ANALYTIC PRO-$p$ GROUPS OF SMALL DIMENSIONS

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Abstract. According to Lazard, every $p$-adic Lie group contains an open pro-$p$ subgroup which is saturable. This can be regarded as the starting point of $p$-adic Lie theory, as one can naturally associate to every saturable pro-$p$ group $G$ a Lie lattice $L(G)$ over the $p$-adic integers.

Essential features of saturable pro-$p$ groups include that they are torsion-free and $p$-adic analytic. In the present paper we prove a converse result in small dimensions: every torsion-free $p$-adic analytic pro-$p$ group of dimension less than $p$ is saturable.

This leads to useful consequences and interesting questions. For instance, we give an effective classification of 3-dimensional soluble torsion-free $p$-adic analytic pro-$p$ groups for $p > 3$. Our approach via Lie theory is comparable with the use of Lazard’s correspondence in the classification of finite $p$-groups of small order.

1. Introduction

In his seminal paper Groupes analytiques $p$-adiques [15] from 1965, Lazard proved that a topological group is $p$-adic analytic if and only if it contains an open pro-$p$ subgroup which is saturable. Indeed, the class of saturable pro-$p$ groups features prominently in $p$-adic Lie theory. This is due to the fact that one can naturally associate to every saturable pro-$p$ group a Lie lattice over the $p$-adic integers $\mathbb{Z}_p$. In fact, Lazard established an isomorphism between the category of saturable pro-$p$ groups and the category of saturable $\mathbb{Z}_p$-Lie lattices.

In the 1980s, Lubotzky and Mann reinterpreted the group-theoretic aspects of Lazard’s work, starting from the new concept of a powerful pro-$p$ group. In their setting, uniformly powerful groups take over the central role which was originally played by saturable pro-$p$ groups; see [3]. It is easily seen that uniformly powerful pro-$p$ groups form a subclass of the class of saturable pro-$p$ groups. Klopsch pointed out that the Sylow pro-$p$ subgroups of many classical groups are saturable, but typically fail to be uniformly powerful. Generalising results of Ilani, he established a one-to-one correspondence between closed subgroups

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of a saturable pro-$p$ group and Lie sublattices of the associated $\mathbb{Z}_p$-Lie lattice; see [12]. This somewhat re-established Lazard’s ‘groupes $p$-saturables’ as key players in the Lie theory of $p$-adic analytic groups and led to applications, for instance, in the subject of subgroup growth; cf. [13]. More recently, González-Sánchez [5] gave a characterisation of saturable pro-$p$ groups, which is more suitable for applications in group theory than Lazard’s original definition: a pro-$p$ group is saturable if and only if it is finitely generated, torsion-free and admits a potent filtration; see Section 2 for details. In general it is a delicate issue to decide whether a given torsion-free $p$-adic analytic pro-$p$ group is saturable or not. Using González-Sánchez’ characterisation, we prove

**Theorem A.** Every torsion-free $p$-adic analytic pro-$p$ group of dimension less than $p$ is saturable. On the other hand, there exists a torsion-free $p$-adic analytic pro-$p$ group of dimension $p$ which is not saturable.

In the present paper Theorem A leads us to several related results, applications and open questions. For instance, it is the basic ingredient for an analogue of Lazard’s classical correspondence between finite $p$-groups and finite nilpotent Lie rings of $p$-power order.

### 1.1. Isomorphism of Categories.

As indicated above, a most remarkable and useful property of a saturable pro-$p$ group is that its underlying set carries naturally the ‘dual’ structure of a $\mathbb{Z}_p$-Lie lattice. Theorem A and a similar result for Lie lattices (see Theorem 4.6) lead to the following specific instance of Lazard’s general Lie correspondence between saturable pro-$p$ groups and saturable $\mathbb{Z}_p$-Lie lattices; cf. [15, IV (3.2.6)] and [12]. Let $\text{Grp}_{<p}$ denote the category of torsion-free $p$-adic analytic pro-$p$ groups of dimension less than $p$, and let $\text{Lie}_{<p}$ denote the category of residually-nilpotent $\mathbb{Z}_p$-Lie lattices of dimension less than $p$. As we recall in Section 2, the Hausdorff series and its inverse give rise to functors $\text{grp} : \text{Lie}_{<p} \to \text{Grp}_{<p}$ and $\text{lie} : \text{Grp}_{<p} \to \text{Lie}_{<p}$. These functors can be understood by means of the familiar exponential and logarithm series; cf. [2, III §7] and [15, IV (3.2)].

**Theorem B.** The functors $\text{grp}$ and $\text{lie}$ are mutually inverse isomorphisms of categories between $\text{Lie}_{<p}$ and $\text{Grp}_{<p}$.

1. The functor $\text{grp}$ sends $\mathbb{Z}_p$-Lie sublattices to closed subgroups, and Lie ideals to closed normal subgroups. Conversely, the functor $\text{lie}$ sends closed subgroups to $\mathbb{Z}_p$-Lie sublattices, and closed normal subgroups to Lie ideals.

2. The functors $\text{grp}$ and $\text{lie}$ restrict to isomorphisms between the subcategories of nilpotent (respectively soluble) objects on both sides. Moreover, they preserve the nilpotency class (respectively derived length), when the groups and Lie lattices involved are nilpotent (respectively soluble).
In Section 7 we illustrate the usefulness of this theorem by giving a concrete application: based on an analysis of conjugacy classes in $\text{SL}_2(\mathbb{Z}_p)$, we obtain an effective classification of 3-dimensional soluble torsion-free $p$-adic analytic pro-$p$ groups for $p > 3$; see Theorem 7.4. Our procedure is comparable to the use of Lazard’s correspondence in classifying finite-$p$ groups of small order, e.g. see [17]. Implicitly we obtain a complete list of 3-dimensional soluble torsion-free $\mathbb{Z}_p$-Lie lattices; this has already been used in [14]. We remark that among the 3-dimensional soluble torsion-free $p$-adic analytic pro-$p$ groups there are several which are not powerful. They complement the insoluble examples of saturable but not uniformly powerful pro-$p$ groups given in [12]. For completeness we also discuss briefly insoluble torsion-free $p$-adic analytic pro-$p$ groups of dimension 3.

1.2. Correspondence between subgroups and Lie sublattices. In view of Theorem A, the borderline groups of dimension $p$ are of special interest. We prove that saturable pro-$p$ groups of dimension at most $p$ are in fact saturable in a ‘strong’ sense.

Proposition C. Every saturable pro-$p$ group $G$ of dimension at most $p$ satisfies $\gamma_p(G) \subseteq \Phi(G)^p$.

We remark that every torsion-free finitely generated pro-$p$ group $G$ satisfying $\gamma_p(G) \subseteq \Phi(G)^p$ is saturable; see Theorem 2.2. Moreover, González-Sánchez has given an example of a saturable pro-$p$ group $G$ of dimension $p + 1$ such that $\gamma_p(G) \not\subseteq \Phi(G)^p$; see [4, Example after Corollary 3.6]. Proposition C shows that no such examples exist in dimensions up to $p$.

Proposition D. Every $p$-adic analytic pro-$p$ group of dimension $p$ which embeds into a saturable pro-$p$ group is itself saturable.

As an immediate consequence of this inconspicuous proposition we obtain a new and conceptually satisfying proof of

Theorem E. Let $G$ be a saturable pro-$p$ group and let $L(G)$ denote the associated saturable $\mathbb{Z}_p$-Lie lattice. Let $K, H \subseteq G$ be closed subsets, and denote them by $L(K), L(H)$ when regarded as subsets of $L(G)$.

(1) Suppose that $H$ is a subgroup of $G$ such that every 2-generated subgroup of $H$ has dimension at most $p$. Then $L(H)$ is a Lie sublattice of $L(G)$. Moreover, if $K$ is a normal subgroup of $H$, then $L(K)$ is a Lie ideal of $L(H)$.

(2) Suppose that $L(H)$ is a Lie sublattice of $L(G)$ such that every 2-generated Lie sublattice of $L(H)$ has dimension at most $p$. Then $H$ is a subgroup of $G$. Moreover, if $L(K)$ is a Lie ideal of $L(H)$, then $K$ is a normal subgroup of $H$.

This Lie correspondence between subgroups of a saturable pro-$p$ group and Lie sublattices of the associated Lie lattice was originally
discovered by Ilani \cite{Ila} in the context of uniform pro-$p$ groups, and subsequently extended by Klopsch \cite{Klo} to cover the more general case. Ilani and Klopsch both missed the central insight which we recorded as Proposition \ref{prop:D} and had to rely on a less transparent line of argument.

1.3. Open questions. Our results raise several open questions which we record together with some partial results. In the present paper we work within the class of all $p$-adic analytic pro-$p$ groups. It would be interesting to develop saturability criteria, which are tailor-made for more specific families of groups. Foremost we have in mind the class of $p$-adic analytic just-infinite pro-$p$ groups, which was systematically studied in \cite{Klo1}. Indeed, the original starting point for our present work was a result of Klopsch in this direction \cite[Theorem 1.3]{Klo}: every insoluble maximal $p$-adic analytic just-infinite pro-$p$ group of dimension less than $p - 1$ is saturable. In Section \ref{sec:6} we consider

Question 1.1. What group theoretic properties ensure that a $p$-adic analytic just-infinite pro-$p$ group is torsion-free or saturable?

In order to formulate a first necessary condition we recall that to every $p$-adic analytic group $G$ one associates a $\mathbb{Q}_p$-Lie algebra $\mathcal{L}(G)$; see Section 2. It is known that the $\mathbb{Q}_p$-Lie algebra attached to an insoluble $p$-adic analytic just-infinite pro-$p$ group $G$ is semisimple; in fact, there exists $k \in \mathbb{N}_0$ such that $\mathcal{L}(G)$ is a direct sum of $p^k$ copies of a simple $\mathbb{Q}_p$-Lie algebra.

**Proposition F.** The $\mathbb{Q}_p$-Lie algebra associated to a saturable insoluble just-infinite pro-$p$ group is simple. Every saturable insoluble just-infinite pro-$p$ group is hereditarily just-infinite.

