A Legendre pair of length 77 using complementary binary matrices with fixed marginals

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Received: 26 January 2021 / Revised: 26 February 2021 / Accepted: 1 March 2021 / Published online: 18 April 2021 © The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2021, corrected publication 2021

Abstract
We provide a search method for Legendre pairs of composite length based on generating binary matrices with fixed row and column sums from compressed, complementary integer vectors. This approach yielded the first construction of a Legendre pair of length 77, as well as the first exhaustive generation of Legendre pairs of length 55.

Keywords Fixed marginal · Legendre pair · Fourier transform

Mathematics Subject Classification 15B34 · 05B20 · 05B10

1 Background

Two vectors, $u$ and $v$, indexed by $\mathbb{Z}_\ell$ are called complementary sequences if there exists some $\lambda$ such that:

$$\sum_{j=0}^{\ell-1} v_j v_{j+g} + u_j u_{j+g} = \lambda \quad \forall g \neq 0, \ g \in \mathbb{Z}_\ell, \quad (1)$$

Communicated by V. D. Tonchev.
see [7]. Specialized classes of complementary sequences such as Golay pairs have applications in physics under multislit spectroscopy, and many telecommunication problems such as OFDM codes and ISI channel estimation [8,12,14].

A Hadamard matrix $H$ is a $n \times n$ matrix of $\pm 1$’s such that $HH^T = nI$. It is easy to show that $n > 2$ must be divisible by 4 for such a matrix to exist. Complementary sequences have long been of interest in combinatorics for generating orthogonal designs such as Hadamard matrices, with applications to error correcting codes, signals analysis, and cryptography [5,7,9].

One notoriously difficult to construct class of complementary sequences are Legendre pairs (LP). The density of a $\{0,1\}$ vector, $v$, of length $\ell$ is $\kappa = \sum_{j=0}^{\ell} v_j$. Two $\{0,1\}$ complementary sequences of length $\ell$ constitute a $\{0,1\}$–LP if $\kappa = \lambda = (\ell + 1)/2$. Common transformations of LP are listed for completeness along with the corresponding constants. $\{0,1\}$–LP with $\kappa = (\ell - 1)/2$ have $\lambda = (\ell - 3)/2$. Transforming to $\{-1,1\}$–LP requires $\kappa = \pm 1$ with $\lambda = -2$.

There exists a construction for Hadamard matrices with size $n = 2\ell + 2$ based on LP known as cocyclic Hadamard matrices [7]. Just as Hadamard matrices are conjectured to exist for all $n$ divisible by 4, LP are conjectured to exist for all odd $\ell$ [2]. The smallest unresolved case for LP has long been $\ell = 77$ [7]. Furthermore, exhaustive searches have only been conducted for $\ell \leq 45$ [7].

A circulant shift of a vector $v$ by $j \in \mathbb{Z}_\ell$, denoted by $c_j(v)$, is the permutation of indices such that $(c_j(v))_g = v_{g-j}$. A necklace is an equivalence class of vectors of length $\ell$ under circulant shifts, $j \in \mathbb{Z}_\ell$ [6,10]. Let $\mathbb{Z}_\ell^\times = \{j \in \mathbb{Z}_\ell \mid \gcd(j, \ell) = 1\}$ be the multiplicative group of $\mathbb{Z}_\ell$. A decimation of a vector $v$ by $k \in \mathbb{Z}_\ell^\times$, denoted by $d_k(v)$, is defined to be $(d_k(v))_g = v_{k^{-1}g}$ where $k^{-1}$ denotes the multiplicative inverse of $k$ in $\mathbb{Z}_\ell^\times$. A decimation class is an equivalence class of vectors under circulant shifts and decimations [7]. For each $u'$ in the decimation class of $u$, there exists some $v'$ in the decimation class of $v$ that is complementary to $u'$ if and only if $u$ and $v$ are complementary sequences [7]. We call $(v,u)$ and $(v',u')$ equivalent complementary sequences provided that not necessarily the same circulant shift yet the same decimation up to sign can be applied to the elements of $\{v,u\}$ to obtain the elements of $\{v',u'\}$.

Computational searches for LP focus on discovering new necessary constraints for generating vectors satisfying Eq. (1), or similarly applying constraints that only remove equivalent LP. A constraint called the PSD-test significantly reduces the search space [7]. The discrete Fourier transform (DFT) of $v$ is $\mu$, where $\mu_k = \sum_{j=0}^{\ell-1} v_j \omega^{jk}_\ell$ for $k = 0, 1, \ldots, \ell-1$ and $\omega^{jk}_\ell = e^{2\pi ij/\ell}$. The power spectral density (PSD) of a vector $v$ with DFT $\mu$ is PSD($v$) where

$$\text{PSD}(v, j) = \mu_j \tilde{\mu}_j \quad \text{for each } j \in \mathbb{Z}_\ell.$$ 

Let $u$ have DFT $v$. By the Weiner–Khinchin theorem, Eq. (1) transforms to

$$\sum_{j=0}^{\ell-1} \mu_j \tilde{\mu}_{j+g} + v_j \tilde{v}_{j+g} = \text{PSD}(v, g) + \text{PSD}(u, g) = \gamma \quad \forall g \neq 0, \ g \in \mathbb{Z}_\ell,$$

where $\gamma = (\ell + 1)/2$ ($\gamma = 2\ell + 2$) for $\{0,1\}$–LP ($\{-1,1\}$–LP) [7]. The PSD-test follows directly from Eq. (2) and states that if $v$ forms an LP with some $u$, then $\text{PSD}(v, g) \leq \gamma$ for all $g \in \mathbb{Z}_\ell/\{0\}$ [7].

