Yangian symmetry, S-matrices and Bethe ansatz for the AdS$_5$ $\times$ S$^5$ superstring

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We discuss the relation between the recently derived bound state S-matrices for the AdS$_5$ $\times$ S$^5$ superstring and Yangian symmetry. We will study the relation between this Yangian symmetry and the Bethe ansatz. In particular we can use it to derive the Bethe equations for bound states.

1 Introduction and motivation

The AdS/CFT correspondence [1] has been the subject of intensive research. One of the best studied examples of this correspondence is $\mathcal{N} = 4$ SYM gauge theory which is conjectured to be dual to superstring theory on AdS$_5$ $\times$ S$^5$. However, testing or proving the duality for these concrete models poses a very challenging problem.

A breakthrough in the understanding of this duality was the discovery of integrable structures. Integrable structures were found both in $\mathcal{N} = 4$ SYM and in the AdS$_5$ $\times$ S$^5$ string sigma model [2, 3]. Because of the huge number of symmetries, integrable theories are special. For example, in scattering processes for integrable theories, the set of particle momenta is conserved and scattering processes always factorize into a sequence of two-body interactions. This implies that in such theories the scattering information is encoded in the two-body S-matrix. However, a full proof of integrability for both $\mathcal{N} = 4$ SYM and superstrings on AdS$_5$ $\times$ S$^5$ is currently still lacking.

Nevertheless, one can assume integrability and try to exploit its consequences to solve the theory. Then one can compare the obtained results to explicit calculations afterwards. This is the current approach and it proved to be quite fruitful. By assuming integrability one can make use of the S-matrix approach. This leads to the conjecture of the “all-loop” Bethe equations describing the gauge theory asymptotic spectrum [4]. Also for the AdS$_5$ $\times$ S$^5$ superstring, a Bethe ansatz for the su(2) sector was proposed [5]. Actually, in both theories, exact two body S-matrices were found [6, 7]. These enabled the use of the Bethe ansatz [6], confirming the conjectured Bethe equations for physical states.

More precisely, the two body S-matrix is almost completely fixed by symmetry. Both the asymptotic spectrum of $\mathcal{N} = 4$ super Yang-Mills theory [6] and the light-cone Hamiltonian [8] for the AdS$_5$ $\times$ S$^5$ superstring exhibit the same symmetry algebra; centrally extended su(2|2). The requirement that the S-matrix is invariant under this algebra determines it up to an overall phase [6] and the choice of representation basis [7]. In a suitable local scattering basis, the S-matrix exhibits most properties of massive two-dimensional
integrable field theories, like crossing symmetry [9] and the Yang-Baxter equation [7]. The overall (dressing) phase appears to be a remarkable feature of the string S-matrix and it has been studied intensely, see e.g. [10].

The string sigma model also contains an infinite number of bound states [11]. These bound states fall into short (atypical) symmetric representations of the centrally extended $su(2|2)$ algebra [12]. Of course, these states scatter via their own S-matrices. Recently a number of these bound states S-matrices have been found [13, 14]. However, invariance under centrally extended $su(2|2)$ is not enough to fix all these S-matrices. One needs to impose the Yang-Baxter equation by hand if one wants to completely fix them up to a phase.

The found asymptotic Bethe ansätze only describe the spectra in the infinite volume limit. For a complete check of this particular case of the AdS/CFT correspondence, the full spectra have to be computed and compared. Away from the asymptotic region wrapping interactions appear. One way to include these is Lüscher’s perturbative approach [15]. In Lüscher’s approach one deals with corrections coming from virtual particles that propagate around the compact direction. These virtual particles can be both fundamental and bound states. This method has proven successful since it recently allowed the computation of the full spectrum of the four-loop Konishi operator with wrapping interactions [14]. The result coincided with the gauge theory computation [16], providing a very non-trivial check of the correspondence.

In this section we will briefly discuss the Yangian of $su(2|2)$ and S-matrices. Then we will sketch how Yangian symmetry enables one to find the asymptotic Bethe equations for bound states [17]. This paper is organized as follows, first of the role of Yangian symmetry in this. More specifically, we will sketch how Yangian symmetry enables crucial for a complete understanding of finite size effects. In this proceedings, we will give a short overview one deals with finite size effects by defining a mirror model [11]. Finite size effects in the original theory correspond to include all (physical) bound states. One of the advantages of this method is that one can still use the asymptotic Bethe ansatz.

