Quasi-Kähler manifolds
with a pair of Norden metrics

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Abstract. The basic class of the non-integrable almost complex manifolds with a pair of Norden metrics are considered. The interconnections between corresponding quantities at the transformation between the two Levi-Civita connections are given. A 4-parametric family of 4-dimensional quasi-Kähler manifolds with Norden metric is characterized with respect to the associated Levi-Civita connection.

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Introduction

It is a fundamental fact that on an almost complex manifold with Hermitian metric (almost Hermitian manifold), the action of the almost complex structure on the tangent space at each point of the manifold is an isometry. There is another kind of metric, called a Norden metric or a $B$-metric on an almost complex manifold, such that action of the almost complex structure is an anti-isometry with respect to the metric. Such a manifold is called an almost complex manifold with Norden metric [1] or with $B$-metric [2]. See also [3] for generalized $B$-manifolds. It is known [1] that these manifolds are classified into eight classes.

The basic class of the non-integrable almost complex manifolds (the quasi-Kählerian ones) with Norden metric is considered in [4]. Its curvature properties are studied and the isotropic Kähler type (a notion from [5]) of the investigated manifolds is introduced and characterized geometrically [4]. A 4-parametric family of 4-dimensional quasi-Kähler manifolds with Norden metric is constructed on a Lie group in [6]. This family is characterized geometrically and there is given the condition for such a 4-manifold to be isotropic Kählerian.
In the present paper we continue studying the quasi-Kähler manifolds with Norden metric. The purpose is to get of the interconnections between corresponding quantities at the transformation between the two Levi-Civita connections. Moreover, characterizing of the known example from [6] with respect to the associated Levi-Civita connection.

1. Almost complex manifolds with Norden metric

1.1. Preliminaries

Let \((M, J, g)\) be a \(2n\)-dimensional almost complex manifold with Norden metric, i.e. \(J\) is an almost complex structure and \(g\) is a metric on \(M\) such that \(J^2X = -X\), \(g(JX, JY) = -g(X, Y)\) for all differentiable vector fields \(X, Y\) on \(M\), i.e. \(X, Y \in \mathfrak{X}(M)\).

The associated metric \(\tilde{g}\) of \(g\) on \(M\) given by \(\tilde{g}(X, Y) = g(X, JY)\) for all \(X, Y \in \mathfrak{X}(M)\) is a Norden metric, too. Both metrics are necessarily of signature \((n, n)\). The manifold \((M, J, \tilde{g})\) is an almost complex manifold with Norden metric, too.

Further, \(X, Y, Z, U, (x, y, z, u)\), respectively will stand for arbitrary differentiable vector fields on \(M\) (vectors in \(T_pM, p \in M\), respectively).

The Levi-Civita connection of \(g\) is denoted by \(\nabla\). The tensor field \(F\) of type \((0, 3)\) on \(M\) is defined by

\[
F(X, Y, Z) = g((\nabla_X J)Y, Z).
\]

It has the following symmetries

\[
F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ).
\]

Further, let \(\{e_i\}\) \((i = 1, 2, \ldots, 2n)\) be an arbitrary basis of \(T_pM\) at a point \(p\) of \(M\). The components of the inverse matrix of \(g\) are denoted by \(g^{ij}\) with respect to the basis \(\{e_i\}\).

The Lie form \(\theta\) associated with \(F\) is defined by \(\theta(z) = g^{ij}F(e_i, e_j, z)\).

A classification of the considered manifolds with respect to \(F\) is given in [1]. All eight classes of almost complex manifolds with Norden metric are characterized there according to the properties of \(F\). The three basic classes are given as follows

\[
W_1 : F(X, Y, Z) = \frac{1}{2 \theta} \left\{ g(X, Y)\theta(Z) + g(X, Z)\theta(Y) 
+ g(X, JY)\theta(JZ) + g(X, JZ)\theta(JY) \right\} ;
\]

\[
W_2 : \mathfrak{S}_{X, Y, Z} F(X, Y, JZ) = 0, \quad \theta = 0;\]

\[
W_3 : \mathfrak{S}_{X, Y, Z} F(X, Y, Z) = 0,
\]

where \(\mathfrak{S}\) is the cyclic sum by three arguments.

