Liao Standard Systems and Nonzero Lyapunov Exponents for Differential Flows

Wenxiang Sun*  Todd Young
School of Mathematical Sciences  Department of Mathematics
Peking University  Ohio University
Beijing 100871, China  Athens 45701, Ohio, USA
sunwx@math.pku.edu.cn  young@math.ohiou.edu
fax: 740-593-9805

October 9, 2018

This paper is dedicated to the memory of Professor Liao Shantao, 1920–1997.

Abstract

Consider a $C^1$ vector field together with an ergodic invariant probability that has $\ell$ nonzero Lyapunov exponents. Using orthonormal moving frames along certain transitive orbits we construct a linear system of $\ell$ differential equations which is a reduced form of Liao’s “standard system”. We show that the Lyapunov exponents of this linear system coincide with all the nonzero exponents of the given vector field with respect to the given probability. Moreover, we prove that these Lyapunov exponents have a persistence property that implies that a “Liao perturbation” preserves both sign and value of nonzero Lyapunov exponents.

Key Words and Phrases: Lyapunov exponent, standard linear system, Liao perturbation

2000 MSC: 37C15, 37A10, 34A26

Running title: Liao Systems and Lyapunov Exponents.

*Supported partly by NNSFC 10171004. The first author also thanks Ohio University for its hospitality during the winter and spring of 2002 when this paper was written.
1 Introduction

Lyapunov exponents measure the asymptotic exponential rate at which infinitesimally nearby points approach or move away from each other as time increases to infinity. For a uniformly hyperbolic system with positive (resp. negative) Lyapunov exponents, its nearby system has positive (resp. negative) Lyapunov exponents as well. Using orthonormal frames moving along certain transitive orbits Liao (see [7]) constructed a system of linear equations, known as a “standard system”. In the hyperbolic case, our Main Theorem together with a result of Liao’s [7, Theorem 2.4.1] shows that the Lyapunov exponents of the standard system coincide with those of the original flow. Professor Liao had conjectured this result. For the complement of uniform hyperbolicity in the space of all $C^1$ systems with $C^1$ topology, understanding dynamics through Lyapunov exponents and SRB measures is incomplete but very important (see Palis [11]). Young [17, 18] constructed open sets of nonuniform hyperbolicity cocycles for certain special systems. In [16] Viana constructed an open set of systems with multidimensional nonhyperbolic attractors which have SRB measures [1]. For a compact surface, Bochi [2] showed that there is a residual set of $C^1$ area preserving diffeomorphisms so that each diffeomorphism in the set is either Anosov or has a zero Lyapunov exponent almost everywhere.

In the 1960’s, Liao (see [7]) constructed a system of linear equations, known as a “standard system”. This system is essentially the variational equations along a typical orbit with respect to a typical orthonormal frame evolving along the orbit. Liao had used the standard system to give independent proofs of the $C^1$ closing lemma [7, Appendix A] and of the topological stability for Anosov flows [7, Chapter 2]. While Liao’s approach is obviously philosophically related to Lyapunov exponents, the connection has never been rigorously shown. In the hyperbolic case, our Main Theorem together with a result of Liao’s [7, Theorem 2.4.1] shows that the Lyapunov exponents of the standard system coincide with those of the original flow. Professor Liao had conjectured this result.

We work with $C^1$ vector fields and develop a reduced form of Liao’s standard systems. We consider a $C^1$ vector field together with an ergodic invariant probability that has $\ell$ nonzero Lyapunov exponents. Using typical moving orthonormal $\ell$-frames along typical transitive orbits of the ergodic measure, and by using a characterization of the Lyapunov spectrum [9, 12] we construct a “reduced standard system” of differential equations and show that its Lyapunov exponents coincide with the nonzero exponents of the original vector field. In the final section we show that the nonzero Lyapunov exponents of the reduced standard system have certain persistence properties.
Now let us describe the main theorem of the present paper. We denote by \( M^n \) a compact smooth \( n \)-dimensional Riemannian manifold and by \( S \) a \( C^1 \) differential system, or in other words, a \( C^1 \) vector field on \( M^n \). As usual \( S \) induces a one-parameter transformation group \( \phi_t: M^n \to M^n, t \in \mathbb{R} \) on the state manifold and therefore a one-parameter transformation group \( \Phi_t = d\phi_t: TM^n \to TM^n, t \in \mathbb{R} \) on the tangent bundle. A probability \( \nu \) on \( M^n \) is \( \phi \)-invariant if it is \( \phi_t \)-invariant for any \( t \in \mathbb{R} \). A \( \phi \)-invariant probability is called \( \phi \)-ergodic if every \( \phi \)-invariant set has zero or full probability. For a compact metric space \( X \) and a topological flow \( \varphi_t \) on it we denote by \( E(X, \varphi) \) the set of all \( \varphi \)-invariant and ergodic probabilities. Let \( \nu \) be a \( \phi \)-invariant and ergodic probability, i.e., \( \nu \in E(M^n, \phi) \). From the Multiplicative Ergodic Theorem (see [4, 10]), there exists a \( \phi_t \)-invariant subset \( B \), with \( \nu \)-full probability, such that for any \( x \in B \) and \( u \in T_xM^n \) the following limit, called Lyapunov exponent,

\[
\lambda := \lim_{t \to \infty} \frac{1}{t} \log \left\| \Phi_t(u) \right\| \quad \text{or} \quad \lambda := \lim_{t \to -\infty} \frac{1}{t} \log \left\| \Phi_t(u) \right\|. \tag{1.1}
\]

It is known that \( \nu \) has at most \( n \) different Lyapunov exponents, where \( n \) indicates the dimension of the state manifold \( M^n \).

**Main Theorem** Suppose that a \( \phi \)-invariant and ergodic probability, \( \nu \in E(M^n, \phi) \), has \( \ell \) simple nonzero Lyapunov exponents

\[
\lambda_1 < \lambda_2 < \ldots < \lambda_\ell, \tag{1.2}
\]

together with \( n - \ell \) zero Lyapunov exponents. Then the reduced standard linear system (defined in Section 4),

\[
\frac{dy}{dt} = yA_{\ell \times \ell}(t), \quad y \in \mathbb{R}^\ell, \quad t \in \mathbb{R}, \tag{1.3}
\]

is well defined and has the following properties:

1. The matrix \( A_{\ell \times \ell}(t) \) is uniformly bounded and continuous with respect to \( t \).

2. There exist \( u_1, u_2, \ldots, u_\ell \in \mathbb{R}^\ell \) such that

\[
\lim_{t \to \infty} \frac{1}{t} \log \| y(t, u_i) \| = \lambda_i,
\]

where \( y(t, v) \) denotes a unique solution of the initial value problem

\[
\frac{dy}{dt} = yA_{\ell \times \ell}(t), \quad y(0, v) = v. \tag{1.4}
\]

3. Consider a perturbation of the linear system

\[
\frac{dy}{dt} = yA_{\ell \times \ell}(t) + f(t, y), \quad \sup_{t \in \mathbb{R}, y \in \mathbb{R}^\ell} \| f(t, y) \| \leq L < \infty, \tag{1.5}
\]
where \( f(t, y) \) is Lipschitz in \( y \). Then there exist \( u_1^*, u_2^*, \ldots, u_\ell^* \in \mathbb{R}^\ell \) such that

\[
\lim_{t \to \infty} \frac{1}{t} \log \| y(t, u_1^*) \| = \lambda_i,
\]

where \( y(t, v) \) denotes a unique solution of the initial value problem

\[
\frac{dy}{dt} = yA_\ell \times \ell(t) + f(t, y), \quad y(0, v) = v.
\]

In Section 2 we review frame bundles and the corresponding one parameter transformation groups induced by a given vector field. In Section 3 we construct a reduced version of Liao’s “qualitative functions” and then use them to present a characterization of the Lyapunov spectrum. In Section 4 we construct the reduced standard linear system of \( \ell \) differential equations on a given probability and establish a relation between the nonzero Lyapunov exponents of this probability and that of the linear system. We complete the proof of the Main Theorem in Section 5. An example in Section 5 illustrates that the original standard linear system of \( n \) differential equations introduced by Liao [7, Chapter 2] fails to satisfy the conclusions of the Main Theorem, and so, it is necessary to develop the reduced standard linear system of \( \ell \) differential equations for the Main Theorem. In Section 7 we present the notion of Liao perturbation and point out by the Main Theorem that a certain type of perturbation, known as “Liao perturbation”, preserves the nonzero Lyapunov exponents.

### 2 One parameter transformation groups

We start from a \( C^1 \) vector field \( S \) on a compact smooth \( n \)-dimensional Riemannian manifold \( M^n \), and its induced one-parameter transformation groups \( \phi_t : M^n \to M^n \), \( t \in \mathbb{R} \) on the state manifold and \( \Phi_t = d\phi_t : TM^n \to TM^n \), on the tangent bundle.

Fix some integer \( \ell, 1 \leq \ell \leq n \). Construct a bundle \( \mathcal{U}_\ell = \bigcup_{x \in M^n} \mathcal{U}_\ell(x) \) of \( \ell \)-frames, where the fiber over \( x \) is

\[
\mathcal{U}_\ell(x) = \{(u_1, \ldots, u_\ell) \in T_xM^n \times \cdots \times T_xM^n : u_1, u_2, \ldots, u_\ell, \text{ are linearly independent}\}.
\]

Let \( p_\ell : \mathcal{U}_\ell \to M^n \) denote the bundle projection. Denote by \( \text{proj}_k : \mathcal{U}_\ell \to TM^n \) the map which sends \( \alpha \in \mathcal{U}_\ell \) to the \( k \)-th vector in \( \alpha \). The vector field \( S \) induces a one-parameter transformation group on \( \mathcal{U}_\ell \), which we denote (with the same notation as the tangent map for the sake of simplicity) by \( \Phi_t, t \in \mathbb{R} \), namely,

\[
\Phi_t(u_1, u_2, \ldots, u_\ell) = (d\phi_t(u_1), d\phi_t(u_2), \ldots, d\phi_t(u_\ell)).
\]
For \( \alpha = (u_1, u_2, \ldots, u_\ell) \in \mathcal{U}_\ell \) and a nondegenerate \( \ell \times \ell \) matrix \( B = (b_{ij}) \) we write

\[
\alpha \circ B = \left( \sum_{i=1}^\ell b_{1i} u_i, \sum_{i=1}^\ell b_{2i} u_i, \ldots, \sum_{i=1}^\ell a_{\ell i} u_i \right).
\]

Then \( \Phi_\ell(\alpha \circ B) = \Phi_\ell(\alpha) \circ B \). By the Gram-Schmidt orthogonalization process there exists a unique upper triangular matrix \( \Gamma(\alpha) \) with diagonal elements 1 such that \( \alpha \circ \Gamma(\alpha) \) is orthogonal.

