EXTREMES OF VECTOR-VALUED GAUSSIAN PROCESSES WITH TREND

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Abstract: Let $X(t) = (X_1(t), \ldots, X_n(t)), t \in T \subset \mathbb{R}$ be a centered vector-valued Gaussian process with independent components and continuous trajectories, and $h(t) = (h_1(t), \ldots, h_n(t)), t \in T$ be a vector-valued continuous function. We investigate the asymptotics of

$$
P \left\{ \sup_{t \in T} \min_{1 \leq i \leq n} (X_i(t) + h_i(t)) > u \right\}
$$

as $u \to \infty$. As an illustration to the derived results we analyze two important classes of $X(t)$: with locally-stationary structure and with varying variances of the coordinates, and calculate exact asymptotics of simultaneous ruin probability and ruin time in a Gaussian risk model.

Key Words: Vector-valued Gaussian Process; Extremes; Conjunction; Piterbarg constant; Pickands constant

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1. Introduction and Preliminaries

Motivated by various applied-oriented problems, the asymptotics of

$$
P \left\{ \sup_{t \in T} (X(t) + h(t)) > u \right\},
$$

as $u \to \infty$, for both $T = [0, T]$ and $T = [0, \infty)$, where $X(t)$ is a centered Gaussian process with continuous trajectories and $h(t)$ is a continuous function, attracted substantial interest in the literature; see e.g. [1–8] and references therein for connections of (1) with problems considered, e.g., in risk theory or fluid queueing models. For example, in the setting of risk theory one usually supposes that $h(t) = -ct$, with $c > 0$ and $X$ has stationary increments. Then, using that $P \{ \sup_{t \in T} (X(t) + h(t)) > u \} = P \{ \inf_{t \in T} (u - X(t) + ct) < 0 \}$, (1) represents ruin probability, with $X(t)$ modelling the accumulated claims amount in time interval $[0, t]$, $c$ being the constant premium rate and $u$, the initial capital. The most celebrated model in this context is the Brownian risk model introduced in the seminal work by Iglehart [9], where $X$ is a standard Brownian motion. Extensions to more general class of Gaussian processes with stationary increments, including fractional Brownian motions, was analyzed in, e.g., [10, 1, 3, 11, 12]. Recent interest in the analysis of risk models has turned to the investigation of multidimensional ruin problems, including investigation of simultaneous ruin probability of some number, say $n$, of independent risk processes

$$
P \left\{ \exists_{t \in T} \forall_{i=1, \ldots, n} (u_i - X_i(t) + c_i t) < 0 \right\},
$$

see, e.g., [13] and [14]. Motivated by this sort of problems, in this paper we investigate multidimensional counterpart of (1), i.e., we are interested in the exact asymptotics of

$$
P \left\{ \exists_{t \in [0, T]} X(t) + h(t) > u \right\} = P \left\{ \sup_{t \in [0, T]} \min_{1 \leq i \leq n} (X_i(t) + h_i(t)) > u \right\},
$$

as $u \to \infty$, $T \in (0, \infty)$, where $X(t) = (X_1(t), \ldots, X_n(t)), t \in T \subset \mathbb{R}$ is an $n$-dimensional centered Gaussian process with mutually independent coordinates and continuous trajectories and $h(t) = (h_1(t), \ldots, h_n(t)), t \in [0, T]$ is a vector-valued continuous function.

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We note that (2) can also be viewed as the probability that the conjunction set \( S_{T,u} := \{ t \in [0,T] : \min_{1 \leq i \leq n} (X_i(t) + h_i(t)) > u \} \) is not empty in Gaussian conjunction problem, since
\[
\mathbb{P} \{ S_{T,u} \neq \emptyset \} = \mathbb{P} \left\{ \sup_{t \in [0,T]} \min_{1 \leq i \leq n} (X_i(t) + h_i(t)) > u \right\},
\]
see, e.g., [15, 16] and references therein.

The main results of this contribution extend recent findings of [16], where the exact asymptotics of (2) for \( h_i \equiv 0, 1 \leq i \leq n \) was analyzed; see also [17] where \( X(t) \) is a multidimensional Brownian motion, \( h_i(t) = c_i t \) and \( T = \infty \), and [18, 19] for LDP-type results. It appears that the presence of the drift function substantially increases difficulty of the problem when comparing it with the analysis given for the driftless case in [16]. More specifically, as advocated in Section 2, it requires to deal with
\[
\mathbb{P} \left\{ \sup_{t \in [0,T]} \min_{1 \leq i \leq n} X_{u,i}(t) > u \right\},
\]
where \( (X_{u,i}(t), t \in [0,T]) \), \( i = 1, \ldots, n \) are families (with respect to \( u \)) of centered threshold-dependent Gaussian processes; see Theorem 2.1.

In Section 3 we apply general results derived in Section 2 to two important families of Gaussian processes, i.e.

i) to locally-stationary processes in the sense of Berman and

ii) to processes with varying variance \( \text{Var}(X_i(t)) \),

\( t \in [0,T] \). Then, as an example to the derived theory, we analyze the probability of simultaneous ruin in Gaussian risk model. Complementary, we investigate the limit distribution of the \textit{simultaneous ruin time}
\[
\tau_u := \inf \{ t \geq 0 : (X(t) + h(t)) > u1 \},
\]
conditioned that \( \tau_u \leq T \), as \( u \to \infty \).

Organization of the rest of the paper: Section 2 is devoted to the main result of this contribution, concerning

the extremes of the threshold-dependent centered Gaussian vector processes. In Section 3 we specify our result
to locally-stationary vector-valued Gaussian processes with trend and non-stationary Gaussian vector-valued
processes with trend. Detailed proofs of all the results are postponed to Section 4. Additionally, in Section 3
we analyze asymptotics of the simultaneous ruin probability.

2. Main Results

We begin with observation that, for sufficiently large \( u \),
\[
(3) \quad \mathbb{P} \left\{ \sup_{t \in [0,T]} \min_{1 \leq i \leq n} (X_i(t) + h_i(t)) > u \right\} = \mathbb{P} \left\{ \exists t \in [0,T] X_u(t) > u1 \right\},
\]
where \( X_u(t) = \left( \frac{uX_1(t)}{u-h_1(t)}, \ldots, \frac{uX_n(t)}{u-h_n(t)} \right) \) is a family of centered vector-valued threshold-dependent Gaussian processes. Since the above rearrangement appears to be useful for the technique of the proof that we use in order to get the exact asymptotics of (2), then in this section we focus on asymptotics of extremes of threshold-dependent vector-valued Gaussian processes.

More specifically, let \( X_u(t) := (X_{u,1}(t), \ldots, X_{u,n}(t)), t \in E(u) \), with \( 0 \in E(u) = (x_1(u), x_2(u)) \), be a family of centered \( n \)-dimensional vector-valued Gaussian processes with continuous trajectories. Let \( \sigma^2_{u,i}(\cdot) \) and \( r_{u,i}(\cdot, \cdot) \) be the variance function and the correlation function of \( X_{u,i}(t) \), \( 1 \leq i \leq n \) respectively. Moreover, we tacitly assume that \( X_{u,i}(t) \), \( 1 \leq i \leq n \) are mutually independent.

We shall impose the following assumptions on \( X_u(t) \):

\textbf{A1:} \( \lim_{u \to \infty } \sigma_u(0) = \sigma > 0 \).
A2: There exist $\lambda_i \in [0, \infty), 1 \leq i \leq n$ with $\max_{1 \leq i \leq n} \lambda_i > 0$ and some continuous functions $f_i(\cdot), 1 \leq i \leq n$ with $f_i(0) = 0$ such that for any $\epsilon \in (0, 1)$, as $u \to \infty$,
\[
\left( \frac{\sigma_{u,i}(0)}{\sigma_{u,i}(t)} - 1 \right) u^2 - f_i(u^{\lambda_i} t) \leq \epsilon(\|f_i(u^{\lambda_i} t)\| + 1), \quad t \in E(u).
\]

A3: There exist $\alpha_i \in (0, 2]$ and $a_i > 0, 1 \leq i \leq n$ such that
\[
\lim_{u \to \infty} \sup_{s,t \in E(u)} \left| \frac{1 - r_{a,i}(t,s)}{a_i |t-s|^\alpha_i} - 1 \right| = 0.
\]

In the following we write $f \in \mathcal{R}_\alpha$ to denote that function $f$ is regularly varying at $\infty$ with index $\alpha$, see $[20-22]$ for the definition and properties of regularly varying functions.

Let $\lambda := \max_{1 \leq i \leq n} \lambda_i$, $\alpha := \min_{1 \leq i \leq n} \alpha_i$, $\tilde{f}(t) := \left( \tilde{f}_1(t), \ldots, \tilde{f}_n(t) \right)$ with
\[
\tilde{f}_i(t) = f_i(t) 1_{\{\lambda_i = \lambda\}}
\]
and suppose that $x_1(u) \in \mathcal{R}_{-\mu_1}$, $x_2(u) \in \mathcal{R}_{-\mu_2}$ with $\mu_1, \mu_2 \geq \lambda$ and
\[
\lim_{u \to \infty} u^\lambda x_1(u) = x_1 \in [-\infty, \infty), \quad \lim_{u \to \infty} u^\lambda x_2(u) = x_2 \in (-\infty, \infty), \quad x_1 < x_2,
\]
\[
\lim_{u \to \infty} u^\lambda x_i(u) = 0, \quad i = 1, 2, \lambda_j < \lambda.
\]

If $|x_1| + |x_2| = \infty$, we additionally assume that
\[
\liminf_{|t| \to \infty} \sup_{t \in [x_1, x_2]} \left( \sum_{i=1}^n \frac{\tilde{f}_i(t)}{\sigma_i^2} \right) / \left( \sum_{i=1}^n \left| \frac{\tilde{f}_i(t)}{\sigma_i^2} \right| \right) > 0.
\]

Assumption (5) means that the negative components of $\tilde{f}(t), 1 \leq i \leq n$ do not play a significant role to the sum in comparison with the positive components.

Moreover, we suppose that $0 \cdot \infty = 0, u^{-\infty} = 0$ for any $u > 0$ and introduce
\[
[x_1, x_2] := \lim_{u \to \infty} f(u)[x_1(u), x_2(u)],
\]
if $\lim_{u \to \infty} f(u)x_1(u) = x_1 \in (-\infty, \infty)$ and $\lim_{u \to \infty} f(u)x_2(u) = x_2 \in (-\infty, \infty)$ with $x_1 < x_2$.

Next we introduce some notation and definition of the Pickands-Piterbarg constants.

Throughout this paper, all the operations on vectors are meant componentwise, for instance, for any given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we write $x > y$ if and only if $x_i > y_i$ for all $1 \leq i \leq n$, write $1/x = (1/x_1, \ldots, 1/x_n)$ if $x_i \neq 0, 1 \leq i \leq n$, and write $xy = (x_1y_1, \ldots, x_ny_n)$. Further we set $0 := (0, \ldots, 0) \in \mathbb{R}^n$ and $1 := (1, \ldots, 1) \in \mathbb{R}^n$.

Define for $S_1, S_2 \in \mathbb{R}$, $S_1 < S_2$, $a = (a_1, a_2, \ldots, a_n)$ with $a_i \geq 0, 1 \leq i \leq n$ and $f(t) = (f_1(t), \ldots, f_n(t))$ with $f_i(t), 1 \leq i \leq n$ being continuous functions
\[
\mathcal{P}^{f}_{\alpha,a}[S_1, S_2] := \int_{\mathbb{R}^{n}} \epsilon^{\sum_{i=1}^n w_i} \mathbb{P} \left\{ \exists t \in [S_1, S_2]: \left( \sqrt{2}aB_\alpha(t) - a|t|^\alpha - f(t) \right) > w \right\} dw
\]
\[
= \int_{\mathbb{R}^{n}} \epsilon^{\sum_{i=1}^n w_i} \mathbb{P} \left\{ \sup_{t \in [S_1, S_2]} \left( \min_{1 \leq i \leq n} \sqrt{2}a_iB_{\alpha,i}(t) - a_i|t|^\alpha - f_i(t) - w_i \right) > 0 \right\} dw \in (0, \infty),
\]
where $B_\alpha(t), t \in \mathbb{R}$ is an $n$-dimensional vector-valued standard fractional Brownian motion (fBm) with mutually independent coordinates $B_{\alpha,i}(t)$ and common Hurst index $\alpha/2 \in (0, 1]$.

Let $\mathcal{P}^{f}_{\alpha,a}[0, \infty) := \lim_{S_2 \to \infty} \mathcal{P}^{f}_{\alpha,a}[0, S_2]$, $\mathcal{P}^{f}_{\alpha,a}(-\infty, \infty) := \lim_{S_1 \to -\infty, S_2 \to \infty} \mathcal{P}^{f}_{\alpha,a}[S_1, S_2]$.