In other words, knowledge of the bound states, their S-matrices and the corresponding Bethe ansätze is crucial for a complete understanding of finite size effects. In this proceedings, we will give a short overview of the role of Yangian symmetry in this. More specifically, we will sketch how Yangian symmetry enables one to find the asymptotic Bethe equations for bound states [17]. This paper is organized as follows, first of the role of Yangian symmetry in this. More specifically, we will sketch how Yangian symmetry enables one to find the asymptotic Bethe equations for bound states [17].

The approach by Lüscher is closely related to the thermodynamic Bethe ansatz (TBA). In the TBA one deals with finite size effects by defining a mirror model [11]. Finite size effects in the original theory correspond to finite temperature effects in the infinite volume for the mirror theory. Here again, one needs to include all (physical) bound states. One of the advantages of this method is that one can still use the asymptotic Bethe ansatz.

2 The Yangian of $su(2|2)$ and S-matrices

In this section we will briefly discuss the Yangian of $su(2|2)$ and bound state representations. For more details see e.g [17] and references therein.

2.1 The algebra in superspace

The superalgebra $su(2|2)$ is the symmetry algebra of the AdS$_5 \times S^5$ superstring and it also is the symmetry algebra of the spin chain connected to $N=4$ SYM. The algebra consists of bosonic generators $R, L$, generating two copies of $su(2)$, supersymmetry generators $Q, G$ and central charges $H, C, C^\dagger$. The non-trivial commutation relations are given by

\[
\begin{align*}
\{L_a^b, J_c^d\} &= \delta_c^d J_a^b - \frac{1}{2} \delta_a^b J_c^d \\
\{J_a^b, J_c^d\} &= -\delta_a^d J_b^c + \frac{1}{2} \delta_a^c J_b^d \\
\{Q_a^\alpha , Q_b^\beta \} &= \epsilon_{\alpha \beta} e^{ab} C \\
\{G_a^\alpha , G_b^\beta \} &= \epsilon_{\alpha \beta} e_{ab} C^\dagger \\
\{Q_a^\alpha , G_b^\beta \} &= \delta_a^b R_{\alpha \beta} + \delta_a^d L_{\alpha \beta} + \frac{1}{2} \delta_{a \beta} \delta_{d \alpha} H.
\end{align*}
\]

The generators $Q, G$ are conjugate. For the gauge fixed AdS$_5 \times S^5$ superstring, $H$ corresponds to the lightcone Hamiltonian and the central charge $C$ depends on the world-sheet momentum in the following way $C = ig\zeta (e^{ip} - 1)$.

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A convenient way to describe representations of bound states is the superspace formalism [13]. Consider the vector space of analytic functions of two bosonic variables $w_a$ and two fermionic variables $\theta_\alpha$. The $su(2|2)$ representation that describes $\ell$-particle bound states of the light-cone string theory on $AdS_5 \times S^5$ is $4\ell$ dimensional and is spanned by monomials of degree $\ell$. This representation is called the atypical totally symmetric representation, and it describes $\ell$-particle bound states [13]. In this representation, the algebra generators are represented by differential operators, depending on parameters, $a, b, c, d$.

\[
L^b_a = w_a \frac{\partial}{\partial w_b} - \frac{1}{2} \delta^b_a w_c \frac{\partial}{\partial w_c}, \quad \mathcal{R}^\beta_\alpha = \theta_\alpha \frac{\partial}{\partial \theta_\beta} - \frac{1}{2} \delta^\beta_\alpha \theta_\gamma \frac{\partial}{\partial \theta_\gamma},
\]

(2)

The central charges are given by

\[
\mathbb{C} = ab \left( w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right), \quad \mathbb{H} = (ad + bc) \left( w_a \frac{\partial}{\partial w_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right).
\]

(3)

