Worldline Path Integrals for
Gauge Fields and Quantum Computing

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Abstract

We study different aspects the worldline path integrals with gauge fields using quantum computing. We use the Variational Quantum Eigensolver (VQE) and Evolution of Hamiltonian (EOH) quantum algorithms and IBM QISKit to perform our computations. We apply these methods to the path integral of a particle moving in a Abelian and non-Abelian background gauge field associated with a constant magnetic field and the field of a chromo-magnetic field. In all cases we found excellent agreement with the classical computation. We also discuss the insertion of vertex operators into the worldline path integrals to study scattering and show how to represent them using unitary operators and quantum gates on near term quantum computers.
1 Introduction

Recently there has been a great deal of interest in the quantum simulation of non-Abelian gauge fields. Most of these studies are restricted to lower dimensions because of the large amount of qubits and large depth quantum circuits that are involved in performing a quantum field simulation. In this paper we explore quantum simulation of gauge fields using the worldline formalism. We study the quantum simulation of a particle moving in a constant magnetic field in both Cartesian and polar coordinates. We also study the world line Hamiltonian for a non-Abelian gauge field and discuss the use of vertex operators for the quantum simulation of nonlinear theories. We use the Variational Quantum Eigensolver (VQE) quantum algorithm and Evolution of Hamiltonian (EOH) quantum algorithm to obtain our results and obtain excellent agreement with classical computations.

2 Quantum computer simulation for a constant magnetic field

One example of a worldline path integral in particle propagation in a constant electromagnetic field. For a constant magnetic field the non-relativistic Lagrangian is given by [1][2][3]:

\[ L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{eB}{2c}(xy - \dot{x}y) \]  

(2.1)

Where \( B \) is the magnitude of the magnetic field in the \( z \) direction. This is written in terms of the vector potential as:

\[ L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + A_x(x, y, z)\dot{x} + A_y(x, y, z)\dot{y} + A_z(x, y, z)\dot{z} \]  

(2.2)

and with vector potential given by:

\[ A = (\frac{-eB}{2c}y, \frac{eB}{2c}x, 0) \]  

(2.3)

one has the Hamiltonian:

\[ H = \frac{1}{2m}(p - \frac{q}{c}A(x))^2 \]  

(2.4)

which can be written:

\[ H = \frac{1}{2}\left(\frac{px + B}{2}y\right)^2 + \frac{1}{2}\left(\frac{py - B}{2}x\right)^2 \]  

(2.5)

for a constant magnetic field the energy eigenvalues are:

\[ E_n = |B|(n + \frac{1}{2}) \]  

(2.6)
with ground state wave function:

\[ \psi_0(x, y) = \sqrt{\frac{|B|}{2\pi}} e^{-\frac{|B|}{4}(x^2+y^2)} \]  

(2.7)

Expanding out the terms in $B$ the Hamiltonian can be written:

\[ H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2} \left( \frac{B}{2} \right)^2 (x^2 + y^2) - \frac{B}{2} (xp_y - yp_x) \]  

(2.8)

which shows the relation to the two dimensional isotropic simple harmonic oscillator. The Kernel function

\[ K(x_i, y_i, x_f, y_f; t) = \langle x_f, y_f | e^{-iHt} | x_i, y_i \rangle \]  

(2.9)

can be evaluated using path integrals and in Cartesian coordinates is given by:

\[ K = \left( \frac{1}{2\pi T} \right)^{3/2} \left( \frac{BT/2}{\sin(BT/2)} \right) e^{\frac{(x_f-x_i)^2}{2T}} \exp \left[ \frac{iBT/2}{\tan(BT/2)} \left( (x_f-x_i)^2 + (y_f-y_i)^2 + B(y_fx_i - x_fy_i) \right) \right] \]  

(2.10)

One can also use polar coordinates to study the particle in a constant magnetic field. In terms of polar coordinates the Hamiltonian is:

\[ H = \frac{1}{2} \rho^{-1/2} p_{\rho} p_{\rho} \rho^{-1/2} + \frac{1}{2} \left( \frac{B}{2} \right)^2 \rho^2 + \frac{1}{2} \frac{L_z^2}{\rho^2} - \frac{B}{2} L_z \]  