Recently, a set of constraints was developed that enforces Eq. (1) on indices $g \notin \mathbb{Z}_\ell^\times$ based on $\delta$–modular compression. For a vector, $v$, of length, $\ell = \delta_1\delta_2$, the $\delta_1$-compression of $v$ is vector $q$ of length $\delta_1$ such that $q_g = \sum_{j=0}^{\delta_1-1} v_{g+j\delta_1}$ for $g \in \mathbb{Z}_{\delta_1}$ [5]. We also say $v$ $\delta_1$-

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A Legendre pair of length 77 using complementary compresses to \( q \). It follows that if \( u \) and \( v \) are complementary sequences with \( \delta_1 \)-compressions \( p \) and \( q \), then \( p \) and \( q \) are also complementary sequences \([5]\). That is, they satisfy Eqs. (1) and (2). Moreover, for a \( \{0, 1\}\)-LP

\[
\sum_{j=0}^{\delta_1-1} p_j p_{j+g} + q_j q_{j+g} = \delta_2 \lambda \quad \forall g \neq 0, \ g \in \mathbb{Z}_{\delta_1}
\]

and

\[
\sum_{j=0}^{\delta_1-1} p_j^2 + q_j^2 = (\delta_2 - 1)\lambda + 2\kappa,
\]

implying,

\[
PSD(p, g) + PSD(q, g) = \gamma \quad \forall g \neq 0, \ g \in \mathbb{Z}_{\delta_1},
\]

see \([5]\).

This constraint set essentially divides the problem into two stages. The first is the generation of all complementary, integer vectors of length \( \delta_1 \) with elements in \( \{0, 1, \ldots, (\ell/\delta_1)\} \). The second stage is the decompression of these compressed pairs, followed by a search for binary complementary sequences. This relation was used to great effect in the search for complementary sequences known as periodic Golay pairs \([6]\).

We expand the concept of \( \delta \)-modular compression to increase the set of constraints when \( \ell = \delta_1 \delta_2 \) with \( \gcd(\delta_1, \delta_2) = 1 \). This is the case for the remaining unsolved cases for LP existence with \( \ell < 200 \), excluding 169 = \( 13^2 \). The cases for

\[
\ell \in \{77, 85, 87, 115, 117, 129, 133, 145, 147, 159, 161, 169, 175, 177, 185, 187, 195\}
\]

constitute the current list of unsolved cases \([2]\). The result of this expansion are pairs of binary matrices with fixed row and column sums, also known as binary matrices with fixed marginals \([3]\). We prove various features on the spaces of these matrices with respect to decimation classes to further reduce the space of compressed complementary sequences. We conclude with the first exhaustive generation of LP of size \( \ell = 55 \) and the first discovery of a size \( \ell = 77 \) LP.

2 \( \delta \)-Modular compression

The PSD vector of \( v \) contains the PSD vector of its \( \delta \)-modular compression \([5]\). To prove this and facilitate understanding, it is expedient to first prove a stronger result regarding the DFT.

**Theorem 2.1** Let \( v \) be a vector of length \( \ell = \delta_1 \delta_2 \), and \( q \) be the \( \delta_1 \)-compression of \( v \). Let \( \mu \) be the DFT of \( v \), and \( \nu \) be the DFT of \( q \). Then \( \nu_g = \mu_{g \delta_2} \).

**Proof** By definition, \( \mu_{g \delta_2} = \sum_{j=0}^{\ell-1} v_j \omega_{\ell}^{g \delta_2 j} \). Notice \( \omega_{\ell}^{\delta_2} = e^{2\pi i \delta_2 / \ell} = e^{2\pi i / \delta_1} = \omega_{\delta_1} \). Then

\[
\mu_{g \delta_2} = \sum_{j=0}^{\ell-1} v_j \omega_{\delta_1}^{j} \omega_{\delta_1}^{g \delta_2 j}. \]

By definition, \( \omega_{\delta_1}^{j} = \omega_{\delta_1}^{(j+\delta_1)} \) and

\[
\mu_{g \delta_2} = \sum_{j=0}^{\delta_1-1} \omega_{\delta_1}^{g j} \left( \sum_{k=0}^{\delta_2-1} \nu_k \delta_1 + j \right). \]
Corollary 2.2 Let vectors \( u \) and \( v \) of length \( \ell = \delta_1 \delta_2 \) be complementary sequences with PSD constant \( \gamma \). If \( p \) and \( q \) are the respective \( \delta_1 \)-compressions of \( u \) and \( v \), then \( p \) and \( q \) are complementary sequences with PSD constant \( \gamma \).

