Analysis of Collision Probability in a Parallel Machines Model When the Number of Jobs Is Constrained

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Abstract

This paper analyses collision probability in a production line. Collision probability is the probability of a collision occurring between jobs in a production line. One element that determines collision probability is the time interval between materials which are being fed into a production line. When the time interval is shortened, the probability of a collision occurring increases. However, in order to reduce production completion time, this time interval should be as short as possible. When determining the appropriate time interval, evaluating collision probability for a given time interval is important. Methods to compute collision probability in an in-line and a parallel machines model already exist. Moreover, in an in-line machines model, an approximate formula of collision probability has also been presented. In a parallel machines model, there are no results on analysis of collision probability. This paper shows a theoretical formula of collision probability in a parallel machines model when the number of jobs is constrained. Further, when the processing time follows an Erlang distribution, we show that this theoretical formula of collision probability can be described as a closed form. In addition, when the processing time follows an exponential distribution, we show a concrete closed form of collision probability. Finally, from computational experimentation, we compare the results obtained from the closed form presented with a known simulation method.

Key words: parallel machines model, collision probability, Erlang distribution, exponential distribution, closed form

1 Introduction

When scheduling and controlling of production processes, many different factors must be considered, such as customer orders, demand forecasts, working shifts, production lines performance, etc. Therefore, production scheduling and control problems are complex. For most of these problems, exact optimization methods cannot be applied, and heuristic methods are implemented as a reasonable approach [1]. We focus on production lines efficiency in this paper. One goal tasks for companies having production lines is to maximize the number of products completed in a given time. To this end, the time interval between materials being fed a line should be as short as possible. However, when the time interval is shortened, the probability of a collision occurring between materials increases. In a manufacturing process, such a collision might result in losses in many areas. For example, financial or time losses might be incurred if a manufacturing system shuts down. Moreover, a shutdown might also increase labor costs due to the recovery required. Therefore, when the time interval is given, evaluating collision probability is considered to be important.

Production line efficiency is a key area in the fields of scheduling and queueing theory. In scheduling theory, research that considers parallel machines already exists [2]. Although, for parallel machines models, the research discusses production efficiency, it does not discuss collision probability. Likewise, queueing theory research also includes parallel machines models. Concretely, for parallel machines models, the analytical results relating to loss probability, average waiting time, etc. exist in a steady state [7]. On the other hand, collision probability is an evaluation item in an unsteady state, and current analytical research relating to collision probability considers on in-line machines model only [3, 4].

The concept of collision probability was first introduced in reference [3]. Reference [3] gave an approximate
collision probability in an in-line machines model. From the discussion in [3, 4], theoretical analysis of collision probability appears to be hard to carry out. As analysis of collision probability is difficult in an in-line machines model, and so it is practicable to compute collision probability using computer simulations, and in fact, such an approach has already been. In a parallel machines model, too, computer simulation methods to compute collision probability have been proposed [8].

In this paper, we derive a theoretical formula of collision probability in a parallel machines model, when the number of jobs is constrained. Further, when the processing time follows an Erlang distribution, this paper proves that collision probability can be written in a closed form formula for any number of jobs. Moreover, when processing time follows an exponential distribution and the number of jobs is constrained, we show a closed form for collision probability. Finally, collision probabilities obtained by theoretical formula and simulation method are compared.

2 A Parallel Machines Model

The following notations are used:

- \( J_1, J_2, \ldots, J_n \): \( n \) jobs to be processed.
- \( T_i \) (\( > 0 \)): Processing time of \( J_i \).
- \( t_{\text{tact}} \) (\( > 0 \)): Tact time, i.e., the time difference between the time instants of feeding \( J_i \) and \( J_{i+1} \) for all \( 1 \leq i \leq n-1 \) at the entrance to the line.

A parallel machines model is assumed, where the number of machines is \( m \). The processing time \( T_i \) is a random variable that follows a continuous probability distribution, and all \( T_i \) (\( 1 \leq i \leq n \)) are independent of each other. Let \( f \) denote the probability density function of this distribution, where \( f(t) = 0 \) for \( t \leq 0 \). The assumption \( f(t) = 0 \) for \( t \leq 0 \) is reasonable since processing time never a negative. Jobs are fed one by one into the line at the entrance with the same time interval, \( t_{\text{tact}} \). The feeding order of jobs into the line is \( J_1, J_2, \ldots, J_n \). Each job is first delivered to an idle machine. As soon as the machine receives the job, it starts processing. After processing, the job is then delivered to the exit. For simplicity, we assume that the delivery time is nil.

If a job is fed into the line when all machines are processing, a collision occurs. The collision probability is the probability of at least one collision occurring. Given the tact times \( t_{\text{tact}} \), the number of jobs \( n \), and the number of machines \( m \), we denote the collision probability by \( p_{m,n}(t_{\text{tact}}) \).

3 Analysing Collision Probability

In this section, we derive collision probabilities for the cases of \( n = m + 1 \), \( n = m + 2 \), and \( n = m + 3 \). First for any integer \( c \), we define a function \( F_c \) as follows:

\[
F_c = \int_0^{ct_{\text{tact}}} f(t) \, dt.
\]

Next, the probability of a collision occurring when feeding \( J_i \) under the condition that a collision does not occur when feeding \( J_1, \ldots, J_{i-1} \) is denoted by \( r_{m,i}(t_{\text{tact}}) \). The collision probability \( p_{m,n}(t_{\text{tact}}) \) is written as follows:

\[
p_{m,n}(t_{\text{tact}}) = \sum_{i=1}^{n} r_{m,i}(t_{\text{tact}}).
\]

When \( n \leq m \), no collision occurs. Moreover, \( r_{m,i}(t_{\text{tact}}) = 0 \) holds for \( i \leq m \). Therefore,

\[
p_{m,n}(t_{\text{tact}}) = \begin{cases} 0, & n \leq m, \\ \sum_{i=m+1}^{n} r_{m,i}(t_{\text{tact}}), & n > m. \end{cases}
\]

3.1 When \( n = m + 1 \)

In this subsection, we derive the collision probability \( p_{m,m+1}(t_{\text{tact}}) \), when \( n = m + 1 \). Since we assume that \( n = m + 1 \), Eq. (1) is written as follows:

\[
p_{m,m+1}(t_{\text{tact}}) = r_{m,m+1}(t_{\text{tact}}).
\]

When \( J_1, J_2, \ldots, J_m \), are being processed, \( r_{m,m+1}(t_{\text{tact}}) \) is the collision probability when feeding \( J_{m+1} \). Thus,

\[
m t_{\text{tact}} < (i-1) t_{\text{tact}} + T_i \quad (1 \leq i \leq m).
\]

The probability that satisfies Eq. (3) is \( 1 - F_{m+1} \), for \( 1 \leq i \leq m \). Hence, the \( r_{m,m+1}(t_{\text{tact}}) \) is written as follows:

\[
r_{m,m+1}(t_{\text{tact}}) = \prod_{i=1}^{m} (1 - F_i).
\]

Note that, from Eq. (2), the above equation is the collision probability when \( n = m + 1 \).

