Quantitative Evaluation of a Linear Reduced-order Model based on
Particle-image-velocimetry Data of Flow Field around Airfoil

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Abstract
A quantitative evaluation method for a reduced-order model of the flow field around a NACA0015 airfoil based on particle image velocimetry (PIV) data is proposed in this paper. In a previous work, the velocity field data obtained by the time-resolved PIV measurement were decomposed into significant modes by proper orthogonal decomposition (POD) technique, and a linear reduced-order model was then constructed by the linear regression of the time advancement of the first ten POD modes. The present evaluation method can be used to evaluate the estimation error and determine the reproducibility of the model. In this study, the model was constructed using different numbers of POD modes for order-reduction of the fluid data and different methods of estimating the linear coefficients, and the effects of these conditions on the model performance were quantitatively evaluated. The proposed method specifies the conditions that realize the best reproducibility. Moreover, it was demonstrated that the model performance depends on the configuration of the flow fields that are the target of the model, and the reproducibility is high at high angles of attack.

1. Introduction
Flow separation control by active flow control devices has recently attracted a great deal of attention, and the control performance of such devices, represented by dielectric barrier discharge plasma actuators, has been investigated in many studies (Corke et al. 2007; Little et al. 2010; Aono et al. 2017). It has been demonstrated that the control input (e.g., the burst-mode frequency) influences the effectiveness of the control and the input should be adapted to the flow configuration. In other words, for the effective control of an unsteady flow, such as a flow field on an airfoil, the control input should be determined based on the state of the flow field. Therefore, we are aiming to construct an optimal feedback flow control system to determine the control input by taking the system output into consideration. The construction of the system requires an observer with a model which estimates the state of the flow field from the limited system outputs.

In the past, we have constructed a linear reduced-order model which estimates the time advancement of low-dimensionalized flow fields around a NACA0015 airfoil based on particle image velocimetry (PIV) data and qualitatively investigated the reproducibility of the model (Nankai et al. 2019). In this previous study, the low-dimensional description of the flow field data were obtained by proper orthogonal decomposition (POD) technique in order to reduce the computational cost of the estimation and de-noise the data. In addition, the model described by a linear equation can be directly adapted to modern control theory. The model equation is based on the concept of dynamic mode decomposition (DMD) proposed by Schmid (2010). They showed that the linear equation can approximately express the dominant flow structures. Many studies have also applied POD or DMD technique to the experimental fluid data for the reduced-order modeling (Semeraro et al. 2012; Schmid et al. 2012; Suzuki 2014). It has been shown that the linear reduced-order model reproduces the original data near the initial time and the reproducibility is improved as the angle of attack increases. However, this evaluation was performed only by qualitative observation in the previous work.
In the present study, we focus on the estimation error and propose a quantitative evaluation method of the reproducibility. The effects of parameters the number of POD modes used in the model and the method for computing the coefficient matrix of the model equation on the reproducibility were investigated based on the evaluation. In addition, the previous evaluation of the model performance by qualitative observation was verified quantitatively.

2. Review of Previous Work

2.1 Linear Reduced-order Model

The construction of the linear reduced-order model starts with the derivation of a low-dimensional description of the velocity field data acquired by time-resolved PIV. First, the data matrix \( X \) is constructed by sorting the fluctuations of two-dimensional velocity components \( u(t) \) and \( v(t) \):

\[
X = \begin{bmatrix}
u(1) & u(2) & \cdots & u(N) \\
v(1) & v(2) & \cdots & v(N)
\end{bmatrix}
\] (1)

POD analysis is then applied to \( X \), and the POD modes are obtained as:

\[
X = \sum_{k=1}^{N} \sigma_k \phi_k(x) \psi_k(n),
\] (2)

where \( \sigma_k, \phi_k(x) \) and \( \psi_k(n) \) are the singular value, the spatial mode and the temporal mode, respectively. The modal analysis by POD provides the orthogonal bases that express the original data with the utmost efficiency. \( \sigma_k \) represents the energy contained in each POD mode corresponding to the amount of information included in the original data. Therefore, the degrees of freedom of the data are reduced with minimal information loss by truncating less-energetic POD modes. The low-dimensionalized data matrix \( X_{\text{low}} \) is then reconstructed using the \( r \) most energetic POD modes as follows:

\[
X_{\text{low}} = \sum_{k=1}^{r} \sigma_k \phi_k(x) \psi_k(n).
\] (3)

In our previous study, \( r \) was set to ten (\( r = 10 \)) from the viewpoint of reducing the computational cost required for estimation by the model (Nankai et al. 2019). Expressing the flow field data with ten POD modes means that the complexity of the multiplication required to estimate the velocity field at next time step in the present case is reduced to approximately hundred-thousandth (Nonomura et al. 2018).

