LOWER BOUND FOR THE REMAINDER IN THE PRIME-PAIR CONJECTURE

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Abstract. Taking \( r > 0 \) let \( \pi_{2r}(x) \) denote the number of prime pairs \((p, p + 2r)\) with \( p \leq x \). The prime-pair conjecture of Hardy and Littlewood (1923) asserts that \( \pi_{2r}(x) \sim 2C_{2r} \text{li}_2(x) \) with an explicit constant \( C_{2r} > 0 \). A heuristic argument indicates that the remainder \( e_{2r}(x) \) in this approximation cannot be of lower order than \( x^\beta \), where \( \beta \) is the supremum of the real parts of zeta’s zeros. The argument also suggests an approximation for \( \pi_{2r}(x) \) similar to one of Riemann for \( \pi(x) \).

1. Introduction

For \( r \in \mathbb{N} \) let \( \pi_{2r}(x) \) denote the number of prime pairs \((p, p + 2r)\) with \( p \leq x \). The famous prime-pair conjecture (PPC) of Hardy and Littlewood asserts that for \( x \to \infty \),

\[
\pi_{2r}(x) \sim 2C_{2r} \text{li}_2(x) = 2C_{2r} \int_2^x \frac{dt}{\log^2 t} \sim 2C_{2r} \frac{x}{\log^2 x}. \tag{1.1}
\]

Here \( C_2 \) is the ‘twin-prime constant’,

\[
C_2 = \prod_{p \text{ prime}, p > 2} \left( 1 - \frac{1}{(p-1)^2} \right) \approx 0.6601618, \tag{1.2}
\]

and the general ‘prime-pair constant’ \( C_{2r} \) is given by

\[
C_{2r} = C_2 \prod_{p \mid r, p > 2} \frac{p-1}{p-2}. \tag{1.3}
\]

No proof of (1.1) is in sight, but our arguments make it plausible that the best asymptotic estimate for the remainder

\[
e_{2r}(x) \overset{\text{def}}{=} \pi_{2r}(x) - 2C_{2r} \text{li}_2(x) \tag{1.4}
\]
cannot be as small as \( x^{1/2} / \log^2 x \); see Section 10.

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For the following we set

\[ \beta \stackrel{\text{def}}{=} \sup_{\rho} \Re \rho, \]

where \( \rho \) runs over the complex zeros of \( \zeta(s) = \zeta(\sigma + i\tau) \). Recall that Riemann’s Hypothesis (RH) asserts that \( \beta = 1/2 \). For the case of the prime number theorem it is known that the remainder

\[ e(x) \stackrel{\text{def}}{=} \pi(x) - \text{li}(x) = \sum_{\rho \leq x} 1 - \int_{2}^{x} \frac{dt}{\log t} \]

is \( O(x^{\beta+\varepsilon}) \) for every \( \varepsilon > 0 \), but cannot be \( O(x^{\beta-\varepsilon}) \) for any \( \varepsilon > 0 \). Indeed, a formula from Riemann’s work suggests the approximation

\[ \pi(x) = \text{li}(x) - (1/2) \text{li}(x^{1/2}) - \sum_{\rho} \text{li}(x^\rho) + O(x^b) \]

for any \( b > \max\{1/3, \beta/2\} \). Here the sum over \( \rho \) is a limit of ‘symmetric’ partial sums; it becomes significant for very large \( x \). In 1895 von Mangoldt obtained the following formula, from which he derived a proof of (1.6); cf. Davenport [5], Edwards [7]:

\[ \psi(x) = \sum_{n \leq x} \Lambda(n) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \sum_{k} \frac{x^{-2k}}{2k}. \]

The formula is exact for all \( x > 1 \) where \( \psi(x) \) is continuous.

For prime pairs \((p, p + 2r)\) one would expect that

\[ e_{2r}(x) \ll x^{\beta+\varepsilon} \quad \text{for every} \quad \varepsilon > 0, \quad \text{but} \]
\[ e_{2r}(x) \ll x^{\beta-\varepsilon} \quad \text{for no} \quad \varepsilon > 0. \]

Here the symbol \( \ll \) is shorthand for the \( O \)-notation. Some time ago, Dan Goldston [9] suggested that the author’s complex method (now in [14]) might provide a good lower bound for \( e_{2r}(x) \). In this note we use such an approach to obtain a conditional proof for

**Metatheorem 1.1.** Statement (1.9) is correct.

For our analysis we introduce an analog to \( \psi(x) \):

\[ \psi_{2r}(x) \stackrel{\text{def}}{=} \sum_{n \leq x} \Lambda(n)\Lambda(n + 2r). \]
It is not difficult to see that the PPC (1.11) is equivalent to the asymptotic relation
\begin{equation} \psi_{2r}(x) \sim 2C_{2r}x \quad \text{as} \quad x \to \infty. \end{equation}

For our subsequent analysis it is convenient to work with the following series of Dirichlet-type, where \(s = \sigma + i\tau\):
\begin{equation} D_{2r}(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\Lambda(n)\Lambda(n+2r)}{n^s(n+2r)^s} = \int_1^{\infty} \frac{d\psi_{2r}(t)}{t^s(t+2r)^s} \quad (\sigma > 1/2). \end{equation}

Note that for the boundary behavior of \(D_{2r}(s)\) as \(\sigma \searrow 1/2\), the denominators \(n^s(n+2r)^s\) may be replaced by \(n^{2s}\). Hence by a two-way Wiener–Ikehara theorem for Dirichlet series with positive coefficients, the PPC in the form (1.11) is true if and only if the difference
\begin{equation} G_{2r}(s) = D_{2r}(s) - \frac{2C_{2r}}{2s-1} \end{equation}
has ‘good’ boundary behavior as \(\sigma \searrow 1/2\). That is, \(G_{2r}(\sigma + i\tau)\) should tend to a distribution \(G_{2r}\{(1/2) + i\tau\}\) which is locally equal to a pseudofunction. By a pseudofunction we mean the distributional Fourier transform of a bounded function which tends to zero at infinity; see [13]. It cannot have poles and is locally given by Fourier series whose coefficients tend to zero. In particular \(D_{2r}(s)\) itself would have to show pole-type behavior, with residue \(C_{2r}\), for angular approach of \(s\) to \(1/2\) from the right; there should be no other poles on the line \(\{\sigma = 1/2\}\).

Heuristic arguments make it plausible that \(D_{2r}(s)\) has a meromorphic extension to some half-plane \(H_\varepsilon = \{\sigma > (\beta - \varepsilon)/2\}\) where \(\beta = \sup \Re \rho\):

**Metatheorem 1.2.** For every \(r \in \mathbb{N}\) there is a number \(\varepsilon > 0\) such that
\begin{equation} D_{2r}(s) = \frac{2C_{2r}}{2s-1} - 4C_{2r} \sum_{\rho} \frac{1}{2s-\rho} + H_{2r}(s), \end{equation}
where \(H_{2r}(s)\) is holomorphic in \(H_\varepsilon\).

Our approach would take care of Metatheorem 1.1 in the case \(\beta > 1/2\). Metatheorem 1.2 suggests the following approximation for \(\psi_{2r}(x)\):

**Metatheorem 1.3.** For each \(r\) there is a number \(\eta > 0\) such that
\begin{equation} \psi_{2r}(x) = 2C_{2r}x - 4C_{2r} \sum_{\rho} x^\rho/\rho + O(x^{\beta-\eta}). \end{equation}
The case $\beta = 1/2$ of Metatheorem 1.1 is more subtle. It requires consideration of the function
\begin{equation}
(1.16) \quad \theta_{2r}(x) \overset{\text{def}}{=} \sum_{p, p+2r \text{ prime}; p \leq x} \log^2 p = \int_2^{x^+} (\log^2 t) d\pi_{2r}(t),
\end{equation}
and the associated Dirichlet series
\begin{equation}
(1.17) \quad D^0_{2r}(s) \overset{\text{def}}{=} \sum_{p, p+2r \text{ prime}} \frac{\log^2 p}{p^{2s}} = \int_1^{\infty} \frac{d\theta_{2r}(t)}{t^{2s}}.
\end{equation}

Here our arguments suggest Metatheorem 1.4. If $\beta > 1/2$ there is a representation for $D^0_{2r}(s)$ similar to the one for $D_{2r}(s)$. However, if $\beta = 1/2$ one has
\begin{equation}
(1.18) \quad D^0_{2r}(s) = \frac{2C_{2r}}{2s-1} - \frac{4C_{2r}^*}{4s-1} - 4C_{2r} \sum_{\rho} \frac{1}{2s-\rho} + H^0_{2r}(s),
\end{equation}
with constants $C_{2r}^* > 0$ and a function $H^0_{2r}(s)$ that is holomorphic for $\sigma > 1/4$ and has ‘good’ boundary behavior as $\sigma \downarrow 1/4$.

