ON THE SMALL TIME ASYMPTOTICS OF STOCHASTIC LADYZHENSKAYA-SMAGORINSKY EQUATIONS WITH DAMPING PERTURBED BY MULTIPLICATIVE NOISE

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ABSTRACT. The Ladyzhenskaya-Smagorinsky equations model turbulence phenomena, and are given by

\[
\frac{\partial u}{\partial t} - \mu \text{div} \left( (1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right) + (u \cdot \nabla) u + \nabla p = f, \quad \nabla \cdot u = 0,
\]

for \( p \geq 2 \), in a bounded domain \( \mathcal{O} \subset \mathbb{R}^d \) \( (2 \leq d \leq 4) \). In this work, we consider the stochastic Ladyzhenskaya-Smagorinsky equations with the damping \( \alpha u + \beta |u|^{r-2} u \), for \( r \geq 2 \) \((\alpha, \beta \geq 0)\), subjected to multiplicative Gaussian noise. We show the local monotonicity \((p \geq \frac{d}{2} + 1, \ r \geq 2)\) as well as global monotonicity \((p \geq 2, \ r \geq 4)\) properties of the linear and nonlinear operators, which along with an application of stochastic version of Minty-Browder technique imply the existence of a unique pathwise strong solution. Then, we discuss the small time asymptotics by studying the effect of small, highly nonlinear, unbounded drifts (small time large deviation principle) for the stochastic Ladyzhenskaya-Smagorinsky equations with damping.

1. Introduction

In the works [31, 32, 33], etc, Olga Ladyzhenskaya proposed a new set of equations, which describe turbulence phenomena. Let \( \mathcal{O} \subset \mathbb{R}^d \) \((2 \leq d \leq 4)\) be a bounded domain with a smooth boundary \( \partial \mathcal{O} \). Let \( u(t, x) \in \mathbb{R}^d \) be the velocity field at time \( t \) and position \( x \), \( p(t, x) \in \mathbb{R} \) be the pressure field, \( f(t, x) \in \mathbb{R}^d \) be an external forcing. The model described by Ladyzhenskaya is given by (6)

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nabla \cdot \mathbb{T}(u, p) &= f, \quad \text{in } \mathcal{O} \times (0, T), \\
\nabla \cdot u &= 0,
\end{aligned}
\]

where \( \mathbb{T} \) is the stress tensor

\[
\mathbb{T} = -p I + \nu_T(u) D u, \quad D u = \frac{1}{2}(\nabla u + (\nabla u)^\top),
\]

\[
\nu_T(u) = \nu_0 + \nu_1 |D u|^{p-2},
\]

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and \( \nu_0, \nu_1 \) are strictly positive constants. The system (1.1) is supplemented by the following initial and boundary conditions:

\[
\begin{align*}
\mathbf{u} &= \mathbf{0}, \quad \text{on } \partial \Omega \times [0, T), \quad (1.4) \\
\mathbf{u}(0) &= \mathbf{u}_0 \quad \text{in } \Omega. \quad (1.5)
\end{align*}
\]

The system (1.1) satisfies the Stokes principle (cf. Appendix 1, [6]). Note that the equation (1.2) tells us that the stress tensor \( \mathbb{T} \) depends on the symmetric part \( \mathcal{D}\mathbf{u} \) of the gradient of the velocity has a polynomial growth of \( p \)-order, for \( p > 2 \). Taking \( p = 3 \), we obtain the classical Smagorinsky turbulence model introduced in [58] (see [6, 30, 53], etc). J.-L. Lions in [34] and [35] (Chapter 2, Section 5) considered the case in which \( \mathcal{D}\mathbf{u} \) is replaced by \( \nabla \mathbf{u} \), but in this case the Stokes principle is not satisfied. The existence of a global weak solution to the system (1.1)-(1.5) is known due to Ladyzhenskaya for \( p \geq 1 + \frac{2d}{d+2} \) and its uniqueness for \( p \geq 1 + \frac{d}{2} \). For further results on the global solvability results (the existence and uniqueness of weak as well as strong solutions) on the deterministic Ladyzhenskaya-Smagorinsky equations, the interested readers are referred to see [6, 31, 32, 33, 35, 42, 43, 44, 60], etc.

In this work, we consider a stochastic counterpart of the following form of Ladyzhenskaya-Smagorinsky equations (cf. [34, 51]) with damping:

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} - \mu \text{div} \left( (1 + |\nabla \mathbf{u}|^2)^{\frac{p-2}{2}} \nabla \mathbf{u} \right) + (\mathbf{u} \cdot \nabla) \mathbf{u} + \alpha \mathbf{u} + \beta |\mathbf{u}|^{r-2} \mathbf{u} \\
+ \nabla p &= \mathbf{f}, \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega \times (0, T), \\
\mathbf{u} &= \mathbf{0}, \quad \text{on } \partial \Omega \times [0, T), \\
\mathbf{u}(0) &= \mathbf{u}_0 \quad \text{in } \Omega, \quad (1.6)
\end{align*}
\]

where \( \alpha, \beta \geq 0 \) are damping parameters. The model under our consideration can be described as

\[
\begin{align*}
\mathbf{d}\mathbf{u}(t) + \left[ -\mu \text{div} \left( (1 + |\nabla \mathbf{u}(t)|^2)^{\frac{p-2}{2}} \nabla \mathbf{u}(t) \right) + (\mathbf{u}(t) \cdot \nabla) \mathbf{u}(t) + \alpha \mathbf{u}(t) + \beta |\mathbf{u}(t)|^{r-2} \mathbf{u}(t) \\
+ \nabla p(t) \right] dt &= \mathbf{f}(t) dt + \sum_{k=1}^{\infty} \sigma_k(t, \mathbf{u}(t)) dW_k(t), \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot \mathbf{u}(t) &= 0, \quad \text{in } \Omega \times (0, T), \\
\mathbf{u}(t) &= \mathbf{0} \quad \text{on } \partial \Omega \times [0, T), \\
\mathbf{u}(0) &= \mathbf{u}_0 \quad \text{in } \Omega, \quad (1.7)
\end{align*}
\]

where \( \{W_k(t)\}_{k \geq 1} \) is a sequence of independent \( \mathcal{F}_t \)-adapted Brownian motions defined on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). As \( \alpha \) does not play a major role in our analysis, we fix \( \alpha = 0 \) in the rest of the paper. By showing the local monotonicity \( (p \geq \frac{d}{2} + 1, \ r \geq 2) \) as well as global monotonicity \( (p \geq 2, \ r \geq 4, \ 2\beta\mu \geq 1 \text{ for } r = 4) \), hemicontinuity properties of the linear and nonlinear operators and a stochastic generalization of the Minty-Browder technique, our first goal in this work is to establish the existence of a unique pathwise strong solution. Our second goal is to discuss the small time asymptotics by studying the effect of small, highly nonlinear, unbounded drifts (small time large deviation principle) for the system (1.7). More specifically, our focus on the limiting behavior of the strong solution to the Ladyzhenskaya-Smagorinsky equations with damping in a time interval \([0, t]\) as \( t \) tends
to zero. An inspiration for considering such problems arises from Varadhan’s identity

$$\lim_{t \to 0} 2t \log P\{u(0) \in B, \ u(t) \in C\} = -d^2(B, C),$$

where $u$ is the strong solution to the Ladyzhenskaya-Smagorinsky equations with damping and $d$ is an appropriate Riemann distance associated with the diffusion generated by $u$.

The existence of martingale weak solution for an incompressible non-Newtonian fluid equations (stochastic power law fluids) driven by a Brownian motion is proved in [8]. The existence and uniqueness of weak solutions for stochastic power law fluids is obtained in [63]. The existence and uniqueness of strong solutions to coercive and locally monotone stochastic partial differential equations (SPDEs) like stochastic equations of non-Newtonian fluids driven by Lévy noise is proved in [9]. The local and global existence and uniqueness of solutions for general nonlinear evolution equations with coefficients satisfying some local monotonicity and generalized coercivity conditions is established in [39]. The existence of random dynamical systems and random attractors for a large class of locally monotone SPDEs perturbed by additive Lévy noise, which includes stochastic power law fluids, is obtained in [24]. The existence of global pathwise strong solutions for the two dimensional stochastic non-Newtonian incompressible fluid equations is showed in [28]. The existence and uniqueness of pathwise strong solutions to stochastic 2D Navier-Stokes equations governed by Gaussian and pure jump noise is obtained in [47, 48], respectively. The existence and uniqueness of strong solutions to stochastic 3D tamed Navier-Stokes equations governed by Lévy noise, which includes stochastic power law fluids, is obtained in [56, 17], respectively. The existence of global pathwise strong solutions of stochastic equations with a monotone operator, of the Ladyzhenskaya-Smagorinsky type, governed by a general Lévy noise is established in [51].

Large deviation theory gained its attention in the past few decades due to its wide range of applications in the areas like mathematical finance, risk management, fluid mechanics, statistical mechanics, quantum physics, etc (cf. [7, 11, 13, 16, 17, 20, 22, 46, 49, 50, 55, 60, 61, 65, 66], etc and the references therein). The small time large deviation principle (LDP) examines the asymptotic behavior of the tails of a family of probability measures at a given point in space when the time is very small. That is, the limiting behavior of the solution in a time interval $[0, t]$ as $t \to 0$. In this direction, the first and most celebrated work is due to Varadhan [65], where he considered the small-time asymptotics for finite-dimensional diffusion processes. For the small-time LDP for infinite dimensional diffusion processes, the interested readers are referred to see [11, 12, 13, 29, 67, 70], etc. The small time LDP for stochastic 2D Navier-Stokes equations, stochastic 3D tamed Navier-Stokes equations, stochastic quasi-geostrophic equations in the sub-critical case, stochastic two-dimensional non-Newtonian fluids, 3D stochastic primitive equations, SPDEs with locally monotone coefficients, stochastic convective Brinkman-Forchheimer equations, scalar stochastic conservation laws, are established in [68, 56, 40, 36, 18, 38, 50, 19], respectively. Even though the work [38] covers the case of SPDEs with locally monotone coefficients like stochastic power law fluid equations, it won’t cover the system under our consideration for arbitrary values of $r$ (for example $r > \frac{d}{2}$). Therefore, we need a separate analysis for the small noise asymptotics of the system (1.7).

The rest of the paper is organized as follows: In the next section, we provide the essential functional setting to obtain the global solvability results of the system (1.7). Then we discuss the local monotonicity ($p \geq \frac{d}{2} + 1$, $r \geq 2$), global monotonicity ($p \geq 2$, $r \geq 4$, $2\beta \mu \geq 1$,
for \( r = 4 \) and demicontinuity properties of the linear and nonlinear operators (Lemmas 2.2, 2.4, 2.7). Using these properties and under proper assumptions on the noise coefficient, we establish the existence of a pathwise strong solution to the system \((1.7)\) in section 3 (Theorem 3.4). Under an additional assumption on the noise coefficient, we discuss the small time asymptotics of the system \((1.7)\) for \( p \geq \frac{d}{2} + 1, \ r \geq 2 \) or \( p \geq 2, \ r \geq 4 \ (\beta \mu > 1 \ for \ r = 4)\), by studying the effect of small, highly nonlinear, unbounded drifts (small time LDP) in the final section (Theorem 4.2).

2. Mathematical Formulation

This section is devoted for the necessary function spaces needed to obtain the global solvability results of the system \((1.7)\). Furthermore, we discuss important properties of the nonlinear operators.

2.1. Function spaces. Let \( \mathcal{O} \subset \mathbb{R}^d \) (\( 2 \leq d \leq 4 \)) be a bounded domain with smooth boundary \( \partial \mathcal{O} \subset \mathbb{C}^m \), for some \( m \geq 3 \). Let \( C_0^\infty(\mathcal{O}; \mathbb{R}^d) \) denote the space of all infinitely differentiable functions \((\mathbb{R}^d\text{-valued})\) with compact support in \( \mathcal{O} \subset \mathbb{R}^d \). The Lebesgue spaces are denoted by \( L^r(\mathcal{O}) = L^r(\mathcal{O}; \mathbb{R}^d) \) and Sobolev spaces are represented by \( \mathbb{W}^{k,p}(\mathcal{O}) := W^{k,p}(\mathcal{O}; \mathbb{R}^d) \) and \( \mathbb{H}^k(\mathcal{O}) := \mathbb{W}^{k,2}(\mathcal{O}) \). We define

\[
\mathcal{V} := \{ \mathbf{u} \in C_0^\infty(\mathcal{O}, \mathbb{R}^d) : \nabla \cdot \mathbf{u} = 0 \},
\]

\[
\mathcal{H} := \text{the closure of } \mathcal{V} \text{ in the Lebesgue space } L^2(\mathcal{O}),
\]

\[
\mathcal{V}_2 := \text{the closure of } \mathcal{V} \text{ in the Sobolev space } \mathbb{H}_0^1(\mathcal{O}),
\]

\[
\mathcal{L}^r := \text{the closure of } \mathcal{V} \text{ in the Lebesgue space } L^r(\mathcal{O}),
\]

\[
\mathcal{V}_p := \text{the closure of } \mathcal{V} \text{ in the Sobolev space } \mathbb{W}^{1,p}(\mathcal{O}),
\]

for \( p, r \in (2, \infty) \). Then under the above smoothness assumptions on the boundary, we characterize the spaces \( \mathcal{H}, \mathcal{V}_2, \mathcal{L}^r \) and \( \mathcal{V}_p \) as \( \mathcal{H} = \{ \mathbf{u} \in L^2(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} \big|_{\partial \mathcal{O}} = 0 \} \), with norm \( \| \mathbf{u} \|_{\mathcal{H}} := \int_\mathcal{O} |\mathbf{u}(x)|^2 \, dx \), where \( \mathbf{n} \) is the outward normal to \( \partial \mathcal{O} \), and \( \mathbf{u} \cdot \mathbf{n} \big|_{\partial \mathcal{O}} = 0 \) should be understood in the sense of trace in \( \mathbb{H}^{-1/2}(\partial \mathcal{O}) \) (cf. Theorem 1.2, Chapter 1, [62]), \( \mathcal{V}_2 = \{ \mathbf{u} \in \mathbb{H}_0^1(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0 \} \), with norm \( \| \mathbf{u} \|_{\mathcal{V}_2}^2 := \int_\mathcal{O} |\nabla \mathbf{u}(x)|^2 \, dx \), \( \mathcal{L}^r = \{ \mathbf{u} \in L^r(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} \big|_{\partial \mathcal{O}} = 0 \} \), with norm \( \| \mathbf{u} \|_{\mathcal{L}^r}^r := \int_\mathcal{O} |\mathbf{u}(x)|^r \, dx \), and \( \mathcal{V}_p = \{ \mathbf{u} \in \mathbb{W}^{1,p}(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0 \} \), with norm \( \int_\mathcal{O} |\nabla \mathbf{u}(x)|^p \, dx \), respectively. Let \( \langle \cdot, \cdot \rangle \) denote the inner product in the Hilbert space \( \mathcal{H} \) and \( \langle \cdot, \cdot \rangle \) denote the induced duality between the spaces \( \mathcal{V}_2 \) and its dual \( \mathcal{V}_2' \), \( \mathcal{V}_p \) and its dual \( \mathcal{V}_p' \), and \( \mathcal{L}^r \) and its dual \( \mathcal{L}^r' \), where \( \frac{1}{r} + \frac{1}{r'} = 1 \). Note that \( \mathcal{H} \) can be identified with its own dual \( \mathcal{H}' \). We endow the space \( \mathcal{V}_p \cap \mathcal{L}^r \) with the norm \( \| \mathbf{u} \|_{\mathcal{V}_p} + \| \mathbf{u} \|_{\mathcal{L}^r} \), for \( \mathbf{u} \in \mathcal{V}_p \cap \mathcal{L}^r \) and its dual \( \mathcal{V}_p' + \mathcal{L}^r' \) with the norm

\[
\inf \left\{ \max \left( \| \mathbf{v}_1 \|_{\mathcal{V}_p'}, \| \mathbf{v}_2 \|_{\mathcal{L}^r'} \right) : \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2, \ \mathbf{v}_1 \in \mathcal{V}_p', \ \mathbf{v}_2 \in \mathcal{L}^r' \right\}.
\]

