On the Minimum Cycle Cover problem on graphs with bounded co-degeneracy

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Abstract. In 2017, Knop, Koutecký, Masařík, and Toufar [WG 2017] asked about the complexity of deciding graph problems Π on the complement of G considering a parameter p of G, especially for sparse graph parameters such as treewidth. In 2021, Duarte, Oliveira, and Souza [MFCS 2021] showed some problems that are FPT when parameterized by the treewidth of the complement graph (called co-treewidth). Since the degeneracy of a graph is at most its treewidth, they also introduced the study of co-degeneracy (the degeneracy of the complement graph) as a parameter. In 1976, Bondy and Chvátal [DM 1976] introduced the notion of closure of a graph: let ℓ be an integer; the (n+ℓ)-closure, cl_{n+\ell}(G), of a graph G with n vertices is obtained from G by recursively adding an edge between pairs of nonadjacent vertices whose degree sum is at least n+ℓ until no such pair remains. A graph property Υ defined on all graphs of order n is said to be (n+ℓ)-stable if for any graph G of order n that does not satisfy Υ, the fact that uv is not an edge of G and that G+uv satisfies Υ implies d(u)+d(v)<n+ℓ. Duarte et al. [MFCS 2021] developed an algorithmic framework for co-degeneracy parameterization based on the notion of closures for solving problems that are (n+ℓ)-stable for some ℓ bounded by a function of the co-degeneracy. In 2019, Jansen, Kozma, and Nederlof [WG 2019] relax the conditions of Dirac’s theorem and consider input graphs G in which at least n−k vertices have degree at least n 2 , and present an FPT algorithm concerning to k, to decide whether such graphs G are Hamiltonian. In this paper, we first determine the stability of the property of having a bounded cycle cover. After that, combining the framework of Duarte et al. [MFCS 2021] with some results of Jansen et al. [WG 2019], we obtain a 2O(k) · nO(1)-time algorithm for Minimum Cycle Cover on graphs with co-degeneracy at most k, which generalizes Duarte et al. [MFCS 2021] and Jansen et al. [WG 2019] results concerning the Hamiltonian Cycle problem.

Keywords: degeneracy, complement graph, cycle cover, FPT, kernel

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1 Introduction

Graph width parameters are useful tools for identifying tractable classes of instances for NP-hard problems and designing efficient algorithms for such problems on these instances. Treewidth and clique-width are two of the most popular graph width parameters. An algorithmic meta-theorem due to Courcelle, Makowsky, and Rotics [7] states that any problem expressible in the monadic second-order logic on graphs (MSO₁) can be solved in FPT time when parameterized by the clique-width of the input graph. In addition, Courcelle [5] states that any problem expressible in the monadic second-order logic of graphs with edge set quantifications (MSO₂) can be solved in FPT time when parameterized by the treewidth of the input graph. Although the class of graphs with bounded treewidth is a subclass of the class of graphs with bounded clique-width [4], the MSO₂ logic on graphs extends the MSO₁ logic, and there are MSO₂ properties like “G has a Hamiltonian cycle” that are not MSO₁ expressible [6]. In addition, there are problems that are fixed-parameter tractable when parameterized by treewidth, such as MaxCut, Largest Bond, Longest Cycle, Longest Path, Edge Dominating Set, Graph Coloring, Clique Cover, Minimum Path Cover, and Minimum Cycle Cover that cannot be FPT when parameterized by clique-width [12,15,16,17,18], unless FPT = W[1].

For problems that are fixed-parameter tractable concerning treewidth, but intractable when parameterized by clique-width, the identification of tractable classes of instances of bounded clique-width and unbounded treewidth becomes a fundamental quest [11]. In 2016, Dvořák, Knop, and Masařík [13] showed that k-Path Cover is FPT when parameterized by the treewidth of the complement of the input graph. This implies that Hamiltonian Path is FPT when parameterized by the treewidth of the complement graph. In 2017, Knop, Koutecký, Masařík, and Toufar (WG 2017, [21]) asked about the complexity of deciding graph problems Π on the complement of G considering a parameter p of G (i.e., with respect to p(G)), especially for sparse graph parameters such as treewidth. In fact, the treewidth of the complement of the input graph, proposed be called co-treewidth in [11], seems a nice width parameter to deal with dense instances of problems that are hard concerning clique-width. MaxCut, Clique Cover, and Graph Coloring are example of problems W[1]-hard concerning clique-width but FPT-time solvable when parameterized by co-treewidth (see [11]).

The degeneracy of a graph G is the least k such that every induced subgraph of G contains a vertex with degree at most k. Equivalently, the degeneracy of G is the least k such that its vertices can be arranged into a sequence so that each vertex is adjacent to at most k vertices preceding it in the sequence. It is well-known that the degeneracy of a graph is upper bounded by its treewidth; thus, the class of graphs with bounded treewidth is also a subclass of the class of graphs with bounded degeneracy. In [11], Duarte, Oliveira, and Souza presented an algorithmic framework to deal with the degeneracy of the complement graph, called co-degeneracy, as a parameter.

