The Stochastic Strichartz estimates and stochastic nonlinear Schrödinger equations driven by Lévy noise

JIAHUI ZHU\textsuperscript{a}, ZDZISLAW BRZEŽNIK\textsuperscript{b}, WEI LIU\textsuperscript{c,1}

\textsuperscript{a}. School of Science, Zhejiang University of Technology, Hangzhou 310019, China
\textsuperscript{b}. Department of Mathematics, University of York, York YO10 5DD, UK
\textsuperscript{c}. School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, China

Abstract. We establish a general version of the stochastic Strichartz estimate for the stochastic convolution driven by jump noise and prove the existence and uniqueness of global mild solutions to the stochastic nonlinear Schrödinger equation with a nonlinear multiplicative jump noise in the Marcus canonical form.

Keywords. Schrödinger equation, Lévy noise, Stochastic Strichartz estimate, Marcus canonical integral

AMS Subject Classification. 60H15; 60J75; 35B65; 46B09

1. Introduction

In this paper, we study the following stochastic nonlinear Schrödinger equation in Marcus form in $L^2(\mathbb{R}^n)$

\begin{equation}
\begin{aligned}
du(t) &= i(\Delta u(t) - f(u(t)))dt - i\sum_{j=1}^{m} g_j(u(t-)) \diamond dL_j(t), \quad t > 0, \\
u(0) &= u_0,
\end{aligned}
\end{equation}

where $L(t) = (L_1(t), \cdots, L_m(t))$ is an $\mathbb{R}^m$-valued pure jump Lévy process $L(t) = \int_0^t \int_B z\tilde{N}(ds,dz)$, $B = \{x \in \mathbb{R}^m : 0 < |x| \leq 1\}$, with intensity measure $\nu$ and $g(u)h := \sum_{j=1}^{m} g_j(u)h_j$, $h \in \mathbb{R}^m$ and $g_j : \mathbb{C} \to \mathbb{C}$.

The nonlinear Schrödinger equation (NLS) is a fundamental model for describing wave propagation which appears in various fields, including nonlinear optics, nonlinear water propagation, quantum physics, Bose-Einstein Condensate, plasma physics and molecular biology etc. The global existence results of deterministic Schrödinger equation are essentially obtained by a fixed point argument in a suitable mixed space along with Strichartz estimates and conservation laws. In a wide range of physical and engineering models it may be appropriate to incorporate some types of random perturbations which may be caused by the influence of thermal fluctuations or inhomogeneous media etc. In [12] de Bouard and Debussche studied the existence and uniqueness of global $L^2(\mathbb{R}^n)$-valued solutions to the stochastic NLS with linear multiplicative Stratonovich noise. In a subsequent paper [13], de Bouard and Debussche proved the global existence and uniqueness of solutions in the case of a Stratonovich noise with paths in $H^{1,2}(\mathbb{R}^n)$. Brzeźniak and Millet in [9] established a general version of Strichartz estimates for stochastic convolution and proved the global existence and uniqueness to the Stochastic NLS with nonlinear Stratonovich noise in $H^{1,2}(\mathbb{R}^n)$. Brzeźniak and Millet in [9] established a general version of Strichartz estimates for stochastic convolution and proved the global existence and uniqueness to the Stochastic NLS with nonlinear Stratonovich noise in $H^{1,2}(\mathbb{R}^n)$. Brzeźniak and Millet in [9] established a general version of Strichartz estimates for stochastic convolution and proved the global existence and uniqueness to the Stochastic NLS with nonlinear Stratonovich noise in $H^{1,2}(\mathbb{R}^n)$. Brzeźniak and Millet in [9] established a general version of Strichartz estimates for stochastic convolution and proved the global existence and uniqueness to the Stochastic NLS with nonlinear Stratonovich noise in $H^{1,2}(\mathbb{R}^n)$.
Stratonovich noise in $L^2(\mathbb{R}^n)$ for subcritical and critical nonlinearities. By using a modified Faedo-Galerkin method, Brzeźniak et al. [7] constructed a martingale solution for a stochastic Schrödinger equation with multiplicative Wiener noise in an abstract framework and showed the pathwise uniqueness for the 2D manifolds with bounded geometry by means of Strichartz estimates. Barbu et al. [1, 2] proved the global well-posed result of stochastic nonlinear Schrödinger equations with linear multiplicative Wiener noise via the rescaling approach and an application of the Strichartz estimates. See also [3] and [4] for some further studies on this topic.

When studying stochastic Schrödinger equation perturbed by Gaussian noise, Stratonovich form has its merits as the conservation law is still preserved by the solution of the equation. Besides, Stratonovich form has two other important properties. The first is that the Stratonovich form obeys the classical rules of differentiation as in ordinary calculus. The second property is that it is consistent with Wong-Zakai type approximation which is important from a modeling point of view. However, these remarkable properties of Stratonovich form are violated if the driving process includes jumps because the higher order integrals do not vanish in the jump noise case. Marcus in [16, 17] introduced a new type of integral which pertains the same preferable properties as that of Stratonovich integral in the continuous case. We are thus motivated to study stochastic NLS of the Marcus canonical form (1.1). One most delightful property of the Marcus canonical form in (1.1) is that it allows the equation to preserve the $L^2(\mathbb{R}^n)$-norm of the solution. This ensures the non-blow-up of the solution in finite time.

In spite of quite a number of contributions on stochastic NLS with Gaussian noise, the theory is much less well-developed in the case where the driving noise has jumps. Recently de Bouard and Hausenblas in [14] studied the existence of a martingale solution of the stochastic NLS driven by a Lévy-type noise but they considered the case of linear multiplicative noise in $H^{1,2}(\mathbb{R}^n)$. Brzeźniak et al. in [6] constructed a martingale solution of the stochastic NLS with a multiplicative jump noise by using a variant of the Faedo-Galerkin method.

Comparing our work with the last cited paper [6], we would like to point out three main differences. Firstly, the current paper establishes the stochastic Strichartz estimates for the stochastic NLS driven by Poisson random measures and use them to prove the existence and uniqueness of strong solutions to stochastic nonlinear Schrödinger equation, while the other paper proves the existence (but not the uniqueness) of weak martingale solutions to stochastic NLS using the compactness method and the generalization of the Skorokhod-Jakubowski theorem from [5] and [18]. Secondly, the current paper deals with solutions with the initial data from the $L^2$ space, while the other paper deals with solutions with initial data belongs to the energy space $H^1$. Thirdly, the current paper is setup in the whole Euclidean space while the other paper is on a compact Riemannian manifold.

The main common feature of both papers is the use of stochastic equations w.r.t. the Poisson random measure in the Marcus canonical form, in comparison with [14]. An important consequence of this is that the solutions constructed in both papers have their mass, i.e. the $L^2$-norm, a.s. preserved. Let us point out that Brzeźniak and Manna [8] recently proved the existence of a weak martingale solution for a stochastic Landau-Lifshitz-Gilbert equation with pure jump noise also in the Marcus canonical form.

The purpose of this paper is to prove the following global existence and uniqueness result to (1.1) in $L^2(\mathbb{R}^n)$ with the use of a new version of Strichartz inequality for stochastic convolution driven by jump noise.
Theorem 1.1. Let $p \geq 2$, $0 < \sigma < \frac{2}{n}$, $(p, r)$ be an admissible pair with $r = 2\sigma + 2$ and $u_0 \in L^p(\Omega, F_0; L^2(\mathbb{R}^n))$.

Under Assumption 2.1, there exists a unique global mild solution $(u(t))_{t \in [0, T]}$ of (1.1) such that

$$u \in L^p(\Omega; D(0, T; L^2(\mathbb{R}^n))) \cap L^p(0, T; L^r(\mathbb{R}^n)).$$

Moreover, we have for all $t \in [0, T]$,

$$\|u(t)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}, \quad \mathbb{P}\text{-a.s.}$$

We will generalize the Strichartz estimate in [9] to the case of stochastic convolution driven by Lévy noise, see Proposition 2.6. This Strichartz estimate for jump noise is new and novel. Compared to papers by de Bouard and Debussche [12, 13], our Strichartz estimate holds for arbitrary admissible pair $(p, r)$ and hence avoids the additional restriction $0 < \sigma < \frac{1}{m}$ for $n \geq 3$ on $\sigma$. This general version of Strichartz estimate also allows us to obtain the existence and uniqueness results to the stochastic NLS with nonlinear noise which differs from the linear noise results from [14]. As the nonlinear term is not Lipschitz, we shall truncate the nonlinear term and combine the Strichartz estimates for the approximate solution to construct a solution of the truncated problem by using a fixed point argument in the space $D(0, T; L^2(\mathbb{R}^n)) \cap L^p(0, T; L^r(\mathbb{R}^n))$. Then we use the solution of the truncated problem to construct a maximal local solution to the original equation. A novel aspect of our equation proving the local existence and uniqueness of (2.1). In the last section, we first prove that the existence and uniqueness of global solutions.

The rest of the paper is organized as follows. In Section 2 we introduce some notations and assumptions and investigate the stochastic Strichartz estimate. Section 3 is devoted to studying the truncated equation and proving the local existence and uniqueness of (2.1). In the last section, we first prove that the $L^2$-norm of the solution in (1.1) is conserved and get a uniform estimate of the solutions of the truncated equations in $L^p(\Omega; L^p(0, T; L^r(\mathbb{R}^n)))$. Finally, we establish the existence and uniqueness of global solutions.

2. Notations and Strichartz estimates

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, be a filtered probability space satisfying the usual hypothesis. Let $L(t) = (L_1(t), \ldots, L_m(t))$, $t \geq 0$ be an $\mathbb{R}^m$-valued pure jump Lévy process, i.e. $L(t) = \int_0^t \int_B z \tilde{N}(ds, dz)$, where $B = \{z \in \mathbb{R}^m : 0 < |z| \leq 1\}$ and $N$ is a time homogeneous Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^m - \{0\})$ with $\sigma$-finite intensity measure $\nu$ satisfying $\int_B |z|^2 \nu(dz) < \infty$. 

Based on the definition of Marcus canonical integral from \[16, 17\], equation (1.1) with the notation $\circ$ is defined by

$$du(t) = i[\Delta u(t) - f(u(t))]dt + \int_B \left[ \Phi(z, u(t-)) - u(t-) \right] N(dtt, dz)$$

$$+ \int_B \left[ \Phi(z, u(t)) - u(t) + i \sum_{j=1}^{m} z_j g_j(u(t)) \right] \nu(dz)dt, \quad t > 0,$$

(2.1)

with $\Phi(z, x)$ being the value at time $t = 1$ of the solution of the following equation

$$\frac{\partial \Phi}{\partial t}(t, z, x) = -i \sum_{j=1}^{m} z_j g_j(\Phi(t, z, x)), \quad \Phi(0, z, x) = x.$$ 

(2.2)

We make the following assumptions.

**Assumption 2.1.**

1. Let $n \in \mathbb{N}$ and $(S_t)_{t \in \mathbb{R}}$ denote the group of isometries on $L^2(\mathbb{R}^n)$ generated by $i\Delta$.

2. For each $1 \leq j \leq m$, there exists a function $\tilde{g}_j : [0, \infty) \to \mathbb{R}$ of class $C^1$ such that $g_j$ is given by

$$g_j(y) = \tilde{g}_j(|y|^2)y, \quad y \in \mathbb{C}.$$ We also assume there exist constants $L_1, L_2 > 0$ such that for all $x, y \in \mathbb{C}$

$$\max_{1 \leq j \leq m} |g_j(x) - g_j(y)| \leq L_1|x - y|,$$ 

(2.3)

$$\max_{1 \leq j, k \leq m} |g_j'(x)g_k(x) - g_j(y)g_k(y)| \leq L_2|x - y|.$$ 

(2.4)

3. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(y) = |y|^{2\sigma}y$, $y \in \mathbb{C}$, for some $\sigma > 0$.

Note that we identify $\mathbb{C}$ with $\mathbb{R}^2$ and denote $\langle \cdot, \cdot \rangle$ (resp. $|\cdot|$) the scalar product (resp. the Euclidian norm) in $\mathbb{C} \cong \mathbb{R}^2$.

