Basis dependence of approximative energy levels in a strongly driven two-level system

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Abstract. We introduce the diabatic and the adiabatic bases of a strongly driven superconducting two-level system. The multiphoton resonance conditions, the Rabi couplings, and the energy levels are calculated in both bases using the rotating wave approximation, and the results are compared with the corresponding numerical values. We show the basis dependence of the approximate energy levels and deduce the validity regions for both bases. We demonstrate a parameter region where neither of the bases is sufficient for calculations in the rotating wave approximation.

1. Introduction

In a typical Landau-Zener-problem \cite{1, 2}, it is considered a two-level system having an avoided crossing in energy levels as a function of external parameters, see Fig. 1a. Starting far from the avoided crossing and passing it slowly by changing the parameters adiabatically, the system will remain in its initial state (e.g. the adiabatic ground state) in accordance with the adiabatic theorem \cite{3}. In contrast, if the avoided crossing is passed fast, it happens a partial or full tunneling from the initial state to the other state (e.g. from the adiabatic ground state to the adiabatic excited state). Thus, the system is preferably described with the adiabatic eigenstates in the slow passage and in the fast passage the more natural basis would be the diabatic one. We introduce the diabatic and the adiabatic bases in a strongly driven two-level system and study the applicability of them in the calculation of energy levels in the rotating wave approximation (RWA).

Our quantum two-level system is a single-Cooper-pair transistor driven with a microwave field via the non-linear Josephson energy (see Ref. \cite{4} and Fig. 1b). When $E_{J0}$ and $d$ denote the total Josephson energy and the asymmetry, respectively, the Hamiltonian in the charge basis is

$$\hat{H}_q(t) = -\frac{E_{J0}}{2} [\hat{\sigma}_x \cos \varphi(t) - \hat{\sigma}_y d \sin \varphi(t)].$$

(1)

The magnetic flux $\Phi$ penetrating the superconducting loop is written in the dimensionless units as $\varphi = e\Phi/h$. The flux is driven so that $\varphi(t) = \varphi_b + \varphi_L \cos \omega_L t$, where $\varphi_b$ is the time-independent dc-bias and the time-dependency is characterized with the amplitude $\varphi_L$ and the angular frequency $\omega_L$, which all can be controlled experimentally.

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Figure 1. (a) The diabatic $E_{\phi}^{\dagger}(\phi)$ (---) and the adiabatic $E_{\pm}(\phi)$ (-----) eigenenergies are shown as a function of flux $\phi$. The cyclic time-dependence of the flux $\phi(t)$ is visualized in the case of passing of the avoided crossing. (b) Schematic of the single-Cooper-pair transistor qubit consisting of a superconducting loop interrupted by two Josephson junctions that separate a small superconducting island. The gate voltage $V_g$ is adjusted so that the two charge states $|0\rangle$ and $|1\rangle$ (eigenstates of $\hat{\sigma}_z$) differing by one Cooper pair are electrostatically degenerate.

### 2. The Hamiltonian

Let us first study $\hat{H}_q$ (1) by ignoring the time-dependence. In the limit of small asymmetry $d$, we may diagonalize the $\hat{\sigma}_z$-term and treat the asymmetry-part as a weak perturbation. We call the eigenstates of the $\hat{\sigma}_z$-term the diabatic basis and $E_{\pm}^{\dagger}(\phi) = \pm \frac{E_{J0}}{2} \cos \phi$ denote the corresponding unperturbed energies, (cf. Fig. 1a). In contrast, if $d$ can not be assumed small, we may diagonalize the total $\hat{H}_q$ (1). We refer to the eigenstates of $\hat{H}_q$ as the adiabatic basis and $E_{\pm}(\phi) = \pm \frac{E_{J0}}{2} \sqrt{\cos^2 \phi + d^2 \sin^2 \phi}$ denote the corresponding eigenenergies, (cf. Fig. 1a).

