Relationship of field-theory based single boson exchange potentials to current ones

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November 16, 2017

Abstract

It is shown that field-theory based single boson exchange potentials cannot be identified to those of the Yukawa or Coulomb type that are currently inserted in the Schrödinger equation. The potential which is obtained rather corresponds to this current single boson exchange potential corrected for the probability that the system under consideration is in a two-body component, therefore missing contributions due to the interaction of these two bodies while bosons are exchanged. The role of these contributions, which involve at least two boson exchanges, is examined. The conditions that allow one to recover the usual single boson exchange potential are given. It is shown that the present results have some relation: i) to the failure of the Bethe-Salpeter equation in reproducing the Dirac or Klein-Gordon equations in the limit where one of the constituent has a large mass, ii) to the absence of corrections of relative order $\alpha \log \frac{1}{\alpha}$ to a full calculation of the binding energy in the case of neutral massless bosons or iii) to large corrections of wave-functions calculated perturbatively in some light-front approaches.

PACS numbers: 13.40.Fn, 13.75.Cs, 25.45.De
Keywords: potential, boson exchange, field theory

1 Introduction

It is well known that the Bethe-Salpeter equation $[1]$, in both the ladder approximation and in the limit where one of the two interacting particles has a large mass, does not

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reproduce the Dirac or Klein-Gordon equations [3]. These are important benchmarks, as they accurately describe the interaction of charged particles with spin 1/2 (electrons, muons,...) or spin 0 (pions,...) in the Coulomb field produced by a nucleus. The remedy to the above failure resides in the contribution of second order crossed diagrams (see refs [2, 3] for some review). At first sight, taking into account these diagrams is somewhat surprising as both the Bethe-Salpeter and the Dirac or Klein-Gordon equations imply the iteration of what looks like a single boson exchange, allowing one to sum up the contribution of ladder diagrams. There is no obvious indication that these equations explicitly involve crossed diagrams. One can thus expect that the role of crossed diagrams in restoring the equivalence of the Bethe-Salpeter and Dirac or Klein-Gordon equations is partly fortuitous and results from some cancellation of contributions, which may not have a general character.

It is also well known that the Bethe-Salpeter equation or the light-front approach, when applied to the Wick-Cutkosky model (two scalar particles exchanging a massless spin 0 particle [4, 5], leads to corrections of the binding energy with an unusual character [2, 3]. In the non-relativistic limit, this model is equivalent to the Coulomb problem and the binding energy can be expressed in terms of the quantity, \( m \alpha^2 \), where \( \alpha \) can be identified to the similar quantity used in QED (\( \alpha \) is equal to the ratio of the coupling constant \( g^2 \), generally referred to in the model, with the quantity \( 4m^2 \pi \)). With these definitions, the corrections are of order \( m \alpha^3 \log \frac{1}{\alpha} \) (relative order \( \alpha \log \frac{1}{\alpha} \) with respect to the binding energy). These ones have been determined theoretically [3] and are present in some numerical calculations [4, 5, 6]. It has been suggested that the difference with the Coulomb interaction case, where these corrections are known to be absent, would be due to the spin 1 nature of the photon [3].

A third point we would like to mention deals with the determination of a relativistic wave function in the light-front approach for a realistic physical system. This represents a tremendous task, especially in the case of the NN system where the exchange of several mesons with different spins or isospins has to be considered. Present perturbative calculations using current deuteron wave functions as a zeroth order require a large renormalization of the first order wave function, by about 20% [7, 8]. This is much larger than the corrections due to incorporating relativistic kinematical effects, such as those due to factors \( \left( \frac{m}{E} \right)^2 \), which amount to a few percent [9].

This is also much larger than corrections due to renormalization effects, which are related to the meson in flight content of the deuteron state and have been found to be of the order of 3 − 4% [10, 11]. Furthermore, the use of this wave function calculated perturbatively has revealed bad features such as an unacceptable ratio of the deuteron asymptotic normalizations \( A_D \) and \( A_S \), \( \frac{A_D}{A_S} \), or a deuteron quadrupole moment, both exceeding experiment by 20 − 25%. These discrepancies are too large to be considered as conventional relativistic corrections of the order \( (\frac{v}{c})^2 \). They rather suggest that the comparison is biased by the absence of essential ingredients in the calculation, which should be first completed to provide physically relevant results.

Obviously, as mentioned in [14], one expects these difficulties to be removed when the problem is treated in its totality, implying the determination of an appropriate nucleon-nucleon interaction model, as made in the full (energy dependent) Bonn model [12]. We
nevertheless believe that it is important to understand the origin of the large renor-
malization as well as of unexpected results in relation with the deuteron D-state. On the
one hand, the difficulty in determining a nucleon-nucleon interaction model invites to rely
as much as possible on what has already been achieved in the past. On the other, this
understanding should avoid to blindly absorb in the parameters of a fitted single meson
exchange potential contributions that are too specific to be treated in a so crude way.
This remark is relevant for the $2\pi$-exchange part. Although it concerns a case which is
less realistic than the NN one, a relatively large correction of the wave function was also
found in the light-front approach applied to the Wick-Cutkosky model in the limit of a
small coupling $\mathbf{[3]}$. This correction is located in the low momentum range of the wave
function and, contrary to that one anticipated in $\mathbf{[5]}$, which rather occurs in the high
momentum domain, it does not go to 0 with the momentum. For some part, it can be
accounted for by a renormalization of the coupling constant, $\alpha \rightarrow \alpha (1 - \frac{2}{\pi} \alpha \log \frac{1}{\alpha})$.

All the problems briefly sketched above have been raised in the framework of relativistic
descriptions of a two-body system, such as the Bethe-Salpeter equation, or the light-front
equation as used in $\mathbf{[7, 14, 15]}$. What we would like to show here is that they are rather
due to the field-theory treatment which underlies them and that they have a much more
general character, with a close relation to dynamical effects in the field of the many body
problem and the related derivation of effective degrees of freedom. Essentially, the effects
under consideration rather have a static character and are different from usual relativistic
effects that are velocity dependent and represent a higher order effect. To emphasize our
point, we will essentially show how the same problems occur and are solved in an approach
which is mainly a non-relativistic one. While doing so, we are aware that the distinction is
not straightforward. A field-theory treatment, which implies the creation and destruction
of particles, is one of the ingredients that are required to fulfill the Lorentz covariance
of some approaches like that followed in $\mathbf{[13, 14]}$. However, as evidenced by the very
existence of relativistic quantum mechanics approaches $\mathbf{[16]}$, this property can be ensured
without relying on field theory. On the other hand, a consequence of the present work
should be to provide a better zeroth order approximation for perturbative calculations
using energy dependent interaction models and aiming to study relativistic effects.

The plan of the paper is as follows. In the second section, starting from a field-theory
based single boson exchange contribution to the two-body interaction, we remind how one
can derive an energy independent effective interaction to be employed in a Schrödinger
equation. The difference with the current single boson exchange potential of the Yukawa
or Coulomb type is emphasized. The third section is devoted to the corrections that the
above difference produces, namely some renormalization of the two-body wave function
and $\alpha^3 \log \frac{1}{\alpha}$ terms in the binding energy in the case of an exchange of a massless (scalar)
boson. In the fourth part, we consider the contribution due to including two boson
exchanges, arising especially from the crossed diagram. It is shown how this allows one
to recover current single boson exchange potentials in the case where the boson is unique
(neutral) and couples predominantly in a scalar way with the constituent particles as
in the Dirac or Klein Gordon equations. A simple explanation for the role played by
this contribution is proposed. Some attention is given to the massless case, with regard
to possible $m \alpha^3 \log \frac{1}{\alpha}$ corrections to the binding energy, and to the nucleon-nucleon
interaction case, with regard to the spin and isospin of the exchanged mesons. Some of
the results presented here may be partly known in various domains of physics, but do not
seem to be (or have been forgotten) in the domain of intermediate energy physics, at the
border line of particle and nuclear physics. While they may not be quite new, we believe
that reminding them here will make the paper more consistent.

2 Contribution of the field-theory based single boson
exchange to the current two-body effective inter-
action

![Diagram]

Figure 1: Time-ordered single boson exchange contributions to the two-body interaction with
indication of the kinematics in the center of mass.

Usual two-body interaction models suppose the instantaneous propagation of the ex-
changed bosons, which results in the conservation of the number of particles. In a field
theory approach, the number of particles is not conserved and the two constituent particles
may be accompanied by one, two,... bosons in flight (beside the excitation of anti-particles
ignored in the present work). In this approach, which has a more fundamental basis [12]
and is much more ambitious, the lowest order interaction of two constituent particles,
possibly bare, can be obtained by summing the contributions of the two time ordered
diagrams shown in Fig. 1. In the center of mass system, it reads:

\[ V_E(\vec{p}_i, \vec{p}_f) = \sum \frac{g_v^2 O_v^1 O_v^2}{2\omega_k^v} \left( \frac{1}{E - \omega_k^v - E_i - E_f} + \frac{1}{E - \omega_k^v - E_i - E_f} \right) . \] (1)

In this equation, the summation on the index \( v \) is carried over the different species of
bosons, if any. The quantities \( g_v \) and \( O_v \) represent the coupling constant relative to the
interaction of the boson with the constituent particles and its vertex function. This one
may involve the spin, the isospin and some form factor. It also contains normalization
factors relative to the constituent particles: \( \sqrt{E_i} \) and \( \sqrt{E_f} \), so that \( g \) will be in what follows
a dimensionless constant. The quantity, \( \omega_k^v, E_i, E_f \) represent the on-mass shell energy of
the exchanged boson and the constituent particles. They are defined as:

\[ \begin{align*}
\omega_k^v &= \sqrt{\mu_v^2 + \vec{k}^2}, \\
E_i &= \sqrt{m^2 + \vec{p}_i^2}, \\
E_f &= \sqrt{m^2 + \vec{p}_f^2},
\end{align*} \] (2)
where \( \vec{k}, \vec{p}_i, \vec{p}_f \) represent the 3-momenta of the particles under consideration (in the center of mass for the two last ones) (see Fig. 1 for the kinematics). As to \( E \), it represents the energy of the system we are interested in. Its presence in Eq. (2) is a characteristic feature of the coupling of the two-body component to components that are not explicitly specified with three or more bodies in the general case.

