Existence of almost periodic solutions to some nonautonomous higher-order stochastic difference equations

Abstract: The paper studies the existence of almost periodic solutions to some nonautonomous higher-order stochastic difference equation of the form:

\[ X(t + n) + \sum_{r=1}^{n-1} A_r(t)X(t + r) + A_0(t)X(t) = f(t, X(t)), \]

\( n \in \mathbb{Z} \), by means of discrete dichotomy techniques.

Keywords: almost periodic sequence, high-order stochastic difference equation, exponential dichotomy

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1 Introduction

The study of almost periodicity which generalizes the notion of periodicity is an area of interest in its own right and has sundry applications in fields like Physics. For a study of almost periodic and almost automorphic sequences we refer the reader to (Bezandry and Diagana [2], Bezandry et al. [3], Corduneanu [4], Diagana, Diagana et al. [6], Han and Hong [8], Hong and Nunez [10]) and references therein. Almost periodicity is also of importance in the study of stochastic processes.

In Bezandry et al. [3], the notion of almost periodicity in mean was introduced and used to study the existence and uniqueness of almost periodic solutions to the stochastic Beverton-Holt equation.

The main motivation of this paper comes from a paper by Diagana [5] in which discrete dichotomy techniques were utilized to find sufficient conditions for the existence of almost automorphic solutions to some higher-order nonautonomous systems of difference equations.

In this paper, we extend Diagana’s results to stochastic case. More precisely, we study the existence of almost periodic solutions to the class of higher-order nonautonomous stochastic difference equations of the form:

\[ X(t + n) + \sum_{r=1}^{n-1} A_r(t)X(t + r) + A_0(t)X(t) = f(t, X(t)), \quad t \in \mathbb{Z}, \tag{1.1} \]

on \( \mathbb{R}^k \), where \( A_r(t), \quad r = 0, \ldots, n - 1 \) are sequences of independent invertible almost periodic \( k \times k \) random matrices, and the forcing term \( f : \mathbb{Z} \times \mathbb{R}^k \to \mathbb{R}^k \) is almost periodic and satisfies the following condition:

(H.O) \( f : (t, x) \to f(t, x) \) is Lipschitz in \( x \in \mathbb{R}^k \) uniformly \( t \in \mathbb{Z} \), that is, there exists \( L > 0 \) such that

\[ E\|f(t, U) - f(t, V)\| \leq L E\|U - V\| \]
for all $U$, $V \mathbb{R}^k$-valued random variables with finite expectation and $t \in \mathbb{Z}$. We assume that for each fixed $r$, the $A_r(t)$'s are independent and independent of $X(0)$. This assumption together with Eq.(1.1) imply that the sequence

$\{(A_0(t), \ldots, A_{n-1}(t))\}_{t \in \mathbb{Z}}$ is independent of the sequence $\{X(t)\}_{t \in \mathbb{Z}}$.

For that, the main idea consists in rewriting Eq.(1.1) as a nonautonomous first-order system of stochastic difference equations on $(\mathbb{R}^k)^n = \mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k$.

Indeed, setting $Z(t) := ((X(t), X(t+1), \ldots, X(t+n-1))^T$, where the symbol $T$ stands for the transpose operation and if $I$ denotes the identity matrix of $\mathbb{R}^k$, then Eq.(1.1) can be rewritten in $(\mathbb{R}^k)^n$ in the following form

$$Z(t+1) = A(t)Z(t) + F(t, Z(t)), \ t \in \mathbb{Z}, \ (1.2)$$

and its corresponding homogeneous equation

$$Z(t+1) = A(t)Z(t), \ t \in \mathbb{Z}, \ (1.3)$$

where $A(t)$ is the family of time-dependent sequence matrices defined by

$$A(t) = \begin{pmatrix}
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
\vdots & \vdots & \vdots & \vdots \\
-A_0(t) & -A_1(t) & \ldots & -A_{n-1}(t)
\end{pmatrix}$$

and the function $F$ appearing in Eq.(1.2) is defined by $F(t, Z) = (0, 0, \ldots, f(t, X))^T$.

