Abstract. In order to apply the shell model to most nuclei, an effective (“residual”) interaction between nucleons should be added to the single nucleon Hamiltonian. It cannot be the bare interaction which would introduce strong short range correlations into the shell model wave functions of nucleons moving independently in a potential well. For many years, many authors have been developing methods for calculating from the bare interaction, an effective interaction which may be used in the shell model. Only recently, the results seem to agree with experiment. During these years, successful shell model calculations have been carried out by using effective interactions determined consistently from measured energies of nuclei. Some old and some new results exhibit the success of this approach. In principle, shell model wave functions may be determined by a renormalized Hamiltonian which contains the effective interactions rather than the bare ones. Shell model wave functions may be used to calculate other observables if matrix elements are taken of renormalized operators. Such a program could succeed irrespective of the relation between shell model wave functions and the real ones. This would certainly be the case for ab initio calculations where a central potential well is not assumed and derivation of nuclear structure is attempted by using many body theory of nucleons interacting by the bare interaction. Still, there are observables which have been successfully calculated by shell model wave functions from real, un-renormalized operators. Several examples are briefly described below. This fact may be taken as some evidence that the “wounds” inflicted on shell model wave functions by the short range correlations occupy only a small volume and that there is a considerable overlap between shell model wave functions and the real ones. The calculated overlap between shell model and real wave functions may be obtained from derivations of the effective interaction. It is more difficult to see how the shell model may emerge as a good approximation from ab initio calculations. The question is whether the simplicity of the shell model as a good approximation will emerge from those complex calculations.

1. Introduction

It is a great honor and pleasure to open the 11th Spring Seminar on Nuclear Physics. It is intended to pay tribute to our friend and colleague Aldo Covello who established a series of superb conferences which have been highly interesting, informative and stimulating. We wish him and Renata, many years of good health, fruitful activity and much happiness. The present conference was organized by his younger colleagues and it followed very successfully the great tradition which he established. We are very grateful to them for their efforts.

The subject of the conference this year is the Shell Model and Nuclear Structure. There are several approaches to the shell model, like the situation of the proverbial elephant and the blind men. One approach is the simple shell model with (two-body) effective interactions determined from...
experiment. Other efforts aim to derive the effective interaction to be used in the shell model, from the interaction between free nucleons. In a more ambitious approach no shell structure is adopted. It begins \textit{ab initio}, from the “bare” interaction between free nucleons and attempts to obtain good approximations to the real wave functions of nuclei. In the following, a brief description of these approaches is given. In the shell model, nucleons are taken to move independently in a central potential well. Magic nuclei contain closed shells of protons and neutrons. There is only one shell model wave function of each of these nuclei and it is well defined. This is the case also for nuclei in which there is one (valence) nucleon outside closed shells or one nucleon missing from them (one hole state). If there are several valence nucleons or holes, there are several states of the system and they are all degenerate in the central potential well. If an interaction is included in the shell model Hamiltonian, the eigenvalues are no longer degenerate and the various eigenstates have definite wave functions with well defined values of total spin $J$ and projection $M$.

In the early days, the rather mild potentials used in that time to describe the interaction between free nucleons, were used in the shell model. The agreement between calculated energies and experimental ones was at best only qualitative. Later, it was established by experiments, that the bare interaction is too strong and singular at short distances and hence, cannot be used in the shell model. Thus, it turned out that shell model wave functions may be used only with a strongly renormalized interaction. Matrix elements of the bare interaction between the real wave functions could be hopefully replaced by matrix elements of the renormalized, or effective interaction between shell model wave functions. More than 50 years ago, Brueckner introduced the G-matrix and he was followed by many authors who developed methods of many-body theory to be applied to actual (finite) nuclei. Some of those authors who made important contributions to this approach are among the participants in this conference. This is a very difficult problem and only recently some reliable results were obtained, like those of Covello et al. [1]

It should be realized that in addition to the computational difficulties, there is an additional problem which is more serious in the \textit{ab initio} approach. There is now extensive and detailed experimental information about the interaction between free nucleons. Most of it is obtained from scattering experiments which yield information about “on-shell” features of the interaction. There are many theoretical models which reproduce rather well the experimental data. There is no theory, however, from which this interaction could be derived. The QCD theory which gives excellent results for high energy phenomena is not very helpful in the low energy, non perturbative domain. This subject will be discussed in this conference.

