Nonmicroscopic and microscopic descriptions of condensate states in the $^{12}$C and $^{16}$O nuclei

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Abstract. $^{12}$C and $^{16}$O nuclei are investigated within nonmicroscopic and microscopic theoretical frameworks, respectively. For the $^{12}$C nucleus viewed as a 3α system, 3-body Faddeev equations are solved in configuration space. Positions of the resonant states are obtained through the complex scaling method. We show that the nonlocal potential developed by Z. Papp and S. Moszkowski appears to be well-adapted to study 3α system. In particular, we show evidence for $^{12}$C states of positive-parity which share common features with the well-known $0^+_2$ Hoyle state, currently interpreted as a condensate state. For the $^{16}$O nucleus, a $^{12}$C+α multicluster generator coordinate method is used to solve the 16-nucleon problem. The $^{16}$O nucleus is described by four α cluster. We comment the difficulty to interpret broad resonant states. The phase-shift analysis of the $^{12}$C($0^+_2$) + α channel reveals the existence of two $0^+$ states located above the 4α threshold which can be interpreted as condensate states.

1. Introduction

Owing to the compactness and strong binding of the α particles, the idea that they tend to keep their own identity inside the nuclei is at the origin of cluster models. According to the description of the α as an elementary or as a composite particle, nonmicroscopic and microscopic theoretical frameworks have been developed. The first applications were devoted to the study of so-called α-particle nuclei, that is $^{4n}$X type nuclei, with nucleon number $A = 4n$. In nonmicroscopic approaches, α being boson, nuclear states are described by $n\alpha$ symmetrized wave functions. Phenomenological potentials between the α’s have therefore to be considered. In microscopic approaches, the nucleus is described at the nucleon scale. It is the analysis of the antisymmetrized wave functions described in adapted basis which allows to show evidence for clustering effects.

The transition from descriptions in terms of $A$ fermions toward those in terms of $n$ bosons is far from being obvious. A current challenge in nuclear structure is to show evidence for states in which α clustering is expected to be strongly dominant such as in so-called condensate states [1, 2]. A typical example is the $0^+_2$ in the $^{12}$C nucleus, known as the Hoyle state [1]. Condensate states being usually located near the relevant $\alpha$ breakup threshold, a proper treatment of many-body resonances is necessary in most cases. This subject remains however a difficult theoretical topic.

In this publication, we present recent calculations performed within either a nonmicroscopic theoretical framework [3] or a microscopic one [4]. The nonmicroscopic calculation based on the Faddeev equations [5] is used to study the $^{12}$C viewed as a 3α system (section 1). The...
chosen $\alpha\alpha$ interaction is the nonlocal potential developed by Z. Papp and S. Moszkowski [6]. The Complex Scaling Method (CSM) is used to analyze the resonances [7, 8]. The microscopic calculation is devoted to the $^{16}\text{O}$ nucleus (section 2). A multicluster $^{12}\text{C} + \alpha$ model is used to solve the 16-nucleon problem. The total wave functions of $^{16}\text{O}$ are fully antisymmetrized. They are obtained by the Generator Coordinate Method (GCM) combined with the Microscopic R Matrix method (MRM) [9]. This method allows an exact treatment of the asymptotic behavior of wave functions. We show the importance to calculate phase shifts in each channel of reaction to interpret broad resonances. In both approaches, we focus here on the existence of states showing a clear $\alpha\alpha$ structure interpreted here as the signature of $\alpha$ condensate states.

2. Nonmicroscopic calculations of $3\alpha$ bosonic states in $^{12}\text{C}$

One of the most important challenges in nonmicroscopic calculations is to replace the forces acting between nucleons by phenomenological potentials acting between the $\alpha$-particles. These forces must be able to simulate not only the internal structure of the composite cluster but also, and most importantly, the Pauli principle between the single nucleons. In such a context, several local and nonlocal $\alpha\alpha$ potentials have been developed [3]. Among the well known local interactions, let us cite the Ali and Bodmer potential developed in the 1960’s [10] and the B. Buck et al. potential [11]. However non-microscopic calculations based on these interactions fail to describe bound states of the $^{12}\text{C}$ nucleus and thus reveal the necessity to introduce phenomenological multi-body forces involving additional parametrization [12].

In this paper, we report on calculations performed with the nonlocal Papp and Moszkowski potential [6], which is of fish-bone type [13]. This potential represents a very accurate parametrization of the $\alpha\alpha$ system. It takes into consideration not only fully Pauli forbidden states of relative motion between the clusters but also partially Pauli forbidden states and thus may be considered as an extension of the Orthogonality Condition Model (OCM) [14].

