On the Nonlocal Equations and Nonlocal Charges Associated with the Harry Dym Hierarchy

J.C. Brunelli\textsuperscript{a,\textasteriskcentered}, G.A.T.F. da Costa\textsuperscript{b,\textdagger}

\textsuperscript{a}Departamento de Física, CFM, Universidade Federal de Santa Catarina, Campus Universitário, Trindade, C.P. 476, CEP 88040-900, Florianópolis, SC, Brazil
\textsuperscript{b}Departamento de Matemática, CFM, Universidade Federal de Santa Catarina, Campus Universitário, Trindade, CEP 88040-900, Florianópolis, SC, Brazil

January 3, 2022

Abstract

A large class of nonlocal equations and nonlocal charges for the Harry Dym hierarchy is exhibited. They are obtained from nonlocal Casimirs associated with its bi-Hamiltonian structure. The Lax representation for some of these equations is also given.

1 Introduction

The following nonlinear partial differential equation

\[ w_t = (w^{-1/2})_{xxx}, \] (1)

is known as the Harry Dym equation (see \cite{1} for a review). It can also be written in the following equivalent forms

\[ v_t = \frac{1}{4} v^3 v_{xxx}, \]
\[ u_t = (u_{xx}^{-1/2})_x, \]

\textsuperscript{*}Supported by CNPq, Brazil. brunelli@fsc.ufsc.br.
\textsuperscript{†}gatcosta@mtm.ufsc.br.
for $v = -2^{1/3}w^{-1/2}$ and $u_{xx} = w$, respectively. This equation was obtained by Harry Dym and Martin Kruskal as an evolution equation solvable by a spectral problem based on the string equation instead of the Schrödinger equation. This result was reported in [2] and rediscovered independently in [3, 4]. The Harry Dym equation share many of the properties typical of the soliton equations. It is a completely integrable equation which can be solved by the inverse scattering transform [5, 6, 7]. It has a bi-Hamiltonian structure and an infinite number of conservation laws and infinitely many symmetries [8, 9].

The nonlinear hyperbolic equation, which we call the Hunter-Zheng equation,

$$
(u_t + uu_x)_{xx} = \frac{1}{2}(u_x^2)_x,
$$

or the nonlocal version

$$
\begin{align*}
\frac{w_t}{w_x} &= - (\partial^{-1}w)w_x - 2(\partial^{-1}w)w \\
&\quad = (\partial^{-2}w)w_x - \left(\partial^{-1}w\right)w
\end{align*}
$$

for $u_{xx} = w$, has the same bi-Hamiltonian structure as the Harry Dym equation. The complete integrability of (2) was established in [10] as well its connection with the Camassa-Holm equation [11]; the former is the high-frequency limit of the latter. Due to the presence of the antiderivative $\partial^{-1}$ the Hunter-Zheng equation (2) is nonlocal.

As we will describe the Harry Dym equation (1) and the Hunter-Zheng equation (2) belong to the same hierarchy of flows, which we will call the Harry Dym hierarchy. The Hunter-Zheng equation is a member of the positive order equations in this hierarchy while the Harry Dym belongs to the negative order equations. Hierarchies of negative order equations were considered previously through the use of the negative powers of the recursion operator [12, 13, 14]. Usually, while the positive order equations are local the negative order ones are nonlocal. For the Harry Dym hierarchy we have the opposite situation, all the positive order equations are nonlocal.

We will show the existence of two new hierarchies of integrable nonlocal equations associated with the Harry Dym hierarchy besides the Hunter-Zheng one. In fact the existence of two nonlocal Casimirs operators implies some sort of degeneracy for the positive order equations.

The paper is organized as follows. In Sec. 2 we review the bi-Hamiltonian formulation of integrable evolution equations emphasizing the role played by the Casimirs or distinguished functionals of the Hamiltonian operators. We find one more nonlocal Casimir for the modified
Kortweg-de Vries equation. In Sec. 3 we obtain new nonlocal equations as well nonlocal charges for the Harry Dym hierarchy. In Sec. 4 we discuss the Lax representation of these equations. The conclusions are given in Sec. 5.