Let, for $a > 0$,
\[
\mathcal{K}_{\alpha,a} = \lim_{T \to \infty} \frac{1}{T} \mathcal{P}^{0}_{\alpha,a}[0, T].
\]
In this section we apply Theorem 2.1 for some constants \((7)\) which are satisfied in our setup; see [16, 23, 24]. We refer to, e.g., [24–26, 2, 4, 27–35] for properties of the above constants.

Throughout this paper we write \(f(u) = h(u)(1 + o(1))\) or \(f(u) \sim h(u)\) if \(\lim_{u \to \infty} \frac{f(u)}{h(u)} = 1\) and \(f(u) = o(h(u))\) if \(\lim_{u \to \infty} \frac{f(u)}{h(u)} = 0\). Let \(\Psi(\cdot)\) denote the tail distribution of an \(N(0, 1)\) random variable, \(\Gamma(\cdot)\) denote the Euler Gamma function and \(I_{\{a=b\}} := (\mathbb{I}_{a_2=b_1}, \ldots, \mathbb{I}_{a_n=b_n})\) with \(\mathbb{I}_{\cdot}\) being the indicator function.

**Theorem 2.1.** Let \(X_u(t), t \in E(u)\) be a family of centered vector-valued Gaussian processes with continuous trajectories and independent coordinates satisfying A1–A3 and (4)–(5) holds. Let further \(m_u\) be a vector function of \(u\) with \(\lim_{u \to \infty} \frac{m_u}{u} = 1\) and for \(j \in \{1 \leq i \leq n : \lambda_i = \lambda\}\), \(f_j(t)\) be regularly varying at \(\pm \infty\) with positive index. Then we have

\[
\mathbb{P}\left\{ \exists t \in E(u) \mid X_u(t) > m_u \right\} \sim u^{\left(\frac{2}{\alpha} - \lambda\right)_+} \prod_{i=1}^n \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \times \begin{cases} \mathcal{H}_{\alpha, \alpha} I_{\{\alpha=\alpha\}} \int_{x} e^{- \sum_{i=1}^{n} \frac{f(u)}{\alpha_i}} dt, & \text{if } \lambda < 2/\alpha, \\ \mathcal{P}_{\alpha, \alpha} I_{\{\alpha=\alpha\}} [x_1, x_2], & \text{if } \lambda = 2/\alpha, \\ \int_{\mathbb{R}^n} e^{\sum_{i=1}^{n} \frac{f(u)}{\alpha_i}} \mathbb{I}_{\{x_1 > x_2\}} \mathbb{I}_{\{u > 0\}} dw, & \text{if } \lambda > 2/\alpha. \end{cases}
\]

### 3. Applications

In this section we apply Theorem 2.1 to the analysis of the exact asymptotics of

\[
\mathbb{P}\left\{ \exists t \in [0,T] \mid (X(t) + h(t)) > u1 \right\},
\]
as \(u \to \infty\). We distinguish two classes of processes \(X\): processes with non-stationary coordinates and processes with locally-stationary coordinates, including strictly stationary case.

#### 3.1. Non-stationary coordinates.

Let \(X(t), t \geq 0\) be a centered vector-valued Gaussian process with independent coordinates. Suppose that \(\sigma_i(\cdot), 1 \leq i \leq n\) attains its maximum on \([0, T]\) at the unique point \(t_0 \in [0, T]\), and further

\[
\sigma_i(t) = \sigma_i(t_0) - b_i |t - t_0|^\beta_i (1 + o(1)), \quad t \to t_0
\]

with \(b_i > 0, \beta_i > 0\), and

\[
r_i(s, t) = 1 - a_i |t - s|^\alpha_i (1 + o(1)), \quad s, t \to t_0
\]

for some constants \(a_i > 0\) and \(\alpha_i \in (0, 2]\). We further assume that there exists \(\mu_1 > 0\) such that

\[
\max_{i=1, \ldots, n} \sup_{s \neq t, t \in [0, T]} \mathbb{E} \left( \frac{(X_i(t) - X_i(s))^2}{|t - s|^\mu_1} \right) < \infty.
\]

Let \(h(t)\) be a continuous vector function over \([0, T]\) satisfying

\[
h_i(t) = h_i(t_0) - c_i |t - t_0|^\gamma_i (1 + o(1)), \quad t \to t_0
\]

with \(c_i < 0\) and \(\gamma_i \geq \frac{\beta_i}{2}\); and \(c_i \geq 0\) and \(\gamma_i > 0\). Moreover, there exists \(\mu_2 > 0\) such that

\[
\max_{i=1, \ldots, n} \sup_{s \neq t, t \in [0, T]} \frac{|h_i(t) - h_i(s)|}{|t - s|^\mu_2} < \infty.
\]

**Theorem 3.1.** Suppose that \(X(t), t \geq 0\) is a centered vector-valued Gaussian process with independent coordinates satisfying (6)–(8), and \(h(t), t \geq 0\) is a continuous vector function over \([0, T]\) satisfying (9)–(10). Then

\[
\mathbb{P}\left\{ \exists t \in [0,T] \mid (X(t) + h(t)) > u1 \right\} \sim u^{\left(\frac{2}{\alpha} - \frac{\beta_i}{2}\right)_+} \prod_{i=1}^n \Psi \left( \frac{u - h_i(t_0)}{\sigma_i(t_0)} \right)
\]
Suppose that for each $\gamma$, $P(f_i(t) = \gamma \| f_{i(2\gamma)} \| = 0)$, and let $\sigma(0) = (\sigma_1(0), ..., \sigma_n(0))$, $f = (f_1, ..., f_n)$ with $f_i(t) = \frac{b_i}{\sigma_i(t)}|t|^\beta_1 + \frac{c_i}{\sigma_i(t)}|t|^\gamma t_{2\gamma}$, and
\[
q = \begin{cases} 
-\infty, & \text{if } t_0 \in (0, T), \\
0, & \text{if } t_0 = 0 \text{ or } t_0 = T.
\end{cases}
\]

Remark 3.2. If $n = 1$ and $h_1(t) \equiv 0$, then Theorem 3.1 covers the classical Piterbarg-Prisjažnuk result; see [36].

In the following corollary we apply Theorem 3.1 for the analysis of exact asymptotics of $\tau_u = \inf\{t \geq 0 : (X(t) + h(t)) > u1\}$, as $u \to \infty$, conditioned that $\tau_u \leq T$.

Corollary 3.3. Under the same assumptions as in Theorem 3.1 with $t_0 = T$, we have for $x \in (0, \infty)$, as $u \to \infty$,
\[
P\left\{ (T - \tau_u)u^{2/\beta} \leq x | \tau_u \leq T \right\} \sim \begin{cases} 
\frac{\int_0^\infty e^{-\sum_{i=1}^n f_i(t)} dt}{\int_0^\infty e^{-\sum_{i=1}^n f_i(t)} dt}, & \text{if } \alpha < \beta, \\
\frac{\mathcal{P}^f_{\alpha, \sigma(0)}[0, x]}{\mathcal{P}^f_{\alpha, \sigma(0)}[0, \infty]}, & \text{if } \alpha = \beta, \\
1, & \text{if } \alpha > \beta.
\end{cases}
\]

We give a short proof of Corollary 3.3 in Appendix.

3.2. Locally-stationary coordinates. Suppose that for each $i = 1, ..., n$, $X_i$ is a centered locally-stationary Gaussian process with continuous trajectories, that is process with unit variance and correlation function $r_i(\cdot, \cdot), 1 \leq i \leq n$ satisfying
\[
r_i(t_1 + s) = 1 - a_i(t_1)|s|^\alpha + O(|s|^{\alpha}), \quad s \to 0
\]
uniformly with respect to $t \in [0, T]$, where $\alpha_i \in (0, 2)$, and $a_i(t) \in (0, \infty)$ is a positive continuous function on $[0, T]$. Further, we suppose that
\[
r_i(s, t) < 1, \quad \forall s, t \in [0, T] \text{ and } s \neq t.
\]

We refer to e.g., [37–40] for the investigation of extremes of one-dimensional locally-stationary Gaussian processes under the above conditions.

Denote by
\[
H = \bigcap_{i=1}^n \{s \in [0, T] : h_i(s) = h_{m,i} = \max_{t \in [0, T]} h_i(t) \}.
\]

Theorem 3.4. Let $X(t), t \in [0, T]$ be a locally stationary vector-valued Gaussian process satisfying (13) and (14). Moreover, assume that $h(t)$ is a vector function satisfying (10) and $\alpha = \min_{1 \leq i \leq n} \alpha_i$.

i) If $H = \{t_0\}$ and (9) holds with $c_i \geq 0$ and $\max_{1 \leq i \leq n} c_i > 0$, then
\[
P\left\{ \exists t \in [0, T] \left( X(t) + h(t) \right) > u1 \right\} \sim u^{\frac{2}{\alpha} - \frac{1}{2}} \prod_{i=1}^n \Psi (u - h_{m,i}) \begin{cases} 
\mathcal{H}_{\alpha, \omega(0)}[\alpha = 1 \cdot ] \int_q^\infty e^{-\sum_{i=1}^n f_i(x)} dx, & \text{if } \alpha < 2\gamma, \\
\mathcal{P}_{\alpha, \omega(0)}[\alpha = 1 \cdot ] [q, \infty), & \text{if } \alpha = 2\gamma, \\
1, & \text{if } \alpha > 2\gamma.
\end{cases}
\]

where $\gamma = \min_{1 \leq i \leq n} (\gamma_i I_{\{\gamma_i > 0\}} + \infty I_{\{\gamma_i = 0\}})$, $f_i(t) = c_i(t)^\gamma I_{\{\gamma_i = 0\}}$, and $q$ is given by (11).

ii) If $H = [A, B] \subset [0, T]$ with $A > B$, then
\[
P\left\{ \exists t \in [0, T] \left( X(t) + h(t) \right) > u1 \right\} \sim \int_A^B \mathcal{H}_{\alpha, \omega(t)}[\alpha = 1 \cdot ] dt u^{\frac{2}{\alpha} - 1} \prod_{i=1}^n \Psi (u - h_{m,i}).
\]
Similarly to Corollary 3.3, we get the asymptotics of $\tau_u$ for locally-stationary coordinates of $X$.

**Corollary 3.5.** Under the same assumptions as in i) of Theorem 3.4, with $t_0 = T$, we have for $x \in (0, \infty)$, as $u \to \infty$,

$$
P \left\{ (T - \tau_u)^{1/\gamma} \leq x \mid \tau_u \leq T \right\} \sim \begin{cases} 
\int_0^x e^{-\sum_{i=1}^n f_i(t)} dt / \int_0^\infty e^{-\sum_{i=1}^n f_i(t)} dt, & \text{if } \alpha < 2\gamma, \\
\mathcal{P}_\alpha \{a(t_0)I_{(a=1)} [0, x] / \mathcal{P}_\alpha [0, x), & \text{if } \alpha = 2\gamma, \\
1, & \text{if } \alpha > 2\gamma.
\end{cases}
$$

3.3. **A simultaneous ruin model.** Consider portfolio $U(t) = (U_1(t), \ldots, U_n(t))$, where

$$
U(t) = ud + ct - B_\alpha(t), \quad t \geq 0,
$$

with $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$, $d = (d_1, \ldots, d_n) > 0$ and $B_\alpha(t)$, $1 \leq i \leq n$, independent standard fractional Brownian motions with variance $\text{Var}(B_\alpha(t)) = t^{\alpha_i}$ for $\alpha_i \in (0, 2]$, $1 \leq i \leq n$, respectively. The corresponding simultaneous ruin probability over $[0, T]$ is defined as

$$
P \left\{ \exists t \in [0, T] \mid U(t) < 0 \right\}
$$

and the simultaneous ruin time $\tau_u := \inf \{ t \geq 0 : U(t) < 0 \}$. We refer to, e.g., [10] for theoretical justification of the use of fractional Brownian motion as the approximation of the claim process in risk theory.

In the following proposition we present exact asymptotics of the simultaneous ruin probability and the conditional simultaneous ruin time $\tau_u \mid \tau_u < T$, as $u \to \infty$.