Metatheorems 1.2 and 1.4 lead to plausible approximations for $\theta_{2r}(x)$ and finally, $\pi_{2r}(x)$:

**Metatheorem 1.5.** There are constants $C^*_{2r} > 0$ such that
\begin{equation}
(1.19) \quad \pi_{2r}(x) = 2C_{2r} \text{li}_2(x) - C^*_{2r} \text{li}_2(x^{1/2}) - 4C_{2r} \sum_{\rho} \rho \text{li}_2(x^{\rho}) + o(x^{\beta/\log^2 x}).
\end{equation}

The constants $C^*_{2r}$ come from the special case of the Bateman–Horn conjecture [1], [2] that involves the prime pairs $(p, p^2 \pm 2r)$: the number $\pi^*_2r(x)$ of such pairs with $p \leq x$ should satisfy an asymptotic relation
\begin{equation}
(1.20) \quad \pi^*_2r(x) \sim 2C^*_{2r} \text{li}_2(x) \quad \text{as} \quad x \to \infty,
\end{equation}
with certain specific constants $C^*_{2r}$. The analysis in Sections 8–10, which includes computations by Fokko van de Bult [3], supports and utilizes

**Metatheorem 1.6.** The Bateman–Horn constants $C^*_{2r}$ in (1.20) have mean value one (just like the Hardy–Littlewood constants $C_{2r}$).
2. Auxiliary functions

Integration by parts shows that the estimate $e_{2r}(x) \ll x^{\beta-\varepsilon}$ with small $\varepsilon > 0$ would be equivalent to the inequality

$$e'_{2r}(x) \overset{\text{def}}{=} \theta_{2r}(x) - 2C_{2r}x \ll x^{\beta-\varepsilon} \log^2 x.$$  \hspace{1cm} (2.1)

Note that (1.17) and (2.1) would imply holomorphy of the difference

$$C^0_{2r}(s) = D^0_{2r}(s) - \frac{2C_{2r}}{2s-1} \text{ for } \sigma = \text{Re } s > (\beta - \varepsilon)/2.$$  \hspace{1cm} (2.2)

Comparison of the series for $D^0_{2r}(s)$ and $D_{2r}(s)$ will show that the difference $D_{2r}(s) - D^0_{2r}(s)$ is holomorphic for $\sigma > 1/4$; cf. Lemma 7.1 below. Hence an estimate $e_{2r}(x) \ll x^{\beta-\varepsilon}$ would imply holomorphy of the difference $G_{2r}(s)$ in (1.13) for $\sigma > (\beta - \varepsilon)/2$, provided $\beta - \varepsilon \geq 1/2$.

We need precise information on the function $D_{0}(s)$ derived from (1.12).

**Lemma 2.1.** For $\sigma > \frac{1}{2}$ one has

$$D_{0}(s) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \frac{\Lambda^2(k)}{k^{2s}} = \frac{1}{2} \frac{d}{ds} \left\{ \frac{\zeta'(2s)}{\zeta(2s)} - \frac{1}{2} \frac{\zeta'(4s)}{\zeta(4s)} \right\} + H_0(s),$$  \hspace{1cm} (2.3)

where $H_0(s)$ has an analytic continuation to the half-plane $\{ \sigma > 1/6 \}$. This gives a meromorphic continuation of $D_{0}(s)$:

$$D_{0}(s) = \frac{1}{(2s-1)^2} - \frac{1}{(4s-1)^2} - \sum_{\rho} \left\{ \frac{1}{(2s-\rho)^2} - \frac{1}{(4s-\rho)^2} \right\} + H_1(s),$$  \hspace{1cm} (2.4)

where $H_1(s)$ is holomorphic for $\sigma > 1/6$.

**Proof.** Taking $x = \text{Re } z > 1$ one has

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{k} \frac{\Lambda(k)}{k^z} = \sum_{p} (\log p) \left( \frac{1}{p^z} + \frac{1}{p^{2z}} + \frac{1}{p^{3z}} + \cdots \right).$$  \hspace{1cm} (2.5)

It follows that

$$\sum (\log p) p^{-z} = \sum \Lambda(k) k^{-z} - \sum \Lambda(k) k^{-2z} + g_1(z) = -\zeta'(z)/\zeta(z) + \zeta'(2z)/\zeta(2z) + g_1(z),$$

where $g_1(z)$ is holomorphic for $x > 1/3$. Hence by differentiation,

$$\sum (\log^2 p) p^{-z} = \frac{d}{dz} \left\{ \frac{\zeta'(z)}{\zeta(z)} - \frac{\zeta'(2z)}{\zeta(2z)} \right\} - g_1'(z).$$
\[ \sum_{k} \Lambda^2(k)k^{-z} = \sum (\log^2 p)p^{-z} + \sum (\log^2 p)p^{-2z} + g_2(z) \]
\[ = \frac{d}{dz} \left\{ \zeta'(z) - \frac{1}{2} \zeta'(2z) \right\} + g_2(z), \]

where \( g_2(z) \) is also holomorphic for \( x > 1/3 \). Finally use a standard formula for \( (\zeta'/\zeta)(\cdot) \):

\[ (2.6) \quad \frac{\zeta'(z)}{\zeta(z)} = b - \frac{1}{z - 1} - \frac{1}{2} \frac{\Gamma'(1 + z/2)}{\Gamma(1 + z/2)} + \sum_{\rho} \left( \frac{1}{z - \rho} + \frac{1}{\rho} \right), \]

cf. Titchmarsh \[16\], and set \( z = 2s \).

We need the representation in Theorem 3.1 below. It involves sufficiently smooth even sieving functions \( E^\lambda(\nu) = E(\nu/\lambda) \) depending on a parameter \( \lambda > 0 \). The basic functions \( E(\nu) \) have \( E(0) = 1 \) and support \([-1, 1] \); we require that \( E, E' \) and \( E'' \) are absolutely continuous with \( E''' \) of bounded variation. An example involving the Jackson kernel for \( \mathbb{R} \) is given by

\[ E^\lambda(\nu) = E^\lambda_J(\nu) = \frac{3}{4\pi} \int_0^\infty \frac{\sin^4(\lambda t/4)}{\lambda^3 t^4} \cos \nu t \, dt \]
\[ = \begin{cases} 
1 - 6(\nu/\lambda)^2 + 6(|\nu|/\lambda)^3 & \text{for } |\nu| \leq \lambda/2, \\
2(1 - |\nu|/\lambda)^3 & \text{for } \lambda/2 \leq |\nu| \leq \lambda, \\
0 & \text{for } |\nu| \geq \lambda.
\end{cases} \]

An important role is played by a Mellin transform associated with the Fourier transform \( \hat{E}^\lambda(t) \). For \( 0 < x = \text{Re} \, z < 1 \)

\[ M^\lambda(z) \overset{\text{def}}{=} \frac{1}{\pi} \int_0^\infty \hat{E}^\lambda(t)t^{-z} \, dt = \frac{2}{\pi} \int_0^\infty t^{-z} \, dt \int_0^\lambda E^\lambda(\nu)(\cos \nu t) \, d\nu \]
\[ = \frac{2}{\pi} \int_0^\lambda E(\nu/\lambda) \, d\nu \int_0^\infty \cos \nu t t^{-z} \, dt \]
\[ = \frac{2}{\pi} \Gamma(1 - z) \sin(\pi z/2) \int_0^\lambda E(\nu/\lambda) \nu^{z-1} \, d\nu \]
\[ = \frac{2\lambda^z}{\pi} \Gamma(1 - z) \sin(\pi z/2) \int_0^1 E(\nu) \nu^{z-1} \, d\nu \]
\[ = \frac{2\lambda^z}{\pi} \Gamma(-z - 3) \sin(\pi z/2) \int_0^{1+} \nu^{z+3} \, dE'''(\nu). \]
In the special case of $E^{J}_{\lambda}(\cdot)$ one finds
\[M^{J}_{\lambda}(z) = \frac{3}{4\pi} \int_{0}^{\infty} \frac{\sin^{4}(\lambda t/4)}{\lambda^{3}(t/4)^{4}} t^{-z} dt = \frac{24}{\pi} \lambda^{z}(1 - 2^{-z-1})\Gamma(-z - 3) \sin(\pi z/2).\]

