Abstract. We study the dynamics and statics of a dilute batch minority game with random external information. We focus on the case in which the number of connections per agent is infinite in the thermodynamic limit. The dynamical scenario of ergodicity breaking in this model is different from the phase transition in the standard minority game and is characterized by the onset of long-term memory at finite integrated response. We demonstrate that finite memory appears at the de Almeida–Thouless (AT) line obtained from the corresponding replica calculation, and compare the behaviour of the dilute model with the minority game with market impact correction, which is known to exhibit similar features.

Keywords: random graphs, networks, interacting agent models, stochastic processes

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1. Introduction

The minority game (MG) has attracted much interest in the statistical physics community over recent years. It models a simple market in which the trading agents at each time step have to make a decision on whether to buy or sell based on publicly available information, such as the past market history, political circumstances or the weather forecast. Each agent aims to make profit by making the opposite choice to the majority of agents. The interaction between the agents is indirect; the players cannot identify the actions of individual other agents, but react to the total cumulative outcome of all actions in the market. In order to take their trading decisions each agent holds a pool of strategies, which can be understood as look-up tables, mapping the value of the publicly available information onto a trading decision. The identification of the best strategy is based on virtual scores that the agents allocate to each of their strategies in order to keep track of their performance. At each round of the game each agent updates the score of each of his or her strategies, increasing the scores of strategies which would have predicted the correct minority decision. Exhaustive reviews on MGs can be found in [1, 2].

Although the update rules seem simple, the MG displays remarkably complex features. The behaviour of the fluctuations of the total bid, the so-called volatility, is non-trivial [3, 6, 7, 4, 5] and a phase transition separating an ergodic and a non-ergodic phase was identified [10, 5, 9, 11, 8]. The main control parameter in MGs is the ratio $\alpha = P/N$ of the number $P$ of possible values for the public information over the number $N$ of players.

From the point of view of statistical mechanics, the MG is an extremely interesting model. It contains the ingredients of a disordered mean-field system and can be

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approached with the techniques of spin-glass physics, if the thermodynamic limit of an infinite number of players is considered.

Although the model lacks detailed balance, it was shown that the stationary state of the adaptive dynamics can be captured by the minima of a disordered Hamiltonian $H$ for $\alpha > \alpha_c = 0.3374 \ldots$ [12]. Its ground states were calculated using the replica method [5, 9, 10], and the location of the ergodic/non-ergodic phase transition, $\alpha_c$, was identified analytically as the point where the static susceptibility obtained within a replica symmetric ansatz becomes infinite. While replica symmetry is not broken even in the non-ergodic phase of the standard MG, later studies of MGs in which the agents correct for their own contribution to the total bid then revealed the presence of phases with replica symmetry breaking [13].

Dynamical studies of the MG have been conducted in [8, 14] using the exact generating functional technique of [15], and enabled a full understanding of the game in the ergodic phase. The phase transition in the standard MG was identified as the point $\alpha = \alpha_c$ where the assumption that the dynamical susceptibility (or integrated response function) is finite breaks down. A review of dynamical analyses of MGs can be found in [16]. It was also shown that MGs with market impact correction exhibit a different type of phase transition [13]. While the integrated response remains finite in such models, long-term memory was identified as the source of ergodicity breaking, and it was seen that the onset of long-term memory coincides with the AT line, at which the replica symmetric solution becomes unstable [17, 13].

On the technical level the MG is closely related to the Hopfield model of neural networks [18]; the update rules of the MG are similar to ‘anti-Hebbian’ learning rules for neurons. While the quenched patterns in the Hopfield model form attractors of the learning dynamics, the agents in the MG try to avoid such fixed points. Up to an additional random field the above Hamiltonian of the MG is therefore essentially identical to the Hamiltonian of the Hopfield model with negative sign. The energy landscape of a suitably defined majority game coincides with that of the Hopfield model, although the dynamical rules of the game differ from the Glauber dynamics considered in neural networks [19]. The possibility of retrieval states in majority games is discussed in [20].

In analogy to work on dilute neural networks, in which not every neuron is connected to every other neuron, we study the effects of dilution on the phase behaviour of the MG in this paper. We restrict ourselves to the case in which the average number of connections per agent scales with the number of agents in such a way that it becomes infinite in the thermodynamic limit, as opposed to the case of finite connectivity. In terms of the decision making of the agents this corresponds to a game in which the agents do not use the global aggregate bid to update the scores of their strategies, but only the cumulative bid of an individual subset of neighbours to which they are connected. Related games in which players adjust their decisions depending on the behaviour of agents in a local neighbourhood have been discussed for example in [21]–[23]. MG-type models with inter-agent communication on random graphs have also been studied analytically using a so-called ‘crowd–anticrowd theory’ in [24, 25].

2. Model definitions

Before introducing the dilution we will briefly recall the dynamical rules of conventional MGs as studied for example in [8, 26]. The $N$ agents in the MG will be labelled with
Roman indices. At each round $t$ of the game each agent $i$ takes a trading decision $b_i(t) \in \mathbb{R}$ (a ‘bid’) in response to the observation of public information $I_{\mu(t)}$. While in the original version of the MG the information coded for the actual market history $[27,28]$, we will here assume that $\mu(t)$ is chosen randomly and independently from a set with $P = \alpha N$ possible values, i.e. $\mu(t) \in \{1, \ldots, \alpha N\}$. This is the so-called MG with random external information. One then defines the rescaled total market bid at round $t$ as $A(t) = N^{-1/2} \sum_i b_i(t)$. Each agent $i$ holds a pool of $S \geq 2$ fixed trading strategies (look-up tables) $R_{ia} = (R_{ia}^1, \ldots, R_{ia}^S)$, with $a = 1, \ldots, S$. If agent $i$ decides to use strategy $a$ in round $t$ of the game, his or her bid at this stage will be $b_i(t) = R_{ia}^{\mu(t)}$. All strategies $R_{ia}$ are chosen randomly and independently before the start of the game; they represent the quenched disorder of this problem. The behaviour of the MG was found not to depend much on the value of $S [27,29]$, nor on whether bids are discrete or continuous [4]. For convenience, we choose $S = 2$ and $R_{ia} \in \{-1, 1\}^P$ in this paper. The agents decide which strategy to use based on points $p_{ia}(t)$ which they allocate to each of their strategies. These virtual scores are based on their success had they always played that particular strategy:

$$p_{ia}(t + 1) = p_{ia}(t) - R_{ia}^{\mu(t)} A(t).$$

(1)

Strategies which would have produced a minority decision are thus rewarded. At each round $t$ each player $i$ then uses the strategy in his or her arsenal with the highest score, i.e. $b_i(t) = R_{ia}^{\mu(t)}$, where $\hat{a}_i(t) = \arg \max_a p_{ia}(t)$. For $S = 2$ the rules (1) can then be simplified upon introducing the point differences $q_i(t) = \frac{1}{2}[p_{i1}(t) - p_{i2}(t)]$. Thus agent $i$ plays strategy $R_{i1}$ in round $t$ if $q_i(t) > 0$, and $R_{i2}$ if $q_i(t) < 0$, so that his or her bid in round $t$ reads $b_i(t) = \omega_i^{\mu(t)} + \text{sgn}[q_i(t)]\xi_i^{\mu(t)}$, where $\omega_i = \frac{1}{2}[R_{i1} + R_{i2}]$ and $\xi_i = \frac{1}{2}[R_{i1} - R_{i2}]$.

