LOCAL WELL-POSEDNESS OF VISCOUS SURFACE WAVE
WITHOUT SURFACE TENSION
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ABSTRACT. We consider an incompressible viscous flow without surface tension in a finite-depth domain of three dimension, with free top boundary. This system is governed by a Navier-Stokes equation in a moving domain and a transport equation for the top boundary. Traditionally, we consider this problem in Lagrangian coordinate and perturbed linear form. In [1], I. Tice and Y. Guo introduced a new framework using geometric structure in Eulerian coordinate to study both local and global wellposedness of this system. Following this path, we extend their result in local wellposedness from small data case to arbitrary data case. Other than the geometric energy estimate and time-dependent Galerkin method introduced in [1], we utilize a few new techniques: (1) using parameterized Poisson integral to construct a nontrivial transform between fixed domain and moving domain; (2) using bootstrapping argument to prove a comparison result for steady Navier-Stokes equation for arbitrary data of free surface.

1. INTRODUCTION

1.1. Problem Presentation. We consider a viscous incompressible flow in a moving domain.

\[ \begin{align*}
\Omega(t) &= \{ y \in \Sigma \times R \mid -b(y_1, y_2) < y_3 < \eta(y_1, y_2, t) \} \\
\text{Here we can take either } \Sigma &= R^2 \text{ or } \Sigma = (L_1 T) \times (L_2 T) \text{ for which } T \text{ denotes the 1-torus and } L^1, L^2 > 0 \text{ the periodicity lengths. The lower boundary } b \text{ is fixed and given satisfying} \\
0 < b(y_1, y_2) < \bar{b} = \text{constant} \quad \text{and} \quad b \in C^\infty(\Sigma) \\
\text{When } \Sigma = R^2, \text{ we further require that } b(y_1, y_2) \to b_0 = \text{positive constant as } |y_1| + |y_2| \to \infty \text{ and} \\
b - b_0 \in H^s(\Sigma) \text{ for any } s \geq 0. \text{ When } \Sigma = (L_1 T) \times (L_2 T), \text{ we just denote } b_0 = 1/2(\max\{b(y_1, y_2)\} + \min\{b(y_1, y_2)\}). \text{ It is easy to see this also implies } b - b_0 \in H^s(\Sigma) \text{ for any } s \geq 0 \text{ since } \Sigma \text{ is a bounded domain. We denote the initial domain } \Omega(0) = \Omega_0. \text{ For each } t, \text{ the flow is described by velocity and pressure } (u, p) : \Omega(t) \to R^3 \times R \text{ which satisfies the incompressible Navier-Stokes equation} \\
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u & \text{in } \Omega(t) \\
\nabla \cdot u = 0 & \text{in } \Omega(t) \\
(pI - \mu \nabla\nabla u) \nu = g \eta \nu & \text{on } \{ y_3 = \eta(y_1, y_2, t) \} \\
u = 0 & \text{on } \{ y_3 = \eta(y_1, y_2) \} \\
\partial_t \eta = u_3 - u_1 \partial_{y_1} \eta - u_2 \partial_{y_2} \eta & \text{on } \{ y_3 = \eta(y_1, y_2, t) \} \\
u(t = 0) = u_0 & \text{in } \Omega_0 \\
\eta(t = 0) = \eta_0 & \text{on } \Sigma
\end{cases}
\end{align*} \]

for \( \nu \) the outward-pointing unit normal vector on \( \{ y_3 = \eta \} \), \( I \) the \( 3 \times 3 \) identity matrix, \( (\nabla u)_{ij} = \partial_i u_j + \partial_j u_i \) the symmetric gradient of \( u \), \( g \) the gravitational constant and \( \mu > 0 \) the viscosity. As described in [1], the fifth equation in (1.1.3) implies that the free surface is convected with the fluid. Note that in (1.1.3), we have make the shift of actual pressure \( \bar{p} \) by constant atmosphere pressure \( p_{atm} \) according to \( p = \bar{p} + g y_3 - p_{atm} \).

We will always assume the natural condition that there exists a positive number \( \rho \) such that \( \eta_0 + b \geq \rho > 0 \) on \( \Sigma \), which means that the initial free surface is always strictly separated from the bottom. Also without loss of generality, we may assume that \( \mu = g = 1 \), which in fact will not infect our proof. In the following, we will use the term “infinite case” when \( \Sigma = R^2 \) and “periodic case” when \( \Sigma = (L_1 T) \times (L_2 T) \).
1.2. Previous Results. Traditionally, this problem is studied in several different settings according to whether we consider the viscosity and surface tension and the different choices of domain. Under each setting, many different kinds of framework have been developed.

In the viscous case without surface tension, the local wellposedness of equation (1.1.3) was proved by Solonnikov and Beale. Solonnikov [11] employed the framework of Hölder spaces to study the problem in a bounded domain, all of whose boundary is free. Beale [2] utilized the framework of $L^2$ spaces to study the domain like ours. Both of them took the equation as a perturbation of the parabolic equation and make use of the regularity results for linear equations. Abel [20] extended the result to the framework of $L^p$ spaces.

Many authors have also considered the effect of surface tension, which can help stabilize the problem and gain regularity. However, most of these results are related to global wellposedness for small data.

Almost all of above results are built in the Lagrangian coordinate and ignore the natural energy structure of the problem. In [1], [9] and [10], Y. Guo and I. Tice introduced the geometric energy framework and prove both the local and global wellposedness in small data. The main idea in their proof is that instead of considering perturbation of the parabolic equation, we restart to prove the regularity under time-dependent basis. In our paper, we will follow this path and employ a similar argument.

For the inviscid case, traditionally, we replace the no-slip condition with no penetration condition $u \cdot \nu = 0$ on $\Sigma_b$. It is often assumed that the initial fluid is irrotational and this curl-free condition will be conserved in the later time. This allows to reformulate the problem to one only on the free surface. The local wellposedness in this framework was proved by Wu [12, 13] and Lannes [14]. Local wellposedness without irrotational assumption was proved by Zhang-Zhang [15], Christodoulou-Lindblad [16], Lindblad [17], Coutand-Shkoller [18] and Shatah-Zheng [19]. In the viscous case, vorticity will be naturally introduced at the free surface, so the surface formulation will not work.

1.3. Geometric Formulation. In order to work in a fixed domain, we want to flatten the free surface via a coordinate transformation. Beale introduced a flatten transform in [2] and we will use a slightly modified version. We define a fixed domain

\[
\Omega = \{x \in \Sigma \times R \mid -b_0 < x_3 < 0\}
\]

for which we will write the coordinate $x \in \Omega$. In this slab, we take $\Sigma : \{x_3 = 0\}$ as the upper boundary and $\Sigma_b : \{x_3 = -b_0\}$ as the lower boundary. Simply denote $x' = (x_1, x_2)$ and $x = (x_1, x_2, x_3)$. Then we define

\[
\hat{\eta}^\epsilon = P^\epsilon \eta = \text{The parameterized harmonic extension of } \eta
\]

where $P^\epsilon \eta$ is defined as (A.3.1) when $\Sigma = R^2$ and as (A.4.1) when $\Sigma = (L_1T) \times (L_2T)$.

Consider the transform from $\Omega$ to $\Omega(t)$:

\[
\Phi^\epsilon : (x_1, x_2, x_3) \mapsto (x_1, x_2, \frac{b}{b_0} x_3 + \hat{\eta}^\epsilon (1 + \frac{x_3}{b_0})) = (y_1, y_2, y_3)
\]

This transform maps $\Omega$ into $\Omega(t)$ and its Jacobian matrix

\[
\nabla \Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A^\epsilon & B^\epsilon & J^\epsilon \end{pmatrix}
\]

and the transform matrix

\[
A^\epsilon = ((\nabla \Phi^\epsilon)^{-1})^T = \begin{pmatrix} 1 & 0 & -A^\epsilon K^\epsilon \\ 0 & 1 & -B^\epsilon K^\epsilon \\ 0 & 0 & K^\epsilon \end{pmatrix}
\]
where
\[
\begin{align*}
A^c &= x_3 \partial_1 b/b_0 + \partial_1 \tilde{\eta}^c \tilde{b} \\
B^c &= x_3 \partial_2 b/b_0 + \partial_2 \tilde{\eta}^c \tilde{b} \\
J^c &= b/b_0 + \tilde{\eta}^c / b_0 + \partial_3 \tilde{\eta}^c \tilde{b} \\
K^c &= 1/J^c \\
\tilde{b} &= 1 + x_3/b_0
\end{align*}
\]
(1.3.6)

By our assumption on initial data, there always exists \( \delta > 0 \), such that \( \eta_0(x') + b(x') \geq 2b_0 \delta > 0 \) for arbitrary \( x' \in \Sigma \). Based on lemma (A.6) and lemma (A.9), if we take \( \epsilon \leq \delta^2/(4C^2 ||\eta_0||_{H^{5/2}}^2) \), we have
\[
||\partial_3 \tilde{\eta}_0^c ||_{L^\infty} \leq \delta/2
\]
(1.3.7)

where \( \tilde{\eta}_0^c \) denotes \( \tilde{\eta}^c \) at \( t = 0 \). Furthermore, by fundamental theorem of calculus, we can estimate for any \( x_3 \in [-b_0, 0] \) and \( x' \in \Sigma \)
\[
\left| \tilde{\eta}_0^c(x', x_3) - \tilde{\eta}_0^c(x', 0) \right| \leq \int_{-b_0}^0 \left| \partial_3 \tilde{\eta}_0^c(x', z) \right| dz \leq b_0 ||\partial_3 \tilde{\eta}_0^c ||_{L^\infty} \leq b_0 \delta/2
\]
(1.3.8)

Let \( J_0^c \) denote \( J^c \) at \( t = 0 \). Since \( \tilde{\eta}_0^c(x', 0) = \eta_0(x') \), we have
\[
J_0^c = b/b_0 + \tilde{\eta}_0^c/b_0 + \partial_3 \tilde{\eta}_0^c = (b + \eta_0)/b_0 + (\tilde{\eta}_0^c - \eta_0)/b_0 + \partial_3 \tilde{\eta}_0^c \tilde{b}
\]
(1.3.9)

This means for given initial data \( \eta_0 \), we can always find \( 0 < \epsilon \leq \delta^2/(4C^2 ||\eta_0||_{H^{5/2}}^2) \) such that \( J_0^c > \delta > 0 \), then this nonzero Jacobi implies the transform \( \Phi^c \) makes sense at \( t = 0 \). If we further assume \( \eta_0 \in H^{7/2}(\Sigma) \), we can conclude \( \Phi^c \) is a \( C^1 \) diffeomorphism from \( \Omega \) to \( \Omega_0 \).

In the following, we will always choose \( \epsilon = \delta^2/(4C^2 ||\eta_0||_{H^{5/2}}^2) \) and for simplicity, we just write \( \tilde{\eta} \) instead of \( \tilde{\eta}^c \), while the same fashion applies to \( A, \Phi, A, B, J \) and \( K \). For our special choice of \( \epsilon \) satisfying above condition, it is guaranteed that \( J_0 > \delta > 0 \) and \( \Phi \) is a \( C^1 \) diffeomorphism at \( t = 0 \).

Define some transformed operators as follows.
\[
(\nabla A f)_i = A_{ij} \partial_j f \\
\nabla A \cdot \tilde{g} = A_{ij} \partial_j g_i \\
\Delta A f = \nabla A \cdot \nabla A f \\
\nabla \cdot \tilde{\eta} = ( - \partial_1 \eta, - \partial_2 \eta, 1) \\
(\mathbb{D}_A u)_{ij} = A_{ik} \partial_k u_{ij} + A_{jk} \partial_k u_{ij} \\
S_A(p, u) = p I - \mathbb{D}_A u
\]
(1.3.10)

where the summation should be understood in the Einstein convention. If we extend the divergence \( \nabla A \cdot \) to act on symmetric tensor in the natural way, then a straightforward computation reveals that \( \nabla A \cdot S_A(p, u) = \nabla A p - \Delta A u \) for vector fields satisfying \( \nabla A \cdot u = 0 \).

In our new coordinate, the original equation system (1.1.3) becomes
\[
\begin{align*}
\partial_t u - \partial_1 \tilde{\eta} b K \partial_3 u + u \cdot \nabla A u - \Delta A u + \nabla A p = 0 \quad &\text{in} \quad \Omega \\
\nabla A \cdot u = 0 \quad &\text{in} \quad \Omega \\
S_A(p, u) N = \tilde{\eta} N \quad &\text{on} \quad \Sigma \\
u = 0 \quad &\text{on} \quad \Sigma_b \\
u(x, 0) = u_0(x) \quad &\text{in} \quad \Omega \\
\partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 \quad &\text{on} \quad \Sigma \\
\eta(x', 0) = \eta_0(x') \quad &\text{on} \quad \Sigma
\end{align*}
\]
(1.3.11)

where we can split the system into a Navier-Stokes equation and a transport equation.

Since \( A \) is determined by \( \eta \) through the transform, so all the quantities above is related to \( \eta \), i.e., the geometric structure of the free surface. This is the central idea of our proof. It is noticeable that in proving local wellposedness of above equation system, we must verify \( \Phi(t) \) is a \( C^1 \) diffeomorphism for any \( t \in [0, T] \), where the theorem holds.
1.4. Main Theorem. In this paper, we will prove the local wellposedness for higher order regularity.

Theorem 1.1. Let \( N \geq 3 \) be an integer. Assume the initial data \( \eta_0 + b \geq 2h_0 \delta > 0 \). Suppose that \( u_0 \) and \( \eta_0 \) satisfy the estimate \( \| u_0 \|_{H^{2N}} + \| \eta_0 \|_{H^{2N+1/2}(\Sigma)} < \infty \) as well as the \( N \)th compatible condition. Then there exists \( 0 < T_0 < 1 \) such that for \( 0 < T < T_0 \), there exists a unique solution \((u,p,\eta)\) to system (1.3.11) on the interval \([0,T]\) that achieves the initial data. Furthermore, the solution obeys the estimate

\[
(1.4.1) \quad \left( \sum_{j=0}^{N} \sup_{0 \leq t \leq T} \| \partial_t^j u \|_{H^{2N-2j}}^2 + \sum_{j=0}^{N} \int_0^T \| \partial_t^j u \|_{H^{2N-2j+1}}^2 + \| \partial_t^{N+1} u \|_{H^N}^2 \right) \\
+ \left( \sum_{j=0}^{N-1} \sup_{0 \leq t \leq T} \| \partial_t^j p \|_{H^{2N-2j-1}}^2 + \sum_{j=0}^{N} \int_0^T \| \partial_t^j p \|_{H^{2N-2j}}^2 \right) \\
+ \left( \sup_{0 \leq t \leq T} \| \eta \|_{H^{2N+1/2}(\Sigma)}^2 + \sum_{j=1}^{N} \sup_{0 \leq t \leq T} \| \partial_t^j \eta \|_{H^{2N-2j+3/2}(\Sigma)}^2 \right) \\
+ \int_0^T \| \eta \|_{H^{2N+1/2}(\Sigma)}^2 + \int_0^T \| \partial_t \eta \|_{H^{2N-1/2}(\Sigma)}^2 + \sum_{j=2}^{N+1} \int_0^T \| \partial_t^j \eta \|_{H^{2N-2j+5/2}(\Sigma)}^2 \right) \\
\leq C(\Omega_0, \delta) P(\| u_0 \|_{H^{2N}} + \| \eta_0 \|_{H^{2N+1/2}(\Sigma)})
\]

where \( C(\Omega_0, \delta) \) is a positive constant depending on the initial domain \( \Omega_0 \) and separation quantity \( \delta \) and \( P(\cdot) \) is a single variable polynomial. The solution is unique among functions that achieves the initial data and for which the left hand side of the estimate is finite. Moreover, \( \eta \) is such that the mapping \( \Phi(t) \) defined by (1.3.3) is a \( C^{2N-2} \) diffeomorphism for each \( t \in [0,T] \).

Remark 1.2. In above theorem, \( H^k \) denotes the usual Sobolev space in \( \Omega \) and \( H^k(\Sigma) \) denotes the usual Sobolev space on \( \Sigma \), while we will state the definition of \( \| \cdot \|_{\cdot} \) later.

Remark 1.3. Because the map \( \Phi(t) \) is a \( C^{2N-2} \) diffeomorphism, then we may change variable to \( y \in \Omega(t) \) to produce solution of (1.1.3).

Remark 1.4. Since our paper is a natural extension of results in [1] from small data to arbitrary data and we employ the same framework, then many lemmas and theorems are quite similar except for some minor corrections. Hence, we will not always give the detailed proof of these results. Instead, we will refer to the corresponding results in [1] and merely point out the differences when necessary.

1.5. Convention and Terminology. We now mention some of the definition, bits of notation and conventions we will use throughout the paper.

1. We will employ Einstein summation convention to sum up repeated indices for vector and tensor operations.

2. Throughout the paper \( C > 0 \) will denote a constant only depend on the parameter of the problem, \( N \) and \( \Omega \), but does not depend on the data. They are referred as universal and can change from one inequality to another one. When we write \( C(z) \), it means a certain positive constant depending on quantity \( z \).

There are two exceptions to above rules. The first one is that in the elliptic estimates and Korn’s inequality, there are constants depending on the initial domain \( \Omega_0 \). Although this should be understood as depending on the initial free surface \( \eta_0 \), since its dependent relation is given implicitly and cannot be simplified further, we will also call them universal. The second one is that we will also call the constant \( C(\delta) \) universal, though \( \delta \) is determined by initial domain \( \Omega_0 \). To note that apart from these, all the other constants related to initial data \( \Omega_0 \), \( u_0 \) and \( \eta_0 \) should be specified in detail.

3. We will employ notation \( a \lesssim b \) to denote \( a \leq Cb \), where \( C \) is a universal constant as defined above.
(4) We will write $P(z_1, \ldots, z_n)$ to denote the polynomial with respect to finite arguments $z_i$ for $i = 1, \ldots, n$. The detailed properties of the polynomial will be specified in the context. This notation is used to denote some very complicated polynomial expressions, however whose details we do not really care. This kind of polynomial may change from line to line.

(5) We always write $Df$ to denote the horizontal derivative of $f$ and $\nabla f$ for the full derivative.

(6) For convenience, we will typically write $H^0 = L^2$, except for notation $L^2([0, T]; H^k)$. We write $H^k(\Omega)$ with $k \geq 0$ and $H^s(\Sigma)$ with $s \in \mathbb{R}$ for standard Sobolev space. Same style of notation also holds for $L^2([0, T]; H^k)$ and $L^\infty([0, T]; H^k)$. When we write $\|\cdot\|_{H^k}$, this always means the Sobolev norm in $\Omega$, otherwise, we will point out the exact space it stands for, e.g. $\|\cdot\|_{H^k(\Sigma)}$. A similar fashion is adopted for $\|\cdot\|_{L^2 H^k}$ and $\|\cdot\|_{L^\infty H^k}$.

1.6. Structure of This Paper. Our proof of the main theorem is based on an iteration argument for nonlinear terms. Hence, in Section 2, we will first prove the wellposedness to higher regularity for the linear equation

$$\begin{cases}
\partial_t u - \Delta_A u + \nabla_A u = F & \text{in } \Omega \\
\nabla_A \cdot u = 0 & \text{in } \Omega \\
(pI - D_A u)N = H & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma_h \\
u(t = 0) = u_0 & \text{in } \Omega
\end{cases}$$

(1.6.1)

In Section 3, we prove the wellposedness for transport equation

$$\begin{cases}
\partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{on } \Sigma \\
\eta(t = 0) = \eta_0 & \text{on } \Sigma
\end{cases}$$

(1.6.2)

and also the estimate for forcing terms in nonlinear problems. Finally, in Section 4, we will employ an iteration argument to complete the proof.

Throughout the paper, we always assume $N \geq 3$ is an integer. We consider both infinite case and periodic case simultaneously and won’t differentiate between them unless necessary.

2. Linear Navier-Stokes Equation

2.1. Introduction. In this section, we will concentrate on proving the wellposedness of the linear Navier Stokes equation

$$\begin{cases}
\partial_t u - \Delta_A u + \nabla_A u = F & \text{in } \Omega \\
\nabla_A \cdot u = 0 & \text{in } \Omega \\
(pI - D_A u)N = H & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma_h \\
u(t = 0) = u_0 & \text{in } \Omega
\end{cases}$$

(2.1.1)

Throughout this section, we will assume for any $t$ in the finite time interval $[0, T]$ which we will specify later, the condition $J(t) > \delta/2 > 0$ holds, where $J(t)$ is the Jacobi of transform $\Phi(t)$.

2.2. Preliminaries.

2.2.1. Transform Estimates. In order to study the linear problem in the slab domain, we will employ the idea that transforming the variable-coefficient problem into a constant-coefficient problem through diffeomorphism. So before estimating, we need to confirm the mapping $\Phi$ is an isomorphism from $\Omega$ to $\Omega(t)$ and determine the relation of corresponding norms between these two spaces. We borrow the idea from lemma 2.3.1 of [1].

Lemma 2.1. Let $\Psi : \Omega \rightarrow \Omega'$ be a $C^1$ diffeomorphism satisfying $\Psi \in H^{k+1}_{loc}$, the Jacobi $J = \det(\nabla \Psi) > \delta > 0$ a.e. in $\Omega$ and $\nabla \Psi - I \in H^k(\Omega)$ for an integer $k \geq 3$. If $v \in H^m(\Omega')$, then $v \circ \Psi \in H^m(\Omega)$ for $m = 0, 1, \ldots, k + 1$, and

$$\|v \circ \Psi\|_{H^m(\Omega)} \lesssim C_1 \|v\|_{H^m(\Omega')}$$

(2.2.1)
where
\[ C_1 = \left(1 + \frac{1}{\delta}\right) \left(1 + \|\nabla \Psi - I\|_{H^k(\Omega)}\right)^k \]

Similarly, for \( u \in H^m(\Omega), u \circ \Psi^{-1} \in H^m(\Omega') \) for \( m = 0, 1, \ldots, k + 1 \) and
\[ \|u \circ \Psi^{-1}\|_{H^m(\Omega')} \lesssim C_2\|u\|_{H^m(\Omega)} \]
where
\[ C_2 = \left(1 + \frac{1}{\delta}\right) \left(1 + \|\nabla \Psi - I\|_{H^k(\Omega)}\right)^{2k+2} \]

Let \( \Sigma' = \Psi(\Sigma) \) denote the upper boundary of \( \Omega' \). If \( v \in H^{m-1/2}(\Sigma') \) for \( m = 1, \ldots, k - 1 \), then \( v \circ \Psi \in H^{m-1/2}(\Sigma) \), and
\[ \|v \circ \Psi\|_{H^{m-1/2}(\Sigma)} \lesssim C_3\|v\|_{H^{m-1/2}(\Sigma')} \]
where
\[ C_3 = C_1 C(\|\nabla \Psi - I\|_{H^k(\Omega)}) \]

If \( u \in H^{m-1/2}(\Sigma) \) for \( m = 1, \ldots, k - 1 \), then \( u \circ \Psi^{-1} \in H^{m-1/2}(\Sigma') \) and
\[ \|u \circ \Psi^{-1}\|_{H^{m-1/2}(\Sigma')} \lesssim C_4\|u\|_{H^{m-1/2}(\Sigma)} \]
where
\[ C_4 = C_2 C(\|\nabla \Psi - I\|_{H^k(\Omega)}) \]

In all above, \( C(\|\nabla \Psi - I\|_{H^k(\Omega)}) \) is a constant depending on \( \|\nabla \Psi - I\|_{H^k(\Omega)} \).

**Proof.** Let \( x \) denote the coordinate in \( \Omega \) and \( y \) in \( \Omega' \). A direct computation shows that, for the case \( m = 0 \), if we have \( v \in L^2(\Omega') \), then
\[ \|v \circ \Psi\|_{L^2(\Omega)}^2 = \int_{\Omega'} |v \circ \Psi(x)|^2 \, dx = \int_{\Omega'} |v|^2 |J(x)|^{-1} \, dy \lesssim \|v\|_{L^2(\Omega')}^2 \]
\[ \lesssim \left(1 + \frac{1}{\delta}\right) \|v\|_{L^2(\Omega')}^2 \]

For \( m = 1 \), \( \partial_i v \in L^2(\Omega') \), then
\[ \frac{\partial}{\partial x_i}(v \circ \Psi)(x) = \frac{\partial v}{\partial y_j} \circ \Psi(x) \frac{\partial \Psi_j}{\partial x_i}(x) \]

Hence, we have the estimate
\[ \left\| \frac{\partial}{\partial x_i}(v \circ \Psi) \right\|_{L^2(\Omega)}^2 = \left\| \frac{\partial v}{\partial y_j} \circ \Psi(x) \frac{\partial \Psi_j}{\partial x_i}(x) \right\|_{L^2(\Omega)}^2 \lesssim \left(1 + \frac{1}{\delta}\right) \left\| \frac{\partial v}{\partial y_j} \right\|_{L^2(\Omega')}^2 \left\| \frac{\partial \Psi_j}{\partial x_i}(x) \right\|_{L^\infty(\Omega)}^2 \]
\[ \leq \left(1 + \frac{1}{\delta}\right) \left(1 + \|\nabla \Psi - I\|_{H^k(\Omega)}\right)^2 \left\| \frac{\partial v}{\partial y_j} \right\|_{L^2(\Omega')}^2 \]
\[ \lesssim \left(1 + \frac{1}{\delta}\right) \left(1 + \|\nabla \Psi - I\|_{H^k(\Omega)}\right)^2 \|v\|_{H^1(\Omega')}^2 \]

Similarly, it is easy to show the result when \( m = 2 \), i.e. if \( \partial v \in H^1(\Omega') \), then
\[ \left\| \frac{\partial}{\partial x_i}(v \circ \Psi) \right\|_{H^1(\Omega)}^2 \leq \left(1 + \frac{1}{\delta}\right) \left(1 + \|\nabla \Psi - I\|_{H^k(\Omega)}\right)^4 \|v\|_{H^2(\Omega')}^2 \]
For $m = 3, \ldots, k + 1$, inductively, we can assume the statement is valid for $m \leq m_0$ and try to justify it in $m_0 + 1$ case.

\begin{equation}
(2.2.10) \quad \frac{\partial}{\partial x_i} (v \circ \Psi)(x) = \frac{\partial v}{\partial y_j} \circ \Psi(x) \frac{\partial \Psi_j}{\partial x_i}(x) = \frac{\partial v}{\partial y_j} \circ \Psi(x) + \frac{\partial v}{\partial y_j} \circ \Psi(x) \left( \frac{\partial \Psi_j}{\partial x_i}(x) - I_{ij} \right)
\end{equation}

Induction hypothesis tells us if $v \in H^{m_0+1}$, then we have

$$\frac{\partial v}{\partial y_j} \circ \Psi \in H^{m_0}$$

Since we have the multiplicative embedding $H^{m_0} \times H^k \hookrightarrow H^{m_0}$ for $m_0 \geq 2$ and $k \geq 3$, we deduce that

$$\frac{\partial}{\partial x_i} (v \circ \Psi) \in H^{m_0}$$

which implies $v \circ \Psi \in H^{m_0+1}$. Moreover, we have the estimate

\begin{align*}
\left\| \frac{\partial}{\partial x_i} (v \circ \Psi) \right\|_{H^{m_0}(\Omega)}^2 & \leq \left\| \frac{\partial v}{\partial y_j} \circ \Psi(x) \right\|_{H^{m_0}(\Omega)}^2 + \left\| \frac{\partial v}{\partial y_j} \circ \Psi(x) \left( \frac{\partial \Psi_j}{\partial x_i}(x) - I_{ij} \right) \right\|_{H^{m_0}(\Omega)}^2 \\
& \lesssim \left( 1 + \| \nabla \Psi - I \|_{H^k(\Omega')} \right)^2 \left\| \frac{\partial v}{\partial y_j} \circ \Psi(x) \right\|_{H^{m_0}(\Omega)}^2 \\
& \lesssim \left( 1 + \frac{1}{\delta} \right)^2 \left( 1 + \| \nabla \Psi - I \|_{H^k(\Omega')} \right)^2 \| v \|_{H^{m_0+1}(\Omega')}^{2m_0+2}
\end{align*}

which implies that

$$\| v \circ \Psi \|_{H^{m_0+1}(\Omega)} \lesssim \left( 1 + \frac{1}{\delta} \right)^{m_0+1} \| v \|_{H^{m_0+1}(\Omega')}$$

Therefore, (2.2.1) holds. A similar argument justifies (2.2.3), utilizing the fact $\nabla \Psi^{-1}(y) = (\nabla \Psi)^{-1} \circ \Psi^{-1}(y)$. Since $\Psi \in H^{k+1}_{loc}$, we have $\Sigma'$ is locally the graph of a $C^{k-1,1/2}$ function. Based on [6], there exists an extension operator $E : H^{m-1/2}(\Sigma') \to H^m(\Omega')$ for $m = 1, \ldots, k - 1$ with the norm of the operator depending on $C(\| \nabla \Psi - I \|_{H^k(\Omega')})$. For $v \in H^{m-1/2}(\Sigma')$, let $V = Ev \in H^m(\Omega')$. Furthermore, we have $V \circ \Psi \in H^m(\Omega)$ and $v \circ \Psi = V \circ \Psi|_{\Sigma} \in H^{m-1/2}$. Moreover,

\begin{equation}
(2.2.11) \quad \| v \circ \Psi \|_{H^{m-1/2}(\Sigma)} \lesssim \| V \circ \Psi \|_{H^m(\Omega)} \lesssim C_1 \| V \|_{H^m(\Omega')} \lesssim C_2 \| v \|_{H^{m-1/2}(\Sigma')}
\end{equation}

which is (2.2.5). Similarly, (2.2.7) follows. \hfill \square

**Remark 2.2.** Based on our assumptions on $\bar{\eta}$ and $b$, it is easy to see $\Phi$ defined in 1.3.3 is a $C^1$ diffeomorphism satisfying the hypothesis in lemma 2.1. Moreover, the universal bounding constant $C$ now can be taken in a succinct form

\begin{equation}
(2.2.12) \quad C = C(\| \eta \|_{H^{k+1/2}(\Sigma)})
\end{equation}

to replace $C_1$ in the estimate. If we only consider the estimate in the domain $\Omega$ and need to specify the constant, we can take the explicit form

\begin{equation}
(2.2.13) \quad C = (1 + \| \eta \|_{H^{k+1/2}(\Sigma)})^{2k+2}
\end{equation}

to replace $C_1$ and $C_2$.