The function $M^{\lambda}(z)$ extends to a meromorphic function for $x > -3$ with simple poles at the points $z = 1, 3, \ldots$. The residue of the pole at $z = 1$ is $-2(\lambda/\pi)A^{E}$ with $A^{E} = \int_{1}^{\infty} E(\nu) d\nu$, and $M^{\lambda}(0) = 1$. Furthermore, the standard order estimates
\[\Gamma(z) \ll |y|^{x-1/2}e^{-\pi|y|/2}, \quad \sin(\pi z/2) \ll e^{\pi|y|/2}\]
for $|x| \leq C$ and $|y| \geq 1$ imply the useful majorization
\[M^{\lambda}(x + iy) \ll \lambda^{x}(|y| + 1)^{-x-7/2} \quad \text{for} \quad -3 < x \leq C, \quad |y| \geq 1.\]

3. A BASIC REPRESENTATION

The following result is related to Theorem 3.1 in [14], but more precise. It will be verified in Section 6.

**Theorem 3.1.** For any $\lambda > 0$ and $s = \sigma + i\tau$ with $1/2 < \sigma < 1$ there is a meromorphic representation
\[D_{0}(s) + 2 \sum_{0 < 2r < \lambda} E(2r/\lambda)D_{2r}(s) = V^{\lambda}(s) + \Sigma^{\lambda}(s) + H^{\lambda}(s).\]
Here $D_{2r}(s)$ is given by (1.12), also for $r = 0$; the functions $D_{2r}(s)$ are holomorphic for $\sigma > 1/2$. The function $D_{0}(s)$ has a purely quadratic pole at $s = 1/2$; see (2.4). On the basis of the PPC one expects that for $r \geq 1$, the function $D_{2r}(s)$ has a first-order pole at $s = 1/2$ with residue $C_{2r}$. The functions $V^{\lambda}(s)$ and $\Sigma^{\lambda}(s)$ are described in (3.2)–(3.4) below. The error term $H^{\lambda}(s)$ is holomorphic for $0 < \sigma < 1$.

The function $V^{\lambda}(s)$ is given by the sum
\[\Gamma^{2}(1 - s)M^{\lambda}(2 - 2s) - 2\Gamma(1 - s)\frac{\zeta'(s)}{\zeta(s)} M^{\lambda}(1 - s) \sin(\pi s/2) + W^{\lambda}(s),\]
(3.2)
where $W^{\lambda}(s) = -2\Gamma(1 - s) \sum_{\rho} \Gamma(\rho - s)M^{\lambda}(1 + \rho - 2s) \sin(\pi \rho/2)$. 
Here $\rho$ runs over the complex zeros of $\zeta(s)$. The combination $V^\lambda(s)$ is meromorphic for $0 < \sigma < 1$, with poles at $s = 1/2$ and the points $s = \rho/2$; the apparent poles at the points $s = \rho$ cancel each other. The simple poles at $s = 1/2$ and $s = \rho/2$ have residues

\begin{equation}
A^E \lambda, \quad \text{and} \quad -2A^E \lambda,
\end{equation}

respectively, with $A^E = \int_0^1 E(\nu)d\nu$.

The function $\Sigma^\lambda(s)$ is given by the sum

\begin{equation}
\left\{ \frac{\zeta'(s)}{\zeta(s)} \right\}^2 + 2 \frac{\zeta'(s)}{\zeta(s)} \sum_{\rho} \Gamma(\rho - s)M^\lambda(\rho - s) \cos\{\pi(\rho - s)/2\} + \sum_{\rho, \rho'} \Gamma(\rho - s)\Gamma(\rho' - s)M^\lambda(\rho + \rho' - 2s) \cos\{\pi(\rho - \rho')/2\}. \tag{3.4}
\end{equation}

Here $\rho$ and $\rho'$ independently run over the complex zeros of $\zeta(s)$. It is convenient to denote the sum of the first two terms by $\Sigma^\lambda_1(s)$; for $0 < \sigma \leq 1$ it has poles at $s = 1$ and at the points $\rho$. The double series defines a function which we call $\Sigma^\lambda_2(s)$. Under RH the series is absolutely convergent for $1/2 < \sigma < 3/2$. Indeed, setting $\rho = (1/2) + i\gamma$, $\rho' = (1/2) + i\gamma'$ and $s = \sigma + i\tau$, the inequalities (2.8), (2.9) show that the terms in the double series are majorized by

\begin{equation}
C(\lambda, \tau)(|\gamma| + 1)^{-\sigma}(|\gamma'| + 1)^{-\sigma}(|\gamma + \gamma'| + 1)^{-1+2\sigma-7/2}. \tag{3.5}
\end{equation}

Observing that the number of zeros $\rho = (1/2) \pm i\gamma$ with $n < \gamma \leq n + 1$ is $O(\log n)$, the convergence now follows from a discrete analog of Lemma 5.1 below.

If $\beta = \sup \text{Re} \rho > 1/2$ there is absolute convergence for $\beta < \sigma < 2 - \beta$. For $1/2 < \sigma \leq \beta$ the double sum may be interpreted as a limit of sums over the zeros $\rho, \rho'$ whose imaginary part has absolute value less than $R$, as $R \to \infty$ through suitable values; see [14]. By (3.1) the apparent poles of $\Sigma^\lambda(s)$ at the points $s = \rho$ with $\text{Re} \rho > 1/2$ must cancel each other. Formally, there is cancellation also at the other points $\rho$. 


4. Metatheorem 1.1 for $\beta > 1/2$ and Metatheorem 1.2

Taking $1/2 < \sigma < 1$, formulas (3.1)–(3.5) show that

$$
\Sigma^*_\lambda(s) \overset{\text{def}}{=} \Sigma^\lambda(s) - D_0(s) = 2 \sum_{0 < 2r \leq \lambda} E(2r/\lambda)D_{2r}(s)
$$

(4.1)

$$
- \frac{A^E\lambda}{s - 1/2} + 2A^E\lambda \sum_\rho \frac{1}{s - \rho/2} + H^\lambda_*(s),
$$

with a ‘symmetric’ sum over $\rho$ and a remainder $H^\lambda_*(s)$ that is holomorphic for $0 < \sigma < 1$. Recall from Section 2 that an inequality $e_{2r}(x) \ll x^{\beta - \varepsilon}$ with $\beta - \varepsilon \geq 1/2$ would imply holomorphy of the difference

$$
G_{2r}(s) = D_{2r}(s) - \frac{C_{2r}}{s - 1/2}
$$

(4.2)

for $\sigma > (\beta - \varepsilon)/2$. Hence if such holomorphy leads to a contradiction, so does (1.9). This would prove Metatheorem 1.1 for the case $\beta > 1/2$.