**Remark 2.1.**

1. Let us introduce the linear operator $I : \mathbb{R}^2 \ni (y_1, y_2) \mapsto (-y_2, y_1) \in \mathbb{R}^2$ and identify the operator of multiplication by the imaginary unit $i$ with the operator $I$. Define $\phi_j(y) := (ig_j(y) = \tilde{g}_j(|y|^2)Iy.$ Then we have

$$[\phi_j(y)](x) = 2\tilde{g}_j(|y|^2)(y, x)Iy + \tilde{g}_j(|y|^2)Ix, \quad f, y \in \mathbb{R}^2.$$ 

From this it follows that

$$[(ig_j)'(y)](-ig_k(y)) = 2\tilde{g}_j'(|y|^2)(y, -Ig_k(y))Iy - \tilde{g}_j(|y|^2)I^2g_k(y)$$

$$= -2\tilde{g}_j'(|y|^2)\tilde{g}_k(|y|^2)yIy + \tilde{g}_j(|y|^2)\tilde{g}_k(|y|^2)y$$

$$= \tilde{g}_j(|y|^2)\tilde{g}_k(|y|^2)y$$

$$= m_{j,k}(y),$$

where $m_{j,k}(y) = \tilde{g}_j(|y|^2)\tilde{g}_k(|y|^2)y$. Then the above condition (2.4) can be replaced by

$$\max_{1 \leq j, k \leq m} |m_{j,k}(x) - m_{j,k}(y)| \leq L_2|x - y|, \quad x, y \in \mathbb{C}.$$ 

For Assumption 2.1(2) to hold, it is sufficient that each $\tilde{g}_j \in C^2([0, \infty); \mathbb{R})$ and $\sup_{y > 0}(1 + \theta)|\tilde{g}_j'(\theta)| < \infty$, for $j = 1, \ldots, m$.

2. Under Assumption 2.1, there exists a measurable mapping $\Phi : \mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{C} \to \mathbb{C}$ such that, for each $z \in \mathbb{R}^m$, $x \in \mathbb{C}$, the function $t \mapsto \Phi(t, z, x)$ is continuously differentiable and solves (2.2).
We now state the following famous deterministic Strichartz estimates, we refer the reader to e.g. [11, Theorem 2.3.3] for the proof and details. Let us first recall the definition of an admissible pair. We say a pair $(p,r)$ is admissible if

$\frac{2}{p} = n\left(\frac{1}{2} - \frac{1}{r}\right)
$ 

and

\[
\begin{cases}
2 \leq r \leq \infty, & \text{if } n = 1, \\
2 \leq r < \infty, & \text{if } n = 2, \\
2 \leq r \leq \frac{2n}{n-2}, & \text{if } n \geq 3.
\end{cases}
\]

Notice that $(\infty,2)$ and $(2, \frac{2n}{n-2})$, $n \geq 3$ are always admissible.

Throughout the paper, the symbol $C$ will denote a positive generic constant whose value may change from place to place. If a constant depends on some variable parameters, we will put them in subscripts.

**Proposition 2.2.** Let $(p,r)$ and $(\gamma,\rho)$ be two admissible pairs and let $\gamma',\rho'$ be conjugates of $\gamma$ and $\rho$. Then

1. for every $\phi \in L^2(\mathbb{R}^n)$, the function $t \mapsto S_t \phi$ belongs to $L^p(\mathbb{R}; L^r(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}; L^2(\mathbb{R}^n))$ and there exists a constant $C$ such that

\[
\|S_t \phi\|_{L^p(\mathbb{R}; L^r(\mathbb{R}^n))} \leq C\|\phi\|_{L^2(\mathbb{R}^n)}. \tag{2.5}
\]

2. Let $I$ be an interval of $\mathbb{R}$ and $J = \bar{I}$ with $0 \in J$. Then for every $f \in L^{\gamma'}(I; L^{\rho'}(\mathbb{R}^n))$, the function $t \mapsto \Phi_f(t) = \int_0^t S_t S_{t-s} f(s)ds$ belongs to $L^p(I; L^{r}(\mathbb{R}^n)) \cap L^\infty(J; L^2(\mathbb{R}^n))$ and there exists a constant $C$ independent of $I$ such that

\[
\|\Phi_f\|_{L^p(I; L^{r}(\mathbb{R}^n))} \leq C\|f\|_{L^{\gamma'}(I; L^{\rho'}(\mathbb{R}^n))}; 
\]

\[
\|\Phi_f\|_{L^p(I; L^{r}(\mathbb{R}^n))} \leq C\|f\|_{L^{\gamma'}(I; L^{\rho'}(\mathbb{R}^n))}. \tag{2.6}
\]

**Remark 2.2.** Note that inequality (2.5) is consistent with Assumption 3.1 in [9]. Take $v \in L^\infty(0,T; L^2(\mathbb{R}^n))$, by (2.5) we have

\[
\|S_{-s} 1_{[s,T]}(\cdot)v_s\|_{L^p(0,T; L^{r}(\mathbb{R}^n))} = \left( \int_s^T \|S_t S_{t-s} v_s\|^p R(\mathbb{R}^n) dt \right)^{\frac{1}{p}} \leq \left( \int_0^T \|S_t S_{t-s} v_s\|^p R(\mathbb{R}^n) dt \right)^{\frac{1}{p}} \leq C\|v_s\|_{L^2(\mathbb{R}^n)}. 
\]

It follows that

\[
\left\| \int_0^t S_{t-s} v_s ds \right\|_{L^p(0,T; L^{r}(\mathbb{R}^n))} = \left\| \int_0^T 1_{[s,T]}(t)S_{t-s} v_s ds \right\|_{L^p(0,T; L^{r}(\mathbb{R}^n))} 
\]

\[
\leq \int_0^T \|1_{[s,T]}(t)S_{t-s} v_s\|_{L^p(0,T; L^{r}(\mathbb{R}^n))} ds 
\]

\[
\leq C \int_0^T \|v_s\|_{L^2(\mathbb{R}^n)} ds 
\]

\[
\leq TC\|v\|_{L^\infty(0,T; L^2(\mathbb{R}^n))}. 
\]

This inequality will play a key role later.

Let $p,r \in [2,\infty)$ with $\frac{2}{p} = n\left(\frac{1}{2} - \frac{1}{r}\right)$, that is $(p,r)$ is an admissible pair. For $0 \leq t_1 \leq t_2$, let us denote $D(t_1,t_2; L^2(\mathbb{R}^n))$ the space of all right continuous functions with left-hand limits from $[t_1,t_2]$ to $L^2(\mathbb{R}^n)$ and

\[
Y_{[t_1,t_2]} := D(t_1,t_2; L^2(\mathbb{R}^n)) \cap L^p(t_1,t_2; L^r(\mathbb{R}^n)). \tag{2.8}
\]
Then $Y_{[t_1,t_2]}$ is a Banach space with norm defined by
\[
\|u\|_{Y_{[t_1,t_2]}} := \sup_{s \in [t_1,t_2]} \|u(s)\|_{L^2(\mathbb{R}^n)} + \left( \int_{t_1}^{t_2} \|u(s)\|_{L^r(\mathbb{R}^n)}^r \, ds \right)^{\frac{1}{r}}.
\] (2.9)
For the simplicity of notation, we write $Y_t$ instead of $Y_{[0,t]}$. Notice that $Y_t$, $t \geq 0$ is a non-decreasing family of Banach spaces. That is if $0 < s < t$ and $u \in Y_s$, then $u|_{[0,s]} \in Y_s$ and $\|u|_{[0,s]}\|_{Y_s} \leq \|u\|_{Y_t}$. Let $\tau > 0$ be a stopping time. We call $\tau$ an accessible stopping time if there exists an increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times such that $\tau_n < \tau$ and $\tau_n \to \tau$ $\mathbb{P}$-a.s. as $n \to \infty$ and we call $(\tau_n)_{n \in \mathbb{N}}$ an approximating sequence for $\tau$.

Let $M^p_F(Y_\tau) := L^p(\Omega; D(0,\tau; L^2(\mathbb{R}^n)) \cap L^p(0,\tau; L^r(\mathbb{R}^n)))$ denote the space of all $L^2(\mathbb{R}^n)$-valued adapted càdlàg and $L^r(\mathbb{R}^n)$-valued progressively measurable processes $u : [0,T] \times \Omega \to L^2(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ satisfying
\[
\|u\|_{M^p_F(Y_\tau)} := \mathbb{E}\|u\|_{Y_\tau}^p = \mathbb{E}\left( \sup_{s \in [0,\tau]} \|u(s)\|_{L^2(\mathbb{R}^n)}^p + \int_0^\tau \|u(s)\|_{L^r(\mathbb{R}^n)}^r \, ds \right) < \infty.
\]

Now we introduce the definitions of local solutions and maximal local solutions, see e.g. [19] for details.

**Definition 2.3.** A local mild solution to equation (2.1) is an $L^2(\mathbb{R}^n)$-valued càdlàg $\mathbb{F}$-adapted process $u(t)$, $t \in [0,\tau)$, where $\tau$ is an accessible stopping time with an approximating sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times such that for every $n \in \mathbb{N}$,

(i) $(u(t))_{t \in [0,\tau_n]}$ belongs to $M^p_F(Y_{\tau_n})$;
(ii) for every $t \in [0,T)$, the following equality holds
\[
u(t \wedge \tau_n) = S_{t \wedge \tau_n} u_0 - \int_0^{t \wedge \tau_n} S_{t \wedge \tau_n - s} (f(u(s))) \, ds + I_{\tau_n}(\Phi(z,u) - u)(t \wedge \tau_n)
+ \int_0^{t \wedge \tau_n} \int_{B} S_{t \wedge \tau_n - s} \left[ \Phi(z,u(s)) \right] - u(s) + \sum_{j=1}^{N} z_j g_j(u(s)) \right] \nu(dz) \, ds,
\] $\mathbb{P}$-a.s.

where $I_{\tau_n}(\Phi(z,u) - u)$ is a process defined by
\[
I_{\tau_n}(\Phi(z,u) - u)(t) = \int_0^t \int_{B} S_{t - s} \left[ \Phi(z,u(s \wedge \tau_n -)) \right] - u(s \wedge \tau_n -) \right] \nu(dz,ds).
\]

A local mild solution $(u(t))_{t \in [0,\tau)}$ is called unique, if for any other local mild solution $(v(t))_{t \in [0,\tau)}$ of (2.1), we have
\[
\mathbb{P}(u(t) = v(t), \forall t \in [0,\tau \wedge \sigma)) = 1.
\]

A local mild solution $u = (u(t))_{0 \leq t < \tau}$ is called a maximal local mild solution if for any other local mild solution $(v(t))_{t \in [0,\sigma)}$ satisfying $\sigma \geq \tau$ a.s. and $v|_{[0,\sigma)}$ is equivalent to $u$, one has $\sigma = \tau$ a.s.

A local mild solution $(u(t))_{t \in [0,\tau)}$ is a global mild solution if $\tau = T$, $\mathbb{P}$-a.s. and $u \in M^p_F(Y_T)$.

Now let us explore the stochastic Strichartz estimates. First we recall the definition of martingale type 2 Banach space. We say a real separable Banach space $(E, \| \cdot \|_E)$ is of martingale type 2 if there is a constant $K(E) > 0$ such that for all $E$-valued discrete martingales $(M_n)_{n=0}^N$ the following inequality holds
\[
\sup_n E \|M_n\|^2_E \leq K(E) \sum_{n=0}^N E \|M_n - M_{n-1}\|^2_E,
\]
where we set $M_{-1} = 0$ as usual. Note that all $L^p$ spaces, $p \geq 2$ are of martingale type 2.