### 2.2.2. Functional Spaces.

Now we introduce several functional spaces. We write $H^k(\Omega)$ and $H^k(\Sigma)$ for usual Sobolev space of either scalar or vector functions. Define

\begin{align*}
W(t) & = \{ u(t) \in H^1(\Omega) : u(t)|_{\Sigma} = 0 \} \\
X(t) & = \{ u(t) \in H^1(\Omega) : u(t)|_{\Sigma} = 0, \nabla A(t) \cdot u(t) = 0 \}
\end{align*}


Moreover, let $L^2([0, T]; H^k(\Omega))$ and $L^2([0, T]; H^k(\Sigma))$ denote the usual time-involved Sobolev space. Define

\begin{equation}
\mathcal{W} = \{ u \in L^2([0, T]; H^1(\Omega)) : u|_{\Sigma_b} = 0 \}
\end{equation}

\begin{equation}
\mathcal{X} = \{ u \in L^2([0, T]; H^1(\Omega)) : u|_{\Sigma_b} = 0, \nabla \cdot u = 0 \}
\end{equation}

It is easy to see $W$, $X$, $\mathcal{W}$ and $\mathcal{X}$ are all Hilbert spaces with the inner product defined exactly the same as in $H^1(\Omega)$ or $L^2([0, T]; H^1(\Omega))$. Hence, in the subsequence, we will take the norms of these spaces the same as $\| \cdot \|_{H^1}$ or $\| \cdot \|_{L^2H^1}$.

Furthermore, we define the functional space for zero upper boundary.

\begin{equation}
V(t) = \{ u(t) \in H^1(\Omega) : u(t)|_{\Sigma} = 0 \}
\end{equation}

Also we will need an orthogonal decomposition $H^0(\Omega) = Y(t) \oplus Y^\perp(t)$, where

\begin{equation}
Y^\perp(t) = \{ \nabla_A(t) \varphi(t) : \varphi(t) \in V(t) \}
\end{equation}

In our use of these spaces, we will often drop the $(t)$ when there is no potential of confusion. Finally we will define the regular divergence-free space in $H^1(\Omega)$.

\begin{equation}
X_0 = \{ u \in H^1(\Omega) : u|_{\Sigma_b} = 0, \nabla \cdot u = 0 \}
\end{equation}

\begin{equation}
X_0 = \{ u \in L^2([0, T]; H^1(\Omega)) : u|_{\Sigma_b} = 0, \nabla \cdot u = 0 \}
\end{equation}

Sometimes, we will consider the functional space with the similar properties as defined above in transformed space $\Omega'$. If that is the case, we will employ the notation $W(\Omega')$ or $X(\Omega')$ to specify the space they live in. The same fashion can be applied to other functional spaces.

We will use $\langle \cdot, \cdot \rangle_{H^0}$ to denote the normalized inner product in $H^0$, i.e.

\begin{equation}
\langle u, v \rangle_{H^0} = \int_\Omega Ju \cdot v
\end{equation}

and $\langle \cdot, \cdot \rangle_{L^2H^0}$ to denote the normalized inner product in $L^2([0, T]; H^0)$, i.e.

\begin{equation}
\langle u, v \rangle_{L^2H^0} = \int_0^t \int_\Omega Ju \cdot v
\end{equation}

Also we will use $\langle \cdot, \cdot \rangle_W$ and $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ to denote the regular inner product in $H^1$ if both arguments are in $W$ or $\mathcal{W}$. Moreover, we define the regular inner product on $\Sigma$ as follows.

\begin{equation}
\langle u, v \rangle_{H^0(\Sigma)} = \int_\Sigma u \cdot v
\end{equation}

\begin{equation}
\langle u, v \rangle_{L^2H^0(\Sigma)} = \int_0^t \int_\Sigma u \cdot v
\end{equation}

The following result gives a version of Korn’s type inequality for initial domain $\Omega_0$. Here we employ Beale’s idea in [2].

**Lemma 2.3.** Suppose $\Omega_0$ is the initial domain and $\eta_0 \in H^{5/2}(\Sigma)$. Then

\begin{equation}
\| v \|^2_{H^1(\Omega_0)} \lesssim \| \nabla v \|^2_{H^0(\Omega_0)}
\end{equation}

for any $v \in H^1(\Omega_0)$ and $v = 0$ on $\{ y_3 = -b \}$.

**Proof.** In the periodic case, $\Sigma$ is bounded and $v|_{\Sigma_b} = 0$, so naturally Korn’s inequality is valid (see proof of lemma 2.7 in [2]). Then we consider the infinite case. First we prove the decaying property of initial surface $\eta_0$, i.e. for $|\alpha| \leq 1$, $|\partial^\alpha \eta_0| \to 0$ as $|x| \to \infty$ in the slab. For horizontal derivative $\partial^\alpha$,

\begin{equation}
\partial^\alpha \eta_0 = \int_{\mathbb{R}^2} (2\pi i \xi)^\alpha e^{2\pi i x \cdot \xi} e^{2\pi i \xi \cdot \eta_0(\xi)} d\xi
\end{equation}
where $\hat{\eta}_0$ is the Fourier transform of $\eta_0$ in $R^2$. Then

$$\int_{R^2} \left| (2\pi i \xi)^\alpha e^{\delta x_3 |\xi|} \hat{\eta}_0(\xi) \right| |d\xi| \lesssim \int_{R^2} \left| \xi^\alpha e^{\delta x_3 |\xi|} \right| (1 + |\xi|^{5/4}) \left( \frac{1}{1 + |\xi|^{5/4}} \right) d\xi$$

$$\leq \left( \int_{R^2} |\xi|^{2\alpha} |\hat{\eta}_0(\xi)|^2 (1 + |\xi|^{5/4})^2 d\xi \right)^{1/2} \left( \int_{R^2} \left( \frac{\epsilon^2 e^{\delta x_3 |\xi|}}{1 + |\xi|^{5/4}} \right) d\xi \right)^{1/2}$$

$$\lesssim \|\eta_0\|_{H^{\alpha+5/4}} \left( \int_{R^2} \left( \frac{\epsilon^2 e^{\delta x_3 |\xi|}}{1 + |\xi|^{5/4}} \right) d\xi \right)^{1/2} \lesssim \infty$$

The last inequality is valid since $x_3 < 0$. Hence, $(2\pi i \xi)^\alpha e^{\delta x_3 |\xi|} \hat{\eta}_0(\xi) \in L^1(R^2)$. A similar proof can justify the result if $\partial^\alpha = \partial_3$. Since $|x| \to \infty$ naturally implies $|x'| \to \infty$ in the slab, the Riemann-Lebesgue lemma implies $|\partial^\alpha \hat{\eta}_0| \to 0$ as $|x| \to \infty$.

We construct a mapping $\tilde{\sigma} = \Phi(0)$ as defined in (1.3.3), which maps $\Omega = \{R^3 : -b_0 < x_3 < 0\}$ to $\Omega_0$. Denote $\tilde{\sigma}(x) = x + \sigma(x)$. The above decaying property and our assumption on $b$ leads to $\partial^{\alpha} \tilde{\sigma}(x) \to 0$ as $|x| \to \infty$ for $|\alpha| \leq 1$.

We partition the slab $\Omega$ into cubes

$$Q_{j,k} = \{x : j < x_1 < j + 1, k < x_2 < k + 1, -b_0 < x_3 < 0\}$$

In each cube, we have Korn’s inequality

$$\|v\|_{H^1(Q)}^2 \leq C(\|Dv\|_{H^0(Q)}^2 + \|v\|_{H^0(Q)}^2)$$

Employing the compactness argument as Beale did, under the condition $v = 0$ on $\{x_3 = -b_0\}$, we can strengthen this result to

$$\|v\|_{H^1(Q)}^2 \leq C \|Dv\|_{H^0(Q)}^2$$

This argument relies on the fact that for such $v$

$$\|v\|_{H^0(Q)}^2 \leq C \|Dv\|_{H^0(Q)}^2$$

Now suppose $D \subseteq \Omega_0$ is the image of union of such cubes contained in $\{|x| \geq R\}$ for some $R$. Applying last estimate to $v = u \circ \tilde{\sigma}$ and then transform to $\Omega_0$, we have

$$\|u\|_{H^1(D)} \leq C \|Dv\|_{H^0(D)} + \epsilon \|u\|_{H^1(D)}$$

where $\epsilon < 1$ when $R$ is large enough based on the decaying property. Then we use Korn’s inequality in the bounded domain $\Omega_0 - D$ and combine with above estimate to obtain the estimate required. \hfill $\square$

**Remark 2.4.** In the final step of the proof, the bounding constant in Korn’s inequality depends on the domain $\Omega_0$. However, based on our notation rule, we still use $\lesssim$ to denote it.

Next, we need to show the equivalence of certain quantities in $\Omega$.

**Lemma 2.5.** There exists a $0 < \epsilon_0 < 1$, such that if $\|\eta - \eta_0\|_{H^{5/2}} < \epsilon_0$, then the following relation

$$\|u\|_{H^0}^2 \lesssim \int_{\Omega} J |u|^2 \lesssim (1 + \|\eta_0\|_{H^{5/2}(\Sigma)}) \|u\|_{H^0}^2$$

$$\frac{1}{(1 + \|\eta_0\|_{H^3(\Sigma)})^3} \|u\|_{H^1}^2 \lesssim \int_{\Omega} J |\nabla u|^2 \lesssim (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^2 \|u\|_{H^1}^2$$

holds for all $u \in W$.

**Proof.** (2.2.24) is just a natural corollary of relation $\delta \lesssim \|u\|_{L^\infty} \lesssim (1 + \|\nabla \eta\|_{L^\infty}) \lesssim (1 + \|\eta\|_{H^{5/2}}) \lesssim (1 + \|\eta_0\|_{H^{5/2}})$ based on our assumption and Sobolev embedding.

To derive (2.2.25), notice that

$$\int_{\Omega} J |\nabla u|^2 = \int_{\Omega} J |\nabla A_0 u|^2 + \int_{\Omega} J (\nabla A u + \nabla A_0 u) : (\nabla A u - \nabla A_0 u)$$
For the first term on the right-handed side, based on the integral substitution and Korn’s inequality we proved above, we have
\[
\int_{\Omega} J |D_{\mathcal{A}} u|^2 \, dx \geq \frac{1}{1 + \|\eta_0\|_{H^{5/2}(\Sigma)}} \int_{\Omega} J_0 |D_{\mathcal{A}} u|^2 \, dx = \frac{1}{1 + \|\eta_0\|_{H^{5/2}(\Sigma)}} \int_{\Omega_0} |D v|^2 \, dy
\]
\[
\geq \frac{1}{1 + \|\eta_0\|_{H^{5/2}(\Sigma)}} \|v\|^2_{H^1(\Omega_0)} \geq \frac{1}{(1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3} \|u\|^2_{H^1}
\]
For the second term on the right-handed side, using Korn’s inequality in slab \(\Omega\), we can naturally estimate
\[
\left| \int_{\Omega} J(D_{\mathcal{A}} u + D_{\mathcal{A}_0} u) : (D_{\mathcal{A}} u - D_{\mathcal{A}_0} u) \right| \lesssim \|J\|_{L^\infty} \|\mathcal{A} + \mathcal{A}_0\|_{L^\infty} \|\mathcal{A} - \mathcal{A}_0\|_{L^\infty} \int_{\Omega} |\nabla u|^2
\]
\[
\lesssim \epsilon_0 (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3 \int_{\Omega} |D u|^2 \lesssim \epsilon_0 (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3 \|u\|^2_{H^1(\Omega)}
\]
For given initial data, we can always take \(\epsilon_0\) sufficiently small to absorb the second term into the first one. Then we have
\[
\int_{\Omega} J |D_{\mathcal{A}} u|^2 \geq \frac{1}{(1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3} \|u\|^2_{H^1}
\]
The first part of (2.2.25) is verified. For the second part, notice that
\[
\int_{\Omega} J |D_{\mathcal{A}} u|^2 \lesssim (1 + \|\eta\|_{H^{5/2}}) \int_{\Omega} |D_{\mathcal{A}} u|^2 \lesssim (1 + \|\eta_0\|_{H^{5/2}}) \max\{1, \|AK\|^2_{L^\infty}, \|BK\|^2_{L^\infty}, \|K\|^2_{L^\infty}\} \|u\|^2_{H^1}
\]
Since it is easy to see
\[
\max\{1, \|AK\|^2_{L^\infty}, \|BK\|^2_{L^\infty}, \|K\|^2_{L^\infty}\} \lesssim 1 + (1 + \|\nabla \eta\|^2_{L^\infty}) \|K\|^2_{L^\infty} \lesssim (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^2
\]
then the second part of (2.2.25) follows.

\textbf{Remark 2.6.} Throughout this section, we can always assume the restriction of \(\eta\) depending on \(\epsilon_0\) is justified, and finally we will verify this condition for \(t \in [0, T]\) in the nonlinear part.

Now we present a lemma about the differentiability of norms in time-dependent space.

\textbf{Lemma 2.7.} Suppose that \(u \in \mathcal{W}\) and \(\partial_t u \in \mathcal{W}^*\). Then the mapping \(t \to \|u(t)\|^2_{H_0}\) is absolute continuous, and
\[
\frac{d}{dt} \|u(t)\|^2_{H_0} = 2(\partial_t u(t), u(t)) + \int_{\Omega} |u(t)|^2 \partial_t J(t)
\]
Moreover, \(u \in C^0([0, T]; H^0(\Omega))\). If \(v \in \mathcal{W}\) and \(\partial v \in \mathcal{W}^*\) as well, we have
\[
\frac{d}{dt} \int_{\Omega} u(t) \cdot v(t) = (\partial_t u(t), v(t)) + (\partial_t v(t), u(t)) + \int_{\Omega} u(t) \cdot v(t) \partial_t J(t)
\]
\textbf{Proof.} This is exactly the same result as lemma 2.4 in [1], so we omit the proof here.

Next we show the estimate for \(H^{-1/2}\) boundary functions.

\textbf{Lemma 2.8.} If \(v \in H^0\) and \(\nabla A \cdot v \in H^0\), then \(v \cdot \mathcal{N} \in H^{-1/2}(\Sigma)\), \(v \cdot v \in H^{-1/2}(\Sigma_b)\) (\(\nu\) is the outer normal vector on \(\Sigma_b\)) and satisfies the estimate
\[
\|v \cdot \mathcal{N}\|_{H^{-1/2}(\Sigma)} + \|v \cdot v\|_{H^{-1/2}(\Sigma_b)} \lesssim (1 + \|\eta\|_{H^{5/2}})^2 \left(\|v\|_{H^0} + \|\nabla A \cdot v\|_{H^0}\right)
\]
A similar argument can justify the case for \( L \) Pressure as a Lagrangian Multiplier.

2.2.3. The space \( X \) present here.

Remark 2.9. Recall the space \( Y(t) \subset H^0 \). It can be shown that if \( v \in Y(t) \), then \( \nabla_A \cdot v = 0 \) in the weak sense, such that lemma 2.8 implies that \( v \cdot N \in H^{-1/2}(\Sigma) \) and \( v \cdot v \in H^{-1/2}(\Sigma) \). Moreover, since the elements of \( Y(t) \) are orthogonal to each \( \nabla_A \phi \) for \( \phi \in V(t) \), we find that \( v \cdot v = 0 \) on \( \Sigma_b \).

We want to connect the divergence-free space and divergence-\( A \)-free space, so we define

\[
M = M(t) = K \nabla \Phi = \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ AK & BK & 1 \end{pmatrix}
\]

Note that \( M \) is invertible and \( M^{-1} = JA^T \). Since \( \partial_j(JA_{ij}) = 0 \) for \( i = 1, 2, 3 \), we have the following relation

\[
p = \nabla_A \cdot v \Leftrightarrow Jp = J\nabla_A \cdot v = JA_{ij} \partial_j v_i = \partial_j(JA_{ij} v_i) = \partial_j(JA^T v)_j = \partial_j(M^{-1}v)_j = \nabla \cdot (M^{-1}v)
\]

Then if \( \nabla_A \cdot v = 0 \), we have \( \nabla \cdot (M^{-1}v) = 0 \). \( M \) induces a linear operator \( M_t : u \rightarrow M_t(u) = M(t)u \). It has the following property.

Lemma 2.10. For each \( t \in [0, T] \), \( k = 0, 1, 2 \), \( M_t \) is a bounded linear isomorphism from \( H^k(\Omega) \) to \( H^k(\Omega) \) and \( X_0 \) to \( X \). The bounding constant is given by \( (1 + ||\eta(t)||_{H^{3/2}(\Sigma)})^2 \). If we further define \( M \) by \( M(u(t)) = M_t u(t) \), then it is a bounded linear isomorphism from \( L^2([0, T]; H^k(\Omega)) \) to \( L^2([0, T]; H^k(\Omega)) \) and from \( X_0 \) to \( X \). The bounding constant is given by \( (1 + \sup_{0 \leq t \leq T} ||\eta(t)||_{H^{3/2}(\Sigma)})^2 \).

Proof. It is easy to see for each \( t \in [0, T] \),

\[
||M_t u||_{H^k} \lesssim ||M_t||_{C^2} ||u||_{H^k} \lesssim (1 + ||\eta(t)||_{H^{3/2}(\Sigma)})^2 ||u||_{H^k}
\]

for \( k = 0, 1, 2 \), which implies \( M_t \) is a bounded operator from \( H^k \) to \( H^k \). Since \( M_t \) is invertible, we can estimate \( ||M^{-1}v||_{H^k} \lesssim (1 + ||\eta(t)||_{H^{3/2}(\Sigma)})^2 ||v||_{H^k} \). Hence \( M_t \) is an isomorphism between \( H^k \). Also above analysis implies \( M_t \) maps divergence-free function to divergence-\( A \)-free function. Then it is also an isomorphism from \( X_0 \) to \( X(t) \).

A similar argument can justify the case for \( L^2([0, T]; H^k) \) and \( X \).

2.2.3. Pressure as a Lagrangian Multiplier. It is well-known that in usual Navier-Stokes equation, pressure can be taken as a Lagrangian multiplier. In our new settings now, proposition 2.2.9 in [1] gives construction of pressure from transformed equation 1.3.11, which is valid for small free surface. So our result in arbitrary initial data is not completely reliable, which we will present here.

For \( p \in H^0 \), we define the functional \( S_t \in W^* \) by \( S_t(v) = \langle p, \nabla_A \cdot v \rangle_{H^0} \). By the Riesz representation theorem, there exists a unique \( Q_t(p) \in W \) such that \( S_t(v) = \langle Q_t(p), v \rangle_W \) for all \( v \in W \). This defines a linear operator \( Q_t : H^0 \rightarrow W \), which is bounded since we may take \( v = Q_t(p) \) to see

\[
||Q_t(p)||^2_W = S_t(v) = \langle p, \nabla_A \cdot v \rangle_{H^0} \lesssim (1 + ||\eta(t)||_{H^{3/2}(\Sigma)}) ||p||_{H^0} ||\nabla_A \cdot v||_{H^0} \lesssim (1 + ||\eta(t)||_{H^{3/2}(\Sigma)})^3 ||p||_{H^0} ||v||_W = (1 + ||\eta(t)||_{H^{3/2}(\Sigma)})^3 ||p||_{H^0} ||Q_t(p)||_W
\]
so we have $\|Q_t(p)\|_W \lesssim (1 + \|\eta(t)\|_{H^{5/2}(\Sigma)})^3 \|p\|_{H^0}$. Similarly, for $S \in W^*$, we may also define a bounded linear operator $Q : L^2([0, T]; H^0) \to W$ via the relation $(p, \nabla_A \cdot v)_{L^2_H} = \langle Q(p), v \rangle_W = S(v)$ for all $v \in W$. Similar argument shows $\|Q(p)\|_W \lesssim (1 + \sup_{0 \leq t \leq T} \|\eta(t)\|_{H^{5/2}(\Sigma)})^3 \|p\|_{L^2 H^0}$.

**Lemma 2.11.** Let $p \in H^0$, then there exists a $v \in W$ such that $\nabla_A \cdot v = p$ and $\|v\|_W \lesssim (1 + \|\eta(t)\|_{H^{5/2}(\Sigma)})^6 \|p\|_{H^0}$. If instead $p \in L^2([0, T]; H^0)$, then there exists a $v \in W$ such that $\nabla_A \cdot v = p$ for a.e. $t$ and $\|v\|_W \lesssim (1 + \sup_{0 \leq t \leq T} \|\eta(t)\|_{H^{5/2}(\Sigma)})^6 \|p\|_{L^2 H^0}$.

*Proof.* In the proof of lemma 3.3 of [2], it is established that for any $q \in L^2(\Omega)$, the problem $\nabla \cdot u = q$ admits a solution $u \in W$ such that $\|u\|_{H^1} \lesssim \|q\|_{H^0}$. A simple modification of this proof in infinite case can be applied to periodic case. Let define $q = Jp$, then

$$
\|q\|_{H^0}^2 = \int_\Omega |q|^2 = \int_\Omega |p|^2 J^2 \leq \|J\|_{L^\infty}^2 \|p\|_{H^0}^2 \lesssim (1 + \|\eta(t)\|_{H^{5/2}(\Sigma)})^2 \|p\|_{H^0}^2
$$

Hence, we know $v = M(t)u \in W$ satisfies $\nabla_A v = p$ and

$$
\|v\|_W^2 \lesssim (1 + \|\eta(t)\|_{H^{5/2}(\Sigma)})^4 \|v\|_W^2 \lesssim (1 + \|\eta(t)\|_{H^{5/2}(\Sigma)})^4 \|q\|_{H^0}^2,
$$

A similar argument may justifiy the case for $W$. \qed

With this lemma in hand, we can show the range of operator $Q_t$ and $Q$ is closed in $W$ and $\mathcal{W}$.

**Lemma 2.12.** $R(Q_t)$ is closed in $W$, and $R(Q)$ is closed in $\mathcal{W}$.

*Proof.* For $p \in H^0$, let $v \in W$ be the solution of $\nabla_{A_0} \cdot v = p$ provided by lemma 2.11. Then

$$
\|p\|_{H^0}^2 \lesssim \langle p, \nabla_A \cdot v \rangle_{H^0} = \langle Q_t(p), v \rangle_W \lesssim \|Q_t(p)\|_W \|v\|_W \lesssim \|Q_t(p)\|_W (1 + \|\eta(t)\|_{H^{5/2}(\Sigma)})^6 \|p\|_{H^0}
$$

such that

$$
\frac{1}{(1 + \|\eta(t)\|_{H^{5/2}(\Sigma)})^3} \|Q_t(p)\|_W \lesssim \|p\|_{H^0} \lesssim (1 + \|\eta(t)\|_{H^{5/2}(\Sigma)})^6 \|p\|_{H^0}.
$$

Hence $R(Q_t)$ is closed in $W$. A similar analysis shows that $R(Q)$ is closed in $\mathcal{W}$. \qed

Now we can perform a decomposition of $W$ and $\mathcal{W}$.

**Lemma 2.13.** We have that $W = X \oplus R(Q_t)$, i.e. $X^\perp = R(Q_t)$. Also, $\mathcal{W} = X \oplus R(Q)$, i.e. $X^\perp = R(Q)$.