When the contribution of \( V_E \) to the Born scattering amplitude is considered, the energy conservation relation, \( E = 2 E_i = 2 E_f \), is fulfilled and it can then be checked that the two terms in Eq. (1) sum up to reproduce the standard Feynman expression for the propagator of a boson. Contrary to this one however, Eq. (1) provides a well determined procedure for extrapolating the interaction off-energy shell. The dependence of the interaction, \( V_E \), on the energy \( E \), which has the same origin as the one present in the full Bonn model of the \( NN \) interaction [12] (not to be confused with the other Bonn models denoted Q, R, QA, QB, QC, RA, RB [12, 17]), is at the same time a distinctive feature of a field-theory approach, but also a source of difficulties [2]. Indeed, inserting \( V_E \) in a Schrödinger equation leads to solutions that are not orthogonal in the case of states with different energies. This violates an important property expected from quantum mechanics, therefore suggesting that this way of proceeding is wrong or, at least, incomplete. While this incompleteness may be remedied by inserting contributions due to the many body components that the energy dependence of \( V_E \) implies (these may be 2 constituent particles plus 1, 2,... bosons in flight), most often, the use of energy dependent potentials has been discarded, not without any reason [13]. Another way to remedy these difficulties is to derive from \( V_E \), Eq. (1), an energy independent, but effective interaction. Many works along these lines have been performed in the literature [19, 20, 21, 22]. Here, we will proceed in a way which is perhaps more appropriate to emphasize the physics hidden behind the above energy dependence [13]. The interaction \( V_E \), Eq. (1), is expanded in powers of \( 1/\omega_k \). Starting from:

\[
\frac{1}{E - \omega_k - \vec{p}_i^2/2m - \vec{p}_f^2/2m} = \frac{1}{\omega_k} - \frac{E - \vec{p}_i^2/2m - \vec{p}_f^2/2m}{\omega_k^2} + \cdots ,
\]

where the energies of the total system and of the constituent particles are, from now on, replaced by the non-relativistic counterparts, one gets:

\[
V_E = V_0 - \frac{1}{2} \{ (E - \vec{p}_i^2/m), V_1 \} \cdots .
\]

A different expansion around some energy, \( E_0 \), (renormalization point) could also be performed. More appropriate to the framework of the renormalization theory, it is less convenient here where a comparison with standard descriptions of two-body interactions is intended. The difference, which involves higher order terms in Eq. (3), is irrelevant for the present work, essentially devoted to a non-relativistic description of two-body systems.

The first term in the r.h.s. of Eq. (4), which stems from the similar one in Eq. (3), can be identified to a usual boson exchange interaction. In momentum space, it reads:

\[
V_0(k) = -\sum_v \frac{g_v^2 O_v^1 O_v^2}{\mu_v^2 + k^2}.
\]
In configuration space (for the simplest case where the vertex functions \(O_v\) do not depend on momenta and where form factors are neglected), its expression is given by:

\[
V_0(r) = -\sum_v g_v^2 O_v^1 O_v^2 \frac{e^{-\mu r}}{4\pi r}.
\] (6)

The quantity \(V_1\) in Eq. (4), which is related to \(V_E\) by the relation:

\[
V_1(k) = -\frac{\partial V_E}{\partial E}, \quad \text{(up to } \frac{1}{\omega^4} \text{ terms)},
\] (7)

is given, in momentum space, by:

\[
V_1(k) = \sum_w g_w^2 O_w^1 O_w^2 \frac{1}{(\omega_{k}^w)^3}.
\] (8)

The corresponding expression in configuration space, in the same approximation as for Eq. (6), reads:

\[
V_1(r) = \sum_w g_w^2 O_w^1 O_w^2 \frac{1}{2\pi^2} \int \frac{dk k^2 j_0(kr)}{(\omega_{k}^w)^3},
\] (9)

With the expansion of \(V_E\) we made together with the assumption of local vertex functions, the dimensionless quantity \(V_1(r)\) given by Eq. (9) is local. This feature greatly facilitates the interpretation of its role but will not be necessarily used throughout this paper. In the case of massless bosons, like in QED or in the Wick-Cutkosky model, the integral in Eq. (9) diverges logarithmically and some slight modification of the expansion given by Eq. (3) should be performed (see Sect. 3.2).

With the above definitions, the equation that the wave function describing the two-body components, \(\psi(r)\), has to satisfy reads:

\[
\left( V_0(r) + \frac{1}{2} \left\{ \frac{\vec{p}^2}{m}, (1 + V_1(r)) \right\} - E(1 + V_1(r)) \right) \psi(r) = 0.
\] (10)

By multiplying on the left by \((1 + V_1(r))^{-1/2}\) and making the substitution

\[
\psi(r) = (1 + V_1(r))^{-1/2} \phi(r),
\] (11)

one gets an equation that can now be identified to a Schrödinger equation:

\[
\left( V(r) + \frac{\vec{p}^2}{m} - E \right) \phi(r) = 0,
\] (12)

where

\[
V(r) = (1 + V_1(r))^{-1/2} V_0(r) (1 + V_1(r))^{-1/2} + \ldots.
\] (13)

The dots in Eq. (13) represent a higher order contribution in both \(V_1(r)\) and \(\frac{1}{m}\), whose expression, in the limit of a local operator, is given by:

\[
-\frac{1}{4m} \frac{(\vec{\nabla} V_1(r))^2}{(1 + V_1(r))^2}.
\]
Being of the order of relativistic contributions neglected here (Z-diagrams for instance), it is not considered in the following. The derived potential, \( V(r) \), can thus be approximated by:

\[
V(r) = V_0(r) - \frac{1}{2} \{ V_1(r), V_0(r) \} + \cdots.
\]  

(14)

This expansion may be useful for a discussion of corrections order by order. The full expression (13) is nevertheless more appropriate in the case where an expansion in terms of \( V_1 \) is not legitimate.

At the lowest order in \( V_1 \), the correction to the potential \( V_0 \) reads in momentum space:

\[
\Delta V(\vec{p}_1, \vec{p}_f) = \frac{1}{2} \int \frac{d\vec{p}}{(2\pi)^3} \left( \sum_w \frac{g_w^2 O^1_w O^2_w}{(\mu^2_w + (\vec{p}_i - \vec{p})^2)^3} \sum_v \frac{g_v^2 O^1_v O^2_v}{\mu^2_v + (\vec{p}_f - \vec{p})^2} \right.
\]

\[
+ \sum_v \frac{g_v^2 O^1_v O^2_v}{\mu^2_v + (\vec{p}_i - \vec{p})^2} \sum_w \frac{g_w^2 O^1_w O^2_w}{(\mu^2_w + (\vec{p}_f - \vec{p})^2)^3} \right).
\]  

(15)

After making the change \( w \leftrightarrow v \) in the second term and using quantities \( \omega^w_i \) and \( \omega^v_f \) defined analogously to Eq. (2):

\[
\omega^w_i = (\mu^2_w + (\vec{p}_i - \vec{p})^2)^{\frac{1}{2}}, \quad \omega^v_f = (\mu^2_v + (\vec{p}_f - \vec{p})^2)^{\frac{1}{2}},
\]  

(16)

the above expression may be written in a simpler form:

\[
\Delta V(\vec{p}_1, \vec{p}_f) = \frac{1}{2} \int \frac{d\vec{p}}{(2\pi)^3} \sum_{w,v} \left( \frac{g_w^2 g_v^2 O^1_w O^2_w O^1_v O^2_v}{\omega^w_i \omega^v_f} \left( \frac{1}{\omega^w_i} + \frac{1}{\omega^v_f} \right) \right).
\]  

(17)

This correction, which contains twice the coupling \( g^2 \) appears as a second order boson exchange contribution and, for a unique spinless scalar boson, tends to have a repulsive character. For consistency, it should be considered with contributions of the same order (to be discussed in Sect. 4). The appearance of second order contributions in Eq. (13) may look surprising as the starting point given by the field-theory motivated interaction, Eq. (1), was a first order one. It originates from the off-energy shell behavior of this interaction, which, after elimination, transforms into a higher order correction. This currently occurs when changing the degrees of freedom used in describing some system to effective (or dressed) ones (see [28] for another example in relation with the NN interaction). Notice that \( V_0(r) \) and \( V_1(r) \) in Eq. (14), or factors \( O^1_w O^2_w \) and \( O^1_v O^2_v \) in Eq. (15), may not always commute. This respectively occurs for momentum dependent (non local) interactions or spin-isospin dependent ones. In the following, we will sometimes disregard these cases, essentially for simplifying the discussion and providing a more sensible presentation of the role of corrections associated with \( V_1(r) \).

The operator, \( V_1(r) \) \((V_1(r)/(1 + V_1(r))\) more exactly) has a simple interpretation. It represents the probability that the system be in a component where the two constituent particles are accompanied by a boson in flight. The interaction \( V(r) \) in Eq. (12) therefore corresponds to the interaction due to the component with two constituent particles only. Consistently with contributions of diagrams of Fig. 1 retained until now, it does not contain any interaction of these particles while a boson is in flight (see Fig. 2 for a graphical representation). As to the reference to dressed particles, it makes it clear that
higher order corrections should appear in the effective potential, as mentioned above. These corrections have some relationship to the renormalization presented in a more abstract way in \[19, 20, 24, 26, 22\]. Let’s also mention the consistency of: i) the above interpretation of \(V_1(r)\), ii) the general definition of the normalization for an energy dependent potential like that given by Eq. (10) and iii) the relation given by Eq. (11):

\[
N = \int d\vec{r} \psi^2(r) \left( 1 - \frac{\partial V_E}{\partial E} \right) = \int d\vec{r} \psi^2(r) (1 + V_1(r)) = \int d\vec{r} \phi^2(r),
\]

Not surprisingly, there is a relationship to the renormalization in the field of the many problem. The quantity, \((1 + V_1(r))^{-1}\), is quite similar to the spatially dependent spectroscopic factor, which is introduced in the domain of mean field approximations to account for the effect of degrees of freedom (nuclear correlations) that have been eliminated (see ref. \[27\] for a review). Its effect, or what accounts for it to be more correct, turned out to be essential to preserve the orthogonality of many body wave functions when incorporating contributions due to the same degrees of freedom \[28\].

At this point, it thus appears that the interaction \(V(r)\), which is obtained from a field-theory based single boson exchange interaction does not identify to the single boson exchange potential \(V_0(r)\) which, most often, is referred to and is used in the Schrödinger equation, or an equivalent one.

In the case of the strong \(NN\) interaction, the above problem may not be a serious source of concern. Its short range part has some phenomenological character and through the fit of the potential parameters to experimental data, it may be possible to account in some part for the correction to \(V_0(r)\) approximately given by \(\frac{1}{2}\{V_1(r), V_0(r)\}\), which also has a short range character. Only the longest range part of this term which corresponds to two pion exchange and has a complicated spin isospin structure, may not be well reproduced.

In the case of the Coulomb potential, where the fundamental interaction is known, there is no uncertainty similar to that affecting the short range part of the \(NN\) interaction. As contributions provided by the term, \(\frac{1}{2}\{V_1(r), V_0(r)\}\), are known to be absent from the energy spectrum of atomic bound states involving electrons or other charged particles, the derivation of the potential given in this section has to be completed in any case. This will be done in Sect. 4, while contributions will be estimated in the next section.

It is worthwhile to mention that removing the energy dependence of the potential \(V_E\) is
quite similar to the Foldy-Wouthuysen transformation for the Dirac equation. Like this one, it is not merely a mathematical operation. From a state which is a superposition of a two-body and a two-body plus one boson in flight components, which respectively play the role of the large and small components in the Dirac spinors, it allows one to build a state with a unique two-body (dressed) component. This one represents a frozen coherent superposition of the bare components, with the consequence that the constituent particles have now acquired in the transformation an effective character. The change in the nature of the constituents is better seen by working in Fock space [29] (see also the appendix for an example). Quite similarly to the Foldy-Wouthuysen transformation, the developments presented in the present section make sense around some given energy (states close to threshold in the present work). On the one hand, the choice of this energy, which plays the role of the renormalization point in the renormalization theory, could be optimized. On the other hand, by removing higher order energy dependent terms in the expansion given by Eq. (3), one can enlarge the energy domain at will, as far as one does not cross the inelasticity threshold where the frozen degrees of freedom may show up. While doing so however, the effective (or renormalized) interaction $V(r)$ in Eq. (13) acquires a non-local character whose role will steadily increases when approaching this threshold.

