Random Fields, Topology, and The Imry-Ma Argument

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We consider $n$-component fixed-length order parameter interacting with a weak random field in $d = 1, 2, 3$ dimensions. Relaxation from the initially ordered state and spin-spin correlation functions have been studied on lattices containing hundreds of millions sites. At $n - 1 < d$ presence of topological structures leads to metastability, with the final state depending on the initial condition. At $n - 1 > d$, when topological objects are absent, the final, lowest-energy, state is independent of the initial condition. It is characterized by the exponential decay of correlations that agrees quantitatively with the theory based upon the Imry-Ma argument. In the borderline case of $n - 1 = d$, when topological structures are non-singular, the system possesses a weak metastability with the Imry-Ma state likely to be the global energy minimum.

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Almost forty years ago Imry and Ma argued [1] that for a model described by the Hamiltonian

$$\mathcal{H} = \int d^d r \left[ \frac{\alpha}{2} (\nabla S)^2 - h \cdot S \right]$$

(1)

the random field, $h(r)$, regardless of its strength $h$, destroys the long-range order associated with the continuous-symmetry $n$-component order parameter $S$ below $d = 4$ spatial dimensions. In the resulting Imry-Ma state there is only short-range ordering within randomly oriented (Imry-Ma) domains of average size $R_f$ which depends on the strength of the random field. A similar argument had been made earlier by Larkin [2] for pinned flux lattices in superconductors. Aizenman and Wehr published a rigorous analytical proof of the destruction of the long-range order by the random field [3]. Countless papers applied the Imry-Ma argument to random magnets [4], arrays of magnetic bubbles [5], superconductors [6], charge-density waves [7], liquid crystals [8], etc. Soon, however, the picture of Imry-Ma domains became subject of a significant controversy. Theoretical work based upon renormalization group and replica-symmetry breaking methods questioned the validity of the argument for distances beyond $R_f$ [9]. Numerical work [10-15] added to the controversy by revealing glassy properties of the random-field model: The final state seemed to depend on the initial condition, making the statement about the destruction of the long-range order meaningless. It was suggested in Ref. [15] that the high energy cost of vortices in two dimensions and vortex loops in three dimensions was preventing the spins in the $xy$ random-field model from relaxing to a disordered state from the initially ordered state.

In this Letter we show more generally that the long-range behavior of random-field systems is controlled by topology. The latter, for the model described by the Hamiltonian (1) with $n$-component $S$ and $n$-component $h$, depends on the relation between $n$ and $d$. The condition $S^2 = S^2_0 = const$ leaves $n - 1$ components of the field independent. At $n - 1 < d$, mapping of $n - 1$ independent parameters describing the field $S$ onto $d' < d$ spatial dimensions provides topological defects with singularities. They are vortices in the $xy$ model ($n = 2$) in $2d$, vortex loops in the $xy$ model in $3d$, and hedgehogs in the Heisenberg model ($n = 3$) in $3d$. Energy barriers associated with creation/annihilation of these topological defects and their pinning by the random field make the final state of the system strongly dependent on the initial condition, thus invalidating the Imry-Ma argument. Moreover, as we shall see, the Imry-Ma state necessarily contains singularities that make its energy higher than that of the ordered state.

In the opposite case of $n - 1 > d$ the mapping of the $S$-space onto the $r$-space that generates topological objects is impossible. They are absent together with the energy barriers and pinning. The stable state of the system is unique and independent of the initial condition. In this case the long-range order is destroyed in a manner that agrees quantitatively with the Imry-Ma picture. This applies to the Heisenberg model with $n = 3$ (and greater) in one dimension, $n = 4$ (and greater) in two dimensions, and $n = 5$ (and greater) in $3d$.

The case of $n - 1 = d$ is the borderline between the above two cases. It corresponds to non-singular topological objects: Kinks in the $xy$ model in $1d$, skyrmions in the Heisenberg model with $n = 3$ in $2d$, and similar non-singular solutions for $n = 4$ in $3d$. They are characterized by the topological charge. Its conservation is important as it is only weakly violated by the discreteness of the lattice and weak random field. Possession of a pinned topological charge by the Imry-Ma state prevents the system from relaxing to this state from any initial state that has a different topological charge.