Suppose now that for all $r \leq \lambda/2$ and some $\varepsilon > 0$, the differences $G_{2r}(s)$ are holomorphic in the strip $S_\varepsilon$ given by $(\beta - \varepsilon)/2 < \sigma < 1$. Then by (1.1), the function $\Sigma^\lambda_*(s)$ has a meromorphic continuation [also called $\Sigma^\lambda_*(s)$] to $S_\varepsilon$, with poles at $s = 1/2$ and some points $\rho/2$. The pole at 1/2 will have residue

$$
R(1/2, \lambda) = 2 \sum_{0 < 2r \leq \lambda} E(2r/\lambda)C_{2r} - \lambda \int_0^1 E(\nu) d\nu.
$$

(4.3)

At this point we use the fact that the prime-pair constants $C_{2r}$ have mean value one. Good estimates were obtained by Bombieri–Davenport and Montgomery; these were later improved by Friedlander and Goldston [8] to

$$
S_m = \sum_{r=1}^m C_{2r} = m - (1/2) \log m + O\{\log^{2/3}(m + 1)\}.
$$

(4.4)

It follows that $R(1/2, \lambda)$ is $o(\lambda)$ as $\lambda \to \infty$, and even $O(\log \lambda)$. Hence by (3.4) the residue at $s = 1/2$ of (the meromorphic continuation of) the double sum $\Sigma^\lambda_*(s)$ also is $o(\lambda)$. [By (2.4) the pole of $D_0(s)$ at $s = 1/2$ is purely quadratic.] The estimate $o(\lambda)$ is not surprising if one observes that $\lambda$ occurs in the terms of $\Sigma^\lambda_*(s)$ only as a factor $\lambda^{\rho + \rho' - 2s}$, cf. (2.5). For $\sigma > 1/2$ the exponents have real part $\leq 2\beta - 1$, which is less than 1 if $\beta < 1$. 

If the latter kind of heuristic has general validity, the residues $R(\rho/2, \lambda)$ of the poles of $\Sigma^\lambda_s(s)$ or $\Sigma^2_\lambda(s)$ at the points $\rho/2$ in $S_\varepsilon$ must also be $o(\lambda)$ (at least when $\beta < 1$ and $\varepsilon$ is small). In view of (4.1) this would imply that many of the functions $D_{2r}(s)$ must become singular at points $s = \rho/2$ in $S_\varepsilon$, which would contradict our assumption on the differences $G_{2r}(s)$.

What would be a reasonable hypothesis on the form of the singularities? Let us start with $0 < \lambda \leq 4$ and suppose that $G_{2}(s) = D_{2}(s) - C_2/(s - 1/2)$ is holomorphic in $S_\varepsilon$. The residue $R(1/2, \lambda)$ will equal $2E(2/\lambda)C_2 - A^E\lambda$. Thus it changes character as $\lambda$ passes through the value 2: it will be linear in $\lambda$, of the form $-A^E\lambda$, for $\lambda \leq 2$, and this linear term is augmented by the nonlinear term $2E(2/\lambda)C_2$ as $\lambda$ enters the interval $(2, 4]$. It is plausible that the poles of $\Sigma^\lambda_s(s)$ at the points $\rho/2$ in $S_\varepsilon$ will be affected in a corresponding manner. More precisely, the residues $R(\rho/2, \lambda)$ should change from the linear form $-A^E\lambda$ to $2A^E\lambda - 4E(2/\lambda)C_2$ as $\lambda$ enters the interval $(2, 4]$. If that is correct, the function $D_{2}(s)$ must have first-order poles at the points $\rho/2$ in $S_\varepsilon$ with residue $-2C_2$. The combination

$$D_{2}(s) =\frac{C_2}{s - 1/2} + 2C_2 \sum_{\rho \in S_\varepsilon} \frac{1}{s - \rho/2}$$

would be holomorphic in $S_\varepsilon$.

Next taking $4 < \lambda \leq 6$ (and if desired, using a modified function $E(\nu)$ which vanishes on $[-1/2, 1/2]$, say), one may pass to the case $r = 2$, etc. Thus one is led to the postulate that each function $D_{2r}(s)$ has poles at the points $\rho/2$ in some strip $S_\varepsilon$ with residue $-2C_{2r}$. If this is correct, the residue of $\Sigma^\lambda_s(s)$ or $\Sigma^2_\lambda(s)$ at the poles $\rho/2$ in $S_\varepsilon$ will be

$$R(\rho/2, \lambda) = -4 \sum_{0 < 2r \leq \lambda} E(2r/\lambda)C_{2r} + 2\lambda \int_{0}^{1} E(\nu)d\nu.$$

Since the constants $C_{2r}$ have average 1 this would be consistent with the earlier argument that $R(\rho/2, \lambda)$ should be $o(\lambda)$.

It follows that Metatheorem 1.2 is altogether plausible, and this suggests Metatheorem 1.3.
5. Integral representations

Setting \( z = x + iy \) (and later \( w = u + iv \)), we write \( L(c) \) for the ‘vertical line’ \( \{ x = c \} \); the factor \( 1/(2\pi i) \) in complex integrals will be omitted. Thus

\[
\int_{L(c)} f(z) dz \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(z) dz.
\]

Since it is important for us to have absolutely convergent integrals, we often have to replace a line \( L(c) \) by a path \( L(c, B) = L(c_1, c_2, B) \) with suitable \( c_1 < c_2 \) and \( B > 0 \):

\[
L(c, B) = \begin{cases} 
\text{the half-line} & \{ x = c_1, -\infty < y \leq -B \} \\
+ \text{the segment} & \{ c_1 \leq x \leq c_2, y = -B \} \\
+ \text{the segment} & \{ x = c_2, -B \leq y \leq B \} \\
+ \text{the segment} & \{ c_2 \geq x \geq c_1, y = B \} \\
+ \text{the half-line} & \{ x = c_1, B \leq y < \infty \};
\end{cases}
\]

(5.1)

cf. Figure 1. Thus, for example,

\[
\cos \alpha = \int_{L(c, B)} ^{-\infty < y \leq -B} \Gamma(z) \alpha^{-z} \cos(\pi z/2) dz \quad (\alpha > 0),
\]

Figure 1. The path \( L(c_1, c_2, B) \)
with absolute convergence if $c_1 < -1/2$ and $c_2 > 0$. Similarly for $\sin \alpha$. For the combination
\[ \cos(\alpha - \beta)t = \cos \alpha t \cos \beta t + \sin \alpha t \sin \beta t \]
with $\alpha, \beta, t > 0$, one can now write down an absolutely convergent repeated integral. In [14] it was combined with (2.7) to obtain a repeated complex integral for the sieving function $E^\lambda(\alpha - \beta)$ in which $\alpha > 0$ and $\beta > 0$ occur separately. Taking $-3 < c_1 + c_1' < 0$, $c_2, c_2' > 0$, $c_2 + c_2' < 1$ it was found that
\[ E^\lambda(\alpha - \beta) = \int_{L(c,B)} \Gamma(z)\alpha^{-z} \int_{L(c',B)} \Gamma(w)\beta^{-w} \cdot M^\lambda(z + w) \cos\left\{\pi(z - w)/2\right\} dw. \]
(5.2)

We then considered the following integral:
\[ T^\lambda(s) = \int_{L(c,B)} \Gamma(z)\frac{\zeta'(z + s)}{\zeta(z + s)} \int_{L(c,B)} \Gamma(w)\frac{\zeta'(w + s)}{\zeta(w + s)} \cdot M^\lambda(z + w) \cos\left\{\pi(z - w)/2\right\} dw, \]
(5.3)
with suitable paths of integration and for appropriate $s$; cf. Section 6. Next, substituting the Dirichlet series for $(\zeta'/\zeta)(Z)$, formula (5.2) led to the expansion
\[ T^\lambda(s) = \sum k,l \Lambda(k)\Lambda(l)k^{-s}l^{-s}E^\lambda(k - l) \]
\[ = D_0(s) + 2 \sum_{0 < d \leq \lambda} \sum_k \Lambda(k)\Lambda(k + d)k^{-s}(k + d)^{-s}E^\lambda(d) \]
\[ = D_0(s) + 2 \sum_{0 < 2r \leq \lambda} E(2r/\lambda)D_{2r}(s) + H^\lambda_2(s), \]
(5.4)
where $H^\lambda_2(s)$ is holomorphic for $\sigma > 0$. Indeed, for odd numbers $d$, the product $\Lambda(k)\Lambda(k + d)$ can be $\neq 0$ only if either $k$ or $k + d$ is of the form $2^a$ for some $\alpha > 0$. Thus $T^\lambda(s)$ was extended to a holomorphic function on the half-plane \{\sigma > 1/2\}.