3.2 When \( n = m + 2 \)

We analyse the collision probability when \( n = m + 2 \). Since \( n = m + 2 \), Eq. (1) is written as follows:

\[
p_{m,m+2}(t_{\text{tact}}) = r_{m,m+1}(t_{\text{tact}}) + r_{m,m+2}(t_{\text{tact}}).
\]
Assuming that no collision occurs when feeding \(J_1, \ldots, J_{m+1}\), and a collision occurs when feeding \(J_{m+2}\), then \(J_{m+1}\) must be being processed when feeding \(J_{m+2}\). Therefore, one of \(J_1, \ldots, J_m\) must be finished before feeding \(J_{m+1}\). We suppose such a job finished as \(J_i\). Thus,

\[
\begin{cases}
(m + 1)t_{\text{tact}} < m t_{\text{tact}} + T_{m+1}, \\
m t_{\text{tact}} \geq (i - 1)t_{\text{tact}} + T_i, \\
(m + 1)t_{\text{tact}} < (j - 1)t_{\text{tact}} + T_j, \\
(j = 1, \ldots, m, \ j \neq i)
\end{cases}
\]

The probability that satisfies Eq. (6) is written as

\[
(1 - F_i)F_{m+1-i} \prod_{j=2, j \neq m+2-i}^{m+1} (1 - F_j)
\]

Therefore, \(r_{m, m+2}(t_{\text{tact}})\) is written as

\[
r_{m, m+2}(t_{\text{tact}}) = \sum_{i=1}^{m} \left\{ F_i \prod_{j=1, j \neq i+1}^{m+1} (1 - F_j) \right\}.
\]

From Eqs. (4) and (7), Eq. (5) is expressed as follows:

\[
p_{m, m+2}(t_{\text{tact}}) = \prod_{i=1}^{m} (1 - F_i) + \sum_{i=1}^{m} \left\{ F_i \prod_{j=1, j \neq i+1}^{m+1} (1 - F_j) \right\}.
\]

### 3.3 When \(n = m + 3\)

Assuming that \(n = m + 3\), Eq. (1) becomes as follows:

\[
p_{m, m+3}(t_{\text{tact}}) = r_{m, m+1}(t_{\text{tact}}) + r_{m, m+2}(t_{\text{tact}}) + r_{m, m+3}(t_{\text{tact}}).
\]

To derive \(r_{m, m+3}(t_{\text{tact}})\), let \(A = (a_{i,j})\) be a \((n \times n)\)-matrix, where the \(ij\)-component \(a_{i,j}\) of \(A\) is defined by

\[
a_{i,j} = \begin{cases} 
0, & \text{if } i \neq j \text{ and } J_i \text{ is not processed when feeding } J_j, \\
1, & \text{if } i = j \text{ or } J_i \text{ is processed when feeding } J_j.
\end{cases}
\]

For example, when \(m = 2, n = 5\), \(J_1\) and \(J_3\) are being processed and a collision occurs when feeding \(J_5\), then

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

A matrix \(A\) shows that if the sum of components in the \(i\)-th column is larger than \(m\) then collision occurs when feeding \(J_i\).

Moreover, let \(I_{\alpha, \beta}\), which is the subset of an integer set for two integers \(\alpha, \beta (1 \leq \alpha < \beta \leq m)\), be defined as

\[
I_{\alpha, \beta} = \{1, 2, \ldots, m + 2\} \setminus \{m + 3 - \alpha, m + 3 - \beta\}
\]

Assuming that no collision occurs when feeding \(J_1\), \(\ldots, J_{m+2}\), and a collision occurs when feeding \(J_{m+3}\), then \(J_{m+2}\) must be being processed when feeding \(J_{m+3}\). Therefore, any two jobs among \(J_1, \ldots, J_{m+1}\) must be finished before feeding \(J_{m+2}\). We suppose that \(J_{m+1}\) and \(J_{\alpha}\), where \(\alpha\) is any value in the set \(\{1, 2, \ldots, m\}\), are finished before feeding \(J_{m+3}\). The probability of this occurring is defined by \(r_{m, m+3}(t_{\text{tact}})\). If \(J_{\alpha}\) is being processed when feeding \(J_{\alpha+1}, \ldots, J_m\), then the matrix \(A\) is

\[
\begin{pmatrix}
\downarrow m \\
\alpha & 0 & 1 & \ldots & \ldots & \ldots & 1 & 1 & 1 & 1 \\
0 & 1 & \ldots & \ldots & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \ldots & \ldots & \ldots & 1 & 1 & 1 & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & \ldots & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & \ldots & \ldots & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \ldots & \ldots & \ldots & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & \ldots & \ldots & \ldots & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & \ldots & \ldots & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

When \(J_{\alpha}\) is finished before \(J_m\) is fed into the line, a similar collision (i.e. when feeding \(J_{m+3}\), \(J_{\alpha}\) and \(J_{m+1}\) are finished and other jobs are being processed) occurs. Therefore, the processing times which satisfy supposition are as follows:

\[
\begin{cases}
(m + 1)t_{\text{tact}} \geq m t_{\text{tact}} + T_{m+1}, \\
m t_{\text{tact}} \geq (\alpha - 1)t_{\text{tact}} + T_{\alpha}, \\
(m + 2)t_{\text{tact}} < (i - 1)t_{\text{tact}} + T_i, \\
(i = 1, \ldots, m + 2, \ i \neq \alpha, m + 1)
\end{cases}
\]

