Lipschitz stability for the inverse conductivity problem for a conformal class of anisotropic conductivities

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Abstract

We consider the stability issue of the inverse conductivity problem for a conformal class of anisotropic conductivities in terms of the local Dirichlet–Neumann map. We extend here the stability result obtained by Alessandrini and Vessella (Alessandrini G and Vessella S 2005 Lipschitz stability for the inverse conductivity problem \textit{Adv. Appl. Math.} \textbf{35} 207–241), where the authors considered the piecewise constant isotropic case.

Keywords: Lipschitz stability, anisotropic conductivity, conformal class

1. Introduction

In the present paper we study the stability issue for the inverse conductivity problem in the presence of anisotropic conductivity which is \textit{a priori} known to depend linearly on an unknown piecewise-constant function. Let us start by recalling the basic formulation of the inverse conductivity problem.

In absence of internal sources, the electrostatic potential $u$ in a conducting body, described by a domain $\Omega \subset \mathbb{R}^n$, is governed by the elliptic equation

$$\text{div} (\sigma \nabla u) = 0 \quad \text{in} \quad \Omega,$$  

where the symmetric, positive definite matrix $\sigma = \sigma(x)$, $x \in \Omega$ represents the (possibly anisotropic) electric conductivity. The inverse conductivity problem consists of finding $\sigma$ when the so called Dirichlet–Neumann (D–N) map
\( \Lambda_\sigma : u \big|_{\partial \Omega} \in H^{1/2}(\partial \Omega) \to \sigma \nabla u \cdot \nu \big|_{\partial \Omega} \in H^{-1/2}(\partial \Omega) \)

is given for any \( u \in H^1(\Omega) \) solution to (1.1). Here, \( \nu \) denotes the unit outer normal to \( \partial \Omega \). If measurements can be taken only on one portion \( \Sigma \) of \( \partial \Omega \), then the relevant map is called the local D–N map. Let \( \Sigma \) be a non-empty open portion of \( \partial \Omega \) and let us introduce the subspace of \( H^{1/2}(\partial \Omega) \)

\[
H^1_0(\Sigma) = \left\{ f \in H^{1/2}(\partial \Omega) \mid \text{supp } f \subset \Sigma \right\}.
\]

The local D–N map is given, in its weak formulation, as the operator \( \Lambda_{\Sigma}^* \) such that

\[
\langle \Lambda_{\Sigma}^* u, \phi \rangle = \int_{\Omega} \sigma \nabla u \cdot \nabla \phi,
\]

for any \( u, \phi \in H^1(\Omega), u|_{\partial \Omega}, \phi|_{\partial \Omega} \in H^1_0(\Sigma) \) and \( u \) is a weak solution to (1.1).

The problem of recovering the conductivity of a body by taking measurements of voltage and current on its surface has come to be known as electrical impedance tomography (EIT). Different materials display different electrical properties, so that a map of the conductivity \( \sigma(x), x \in \Omega \) can be used to investigate internal properties of \( \Omega \). EIT has many important applications in fields such as geophysics, medicine and non-destructive testing of materials.

The first mathematical formulation of the inverse conductivity problem is due to Calderón [C], where he addressed the problem of whether it is possible to determine the (isotropic) conductivity \( \sigma = \gamma I \) by the D–N map. Although Calderón studied the problem of determining \( \sigma \) from the knowledge of the quadratic form

\[
Q_{\gamma}(u) = \int_{\Omega} \gamma |\nabla u|^2,
\]

where \( u \) is a solution to (1.1), it is well known that the knowledge of \( Q_{\sigma} \) is equivalent to the knowledge of \( \Lambda_{\sigma} \) by

\[
Q_{\gamma}(u) = \langle \Lambda_{\sigma} u, u \rangle, \quad \text{for every } u \in H^1(\Omega),
\]

where \( \sigma = \gamma I \). Here \( \langle \cdot, \cdot \rangle \) denotes the dual pairing between \( H^{1/2}(\partial \Omega) \) and its dual \( H^{-1/2}(\partial \Omega) \), with respect to the \( L^2 \) scalar product. Reference [C] opened the way to the solution to the uniqueness issue where one is asking whether \( \sigma \) can be determined by the knowledge of \( \Lambda_{\sigma} \) (or \( \Lambda_{\Sigma}^* \) in the case of local measurements). As main contributions in this respect we mention the papers by Kohn and Vogelius [K-Vo1, K-Vo2], Sylvester and Uhlmann [S-U] and Nachman [Na]. We refer to [Bo, Ch-I-N] and [U] for an overview of recent developments regarding the issues of uniqueness and reconstruction of the conductivity.