To verify the absolute convergence of the repeated integral in (5.2) we substituted $z = x + iy$, $w = u + iv$, and used the inequalities (2.8), (2.9) together with a simple lemma:

**Lemma 5.1.** For real constants $a$, $b$, $c$, the function
\[ \phi(y, v) = (|y| + 1)^{-a}(|v| + 1)^{-b}(|y + v| + 1)^{-c} \]
is integrable over $\mathbb{R}^2$ if and only if $a + b > 1$, $a + c > 1$, $b + c > 1$ and $a + b + c > 2$.

For the convergence of the repeated integral in (5.3) we also used the fact that the quotient $(\zeta'/\zeta)(Z)$ grows at most logarithmically in $Y$ for $X \geq 1$, and for $X \neq 1/2$ under RH; cf. (2.6) and Titchmarsh [16]. The holomorphy of the integral for $T^\lambda(s)$ then followed from locally uniform convergence in $s$.

The following sections serve as preparations for the case $\beta = 1/2$ of Theorem 1.1 so that RH is satisfied.

6. Derivation of Theorem 3.1 under RH

Changing variables in (5.3) one obtains

$$T^\lambda(s) = \int_{L(c,B)} \Gamma(z - s) \frac{\zeta'(z)}{\zeta(z)} \; dz \int_{L(c,B)} \Gamma(w - s) \frac{\zeta'(w)}{\zeta(w)} \; \cdot \cdot \cdot M^\lambda(z + w - 2s) \cos\{\pi(z - w)/2\} \; dw,$$

with new paths $L(c, B)$ and the point $s$ to the left of them. Using Cauchy’s theorem and assuming RH, one may take $c_1 = (1/2) + \eta$, $c_2 = 1 + \eta$ with small $\eta > 0$ and $(1/2) + \eta < \sigma < 1 + \eta$, $|\tau| < B$. [Without RH one could take $c_1 = 1$, $c_2 = 3/2$ and $1 < \sigma < 3/2$.] The absolute convergence of the repeated integral follows from Lemma 5.1.

We now move the paths of integration across the poles of the integrand, the points where $z$ or $w$ is equal to 1, $s$ or $\rho$. For the transition one may use quasi-rectangular contours $W_R$, see Figure 2, where $R$ runs through a sequence $R_n \in (n, n + 1)$ such that the horizontal segments at level $\pm R$ are as far from zeros of the zeta function as possible. Moving the $w$-path to a line $L(d_1)$ with $d_1 \approx 0$, one gets

$$T^\lambda(s) = \int_{L(c,B)} \cdots dz \int_{L(d_1)} \cdots dw + U^\lambda(s) = T^\lambda_*(s) + U^\lambda(s),$$

say, where by the residue theorem

$$U^\lambda(s) = \int_{L(c,B)} \Gamma(z - s) \frac{\zeta'(z)}{\zeta(z)} J(z, s) \; dz,$$
Figure 2. Upper half of $W_R$

with

\[
J(z, s) = -\Gamma(1 - s)M^\lambda(z + 1 - 2s)\cos\{\pi(z - 1)/2\} \\
+ \frac{\zeta'(s)}{\zeta(s)} M^\lambda(z - s)\cos\{\pi(z - s)/2\} \\
+ \sum_\rho \Gamma(\rho - s)M^\lambda(z + \rho - 2s)\cos\{\pi(z - \rho)/2\}.
\]

Observe that for given $s$ with $1/2 < \sigma < 1$, $|\tau| < B$ and small $\eta$, the function $J(z, s)$ is holomorphic in $z$ on and between the paths $L(c, B)$ and $L(d_1)$. Defining $J(z, s)$ for $z \in L(c, B)$ by continuity at the points $s = 1$ and $s = \rho$, it becomes holomorphic in $s$ for $c_2/2 < \sigma < c_2$. Indeed, the poles at the point $s = 1$ cancel each other, as do the poles at the points $s = \rho$.

What conditions do $c$, $d$ and $s$ have to satisfy? The double integral for $T_\lambda^*(s)$ must be absolutely convergent, which requires $\sigma > (c_1 + d_1)/2$; cf. Lemma 5.1. Also, one should not cross a pole of $M^\lambda(\cdot)$ during the shifting operation. Thus $x + u - 2\sigma$ should remain less than 1. Taking $\eta$ small, this allows values of $\sigma$ close to $1/2$. Since we ultimately want to consider values of $\sigma$ around $1/4$, we take $d_1 < 0$. Varying $c$ and $d$, the double integral will define $T_\lambda^*(s)$ as a holomorphic function for $0 < \sigma < 1$ and $|\tau| < B$.

We next consider the single integral for $U^\lambda(s)$. Moving the path $L(c, B)$ across the points $z = 1$, $z = s$ and $z = \rho$ to the line $L(d_1)$, we obtain the
decomposition

\[ U^\lambda(s) = \int_{L(d_1)} \Gamma(z - s) \frac{\zeta'(z)}{\zeta(z)} J(z, s) dz \]

(6.5) \[ + \left\{ - \Gamma(1 - s)J(1, s) + \frac{\zeta'(s)}{\zeta(s)} J(s, s) + \sum_{\rho'} \Gamma(\rho' - s)J(\rho', s) \right\}. \]

Working out the residue with the aid of (6.4) one obtains nine terms. Five of these combine into the function \( V^\lambda(s) \) of (3.2). Using the pole-type behavior of \( M^\lambda(Z) \) at the point \( Z = 1 \) (Section 2), the first term in \( V^\lambda(s) \) provides an important pole at the point \( s = 1/2 \):

(6.6) \[ \Gamma^2(1 - s)M^\lambda(2 - 2s) = \frac{A^E \lambda}{s - 1/2} + H^\lambda_3(s), \]

where \( H^\lambda_3(s) \) is holomorphic for \( 0 < \sigma < 1 \). The other terms in \( V^\lambda(s) \) only present simple poles at the points \( s = \rho/2 \). A short computation shows that the residues at those poles are all equal to \(-2A^E \lambda\). The four remaining terms coming from the big residue \{\cdots\} provide the function \( \Sigma^\lambda(s) \) of (3.4).

It remains to consider the single integral along \( L(d_1) \) in (6.3), let us call it \( U^\lambda_1(s) \), which we want to define a holomorphic function in a relatively wide strip. For that we need absolute convergence of the ‘double sum’, formed by the \( y \)-integral along \( L(d_1) \) and the sum over \( \rho \) in (6.4). With \( s = \sigma + i\tau \) and \( \text{Im} \rho = \gamma \), the standard estimates give the following majorant for the integrand:

\[ C(\tau)(|y| + 1)^{d_1 - \sigma - 1/2} \log(|y| + 1) \cdot \sum_{\gamma} (|\gamma| + 1)^{-\sigma \lambda^{d_1 + 1 - 2\sigma}} (|y + \gamma| + 1)^{-d_1 + 2\sigma - 4}. \]

Taking \( d_1 = -1/2 \), the analog of Lemma 5.1 for the integral of a sum proves the absolute convergence and holomorphy of the integral when \( 0 < \sigma < 1 \).

Combination of the above results with (5.4) will verify Theorem 3.1 under RH.

7. The differences \( D_{2r}(s) - D_{2r}^0(s) \) and \( \psi_{2r}(x) - \theta_{2r}(x) \)

To treat the case \( \beta = 1/2 \) of Theorem 1.1 one has to work with the function \( D_{2r}^0(s) \) of (1.17) instead of \( D_{2r}(s) \). In the following \( p \) and \( q \) are
primes that run over all pairs of the type indicated with the sums. By the definition of $\Lambda(\cdot)$,

$$D_{2r}(s) = \sum \frac{\Lambda(n)\Lambda(n+2r)}{n^s(n+2r)^s}$$

(7.1)

where $H_{1,r}(s)$ is holomorphic for $\sigma > 1/6$. The final sum comes from the cases $n = p$, $n + 2r = q^2$ and $n = q^2$, $n + 2r = p$. There are only finitely many $n$ of the form $p^2$ such that $n + 2r = q^2$. The function $g_{1,r}(s)$ includes these and the cases where $n$ or $n + 2r$ is a prime power with exponent $\geq 3$.