*Proof.* By lemma 2.12, $R(Q_t)$ is closed subspace of $W$, so it suffices to show $R(Q_t)^\perp = X$. Let $v \in R(Q_t)^\perp$, then for all $p \in H^0$, we have

$$
\langle p, \nabla_A \cdot v \rangle_{H^0} = \langle Q_t(p), v \rangle_W = 0
$$

and hence $\nabla_A \cdot v = 0$, which implies $R(Q_t)^\perp \subseteq X$. On the other hand, suppose $v \in X$, then $\nabla_A \cdot v = 0$ implies

$$
\langle Q_t(p), v \rangle_W = \langle p, \nabla_A \cdot v \rangle_{H^0} = 0
$$

for all $p \in H^0$. Hence $V \in R(Q_t)^\perp$ and we see $X \subseteq R(Q_t)^\perp$. So we finish the proof. A similar argument can show $W = X \oplus R(Q)$. \qed

**Proposition 2.14.** If $\lambda \in W^*$ such that $\lambda(v) = 0$ for all $v \in X$, then there exists a unique $p(t) \in H^0$ such that

(2.2.31) \hspace{1cm} \langle p(t), \nabla_A \cdot v \rangle_{H^0} = \lambda(v) \hspace{1cm} \text{for all } v \in W

and $\|p(t)\|_{H^0} \lesssim (1 + \|\eta(t)\|_{H^{5/2}(\Sigma)})^6 \|\lambda\|_{H^*}$. If $\Lambda \in W^*$ such that $\Lambda(v) = 0$ for all $v \in X$, then there exists a unique $p \in L^2([0, T]; H^0)$ such that

(2.2.32) \hspace{1cm} \langle p, \nabla_A \cdot v \rangle_{L^2_{H^0}} = \Lambda(v) \hspace{1cm} \text{for all } v \in W

and $\|p\|_{L^2_{H^0}} \lesssim (1 + \sup_{0 \leq t \leq T} \|\eta(t)\|_{H^{5/2}(\Sigma)})^6 \|\Lambda\|_{H^*}$. \hfill \qed
Proof. If \( \lambda(v) = 0 \) for all \( v \in X \), then the Riesz representation theorem yields the existence of a unique \( u \in X^\perp \) such that \( \lambda(v) = \langle u, v \rangle_W \) for all \( v \in W \). By lemma 2.13, \( u = Q(p) \) for some \( p \in H^0 \). Then \( \lambda(v) = \langle Q(p), v \rangle_W = \langle p(t), \nabla_A \cdot v \rangle_{\mathcal{H}^0} \) for all \( v \in W \).

As for the estimate, by lemma 2.11, we may find \( v \in W \) such that \( \nabla_A \cdot v = p \) and \( \|v\|_W \lesssim (1 + \|\eta(t)\|_{H^{3/2}(\Sigma)})^6 \|p\|_{H^0} \). Hence,

\[
\|p\|_{H^0} \lesssim \langle p, \nabla_A \cdot v \rangle_{\mathcal{H}^0} = \lambda(v) \lesssim \|\lambda\|_W (1 + \|\eta(t)\|_{H^{3/2}(\Sigma)})^6 \|p\|_{H^0}
\]

So the desired estimate holds and a similar argument can show the result for \( \Lambda \). \hfill \Box

2.3. Elliptic Estimates. In this section, we will study two types of elliptic problems, which will be employed in deriving the linear estimates. For both equations, we will present wellposedness theorems to the higher regularity.

2.3.1. \( A \)-Stokes Equation. Let us consider the stationary Navier-Stokes problem.

\[
\begin{cases}

\nabla_A \cdot S_A(p, u) = F & \text{in } \Omega \\
\nabla_A \cdot u = G & \text{in } \Omega \\
S_A(p, u)N = H & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma_b
\end{cases}
\]

Since this problem is stationary, we will temporarily ignore the time dependence of \( \eta, A \), etc. Before discussing the higher regularity result for this equation, we should first define the weak formulation. Our method is quite standard; we will introduce the unknowns to \( w \) and \( \bar{u} \) such that the map \( \nabla_A \cdot w = 0 \) and satisfies

\[
\frac{1}{2} \langle \nabla_A w, \nabla_A v \rangle_{\mathcal{H}^0} - \langle p, \nabla_A \cdot v \rangle_{\mathcal{H}^0} = \langle F, v \rangle_{\mathcal{H}^0} - \langle H, v \rangle_{\mathcal{H}^0(\Sigma)}
\]

for all \( v \in W \).

Lemma 2.15. Suppose \( F \in W^*, G \in H^0 \) and \( H \in H^{-1/2}(\Sigma) \), then there exists a unique weak solution \( (u, p) \in W \times H^0 \) to 2.3.1.

Proof. By lemma 2.11, there exists a \( \bar{u} \in W \) such that \( \nabla_A \cdot \bar{u} = F^2 \). Naturally, we can switch the unknowns to \( w = u - \bar{u} \) such that in the weak formulation \( w \) is such that \( \nabla_A \cdot w = 0 \) and satisfies

\[
\frac{1}{2} \langle \nabla_A w, \nabla_A v \rangle_{\mathcal{H}^0} - \langle p, \nabla_A \cdot v \rangle_{\mathcal{H}^0} = -\frac{1}{2} \langle \nabla_A \bar{u}, \nabla_A v \rangle_{\mathcal{H}^0} + \langle F, v \rangle_{\mathcal{H}^0} - \langle H, v \rangle_{\mathcal{H}^0(\Sigma)}
\]

for all \( v \in W \).

To solve this problem, we may restrict our test function to \( v \in X \) such that the pressure term vanishes. A direct application of Riesz representation theorem to the Hilbert space whose inner product is defined as \( \langle u, v \rangle = \langle \nabla_A u, \nabla_A v \rangle_{\mathcal{H}^0} \) provides a unique \( w \in X \) such that

\[
\frac{1}{2} \langle \nabla_A w, \nabla_A v \rangle_{\mathcal{H}^0} = -\frac{1}{2} \langle \nabla_A \bar{u}, \nabla_A v \rangle_{\mathcal{H}^0} + \langle F, v \rangle_{\mathcal{H}^0} - \langle H, v \rangle_{\mathcal{H}^0(\Sigma)}
\]

for all \( v \in X \).

In order to introduce the pressure, we can define \( \lambda \in W^* \) as the difference of the left and right hand sides in (2.3.4). So \( \lambda(v) = 0 \) for all \( v \in X \). Then by proposition 2.14, there exists a unique \( p \in H^0 \) satisfying \( \langle p, \nabla_A \cdot v \rangle_{\mathcal{H}^0} = \lambda(v) \) for all \( v \in W \), which is equivalent to (2.3.3). \hfill \Box

The regularity gain available for solution to (2.3.1) is limited by the regularity of the coefficients of the operator \( \Delta_A, \nabla_A \) and \( \nabla_A \cdot \), and hence by the regularity of \( \eta \). In the next lemma, we will present some preliminary elliptic estimates.

Lemma 2.16. Suppose that \( \eta \in H^{k+1/2}(\Sigma) \) for \( k \geq 3 \) such that the map \( \Phi \) defined in (1.3.3) is a \( C^1 \) diffeomorphism of \( \Omega \) to \( \Omega' = \Phi(\Omega) \). If \( F \in H^0(\Omega) \), \( G \in H^1(\Omega) \) and \( H \in H^{1/2}(\Sigma) \), then the
equation (2.3.1) admits a unique strong solution \((u, p) \in H^2 \times H^1\), i.e. \((u, p)\) satisfies (2.3.1) in strong sense. Moreover, for \(r = 2, \ldots, k - 1\) we have the estimate
\[
\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim C(\eta) \left( \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)} \right)
\]
whenever the right hand side is finite, where \(C(\eta)\) is a constant depending on \(\|\eta\|_{H^{k+1/2}(\Sigma)}\).

**Proof.** This lemma is exactly the same as lemma 3.6 in [1], so we omit the proof here. \(\square\)

Notice that the estimate (2.3.5) can only go up to \(k - 1\) order, which does not fully satisfy our requirement. Hence, in the following we will employ approximating argument to improve this estimate. For clarity, we divide it into two steps. In the next lemma, we first prove that the constant can actually only depend on the initial free surface.

**Lemma 2.17.** Let \(k \geq 6\) be an integer and suppose that \(\eta \in H^{k+1/2}(\Sigma)\) and \(\eta_0 \in H^{k+1/2}(\Sigma)\). Then there exists \(\epsilon_0 > 0\) such that if \(\|\eta - \eta_0\|_{H^{k-3/2}(\Sigma)} \leq \epsilon_0\), solution to (2.3.1) satisfies
\[
\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim C(\eta_0) \left( \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)} \right)
\]
for \(r = 2, \ldots, k - 1\), whenever the right hand side is finite, where \(C(\eta_0)\) is a constant depending on \(\|\eta_0\|_{H^{k+1/2}(\Sigma)}\).

**Proof.** Based on lemma 2.16, we have the estimate
\[
\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim C(\eta) \left( \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)} \right)
\]
for \(r = 2, \ldots, k - 1\), whenever the right hand side is finite, where \(C(\eta)\) is a constant depending on \(\|\eta\|_{H^{k+1/2}(\Sigma)}\). Define \(\xi = \eta - \eta_0\), then \(\xi \in H^{k+1/2}(\Sigma)\). Let us denote \(A_0\) and \(N_0\) of quantities in terms of \(\eta_0\). We rewrite the equation 2.3.1 as a perturbation of initial status
\[
\begin{aligned}
\nabla_{A_0} \cdot S_{A_0}(p, u) &= F + F^0 \quad \text{in} \quad \Omega \\
\nabla_{A_0} \cdot u &= G + G^0 \quad \text{in} \quad \Omega \\
S_{A_0}(p, u)N_0 &= H + H^0 \quad \text{on} \quad \Sigma \\
u &= 0 \quad \text{on} \quad \Sigma_b
\end{aligned}
\]
where
\[
\begin{aligned}
F^0 &= \nabla_{A_0-A} \cdot S_A(p, u) + \nabla_{A_0} \cdot S_{A_0-A}(p, u) \\
G^0 &= \nabla_{A_0-A} \cdot u \\
H^0 &= S_{A_0}(p, u)(N_0 - N) + S_{A_0-A}(p, u)
\end{aligned}
\]
Suppose that \(\|\xi\|_{H^{k-3/2}(\Sigma)} \leq 1\), which implies \(\|\xi\|_{H^{k-3/2}(\Sigma)} \leq \|\xi\|_{H^{k-3/2}(\Sigma)} < 1\) for any \(l > 1\). A straightforward calculation reveals that
\[
\begin{aligned}
\|F^0\|_{H^{r-2}} &\leq C(1 + \|\eta_0\|_{H^{k+1/2}(\Sigma)})^4 \|\xi\|_{H^{k-3/2}(\Sigma)} (\|u\|_{H^r} + \|p\|_{H^{r-1}}) \\
\|G^0\|_{H^{r-1}} &\leq C(1 + \|\eta_0\|_{H^{k+1/2}(\Sigma)})^2 \|\xi\|_{H^{k-3/2}(\Sigma)} (\|u\|_{H^r}) \\
\|H^0\|_{H^{r-3/2}(\Sigma)} &\leq C(1 + \|\eta_0\|_{H^{k+1/2}(\Sigma)})^2 \|\xi\|_{H^{k-3/2}(\Sigma)} (\|u\|_{H^r} + \|p\|_{H^{r-1}})
\end{aligned}
\]
for \(r = 2, \ldots, k - 1\) and \(C\) includes the term related to \(\delta\).

Since the initial surface function \(\eta_0\) satisfies all the requirement of lemma 2.16, we arrive at the estimate that for \(r = 2, \ldots, k - 1\)
\[
\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim C(\eta_0) \left( \|F + F^0\|_{H^{r-2}} + \|G + G^0\|_{H^{r-1}} + \|H + H^0\|_{H^{r-3/2}(\Sigma)} \right)
\]
where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+1/2}(\Sigma)}$. Combining all above, we will have
\begin{equation}
\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim C(\eta_0) \left( \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)} \right) + C(\eta_0)(1 + \|\eta_0\|_{H^{k+1/2}(\Sigma)})^4 \|\xi\|_{H^{k-3/2}(\Sigma)} (\|u\|_{H^r} + \|p\|_{H^{r-1}})
\end{equation}
So if
\begin{equation}
\|\xi\|_{H^{k-3/2}(\Sigma)} \leq \min \left\{ \frac{1}{2}, \frac{1}{4CC(\eta_0)(1 + \|\eta_0\|_{H^{k+1/2}(\Sigma)})^4} \right\}
\end{equation}
we can absorb the extra term in right hand side into left hand side and get a succinct form
\begin{equation}
\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim C(\eta_0) \left( \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)} \right)
\end{equation}
for $r = 2, \ldots, k - 1$.

Note that the above lemma only concerns about regularity up to $k - 1$ and we actually need two more order. Then the next result allows us to achieve this with a bootstrapping argument.

**Proposition 2.18.** Let $k \geq 6$ be an integer, and suppose that $\eta \in H^{k+1/2}(\Sigma)$ as well as $\eta_0 \in H^{k+1/2}(\Sigma)$ satisfying $\|\eta - \eta_0\|_{H^{k+1/2}(\Sigma)} \leq \epsilon_0$. Then solution to 2.3.1 satisfies
\begin{equation}
\|u\|_{H^r} + \|p\|_{H^{r-1}} \lesssim C(\eta_0) \left( \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)} \right)
\end{equation}
for $r = 2, \ldots, k + 1$, whenever the right hand side is finite, where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+1/2}(\Sigma)}$.

**Proof.** If $r \leq k - 1$, then this is just the conclusion of lemma 2.17, so our main aim is to gain two more regularity for $r = k$ and $r = k + 1$. In the following, we first define an approximate sequence for $\eta$. In the case that $\Sigma = R^2$, we let $\rho \in C^\infty_0(R^2)$ be such that $\text{supp}(\rho) \subset B(0, 2)$ and $\rho = 1$ for $B(0, 1)$. For $m \in \mathbb{N}$, define $\eta^m$ by $\hat{\eta}^m(\xi) = \rho(\xi/m)\hat{\eta}(\xi)$ where $\hat{\cdot}$ denotes the Fourier transform. For each $m$, $\eta^m \in H^j(\Sigma)$ for arbitrary $j \geq 0$ and also $\eta^m \rightharpoonup \eta$ in $H^{k+1/2}(\Sigma)$ as $m \to \infty$. In the periodic case, we define $\eta^m$ by throwing away the higher frequencies: $\hat{\eta}^m(n) = 0$ for $|n| \geq m$. Then $\eta^m$ has the same convergence property as above. Let $A^m$ and $\mathcal{N}^m$ be defined in terms of $\eta^m$.

Consider the problem (2.3.1) with $A$ and $\mathcal{N}$ replaced by $A^m$ and $\mathcal{N}^m$. Since $\eta^m \in H^{k+5/2}$, we can apply lemma (2.16) to deduce the existence of $(u^m, p^m)$ that solves
\begin{equation}
\begin{cases}
\nabla_{A^m} \cdot S_{A^m}(p^m, u^m) = F & \text{in } \Omega \\
\nabla \cdot u^m = G & \text{in } \Omega \\
S_{A^m}(p^m, u^m)\mathcal{N}_0 = H & \text{on } \Sigma \\
u^m = 0 & \text{on } \Sigma_b
\end{cases}
\end{equation}
and such that
\begin{equation}
\|u^m\|_{H^r} + \|p^m\|_{H^{r-1}} \lesssim C(\|\eta^m\|_{H^{k+5/2}(\Sigma)}) \left( \|F\|_{H^{r-2}} + \|G\|_{H^{r-1}} + \|H\|_{H^{r-3/2}(\Sigma)} \right)
\end{equation}
for $r = 2, \ldots, k + 1$. We can rewrite above equation in the following shape, as long as we split the $\mathcal{D}_{A^m}u^m$ term.
\begin{equation}
\begin{cases}
-\Delta_{A^m}u^m + \nabla_{A^m} p^m = F + \nabla_{A^m} G & \text{in } \Omega \\
\nabla_{A^m} \cdot u^m = G & \text{in } \Omega \\
(p^m I - \mathcal{D}_{A^m}u^m)\mathcal{N}^m = H & \text{on } \Sigma \\
u^m = 0 & \text{on } \Sigma_b
\end{cases}
\end{equation}
In the following, we will prove an improved estimate for $(u^m, p^m)$ in terms of $\|\eta^m\|_{H^{k+1/2}(\Sigma)}$. We divide the proof into several steps.
Step 1: Preliminaries
To abuse the notation, within this bootstrapping procedure, we always use \((u^m, p^m, \eta^m)\) to make the expression succinct, but in fact they should be understood as the approximate sequence. Also it is easy to see the term \(\nabla A^m G\) will not affect the shape of estimate because \(\|\nabla A^m G\|_{H^{k-1}} \lesssim \|\eta^m\|_{H^{k+1/2}(\Sigma)} \|G\|_{H^k}\). Hence, we still write \(F\) here to indicate the forcing term in first equation.

We write explicitly each terms in above equations, which will be hired in the following.

\begin{align}
\dot{u}_1 u_1 + \partial_{22} u_1 + (1 + A^2 + B^2)K^2\partial_{33} u_1 - 2AK\partial_{13} u_1 - 2BK\partial_{23} u_1 \\
&+ (AK\partial_3(\partial z) + BK\partial_3(BK) - \partial_1(AK) - \partial_2(BK) + K\partial_3 K)\partial_{13} u_1 + \partial_1 p - AK\partial_3 p = F_1 \\
\dot{u}_2 u_2 + \partial_{22} u_2 + (1 + A^2 + B^2)K^2\partial_{33} u_2 - 2AK\partial_{13} u_2 - 2BK\partial_{23} u_2 \\
&+ (AK\partial_3(\partial z) + BK\partial_3(BK) - \partial_1(AK) - \partial_2(BK) + K\partial_3 K)\partial_{13} u_2 + \partial_2 p - BK\partial_3 p = F_2 \\
\dot{u}_3 u_3 + \partial_{22} u_3 + (1 + A^2 + B^2)K^2\partial_{33} u_3 - 2AK\partial_{13} u_3 - 2BK\partial_{23} u_3 \\
&+ (AK\partial_3(\partial z) + BK\partial_3(BK) - \partial_1(AK) - \partial_2(BK) + K\partial_3 K)\partial_{13} u_3 + K\partial_3 p = F_3 \\
\dot{u}_1 - AK\partial_{31} u_1 + \partial_{22} u_2 - BK\partial_{31} u_2 + K\partial_{33} u_2 = G
\end{align}

where for all above, \(A, B\) and \(K\) should be understood in terms of \(\eta^m\). For convenience, we define

\begin{align}
\mathcal{Z} = C(\eta_0)P(\eta)\left( \|F\|_{H^{k-1}}^2 + \|G\|_{H^k}^2 + \|H\|_{H^{k-1/2}(\Sigma)}^2 \right)
\end{align}

where \(C(\eta_0)\) is a constant depending on \(\|\eta_0\|_{H^{k+1/2}(\Sigma)}\) and \(P(\eta)\) is the polynomial of \(\|\eta\|_{H^{k+1/2}(\Sigma)}\).

Step 2: \(r = k\) case
For \(k - 1\) order elliptic estimate, we have

\begin{align}
\|u\|_{H^{k-1}}^2 + \|p\|_{H^{k-2}}^2 \leq C(\eta_0)\left( \|F\|_{H^{k-3}}^2 + \|G\|_{H^{k-2}}^2 + \|H\|_{H^{k-5/2}(\Sigma)}^2 \right) \lesssim \mathcal{Z}
\end{align}

By lemma 2.17, the bounding constant \(C(\eta_0)\) only depend on \(\|\eta_0\|_{H^{k+1/2}}\).

For \(i = 1, 2\), since \((\partial_i u, \partial_i p)\) satisfies the equation

\begin{align}
\left\{ \begin{array}{ll}
-\Delta A(\partial_i u) + \nabla A(\partial_i p) = \bar{F} & \text{in } \Omega \\
\nabla A \cdot (\partial_i u) = \bar{G} & \text{in } \Omega \\
(\partial_i p)I - D_A(\partial_i u)N = \bar{H} & \text{on } \Sigma \\
\partial_i u = 0 & \text{on } \Sigma_b
\end{array} \right.
\end{align}

where

\begin{align*}
\bar{F} &= \partial_1 F + \nabla_{\partial_1 A} \cdot \nabla A u + \nabla_A \cdot \nabla_{\partial_1 A} u - \nabla_{\partial_1 A} p \\
\bar{G} &= \partial_1 G - \nabla_{\partial_1 A} \cdot u \\
\bar{H} &= \partial_1 H - (p I - D_A u) \partial_1 N + D_{\partial_1 A} u N
\end{align*}

Employing \(k - 1\) order elliptic estimate, we have

\begin{align}
\|\partial_i u\|_{H^{k-1}}^2 + \|\partial_i p\|_{H^{k-2}}^2 \lesssim C(\eta_0)\left( \|\bar{F}\|_{H^{k-3}}^2 + \|\bar{G}\|_{H^{k-2}}^2 + \|\bar{H}\|_{H^{k-5/2}(\Sigma)}^2 \right)
\end{align}

Since except for the derivatives of \(F, G\) and \(H\), all the other terms on the right hand side has the form \(\|A \cdot B\|_{H^r}\), in which \(A = \partial^2 \eta\) and \(B = \partial^3 u\) or \(\partial^3 p\). These kinds of estimates can be achieved by lemma A.1. Because this lemma will be repeated used in the following estimates, we will not mention it every time and all of the following estimate can be derived in the same
Combining (2.3.29), (2.3.30) and (2.3.32), it is easy to see we have
\[
\left\| \tilde{F} \right\|_{H^{3-k}}^2 + \left\| \tilde{G} \right\|_{H^{k-2}}^2 + \left\| \tilde{H} \right\|_{H^{k-5/2}}^2 
\leq \left\| F \right\|_{H^{3-k}}^2 + \left\| G \right\|_{H^{k-1}}^2 + \left\| H \right\|_{H^{k-3/2}}^2 + C(\eta_0) P(\eta) \left( \left\| u \right\|_{H^{k-1}}^2 + \left\| p \right\|_{H^{k-2}}^2 \right) 
\leq Z
\]

In detail, this means
\[
\left\| \partial_1 u \right\|_{H^{k-1}}^2 + \left\| \partial_1 p \right\|_{H^{k-2}}^2 + \left\| \partial_2 u \right\|_{H^{k-1}}^2 + \left\| \partial_2 p \right\|_{H^{k-2}}^2 \leq Z
\]

Hence, most parts in \( \left\| u \right\|_{H^k} + \left\| p \right\|_{H^{k-1}} \) has been covered by this estimate, except those with highest order derivative of \( \partial_3 \).

Multiplying \( A \) to (2.3.21) and adding it to (2.3.19) will eliminate the \( \partial_3 p \) term and get
\[
(\partial_1 u_1 + A\partial_1 u_3) + (\partial_2 u_1 + A\partial_2 u_3) + (1 + A^2 + B^2) K^2 (\partial_{33} u_1 + A\partial_{33} u_3) - 2AK(\partial_{13} u_1 + A\partial_{13} u_3) - 2BK(\partial_{23} u_1 + A\partial_{23} u_3) + (AK\partial_3 AK) + BK\partial_3 BK - \partial_1 AK - \partial_2 BK + K\partial_3 K)(\partial_1 u_1 + A\partial_1 u_3) + \partial_3 p = F_1 + A F_3
\]

Then taking derivative \( \partial_3^{k-2} \) on both sides and focus in the term \((1 + A^2 + B^2)K^2(\partial_3^k u_1 + A\partial_3^k u_3), \) the estimate of all the other terms in \( H^0 \) norm implies that
\[
\left\| \partial_3^k u_1 + A\partial_3^k u_3 \right\|_{H^0}^2 \leq Z
\]

Similarly, we have
\[
\left\| \partial_3^k u_2 + B\partial_3^k u_3 \right\|_{H^0}^2 \leq Z
\]

Rearrange the terms in (2.3.22), we get
\[
K(1 + A^2 + B^2) \partial_3 u_3 = G - \partial_1 u_1 - \partial_2 u_2 + AK(\partial_3 u_1 + A\partial_3 u_3) + BK(\partial_3 u_2 + B\partial_3 u_3)
\]

Taking derivative \( \partial_3^{k-1} \) on both sides, focusing in the term \( K(1 + A^2 + B^2)\partial_3^k u_3 \) employing all of the estimate we have known, we can show
\[
\left\| \partial_3^k u_3 \right\|_{H^0}^2 \leq Z
\]

Combining (2.3.29), (2.3.30) and (2.3.32), it is easy to see we can get
\[
\left\| \partial_3^k u_1 \right\|_{H^0}^2 + \left\| \partial_3^k u_2 \right\|_{H^0}^2 \leq Z
\]

Plugging this to (2.3.21) and taking derivative \( \partial_3^{k-2} \) on both sides, we get
\[
\left\| \partial_3^{k-1} p \right\|_{H^0}^2 \leq Z
\]

Combining this with all above estimate, we have proved
\[
\left\| u \right\|_{H^k}^2 + \left\| p \right\|_{H^{k-1}}^2 \leq Z
\]

Therefore, we have proved the case \( r = k \).