Let $E$ be a separable Banach space of martingale type 2 and let $\xi : [0,T] \times \Omega \times Z \to E$ be an $E$-valued $\mathbb{F}$-predictable process in $L^2([0,T] \times \Omega \times Z)$. For detailed discussion of stochastic integral with respect to Poisson
random measure in martingale type 2 Banach space, we refer to [20, 21]. The Burkholder inequality holds in this framework, i.e. there exists a generic constant \( C_p \) depending only on \( p \) and the constant \( K(E) \) from the martingale type 2 condition such that (see e.g. [21, Corollary 2.4])

\[
\begin{align*}
E \sup_{t \in [0,T]} \left( \int_0^t \int_Z \xi(s,z)\bar{N}(ds,dz) \right)^p_E & \leq C_p E \left( \int_0^T \int_Z \|\xi(s,z)\|^p_E \nu(dz) ds \right) \\
& + C_p E \left( \int_0^T \int_Z \|\xi(s,z)\|^2 \nu(dz) ds \right)^{\frac{p}{2}} \text{ for all } 2 \leq p < \infty. \tag{2.10}
\end{align*}
\]

**Lemma 2.4.** Let \( F \) be a martingale type 2 Banach space and \( p \in [2, \infty) \). Then there exists a constant \( C_p \) such that for all \( T \in (0, \infty) \) and all \( L^p(0,T;F) \)-valued predictable process \( \xi \),

\[
\begin{align*}
E \sup_{t \in [0,T]} \left( \int_0^t \int_Z \xi(s,z)\bar{N}(ds,dz) \right)^p_{L^p(0,T;F)} & \leq C_p E \left( \int_0^T \int_Z \|\xi(s,z)\|^p_{L_p(0,T;F)} \nu(dz) ds \right) \\
& + C_p E \left( \int_0^T \int_Z \|\xi(s,z)\|^2 \nu(dz) ds \right)^{\frac{p}{2}}. \tag{2.11}
\end{align*}
\]

**Proof.** The proof of this lemma is an immediate consequence of the facts that \( L^p(\mathbb{R}^+;F) \) is a martingale type 2 Banach space, and since \( L^p(0,T;F) \) is isometrically identified with a closed subspace of \( L^p(\mathbb{R}^+;F) \), it is still a martingale type 2 Banach space. \( \square \)

**Proposition 2.5.** Let \( \xi : [0,T] \times \Omega \times Z \to L^2(\mathbb{R}^n;\mathbb{C}) \) be an \( L^2(\mathbb{R}^n;\mathbb{C}) \)-valued predictable process. For all \( q \geq 2 \), we have

\[
\begin{align*}
E \sup_{t \in [0,T]} \left( \int_0^t \int_Z S_{t-s} \xi(s,z)\bar{N}(ds,dz) \right)^q_{L^2(\mathbb{R}^n)} & \leq C_q E \left( \int_0^T \int_Z \|\xi(s,z)\|^q_{L^2(\mathbb{R}^n)} \nu(dz) ds \right) \\
& + C_q E \left( \int_0^T \int_Z \|\xi(s,z)\|^2 \nu(dz) ds \right)^{\frac{q}{2}}. \tag{2.12}
\end{align*}
\]

**Proof.** Note that \( (S_t)_{t \in \mathbb{R}} \) is a unitary group in \( L^2(\mathbb{R}^n) \), so \( (S_t)_{t \in \mathbb{R}} \) is a \( C_0 \)-group of contractions. The required result (2.12) follows from a straightforward application of the maximal inequality in [21, Theorem 3.1]. \( \square \)

**Proposition 2.6.** Let \( (p,r) \) be an admissible pair and \( p, r \in [2, \infty) \). Then for all \( q \geq p \) and all \( \mathbb{F} \)-predictable process \( \xi : [0,T] \times \Omega \times Z \to L^r(\mathbb{R}^n;\mathbb{C}) \) in \( L^n(\Omega; L^2([0,T] \times Z; L^2(\mathbb{R}^n))) \cap L^n([0,T] \times Z; L^2(\mathbb{R}^n))) \), we have

\[
\begin{align*}
E \left( \int_0^T \int_Z S_{t-s} \xi(s,z)\bar{N}(ds,dz) \right)^q_{L^p(0,T;L^r(\mathbb{R}^n))} & \leq C_q E \left( \int_0^T \int_Z \|\xi(s,z)\|^q_{L^2(\mathbb{R}^n)} \nu(dz) ds \right)^{\frac{q}{2}} \\
& + C_q E \left( \int_0^T \int_Z \|\xi(s,z)\|^2 \nu(dz) ds \right)^{\frac{q}{2}}. \tag{2.13}
\end{align*}
\]

**Proof.** For an \( L^r(\mathbb{R}^n) \)-valued \( \mathbb{F} \)-predictable process \( \xi \), we define an \( L^p(0,T;L^r(\mathbb{R}^n)) \)-valued process \( \Upsilon \) as follows

\[
\Upsilon_{s,z} := \left\{ \left[ 0, T \right] \ni t \mapsto 1_{[s,T]}(t) S_{t-s} \xi(s,z) \right\}, \quad s \in [0,T], z \in Z. \tag{2.14}
\]

Note that the mapping \( \Gamma_s : L^2(\mathbb{R}^n) \ni x \mapsto 1_{[s,T]}(\cdot) S_{t-s} x \in L^p(0,T;L^r(\mathbb{R}^n)) \) is linear and continuous and \( \Upsilon_{s,z}(\omega) = \Gamma_s \circ \xi(s,z,\omega) \), \((s,z,\omega) \in [0,T] \times Z \times \Omega\). So the process \( \Upsilon_{s,z} \) is \( \mathbb{F} \)-predictable. Applying Burkholder’s inequality (2.11) gives

\[
\begin{align*}
E \left( \int_0^T \int_Z S_{t-s} \xi(s,z)\bar{N}(ds,dz) \right)^q_{L^p(0,T;L^r(\mathbb{R}^n))} & = E \left( \int_0^T \int_0^T 1_{[s,T]}(t) S_{t-s} \xi(s,z)\bar{N}(ds,dz) \right)^p_{L^r(\mathbb{R}^n)} dt \right)^{\frac{q}{2}}
\end{align*}
\]
In this section, we will construct a local solution of equation (2.1). Since the nonlinear term is not Lipschitz, let

\[ \theta \in L^p(0,T;L^r(\mathbb{R}^n)) \]

be given. By using the Strichartz inequalities (2.5), we have

\[ \| Y_{s,z}(\cdot) \|_{L^p(0,T;L^r(\mathbb{R}^n))} = \| 1_{[s,T]}(\cdot) S_{-s} \xi(s,z) \|_{L^p(0,T;L^r(\mathbb{R}^n))} \leq C \| S_{-s} \xi(s,z) \|_{L^p(0,T;L^r(\mathbb{R}^n))} \]

and

\[ \| S_{-s} \xi(s,z) \|_{L^2(\mathbb{R}^n)} \leq C \| \xi(s,z) \|_{L^2(\mathbb{R}^n)}. \]

Inserting back gives

\[ \mathbb{E} \left[ \int_0^T \int Z S_{-s} \xi(s,z) \tilde{N}(ds,dz) \right]^q \leq C \mathbb{E} \left[ \int_0^T \int Z \| \xi(s,z) \|_{L^2(\mathbb{R}^n)}^2 \nu(dz) ds \right]^{\frac{q}{2}} + C \mathbb{E} \left[ \int_0^T \int Z \| \xi(s,z) \|_{L^2(\mathbb{R}^n)}^q \nu(dz) ds \right]. \]

Put \( T_0 > 0 \). Denote by \( \mathbb{F}_{T_0} := (\mathcal{F}_{t+T_0})_{t \geq 0} \) the shifted filtration. Define a new process by

\[ N_{T_0}(t,A) := N(t+T_0,A) - N(T_0,A) \]

for each \( t \geq 0 \) and \( A \in Z \). It is easy to verify that \( N_{T_0}(t,A) \) is a Poisson random measure with respect to \( \mathbb{F}_{T_0} \) with the same intensity measure \( \nu \) and

\[ \int_0^t \int Z S_{-s} g(T_0+s,z) \tilde{N}_{T_0}(ds,dz) = \int_{T_0}^{T_0+t} \int Z S_{T_0+t-s} g(s,z) \tilde{N}_{T_0}(ds,dz). \] \hspace{1cm} (2.15)

**Corollary 2.1.** Let \( T_1 > 0 \). Assume that \((p,r)\) is an admissible pair and \( p, r \in [2, \infty) \). Then for all \( q \geq p \) and all \( \mathbb{F}_{T_0} \)-predictable process \( \xi : [0,T_1] \times \Omega \times Z \to L^2(\mathbb{R}^n) \) in \( L^q(\Omega; L^2([0,T_1] \times Z); L^r(\mathbb{R}^n)) \cap L^q(\Omega; L^2([0,T_1] \times Z); L^2(\mathbb{R}^n)) \),

\[ \mathbb{E} \left[ \int_0^T \int Z S_{-s} \xi(s,z) \tilde{N}_{T_0}(ds,dz) \right]^q \leq C \mathbb{E} \left[ \int_0^{T_1} \int Z \| \xi(s,z) \|_{L^2(\mathbb{R}^n)}^2 \nu(dz) ds \right]^{\frac{q}{2}} + \mathbb{E} \left[ \int_0^{T_1} \int Z \| \xi(s,z) \|_{L^2(\mathbb{R}^n)}^q \nu(dz) ds \right]. \] \hspace{1cm} (2.16)

3. A Truncated Equation

In this section, we will construct a local solution of equation (2.1). Since the nonlinear term is not Lipschitz, we will use a similar truncation argument as in [9, 12, 15] and approximate the original equation by truncating the nonlinear term as follows. First we define a truncation function \( \theta \). Let \( \theta : \mathbb{R}_+ \to [0,1] \) be a non-increasing \( C_0^\infty \) function such that \( 1_{[0,1]} \leq \theta \leq 1_{[0,2]} \) and \( \inf_{x \in \mathbb{R}_+} \theta'(x) \geq -1 \). For \( R \geq 1 \), set \( \theta_R(\cdot) = \theta(\frac{\cdot}{R}) \).
Now let us estimate the deterministic term $\Psi$.

**Remark 3.1.** If $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non-decreasing function, then for every $x, y \in \mathbb{R}$,

$$\theta_R(x)h(x) \leq h(2R), \quad |\theta_R(x) - \theta_R(y)| \leq \frac{1}{R}|x - y|.$$  

Let us fix $R \geq 1$. Now we will prove the existence and uniqueness of the global solution $u^R$ to the following truncated equation

$$u(t) = S_t u_0 - i \int_0^t S_{t-s} (\theta_R(||u||_{Y_t})) f(u(s)) ds + \int_0^t \int_B S_{t-s} \theta_R(||u||_{Y_t}) \left[ \Phi(z, u(s-)) - u(s-) \right] \tilde{N}(ds, dz)$$

$$+ \int_0^t \int_B S_{t-s} \theta_R(||u||_{Y_t}) \left[ \Phi(z, u(s)) - u(s) + i \sum_{j=1}^m z_j g_j(u(s)) \right] \nu(dz) ds, \quad 0 \leq t \leq T. \quad (3.1)$$

For the simplicity of presentation, we shall adopt the following notations for $t \in [0, T]$,

$$[\Psi^R_1(u)](t) = -i \int_0^t S_{t-s} (\theta_R(||u||_{Y_t})) f(u(s)) ds,$$

$$[\Psi^R_2(u)](t) = \int_0^t \int_B S_{t-s} \theta_R(||u||_{Y_t}) \left[ \Phi(z, u(s-)) - u(s-) \right] \tilde{N}(ds, dz),$$

$$[\Psi^R_3(u)](t) = \int_0^t \int_B S_{t-s} \theta_R(||u||_{Y_t}) \left[ \Phi(z, u(s)) - u(s) + i \sum_{j=1}^m z_j g_j(u(s)) \right] \nu(dz) ds.$$

Now let us estimate the deterministic term $\Psi^R_1(u)$.

**Proposition 3.1.** Assume that $0 < \sigma < \frac{2}{n}$ and $r = 2\sigma + 2$. Then $\Psi^R_1$ maps from $Y_T$ into itself and for all $u_1, u_2 \in Y_T$ we have

$$||\Psi^R_1(u_1) - \Psi^R_1(u_2)||_{Y_T} \leq C_{\sigma} R^{2\sigma} T^{1 - \frac{2\sigma}{n}} ||u_1 - u_2||_{Y_T}. \quad (3.2)$$

**Proof.** Take $u \in Y_T$. Let us define $\tau = \inf\{t \geq 0 : ||u||_{Y_t} \geq 2R\} \land T$. Observe that $\theta_R(||u||_{Y_t}) = 0$ for $||u||_{Y_t} \geq 2R$.