In contrast, for any simple $\mathbb{Q}_p$-Lie algebra $\mathcal{L}$ and for any $p$-power $p^k$, $k \in \mathbb{N}_0$, there exists a torsion-free $p$-adic analytic just-infinite pro-$p$ group whose associated $\mathbb{Q}_p$-Lie algebra is isomorphic to the direct sum of $p^k$ copies of $\mathcal{L}$.

In particular, the proposition yields just-infinite examples of non-saturable torsion-free $p$-adic analytic pro-$p$ groups of dimension $3p$. This indicates natural limitations of a possible extension of Theorem \ref{thm:A}.

A trivial consequence of Proposition \ref{prop:D} is that saturable pro-$p$ groups of dimension at most $p$ have the property that every one of their open subgroups is again saturable. It is a challenging problem to find other natural families of pro-$p$ groups which are ‘hereditarily saturable’. This motivates

Question 1.2. Are all open subgroups of a saturable just-infinite pro-$p$ group saturable?

In view of \cite[Theorem 1.1]{Klo} we may specialise our question further to special linear groups over the ring of integers of a $p$-adic field as follows. Let $\mathfrak{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with
ramification index $e$, and let $n \in \mathbb{N}$ with $ne \leq p - 2$. Are all open subgroups of a Sylow pro-$p$ subgroup of $\text{SL}_n(\mathbb{O})$ saturable?

In a different direction, it would be interesting to explore further the borderline cases where Lie theoretic methods break down. In light of Theorem A it is natural to ask

**Question 1.3.** What are the torsion-free $p$-adic analytic pro-$p$ groups $G$ of dimension $p$ which are not saturable, i.e. which fail to satisfy the condition $\gamma_p(G) \subseteq \Phi(G)^p$? Can we classify them? Similarly, what are the residually-nilpotent $\mathbb{Z}_p$-Lie lattices of dimension $p$ which are not saturable?

Based on the examples that we have at present, it is conceivable that one can go beyond a Lie theory founded on Lazard’s notion of saturability. Pink has interesting results which support this idea; cf. [18].

**Question 1.4.** Is there a (natural) correspondence between residually-nilpotent $\mathbb{Z}_p$-Lie lattices of dimension $p$ and torsion-free $p$-adic analytic pro-$p$ groups of dimension $p$?

**Organisation.** In Section 2 we recall the characterisations of saturable pro-$p$ groups and saturable $\mathbb{Z}_p$-Lie lattices in terms of potent filtrations. Subsequently, we summarise Lazard’s correspondence between saturable pro-$p$ groups and saturable $\mathbb{Z}_p$-Lie lattices. In Section 3 we prove an auxiliary result concerning isolated subgroups of saturable pro-$p$ groups. In Section 4 we prove Theorem A, Proposition C and Proposition D. As immediate consequences we obtain Theorems E and B. In Section 5 we provide in the context of $\mathbb{Z}_p$-Lie lattices analogues of the classical Levi splitting and the decomposition of semisimple Lie algebras into simple summands. From this we prove the first part of Proposition F. In Section 6 we look at $p$-adic analytic just-infinite pro-$p$ groups and complete the proof of Proposition F. In Section 7 we describe torsion-free $p$-adic analytic pro-$p$ groups of dimension at most 3 by a procedure which is comparable to the use of Lazard’s correspondence in classifying finite-$p$ groups of small order. Our main result is Theorem 7.4 which provides an effective classification of 3-dimensional soluble torsion-free $p$-adic analytic pro-$p$ groups for $p > 3$.

**Notation.** Throughout, $p$ denotes an odd prime. Let $G$ be a group. If $H \subseteq G$ and $n \in \mathbb{N}$, then $H^n := \langle h^n \mid h \in H \rangle$. The Frattini subgroup of $G$ is denoted by $\Phi(G)$. In particular, if $G$ is a pro-$p$ group, then $\Phi(G) = G^p[G,G]$. As customary, the terms of the lower central series of $G$ are denoted by $\gamma_i(G)$, $i \in \mathbb{N}$.

As a rule we do not distinguish notationally between the group commutator $[x, y]$ of group elements $x, y$ and the Lie bracket $[x, y]$ of Lie lattice elements $x, y$. Both, group commutators and Lie brackets are left-normed. For $k \in \mathbb{N}$, the abbreviation $[N, k \ M]$ stands for $[N, M, \ldots, M]$ with $M$ occurring $k$ times.
2. Potent filtrations and Lie correspondence

In the present section we recall González-Sánchez’ characterisations of saturable pro-$p$ groups and saturable Lie lattices, based on the concept of potent filtrations. Then we summarise Lazard’s correspondence between saturable pro-$p$ groups and saturable $\mathbb{Z}_p$-Lie lattices.

Let $G$ be a pro-$p$ group, and let $N$ be a closed normal subgroup of $G$. A potent filtration of $N$ in $G$ is a descending series $(N_i)_{i\in\mathbb{N}}$ of closed normal subgroups of $G$ such that (i) $N_1 = N$, (ii) $\bigcap_{i\in\mathbb{N}} N_i = 1$, (iii) $[N_i, G] \subseteq N_{i+1}$ and $[N_i, p^{-1} G] \subseteq N_i^p$ for all $i \in \mathbb{N}$. We say that $N$ is $PF$-embedded in $G$ if there exists a potent filtration of $N$ in $G$. The group $G$ is a $PF$-group, if $G$ is $PF$-embedded in itself.

We recall basic properties of $PF$-embedded subgroups, which follow essentially from the Hall-Petrescu collection formula [7, §9]; see [4, Proposition 3.2 and Theorem 3.4] and [5, Proposition 2.2].

Lemma 2.1 (Fernández-Alcober, González-Sánchez, Jaikin-Zapirain). Let $G$ be a pro-$p$ group, and let $N, M$ be $PF$-embedded subgroups of $G$. Then

1. $NM, N^p$ and $[N, k^p G]$ are $PF$-embedded in $G$ for all $k \in \mathbb{N}$;
2. $[N^p, G] = [N, G]^p$;
3. $N^p = \{ x^p \mid x \in N \}$;
4. if $G$ is torsion-free and $x \in G$ with $x^p \in N^p$, then $x \in N$; moreover, if $x, y \in N$ such that $x^p = y^p$, then $x = y$.

We also state González-Sánchez’ characterisation of saturable pro-$p$ groups; see [5, Theorem 3.4 and Corollary 5.4].

Theorem 2.2 (González-Sánchez). Let $G$ be a torsion-free finitely generated pro-$p$ group. Then $G$ is saturable if and only if $G$ (or equivalently $G/\Phi(G)^p$) is a $PF$-group.

In particular, if $\gamma_p(G) \subseteq \Phi(G)^p$, then $G$ is saturable.

If $G$ is a torsion-free finitely generated pro-$p$ group satisfying $\gamma_p(G) \subseteq \Phi(G)^p$, then it is particularly easy to write down a potent filtration $G_i$, $i \in \mathbb{N}$: one simply takes the lower $p$-series

\[ G_1 := G, \quad \text{and} \quad G_i := [G_{i-1}, G]G_i^{p_{i-1}} \text{ for } i \geq 2. \]

Indeed, $[G_{1, p^{-1} G}] = \gamma_p(G) \subseteq \Phi(G)^p = G_2^p$ and, using induction and Lemma 2.1, we see that for $i \geq 2$,

\[ [G_{i, p^{-1} G}] = [[G_{i-1}, G]G_i^{p_{i-1}}] \subseteq [G_{i-1, p^{-1} G}, G_i]G_i^{p_{i-1}} \]
\[ \subseteq [G_i^p, G_i^p[G_{i-1, p^{-1} G}, G_i^p, G_i]^p] \subseteq [G_i, G_i^p[G_i^p, G_i]^p] \subseteq G_{i+1}. \]

Similarly as for groups, one can study saturable $\mathbb{Z}_p$-Lie lattices and Lie lattices admitting a potent filtration by ideals; for details see [15, I (2)], summarised in [12, §2], and [5, §4]. The analogue of Theorem 2.2 states that a $\mathbb{Z}_p$-Lie lattice $L$ is saturable if and only if $L$ (or equivalently $L/(p[L, L] + p^2L)$) admits a potent filtration.
Lazard’s correspondence between saturable pro-$p$ groups and saturable $\mathbb{Z}_p$-Lie lattices can be obtained via exponential and logarithm maps inside certain Hopf algebras. Alternatively, the passage from Lie lattice to group structure can be described in terms of the Hausdorff series (sometimes called Baker-Campbell-Hausdorff series, cf. [2, Historical Note, V]) and its inverse.

The Hausdorff series is defined as $\Phi(X, Y) := \log((\exp X)(\exp Y)) \in \mathbb{Q}[[X, Y]]$. For our purposes one writes $\Phi(X, Y)$ as a series of Lie words in independent variables $X, Y$. For instance, modulo terms of weight 4 or higher, we have

$$\Phi(X, Y) \equiv X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, Y, Y] - [X, Y, X]);$$

cf. [2, II §8]. Given a saturable $\mathbb{Z}_p$-Lie lattice $L$, the saturable pro-$p$ group corresponding to $L$ has underlying set $L$ and the group product of $x, y \in L$ is given by $xy := \Phi(x, y)$. Conversely, given a saturable pro-$p$ group $G$, one defines on the set $G$ the structure of a saturable $\mathbb{Z}_p$-Lie lattice $L(G)$ as follows. For $\lambda \in \mathbb{Z}_p$ and $x, y \in G$ one sets

$$\lambda x := x^\lambda, \quad x + y := \lim_{n \to \infty} (x^{p^n} y^{p^n})^{p^{-n}}, \quad [x, y]_{\text{Lie}} := \lim_{n \to \infty} [x^{p^n}, y^{p^n}]^{p^{-2n}}.$$

See [15, IV (3.2.6)] or [12, §2]; the latter contains a more detailed summary. The $\mathbb{Q}_p$-Lie algebra associated to a compact $p$-adic analytic group $G$ is defined as $\mathcal{L}(G) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L(U)$, where $U$ is an open saturable pro-$p$ subgroup of $G$. Clearly, $\mathcal{L}(G)$ only depends on the commensurability class of $G$.