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3 Originally this required a clique-width expression as part of the input.
Although the notion of co-parameters is as natural as their complementary versions, just a few studies have ventured into the world of dense instances with respect to sparse parameters of their complements. Also, note that would be natural to consider “co-clique-width” parameterization, but Courcelle and Olariu proved that for every graph $G$ its clique-width is at most twice the clique-width of $G$. Thus, the co-clique-width notion is redundant from the point of view of parameterized complexity. Therefore, in the sense of being a useful parameter for many NP-hard problems in identifying a large and new class of (dense) instances that can be efficiently handled, the co-degeneracy seems interesting because it is incomparable with clique-width and stronger than co-treewidth.

In [11], Duarte, Oliveira, and Souza developed an algorithmic framework for co-degeneracy parameterization based on the notion of Bondy-Chvátal closure for solving problems that have a “bounded” stability concerning some closure. More precisely, for a graph $G$ with $n$ vertices, and two distinct nonadjacent vertices $u$ and $v$ of $G$ such that $d(u) + d(v) \geq n$, Ore’s theorem states that $G$ is hamiltonian if and only if $G + uv$ is hamiltonian. In 1976, Bondy and Chvátal generalized Ore’s theorem and defined the closure of a graph:

- let $\ell$ be an integer; the $(n + \ell)$-closure, $\text{cl}_{n+\ell}(G)$, of a graph $G$ is obtained from $G$ by recursively adding an edge between pairs of nonadjacent vertices whose degree sum is at least $n + \ell$ until no such pair remains.

Bondy and Chvátal showed that $\text{cl}_{n+\ell}(G)$ is uniquely determined from $G$ and that $G$ is hamiltonian if and only if $\text{cl}_{n}(G)$ is hamiltonian.

A property $\mathcal{Y}$ defined on all graphs of order $n$ is said to be $(n + \ell)$-stable if for any graph $G$ of order $n$ that does not satisfy $\mathcal{Y}$, the fact that $uv$ is not an edge of $G$ and that $G + uv$ satisfies $\mathcal{Y}$ implies $d(u) + d(v) < n + \ell$. In other words, if $uv \notin E(G)$, $d(u) + d(v) \geq n + \ell$ and $G + uv$ has property $\mathcal{Y}$, then $G$ itself has property $\mathcal{Y}$ (c.f. [3]). The smallest integer $n + \ell$ such that $\mathcal{Y}$ is $(n + \ell)$-stable is the stability of $\mathcal{Y}$, denoted by $s(\mathcal{Y})$. Note that Bondy and Chvátal showed that Hamiltonicity is $n$-stable. A survey on the stability of graph properties can be found in [3].

In [11], based on the fact that the class of graphs with co-degeneracy at most $k$ is closed under completion (edge addition), it was proposed the following framework for determining whether a graph $G$ satisfies a property $\mathcal{Y}$ in FPT time regarding the co-degeneracy of $G$, denoted by $k$:

1. determine an upper bound for $s(\mathcal{Y})$ - the stability of $\mathcal{Y}$;
2. If $s(\mathcal{Y}) \leq n + \ell$ where $\ell \leq f(k)$ (for some computable function $f$) then
   (a) set $G = \text{cl}_{n+\ell}(G)$;
   (b) since $G = \text{cl}_{n+\ell}(G)$ and $G$ has co-degeneracy $k$ then $G$ has co-vertex cover number (distance to clique) at most $2k + \ell + 1$ (see [11]);
   (c) at this point, it is enough to solve the problem in FPT-time concerning co-vertex cover parameterization.

\footnote{A parameter $y$ is stronger than $x$, if the set of instances where $x$ is bounded is a subset of those where $y$ is bounded.}
In [11], using such a framework, it was shown that Hamiltonian Path, Hamiltonian Cycle, Longest Path, Longest Cycle, and Minimum Path Cover are all fixed-parameter tractable when parameterized by co-degeneracy. Note that Longest Path and Minimum Path Cover are two distinct ways to generalize the Hamiltonian Path problem just as Longest Cycle and Minimum Cycle Cover generalize the Hamiltonian Cycle problem. However, the Minimum Cycle Cover problem seems to be more challenging than the others concerning co-degeneracy parameterization, even because the stability of having a cycle cover of size at most $r$, to the best of our knowledge, is unknown.

In the Minimum Cycle Cover problem, we are given a simple graph $G$ and asked to find a minimum set $S$ of vertex-disjoint cycles of $G$ such that each vertex of $G$ is contained in one cycle of $S$, where single vertices are considered trivial cycles. Note that each nontrivial cycle has size at least three. In this paper, our focus is on Minimum Cycle Cover parameterized by co-degeneracy.