The probability that satisfies Eq. (10) is written as

\[
F_i F_{m+1-i} \prod_{i \in I_{\alpha, m+1}} (1 - F_i).
\]
Thus, \( r_{m,m+3}(t_{tact}) \) is written as
\[
  r_{m,m+3}(t_{tact}) = \sum_{\alpha=1}^{m} \left\{ F_{m+1-\alpha} F_1 \prod_{i \in I_{m+1}} (1 - F_i) \right\}. \tag{11}
\]

If \( m > 1 \) then, we should derive probability which \( J_\alpha \) and \( J_\beta \) (\( 1 \leq \alpha < \beta \leq m \)) are finished, when feeding \( J_{m+3} \). This probability is defined by \( r_{m,m+3}(t_{tact}) \). We assume that \( J_\alpha \) and \( J_\beta \) are finished, when feeding \( J_{m+3} \). If \( J_\alpha \) being processed when feeding \( J_{\alpha+1}, \ldots, J_m \), and \( J_\beta \) being processed when feeding \( J_{\beta+1}, \ldots, J_m \), then matrix \( A \) is
\[
\begin{align*}
\downarrow m \\
\alpha \rightarrow & \begin{pmatrix} 1 & \cdots & \cdots & \cdots & 1 & 1 & 1 & 1 \\ 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m+1 & \rightarrow & 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \\ m+2 & \rightarrow & 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \\ m+3 & \rightarrow & 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 
\end{pmatrix}
\end{align*}
\]

When \( J_\alpha \) and \( J_\beta \) are finished before \( J_m \) is fed into the line, a similar collision occurs. In this case, inequality is as follows the processing times satisfy the following constraints:
\[
\begin{align*}
m t_{tact} & \geq (\alpha - 1) t_{tact} + T_\alpha, \\
m t_{tact} & \geq (\beta - 1) t_{tact} + T_\beta, \\
(m+2) t_{tact} & < (i-1) t_{tact} + T_i \\
& (i = 1, \ldots, m+2, \ i \neq \alpha, \beta). \tag{12}
\end{align*}
\]

Even if either job \( J_\alpha \) or \( J_\beta \) is being processed when feeding \( J_{\alpha+1} \), then a collision occurs when feeding \( J_{m+1} \). Therefore, we add the probability of satisfying that following constraints to the probability satisfying Eq. (12):
\[
\begin{align*}
(m+1) t_{tact} & \geq (\alpha - 1) t_{tact} + T_\alpha, \\
m t_{tact} & \geq (\beta - 1) t_{tact} + T_\beta, \\
(m+2) t_{tact} & < (i-1) t_{tact} + T_i \\
& (i = 1, \ldots, m+2, \ i \neq \alpha, \beta). \tag{13}
\end{align*}
\]

The probability that satisfying Eqs. (12), (13), and (14) is written as
\[
\begin{align*}
F_{m+1-\alpha} F_{m+1-\beta} \prod_{i \in I_{m+1}} (1 - F_i) \\
+ (F_{m+2-\alpha} - F_{m+1-\alpha}) F_{m+1-\beta} \\
\times \prod_{i \in I_{m+1}} (1 - F_i) \\
+ F_{m+1-\alpha} (F_{m+2-\beta} - F_{m+1-\beta}) \\
\times \prod_{i \in I_{m+1}} (1 - F_i) \\
= \{ F_{m+2-\alpha} F_{m+1-\beta} + F_{m+1-\alpha} (F_{m+2-\beta} - F_{m+1-\beta}) \} \\
\times \prod_{i \in I_{m+1}} (1 - F_i).
\end{align*}
\]

Thus, \( r_{m,m+3}(t_{tact}) \) is
\[
\begin{align*}
r_{m,m+3}(t_{tact}) = \sum_{\alpha=1}^{m-1} \sum_{\beta=\alpha+1}^{m} \left\{ F_{m+2-\alpha} F_{m+1-\beta} \\
+ F_{m+1-\alpha} (F_{m+2-\beta} - F_{m+1-\beta}) \right\} \\
\times \prod_{i \in I_{m+1}} (1 - F_i) \tag{15},
\end{align*}
\]

where this probability excludes \( m = 1 \). Therefore, if \( m = 1 \) then we define \( r_{m,m+3}(t_{tact}) = 0 \). Thus,
\[
r_{m,m+3}(t_{tact}) = r_{m+1,m+3}(t_{tact}) + r_{m,m+3}(t_{tact}). \tag{16}
\]

Therefore, if \( m > 1 \), then from Eqs. (4), (7), (9), (11), (15), and (16), \( p_{m,m+3}(t_{tact}) \) is written as follows:
\[
\begin{align*}
\sum_{\alpha=1}^{m} \sum_{\beta=\alpha+1}^{m} \left\{ F_{m+2-\alpha} F_{m+1-\beta} \\
+ F_{m+1-\alpha} (F_{m+2-\beta} - F_{m+1-\beta}) \right\} \\
\times \prod_{i \in I_{m+1}} (1 - F_i)
\end{align*}
\]

and
\[
\begin{align*}
\sum_{\alpha=1}^{m} \sum_{\beta=\alpha+1}^{m} \left\{ F_{m+2-\alpha} F_{m+1-\beta} \\
+ F_{m+1-\alpha} (F_{m+2-\beta} - F_{m+1-\beta}) \right\} \\
\times \prod_{i \in I_{m+1}} (1 - F_i)
\end{align*}
\]

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5 Closed Formula under Earlang Distribution

In this section, we prove that the collision probability \( p_{m,n}(t_{\text{tact}}) \) can be expressed by a closed form formula, when the processing time follows an Earlang distribution. Chiha, Fujisawa, Sekiguchi, and Ibaraki [4] shows an Earlang distribution is flexible enough to represent actual processing times. An Earlang distribution is defined by two parameters, a positive integer \( k \), and a positive real number \( \lambda \). The probability density function of this distribution is defined as

\[
f(x; k, \lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} \quad (x > 0).
\]

If processing time \( T_i \) follows an Earlang distribution, then

\[
F_e = 1 - \exp(-c \lambda t_{\text{tact}}) = 1 - \sum_{i=1}^{k-1} \frac{(c \lambda t_{\text{tact}})^i}{i!}.
\]

\( p_{m,m+1}(t_{\text{tact}}) \), \( p_{m,m+2}(t_{\text{tact}}) \), and \( p_{m,m+3}(t_{\text{tact}}) \) are consist of additions and multiplications of \( F_i \) and \( 1 - F_i \) \( (i = 1, \ldots, m + 2) \). Therefore, these probabilities can be expressed by a closed form formula.