To illustrate the validity of the above arguments, we have numerically studied the discrete counterpart of the Hamiltonian (1) with the nearest-neighbor exchange in-
teraction in the presence of the external field $H$, 

$$\mathcal{H} = -\frac{1}{2} \sum_{ij} J_{ij} s_i \cdot s_j - \sum_i h_i \cdot s_i - H \cdot \sum_i s_i,$$  \hspace{1cm} (2)$$
on lattices containing hundreds of millions spins $s_i$ of length $s$. The relation between parameters of the continuous and discrete models is $\alpha = J a^{d+2}$, $S_0 = s/a^d$, where $a$ is the lattice parameter. We consider hypercubic lattices with periodic boundary conditions containing $L^d$ spins; $L$ being the linear size of the system. In computations we use $J = s = a = 1$ and $h = H_R$. Our numerical method combines sequential rotations of spins towards the direction of the local effective field, 

$$H_{i,\text{eff}} = \sum_j J_{ij} s_j + h_j + H_i,$$

with energy-conserving spin flips: $s_i \rightarrow 2(s_i \cdot H_{i,\text{eff}})H_{i,\text{eff}}/H_i^2 - s_i$ (so-called over-relaxation), applied with probabilities $\alpha$ and $1 - \alpha$ respectively; $\alpha$ playing the role of the relaxation constant. Refs. [14, 15] show high efficiency of this method for glassy systems under the condition $\alpha \ll 1$ which is physically equivalent to slow cooling. Relaxation of the per-site magnetization, 

$$m = \sqrt{\mathbf{m} \cdot \mathbf{m}},$$

where 

$$\mathbf{m} = (sN)^{-1} \sum_i \mathbf{S}_i,$$

out of a collinear state is shown for different $d$ and $n$ in Fig. 1.

In one dimension, $n = 2$ is the marginal case. Numerical analysis of different spin configurations shows that the Imry-Ma-like state with $m = 0$ has the lowest energy. This state, however, cannot be achieved through relaxation from the initially ordered state without forming non-singular kinks or antikinks associated with the full clockwise or counterclockwise rotations of the spin as one moves along the spin chain. They become pinned by the random field. The difference in the number of kinks and antikinks is a conserved topological charge. While the system tends to disorder, it cannot do so completely because it requires changing the topological charge. However, for three-component spins in one dimension, topologically stable objects are absent and the system disorders completely as is illustrated by Fig. 1.

Two-component spins in two dimensions form well-known topological singularities – vortices in the $xy$ model [10, 15, 16]. Here again the system wants to relax to the Imry-Ma-like state with $m = 0$ but cannot do it without forming vortices that cost energy, which explains the curve in Fig. 1b for the $xy$ model in 2d.

In the marginal case of $d = 2$, $n = 3$ the model possesses non-singular topological objects – skyrmions [16]. In the absence of the random field the difference in the number of skyrmions and antiskyrmions is a conserved topological charge. Skyrmions on the lattice tend to collapse [17]. However, pinning by the random field stabilizes them. We have checked numerically that for $d = 2$, $n = 3$ the Imry-Ma state with $m = 0$ has the lowest energy. However, conservation of the topological charge prevents the system from relaxing to this state from almost any initial condition. This effect is responsible for a small but finite magnetization obtained by the relaxation from the initially ordered state, see Fig. 1b. However, for a four-component spin in two dimensions, topological objects are absent and the system relaxes to the state with a zero magnetic moment, see Fig. 1b.

Relaxation in a three-dimensional case is illustrated by
Figure 2: Topological singularities in the random-field spin model in three dimensions obtained by relaxation from random initial orientation of spins: (a) Pinned vortex loops of the \( xy \) \((n = 2)\) model; (b) Pinned hedgehogs of the Heisenberg \((n = 3)\) model. \( f_S \) is fraction of the lattice interstitial (body centered) sites occupied by singularities.