(2.11)

and the eigenvalues and eigenfunctions are given by

\[ E_{n,m} = (n + 1 - m) \frac{B}{2} \]  

(2.12)

and

\[ \psi_{n,m}(\rho, \varphi, t) = \left[ \frac{n! (B/2)^{1/2}}{\pi (n + |m|)!} \right]^{1/2} \left( \frac{B}{2} \right)^{1/2} \rho^{|m|} e^{-\frac{B}{4} \rho^2} L_n^{[m]} \left( \frac{B}{2 \rho^2} \right) e^{im\varphi} e^{iE_{n,m}t} \]  

(2.13)

where $L_n^{\alpha}$ are generalized Laguerre polynomials. In terms of polar coordinates the K function is:

\[ K = \left( \frac{1}{2\pi i} \right) \frac{B/2}{\sin(BT/2)} e^{-\frac{B}{4} \left( \rho_f^2 + \rho_i^2 \right)} \left( \frac{B}{2 \rho_f^2} \right)^{1/2} \sum_{m=-\infty}^{\infty} e^{im(\varphi_f - \varphi_i + \frac{B}{2} T)} I_{|m|} \left[ -i \frac{B}{2 \rho_f \rho_i} \frac{1}{\sin(BT/2)} \right] \]  

(2.14)

where $I_{\nu}(z)$ is a modified Bessel function. Having reviewed the energies and Kernel functions for motion of particle in a constant magnetic field we now turn the the quantum computation of these quantities in terms of qubits and quantum circuits.
2.1 Quantum computation with the Variational Quantum Eigensolver

The Variational Quantum Eigensolver (VQE) is a hybrid classical quantum algorithm that can be run on near term quantum hardware because the circuit depth can be kept small for the algorithm, at least for simple variational wave forms [4]. Like the traditional variational method one optimizes

$$E_0(\theta_i) = \frac{\langle \psi(\theta_i) | H | \psi(\theta_i) \rangle}{\langle \psi(\theta_i) | \psi(\theta_i) \rangle}$$

(2.15)

with variational parameters $\theta_i$ to obtain an upper bound on the value of the ground state energy of a Hamiltonian. One uses an optimizer to determine the minimum of the energy expectation value and the expectation value is evaluated on the quantum computer. QISKit provides several optimizers. In this paper we use the The the statevector simulator with the Sequential Least SQuares Programming (SLSQP) optimizer. For the VQE the variational wave functions are represented as unitary matrices acting on qubits through quantum gates and the the parameters are rotation angles associated with these gates.

For the Hamiltonian (2.5)

$$H = \frac{1}{2} \left( p_x + \frac{B}{2} y \right)^2 + \frac{1}{2} \left( p_y - \frac{B}{2} x \right)^2$$

(2.16)

and using the Harmonic oscillator basis to map the the Hamiltonian in terms of qubits we use

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{1} & 0 & \cdots & 0 \\ \sqrt{1} & 0 & \sqrt{2} & \cdots & 0 \\ 0 & \sqrt{2} & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 0 & \sqrt{N-1} \\ 0 & 0 & \cdots & \sqrt{N-1} & 0 \end{bmatrix}$$

(2.17)

while for the momentum operator we have:

$$P = \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & -\sqrt{1} & 0 & \cdots & 0 \\ \sqrt{1} & 0 & -\sqrt{2} & \cdots & 0 \\ 0 & \sqrt{2} & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 0 & -\sqrt{N-1} \\ 0 & 0 & \cdots & \sqrt{N-1} & 0 \end{bmatrix}$$

(2.18)

taking $N = 16$ we can represent the operators using tensor products with each factor in the tensor products as below:

$$x = Q \otimes I_{16}$$
$$y = I_{16} \otimes Q$$
$$p_x = P \otimes I_{16}$$
$$p_y = I_{16} \otimes P$$

(2.19)
We obtain the result for $E_0$ from the VQE for the above two boson Hamiltonian is oscillator basis where each boson is represented by a $16 \times 16$ matrix the total Hamiltonian uses a 256 matrix and is represented using eight qubits. The results are shown in figure 1 and table 1 and agree well with the exact value.

