FIRST-CLASS APPROACHES TO MASSIVE 2-FORMS

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Abstract
Massive 2-forms are analyzed from the point of view of the Hamiltonian quantization using the gauge-unfixing approach and respectively the Batalin–Fradkin method. Both methods finally output the manifestly Lorentz covariant path integral for 1- and 2-forms with Stueckelberg coupling.

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1 Introduction
Models with p-form gauge fields (antisymmetric tensor fields of various orders) play an important role in string and superstring theory, supergravity and the gauge theory of gravity [1]–[5]. Antisymmetric tensor fields of various orders are included within the supergravity multiplets of many supergravity theories [3]–[4], especially in 10 or 11 dimensions. Moreover, p-forms have a special place in the theory of p-branes [5], where (p+1)-forms couple naturally to p-branes. In fact, it is known that the configuration space for closed p-branes is nothing but the space of all closed p-manifolds embedded in space-time, in which background rank-(p+1) antisymmetric tensor fields should be analyzed in connection with their geometric aspects. Interacting p-form gauge theories have been analyzed from the redundant Hamiltonian BRST point of view in [6], where the ghost and auxiliary field structures required by the antifield BRST formalism are derived. Finally, it is worth to notice that a U(1) gauge theory defined in the configuration space for closed p-branes yields the gauge theory of a massless rank-(p+1) antisymmetric tensor field plus the Stueckelberg formalism for a massive vector field. From the point of view of Hamiltonian second-class constrained systems, the Stueckelberg formalism has been largely used at the quantization of massive vector field following various schemes [7]–[16].

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The main aim of this paper is to quantize massive 2-forms using two different methods: gauge-unfixing [17]–[18] and Batalin–Fradkin [19]–[21]. The first approach (gauge unfixing method) [17]–[18] relies on separating the second-class constraints into two subsets, one of them being first-class and the other providing some canonical gauge conditions for the first-class subset. Starting from the canonical Hamiltonian of the original second-class system, one constructs a first-class Hamiltonian with respect to the first-class subset through an operator that projects any smooth function defined on the phase-space into a function that is in strong involution with the first-class subset. A systematic BRST treatment of the gauge-unfixed method has been realized in Refs. [22] and [23].

The second approach (Batalin–Fradkin method) [19]–[21] relies on enlarging the original phase-space and constructing a first-class constraint set and a first-class Hamiltonian, with the property that they coincide with the original second-class constraints and respectively with the starting canonical Hamiltonian if one sets all the extravariables equal to zero.

This paper is organized in four sections. In Section 2 we start from a bosonic second-class constrained system and briefly expose the above mentioned methods of constructing first-class systems equivalent with the original theory. In Section 3 we apply both methods to massive 2-forms and meanwhile obtain the path integrals corresponding to the first-class systems associated with this model. After integrating out the auxiliary fields and performing some field redefinitions, we discover nothing but the manifestly Lorentz covariant path integrals corresponding to the Lagrangian formulation of the first-class systems, which reduce to the Lagrangian path integral for Stückelberg-coupled 1- and 2-forms. Section 4 ends the paper with the main conclusions.

2 First-Class Approaches to Second-Class Constrained Systems

The starting point is a bosonic dynamic system with the phase-space locally parameterized by \( n \) canonical pairs \( z^a = (q^i, p_i) \), endowed with the canonical Hamiltonian \( H_c \), and subject to the purely second-class constraints

\[
\chi_{\alpha_0} (z^a) \approx 0, \quad \alpha_0 = \frac{1}{2M_0},
\]

where “≈” represents the weak equality symbol. The idea is to associate a first-class system with the original second-class theory that satisfies the following requirements: its number of physical degrees of freedom coincides with that of the original second-class theory, the algebras of classical observables are isomorphic, the first-class Hamiltonian (governing the dynamics of the first-class system) restricted to the constraint surface (1) reduces to the original canonical Hamiltonian \( H_c \). The construction of such a first-class system, equivalent to a given, second-class one, can proceed in several ways. As announced in the introduction, we chose two of them. One is based on interpreting the second-class constraint set as stemming from the gauge-fixing of a first-class constraint
set [17–18] and the other on enlarging (in an appropriate manner) the original phase-space and constructing a first-class constraint sets that reduces to (11) in the zero limit of all extravariables [19–21].