To form a representation, the parameters $a, b, c, d$ must satisfy $ad - bc = 1$. One can also express these parameters in terms of particle momentum $p$ and coupling $g$:

\[
a = \sqrt{\frac{g}{2\ell}} \eta, \quad b = \sqrt{\frac{g}{2\ell}} \zeta \left( \frac{x^+}{x^-} - 1 \right), \quad c = -\sqrt{\frac{g}{2\ell}} \frac{\eta}{i\xi} \zeta x^+, \quad d = \sqrt{\frac{g}{2\ell}} \frac{x^+}{i\eta} \left( 1 - \frac{x^-}{x^+} \right),
\]

(4)

where $x^\pm$ are related to the particle momentum and coupling constant via

\[
x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2\ell}{g}, \quad \frac{x^+}{x^-} = e^{ip}.
\]

(5)

The parameters $\eta$ parameterize the scattering basis and are given by

\[
\eta = e^{i\xi} \eta(p), \quad \eta(p) = e^{ip} \sqrt{ix^- - ix^+}, \quad \zeta = e^{2i\xi}.
\]

(6)

The fundamental representation is obtained by taking $\ell = 1$.

The double Yangian $DY(g)$ of a (simple) Lie algebra $g$ is a deformation of the universal enveloping algebra $U(g[u, u^{-1}])$ of the loop algebra $g[u, u^{-1}]$. The Yangian is generated by a tower of generators $J^A_n$, $n \in \mathbb{Z}$ that satisfy the commutation relations

\[
[J^A_m, J^B_n] = F^{AB}_C J^C_{m+n} + \mathcal{O}(\hbar),
\]

(7)

where $F^{AB}_C$ are the structure constants of the Lie algebra $g$. The level 0 generators $J^A_0$ span the Lie-algebra itself. We are interested in the Yangian of centrally extended $su(2|2)$. This Yangian can be supplied with a coproduct structure [18, 19]. The coproduct of the $su(2|2)$ operators is given by:

\[
\Delta(J^A_0) = J^A_0 \otimes 1 + 1 \otimes J^A_0.
\]

(8)

We refer to [18] for explicit formulas for the Yangian generators.

An important representation of the Yangian is the evaluation representation. It consists of states $|u\rangle$, with action $J^A_n|u\rangle = u^n J^A_0 |u\rangle$. In this representation the coproduct structure is fixed in terms of the coproducts of $J^A_0, J^A_1$. We will work in this representation and identify $J^1_0 \equiv \mathbb{J} = \frac{d}{du} u |u\rangle$ for the $su(2|2)$ Yangian. One finds that $u$ is dependent on the parameters $x^\pm$ via $u_j = x^+_j + \frac{1}{x^-_j} - \frac{i}{g}$.

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2.2 S-matrices and symmetry

Requiring invariance under centrally extended $su(2|2)$ proved to be enough to fix the fundamental $S$-matrix up to a phase factor [6,7]. Symmetry invariance means that for any generator $J^A$ from $su(2|2)$, the $S$-matrix should satisfy

$$S \Delta(J^A) = \Delta^{op}(J^A) S, \quad (9)$$

where $\Delta^{op} = P \Delta$, with $P$ the graded permutation.

When one uses this to compute bound state $S$-matrices one finds that invariance under centrally extended $su(2|2)$ is no longer sufficient. In the case when one scatters two 2-particle bound states, the $S$-matrix is still dependent on an additional parameter. This parameter can be fixed by insisting that the $S$-matrix satisfies the Yang-Baxter equation [13].

However, there is an alternative to the Yang-Baxter equation to completely fix the $S$-matrix. It was shown that the fundamental $S$-matrix actually has a larger symmetry group than just $su(2|2)$, namely the Yangian of $su(2|2)$ [18]. This symmetry can again be understood in terms of coproducts of generators, i.e. the $S$-matrix satisfies

$$S \Delta(\hat{J}^A) = \Delta^{op}(\hat{J}^A) S, \quad (10)$$

Then it was also shown that the bound state $S$-matrices respect Yangian symmetry [20]. Moreover, it was found that the bound state $S$-matrices were fixed by Yangian symmetry without reference to the Yang-Baxter equation. Hence, bound state $S$-matrices, up to an overall phase, are completely fixed by invariance under the full Yangian symmetry rather than under $su(2|2)$ alone. It seems that the Yangian of centrally extended $su(2|2)$ is the fundamental symmetry group underlying the scattering processes of the $AdS_5 \times S^5$ superstring.