Step 3: \( r = k + 1 \) case: first loop
For \( i, j = 1, 2 \), since \( (\partial_{ij} u, \partial_{ij} p) \) satisfies the equation
\[
\begin{cases}
-\Delta_A (\partial_{ij} u) + \nabla_A (\partial_{ij} p) = \tilde{F} & \text{in } \Omega \\
\nabla_A \cdot (\partial_{ij} u) = \tilde{G} & \text{in } \Omega \\
((\partial_{ij} p) I - \nabla_A (\partial_{ij} u)) \cdot \mathcal{N} = \tilde{H} & \text{on } \Sigma \\
(\partial_{ij} u) = 0 & \text{on } \Sigma_h
\end{cases}
\]
where
\[ \tilde{F} = \partial_j F + \nabla_{\partial_j A} \cdot \nabla A u + \nabla A \cdot \nabla_{\partial_j A} u + \nabla_{\partial_j A} \cdot \nabla A u + \nabla_{\partial_j A} \cdot \nabla A (\partial_j u) + \nabla \cdot \nabla_{\partial_j A} (\partial_j u) + \nabla A \cdot \nabla_{\partial_j A} (\partial_j u) - \nabla_{\partial_j A} p - \nabla_{\partial_j A} (\partial_j p) - \nabla_{\partial_j A} (\partial_j p) \]
\[ \tilde{G} = \partial_j G - \nabla_{\partial_j A} \cdot u - \nabla_{\partial_j A} \cdot (\partial_j u) - \nabla_{\partial_j A} \cdot (\partial_j u) \]
\[ \tilde{H} = \partial_j H - (p I - \mathbb{D}_A u) \partial_j N - (p I - \mathbb{D}_A (\partial_j u) - \mathbb{D}_A (\partial_j u) \partial_j N - ((\partial_j p) I - \mathbb{D}_A (\partial_j u) - \mathbb{D}_A (\partial_j u)) \partial_j N + (\mathbb{D}_{\partial_j A} u + \mathbb{D}_{\partial_j A} (\partial_j u) + \mathbb{D}_{\partial_j A} (\partial_j u)) N \]
Employing \( k - 1 \) order elliptic estimate, we have
\[
\| \partial_j u \|^2_{H_k - 1} + \| \partial_j p \|^2_{H_k - 2} \lesssim \| \tilde{F} \|^2_{H_k - 3} + \| \tilde{G} \|^2_{H_k - 2} + \| \tilde{H} \|^2_{H_k - 5/2} \lesssim \mathcal{Z}
\]
where the forcing estimate can be taken in a similar argument as in \( r = k \) case.
In detail, this is actually
\[
\| \partial_{11} u \|^2_{H_k - 1} + \| \partial_{12} u \|^2_{H_k - 1} + \| \partial_{22} u \|^2_{H_k - 1} + \| \partial_{11} p \|^2_{H_k - 2} + \| \partial_{12} p \|^2_{H_k - 2} + \| \partial_{22} p \|^2_{H_k - 2} \lesssim \mathcal{Z}
\]

Step 4: \( r = k + 1 \) case: second loop
Similar to \( r = k \) case argument, for \( i = 1, 2 \), taking derivative \( \partial_3^{k-2} \partial_i \) on both sides of (2.3.29) and focus in the term \( \partial_3^k \partial_i u_1 + A \partial_3^k \partial_i u_3 \), we get
\[
\| \partial_3^k \partial_i u_1 + A \partial_3^k \partial_i u_3 \|^2_{H_0} \lesssim \mathcal{Z}
\]
Similarly, we have
\[
\| \partial_3^k \partial_i u_2 + B \partial_3^k \partial_i u_3 \|^2_{H_0} \lesssim \mathcal{Z}
\]
Plugging in this result to (2.3.22) and taking derivative \( \partial_3^{k-2} \partial_i \) on both sides, we will get
\[
\| \partial_3^k \partial_i u_3 \|^2_{H_0} \lesssim \mathcal{Z}
\]
An easy estimate for these three terms implies
\[
\| \partial_3^k \partial_i u \|^2_{H_0} \lesssim \mathcal{Z}
\]
Plugging this to (2.3.21) and taking derivative \( \partial_3^{k-2} \partial_i \) on both sides, we get
\[
\| \partial_3^{k-1} \partial_i p \|^2_{H_0} \lesssim \mathcal{Z}
\]
Combining all above estimate with estimate of previous steps, we have shown
\[
\| \partial_{13} u \|^2_{H_k - 1} + \| \partial_{23} u \|^2_{H_k - 1} + \| \partial_{13} p \|^2_{H_k - 2} + \| \partial_{23} p \|^2_{H_k - 2} \lesssim \mathcal{Z}
\]
In detail, this is actually
\[
\| \partial_{13} u \|^2_{H_k - 1} + \| \partial_{23} u \|^2_{H_k - 1} + \| \partial_{13} p \|^2_{H_k - 2} + \| \partial_{23} p \|^2_{H_k - 2} \lesssim \mathcal{Z}
\]

Step 5: \( r = k + 1 \) case: third loop
Again we use the same trick as above. Taking derivative \( \partial_3^{k-1} \) on both sides of (2.3.29) and focusing in the term \( \partial_3^{k+1} u_1 + A \partial_3^{k+1} u_3 \), we can bound \( \partial_3^{k+1} u_1 - A \partial_3^{k+1} u_3 \). Similarly, we control \( \partial_3^{k+1} u_2 - B \partial_3^{k+1} u_3 \). Plugging in this result to (2.3.22) and taking derivative \( \partial_3^{k-1} \) on both sides, we can estimate \( \partial_3^{k+1} u_3 \). Then we have
\[
\| \partial_3^{k+1} u \|^2_{H_0} \lesssim \mathcal{Z}
\]
Plugging this to (2.3.21) and taking derivative $\partial^k_k$ on both sides, we get

\begin{equation}
(2.3.47) \quad \left\| \partial^k_k P \right\|^2_{H^0} \lesssim Z
\end{equation}

Combining this will all above estimate, we get

\begin{equation}
(2.3.48) \quad \left\| \partial^k_k u \right\|^2_{H^{k-1}} + \left\| \partial^k_k p \right\|^2_{H^{k-2}} \lesssim Z
\end{equation}

Step 6: $r = k + 1$ case: conclusion

To synthesize, (2.3.38), (2.3.45) and (2.3.48) imply that all the second order derivative of $u$ in $k - 1$ norm and $p$ in $k - 2$ norm is controlled, so we naturally have the estimate

\begin{equation}
(2.3.49) \quad \left\| u \right\|^2_{H^{k+1}} + \left\| p \right\|^2_{H^k} \lesssim C(\eta_0) P(\eta) \left( \left\| F \right\|^2_{H^{k-1}} + \left\| G \right\|^2_{H^k} + \left\| H \right\|^2_{H^{k-1/2}} \right)
\end{equation}

This is just what we desired.

Now let us go back to original notation, since $P(\cdot)$ is a fixed polynomial, this gives an estimate

\begin{equation}
(2.3.50) \quad \left\| u^m \right\|^2_{H^{k+1}} + \left\| p^m \right\|^2_{H^k} \lesssim C(\eta_0) P(\eta^m) \left( \left\| F \right\|^2_{H^{k-1}} + \left\| G \right\|^2_{H^k} + \left\| H \right\|^2_{H^{k-1/2}(\Sigma)} \right)
\end{equation}

where $C(\eta_0)$ depends on $\left\| \eta_0 \right\|_{H^{k+1/2}(\Sigma)}$.

This bound implies that the sequence $(u^m, p^m)$ is uniformly bounded in $H^{k+1} \times H^k$, so we can extract weakly convergent subsequence $u^m \to u^0$ and $p^m \to p^0$.

In the second equation of (2.3.16), we multiplied both sides by $J^m w$ for $w \in C_0^\infty$ to see that

\begin{equation}
(2.3.51) \quad \int_{\Omega} G w J^m = \int_{\Omega} (\nabla A^m \cdot u^m) w J^m = - \int_{\Omega} u^m \cdot (\nabla A^m w) J^m \to - \int_{\Omega} u^0 \cdot (\nabla A w) J = \int_{\Omega} (\nabla A \cdot u^0) J
\end{equation}

which implies $\nabla A \cdot u^0 = G$. Then we multiply the first equation by $w J^m$ for $w \in W$ and integrate by parts to see that

\begin{equation}
(2.3.52) \quad \frac{1}{2} \int_{\Omega} D_{A^m} u^m : D_{A^m} w J^m - \int_{\Omega} p^m \nabla_{A^m} \cdot w J^m = \int_{\Omega} F \cdot w J^m - \int_{\Sigma} H \cdot w
\end{equation}

Passing to the limit $m \to \infty$, we deduce that

\begin{equation}
(2.3.53) \quad \frac{1}{2} \int_{\Omega} D_A u^0 : D_A w J - \int_{\Omega} p^0 \nabla_A \cdot w J = \int_{\Omega} F \cdot w J - \int_{\Sigma} H \cdot w
\end{equation}

which reveals, upon integrating by parts again, $(u^0, p^0)$ satisfies (2.3.1). Since $(u, p)$ is the unique strong solution to (2.3.1), we have $u^0 = u$ and $p^0 = p$. This, weakly lower semi-continuity and estimate (2.3.50) imply (2.3.54).

**Remark 2.19.** The key part of this proposition is that as long as we can deduce $\eta_0 \in H^{k+1/2}(\Sigma)$ and $\eta \in H^{k+1/2}(\Sigma)$, we can achieve $u \in H^{k+1}$ and $p \in H^k$, which is the highest possible regularity we can expect.

**Remark 2.20.** By our notation rule, since $C(\eta_0)$ depends on $\Omega_0$ and is given implicitly, we can take it as a universal constant, so the estimate may be rewritten as follows.

\begin{equation}
(2.3.54) \quad \left\| u \right\|^r_{H^r} + \left\| p \right\|^r_{H^{r-1}} \lesssim \left\| F \right\|^r_{H^{r-2}} + \left\| G \right\|^r_{H^{r-1}} + \left\| H \right\|^r_{H^{r-3/2}(\Sigma)}
\end{equation}

for $r = 2, \ldots, k + 1$. 
2.3.2. A-Poisson Equation. We consider the elliptic problem

$$\begin{cases}
\Delta_A p = f & \text{in } \Omega \\
p = g & \text{on } \Sigma \\
\nabla_A p \cdot \nu = h & \text{on } \Sigma_b
\end{cases}$$

(2.3.55)

For the weak formulation, we suppose \( f \in V^* \), \( g \in H^{1/2}(\Sigma) \) and \( h \in H^{-1/2}(\Sigma_b) \). Let \( \tilde{p} \in H^1 \) be an extension of \( g \) such that \( \text{supp}(\tilde{p}) \subset \{ -\inf(b)/2 < x_3 < 0 \} \). Then we can switch the unknowns to \( q = p - \tilde{p} \). Hence, the weak formulation of (2.3.55) is

$$\langle \nabla_A q, \nabla_A \phi \rangle_{\Sigma} = -\langle \nabla_A \tilde{p}, \nabla_A \phi \rangle_{\Sigma} - \langle f, \phi \rangle_{V^*} + \langle h, \phi \rangle_{-1/2}$$

(2.3.56)

for all \( \phi \in V \), where \( \langle \cdot, \cdot \rangle_{V^*} \) denotes the dual pairing with \( V \) and \( \langle \cdot, \cdot \rangle_{-1/2} \) denotes the dual pairing with \( H^{-1/2}(\Sigma_b) \). The existence and uniqueness of a solution to (2.3.56) is given by standard argument for elliptic equation.

If \( f \) has a more specific fashion, we can rewrite the equation to accommodate the structure of \( f \). Suppose the action of \( f \) on an element \( \phi \in V \) is given by

$$\langle f, \phi \rangle_{V^*} = \langle f_0, \phi \rangle_{H^0} + \langle F_0, \nabla_A \phi \rangle_{H^0}$$

(2.3.57)

where \( (f_0, F_0) \in H^0 \times H^0 \) with \( \|f_0\|_{H^0} + \|F_0\|_{H^0} = \|f\|_{V^*} \). Then we rewrite (2.3.56) into

$$\langle \nabla_A p + F_0, \nabla_A \phi \rangle_{H^0} = -\langle f_0, \phi \rangle_{H^0} + \langle h, \phi \rangle_{-1/2}$$

(2.3.58)

Hence, it is possible to say \( p \) is a weak solution to equation

$$\begin{cases}
\nabla_A \cdot (\nabla_A p + F_0) = f_0 \\
p = g \\
(\nabla_A p + F_0) \cdot \nu = h
\end{cases}$$

(2.3.59)

This formulation will be used to construct the higher order initial conditions in later sections.

**Lemma 2.21.** Suppose that \( \eta \in H^{k+1/2}(\Sigma) \) for \( k \geq 3 \) such that the map \( \Phi \) is a \( C^1 \) diffeomorphism of \( \Omega \) to \( \Omega' = \Phi(\Omega) \). If \( f \in H^0, g \in H^{3/2} \) and \( h \in H^{1/2} \), then the equation 2.3.55 admits a unique strong solution \( p \in H^2 \). Moreover, for \( r = 2, \ldots, k - 1 \), we have the estimate

$$\|p\|_{H^r} \lesssim C(\eta) \left( \|f\|_{H^{r-2}} + \|g\|_{H^{r-1/2}(\Sigma)} + \|h\|_{H^{r-3/2}(\Sigma_b)} \right)$$

(2.3.60)

whenever the right hand side is finite, where \( C(\eta) \) is a constant depending on \( \|\eta\|_{H^{k+1/2}(\Sigma)} \).

**Proof.** This is exactly the same as lemma 3.8 in [1], so we omit the proof here. \( \square \)

Next, we will prove the bounding constant for the estimate can actually only depend on the initial surface. Since we do not need optimal regularity result for this equation, we do not need the bootstrapping argument now.

**Proposition 2.22.** Let \( k \geq 6 \) be an integer suppose that \( \eta \in H^{k+1/2}(\Sigma) \) and \( \eta_0 \in H^{k+1/2}(\Sigma) \). There exists \( \epsilon_0 > 0 \) such that if \( \|\eta - \eta_0\|_{H^{k-3/2}} \leq \epsilon_0 \), then solution to 2.3.55 satisfies

$$\|p\|_{H^r} \lesssim C(\eta_0) \left( \|f\|_{H^{r-2}} + \|g\|_{H^{r-1/2}(\Sigma)} + \|h\|_{H^{r-3/2}(\Sigma_b)} \right)$$

(2.3.61)

for \( r = 2, \ldots, k - 1 \), whenever the right hand side is finite, where \( C(\eta_0) \) is a constant depending on \( \|\eta_0\|_{H^{k+1/2}(\Sigma)} \).

**Proof.** The proof is similar to lemma 2.17. We rewrite the problem as a perturbation of the initial status.

$$\begin{cases}
\Delta_{A_0} p^m = f + f^m & \text{in } \Omega \\
p^m = g + g^m & \text{on } \Sigma \\
\nabla_{A_0} p^m \cdot \nu = h + h^m & \text{on } \Sigma_b
\end{cases}$$

(2.3.62)

The constant in this elliptic estimate only depends on the initial free surface. We may estimate \( f^m, g^m \) and \( h^m \) in terms of \( p^m, \eta_0 \) and \( \xi^m = \eta^m - \eta_0 \). The smallness of \( \xi^m \) implies that we can absorb these terms into left hand side, which is actually (2.3.63). \( \square \)
Remark 2.23. Similarly, we can also take this constant $C(\eta_0)$ as a universal constant and rewrite the estimate as follows.

\[
\|p\|_{H^r} \lesssim \|f\|_{H^{r-2}} + \|g\|_{H^{r-1/2}(\Sigma)} + \|h\|_{H^{r-3/2}(\Sigma_b)}
\]

for $r = 2, \ldots, k - 1$.

2.4. Linear Estimates.

\[
\begin{aligned}
&\begin{cases}
\partial_t u - \Delta u + \nabla \cdot u = F & \text{in } \Omega \\
\nabla \cdot u = 0 & \text{in } \Omega \\
(pI - \mathbb{D}u)\nu = H & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma_b
\end{cases}
\end{aligned}
\]

(2.4.1)

In this section, we will study the linear Navier-Stokes problem (2.4.1). Following the path of I. Tice and Y. Guo in [1]. We will employ two notions of solution: weak and strong. Then we prove the wellposedness theorem from lower regularity to higher regularity. To note that, most parts of the results and proofs here are identical to section 4 of [1], so we won’t give all the details. The only difference is the bounding constants related to free surface since our surface has less restrictions.

In this section, we always assume $P(z)$ as a polynomial of $z$.

2.4.1. Weak Solution and Strong Solution. The weak formulation of linear problem (2.4.1) is as follows.

\[
\begin{aligned}
&\begin{cases}
\langle \partial_t u, v \rangle_{L^2} + \frac{1}{2} \langle \mathbb{D}u, \mathbb{D}v \rangle_{L^2} - \langle p, \nabla \cdot v \rangle_{L^2} = \langle F, v \rangle_{L^2} - \langle H, v \rangle_{L^2}(\Sigma)
\end{cases}
\end{aligned}
\]

(2.4.2)

Definition 2.24. Suppose that $u_0 \in H^0$, $F \in \mathcal{W}^*$ and $H \in L^2([0, T]; H^{-1/2}(\Sigma))$. If there exists a pair $(u, p)$ achieving the initial data $u_0$ and satisfies $u \in \mathcal{W}, p \in L^2([0, T]; H^0)$ and $\partial_t u \in \mathcal{W}^*$, such that the (2.4.2) holds for any $v \in \mathcal{W}$, we call it a weak solution.

If further restricting the test function $v \in \mathcal{X}$, we have a pressureless weak formulation.

\[
\begin{aligned}
&\begin{cases}
\langle \partial_t u, v \rangle_{L^2} + \frac{1}{2} \langle \mathbb{D}u, \mathbb{D}v \rangle_{L^2} - \langle p, \nabla \cdot v \rangle_{L^2} = \langle F, v \rangle_{L^2} - \langle H, v \rangle_{L^2}(\Sigma)
\end{cases}
\end{aligned}
\]

(2.4.3)

Definition 2.25. Suppose that $u_0 \in \mathcal{Y}(0)$, $F \in \mathcal{X}^*$ and $H \in L^2([0, T]; H^{-1/2}(\Sigma))$. If there exists a function $u$ achieving the initial data $u_0$ and satisfies $u \in \mathcal{X}$ and $\partial_t u \in \mathcal{X}^*$, such that the (2.4.3) holds for any $v \in \mathcal{X}$, we call it a pressureless weak solution.

Remark 2.26. It is noticeable that in $\Omega$, $\mathcal{W}^* \subset \mathcal{X}^*$ and $\|u\|_{\mathcal{X}^*} \leq \|u\|_{\mathcal{W}^*}$; on $\Sigma$, for $u \in \mathcal{X}$, we have $u|\Sigma| \in L^2([0, T]; H^{1/2}(\Sigma))$.

Since our main aim is to prove higher regularity of the linear problem, so the weak solution is a natural byproduct of the proof for strong solution in next subsection. Hence, we will not give the existence proof here and only present the uniqueness.

Lemma 2.27. Suppose $u$ is a pressureless weak solution of (2.4.1). Then for a.e. $t \in [0, T]$,

\[
\frac{1}{2} \int_0^t \int_\Omega |u(t)|^2 + \frac{1}{2} \int_0^t \int_\Omega |\mathbb{D}u|^2 = \frac{1}{2} \int_0^t \int_\Omega |u_0|^2 + \frac{1}{2} \int_0^t \int_\Omega |u|^2 \partial_t J + \int_0^t \int_\Omega \langle F, u \rangle - \int_0^t \int_\Sigma H \cdot u
\]

Also

\[
\|u\|_{L^\infty L^0} + \|u\|_{L^2 H^1} \lesssim C_0(\eta)(\|u_0\|_{H^0} + \|F\|_{\mathcal{X}^*}^2 + \|H\|_{L^2 H^{-1/2}(\Sigma)}^2)
\]

(2.4.5)

where

\[
C_0(\eta) = P(1 + \|\eta\|_{H^{5/2}(\Sigma)} + \|\partial_t \eta\|_{H^{5/2}(\Sigma)}) \exp \left( TP(1 + \|\partial_t \eta\|_{H^{5/2}(\Sigma)}) \right)
\]

(2.4.6)

Proof. The proof is almost the same as that of lemma 4.1 in [1]. The only difference is that we should replace the lemma 2.1 in [1] used in that proof by our lemma 2.5 now. \qed
Proposition 2.28. If a pressureless weak solution exists, then it is unique.

Proof. If \( u^1 \) and \( u^2 \) are both pressureless weak solutions to (2.4.1), then \( w = u^1 - u^2 \) is a pressureless weak solution with \( F = H = w(0) = 0 \). Then the estimate (2.4.5) implies that \( w = 0 \). Hence the solution is unique. \( \square \)

Next, we define the strong solution.

Definition 2.29. Suppose that

\[
\begin{align*}
  & u_0 \in H^2 \cap X(0) \\
  & F \in L^2([0, T]; H^1) \cap C^0([0, T]; H^0) \\
  & H \in L^2([0, T]; H^{3/2}(\Sigma)) \cap C^0([0, T]; H^{1/2}(\Sigma)) \\
  & \partial_t F \in X^* \\
  & \partial_t H \in L^2([0, T]; H^{-1/2}(\Sigma))
\end{align*}
\]

If there exists a pair \((u, p)\) achieving the initial data \( u_0 \) and satisfies

\[
\begin{align*}
  & u \in L^2([0, T]; H^3) \cap C^0([0, T]; H^2) \cap \mathcal{X} \\
  & \partial_t u \in L^2([0, T]; H^1) \cap C^0([0, T]; H^0) \\
  & p \in L^2([0, T]; H^2) \cap C^0([0, T]; H^1)
\end{align*}
\]

such that they satisfies (2.4.1) in the strong sense, we call it a strong solution.

2.4.2. Lower Regularity Theorem.

Theorem 2.30. Assume the initial data and forcing terms satisfies the condition (2.4.7). Define the divergence-\(A\) preserving operator \( D_t u \) by

\[
D_t u = \partial_t u - Ru \quad \text{for} \quad R = \partial_t M M^{-1}
\]

where \( M \) is defined in (2.2.30). Furthermore, define the orthogonal projection onto the tangent space of the surface \( \{x_3 = \eta_0\} \) according to

\[
\Pi_0 (v) = v - (v \cdot \mathbf{N}_0) \mathbf{N}_0 |\mathbf{N}_0|^{-2} \quad \text{for} \quad \mathbf{N}_0 = (-\partial_1 \eta_0, -\partial_2 \eta_0, 1)
\]

Suppose the initial data satisfy the compatible condition

\[
\Pi_0 (H(0) + \mathbb{D}_A_0 u_0 \mathbf{N}_0) = 0
\]

Then the equation (2.4.1) admits a unique strong solution \((u, p)\) which satisfies the estimate

\[
\begin{align*}
  & \|u\|_{L^2_t H^3}^2 + \|\partial_t u\|_{L^2_t H^1}^2 + \|\partial_t^2 u\|_{W^*}^2 + \|u\|_{L^\infty_t H^0}^2 + \|\partial_t u\|_{L^\infty_t H^0}^2 + \|p\|_{L^2_t H^2}^2 + \|p\|_{L^\infty_t H^1}^2 \\
  \lesssim & L(\eta) \left( \|u_0\|_{H^2}^2 + \|F(0)\|_{H^0}^2 + \|H(0)\|_{H^{1/2}}^2 + \|F\|_{L^2_t H^1}^2 + \|\partial_t F\|_{X^*}^2 + \|H\|_{L^2_t H^{3/2}}^2 + \|\partial_t H\|_{L^2_t H^{-1/2}}^2 \right)
\end{align*}
\]

where

\[
L(\eta) = P(1 + K(\eta)) \exp \left( T P(1 + K(\eta)) \right)
\]

for

\[
K(\eta) = \sup_{0 \leq t \leq T} \left( \|\eta(t)\|_{H^{9/2}(\Sigma)} + \|\partial_t \eta(t)\|_{H^{7/2}(\Sigma)} + \|\partial_t^2 \eta(t)\|_{H^{5/2}(\Sigma)} \right)
\]

The initial pressure \( p(0) \in H^1 \), is determined in terms of \( u_0, \eta_0, F(0) \) and \( H(0) \) as the weak solution to

\[
\begin{align*}
  & \nabla_{A_0} \cdot (\nabla_{A_0} p(0) - F(0)) = -\nabla_{A_0} (R(0) u_0) \quad \text{in} \quad \Omega \\
  & p(0) = (H(0) + \mathbb{D}_A_0 u_0 \mathbf{N}_0) \cdot \mathbf{N}_0 |\mathbf{N}_0|^{-2} \quad \text{on} \quad \Sigma \\
  & (\nabla_{A_0} p(0) - F(0)) \cdot \nu = \Delta_{A_0} u_0 \cdot \nu \quad \text{on} \quad \Sigma_0
\end{align*}
\]

in the sense of (2.3.59).

Also, \( D_t u(0) = \partial_t u(0) - R(0) u_0 \) satisfies

\[
D_t u(0) = \Delta_{A_0} u_0 - \nabla_{A_0} p(0) + F(0) - R(0) u_0 \in \mathcal{Y}(0).
\]
Moreover, \((D_tu, \partial_t p)\) satisfies
\[
\begin{align*}
\partial_t(D_tu) - \Delta_A(D_tu) + \nabla_A(\partial_t p) &= D_tF + GF \quad \text{in} \quad \Omega \\
\nabla_A \cdot (D_tu) &= 0 \quad \text{in} \quad \Omega \\
S_A(\partial_t p, D_tu)N &= \partial_t H + GH \quad \text{on} \quad \Sigma \\
D_tu &= 0 \quad \text{on} \quad \Sigma_b
\end{align*}
\]
in the weak sense of (2.4.3), where
\[
G^F = -(R + \partial_t JK)\Delta_A u - \partial_t Ru + (\partial_t JK + R + R^T)\nabla_A p + \nabla_A \cdot (\mathbb{D}_A(Ru) - R \Omega \mathbb{D}_A u + \mathbb{D}_\partial A u)
\]
\[
G^H = \mathbb{D}_A(Ru)N - (pI - \mathbb{D}_A u)\partial_t N + \mathbb{D}_\partial A u N
\]

Proof. Since this theorem is almost identical to theorem 4.3 in [1], we will only point out the differences here without giving details. In order for contrast, we use the same notation and statement as [1] here.

1. The only difference in the assumption part is that in theorem 4.3 of [1], \(K(\eta)\) is sufficiently small, but our \(K(\eta)\) can be arbitrary. Hence, in [1], we can always bound \(P(1 + K(\eta)) \leq 1 + K(\eta)\), but now we have to keep this term as a polynomial (this is the only difference of \(L(\eta)\) between these two theorems).

2. It is noticeable that each lemma used in proving theorem 4.3 of [1] has been recovered or slightly modified in our preliminary section. To note that these kind of modification merely involves the bounding constant due to the arbitrary \(K(\eta)\), but does not change the conclusion and the shape of estimate at all. Therefore, all these modification will only contribute to \(P(1 + K(\eta))\) term.

3. In elliptic estimate, we now use proposition 2.18 to replace proposition 3.7 in [1], which gives exactly the same estimates.

Based on all above, we can easily repeat the proving process in [1] with no more work, so our result naturally follows. \(\square\)

2.4.3. Initial Data and Compatible Condition. Before starting to discuss the higher regularity result, we first need to determine the initial data to higher order derivatives and the corresponding compatible conditions. Although this part is also identical to section 4.3 of [1], for clarity, we repeat this process here.