We first note that \( \mathcal{V} \subset \mathcal{V}_p \cap \mathcal{L}^r \subset \mathcal{H} \) and \( \mathcal{V} \) is dense in \( \mathcal{H}, \mathcal{V}_p \) and \( \mathcal{L}^r \), and hence \( \mathcal{V}_p \cap \mathcal{L}^r \) is dense in \( \mathcal{H} \). We have the following continuous embedding also:

\[
\mathcal{V}_p \cap \mathcal{L}^r \hookrightarrow \mathcal{H} \equiv \mathcal{H}' \hookrightarrow \mathcal{V}_p' + \mathcal{L}^r'.
\]
One can define equivalent norms on $V_p \cap \widetilde{L}^r$ and $V_p + \widetilde{L}^r$ as (see [3])

$$\|u\|_{V_p \cap \widetilde{L}^r} = \left(\|u\|_{V_p}^2 + \|u\|_{\widetilde{L}^r}^2\right)^{1/2}$$
and

$$\|u\|_{V_p + \widetilde{L}^r} = \inf_{u = v + w} \left(\|v\|_{V_p}^2 + \|w\|_{\widetilde{L}^r}^2\right)^{1/2}.$$ 

2.2. The operator $A$. Let $P_p : \mathbb{L}^p(\mathcal{O}) \to \mathbb{L}^p$ denote the Helmholtz-Hodge projection ([23]). For $p = 2$, $P := P_2$ becomes an orthogonal projection and for $2 < p < \infty$, it is a bounded linear operator. Moreover, $P$ maps $\mathbb{H}^{m-1}(\mathcal{O})$ into itself and is bounded if $\mathcal{O}$ is of class $C^m$ (Remark 1.6, [62]). Let us define the operator $A : V_p \to V_p$ by

$$A(u) = -P \text{div}\left((1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\right),$$

(2.1)
for all $u \in V_p$. For $v \in V_p$, it can be easily seen that

$$|\langle A(u), v \rangle| = |\langle (1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u, \nabla v \rangle| \leq 2^{\frac{p-2}{2}} \left(\|\nabla u\|_H \|\nabla v\|_H + \|\nabla u\|_{L^p}^{p-1} \|\nabla v\|_{L^p}\right)$$

$$\leq C \left(1 + \|u\|_{V_p}^{p-2}\right) \|u\|_{V_p} \|v\|_{V_p},$$

for all $v \in V_p$, so that $\|A(u)\|_{V_p} \leq C \left(1 + \|u\|_{V_p}^{p-2}\right) \|u\|_{V_p}$. Since

$$2\langle (1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u, \nabla u \rangle \geq \|\nabla u\|_H^2 + \|\nabla u\|_{L^p}^p,$$
we have

$$\langle A(u), u \rangle = \langle (1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u, \nabla u \rangle \geq \frac{1}{2} \left[\|\nabla u\|_H^2 + \|\nabla u\|_{L^p}^p\right].$$

(2.2)

Using Taylor’s formula and Hölder’s inequalities, we find

$$|\langle A(u) - A(v), w \rangle|$$

$$\leq \left|\langle (1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u - \nabla v, \nabla w \rangle\right|$$

$$+ \left|\langle (1 + |\nabla u|^2)^{\frac{p-2}{2}} - (1 + |\nabla v|^2)^{\frac{p-2}{2}}, \nabla v, \nabla w \rangle\right|$$

$$\leq \left\{2^{\frac{p-2}{2}} (1 + \|\nabla u\|_{L^p})
+ (p - 2) 2^{\frac{p-4}{2}} \left((1 + \|\nabla u\|_{L^p}^{p-4} + \|\nabla v\|_{L^p}^{p-4}) \|\nabla v\|_{L^p} \|\nabla v\|_{L^p}\right)\right\}$$

$$\times \|\nabla (u - v)\|_{L^p} \|\nabla w\|_{L^p},$$

(2.3)
for all $w \in V_p$. Thus the operator $A(\cdot) : V_p \to V_p'$ is a locally Lipschitz operator.

2.3. The bilinear operator $B(\cdot)$. Let us define the trilinear form $b(\cdot, \cdot, \cdot) : V_2 \times V_2 \times V_2 \to \mathbb{R}$ by

$$b(u, v, w) = \int_{\mathcal{O}} (u(x) \cdot \nabla) v(x) \cdot w(x) dx = \sum_{i,j=1}^n \int_{\mathcal{O}} u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x) dx.$$ 

If $u, v$ are such that the linear map $b(u, v, \cdot)$ is continuous on $V_2$, the corresponding element of $V'_2$ is denoted by $B(u, v)$. We also represent $B(u) = B(u, u) = P[(u \cdot \nabla) u]$. An integration
by parts gives
\[ \begin{cases} b(u, v, v) = 0, & \text{for all } u, v \in \mathbb{V}_2, \\ b(u, v, w) = -b(u, w, v), & \text{for all } u, v, w \in \mathbb{V}_2. \end{cases} \]

Making use of Hölder’s and Sobolev’s inequalities, we obtain
\[ |\langle B(u, u), v \rangle| = |b(u, v, u)| \leq \| v \|_{\mathbb{V}_2} \| u \|_{\mathbb{L}^2}^2 \leq C \| u \|_{\mathbb{V}_2}^2 \| v \|_{\mathbb{V}_2}, \tag{2.4} \]
for all \( u, v \in \mathbb{V}_2 \) and since \( 2 \leq d \leq 4 \), we get \( \| B(u) \|_{\mathbb{V}_2} \leq C \| u \|_{\mathbb{V}_2}^2 \). One can also show that
\[ \| B(u) - B(v) \|_{\mathbb{V}_2} \leq C(\| u \|_{\mathbb{V}_2} + \| v \|_{\mathbb{V}_2}) \| u - v \|_{\mathbb{V}_2}, \]
and hence the operator \( B(\cdot) : \mathbb{V}_2 \to \mathbb{V}_2 \) is locally Lipschitz. Using Hölder’s and Sobolev’s inequalities, we obtain
\[ |\langle B(u) - B(v), w \rangle| \leq |\langle B(u - v, u), w \rangle| + |\langle B(v, u - v), w \rangle| \leq C \| u - v \|_{\mathbb{V}_p} (\| u \|_{\mathbb{V}_p} + \| v \|_{\mathbb{V}_p}) \| w \|_{\mathbb{V}_p}, \]
for all \( u, v, w \in \mathbb{V}_p \) and \( p \geq \frac{4d}{d-2} \). Thus the operator \( B(\cdot) : \mathbb{V}_p \to \mathbb{V}_p \) is a locally Lipschitz operator. Using Hölder’s and interpolation inequalities, we get
\[ |\langle B(u, v), v \rangle| = |b(u, v, u)| \leq \| u \|_{\mathbb{L}^r} \| u \|_{\mathbb{L}^\frac{2r}{r-2}} \| v \|_{\mathbb{V}_2} \leq \| u \|_{\mathbb{L}^\frac{2r}{r-2}} \| u \|_{\mathbb{H}^\frac{r-2}{2}} \| v \|_{\mathbb{V}_2}, \tag{2.5} \]
for all \( v \in \mathbb{V}_2 \cap \mathbb{L}^r \) and \( r > 4 \). Thus, for \( r > 4 \), we have
\[ \| B(u) \|_{\mathbb{V}_2 + \mathbb{L}^\frac{2r}{r-2}} \leq \| u \|_{\mathbb{L}^\frac{2r}{r-2}} \| u \|_{\mathbb{H}^\frac{r-2}{2}}. \tag{2.6} \]

For \( u, v \in \mathbb{V}_2 \cap \mathbb{L}^r \), we also find
\[ \| B(u) - B(v) \|_{\mathbb{V}_2 + \mathbb{L}^\frac{2r}{r-2}} \leq \left( \| u \|_{\mathbb{L}^\frac{2r}{r-2}} \| u \|_{\mathbb{H}^\frac{r-2}{2}} + \| v \|_{\mathbb{L}^\frac{2r}{r-2}} \| v \|_{\mathbb{H}^\frac{r-2}{2}} \right) \| u - v \|_{\mathbb{L}^r}, \tag{2.7} \]
for \( r > 4 \), so that the operator \( B(\cdot) : \mathbb{V}_2 \cap \mathbb{L}^r \to \mathbb{V}_2 + \mathbb{L}^\frac{2r}{r-2} \) is a locally Lipschitz operator. For \( r = 4 \), a calculation similar to (2.7) yields
\[ \| B(u) - B(v) \|_{\mathbb{V}_2 + \mathbb{L}^4} \leq (\| u \|_{\mathbb{L}^4} + \| v \|_{\mathbb{L}^4}) \| u - v \|_{\mathbb{L}^4}, \]
hence \( B(\cdot) : \mathbb{V}_2 \cap \mathbb{L}^4 \to \mathbb{V}_2 + \mathbb{L}^4 \) is a locally Lipschitz operator.

2.4. The nonlinear operator \( C(\cdot) \). Let us define the operator \( \mathcal{C} : \mathbb{V}_2 \cap \mathbb{L}^r \to \mathbb{V}_2 + \mathbb{L}^\frac{2r}{r-2} \) by \( \mathcal{C}(u) := \mathcal{P}(|u|^{-2}u) \). It is immediate that \( \langle \mathcal{C}(u), u \rangle = \| u \|_{\mathbb{L}^r}^2 \), for all \( u \in \mathbb{V}_2 \cap \mathbb{L}^r \). It has been shown in [47] that
\[ \langle \mathcal{C}(u) - \mathcal{C}(v), w \rangle \leq (r - 1)(\| u \|_{\mathbb{L}^r} + \| v \|_{\mathbb{L}^r})^{r-2} \| u - v \|_{\mathbb{L}^r} \| w \|_{\mathbb{L}^r}, \tag{2.8} \]
for all \( u, v, w \in \mathbb{V}_2 \cap \mathbb{L}^r \). Thus the operator \( \mathcal{C}(\cdot) : \mathbb{L}^r \to \mathbb{L}^\frac{2r}{r-2} \) is locally Lipschitz. For any \( r \in [1, \infty) \), we have (see [47])
\[ \langle \mathcal{C}(u) - \mathcal{C}(v), u - v \rangle \geq \frac{1}{2} \| |u|^{-\frac{2}{r}}(u - v) \|_{\mathbb{H}^\frac{r}{2}}^2 + \frac{1}{2} \| |v|^{-\frac{2}{r}}(u - v) \|_{\mathbb{H}^\frac{r}{2}}^2 \geq 0, \tag{2.9} \]
for \( r \geq 1 \). Furthermore
\[ \| u - v \|_{\mathbb{L}^r} \leq 2^{r-3} \left[ \| |u|^{-\frac{2}{r}}(u - v) \|_{\mathbb{H}^\frac{r}{2}}^2 + \| |v|^{-\frac{2}{r}}(u - v) \|_{\mathbb{H}^\frac{r}{2}}^2 \right], \tag{2.10} \]
for \( r \geq 1 \) (replace \( 2^{r-3} \) with 1, for \( 2 \leq r \leq 3 \)), so that
\[
\langle C(u) - C(v), u - v \rangle \geq \frac{1}{2^{r-2}} \| u - v \|_{L^r}^r,
\]
for all \( u, v \in \dot{L}^r \).

2.5. **Monotonicity.** Let us now discuss the monotonicity as well as the hemicontinuity properties of the linear and nonlinear operators, which plays a crucial role in the global solvability of the system \([16]\). Similar kind of analysis has been carried in \([47, 60]\), etc also.

**Definition 2.1** \([11]\). Let \( X \) be a Banach space and let \( X' \) be its topological dual. An operator \( G : D \to X' \), \( D = D(G) \subset X \) is said to be monotone if
\[
\langle G(x) - G(y), x - y \rangle \geq 0, \quad \text{for all } x, y \in D.
\]
The operator \( G(\cdot) \) is said to be hemicontinuous, if for all \( x, y \in X \) and \( w \in X' \),
\[
\lim_{\lambda \to 0} \langle G(x + \lambda y), w \rangle = \langle G(x), w \rangle.
\]
The operator \( G(\cdot) \) is called demicontinuous, if for all \( x \in D \) and \( y \in X \), the functional \( x \mapsto \langle G(x), y \rangle \) is continuous, or in other words, \( x_k \to x \) in \( X \) implies \( G(x_k) \wto G(x) \) in \( X' \). Clearly demicontinuity implies hemicontinuity.

**Lemma 2.2.** Let \( u, v \in \dot{V}_p \cap \dot{L}^4 \), for \( p \geq 2 \) and \( r > 4 \). Then, for the operator
\[
G(u) = -\mu A(u) + B(u) + \beta C(u),
\]
we have
\[
\langle (G(u) - G(v), u - v) + \eta \| u - v \|_{H^r}^2 \rangle \geq 0,
\]
where
\[
\eta = \frac{p - 4}{2\mu(p - 2)} \left( \frac{2}{\beta \mu(p - 2)} \right)^{\frac{1}{p+2}}.
\]
That is, the operator \( G + \eta I \) is a monotone operator from \( \dot{V}_p \) to \( \dot{V}_p \).

For \( r = 4 \) with \( 2\beta \mu \geq 1 \), the operator \( G(\cdot) : \dot{V}_p \cap \dot{L}^4 \to \dot{V}_p' + \dot{L}^4 \) is globally monotone, that is, for all \( u, v \in \dot{V}_p \cap \dot{L}^4 \), we have
\[
\langle G(u) - G(v), u - v \rangle \geq 0.
\]

**Proof.** Let us estimate \(-\mu \langle A(u) - A(v), u - v \rangle\) using an integration by parts as
\[
-\mu \langle A(u) - A(v), u - v \rangle
= -\mu \left( \text{div} \left( (1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right) - \text{div} \left( (1 + |\nabla v|^2)^{\frac{p-2}{2}} \nabla v \right), u - v \right)
= \mu \left( (1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u - (1 + |\nabla v|^2)^{\frac{p-2}{2}} \nabla v, \nabla (u - v) \right)
= \mu \left( \int_0^1 \frac{d}{d\theta} \left[ (1 + |\nabla (\theta u + (1 - \theta) v)|^2)^{\frac{p-2}{2}} \nabla (\theta u + (1 - \theta) v) \right] d\theta, \nabla (u - v) \right)
= \mu \left( \int_0^1 (1 + |\nabla (\theta u + (1 - \theta) v)|^2)^{\frac{p-2}{2}} d\theta \nabla (u - v), \nabla (u - v) \right)
\]
\[ + \mu(p-2) \left( \int_0^1 (1 + |\nabla(\theta u + (1 - \theta v))|^2)^{\frac{p-2}{2}} |\nabla(\theta u + (1 - \theta v))|^2 d\theta \times \nabla(u - v), \nabla(u - v) \right) \]
\[ \geq \mu \left( \int_0^1 (1 + |\nabla(\theta u + (1 - \theta v))|^2)^{\frac{p-2}{2}} d\theta |\nabla(u - v), \nabla(u - v) \right) \]
\[ \geq \mu \|\nabla(u - v)\|^2_{H^1}. \quad (2.15) \]

From (2.9), we easily have
\[ \beta \langle C(u) - C(v), u - v \rangle \geq \frac{\beta}{2} \|v\|^{\frac{p-2}{2}} \|u - v\|^2_{H^1}. \quad (2.16) \]

Since \( \langle B(u, u - v), u - v \rangle = 0 \), we get \( \langle B(u) - B(v), u - v \rangle = -\langle B(u - v, u - v), v \rangle \), and a calculation similar to the proof of Theorem 2.2, \[47\] yields
\[ |\langle B(u - v, u - v), v \rangle| \]
\[ \leq \frac{\mu}{2} \|u - v\|^2_v + \frac{\beta}{2} \|v\|^{\frac{r-2}{2}} \|u - v\|^2_{H^1} + \frac{r-4}{2\mu(r-2)} \left( \frac{2}{\beta \mu(r-2)} \right)^{\frac{2}{r-4}} \|u - v\|^2_{H^1}. \quad (2.17) \]

Combining (2.15), (2.16) and (2.17), we get
\[ \langle (G(u) - G(v), u - v) \rangle + \frac{r-4}{2\mu(r-2)} \left( \frac{2}{\beta \mu(r-2)} \right)^{\frac{2}{r-4}} \|u - v\|^2_{H^1} \geq \mu \|u - v\|^2_{V} \geq 0, \quad (2.18) \]

for \( r > 4 \) and the estimate (2.12) follows.