### 3. Isolators in saturable pro-$p$ groups

An effective tool, used in [12], consists in embedding a given subgroup of a saturated pro-$p$ group into its saturated closure; also cf. [15, IV (3.2.6)]. We now explain this procedure using potent filtrations.

Let $G$ be a closed subgroup of a pro-$p$ group $S$. The isolator of $G$ in $S$ is defined as the closed subgroup

$$\text{iso}_S(G) := \langle g \in S \mid \exists k \in \mathbb{N} : g^{p^k} \in G \rangle.$$

Now suppose that $S$ is saturable. Then Theorem 2.2 shows that $S$ admits a potent filtration $S_i$, $i \in \mathbb{N}$. Note that $S_i^{p^i} = S_i^{(p)}$ for all $i \in \mathbb{N}$, by Lemma 2.1, and thus

$$[(\text{iso}_S(G) \cap S_i),p^{-1} \text{iso}_S(G)] \subseteq \text{iso}_S(G) \cap S_i^{(p)} = (\text{iso}_S(G) \cap S_{i+1})^{(p)}.$$

This implies that $\text{iso}_S(G) \cap S_i$, $i \in \mathbb{N}$, is a potent filtration of $\text{iso}_S(G)$. By Theorem 2.2, the group $\text{iso}_S(G)$ is saturable.

If $G$ is a normal subgroup of $S$, the isolator $\text{iso}_S(G)$ is also a normal subgroup of $S$, in particular

$$[(\text{iso}_S(G) \cap S_i),p^{-1} S] \subseteq \text{iso}_S(G) \cap S_i^{(p)} = (\text{iso}_S(G) \cap S_{i+1})^{(p)}.$$
Hence $\text{iso}_S(G) \cap S_i, i \in N,$ is a potent filtration of $S$ starting at $\text{iso}_S(G),$ and thus $\text{iso}_S(G)$ is PF-embedded in $S.$

At this stage it is advantageous to briefly consider the analogous situation for $\mathbb{Z}_p$-Lie lattices. Let $L$ be a Lie sublattice of a $\mathbb{Z}_p$-Lie lattice $S$. Then the isolator of $L$ in $S$ is the Lie sublattice $\text{iso}_S(L) = S \cap (\mathbb{Q}_p \otimes L).$ We observe that the concept is a priori easier to handle than for groups. For instance, it is immediate that $|\text{iso}_S(L) : L| < \infty.$

Similar arguments as the ones given above in the context of groups lead to

**Proposition 3.1.** Let $L$ be a Lie sublattice of a saturable $\mathbb{Z}_p$-Lie lattice $S$. Then the isolator $\text{iso}_S(L)$ is saturable and $|\text{iso}_S(L) : L| < \infty$. In particular, $L$ and $\text{iso}_S(L)$ have the same dimension.

Furthermore, if $L$ is a Lie ideal of $S$, then $\text{iso}_S(L)$ is PF-embedded in $S$.

Our aim is to prove the corresponding result for groups, and we return to the situation where $G$ is a subgroup of a saturable pro-$p$ group $S$. Recall that the group $S$ and its associated $\mathbb{Z}_p$-Lie lattice $L(S)$ are defined on the same underlying set and that taking $p$th powers in $S$ has the same effect as multiplying by $p$ in $L(S)$; cf. Section 2. The group $G$ contains an open subgroup $H$ which is saturable, and clearly $\text{iso}_S(G) = \text{iso}_S(H)$. By Lazard’s correspondence $L(H)$, i.e. $H$ considered as a subset of $L(G)$, forms a Lie sublattice of $L(S)$. By Proposition 3.1, the Lie isolator $\text{iso}_{L(S)}(L(H))$ is saturable and corresponds to a subgroup of $S$. Indeed, this latter subgroup is easily identified as the group isolator $\text{iso}_S(H)$. We summarise our findings in

**Proposition 3.2.** Let $G$ be a closed subgroup of a saturable pro-$p$ group $S$. Then $\text{iso}_S(G) = \{g \in S \mid \exists k \in \mathbb{N} : g^{p^k} \in G\}$. Moreover, $\text{iso}_S(G)$ is saturable and $|\text{iso}_S(G) : G| < \infty$. In particular, $G$ and $\text{iso}_S(G)$ have the same dimension.

Furthermore, if $G$ is a normal subgroup of $S$, then $\text{iso}_S(G)$ is PF-embedded in $S$.

4. Groups of dimension at most $p$

In the present section we prove Theorem A, Proposition C and Proposition D. As immediate consequences we obtain Theorems E and E.

**Proposition 4.1.** Let $G$ be a torsion-free $p$-adic analytic pro-$p$ group of dimension less than $p$. Then $\gamma_p(G) \subseteq \Phi(G)^p$; in particular, $G$ is a PF-group.

**Proof.** If $p = 2$, then $\mathbb{Z}_2$ is the only 1-dimensional torsion-free pro-$p$ group and the proposition is obviously true. Let us suppose that $p \geq 3$. By Theorem 2.2, the assertion $\gamma_p(G) \subseteq \Phi(G)^p$ which we are about to prove will imply that $G$ is a PF-group. Clearly, we may assume that
$G$ is not the trivial group. Since $G$ is $p$-adic analytic, there exists a proper open normal subgroup $N$ of $G$ which is saturable and hence a PF-group. The chief factors of the finite $p$-group $G/N$ are cyclic of order $p$. Arguing by induction on the composition length of $G/N$, we may assume that $[G : N] = p$.

Since $N$ is saturable, we have $|N : N^p| = p^{|\dim(N)|} = p^{|\dim(G)|} \leq p^{p-1}$; see [15, III (3.1)]]. This shows that $|G : N^p| \leq p$, and consequently $\gamma_p(G) \subseteq N^p$.

**Case 1:** $\gamma_{p-1}(G) \subseteq N^p$. Then, a fortiori, $\gamma_{p-1}(G) \leq G^p$ and the Hall-Petrescu formula [11 §9] shows that, modulo $[G,G]^p$,

$$\gamma_p(G) = [\gamma_{p-1}(G),G] \subseteq [G^p,G] \subseteq \gamma_{p+1}(G);$$

cf. [6, proof of Theorem 3.1]. Thus $\gamma_p(G) \subseteq [G,G]^p \subseteq \Phi(G)^p$, as wanted.

**Case 2:** $\gamma_{p-1}(G) \not\subseteq N^p$. Then $G/N^p$ is a finite $p$-group of maximal class: $G/N^p$ has order $p^a$ and nilpotency class $p - 1$. Put \( N_1 := N \) and \( N_i := \gamma_i(G)N \) for $i \in \{2, \ldots, p-1\}$. Then

$$G \supset N = N_1 \supset N_2 \supset \ldots \supset N_{p-1} \supset N^p$$

is a central descending series of closed normal subgroups such that each term has index $p$ in its predecessor. Moreover, every normal subgroup of $G$ which is strictly contained in $N$ must be contained in $N_2 = \gamma_2(G)N^p$.

First suppose that $\gamma_p(G) \not\subseteq N^p$. The set of elements of order $p$ in the regular $p$-group $N/\gamma_p(G)$ forms a subgroup. Hence $M := \{x \in N \mid x^p \in \gamma_p(G)\}$ forms a subgroup of $G$ which, by assumption, is strictly contained in $N$. By our observation, $M$ is thus contained in $N_2$.

Furthermore, according to Lemma 2.1(3), we have $N^p = \{x^p \mid x \in N\}$, and hence $M^p = \gamma_p(G)$. This gives $\gamma_p(G) = M \subseteq N_2^p \subseteq \Phi(G)^p$, as wanted.

We claim that the remaining case $\gamma_p(G) = N^p$ actually does not occur. For a contradiction, suppose that $\gamma_p(G) = N^p$. We define recursively $N_i := N_{i-1}^{p+1}$ for $i \in \mathbb{N}$ with $i \geq p$. Note that $N_i = \gamma_i(G)$ for $i \in \{2, \ldots, p\}$.

**Subclaim:** $N_i$ is PF-embedded in $N$ for all $i \in \mathbb{N}$.

**Subproof.** This is true for $i = 1$ as $N = N_1$ is a PF-group, and by Lemma 2.1(1) it now suffices to prove the assertion for $i \in \{2, \ldots, p-1\}$. Let $i$ be in this range. It follows from Lemma 2.1 (1),(2) that $[N_{i-1}^p, N_{i-1}] \subseteq \gamma_i(N)^p$. This gives

$$[N_{i,p-1} - N] = [\gamma_i(G),p-1 N] \subseteq [\gamma_p(G),i-1 N] \subseteq [N_{i-1}^p, N_{i-1}] \subseteq \gamma_i(N)^p.$$ 

Now, $\gamma_i(N)$ is PF-embedded in $N$ by Proposition 2.1(1), hence we can complete the inclusion $N_i = \gamma_i(G) \supseteq \gamma_i(N)$ to a potent filtration of $N_i$ in $N$. This proves the subclaim.

**Subclaim:** $|N_i : N_{i+1}| = p$ for all $i \in \mathbb{N}$. 


Subproof. This is clear for $i \in \mathbb{N}$ with $i \leq p - 1$. Now let $i \geq p$ and argue by induction. By the first subclaim $N_j$ is a PF-group for every $j \in \mathbb{N}$; thus Lemma 2.1 (3) shows that $N_{j+p-1} = N_j^p = \{x^p \mid x \in N_j\}$. From [12, Lemma A.3] it follows by induction that $|N_i : N_{i+1}| = |N_{i+p+1} : N_{i+p+2}| = p$.

Subclaim: $N_i = \gamma_i(G)$ for all $i \in \mathbb{N}$ with $i \geq 2$.