The paper is organized as follows. In Section 2, we recall a basic theory of almost periodic random sequences on $\mathbb{Z}$. In Section 3, we apply the techniques developed in Section 2 to find some sufficient conditions for the existence of the almost periodic solution to some semi-linear system of stochastic difference equations. In Section 4, we study some second-order stochastic difference equations to illustrate our main result.

### 2 Preliminaries

In this section we review a basic theory for almost periodic random sequences. To facilitate our task, we first introduce the notations needed in the sequel.

Let $(\mathcal{B}, \| \cdot \|)$ be a Banach space and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space. Throughout the rest of the paper, $\mathcal{Z}$ denotes the set of all integers. Define $L^1(\Omega; \mathcal{B})$ to be the space of all $\mathcal{B}$-valued random variables $V$ such that

$$E\|V\| := \left( \int_D \|V(\omega)\| \,d\mathbf{P}(\omega) \right) < \infty. \quad (2.1)$$

It is then routine to check that $L^1(\Omega; \mathcal{B})$ is a Banach space when it is equipped with its natural norm $\| \cdot \|_1$ defined by, $\|V\|_1 := E\|V\|$ for each $V \in L^1(\Omega; \mathcal{B})$.

Let $X = \{X(t)\}_{t \in \mathbb{Z}}$ be a sequence of $\mathcal{B}$-valued random variables satisfying $E\|X(t)\| < \infty$ for each $t \in \mathcal{Z}$. Thus, interchangeably we can, and do, speak of such a sequence as a function, which goes from $\mathcal{Z}$ into $L^1(\Omega; \mathcal{B})$.

This setting requires the following preliminary definitions.

**Definition 2.1.** An $L^1(\Omega; \mathcal{B})$-valued random sequence $X = \{X(t)\}_{t \in \mathcal{Z}}$ is said to be Bohr almost periodic in mean if for each $\varepsilon > 0$ there exists $N_0(\varepsilon) > 0$ such that among any $N_0$ consecutive integers there exists at least an integer $p > 0$ for which

$$E\|X(t+p) - X(t)\| < \varepsilon, \ \forall \ t \in \mathcal{Z}.$$
An integer \( p > 0 \) with the above-mentioned property is called an \( \varepsilon \)-almost period for \( X \). The collection of all \( \mathbb{B} \)-valued random sequences \( X = \{X(t)\}_{t \in \mathbb{Z}} \) which are Bohr almost periodic in mean is then denoted by \( AP(\mathbb{Z}; L^1(\Omega; \mathbb{B})) \).

**Definition 2.2.** A \( \mathbb{B} \)-valued random sequence \( X = \{X(t)\}_{t \in \mathbb{Z}} \) is said to be almost periodic in probability if for each \( \varepsilon > 0 \), and \( \eta > 0 \) there exists \( N_0(\varepsilon, \eta) > 0 \) such that among any \( N_0 \) consecutive integers there exists at least an integer \( p > 0 \) for which

\[
P\{ \omega \in \Omega : \|X(\omega, t + p) - X(\omega, t)\| > \varepsilon \} < \eta, \ \forall \ t \in \mathbb{Z}.
\]

**Theorem 2.3.** If \( X \) is almost periodic in mean, then it is almost periodic in probability and there also exists a constant \( M > 0 \) such that \( E\|X(t)\| \leq M \) for all \( t \in \mathbb{Z} \). Conversely, if \( X \) is almost periodic in probability and the sequence \( \{\|X(t)\|, \ t \in \mathbb{Z}\} \) is uniformly integrable, then \( X \) is almost periodic in mean.

Let \( k = \{k(i)\}_{i \in \mathbb{Z}} \), and denote \( T_kX(\omega, t) := \lim_{n \to \infty} X(\omega, t + k(i)) \) for each \( \omega \in \Omega \) and each \( t \in \mathbb{Z} \) if it exists.

**Definition 2.4.** A \( \mathbb{B} \)-valued random sequence \( X = \{X(t)\}_{t \in \mathbb{Z}} \) satisfies Bochner’s almost sure uniform double sequence criterion if, for every pair of sequences \( (k) \) and \( (l) \), there exists a measurable subset \( \Omega_1 \subset \Omega \) with \( \mathbb{P}(\Omega_1) = 1 \) and there exist subsequences \( k = \{k(i)\} \subset \{k(i)\} \) and \( l = \{l(i)\} \subset \{l(i)\} \) respectively, with the same indexes (independent of \( \omega \)) such that, for every \( t \in \mathbb{Z} \),

\[
T_kT_lX(\omega, t) = T_{k+l}X(\omega, t), \ \forall \ \omega \in \Omega_1.
\]

(In this case, \( \Omega_1 \) depends on the pair of sequences \( (k) \) and \( (l) \)).

