Analysis of Flux Corrected Transport Schemes for Evolutionary Convection-Diffusion-Reaction Equations

Abhinav Jha †, Naveed Ahmed‡

Abstract

We report in this paper the analysis for the linear and nonlinear version of the flux corrected transport (FEM-FCT) schemes in combination with the backward Euler time-stepping scheme applied to a time-dependent convection-diffusion-reaction problems. We present the stability and error estimates for the linear and nonlinear FEM-FCT scheme. Numerical results confirm the theoretical predictions.

Keywords: evolutionary convection-diffusion-reaction equations; finite element method flux corrected transport; finite element error analysis

AMS subject classifications: 65N12, 65N30

1 Introduction

The time-dependent convection diffusion reaction equations are used to model many physical processes arising in computational fluid dynamics. When convection dominates the diffusion, we have the presence of layers in the interior and the boundary of the domain.

To overcome the difficulty of instabilities, stabilization schemes are applied. One of the most popular methods is the Streamline Upwind Petrov-Galerkin (SUPG) method which was introduced in [HB79] for the steady state problems. However, the drawback of the SUPG method for time-dependent problem is the fact that for ensuring the strong consistency the time derivative, the second order derivative and the source term have to be included into the stabilization term. The SUPG method in space combined with different time discretization was investigated in [Bur10]. In [JN11], the SUPG method is combined with the backward Euler and Crank-Nicolson methods. It turns out that the stabilization vanishes if the time step length approaches zero. In the case of time independent convection and reaction coefficients, error estimates are derived which allow the stabilization parameters to be chosen similar to the steady-state case.

Alternative to SUPG are the symmetric stabilization schemes such as the local projection stabilization (LPS) [AMTX11, BB04, BB06, AM15], the continuous interior penalty method (CIP) [BF09], the subgrid scale modeling (SGS)
A comparison of the SUPG method with the LPS method can be found in \cite{AM15, AJ15}.

Another stabilization approach is the so-called algebraic schemes introduced in \cite{Zal79} and was combined with finite element discretization in \cite{PC86} which works on an algebraic level rather than a variational level. In literature, these schemes are referred to as flux-corrected transport (FEM-FCT). Some of the prominent work in this direction has been done by Kuzmin and co-authors and can be found in \cite{Kuz06, Kuz07, LKSM17, Loh19}. It has been noted in \cite{JR10} that the FEM-FCT schemes performed better than the SUPG schemes. A comparison for different stabilization schemes and discretization can be found in \cite{ACF+11}.

This technique’s steady-state counterpart is referred to as the algebraic flux correction (AFC) schemes introduced in \cite{Kuz07}. The numerical analysis of the AFC schemes has been developed recently in \cite{BJK16} and a modification to the limiter definition is presented in \cite{BJK17}. This paper provided a new insight for the AFC schemes as this was the first finite element error analysis for the schemes. The stabilization parameters are referred to as limiters, and a comprehensive comparison of the results based on different limiters is presented in \cite{BJKR18}. One of the drawbacks of the AFC schemes is their nonlinear nature. A study on the solvers for these schemes can be found in \cite{JJ19, JJ20}. This nonlinearity also arises in the FEM-FCT schemes, and a study for different types of solvers is presented in \cite{JN12}. The FCT schemes have been successfully applicable only to lower-order elements and we consider the same in our analysis.

As mentioned above, the FEM-FCT schemes are nonlinear in nature when combined with a $\theta-$scheme, but one can also linearize the scheme using the solution at the previous time step. A comparison of the linear and the nonlinear schemes are present in \cite{JS08} where it is shown that the linear FEM-FCT has a better ratio of accuracy and efficiency. The solvability of the nonlinear FEM-FCT schemes has been presented recently in \cite{JKK21} and the existence and uniqueness of the solution are shown in \cite{JK21} where the analysis from \cite{BJK16} is extended. This paper presents the finite element error analysis of the FEM-FCT schemes with backward Euler as the time-stepping. To the best of our knowledge, this is the first work in this direction. The positivity of the solution has already been discussed in \cite{Kuz09} where the positivity is independent of the choice of limiters’. Here we will be considering both the linear and the nonlinear version of the FEM-FCT schemes mentioned in \cite{Kuz09}. We assume the limiter’s general properties in order to obtain optimal order of convergence on the time and space in the $L^2$ and the $H^1$ norm. The analysis is performed in the system’s natural norm, which we refer to as the FCT norm.

The work’s main findings are the CFL-like stability condition, the optimal convergence rate for the $L^2$ and the $H^1$ norm, and the sub-optimal convergence rate for the FCT norm. Numerical simulations verify the analytical findings. The optimal convergence rate in the $L^2$ and the $H^1$ norm and the sub-optimal convergence rate in the FCT norm are proved.

The structure of the paper is as follows: In Sec. 2 we introduce the FEM-FCT scheme and give an example of the limiter that will be used in the simulations. In Sec. 3 we provide a variational formulation of the scheme and prove the stability of the linear as well as the nonlinear version of the scheme in the FCT norm. In Sec. 4 we prove the finite element error analysis of the scheme using standard interpolation estimates. Next, in Sec. 5 we provide the numerical simulations
of the scheme on three different types of the grid. Lastly, in Sec. 6 we present a summary and provide an outlook.

2 Preliminaries

Let $\Omega \in \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with boundary $\partial \Omega$ and $[0, T]$ be a bounded time interval. Consider the evolutionary convection-diffusion-reaction equation: Find $u : (0, T] \times \Omega \rightarrow \mathbb{R}$ such that

$$
\begin{align*}
    u' - \varepsilon \Delta u + b \cdot \nabla u + cu &= f &\text{in } (0, T] \times \Omega, \\
    u &= 0 &\text{in } (0, T] \times \partial \Omega, \\
    u(0, x) &= u_0 &\text{in } \Omega.
\end{align*}
$$

Here $0 < \varepsilon \ll 1$ is a diffusivity parameter, $b(t, x)$ is the convection field, $c(t, x)$ is the reaction coefficient, $f(t, x)$ is the given outer source of unknown quantity $u$, and $u_0(x)$ is the initial condition. The use of homogeneous Dirichlet boundary conditions in the system (1) is just for simplicity of presentation. Furthermore, it is assumed that there exists a positive constant $c_0$ such that

$$
c(t, x) - \frac{1}{2} \nabla \cdot b(t, x) \geq c_0 > 0
$$

which guarantees the unique solvability of Eq. (1) (see [RST08]).

This paper’s main topic is the analysis of the FEM-FCT schemes described in [Kuz09]. To keep the paper self-contained, a short presentation of these schemes will be given here. Consider a spatial discretization of Eq. (1) using the FEM with piecewise linear elements. For temporal discretization, a backward Euler scheme is used. These schemes, in algebraic form, lead at the discrete-time $t_n$ to an equation of the form

$$
M_C u^n + \tau A u^n = \tau f^n + M_C u^{n-1},
$$

where $\tau$ is the time step length. The matrix $M_C$ is the consistent mass matrix, and the stiffness matrix $A$ is the sum of diffusion, convection, and reaction. Furthermore, the notations $u^n$, $f^n$ stand for the vectors unknown coefficients of the finite element method.

In order to satisfy the maximum principle for the discrete problem, the system matrix should be an $M$-matrix. The sufficient condition for a matrix to be an $M$-matrix is that all the diagonal entries are positive, all off-diagonal entries are non-positive, and the row sum is positive [Cia70, Theorem 2]. To achieve this, we modify the left hand side of Eq. (3) such that the system matrix corresponds to an $M$-matrix. Let us define the artificial diffusion matrix $D$ such that

$$
d_{ij} = -\max\{a_{ij}, 0, a_{ji}\} \quad \text{for } i \neq j, \\
d_{ii} = -\sum_{j=1, j \neq i}^{N} d_{ij},
$$

where $N$ is the number of degrees of freedom and lumped mass matrix $M_L$

$$
M_L = \text{diag } (m_i), \quad m_i = \sum_{j=1}^{N} m_{ij}.
$$
Then, the matrix $A = A + D$ and $M_L$ satisfies the condition for a $M$-matrix and the following scheme is a stable low-order scheme

$$M_L u^n + \tau A u^n = \tau f^n + M_L u^{n-1} + f^*(u^n, u^{n-1}). \quad (6)$$

In the next step, one needs to define an appropriate ansatz for $f^*(u^n, u^{n-1})$. To this end, subtracting Eq. (3) from Eq. (6) one get the residual vector

$$r = (M_L - M_C) u^n + (A - A) u^n \tau - (M_L - M_C) u^{n-1}.$$ 

The idea is to limit these modifications by introducing solution dependent limiters $\alpha_{ij}$ such that

$$f_i^*(u^n, u^{n-1}) = \sum_{j=1}^{N} \alpha_{ij} f_{ij} \quad i = 1, \ldots, N.$$ 

The basic idea of FEM-FCT is to find these weights in such a way that they are close to 1 in smoother regions (i.e., we recover the Galerkin FEM) and close to 0 in the vicinity of layers (to recover the low-order scheme). The contribution of $f_i^*(u^n, u^{n-1})$ stems from a decomposition of the residual vector

$$r_i = \sum_{j=1}^{N} f_{ij} = \sum_{j=1}^{N} m_{ij}(u^n_j - u^n_{j-1}) - \sum_{j=1}^{N} m_{ij}(u^{n-1}_i - u^{n-1}_j) + \sum_{j=1}^{N} d_{ij}(u^n_j - u^n_i) \tau$$

for $i = 1, \ldots, N$. The above representation is derived from the definition of $D$ and $M_L$. The number $f_{ij}$ are referred as fluxes.