Corollary 2.2 follows directly from Theorem 2.1. The benefit of this approach arises from the simpler problem of generating complementary positive integer vectors with appropriate density. These sequences are decompressed into binary vectors satisfying \( \delta_1 \)-equality constraints given by Eq. (2), as well as the required density. It is well known if \( u \) and \( v \) are complementary, then \( u \) and \( c_j(v) \) are also complementary for any \( j \). The equivalence class of circulant shifts of a vector is known as a *necklace*. Each cyclic shift of \( v \) is distinct when \( \gcd(\ell, \kappa) = 1 \), as is the case with LP [10]. It follows that \( \gcd(\delta_1, \kappa) = 1 \) for any \( \delta_1 \) dividing \( \ell \), and each compressed necklace has exactly \( \delta_1 \) distinct vectors. Moreover, reducing compressed vectors to necklace representatives will not preclude any LP.

This is partially true for decimation class representatives as well. Reducing to decimation class representatives of the compressed vectors will not preclude any decimation class forming an LP with another. However, the resultant vectors from decompression may require decimations to form the LP.

**Theorem 2.3** Let \( v \) be a vector of length \( \ell = \delta_1 \delta_2 \) such that \( \gcd(\delta_1, \delta_2) = 1 \), and \( q \) be the \( \delta_1 \)-compression of \( v \). For any \( (j, h) \in \mathbb{Z}_{\delta_1} \times \mathbb{Z}_{\delta_1}^\times \), there exists \((k, g) \in \mathbb{Z}_{\ell} \times \mathbb{Z}_{\ell}^\times \) such that \( c_k(d_g(v)) \) has \( \delta_1 \)-compression \( c_j(d_h(q)) \).

Theorem 2.3 guarantees that if \( v \) \( \delta_1 \)-compresses to \( q \), each member in the decimation class of \( v \) will \( \delta_1 \)-compress to some member in the decimation class of \( q \). This implies that a search for combinatorial objects that are equivalent up to decimation classes may be restricted to decimation class representatives of the compressions to reduce the number of duplicate vectors from each decimation class. The number of duplicates corresponding to decimations relies on the size of each class’ *multiplier* subgroup, where \( g \in \mathbb{Z}_{\ell}^\times \) is called a multiplier of \( v \) if there exists some \( j \in \mathbb{Z}_{\ell} \) such that \( c_j(d_g(v)) = v \).

**Corollary 2.4** Let \( v \) be a vector of length \( \ell = \delta_1 \delta_2 \) such that \( \gcd(\delta_1, \delta_2) = 1 \), and \( q \) be its \( \delta_1 \)-compression. Then \( c_j(d_k(v)) \) \( \delta_1 \)-compresses to \( q \) for some \( j \in \mathbb{Z}_{\ell} \) if and only if \( k \text{ mod } \delta_1 \) is a multiplier of \( q \).

Corollary 2.4 follows directly from the definition of \( \delta_1 \)-compression. We denote the set of vectors in the decimation class containing \( v \) with \( \delta_1 \)-compression \( q \) as \( D_{v,q} \). The number of vectors within the same decimation class having the same \( \delta_1 \)-compression is derived in Corollary 2.5.

**Corollary 2.5** Let \( v \) be a vector with density relatively prime to length \( \ell = \delta_1 \delta_2 \) where \( \gcd(\delta_1, \delta_2) = 1 \), and \( q \) be its \( \delta_1 \)-compression. Let \( G_v \leq \mathbb{Z}_{\ell}^\times \) and \( H \leq \mathbb{Z}_{\delta_1}^\times \) be the group of multipliers of \( v \) and \( q \) respectively. Then the number of vectors within the decimation class of \( v \) that \( \delta_1 \)-compress to \( q \) is

\[
|D_{v,q}| = \delta_2 \frac{\phi(\delta_2)|H|}{|G_v|}.
\]

**Proof** Since \( \gcd(\delta_1, \kappa) = 1 \) there exists \( \delta_2 \) circulant shifts of \( v \) that \( \delta_1 \)-compress to \( q \). It remains to show the duplicity of necklaces. It is well known there exists an isomorphism
Let $Q$ be the set of vectors that $\delta_1$-compress to $q$. A simple counting argument proves the size of this set is

$$|Q| = \prod_{j=0}^{\delta_1-1} \left( \frac{\phi(\delta_2)}{|G_{\delta_2}|} \right).$$

Let $d_j$ denote the lexicographically smallest vector within its respective decimation class with $\delta_1$-compression $q$. By Corollary 2.5,

$$|Q| = \prod_{j=0}^{\delta_1-1} \left( \frac{\phi(\delta_2)}{|G_{\delta_2}|} \right) = \sum_{d_j \in Q} |D_{d_j,q}| = |H| \sum_{d_j \in Q} \frac{\delta_2 \phi(\delta_2)}{|G_{d_j}|} = \sum_{d_j \in Q} \frac{|D_{d_j}|}{|D_q|}.$$

It follows that generating vectors through decompression of integer decimation class representatives causes significant duplicity for the representatives of the corresponding binary decimation classes. This duplicity can be reduced by implementing simultaneous decompression on relatively prime factors of $\ell$.