Figure 3: Hysteresis curves of the random field spin model in two dimensions for \( n = 2, 3, 4 \). The model with non-singular topological structures pinned by the random field which are similar to skyrmions in 2d. In this case the final magnetic moment is still non-zero but small, see Fig. 2. We again find that the energy of the Imry-Ma-like state \( m = 0 \) state for \( d = 3 \), \( n = 4 \) is lower than that of the \( m \neq 0 \) state. However, the difference in the topological charge prevents the system from relaxing to the Imry-Ma state from almost any initial state.

The model with five-component spins in 3d does not possess any topologically stable structures. The relaxation of the system from the ordered initial state is unobstructed by any topological arguments and the system ends up in the state with \( m = 0 \), Fig. 1c.

The relation between topology and metastability in, e.g., two spatial dimensions is further illustrated by the hysteresis curves in Fig. 3. The model with \( n = 2 \), that possesses \( xy \) vortices with singularities, is characterized by sizable hysteresis loop which is indicative of strong metastability. The loop becomes thin for \( n = 3 \) when non-singular skyrmions are present. It disappears completely, resulting in a reversible magnetic behavior, at \( n = 4 \) when topological objects are absent. Similar behavior for different \( n \) has been observed in 3d.

To get a better understanding of how topology modifies the Imry-Ma argument and leads to metastability let us recall that in that argument the order parameter \( S \) follows the direction of the average random field, \( \bar{h} \), on a scale \( R_f \). The energy of the random field in Eq. (1) is proportional to \( -\bar{h}S_0 \sim -hS_0/R_f^{d/2} \), while the non-uniformity energy is proportional to \( \alpha S_0^2/R_f^2 \). Minimization of the total energy with respect to \( R_f \) then leads to the rotation of \( S \) by a significant angle on a scale \( R_f \propto (Js/h)^{2/(4-d)} \). For \( R \gtrsim R_f \) correlations should be completely destroyed, thus the state of the system should be disordered. This famous argument, however, does not account for the energy associated with unavoidable singularities at \( n - 1 < d \).
Figure 4: Emergence of vortices and antivortices at the intersections of lines corresponding to $\bar{h}_x = 0$ (red) and $\bar{h}_y = 0$ (blue) in the random-field $2d$ $xy$ model. Picture reflects numerical averaging of $h$ within finite range according to the prescription of the Imry-Ma model for a particular realization of the random field. Arrows show spins on lattice sites. Similar structures emerge after relaxation from random orientation of spins, with the positions of vortices depending on the initial state.

To show this, consider components of the averaged random field $\bar{h}_\alpha$, $\alpha = 1, \ldots, n$. Since $\bar{h}_\alpha$ are sums of many random numbers, they are statistically independent and have Gaussian distribution. In about a half of the space $\bar{h}_\alpha > 0$, in the other half $\bar{h}_\alpha < 0$. Boundaries between these regions are subspaces of dimension $d - 1$, where $\bar{h}_\alpha = 0$. Intersection of all these subspaces, that is, $\bar{h} = 0$, is unavoidable and forms a subspace of dimension $d - n$, if $n \leq d$. It is easy to see that subspaces with $\bar{h} = 0$ are singularities in the spin field $S$. Since $S^2 = \text{const}$, crossing subspaces $\bar{h} = 0$ makes all components of $S$ change the direction.

