Research Article

On the Open Problem Related to Rank Equalities for the Sum of Finitely Many Idempotent Matrices and Its Applications

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Tian and Styan have shown many rank equalities for the sum of two and three idempotent matrices and pointed out that rank equalities for the sum $P_1 + \cdots + P_k$ with $P_1, \ldots, P_k$ be idempotent ($k > 3$) are still open. In this paper, by using block Gaussian elimination, we obtained rank equalities for the sum of finitely many idempotent matrices and then solved the open problem mentioned above. Extensions to scalar-potent matrices and some related matrices are also included.

1. Introduction

Let $\mathbb{C}^{m \times n}$ and $GL_n(\mathbb{C})$ be the sets of $m \times n$ complex matrices and $n \times n$ nonsingular matrices, respectively. The $n \times n$ identity matrix is denoted by $I_n$ or simply by $I$ if the size is immaterial. Let $\mathbb{Z}^+$ be the set of all the positive integer numbers. The symbols $r(A)$ and $A^T$ stand for the rank and transpose of $A \in \mathbb{C}^{m \times n}$, respectively, while $\text{tr} A$ denotes the trace of a square matrix $A$. A matrix $A \in \mathbb{C}^{n \times n}$ is said to be idempotent, if $A^2 = A$, and scalar-potent (determined by $\lambda$), if $A^2 = \lambda A$, for some $(0 \neq) \lambda \in \mathbb{C}$ (see, e.g., [1]). When $\lambda = 1$, it coincides with the definition of an idempotent matrix.

As one of the fundamental building blocks in matrix theory, idempotent matrices are very useful in many contexts and have been extensively studied in the literature (see, e.g., [1–6]). Here we focus on the research on the rank of the sum of idempotent matrices.

Gröss and Trenkler have studied rank of the sum of two idempotent matrices (see [3, Theorem 3]). Also, Tian and Styan have shown a rank equality for two idempotent matrices as follows.

**Proposition 1** (see [1, Theorem 2.4] and [2, Theorem 2.1]). Let $P, Q \in \mathbb{C}^{n \times n}$ be idempotent. Then

$$r (P + Q) = r \left( \begin{array}{cc} P & Q \\ \hline Q & 0 \end{array} \right) - r (Q).$$

By (3), Tian and Styan have extended the rank equality for the sum of idempotent matrices to the scalar-potent matrices (see, e.g., [1]).

**Proposition 2** (see [1, P110]). Let $P, Q \in \mathbb{C}^{n \times n}$ be scalar-potent matrices determined by nonzero complexes $\lambda, \mu$. Then

$$r (\mu P + \lambda Q) = r \left( \begin{array}{cc} P & Q \\ \hline Q & 0 \end{array} \right) - r (Q),$$

$$P^2 = \lambda P, \quad Q^2 = \mu Q, \quad \lambda \mu \neq 0.$$  (2)

Later, Tian and Styan considered the rank equality for the sum of three idempotent matrices in [2] as follows.

**Proposition 3** (see [2, P95]). Let $P_1, P_2, P_3 \in \mathbb{C}^{n \times n}$ be idempotent. Then

$$r (P_1 + P_2 + P_3) = r \left( \begin{array}{ccc} \frac{1}{2} P_1 & P_2 & P_3 \\ P_2 & 0 & P_2 P_3 \\ P_3 & P_2 P_3 & 0 \end{array} \right) - r (P_2) - r (P_3).$$  (3)

By (3), Tian and Styan have induced many useful results, for example, if $P_1$, $P_2$, $P_3$ are idempotent and $P_1 + P_2 + P_3 = 0$, then $P_1 = P_2 = P_3 = 0$. The literatures [2, 4–6]
show that establishing various kinds of rank equalities for $k$ idempotent matrices is interesting. Tian and Styan pointed out that rank equalities for the sum $P_1 + \cdots + P_k$ with $P_1, \ldots, P_k$ be idempotent ($k > 3$) are still open (see [2, P95]).

In this paper, by applying block Gaussian elimination, rank equalities for the sum of finitely many idempotent matrices are obtained. These results generalize (3) and solve the open problem proposed by Tian and Styan (see, e.g., [2]). Also, new rank equalities for finitely many idempotent matrices are given. The rank equality (3) is generalized to $k$ scalar-potent matrices as well.