### 3 Simultaneous decompression

**Definition 3.1** Let $\ell = \prod_{j=1}^{n} \delta_j$ such that $\gcd(\delta_i, \delta_j) = 1$ for $i \neq j$. Let $q^j$ be a vector of length $\delta_j$. Vector $v$ of length $\ell$ is defined to be a simultaneous decompression of $\{q^1, q^2, \ldots, q^n\}$ if $q^j = \sum_{k=0}^{((\ell/\delta_j)-1)} v_{g+k\delta_j}$, for each $j \in \{1, 2, \ldots, n\}$ and $g \in \{1, 2, \ldots, \delta_j\}$.

For example, let $\ell = 35$, $\delta_1 = 7$, $\delta_2 = 5$, $n = 2$, $q^1 = [4, 2, 1, 4, 3, 3, 1]^T$, and $q^2 = [5, 2, 3, 4, 4]^T$. One possible simultaneous decompression of $q^1$, and $q^2$ is

$$v^T = [1, 1, 0, 1, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0]$$

as $v$ $\delta_1$-compresses to $q^1$ and $\delta_2$-compresses to $q^2$. In Eq. (4), the top row shows indices $\{1, \ldots, 17\}$ and the bottom shows $\{18, \ldots, 35\}$. Similarly,

$$u^T = [1, 1, 0, 1, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0]$$

simultaneously compresses to $p^1 = [3, 4, 3, 2, 2, 1, 3]^T$ and $p^2 = [6, 3, 2, 4, 3]^T$.

Each theorem in the paper for single decompression trivially holds for simultaneous decompression. The linearly independent set of simultaneous decompression constraints
are

\[
\begin{bmatrix}
1_{\ell}^\top & \otimes & 0_{\delta_1-1} & I_{\delta_1-1} \\
1_{\ell}^\top & \otimes & 0_{\delta_2-1} & I_{\delta_2-1} \\
& & \ddots & \ddots \\
1_{\ell}^\top & \otimes & 0_{\delta_n-1} & I_{\delta_n-1}
\end{bmatrix} v = \begin{bmatrix}
k \, 1_{\ell}^\top \, \theta_y \\
q_1,_{\ell}^\top \, h_1 \\
q_2,_{\ell}^\top \, h_2 \\
\vdots \\
q_n,_{\ell}^\top \, h_n
\end{bmatrix},
\] (5)

where \( \kappa = \sum_{i=0}^{\ell-1} v_i \) is the density constraint, \( \theta_y \) denotes a zero vector of length \( \gamma \), \( q^i,_{\ell} \) denotes all the \( \delta_i \)-compression excluding the 0th index, and \( \otimes \) is the Kronecker product. It follows from Eq. (5) that the number of linearly independent constraints defining simultaneous decompression is \( 1 + \sum_{i=1}^n (\delta_i - 1) \). Similarly, if \( q^i \) and \( p^i \) are complementary sequences with PSD constant \( \gamma \) for each \( i \in \{1, 2, \ldots, n\} \), then each vector generated through simultaneous decompression of all \( q^i \) and \( p^i \) respectively will satisfy \( \sum_{j=1}^n (\delta_j - 1) \) of the constraints defined by Eq. (2). This is the case for pairs \( q^1, p^1 \) and \( q^2, p^2 \) with PSD constant \( \gamma = 18 = (35 + 1)/2 \).

Notice from Eq. (5) the number of constraints set by simultaneous decompression increases as \( \delta_i \) become imbalanced, with the most extreme example of this being \( \delta_1 = \ell \) and \( \delta_2 = 1 \). The trade-off of this approach is the efficiency of generating complementary compressed sequences versus the efficiency of decompression. Imbalanced factors increase complexity of generating complementary compressed sequences while relatively balanced factors, such as \( \delta_1 = 7, \delta_2 = 5 \), have greater complexity for generating decompressions.

The development of theory for simultaneous decompression will proceed similarly to single decompression. The effects of circulant shifts and decimations will be addressed when considering simultaneous compressions, followed by an equation to determine the unicity of decimation classes through simultaneous decompression.