Based on single parameter set for each partial wave this potential reproduces together the $\ell = 0, \ell = 2, \ell = 4$ $\alpha\alpha$ phase shifts up to 20 MeV, the $E_{2\alpha} = 91.6$ keV resonance in the $^{8}\text{Be}$ nucleus as well as the ground-state binding-energy of the $^{12}\text{C}$.

For the $^{12}\text{C}$ nucleus viewed as a $3\alpha$ system, 3-body Faddeev equations [5] are solved in configuration space. Resonant state energies and widths are determined by applying the CSM [7]. Complex scaling operation rotates the continuous spectra clockwise into the complex energy plane by an angle $2\theta$ (where $\theta$ is the complex scaling angle parameter used in the calculation), whereas positions of bound or resonant states remain unaffected. If $\theta > \arctan(-\text{Im}(E_{res})/\text{Re}(E_{res}))$ complex scaled wave-functions of the resonant states become exponentially convergent and thus may be described using bound-state like techniques. In Figure 1 we present spectra obtained for $0^{+}$ $^{12}\text{C}$ nucleus. Here, huge three-body matrices are not diagonalised. We are searching for the eigenvalues with the smallest $|\text{Im}(E_{res})/\text{Re}(E_{res})|$ value by using iterative methods and thus only small part of the eigenvalues are presented in this figure. We can identify two types of the continuum states which set the branches making $2\theta$ angle with the real axis: one branch is associated with three-alpha particle continuum starting from the origin, the other branches are based on the $^{8}\text{Be}+\alpha$ continuum and start from the respective positions of $^{8}\text{Be}$ states [8]. In Figure 1 the branch of the three-alpha particle continuum is difficult to dissociate from the $^{8}\text{Be}(0^{+}) + \alpha$, $^{8}\text{Be}(2^{+}) + \alpha$ branches. Since both $^{8}\text{Be}(0^{+})$ and $^{8}\text{Be}(2^{+})$ resonances are situated very close to the three-alpha particle continuum branch. On the contrary, the $^{8}\text{Be}(0^{+}) + \alpha$ branch is clearly visible. In addition to these continuum states we identify one bound and three resonant states of the $^{12}\text{C}$ nucleus.

In Figure 2, we summarize $^{12}\text{C}$ positive parity spectrum. A very interesting result related to our study is the existence of a set of states $\{0^{+}_{2}, 2^{+}_{2}, 0^{+}_{4}, 0^{+}_{6}, 4^{+}_{2}\}$ which shares common feature with the well reproduced $0^{+}_{2}$ Hoyle state. Indeed, they all show similarities in the wave functions and can be shown to belong to same rotational bands. We refer to the reference [3] for detailed
discussions. Let us just mention that for the Hoyle state we get a rms radius of 3.98 fm compatible with the analysis of electron inelastic scattering data [15]. Furthermore, the $2^+_2$ state at 4.42 MeV with a width of 0.66 MeV is interpreted as the $2^+$ member of the Hoyle-state band. Its energy is however overestimated by $\approx 2$ MeV as compared to the measure of Freer et al. [16]. We also assign the $4^+_2$ state to a possible $4^+$ member of the Hoyle-state band.

Figure 1. The $0^+ 12\text{C}$ states obtained using the CSM. The bound and resonant states of $12\text{C}$ are identified with circles. States representing discretized continuum form the bands making $2\theta$ angle with the real energy axis. In this calculation complex scaling angle $\theta = 8^\circ$ has been used.

Figure 2. Ground-State band and positive-parity condensate states (bold) obtained with the nonlocal Papp and Moszkowski potential (PM). Corresponding experimental data (Exp.) are taken from Ref. [16] for the $2^+_2$ and from Ref. [17] for the other data.

3. Microscopic calculations of $0^+$ condensate states in $^{16}\text{O}$

3.1. The multicluster model

In the present investigation, a multicluster generator coordinate method is applied to a microscopic study of the $^{16}\text{O}$ nucleus [4]. Details on the related theoretical framework and on the multicluster description can be found in Ref. [9, 18] and in references therein. We restrict ourselves to the particularities of the four-$\alpha 12\text{C}+\alpha$ model used here.

A schematic representation of the four-$\alpha$ cluster model is given in Figure 3. The $^{12}\text{C}$ is described by 3 $\alpha$ clusters located at the apexes of an isosceles triangle. The fourth cluster
is devoted to the $\alpha$ particle. The side $R_S$ and the angle $\theta$ of the isosceles triangle are the generator coordinates associated with the $^{12}\text{C}$ nucleus. The generator coordinate $R$ is defined as the distance between the center of mass of the triangle and the fourth cluster. It controls the relative motion. This model generalizes a previous approach where the $^{12}\text{C}$ nucleus was described only in terms of equilateral triangles [19].

