Odd supersymmetric Kronecker elliptic function 
and Yang-Baxter equations

A. Levin§† M. Olshanetsky‡♭ A. Zotov♦‡§♮

§ - National Research University Higher School of Economics, Russian Federation, Usacheva str. 6, Moscow, 119048, Russia
† – ITEP, B. Cheremushkinskaya str. 25, Moscow, 117218, Russia
♭ – Institute for Information Transmission Problems RAS (Kharkevich Institute), Bolshoy Karetny per. 19, Moscow, 127994, Russia
♦ – Steklov Mathematical Institute of Russian Academy of Sciences, Gubkina str. 8, Moscow, 119991, Russia
♭ – Moscow Institute of Physics and Technology, Insttitutskii per. 9, Dolgoprudny, Moscow region, 141700, Russia

E-mails: alevin2@hse.ru, olshanet@itep.ru, zotov@mi-ras.ru

Abstract

We introduce an odd supersymmetric version of the Kronecker elliptic function. It satisfies the genus one Fay identity and supersymmetric version of the heat equation. As an application we construct an odd supersymmetric extensions of the elliptic $R$-matrices, which satisfy the classical and the associative Yang-Baxter equations.

1 Introduction

Consider an elliptic curve $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z})$ with moduli $\tau$, $\text{Im}(\tau) > 0$. The elliptic Kronecker function [12]

$$\phi(h, z; \tau) \equiv \phi(h, z) = \frac{\vartheta'(0) \vartheta(h + z)}{\vartheta(h) \vartheta(z)}$$ (1.1)

is defined on $\Sigma_\tau$ in terms of the theta-function

$$\vartheta(z; \tau) \equiv \vartheta(z) = \sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau \left( k + \frac{1}{2} \right)^2 + 2 \pi i (z + \frac{1}{2}) \left( k + \frac{1}{2} \right) \right) , \quad (1.2)$$
which has simple zero at \( z = 0 \) due to skew-symmetry \( \vartheta(z) = -\vartheta(-z) \). Therefore, the function (1.1) has a simple pole at \( z = 0 \)

\[
\text{Res}_{z=0} \phi(h, z) = 1.
\]

The quasi-periodic behavior on the lattice \( \mathbb{Z} \oplus \tau \mathbb{Z} \) is as follows:

\[
\phi(h, z+1) = \phi(h, z), \quad \phi(h, z+\tau) = e^{-2\pi h} \phi(h, z).
\]

These two properties (1.3), (1.4) fix the Kronecker function explicitly as it is given in (1.1).

For our purposes the most important property of (1.1) is that it satisfies the following quadratic relation called the genus one Fay trisecant identity [3]:

\[
\phi(h_1, z_{12})\phi(h_2, z_{23}) = \phi(h_2, z_{13})\phi(h_1 - h_2, z_{12}) + \phi(h_2 - h_1, z_{23})\phi(h_1, z_{13}).
\]

(1.5)

The next important property of the Kronecker function is that it satisfies the heat equation:

\[
2\pi i \partial_\tau \phi(h, z; \tau) = \partial_z \partial_h \phi(h, z; \tau).
\]

(1.6)

This one follows from the heat equation for the theta-function (1.2): 4\pi i \partial_\tau \vartheta(z; \tau) = \partial^2_z \vartheta(z; \tau).

Using the skew-symmetry

\[
\phi(h, z_{12}) = -\phi(-h, z_{21})
\]

(1.7)

rewrite (1.5) in the form

\[
\phi(h_1, z_{12})\phi(h_2, z_{23}) + \phi(-h_2, z_{31})\phi(h_1 - h_2, z_{12}) + \phi(h_2 - h_1, z_{23})\phi(-h_1, z_{31}) = 0,
\]

(1.8)

so that the Kronecker function is a scalar representation of the Fomin-Kirillov algebra [4] of \( A_2 \) type. Relation (1.5) or (1.8) and its degenerations are widely used in classical and quantum integrable systems [6, 1, 5, 9]. The main reason is that it underlies the Yang-Baxter equations.