In order to have a conservative scheme, the limiters $\alpha_{ij}$ have to be symmetric i.e.,

$$\alpha_{ij} = \alpha_{ji}, \quad (7)$$

Now, we can rewrite Eq. (6) in an algebraic form

$$\sum_{j=1}^{N} m_{ij}(u^n_j - u^n_{j-1}) + \tau \sum_{j=1}^{N} a_{ij} u^n_j$$

$$+ \sum_{j=1}^{N} m_{ij}(1 - \alpha_{ij}) \left[(u^n_i - u^n_{i-1}) - (u^n_j - u^n_{j-1})\right]$$

$$+ \tau \sum_{j=1}^{N} d_{ij}(1 - \alpha_{ij})(u^n_j - u^n_i) = f^n_i \tau \quad \text{for } i = 1, \ldots, N, \quad (8)$$

where $\alpha_{ij} \in [0, 1], i, j = 1, \ldots, N$ satisfy Eq. (7).

The scheme Eq. (8) can be handled in two different ways. The nonlinear version of the FEM-FCT scheme utilizes an explicit solution $\bar{u}$ with the forward Euler scheme at $t_n - \tau/2$

$$\bar{u} = u^{n-1} - \frac{\tau}{2} M_L^{-1} (Au^{n-1} - f^{n-1}). \quad (9)$$

Here $\bar{u}$ will be used as a prelimiting of the fluxes for the nonlinear scheme. If $f_{ij}(\bar{u}_i - \bar{u}_j) < 0$ then set $f_{ij} = 0$, which is prescribed in [KM05, Kuz09].
The linear FEM-FCT is a special case presented in [Kuz09]. Consider the vector \( u_n \) in the flux \( f_{ij} \) which is replaced by an approximation obtained with an explicit scheme. Using \( u_n^{n-1/2} = (u_n^+ + u_n^-)/2 \) in the definition of \( f_{ij} \) leads to

\[
f_{ij} = 2m_{ij}(u_i^{n-1/2} - u_i^{n-1}) - 2m_{ij}(u_j^{n-1/2} - u_j^{n-1}) \\
+ 2\tau d_{ij}(u_i^{n-1/2} - u_i^{n-1/2}) + \tau d_{ij}(u_i^{n-1} - u_j^{n-1}).
\]

We can approximate \( u_n^{n-1/2} \) by the forward Euler method in the same way as the pre-limiting in the nonlinear scheme. Inserting Eq. (9) leads to

\[
f_{ij} = \tau m_{ij}(\nu_i^{n-1/2} - \nu_j^{n-1/2}) + \tau d_{ij}\left[u_i^{n-1} - u_i^{n-1} + \tau(\nu_i^{n-1/2} - \nu_i^{n-1/2})\right]
\]

where \( \nu_i^{n-1/2} = (M_L^{-1}(f_{n-1} - A u_n^{n-1}))_i \).

Note that both methods use an explicit method as predictor, which results in a CFL condition for these methods, for details see [KM05, Kuz09].

2.1 Limiters

In this section we give an example of limiter that will be used in the simulations. We follow the definition of the Zalesak algorithm presented in [JN12].

1. Compute

\[
P_i^+ = \sum_{j=1, j\neq i}^N f_{ij}^+ \quad P_i^- = \sum_{j=1, j\neq i}^N f_{ij}^-.
\]

2. Compute

\[
Q_i^+ = \max\left\{0, \max_{i=1, \ldots, N, j \neq i} (\bar{u}_j - \bar{u}_i)\right\}, \\
Q_i^- = \min\left\{0, \min_{i=1, \ldots, N, j \neq i} (\bar{u}_j - \bar{u}_i)\right\}.
\]

3. Compute

\[
R_i^+ = \min\left\{1, \frac{m_i Q_i^+}{P_i^+}\right\}, \\
R_i^- = \min\left\{1, \frac{m_i Q_i^-}{P_i^-}\right\}.
\]

If the \( P_i^+ \) or \( P_i^- \) is zero, we set \( R_i^+ = 1 \) or \( R_i^- = 1 \), respectively.

4. Compute

\[
\alpha_{ij} = \begin{cases} \min\{R_i^+, R_j^+\} & \text{if } f_{ij} > 0 \\ \min\{R_i^-, R_j^-\} & \text{otherwise}, \end{cases}
\]

where \( f_{ij}^+ = \max\{f_{ij}, 0\} \) and \( f_{ij}^- = \min\{f_{ij}, 0\} \).
3 Stability of the FEM-FCT Methods

This section studies the stability of the fully discrete version of the linear and nonlinear FEM-FCT schemes. To this end, we first write Eq. (1) into a variational form, and then using the finite element discretization, we mention the variational formulation of the nonlinear algebraic problem.

Let \( V \subset H^1_0(\Omega) \). A variational formulation of Eq. (1) reads: Find \( u : (0, T] \rightarrow V \) with such that

\[
(u', v) + \varepsilon (\nabla u, \nabla v) + (b \cdot \nabla u + cu, v) = (f, v) \quad \forall \; v \in V.
\] (10)

Here, \((\cdot, \cdot)\) denotes the inner product in \( L^2(\Omega)^d \). For the finite element discretization of Eq. (10), \( V \) is replaced by the finite-dimensional space \( V_h \), where \( h \) represents the mesh size of the underlying triangulation \( \{T_h\} \) of \( \Omega \). We consider in this paper the conforming finite element method and, therefore, \( V_h \subset V \).

The numerical analysis of the AFC schemes has been derived for first-order elements on triangles, i.e., we choose \( V_h = P_1 \).

The variational formulation for the nonlinear FEM-FCT scheme reads: Find \( u_h : (0, T] \rightarrow V_h \) with \( u_h(0) = u_{h,0} \) such that

\[
(u'_h, v_h) + a_h(u_h, v_h) + d^D_h(u_h; u_h, v_h) + d^M_h(u_h; u_h, v_h) = (f, v_h) \quad \forall \; v_h \in V_h,
\] (11)

where

\[
a_h(u_h, v_h) = \varepsilon (\nabla u_h, \nabla v_h) + (b \cdot \nabla u_h, v_h) + (cu_h, v_h),
\]

\[
d^D_h(u_h; u_h, v_h) = \sum_{i,j=1}^N (1 - \alpha_{ij}) d_{ij}(u_{hj} - u_{hi}) v_{hi},
\]

\[
d^M_h(u_h; u_h, v_h) = \sum_{i,j=1}^N (1 - \alpha_{ij}) m_{ij} (u'_{hi} - u'_{hj}) v_{hi},
\] (12)

and \( u'_h \) represents the time derivative of \( u \).

Note that \( u_{h,0} \in V_h \) is a suitable approximation of \( u_0 \) in the finite dimensional space \( V_h \).

The backward Euler scheme as the temporal discretization of Eq. (11) lead at the discrete-time \( t_n \) to an equation of the form

\[
(u^n_h, v_h) + \tau \left[ a_h(u^n_h, v_h) + d^D_h(u^n_h; u^n_h, v_h) + d^M_h(u^n_h; u^n_h, v_h) \right]

= \tau (f^n, v_h) + (u^{n-1}_h, v_h) \quad \forall \; v_h \in V_h.
\] (13)

Similarly, the discrete version of the linear FEM-FCT scheme after the temporal discretization leads to the fully discrete problem

\[
(u^n_h, v_h) + \tau \left[ a_h(u^n_h, v_h) + d^D_h(u^n_h; u^n_h, v_h) + d^M_h(u^n_h, v_h) \right]

= \tau (f^n, v_h) + \tau (f^*, v_h) + (u^{n-1}_h, v_h) \quad \forall \; v_h \in V_h.
\] (14)
where

\[
d^D_h(u_h, v_h) = \sum_{i,j=1}^{N} d_{ij}(u_{hj} - u_{hi})v_{hi},
\]

\[
d^M_h(u_h, v_h) = \sum_{i,j=1}^{N} m_{ij} (u'_{hi} - u'_{hj}) v_{hi},
\]

\[
(f^*, v_h) = \sum_{i,j=1}^{N} \alpha_{ij} \left( (\tau d_{ij} - m_{ij}) (v_{hj}^{n-1/2} - v_{hi}^{n-1/2}) \right) v_{hi}
\]

\[
+ \sum_{i,j=1}^{N} d_{ij} \alpha_{ij} (u_{hj}^{n-1} - u_{hi}^{n-1}) v_{hi}.
\]

For the analysis, some preliminaries are introduced. Assuming that the meshes are quasi-uniform, the following inverse inequality (see [BS08, Lemma 4.5.3]) holds for each \(v_h \in \mathcal{V}_h\)

\[
\|v_h\|_{W^{m,q}(K)} \leq C_{inv} h^{-m-d(1/q'-1/q)} \|v_h\|_{W^{1,q'}(K)} ,
\]

where \(0 \leq l \leq m \leq 1, 1 \leq q' \leq q \leq \infty\), and \(\|\cdot\|_{W^{m,q}(K)}\) is the norm in \(W^{m,q}(K)\). The norm and the semi-norm in \(W^{m,q}\) are given by \(\|\cdot\|_{m,q}\) and \(|\cdot|_{m,q}\), respectively. In case \(q = 2\), we write \(H^m(\Omega)\), \(\|\cdot\|_m\), and \(|\cdot|_m\) instead of \(W^{m,q}(K)\), \(\|\cdot\|_{m,q}\), and \(|\cdot|_{m,q}\), respectively.

Consider \(v \in H^1(K)\) and \(E \subset \partial K\), then the following trace inequality holds (see [Ver98, Lemma 3.1])

\[
\|v\|_{0,E} \leq C_{T_1} h^{-1/2} \|v\|_{0,K} + C_{T_2} h^{1/2} \|
abla v\|_{0,K},
\]

where \(C_{T_1}\) and \(C_{T_2}\) are constants independent of \(h\).

Let \(m_{ij}\) be an element of the mass matrix \(M_C\), then the estimate

\[
|m_{ij}| \leq C_{m_{ij}} h^d,
\]

holds true, where \(C_{m_{ij}}\) is independent of \(h\).