Let $\phi_\alpha(S)$ be a Slater determinant for the $\alpha$ particle, involving $0s$ harmonic-oscillator wave functions with parameter $b = 1.36$ fm, and centered at $S$, we then consider four $\alpha$ particles located at $S_i$, where $i$ labels the $i$th apex of the multicluster model. $S_i$ is a function of the generator coordinates. In these conditions, a basic GCM $^{16}\text{O}$ basis function can be written as

$$\phi(R_S, \theta, R) = A \phi^{-1}_{\text{cm}} \phi(S1)\phi(S2)\phi(S3)\phi(S4),$$

(1)

where $A$ is the antisymmetrization operator. Here, the center of mass function $\phi_{\text{c.m.}}$ is removed. The total wave functions of the $^{16}\text{O}$ nucleus are then expressed as a sum of $^{12}\text{C}+\alpha$ multichannel functions. More precisely, in partial wave $JM\pi$, they can be written as

$$\Psi^{JM\pi} = \sum_{c,I} \int f_{cII}^{J\pi}(R) \Phi_{cII}^{J\pi}(R) \, dR,$$

(2)

where the $\Phi_{cII}^{J\pi}(R)$ functions are obtained from the $\phi(R_S, \theta, R)$ functions by usual techniques of projection. Here, label $c$ represents the various $^{12}\text{C}+\alpha$ channels. $I$ is the $^{12}\text{C}$ spin, $\ell$, the relative angular momentum, and $J = I + \ell$, the $^{16}\text{O}$ spin. In our calculations, two $I^\pi$ values are considered: $0^+$ and $2^+$. The $^{16}\text{O}$ parity is then given by $\pi = (-1)^\ell$. The generator functions $f_{cII}^{J\pi}(R_S, \theta, R)$ are determined by solving the MRM which restores the correct Coulomb behavior of the wave functions. This step is crucial to treat properly resonating states.

![Figure 3. Schematic representation of the GCM basis wave functions used in the four-alpha multicluster model.](image)

Our calculations have been performed with the Volkov V2 [20] and the Minnesota [21] nucleon-nucleon interactions. Both contain one adjustable parameter which is tuned to reproduce the $0^+_2$ state located at $E_{\text{c.m.}} = -1.11$ MeV below the $^{12}\text{C}+\alpha$ threshold. Indeed, previous studies show that this state can be well reproduced by cluster models [22]. Value of the parameter is $m = 0.63726$ for V2 and $u = 0.82225$ for Minnesota, respectively.

3.2. The $^{12}\text{C}(0^+_2) + \alpha$ channel

In practice, Eq. (2) becomes a finite sum over the generator coordinates. $R$ takes values from $R = 2.0$ fm to $R = 10.4$ fm in steps of $1.2$ fm. The generator coordinates associated with the isosceles triangles are chosen in order to optimize the $^{12}\text{C}(0^+_2) - ^{12}\text{C}(0^+_4)$ energy gap. This point is crucial. Indeed, the interpretation of $^{16}\text{O}$ resonances in terms of condensate of $\alpha$ particles is linked to the importance of the $^{12}\text{C}(0^+_4) + \alpha$ channel in the corresponding wave functions. A good description of this channel is then necessary. In our calculation, seven different triangles have been considered. The corresponding values of $R_S$ and $\theta$ are gathered in Table 1. They
Table 1. Generator coordinates of the isosceles triangles considered in the multicluster calculations.

| T   | $R_S$ (fm) | $\theta$ (degree) |
|-----|------------|-------------------|
| T1  | 2.1        | 28.1              |
| T2  | 3.0        | 100.0             |
| T3  | 4.0        | 30.0              |
| T4  | 5.0        | 45.0              |
| T5  | 6.0        | 30.0              |
| T6  | 4.0        | 110.0             |
| T7  | 1.0        | 60.0              |

have been chosen step by step in order to minimize the ground state and the gap energy. We can notice that most of the triangles are isosceles and that the chain configuration ($\theta = 180$ degree) is not selected.

The evolution of the $^{12}$C(0$^+_2$) - $^{12}$C(0$^+_1$) energy gap as a function of the number of triangles introduced in the GCM variational basis is given in Figure 4. We can notice the clear improvement obtained when the number of triangles increases. For seven triangles, the result becomes close to experiment for both interactions.