In this paper we deal with two types of the equations (see details in Section 3):

- classical Yang-Baxter equation

\[
[r_{12}(z_{12}), r_{13}(z_{13})] + [r_{12}(z_{12}), r_{23}(z_{23})] + [r_{13}(z_{13}), r_{23}(z_{23})] = 0.
\]

(1.9)

- associative Yang-Baxter equation [1, 11]

\[
R_{12}^{h_1}(z_{12})R_{23}^{h_2}(z_{23}) = R_{13}^{h_0}(z_{13})R_{12}^{h_1-h_2}(z_{12}) + R_{23}^{h_2-h_1}(z_{23})R_{13}^{h_1}(z_{13}).
\]

(1.10)

**Purpose of the paper** is to construct supersymmetric generalization of the Kronecker function (1.2) in such a way that the Fay identity (1.5) or (1.8) remains valid. The notion of supersymmetric elliptic curve (supertorus with odd spin structure) together with definitions of supersymmetric version of elliptic functions was introduced in [8] and [11]. See also [2] for applications and further developments. The supersymmetric version of the Kronecker function can be defined in different ways. For example, one can define it through the ratio (1.1) of supersymmetric theta-functions proposed in papers [8, 11]. But the Fay identity is not valid for this type generalization. In this paper we suggest an alternative construction, which solves the problem.
Let $\zeta_k, \mu, \omega$ be a set of Grassmann variables, that is

$$\zeta_k^2 = \mu^2 = \omega^2 = 0, \quad [\zeta_k, \zeta_l]_+ = [\zeta_k, \mu]_+ = [\mu, \mu]_+ = [\zeta_k, \omega]_+ = [\omega, \mu]_+ = 0. \quad (1.11)$$

Introduce the following odd function:

$$\Phi(h, z_1, z_2; \tau | \mu, \zeta_1, \zeta_2, \omega) \equiv \Phi^{h|\mu}(z_1, z_2| \zeta_1, \zeta_2) = (\zeta_1 - \zeta_2)\phi(h, z_{12}) +$$

$$+ \omega \partial_1 \phi(h, z_{12}) + 2\pi i \zeta_1 \zeta_2 \omega \partial_\tau \phi(h, z_{12}) + \zeta_1 \zeta_2 \mu \partial_1 \phi(h, z_{12}) + \frac{1}{2}(\zeta_1 + \zeta_2)\mu \omega \partial_1^2 \phi(h, z_{12}), \quad (1.12)$$

where

$$\partial_1 \phi(x, y) = \partial_x \phi(x, y), \quad \partial_2 \phi(x, y) = \partial_y \phi(x, y). \quad (1.13)$$

These notations will be used in what follows. Also, by definition

$$\Phi^{h|0}(z_1, z_2| \zeta_1, \zeta_2) = (\zeta_1 - \zeta_2)\phi(h, z_{12}) + \omega \partial_1 \phi(h, z_{12}) + 2\pi i \zeta_1 \zeta_2 \omega \partial_\tau \phi(h, z_{12}), \quad (1.14)$$

which is (1.12) without two last terms. The third term in (1.12) or (1.14) can be transformed via (1.6) as $2\pi i \zeta_1 \zeta_2 \omega \partial_\tau \phi(h, z_{12}) = \zeta_1 \zeta_2 \omega \partial_1 \partial_2 \phi(h, z_{12})$. Instead of skew-symmetry (1.7) we now have the symmetry property

$$\Phi^{h|\mu}(z_1, z_2| \zeta_1, \zeta_2) = \Phi^{-h|\mu}(z_2, z_1| \zeta_2, \zeta_1). \quad (1.15)$$

We will prove that (1.12) and (1.14) satisfy the Fay identity in the form (1.8). Namely,

$$\Phi^{h_1|\mu_1}(z_1, z_2| \zeta_1, \zeta_2) \Phi^{h_2|\mu_2}(z_2, z_3| \zeta_2, \zeta_3) + \Phi^{-h_2|-\mu_2}(z_3, z_1| \zeta_3, \z_1) \Phi^{h_1-h_2|\mu_1-\mu_2}(z_1, z_2| \zeta_1, \zeta_2) +$$

$$+ \Phi^{h_2-h_1|\mu_2-\mu_1}(z_2, z_3| \zeta_2, \zeta_3) \Phi^{-h_1-\mu_1}(z_3, z_1| \zeta_3, \zeta_1) = 0. \quad (1.16)$$

Then we show that (1.12) satisfies the following relation:

$$\left(\partial_\omega + 2\pi i (\zeta_1 + \zeta_2) \partial_\tau \right) \Phi^{h|\mu}(z_1, z_2| \zeta_1, \zeta_2) = \left(\partial_{\zeta_1} + \zeta_1 \partial_{z_1} - \frac{1}{2} \mu \partial_h \right) \partial_h \Phi^{h|\mu}(z_1, z_2| \zeta_1, \zeta_2), \quad (1.17)$$

which we call the supersymmetric version of the heat equation.