The above update rule for the score differences $\{q_i\}$ can be written as

$$q_i(t + 1) = q_i(t) - \xi_i^{\mu(t)} \left[ \Omega^{\mu(t)} + \frac{1}{\sqrt{N}} \sum_j \xi_j^{\mu(t)} \text{sgn}[q_j(t)] \right]$$

(2)

with $\Omega = N^{-1/2} \sum_j \omega_j$. Equation (2) defines the standard (or so-called ‘on-line’) MG. Alternatively, it is common in studies of the MG to define the dynamics in terms of an average over all possible values of the external information in (2). This corresponds to updating the $\{q_i\}$ only every $O(N)$ time steps, and leads to the so-called ‘batch’ MG [8]:

$$q_i(t + 1) = q_i(t) - \frac{2}{N} \sum_j \left[ \sum_{\mu=1}^P \xi_i^\mu \xi_j^\mu s_j(t) + \sum_{\mu=1}^P \xi_i^\mu \omega_j^\mu \right].$$

(3)

Here $s_j(t) = \text{sgn}[q_j(t)]$. We can now introduce a dilute version of the batch MG by modifying the update rules as follows:

$$q_i(t + 1) = q_i(t) - \frac{2}{N} \sum_j c_{ij} \left[ \sum_{\mu=1}^P \xi_i^\mu \xi_j^\mu s_j(t) + \sum_{\mu=1}^P \xi_i^\mu \omega_j^\mu \right].$$

(4)

Here, the $c_{ij} \in \{0, 1\}$ ($i, j = 1, \ldots, N$) are additional quenched random variables. As before, the disorder corresponding to the strategy assignments is contained in the variables $\{\xi_i^\mu, \omega_i^\mu\}$. $c$ is a new control parameter and denotes the probability for a given link $c_{ij}$
to be present, \( P(c_{ij} = 1) = c \). It hence controls the degree of dilution; on average any given agent is connected to \( cN \) other agents. For \( c = 1 \) the fully connected batch MG is recovered. In the context of neural networks the limit \( \lim_{N \to \infty} c = 0 \), while still \( \lim_{N \to \infty} cN = \infty \) is often referred to as the ‘extremely dilute’ limit [30]–[32]. We will specify the further details of the distribution from which the \( \{c_{ij}\} \) are drawn below.

In terms of the updating of strategy scores the dilution means that a given agent \( i \) no longer uses the total re-scaled bid \( A(t) = N^{-1/2} \sum_j b_j(t) \) to evaluate the performance of his or her strategies, but takes into account only the cumulative bid \( A_i(t) = N^{-1/2} \sum_{\{j: c_{ij} \neq 0\}} b_j(t) \) of the agents \( j \) to which he or she is connected, i.e. for which \( c_{ij} = 1 \). Such so-called ‘local’ MGs and related models have been studied in [21, 22, 24, 25]. In local MGs the perceived market history may be different for different agents. Accordingly, if ourselves to the statistical mechanics analysis of the model defined by the update rule (4).

It remains to specify the distribution from which the dilution variables \( \{c_{ij}\} \) are drawn. We will choose \( c_{ii} = 1 \), and will assume that \( c_{kl} \) is independent of \( c_{ij} \) whenever \((k, l) \notin \{(i, j), (j, i)\} \). The parameters characterizing the distribution of the dilution variables are then the expectation value and covariance

\[
c = \langle c_{ij} \rangle_c = \langle c_{ji} \rangle_c, \quad (c_{ij}, c_{ji})_c - c^2 = \Gamma c (1 - c);
\]

note that the \( \{c_{ij}\} \) only take values 0 and 1, so \( \langle c_{ij}^2 \rangle_c = c \). We write averages over the dilution variables as \( \langle \cdots \rangle_c \). The parameter \( \Gamma \) controls the symmetry of the dilution and allows a smooth interpolation between the case of fully symmetric dilution \( c_{ij} = c_{ji} \) for \( \Gamma = 1 \) and the fully uncorrelated case \( \Gamma = 0 \), where \( c_{ij} \) and \( c_{ji} \) are chosen independently. If \( c = 1 \), then all the links are present and \( \Gamma \) becomes meaningless. In closed form the corresponding distribution for any pair \( c_{ij} \), \( c_{ji} \) with \( i < j \) can be written as

\[
P(c_{ij}, c_{ji}) = c[c(1 - \Gamma) + \Gamma] \delta_{c_{ij},1} \delta_{c_{ji},1} + c(1 - c)(1 - \Gamma)(\delta_{c_{ij},1} \delta_{c_{ji},0} + \delta_{c_{ij},0} \delta_{c_{ji},1}) + (1 - c)[(1 - c)(1 - \Gamma) + \Gamma] \delta_{c_{ij},0} \delta_{c_{ji},0}.
\]

Taking into account that on average there are \( cN \) connections per agent, the appropriate control parameter is now \( \alpha = P/(cN) \), where \( P \) is as before the number of possible values of the information, i.e. we have \( \mu \in \{1, \ldots, \alpha cN\} \).

3. Dynamics

3.1. The generating functional and the single-effective-agent process

We will start by studying the dynamics of the dilute batch minority game. The now standard approach is based on dynamical functionals [8, 26] and has been applied to different versions of the MG in [17, 33, 20, 35, 34]. Before we introduce the generating

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where $D \psi$ representaitve agent. After setting the source fields Markovian processes into an equivalent effective non-Markovian process for a single along the lines of [8] in order to perform the average over the

standard calculation. One first averages over the dilution variables

strategies assigned at the beginning of the game. We will not report the details of the
given by

and have added the perturbation fields $\{ \theta_i(t) \}$ to generate response functions.
The dynamical partition function is then introduced as

where $\{ c_{ij} \}$ and $\{ \xi^i_\mu \}$ have been calculated, and then proceeds along the lines of [8] in order to perform the average over the $\{ \xi^i_\mu \}$ and $\{ \omega^\mu_j \}$, i.e. over the strategies assigned at the beginning of the game. We will not report the details of the calculation here, but refer the reader to [30,32] where similar averaging procedures are discussed in the context of dilute neural networks.