The contribution considered here may have been considered as a relativistic effect and, as such, as a new contribution to be accounted for. It implies a non-instantaneous propagation of the exchanged boson, but this feature is also pertinent to a two-body system that, under the effect of some interaction, undergoes a transition from a low momentum component to a high momentum one and back, or a two constituents system (NN for instance) that undergoes a transition to a component involving their internal excitation ($\Delta \Delta$) and back. These processes, which can give rise to an effective interaction, are typically part of a non-relativistic description of a two-body system. Contrary to current relativistic effects that are of the order $(v/c)^2$, they have a static character and don’t vanish when the mass of the constituents goes to infinity.

The current relativistic effects and the contribution of interest here can be related to different contributions in the expression which, for instance, allows one to calculate some amplitude from an interaction acting on an input amplitude:

$$I = \int dq \int dq_0 \frac{f(q_0, \ldots)}{((P_0 + q_0)^2 - m^2 - \ldots)((P_0 - q_0)^2 - m^2 - \ldots)} \frac{1}{\omega^2 - q_0^2}. \quad (19)$$

In this expression, where one recognizes the propagators of the constituent particles as well as the propagator of the exchanged boson, $q_0$ and $P_0$ represent the time component of the 4-momentum carried by the boson and the system under consideration. Contributions to the integral come from the poles in the integrand. The poles arising from the constituent propagators generally imply a small value of $q_0$, determined by the energy transfer of a constituent in the initial state to a constituent in the final state. They give rise to the above mentioned relativistic effects of the order $\omega^{-4}$. The pole due to the boson propagator implies a value of $q_0$ that by no means is small. It is responsible for the effect mainly considered here, of the order $\omega^{-3}$. While it is conceivable that the first effect disappears in the limit of an instantaneous propagation of the boson, which supposes that $q_0$ can be neglected in the boson propagator in Eq. (19), this is difficult to imagine for the second one. It is also noticed that the two contributions are on a quite different footing. The second one disappears when the boson propagator in Eq. (19) is replaced by the
propagator \((\mu^2 - q^2 + \frac{(Pq)^2}{2})\), which is equivalent for a physical process (Born amplitude). On the other hand, as it can be seen from the expression of the denominator in Eq. (3), the effect which is considered here is of the order \(\mu/m\), i.e. a recoil effect.

### 3 Estimates of the contributions due to the term \(\frac{1}{2}\{V_1, V_0\}\)

We present in this section results of perturbative calculations of two-body wave functions with and without including in the Schrödinger equation the contribution due to the term \(\frac{1}{2}\{V_1, V_0\}\). At the same time, we will discuss the relationship of the wave functions so obtained with those calculated from an energy dependent interaction. The aim is to show that a large part of the physics which is at the origin of this energy dependence can be accounted for in an energy independent scheme (where the underlying degrees of freedom have acquired an effective character). In practice, this will be done for the NN interaction case and the Wick-Cutkosky model.

The wave functions of interest are given in momentum space by:

\[
\tilde{\phi}_0(\vec{p}) = -\frac{m}{p^2 + \kappa^2} \int \frac{d\vec{p}'}{(2\pi)^3} V_0(\vec{p}, \vec{p}') \phi_0(\vec{p}'), \tag{20}
\]

\[
\tilde{\phi}(\vec{p}) = -\frac{m}{p^2 + \kappa^2} \int \frac{d\vec{p}'}{(2\pi)^3} V(\vec{p}, \vec{p}') \phi_0(\vec{p}'). \tag{21}
\]

where \(V(\vec{p}, \vec{p}')\) differs from \(V_0(\vec{p}, \vec{p}')\) by the correction given by Eq. (15). The quantity \(\kappa^2\) is related to the binding energy by the relation \(|E| = \frac{\kappa^2}{m}\). By comparing the results, we can estimate the effect of this correction. In principle, if \(\phi_0(\vec{p})\) is chosen as the solution of the Schrödinger equation with the potential \(V_0(\vec{p}, \vec{p}')\), \(\tilde{\phi}_0(\vec{p})\) is equal to \(\phi_0(\vec{p})\) (strictly speaking, this is not anymore a perturbative calculation but, rather, a consistency check). In some cases however, some terms in the original potential model that was used to calculate \(\phi_0(\vec{p})\), may be neglected. It is then more appropriate to look at the effect of the correction by comparing \(\tilde{\phi}(\vec{p})\) and \(\tilde{\phi}_0(\vec{p})\), rather than \(\phi(\vec{p})\) and \(\phi_0(\vec{p})\), hence the introduction of \(\tilde{\phi}_0(\vec{p})\). On the other hand, while a schematic description of effects under consideration here has been given in the previous section, in practice some of the calculations presented in the present section, with the Bonn models, have been performed with the full complexity of the interaction. This involves in particular the non-locality due to describing spin 1/2 particles by Dirac spinors and normalization factors appropriate to the definition of the potential \(V\) (together with the corresponding equation). For these cases, the numerical accuracy has been pushed to the point where one can reproduce the numerical deuteron wave function at a level where the discrepancy cannot be seen in the figures presented below.

Other quantities of interest correspond to perturbative solutions of the Schrödinger equation where the potential \(V_0(\vec{p}, \vec{p}')\) is replaced by the energy dependent potential given by Eq. (1):

\[
\tilde{\psi}(\vec{p}) = -\frac{m}{p^2 + \kappa^2} \int \frac{d\vec{p}'}{(2\pi)^3} V_E(\vec{p}, \vec{p}') \psi_0(\vec{p}'). \tag{22}
\]
Making an expansion up to the first order term in $V_1$, the above equation may also read:

$$\tilde{\psi}(\vec{p}) \simeq -\frac{m}{\vec{p}^2 + \kappa^2} \int \frac{d\vec{p}'}{(2\pi)^3} \left( V_0(\vec{p}, \vec{p}') + \frac{1}{2} \left( \frac{\vec{p}^2 + \kappa^2}{m} V_1(\vec{p}, \vec{p}') + \frac{\vec{p}'^2 + \kappa^2}{m} \right) \right) \psi_0(\vec{p}')$$

$$\simeq -\frac{m}{\vec{p}^2 + \kappa^2} \int \frac{d\vec{p}'}{(2\pi)^3} \left( V_0(\vec{p}, \vec{p}') + \frac{1}{2} V_1(\vec{p}, \vec{p}') \frac{\vec{p}'^2 + \kappa^2}{m} \right) \psi_0(\vec{p}')$$

$$- \frac{1}{2} \int \frac{d\vec{p}'}{(2\pi)^3} V_1(\vec{p}, \vec{p}') \psi_0(\vec{p}') \tag{23}$$

Equation (22) (together with Eq. (23)) differs from Eq. (21) by the fact that the first one involves the full energy dependent interaction $V_E$, while the second one involves the corresponding energy independent potential. It may be of interest to compare the outputs of these equations. It is remembered that $\phi_0(\vec{p})$ and $\psi_0(\vec{p})$ differ by an operatorial factor, $(1 + V_1)^{1/2}$, see Eq. (11). Two cases are to be considered:

i) The first one assumes that $\psi_0$ (and not $\phi_0$) is solution of the Schrödinger equation with the potential $V_0$. One thus gets:

$$\tilde{\psi} = \psi_0 + \frac{1}{2} \frac{m}{\vec{p}^2 + \kappa^2} \int \frac{d\vec{p}'}{(2\pi)^3} V_0 \psi_0 - \frac{1}{2} \int \frac{d\vec{p}'}{(2\pi)^3} V_1 \psi_0 \tag{24}$$

This is formally close to the choice made in refs. [10, 7].

ii) The second case assumes that $\phi_0(\vec{p})$ is solution of the Schrödinger equation with the potential $V_0$. According to Eq. (11), the zeroth order wave function to be inserted in Eq. (23) is related to the above one by the relation $\psi_0 = (1 + V_1)^{-1/2} \phi_0$. The wave function, $\tilde{\psi}(\vec{p})$, then reads:

$$\tilde{\psi}(\vec{p}) \simeq -\frac{m}{\vec{p}^2 + \kappa^2} \int \frac{d\vec{p}'}{(2\pi)^3} \left( V_0(\vec{p}, \vec{p}') - \frac{1}{2} \{V_0, V_1\} \right) \phi_0(\vec{p}')$$

$$- \frac{1}{2} \int \frac{d\vec{p}'}{(2\pi)^3} V_1(\vec{p}, \vec{p}') \psi_0(\vec{p}')$$

$$\simeq (1 - \frac{1}{2} V_1) \tilde{\phi}(\vec{p}) \tag{25}$$

This relation is in agreement with Eq. (11), which relates the wave functions of the energy dependent and energy independent schemes, $\psi \simeq (1 + V_1)^{-1/2} \phi \simeq (1 - \frac{1}{2} V_1) \phi$.

We now consider successively the Nucleon Nucleon interaction and the Wick Cutkosky (or QED) models.

### 3.1 Nucleon Nucleon case

We estimated corrections to the wave function arising from the term $\frac{1}{2} \{V_0, V_1\}$, starting from a zeroth order wave function given by the Paris model [30]. This was done perturbatively, using expressions given in the introduction of this section. In a first step,
was approximated by a sum of a few well known meson exchanges (π, ρ, ω, σ), whose contributions in the standard non-relativistic limit were retained. The various parameters were chosen so that to reproduce the deuteron wave function of the model at large distances and as accurately as possible at short distances. Results obtained in this way have shown that the deuteron S wave function was strongly reduced, more than obtained in refs. [10, 7]. The D wave was less changed, resulting in a relative enhancement of the D wave with respect to the S wave, qualitatively in agreement with the findings of these works. Due to the large effects, and to avoid to be misled by a hidden bias, we turned to a more complete calculation, with V₀ taken as identical to the potential used to calculate the deuteron wave function. In this order, we use meson exchange models which, contrary to the Paris model, allow one to easily derive the operator, V₁. The choice of the Bonn Q model will make possible a comparison with earlier work [10]. As this model provides a wrong prediction of the mixing angle, ∈₁, for energies beyond 100 MeV, we also use the Bonn QB model which provides a correct prediction with this respect and is generally considered as a quite reasonable model.

In Fig. 3, we first show the effect of introducing somewhat arbitrarily extra normalization factors \( \sqrt{E} \sqrt{E'} \) in the integrand of Eq. (20). Calculations are performed with the Bonn QB model for both the deuteron S and D waves. This can provide an estimate of the size of typical relativistic effects. The largest part, 20% in the range \( p = 1 \) GeV/c [11], is given by the factor, \( \sqrt{E} \), which can be factorized out in Eq. (20). As the presence or absence of such a factor is dictated by unitarity conditions and is accounted for in many nucleon nucleon interaction models, the effect may not be a real one however. Some effect is also due to the other factor, \( \sqrt{E'} \), whose relevance was mentioned in [14]. It gives rise to a decrease of the S wave of 4% at small momenta, which has the size of corrections generally expected from relativistic effects involving the internal structure, and of the order of 10-20% in the range \( p = 1 \) GeV/c. The effect of the factor, \((1 + V₁)^{-\frac{1}{2}}\), relating the wave functions in the energy dependent and energy independent schemes, is also shown in Fig. 3. Due to the difficulty in performing a full calculation, which has to do with the non-locality of the operator, we can only display the effect due to first and second order corrections in V₁. This however has the advantage of providing an estimate of some contributions neglected here. As expected, wave functions at small momenta are not modified. At intermediate momenta, some decrease is observed for both the S and D waves, which is compensated at the level of the norm by the contribution due to the meson in flight content of the order of 3-4%. The results roughly agree with those obtained in ref. [13], which were calculated exactly but with a localized form of V₁, or with that obtained for the full Bonn potential [12]. Altogether, results presented in Fig. 3 show that neither relativistic effects, nor those dealing with the relation of the wave functions of the energy dependent and energy independent schemes can explain a large change of the deuteron S and D waves at small momenta (and equivalently at large distances).