![Figure 4. Evolution of the $^{12}$C(0$^+_2$) - $^{12}$C(0$^+_1$) energy gap as a function of the number of triangles introduced in the GCM calculations.](image)

3.3. Many-body resonance analysis

In this sub-section, we comment the difficulties to interpret broad resonances. In Table 2, we show typical calculations performed with the V2 interaction for the 0$^+_1$ states. Results obtained with a bound-state approximation are given in the first column. The MRM output involving the correction of the asymptotic behavior, is given in the second column for the energies and the third for the widths, respectively. As expected, we get similar results for the two bound states. Then we get resonance energies which appear to be close to the bound-state values. The corresponding widths calculated with usual techniques based on Breit-Wigner approximation [23] are however (very) broad. The interpretation of such states as physical ones is then questionable. The way to disentangle between physical and non physical states is to compute the phase shifts in each channel of reaction. Results obtained for 0$^+_1$ partial waves obtained with V2 are given in Figure 5. The stability between six and seven triangle calculations is also tested. We find that only three resonances can be considered as physical. They are indicated with a X in Table 2. Furthermore, the analysis of reduced $\alpha$ widths $\theta^2_{\alpha}$ shows that the resonances at $E_{c.m.} = 9.742$ MeV and at $E_{c.m.} = 11.498$ MeV have dominant components in the $^{12}$C(0$^+_2$) + $\alpha$ channel. This result supports an interpretation in terms of $\alpha$ condensates, in agreement with calculations based on a 4$\alpha$ THSR wave-function by Y. Funaki et al. [24] Corresponding values are gathered in Table 3. This study illustrates the importance to compute phase shifts. It also shows that bound-state approximations are generally not well appropriate to describe resonances.
Table 2. $^{16}$O $0^+$ states obtained with a bound-state approximation (BSA) and with the MRM. The corresponding total width is given in the third column.

| $E^{BSA}_{c.m.}$ (MeV) | $E^{MRM}_{c.m.}$ (MeV) | $\Gamma$ (MeV) |
|------------------------|-------------------------|----------------|
| 1                      | -14.045                 | -14.045        |
| 2                      | -1.112                  | -1.112         |
| 3                      | 2.257                   | 2.200          |
| 4                      | 3.686                   | 3.818          |
| 5                      | 5.973                   | 6.369          |
| 6                      | 6.635                   | 6.859          |
| 7                      | 9.060                   | 9.724          |
| 8                      | 9.705                   | 10.137         |
| 9                      | 11.083                  | 11.498         |
| 10                     | 11.676                  | 13.897         |
| 11                     | 13.810                  | 13.997         |
| 12                     | 14.226                  | 15.701         |

Table 3. Three dominant reduced $\alpha$ widths (in %) for the two resonances interpreted as $\alpha$ condensate states obtained with V2.

| $0^+_I E_{c.m.}$ | $I, \ell$ Channel | $\theta^\circ_c$ |
|------------------|-------------------|------------------|
| 9.72 MeV         | $I = 0^+_1 \ell = 0$ | $6.4 \times 10^{-2}$ |
|                  | $I = 0^+_2 \ell = 0$ | $6.8 \times 10^{-2}$ |
|                  | $I = 2^+_1 \ell = 0$ | $2.4 \times 10^{-2}$ |
| 11.49 MeV        | $I = 0^+_1 \ell = 0$ | $1.5 \times 10^{-2}$ |
|                  | $I = 0^+_2 \ell = 0$ | $7.7 \times 10^{-1}$ |
|                  | $I = 2^+_1 \ell = 2$ | $4.1 \times 10^{-2}$ |

3.4. The $^{16}$O spectrum

In Figure 6, we show the $^{16}$O $0^+$ spectra obtained with our multicluster approach. The $0^+_1$ binding energy is overestimated by both the V2 and the Minnesota interactions and are not reproduced on the picture. This problem is well known in cluster model [22]. The calculated $0^+_1$ resonance at $E_{c.m.} = 2.2$ MeV (with V2) with a width of $\Gamma = 0.46$ MeV cannot be directly assigned. The calculated $0^+_1$ and $0^+_3$ resonances previously discussed as possible condensate states are located above the $4\alpha$ threshold. Experiment reveals the presence of a $0^+$ state located at $E_{c.m.} = 7.935$ MeV with a width of $\Gamma = 0.166$ MeV [25] which could correspond to one of them. Furthermore, if we compare with the band measured by Chevallier et al. [27], we find that the extrapolated $0^+$ state at $E_{c.m.} = 9.64$ MeV is very close to the lowest condensate energy obtained with both interactions.
4. Conclusion
In this paper, we have seen that both nonmicroscopic and microscopic approaches are able to show the existence of condensate states.

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