Continuing one obtains a sum over the prime pairs $(p, p + 2r)$ and a sum over prime pairs $(q, q^2 \pm 2r)$:

**Lemma 7.1.** One has

$$D_{2r}(s) = \sum_{p: p + 2r \text{ prime}} \frac{\log^2 p}{p^{2s}} + 2 \sum_{q: q^2 \pm 2r \text{ prime}} \frac{\log^2 q}{q^{4s}} + H_{2,r}(s)$$

(7.2)

where $D^*_{2r}(s)$ and $H_{2,r}(s)$ are holomorphic for $\sigma > 1/4$, and $\sigma > 1/6$, respectively.

We also consider corresponding partial sums, $\theta_{2r}(x)$ from (1.16) and

$$\theta^*_{2r}(x) \overset{\text{def}}{=} \sum_{q \leq x: q^2 \pm 2r \text{ prime}} \log^2 q.$$  

(7.3)

A sieving argument would show that $\theta^*_{2r}(x) = O(x)$; cf. [2], [10], [12].

**Lemma 7.2.** By (1.16) and (7.1) – (7.3),

$$\psi_{2r}(x) = \sum_{n \leq x} \Lambda(n)\Lambda(n + 2r)$$

(7.4)

$$= \theta_{2r}(x) + 2\theta^*_{2r}(x^{1/2}) + O(x^{(1/3)} \log^2 x).$$

We can now formulate a refinement of Theorem 3.1. In view of (7.2) the discussion in Section 6 shows the following.
Theorem 7.3. For \( \lambda > 0 \) and \( 1/2 < \sigma < 1 \), one has

\[
T^\lambda(s) = D_0(s) + 2 \sum_{0 < 2r \leq \lambda} E(2r/\lambda) \{D^0_{2r}(s) + 2D^*_{2r}(s)\} + H^\lambda_4(s)
\]

(7.5)

where the error terms \( H^\lambda_j(s) \) are holomorphic for \( 1/6 < \sigma < 1 \).

We wish to use (7.5) for the study of the prime-pair functions \( D^{0}_{2r}(s) \) when \( \beta = 1/2 \), and for that we need information on the functions \( D^*_{2r}(s) \) near the line \( L(1/4) = \{ \sigma = 1/4 \} \). This requires the consideration of prime pairs \( (p, p^2 \pm 2r) \).

8. Prime pairs \( (p, p^2 \pm 2r) \)

Let \( f(p) = p^2 - 2r \) with \( r \in \mathbb{Z} \setminus 0 \), and define

\[
\pi_f(x) = \# \{ p \leq x : f(p) \text{ prime} \}.
\]

Does \( \pi_f(x) \) tend to infinity as \( x \to \infty \)? Not if \( f(n) \) can be factored, nor if \( r \equiv 2 \pmod{3} \), for then \( p^2 - 2r \) is divisible by 3 when \( p \not\equiv 3 \). However, if \( f(n) \) is irreducible and for every prime \( p \), there is a positive integer \( n \) such that \( p \) does not divide \( nf(n) \), one would expect that \( \pi_f(x) \to \infty \) as \( x \to \infty \). This is a very special case of what is usually called Schinzel’s conjecture [15]. More generally, let \( f(n) \) be any polynomial of degree \( d \) with integer coefficients. For irreducible \( f(n) \) we set

\[
N_f(p) = \# \{ n, 1 \leq n \leq p : nf(n) \equiv 0 \pmod{p} \},
\]

and define

\[
C(f) = \prod_p \left(1 - \frac{1}{p}\right)^{-2} \left(1 - \frac{N_f(p)}{p}\right).
\]

The product will converge, but \( C(f) \) may be zero; if \( f(n) \) can be factored, we define \( C(f) = 0 \). Then a special case of the general conjecture of Bateman and Horn [1], [2] asserts the following:

Conjecture 8.1. As \( x \to \infty \), one has

\[
\pi_f(x) \sim \frac{C(f)}{d} \text{li}_2(x) = \frac{C(f)}{d} \int_2^x \frac{dt}{\log^2 t}.
\]

(8.4)
\begin{table}
\begin{tabular}{cccc}
\(x\) & \(\pi^*_2(x)\) & \(L^*_2(x)\) & \(\rho(x)\) \\
10 & 4 & & \\
10^2 & 13 & & \\
10^3 & 52 & & \\
10^4 & 259 & 274 & 0.945 \\
10^5 & 1595 & 1599 & 0.997 \\
10^6 & 10548 & 10560 & 0.999 \\
10^7 & 74914 & 75223 & 0.996 \\
10^8 & 563533 & 563804 & 0.9995 \\
\end{tabular}
\caption{Counting prime pairs \((p, p^2 - 2)\)}
\end{table}

Cf. Davenport and Schinzel \cite{Davenport1974}, and Hindry and Rivoal \cite{Hindry2007}. In the special case of the polynomial
\begin{equation}
\label{eq:8.5}
f_{2r}(n) = n^2 - 2r \quad (r \in \mathbb{Z} \setminus 0),
\end{equation}
one finds that for \(p \nmid 2r\), using the Legendre symbol,
\begin{equation}
\label{eq:8.6}
N_{f_{2r}}(p) - 2 = \left(\frac{2r}{p}\right) = \chi(p).
\end{equation}
Here \(\chi(p)\) generates a real character (different from the principal character) belonging to a modulus \(m = m_{2r}\). The convergence of the product for \(C(f_{2r})\) thus follows from the known convergence of series \(\sum_p \chi(p)/p\).

Fokko van de Bult \cite{van2008} has computed
\begin{equation}
\label{eq:8.7}
C(f_2) \approx 3.38,
\end{equation}
and counted
\[\pi^*_2(x) = \pi_{f_2}(x) = \#\{p \leq x : p^2 - 2 \text{ prime}\}\]
for \(x = 10, 10^2, \cdots, 10^8\). His results are in excellent agreement with Conjecture \ref{conj:8.1}. In the table the number \(\pi^*_2(x)\) is compared to rounded values
\[L^*_2(x) = \int_2^x \frac{dt}{\log^2 t}.
\]
The table also gives some ratios
\[\rho(x) = \pi^*_2(x)/L^*_2(x).
\]
These seem to converge to 1 rather quickly!
We can now discuss the functions
\[
D^*_r(s) = \int_1^\infty \frac{d\theta^*_r(t)}{t^{4s}} \quad (r \in \mathbb{N})
\]
of (7.2). Assuming that the Bateman–Horn conjecture is true for the polynomials \( f_{\pm 2r}(n) = n^2 \pm 2r \), one obtains the following asymptotic relation for the functions \( \theta^*_r(x) \) of (7.3):
\[
\theta^*_r(x) = \sum_{q \leq x; q^2 \pm 2r \text{ prime}} \log^2 q \sim \frac{C(f_{2r}) + C(f_{-2r})}{2} x.
\]
For us it will be convenient to write this relation in the form
\[
\theta^*_r(x) \sim 2C^*_r x.
\]
By the two-way Wiener–Ikehara theorem of [13] and integration by parts, relation (8.10) is equivalent to the statement that the difference
\[
G^*_r(s) = D^*_r(s) - \frac{2C^*_r}{4s - 1}
\]
has good (that is, pseudofunction) boundary behavior as \( \sigma \searrow 1/4 \). In particular \( D^*_r(s) \) must have a first-order pole at \( s = 1/4 \) with residue \( (1/2)C^*_r \), and no other poles on the line \( \{ \sigma = 1/4 \} \).

Before returning to the proof of Theorem 1.1 we give a supporting argument for Metatheorem 1.6 which asserts that the constants \( C^*_r \) have mean value one.