2. Main Results

Before showing main results, we need some preparations.

**Lemma 4.** Let $A_1, \ldots, A_k \in \mathbb{C}^{n \times n}$. Then

$$r \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix} = \sum_{i=1}^{k} r(A_i) + r \left( \sum_{i=1}^{k} A_i \right),$$

(4)

for any $k \in \mathbb{Z}^+$.  

**Proof.** Let

$$G = \begin{pmatrix} I_n & 0 & \cdots & 0 \\ 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -I_n & -I_n & \cdots & -I_n \end{pmatrix} \in \mathbb{C}^{n(k+1) \times n(k+1)},$$

$$S = \begin{pmatrix} I_m & 0 & \cdots & 0 \\ 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_m \end{pmatrix} \in \mathbb{C}^{m(k+1) \times m(k+1)}.$$  

It is evident that $G$ and $S$ are nonsingular.

By calculation,

$$G \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix} S = \text{diag} \left( A_1, \ldots, A_k, -\sum_{i=1}^{k} A_i \right);$$

since $G$ and $S$ are nonsingular, hence

$$r \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix} = r \left( \text{diag} \left( A_1, \ldots, A_k, -\sum_{i=1}^{k} A_i \right) \right),$$

(7)

This completes the proof. 

The proof method of Lemma 4 is inspired by Marsaglia and Styan [5, Theorem 9]. By (4), we get the rank equality for the sum of finitely many idempotent matrices; it is different from the one of three idempotent matrices (3) given by Tian and Styan. Consequently, to find the generalization of Proposition 3 and solve the open problem given by Tian and Styan (see, e.g., [2]), it is necessary to seek a new method different from Lemma 4.

**Lemma 5** (see [7, Problem 4.9]). Let $P \in \mathbb{C}^{n \times n}$ be idempotent. Then $r(P) = \text{tr} P$.

In this section, from now on, for $A_1, A_2, \ldots, A_k \in \mathbb{C}^{n \times n}$, one denotes

$$W(A_1, A_2, \ldots, A_k) = \begin{pmatrix} \frac{1}{2} A_1 & A_2 & A_3 & \cdots & A_{k-1} & A_k \\ A_2 & 0 & A_3 & \cdots & A_{k-1} & A_k \\ A_3 & A_2 & 0 & \cdots & A_{k-1} & A_k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{k-1} & A_{k-1} & A_k & \cdots & 0 & A_k \\ A_k & A_{k-1} & A_k & \cdots & A_k & 0 \end{pmatrix} \in \mathbb{C}^{n \times n \times n}.$$  

(8)

**Theorem 6.** For any $k \in \mathbb{Z}^+$, let $P_1, \ldots, P_k \in \mathbb{C}^{n \times n}$ be idempotent. Then

$$r \begin{pmatrix} \sum_{i=1}^{k} P_i \end{pmatrix}$$

$$= r (W(P_1, \ldots, P_k) - \sum_{i=2}^{k} r(P_i))$$

$$= r (W(P_1, \ldots, P_k)) - \sum_{i=2}^{k} \text{tr} P_i$$

$$= r (W(P_1, \ldots, P_k)) - \text{tr} \left( \sum_{i=2}^{k} P_i \right).$$

(9)
Proof. From Lemma 4, it follows that
\[
\begin{pmatrix}
    P_1 & 0 & \cdots & 0 & P_1 \\
    0 & P_2 & \cdots & 0 & P_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & P_k & P_k \\
    P_1 & P_2 & \cdots & P_k & 0
\end{pmatrix}
= \sum_{i=1}^{k} r(P_i) + r\left(\sum_{i=1}^{k} P_i\right).
\] (10)

On the other hand, by block Gaussian elimination, we will see that
\[
\begin{pmatrix}
    P_1 & 0 & \cdots & 0 & P_1 \\
    0 & P_2 & \cdots & 0 & P_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & P_k & P_k \\
    P_1 & P_2 & \cdots & P_k & 0
\end{pmatrix}
= r(W(P_1, \ldots, P_k)) + r(P_1).
\] (11)