The special class \(W_0\) of the Kähler manifolds with Norden metric belonging to any other class is determined by the condition \(F = 0\).
The only class of the three basic classes, where the almost complex structure is not integrable, is the class \( \mathcal{W}_3 \) – the class of the quasi-Kähler manifolds with Norden metric. Let us remark that the definitional condition from (1.3) implies the vanishing of the Lie form \( \theta \) for the class \( \mathcal{W}_3 \).

1.2. Curvature properties

Let \( R \) be the curvature tensor field of \( \nabla \) defined by \( R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \). The corresponding tensor field of type \( (0,4) \) is determined by \( R(X,Y,Z,U) = g(R(X,Y)Z,U) \). The Ricci tensor \( \rho \) and the scalar curvature \( \tau \) are defined as usual by \( \rho(y,z) = g^{ij} R(e_i,y,z,e_j) \) and \( \tau = g^{ij} \rho(e_i,e_j) \).

It is well-known that the Weyl tensor \( W \) on a \( 2n \)-dimensional pseudo-Riemannian manifold \( (n \geq 2) \) is given by

\[
W = R - \frac{1}{2n-2} \psi_1(\rho) - \frac{\tau}{2n-1} \pi_1,
\]

where

\[
\psi_1(\rho)(x,y,z,u) = g(y,z)\rho(x,u) - g(x,z)\rho(y,u) + \rho(y,z)g(x,u) - \rho(x,z)g(y,u);
\]

\[
\pi_1 = \frac{1}{2} \psi_1(g) = g(y,z)g(x,u) - g(x,z)g(y,u).
\]

Moreover, the Weyl tensor \( W \) is zero if and only if the manifold is conformally flat.

1.3. Isotropic Kähler manifolds

The square norm \( \|\nabla J\|^2 \) of \( \nabla J \) is defined in [5] by

\[
\|\nabla J\|^2 = g^{ij} g^{kl} g((\nabla e_i J) e_k, (\nabla e_j J) e_l).
\]

Having in mind the definition (1.1) of the tensor \( F \) and the properties (1.2), we obtain the following equation for the square norm of \( \nabla J \)

\[
\|\nabla J\|^2 = g^{ij} g^{kl} g^{pq} F_{ikp} F_{jlp},
\]

where \( F_{ikp} = F(e_i, e_k, e_p) \).

An almost complex manifold with Norden metric \( g \) satisfying the condition \( \|\nabla J\|^2 = 0 \) is called an isotropic Kähler manifold with Norden metric [4].

Remark 1.1. It is clear, if a manifold belongs to the class \( \mathcal{W}_0 \), then it is isotropic Kählerian but the inverse statement is not always true.
2. The transformation $\nabla \rightarrow \tilde{\nabla}$ between the Levi-Civita connections corresponding to the pair of Norden metrics

2.1. The interconnections between corresponding tensors at $\nabla \rightarrow \tilde{\nabla}$

Let $\nabla$ be the Levi-Civita connection of $g$. Then the following well-known condition is valid

$$2g(\nabla xy, z) = xg(y, z) + yg(x, z) - zg(x, y) + g([x, y], z) + g([z, x], y) + g([z, y], x).$$  (2.1)

Let $\tilde{\nabla}$ be the Levi-Civita connection of $\tilde{g}$. Then the corresponding condition of (2.1) holds for $\tilde{\nabla}$ and $\tilde{g}$. These two equations imply immediately

$$g(\nabla xy - \tilde{\nabla}xy, z) = \frac{1}{2}\{F(Jz, x, y) - F(x, y, Jz) - F(y, x, Jz)\}.$$  (2.2)

Having in mind the properties (1.2) of $F$, the condition (2.2) implies the following two equations

$$\tilde{F}(x, y, z) = \frac{1}{2}\{F(Jy, z, x) + F(y, z, Jx) + F(z, Jx, y) + F(Jz, x, y)\},$$

$$\mathcal{S}_{x,y,z} \tilde{F}(x, y, z) = \mathcal{S}_{x,y,z} F(Jx, y, z),$$

where $\tilde{F}$ is the corresponding structural tensor of $F$ for $(M, J, \tilde{g})$.