For $n = 2$ in $2d$ subspaces $\bar{h} = 0$ are points and the corresponding singularities are vortices or antivortices. A spin field in the $2d$ $xy$ model generated in accordance with the Imry-Ma prescription is shown in Fig. 4. The red line corresponds to $\bar{h}_x = 0$ and thus spins directed along the $y$-axis. The blue line corresponds to $\bar{h}_y = 0$ and thus spins directed along the $x$-axis. At the intersections of red and blue lines the spins can look neither in the $x$ nor in the $y$-direction. This generates topological defects – vortices or antivortices. For $n = 3$ in $3d$ subspaces $\bar{h} = 0$ are lines and the singularities are vortex lines or loops. For $n = 3$ in $3d$ subspaces $\bar{h} = 0$ are points and the singularities are hedgehogs. They emerge at the intersection of surfaces corresponding to $\bar{h}_x = 0$, $\bar{h}_y = 0$, and $\bar{h}_z = 0$, see Fig. 5. Singularities push the energy of the Imry-Ma state up and (according to our numerical results) make it higher than the energy of the ordered state. However, for $n > d$, the averaged random field is non-zero everywhere and the spin field $S$ is non-singular. Consequently, at $n > d$ the $m = 0$ state has the lowest energy in accordance with the Imry-Ma argument and the Aizenman-Wehr theorem that assume continuity of the spin field. Still at $n = d + 1$ the presence of non-singular topological objects and conservation of topological charge prevents the ordered state from relaxing to the $m = 0$ Imry-Ma state. Only at $n > d + 1$, when the spin-field is continuous and topological objects are absent the system relaxes to the Imry-Ma state.

To get a better idea of the long-range correlations in the random-field model we have computed the spin-spin correlation function in the final state obtained through relaxation from the initially ordered state. Fig. 6 shows
the 3d spin-spin correlation function for \( n = 2, 3, 4 \). Note that the height of the plateau at large distances equals \( m^2 \) for \( R_f \ll L \). For \( n = 2, 3 \) the ferromagnetic order persists at all distances. For \( n = 4 \) the long-range correlation plateau becomes very low in accordance with Fig. [4]. The 3d correlation function for \( n = 5 \) is shown in Fig. [7]. Here, in accordance with the Imry-Ma picture, correlations are fully destroyed at large distances.

We have analytically derived from Eq. [1] the short-distance form of the spin-spin correlation function in 3d, 

\[
\langle \mathbf{s}(\mathbf{r}_1) \cdot \mathbf{s}(\mathbf{r}_2) \rangle \cong 1 - |\mathbf{r}_1 - \mathbf{r}_2|/R_f,
\]

where

\[
R_f = \frac{8\pi a^2 S_0^2}{h^2 a^4 (1 - 1/n)} = \frac{8\pi}{(1 - 1/n)} \left( \frac{J_s}{h} \right)^2.
\]

In fact, this short-range form of the correlation function agrees with our numerical results for all \( n \) in 3d. For \( n \geq 5 \) the spin-spin correlation function at all distances can be very well fitted by \( \langle \mathbf{s}(\mathbf{r}_1) \cdot \mathbf{s}(\mathbf{r}_2) \rangle = \exp(-|\mathbf{r}_1 - \mathbf{r}_2|/R_f) \) that we will loosely call “theoretical”. A good agreement with this formula is illustrated by Fig. [7]. So far we have been able to prove analytically the numerically confirmed exponential decay of the correlation function in 3d only for the mean-spherical model which corresponds to \( n = \infty \). However, the observed exponential behavior of \( \langle \mathbf{s}(\mathbf{r}_1) \cdot \mathbf{s}(\mathbf{r}_2) \rangle \) and the observed \( 1/h^2 \) dependence of \( R_f \) on the strength of the random field for small \( h \) at \( n = 5, d = 3 \) (see Fig. [5]) present clear evidence of the onset of the Imry-Ma state in the absence of topological objects.

In conclusion, we have demonstrated that the long-range order in random-field systems is controlled by topology. For the \( n \)-component spin in \( d \) dimensions the presence of topological structures at \( n - 1 \leq d \) invalidates the Imry-Ma argument which suggests destruction of the long-range order by the arbitrary weak random field. For \( n - 1 > d \), when topological structures are absent, the long-range behavior of the system agrees quantitatively with the Imry-Ma picture. This finding solves the controversy that existed in the field for over four decades. It provides the guiding principle for assessing the long-range behavior of various systems with quenched randomness and continuous-symmetry order parameter.

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