In Figs. 4 and 5, we show the results of a perturbative calculation tending to estimate the effect of corrections to the potential due to the extra term, \( \frac{1}{2}\{V₀, V₁\} \) (dotted line). In this order, we insert as inputs in Eq. (21) the wave functions and potentials corresponding to the Bonn-Q and Bonn-QB NN interaction models. The resulting wave functions for these two models and for both S and D waves are drawn in Figs. 4 and 5, together with the unperturbed ones. It is observed that the global structure of the wave functions remains
Figure 3: Sensitivity to relativistic kinematics and underlying schemes (dependence or independence of the total energy). The input wave function (continuous line) and the potential are those of the Bonn-QB model. The S and D components are presented respectively in the top and bottom parts of the figure. To better emphasize the small momentum behavior, we provide in the insets the ratio of the various calculations to the exact ones.

especially unchanged for the Bonn-Q model and the S wave of the Bonn-QB model. The position of the minima is only slightly shifted. Quantitatively, the most important feature concerns the decrease of the S wave at low momenta. This one is largely due to a decrease of the contribution produced by the tensor force acting on the deuteron D wave. The effect, which is better seen in the insets of Figs. 4 and 5, is given by a factor 0.73 for the Bonn-Q model and 0.63 for the Bonn-QB model. It shows that the wave function accounting for the renormalization of the interaction by the implicit meson in flight probability, see Eq. (13), strongly differs from the original one in a momentum domain, $p \simeq \kappa$ ($= 0.232 \text{fm}^{-1}$), which is essential for the deuteron description. In all presented calculations, the D wave at small momenta is much less affected, resulting in an enhancement of this one when the complete perturbed wave function, Eq. (21), is
Figure 4: Effect of the $\frac{1}{2}\{V_1,V_0\}$ term, Eq. (14), on the calculation of the S component of the deuteron wave function. The input wave function (continuous line) and the potential are those of the Bonn-Q and Bonn-QB models (respectively upper and lower parts of the figure). The dotted and dash-dotted lines correspond to outputs of the energy independent and energy dependent schemes respectively. The low momentum results are shown in the insets as a ratio to the exact results.

normalized to unity. The comparison of the results obtained with the two Bonn models does not show significant qualitative differences at low momenta, despite the fact that the model Bonn-Q does badly for observables in relation with the tensor force. This rough agreement indicates the degree of reliability one can give to the estimate of the effects under consideration in the present work.

The comparison of present results with those of refs. [10, 7] evidences some difference: their suppression of the S wave, roughly a factor 0.8, is not as large as here. Furthermore, the “relativistic” wave functions in the first reference, $f_1$ and $f_2$, significantly differ from their non relativistic limits respectively given by the usual S and D waves: a new zero
appears in $f_1$ while none is seen in $f_2$ in the range extending to $p = 1.5 \text{ GeV/c} (\approx 7.5 \text{ fm}^{-1})$. To identify the origin of these differences, which may have many sources and among them genuine relativistic effects not accounted for here, we did a calculation whose spirit is closer to theirs. Using Eq. (22), we took the energy dependent potential given by Eq. (1) together with the unperturbed wave function given by the Bonn-Q model. Calculations corresponding to an expansion of $V_E$ up to the first order term in $V_1$ have also been performed. They may allow one to test the validity of this approximation, which was made in deriving the potential appropriate to an energy independent picture, Eq. (13). Results are presented in Fig. 6 for both the S and D waves, together with the unperturbed wave functions. As for calculations presented in Figs. 4 and 5 for the same Bonn model, we do not observe major changes in the global structure of the wave function. The position of the zeros is only shifted a little more, as is the position of the same zeros in Fig. 3 for the Bonn-QB model. Quantitatively, the S wave at low momenta is also reduced, by a factor 0.89 for the full operator $V_E$ and a factor 0.85 for its expanded form limited to the first order term in $V_1$. The decrease compares to that obtained in refs. [10, 7], but is...
Figure 6: Effect of the energy dependent part of the potential, Eq. (1), on the calculation of the S and D components of the deuteron wave function with various approximations: full kernel (dotted line, $V_E$) and limited expansion up to the first order in $V_1$ (dash-dotted line, L.E.). The input wave function (continuous line) and the potential are those of the Bonn-Q model. The S and D components are presented respectively in the upper and lower parts of the figure. The low momentum part is shown in the insets as a ratio to the exact results.

slightly smaller. We checked that this can be ascribed for a large part to the kernel of the interaction in the underlying light-front approach, which has some similarity with ours, Eq. (1), but also some difference. In any case, the decrease is significantly less than the one obtained previously in the energy dependent scheme (0.73), suggesting a bias in one of the approaches whose results are compared.

The difference in the reduction can be traced back to the extra contribution due to the term $\frac{1}{2} V_0 V_1$, which is present in Eq. (21), but absent in Eq. (24). This term is a priori different from the other one, $\frac{1}{2} V_1 V_0$, which is present in both equations. They become equal to each other in the approximation where $V_0$ and $V_1$ are local and commute. This
can explain the rough factor 2 difference between corrections calculated from Eqs. (21) and (24). As explained in the introduction to this chapter, it would be more appropriate in a calculation of the wave function for the energy dependent scheme to use the function, \( \psi_0 \simeq (1 + V_1)^{-1/2} \phi_0 \), as an unperturbed one. When this is done, the above discrepancy disappears, the remaining difference being provided by the operator, \( (1+V_1)^{-1} \simeq 1 - \frac{1}{2} V_1 \). Results calculated in this way at the first order in \( V_1 \) are presented in Figs. 4 and 5 (dash-dotted line). As expected, the wave functions calculated perturbatively in both schemes coincide at small momenta. The differences at higher momenta are similar to those already mentioned in relation with the discussion of results presented in Fig. 3 (effects of the factor \( (1 + V_1)^{-1/2} \simeq 1 - \frac{1}{2} V_1 \)).

None of our results for the Bonn-Q model presents a qualitative change of the wave function similar to that obtained in ref. [10], namely an appearance and a disappearance of a zero in the S and D waves respectively (below \( p = 1.5 \text{ GeV}/c \simeq 7.5 \text{ fm}^{-1} \)). As mentioned by the authors, this feature has been considered as a genuine feature of the relativistic wave functions in their approach. Surprised by the fact that we could reproduce other features of their calculation, especially the decrease of the S wave at low momenta, translating into a relative increase of the D wave after normalization, we performed further checks. We found that the parametrized wave function given in ref. [12], whose behavior at high momenta differs from what is theoretically expected, was not accurate enough to allow the authors to make a so strong statement. The wave functions that come out from a consistency check, Eq. (20), differ from the input by an amount which, in the range \( p \geq 0.8 \text{ GeV}/c \simeq 4.0 \text{ fm}^{-1} \), is of the same order as the effect they were interested in. Furthermore, the potential effectively used differs from the original Bonn-Q model in the treatment of the \( \rho \) meson nucleon tensor coupling. This involves a term containing the time component of the 4-momentum carried by the meson, \( q_0 \). We re-calculated what should be the reference wave function for their study, using the parametrized wave function on the one hand and their \( \rho \) meson nucleon coupling. Results, which are presented in Fig. 7 (dash-dotted line), are qualitatively quite close to their so called relativistic components, \( f_1 \) and \( f_2 \). The extra zero in the former and the absence of a zero in the latter, in the range \( p \leq 1.5 \text{ GeV}/c \simeq 7.5 \text{ fm}^{-1} \), thus appear as the consequence of approximations in the calculations. Other details relative to the role of various ingredients are shown in the figure.

3.2 Wick-Cutkosky model, QED

The exchange of zero mass bosons, which applies to the Wick-Cutkosky model or to QED, especially the Coulomb potential, deserves some attention. Indeed, the quantity, \( V_1(r) \), defined by Eq. (9) is then scale independent and, being proportional to \( \int dx (\sin x) / x^2 \), is logarithmically divergent, requiring to come back on the definition of \( V_1(r) \).

Two questions are of interest here: the presence of \( \alpha \log \frac{1}{\alpha} \) corrections to the binding energy [3] and some correction of the order \( p/m \) to the wave function [13] which, both, have been attributed to a relativistic approach.

To determine \( V_1(r) \), we started from an \( 1/\omega_k \) expansion of the propagator \( (E - \omega_k - \)
Figure 7: Consistency check and sensitivity to the deuteron input wave function for the Bonn-Q model (upper and lower parts for the S and D waves respectively). The continuous and dotted lines respectively represent the exact (numerical) and parametrized wave functions (denoted exact and param. in the figure). The wave function recalculated using the exact wave function cannot be distinguished from the original one while that one using the parametrized wave function (param.-recalcalc.), represented by the short-dashed line, evidences some discrepancy. The same wave functions recalculated using the ρ meson nucleon tensor coupling employed in ref. [10] are respectively represented by the long-dash and dash-dotted lines (denoted exact recalcalc.(with q0) and param.-recalcalc.(with q0)).

\[ \frac{\vec{p}_i^2 - \vec{p}_f^2}{2m} \left( \frac{1}{\omega_k} \right)^2, \] Eq. (3), with the idea to only retain the first order term in \( (1/\omega_k)^2 \). The presence of a singularity in the higher order terms when the boson mass is zero, which implies that \( \omega_k \to 0 \) for \( k \to 0 \), supposes to proceed differently. The difficulty can be overcome by removing from the full propagator, Eq. (3), that part which gives rise to the
usual zero mass single boson exchange contribution to the two-body interaction:

\[
\frac{1}{E - \omega_k - \frac{p_i^2}{2m} - \frac{p_f^2}{2m}} = - \frac{1}{\omega_k} \frac{E - \frac{p_i^2}{2m} - \frac{p_f^2}{2m}}{\omega_k(E + \frac{p_i^2}{2m} + \frac{p_f^2}{2m})}.
\]

(26)

Now, the extra quantity at the denominator of the second term in the r.h.s. of the above equation, \(-E + \frac{p_i^2}{2m} + \frac{p_f^2}{2m}\), can be bounded from below by the absolute value of the binding energy of the system under consideration, \(\frac{\kappa^2}{m}\), where \(\kappa = \sqrt{\frac{m}{|E_{b.e.}|}}\). This, which corresponds to summing up an infinite set of terms in Eq. (3), makes the second term in the above equation less singular than in the former, allowing one to deal with it rather safely. Being positive, the extra terms, \(\frac{p_i^2}{2m}\), don’t introduce poles and, on the other hand, their size is expected to be the same as that of the term \(\frac{\kappa^2}{m}\), which we will account for by multiplying this one by a factor 2. After making these approximations, the developments from Eq. (3) to Eq. (9) can be performed again with the result:

\[
V_1(r) = \frac{2}{\pi} \alpha \int \frac{dk}{\omega_k^2(\omega_k + 2\frac{\kappa^2}{m})} j_0(kr).
\]

(27)

which is seen to be a function of the dimensionless variable \(2\frac{\kappa^2}{m}r\). The coupling has been particularized to the case of a spinless particle (the time component for the photon) and the coupling \(g^2\) has been expressed in terms of the coupling, \(\alpha\), often used in QED, \(g^2 = 4\pi\alpha\). In terms of this coupling, the quantity \(\kappa\) for the ground state of the system of two equal mass particles is given by \(\kappa = \frac{m\alpha}{\kappa^2}\).