Subproof. Let $i \in \mathbb{N}$ with $i \geq 2$, and argue by induction. For $i \leq p$ the assertion holds, as noted above. Suppose now that $i > p$. By induction, we have $N_i = N_i^p = \gamma_i(G)^p = [\gamma_i(G), G]^p = [N_{i-1}, G]^p$ and similarly $\gamma_i(G) = [\gamma_i(G), G] = [N_{i-1}, G] = [N_{i-p}, G]$. The Hall-Petrescu formula [7, §9] yields $[N_{i-1}, G]^p \equiv [N_{i-p}, G]$ modulo $[G, p, N_{i-p}]$ (cf. [4, Theorem 2.4]), thus

$$N_i \equiv \gamma_i(G) \pmod{[G, p, N_{i-p}]}.$$

By induction, $[G, p, N_{i-p}] = [G, \gamma_{i-p}(G)] \subseteq \gamma_i(G)$. Again by induction and since $N_{i+p-1}$ is PF-embedded in $N$, we also obtain

$$[G, p, N_{i-p}] = [G, \gamma_{i-p}(G), N_{i-p}] \subseteq [\gamma_{i-p}(G), N_{i-p}] \subseteq [N_{i-p+1}, N_{i-p+1}] = N_{i+p} = N_{i+1} \subseteq N_i.$$

Therefore it follows that $N_i = \gamma_i(G)$, as claimed.

The last two subclaims show that $G$ is the unique pro-$p$ group of maximal class; cf. [16] Section 7.4. But in this case $G$ has elements of order $p$, a contradiction.

The following example indicates that the condition on the dimension in Proposition 4.1 is sharp.

Example 4.2. We display a torsion-free $p$-adic analytic pro-$p$ group of dimension $p$ which is not a PF-group.

Let $M = \langle x_1, \ldots, x_{p-1} \rangle \cong \mathbb{Z}_{p}^{p-1}$ be a free abelian pro-$p$ group of rank $p - 1$, and let $A = \langle \alpha \rangle \cong \mathbb{Z}_p$. Consider the semidirect product $G := A \rtimes M$, with respect to the action of $A$ on $M$ defined by

$$x^\alpha_i = \begin{cases} x_ix_{i+1} & \text{if } 1 \leq i \leq p - 2, \\ x_{p-1}x_1^p & \text{if } i = p - 1. \end{cases}$$

Clearly, $G$ is a torsion-free $p$-adic analytic pro-$p$ group. Moreover, it is easily seen that $[M, p-1 G] = M^p$. For a contradiction, suppose that $G$ is a PF-group and let $(G_i)_{i \in \mathbb{N}}$ be a potent filtration of $G$. Then $M^p \subseteq [G, p-1 G] \subseteq G^p$ implies that $M \subseteq G_2$. Inductively, it follows that $M \subseteq G_i$ for all $i \in \mathbb{N}$. This contradicts the fact that $\bigcap G_i = 1$. ◊

Theorem A follows directly from Proposition 4.1 and Example 4.2.

Corollary 4.3. Let $G$ be a saturable pro-$p$ group, and let $N$ be a normal subgroup of $G$ of dimension less than $p$. Then $N$ is PF-embedded in $G$.  

Proof. By Theorem 2.2, \( G \) is a PF-group, and by Proposition 4.1, \( N \) is a PF-group and hence saturable. Let \( (G_i)_{i \in \mathbb{N}} \) be a potent filtration of \( G \). Define \( N_i := N \cap G_i \) for \( i \in \mathbb{N} \). Since \( N \) is saturable of dimension less than \( p \), we have \([N : N^p] \leq p^{p-1} \); see [15, III (3.1)]. Therefore \([N_{i-1} : N^p] \subseteq N^p \) and hence \([N_{i-1} : N^p] \subseteq N^p \cap G_{i+1}^p \). Applying Lemma 2.1 (3), (4), we obtain \( N^p \cap G_{i+1}^p = (N \cap G_{i+1})^p = N_{i+1}^p \) for all \( i \in \mathbb{N} \). Thus \((N_i)_{i \in \mathbb{N}} \) is a potent filtration of \( N \) in \( G \).

Proposition 4.1 implies that every saturable pro-\( p \) group \( G \) of dimension less than \( p \) satisfies \( \gamma_p(G) \subseteq \Phi(G)^p \). In order to complete the proof of Proposition 4.1, it suffices to show

**Proposition 4.4.** Let \( G \) be a saturable pro-\( p \) group of dimension \( p \). Then \( \gamma_p(G) \subseteq \Phi(G)^p \).

Proof. Consider the finite \( p \)-group \( H := G/\Phi(G)^p \gamma_{p+1}(G) \). By definition this is a PF-group of nilpotency class at most \( p \), and our task is to prove that, in fact, \( \gamma_p(H) = 1 \).

Note that \(|H : H^p| \leq |G : G^p| = p^{\dim(G)} = p^p\). This shows that \( \gamma_p(H) \subseteq H^p \). First suppose that \(|H : \Phi(H)| \geq p^3\). As \( H \) is a PF-group of dimension \( p \), we deduce that \( \gamma_p(H) \subseteq H^p \), and Lemma 2.1 (2) implies

\[
\gamma_p(H) = [\gamma_{p-1}(H), H] \subseteq [H^p, H] = [H, H]^p \subseteq \Phi(H)^p = 1,
\]
as wanted.

Now suppose that \(|H : \Phi(H)| = p^2\), i.e. that \( H \) is 2-generated. For a contradiction, we assume that \( \gamma_p(H) \neq 1 \). Choose \( z \in \gamma_p(H) \setminus \{1\} \). Since \( \gamma_p(H) \subseteq H^p \), Lemma 2.1 (3) shows that \( Y = \{y \in H \setminus y^p = z\} \neq \emptyset \).

Since \( Y \cap \Phi(H) = \emptyset \), every \( y \in Y \) forms part of a generating pair of \( H \). Lemma 2.1 (2) shows that \([H^p, H] \subseteq [H, H]^p \subseteq \Phi(H)^p = 1\), hence

\[
[\Phi(H)^p, H] \subseteq [H^p, H][[H, H], H^p] = \gamma_{p+1}(H) = 1.
\]

This implies that \( \gamma_p(H) = [y_{p-1}, H] \) for every \( y \in Y \).

Let \((H_i)_{i \in \mathbb{N}} \) be a potent filtration of \( H \). To obtain the desired contradiction, we show that \( Y \cap H_i \neq \emptyset \) for all \( i \in \mathbb{N} \). Clearly, the assertion is true for \( i = 1 \). Now consider \( i \geq 2 \). By induction, we find \( y \in Y \cap H_{i-1} \), and hence \( z \in \gamma_p(H) = [y_{p-1}, H] \subseteq [H_{i-1}, H_{i-1}] = H_{i-1}^p \). Hence Lemma 2.1 (3) implies that \( Y \cap H_i \neq \emptyset \).

The Lie correspondence stated as Theorem E connects subgroups and Lie sublattices of dimension at most \( p \). A satisfying explanation for the underlying phenomenon comes from the next result, which we stated as Proposition D in the introduction.

**Corollary 4.5.** Every \( p \)-adic analytic pro-\( p \) group of dimension \( p \) which embeds into a saturable pro-\( p \) group is itself saturable.
Proof. Let $G$ be a subgroup of a saturable pro-$p$ group $S$ and suppose that $\dim(G) \leq p$. By Proposition 3.2 we may assume that $G$ has finite index in $S$ and by induction we can further assume that $|S:G| = p$. Proposition 3.3 then shows that $\gamma_p(S) \subseteq \Phi(S)^p$. As explained after Theorem 2.2 the lower $p$-series

$$S_1 := S, \quad \text{and} \quad S_i := [S_{i-1}, S]S_{p-1}^i \text{ for } i \geq 2$$

is a potent filtration of $S$. Note that $|S:G| = p$ implies $S_2 = \Phi(S) \subseteq G$. Hence $G \supseteq S_2 \supseteq S_3 \supseteq \ldots$ supplies a potent filtration of $G$ and $G$ is saturable. □

The results in the present section, notably Proposition 4.1 and Corollary 4.3, have counterparts for $\mathbb{Z}_p$-Lie lattices; the corresponding proofs are very similar to the ones for pro-$p$ groups – in fact, they are somewhat easier.

**Theorem 4.6.** Every residually-nilpotent $\mathbb{Z}_p$-Lie lattice of dimension less than $p$ is saturable.

Furthermore, if $L$ is a saturable $\mathbb{Z}_p$-Lie lattice and $M$ is a Lie ideal of $L$ of dimension less than $p$, then $M$ is PF-embedded in $L$.

In view of [5, §4], we can now combine the last theorem, Proposition 4.1 and Corollary 4.3 to deduce: there is a correspondence between torsion-free $p$-adic analytic pro-$p$ groups of dimension less than $p$ and residually-nilpotent $\mathbb{Z}_p$-Lie lattices of dimension less than $p$ with the additional properties stated in Theorem B.

For completeness we record the analogue of Example 4.2.

**Example 4.7.** We display a $\mathbb{Z}_p$-Lie lattice of dimension $p$ which is not saturable. Consider the Lie lattice $L = \mathbb{Z}_p x + \mathbb{Z}_p y_1 + \ldots + \mathbb{Z}_p y_{p-1}$ of dimension $p$ where $[y_i, y_j] = 0$ for $1 \leq i, j \leq p - 1$ and

$$[y_i, x] = \begin{cases} 
y_{i+1} & \text{if } i < p - 1, \\
py_1 & \text{if } i = p - 1.
\end{cases}$$

It is easy to check that, indeed, $L$ is not saturable.

Furthermore, it is impossible to associate a pro-$p$ group to the Lie lattice $L$ by direct application of the Hausdorff series. The problem in this specific case is that, although all homogeneous summands which appear in the formula exist, their sum does not necessarily converge (for example, try to compute $\Phi(x, y_1)$). □

Note that the Lie lattice in Example 4.7 and the pro-$p$ group in Example 4.2 are constructed in a very similar way. This observation leads naturally to Questions 1.3 and 1.4 in the introduction.
5. The structure of saturable groups

In the present section we apply the notion of isolators to derive for \( \mathbb{Z}_p \)-Lie lattices analogues of the classical Levi splitting and the decomposition of semisimple Lie algebras into simple summands. Subsequently, we relate these results to saturable pro-\( p \) groups and prove the first part of Proposition 3.1.

