $\lambda$—free upper bounds for $1 \leq p \leq +\infty$. Therefore, interpolation gives the following results.

**Corollary 1.3** For $p > 4$, we have the sharp $L^p$ decay estimates

$$\|T_\lambda^\psi f\|_{L^p(\mathbb{R}^2)} \lesssim C_{\psi} \lambda^{-3/2p} \|f\|_{L^p(\mathbb{R})},$$

$$\|\tilde{T}_\lambda^\psi f\|_{L^p(\mathbb{R}^2)} \lesssim C_{\psi, a, b, c, d} \lambda^{-3/2p} \|f\|_{L^p(\mathbb{R})}.$$

For $1 \leq p < 4$, we have the $L^p$ decay estimates

$$\|T_\lambda^\psi f\|_{L^p(\mathbb{R}^2)} \lesssim C_{\psi} \lambda^{-1/2 + 4/2p} \|f\|_{L^p(\mathbb{R})},$$

$$\|\tilde{T}_\lambda^\psi f\|_{L^p(\mathbb{R}^2)} \lesssim C_{\psi, a, b, c, d} \lambda^{-1/2 + 1/2p} \|f\|_{L^p(\mathbb{R})}.$$
**Notation:** In this paper, $A \lesssim B$ means that there exists an absolute constant $C$ such that $A \leq CB$.

### 2 Outline of the Proof

The strategy of proving Theorem 1.1 is similar to what have been done to Fourier restriction estimates. We use broad-narrow method to divide $T_\lambda^\psi f(x, y)$ into broad part and narrow part. Then we establish a rescaling lemma to deal with the narrow part. By employing the classical result of Phong–Stein [23] in $(1 + 1)$—dimensional case, we establish a bilinear estimate for the broad part. Last, we combine the argument above and derive an induction relation which leads to the final result.

Before the formal argument, we need some uniform constraints on the support function. Denote the class of functions $U$ by

$$U := \{\psi(x, y, t) : \supp \psi \subset [-1, 1]^3, \left| \frac{\partial_j^j}{\partial t} \psi \right| \leq 1, \quad j = 0, 1, 2, 3\}.$$ 

Based on this, we further write

$$Q_p(\lambda) := \sup_{\psi \in U} \sup_{\|f\|_{L_p(\mathbb{R})} \leq 1} \|T_\lambda^\psi f\|_{L_p(\mathbb{R}^2)}.$$ 

To prove Theorem 1.1, it suffices to show the following result.

**Theorem 2.1**

$$Q_4(\lambda) \lesssim \lambda^{-\frac{3}{8}}.$$ 

To prove this theorem, it suffices to give the following iteration relation.

**Lemma 2.2** For any large number $K > 10^4$, we have

$$Q_4(\lambda) \lesssim K^\frac{9}{8} \lambda^{-\frac{3}{8}} + K^{-\frac{1}{2}} Q_4(\lambda/K).$$

We now explain how Lemma 2.2 implies Theorem 2.1. We write down each step of the iteration

$$Q_4(\lambda) \lesssim K^\frac{9}{8} \lambda^{-\frac{3}{8}} + K^{-\frac{1}{2}} Q_4(\lambda/K),$$

$$Q_4(\lambda/K) \lesssim K^\frac{9}{8} (\lambda/K)^{-\frac{3}{8}} + K^{-\frac{1}{2}} Q_4(\lambda/K^2),$$

$$\ldots$$

$$Q_4(\lambda/K^M) \lesssim K^\frac{9}{8} \left(\lambda/K^M\right)^{-\frac{3}{8}} + K^{-\frac{1}{2}} Q_4(\lambda/K^{M+1}).$$
where $M \approx \log_K \lambda$. This means

$$Q_4(\lambda) \lesssim K^{\frac{9}{8}} \lambda^{-\frac{3}{8}} + K^{-\frac{M}{2}} Q_4(\lambda/K^{M+1})$$

$$\approx K^{\frac{9}{8}} \lambda^{-\frac{3}{8}} + \lambda^{-\frac{1}{2}} Q_4(1)$$

$$\lesssim \lambda^{-\frac{3}{8}}.$$

The last inequality follows from the induction.

It should be pointed out, using induction to prove oscillatory estimates had appeared previously in [2] or even earlier in [20].

### 3 Optimality

Now we give an example to show that the decay estimates given in Theorem 1.1, Corollary 1.2 and the first half part of Corollary 1.3 are sharp.