From (2.9), we infer that
\[ \beta \langle C(u) - C(v), u - v \rangle \geq \frac{\beta}{2} \|v(u - v)\|^2_{H^1}. \quad (2.19) \]

We estimate \( |\langle B(u - v, u - v), v \rangle| \) using Hölder’s and Young’s inequalities as
\[ |\langle B(u - v, u - v), v \rangle| \leq \|v(u - v)\|_{H^1} \|u - v\|_v \leq \mu \|u - v\|^2_v + \frac{1}{4\mu} \|v(u - v)\|^2_{H^1}. \quad (2.20) \]

Combining (2.15), (2.19) and (2.20), we obtain
\[ \langle G(u) - G(v), u - v \rangle \geq \frac{1}{2} \left( \beta - \frac{1}{2\mu} \right) \|v(u - v)\|^2_{H^1} \geq 0, \quad (2.21) \]

provided \( 2\beta \mu \geq 1 \).

**Remark 2.3.** One can estimate the term \( -\mu \langle A(u) - A(v), u - v \rangle \) in the following way also:
\[ -\mu \langle A(u) - A(v), u - v \rangle \]
\[ = -\mu \left\langle \text{div} \left( (1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right) - \text{div} \left( (1 + |\nabla v|^2)^{\frac{p-2}{2}} \nabla v \right), u - v \right\rangle \]
\[ = \mu \left( (1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u - (1 + |\nabla v|^2)^{\frac{p-2}{2}} \nabla v, \nabla(u - v) \right) \]
\[ = \mu \left( (1 + |\nabla u|^2)^{\frac{p-2}{2}}, |\nabla(u - v)|^2 \right) + \mu \left( (1 + |\nabla v|^2)^{\frac{p-2}{2}}, |\nabla(u - v)|^2 \right) \]
Lemma 2.4. Let $p > \frac{d}{2}$, $r \geq 2$ and $u, v \in \mathbb{V}_p \cap \mathbb{L}^r$. Then, for the operator $G(u) = -\mu A(u) + B(u) + \beta C(u)$, we have

$$
\langle (G(u) - G(v), u - v \rangle + \eta N \frac{2r}{p-r} \|u - v\|_H^2 \geq 0.
$$
for all \( v \in \mathbb{B}_N := \{ z \in \mathbb{V}_p : \| z \|_{\mathbb{V}_p} \leq N \} \), where

\[
\tilde{\eta} = C \frac{2p}{2p-d} \left( \frac{2p-d}{2p} \right) \left( \frac{d}{\mu p} \right)^{\frac{d}{2p-d}},
\]  

(2.27)

and \( C \) is the constant appearing in Gagliardo-Nirenberg’s inequality.

**Proof.** We estimate \( |\langle B(u - v, u - v)\rangle| \) using Hölder’s, Gagliardo-Nirenberg’s and Young’s inequalities as

\[
|\langle B(u - v, u - v)\rangle| \leq \| \nabla v \|_{L^p} \| u - v \|_{L^p}^{2} \leq C \| v \|_{\mathbb{V}_p} \| u - v \|_{\mathbb{V}_p} \| u - v \|_{H} \leq \frac{\mu}{2} \| u - v \|_{\mathbb{V}_p}^{2} + C\frac{2p}{2p-d} \left( \frac{2p-d}{2p} \right) \left( \frac{d}{\mu p} \right)^{\frac{d}{2p-d}} \| v \|_{\mathbb{V}_p} \| u - v \|_{H}^{2}.
\]

(2.28)

Combining (2.15), (2.9) and (2.28), we deduce that

\[
\langle (G(u) - G(v), u - v) \rangle + C\frac{2p}{2p-d} \left( \frac{2p-d}{2p} \right) \left( \frac{d}{\mu p} \right)^{\frac{d}{2p-d}} \| v \|_{\mathbb{V}_p} \| u - v \|_{H} \geq \frac{\mu}{2} \| u - v \|_{\mathbb{V}_p}^{2} \geq 0.
\]

Let \( \mathbb{B}_N \) be an \( \mathbb{V}_p \)-ball of radius \( N \), that is, \( \mathbb{B}_N := \{ z \in \mathbb{V}_p : \| z \|_{\mathbb{V}_p} \leq N \} \). Thus for all \( v \in \mathbb{B}_N \), we have

\[
\langle (G(u) - G(v), u - v) \rangle + C\frac{2p}{2p-d} \left( \frac{2p-d}{2p} \right) \left( \frac{d}{\mu p} \right)^{\frac{d}{2p-d}} N \frac{2p}{2p-d} \| u - v \|_{H} \geq 0,
\]

(2.29)

and hence the operator \( G(\cdot) \) is locally monotone. \( \square \)

**Remark 2.5.** 1. It should be noted that if \( v \in L^p(0,T;\mathbb{V}_p) \), then

\[
\int_{0}^{T} \| v(t) \|_{\mathbb{V}_p}^{\frac{2p}{2p-d}} \, dt \leq T^{\frac{2p-(d+2)}{2p-d}} \left( \int_{0}^{T} \| v(t) \|_{\mathbb{V}_p}^{p} \, dt \right)^{\frac{2p-d}{2p-(d+2)}} < +\infty,
\]

(3.30)

for \( p \geq \frac{d}{2} + 1 \).

2. Using Gagliardo-Nirenberg’s and Hölder’s inequalities, we find

\[
\int_{0}^{T} \| \nabla v(t) \|_{L^p}^{2} \, dt \leq C \int_{0}^{T} \| v(t) \|_{\mathbb{V}_p}^{p} \left( \int_{0}^{T} \| v(t) \|_{H}^{2} \, dt \right)^{\frac{2(p+d-2d)}{2p+d-2d}} \, dt
\]

\[
\leq CT^{\frac{2p+d-2d}{2p+d-2d}} \sup_{t \in [0,T]} \| v(t) \|_{H}^{2} \left( \int_{0}^{T} \| v(t) \|_{\mathbb{V}_p}^{p} \, dt \right)^{\frac{d(r-2)}{2p+d-2d}} < +\infty,
\]

for \( 2 \leq r \leq p(\frac{d}{2} + 1) \), so that we have the following embedding

\[
L^\infty(0,T;\mathbb{H}) \cap L^p(0,T;\mathbb{V}_p) \subset L^r(0,T;\mathbb{L}^r).
\]

**Remark 2.6.** 1. One can estimate \( |\langle B(u - v, u - v)\rangle| \) in the following way also:

\[
|\langle B(u - v, u - v)\rangle| \leq \| u - v \|_{\mathbb{V}_p} \| u - v \|_{H} \| v \|_{L^\infty}
\]

\[
\leq \frac{\mu}{2} \| u - v \|_{\mathbb{V}_p}^{2} + \frac{1}{2\mu} \| v \|_{L^\infty}^{2} \| u - v \|_{H}^{2}.
\]
Lemma 2.7. The operator $G: \mathbb{V}_p \cap \widetilde{L}^r \to \mathbb{V}_p + \widetilde{L}^r$ is demicontinuous.

Proof. Let us take a sequence $u^n \to u$ in $\mathbb{V}_p \cap \widetilde{L}^r$, that is, $\|u^n - u\|_{\mathbb{V}_p} + \|u^n - u\|_{\widetilde{L}^r} \to 0$ as $n \to \infty$. For any $v \in \mathbb{V}_p \cap \widetilde{L}^r$, we consider

$$
\langle G(u^n) - G(u), v \rangle = \mu \langle A(u^n) - A(u), v \rangle + \langle B(u^n) - B(u), v \rangle - \beta \langle C(u^n) - C(u), v \rangle.
$$

Next, we consider $\langle A(u^n) - A(u), v \rangle$ from (2.32) and estimate it using (2.33) as

$$
\langle A(u^n) - A(u), v \rangle \\
= |\langle (1 + |\nabla u^n|^2)^{\frac{p-2}{2}} \nabla u^n - (1 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u, \nabla v \rangle| \\
\leq \left\{ 2^{\frac{p-2}{2}} (1 + \|\nabla u^n\|_{\mathbb{V}_p}) \\
+ (p - 2)2^{\frac{p-4}{2}} \left[ (1 + \|\nabla u^n\|_{\mathbb{V}_p}^4 + \|\nabla u\|_{\mathbb{V}_p}^4) \left( \|\nabla u^n\|_{\mathbb{V}_p}^2 + \|\nabla u\|_{\mathbb{V}_p}^2 \right) \right] \\
\times \|\nabla(u_n - u)\|_{\mathbb{V}_p} \|\nabla v\|_{\mathbb{V}_p}
\right\} \\
c \to 0, \quad \text{as} \quad n \to \infty,
$$

since $u^n \to u$ in $\mathbb{V}_p$. We estimate the term $\langle B(u^n) - B(u), v \rangle$ from (2.32) using Hölder's and Sobolev's inequalities as

$$
\langle B(u^n) - B(u), v \rangle \leq \|B(u^n) - u\|_{\mathbb{V}_2} + \|B(u^n) - u\|_{\mathbb{V}_2} \|v\|_{\mathbb{V}_2} \\
\to 0, \quad \text{as} \quad n \to \infty,
$$

since $u^n \to u$ in $\mathbb{V}_2$ and $u^n, u \in \mathbb{V}_2$. Finally, we estimate the term $\langle C(u^n) - C(u), v \rangle$ from (2.32) using Taylor's formula and Hölder's inequality as

$$
\langle C(u^n) - C(u), v \rangle \leq (r - 1)\|u^n - u\|_{\mathbb{V}_2} \left( \|u^n\|_{\mathbb{V}_r}^2 + \|u\|_{\mathbb{V}_r}^2 \right)^{r-2} \|v\|_{\mathbb{V}_r} \\
\to 0, \quad \text{as} \quad n \to \infty,
$$

for $p > d$.

2. Using Hölder’s, Gagliardo-Nirenberg’s and Young’s inequalities, one can estimate $|\langle B(u - v, u - v), v \rangle|$ as

$$
|\langle B(u - v, u - v), v \rangle| \leq \|u - v\|_{\mathbb{V}_2} \|v\|_{\mathbb{V}_2} \|u - v\|_{\mathbb{V}_2} \\
\leq \frac{\mu}{2} \|u - v\|^2_{\mathbb{V}_2} + \frac{1}{2\mu} \|v\|^2_{\mathbb{V}_2} \|u - v\|^2_{\mathbb{V}_2},
$$

provided $\frac{4}{2} < p < d$. From (2.30), we infer that $\int_0^T \|v(t)\|_{\mathbb{V}_p}^{\frac{2p}{p-d}} \mathrm{d}t < \infty$, for $1 + \frac{4}{2} \leq p < d$. For more estimates on the bilinear operator, the interested readers are referred to see [43].
since \( \mathbf{u}^n \to \mathbf{u} \) in \( \tilde{L}^r \) and \( \mathbf{u}^n, \mathbf{u} \in \mathcal{V}_2 \cap \tilde{L}^r \). From the above convergences, it is immediate that 
\( \langle G(\mathbf{u}^n) - G(\mathbf{u}), \mathbf{v} \rangle \to 0 \), for all \( \mathbf{v} \in \mathcal{V}_p \cap \tilde{L}^r \). Hence the operator \( G: \mathcal{V}_p \cap \tilde{L}^r \to \mathcal{V}_p' + \tilde{L}^{r-r} \) is demicontinuous, which implies that the operator \( G(\cdot) \) is hemicontinuous. \( \square \)

3. **Stochastic Ladyzhenskaya-Smagorinsky Equations with Damping**

In this section, we consider the stochastic Ladyzhenskaya-Smagorinsky equations with damping perturbed by multiplicative Gaussian noise and discuss global solvability results. Let us first state the assumptions on the noise coefficient, so that we obtain the existence and uniqueness of strong solution to the system \((1.7)\).

**Hypothesis 3.1.** The noise coefficient \( \sigma_k(\cdot, \cdot), k \geq 1 \) satisfies:

(H.1) (Growth condition). There exists a positive constant \( K \) such that for all \( t \in [0, T] \) and \( \mathbf{u} \in \mathbb{H} \),

\[
\sum_{k=1}^{\infty} \| \sigma_k(t, \mathbf{u}) \|_{\mathbb{H}}^2 \leq K \left( 1 + \| \mathbf{u} \|_{\mathbb{H}}^2 \right),
\]

(H.2) (Lipschitz condition). There exists a positive constant \( L \) such that for any \( t \in [0, T] \) and all \( \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{H} \),

\[
\sum_{k=1}^{\infty} \| \sigma_k(t, \mathbf{u}_1) - \sigma_k(t, \mathbf{u}_2) \|_{\mathbb{H}}^2 \leq L \| \mathbf{u}_1 - \mathbf{u}_2 \|_{\mathbb{H}}^2.
\]

Note that the condition (H.1) implies that for every \( \mathbf{u} \in \mathbb{H} \), the linear map \( \sigma(\cdot, \mathbf{u}) := \{ \sigma_k(\cdot, \mathbf{u}) \}_{k \in \mathbb{N}} : \ell_2 \to \mathbb{H} \) defined by

\[
\sigma(\cdot, \mathbf{u}) h := \sum_{k=1}^{\infty} \sigma_k(\cdot, \mathbf{u}) h_k, \quad h = \{ h_k \}_{k \in \mathbb{N}}
\]

is in \( L_2(\ell_2; \mathbb{H}) \), that is, \( \sigma(\cdot, \mathbf{u}) \) is a Hilbert-Schmidt operator from \( \ell_2 \) to \( \mathbb{H} \). For an orthonormal basis \( \{ e_k \}_{k \in \mathbb{N}} \in \ell_2 \) with \( e_k = (0, \ldots, 1, \ldots) \), we have

\[
\| \sigma(t, \mathbf{u}) \|_{L_2(\ell_2; \mathbb{H})}^2 = \sum_{k=1}^{\infty} \| \sigma_k(t, \mathbf{u}) e_k \|_{\mathbb{H}}^2 = \sum_{k=1}^{\infty} \| \sigma_k(t, \mathbf{u}) \|_{\mathbb{H}}^2 \leq K \left( 1 + \| \mathbf{u} \|_{\mathbb{H}}^2 \right) < \infty,
\]

for all \( t \in [0, T] \).

Taking Helmholtz-Hodge projection onto the system \((1.7)\), we obtain

\[
\begin{array}{l}
d\mathbf{u}(t) + [\mu \mathbf{A}(\mathbf{u}(t)) + B(\mathbf{u}(t)) + \beta \mathbf{C}(\mathbf{u}(t))]dt = \mathbf{f}(t)dt + \sum_{k=1}^{\infty} \sigma_k(t, \mathbf{u}(t))dW_k(t), \\
\mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{H},
\end{array}
\]

for \( t \in (0, T) \). For simplicity, we used the notation \( \mathbf{f} \) for \( \mathcal{P} \mathbf{f} \) and \( \sigma_k \) for \( \mathcal{P} \sigma_k \).

**Definition 3.2** (Global strong solution). Let \( \mathbf{u}_0 \in \mathbb{H} \) and \( \mathbf{f} \in L^{\frac{r}{r-1}}(0, T; \mathcal{V}_p') \) be given. An \( \mathbb{H} \)-valued \( \mathcal{F}_t \)-adapted stochastic process \( \mathbf{u}(\cdot) \) is called a strong solution to the system \((3.2)\) if the following conditions are satisfied:
(i) The process \( u \in L^{2p}(\Omega; L^\infty(0, T; H)) \cap L^p(\Omega; L^2(0, T; V_p)) \cap L^r(\Omega; L^r(0, T; \tilde{L}^r)) \) and \( u(\cdot) \) has a \( V_p \cap \tilde{L}^r \)-valued modification, which is progressively measurable with continuous paths in \( H \) and \( u \in C([0, T]; H) \cap L^2(0, T; V_p) \cap L^r(0, T; \tilde{L}^r), \ P\text{-a.s.} \).