The estimation target of the present model is the time fluctuation of the POD modes, namely \( \sigma_k \psi_k(n) \). The spatial modes \( \phi_k \) are only used to visualize the estimated POD modes as reconstructed velocity fields. Thus, \( \sigma_k \) and \( \psi_k(n) \) are used to reconstruct the reduced data matrix for constructing the model:

\[
\tilde{X}_{\text{low}} = \sum_{k=1}^{N} \sigma_k \psi_k(n)
\]

\[
\begin{bmatrix}
z_1(1) & z_1(2) & \cdots & z_1(N-1) & z_1(N) \\
z_2(1) & z_2(2) & \cdots & z_2(N-1) & z_2(N) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
z_r(1) & z_r(2) & \cdots & z_r(N-1) & z_r(N) \\
z_1(1) & z_2(2) & \cdots & z_r(N-1) & z_r(N)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
z_1(1) & z_2(2) & \cdots & z_r(N-1) & z_r(N)
\end{bmatrix}
\]

where \( z(n) \) consists of the strength of the POD modes (POD-mode coefficients) at the \( n \)th time step.

The standard model equation is defined based on the concept of DMD as follows:

\[
z(n) =Az(n-1), \quad (n=2,3,\ldots,N)
\] (5)

The model is constructed by computing the coefficient matrix \( A \) from the training dataset. In our previous work, \( A \) is computed by the least squares (LS) method, which corresponds to the exact DMD method (Tu et al. 2013), as follows:

\[
A = X_n X_{n-1}^\dagger \left( (X_{n-1} X_{n-2})^\dagger \right),
\] (6)
where the columns of the matrices $X_N$ and $X_{N-1}$ are collections of snapshots of POD-mode coefficients, as

$$X_{N-1} = [z(1) \ z(2) \ \ldots \ z(N-1)],$$  \quad (7)

$$X_N = [z(2) \ z(3) \ \ldots \ z(N)].$$  \quad (8)

The estimation of each POD mode by the model at an arbitrary time step is performed recursively based on the original data at the first time step. The estimated POD-mode coefficients at the $n$th time step are calculated as:

$$\hat{z}(n) = A^{n-1}z(1).$$  \quad (9)

These estimated instantaneous POD-mode coefficients are visualized as velocity fields obtained by multiplying them by the spatial modes $\phi_k$, which are omitted in the construction of the model. This is described by

$$\hat{X}_{low} = \sum_{k=1}^{\infty} \phi_k(x)\hat{z}_k(n).$$  \quad (10)

### 2.2 Experimental Setup

The wind tunnel testing was conducted in the Tohoku-university Basic Aerodynamic Research Wind Tunnel (T-BART) with a closed test section of 300 mm × 300 mm cross section. The airfoil of the test model has a NACA0015 profile with a chord length $c$ of 100 mm and a span width of 300 mm. The model was fabricated using stereolithography, which is a high-precision three-dimensional printing method. The time-resolved PIV measurement was conducted according to the test conditions given in Table 1.

Figure 1 shows a schematic of the PIV measurement system. The airfoil model was vertically fixed on the test section. The tracer particles were a 50% aqueous solution of glycerin with an estimated diameter of a few micrometers. The particle images were acquired using a double pulse laser (LDY-303PIV, Litron) and a high-speed camera (SA-X2, Photron) that were synchronized with each other.

The parameters of the PIV measurement are summarized in Table 2. DynamicStudio 5.1 (Dantec Dynamics) was used to acquire the particle images with a size of 1024 × 1024 pixels and calculate the time-resolved data of the two-dimensional velocity vectors using an adaptive PIV algorithm with an interrogation area of 8 × 8 pixels. Moving average validation was employed to smooth out each vector using 3 × 3 vectors around it.

### 2.3 Results

#### 2.3.1 Particle Image Velocimetry Measurement

The time-averaged vorticity fields and streamlines at each angle of attack are shown in Fig. 2. In this study, the calculated velocity data near the airfoil and behind the laser light were not used because their reliability is reduced by the presence of reflections and a lack of tracer particles. The black and gray regions in Fig. 2 represent the masked region and the position of the airfoil, respectively. Figure 2 shows that the flow separation is captured at $\alpha = 12^\circ$; thus, the data at $\alpha = 11^\circ$ were not used in the construction of the model because the target of the model is velocity fluctuations produced by the flow separation.