**Lemma 1.** ([Jha20, Lemma 5.23]) Let \(E\) be an edge with length \(h_E\) and \(v\) be a linear function on \(E\), then

\[
\|\nabla v \cdot t_E\|_{0,E}^2 \leq \|\nabla v\|_{0,E}^2,
\]

where \(t_E\) is the tangent unit vector to \(E\).

The next lemma states the coercivity of FEM-FCT scheme and the proof can be found in [BJK16].

**Lemma 2.** Let Eq. (2) be satisfied. Then the bilinear form

\[
a_{FCT}(u, v) = a_h(u, v) + d_h(u; u, v)
\]

associated with the FEM-FCT scheme is coercive with respect to the \(\|\cdot\|_{FCT}\) norm, i.e.,

\[
a_{FCT}(u, u) \geq C_{FCT} \|u\|_{FCT}^2,
\]

where \(C_{FCT}\) is the coercive constant and the \(\|\cdot\|_{FCT}\) norm is defined by

\[
\|u_h\|_{FCT} := (\varepsilon \|u_h\|^2_1 + c_0 \|u_h\|^2_0 + d_h(u_h; u_h, u_h))^{1/2}.
\]
For linear elements \( u, v \in V_h \), one can represent \( d_h^D(\cdot, \cdot) \) through an edge formulation (see [BJKR18, Eq. (16)])

\[
d_h^D(w; u, v) = \sum_{E \in \mathcal{E}_h} (1 - \alpha_E(w)) |d_E| h_E (\nabla u \cdot t_E, \nabla v \cdot t_E)_E,
\]

where \( \mathcal{E}_h \) is the set of all edges, \( d_E \) denotes \( d_{ij} \), and \( \alpha_E \) denotes \( \alpha_{ij} \) along the edge \( E \) with endpoints \( x_i \) and \( x_j \).

In the same way, we can represent \( d_h^M(\cdot, \cdot) \) also through an edge formulation

\[
d_h^M(w; u, v) = \sum_{E \in \mathcal{E}_h} (1 - \alpha_E(w)) |m_E| h_E (\nabla u \cdot t_E, \nabla v \cdot t_E)_E,
\]

where \( m_E \) denotes \( m_{ij} \). Similarly we can represent \( d_h^M(\cdot, \cdot) \) (and \( d_h^D(\cdot, \cdot) \)) in an edge formulation. We note that we have a predictor-corrector scheme for the FEM-FCT algorithm and hence, we need to prove stability estimates for both the predictor as well as the corrector step.

The forward Euler (FE) scheme at time \( t_{n+1} - (\tau)/2 \) gives

\[
\tilde{u}^n = u^{n-1} - \frac{\tau}{2} M_L^{-1}(A u^{n-1} - f_{n-1}).
\]

Our variational formulation for the FE scheme looks like: Find \( \tilde{u}_h^n \in V_h \) such that

\[
(u_{FE}, v) + a(u_h^{n-1}, v) + d_h^D(u_h^{n-1}, v) + d_h^M(u_{FE}, v) = (f_{n-1}, v),
\]

where

\[
u_{FE} = \frac{1}{2} \left( \frac{\tilde{u}_h^n - u_h^{n-1}}{\tau} \right) \quad \text{and},
\]

\[
d_h^M(u_{FE}, v) = \frac{2}{\tau} \sum_{i,j=1}^N m_{ij} \left( \tilde{u}_h^i - \tilde{u}_h^j - u_h^{i-1} + u_h^{j-1} \right) v^i.
\]

**Theorem 1.** Let Eq. (2) hold. With the additional condition

\[
\tau \leq C h^2,
\]

then the solution \( \tilde{u}_h^n \) of Eq. (22) satisfies at \( t_n = n \tau \)

\[
\|\tilde{u}_h^n\|^2_0 + C \tau \sum_{m=1}^n \|u_m^n\|^2_{FCT}
\]

\[
\leq \|\tilde{u}_h^n\|^2_0 + \sum_{E \in \mathcal{E}_h} |m_E| h_E \|\nabla u_h^0 \cdot t_E\|^2_0 + C \tau \sum_{m=1}^n \|f_m\|^2_0,
\]

where \( C \) is a constant that do not depend on \( h \) and \( \tau \).

**Proof.** Taking \( v = \tilde{u}_h^n \) in Eq. (22) and using

\[
(u_h^n - u_h^{n-1}, \tilde{u}_h^n) = \frac{1}{2} \left( \|\tilde{u}_h^n\|^2_0 - \|u_h^{n-1}\|^2_0 + \|u_h^n - u_h^{n-1}\|^2_0 \right),
\]

8
adding \( a(\bar{u}_h^n, \bar{u}_h^n) \) and \( d_h(\bar{u}_h^n, \bar{u}_h^n) \) on both sides and using the Cauchy-Schwarz inequality we get

\[
\frac{1}{\tau} \left( \|\bar{u}_h^n\|_0^2 - \|u_h^{n-1}\|_0^2 + \|\bar{u}_h^n - u_h^{n-1}\|_0^2 \right) + a(\bar{u}_h^n, \bar{u}_h^n) + d_h^D(\bar{u}_h^n, \bar{u}_h^n) + \delta_h^M(u^{FE}, \bar{u}_h^n) \\
\leq \|f_n-1\|_0\|\bar{u}_h^n\|_0 + |a(\bar{u}_h^n - u_h^{n-1}, u_h^n)| + |d_h^D(\bar{u}_h^n - u_h^{n-1}, \bar{u}_h^n)|. \quad (25)
\]

Now using the Cauchy-Schwarz inequality, Eq. [15], and Young’s inequality

\[
|a(\bar{u}_h^n - u_h^{n-1}, \bar{u}_h^n)| = |\varepsilon (\nabla (\bar{u}_h^n - u_h^{n-1}), \nabla \bar{u}_h^n) + (\mathbf{b} \cdot \nabla (\bar{u}_h^n - u_h^{n-1}), \bar{u}_h^n)| \\
\leq \varepsilon |\bar{u}_h^n - u_h^{n-1}|_1 |\bar{u}_h^n|_1 + \|\mathbf{b}\|_\infty \|\bar{u}_h^n - u_h^{n-1}\|_1 \|\bar{u}_h^n\|_0 \\
+ \|\varepsilon\|_\infty \|\bar{u}_h^n - u_h^{n-1}\|_0 \|\bar{u}_h^n\|_0 \\
\leq \left( \varepsilon^{1/2} C_{inv}^2 h^{-1} + \|\mathbf{b}\|_\infty C_{inv} h^{-1} + \|\varepsilon\|_\infty \right) \\
\times \|\bar{u}_h^n - u_h^{n-1}\|_0 \|\bar{u}_h^n\|_{FCT} \\
\leq \left( \varepsilon^{1/2} C_{inv}^2 h^{-1} + \|\mathbf{b}\|_\infty C_{inv} h^{-1} + \|\varepsilon\|_\infty \right)^2 \\
\times \frac{3}{2} \|\bar{u}_h^n - u_h^{n-1}\|_0^2 + \|\bar{u}_h^n\|_{FCT}^2. \quad (26)
\]

Next using the edge formulation Eq. [20], the Cauchy-Schwarz inequality, the trace inequality, the inverse estimate, and Young’s inequality we get

\[
|d_h^D(\bar{u}_h^n - u_h^{n-1}, \bar{u}_h^n)| \leq \sum_{E \in E_h} |d_E| h_E \|\nabla (\bar{u}_h^n - u_h^{n-1})\|_{0,E} \|\nabla \bar{u}_h^n\|_{0,E} \\
\leq \sum_{E \in E_h} |d_E| h_E \|\nabla (\bar{u}_h^n - u_h^{n-1})\|_{0,E} \|\nabla \bar{u}_h^n\|_{0,E} \\
\leq \max_{E \in E_h} \left( |d_E| \left( \sum_{E \in E_h} h_E \|\nabla (\bar{u}_h^n - u_h^{n-1})\|_{0,E}^2 \right)^{1/2} \\
\times \left( \sum_{E \in E_h} h_E \|\nabla \bar{u}_h^n\|_{0,E}^2 \right)^{1/2} \right) \\
\leq \max_{E \in E_h} \left( |d_E| \|\nabla (\bar{u}_h^n - u_h^{n-1})\|_{0} \|\nabla \bar{u}_h^n\|_{0} \right) \\
\leq C_{dE} C_{inv} h^{-1} \|\bar{u}_h^n - u_h^{n-1}\|_0 \|\nabla \bar{u}_h^n\|_0 \\
\leq C_{dE}^2 \left( \frac{3 h^{-2}}{2} \|\bar{u}_h^n - u_h^{n-1}\|_0^2 \right) + \|\bar{u}_h^n\|_{FCT}^2, \quad (27)
\]

where \( C_{dE} \) is the maximum of \(|d_E|\) and independent of \( h \).