**Proposition 3.6.** For $T \in (0, \infty)$, $\alpha = \min_{1 \leq i \leq n} \alpha_i$, $b_i = \frac{d_i^2}{2T^{2\alpha_i}}$ and $f_i(t) = \frac{\alpha_i d_i^2}{2T^{\alpha_i+1}} t$, as $u \to \infty$, we have

$$
P \left\{ \exists t \in [0, T] \mid U(t) < 0 \right\} \sim u^{(\alpha - 2)n} \prod_{i=1}^n \Psi \left( \frac{d_i u + c_i T}{T^{\alpha_i/2}} \right)
$$

and for $x \in (0, \infty)$

$$
P \left\{ (T - \tau_u)^{1/\gamma} \leq x \mid \tau_u \leq T \right\} \sim \begin{cases} 
1 - e^{-\left( \sum_{i=1}^n \frac{\alpha_i d_i^2}{2^{\alpha_i+1}} \right) x}, & \text{if } \alpha < 1, \\
\mathcal{P}_\alpha \{a(t_0)I_{(a=1)} [0, x] / \mathcal{P}_\alpha [0, x), & \text{if } \alpha = 1, \\
1, & \text{if } \alpha > 1.
\end{cases}
$$

Specifically, Proposition 3.6 allows us to get exact asymptotics for multidimensional counterpart of the classical Brownian risk model [9]. For simplicity we focus on 2-dimensional case. Let $B(t) := (B^{(1)}(t), B^{(2)}(t))$, where $B^{(1)}(t)$ and $B^{(2)}(t)$ are two independent standard Brownian motions, $c = (c_1, c_2) \in \mathbb{R}^2$ and $d = (d_1, d_2) \in \mathbb{R}^2$. Then we have, as $u \to \infty$,

$$
P \left\{ \exists t \in [0, T] \mid \begin{pmatrix} d_1 u + c_1 t - B^{(1)}(t) \\ d_2 u + c_2 t - B^{(2)}(t) \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \sim \mathcal{P}_{1,b}^b \{0, \infty\} \Psi \left( \frac{d_1 u + c_1 T}{T^{1/2}} \right) \Psi \left( \frac{d_2 u + c_2 T}{T^{1/2}} \right)
$$

and for $x \in (0, \infty)$

$$
P \left\{ (T - \tau_u)^{1/\gamma} \leq x \mid \tau_u \leq T \right\} \sim \mathcal{P}_{1,b}^b \{0, x\} \mathcal{P}_{1,b}^b [0, x),
$$

where $b = \left( \frac{d_1^2}{2T^{2\alpha_1}}, \frac{d_2^2}{2T^{2\alpha_2}} \right)$. 
Before proceeding to the proof of Theorem 2.1, we present two lemmas which play an important role in the proof of Theorem 2.1. The first one is a vector-valued version of the uniform Pickands-Piterbarg lemma while the second one gives an upper bound for the double maximum of vector-valued Gaussian process. Hereafter, we denote by \( \mathbb{C}, l \in \mathbb{N} \) some positive constants that may differ from line to line. Moreover, the notation \( f(u, S, \epsilon) \sim g(u) \) as \( u \to \infty, S \to \infty, \epsilon \to 0 \), means that \( \lim_{u \to 0} \lim_{S \to \infty} \lim_{u \to \infty} f(u, S, \epsilon) g(u) = 1. \)

For \( b \geq 0, \lambda_i \in [0, \infty) \), and \( -\infty < S_1 < S_2 < \infty \), define a vector-valued Gaussian process \( Z_u(t) = (Z_{u,1}(t), \ldots, Z_{u,n}(t)) \) by

\[
Z_{u,i}(t) = \frac{\xi_i(t)}{1 + b_i u^{-2} f_i(u, t)}, \quad t \in [S_1, S_2], \quad i = 1, \ldots, n,
\]

where \( \xi(t) = (\xi_1(t), \ldots, \xi_n(t)), t \in \mathbb{R} \) is a vector-valued Gaussian process with independent stationary coordinates, continuous sample paths, unit variance and correlation function \( r_i(\cdot) \) on \( i \)-th coordinate, \( 1 \leq i \leq n \), satisfying

\[
1 - r_i(t) = a_i \left| t \right|^{\alpha_i} (1 + o(1)),
\]

for \( a_i > 0 \) and \( \alpha_i \in (0, 2] \), and \( f_i(t), 1 \leq i \leq n \) are some continuous functions. We suppose that the threshold vector \( m_u(k) = (m_{u,1}(k), \ldots, m_{u,n}(k)) \) satisfies

\[
\lim_{u \to \infty} \sup_{k \in K_u} \left| \frac{1}{u} m_u(k) - c \right| = 0, \quad c > 0,
\]

with \( K_u \) an index set.

Denote by

\[
\alpha = \min_{1 \leq i \leq n} \alpha_i, \quad \lambda = \max_{1 \leq i \leq n} (\lambda_i \mathbb{1}_{\{b_i \neq 0\}}) > 0, \quad \bar{f}(t) = (\bar{f}_1(t), \ldots, \bar{f}_n(t)), \quad \text{with} \quad \bar{f}_i(t) = f_i(t) \mathbb{1}_{\{\lambda_i = \lambda\}}.
\]

**Lemma 4.1.** Let \( Z_u(t) \) be defined in (18) and \( m_u(k) \) satisfy (20).

i) If \( \lambda \leq 2/\alpha \), then

\[
\lim_{u \to \infty} \sup_{k \in K_u} \left| \frac{1}{u} m_u(k) - c \right| = 0,
\]

where

\[
\mathcal{R}_\lambda[S_1, S_2] = \begin{cases} \mathcal{P} \mathcal{C}^2 \bar{f}_{\alpha, \alpha c^2 \mathbb{1}_{I_{(\alpha = 1)}}}[S_1, S_2], & \text{if } \lambda = 2/\alpha, \\ \mathcal{P} \mathcal{C}^2 \bar{f}_{\alpha c^2 \mathbb{1}_{I_{(\alpha = 1)}}}[S_1, S_2], & \text{if } \lambda < 2/\alpha, \\ \mathcal{H}_{\alpha, \alpha c^2 \mathbb{1}_{I_{(\alpha = 1)}}}[S_1, S_2], & \text{if } b = 0. 
\end{cases}
\]

ii) If \( \lambda > 2/\alpha \), then

\[
\lim_{u \to \infty} \sup_{k \in K_u} \left| \frac{1}{u} m_u(k) - c \right| = 0,
\]

**Proof.** i) Suppose that \( \lambda \leq 2/\alpha \). Conditioning on \( \{\xi(0) = m_u(k) - \frac{w}{m_u(k)}\} \), \( w \in \mathbb{R}^n \), we have for all \( u \) large enough

\[
\mathbb{P} \left\{ \exists t \in [u^{-\alpha} S_1, u^{-\alpha} S_2] \mathbb{Z}(t) > m_u(k) \right\} \\
= \frac{1}{\prod_{i=1}^{n} \psi(m_u(k))} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \sum_{i=1}^{n} \left( m_u(k) - \frac{w}{m_u(k)} \right)^2} \times \mathbb{P} \left\{ \exists t \in [S_1, S_2] \mathbb{Z}(u^{-2/\alpha} t) > m_u(k) \mid \xi(0) = m_u(k) - \frac{w}{m_u(k)} \right\} dw
\]
= \left( \prod_{i=1}^{n} e^{-\frac{(m_{u,i}(k))^2}{2}} \right) \int_{\mathbb{R}^n} e^{\sum_{i=1}^{n} \left( w_i - \frac{w_i^2}{2(m_{u,i}(k))^2} \right)} \mathbb{P}\left\{ \exists t \in [S_1, S_2] \mathcal{X}_u^w(t, k) > w \right\} \, dw

= \left( \prod_{i=1}^{n} e^{-\frac{(m_{u,i}(k))^2}{2}} \right) I_{u,k},

where \( \mathcal{X}_u^w(t, k) = (\mathcal{X}_{u,1}^w(t, k), \ldots, \mathcal{X}_{u,n}^w(t, k)) \) with

\[
\mathcal{X}_{u,i}^w(t, k) = m_{u,i}(k)(Z_{u,i}(u^{-2/\alpha} t) - m_{u,i}(k)) + w_i \xi_{i}(0) = m_{u,i}(k) - \frac{w_i}{m_{u,i}(k)}.
\]

By (20), it follows that

\[
\lim_{u \to \infty} \sup_{k \in K_u} \left| \frac{1}{n} \sum_{i=1}^{n} e^{-\frac{(m_{u,i}(k))^2}{2}} I_{u,k} \right| = 0.
\]

Thus in order to establish the proof, it suffices to prove that

\[
\lim_{u \to \infty} \sup_{k \in K_u} \left| I_{u,k} - \mathcal{R}_\lambda^f[S_1, S_2] \right| = 0.
\]

It follows that, for each \( W > 0 \), with \( \tilde{W}^n = [-W, W]^n \) and \( \tilde{W}^n_j = \{ w \in \mathbb{R}^n | w_j \in (-\infty, -W) \cup (W, \infty) \} \),

\[
\sup_{k \in K_u} \left| I_{u,k} - \mathcal{R}_\lambda^f[S_1, S_2] \right| \leq \sup_{k \in K_u} \left| \int_{\tilde{W}^n} \left[ e^{\sum_{i=1}^{n} \left( w_i - \frac{w_i^2}{2m_{u,i}(k)^2} \right)} \mathbb{P}\left\{ \exists t \in [S_1, S_2] \mathcal{X}_u^w(t, k) > w \right\} - e^{\sum_{i=1}^{n} w_i} \mathbb{P}\left\{ \exists t \in [S_1, S_2] \zeta(t) > w \right\} \right] \, dw \right|

+ \sum_{j=1}^{n} \sup_{k \in K_u} \left| \int_{\tilde{W}^n_j} \left[ e^{\sum_{i=1}^{n} w_i} \mathbb{P}\left\{ \exists t \in [S_1, S_2] \mathcal{X}_u^w(t, k) > w \right\} \right] \, dw \right|

+ \sum_{j=1}^{n} \int_{\tilde{W}^n_j} \left[ e^{\sum_{i=1}^{n} w_i} \mathbb{P}\left\{ \exists t \in [S_1, S_2] \zeta(t) > w \right\} \right] \, dw

:= I_1(u) + I_2(u) + I_3(u),

where \( \zeta(t) = (c \sqrt{2} a_\alpha - a c^2 t^{\alpha})I_{\{\alpha=\alpha_1\}} - c^2 \tilde{f}(t)I_{\{\lambda=2/\alpha\}}. \)

Next, we give upper bounds for \( I_i(u), i = 1, 2, 3 \). We begin with the weak convergence of process \( \mathcal{X}_u^w(t, k) \).

**Weak convergence of \( \mathcal{X}_u^w(t, k) \).** Direct calculation shows that

\[
\mathbb{E}\left\{ (1 + b_i u^{-2} f_i(u^{\lambda_i} t)) \mathcal{X}_{u,i}^w(t, k) \right\} = -m_{u,i}(k) \left( 1 - r_i(u^{-2/\alpha} t) + b_i u^{-2} f_i(u^{\lambda_i - 2/\alpha} t) \right)

+ w_i \left( 1 - r_i(u^{-2/\alpha} t) + b_i u^{-2} f_i(u^{\lambda_i - 2/\alpha} t) \right),
\]

and

\[
\text{Var} \left( (1 + b_i u^{-2} f_i(u^{\lambda_i} t)) \mathcal{X}_{u,i}^w(t, k) \right) - (1 + b_i u^{-2} f_i(u^{\lambda_i} t')) \mathcal{X}_{u,i}^w(t', k) \right) = m_{u,i}(k) \left( \text{Var} \left( \xi_i(u^{-2/\alpha} t') - \xi_i(u^{-2/\alpha} t') - (r_i(u^{-2/\alpha} t') - r_i(u^{-2/\alpha} t'))^2 \right) \right).
\]

By (19) and (20), it follows that

\[
\mathbb{E}\left\{ (1 + b_i u^{-2} f_i(u^{\lambda_i} t)) \mathcal{X}_{u,i}^w(t, k) \right\} \to -c_i^2 a_\alpha |t|^{\alpha_2} \mathbb{I}_{\{\alpha_2=\alpha_1\}} - c_i^2 \left( \tilde{f}(t) \mathbb{I}_{\{\lambda=2/\alpha\}} \right),
\]

as \( u \to \infty \), uniformly with respect to \( t \in [S_1, S_2], k \in K_u, w_i \in [-W, W] \). Moreover, for any \( t, t' \in [S_1, S_2] \) uniformly with respect to \( k \in K_u \), any \( w_i \in \mathbb{R} \),

\[
\text{Var} \left( (1 + b_i u^{-2} f_i(u^{\lambda_i} t)) \mathcal{X}_{u,i}^w(t, k) \right) \to 2c_i^2 a_\alpha |t - t'|^{\alpha_2} \mathbb{I}_{\{\alpha_2=\alpha_1\}},
\]

as \( u \to \infty \). Combination of (22) and (23) shows that the finite-dimensional distributions of

\[
\{(1 + b_i u^{-2} f_i(u^{\lambda_i} t)) \mathcal{X}_{u,i}^w(t, k) | t \in [S_1, S_2] \}
\]
weakly converge to the finite-dimensional distributions of \( \{ \zeta(t), t \in [S_1, S_2] \} \). Moreover, by (19) we have that there exists a constant \( C > 0 \) such that for all \( t, t' \in [S_1, S_2] \) and all large \( u \)

\[
\sup_{k \in K_u} \text{Var} \left( (1 + b_i u^{-2} f_i(u^\lambda t)) \chi^{w}_{u,i}(t, k) - (1 + b_i u^{-2} f_i(u^\lambda t')) \chi^{w}_{u,i}(t', k) \right) \\
\leq m^2 n^i (k) \text{Var} \left( \xi_i(u^{-2} t) - \xi_i(u^{-2} t') \right) \leq C |t-t'|^\alpha,
\]

which combined with (22) implies that the family of distributions

\[
\mathbb{P} \left\{ (1 + bu^{-2} f(u^\lambda t)) \chi^{w}_{u}(t, k) \in (\cdot) \right\}
\]

is uniformly tight with respect to \( k \in K_u \) and \( w \) in a compact set of \( \mathbb{R}^n \). Consequently,

\[
\{(1 + bu^{-2} f(u^\lambda t)) \chi^{w}_{u}(t, k), t \in [S_1, S_2] \}
\]

weakly converges to \( \{ \zeta(t), t \in [S_1, S_2] \} \).