Define a vector field on \(\Omega\):
\[
G^0(F, v, q) = \Delta_A v - \nabla_A q + F - Rv
\]

Define function \(f\) on \(\Omega\), \(g\) on \(\Sigma\) and \(h\) on \(\Sigma_b\) according to
\[
\begin{align*}
f(F, v) &= \nabla_A \cdot (F - Rv) \\
g(H, v) &= (H + \mathbb{D}_A u N) \cdot N |N|^{-2} \\
h(F, v) &= (F + \Delta_A v) \cdot v
\end{align*}
\]

Define the quantities related to the initial data:
\[
\begin{align*}
\mathcal{H}_0(u, p) &= \sum_{j=0}^N \left\| \partial_t^j u(0) \right\|_{H^2N-2j}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j p(0) \right\|_{H^2N-2j-1}^2 \\
\mathcal{K}_0(u_0) &= \left\| u_0 \right\|_{H^2N}^2
\end{align*}
\]
\[
\begin{align*}
\mathcal{H}_0(\eta) &= \left\| \eta(0) \right\|_{H^{2N+1/2}(\Sigma)}^2 + \sum_{j=1}^N \left\| \partial_t^j \eta(0) \right\|_{H^{2N-2j+3/2}(\Sigma)}^2 \\
\mathcal{K}_0(\eta_0) &= \left\| \eta_0 \right\|_{H^{2N+1/2}(\Sigma)}^2
\end{align*}
\]
\[
\begin{align*}
\mathcal{K}_0(F, H) &= \sum_{j=0}^{N-1} \left\| \partial_t^j F(0) \right\|_{H^{2N-2j-2}}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j H(0) \right\|_{H^{2N-2j-3/2}(\Sigma)}^2
\end{align*}
\]
Furthermore, we need to define mappings $G_1$ on $\Omega$ and $G_2$ on $\Sigma$.

\[ G^1(v, q) = -(R + \partial_t JK)\Delta_A v - \partial_t Rv + (\partial_t JK + R + R^2)\nabla_A q + \nabla_A \cdot (D_A (Rv) - R D_A v + D_{\partial_t A} v) \]

\[ G^2(v, q) = D_A (Rv) \nabla - (q I - D_A v) \partial_t N + D_{\partial_t A} v \partial_t N \]

The mappings above allow us to define the arbitrary order forcing terms. For $j = 1, \ldots, N$, we have

\[ F^0 = F \]

\[ H^0 = H \]

\[ F^j = D_t F^{j-1} + G^1(D_t^{j-1} u, \partial_t^{j-1} p) = D_t^j F + \sum_{l=0}^{j-1} D_t^l G_1(D_t^{j-l-1} u, \partial_t^{j-l-1} p) \]

\[ H^j = D_t H^{j-1} + G^2(D_t^{j-1} u, \partial_t^{j-1} p) = D_t^j H + \sum_{l=0}^{j-1} D_t^l G_2(D_t^{j-l-1} u, \partial_t^{j-l-1} p) \]

Finally, we define the general quantities we need to estimate as follows.

\[ \mathcal{E}(u, p) = \sum_{j=0}^N \|\partial_t^j u\|_{L^\infty H^{2N-2j-1}}^2 + \sum_{j=0}^{N-1} \|\partial_t^j p\|_{L^\infty H^{2N-2j-1}}^2 \]

\[ \mathcal{D}(u, p) = \sum_{j=0}^N \|\partial_t^j u\|_{L^2 H^{2N-2j+1}}^2 + \|\partial_t^{N+1} u\|_{L^\infty H^{N}}^2 + \sum_{j=0}^{N-1} \|\partial_t^j p\|_{L^2 H^{2N-2j}}^2 \]

\[ \mathcal{K}(u, p) = \mathcal{E}(u, p) + \mathcal{D}(u, p) \]

\[ \mathcal{E}(\eta) = \|\eta\|_{L^\infty H^{2N+1/2}(\Sigma)}^2 + \sum_{j=1}^N \|\partial_t^j \eta\|_{L^\infty H^{2N-2j+3/2}(\Sigma)}^2 \]

\[ \mathcal{D}(\eta) = \|\eta\|_{L^2 H^{2N+1/2}(\Sigma)}^2 + \|\partial_t \eta\|_{L^2 H^{2N-1/2}(\Sigma)}^2 + \sum_{j=1}^{N+1} \|\partial_t^j \eta\|_{L^2 H^{2N-2j+5/2}(\Sigma)}^2 \]

\[ \mathcal{K}(\eta) = \mathcal{E}(\eta) + \mathcal{D}(\eta) \]

\[ \mathcal{K}(F, H) = \sum_{j=0}^{N-1} \|\partial_t^j F\|_{L^2 H^{2N-2j-1}}^2 + \|\partial_t^N F\|_{L^\infty H^{N}}^2 + \sum_{j=0}^N \|\partial_t^j H\|_{L^2 H^{2N-2j-1/2}(\Sigma)}^2 \]

\[ + \sum_{j=0}^{N-1} \|\partial_t^j F\|_{L^\infty H^{2N-2j-2}}^2 + \sum_{j=0}^{N-1} \|\partial_t^j H\|_{L^\infty H^{2N-2j-3/2}(\Sigma)}^2 \]

In the following, we will present some preliminary results to estimate the forcing terms and initial data, which will be used both in constructing the higher order initial data and proving the higher regularity theorem.

**Lemma 2.31.** For $m = 1, \ldots, N - 1$ and $j = 1, \ldots, m$, the following estimates hold whenever the right-hand sides are finite:

\[ \|F^j\|_{L^2 \Omega}^2 + \|F^j\|_{L^2 H^{2m-2j+1}}^2 + \|H^j\|_{L^2 \Omega}^2 + \|H^j\|_{L^2 H^{2m-2j+3/2}(\Sigma)}^2 \lesssim P(1 + \mathcal{K}(\eta)) \left( \mathcal{K}(F, H) + \sum_{j=0}^{j-1} \|\partial_t^j u\|_{L^\infty H^{2m-2j+1}}^2 \right) \]

\[ \|F^j\|_{L^\infty \Omega}^2 + \|H^j\|_{L^\infty \Omega}^2 \lesssim P(1 + \mathcal{K}(\eta)) \left( \mathcal{K}(F, H) + \sum_{j=0}^{j-1} \|\partial_t^j u\|_{L^\infty H^{2m-2j+2}}^2 \right) \]
Lemma 2.33. can be easily shown.

Detailed estimation can show this goal can always be achieved. Hence, we can show (2.4.34) and is not to exceed the order of derivative in the definition of right-hand sides in the estimates. A ∥

Lemma 2.32. For j = 1, . . . , N − 1,

\[ \| \mathcal{F}(0) \|^2_{L^2 H^{2N-2-j-2}} + \| \mathcal{H}(0) \|^2_{L^2 H^{2N-2-j-3/2}} \lesssim P(1 + \mathcal{K}_0(\eta))(1 + \mathcal{K}_0(F, H)) \]

Proof. It is similar to proof of lemma 2.31, so we omit it here. □

Lemma 2.33. If k = 0, . . . , 2N − 1 and v is sufficiently regular, then

\[ \| \partial_t v - D_t v \|^2_{L^2 H^k} \lesssim P(1 + \mathcal{K}(\eta)) \| v \|^2_{L^2 H^k} \]

and if k = 0, . . . , 2N − 2 and v is sufficiently regular, then

\[ \| \partial_t v - D_t v \|^2_{L^2 H^k} \lesssim P(1 + \mathcal{K}(\eta)) \| v \|^2_{L^2 H^k} \]

If m = 1, . . . , N − 1 and j = 1, . . . , m, assuming v is sufficiently regular, then

\[ \| \partial_t^{m+1} v - \partial_t D_t^m v \|^2_{L^2 H^j} \lesssim P(1 + \mathcal{K}(\eta)) \left( \sum_{l=0}^{m-1} \| \partial_t^l v \|^2_{L^2 H^{j+l}} + \| \partial_t^l u \|^2_{L^2 H^{j+l}} \right) \]

(2.4.42)
Proof. In a similar fashion as proof of lemma 2.31, we can easily show this.  

**Lemma 2.34.** If \( j = 0, \ldots, N \) and \( v \) is sufficiently regular, then

\[
\| \partial_t^j v(0) - D_t^j v(0) \|_{H^{2N-2j-2}}^2 \lesssim P(1 + \| \eta(0) \|_{H^{2N} (\Sigma)}^2 + \| \partial_t \eta(0) \|_{H^{2N-1} (\Sigma)}^2)
\]

\[
(2.4.44)
\]

\[
(2.4.38), (2.4.44) \text{ and } (2.4.45) \text{ allow us to define }
\]

\[
\text{It is easy to see our construction satisfies this compatible condition for all } j, \text{ so } \nabla_{\eta(0)} D_t^{N-1} u(0) = 0, \text{ which means we can use estimate (2.4.47) to see that } g = g(H^{N-1}(0), D_t^{N-1} u(0)) \in H^{1/2}(\Sigma) \text{ and } h = h(H^{N-1}(0), D_t^{N-1} u(0)) \in H^{-1/2}(\Sigma_b). \text{ We can also define } f_0 = -\nabla_{\eta(0)} (R(0) D_t^{N-1} u(0))
\]

\[
(2.4.45)
\]

\[
(2.4.46)
\]

\[
(2.4.47)
\]

\[
(2.4.48)
\]

\[
(2.4.49)
\]

\[
(\sum_\alpha A_{\alpha}^j v, D_t^j u(0)) = 0
\]

\[
(0)
\]

\[
(2.4.35).
\]

\[
26 \text{ LEI WU}
\]

Now we start to determine the initial data. We assume that \( u_0 \in H^{2N}(\Omega), \eta_0 \in H^{2N+1/2}(\Sigma) \) and \( K_0(F, H) < \infty \). In the following, we will iteratively construct \( D_t^j u(0) \) for \( j = 0, \ldots, N \) and \( \partial_t^j p(0) \) for \( j = 0, \ldots, N - 1 \).

For \( j = 0, F^0(0) = F(0) \in H^{2N-2}, H^0(0) = H(0) \in H^{2N-3/2}(\Sigma) \) and \( D_t^0 u(0) = u_0 \in H^2N \).

Suppose now \( F^l(0) \in H^{2N-2l-2}, H^l(0) \in H^{2N-2l-3/2}(\Sigma) \) and \( D_t^l u(0) \in H^{2N-2l} \) are given for \( 0 \leq l \leq j \in [0, N - 2], \) we will define \( \partial_t^j p(0) \in H^{2N-2j-1} \) as well as \( D_t^{j+1} u(0) \in H^{2N-2j-1}, F^{j+1}(0) \in H^{2N-2j-4} \) and \( H^{j+1}(0) \in H^{2N-2j-7/2}(\Sigma) \). By virtue of (2.4.46), we know that \( f = f(F(0), D_t^l u(0)) \in H^{2N-2j-3}, g = g(H(0), D_t^j u(0)) \in H^{2N-2j-3}(\Sigma) \) and \( h = h(F^j(0), D_t^j u(0)) \in H^{2N-2j-5/2}(\Sigma_b). \) This allows us to define \( \partial_t^j p(0) \) as the solution to (2.3.55) with this choice of \( f, g, h \) and then the proposition 2.22 with \( k = 2N \) implies \( \partial_t^j p(0) \in H^{2N-2j-1} \). The estimate (2.4.38), (2.4.44) and (2.4.45) allow us to define

\[
D_t^{j+1} u(0) = g_0(F^j(0), D_t^j u(0), \partial_t^j p(0)) \in H^{2N-2j-2}
\]

\[
F^{j+1}(0) = D_t F^j(0) + G^1(D_t^j u(0), \partial_t^j p(0)) \in H^{2N-2j-4}
\]

\[
H^{j+1}(0) = \partial_t H^j(0) + G^2(D_t^j u(0), \partial_t^j p(0)) \in H^{2N-2j-7/2}(\Sigma)
\]

Iteratively, we can construct all the data except \( D_t^N u(0) \) and \( \partial_t^{N-1} p(0) \).

In order to complete this construction, we must enforce our compatible condition for \( j = 0, \ldots, N - 1 \). We say that the \( j^{th} \) compatible condition is satisfied if

\[
(2.4.49)
\]

\[
(0)
\]

\[
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\[
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\[
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\[
(0)
\]
with \( f_0 \in H^0 \) and \( F_0 = -F^{N-1}(0) \in H^0 \). Hence, we can define \( \partial_t^{N-1} p(0) \in H^1 \) as the weak solution of (2.3.55) in the sense of (2.3.59). Then we can just define
\[
(2.4.50) \quad D_t^N u(0) = \mathcal{A}(F^{N-1}(0), D_t^{N-1} u(0), \partial_t^{N-1} p(0)) \in H^0
\]

In fact, above construction guarantees that \( D_t^N u(0) \in \mathcal{Y}(0) \). Furthermore, the bounds (2.4.38), (2.4.45) and (2.4.46) provides us with the estimate
\[
(2.4.51) \quad \sum_{j=0}^{N} \left\| D_t^j u(0) \right\|_{H^{2N-2j}}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j p(0) \right\|_{H^{2N-2j-1}}^2 \lesssim P(1 + \mathcal{H}_0(\eta)) \left( \| u_0 \|_{H^{2N}}^{2} + \mathcal{K}_0(F, H) \right)
\]

Owing to estimate (2.4.44), the above estimate also holds when replace \( D_t^j u \) with \( \partial_t^j u \), i.e.
\[
(2.4.52) \quad \mathcal{H}_0(u, p) \lesssim P(1 + \mathcal{H}_0(\eta)) \left( \| u_0 \|_{H^{2N}}^{2} + \mathcal{K}_0(F, H) \right)
\]

### 2.4.4. Higher Regularity Theorem

**Theorem 2.36.** Suppose the initial data \( u_0 \in H^2 \), \( \eta_0 \in H^{2N+1/2}(\Sigma) \) and the initial condition is constructed as above to satisfy the \( j \)th compatible condition for \( j = 0, 1, \ldots, N - 1 \), with the forcing function \( \mathcal{K}(F, H) + \mathcal{K}_0(F, H) < \infty \). Then there exists a universal constant \( T_0(\eta) > 0 \) depending on \( \mathcal{L}(\eta) \) as defined in the following, such that if \( 0 < T < T_0(\eta) \), then there exists a unique strong solution \( (u, p) \) to the equation (2.4.1) on \([0, T]\) such that
\[
(2.4.53) \begin{aligned}
\partial_t^j u &\in L^2([0, T]; H^{2N-2j+1}) \cap C^0([0, T]; H^{2N-2j}) \quad \text{for} \quad j = 0, \ldots, N, \\
\partial_t^j p &\in L^2([0, T]; H^{2N-2j}) \cap C^0([0, T]; H^{2N-2j-1}) \quad \text{for} \quad j = 0, \ldots, N - 1 \\
\partial_t^{N+1} u &\in X^* 
\end{aligned}
\]

Also the pair \((u, p)\) satisfies the estimate
\[
(2.4.54) \quad \mathcal{K}(u, p) \lesssim \mathcal{L}(\eta) \left( \| u_0 \|_{H^{2N}}^{2} + \mathcal{K}_0(F, H) + \mathcal{K}(F, H) \right)
\]

where
\[
(2.4.55) \quad \mathcal{L}(\eta) = P(1 + \mathcal{H}_0(\eta) + \mathcal{K}(\eta)) \exp\left( T \mathcal{P}(1 + \mathcal{K}(\eta)) \right)
\]

Furthermore, the pair \((D_t^j u, \partial_t^j p)\) satisfies the equation
\[
(2.4.56) \begin{aligned}
\partial_t(D_t^j u) - \Delta_A(D_t^j u) + \nabla_A(\partial_t^j p) &= F^j \quad \text{in} \quad \Omega, \\
\nabla_A \cdot (D_t^j u) &= 0 \quad \text{in} \quad \Omega, \\
S_A(\partial_t^j p, D_t^j u)N &= H^j \quad \text{on} \quad \Sigma, \\
D_t^j u &= 0 \quad \text{on} \quad 
\end{aligned}
\]

in the strong sense for \( j = 0, \ldots, N - 1 \) and in the weak sense for \( j = N \).

**Proof.** Similar to lower regularity case, this theorem is almost identical to theorem 4.8 in [1]. Due to the same reason that our free surface \( \eta \) can be arbitrary, the only difference here is that we keep the polynomial of \( \mathcal{K}(\eta) \) and \( \mathcal{H}_0(\eta) \) in the final estimate which cannot be further simplified. Hence, our result easily follows. \( \square \)

### 3. Transport Equation

#### 3.1. Introduction

In this section, we will prove the wellposedness of the transport equation
\[
(3.1.1) \begin{aligned}
\partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta &= u_3 \quad \text{on} \quad \Sigma, \\
\eta(0) &= \eta_0
\end{aligned}
\]

Throughout this section, we can always assume \( u \) is known and satisfies some bounded properties, which will be specified in the following.

#### 3.2. Transport Estimates
3.2.1. Initial Data and Compatible Condition. In order to derive the wellposedness theorem of the transport equation for arbitrary order, we should first determine the initial data of arbitrary order. In this section, we can always assume $\mathcal{H}_0(u,p) < \infty$ and $K(\eta_0) < \infty$, then we try to define $\partial_t^j \eta(0)$ for $j = 1, \ldots, N$.

First, since we know $\eta(0) = \eta_0 \in H^{2N+1/2}(\Sigma)$, we can define $\partial_t \eta(0)$ via the transport equation.

\begin{equation}
\partial_t \eta(0) = u(0) \cdot N(0) \in H^{2N-1/2}(\Sigma)
\end{equation}

Notice that $u(0) \in H^{2N-1/2}(\Sigma)$ by trace theorem and this term will dominate. In above estimate, we mainly employ the fact that $H^r(\Sigma)$ is an algebra for $r \geq 5/2$.

For $j = 2, \ldots, N$, assuming $\eta(0) \in H^{2N+1/2}(\Sigma)$ and $\partial_t^j \eta(0) \in H^{2N-2k+3/2}(\Sigma)$ for $k = 1, \ldots, j-1$, then we try to determine $\partial_t^j \eta(0)$. Considering the transport equation $\partial_t \eta = u \cdot N$ and $u$ is sufficiently regular, we can always take $j - 1$ temporal derivative on both sides.

\begin{equation}
\partial_t^j \eta = \sum_{l=0}^{j-1} \left( \frac{j-1}{l} \right) \partial_t^l u \cdot \partial_t^{j-1-l} N
\end{equation}

Since $\partial_t^l u \in C^0([0, T]; H^{2N-2k-1/2}(\Sigma))$, we can evaluate the right-hand side of above equation in $t = 0$. Then

\begin{equation}
\partial_t^j \eta(0) = \sum_{l=0}^{j-1} \partial_t^l u(0) \cdot \partial_t^{j-1-l} N(0) \in H^{2N-2j+3/2}(\Sigma)
\end{equation}

This completes the construction of initial data. Furthermore, it is easy to see that we have the estimate

\begin{equation}
\mathcal{H}_0(\eta) \lesssim P(1 + \mathcal{H}_0(u,p)) \| \eta_0 \|_{H^{2N+1/2}(\Sigma)}^2
\end{equation}

3.2.2. Wellposedness of Transport Equation. Define the quantity

\begin{equation}
Q(u) = \sum_{j=0}^{N} \| \partial_t^j u \|_{L^2 H^{2N-2j+1}}^2 + \sum_{j=0}^{N-1} \| \partial_t^j u \|_{L^\infty H^{2N-2j}}^2
\end{equation}

Obviously, we have the relation $Q(u) \leq K(u, p)$.

**Theorem 3.1.** Suppose that $K_0(\eta_0) < \infty$ and $Q(u) < \infty$. Then the problem (3.1.1) admits a unique solution $\eta$ satisfying $K(\eta) < \infty$ and achieving the initial data $\partial_t^j \eta(0)$ for $j = 0, \ldots, N$. Moreover, there exists a $0 < T(u) < 1$, depending on $Q(u)$, such that if $0 < T < T$, then we have the estimate

\begin{equation}
K(\eta) \lesssim (1 + K_0(\eta_0)) P(1 + Q(u))
\end{equation}

**Proof.** We divide the proof into several steps.

Step 1: Solvability of the equation:

Similar to proof of theorem 5.4 in [1], based on proposition 2.1 in [3], since $u \in L^2([0, T]; H^{2N+1/2}(\Sigma))$, then the equation (3.1.1) admits a unique solution $\eta \in L^\infty([0, T]; H^{2N+1/2}(\Sigma))$ achieving the initial data $\eta(0) = \eta_0$.

Step 2: Estimate of $E(\eta)$:

By lemma B.1, we have

\begin{equation}
\| \eta \|_{L^\infty H^{2N+1/2}(\Sigma)} \leq \exp \left( C \int_0^T \| u(t) \|_{H^{2N+1/2}(\Sigma)} dt \right) \left( \sqrt{K(\eta_0)} + \int_0^T \| u_3(t) \|_{H^{2N+1/2}(\Sigma)} dt \right)
\end{equation}

for $C > 0$. The Cauchy-Schwarz inequality implies

\begin{equation}
C \int_0^T \| u(t) \|_{H^{2N+1/2}(\Sigma)} dt \lesssim C \int_0^T \| u(t) \|_{H^{2N+1/2}} dt \lesssim \sqrt{T} \sqrt{Q(u)}
\end{equation}
Similar to above argument, we have

$$\exp \left( C \int_0^T \| u(t) \|_{H^{2N+1/2}(\Sigma)} \, dt \right) \leq 2$$

Hence, we have

$$\| \eta \|^2_{L^\infty H^{2N+1/2}(\Sigma)} \lesssim K_0(\eta_0) + T Q(u) \lesssim K_0(\eta_0) + 1$$

Based on the equation, we further have

$$\| \partial_t \eta \|^2_{L^\infty H^{2N-1/2}(\Sigma)} \lesssim \| u \|^2_{L^\infty H^{2N-1/2}(\Sigma)} + \| u \|^2_{L^\infty H^{2N-1/2}(\Sigma)} \| \eta \|^2_{L^\infty H^{2N+1/2}(\Sigma)}$$

$$\lesssim K_0(\eta_0) Q(u) + Q(u) + T Q(u)^2$$

$$\lesssim \left( K_0(\eta_0) + 1 \right) Q(u)$$

The last inequality is based on the choice of $T$ as above. The solution $\eta$ is temporally differentiable to get the equation

$$\partial_t (\partial_t \eta) + u_1 \partial_1 (\partial_t \eta) + u_2 \partial_2 (\partial_t \eta) = \partial_t u_3 - \partial_t u_1 \partial_1 \eta - \partial_t u_2 \partial_2 \eta$$

So we have the estimate

$$\| \partial_t^2 \eta \|^2_{L^\infty H^{2N-5/2}(\Sigma)} \lesssim \| u \|^2_{L^\infty H^{2N-5/2}(\Sigma)} \| \partial_t \eta \|^2_{L^\infty H^{2N-3/2}(\Sigma)} + \| \partial_t u \|^2_{L^\infty H^{2N-5/2}(\Sigma)} \| \eta \|^2_{L^\infty H^{2N-3/2}(\Sigma)}$$

$$\lesssim \left( K_0(\eta_0) + 1 \right) \left( Q(u)^2 + Q(u) \right)$$

In a similar fashion, we can achieve the estimate for $\partial_t^j \eta$ as $j = 1, \ldots, N$.

$$\| \partial_t^j \eta \|^2_{L^\infty H^{2N-2j+3/2}(\Sigma)} \lesssim \left( K_0(\eta_0) + 1 \right) \sum_{i=1}^j Q(u)^i$$

To sum up, we have

$$E(\eta) \lesssim \left( 1 + K_0(\eta_0) \right) P(1 + Q(u))$$

Step 3: Estimate of $D(\eta)$:
A straightforward calculation reveals

$$\| \eta \|^2_{L^2 H^{2N+1/2}(\Sigma)} \lesssim T \| \eta \|^2_{L^\infty H^{2N+1/2}(\Sigma)} \lesssim T \left( K_0(\eta_0) + 1 \right)$$

Similar to above argument, we have

$$\| \partial_t \eta \|^2_{L^2 H^{2N-1/2}(\Sigma)} \lesssim \| u \|^2_{L^2 H^{2N-1/2}(\Sigma)} \| \eta \|^2_{L^\infty H^{2N+1/2}(\Sigma)} + \| u_3 \|^2_{L^2 H^{2N-1/2}(\Sigma)}$$

$$\lesssim \left( TK_0(\eta_0) + 1 \right) Q(u)$$

Furthermore, we have

$$\| \partial_t^2 \eta \|^2_{L^2 H^{2N-3/2}(\Sigma)} \lesssim \| u \|^2_{L^2 H^{2N-3/2}(\Sigma)} \| \partial_t \eta \|^2_{L^2 H^{2N-1/2}(\Sigma)} + \| \partial_t u \|^2_{L^2 H^{2N-3/2}(\Sigma)}$$

$$\lesssim \left( K_0(\eta_0) + 1 \right) \left( Q(u)^2 + Q(u) \right)$$

In the same fashion, we can achieve the estimate for $\partial_t^j \eta$ as $j = 1, \ldots, N + 1$.

$$\| \partial_t^j \eta \|^2_{L^2 H^{2N-2j+5/2}(\Sigma)} \lesssim \left( K_0(\eta_0) + 1 \right) \sum_{i=1}^j Q(u)^i$$
To sum up, we have
\[(3.2.20) \quad \mathcal{D}(\eta) \lesssim (1 + K_0(\eta_0))P(1 + Q(u))\]

Step 4: Synthesis:
It is easy to see the synthesized estimate
\[(3.2.21) \quad K(\eta) = \mathcal{E}(\eta) + \mathcal{D}(\eta) \lesssim (1 + K_0(\eta_0))P(1 + Q(u))\]

Next, we introduce a lemma to describe the difference between \(\eta\) and \(\eta_0\) in a small time period.

**Lemma 3.2.** If \(Q(u) + K_0(\eta_0) < \infty\), then for any \(\epsilon > 0\), there exists a \(\tilde{T}(\epsilon) > 0\) depending on \(Q(u)\) and \(K_0(\eta_0)\), such that for any \(0 < T < \tilde{T}\), we have the estimate
\[(3.2.22) \quad \|\eta - \eta_0\|_{L^\infty H^{2N+1/2}(\Sigma)} \lesssim \epsilon\]

**Proof.** \(\xi = \eta - \eta_0\) satisfies the transport equation
\[(3.2.23) \quad \begin{cases} 
\partial_t \xi + u_1 \partial_1 \xi + u_2 \partial_2 \xi = u_3 - u_1 \partial_1 \eta_0 - u_2 \partial_2 \eta_0 & \text{on} \quad \Sigma \\
\xi(0) = 0
\end{cases}\]

By the transport estimate stated above, we have
\[(3.2.24) \quad \|\xi\|_{L^\infty H^{2N+1/2}(\Sigma)} \leq \exp(C \int_0^T \|u(t)\|_{H^{2N+1/2}(\Sigma)} dt) \left( \int_0^T \|u_3(t) - u_1(t) \partial_1 \eta_0 - u_2(t) \partial_2 \eta_0\|_{H^{2N+1/2}(\Sigma)} dt \right)
\lesssim TQ(u)K_0(\eta_0)\]

Hence, when \(T\) is small enough, our result naturally follows. \(\square\)

3.3. **Forcing Estimates.** Now we need to estimate the forcing terms which appears on the right-hand side of the linear Navier-Stokes equation. The forcing terms is as follows.

\[(3.3.1) \quad F = \partial_t \bar{\eta} \bar{b} K \partial_3 u - u \cdot \nabla A u\]
\[(3.3.2) \quad H = \eta N\]

Recall that we define the forcing quantities as follows.

\[(3.3.3) \quad \mathcal{K}(F, H) = \sum_{j=0}^{N-1} \left\| \partial_t^j F \right\|_{L^2 H^{2N-2j-1}}^2 + \left\| \partial_t^N F \right\|_{L^\infty}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j H \right\|_{L^2 H^{2N-2j-1/2}(\Sigma)}^2\]
\[(3.3.4) \quad \mathcal{K}_0(F, H) = \sum_{j=0}^{N-1} \left\| \partial_t^j F(0) \right\|_{H^{2N-2j-2}}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j H(0) \right\|_{H^{2N-2j-3/2}(\Sigma)}^2\]

In the estimate of Navier-Stokes-transport system, we also need some other forcing quantities.

\[(3.3.5) \quad \mathcal{F}(F, H) = \sum_{j=0}^{N-1} \left\| \partial_t^j F \right\|_{L^2 H^{2N-2j-1}}^2 + \left\| \partial_t^N F \right\|_{L^2 H^0}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j H \right\|_{L^\infty H^{2N-2j-1/2}(\Sigma)}^2\]
\[(3.3.6) \quad \mathcal{H}(F, H) = \sum_{j=0}^{N-1} \left\| \partial_t^j F \right\|_{L^2 H^{2N-2j-1}}^2 + \sum_{j=0}^{N-1} \left\| \partial_t^j H \right\|_{L^2 H^{2N-2j-1/2}(\Sigma)}^2\]
The forcing terms satisfy the estimate

\begin{align}
(3.3.7) & \quad \mathcal{K}(F, H) \lesssim \left(1 + \mathcal{K}(\eta)^2 \right) \left(1 + Q(u)^2 \right) \\
(3.3.8) & \quad \mathcal{F}(F, H) \lesssim \left(1 + \mathcal{K}(\eta)^2 \right) \left(1 + Q(u)^2 \right) \\
(3.3.9) & \quad \mathcal{H}(F, H) \lesssim T \left(1 + \mathcal{K}(\eta)^2 \right) \left(1 + Q(u)^2 \right) \\
(3.3.10) & \quad \mathcal{K}_0(F, H) \lesssim \left(1 + \mathcal{K}_0(\eta_0)^2N \right) \left(1 + \mathcal{K}_0(u_0)^2N \right)
\end{align}

Proof. The proof is standard, just similar to what we did in derive (2.4.34) to (2.4.47), so we omit the details here. In (3.3.7), note the trivial bound

\begin{align}
(3.3.11) & \quad \|\partial_t^n F\|_{\tilde{X}^s}^2 \lesssim \|\partial_t^n F\|_{L^2 H^0}^2
\end{align}

Also in (3.3.9), the key part is the appearance of bounding constant $T$. We first trivially bound it as follows.

\[
\sum_{j=0}^{N-1} \|\partial_t^j F\|_{L^2 H_2^{N-2j-1}}^2 + \sum_{j=0}^{N-1} \|\partial_t^j H\|_{L^2 H_2^{N-2j-1/2}(\Sigma)}^2 
\leq T \left( \sum_{j=0}^{N-1} \|\partial_t^j F\|_{L^\infty H_2^{N-2j-1}}^2 + \sum_{j=0}^{N-1} \|\partial_t^j H\|_{L^\infty H_2^{N-2j-1/2}(\Sigma)}^2 \right)
\]

Then just as previous, we can easily show the result. \qed

Remark 3.4. The reason why we can get the bounding constant $T$ lies in that, in the nonlinear Navier-Stokes equation, the $u$ in nonlinear terms actually has one less derivative than the linear part, which makes it available to use lemma A.14. The appearance of $T$ in (3.3.9) will play a key role in the following nonlinear iteration argument.