2 Bi-Hamiltonian Systems and Casimirs

Central to our discussion is the concept of the bi-Hamiltonian formulation of an integrable evolution equation [8, 9]

\[ u_t = K_1[u] = D_1 \frac{\delta H_2}{\delta u} = D_2 \frac{\delta H_1}{\delta u} . \]

Whenever \( D_1 \) and \( D_2 \) are compatible this implies the existence of an infinite hierarchy of higher order commuting bi-Hamiltonian systems,

\[ u^{(n)}_t = K_n[u] = D_1 \frac{\delta H_{n+1}}{\delta u} = D_2 \frac{\delta H_n}{\delta u} , \quad \text{with } n \in \mathbb{Z} , \quad (3) \]

where the higher order conservation laws \( H_n[u] \) are shared by all members of the hierarchy. This hierarchy of equations can be generated by the recursion operator

\[ R = D_2 D_1^{-1} , \]

since

\[ K_{n+1} = RK_n . \]

Also, using

\[ \frac{\delta H_{n+1}}{\delta u} = R^\dagger \frac{\delta H_n}{\delta u} , \quad (4) \]

where \( R^\dagger = D_1^{-1} D_2 \) is the adjoint of \( R \), as a recursion scheme we can obtain the higher Hamiltonians \( H_n \).

We call any functional \( H_C[u] \) a Casimir (or distinguished functional) of the Hamiltonian operator \( D \) if

\[ D \frac{\delta H_C}{\delta u} = 0 . \]
As a consequence any Hamiltonian system having $\mathcal{D}$ as a Hamiltonian operator,
\[ u_t = \mathcal{D} \frac{\delta H}{\delta u}, \]
has $H_C$ as a conserved charge. In fact
\[ \dot{H}_C = \{H_C, H\} = \int dx \int dy \frac{\delta H}{\delta u(x,t)} \{u(x,t), u(y,t)\} \frac{\delta H}{\delta u(y,t)} \]
\[ = - \int dx \left( \mathcal{D} \frac{\delta H}{\delta u} \right) \frac{\delta H}{\delta u} = 0. \]
When one of the conservation laws in (3) is a Casimir, let us say of $\mathcal{D}_1$, the hierarchy of equations stops except if the Hamiltonian operator $\mathcal{D}_2$ has at least one Casimir.

The Kortweg-de Vries (KdV) equation
\[ u_t = u_{xxx} + 3uu_x \]
has the following series of conservation laws
\[ H_0 = \int dx \ u, \]
\[ H_1 = \int dx \ \frac{1}{2} u^2, \]
\[ H_2 = \int dx \ \frac{1}{2} (u^3 - u_x^2), \]
\[ \vdots \]
and the two compatible Hamiltonian operators
\[ \mathcal{D}_1 = \partial, \]
\[ \mathcal{D}_2 = \partial^3 + u\partial + \partial u. \] (5)
From (3) we get
\[ u_t^{(0)} = \mathcal{D}_1 \frac{\delta H_1}{\delta u} = \mathcal{D}_2 \frac{\delta H_0}{\delta u} = u_x, \]
\[ u_t^{(1)} = \mathcal{D}_1 \frac{\delta H_2}{\delta u} = \mathcal{D}_2 \frac{\delta H_1}{\delta u} = u_{xxx} + 3uu_x, \]
\[ u_t^{(2)} = \mathcal{D}_1 \frac{\delta H_3}{\delta u} = \mathcal{D}_2 \frac{\delta H_2}{\delta u} = u_{xxxx} + 5uu_{xxx} + 10uu_x + 15u^3, \]
\[ \vdots \]
and apparently we can not extend this recursion procedure for negatives values of $n$ in (3) since $H_0$ is a Casimir of $\mathcal{D}_1$ and $\mathcal{D}_2$ appears to have only trivial local distinguished functionals (see Eq. (3)). Now, the modified KdV equation (mKdV)

$$u_t = u_{xxx} + \frac{3}{2} u^2 u_x$$

has the following bi-Hamiltonian form

$$u_t = \mathcal{D}_1 \frac{\delta H_2}{\delta u} = \mathcal{D}_2 \frac{\delta H_1}{\delta u},$$

where

$$\mathcal{D}_1 = \partial ,$$

$$\mathcal{D}_2 = \partial^2 + \partial u \partial^{-1} u \partial ,$$

and

$$H_1 = \int dx \frac{1}{2} u^2 ,$$

$$H_2 = \int dx \left( \frac{1}{8} u^4 - \frac{1}{2} u_x^2 \right).$$

Of course, $H_0 = \int dx u$ is the Casimir of $\mathcal{D}_1$, however $\mathcal{D}_2$ in (3) admits a nontrivial nonlocal Casimir [15]

$$H_C = \int dx \cos(\partial^{-1} u) .$$

Here we will define the skew-adjoint anti-derivative $\partial^{-1}$ acting on functions $u$, which satisfy $u \to 0$ as $|x| \to \infty$, by

$$(\partial^{-1} u) \equiv (\partial^{-1} u)(x) = \int_{-\infty}^{+\infty} dy \epsilon(x-y)u(y) ,$$

where

$$\epsilon(x-y) = \begin{cases} 
 1/2 & \text{for } x > y , \\
 -1/2 & \text{for } x < y .
\end{cases}$$
From now on we will omit the $x$ subscript in (7). It is easy to verify that for functions $A$ and $B$ of $u$ we have the property

\[ \int dx A(\partial^{-1} B) = - \int dx (\partial^{-1} A)B . \]