(ii) The following equality holds for every \( t \in [0, T] \), as an element of \( V_p \cap \tilde{L}^r \), \( P\text{-a.s.} \)

\[
\mathbf{u}(t) = \mathbf{x} - \int_0^t [\mu \mathbf{A}(\mathbf{u}(s)) + \mathbf{B}(\mathbf{u}(s)) + \beta \mathbf{C}(\mathbf{u}(s)) - \mathbf{f}(s)] ds + \sum_{k=1}^\infty \int_0^t \sigma_k(s, \mathbf{u}(s)) dW_k(s),
\]

(iii) The following Itô formula holds true:

\[
\begin{align*}
\| \mathbf{u}(t) \|^2_H + 2\mu \int_0^t \| (1 + |\nabla \mathbf{u}(s)|^2)^{\frac{d-2}{2}} \nabla \mathbf{u}(s) \|^2_B ds & + 2\beta \mu \int_0^t \| \mathbf{u}(s) \|^2_B ds \\
= \| \mathbf{x} \|^2_H + 2\int_0^t \langle \mathbf{f}(s), \mathbf{u}(s) \rangle ds + \sum_{k=1}^\infty \int_0^t \| \sigma_k(s, \mathbf{u}(s)) \|^2_B ds \\
& + 2\sum_{k=1}^\infty \int_0^t (\sigma_k(s, \mathbf{u}(s)), \mathbf{u}(s)) dW_k(s),
\end{align*}
\]

for all \( t \in [0, T] \), \( P\text{-a.s.} \).

An alternative version of the condition (3.3) is to require that for any \( \mathbf{v} \in V_p \cap \tilde{L}^r \):

\[
(\mathbf{u}(t), \mathbf{v}) = (\mathbf{x}, \mathbf{v}) - \int_0^t \langle \mu \mathbf{A}(\mathbf{u}(s)) + \mathbf{B}(\mathbf{u}(s)) + \beta \mathbf{C}(\mathbf{u}(s)) - \mathbf{f}(s), \mathbf{v} \rangle ds \\
+ \sum_{k=1}^\infty \int_0^t (\sigma_k(s, \mathbf{u}(s)) dW_k(s), \mathbf{v}), \ P\text{-a.s.,}
\]

for all \( t \in [0, T] \).

**Definition 3.3.** A strong solution \( \mathbf{u}(\cdot) \) to (3.2) is called a pathwise unique strong solution if \( \tilde{\mathbf{u}}(\cdot) \) is another strong solution, then

\[ P\{ \omega \in \Omega : \mathbf{u}(t) = \tilde{\mathbf{u}}(t), \ for \ all \ t \in [0, T] \} = 1. \]

**Theorem 3.4.** Let \( \mathbf{u}_0 \in H \) and \( \mathbf{f} \in \tilde{L}^r(0, T; V_p^\prime) \) be given. Then for \( p \geq \frac{d}{2} + 1, r \geq 2 \) or \( p \geq 2, r \geq 4 \) (\( 2\beta \mu \geq 1 \) for \( r = 4 \)), under Hypothesis \( 3.7 \), there exists a pathwise unique strong solution \( \mathbf{u}(\cdot) \) to the system (3.2) such that

\[ \mathbf{u} \in L^{2p}(\Omega; L^\infty(0, T; H)) \cap L^p(\Omega; L^2(0, T; V)) \cap L^r(\Omega; L^r(0, T; \tilde{L}^r)), \]

with \( P\text{-a.s.} \), continuous trajectories in \( H \).

**Proof.** For \( p \geq \frac{d}{2} + 1 \) and \( r \geq 2 \), since the operator \( G(\mathbf{u}) = -\mu \mathbf{A}(\mathbf{u}) + \mathbf{B}(\mathbf{u}) + \beta \mathbf{C}(\mathbf{u}) \) satisfies the local monotonicity condition (2.26) as well as demi-continuity condition (Lemma 2.7), one can establish the existence of a strong solution by a localized version (stochastic generalization) of the Minty-Browder technique (see \( [46], [59] \), etc for similar techniques). In order to establish the energy equality (Itô’s formula) (3.4) for \( 2 \leq r \leq \frac{pd}{d-p} \) \( (2 \leq r < \infty \) for \( d = 2 \)), one can use the recent result in Theorem 2.1, \( [26] \). But for \( r \geq \frac{pd}{d-p} \), we need a different technique to establish Itô’s formula. The authors in \( [21] \) were able to construct
functions that can approximate functions defined on smooth bounded domains by elements of eigenspaces of the Stokes operator in such a way that the approximations are bounded and converge in both Sobolev and Lebesgue spaces simultaneously. Note that by taking smooth boundary, say $\partial \Omega \in C^m$, then the eigenfunctions $\{w_j\}_{j=1}^{\infty}$ of the Stokes operator belongs to $\mathbb{V}_m$, for $m \geq 2$. In our case, one can choose $m \geq 3$, so that $\{w_j\}_{j=1}^{\infty} \in \mathbb{V}_p \cap \tilde{\mathbb{L}}^r$, for $p \geq 2$ and $r \geq 2$. For the case $r > \frac{pd}{d-2}$, combining this idea along with the techniques used in Theorem 3.7, [47], one can establish Itô’s formula. For the case $p \geq 2$ and $r \geq 4$ ($2 \beta \mu \geq 1$ for $r = 4$), as the operator $G(\cdot)$ is globally monotone and demicontinuous, one can use the stochastic generalization of the Minty-Browder technique to obtain global solvability results (see [47] for the case of Navier-Stokes equations with damping). The task of establishing the energy equality (Itô’s formula) (3.4) can completed using similar techniques as in Theorem 3.7, [47]. □

4. Small time asymptotics

In this section, we discuss the small time asymptotics of the Ladyzhenskaya-Smagorinsky equations with damping by studying the effect of small, highly nonlinear, unbounded drifts (small time LDP). Let us recall some basics of LDP.

Let $\mathcal{E}, \rho)$ be a complete separable metric space. We are given a family of probability measures $\{\mu_\epsilon\}_{\epsilon > 0}$ on $\mathcal{E}$ and a lower semicontinuous function $R : \mathcal{E} \to [0, \infty]$, not identically equal to $+\infty$ and such that its level sets, $K_M := \{x \in \mathcal{E} : R(x) \leq M\}$, $M > 0$, are compact for arbitrary $M \in [0, +\infty)$. The family $\{\mu_\epsilon\}_{\epsilon > 0}$ is said to satisfy the large deviation principle or to have the large deviation property with respect to the rate function $R$ if

(i) for all closed sets $F \subset \mathcal{E}$, we have
$$\limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(F) \leq - \inf_{x \in F} R(x),$$

(ii) for all open sets $G \subset \mathcal{E}$, we have
$$\liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(G) \leq - \sup_{x \in G} R(x).$$

It is well-known that under Hypothesis 3.1, the system (3.2) has a unique strong solution with paths in $C([0, T]; \mathbb{H}) \cap L^p(0, T; \mathbb{V}_p) \cap L^r(0, T; \tilde{\mathbb{L}}^r)$, $\mathbb{P}$-a.s., that is, $u(\cdot)$ satisfies

$$u(t) = u_0 - \int_0^t \left[ \mu A(u(s)) + B(u(s)) + \beta C(u(s)) \right] ds + \int_0^t f(s) ds$$

$$+ \sum_{k=1}^{\infty} \int_0^t \sigma_k(s, u(s)) dW_k(s), \quad (4.6)$$

$\mathbb{P}$-a.s., for all $t \in [0, T]$. Let us take $\epsilon > 0$. Then by the scaling property of the Brownian motion, it can be easily verified that $u(\epsilon t)$ coincides in law with the solution of the following equation:

$$u^\epsilon(t) = u_0 - \epsilon \int_0^t \left[ \mu A(u^\epsilon(s)) + B(u^\epsilon(s)) + \beta C(u^\epsilon(s)) \right] ds + f(\epsilon s) ds$$
\[ + \sum_{k=1}^{\infty} \sqrt{\varepsilon} \int_0^t \sigma_k(\varepsilon_s, u^c(s))dW_k(s), \]}

\[ \mathbb{P}\text{-a.s., for all } t \in [0, T]. \] In order to prove small time LDP, we need the following additional assumption on the noise coefficient:

**Hypothesis 4.1.** The noise coefficient \( \sigma_k(\cdot, \cdot), k \geq 1 \) satisfies:

\( (H.3) \) (Additional growth condition). There exists positive constants \( \tilde{K}, \tilde{K} \) and \( \overline{K} \) such that for all \( t \in [0, T] \) and \( u \in \mathbb{H}, \)

\[ \sum_{k=1}^{\infty} \|\sigma_k(t, u)\|_{V_2}^2 \leq \tilde{K}(1 + \|u\|_{V_2}^2), \tag{4.8} \]

\[ \sum_{k=1}^{\infty} \|\sigma_k(t, u)\|_{V_p}^2 \leq \tilde{K}(1 + \|u\|_{V_p}^2), \text{ for } p > 2, \tag{4.9} \]

\[ \sum_{k=1}^{\infty} \|\sigma_k(t, u)\|_{L^r}^2 \leq \overline{K}(1 + \|u\|_{L^r}^2), \text{ for } r \geq 2. \tag{4.10} \]

Note that the condition (4.8) implies that for each \( u \in V_2 \), the linear map \( \sigma(\cdot, u) := \{\sigma_k(\cdot, u)\}_{k \in \mathbb{N}} : \ell_2 \to V_2 \) defined by (3.1) is in \( L_2(\ell_2; V_2) \). Note that every Hilbert space \( L^p \) spaces with \( p \geq 2 \) and Sobolev spaces \( W_0^{s,p} \) with \( p \geq 2 \) and \( s \geq 1 \) are 2-smooth Banach spaces. For a Hilbert space \( U \) and a 2-smooth Banach space \( V \), we denote by \( \gamma(U; V) \) for the space of all \( \gamma \)-radonifying operators from \( U \) to \( V \). We remember that an operator \( R \in \gamma(U; V) \) if the series

\[ \sum_{k=1}^{\infty} \gamma_k R(e_k) \]

converges in \( L^2(\tilde{\Omega}; V) \) for any sequence \( \{\gamma_k\}_{k \geq 0} \) of independent Gaussian real-valued random variables on a probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \) and any orthonormal basis \( \{e_k\}_{k \geq 0} \) of \( U \). Then, the space \( \gamma(U; V) \) is endowed with the norm

\[ \|R\|_{\gamma(U; V)} = \left( \mathbb{E} \left[ \sum_{k=1}^{\infty} \gamma_k R(e_k) \right]^2 \right)^{\frac{1}{2}}, \]

which does not depend \( \{\gamma_k\}_{k \geq 0} \) and \( \{e_k\}_{k \geq 0} \), and is a Banach space. If \( V \) is a separable Hilbert space, then \( \gamma(U; V) \) consists of all Hilbert-Schmidt operators of mapping \( U \) into \( V \). The condition (4.9) implies that for every \( u \in V_p \), for some \( p \geq 2 \), the linear map \( \sigma(\cdot, u) := \{\sigma_k(\cdot, u)\}_{k \in \mathbb{N}} : \ell_2 \to V_p \) defined by (3.1) is in \( \gamma(\ell_2; V_p) \), that is, \( \sigma(\cdot, u) \) is a \( \gamma \)-radonifying operator from \( \ell_2 \) to \( V_p \). For the orthonormal basis \( \{e_k\}_{k \in \mathbb{N}} \in \ell_2 \) with \( e_k = (0, \ldots, 1, \ldots) \) and an independent standard Gaussian sequence \( \{\gamma_k\}_{k \in \mathbb{N}} \) on some probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \), using Hölder’s inequality, stochastic Fubini’s theorem, Gaussian formula (Appendix A.1, [52]) and Minkowski’s inequality, we have

\[ \|\sigma(t, u)\|_{\gamma(\ell_2; V_p)}^2 = \lim_{N \to \infty} \mathbb{E} \left[ \left\| \sum_{k=1}^{N} \gamma_k \sigma_k(t, u) e_k \right\|_{V_p}^2 \right]. \]
\[
E = \lim_{N \to \infty} \mathbb{E} \left[ \left( \int_{\mathcal{O}} \left| \sum_{k=1}^{N} \gamma_k \nabla \sigma_k(t, u(x)) \right|^p \, dx \right)^\frac{2}{p} \right]
\]

\[
\leq \lim_{N \to \infty} \left\{ \int_{\mathcal{O}} \left| \sum_{k=1}^{N} \gamma_k \nabla \sigma_k(t, u(x)) \right|^p \, dx \right\}^{\frac{2}{p}}
\]

\[
= \lim_{N \to \infty} \mathbb{A}_p \left\{ \int_{\mathcal{O}} \left( \sum_{k=1}^{N} \left| \nabla \sigma_k(t, u(x)) \right|^2 \right)^{\frac{2}{p}} \, dx \right\}^{\frac{2}{p}}
\]

\[
\leq \lim_{N \to \infty} \mathbb{A}_p \sum_{k=1}^{N} \left( \int_{\mathcal{O}} |\nabla \sigma_k(t, u(x))|^p \, dx \right)^{\frac{2}{p}} = \mathbb{A}_p \sum_{k=1}^{\infty} \|\sigma_k(t, u)\|_{\ell^p}^2
\]

\[
\leq \mathbb{A}_p \hat{K} \left( 1 + \|u\|_{\ell^p}^2 \right) < \infty,
\]

where \( \mathbb{A}_p = \int_{\mathbb{R}} \left( \frac{|\xi|^p}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \right) \, d\xi \). Similarly, one can show that the condition (4.10) implies that for every \( u \in \tilde{L}^r \), for \( r \geq 2 \), the linear map \( \sigma(\cdot, u) := \{ \sigma_k(\cdot, u) \}_{k \in \mathbb{N}} : \ell_2 \to \tilde{L}^r \) defined by (3.1) is in \( \gamma(\ell_2; \tilde{L}^r) \).