4. Navier-Stokes-Transport System

In this section, we will study the nonlinear system

\begin{align}
\begin{cases}
\partial_t u - \partial_t \tilde{\eta} \tilde{b} K \partial_3 u + u \cdot \nabla_A u - \Delta_A u + \nabla_A p = 0 & \text{in } \Omega \\
\nabla_A u = 0 & \text{in } \Omega \\
S_A(p, u) N = \tilde{\eta} N & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma_b \\
u(x, 0) = u_0(x) & \text{on } \Sigma \\
\eta(x', 0) = \eta_0(x') & \text{on } \Sigma
\end{cases}
\end{align}

(4.1)

We will first show a revised version of construction of the initial data and then employ an iteration argument to show the wellposedness of this system.

4.1. Initial Data and Compatible Condition. In the construction of initial data for linear Navier-Stokes equation and transport equation, we always assume the other variables are known at the moment. However, in the Navier-Stokes-transport system, $(u, p)$ and $\eta$ are coupled. That means we have to figure out a coupling construction method for this system.

Define

\begin{align}
(4.1.1) & \quad \mathcal{K}_0 = \mathcal{K}_0(u_0) + \mathcal{K}_0(\eta_0)
\end{align}
We need a more exact enumeration of terms in $H_0(u, p)$, $H_0(\eta)$ and $K_0(F, H)$. For $j = 0, \ldots, N - 1$, we define

$$K_0^j(F, H) = \sum_{l=0}^{j} \left\| \partial_l^j F(0) \right\|_{H^{2N-2l-2}}^2 + \left\| \partial_l^j H(0) \right\|_{H^{2N-2l-3/2}}^2$$

For $j = 1, \ldots, N$, we define

$$H_0^j(\eta) = \| \eta_0 \|_{H^{2N+1/2}}^2$$

$$H_0^j(u, p) = \| u_0 \|_{H^2}^2$$

$$H_0^j(u, p) = \sum_{l=0}^{j} \left\| \partial_l^j u(0) \right\|_{H^{2N-2l-2}}^2 + \sum_{l=0}^{j-1} \left\| \partial_l^j p(0) \right\|_{H^{2N-2l-1}}^2$$

The following lemma records more detailed estimates of above quantities.

**Lemma 4.1.** If we write $P_j(\cdot, \cdot)$ for a polynomial with $P(0, 0) = 0$. Then for $F$ and $H$ defined as in (3.3.1) and $j = 0, \ldots, N - 1$, we have

$$K_0^j(F, H) \leq P_j(H_0^{j+1}(\eta), H_0^j(u, p))$$

For $\partial_l^j F(0)$ and $\partial_l^j H(0)$ defined as in (3.3.1) combined with construction in linear problem, and $j = 0, \ldots, N - 1$, we have

$$\left\| F^j(0) \right\|_{H^{2N-2j-2}}^2 + \left\| H^j(0) \right\|_{H^{2N-2j-3/2}}^2 \leq P_j(H_0^j(\eta), H_0^j(u, p))$$

For $j = 0, \ldots, N$, it holds that

$$\left\| \partial_l^j u(0) - D_l^j u(0) \right\|_{H^{2N-2j}}^2 \leq P_j(H_0^j(\eta), H_0^j(u, p))$$

For $j = 0, \ldots, N - 1$, it holds that

$$\left\| \sum_{l=0}^{j} \left( \sum_{l=0}^{j} \partial_l^j N(0) \cdot \partial_l^{j-l} u(0) \right) \right\|_{H^{2N-2j+3/2}}^2 \leq P_j(H_0^j(\eta), H_0^j(u, p))$$

Also,

$$\left\| u_0 - \mathcal{N}_0 \right\|_{H^{2N-1/2}}^2 \lesssim \left\| u_0 \right\|_{H^2}^2 (1 + \| \eta_0 \|_{H^{2N+1/2}}^2)$$

**Proof.** These bounds are easy to derive by arguing as in lemma 2.31, so we omit the details here.

This lemma allows us to combine all the construction of initial data in previous sections under the condition that $K_0(u_0) + K_0(\eta_0) < \infty$. Define $\partial_l^j \eta(0) = u_0 \cdot \mathcal{N}_0$, so estimate (4.1.11) implies $\| \partial_l^j \eta(0) \|_{H^{2N-1/2}} \lesssim K_0^j$. Then we use this for estimate (4.1.7) and have

$$K_0^j(F, H) \lesssim P(K_0)$$

for a polynomial $P(\cdot)$ satisfying $P(0) = 0$. In the following, this polynomial can change from line to line. Since we do not care about the detailed form of it, we just omit it.

Suppose now for $j \in [0, N - 2]$ and that $\partial_l^j u(0)$ are known for $l = 0, \ldots, j$, $\partial_l^j \eta(0)$ are known for $l = 0, \ldots, j + 1$ and $\partial_l^j p(0)$ are known for $l = 0, \ldots, j - 1$ and that

$$H_0^j(u, p) + H_0^{j+1}(\eta) + K_0^j(F, H) \lesssim P(K_0)$$

According to estimate (4.1.8) and (4.1.9), we know

$$\left\| F^j(0) \right\|_{H^{2N-2j-2}}^2 + \left\| H^j(0) \right\|_{H^{2N-2j-3/2}}^2 + \left\| D_l^j u(0) \right\|_{H^{2N-2j}}^2 \lesssim P(K_0)$$
By virtue of estimate (2.4.45) to (2.4.47), we have

\begin{equation}
(4.1.15) \quad \left\| f(F^j(0), D^j_t u(0)) \right\|_{H^{2N-2j-3} +} + \left\| g(H^j(0), D^j_t u(0)) \right\|_{H^{2N-2j-3/2}}^2 + \left\| h(F^j(0), D^j_t u(0)) \right\|_{H^{2N-2j-5/2}}^2 \lesssim P(K_0)
\end{equation}

This allows us to define $\partial_t^j p(0)$ as the solution to (2.3.55) with $f$, $g$ and $h$ given by $f$, $g$ and $h$ as above. Then proposition 2.22 with $k = 2N$ implies that

\begin{equation}
(4.1.16) \quad \left\| \partial_t^j p(0) \right\|_{H^{2N-2j-1}} \lesssim P(K_0)
\end{equation}

Now we can freely define

\begin{equation}
(4.1.17) \quad D^j_t u(0) = G^0(F^j(0), D^j_t u(0), \partial_t^j p(0)) \in H^{2N-2j-2}
\end{equation}

which satisfies the estimate

\begin{equation}
(4.1.18) \quad \left\| \partial_t^j u(0) \right\|_{H^{2N-2j+1}} \lesssim P(K_0)
\end{equation}

Then we define

\begin{equation}
(4.1.19) \quad \partial_t^j \eta(0) = \sum_{l=0}^{j+1} \left( \begin{array}{c} j+1 \\ l \end{array} \right) \partial_t^l N(0) \cdot \partial_t^{j-l+1} u(0)
\end{equation}

which satisfies the estimate

\begin{equation}
(4.1.20) \quad \left\| \partial_t^j \eta(0) \right\|_{H^{2N-2j+1/2}} \lesssim P(K_0)
\end{equation}

Combining with above estimates, we have

\begin{equation}
(4.1.21) \quad \mathcal{H}_0^{j+1}(u, p) + \mathcal{H}_0^{j+2}(\eta) + \mathcal{K}_0^{j+1}(F, H) \lesssim P(K_0)
\end{equation}

By this iterative method, we can derive that

\begin{equation}
(4.1.22) \quad \mathcal{H}_0^{N-1}(u, p) + \mathcal{H}_0^{N}(\eta) + \mathcal{K}_0^{N-1}(F, H) \lesssim P(K_0)
\end{equation}

Then it remains to define $\partial_t^{N-1} p(0)$ and $D_t^N u(0)$. In order to do this, we need to restate the compatible condition for the initial data. We say $(u_0, \eta_0)$ satisfies the $N^{th}$ order compatible condition if

\begin{equation}
(4.1.23) \quad \left\{ \begin{array}{l}
\nabla A_0 \cdot (D_t^j u(0)) = 0 & \text{in } \Omega \\
D_t^j u(0) = 0 & \text{on } \Sigma_0 \\
\Pi_0(H^j(0) + \nabla A_0 D_t^j u(0) N_0) = 0 & \text{on } \Sigma
\end{array} \right.
\end{equation}

for $j = 0, \ldots, N - 1$.

Then the argument is quite standard, following exactly the path in linear problem, we can define $\partial_t^{N-1} p(0)$ as the weak solution of (2.3.55) with forcing terms defined as in linear problem. Then define

\begin{equation}
(4.1.24) \quad D_t^N u(0) = G^0(F^{N-1}(0), D_t^{N-1} u(0), \partial_t^{N-1} p(0))
\end{equation}

with corresponding estimates.

Then we have a final version of initial estimate.

**Theorem 4.2.** Suppose $(u_0, \eta_0)$ satisfies $K_0 < \infty$. Let $\partial_t^j u(0)$, $\partial_t^j \eta(0)$ for $j = 0, \ldots, N$ and $\partial_t^j p(0)$ for $j = 0, \ldots, N - 1$ defined as above. Then

\begin{equation}
(4.1.25) \quad K_0 \leq \mathcal{H}_0(u, p) + \mathcal{H}_0(\eta) \lesssim P(K_0)
\end{equation}

for a given polynomial $P(\cdot)$ with $P(0) = 0$.

**Proof.** Just summarizing all above estimates in the construction will show the result. \qed
4.2. Construction of Iteration. For given initial data $(u_0, \eta_0)$, by extension lemma A.15, there exists $u^0$ defined in $\Omega \times [0, \infty)$ such that $Q(u^0) \lesssim K_0(u_0)$ achieving the initial data to $N^{th}$ order. By solving transport equation with respect to $u^0$, theorem 3.1 implies that there exists $\eta^0$ defined in $\Omega \times [0, T_0)$ such that $K(\eta^0) \lesssim (1 + K_0(\eta_0))(1 + Q(u^0))^N \lesssim K_0^{2N}$. This is our start point.

For any integer $m \geq 1$, define the approximate solution $(u^m, p^m, \eta^m)$ on the existence interval $[0, T_m)$ by the following iteration.

$$
\begin{align*}
(\partial_t u^m - \Delta u^m - 1 u^m + \nabla A_{m-1} p^m = \partial_t \eta^m - \bar{\partial} K_{m-1} \partial_3 u^{m-1} & \quad \text{in } \Omega \\
\nabla A_{m-1} \cdot u^m = 0 & \quad \text{in } \Omega \\
(p^m I - \mathcal{D} A_{m-1} u^m) \mathcal{N}^m - 1 = \eta^m - \mathcal{N}^m - 1 & \quad \text{on } \Sigma \\
u^m = 0 & \quad \text{on } \Sigma_b \\
\partial_t \eta^m + u^m_1 \partial_1 \eta^m + u^m_2 \partial_2 \eta^m = u^m_3 & \quad \text{on } \Sigma
\end{align*}
$$

(4.2.1)

where $(u^m, \eta^m)$ achieves the same initial data $(u_0, \eta_0)$, and $A^m, N^m, K^m$ are given in terms of $\eta^m$.

This is only a formal definition of iteration. In the following theorems, we will finally prove that this approximate sequence can be defined for any $m \in \mathbb{N}$ and the existence interval $T_m$ will not shrink to 0 as $m \to \infty$.

4.3. Boundedness Theorem.

**Theorem 4.3.** Assume $J^0 > \delta > 0$ and the initial data $(u_0, \eta_0)$ satisfies the $N^{th}$ compatible condition. Then there exists a constant $0 < Z < \infty$ and $0 < T < 1$ depending on $K_0$, such that if $0 < T \leq T$ and $K_0 < \infty$, then there exists an infinite sequence $(u^m, p^m, \eta^m)_{m=0}^\infty$ satisfying the iteration equation (4.2.1) within the existence interval $[0, T)$ and the following properties.

1. The iteration sequence satisfies the estimate

$$
Q(u^m) + K(\eta^m) \leq Z
$$

for arbitrary $m$, where the temporal norm is taken with respect to $T$.

2. For any $m$, $J^m(t) > \delta/2$ for $0 \leq t \leq T$.

**Remark 4.4.** Before we start to prove this boundedness result, it is useful to notice that, based on the linear estimate (2.4,54) and forcing estimate (3.3), this sequence can always be constructed and we can directly derive an estimate

$$
K(u^{m+1}, p^{m+1}) + K(\eta^{m+1}) \leq CP(1 + Q(u^m) + K(\eta^m) + K_0) \exp \left( TP(1 + K(\eta^m)) \right)
$$

(4.3.1)

For a universal constant $C > 0$. Since the initial data can be arbitrarily large, this estimate cannot meet our requirement. Hence, we have to go back to the energy structure and derive a stronger estimate. However, this result naturally implies a lemma, which will be used in the following.

If

$$
Q(u^{m-1}) + K(\eta^{m-1}) \leq Z
$$

(4.3.2)

then

$$
K(u^m, p^m) + K(\eta^m) \leq CP(1 + K_0 + Z)
$$

(4.3.3)

**Proof.** Let’s denote the above two assertions related to $m$ as statement $\mathbb{P}_m$. We use induction to prove this theorem. To note that in the following, we will extensively use the notation $P(\cdot)$, however, these polynomials should be understood as explicitly given, but not necessarily written out here. Also they can change from line to line.

Step 1: $\mathbb{P}_0$ case:

This is only related to the initial data. Obviously, the construction of $u^0$ leads to $Q(u^0) \lesssim K_0$.

By transport estimate (3.1), we have $K(\eta^0) \lesssim P(K_0)$. We can choose $Z \geq P(K_0)$, so the first
assertion is verified.
Define $\xi^0 = \eta^0 - \eta_0$ the difference between free surface at later time and its initial data. Then the estimate in lemma 3.2 implies $\|\xi^0\|_{L^{\infty} H^{5/2}} \lesssim T \mathcal{Z}$. Naturally, $\sup_{t \in [0, T]} |J^0(t) - J^0(0)| \lesssim \|\xi^0\|_{L^{\infty} H^{5/2}} \lesssim T \mathcal{Z}$. Thus if we take $T \leq \delta/(2 \mathcal{Z})$, then $J^0(t) \geq \delta/2$. So the second assertion is also verified. In a similar fashion, we can verify the closeness assumption we made in linear Navier-Stokes equation, i.e. $\eta$ and $\eta_0$ is close enough within $[0, T]$ can also be verified. Hence, $\mathbb{P}_0$ is true.

In the following, we will assume $\mathbb{P}_{m-1}$ is true for $m \geq 1$ and prove $\mathbb{P}_m$ is also true. As long as we can show this, by induction, $\mathbb{P}_n$ is valid for arbitrary $n \in \mathbb{N}$. Certainly, the induction hypothesis and above remark implies $\mathcal{Q}(u^{m-1}) + \mathcal{K}(\eta^{m-1}) \leq \mathcal{Z}$ and $\mathcal{K}(u^m, p^m) + \mathcal{K}(\eta^m) \leq C \mathcal{P}(1 + K_0 + \mathcal{Z})$.

Step 2: $\mathbb{P}_m$ case: estimate of $u^m$ via energy structure

By theorem 2.36, the pair $(D_t^N u^m, \partial_t^N u^m)$ satisfies the equation (2.4.56) in the weak sense, i.e.

\begin{equation}
\begin{cases}
\partial_t (D_t^N u^m) - \Delta_{A^{m-1}} (D_t^N u^m) + \nabla_{A^{m-1}} (\partial_t^N p^m) = F^N & \text{in } \Omega \\
\nabla_{A^{m-1}} \cdot (D_t^N u^m) = 0 & \text{in } \Omega \\
\mathcal{S}_{A^{m-1}} (D_t^N u^m, \partial_t^N p^m) \mathcal{N} = H^N & \text{on } \Sigma \\
D_t^N u^m = 0 & \text{on } \Sigma_0
\end{cases}
\end{equation}

where $F^N$ and $H^N$ is given in terms of $u^m$ and $\eta^{m-1}$. Hence, we have the standard weak formulation, i.e. for any test function $\psi \in \mathcal{X}^{m-1}$, the following holds

\begin{equation}
\langle \partial_t D_t^N u^m, \psi \rangle_{L^2 \mathcal{H}^0} + \frac{1}{2} \langle D_t^N u^m, \partial_t^N u^m, \partial_t^N \psi \rangle_{L^2 \mathcal{H}^0} = \langle D_t^N u^m, F^N \rangle_{L^2 \mathcal{H}^0} - \langle D_t^N u^m, H^N \rangle_{L^2 \mathcal{H}^0} \tag{4.3.5}
\end{equation}

Therefore, when we plug in the test function $\psi = D_t^N u^m$, we have the natural energy structure

\begin{align}
\frac{1}{2} \int_0^T \int_\Omega |D_t^N u^m|^2 + \frac{1}{2} \int_0^T \int_\Omega \nabla_{A^{m-1}} D_t^N u^m|^2 = \\
\frac{1}{2} \int_\Omega |D_t^N u^m(0)|^2 + \frac{1}{2} \int_0^t \int_\Omega \partial_t J |D_t^N u^m|^2 + \int_0^t \int_\Omega J F^N \cdot D_t^N u^m - \int_0^t \int_\Omega H^N \cdot D_t^N u^m
\end{align}

A preliminary estimate is as follows.

$$
LHS \gtrsim \|D_t^N u^m\|_{L^\infty H^0} + \|D_t^N u^m\|_{L^2 H^1}^2
$$

$$
RHS \lesssim P(K_0) + T \mathcal{Z} \|D_t^N u^m\|_{L^\infty H^0}^2 + \sqrt{T} \|F^N\|_{L^2 H^0} \|D_t^N u^m\|_{L^\infty H^0} + \sqrt{T} \|H^N\|_{L^\infty H^{-1/2}(\Sigma)} \|D_t^N u^m\|_{L^2 H^{1/2}(\Sigma)}^2
\lesssim P(K_0) + T \mathcal{Z} \|D_t^N u^m\|_{L^\infty H^0}^2 + \sqrt{T} \|F^N\|_{L^2 H^0} + \sqrt{T} \|H^N\|_{L^\infty H^{-1/2}(\Sigma)} \|D_t^N u^m\|_{L^2 H^1}^2
$$

Taking $T \leq 1/(16 \mathcal{Z}^4)$ and absorbing the extra term on RHS into LHS implies

$$
\|D_t^N u^m\|_{L^\infty H^0} + \|D_t^N u^m\|_{L^2 H^1}^2 \lesssim P(K_0) + \sqrt{T} \|F^N\|_{L^2 H^0} + \sqrt{T} \|H^N\|_{L^\infty H^{-1/2}(\Sigma)} \|D_t^N u^m\|_{L^2 H^1}^2
$$

Drop the $\|D_t^N u^m\|_{L^\infty H^0}^2$ term and we derive further the estimate for $\partial_t^N u^m$

$$
\|\partial_t^N u^m\|_{L^2 H^1}^2 \lesssim P(K_0) + \sqrt{T} \|F^N\|_{L^2 H^0} + \sqrt{T} \|H^N\|_{L^\infty H^{-1/2}(\Sigma)} + \|D_t^N u^m - \partial_t^N u^m\|_{L^2 H^1}^2
$$

Then we need to estimate each term on RHS. For the middle two terms, it suffices to show it is bounded, however, for the last term, we need a temporal constant $T$ within the estimate, which can be done by lemma A.14. Since these estimates is similar to the proof of lemma 2.31, we
will not give the details here.
\[
\|F^N\|_{L^2 H^0}^2 \lesssim P(1 + \mathcal{K}(\eta^{m-1})) \left( \sum_{j=0}^{N-1} \|\partial_t^{j} u^m\|_{L^2 H^2}^2 + \sum_{j=0}^{N-1} \|\partial_t^{j} p^m\|_{L^2 H^1}^2 \right) + \mathcal{F}(F, H)
\]
\[
\lesssim P(1 + \mathcal{K}_0 + Z) + \mathcal{F}(F, H)
\]
\[
\|H^N\|_{L^\infty H^{-1/2}(\Sigma)}^2 \lesssim P(1 + \mathcal{K}(\eta^{m-1})) \left( \sum_{j=0}^{N-1} \|\partial_t^{j} u^m\|_{L^\infty H^2}^2 + \sum_{j=0}^{N-1} \|\partial_t^{j} p^m\|_{L^\infty H^1}^2 \right) + \mathcal{F}(F, H)
\]
\[
\lesssim P(1 + \mathcal{K}_0 + Z) + \mathcal{F}(F, H)
\]
\[
\|D_t^N u^m - \partial_t^N u^m\|_{L^2 H^1} \lesssim T \|D_t^N u^m - \partial_t^N u^m\|_{L^\infty H^1}
\]
\[
\lesssim TP(1 + \mathcal{K}(\eta^{m-1})) \left( \sum_{j=0}^{N-1} \|\partial_t^{j} u^m\|_{L^\infty H^1}^2 \right)
\]
\[
\lesssim TP(1 + \mathcal{K}_0 + Z)
\]

Therefore, to sum up, we have
\[
(4.3.9) \quad \|\partial_t^N u^m\|_{L^2 H^1}^2 \lesssim P(K_0) + \sqrt{T}P(1 + \mathcal{K}_0 + Z) + \sqrt{T}\mathcal{F}(F, H)
\]

Step 3: $F_m$ case: estimate of $u^m$ via elliptic estimate
For $0 \leq n \leq N - 1$, the $n^{th}$ order Navier-Stokes equation is as follows.
\[
\begin{cases}
\partial_t(D_t^n u^m) - \Delta A(D_t^n u^m) + \nabla A(\partial_t^n p^m) = F^n & \text{in } \Omega \\
\nabla_A \cdot (D_t^n u^m) = 0 & \text{in } \Omega \\
S_A(D_t^n u^m, \partial_t^n p^m)\mathcal{N} = H^n & \text{on } \Sigma \\
D_t^n u^m = 0 & \text{on } \Sigma_b
\end{cases}
\]
where $F^n$ and $H^n$ is given in terms of $u^m$ and $\eta^{m-1}$.
A straightforward application of elliptic estimate reveals the fact.
\[
(4.3.11) \quad \|D_t^n u^m\|_{L^2 H^{2N-2n+1}}^2 \lesssim \|\partial_t D_t^n u^m\|_{L^2 H^{2N-2n-1}}^2 + \|F^n\|_{L^2 H^{2N-2n-1}}^2 + \|H^n\|_{L^2 H^{2N-2n-1/2}}^2
\]
A more reasonable form is as follows.
\[
(4.3.12) \quad \|\partial_t^n u^m\|_{L^2 H^{2N-2n+1}}^2 \lesssim \|F^n\|_{L^2 H^{2N-2n-1}}^2 + \|H^n\|_{L^2 H^{2N-2n-1/2}}^2 + \|\partial_t^{n+1} u^m\|_{L^2 H^{2N-2n-1}}^2 + \|\partial_t^n(D_t^n u^m - \partial_t^n u^m)\|_{L^2 H^{2N-2n-1}}^2 + \|D_t^n u^m - \partial_t^n u^m\|_{L^2 H^{2N-2n+1}}^2
\]

Then we give a detailed estimate for each term on RHS. These estimates can be easily obtained as what we did before, so we omit the details here. It is noticeable that the appearance of $T$ is due to lemma A.14.
\[
\|F^n\|_{L^2 H^{2N-2n-1}}^2 \lesssim TP(1 + \mathcal{K}(\eta^{m-1})) \left( \sum_{j=0}^{N-2} \|\partial_t^{j} u^m\|_{L^\infty H^{2N-2j-1}}^2 + \sum_{j=0}^{N-2} \|\partial_t^{j} p^m\|_{L^\infty H^{2N-2j-2}}^2 \right) + \mathcal{H}(F, H)
\]
\[
\lesssim TP(1 + \mathcal{K}_0 + Z) + \mathcal{H}(F, H)
\]
\[
\|H^n\|_{L^2 H^{2N-2n-1/2}(\Sigma)}^2 \lesssim TP(1 + \mathcal{K}(\eta^{m-1})) \left( \sum_{j=0}^{N-2} \|\partial_t^{j} u^m\|_{L^\infty H^{2N-2j-1}}^2 + \sum_{j=0}^{N-2} \|\partial_t^{j} p^m\|_{L^\infty H^{2N-2j-2}}^2 \right) + \mathcal{H}(F, H)
\]
\[
\lesssim TP(1 + \mathcal{K}_0 + Z) + \mathcal{H}(F, H)
\]
\[
\|\partial_t(D_t^n u^m - \partial_t^n u^m)\|_{L^2 H^{2N-2n-1}}^2 \lesssim TP(1 + \mathcal{K}(\eta^{m-1})) \left( \sum_{j=0}^{N-1} \|\partial_t^{j} u^m\|_{L^\infty H^{2N-2j-1}}^2 \right)
\]
\[
\lesssim TP(1 + \mathcal{K}_0 + Z)
\]
Therefore, to sum up, we have

\[
(4.3.13) \quad \sum_{n=0}^{N-1} \| \partial_t^n u^m \|_{L^2 H^{2N-2n+1}}^2 \lesssim TP(1 + K(\eta^m)) \left( \sum_{j=0}^{N-2} \| \partial_t^j u^m \|_{L^\infty H^{2N-2j+1}}^2 \right) \lesssim TP(1 + K_0 + \mathcal{Z})
\]

Step 4: \( \mathbb{P}_m \) case: synthesis of estimate for \( u^m \)
Combining (4.3.9) and (4.3.13) with lemma A.12 implies

\[
(4.3.14) \quad Q(u^m) \lesssim P(K_0) + \sqrt{T}P(1 + K_0 + \mathcal{Z}) + \sqrt{T}\mathcal{F}(F, H) + \mathcal{H}(F, H)
\]

Then, by forcing estimate (3.3), we have

\[
\mathcal{F}(F^0, H^0) \lesssim (1 + K(\eta)^2) \left( 1 + Q(u)^2 \right) \lesssim \mathcal{Z}^4
\]

\[
\mathcal{H}(F^0, H^0) \lesssim T \left( 1 + K(\eta)^2 \right) \left( 1 + Q(u)^2 \right) \lesssim T\mathcal{Z}^4
\]

Hence, to sum up, we achieve the estimate

\[
(4.3.15) \quad Q(u^m) \lesssim P(K_0) + \sqrt{T}P(1 + K_0 + \mathcal{Z})
\]

This is actually

\[
(4.3.16) \quad Q(u^m) \leq CP(K_0) + \sqrt{T}CP(1 + K_0 + \mathcal{Z})
\]

for some universal constant \( C > 0 \). So we can take \( \mathcal{Z} \geq 2CP(K_0) \). If \( T \) is sufficiently small depending on \( \mathcal{Z} \), we can bound \( Q(u^m) \lesssim 2CP(K_0) \lesssim \mathcal{Z} \).