Since \( \ell = \prod_{j=1}^n \delta_j \) for \( \delta_j \) pairwise relatively prime, the mapping

\[
\psi : \mathbb{Z}_\ell \leftrightarrow \mathbb{Z}_{\delta_1} \times \mathbb{Z}_{\delta_2} \times \cdots \times \mathbb{Z}_{\delta_n} = \times_{j=1}^n \mathbb{Z}_{\delta_j}
\] (6)

with \( \psi(1) = (1, 1, \ldots, 1) \) constitutes a well known isomorphism. Let \( v \) be a vector of length \( \ell \) with \( \delta_i \)-compression \( q^i \) for \( i \in \{1, 2, \ldots, n\} \). Use of an isomorphism for simultaneous compression ensures each circulant shift on \( v \) maps to a unique set of circulant shifts on \( \{q^1, q^2, \ldots, q^n\} \). Similarly let \( \chi(j) = (j \mod \delta_1, j \mod \delta_2, \ldots, j \mod \delta_n) \) be the well known isomorphism,

\[
\chi : \mathbb{Z}_\ell^\times \leftrightarrow \mathbb{Z}_{\delta_1}^\times \times \mathbb{Z}_{\delta_2}^\times \times \cdots \times \mathbb{Z}_{\delta_n}^\times = \times_{j=1}^n \mathbb{Z}_{\delta_j}^\times
\] (7)

It follows that decimation \( d_j(v) \) yields a corresponding decimation on each \( \delta_i \)-compression, \( d_j(q^i) \) for \( i \in \{1, 2, \ldots, n\} \). The following theorem determines the number of vectors within the decimation class of \( v \) that simultaneously compress to \( \{q^1, q^2, \ldots, q^n\} \).

**Theorem 3.2** Let \( v \) be a vector of length \( \ell = \prod_{j=1}^n \delta_j \) such that \( \gcd(\delta_i, \delta_j) = 1 \) for \( i \neq j \), and \( q^i \) its \( \delta_i \)-compression. Let \( G_v \leq \mathbb{Z}_\ell^\times \) be the multiplier group of \( v \), and \( H_i \leq \mathbb{Z}_{\delta_i}^\times \) be the multiplier group of \( q^i \). The number of vectors within the decimation class of \( v \) that
simultaneously compress to \( \{q^1, q^2, \ldots, q^n\} \) is
\[
|D_{v, \{q^1, q^2, \ldots, q^n\}}| = \frac{\prod_{i=1}^n |H_i|}{|G_v|}.
\]

**Proof** Since no circulant shift of a vector with density relatively prime to \( \ell \) may result in the same set of simultaneous compressions, it suffices to determine the duplicity of decimations. By Eq. (7), for each \((j_1, j_2, \ldots, j_n)\) there exists exactly one \( j \in \mathbb{Z}_\ell^* \) such that \( \chi(j) = (j_1, j_2, \ldots, j_n) \). If there exists \( i \in \{1, \ldots, n\} \) such that \( g \equiv 1 \pmod{\delta_i} \) is not a multiplier of \( q^i \), then no \( j \in \mathbb{Z}_{\delta_i} \) exists such that \( d_g(q^i) = c_j(q^i) \). Hence, \( d_g(q^i) \) cannot decompress to \( v \).

Let \((h_1, h_2, \ldots, h_n) \in H_1 \times H_2 \times \cdots \times H_n \), and assume there exists some \( g \in \mathbb{Z}_\ell^* \) such that \( \chi(g) = (h_1, h_2, \ldots, h_n) \), and \( g \) is a multiplier of each \( q^i \). If \( g \) is a multiplier of \( v \), by definition \( d_g(v) = c_j(v) \) for some \( j \in \mathbb{Z}_\ell \). Then \( d_g(v) \) simultaneously compresses to \( \{q^1, q^2, \ldots, q^n\} \) if and only if \( j = 0 \).

Now assume \( g \) is not a multiplier of \( v \), implying \( c_j(d_g(v)) \neq v \) for any \( j \in \mathbb{Z}_\ell \). Since \( \psi \) is an isomorphism (Eq. (6)) and \( g \) is a multiplier of each \( q^i \), there exists exactly one circulant shift \( j \in \mathbb{Z}_\ell \) such that \( c_j(d_g(q^i)) = q^i \) for each \( i \in \{1, 2, \ldots, n\} \) and \( c_j(d_g(v)) \) compresses to \( \{q^1, q^2, \ldots, q^n\} \). Thus, distinct vectors within \( D_{v, \{q^1, q^2, \ldots, q^n\}} \) result from simultaneous decompression for each \( (g) \in (H_1 \times H_2 \times \cdots \times H_n) / \chi(G_v) \). implying
\[
|D_{v, \{q^1, q^2, \ldots, q^n\}}| = \frac{|\chi|}{|G_v|} = \frac{\prod_{i=1}^n |H_i|}{|G_v|}.
\]

\( \square \)

It follows from Theorem 3.2 and Corollary 2.5 that \( |D_{v, \{q^1, q^2, \ldots, q^n\}}| \) divides \( |D_{v, q^i}|_{\delta_i} / \ell \) for any \( i \in \{1, 2, \ldots, n\} \). By Theorem 2.3, the search space can be restricted to decimation class representatives for each \( \delta_i \)-compression, \( q^i \).