Since

\[
\lim_{u \to \infty} \max_{1 \leq i \leq n} \sup_{k \in K_u} \sup_{t \in [S_1, S_2]} \left| (1 + b_i u^{-2} f_i(u^\lambda t)) - 1 \right| = 0,
\]

we conclude that

\[
\{ \chi^{w}_{u}(t, k), t \in [S_1, S_2] \}
\]

weakly converges to \( \{ \zeta(t), t \in [S_1, S_2] \} \).

**Upper bound for \( I_1(u) \).** We first show that

\[
c_u(w) := \sup_{k \in K_u} \left| \mathbb{P} \left\{ \exists t \in [S_1, S_2] \chi^{w}_{u}(t, k) > w \right\} - \mathbb{P} \left\{ \exists t \in [S_1, S_2] \zeta(t) > w \right\} \right|
\]

\[
= \sup_{k \in K_u} \left| \mathbb{P} \left\{ \sup_{t \in [S_1, S_2]} \min_{1 \leq i \leq n} (\chi^{w}_{u,i}(t, k) - w_i) > 0 \right\} - \mathbb{P} \left\{ \sup_{t \in [S_1, S_2]} \min_{1 \leq i \leq n} (\zeta_i(t) - w_i) > 0 \right\} \right| 
\]

for almost all \( w \in \mathbb{R}^n \). Let

\[
A := \left\{ v : \mathbb{P} \left\{ \sup_{t \in [S_1, S_2]} \min_{1 \leq i \leq n} (\zeta_i(t) - w_i) > 0 \right\} \text{ is continuous at } v \right\}.
\]

Note that if \( w \in A \), then

\[
\mathbb{P} \left\{ \sup_{t \in [S_1, S_2]} \min_{1 \leq i \leq n} (\zeta_i(t) - w_i) > x \right\}
\]

is continuous with respect to \( x \) at \( x = 0 \). Hence by the continuity of functional sup min, we have that

\[
c_u(w) \to 0,
\]

for \( w \in A \) and \( \text{mes}(A^c) = 0 \). Thus in light of dominated convergence theorem, we have

\[
I_1(u) \leq e^{nW} \int_{w \in \mathbb{R}^n \cap A} c_u(w) dw + W e^{nW} \sup_{w \in \mathbb{R}^n} \left| 1 - e^{-\sum_{j=1}^{n} \frac{w_j^2}{2m^2_n(j)}} \right| \to 0, \quad u \to \infty.
\]

**Upper bound for \( I_2(u) \).** Using (22) and (23), for some \( \delta \in (0, 1/2) \), \( |w_i| > W \) with \( W \) sufficiently large and all \( u \) large we have

\[
\sup_{k \in K_u, t \in [S_1, S_2]} \mathbb{E} \left\{ (1 + b_i u^{-2} f_i(u^\lambda t)) \chi^{w}_{u,i}(t, k) \right\} \leq C_1 + \delta |w_i|
\]

and

\[
\sup_{k \in K_u, t \in [S_1, S_2]} \text{Var} \left( (1 + b_i u^{-2} f_i(u^\lambda t)) \chi^{w}_{u,i}(t, k) \right) \leq C_2.
\]

Moreover, by the mutual independence of \( \chi^{w}_{u,i}(t, k), 1 \leq i \leq n \)

\[
\mathbb{P} \left\{ \exists t \in [S_1, S_2] \chi^{w}_{u}(t, k) > w \right\} = \mathbb{P} \left\{ \sup_{t \in [S_1, S_2]} \min_{1 \leq i \leq n} (\chi^{w}_{u,i}(t, k) - w_i) > 0 \right\}
\]

\[
\leq \mathbb{P} \left\{ \min_{1 \leq i \leq n} \left( \sup_{t \in [S_1, S_2]} \chi^{w}_{u,i}(t, k) - w_i \right) > 0 \right\}
\]
Consequently, it follows that
\[ \sup_{k \in K_u} \int_{|W_j| > W} e^{\sum_{i=1}^n w_i} \mathbb{P} \left\{ \exists_{t \in [S_1, S_2]} \mathcal{X}^w_{u, i}(t, k) > w \right\} \, dw \leq J_1 \times J_2, \]
where by (24) and Theorem 8.1 of [40]

\begin{align*}
J_1 &= \sup_{k \in K_u} \int_{|W_j| > W} e^{w_j} \mathbb{P} \left\{ \sup_{t \in [S_1, S_2]} \mathcal{X}^w_{u, j}(t, k) > w_j \right\} \, dw_j \\
&\leq \sup_{k \in K_u} \int_{|W_j| > W} e^{w_j} \mathbb{P} \left( \sup_{t \in [S_1, S_2]} \left( (1 + b_i u^{-2} f_i(u_i t)) \mathcal{X}^w_{u, j}(t, k) - \mathbb{E} \{ (1 + b_i u^{-2} f_i(u_i t)) \mathcal{X}^w_{u, j}(t, k) \} \right) > (1 - \delta) w_j - \mathbb{C}_1 \right) \, dw_j \\
&\leq e^{-W} + \int_{W_1}^{\infty} e^{w_j} \mathbb{C}_3 w_j^{2/\alpha} \Psi \left( \frac{(1 - \delta) w_j - \mathbb{C}_1}{\mathbb{C}_2} \right) \, dw_j \\
&=: A_1(W) \to 0, W \to \infty,
\end{align*}

and
\begin{align*}
J_2 &= \sup_{k \in K_u} \prod_{i=1}^n \left( \int_{R} e^{w_i} \mathbb{P} \left( \sup_{t \in [S_1, S_2]} \mathcal{X}^w_{u, i}(t, k) > w_i \right) \, dw_i \right) \\
&\leq \sup_{k \in K_u} \prod_{i=1}^n \left( e^{w_i} + \int_{W_1}^{\infty} e^{w_i} \mathbb{C}_4 w_i^{2/\alpha} \Psi \left( \frac{(1 - \delta) w_i - \mathbb{C}_1}{\mathbb{C}_2} \right) \, dw_i \right) \leq C_5,
\end{align*}

with \( W_1 \) some positive constant. Thus we have

\[ I_2(u) \leq n C_5 A_1(W) \to 0, W \to \infty. \]

Upper bound for \( I_3(u) \). Borell-TIS inequality (see, e.g., [41]) implies that
\[ I_3(u) \to 0, u, W \to \infty. \]

Hence (21) follows.

ii) Suppose that \( \lambda > 2/\alpha \). Observe that

\[ \mathbb{P} \left\{ \exists_{t \in [u^{-\lambda} S_1, u^{-\lambda} S_2]} Z_u(t) > m_u(k) \right\} \]

\[ = \prod_{i=1}^n \mathbb{P} \left\{ \sum_{i=1}^n \Psi \left( m_u(k) \right) \right\} \int_{R^n} e^{\sum_{i=1}^n \left( w_i - \frac{u_i^2}{2 m_u(k) \Psi(m_u(k))} \right)} \mathbb{P} \left\{ \exists_{t \in [S_1, S_2]} \mathcal{X}^w_{u, i}(t, k) > w \right\} \, dw,
\]

where \( \mathcal{X}^w_u(t, k) = (\mathcal{X}^w_{u, 1}(t, k), \ldots, \mathcal{X}^w_{u, n}(t, k)) \) with

\[ \mathcal{X}^w_{u, i}(t, k) = m_u(k) (Z_{u, i}(u^{-\lambda} t) - m_{u, i}(k)) + w_i |\xi_i(0) = m_{u, i}(k) - \frac{w_i}{m_{u, i}(k)}. \]

The rest of derivations for this case is the same as given in the proof for case \( \lambda \leq 2/\alpha \), with exception that

\[ \mathbb{E} \{ (1 + b_i u^{-2} f_i(u_i t)) \mathcal{X}^w_{u, j}(t, k) \} \to -c_i^2 f_i(t), u \to \infty, \]

and

\[ \text{Var} (\mathcal{X}^w_{u, i}(t, k) - \mathcal{X}^w_{u, i}(t', k)) \to 0, u \to \infty. \]
Hence we omit the rest of the proof. □

**Lemma 4.2.** Let \( X(t), (t) \in \mathbb{R} \) be a centered vector-valued stationary Gaussian process with independent coordinates \( X_i \)'s. Suppose that for each \( i = 1, \ldots, n \), \( X_i(t) \) has continuous sample paths, unit variance and correlation function \( r_i(z) \), \( 1 \leq i \leq n \), satisfying

\[
0 < 1 - 2a_i|t|^{\alpha_i} \leq r_i(t) \leq 1 - \frac{a_i}{2}|t|^{\alpha_i}, \quad a_i > 0, \quad \alpha_i \in (0, 2],
\]

for all \( t \in [0, \varepsilon] \) with \( 0 < \varepsilon < 1 \) small enough. Let \( K_u \) be an index set. Then we have for any \( m_u(k) \), \( w_u(l) \) such that

\[
\lim_{u \to \infty} \sup_{k \in K_u} \frac{1}{u} m_u(k) - c = 0, \quad \lim_{u \to \infty} \sup_{l \in K_u} \frac{1}{u} w_u(l) - c = 0,
\]

and any \( T(k, l) > S > 1 \) satisfying \( \lim_{u \to \infty} \sup_{k, l \in K_u} T(k, l) u^{-2/\alpha} = 0 \), that

\[
\mathbb{P} \left\{ \exists t \in [0, S] u^{-2/\alpha} X(t) > m_u(k), \exists \epsilon \in [T(k, l), T(k, l) + S] u^{-2/\alpha} X(t) > w_u(l) \right\}
\]

\[
\leq FS^{2n} \exp(-G(T(k, l) - S)^{\alpha}) \prod_{i=1}^{n} \Psi \left( \frac{m_u(k)}{2} \right)
\]

holds uniformly for any \( k, l \in K_u \) and all \( u \) large where \( \alpha = \min_{1 \leq i \leq n}(\alpha_i) \) and \( F, G \) are two positive constants.

**Proof of Lemma 4.2:** By the independence of \( X_i \)'s, we have that

\[
\mathbb{P} \left\{ \exists t \in [0, S] u^{-2/\alpha} X(t) > m_u(k), \exists \epsilon \in [T(k, l), T(k, l) + S] u^{-2/\alpha} X(t) > w_u(l) \right\}
\]

\[
\leq \prod_{i=1}^{n} \left\{ \sup_{t \in [0, S] u^{-2/\alpha}} X_i(t) > m_{u,i}(k), \sup_{t \in [T(k, l), T(k, l) + S] u^{-2/\alpha}} X_i(t) > w_{u,i}(k) \right\}
\]

\[
\leq n \prod_{i=1}^{n} \left\{ \sup_{t \in [0, S] u^{-2/\alpha}} X_i(t) > m_{u,i}(k), \sup_{t \in [T(k, l), T(k, l) + S] u^{-2/\alpha}} X_i(t) > w_{u,i}(k) \right\}.
\]

Application of Lemma 6.3 in [40] (or Theorem 3.1 in [42]) for each term in the above product establishes the claim. □

**Proof of Theorem 2.1:** Let

\[
\pi(u) := \mathbb{P} \left\{ \exists t \in E(u) \ X_u(t) > m_u \right\} = \mathbb{P} \left\{ \exists t \in E(u) \ \frac{X_u(t)}{\sigma_u(t)} > \frac{m_u}{\sigma_u(0)} \right\}.
\]