The Dirac’s theorem from 1952 (see [10]) states that a graph $G$ with $n$ vertices ($n \geq 3$) is Hamiltonian if every vertex of $G$ has degree at least $\frac{n}{2}$. In [20], Jansen, Kozma, and Nederlof relax the conditions of Dirac’s theorem and consider input graphs $G$ in which at least $n - k$ vertices have degree at least $\frac{n}{2}$, and present a $2^{O(k)} \cdot n^{O(1)}$-time algorithm to decide whether $G$ has a Hamiltonian cycle. In 2022, F. Fomin, P. Golovach, D. Sagunov, and K. Simonov [19] presented the following algorithmic generalization of Dirac’s theorem: if all but $k$ vertices of a 2-connected graph $G$ are of degree at least $\delta$, then deciding whether $G$ has a cycle of length at least $\min\{2\delta + k, n\}$ can be done in time $2^{k} \cdot n^{O(1)}$. Besides, in 2020, F. Fomin, P. Golovach, D. Lokshtanov, F. Panolan, S. Saurabh, and M. Zehavi [14] proved that deciding whether a 2-connected $d$-degenerate $n$-vertex $G$ contains a cycle of length at least $d + k$ can be done in time $2^{O(k)} \cdot n^{O(1)}$.

In this paper, we first determine the stability of the property of having a cycle cover of size at most $r$. After that, using the closure framework proposed in [11] together with some results and techniques presented in [20], we show that Minimum Cycle Cover admits a kernel with linear number of vertices when parameterized by co-degeneracy. After that, by designing an exact single-exponential time algorithm for solving Minimum Cycle Cover, we obtain as a corollary a $2^{O(k)} \cdot n^{O(1)}$-time algorithm for the Minimum Cycle Cover problem on graphs with co-degeneracy at most $k$. These results also imply a $2^{O(k)} \cdot n^{O(1)}$-time algorithm for solving Minimum Cycle Cover on graphs $G$ in which at least $n - k$ vertices have degree at least $\frac{n}{2}$, generalizing the Jansen, Kozma, and Nederlof’s result presented in [20] (WG 2019) for the Hamiltonian Cycle problem. Also, the single-exponential FPT algorithm for Minimum Cycle Cover parameterized by co-degeneracy implies that Hamiltonian Cycle can be solved with the same running time, improving the current state of the art for solving the Hamiltonian Cycle problem parameterized by co-degeneracy since the algorithm presented in [11] runs in $2^{O(k \log k)} \cdot n^{O(1)}$ time, where $k$ is the co-degeneracy. Note that our results also imply that Minimum Cycle Cover on co-planar graphs can be solved in polynomial time, which seemed to be unknown in the literature.
2 On the stability of having a bounded cycle cover

Although the stability of several properties has already been studied (c.f. [3]), the stability of the property of having a cycle cover of size at most $r$, to the best of our knowledge, is unknown. Therefore, we show that $s(\mathcal{U}) \leq n$, where $r$ is any positive integer, and $\mathcal{U}$ is the property of having a cycle cover of size at most $r$.

Lemma 1. Let $r$ be a positive integer. A simple graph $G$ with $n$ vertices has a cycle cover of size at most $r$ if and only if its $n$-closure, $\text{cl}_n(G)$, has also a cycle cover of size at most $r$.

Proof. Let $G$ be a simple graph with $n$ vertices, $r$ be a positive integer, and $\mathcal{U}$ be the graph property of having a cycle cover of size at most $r$. Since the claim trivially holds when $r = 0$ or $r \geq n$, we assume that $1 \leq r \leq n - 1$.

First, note that if $G$ has a cycle cover $S$ of size $r$ then the set $S$ is also a cycle cover of $\text{cl}_n(G)$, because $G$ is a spanning subgraph of $\text{cl}_n(G)$.

Now, suppose that $G$ does not have a cycle cover of size at most $r$ but $\text{cl}_n(G)$ has a cycle cover of size at most $r$.

Given that $\text{cl}_n(G)$ is uniquely determined from $G$ [2], the construction of $\text{cl}_{n+i}(G)$ can be seen as an iterative process of adding edges, starting from $G$, where a single edge is added at each step $i$, until no more edges can be added. Let $E_0 = E(G)$. We call by $E_i$ the resulting set of edges after adding $i$ edges during such a process. Therefore, $G_0 = G$, $G_1 = (V, E_1)$, $G_2 = (V, E_2), \ldots, G_t = (V, E_t)$, where $G_t = \text{cl}_n(G)$ is the finite sequence of graphs generated during a construction of the $n$-closure of $G$.

Since $G$ does not have a cycle cover of size at most $r$ but $\text{cl}_n(G)$ has a cycle cover of size at most $r$, by the construction of $\text{cl}_n(G)$, there is a single $i$ ($1 \leq i \leq t$) such that $G_{i-1}$ does not have a cycle cover of size at most $r$ but $G_i$ has a cycle cover of size at most $r$. Let $\{uw\} = E_i \setminus E_{i-1}$.