The soluble radical of a \( \mathbb{Z}_p \)-Lie lattice \( L \) is the unique maximal soluble Lie ideal of \( L \). There is a close connection between the soluble radical of \( L \) and the soluble radical of the \( \mathbb{Q}_p \)-Lie algebra \( \mathbb{Q}_p \otimes L \).

**Proposition 5.1.** Let \( L \) be a \( \mathbb{Z}_p \)-Lie lattice. Then the soluble radical \( R \) of \( L \) is isolated, i.e. \( R = \text{iso}_L(R) \). Moreover, if \( L \) is saturable, then \( R \) is PF-embedded in \( L \).

**Proof.** Clearly, the intersection of \( L \) with any soluble Lie ideal of the \( \mathbb{Q}_p \)-Lie algebra \( \mathbb{Q}_p \otimes L \) yields a soluble Lie ideal of \( L \). Conversely, if \( I \) is a soluble Lie ideal of \( L \), then \( \mathbb{Q}_p \otimes I \) constitutes a soluble Lie ideal of \( \mathbb{Q}_p \otimes L \). Hence the soluble radical of \( \mathbb{Q}_p \otimes L \) is equal to \( \mathbb{Q}_p \otimes R \), and \( R = L \cap (\mathbb{Q}_p \otimes R) \) is isolated. The final claim follows from Proposition 3.1. \( \square \)

The classical Levi splitting of \( \mathbb{Q}_p \)-Lie algebras (cf. [19, Part I, VI.4]) yields a corresponding splitting of \( \mathbb{Z}_p \)-Lie lattices up to finite index.

**Proposition 5.2.** Let \( L \) be a \( \mathbb{Z}_p \)-Lie lattice with soluble radical \( R \). Then there exists a Lie sublattice \( H \) of \( L \) with \( H \cap R = 0 \) such that the semidirect sum \( H + R \) has finite index in \( L \).

**Proof.** Let \( R \) be the soluble radical of the \( \mathbb{Q}_p \)-Lie algebra \( \mathcal{L} := \mathbb{Q}_p \otimes L \). By Levi’s classical splitting theorem, \( \mathcal{L} \) is the semidirect sum of \( R \) and a suitable subalgebra \( \mathcal{H} \). Write \( H := \mathcal{H} \cap L \) and observe that \( R = \mathcal{R} \cap L \) is the soluble radical of \( L \). Then clearly \( H \cap R = 0 \) and, comparing dimensions, we see that the Lie sublattice \( H + R \) has finite index in \( L \). \( \square \)

**Example 5.3.** We display a powerful \( \mathbb{Z}_p \)-Lie lattice which is not the semidirect sum of a suitable Lie sublattice and the soluble radical. This illustrates that in general one cannot hope for a stronger form of Levi splitting in \( \mathbb{Z}_p \)-Lie lattices corresponding to saturable pro-\( p \) groups.

Let \( k \in \mathbb{N} \) with \( k \geq 2 \), and let \( L \) be the 5-dimensional \( \mathbb{Z}_p \)-Lie sublattice of \( \mathfrak{gl}_3(\mathbb{Z}_p) \) with \( \mathbb{Z}_p \)-basis

\[
\mathbf{x} := p^k \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{y} := p^k \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{h} := p^k \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\mathbf{a} := p^{2k-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{b} := p^{2k-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]
Clearly, \([L, L] \subseteq pL\) and the Lie lattice \(L\) is powerful. The \(\mathbb{Q}_p\)-Lie algebra \(\mathbb{Q}_p \otimes L\) is, of course, the semidirect sum of \(\mathfrak{sl}_2(\mathbb{Q}_p)\) and \(\mathbb{Q}_p \oplus \mathbb{Q}_p\) with respect to the natural action. Accordingly the radical of \(L\) is \(R = \mathbb{Z}_p a + \mathbb{Z}_p b\). Let \(\tilde{h} \equiv h\) and \(\tilde{x} \equiv x\) modulo \(R\). Then \([\tilde{h}, \tilde{x}] \equiv [h, x]\) modulo \(p\mathbb{Z}_p R\). Moreover, \([h, x] - 2p^k \tilde{x} \equiv 0\) modulo \(R\), but \([h, x] - 2p^k \tilde{x} \not\equiv 0\) modulo \(p^k R\). This shows that the Lie sublattice generated by \(\tilde{x}\) and \(\tilde{h}\) intersects \(R\) non-trivially. Hence \(R\) does not admit a complement in \(L\). \(\diamondsuit\)

We translate our results for \(\mathbb{Z}_p\)-Lie lattices into corresponding statements about \(p\)-adic analytic pro-\(p\) groups.

**Proposition 5.4.** Let \(G\) be a \(p\)-adic analytic pro-\(p\) group. Then \(G\) has a unique maximal soluble normal subgroup \(R\). Moreover, if \(G\) is saturable, then \(R\) is isolated in \(G\), i.e. \(R = \text{iso}_G(R)\), and \(R\) is PF-embedded in \(G\).

**Proof.** As \(G\) has finite rank, every subgroup of \(G\) is finitely generated. This implies that \(G\) has a unique maximal soluble normal subgroup. Suppose that \(G\) is saturable, and consider the \(\mathbb{Z}_p\)-Lie lattice \(L := \mathcal{L}(G)\) associated to \(G\). Recall that \(G\) and \(L\) are the same as sets. Let \(R\) be the soluble radical of the saturable Lie lattice \(L\). By Proposition 5.1, \(R\) is isolated and PF-embedded in \(L\). In particular, \(R\) is saturable. According to [5, Theorem 4.5 and Corollary 4.7], the subset \(R\) forms a soluble PF-embedded subgroup of the saturable group \(G\). Clearly, \(R\) is also isolated as a group.

We claim that \(R\) is the maximal soluble normal subgroup of \(G\). Let \(N\) be a soluble normal subgroup of \(G\). We have to show that \(N \subseteq R\). By Proposition 3.2 we may assume without loss of generality that \(N\) is saturable. By [5, §4], the subset \(N\) forms a saturable soluble Lie ideal of the \(\mathbb{Z}_p\)-Lie lattice \(L\). This shows that \(N \subseteq R\). \(\square\)

**Proposition 5.5.** Let \(G\) be a saturable pro-\(p\) group with soluble radical \(R\). Then \(G\) is virtually isomorphic to a semidirect product of \(H\) and \(R\), where \(H\) is the direct product of saturable pro-\(p\) groups whose corresponding \(\mathbb{Q}_p\)-Lie algebras are simple.

**Proof.** Consider the \(\mathbb{Q}_p\)-Lie algebra \(\mathcal{L} := \mathcal{L}(G)\). Let \(\mathcal{R}\) denote the soluble radical of \(\mathcal{L}\). Then there exists a semisimple subalgebra \(\mathcal{H}\) of \(\mathcal{L}\) such that \(\mathcal{L}\) is the semidirect sum of \(\mathcal{H}\) and \(\mathcal{R}\). As \(\mathcal{H}\) is semisimple it decomposes as a direct sum \(\mathcal{H} = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_m\) of simple subalgebras.

Now consider the \(\mathbb{Z}_p\)-Lie lattice \(L := \mathcal{L}(G)\) associated to \(G\). We know that \(R := L \cap \mathcal{R}\), the soluble radical of \(L\), is isolated in \(L\) and hence saturable. For \(i \in \{1, \ldots, m\}\) put \(H_i := L \cap \mathcal{H}_i\). Then each \(H_i\) is isolated in \(L\) and hence saturable. From the characterisation of saturable \(\mathbb{Z}_p\)-Lie lattices in terms of potent filtrations (cf. the remark following Theorem [2,2]) it follows that the direct sum \(H := H_1 \oplus \ldots \oplus H_m\) is saturable. Observe that \(H \cap R = \{0\}\) and that the semidirect sum
$H + R$ is open in $L$. The proposition now follows by passing to the saturable subgroups of $G$ corresponding to $H$ and $R$. □

**Corollary 5.6.** Let $G$ be a saturable insoluble just-infinite pro-$p$ group. Then the associated $\mathbb{Q}_p$-Lie algebra $\mathcal{L}(G)$ is simple, and $G$ is hereditarily just-infinite.

**Proof.** For a contradiction suppose that $\mathcal{L} := \mathcal{L}(G)$ is not simple. Since $G$ has no soluble normal subgroups, the soluble radical of $\mathcal{L}$ is trivial. The semisimple Lie algebra $\mathcal{L}$ decomposes as a direct sum of simple Lie algebras $\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_m$ where $m \geq 2$. Then $H_1 := L \cap \mathcal{H}_1$ is an isolated and hence saturable Lie ideal of $L = L(G)$. It corresponds to a non-trivial normal subgroup of infinite index in $G$. This is the required contradiction. □

### 6. Just-infinite groups

As indicated in the introduction, it is natural to ask whether Proposition 4.1 can be strengthened in a more restricted setting. Of particular interest in this context is the class of $p$-adic analytic just-infinite pro-$p$ groups. These groups, also termed $p$-adically simple groups, are studied to great depth in [10]. In the present section we explore what implications our approach has within this more restricted class of $p$-adic analytic pro-$p$ groups. There are two basic questions: when are these groups torsion-free and when are they saturable? In view of Theorem A we are particularly interested in answering these questions in dimension equal to or slightly larger than $p$.

There are two types of $p$-adic analytic just-infinite pro-$p$ groups, soluble and insoluble. The soluble ones are irreducible $p$-adic space groups and have dimension $(p-1)p^r$, $r \in \mathbb{N}_0$; see [16, Section 10]. Hence the smallest dimension above $p-1$ is $(p-1)p$; in particular, dimension $p$ occurs in this context only for $p = 2$. But more significantly, soluble just-infinite pro-$p$ groups simply fail to be torsion-free.