Construct the bundle \( \mathcal{F}_\ell = U_{x \in M^n} \mathcal{F}_\ell(x) \) of \( \ell \)-orthogonal frames, where the fiber over \( x \) is

\[
\mathcal{F}_\ell(x) = \{(u_1, u_2, \ldots, u_\ell) \in \mathcal{U}_\ell(x) \mid \langle u_i, u_j \rangle = 0, \ 1 \leq i \neq j \leq \ell \}.
\] (2.2)

The bundle projection is given by \( q_\ell = p_\ell(\mathcal{F}_\ell) \). The vector field \( S \) then induces a one-parameter transformation group

\[
\chi_\ell: \mathcal{F}_\ell \to \mathcal{F}_\ell : \alpha \mapsto \Phi_\ell(\alpha) \circ \Gamma(\Phi_\ell(\alpha)).
\] (2.3)

If we define \( \pi: \mathcal{U}_\ell \to \mathcal{F}_\ell \) by \( \alpha \mapsto \alpha \circ \Gamma(\alpha) \) then \( \chi_\ell(\alpha) = \pi(\Phi_\ell(\alpha)) \).

Construct a bundle \( \mathcal{F}_\ell^\# = U_{x \in M^n} \mathcal{F}_\ell^\#(x) \) of orthonormal \( \ell \)-frames, where the fiber over \( x \) is

\[
\mathcal{F}_\ell^\#(x) = \{(u_1, u_2, \ldots, u_\ell) \in \mathcal{F}_\ell(x) \mid \|u_i\| = 1, \ i = 1, 2, \ldots, \ell \}.
\] (2.4)

Then \( \mathcal{F}_\ell^\# \) is a compact metrizable space. Let \( \pi^\#: \mathcal{F}_\ell \to \mathcal{F}_\ell^\# \) be given by

\[
\pi^\#(u_1, u_2, \ldots, u_\ell) = \left( \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \ldots, \frac{u_\ell}{\|u_\ell\|} \right).
\]

Setting \( \chi_\ell^\# = \pi^\# \circ (\chi_\ell|_{\mathcal{F}_\ell^\#}) \), we get a one-parameter transformation group \( \chi_\ell^\#: \mathcal{F}_\ell^\# \to \mathcal{F}_\ell^\# \). Let \( q_\ell^\# = q_\ell|_{\mathcal{F}_\ell^\#} \), then \( q_\ell^\# \) is a bundle projection. It is easy to check that the following properties hold:

\[
q_\ell \circ \chi_\ell = \phi_\ell \circ q_\ell, \quad q_\ell^\# \circ \chi_\ell^# = \phi_\ell \circ q_\ell^#, \quad \chi_\ell^\# \circ \pi^\# = \pi^\# \circ \chi_\ell.
\] (2.5)

**Remark 2.1** We point out that the unitary \( \ell \)-bundle \( \mathcal{U}_\ell^\# \) of \( \mathcal{U}_\ell \) is not necessarily a compact metric space. For instance, when \( \ell = 2 \), there are sequences of \( 2 \)-frames \( \{\alpha^m = (u_1^m, u_2^m)\}_{m=1}^\infty \) such that the angle between \( u_1^m \) and \( u_2^m \) goes to zero as \( m \to +\infty \). Such a sequence of frames has no accumulating point inside \( \mathcal{U}_\ell^\# \).

### 3 Qualitative functions

For \( \alpha \in \mathcal{F}_n \), let \( \zeta_{\alpha k}(t) = \|\text{proj}_k \chi_\ell(\alpha)\|, \ k = 1, 2, \ldots, n \). Note that \( \zeta_{\alpha k}(t) > 0 \) for any \( t \in \mathbb{R} \).

**Definition 3.1** For each \( k = 1, 2, \ldots, n \), we call \( \omega_k \) defined by:

\[
\omega_k: \mathcal{F}_n \to \mathbb{R} : \alpha \mapsto \left. \frac{d\zeta_{\alpha k}(t)}{dt} \right|_{t=0}
\] (3.1)
a qualitative function over the orthogonal n-frame bundle \( \mathcal{F}_n \) and call \( \omega_k|\mathcal{F}_n^\# \) a qualitative function over the orthonormal n-frame bundle \( \mathcal{F}_n^\# \).

The qualitative function for vector fields was introduced by Liao in 1963, and it plays an important role in Liao theory [4-8]. Sun introduced its diffeomorphism version and described its relation with Lyapunov exponents in [13, 14], and determined in [15] the entropy of certain classes of Grassmann bundle systems by using these functions.

From the definition it is easy to show that \( \omega_k(\alpha) \) is continuous, \( \omega_k(\chi_t(\alpha)) = \frac{d\zeta_{\alpha k}(t)}{dt} \), and \( \omega_k(\chi_t^\#(\alpha)) = \frac{1}{\zeta_{\alpha k}(t)} \frac{d\zeta_{\alpha k}(t)}{dt} \), so the following lemma is clear.

**Lemma 3.2** For \( \alpha \in \mathcal{F}_n^\# \) and \( k = 1, 2, \ldots, n \), we have that \( \log \zeta_{\alpha k}(T) = \int_0^T \omega_k(\chi_t^\#(\alpha)) \, dt \).

If we denote by \( Q_{\nu}(M^n, \phi) \) the set of all points \( x \in M^n \) that satisfy, for any continuous function \( f \) on \( M^n \),

\[
\lim_{t \to \pm \infty} \frac{1}{t} \int_0^t f(\phi_r(x)) \, d\tau = \int_{M^n} f \, d\mu, \tag{3.2}
\]

then \( Q_{\nu}(M^n, \phi) \) is \( \phi \)-invariant subset with \( \nu \)-full probability. Similarly one can define \( Q_{\mu}(\mathcal{F}_n^\#, \chi^\#) \) for any probability \( \mu \in E(\mathcal{F}_n^\#, \chi^\#) \) with \( q^\#(\mu) = \nu \).

The following is a slight modification of [3] Theorem 4.1. We state it here without proof.

**Lemma 3.3** For any given \( \mu \in E(\mathcal{F}_n^\#, \chi^\#) \) and any permutation

\[
r : \{1, 2, \ldots, n\} \to \{r(1), r(2), \ldots, r(n)\}
\]

there exists \( \tilde{\mu} \in E(\mathcal{F}_n^\#, \chi^\#) \) such that \( q_{\alpha^*}^\#(\mu) = q_{\alpha^*}^\#(\tilde{\mu}) \) and

\[
\int \omega_{r(i)} \, d\mu = \int \omega_i \, d\tilde{\mu}, \quad i = 1, 2, \ldots, n.
\]

Now we fix the positive integer \( \ell, \ell \leq n \), as in the Main Theorem. Define

\[
id_\ell : \mathcal{F}_n^\# \to \mathcal{F}_\ell^\# : \alpha = (v_1, \ldots, v_{n-\ell}, v_{n-\ell+1}, \ldots, v_n) \to \tilde{\alpha} = (v_{n-\ell+1}, \ldots, v_n). \tag{3.3}
\]

Then, \( id_\ell \) is a continuous projection. For \( \tilde{\alpha} \in \mathcal{F}_\ell^\# \), set:

\[
\tilde{\zeta}_k(\tilde{\alpha}) = \zeta_{n-\ell+k} \circ (id_\ell)^{-1}(\tilde{\alpha}), \quad \text{and} \quad \tilde{\omega}_k(\tilde{\alpha}) = \omega_{n-\ell+k} \circ (id_\ell)^{-1}(\tilde{\alpha}). \tag{3.4}
\]

It is clear by the definitions that both \( \tilde{\zeta}_k(\tilde{\alpha}) \) and \( \tilde{\omega}_k(\tilde{\alpha}) \) are independent of the choice of preimages in \( id_\ell^{-1}(\alpha) \). Thus \( \tilde{\zeta}_k, \tilde{\omega}_k : \mathcal{F}_\ell^\# \to \mathbb{R} \) are all well defined. For \( \mu \in E(\mathcal{F}_n^\#, \chi^\#) \) set \( \tilde{\mu} := id_\ell(\mu) \). Then \( \tilde{\mu} \in E(\mathcal{F}_\ell^\#, \chi^\#) \). Take \( \alpha = (u_1, \ldots, u_n) \in Q_{\mu}(\mathcal{F}_n^\#, \chi^\#) \). Then by Lemma 3.2,

\[
\lim_{t \to \pm \infty} \frac{1}{t} \log \zeta_{\alpha(n-\ell+k)}(t) = \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \omega_{n-\ell+k}(\chi_t^\#(\alpha)) \, d\tau = \int_{\mathcal{F}_\ell^\#} \omega_{n-\ell+k} \, d\mu,
\]
for \( k = 1, 2, \ldots, \ell \). Now write \( \tilde{\alpha} := id_{\ell}(\alpha) = (u_{n-\ell+1}, \ldots, u_n) \). Then \( \tilde{\omega}_k(\tilde{\alpha}) = \left. \frac{d\tilde{\zeta}_{\ell k}(t)}{dt} \right|_{t=0} \) and \( \tilde{\omega}_k(\chi^\#_{\ell k}(\tilde{\alpha})) = \frac{1}{\zeta_{\ell k}(t)} \frac{d\zeta_{\ell k}(t)}{dt} \) and thus Lemma 3.2 holds for \( \tilde{\zeta}_{\ell k} \) and \( \tilde{\omega}_k \), \( k = 1, \ldots, \ell \). Observe that for \( \tilde{\alpha} \in Q_{\tilde{\mu}}(F_{\ell}^#, \chi^#) \) we have

\[
\lim_{t \to \infty} \frac{1}{t} \log \tilde{\zeta}_{\ell k}(t) = \int_{F_{\ell}^#} \tilde{\omega}_k d\tilde{\mu} = \int_{F_{\ell}^#} \omega_{n-\ell+k} d\mu = \lim_{t \to \infty} \frac{1}{t} \log \zeta_{\ell k(n-\ell+k)}(t), \quad k = 1, \ldots, \ell.
\]

We remark that the above function \( \tilde{\zeta}_k, F_{\ell}^# \to \mathbb{R} \) is not necessarily the same as \( \zeta_k : F_{\ell}^# \to \mathbb{R} \), and the function \( \tilde{\omega}_k : F_{\ell}^# \to \mathbb{R} \) is not exactly the same as \( \omega_k : F_{\ell}^# \to \mathbb{R} \), where \( \zeta_k, \omega_k : F_{\ell}^# \to \mathbb{R} \) are given in Definition 3.1 with \( n \) replaced by \( \ell \).

