Nonlinear waves in the Hall magnetic hydrodynamics

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Abstract. Nonlinear waves are studied in the framework of the magnetic hydrodynamics with allowance for the Hall effect (HMHD). Along with the classical magnetic hydrodynamics (MHD), this model is one of the widely used models in the plasma theory. When solving the problem in the framework of HMHD, in addition to the usual MHD parameters, it is possible to obtain the electron velocities. In the problems considering the systems of electrodes, the Hall effect allows to distinguish one or another type of polarity. Also, the waves in HMHD (as opposed to MHD) have dispersion. One-dimensional waves are considered propagating in the finite-temperature plasma along the magnetic field. It is shown that there can be both solitary waves and periodic solutions, and their properties are investigated.

1. Equations of magnetic hydrodynamics taking into account the Hall effect (HMHD)

HMHD differs from MHD by taking into account the Hall effect, which results in a change in the Ohm law [1, 2]. In the HMHD model, the wave dispersion makes it possible to search for the solutions in the form of the solitary nonlinear waves, as it is done for the KdF, sin-Gordon, and other equations. In this paper, this approach is used for a set of one-dimensional HMHD equations for the ideal plasma with finite temperature. The case of zero temperature is considered in [3]. Using the traditional notations for the basic quantities, the HMHD equations can be written in the following form:

\[
\begin{align*}
\frac{\partial}{\partial t} n + \text{div}(n\mathbf{V}) &= 0 \\
 m_e n \left( \frac{\partial}{\partial t} \mathbf{V} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) &= -\nabla (P_e + P_i) + \frac{1}{c} [\mathbf{j}, \mathbf{B}] \\
 \frac{\partial}{\partial t} \mathbf{B} &= -c \text{rot} \mathbf{E} \\
 \mathbf{E} &= -\frac{1}{c} [\mathbf{V}, \mathbf{B}] + \frac{1}{en c} [\mathbf{j}, \mathbf{B}] - \frac{\nabla P_e}{en} \\
 \text{rot} \mathbf{B} &= \frac{4\pi}{c} \mathbf{j}, \quad \text{div} \mathbf{B} = 0, \quad \mathbf{j} = en (\mathbf{V} - \mathbf{V}_e) 
\end{align*}
\]

The Ohm law written in form (1d) can be obtained from the equation of motion for electrons

\[
m_e n \left( \frac{\partial}{\partial t} \mathbf{V}_e + (\mathbf{V}_e \cdot \nabla) \mathbf{V}_e \right) = -\nabla P_e - en (E + \frac{1}{c} [\mathbf{V}_e, \mathbf{B}])
\]
Then, in this equation, we omit the inertial term and using Eq. (1e), express the electron velocity \( V_e \) in terms of the ion velocity \( V \) and current \( j \).

### 2. One-dimensional HMHD equations

We will take into account the non-zero temperature using the isothermal approximation \( (T_e = T_i = T) \) and \( P = \frac{2k_B T}{m_i} \rho \), \( \rho = m_i n \). We consider the one-dimensional motion of the plasma along the x-axis. Assuming \( \partial / \partial y = \partial / \partial z = 0 \) and \( B_x = \text{const} \), we obtain the following set of equations:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho U_x}{\partial x} &= 0 \\
\frac{\partial \rho U_x}{\partial t} + \frac{\partial}{\partial x} \left( \rho U_x^2 + \frac{B_x^2}{2} + \frac{B_y^2 + B_z^2}{2} \right) &= 0 , \quad P = \rho \\
\frac{\partial \rho U_y}{\partial t} + \frac{\partial}{\partial x} \left( \rho U_x U_y - B_x B_y \right) &= 0 \\
\frac{\partial \rho U_z}{\partial t} + \frac{\partial}{\partial x} \left( \rho U_x U_z - B_x B_z \right) &= 0 \\
\frac{\partial B_x}{\partial t} - \frac{\partial E_y}{\partial x} &= 0 \quad E_y = -(U_x B_y - U_y B_x) + \xi \frac{B_x}{\rho} \frac{\partial B_y}{\partial x} \\
\frac{\partial B_y}{\partial t} + \frac{\partial E_x}{\partial x} &= 0 \quad E_x = -(U_x B_y - U_y B_x) + \xi \frac{B_x}{\rho} \frac{\partial B_y}{\partial x} .
\end{align*}
\]