2.1 Gauge unfixing (GU) method

Assume that one can split the second-class constraint set (11) into two subsets with equal numbers of independent constraint functions

\[ \chi_{\alpha_0}(z^a) \equiv (G_{\bar{\alpha}_0}(z^a), \bar{C}_{\bar{\beta}_0}(z^a)) \approx 0, \quad \bar{\alpha}_0, \bar{\beta}_0 = \frac{1}{2}, M_0, \]  

(2)
such that

\[ [G_{\bar{\alpha}_0}, G_{\bar{\beta}_0}] = D_{\bar{\alpha}_0\bar{\beta}_0} \bar{G}_{\bar{\alpha}_0}, \]  

(3)
where \( D_{\bar{\alpha}_0\bar{\beta}_0} \) may in principle be functions of \( z^a \). Hereafter the square brackets, [ , ], denote the Poisson brackets on the phase space of the theory. On the one hand, relations (3) yield the subset

\[ G_{\bar{\alpha}_0}(z^a) \approx 0 \]  

(4)
to be first-class. On the other hand, the second-class behaviour of the overall constraint set ensures that

\[ C_{\bar{\alpha}_0}(z^a) \approx 0 \]  

(5)
may be regarded as some gauge-fixing conditions for this first-class set. It is possible to construct a first-class Hamiltonian with respect to (4) with the help of an operator \( \hat{X} \) [24–25] that associates with every smooth function \( F \) on the original phase-space an application \( \hat{X}F \), which is in strong involution with the functions \( G_{\bar{\alpha}_0} \),

\[ \hat{X}F = F - C_{\bar{\alpha}_0} [G_{\bar{\alpha}_0}, F] + \frac{1}{2} C_{\bar{\alpha}_0} C_{\bar{\beta}_0} [G_{\bar{\alpha}_0}, [G_{\bar{\beta}_0}, F]] - \cdots, \]  

(6)
\[ [\hat{X}F, G_{\bar{\alpha}_0}] = 0. \]  

(7)
If we denote by \( S_O \) and \( S_{GU} \) the original and respectively the gauge-unfixed system, then they are classically equivalent since they possess the same number of physical degrees of freedom

\[ N_O = \frac{1}{2} (2n - 2M_0) = N_{GU} \]  

(8)
and, moreover, the corresponding algebras of classical observables are isomorphic

\[ \text{Phys} \left( S_O \right) = \text{Phys} \left( S_{GU} \right). \]  

(9)
Consequently, the two systems become also equivalent at the level of the path integral quantization, which allows one to replace the Hamiltonian path integral of the original second-class theory

\[ Z_O = \int D (z^a, \lambda^{\alpha_0}) \det \left( [G_{\bar{\alpha}_0}, C_{\bar{\beta}_0}] \right) \exp \left[ i \int dt \left( \dot{q}^i p_i - H_c - \lambda^{\alpha_0} \chi_{\alpha_0} \right) \right] \]  

(10)
with that of the gauge-unfixed first-class system

\[
Z_{GU} = \int \mathcal{D}(z^\alpha, \lambda^{\bar{\alpha}a}) \left( \prod_{\bar{\alpha}a} \delta(C^{\bar{\alpha}a}) \right) \left( \det \left( [G_{\bar{\alpha}0}, C^{\bar{\alpha}0}] \right) \right) \times \\
\times \exp \left[ i \int dt \left( \dot{q}^i p_i - \dot{X} H_c - \lambda^{\bar{\alpha}0} G_{\bar{\alpha}0} \right) \right].
\] (11)

In the above \( \lambda^{\alpha_0} \) denote the Lagrange multipliers associated with the constraints (2), while \( \lambda^{\bar{\alpha}_0} \) correspond to the first-class subset (4). For concrete models, the argument of the exponential from the path integral (11) may contain other terms as well, such that the integration measure should be accordingly modified [26].

2.2 Batalin-Fradkin (BF) method

The BF approach [19]–[21] to the problem of constructing a first-class system equivalent to the starting second-class one (subject to the second-class constraints (1)) relies on enlarging the original phase-space with \( 2M (M \geq M_0) \) bosonic variables \( (\zeta^\alpha)_{\alpha = 1, 2M} \) and on further extending the Poisson bracket to the newly added variables through the relations

\[
[\zeta^\alpha, z^a] = 0, \quad [\zeta^\alpha, \zeta^{\beta}] = \omega^{\alpha\beta}.
\] (12)

In the above \( \omega^{\alpha\beta} \) are the elements of a quadratic, antisymmetric and invertible matrix, independent of the extended phase-space variables. The elements of its inverse will be denoted by \( \omega_{\alpha\beta} \) in the sequel. The next step is to construct a set of independent, smooth, real functions defined on the extended phase-space, \( (G_A(z, \zeta))_{A = 1, M_0 + M} \), such that it reduces to the original second-class constraint function set \( (\chi_{\alpha_0}(z))_{\alpha_0 = 1, 2M_0} \) in the limit of setting all the extravariables equal to zero

\[
G_{\alpha_0}(z, 0) \equiv \chi_{\alpha_0}(z), \quad \alpha_0 = 1, 2M_0, \quad G_A(z, 0) \equiv 0, \quad A = 2M_0 + 1, M_0 + M,
\] (13) (14)

and, moreover, the functions \( G_A \) are in (strong) involution

\[
[G_A, G_B] = 0, \quad A, B = 1, M_0 + M.
\] (15)