**Proposition 3.4** Let \( \nu \in E(M^n, \phi) \) be as in the Main Theorem, that is, it supports \( \ell \) nonzero Lyapunov exponents \( \lambda_1 < \ldots < \lambda_\ell \) together with \( n-\ell \) zero Lyapunov exponents. Then there exist two probabilities \( \mu \in E(F_{n}^#, \chi^#) \) and \( \tilde{\mu} \in E(F_{\ell}^#, \chi_{\ell}^#) \), and two subsets \( \Lambda \subset M^n \) and \( W \subset F_{n}^# \) such that

1. \( q^#_{n*}(\mu) = \nu, \quad q^#_{\ell*}(\tilde{\mu}) = \nu, \quad id_{\ell*}(\mu) = \tilde{\mu} \);
2. \( \phi_{\ell}(\Lambda) = \Lambda, \quad \chi^#_{\ell}(W) = W, \quad \nu(\Lambda) = 1, \quad \text{and} \quad \mu(W) = 1 \);
3. For each \( x \in \Lambda \) and \( \alpha \in W \) with \( q^#_{n*}(\alpha) = x \)

\[
\lim_{t \to \pm \infty} \frac{1}{t} \log \zeta_{\ell k(n-\ell+k)}(t) = \int_{F_n^#} \omega_{n-\ell+k} d\mu = \int_{F_{\ell}^#} \tilde{\omega}_k d\tilde{\mu} = \lambda_k
\]

for \( k = 1, 2, \ldots, \ell \).

**Proof.** Take a \( \phi_{t} \)-invariant subset \( \Lambda_1 \subset M^n \) with \( \nu \)-total probability so that at each point the spectrum of all Lyapunov exponents is \( \lambda_1, \ldots, \lambda_\ell \) together with \( n-\ell \) zeros. Furthermore,

\[
\{\lambda_1, \ldots, \lambda_\ell, 0, \ldots, 0\} = \left\{ \int \omega_k d\mu : \mu \in E(F_{n}^#, \chi^#), q^#_{n*}(\mu) = \nu, \ k = 1, 2, \ldots, n \right\}.
\]

The existence of \( \Lambda_1 \) follows from the hypothesis of the present proposition and Theorem 2.2 in [12]. Choose an arbitrary \( \mu_1 \in E(F_{n}^#, \chi^#) \) to cover \( \nu \), i.e., \( q^#_{n*}(\mu_1) = \nu \). We claim that

\[
\{\lambda_1, \ldots, \lambda_\ell, 0, \ldots, 0\} = \left\{ \int \omega_k d\mu_1, k = 1, 2, \ldots, n \right\}.
\]

Observe that \( \mu_1(Q_{\mu_1}(F_{n}^#, \chi^#)) = 1 \) and \( \nu(q^#_{n}Q_{\mu_1}(F_{n}^#, \chi^#)) = 1 \), thus

\[
\nu\left(q^#_{n}Q_{\mu_1}(F_{n}^#, \chi^#) \cap \Lambda_1\right) = 1.
\]
Take \( x \in q_n^\# Q_{\mu_1}(\mathcal{F}_n^\# , \chi^\#) \cap A_1 \) and \( \alpha = (u_1, \ldots, u_n) \in \mathcal{F}_n^\# (x) \cap Q_{\mu_1}(\mathcal{F}_n^\# , \chi^\#) \). Remember that \( \omega_k \) is a continuous function. By Lemma 3.2 we then have

\[
\lim_{t \to \pm \infty} \frac{1}{t} \log \zeta_{\alpha k}(t) = \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \omega_k(\chi^\#_s(\alpha)) \, ds = \int \omega_k \, d\mu_1, \quad k = 1, 2, \ldots, n.
\]

We point out that in the case when index \( k = 1 \) we have

\[
\lim_{t \to \pm \infty} \frac{1}{t} \log \| \Phi_t(u_1) \| = \lim_{t \to \pm \infty} \frac{1}{t} \log \zeta_{\alpha 1}(t) = \int \omega_1 \, d\mu_1.
\]

If we suppose that Equation (3.5) is not true, then there would exist a minimal index \( i_0 > 1 \) such that

\[
\lim_{t \to \pm \infty} \frac{1}{t} \log \| \Phi_t(u_{i_0}) \| \neq \int \omega_k \, d\mu_1, \quad \text{for all} \quad k = 1, 2, \ldots, n.
\]

Note that \( \{ \text{proj}_1 \chi_t^\#(\alpha), \ldots, \text{proj}_n \chi_t^\#(\alpha) \} \) is an orthonormal frame on the tangent space \( T_{\phi_t(x)}M^n \) and \( \langle \Phi_t(u_{i_0}), \text{proj}_j \chi_t^\#(\alpha) \rangle = 0 \) for each \( j = i_0 + 1, \ldots, n \). We can represent \( \frac{\Phi_t(u_{i_0})}{\| \Phi_t(u_{i_0}) \|} \) as

\[
\frac{\Phi_t(u_{i_0})}{\| \Phi_t(u_{i_0}) \|} = a_1(t) \text{proj}_1 \chi_t^\#(\alpha) + a_2(t) \text{proj}_2 \chi_t^\#(\alpha) + \ldots + a_{i_0}(t) \text{proj}_{i_0} \chi_t^\#(\alpha),
\]

where \( |a_k(t)| \leq 1, k = 1, 2, \ldots, i_0 \). Now let us suppose that \( \lim_{t \to \infty} |a_{i_0}(t)| > 0 \). Observe that both \( a_{i_0}(t) \| \Phi_t(u_{i_0}) \| \text{proj}_{i_0} \chi_t^\#(\alpha) \) and \( \text{proj}_{i_0} \chi_t(\alpha) \) express the same projection of \( \Phi_t(u_{i_0}) \) on the direction determined by \( \text{proj}_{i_0} \chi_t^\#(\alpha) \), thus

\[
|a_{i_0}(t)| \| \Phi_t(u_{i_0}) \| = \zeta_{\alpha i_0}(t).
\]

Therefore by Lemma 3.2

\[
\lim_{t \to \infty} \frac{1}{t} \log \| \Phi_t(u_{i_0}) \| = \lim_{t \to \infty} \frac{1}{t} \log \| \Phi_t(u_{i_0}) \|
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \log |a_{i_0}(t)|^{-1} + \lim_{t \to \infty} \frac{1}{t} \log \zeta_{\alpha i_0}(t)
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \log \zeta_{\alpha i_0}(t)
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \int_0^t \omega_{i_0}(\chi_t^\#(\alpha)) \, ds
\]

\[
= \int \omega_{i_0} \, d\mu_1.
\]

This is a contradiction to the choice of \( i_0 \). For the case \( \lim_{t \to \infty} |a_{i_0}(t)| = 0 \), one then gets that

\[
\lim_{t \to \infty} \frac{1}{t} \log \| \Phi_t(u_{i_0}) \|
\]
coincides with \( \int \omega_i \, d\mu_1 \) for some \( i < i_0 \), again a contradiction to the choice of \( i_0 \). Consequently, \( \Box \) holds.

Now there is a permutation \( r : \{1, 2, \ldots, n\} \to \{r(1), r(2), \ldots, r(n)\} \) so that

\[
\int \omega_{r(i)} \, d\mu_1 = 0, \quad i = 1, 2, \ldots, n - \ell,
\]

and

\[
\int \omega_{r(i)} \, d\mu_1 = \lambda_{i-(n-\ell)}, \quad i = n - \ell + 1, n - \ell + 2, \ldots, n.
\]

From Lemma 3.3 there exists a covering probability \( \mu \in E(\mathcal{F}_n^\#, \chi^\#) \) of \( \nu \), \( q_{n^*}(\mu) = q_{n^*}(\mu_1) = \nu \), so that \( \int \omega_i \, d\mu = \int \omega_{r(i)} \, d\mu_1 \), \( i = 1, 2, \ldots, n \).

Define \( W := Q_\mu(\mathcal{F}_n^\#, \chi^\#) \) and \( \Lambda := \Lambda_1 \bigcap q_n^!(W) \). Then \( \nu(\Lambda) = \mu(W) = 1 \), \( \chi^!(W) = W \), \( \phi_t(\Lambda) = \Lambda \), \( t \in \mathbb{R} \), and \( q_n^!(W) = \Lambda \). Define \( \hat{\mu} := id_{\ell*}(\mu) \). Then \( \hat{\mu} \in E(\mathcal{F}_\ell^\#, \chi^\#) \). Clearly \( q_{\ell^*}(\hat{\mu}) = \nu \), and

\[
\int \hat{\omega}_k \, d\hat{\mu} = \int \omega_{n-\ell+k} \, d\mu, \quad k = 1, \ldots, \ell.
\]

Take \( x \in \Lambda \) and \( \alpha \in W \bigcap \mathcal{F}_n^!(x) \), then

\[
\lim_{t \to \pm \infty} \frac{1}{t} \log \zeta_{\alpha(n-\ell+k)}(t) = \int \omega_{n-\ell+k} \, d\mu = \int \hat{\omega}_k \, d\hat{\mu}, \quad k = 1, 2, \ldots, \ell.
\]

This completes the proof of Proposition 3.4.