Since $t \rightarrow ||u||_{Y_t}$ is non-decreasing on $[0, T]$, we have $\theta_R(||u||_{Y_t}) = 0$ for $t \geq \tau$. By applying the Strichartz inequality (2.6) and (2.7), we get

$$\sup_{0 \leq t \leq \tau} ||\Psi^R_1(u)(t)||_{L^2(\mathbb{R}^n)} \leq C ||\theta_R(||u||_{Y_t})||_{L^2([0, \tau]; L^{r'}(\mathbb{R}^n))} f(u)||_{L^{r'}([0, \tau]; L^{r'}(\mathbb{R}^n))},$$

$$||\Psi^R_1(u)||_{L^r([0, \tau]; L^{r'}(\mathbb{R}^n))} \leq C \theta_R(||u||_{Y_T}) f(u)||_{L^{r'}([0, \tau]; L^{r'}(\mathbb{R}^n))}.$$

Collecting the above two estimates and then applying Hölder’s inequality, we obtain

$$||\Psi^R_1(u)||_{Y_T} \leq C \theta_R(||u||_{Y_T}) f(u)||_{L^{r'}([0, \tau]; L^{r'}(\mathbb{R}^n))} \leq C ||u||^{2\sigma+1}_{L^{r'}([0, \tau]; L^{r'}(\mathbb{R}^n))}$$

$$= C \left( \int_0^\tau \left( \int_{\mathbb{R}^n} |u(t,x)|^{r'} dx \right)^\frac{2}{r'} dt \right)^\frac{1}{2} \leq C \tau^{\frac{2\sigma}{n}} \left( \int_0^\tau \left( \int_{\mathbb{R}^n} |u(t,x)|^{r'} dx \right)^\frac{2}{r'} dt \right)^\frac{1}{2} \leq C T^{1 - \frac{2\sigma}{n}} \left( \int_0^\tau \left( \int_{\mathbb{R}^n} |u(t,x)|^{r'} dx \right)^\frac{2}{r'} dt \right)^\frac{1}{2} \leq C T\frac{1}{2} \frac{2\sigma+1}{2} ||u||^{2\sigma+1}_{Y_T},$$
where $1 - \frac{nσ}{2} > 0$. Now take $u_1, u_2 \in Y_T$. Let us define $τ_i = \inf \{t \geq 0 : |u_i|_{Y_1} \geq 2R \} \wedge T, i = 1, 2$. Without loss of generality we may assume that $τ_1 \leq τ_2$. Similarly, by the Strichartz inequality (2.6) and (2.7), we obtain

$$\sup_{0 \leq t \leq T} \|Ψ_t^R(u_1) - Ψ_t^R(u_2)\|_{L^2(Y)} \leq C\|θ_R(\|u_1\|_Y) f(u_1) - θ_R(\|u_2\|_Y) f(u_2)\|_{L^p(0,T;L^r(\mathbb{R}^n))}$$

Combining the above two estimates and applying Remark 3.1, we get

$$\|Ψ_t^R(u_1) - Ψ_t^R(u_2)\|_{L^r(Y_T)} = 2C\|θ_R(\|u_1\|_Y) f(u_1) - θ_R(\|u_2\|_Y) f(u_2)\|_{L^p(0,T;L^r(\mathbb{R}^n))}$$

It is clear that $|f(y)| = |y|^{2σ+1}$ and $|f'(y)| = ((2σ + 1)^2 + 1)^{\frac{1}{2}}|y|^{2σ}$ for $y \in \mathbb{C} \cong \mathbb{R}^2$. By Taylor’s formula and Hölder’s inequality, we infer

$$\|f(u_1) - f(u_2)\|_{L^r(\mathbb{R}^n)} = \|f'(θu_1 + (1 + θ)u_2)|u_1 - u_2|_{L^r(\mathbb{R}^n)}$$

Using again the Hölder inequality gives

$$\|f(u_1) - f(u_2)\|_{L^p(0,τ_1;L^r(\mathbb{R}^n))} \leq C_T^1(2σ)(4R)^{2σ} \|u_1 - u_2\|_{L^p(0,τ_1;L^r(\mathbb{R}^n))}$$

which proves (3.2).

To establish the stochastic Strichartz estimates for the stochastic term $Ψ_t^R$, we need the following two technical lemmas.

**Lemma 3.2.** Under Assumption 2.1, the Marcus map satisfies

$$|Φ(θ, z, y)| = |Φ(0, z, y)| = |y|, \text{ for all } θ \in \mathbb{R}, z \in \mathbb{R}^m, y \in \mathbb{C}. \quad (3.3)$$

**Proof.** Let us fix $z \in \mathbb{R}^m$ and $y \in \mathbb{C}$. Then we have

$$\frac{1}{2} \frac{∂|Φ(θ, z, y)|^2}{∂θ} = Re\left(\frac{∂Φ(θ, z, y)}{∂θ}, Φ(θ, z, y)\right)$$

$$= -Re\left(i \sum_{j=1}^n z_j g_j(Φ(θ, z, y)), Φ(θ, z, y)\right) \quad (3.4)$$

$$\frac{1}{2} \frac{∂|Φ(θ, z, y)|^2}{∂θ} = Re\left(\frac{∂Φ(θ, z, y)}{∂θ}, Φ(θ, z, y)\right)$$

$$= -Re\left(i \sum_{j=1}^n z_j g_j(Φ(θ, z, y)), Φ(θ, z, y)\right) \quad (3.5)$$
where we used the following fundamental identity:

\begin{equation}
Re \langle i u, u \rangle = Re \left[ i \langle u, u \rangle \right] = Re \left[ i |u|^2 \right] = 0, \quad u \in \mathbb{C}.
\end{equation}

\hfill \square

For the simplicity of notation, for each \( s \in [0, T] \), \( z \in \mathbb{R}^m \), \( y \in \mathbb{C} \), we denote

\[ G(s, z, y) := \Phi(s, z, y) - y \]

\[ H(s, z, y) := \Phi(s, z, y) - y + i \sum_{j=1}^{m} z_j g_j(y). \]

For abbreviation, we let \( G(z, y) = G(1, z, y) \) and \( H(z, y) = H(1, z, y) \).

**Lemma 3.3.** Under Assumption 2.1, there exist \( C_{m}^{1}, C_{m}^{2}, C_{m}^{3}, C_{m}^{4} > 0 \) such that for all \( y, y_1, y_2 \in \mathbb{C} \) and all \( z \in \mathbb{R}^m : |z|_{\mathbb{R}^m} \leq 1 \),

\begin{align}
|G(z, y)| &\leq C_{m}^{1} |z|_{\mathbb{R}^m} |y|, \\
|G(z, y_1) - G(z, y_2)| &\leq C_{m}^{2} |z|_{\mathbb{R}^m} |y_1 - y_2|, \\
|H(z, y)| &\leq C_{m}^{3} |z|_{\mathbb{R}^m} |y|, \\
|H(z, y_1) - H(z, y_2)| &\leq C_{m}^{4} |z|_{\mathbb{R}^m}^2 |y_1 - y_2|.
\end{align}

**Proof.** Using the Cauchy-Schwartz inequality we have

\[ |G(s, z, y)| = |\Phi(s, z, y) - y| = \left| \int_0^s \sum_{j=1}^{m} -iz_j g_j(\Phi(a, z, y)) da \right| \]

\[ \leq |z|_{\mathbb{R}^m} \int_0^s \left( \sum_{j=1}^{m} |g_j(\Phi(a, z, y))|^2 \right)^{\frac{1}{2}} da \]

\[ \leq |z|_{\mathbb{R}^m} L_1 m^{\frac{1}{2}} \int_0^s |\Phi(a, z, y)| da \]

\[ \leq |z|_{\mathbb{R}^m} L_1 m^{\frac{1}{2}} |y| s + |z|_{\mathbb{R}^m} L_1 m^{\frac{1}{2}} \int_0^s |\Phi(a, z, y)| da, \]

where in the last two steps we used (2.3) and the definition of \( \mathcal{G} \). Applying the Gronwall inequality yields

\begin{equation}
|G(s, z, y)| \leq sm^{\frac{1}{2}} L_1 |z|_{\mathbb{R}^m} |y| e^{m^{\frac{1}{2}} s L_1 |z|_{\mathbb{R}^m}},
\end{equation}

\hfill (3.13)
which proves (3.9) with $s = 1$. Similarly, we have

$$\left| \mathcal{G}(s, z, y_1) - \mathcal{G}(s, z, y_2) \right| = |\Phi(s, z, y_1) - y_1 - \Phi(s, z, y_2) + y_2|$$

$$\leq |\int_0^s \sum_{j=1}^m -iz_j \left( g_j(\Phi(a, z, y_1)) - g_j(\Phi(a, z, y_2)) \right) da|$$

$$\leq |z|_{\mathbb{R}^m} \int_0^s \left( \sum_{j=1}^m |g_j(\Phi(a, z, y_1)) - g_j(\Phi(a, z, y_2))|^2 \right)^{\frac{1}{2}} da$$

$$\leq m^{\frac{1}{2}} |z|_{\mathbb{R}^m} \int_0^s |\Phi(a, z, y_1) - \Phi(a, z, y_2)| da. \quad (3.14)$$

Taking the second and the last expressions of the above inequality we deduce that

$$|\Phi(s, z, y_1) - \Phi(s, z, y_2)| \leq |y_1 - y_2| + m^{\frac{1}{2}} |z|_{\mathbb{R}^m} \int_0^s |\Phi(a, z, y_1) - \Phi(a, z, y_2)|^2 da.$$ 

Applying the Gronwall inequality we get

$$|\Phi(s, z, y_1) - \Phi(s, z, y_2)| \leq |y_1 - y_2| e^{m^{\frac{1}{2}} |z|_{\mathbb{R}^m} L_1}. \quad (3.15)$$

The required result (3.10) is obtained on inserting (3.15) back into (3.14) and putting $s = 1$. Observe that

$$|H(z, y)| = |\Phi(1, z, y) - y + i \sum_{j=1}^m z_j g_j(y)|$$

$$= \left| \int_0^1 - \sum_{j=1}^m z_j \left[ i g_j(\Phi(a, l, y)) - i g_j(y) \right] da \right|$$

$$= \left| \int_0^1 \sum_{j=1}^m z_j \int_0^a \frac{d(g_j)}{d\Phi}(\Phi(b, z, y)) \left( -i \sum_{k=1}^m z_k g_k(\Phi(b, z, y)) \right) dbda \right|.$$ 

We now apply the Cauchy-Schwarz inequality, Assumption 2.1 and (3.13) to obtain

$$|H(z, y)| = |\Phi(1, z, y) - y + i \sum_{j=1}^m z_j g_j(y)|$$

$$\leq \int_0^1 \int_0^a \left| \sum_{j=1}^m z_j \left( \sum_{k=1}^m |z_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^m \bar{g}_k(\Phi(b, z, y)|^2) \right)^{\frac{1}{2}} dbda$$

$$\leq L_2 |z|_{\mathbb{R}^m} \int_0^1 \int_0^a \left( \sum_{j=1}^m \sum_{k=1}^m |\Phi(b, z, y)|^2 \right)^{\frac{1}{2}} dbda$$

$$\leq mL_2 |z|_{\mathbb{R}^m} \int_0^1 \int_0^a \left( |y| + |\mathcal{G}(b, z, y)| \right) dbda$$

$$= \frac{m}{2} L_2 |z|_{\mathbb{R}^m} |y| + mL_2 |z|_{\mathbb{R}^m} \int_0^1 \int_0^a \bar{g}(\Phi(b, z, y)|^2) \Phi(b, z, y)|^2 |y| dbda$$

$$= \frac{m}{2} L_2 |z|_{\mathbb{R}^m} |y| + Cm^{\frac{1}{2}} L_2 L_1 |z|_{\mathbb{R}^m} |y| \int_0^1 \int_0^a \bar{g}(\Phi(b, z, y)|^2) \Phi(b, z, y)|^2 |y| dbda$$

$$\leq C_m^3 |z|_{\mathbb{R}^m}^2 |y|,$$

where $C_m^3 = \frac{m}{2} L_2 + L_1 L_2 m^{\frac{1}{2}} K_1 \int_0^1 b e^{m^{\frac{1}{2}} L_1 b} dbda$.