#### 2.3.2 POD Analysis

Figure 3 shows the POD-mode energy distributions at $\alpha = 16^\circ$. Figure 3(a) and (b) represents the energy ratio of each POD mode and the amount of energy contained in the first $k$ POD modes, respectively. The first ten POD modes represent approximately 70% of the total energy. Figure 4 displays the streamwise velocity fields of several of the first ten POD modes at $\alpha = 16^\circ$. See our previous report (Nankai et al. 2019) for a more detailed discussion of the POD analysis.

#### 2.3.3 Estimation by the Linear Reduced-order Model

Figure 5 shows the time histories of the first two original and estimated POD-mode coefficients. These results show that the model reproduces the time fluctuation of the original POD-mode coefficients near the initial time. However, as time progresses, the estimated POD-mode coefficients gradually attenuate and finally converge to zero. Additionally, the reproducibility of the model appears to improve as $\alpha$ increases.

3
3. Parameters of the Model

In the construction of the present model, there are some parameters that are considered to affect the model performance, such as the number $r$ of POD modes and the coefficient matrix $A$. For example, the effect of $r$ on the time histories of the estimated POD-mode coefficients is shown in Fig. 6. The results demonstrate that the behavior of the estimated POD modes depends on $r$. In the previous work, the model was constructed under just one condition: with $r = 10$ and $A$ computed by the LS method. In this study, the model was constructed under different sets of conditions, and the effects of $r$ and $A$ on the model performance were investigated.

The coefficient matrix was computed by three additional methods in addition to the exact-DMD-based method by LS applied in the previous work. The first one is the forward-backward method (FB) proposed by Dawson et al. (2016). This method considers the following forward and backward dynamical systems:

$$
\begin{align*}
    z(n) &= A_f z(n-1), \\
    z(n-1) &= A_b z(n).
\end{align*}
$$

The two matrices $A_f$ and $A_b$ are computed by the LS method. Note that $A_f$ corresponds to the standard coefficient matrix $A$ acquired in the previous work; that is, $A = A_f$. This method is hereafter referred to as the “forward (or standard) method”. If these matrices are computed from a linear dynamical system, the forward propagator matrix should be the inverse of the backward matrix. In reality, they have the same type of eigenvalue bias and are only approximate inverses. Dawson et al. (2016) have shown that the corresponding debiased matrix can be estimated by combining them as:

$$
A_{fb} = (A_f (A_b)^{-1})^{1/2}.
$$

The second method is the total least-squares (TLS) method developed by Hemati et al. (2017). The standard LS method minimizes the error with respect to time-shifted data $X_N$; that is, it does not assume noise on $X_N^{-1}$. On the other hand, the TLS method assumes noise on both matrices. The new coefficient matrix $A_{tls}$ is computed by performing a linear fitting in which the Frobenius norms of the errors on $X_{N^{-1}}$ and $X_N$ are minimized, namely solving the following problem:

$$
\begin{align*}
    \min_{A_{tls}, \Delta X_{N^{-1}}, \Delta X_N} \| \Delta X_{N^{-1}} \|, \text{ subject to } (X_N + \Delta X_N) = A_{tls} (X_{N^{-1}} + \Delta X_{N^{-1}}),
\end{align*}
$$

where $\Delta$ is the error component of each data matrix.

These two DMD-based methods have been shown to be effective for debiasing the eigenvalues of the propagator matrix against the effects of the observation noise. In addition to these methods, the coefficient matrix was also computed based on the approach taken by Perret et al. (2006). The following ordinary differential equation (ODE) was assumed:

$$
\dot{z}(t) = Dz(t),
$$

where $D$ is the coefficient matrix of the linear term. A second-order finite difference scheme was adopted, and the time derivatives were estimated in accordance with the approach by Perret et al. (2006):

$$
\dot{z}(t + \Delta t/2) = \frac{z(t + \Delta t) - z(t)}{\Delta t}.
$$

In addition, the POD-mode coefficients were modified to maintain the simultaneity of the samples of the POD-mode coefficients and their time derivatives, as