The case \( n = 2 \) is of particular interest as next five unsolved cases for LP existence are of length \( \{77, 85, 87, 115, 117\} \), each the product of exactly two prime powers [2]. Let
\[
\theta : \{0, 1\}^\ell \mapsto \{0, 1\}^{\delta_1 \times \delta_2}
\]
be the mapping of binary vectors of length \( \ell \) to arrays of size \( \delta_1 \times \delta_2 \), with indices mapped as defined by Eq. (6). When \( n = 2 \), \( \delta_1 = 3, \delta_2 = 5 \), \( A = q^{2T} \) and \( A1_5 = q^1 \). For example, let \( v \) be the vector given in Eq. (4). Then
\[
\theta(v) = A = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

As expected, \( 1_5^T A = q^{2T} \) and \( A1_5 = q^1 \). The problem of generating all such \( A \) for fixed \( q^1 \) and \( q^2 \) is also known as enumerating all binary matrices with fixed marginals (BMFM) [3].

We focus on \( n = 2 \) exclusively for the remainder of the paper.

**Theorem 3.3** Let \( \Omega_{\delta_1}, \Omega_{\delta_2}, \) and \( \Omega_\ell \) be the DFT matrices of sizes \( \delta_1, \delta_2, \) and \( \ell = \delta_1 \delta_2 \) respectively where \( \gcd(\delta_1, \delta_2) = 1 \). There exists unique \( z = \chi^{-1}(\delta_2^{-1}, \delta_1^{-1}) \) for \( (\delta_2^{-1}, \delta_1^{-1}) \in \mathbb{Z}_{\delta_1}^* \times \mathbb{Z}_{\delta_2}^* \) such that
\[
\theta(\Omega_\ell(d_z(v))) = \theta(d_{\delta_1}(\Omega_{\ell}v)) = \Omega_{\delta_1} \theta(v) \Omega_{\delta_2}.
\]
Proof The uniqueness of $z$ is guaranteed by Eq. (7). It follows,

$$\chi(z^{-1}) = (\chi(z))^{-1} = (\delta_2 \bmod \delta_1, \delta_1 \bmod \delta_2).$$

The relation $d_{z^{-1}}(\Omega_{\ell}v) = \Omega_{\ell}(d_z(v))$ for each $z \in \mathbb{Z}_\ell^\times$ is well known [7]. To prove $\theta(d_{z^{-1}}(\Omega_{\ell}v)) = \Omega_{\delta_1} \theta(v) \Omega_{\delta_2}$, let $M = \Omega_{\delta_1} \theta(v) \Omega_{\delta_2}$, and $m_{r,c}$ be the element of $M$ in row $r$, column $c$. Then

$$m_{r,c} = \sum_{j=0}^{\delta_2-1} e^{(c_j \frac{2\pi i}{\delta_2})} \sum_{k=0}^{\delta_1-1} e^{(rk \frac{2\pi i}{\delta_1})} a_{k,j}.$$ 

It follows that

$$m_{(\delta_2^{-1}, \delta_1^{-1})} = \sum_{j=0}^{\delta_2-1} \sum_{k=0}^{\delta_1-1} e^{(yj+rk \frac{2\pi i}{\delta_1})} a_{k,j} = \mu_{\psi^{-1}(x,y)}.$$ 

$\square$

Let pairs $(q^1, p^1)$ and $(q^2, p^2)$ are complementary sequences of sizes $\delta_1$ and $\delta_2$ respectively with PSD constant $\gamma$ and density $\kappa$. Let $v$ and $u$ be solutions to BMFM with constraints $(q^1, q^2)$ and $(p^1, p^2)$ respectively. It follows from Eq. (3) that $PSD(v, g) + PSD(u, g) = \gamma$ for all $g \in \mathbb{Z}_\ell / \{0, \mathbb{Z}_\ell^\times\}$, and $PSD(v, j) + PSD(u, j) = \gamma$ need only be verified for $j \in \mathbb{Z}_\ell^\times$ to determine if $v$ and $u$ are complementary. These indices correspond to submatrix $M^{-1}$ of $M$, where the 0th row and column are removed from $M$. That is, if $M = \Omega_{\delta_1} \theta(v) \Omega_{\delta_2}$ and $N = \Omega_{\delta_1} \theta(u) \Omega_{\delta_2}$, then

$$M^{(-)} \circ M^{-1} + N^{(-)} \circ N^{-1} = \frac{\delta_1 \delta_2 + 1}{2} J_{(\delta_1-1) \times (\delta_2-1)},$$

where $J_{m \times n}$ is the $m \times n$ matrix of ones, $\circ$ is the element-wise Hadamard product, and $M^{-1}$ represents the element-wise complex conjugate of $M^{(-)}$.

The following theorem further reduces the number of necessary PSD constraints from Eq. (3) to $|[\delta : \delta \in \mathbb{Z}_\ell, \delta | \ell]|$.

**Theorem 3.4** Let $v$ and $u$ be integer vectors indexed by $\mathbb{Z}_\ell$ and $\delta_1 \delta_2 = \ell$. If $(PSD(v, \delta_2) + PSD(u, \delta_2)) \in \mathbb{Q}$, then

$$(PSD(v, \delta_2) + PSD(u, \delta_2)) = (PSD(v, r \delta_2) + PSD(u, r \delta_2))$$

for each $r \in \mathbb{Z}_{\delta_1}^\times$. 