As usual, the evaluation of $Z[\psi]$ converts the original system (4) of $N$ coupled Markovian processes into an equivalent effective non-Markovian process for a single representative agent. After setting the source fields $\psi_i(t)$ to zero and assuming that $\theta_i(t) = \theta(t)$ for all $i$, we find the following single-effective-agent equation:

$\alpha \sum_{tt'}(1+c)|I+G|^{-1} s(t')$

\begin{align}
q(t + 1) = q(t) - \alpha(1 - c)s(t) + \theta(t) - \alpha c \sum_{t'}(1 + G)^{-1} s(t') \\
+ \Gamma \alpha(1 - c) \sum_{t'} G_{tt'} s(t') + \theta(t) + \sqrt{\alpha} \eta(t),
\end{align}

where $s(t') = \text{sgn}(q(t'))$. $\eta(t)$ is Gaussian noise with zero mean and temporal correlations given by

\begin{align}
\langle \eta(t) \eta(t') \rangle_s = c((I + G)^{-1}(E + C)(I + G^T)^{-1})_{tt'} + (1 - c)(E + C)_{tt'}.
\end{align}
determined self-consistently according to
\[ C_{tt'} = \langle s(t)s(t') \rangle_s, \quad G_{tt'} = \left\langle \frac{\partial s(t)}{\partial \theta(t')} \right\rangle_s, \]
where \( \langle \cdot \cdot \cdot \rangle_s \) denotes averages over the single-agent noise \( \{\eta(t)\} \). This procedure is exact in the thermodynamic limit \( N \to \infty \) and the order parameters \( C \) and \( G \) can be identified with the disorder-averaged correlation and response functions of the original multi-agent batch process:
\[ C_{tt'} = \lim_{N \to \infty} \frac{1}{N-1} \sum_i \langle s_i(t)s_i(t') \rangle_{\text{paths}}, \quad G_{tt'} = \lim_{N \to \infty} \frac{1}{N-1} \sum_i \frac{\partial}{\partial \theta_i(t')} \langle s_i(t)s_i(t') \rangle_{\text{paths}}. \]

Equations (10)–(12) define a self-consistent problem to be solved for the macroscopic observables \( C \) and \( G \). We note that the symmetry parameter \( \Gamma \) is irrelevant in the case of full connectivity, \( c = 1 \), and that the single-effective-agent problem reduces to that of the standard batch MG [8] in this limit. Because of the retarded interaction terms and the presence of coloured noise, a full analytical solution is in general not feasible beyond the first few time steps. Alternatively one can resort to a Monte Carlo integration of the single-agent process using the procedure of [36]. This method allows one to determine the correlation and response functions numerically without finite-size effects, but becomes more and more costly as the number of time steps is increased.

### 3.2. Stationary state

We now proceed to study the stationary states of the single-effective-agent process. We make the common assumptions of time-translation invariance:
\[ \lim_{t \to \infty} C_{t+t',t} = C(\tau), \quad \lim_{t \to \infty} G_{t+t',t} = G(\tau), \]
finite integrated response:
\[ \lim_{t \to \infty} \sum_{t' \leq t} G_{tt'} = \chi < \infty \]
and weak long-term memory:
\[ \lim_{t \to \infty} G_{tt'} = 0 \quad \forall t' \text{ finite}. \]

Under these assumptions we can follow [8] and perform an average over time of (10). One defines \( s = \lim_{\tau \to \infty} \tau^{-1} \sum_{t \leq \tau} \text{sgn}[q(t)] \) as well as \( \tilde{q} = \lim_{t \to \infty} q(t)/t \) and obtains, after setting \( \theta(t) = 0 \),
\[ \tilde{q} = -Rs + \sqrt{\alpha} \eta, \]
where we have defined
\[ R = -\alpha(1 - c)\left(\Gamma \chi - 1\right) + \frac{\alpha c}{1 + \chi}. \]
\[ \eta = \lim_{\tau \to \infty} \tau^{-1} \sum_{t \leq \tau} \eta(t) \] is a Gaussian random variable with zero mean and variance:

\[ Y = \langle \eta^2 \rangle_\tau = \frac{1+Q}{(1+\chi)^2} + (1-c)(1+Q), \quad (20) \]

where we have introduced the persistent correlation \( Q = \lim_{\tau \to \infty} \tau^{-1} \sum_{t \leq \tau} C(t) \).

Now, as usual in studies of MGs, we can distinguish self-consistently between so-called ‘frozen’ and ‘fickle’ agents. The representative agent is frozen and keeps playing the same strategy if \( |q(t)| \to \infty \) asymptotically, so \( \tilde{q} \neq 0 \). In this case one has \( s = \text{sgn}(\tilde{q}) \).

Provided that \( R > 0 \), this is easily seen to be the case if \( |\eta| > R/\sqrt{\alpha} \). On the other hand, the effective agent is fickle and keeps switching between his or her two strategies when \( \tilde{q} = 0 \), which is the case if \( |\eta| < R/\sqrt{\alpha} \). In this case \( s = (\sqrt{\alpha}/R)\eta \). Given \( Q = \langle s^2 \rangle_\tau \), the persistent part of the correlation function can then be computed self-consistently from

\[ Q = \langle \theta(|\eta| - R/\sqrt{\alpha}) \rangle_\tau + \left( \theta(R/\sqrt{\alpha} - |\eta|) \frac{\alpha \eta^2}{R^2} \right)_\tau, \quad (21) \]

where \( \theta(t) \) is the step function, \( \theta(t) = 1 \) for \( t > 0 \) and \( \theta(t) = 0 \) otherwise. \( \phi = \langle \theta(|\eta| - R/\sqrt{\alpha}) \rangle_\tau \) denotes the fraction of frozen agents. On the other hand, only fickle agents contribute to the response function asymptotically and we have

\[ \chi = \frac{1}{\sqrt{\alpha}} \langle \theta(R/\sqrt{\alpha} - |\eta|) \frac{\partial s}{\partial \eta} \rangle_\tau = \frac{1 - \phi}{R}. \quad (22) \]

Explicitly performing the Gaussian integrals over \( \eta \) in (21) then leads to the following two non-linear, but closed equations for \( Q \) and \( \chi \):

\[ Q = 1 + \text{erf} \left( \frac{R}{\sqrt{2\alpha Y}} \right) \left( \frac{\alpha Y}{R^2} - 1 \right) - \sqrt{\frac{2}{\pi}} \sqrt{\frac{\alpha Y}{R}} \exp \left( -\frac{R^2}{2\alpha Y} \right), \quad (23) \]

\[ \chi = \frac{1}{R} \text{erf} \left( \frac{R}{\sqrt{2\alpha Y}} \right). \quad (24) \]

The fraction of frozen agents reads \( \phi = 1 - \text{erf}(R/\sqrt{2\alpha Y}) \).