A similar argument using (2.4) yields

$$|H(z, y_1) - H(z, y_2)|$$
Proposition 3.4. Here we also used the fact that \( R \) from which we also deduce that \( \Psi \)

Proof. Let us define

\[
|\Phi(1, z, y_1) - y_1 + \sum_{j=1}^{m} z_j g_j(y_1) - \Phi(1, z, y_2) + y_2 - \sum_{j=1}^{m} z_j g_j(y_2)|
\]

\[
= \int_0^1 \sum_{j=1}^{m} z_j \left[ ig_j(\Phi(a, z, y_1)) - ig_j(y_1) - ig_j(\Phi(a, z, y_2)) + ig_j(y_2) \right] da
\]

\[
= \int_0^1 \sum_{j=1}^{m} z_j \left( \frac{d(i g_j)}{d\Phi}(\Phi(b, z, y_1))(-i) \sum_{k=1}^{m} z_k g_k(\Phi(b, z, y_1)) - \frac{d(i g_j)}{d\Phi}(\Phi(b, z, y_2))(-i) \sum_{k=1}^{m} z_k g_k(\Phi(b, z, y_2)) \right) db da
\]

\[
\leq \left| \sum_{j=1}^{m} z_j \int_0^a \left( \sum_{k=1}^{m} z_k \left[ \frac{d(i g_j)}{d\Phi}(\Phi(b, z, y_1))(-i) \sum_{k=1}^{m} z_k g_k(\Phi(b, z, y_1)) \right] \right)^2 \right| db da
\]

\[
\leq L_2 |y_1 - y_2| \int_0^1 \int_0^a \left| \sum_{k=1}^{m} \sum_{k=1}^{m} \left| \Phi(b, z, y_1) - \Phi(b, z, y_2) \right|^2 \right|^{\frac{1}{2}} db da
\]

\[
\leq m L_2 |y_1 - y_2| \int_0^1 \int_0^a |y_1 - y_2| e^{b m^2 |z| |u| L_1} db da
\]

\[
\leq C_m^4 |y_1 - y_2|
\]

where \( C_m^4 = m L_2 \int_0^1 \int_0^a e^{b m^2 |z| |u| L_1} db da \) and the proof is finished. \( \square \)

Proposition 3.4. Under Assumption 2.1, \( \Psi_R^R(u) \) maps \( M_p^p(Y_T) \) into itself and for all \( u_1, u_2 \in Y_T \) we have

\[
\mathbb{E}\|\Psi_R^R(u_1)(t) - \Psi_R^R(u_2)(t)\|_{Y_T}^p \leq C_{p,m}(T \nabla^\frac{t}{T} + T)\mathbb{E}\|u_1 - u_2\|_Y^p.
\]

Proof. Let us define \( \tau = \inf\{ t \geq 0 : |u|_{Y_T} \geq 2R \} \wedge T \). By using Proposition 2.5 and Lemma 3.3 we have

\[
\mathbb{E}\sup_{t \in [0,T]} \|\Psi_R^R(u)(t)\|_{L^p(Y_T)} \leq C_p \mathbb{E} \left( \int_0^T \int_B \|\theta_R\|_{L^p(z, u(s))}(\Phi(z, u(s)) - u(s))\|_{L^p(Y_T)}^p \nu(dz) ds \right)^{\frac{1}{p}}
\]

from which we also deduce that \( \Psi_R^R(u) \) is \( L^2(\mathbb{R}^n) \)-valued adapted with càdlàg modification, see e.g. [21, Theorem 3.1]. Here we also used the fact that \( \int_B |z|_{L^p(Y_T)}^p \nu(dz) \leq \int_B |z|_{L^p(Y_T)}^p \nu(dz) < \infty \) for \( p \geq 2 \).

By applying Proposition 2.6 and Lemma 3.3 we get

\[
\mathbb{E}\|\Psi_R^R(u)\|_{L^p([0,T],L^p(\mathbb{R}^n))} \leq C_p \mathbb{E} \left( \int_0^T \int_B \|\theta_R\|_{L^p(z, u(s))}(\Phi(z, u(s)) - u(s))\|_{L^p(\mathbb{R}^n)}^p \nu(dz) ds \right)^{\frac{1}{p}}
\]
Thus we infer that
\[ \mathbb{E}\left[ \sup_{t \in [0,T]} \| \Psi^R_2(u_1)(t) - \Psi^R_2(u_2)(t) \|^p_{L^2(\mathbb{R}^n)} \right] \]
\[ \leq C_{p,m} K_1(T^{\frac{p}{2}} + T) \mathbb{E}\| u_1^p - u_2^p \|_{Y_T}. \]

Take \( u_1, u_2 \in Y_T \). Let us define \( \tau_i = \inf\{ t \geq 0 : |u_i|_{Y_t} \geq 2R \} \wedge T, i = 1, 2 \). Without loss of generality we may assume that \( \tau_1 \leq \tau_2 \). By Proposition 2.5 and 2.6, Lemma 3.3, applying similar arguments as before we obtain
\[ \mathbb{E}\| \Psi^R_2(u_1)(t) \|^p_{Y_T} \leq C_{p,m} K_1(T^{\frac{p}{2}} + T) \mathbb{E}\| u_1^p - u_2^p \|_{Y_T}. \]
Combining the above estimates gives that
\[ E\| \Psi^R_2(u_1)(t) - \Psi^R_2(u_2)(t) \|_{Y_T} \leq C_{p,m}(T^\frac{p}{2} + T)\| u_1 - u_2 \|_{Y_T}^p. \]

This concludes the proof of Proposition 3.4. \( \square \)

**Proposition 3.5.** Suppose that Assumption 2.1 hold. Then \( \Psi^R_3(u) \) maps \( Y_T \) into itself and for all \( u_1, u_2 \in Y_T \) we have
\[ \| \Psi^R_3(u_1) - \Psi^R_3(u_2) \|_{Y_T} \leq C_m T \| u_1 - u_2 \|_{Y_T}. \]

**Proof.** Similarly, let \( \tau = \inf\{t \geq 0 : |u|_{Y_t} \geq 2R\} \wedge T \). Observe that
\[
\sup_{0 \leq t \leq T} \| \Psi^R_3(u) \|_{L^2(\mathbb{R}^n)} = \sup_{t \in [0,T]} \left\| \int_0^t \int_B S_{t-s}\theta_R(\|u\|_{Y_T})H(z,u(s))\nu(dz)ds \right\|_{L^2(\mathbb{R}^n)}
\leq C \int_0^T \| H(z,u(s)) \|_{L^2(\mathbb{R}^n)}\nu(dz)ds
\leq C_m \int_0^T \left( \int_B |z|_2^2 \nu(dz) \right) \| u(s) \|_{L^2(\mathbb{R}^n)}ds
\leq C_m T \| u \|_{Y_T} \left( \int_B |z|_2^2 \nu(dz) \right).
\]

By applying Remark 2.2 and Lemma 3.3 we obtain
\[
\| \Psi^R_3(u) \|_{L^p(0,T;L^r(\mathbb{R}^n))} = \left\| \int_0^t \int_B S_{t-s}\theta_R(\|u\|_{Y_T})H(z,u(s))\nu(dz)ds \right\|_{L^p(0,T;L^r(\mathbb{R}^n))}
\leq C \int_0^T \left\| \theta_R(\|u\|_{Y_T})H(z,u(s))\nu(dz) \right\|_{L^2(\mathbb{R}^n)}ds
\leq C_m \int_0^T \left( \int_B |z|_2^2 \nu(dz) \right) \| u(s) \|_{L^2(\mathbb{R}^n)}ds
\leq C_m T \| u \|_{Y_T} \left( \int_B |z|_2^2 \nu(dz) \right).
\]

It follows that
\[
\| \Psi^R_3(u) \|_{Y_T} \leq C_m T \| u \|_{Y_T}.
\]

Again by using Remark 2.2 and Lemma 3.3, we can show that
\[
\sup_{0 \leq t \leq T} \| \Psi^R_3(u_1) - \Psi^R_3(u_2) \|_{L^2(\mathbb{R}^n)} \leq C \int_0^T \int_B \| \theta_R(\|u_1\|_{Y_T})H(z,u_1(s)) - \theta_R(\|u_2\|_{Y_T})H(z,u_2(s)) \|_{L^2(\mathbb{R}^n)}\nu(dz)ds
\leq C \int_0^T \int_B \| \theta_R(\|u_1\|_{Y_T})H(z,u_1(s)) - H(z,u_2(s)) \|_{L^2(\mathbb{R}^n)}\nu(dz)ds
+ C \int_0^T \int_B \| \theta_R(\|u_2\|_{Y_T})H(z,u_2(s)) \|_{L^2(\mathbb{R}^n)}\nu(dz)ds
\leq C_m \int_0^T \left( \int_B |z|_2^2 \nu(dz) \right) \| u_1(s) - u_2(s) \|_{L^2(\mathbb{R}^n)}ds.
\]
for each \( u \) then \( \Gamma \)

Proposition 3.6. The proof of Proposition 3.5 is now complete.

We will prove the result by the following two steps. From Proposition 3.1, 3.4 and 3.5, we see that for every \( \mathcal{M}_\nu \), see e.g. [20, 21]. We will identify \( \Psi \) to equation (3.1) restricted to \( [0, T] \). Hence by the Banach fixed point theorem, there exists a unique solution \( u \) that if \( T \) is sufficiently small (depending on \( \sigma, p, m \)) such that

\[
\| \Psi_R^T(u_1) - \Psi_R^T(u_2) \|_{\mathcal{M}_\nu(Y_T)} \leq C_m T \| u_1 - u_2 \|_{Y_T}.
\]

The proof of Proposition 3.5 is now complete.

Proposition 3.6. Let \( p \geq 2, 0 < \sigma < \frac{2}{n} \) and \( (p, r) \) be an admissible pair with \( r = 2\sigma + 2 \). Under Assumption 2.1, for each \( u_0 \in L^p(\Omega; L^2(\mathbb{R}^n)) \) and \( T \in (0, \infty) \), there exists a unique global solution \( u^R \) in \( L^p(\Omega; D(0, T; L^2(\mathbb{R}^n)) \cap L^p(0, T; L^r(\mathbb{R}^n))) \) to equation (3.1).

Proof. We will prove the result by the following two steps.

Step 1. Define an operator by

\[
\Gamma^R(u)(t) := S_t u_0 + \Psi_1^R(u)(t) + \Psi_2^R(u)(t) + \Psi_3^R(u)(t), \quad t \in [0, T].
\]

We will construct a unique solution by the Banach fixed point theorem. Combining Remark 2.2, Proposition 3.1, 3.4 and 3.5, we see that for every \( T \), the operator \( \Gamma^R \) maps from \( M^p_\nu(Y_T) \) into \( M^p_\nu(Y_T) \). Now let us show that if \( T \) is sufficiently small, which will be determined later, then this operator is a strict contraction in the space \( M^p_\nu(Y_T) \). It follows again from Proposition 3.1, 3.4 and 3.5 that

\[
\| \Gamma^R(u_1) - \Gamma^R(u_2) \|_{\mathcal{M}_\nu^p(Y_T)} \leq \left[ C_\sigma R^{2\sigma T} T^{1-\frac{n\sigma}{r}} + C_{p,m} (T^\frac{n}{2} + T) + C_m T \right] \| u_1 - u_2 \|_{\mathcal{M}_\nu^p(Y_T)}.
\]

If we choose \( T_0 \) sufficiently small (depending on \( R, \sigma, p, m \)) such that

\[
C_\sigma R^{2\sigma T_0} T_0^{1-\frac{n\sigma}{r}} + C_{p,m} (T_0^\frac{n}{2} + T_0) + C_m T_0 \leq \frac{1}{2},
\]

then \( \Gamma^R \) is a \( \frac{1}{2}\)-contraction in the space \( M^p_\nu(Y_{T_0}) \). Hence by the Banach fixed point theorem, there exists a unique solution \( u^R \in M^p_\nu(Y_{T_0}) \) satisfying \( u^R = \Gamma^R(u^R) \). Note that one can always find a càdlàg modification of \( \Psi_2^R(u^R) \) in \( L^2(\mathbb{R}^n) \), see e.g. [20, 21]. We will identify \( \Psi_2^R(u^R) \) with this modification. It follows that the solution \( u^R \) is \( L^2(\mathbb{R}^n) \)-valued càdlàg. This is the unique solution in \( M^p_\nu(Y_{T_0}) \) of equation (3.1) restricted to \([0, T_0]\).