$$
z(t + \Delta t/2) = \frac{z(t + \Delta t) + z(t)}{2}.
$$

Equation (15) can then be modified as

$$
\dot{z}(t + \Delta t/2) = Dz(t + \Delta t/2)
\Rightarrow \frac{z(t + \Delta t) - z(t)}{\Delta t} = D \frac{z(t + \Delta t) + z(t)}{2},
$$
\[ z(t + \Delta t) - z(t) = D\Delta t \frac{z(t + \Delta t) + z(t)}{2} \]

\[ \Rightarrow \delta z(t + \Delta t / 2) = D\Delta t \delta z(t + \Delta t / 2) , \]

where the difference \( z(t + \Delta t) - z(t) \) is written as \( \delta z(t + \Delta t/2) \). The matrix \( D\Delta t \) is computed by the LS method using a collection of all of the snapshots of the POD-mode coefficients \( Z \) as

\[
Z = \begin{bmatrix} z(t_1 + \Delta t / 2) & z(t_2 + \Delta t / 2) & \cdots & z(t_{N-1} + \Delta t / 2) \\
\delta z(t_1 + \Delta t / 2) & \delta z(t_2 + \Delta t / 2) & \cdots & \delta z(t_{N-1} + \Delta t / 2) 
\end{bmatrix},
\]

\[
\delta Z = D\Delta t Z .
\]

The following equation is obtained by integrating Eq. (15):

\[ z(t) = e^{\beta t} E , \]

where \( E \) is a constant. Equation (21) indicates that the time history of the POD-mode coefficients can be acquired as follows:

\[ z(t + \Delta t) = e^{(\beta(t + \Delta t))} E = e^{\Delta t \beta} e^{\Delta t} E = e^{\Delta t \beta} z(t) , \]

\[ \Rightarrow z(t_i + n\Delta t) = (e^{\Delta t \beta})^n z(t_i) . \]

Therefore, the new coefficient matrix based on the ODE (ODE-based method) corresponds to the time evolution operator in Eq. (23):

\[ A_{\text{ODE}} = e^{\Delta t \beta} . \]

### 4. Evaluation Method

In this study, the estimation error of the model was investigated, and the reproducibility was evaluated quantitatively based on the results. The new evaluation method enables the specification of the best set of parameters for the construction of the model \((r \text{ and } A)\), as described in Sect. 3) to yield the highest reproducibility. In addition, the quantitative evaluation results were used to confirm the qualitative assessment of the model performance conducted in the previous study.

#### 4.1 Estimation Error

The estimation results by the present model were obtained as the time histories of the POD-mode coefficients, as shown in Fig. 5. First, the difference between the original and estimated POD-mode coefficient values is calculated at each time step as

\[ e_k(n) = z_k(n) - \hat{z}_k(n) . \quad (k = 1, 2, \cdots, r) \]

The instantaneous error can be defined as the root sum of squares of \( e_k \) because the POD bases are orthogonal to each other:

\[ e(n) = \sqrt{\sum_{k=1}^{r} \{e_k(n)\}^2} . \]

The temporal evolution of the error described by Eq. (26) can then be plotted in a graph, as shown in Fig. 7. The vertical axis represents the estimation error \( e \), and the horizontal axis represents the non-dimensionalized time from the initial time step, which is the time step at which the original POD-mode coefficient is given to the model. However, in fact, the instantaneous error varied over a wide range, as shown in Fig. 7, and was difficult to investigate accurately. Therefore, the ensemble average of the estimation results was taken to produce a smooth curve of the temporal evolution of the error. Equation (9) shows that many estimation results can be obtained by substituting the original data for the initial value in the model; that is, \( z(1) \) in Eq. (9) is changed to the value of the original POD modes at an arbitrary time step \( z(p) \):

\[ \hat{z}^{(p)}(n) = A^{n-p} z(p) . \]

Accordingly, we generated as many estimation error curves as possible in the range of the data used for the estimation, as

\[ e_k^{(p)}(n) = z_k(n) - \hat{z}_k^{(p)}(n) , \]
\[ e^{(p)}(n) = \sqrt{\sum_{k=1}^{r} \{e_{k}^{(p)}(n)\}^2}. \]  

The ensemble average of these curves was then taken as

\[ e(n) = \frac{\sum_{p=1}^{q} e^{(p)}(n)}{q}. \]  

The smooth estimation error curve obtained in this way is illustrated in Fig. 8.