The system (23), (24) reduces to the corresponding equations of the standard batch MG [8] in the limit \( c = 1 \). It can be solved numerically for any fixed values of the parameters \( 0 \leq c \leq 1 \) and \( 0 \leq \Gamma \leq 1 \) to obtain the order parameters \( Q \) and \( \chi \) as a function of \( \alpha \). We compare the analytical results for the persistent part of the correlation function with numerical simulations in figure 1. Here, the extremely dilute limit \( \lim_{N \to \infty} c N = \infty \) has been simulated by setting \( c = N^{-1/2} \). We find near perfect agreement for large values of \( \alpha \), but for \( \Gamma > 0 \) systematic deviations are observed in the low-\( \alpha \) region. Using the method of [36] we have performed an explicit iteration of the single-agent equation to confirm that the deviations for small values of \( \alpha \) are not artefacts due to a finite number of agents in the simulations.

In the standard MG, \( c = 1 \), the breakdown of the ergodic theory is associated with the onset of an anomalous response, i.e. the violation of the assumption that the integrated
response $\chi$ be finite. At the same time one expects the assumption of weak long-term memory to be broken as well [37]. No singularities are found, however, in the numerical solution of (23), (24) for $c < 1$. Since no signs of ageing are found in MGs, we are led to the conclusion that the assumption of weak long-term memory breaks down below a critical value of $\alpha$, whereas $\chi$ remains finite. We will refer to this as the onset of memory; similar behaviour was observed in batch MGs with market impact correction [17].

3.3. The onset of memory and the phase diagram

We will now proceed to identify this onset of memory analytically. To this end we follow the lines of [17] and separate time translation invariant (TTI) contributions to the response function from the non-TTI parts and write

$$\lim_{t \to \infty} G_{tt'} = \hat{G}(t - t') + \hat{G}_{tt'}. \quad (25)$$

While perturbations in the initial stages of the dynamics might influence the final state of a given agent, we expect the effect to become independent of $t$ eventually, so we assume $\lim_{t \to \infty} \hat{G}_{tt'} = \hat{G}(t')$. In the stationary state, the point differences of frozen agents have become large, so that only fickle agents are affected by perturbations applied at later times. This is accounted for with $\hat{G}$. Hence we expect that $\lim_{t' \to \infty} \hat{G}(t') = 0$ [17]. Assuming furthermore that $\hat{G}$ is small, we expand the single-effective-agent process to first order in
\( \dot{G} \) and find

\[
q(t+1) = q(t) - \alpha(1-c)s(t) - \alpha c \sum_{t'} (1 + \dot{G})^{-1}_{tt'} s(t') + \Gamma \alpha(1-c) \sum_{t'} \dot{G}_{tt'} s(t') \\
+ \theta(t) + \sqrt{\alpha} \eta(t) + \alpha c \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} \left[ (-\dot{G})^m \dot{G} (-\dot{G})^{n-m-1} \right] \sum_{t'} s(t') \\
+ \Gamma \alpha(1-c) \sum_{t'} \dot{G}(t') s(t') + O(\dot{G}^2).
\] (26)

Writing \( \chi = \sum_t \dot{G}(t) \) one then has after averaging over time

\[
\dot{q} = -\alpha (1-c) s - \frac{\alpha c s}{1 + \chi} + \Gamma \alpha (1-c) \chi s + \sqrt{\alpha} \eta \\
+ \alpha c \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} (-\chi)^m \sum_{t'} \dot{G}(t') \sum_{t''} (-\dot{G})^{n-m-1} s(t'') + \Gamma \alpha(1-c) \sum_{t'} \dot{G}(t') s(t').
\] (27)

As before we have \( \chi = (1/\sqrt{\alpha}) \langle \partial s/\partial \eta \rangle_s \), whereas

\[
\dot{G}(t) = \left\langle \frac{\partial s}{\partial \theta(t)} \right\rangle_s \\
= \gamma^{-1} \left[ \sqrt{\alpha} c \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} (-\chi)^m \sum_{t'} \dot{G}(t') \sum_{t''} (-\dot{G})^{n-m-1} \dot{G}_{tt'} \\
+ \Gamma \sqrt{\alpha} (1-c) \sum_{t'} \dot{G}(t') \dot{G}_{tt'} \right],
\] (28)

where we have introduced \( \gamma = \sqrt{\alpha}(1-c)(1-\Gamma \chi) + \sqrt{\alpha} c/(1 + \chi) \).

We now define \( \dot{\chi} = \sum_t \dot{G}(t) \), and after summing over \( t \) and re-organizing the terms we finally find to first order in \( \dot{G} \)

\[
\dot{\chi} = \sqrt{\alpha} \chi \left[ \frac{c}{1 + \chi} + \Gamma(1-c) \right] \dot{\chi} + O(\dot{G}^2).
\] (29)

Although \( \dot{\chi} = 0 \) is always a solution, a bifurcation occurs when the following condition is fulfilled:

\[
\sqrt{\alpha} \chi \left[ \frac{c}{1 + \chi} + \Gamma(1-c) \right] = \gamma,
\] (30)

so that non-zero solutions for \( \dot{\chi} \) become possible. For fixed \( \Gamma \) this condition marks the continuous onset of long-term memory and defines a line in the \( (\alpha, c) \) plane. We will refer to this line as the memory onset (MO) line in the following, and will discuss the resulting phase diagram in section 5.

An intuitive argument for the different type of phase transition of the dilute MG as compared with the fully connected model can be based on a simple geometrical explanation of the transition in the fully connected standard MG \([9,38]\). In the stationary state only the \((1 - \phi)N\) fickle agents contribute to the integrated response \( \chi \), as perturbations...
The variables $\phi$ spanned by the $\{\xi^a \}$ correspond to minima of the disordered function $H_1 = (1/P) \sum_{i=1}^P \sum_{i,j} \xi_i^a \phi_i \phi_j + 2 \sum_{i,j} \xi_i^a \omega_i^a \phi_i + \sum_{i,j} \omega_i^a \omega_j^a$ \cite{5,9}. The variables $\phi_i \in [-1,1]$ are the average asymptotic values of $s_i(t)$, so that the corresponding agent is frozen whenever $\phi_i = \pm 1$ and remains fickle for $-1 < \phi_i < 1$.

We generalize the above function to the case of dilution and introduce

$$H_c = \frac{1}{2\alpha} \left[ \sum_{ij} \{ J_{ij}^c \phi_i \phi_j + 2h_{ij}^c \phi_i + K_{ij}^c \} \right]. \quad (31)$$

Here we use the same definitions for $J_{ij}^c$ and $h_{ij}^c$ as in the analysis of the dynamics, and introduce new variables $K_{ij}^c = (c_{ij}/c)(2/N) \sum_{p=1}^P \omega_i^p \omega_j^p$.

As usual in disordered systems, it is in general not possible to find a Hamiltonian which is minimized by the dynamics in the case of asymmetric couplings. Therefore, we have to restrict this section to the case of fully symmetric dilution, $c_{ij} = c_{ji}$; i.e. we will assume $\Gamma = 1$ in the above distribution (6) of the connectivity variables.