![Fig. 1. Representation of a graph with $r-1$ cycles, a path of size $h$ and the edge $vw$ that will be added, creating a graph with $r$ cycles.](image-url)

Suppose that $G_i$ has a cycle cover $S_i$ of size at most $r$. For simplicity, we assume that $|S_i| = r$. Therefore, the vertices of $G_{i-1}$ can be covered by a set
formed by \( r - 1 \) cycles \( C_1, C_2, \ldots, C_{r-1} \) and a path \( P \) (the cycle of \( G_i \) that contains the edge \( uw \)). Assume that each cycle \( C_j \) is defined by the sequence \( v_{C_j}^1, v_{C_j}^2, \ldots, v_{C_j}^{x_j} \) of vertices, where \( x_j \) is the number of vertices of \( C_j \). Let \( C = \{C_1, C_2, \ldots, C_{r-1}\} \), and \( P = v_p^1, v_p^2, \ldots, v_p^h \), where \( u = v_p^1 \), \( w = v_p^h \) and \( h \) is the number of vertices of \( P \). Note that \( h \geq 3 \); otherwise \( u = w \), implying that \( P \) is a trivial cycle and \( G_{i-1} \) has a cycle cover of size \( r \). Figure 1 illustrates \( C \) and \( P \).

We partition some vertices of \( G_{i-1} \) into four sets:

\[
X_P = \{v_p^q \mid (v_p^{q-1}, v_p^h) \in E_{i-1} \text{ and } 2 < q < h\},
\]

\[
X_C = \{v_{C_j}^q \mid (v_{C_j}^{(q \mod x_j) + 1}, v_p^h) \in E_{i-1}, 1 \leq q \leq x_j, \text{ and } C_j \in C\},
\]

\[
Y_P = \{v_p^q \mid (v_p^1, v_p^q) \in E_{i-1} \text{ and } 2 < q < h\},
\]

and

\[
Y_C = \{v_{C_j}^q \mid (v_{C_j}^{q-1}, v_p^h) \in E_{i-1}, 1 \leq q \leq x_j, \text{ and } C_j \in C\}.
\]

Note that \( v_{C_j}^1 = v_{C_j}^{(1 \mod x_j) + 1} \) for trivial cycles \( C_j \). Thus, \( X_C \) is well defined. Let \( X = X_P \cup X_C \) and \( Y = Y_P \cup Y_C \).

The set \( X \), is the set of vertices (with the exception of \( v_p^h \)) in which its predecessor in the path or its successor in the cycle is adjacent to \( v_p^h \). Also, the set \( Y \), is the set of vertices adjacent to \( v_p^h \) (with the exception of \( v_{C_j}^q \)). Note that the size of both \( X \) and \( Y \) are bounded by \( n - 3 \), since they exclude the vertices \( v_p^1, v_p^2 \) and \( v_p^h \) of \( P \). Besides that, we can observe that

\[
|X| = d(v_p^h) - 1 \text{ and } |Y| = d(v_p^h) - 1,
\]

where \( d(v) \) is the degree of the vertex \( v \). Therefore, the following holds:

\[
|X| + |Y| = d(v_p^h) + d(v_p^1) - 2
\]

that is,

\[
|X| + |Y| \geq n - 2
\]

since \( d(u) + d(w) \geq n \) where \( u = v_p^1 \), \( w = v_p^h \), and \( \{uw\} = E_i \setminus E_{i-1} \).

However, \( |X \cup Y| \leq n - 3 \) because both \( X \) and \( Y \) exclude \( v_p^1, v_p^2 \) and \( v_p^h \). Therefore, there is at least one vertex that belong to both \( X \) and \( Y \). Note that \( (X_P \cup Y_P) \cap (X_C \cup Y_C) = \emptyset \), since, by definition, the elements of the covering are vertex disjoint.

Therefore, there are two possibilities:

1. There is a vertex \( v_p^q \) belonging to the path \( P \) such that \( v_p^q \in X_P \cap Y_P \). This implies that \( G_{i-1} \) already had a cycle covering exactly the vertices of \( P \) before the addition of the edge \( uw = v_p^1 v_p^h \), which could be formed as follows (see Fig. 2):

\[
v_p^1, v_p^2, \ldots, v_p^{q-1}, v_p^h, v_p^h, v_p^{h-1}, \ldots, v_p^{q+1}, v_p^h, v_p^h;
\]
Fig. 2. Representation of case 1, where the vertex $v^h_P$, highlighted in gray, belongs to $X_P \cap Y_P$.