The paper is organized as follows. In the next Section we derive the expression (1.12) from supersymmetric analogues of the simple pole condition (1.3) and the quasi-periodic boundary condition (1.4). Then we prove the Fay identity (1.16) and the odd supersymmetric version (1.17) of the heat equation. In Section 3 we use the function (1.12) to construct odd elliptic $R$-matrices satisfying supersymmetric versions of the Yang-Baxter equations (1.9)-(1.10). A summary of results is given in the Conclusion.

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$^1$ $[a, b]_+ = ab + ba$ is the anticommutator, while $[a, b]_- = [a, b] = ab - ba$ stands for commutator.
2 Supersymmetric Kronecker function

In order to construct supersymmetric generalization of the Kronecker function let us specify its definition in the ordinary case. As mentioned previously, it is fixed as the ratio of theta-functions (1.1) by two conditions: to have a simple pole at \( z = 0 \) with residue (1.3) and by the quasi-periodic boundary conditions (1.4). Equivalently, the function \( \phi(h, z_1 - z_2) \) is fixed as the Green function \( \phi(h, z_1, z_2) \) of \( \bar{\partial} = \partial_{\bar{z}} \)-operator

\[
\text{Res } \phi(h, z_1, z_2) = 1 \quad \text{or} \quad \bar{\partial}\phi(h, z_1, z_2) = 1 \tag{2.1}
\]

with the boundary conditions

\[
\begin{align*}
\phi(h, z_1 + 1, z_2) &= \phi(h, z_1, z_2), \\
\phi(h, z_1 + \tau, z_2) &= e^{-2\pi i h} \phi(h, z_1, z_2), \\
\phi(h, z_1, z_2 + 1) &= \phi(h, z_1, z_2), \\
\phi(h, z_1, z_2 + \tau) &= e^{2\pi i h} \phi(h, z_1, z_2).
\end{align*} \tag{2.2}
\]

Then the solution of (2.1)-(2.2) is given by \( \phi(h, z_1, z_2) = \phi(h, z_1 - z_2) \).

Odd supersymmetric Kronecker elliptic function. While an elliptic curve \( \Sigma_\tau \) with moduli \( \tau \) is a quotient of \( \mathbb{C} \) (with a coordinate \( z \)) by translations \( z \to z + 1 \), \( z \to z + \tau \), the corresponding super elliptic curve \( \Sigma_{\tau,\omega} \) is a quotient of \( \mathbb{C}^{1|1} \) (with coordinates \( z, \zeta \)) by (super)translations

\[
\begin{align*}
\{ z \to z + 1, \\
\zeta \to \zeta + 2\pi i \zeta \omega, \\
\zeta \to \zeta + 2\pi i \omega .
\end{align*} \tag{2.3}
\]

It is equipped with the covariant derivative \( D_\zeta = \partial_\zeta + \zeta \partial_z \), \( D_\zeta^2 = \partial_z \). In what follow we use the Grassmann variables \( \zeta_k, \omega, \mu_i \) as superpartners to the coordinates \( z_k \), to the moduli \( \tau \) and to \( \mathbb{C} \)-valued parameters of the boundary conditions \( h_i \) respectively:

\[
\begin{align*}
even \text{ variables:} & \quad z_k \quad \tau \quad h_i \\
odd \text{ variables:} & \quad \zeta_k \quad \omega \quad \mu_i
\end{align*} \tag{2.4}
\]

Similarly to (2.1)-(2.2) the supersymmetric Kronecker function (1.12) is defined on a product of two super elliptic curves. It is fixed by the following two conditions:

- it has a simple pole on a diagonal \( z_1 = z_2 \) with residue

\[
\text{Res } \Phi^{h|\mu}(z_1, z_2| \zeta_1, \zeta_2) = \zeta_1 - \zeta_2. \tag{2.5}
\]

- it is a quasi-periodic function with respect to (super)translations (2.3):

\[
\begin{align*}
\Phi^{h|\mu}(z_1 + 1, z_2| \zeta_1, \zeta_2) &= \Phi^{h|\mu}(z_1, z_2 + 1| \zeta_1, \zeta_2), \\
\Phi^{h|\mu}(z_1 + \tau + 2\pi i \zeta_1 \omega, z_2| \zeta_1 + 2\pi i \omega, \zeta_2) &= g_1 \Phi^{h|\mu}(z_1, z_2| \zeta_1, \zeta_2), \\
\Phi^{h|\mu}(z_1, z_2 + \tau + 2\pi i \zeta_2 \omega| \zeta_1, \zeta_2 + 2\pi i \omega) &= g_2 \Phi^{h|\mu}(z_1, z_2| \zeta_1, \zeta_2),
\end{align*} \tag{2.6}
\]

where

\[
g_1 = \exp \left( -2\pi i (h + \mu \zeta_1 + \pi i \mu \omega) \right), \quad g_2 = \exp \left( 2\pi i (h + \mu \zeta_2 - \pi i \mu \omega) \right). \tag{2.7}
\]
So that \( \Phi^{h,\mu}(z_1, z_2 | \zeta_1, \zeta_2) \) is the Green function of the \( \bar{\partial} = \partial_1 \) operator with the boundary conditions (2.6).