Lastly, we have to approximate \( d_h^M(u^{FE}, \bar{u}_h^n) \). We can have an edge representation of this term similar to Eq. [21]

\[
d_h^M(u^{FE}, \bar{u}_h^n) = \sum_{E \in E_h} \frac{2}{\tau} |m_E| h_E \left( \nabla (\bar{u}_h^n - u_h^{n-1}) \cdot \mathbf{t}_E, \nabla \bar{u}_h^n \cdot \mathbf{t}_E \right)_E.
\]
By simple algebraic manipulation we get
\[ d_h^M(u^{FE}, \bar{u}_h^n) = \frac{2}{\tau} \left( \sum_{E \in \mathcal{E}_h} |m_E|h_E \| \nabla (\bar{u}_h^n - u_h^{n-1}) \cdot t_E \|^2_{0, E} \right. \\
- \sum_{E \in \mathcal{E}_h} |m_E|h_E \| \nabla u_h^{n-1} \cdot t_E \|^2_{0, E} \\
+ \sum_{E \in \mathcal{E}_h} |m_E|h_E \| \nabla (\bar{u}_h^n \cdot t_E, \nabla u_h^{n-1} \cdot t_E) \right), \quad (28)\]

and we also have
\[ d_h^M(u^{FE}, \bar{u}_h^n) = \frac{2}{\tau} \left( \sum_{E \in \mathcal{E}_h} |m_E|h_E \| \nabla \bar{u}_h^n \cdot t_E \|^2_{0, E} \right. \\
- \sum_{E \in \mathcal{E}_h} |m_E|h_E \| \nabla (\bar{u}_h^n \cdot t_E, \nabla u_h^{n-1} \cdot t_E) \right). \quad (29)\]

Adding Eq. (28) and Eq. (29) we finally have
\[ d_h^M(u^{FE}, \bar{u}_h^n) = \frac{1}{\tau} \left( \sum_{E \in \mathcal{E}_h} |m_E|h_E \| \nabla \bar{u}_h^n \cdot t_E \|^2_{0, E} \right. \\
+ \sum_{E \in \mathcal{E}_h} |m_E|h_E \| \nabla (\bar{u}_h^n - u_h^{n-1}) \cdot t_E \|^2_{0, E} \\
- \sum_{E \in \mathcal{E}_h} |m_E|h_E \| \nabla u_h^{n-1} \cdot t_E \|^2_{0, E} \right). \quad (30)\]

Substituting Eq. (26), Eq. (27), and Eq. (30) in Eq. (25) and using
\[ C_{FCT} \| \bar{u}_h^n \|^2_{FCT} \leq a(\bar{u}_h^n, \bar{u}_h^n), \]
\[ 0 \leq d_h^M(\bar{u}_h^n) \]
\[ 0 \leq \sum_{E \in \mathcal{E}_h} |m_E|h_E \| \nabla (\bar{u}_h^n - u_h^{n-1}) \cdot t_E \|^2_{0, E}, \]

we get
\[ \| \bar{u}_h^n \|^2_0 + \| \bar{u}_h^n - u_h^{n-1} \|^2_0 + \sum_{E \in \mathcal{E}_h} |m_E|h_E \| \nabla \bar{u}_h^n \cdot t_E \|^2_{0, E} + \tau C_{FCT} \| u_h^n \|^2_{FCT} \]
\[ \leq C \| f_{n-1} \|^2_0 \\
+ C_{FCT} \frac{\tau}{h^2} \left( \| b \|^2_{\text{inv}} + \| c \|^2_{\text{inv}} + \| \sigma \|^2_{\text{inv}} + \frac{C_{dE}}{\varepsilon} \right) \| u_h^n - u_h^{n-1} \|^2_0 \\
+ \sum_{E \in \mathcal{E}_h} |m_E|h_E \| \nabla u_h^{n-1} \cdot t_E \|^2_{0, E} + \| u_h^{n-1} \|^2_0. \]

Summing over \( n = 1, \ldots, N \) and bounding \( \sum_{E \in \mathcal{E}_h} |m_E|h_E \| \nabla u_h^n \cdot t_E \|^2_{0, E} \) by below we have stability whenever \( \tau \leq C h^2 \), where \( C \) is independent of \( h \).
The next theorem provides the stability of the nonlinear FEM-FCT scheme Eq. (13).

**Theorem 2.** Let Eq. (2) hold. With the additional condition
\[
\alpha_E \leq \frac{1}{(C_T, C_{inv} C_{m_E})^2 \min \{ \tau, \frac{\alpha}{2} \}},
\]
then the solution \( u^n_h \) of Eq. (13) satisfies at \( t_n = n\tau \)
\[
\| u^n_h \|^2_0 + C\tau \sum_{m=1}^n \| u^m_h \|^2_{\text{FCT}} \leq \| u_0^0 \|^2_0 + \frac{1}{C_T^2} \| \nabla u_0^0 \|^2_0 + \frac{\tau}{C_0} \sum_{m=1}^n \| f^m \|^2_0
\]
where \( C \) and \( C_T \) are constants that do not depend on \( h \) and \( \tau \).

**Proof.** The proof follows the standard approach. Setting \( v_h = u^n_h \) in Eq. (13), we get
\[
(u^n_h - u_h^{n-1}, u_h^{m}) + \tau [a_h(u_h^{m}, u_h^{m}) + d_h^D(u_h^{m}, u_h^{m}) + d_h^M(u_h^{m}, u_h^{m})] = \tau (f^m, u_h^{m}).
\]
Using \( (a - b, a) = \frac{1}{2} (\| a \|^2 - \| b \|^2 + \| a - b \|^2) \) for \( a, b \in L^2(\Omega) \), Eq. (12), and Eq. (19), it follows that
\[
\frac{1}{2} (\| u_h^{m} \|^2_0 - \| u_h^{m-1} \|^2_0 + \| u_h^{m} - u_h^{m-1} \|^2_0)
+ C_a \tau \| u_h^{m} \|^2_{\text{FCT}} + \tau \sum_{i,j=1}^N m_{ij} (u_{hi}^{m} - u_{hi}^{m}) u_{hi}^{m}
\leq \tau \sum_{i,j=1}^N \alpha_{ij} m_{ij} (u_{hi}^{m} - u_{hi}^{m}) u_{hi}^{m} + \tau (f^m, u_h^{m}).
\]
Using \( u' = (u^m - u^{m-1})/\tau \) and the Cauchy-Schwarz inequality for the last term on the left-hand side of Eq. (33), we get
\[
\tau \sum_{i,j=1}^N m_{ij} (u_{hi}^{m} - u_{hi}^{m}) u_{hi}^{m}
= \sum_{E \in \mathcal{E}_h} |m_E| h_E \left( \nabla (u_h^{m} - u_h^{m-1}) \cdot t_E, \nabla u_h^{m} \cdot t_E \right)_{0,E}
= \sum_{E \in \mathcal{E}_h} |m_E| h_E \left\{ \nabla (u_h^{m} - u_h^{m-1}) \cdot t_E, \nabla (u_h^{m} - u_h^{m-1}) \cdot t_E \right\}_{0,E}
+ \nabla (u_h^{m} - u_h^{m-1}) \cdot t_E, \nabla u_h^{m-1} \cdot t_E \right)_{0,E}
= \sum_{E \in \mathcal{E}_h} |m_E| h_E \left\{ \| \nabla (u_h^{m} - u_h^{m-1}) \|^2_{0,E}
+ \nabla u_h^{m} \cdot t_E, \nabla u_h^{m-1} \cdot t_E \right\}_{0,E} - \| \nabla u_h^{m-1} \cdot t_E \|^2_{0,E}
\]
One can also estimate the same term as follows

\[
\tau \sum_{i,j=1}^{N} m_{ij} \left( u_{hi}^{'m} - u_{hj}^{'m} \right) u_{hi}^m \\
= \sum_{E \in \mathcal{E}_h} |m_E|h_E \left\{ \| \nabla u_{hi}^m \cdot t_E \|_{0,E}^2 - (\nabla u_{hi}^m \cdot t_E, \nabla u_{hi}^{m-1} \cdot t_E)_{0,E} \right\}. 
\]

Adding the above estimate, we get

\[
\tau \sum_{i,j=1}^{N} m_{ij} \left( u_{hi}^{'m} - u_{hj}^{'m} \right) u_{hi}^m \\
= \frac{1}{2} \sum_{E \in \mathcal{E}_h} |m_E|h_E \left\{ \| \nabla (u_{hi}^m - u_{hi}^{m-1}) \|_{0,E}^2 + \| \nabla u_{hi}^m \cdot t_E \|_{0,E}^2 \right\} \\
- \| \nabla u_{hi}^{m-1} \cdot t_E \|_{0,E}^2. 
\]

The first term on the right-hand side of Eq. (33) uses the edge formulation, the Cauchy-Schwarz inequality, estimate Eq. (18) and the Young’s inequality

\[
\tau \sum_{i,j=1}^{N} \alpha_{ij} m_{ij} \left( u_{hi}^{'m} - u_{hj}^{'m} \right) u_{hi}^m \\
= \tau \sum_{E \in \mathcal{E}_h} \alpha_E |m_E| \| \nabla u_{hi}^{'m} \cdot t_E, \nabla u_{hi}^m \cdot t_E \|_E h_E \\
\leq \tau \sum_{E \in \mathcal{E}_h} \alpha_E |m_E| \| \nabla u_{hi}^{'m} \|_{0,E} h_E \\
\times \| \nabla u_{hi}^m \cdot t_E \|_{0,E} h_E \\
\leq \tau \sum_{E \in \mathcal{E}_h} \alpha_E |m_E| \| \nabla u_{hi}^{'m} \|_{0,E} \| \nabla u_{hi}^m \|_{0,E} h_E \\
\leq \tau \sum_{E \in \mathcal{E}_h} \alpha_E |m_E| \frac{h_E}{2} \| \nabla u_{hi}^{'m} \|_{0,E}^2 \\
+ \tau \sum_{E \in \mathcal{E}_h} \alpha_E |m_E| \frac{h_E}{2} \| \nabla u_{hi}^m \|_{0,E}^2.
\]