For any positive integer \( n \), we consider the follow. Assuming that no collision occurs when feeding \( J_1, \ldots, J_{n-1} \), and a collision occurs when feeding \( J_n \). When feeding \( J_j (j = 1, \ldots, n) \), we can determine a \((n \times n)\)-matrix, which defined in subsection 3.3, depending on whether \( J_i (i = 1, \ldots, n) \) is being processed or not. Hence, let \( A = (a_{ij}) \) denote this \((n \times n)\)-matrix. When defining

\[
A_i = \sum_{j=1}^{n} a_{ij},
\]

the processing times satisfying the following constraints:

\[
\begin{align*}
\text{When } A_i &= n + 1 - i, \\
(n-1) t_{\text{tact}} &< (i-1) t_{\text{tact}} + T_i, \\
\text{When } A_i &= n + 1 - i, \\
(A_i + i - 1) t_{\text{tact}} &\geq (i-1) t_{\text{tact}} + T_i \\
&> (A_i + i - 2) t_{\text{tact}}.
\end{align*}
\]

The probability satisfying Eq. (20) is written as

\[
\begin{align*}
\text{When } A_i &= n + 1 - i, \\
1 - F_{n-i} &< F_{A_i} - F_{A_{i-1}}.
\end{align*}
\]

Therefore, probability of a collision occurring which is represented by matrix \( A \) is calculated by multiplications of \( F_i \) and \( 1 - F_i \) \( (i = 1, \ldots, n) \). The probability which satisfies this assumption is calculated by the adding of probabilities of collisions occurring which is represented by matrix \( A \). On the other hand, \( r_{m,n}(t_{\text{tact}}) \) is also the probability which satisfies this assumption. Thus, \( r_{m,n}(t_{\text{tact}}) \) is calculated the additions and multiplications of \( F_i \) and \( 1 - F_i \) \( (i = 1, \ldots, n) \). From Eq. (1), \( p_{m,n}(t_{\text{tact}}) \) is calculated by the additions of \( r_{m,i}(t_{\text{tact}}) \) \( (m + 1 \leq i \leq n) \). Hence, \( p_{m,n}(t_{\text{tact}}) \) is also calculated by the additions and multiplications of \( F_i \) and \( 1 - F_i \) \( (i = 1, \ldots, n) \). Therefore, we conclude that \( p_{m,n}(t_{\text{tact}}) \) can be expressed by a closed form formula.

5 Closed Formula under Exponential Distribution

In section 4, we showed that collision probability can be expressed by a closed form formula, when the processing time follows an Earlang distribution. It naturally follows, therefore, that when when the processing time follows an exponential distribution, collision probability can also be expressed by a closed form formula. In this section, we show a concrete closed form formula of collision probability under exponential distribution, when the number of jobs is constrained. If processing time \( T_i \) follows an exponential distribution, then

\[
F_e = 1 - \exp(-c \lambda t_{\text{tact}}).
\]
5.1 When $n = m + 1$

In this subsection, we derive the collision probability $p_{m,m+1}(t_{\text{tact}})$, when $n = m + 1$. From Eqs. (2), (4), and (21), $p_{m,m+1}(t_{\text{tact}})$ is expressed as follows:

$$p_{m,m+1}(t_{\text{tact}}) = \prod_{i=1}^{m}(1 - F_i)$$

(from Eqs. (2) and (4))

$$= \exp\left(-\sum_{i=1}^{m} F_i\right)$$

(from Eq. (21))

$$= \exp\left(-\frac{m(m+1)}{2} \lambda t_{\text{tact}}\right).$$  (22)

Hence,

$$F_i \prod_{j=1,j\neq i}^{m+1} (1 - F_j)$$

$$= \frac{\prod_{j=1, j\neq i}^{m+1} (1 - F_j t_{\text{tact}})}{1 - F((i+1)t_{\text{tact}})}$$

(from Eq. (21))

$$= \exp(-\frac{m+1}{2} \lambda t_{\text{tact}}) \exp(-i \lambda t_{\text{tact}})$$

$$= \exp\left(-\frac{m(m+3) - 2i}{2} \lambda t_{\text{tact}}\right).$$  (23)

Thus, from Eqs. (7) and (23), $r_{m,m+2}(t_{\text{tact}})$ is written as follows:

$$r_{m,m+2}(t_{\text{tact}}) = \sum_{i=1}^{m} \left\{ F_i \prod_{j=1, j\neq i}^{m+1} (1 - F_j) \right\}$$

(from Eq. (7))

$$= \sum_{j=1}^{m} \left\{ (1 - \exp(-i \lambda t_{\text{tact}})) \right\}$$

5.2 When $n = m + 2$

We derive the collision probability $p_{m,m+2}(t_{\text{tact}})$, when $n = m + 2$. For any $i \in \{1, \ldots, m\}$, the products of Eq. (7) is calculated as follows:

$$\prod_{j=1, j\neq i}^{m+2} (1 - F_j)$$

$$= \frac{\prod_{j=1}^{m+2} (1 - F_j t_{\text{tact}})}{1 - F((i+2)t_{\text{tact}})}$$

(from Eq. (21))

$$= \exp(-\frac{(m+2)(m+3)}{2} \lambda t_{\text{tact}}) \exp(-i \lambda t_{\text{tact}})$$

$$= \exp\left(-\frac{m(m+3) - 2i}{2} \lambda t_{\text{tact}}\right).$$  (24)