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\textbf{Proof} Let \( p \) and \( q \) be the \( \delta_1 \)-modular compression of \( v \) and \( u \) respectively. Then \( \text{PSD}(p, 1) = \text{PSD}(v, \delta_2) \) and \( \text{PSD}(q, 1) = \text{PSD}(u, \delta_2) \). Let \( \omega = e^{2\pi i / \delta_1} \). Then \( \mathbb{Q}(\omega) \) is a field extension of \( \mathbb{Q} \) of degree \( \varphi(\delta_1) \), with \( \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \cong \mathbb{Z}_{\delta_1}^\times \). Recall that \( \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \) is the group of all automorphisms of the field \( \mathbb{Q}(\omega) \). Then for each \( \sigma_r \in \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \)

\[ \sigma_r(\omega) = \omega^r \]

and \( \sigma_r(z) = z \) for \( z \in \mathbb{Q} \) and \( r \in \mathbb{Z}_{\delta_1}^\times \). By definition

\[ \text{PSD}(p, 1) = \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^i p_i \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^j p_i. \]

Then

\[ \text{PSD}(p, 1) = \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^i p_i \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^{-i} p_i. \]

Consequently,

\[ \text{PSD}(p, 1) = \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^i p_i \sigma_{-1} \left( \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^j p_i \right). \] (8)

For \( r \in \mathbb{Z}_{\delta_1}^\times \), applying \( \sigma_r \in \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \) to both sides of Eq. (8) yields

\[ \sigma_r(\text{PSD}(p, 1)) = \sigma_r \left( \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^j p_i \right) \sigma_{-1} \left( \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^j p_i \right). \] (9)

Since \( \text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \) is commutative Eq. (9) implies

\[ \sigma_r(\text{PSD}(p, 1)) = \sigma_r \left( \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^j p_i \right) \sigma_{-1} \left( \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^j p_i \right). \] (10)

Notice

\[ \sigma_r(\text{PSD}(p, 1)) + \sigma_r(\text{PSD}(q, 1)) = \sigma_r(\text{PSD}(p, 1) + \text{PSD}(q, 1)), \]

and

\[ (\text{PSD}(p, 1) + \text{PSD}(q, 1)) \in \mathbb{Q} \]

implies

\[ \sigma_r(\text{PSD}(p, 1) + \text{PSD}(q, 1)) = \text{PSD}(p, 1) + \text{PSD}(q, 1). \]

Then by Eq. (10), we get

\[ \text{PSD}(p, 1) + \text{PSD}(q, 1) = \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^{ri} p_i \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^{ri} p_i + \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^{ri} q_i \sum_{i \in \mathbb{Z}_{\delta_1}} \omega^{ri} q_i \]

for each \( r \in \mathbb{Z}_{\delta_1}^\times \), and

\[ \text{PSD}(v, r\delta_2) + \text{PSD}(u, r\delta_2) = \text{PSD}(v, \delta_2) + \text{PSD}(u, \delta_2). \]

\( \square \)
The following corollary follows directly from Theorem 3.4 by the fact $(\ell + 1)/2 \in \mathbb{Q}$.

**Corollary 3.5** Vectors $u \in \{0, 1\}^\ell$ and $v \in \{0, 1\}^\ell$ constitute an LP if

$$PSD(v, \delta) + PSD(u, \delta) = \frac{\ell + 1}{2} \forall \delta \mid \ell$$

It follows that if $(p^1, q^1)$ and $(p^2, q^2)$ are complementary sequences with simultaneous decompression $v$ and $u$ respectively, then $(v, u)$ constitute an LP if

$$PSD(v, 1) + PSD(u, 1) = \gamma.$$  

### 4 A recursive method

Several algorithms exist for enumerating the BMFM solution space. Snijders’ [11] method recursively sets each matrix element according to a binary branching strategy based on the well known relation

$$\binom{\ell}{k} = \binom{\ell - 1}{k} + \binom{\ell - 1}{k - 1}.$$  

This equation is generalized to BMFM by the following theorem.

**Theorem 4.1** Let $q$ and $p$ be integer vectors of length $\delta_1$ and $\delta_2$ respectively. Let $N(q, p)$ be the size of the solution set to BMFM defined by $p$ and $q$. Then

$$N(q, p) = N(p, q) = \sum_{j=0}^{(\delta_2 - 1)} N(q_{1:\delta_1 - 1}, p - r_j),$$

where $r_j$ is the binary vector of the $j$th ranked subset of size $q_0$ chosen from $\{0, 1, \ldots, \delta_2 - 1\}$.

We call a BMFM infeasible if $N(q, p) = 0$. It follows that a branch may be fathomed during matrix generation if the reduced problem is determined to be infeasible. The next theorem provides the necessary and sufficient condition for determining if a BFMF is feasible ([3], p. 162).