**Proposition 6.1.** Every soluble just-infinite pro-$p$ group other than $\mathbb{Z}_p$ has torsion.

**Proof.** Let $G$ be a soluble just-infinite pro-$p$ group. Then $G$ is virtually abelian. Let $A$ be a maximal abelian normal subgroup of $G$. Then $A$ is torsion-free and self-centralising in $G$. If $G = A$, then $G \cong \mathbb{Z}_p$ and there is nothing further to show. Now assume that $A$ is properly contained in $G$ and write $H := G/A$. Then $V := \mathbb{Q}_p \otimes A$ is a faithful $H$-module. Choose $g \in G$ such that $h := gA$ is central and of order $p$ in $H$. By Clifford’s theorem $V$ decomposes into a direct sum of irreducible $(h)$-submodules of equal dimensions. Since $(h)$ acts faithfully on $V$, the trivial representation does not occur in this decomposition. But clearly $h$ fixes $g^p \in A$, and therefore $g^p = 1$. □

The insoluble $p$-adic analytic just-infinite pro-$p$ groups can be realised as open subgroups of the groups of $Q_p$-rational points of certain semisimple algebraic groups; cf. [10, Section III]. The components of the corresponding semisimple Lie algebras are pairwise isomorphic and their number is a power of $p$. Every simple Lie algebra $L$ over $Q_p$ is absolutely simple over its centroid $\text{End}_L(L)$. The dimension of an insoluble $p$-adic analytic just-infinite pro-$p$ group is therefore of the form $p^kmd$, where $p^k$ is the number of simple components, $m$ is the dimension of the centroid and $d$ is the dimension of an absolutely simple Lie algebra. From the classification of absolutely simple Lie algebras we conclude that dimension $p$ occurs only for $p = 3$.

We note that conversely, every semisimple $Q_p$-Lie algebra with simple components which are pairwise isomorphic and whose number is a power of $p$ occurs as the Lie algebra associated to some insoluble $p$-adic analytic just-infinite pro-$p$ group; cf. [10, Proposition (III.9)]. We deduce a slightly stronger result of relevance in our context.

**Proposition 6.2.** Let $L$ be a simple $Q_p$-Lie algebra and let $k \in \mathbb{N}_0$. Then there exists a torsion-free insoluble $p$-adic analytic just-infinite pro-$p$ group $G$ such that the $Q_p$-Lie algebra associated to $G$ is isomorphic to the direct sum of $p^k$ copies of $L$.

**Proof.** Let $H$ be a torsion-free open pro-$p$ subgroup of $\text{Aut}(L)$ and consider the standard wreath product $W := H \wr C$ where $C = \langle x \rangle$ is a cyclic group of order $p^k$. According to [10, Proposition (III.9)] it suffices to find a torsion-free open subgroup of $W$ which projects onto $C$ under the natural projection.

Choose a non-trivial element $h \in H$ and an open normal subgroup $N \trianglelefteq H$ such that $h^{p^k} \notin N$. The base group of $W$ is the $p^k$-fold product $H \times \ldots \times H$ and accordingly we write its elements as $p^k$-tuples in $H$. Let $G$ be the group generated by $x(h, \ldots, h)$ and the open normal subgroup $N \times \ldots \times N \trianglelefteq W$. Clearly, $G$ is an open subgroup of $W$ projecting onto $C$. It remains to prove that $G$ is torsion-free. As $H$ is torsion-free this follows from the observation that

$$(x(h, \ldots, h))^{p^k} = (h^{p^k}, \ldots, h^{p^k}) \in (H \times \ldots \times H) \setminus (N \times \ldots \times N).$$

□

In [12] it was shown by example that torsion-free insoluble $p$-adic analytic maximal just-infinite pro-$p$ groups need not be saturable. Here we provide further examples: Corollary 5.6 shows that the groups constructed in Proposition 6.2 for $k \geq 1$ are not saturable. As indicated in the introduction, torsion-free $p$-adic analytic just-infinite pro-$p$ groups whose $Q_p$-Lie algebras are isomorphic to a sum of $p$ copies of $\mathfrak{sl}_2(Q_p)$ yield examples of $3p$-dimensional groups which are not saturable.
Moreover, combining Corollary 5.6 and Proposition 6.2 we obtain Proposition F. Another interesting problem concerning saturable just-infinite pro-$p$ groups is recorded in Question 1.2.

7. Pro-$p$ groups of dimension at most 3

Taking advantage of Lazard’s correspondence, finite $p$-groups of reasonably small order can be classified or at least enumerated by solving the respective problem for finite nilpotent Lie rings of $p$-power order; e.g. see [17]. The point is that, because of their linear structure, Lie rings are generally more tractable than groups. Being based on the exponential and logarithm maps, Lazard’s correspondence is restricted to groups and Lie rings of nilpotency class less than $p$.

As an illustration of the method, let us consider non-abelian groups and nilpotent Lie rings of order $p^3$. One easily verifies that there are precisely two isomorphism types of nilpotent Lie rings of order $p^3$, represented by

$L_1 = \langle x, y \rangle_+ \quad \text{where} \quad px = p^2 y = 0 \quad \text{and} \quad [y, x] = py$,

$L_2 = \langle x, y, z \rangle_+ \quad \text{where} \quad px = py = pz = 0 \quad \text{and} \quad [x, y] = z \quad \text{is central.}$

Applying Lazard’s correspondence, for $p \geq 3$ we deduce that there are precisely two non-abelian $p$-groups of order $p^3$, namely

$G_1 = \langle x, y \mid x^p = y^{p^2} = 1 \text{ and } [x, y] = y^p \rangle$,

$G_2 = \langle x, y \mid x^p = y^p = [x, y]^p = 1 \text{ and } [x, y] \quad \text{central} \rangle$.

In the present section we show how a similar approach, based on Theorem [B] leads to a method for classifying torsion-free $p$-adic analytic pro-$p$ groups of dimension less than $p$. We work out the details only in the soluble case up to dimension 3. It goes without saying that, while illustrating the point, our results in this direction are much less sophisticated than the work in [17], say. It can be expected that for groups of dimension 4 and higher the classification problem becomes rather difficult.

Clearly, there is only one torsion-free $p$-adic analytic pro-$p$ group of dimension 1, namely the infinite procyclic group $\langle x \rangle \cong \mathbb{Z}_p$. More generally there is precisely one abelian torsion-free $p$-adic analytic pro-$p$ group in every given dimension. Our task is to construct the non-abelian groups in dimensions 2 and 3 from the corresponding Lie lattices.

7.1. Groups of dimension 2. Consider a non-abelian $\mathbb{Z}_p$-Lie lattice $L$ of dimension 2. Tensoring with $\mathbb{Q}_p$, we obtain a non-abelian Lie algebra $\mathbb{Q}_p \otimes L$ of dimension 2. This Lie algebra can be spanned by elements $x, y$ with the Lie product given by $[y, x] = y$.

From this we can describe all non-abelian $\mathbb{Z}_p$-Lie lattices of dimension 2. Every Lie lattice $L$ of this type can be spanned by elements $x, y$
whose Lie product is given by \([y, x] = p^s y\) with \(s \in \mathbb{N}_0\). Moreover, the invariant \(s = s(L)\) distinguishes the isomorphism class of \(L\), as it determines \(|C_L([L, L]) : [L, L]| = p^s\), the index of the commutator Lie sublattice in its centraliser. Note that \(L\) is residually-nilpotent if and only if \(s > 0\). Theorem \([\text{II}]\) allows us to translate our findings to groups.

**Proposition 7.1.** For \(p \geq 3\) there exists precisely one infinite family of 2-dimensional non-abelian torsion-free \(p\)-adic analytic pro-\(p\) groups, parameterised by \(s \in \mathbb{N}\), namely

\[
G(s) = \langle x, y \mid [x, y] = y^{p^s}\rangle.
\]

We remark that these groups are not only saturable, but even uniformly powerful; cf. \([\text{II}]\) Section 7.3]. Furthermore we have \(G(s + 1) \cong G(s)^p\) for all \(s \in \mathbb{N}\), and the 2-dimensional abelian torsion-free \(p\)-adic analytic pro-\(p\) group can be regarded as a limit of the family \(G(s)\) as \(s\) tends to infinity.

For completeness we also consider the case \(p = 2\), where the Lie correspondence breaks down. It is convenient to define \(2^\infty := 0\) in \(\mathbb{Z}_2\).

**Proposition 7.2.** There exist precisely two infinite families of 2-dimensional torsion-free 2-adic analytic pro-2 groups, each parameterised by \(s \in \mathbb{N} \cup \{\infty\}\) with \(s \geq 2\), namely

\[
G_+(s) = \langle x, y \mid [x, y] = y^{2^s}\rangle \quad \text{and} \quad G_-(s) = \langle x, y \mid [x, y] = y^{-2-2^s}\rangle.
\]

**Proof.** The cyclic subgroups of \(\text{Aut}(\mathbb{Z}_2) \cong \mathbb{Z}_2^\ast = \langle -1 \rangle \times \langle 1 + 4 \rangle\) are precisely the groups \(\langle 1 + 2^s \rangle\) and \(\langle -1 - 2^s \rangle\), where \(s \in \mathbb{N} \cup \{\infty\}\) with \(s \geq 2\). Hence it suffices to show that every torsion-free 2-adic analytic pro-2 group is isomorphic to a semidirect product of \(\mathbb{Z}_2\) by \(\mathbb{Z}_2\).

Let \(G\) be a torsion-free 2-adic analytic pro-2 group. Then \(G\) contains a saturable open subgroup \(H\). By consideration of the associated \(\mathbb{Z}_2\)-Lie lattice one sees that \(H\) is of the form \(\mathbb{Z}_2 \times \mathbb{Z}_2\), hence isomorphic to one of the groups \(G_+(s)\), \(G_-(s)\) for suitable \(s \in \mathbb{N} \cup \{\infty\}\) with \(s \geq 2\).

Arguing by induction on \(|G : H|\), we may assume that \(|G : H| = 2\).