The small distance behavior of \(V_1(r)\) is dominated by the logarithmic divergence of the integral in Eq. (27):

\[
V_1(r)_{r \to 0} = \frac{2}{\pi} \alpha \left(1 - \gamma + \log(\frac{m}{2\kappa^2 r})\right) + O(\frac{\kappa^2 r}{m}).
\]

(28)

where \(\gamma\) represents the Euler constant, \(\gamma = 0.577\ldots\). At very large distances \((r >> \frac{m}{\kappa^2})\), \(V_1(r)\) behaves like:

\[
V_1(r)_{r \to \infty} = \alpha \frac{m}{2\kappa^2 r}.
\]

(29)

A useful and approximate interpolating expression may be given by:

\[
V_1(r) \simeq \frac{2}{\pi} \alpha \log \left(1 + \frac{m}{2\kappa^2 r} e^{1-\gamma}\right).
\]

(30)

For small \(r\), it is exact and for \(r \to \infty\), it provides a slight underestimate by a factor \(e^{1-\gamma}/(\pi/2) = 0.97\ldots\).

Solving the Schrödinger equation with the potential \(V(r)\) together with \(V_1(r)\) given by Eq. (28) may be done approximately by noticing that \(V_1(r)\) is a smoothly varying function in the range \(r \simeq \frac{1}{\kappa} >> \frac{1}{m}\), corresponding to \(\alpha << 1\). It can then be approximated by a constant number:

\[
V_1(r)_{r \approx \kappa^{-1}} = \frac{2}{\pi} \alpha \log \frac{1}{\alpha}.
\]

(31)
With the above approximation, the correction to the potential given by \( \frac{1}{2} \{ V_1(r), V_0(r) \} \) can be incorporated in the Coulomb potential, \( V_0(r) = -\frac{e}{r} \), whose coupling is renormalized as follows:

\[
\alpha \to \alpha \left( 1 - \frac{2}{\pi} \alpha \log \frac{1}{\alpha} \right). \tag{32}
\]

The correction to the binding energy at the lowest order immediately follows:

\[
E_{b.e.} = -m \frac{\alpha^2}{2n^2} \to E_{b.e.} = -m \frac{\alpha^2}{2n^2} \left( 1 - \frac{4}{\pi} \alpha \log \frac{1}{\alpha} \right). \tag{33}
\]

In all the previous discussion, we have been mainly concerned with the largest correction, of the order \( \alpha \log \frac{1}{\alpha} \). We have not paid much attention to the precise value of some factors. Quantities such as \( \frac{\vec{p}^2}{2m} \) or \( r \) have been replaced by approximate values in getting Eqs. (31, 32), which may affect the coefficient of \( \frac{1}{\alpha} \) in the log term. This can be seen to correspond to a correction to the binding energy of the relative order \( \alpha \) in Eq. (33), but does not change the correction of the order \( \alpha \log \frac{1}{\alpha} \), which dominates in the limit \( \alpha \to 0 \).

Going beyond requires some care, but one can anticipate that the replacement of the factor \( 2\kappa^2m \) appearing in Eqs. (27-30) by the potential \( \alpha/r \), which, acting on the wave function, provides the potential \( \alpha/r, V_1(r) \) becomes a constant. Expressions with increasing accuracy would be successively given by:

\[
V_1(r) \simeq \frac{2}{\pi} \alpha \log \frac{1}{\alpha} \text{ [up to 20% for } \alpha \leq 0.1],
\]

\[
\simeq \frac{2}{\pi} \alpha (1 - \gamma + \log \frac{1}{\alpha}) \text{ [up to 25% for } \alpha \leq 0.5],
\]

\[
\simeq \frac{2}{\pi} \alpha \log \left( 1 + \frac{e^{1-\gamma}}{\alpha} \right) \text{ [up to 4% } \forall \alpha]. \tag{34}
\]

The first expression is identical to Eq. (31), which was obtained on a slightly different basis. The second and third ones could be used to improve Eqs. (32) and (33), when the first one is out of its range of validity (limited to \( \alpha \leq 0.1 \)). A further improvement consists in expressing the quantity, \( 2\kappa^2m \), in terms of the renormalized potential, \( \alpha_{eff}/r \).

The effective coupling, \( \alpha_{eff} \), which should replace the factor \( \alpha \) in the log term appearing in Eq. (34), is determined by the equation:

\[
\alpha_{eff} = \frac{\alpha}{1 + \alpha J} = \alpha (1 - \alpha_{eff} J), \tag{35}
\]

where \( J = \frac{2}{\pi} \log \frac{1}{\alpha_{eff}}, \frac{2}{\pi} (1 - \gamma + \log \frac{1}{\alpha_{eff}}), \frac{2}{\pi} \log(1 + \frac{e^{1-\gamma}}{\alpha_{eff}}) \) or a better expression depending on the desired accuracy.

The correction to the binding energy in Eq. (33) is identical to that found in [3]. This result, which has been obtained here in a picture that is not especially a relativistic one, tends to show that the correction has essentially its source in the field theory which underlies the present approach as well as that followed by Feldman et al. [3] and many other authors [13, 4]. Results for the binding energy of the lowest states calculated with the renormalized interaction have also been obtained [33]. They explain the bulk of the
departures from the instantaneous approximation results derived either from the Bethe-Salpeter equation [32] or from energy dependent approaches [33, 34, 9].

It is also interesting to consider various wave functions corresponding to different approximations or different schemes. In comparison with the NN interaction case discussed previously, one can go a step further here by using the renormalized interaction which, in a first approximation, differs from the bare one by changing the coupling \( \alpha \) into \( \alpha_{\text{eff}} \), Eq. (35). In Fig. 8, we present wave functions pertinent to the lowest \( s \)-state. These are the usual non-relativistic wave function:

\[
\phi_0(\vec{p}) = \sqrt{4\pi \frac{4 \kappa_0^2}{(\kappa_0^2 + p^2)^2}} \quad \text{with} \quad \kappa_0 = \frac{1}{2} m \alpha, \quad (36)
\]

the non-relativistic wave function with the renormalized interaction:

\[
\phi(\vec{p}) = \sqrt{4\pi \frac{4 \kappa^2}{(\kappa^2 + p^2)^2}} \quad \text{with} \quad \kappa = \frac{1}{2} m \alpha_{\text{eff}}, \quad (37)
\]

the wave function corresponding to the energy dependent scheme obtained by using Eq. (11), \( \psi_0 = (1 + V_1)^{-\frac{1}{2}} \phi_0 \):

\[
\psi_0(\vec{p}) = \sqrt{\frac{\alpha_{\text{eff}}}{\alpha}} \phi(\vec{p}), \quad (38)
\]

and the same wave function calculated perturbatively using Eq. (22), of which analytic expression is given by:

\[
\tilde{\psi}(\vec{p}) = \sqrt{4\pi \frac{4 \kappa^2}{(\kappa^2 + p^2)^2}} \sqrt{\frac{\alpha_{\text{eff}}}{\alpha}} \times \alpha_{\text{eff}} \alpha \left( 1 - \frac{5 \kappa^2}{4 m^2} - \frac{p^2}{4 m^2} + \frac{2}{\pi p} \frac{(3 \kappa^2 - p^2)^2 + 3 \kappa^2 + p^2}{2 m^2} \right) \arctg \left( \frac{\kappa}{p} \right) \left( 1 - \frac{p}{m} + \frac{\kappa^2 + p^2}{4 m^2} \right) \left( 1 + \frac{p}{m} + \frac{\kappa^2 + p^2}{4 m^2} \right) \log 2 + \frac{\kappa}{\pi m} \left( \frac{2 - \kappa^2 + p^2}{4 m^2} \right) \log 2 + \frac{\kappa}{\pi p} \left[ \sqrt{1 - \frac{2 \kappa^2 + p^2}{m^2}} \log \left( \frac{1 - \frac{p}{m} + \sqrt{1 - \frac{2 \kappa^2 + p^2}{m^2}}}{1 + \frac{p}{m} + \frac{\kappa^2 + p^2}{4 m^2}} \right) \right] \right), \quad (39)
\]

where the square root (and log) functions should be appropriately changed when their argument gets negative (or imaginary).

In writing Eq. (39), we have emphasized at the first line factors which together form the unperturbed wave function, \( \psi_0(\vec{p}) \), see Eqs. (37) and (38). The factor that accounts for the improvement obtained by using the complete kernel of the interaction, Eq. (1), is given at the following lines. For the numerical study, whose results are presented in Fig. 8, the value of \( \alpha \) has been taken to be 0.5, which corresponds to \( \alpha_{\text{eff}} = 0.32 \).

Not surprisingly, \( \phi_0(\vec{p}) \) and \( \phi(\vec{p}) \) differ significantly, showing the indirect importance of the energy dependence of the interaction, \( V_E \), which was accounted for by employing a
renormalized interaction in the energy independent scheme. The wave function appropriate to the energy dependent scheme, \( \tilde{\psi}(\vec{p}) \), is quite close to the corresponding unperturbed one, \( \psi_0(\vec{p}) \). The discrepancy, which can hardly be seen in the figure, is less than 1% in the domain of momenta covering 95% of its contribution to the normalization. This indicates that the way \( \alpha_{\text{eff}} \) has been derived is essentially correct. These wave functions strongly differ from that one obtained with the bare potential in the energy independent scheme, \( \phi_0(\vec{p}) \). A similar result was obtained in the light front approach in refs. [8, 9] and most probably with the trial wave function employed in ref. [34]. The point is of relevance for works that have some relationship to the present one [15], where only an effect in the high momentum range was emphasized. This second effect, which implies corrections of the order \( p/m \), is also present here (see below). It certainly explains some discrepancy between \( \tilde{\psi}(\vec{p}) \) and its unperturbed counterpart \( \psi_0(\vec{p}) \), but the effect shows up at high momenta, outside the range shown in Fig. 8 (4% at \( \frac{p}{m} = 0.5 \), 10% at \( \frac{p}{m} = 0.8 \) and 20% at \( \frac{p}{m} = 1.3 \)). We also looked at the contribution that the operator \( V_1 \) provides for the normalization, see Eq. (18). The operator has been chosen according to calculations presented in this section. It is given by Eq. (27) together with the full propagator appearing in Eq. (26) (instead of \( -\frac{\partial V}{\partial E} \), which it is equal to in some limit). Discarding any theoretical improvement, the contribution was found to be close to what was approximately expected, \( 1 - \frac{\alpha_{\text{eff}}}{\alpha} \), (0.345 instead of 0.360). This indicates, in a particular case, that the derivation of the effective interaction, \( V(r) \), Eq. (13), its solution, \( \phi(r) \), Eq. (12), the relation of this solution with that for the energy dependent scheme, \( \psi(r) \), Eq. (11), and the normalization condition, Eq. (18) are essentially consistent with each
other. The remaining discrepancies, which are at the level of a few % in comparison with the major effect of 30-40% related to the renormalization of the interaction, are ascribed to the approximations made in dealing with the denominator of the second term in the r.h.s. of Eq. (26).