In view of A2-A3 and by Gordon inequality (see, e.g., Lemma 5.1 in [16]), we have that for \( \epsilon \in (0, 1) \) and \( u \) sufficiently large

\[
\mathbb{P} \left\{ \exists t \in E(u) \ Z_{u,-\epsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\} \leq \pi(u) \leq \mathbb{P} \left\{ \exists t \in E(u) \ Z_{u,\epsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\},
\]

where

\[
Z_{u,\pm\epsilon}(t) = \frac{Y_{\pm\epsilon}(t)}{w_{u,\pm\epsilon}(t)}, \quad t \in \mathbb{R},
\]

with \( Y_{\pm\epsilon}(t), t \in \mathbb{R} \) being homogeneous vector-valued Gaussian processes with independent coordinates \( Y_{i,\pm\epsilon}(t), t \in \mathbb{R} \) having continuous trajectories, unit variance and correlation function satisfying

\[
r_{i,\pm\epsilon}(t) = e^{-(1+\epsilon)a_i|t|^{\alpha_i}},
\]

and \( w_{u,\pm\epsilon}(t) = (w_{u,\pm\epsilon}(t), \ldots, w_{u,n,\pm\epsilon}(t)) \) with

\[
w_{u,i,\pm\epsilon}(t) = 1 + u^{-2} \left( f_i(u^{\lambda_i} t) \pm \epsilon \right) \left( f_i(u^{\lambda_i} t) \pm \epsilon \right), \quad \epsilon \in (0, 1).
\]
Next, we use the double-sum method to derive an upper and a lower bound of (27) and then show that they are asymptotically tight. We distinguish three scenarios: $\lambda < 2/\alpha$, $\lambda = 2/\alpha$ and $\lambda < 2/\alpha$.

\begin{itemize}
\item[$\diamond$] Case $\lambda < 2/\alpha$. For any $S > 0$, let
\begin{align}
I_k(u) &= \left[ku^{-2/\alpha}S, (k+1)u^{-2/\alpha}S\right), \quad k \in \mathbb{Z}, \quad N_1(u) = \left\lceil \frac{x_1(u)}{Su^{-2/\alpha}} \right\rceil - \left\lfloor \frac{x_1(u)}{Su^{-2/\alpha}} \right\rfloor, \\
N_2(u) &= \left\lceil \frac{x_2(u)}{Su^{-2/\alpha}} \right\rceil + \mathbb{I}_{\{x_2 \leq 0\}}, \quad v_{u,\pm\varepsilon}(k) = (v_{u,1,\pm\varepsilon}(k), \ldots, v_{u,n,\pm\varepsilon}(k)),
\end{align}
with

\begin{align*}
v_{u,i,\pm\varepsilon}(k) &= \frac{m_{u,i}}{\sigma_{u,i}(0)} \sup_{s \in I_k(u)} w_{u,i,\pm\varepsilon}(s), \quad v_{u,i,\pm\varepsilon}(k) = \frac{m_{u,i}}{\sigma_{u,i}(0)} \inf_{s \in I_k(u)} w_{u,i,\pm\varepsilon}(s).
\end{align*}

For $u$ large enough, in view of (27) we have

\begin{align*}
\pi(u) \leq & \Pr\left\{ \exists t \in E(u) Z_{u,+\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\} \leq \sum_{k = N_1(u)}^{N_2(u)} \Pr\left\{ \exists t \in I_k(u) Z_{u,+\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\}, \\
\pi(u) \geq & \Pr\left\{ \exists t \in E(u) Z_{u,-\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\} \geq \sum_{k = N_1(u)+1}^{N_2(u)-1} \Pr\left\{ \exists t \in I_k(u) Z_{u,-\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\} - 2 \sum_{i=1}^{N_2(u)} \Lambda_i(u),
\end{align*}
where

\begin{align*}
\Lambda_1(u) &= \sum_{k = N_1(u)}^{N_2(u)} \Pr\left\{ \exists t \in I_k(u) Z_{u,-\varepsilon}(t) > \frac{m_u}{\sigma_u(0)}, \exists t \in I_{k+1}(u) Z_{u,-\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\}, \\
\Lambda_2(u) &= \sum_{N_1(u) \leq k \leq l \leq N_2(u), l \geq k+2} \Pr\left\{ \exists t \in I_k(u) Z_{u,-\varepsilon}(t) > \frac{m_u}{\sigma_u(0)}, \exists t \in I_l(u) Z_{u,-\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\}.
\end{align*}

\textbf{Asymptotics of $\pi(u)$}. By stationarity of $Y_{+\varepsilon}$ and Lemma 4.1, we have that

\begin{align*}
\pi(u) &\leq \sum_{k = N_1(u)}^{N_2(u)} \Pr\left\{ \exists t \in I_k(u) Y_{+\varepsilon}(t) > v_{u,-\varepsilon}(k) \right\} \\
&\leq \sum_{k = N_1(u)}^{N_2(u)} \Pr\left\{ \exists t \in I_k(u) Y_{+\varepsilon}(t) > v_{u,-\varepsilon}(k) \right\} \\
&\sim \mathcal{H}_{\alpha,(1+\varepsilon)\frac{m_u}{\sigma_u(0)}} \left[ 0, S \right] \sum_{k = N_1(u)}^{N_2(u)} \prod_{i=1}^{n} \Psi(v_{u,i,-\varepsilon}(k)), \ u \to \infty.
\end{align*}

Furthermore,

\begin{align*}
&\sum_{k = N_1(u)}^{N_2(u)} \prod_{i=1}^{n} \Psi(v_{u,i,-\varepsilon}(k)) \\
&\sim \sum_{k = N_1(u)}^{N_2(u)} \left( \prod_{i=1}^{n} \Psi\left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right) \sum_{k = N_1(u)}^{N_2(u)} \left( \frac{1}{\sqrt{2\pi v_{u,i,-\varepsilon}(k)}} \exp\left( -\frac{v_{u,i,-\varepsilon}^2(k)}{2} \right) \right) \\
&\sim \left( \prod_{i=1}^{n} \Psi\left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right) \sum_{k = N_1(u)}^{N_2(u)} \left( \frac{1}{\sqrt{2\pi v_{u,i,-\varepsilon}(k)}} \exp\left( -\frac{v_{u,i,-\varepsilon}^2(k)}{2} \right) \right) \\
&\sim \left( \prod_{i=1}^{n} \Psi\left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right) \sum_{k = N_1(u)}^{N_2(u)} \left( \frac{1}{\sqrt{2\pi v_{u,i,-\varepsilon}(k)}} \exp\left( -\frac{v_{u,i,-\varepsilon}^2(k)}{2} \right) \right) \\
&\leq \left( \prod_{i=1}^{n} \Psi\left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right) S^{-1} u^{\frac{2}{\alpha} - \lambda} \int_{x_1}^{x_2} \exp\left( -\frac{n \int_0^t \frac{\varepsilon}{\sigma_i^2} dt}{\sigma_i^2} \right) dt,
\end{align*}

(29)
where $\tilde{f}_i(t) = f_i(t) - \varepsilon \left| f_i(t) \right| - \varepsilon$. In order to prove (29), we note that for $-\infty < x_1 < x_2 < \infty$,

$$
\sum_{k=0}^{N_2(u)} \exp \left( - \sum_{i=1}^{n} m_{u,i}^{-2} \inf_{s \in \mathbb{R}^k} \left( f_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) - \varepsilon \left| f_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) \right| - \varepsilon \right) \right)
$$

$$
\sim S^{-1} u^{2/\alpha - \lambda} \int_{x_1}^{x_2} \exp \left( - \sum_{i=1}^{n} \frac{\tilde{f}_i(t)}{\sigma_i^2} \right) dt, \quad u \to \infty,
$$

which implies that (29) holds for $-\infty < x_1 < x_2 < \infty$. Next we assume that $-\infty < x_1 < x_2 = \infty$. Let $y$ be a positive constant satisfying $x_1 < y < \infty$ and $N(u, y) = \left[ \frac{uy^{2/\alpha - \lambda}}{y} \right]$. Then it follows that

$$
\sum_{k=0}^{N_2(u)} \exp \left( n \sum_{i=1}^{n} m_{u,i}^{-2} \inf_{s \in \mathbb{R}^k} \left( f_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) - \varepsilon \left| f_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) \right| - \varepsilon \right) \right)
$$

$$
\sim S^{-1} u^{2/\alpha - \lambda} \int_{x_1}^{y} \exp \left( - \sum_{i=1}^{n} \frac{\tilde{f}_i(t)}{\sigma_i^2} \right) dt, \quad u \to \infty.
$$

(30)

By Potter’s Theorem (Theorem 1.5.6 in [43]) and the fact that for $j \in \{1 \leq i \leq n : \lambda_i = \lambda\}$, $f_j(t)$ is regularly varying at $\infty$ with positive index, we have that for any $\eta > 0$ and sufficiently large $y$ and $u$

$$
\left| \frac{\sigma_i^2}{\sigma_{u,i}^2(0)} \sum_{\lambda_i = \lambda} m_{u,j}^{-2} \inf_{s \in \mathbb{R}^k} \left( f_j(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) - \varepsilon \left| f_j(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) \right| - \varepsilon \right) \right|
$$

$$
\leq \eta \sum_{\lambda_i = \lambda} |\tilde{f}_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_k)|
$$

holds for all $k > N(u, y)$. Then we have that for $k > N(u, y)$

$$
\left| \sum_{\lambda_i = \lambda} m_{u,j}^{-2} \inf_{s \in \mathbb{R}^k} \left( f_j(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) - \varepsilon \left| f_j(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) \right| - \varepsilon \right) - \sum_{\lambda_i = \lambda} \frac{\tilde{f}_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_k)}{\sigma_i^2} \right|
$$

$$
\leq \eta \sum_{\lambda_i = \lambda} \left| \frac{\tilde{f}_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_k)}{\sigma_i^2} \right|
$$

Using (4), it follows that

$$
\lim_{u \to \infty, \sup_{N_1(u) \leq k \leq N_2(u)}} \left| \sum_{\lambda_i < \lambda} m_{u,j}^{-2} \inf_{s \in \mathbb{R}^k} \left( f_j(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) - \varepsilon \left| f_j(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) \right| - \varepsilon \right) - \sum_{\lambda_i < \lambda} \frac{\tilde{f}_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_k)}{\sigma_i^2} \right| = 0.
$$

Hence, for sufficiently large $y$ and $u$ we have that

$$
\sum_{i=1}^{n} m_{u,i}^{-2} \inf_{s \in \mathbb{R}^k} \left( f_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) - \varepsilon \left| f_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) \right| - \varepsilon \right) \geq \sum_{i=1}^{n} \frac{\tilde{f}_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_k)}{\sigma_i^2} - \eta \sum_{i=1}^{n} \frac{|\tilde{f}_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_k)|}{\sigma_i^2}
$$

holds for $k > N(u, y)$. Combining the above with (5) implies that

$$
\sum_{k=N(u,y)+1}^{N_2(u)} \exp \left( - \sum_{i=1}^{n} m_{u,i}^{-2} \inf_{s \in \mathbb{R}^k} \left( f_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) - \varepsilon \left| f_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_s) \right| - \varepsilon \right) \right)
$$

$$
\leq \sum_{k=N(u,y)+1}^{N_2(u)} \exp \left( - \sum_{i=1}^{n} \frac{\tilde{f}_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_k)}{\sigma_i^2} + \eta \sum_{i=1}^{n} \frac{|\tilde{f}_i(u^{\lambda_{i}} - \frac{\varphi}{\sigma_i} S_k)|}{\sigma_i^2} \right)
$$

$$
\leq u^{\frac{\alpha}{\alpha - \lambda}} S^{-1} \int_{x_1}^{y} \exp \left( - \sum_{i=1}^{n} \frac{\tilde{f}_i(t)}{\sigma_i^2} + \eta \sum_{i=1}^{n} \frac{|\tilde{f}_i(t)|}{\sigma_i^2} \right) dt,
$$

which together with (30) and the arbitrariness of $\eta > 0$ confirms that (29) holds. For other cases of $x_1$ and $x_2$, we can similarly show that (29) is satisfied. By (4) and (5), we have that

$$
\int_{x_1}^{x_2} \exp \left( - \sum_{i=1}^{n} \frac{\tilde{f}_i(t)}{\sigma_i^2} \right) dt < \infty.
$$
Consequently,

\begin{equation}
\pi(u) \leq \mathcal{H}_{a, \frac{\alpha}{2\sigma}} I_{(\alpha=\alpha)} u^{2/\alpha-\lambda} \int_{x_1}^{x_2} \exp \left( -\sum_{i=1}^{n} \frac{f_i(t)}{\sigma_i^2} \right) dt \left( \prod_{i=1}^{n} \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right),
\end{equation}

as \( u \to \infty, S \to \infty, \varepsilon \to 0 \). Analogously, we have

\begin{equation}
\geq \mathcal{H}_{a, \frac{\alpha}{2\sigma}} I_{(\alpha=\alpha)} u^{2/\alpha-\lambda} \int_{x_1}^{x_2} \exp \left( -\sum_{i=1}^{n} \frac{f_i(t)}{\sigma_i^2} \right) dt \left( \prod_{i=1}^{n} \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right),
\end{equation}

as \( u \to \infty, S \to \infty, \varepsilon \to 0 \).