In Eqs. (2), we used new (arbitrary) units of measurement:

\[
L_0, \; n_0, \; P_0, \; B_0, \; V_0 = B_0 / \sqrt{4\pi m_i n_0} , \; t_0 = L_0 / V_0 ; \quad \text{in addition, two dimensionless parameters have appeared}
\]

\[
\beta = \frac{8\pi P_0}{B_0^2} , \quad \text{and} \quad \xi = \frac{c}{e L_0} \left( \frac{m_i}{4\pi n_0} \right)^{1/2} .
\]

The parameter \( \beta \) is the so-called "plasma beta". The parameter \( \xi \neq 0 \) corresponds to the case of taking into account the Hall effect [2]. For \( L_0 = 5 cm , \; n_0 = 2 \cdot 10^{14} \; cm^{-3} \), the corresponding parameter is \( \xi = 0.322 \).

### 3. One-dimensional equations for traveling wave in HMHD

We are trying the solutions to set of Eqs. (2) in the form of traveling waves. The arguments of all functions are \( \chi = x - at \), \( a = \text{const} \). We obtain the integrals of motion and two first-order differential equations (4):

\[
\begin{align*}
\Pi &= -a \rho + \rho U_x \quad N_x = -a \rho U_x + \rho U_x^2 + \beta \rho / 2 + (B_y^2 + B_z^2) / 2 , \quad N_y = -a \rho U_y + \rho U_x U_y - B_x B_y \\
N_z &= -a \rho U_z + \rho U_x U_z - B_x B_z , \quad E_{y0} = -a B_z + E_y , \quad E_{z0} = a B_y + E_z \\
\xi \frac{B_y}{\rho} (B_y) &= E_y + (U_x B_y - U_y B_x)
\end{align*}
\]

(4)
If \( \xi_i = 0 \) (MHD model), then the set of differential equations (4) will vanish and we obtain the constant solution.

Now we fix the unit of the magnetic field \( B_i = 1 \), go to the coordinate system moving with the wave \( V_x = U_x - a \) and make the change of variables \( d\chi = \xi_i V_x \, d\tau \). Thus, we obtain set of Eqs. (4) in the following form:

\[
\frac{d B_x}{d\tau} = - (\Pi V_x - 1) B_z \tag{5}
\]
\[
\frac{d B_z}{d\tau} = (\Pi V_x - 1) B_y + Q
\]

Here

\[
V_x = \frac{1}{4\Pi} \left( (R^2 - B^2) \pm \sqrt{(R^2 - B^2)^2 - S^2} \right), \quad B^2 = B_y^2 + B_z^2, \tag{6}
\]

\[
Q = \Pi E_0 - N_y, \quad R^2 = 2(N_x - a\Pi), \quad S^2 = 8\beta\Pi^2.
\]

Formula (6) is written under the assumption that \( \Pi > 0 \) (in this case, \( V_x > 0 \)). Therefore, the maximum transverse field can be estimated as \( B_y^2 + B_z^2 < R^2 - S \).