**Corollary 3.5** For any given \( \hat{\alpha} = (u_{n-\ell+1}, \ldots, u_n) \in Q_{\hat{\mu}}(\mathcal{F}_\ell^\#, \chi^\#) \) with \( q_{\ell^*}(\hat{\alpha}) \in \Lambda \) we have

\[
\lim_{t \to -\infty} \frac{1}{t} \log \|\Phi_t(u_i)\| < 0, \quad i = n - \ell + 1, \ldots, n - \ell + p,
\]

and,

\[
\lim_{t \to +\infty} \frac{1}{t} \log \|\Phi_t(u_i)\| > 0, \quad i = n - \ell + p + 1, \ldots, n,
\]

where \( p \) satisfies: \( \lambda_1 < \ldots < \lambda_p < 0 < \lambda_{p+1} < \ldots < \lambda_\ell \).

**Proof.** For \( n - \ell + p + 1 \leq i_0 \leq n \), we have by Lemma 3.2 that,

\[
0 < \int_{\mathcal{F}_n^\#} \omega_{i_0} \, d\mu
\]

\[
= \lim_{t \to +\infty} \frac{1}{t} \int_0^t \omega_{i_0}(\chi_t^!(\alpha)) \, dt
\]

\[
= \lim_{t \to +\infty} \frac{1}{t} \log \zeta_{\alpha i_0}(t)
\]

\[
\leq \lim_{t \to +\infty} \frac{1}{t} \log \|\Phi_t(i_{k_0})\|.
\]

For \( n - \ell + p + 1 \leq i_0 \leq n \), we may deduce a similar inequality. \( \Box \)
4 Reduced standard linear systems of $\ell$ differential equations

We start this section from the $\phi$-invariant, ergodic probability $\nu \in E(M^n, \phi)$ assumed in the Main Theorem together with its two covering probabilities $\mu \in E(F_n^\#, \chi^\#)$ and $\tilde{\mu} \in E(F_n^\#, \chi^\#)$ and the corresponding total probability subsets $\Lambda \subset M^n$ and $W \subset F_n^\#$ as in Proposition 3.4. Take a point $x \in \Lambda$ and an orthonormal frame $\alpha \in W \cap F_n^\#(x)$. Then

$$
\lim_{t \to \infty} \frac{1}{t} \log \zeta_{\alpha k}(t) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \omega_k(\chi^\#(\alpha))d\tau \\
= \int_{F_n^\#} \omega_k d\mu \\
= \int_{F_n^\#} \omega_{k-(n-\ell)} d\tilde{\mu}, \quad k = n - \ell + 1, \ldots, n.
$$

In this section we will construct the reduced standard linear system needed in the Main Theorem along the orbit orb($x, \phi$) with respect to the given orthonormal frame $\alpha \in F_n^\#(x)$, by developing the technique in [4]. Since $\chi^\#(\alpha)$ is an orthonormal frame at $T_{\phi_t}(x)M^n$, there exists an $n \times n$ matrix $B_\alpha(t)$ such that $\Phi_t(\alpha) = \chi^\#_t(\alpha) \circ B_\alpha(t)$. Define $R_\alpha(t) = \frac{dB_\alpha(t)}{dt} \circ B_\alpha(t)^{-1}$. Define a diagonal matrix

$$
\zeta_\alpha(t) = \text{diag} (\zeta_{\alpha 1}(t), \zeta_{\alpha 2}(t), \ldots, \zeta_{\alpha n}(t)).
$$

From Gram-Schmidt orthogonalization, $\chi^\#_t(\alpha) = \Phi_t(\alpha) \circ \Gamma(\Phi_t(\alpha)) \circ \zeta^{-1}_\alpha(t)$, or, $\Phi_t(\alpha) = \chi^\#_t \circ \zeta_\alpha(t) \circ \Gamma(\Phi_t(\alpha))^{-1}$, where $\Gamma(\Phi_t(\alpha))$ is an $n \times n$ upper triangular matrix with elements 1 on the diagonal. So $B_\alpha(t) = \zeta_\alpha(t) \circ \Gamma(\Phi_t(\alpha))^{-1}$, which is differentiable with respect to $t \in \mathbb{R}$. Observe

$$
\frac{1}{\zeta_{\alpha k}(t)} \frac{d\zeta_{\alpha k}(t)}{dt} = \omega_k(\chi^\#_t(\alpha)), \quad k = 1, \ldots, n,
$$

and

$$
\frac{dB_\alpha(t)}{dt} \circ B_\alpha(t)^{-1} = \begin{pmatrix} \frac{1}{\zeta_{\alpha 1}(t)} \frac{d\zeta_{\alpha 1}(t)}{dt} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \frac{1}{\zeta_{\alpha n}(t)} \frac{d\zeta_{\alpha n}(t)}{dt} \end{pmatrix}.
$$

Thus

$$
R_\alpha(t) = \begin{pmatrix} \omega_1(\chi^\#_t(\alpha)) & \cdots & 0 \\
\omega_2(\chi^\#_t(\alpha)) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \omega_n(\chi^\#_t(\alpha)) \end{pmatrix}. \quad (4.1)
$$
Now let us denote $R_\alpha(t)^T$ by $(r_{ij}(t))_{n \times n}$, where $r_{ij}(t) = 0$ if $i < j$; $r_{ii}(t) = \omega_i(\chi_i^\#(\alpha))$, $i,j = 1,\ldots,n$. Set $\tilde{\alpha} := id_\ell(\alpha)$. Recall from Section 3 that, $\tilde{\omega}_i(\tilde{\alpha}) = \omega_i(n^{-\ell}+i) \circ id_\ell^{-1}(\tilde{\alpha})$, for $i = 1,\ldots,\ell$.

We define a triangular $\ell \times \ell$ matrix $A_{\ell \times \ell}(t) = (a_{ij}(t))_{\ell \times \ell}$ as follows: $a_{ij}(t) = 0$ if $i < j$; $a_{ij}(t) = r_{(n-\ell-i)(n-\ell-j)}(t)$ if $i > j$; $a_{ii}(t) = \tilde{\omega}_i(\chi_i^\#(\tilde{\alpha})), i,j = 1,\ldots,\ell$. Thus

$$A_{\ell \times \ell}(t) = \begin{pmatrix}
\tilde{\omega}_1(\chi_1^\#(\tilde{\alpha})) \\
\tilde{\omega}_2(\chi_2^\#(\tilde{\alpha})) & \ddots \\
& \ddots & \tilde{\omega}_\ell(\chi_\ell^\#(\tilde{\alpha}))
\end{pmatrix}, \quad (4.2)\]$$

where $\alpha \in W \cap \mathcal{F}_n^#(x)$ and $x \in \Lambda$.

**Definition 4.1** We call

$$\frac{dy}{dt} = y A_{\ell \times \ell}(t),$$

the reduced standard linear system of $\ell$ differential equations for the given system $(M^n, S, \nu)$ with respect to an orthonormal $n$-frame $\alpha$, where $A_{\ell \times \ell}(t)$ is given by (4.2).

**Proof of the Main Theorem (1.) (2.).** For (1) it is sufficient to show $A_{\ell \times \ell}(t)$ is uniformly bounded. In [4] Liao proved that $\sup_{i \in \mathbb{R}} \| R_\alpha(t) \| < \infty$, from which it is easy to get

$$\sup_{t \in \mathbb{R}} \| A_{\ell \times \ell}(t) \| \leq \sup_{t \in \mathbb{R}} \| R_\alpha(t) \| < \infty.$$ 

Now we prove the Main Theorem (2.) by showing the following proposition.

**Proposition 4.2** Let $\nu \in E(M^n, \phi)$ be as in the Main Theorem. Let us take covering probabilities $\mu \in E(\mathcal{F}_{n}^#, \chi^#)$, and $\tilde{\mu} \in E(\mathcal{F}_l^#, \chi^#)$, satisfying $q_{\mu}^#(\mu) = q_{\tilde{\mu}}^#(\tilde{\mu}) = \nu$, and take a $\mu$-total probability subset $W \subset \mathcal{F}_n^#$ and a $\nu$-total probability subset $\Lambda \subset M^n$ as in Proposition 3.3. Take $x \in \Lambda$ and $\alpha \in W \cap \mathcal{F}_n^#(x)$ and construct the reduced standard linear system (4.3) of $\ell$ differential equations as in Definition 4.1. For a coordinate vector $e_i = (0,\ldots,0,1(i),0,\ldots,0) \in \mathbb{R}^\ell$ denote by $\tilde{y}(t,e_i)$ a unique solution of the initial value problem (1.3) with $y(0,e_i) = e_i$. Then

$$\lim_{t \to \pm \infty} \frac{1}{t} \log \| \tilde{y}(t,e_i) \| = \lambda_i, \quad i = 1,\ldots,\ell.$$

**Proof.** Solving the initial value problem

$$\frac{dy_t}{dt} = \tilde{\omega}_i(\chi_i^\#(\tilde{\alpha})) y_t, \quad y(0) = e_\ell,$$

we get

$$y_t(t,e_\ell) = e_\ell \int_0^t \tilde{\omega}_i(\chi_i^\#(\tilde{\alpha})) \,dr.$$

11
This equality together with Proposition 3.4 implies the following

\[
\lim_{t \to \infty} \frac{1}{t} \log |y(t, e_{\ell})| = \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} \tilde{\omega}_t(\chi_\tau^\#(\tilde{\alpha}))d\tau
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} \omega_n(\chi_\tau^\#(\alpha))d\tau
\]

\[
= \int_{\mathbb{R}^n} \omega_n d\mu
\]

\[
= \int_{\mathbb{R}^n} \tilde{\omega}_t d\tilde{\mu}
\]

\[
= \lambda_{\ell}.
\]