There is a great promise but there are very great difficulties in the \textit{ab initio} approach. Its success depends on the knowledge of the bare interaction and the accuracy of the necessary approximations of the nuclear many-body problem. Reliable information on the (real) nuclear wave functions would enable the calculation of nuclear states and their energies, also of nuclei which are inaccessible experimentally. It will also enable the calculation of various moments and transitions, electromagnetic and weak ones including neutrino-less beta decay. There will be a talk about this approach by one of its originators. An intriguing question about this (no-core shell model) approach is whether the simplicity of the shell model will emerge from the very complicated calculations. This simplicity is demonstrated in the following by a few examples, some old and some new ones.

2. The simple shell model

Without reliable theoretical derivation, matrix elements of the effective interaction were extracted from energies of nuclear states in a consistent way. Restriction to two-body interactions leads to matrix elements between $n$-nucleon states which are linear combinations of two-nucleon matrix elements. Thus, energies of nuclear states can be expressed in terms of a smaller set of two-nucleon matrix elements. If there are sufficient measured energies, these matrix elements may be determined. If nuclear energies, calculated with these matrix elements are in good agreement with the measured ones, this procedure is consistent. The two-nucleon matrix elements thus determined, may be used in calculating and predicting energies which have not yet been measured.
This method has been extensively used in many cases. In order to obtain more detailed and accurate information, more configurations have been included. In recent years, many millions of nuclear states are included in large scale shell model calculations. They need very many two-nucleon matrix elements and single nucleon energies. There are not enough measured energies to determine all of them but the important ones are determined from experiment [2]. In the following, however, only simple calculations will be presented, demonstrating how the very simple shell model may be used in many cases.

Using simple shell model configurations, only a few matrix elements could be determined. Still, some general features of the effective interaction were extracted from simple cases. As has been well known, the $T = 1$ interaction is attractive and strong in $f^*$ states. In other states, however, it is weak on the average and its average is repulsive in $f^*$-configurations. In $j^n$-configurations it leads to eigenstates with good seniority [3]. The average $T = 0$ interaction which acts between protons and neutrons, is strong and attractive. It breaks seniority in a major way leading to rotational spectra in many nuclei with both valence protons and neutrons.

A direct consequence of these features is that the potential well of the shell model is created by the attractive proton-neutron interaction. It determines its depth and its shape. Hence, energies of proton orbits in the well (single nucleon energies) are determined by the occupation numbers of neutrons and $\nu$. A clear demonstration of this conclusion is offered by the prediction of the $^{11}$Be ground state spin [4]. Some examples presented below, demonstrate these features.

3. Some shell model applications to nuclei

Some properties of the $T = 1$ interaction may be seen in $j^n$-configurations of identical nucleons. Ground states have maximum $J = 0$ pairing and hence, lowest seniorities, $\nu = 0$ ($J = 0$) for even $n$ values and $\nu = 1$ ($J = j$) for odd ones. There are many two-body interactions which are diagonal in seniority and all of them yield a simple expression for ground state energies. Binding energies are given by the following expression.

$$BE(j^n) = BE(n = 0) + nC + \alpha n(n - 1)/2 + \beta(n/2) .$$ (1)

Equation (1) includes the binding energy of the nucleus with closed shells, without $j$-nucleons, and $n$ single nucleon energies. The contribution of the mutual interaction of the $j$-nucleons is given by the quadratic term in $n$ and the pairing term. The coefficients of these terms are linear combinations of two-nucleon matrix elements

$$\alpha = \frac{[2j+2]\nu_2-V_0}{2j+1}, \quad \beta = (2j+2)V_0-V_2,\frac{2j+1}{2j+2}$$ (2)

where

$$V_0 = \langle j^2 \nu = 0, J = 0 | V | j^2 \nu = 0, J = 0 \rangle$$

$$\bar{V}_2 = \Sigma (2J+1)(j^2 \nu = 2, J | V | j^2 \nu = 2, J) / \Sigma (2J+1).$$ (3)