In order to minimize the above function $H$ in terms of the variables $\{\phi_i\}$, we resort to the replica method and compute the disorder average of the replicated partition function

$$Z^n = \int_{-1}^1 \left( \prod_a d\phi_a^o \right) \exp \left[ -\frac{\beta}{2\alpha} \sum_a \sum_{ij} \{ J_{ij}^c \phi_i^a \phi_j^a + 2h_{ij}^c \phi_i^a + K_{ij}^c \} \right] \quad (32)$$

at some ‘annealing temperature’ $\beta^{-1}$. We will eventually take the limit $\beta \to \infty$. The superscript $a$ is a replica index, $a = 1, \ldots, n$. The minima of $H_c$ are then found as

$$\min_{\{\phi_i\}} H_c = - \lim_{\beta \to \infty} \lim_{N \to \infty} \lim_{n \to 0} \frac{Z^n - 1}{\beta N n}. \quad (33)$$

The computation of $Z^n$ is again lengthy, but straightforward and is essentially a combination of the corresponding calculations for the fully connected MG \cite{5,9,10} and...
replica analyses of dilute neural networks [31, 32]. We will therefore only report the final result:

\[
\overline{Z}^n = \int DQ DS \exp(-\beta n N f(Q, S) + \mathcal{O}(N^0) + \mathcal{O}(n^2)),
\]

(34)

with \( DQ = \prod_{ab} [\sqrt{N} dQ_{ab} / \sqrt{2\pi}] \) and similarly for \( DS \). Here the replicated ‘free energy’ is given by

\[
f(Q, S) = \frac{ac}{2\beta n} \log \det T + \frac{\alpha \beta}{2n} \sum_{ab} S_{ab} Q_{ab}
\]

\[- \frac{1}{\beta n} \log \left[ \int_{-1}^{1} \left( \prod_{a} d\phi^a \right) \exp \left( \frac{\alpha \beta^2}{2} \sum_{ab} S_{ab} \phi^a \phi^b \right) \right] \]

\[- \frac{\beta}{4\alpha n} (1-c) \sum_{ab} Q_{ab} Q_{ab} - \frac{c-1}{2n} \sum_{a} Q_{aa} - \frac{\beta}{\alpha} \frac{(1-c)}{2n} \sum_{ab} Q_{ab} + \frac{1-c}{2}. \]

(35)

\( Q \) is the overlap matrix defined by \( Q_{ab} = N^{-1} \sum_i \phi_i^a \phi_i^b \), and the \( S_{ab} \) are the conjugate Lagrange multipliers. The \( n \times n \) matrix \( T \) is given by \( T_{ab} = \delta_{ab} + (\beta/\alpha)(1 + Q_{ab}) \).

4.2. The replica symmetric ansatz

We will now make a replica symmetric ansatz corresponding to the assumption of ergodicity in the dynamics and will set

\[ Q_{ab} = q + (Q - q) \delta_{ab}, \quad S_{ab} = s + (S - s) \delta_{ab}. \]  

(36)

Insertion into (35) and taking the limit \( n \to 0 \) gives the replica symmetric free energy

\[ f_{RS}(Q, S) = \frac{ac}{2\beta} \log \left[ 1 + \frac{\beta}{\alpha} (Q - q) \right] + \frac{ac}{2} \frac{(1+q)}{\alpha + \beta (Q - q) } \]

\[ + \frac{\alpha \beta}{2} (SQ - sq) - \frac{1}{\beta} \left\langle \log \int_{-1}^{1} d\phi \exp (-\beta V_z(\phi)) \right\rangle_z \]

\[- \frac{\beta}{4\alpha} (1-c)(Q^2 - q^2) - \frac{c-1}{2} Q + \frac{\beta}{\alpha} \frac{(1-c)}{2} (q - Q) + \frac{1-c}{2}, \]  

(38)

where \( V_z(\phi) \) is an effective potential defined by \( V_z(\phi) = -\sqrt{\alpha s \phi} - (\alpha \beta/2)(S - s) \phi^2 \). \( \langle \cdots \rangle_z \) denotes the average over the standard Gaussian variable \( z \).

The integrations in (34) are then performed by the method of steepest descent in the thermodynamic limit and the corresponding saddle-point equations are obtained by working out the variations of \( f_{RS} \) with respect to the parameters \( Q, q, S \) and \( s \). Similar to [10], one finds

\[ s = c \frac{1+q}{(\alpha + \beta(Q - q))^2} + \frac{1-c}{\alpha^2} (1+q) \]  

(39)

\[ Q = \langle \langle \phi^2 | z \rangle_{\phi} \rangle_z \]  

(40)

\[ \beta(S - s) = -\frac{c}{\alpha + \beta(Q - q)} + \frac{\beta(1-c)}{\alpha^2} (Q - q) + \frac{c-1}{\alpha} \]  

(41)

\[ \beta(Q - q) = \frac{1}{\sqrt{\alpha s}} \langle z \langle \phi | z \rangle_{\phi} \rangle_z \]  