Solving the initial value problem

\[
\frac{dy_{\ell-1}}{dt} = \tilde{\omega}_{\ell-1}(\chi_t^\#(\tilde{\alpha}))y_{\ell-1} + e_{\ell} r_{t-1}(t) e_{\ell} e_{\ell} \tilde{\omega}_t(\chi_\tau^\#(\tilde{\alpha}))d\tau, \quad y_{\ell-1}(0, e_{\ell-1}) = e_{\ell-1}
\]

we get

\[
y_{\ell-1}(t, e_{\ell-1}) = e_{\ell-1} e_{\ell} e_{\ell} \tilde{\omega}_{\ell-1}^{\ast}(\chi_t^\#(\tilde{\alpha}))d\tau + e_{\ell} e_{\ell} e_{\ell} e_{\ell} \tilde{\omega}_{\ell-1}^{\ast}(\chi_t^\#(\tilde{\alpha}))d\tau \int_{0}^{t} r_{t-1}(\tau) e_{\ell-1} e_{\ell} e_{\ell} \tilde{\omega}_t(\chi_\tau^\#(\tilde{\alpha}))d\tau e_{\ell} e_{\ell} e_{\ell} e_{\ell} \tilde{\omega}_{\ell-1}(\chi_\tau^\#(\tilde{\alpha}))d\tau d\tau.
\]

Thus,

\[
\lim_{t \to \infty} \frac{1}{t} \log |y_{\ell-1}(t, e_{\ell})| = \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} \tilde{\omega}_{\ell-1}(\chi_t^\#(\tilde{\alpha}))d\tau
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} \omega_{n-1}(\chi_\tau^\#(\alpha))d\tau
\]

\[
= \int_{\mathbb{R}^n} \omega_{n-1} d\mu
\]

\[
= \int_{\mathbb{R}^n} \tilde{\omega}_{\ell-1} d\tilde{\mu}
\]

\[
= \lambda_{\ell-1}.
\]

By repeating this procedure we will obtain:

\[
\lim_{t \to \infty} \frac{1}{t} \log |y_{j}(t, e_{j})| = \lambda_{j},
\]

for \( j = 1, \ldots, \ell \).

From the form of the functions \( y_{\ell}(t, e_{\ell}), \ldots, y_{1}(t, e_{1}) \), which depend linearly on the initial values \( e_{1}, e_{2}, \ldots, e_{\ell} \), we get easily

\[\|\tilde{y}(t, e_{i})\| = |y_{i}(t, e_{i})|\].
Therefore
\[ \lim_{t \to \infty} \frac{1}{t} \log \|\tilde{y}(t, e_i)\| = \lim_{t \to \infty} \frac{1}{t} \log |y_i(t, e_i)| = \lambda_i. \]

This proves the proposition and thus proves parts (1.) and (2.) of the Main Theorem.

5 Proof of the Main Theorem (3.)

In this section we will complete the proof of the Main Theorem.

Let \( \nu \in E(M^n, \phi) \) denote the given probability in the Main Theorem. Let \( \mu \in E(F^\#, \chi^\#) \) and \( \tilde{\mu} \in E(F^\#, \chi^\#) \) be the covering probabilities, and let \( \Lambda \subset M^n \) and \( W \subset F^\# \) be the two total probability sets, \( q^\#_\nu(W) = \Lambda, \) as in Proposition 3.4. Write

\[ \vartheta_i(\tilde{\mu}) = \int_{F^\#} \tilde{\omega}_i d\tilde{\mu}, \quad i = 1, 2, \ldots, \ell. \]

Then we have that \( \vartheta \) satisfies:

\[ \vartheta_1(\tilde{\mu}) = \lambda_1 < \vartheta_2(\tilde{\mu}) = \lambda_2 < \ldots < \vartheta_\ell(\tilde{\mu}) = \lambda_\ell. \]

Let \( T_1 \geq 1 \) be a fixed constant and let \( T_{i+1} = 2T_i, \ i = 1, 2, \ldots. \)

Recall from Section 3 the projection map (3.3).

Definition 5.1 For \( \eta > 0 \) we denote by \( D(\vartheta, \eta) \) the set of all \( \tilde{\gamma} \in id_\ell(W) \) with the property that for each integer \( i \geq 1 \) there exist an integer \( c = c(\tilde{\gamma}, i, \eta) \geq i \) and a sequence

\[ \ldots < s(-2) < s(-1) < s(0) = 0 < s(1) < s(2) < \ldots \]

such that

\[ \lim_{j \to -\infty} s(j) = -\infty, \quad \lim_{j \to +\infty} s(j) = +\infty, \]

for

\[ 1 \leq \left| \frac{\delta T}{l} \sum_{\tau=0}^{l-1} \max_{k=1,2,\ldots,\ell} \left| \vartheta_k(\tilde{\mu}) - \frac{1}{\delta T} \int_{s \delta T}^{(s+1)\delta T} \tilde{\omega}_k(\chi^\#_{c+s(j)T}(\tilde{\gamma})) dt \right| < \eta, \]

\[ l = 1, 2, \ldots; \ j = 0, \pm 1, \pm 2, \ldots; \delta = \pm 1. \]

Lemma 5.2 \( \tilde{\mu}(D(\vartheta, \eta)) > 0. \)

Proof. This is a partial result of [8, Theorem 2.1], where Liao gave a general proof. We present a proof of our case here for convenience to readers. Set

\[ h_k(\tilde{\gamma}, T, \delta) = |\vartheta_k(\tilde{\mu}) - \frac{1}{\delta T} \int_{0}^{\delta T} \tilde{\omega}_k(\chi^\#_{c+s(j)T}(\tilde{\gamma})) dt|, \ 0 < T < +\infty, \ \delta = \pm 1, \ \tilde{\gamma} \in F^\#, \]

\[ h(\tilde{\gamma}, T, \delta) = \max_{k=1,2,\ldots,\ell} h_k(\tilde{\gamma}, T, \delta). \]
When \( \tilde{\gamma} \in id_\ell(W) \) we choose and fix \( \gamma \in W \) with \( id_\ell(\gamma) = \tilde{\gamma} \). We have from Proposition 3.4, for \( k = 1, 2, \ldots, \ell, \)

\[
\lim_{T \to \pm \infty} \frac{1}{T} \int_0^T \tilde{\omega}_k(\chi_t^\#(\tilde{\gamma})) dt = \lim_{T \to \pm \infty} \frac{1}{T} \int_0^{T} \omega_{n-\ell+k}(\chi_t^\#(\gamma)) dt \\
= \vartheta_k(\tilde{\mu}) \\
= \lambda_k.
\]

By Chapter 6 in [9],

\[
\lim_{T \to \infty} \int_{F_\ell^\#} \left| \partial_k(\tilde{\mu}) - \frac{1}{\delta T} \int_{\alpha}^{\alpha+T} \tilde{\omega}_k(\chi_t^\#(\tilde{\gamma})) dt \right| \tilde{d}\tilde{\mu} = 0,
\]

where the convergence is uniform with respect to the choice of \( \alpha \in \mathbb{R} \). For \( \delta = 1 \) and \( \alpha = 0 \) we get

\[
\lim_{T \to \infty} \int_{F_\ell^\#} h_k(\tilde{\gamma}, T, +1) \tilde{d}\tilde{\mu} = 0, \quad k = 1, \ldots, \ell.
\]

For \( \delta = -1 \), taking \( \alpha = -T \), we get

\[
\lim_{T \to \infty} \int_{F_\ell^\#} h_k(\tilde{\gamma}, T, -1) \tilde{d}\tilde{\mu} = 0, \quad k = 1, \ldots, \ell.
\]

Therefore

\[
\lim_{T \to \infty} \int_{F_\ell^\#} h(\tilde{\gamma}, T, \delta) \tilde{d}\tilde{\mu} = 0, \quad \delta = \pm 1.
\]

For \( \eta > 0 \) one can thus take an integer \( d = d(\eta) > 0 \) such that

\[
\int_{F_\ell^\#} h(\tilde{\gamma}, T_d, \delta) \tilde{d}\tilde{\mu} < \eta
\]

\[ \text{for } \delta = \pm 1. \]

Let us consider a \( \tilde{\mu} \)-preserving homeomorphism \( \rho = \chi_T^\#: F_\ell^\# \to F_\ell^\# \). Applying the Birkhoff Ergodic Theorem to the homeomorphism \( \rho^\delta \) and continuous function \( h(\tilde{\gamma}; T, \delta), \delta = \pm 1 \), there is a \( \tilde{\mu} \)-measurable subset \( X \subset F_\ell^\# \) with \( \tilde{\mu}(X) = 1 \) such that for any \( \tilde{\gamma} \in X, \delta = \pm 1 \), the following limit exists

\[
\lim_{l \to \infty} \frac{1}{l} \sum_{\tau=0}^{l-1} h(\rho^{\delta\tau}(\tilde{\gamma}), T_d, \delta) = h^*(\tilde{\gamma}, T_d, \delta).
\]

Moreover,

\[
\int_{F_\ell^\#} h^*(\tilde{\gamma}, T_d, \delta) \tilde{d}\tilde{\mu} = \int_{F_\ell^\#} h(\tilde{\gamma}, T_d, \delta) \tilde{d}\tilde{\mu} < \frac{\eta}{30}, \quad \delta = \pm 1.
\]

This implies that the set \( \{ \tilde{\gamma} \in F_\ell^\# | h^*(\tilde{\gamma}, T_d, \delta) > \frac{\eta}{2} \} \) is \( \tilde{\mu} \)-measurable and has \( \tilde{\mu} \)-probability less than or equal to \( \frac{1}{12}, \delta = \pm 1 \). Applying Egoroff’s Theorem (see, for example, [3]) there exists a subset \( Y \) of \( X \) with \( \tilde{\mu}(Y) \geq \frac{1}{2} > 0 \) such that

\[
\frac{1}{l} \sum_{\tau=0}^{l-1} h(\rho^{\delta\tau}(\tilde{\gamma}), T_d, \delta) < \eta, \quad \forall \tilde{\gamma} \in Y, l \geq \bar{l}.
\]
Therefore
\[
\frac{1}{l} \sum_{\tau = 0}^{l-1} \max_{k=1,2,\ldots,l} \left| \vartheta_k(\tilde{\mu}) - \frac{1}{\delta T_d} \int_{\delta \tau T_d}^{\delta(T+1)T_d} \tilde{\omega}_k(\chi_{t+\delta \tau T_d}(\tilde{\gamma})) dt \right| < \eta,
\]
or
\[
\frac{1}{l} \sum_{\tau = 0}^{l-1} \max_{k=1,2,\ldots,l} \left| \vartheta_k(\tilde{\mu}) - \frac{1}{\delta T_d} \int_{\delta \tau T_d}^{\delta(T+1)T_d} \tilde{\omega}_k(\chi_{t+\delta \tau T_d}(\tilde{\gamma})) dt \right| < \eta, \quad \forall \tilde{\gamma} \in Y, \quad l \geq \tilde{l}.
\]
From the Poincaré Recurrence Theorem, let us take a subset \( Y' \) of \( Y \), \( \tilde{\mu}(Y') = \tilde{\mu}(Y) \), with the property that for each \( \tilde{\gamma} \in Y' \) there exists a sequence \( \{ s(j) \} \) of the form (5.1), so that \( \rho \varphi(s(j))(\tilde{\gamma}) \in Y \).