The summations in (3) are on all $j^2$ states with even values of $J > 0$. The expression (1) of binding energies yields a very simple result for single nucleon separation energies. They should lie on two straight and parallel lines, one for odd and the other for even numbers $n$. In figure 1, measured single neutron separation energies of calcium isotopes are plotted as a function of $n$. In these nuclei, valence neutrons are taken to occupy the $1f_{7/2}$ orbit.
The agreement between measured energies and the expression (1) is rather good. The coefficient of the pairing term is large and attractive whereas the coefficient of the quadratic term is small and repulsive. From figure 1 follows that the magic nucleus $^{48}\text{Ca}$ is not “tightly bound”. In fact, it is less tightly bound than its preceding even-even neighbours. Magic nuclei have extra stability due to nuclei following them being less bound. This fact has been known for many years but references are still published to “an enhanced binding” of magic nuclei.

An important question may be raised in view of figure 1. The experimental data agree rather well with the formula which is valid for two-nucleon effective interaction between the valence nucleons. Where are the three-body forces? Such forces are used by many authors who start from the bare interaction. Also, if there are no such forces in the bare interaction, they may well be introduced into the effective interaction by the renormalization. There is no clear evidence for three-body interactions between valence nucleons also in other cases, including the large scale shell model calculations. Still, three-body interactions could contribute implicitly. Single nucleon energies, usually taken from experiment, could include contributions from such interaction between two core nucleons with each valence nucleon. The two-nucleon matrix elements extracted from experiment could include contributions from three-nucleon interactions between a core nucleon and two valence ones. Three-nucleon terms, proportional to $n^3$, as well as $n$ and $n^2$ terms, could simply arise from polarization of the core by the valence nucleons. The first term in (1), $\text{BE}(n = 0)$ may not be constant as the valence orbit is being filled. It may depend on $n$ and yield $n^3$ terms in addition to terms proportional to $n$ and $n^2$. Before reliable calculations will be available, there is no way to determine such possible contributions. Still, the fact that three-body interactions between valence nucleons do not appear, may indicate that they are rather weak and do not depend strongly on the nuclear states.

In the shell model, magic nuclei have closed (fully occupied) shells. Shells may include one $j$-orbit or several orbits whose energies are close and hence, the effective interaction causes mixing of configurations. In either case, the first excited state of magic nuclei is due to a nucleon elevated to a higher orbit and hence, it is much higher than in other nuclei. Another feature of a magic nucleus is that if a nucleon is added to it, it must occupy a higher shell. Hence, its separation energy is lower than in preceding nuclei. This effect is clearly demonstrated in figure 1 for $^{49}\text{Ca}$ and $^{50}\text{Ca}$. The magicity of $^{48}\text{Ca}$ is seen also in its high first excited state compared to $^{46}\text{Ca}$ as shown in figure 2.
The data in figure 1 are rather old but recently, it became possible to investigate experimentally calcium isotopes beyond $^{48}\text{Ca}$ [5]. To assign shell model configurations of those nuclei it is useful to look at the single neutron energies of the orbits higher than $1f_{7/2}$. The lowest orbits may be seen in figure 3, in which levels of $^{49}\text{Ca}$ are shown.

Thus, the expected orbit to be occupied by valence neutrons is $2p_{3/2}$. Non-diagonal matrix elements between this orbit and the $2p_{1/2}$ and $1f_{5/2}$ orbits were determined many years ago from energies of nickel isotopes [6]. They are not expected to cause appreciable mixing of higher configurations into the $(1p_{3/2})^{n}$ configuration. To see whether this expectation is correct, we look at the neutron separation energies of calcium isotopes which were recently measured (figure 4).
The neutron separation energies in figure 4 agree well with valence neutrons occupying a \( j = 3/2 \) orbit which is full in \(^{52}\text{Ca}\). The first excited \( J = 2 \) level in that nucleus is considerably higher than in \(^{50}\text{Ca}\). In fact, the position of the \( J = 2 \) level in \(^{50}\text{Ca}\) is consistent with the measured binding energies. According to (2), it is equal to

\[
V_2 - V_0 = \frac{4\beta}{5} = 1.36 \times 0.8 = 1.088 \text{ MeV}.
\]

This value is in fair agreement with the measured 1.027 MeV.