The last equation, the definition condition (1.3) of the class $\mathcal{W}_3$ and the properties (1.2) of $F$ imply

**Theorem 2.1.** The manifold $(M, J, \tilde{g})$ is quasi-Kählerian with Norden metric if and only if the manifold $(M, J, g)$ is also quasi-Kählerian with Norden metric. □

Further, we consider only quasi-Kähler manifolds with Norden metric. In this case the condition (2.2) and the definition (1.3) of $\mathcal{W}_3$ imply

$$\tilde{\nabla}xy = \nabla xy + T(x, y),$$  (2.3)

where

$$T(x, y) = (\nabla xJ)Jy + (\nabla yJ)Jx.$$  (2.4)

Hence, by direct computations, we get

**Theorem 2.2.** Let $(M, J, g)$ and $(M, J, \tilde{g})$ be quasi-Kähler manifolds with Norden metric. Then the following equivalent interconnections are valid

$$\tilde{F}(x, y, z) = -F(Jx, y, z),$$  (2.5)

$$\left(\tilde{\nabla}xJ\right) y = -\left(\nabla JxJ\right) Jy.$$  (2.6)

Let $\tilde{R}$ be the curvature tensor of $\tilde{\nabla}$. It is well-known in the case of transformation of the connection of type (2.6) that corresponding curvature tensors have the following interconnection

$$\tilde{R}(x, y)z = R(x, y)z + Q(x, y)z,$$  (2.7)
where
\[ Q(x,y)z = (\nabla x T)(y,z) - (\nabla y T)(x,z) + T(x, T(y,z)) - T(y, T(x,z)). \] (2.8)

By the definition, the corresponding curvature tensor of type (0,4) is
\[ \tilde{\mathcal{R}}(x,y,z,u) = \tilde{g}(\tilde{\mathcal{R}}(x,y)z,u). \] (2.9)

**Theorem 2.3.** Let \((M,J,g)\) and \((M,J,\tilde{g})\) be quasi-Kähler manifolds with Norden metric. Then the corresponding curvature tensors at the transformation \(\nabla \rightarrow \tilde{\nabla}\) satisfy the following equation
\[ \tilde{\mathcal{R}}(x,y,z,u) = \mathcal{R}(x,y,z,Ju) - g((\nabla y J)z + (\nabla z J)y, (\nabla x J)Ju + (\nabla u J)Jx) \] (2.10)
\[ + g((\nabla z J)z + (\nabla J z)x, (\nabla J J)Ju + (\nabla J J)Jy). \]

**Proof.** Let us denote \(\tilde{T}(y,z,u) = \tilde{g}(\tilde{T}(y,z),u).\) According to (2.4), the properties of \(F\) and the condition \(\mathcal{S} F = 0,\) we get \(\tilde{T}(y,z,u) = F((Ju,y,z)).\) Having in mind (2.4) and \(\nabla g = 0,\) we obtain
\[ g((\nabla x T)(y,z),u) = (\nabla x T)(y,z,u). \]

Then, by direct computations, we have
\[ (\nabla x T)(y,z,u) - (\nabla y T)(x,z,u) = (\nabla x F)(Ju,y,z) - (\nabla y F)(Ju,x,z) \] (2.11)
\[ + F((\nabla J u)Jx, y) - F((\nabla J u)Ju, x), \]
\[ T(x, T(y,z)) - T(y, T(x,z)) = g((\nabla J u)Jx, (\nabla y J)Jz + (\nabla z J)y) \] (2.12)
\[ - g((\nabla J u)Jy, (\nabla J J)z + (\nabla z J)Jx). \]

The equations (2.7), (2.8), (2.9), (2.11), (2.12) imply immediately the interconnection (2.10) of the statement. \(\square\)