Regarding the stability, Alessandrini proved in [A] that, assuming \( n \geq 3 \) and \textit{a priori} bounds on \( \gamma \) of the form

\[
\|\gamma\|_{H^{s}(\Omega)} \leq E, \quad \text{for some } s > \frac{n}{2} + 2, \quad (1.4)
\]

\( \gamma \) depends continuously on \( \Lambda_{\sigma} \) with a modulus of continuity of logarithmic type. In [A1, A2] the same author subsequently proved that a similar stability estimate holds when the \textit{a priori} bound (1.4) is replaced by

\[
\|\gamma\|_{W^{1,\infty}(\Omega)} \leq E. \quad (1.5)
\]

Logarithmic type stability estimates have recently been proved for \( \gamma \in C^{1,\epsilon}(\bar{\Omega}) \) and \( n \geq 3 \) in [Ca-G-R]. For the two-dimensional case, logarithmic type stability estimates were obtained in [B-B-R, B-F-R, Liu]. Unfortunately, all the above results share the common
inconvenient logarithmic type of stability which cannot be avoided \[A3\]. In fact Mandache
\[Ma\] showed that the logarithmic stability is the best possible, in any dimension \(n \geq 2\) if
\(a\) priori assumptions of the form
\[
\|\gamma\|_{C^1(\Omega)} \leq E
\]
for any \(k = 0, 1, 2, \ldots\) are assumed. It seems therefore reasonable to think that, in order to
restore stability in a really (Lipschitz) stable fashion, one needs to replace in some way the
\(a\) priori assumptions expressed in terms of regularity bounds such as (1.6), with \(a\) priori
pieces of information of a different type. Alessandrini and Vessella showed in [A-V] that \(\gamma\)
depends in a Lipschitz continuous fashion upon the local D–N map, by assuming that \(\gamma\) is a
function \(a\) priori known to be piecewise constant
\[
\gamma(x) = \sum_{j=1}^{N} \gamma_j \chi_{D_j}(x),
\]
where each subdomain of \(\Omega\), \(D_j, j = 1, \ldots, N\) is given and each number \(\gamma_j, j = 1, \ldots, N\) is
unknown. From a medical imaging point of view, each \(D_j\) may represent the area occupied by
different tissues or organs and one can think that the geometrical configuration of each \(D_j\) is
given by means of other imaging techniques such as MRI for example. Since most tissues in the
human body are anisotropic, the present authors, motivated by the work in [A-V] and its
medical application, consider here the more general case of an anisotropic conductivity of type
\[
\sigma(x) = \gamma(x) A(x),
\]
where \(A(x)\) is a known, matrix valued function which is Lipschitz continuous and \(\gamma(x)\) is
of type (1.7). Anisotropic conductivity appears in nature, for example as a homogenization
limit in layered or fibrous structures such as rock stratum or muscle, as a result of crystalline
structure or of deformation of an isotropic material, therefore the case treated in this paper
seems to be a natural extension of [A-V] relevant to several applications.

It is well known that since Tartar’s observation [K-Vo1] that any diffeomorphism of \(\Omega\)
which keeps the boundary points fixed has the property of leaving the D–N map unchanged,
whereas \(\sigma\) is modified, different lines of research have been pursued. One direction has been
to find the conductivity up to a diffeomorphism which keeps the boundary fixed (see
\[LU, Na, S, La-U\]). Another direction has been the one to assume that the anisotropic
conductivity is \(a\) priori known to depend on a restricted number of spatially-dependent
parameters (see \[A, A-G, A-G1, G-L, L, \]). Nevertheless the problem of proving uniqueness
for anisotropic conductivities still represents an open problem in dimension \(n \geq 3\), where
the two-dimensional case can be considered settled, in fact it has been proved in [A-La-P] that an
anisotropic \(L^\infty\)-conductivity can be determined in dimension 2 up to a \(W^{1,2}\)-diffeomorphism.