4. Generalized seniority

The behavior of first excited \( J = 2 \) states in figure 2 of nickel isotopes is very different from that in calcium isotopes. There is no trace of the “magic number” \( n = 34 \) in \(^{52}\text{Ca}\). It seems that in the case of nickel, there is mixing between configurations of the various orbits which are filled at \( n = 40 \) where the first excited \( J = 2 \) is higher than in lighter nuclei. The different behaviour can be traced to the difference in single nucleon levels between \(^{49}\text{Ca}\) and \(^{57}\text{Ni}\) shown in figure 3. This difference is due to the attractive interaction between valence neutrons and the 8 protons occupying the \( 1f_{7/2} \) orbit in nickel.

The neutron separation energy from \(^{49}\text{Ca}\) is 5.146 MeV whereas from \(^{57}\text{Ni}\) it is 10.250 MeV. Hence, the average interaction between a \( 1f_{7/2} \) proton and a \( 2p_{3/2} \) neutron is 0.638 MeV. The average attraction between a \( 1f_{7/2} \) and a \( 1f_{7/2} \) neutron is stronger, which accounts for the lowering of the \( 5/2 \) state in \(^{57}\text{Ni}\). It is equal to \( \{(10.250-5.146)+(3.585-0.769)\}/8 = 0.990 \) MeV. This attraction is close to the average attraction, not considered here, between a \( 1f_{7/2} \) proton and a \( 1f_{7/2} \) neutron, 1.068 MeV. Similarly, the average attraction between a \( 1f_{7/2} \) proton and a \( 2p_{1/2} \) neutron is calculated to be 0.752 MeV, a little stronger than with its spin orbit partner, \( 2p_{3/2} \).

In view of the single nucleon energies in figure 3, it is expected that configurations will be strongly admixed in nickel isotopes heavier than \(^{57}\text{Ni}\). There are some similarities in the energies of nickel isotopes heavier than \(^{57}\text{Ni}\) and those in the seniority scheme. Some are related to binding energies and some to positions of first excited \( J = 2 \) states. The spacings of the latter and the \( J = 0 \) ground states are certainly not equal but they do not vary much compared to levels of odd Ni isotopes of which some are presented in figure 5.

![Figure 5. Low lying levels of odd isotopes of Nickel.](image-url)
States of $j^n$-configurations of identical nucleons have well defined seniorities. Where such configurations are mixed, some of their states may be analogous to those of seniority. In a single $j$-orbit, the seniority scheme is defined by using the pair creation operator

$$ S_j^+ = \frac{1}{2} \sum (-1)^{j-m} a_{jm}^+ a_{j-m}^+ . $$

In *generalized seniority* this role is played by a linear combination

$$ S^+ = \sum a_j S_j^+ $$

which creates an eigenstate of the shell model Hamiltonian, $HS^+|0>$ = $VS^+|0>$. If also $(S^+)^2|0> = W(S^+)$, then the following relation is obeyed

$$ [[H,S^+],S^+] = W(S^+)^2 . $$

If this is the case, all analogs of seniority $\nu = 0$ states are eigenstates

$$ H(S^+)^N|0> = \left\{ NV + \frac{1}{2} WN(N-1) \right\} (S^+)^N|0> . $$

From (7) follows that nucleon *pair* separation energies as a function of $N$ lie on a straight line. In the case of nickel isotopes this is fairly the case as plotted in figure 6.