In the last step one generates a smooth, real function, defined on the extended phase-space, \( H_{BF} = H_{BF}(z, \zeta) \), with the properties that \( H_{BF} \) reduces to \( H_c \) in the limit of setting all the extravariables equal to zero

\[
H_{BF}(z, 0) \equiv H_c(z)
\] (16)

and is in involution with the first-class constraint functions \( (G_A)_{A = 1, M_0 + M} \)

\[
[H_{BF}, G_A] = V_A \quad B G_B, \quad A = 1, M_0 + M.
\] (17)
The previous steps unravel a dynamic system subject to the first-class constraints

\[ G_A (z, \xi) \approx 0, \quad A = 1, M_0 + M, \tag{18} \]

whose evolution is governed by the first-class Hamiltonian \( H_{BF} = H_{BF} (z, \xi) \). Denoting by \( S_{BF} \) the BF first-class system, it follows that it is classically equivalent to the original theory \( S_O \) since both of them display the same number of physical degrees of freedom

\[ N_O = \frac{1}{2} (2n - 2M_0) = \frac{1}{2} [2n + 2M - 2 (M_0 + M)] \equiv N_{BF} \tag{19} \]

and, in addition, the corresponding algebras of classical observables are isomorphic

\[ \text{Phys} (S_O) = \text{Phys} (S_{BF}). \tag{20} \]

In turn, the above isomorphism renders the two systems equivalent also at the level of the path integral quantization and hence allows the replacement of the Hamiltonian path integral for the original second-class theory with that of the BF first-class system.

### 3 The Model

We start from the Lagrangian action of massive 2-forms in \( D \) space-time dimensions (\( D \geq 3 \)) \[6\]–\[27\]

\[ S^L_0 [A_{\mu \nu}] = \int d^D x \left( -\frac{1}{12} F_{\mu \nu \rho} F^{\mu \nu \rho} - \frac{m^2}{4} A_{\mu \nu} A^{\mu \nu} \right), \tag{21} \]

with \( m \) the mass of \( A_{\mu \nu} \) and \( F_{\mu \nu \rho} \) the field strength of the 2-form, defined in the standard manner as \( F_{\mu \nu \rho} = \partial_\mu A_{\nu \rho} \equiv \partial_\mu A_{\rho \nu} + \partial_\rho A_{\nu \mu} + \partial_\nu A_{\rho \mu} \). Everywhere in this paper the notation \( \mu \nu \ldots \rho \) signifies complete antisymmetry with respect to the (Lorentz) indices between brackets, with the conventions that the minimum number of terms is always used and the result is never divided by the number of terms. We work with the Minkowski metric tensor of ‘mostly minus’ signature \( \sigma^{\mu \nu} = \sigma^{\nu \mu} = \text{diag} (+ - - -) \). In the sequel we denote by \( \pi^{\mu \nu} \) the canonical momenta respectively conjugated with \( A_{\mu \nu} \). For definiteness, we work with the non-vanishing fundamental Poisson brackets

\[ [A_{\alpha i} (x), \pi_{0 j} (y)]_{x^0 = y^0} = \delta_i^j \delta (x - y), \tag{22} \]

\[ [A_{ij} (x), \pi_{kl} (y)]_{x^0 = y^0} = \frac{1}{2} \delta [i] \delta [k] \delta (x - y). \tag{23} \]

By performing the canonical analysis of this model \[28\]–\[29\], there result the constraints

\[ \chi^{(1)i} = \pi^{0i} \approx 0, \tag{24} \]

\[ \chi^{(2)} = 2 \partial_\nu \pi_{\nu j} - m^2 A_{0i} \approx 0. \tag{25} \]
along with the canonical Hamiltonian

$$H_c(x^0) = \int d^{D-1}x \left(-\pi_{ij}\pi^{ij} + \frac{1}{12} F_{ij}F^{ij} + \frac{m^2}{4} A_{\mu\nu} A^{\mu\nu} - 2A_0\partial_j\pi^{ji}\right).$$

The constraints (24) and (25) are second-class and irreducible (see Ref. [30], Chapter 1, Subsection 1.3.4), with the matrix of the Poisson brackets among the constraint functions expressed by

$$\left(\chi_{\alpha_0}(x),\chi_{\beta_0}(y)\right)_{x^0=y^0} = \begin{pmatrix} 0 & m^2\delta^i_j \\ -m^2\delta^j_i & 0 \end{pmatrix} \delta(x-y),$$

so the matrix (27) is invertible. The number of physical degrees of freedom per space point is equal to $N_O = (D-1)(D-2)/2$.