Now, using the bounds Eq. (17) and Eq. (31), the local trace inequality Eq. (16)
and an inverse inequality Eq. (15) gives
\[ \tau \sum_{i,j=1}^{N} \alpha_{ij} m_{ij} (u_{hi} - u'_{hj}) u_{hi}^m \]
\[ = \sum_{E \in \mathcal{E}_h} \frac{h^{3+q}}{2(C_t C_{inv})} \| \nabla u'_{h} \|_{0,E}^2 \]
\[ + \frac{c_0 h^{3+q}}{4(C_t C_{inv})^2} \| \nabla u_{h}^m \|_{0,E}^2 \]
\[ \leq \sum_{K \in \mathcal{T}_h} \frac{\tau}{2} \| u'_{h} \|_{0,K}^2 + \sum_{K \in \mathcal{T}_h} \frac{c_0 \tau}{4} \| u_{h}^m \|_{0,K}^2 \]
\[ = \sum_{K \in \mathcal{T}_h} \frac{1}{2} \| u_{h}^m - u_{h}^{m-1} \|_{0,K}^2 \]
\[ + \sum_{K \in \mathcal{T}_h} \frac{c_0 \tau}{4} \| u_{h}^m \|_{0,K}^2 \]
\[ \leq \frac{1}{2} \| u_{h}^m - u_{h}^{m-1} \|_{0}^2 + \frac{c_0 \tau}{4} \| u_{h}^m \|_{0}^2. \]

The estimate for the second term on the right-hand side of Eq. (33) uses the Cauchy-Schwarz inequality, and the Young’s inequality to get
\[ \tau (f^m, u_{h}^m) \leq \frac{\tau}{c_0} \| f^m \|_{0}^2 + \frac{c_0 \tau}{4} \| u_{h}^m \|_{0}^2. \]

Collecting the above estimates in Eq. (33) and the fact that the first term in Eq. (34) is positive and can bounded by zero, we get
\[ \frac{1}{2} \| u_{h}^n \|_{0} + C \tau \| u_{h}^m \|_{FCT}^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_h} |m_E| h_E \| \nabla u_{h}^m \cdot t_E \|_{0,E}^2 \]
\[ \leq \frac{1}{2} \| u_{h}^{m-1} \|_{0}^2 + \frac{\tau}{c_0} \| f^m \|_{0}^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_h} |m_E| h_E \| \nabla u_{h}^{m-1} \cdot t_E \|_{0,E}^2. \]

Summing over \( m = 1, \ldots, n \) leads to
\[ \frac{1}{2} \| u_{h}^n \|_{0} + C \tau \sum_{m=1}^{n} \| u_{h}^{m} \|_{FCT}^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_h} |m_E| h_E \| \nabla u_{h}^{m} \cdot t_E \|_{0,E}^2 \]
\[ \leq \frac{1}{2} \| u_{h}^{0} \|_{0}^2 + \frac{\tau}{c_0} \| f^{0} \|_{0}^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_h} |m_E| h_E \| \nabla u_{h}^{0} \cdot t_E \|_{0,E}^2. \]

and then bounding the terms
\[ \frac{1}{2} \sum_{E \in \mathcal{E}_h} |m_E| h_E \| \nabla u_{h}^{m} \cdot t_E \|_{0,E}^2 \geq 0 \text{ and,} \]
\[ \frac{1}{2} \sum_{E \in \mathcal{E}_h} |m_E| h_E \| \nabla u_{h}^{0} \cdot t_E \|_{0,E}^2 \leq \frac{1}{C_{T_1}^2} \| \nabla u_{h}^{0} \|_{0}^2 \]
gives the statement of the theorem.

\textbf{Remark 1.} The assumption Eq. (31) on the limiter is only needed in the region where the convection is dominant. For the regions away from layers, the value of \( \alpha_E \) is close to 1 and the effect of stabilization vanishes.

13
The stability of the linear FEM-FCT scheme is given in the next theorem.

**Theorem 3.** Let Eq. (2) and the conditions of Theorem 1 be fulfilled. Then, the solution of Eq. (14) satisfies at \( t_n = n \tau \)

\[
\|u_h^n\|_0^2 + C \tau \sum_{m=1}^n \|u_h^m\|_{FCT}^2 \\
\leq \|u_h^0\|_0^2 + \frac{2\tau}{c_0} \sum_{m=1}^n \left( \|f_j^m\|_0^2 + \|f^{*(j-1)}\|_0^2 \right) + \frac{1}{C_T^2} \|\nabla u_h^0\|_0^2. \tag{35}
\]

**Proof.** The proof starts with the same lines of arguments as Theorem 2. We have for \( v_h = u_h^n \)

\[
\frac{1}{2} \left( \|u_h^m\|_0^2 - \|u_h^{m-1}\|_0^2 + \|u_h^m - u_h^{m-1}\|_0^2 \right) + C_a \tau \|u_h^m\|_{FCT}^2 + \tau \sum_{i,j} m_{ij} (u_{hi}^m - u_{hj}^m) u_{hi}^m \\
\leq \tau (f^m, u_h^m) + \tau (f^{*(m-1)}, u_h^m). \tag{36}
\]

Comparing this with Eq. (35), it can be seen that the difference is only the right hand side.

Applying the Cauchy-Schwarz inequality followed by Young’s inequality for the first term on the right hand side gives

\[
\tau (f^m, u_h^m) \leq \frac{\tau}{c_0} \|f^m\|_0^2 + \frac{c_0 \tau}{4} \|u_h^m\|_0^2.
\]

Inserting this estimate and Eq. (34) in Eq. (36) and ignoring the terms with positive contribution to get

\[
\frac{1}{2} \|u_h^m\|_0^2 + C_a \tau \|u_h^m\|_{FCT}^2 + \frac{1}{2} \sum_{E \in E_h} |m_E| h_E \|\nabla u_h^m \cdot t_E\|^2_{0,E} \\
\leq \frac{1}{2} \|u_h^{m-1}\|_0^2 + \frac{1}{2} \sum_{E \in E_h} |m_E| h_E \|\nabla u_h^{m-1} \cdot t_E\|^2_{0,E} + \frac{\tau}{c_0} \|f^m\|_0^2 + \tau (f^{*(m-1)}, u_h^m).
\]

To approximate the last term we will use the stability of the predictor step.

\[
(f^{*(m-1)}, u_h^m) = \sum_{i,j=1}^N \alpha_{ij} f_{ij} u_{hi}^{mi} \\
= \sum_{i,j=1}^N \alpha_{ij} \left( 2 m_{ij} \left[ \tilde{u}_{hi}^{mi} - u_h^{(m-1)i} - u_{hi}^{(m-1)j} \right] u_{hi}^{mi} \\
+ \tau d_{ij} \left[ 2 \tilde{u}_h^{im} - 2 \tilde{u}_h^{im} + u_h^{(m-1)i} - u_h^{(m-1)j} \right] u_{hi}^{mi} \right) \\
= \sum_{i,j=1}^N \alpha_{ij} \left( 2 (-m_{ij} + \tau d_{ij}) \left[ \tilde{u}_h^{mj} - \tilde{u}_h^{mi} \right] u_{hi}^{mi} \\
+ (-2 m_{ij} + \tau d_{ij}) \left( u_h^{(n-1)i} - u_h^{(n-1)j} \right) u_{hi}^{mi} \right).
\]

14
Adding and subtracting \((-2m_{ij} + \tau d_{ij}) (u^{n_i} - u^{m_j}) u^{m_i}_h\) on the right-hand side, we get
\[
(f^*, u^n_h) = \sum_{i,j=1}^{N} \alpha_{ij} f_{ij} u^{m_i}_h
\]
\[
= \sum_{i,j=1}^{N} \alpha_{ij} \left( 2 (-m_{ij} + \tau d_{ij}) \left( \bar{u}^{m_j}_h - \bar{u}^{m_i}_h \right) u^{m_i}_h \right)
\]
\[
+ (-2m_{ij} + \tau d_{ij}) \left( u^{m_j}_h - u^{m_i}_h + u^{(m-1)i}_h - u^{(m-1)j}_h \right) u^{m_i}_h
\]
\[
+ (-2m_{ij} + \tau d_{ij}) \left( u^{m_i}_h - u^{m_j}_h \right) u^{m_i}_h.
\]

The last term in the last equation can be taken to the left side. We note that all the terms can be written in edge formulation. Bounding \((-2m_{ij} + \tau d_{ij}) (u^{m_i}_h - u^{m_j}_h) u^{m_i}_h\) by below using [BJK16, Lemma 1]. To bound the rest of terms we use the Cauchy-Schwarz inequality, Eq. (15), and Young’s inequality,
\[
\tau (f^{*(m-1)}, u^n) \leq 2\tau \sum_{E \in \mathcal{E}_h} \alpha_E |m_E + \tau d_E| h_E ||\nabla \bar{u}^m_h||^2_{0,E} \leq \tau \sum_{E \in \mathcal{E}_h} \alpha_E |2m_E + \tau d_E| h_E
\]
\[
\times ||\nabla (u^{(m-1)}_h - u^m_h)||_{0,E} ||\nabla u^m_h||_{0,E}
\]
\[
\leq 2\tau \max_{E \in \mathcal{E}_h} \left( |m_E + \tau d_E| \right) \left( \sum_{E \in \mathcal{E}_h} h_E ||\nabla \bar{u}^m_h||^2_{0,E} \right)^{1/2}
\]
\[
\times \left( \sum_{E \in \mathcal{E}_h} h_E ||\nabla u^m_h||^2_{0,E} \right)^{1/2} + \tau \max_{E \in \mathcal{E}_h} (|2m_E + \tau d_E|)
\]
\[
\times \left( \sum_{E \in \mathcal{E}_h} h_E ||\nabla (u^{m-1}_h - u^m_h)||^2_{0,E} \right)^{1/2}
\]
\[
\times \left( \sum_{E \in \mathcal{E}_h} h_E ||\nabla u^m_h||^2_{0,E} \right)^{1/2}
\]
\[
\leq \tau C ||\nabla \bar{u}^m_h||_0 ||\nabla u^m_h||_0 + \tau C ||\nabla (u^{m-1}_h - u^m_h)||_0 ||\nabla u^m_h||_0
\]
\[
\leq \tau CC_{\text{inv}} h^{-1} ||\bar{u}^m_h||_0 ||\nabla u^m_h||_0
\]
\[
+ \tau CC_{\text{inv}} h^{-1} ||\nabla (u^{m-1}_h - u^m_h)||_0 ||\nabla u^m_h||_0
\]
\[
\leq \frac{4\tau C}{\varepsilon} ||u^m_h||_0^2 + \frac{4\tau C}{\varepsilon} ||\nabla (u^{m-1}_h - u^m_h)||_0^2 + \frac{\varepsilon}{8} ||u^m_h||_a^2.
\]