**Theorem 2.5.** The following properties of \( X \) are equivalent:

(i) \( X \) satisfies Bochner’s almost sure uniform double sequence criterion.

(ii) \( X \) is almost periodic in probability.

The proof of the theorem can be seen in Bedouhene et al. [1] for instance.

**Theorem 2.6.** If \( X \) satisfies Bochner’s almost sure uniform double sequence criterion and the sequence \( \{\|X(t)\|, \ t \in \mathbb{Z}\} \) is uniformly integrable, then \( X \) is almost periodic in mean.

Let \( (\mathbb{B}_1, \|\cdot\|_1) \) and \( (\mathbb{B}_2, \|\cdot\|_2) \) be Banach spaces and let \( L^1(\Omega; \mathbb{B}_1) \) and \( L^1(\Omega; \mathbb{B}_2) \) be their corresponding \( L^1 \)-spaces, respectively.

**Definition 2.7.** A function \( F : \mathbb{Z} \times L^1(\Omega; \mathbb{B}_1) \to L^1(\Omega; \mathbb{B}_2) \), \( (t, U) \to F(t, U) \) is said to be almost periodic in mean in \( t \in \mathbb{Z} \) uniformly in \( U \in K \) where \( K \subset L^1(\Omega; \mathbb{B}_1) \) is a compact if for any \( \varepsilon > 0 \), there exists a positive integer \( l(\varepsilon, K) \) such that among any \( l \) consecutive integers there exists at least a integer \( p \) with the following property

\[
E\|F(t + p, U) - F(t, U)\| < \varepsilon
\]

for each random variable \( U \in L^1(\Omega; \mathbb{B}_1) \) and \( t \in \mathbb{Z} \).

Here again, the number \( p \) will be called an \( \varepsilon \)-translation of \( F \) and the set of all \( \varepsilon \)-translations of \( F \) is denoted by \( E(\varepsilon, F, K) \).

Let \( UB(\mathbb{Z}; L^1(\Omega; \mathbb{B})) \) denote the collection of all uniformly bounded \( L^1(\Omega; \mathbb{B}) \)-valued random sequences \( X = \{X(t)\}_{t \in \mathbb{Z}} \). It is then easy to check that the space \( UB(\mathbb{Z}; L^1(\Omega; \mathbb{B})) \) is a Banach space when it is equipped with the norm:

\[
\|X\|_\infty = \sup_{t \in \mathbb{Z}} E\|X(t)\|.
\]

**Lemma 2.8.** \( AP(\mathbb{Z}; L^1(\Omega; \mathbb{B})) \subset UB(\mathbb{Z}; L^1(\Omega; \mathbb{B})) \) is a closed space.
In view of the above, the space \( AP(\mathbb{Z}; L^1(\Omega; \mathbb{B})) \) of almost periodic random sequences equipped with the sup norm \( \| \cdot \|_\infty \) is also a Banach space.

We now state the following composition result.

**Theorem 2.9.** Let \( \| \cdot \| \) be almost periodic in mean in \( t \in \mathbb{Z} \) uniformly in \( U \in L^1(\Omega; \mathbb{B}_1) \). If in addition, \( F \) is Lipschitz in \( U \in \mathbb{K} \), where \( \mathbb{K} \subset L^1(\Omega; \mathbb{B}_1) \) is compact, (That is, there exists \( L > 0 \) such that

\[
E \| F(t, U) - F(t, V) \| \leq M \| U - V \|_1 \quad \forall U, V \in L^1(\Omega; \mathbb{B}_1), \ t \in \mathbb{Z}
\]

then for any almost periodic random sequence \( X = \{X(t)\}_{t \in \mathbb{Z}} \), then the \( L^1(\Omega; \mathbb{B}_1) \)-valued random sequence \( Y(t) = F(t, X(t)) \) is almost periodic in mean.