### 3.1 GU method

According to the GU method exposed in subsection 2.1, one may consider either of the constraints (24) or (25) as the first-class constraint set and the remaining constraints (25 or respectively 24) as the corresponding canonical gauge-fixing conditions. The first choice (24) are first-class and (25) their associated gauge-fixing conditions) yields a path integral that cannot be written in a manifestly covariant form. This can be shown for instance along a line similar to that employed in [31] with respect to the Proca field, and therefore we will avoid this choice. Thus, we adhere to the second choice and redefine the first-class constraints (25) as

$$G^i \equiv -\frac{1}{m^2} \left(2\partial_j\pi^{ji} - m^2 A^{0i}\right) \approx 0.$$ 

The first-class Hamiltonian with respect to (28) follows from relation (0), with $H_c$ expressed by (25), and reads as

$$\hat{X}H_c(y^0) = H_c(y^0) - \int d^{D-1}y \chi^{(1)}_i(y) \left[G^i(y),H_c(y^0)\right] + \frac{1}{2} \int d^{D-1}y d^{D-1}z \chi^{(1)}_i(y) \chi^{(1)}_j(y^0,z) \left[G^i(y),G^j(y^0,z),H_c(y^0)\right] - \cdots$$

$$= H_c(y^0) - \int d^{D-1}y \left[\pi_{0j}\partial_j A^{0i} - \frac{1}{2m^2} \partial_i\pi_{j0}\partial^i\pi^{j0}\right].$$

Clearly, the first-class constraint set (28) is irreducible (all the equations are independent). This ends the GU procedure. In the sequel we will improve it by passing to another first-class system (equivalent with the original, second-class one at both classical and path integral levels) such that the corresponding path integral takes a manifestly Lorentz covariant form.

It is well known that any irreducible set of constraints can always be replaced by a reducible one by introducing constraints that are consequences of the ones
already at hand (see Ref. [30], Chapter 1, Subsection 1.1.8). In view of this, we supplement (28) with one more constraint, \( G ≡ −m^2 \partial_i G^i ≈ 0 \), such that the new constraint set

\[
G^i ≡ −\frac{1}{m^2} (2\partial_j \pi^{ji} − m^2 A^{0i}) ≈ 0, \quad (30)
\]

\[
G ≡ −m^2 \partial_i A^{0i} ≈ 0 \quad (31)
\]

remains first-class and, moreover, becomes off-shell first-order reducible. This means that there exists a single relation among the constraint functions involved in (30) and (31) which is strongly equal to zero. In other words, if we organize the constraint functions (30) and respectively (31) into a column vector \( G^\kappa \), then there exists a row vector \( Z^\kappa \) (first-order reducibility functions) such that \( Z^\kappa G^\kappa = 0 \) in condensed De Witt notations. Indeed, it is simple to check that one can choose

\[
(Z^\kappa) = \left( \partial_i \frac{1}{m^2} \right). \quad (32)
\]

Obviously, (29) is still a first-class Hamiltonian with respect to the reducible first-class constraint set (30) and (31). This procedure preserves the classical equivalence with the first-class theory from the GU method since it merely adds to it a combination of existing first-class constraints, so it does not change either the number of physical degrees of freedom or the classical observables, and keeps the first-class Hamiltonian, such that the evolution is not affected. As a result, the GU and first-order reducible first-class systems remain equivalent also at the level of the Hamiltonian path integral quantization. This further implies, given the established equivalence between the GU first-class system and the original second-class theory, that the first-order reducible first-class system is completely equivalent with the original second-class theory.

At this stage, it is useful to make the canonical transformation

\[
A_{0i} \rightarrow −\frac{1}{m^2} \Pi_i, \quad \pi^{0i} \rightarrow m^2 B^i, \quad (33)
\]

which induces the non-vanishing Poisson brackets

\[
\left[ B^i (x), \Pi_j (y) \right]_{x^0 = y^0} = \delta^i_j \delta (x − y). \quad (34)
\]

It is important to remark that canonical transformations do not change either the first-class behaviour or the reducibility. Consequently, the constraints (30) and (31) become

\[
G^i ≡ −\frac{1}{m^2} (2\partial_j \pi^{ji} + \Pi^i) ≈ 0, \quad (35)
\]

\[
G ≡ \partial_i \Pi^i ≈ 0 \quad (36)
\]

and remain first-class, while the first-class Hamiltonian (29) takes the form

\[
H_{GU} (y^0) = \int d^{D−1} y \left[ −\pi_{ij} \pi^{ij} + \frac{1}{12} F_{ijk} F^{ijk} + \frac{m^2}{4} A_{ij} A^{ij} + \frac{m^2}{2} A_{ij} \partial_j B^i \right]
\]
and is of course a first-class Hamiltonian with respect to (35) and (36). In addition, (32) remain first-order reducibility functions for the constraint set (30) and (31).