| Hamiltonian                  | No. of Qubits | Exact Result | VQE Result     |
|-----------------------------|---------------|--------------|----------------|
| Constant Magnetic Cartesian | 8             | 1.0          | 1.0000003553   |
| Constant Magnetic Polar     | 4             | 0.9980452    | 0.99859198     |

Table 1: VQE results for a particle moving in a constant magnetic field using Cartesian and Polar coordinates. The Hamiltonian were mapped to 8-qubit operators for Cartesian coordinates and 4-qubit operators for Polar coordinates. The quantum circuit for each simulation utilized an $R_y$ variational form, with a fully entangled circuit of depth 3. The backend used was a statevector simulator. The Sequential Least SQuares Programming (SLSQP) optimizer was used, with a maximum of 600 iterations.

For the polar coordinates form of the Hamiltonian

$$H = \frac{1}{2} \rho^{-1/2} p_\rho p_\rho \rho^{-1/2} + \frac{1}{2} \left( \frac{B}{2} \right)^2 \rho^2 + \frac{1}{2} \mu^2 - \frac{B}{2} L_z$$  \hspace{1cm} (2.20)$$

In this case we don’t need a tensor product to construct the Hamiltonian and 4 qubits suffice for the quantum computation. The results of the VQE computations are shown in
2.2 Quantum computation with the Evolution of Hamiltonian algorithm.

A second quantum algorithm we applied to worldline path integrals of gauge fields is the Evolution of Hamiltonian algorithm. The Evolution of Hamiltonian (EOH) algorithm evaluates

$$K(X_i, X_f; T) = \langle \psi(X_f) \vert e^{-itH} \vert \psi(X_i) \rangle$$

the transition amplitude between an initial state and a final state with $H$ the Hamiltonian and $t$ the time of the evolution and is depicted in figure 2. When applying the EOH algorithm it is convenient to use the position basis in which the position operator is diagonal instead of the oscillator basis described above. In the position basis the momentum matrix is dense and constructed from the position operator using a Sylvester matrix $F$. In the

Figure 2: (top) Evolution of Hamiltonian for particle moving in constant magnetic field in terms of a quantum circuit where the green rectangle indicated the unitary operator for time evolution (bottom) Feynman propagator of Kernel function in terms of a Feynman graph corresponding to the top circuit.

figure 1 and table 2 and are in excellent agreement with the classical computations. This is possible because of the similarity of the particle in a constant magnetic field and the two dimensional harmonic oscillator for which the variational forms in QISKit has a strong overlap with the ground state wave function.
position basis the position matrix is:

\[(Q_{pos})_{j,k} = \sqrt{\frac{2\pi}{4N}}(2j - (N + 1))\delta_{j,k}\]  \hspace{1cm} (2.22)

and the momentum matrix is:

\[P_{pos} = F^\dagger Q_{pos} F\] \hspace{1cm} (2.23)

where

\[F_{j,k} = \frac{1}{\sqrt{N}}e^{\frac{2\pi i}{4N}(2j-(N+1))(2k-(N+1))}\] \hspace{1cm} (2.24)

Then the position and momentum operators are defined similarly to the above in terms of tensor products with \(Q\) replaced by \(Q_{pos}\) and \(P\) replaced by \(P_{pos}\). The results of the EOH computation using the Trotter-Suzuki approximation in QISKit of the particle in a constant magnetic field are shown in figure 3. The multiple curves in the graphs indicate the various transition probabilities from initial position \((x_i, y_i) = (0, 0)\) to a final position \((x_f, y_f)\) and are in excellent agreement with the classical results obtained by directly exponentiating the Hamiltonian matrix.

3 Quantum computer simulation for a non-Abelian gauge field

The quantum computer simulation of a particle in a background non-Abelian gauge field is more complicated than the quantum simulation of a particle in an Abelian gauge field discussed above. This is because the part of the Lagrangian involving the non-Abelian gauge field has to be path ordered for gauge invariance and this is difficult to implement directly. Instead one introduces worldline fermions into the 0+1 dimensional path integral associated with the non-Abelian symmetry group to implement the path ordering.
Then one can treat the systems as a more traditional Lagrangian system and can implement quantum simulation more straightforwardly. The Lagrangian of a particle moving in a non-Abelian gauge field then is given by

\[
L = \frac{1}{2} \dot{x}^2 + \psi^a \dot{\psi}^a + f_{abc} A_a^i(x) \dot{x}^i \psi^b \psi^c
\]  