**Figure 6.** Neutron pair separation energies from *even* Ni isotopes.

In the seniority scheme in a single $j$-orbit, spacings between levels with seniorities $\nu \leq n$ are constant, independent of $n$. In the case of generalized seniority, this is the case only if all coefficients in (5) are equal. Still, also in the case of unequal coefficients, there may be analogs of $S^+$ states which are eigenstates. Consider an eigenstate with $J = 2$ in the two nucleon configuration. It is created by an operator

$$ D_M^+ = \sum \beta_{jj'} D_{M,jj'}^+ = \sum \beta_{jj'} \left( 1 + \delta_{jj'} \right)^{-\frac{1}{2}} (jmj'm'|jj'J = 2,M)a_{jm}^+ a_{j'm'}^+ , $$
i.e. $HD_M^\dagger |0 >= V_2 D_M^\dagger |0 >$. If also $HS^+ D_M^\dagger |0 >$ is an eigenstate, than the condition

$$[[H, S^+], D_M^\dagger] = WS^+ D_M^\dagger$$

is obeyed. In this case, the analogs of $v = 2, J = 2$ states are eigenstates. The eigenvalues are simply given by

$$H(S^+)^{N-1} D_M^\dagger |0 >= \left( (N - 1)V + V_2 + \frac{1}{2} WN(N - 1) \right) (S^+)^{N-1} D_M^\dagger |0 >.$$  

The difference between the eigenvalues in (10) and those in (7) for the same value of $N$, is constant, independent of $N$. Thus, the positions of the special eigenstates in (10) above the ground states in (7) are constant and equal to $V_2 - V$. This is only approximately seen in nickel isotopes. There are more impressive examples which will not be considered here.

The examples above exhibit the fact that the simple shell model may be very useful in some cases. It seems to be a very useful approximation even for some nuclei away from the valley of stability.

5. Shell model wave function

In view of the complication in deriving the effective interaction, not much attention has been paid to the emerging wave functions. Shell model wave functions have been considered just as model wave functions. Since they do not include the important short range correlations, they could be very different from the real wave functions of nuclei. The short range correlation introduced by the strong and short range bare interaction, were described as “wounds” inflicted on the wave functions of nucleons moving independently in a central potential well. Still, such wounds may be limited to short distances and not cause too big changes in the shell model wave functions. The correlations are important for “short range” observables, but perhaps not for other (“long range”) ones. There is evidence, some of it mentioned below, that there are observables which have been successfully calculated by using shell model wave functions with un-normalized operators.

It is interesting to have some idea about the possible extension of the wounds. The change in the wave function of two nucleons is more appreciable when they are close. It is possible to find out how many encounters of two nucleons occur at zero range by calculating the expectation value of the zero range $\delta$-potential in given nuclear states. This may be carried out in some simple cases.

Consider $n$ identical nucleons in a single $j$-orbit. As mentioned above, eigenstates of this system are states of the seniority scheme. The $\delta$-potential interaction is diagonal in this scheme. Its eigenvalues are given by (1) without the quadratic term. Hence, the eigenvalues in ground states are equal to $nV_0/2$. Thus, the number of zero range encounters is linear in $n$, considerably smaller than the total number, $n(n-1)/2$.

In the case of a harmonic oscillator potential well, the calculation above has a more precise meaning. There, any wave function of two nucleons may be expressed as a linear combination with a finite number of terms, of products of two oscillator wave functions. One is a function of their center-of-mass, $(\vec{r}_1 + \vec{r}_2)/2$ and the other, a function of their distance $\vec{r} = \vec{r}_2 - \vec{r}_1$ [7]. The matrix elements of $\delta(r)$ between wave functions with $I > 0$ vanish since these wave functions vanish at $r = 0$. The only contribution to the eigenvalue $nV_0/2$ is due to $s$-states wave functions of $r$. The state $j^2 = 0$ has in its expansion $s$-states but their weight is always smaller than 1 due to higher $l$-states in the expansion. Hence, the eigenvalue $nV_0/2$ is an upper bound of the number of $s$-states which suffer the most severe wounds.

Another simple case is offered by protons and neutrons occupying a single $j$-orbit. In this case, no simple prescription is available for the structure of such a system. There is, however, only one wave function when both proton and neutron $j$-orbits are completely filled. This wave function may be expressed by the seniority scheme where for a simple $\delta$-potential, the eigenvalue is calculated to be
equal to \(5(2j+1)V_0/2\). It is much smaller than the total number of matrix elements which is 
\((4j+2)(4j+1)/2\). The eigenvalue includes the contribution of the proton-proton and neutron-neutron 
interactions, \(2\times(2j+1)V_0/2\). The proton-neutron interactions contribute, in this case, \(3(2j+1)V_0/2\).