We now consider more closely how the wave function of the energy dependent scheme given by Eqs. (1, 10) relates to the energy independent one given by Eqs. (12, 14) in the high momentum range. This relation in the configuration space,

\[ \psi(\vec{r}) = (1 + V_1(r))^{-1/2} \phi(\vec{r}) \simeq \left(1 - \frac{1}{2} V_1(r)\right) \phi(\vec{r}), \quad (40) \]

reads in momentum space:

\[ \psi(\vec{p}) \simeq \phi(\vec{p}) - \frac{1}{2} \int \frac{d\vec{p}'}{(2\pi)^3} \frac{1}{(\vec{p} - \vec{p}')^2 + \mu^2} \phi(\vec{p}'), \quad (41) \]

The high momentum regime of this equation is obviously that one where relativistic effects are expected to show up.

Neglecting a possible renormalization of the interaction as given by Eq. (13), or assuming it can be accounted for effectively using Eq. (34), the solution of the Schrödinger equation for a Coulomb or Yukawa type interaction, \( \phi(\vec{p}) \), obeys the equation:

\[ \phi(\vec{p}) = \frac{4\pi \alpha m}{p^2 + \kappa^2} \int \frac{d\vec{p}'}{(2\pi)^3} \frac{1}{(\vec{p} - \vec{p}')^2 + \mu^2} \phi(\vec{p}'), \quad (42) \]

while \( V_1(\vec{p}, \vec{p}') \) is given by:

\[ V_1(\vec{p}, \vec{p}') = \frac{4\pi \alpha}{((\vec{p} - \vec{p}')^2 + \mu^2)^{3/2}}. \quad (43) \]

In the limit of large \( p \), Eq. (42) indicates that \( \phi(\vec{p}) \) behaves like:

\[ \phi(\vec{p})_{p \to \infty} = \frac{4\pi \alpha m}{p^2} \int \frac{d\vec{p}'}{(2\pi)^3} \phi(\vec{p}'), \quad (44) \]

where, up to some factor, the integral represents the configuration space wave function at the origin. The second term in Eq. (41) can be similarly calculated, using the expression (43) of \( V_1(\vec{p}, \vec{p}') \):

\[ \delta \phi(\vec{p})_{p \to \infty} = -\frac{1}{2} \frac{4\pi \alpha}{p^2} \int \frac{d\vec{p}'}{(2\pi)^3} \phi(\vec{p}'). \quad (45) \]

Gathering results of Eqs. (44) and (45), one obtains:

\[ \psi(\vec{p})_{p \to \infty} \simeq \left(1 - \frac{p}{2m}\right) \phi(\vec{p})_{p \to \infty}. \quad (46) \]

This result supposes that \( V_1 \) is small enough so that the first order expansion given in Eq. (40) is valid. It is actually limited to the range \( \kappa < p < m \) and neglects higher order effects in the coupling, \( \alpha \). A result which is better with both respects, but limited to a zero mass boson, is given by Eq. (39) (see the second line).
The above effect may be compared to that one attributed to relativity in [15]. In this work, the light front wave function was found to be expressed in terms of the non-relativistic one as:

\[
\psi(\vec{p})_{l.f.} = \frac{1}{1 + \frac{\vec{p} \cdot \vec{n}}{E_p}} \phi(\vec{p})_{n.r.},
\]

(47)

where \( \vec{n} \) represents the orientation of the light front. In the non-relativistic limit, where \( \frac{\vec{n}}{m} < 1 \), and after making an average over the orientation of \( \vec{n} \), the above result reads:

\[
\psi(\vec{p})_{l.f.} \simeq \frac{1}{2m} \phi(\vec{p})_{n.r.},
\]

(48)

which is identical to what was obtained above by an approach of which full relativity is absent. This indicates that the correction of the order \( \frac{\vec{n}}{m} \) is not characteristic of a relativistic approach, but rather of the field theory which underlies Eqs. (1, 10, 11) on the one hand and (46) on the other. Relativity in this last equation would imply higher order terms in \( \frac{\vec{n}}{m} \), for instance in the replacement of \( m \) by \( E_p \) or in the non-isotropic dependence of \( \psi(\vec{p})_{l.f.} \) with respect to the orientation of \( \vec{n} \). In both cases, they will manifest by effects of the order \( \frac{\vec{n}^2}{m^2} \), as usually expected. Notice that the above \( \frac{\vec{n}}{m} \) correction has for some part an elusive character. In the low momentum domain, it disappears from a correct definition of the norm, see Eq. (18), in the same way that renormalization effects do not show up in the calculation of various observables at the lowest order in \( 1/\omega^3 \) [29]. It may nevertheless show up in the ultra relativistic regime where \( \frac{\vec{n}}{m} \gg 1 \) [35], but, most probably, this is a feature pertinent to the energy dependent scheme.

4 Extra contributions due to two boson exchanges

We consider in this section the role of genuine two boson exchanges with respect to the correction brought about by the term in Eq. (14). We derive first the general expression for these contributions and subsequently look at particular cases depending on whether bosons are neutral, charged or massless.

4.1 General case

The contributions, which have the same order as the correction given by Eq. (15), and are due to two boson exchanges, are calculated using standard perturbation theory. Only those which have some relevance for our purpose are retained. The corresponding time-ordered diagrams are shown in Figs. 9, 10, 11. They involve a common normalization factor, \( 1/(\omega_i \omega_f) \), a common factor corresponding to two boson propagation, \( 1/(\omega_i + \omega_f) \), and two factors corresponding to single boson propagators, \( 1/(\omega_i) \) or/and \( 1/(\omega_f) \).

The first ones, Fig. 9, have a box type character, and have a topological structure similar to that implied by Eqs. (15, 17):

\[
\Delta V_l(\vec{p}_i, \vec{p}_f) = -\frac{1}{2} \int \frac{d\vec{p}}{(2\pi)^3} \sum_{v,w} \left( \frac{g_v^2 g_w^2}{\omega_i^v \omega_f^w} O_{v}^{1} O_{w}^{1} O_{v}^{2} O_{w}^{2} \frac{1}{\omega_i^v \omega_f^w} \right).
\]

(49)
The second ones, Fig. 10, have a crossed type character. They involve the two time-ordered diagrams corresponding to Fig. 1, while another boson is exchanged. They provide most of the contribution to the interaction of the two constituent particles while a boson is in-flight, Fig. 2. Their contribution is given by:

\[
\Delta V_{II}(\vec{p}_i, \vec{p}_f) = -\frac{1}{2} \int \frac{d\vec{p}}{(2\pi)^3} \sum_{v,w} \left( \frac{g_v^2 g_w^2 O_v^1 O_w^1 O_v^2 O_w^2}{\omega_i^v \omega_f^v (\omega_i^w + \omega_f^w)} \left( \frac{1}{\omega_i^w} + \frac{1}{\omega_f^w} \right) \right).
\]

(50)

The third ones, Fig. 11, have also a crossed type character. Except for a different spin-isospin structure, their contribution is identical to that shown in Fig. 9 and is given by:

\[
\Delta V_{III}(\vec{p}_i, \vec{p}_f) = -\frac{1}{2} \int \frac{d\vec{p}}{(2\pi)^3} \sum_{v,w} \left( \frac{g_v^2 g_w^2 O_v^1 O_w^1 O_v^2 O_w^2}{\omega_i^v \omega_f^v (\omega_i^w + \omega_f^w)} \left( \frac{1}{\omega_i^w} + \frac{1}{\omega_f^w} \right) \right).
\]

(51)

Notice that the ordering of \(O_v^1, O_v^2, O_w^1\) and \(O_w^2\) in Eqs. (50, 51) is different from that in Eqs. (15, 49).

In deriving Eqs. (49-51), we neglected contributions involving the energies of the constituents, consistently with our intent to only consider second order effects in boson exchanges. They may obviously be accounted for, which supposes to extend the analysis we developed for one boson exchange, Eq. (1), to two boson exchanges.

On the other hand, we did not consider the excitations of a constituent and its antiparticle (negative energy state). These ones, which imply Z-type diagrams (see Fig. 12), should be in any case accounted for, but having a different mathematical structure (the energy of the excited pair at the denominator and different vertex functions), they don’t interfere with the discussion presented here. In the case of infinitely massive constituent particles and exchanged bosons coupling in a scalar way to them, their contribution should vanish.

The momenta in Eqs. (49-51) refer to those of particle 1, thus introducing an apparent asymmetry between particles 1 and 2. It can be verified that for the first contribution, \(\Delta V_I\), the reference to particle 2 would be accounted for by the change of sign of the momenta, \(\vec{p}_i, \vec{p}_f\) and \(\vec{p}\), which does not provide any change of these equations. For the contributions produced by the crossed diagrams, \(\Delta V_{II}\) and \(\Delta V_{III}\), Figs. 10, 11, the
Figure 10: Selected contributions to the two-body interaction due to two boson exchange (crossed-type diagrams). Apart from a common factor $1/(\omega_i\omega_f(\omega_i + \omega_f))$, they contain a factor $1/\omega_i^2$ (a, b) or $1/\omega_f^2$ (c, d). These provide the dominant contribution to diagrams displayed in Fig. 2.

argument is not so straightforward, due to the fact that the momentum of the second particle in the intermediate state is not $-\vec{p}$, but $\vec{p} - \vec{p}_i - \vec{p}_f$ (in the c.m. system). The symmetry is recovered by making a change of variable.

4.2 Neutral and spinless boson case

The results for the case of the exchange of a neutral boson coupling in a scalar way and non-relativistically to constituents stem from Eqs. (49-51). This applies especially to the $\sigma$ and $\omega$ mesons coupling to nucleons and to the photon coupling to charged particles (Coulomb part). In both cases, their contribution provides a large, if not the dominant part of the interaction. Taking into account that the vertex functions, $O_v$ and $O_w$, commute, it is easy to sum up the different contributions, $\Delta V_I$, $\Delta V_{II}$ and $\Delta V_{III}$, with the result that the common factor, $(\omega_i^w + \omega_f^v)^{-1}$, cancels out:

$$\Delta V_I + \Delta V_{II} + \Delta V_{III} = -\frac{1}{2} \int \frac{d\vec{p}}{(2\pi)^3} \sum_{w,v} \left( \frac{g_w^2 g_v^2 O_w^1 O_w^2 O_v^1 O_v^2}{\omega_i^w \omega_f^v} \left( \frac{1}{\omega_i^w} + \frac{1}{\omega_f^v} \right) \right). \quad (52)$$

Remembering that, in the approximation under consideration, the vertex functions, $O$, reduce to unity, this total contribution has an attractive character. It just cancels the contribution which arises from removing the energy dependence from Eqs. (1, 4, 10) and corresponds to the renormalized potential $V$, and the associated wave function, $\phi$. 