**Upper bound for \( \Lambda_1(u) \).** It follows that

\[
\Lambda_1(u) = \sum_{k=N_1(u)}^{N_2(u)-1} \left( P \left\{ \exists \ell \in I_k(u) Z_{u,-\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\} + P \left\{ \exists \ell \in I_{k+1}(u) Z_{u,-\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\} \right)
\]

\[
- P \left\{ \exists \ell \in I_k(u), \ell \in I_{k+1}(u) Z_{u,-\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\}
\]

\[
\leq \sum_{k=N_1(u)}^{N_2(u)} \left( P \left\{ \exists \ell \in I_k(u) Y_{-\varepsilon}(t) > \tilde{u}_{u,+\varepsilon}(k) \right\} + P \left\{ \exists \ell \in I_{k+1}(u) Y_{-\varepsilon}(t) > \tilde{u}_{u,+\varepsilon}(k) \right\} \right)
\]

\[
- P \left\{ \exists \ell \in I_k(u), \ell \in I_{k+1}(u) Y_{-\varepsilon}(t) > \tilde{u}_{u,+\varepsilon}(k) \right\}
\]

\[
\sim \left( 2\mathcal{H}_{a,1-\varepsilon} I_{(\alpha=\alpha)} [0,S] - \mathcal{H}_{a,2\varepsilon} I_{(\alpha=\alpha)} [0,2S] \right) \sum_{k=N_1(u)}^{N_2(u)} \left( \prod_{i=1}^{n} \Psi(v_{u,i,+\varepsilon}(k)) \right)
\]

\begin{equation}
= o \left( u^{2/\alpha-\lambda} \prod_{i=1}^{n} \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right), \quad u \to \infty, S \to \infty, \varepsilon \to 0,
\end{equation}

where

\[
\tilde{u}_{u,i,+\varepsilon}(k) = \min \left( \inf_{s \in I_k(u)} w_{u,i,+\varepsilon}(s), \inf_{s \in I_{k+1}(u)} w_{u,i,+\varepsilon}(s) \right)
\]

and

\[
\tilde{u}_{u,i,+\varepsilon}(k) = \max \left( v_{u,i,+\varepsilon}(k), v_{u,i,+\varepsilon}(k+1) \right).
\]

**Upper bound for \( \Lambda_2(u) \).** In light of Lemma 4.2, we have that

\[
\Lambda_2(u) = \sum_{N_1(u) \leq k, l \leq N_2(u), l \geq k+2} P \left\{ \exists \ell \in I_k(u) Z_{u,-\varepsilon}(t) > \frac{m_u}{\sigma_u(0)}, \exists \ell \in I_l(u) Z_{u,-\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\}
\]

\[
\leq \sum_{N_1(u) \leq k, l \leq N_2(u), l \geq k+2} P \left\{ \exists \ell \in I_k(u) Y_{-\varepsilon}(t) > \tilde{v}_{u,+\varepsilon}(k), \exists \ell \in I_l(u) Y_{-\varepsilon}(t) > \tilde{v}_{u,+\varepsilon}(l) \right\}
\]

\[
\leq \sum_{N_1(u) \leq k, l \leq N_2(u), l \geq k+2} P \left\{ \exists \ell \in I_k(u) Y_{-\varepsilon}(t) > \tilde{v}_{u,+\varepsilon}(k), \exists \ell \in I_{l-1}(u) Y_{-\varepsilon}(t) > \tilde{v}_{u,+\varepsilon}(l) \right\}
\]

\[
\leq \sum_{N_1(u) \leq k, l \leq N_2(u), l \geq k+2} C_1 S^{2n} \exp(-C_2((l - k - 1)S)^n) \prod_{i=1}^{n} \Psi \left( \frac{\tilde{v}_{u,i,-\varepsilon}(k) + \tilde{v}_{u,i,-\varepsilon}(l)}{2} \right)
\]

\[
\leq 2 \sum_{l=1}^{\infty} C_1 S^{2n} \exp(-C_2(lS)^n) \sum_{k=N_1(u)}^{N_2(u)} \prod_{i=1}^{n} \Psi (\tilde{v}_{u,i,-\varepsilon}(k))
\]

\[
\leq S^{2n} \exp(-C_3 S^n) u^{2/\alpha-\lambda} \prod_{i=1}^{n} \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right)
\]

\begin{equation}
= o \left( u^{2/\alpha-\lambda} \prod_{i=1}^{n} \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right), \quad u \to \infty, S \to \infty,
\end{equation}

where \( \tilde{v}_{u,i,+\varepsilon}(k) = \max (v_{u,i,+\varepsilon}(k), v_{u,i,+\varepsilon}(k+1)) \).
where
\[
\bar{v}_{u,i,+\epsilon}(k) = \frac{m_{u,i}}{\sigma_{u,i}(0)} \inf_{s \in I_k(u)} w_{u,i,+\epsilon}(s).
\]
Combination of (29)-(34) leads to
\[
\pi(u) \sim \mathcal{H}_{\alpha, \frac{u}{\sigma^2(\alpha)}} \left( u^{2/\alpha - \lambda} \int_{x_1}^{x_2} \exp \left( -\frac{n}{i=1} \sum \tilde{f}(t) \frac{dt}{\sigma^2_i} \right) \prod_{i=1}^n \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right), \quad u \to \infty.
\]

\(\diamond\) Case \(\lambda = 2/\alpha\). Without loss of generality we assume that \(x_1 = -\infty\) and \(x_2 = \infty\). The cases \(x_1 > -\infty\) and \(x_2 < \infty\) can be dealt with analogously. In what follows, we use notation introduced in (28) and set \(\tilde{I}(u) = I_0(u) \cup I_{-1}(u)\). Observe that for large \(u\)
\[
\pi(u) \geq \mathbb{P} \left\{ \exists t \in \tilde{I}(u) \right\} \left( Z_{u,-\epsilon}(t) > \frac{m_u}{\sigma_u(0)} \right),
\]
\[
\pi(u) \leq \mathbb{P} \left\{ \exists t \in \tilde{I}(u) \right\} \left( Z_{u,+\epsilon}(t) > \frac{m_u}{\sigma_u(0)} \right) + \sum_{k=1}^{N_2(u)} \mathbb{P} \left\{ \exists t \in I_k(u) \right\} \left( Z_{u,+\epsilon}(t) > \frac{m_u}{\sigma_u(0)} \right).
\]

Lemma 4.1 yields that
\[
\mathbb{P} \left\{ \exists t \in \tilde{I}(u) \right\} \left( Z_{u,+\epsilon}(t) > \frac{m_u}{\sigma_u(0)} \right) \sim \mathcal{P}_{\alpha, \frac{u}{\sigma^2(\alpha)}} [-S, S] \prod_{i=1}^n \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right),
\]
as \(u \to \infty, \epsilon \to 0\). Moreover, in light of Lemma 4.1 and (5) we have
\[
\sum_{k=1}^{N_2(u)} \mathbb{P} \left\{ \exists t \in I_k(u) \right\} \left( Z_{u,+\epsilon}(t) > \frac{m_u}{\sigma_u(0)} \right) \sim \mathcal{H}_{\alpha, (1+\epsilon)} \frac{u}{\sigma^2(\alpha)} \left( [0, S] \sum_{k=1}^{N_2(u)} \prod_{i=1}^n \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right)
\]
\[
\sim \mathcal{H}_{\alpha, (1+\epsilon)} \frac{u}{\sigma^2(\alpha)} \left( [0, S] \prod_{i=1}^n \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right) \sum_{k=1}^{N_2(u)} \exp \left( -\frac{n}{i=1} \sum \inf_{s \in [k-1]} \frac{\tilde{f}_i(S)}{\sigma^2_i} \right)
\]
\[
\leq \mathcal{H}_{\alpha, (1+\epsilon)} \frac{u}{\sigma^2(\alpha)} \left( \prod_{i=1}^n \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right) S e^{-\eta \ln S} = o \left( \prod_{i=1}^n \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right), \quad u \to \infty, \epsilon \to 0, S \to \infty,
\]
where \(\eta \in (1, \infty)\) is a constant. Inserting (37)-(38) into (35)-(36) and letting \(S \to \infty\), we obtain that
\[
\pi(u) \sim \mathcal{P}_{\alpha, \frac{u}{\sigma^2(\alpha)}} \prod_{i=1}^n \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right), \quad u \to \infty.
\]
This establishes the claim.

\(\diamond\) Case \(\lambda > \frac{2}{\alpha}\). Without loss of generality we assume that \(x_1 = -\infty\) and \(x_2 = \infty\). For any \(S > 0\), define
\[
J_k(u) = \left[ ku^{-\lambda} S \right], k \in \mathbb{Z}, \quad \bar{J}(u) = J_0(u) \cup J_{-1}(u),
\]
\[
K_1(u) = \left[ \frac{x_1(u)}{S u^{-\lambda}} \right] - \mathbb{I}_{(x_1 \leq 0)}, \quad K_2(u) = \left[ \frac{x_2(u)}{S u^{-\lambda}} \right] + \mathbb{I}_{(x_2 \leq 0)}, \quad \Psi_{u, \pm \epsilon}(k) = (\Psi_{u,1, \pm \epsilon}(k), \ldots, \Psi_{u,n, \pm \epsilon}(k)),
\]
with
\[ v_{u,i,+\varepsilon}(k) = \frac{m_{u,i}}{\sigma_{u,i}(0)} \sup_{s \in J_k(u)} w_{u,i,+\varepsilon}(s), \quad v_{u,i,-\varepsilon}(k) = \frac{m_{u,i}}{\sigma_{u,i}(0)} \inf_{s \in J_k(u)} w_{u,i,-\varepsilon}(s). \]

Then for \( u \) large enough, we have
\begin{align*}
(39) \quad &\pi(u) \geq \mathbb{P}\left\{ \exists t \in J(u) \, Z_{u,-\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\} , \\
(40) \quad &\pi(u) \leq \mathbb{P}\left\{ \exists t \in J(u) \, Z_{u,+\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\} + \sum_{k=K_1(u)}^{K_2(u)} \mathbb{P}\left\{ \exists t \in J_k(u) \, Z_{u,+\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\} .
\end{align*}

It follows from Lemma 4.1 that
\begin{align*}
(41) \quad &\mathbb{P}\left\{ \exists t \in J(u) \, Z_{u,\pm\varepsilon}(t) > \frac{m_u}{\sigma_u(0)} \right\} \sim \int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i \left( \sum_{\varepsilon = \pm 1} \sup_{s \in [-s,s]} Z_{u,\varepsilon}(t) \right) dw} \prod_{i=1}^n \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) , \\
(42) \quad &\pi(u) \sim \left( \int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i \left( \sum_{\varepsilon = \pm 1} \sup_{s \in [-s,s]} Z_{u,\varepsilon}(t) \right) dw} \prod_{i=1}^n \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right), \quad u \to \infty, \quad S \to \infty.
\end{align*}

Inserting (41)-(42) into (39)-(40) and letting \( S \to \infty \) and \( \varepsilon \to 0 \) we derive that
\[ \pi(u) \sim \left( \int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i \left( \sum_{\varepsilon = \pm 1} \sup_{s \in [-s,s]} Z_{u,\varepsilon}(t) \right) dw} \prod_{i=1}^n \Psi \left( \frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right), \quad u \to \infty. \]

This completes the proof. \( \square \)

**Proof of Theorem 3.1**: We first focus on the case of \( t_0 \in (0,T) \). Set
\[ E(u) = [-\delta(u), \delta(u)], \quad D(u) := [t_0 - \theta, t_0 + \theta] \setminus (t_0 + E(u)), \]
where \( \theta \in (0, \frac{1}{2}) \) is a small constant and \( \delta(u) = \left( \frac{(\ln u)^q}{u} \right)^{2/\beta} \) with \( q > 1 \), \( \beta = \min_{1 \leq i \leq n} \beta_i^* \) and \( \beta_i^* = \min \left( \beta_i^*, 2\gamma_i I_{\{c_i \neq 0\}} + \infty I_{\{c_i = 0\}} \right) \). Then it follows that
\[ \Pi_1(u) \leq \mathbb{P}\left\{ \exists t \in [0,T] \, (X(t) + h(t)) > u1 \right\} \leq \Pi_1(u) + \Pi_2(u) + \Pi_3(u), \]
where
\[ \Pi_1(u) = \mathbb{P}\left\{ \exists t \in E(u) \, (X(t_0 + t) + h(t_0 + t)) > u1 \right\}, \quad \Pi_2(u) = \mathbb{P}\left\{ \exists t \in D(u) \, (X(t) + h(t)) > u1 \right\} , \quad \Pi_3(u) = \mathbb{P}\left\{ \exists t \in [0,T] \setminus [t_0 - \theta, t_0 + \theta] \, (X(t) + h(t)) > u1 \right\} . \]