9. A FUNCTION \( T^\lambda_2(s) \). METATHEOREM 1.6

Using paths specified below we will study the function
\[
T^\lambda_2(s) = \int_{L(c,B)} \Gamma(z-s) \frac{\zeta'(2z)}{\zeta(2z)} \int_{L(c',B)} \Gamma(w-s) \frac{\zeta'(w)}{\zeta(w)} \cdot M^\lambda(z+w-2s) \cos\{\pi(z-w)/2\} \, dw.
\]
(9.1)

Here analogs to (5.3), (5.4) provide the following expansion for \( \lambda > 0 \), cf. (7.2):
\[
T^\lambda_2(s) = \sum_{k,l} \Lambda(k)\Lambda(l) k^{-2s-l-s} E^\lambda(k^2 - l)
\]
(9.2)
\[
= D^*_0(2s) + 2 \sum_{0 < 2r \leq \lambda} E(2r/\lambda) D^*_2r(s) + H^\lambda_0(s),
\]
where $D^*_0(s) = \sum_{\rho} (\log^2 p)/p^{2s}$ and $H^\lambda_6(s)$ is holomorphic for $\sigma > 1/5$. Comparison with $D_0(s)$ in (2.3) shows that

$$D^*_0(s) = \frac{1}{(2s-1)^2} - \sum_{\rho} \frac{1}{(2s-\rho)^2} - \frac{2}{(4s-1)^2} + H_7(s),$$

where $H_7(s)$ is holomorphic for $\sigma > \beta/4$. Formula (9.2) may be used to define $T^\lambda_2(s)$ as a holomorphic function for $\sigma > 1/4$.

In (9.1), assuming RH, one may take $c_1 = (1/4) + \eta$, $c_2 = (1/2) + \eta$ and $c'_1 = (1/2) + \eta$, $c'_2 = 1 + \eta$ with small $\eta > 0$. Varying $\eta$, the integral thus represents $T^\lambda_2(s)$ as a holomorphic function for $3/8 < \sigma < 1$ and $|\tau| < B$.

We now move the $w$-path $L(c', B)$ across the poles at the points $w = 1$, $s$ and $\rho$ to the path $L(d, B)$, where $d_1 = -1/2$ and $d_2 = 0$. Then the residue theorem gives

$$T^\lambda_2(s) = \int_{L(c, B)} \cdots dz \int_{L(0)} \cdots dw + U^\lambda_2(s) = T^\lambda_2(s) + U^\lambda_2(s),$$

say, where

$$U^\lambda_2(s) = \int_{L(c, B)} \Gamma(z - s) \frac{\zeta'(2z)}{\zeta(2z)} J(z, s) dz,$$

with $J(z, s)$ as in (6.4). Recall that the apparent poles of $J(z, s)$ at the points $s = 1$ and $s = \rho$ cancel out.

We next move the $z$-path $L(c, B)$ in the integral for $U^\lambda_2(s)$ to $L(d, B)$. Picking up residues at $z = s$, $1/2$ and the zeros $\rho'/2$ of $\zeta(2z)$, the result is

$$U^\lambda_2(s) = \int_{L(d, B)} \Gamma(z - s) \frac{\zeta'(2z)}{\zeta(2z)} J(z, s) dz + V^\lambda_2(s)$$

$$= U^\lambda_2(s) + V^\lambda_2(s),$$

say, where

$$V^\lambda_2(s) = \frac{\zeta'(2s)}{\zeta(2s)} J(s, s) - (1/2)\Gamma\{(1/2) - s\}J(1/2, s)$$

$$+ \sum_{\rho'} (1/2)\Gamma\{(\rho'/2) - s\}J(\rho'/2, s).$$

The integrals for $T^\lambda_2(s)$ and $U^\lambda_2(s)$ in (9.4) and (9.6) will define holomorphic functions for $1/4 \leq \sigma < 1$.

Let $\mathcal{S}$ denote the strip $\{1/4 < \sigma < 1/2\}$. We have to know the boundary behavior of $T^\lambda_2(s)$ as $\sigma \searrow 1/4$. What sort of poles on the line $L(1/4) =$
\{\sigma = 1/4\} will result from the three products in the formula for \(V^\lambda_2(s)\)? The first product involves \(J(s,s)\), which by \((6.4)\) is holomorphic on \(L(1/4)\), and \((\zeta'/\zeta)(2s)\), which has poles at the points \(s = \rho'/2\). The resulting poles have principal parts

\[(9.8) \quad \frac{(1/2)J(\rho'/2,\rho'/2)}{s - \rho'/2}.\]

Turning to the second product, the function \(J(1/2,s)\) is holomorphic on \(L(1/4)\), except for a simple pole at \(s = 1/4\) due to the pole of \(M^\lambda(Z)\) for \(Z = 1\). The other factor is \(-(1/2)\Gamma\{(1/2) - s\}\), and by a short calculation, cf. \((2.7)\), the principal part of the pole at \(s = 1/4\) works out to

\[(9.9) \quad \frac{(1/2)A^E\lambda}{s - 1/4}, \quad \text{where} \quad A^E = \int_0^1 E(\nu)d\nu.\]

In the third product the function \(J(\rho'/2,s)\) is holomorphic on \(L(1/4)\). However, the factors \((1/2)\Gamma\{(\rho'/2) - s\}\) introduce poles at the points \(s = \rho'/2\). The poles in the product have principal part

\[(9.10) \quad \frac{-(1/2)J(\rho'/2,\rho'/2)}{s - \rho'/2},\]

hence they cancel the poles at the points \(s = \rho'/2\) in \((9.8)\). The third product also generates a double series \(\Sigma^\lambda_{2,2}(s)\):

\[(9.11) \quad \Sigma^\lambda_{2,2}(s) \overset{\text{def}}{=} \sum_{\rho, \rho'} (1/2)\Gamma(\rho - s)\Gamma\{(\rho'/2) - s\} \cdot M^\lambda(\rho - 2s + \rho'/2) \cos\{\pi(\rho - \rho'/2)\} \cdot M^\lambda(\rho - \rho'/2)cos\{\pi(\rho - \rho'/2)\}.

The series is absolutely convergent for \(3/8 < \sigma < 1/2\). Its sum will have an analytic continuation to \(\mathcal{S}\), also denoted \(\Sigma^\lambda_{2,2}(s)\), but we do not know much about its behavior near the line \(L(1/4)\); see below.

In support of the hypothesis that the poles of \(V^\lambda_2(s)\) at the points \(s = \rho'/2\) cancel out one may analyze an integral \(T^\lambda_{2,2}(s)\) related to \(T^\lambda_{2}(s)\). It is obtained from \((9.11)\) by interchanging the roles of \((\zeta'/\zeta)(2\cdot)\) and \((\zeta'/\zeta)(\cdot)\). The new integral is of course equal to \(T^\lambda_{2}(s)\). In the analysis the role of
\[ J(z, s) \text{ is now taken by} \]
\[ J_2(z, s) = \frac{\zeta'{}(2s)}{\zeta{}(2s)} M^\lambda(z - s) \cos\{\pi(z - s)/2\} \]
\[ - (1/2) \Gamma\{(1/2) - s\} M^\lambda\{z + (1/2) - 2s\} \cos\{\pi(z - 1/2)/2\} \]
\[ + (1/2) \sum_{\rho} \Gamma\{(\rho/2) - s\} M^\lambda\{z + (\rho/2) - 2s\} \cos\{\pi(z - \rho/2)/2\}. \]

(9.12)

Here the apparent poles at the points \( s = 1/2 \) and \( s = \rho/2 \) cancel out.

**Summary 9.1.** Assume RH. Combination of (9.2) and the subsequent results shows that for \( 3/8 < \sigma < 1/2 \),
\[ T_2^\lambda(s) = D_0^*(2s) + 2 \sum_{0 < 2r \leq \lambda} E(2r/\lambda) D_{2r}^*(s) + H_s^\lambda(s) \]
\[ = \frac{(1/2) A^E \lambda}{s - 1/4} + \Sigma_{2,2}^\lambda(s) + H_\sigma^\lambda(s), \]

(9.13)

where \( H_s^\lambda(s) \) and \( H_\sigma^\lambda(s) \) are holomorphic for \( 1/4 < \sigma < 1/2 \).