Step 2. We will extend the solution to \([0, T]\) by induction. Define \( j = \left\lfloor \frac{T}{T_0} \right\rfloor + 1 \). Assume that for some \( k \in \{1, 2, \cdots, j\} \) there exists \( u^R_k \in M^p_\nu(Y_{T_0}) \) such that

\[
u^R_k = \Gamma^R(u^R_k) \text{ on } [0, kT_0].\]
First let us define a new cutoff function by

\[ \Theta^R_k(u)(t) := \theta_R(\phi(u)(t)), \]

where \( \phi(u)(t) = \left( \| u^R_k \|_{L^p(0,kT_0; L^r(\mathbb{R}^n))} + \| u^p \|_{L^p(0,T; L^r(\mathbb{R}^n))} \right)^{\frac{1}{p}} + \max \left\{ \sup_{0 \leq t \leq kT_0} \| u^R_k(t) \|_{L^2(\mathbb{R}^n)}, \sup_{0 \leq s \leq t} \| u(s) \|_{L^2(\mathbb{R}^n)} \right\}. \]

Consider the following operator

\[ \Gamma^R_k(u)(t) := S_t u^R_k(kT_0) + \Psi^R_1(u)(t) + \Psi^R_2(u)(t) + \Psi^R_3(u)(t), \quad u \in M^{p}_{p,kT_0}(Y_{T_0}). \]

where for \( t \in [0,T_0] \) and \( u \in M^{p}_{p,kT_0}(Y_{T_0}), \)

\[ \left[ \Psi^R_k(u)(t) \right](t) = -i \int_0^t S_{t-s}(\Theta^R_k(u)(s))f(u(s))ds, \]

\[ \left[ \Psi^R_2(u)(t) \right](t) = \int_0^t \int_B S_{t-s}(\Theta^R_k(u)(s))\left[ \Pi(z, u_{-}) - u_{-} \right] \hat{N}^k T_0(ds, dl), \]

\[ \left[ \Psi^R_3(u)(t) \right](t) = \int_0^t \int_B S_{t-s}(\Theta^R_k(u)(s))\left[ \Pi(z, u_{s}) - u_{s} + i \sum_{j=1}^m z_j g_j(u_{s}) \right] \nu(dz)ds. \]

All the arguments in Step 1 can be reproduced. Take \( v_1, v_2 \in M^{p}_{p,kT_0}(Y_{T_0}). \) Let us define \( \tau_i = \inf\{ t \geq 0 : \phi(v_i)(t) \geq 2R \} \wedge T_0, \quad i = 1, 2. \) Without loss of generality we may assume that \( \tau_1 \leq \tau_2. \) By following the same line of argument as used in the proof of Proposition 3.1, we infer that

\[ \| \Psi^R_k(v_1) - \Psi^R_k(v_2) \|_{Y_{T_0}} \leq 2C \left\| \Theta^R_k(v_1)(s)f(v_1(s)) - \Theta^R_k(v_2)(s)f(v_2(s)) \right\|_{L^p(0,T; L^r(\mathbb{R}^n))} \]

\[ \leq 2C \left\| \Theta^R_k(v_1)(s) - \Theta^R_k(v_2)(s) \right\|_{L^p(0,T; L^r(\mathbb{R}^n))} \]

\[ + 2C \left\| f(v_1(s)) - f(v_2(s)) \right\|_{L^p(0,T; L^r(\mathbb{R}^n))} \]

\[ \leq \frac{2C}{R} \| v_1 - v_2 \|_{Y_{T_0}} \| f(v_2(s)) \|_{L^p(0,T; L^r(\mathbb{R}^n))} \]

\[ + 2C \| v_2 \|_{L^p(0,T; L^r(\mathbb{R}^n))} \| f(v_1(s)) - f(v_2(s)) \|_{L^p(0,T; L^r(\mathbb{R}^n))} \]

\[ \leq 4C_\sigma T_0^{1-\sigma} \| v_1 - v_2 \|_{Y_{T_0}} + 4C_\sigma T_0^{1-\sigma} (4R)^{2\sigma} \| v_1 - v_2 \|_{Y_{T_0}} \]

\[ \leq C_\sigma R^{2\sigma} T_0^{1-\sigma} \| v_1 - v_2 \|_{Y_{T_0}}. \]

Now using the same argument as in the proof of Proposition 3.4 and 3.5 and applying Corollary 2.1, we obtain

\[ \mathbb{E} \left\| \Psi^R_k(v_1) - \Psi^R_k(v_2) \right\|_{M^{p}_{p,kT_0}(Y_{T_0})} \leq C_{p,m}(T_0^\sigma + T) \mathbb{E} \| v_1 - v_2 \|_{M^{p}_{p,kT_0}(Y_{T_0})} \]

(3.17)

\[ \| \Psi^R_k(v_1) - \Psi^R_k(v_2) \|_{Y_{T_0}} \leq C_{m,T} \| v_1 - v_2 \|_{Y_{T_0}}. \]

(3.18)

Hence we conclude that

\[ \| \Gamma^R_k(v_1) - \Gamma^R_k(v_2) \|_{M^{p}_{p,kT_0}(Y_{T_0})} \leq \left[ C_\sigma R^{2\sigma} T_0^{1-\sigma} + C_{p,m}(T_0^\sigma + T_0) + C_{m,T} \right] \| v_1 - v_2 \|_{M^{p}_{p,kT_0}(Y_{T_0})}. \]

Note that the constant is the same as in the first step. It follows that \( \Gamma^R_k \) is a \( \frac{1}{2} \)-contraction in the space \( M^{p}_{p,kT_0}(Y_{T_0}). \). Let \( v^R_{k+1} \) be the unique solution satisfying \( v^R_{k+1} = \Gamma^R_k(v^R_{k+1}). \) Then we construct a solution as follows

\[ u^R_{k+1}(t) = \begin{cases} u^R_t(t), & \text{for } t \in [0,kT_0] \\ v^R_{k+1}(t-kT_0), & \text{for } t \in [kT_0,(k+1)T_0] \end{cases} \]

and so on, recursively. Notice that \( u^R_{k+1} \) is \( F \)-adapted, càdlàg in \( L^2(\mathbb{R}^n) \) and we have \( \mathbb{E} \| u^R_{k+1} \|_{Y_{(k+1)T_0}} < \infty. \)

Therefore, we obtain that \( u^R_{k+1} \in M^p(Y_{(k+1)T_0}). \)
Now we shall show that $u^R_{k+1}$ is a fixed point of $\Gamma^R$ in $M^p_Y(Y_{(k+1)T_0})$. Let $t \in [kT_0, (k+1)T_0]$, Define $\hat{t} := t - kT_0$. Then we have

$$
u^R_{k+1}(t) = \nu^R_{k+1}(\hat{t}) = \Gamma^R_{k}(\nu^R_{k+1}(\hat{t}))$$

$$= S_t \nu^R_k(kT_0) + \Psi^{R,k}_1(\nu^R_{k+1}(\hat{t})) + \Psi^{R,k}_2(\nu^R_{k+1}(\hat{t})) + \Psi^{R,k}_3(\nu^R_{k+1}(\hat{t}))$$

$$= S_t \nu^R_k(kT_0) + S_t \Psi^R_1(\nu^R_k)(kT_0) + \Psi^{R,k}_1(\nu^R_{k+1}(\hat{t})) + S_t \Psi^R_2(\nu^R_k)(kT_0) + \Psi^{R,k}_2(\nu^R_{k+1}(\hat{t}))$$

$$+ S_t \Psi^R_3(\nu^R_k)(kT_0) + \Psi^{R,k}_3(\nu^R_{k+1}(\hat{t})).$$

Observe that $\theta_R(\|u^R_k\|_{Y_\tau}) = \theta(\|u^R_{k+1}\|_{Y_\tau})$ for $s \in [0, kT_0]$ and $\Theta^R_k(\nu^R_{k+1}(s)) = \theta_R(\|u^R_{k+1}\|_{Y_{(k+1)T_0}})$ for $s \in [0, T_0]$. It follows that

$$S_t \Psi^R_1(\nu^R_k)(kT_0) + \Psi^{R,k}_1(\nu^R_{k+1}(\hat{t})) = -i \int_0^{kT_0} S^R_{T_0-s}(\theta_R(\|u^R_k\|_{Y_\tau}) f(\nu^R_k(s))) ds$$

$$- i \int_0^{kT_0} S^R_{T_0-s}(\theta_R(\|u^R_k\|_{Y_\tau}) f(\nu^R_k(s))) ds$$

$$= - i \int_0^{kT_0} S^R_{T_0-s}(\theta_R(\|u^R_k\|_{Y_\tau}) f(\nu^R_k(s))) ds$$

$$= - i \int_0^{kT_0} S^R_{T_0-s}(\theta_R(\|u^R_k\|_{Y_\tau}) f(\nu^R_k(s))) ds$$

$$= - i \int_0^{kT_0} S^R_{T_0-s}(\theta_R(\|u^R_k\|_{Y_\tau}) f(\nu^R_k(s))) ds$$

$$= - \Psi^R_1(\nu^R_{k+1}(\hat{t})).$$

Similarly, by using (2.15), we can prove

$$S_t \Psi^R_2(\nu^R_k)(kT_0) + \Psi^{R,k}_2(\nu^R_{k+1}(\hat{t})) = \Psi^R_2(\nu^R_{k+1}(\hat{t}))$$

$$S_t \Psi^R_3(\nu^R_k)(kT_0) + \Psi^{R,k}_3(\nu^R_{k+1}(\hat{t})) = \Psi^R_3(\nu^R_{k+1}(\hat{t})).$$

Therefore, we infer for $t \in [kT_0, (k+1)T_0]$

$$u^R_{k+1}(t) = S_{t-s} u_0 + \Psi^R_1(\nu^R_k(t)) + \Psi^R_2(\nu^R_k(t)) + \Psi^R_3(\nu^R_k(t)),$$

which shows that $u^R_{k+1}$ is a fixed point of $\Gamma^R$ in $M^p_Y(Y_{(k+1)T_0})$. Therefore $u^R := u^R_j$ is the unique solution to (3.1) on $[0, T]$.

By using the above results for the truncated problem (3.1), we can derive the existence and uniqueness of local mild solutions for the original equation (2.1). The following arguments are standard. One can also see [19, Proposition 1] for analogous arguments of proving the existence result for maximal local mild solutions to stochastic nonlinear beam equations.

**Proposition 3.7.** For each $k \in \mathbb{N}$, let $u_k \in M^p_Y(Y_T)$ be the solution of (3.1) with $R$ replaced by $k$. Define a stopping time $\tau_k$ by

$$\tau_k = \inf\{t \in [0, T] : \|u^k\|_{Y_\tau} \geq k\},$$

with the usual convention $\inf\emptyset = T$. Then

(1) For $k \leq n$, we have $0 < \tau_k \leq \tau_n$, $\mathbb{P}$-a.s. and $u^k(t) = u^n(t)$ $\mathbb{P}$-a.s. for $t \in [0, \tau_k]$. 

(2) Define $u(t) = u^k(t)$ for $t \in [0, \tau_k]$ and $\tau_\infty = \lim_{n \to \infty} \tau_n$. Then $(u(t))_{t \in [0, \tau_\infty]}$ is a maximal local mild solution of (2.1).

(3) The solution $u$ is unique.

Proof. For any $n \in \mathbb{N}$, by Proposition 3.6, there exists a unique global solution $u^n$ in $M^p(Y_T)$ to equation (3.1) which satisfies

$$u^n(t) = S_t u_0 - i \int_0^t S_{t-s} (\|u^n\|_{Y_T}) f(u^n(s)) ds + \int_0^t \int_{B} S_{t-s} (\|u^n\|_{Y_T}) \left[ \Phi(z, u^n(s-)) - u^n(s-) \right] \tilde{N}(ds, dz) + \int_0^t \int_{B} S_{t-s} (\|u^n\|_{Y_T}) \left[ \Phi(z, u^n(s)) - u^n(s) + i \sum_{j=1}^m z_j g_j(u^n(s)) \right] \nu(dz) ds, \quad \mathbb{P}\text{-a.s.}$$

(3.19)

for $t \in [0, T]$. For $k \leq n$, set $\tau_{k,n} = \tau_k \wedge \tau_n$. Hence by the definition of $\theta_n$, we have $\theta_n(\|u^n\|_{Y_T}) = 1$ and $\theta_k(\|u^k\|_{Y_T}) = 1$, for $t \in [0, \tau_{k,n})$. It follows that on $[0, \tau_{k,n})$ we have

$$u^l(t) = S_t u_0 - i \int_0^t S_{t-s} f(u^l(s)) ds + \int_0^t \int_{B} S_{t-s} \left[ \Phi(z, u^l(s-)) - u^l(s-) \right] \tilde{N}(ds, dz)$$

$$+ \int_0^t \int_{B} S_{t-s} \left[ \Phi(z, u^l(s)) - u^l(s) + i \sum_{j=1}^m z_j g_j(u^l(s)) \right] \nu(dz) ds, \quad \mathbb{P}\text{-a.s.}$$

(3.20)

The uniqueness of the solution to equation (3.1) implies that $u^k(t) = u^n(t) \mathbb{P}\text{-a.s.}$ on $\{ t < \tau_{k,n} \}$. Let $\Phi(u^l)$ denote the right hand side of (3.20). Note that the value of $\Phi(u^l)$ at $\tau_{k,n}$ depends on only the values of $u^l$ on $[0, \tau_{k,n})$. Hence we may extend the process $u^l$ from the interval $[0, \tau_{k,n})$ to the closed interval $[0, \tau_{k,n}]$ by setting

$$u^l(\tau_{k,n}) = \Phi(u^l)(\tau_{k,n}) = S_{\tau_{k,n}} u_0 - i \int_{0}^{\tau_{k,n}} S_{\tau_{k,n} - s} f(u^l(s)) ds + I_{\tau_{k,n}} (\Phi(z, u^l) - u^l(\tau_{k,n})))$$

$$+ \int_{0}^{\tau_{k,n}} \int_{B} S_{\tau_{k,n} - s} \left[ \Phi(z, u^l(s)) - u^l(s) + i \sum_{j=1}^m z_j g_j(u^l(s)) \right] \nu(dz) ds, \quad \mathbb{P}\text{-a.s.}$$