In Eq. (37) the last term can be taken to the left side and the second term again is bounded by the inequality \(\tau \leq Ch^2\) from the predictor step.
Summing over the time steps $m = 1, \ldots, n$ gives

$$
\left\| u^n_h \right\|_{0}^2 + C_n \tau \sum_{m=1}^{n} \left\| \left. u^m_j \right|_{\text{FCT}} \right\|_{0}^2 + \sum_{E \in \mathcal{E}_h} |m_E| h_E \| \nabla u^n_h \cdot t_E \|_{0,E}^2 \\
\leq \left\| u^0_h \right\|_{0}^2 + \sum_{E \in \mathcal{E}_h} |m_E| h_E \| \nabla u^n_h \cdot t_E \|_{0,E}^2 + \frac{2\tau}{C_0} \sum_{m=1}^{n} \left( \left\| f^m \right\|_{0}^2 + C \| \bar{u}^m_h \|_{0}^2 \right).
$$

Finally, ignoring the terms with positive contribution, bounding the last term from above by the predictor step stability condition, and the application of the local trace inequality gives the statement of the theorem.

\[ \square \]

## 4 Finite Element Error Analysis

This section details the error analysis of the linear and nonlinear FEM-FCT schemes. Sufficient regularity condition for the solution $u$ of Eq. (1) is assumed for a priori analysis. The analysis of both schemes starts by decomposing the error into an interpolation error and the difference of interpolation and the solution i.e.,

$$
u^n_h - u(t_n) = (u^n_h - \Pi_h u(t_n)) + (\Pi_h u(t_n) - u(t_n)),
$$

where $\Pi_h$ is a stable interpolation operator.

The following error estimates are used to get the estimates for the terms involving interpolation error

$$
\| u - \Pi_h u \|_0 + h \| u - \Pi_h u \|_1 \leq C h^2 \| u \|_2 \quad \text{(38)}
$$

for $u \in V \cap H^2(\Omega)$ (see [BS08]).

In the analysis below, the following semi-norm property will be used

$$
|d_D^h(w; z, v)|^2 \leq d_D^h(w; z, z)d_D^h(w; v, v), \quad \text{(39)}
$$

which is valid for $d_D^h(\cdot, \cdot)$ as well see [BJK16, Eq. (41)].

In order to get the bounds for the term $d_D^h(\cdot, \cdot, \cdot)$, we will use the following lemma, for proof we refer to [BJK16, Lemma 16].

**Lemma 3.** Let the matrix $D$ be defined by Eq. (4). Then, there exists a constant $C$ that does not depend on $\tau$, $h$ and the data of Eq. (1) such that

$$
d_D^h(w_h, \Pi_h u, \Pi_h v) \leq C(\varepsilon + \|b\|_{0,\infty,\Omega}h + \|c\|_{0,\infty,\Omega}h^2)\|\Pi_h u\|_1^2 \quad \forall w_h \in V_h, \, u \in C(\Omega). \quad \text{(40)}
$$

This lemma also holds true for time dependent $b$ and $c$, as we assume them to be bounded in time as well. If we follow the same procedure as in the proof of [BJK16, Lemma 16], we can get the following estimate.

**Lemma 4.** Let the matrix $M_L$ be defined by Eq. (5) and let Eq. (17) hold. Then, there exists a constant $C$ that does not depend on $\tau$, $h$ and the data of Eq. (1) such that

$$
d_M^h(w_h, \Pi_h u, \Pi_h v) \leq C h \|\Pi_h u\|_1^2 + C \|\Pi_h v\|_0^2 \quad \forall w_h \in V_h, \, u, v \in C(\Omega). \quad \text{(41)}
$$

---

16
Applying the stability techniques to the above estimate, one gets Young’s inequality, the triangular inequality and the interpolation error estimate. We detail the estimates here. The application of the Cauchy-Schwarz inequality, for the solution $u$ of the Eq. (1).

Proof. From the stability result of Theorem 2 we have for $v_h = e_h^m$

$$
(e_h^m - e_h^{m-1}, e_h^m) + \tau \left[ a_h(e_h^m, e_h^m) + d_h^P(u_h; e_h^m, e_h^m) + d_h^m(u_h; e_h, e_h^m) \right] \\
= \tau \left[ f^m, e_h^m \right] - \left( \Pi_h^m u - \Pi_h^{m-1} u, e_h^m \right) \\
- \tau \left[ a_h(\Pi_h^m u, e_h^m) + d_h^P(u_h; \Pi_h^m u, e_h^m) + d_h^m(u_h; \Pi_h u, e_h^m) \right].
$$

Applying the stability techniques to the above estimate, one gets

$$
\|e_h^m\|^2 + C \tau \sum_{m=1}^n \|e_h^m\|^2_{\text{FCT}} \leq \|e_h^0\|^2 + \frac{1}{C_1^2} \|\nabla e_h^0\|^2 + \sum_{m=1}^n (N_1^m, e_h^m) \\
+ \sum_{m=1}^n (N_2^m, e_h^m) - \tau \sum_{m=1}^n \left( d_h^P(u_h; \Pi_h^m u, e_h^m) + d_h^m(u_h; \Pi_h u, e_h^m) \right),
$$

where $N_1^m$ and $N_2^m$ are given by

$$
(N_1^m, e_h^m) = \tau \left( u'(t_m) - \left( \Pi_h^m u - \Pi_h^{m-1} u \right) \right),
$$

$$
(N_2^m, e_h^m) = \tau \varepsilon \left( \nabla (u'(t_m) - \Pi_h^m u), \nabla e_h^m \right).
$$

The estimates for the term $N_1$ can also be found in [11]. For completeness, we detail the estimates here. The application of the Cauchy-Schwarz inequality, Young’s inequality, the triangular inequality and the interpolation error estimates Eq. (38) gives

$$
\|N_1^m, e_h^m\| \leq \frac{\tau}{2c_0} \|N_1^m\|_0 + \frac{\tau c_0}{2} \|e_h^m\|_0 \\
\leq C h^2 \left[ \|u'(t_m)\|^2_0 + \|e_h^m\|^2_\infty \|u(t_m)\|_2^2 + \|b\|_\infty^2 \|u(t_m)\|^2_\infty \right] \\
+ C \tau \left( \|\Pi_h^m u - \Pi_h^{m-1} u\|_0^2 + \frac{\tau c_0}{2} \|e_h^m\|_0^2 \right).
$$
The term with the backward difference can be estimated using the Taylor formula with integral remainder form, the property that the time derivative and the interpolation $\Pi_h$ commute, the Cauchy-Schwarz inequality and the stability of $\Pi_h$, one gets

$$\left\| \Pi_h^m u(t_m) - \frac{\Pi_h^n u - \Pi_h^{n-1} u}{\tau} \right\|^2_0 \leq \frac{1}{\tau^2} \left\| \int_{t_{m-1}}^{t_m} (t - t_{m-1}) \Pi_h^n u'' \right\|_0^2 \leq \frac{1}{\tau^2} \left( \left( \int_{t_{m-1}}^{t_m} (t - t_{m-1})^2 dt \right)^{1/2} \left( \int_{t_{m-1}}^{t_m} \| \Pi_h^n u'' \|_0^2 dt \right) \right) \leq C \tau \int_{t_{m-1}}^{t_m} \| \Pi_h^n u'' \|_1^2 = C \tau \| \Pi_h^n u'' \|_{L^2(t_{m-1}, t_m ; H^1)}.$$  

It follows that

$$\left| (N_1^m, e_h^m) \right| \leq C h^2 \left[ \| u'(t_m) \|_2^2 + \| e \|_{L^\infty}^2 \| u(t_m) \|_2^2 + \| b \|_{L^\infty}^2 \| u(t_m) \|_2^2 \right] + C \tau^2 \| \Pi_h^n u'' \|_{L^2(t_{m-1}, t_m ; H^1)}^2 + \frac{\tau C_0}{2} \| e_h^m \|_0^2.$$  

For $N_2^m$, the application of Cauchy-Schwarz inequality, Young’s inequality and Eq. (38) gives

$$(N_2^m, e_h^m) \leq \frac{\varepsilon \tau h^2}{2} \| u(t_m) \|_2^2 + \frac{\varepsilon \tau}{2} \| e_h^m \|_1^2.$$  

For the fifth term on the right-hand side of Eq. (44), the seminorm property Eq. (39), Young’s inequality and Lemma 3 give

$$\tau | d_h^M (u_h^m; \Pi_h^n u, e_h^m) | \leq \tau | d_h^D (u_h^m; \Pi_h^n u, \Pi_h^n u) | + \tau | d_h^D (u_h^m; e_h^m, e_h^m) | \leq C \tau \left( \varepsilon + \| b \|_{L^\infty} h + \| e \|_{L^\infty} h^2 \right) \| \Pi_h^n u \|_1^2 + \frac{\tau C_0}{2} \| e_h^m \|_0^2.$$  

Similarly, for the sixth term on the right-hand side of Eq. (44) using Lemma 4 we obtain

$$\tau | d_h^M (u_h^m; \Pi_h^n u, e_h^m) | \leq C \tau h^2 \| \Pi_h u'(t_m) \|_1^2 + \frac{C \tau}{2} \| e_h^m \|_0^2.$$  

Collecting all the estimates in Eq. (44), absorbing the similar terms to the left-hand side, and using $e_h^0 = 0$, we get

$$\| e_h^n \|_0^2 + C \tau \sum_{m=1}^n \| e_h^m \|_{\text{FCT}}^2 \leq \sum_{m=1}^n \left[ C h^2 \left( \| u'(t_m) \|_2^2 + \| b \|_{L^\infty}^2 \right) + \| e \|_{L^\infty}^2 \| u(t_m) \|_2^2 \right] + C \tau^2 \| u'' \|_{L^2(t_{m-1}, t_m ; H^1)}^2 + C \tau^2 \| u'(t_m) \|_1^2.$$  

The statement of the theorem then follows by applying the triangular inequality and the interpolation error estimates.
In the next theorem, an error estimate for the linear FEM-FCT scheme is derived.