Let \( \mu_{u_0}^\varepsilon \) be the law of \( u^\varepsilon(\cdot) \) on \( C([0, T]; \mathbb{R}) \). Let us set 

\[
\mathcal{H} := \left\{ h = (h_1, \ldots, h_k, \ldots); h(\cdot) : [0, T] \to \ell^2 \text{ such that} \right. \\
\left. h \text{ is absolutely continuous and } \sum_{k=1}^{\infty} \int_0^T h_k^2(t) \, dt < \infty \right\}.
\]

For \( h \in \mathcal{H} \), let \( u^h(t) \) be the unique solution of the following deterministic equation:

\[
\begin{cases}
\frac{d u^h(t)}{dt} = \sum_{k=1}^{\infty} \sigma_k(t, u^h(t)) \dot{h}_k(t) \, dt, \\
u^h(0) = u_0.
\end{cases}
\]

(4.11)

For \( h(t) = \sum_{k=1}^{\infty} h_k(t) e_k \in \mathcal{H} \), we define

\[
I(h) = \frac{1}{2} \sum_{k=1}^{\infty} \int_0^T \left[ \dot{h}_k(t) \right]^2 \, dt.
\]

For \( g \in C([0, T]; \mathbb{R}) \), we also define

\[
\Gamma_g := \left\{ h \in \mathcal{H} : g(t) = u_0 + \sum_{k=1}^{\infty} \int_0^t \sigma_k(s, g(s)) \dot{h}_k(s) \, ds, \ 0 \leq t \leq T \right\}.
\]
Furthermore, we define
\[
R(g) = \begin{cases} 
\inf_{h \in \mathcal{H}^g} I(h) & \text{if } \Gamma_g \neq \emptyset, \\
+\infty & \text{if } \Gamma_g = \emptyset.
\end{cases}
\] (4.12)

Then we have the following result:

**Theorem 4.2.** For \( p \geq \frac{d}{2} + 1 \), \( r \geq 2 \) or \( p \geq 2 \), \( r \geq 4 \) (\( \beta \mu > 1 \) for \( r = 4 \)), under Hypotheses 3.1 and 4.1, \( \mu^\varepsilon \) satisfies a large deviation principle with the rate function \( R(\cdot) \), that is,

(i) For any closed set \( F \subset C([0, T]; \mathbb{H}) \),
\[
\limsup_{\varepsilon \to 0} \varepsilon \log \mu^\varepsilon (F) \leq - \inf_{g \in F} R(g).
\]

(ii) For any open set \( G \subset C([0, T]; \mathbb{H}) \),
\[
\liminf_{\varepsilon \to 0} \varepsilon \log \mu^\varepsilon (G) \geq - \sup_{g \in G} R(g).
\]

**Proof.** Let \( v^\varepsilon(\cdot) \) be the solution of the stochastic equation
\[
v^\varepsilon(t) = u_0 + \sum_{k=1}^{\infty} \sqrt{\varepsilon} \int_0^t \sigma_k(\varepsilon s, v^\varepsilon(s)) dW_k(s),
\] (4.13)
and \( \tilde{\mu}^\varepsilon \) be the law of \( v^\varepsilon(\cdot) \) on the \( C([0, T]; \mathbb{H}) \). Then from Theorem 12.9, [14], we infer that \( v^\varepsilon(\cdot) \) satisfies a large deviation principle with the rate function \( R(\cdot) \). Our main task is to show that two families of the probability measures \( \mu^\varepsilon \) and \( \tilde{\mu}^\varepsilon \) are exponentially equivalent, that is, for any \( \delta > 0 \),
\[
\lim_{\varepsilon \to 0} \varepsilon \log P\left\{ \sup_{t \in [0, T]} \| u^\varepsilon(t) - v^\varepsilon(t) \|_{\mathbb{H}}^2 > \delta \right\} = -\infty.
\] (4.14)

Theorem 4.2.13, [16] states that if one of the two exponentially equivalent families satisfies a large deviation principle, so does the other. Then our Theorem follows from (4.14) and Theorem 4.2.13.

Now, it is left to prove (4.14) only. Due to the presence of the nonlinear operators \( A, B \) and \( C \), the proof of (4.14) not easy and we divide the proof into several lemmas. The following result is an estimate of the probability that the solution of (4.7) leaves an energy ball.

**Lemma 4.3.** For \( f \in L^{\infty}(0, T; \mathbb{V}_p^\varepsilon) \), under Hypothesis 3.7, we have
\[
\lim_{M \to \infty} \sup_{0 < \varepsilon \leq 1} \varepsilon \log P\left\{ \right. \left| u^\varepsilon \right|_{\mathbb{V}_p, L^r}(T) > M \left. \right\} = -\infty,
\] (4.15)
where
\[
\left( \left| u^\varepsilon \right|_{\mathbb{V}_p, L^r}(T) \right)^2 = \sup_{0 \leq t \leq T} \| u^\varepsilon(t) \|_{\mathbb{H}}^2 + \varepsilon \mu \int_0^T \| u^\varepsilon(t) \|_{\mathbb{V}_p}^2 dt + \frac{\varepsilon \mu}{2} \int_0^T \| u^\varepsilon(t) \|_{\mathbb{V}_p}^2 dt + \varepsilon \beta \int_0^T \| u^\varepsilon(t) \|_{\mathbb{V}_p}^2 dt.
\]

**Proof.** Applying the infinite dimensional Itô’s formula to the process \( \| u^\varepsilon(\cdot) \|_{\mathbb{H}}^2 \) (25, 45, 47), we obtain
\[
\| u^\varepsilon(t) \|_{\mathbb{H}}^2 + 2\mu \varepsilon \int_0^t \left( \left( 1 + | \nabla u(s) |^2 \right)^{\frac{p-2}{2}} \nabla u(s), \nabla u(s) \right) ds + 2\beta \int_0^t \| u^\varepsilon(s) \|_{\mathbb{V}_p}^2 ds,
\]
where

\[ \text{Substituting (4.17)-(4.18) in (4.16), we deduce that} \]

\[ 2\left(1 + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u, \nabla u \geq \|
abla u\|^2_H + \|
abla u\|^p_{L^p}. \]

We estimate \( I_1 \) using Cauchy-Schwarz’s and Young’s inequalities, and \( I_2 \) using Hypothesis [3.1] (H.1) as

\[ |I_1| \leq 2\varepsilon \int_0^t \|u^\varepsilon(s)\|_{V_p} \|f(\varepsilon s)\|_{V_p} ds \]

\[ \leq \frac{\varepsilon \mu}{2} \int_0^t \|u^\varepsilon(s)\|_{V_p} ds + \frac{2^{\frac{p+1}{p}}(p-1)}{p} \left( \frac{1}{\mu p} \right)^{\frac{1}{p-1}} \int_0^t \|f(s)\|_{V_p}^\varepsilon ds, \]

\[ |I_2| \leq \varepsilon K \int_0^t (1 + \|u^\varepsilon(s)\|_{H}^2) ds \leq \varepsilon K t + \varepsilon K \int_0^t \|u^\varepsilon(s)\|_{H}^2 ds. \]

Substituting (4.17)-(4.18) in (4.16), we deduce that

\[ \|u^\varepsilon(t)\|^2_H + \varepsilon \mu \int_0^t \|u(s)\|_{V_2}^2 ds + \frac{\varepsilon \mu}{2} \int_0^t \|u(s)\|_{V_p}^p ds + 2\beta \varepsilon \int_0^t \|u(s)\|_{L_p}^r ds \]

\[ \leq \left( \|x\|^2_H + \varepsilon K t + C_{p,\mu} \int_0^{\varepsilon t} \|f(s)\|_{V_p}^{\varepsilon} ds \right) + \varepsilon K \int_0^t \|u^\varepsilon(s)\|_{H}^2 ds \]

\[ + 2\sqrt{\varepsilon} \sum_{k=1}^\infty \int_0^t (\sigma_k(\varepsilon s, u^\varepsilon(s)), u^\varepsilon(s))dW_k(s), \]

where \( C_{p,\mu} = \frac{2^{\frac{p+1}{p}}(p-1)}{p} \left( \frac{1}{\mu p} \right)^{\frac{1}{p-1}} \). Therefore, taking supremum over time \( t \in [0, T] \), we get

\[ \left( |u^\varepsilon|_{V_p,L^r}^T(t) \right)^2 \leq \left( \|x\|^2_H + \varepsilon K T + C_{p,\mu} \int_0^{T\varepsilon} \|f(t)\|_{V_p}^{\varepsilon} dt \right) + \varepsilon K \int_0^T \left( |u^\varepsilon|_{V_p,L^r}^T(t) \right)^2 dt \]

\[ + 2\sqrt{\varepsilon} \sup_{t \in [0, T]} \left| \sum_{k=1}^\infty \int_0^t (\sigma_k(\varepsilon s, u^\varepsilon(s)), u^\varepsilon(s))dW_k(s) \right|. \]

Thus, for \( q \geq 2 \), we obtain

\[ \left\{ \mathbb{E} \left[ \left( |u^\varepsilon|_{V_p,L^r}^T(t) \right)^{2q} \right] \right\}^{\frac{1}{q}} \]

\[ \leq 2 \left( \|x\|^2_H + \varepsilon K T + C_{p,\mu} \int_0^{T\varepsilon} \|f(t)\|_{V_p}^{\varepsilon} dt \right) + \varepsilon K \left\{ \mathbb{E} \left[ \left( \int_0^T \left( |u^\varepsilon|_{V_p,L^r}^T(t) \right)^2 dt \right)^q \right] \right\}^{\frac{1}{q}} \]

\[ + 4\sqrt{\varepsilon} \left\{ \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \sum_{k=1}^\infty \int_0^t (\sigma_k(\varepsilon s, u^\varepsilon(s)), u^\varepsilon(s))dW_k(s) \right| \right]^q \right\}^{\frac{1}{q}}. \]
In order to estimate the stochastic integral term, we use the following important result (cf. [5, 13]). There exists a universal constant $C > 0$ such that, for any $q \geq 2$ and for any continuous martingale $M = \{M_t\}_{t \geq 0}$ with $M_0 = 0$, we have

$$\|M_t^r\|_{L^q} \leq C q^{\frac{1}{2q}} \|M_t\|_{L^q}, \quad (4.20)$$

where $M_t^r = \sup_{s \in [0, t]} |M_s|$, and $\{\langle M_t \rangle\}_{t \geq 0}$ is the quadratic variation process of $M$. Making use of this result and Minkowski’s integral inequality, we get

$$4\sqrt{\varepsilon} \left\{ \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \sum_{k=1}^{\infty} \int_0^t (\sigma_k(\varepsilon s, u^\varepsilon(s)), u^\varepsilon(s)) dW_k(s) \right|^q \right] \right\}^{\frac{1}{q}} \leq 4C \sqrt{\varepsilon} \left\{ \mathbb{E} \left[ \left( \int_0^T \sum_{k=1}^{\infty} \|\sigma_k(\varepsilon t, u^\varepsilon(t))\|_{H}^2 \|u^\varepsilon(t)\|_H^2 dt \right)^{\frac{q}{2}} \right] \right\}^{\frac{1}{q}} \leq 4CK \sqrt{q} \left\{ \mathbb{E} \left[ \left( \int_0^T (1 + \|u^\varepsilon(t)\|_{H}^2) \|u^\varepsilon(t)\|_H^2 dt \right)^{\frac{q}{2}} \right] \right\}^{\frac{1}{q}} \leq 4CK \sqrt{2q} \left\{ \mathbb{E} \left[ \left( \int_0^T (1 + \|u^\varepsilon(t)\|_{H}^2) dt \right)^{\frac{q}{2}} \right] \right\}^{\frac{1}{q}} \leq 8CK \sqrt{q} \left\{ \int_0^T (1 + \mathbb{E}(\|u^\varepsilon(t)\|_{H}^{2q})) dt \right\}^{\frac{1}{2}}. \quad (4.21)$$

Moreover, by using Minkowski’s integral inequality, we have

$$2\varepsilon K \left\{ \mathbb{E} \left[ \left( \int_0^T (|u^\varepsilon|_{H, \tilde{L}^r}^2) dt \right)^q \right] \right\}^{\frac{1}{q}} \leq 2\varepsilon K \int_0^T \mathbb{E} \left[ |u^\varepsilon|_{H, \tilde{L}^r}^{2q} \right] dt. \quad (4.22)$$

Substituting (4.21)-(4.22) in (4.19), we deduce that

$$\left\{ \mathbb{E} \left[ \left( |u^\varepsilon|_{H, \tilde{L}^r}^2(T) \right)^{2q} \right] \right\}^{\frac{1}{q}} \leq 8\left( \|x\|_{H}^2 + \varepsilon KT + C_{p, \mu} \int_0^T \|f(t)\|_{L^p} dt \right)^2 + 8\varepsilon^2 K^2 \int_0^T \mathbb{E} \left[ |u^\varepsilon|_{H, \tilde{L}^r}^{2q} \right] dt + 128C^2 K^2 q \varepsilon T + 128C^2 K^2 q \varepsilon^2 \int_0^T \mathbb{E} \left[ |u^\varepsilon|_{H, \tilde{L}^r}^{2q} \right] \frac{2}{3} dt. \quad (4.23)$$

An application of Gronwall’s inequality yields

$$\left\{ \mathbb{E} \left[ \left( |u^\varepsilon|_{H, \tilde{L}^r}^2(T) \right)^{2q} \right] \right\}^{\frac{1}{q}}$$
\[
\begin{align*}
&\leq \left\{ 8 \left( \|x\|^2_H + \varepsilon KT + C_{p,\mu} \int_0^{T\varepsilon} \|f(t)\|_{V_p}^p \ dt \right)^2 + 128C^2K^2q\varepsilon T \right\} \\
&\times \exp\left\{ 8\varepsilon^2K^2T + 128C^2K^2q\varepsilon T \right\}. \quad (4.24)
\end{align*}
\]
By Markov’s inequality, we have
\[
P \left\{ \left( |u^\varepsilon|_{V_p,L^r}^H(T) \right)^2 > M \right\} \leq M^{-q}\mathbb{E} \left[ \left( |u^\varepsilon|_{V_p,L^r}^H(T) \right)^{2q} \right].
\]
Letting \( q = \frac{1}{\varepsilon} \) in (4.24), we get
\[
\varepsilon \log P \left\{ \left( |u^\varepsilon|_{V_p,L^r}^H(T) \right)^2 > M \right\} \leq - \log M + \log \left\{ \mathbb{E} \left[ \left( |u^\varepsilon|_{V_p,L^r}^H(T) \right)^{2q} \right] \right\} \frac{1}{q}
\]
\[
\leq - \log M + \log \left\{ 8 \left( \|x\|^2_H + \varepsilon KT + C_{p,\mu} \int_0^{T\varepsilon} \|f(t)\|_{V_p}^p \ dt \right)^2 + 128C^2K^2T \right\} \\
+ 4\varepsilon^2K^2T + 64C^2K^2.
\]
(4.25)

Therefore, it is immediate that
\[
\sup_{0<\varepsilon\leq 1} \varepsilon P \left\{ \left( |u^\varepsilon|_{V_p,L^r}^H(T) \right)^2 > M \right\} \leq - \log M + \log \left\{ 8 \left( \|x\|^2_H + KT + C_{p,\mu} \int_0^{T} \|f(t)\|_{V_p}^p \ dt \right)^2 + 128C^2K^2T \right\} \\
+ 4K^2T + 64C^2K^2.
\]
Letting \( M \to \infty \) in the above expression leads to the estimate (4.13). \( \square \)

Since \( V \subset V_p \cap \tilde{L}^r \subset H \) and \( V \) is dense in \( H \) implies that \( V_p \cap \tilde{L}^r \) is also dense in \( H \). Let \( \{u^n_0\} \) be a sequence in \( V_p \cap \tilde{L}^r \) such that \( \|u^n_0 - u_0\|_H \to 0 \) as \( n \to \infty \). Let \( u^n_\varepsilon(\cdot) \) be the solution of (4.17) with the initial value \( u^n_0 \). From the proof of Lemma 4.3, it follows that
\[
\lim_{M \to \infty} \sup_{0<\varepsilon\leq 1} \varepsilon \log P \left\{ \left( |u^n_\varepsilon|_{V_p,L^r}^H(T) \right)^2 > M \right\} = -\infty. \quad (4.26)
\]
Let \( v^n_\varepsilon(\cdot) \) be the solution of (4.13) with the initial value \( u^n_0 \). Then, we have the following result:

**Lemma 4.4.** Under hypothesis 4.1, for any \( n \in \mathbb{Z}^+ \), we have
\[
\lim_{M \to \infty} \sup_{0<\varepsilon\leq 1} \varepsilon \log P \left\{ \sup_{0 \leq t \leq T} \|v^n_\varepsilon(t)\|_{V_2} \geq M \right\} = -\infty, \quad (4.27)
\]
\[
\lim_{M \to \infty} \sup_{0<\varepsilon\leq 1} \varepsilon \log P \left\{ \sup_{0 \leq t \leq T} \|v^n_\varepsilon(t)\|_{V_p}^p \geq M \right\} = -\infty, \text{ and} \quad (4.28)
\]
\[
\lim_{M \to \infty} \sup_{0<\varepsilon\leq 1} \varepsilon \log P \left\{ \sup_{0 \leq t \leq T} \|v^n_\varepsilon(t)\|_{L^r} \geq M \right\} = -\infty. \quad (4.29)
\]
for \( p \geq 2 \) and \( r \geq 2 \).