(42)
In these expressions we have introduced the average
\[ \langle f(\phi) | z \rangle_\phi = \frac{\int_{-1}^{1} d\phi f(\phi) \exp(-\beta V_z(\phi))}{\int_{-1}^{1} d\phi \exp(-\beta V_z(\phi))}. \quad (43) \]
We will be looking for solutions with \( \lim_{\beta \to \infty} Q = \lim_{\beta \to \infty} q \) and \( \lim_{\beta \to \infty} S = \lim_{\beta \to \infty} s \) and will also use the quantities
\[ \chi = \frac{\beta}{\alpha} (Q - q), \quad \zeta = -\sqrt{\frac{\alpha}{s}} \beta (S - s), \quad (44) \]
which will remain finite in this limit. The remaining averages over \( \phi \) are easily evaluated in the limit \( \beta \to \infty \) as the corresponding integrals over \( \phi \) are dominated by the minimum of the potential \( V_z(\phi) \) in \( \phi \in [-1, 1] \). We find that the minimum is at \( \phi = 1 \) for \( z \geq \zeta \) and at \( \phi = -1 \) for \( z \leq -\zeta \). This is the analogue of the corresponding condition on the time-averaged single-particle noise for frozen agents in the generating functional calculation. For \(-\zeta < z < \zeta \) the minimum is found at \( \phi = z/\zeta \), so that the agent is fickle. Working out the remaining integrals we then find the following two equations for the order parameters \( \chi \) and \( Q \) \[10\]:
\[ Q = 1 - \sqrt{\frac{2}{\pi}} \frac{\exp(-\zeta^2/2)}{\zeta} \left( 1 - \frac{1}{\zeta^2} \right) \text{erf} \left( \frac{\zeta}{\sqrt{2}} \right), \quad (45) \]
\[ \sqrt{\alpha s} \beta (Q - q) = \frac{1}{\zeta} \text{erf} \left( \frac{\zeta}{\sqrt{2}} \right). \quad (46) \]
In order to make contact with the results obtained from the analysis of the dynamics we define
\[ R = -\alpha^2 \beta (S - s) = -\alpha (1 - c) (\chi - 1) + \frac{\alpha c}{1 + \chi}, \quad (47) \]
\[ Y = \alpha^2 s = c \frac{1 + Q}{(1 + \chi)^2} + (1 - c) (1 + Q), \quad (48) \]
so that \( \zeta = R/\sqrt{\alpha Y} \) and find
\[ Q = 1 - \sqrt{\frac{2}{\pi}} \frac{\sqrt{\alpha Y}}{R} \exp \left( -\frac{R^2}{2\alpha Y} \right) \left( 1 - \frac{\alpha Y}{R^2} \right) \text{erf} \left( \frac{R}{\sqrt{2\alpha Y}} \right), \quad (49) \]
\[ \chi = \frac{1}{R} \text{erf} \left( \frac{R}{\sqrt{2\alpha Y}} \right). \quad (50) \]
Note that this set of equations is identical to the one found from the dynamics (cf (23), (24)) for the case of fully symmetric dilution, \( \Gamma = 1 \). Finally, the replica symmetric free energy \( (37) \) can be simplified upon using the saddle point equations and we find
\[ \lim_{\beta \to \infty} f_{RS} = \frac{c}{2} \frac{1 + Q}{(1 + \chi)^2} + \frac{1 - c}{2} (1 + Q) (1 - 2\chi). \quad (51) \]
We compare this prediction of the replica symmetric theory with simulations in figure 2 and find very good agreement in the phase with only weak long-term memory. For \( c = 1 \) the result of equation (51) reduces to the replica symmetric free energy of the fully connected model as reported in \[10\]. Note that in this case \( H_c \) can be identified as the so-called predictability of the market and is a positive definite quadratic form and hence non-negative. A similar interpretation of \( H_c \) is not straightforward in the dilute model, and in particular \( H_c \) need not be non-negative for \( c < 1 \). It is however instructive to
Figure 2. Left: the stationary value of the ‘energy’ $H_c$ as a function of $\alpha$ for $c = 1$ (circles), $c = 0.75$ (squares), $c = 0.5$ (diamonds) and $c = 0$ (triangles). Data for $c > 0$ are from simulations with $N = 500$ agents, run for 500 time steps and averaged over 20 realizations of the disorder. Data for $c = 0$ are for $N = 1000$ and 10 realizations. Right: a zoom for large values of $\alpha$; the solid curves are the predictions of the replica symmetric theory, equation (51), while the vertical dashed lines mark the onset of the AT instability; see equation (69).

study the predictability

$$H = \lim_{N \to \infty} \frac{1}{P} \sum_{\mu=1}^{P} \langle A | \mu \rangle^2 = \lim_{N \to \infty} \frac{1}{PN} \sum_{\mu=1}^{N} \sum_{ij} (\omega_{\mu}^i + \xi_{\mu}^{\phi_i})(\omega_{\mu}^j + \xi_{\mu}^{\phi_j})$$

(52)

also for the case $c < 1$. Here $\langle A | \mu \rangle$ denotes an average over the total bid conditioned on a pattern $\mu$ in the stationary state [39,5]. We display data from simulations in figure 3. An analytical result for the predictability in the ergodic state of the fully connected batch MG is given in [8] and reads

$$H = \frac{1}{2} \frac{1 + Q}{(1 + \chi)^2}.$$ (53)

We note that this expression can carried over to the dilute case $c < 1$ and that the numerical data agrees very well with the prediction of (53); see figure 3.

While the two phases in the fully connected model correspond to $H_{c=1} = 0$ in the regime of low $\alpha$ and to a predictable phase ($H_{c=1} > 0$) above $\alpha_c(c = 1) = 0.3374 \ldots$, we observe that no unpredictable (or so-called symmetric) phase is present for dilution parameters $c < 1$. Although figure 3 only depicts results for some combinations of the parameters $c$ and $\Gamma$, we have verified that $H$ is strictly positive also for other values of $\Gamma$ as long as $c < 1$. This absence of a symmetric phase with vanishing predictability is also observed in MGs with market impact correction [40,9].

2 In the fully connected case the same expression for the predictability $H = H_{c=1}$ can also be obtained from the replica analysis of the statics, as mentioned in the main text. Note, however, that the result of [8] is obtained from a generating functional analysis of the dynamics of the fully connected game. Equation (53) therefore applies not only to the case of symmetric interactions between the agents, but also for arbitrary symmetry parameters $\Gamma < 1$ (where no replica theory is available due to the lack of symmetry in the couplings $\{J_{ij}\}$).
4.3. Calculation of the AT line

We will now study the stability of the ground states of $H_c$ obtained within the replica symmetric approximation against small fluctuations which break replica symmetry. To this end we follow the procedure first proposed by de Almeida and Thouless [41] and study the eigenvectors of the matrix of second derivatives of the free energy with respect to the \{Q_{ab}\} and \{S_{ab}\}. This results in an instability condition marking the onset of replica symmetry breaking and defines a line in the ($\alpha$, $c$) plane, the so-called AT line. Our analysis follows the lines of [13,37], where the AT line has been computed for the fully connected MG with market impact correction.

The stability matrix of second derivatives of the free energy has dimension $n(n-1) \times n(n-1)$ and is given by

$$\Sigma = \begin{pmatrix} A & C \\ C & B \end{pmatrix},$$

where the sub-matrices $A$, $B$ and $C$ read

$$A_{(ab)(cd)} = \frac{\partial^2(nf)}{\partial Q_{ab} \partial Q_{cd}}, \quad B_{(ab)(cd)} = \frac{\partial^2(nf)}{\partial S_{ab} \partial S_{cd}}, \quad C_{(ab)(cd)} = \frac{\partial^2(nf)}{\partial Q_{ab} \partial S_{cd}}.$$ (55)

The derivatives with respect to the \{Q_{ab}\} are found as [13,37]

$$A_{(ab)(ab)} = -c \frac{\beta}{\alpha} [C_1^2 + C_2^2] - \frac{\beta}{\alpha} (1 - c)$$

$$A_{(ab)(ac)} = -c \frac{\beta}{\alpha} C_1 [C_1 + C_2]$$

$$A_{(ab)(cd)} = -c \frac{2 \beta}{\alpha} C_1^2.$$ (58)

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where $C_1 = -(\beta(1 + Q)/\alpha(1 + \chi)^2)$ and $C_2 = C_1 + (1/(1 + \chi))$. Furthermore, we find

$$B_{(ab)(ab)} = -\alpha^2 \beta^2 \left[ \langle \langle \phi^2 | z \rangle^2 \phi \rangle_{z} - \langle \langle \phi | z \rangle^2 \phi \rangle_{z} \right]$$