**Theorem 5.** Let \( b \in L^\infty(0, T; (L^\infty)^d), \nabla \cdot b, c \in L^\infty(0, T; L^\infty) \) for the coefficients in Eq. (10). Further, assume that the solution \( u \) satisfies the regularity assumption Eq. (42). Then, the error \( u^n_h - u(t_n) \) satisfies

\[
\|u^n_h - u(t_n)\|_0^2 + \tau \sum_{m=1}^{n} \|u^m_h - u(t_m)\|_{FCT}^2 \leq C\left[h^3 + \tau^2 + (1 + \varepsilon)h^2 + (1 + \tau)\left(\varepsilon + \|b\|_\infty h + \|c\|_\infty h^2\right)\right]
\]

where \( C \) is a constant that depends on the \( u, u', \) and \( u'' \).

**Proof.** The error analysis for the error in the linear FEM-FCM starts by taking \( v_h = e^m_h \) in the stability Theorem 2, using \( e^0_h = 0 \), and applying similar estimates to get

\[
\|e^m_h\|_0^2 + \tau C \sum_{m=1}^{n} \|e^m_h\|_{FCT}^2 \leq \tau \sum_{m=1}^{n} (f^*(m-1), e^m_h) + \sum_{m=1}^{n} (N^1_1 + N^2_2, e^m_h)
\]

\[
-\tau \sum_{m=1}^{n} d^D_h (\Pi^m_h u, e^m_h) - \tau \sum_{m=1}^{N} d^M_h (\Pi_h u, e^m_h).
\]

(46)

The second term on the right-hand side is estimated in Theorem 4. Also, the bounds for the last two terms can be derived by using the same arguments as in Theorem 4

\[
\tau d^D_h (\Pi^m_h u, e^m_h) \leq C \tau \left(\varepsilon + \|b\|_\infty h + \|c\|_\infty h^2\right) \|\Pi^m_h u\|_1^2 + \frac{\tau}{2} d^D_h (e^m_h, e^m_h)
\]

\[
\tau d^M_h (\Pi^m_h u, e^m_h) \leq C \tau h^2 \|\Pi^m_h u(t_m)\|_1^2 + \tilde{C} \tau \|e^m_h\|_0^2.
\]

For the first term on the right-hand side of Eq. (46), we note that it can be written in an edge formulation similar to Eq. (21). So we get,

\[
\left(f^*(m-1), e^m_h\right)
\]

\[
= \sum_{i,j=1}^{N} \alpha_{ij} \left(\tau d_{ij} - m_{ij}\right)(\Pi^{m-1/2}_h \nu_j - \Pi^{m-1/2}_h \nu_i) \right) e^m_{h_i}
\]

\[
+ \sum_{i,j=1}^{N} \alpha_{ij} d_{ij}(\Pi^{m-1}_h u_b_j - \Pi^{m-1}_h u_b_i) e^m_{h_i}
\]

\[
= \sum_{E \in E_h} \alpha_{E} \cdot e^m_{h_E} \left(\nabla \Pi^{m-1/2}_h \nu \cdot t_E, \nabla e^m_{h} \cdot t_E\right)
\]

\[
+ \sum_{E \in E_h} \alpha_{E} \cdot e^m_{h_E} \left(\nabla \Pi^{m-1}_h u \cdot t_E, \nabla e^m_{h} \cdot t_E\right).
\]
Using the triangle inequality, $\alpha_E \leq 1$, Cauchy-Schwarz inequality, Young’s inequality, and lemma 3, it follows that

$$\left( f^{n-1} \right)_h \leq \sum_{E \in \mathcal{E}_h} (\tau |d_E| + |m_E||h_E|\Pi^{-1/2}_h \nu \cdot t_E|,E) |e^{n}_h \cdot t_E|,E + 1 \frac{d^D}{h}(e^m_h, e^m_h).$$

Note that the second and the last term are the same, so adding them and using lemma 3 for the first term and lemma 4 for the third and fourth term, we finally get

$$\left( f^{n-1} \right)_h \leq \sum_{E \in \mathcal{E}_h} (\tau^2 |d_E|h_E|\Pi^{-1/2}_h \nu \cdot t_E|,E) + \frac{1}{4} \sum_{E \in \mathcal{E}_h} |m_E|h_E|e^m_h \cdot t_E|^2 + C C \sum_{E \in \mathcal{E}_h} |m_E|h_E|e^m_h \cdot t_E|^2 + \frac{1}{4} d^D(h^m_h, h^m_h).$$

Collecting all the estimates in Eq. (46), absorbing the similar terms to the left-hand side, using $e^0_h = 0$, the statement of the theorem follows by applying the triangular inequality and the interpolation error estimates.

Remark 2. We observe that both in Theorem 4 and Theorem 5 we have $O(\tau)$ convergence for the time-discretization and $O\left(h^{0.5}\right)$ in the space discretization for the FCT norm in the convection-dominated regime. It has been noted in [BJK16] that for shock capturing methods such as the FEM-FCT schemes $O\left(h^{0.5}\right)$ convergence is expected.

## 5 Numerical Simulation

In this section we present the numerical study to assess the accuracy of the FEM-FCT scheme. We consider an example which possess a smooth solution to address the accuracy of the scheme. All computations were performed using ParMooN WBA-16.

We present the numerical studies for both linear and nonlinear FEM-FCT schemes. To solve the nonlinear problem, we used a fixed point iteration and the nonlinear iteration were stopped if the the Euclidean norm of the residual was smaller than $10^{-9}$.

The calculations have been performed on three different types of triangular grids with $P_1$ finite elements. The initial grids (level 0) are given in Figure 1. Grid 1 is a structured grid, also referred as Friedrichs–Keller type of grid. Grid 2 is an unstructured grid. Uniform refinement was performed for grid 1 and grid 2. Grid 3 is constructed from grid 1. First, the direction of diagonals in even rows...
were changed. Then the interior nodes are shifted to the right by a tenth of the horizontal mesh width, such that for the diagonal edge of grid 3, the sum of opposite angles is greater than $\pi/2$.

Numerical simulations for an example with dominating temporal error was also performed and a first order of convergence in time is observed. For brevity the detailed results are not presented here.

The problem Eq. (1) is considered on $\Omega = (0,1)^2$ with $\varepsilon = 10^{-8}$, with $b= (2,3)$, $c = 1$, and the right-hand side $f$ and the boundary conditions are chosen such that

$$u(t,x,y) = 100tx^2(1-x^2)y(1-y)(1-2y)$$

is the solution of Eq. (2).

In this example, we are interested in the convergence order in space. To this end, we keep the convergence order in time small by using the backward Euler method with time step length $\tau = 10^{-3}$. We use the notation $\|e_u\|_{L^2(FCT)}$ with $e_u = u - u_h$ to represent the time integrated norm of $\|\cdot\|_{FCT}$ and similarly $\|e_h\|_{L^2(d_h)}$ of $d_h(\cdot,\cdot,\cdot)$.

We first present results for constant $\alpha_{ij}$. We chose $\alpha_{ij} = 0.5$ for the interior nodes and set them according to Zalesak limiter for the boundary nodes. The results for the linear and the non-linear FEM-FCT on grid 3 are presented in Fig. 2. We note that we get convergence order of $O(h^{0.5})$ in the FCT norm for both the schemes which were predcited by our analysis and shows that the estimate is sharp.
Now we present results, with limiters defined by the Zalesak algorithm defined in Sec. 2.1. For grid 1, we observe that we have an optimal order of convergence on the $L^2$ and $H^1$ norm, whereas, for the FCT norm, we obtain an $O(h^2)$. The same behavior was observed for the AFC schemes in [BJK16]. The reason for this is because we only considered general properties of the limiter in the analysis.

On grid 2, we have a similar order of convergence as compared to grid 1. We have presence of non-Delaunay triangulations, but still, the order remains the same.

Lastly, we present results for grid 3. We observe a reduction of order as compared to the previous two grids for the $L^2$ norm and the FCT norm. Moreover, the $H^1$ norm does not converge and tends to zero as $h \to 0$.

Altogether, in the convection-dominated regime, we observe a higher order of error reduction for the Zalesak limiters than in the worst case that was considered in the analysis.

We want to make some remarks on the limiter’s stability condition in Eq. (31). It can be noted that the assumption is strong as the time-step length bounds the limiter, and for a small $\tau$, one may have to choose small values of $\alpha_{ij}$. This inequality has certain constants and requires a deeper analysis of the constants to get a better understanding.
6 Summary

In this paper, we presented the first finite element error analysis for the flux corrected transport techniques introduced in [Kuz09] in the natural norm of the system referred as the FCT norm. The Galerkin FEM was used for the space discretization coupled with the backward Euler time discretization. Numerical studies are presented in two dimensions for three different type of grids. The main finding of the analysis and the numerical simulations are given below:

1. We proved conditional stability of the FEM-FCT algorithm with the restriction in time step coming from the predictor step of forward Euler (see Eq. (23)) for both the linear and the nonlinear schemes.

2. For the analysis of the linear FEM-FCT algorithm, there was no restriction on the choice of the limiter, and only general assumptions, i.e., $\alpha_{ij} \in [0, 1]$ and Eq. (7) are considered.