Due to the equivalence between the first-order reducible first-class system and the original second-class theory argued previously, one can replace the Hamiltonian path integral of massive 2-forms with that associated with the reducible first-class system. The first-class Hamiltonian (37) outputs the argument of the exponential from the Hamiltonian path integral of the reducible first-class system as

\[
S_{GU} = \int d^Dx \left[ (\partial_t A_{ij}) \pi^{ij} + (\partial_t B_i) \Pi^i + \pi_{ij} \pi^{ij} - \frac{1}{12} F_{ijk} F^{ijk} - \frac{m^2}{4} A_{ij} A^{ij} \right.
\]

\[
- \frac{m^2}{2} A_{ij} \partial^i B^j - \frac{m^2}{4} \partial_t B_{ij} \partial^i B^j + \frac{1}{2m^2} \Pi_i \Pi^i - \frac{1}{m^2} \Pi_i \left( 2 \partial_j \pi^{ij} + \Pi^i \right)
\]

\[
+ \frac{1}{m^2} \lambda_i \left( 2 \partial_j \pi^{ij} + \Pi^i \right) - \lambda \left( \partial_i \Pi^i \right) \right],
\]

where \( \lambda_i \) and \( \lambda \) denote the Lagrange multipliers respectively corresponding to the first-class constraints (30) and (31). If we perform the transformation

\[
\Pi^i \rightarrow \bar{\Pi}^i, \quad \lambda_i \rightarrow \bar{\lambda}_i = \lambda_i - \Pi_i
\]

in the path integral, the argument of the exponential becomes

\[
S'_{GU} = \int d^Dx \left[ (\partial_t A_{ij}) \pi^{ij} + (\partial_t B_i) \Pi^i + \pi_{ij} \pi^{ij} - \frac{1}{12} F_{ijk} F^{ijk} - \frac{m^2}{4} A_{ij} A^{ij} \right.
\]

\[
- \frac{m^2}{2} A_{ij} \partial^i B^j - \frac{m^2}{4} \partial_t B_{ij} \partial^i B^j + \frac{1}{2m^2} \Pi_i \Pi^i
\]

\[
+ \frac{1}{m^2} \bar{\lambda}_i \left( 2 \partial_j \pi^{ij} + \Pi^i \right) - \bar{\lambda} \left( \partial_i \Pi^i \right) \right].
\]

At this stage, the reducible first-class system is endowed with the Hamiltonian path integral

\[
Z_{GU} = \int \mathcal{D} \left( \text{fields} \right) \mu ([A_{ij}], [B_i]) \exp \left( iS'_{GU} \right),
\]

where by ‘\text{fields}’ we denoted the present fields, the associated momenta and the Lagrange multipliers, and by ‘\( \mu ([A_{ij}], [B_i]) \)’ the integration measure associated with the model subject to the reducible first-class constraints (30) and (31). This measure includes some suitable canonical gauge conditions, is independent of gauge-fixing conditions [32] and is chosen such that (11) is convergent [26]. A set of canonical gauge conditions associated to the first-class constraints (30) and (31) reads as

\[
\bar{C}_i \equiv \partial^j A_{ji} + B_i \approx 0,
\]
\[ \bar{C} \equiv \partial^i B_i \approx 0. \] (43)

In order to infer from (41) a path integral that leads, after integrating out the auxiliary variables, a manifestly Lorentz covariant functional in its exponential, we enlarge the original phase-space with the Lagrange multipliers \( \lambda \) and \( \lambda \) respectively associated with the first-class constraints (30) and (31) (Chapter 11, Subsection 11.3.2) and with their canonical momenta \( p^i \) and \( p \).