(3.21)

where $I_{\tau_{k,n}} (\Phi(z, u) - u)$ is a process defined by

$$I_{\tau_{k,n}} (\Phi(z, u^l) - u^l)(t) = \int_0^t \int_{1[0, \tau_{k,n}]} S_{t-s} \left[ \Phi(z, u^l(s \wedge \tau_{k,n}^-)) - u^l(s \wedge \tau_{k,n}^-) \right] \tilde{N}(ds, dz), \quad 0 \leq t \leq T.$$

Therefore, combining the above two equalities (3.20) and (3.21), we deduce that the stopped process $u^l(\cdot \wedge \tau_n)$ satisfies

$$u^l(t \wedge \tau_{k,n}) = S_{t \wedge \tau_{k,n}} u_0 - i \int_0^{t \wedge \tau_{k,n}} S_{t \wedge \tau_{k,n} - s} f(u^l(s)) ds + I_{\tau_{k,n}} (\Phi(z, u^l) - u^l(t \wedge \tau_{k,n}))$$

$$+ \int_0^{t \wedge \tau_{k,n}} \int_{B} S_{t \wedge \tau_{k,n} - s} \left[ \Phi(z, u^l_s) - u^l_s + i \sum_{j=1}^m z_j g_j(u^l_s) \right] \nu(dz) ds, \quad \mathbb{P}\text{-a.s.}$$

(3.22)

Since $\Delta u^l(\tau_{k,n}) = \int_{B} 1_{[0, \tau_{k,n}]} S_{\tau_{k,n} - s} \left[ \Phi(z, u^l_{s \wedge \tau_{k,n}^-}) - u^l_{s \wedge \tau_{k,n}^-} \right] \tilde{N}(\{\tau_{k,n}\}, dz)$, for $l = k, n$ and $\Phi(z, u^k_{s \wedge \tau_{k,n}^-}) - u^k_{s \wedge \tau_{k,n}^-}$ coincides with $\Phi(z, u^n_{s \wedge \tau_{k,n}^-}) - u^n_{s \wedge \tau_{k,n}^-}$ on $[0, \tau_{k,n}]$, we infer that

$$u^k = u^n \text{ on } [0, \tau_{k,n}].$$

(3.23)

Hence, by the contradiction argument, we can show that a.s.

$$\tau_k \leq \tau_n \text{ if } k < n.$$

So the limit $\lim_{n \to \infty} \tau_n =: \tau_\infty$ exists a.s. Let us denote $\Omega_0 = \{ \omega : \lim_{n \to \infty} \tau_n = \tau_\infty \}$ and note that $\mathbb{P}(\Omega_0) = 1$. 

---

Stochastic nonlinear Schrödinger equations 19
Now we define a local process \( (u(t))_{0 \leq t < \tau_\infty} \) as follows. If \( \omega \notin \Omega_0 \), set \( u(t, \omega) = 0 \), for \( 0 \leq t < \tau_\infty \). If \( \omega \in \Omega_0 \), then for every \( t < \tau_\infty(\omega) \), there exists a number \( n \in \mathbb{N} \) such that \( t \leq \tau_n(\omega) \) and we set \( u(t, \omega) = u^n(t, \omega) \). In view of (3.23) this process is well defined, \( (u(t))_{t \in [0, \tau_n]} \in M^p_\mathcal{F}(\mathcal{Y}_n) \) and it satisfies for \( t \in [0, T] \)

\[
u(t, \omega) = S_{t \wedge \tau_n} u_0 - i \int_0^{t \wedge \tau_n} S_{t \wedge \tau_n - s} f(u(s)) ds + I_{\tau_n}(\Phi(z, u) - u)(t \wedge \tau_n)
+ \int_0^{t \wedge \tau_n} \int_B S_{t \wedge \tau_n - s} \left[ \Phi(z, u(s)) - u(s) + i \sum_{j=1}^m z_j g_j(u(s)) \right] \nu(dz) ds, \quad \mathbb{P}\text{-a.s.}
\]

(3.24)

where we used the fact that because of (3.23), for all \( 0 \leq t \leq T \),

\[ I_{\tau_n}(\Phi(z, u) - u)(t) = I_{\tau_n}(\Phi(z, u^n) - u^n)(t). \]

Furthermore, by the definition of the sequence \( \{\tau_n\}_{n=1}^\infty \) we infer that a.s. on the set \( \{\tau_\infty < \infty\} \),

\[
\lim_{t \to \tau_\infty} \|u\|_{L^p} = \lim_{n \to \infty} \|u\|_{L^p_{\tau_n}} \geq \lim_{n \to \infty} n = \infty \quad \mathbb{P}\text{-a.s.}
\]

(3.25)

Using arguments similar to that in the proof of [19, Proposition 1], we can prove that \( (u(t))_{t \in [0, \tau_\infty]} \) is a maximal local mild solution of (2.1). The uniqueness of the solution follows from the construction of the solution and the uniqueness of the solution to the truncated equation.

\[ \square \]

4. Existence and Uniqueness of Global Solutions to the Stochastic NLS in the Marcus form

In this section, we shall prove the global existence of the original equation (1.1). To do that, first we show the \( L^2(\mathbb{R}^n) \)-norm of the solution is preserved. Then we establish uniform bounds for solutions of (3.1) in \( L^p([0, T); L^r(\mathbb{R}^n)) \) by using again the deterministic and stochastic Strichartz inequalities.

**Proposition 4.1.** Assume that Assumption 2.1 holds. Let \( p \geq 2, 0 < \sigma < \frac{2}{n}, 1 \leq n < \infty, (p, r) \) be an admissible pair with \( r = 2\sigma + 2 \) and \( u_0 \in L^p(\Omega; L^2(\mathbb{R}^n)) \). For \( k \in \mathbb{N} \) let \( u^k \) be the global mild solution of equation (3.1) with \( R \) replaced by \( k \). Then we have for all \( t \in [0, T] \),

\[ \|u^k(t)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)} \quad \mathbb{P}\text{-a.s.} \]

**Proof.** According to the Yosida approximating argument, we can always approximate (3.1) by equations having strong solutions. To do this, let us introduce

\[
\begin{align*}
f^k(\lambda, t, \omega) &= \lambda(\lambda - A)^{-1} f(u^k(t, \omega)), \\
G^k(\lambda, t, \omega) &= \lambda(\lambda - A)^{-1} [\Phi(z, u^k(t-)) - u^k(t-)], \\
H^k(\lambda, t, \omega) &= \lambda(\lambda - A)^{-1} [\Phi(z, u^k(t-)) - u^k(t-) + i \sum_{j=1}^m z_j g_j(u^k(t))].
\end{align*}
\]

Then the equation

\[
\begin{align*}
du^k(t) &= i\Delta u^k(t) + \theta_\epsilon(\|u^k\|_{L^p}) f^k(t, \omega) dt + \int_B G^k_{\lambda}(z, t) \mathcal{N}(dt, dz) + \int_B H^k_{\lambda}(z, t) \nu(dz) dt, \\
u^k(0) &= \lambda(\lambda - A)^{-1} u(0),
\end{align*}
\]

...
has a unique strong solution. By the properties of Yosida approximations, it’s easy to see that
\[
\lim_{\lambda \to \infty} \mathbb{E} \sup_{t \in [0,T]} \|u^k(t) - u^k(t)\|_{L^2(\mathbb{R}^n)}^2 = 0.
\]
(4.1)
Define a function \( \psi : L^2(\mathbb{R}^n) \ni u \mapsto \frac{1}{4} |u|^2_{L^2(\mathbb{R}^n)} = \frac{1}{2} \int_{\mathbb{R}^n} u(x)^2 dx \in \mathbb{R} \). Then we have
\[
\psi'(u)(v) = Re\langle u, v \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} Re\langle u(x)v(x) \rangle dx.
\]
Applying Itô formula to the function \( \psi \) and the strong solution \( u^k_t \), we obtain
\[
\psi(u^k_t(t)) - \psi(u^k_t(0)) = \int_0^t \langle \psi'(u^k_t(s)) \rangle, i(\Delta u^k_t(s) - \theta_k(\|u^k_t\|_{Y^*})f^k_t(s)) \rangle_{L^2(\mathbb{R}^n)} ds
\]
\[
+ \int_0^t \int_B \left[ \psi(u^k_t(s)) + G^k_{\lambda}(z, t) \right] \psi'(u^k_t(s)) \nu(ds, dz)
\]
\[
+ \int_0^t \int_B \left[ \psi(u^k_t(s)) + G^k_{\lambda}(z, t) \right] - \psi(u^k_t(s)) - \langle \psi'(u^k_t(s)) \rangle, G^k_{\lambda}(z, t) \rangle_{L^2(\mathbb{R}^n)} \nu(ds, dz) ds
\]
\[
+ \int_0^t \int_B \left[ \psi(u^k_t(s)), H^k_{\lambda}(z, t) \right] \nu(ds, dz)
\]
\[
\int_0^t \int_B \left[ \psi(u^k_t(s)) + G^k_{\lambda}(z, t) \right] - \psi(u^k_t(s)) - \langle \psi'(u^k_t(s)) \rangle, G^k_{\lambda}(z, t) \rangle_{L^2(\mathbb{R}^n)} \nu(ds, dz) ds
\]
\[
= - \int_0^t \int_B \left[ \psi(u^k_t(s)) + G^k_{\lambda}(z, t) \right] - \psi(u^k_t(s)) - \langle \psi'(u^k_t(s)) \rangle, G^k_{\lambda}(z, t) \rangle_{L^2(\mathbb{R}^n)} \nu(ds, dz) ds
\]
where we used the fact that \( Re\langle u(s), i\Delta u(s) \rangle_{L^2(\mathbb{R}^n)} = 0 \), since \( i\Delta \) is self-adjoint in \( L^2(\mathbb{R}^n) \).

By using (4.1), the Itô continuity of the stochastic integral and the Lebesgue dominated convergence theorem, we get as \( \lambda \to \infty \) (passing to a subsequence if necessary)
\[
\int_0^t \int B \left[ \frac{1}{2} ||u^k(s) - G^k_{\lambda}(z, t)||^2_{L^2(\mathbb{R}^n)} - \frac{1}{2} ||u^k(s)||^2_{L^2(\mathbb{R}^n)} \right] \mathbb{N}(ds, dz)
\]
\[
\int_0^t \int B \left[ \frac{1}{2} ||u^k(s) + G^k_{\lambda}(z, t)||^2_{L^2(\mathbb{R}^n)} - \frac{1}{2} ||u^k(s)||^2_{L^2(\mathbb{R}^n)} \right] \mathbb{N}(ds, dz)
\]
\[
\int_0^t \int B \left[ \frac{1}{2} ||u^k(s) - G^k_{\lambda}(z, t)||^2_{L^2(\mathbb{R}^n)} + Re\langle u^k(s), i \sum_{j=1}^m z_j g_j(u^k(s)) \rangle_{L^2(\mathbb{R}^n)} \right] \nu(ds, dz) ds
\]
\[
\int_0^t \int_B \left[ \frac{1}{2} \| \Phi(z, u^k(s-)) \|^2_{L^2(\mathbb{R}^n)} - \frac{1}{2} \| u^k(s-) \|_{L^2(\mathbb{R}^n)} + \Re \langle u^k(s), i \sum_{j=1}^m z_j g_j(u^k(s)) \rangle_{L^2(\mathbb{R}^n)} \right] \nu(dz)ds
\]

Therefore, using these limiting results and (4.1), we obtain
\[
\psi(u^k(t)) - \psi(u^k(0)) = - \int_0^t \Re \langle u^k(s), i \theta_k(\| u^k \|_{Y_\nu}) f(u^k(s)) \rangle_{L^2(\mathbb{R}^n)} ds
\]
\[
+ \int_0^t \int_B \left[ \frac{1}{2} \| \Phi(z, u^k(s-)) \|^2_{L^2(\mathbb{R}^n)} - \frac{1}{2} \| u^k(s-) \|_{L^2(\mathbb{R}^n)} \right] \tilde{N}(ds, dz)
\]
\[
+ \int_0^t \int_B \left[ \frac{1}{2} \| \Phi(z, u^k(s-)) \|^2_{L^2(\mathbb{R}^n)} - \frac{1}{2} \| u^k(s-) \|_{L^2(\mathbb{R}^n)} + \Re \langle u^k(s), i \sum_{j=1}^m z_j g_j(u^k(s)) \rangle_{L^2(\mathbb{R}^n)} \right] \nu(dz)ds.
\]

Let us observe that by (3.3)
\[
\| \Phi(z, u) \|^2_{L^2(\mathbb{R}^n)} = \| u \|^2_{L^2(\mathbb{R}^n)}, \quad \text{for all } u \in L^2(\mathbb{R}^n), \ z \in \mathbb{R}^m,
\]
and the following fundamental identity:
\[
\Re (iu, u) = \Re [i(u, u)] = \Re [i|u|^2], \quad u \in \mathbb{C}.
\]

It follows that
\[
\Re \langle u^k(s), i \sum_{j=1}^m z_j g_j(u^k(t)) \rangle_{L^2(\mathbb{R}^n)} = 0.
\]

and
\[
\Re \langle u^k(s), i \theta_k(\| u^k \|_{Y_\nu}) f(u^k(t)) \rangle_{L^2(\mathbb{R}^n)} = 0.
\]

Consequently, we obtain
\[
\psi(u^k(t)) - \psi(u_0) = 0,
\]
which finishes the proof. \(\square\)

We are finally ready to present the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let \((u^k)_{k \in \mathbb{N}}\) be the sequence of solutions of (3.1) with \(R\) replaced by \(k\) as in Proposition 3.7.