### 3 Existence of almost periodic solutions

Let \( (\mathcal{L}(\mathbb{B}), \| \cdot \|) \) denote the Banach algebra of bounded linear operators on a Banach space \( \mathbb{B} \) equipped with its operator-norm.

Let \( \{A(t)\}_{t \in \mathbb{Z}} \) be a family of bounded linear invertible operators on \( \mathbb{B} \) and consider the first-order system of stochastic difference equations given by

\[
X(\omega, t + 1) = A(\omega, t)X(\omega, t) + g(\omega, t), \quad t \in \mathbb{Z}, \ \omega \in \Omega
\] (3.1)

where \( g : \Omega \times \mathbb{Z} \rightarrow \mathbb{B} \) is almost periodic in mean, and its corresponding homogeneous equation

\[
X(\omega, t + 1) = A(\omega, t)X(\omega, t), \quad t \in \mathbb{Z}, \ \omega \in \Omega.
\] (3.2)

Our settings requires the following assumptions:

(H.1) \( t \rightarrow A(t) \) is almost periodic in mean.

(H.2) \( t \rightarrow g(t) \) is almost periodic in mean.

The evolution family \( \Phi(t, s) \) associated with Eq. (3.1) is given by

\[
\Phi(t, s) = \begin{cases} 
\prod_{l=s}^{t-1} A(l) & \text{for all } s < t \in \mathbb{Z} \\
\prod_{l=t}^{t-1} A^{-1}(l) & \text{for all } s > t \in \mathbb{Z} 
\end{cases}
\]

and \( \Phi(t, t) = I \).

**Definition 3.1.** Eq.(3.2) is said to have a regular discrete dichotomy if there exist random projections \( P(t) \in \mathcal{L}(\mathbb{B}) \) for all \( t \in \mathbb{Z} \) and positive constants \( M \) and \( \beta \in (0, 1) \) such that the following four conditions are satisfied:

(i) \( A(t)P(t) = P(t + 1)A(t) \);

(ii) The matrix \( A(t)|_{R(I-P(t))} \) is an isomorphism from \( R(I-P(t)) \) onto \( R(I-P(t+1)) \);

(iii) \( \| \Phi(r, t)P(r)X \|_1 \leq M \beta^{t-r} \| X \|_1 \), for \( 0 \leq r \leq t \), \( X \in L^1(\Omega, \mathbb{B}) \);

(iv) \( \| \Phi(r, t)(I-P(t))X \|_1 \leq M \beta^{t-r} \| X \|_1 \), for \( 0 \leq r \leq t \), \( X \in L^1(\Omega, \mathbb{B}) \).

By repeated application of [(i), Definition 3.1]), we obtain

\[
P(t)\Phi(t, s) = \Phi(t, s)P(s) \quad t \geq s.
\]

Define the hull \( H(X) \) of a random sequence \( X \) as follows:
Definition 3.2. The set
\[ H(X) = \{ \tilde{X} \mid \text{there exists a sequence } k \subset \mathbb{Z}, \text{with } T_k X = \tilde{X} \} . \]

Similarly, for a matrix function \( A(n) \), we define
\[ H(A) = \{ \tilde{A} \mid \text{there exists a sequence } k \subset \mathbb{Z}, \text{with } T_k A = \tilde{A} \} . \]

where \( T_k A = \tilde{A} \) means that \( \lim_{i \to \infty} A(t + l(i)) = \tilde{A}(t) \).

Theorem 3.3. Suppose that Eq. (3.2) has a regular discrete dichotomy and \( \tilde{A}(t) \in H(A(t)) \). Then the system
\[ X(t + 1) = \tilde{A}(t)X(t) \]

satisfies a regular discrete dichotomy with same projections and constants.

Let us now state the main results of this paper. For linear stochastic difference equations, we obtain the following theorem.