2. There is a vertex $v^q_C$ belonging to a cycle $C_j \in C$ such that $v^q_C \in X_C \cap Y_C$. In this case, $G_{i-1}$ has a larger cycle that can be obtained by merging $C_j$ with the path $P$ as follows (see Fig. 3):

$$v^q_{P,j}, v^{(q \mod x_j)+1}_{C_j}, v^{(q \mod x_j)+2}_{C_j}, \ldots, v^q_{P,j}, v^1_P, v^2_P, \ldots, v^h_P, v^{(q \mod x_j)+1}_{C_j}.$$

Fig. 3. Representation of case 2, where the vertex $v^q_C$, highlighted in gray, belongs to $X_C \cap Y_C$.

In the first case $G_{i-1}$ has a cycle cover of size $r$, while in the second case $G_{i-1}$ has a cycle cover of size $r - 1$. Both cases contradict the hypothesis that $G_{i-1}$ does not have a cycle cover of size at most $r$.

Therefore, there is no $1 \leq i \leq t$ such that $G_{i-1}$ does not have a cycle cover of size at most $r$ and $G_i$ has such a cycle cover. Thus, if $G_t = cl_n(G)$ has a cycle cover of size at most $r$ then $G_0 = G$ also has a cycle cover of size at most $r$. □

Lemma 1 states that for any positive integer $r$, the graph property $\Upsilon$ of having a cycle cover of size at most $r$ satisfies that $s(\Upsilon) \leq n$. We remark that such a bound is tight since whenever $r = 1$, the target $\Upsilon$ is the Hamiltonicity property, which is well known to have stability (exactly) equal to $n$ (c.f. [2]).

Now, observe that the class of graphs with co-degeneracy at most $k$ is closed under completion (edge addition), in the same way as the class of graphs with degeneracy at most $k$ is closed under edge removals. Recall that $cl_n(G)$ is uniquely determined from a $n$-vertex graph $G$ and it can be constructed in polynomial
time. Therefore, by Lemma [1] we may assume that $G = cl_n(G)$ whenever $G$ is an instance of Minimum Cycle Cover parameterized by co-degeneracy.

We call by co-vertex cover any set of vertices whose removal makes the resulting graph complete, i.e., a vertex cover in the complement graph. The co-vertex cover number of a graph $G$, co-vc($G$), is the size of its minimum co-vertex cover.

The following theorem is a key tool for this work.

**Theorem 1 ([11]).** Let $\ell \geq 0$ be an integer. If a graph $G$ has co-degeneracy $k$ and $G = cl_n(G)$ then $G$ has co-vertex cover number bounded by $2k + \ell + 1$. In addition, a co-vertex cover of $G$ with size at most $2k + \ell + 1$ can be found in polynomial time.

From Lemma [1] and Theorem [1] the problem of solving Minimum Cycle Cover on instances $G$ with co-degeneracy $k$ can be reduced in polynomial time to the problem of solving Minimum Cycle Cover on instances $G' = cl_n(G)$ with co-vertex cover number at most $2k + 1$. Therefore, in the next section we will focus on parameterization by the co-vertex cover number.

### 3 Polynomial kernelization

In [20], Jansen, Kozma, and Nederlof showed that given a graph $G$ with $n$ vertices such that at least $n - k$ vertices of $G$ have degree at least $\frac{n}{2}$, there is a deterministic algorithm that constructs in polynomial time a graph $G'$ with at most $3k$ vertices, such that $G$ is Hamiltonian if and only if $G'$ is Hamiltonian. In other words, they showed that the HAMILTONIAN CYCLE problem parameterized by such a $k$ has a kernel with a linear number of vertices.

First, we remark that such a parameterization that aims to explore a “distance measure” ($k$) of a given graph $G$ from satisfying the Dirac property, when applied to problems that are $n$-stable (such as HAMILTONIAN CYCLE and Minimum Cycle Cover) can be polynomial-time reduced to the case where the co-degeneracy is bounded by $k$. Since for such problems one can consider only instances $G'$ such that $G' = cl_n(G')$, from a graph $G$ with $n$ vertices such that at least $n - k$ vertices of $G$ have degree at least $\frac{n}{2}$, we obtain an instance $G' = cl_n(G)$ having a clique of size at least $n - k$.

Therefore, in the following, we extend the “relaxed” Dirac result from [20] by considering co-degeneracy and the Minimum Cycle Cover problem.

**Theorem 2.** There is a polynomial-time algorithm that, given a graph $G$ and a nonempty set $S \subseteq V(G)$ such that $G - S$ is a clique, outputs an induced subgraph $G'$ of $G$ on at most $3|S|$ vertices such that $G$ has a cycle cover of size at most $r$ if and only if $G'$ has a cycle cover of size at most $r$.

**Proof.** Let $G = (V, E)$ be a graph having a co-vertex cover $S$. Let $C = V(G) \setminus S$. If $|C| \leq 2|S|$ then by setting $G' = G$ the claim holds. Now, assume that $|C| > 2|S|$.