From Eqs. (2), (5), (22), and (24) the collision probability when $n = m + 2$ is derived as follows:

$$p_{m,m+2}(t_{\text{tact}})$$

$$= r_{m,m+1}(t_{\text{tact}}) + r_{m,m+2}(t_{\text{tact}})$$

(from Eq. (5))

$$= \exp\left(-\frac{m(m+1)}{2} \lambda t_{\text{tact}}\right)$$

$$+ \exp\left(-\frac{m(m+1)}{2} \lambda t_{\text{tact}}\right) \exp(-m \lambda t_{\text{tact}}) - 1$$

$$- m \exp\left(-\frac{m(m+3) - 2i}{2} \lambda t_{\text{tact}}\right)$$

(from Eqs. (2), (22), and (24))

$$= \exp\left(-\frac{m(m+1)}{2} \lambda t_{\text{tact}}\right)$$

$$+ \exp(-m \lambda t_{\text{tact}}) + \exp(-\lambda t_{\text{tact}}) - 2$$

$$- m \exp\left(-\frac{m(m+3) - 2i}{2} \lambda t_{\text{tact}}\right).$$  (25)

5.3 When $n = m + 3$

We derive the collision probability $p_{m,m+3}(t_{\text{tact}})$, when $n = m + 3$. From Eqs. (11) and (21), $r_{m,m+3}^{(1)}(t_{\text{tact}})$ is written as follows:

$$r_{m,m+3}^{(1)}(t_{\text{tact}})$$

$$= \exp\left(-\frac{(m+2)(m+3)}{2} \lambda t_{\text{tact}}\right)$$

$$\times \{ \exp((m+4) \lambda t_{\text{tact}}) - (m+1) \exp(4 \lambda t_{\text{tact}}) + m \exp(3 \lambda t_{\text{tact}}) \}.$$  (26)

The proof of Eq. (26) is given in Appendix A.1.

If $m > 1$, then $r_{m,m+3}^{(2)}(t_{\text{tact}})$ is expressed as the sum of two formulae:

$$r_{m,m+3}^{(2)}(t_{\text{tact}})$$
Moreover, for any positive integer \( m \), \( \xi_{m,m+3}(t_{tact}) \) and \( \zeta_{m,m+3}(t_{tact}) \) become as follows:

\[
\xi_{m,m+3}(t_{tact}) = m \sum_{\alpha+1}^{m-1} \sum_{\beta=\alpha+1}^{m} \left( F_{m+2-\alpha}F_{m+1-\beta} + F_{m+1-\alpha}(F_{m+2-\beta} - F_{m+1-\beta}) \right) \times \prod_{i \in I_{a,\beta}} (1 - F_i) \]

\[
\zeta_{m,m+3}(t_{tact}) = \exp \left( -\frac{(m + 2)(m + 3)}{2} \right) \lambda_{tact} \right) \times \exp((m + 5)\lambda_{tact}) \right) - \exp(5\lambda_{tact}) \right) \times \exp((m + 4)\lambda_{tact}) \right) \times (1 - \exp(\lambda_{tact}))^2 \right) + m \exp((m + 6)\lambda_{tact}) \right) \times (1 - \exp(\lambda_{tact}))^2 \right) + \frac{m(m - 1)}{2} \exp(3\lambda_{tact}) \right) \}
\]

The proofs of Eqs. (30) and (31) are given in Appendixes A.2 and A.3.

\( r_{m,m+3}(t_{tact}) \) is written as follows:

\[
r_{m,m+3}(t_{tact}) = r_{m,m+3}^{(1)}(t_{tact}) + r_{m,m+3}^{(2)}(t_{tact})
\]

from Eq. (16)
The proof of Eq. (33) is given in Appendix A.4. Therefore, \( r_{m,m+3}(t_{\text{tact}}) \) becomes as follows:

\[
\begin{align*}
&= \exp\left( -\frac{(m+2)(m+3)}{2} \lambda t_{\text{tact}} \right) \\
&\quad \times \frac{1}{(1-\exp(\lambda t_{\text{tact}}))^2} \times \left[ m^2 \exp(3\lambda t_{\text{tact}}) - \frac{3m(m+1)}{2} \exp(4\lambda t_{\text{tact}}) \
- \frac{m^2 - m - 2}{2} \exp(5\lambda t_{\text{tact}}) \
+ \frac{3m(m+1)}{2} \exp(6\lambda t_{\text{tact}}) \
- \frac{m(m+1)}{2} \exp(7\lambda t_{\text{tact}}) \
+ m \exp((m+4)\lambda t_{\text{tact}}) \
- \exp((m+5)\lambda t_{\text{tact}}) \
- (m+1) \exp((m+6)\lambda t_{\text{tact}}) \
+ \exp(2(m+3)\lambda t_{\text{tact}}) \right].
\end{align*}
\]

(34)

The proof of Eq. (34) is given in Appendix A.5. Thus, from Eqs. (2), (9), (22), (24), and (34) the collision probability \( p_{m,m+3}(t_{\text{tact}}) \) when \( n = m + 3 \) is written as follows:

\[
\begin{align*}
p_{m,m+3}(t_{\text{tact}})
&= r_{m,m+1}(t_{\text{tact}}) + r_{m,m+2}(t_{\text{tact}}) + r_{m,m+3}(t_{\text{tact}}) \\
&\quad \text{(from Eq. (9))}
&= \exp\left( -\frac{m(m+1)}{2} \lambda t_{\text{tact}} \right) \\
&\quad \times \exp(-m\lambda t_{\text{tact}}) + \exp(-\lambda t_{\text{tact}}) - 2 \\
&\quad \times \exp(-\exp(-\lambda t_{\text{tact}}) - 1) \\
&\quad - m \exp(-\frac{m(m+3)}{2} \lambda t_{\text{tact}}) \\
&\quad \exp\left( -\frac{(m+2)(m+3)}{2} \lambda t_{\text{tact}} \right) \\
&\quad \times \frac{1}{(1-\exp(\lambda t_{\text{tact}}))^2} \times \left[ m^2 \exp(3\lambda t_{\text{tact}}) - \frac{3m(m+1)}{2} \exp(4\lambda t_{\text{tact}}) \
- \frac{m^2 - m - 2}{2} \exp(5\lambda t_{\text{tact}}) \
+ \frac{3m(m+1)}{2} \exp(6\lambda t_{\text{tact}}) \
- \frac{m(m+1)}{2} \exp(7\lambda t_{\text{tact}}) \
+ m \exp((m+4)\lambda t_{\text{tact}}) \
- \exp((m+5)\lambda t_{\text{tact}}) \
- (m+1) \exp((m+6)\lambda t_{\text{tact}}) \
+ \exp(2(m+3)\lambda t_{\text{tact}}) \right].
\end{align*}
\]

(35)

### 6 Numerical Results

In this section, we compare the numerical and simulation results of collision probability.