(3.1)

where \(\psi^a\) are world line fermion fields which take into account the path ordering and \(f_{abc}\) are the structure constant of the Lie group. The Hamiltonian is then:

\[
H = \frac{1}{2} \left( p_i + f_{abc} A_a^i(x) \psi^b \psi^c \right)^2
\]  

(3.2)

In a background field expansion about a reference point \(x_0\) and a background field expansion using normal coordinates the Hamiltonian can be written in terms of the field strength as:

\[
H = \frac{1}{2} \left( p_i + f_{abc} F^a_{ij}(x_0) \psi^b \psi^c \right)^2
\]  

(3.3)

a form similar to the Abelian case discussed above.

To proceed further one needs a solution to the non-Abelian background gauge field equations of motion or the Yang-Mills equation [14] [15] [16] [17] [18]. For the non-Abelian \(SU(2)\) gauge field an ansatz used for the Wu-Yang is given by:

\[
A_a^i(x) = \varepsilon^i_{ab} x^b \frac{f(r)}{r^2}
\]  

(3.4)

defining \(g = f - 1\) the equations of motion reduce to:

\[
g'' = \frac{g}{r^2} (g^2 - 1)
\]  

(3.5)

which for small \(r\) has the expansion

\[
g \approx 1 - r^2 + \frac{3}{10} r^4 + \ldots
\]  

(3.6)

and large \(r\) we have:

\[
g \approx 1 - \frac{1}{r} + \frac{3}{4} \frac{1}{r^2} + \ldots
\]  

(3.7)

These expressions can be used to study the quantum simulation of a particle in a background non-Abelian gauge field but other forms can be studied as well depending on the application.
3.1 Quantum computation with the VQE for Hamiltonian for a non-Abelian gauge field

To map the Hamiltonian for a particle moving in a non-Abelian gauge field in terms of qubits we use $P$ and $Q$ from (2.17) and (2.18) with $N = 4$ and form the tensor products:

\begin{align*}
x &= Q \otimes I_4 \otimes I_4 \otimes I_8 \\
y &= I_4 \otimes Q \otimes I_4 \otimes I_8 \\
z &= I_4 \otimes I_4 \otimes Q \otimes I_8 \\
p_x &= P \otimes I_4 \otimes I_4 \otimes I_8 \\
p_y &= I_4 \otimes P \otimes I_4 \otimes I_8 \\
p_z &= I_4 \otimes I_4 \otimes P \otimes I_8 \\
\end{align*}

(3.8)

For $SU(2)$ we have three worldline fermions which are represented by

\begin{align*}
\psi_1 &= I_{64} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes I_2 \otimes I_2 \\
\psi_2 &= I_{64} \otimes I_2 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes I_2 \\
\psi_3 &= I_{64} \otimes I_2 \otimes I_2 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{align*}

(3.9)

For the quantum computation for non-Abelian gauge field we use the Hamilton

\begin{align*}
H &= \frac{1}{2}(p_x + B(-y\psi_1^2 + z\psi_3^3 \psi_1^1))^2 + \frac{1}{2}(p_y + B(-z\psi_2^2 \psi_3^3 + x\psi_1^1 \psi_2^2))^2 + \frac{1}{2}(p_z + B(-x\psi_3^3 \psi_1^1 + y\psi_2^2 \psi_3^3))^2 \\
\end{align*}

(3.10)

where $B = -\frac{\mu m}{\hbar^2}$.

For three bosons $x_i$ represented as $4 \times 4$ so six qubits are used to represent the bosons and for three fermions $\psi_i$ we use three qubits. The full Hamiltonian for a particle moving in an $SU(2)$ gauge field uses nine qubits and is represented by a $512 \times 512$ matrix. The results using the VQE algorithm and the SLSQP optimizer are shown in figure 4 and table 2. The results are not as accurate as for the Abelian gauge field. One possible reason is that the variational form used in the optimization is parametrized by angles associated with quantum gates does not overlap as strongly with the ground state for non-Abelian backgrounds as it does in the Abelian case.