**Proposition 2.1** The boundary conditions (2.6)-(2.7) holds true for the function (1.12). For the truncated Kronecker function (1.14) the transition function (2.7) is given by \( g = \exp(-2\pi i \hbar) \).

**Proof:** Let us prove the statement for the truncated function. Consider the first term in (1.14). Under translations (2.3) it transforms as

\[
(\zeta_1 - \zeta_2) \phi(h, z_{12}) \rightarrow (\zeta_1 - \zeta_2 + 2\pi i \omega) \phi(h, z_{12} + \tau + 2\pi i \zeta_1 \omega) = \\
= (\zeta_1 - \zeta_2 + 2\pi i \omega) \left( \phi(h, z_{12} + \tau) + 2\pi i \zeta_1 \omega \partial_2 \phi(h, z_{12} + \tau) \right) = \\
= \exp(-2\pi i \hbar) \left( (\zeta_1 - \zeta_2) \phi(h, z_{12}) + 2\pi i \omega \phi(h, z_{12}) + 2\pi i \zeta_1 \zeta_2 \omega \phi(h, z_{12}) \right).
\]

(2.8)

Therefore, the first term in (1.12) is not quasi-periodic. As a result of (super)translation it acquires additional unwanted terms proportional to \( \omega \phi(h, z_{12}) \) and \( \zeta_1 \zeta_2 \omega \phi(h, z_{12}) \). They are compensated by the contributions coming from the second and the third terms of (1.12). Indeed, using (2.2) one can easily verify that

\[
\omega \partial_1 \phi(h, z_{12}) \rightarrow \omega \exp(-2\pi i \hbar) \left( \partial_1 \phi(h, z_{12}) - 2\pi i \phi(h, z_{12}) \right),
\]

(2.9)

\[
\zeta_1 \zeta_2 \omega \partial_\tau \phi(h, z_{12}) \rightarrow \zeta_1 \zeta_2 \omega \exp(-2\pi i \hbar) \left( \partial_\tau \phi(h, z_{12}) - \partial_2 \phi(h, z_{12}) \right).
\]

Then, summing up (2.8)-(2.9) we conclude that the truncated function is quasi-periodic with the multiplicator \( \exp(-2\pi i \hbar) \). The rest of the proof for the function (1.12) also uses

\[
\partial_1^2 \phi(h, z_{12} + \tau) = \exp(-2\pi i \hbar) \left( \partial_1^2 \phi(h, z_{12}) - 4\pi i \partial_1 \phi(h, z_{12}) - 4\pi^2 \phi(h, z_{12}) \right).
\]

(2.10)

The calculations are performed in a similar way. ■

Conversely, one can derive (1.12), (1.14) from conditions (2.5)-(2.7). For example, to reproduce the truncated function (1.14) from (2.5)-(2.7) with \( g = \exp(-2\pi i \hbar) \) one should start with the first term in (1.14). Its presence in the final expression follows from (2.5) and (1.4). Under the translations (2.3) is transformed as given in (2.8). In order to compensate the unwanted terms one should use the terms from (2.9). A more general answer (1.12) is reproduced in the same way.

**Fay identity.** Let us prove the key property of the supersymmetric Kronecker function.

**Proposition 2.2** The Fay identity (1.16) holds true for the function (1.13).

**Proof:** The verification of the statement is a tedious but straightforward calculation. One should substitute the definition (1.12) into (1.16) and write down the coefficients behind all possible monomials of (distinct) Grassmann variables. For example, the coefficient behind monomials \( \zeta_1 \zeta_2, \zeta_2 \zeta_3 \) and \( \zeta_3 \zeta_1 \) is given by (the l.h.s. of) the ordinary Fay identity (1.8). The coefficients
behind \( \zeta_1 \omega \), \( \zeta_2 \omega \) and \( \zeta_3 \omega \) are obtained from (1.8) by the action of operators \( \partial_{\hbar_2} \), \( \partial_{\hbar_1} + \partial_{\hbar_2} \) and \( \partial_{\hbar_3} \) respectively. All other coefficients are also identities, which follows from the Fay identity (1.8) by taking some derivatives. ■