Suppose $\psi$ is a nonnegative cut-off function and satisfies

$$\psi(x, y, t) = \begin{cases} 0, & |(x, y, t)| \geq 1, \\ 1, & |(x, y, t)| \leq \frac{1}{2}. \end{cases} \quad (3.1)$$

We choose the test function as

$$f(t) = \chi_{[0,1]}(t). \quad (3.2)$$

Assume the priori estimate

$$\|T_\lambda^{\psi} f\|_{L^p(\mathbb{R}^2)} \lesssim C_{\psi} \lambda^{-\delta} \| f \|_{L^p(\mathbb{R})}.$$ 

Then for the specific function (3.2), we see that

$$\left[ \iint \left| \int_0^1 e^{i\lambda(x^2 t + y^2 t^2)} \psi(x, y, t) \, dt \right|^p \, dx \, dy \right]^\frac{1}{p} \lesssim C_{\psi} \lambda^{-\delta}.$$

With the support function (3.1), we have

$$\lambda^{-\frac{3}{2p}} \lesssim \left[ \int_{|y| \leq \lambda^{-1}} \int_{|x| \leq \lambda^{-\frac{1}{2}}} \left| \int_0^1 e^{i\lambda(x^2 t + y^2 t^2)} \psi(x, y, t) \, dt \right|^p \, dx \, dy \right]^\frac{1}{p} \lesssim C_{\psi} \lambda^{-\delta}.$$

Since the inequality holds for arbitrarily large $\lambda$, then it requires

$$\delta \leq \frac{3}{2p}.$$

This ultimately shows the optimality.
4 Proof of Theorem 2.1

Broad-narrow method was introduced by Bourgain–Guth in [5] which efficiently reduces the linear restriction estimates to multilinear estimates which are well understood. As usual, we decompose the test function into $K$ parts

$$f(t) = \sum_{j=0}^{K-1} f_j(t),$$

where $\text{supp } f_j \subset \left[ \frac{j}{K}, \frac{j+1}{K} \right] \cup \left( -\frac{j+1}{K}, -\frac{j}{K} \right)$. For a positive number $\alpha \in (0, 1)$, we say a point $(x, y) \in [-1, 1]^2$ is $\alpha$–broad if

$$\max_j \left| T_\lambda f_j(x, y) \right| \leq \alpha \left| T_\lambda f(x, y) \right|,$$

otherwise, the point $(x, y)$ is called narrow. The choice of $\alpha$ is independent of $K$, it depends on whether we can get the bilinear bound (4.4) or not, actually, $\alpha = 10^{-1}$ suffices.

We write

$$Br_\alpha(T_\lambda f)(x, y) = \begin{cases} \left| T_\lambda f(x, y) \right| & \text{if } (x, y) \text{ is } \alpha \text{– broad;} \\
0 & \text{if } (x, y) \text{ is narrow.} \end{cases}$$

Therefore, pointwise we have

$$\left| T_\lambda f(x, y) \right| \leq Br_\alpha(T_\lambda f)(x, y) + \alpha^{-1} \max_j \left| T_\lambda f_j(x, y) \right|.$$

This implies the following inequality

$$\int \int \left| T_\lambda f(x, y) \right|^4 \, dx \, dy \leq \int \int \left| Br_\alpha(T_\lambda f)(x, y) \right|^4 \, dx \, dy + \alpha^{-4} \sum_{j=0}^{K-1} \int \int \left| T_\lambda f_j(x, y) \right|^4 \, dx \, dy. \tag{4.1}$$

We deal with the first term using bilinear estimates and the latter one using a rescaling estimate we now turn to.

4.1 Rescaling Argument

This part is devoted to proving a degenerate rescaling estimate which is basically the same as the parabolic rescaling estimate. Now we state the main result.
Lemma 4.1

\[
\sup_{\psi \in \mathcal{U}} \| T^\psi_\lambda f_j \|_{L^4(\mathbb{R}^2)} \lesssim K^{-\frac{1}{2}} Q_4(\lambda / K) \| f_j \|_{L^4(\mathbb{R})}. \tag{4.2}
\]

**Proof** For a cut-off function \( \phi \), we may assume \( f_j(t) = f(t)\phi_j(t) \) where \( \phi_j(t) = \phi \left( K \left( t - \frac{j}{K} \right) \right) \), then

\[
T^\psi_\lambda f_j(x, y) = \int_{\mathbb{R}} e^{i\lambda(x^2 t + y^2 t)} \psi(x, y, t) \phi_j(t) f(t) dt,
\]

\[
= \int_{\mathbb{R}} e^{i\lambda(x^2 t + y^2 t)} \psi(x, y, t) \phi \left( K \left( t - \frac{j}{K} \right) \right) f(t) dt
\]

\[
= \int_{\mathbb{R}} e^{i\lambda \left( \frac{x^2 + y^2}{K^2} \right)} \psi(x, y, \frac{s}{K}) \phi(s - j) f \left( \frac{s}{K} \right) ds / K,
\]

\[
= \int_{\mathbb{R}} e^{i\lambda \left( \frac{x^2 + y^2}{K^2} \right)} \psi(x, y, \frac{s}{K}) \phi(s - j) f \left( \frac{s}{K} \right) ds / K.
\]