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5 co-vc($G$) is also called the distance to clique of $G$, and a co-vertex cover set is also called a clique modulator.
As in [20], let $S' = \{v_1, v_2 : v \in S\}$ be a set containing two representatives for each vertex of $S$. We construct a bipartite graph $H$ on vertex set $C \cup S'$, where for each edge $cv \in E(G)$ with $c \in C$ and $v \in S$, we add the edges $cv_1, cv_2$ to $E(H)$.

Now, we compute a maximum matching $M \subseteq E(H)$ of $H$. Let $C^*$ be the subset of vertices of $C$ saturated (matched) by $M$. If $|C^*| \geq |S| + 1$ then set $C' = C^*$; otherwise, let $C' \subseteq C$ be a superset of $C^*$ with size $|S| + 1$. Finally, set $G' = G[C' \cup S]$.

Note that $G'$ has at most $3|S|$ vertices, because $C'$ has at most $2|S|$ vertices.

First, suppose that $G'$ has cycle cover $Q'$ of size at most $r$. Since $G'$ is a subgraph of $G$, the set $Q'$ is a set of vertex disjoint cycles of $G$ covering $S \cup C' \subseteq V(G)$. Thus, only vertices of $C \setminus C'$ are not covered by $Q'$. However, since the size of $C'$ is greater than the size of $S$, there is at least one cycle $Q_j \in Q'$ that either is a single vertex of $C$ or contains an edge between vertices of $C$. If $|Q_j| = 1$ then we can replace it by a cycle containing all the vertices of $(C \setminus C') \cup Q_j$. If $Q_j$ has an edge $uv$ such that $u, v \in C$, then we can replace this edge by a $uv$-path containing the vertices of $C \setminus C'$ as internal vertices. In both cases we obtain a cycle cover of size at most $|Q'|$ in the graph $G$.

At this point, it remains to show that if $G$ has a cycle cover of size $r$ then $G'$ has a cycle cover of size at most $r$.

Using a strategy similar to that in [20], we first present a structure that implies cycle covers of size at most $r$ in $G'$. For a vertex set $S^*$ in a graph $G^*$, we define a cycle-path cover of $S^*$ in $G^*$ as a set $L$ of pairwise vertex-disjoint simple paths or cycles such that each vertex of $S^*$ belongs to exactly one element of $L$, i.e., $L$ can be seen as a subgraph with maximum degree two which contains every vertex of $S^*$. For a vertex set $C^*$ in $G^*$, we say that a cycle-path cover $L$ has $C^*$-endpoints if the endpoints of each path $P \in L$ belong to $C^*$.

**Claim 1** If $G'$ has a cycle-path cover of $S$ having $C'$-endpoints and containing at most $r - 1$ cycles, then $G'$ has a cycle cover of size at most $r$.

**Proof.** We have two cases to analyse: if the cycle-path cover of $S$ contains only cycles, as the number of cycles is at most $r - 1$, then we can add a new cycle formed by the vertices not yet covered; if the cycle-path cover contains some paths, by vertex disjointness, all the paths have different endpoints in $C'$, and, since $C'$ is a clique, we can connect such endpoints in such a way as to form a single cycle containing these paths as subgraphs, after that, an edge $uv$ of such a cycle having $u, v \in C'$ can replaced by a $uv$-path containing as internal vertices the vertices of $G'$ that are not in such a cycle-path cover of $S$. In both cases, we conclude that $G'$ has a cycle cover of size at most $r$. \qed

Now, considering the bipartite graph $H$ and its maximum matching $M$, let $U_C$ be the set of vertices of $C$ that are not saturated by $M$, and let $R$ be the vertices of $H$ that are reachable from $U_C$ by an $M$-alternating path in $H$ (which starts with a non-matching edge). Set $R_C = R \cap C$ and $R_{S'} = R \cap S'$.
By Claim 1 it is enough to show that if $G$ has a cycle cover of size $r$ then $G'$
has a cycle-path cover of $S$ having $C'$-endpoints and containing at most $r - 1$
cycles. For that, we consider Claim 2 presented in [20].

Claim 2 ([20]) The sets $R, R_C, R_{S'}$ satisfy the following.

1. Each $M$-alternating path in $H$ from $U_C$ to a vertex in $R_{S'}$ (resp. $R_C$) ends
   with a non-matching (resp. matching) edge.
2. Each vertex of $R_{S'}$ is matched by $M$ to a vertex in $R_C$.
3. For each vertex $x \in R_C$ we have $N_H(x) \subseteq R_{S'}$.
4. For each vertex $v \in S$ we have $v_1 \in R_{S'}$ if and only if $v_2 \in R_{S'}$.
5. For each vertex $v \in S \setminus R_{S'}$, we have $N_H(v) \cap R_C = \emptyset$ and each vertex of
   $N_H(v)$ is saturated by $M$.

Lemma 2. If $G$ has a cycle cover of size at most $r$, then $G'$ has a cycle-path
cover of $S$ having $C'$-endpoints and containing at most $r - 1$ cycles.