Theorem 3.4. Under assumptions (H.1)-(H.2), if Eq. (3.2) has a regular discrete dichotomy and \( \tilde{A}(t) \in H(A(t)) \), then Eq. (3.1) has an almost periodic solution given by
\[
\hat{X}(t) = \sum_{r=-\infty}^{t-1} \Phi(t, r + 1)P(r + 1)g(r) - \sum_{r=t}^{\infty} \Phi(t, r + 1)(I - P(r + 1))g(r), \quad (3.3)
\]

Proof. It is not hard to show that \( \hat{X}(t) \) defined by Eq. (3.3) is a solution of Eq. (3.2). Moreover,
\[
E]\|\hat{X}(t)\| \leq \sum_{r=-\infty}^{t-1} E\|\Phi(t, r + 1)P(r + 1)g(r)\| + \sum_{r=t}^{\infty} E\|\Phi(t, r + 1)(I - P(r + 1))g(r)\|
\leq \left\{ \sum_{r=-\infty}^{t-1} M\beta^{t-r-1} + \sum_{r=t}^{\infty} M\beta^{t-r-1} \right\} \sup_{s \in \mathbb{Z}} E\|g(r)\|
\leq M \frac{1 + \beta}{1 - \beta} \|g\|_{\infty}.
\]

This implies that \( \{\|\hat{X}(t)\|, t \in \mathbb{Z}\} \) is uniformly integrable. Now, to prove the almost periodicity of \( \hat{X}(t) \), it suffices by Theorem 2.6 to show that \( \hat{X}(\cdot) \) satisfies Bochner’s almost sure uniform double sequence criterion. To this end, let \( k' = (k'_1) \) and \( \ell' = (\ell'_1) \) be arbitrary sequences of nonnegative integers and then choose a measurable set \( \Omega_1 \subset \Omega \) with \( P(\Omega_1) = 1 \). Let \( k(t) \subset (k'_1) \) and \( \ell(t) \subset (\ell'_1) \) be their common subsequences such that for each \( \omega \in \Omega_1 \), \( (T_{k(t)} A) \omega = (T_{k'_1} A) (T_{\ell(t)} g) \omega \) and \( (T_{k(t)} g) \omega = (T_{k'_1} g) (T_{\ell(t)} g) \omega \). For simplicity, we omit \( \omega \) in what follows. Then we have
\[
\hat{X}(t + k_i) = \sum_{r=-\infty}^{t+k_i-1} \Phi(t + k_i, r + 1)P(r + 1)g(r) - \sum_{r=t+k_i}^{\infty} \Phi(t + k_i, r + 1)(I - P(r + 1))g(r)
\]
\[
= \sum_{r=-\infty}^{t-1} \Phi(t + k_i, r + k_i + 1)P(r + k_i + 1)g(r + k_i) - \sum_{r=t}^{\infty} \Phi(t + k_i, r + k_i + 1)(I - P(r + k_i + 1))g(r + k_i)
\]
\[
= \sum_{r=-\infty}^{t-1} A(t + k_i - 1) \cdots A(r + k_i + 1)P(r + k_i + 1)g(r + k_i) - \sum_{r=t}^{\infty} A(t + k_i - 1) \cdots A(r + k_i + 1)(I - P(r + k_i + 1))g(r + k_i).\]
Thus, taking the limit of the above expression as \(i \to \infty\) and recalling the fact that \(\lim_{i \to \infty} \tilde{X}(t + k_i) = (T_k \tilde{X})(t)\), we can then write

\[
(T_k \tilde{X})(t) = \sum_{r=-\infty}^{t-1} \tilde{A}(t-1) \cdots \tilde{A}(r+1) \tilde{P}(r+1) \tilde{g}(r) \\
- \sum_{r=t}^{\infty} \tilde{A}(t-1) \cdots \tilde{A}(r+1)[I - \tilde{P}(r+1)] \tilde{g}(r) \\
= \sum_{r=-\infty}^{t-1} (T_k \tilde{A})(t-1) \cdots (T_k \tilde{A})(r+1)(T_k \tilde{P})(r+1)(T_k \tilde{g})(r) \\
- \sum_{r=t}^{\infty} (T_k \tilde{A})(t-1) \cdots (T_k \tilde{A})(r+1)[I - (T_k \tilde{P})(r+1)](T_k \tilde{g})(r).
\]

Moreover,

\[
(T_k T_k \tilde{X})(t) = \sum_{r=-\infty}^{t-1} (T_k T_k \tilde{A})(t-1) \cdots (T_k T_k \tilde{A})(r+1)(T_k T_k \tilde{P})(r+1)(T_k T_k \tilde{g})(r) \\
- \sum_{r=t}^{\infty} (T_k T_k \tilde{A})(t-1) \cdots (T_k T_k \tilde{A})(r+1)[I - (T_k T_k \tilde{P})(r+1)](T_k T_k \tilde{g})(r) \\
= (T_k T_k \tilde{X})(t),
\]
as desired. \(\square\)