Set \( \psi_K(x, y, s) = \psi \left( x, y, \frac{s}{K} \right) \) and \( f_j, K(s) = \phi(s - j) f \left( \frac{s}{K} \right) \), it can be verified that \( \psi_K \in \mathcal{U} \). Therefore,

\[
\left| T^\psi_\lambda f_j(x, y) \right| = \frac{1}{K} \left| T^\psi_{\lambda, K} f_j, K(x, y / K) \right|.
\]

So we have

\[
\| T^\psi_\lambda f_j \|_{L^4(\mathbb{R}^2)}^4 = \frac{1}{K^4} \cdot K \| T^\psi_{\lambda, K} f_j, K \|_{L^4(\mathbb{R}^2)}^4
\]

\[
\leq \frac{1}{K^3} Q_4 \left( \frac{\lambda}{K} \right) K \| f_j, K \|_{L^4(\mathbb{R})}^4
\]

\[
= \frac{1}{K^2} Q_4 \left( \frac{\lambda}{K} \right) \| f_j \|_{L^4(\mathbb{R})}^4.
\]

This implies the inequality (4.2). \( \square \)

Consequently, the latter term in RHS of (4.1) is bounded from above by

\[
10^4 K^{-2} Q_4(\lambda / K) \| f \|_{L^4(\mathbb{R})}^4. \tag{4.3}
\]

### 4.2 Bilinear Estimate

We now deal with the first term in RHS of (4.1). With the assumption \( \alpha = 10^{-1} \), we know that for each \( \alpha \)--broad point \( (x, y) \) there exist \( j, k \) with \( |j - k| \geq 2 \) such that

\[
\left| T^\psi_\lambda f(x, y) \right| \leq K \left| T^\psi_\lambda f_j(x, y) \right|^{\frac{1}{2}} \left| T^\psi_\lambda f_k(x, y) \right|^{\frac{1}{2}}. \tag{4.4}
\]
Notice that the indices \( j, k \) depend on the point \((x, y)\), we use summation over all indices to eliminate such dependence. For each \( \alpha \)–broad point \((x, y)\), we always have
\[
|T_\lambda^\psi f(x, y)|^4 \leq K^4 \sum_{|j-k| \geq 2} |T_\lambda^\psi f_j(x, y)|^2 |T_\lambda^\psi f_k(x, y)|^2.
\]
This implies
\[
\iint |B_{\lambda\alpha}(T_\lambda^\psi f)(x, y)|^4 \, dx \, dy \\
\leq K^4 \sum_{|j-k| \geq 2} \iint |T_\lambda^\psi f_j(x, y)|^2 |T_\lambda^\psi f_k(x, y)|^2 \, dx \, dy \\
= K^4 \sum_{|j-k| \geq 2} \iint |T_\lambda^\psi f_j(x, y)T_\lambda^\psi f_k(x, y)|^2 \, dx \, dy \\
= K^4 \sum_{|j-k| \geq 2} \iint \left\| \int \int e^{i\lambda(x^2t+yt^2+x^2s+ys^2)} \psi(x, y, t)\psi(x, y, s)f_j(t)f_k(s) \, dt \, ds \right\|^2 \, dx \, dy.
\]
By changing variables
\[
u = t + s, \quad v = t^2 + s^2,
\]
then
\[
\iint e^{i\lambda(x^2t+yt^2+x^2s+ys^2)} \psi(x, y, t)\psi(x, y, s)f_j(t)f_k(s) \, dt \, ds \\
= \iint e^{i\lambda(x^2u+yv)} \psi(x, y, u)\psi(x, y, s) f_j(t(u, v))f_k(s(u, v)) \frac{du \, dv}{2|t-s|}.
\]
Write
\[
F_{j,k}(u, v) = \frac{f_j(t(u, v))f_k(s(u, v))}{2|t-s|}, \quad \psi(x, y, u, v) = \psi(x, y, t(u, v))\psi(x, y, s(u, v)),
\]
and transform the integral above to
\[
\iint e^{i\lambda(x^2u+yv)} \psi(x, y, u, v)F_{j,k}(u, v) \, du \, dv.
\]
Return to the broad part
\[
\iint |B_{\lambda\alpha}(T_\lambda^\psi f)(x, y)|^4 \, dx \, dy \\
\leq K^4 \sum_{|j-k| \geq 2} \iint \left\| \int \int e^{i\lambda(x^2u+yv)} \psi(x, y, u, v)F_{j,k}(u, v) \, du \, dv \right\|^2 \, dx \, dy.
\]
Observe that the phase function in the integrand can be viewed as separation of variables. Specifically, we write