Proof. Let $F$ be a cycle cover of size at most $r$ of $G$. Consider $F$ as a 2-regular
subgraph of $G$. Let $F_1 = F[S]$ be the subgraph of $F$ induced by $S$. Since $F$ is
a spanning subgraph of $G$, and $S \subseteq V(G')$, it follows that $F_1$ is a cycle-path
cover of $S$ in $G'$. At this point, we need to extend it to have $C'$-endpoints. As
in [20], we do that by inserting edges into $F_1$ to turn it into a subgraph $F_2$ of
$G'$ in which each vertex of $S$ has degree exactly two. This structure $F_2$ must be
a cycle-path cover of $S$ in $G'$ with $C'$-endpoints, since the degree-two vertices $S$
cannot be endpoints of the paths.

Setting $F_2 = F_1, R_S = \{ v \in S : v_1 \in R_{S'} \text{ or } v_2 \in R_{S'} \}$, we proceed as follows.

1. For each vertex $v \in R_S$, we have $v_1, v_2 \in R_{S'}$, by Claim 2(4), which implies
   by Claim 2(2) that both $v_1$ and $v_2$ are matched to distinct vertices $x_1, x_2$ in
   $R_C$. If $v$ has degree zero in subgraph $F_1$, then add the edges $vx_1, vx_2$ to $F_2$.
   If $v$ has degree one in $F_2$ then only add the edge $vx_1$, (we do not add edges
   if $v$ already has degree two in $F_1$)

2. For each vertex $v \in S \setminus R_S$, it holds that $N_G(v) \cap R_C = \emptyset$. This follows
   from the fact that $N_G(v) = N_H(v_1) = N_H(v_2)$ and Claim 2(5). Note
   that $v \notin R_S$ implies $v_1, v_2 \notin R_{S'}$. Hence the (up to two) neighbors that
   $v \in S \setminus R_S$ has in $C$ on the cycle cover $F$ do not belong to $R_C$ (see also
   Claim 2(3)). In addition, Claim 2(5) ensures that all vertices of $N_G(v)$ are
   saturated by $H$ and hence belong to $C'$. Thus, for each vertex $v \in S \setminus R_S$, for
each edge from $v$ to $C \cap C'$ incident on $v$ in $F$, we insert the corresponding
edge into $F_2$.

It is clear that the above procedure produces a subgraph $F_2$ in which all
vertices of $S$ have degree exactly two. By Claim 2(5), we have that a vertex
c $\in C$ does not have edges added in $F_2$ by both previous steps, thus each vertex
c $\in C$ added in $F_2$ has degree at most two in it because $c$ has at most one edge
in the matching $M$ (see Step 1), while $c$ has two edges in the cycle cover $F$ (see
Step 2).

At this point, we know that $F_2$ is a cycle-path cover of $S$ having $C'$-endpoints.
It remains to show that it contains at most $r - 1$ cycles.
Claim 3 Every cycle of $F_2$ is a cycle of $F$.

Proof. Suppose that $F_2$ has a cycle $Q$ that is not in $F$. As $F_2$ is formed from $F_1$, the edges in $Q$ between the vertices of $S$ are also edges of $F$. Furthermore, by construction, the added edges from $F_1$ to obtain $F_2$ are the edges incident to the vertices of $S$. Therefore, there is no edge between the vertices of the clique $C$ in $Q$. By Claim 2, we have that a vertex $c \in C$ cannot be incident to two edges of $F_2$ being one added by Step 1 and the other by Step 2 of the construction. Since these steps are mutually exclusive with respect to a vertex $c \in C$, and given that $c$ has degree two in $Q$ (since $Q$ is a cycle), we have that the edges of each vertex $c \in C \cap Q$ were added by Step 2 of the construction (Step 1 adds only one edge of the matching). However, by construction, the edges in $Q$ incident to a vertex $c \in C$ are the edges in $F$. Therefore, every edge of $Q$ is contained in $F$, contradicting the hypothesis that $Q$ is not contained in $F$. ⊓⊔

By hypothesis, $F$ has at most $r$ cycles. Since $|C| > |S|$, it holds that at least one cycle of $F$ must have an edge between vertices of $C$. Thus, at least one cycle of $F$ is not completely contained in $F_2$, which implies, by Claim 3, that $F_2$ has at most $r - 1$ cycles. Therefore, $F_2$ is a cycle-path cover of $S$ having $C'$-endpoints which contains at most $r - 1$ cycles. This concludes the proof of Lemma 2. ⊓⊔

By Lemma 2 and Claim 1, it holds that if $G$ has a cycle cover of size at most $r$ then $G'$ has a cycle cover of size at most $r$. Since the reduction can be performed in polynomial time, and $|V(G')| = 3|S|$, we conclude the proof of Theorem 2. ⊓⊔

Corollary 1. Minimum Cycle Cover parameterized by co-degeneracy admits a kernel with at most $6k + 3$ vertices, where $k = \text{co-deg}$.