Let us slightly rewrite the definition of the Kronecker function (1.12) using the heat equation (1.6) for its third term:

\[
\Phi(\hbar, z_1, z_2; \tau| \mu, \zeta_1, \zeta_2; \omega) = \left[ (\zeta_1 - \zeta_2) + \omega \partial_{\hbar} + \zeta_1 \zeta_2 \omega \partial_{\hbar} \partial_{z_1} + \zeta_1 \zeta_2 \mu \partial_{\hbar} + \frac{1}{2} (\zeta_1 + \zeta_2) \mu \omega \partial_{\hbar}^2 \right] \phi(\hbar, z_1 - z_2). \tag{2.11}
\]

This formula can be applied to the rational and trigonometric degenerations, where the moduli \( \tau \) is absent. For example, in the trigonometric case the function (1.1) turns into

\[
\phi(\hbar, z_1) = \coth(\hbar) + \coth(z_1 - z_2). \tag{2.12}
\]

Similarly, for the rational degeneration \( \phi(\hbar, z_1) = 1/\hbar + 1/z_1 \) we have

\[
\phi(\hbar, z_1) = (\zeta_1 - \zeta_2) \left( \frac{1}{\hbar} + \frac{1}{z_1 - z_2} \right) - \frac{\omega + \zeta_1 \zeta_2 \mu}{\hbar^2} + \frac{(\zeta_1 + \zeta_2) \mu \omega}{\hbar^3}. \tag{2.13}
\]

The functions (2.12) and (2.13) also satisfy the Fay identity (1.16) since the functions \( \coth(\hbar) + \coth(z_1 - z_2) \) satisfy (1.8).

Heat equation. The result is as follows.

**Proposition 2.3** The function \( \Phi(\hbar, z_1, z_2; \tau| \mu, \zeta_1, \zeta_2; \omega) \) (1.12) satisfies the odd supersymmetric heat equation (1.17). Similarly, for the function (1.14) we have

\[
\left( \partial_{\omega} + 2\pi i (\zeta_1 + \zeta_2) \partial_{\tau} \right) \Phi^{h|0}(z_1, z_2| \zeta_1, \zeta_2) = \left( \partial_{\zeta_1} + \zeta_1 \partial_{z_1} \right) \partial_{\hbar} \Phi^{h|0}(z_1, z_2| \zeta_1, \zeta_2). \tag{2.14}
\]

The proof is straightforward. It uses the ordinary heat equation (1.6) only.

### 3 Yang-Baxter equations

The construction of elliptic \( R \)-matrix uses special basis in Mat\( (N, \mathbb{C}) \) [1]. The pair of matrices

\[
Q_{kl} = \delta_{kl} \exp \left( \frac{2\pi i}{N} k \right), \quad \Lambda_{kl} = \delta_{k-l+1=0 \text{mod } N}, \quad Q^N = \Lambda^N = 1_N \tag{3.1}
\]

provides the finite-dimensional representation of the Heisenberg group due to

\[
\exp \left( \frac{2\pi i}{N} a_1 a_2 \right) Q^{a_1} \Lambda^{a_2} = \Lambda^{a_2} Q^{a_1}, \quad a_1, a_2 \in \mathbb{Z}. \tag{3.2}
\]
Then the basis in Mat\((N, \mathbb{C})\) is given by the following set of \(N^2\) matrices:

\[
T_a = T_{a_1a_2} = \exp\left(\frac{\pi i}{N} a_1a_2\right)Q^{a_1A^{a_2}}, \quad a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N.
\] (3.3)

From (3.2) we have

\[
T_a T_{\beta} = \kappa_{\alpha, \beta} T_{a+\beta}, \quad \kappa_{\alpha, \beta} = \exp\left(\frac{\pi i}{N}(\beta_1a_2 - \beta_2a_1)\right),
\] (3.4)

where \(\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)\). Next, define the set of \(N^2\) basis functions numerated by the index \(a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N\):

\[
\varphi_a(h + \Omega, z) = \exp(2\pi i \frac{a_2}{N} z) \phi(h + \Omega, z), \quad \Omega = \frac{a_1 + a_2\tau}{N}.
\] (3.5)

Finally, the quantum Baxter-Belavin’s elliptic \(R\)-matrix is of the form:

\[
R^{\beta}_{12}(z) = \sum_{\alpha} T_{\alpha} \otimes T_{-\alpha} \varphi_{\alpha}(h + \Omega, z).
\] (3.6)