**Case 1:** \(H\) is abelian. Write \(H = \mathbb{Z}_2x + \mathbb{Z}_2y \cong \mathbb{Z}_2 \times \mathbb{Z}_2\) and \(G = \langle g \rangle H\). If \(g\) acts trivially on \(H\), then \(G\) is abelian and there is nothing further to prove. Hence suppose that \(g\) acts on \(H\) as an involution. The finite 2-subgroups of \(\text{GL}_2(\mathbb{Z}_2)\) are known; cf. \([\text{I6}]\) Section 10]. Carrying out if necessary a change of basis, we may assume that the action of \(g\) on \(H\) with respect to the basis \((x, y)\) is given by one of the following pairs of equations

\[
\begin{align*}
(a) \quad x^g &= x^{-1} \quad \text{and} \quad y^g = y, \\
(b) \quad x^g &= x^{-1} \quad \text{and} \quad y^g = y^{-1}, \\
(c) \quad x^g &= y \quad \text{and} \quad y^g = x.
\end{align*}
\]

Suppose that \(g\) acts on \(H\) according to the equations (a). Then \(g^2 \in Z(G) = \langle y \rangle\), and since \(G\) is torsion-free, \(g^2\) must be a generator of the
procyclic group $\langle y \rangle$. This shows that $G = \langle g \rangle \ltimes \langle x \rangle$ is isomorphic to a semidirect product of $\mathbb{Z}_2$ by $\mathbb{Z}_2$.

The second and third type of action do not actually arise. If the action of $g$ on $H$ was given by the equations (b), then $g^2 \in Z(G) = 1$ and $G$ would not be torsion-free.

Now assume for a contradiction that $g$ acts on $H$ according to the equations (c). Then $g^2 \in Z(G) = \langle xy \rangle$, and since $G$ is torsion-free, $g^2$ must be a generator of the procyclic group $\langle xy \rangle$. Without loss of generality, we may assume that $g^2 = xy$. Then $gy^{-1} \neq 1$, but $(gy^{-1})^2 = g^2x^{-1}y^{-1} = 1$, in contradiction to $G$ being torsion-free.

**Case 2:** $H$ is non-abelian. In this case $H = \langle x \rangle \ltimes \langle y \rangle$ is a split extension, where

\begin{equation}
    x^{-1}yx = y^{\pm (1+2^s)} \neq y \quad \text{for suitable } s \in \mathbb{N} \cup \{\infty\} \text{ with } s \geq 2.
\end{equation}

We observe that the 1-dimensional subgroup $N := \langle y \rangle = \text{iso}_H([H, H])$ is characteristic in $H$ and hence normal in $G$. The quotient $G/N$ is, of course, 1-dimensional. In particular, if $G/N$ is torsion-free, then the exact sequence $1 \to N \to G \to G/N \cong \mathbb{Z}_2 \to 1$ splits, and $G$ is isomorphic to a semidirect product of $\mathbb{Z}_2$ and $\mathbb{Z}_2$. Similarly, if $G/N$ is abelian, then writing $M := \text{iso}_G(N) \cong \mathbb{Z}_2$, the exact sequence $1 \to M \to G \to G/M \cong \mathbb{Z}_2 \to 1$ splits, and $G$ is isomorphic to a semidirect product of $\mathbb{Z}_2$ and $\mathbb{Z}_2$.

Hence we assume, for a contradiction, that $G/N$ is non-abelian and has torsion. Since $H/N$ is torsion-free, we find $g \in G$ with $G = \langle g \rangle H$ and $g^2 \in N$. As $G$ is torsion-free, the 1-dimensional group $\langle g \rangle N$ is procyclic and $g^2$ must be a generator of $N = \langle y \rangle$. Without loss of generality we may assume that $g^2 = y$. Being non-abelian, $G/N = \langle gN \rangle \ltimes \langle xN \rangle$ is isomorphic to the infinite pro-2 dihedral group. Hence we find $k \in \mathbb{Z}$ such that $g^{-1}xg = x^{-1}y^k = x^{-1}g^2k$. Writing $w := g^{-1}x \neq 1$, this gives $w^2 = g^{2(k-1)} = y^{k-1}$.

By (7.1) this implies

\[ w^2 = w^{-1}g^{2(k-1)}w = x^{-1}y^{k-1}x = y^{\pm (1+2^s)(k-1)} = (w^2)^{\pm (1+2^s)} \]

where $\pm (1+2^s) \neq 1$. Hence $w$ has finite order, in contradiction to $G$ being torsion-free. \qed

7.2. **Soluble groups of dimension 3.** Again we aim for a classification of groups by first describing the relevant Lie lattices. Since the correspondence between saturable $\mathbb{Z}_p$-Lie lattices and saturable pro-$p$ groups breaks down for $p \leq 3$ in dimension 3, we avoid from the beginning the extra work required to deal with the prime 2: throughout the present subsection, let $p$ denote an odd prime.

Consider a soluble $\mathbb{Z}_p$-Lie lattice $L$ of dimension 3. Tensoring with $\mathbb{Q}_p$, we obtain a soluble Lie algebra $\mathcal{L} := \mathbb{Q}_p \otimes L$ of dimension 3. According to [9] Chapter 1 §4, the Lie algebra $\mathcal{L}$ has an abelian ideal of dimension 2. Consequently $\mathcal{L}$ decomposes as the semidirect sum of
a 1-dimensional subalgebra and such a 2-dimensional abelian ideal. If $[\mathcal{L}, \mathcal{L}]$ is central, then $\mathcal{L}$ is nilpotent and consequently either abelian or isomorphic to the so-called Heisenberg Lie algebra. If $[\mathcal{L}, \mathcal{L}]$ is not central, then $\mathcal{L}$ fails to be nilpotent and the centraliser $C_L([\mathcal{L}, \mathcal{L}])$ is the unique 2-dimensional abelian ideal of $\mathcal{L}$.

Our discussion shows that, in any case, the original Lie lattice $L$ has an abelian ideal $I$ of dimension 2 such that $L/I$ is a Lie lattice of dimension 1. Write $I = \mathbb{Z}_p y_1 + \mathbb{Z}_p y_2$, and choose an additive complement $K = \mathbb{Z}_p x$ to $I$ in $L$. Then $L = \mathbb{Z}_p x + \mathbb{Z}_p y_1 + \mathbb{Z}_p y_2$ decomposes as a semidirect sum $K \rtimes I$; the Lie bracket is determined by the equations

\[
[y_1, y_2] = 0, \quad [y_1, x] = a_{11} y_1 + a_{12} y_2, \quad [y_2, x] = a_{21} y_1 + a_{22} y_2
\]

for suitable coefficients $a_{ij} \in \mathbb{Z}_p$. Define

\[
A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{Z}_p).
\]

The Lie lattice $L$ is abelian if and only if $A$ is the null matrix. Similarly, it is nilpotent if and only if $A$ is nilpotent. These cases are fairly easy to deal with. On the other hand, if $L$ is not nilpotent, then $I = C_L([L, L])$ is in fact uniquely determined. The matrix $A$, however, generally depends on the particular basis $(x, y_1, y_2)$. Changing this basis corresponds to conjugating and scaling the matrix $A$. This introduces an equivalence relation on the set $\mathfrak{gl}_2(\mathbb{Z}_p)$: the multiplicative-similarity class of $A$,

\[
A_L := \{ uB^{-1}AB \mid u \in \mathbb{Z}_p^* \text{ and } B \in \text{GL}_2(\mathbb{Z}_p) \},
\]

does not depend on the chosen basis $(x, y_1, y_2)$. Moreover, by our observations the invariant $A_L$ characterises and parameterises the isomorphism class of the Lie lattice $L$.

**Proposition 7.3** (Representatives for multiplicative-similarity classes). Every nilpotent matrix $A \in \mathfrak{gl}_2(\mathbb{Z}_p)$ is multiplicatively-similar to precisely one of the matrices

\[
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad p^s \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad s \in \mathbb{N}_0.
\]

Let $p \in \mathbb{Z}_p^*$, not a square modulo $p$. Then every non-nilpotent matrix $A \in \mathfrak{gl}_2(\mathbb{Z}_p)$ is multiplicatively-similar to precisely one matrix of the form $p^s A_0$, where $s \in \mathbb{N}_0$ and $A_0$ is one of the following core matrices:

1. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;
2. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + p^r \begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix}$, where $r \in \mathbb{N}$ and $d \in \mathbb{Z}_p$;
3. $\begin{pmatrix} 0 & d \\ 1 & p^r \end{pmatrix}$, where $r \in \mathbb{N}_0$ and $d \in \mathbb{Z}_p$;
\[
(4) \begin{pmatrix} 0 & p^r \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \rho p^r \\ 1 & 0 \end{pmatrix}, \text{ where } r \in \mathbb{N}_0.
\]

**Proof.** Recall that we assume \( p > 2 \) throughout. From the analysis in \[1\], we deduce that the \( \text{GL}_2(\mathbb{Z}_p) \)-conjugacy class of a matrix \( A \in \mathfrak{gl}_2(\mathbb{Z}_p) \) which is not scalar modulo \( p \) is uniquely determined by its characteristic polynomial, i.e., by its trace and determinant.

Clearly, the null matrix forms a multiplicative-similarity class by itself. Thus it suffices to consider a non-zero matrix \( A \in \mathfrak{gl}_2(\mathbb{Z}_p) \) and its multiplicative-similarity class \( A \). Write \( A = p^s A_0 \) where \( s \in \mathbb{N}_0 \) is chosen so that \( A_0 \not\equiv 0 \) modulo \( p \). Note that \( s \) is in fact an invariant of the class \( A \), independent of the particular representative \( A \).

It is enough to concentrate on \( A_0 \) and its multiplicative-similarity class \( A_0 \). We distinguish four cases and show that \( A_0 \) is multiplicatively-similar to one of the listed core matrices. At the same time it will become clear that the listed matrices are pairwise not multiplicatively-similar.