Thus, we have two variants of set of Eqs. (5) (two signs in formula (6)). This set of Eqs. turned out to be the Hamiltonian one

\[
\frac{d B_y}{d\tau} = - \frac{\partial H}{\partial B_z}, \tag{7}
\]
\[
\frac{d B_z}{d\tau} = \frac{\partial H}{\partial B_y}
\]

and its Hamiltonian is as follows:

\[
H(B_y, B_z) = \frac{B_y^4 + B_z^4}{16} - \frac{B_y^2 B_z^2}{8} + \left( \frac{1}{4} R^2 - 1 \right) B_z^2 + Q B_z + \frac{1}{16} [(R^2 - B_y^2 - B_z^2) \sqrt{(R^2 - B_y^2 - B_z^2)^2 - S^2}] \ln \left[ R^2 - B_y^2 - B_z^2 + \sqrt{(R^2 - B_y^2 - B_z^2)^2 - S^2} \right]. \tag{8}
\]

In formula (6), the branch corresponding to the sign (+) will be called the first branch. The branch corresponding to the sign (−) is the second one. If \( S = 0 \), only the first branch will remain.

The level lines of the Hamiltonian are the phase curves of the set of Eqs. (5) on the \((B_y, B_z)\) plane. The singular points of this set of equations \((B_y = b, B_z = 0)\) can be found from the following equation (the singular point of the Hamiltonian system on the plane can be either the center or the saddle):

\[
\frac{1}{4} \left( R^2 - b^2 \pm \sqrt{(R^2 - b^2)^2 - S^2} \right) - 1 \right] b + Q = 0 \tag{9}
\]

This equation has to be solved numerically. At \( \beta = 0 \) (zero temperature), this equation can be simplified.
The qualitative properties of the phase portrait (only a part of the phase plane has the physical meaning: \( B_y^2 + B_z^2 < R^2 - S \)), as well as the \( b \) root value, depend on the \( R, Q, S \) parameters included in this equation. There may be different cases: the singular points are only centers, centers and the saddle, etc. It seems impossible to separate these cases analytically by introducing some formulas and inequalities. In each particular case, the singular points can be easily determined numerically by constructing the level lines of Hamiltonian (8).

4. Results

As an example, the phase portraits of the first and second branches are shown in Figures 1 and 2 for two sets of parameters \( R, Q, S \) (the scales in these Figures are different)

![Figure 1. Phase portrait of the first branch: \( R = 3, Q = -2, S = 3 \).](image1)

![Figure 2. Phase portrait of the second branch: \( R = 7, Q = -2, S = 10 \).](image2)

Of course, these examples show only the singular point configurations of interest in terms of further consideration. In Figure 1, there is the center and the saddle; in Figure 2, there are two centers and the saddle. In both cases, there are separatrix lines that start from the singular points and return to the same points (the homoclinic phase curves). Obviously, the solitary waves correspond to such separatrix lines. The closed phase curves located inside the separatrix correspond to the periodic solutions.

In further consideration, we will focus on the solitary waves. We will construct the "spatial" profiles of the main parameters for the two phase portraits shown above.

Let us accept the following normalization of the basic parameters of the singular point (saddle). This point will be the starting point for numerical integration of set of Eqs. (5). Let \( B_y = b, B_z = 0 \) at "infinite" \( \tau \) (\( b \) is found numerically from Eq. (9) for given parameters \( R, Q, S \)). Let us set \( \rho = 1 \) (the choice of the density unit) and \( U_x = 0, V_x = -\alpha \) at this point. Then \( \Pi = -\alpha \) and \( V_x = -\alpha \). Next, we can calculate \( \alpha \) and \( \beta \):

\[
a^2 = \frac{1}{4} [(R^2 - b^2) \pm \sqrt{(R^2 - b^2)^2 - S^2}] \quad \text{and} \quad \beta = \frac{1}{8} \frac{S^2}{\alpha^2}.
\]

From formula (10), we can obtain the following simple condition for \( \beta \): \( \beta \leq R^2 - b^2 \).

Numerical integration of set of equations (5) or (7) over \( \tau \) was performed with a considerably small step. The solution "is launched" from the starting point along the unstable direction. It is necessary to make sure that the resulting phase curve does not noticeably deviate from the separatrix.
Figures 3 and 4 show the results of this integration (the profiles of $B_y, B_z, V_x, \rho$ are shown).