**2.2. The invariant tensors of the transformation \(\nabla \rightarrow \tilde{\nabla}\)**

Having in mind the tensor \(T\) determined by (2.4), we denote the corresponding tensor \(\tilde{T}\) of \(\tilde{\nabla}.\) Since we suppose the conditions (2.2) and \(\mathcal{S} F = 0,\) we get \(T(y,z,u) = F(Ju,y,z).\) Having in mind (2.4) and \(\nabla g = 0,\) we obtain
\[ g((\nabla x T)(y,z),u) = (\nabla x T)(y,z,u). \]

Then, by direct computations, we have
\[ (\nabla x T)(y,z,u) - (\nabla y T)(x,z,u) = (\nabla x F)(Ju,y,z) - (\nabla y F)(Ju,x,z) \] (2.13)
\[ + F((\nabla J u)Jx, y) - F((\nabla J u)Ju, x), \]
\[ T(x, T(y,z)) - T(y, T(x,z)) = g((\nabla J u)Jx, (\nabla y J)Jz + (\nabla z J)y) \]
\[ - g((\nabla J u)Jy, (\nabla J J)z + (\nabla z J)Jx). \]

The equations (2.7), (2.8), (2.9), (2.11), (2.12) imply immediately the interconnection (2.10) of the statement. \(\square\)
Therefore, the following theorem is valid.

**Theorem 2.4.** Let \((M, J, g)\) and \((M, J, \tilde{g})\) be quasi-Kähler manifolds with Norden metric. The following two tensors are invariant with respect to the transformation \(\nabla \to \tilde{\nabla}\)

\[
S(x, y) = \nabla_x y + \frac{1}{2} T(x, y), \quad P(x, y)z = R(x, y)z + \frac{1}{2} Q(x, y)z.
\]

\[\square\]

3. The Lie group as a 4-dimensional \(\mathcal{W}_3\)-manifold and its characteristics

In [6] is constructed an example of a 4-dimensional Lie group equipped with a quasi-Kähler structure and Norden metric \(g\). There it is characterized with respect to the Levi-Civita connection \(\nabla\). Now, we recall the facts known from [6] and give their corresponding ones with respect to the Levi-Civita connection \(\tilde{\nabla}\).

**Theorem 3.1 ([6]).** Let \((G, J, g)\) be a 4-dimensional almost complex manifold with Norden metric, where \(G\) is a connected Lie group with a corresponding Lie algebra \(g\) determined by the global basis of left invariant vector fields \(\{X_1, X_2, X_3, X_4\}\); \(J\) is an almost complex structure defined by

\[
JX_1 = X_3, \quad JX_2 = X_4, \quad JX_3 = -X_1, \quad JX_4 = -X_2;
\]

\((3.1)\)

\(g\) is an invariant Norden metric determined by

\[
g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1,
\]

\((3.2)\)

and

\[
g([X, Y], Z) + g([X, Z], Y) = 0.
\]

\((3.3)\)

Then \((G, J, g)\) is a quasi-Kähler manifold with Norden metric if and only if \(G\) belongs to the 4-parametric family of Lie groups determined by the conditions

\[
[X_1, X_3] = \lambda_2 X_2 + \lambda_4 X_4, \quad [X_2, X_4] = \lambda_1 X_1 + \lambda_3 X_3,
\]

\[
[X_2, X_3] = -\lambda_2 X_1 - \lambda_3 X_4, \quad [X_3, X_4] = -\lambda_4 X_1 + \lambda_3 X_2,
\]

\((3.4)\)

\[\square\]

Theorem 2.1 and Theorem 3.1 imply the following

**Theorem 3.2.** The manifold \((G, J, \tilde{g})\) is a quasi-Kähler manifold with Norden metric. \[\square\]
The components of the Levi-Civita connection $\nabla$ are determined \((6)\) by \((3.4)\) and
\[
\nabla X_i X_j = \frac{1}{2} [X_i, X_j] \quad (i, j = 1, 2, 3, 4). \tag{3.5}
\]
According to \((2.3)\), \((2.4)\) and \((3.5)\), we receive the following equation for the Levi-Civita connection $\tilde{\nabla}$
\[
\tilde{\nabla}_i X_j = \frac{1}{2} \{ [X_i, X_j] + J[X_i, J X_j] - J[J X_i, X_j] \} \quad (i, j = 1, 2, 3, 4). \tag{3.6}
\]
The components of $\tilde{\nabla}$ are determined by \((3.6)\) and \((3.4)\).