**Asymptotics of \( \Pi_1(u) \)**. In order to derive the asymptotics of \( \Pi_1(u) \), we check the assumptions in Theorem 2.1. For this purpose, rewrite
\[ \Pi_1(u) = \mathbb{P}\left\{ \exists t \in E(u) \, X_u(t) > u1 \right\}, \quad \text{with} \quad X_u(t) = \frac{X(t_0 + t)}{1 - h(t_0 + t)/u}. \]

It follows straightforwardly that \( \sigma_u(t) = \frac{\sigma_t(t_0 + t)}{1 - h(t_0 + t)/u} \) satisfies \( \lim_{u \to \infty} \sigma_u(0) = \sigma(t_0) > 0 \) implying that \( A1 \) holds. Next we verify \( A2 \). Direct calculation shows that
\[ \frac{\sigma_{u,i}(0)}{\sigma_{u,i}(t)} - 1 = \frac{1}{\sigma_i(t_0 + t)} (\sigma_i(t_0) - \sigma_i(t_0 + t)) + \frac{1}{u - h_i(t_0)} \frac{\sigma_i(t_0)}{\sigma_i(t_0 + t)} (h_i(t_0) - h_i(t_0 + t)). \]
Thus by (6) and (9) we have that for all $u$ large
\begin{equation}
\frac{\sigma_{u,i}(0)}{\sigma_{u,i}(t)} = 1 + \left( \frac{b_i}{\sigma_i(t_0)} |t|^\beta_i + \frac{c_i}{u - h_i(t_0)} |t|^\gamma_i \right) (1 + o(1)), \quad t \to 0.
\end{equation}

Denote by $\tilde{f}_i(t) = \frac{b_i}{\sigma_i(t_0)} |t|^\beta_i \mathbb{I}_{\{\beta_i = \beta_i^*\}} + \frac{c_i}{u - h_i(t_0)} |t|^\gamma_i \mathbb{I}_{\{\beta_i^* = 2\gamma_i\}}$. Then we have
\begin{equation}
\lim_{u \to \infty} \sup_{t \in E(u)} \left| \frac{\left( \frac{\sigma_{u,i}(0)}{\sigma_{u,i}(t)} - 1 \right) u^2 - \tilde{f}_i(u^{2/\beta_i^*})}{|\tilde{f}_i(u^{2/\beta_i^*})| + 1} \right| = 0,
\end{equation}
which confirms that A2 is satisfied. Apparently, A3 follows by (7). Thus we conclude that A1-A3 are satisfied.

Also, (4) holds with $x_1 = -\infty$ and $x_2 = \infty$. Therefore, in light of Theorem 2.1, we have, as $u \to \infty$,
\begin{equation}
\Pi_1(u) \sim u^{\frac{d}{2} - \frac{d}{\hat{g}}} \prod_{i=1}^n \Psi \left( \frac{u - h_i(t_0)}{\sigma_i(t_0)} \right) \begin{cases} \mathcal{H}_{\alpha} \mathcal{L}(\alpha = 1) \int_{-\infty}^{\infty} e^{-\sum_{i=1}^n f_i(x) dx}, & \text{if } \alpha < \beta, \\ \mathcal{P}_{\alpha} \mathcal{L}(\alpha = 1) (-\infty, \infty), & \text{if } \alpha = \beta, \\ 1, & \text{if } \alpha > \beta, \end{cases}
\end{equation}
where $f_i(t) = \frac{b_i}{\sigma_i(t_0)} |t|^\beta_i \mathbb{I}_{\{\beta_i = \beta_i^*\}} + \frac{c_i}{u - h_i(t_0)} |t|^\gamma_i \mathbb{I}_{\{\beta_i^* = 2\gamma_i\}}, 1 \leq i \leq n$.

**Upper bound for $\Pi_2(u)$**. Observe that
\begin{equation}
\Pi_2(u) = \mathbb{P} \{ \exists t \in E(u) (X(t) + h(t)) > u \} \leq \mathbb{P} \left\{ \sup_{t \in [-\theta, \theta] \setminus E(u)} Y_u(t) > u \right\},
\end{equation}
where
\begin{equation}
Y_u(t) = \sum_{i=1}^n G_{u,i}(t) X_i(t_0 + t), \quad t \in [-t_0, T - t_0],
\end{equation}
with
\begin{align*}
G_{u,i}(t) &:= \left( \prod_{j=1, j \neq i}^n \sigma_j^2(t_0 + t) \sigma_j(t_0 + t) \right) \frac{1}{1 - h_i(t_0 + t)/u}, \quad t \in [-t_0, T - t_0], \\
A_u(t) &:= \sum_{k=1}^n \left( \prod_{j=1, j \neq k}^n \frac{\sigma_j^2(t)}{1 - h_j(t)/u^2} \right), \quad t \in [0, T].
\end{align*}

In order to analyze the variance of $Y_u$, we introduce $g_u(t) = \sum_{i=1}^n \frac{1}{\sigma_{u,i}^2(t)}$. Using (43) we have that
\begin{align*}
g_u(t) - g_u(0) &= \sum_{i=1}^n \frac{1}{\sigma_{u,i}^2(t)} - \sum_{i=1}^n \frac{1}{\sigma_{u,i}^2(0)} \\
&= \sum_{i=1}^n \frac{(\sigma_{u,i}(0) - \sigma_{u,i}(t))(\sigma_{u,i}(0) + \sigma_{u,i}(t))}{\sigma_{u,i}^2(0)}, \\
&\geq C_0 \sum_{i=1}^n \frac{1}{\sigma_{u,i}^2(t_0)} \left( \frac{b_i}{\sigma_i(t_0)} |t|^\beta_i + \frac{c_i}{u} |t|^\gamma_i \right) \\
&\geq C (\ln u)^q \frac{1}{u^2}
\end{align*}
holds for all $t \in [-\theta, \theta] \setminus E(u)$ with a positive constant $C$. Consequently,
\begin{equation}
\sup_{t \in [-\theta, \theta] \setminus E(u)} \text{Var}(Y_u(t)) = \sup_{t \in [-\theta, \theta] \setminus E(u)} \left( \sum_{i=1}^n \frac{(1 - h_i(t_0 + t)/u)^2}{\sigma_i^2(t_0 + t)} \right)^{-1} = \sup_{t \in [-\theta, \theta] \setminus E(u)} \frac{1}{g_u(t)} \leq \frac{1}{g_u(0) + \frac{C (\ln u)^q}{u^2}}.
\end{equation}

By (10) and the fact that in view of (8),
\begin{equation}
(\sigma_i(t) - \sigma_i(s))^2 \leq \mathbb{E} \{(X_i(t) - X_i(s))^2 \} \leq C_1 |t - s|^\mu_i, \quad s, t \in [0, T],
\end{equation}
we have that there exists $\mu_3 > 0$ such that
\[
\max_{i=1,\ldots,n} (G_{u,i}(t) - G_{u,i}(s))^2 \leq C_2|t-s|^{\mu_3}, \ s, t \in [0,T],
\]
which together with (8) implies that
\[
\mathbb{E}(Y_u(t) - Y_u(s))^2 = \mathbb{E}\left(\sum_{i=1}^n G_{u,i}(t)X_i(t) - \sum_{i=1}^n G_{u,i}(s)X_i(s)\right)^2 = \sum_{i=1}^n \mathbb{E}(G_{u,i}(t)X_i(t) - G_{u,i}(s)X_i(s))^2
\]
\[
\leq 2\sum_{i=1}^n \sigma_i^2(t) (G_{u,i}(t) - G_{u,i}(s))^2 + 2\sum_{i=1}^n G_{u,i}(s)^2 \mathbb{E}(X_i(t) - X_i(s))^2
\]
(49)
\[
\leq C_3|t-s|^{\mu_4}, \ s, t \in [0,T]
\]
with $\mu_4 > 0$. Consequently Piterbarg inequality (Theorem 8.1 in [40]) gives that
\[
\Pi_2(u) \leq \mathbb{P}\left\{\sup_{t \in [-\theta,\theta]} Y_u(t) > u\right\} \\
\leq C_4 u^{2/\mu_4} \Psi(\sqrt{u^2 g_u(0) + C(u^n)^q})
\]
\[
= o\left(u^{(2-\frac{2}{\mu_4}) + \frac{n}{\mu_4}} \prod_{i=1}^n \Psi\left(\frac{u - h_i(t_0)}{\sigma_i(t_0)}\right)\right), \ u \to \infty.
\]

**Upper bound for $\Pi_3(u)$.** Note that there exists $\epsilon \in (0,1)$ such that
\[
\sup_{t \in [0,T] \setminus [\theta - t_0, t_0 + \theta]} \sigma_i(t) \leq (1 - \epsilon)\sigma_i(t_0), \ 1 \leq i \leq n.
\]
Thus
\[
\sup_{t \in [0,T] \setminus [-\theta,\theta]} \text{Var}(Y_u(t)) = \left(\inf_{t \in [0,T] \setminus [-\theta,\theta]} g_u(t)\right)^{-1} \leq (1 - \epsilon/2)^{-2} \left(\sum_{i=1}^n \frac{1}{\sigma_i^2(t_0)}\right)^{-1},
\]
which together with (49) and Piterbarg inequality (Theorem 8.1 in [40]) implies that
\[
\Pi_3(u) = \mathbb{P}\left\{\exists t \in (0,T) \setminus \{t_0\} \left(\mathbf{X}(t) + \mathbf{h}(t)\right) > u\mathbf{1}\right\}
\]
\[
\leq \mathbb{P}\left\{\sup_{t \in [0,T] \setminus \{t_0\}} Y_u(t) > u\right\}
\]
\[
\leq C_5 u^{2/\mu_4} \Psi\left((1 - \epsilon/2) \left(\sum_{i=1}^n \frac{1}{\sigma_i^2(t_0)}\right)^{1/2} u\right)
\]
\[
= o(\Pi_1(u)), \ u \to \infty.
\]
Therefore, we conclude that
\[
\mathbb{P}\left\{\exists t \in [0,T] \left(\mathbf{X}(t) + \mathbf{h}(t)\right) > u\mathbf{1}\right\} \sim \Pi_1(u), \ u \to \infty,
\]
which combined with (45) establishes the claim.

The case of $t_0 = 0 \ (t_0 = T)$ can be dealt with using the same argument as above with the only difference that one has to substitute $E(u)$ by $[0,\delta(u)]$ (or by $[-\delta(u),0]$).

Thus the proof is complete. □

**Proof of Theorem 3.4:** i) We provide the proof only for case $t_0 \in (0,T)$, since cases $t_0 = 0$ and $t_0 = T$ can be established analogously. Let $E(u) = [-\delta(u),\delta(u)]$, where $\delta(u) = \left(\frac{(\ln u)^q}{u}\right)^{1/q}$ with $q > 1$. It follows that
\[
\Pi(u) \leq \mathbb{P}\left\{\exists t \in [0,T] \left(\mathbf{X}(t) + \mathbf{h}(t)\right) > u\mathbf{1}\right\} \leq \Pi(u) + \Pi_1(u),
\]
where
\[ \Pi(u) = \mathbb{P} \{ \exists t \in E(u) \mid X(t_0 + t) + h(t_0 + t) > u1 \}, \quad \Pi_1(u) = \mathbb{P} \{ \exists t \in [0, T] \mid (t_0 + E(u)) (X(t) + h(t)) > u1 \}. \]

In order to derive the asymptotics of \( \Pi(u) \) we apply Theorem 2.1 by checking conditions A1-A3. Set \( \sigma_{u,i}(t) = \frac{1}{1-h_i(t_0+t)/u} \) and then \( \lim_{u \to \infty} \sigma_{u,i}(0) = 1 \), which indicates that A1 holds. By the fact that
\[ \frac{\sigma_{u,i}(0)}{\sigma_{u,i}(t)} - 1 = \frac{b_i(t_0) - b_i(t_0 + t)}{u - h_i(t_0)}, \]
and (9), we have
\[ \lim_{u \to \infty} \sup_{t \not= 0} \left| \frac{\left( \frac{\sigma_{u,i}(0)}{\sigma_{u,i}(t)} - 1 \right) u^2 - c_i |u|^\gamma |t|^{\gamma_i} + 1}{c_i |u|^\gamma |t|^{\gamma_i}} \right| = 0. \]