It satisfies the quantum Yang-Baxter equation and the associative Yang-Baxter equation (1.10) [10]. Similarly, the classical Belavin-Drinfeld-Sklyanin \(r\)-matrix

\[
r_{12}(z) = \sum_{\alpha \neq 0} T_{\alpha} \otimes T_{-\alpha} \varphi_{\alpha}(\Omega, z)
\] (3.7)

satisfies the classical Yang-Baxter equation (1.9), which is based on the Fay identity (1.5) or (1.8) written as relations for the functions (3.5):

\[
\varphi_{\alpha}(\Omega, z_{12}) \varphi_{\beta}(\Omega_{\beta}, z_{23}) + \varphi_{\beta}(-\Omega, z_{31}) \varphi_{\beta-\alpha}(\Omega_{\alpha-\beta}, z_{12}) + \\
+ \varphi_{\beta-\alpha}(\Omega_{\beta-\alpha}, z_{23}) \varphi_{\alpha-\beta}(-\Omega_{\alpha}, z_{31}) = 0, \quad \alpha, \beta, \alpha - \beta \neq 0.
\] (3.8)

In the same way the associative Yang-Baxter equation (1.10) is based on a more general relation

\[
\varphi_{\alpha}(h + \Omega, z_{12}) \varphi_{\beta}(\eta + \Omega, z_{23}) + \varphi_{\beta}(-\eta + \Omega, z_{31}) \varphi_{\beta-\alpha}(h - \eta + \Omega_{\alpha-\beta}, z_{12}) + \\
+ \varphi_{\beta-\alpha}(\eta - h + \Omega_{\beta-\alpha}, z_{23}) \varphi_{\alpha-\beta}(-h - \Omega_{\alpha}, z_{31}) = 0.
\] (3.9)

**Supersummetric basis functions.** In order to construct supersymmetric generalizations of (3.6) and (3.7) we need an analogue of the basis functions (3.5). They have the following form:

\[
\Phi_{\alpha}^{h+\Omega, |\mu}(z_1, z_2 | z_1, \zeta_2) := \exp\left(2\pi i \frac{\alpha_2}{N} (z_1 - z_2 + \zeta_1\zeta_2)\right)\Phi_{\alpha}^{h+\Omega, |\mu}(z_1, z_2 | z_1, \zeta_2) = \\
= \left(1 + 2\pi i \frac{\alpha_2}{N} \zeta_1\zeta_2\right)\Phi_{\alpha}^{h+\Omega, |\mu}(z_1, z_2 | z_1, \zeta_2) = \\
= \Phi_{\alpha}^{h+\Omega, |\mu}(z_1, z_2 | z_1, \zeta_2) + 2\pi i \frac{\alpha_2}{N} \zeta_1\zeta_2 \omega \partial_1 \phi(h + \Omega_{\alpha}, z_{12}).
\] (3.10)

Equivalently, the set of functions is written in the forms

\[
\Phi_{\alpha}^{h+\Omega, |\mu}(z_1, z_2 | z_1, \zeta_2) = \exp\left(2\pi i \frac{\alpha_2}{N} (z_1 - z_2)\right)\Phi_{\alpha}^{h+\Omega, |\mu+2\pi i \frac{\alpha_2}{N} \omega}(z_1, z_2 | z_1, \zeta_2)
\] (3.11)
or
\[ \Phi^h_{\alpha|\Omega\alpha,|\mu}(z_1, z_2| z_1, z_2) = \exp \left( 2\pi i \frac{\Omega_\alpha}{N}(z_1 - z_2) \right) \Phi^h_{\alpha|\Omega\alpha,|\mu}(z_1, z_2| z_1, z_2), \] (3.12)

where
\[ \Phi^h_{\alpha|\Omega\alpha,|\mu}(z_1, z_2| z_1, z_2) = (\zeta_1 - \zeta_2)\varphi_\alpha(h + \Omega_\alpha, z_{12}) + \omega \partial_1 \varphi_\alpha(h + \Omega_\alpha, z_{12}) + 
+ 2\pi \nu \zeta_1 \zeta_2 \omega \frac{d}{d\tau} \varphi_\alpha(h + \Omega_\alpha, z_{12}) + \zeta_1 \zeta_2 \mu \partial_1 \varphi_\alpha(h + \Omega_\alpha, z_{12}) + 
+ \frac{1}{2} (\zeta_1 + \zeta_2) \mu \omega \partial_1^2 \varphi_\alpha(h + \Omega_\alpha, z_{12}). \] (3.13)

In the third term of (3.13) the full derivative with respect to \( \tau \) includes also the partial derivative with respect to the first argument of \( \varphi_\alpha(h + \Omega_\alpha, z_{12}) \), depending on \( \tau \) through \( \Omega_\alpha \) (3.5).