Figure 11: Selected contributions to the two-body interaction due to two boson exchange (crossed-type diagrams). Apart from a common factor $1/(\omega_i \omega_f (\omega_i + \omega_f))$, they involve a factor $1/\omega_i \omega_f$ (a,b).

Figure 12: Example of time-ordered two boson exchange contributions with negative energies for the constituents (Z-type diagrams). These contributions, which should be incorporated in a full description of the two-body interaction, don’t interfere with those considered in this work.

This result is important in the sense that it indicates that usual single boson exchange potentials, that are used in the Schrödinger equation and are described by Yukawa potentials, already account, in a hidden way, for part of the two boson exchange contribution, including crossed diagrams. By the same token, they also account for the renormalization and recoil effects related with the energy dependence of the genuine interaction, $V_E$, but this may be true only at the lowest order in $1/\omega$.

Neglecting some non-locality, a physical interpretation of the above contribution to the two-body interaction may be obtained by going back to its expression in configuration space:

$$\Delta V_I + \Delta V_{II} + \Delta V_{III} = V_1(r) V_0(r),$$  \hspace{1cm} (53)

where $V_0(r)$ and $V_1(r)$ have been defined previously (Eqs. 6, 9). It corresponds to the interaction between the two constituents, $V_0(r)$, multiplied by the boson in flight probability, which is given by $V_1(r)$. It is not a surprise therefore if it cancels the normalization correction to the interaction potential, $\frac{i}{2} \{V_1(r), V_0(r)\}$, which was corresponding to missing this contribution. Obviously, as already mentioned, the cancellation holds provided that the two constituent particles in the intermediate state with a boson in flight have the same spin, isospin or angular momentum as the initial or final states. A little bit of
caution is nevertheless in order in identifying Eq. (53) with the contribution of Fig. 2, which at first sight could be written as the interaction of the two constituent particles, $V_0(r)$, multiplied by the probability to have a boson in flight. The argument misses the point that, in order to get $V_0(r)$ in Eq. (53), all relative times corresponding to an emission of a boson by one constituent and its absorption by the other have to be considered. This cannot be achieved by crossed diagrams alone as the time at which the absorption takes place (in Fig. 10a for instance) is necessarily bounded by the time at which the first emitted boson is absorbed. This is the place where the contribution of the box diagram (Fig. 9a) is relevant as it precisely provides the missing time ordered contribution. In view of this, one can imagine that the cancellation of the norm correction to the potential together with that one involving the interaction of the two constituents while one boson is in flight (part of Fig. 2) can be generalized to an undetermined number of bosons in flight, within the conditions under which Eq. (53) has been derived (locality of the interaction, spin- and isospin-less particles, no Z diagram). A schematic proof is given in the appendix.

Figure 13: Time ordered diagrams showing the relationship of two boson exchange contributions considered here (bottom part) with recoil-norm contributions associated to the interaction with an external probe (represented by a thin line in the top part). The different diagrams, a, b and c, correspond to crossed, box and renormalization contributions. The double bar on the boson line in the diagram c) reminds that only the part contributing to the norm has to be retained. The bottom diagrams are obtained by bringing in the top ones the line 3 in coincidence with the line 1.

The above cancellation has been obtained in the case of constituents with equal masses. There is no difficulty to check that it also holds with unequal masses within the same approximations. In the present approach, this is a first step in allowing one to recover in particular the usual Coulomb interaction of an electron (light particle) in the field of a nucleus (heavy particle).

The cancellation of contributions given by Eq. (52) with those due to the “renormal-
ization” of both the interaction and the wave function, Eq. (17), reminds the norm-recoil cancellation discussed in the literature in the 70’s. This one however concerns the interaction of a two-body system (in the simplest case) with an external probe and thus involves at least three bodies. We examined this situation and indeed find some relationship. The difference with the present case is that one of the two bosons exchanged between the two bodies in Figs. 9, 10, 11 is attached to a third body, the electron for instance for an electromagnetic probe, see Fig. 13. Apart from a possible factor related to the number of constituents, there is a one to one correspondence. The demonstration is achieved by noticing that all time ordered diagrams are summed up while the two bosons attached to the same constituent line interact independently with the two other lines, with the consequence that the corresponding vertices necessarily commute.

4.3 Massless neutral and spinless bosons

The logarithmically divergent character of the extra contribution to the interaction given by Eq. (17) in the limit of a zero mass boson also occurs for the contribution given by Eq. (52). Like there, to overcome the difficulty, one has to take into account the binding energy in the boson propagators appearing in Eqs. (49-51). A priori, the corrections differ from those appearing in Eq. (27), preventing contributions of the two neutral boson exchanges to cancel that one due to the renormalization of the wave function. However, we saw that the dominant correction arising from this renormalization, of relative order $\alpha \log \frac{1}{\alpha}$, was unaffected by the precise value of the coefficient to be inserted in front of this binding energy (2 in Eq. (27)). As a result, this particular correction cancels out when considering both the renormalization of the wave function and the two boson exchange contribution.

Remaining corrections to the interaction may therefore be of relative order $\alpha$. At first sight, it is not clear whether these ones should also cancel out from the consideration of the above contributions alone (assuming they should cancel at all as expected from relativistic calculations where the correction to the dominant order is often believed to be of the order $\alpha^2$ rather than $\alpha$). The problem is that the long range of the force makes it difficult to discuss the contributions order by order, the binding energy introduced to remove the logarithmic divergence in Eq. (27) accounting for some of them. However, on the basis of the result presented in the appendix and assuming we have not missed some mathematical sublety, we expect that these corrections of relative order $\alpha$ with respect to the dominant term should vanish. It is obviously out of question to discuss here the next correction of relative order $\alpha^2$, which would also suppose to precise the framework (equation) to be used as well as to account for other radiative corrections.

4.4 Residual contribution for bosons with spin or charge

In the case of the exchange of bosons carrying some charge (isospin or color) or some spin, or coupling to the spin of constituents, the vertex functions, $O$, in Eqs. (49-51) do not commute. Some residual contribution then remains when adding the contribution of two boson exchanges, Eqs. (49-51), and the contribution arising from the renormalization, Eq.
(17). It occurs for bosons currently referred to as the pion in the field of nuclear physics or the gluon in QCD. Such contributions are generally ignored. Part of them may be accounted for in a phenomenological approach, by fitting parameters of a potential to NN scattering cross sections for instance. While one cannot underestimate the power of this procedure, it should be noticed that the resulting two pion contribution has certainly a very complicated structure and it is not sure that it can be easily approximated by single meson exchanges. One can thus imagine that these last approaches, by trying to reproduce fine contributions which their simplicity does not permit in principle, introduce some bias in other sectors of the interaction. With this respect, we notice that the Paris model of the NN interaction is the only one where the two pion exchange contribution is correctly accounted for [30]. It however suffers from some drawbacks, such as neglecting terms of the order \((p/m)^4\), which now turn out to be relevant in determining a parametrization of the model [23]. The two pion exchange contribution was also considered in the full Bonn model [12], but its energy dependence and the problems this feature are raising have led to its discarding. It is finally worthwhile to mention the description of the Nucleon Nucleon system from the Bethe-Salpeter equation [36] or from other relativistic approaches [37]. Based on single meson exchange, they incorporate contributions discussed in the first part of this work (Sect. 2), but neglect contributions corresponding to the crossed diagrams of Figs. 10 and 11, thus introducing another kind of bias.

We tentatively examined contributions due to the above residual two meson exchange contribution in the case of the NN interaction. The idea is to get an order of magnitude and, consequently, to know whether one has to worry about them. In this aim, we neglect the contribution of the box diagram, Eq. (49) and a contribution of the same magnitude for the crossed diagram. The terms left out are of the order \(\omega_i \omega_f/\omega_i + \omega_f\), where \(\omega_i\) and \(\omega_f\) refer to the on-mass shell energies of the two exchanged mesons. This is at most equal to 1/4 \((\omega_i = \omega_f)\), which gives some estimate of what is neglected. Due to possible destructive interferences, actual corrections may be larger however. With the above approximation, the expression of the two meson exchange contributions given by Eqs. (49-51) becomes identical to that of the norm correction to the energy independent interaction, Eq. (17), except for a different ordering of the spin and isospin operators. Furthermore, we concentrated on the contribution of isospin 1 mesons, which in particular contain the pion whose contribution to the NN interaction is known to be important, and pick up the part whose spin structure is the same as for the term \(\frac{1}{2}\{V_1, V_0\}\). The residual contribution under consideration thus reads

\[
V_{\text{res.}}^{NN} = - \left[ (\vec{\tau}_1 \cdot \vec{\tau}_2)(\vec{\tau}_1 \cdot \vec{\tau}_2) - \vec{\tau}_1 (\vec{\tau}_1 \cdot \vec{\tau}_2) \vec{\tau}_2 \right]_{\text{sym.}} \frac{1}{2} \{\hat{V}_1, \hat{V}_0\} = 4 (\vec{\tau}_1 \cdot \vec{\tau}_2) \frac{1}{2} \{\hat{V}_1, \hat{V}_0\},
\]

where the first and second terms in the squared bracket respectively arise from the renormalization of the interaction (generalization of Eq. (15)) and the crossed diagrams (generalization of Eq. (53)). The \(\hat{\text{symbol}}\) indicates that only the isospin 1 meson contribution to the full interaction is considered, the corresponding product of isospin matrices being factored out.

The relative effect of the two terms is immediately obtained. For the deuteron, of which isospin is zero, they are respectively proportional to 9 and 3. This evidences a construc-
tive interference, contrary to the neutral spinless meson case where the interference was destructive and complete. Amazingly, the second term in the bracket on the r.h.s. of Eq. (54) involves the interaction of two nucleons in \( ^1S_0, ^3P_0 \ldots \) states, which may look surprising. One can convince oneself however that nothing is wrong as this state is coupled to a meson in flight which, beside some angular momentum, carries isospin 1, allowing one to get the total expected isospin, 0.

Figure 14: Effect of a residual NN interaction due to isospin 1 mesons: contributions to the S and D wave components of the deuteron state shown in the figure are calculated perturbatively.

The effect of the residual interaction, Eq. (54), can be seen in Fig. 14 for both the S and D wave components of the deuteron state (Bonn-QB model). Not much is seen at very low momentum for this interaction model. For the S wave, this may be accidental as a large effect was obtained with the Bonn-Q model, comparable to the total correction obtained previously from the term \( \frac{1}{2} \{ V_1, V_0 \} \) (see Figs. 4 and 5). At intermediate momenta, the zero at 2 fm\(^{-1}\) is washed out in both cases while at large momenta the contribution blows up. It is not clear whether this is a real effect or the result of picking up a part of the potential in a domain where what comes from the exchange of mesons with different isospin is not well determined.