### 4.2 Reproducibility

The reproducibility of the model was evaluated quantitatively based on the estimation error curve. First, the forward model, which is the standard model, was considered. It was expected that the convergence value of the estimation error could be determined by the original POD-mode coefficients, because the estimated POD-mode coefficients ultimately approach zero and the error becomes equal to the deviation of the mode coefficient around zero:

\[
\lim_{n \to \infty} e(n) = \lim_{n \to \infty} \sqrt{\sum_{k=1}^{r} \{z_{k}^{(n)} - \hat{z}_{k}(n)\}^2} = \lim_{n \to \infty} \sqrt{\sum_{k=1}^{r} \{z_{k}^{(n)} - 0\}^2} = \lim_{n \to \infty} \sqrt{\sum_{k=1}^{r} \{z_{k}^{(n)}\}^2}.
\]  

Therefore, the convergence value of the error was defined as the root mean square (RMS) of the original POD-mode coefficients \(z_{RMS}\), as given by

\[ z_{RMS} = \sqrt{\frac{\sum_{k=1}^{r} \sum_{n=1}^{N} \{z_{k}^{(n)}\}^2}{N}}. \]  

The “permissive time range” \(n_{perm}\) is defined as the time step at which the error reaches 63.2% of \(z_{RMS}\), i.e., \(e(n_{perm}) = 0.632z_{RMS}\), under the simple assumption of a first-order lag system, as illustrated in Fig. 9. The reproducibility of the model is described as \(n_{perm}\) (nondimensionalized as \((tU/c)_{perm}\)).

## 5. Results and Discussion

### 5.1 Effects of Modeling Parameters

The linear reduced-order model was constructed with different sets of the parameters \(r\) and \(A\), and their effects on the model performance were investigated. The number of POD modes was varied from \(r = 2\) to \(r = 100\), and the coefficient matrix was computed using the four methods described in Section 3.

The effect of \(r\) on the model performance are shown in Fig. 10. Figure 10(a) displays the temporal evolution of the estimation error under different values of \(r\), and Fig. 10(b) displays the relationship between the reproducibility and \(r\). In this study, \(r\) was varied up to one-hundred, which corresponds to 10% of the total number of POD modes from the viewpoint of the computational cost for the estimation. These results show that the reproducibility does not change monotonically with respect to \(r\) and reaches a maximum at a specific value. In addition, the performance was highly sensitive at small \(r\) and did not significantly change when \(r\) was large. It is noteworthy that increasing the number of POD modes is not effective for improving the model performance. Figure 11 displays the relationship between the value of \(r\) at which the reproducibility was maximized and the amount of energy contained in the first \(r\) POD modes at different angles of attack. The square symbols on the curves of the POD-mode energy distributions indicate the amount of energy contained in the low-dimensionalized data when \(r\) was...
selected to maximize the reproducibility. Figure 11 shows that the reproducibility was maximized when the energy ratio was approximately 60%, except in the case of $\alpha = 12^\circ$. This result appears to be associated with the configuration of the flow separation. The separated region at $\alpha = 12^\circ$ was smaller than those in other cases, as shown in Fig. 2, and its flow configuration is considered to be different from those in other cases, e.g., a flow reattachment may have occurred.

Figures 12-14 demonstrate the effects of $A$ on the model performance under different values of $r$ at $\alpha = 16^\circ$. Figures 12-14(a) and (b) display the time histories of the POD-mode coefficients estimated by the models, and Figs. 12-14(c) show the eigenvalue distributions of the coefficient matrices. The behavior of the estimated POD-mode coefficients is determined by the eigenvalues of the coefficient matrix. It has been shown that the amplification factor, which shows how the mode evolves in time, corresponds to the magnitude of the eigenvalues $\lambda$ and the frequency of the time fluctuation of each mode is represented by the argument of $\lambda$ (Taira et al. 2017). Additionally, Figs. 12-14(d) present the temporal evolution of the estimation error of each model. For all values of $r$, the POD modes estimated by the forward (standard) model attenuate and diminish to zero. This is also demonstrated by the eigenvalues of $A_I$ (i.e., the magnitudes of all eigenvalues is less than unity).