Now

\[ \frac{\delta H_C}{\delta u} = \partial^{-1} \left( \sin(\partial^{-1} u) \right) \]

and it is easy to show that $\mathcal{D}_2(\delta H_C/\delta u) = 0$. From (8) we get, for $n = -1, -2, \ldots$, a hierarchy of negative order equations

\[
\begin{align*}
    u_t^{(0)} &= \mathcal{D}_1 \frac{\delta H_1}{\delta u} = \mathcal{D}_2 \frac{\delta H_0}{\delta u} = u_x , \\
    u_t^{(-1)} &= \mathcal{D}_1 \frac{\delta H_0}{\delta u} = \mathcal{D}_2 \frac{\delta H_{-1}}{\delta u} = 0 , \\
    u_t^{(-2)} &= \mathcal{D}_1 \frac{\delta H_{-1}}{\delta u} = \mathcal{D}_2 \frac{\delta H_{-2}}{\delta u} = \sin(\partial^{-1} u) , \\
    &\vdots
\end{align*}
\]

where $H_{-1} \equiv H_C$. Introducing the potential function $\psi_x = u$ the last equation in (8) is the sine-Gordon equation

\[ \psi_{xt} = \sin \psi . \]

Besides $H_0$ and $H_C$ we have found that the Hamiltonian operator $\mathcal{D}_2$ also has the following nonlocal Casimir

\[ H'_C = \int dx \sin(\partial^{-1} u) \left( \partial^{-1} \cos(\partial^{-1} u) \right) , \]

which will generate another negative order hierarchy of equations.

Returning to the KdV equation if we set

\[ u = -2\psi^{-1} \psi_{xx} , \]

the second Hamiltonian structure in (8) can be written as

\[ \mathcal{D}_2 = \psi^{-2} \partial \psi^2 \partial \psi^{-2} , \]
and it follows that
\[
\psi^2, \quad \psi^2 (\partial^{-1} \psi^{-2}), \quad \psi^2 \partial^{-1} (\psi^{-2} (\partial^{-1} \psi^{-2}))
\]
are non trivial kernels of $\mathcal{D}_2$. We will not discuss this system here but the Casimirs associated with (9) will give rise to nonlocal KdV hierarchies of equations (see [12, 13, 14], and references therein) and nonlocal charges. Instead, we will perform this analysis in a systematic way for the Harry Dym hierarchy in the next section.

3 The Harry Dym Hierarchy

The Harry Dym equation (1) is a completely integrable bi-Hamiltonian system [8, 9]
\[
\frac{\partial w}{\partial t} = \mathcal{D}_1 \frac{\delta H_{-1}}{\delta w} = \mathcal{D}_2 \frac{\delta H_{-2}}{\delta w},
\]
where
\[
\mathcal{D}_1 = \partial^3, \\
\mathcal{D}_2 = w \partial + \partial w,
\]
and
\[
H_{-1} = \int dx \left(2 w^{1/2}\right), \\
H_{-2} = \int dx \left(\frac{1}{8} w^{-5/2} w_x^2\right).
\]
It is well known that $H_{-1}$ is a Casimir of $\mathcal{D}_2$ and that
\[
H_0 = -\int dx \, w \quad (11)
\]
is a Casimir of $\mathcal{D}_1$. So, we can consider equations going “up” and “down” in the equation (3). However, we also have the following nonlocal Casimirs for $\mathcal{D}_1$
\[
H_0^{(1)} = \int dx \, (\partial^{-1} w), \\
H_0^{(2)} = \int dx \, (\partial^{-2} w).
\]
In this way (3) gets degenerated for $n > 0$

\[
\begin{align*}
\frac{\partial \delta H_2^{(\alpha)}}{\partial \delta w} &= D_1 \frac{\delta H_1^{(\alpha)}}{\delta w}, \\
\frac{\partial \delta H_1^{(\alpha)}}{\partial \delta w} &= D_2 \frac{\delta H_0^{(\alpha)}}{\delta w}, \\
\frac{\partial \delta H_0^{(\alpha)}}{\partial \delta w} &= D_1 \frac{\delta H_{-1}}{\delta w} = 0, \\
\frac{\partial \delta H_{-1}}{\partial w} &= D_2 \frac{\delta H_{-2}}{\delta w} = (w^{-1/2})_{xxx}, \\
\frac{\partial \delta H_{-2}}{\partial w} &= \frac{\delta H_{-3}}{\delta w} = (w^{-1/2})_{xxx}, \\
\end{align*}
\]

(13)

where $\alpha = 0, 1, 2$ and $H_0^{(0)} = H_0$.