Observe that the (analytic continuation of the) sum \( \Sigma_{2,2}^\lambda(s) \) must have a second-order pole at the point \( s = 1/4 \). Indeed, \( D_0^*(2s) \) has a quadratic pole at \( s = 1/4 \), see (9.3), and by sieving, the functions \( D_{2r}^*(s) \) cannot have a worse singularity at \( s = 1/4 \) than a first-order pole. In Section 8 it was made plausible that the functions \( D_{2r}^*(s) \) indeed have a first-order pole at \( s = 1/4 \). What can we say about the mean value of the residues \( (1/2) C_{22r}^* \), or of the numbers \( C_{2r}^* \)? By (9.13) and (8.10) the residue of \( \Sigma_{2,2}^\lambda(s) \) at \( s = 1/4 \) is equal to
\[ R^*(\lambda) = \sum_{0 < 2r \leq \lambda} E(2r/\lambda) C_{2r}^* - (\lambda/2) \int_0^1 E(\nu) d\nu. \]

(9.14)

Now it is plausible that this residue is \( o(\lambda) \) as \( \lambda \to \infty \). Indeed, \( \lambda \) occurs in the terms of \( \Sigma_{2,2}^\lambda(s) \) only as a factor \( \lambda^{\rho - 2s + \rho'/2} \); cf. the considerations in Section 4. Assuming \( R^*(\lambda) = o(\lambda) \), and letting \( E(\nu) \leq 1 \) approach the constant function 1 on \([0, 1]\), it follows from (9.14) that
\[ \sum_{0 < r \leq \lambda/2} C_{2r}^* \sim \lambda/2 \quad \text{as} \quad \lambda \to \infty. \]

(9.15)

Thus the numbers \( C_{2r}^* \) should have mean value 1, as asserted in Metatheorem 1.6. The metatheorem is supported by numerical evidence: a computation
of the first fifteen constants $C_{2r}$ by Fokko van de Bult [3] gave their average as 0.98.

Remark 9.2. Simple adaptation of our heuristics and accompanying numerical results indicate that relation (9.15) and Metatheorem 1.6 can be extended to the case of prime pairs $(p, p^k \pm 2r)$ with $k \geq 3$; see [4].

10. Metatheorem 1.1 for $\beta = 1/2$ and Metatheorem 1.4

Taking $1/2 < \sigma < 1$, Theorem 7.3 shows that

$$\Sigma_\lambda^\lambda(s) = \Sigma_\lambda^\lambda(s) - D_0(s) = 2 \sum_{0 < 2r \leq \lambda} E(2r/\lambda)\{D_0^0(2r,s) + 2D_2^*(s)\}$$

(10.1)

where $A^E = \int_0^1 E(\nu) d\nu$. The error term $H_\lambda^*(s)$ is holomorphic for $1/6 < \sigma < 1$. To complete the proof of Theorem 1.1 we have to deal with the case $\beta = 1/2$, so that RH holds. Suppose now that for $2r \leq \lambda$ and $x \to \infty$,

$$\theta_{2r}(x) - 2C_{2r}x \ll x^{1/2}/\log^2 x.$$  

(10.2)

Then the corresponding functions $G_0^0(2r,s) = D_0^0(2r,s) - 2C_{2r}/(2s - 1)$ of (2.2) have continuous boundary values for $\sigma \searrow 1/4$; cf. (1.17).

On the basis of Section 8 we may plausibly assume that the functions $G_2^*(2r,s) = D_2^*(2r,s) - 2C_{2r}/(4s - 1)$ show ‘good’ (pseudofunction) boundary behavior for $\sigma \searrow 1/4$. Hence by (10.1), the function $\Sigma_\lambda^\lambda(s)$ would have a ‘good’ extension to the strip $1/4 \leq \sigma < 1$, apart from first-order poles at $s = 1/2, 1/4$ and the points $\rho/2$. ‘Good’ meaning: holomorphy for $\sigma > 1/4$ and good boundary behavior after subtraction of the poles. As in Section 4, the pole at $s = 1/2$ of $\Sigma_\lambda^\lambda(s)$, or of the double sum $\Sigma_\lambda^\lambda(s)$ in (3.1), will have residue $R(1/2, \lambda)$ as in (4.3). By the mean-value property of the constants $C_{2r}$ this residue is $o(\lambda)$ as $\lambda \to \infty$. We recall that this was not surprising because $\lambda$ occurs in the terms of $\Sigma_\lambda^\lambda(s)$ only as a factor $\lambda^\rho + \rho' - 2s$.

Since by our assumption (10.2) the functions $D_0^0(2r,s)$ would have no pole at $s = 1/4$, the pole of $\Sigma_\lambda^\lambda(s)$ or $\Sigma_\lambda^\lambda(s)$ at that point would have residue

$$R(1/4, \lambda) = 2 \sum_{0 < 2r \leq \lambda} E(2r/\lambda)C_{2r}^* = (\lambda/2) \int_0^1 E(\nu)d\nu + R^*(\lambda),$$

(10.3)

with $R^*(\lambda)$ as in (9.14). In Section 9 it was made plausible that $R^*(\lambda) = o(\lambda)$ as $\lambda \to \infty$. We used both numerical evidence and the argument that
the terms of the double sum $\Sigma_{2,2}^{\lambda}(s)$ contain $\lambda$ only as a factor $\lambda^{\rho-2s+\rho'/2}$. However, the latter argument would also suggest that $R(1/4, \lambda) = o(\lambda)$. Indeed, the terms in the double series $\Sigma_{2}^{\lambda}(s)$ of (3.4) contain $\lambda$ only as a factor $\lambda^{\rho+\rho'-2s}$!

The contradiction indicates that assumption (10.2) is false, and that formula (10.3) for $R(1/4, \lambda)$ is incorrect. It is most likely that the functions $D_{2r}^{0}(s)$ have poles at the point $s = 1/4$, and that these poles more or less cancel those of the functions $2D_{2r}^{*}(s)$. Thus the true residue $R(1/4, \lambda)$ of $\Sigma_{2}^{\lambda}(s)$ at the point $s = 1/4$ may still be $o(\lambda)$ as $\lambda \to \infty$. Note also that by (10.1), the (true) residue $R(1/4, \lambda)$ is equal to 0 for $0 < \lambda \leq 2$. Combining our observations, the simplest hypothesis would be that $\Sigma_{s}^{\lambda}(s)$ does not have a pole at $s = 1/4$ for any value of $\lambda!$ Letting $\lambda$ increase from 2 on, it would follow that $D_{2r}^{0}(s)$ has a pole at $s = 1/4$ with residue $-C_{2r}^{*}$ for every $r$. This contradiction to (10.2) would establish Metatheorem 1.1!

One could also argue on the basis of the points $s = \rho/2$. Since $D_{2r}^{*}(s)$ would have no poles at those points, assumption (10.2) would require poles of $\Sigma_{s}^{\lambda}(s)$ or $\Sigma_{2}^{\lambda}(s)$ at $s = \rho/2$ with residue $2A^{E}\lambda$. But this would contradict the assumption that the residues are $o(\lambda)$ which was reasonable because the terms of $\Sigma_{2}^{\lambda}(s)$ contain $\lambda$ only as a factor $\lambda^{\rho+\rho'-2s}$. Thus (10.2) must be incorrect for many values of $r$. The simplest explanation of a residue $o(\lambda)$ for $\Sigma_{2}^{\lambda}(s)$ would be that the functions $D_{2r}^{0}(s)$ have poles at $s = \rho/2$ with residue $-2C_{2r}$. Indeed, we know that

$$-2 \sum_{0 < 2r \leq \lambda} E(2r/\lambda)2C_{2r} + 2A^{E}\lambda = o(\lambda) \quad \text{as} \quad \lambda \to \infty.$$ 

We now turn to Metatheorem 1.4. Using Lemma 7.1 the preceding arguments make it plausible that, indeed,

$$D_{2r}^{0}(s) = D_{2r}(s) - 2D_{2r}^{*}(s) - H_{2r}(s)$$

$$= \frac{2C_{2r}}{2s-1} - \frac{4C_{2r}^{*}}{4s-1} - 4C_{2r} \sum_{\rho} \frac{1}{2s-\rho} + H_{2r}^{0}(s),$$

where $H_{2r}^{0}(s)$ is holomorphic for $\sigma > 1/4$ and has good boundary behavior for $\sigma \searrow 1/4$.

In the case $\beta = 1/2$ Metatheorem 1.4 suggests the approximation

$$\theta_{2r}(x) = 2C_{2r}x - 4C_{2r}^{*}x^{1/2} - 4C_{2r} \sum_{\rho} x^{\rho}/\rho + o(x^{1/2}).$$
Finally, to arrive at Metatheorem 1.5 one would use the formula

\[ \pi_{2r}(x) = \int_2^x \frac{d\theta_{2r}(t)}{\log^2 t}. \]

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