This gives rise to
\[
\frac{1}{l} \sum_{\tau = 0}^{l-1} \max_{k=1,2,\ldots,l} \left| \vartheta_k(\tilde{\mu}) - \frac{1}{\delta T_d} \int_{\delta \tau T_d}^{\delta(T+1)T_d} \tilde{\omega}_k(\chi_{t+\delta \tau T_d}(\tilde{\gamma})) dt \right| < \eta, \quad l \geq \tilde{l}, \quad j = 0, \pm 1, \pm 2, \ldots, \quad \delta = \pm 1.
\]

Denote by \( \xi(\gamma) \) a character function for \( Y \) on \( \mathcal{F}_d^\# \). Let us consider a \( \tilde{\mu} \)-preserving homeomorphism
\[
\psi : \mathcal{F}_d^\# \rightarrow \mathcal{F}_d^\#.
\]
Set
\[
\bar{\xi}(\tilde{\gamma}, \psi, \delta) := \limsup_{l \to \infty} \frac{1}{l} \sum_{\tau = 0}^{l-1} \xi(\psi^{\delta \tau}(\tilde{\gamma})), \quad \tilde{\gamma} \in \mathcal{F}_d^\#, \quad \delta = \pm 1.
\]
Then \( \bar{\xi} \) is a Baire function. Let
\[
E(\eta, \psi, \delta) = \{ \tilde{\gamma} \in \mathcal{F}_d^\# | \bar{\xi}(\tilde{\gamma}, \psi, \delta) > 0 \},
\]
\[
E(\eta, \psi) = E(\eta, \psi, -1) \cap E(\eta, \psi, +1).
\]
By the Birkhoff Ergodic Theorem there exists a subset \( Z \subset \mathcal{F}_d^\# \), \( \tilde{\mu}(Z) = 1 \), such that for all \( \tilde{\gamma} \in Y \cap Z \), the limit exists
\[
\lim_{l \to \infty} \frac{1}{l} \sum_{\tau = 0}^{l-1} \xi(\psi^{\delta \tau}(\tilde{\gamma})) = \xi^*(\tilde{\gamma}, \psi, \delta).
\]
Since
\[
1 \geq \bar{\xi}(\tilde{\gamma}, \psi, \delta) \geq \xi^*(\tilde{\gamma}, \psi, \delta) \geq 0,
\]
then
\[
\mu(E(\eta, \psi, \delta)) \geq \int \bar{\xi}(\tilde{\gamma}, \psi, \delta) d\tilde{\mu} \geq \int \xi^*(\tilde{\gamma}, \psi, \delta) d\tilde{\mu} = \tilde{\mu}(Y \cap Z) = \tilde{\mu}(Y) \geq \frac{3}{4},
\]

15
for both $\delta = 1$ and $\delta = -1$, which implies then $\bar{\mu}(E(\eta, \psi)) \geq \frac{1}{2}$.

Now for each integer $i \geq 1$ take $\psi$ as

$$\psi_i = \chi_{T_c}$$

where $T = T_c$, $c = c(i, \eta) \geq 1$ with $2^{c(i, \eta)} - d \geq \bar{l}$. Moreover we take $c(i, \eta) < c(i + 1, \eta)$, $i = 1, 2, 3, \ldots$. Then $T_{c(i, \eta)} = 2^{c(i, \eta)} - d T_d \geq \bar{l} T_d$. Write $F(\eta, \psi_i) = Y' \cap E(\eta, \psi_i)$ then $\bar{\mu}(F(\eta, \psi_i)) \geq \frac{1}{4}$. Take $\tilde{\gamma} \in F(\eta, \psi_i)$. Then there is a sequence $\{s(j)\}$ of the form (5.1) such that $\xi(\psi_i^{s(j)}(\tilde{\gamma})) = 1$, namely, $\psi_i^{s(j)}(\tilde{\gamma}) \in Y$, $j = 0, \pm 1, \pm 2, \ldots$. Recall by definition $T_c = 2^{c-d} T_d$. We have

$$\frac{1}{l} \sum_{k=1}^{l-1} \max_{\tau=0}^{l-1} \left| \vartheta_{k}(\mu) - \frac{1}{T_c} \int_{\delta \tau T_c}^{\delta (\tau + 1) T_c} \bar{\omega}_k(\chi_{T_c}^{\#}(\psi^{s(j)}(\tilde{\gamma}))) d\tau \right|$$

$$= \frac{1}{l} \sum_{k=1}^{l-1} \max_{\tau=0}^{l-1} \left| \vartheta_{k}(\mu) - \frac{1}{T_d} \int_{\delta \tau T_d}^{\delta (\tau + 1) T_d} \bar{\omega}_k(\chi_{T_d}^{\#}(\psi^{s(j)}(\tilde{\gamma}))) d\tau \right|$$

$$\leq \frac{1}{2^{c-d}} \sum_{k=1}^{l-1} \max_{\tau=0}^{l-1} \left| \vartheta_{k}(\mu) - \frac{1}{T_d} \int_{\delta \tau T_d}^{\delta (\tau + 1) T_d} \bar{\omega}_k(\chi_{T_d}^{\#}(\psi^{s(j)}(\tilde{\gamma}))) d\tau \right|$$

$$= \frac{1}{2^{c-d}} \sum_{k=1}^{l-1} \max_{\tau=0}^{l-1} \left| \vartheta_{k}(\mu) - \frac{1}{T_d} \int_{\delta \tau T_d}^{\delta (\tau + 1) T_d} \bar{\omega}_k(\chi_{T_d}^{\#}(\psi^{s(j)}(\tilde{\gamma}))) d\tau \right|$$

$$= \frac{1}{2^{c-d}} \sum_{k=1}^{l-1} \max_{\tau=0}^{l-1} h(\rho^{\delta \tau}(\psi^{s(j)}(\tilde{\gamma})), T_d, \delta)$$

$$< \eta,$$

where $\tilde{\gamma} \in F(\eta, \psi_i)$; $l = 1, 2, \ldots$, $j = 0, \pm 1, \pm 2, \ldots$, and $\delta = \pm 1$. Letting $D(\vartheta, \eta) := F(\eta, \psi_i)$ we complete the proof of Lemma 5.1. \hfill \Box

**Corollary 5.3** Set $F(\eta) = \bigcap_{i=1,2,\ldots} F(\eta, \psi_i)$, where $F(\eta, \psi_i)$ is as in the proof of Lemma 5.2. Then $F(\eta)$ is a Borel subset. Since $\bar{\mu}(F(\eta, \psi_i)) \geq \frac{1}{4}$ and $c(i, \eta) < c(i + 1, \eta)$, and thus $F(\eta, \psi_i) \supset F(\eta, \psi_{i+1})$, $i = 1, 2, \ldots$, we then have

$$\mu(F(\eta)) \geq \frac{1}{4} > 0.$$

**Theorem 5.4** (Liao, [8]) Consider two systems

$$\frac{dy}{dt} = yC(t) + f(t, y), \ (t, y) \in \mathbb{R} \times \mathbb{R}^\ell \quad (5.2)$$

$$\frac{dy}{dt} = yC(t) \quad (t, y) \in \mathbb{R} \times \mathbb{R}^\ell. \quad (5.3)$$

Let the following (i)(ii) and (iii) hold.
(i). For any $t \in \mathbb{R}$, $C(t) = (c_{ij})_{\ell \times \ell}$ is a lower triangular $\ell \times \ell$ matrix. $C(t)$ is continuous with respect to $t$ and uniformly bounded.

(ii). There exist constants $\lambda > 0$, $T > 0$, $c = \frac{1}{10} \min \{1, \lambda\}$, and a bi-infinite sequence $\{s(j)\}$ of the form (5.4) so that for some integer $p \in \mathbb{N}$, $\ell > 0$ the following inequalities hold

$$-\lambda < \frac{1}{T} \sum_{\tau = 0}^{\ell-1} \max_{k=1,2,...,p} \frac{1}{\delta T} \int_{\delta \tau T}^{\delta(\tau+1)T} c_{kk}(t+s(j)T)dt < -\lambda + c,$$

$$\lambda - c < \frac{1}{T} \sum_{\tau = 0}^{\ell-1} \min_{k=p+1,...,\ell} \frac{1}{\delta T} \int_{\delta \tau T}^{\delta(\tau+1)T} c_{kk}(t+s(j)T)dt < \lambda$$

$$j = 0, \pm 1, \pm 2, \ldots; l = 1, 2, \ldots; \delta = \pm 1.$$

(iii). Vector function $f(t, y)$ is continuous with $(t, y)$ and is uniformly bounded and Lipschitz with respect to $y$.

Then, for each $u^* \in \mathbb{R}^\ell$ there exists uniquely $u \in \mathbb{R}^\ell$ so that the solutions $y(t, u^*)$ and $y(t, u)$ of the initial value problem (5.4), (6.3) with initial conditions $y(0; u^*) = u^*$ and $y(0; u) = u$, respectively, satisfy the following relation.