Using (4) and (13) as a recursion scheme we can obtain, after a straightforward but tedious calculation, the first few Hamiltonian functionals and flows for the Harry Dym hierarchy equations. For $n \leq 0$ the first conserved charges, some of them already calculated in [10], are

\[
\begin{align*}
H_0 &= H_0^{(0)} = \int dx (-w), \\
H_{-1} &= \int dx 2w^{1/2}, \\
H_{-2} &= \int dx \frac{1}{8} w^{-5/2}w_x^2, \\
H_{-3} &= \int dx \frac{1}{16} (\frac{35}{16} w^{-11/2}w_x^4 - w^{-7/2}w_{xx}^2), \\
H_{-4} &= \int dx \frac{1}{32} (\frac{5005}{128} w^{-17/2}w_x^6 - \frac{231}{8} w^{-13/2}w_x^2w_{xx}^2 + 5w^{-11/2}w_{xxx}^3 \\
&\quad + w^{-9/2}w_{xxx}^2), \\
\end{align*}
\]

(14)
and the first flows are

\[ w_t^{(0)} = 0, \]
\[ w_t^{(-1)} = (w^{-1/2})_{xxx}, \]
\[ w_t^{(-2)} = \frac{1}{4}\left(\frac{5}{4}w^{-7/2}w_x^2 - w^{-5/2}w_{xx}\right)_{xxx}, \]
\[ w_t^{(-3)} = \frac{1}{16}\left(\frac{1155}{32}w^{-13/2}w_x^4 - \frac{231}{4}w^{-11/2}w_x^2w_{xx} + \frac{21}{2}w^{-9/2}w_{xx}^2 + 14w^{-9/2}w_{xxx}w_x - 2w^{-7/2}w_{xxxx}\right)_{xxx}. \]

(15)

For \( n > 0 \) we have to consider the three cases \( \alpha = 0, 1, 2 \) separately. So, for \( \alpha = 0 \) we get the nonlocal conserved charges

\[ H_1^{(0)} = \int dx \frac{1}{2} (\partial^{-1}w)^2, \]
\[ H_2^{(0)} = \int dx \frac{1}{2} (\partial^{-2}w)(\partial^{-1}w)^2, \]
\[ H_3^{(0)} = \int dx \left[ \frac{1}{4}(\partial^{-2}w)^2(\partial^{-1}w)^2 + \frac{1}{8}(\partial^{-1}(\partial^{-1}w)^2)^2 \right], \]
\[ H_4^{(0)} = \int dx \left[ \frac{1}{12}(\partial^{-2}w)^3(\partial^{-1}w)^2 - \frac{1}{4}(\partial^{-1}w)^2 \partial^{-2}((\partial^{-2}w)(\partial^{-1}w)^2) \right. \]
\[ - \left. \frac{1}{8}(\partial^{-2}w)(\partial^{-1}(\partial^{-1}w)^2)^2 \right], \]
\[ H_5^{(0)} = \int dx \left[ \frac{1}{48}(\partial^{-2}w)^4(\partial^{-1}w)^2 - \frac{1}{8}(\partial^{-2}w)^2(\partial^{-1}w)^2(\partial^{-2}(\partial^{-1}w)^2) \right. \]
\[ - \left. \frac{1}{16}(\partial^{-2}w)^2(\partial^{-1}(\partial^{-1}w)^2)^2 + \frac{1}{16}(\partial^{-2}(\partial^{-1}w)^2)^2(\partial^{-1}w)^2 \right. \]
\[ + \left. \frac{1}{16}(\partial^{-1}w)^2(\partial^{-2}(\partial^{-1}w)^2)^2 + \frac{1}{8}(\partial^{-1}((\partial^{-1}w)(\partial^{-1}w))^2)^2 \right]. \]

(16)
and the first flows are
\[
\begin{align*}
  w_t^{(1,0)} &= -w_x , \\
  w_t^{(2,0)} &= -(\partial^{-2} w) w_x - 2(\partial^{-1} w) w , \\
  w_t^{(3,0)} &= -\frac{1}{2} (\partial^{-2} w)^2 w_x - 2(\partial^{-2} w)(\partial^{-1} w) w + \frac{1}{2} (\partial^{-2}(\partial^{-1} w)^2) w_x \\
  & \quad + w (\partial^{-1}(\partial^{-1} w)^2) ,
\end{align*}
\]  

Note that the flow \( w_t^{(2,0)} \) is the Hunter-Zheng equation (4).