**Proof.** Applying infinite dimensional Itô’s formula to the process \( \|v_n^\varepsilon(\cdot)\|^2_{V_p} \), we get

\[
\|\nabla v_n^\varepsilon(t)\|^2_H = \|\nabla u_0^\varepsilon\|^2_H + \varepsilon \sum_{k=1}^\infty \int_0^t \|\nabla \sigma_k(\varepsilon s, v_n^\varepsilon(s))\|^2_H ds
+ 2\sqrt{\varepsilon} \sum_{k=1}^\infty \int_0^t (\nabla \sigma_k(\varepsilon s, v_n^\varepsilon(s)), \nabla v_n^\varepsilon(s)) dW_k(s),
\]

(4.30)

\( \mathbb{P} \)-a.s., for all \( t \in [0, T] \). By using Hypothesis 3.1 (H.1) and (4.20), we obtain

\[
\left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|v_n^\varepsilon(t)\|_{V_2}^{2q} \right] \right\}^{\frac{1}{2q}}
\leq 2\|u_0^\varepsilon\|_{V_2}^4 + 2\varepsilon^2 \left\{ \mathbb{E} \left[ \left( \sum_{k=1}^\infty \int_0^T \|\nabla \sigma_k(\varepsilon s, v_n^\varepsilon(s))\|^2_H ds \right)^q \right] \right\}^{\frac{1}{2q}}
+ 8\varepsilon \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \sum_{k=1}^\infty \int_0^t (\nabla \sigma_k(\varepsilon s, v_n^\varepsilon(s)), \nabla v_n^\varepsilon(s)) dW_k(s) \left] \right.^q \right\}^{\frac{1}{2q}}
\leq 2\|u_0^\varepsilon\|_{V_2}^4 + 2\varepsilon^2 \hat{K}^2 \left\{ \mathbb{E} \left[ \left( \int_0^T (1 + \|v_n^\varepsilon(s)\|_{V_2}^{2q}) ds \right)^q \right] \right\}^{\frac{1}{2q}}
+ 8Cq\varepsilon \left\{ \mathbb{E} \left[ \left( \sum_{k=1}^\infty \int_0^T \|v_n^\varepsilon(s)\|_{V_2}^2 \|\nabla \sigma_k(\varepsilon s, v_n^\varepsilon(s))\|^2_H ds \right)^q \right] \right\}^{\frac{1}{2q}}
\leq 2\|u_0^\varepsilon\|_{V_2}^4 + 4\varepsilon^2 \hat{K}^2 T \left\{ T + \int_0^T \mathbb{E} \left[ \sup_{0 \leq r \leq s} \|v_n^\varepsilon(r)\|_{V_2}^{2q} \right] \right\}^{\frac{1}{2q}} ds
+ 16Cq\varepsilon \hat{K} \left\{ T + \int_0^T \mathbb{E} \left[ \sup_{0 \leq r \leq s} \|v_n^\varepsilon(r)\|_{V_2}^{2q} \right] \right\}^{\frac{1}{2q}} ds
\]  

(4.31)

An application of Gronwall’s inequality yields

\[
\left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|v_n^\varepsilon(t)\|_{V_2}^{2q} \right] \right\}^{\frac{1}{2q}} \leq \left\{ 2\|u_0^\varepsilon\|_{V_2}^4 + 4\varepsilon^2 \hat{K}^2 T^2 + 16Cq\varepsilon \hat{K} T \right\} e^{4\varepsilon^2 \hat{K}^2 T^2 + 16Cq\varepsilon \hat{K} T},
\]

and the proof of (4.27) can be completed by using similar arguments as in the proof of Lemma 4.3.

For some \( p \geq 2 \), applying the infinite dimensional formula to the process \( \|v_n^\varepsilon(\cdot)\|_{V_p}^p \), we obtain (Theorem A.1, [10])

\[
\|v_n^\varepsilon(t)\|_{V_p}^p = \|u_0^\varepsilon\|_{V_p}^p + \frac{p(p-1)}{2} \sum_{k=1}^\infty \int_0^t \langle |\nabla v_n^\varepsilon(s)|^{p-2}, |\nabla \sigma_k(\varepsilon s, v_n^\varepsilon(s))|^2 \rangle ds
+ p\sqrt{\varepsilon} \sum_{k=1}^\infty \int_0^t \langle |\nabla v_n^\varepsilon(s)|^{p-2} \nabla v_n^\varepsilon(s), \nabla \sigma_k(\varepsilon s, v_n^\varepsilon(s)) \rangle dW_k(s),
\]

(4.32)
for all \( t \in [0, T] \), \( \mathbb{P} \)-a.s. For \( q \geq 2 \), using Hypothesis 4.1 (H.3) and (4.20), we find
\[
\left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| v_n^\varepsilon(t) \right\|_{L_p}^q \right] \right\}^{\frac{2}{q}} \leq 2 \left\| u_0^p \right\|_{L_p}^{2p} + p^2 (p-1)^2 \varepsilon^2 \left\{ \mathbb{E} \left[ \left( \sum_{k=1}^{\infty} \int_0^T \left\| \nabla v_n^\varepsilon(s) \right\|_{L_p}^{p-2} \left\| \nabla \sigma_k(\varepsilon s, v_n^\varepsilon(s)) \right\|_{L_2}^2 \, ds \right] \right\}^{\frac{q}{2}}
\]
\[
+ 2p^2 \varepsilon \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \sum_{k=1}^{\infty} \int_0^t \left\| \nabla v_n^\varepsilon(s) \right\|_{L_p}^{p-2} \left\| \nabla \sigma_k(\varepsilon s, v_n^\varepsilon(s)) \right\|_{L_2}^2 \, ds \right] \right\}^{\frac{q}{2}}
\]
\[
\leq 2 \left\| u_0^p \right\|_{L_p}^{2p} + p^2 (p-1)^2 \varepsilon^2 \mathcal{K}^2 \left\{ \mathbb{E} \left[ \left( \int_0^T \left\| \nabla v_n^\varepsilon(s) \right\|_{L_p}^{2(p-1)} \right. \left. \left(1 + \left\| \nabla v_n^\varepsilon(s) \right\|_{L_2}^2 \right) \, ds \right] \right\}^{\frac{2}{q}}
\]
\[
+ 2Cp^2 q \varepsilon \mathcal{K} \left\{ \mathbb{E} \left[ \left( \int_0^T \left\| \nabla v_n^\varepsilon(s) \right\|_{L_p}^{2(p-1)} \left(1 + \left\| \nabla v_n^\varepsilon(s) \right\|_{L_2}^2 \right) \, ds \right] \right\}^{\frac{2}{q}}
\]
\[
\leq 2 \left\| u_0^p \right\|_{L_p}^{2p} + p^2 (p-1)^2 \varepsilon^2 \mathcal{K}^2 \left\{ \mathbb{E} \left[ \left( \sup_{0 \leq r \leq s} \left\| v_n^\varepsilon(r) \right\|_{L_p}^{p} \right] \right\}^{\frac{2}{q}}
\]
\[
+ 4Cp^2 q \varepsilon \mathcal{K} \left\{ T + \int_0^T \mathbb{E} \left( \sup_{0 \leq r \leq s} \left\| v_n^\varepsilon(r) \right\|_{L_p}^{p} \right] \right\}^{\frac{2}{q}} \, ds \right\}.
\] (4.33)

An application of Gronwall’s inequality in (4.33) yields
\[
\left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| v_n^\varepsilon(t) \right\|_{L_p}^q \right] \right\}^{\frac{2}{q}} \leq 2 \left\| u_0^p \right\|_{L_p}^{2p} + p^2 (p-1)^2 \varepsilon^2 \mathcal{K}^2 T + 4Cp^2 q \varepsilon \mathcal{K} T \, e^{2p^2 (p-1)^2 \varepsilon^2 \mathcal{K}^2 T + 4Cp^2 q \varepsilon \mathcal{K} T},
\] (4.34)

and the rest of the proof of (4.28) can be completed by using similar arguments as in the proof of Lemma 4.3.

For some \( r \geq 2 \), applying the infinite dimensional formula to the process \( \left\| v_n^\varepsilon(\cdot) \right\|_{L_p}^r \), using Hypothesis 4.1 (H.3), and then applying similar arguments as in the previous case one can obtain the estimate (4.29).

\[ \text{Lemma 4.5.} \quad \text{For any } \delta > 0, \text{ we have} \]
\[
\lim_{n \to \infty} \sup_{0 \leq \varepsilon \leq 1} \varepsilon \log \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \left\| v_n^\varepsilon(t) - \varepsilon^\delta \right\|_{L_2}^2 > \delta \right\} = -\infty. \] (4.35)

\[ \text{Proof.} \quad \text{Applying infinite dimensional Itô’s formula to the process } \left\| v_n^\varepsilon(\cdot) - \varepsilon^\delta \right\|_{L_2}^2, \text{ we find} \]
\[
\left\| v_n^\varepsilon(t) - \varepsilon^\delta \right\|_{L_2}^2
\]
Lemma 4.6. Proof. For \( \epsilon > 0 \), it is clear that

\[
= \| u_0^n - u_0 \|^2_{H^2} + \epsilon \sum_{k=1}^{\infty} \int_0^t \| \sigma_k(\epsilon s, v^\epsilon_n(s)) - \sigma_k(\epsilon s, v^\epsilon(s)) \|^2_{H^2} ds
\]

\[
+ 2\sqrt{\epsilon} \sum_{k=1}^{\infty} \int_0^t (\sigma_k(\epsilon s, v^\epsilon_n(s)) - \sigma_k(\epsilon s, v^\epsilon(s)), v^\epsilon_n(s) - v^\epsilon(s)) dW_k(s),
\]

\( \mathbb{P} \)-a.s., for all \( t \in [0, T] \). Using calculations similar to (4.31) yield

\[
\left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| v^\epsilon_n(t) - v^\epsilon(t) \|_{V_2}^{2q} \right] \right\}^{\frac{2}{q}} \leq 2 \| u_0^n - u_0 \|_{V_2}^{4} e^{(\epsilon^2 L^2 T + 16Cq \epsilon L^2) T}.
\]

Taking \( q = \frac{2}{\epsilon} \) and applying Markov’s inequality, we obtain

\[
\sup_{0 < \epsilon \leq 1} \epsilon \log \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \| v^\epsilon(t) - v^\epsilon_n(t) \|_{H^2} > \delta \right\}
\]

\[
\leq \sup_{0 < \epsilon \leq 1} \epsilon \log \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| v^\epsilon(t) - v^\epsilon_n(t) \|_{H^2}^{2q} \right] \right\}^{\frac{1}{q}}
\]

\[
\leq (4L^2 T + 32CL^2) T + \log(2\| u_0^n - u_0 \|_{V_2}^{4}) - 2 \log \delta
\]

\[
\rightarrow -\infty \quad \text{as} \quad n \rightarrow \infty,
\]

for any \( \delta > 0 \), which completes the proof.

Lemma 4.6. For any \( \delta > 0 \), we have

\[
\lim_{n \to \infty} \sup_{0 < \epsilon \leq 1} \epsilon \log \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \| u^\epsilon_n(t) - u^\epsilon(t) \|_{H^2} > \delta \right\} = -\infty.
\]

Proof. For \( M > 0 \), let us define a sequence of stopping times as

\[
\tau_{\epsilon,M} := \inf_{t \geq 0} \left\{ t : \epsilon \int_0^t [\| u^\epsilon(s) \|_{V_2}^{2} + \| u^\epsilon(s) \|_{H^2}^{4}] ds > M \right. \quad \text{or} \right. \left. \| u^\epsilon(t) \|_{H^2}^{2} > M \right\}.
\]

It is clear that

\[
\mathbb{P} \left\{ \sup_{0 \leq t \leq T} \| u^\epsilon_n(t) - u^\epsilon(t) \|_{H^2}^{2} > \delta, \left( \| u^\epsilon \|_{V_2, L^2}(T) \right)^{2} \leq M \right\}
\]

\[
\leq \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \| u^\epsilon_n(t) - u^\epsilon(t) \|_{H^2}^{2} > \delta, \tau_{\epsilon,M} \geq T \right\}
\]

\[
\leq \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \| u^\epsilon_n(t) - u^\epsilon(t) \|_{H^2}^{2} > \delta \right\}.
\]

(4.39)
Case 1: $p \geq \frac{d}{2} + 1$ and $r \geq 2$. Let $k$ be a positive constant. For $p \geq \frac{d}{2} + 1$ and $r \geq 2$, applying Itô’s formula to $e^{-ke \int_0^{t \wedge \tau,s,M} \| u^\epsilon(s) \|^2_{V_p} ds} \| u^\epsilon_n(t) - u^\epsilon(t) \|^2_H$, we find

$$e^{-ke \int_0^{t \wedge \tau,s,M} \| u^\epsilon(s) \|^2_{V_p} ds} \| u^\epsilon_n(t \wedge \tau,s,M) - u^\epsilon(t \wedge \tau,s,M) \|^2_H$$

$$= -2 \mu \epsilon \int_0^{t \wedge \tau,s,M} e^{-ke \int_0^r \| u^\epsilon(r) \|^2_{V_p} dr} \langle A(u^\epsilon_n(s) - u^\epsilon(s)), u^\epsilon_n(s) - u^\epsilon(s) \rangle ds$$

$$+ 2 \beta \epsilon \int_0^{t \wedge \tau,s,M} e^{-ke \int_0^r \| u^\epsilon(r) \|^2_{V_p} dr} \langle C(u^\epsilon_n(s) - C(u^\epsilon(s)), u^\epsilon_n(s) - u^\epsilon(s) \rangle ds$$

$$= \| u_0^\epsilon - u_0 \|^2_H - k \epsilon \int_0^{t \wedge \tau,s,M} e^{-ke \int_0^r \| u^\epsilon(r) \|^2_{V_p} dr} \| u^\epsilon(s) \|^2_{V_p} ds$$

$$\leq 2 \epsilon \int_0^{t \wedge \tau,s,M} e^{-ke \int_0^r \| u^\epsilon(r) \|^2_{V_p} dr} \| \sigma_j(\epsilon_s, u^\epsilon_n(s) - \sigma_j(\epsilon_s, u^\epsilon(s)) \|^2_H ds$$

$$+ 2 \sqrt{\epsilon} \sum_{j=1}^\infty \int_0^{t \wedge \tau,s,M} e^{-ke \int_0^r \| u^\epsilon(r) \|^2_{V_p} dr} \times (\| \sigma_j(\epsilon_s, u^\epsilon_n(s) - \sigma_j(\epsilon_s, u^\epsilon(s)) \|^2_H ds$$

for all $t \in [0, T]$, $P$-a.s. From (2.15), we infer that

$$-2 \mu \epsilon \langle A(u^\epsilon_n - u^\epsilon), u^\epsilon_n - u^\epsilon \rangle \geq 2 \mu \epsilon \| \nabla (u^\epsilon_n - u^\epsilon) \|^2_H,$$

and from (2.9), we obtain

$$2 \beta \epsilon \langle C(u^\epsilon_n) - C(u^\epsilon), u^\epsilon_n - u^\epsilon \rangle \geq 0.$$

For $p \geq \frac{d}{2} + 1$ and $r \geq 2$, we estimate $-2 \epsilon (B(u^\epsilon_n) - B(u^\epsilon), u^\epsilon_n - u^\epsilon)$ in a similar way as in (2.28) as

$$-2 \epsilon (B(u^\epsilon_n) - B(u^\epsilon), u^\epsilon_n - u^\epsilon) \leq \mu \epsilon \| \nabla (u^\epsilon_n - u^\epsilon) \|^2_H + \epsilon \eta \| u^\epsilon \|^2_{V_p} \| u^\epsilon_n - u^\epsilon \|^2_H,$$

where $\eta$ is defined in (2.27). Let us choose $k > \eta$. Combining (4.41)-(4.43) and then substituting it in (4.40), we find

$$e^{-ke \int_0^{t \wedge \tau,s,M} \| u^\epsilon(s) \|^2_{V_p} ds} \| u^\epsilon_n(t \wedge \tau,s,M) - u^\epsilon(t \wedge \tau,s,M) \|^2_H$$

$$\leq \| u_0^\epsilon - u_0 \|^2_H + \epsilon L \int_0^{t \wedge \tau,s,M} e^{-ke \int_0^r \| u^\epsilon(r) \|^2_{V_p} dr} \| u^\epsilon_n(s) - u^\epsilon(s) \|^2_H ds$$

$$+ 2 \epsilon \sum_{j=1}^\infty \int_0^{t \wedge \tau,s,M} e^{-ke \int_0^r \| u^\epsilon(r) \|^2_{V_p} dr} \times (\| \sigma_j(\epsilon_s, u^\epsilon_n(s) - \sigma_j(\epsilon_s, u^\epsilon(s)) \|^2_H ds$$

$$+ 2 \sqrt{\epsilon} \sum_{j=1}^\infty \int_0^{t \wedge \tau,s,M} e^{-ke \int_0^r \| u^\epsilon(r) \|^2_{V_p} dr} \times (\| \sigma_j(\epsilon_s, u^\epsilon_n(s) - \sigma_j(\epsilon_s, u^\epsilon(s)) \|^2_H ds$$