In fact, let us write the matrix
\[
\begin{pmatrix}
    P_1 & 0 & \cdots & 0 & P_1 \\
    0 & P_2 & \cdots & 0 & P_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & P_k & P_k \\
    P_1 & P_2 & \cdots & P_k & 0
\end{pmatrix}
\]
as the quadripartitioned matrix
\[
\begin{pmatrix}
    M_{11} & M_{12} \\
    M_{21} & 0
\end{pmatrix}
\in \mathbb{C}^{n(k+1)\times n(k+1)}.
\] (13)

where
\[
M_{11} = \text{diag}(P_1, \ldots, P_k) \in \mathbb{C}^{nk\times nk},
\]
\[
M_{12} = (P_1^T, \ldots, P_k^T)^T \in \mathbb{C}^{nk\times n},
\]
\[
M_{21} = (P_1, \ldots, P_k) \in \mathbb{C}^{nk\times n}.
\]

By (11) and (13), it suffices to show
\[
r\left(\begin{pmatrix}
    M_{11} & M_{12} \\
    M_{21} & 0
\end{pmatrix}\right) = r(W(P_1, \ldots, P_k)) + r(P_1).
\] (15)

Direct calculations to (13) show that
\[
\begin{pmatrix}
    I_{nk} & W_{12} \\
    0 & I_n
\end{pmatrix}
\begin{pmatrix}
    M_{11} & M_{12} \\
    M_{21} & 0
\end{pmatrix}
= \begin{pmatrix}
    M_{11} + W_{12}M_{21} & M_{12} \\
    M_{21} & 0
\end{pmatrix},
\] (16)

where
\[
W_{12} = (0, -P_2^T, \ldots, -P_k^T)^T \in \mathbb{C}^{nk\times n},
\]
\[
\begin{pmatrix}
    I_{nk} & W_{12} \\
    0 & I_n
\end{pmatrix}
\in GL_{n(k+1)}(\mathbb{C}).
\] (17)

If we define \(X_{11} = M_{11} + W_{12}M_{21}\), by (14) and (17), we get
\[
X_{11} = \begin{pmatrix}
    P_1 & 0 & \cdots & 0 & 0 \\
    -P_2P_1 & 0 & \cdots & -P_2P_{k-1} & -P_2P_k \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    -P_{k-1}P_1 & -P_{k-1}P_2 & \cdots & 0 & -P_{k-1}P_k \\
    -P_kP_1 & -P_kP_2 & \cdots & -P_kP_{k-1} & 0
\end{pmatrix}.
\] (18)

Moreover, let
\[
S_{21} = \begin{pmatrix}
    -\frac{1}{2}E_n, 0, & \cdots, & 0, 0
\end{pmatrix}, \quad G_{21} = (P_1, 0, \ldots, 0),
\]
\[
R_{12} = \begin{pmatrix}
    -\frac{1}{2}P_1^T, 0, & \cdots, & 0
\end{pmatrix}^T;
\] (19)

then
\[
\begin{pmatrix}
    I_{nk} & 0 & \cdots & 0 \\
    S_{21} & I_n
\end{pmatrix}, \quad \begin{pmatrix}
    I_{nk} & 0 \\
    G_{21}
\end{pmatrix}, \quad \begin{pmatrix}
    I_{nk} & R_{12}
\end{pmatrix}
\]
are nonsingular.
(20)

By (14) and (19), we see that
\[
X_{11} + M_{12}G_{21}
\]
\[
= \begin{pmatrix}
    2P_1 & 0 & \cdots & 0 & 0 \\
    0 & 0 & \cdots & -P_2P_{k-1} & -P_2P_k \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & -P_{k-1}P_2 & \cdots & 0 & -P_{k-1}P_k \\
    0 & -P_kP_2 & \cdots & -P_kP_{k-1} & 0
\end{pmatrix}
\] (21)

\in \mathbb{C}^{nk\times nk}.