**Step 1.** We first prove that \(u^k\) is uniformly bound in \(L^p(\Omega, L^p(0, T; L^2(\mathbb{R}^n)))\), i.e. for some uniform constant \(C > 0\),
\[
\sup_k \mathbb{E} \| u^k \|^p_{L^p(0, T; L^2(\mathbb{R}^n)))} \leq C.
\]

Recall that \(u_k\) is given by
\[
u(t) = S_t u_0 + \Psi_k^1(u^k(t)) + \Psi_k^2(u^k(t)) + \Psi_k^3(u^k(t)), \quad t \in [0, T].
\]

By Proposition 3.4, we have
\[
\mathbb{E} \| \Psi_k^1(u^k(t)) \|^p_{L^p(0, T; L^2(\mathbb{R}^n)))} \leq C_{p, m}(T + T^2 \mathbb{E} \| u^k \|^p_{L^\infty([0, T]; L^2(\mathbb{R}^n)))})
\]
\[
= C_{p, m}(T + T^2 \mathbb{E} \| u_0 \|^p_{L^2(\mathbb{R}^n)}) < \infty.
\]
Hence \( \| \Psi_2(u^k)(t) \|_{L^p(0,T;L^r(\mathbb{R}^n))}^p < \infty \) P-a.s. Denote \( M_1 = \| \Psi_2(u_k)(t) \|_{L^p(0,T;L^r(\mathbb{R}^n))}^p \) which may depend on \( \omega \).

Applying similar arguments as in Propositions 3.1, 3.4 and 3.5, we have

\[
\begin{align*}
\|u_k\|_{L^p(0,T;L^r(\mathbb{R}^n))}^p \\
&\leq |S_{t_0}|u_0|_{L^p(0,T;L^r(\mathbb{R}^n))}^p + \|\Psi_1(u_k)\|_{L^p(0,T;L^r(\mathbb{R}^n))}^p + \|\Psi_2(u_k)(t)\|_{L^p(0,T;L^r(\mathbb{R}^n))}^p + \|\Psi_3(u_k)(t)\|_{L^p(0,T;L^r(\mathbb{R}^n))}^p \\
&\leq C\|u_0\|^p_{L^2(\mathbb{R}^n)} + CT(1 - \frac{m}{p})\|u_k\|_{L^p(0,T;L^r(\mathbb{R}^n))}^{p(2r+1)} + M_1 + C_mT^p\|u_k\|^p_{L^\infty(0,T;L^2(\mathbb{R}^n))} \\
&= C\|u_0\|^p_{L^2(\mathbb{R}^n)} + CT(1 - \frac{m}{p})\|u_k\|_{L^p(0,T;L^r(\mathbb{R}^n))}^{p(2r+1)} + M_1 + C_mT^p\|u_0\|^p_{L^2(\mathbb{R}^n)} \\
&= \left[(C + C_mT^p)\|u_0\|^p_{L^2(\mathbb{R}^n)} + M_1\right] + CT(1 - \frac{m}{p})\|u_k\|_{L^p(0,T;L^r(\mathbb{R}^n))}^{p(2r+1)} \\
&= M_2 + CT(1 - \frac{m}{p})\|u_k\|_{L^p(0,T;L^r(\mathbb{R}^n))}^{p(2r+1)}
\end{align*}
\]

where \( M_2 = (C + C_mT^p)\|u_0\|^p_{L^2(\mathbb{R}^n)} + M_1 \). By chosen \( T \) small enough, for instance

\[
T < (4C(2M_2)^{2\sigma})^{-\frac{1}{\frac{1}{m}-\frac{1}{p}}},
\]

such that there exists a positive number \( x \) satisfying

\[
M_2 + CT(1 - \frac{m}{p})\|x\|_{L^p(0,T;L^r(\mathbb{R}^n))}^{p(2r+1)} - x^p \geq 0 \quad \text{and} \quad 0 < x^p \leq 2M_2.
\]

It follows that for \( 1 \leq s \leq T \),

\[
\|u_k\|_{L^p(0,T;L^r(\mathbb{R}^n))}^p \leq x^p \leq 2M_2.
\]

Therefore, we obtain

\[
\mathbb{E}\|u_k\|_{L^p(0,T;L^r(\mathbb{R}^n))}^p \leq 2(C + C_mT^p)\mathbb{E}\|u_0\|^p_{L^2(\mathbb{R}^n)} + 2\mathbb{E}M_1 \\
= 2(C + C_mT^p)\mathbb{E}\|u_0\|^p_{L^2(\mathbb{R}^n)} + 2\mathbb{E}\Psi_2(u_k)(t)\|_{L^p(0,T;L^r(\mathbb{R}^n))}^p \\
\leq 2(C + C_mT^p + C_{p,m}(T + T^\frac{1}{p}))\mathbb{E}\|u_0\|^p_{L^2(\mathbb{R}^n)} := \rho.
\]

Here the constant \( \rho \) is independent of \( k \). Hence we proved (4.4).

**Step 2.** Recall from Proposition 3.7 that \( \tau_k \) is defined by \( \tau_k = \inf\{t \in [0,T] : \|u^k\|_{Y_1} \geq k\} \). By using the result we obtained in the first step, we deduce that

\[
P(\tau_k = T) = \mathbb{P}\left\{ \sup_{0 \leq t \leq T} \|u^k(t)\|_{L^2(\mathbb{R}^n)} + \|u^k\|_{L^p(0,T;L^r(\mathbb{R}^n))} \leq k \right\} \\
\geq 1 - \frac{2p\mathbb{E}\|u^k\|^p_{L^2(\mathbb{R}^n)} + 2p\mathbb{E}\|u^k\|^p_{L^p(0,T;L^r(\mathbb{R}^n))}}{k^p} \\
\geq 1 - \frac{2p\rho}{k^p}.
\]

Hence we have

\[
P(\tau_\infty = T) = \mathbb{P}\left( \bigcup_k \{\tau_k = T\} \right) = \lim_{k \to \infty} P(\tau_k = T) = 1.
\]

This shows \( (u(t))_{t \in [0,T]} \) is a global mild solution. Moreover, we have \( \|u(t)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)} \) for all \( t \in [0,T] \), P-a.s. and \( u \in L^p(\Omega; D(0,T;L^2(\mathbb{R}^n))) \cap L^p(\Omega; L^p(0,T;L^r(\mathbb{R}^n))) \) by Proposition 3.6 and 3.7.

**Acknowledgements 1.** A preliminary version of this paper (continuing Proposition 2.6 about stochastic Strichartz estimates) was presented by Zdzisław Brzeźniak at the workshop “Stochastic Analysis, Lévy processes and (B)SDEs” held in Innsbruck in October 2011 and organized by C. Geis, S. Geis and E. Hausenblas.
References

[1] Barbu V., Röckner M. and Zhang D., Stochastic nonlinear Schrödinger equations with linear multiplicative noise: rescaling approach, Journal of Nonlinear Science, 24 (3):383–409, 2014.

[2] Barbu V., Röckner M. and Zhang D., The stochastic logarithmic Schrödinger equation, J. Math. Pures Appl. 107:123–149, 2017.

[3] Barbu V., Röckner M. and Zhang D., Optimal bilinear control of nonlinear stochastic Schrödinger equations driven by linear multiplicative noise, Ann. Probab., 46(4):1957–1999, 2018.

[4] Herr S., Röckner M. and Zhang D., Scattering for stochastic nonlinear Schrödinger equations, Comm. Math. Phys., 368(2), 843–884, 2019.

[5] Brzeźniak Z., Hausenblas E. and Razafimandimby P.A.; Stochastic reaction-diffusion equations driven by jump processes, Potential Anal., 49:131–201, 2018.

[6] Brzeźniak Z., Hornung F. and Manna U., Weak martingale solutions for the stochastic nonlinear Schrödinger equation driven by pure jump noise, Stochastics and Partial Differential Equations: Analysis and Computations, 8, 2020, In press.

[7] Brzeźniak Z., Hornung F. and Weis L., Martingale solutions for the stochastic nonlinear Schrödinger equation in the energy space, Probab. Theory Related Fields, 174:1273–1338, 2019.

[8] Brzeźniak Z. and Manna U., Weak solutions of a stochastic Landau-Lifshitz-Gilbert equation driven by pure jump noise. Comm. Math. Phys. 371(3): 1071–1129, 2019.

[9] Brzeźniak Z. and Millet A.: On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact Riemannian manifold, Potential Anal., 41(2):269–315, 2014.

[10] Burq N., Gérard P. and Tzvetkov N., Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. Amer. J. Math. 126(3):569–605, 2004.

[11] Cazenave T., Semilinear Schrödinger equations, American Mathematical Society, 2003.

[12] de Bouard A., Debussche A., A stochastic nonlinear Schrödinger equation with multiplicative noise, Commun. Math. Phys. 205(1):161–181, 1999.

[13] de Bouard A. and Debussche A., The stochastic nonlinear Schrödinger equation in $H^1$, Stoch. Anal. Appl. 21(1):97–126, 2003.

[14] de Bouard A. and Hausenblas E., The nonlinear Schrödinger equation driven by jump processes, Journal of Mathematical Analysis and Applications, 475(1):215–252, 2019.

[15] Hornung F., The nonlinear stochastic Schrödinger equation via stochastic Strichartz estimates. Journal of Evolution Equations, 18(3):1085–1114, 2018.

[16] Marcus S., Modeling and analysis of stochastic differential equations driven by point processes, IEEE Transactions on Information Theory, 24(2):164–172, 1978.

[17] Marcus S., Modeling and approximation of stochastic differential equations driven by semimartingales, Stochastics: An International Journal of Probability and Stochastic Processes, 4(3):223–245, 1981.

[18] Motyl E., Stochastic Navier-Stokes equations driven by Lévy noise in unbounded 3D domains, Potential Anal. 38: 863–912, 2012.

[19] Zhu J. and Brzeźniak, Z., Nonlinear stochastic partial differential equations of hyperbolic type driven by Lévy-type noises, Discrete & Continuous Dynamical Systems, Series B, 21:3269–3299, 2016.

[20] Zhu J., Brzeźniak Z. and Hausenblas E., Maximal inequalities for stochastic convolutions driven by compensated Poisson random measures in Banach spaces, Annales de l’Institut Henri Poincaré Probabilités et Statistiques, 53:937–956, 2017.

[21] Zhu J., Brzeźniak Z. and Liu W., Maximal inequalities and exponential estimates for stochastic convolutions driven by Lévy-type processes in Banach spaces with application to stochastic quasi-geostrophic equations, SIAM Journal on Mathematical Analysis, 51:2121–2167, 2019.

[22] Zhu J., Brzeźniak Z. and Liu W., $L^p$-solutions for stochastic Navier-Stokes equations with jump noise, Statistics & Probability Letters, 155:108563, 2019.