Step 5: \( \mathbb{P}_m \) case: estimate of \( \eta^m \) via transport estimate
Employing transport estimate in lemma 3.1

\[
(4.3.17) \quad K(\eta^m) \lesssim K_0 P(Q(u^m))
\]

Hence, we can take \( \mathcal{Z} = 2K_0P(2CP(K_0)) \), then we have

\[
(4.3.18) \quad K(\eta^m) \lesssim \mathcal{Z}
\]

Step 6: \( \mathbb{P}_m \) case: estimate of \( J^m(t) \)
Define \( \xi^m = \eta^m - \eta_0 \), then transport estimate in lemma 3.2 implies \( \| \xi^m \|_{L^\infty H^{5/2}} \lesssim T\mathcal{Z} \). Naturally, \( \sup_{t \in [0,T]} |J^m(t) - J^m(0)| \lesssim \| \xi^m \|_{L^\infty H^{5/2}} \lesssim T\mathcal{Z} \). Thus if we take \( T \leq \delta/(2\mathcal{Z}) \), then \( J^m(t) \geq \delta/2 \). A similar argument can justify the closeness assumption of \( \eta^m \) and \( \eta_0 \) in studying linear Navier-Stokes equation.

Synthesis:
Above estimates reveals that if we take \( \mathcal{Z} = CP(K_0) \) where the polynomial can be given explicitly by summarizing all above and \( T \) small enough depending on \( \mathcal{Z} \), we have

\[
(4.3.19) \quad Q(u^m) + K(\eta^m) \leq \mathcal{Z}
\]

and

\[
(4.3.20) \quad J^m(t) \geq \delta/2 \quad \text{for} \quad t \in [0, T]
\]

Therefore, \( \mathbb{P}_m \) is verified. By induction, we conclude that \( \mathbb{P}_n \) is valid for any \( n \in \mathbb{N} \). □
Theorem 4.5. Assume exactly the same condition as Theorem (4.3), then actually we have the estimate
\[ K(u^m, p^m) + K(\eta^m) \lesssim P(1 + K_0 + Z) \]

Proof. This is a natural corollary of above theorem. Since we know \( Q(u^{m-1}) \lesssim Z \), by above remark, it implies \( K(u^m, p^m) \) is bounded for any \( m \in \mathbb{N} \). For convenience, in the following we also call this bound \( Z \).

\[ \square \]

Remark 4.6. In above proof, the bounding polynomial is changing from line to line, however, it can always be explicitly given and does not depend on the data and iterative index \( m \). Since it is too complicated and does not help us a lot to understand the problem, we omit its expression here.

4.4. Contraction Theorem. Define
\[ M(v, q; T) = \|v\|_{L^\infty H^2}^2 + \|v\|_{L^2 H^3}^2 + \|\partial_t v\|_{L^\infty H^0}^2 + \|\partial_t v\|_{L^2 H^1}^2 + \|q\|_{L^2 H^2}^2 \]
\[ M(\zeta; T) = \|\zeta\|_{L^\infty H^{3/2}(\Sigma)}^2 + \|\partial_t \zeta\|_{L^\infty H^{3/2}(\Sigma)}^2 + \|\partial_t^2 \zeta\|_{L^2 H^{1/2}(\Sigma)}^2 \]

Theorem 4.7. For \( j = 1, 2 \), Suppose that \( v^j, q^j, w^j \) and \( \zeta^j \) achieve the same initial condition for different \( j \) and satisfy
\[ \begin{cases} 
\partial_t v^j - \Delta A_i v^j + \nabla A_i q^j = \partial_t \zeta^j b K_j \partial_3 w_j - w_j \cdot \nabla A_i w^j & \text{in } \Omega \\
\nabla A_i \cdot v^j = 0 & \text{in } \Omega \\
S_A(q^j, v^j) N^j = \zeta^j N^j & \text{on } \Sigma \\
v^j = 0 & \text{on } \Sigma_b \\
\partial_t \zeta^j = w^j \cdot N^j & \text{in } \Omega 
\end{cases} \]

where \( A_i, N^j \) and \( K^j \) are in terms of \( \zeta^j \). Suppose \( K(w^j, 0), K(v^j, q^j) \) and \( K(\zeta^j) \) is bounded by \( Z \). Then there exists \( 0 < \bar{T} < 1 \) such that for any \( 0 < T < \bar{T} \), we have the following contraction relation
\[ M(v^1 - v^2, q^1 - q^2; T) \leq \frac{1}{2} M(w^1 - w^2, 0; T) \]
\[ M(\zeta^1 - \zeta^2; T) \lesssim M(w^1 - w^2, 0; T) \]

Proof. We divide this proof into several steps.

Step 1: Lower order equations define \( v = v^1 - v^2, q = q^1 - q^2, w = w^1 - w^2 \) and \( \zeta = \zeta^1 - \zeta^2 \), which has trivial initial condition. Then they satisfy the equation as follows.
\[ \begin{cases} 
\partial_t v + \nabla A_i \cdot S_A(q, v) = H^1 + \nabla A_i \cdot (\mathbb{D}(A_i - A^2) v^2) & \text{in } \Omega \\
\nabla A_i \cdot v = H^2 & \text{in } \Omega \\
S_A(q, v) N^1 = H^3 + \mathbb{D}(A_i - A^2) v^2 N^1 & \text{on } \Sigma \\
v = 0 & \text{on } \Sigma_b 
\end{cases} \]

where
\[ \begin{align*}
H^1 &= \nabla \cdot (\mathbb{D}_A v^2) - (A_i - A^2) \nabla p^2 + \partial_t \zeta^2 \hat{b} K^2 \partial_3 w^2 + \partial_t \zeta^2 \hat{b} K^2 \partial_3 w^2 \\
&\quad + \partial_t \zeta^2 \hat{b} \left(K^2 - K^2 \right) \partial_3 w^2 - w \cdot \nabla A_i w^1 - w^2 \cdot \nabla A_i w - w^2 \cdot \nabla A_i w^2 \\
H^2 &= -\nabla \cdot (A_i - A^2) \cdot v^2 \\
H^3 &= -q^2 (N^1 - N^2) + \mathbb{D}_A v^2 (N^1 - N^2) - \mathbb{D}_A (A_i - A^2) v^2 (N^1 - N^2) + \zeta N^1 + \zeta^2 (N^1 - N^2)
\end{align*} \]
The solutions are sufficiently regular for us to differentiate in time, which is the following equation.
\[ \begin{cases} 
\partial_t (\partial_t v) + \nabla A_i \cdot S_A(\partial_t q, \partial_t v) = \hat{H}^1 + \nabla A_i \cdot (\mathbb{D}(\partial_t A_i - \partial_t A^2) v^2) & \text{in } \Omega \\
\nabla A_i \cdot \partial_t v = \hat{H}^2 & \text{in } \Omega \\
S_A(\partial_t q, \partial_t v) N^1 = \hat{H}^3 + \mathbb{D}(\partial_t A_i - \partial_t A^2) v^2 N^1 & \text{on } \Sigma \\
\partial_t v = 0 & \text{on } \Sigma_b 
\end{cases} \]
where

\( H^1 = \partial_t H^1 + \nabla_{\partial_t A^1} \cdot (\nabla_{A^1 - A^2} v)^2 + \nabla_{A^1} \cdot (\nabla_{A^1 - A^2} \partial_t v^2) + \nabla_{\partial_t A^1} \cdot (\nabla_{\partial_t A^1} v) + \nabla_{A^1} (\nabla_{\partial_t A^1} v) - \nabla_{\partial_t A^1} q \)

\( H^2 = \partial_t H^2 - \nabla_{\partial_t A^1} \cdot v \)

\( H^3 = \partial_t H^3 + \nabla_{(A^1 - A^2)} \partial_t v^2 \lambda^1 + \nabla_{(A^1 - A^2)} v^2 \partial_t \lambda^1 - S_{A^1}(q, v) \partial_t \lambda^1 + \nabla_{\partial_t A^1} v \lambda^1 \)

Step 2: Energy evolution for \( \partial_t v \)

Multiply \( J^1 \partial_t v \) on both sides to get the natural energy structure:

\[
\frac{1}{2} \int_\Omega |\partial_t v|^2 J^1 + \frac{1}{2} \int_\Omega \int_\Omega |\nabla_{A^1} \partial_t v|^2 J^1 = \int_0^t \int_\Omega J^1 \tilde{H}^1 \cdot \partial_t v + \int_0^t \int_\Omega J^1 \tilde{H}^2 \partial_t q - \int_0^t \int_\Sigma \tilde{H}^3 \cdot \partial_t v + \frac{1}{2} \int_0^t \int_\Omega |\partial_t v|^2 \partial_t J^1 - \frac{1}{2} \int_0^t \int_\Omega \int_\Omega J^1 \nabla_{(\partial_t A^1 - \partial_t A^2)} v^2 : \nabla_{A^1} \partial_t v
\]

LHS is simply the energy and dissipation term, so we focus on estimate of RHS.

\[
\int_0^t \int_\Omega J^1 \tilde{H}^1 \cdot \partial_t v \leq \int_0^t \left( \left\| \tilde{H}^1 \right\|_{L^2 H^0} \left\| \partial_t v \right\|_{H^0} \right) \leq Z \left\| \partial_t v \right\|_{L^\infty H^0} \sqrt{T} \left\| \tilde{H}^1 \right\|_{L^2 H^0}
\]

\[
\int_0^t \int_\Omega J^1 \tilde{H}^2 \partial_t q \leq \int_0^t \int_\Omega J^1 q \tilde{H}^2 - \int_0^t \int_\Omega (\partial_t J^1 q \tilde{H}^2 + J^1 q \partial_t \tilde{H}^2)
\]

\[
\leq \int_0^t \int_\Omega J^1 q \tilde{H}^2 + \|q\|_{L^\infty H^0} \sqrt{T} \left\| \tilde{H}^2 \right\|_{L^2 H^0} + \|q\|_{L^\infty H^0} \sqrt{T} \left\| \partial_t \tilde{H}^2 \right\|_{L^2 H^0}
\]

\[
\leq \int_0^t \int_\Omega J^1 q \tilde{H}^2 + \sqrt{T} \sqrt{\|q\|_{L^\infty H^0} \left\| \tilde{H}^2 \right\|_{L^2 H^0} + \left\| \partial_t \tilde{H}^2 \right\|_{L^2 H^0}}
\]

\[
\int_0^t \int_\Sigma \tilde{H}^3 \cdot \partial_t v \leq \int_0^t \left( \left\| \tilde{H}^3 \right\|_{H^{-1/2}(\Sigma)} \left\| \partial_t v \right\|_{H^{1/2}(\Sigma)} \right) \leq \sqrt{T} \left\| \tilde{H}^3 \right\|_{L^\infty H^{-1/2}(\Sigma)} \left\| \partial_t v \right\|_{L^2 H^{1/2}(\Sigma)}
\]

\[
\leq \sqrt{T} \sqrt{\|q\|_{L^\infty H^0} \left\| \tilde{H}^3 \right\|_{L^\infty H^{-1/2}(\Sigma)}}
\]

\[
\int_0^t \int_\Omega |\partial_t v|^2 \partial_t J^1 \leq \left\| \partial_t J^1 \right\|_{L^\infty H^2} T \left\| \partial_t v \right\|_{L^\infty H^0}^2 \leq T Z \|q\|_{L^\infty}
\]

So we need several estimate on \( \tilde{H}^i \). For the following terms, it suffices to show they are bounded. We use the usual way to estimate these quadratic terms. Since we have repeatedly used this method, we will not give the details here.

\[
\left\| \tilde{H}^1 \right\|_{L^2 H^0} \leq Z \left( \|\zeta\|_{L^2 H^{3/2}} + \|\partial_t \zeta\|_{L^2 H^{1/2}} + \|\partial_t^2 \zeta\|_{L^2 H^{-1/2}} + \|w\|_{L^2 H^1} + \|\partial_t w\|_{L^2 H^1} \right)
\]

\[
\leq Z \left( \sqrt{\|q\|_{L^\infty}} + \sqrt{\|q\|_{L^\infty}} + \sqrt{\|q\|_{L^\infty}} \right)
\]

\[
\left\| \tilde{H}^2 \right\|_{L^2 H^0} \leq Z \left( \|\partial_t \zeta\|_{L^2 H^{1/2}} + \|\zeta\|_{L^2 H^{1/2}} + \|v\|_{L^2 H^1} \right) \leq Z \left( \sqrt{\|q\|_{L^\infty}} + \sqrt{\|q\|_{L^\infty}} \right)
\]

\[
\left\| \partial_t \tilde{H}^2 \right\|_{L^2 H^0} \leq Z \left( \|\partial_t^2 \zeta\|_{L^2 H^{1/2}} + \|\partial_t \zeta\|_{L^2 H^{1/2}} + \|\zeta\|_{L^2 H^{1/2}} + \|\partial_t v\|_{L^2 H^1} + \|v\|_{L^2 H^1} \right)
\]

\[
\leq Z \left( \sqrt{\|q\|_{L^\infty}} + \sqrt{\|q\|_{L^\infty}} \right)
\]
\[ \left\| \vec{H}^3 \right\|_{L^\infty H^{-1/2}(\Sigma)} \lesssim Z \left( \left\| \zeta \right\|_{L^\infty H^{1/2}} + \left\| \partial_t \zeta \right\|_{L^\infty H^{1/2}} + \left\| q \right\|_{L^\infty H^1} + \left\| v \right\|_{L^\infty H^1} \right) \]
\[ \lesssim Z \left( \sqrt{\eta v} + \sqrt{\eta r} \right) \]

Also we have the following estimate.
\[ \int_\Omega J^1 q \vec{H}^2 \lesssim Z \left\| q \right\|_{L^\infty H^0} \left\| \vec{H}^2 \right\|_{L^\infty H^0} \]
\[ \lesssim Z \left\| q \right\|_{L^\infty H^0} \left( \left\| \zeta \right\|_{L^\infty H^{1/2}} + \left\| \partial_t \zeta \right\|_{L^\infty H^{1/2}} + \left\| v \right\|_{L^\infty H^1} \right) \]
\[ \lesssim Z \sqrt{\eta v} \left( \left\| \zeta \right\|_{L^\infty H^{1/2}} + \left\| \partial_t \zeta \right\|_{L^\infty H^{1/2}} + \left\| v \right\|_{L^\infty H^1} \right) \]

\( \zeta \) satisfies the equation
\[ \begin{cases} \partial_t \zeta + w_1^1 \partial_1 \zeta + w_2^2 \partial_2 \zeta = -N^2 \cdot w \\ \zeta(0) = 0 \end{cases} \]

Employing transport estimate and the boundedness of higher order norms, we have
\[ \left\| \zeta \right\|_{L^\infty H^{1/2}} \lesssim \sqrt{T} Z \left\| w \right\|_{L^2 H^{1/2}} \lesssim \sqrt{T} Z \sqrt{\eta v} \]

\( \partial_t \zeta \) satisfies the equation
\[ \begin{cases} \partial_t (\partial_t \zeta) + w_1^1 \partial_1 (\partial_t \zeta) + w_2^2 \partial_2 (\partial_t \zeta) = -N^2 \cdot \partial_t w - \partial_t N^2 \cdot w - \partial_t w_1^1 \partial_1 \zeta - \partial_t w_2^2 \partial_2 \zeta \\ \partial_t (\zeta(0)) = 0 \end{cases} \]

Similar argument as above shows that
\[ \left\| \partial_t \zeta \right\|_{L^\infty H^{1/2}} \lesssim \sqrt{T} Z \left( \left\| \partial_t \zeta \right\|_{L^2 H^{1/2}} + \left\| w \right\|_{L^2 H^{1/2}} \right) \lesssim \sqrt{T} Z \sqrt{\eta v} \]

Combining above transport estimate and lemma A.12, we have the final version
\[ \int_\Omega J^1 q \vec{H}^2 \lesssim \sqrt{T} Z \sqrt{\eta v} \sqrt{\eta r} + Z \sqrt{\eta v} \left\| v \right\|_{L^\infty H^1} \]
\[ \lesssim \sqrt{T} Z \sqrt{\eta v} \sqrt{\eta r} + Z \sqrt{\eta v} \left( \left\| v \right\|_{L^2 H^2} + \left\| \partial_t v \right\|_{L^2 H^0} \right) \]
\[ \lesssim \sqrt{T} Z \sqrt{\eta v} \sqrt{\eta r} + \sqrt{T} Z \sqrt{\eta v} \left( \left\| v \right\|_{L^\infty H^2} + \left\| \partial_t v \right\|_{L^\infty H^0} \right) \]
\[ \lesssim \sqrt{T} Z \left( \sqrt{\eta v} \sqrt{\eta r} + \eta v \right) \]

Then we consider simplify the last term in RHS of energy structure.
\[ \int_0^t \int_\Omega J^1 \mathbb{D}(\partial_t A^1 - \partial_t A^2) v^2 : \mathbb{D}_A \partial_t v \lesssim \frac{1}{4\epsilon} \left\| J^1 \mathbb{D}(\partial_t A^1 - \partial_t A^2) v^2 \right\|_{L^2 H^0}^2 + \epsilon \left\| \mathbb{D}_A \partial_t v \right\|_{L^2 H^0}^2 \]
where \( \epsilon \) should be determined in order for the second term in RHS to be absorbed in LHS.
\[ \left\| J^1 \mathbb{D}(\partial_t A^1 - \partial_t A^2) v^2 \right\|_{L^2 H^0}^2 \lesssim \sqrt{T} \left\| J^1 \mathbb{D}(\partial_t A^1 - \partial_t A^2) v^2 \right\|_{L^\infty H^0}^2 \lesssim \sqrt{T} Z \eta r \]
To summarize, using Cauchy inequality
\[ (4.4.9) \quad \left\| \partial_t v \right\|_{L^\infty H^0}^2 + \left\| \partial_t v \right\|_{L^2 H^1}^2 \lesssim \sqrt{T} Z (\eta r + \eta v + \eta r) \]

Step 3: Elliptic estimate for \( v \)

Based on standard elliptic regularity theory, we have the estimate
\[ \left\| v \right\|_{H^{r+2}}^2 + \left\| q \right\|_{H^{r+1}} \lesssim \left\| \partial_t v \right\|_{H^r}^2 + \left\| H^1 \right\|_{H^r}^2 + \left\| H^2 \right\|_{H^{r+1}}^2 + \left\| H^3 \right\|_{H^{r+2}(\Sigma)}^2 + \left\| \nabla A^1 \cdot (\mathbb{D}(A^1 - A^2) v^2) \right\|_{H^r}^2 + \left\| \mathbb{D}(A^1 - A^2) v^2 N^1 \right\|_{H^{r+2}(\Sigma)}^2 \]
set $r = 0$ and take $L^\infty$ on both sides
\[ \|v\|_{L^\infty H^3}^2 + \|q\|_{L^\infty H^1}^2 \lesssim \|\partial_t v\|_{L^\infty H^0}^2 + \|H^1\|_{L^\infty H^1}^2 + \|H^2\|_{L^\infty H^1}^2 + \|H^3\|_{L^\infty H^2(\Sigma)}^2 + \|\nabla_A^1 \cdot (\mathcal{D}(A^1 - A^2) v^2)\|_{L^\infty H^0}^2 + \|\mathcal{D}(A^1 - A^2) v^2 N^1\|_{L^\infty H^2(\Sigma)}^2 \]

set $r = 1$ and take $L^2$ on both sides
\[ \|v\|_{L^2 H^3}^2 + \|q\|_{L^2 H^1}^2 \lesssim \|\partial_t v\|_{L^2 H^1}^2 + \|H^1\|_{L^2 H^1}^2 + \|H^2\|_{L^2 H^2}^2 + \|H^3\|_{L^2 H^3(\Sigma)}^2 + \|\nabla_A^1 \cdot (\mathcal{D}(A^1 - A^2) v^2)\|_{L^2 H^1}^2 + \|\mathcal{D}(A^1 - A^2) v^2 N^1\|_{L^2 H^3(\Sigma)}^2 \]

We need to estimate all the RHS terms. We can employ the usual way to estimate quadratic terms in $L^2 H^k$ norm, however, for $L^\infty H^k$ norm, we use lemma A.12. Then we have the estimate
\[ \text{RHS} \lesssim \|\partial_t v\|_{L^2 H^1}^2 + \|\partial_t v\|_{L^\infty H^0}^2 + T Z \left( \mathcal{N} w + \mathfrak{M} \right) \]

To summarize, we have
\[ (4.4.10) \]
\[ \|v\|_{L^\infty H^2}^2 + \|q\|_{L^\infty H^1}^2 + \|v\|_{L^2 H^3}^2 + \|q\|_{L^2 H^1}^2 \lesssim \|\partial_t v\|_{L^2 H^1}^2 + \|\partial_t v\|_{L^\infty H^0}^2 + T Z \left( \mathcal{N} w + \mathfrak{M} \right) \]

In total of (4.4.9) and (4.4.10), we get the succinct form of estimate
\[ (4.4.11) \]
\[ \mathcal{N} w \lesssim \sqrt{T} Z (\mathcal{N} w + \mathfrak{M} w + \mathfrak{M}) \]

Step 4: Transport estimate for $\zeta$
\[ \zeta \text{ satisfies the equation } \]
\[ \begin{cases} \partial_t \zeta + w_1^1 \partial_1 \zeta + w_1^2 \partial_2 \zeta = -N^2 \cdot w \\ \zeta(0) = 0 \end{cases} \]

A straightforward application of transport estimate can show
\[ \|\zeta\|_{L^\infty H^{3/2}(\Sigma)}^2 \lesssim \exp \left( C \int_0^\ell \|w_1\|_{H^3} \right) \int_0^\ell \|N^2 \cdot w\|_{H^{5/2}(\Sigma)} \]
\[ \lesssim \sqrt{T} Z \|w\|_{L^2 H^{5/2}(\Sigma)}^2 \lesssim \sqrt{T} Z \mathcal{N} w \]

Similar to the proof of transport estimate in theorem 3.1, we can use the transport equation to estimate the higher order derivatives.
\[ \|\partial_1 \zeta\|_{L^\infty H^{3/2}(\Sigma)}^2 \lesssim \|\zeta\|_{L^\infty H^{5/2}(\Sigma)}^2 \|w\|_{L^\infty H^3(\Sigma)}^2 + \|\zeta\|_{L^\infty H^{5/2}(\Sigma)}^2 \|w_1\|_{L^\infty H^{3/2}(\Sigma)}^2 \lesssim Z \mathcal{N} w \]

In the same fashion, we can easily show
\[ \|\partial_2 \zeta\|_{L^2 H^{1/2}}^2 \lesssim Z \mathcal{N} w \]

To sum up, we have the estimate
\[ (4.4.12) \]
\[ \mathfrak{M} \lesssim \mathcal{N} w \]

Synthesis:
In (4.4.11), for $T$ sufficiently small, we can easily absorbed all $\mathcal{N} w$ term from RHS to LHS and replace all $\mathfrak{M}$ with $\mathcal{N} w$ to achieve
\[ (4.4.13) \]
\[ \mathcal{N} w \lesssim \sqrt{T} Z \mathcal{N} w \]

Certainly, the smallness of $T$ can guarantee the contraction. \qed
4.5. **Local Wellposedness Theorem.** Now we can combine theorem 4.5 and 4.7 to produce a unique strong solution to the equation system (1.3.11).

**Theorem 4.8.** Assume that \((u^0, \eta^0)\) satisfies \(\mathcal{K}_0 < \infty\) and that the initial data are constructed to satisfy the \(N^{th}\) compatible condition. Then there exist \(0 < T_0 < 1\) such that if \(0 < T < T_0\), then there exists a solution triple \((u, p, \eta)\) to the Navier-Stokes equation on the time interval \([0, T]\) that achieves the initial data and satisfies

\[
\mathcal{K}(u, p) + \mathcal{K}(\eta) \leq CP(\mathcal{K}_0)
\]

for a universal constant \(C > 0\) and a polynomial \(P(\cdot)\). The solution is unique among functions that achieve the initial data and satisfy \(\mathcal{K}(u, p) + \mathcal{K}(\eta) < \infty\). Moreover, \(\eta\) is such that the mapping \(\Phi\) is a \(C^{2N-2}\) diffeomorphism for each \(t \in [0, T]\).

**Proof.** We divide the proof into several steps in the following. We first construct an approximate sequence as in (4.2.1) and assume \(T\) is sufficiently small to guarantee the boundedness and contraction theorem is valid.

Step 1: The approximating sequence

By boundedness theorem, we can construct infinite triple sequence \((u^m, p^m, \eta^m)\) satisfying the boundedness estimate. Hence the uniform bound allow us to take weak and weak* limits, up to the extraction of a subsequence.

\[
\begin{align*}
\partial_t^j u^m &\rightarrow \partial_t^j u \quad \text{weakly in } L^2([0,T];H^{2N-2j+2}(\Omega)) \quad \text{for } j = 0, \ldots, N \\
\partial_t^j u^{N+1} &\rightarrow \partial_t^{N+1} u \quad \text{weakly in } \mathcal{X}^* \\
\partial_t^j p^m &\rightarrow \partial_t^j p \quad \text{weakly in } L^\infty([0,T];H^{2N-2j}(\Omega)) \quad \text{for } j = 0, \ldots, N - 1 \\
\partial_t^j \eta^m &\rightarrow \partial_t^j \eta \quad \text{weakly in } L^\infty([0,T];H^{2N-2j+3/2}(\Omega)) \quad \text{for } j = 0, \ldots, N
\end{align*}
\]

and

\[
\begin{align*}
\partial_t^j \eta^m &\rightarrow \partial_t^j \eta \quad \text{weakly in } L^2([0,T];H^{2N-1/2}(\Sigma)) \\
\partial_t^j \eta^m &\rightarrow \partial_t^j \eta \quad \text{weakly in } L^\infty([0,T];H^{2N+1/2}(\Sigma)) \\
\partial_t^j \eta^m &\rightarrow \partial_t^j \eta \quad \text{weakly in } L^\infty([0,T];H^{2N-2j+3/2}(\Sigma)) \quad \text{for } j = 0, \ldots, N
\end{align*}
\]

According to the weak and weak* lower semi-continuity of the norm, we have that for the limit function

\[
\mathcal{K}(u, p) + \mathcal{K}(\eta) \lesssim P(\mathcal{K}_0)
\]

Furthermore, since the collection of triple \((v, q, \zeta)\) is closed under the norms \(\mathcal{H}_0(u, p)\) and \(\mathcal{H}_0(\eta)\) for initial data and \((u^m, p^m, \eta^m)\) always achieves the same initial data, we deduce that the limit \((u, p, \eta)\) also achieves this.