Then by applying (14) and (19) yields
\[
(X_{11} + M_{12}G_{21}) R_{12} = (0, P_2^T, \ldots, P_k^T)^T \in \mathbb{C}^{nk\times nk}.
\] (22)

Thus,
\[
S_{21}X_{11} + M_{21} + S_{21}M_{12}G_{21}
\]
\[
= S_{21}(X_{11} + M_{12}G_{21}) + M_{21}
\]
\[
= \begin{pmatrix}
    -\frac{1}{2}I, 0, & \cdots, & 0 \\
    X_{11} + M_{12}G_{21}
\end{pmatrix}
\]
\[
\times \begin{pmatrix}
    2P_1 & 0 & \cdots & 0 & 0 \\
    0 & 0 & \cdots & -P_2P_{k-1} & -P_2P_k \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & -P_{k-1}P_2 & \cdots & 0 & -P_{k-1}P_k \\
    0 & -P_kP_2 & \cdots & -P_kP_{k-1} & 0
\end{pmatrix}
\] (23)

\[
+ (P_1, P_2, \ldots, P_k)
\]
\[
= (0, P_2, \ldots, P_k).
\]

Hence it follows from (14) and (19) that
\[
(S_{21}X_{11} + M_{21} + S_{21}M_{12}G_{21}) R_{12} + S_{21}M_{12} = -\frac{1}{2}P_1.
\] (24)
Consequently, from (16)–(24), it follows that

\[
\begin{pmatrix}
I_{n_k} & 0 \\
S_{21} & I_n
\end{pmatrix}
\begin{pmatrix}
I_{n_k} & W_{12} \\
M_{11} & M_{12}
\end{pmatrix}
\begin{pmatrix}
M_{21} & M_{12} \\
G_{21} & I_n
\end{pmatrix}
\begin{pmatrix}
I_{n_k} & R_{12} \\
I_{n_k} & 0
\end{pmatrix}
\begin{pmatrix}
I_{n_k} & 0 \\
G_{21} & I_n
\end{pmatrix}
\begin{pmatrix}
I_{n_k} & 0 \\
S_{21} & I_n
\end{pmatrix}

= \begin{pmatrix}
I_{n_k} & 0 \\
S_{21} & I_n
\end{pmatrix}
\begin{pmatrix}
X_{11} & M_{12}G_{21} \\
S_{21}X_{11} + M_{21} + S_{21}M_{12}G_{21}
\end{pmatrix}
\begin{pmatrix}
I_{n_k} & 0 \\
G_{21} & I_n
\end{pmatrix}
\begin{pmatrix}
I_{n_k} & R_{12} \\
I_{n_k} & 0
\end{pmatrix}
\begin{pmatrix}
I_{n_k} & 0 \\
G_{21} & I_n
\end{pmatrix}
\begin{pmatrix}
I_{n_k} & 0 \\
S_{21} & I_n
\end{pmatrix}

= \begin{pmatrix}
X_{11} + M_{12}G_{21} & (X_{11} + M_{12}G_{21})R_{12} + M_{12} \\
S_{21}X_{11} + M_{21} + S_{21}M_{12}G_{21} & (S_{21}X_{11} + M_{21} + S_{21}M_{12}G_{21})R_{12} + S_{21}M_{12}
\end{pmatrix}

\begin{pmatrix}
2P_1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & -P_kP_{k-1} & -P_kP_k & P_k \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & -P_kP_{k-1}P_k & \cdots & 0 & -P_kP_k & P_{k-1} \\
0 & -P_kP_2 & \cdots & -P_kP_{k-1} & 0 & P_k \\
0 & P_2 & \cdots & P_{k-1} & P_k & -\frac{1}{2}P
\end{pmatrix}

= \begin{pmatrix}
2P_1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & -P_kP_{k-1} & -P_kP_k & P_k \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & -P_kP_{k-1}P_k & \cdots & 0 & -P_kP_k & P_{k-1} \\
0 & -P_kP_2 & \cdots & -P_kP_{k-1} & 0 & P_k \\
0 & P_2 & \cdots & P_{k-1} & P_k & -\frac{1}{2}P
\end{pmatrix}

= \text{diag}(2P_1, Z),

where

\[
Z = \begin{pmatrix}
0 & \cdots & -P_kP_{k-1} & -P_kP_k \\
-\frac{1}{2}P_1 & \cdots & -P_k & -\frac{1}{2}P_1 \\
-\frac{1}{2}P_1 & \cdots & -P_kP_{k-1} & -\frac{1}{2}P_1 \\
(P_2, \ldots, P_{k-1}, P_k)
\end{pmatrix} 
\in \mathbb{C}^{nk \times nk}.