Consider the semilinear stochastic difference equations given by

\[
Z(\omega, t + 1) = \tilde{A}(\omega, t)Z(\omega, t) + F(t, Z(\omega, t)), \quad t \in \mathbb{Z} , \ \omega \in \Omega
\]
and its corresponding homogeneous equation

\[
Z(\omega, t + 1) = \tilde{A}(\omega, t)Z(\omega, t), \quad t \in \mathbb{Z} , \ \omega \in \Omega,
\]
where \(F : \mathbb{Z} \times \mathbb{B}^n \to \mathbb{B}^n\).

In order to state similar results for the nonlinear case (3.4), we need the following assumption:

(H.3) \(F : (t, w) \to F(t, w)\) is almost periodic in mean in \(t \in \mathbb{Z}\) uniformly in \(w \in \Omega\) where \(\Omega \subset \mathbb{B}^n\) is an arbitrary bounded subset. In addition, we assume that there exists a constant \(L > 0\) such that

\[
E\|F(t, U) - F(t, V)\| \leq L \cdot E\|U - V\|_{\mathbb{B}^n}, \quad \forall \ U, \ V \in L^1(\Omega, 0), \ t \in \mathbb{Z}.
\]

Under these conditions on \(\tilde{A}\) and \(F\), we have the following theorem

**Theorem 3.5.** Under assumptions (H.1)-(H.3), if the linear stochastic difference equation Eq.(3.5) corresponding to Eq.(3.4) has a regular discrete dichotomy with dichotomy constants \(M > 0\) and \(\beta \in (0, 1)\), then Eq.(3.4) has a unique almost periodic solution

\[
Z(t) = \sum_{r=-\infty}^{t-1} \Phi(t, r+1)P(r+1)F(r, Z(r)) - \sum_{r=t}^{\infty} \Phi(t, r+1)(I - P(r+1))F(r, Z(r)),
\]
provided that

\[
ML \frac{\beta + 1}{1 - \beta} < 1.
\]

**Proof.** Consider the Banach space \(AP(\mathbb{Z}; L^1(\Omega, \mathbb{B}^n))\) with the super norm. By Theorem 2.9, if \(\varphi \in AP(\mathbb{Z}; L^1(\Omega, \mathbb{B}^n))\), then \(F(\cdot, \varphi(\cdot)) \in AP(\mathbb{Z}; L^1(\Omega, \mathbb{B}^n))\). Now, define

\[
\gamma : AP(\mathbb{Z}; L^1(\Omega, \mathbb{B}^n)) \to AP(\mathbb{Z}; L^1(\Omega, \mathbb{B}^n))
\]
be the nonlinear operator defined by

$$(\Gamma \varphi)(t) = \sum_{r=-\infty}^{t-1} \Phi(t, r+1)P(r+1)F(r, \varphi(r)) - \sum_{r=t}^{\infty} \Phi(t, r+1)(I - P(r+1))F(r, \varphi(r)),$$

By Theorem 3.4, $\Gamma$ is well defined. Now, let $\varphi, \psi \in AP(\mathbb{Z}; L^1(\Omega, \mathbb{B}))$ having the same property as $Z$ defined in Eq.(3.4). Since $\{A(r), r = 0, \ldots, n-1\}$ are independent and independent of $\varphi$ and $\psi$, one can easily see that

$$E\|\Gamma \varphi - \Gamma \psi\| \leq \sum_{r=-\infty}^{t-1} \left( M\beta^{-r} E\|F(r, \varphi(r)) - F(r, \psi(r))\| \right) + \sum_{r=t}^{\infty} \left( M\beta^{-r} E\|F(r, \varphi(r)) - F(r, \psi(r))\| \right) \leq L \left( \sum_{r=-\infty}^{t-1} M\beta^{-r} + \sum_{r=t}^{\infty} M\beta^{-r} \right) \sup_{r \in \mathbb{Z}} E\|\varphi(r) - \psi(r)\|_{\mathbb{B}^n}. $$

Thus,

$$\|\Gamma \varphi - \Gamma \psi\|_{\infty} \leq ML \frac{\beta + 1}{1 - \beta} \|\varphi - \psi\|_{\infty}. $$

$\Gamma$ is a contraction provided that $ML \frac{\beta + 1}{1 - \beta} < 1$. Using the Banach fixed point theorem, we obtain that $\Gamma$ has a unique fixed point $Z$, which is the unique almost periodic solution of Eq.(3.4).