#### 6.1 Simulation Algorithm

To compute collision probability, we used the collision check algorithm which determines whether a collision occurs or not. In this collision check algorithm, the flag search method presented in [8] was used. The flag search method is a method which searches for a zero in a 0/1 sequence of any length. When feeding a job, the status of parallel machines can be represented by a 0/1 sequence. If a machine is idle, it is assigned the number zero; otherwise, it is assigned the number one. In the flag search, if a zero exists, the job is processed on any machine corresponding to the zero; otherwise, a collision occurs. Given the tact time \( t_{\text{tact}} \), the number of jobs \( n \), and the number of machines \( m \), the simulation algorithm for computing the collision probability can be made by repeating the flag search \( c \) times, which was previously presented in [8].

**Simulation algorithm for computing collision probability**

**Step 1.** \( \text{loop} := 1 \).

**Step 2.** Generate processing times \( t_i (i = 1, 2, \ldots, n) \) randomly from an probability distribution.

**Step 3.** By applying the flag search algorithm, check whether a collision occurs. \( \text{loop} := \text{loop} + 1 \). If \( \text{loop} \leq c \), then go to Step 2; otherwise go to Step 4.

**Step 4.** Output the collision probability (i.e. the number of collisions in Step 3/c).
The simulation algorithm used Mersenne Twister [9] as a pseudorandom generator. Throughout all the simulations, the number of iterations was set to $c = 50,000$.

6.2 Numerical Results

Based on the formula presented in Section 3, 4, and 5, we obtained numerical results using Python [10]. The numerical and simulation results are shown in Figs. 1-4. In Figs. 1 and 2, the processing times follow an exponential distribution that has an expectation of 10. In Fig. 1, the number of machines $m = 1$; in Fig. 2, $m = 50$. In Figs. 3 and 4, the processing times follow an Erlang distribution, where the expectation and variance are 1 and 0.2, respectively. In Fig. 3, the number of machines $m = 1$; in Fig. 4, $m = 50$. We were able to confirm that the numerical and simulation results are almost the same.

7 Conclusions

In this paper, we focused on the analysis of collision probability in a parallel machines model. Concretely, we first derived a theoretical formula of collision probability when processing time follows a general distribution and the number of jobs is constrained. Next, we showed that collision probability can be described as a closed form formula, when the processing time follows an Erlang distribution for any number of jobs. Moreover, when processing time follows an exponential distribution and the num-
ber of jobs is constrained, we presented a concrete closed form formula of collision probability. Addressing collision probability for any the number of jobs in a parallel machines model would be a natural and logical possible area of future work.

A Appendix

A.1 Proof of Eq. (26)

When processing time $T_i$ follows an exponential distribution, we proof that Eq. (26) holds for any positive integer $m$. From Eqs. (11) and (21), $r_{m,m+3}^{(1)}(t_{\text{tact}})$ is written as follows:

$$
(1 - \exp(-\lambda t_{\text{tact}})) \times \exp(-m \lambda t_{\text{tact}}) \times \prod_{i \in I_m} \exp(-i \lambda t_{\text{tact}})
$$

A.2 Proof of Eq. (30)

When processing time $T_i$ follows an exponential distribution, we proof that Eq. (30) holds for any positive integer $m$. If $m > 1$, then from Eqs. (21) and (27), $\xi_{m,m+3}(t_{\text{tact}})$ is written as follows:

$$
(1 - \exp(-\lambda t_{\text{tact}})) \times \exp(-m \lambda t_{\text{tact}}) \times \prod_{i \in I_m} \exp(-i \lambda t_{\text{tact}})
$$
Analysis of Collision Probability in a Parallel Machines Model When the Number of Jobs Is Constrained