From (10) we can construct the recursion operator
\[
R = D_2 D_1^{-1} = 2w\partial^{-2} + w_x \partial^{-3} ,
\]  
and since
\[
(\partial w + w\partial)^{-1} = \frac{1}{2} w^{-1/2} \partial^{-1} w^{-1/2}
\]
we have
\[
R^{-1} = \frac{1}{2} \partial^3 w^{-1/2} \partial^{-1} w^{-1/2} .
\]
The flows (17) and (17) can now be expressed as the action of powers of \( R \) acting on the seed equation
\[
w_t^{(1,0)} = D_2 \left( \frac{\delta H_0^{(0)}}{\delta w} \right) = -w_x
\]
\[
\begin{align*}
  w_t^{(n)} &= R^{n-1}(-w_x) , \quad n = 0, -1, -2, \ldots , \\
  w_t^{(n,0)} &= R^{n-1}(-w_x) , \quad n = 1, 2, 3, \ldots .
\end{align*}
\]  

To be able to perform the steps in the iteration given in (19-20) we must point out that for any function \( f(x,t) \) we can use instead of (7) the representation (16, 12, 13, 14)
\[
(\partial^{-1} f)(x,t) = \int_{-\infty}^{+\infty} dy \epsilon(x-y)f(y,t) + c(t) ,
\]
where \( c(t) \) is a function of \( t \). Here we set \( c = 0 \) for any \( f \), except \( f = 0 \) where we use \( c = 2 \). In this way
\[
w_t^{(0)} = R^{-1}(-w_x) = 0
\]
and
\[ w_t^{(-1)} = R^{-2}(-w_x) = \frac{1}{2} \partial^3 w^{-1/2} \partial^{-1} 0 = (w^{-1/2})_{xxx}, \]
resulting in the Harry Dym equation.

For \( n > 0 \) and \( \alpha = 1, 2 \) we have the seed equations
\[
\begin{align*}
  w_t^{(1,1)} &= D_2 \frac{\delta H_0^{(1)}}{\delta w} = -2w - x w_x, \\
  w_t^{(1,2)} &= D_2 \frac{\delta H_0^{(2)}}{\delta w} = 2xw + \frac{x^2}{2} w_x,
\end{align*}
\]
where we have used \((\partial^{-n}1) = x^n/n\), for \( n > 0 \). The respective flows follow from the analogue of (20) and they read for \( \alpha = 1 \)
\[
\begin{align*}
  w_t^{(1,1)} &= -2w - x w_x, \\
  w_t^{(2,1)} &= -2wx(\partial^{-1}w) - w_x \left( \partial^{-1}(x(\partial^{-1}w)) \right), \\
  w_t^{(3,1)} &= (2w + w_x \partial^{-1}) \left( \frac{x}{2} \partial^{-1}(\partial^{-1}w)^2 - (\partial^{-1}w)\partial^{-1}(x(\partial^{-1}w)) \right), \\
  \vdots
\end{align*}
\]
and for \( \alpha = 2 \)
\[
\begin{align*}
  w_t^{(1,2)} &= 2xw + \frac{x^2}{2} w_x, \\
  w_t^{(2,2)} &= 2w \left( \frac{1}{2} x^2 (\partial^{-1}w) - (\partial^{-3}w) \right) + w_x \partial^{-1} \left( \frac{1}{2} x^2 (\partial^{-1}w) - (\partial^{-3}w) \right), \\
  w_t^{(3,2)} &= (2w + w_x \partial^{-1}) \left( \frac{1}{2} \partial^{-1}(\partial^{-2}w)^2 + \frac{1}{2} \partial^{-3}(\partial^{-2}w)^2 - (\partial^{-4}w)(\partial^{-1}w) \\
  &\quad - \frac{x^2}{4} \partial^{-1}(\partial^{-1}w)^2 + \frac{1}{2}(\partial^{-1}w)\partial^{-1}(x^2(\partial^{-1}w)) \right), \\
  \vdots
\end{align*}
\]
Again using (11) and (13) recursively we obtain the nonlocal conserved charges
\[ H_1^{(1)} = -\frac{1}{2} \int dx \partial^{-1}(\partial^{-1}w)^2, \]
\[ H_2^{(1)} = \frac{1}{2} \int dx \partial^{-1} ((\partial^{-1}w)\partial^{-1}(\partial^{-1}w)^2), \]
\[ H_3^{(1)} = -\frac{1}{2} \int dx \partial^{-1} \left[ \frac{1}{4} (\partial^{-1}(\partial^{-1}w)^2)^2 + (\partial^{-1}w)\partial^{-1} ((\partial^{-1}w)\partial^{-1}(\partial^{-1}w)^2) \right], \]
\[ \vdots \]
\[ \tag{24} \]