4 An exact single-exponential time algorithm

By Corollary 1, it holds that an exact and deterministic single-exponential time algorithm for Minimum Cycle Cover is enough to obtain an FPT algorithm for Minimum Cycle Cover with single-exponential dependency concerning the co-degeneracy of the input graph. In [9], using the Cut&Count technique, M. Cygan, J. Nederlof, Ma. Pilipczuk, Mi. Pilipczuk, J. Rooij and J. Wojtaszczyk produces a $2^{O(tw)} \cdot |V|^{O(1)}$ time Monte Carlo algorithm for Minimum Cycle Cover (Undirected Min Cycle Cover in [9]), where $tw$ is the treewidth of the input graph. In [1], H. Bodlaender, M. Cygan, S. Kratsch, J. Nederlof presented two approaches to design deterministic $2^{O(tw)} \cdot |V|^{O(1)}$-time algorithms for some connectivity problems, and claimed that such approaches can be apply to all problems studied in [9].

Although such approaches can be used to solve Minimum Cycle Cover by a single-exponential time algorithm, in order to present a simpler deterministic procedure, below we present a simple and deterministic dynamic programming based on modifying the Bellman–Held–Karp algorithm.

Theorem 3. Minimum Cycle Cover can be solved in $O(2^n \cdot n^3)$ time.
Proof. Given a graph $G = (V, E)$ with an isolated vertex $w$, a vertex subset $X \subseteq V$, $s, t \in X$, and a Boolean variable $P2$, we denote by $M[X, s, t, P2]$ the size of a minimum set $S$ of vertex-disjoint cycles but one nonempty vertex-disjoint $st$-path of $G[X]$ such that

- every vertex of $X$ is in an element of $S$;
- the $st$-path is not a $P2$ if the variable $P2 = 0$;
- the $st$-path is a $P2$ if the variable $P2 = 1$.

Note that $M[V, w, w, 0]$ represents the size of a minimum cycle cover of $G$.

In essence, the $st$-path represents the open cycle that is still being built. The variable $P2$ is a control variable to avoid $P2$ as cycles of size two. At each step, we can interpret that the algorithm either lengthens the path by adding a new endpoint or closes a cycle and opens a new trivial path. As we can reduce the MINIMUM CYCLE COVER problem to the case where the graph has an isolated vertex $w$, we assume that this is the case and consider $w \in X$ just when $X = V$.

Our recurrence is as follows.

If $X = \{v\}$ then $M[X, s, t, P2] = 1$ for $s = t = v$ and $P2 = 0$;

otherwise, it is $\infty$.

If $|X| \geq 2$ then $M[X, s, t, P2]$ is equal to

$$
\begin{cases}
\infty & \text{if } s = t, P2 = 1 \\
\min_{s', t' \in X \setminus \{t\} : s' \not\in E \text{ or } s' = t'} \left( M[X \setminus \{t\}, s', t', 0] \right) + 1 & \text{if } s = t, P2 = 0 \\
\infty & \text{if } s \not= t, P2 = 1, st \notin E \\
M[X \setminus \{t\}, s, s, 0] & \text{if } s \not= t, P2 = 1, st \in E \\
\min_{t' \in X \setminus \{t\} \text{ s.t. } t' \in E, P2 \in \{0, 1\}} \left( M[X \setminus \{t\}, s, t', P2]\right) & \text{if } s \not= t, P2 = 0
\end{cases}
$$

The size of the table is bounded by $(2^n - 1) \cdot n^2 \cdot 2$ where $n$ is the number of vertices of the graph. Regarding time complexity, we have three cases: when $P2 = 1$ the recurrences can be computed in $O(1)$ time; when $s = t$ and $P2 = 0$ the recurrence can be computed in $O(n^2)$ time, and since there are at most $(2^n - 1) \cdot n + 1$ cells in this case, the total amount of time taken to compute those cells is $O(2^n \cdot n^3)$; finally, when $s \not= t$ and $P2 = 0$ the recurrence can be computed in $O(n)$ time, but there are $O(2^n \cdot n^2)$ cells in this case, implying into a total amount of $O(2^n \cdot n^3)$ time to compute all these cells. Therefore, the dynamic programming algorithm can be performed in $O(2^n \cdot n^3)$ time. Note that, in addition to determining the size of a minimum cycle cover, one can find it with the same running time. Also, the correctness of the algorithm is straightforward. \hfill \Box

Corollary 2. MINIMUM CYCLE COVER can be solved in $2^{O(\text{co-deg})} \cdot n^{O(1)}$ time.

By Corollary 2 it follows that MINIMUM CYCLE COVER on co-planar graphs can be solved in polynomial time, which seems to be unknown in the literature.

Corollary 3. MINIMUM CYCLE COVER on graphs $G$ in which at least $n - k$ vertices have degree at least $\frac{k}{2}$ can be solved in $2^{O(k)} \cdot n^{O(1)}$ time.
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