4 Vertex operators and quantum simulation associated with non-linear field equations

One difficulty with quantum simulation is the the quantum circuits represent linear operators as a series of Unitary gates. This works well in simulating linear equations like the Schrodinger equation. However nonlinear equations are also important in physics such
Figure 4: (Left) VQE computation of ground state energy of a particle moving in a non-Abelian gauge field with $g_m = 2$ and the SLSQP optimizer (right) VQE computation of ground state energy of a particle moving in a non-Abelian gauge field with $g_m = .2$ and also using the SLSQP optimizer.

| Hamiltonian               | No. of Qubits | Exact Result   | VQE Result    |
|---------------------------|---------------|----------------|---------------|
| Magnetic Monopole $g_m = 2$ | 9             | -2.53854786    | -1.89074415   |
| Magnetic Monopole $g_m = .2$ | 9             | 0.31120022     | 0.32279234    |

Table 2: VQE results for a particle moving in a non-Abelian gauge field and using the oscillator basis. All of the Hamiltonian were mapped to 9-qubit operators. The quantum circuit for each simulation utilized an $R_y$ variational form, with a fully entangled circuit of depth 3. The backend used was a statevector simulator. The Sequential Least SQuares Programming (SLSQP) optimizer was used, with a maximum of 600 iterations.

as the Yang-Mills equation. To represent the nonlinear aspects of of these equations on a quantum computer one can use Vertex operators which are unitary operators and can be realized in terms of gates. This modifies the the Evolution of Hamiltonian algorithm by considering worldline actions on a graph which splits at a vertex at which point the vertex operator is inserted. This method allows the calculation of scattering amplitudes using the worldline formalism and requires far fewer qubits that a full quantum field theory simulation on a quantum computer.

For example for the $SU(2)$ Yang-Mills one needs to add an additional $x$ field in the world line action corresponding to the rank of the group and two vertex operators $V_1$ and $V_2$ associated with the roots of the Lie algebra of which there are two. For $SU(3)$ the rank is two so we add two additional $x$ fields and six vertex operators $V_i$ for $i = 1, 2, \ldots, 6$. 

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4.1 Nonlinear Klein-Gordon equation

As a simple example consider the case of the non-linear Klein-Gordon equation \[19\]. The nonlinear Klein-Gordon equation with interacting potential \(V(\phi)\) is:

\[
\partial_t^2 \phi - \nabla^2 \phi + m^2 \phi + V'(\phi) = 0
\] (4.1)

For a quartic interaction potential this equation has a solution in terms of the Jacobi elliptic functions.

The Effective action for the nonlinear Klein-Gordon field has a representation in term of a worldline path integral as:

\[
\Gamma[\phi] = \frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int D\phi \left( e^{-\int_0^T d\tau \left( \frac{1}{2} \dot{x}^2 + V''(\phi(x(\tau))) \right)} \right)
\] (4.2)

where \(m\) is the mass of the Klein-Gordon field. The Hamiltonian associated with this world line action is:

\[
H = p^\mu p_\mu + m^2 + V''(\phi)
\] (4.3)

For \(\lambda \phi^4\) potential scattering amplitudes can be determined by inserting factors

\[
V(p_k) = \lambda e^{i p_k x(\tau_k)}
\] (4.4)

into the world line path integral where \(p_k\) denotes the energy-momentum of \(k\)th particle.

\[
V_{\text{scalar}}(p) = \int_0^T d\tau e^{i p \cdot x(\tau)}
\] (4.5)

4.2 Nonlinear Euler-Heisenberg equation

Another system of nonlinear equations that can be realized on a quantum computer using this method is the Non-linear Euler-Heisenberg equation \[19\] \[20\] \[21\] \[22\] \[23\] \[24\] \[25\] \[26\] \[27\] \[28\]. In the weak field limit the Euler-Heisenberg Lagrangian density is given by:

\[
L = \frac{1}{2} (E^2 - B^2) + \frac{2 \alpha^2}{45 m^4} \left[ (E^2 - B^2)^2 + 7 (E \cdot B)^2 \right]
\] (4.6)

The origin of the quartic terms are quantum corrections from fermion loop diagrams.