and
\[ H_1^{(2)} = -\frac{1}{2} \int dx \left[ (\partial^{-2}w)^2 + (\partial^{-2}(\partial^{-1}w)^2) \right], \]
\[ H_2^{(2)} = \frac{1}{2} \int dx \left[ \frac{1}{3} (\partial^{-2}w)^3 + \partial^{-2} ((\partial^{-1}w) (\partial^{-1}(\partial^{-1}w)^2)) + (\partial^{-4}w)(\partial^{-1}w)^2 \right], \]
\[ H_3^{(2)} = -\frac{1}{2} \int dx \left[ \frac{1}{12}(\partial^{-1}w)^4 + \frac{1}{4} (\partial^{-2}(\partial^{-1}w)^2)^2 + \frac{1}{4} \partial^{-2} (\partial^{-1}(\partial^{-1}w)^2)^2 \right. \]
\[ + \frac{1}{2}(\partial^{-1}w)^2\partial^{-2}(\partial^{-2}w)^2 - (\partial^{-1}w)^2\partial^{-1} ((\partial^{-4}w)(\partial^{-1}w)) \]
\[ + \partial^{-2} ((\partial^{-1}w)\partial^{-1} ((\partial^{-1}w) (\partial^{-1}(\partial^{-1}w)^2))) \right], \]
\[ \vdots \]
\[ \tag{25} \]

These hierarchies of integrable equations become extremely nonlocal as we proceed further in the recursion. \( w_t^{(2,1)} \) and \( w_t^{(2,2)} \) are equations of the Hunter-Zheng type. It is interesting to note that the equations (17), (22) and (23) are in the positive direction (positive flows) of recursion while the Harry Dym equations (15) are in the negative direction (negative flows). Of course this is due to the fact that the recursion operator (18) is completely nonlocal. That is to be compared with the usual situation we have for the KdV recursion operator \( R = \partial^2 + 2u + u_x\partial^{-1} \).

After obtaining Equations (22) and (23) we became aware of the papers [17] and [18] where these equations are given implicitly in a recursion form. However, we have shown here their origin from the Casimirs (11) and (12).
4 Lax Pairs

The equations in the Harry Dym hierarchy are integrable since they are bi-Hamiltonian. Therefore we hope to find a Lax representation for all of them. In fact, the Lax pair for the Harry Dym equation [1] is given by [1, 19, 20]

\[ L = \frac{1}{w} \partial^2, \]

\[ B = -2w^{-3/2} \partial^3 + \frac{3}{2} w^{-5/2} w_x \partial^2, \]

\[ \frac{\partial L}{\partial t} = [B, L]. \]  

Calculating the square-root of \( L \) (aided by a computer algebra program) we obtain

\[ L^{1/2} = \beta \partial + a_0 + a_1 \partial^{-1} + a_2 \partial^{-2} + a_3 \partial^{-3} + a_4 \partial^{-4} + a_5 \partial^{-5} + O(\partial^{-6}), \]

where

\[ \beta = w^{-1/2}, \]

\[ a_0 = -\frac{1}{2} \beta_x, \]

\[ a_1 = \frac{1}{2^2 \beta_{xx}} - \frac{1}{2^3 \beta_x^2 \beta}, \]

\[ a_2 = \frac{1}{2^3 \beta_{xxx}} - \frac{3}{2^4 \beta_x \beta_{xx}} \beta^{-2} + \frac{3}{2^3 \beta_x^3 \beta}, \]

\[ a_3 = \frac{1}{2^4 \beta_{xxxx}} - \frac{3}{2^4 \beta_x \beta_{xx}} \beta^{-1} + \frac{37}{2^5 \beta_x \beta_{xx}} \beta^2 - \frac{61}{2^7 \beta_x^4 \beta} - \frac{13}{2^5 \beta_x^2 \beta}, \]

\[ a_4 = \frac{1}{2^5 \beta(5)} + \frac{5 \cdot 7}{2^5 \beta_x \beta_{xx}} \beta^{-1} - \frac{5}{2^4 \beta_x^3 \beta} \beta - \frac{3 \cdot 5 \cdot 7}{2^6 \beta_x^3 \beta} \beta^{-2} - \frac{3 \cdot 5 \cdot 13}{2^6 \beta_x \beta_{xx}} \beta^{-2} + \frac{3^2 \cdot 5 \cdot 7}{2^6 \beta_x^3 \beta} \beta^{-2} - \frac{3 \cdot 5 \cdot 29}{2^8 \beta_x^3 \beta} \beta^{-4}, \]