The present work is concerned about the stability issue for the Calderon problem and it
represents the first result of Lipschitz stability estimates for conductivities of anisotropic type
as far as the authors know, therefore the simplest possible extension to the work done in [A-V]
to anisotropic conductivities has been considered in this manuscript. The paper improves
upon the results obtained in [A-V] in the sense that the global Lipschitz stability estimate
obtained there is here adapted to a special anisotropic type of conductivity. The class of
anisotropic conductivities considered here includes some piecewise constant anisotropic ones
but not all. This is due to the fact that the construction of the fundamental solution for the
conductivity equation with matrix valued constant coefficients (see (4.81)) would require a
different type of study to the one carried out in [A-V] and in the present work. The precise
assumptions shall be illustrated in section 2. The authors hope that the stability estimate
obtained here for this simple case of anisotropy will stimulate further study of the stability issue for anisotropic conductivities.

For related results in the anisotropic case we also refer to [Be, D-Ke-S-U, D-Ku-L-S, F-K-R, H-S, K, N-S]. We also recall [Be-Fr, Be-Fr-V, B-dH-Q] where similar Lipschitz stability results have been obtained for complex conductivity, the Lamé parameters and for a Schrödinger type of equation respectively.

For a more in-depth description and consideration of the stability issue and related open problems in the inverse conductivity problem we refer to [A3] and [A-V].

Our approach follows the one by Alessandrini and Vessella [A-V] of constructing singular solutions and studying their asymptotic behaviour when the singularity approaches the discontinuity interfaces for the conductivity. However, in order to deal with the present structure of the conductivity we had to develop original asymptotic analysis estimates and an accurate quantitative control of the error terms which represent a novel feature in the treatment of anisotropic type of conductivity.

The paper is organized as follows. Our main assumptions and our main result (theorem 2.1) are contained in section 2, where the proof of theorem 2.1 is contained in section 3. This section also lists the two main results (theorem 3.4 and proposition 3.5) needed to build the machinery for the proof of theorem 2.1. Theorem 3.4 provides original asymptotic estimates for the Green function of the conductivity equation, for conductivities belonging to a special anisotropic conformal class \( \mathcal{C} \), at the interfaces between the given domains \( D_i \) where the conductivity is discontinuous. Proposition 3.5 provides estimates of unique continuation of the solution to the conductivity equation for conductivities in \( \mathcal{C} \). Section 4 is devoted to the proofs of theorem 3.4 and proposition 3.5. For the proof of theorem 3.4 we provide the explicit form of the fundamental solution for the conformal anisotropic two-phase case with flat interface. The proof of proposition 3.5 is a straightforward consequence proposition 4.3 which we state in this section. The proof of the latter is independent from the presence of anisotropy in the conductivity, therefore we refer to [A-V] for a full proof of it. In this paper we point out the main facts on which the proof is based on only.

2. Main result

2.1. Notation and definitions

In several places within this manuscript it will be useful to single out one coordinate direction. To this purpose, the following notations for points \( x \in \mathbb{R}^n \) will be adopted. For \( n \geq 3 \), a point \( x \in \mathbb{R}^n \) will be denoted by \( x = (x', x_n) \), where \( x' \in \mathbb{R}^{n-1} \) and \( x_n \in \mathbb{R} \). Moreover, given a point \( x \in \mathbb{R}^n \), \( B_r(x), B'_r(x) \) the open balls in \( \mathbb{R}^n, \mathbb{R}^{n-1} \) centered at \( x, x' \), respectively with radius \( r \) and by \( Q_r(x) \) the cylinder

\[
Q_r(x) = B'_r(x') \times (x_n - r, x_n + r).
\]

We shall also denote

\[
\mathbb{R}_+^n = \{(x', x_n) \in \mathbb{R}^n | x_n > 0 \}; \quad \mathbb{R}_-^n = \{(x', x_n) \in \mathbb{R}^n | x_n < 0 \};
\]

\[
B_r^+ = B_r \cap \mathbb{R}_+^n; \quad B_r^- = B_r \cap \mathbb{R}_-^n;
\]

\[
Q_r^+ = Q_r \cap \mathbb{R}_+^n; \quad Q_r^- = Q_r \cap \mathbb{R}_-^n.
\]

where \( B_r, Q_r \) denote \( B_r(0), Q_r(0) \), respectively.