\]

Since \(\begin{pmatrix} 0 & I_{n_k} \\ -I_{n_k} & 0 \end{pmatrix}\) and \(\begin{pmatrix} I_{n_k} & 0 \\ 0 & -I_{n_k} \end{pmatrix}\) are nonsingular, we get

\[
r \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & 0 \end{pmatrix} = r(P_1) + r(Z).
\]

(28)

Combining (13) with (29) together with Lemma 5 yields the desired results. \(\square\)

When \(k = 3\), Theorem 6 leads to Proposition 3 at once, and when \(k = 2\), it leads to Proposition I; for the idempotent matrices \(P\) and \(Q\), it follows that

\[
\begin{pmatrix}
2I_n & 0 \\
0 & I_n
\end{pmatrix} W(P, Q) \begin{pmatrix}
I_n & 0 \\
0 & 2I_n
\end{pmatrix}
\begin{pmatrix}
2I_n & 0 \\
0 & I_n
\end{pmatrix} W(P, Q) \begin{pmatrix}
I_n & 0 \\
0 & 2I_n
\end{pmatrix} = \begin{pmatrix}
P & Q \\
0 & 2I_n
\end{pmatrix}.
\]

(30)
For the sum of two idempotent matrices, Tian and Styan have given out many rank equalities (see [1, Theorem 2.4] and [2, Theorems 2.1, 2.3, 2.4, and 2.7]). Let $P_1, P_2, \ldots, P_k \in C^{n \times n}$ be idempotent; using Theorem 6 together with [6, Theorem 6] and [6, (26)] yields the equalities as follows:

$$r \left( \sum_{i=1}^{k} P_i \right)$$

$$= r \left( W \left( P_1, \ldots, P_k \right) \right) + r (P_1)$$

$$+ r \left( \begin{pmatrix} P_1 - P_2 & \ldots & P_1 - P_k \\ P_1 - P_3 & \ldots & P_1 - P_k \\ \vdots & \ddots & \vdots \\ P_1 - P_k & \ldots & P_{k-1} - P_k \end{pmatrix} \right)$$

$$- r \left( \begin{pmatrix} P_1 & \ldots & P_1 \ P_2 & \ldots & P_{k-1} \ \vdots & \ddots & \vdots \\ P_k & \ldots & P_{k-1} \end{pmatrix} \right)$$

(31)

$$r \left( P + \sum_{i=1}^{k} P_i \right)$$

$$= r \left( W \left( P, P_1, \ldots, P_k \right) \right) + r (P).$$

$$+ r \left( \begin{pmatrix} P - kP_1 & P & \ldots & P \\ P & P - kP_2 & \ldots & P \\ \vdots & \ddots & \ddots & \vdots \\ P & P & \ldots & P - kP_k \end{pmatrix} \right)$$

$$- r \left( \begin{pmatrix} P & \ldots & P \ P & \ldots & P \\ \vdots & \ddots & \vdots \ P & \ldots & P \end{pmatrix} \right)$$

(32)

Theorem 6 together with (31) and (32) indicates that the sum of $k \geq 3$ idempotent matrices has various kinds of rank equalities, as shown in the discussions in the literature [1, 2].

In view of (11), by applying Lemma 5, we see the following.

**Corollary 7.** For any $k \in \mathbb{Z}^+$, let $P_1, \ldots, P_k \in C^{n \times n}$ be idempotent. Then

$$r \left( \begin{pmatrix} P_1 & 0 & \ldots & 0 \\ 0 & P_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & P_k \end{pmatrix} \right) = r \left( W \left( P_1, \ldots, P_k \right) \right)$$

$$= r (P_1) = \text{tr} P_1.$$  

This immediately implies that the difference of the ranks of two block matrices in the left side of (33) is always equal to $r(P_i)$ or $\text{tr} P_i$, independently on the choice of $k$, when $k \geq 3$.

### 3. The Rank Formulas for the Sum of $k$ Scalar-Potent Matrices and Applications

Theorem 6 can easily be extended to scalar-potent matrices; in fact,

$$\left( \frac{1}{\lambda_i} P_i \right)^2 = \frac{1}{\lambda_i} P_i, \quad r \left( \frac{1}{\lambda_i} P_i \right) = r (P_i),$$

$$P_i^2 = \lambda_i P_i, \quad i = 1, \ldots, k.$$  

So $(1/\lambda_i)P_i$ is idempotent.