\[ a_5 = \frac{1}{26} \beta(6) - \frac{7 \cdot 17}{2^7 \beta_x \beta_{xx}} \beta^{-1} - \frac{19}{2^4 \beta_x \beta_{xx}} \beta^{-1} + \frac{43}{2^2 \beta_x^3 \beta \beta} \beta^{-2} - \frac{3 \cdot 5}{2^6 \beta_x \beta(5)} \beta^{-1} + \frac{241}{2^7 \beta_x^2 \beta_{xx}} \beta^{-2} - \frac{569}{2^6 \beta_x^3 \beta_{xx}} \beta^{-3} + \frac{413}{2^7 \beta_x^3 \beta} \beta^{-2} - \frac{3 \cdot 1973}{2^8 \beta_x^2 \beta_{xx}} \beta^{-3} + \frac{3 \cdot 4493}{2^9 \beta_x^3 \beta_{xx}} \beta^{-4} - \frac{7 \cdot 17 \cdot 67}{2^{10} \beta_x^4 \beta} \beta^{-5}, \]
where $\beta(n) = \frac{d^n \beta}{dx^n}$. Now it can be easily recognized that

$$B = -2 \left( L^{3/2} \right)_{\geq 2},$$

and $\left( \right)_{\geq 2}$ stands for the differential part of the pseudodifferential operator with terms $\partial^n$, $n \geq 2$. In this way (27) assumes the nonstandard Lax representation

$$\frac{\partial L}{\partial t} = -2 \left[ \left( L^{3/2} \right)_{\geq 2}, L \right]. \quad (28)$$

Similarly the whole negative hierarchy of equations (15) can be obtained from

$$\frac{\partial L}{\partial t} = -2^n \left[ \left( L^{2n+1} \right)_{\geq 2}, L \right], \quad n = 0, 1, 2, \ldots . \quad (29)$$

The charges (14) (except by multiplicative constant factors) follows from

$$H_{-(n+1)} = \text{Tr} L^{2n+1}, \quad n = 1, 2, 3, \ldots , \quad (30)$$

where “Tr” is the usual Adler trace [21]. We have used (29) and (30) to perform a check of the equations (15) and charges (14), respectively.

Through “gauge transformations” [22] the Lax representation for the Harry Dym hierarchy (27) and (28) can be brought in other forms. For instance, the Harry Dym equation also follows from

$$L' = \partial L \partial^{-1},$$

with the nonstandard Lax representation

$$\frac{\partial L'}{\partial t} = -2 \left[ \left( L^{3/2} \right)_{\geq 1}, L' \right], \quad (31)$$

or even from a standard Lax representation

$$\frac{\partial L''}{\partial t} = -2 \left[ \left( L^{n3/2} \right)_{\geq 0}, L'' \right],$$

with

$$L'' = w^{1/2} L w^{-1/2}.$$
So, for the negative flows of the Harry Dym hierarchy we have a complete Lax representation. However, for the positive flows the Lax representation picture is not so complete. It is easy to check that the Lax operator (26) with
\[
\frac{\partial L}{\partial t} = -2[B^{(i,\alpha)}, L], \quad i = 1, 2, 3, \quad \alpha = 0, 1, 2 ,
\]
(32)
yields the first equations \(w^{(1,\alpha)}\), for the positive flows (17), (22) and (23), if we choose
\[
B^{(1,0)} = \frac{1}{2} \partial ,
\]
\[
B^{(1,1)} = -\frac{1}{4} + \frac{1}{2} x \partial ,
\]
\[
B^{(1,2)} = \frac{1}{4} x - \frac{1}{4} x^2 \partial ,
\]
respectively. For the equations \(w^{(2,\alpha)}\) in (17), (22) and (23) we have obtained, for \(i = 2\) in (32), the operators
\[
B^{(2,0)} = \frac{1}{4}(\partial^{-2} w) \partial + \frac{1}{4} \partial^{-1}(\partial^{-2} w) \partial^2 ,
\]
\[
B^{(2,1)} = \frac{1}{4} \left( \partial^{-1}(x \partial^{-1} w) \right) \partial + \frac{1}{4} \partial^{-1} \left( \partial^{-1}(x \partial^{-1} w) \right) \partial^2 - \frac{1}{4} \partial^{-1}(\partial^{-1} w) ,
\]
\[
B^{(2,2)} = \frac{1}{4} \left( \partial^{-1} \left( \frac{1}{2} x^2 \partial^{-1} w - \partial^{-3} w \right) \right) \partial + \frac{1}{4} \partial^{-1} \left( \partial^{-1} \left( \frac{1}{2} x^2 \partial^{-1} w - \partial^{-3} w \right) \right) \partial^2
\]
\[
- \frac{1}{4} \partial^{-1} x (\partial^{-1} w) + \frac{1}{4} \partial^{-1} (\partial^{-2} w) + \frac{1}{8} \partial^{-2} x w .
\]
In fact \(B^{(2,0)}\) was first obtained in \([10]\) but in the nonstandard Lax representation \((31)\). An interesting question is how, if possible at all, \(B^{(i,\alpha)}\) in \((32)\) and the nonlocal charges \(H^{(\alpha)}_{n}\) in \((14)\), \((24)\) and \((23)\) can be obtained from the same Lax operator \((26)\), i.e., what are the analogues of equations \((29)\) and \((30)\) for the positive Harry Dym flows? In the literature we can find Lax representations for equations obtained by the inverse recursion operator \([23, 24]\), such as of the Harry Dym type. However, these Lax representations are not given only in terms of \(L\) and use some sort of ansatz. Also, the authors do not try to obtain the relation between the nonlocal charges and the Lax operator. These intriguing points are under investigation and will appear elsewhere.
5 Conclusion