(59)

$$B_{(ab)(ac)} = -\alpha^2 \beta^2 \left[ \langle \langle \phi^2 | z \rangle \phi \langle \phi | z \rangle^2 \phi \rangle_{z} - \langle \langle \phi | z \rangle^2 \phi \rangle_{z} \right]$$

(60)

$$B_{(ab)(cd)} = -\alpha^2 \beta^2 \left[ \langle \langle \phi^2 | z \rangle^4 \phi \rangle_{z} - \langle \langle \phi | z \rangle^2 \phi \rangle_{z} \right]$$

(61)

and finally one has

$$C_{(ab)(cd)} = \alpha \beta \delta_{ab} \delta_{bd}.$$  

(62)

We now study fluctuations $\delta Q_{ab}$ and $\delta S_{ab}$ around the replica symmetric solution, which are of the form $\delta S_{ab} = x \delta Q_{ab}$, with $x$ a proportionality constant [13]. We will assume that the perturbation is symmetric, i.e. $\delta Q_{ab} = \delta Q_{ba}$ and $\delta S_{ab} = \delta S_{ba}$, and that it does not affect the diagonal elements, so $\delta Q_{aa} = \delta S_{aa} = 0$. We will here consider fluctuations of the form

$$\delta Q_{12} = \frac{1}{2}(3 - n)(2 - n)y$$

(63)

$$\delta Q_{1b} = \delta Q_{2b} = \frac{1}{2}(3 - n)y \quad b \notin \{1, 2\}$$

(64)

$$\delta Q_{ab} = y \quad a, b \notin \{1, 2\},$$

(65)

corresponding to the so-called ‘replicon’ mode [41, 42].

In the limit $n \to 0$ the corresponding eigenvalues are found to be [13, 37]

$$\lambda_{\pm} = -\frac{1}{2}(u + v) \pm \frac{1}{2} \sqrt{(u - v)^2 + 4},$$

(66)

where

$$u = \frac{c}{\alpha^2(1 + \chi)^2} + \frac{1 - c}{\alpha^2}, \quad v = \alpha \beta^2 \left[ \langle \langle \phi^2 | z \rangle \phi - \langle \phi | z \rangle^2 \phi \rangle_{z} \right].$$

(67)

Now, $\lambda_{-}$ never changes sign, so that the instability sets in when $\lambda_{+} = 0$. This happens when $uv = 1$, i.e. when

$$\lim_{\beta \to \infty} \beta^2 \left[ \langle \langle \phi^2 | z \rangle \phi - \langle \phi | z \rangle^2 \phi \rangle_{z} \right] = \left( \frac{c}{\alpha(1 + \chi)^2} + \frac{1 - c}{\alpha} \right)^{-1}.$$  

(68)

It remains to evaluate the averages on the left-hand side. This is conveniently done by introducing $F_{z}(h) = \beta^{-1} \log \int_{1}^{0} d\phi e^{-\beta V_{z}(\phi) + h \beta \phi}$ and upon taking derivatives with respect to $h$ [37]. After some algebra we finally find the following condition for the AT line:

$$\chi \left( \frac{c}{(1 + \chi)^2} + 1 - c \right) = \frac{c}{1 + \chi} + (1 - c)(1 - \chi).$$

(69)

This coincides with the MO line obtained from the dynamics (30), provided that we set $\Gamma = 1$ in the dynamical calculation. A comparison of equations (51) and (69) demonstrates that the minima of $H_c$ obtained from the replica symmetric theory become unstable precisely at the zero crossing of $H_c$, as illustrated in figure 2.
Figure 4. The phase diagram in the \((\alpha, c)\) plane. The lines separate phases with memory at low \(\alpha\) from phases without long-term memory and indicate the location of the transition for \(\Gamma = 0.25\) (dot–dashed curve), \(\Gamma = 0.5\) (dashed), \(\Gamma = 0.75\) (dotted) and \(\Gamma = 1.0\) (solid curve). The solid curve (\(\Gamma = 1\)) coincides with the AT line obtained from the replica calculation. The cross marks the critical point of the fully connected MG at \(\alpha_c = 0.3374\ldots\).

5. The phase diagram and discussion

The resulting phase diagram is shown in figure 4. We report the transition lines in the \((\alpha, c)\) plane for different fixed values of \(\Gamma > 0\). They separate an ergodic phase with weak long-term memory at high values of \(\alpha\) from a phase with memory for low \(\alpha\). For \(\Gamma = 1\) the replica symmetric ground state of \(H_c\) is stable to the right of the MO (=AT) line. Assuming that \(\chi\) remains finite we find that the condition for the MO line (30) is given by \(1 - c + c/(1 + \chi)^2 = 0\) for \(\Gamma = 0\) and that hence no onset of memory occurs in the case of fully asymmetric dilution as the left-hand side is always positive. We find that the values of \(\alpha_c(c, \Gamma)\) approach the critical point \(\alpha_c = 0.3374\ldots\) of the fully connected MG continuously in the limit \(c \to 1\). Note that the parameter \(\Gamma\) is meaningless in the fully connected case \(c = 1\).

To give further support for the picture of memory onset we follow [17] and measure the distance \(d = N^{-1} \sum_i (\phi_i - \phi'_i)^2\) between two microscopic stationary states obtained from simulations of two copies of the system for an identical realization of the disorder, but starting from slightly different initial conditions. In this expression \(\phi_i\) and \(\phi'_i\) are the long-time averages of \(\text{sgn}[q_i]\) and \(\text{sgn}[q'_i]\) in the stationary state, respectively. In figure 5 we report the average distance \(d\) as a function of \(\alpha\) for different values of the parameters \((c, \Gamma)\). For large \(\alpha\) the two copies end up in the same microscopic state, so \(d = 0\), indicating that the long-term memory is weak. Below the predicted onset of memory the assumption of weak long-term memory is broken, and we find \(d > 0\), so that the final microscopic stationary state depends on the initial conditions. While the data reported in figure 5 qualitatively support the picture of a continuous onset of memory, a verification of the precise point of memory onset by measuring \(d\) in simulations turns out to be difficult due to finite-size effects. Finally, we find that the two copies end up in the same state if the perturbation is applied at much later times, so we do indeed conclude that \(\lim_{t' \to \infty} \hat{G}(t') = 0\).
Figure 5. Average distance $d$ between the stationary states of two identical copies of the system as a function of $\alpha$. All runs are started from zero initial conditions. Open markers are obtained by applying a small perturbation at $t = 0$ (circles: $c = 0.5$, squares: $c = 0$; $\Gamma = 1$ in both cases). The vertical dashed lines indicate the location of the onset of memory, equation (30). The full markers are obtained by perturbing the system at a later time $t = 500$. All simulations are run up to 500 batch steps after the perturbation is applied. 10 runs with different random perturbations are generated per disorder sample and the average is taken over all pairwise distances. Results are then averaged over at least 20 realizations of the disorder.