\[ \int \int e^{i \lambda (x^2 u + y v)} \psi(x, y, u, v) F_{j,k}(u, v) \, du \, dv \]

\[ = \int e^{i \lambda x^2 u} \left[ \int e^{i \lambda y v} \psi(x, y, u, v) F_{j,k}(u, v) \, dv \right] \, du. \]

Iterating the \((1 + 1)-\)dimensional result of Phong–Stein [23], we can see that

\[ \int \int \left| \int \int e^{i \lambda (x^2 u + y v)} \psi(x, y, u, v) F_{j,k}(u, v) \, dv \right|^2 \, du \, dv \]

\[ \leq C \lambda^{-\frac{1}{2}} \int \int \left| \int e^{i \lambda y v} \psi(x, y, u, v) F_{j,k}(u, v) \, dv \right|^2 \, du \, dv \]

\[ \leq C \lambda^{-\frac{1}{2}} \int \int |F_{j,k}(u, v)|^2 \, du \, dv. \]

Thus we arrive at

\[ \int \int |Br_{\alpha}(T_{\psi} f)(x, y)|^4 \, dx \, dy \leq C \lambda^{-\frac{3}{2}} K^4 \sum_{|j-k| \geq 2} \|F_{j,k}(u, v)\|_{L^2(du \, dv)}^2. \quad (4.5) \]

It should be noted that the constant \(C\) depends on the upper bounds of \(\left| \partial_u^j \psi(x, y, u, v) \right| \) and \(\left| \partial_v^j \psi(x, y, u, v) \right| \) for \(j = 0, 1, 2\), by omitting some cumbersome details we can verify that

\[ \left| \partial_u^j \psi(x, y, u, v) \right| \lesssim 1, \quad \left| \partial_v^j \psi(x, y, u, v) \right| \lesssim 1, \]

and this is why the class \(U\) needs third derivatives. Since the supports of \(f_j\) and \(f_k\) are essentially separated, then

\[ \|F(u, v)\|_{L^2(du \, dv)}^2 = \int \int \left| \frac{f_j(t(u, v)) f_k(s(u, v))}{2(t(u, v) - s(u, v))} \right|^2 \, du \, dv \]

\[ = \int \int \left| \frac{f_j(t) f_k(s)}{|2(t - s)|} \right|^2 \, dt \, ds \]

\[ \lesssim \frac{K}{|j - k|} \|f_j\|_{L^2}^2 \|f_k\|_{L^2}^2. \]

So return to (4.5), we know
\[ \sup_{\psi \in U} \int \int \left| \mathcal{B}_\alpha (T_\psi f)(x, y) \right|^4 \, dx \, dy \lesssim \lambda^{-\frac{3}{2}} K^4 \sum_{|j-k| \geq 2} \frac{K}{|j-k|} \| f_j \|_{L^2}^2 \| f_k \|_{L^2}^2 \]
\[ \lesssim \lambda^{-\frac{3}{2}} K^4 \sum_{|j-k| \geq 2} \frac{1}{|j-k|} \| f_j \|_{L^4}^2 \| f_k \|_{L^4}^2 \]
\[ \lesssim \lambda^{-\frac{3}{2}} K^4 \left( \sum_{|j-k| \geq 2} \frac{1}{|j-k|^2} \right)^{\frac{1}{2}} \left( \sum_{|j-k| \geq 2} \| f_j \|_{L^4}^2 \| f_k \|_{L^4}^2 \right)^{\frac{1}{2}} \]
\[ \lesssim \lambda^{-\frac{3}{2}} K^2 \| f \|_{L^4}^4. \quad (4.6) \]

Thus, (4.6) together with (4.3) implies

\[ Q_4^4(\lambda) \lesssim K^{\frac{9}{2}} \lambda^{-\frac{3}{2}} + K^{-2} Q_4^4(\lambda/K). \]

This leads to Lemma 2.2. Thus we complete the proof of Theorem 1.1.

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