Using approximations similar to those made in getting Eq. (54), one can also get an expression for the residual interaction due to the gluon exchange (Coulomb part). It reads

\[
V_{qq}^{\text{res.}} = - \left[ (\lambda_1, \lambda_2)(\lambda_1, \lambda_2) - \lambda_1 (\lambda_1, \lambda_2) \lambda_2 \right]_{\text{sym.}} \frac{1}{2} \{ \hat{V}_1, \hat{V}_0 \}
\]

\[
= 6 (\lambda_1, \lambda_2) \frac{1}{2} \{ \hat{V}_1, \hat{V}_0 \},
\]

where the \( \lambda \) matrices act in the color space. The \( \hat{\cdot} \) symbol now indicates that only the Coulomb part of the interaction should be considered. Using Eq. (34), the above correction amounts to renormalize the Coulomb part of the gluon exchange by a factor roughly equal to 0.5 for \( \alpha_s = 0.5 \). The size of the effect certainly requires a more careful and complete examination, especially in view of the fact that it reduces the role of the Coulomb part of the one gluon exchange in explaining the meson and baryon spectroscopy. It does not support the result of some phenomenological studies that are favoring larger values of \( \alpha_s \), in particular in relation with the position of radial excitations [38, 39].
Notice that the residual interaction, Eq. (55), involves $\alpha \log \frac{1}{\alpha}$ corrections, thus indicating that they don’t cancel in all cases as the example of the Coulomb interaction may a priori suggest. These corrections have some relationship with the well known log corrections that account for the renormalization of the weak interaction of quarks due to QCD effects.

5 Conclusion

In this work, we have studied the relationship of energy dependent and energy independent two-body interaction models. The energy dependence of the first ones, based on a field theory approach, has obviously some theoretical support but, as is well known, is also a source of difficulties. The energy independent models are partly motivated by their empirical success, starting with the success of the Coulomb interaction in describing the dominant part of the electromagnetic interaction.

Our study has concentrated on the effect of possible corrections to wave functions. Essentially, we showed that effects accounted for by an energy dependent model could be incorporated to some extent in an energy independent approach. The resulting interaction is different from the usual ones, of the Yukawa or Coulomb type. These ones turn out to be renormalized by the probability that the system under consideration be in a two-body component, excluding that part involving bosons in flight. Far from raising some problem, we, on the contrary, believe that this result provides a useful intermediate step in the derivation of effective two-body interaction models to be employed in an energy independent scheme. Indeed, beside the above contribution, we expect in any case some contribution due to two crossed boson exchanges. For neutral spinless bosons, this one cancels the contribution due to the renormalization of the interaction together with an extra smaller two boson box contribution. This result, which holds at the zeroth order in the inverse of the constituent masses, possibly different and finite, allows one to recover the well known Yukawa or Coulomb interactions (at least for some part in the last case).

It confirms what has been obtained by other methods for these interactions [10, 11, 12, 13, 14], perhaps providing a more intuitive explanation. It is not however clear whether the conditions required to obtain the cancellation are identical in all cases, authorizing some scepticism about the result [15]. From present results, from those quoted above [14] and from results to come concerning bound states [31], one can thus make the following conjecture for neutral spinless bosons. The description of low energy states obtained from solving the Bethe-Salpeter equation with a single boson exchange interaction as well as that obtained with an energy dependent interaction (including on the light front) has a strong relationship to a description obtained in an energy independent picture with a renormalized Yukawa or Coulomb interaction. Physics described by a Yukawa or Coulomb interaction in an energy independent scheme rather has a strong relationship with that obtained from the Bethe-Salpeter equation or an energy dependent scheme together with an interaction including crossed boson exchange beside single boson exchange. In the case of bosons with charge or spin, some residual contribution is left, which by no means seems to be negligible for hadronic systems of current interest in nuclear or particle physics.
While contributions related to the energy dependence have to be considered and cannot be separated from these other ones allowing one to recover the Coulomb interaction for instance, what has to be done in practice with them depends on whether one looks at the two-body interaction from a theoretical or a phenomenological point of view. In the first case, they are part of a series of contributions that have to be determined with more or less accuracy depending on the domain and especially on the size of the coupling. In the other case, as in the NN strong interaction one, a large body of data has allowed one to determine phenomenological interaction models, accounting for a large part of the new contributions. Dealing with them requires some care, beyond that one mentioned in the literature when employing wave functions issued from an energy dependent model fitted to NN scattering data [13, 46, 47, 48, 29].

In the energy independent scheme and for the simplest case, this supposes to cancel the new contribution with another one equal to it but of opposite sign: nothing is changed with respect to the original model. If one considers that the new contribution has a genuine character, what we believe, then a change of the parameters determining the shorter range part of the models has to be performed. In the energy dependent models, the same can be done but in any case a readjustment of the parameters is required. Furthermore one has to keep in mind that wave functions in both schemes are not equivalent and cannot be used on the same footing as input as well as output. Thus, the zeroth order unperturbed wave function in an energy dependent scheme should not be taken as that one in an energy independent scheme as one may expect at first sight [14, 7, 10], but rather as the product of this one by the operator, \((1 + V_1)^{-\frac{1}{2}}\), where \(V_1\) is related to the derivative of the interaction with respect to the energy, \(V_1 = -\frac{\partial V}{\partial E}\).

The present work has been essentially devoted to the first order contributions issued from the energy dependence of field-theory based interaction models. In the case of a phenomenological approach to the NN strong interaction, they turn out to be largely included in these models, as already mentioned. They have an elusive character for some part and should not provide much effect in analyzing physical results at low or moderate energies. In view of the present work, the large effects found in refs [11, 4] thus appear as resulting from accounting roughly for twice the same contribution. Beyond first order corrections, some effect in relation with boson in flight should show up. They are likely to require studies more refined than those presented here.

Acknowledgement One of the authors (B.D.) is very grateful to T. Frederico for discussions which led to the developments presented in this work and to students, P. Gaspard and I. Pfeiffer, whose exploratory work was quite useful for the present one. We would also like to thank R. Machleidt for providing us with the numerical Bonn wave functions as well as J. Carbonell and V.A. Karmanov for communicating their own results.
A Summation of all contributions to the interaction at the zeroth order in $\frac{1}{m}$

We here discuss the summation of a subset of diagrams contributing to the two body interaction in the non-relativistic limit. Our intent is to show that the cancellation of second order contributions arising mainly from renormalization and crossed boson exchange has a somewhat general character for neutral spinless bosons, leaving the single boson exchange contribution as the only effective one. Before entering into the details, let’s mention that this result has been extended to three bosons. Beside all irreducible boson exchange diagrams, it supposes to take into account corrections of the order $V^2_1$ in the renormalized interaction, Eq. (13), as well as second order contributions in the expansion of the energy denominators given by Eq. (3).

The developments follow those presented for instance in ref. [29], but with a difference as to eliminating the elementary interaction of the constituents with the bosons. This one will be carried only on the part of the Hamiltonian containing the boson mass term, $H_0(b)$, and the interaction term, $H_I(c, b)$. The residual interaction of constituents with the bosons, generated by the transformation of the kinetic Hamiltonian of the constituents, $H_0(c)$, will be of the first order in the inverse of their mass, implying second order contributions for the interaction between the constituents, hence the announced result.

The schematic Hamiltonian we start from may be written as:

$$H(c, b) = H_0(c) + H_0(b) + H_I(c, b),$$

with

$$H_0(c) = \sum_{\vec{p}} E_{\vec{p}} c_{\vec{p}}^+ c_{-\vec{p}}, \quad H_0(b) = \sum_{\vec{k}} \omega_{\vec{k}} b_{\vec{k}}^+ b_{\vec{k}}, \quad H_I(c, b) = g \sum_{\vec{p}, \vec{k}} c_{\vec{p}+\vec{k}}^+ c_{-\vec{p}} \frac{(b_{\vec{k}}^+ + b_{\vec{k}}^+)}{\sqrt{2\omega_{\vec{k}}}}.$$

where the destruction operators, $c_{\vec{p}}$ and $b_{\vec{k}}$, (and the corresponding creation operators), respectively refer to the constituents and the bosons, whose exchange is responsible for the interaction between constituents.

We now make the substitution

$$[c^+, c, b^+, b] = e^{S(C^+, C, B^+, B)} [C^+, C, B^+, B] e^{-S(C^+, C, B^+, B)},$$

where $S(C^+, C, B^+, B)$ is an antihermitic operator given by :

$$S(C^+, C, B^+, B) = \sum_{\vec{p}, \vec{k}} g \frac{1}{\sqrt{2\omega_{\vec{k}}}} C_{\vec{p}}^+ C_{\vec{k}} \left( \frac{B_{\vec{k}}^+ - B_{\vec{k}}^+}{\omega_{\vec{k}}} \right).$$

The above transformation, which allows one to express the bare degrees of freedom ($c$, $b$) in terms of the effective ones ($C$, $B$), is a unitary one, leaving commutation relations unchanged. It is no more than a change of basis in the Fock space. With the above substitution, the Hamiltonian is written:

$$H(c, b) = e^S H(C, B) e^{-S} = H(C, B) + [S, H(C, B)] + \frac{1}{2} [S, [S, H(C, B)]] + ...$$

(59)
Concentrating on that part involving the boson mass and interaction terms in $H(c, b)$, one can show that the choice made in Eq. (58) allows one to remove the interaction term linear in the coupling $g$. For this purpose, one uses the relation:

$$H_I(C, B) + [S, H_0(B)] = 0.$$  \hspace{1cm} (60)

Taking into account this relation, the next contribution of the second order in the coupling $g$ involves the double commutator, $[S, [S, H_0(B)]]$. There are two terms arising from the second and third terms in Eq. (59), with the factors $-1$ and $\frac{1}{2}$ respectively. The resulting contribution is nothing but the usual two-body interaction between the constituents:

$$H_{2body}(C) = -\frac{1}{2} \sum_{\vec{p}, \vec{p}', \vec{k}} g^2 C^+_{\vec{p}+\vec{k}} C^-_{\vec{p}} C^+_{\vec{p}'-\vec{k}} C^-_{\vec{p}'}. \hspace{1cm} (61)$$

Beyond the double commutator, it can be checked that the multiple commutators of $S$ with $H_0(B)$ vanish, leaving the two-body interaction, Eq. (61), as the only contribution from the boson mass and interaction terms of the original Hamiltonian. It is identical to the single boson exchange potential in the instantaneous approximation. Notice that to obtain this result, $S$ has to commute with $H_{2body}(C)$, which is only fulfilled for spin- and charge-less bosons. While the above contribution is quite simple, the contribution arising from the mass term of the constituents, $H_0(c)$, is not. At the lowest order in the coupling $g$, it produces an interaction term given by:

$$\tilde{H}_I(C, B) = \sum_{\vec{p}, \vec{k}} \frac{g}{\sqrt{2\omega_k}} C^+_{\vec{p}+\vec{k}} C^-_{\vec{p}} \left( B^+_{\vec{p}+\vec{k}} - B^-_{\vec{p}} \right) \left( \frac{E_{\vec{p}} - E_{\vec{p}+\vec{k}}}{\omega_k} \right). \hspace{1cm} (62)$$

As seen from the above expression, the effect the interaction may produce is of the order of the difference in the energies of the constituents, namely $\frac{\vec{p}^2}{m}$, which means an effect of the order $(\frac{v}{c})^2$ for the interaction of the constituents. This is a typical relativistic correction of the order $(\frac{v}{c})^2$, expected in any case. As shown in the text of the paper in a limited case, the various corrections to the effective two-body interaction beyond the instantaneous approximation therefore cancel out at the zeroth order in $\frac{1}{m}$ for spin- and charge-less bosons.

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