\[ \begin{align*}
& \times \frac{1}{\exp(-(m + 3 - \beta)\lambda t_{tact})} \\
& = \exp\left( -\frac{(m + 2)(m + 3)}{2} \lambda t_{tact} \right) \\
& \times \sum_{\alpha = 1}^{m-1} \frac{1 - \exp(-(m + 2 - \alpha)\lambda t_{tact})}{\exp(-(m + 3 - \alpha)\lambda t_{tact})} \\
& \times \sum_{\beta = \alpha + 1}^{m} \frac{1 - \exp(-(m + 1 - \beta)\lambda t_{tact})}{\exp(-(m + 3 - \beta)\lambda t_{tact})} \\
& = \exp\left( -\frac{(m + 2)(m + 3)}{2} \lambda t_{tact} \right) \\
& \times \sum_{\alpha = 1}^{m-1} \left( \exp((m + 3 - \alpha)\lambda t_{tact}) - \exp(\lambda t_{tact}) \right) \\
& \times \sum_{\beta = \alpha + 1}^{m} \left( \exp((m + 3 - \beta)\lambda t_{tact}) - \exp(\lambda t_{tact}) \right) \\
& \times \left\{ \frac{\exp(3\lambda t_{tact}) - \exp((m + 3 - \alpha)\lambda t_{tact})}{1 - \exp(\lambda t_{tact})} \right\} \\
& - (m - \alpha) \exp(2\lambda t_{tact}) \\
& = \exp\left( -\frac{(m + 2)(m + 3)}{2} \lambda t_{tact} \right) \\
& \times \sum_{\alpha = 1}^{m-1} \left\{ \frac{\exp((m + 6 - \alpha)\lambda t_{tact})}{1 - \exp(\lambda t_{tact})} \right\} \\
& - \exp(2(m + 3 - \alpha)\lambda t_{tact}) - \exp(4\lambda t_{tact}) \\
& + \exp((m + 4 - \alpha)\lambda t_{tact}) - \exp(4\lambda t_{tact}) \\
& -(m - \alpha) \exp((m + 5 - \alpha)\lambda t_{tact}) \\
& + (m - \alpha) \exp(3\lambda t_{tact}) \\
& = \exp\left( -\frac{(m + 2)(m + 3)}{2} \lambda t_{tact} \right) \\
& \times \sum_{\alpha = 1}^{m-1} \frac{\exp((m + 6 - \alpha)\lambda t_{tact})}{1 - \exp(\lambda t_{tact})} \\
& - \sum_{\alpha = 1}^{m-1} \exp(2(m + 3 - \alpha)\lambda t_{tact}) \\
& - \sum_{\alpha = 1}^{m-1} \exp(4\lambda t_{tact}) \\
& + \sum_{\alpha = 1}^{m-1} \exp((m + 4 - \alpha)\lambda t_{tact}) \\
& - \sum_{\alpha = 1}^{m-1} (m - \alpha) \exp((m + 5 - \alpha)\lambda t_{tact}) \\
& - \sum_{\alpha = 1}^{m-1} (m - \alpha) \exp(3\lambda t_{tact}) \\
& = \exp\left( -\frac{(m + 2)(m + 3)}{2} \lambda t_{tact} \right) \\
& \times \left\{ \frac{\exp(7\lambda t_{tact}) - \exp((m + 6)\lambda t_{tact})}{(1 - \exp(\lambda t_{tact}))^2} \right\} \\
& - \frac{\exp(8\lambda t_{tact}) - \exp(2(m + 3)\lambda t_{tact})}{(1 - \exp(\lambda t_{tact}))^2} \\
& - \frac{(m - 1) \exp(4\lambda t_{tact})}{(1 - \exp(\lambda t_{tact}))^2} \\
& + \frac{\exp(5\lambda t_{tact}) - \exp((m + 4)\lambda t_{tact})}{(1 - \exp(\lambda t_{tact}))^2} \\
& + \frac{\exp(6\lambda t_{tact}) - m \exp((m + 5)\lambda t_{tact})}{(1 - \exp(\lambda t_{tact}))^2} \\
& + \frac{(m - 1) \exp((m + 6)\lambda t_{tact})}{(1 - \exp(\lambda t_{tact}))^2} \\
& + \frac{m(m - 1)}{2} \exp(3\lambda t_{tact}) \\
& = \exp\left( -\frac{(m + 2)(m + 3)}{2} \lambda t_{tact} \right) \\
& \times \left\{ \frac{\exp(7\lambda t_{tact}) - \exp(6\lambda t_{tact}) + \exp(5\lambda t_{tact})}{(1 - \exp(\lambda t_{tact}))^2} \right\} \\
& - \frac{\exp((m + 4)\lambda t_{tact})}{(1 - \exp(\lambda t_{tact}))^2} \\
& + \frac{m \exp((m + 5)\lambda t_{tact})}{(1 - \exp(\lambda t_{tact}))^2} \\
& + \frac{-m \exp((m + 6)\lambda t_{tact})}{(1 - \exp(\lambda t_{tact}))^2} \\
& + \frac{\exp(8\lambda t_{tact}) - \exp(2(m + 3)\lambda t_{tact})}{(1 - \exp(\lambda t_{tact}))^2} \\
& - \frac{(m - 1) \exp(4\lambda t_{tact})}{(1 - \exp(\lambda t_{tact}))^2} \\
& - \frac{m(m - 1)}{2} \exp(3\lambda t_{tact}) \\
\end{align*} \]
If we substitute 1 for \( m \) in Eq. (35), we obtain

\[
\xi_{1,4}(t_{\text{tact}}) = \exp(-6\lambda_{\text{tact}})
\]

\[
= \frac{\exp(7\lambda_{\text{tact}}) - \exp(6\lambda_{\text{tact}}) + \exp(5\lambda_{\text{tact}})}{(1 - \exp(\lambda_{\text{tact}}))^2}
\]

\[
- \exp(5\lambda_{\text{tact}}) + \exp(6\lambda_{\text{tact}}) - \exp(7\lambda_{\text{tact}})
\]

\[
= 0.
\]

Therefore, Eq.(35) holds for any positive integer \( m \).

### A.3 Proof of Eq. (31)

When processing time \( T_i \) follows an exponential distribution, we prove that Eq. (31) holds for any positive integer \( m \). If \( m > 1 \), then from Eqs. (21) and (28), \( \zeta_{m,m+3}(t_{\text{tact}}) \) is written as follows:

\[
\zeta_{m,m+3}(t_{\text{tact}})
= \sum_{\alpha=1}^{m-1} \sum_{\beta=\alpha+1}^{m} \left\{ F_{m+1-\alpha}F_{m+2-\beta} - F_{m+1-\beta} \right\} (\text{from Eq. (28)})
\]

\[
= \sum_{\alpha=1}^{m-1} \sum_{\beta=\alpha+1}^{m} \left\{ 1 - \exp(-(m + 1- \alpha)\lambda_{\text{tact}}) \right\}
\]

\[
\times \exp(-(m + 1 - \beta)\lambda_{\text{tact}})
\]

\[
\times \exp(-(m + 2 - \beta)\lambda_{\text{tact}})
\]

\[
\times \prod_{i \in \{t_{\alpha,\beta}\}} (1 - F_i)
\]

\[
= \exp\left( -\frac{(m + 2)(m + 3)}{2} \lambda_{\text{tact}} \right)
\]

\[
\times \left\{ \exp((m + 5)\lambda_{\text{tact}}) - \exp(5\lambda_{\text{tact}}) \right\}
\]

\[
+ \frac{m\exp((m + 4)\lambda_{\text{tact}})}{1 - \exp(\lambda_{\text{tact}})}
\]

\[
- m\exp(\lambda(m + 5)t_{\text{tact}})
\]

\[
- \frac{m(m - 1)}{2} (\exp(4\lambda_{\text{tact}}) - \exp(3\lambda_{\text{tact}}))
\]
\[ \begin{align*}
&= \exp \left( -\frac{(m + 2)(m + 3)}{2} \lambda_{tact} \right) \\
&\quad \times \left\{ \exp((m + 5)\lambda_{tact}) - \exp(5\lambda_{tact}) \right. \\
&\quad \left. +m \exp((m + 4)\lambda_{tact}) \\
&\quad - \frac{m(m - 1)}{2} (\exp(4\lambda_{tact}) - \exp(3\lambda_{tact})) \right\}.
\end{align*} \]