Using the Fay identity (1.16) it is easy to show that the set of functions (3.10) satisfy the following direct analogue of (3.9):
\[ \Phi^h_{\alpha|\Omega\alpha,|\mu_1}(z_1, z_2| \zeta_1, \zeta_2) \Phi^{h_2+\Omega\beta,|\mu_2}(z_2, z_3| \zeta_2, \zeta_3) + 
+ \Phi^{-h_2-\Omega\beta,|\mu_2}(z_3, z_1| \zeta_3, \zeta_1) \Phi^{h_1-\Omega\alpha,|\mu_1}(z_1, z_2| \zeta_1, \zeta_2) + 
+ \Phi^{h_2-\Omega\alpha,|\mu_2}(z_2, z_3| \zeta_2, \zeta_3) \Phi^{-h_1+\Omega\beta,|\mu_1}(z_3, z_1| \zeta_3, \zeta_1) = 0. \] (3.14)

In the same way for \( \alpha, \beta, \alpha - \beta \neq (0, 0) \) we also have
\[ \Phi^{\Omega\alpha,|0}(z_1, z_2| \zeta_1, \zeta_2) \Phi^{\Omega\beta,|0}(z_2, z_3| \zeta_2, \zeta_3) + \Phi^{-\Omega\beta,|0}(z_3, z_1| \zeta_3, \zeta_1) \Phi^{\Omega\alpha,|0}(z_1, z_2| \zeta_1, \zeta_2) + 
+ \Phi^{\Omega\beta,-\Omega\alpha,|0}(z_2, z_3| \zeta_2, \zeta_3) \Phi^{-\Omega\alpha,|0}(z_3, z_1| \zeta_3, \zeta_1) = 0. \] (3.15)

**Classical super Yang-Baxter equation.** The odd supersymmetric analog of the classical \( r \)-matrix (3.7) as follows:
\[ r_{12}(z_1, z_2| \zeta_1, \zeta_2) = \sum_{\alpha \neq 0} T_\alpha \otimes T_{-\alpha} \Phi^{\Omega\alpha,|0}(z_1, z_2| \zeta_1, \zeta_2). \] (3.16)

For the odd \( r \)-matrix the classical Yang-Baxter equations were studied in [7, 5]. The super version of the equation (1.9) contains anticommutators instead of commutators:
\[ [r_{12}, r_{13}]_+ + [r_{12}, r_{23}]_+ + [r_{13}, r_{23}]_+ = 0. \] (3.17)

The following statement holds true.

**Proposition 3.1** The odd supersymmetric analog of the classical \( r \)-matrix (3.16) satisfies equation (3.17), where \( r_{ab} = r_{ab}(z_a, z_b| \zeta_a, \zeta_b). \)

The proof is similar to the one given below for a more general \( R \)-matrix.

\(^2\)Let us remark that we do not consider super Lie algebras (or groups) as it is discussed in [7]. We deal with \( GL_N \) \( R \)-matrices in fundamental representation.
**Associative Yang-Baxter-equation.** Using the skew-symmetry $R^h_{12}(z) = -R^h_{21}(-z)$ of (3.6) let us rewrite equation (1.10) in the form

$$R^h_{12}(z_{12}) R^h_{23}(z_{23}) + R^{-h}_{31}(z_{31}) R^h_{12} - R^h_{23}(z_{23}) R^{-h}_{31}(z_{31}) = 0, \quad (3.18)$$

which is similar to (1.8). The odd supersymmetric analog of the quantum elliptic Baxter-Belavin $R$-matrix (3.6) is as follows:

$$R^h_{12}(z_{12}, z_{23}) = \sum_{\alpha, \beta} T_{\alpha} \otimes T_{-\alpha} \Phi^h_{\alpha + \Omega_{\alpha}} (z_{12}, z_{23}) \cdot (3.19)$$

**Proposition 3.2** The odd supersymmetric analog of the quantum $R$-matrix (3.19) satisfies the following equation:

$$R^h_{12} (z_{12}, z_{23}) + R^h_{23} (z_{23}) + R^h_{31} (z_{31}) + R^h_{12} (z_{12}, z_{23}) = 0. \quad (3.20)$$

where $R^h_{ab} = R^h_{ab} (z_a, z_b | \zeta_a, \zeta_b)$.