The nonzero components of the tensor $F$ are: \(6\)
\[-F_{122} = -F_{144} = 2F_{212} = 2F_{221} = 2F_{234},\]
\[= 2F_{243} = 2F_{414} = -2F_{423} = -2F_{432} = 2F_{441} = \lambda_1,\]
\[2F_{112} = 2F_{121} = 2F_{134} = 2F_{143} = -2F_{211},\]
\[= -2F_{233} = -2F_{314} = 2F_{323} = 2F_{332} = -2F_{341} = \lambda_2,\]
\[2F_{214} = -2F_{223} = -2F_{232} = 2F_{241} = F_{322},\]
\[= F_{344} = -2F_{412} = -2F_{421} = -2F_{434} = -2F_{443} = \lambda_3,\]
\[-2F_{114} = 2F_{123} = 2F_{132} = -2F_{141} = -2F_{312},\]
\[= -2F_{321} = -2F_{334} = -2F_{343} = F_{411} = F_{433} = \lambda_4,\] (3.7)
where $F_{ijk} = F(X_i, X_j, X_k)$. We receive the nonzero components $\tilde{F}(X_i, X_j, X_k) = \tilde{F}_{ijk}$ of the tensor $\tilde{F}$ from \((2.5)\) using \((3.1)\) and \((3.7)\). They are the following
\[-2\tilde{F}_{214} = 2\tilde{F}_{223} = 2\tilde{F}_{232} = -2\tilde{F}_{241} = -\tilde{F}_{322},\]
\[= -\tilde{F}_{344} = 2\tilde{F}_{412} = 2\tilde{F}_{421} = 2\tilde{F}_{434} = 2\tilde{F}_{443} = \lambda_1,\]
\[2\tilde{F}_{114} = -2\tilde{F}_{123} = -2\tilde{F}_{132} = 2\tilde{F}_{141} = 2\tilde{F}_{312},\]
\[= 2\tilde{F}_{321} = 2\tilde{F}_{334} = 2\tilde{F}_{343} = -2\tilde{F}_{411} = -2\tilde{F}_{433} = \lambda_2,\]
\[-\tilde{F}_{122} = -\tilde{F}_{144} = 2\tilde{F}_{212} = 2\tilde{F}_{221} = 2\tilde{F}_{234},\]
\[= 2\tilde{F}_{243} = 2\tilde{F}_{414} = -2\tilde{F}_{423} = -2\tilde{F}_{432} = 2\tilde{F}_{441} = \lambda_3,\]
\[2\tilde{F}_{112} = 2\tilde{F}_{121} = 2\tilde{F}_{134} = 2\tilde{F}_{143} = -\tilde{F}_{211},\]
\[= -\tilde{F}_{233} = -2\tilde{F}_{314} = 2\tilde{F}_{323} = 2\tilde{F}_{332} = -2\tilde{F}_{341} = \lambda_4.\] (3.8)

The square norm of $\nabla J$ with respect to $g$, defined by \((1.6)\), has the form \(6\)
\[\|\nabla J\|^2 = 4 (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2).\]
Then the manifold \((G, J, g)\) is isotropic Kählerian if and only if the condition \(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0\) holds. \[\] Analogously, the square norm \(\left\|\tilde{\nabla} J\right\|^2\) of \(\tilde{\nabla} J\) with respect to \(\tilde{g}\) is defined by

\[
\left\|\tilde{\nabla} J\right\|^2 = \tilde{g}^{ij} \tilde{g}^{kl} \tilde{g} \left(\left(\tilde{\nabla}_{e_i} J\right) e_k, \left(\tilde{\nabla}_{e_j} J\right) e_l\right),
\]

where \(\tilde{g}^{ij}\) are the components of the inverse matrix of \((\tilde{g}_{ij})\). The last equation has the following form in terms of \(\tilde{\nabla}\)

\[
\left\|\tilde{\nabla} J\right\|^2 = \tilde{g}^{ij} \tilde{g}^{kl} \tilde{g}^{pq} \tilde{F}_{ikp} \tilde{F}_{jql}.
\]

Hence by (3.8) we obtain

\[
\left\|\nabla J\right\|^2 = -8 (\lambda_1 \lambda_3 + \lambda_2 \lambda_4).
\]

The last equation implies the following

**Proposition 3.3.** The manifold \((G, J, \tilde{g})\) is isotropic Kählerian if and only if the condition \(\lambda_1 \lambda_3 + \lambda_2 \lambda_4 = 0\) holds.