**Theorem 8.** For any given $k \in \mathbb{Z}^+$, let $P_i \in C^{n \times n}$ be scalar-potent (determined by $\lambda_i(\neq 0)$, $i = 1, \ldots, k$). Then

$$r \left[ \sum_{i=1}^{k} \left( \prod_{j \neq i} \lambda_j \right) P_i \right]$$

$$= r \left( W \left( \frac{1}{\lambda_1} P_1, \frac{1}{\lambda_2} P_2, \ldots, P_k \right) \right) - \sum_{i=2}^{k} r (P_i)$$

$$= r \left( W \left( \frac{1}{\lambda_1} P_1, \frac{1}{\lambda_2} P_2, \ldots, P_k \right) \right) - \sum_{i=2}^{k} \frac{1}{\lambda_i} \text{tr} P_i.$$  

(35)

**Proof.** By (8) and (34), we get
On the other hand, using (36), we obtain
\[ W \left( \frac{1}{\lambda_1} P_1, P_2, \ldots, P_k \right) = GW \left( \frac{1}{\lambda_1} P_1, \ldots, \frac{1}{\lambda_k} P_k \right) G, \]  
(37)

with \( G = \text{diag}(I_n, \lambda_2 I_2, \ldots, \lambda_k I_k) \). Since \( G \) is nonsingular, by (37), we can write
\[ r\left( W \left( \frac{1}{\lambda_1} P_1, P_2, \ldots, P_k \right) \right) = r\left( W \left( \frac{1}{\lambda_1} P_1, \frac{1}{\lambda_2} P_2, \ldots, \frac{1}{\lambda_k} P_k \right) \right). \]  
(38)

From (34), \((1/\lambda_i)P_i\) is idempotent. Using Theorem 6 together with (38) yields the equality
\[ r\left( \sum_{i=1}^{k} \frac{1}{\lambda_i} P_i \right) = r\left( W \left( \frac{1}{\lambda_1} P_1, \frac{1}{\lambda_2} P_2, \ldots, \frac{1}{\lambda_k} P_k \right) \right) - \sum_{i=1}^{k} r\left( \frac{1}{\lambda_i} P_i \right). \]  
(39)

We note that
\[ r\left( \sum_{i=1}^{k} \frac{1}{\lambda_i} P_i \right) = r \left[ \prod_{i=1}^{k} \lambda_i \left( \sum_{i=1}^{k} \frac{1}{\lambda_i} P_i \right) \right]. \]  
(40)

From (39) and (40), we get the desired result since \( r(P_i) = r((1/\lambda_i)P_i) = (1/\lambda_i) \text{tr} P_i \).

If \( P = I \), it coincides with the definition of a quadratic matrix (see, e.g., [9]). In view of [10, Lemma 1] and [11, Lemma 2.2], we conclude that (42) can be expressed equivalently as
\[ \Omega_n(P) = \{ A \in \mathbb{C}^{n \times n} : (A - aP)(A - bP) = 0, \]  
(43)

\[ AP = PA = A, \forall a, b \in \mathbb{C} \} . \]

If \( A \in \Omega_n(P) \), then from (42) and (43), we see that
\[ a = \frac{1}{2} \left( \alpha + \sqrt{\alpha^2 + 4\beta} \right), \]  
(44)

\[ b = \frac{1}{2} \left( \alpha - \sqrt{\alpha^2 + 4\beta} \right). \]

**Lemma 9.** For any given idempotent matrix \( P \), if \( A \in \Omega_n(P) \) satisfies \( (A - aP)(A - bP) = 0 \) with \( a \neq b \), then \( A - aP \) is a scalar-potent matrix determined by \( b - a \).

**Proof.** For the matrix \( P \), there exists a nonsingular matrix \( S \) such that \( P = S \text{diag}(I_k, 0)S^{-1} \). From \( AP = PA \), we can write \( A = S \text{diag}(A_1, A_2)S^{-1} \) being \( A_1 \in \mathbb{C}^{k \times k} \). From \( AP = A \), we get \( A_2 = 0 \); namely, \( A = S \text{diag}(A_1, 0)S^{-1} \). We have \((A_1 - aI_k)(A_1 - bI_k) = 0 \). It is seen from the fact that a matrix is diagonalizable if and only if its minimal polynomial has simple roots (see [12, Corollary 3.3.10]). Thus, there exists a nonsingular matrix \( S_1 \) such that \( A_1 = S_1 \text{diag}(aI_k, bI_k, 0)S_1^{-1} \). Let \( R = \text{diag}(S_1, I) \); then \( R \) is nonsingular and
\[ P = R \text{diag}(I_k, 0) R^{-1}, \]  
(45)

\[ A = R \text{diag}(aI_k, bI_k, 0) R^{-1}. \]  
Hence
\[ A - aP = R \text{diag}(0, (b - a) L_{k-1}, 0) R^{-1}, \]  
(46)

\[ = (b - a) R \text{diag}(0, I_{k-1}, 0) R^{-1}. \]