Step 2: Contraction

Now we want to improve the result from weak convergence to strong convergence in the norm \(\sqrt{\mathcal{M}(\eta; T)} + \mathcal{M}(u, p, T)\). In the contraction theorem, for \(m \geq 1\), set \(v^1 = u^{m+2}, v^2 = u^{m+1}, w^1 = u^{m+1}, w^2 = u^m, q^1 = p^{m+2}, q^2 = p^{m+1}, \zeta^1 = \eta^{m+1}, \zeta^2 = \eta^m\). Our construction satisfies all the requirements of contraction theorem, so we can naturally deduce that

\[
\mathcal{M}(u^{m+2} - u^{m+1}; p^{m+2} - p^{m+1}; T) \leq \frac{1}{2} \mathcal{M}(u^{m+1} - u^m; p^{m+1} - p^m; T)
\]

and

\[
\mathcal{M}(\eta^{m+1} - \eta^m; T) \lesssim \mathcal{M}(u^{m+1} - u^m; p^{m+1} - p^m; T)
\]
These estimates implies \((u^m, p^m)\) is Cauchy under the norm \(\sqrt{\mathcal{M}(\cdot, \cdot)}\) and \(\eta^m\) is Cauchy under the norm \(\sqrt{\mathcal{M}(\cdot, \cdot)}\). So as \(m \to \infty\), we have

\[
\begin{cases}
  u^m \to u & \text{in } L^2([0, T]; H^3(\Omega) \cap L^\infty([0, T]; H^2(\Omega))
  \\
  \partial_t u^m \to \partial_t u & \text{in } L^2([0, T]; H^1(\Omega) \cap L^\infty([0, T]; H^0(\Omega))
  \\
  p^m \to p & \text{in } L^2([0, T]; H^2(\Omega) \cap L^\infty([0, T]; H^1(\Omega))
  \\
  \eta^m \to \eta & \text{in } L^\infty([0, T]; H^{5/2}(\Sigma))
  \\
  \partial_t \eta^m \to \partial_t \eta & \text{in } L^\infty([0, T]; H^{3/2}(\Sigma))
  \\
  \partial_t^2 \eta^m \to \partial_t^2 \eta & \text{in } L^2([0, T]; H^{1/2}(\Sigma))
\end{cases}
\]

(4.5.7)

Step 3: Interpolation
Since the triple \((u^m, p^m, \eta^m)\) is bounded under higher order norms and strongly convergent under lower order norms, we can use an interpolation argument to get better convergence properties of it. The standard interpolation theory implies the following relation.

\[
\left\| \partial_t^k f \right\|_{L^2 H^0} \leq C(T) \left\| f \right\|_{L^2 H^0}^{\theta} \left\| \partial_t f \right\|_{L^2 H^0}^{1-\theta}
\]

for \(j > k > 0\), \(\theta = \theta(j, k) \in (0, 1)\) and \(C(T)\) a constant depending on \(T\), which reveals that

\[
\begin{cases}
  \partial_t^j u^m \to \partial_t^j u & \text{in } L^2([0, T]; H^{0+j}(\Omega) \text{ for } j = 0, \ldots, N - 1
  \\
  \partial_t^j p^m \to \partial_t^j p & \text{in } L^2([0, T]; H^{0+j}(\Omega) \text{ for } j = 0, \ldots, N - 2
  \\
  \partial_t^j \eta^m \to \partial_t^j \eta & \text{in } L^2([0, T]; H^j(\Sigma) \text{ for } j = 0, \ldots, N
\end{cases}
\]

(4.5.9)

Then we use spacial interpolation between \(H^0\) and \(H^k\) to deduce that

\[
\begin{cases}
  \partial_t^j u^m \to \partial_t^j u & \text{in } L^2([0, T]; H^{2N-2j}(\Omega) \text{ for } j = 0, \ldots, N - 1
  \\
  \partial_t^j p^m \to \partial_t^j p & \text{in } L^2([0, T]; H^{2N-2j-1}(\Omega) \text{ for } j = 0, \ldots, N - 2
  \\
  \partial_t^j \eta^m \to \partial_t^j \eta & \text{in } L^2([0, T]; H^j(\Sigma) \text{ for } j = 0, \ldots, N
\end{cases}
\]

(4.5.10)

By lemma A.12, we have the convergence in \(L^\infty\) and \(C^0\) norm.

\[
\begin{cases}
  \partial_t^j u^m \to \partial_t^j u & \text{in } C^0([0, T]; H^{2N-2j-1}(\Omega) \text{ for } j = 0, \ldots, N - 2
  \\
  \partial_t^j p^m \to \partial_t^j p & \text{in } C^0([0, T]; H^{2N-2j-2}(\Omega) \text{ for } j = 0, \ldots, N - 3
  \\
  \partial_t^j \eta^m \to \partial_t \eta & \text{in } C^0([0, T]; H^{2N-1/2}(\Sigma)
  \\
  \partial_t \eta^m \to \partial_t \eta & \text{in } C^0([0, T]; H^{2N-3/2}(\Sigma)
  \\
  \partial_t^j \eta^m \to \partial_t^j \eta & \text{in } C^0([0, T]; H^{2N-2j+1}(\Sigma) \text{ for } j = 0, \ldots, N - 1
\end{cases}
\]

(4.5.11)

Step 4: Passing to the limit
The strong convergence result as above are more than sufficient to pass to the limit in the equation (4.2.1). Naturally, the limit triple \((u, p, \eta)\) is the strong solution to the problem (1.3.11).

Step 5: Uniqueness
Suppose there are two solutions \((u^i, p^i, \eta^i)\) for \(i = 1, 2\), then by contraction theorem, we have

\[
\mathcal{M}(u^1 - u^2, p^1 - p^2; T) \leq \frac{1}{2} \mathcal{M}(u^1 - u^2, p^1 - p^2; T)
\]

(4.5.12)

and

\[
\mathcal{M}(\eta^1 - \eta^2; T) \leq \mathcal{M}(u^1 - u^2, p^1 - p^2; T)
\]

(4.5.13)

which implies \(u^1 = u^2, p^1 = p^2\) and \(\eta^1 = \eta^2\).
Step 6: Diffeomorphism
Since \( J(t) > \delta/2 > 0 \), so it is sufficient to guarantee diffeomorphism. The smoothness of \( \eta \) implies it is in fact a \( C^{2N-2} \) diffeomorphism.

\[ \square \]

## Appendix A. Analytic Tools

### A.1. Products in Sobolev Space.
We will need some estimates of the products of functions in Sobolev spaces. Since these results have been proved in lemma A.1 and lemma A.2 of [1], we will present the statement of the lemmas here without proof.

**Lemma A.1.** Let \( U \) denote either \( \Sigma \) or \( \Omega \).

1. Let \( 0 \leq r \leq s_1 \leq s_2 \) be such that \( s_1 > n/2 \). Let \( f \in H^{s_1}(U) \), \( g \in H^{s_2}(U) \). Then \( fg \in H^r(U) \) and

\[
\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}} \tag{A.1.1}
\]

2. Let \( 0 \leq r \leq s_1 \leq s_2 \) be such that \( s_2 > r + n/2 \). Let \( f \in H^{s_1}(U) \), \( g \in H^{s_2}(U) \). Then \( fg \in H^r(U) \) and

\[
\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}} \tag{A.1.2}
\]

3. Let \( 0 \leq r \leq s_1 \leq s_2 \) be such that \( s_2 > r + n/2 \). Let \( f \in H^{-r}(\Sigma) \), \( g \in H^{s_2}(\Sigma) \). Then \( fg \in H^{-s_1}(\Sigma) \) and

\[
\|fg\|_{H^{-s_1}} \lesssim \|f\|_{H^{-r}} \|g\|_{H^{s_2}} \tag{A.1.3}
\]

**Lemma A.2.** Suppose that \( f \in C^1(\Sigma) \) and \( g \in H^{1/2}(\Sigma) \). Then \( fg \in H^{1/2}(\Sigma) \) and

\[
\|fg\|_{H^{1/2}} \lesssim \|f\|_{C^1} \|g\|_{H^{1/2}} \tag{A.1.4}
\]

### A.2. Identities involving \( A \).
We now record some identities involving \( A \). Since they are direct consequences of the definition of \( A \), we won’t give detailed proof here. Lemma A.3 in [1] shows the similar results.

**Lemma A.3.** The following hold.

1. For each \( j = 1, 2, 3 \), we have that \( \partial_k(JA_{jk}) = 0 \).
2. On \( \Sigma \) we have that \( JAe_3 = N \), while on \( \Sigma_b \) we have that \( JAe_3 = \nu \).
3. Let \( R \) defined by (2.4.9). Then \( R^T N = -\partial_t N \) on \( \Sigma \).

### A.3. Poisson Integral in Infinite Case.
For a function \( f \) defined on \( \Sigma = R^2 \), the parameterized Poisson integral is defined by

\[
\mathcal{P}^\epsilon f(x', x_3) = \int_{R^2} \hat{f}(\xi)e^{\epsilon |\xi| x_3} e^{2\pi i x' \cdot \xi} d\xi \tag{A.3.1}
\]

where \( \hat{f}(\xi) \) denotes the Fourier transform of \( f(x') \) in \( R^2 \) and \( 0 < \epsilon < 1 \) is the parameter. Although \( \mathcal{P}^\epsilon f \) is defined on \( R^2 \times (-\infty, 0) \), we will only consider the part \( R^2 \times (-b_0, 0) \) here.

**Lemma A.4.** Let \( \mathcal{P}^\epsilon f \) be the parameterized Poisson integral of function \( f \) which is in homogeneous Sobolev space \( H^{q-1/2}(\Sigma) \) for \( q \in \mathbb{N} \). Then we have

\[
\|\nabla^q \mathcal{P}^\epsilon f\|^2_{H^0} \leq \frac{C}{\epsilon} \|f\|^2_{H^{q-1/2}} \tag{A.3.2}
\]

where \( C > 0 \) is a constant independent of \( \epsilon \). In particular, we have

\[
\|\mathcal{P}^\epsilon f\|^2_{H^0} \leq \frac{C}{\epsilon} \|f\|^2_{H^{q-1/2}(\Sigma)} \tag{A.3.3}
\]
Hence, we have
\[ (A.3.8) \]
By definition, it is easy to see that the straight application of lemma A.4, so we only need to verify the second inequality in (A.3.7).

We can simply bound as follows.

\[ \text{Proof.} \]

**Lemma A.6.** Let \( P^s f \) be the parameterized Poisson integral of function \( f \). Then for \( q \in \mathbb{N} \) and \( s > 1 \), we have
\[ (A.3.5) \]
\[ \| \nabla^q P^s f \|_{L^\infty}^2 \leq C \| f \|_{H^{q+s}}^2 (\Sigma) \]

**Proof.** A simple application of Sobolev embedding and lemma A.4 reveals
\[ \| \nabla^q P^s f \|_{L^\infty}^2 \leq C \| \nabla^q P^s f \|_{H^{q+s}}^2 \leq C \| f \|_{H^{q+s}}^2 (\Sigma) \]

The following lemma illustrate the specialty of derivative in vertical direction.

**Lemma A.6.** Let \( P^s f \) be the parameterized Poisson integral of function \( f \). Then we have
\[ (A.3.6) \]
\[ \| \partial_3 P^s f \|_{L^\infty}^2 \leq C \| f \|_{H^{q+s}}^2 (\Sigma) \]

where \( C > 0 \) is a constant independent of \( \epsilon \).

**Proof.** We can simply bound as follows.
\[ (A.3.7) \]
\[ \| \partial_3 P^s f \|_{L^\infty}^2 \leq C \| \partial_3 P^s f \|_{H^2}^2 \leq C \| f \|_{H^{q+s}}^2 (\Sigma) \]

Note that the first inequality is based on Sobolev embedding theorem and the third one is a straightforward application of lemma A.4, so we only need to verify the second inequality in detail.

By definition, it is easy to see
\[ (A.3.8) \]
\[ \partial_3 P^s f = \epsilon \int_{R^2} |\xi| \hat{f}(\xi) e^{i|\xi|x_3} e^{2\pi i x_3 \cdot \xi} d\xi \]

Hence, we have
\[ \| \partial^3 P^s f \|_{H^0}^2 = \epsilon^2 \int_{R^2} \int_{-b_0}^0 |\xi|^2 \hat{f}(\xi) e^{2\pi i |\xi|x_3} d\xi d\xi d\xi \]  
\[ \leq \epsilon^2 \frac{(2\pi)^2}{(2\pi)^2} \| \partial_3 P^s f \|_{H^0}^2 \]

Similarly, we can show that for \( i, j = 1, 2, 3 \)
\[ \| \partial_{3i} P^s f \|_{H^0}^2 \leq \epsilon^2 \frac{(2\pi)^2}{(2\pi)^2} \| \partial_{111} P^s f \|_{H^0}^2 \]

\[ \| \partial_{3i} P^s f \|_{H^0}^2 \leq \epsilon^2 \frac{(2\pi)^2}{(2\pi)^2} \| \partial_{111} P^s f \|_{H^0}^2 \]
A.4. Poisson Integral in Periodic Case. Suppose $\Sigma = (L_1 T) \times (L_2 T)$, we define the parameterized Poisson integral as follows.

$$(A.4.1) \quad \mathcal{P}^{\epsilon} f(x', x_3) = \sum_{n \in (L_1^{-1}Z) \times (L_2^{-1}Z)} e^{2\pi i n \cdot x'} e^{\epsilon |n| x_3} \hat{f}(n)$$

where

$$(A.4.2) \quad \hat{f}(n) = \int_{\Sigma} f(x') \frac{e^{-2\pi i n \cdot x'}}{L_1 L_2} \, dx'$$

Lemma A.7. Let $\mathcal{P}^{\epsilon} f$ be the parameterized Poisson integral of function $f$ which is in homogeneous Sobolev space $H^{q-1/2}(\Sigma)$ for $q \in \mathbb{N}$. Then we have

$$(A.4.3) \quad \| \nabla^q \mathcal{P}^{\epsilon} f \|_{H^0}^2 \leq \frac{C}{\epsilon} \| f \|_{H^{q-1/2}}^2$$

where $C > 0$ is a constant independent of $\epsilon$. In particular, we have

$$(A.4.4) \quad \| \mathcal{P}^{\epsilon} f \|_{H^s}^2 \leq \frac{C}{\epsilon} \| f \|_{H^{q-1/2}(\Sigma)}^2$$

Proof. By Fubini theorem and Parseval identity, we may bound

$$\| \nabla^q \mathcal{P}^{\epsilon} f \|_{H^0}^2 \leq (2\pi)^{2q} \sum_{n \in (L_1^{-1}Z) \times (L_2^{-1}Z)} \int_{-b_0}^{0} |n|^{2q} e^{2\pi i n x_3} \left| \hat{f}(n) \right|^2 \, dx_3$$

$$= (2\pi)^{2q} \sum_{n \in (L_1^{-1}Z) \times (L_2^{-1}Z)} |n|^{2q} \left| \hat{f}(n) \right|^2 \left( \frac{1 - e^{-2\pi b_0 |n|}}{2\pi |n|} \right)$$

$$\leq \frac{(2\pi)^{2q}}{2\epsilon} \sum_{n \in (L_1^{-1}Z) \times (L_2^{-1}Z)} |n|^{2q-1} \left| \hat{f}(n) \right|^2 \leq \frac{\pi}{\epsilon} \| f \|_{H^{q-1/2}}^2$$

We can simply take $C = \pi$ to achieve the estimate. Furthermore, considering

$$\frac{1 - e^{-2\pi b_0 |n|}}{2\pi |n|} \leq \min \left\{ \frac{1}{2\pi |n|}, b_0 \right\}$$

and $0 < \epsilon < 1$, we have

$$(A.4.5) \quad \| \mathcal{P}^{\epsilon} f \|_{H^0}^2 \leq \frac{\pi b_0}{\epsilon} \| f \|_{H^0(\Sigma)}^2$$

Hence, (A.4.4) easily follows. □

Similar to infinite case, we give an $L^\infty$ estimate for Poisson integral.

Lemma A.8. Let $\mathcal{P}^{\epsilon} f$ be the parameterized Poisson integral of function $f$. Then for $q \in \mathbb{N}$ and $s > 1$, we have

$$\| \nabla^q \mathcal{P}^{\epsilon} f \|_{L^\infty}^2 \leq C \| f \|_{H^{q+s}(\Sigma)}^2$$

Proof. A simple application of Sobolev embedding and lemma (A.7) reveals

$$(A.4.6) \quad \| \nabla^q \mathcal{P}^{\epsilon} f \|_{L^\infty}^2 \leq C \| \nabla^q \mathcal{P}^{\epsilon} f \|_{H^{q+1/2}}^2 \leq C \| f \|_{H^{q+s}(\Sigma)}^2$$

We still need the following lemma to illustrate the specialty of derivative in vertical direction.
Hence, we have

\[ \| \partial_3 \mathcal{P}^e f \|_{L^\infty}^2 \leq C \epsilon \| f \|_{H^{5/2}(\Sigma)}^2 \]

where \( C > 0 \) is a constant independent of \( \epsilon \).

**Proof.** We can simply bound as follows.

\[ \| \partial_3 \mathcal{P}^e f \|_{H^0}^2 \leq C \| \partial_3 \mathcal{P}^e f \|_{H^2}^2 \leq C \epsilon^2 \| \mathcal{P}^e f \|_{H^3} \leq C \epsilon \| f \|_{H^{5/2}(\Sigma)}^2 \]

Note that the first inequality is based on Sobolev embedding theorem and the third one is a straightforward application of lemma (A.4), so we only need to verify the second inequality in details. By definition, it is easy to see

\[ \partial_3 \mathcal{P}^e f = \epsilon \sum_{n \in (L_1^{-1}Z) \times (L_2^{-1}Z)} |n| \hat{f}(\xi) e^{i|n|x_3} e^{2\pi i x' \cdot n} \]

Hence, we have

\[ \| \partial_3 \mathcal{P}^e f \|_{H^0}^2 = \epsilon^2 \sum_{n \in (L_1^{-1}Z) \times (L_2^{-1}Z)} \int_{-b_0}^0 |n|^2 \left| \hat{f}(n) \right|^2 e^{2\pi |n|x_3} dx_3 \]

\[ = \frac{\epsilon^2}{(2\pi)^2} \left( \frac{(2\pi)^2}{2} \sum_{n \in (L_1^{-1}Z) \times (L_2^{-1}Z)} \int_{-b_0}^0 |n|^2 \left| \hat{f}(n) \right|^2 e^{2\pi |n|x_3} dx_3 \right) \]

\[ \leq \frac{\epsilon^2}{(2\pi)^2} \| \partial_3 \mathcal{P}^e f \|_{H^0}^2 \]

Similarly, we can show that for \( i, j = 1, 2, 3 \)

\[ \| \partial_{3i} \mathcal{P}^e f \|_{H^0}^2 \leq \frac{\epsilon^2}{(2\pi)^4} \| \partial_{1i} \mathcal{P}^e f \|_{H^0}^2 \]

\[ \| \partial_{3ij} \mathcal{P}^e f \|_{H^0}^2 \leq \frac{\epsilon^2}{(2\pi)^6} \| \partial_{11i} \mathcal{P}^e f \|_{H^0}^2 \]

So we have

\[ \| \partial_{3i} \mathcal{P}^e f \|_{H^0}^2 \leq C \epsilon^2 \| \mathcal{P}^e f \|_{H^3} \]

\[ \square \]

**A.5. Poincare-Type Inequality.** We need several Poincare-type inequality in \( \Omega \). Since all these lemmas have been proved in lemma A.13-A.14 in [1], we will only give the statement here without proof.

**Lemma A.10.** It holds that \( \| f \|_{H^0} \lesssim \| f \|_{H^1} \lesssim \| \nabla f \|_{H^0} \) for all \( f \in H^1(\Omega) \) such that \( f = 0 \) on \( \Sigma_b \).

We also need one type of Korn’s inequality for both infinite case and periodic case.

**Lemma A.11.** If \( \tilde{u} = 0 \) on \( \Sigma_b \), then \( \| \tilde{u} \|_{H^1} \lesssim \| \nabla \tilde{u} \|_{H^0} \).

**A.6. Continuity and Temporal Derivative.** In the following, we give two important lemmas to connect \( L^2 H^k \) norm and \( L^\infty H^k \) norm.

**Lemma A.12.** Suppose that \( u \in L^2([0, T]; H^{s_1}(\Omega)) \) and \( \partial_t u \in L^2([0, T]; H^{s_2}(\Omega)) \) for \( s_1, s_2 \geq 0 \) and \( s = (s_1 + s_2)/2 \). Then \( u \in C^0([0, T]; H^s(\Omega)) \) and satisfies the estimate

\[ \| u \|_{L^2 H^s} \leq \| u(0) \|_{H^s} + \| u \|_{L^2 H^{s_1}} + \| \partial_t u \|_{L^2 H^{s_2}} \]

where the \( L^2 H^k \) norm and \( L^\infty H^k \) norm are evaluated in \([0, T]\).
Proof. Considering the extension theorem in Sobolev space, we only need to prove this result in the \( R^n \) case. The periodic case can be derived in a similar fashion. Using Fourier transform,

\[
\partial_t \|u(t)\|_{H^s}^2 = 2\Re \left( \int_{R^n} \langle \xi \rangle^{2s} \hat{u}(\xi, t) \partial_t \hat{u}(\xi, t) d\xi \right) \leq 2 \int_{R^n} \langle \xi \rangle^{2s} |\hat{u}(\xi, t)| |\partial_t \hat{u}(\xi, t)| d\xi
\]

\[
= 2 \int_{R^n} |\langle \xi \rangle^{s+1} \hat{u}(\xi, t)| |\langle \xi \rangle^{s} \partial_t \hat{u}(\xi, t)| d\xi \leq \int_{R^n} \langle \xi \rangle^{2s+1} |\hat{u}(\xi, t)|^2 d\xi + \int_{R^n} \langle \xi \rangle^{2s} |\partial_t \hat{u}(\xi, t)|^2 d\xi
\]

So integrate with respect to time on \([0, t]\)

\[
\|u(t)\|_{H^s}^2 \leq \|u(0)\|_{H^s}^2 + \|\partial_t u(t)\|_{L^2_{H^s}}^2
\]

□

In [1], lemma A.4 provides us another form of estimate, which will be employed to prove the relation between the topologies generated by these norms.

**Lemma A.13.** Under exactly the same condition as above lemma, we have the estimate

\[
\|u\|_{L^\infty_{H^s}}^2 \lesssim \left( 1 + \frac{1}{T} \right) \left( \|u\|_{L^2_{H^s}}^2 + \|\partial_t u\|_{L^2_{H^s}}^2 \right)
\]

The following lemma shows the estimate in another direction.

**Lemma A.14.** For any \( u \in L^\infty([0, T]; H^k) \) within \([0, T]\), we must have \( u \in L^2([0, T]; H^k) \) and satisfies the estimate

\[
\|u\|_{L^2_{H^k}}^2 \leq T \|u\|_{L^\infty_{H^k}}^2
\]

Proof. The result is simply based on the definition of these two norms. □

**A.7. Extension Theorem.** The following are two extension theorems which will be used to construct start point of iteration from initial data in proving wellposedness of Naiver-Stokes-transport system.

**Lemma A.15.** Suppose that \( \partial_j^t u(0) \in H^{2N-2j}(\Omega) \) for \( j = 0, \ldots, N \), then there exists a extension \( u \) achieving the initial data, such that

\[
\partial_j^t u \in L^2([0, \infty); H^{2N-2j+1}(\Omega)) \cap L^\infty([0, \infty); H^{2N-2j}(\Omega))
\]

for \( j = 0, \ldots, N \). Moreover,

\[
\sum_{j=0}^N \|\partial_j^t u\|_{L^2_{H^{2N-2j+1}}}^2 + \|\partial_j^t u\|_{L^\infty_{H^{2N-2j}}}^2 \lesssim \sum_{j=0}^N \|\partial_j^t u(0)\|_{H^{2N-2j}}^2
\]

Proof. The same as lemma A.5 in [1]. □

**Lemma A.16.** Suppose that \( \partial_j^t p(0) \in H^{2N-2j-1}(\Omega) \) for \( j = 0, \ldots, N - 1 \), then there exists a extension \( p \) achieving the initial data, such that

\[
\partial_j^t p \in L^2([0, \infty); H^{2N-2j}(\Omega)) \cap L^\infty([0, \infty); H^{2N-2j-1}(\Omega))
\]

for \( j = 0, \ldots, N - 1 \). Moreover,

\[
\sum_{j=0}^{N-1} \|\partial_j^t p\|_{L^2_{H^{2N-2j}}}^2 + \|\partial_j^t p\|_{L^\infty_{H^{2N-2j-1}}}^2 \lesssim \sum_{j=0}^{N-1} \|\partial_j^t p(0)\|_{H^{2N-2j-1}}^2
\]

Proof. The same as lemma A.6 in [1]. □
Appendix B. Estimates for Fundamental Equations

B.1. Transport Estimates. Let Σ be either infinite or periodic. Consider the equation

\begin{equation}
\begin{cases}
\partial_t \eta + u \cdot D \eta = g & \text{in } \Sigma \times (0, T) \\
\eta(t=0) = \eta_0 & \text{on } \Sigma \times \{0\}
\end{cases}
\end{equation}

We have the following estimate of the regularity solution to this equation, which is a particular case of a more general result proved in proposition 2.1 of [3]. Note that the result in [3] is stated for \( \Sigma = \mathbb{R}^2 \), but the same result holds in the periodic case as described in [7].

Lemma B.1. Let \( \eta \) be a solution to equation (B.1.1). Then there exists a universal constant \( C > 0 \) such that for any \( s \geq 3 \)

\begin{equation}
\| \eta \|_{L^\infty H^s} \leq \exp \left( C \int_0^t \| Du(r) \|_{H^{3/2}} \, dr \right) \left( \| \eta_0 \|_{H^s} + \int_0^t \| g(r) \|_{H^s} \, dr \right)
\end{equation}

Proof. Use \( p = p_2 = 2, N = 2 \) and \( \Sigma = s \) in proposition 2.1 of [7], along with the embedding \( H^{3/2} \hookrightarrow B^{2}_{2,\infty} \cap L^\infty \).

B.2. Elliptic Estimates. We need the following elliptic estimate for Stokes equation.

Lemma B.2. Suppose \((u, p)\) solves the equation

\begin{equation}
\begin{cases}
-\Delta u + \nabla p = \phi & \text{in } \Omega \\
\nabla \cdot u = \psi & \text{in } \Omega \\
(pI - \nabla u) e_3 = \varphi & \text{on } \Sigma \\
u = 0 & \text{on } \Sigma_b
\end{cases}
\end{equation}

Then for \( r \geq 2 \),

\[ \| u \|_{H^r}^2 + \| p \|_{H^{r-1}}^2 \lesssim \| \phi \|_{H^{r-2}}^2 + \| \psi \|_{H^{r-1}}^2 + \| \varphi \|_{H^{r-3/2}}^2 \]

Proof. The same as lemma A.15 in [1].

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