By taking into account the time-dependence in $\phi(t)$, $\hat{H}_q$ (1) is given in the charge basis as

$$\hat{H}_q(t) = -\frac{E_{J0}}{2}(\cos(\phi_L \cos \omega_L t)[\hat{\sigma}_x \cos \phi_b - \hat{\sigma}_y d \sin \phi_b] - 2 \sin(\phi_L \cos \omega_L t)[\hat{\sigma}_x \sin \phi_b + \hat{\sigma}_y d \cos \phi_b]).$$

When the amplitude is approximately restricted to values $\phi_L \lesssim 1$, we, aiming for a qualitative understanding, further reduce the nested sinusoidal terms using the Jacobi-Anger expansion [5]

$$\hat{H}_q(t) = -\frac{E_{J0}}{2} J_0(\phi_L) [\hat{\sigma}_x \cos \phi_b - \hat{\sigma}_y d \sin \phi_b] - 2 J_1(\phi_L) \cos(\omega_L t)[\hat{\sigma}_x \sin \phi_b + \hat{\sigma}_y d \cos \phi_b],$$

(3)

where $J_i(\phi_L)$ denote Bessel functions. Our main intention is to bring $\hat{H}_q(t)$ (3) in a form where it could be analyzed in terms of RWA. This procedure is shown in the adiabatic basis in Ref. [4], whereas we will shortly rephrase it here using the diabatic basis.

First, we express $\hat{H}_q$ (3) in the diabatic basis. The residual time-dependence in the longitudinal ($\hat{\sigma}_x$) direction is transformed into the transverse ($\hat{\sigma}_x$) direction by using the unitary transformation $\hat{U}(t) = \exp \left( -i E_{J0} J_1(\phi_L) \sin \phi_b \sin(\omega_L t) \hat{\sigma}_x / \hbar \omega_L \right)$. The resulting effective Hamiltonian $\hat{H}_{\text{ldd}} = \hat{U}^{\dagger} \hat{H} U + i\hbar (\partial_t \hat{U}^{\dagger}) \hat{U}$ can be expressed using the longitudinally dressed diabatic (ldd) basis states as

$$\hat{H}_{\text{ldd}} = \frac{\hbar}{2} \left[ \omega_{\text{ldd}} \hat{\sigma}_z + \sum_{k=-\infty}^{\infty} \left( \Omega_k^{\text{ldd}} \hat{\sigma}_+ + \Omega_{-k}^{\text{ldd}} \hat{\sigma}_- \right) \right] e^{i \omega_L t},$$

(4)
the avoided crossing better follows the numerical resonance tracks. The oscillation either does not reach $H$ in the RWA, 3. The rotating wave approximation the Rabi couplings $\omega$ numerical solution for the quasienergy splitting $\Delta$, (a) Diabatic (blue) and adiabatic (green) resonance condition overlaid on the full (b) The diabatic (blue) $\Omega_{\text{ldd}}$ and adiabatic (green) $\Omega_{\text{lda}}$ coupling strengths followed on the corresponding resonance conditions with the marker coding: $k = 1$ (——), 2 (—— — — —), and 3 (—— — —). The calculations are done with the parameters [4]: $E_{30}/\hbar = 27.0$ GHz, $d = 0.19$, and $\omega_L/2\pi = 6.11$ GHz. The gray dotted lines denote the locations of the panels in Fig. 3a-b.

where the energy splitting $\hbar \omega_{\text{ldd}} = (E_1(\varphi_b) - E_L(\varphi_b))J_0(\varphi_L)$ and the Rabi couplings

$$\hbar \Omega_{\text{ldd}}^{\text{ldd}} = (dE_{10}J_0(\varphi_L) \sin \varphi_b + \hbar \omega_L d \cot \varphi_b) J_k \left( \frac{2E_{30}J_1(\varphi_L) \sin \varphi_b}{\hbar \omega_L} \right). \quad (5)$$

Using the longitudinally dressed adiabatic (lda) basis states, we obtain $\hat{H}_{\text{lda}}$ of the exactly same form as $\hat{H}_{\text{ldd}}$ (4), but expressed with the energy splitting $\hbar \omega_{\text{lda}} = (E_+(\varphi_b) - E_-(\varphi_b))J_0(\varphi_L)$ and the Rabi couplings

$$\hbar \Omega_{\text{lda}}^{\text{lda}} = \frac{2d\hbar \omega_L}{\sin \varphi_b (d^2 - 1)} J_k \left( \frac{J_1(\varphi_L)E_{10}(d^2 - 1) \sin(2\varphi_b)}{\hbar \omega_L \sqrt{\cos^2 \varphi + d^2 \sin^2 \varphi}} \right). \quad (6)$$