3 The Bethe ansatz and Yangian symmetry

For integrable systems, the number of particles and the set of momenta is conserved. Let us consider $K$ excitations with momenta $p_1, \ldots, p_K$. We are dealing with closed strings and hence we need to impose periodicity. A way to do this is by using the so-called Bethe ansatz. This gives certain restrictions on the particle momenta, formulated in terms of the Bethe equations.

In this approach one assumes that there are regions where the particle coordinates $x_i$ are far apart in the sense that the particles behave as free particles. In these asymptotic regions, one can make a plane-wave ansatz for the wave function. Consider permutations $P, Q$, then the ansatz for the wave function is of the form of a generalized Bethe ansatz [21]

$$|p_1, \ldots, p_K\rangle = \sum_P \int dx \left\{ A_{a_1, \ldots, a_K}^{P\mid Q} e^{ip_i x_j} \right\} \phi^{a_1}(x_1) \ldots \phi^{a_K}(x_K), \quad (11)$$

where $\phi^{a_i}(x_i)$ creates a particle of type $a_i$ at position $x_i$. This just corresponds to making a linear combination of plane waves for each ordering of the positions and momenta of the particles. The interactions are described by the $S$-matrix and they allow particles to cross the various regions and in this way relate the coefficients

$$A^{P\mid Q} = S_{a_1, a_j} A^{P'\mid Q'}, \quad (12)$$

where the regions $P\mid Q$ and $P'\mid Q'$ differ by permuting particles $i, j$. Therefore, the scattering data give relations on the coefficients from adjacent free regions. The Bethe equations are now of the form

$$S_{kk-1} \ldots S_{kk+1} A^{P\mid Q} = e^{ip_L} A^{P\mid Q}. \quad (13)$$
The invariance of the S-matrix under Yangian symmetry implies that
this\[6\]. The problem becomes more involved upon inserting multiple fermions. Here a new S-matrix
describing these states will depend on the particle momentum and one must check whether this construction
next thing to consider is a fermion inserted in this vacuum and treat it as an excitation. The coefficients
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should be unchanged up to a phase factor. The coordinate Bethe ansatz now proceeds by making an
ansatz for the coefficients \( A^{(i)} \) in such a way that they solve (12) and then reading off the Bethe equations
for the particle momenta.

One first defines a vacuum \( A^{(i)}|0\rangle := |0\rangle \). We restrict to the undressed S-matrix, i.e., \( S|0\rangle = |0\rangle \). The
next thing to consider is a fermion inserted in this vacuum and treat it as an excitation. The coefficients
describing these states will depend on the particle momentum and one must check whether this construction
respects (12). It turns out that this gives some restrictions on the coefficients, which can be solved from this
[6]. The problem becomes more involved upon inserting multiple fermions. Here a new S-matrix \( S^I \) is
introduced. Plugging this into (12) again allows one to find explicit (unique) solutions for the coefficients,
including \( S^I \). We can deal with \( S^I \) by using an additional Bethe ansatz. This is called nesting. One can
repeat the above procedure to deal with this. For details we refer to [6]. This procedure results in the
well-known Bethe equations describing the asymptotic spectrum of the AdS\(_5 \) \( \times \) S\(_5\) superstring:

\[
e^{ip_k L} = \prod_{l=1}^{K^I} \left[ S_0(p_k, p_l) \frac{x_k^+ - x_l^+}{x_k^+ - x_l^-} \sqrt{x_k^+ x_l^+} \right] \frac{1}{2} \prod_{\alpha=1}^{2} \prod_{l=1}^{K^I} \frac{y_{k,\alpha} - y_{l,\alpha}}{y_{k,\alpha} - y_{l,\alpha}^*} \frac{1}{\sqrt{y_{k,\alpha} - y_{l,\alpha}^*}} \frac{1}{\sqrt{x_k^+}} \frac{1}{\sqrt{x_l^+}}
\]

with \( \alpha = 1, 2 \) and \( S_0(p_k, p_l) \) the overall phase of the S-matrix.