In contrast, the POD modes estimated by the other three models do not attenuate, and they seem to reproduce the low-frequency component of the time fluctuation of the original POD modes better than those obtained by the forward (standard) method. However, the amplitude is not consistent with that of the original POD modes, and the phase of the fluctuations shifts gradually. They are presumed to cause the very poor reproducibility at some time steps. In addition, the estimation error curve indicates that the error increases as time advances. This means that although the performance of the forward (standard) model and the new models differs greatly in terms of their attenuation behavior, the reproducibility of all models diminishes as time progresses. The three new models show similar behavior in the case with $r = 10$, as shown in Fig. 12, and they display increasingly different behaviors as $r$ becomes large, as illustrated in Figs. 13 and 14. When $r$ was to fifty, the amplitude of the POD modes estimated by new models increased, and in particular, the TLS model had a large amplification factor. Moreover, in the case of $r = 100$, the performance of the models became more discriminating. The eigenvalues of $A_{th}$ were scattered around the unit circle, and their arguments were much larger than those of the other coefficient matrices. The results demonstrate that $A_{th}$ includes unstable eigenvalues and the POD-mode coefficients estimated by the TLS method are likely to diverge. Meanwhile, the eigenvalues of $A_{fb}$ and $A_{ODEB}$ were mostly located on the unit circle. Nevertheless, in some cases, the magnitudes of a few eigenvalues of $A_{th}$ were much greater or less than unity, as shown in Fig. 14(c). In addition, the arguments of the eigenvalues indicate that the FB model produces higher-frequency oscillations than the forward (standard) and ODE-based models. The features of the eigenvalue spectra are consistent with results obtained in previous works (Kutz et al. 2016).

Figure 15 shows the reproducibility under each considered value of $r$. The three new models showed similar reproducibility in the low $r$ region (approximately $r < 30$, corresponding to more than 80% of the total energy) with differences in their reproducibility gradually increasing as $r$ increases. The estimation results shown in Sect. 4.1 and Fig. 15 demonstrate that the TLS model is likely to diverge because of its high amplification factor, and its reproducibility was very low at large $r$. Furthermore, the reproducibility of the FB model was lower than that of the ODE-based method. This is considered to be because the eigenvalue distribution of $A_{th}$ is more unstable than that of $A_{ODEB}$. The eigenvalues of the ODE-based model stably lie on the unit circle, and the reproducibility does not significantly drop even with increasing $r$. These results illustrate that the ODE-based model has the best performance of the additional models. However, the reproducibility of the ODE-based model is lower than that of the forward (standard) model as shown in Fig. 15. The performance of present models worsened as time advanced; specifically, the estimation error increases over time, as indicated by the estimation error curve. Therefore, the evaluation results reveal that the evolution of the estimation error in the forward (standard) model is the gentlest.

In conclusion, the forward (standard) model shows the best reproducibility of the present linear models. The parameters for modeling that maximize the reproducibility within the range of $2 \leq r \leq 100$ are provided in Table 3. On the other
hand, in terms of the attenuation of the model, i.e., the magnitudes of the eigenvalues of the coefficient matrices, the ODE-based model works the best. Furthermore, Fig. 15 also demonstrates that the present models show the best performance under the condition of the same $r$.

5.2 Effects of Angle of Attack

In addition, the dependence of the reproducibility on $\alpha$ at $r = 10$ is shown in Fig. 16. The result demonstrates that the reproducibility becomes higher as $\alpha$ increases. The size of the vortex structure produced by the flow separation depends on $\alpha$, as illustrated in Fig. 2. This implies that a larger flow structure seems to be expressed better by the linear system. It corresponds to the fact that the present linear model can reproduce low-order POD modes better than high-order POD modes, as described in our previous paper (Nankai et al. 2019). This is because low-order POD modes express larger flow structures than high-order POD modes as shown in Fig. 4.

6. Conclusions

The estimation performance of linear reduced-order models based on PIV data of the flow field around a NACA0015 airfoil were quantitatively investigated in this study. A method of evaluating the model reproducibility based on the estimation error was proposed, and the effects of the modeling parameters, namely the number $r$ of POD modes and the coefficient matrix $A$, were explored. Additional coefficient matrices were introduced based on the concept of dynamic mode decomposition (FB method and TLS method) and a method developed in a previous study (ODE-based method) in addition to the conventional standard method (forward method). Moreover, the dependence of the angle of attack on the model performance, which was discussed in a previous work, was verified quantitatively.