3. For the nonlinear FEM-FCT algorithm an additional assumption on the limiter is required, i.e., Eq. (31). This assumption is strong but is only required in the convection-dominated regime.

4. From the analysis, one expects $O(\tau)$ convergence in the FCT norm for time-dominated problems and $O(h^{1.5})$ convergence for the convection-dominated problems.

5. We obtained the predicted order of convergence in the FCT norm for constant limiter on a particular grid. For the other grids optimal order was obtained for the $L^2$, $H^1$, and the FCT norm. For the FCT norm the experimental order of convergence is better than the obtained order of convergence for grid 1 and grid 2. This is expected as we assumed the general properties of the limiters for the analysis.

In the future, it would be interesting to use a general theta scheme instead of the backward Euler discretization, namely the Crank-Nicolson scheme. For our analysis, we assumed the limiter’s general properties and would like to investigate a deeper analysis for the limiter’s specific choices, e.g., [Zai79] [Kuz20]. Further a deeper analysis of the constants appearing in the stability estimate to
obtain sharper bounds would be beneficial. Lastly, the efficient solution of the nonlinear problem for the FEM-FCT algorithm remains an open question, and one should investigate sophisticated solvers and parallel techniques for obtaining the solution. For more open questions in stabilized techniques for Eq. (I), we refer to [JKN18].

References

[ACF+11] M. Augustin, A. Caiazzo, A. Fiebach, J. Fuhrmann, V. John, A. Linke, and R. Umla. An assessment of discretizations for convection-dominated convection-diffusion equations. *Comput. Methods Appl. Mech. Engrg.*, 200(47-48):3395–3409, 2011.

[AJ15] N. Ahmed and V. John. Adaptive time step control for higher order variational time discretizations applied to convection-diffusion-reaction equations. *Comput. Methods Appl. Mech. Engrg.*, 285:83–101, 2015.

[AM15] N. Ahmed and G. Matthies. Higher order continuous Galerkin-Petrov time stepping schemes for transient convection-diffusion-reaction equations. *ESAIM Math. Model. Numer. Anal.*, 49(5):1429–1450, 2015.

[AMTX11] N. Ahmed, G. Matthies, L. Tobiska, and H. Xie. Discontinuous Galerkin time stepping with local projection stabilization for transient convection-diffusion-reaction problems. *Comput. Methods Appl. Mech. Engrg.*, 200(21-22):1747–1756, 2011.

[BB04] R. Becker and M. Braack. A two-level stabilization scheme for the Navier-Stokes equations. In *Numerical mathematics and advanced applications*, pages 123–130. Springer, Berlin, 2004.

[BB06] M. Braack and E. Burman. Local projection stabilization for the Oseen problem and its interpretation as a variational multiscale method. *SIAM J. Numer. Anal.*, 43(6):2544–2566, 2006.

[BF09] E. Burman and M. A. Fernández. Finite element methods with symmetric stabilization for the transient convection-diffusion-reaction equation. *Comput. Methods Appl. Mech. Engrg.*, 198(33-36):2508–2519, 2009.

[BH82] A. N. Brooks and T. J. R. Hughes. Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Comput. Methods Appl. Mech. Engrg.*, 32(1-3):199–259, 1982. FENOMECH '81, Part I (Stuttgart, 1981).

[BJK16] G. R. Barrenechea, V. John, and P. Knobloch. Analysis of algebraic flux correction schemes. *SIAM J. Numer. Anal.*, 54(4):2427–2451, 2016.
[BJK17] G. R. Barrenechea, V. John, and P. Knobloch. An algebraic flux correction scheme satisfying the discrete maximum principle and linearity preservation on general meshes. Math. Models Methods Appl. Sci., 27(3):525–548, 2017.

[BJKR18] G. R. Barrenechea, V. John, P. Knobloch, and R. Rankin. A unified analysis of algebraic flux correction schemes for convection–diffusion equations. SeMA J., 75(4):655–685, 2018.

[BS08] S. C. Brenner and L. R. Scott. The mathematical theory of finite element methods, volume 15 of Texts in Applied Mathematics. Springer, New York, third edition, 2008.

[Bur10] E. Burman. Consistent SUPG-method for transient transport problems: stability and convergence. Comput. Methods Appl. Mech. Engrg., 199(17-20):1114–1123, 2010.

[Cia70] P. G. Ciarlet. Discrete maximum principle for finite-difference operators. Aequationes Math., 4:338–352, 1970.

[Gue99] J. Guermond. Stabilization of Galerkin approximations of transport equations by subgrid modeling. M2AN Math. Model. Numer. Anal., 33(6):1293–1316, 1999.

[HB79] T. J. R. Hughes and A. Brooks. A multidimensional upwind scheme with no crosswind diffusion. In Finite element methods for convection dominated flows (Papers, Winter Ann. Meeting Amer. Soc. Mech. Engrs., New York, 1979), volume 34 of AMD, pages 19–35. Amer. Soc. Mech. Engrs. (ASME), New York, 1979.

[Jha20] A. Jha. Numerical Algorithms for Algebraic Stabilizations of Scalar Convection-Dominated Problems. PhD thesis, 2020.

[JJ19] A. Jha and V. John. A study of solvers for nonlinear AFC discretizations of convection-diffusion equations. Comput. Math. Appl., 78(9):3117–3138, 2019.

[JJ20] A. Jha and V. John. On basic iteration schemes for nonlinear afc discretizations. In Gabriel R. Barrenechea and John Mackenzie, editors, Boundary and Interior Layers, Computational and Asymptotic Methods BAIL 2018, pages 113–128, Cham, 2020. Springer International Publishing.

[JK21] V. John and P. Knobloch. Existence of solutions of a finite element flux-corrected-transport scheme. Appl. Math. Lett., 115:106932, 6, 2021.

[JKK21] V. John, P. Knobloch, and P. Korsmeier. On the solvability of the nonlinear problems in an algebraically stabilized finite element method for evolutionary transport-dominated equations. Math. Comp., 90(328):595–611, 2021.

[JKN18] V. John, P. Knobloch, and J. Novo. Finite elements for scalar convection-dominated equations and incompressible flow problems: a never ending story? Comput. Vis. Sci., 19(5-6):47–63, 2018.
[JN11] V. John and J. Novo. Error analysis of the SUPG finite element discretization of evolutionary convection-diffusion-reaction equations. *SIAM J. Numer. Anal.*, 49(3):1149–1176, 2011.

[JM] V. John and J. Novo. On (essentially) non-oscillatory discretizations of evolutionary convection-diffusion equations. *J. Comput. Phys.*, 231(4):1570–1586, 2012.

[JR10] V. John and M. Roland. On the impact of the scheme for solving the higher dimensional equation in coupled population balance systems. *Internat. J. Numer. Methods Engrg.*, 82(11):1450–1474, 2010.

[JS08] V. John and E. Schmeyer. Finite element methods for time-dependent convection-diffusion-reaction equations with small diffusion. *Comput. Methods Appl. Mech. Engrg.*, 198(3-4):475–494, 2008.

[KM05] D. Kuzmin and M. Möller. Algebraic flux correction. I. Scalar conservation laws. In *Flux-corrected transport*, Sci. Comput., pages 155–206. Springer, Berlin, 2005.

[Kuz06] D. Kuzmin. On the design of general-purpose flux limiters for finite element schemes. I. Scalar convection. *J. Comput. Phys.*, 219(2):513–531, 2006.

[Kuz07] D. Kuzmin. Algebraic flux correction for finite element discretizations of coupled systems. *Computational Methods for Coupled Problems in Science and Engineering II*, 01 2007.

[Kuz09] D. Kuzmin. Explicit and implicit FEM-FCT algorithms with flux linearization. *J. Comput. Phys.*, 228(7):2517–2534, 2009.

[Kuz20] D. Kuzmin. Monolithic convex limiting for continuous finite element discretizations of hyperbolic conservation laws. *Comput. Methods Appl. Mech. Engrg.*, 361:112804, 28, 2020.

[Lay02] W. Layton. A connection between subgrid scale eddy viscosity and mixed methods. *Appl. Math. Comput.*, 133(1):147–157, 2002.

[LKSM17] C. Lohmann, D. Kuzmin, J. N. Shadid, and S. Mabuza. Flux-corrected transport algorithms for continuous Galerkin methods based on high order Bernstein finite elements. *J. Comput. Phys.*, 344:151–186, 2017.

[Loh19] C. Lohmann. *Physics-Compatible Finite Element Methods for Scalar and Tensorial Advection Problems*. Springer Fachmedien Wiesbaden, 2019.

[PC86] A. K. Parrott and M.A. Christie. Fct applied to the 2-d finite element solution of tracer transport by single phase flow in a porous medium. In *Proc. ICFD Conf. on Numerical Methods in Fluid Dynamics, Oxford University Press*, volume 609, 1986.
[RST08] H.G. Roos, M. Stynes, and L. Tobiska. *Robust numerical methods for singularly perturbed differential equations*, volume 24 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, second edition, 2008. Convection-diffusion-reaction and flow problems.

[Ver98] R. Verfürth. A posteriori error estimators for convection-diffusion equations. *Numer. Math.*, 80(4):641–663, 1998.

[WBA+16] U. Wilbrandt, C. Bartsch, N. Ahmed, N. Alia, F. Anker, L. Blank, A. Caiazzo, S. Ganesan, S. Giere, G. Matthies, R. Meesala, A. Shamim, J. Venkatesan, and V. John. Parmoon – a modernized program package based on mapped finite elements. *Computers and Mathematics with Applications*, 74:74–88, 2016.

[Zal79] S. T. Zalesak. Fully multidimensional flux-corrected transport algorithms for fluids. *J. Comput. Phys.*, 31(3):335–362, 1979.