Now, it is evident that \((A - aP)^2 = (b - a)(A - aP)\).

**Theorem 10.** For any given idempotent matrices \( P_1, \ldots, P_k \in \mathbb{C}^{n \times n} \) and any \( k \in \mathbb{Z}^+ \), if \( A_i \in \Omega_n(P_i) \) satisfies \( (A_i - a_i P_i)(A_i - b_i P_i) = 0 \) with \( a_i \neq b_i, i = 1, \ldots, k \), then
\[ r \left[ \prod_{i=1}^{k} \left( b_j - a_j \right) (A_i - a_i P_i) \right] \]
\[ = r \left( W \left( \frac{1}{b_1 - a_1} (A_1 - a_1 P_1), A_2 - a_2 P_2, \ldots, A_k - a_k P_k \right) \right) \]
\[ - \sum_{i=2}^{k} r(A_i - a_i P_i) \]
\[ = r \left( W \left( \frac{1}{b_1 - a_1} (A_1 - a_1 P_1), A_2 - a_2 P_2, \ldots, A_k - a_k P_k \right) \right) \]
\[ - \sum_{i=2}^{k} \frac{1}{b_i - a_i} \text{tr}(A_i - a_i P_i). \]  
(47)
Proof. For the idempotent matrices $P_i$, by applying Lemma 9, we see that $A_i - a_i P_i$ is a scalar-potent matrix determined by $b_i - a_i$; then results follow from Theorem 8.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] Y. Tian and G. P. H. Styan, "Rank equalities for idempotent and involutory matrices," Linear Algebra and Its Applications, vol. 335, pp. 101–117, 2001.
[2] Y. Tian and G. P. H. Styan, "Rank equalities for idempotent matrices with applications," Journal of Computational and Applied Mathematics, vol. 191, no. 1, pp. 77–97, 2006.
[3] J. Gross and G. Trenkler, "Nonsingularity of the difference of two oblique projectors," SIAM Journal on Matrix Analysis and Applications, vol. 21, no. 2, pp. 390–395, 1999.
[4] W. G. Cochran, "The distribution of quadratic forms in a normal system, with applications to the analysis of covariance," Mathematical Proceedings of the Cambridge Philosophical Society, vol. 30, pp. 178–191, 1934.
[5] G. Marsaglia and G. P. H. Styan, "Equalities and inequalities for ranks of matrices," Linear and Multilinear Algebra, vol. 2, pp. 269–292, 1974.
[6] Y. Tian and G. P. H. Styan, "When does $\text{rank}(ABC) = \text{rank}(AB) + \text{rank}(BC) - \text{rank}(B)$ hold," International Journal of Mathematical Education in Science and Technology, vol. 33, pp. 127–137, 2002.
[7] F. Zhang, Matrix Theory: Basic Results and Techniques, Springer, New York, NY, USA, 1999.
[8] R. W. Farebrother and G. Trenkler, "On generalized quadratic matrices," Linear Algebra and Its Applications, vol. 410, pp. 244–253, 2005.
[9] M. Aleksiejczyk and A. Smoktunowicz, "On properties of quadratic matrices," Mathematica Pannonica, vol. 11, no. 2, pp. 239–248, 2000.
[10] S. Y. Liu, Z. P. Yang, and Y. P. Xie, "Invariance of rank and nullity for linear combinations of generalized quadratic matrix," Journal of Jinlin University, vol. 49, no. 6, pp. 993–996, 2011.
[11] Z. Yang, X. Feng, M. Chen, C. Deng, and J. J. Koliha, "Fredholm stability results for linear combinations of $m$-potent operators," Operators and Matrices, vol. 6, no. 1, pp. 193–199, 2012.