It was demonstrated that the reproducibility and $r$ do not have a simple correlation. The reproducibility does not increase much with $r$; additionally, the reproducibility of the three additional models worsens with increasing $r$. In other words, increasing $r$ does not contribute greatly to the improvement of the reproducibility. The best condition for model performance regarding the value of $r$ appears to be related to the partial amount of energy contained in low-dimensionalized fluid data and the state of the flow fields. The POD-mode coefficients estimated by the new models do not diminish, which is in contrast to those of the forward model. However, the estimation error of every model was shown to increase over time. The forward model showed the lowest growth rate of the error and the best reproducibility. Meanwhile, the eigenvalue distributions of the coefficient matrices of the new models demonstrate that their amplification factors are better than that of the forward model; in particular, the ODE-based method shows better performance. The eigenvalues of the ODE-based model were stably located on the unit circle even as they increase in number with increasing $r$.

Furthermore, the supposition that the reproducibility becomes larger as $\alpha$ increases, as discussed in the previous work, was quantitatively confirmed by the evaluation method. This result strengthens the hypothesis that the linear system reproduces temporal fluctuations in large flow structures better than fluctuations in small ones.

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Table 1. Test conditions

| $U$ [m/s] | $\alpha$ [$^\circ$] | 11 | 12 | 14 | 16 | 18 | 20 |
|-----------|---------------------|----|----|----|----|----|----|
|           | $Re_c$              | $6.4 \times 10^4$ |

Fig. 1 Schematic of the PIV measurement system

Table 2. PIV measurement condition

| Laser | Double pulse lasers |
|-------|---------------------|
| Time between pulses [μs] | 100 |
| Sampling rate [Hz] | 5000 |
| Particle image resolution [pixel × pixel] | $1024 \times 1024$ |
| Total number of image pairs, $N$ | 1000 |

Fig. 2 Time-averaged vorticity fields and streamlines
Fig. 3 POD-mode energy distributions ($\alpha = 16^\circ$)

Fig. 4 Velocity fields of POD modes ($\alpha = 16^\circ, u$)

(a)$\frac{\sigma_k^2}{\sum_k \sigma_k^2}$ vs $k$
(b) $\sum_i \frac{\sigma_i^2}{\sum \sigma_i^2}$ vs $k$

Original POD mode
Estimated POD mode

(i) Mode 1
(ii) Mode 2

(a) $\alpha = 12^\circ$
(i) Mode 1

(ii) Mode 2

(b) $\alpha = 16^\circ$

Fig. 5 Time histories of the POD-mode coefficient

Fig. 6 Effect of $r$ on the estimation results ($\alpha = 16^\circ$, $A_f$)

Fig. 7 Estimation error of the model ($\alpha = 16^\circ$, $r = 10$, $A_f$)
Fig. 8 Schematic of the ensemble averaging procedure for the estimation error investigation

Fig. 9 Schematic of the derivation of the model reproducibility

$q$: Number of samples for averaging
$N$: Total number of time series data
Fig. 10 Effect of $r$ on the model performance ($\alpha = 16^\circ$, $A_t$)

Fig. 11 Partial amount of energy contained in the POD modes

(Symbols indicate the number of POD modes at which the reproducibility is maximized)
Fig. 12 Effect of $A$ on the model performance ($\alpha = 16^\circ, r = 10$)

Fig. 13 Effect of $A$ on the model performance ($\alpha = 16^\circ, r = 50$)
(c) Eigenvalue distribution

\( A_{\text{tls}} \) has several eigenvalues located in the second quadrant

(d) Estimation error

Fig. 14 Effect of \( A \) on the model performance (\( \alpha = 16^\circ, r = 100 \))

(Results of the TLS model are omitted from (a) and (b) because they diverge so drastically that the other results cannot be observed clearly; accordingly, the estimation error of this method is very large from the beginning and is not included in the range of the graph in (d))

Fig. 15 Effect of \( r \) on the reproducibility of the model (\( \alpha = 16^\circ \))
Fig. 16 Effect of $\alpha$ on reproducibility of the model ($r = 10$)

Table 3. Modeling conditions that realize the highest reproducibility

| $\alpha$ [°] | Model | $r$ ($2 \leq r \leq 100$) | Energy ratio [%] | $(tU/c)_{max}$, subject to $2 \leq r \leq 100$ |
|-------------|-------|-----------------|----------------|----------------------------------|
| 12          | Forward | 2               | 31.1           | 0.328                            |
| 14          | Forward | 5               | 53.0           | 0.382                            |
| 16          | Forward | 7               | 63.1           | 0.471                            |
| 18          | Forward | 4               | 56.4           | 0.648                            |
| 20          | Forward | 4               | 54.7           | 0.592                            |