Let $B = \mathbb{R}^k$ be the k-dimensional space of real numbers equipped with Euclidean topology.

**Corollary 3.6.** Under assumptions (H.0)-(H.1), if the linear stochastic difference equation Eq.(1.3) corresponding to Eq.(1.2) has a discrete regular dichotomy with dichotomy constants $M > 0$ and $\beta \in (0, 1)$, then Eq.(1.1) has a unique almost periodic solution whenever $L$ satisfies (3.7).

**Proof.** Following the same lines as in the proof of Theorem 3.5 it follows that Eq.(1.2) has a unique almost periodic solution given by the mapping

$$t \rightarrow Z(t) := (X(t), X(t+1), X(t+2), \ldots, X(t+n-1))^T$$

whenever $L$ satisfies (3.7). Therefore, Eq.(1.1) has a unique almost periodic solution $t \rightarrow X(t)$ whenever $L$ satisfies (3.7).

## 4 Almost periodic solutions to a second-order stochastic difference equations

Let $B = \mathbb{R}$ the set of real numbers equipped with natural absolute value. To illustrate Corollary 3.6, we study the existence of almost periodic solutions to a second-order stochastic difference equations of the form:

$$X(t+2) + b(t)X(t+1) + a(t)X(t) = f(t, X(t)), \; t \in \mathbb{Z}$$

(4.1)

where the function $f : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ is almost periodic and satisfies

(H.4) The function $(t, x) \rightarrow f(t, x)$ is Lipschitz in $x \in \mathbb{R}$ uniformly in $t \in \mathbb{Z}$, that is, there exists $L > 0$ such that

$$E|f(t, X) - f(t, X')| \leq L E|X - X'|$$

for all $X, X' \in L^1(\Omega, \mathbb{R})$ and $t \in \mathbb{Z}$.

We also assume that the real random variables $a(t)'s, b(t)'s$ appearing in Eq.(4.1) are independent and independent of $X(0)$. This assumption together with Eq.(4.1) imply that the sequence $\{a(t), b(t)\}_{t \in \mathbb{Z}}$ is independent of the sequence $\{X(t)\}_{t \in \mathbb{Z}}$.
Setting \( Z(t) := (X(t), X(t + 1)) \), note that Eq.(4.1) can be rewritten in \( \mathbb{R}^2 \) as follows

\[
Z(t + 1) = A(t)Z(t) + F(t, Z(t)), \quad t \in \mathbb{Z},
\]

and its corresponding homogeneous equation

\[
Z(t + 1) = A(t)Z(t), \quad t \in \mathbb{Z},
\]

where \( A(t) \) is the family of time-dependent sequence matrices defined by

\[
A(t) = \begin{pmatrix}
0 & I \\
-a(t) & -b(t)
\end{pmatrix}
\]

and the function \( F \) appearing in Eq.(4.2) is defined by \( F(t, Z) = (0, f(t, X))^T \).

We adopt the following assumptions:

(H.5) The sequences \( a, b : \mathbb{Z} \rightarrow \mathbb{R} \) are periodic in the following sense: there exists \( T \in \mathbb{Z} \) such that

\[
a(t + T) = a(t) \quad \text{and} \quad b(t + T) = b(t)
\]

for all \( t \in \mathbb{Z} \), almost surely.

(H.6) There exist \( a_0, b_0 > 0 \) such that

\[
\inf_{t \in \mathbb{Z}} a(t) = a_0 \quad \text{and} \quad \inf_{t \in \mathbb{Z}} b(t) = b_0,
\]

almost surely.

(H.7) \( b(t) \neq 2 \sqrt{a(t)} \) for all \( t \in \mathbb{Z} \), almost surely.