(4.44)
where we have used Hypothesis 3.1 (H.2) also. Using (4.20), for \(q \geq 2\), one can deduce that

\[
\left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T \land \tau, M} \left( e^{-k \varepsilon \int_0^t \| u^e(s) \|_{V_p}^{2p-\delta} \, ds} \| \mathbf{u}_n(t) - \mathbf{u}^e(t) \|_2^q \right) \right] \right\}^{\frac{2}{q}}
\leq 2 \| \mathbf{u}_0^n - \mathbf{u}_0 \|_2^4 + 2 \varepsilon^2 L^2 \int_0^T \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t \land \tau, M} \left( e^{-k \varepsilon \int_0^s \| u^e(r) \|_{V_p}^{2p-\delta} \, dr} \| \mathbf{u}_n(s) - \mathbf{u}^e(s) \|_2^2 \right) \right] \right\}^{\frac{2}{q}} \, dt

+ 8C \varepsilon qL^2 \int_0^T \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t \land \tau, M} \left( e^{-k \varepsilon \int_0^s \| u^e(r) \|_{V_p}^{2p-\delta} \, dr} \| \mathbf{u}_n(s) - \mathbf{u}^e(s) \|_2 \right) \right] \right\}^{\frac{2}{q}} \, dt
\]

\[
\leq 2 \| \mathbf{u}_0^n - \mathbf{u}_0 \|_2^4 + 2 \varepsilon^2 L^2 \int_0^T \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t \land \tau, M} \left( e^{-k \varepsilon \int_0^s \| u^e(r) \|_{V_p}^{2p-\delta} \, dr} \| \mathbf{u}_n(s) - \mathbf{u}^e(s) \|_2 \right) \right] \right\}^{\frac{2}{q}} \, dt

+ 8C \varepsilon qL^2 \int_0^T \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t \land \tau, M} \left( e^{-k \varepsilon \int_0^s \| u^e(r) \|_{V_p}^{2p-\delta} \, dr} \| \mathbf{u}_n(s) - \mathbf{u}^e(s) \|_2 \right) \right] \right\}^{\frac{2}{q}} \, dt. \tag{4.45}
\]

An application of Gronwall’s inequality in (4.45) gives

\[
\left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T \land \tau, M} \left( e^{-k \varepsilon \int_0^t \| u^e(s) \|_{V_p}^{2p-\delta} \, ds} \| \mathbf{u}_n(t) - \mathbf{u}^e(t) \|_2 \right) \right] \right\}^{\frac{2}{q}}
\leq 2 \| \mathbf{u}_0^n - \mathbf{u}_0 \|_2^4 e^{2 \varepsilon^2 L^2 T + 8C \varepsilon qL^2 T}. \tag{4.46}
\]

Therefore, we easily have

\[
\left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T \land \tau, M} \left( \| u^e_n(t) - u^e(t) \|_2 \right) \right] \right\}^{\frac{2}{q}}
\leq e^{2 \varepsilon M \frac{q^2}{2} - \frac{q^2 - 2q - d}{2p-\delta} T^2 - \frac{q^2 - 2q - d}{2p-\delta} \int_0^T \| u^e_n(t) - u^e(t) \|_2^2 \, dt}
\]

\[
\leq 2 \varepsilon^2 L^2 T + 8C \varepsilon qL^2 T, \tag{4.47}
\]

for \(p \geq \frac{d}{2} + 1\). For any fixed \(M\), taking \(q = \frac{2}{\varepsilon}\), we get

\[
\sup_{0 \leq t \leq 1} \varepsilon \log \mathbb{P} \left\{ \sup_{0 \leq t \leq T \land \tau, M} \| u^e_n(t) - u^e(t) \|_2 > \delta \right\}
\leq \sup_{0 \leq t \leq 1} \varepsilon \log \mathbb{E} \left[ \sup_{0 \leq t \leq T \land \tau, M} \left( \| u^e_n(t) - u^e(t) \|_2^q \right) \right]^\frac{2}{q}
\leq 2kM \frac{q^2}{2} - \frac{q^2 - 2q - d}{2p-\delta} T^2 - \frac{q^2 - 2q - d}{2p-\delta} \int_0^T \| u^e_n(t) - u^e(t) \|_2^2 \, dt
\]

\[
+ 2 \varepsilon^2 L^2 T + 16CL^2 T - 2 \log \delta + \log (2 \| u_0^n - u_0 \|_2^4)\]
→ −∞ as n → ∞. \hfill (4.48)

Using Lemma 4.3, we infer that, for any R > 0, there exists a constant M > 0 such that for any ε ∈ (0, 1], the following inequality holds:

\[
P\left\{ \left( u^\varepsilon_{\mathcal{H}}(T) \right)^2 > M \right\} \leq e^{-\frac{R}{\varepsilon}}. \tag{4.49}\]

For such an M, combining (4.39) and (4.48), there exists a positive integer N, such that for any n ≥ N,

\[
\sup \varepsilon \log P\left\{ \sup_{0 \leq t \leq T \wedge \tau_{r,c,M}} \| u^\varepsilon_n(t) - u^\varepsilon(t) \|^2_{\mathcal{H}} > \delta, \left( u^\varepsilon_{\mathcal{H}}(T) \right)^2 > M \right\} \leq -R. \tag{4.50}\]

Combining (4.49) and (4.50), one can obtain the existence of a positive integer N, such that for any n ≥ N, ε ∈ (0, 1],

\[
\sup \varepsilon \log P\left\{ \sup_{0 \leq t \leq T} \| u^\varepsilon_n(t) - u^\varepsilon(t) \|^2_{\mathcal{H}} > \delta \right\} \leq 2e^{-\frac{R}{\varepsilon}}. \tag{4.51}\]

The arbitrariness of R completes the proof for \( p \geq \frac{d}{2} + 1 \) and \( r \geq 2 \).

**Case 2:** \( p \geq 2 \) and \( r \geq 4 \). For the case \( p \geq 2 \) and \( r \geq 4 \), we first apply Itô’s formula to the process \( \| u^\varepsilon_n(\cdot) - u^\varepsilon(\cdot) \|^2_{\mathcal{H}} \) to find

\[
\| u^\varepsilon_n(t) - u^\varepsilon(t) \|^2_{\mathcal{H}} = 2\mu \varepsilon \int_0^t \mathcal{A}(u^\varepsilon_n(s)) - \mathcal{A}(u^\varepsilon(s)), u^\varepsilon_n(s) - u^\varepsilon(s) \rangle ds
\]

\[+ 2\beta \varepsilon \int_0^t \mathcal{C}(u^\varepsilon_n(s)) - \mathcal{C}(u^\varepsilon(s)), u^\varepsilon_n(s) - u^\varepsilon(s) \rangle ds\]

\[= \| u^\varepsilon_0 - u_0 \|^2_{\mathcal{H}} - 2\varepsilon \int_0^t \langle B(u^\varepsilon_n(s)) - B(u^\varepsilon(s)), u^\varepsilon_n(s) - u^\varepsilon(s) \rangle ds\]

\[+ \varepsilon \sum_{k=1}^{\infty} \int_0^t \| \sigma_k(\varepsilon s, u^\varepsilon_n(s)) - \sigma_k(\varepsilon s, u^\varepsilon(s)) \|^2_{\mathcal{H}} ds\]

\[+ 2\sqrt{\varepsilon} \sum_{k=1}^{\infty} \int_0^t \langle (\sigma_k(\varepsilon s, u^\varepsilon_n(s)) - \sigma_k(\varepsilon s, u^\varepsilon(s))), u^\varepsilon_n(s) - u^\varepsilon(s) \rangle dW_k(s), \tag{4.52}\]

for all \( t \in [0, T] \), \( P \)-a.s. One can estimate \( 2\beta \varepsilon \langle \mathcal{C}(u^\varepsilon_n) - \mathcal{C}(u^\varepsilon), u^\varepsilon_n - u^\varepsilon \rangle \) as (see (2.9))

\[2\beta \varepsilon \langle \mathcal{C}(u^\varepsilon_n) - \mathcal{C}(u^\varepsilon), u^\varepsilon_n - u^\varepsilon \rangle \geq \beta \varepsilon \| u^\varepsilon_n - u^\varepsilon \|^2_{\mathcal{H}} + \beta \varepsilon \| u^\varepsilon - u^\varepsilon \|^2_{\mathcal{H}}. \tag{4.53}\]

A calculation similar to (2.17) gives

\[-2\varepsilon \langle B(u^\varepsilon_n) - B(u^\varepsilon), u^\varepsilon_n - u^\varepsilon \rangle \leq \mu \varepsilon \| \nabla(u^\varepsilon_n - u^\varepsilon) \|^2_{\mathcal{H}} + \beta \varepsilon \| u^\varepsilon - u^\varepsilon \|^2_{\mathcal{H}} + \varepsilon \eta \| u^\varepsilon_n - u^\varepsilon \|^2_{\mathcal{H}}, \tag{4.54}\]

where \( \eta \) is defined in (2.13). Combining (4.41), (4.53)-(4.54) and then substituting it in (4.52), we find

\[
\| u^\varepsilon_n(t) - u^\varepsilon(t) \|^2_{\mathcal{H}}
\]
\[ \leq \| u_0^n - u_0 \|_{H^2}^2 + \varepsilon(\eta + L) \int_0^t \| u_n^\varepsilon(s) - u^\varepsilon(s) \|_{H^2}^2 ds \]
\[ + 2\sqrt{\varepsilon} \sum_{k=1}^{\infty} \int_0^t \left( \left| \left( \sigma_k(\varepsilon s, u_n^\varepsilon(s)) - \sigma_k(\varepsilon s, u^\varepsilon(s)), u_n^\varepsilon(s) - u^\varepsilon(s) \right) dW_k(s) \right| \right), \quad (4.55) \]
where we have used Hypothesis 3.1 (H.2) also. A calculation similar to (4.45) yields
\[ \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| u_n^\varepsilon(t) - u^\varepsilon(t) \|_{H^2}^{2q} \right] \right\}^{\frac{1}{q}} \leq 2 \| u_0^n - u_0 \|_{H^2}^{4} \leq 2^2 \varepsilon + (\eta + L)^2 \int_0^T \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| u_n^\varepsilon(s) - u^\varepsilon(s) \|_{H^2}^{2q} \right] \right\}^{\frac{2}{q}} dt \]
\[ + 8C\varepsilon q L^2 \int_0^T \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t} \| u_n^\varepsilon(s) - u^\varepsilon(s) \|_{H^2}^{2q} \right] \right\}^{\frac{2}{q}} dt. \quad (4.56) \]
An application of Gronwall’s inequality gives
\[ \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| u_n^\varepsilon(t) - u^\varepsilon(t) \|_{H^2}^{2q} \right] \right\}^{\frac{2}{q}} \leq 2 \| u_0^n - u_0 \|_{H^2}^{4} e^{2\varepsilon^2(\eta + L)^2 + 8C\varepsilon q L^2}. \quad (4.57) \]

Arguing similarly as in (4.36), one can complete the proof.

For \( r = 4 \) and \( 2\beta \mu \geq 1 \), one can estimate \(-2\varepsilon \langle B(u_n^\varepsilon) - B(u^\varepsilon), u_n^\varepsilon - u^\varepsilon \rangle \) as (see (2.20))
\[ -2\varepsilon \langle B(u_n^\varepsilon) - B(u^\varepsilon), u_n^\varepsilon - u^\varepsilon \rangle \leq 2\mu \varepsilon \| \nabla (u_n^\varepsilon - u^\varepsilon) \|_{H^2}^2 + \frac{\varepsilon}{2\mu} \| \nabla \varepsilon \|_{H^2}^2 \| u_n^\varepsilon - u^\varepsilon \|_{H^2}^2, \quad (4.58) \]
and the proof can be completed in a similar way as in the case of \( p \geq 2 \) and \( r > 4 \).

**Remark 4.7.** For the case \( p \geq 2 \) and \( r \geq 4 \), it can be easily seen that the stopping time arguments are not needed in the proof of Lemma 4.6.

**Lemma 4.8.** For any \( \delta > 0 \), and every positive integer \( n \),
\[ \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P} \left\{ \sup_{0 \leq t \leq T} \| u_n^\varepsilon(t) - v_n^\varepsilon(t) \|_{H^2}^2 > \delta \right\} = -\infty. \quad (4.59) \]