We conclude this section by reporting results for the magnitude of the fluctuations of the total bid $A(t)$ in figure 6. $\sigma^2 = \langle A^2 \rangle_{\text{time}}$ measures the global efficiency of the market; the better supply and demand match, the smaller $\sigma^2$ and the more efficient the market. By construction, $\langle A \rangle_{\text{time}} = 0$. Note that in our model $\sigma^2$ measures the fluctuations of the total bid and not of the local bids $A_i$ perceived by the individual agents. The volatility in the stationary state is found as [8]

$$\sigma^2 = \frac{1}{2} \lim_{t \to \infty} [(I + G)^{-1}(E + C)(I + G^T)^{-1}]_{tt}. \quad (70)$$

Following the lines of [8] it is possible to find an approximate expression for the volatility in terms of the persistent order parameters $\phi$ and $\chi$, which holds in the ergodic phase:

$$\sigma^2 = \frac{1 + \phi}{2(1 + \chi)^2} + \frac{1}{2}(1 - \phi). \quad (71)$$

As shown in figure 6 this approximation is in very good agreement with numerical simulations of the batch process.

While the stationary volatility in the fully connected MG started from zero initial conditions exhibits a minimum at $\alpha_c = 0.33 \ldots$, we find an increasing function $\sigma^2(\alpha)$ for diluted games; see the left panel of figure 6. Although only the case $c = 0.25, \Gamma = 0.75$ is reported in figure 6, we find very similar curves for other values of $c < 1$ and $\Gamma$. Analogous behaviour of $\sigma^2$ was reported for games with market impact correction [9]. We observe that despite the memory effects no significant dependence of the volatility on the initial conditions is found in dilute games. This is in sharp contrast with the standard MG
Figure 6. Left: volatility $\sigma^2$ versus $\alpha$ for $c = 0.25, \Gamma = 0.75$ (squares). Markers represent numerical simulations with $N = 1000$ players, averaged over 20 samples of the disorder. Open symbols are for tabula rasa starts, $q_i(0) = 0$, full markers for biased starts, $|q_i(0)| = 10$. The vertical dashed line marks the breakdown of the ergodic theory for $c = 0.25, \Gamma = 0.75$. The solid curve to the right is the approximation of equation (71) for $c = 0.25, \Gamma = 0.75$. The circles are for comparison and show the results of the standard batch MG, $c = 1$. $\sigma^2 = 1$ is the random trading limit. Right: $\sigma^2$ versus $c$ at fixed $\alpha = 0.2$ for $\Gamma = 0$ (circles), $\Gamma = 0.5$ (squares) and $\Gamma = 1.0$ (diamonds), simulations started from zero initial conditions. The solid curve is the approximation of (71) for the case $\Gamma = 0$, in which the long-term memory is weak at $\alpha = 0.2$, so that the ergodic theory applies. For $\Gamma = 0.5$ and 1 the fixed value $\alpha = 0.2$ is below the onset of memory.

Finally, we have verified in numerical simulations that the behaviour of the model does not change much if the batch update rule (4) is replaced by a dilute version of the on-line MG, with random external information (see equation (2) for the definition of the fully connected on-line MG). We do not report the numerical data here, but would just like to remark that order parameters in the ergodic stationary state such as the persistent correlation $Q$ follow the same curves as the corresponding observables of the batch game (as displayed in figure 1), and that the phase diagram does not appear to be sensitive to the choice of on-line or batch learning rules. The volatilities of the dilute on-line and batch games differ slightly in their quantitative values, but their qualitative behaviour as a function of $\alpha$, $c$ and $\Gamma$ coincides. In particular, for on-line games, we find only low-volatility solutions for low $\alpha$ and $c < 1$, and a jump in the volatility as $c \uparrow 1$ for small enough fixed values of $\alpha$, similar to the case for the data displayed for the batch game in figure 6(b). These findings are consistent with the results of [26], where it was observed...
that order parameters such as $Q$ and $\phi$ in the stationary state are identical in the fully connected on-line and batch MGs, and that only slight quantitative differences are present in their respective volatilities.

6. Conclusions

In summary, we have presented static and dynamical analyses of the dilute batch MG with random external information. Using a generating functional approach we first solved the dynamics of the game. Assuming time translation invariance, finite integrated response and weak long-term memory, we have computed the persistent order parameters in the stationary state. Numerical simulations confirm these findings convincingly. Standard methods can be employed to obtain a satisfactory approximation for the magnitude of the global market fluctuations of the model.

The study of the dynamics is complemented by replica analysis of the statics of the game with fully symmetric dilution. We find that the resulting equations for the order parameters agree with the ones obtained from the dynamics.

Like in studies of MGs with market impact correction, a condition for the breakdown of weak long-term memory can be derived from the dynamics, and a continuous onset of memory is found at finite integrated response. The resulting MO line agrees with the AT line obtained from the statics, marking the onset of instability of the replica symmetric theory. While the microscopic stationary state of the present model below the transitions depends on the state from which the dynamics is started, macroscopic observables such as the volatility appear to be mostly insensitive to initial conditions. This indicates that below the transition many microscopic stationary states exist, but all with the same or nearly the same global order parameters. Simulations of the MG with market impact correction reveal similar behaviour. In both models, the dilute MG presented in this paper and the model with market impact correction, the microscopic update rules are such that the players use individual bids $A_i(t)$ to update their strategy scores, as opposed to the common global bid $A(t)$ in the fully connected model without impact correction. On the basis of the above geometrical considerations one might speculate that other variants of the MG with this property might also exhibit replica symmetry breaking and the onset of long-term memory at finite integrated response.

The dilute model shares two other features with the MG with market impact correction: the discontinuity in $\sigma^2$ as the special case of the conventional game is approached in its non-ergodic phase and the absence of a symmetric phase with vanishing predictability.

We believe that the dilute model is an intrinsically interesting model from the point of view of statistical mechanics and hope that the present contribution might serve as a starting point for further analytical studies. For example, an approximate or exact solution might be attempted in the regime of broken weak long-term memory, but with finite integrated response. It might also be interesting to apply recently developed methods to study the dynamics of disordered systems with finite connectivity [43, 44] to MGs with a finite number of connections per agent and to understand the consequences for the phase diagram.

While the present model is designed to be accessible using the tools of equilibrium and non-equilibrium statistical mechanics and was not primarily devised as a realistic model of
a market, there is currently much interest in the interplay between local connectivity and global competition in networks of agents [25, 24]. The work on the dilute MG presented in this paper should therefore be seen as an intermediate step towards a better understanding of more elaborate models. Further developments of the current analytical theory might for example include dynamically evolving networks, inter-agent communication or the introduction of real local histories.

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