This confirms that A2 is satisfied. Moreover, (13) implies that
\[ \lim_{u \to \infty} \sup_{t \not= s} \left| \frac{1 - r_i(t_0 + t, t_0 + s)}{a_i(t_0)|t - s|^{\alpha_i} + 1} - 1 \right| = 0, \]
which means that A3 holds. Also, we have that (4) holds with \( x_1 = -\infty \) and \( x_2 = \infty \). Therefore, by Theorem 2.1
\[
\Pi(u) \sim u^{\frac{2}{\gamma} - \frac{\gamma_i}{2}} \prod_{i=1}^{n} \Psi(u - h_{m,i}) \left\{ \begin{array}{ll}
\mathcal{H}_{\alpha, \alpha_0 I(\alpha = 0)} \int_{-\infty}^{\infty} e^{-\sum_{i=1}^{n} f_i(x)} dx, & \text{if } \alpha < 2; \\
\mathcal{P}^{f}_{\alpha, \alpha_0 I(\alpha = 0)}(-\infty, \infty), & \text{if } \alpha = 2; \\
1, & \text{if } \alpha > 2,
\end{array} \right.
\]
where \( \gamma = \min_{1 \leq i \leq n} \left( \gamma_i I(c_i \not= 0) + \infty I(c_i = 0) \right) \), \( f_i(t) = c_i |t|^\gamma I(\gamma = \gamma_i), 1 \leq i \leq n \). Next we show that \( \Pi_1(u) = o(\Pi(u)), u \to \infty \). Observe that
\[ \Pi_1(u) = \mathbb{P} \{ \exists t \in [0, T] \mid (t_0 + E(u)) (X(t) + h(t)) > u1 \} \leq \mathbb{P} \left\{ \sup_{t \in [0, T] \setminus (t_0 + E(u))} Y_u(t) > u \right\}, \]
where
\[ Y_u(t) = \sum_{i=1}^{n} G_{a,i}(t) X_i(t_0 + t), \quad t \in [-t_0, T - t_0], \]
with
\[ G_{a,i}(t) := \left( \prod_{j=1, j \not= i}^{n} \left( 1 - h_j(t_0 + t)/u \right)^2 \right) \frac{1}{A_u(t_0 + t)} \frac{1}{1 - h_i(t_0 + t)/u}, \quad t \in [-t_0, T - t_0], \]
\[ A_u(t) = \sum_{k=1}^{n} \left( \prod_{j=1, j \not= k}^{n} \frac{1}{1 - h_j(t)/u} \right), \quad t \in [0, T]. \]

Let
\[ g_u(t) = \sum_{i=1}^{n} \frac{1}{\sigma_{u,i}(t)} = \sum_{i=1}^{n} (1 - h_i(t_0 + t)/u)^2. \]
Then by (9) and the fact that \( \min_{1 \leq i \leq n} c_i > 0 \), we have for \( \theta > 0 \) sufficiently small and \( u \) sufficiently large
\[ g_u(t) - g_u(0) = \sum_{i=1}^{n} (1 - h_i(t_0 + t)/u)^2 - \sum_{i=1}^{n} (1 - h_i(t_0)/u)^2 \]
\[ \geq \sum_{i=1}^{n} \frac{h_i(t_0) - h_i(t_0 + t)}{u} \]
\[ \geq C_1 |t|^\gamma \geq C_1 \frac{(\ln u)^\gamma}{u^2}, \quad t \in [t_0 - \theta, t_0 + \theta] \setminus (t_0 + E(u)). \]
Consequently, there exists $C > 0$ such that
\[
\sup_{t \in [t_0, t_0 + \theta] \setminus (t_0 + E(u))} \Var(Y_u(t)) = \sup_{t \in [t_0, t_0 + \theta] \setminus (t_0 + E(u))} \frac{1}{g_u(t)} \leq \frac{1}{g_u(0) + \frac{C}{u} g_u(0)}. \tag{54}
\]
Moreover, for $\theta > 0$ sufficiently small and $u$ sufficiently large
\[
g_u(t) - g_u(0) \geq \frac{\sum_{i=1}^{n} h_i(t_0) - \sum_{i=1}^{n} h_i(t_0 + t)}{u} \geq \frac{C_2}{u}, \quad t \in [0, T] \setminus [t_0 - \theta, t_0 + \theta]. \tag{52}
\]
Thus there exists $C_1 > 0$ such that
\[
\sup_{t \in [0, T] \setminus [t_0 - \theta, t_0 + \theta]} \Var(Y_u(t)) \leq \frac{1}{g_u(0) + \frac{C_2}{u} g_u(0)} \cdot [0, T] \setminus (t_0 + E(u)),
\]
with $C_2 > 0$. Moreover, in light of (10) and (13), we have that
\[
\mathbb{E} \left( (Y_u(t) - Y_u(s))^2 \right) \leq C_3 |t - s|^\mu, \quad s, t \in [0, T]
\]
for $\mu > 0$. Piterbarg inequality (Theorem 8.1 in [40]) leads to
\[
\Pi_1(u) \leq \mathbb{P} \left\{ \sup_{t \in [0, T] \setminus (t_0 + E(u))} Y_u(t) > u \right\} \leq C_3 u^{2/\mu} \Psi \left( u \sqrt{g_u(0) + \frac{C_2}{u^2} g_u(0)} \right) = o(\Pi(u)), \quad u \to \infty.
\]
This establishes the claim.

ii) Without loss of generality, we assume that $0 < A < B < T$. Then for $\epsilon > 0$ sufficiently small
\[
\mathbb{P} \left\{ \exists t \in [A, B] \left( X(t) + h(A) > u1 \right) \right\} \leq \mathbb{P} \left\{ \exists t \in [0, T] \left( X(t) + h(t) > u1 \right) \right\}
\leq \mathbb{P} \left\{ \exists t \in [0, A-\epsilon] \left( X(t) + h(t) > u1 \right) \right\} + \mathbb{P} \left\{ \exists t \in [A-\epsilon, B+\epsilon] \left( X(t) + h(A) > u1 \right) \right\}
+ \mathbb{P} \left\{ \exists t \in [B+\epsilon, T] \left( X(t) + h(t) > u1 \right) \right\}
\]
In view of (13) and (14) and by Theorem 4.1 in [16], we have that for any $0 \leq x < y \leq T$
\[
\mathbb{P} \left\{ \exists t \in [x, y] \left( X(t) + h(A) > u1 \right) \right\} = \mathbb{P} \left\{ \exists t \in [x, y] \left( X(t) > u1 - h(A) \right) \right\}
\sim u^2 \int_x^y \mathcal{H}_{\alpha, \alpha}(t) \prod_{i=1}^{n} \Psi \left( u - h_{m,i} \right) dt, \quad u \to \infty,
\]
where $\int_x^y \mathcal{H}_{\alpha, \alpha}(t) \prod_{i=1}^{n} \Psi \left( u - h_{m,i} \right) dt$ is a finite and positive constant (see [16]). Next we show that $\mathbb{P} \left\{ \exists t \in [0, A - \epsilon] \left( X(t) + h(t) > u1 \right) \right\}$ is negligible. Rewrite
\[
\mathbb{P} \left\{ \exists t \in [0, A - \epsilon] \left( X(t) + h(t) > u1 \right) \right\} = \mathbb{P} \left\{ \exists t \in [0, A - \epsilon] Y_u(t) > u1 \right\},
\]
where $Y_u$ is defined in (50). Note that (53) still holds in the case considered with $[0, A - \epsilon]$ instead of $[0, T] \setminus [t_0 - \theta, t_0 + \theta]$. Therefore, in view of (54), by Piterbarg inequality we have that
\[
\mathbb{P} \left\{ \exists t \in [0, A - \epsilon] Y_u(t) > u1 \right\} \leq C_4 u^{2/\mu} \Psi \left( u \sqrt{g_u(0) + \frac{C_1}{u}} \right) = o \left( u^2 \prod_{i=1}^{n} \Psi \left( u - h_{m,i} \right) \right), \quad u \to \infty.
\]
Analogously,
\[
\mathbb{P} \left\{ \exists t \in [B+\epsilon, T] \left( X(t) + h(t) > u1 \right) \right\} = o \left( u^2 \prod_{i=1}^{n} \Psi \left( u - h_{m,i} \right) \right), \quad u \to \infty.
\]
Therefore, we conclude that as $u \to \infty$

$$u^2 \int_A^B \mathcal{H}_{\alpha,a(t)}I_{(\alpha=1)} dt \prod_{i=1}^n \Psi(u - h_{i,i}) \leq \mathbb{P} \{ \exists t \in [0,T] \left( X(t) + h(t) > u1 \right) \}
\leq u^2 \int_{A-\epsilon}^{B+\epsilon} \mathcal{H}_{\alpha,a(t)}I_{(\alpha=1)} dt \prod_{i=1}^n \Psi(u - h_{i,i}) .$$

We establish the claim by letting $\epsilon \to 0$ in the above inequalities. This completes the proof. \hfill \Box

**Proof of Proposition 3.6:** We notice that

$$p(u) = \mathbb{P} \{ \exists t \in [0,T] \left( B_\alpha(t) - ct > ud \right) \} = \mathbb{P} \left\{ \exists t \in [0,T] \left( \frac{1}{d} B_\alpha(t) - \frac{ct}{d} > u1 \right) \right\} ,$$

and the variance function $\sigma_i^2(t)$ and correlation function $r_i(s,t)$ of $\frac{B_\alpha(t)}{d_i}$ satisfy

$$r_i(s,t) = 1 - \frac{1}{2T^{\alpha_i}} |t-s|^{\alpha_i} (1 + o(1)), s, t \to T, \quad \sigma_i(t) = \frac{T^{\alpha_i/2}}{d_i} - \frac{\alpha_i}{2d_i} T^{\alpha_i/2-1}(T-t)(1 + o(1)), t \to T,$$

where $T$ is the unique maximum point of $\sigma_i(t), 1 \leq i \leq n$ over $[0,T]$. Moreover,

$$-\frac{c_i t}{d_i} = -\frac{c_i T}{d_i} + \frac{c_i}{d_i} |t-T|, \quad t \to T.$$

Therefore, in light of Theorem 3.1 and Corollary 3.3, we have that

$$\mathbb{P} \left\{ \exists t \in [0,T] \left( B_\alpha(t) - ct > ud \right) \right\} \sim u^{(\frac{2}{2-\beta}) + \prod_{i=1}^n \Psi \left( \frac{d_i u + c_i T}{T^{\alpha_i/2}} \right)} \left\{ \mathcal{H}_{\alpha,a}I_{(\alpha=1)} \int_0^\infty e^{-\sum_{i=1}^n f_i(t)} dt, \quad \text{if } \alpha < 1, \right.$$  
$$\frac{p_{\alpha,a}I_{(\alpha=1)}}{\sigma_i} [0,1], \quad \text{if } \alpha = 1, \right.$$  
$$1, \quad \text{if } \alpha > 1, \right.$$  

and

$$\mathbb{P} \left\{ (T - \tau_{\alpha}) u^2 \leq x | \tau_{\alpha} \leq T \right\} \sim \left\{ \begin{array}{ll}
1 - e^{-\left( \sum_{i=1}^n \frac{\alpha_i d_i^2}{2^{\alpha_i + 1}} \right) x}, & \text{if } \alpha < 1, \\
\frac{p_{\alpha,a}I_{(\alpha=1)}}{\sigma_i} [0,1], & \text{if } \alpha = 1, \\
1, & \text{if } \alpha > 1, \end{array} \right.$$  

where $\alpha = \min_1 \leq i \leq n \alpha_i$, $\varsigma = (\varsigma_1, \ldots, \varsigma_n)$ with $\varsigma_i = \frac{d_i^2}{2^{\alpha_i+1}}$ and $f_i(t) = \frac{\alpha_i d_i^2}{2^{\alpha_i+1}} |t|.$ \hfill \Box

5. **APPENDIX**

**Proof of Corollary 3.3:** By definition,

$$\mathbb{P} \left\{ (T - \tau_{\alpha}) u^{2/\beta} \leq x | \tau_{\alpha} \leq T \right\} = \frac{\mathbb{P} \left\{ \exists t \in [T - u^{-2/\beta} x, T] \left( X(t) + h(t) > u1 \right) \right\}}{\mathbb{P} \left\{ \exists t \in [0,T] \left( X(t) + h(t) > u1 \right) \right\}}$$

The asymptotics of denominator in (55) follows by Theorem 3.1. In order to get the asymptotics of nominator of (55) we follow the same argument as in the proof of Theorem 3.1 (part related with the asymptotics of $\Pi_1(u)$), which leads to

$$\mathbb{P} \left\{ \exists t \in [T - u^{-2/\beta} x, T] \left( X(t) + h(t) > u1 \right) \right\} \sim u^{(\frac{2}{2-\beta}) + \prod_{i=1}^n \Psi \left( \frac{u - h_{i,i}(t_0)}{\sigma_i(t_0)} \right)} \times \left\{ \begin{array}{ll}
\mathcal{H}_{\alpha,a}I_{(\alpha=1)} \int_{-x}^0 e^{-\sum_{i=1}^n f_i(x)} dx, & \text{if } \alpha < \beta, \\
\frac{p_{\alpha,a}I_{(\alpha=1)}}{\sigma_i(t_0)} [-x,0], & \text{if } \alpha = \beta, \\
1, & \text{if } \alpha > \beta, \end{array} \right.$$  

which completes the proof. \hfill \Box
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