(a). There is a integer sequence

$$\ldots < m(-2) < m(-1) < m(0) = 0 < m(1) < m(2) < \ldots$$

$$\lim_{j \to -\infty} m(j) = -\infty, \quad \lim_{j \to +\infty} m(j) = +\infty$$

so that

$$\sup_{k \in \mathbb{Z}} \|y(m(k)T, u) - y(m(k)T, u^*)\| < \infty.$$  

(b). The map $\Delta^* : \mathbb{R}^\ell \to \mathbb{R}^\ell$, $u^* \to u$ is surjective.

(c). There exist constants $C^* > 0$ and $d > 0$ so that

$$\|y(t, \Delta^*(u^*)) - y(t, u^*)\| \leq C^* \exp(2c|t - s(j)T| + d), \quad j = 0, \pm 1, \pm 2, \ldots.$$  

Proof. (a) and (b) are Theorem 3.1 in [S], its Corollary 1 is (c).

Proof of the Main Theorem (3.). For $\nu \in E(M^n, \phi)$ let us consider all its $\ell$ nonzero Lyapunov exponents $\lambda_1 < \ldots < \lambda_p < \lambda_{p+1} < \ldots < \lambda_\ell$, where $\lambda_p < 0 < \lambda_{p+1}$. We recall again from Proposition 3.3.1 the covering probabilities $\mu \in E(F_\#, \chi^\#), \tilde{\mu} \in E(F_\#^\ell, \chi^\#), q_{\nu^*}(\mu) = \nu = q_{\tilde{\nu}}(\tilde{\mu})$, and the subsets $W \subset F_\#^\ell$ and $\Lambda \subset M^n$ with $q_{\nu^*}(W) = \Lambda$. And consider continuous functions $\tilde{\zeta}_\ell, \tilde{\omega}_k : F_\#^\ell \to \mathbb{R}$ as in Section 3. Take an arbitrary positive real $\lambda$ with $\lambda_p < \lambda < \lambda_{p+1}$ and

$$\lambda < \frac{1}{2} \min_{1 \leq i, j \leq \ell} \{|\lambda_i - \lambda_j|, |\lambda_i - 0|\}.$$
Write $c := \frac{1}{4} \min\{1, \lambda\}$ as in Theorem 5.4 and write $\eta := \frac{c}{4}$ as in Lemma 5.2. We take and fix an orthonormal $\ell$-frame

$$\tilde{\alpha} \in F(\eta),$$

where $F(\eta)$ is defined in the Corollary 5.3. Recall by construction $F(\eta) \subset id_\ell(W)$, one can take $\alpha \in W$ with $id_\ell(\alpha) = \tilde{\alpha}$. By using the moving orthonormal $n$-frame

$$\{\chi^\#_l(\alpha); t \in \mathbb{R}\}$$

we can construct as in Section 4 a reduced standard linear system (4.3) of $\ell$ differential equations. As in Section 4 we can prove the Main Theorem(i)(ii) with respect to this linear system of differential equations.

Now let us consider a perturbed system (1.5) where $f(t, y)$ is Lipschitz and uniformly bounded. Observe that the $kk$-th entry of the matrix $A_{\ell \times \ell}(t)$ is

$$a_{kk}(t) = \tilde{\omega}_k(\chi^\#_l(\tilde{\alpha})), \quad k = 1, 2, \ldots, \ell.$$

Since $\tilde{\alpha} \in F(\eta)$ and $p \leq \ell$ there exist, by Lemma 5.2 and Corollary 5.3 a positive number $T > 0$ and a sequence $\{s(j)\}$ of the form (5.1) such that

$$\frac{1}{t} \sum_{\tau=0}^{t-1} \max_{k=1,2,\ldots,p} |\vartheta_k(\tilde{\mu}) - \frac{1}{\delta T} \int_{\tau \delta T}^{(\tau+1)\delta T} \tilde{\omega}_k(\chi^\#_{l+s(j)}T(\tilde{\gamma}))dt| < \eta,$$

$l = 1, 2, \ldots; j = 0, \pm 1, \pm 2, \ldots; \delta = \pm 1$.

Observe $\lambda_k = \vartheta_k(\tilde{\mu}) < \lambda$, and so we get

$$\frac{1}{t} \sum_{\tau=0}^{t-1} \max_{k=1,2,\ldots,p} \frac{1}{\delta T} \int_{\tau \delta T}^{(\tau+1)\delta T} \tilde{\omega}_k(\chi^\#_{l+s(j)}T(\tilde{\gamma}))dt - \lambda$$

$$= \frac{1}{t} \sum_{\tau=0}^{t-1} \max_{k=1,2,\ldots,p} \left( \frac{1}{\delta T} \int_{\tau \delta T}^{(\tau+1)\delta T} \tilde{\omega}_k(\chi^\#_{l+s(j)}T(\tilde{\gamma}))dt - \lambda \right)$$

$$\leq \frac{1}{t} \sum_{\tau=0}^{t-1} \max_{k=1,2,\ldots,p} \left( \frac{1}{\delta T} \int_{\tau \delta T}^{(\tau+1)\delta T} \tilde{\omega}_k(\chi^\#_{l+s(j)}T(\tilde{\gamma}))dt - \lambda_k \right)$$

$$\leq \frac{1}{t} \sum_{\tau=0}^{t-1} \max_{k=1,2,\ldots,p} |\vartheta_k(\tilde{\mu}) - \frac{1}{\delta T} \int_{\tau \delta T}^{(\tau+1)\delta T} \tilde{\omega}_k(\chi^\#_{l+s(j)}T(\tilde{\gamma}))dt|$$

$$\leq \eta$$

$$= \frac{c}{2}.$$
Therefore

\[-\lambda < \frac{1}{l} \sum_{\tau = 0}^{l-1} \max_{k=1,2,\ldots,p} \frac{1}{\delta T} \int_{\delta T}^{(\tau+1)T} \tilde{\omega}_k(\chi_{\tau+s(j)}^T(\gamma)) dt \leq -\lambda + c,\]

for \(j = 0, \pm 1, \pm 2, \ldots; l = 1, 2, \ldots; \delta = \pm 1.\) Similarly,

\[\lambda - c < \frac{1}{l} \sum_{\tau = 0}^{l-1} \min_{k=p+1,\ldots,\ell} \frac{1}{\delta T} \int_{\delta T}^{(\tau+1)T} \tilde{\omega}_k(\chi_{\tau+s(j)}^T(\gamma)) dt < \lambda,\]

for \(j = 0, \pm 1, \pm 2, \ldots; l = 1, 2, \ldots; \delta = \pm 1.\)

Now we apply Theorem 5.4 to complete the Main Theorem. Since \(\Delta^*\) in Theorem 5.4 is surjective, for

\[u_k = (0, \ldots, 0, 1(k), 0, \ldots, 0) \in \mathbb{R}^\ell\]

there exist \(u_k^* \in \mathbb{R}^\ell\) so that \(\Delta^*(u_k^*) = u_k, \ k = 1, \ldots, \ell.\) From Theorem 5.4 the solution \(y(t, u_k)\) of the initial value problem

\[\frac{dy}{dt} = yA_{\ell \times \ell}(t), \quad y(0, u_k) = u_k\]

and the solution \(y(t, u_k^*)\) of the initial value problem

\[\frac{dy}{dt} = yA_{\ell \times \ell}(t) + f(t, y), \quad y(0, u_k^*) = u_k^*\]

satisfy the relation

\[\|y(t, u_k^*) - y(t, u_k)\| \leq C^* \exp(2c(|t - s(j)T| + d))\]

for some constants \(C^* > 0\) and \(d > 0.\) Letting \(j = 0\) and thus \(s(j) = 0\) it follows

\[\|y(t, u_k^*)\| \leq \|y(t, u_k)\| + C^* \exp(2c|t| + d)\]

\[\leq \|y(t, u_k)\| \times C^* \exp(2c|t| + d)\]

for \(|t| \geq \bar{t} > 0.\) This yields by Proposition 4.2

\[\limsup_{t \to \infty} \frac{1}{t} \log \|y(t, u_k^*)\| \leq \limsup_{t \to \infty} \frac{1}{t} \log \|y(t, u_k)\| + 2c\]

\[= \lambda_k + 2c,\]

where we recall \(c = \frac{1}{16} \min\{1, \lambda\}.\) Since \(\lambda\) and thus \(c\) can be taken small enough, we get

\[\limsup_{t \to \infty} \frac{1}{t} \log \|y(t, u_k)\| \leq \lambda_k, \quad k = 1, 2, \ldots, \ell.\]

Now one can easily get

\[\|y(t, u_k)\| \leq \|y(t, u_k^*)\| \times C^* \exp(2c|t| + d)\]
for $|t| \geq \bar{t} > 0$. This gives rise to
\[
\lambda_k = \liminf_{t \to \infty} \frac{1}{t} \log \|y(t, u_k)\| \\
\leq \liminf_{t \to \infty} \frac{1}{t} \log \|y(t, u_k^*)\| + 2c.
\]
Thus
\[
\lambda_k - 2c < \liminf_{t \to \infty} \frac{1}{t} \log \|y(t, u_k^*)\| \\
\leq \limsup_{t \to \infty} \frac{1}{t} \log \|y(t, u_k)\| \\
< \lambda_k + 2c.
\]
Since $c$ can be taken small enough, we get
\[
\lim_{t \to \infty} \frac{1}{t} \log \|y(t, u_k^*)\| = \lambda_k, \quad k = 1, 2, \ldots, \ell.
\]
This completes the proof of the Main Theorem. □