Next, we show that Eq.(4.3) has a regular discrete dichotomy. For that, let’s compute the eigenvalues of \( A(t) \).

\[
P_1(\lambda) = \det(A(t) - \lambda I_{\mathbb{R}^2}) = \lambda^2 + b(t)\lambda + a(t)
\]

for all \( t \in \mathbb{Z} \). Then, the characteristic equation is given by

\[
\lambda^2 + b(t)\lambda + a(t) = 0
\]

with discriminant given by \( D(t) = b^2(t) - 4a(t) \) for all \( t \in \mathbb{Z} \).

Clearly, (H.7) yields either \( D(t) > 0 \) or \( D(t) < 0 \) for all \( t \in \mathbb{Z} \).

Under assumptions (H.6)-(H.7), we have:

1. If \( D(t) > 0 \) for all \( t \in \mathbb{Z} \), then the eigenvalues of \( A(t) \) are given by

\[
\lambda_1(t) = -\frac{b(t) + \sqrt{b^2(t) - 4a(t)}}{2}
\]

and

\[
\lambda_2(t) = -\frac{b(t) - \sqrt{b^2(t) - 4a(t)}}{2}.
\]

Moreover, it can be shown easily that \( \lambda_1(t), \lambda_2(t) < 0 \) for all \( t \in \mathbb{Z} \).

2. If \( D(t) < 0 \) for all \( t \in \mathbb{Z} \), then the eigenvalues of \( A(t) \) are given by

\[
\lambda_1(t) = -\frac{b(t) + i \sqrt{4a(t) - b^2(t)}}{2}
\]

and

\[
\lambda_2(t) = -\frac{b(t) - i \sqrt{4a(t) - b^2(t)}}{2}.
\]

Moreover, it can be shown easily that \( \Re \lambda_1(t), \Re \lambda_2(t) < 0 \) for all \( t \in \mathbb{Z} \).
In view of the above, it follows that

\[ \Phi(t, s) = \prod_{r=s}^{t-1} A(r) = A(s)A(s + 1)A(s + 2) \ldots A(t - 2)A(t - 1) \]

for all \((t, s) \in \mathcal{I}\), where \(\mathcal{I} = \{(t, s) \in \mathbb{Z} \times \mathbb{Z} : t \geq s\}\),

has an exponential dichotomy which yields (see Henry [9]) that Eq.\((4.3)\) has a discrete dichotomy.

Also, using \((H.5)\), one can easily show that

\[ A(t + T) = A(t) \]

almost surely which, in turn, implies that

\[ \Phi(t + T, s + T) = \Phi(t, s). \]

for all \((t, s) \in \mathcal{I}\) almost surely.

The techniques used in the proof of Corollary 3.6 allow us to obtain the following.

**Theorem 4.1.** Under assumptions \((H.4)-(H.5)-(H.6)-(H.7)\), if Eq. \((4.3)\) has a regular discrete dichotomy with dichotomy constants \(M > 0\) and \(\beta \in (0, 1)\), Eq.\((4.1)\) has a unique almost periodic solution whenever \(L\) satisfies \((3.7)\).

**Example 4.2.** Let in Eq.\((4.1)\), \(a(t) = \alpha \left[ 2 + \sin \left( \frac{\pi}{2} t \right) \right] \), \(b(t) = \beta \left[ 4 + \cos \left( \frac{\pi}{2} t \right) \right] \), and \(f(t, X) = \xi_t \tan^{-1} X\), where \(\alpha, \beta\) are random variables taking their values in \((0, 1)\), and \(\xi = \{\xi_t\}_{t \in \mathbb{Z}}\) is an almost periodic sequence of real random variables. We assume that \(\alpha, \beta, \) and \(\xi\) are mutually independent and independent of \(X(0)\).

Clearly, \((H.4)-(H.5)-(H.6)-(H.7)\) are satisfied. In addition, \(A\) is periodic with period 4 and \(D(t) = b^2(t) - 4a(t) > 0\) for all \(t \in \mathbb{Z}\). Finally,

\[ E|f(t, X) - f(t, Y)| \leq L E|X - Y| \]

where \(L\) is a positive number such that \(E|\xi_t| \leq L\) for all \(t \in \mathbb{Z}\).

We can now conclude that all hypotheses of Theorem 4.1 are satisfied.

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