Figure 3. First branch: 
$R = 7, Q = -2, S = 10, b = 2.131$.

Figure 4. Second branch: 
$R = 7, Q = -2, S = 10, b = -5.251$.

The difference in the solitary wave properties is particularly visible when considering the profiles of the $V_x$ velocity and $\rho$ density. In the wave corresponding to the first branch, the density decreases, while the velocity increases (Figure 3). In the second branch (Figure 4), on the contrary, the deceleration and an increase in density are observed.

Analyzing Tables 1 and 2, which demonstrate how the wave parameters depend on the parameters $S$ and $\beta$, we can learn about a number of solitary wave properties.

| Table 1. First branch. | Table 2. Second branch. |
|------------------------|-------------------------|
| $R = 3, Q = -2$        | $R = 7, Q = -2$         |
| $S$ | $b$ | $\beta$ | $a$   | $S$ | $b$ | $\beta$ | $a$   |
| 0  | 2.29 | 0  | 1.37 | 0  | 5   | -6.56 | 10   | 0.83 |
| 3  | 2.13 | 0.07 | 1.39 | 6  | -6.41 | 13   | 0.82 |
| 4  | 2.0  | 1.0  | 1.41 | 8  | -5.97 | 24   | 0.81 |
| 6  | 1.57 | 1.98 | 1.51 | 10 | -5.25 | 40   | 0.78 |

The phase velocity $a$ of the waves of the first branch is higher than unity (the Alfvén velocity) and it increases with increasing temperature $\beta$. The phase velocity of the waves of the second branch is less than unity and it decreases with increasing temperature. We also indicate here the numerically calculated ranges of the $S$ parameter, within which, at the given $R, Q$ parameters, the closed separatrix forms. For the first branch, at $R = 3, Q = -2$, the range of the $S$ parameter is $0 \leq S \leq 7$. For the second branch, at $R = 7, Q = -2$, it is $5 \leq S \leq 11$.

The phase curves inside the closed separatrix lines correspond to the periodic solutions of set of Eqs. (5). We illustrate their properties by only two examples.

The profiles of four parameters $B_y, B_z, V_x, \rho$ are shown in Figures 5 (the first branch) and 6 (the second branch). The coordinates of the starting point on the $(B_y, B_z)$ plane are also specified: $(b, 0)$. 

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**Figure 3.** First branch:
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**Figure 4.** Second branch: $R = 7, Q = -2, S = 10, b = -5.251$.

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**Table 1.** First branch.
$R = 3, Q = -2$.

| $S$ | $b$ | $\beta$ | $a$ |
|-----|-----|---------|-----|
| 0   | 2.29| 0       | 1.37|
| 3   | 2.13| 0.07    | 1.39|
| 4   | 2.0 | 1.0     | 1.41|
| 6   | 1.57| 1.98    | 1.51|

**Table 2.** Second branch.
$R = 7, Q = -2$.

| $S$ | $b$ | $\beta$ | $a$ |
|-----|-----|---------|-----|
| 5   | -6.56| 10       | 0.83|
| 6   | -6.41| 13       | 0.82|
| 8   | -5.97| 24       | 0.81|
| 10  | -5.25| 40       | 0.78|

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The profiles of four parameters $B_y, B_z, V_x, \rho$ are shown in Figures 5 (the first branch) and 6 (the second branch). The coordinates of the starting point on the $(B_y, B_z)$ plane are also specified: $(b, 0)$.
In the wave corresponding to the first branch (with the phase velocity $a = 1.78$), the density is always less than unity, while in the wave corresponding to the second branch ($a = 0.62$), it is always higher than unity.

5. Conclusions
In this paper, we studied the nonlinear waves in the framework of HMHD (the Hall magnetic hydrodynamics). One-dimensional waves propagating along the magnetic field were studied in the isothermal approximation. It is ascertained that there are two branches of the waves (fast and slow). The existence of both solitary and periodic waves is demonstrated and their basic properties are studied.

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