It is important to note that in this derivation the explicit form of the S-matrix was used. However, not
all bound state S-matrices are known and hence the above procedure cannot be straightforwardly applied
to bound states.

3.1 Bethe ansatz and Yangian symmetry

The Bethe ansatz can be reformulated by considering coproducts of (Yangian) symmetry generators. This
formulation allows us to find solutions to (12) without knowing the explicit form of the bound state S-
matrix. For details and explicit formulae, we refer to [17].

Again one starts by defining a vacuum. The first non-trivial step is to consider a fermion in this vacuum.
The remarkable fact is that, when restricted to two sites, one can find functions \( K_0, K_1 \) such that one can
write this coefficient as

\[
( K_0(p_1, p_2) \Delta Q_{\alpha}^1 + K_1(p_1, p_2) \Delta Q_{\alpha}^1 ) |0\rangle.
\]

The invariance of the S-matrix under Yangian symmetry implies that

\[
S|\alpha\rangle = ( K_0(p_1, p_2) \Delta^{op} Q_{\alpha}^1 + K_1(p_1, p_2) \Delta^{op} Q_{\alpha}^1 ) |0\rangle,
\]

Fig. 1 One particle is moved around the circle, scattering
with the other particles.

These equations can be interpreted in the following way. When a particle is moved around the circle, it
meets the other particles and it scatters with them, see Fig. 1. When the particle has moved around, the
system should be unchanged up to a phase factor. The coordinate Bethe ansatz now proceeds by making an
ansatz for the coefficients \( A^{(i)}|0\rangle \) in such a way that they solve (12) and then reading off the Bethe equations
for the particle momenta.
since $\mathbb{S}(0) = |0\rangle$. However, this means that (12) corresponds to requiring that $K_0$ and $K_1$ are symmetric under interchanging $p_1 \leftrightarrow p_2$. Note that this is solely based on symmetry algebra arguments and is valid for any bound state number. In other words, (12) which appears in the Bethe ansatz proves to be closely related to the symmetry properties of the S-matrix. The symmetry properties of $K_0, K_1$ can then be used to extract information about the Bethe equations.

A similar treatment can be done for two fermions. This again gives rise to solutions for the factors that appear in the Bethe equations. In particular one finds that the auxiliary S-matrix, $S_{II}$ remains unchanged. This completely fixes the Bethe equations.

In conclusion, by making use of coproducts and Yangian symmetry, we have found a way, independent of the explicit form of the S-matrix, to find the Bethe equations of string bound states. They are explicitly given by

$$e^{ip_k L} = \prod_{l=1, l \neq k}^{K^1} \left( S_0(p_k, p_l) x_k^+ - x_l^- \sqrt{\frac{x_k^+ x_l^-}{x_l^+ x_k^-}} \right) ^2 \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{II}(\alpha)} \left( x_k^- - y_l^{(-)}(\alpha) \right) \sqrt{\frac{x_k^+}{x_k^-}}$$

which

$$1 = \prod_{l=1}^{K^1} \frac{g}{y_l^{(+)}(\alpha)} - x_l^- = \prod_{l=1}^{K^{II}(\alpha)} \frac{g}{y_k^{(+)}(\alpha)} + \frac{1}{y_l^{(-)}(\alpha)} - \frac{1}{y_k^{(-)}(\alpha)} + i/g$$

$$1 = \prod_{l=1}^{K^{II}(\alpha)} \frac{g}{y_k^{(+)}(\alpha)} + \frac{1}{y_l^{(-)}(\alpha)} - \frac{1}{y_k^{(-)}(\alpha)} - i/g$$

with

$$x_k^+ + \frac{1}{x_k^-} - x_k^- = \frac{2i\ell_k}{g}, \quad \frac{x_k^+}{x_k^-} = e^{ip_k}, \quad \alpha = 1, 2.$$  

Note that apart from the parameters $x^\pm$, the phase factor $S_0(p_k, p_l)$ also implicitly depends on the bound states number [13]. The found Bethe equations coincide with the Bethe equations one expects from a fusion procedure.

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