**Proof.** For \( M > 0 \), let us define the following stopping times:
\[ \tau_{\varepsilon,M}^{n,1} := \inf_{t \geq 0} \left\{ t : \varepsilon \int_0^t \| u_n^\varepsilon(s) \|_{V^2}^2 + \| u_n^\varepsilon(s) \|_{H^2}^p ds > M \text{ or } \| u_n^\varepsilon(t) \|_{H^2}^2 > M \right\}, \]
\[ \tau_{\varepsilon,M}^{n,2} := \inf_{t \geq 0} \left\{ t : \| \nabla v_n^\varepsilon(t) \|_{L^2}^2 > M \text{ or } \| \nabla v_n^\varepsilon(t) \|_{L^p}^p > M \text{ or } \| v_n^\varepsilon(t) \|_{L^q}^q > M \right\}, \]
and \( \tau_{\varepsilon,M} = \tau_{\varepsilon,M}^{n,1} \wedge \tau_{\varepsilon,M}^{n,2} \). Applying Itô’s formula to \| u_n^\varepsilon(\cdot) - v_n^\varepsilon(\cdot) \|^2_{H^2}, \) we get
\[ \| u_n^\varepsilon(t \wedge \tau_{\varepsilon,M}) - v_n^\varepsilon(t \wedge \tau_{\varepsilon,M}) \|^2_{H^2} - 2\mu \varepsilon \int_0^{t \wedge \tau_{\varepsilon,M}} \langle A(\nabla u_n^\varepsilon(s)) - A(\nabla v_n^\varepsilon(s)), u_n^\varepsilon(s) - v_n^\varepsilon(s) \rangle ds \]
\[ + 2\beta \varepsilon \int_0^{t \wedge \tau_{\varepsilon,M}} \langle C(\nabla u_n^\varepsilon(s)) - C(\nabla v_n^\varepsilon(s)), u_n^\varepsilon(s) - v_n^\varepsilon(s) \rangle ds \]
\[ = 2\mu \varepsilon \int_0^{t \wedge \tau_{\varepsilon,M}} \langle A(\nabla u_n^\varepsilon(s)), u_n^\varepsilon(s) - v_n^\varepsilon(s) \rangle ds - 2\varepsilon \int_0^{t \wedge \tau_{\varepsilon,M}} \langle B(u_n^\varepsilon(s)), u_n^\varepsilon(s) - v_n^\varepsilon(s) \rangle ds. \]
\[
-2\beta\varepsilon\int_0^{t\wedge\tau^n_{\varepsilon,M}} \langle C(\nu^n_s), \nu^n_s - \nu^n(s) \rangle ds + 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \langle f(\varepsilon s), \nu^n(s) - \nu^n(s) \rangle ds
\]
\[
+\varepsilon \sum_{k=1}^{\infty} \int_0^{t\wedge\tau^n_{\varepsilon,M}} ||\sigma_k(\varepsilon s, \nu^n(s)) - \sigma_k(\varepsilon s, \nu^n(s))||^2_{H} ds
\]
\[
+2\sqrt{\varepsilon} \sum_{k=1}^{\infty} \int_0^{t\wedge\tau^n_{\varepsilon,M}} ((\sigma_k(\varepsilon s, \nu^n(s)) - \sigma_k(\varepsilon s, \nu^n(s))), \nu^n(s) - \nu^n(s)) dW_k(s),
\]
for all \(t \in [0, T]\), \(P\)-a.s. Note that
\[
\langle B(u), u - v \rangle = \langle B(u - v, u) \rangle - \langle B(v, u - v), v \rangle,
\]
for all \(u, v \in \mathbb{V}_2\). Using (2.9) and (2.23) in (4.60), we obtain
\[
\|\nu^n(t \wedge \tau^n_{\varepsilon,M}) - \nu^n(t \wedge \tau^n_{\varepsilon,M})\|_{H}^2 + \mu\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \|\nu^n(s) - \nu^n(s)\|_{H}^2 ds
\]
\[
+\frac{\mu\varepsilon}{2} \int_0^{t\wedge\tau^n_{\varepsilon,M}} \|\nabla \nu^n(s) - \nabla \nu^n(s)\|_{H}^2 ds
\]
\[
+\frac{\mu\varepsilon}{2} \int_0^{t\wedge\tau^n_{\varepsilon,M}} \|\nabla \nu^n(s) - \nabla \nu^n(s)\|_{H}^2 ds
\]
\[
+\beta\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \|\nu^n(s) - \nu^n(s)\|_{H}^2 ds
\]
\[
+\beta\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \|\nu^n(s) - \nu^n(s)\|_{H}^2 ds
\]
\[
\leq -2\mu\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} ((1 + |\nabla \nu^n(s)|^2)\frac{2\varepsilon^2}{2} \nabla \nu^n(s), \nabla \nu^n(s) - \nabla \nu^n(s)) ds
\]
\[
-2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \langle B(u^n(s) - v^n(s), u^n(s), u^n(s) - v^n(s)) ds
\]
\[
+2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \langle B(v^n(s), u^n(s) - v^n(s)), v^n(s) ds
\]
\[
-2\beta\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \langle C\nu^n(s), u^n(s) - v^n(s) ds + 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \langle f(\varepsilon s), u^n(s) - v^n(s) ds
\]
\[
+\varepsilon \sum_{k=1}^{\infty} \int_0^{t\wedge\tau^n_{\varepsilon,M}} ||\sigma_k(\varepsilon s, u^n(s)) - \sigma_k(\varepsilon s, u^n(s))||^2_{H} ds
\]
\[
+2\sqrt{\varepsilon} \sum_{k=1}^{\infty} \int_0^{t\wedge\tau^n_{\varepsilon,M}} ((\sigma_k(\varepsilon s, u^n(s)) - \sigma_k(\varepsilon s, u^n(s))), u^n(s) - v^n(s)) dW_k(s).
\]

We estimate \(-2\mu\varepsilon ((1 + |\nabla \nu^n|^2)\frac{2\varepsilon^2}{2} \nabla \nu^n(s), \nabla \nu^n(s) - \nabla \nu^n(s))\) as
\[
-2\mu\varepsilon ((1 + |\nabla \nu^n|^2)\frac{2\varepsilon^2}{2} \nabla \nu^n(s), \nabla \nu^n(s) - \nabla \nu^n(s))
\]
\[
\leq 2\mu\varepsilon ((1 + |\nabla \nu^n|^2)\frac{2\varepsilon^2}{2} \nabla \nu^n(s), ((1 + |\nabla \nu^n|^2)\frac{2\varepsilon^2}{2} \nabla \nu^n(s)) ds
\]

Combining (4.62)-(4.66) and then substituting it in (4.61), we find
\[
\begin{align*}
\text{where} \\
(2.28) \text{ gives} \\
\text{Case 1:} \quad p \geq \frac{d}{2} + 1 \quad \text{and} \quad r \geq 2. \quad \text{For the case } p \geq \frac{d}{2} + 1 \quad \text{and} \quad r \geq 2, \quad \text{an estimate similar to (2.28)} \quad \text{gives}
\end{align*}
\]
where \( \tilde{C}_{p,\mu} = 2^{\frac{d}{p-1}} \left( \frac{(p-1)}{p} \right) \left( \frac{1}{\mu} \right)^{\frac{1}{p-1}} \).

\[
-2\beta \varepsilon \langle C(\nu^\varepsilon_n), u^\varepsilon_n - v^\varepsilon_n \rangle \leq 2\beta \varepsilon \|v^\varepsilon_n\|_{\mathbb{L}^r}\|v^\varepsilon_n - v^\varepsilon_n\|_{\mathbb{H}} \\
\leq \beta \varepsilon \|v^\varepsilon_n\|_{\mathbb{L}^r}^2 + \beta \varepsilon \|v^\varepsilon_n\|_{\mathbb{L}^r}^2,
\]

\[
2\varepsilon \langle f, u^\varepsilon_n - v^\varepsilon_n \rangle \leq 2\varepsilon \|f\|_{\mathbb{L}^r} \|u^\varepsilon_n - v^\varepsilon_n\|_{\mathbb{L}^r} \leq \frac{\mu \varepsilon}{2^{p-1}} \|u^\varepsilon_n - v^\varepsilon_n\|_{\mathbb{L}^r} + \tilde{C}_{p,\mu} \varepsilon \|f\|_{\mathbb{L}^r}^{\frac{p}{p-1}},
\]

Combining (4.62)-(4.66) and then substituting it in (4.61), we find
\[
\begin{align*}
\|u^\varepsilon_n(t \wedge \tau^n_{\varepsilon,M}) - v^\varepsilon_n(t \wedge \tau^n_{\varepsilon,M})\|_{\mathbb{H}}^2 \\
\leq 2^{p+1} \mu \varepsilon \int_0^{t \wedge \tau^n_{\varepsilon,M}} (\|\nabla v^\varepsilon_n(s)\|_{\mathbb{H}}^2 + \|\nabla v^\varepsilon_n(s)\|_{\mathbb{L}^p}^p) ds + \beta \varepsilon \int_0^{t \wedge \tau^n_{\varepsilon,M}} \|v^\varepsilon_n(s)\|_{\mathbb{L}^r}^r ds \\
+ \tilde{C}_{p,\mu} \int_0^{t \wedge \tau^n_{\varepsilon,M}} \|f(s)\|_{\mathbb{L}^r}^p ds + \frac{\mu \varepsilon}{\mu} \int_0^{t \wedge \tau^n_{\varepsilon,M}} \|v^\varepsilon_n(s)\|_{\mathbb{L}^r}^4 ds \\
+ \tilde{C}_{p,\mu} \int_0^{t \wedge \tau^n_{\varepsilon,M}} \|u^\varepsilon_n(s)\|_{\mathbb{L}^r}^{2\mu} ds + \|u^\varepsilon_n(s) - v^\varepsilon_n(s)\|_{\mathbb{H}}^2 + \varepsilon L \int_0^{t \wedge \tau^n_{\varepsilon,M}} \|u^\varepsilon_n(s) - v^\varepsilon_n(s)\|_{\mathbb{H}}^2 ds \\
+ 2\sqrt{\varepsilon} \sum_{k=1}^{\infty} \int_0^{t \wedge \tau^n_{\varepsilon,M}} |(\sigma_k(\varepsilon s, u^\varepsilon_n(s)) - \sigma_k(\varepsilon s, v^\varepsilon_n(s)), u^\varepsilon_n(s) - v^\varepsilon_n(s))| dW_k(s) \bigg|,
\end{align*}
\]
Let us define $M. T. MOHAN$

$$
+ C_{p, \mu} \int_0^{\tau^n_{\varepsilon,M}} ||f(s)||^\frac{p}{p-1} ds + \frac{C \varepsilon}{\mu} \int_0^{\tau^n_{\varepsilon,M}} ||v^\varepsilon_n(s)||^\frac{p}{p-2} ds
$$

$$
+ 2\sqrt{\varepsilon} \sum_{k=1}^{\infty} \int_0^{\tau^n_{\varepsilon,M}} (|\sigma_k(\varepsilon, s, u^\varepsilon_n(s)) - \sigma_k(\varepsilon, s, v^\varepsilon_n(s))|, u^\varepsilon_n(s) - v^\varepsilon_n(s)) dW_k(s)
$$

$$
\times \exp \left\{ \varepsilon L + \tilde{\eta} \varepsilon t \left( \int_0^{\tau^n_{\varepsilon,M}} ||u^\varepsilon_n(s)||^\frac{p}{p-2} ds \right)^{2p-2} \right\}
$$

$$
\leq e^{\varepsilon L + \tilde{\eta} \varepsilon t} 2^{p-2} \left( \int_0^{\tau^n_{\varepsilon,M}} ||u^\varepsilon_n(s)||^\frac{p}{p-2} ds \right)^{2p-2} \left\{ 2^{p+1} \mu \varepsilon M t + \beta \varepsilon M t + C_{p, \mu} \int_0^{\tau^n_{\varepsilon,M}} ||f(s)||^\frac{p}{p-1} ds + \frac{C \varepsilon M^2 t}{\mu} \right\}
$$

$$
+ 2\sqrt{\varepsilon} \sum_{k=1}^{\infty} \int_0^{\tau^n_{\varepsilon,M}} (|\sigma_k(\varepsilon, s, u^\varepsilon_n(s)) - \sigma_k(\varepsilon, s, v^\varepsilon_n(s))|, u^\varepsilon_n(s) - v^\varepsilon_n(s)) dW_k(s) \right\}, \quad (4.68)
$$

for $p \geq 1 + \frac{d}{2}$, where we have used the definition of stopping time also. Using the similar techniques as in Lemma 4.5 we obtain

$$
\left[ \mathbb{E} \left( \sup_{0 \leq s \leq \tau^n_{\varepsilon,M}} ||u^\varepsilon_n(s) - v^\varepsilon_n(s)||^2_{H^2} \right) \right]^{\frac{2}{2}}
$$

$$
\leq e^{2\varepsilon L + \tilde{\eta} \varepsilon t} 2^{p-2} \left( \int_0^{\tau^n_{\varepsilon,M}} ||u^\varepsilon_n(s)||^\frac{p}{p-2} ds \right)^{2p-2} \left\{ 2^{p+3} \mu^2 \varepsilon^2 M^2 T^2 + 2 \beta^2 \varepsilon^2 M^2 T^2
$$

$$
+ 2C_{p, \mu} \left( \int_0^{\tau^n_{\varepsilon,M}} ||f(s)||^\frac{p}{p-1} ds \right)^2 \right\}^{\frac{2}{2}} + \frac{C \varepsilon M^4 T^2}{\mu^2}
$$

$$
+ 16 Cq \varepsilon L \int_0^{\tau^n_{\varepsilon,M}} \left[ \mathbb{E} \left( \sup_{0 \leq r \leq \tau^n_{\varepsilon,M}} ||u^\varepsilon_n(r) - v^\varepsilon_n(r)||^2_{H^2} \right) \right]^{\frac{2}{2}} ds \right\}. \quad (4.69)
$$

Let us define $C_{\varepsilon,M,L,T} = e^{2\varepsilon L + \tilde{\eta} \varepsilon t} 2^{p-2} \left( \int_0^{\tau^n_{\varepsilon,M}} ||u^\varepsilon_n(s)||^\frac{p}{p-2} ds \right)^{2p-2} \left\{ 2^{p+3} \mu^2 \varepsilon^2 M^2 T^2 + 2 \beta^2 \varepsilon^2 M^2 T^2
$$

$$
+ 2C_{p, \mu} \left( \int_0^{\tau^n_{\varepsilon,M}} ||f(s)||^\frac{p}{p-1} ds \right)^2 \right\}^{\frac{2}{2}} + \frac{C \varepsilon M^4 T^2}{\mu^2} e^{C_{\varepsilon,M,L,T}}. \quad (4.70)
$$

From Lemmas 4.3 and 4.4 we infer that, for any $R > 0$, there exists an $M > 0$ such that

$$
\sup_{0 \leq \varepsilon \leq 1} \varepsilon \log \mathbb{P} \left\{ \left( ||u^\varepsilon_n||_{H_p^2}(T) \right)^2 > M \right\} \leq -R, \quad (4.71)
$$

$$
\sup_{0 \leq \varepsilon \leq 1} \varepsilon \log \mathbb{P} \left\{ \sup_{0 \leq t \leq T} ||v^\varepsilon_n(t)||_{V^2_{2}} > M \right\} \leq -R, \quad (4.72)
$$
Combining (4.71)-(4.74) and (4.76), we infer that there exists a constant \( \eta \) any

Therefore, there exists an \( \varepsilon_0 \) such that for any \( \varepsilon \) satisfying \( 0 < \varepsilon \leq \varepsilon_0 \),

Combining (4.71)-(4.74) and (4.76), we infer that there exists a constant \( \varepsilon_0 \) such that for any \( \varepsilon \) satisfying \( 0 < \varepsilon \leq \varepsilon_0 \),

The arbitrariness of \( R \) completes the proof.

**Case 2:** \( p \geq 2 \) and \( r \geq 4 \). For \( p \geq 2 \) and \( r \geq 4 \), an estimate similar to (2.17) yields

where \( \eta_1 = \frac{(r-4)}{(r-2)} \left( \frac{8}{(\beta \mu (r-2))} \right)^{\frac{2}{r}} \). For \( p \geq 2 \), \( \beta \mu > 1 \) and \( r = 4 \), we estimate

where

\[
-2\varepsilon \langle B(\mathbf{u}_n^\varepsilon - \mathbf{v}_n^\varepsilon, \mathbf{u}_n^\varepsilon - \mathbf{v}_n^\varepsilon) \rangle \leq \theta \mu \varepsilon \left\| \mathbf{u}_n^\varepsilon - \mathbf{v}_n^\varepsilon \right\|_{L^2}^2 + \frac{\varepsilon}{\theta \mu} \left\| \mathbf{u}_n^\varepsilon \left( \mathbf{u}_n^\varepsilon - \mathbf{v}_n^\varepsilon \right) \right\|_{L^2}^2.
\]
for some $0 \leq \theta < 1$. The rest of the proof can be completed in a similar fashion as in the case of $p \geq \frac{d}{2} + 1$ and $r \geq 2$.

\textbf{Proof of (4.14).} Making use of Lemmas 4.5 and 4.6, we have for any $R > 0$, there exists an $N_0$ such that

$$P\left\{ \sup_{0 \leq t \leq T} \| u^\varepsilon (t) - u^\varepsilon_{N_0} (t) \|_H^2 > \delta \right\} \leq e^{-R/\varepsilon}, \text{ for any } \varepsilon \in (0, 1],$$

and

$$P\left\{ \sup_{0 \leq t \leq T} \| v^\varepsilon (t) - v^\varepsilon_{N_0} (t) \|_H^2 > \delta \right\} \leq e^{-R/\varepsilon}, \text{ for any } \varepsilon \in (0, 1],$$

From Lemma 4.8 we infer that for such $N_0$, there exists na $\varepsilon_0$ such that

$$P\left\{ \sup_{0 \leq t \leq T} \| u^\varepsilon_{N_0} (t) - v^\varepsilon_{N_0} (t) \|_H^2 > \delta \right\} \leq e^{-R/\varepsilon}, \text{ for any } \varepsilon \in (0, \varepsilon_0],$$

Therefore, for any $\varepsilon \in (0, \varepsilon_0]$, we deduce that

$$P\left\{ \sup_{0 \leq t \leq T} \| u^\varepsilon (t) - v^\varepsilon (t) \|_H^2 > \delta \right\} \leq 3e^{-R/\varepsilon}.$$

The arbitrariness of $R$ implies that

$$\lim_{\varepsilon \to 0} \varepsilon \log P\left\{ \sup_{0 \leq t \leq T} \| u^\varepsilon (t) - v^\varepsilon (t) \|_H^2 > \delta \right\} = -\infty,$$

which completes the proof. ■

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