**Example.** When $\ell < n - 1$, the system $(M^n, \phi, \nu)$ is not hyperbolic. In this case the Main Theorem does not hold for the linear system [7, Chapter 2]
\[
dy \frac{dt}{dt} = y R_\alpha(t)^T
\]
of $n$ first order differential equations based on $\alpha \in \mathcal{F}_n^\#$, where $R_\alpha(t)$ is defined as in Section 4 (see also [7, Chapter 2]). This is illustrated by the following example. Let $n = 2$, $\ell = 1$. Take $\alpha = (u_1, u_2)$ as in Section 3. Then the linear system based on $\alpha$ is
\[
\left( \begin{array}{c}
\frac{dy_1}{dt} \\
\frac{dy_2}{dt}
\end{array} \right) = (y_1, y_2) \left( \begin{array}{cc}
\omega_1(\chi_t^\#(\alpha)) \\
\omega_2(\chi_t^\#(\alpha))
\end{array} \right).
\]
We consider the case when $\lim_{t \to \pm \infty} \frac{1}{t} \int_0^t \omega_1(\chi_t^\#(\alpha)) dt = \lambda < 0$ and $\omega_2(\chi_t^\#(\alpha)) \equiv 0 \ \forall t \in \mathbb{R}$. Let us consider a perturbed system
\[
\left( \begin{array}{c}
\frac{dy_1}{dt} \\
\frac{dy_2}{dt}
\end{array} \right) = (y_1, y_2) \left( \begin{array}{cc}
\omega_1(\chi_t^\#(\alpha)) \\
\omega_2(\chi_t^\#(\alpha))
\end{array} \right) + \left( \begin{array}{c}a \\
a\end{array} \right),
\]
where $a > 0$ is a small constant. We get $y_2(t) = at$, and thus get
\[
0 > \lim_{t \to \pm \infty} \frac{1}{t} \log \|(y_1(t), y_2(t))\| \geq \lim_{t \to +\infty} \frac{1}{t} \log |at| = 0,
\]
which is a contradiction.
6 A persistence property for Liao perturbations

A nearby $C^1$ vector field, while perturbing a given one, keeps neither value nor sign of Lyapunov exponents, in general. However, if we perturb the given $C^1$ vector field by a “Liao perturbation”, we will show in this section that the perturbed vector field will keep both sign and value of the nonzero Lyapunov exponents. The class of Liao perturbations is constructed using the standard system of the given vector field.

Recall that $S$ is the $C^1$ vector field on $M^n$ given in Section 1. It reduces then in Section 2 the flows $\phi: M^n \to M^n$, $\chi^#: F^# \to F^#$. Let $\nu \in E(M^n, \phi)$ denote the probability in the Main Theorem. Let $\eta > 0$ be small and let $F(\eta)$ be as in the Corollary 5.3, $\hat{\mu}(F(\eta)) > \frac{1}{4}$. From Lemma 5.2, $F(\eta) \subset id_\ell(W)$. Recall from Section 3 the projection map $id_\ell: F^# \to F^#_\ell$.

Now we recall briefly the Liao standard system for a perturbation vector field [7, Chapter 2] with respect to the orthonormal $n$-frame $\beta$ we chose. Let us take and fix $x \in \Lambda$ and $\beta \in W$ so that $q^\#_n(\beta) = x$ and $\tilde{\beta} := id_\ell(\beta) \in F(\eta)$. Construct a standard map $P_\beta: \mathbb{R} \times \mathbb{R}^n \to M^n$

$$P_\beta(t,y) = \exp(\sum_{i=1}^n y^i \text{proj}_i \chi^#_t(\beta)), \quad y = (y^1, \ldots, y^n).$$

As $M^n$ is a compact $C^\infty$ Riemannian manifold, the exponential map $\exp: TM^n \to M^n$ is $C^\infty$ and there exists a constant $\zeta_0 > 0$ such that for any $x \in M^n$, $\exp$ maps $\{u \in T_x M^n | \|u\| < \zeta_0\}$ differentially into a neighborhood of $x$ on $M^n$. Let $X$ be a $C^1$ vector field, a perturbation to the given vector field $S$. Fixing $t \in \mathbb{R}$, there exists a unique tangent vector field $X_\beta(t,y)$ on

$$B_0 = \{y \in \mathbb{R}^n | \|y\| < \zeta_0\}$$

so that

$$dP_\beta(X_\beta(t,y)) = dP_\beta(0,X_\beta(t,y))$$

$$= X(P_\beta(t,y)) - dP_\beta(\frac{\partial}{\partial t}|_{t,y}).$$

The system

$$\frac{dy}{dt} = X_\beta(t,y)$$

can be written as

$$\frac{dy}{dt} = yR_\beta(t)^T + \tilde{f}(t,y), \quad (6.1)$$

where $R_\beta(t)^T = (r_{ij})_{n \times n}$ is defined in Section 4, $r_{ii}(t) = \omega_i(\chi^#_t(\beta))$, for $i = 1, \ldots, n$. The vector function $\tilde{f}(t,y)$ is bounded and Lipschitz. The system $\text{(6.1)}$, called the Liao standard system of
based on \((S,\nu)\), was employed by Liao to prove the \(C^1\) closing lemma \([7, \text{Appendix A}]\) and topological stability for Anosov flows \([7, \text{Chapter 2}]\).

Based on the Liao standard system, we now introduce the terminology of Liao perturbation to the given vector field \((M^n, S, \nu)\) in our Main Theorem. We define a triangular \(\ell \times \ell\) matrix \(A_{\ell \times \ell}(t) = (a_{ij}(t))_{\ell \times \ell}\) as follows: 

\[
a_{ij}(t) = 0 \text{ if } i < j; \quad a_{ij}(t) = r_{(n-\ell+i)(n-\ell+j)}(t) \text{ if } i > j; \quad a_{ii}(t) = \tilde{\omega}_i(\chi^#_i(\tilde{\beta})),
\]

\(i, j = 1, \ldots, \ell\). And define a vector function

\[
f_i(t, y) = f_{n-\ell+i}(t, (0, \ldots, 0(n-\ell), y^1, \ldots, y^\ell), \quad i = 1, \ldots, \ell.
\]

We then call the system

\[
\frac{dy}{dt} = yA_{\ell \times \ell}(t) + f(t, y),
\]

(6.2)

a reduced standard system of the perturbation vector field \(X\) based on \((M^n, S, \nu)\). Simply, we call the system \((6.2)\) a Liao perturbation of \((M^n, S, \nu)\).

From our Main Theorem we easily summarize the effect of Liao perturbations on nonzero Lyapunov exponents

**Theorem 6.1** Let \(S\) be a \(C^1\) vector field on \(M^n\) and let \(\nu \in E(M^n, \phi)\) be a probability that has \(\ell\) nonzero Lyapunov exponents \(\lambda_1 < \ldots < \lambda_\ell\) together with \(n - \ell\) zero Lyapunov exponents. Then there exists a \(C^1\) neighborhood \(X^1\) of \(S\) on the space of all \(C^1\) vector fields on \(M^n\), so that for each \(X \in X^1\), its reduced standard system \((6.2)\) based on \((S, \nu)\) has \(\lambda_1, \ldots, \lambda_\ell\) as Lyapunov exponents. In other words, Liao perturbation preserves the nonzero Lyapunov exponents.

**Remark 6.2** Because the Lyapunov exponents are constant on \(\Lambda\) and \(F(\eta) \subset id_{\ell}(W)\), from Proposition \([3.4]\) and the Main Theorem, Theorem 6.1 is independent of the choice of \(x \in \Lambda\) and \(\tilde{\beta} \in F(\eta)\) and thus the reduced standard system based on \((S, \nu)\).

**References**

[1] J. F. Alves, SRB measures for nonhyperbolic systems with multidimensional expansions, preprint, IMPA, Brasil, 1998.

[2] J. Bochi, Genericity of zero Lyapunov exponents, preprint, IMPA, Brazil, 2000.

[3] H. Fedrer, *Geometric measure theory*, Springer-Verlag, 1969.

[4] S. T. Liao, Certain ergodic property theorem for a differential systems on a compact differentiable manifold, *Acta Scientiarum Naturalium Universitatis Pekinesis* 9, 241–265, 309–327 (in Chinese) (1963). Its English version appears as Chapter 1 in the book of S. T. Liao, *Qualitative theory on differentiable dynamical systems*, Science Press, Beijing, New York, 1996.
[5] S. T. Liao, An ergodic property theorem for a differential system, *Science in China* **16**, 1-24 (1973).

[6] S. T. Liao, On characteristic exponents construction of a new Borel set for multiplicative ergodic theorem for vector fields, *Acta Scientiarum Naturalium Universitatis Pekinesis* **29**, 177-302 (1992).

[7] S. T. Liao, *Qualitative theory on differentiable dynamical systems*, Science Press, Beijing, New York, 1996

[8] S. T. Liao, Notes on a study of bundle dynamical systems(II), part 1, and part 2, *Appl. Math. Mechanics (English Edition)* **17**, 805-818 (1996), **18**, 421-440 (1997).

[9] B. B. Nemytskii, B. B. Stepanov, *Qualitative theory of differential equations*, Princeton University Press, 1960

[10] V. I. Oseledec, A multiplicative ergodic theorem, Lyapunov characteristic number for dynamical systems, *Trans. Moscow Math. Soci.* **19**, 197-231 (1968).

[11] Jacob Palis, A global view of dynamics and conjecture on the denseness of finitude of attractors, *Asterisque* **261**, 339-351 (1999).

[12] W. Sun, Characteristic spectrum for differential systems, *J. Diff. Equations* **147**, 184-194 (1998).

[13] W. Sun, Qualitative functions and characteristic spectra for diffeomorphisms, *Far East J. Appl. Math.* **2**, 169-182 (1998).

[14] W. Sun, Characteristic spectra for paralleloptope cocycles, in *Dynamical Systems*, World Scientific, Singapore, 256-265, 1999.

[15] W. Sun, Entropy of orthonormal n-frame flows, *Nonlinearity* **14**, 892-842 (2001).

[16] M. Viana, Multidimensional nonhyperbolic attractors, *Publ. Math. IHES* **85**, 63-96 (1997).

[17] L. -S. Young, Some open sets of nonuniformly hyperbolic cocycles, *Erg. Th. & Dynam. Sys. 13*, 409-415 (1993).

[18] L. -S. Young, Lyapunov exponents for some quasi-periodic cocycles, *Erg. Th. & Dynam. Sys. 17*, 483-501 (1997).