3. The rotating wave approximation

In the RWA, $\hat{H}_{\text{ldd}}$ (4) is brought to the following form and all but the resonant term is ignored

$$\hat{H}_{\text{ldd}}(t) = \frac{\hbar}{2} \left( \sum_{k=\infty}^{\infty} \Omega_k^{\text{ldd}} e^{-i(\omega_{\text{ldd}} - \omega_L)t} \hat{\sigma}_+ + \text{h.c.} \right) \approx \frac{\hbar}{2} \left( \Omega_k^{\text{ldd}} e^{-i(\omega_{\text{ldd}} - \omega_L)t} \hat{\sigma}_+ + \text{h.c.} \right), \quad (7)$$

which can be done when the multiphoton resonance condition holds $\omega_{\text{ldd}} \approx k \omega_L$. This provides the first test for the bases. The superiority between the bases is deduced by comparing the resonance conditions separately with their numerical counterparts (Fig. 2a). In the limit of small driving amplitude $\varphi_L \lesssim 0.5$ and near the avoided crossing of $\varphi_L = \pi/2$, the adiabatic basis (green) better follows the numerical resonance tracks. The oscillation either does not reach the avoided crossing $|\varphi_L - \pi/2| > \varphi_L$ or the passing through the avoided crossing is slow with these parameters. However, with moderate driving amplitudes $\varphi_L \gtrsim 0.5$, the diabatic resonance condition (blue) more closely follows the numerics in the whole range of the bias $\varphi_L$. In this parameter region, the system operates in the fast passage limit and follows the diabatic states.

In addition to resonance conditions, it is useful to compare the Rabi couplings $\Omega_k^{\text{ldd}}$ and $\Omega_k^{\text{lda}}$ (Fig. 2b). In the both bases, the couplings clearly exceed the validity boundary $\Omega_k/\omega_L \approx 0.1$ of
Figure 3. Energy levels calculated near the resonances $k = 1, 2$ with the parameters of Fig. 2 at the driving amplitudes: (a) $\varphi_L = 0.1$ and (b) $\varphi_L = 1.0$. The blue and green color denote the diabatic and adiabatic basis, respectively. We use the following markings for the different energy levels: dressed $\pm \hbar \omega_{\text{d},d}/2$ (– – –), RWA $E_{\pm}^{\text{d},d}$ (○), full numerical $\pm \Delta q/2$ (——).

the RWA. However, one can deduce that the diabatic couplings are in general smaller than the adiabatic ones, which favors the superiority of the diabatic basis, especially when $\varphi_L \gtrsim 0.5$.

The eigenvalues $E_{\pm}^{\text{d},d} = \pm \frac{\hbar}{2} \sqrt{(\omega_{\text{d},d} - k\omega_L)^2 + (\Omega_k^{\text{d},d})^2}$ of $\hat{H}_{\text{d},d}$ (7) in the RWA are shown in Fig. 3 in the both bases together with the energy levels of the dressed states and the full numerical solution. With low driving amplitude $\varphi_L$, the adiabatic RWA works perfectly while the diabatic energies clearly fail (Fig. 3a). In the high driving amplitude, the diabatic dressed states give nearly correct the resonance condition (Fig. 3b). Though, the RWA does not work in either basis. The deviations of the RWA solutions from the full numerical solution are interpreted as the means of the generalized Bloch-Siegert shift [4]. Thus, these shifts are also basis dependent.

4. Conclusion
We have analyzed the multiphoton resonances, the Rabi couplings, and the energy levels of the strongly driven two-level system [4]. The system is naturally described using the adiabatic basis in the low driving limit and using the diabatic basis in the high driving limit. The RWA breaks down in the either both bases with large Rabi couplings, which indicates the need of higher order approximations or numerical solution [6]. We emphasize the basis dependence of the RWA.

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