We have given a unified picture of the Harry Dym hierarchy of equations which includes local as well a series of three nonlocal hierarchies of equations. We have shown, using the bi-Hamiltonian formulation of integrable models, how the nonlocal Casimirs leads to these nonlocal equations and also to three series of nonlocal charges. Some of the nonlocal equations and nonlocal charges obtained in this way are new. This procedure can also be applied for the KdV and mKdV equations since both equations, accordingly with the discussion in Sec. 2 also have three Casimirs associated with the third order Hamiltonian operator $\mathcal{D}_2$. We believe that the treatment given here for the Harry Dym hierarchy unifies, within the bihamiltonian formulation of the integrable models, some of the results scattered in the literature. We have also tried to understand these nonlocal equations and charges from a Lax representation. Even though we have found explicitly Lax pairs for some of the positive flows a unique Lax representation is still missing.

References

[1] W. Hereman, P. P. Banerjee and M. R. Chatterjee, J. Phys. A22 (1989) 241.
[2] M. D. Kruskal, Lecture Notes in Physics, vol. 38 (Springer, Berlin, 1975) p. 310.
[3] P. C. Sabatier, Lett. Nuovo Cimento 26 (1979) 477; 483.
[4] L. Yi-Shen, Lett. Nuovo Cimento 70 (1982) 1.
[5] M. Wadati, Y. H. Ichikawa and T. Shimizu, Prog. Theor. Phys. 64 (1980) 1959.
[6] M. Wadati, K. Konno and Y. H. Ichikawa, J. Phys. Soc. Japan 47 (1979) 1698.
[7] F. Calogero and A. Degasperis, Spectral Transform and Solitons 1 (North Holland, Amsterdam, 1982).
[8] F. Magri, J. Math. Phys. 19 (1978) 1156.
[9] P. J. Olver, Applications of Lie Groups to Differential Equations, second edition (Springer, Berlin, 1993).
[10] J. K. Hunter and Y. Zheng, Physica D79 (1994) 361.

[11] R. Camassa and D. Holm, Phys. Rev. Lett. 71 (1993) 1661.

[12] J. M. Verosky, J. Math. Phys. 32 (1991) 1733.

[13] V. A. Andreev and M. V. Shmakova, J. Math. Phys. 34 (1993) 3491.

[14] S. Y. Lou, J. Math. Phys. 35 (1994) 2390.

[15] P. J. Olver and P. Rosenau, Phys. Rev. 53E (1996) 1900.

[16] J. A. Sanders and J. P. Wang, Physica D149 (2001) 1.

[17] S. Y. Lou, Chaos, Solitons & Fractals 4 (1994) 1961.

[18] B. Fuchssteiner and S. Carillo, Physica A154 (1989) 467.

[19] F. Gesztesy and K. Unterkofler, Rep. Math. Phys. 31 (1992) 113.

[20] B. Konopelchenko and W. Oevel, Publ. RIMS, Kyoto Univ. 29 (1993) 581.

[21] M. Adler, Invent. Math. 50 (1979) 219.

[22] H. Aratyn, E. Nissimov, S. Pacheva and I. Vaysburd, Phys. Lett. B294 (1992) 167.

[23] R. Zhou, J. Math. Phys. 36 (1995) 4220.

[24] Z. Qiao, Physica A252 (1998) 377.