**Proof:** The proof is similar to the ordinary case. One should multiply the l.h.s. of the identity (3.14) by Mat($N, \mathbb{C}$) valued element $\kappa_{\beta, \alpha} T_\alpha \otimes T_{-\alpha} \otimes T_{-\beta}$, and then sum up over indices $\alpha, \beta \in \mathbb{Z}_N \times \mathbb{Z}_N$. In order to prove it let us write down the first term from the l.h.s. of (3.20). Using (3.4) we have

$$R^h_{12} (z_{12}, z_{23}) = \sum_{\alpha, \beta} \kappa_{-\alpha, \beta} T_\alpha \otimes T_{-\alpha} \otimes T_{-\beta} \Phi^h_{\alpha + \Omega_{\alpha}} (z_{12}, z_{23}) \cdot (3.21)$$

The second term from the l.h.s. of (3.20) has the form:

$$R^{-h}_{31} (z_{31}) R^h_{12} (z_{12}, z_{23}) = \sum_{\alpha, \beta} \kappa_{\beta, \alpha} T_{\alpha} \otimes T_{-\alpha} \otimes T_{-\beta} \Phi^{-h}_{\beta - \Omega_{\beta}} (z_{31}, z_{12}) \cdot (3.22)$$

And the third term from the l.h.s. of (3.20) is of the form:

$$R^h_{23} (z_{23}) R^h_{31} (z_{31}) = \sum_{\alpha, \beta} \kappa_{-\alpha, \beta} T_{\alpha} \otimes T_{-\alpha} \otimes T_{-\beta} \Phi^{-h}_{\beta - \Omega_{\beta}} (z_{23}, z_{12}) \cdot (3.23)$$

The statement of the Proposition then follows from $\kappa_{\beta, \alpha} = \kappa_{-\alpha, \beta} = \kappa_{\beta, \alpha - \beta} = \kappa_{\alpha - \beta, \alpha}$. The latter comes from (3.4).
4 Conclusion

We introduced the odd supersymmetric version of the elliptic Kronecker function (1.1):

$$\Phi(ℏ, z; τ | μ, ζ; ω) = \left[ (ζ_1 - ζ_2) + ω∂_ℏ + 2πiζ_1ζ_2ω∂_τ + ζ_1ζ_2μ∂_ℏ + \frac{1}{2}(ζ_1 + ζ_2)μω∂_ℏ^2 \right] \phi(ℏ, z_1 - z_2).$$  

(4.1)

It satisfies the Fay identity (1.16) and the supersymmetric version of the heat equation (1.17). Both equations also hold true for the truncated function (1.14). In this case one should replace $μ$ with 0 in (1.16), (1.17).

Using (4.1) we constructed supersymmetric extension of the elliptic $R$-matrix (3.19). It follows from (3.13) that similarly to (4.1) it can be represented in the form:

$$R^{h}_{12}(z_1, z_2 | ζ_1, ζ_2) = \left[ (ζ_1 - ζ_2) + ω∂_ℏ + 2πiζ_1ζ_2ω∂_τ + ζ_1ζ_2μ∂_ℏ + \frac{1}{2}(ζ_1 + ζ_2)μω∂_ℏ^2 \right] R^{h}_{12}(z_{12}).$$  

(4.2)

In fact, (4.2) contains (4.1) as particular case (when $N = 1$). By changing the third term via the heat equation (1.6) we get

$$R^{h}_{12}(z_1, z_2 | ζ_1, ζ_2) = \left[ (ζ_1 - ζ_2) + ω∂_ℏ + ζ_1ζ_2ω∂_ℏ∂_z_1 + ζ_1ζ_2μ∂_ℏ + \frac{1}{2}(ζ_1 + ζ_2)μω∂_ℏ^2 \right] R^{h}_{12}(z_{12}),$$  

(4.3)

which is applicable in trigonometric and rational cases corresponding to nodal and cuspidal degenerations of the elliptic curve.

Finally, the $R$-matrix (4.3) was proved to satisfy the associative Yang-Baxter equation written as

$$R^{h_1|μ_1}_{12} R^{h_2|μ_2}_{23} R^{h_3|μ_3}_{31} = R^{h_1|μ_1}_{12} R^{h_2|μ_2}_{23} R^{h_3|μ_3}_{31} = 0.$$  

(4.4)

In the same way the supersymmetric version of the classical $r$-matrix (3.16) was shown to solve the classical super Yang-Baxter equation (3.17).

The supersymmetric heat equation together with the Fay identity can be used for construction of the Knizhnik-Zamolodchikov-Bernard equations on supersymmetric elliptic curves. We will discuss it in our next papers.

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