**Case 1:** \( A_0 \) is scalar. Multiplying by a suitable unit \( u \in \mathbb{Z}_p^* \) (uniquely determined), we see that \( A_0 \) is multiplicatively-similar to \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

**Case 2:** \( A_0 \) is scalar modulo \( p \), but not scalar. Let \( r \in \mathbb{N} \) such that \( A_0 \) is scalar modulo \( p^r \), but not scalar modulo \( p^{r+1} \). Note that \( r \) is in fact an invariant of the class \( A_0 \), independent of the particular representative \( A_0 \). Multiplying by a suitable unit \( u_1 \in \mathbb{Z}_p^* \) (uniquely determined modulo \( p^r \)) we see that \( A_0 \) is multiplicatively-similar to \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + p^r A_1 \). Multiplying by a suitable one-unit \( u_2 \in 1 + p^r \mathbb{Z}_p \) (uniquely determined), we see that \( A_0 \) is multiplicatively-similar to \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + p^r A_2 \) where \( A_2 \) has trace \( 0 \). Define \( d := -\det(A_2) \in \mathbb{Z}_p \). Conjugating by a suitable element of \( \text{GL}_2(\mathbb{Z}_p) \), we see that \( A_0 \) is multiplicatively-similar to \( \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \).

**Case 3:** \( A_0 \) is not scalar modulo \( p \) and has non-zero trace. Scaling by a unit \( u \in \mathbb{Z}_p^* \) (uniquely determined), we find that \( A_1 := u A_0 \) has trace \( p^r \) for suitable \( r \in \mathbb{N}_0 \). Note that \( r \) is in fact an invariant of the class \( A_0 \), independent of the particular representative \( A_0 \). Put \( d := -\det(A_2) \). Conjugating by a suitable element of \( \text{GL}_2(\mathbb{Z}_p) \), we see that \( A_0 \) is multiplicatively-similar to \( \begin{pmatrix} 0 & d \\ 1 & p^r \end{pmatrix} \).

**Case 4:** \( A_0 \) is not scalar modulo \( p \) and has zero trace. Scaling by a unit \( u \in \mathbb{Z}_p^* \), we find that \( A_1 := u A_0 \) retains trace \( 0 \) and satisfies \( \det(A_1) = 0 \) or \( \det(A_1) \in \{-p^r, -\rho p^r\} \) for suitable \( r \in \mathbb{N}_0 \). Note that in the latter case \( r \) is an invariant of the class \( A_0 \), independent of the particular representative \( A_0 \). Conjugating by a suitable element of \( \text{GL}_2(\mathbb{Z}_p) \), we see that \( A_0 \) is multiplicatively-similar to one of the matrices \( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \), \( \begin{pmatrix} 0 & \rho p^r \\ 1 & 0 \end{pmatrix} \), \( \begin{pmatrix} 0 & \rho p^r \\ 1 & 0 \end{pmatrix} \).

Let \( A \) be one of the matrices listed in Proposition [7.3] and let \( L \) be the \( \mathbb{Z}_p \)-Lie lattice associated to the multiplicative-similarity class \( A \) of \( A \). Then \( L \) is residually-nilpotent if and only if \( A^2 \equiv_p 0 \). Under
the hypothesis $p > 3$ Theorem 13 allows us to translate our findings to

For this it is convenient to embed $L$ as a Lie sublattice into $\mathfrak{gl}_3(\mathbb{Z}_p)$ by mapping

$$x \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad y_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad y_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

If $L$ is residually-nilpotent, the corresponding group $G$ can be obtained by applying the ordinary exponential map: $G = H \ltimes N$ is the semidirect product of $H \cong \mathbb{Z}_p$ and a free abelian pro-$p$ group $N \cong \mathbb{Z}_p \times \mathbb{Z}_p$. The action of $H$ on $N$ is given by $\exp(A) = \sum_{n=0}^{\infty} A^n/n!$.

Alternatively, we can use Proposition 7.3 directly to write down finite presentations for representatives $G = H \ltimes N$ of the isomorphism classes of 3-dimensional soluble torsion-free pro-$p$ groups. Namely, the action of $H$ on $N$ can be given by $1 + A$ rather than $\exp(A)$. Here $A$ runs again through those matrices listed in Proposition 7.3 which are nilpotent modulo $p$.

**Theorem 7.4.** For $p > 3$ the following constitutes a complete and irredundant list of 3-dimensional torsion-free pro-$p$ analytic pro-$p$ groups (up to isomorphism):

1. the abelian group, isomorphic to $\mathbb{Z}_p^3$,
   $$G_0(\infty) = \langle x_1, x_2, x_3 \mid [x_1, x_2] = [x_1, x_3] = [x_2, x_3] = 1 \rangle.$$

2. the two-step nilpotent groups, parameterised by $s \in \mathbb{N}_0$,
   $$G_0(s) = \langle x, y, z \mid [x, y] = z^{p^s}, [x, z] = [y, z] = 1 \rangle;$$
   these are open subgroups of the Heisenberg group, viz. $G_0(0)$.

3. a family of non-nilpotent groups, parameterised by $s \in \mathbb{N}$,
   $$G_1(s) = \langle x, y_1, y_2 \mid [y_1, y_2] = 1, [y_1, x] = y_1^{p^s}, [y_2, x] = y_2^{p^s} \rangle.$$

4. a family of non-nilpotent groups, parameterised by $s, r \in \mathbb{N}$ and $d \in \mathbb{Z}_p$,
   $$G_2(s, r, d) = \langle x, y_1, y_2 \mid [y_1, y_2] = 1, [y_1, x] = y_1^{p^s} y_2^{p^s+r+d}, [y_2, x] = y_1^{p^s+r} y_2^{p^s} \rangle.$$

5. a family of non-nilpotent groups, parameterised by $s, r \in \mathbb{N}_0$ and $d \in \mathbb{Z}_p$ such that (a) $s \geq 1$ or (b) $r \geq 1$ and $d \in p\mathbb{Z}_p$,
   $$G_3(s, r, d) = \langle x, y_1, y_2 \mid [y_1, y_2] = 1, [y_1, x] = y_2^{p^sd}, [y_2, x] = y_1^{p^s} y_2^{p^s+r} \rangle.$$
(6) two families of non-nilpotent groups, parameterised by $s, r \in \mathbb{N}_0$ with $s + r \geq 1$,

\[
G_4(s, r) = \langle x, y_1, y_2 \mid [y_1, y_2] = 1, [y_1, x] = y_2^{p^{s+r}}, [y_2, x] = y_1^{p^{s}} \rangle,
\]

\[
G_5(s, r) = \langle x, y_1, y_2 \mid [y_1, y_2] = 1, [y_1, x] = y_2^{p^{s+r}}, [y_2, x] = y_1^{p^{r}} \rangle.
\]

We remark that among these groups there are several which are not uniformly powerful. These complement the insoluble examples of saturable but not uniformly powerful pro-$p$ groups given in [12].

7.3. Insoluble groups of dimension 3. Tensoring an insoluble $\mathbb{Z}_p$-Lie lattice $L$ with $\mathbb{Q}_p$, we obtain an insoluble Lie algebra $\mathcal{L} := \mathbb{Q}_p \otimes L$. In dimension 3 the Lie algebra $\mathcal{L}$ is necessarily simple of type $A_1$ and only two forms occur. They are represented by $\mathfrak{sl}_2(\mathbb{Q}_p)$ and $\mathfrak{sl}_1(\mathbb{D}_p)$, where $\mathbb{D}_p$ denotes a central division algebra of index 2 over $\mathbb{Q}_p$.

Suppose that $p > 3$, so that Theorems A and B become applicable. Any saturable 3-dimensional insoluble pro-$p$ group $G$ acts faithfully on itself, regarded as a Lie lattice $L$. This adjoint action provides an embedding of $G$ as an open subgroup into $\text{Aut}(\mathcal{L})$ where $\mathcal{L} = \mathbb{Q}_p \otimes L$ as before.

It is known that the open maximal pro-$p$ subgroups of $\text{Aut}(\mathcal{L})$ form a unique conjugacy class; see [10, Lemma (III.16)]. Any open maximal pro-$p$ subgroup of $\text{Aut}(\mathfrak{sl}_2(\mathbb{Q}_p))$ is isomorphic to the group

\[
\text{SL}_2^\rho(\mathbb{Z}_p) := \{ g \in \text{SL}_2(\mathbb{Z}_p) \mid g \text{ upper uni-triangular modulo } p \},
\]

a Sylow pro-$p$ subgroup of $\text{SL}_2(\mathbb{Z}_p)$; cf. [10] Lemma (XI.4)]. Denote the maximal $\mathbb{Z}_p$-order of $\mathbb{D}_p$ by $\Delta_p$. Then any open maximal pro-$p$ subgroup of $\text{Aut}(\mathfrak{sl}_1(\mathbb{D}_p))$ is isomorphic to $\text{SL}_1^\rho(\Delta_p)$, a Sylow pro-$p$ subgroup of $\text{SL}_1(\mathbb{D}_p)$; see [11] Proof of Proposition 10.4]. For $p > 3$, both of these groups are torsion-free and hence saturable; cf. [12] Theorem 1.3].

In view of Theorem A the 3-dimensional insoluble torsion-free $p$-adic analytic pro-$p$ groups are precisely the open subgroups of $\text{SL}_2^\rho(\mathbb{Z}_p)$ and $\text{SL}_1^\rho(\Delta_p)$. By Theorem B they correspond to open Lie sublattices of the corresponding $\mathbb{Z}_p$-Lie lattices

\[
\mathfrak{sl}_2(\mathbb{Z}_p) \cong \langle x, y, h \mid [x, y] = h, [x, h] = -2px, [y, h] = 2py \rangle,
\]

\[
\mathfrak{sl}_1(\Delta_p) \cong \langle x, y, z \mid [x, y] = pz, [x, z] = p\rho y, [y, z] = -x \rangle.
\]

Here $\rho \in \mathbb{Z}_p^\ast$ is not a square modulo $p$. We refrain from giving an explicit classification up to isomorphism, but the following fact is perhaps worth pointing out (cf. [10] Lemma (III.11)):

an insoluble just-infinite $p$-adic analytic pro-$p$ group is never isomorphic to one of its proper subgroups. An interesting treatment of pro-$p$ subgroups of $\text{SL}_2(\mathbb{Z}_p)$, valid for $p > 2$, can be found in [18].
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