We add the constraints

\[ p^i \approx 0, \quad p \approx 0, \] (44)

such that the constraint set (40), (31), and (44) is again first-class and off-shell first-order reducible. Adding the supplementary first-class constraints (44) does not alter the established equivalence with the original second-class theory. Consequently, the argument of the exponential from the Hamiltonian path integral for the first-class theory with the phase-space locally parameterized by the fields/momenta \( \{ A_{ij}, B_i, \lambda_i, \lambda, \pi^{ij}, \Pi^i, p^i, p \} \) and subject to the first-class constraints (30), (31), and (44) reads as

\[
S''_{GU} = \int d^Dx \left[ (\partial_0 A_{ij}) \pi^{ij} + (\partial_0 B_i) \Pi^i + (\partial_0 \lambda_i) p^i + (\partial_0 \lambda) p + \pi_{ij} \pi^{ij} - \frac{1}{12} F_{ijk} F^{ijk} - \frac{m^2}{4} A_{ij} A^{ij} - \frac{m^2}{2} A_{ij} \partial^{[i} B^{j]} - \frac{m^2}{4} \partial_{[i} B_{j]} \partial^{[i} B^{j]} \right.
\]
\[
+ \frac{1}{2m^2} \pi_{ij} \pi^{ij} + \frac{1}{m^2} \lambda_i (2 \partial_i \pi^{ij} + \Pi^i) - \lambda (\partial_i \Pi^i - \Lambda_i p^i - \Lambda p) \right]. \quad (45)
\]

Performing in (45) the integration over \( \pi^{ij}, \Pi^i, p^i, p, \Lambda_i, \) and \( \Lambda, \) the argument of the exponential from the Hamiltonian path integral becomes

\[
S''_{GU} = \int d^Dx \left[ -\frac{1}{12} F_{ijk} F^{ijk} - \frac{m^2}{4} A_{ij} A^{ij} - \frac{m^2}{2} A_{ij} \partial^{[i} B^{j]} - \frac{m^2}{4} \partial_{[i} B_{j]} \partial^{[i} B^{j]} \right.
\]
\[
- \frac{1}{4} \left( \partial_0 A_{ij} - \frac{1}{2m} \partial_{[i} \lambda_{j]} \right) \left( \partial^0 A^{ij} - \frac{1}{2m^2} \partial_{\lambda_i} \lambda_j \right) - \lambda_i (\partial^0 \Pi^i + \lambda \partial^i) \\
- \frac{1}{2m^2} \pi_{ij} \pi^{ij} - \frac{m^2}{2} (\partial_0 B_i + \partial_i \lambda) (\partial^0 B^i + \partial^i \lambda) \right]. \quad (46)
\]

If we make the notations

\[ \frac{1}{m^2} \lambda_i \equiv -\tilde{A}_{i0}, \quad \lambda \equiv -B_0, \] (47)

then (46) can be written as

\[
S''_{GU} = \int d^Dx \left[ -\frac{1}{12} F_{ijk} F^{ijk} - \frac{m^2}{4} A_{ij} A^{ij} - \frac{m^2}{2} A_{ij} \partial^{[i} B^{j]} - \frac{m^2}{4} \partial_{[i} B_{j]} \partial^{[i} B^{j]} \right.
\]
\[
- \frac{1}{4} \left( \partial_0 A_{ij} + \partial_{[i} \tilde{A}_{j]} \right) \left( \partial^0 A^{ij} + \partial^i \tilde{A}^{j0} \right) - m^2 \tilde{A}_{i0} \left( \partial^0 B^i - \partial^0 B^i \right) \\
- \frac{m^2}{2} \tilde{A}_{i0} \tilde{A}^{i0} - \frac{m^2}{2} (\partial_0 B_i - \partial_i B_0) \left( \partial^0 B^i - \partial^0 B^i \right) \right], \quad (48)
\]
or, equivalently, as
\[
S''_{GU} = \int d^Dx \left[ -\frac{1}{12} F_{ijk} F^{ijk} - \frac{1}{4} \tilde{F}_{0ij} F^{0ij} - \frac{m^2}{4} A_{ij} \tilde{A}^{ij} - \frac{m^2}{2} \tilde{A}_{i0} \tilde{A}^{i0} \right. \\
- \frac{m^2}{2} A_{ij} F^{ij} - m^2 \tilde{A}_{i0} F^{i0} - \frac{m^2}{4} F_{ij} F^{ij} - \frac{m^2}{2} F_{0i} F^{0i} \right],
\]
where
\[
\tilde{F}_{0ij} = \partial_0 A_{ij} + \partial_i \tilde{A}_{j0}, \\
F_{ij} = -\frac{1}{m} \partial_i B_{j}, \\
F_{0i} = -\frac{1}{m} (\partial_0 B_i - \partial_i B_0).
\]
The functional (49) associated with the reducible first-class system takes now a manifestly Lorentz covariant form
\[
S''_{GU} [\tilde{B}_\mu, \tilde{A}_{\mu\nu}] = \int d^Dx \left[ -\frac{1}{12} \tilde{F}_{\mu\nu\rho} \tilde{F}^{\mu\nu\rho} - \frac{1}{4} (F_{\mu\nu} - m \tilde{A}_{\mu\nu}) (F^{\mu\nu} - m \tilde{A}^{\mu\nu}) \right],
\]
with
\[
\tilde{A}_{\mu\nu} = -\tilde{A}_{\nu\mu}, \\
\tilde{A}_{\mu\nu} \equiv (A_{0j}, A_{jk}), \\
\tilde{F}_{\mu\nu\rho} = \partial_{[\mu} A_{\nu\rho]}, \\
\tilde{B}_\mu = -\frac{1}{m} B_\mu, \\
F_{\mu\nu} = \partial_{[\mu} B_{\nu]}.
\]
and describes precisely the (Lagrangian) Stückelberg coupling between the one-form \( \tilde{B}_\mu \) and the two-form \( \tilde{A}_{\mu\nu} \).