For scalar electrodynamics the worldline representation of the Euler-Heisenberg effective action is:

\[
\Gamma[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int D\phi \left( e^{-\int_0^T d\tau (\frac{1}{2} \dot{x}^2 + i e A^\mu(x(\tau)))} \right)
\] (4.7)

The Hamiltonian associated with this world line action is:

\[
H = (p^\mu + e A^\mu(x)) (p_\mu + e A_\mu) + m^2 + V''(\phi)
\] (4.8)
The vertex operator is:

\[ V^{A}_{\text{scalar}}(\epsilon, p) = \int_{0}^{T} d\tau \epsilon_{\mu} \dot{x}^{\mu} e^{ipx(\tau)} \]  

where \( \epsilon \) is the polarization of the photon. For spinor electrodynamics the worldline representation of the effective action is more complicated and one needs to introduce worldline Grassmann variables. It is given by:

\[ \Gamma[A] = -\frac{1}{2} \int_{0}^{\infty} \frac{dT}{T} e^{-m^{2}T} \int Dx(\tau) D\psi(\tau) e^{-\int_{0}^{T} d\tau \left( \frac{1}{2} \dot{x}^{2} + \frac{1}{2} \psi_{\mu} \dot{\psi}^{\mu} + ie \dot{x}^{\mu} A_{\mu}(x(\tau)) - ie \psi_{\mu} F_{\mu\nu}(x(\tau)) \psi^{\nu} \right)} \]

\[ \text{(4.10)} \]

The vertex operator is:

\[ V^{A}_{\text{spinor}}(\epsilon, p) = \int_{0}^{T} d\tau (\epsilon_{\mu} \dot{x}^{\mu} + 2i \epsilon_{\mu} \psi^{\mu} k_{\nu} \psi^{\nu}) e^{ipx(\tau)} \]  

\[ \text{(4.11)} \]

4.3 Quantum computation with Vertex operators

One can use quantum computation with vertex operators. First construct a circuit from a EOH for a free particle for time \( \tau_{1} - \epsilon \) followed by a Unitary insertion of a Vertex operator for a time \( \tau - \epsilon \) to \( \tau + \epsilon \) and then finally we have a EOH from \( \tau + \epsilon \) to \( T \). Schematically we have \( K(0, \tau - \epsilon)V(\tau - \epsilon, \tau + \epsilon)K(\tau + \epsilon, T) \) as described in figure 5. One can also add more insertions of N vertex operators to model scattering of two scalar particles with a nonlinear \( \phi^{3} \) interaction \[29] \[30] \[31] \[32]. For a three scalar particle amplitude given by:
one can insert the unitary vertex operator
$$\hat{V}_{op}(\vec{p}_2) = e^{ip_2\hat{X}}$$
into a quantum circuit and representing in the initial momentum states in terms of qubits we find the results in figure 6. Note the peak in $p_2$ as a function of the initial and final momentum difference which represents a discrete approximation to the Dirac delta function. Four qubits were used in this computation so we have sixteen possible initial momentum states in the discretization. The results of the quantum computation using the Trotter-Suzuki approximation for the insertion of the unitary vertex operator, where the evolution parameter was taken to be $p_2$. The insertion of the vertex operator leads to a $g\phi^3$ interaction potential in the effective action for $\phi$ and a quadratic term in its non-linear Klein-Gordon equation of motion.

5 Conclusions

In this paper we applied the VQE and EOH quantum algorithms on an IBM quantum simulator to the problem of simulating particle propagation in an magnetic field. We compared the simulation in both Cartesian and polar coordinates. We also discussed the
quantum simulation of a particle in a chromo-magnetic field governed by the the non-linear Yang-Mills equation. Finally we gave example of the use of vertex operators on quantum computers to simulate the scattering in various non-linear theories and gave examples in the case of the nonlinear Klein-Gordon equation. The world line approach has several advantages over for quantum field simulation including the use of fewer qubits and being able to do simulations in 3+1 dimensions. It will interesting to explore worldline path integrals on near term quantum computers. The quantum simulation of particles in background gauge fields also has applications to Kaluza-Klein theory and quantum gravity which will also be interesting to examine through quantum computing. The literature on the worldline formalism is extensive and it will be of interest to adapt these sophisticated techniques to quantum computing and explore possible advantages of quantum computing for chiral fermions, background gravitational fields and non-Abelian gauge fields.

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