Some indications that shell model wave functions are real to a considerable extent, are listed below. The analysis of pick-up and stripping direct reactions has always been based on shell model wave functions. They will, however, not be considered here.

Some direct evidence is offered by the rather large extension (“halo”) of the \(^{11}\text{Be}\) nucleus. It is due to the \(2s_{1/2}\) wave function of the valence neutron in the \(1/2^+\) ground state. It has a rather small separation energy and no centrifugal barrier. Hence, the extension of the wave function is indeed rather large.

Striking evidence is offered by measurements of charge distributions due to valence protons. Precise measurements were carried out by Frois et al [8] of the difference between the charge distributions of the \(^{206}\text{Pb}\) and \(^{205}\text{Tl}\) nuclei. The result has a remarkable similarity to the probability distribution of a proton in a \(3s_{1/2}\) orbit. This is the last orbit to be filled in the 50 - 82 shell. It is interesting to see if such a result could be obtained by many body theories.

Some evidence for the reality of shell model wave functions is offered by energy differences of mirror states in nuclei. Differences of electrostatic energies have been successfully calculated using shell model wave functions and the un-normalized Coulomb potential. Here, the dependence of Coulomb energy differences on the wave functions of valence nucleons is considered. A simple example, the difference between mirror levels of \(^{17}\text{O}\) and \(^{17}\text{F}\) nuclei, is shown in figure 7.

At a first glance at figure 7 it may seem that the Coulomb energy difference of the \(1/2^+\) states is larger than that of the \(5/2^+\) ground states. Actually, the opposite is true. The wrong impression is due to the fact that in figure 7, the ground states positons were plotted at the same energy. The energy of the \(^{17}\text{F}\) ground state is higher by 3.54 MeV than the \(^{17}\text{O}\) binding energy. The energy difference between the \(1/2^+\) states is smaller by 0.871 \(- 0.495 = 0.376\) MeV than the difference between ground state energies. This difference is due to the \(2s_{1/2}\) wave function of the valence nucleon being more extended than its \(1d_{5/2}\) wave function in the ground states.

The dependence of Coulomb energy differences on the shell model wave functions of the state considered, is evident in many cases. An example is shown in figure 8 where mirror levels of \(^{14}\text{C}\) and \(^{14}\text{N}\) are shown. Also in this case, the ground state of \(^{14}\text{C}\) is plotted at the same energy as the corresponding \(T = 1, J = 0\) state of \(^{14}\text{N}\) which lies 2.31 MeV above the \(T = 0, J = 1\) ground state.
According to the simple shell model, the negative parity states in figure 8 are due to coupling of a \(2p_{1/2}\) nucleon to a nucleon in the \(2s_{1/2}\) or the \(1d_{5/2}\) orbit. The states with \(J = 2\) and \(J = 3\) are \(p_{1/2}d_{5/2}\) states. As in the preceding example, the differences in Coulomb energy of these states are smaller than that of the ground states which mainly contain \(p\)-shell nucleons. The \(J = 0\) and \(J = 1\) states are \(p_{1/2}s_{1/2}\) states and their Coulomb energies differences are smaller than those of the ground states. They are also smaller than the states with \(d_{5/2}\) nucleons, as is seen in the case of \(A = 17\) nuclei.

6. Conclusion – will simplicity arise from complexity?

Shell model wave functions, of independent motion of individual nucleons, cannot be the real wave functions of the nucleons. They do not include short range correlations which are due to the strong bare interaction. The simple shell model, however, has considerable predictive power and its wave functions seem to have some reality. It is a very good approximation of nuclear structure and may be used for many aspects. Examples were given above as a reminder of its power and simplicity.

In ab initio calculations the starting point is the bare interaction leading to admixtures of many states. In one of these approaches, the No Core Shell Model, harmonic oscillator wave functions are used just as a convenient scheme. States from higher and higher oscillator shells are included, their number limited only by computational difficulties.

Will the simplicity of the shell model as a good approximation, emerge from the complexity of such calculations? This should be a challenge to every many-body theory of nuclear structure. A reliable many-body theory should explain why the simple shell model works so well, which interactions lead to it and where it becomes useless.

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