3.2 BF method

In the sequel we apply the BF method exposed in subsection 2.2 to massive 2-forms. In view of this, we enlarge the original phase-space by adding the bosonic fields/momenta \((B^\mu, \Pi_\mu)_{\mu=0,1,2,3}\), endowed with the non-vanishing Poisson brackets
\[
[B^\mu(x), \Pi_\nu(y)]_{x^0=y^0} = \delta^\mu_\nu \delta (x - y).
\]
The constraints (18) gain in this case the concrete form
\[
G^A \equiv (G^{(1)}_j, G^{(2)}_j, G) \approx 0,
\]
where
\[
G^{(1)}_j \equiv \chi^{(1)}_j + m B^j \approx 0, \\
G^{(2)}_j \equiv \chi^{(2)}_j - m \Pi_j \approx 0, \\
G \equiv \Pi_0 \approx 0.
\]
It is easy to check that they form an Abelian and irreducible first-class constraint set. The first-class Hamiltonian complying with the general requirements (16) and (17) is expressed by
\[
H_{BF}(x^0) = H_c(x^0) + \int d^{D-1}x \left[ \frac{1}{2} \Pi^i \Pi_i - \frac{1}{m} \Pi^j \chi^{(2)}_j \right]
\]

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Consequently, the Hamiltonian gauge algebra relations (59) are given by

\[
[H_{BF}(x^0), G^{(1)}_j(x)] = 0 = [H_{BF}(x^0), G^{(2)}_j(x)],
\]

(60)

\[
[H_{BF}(x^0), G(x)] = \frac{1}{m} \partial^j G^{(2)}_j(x).
\]

(61)

In the following we analyze the Hamiltonian path integral for the above BF first-class system, equivalent with that of massive 2-forms. Imposing some appropriate gauge-fixing conditions

\[
C_A \equiv (C^{(1)}_j, C^{(2)}_j, C) \approx 0,
\]

(62)

the Hamiltonian path integral takes the form

\[
Z_{BF} = \int \mathcal{D}(A^{\mu\nu}, \pi^{\mu\nu}, B^\mu, \Pi_\mu) \left( \prod_{A,B} \delta \left( G_A \right) \delta \left( C_B \right) \right) (\det ([G_A', C_B'])) \exp (i S_{BF}),
\]

where

\[
S_{BF} = \int d^D x \left( \pi_{0j} \dot{A}_0^j + \pi_{jk} \dot{A}_k^j + \Pi_\mu \dot{B}^\mu - H_{BF} \right).
\]

(63)

In this situation an example of canonical gauge conditions is

\[
C^{(1)}_j \equiv \Pi_j \approx 0, \quad C^{(2)}_j \equiv B^j \approx 0, \quad G \equiv B^0 \approx 0.
\]

(64)

By performance a Fourier representation of the factors \( \delta \left( G_A \right) \) from (62), it becomes

\[
Z_{BF} = \int \mathcal{D}(A^{\mu\nu}, \pi^{\mu\nu}, B^\mu, \Pi_\mu, \lambda^{(1)}_j, \lambda^{(2)}_j, \lambda) \left( \prod_{A} \delta \left( C_A \right) \right) \times
\]

\[
(\det ([G_A', C_B'])) \exp (i S'_{BF}),
\]

(65)

with

\[
S'_{BF} = \int d^D x \left( \pi_{0j} \dot{A}_0^j + \pi_{jk} \dot{A}_k^j + \Pi_\mu \dot{B}^\mu - H_{BF} - 2 \sum_{m=1}^2 \lambda^{(m)j}_j G^{(m)}_j - \lambda G \right).
\]

(66)