**Remark 3.4.** As is known from [6], \((G, J, g)\) is isotropic Kählerian if and only if the set of vectors with the coordinates \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) at an arbitrary point \(p \in G\) describes the isotropic cone in \(T_p G\) with respect to \(g\). Obviously, \((G, J, \tilde{g})\) is isotropic Kählerian if and only if the the same set of vectors describes the isotropic cone in \(T_p G\) but with respect to \(\tilde{g}\).

**Remark 3.5.** Let us note that two manifolds \((G, J, g)\) and \((G, J, \tilde{g})\) can be isotropic Kählerian by independent way. For example:

(i) if \(\lambda_3 = \lambda_2 = -\lambda_1\), then both \((G, J, g)\) and \((G, J, \tilde{g})\) are isotropic Kähler manifolds;

(ii) if \(\lambda_3 = \lambda_1 \neq 0\) and \(\lambda_4 = -\lambda_3\), then \((G, J, g)\) is an isotropic Kähler manifold but \((G, J, \tilde{g})\) is not isotropic Kählerian;

(iii) if \(\lambda_2 = \lambda_1 \neq 0\), \(\lambda_4 = -\lambda_3\) and \(|\lambda_1| \neq |\lambda_3|\), then \((G, J, g)\) is not an isotropic Kähler manifold but \((G, J, \tilde{g})\) is isotropic Kählerian.

The components \(R_{i j k s} = R(X_i, X_j, X_k, X_s)\) \((i, j, k, s = 1, 2, 3, 4)\) of the curvature tensor \(R\) on \((G, J, g)\) are:

\[
\begin{align*}
R_{1221} &= -\frac{1}{4} \left(\lambda_1^2 + \lambda_2^2\right), & R_{1331} &= \frac{1}{4} \left(\lambda_2^2 - \lambda_3^2\right), \\
R_{1441} &= -\frac{1}{4} \left(\lambda_1^2 - \lambda_2^2\right), & R_{2332} &= \frac{1}{4} \left(\lambda_2^2 - \lambda_3^2\right), \\
R_{2442} &= \frac{1}{4} \left(\lambda_1^2 - \lambda_3^2\right), & R_{3443} &= \frac{1}{4} \left(\lambda_2^2 + \lambda_3^2\right), \\
R_{1341} &= R_{2342} = -\frac{1}{4} \lambda_1 \lambda_2, & R_{2132} &= -R_{4134} = \frac{1}{4} \lambda_1 \lambda_3, \\
R_{1231} &= -R_{4234} = \frac{1}{4} \lambda_1 \lambda_4, & R_{2142} &= -R_{3143} = \frac{1}{4} \lambda_2 \lambda_3, \\
R_{1241} &= -R_{3243} = \frac{1}{4} \lambda_2 \lambda_4, & R_{3123} &= R_{4124} = \frac{1}{4} \lambda_3 \lambda_4.
\end{align*}
\]

(3.9)
We get the nonzero components $\tilde{R}_{i j k s} = \tilde{R}(X_i, X_j, X_k, X_s)$, $(i, j, k, s = 1, 2, 3, 4)$ of $\tilde{R}$ on $(G, J, \tilde{g})$, having in mind (3.7), (3.9), (2.10), (3.1) as follows:

\[
\begin{align*}
\tilde{R}_{1221} &= -\tilde{R}_{1441} = -\tilde{R}_{2332} = \tilde{R}_{3443} = \lambda_1 \lambda_3 + \lambda_2 \lambda_4, \\
\tilde{R}_{1331} &= -\frac{1}{2} \lambda_2 \lambda_4, \quad \tilde{R}_{2442} = -\frac{1}{2} \lambda_1 \lambda_3, \\
\tilde{R}_{1234} &= \tilde{R}_{1432} = \frac{1}{4} (\lambda_1 \lambda_3 + \lambda_2 \lambda_4), \\
\tilde{R}_{1241} &= \frac{1}{4} (4 \lambda_1^2 + 2 \lambda_2^2 + \lambda_3^2 - 2 \lambda_4^2), \\
\tilde{R}_{2132} &= \frac{1}{4} (2 \lambda_1^2 + 4 \lambda_2^2 - 2 \lambda_3^2 + \lambda_4^2), \\
\tilde{R}_{4134} &= \frac{1}{4} (-2 \lambda_1^2 + \lambda_2^2 + 2 \lambda_3^2 + 4 \lambda_4^2), \\
\tilde{R}_{3243} &= \frac{1}{4} (\lambda_1^2 - 2 \lambda_2^2 + 4 \lambda_3^2 + 2 \lambda_4^2), \\
\tilde{R}_{1231} &= \tilde{R}_{2142} = -\frac{1}{4} (2 \lambda_1 \lambda_2 + 3 \lambda_3 \lambda_4), \\
\tilde{R}_{1341} &= \tilde{R}_{4124} = -\frac{1}{4} (2 \lambda_1 \lambda_4 - 3 \lambda_2 \lambda_3), \\
\tilde{R}_{3143} &= \tilde{R}_{4234} = -\frac{1}{4} (3 \lambda_1 \lambda_2 + 2 \lambda_3 \lambda_4), \\
\tilde{R}_{3123} &= \tilde{R}_{2334} = \frac{1}{4} (3 \lambda_1 \lambda_4 - 2 \lambda_2 \lambda_3).
\end{align*}
\]

The scalar curvature $\tau$ on $(G, J, g)$ is [6]

\[
\tau = \frac{3}{2} \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right),
\]

and for the scalar curvature $\tilde{\tau}$ on $(G, J, \tilde{g})$ we obtain

\[
\tilde{\tau} = 5 \left( \lambda_1 \lambda_3 + \lambda_2 \lambda_4 \right).
\]

The manifold $(G, J, g)$ is isotropic Kählerian if and only if it has a zero scalar curvature $\tau$. [6] Proposition 3.3 and equation (3.11) imply

**Proposition 3.6.** The manifold $(G, J, \tilde{g})$ is isotropic Kählerian if and only if it has zero scalar curvature $\tilde{\tau}$.

**Remark 3.7.** Let us note that two manifolds $(G, J, g)$ and $(G, J, \tilde{g})$ can be scalar flat by independent way.

Let us recall, that the manifold $(G, J, g)$ has vanishing Weyl tensor $W$ [6]. Now, let us consider the Weyl tensor $\tilde{W}$ on $(G, J, \tilde{g})$ determined by analogy to $W$ by (1.2) and (1.3). Taking into account (3.11), (3.12), (3.10) and (3.11), we receive the following nonzero components of $\tilde{W}$:

\[
\begin{align*}
\tilde{W}_{1221} &= -\tilde{W}_{1441} = -\tilde{W}_{2332} = \tilde{W}_{3443} = 3\tilde{W}_{1234} = 3\tilde{W}_{1432} \\
&= -\frac{3}{4} \tilde{W}_{1331} = -\frac{3}{4} \tilde{W}_{2442} = \lambda_1 \lambda_3 + \lambda_2 \lambda_4.
\end{align*}
\]

According to Proposition 3.3 and Equations (3.11), (3.12), we establish the truthfulness of the following

**Theorem 3.8.** The following conditions are equivalent for the manifold $(G, J, \tilde{g})$:

(i) $(G, J, \tilde{g})$ is an isotropic Kähler manifold;
(ii) the Weyl tensor $\tilde{W}$ vanishes;

(iii) the condition $\lambda_1\lambda_3 + \lambda_2\lambda_4 = 0$ holds;

(iv) the scalar curvature $\tilde{\tau}$ vanishes.

□

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