A.5 Proof of Eq. (34)

When processing time \( T_i \) follows an exponential distribution, we prove that Eq. (34) holds for any positive integer \( m \). From Eqs. (30), (32), and (33), \( r_{m,m+3}(t_{tact}) \) is expressed as follows:

\[ r_{m,m+3}(t_{tact}) = (r_{m,m+3}(t_{tact}) + \zeta_{m,m+3}(t_{tact})) \]

\[ + \zeta_{m,m+3}(t_{tact}) \quad \text{(from Eq. (32))} \]

\[ = \exp \left( -\frac{(m + 2)(m + 3)}{2} \lambda_{tact} \right) \\
\times \left\{ \exp((m + 5)\lambda_{tact}) - \exp(5\lambda_{tact}) \right. \\
\quad + (m + 1) \exp((m + 4)\lambda_{tact}) \\
\quad - \frac{m^2 + m + 2}{2} \exp(4\lambda_{tact}) \\
\quad + \frac{(m + 1)}{2} \exp(3\lambda_{tact}) \right\}. \]

A.4 Proof of Eq. (33)

When processing time \( T_i \) follows an exponential distribution, we prove that Eq. (33) holds for any positive integer \( m \). From Eqs. (26) and (31), \( r_{m,m+3}(t_{tact}) + \zeta_{m,m+3}(t_{tact}) \) is expressed as follows:

\[ r_{m,m+3}^{(1)}(t_{tact}) + \zeta_{m,m+3}(t_{tact}) \]

\[ = \exp \left( -\frac{(m + 2)(m + 3)}{2} \lambda_{tact} \right) \\
\times \left\{ \exp((m + 5)\lambda_{tact}) - \exp(5\lambda_{tact}) \right. \\
\quad + (m + 1) \exp((m + 4)\lambda_{tact}) \\
\quad - \frac{m^2 + m + 2}{2} \exp(4\lambda_{tact}) \\
\quad + \frac{(m + 1)}{2} \exp(3\lambda_{tact}) \right\} \]

\[ \text{(from Eqs. (26) and (31))} \]

\[ = \exp \left( -\frac{(m + 2)(m + 3)}{2} \lambda_{tact} \right) \]

\[ \times \left\{ \exp((m + 5)\lambda_{tact}) - \exp(5\lambda_{tact}) \right. \\
\quad + (m + 1) \exp((m + 4)\lambda_{tact}) \\
\quad - \frac{m^2 + m + 2}{2} \exp(4\lambda_{tact}) \\
\quad + \frac{(m + 1)}{2} \exp(3\lambda_{tact}) \right\}. \]
\begin{align*}
\times \frac{1}{(1 - \exp(\lambda_{t_{\text{tact}}}))^2(1 + \exp(\lambda_{t_{\text{tact}}}))} \\
\times \left[ (\exp((m + 5)\lambda_{t_{\text{tact}}}) - \exp((m + 7)\lambda_{t_{\text{tact}}}) \\
- \exp(5\lambda_{t_{\text{tact}}}) + \exp(7\lambda_{t_{\text{tact}}}) \right) \\
+ (m + 1)(\exp((m + 4)\lambda_{t_{\text{tact}}}) \\
- \exp((m + 5)\lambda_{t_{\text{tact}}}) \\
- \exp((m + 6)\lambda_{t_{\text{tact}}}) + \exp((m + 7)\lambda_{t_{\text{tact}}})) \\
\frac{m^2 + m + 2}{2} \left( \exp(4\lambda_{t_{\text{tact}}}) - \exp(5\lambda_{t_{\text{tact}}}) \\
- \exp(6\lambda_{t_{\text{tact}}}) + \exp(7\lambda_{t_{\text{tact}}}) \right) \\
\frac{m(m + 1)}{2} \left( \exp(3\lambda_{t_{\text{tact}}}) - \exp(4\lambda_{t_{\text{tact}}}) \\
- \exp(5\lambda_{t_{\text{tact}}}) + \exp(6\lambda_{t_{\text{tact}}}) \right) \\
\left\{ \exp(8\lambda_{t_{\text{tact}}}) + \exp(5\lambda_{t_{\text{tact}}}) \\
- \exp((m + 4)\lambda_{t_{\text{tact}}}) \\
+ (m - 1)\exp((m + 5)\lambda_{t_{\text{tact}}}) \\
+ m\exp((m + 7)\lambda_{t_{\text{tact}}}) \right\} \\
- \left( \exp(8\lambda_{t_{\text{tact}}}) - \exp(2(m + 3)\lambda_{t_{\text{tact}}}) \right) \\
-(m - 1)(\exp(4\lambda_{t_{\text{tact}}}) - \exp(6\lambda_{t_{\text{tact}}}) \\
+ \frac{m(m - 1)}{2} \left( \exp(3\lambda_{t_{\text{tact}}}) - \exp(4\lambda_{t_{\text{tact}}}) \\
- \exp(5\lambda_{t_{\text{tact}}}) + \exp(6\lambda_{t_{\text{tact}}}) \right) \\
\exp \left( - \frac{(m + 2)(m + 3)}{2} \lambda_{t_{\text{tact}}} \right) \right] \\
\times \frac{1}{(1 - \exp(\lambda_{t_{\text{tact}}}))^2(1 + \exp(\lambda_{t_{\text{tact}}}))} \\
\times \left\{ m^2 \exp(3\lambda_{t_{\text{tact}}}) - \frac{3m(m + 1)}{2} \exp(4\lambda_{t_{\text{tact}}}) \\
- \frac{m^2 - m - 2}{2} \exp(5\lambda_{t_{\text{tact}}}) \\
+ \frac{3m(m + 1)}{2} \exp(6\lambda_{t_{\text{tact}}}) \right\} \\
\times \frac{-m(m + 1)}{2} \exp(7\lambda_{t_{\text{tact}}}) \\
+ m\exp((m + 4)\lambda_{t_{\text{tact}}}) - \exp((m + 5)\lambda_{t_{\text{tact}}}) \\
-(m + 1)\exp((m + 6)\lambda_{t_{\text{tact}}}) \\
+ \exp(2(m + 3)\lambda_{t_{\text{tact}}}) \right\}.
\end{align*}

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