Employing in (65) the change of variables

\[
\Pi_j \rightarrow \Pi'_j = \Pi_j + m A_{0j}, \quad \pi_{0j} \rightarrow \pi'_{0j} = \pi_{0j} + m B_j
\]

(67)

and integrating over the momenta \( \pi_{jk} \), the argument of the exponential from the path integral reads as

\[
S''_{BF} = \int d^D x \left[ \pi'_{0j} \dot{A}_0^j + \Pi'_j \dot{B}^j + \Pi_0 \dot{B}^0 - \frac{1}{12} F_{ijk} F^{ijk} - \frac{1}{2} \Pi'_j \Pi'^j \right]
\]

(68)
\[ -\frac{1}{4} \left( A_{ij} - \frac{1}{m} \partial_{[i} \Pi'_{j]} - \partial_{[i} \lambda^{(2)[j]} \right) \left( A_{ij} - \frac{1}{m} \partial^{[i} \Pi^{j]} - \partial^{[i} \lambda^{(2)[j]} \right) \]

\[ -\frac{1}{4} \left( \partial_{[i} B_{j]} - m A_{ij} \right) \left( \partial^{[i} B^{j]} - m A^{ij} \right) + B^0 \partial^i \Pi'_j - \lambda^{(1)} \pi'_{0j} - m \lambda^{(2)} \Pi'_{0j} - \lambda \Pi \right]. \]

Making in the last form of the path integral the change of variables

\[ \Pi'_{j} \rightarrow \tilde{A}_{0j} \equiv \frac{1}{m} \Pi'_j + \lambda^{(2)}_j, \quad \lambda^{(2)}_j \rightarrow \lambda^{(2)}_j \]

and using the notations \[ \text{[53]} \], the argument of the exponential from the path integral is turned into

\[ S''_{BF} = \int d^D x \left[ \pi'_{0j} A^{0j} + m \left( \tilde{A}_{0j} - \lambda^{(2)}_j \right) \tilde{B}^j + \Pi_0 \tilde{B}^0 - \frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} \right. \]

\[ -\frac{1}{4} \left( \partial_{[i} B_{j]} - m A_{ij} \right) \left( \partial^{[i} B^{j]} - m A^{ij} \right) - \frac{m^2}{2} \left( \tilde{A}_{0j} A^{0j} - \lambda^{(2)}_j \lambda^{(2)}_j \right) \]

\[ -m \left( \tilde{A}_{0j} - \lambda^{(2)}_j \right) \partial^j B^0 - \lambda^{(1)} \pi'_{0j} - \lambda \Pi_0 \right]. \] (70)

Finally, we integrate in the path integral over \( \lambda^{(2)}_j, \pi'_{0j}, \lambda^{(1)}_j, \Pi_0, \lambda, \) and \( A^{0j} \), such that the argument of the exponential reduces to

\[ \tilde{S}_{BF} \left[ B_\mu, \tilde{A}_{\mu\nu} \right] = \int d^D x \left[ -\frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} - \frac{1}{4} \left( F_{\mu\nu} - m \tilde{A}_{\mu\nu} \right) \left( F^{\mu\nu} - m \tilde{A}^{\mu\nu} \right) \right. \]

\[ \left. \right], \]

where \( F_{\mu\nu} = \partial_{[\mu} B_{\nu]} \). It is now obvious that the path integral of the BF first-class system takes a manifestly Lorentz covariant form and describes again the (Lagrangian) St"uckelberg coupling between the one-form \( B_\mu \) and the two-form \( A_{\mu\nu} \).

### 4 Conclusion

In this paper we analyzed massive 2-form fields from the point of view of gauge-unfixing and respectively Batalin–Fradkin methods. The first approach (GU) relies on separating the (independent) second-class constraints into two subsets, of which one is first-class and the other a set of canonical gauge conditions. Starting from the original canonical Hamiltonian, we generated a first-class Hamiltonian with respect to the first-class constraint subset. Finally, we built the Hamiltonian path integral of the GU first-class system and then eliminated the auxiliary fields and performed some variable redefinitions such that the path integral finally takes a manifestly Lorentz covariant form. The second approach (BF) involves an appropriate extension of the original phase-space and then the construction of a first-class system on the extended phase-space that reduces to the original, second-class theory in the zero limit of all extravariabes. The
Hamiltonian path integral of the BF first-class system leads again, after integrating out some of the variables and performing some field redefinitions, to a manifestly Lorentz covariant form. It is interesting to remark that both approaches require an appropriate extension of the phase-space in order to render a manifestly covariant path integral. Both procedures allowed the identification of the Lagrangian path integral for Stückelberg-coupled 1- and 2-forms.

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