**Theorem 4.2** Let $q$ and $p$ be integer vectors of length $\delta_1$ and $\delta_2$ respectively. $N(q, p) > 0$ if and only if $t_{q,p}(I, J) \geq 0$ for all $I \subset \{1, \ldots, \delta_1\}$ and $J \subset \{1, \ldots, \delta_2\}$, where

$$t_{q,p}(I, J) = |I||J| + \sum_{i \notin I} q_i - \sum_{j \in J} p_j.$$  

A necessary first step to employing a decompression based search is to locate all pairs of candidate compressed sequences that satisfy Eq. (3). By Eq. (7), the search for compressed complementary sequences can be initially reduced to searching across decimation class representatives. It is well known that if $v$ and $u$ are complementary sequences, then $v$ and $d_{-1}(u)$ are also complementary [7]. This holds true for compressed sequences as well. Rather, if $p^1$ and $q^1$ are compressed complementary sequences of size $\delta_1$, there exists $S \subset \mathbb{Z}_{\delta_1}^\times$ such that each $s \in S$ is not a multiplier of $q^1$ but $PSD(d_s(q^1)) = PSD(q^1)$. Similarly, there exists $T \subset \mathbb{Z}_{\delta_2}^\times$ such that each $t \in T$ is not a multiplier of $q^2$ but $PSD(d_t(q^2)) = PSD(q^2)$. It follows that not all LP with elements from two decimation classes will arise as simultaneous decompressions from a single complementary pair of decimation class representatives.
There are two options for handling this required duplicity of decimation class representation. The first approach is to determine all \( j \in \mathbb{Z}_\ell \times \ell \) such that \((j \mod \delta_1) \in S\) and \((j \mod \delta_2) \in T\). Each decompression of \( p_1, p_2 \) must be decimated by each \( j \) and compared to the corresponding \( q_1, q_2 \) to determine if the decompressions, or decimations thereof, are complementary. The second approach is to conduct simultaneous decompression on each decimation of the set of complementary sequences,

\[
\{(p_1, p_2), (d_s(q_1), d_t(q_2))\} \quad \text{for each} \quad (s, t) \in (S \cup 1) \times (T \cup 1).
\] (11)

Notice only one simultaneous decompression in each pair must be decimated. We chose the latter approach as it has greater performance for conducting calculations in parallel.

We use the recursive method discussed by Brualdi ([3], pg.195) based on Theorem 4.1 to enumerate the set of decompressions for fixed marginal vectors \( q_1, q_2 \). This method recursively sets rows and columns of \( A \), thereby reducing the BMFM’s dimensions. The method may be summarized as three stages. The first stage determines if the BMFM is infeasible by Theorem 4.2 and the search branch may be fathomed. The second stage determines if there is a trivial assignment, such as a row that must be filled with ones or a column that must be filled with zeros. In this same stage, the method determines if the trivial assignments complete the matrix. The third stage selects a row/column to branch on and generates all possible binary vectors of the corresponding size and density.

We include DFT calculations at stages two and three of this method to improve computational efficiency of implementing the PSD test on completed decompressions. PSD-based comparisons were used for our partial search over \( \ell = 77 \). Our enumerative LP search uses Eq. (1) for comparisons. The latter stores two integer digits for each index and eliminates errors due to computational precision as \( \sum_{j=0}^{\ell-1} v_j v_{j+\delta} \) is necessarily integer. However, \( O(\ell^2) \) calculations must be conducted for each decompression, reducing overall efficiency.

5 Results

5.1 Enumeration for \( \ell = 55 \)

An exhaustive search for LP of length \( \ell = 55 \) was conducted. We identified 17 inequivalent 5-compression pairs and 2051 inequivalent 11-compression pairs satisfying Eq. (3). These were expanded to 31 and 3038 pairs respectively per the decimations defined in Eq. (11). Each 5 and 11 pair combination yield 4 BMFM for a total of 376, 712. The total time required to generate the solution set of all BMFM is 793.3 h on a 16 cores 2.4 Ghz computer running 128 threads, or 101, 542.4 CPU hours (Fig. 1).

A total of 18,042 inequivalent LP were identified spanning 36,050 decimation classes. Known solutions were verified (up to decimation class equivalence) to be in the solution set [2,4,13].

The correlation energy of solutions \( (u, v) \), defined as:

\[
\rho_v = \sum_{j=1}^{\ell-1} P_j(v)^2
\]

have a range of [5952, 8262] with a mode at 6942. The difference in observed consecutive correlation energy values is \( 2\ell \), the theoretical minimum change. For comparison, the correlation energy of known \( \ell = 55 \) LP are (6722) [2], (7382) [4], and (7052, 7162, 7602) [13].
The tightest known lower bound on minimum correlation energy is 5292, and is known not to be tight for $\ell = 55$ as there exists no $(55, 28, 14)$ difference set. Rather, no binary vector of size $\ell = 55$ and density $\kappa = 28$ is self-complementary [1].

5.2 Results for $\ell = 77$

A partial search for LP of length 77 with $\delta_1 = 7$ and $\delta_2 = 11$ produced the LP in Table 1. This search was conducted with 168 threads on an Intel i5-3570 CPU, 4 core, 3.40 GHz processor, and required approximately 182,280 CPU hours.

Acknowledgements This research was sponsored by AFIT Graduate School of Engineering and Management’s Faculty Research Council, grant 2019-213. The views expressed in this article are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government.

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