In the sequel, we shall make a repeated use of quantitative notions of smoothness for the boundaries of various domains. Let us introduce the following notation and definitions.
Definition 2.1. Let $\Omega$ be a domain in $\mathbb{R}^n$. We say that a portion $\Sigma \subseteq \partial \Omega$ is of Lipschitz class with constants $r_0, L$ if for any $P \in \Sigma$ there exists a rigid transformation of $\mathbb{R}^n$ under which we have $P \equiv 0$ and

$$\Omega \cap Q_{r_0} = \left\{ x \in Q_{r_0} : x_0 = \varphi(x') \right\},$$

where $\varphi$ is a Lipschitz function on $B_{r_0}^c$ satisfying

$$\varphi(0) = \left| \nabla \varphi(0) \right| = 0; \quad \| \varphi \|_{C^1(B_{r_0})} \leq L r_0.$$

It is understood that $\partial \Omega$ is of Lipschitz class with constants $r_0, L$ as a special case of $\Sigma$, with $\Sigma = \partial \Omega$. Here and in the sequel $B_{r_0}^c$ denotes $B_{r_0}(0)$.

Definition 2.2. Let $\Omega$ be a domain in $\mathbb{R}^n$. Given $\alpha, \alpha \in (0, 1]$, we say that a portion $\Sigma \subseteq \partial \Omega$ is of class $\alpha$ with constants $r_0, M$ if for any $P \in \Sigma$ there exists a rigid transformation of $\mathbb{R}^n$ under which we have $P = 0$ and

$$\Omega \cap Q_{r_0} = \left\{ x \in Q_{r_0} : x_0 = \varphi(x') \right\},$$

where $\varphi$ is a $C^{1, \alpha}$ function on $B_{r_0}^c$ satisfying

$$\varphi(0) = \left| \nabla \varphi(0) \right| = 0; \quad \| \varphi \|_{C^{1, \alpha}(B_{r_0})} \leq M r_0,$$

where we denote

$$\| \varphi \|_{C^{1, \alpha}(B_{r_0})} = \| \varphi \|_{L^\infty(B_{r_0})} + n \| \nabla \varphi \|_{L^\infty(B_{r_0})} + n \alpha + \sup_{x, y \in B_{r_0}, x \neq y} \frac{\| \varphi(x) - \varphi(y) \|}{|x - y|^\alpha}.$$ 

Let us rigorously define the local D–N map.

Definition 2.3. Let $\Omega$ be a domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial \Omega$ and $\Sigma$ an open non-empty subset of $\partial \Omega$. Assume that $\sigma \in L^\infty(\Omega, \text{Sym}_n)$ satisfies the ellipticity condition

$$\lambda |\xi|^2 \leq \sigma(x) \xi \cdot \xi \leq \Lambda |\xi|^2, \quad \text{for almost every } x \in \Omega, \quad \text{for every } \xi \in \mathbb{R}^n. \quad (2.1)$$

The local D–N map associated to $\sigma$ and $\Sigma$ is the operator

$$\Lambda^\Sigma_{\sigma} : H^{1/2}_{co}(\Sigma) \rightarrow H^{-1/2}_{co}(\Sigma) \quad (2.2)$$

defined by

$$\langle \Lambda^\Sigma_{\sigma} g, \eta \rangle = \int_{\Omega} \sigma(x) \nabla u(x) \cdot \nabla \phi(x) \, dx, \quad (2.3)$$

for any $g, \eta \in H^{1/2}_{co}(\Sigma)$, where $u \in H^1(\Omega)$ is the weak solution to

\[
\begin{aligned}
\div (\sigma(x) \nabla u(x)) &= 0, & & \text{in } \Omega, \\
u &= g, & & \text{on } \partial \Omega,
\end{aligned}
\]

and $\phi \in H^1(\Omega)$ is any function such that $\phi|_{\partial \Omega} = \eta$ in the trace sense. Here we denote by $\langle \cdot, \cdot \rangle$ the $L^2(\partial \Omega)$-pairing between $H^{1/2}_{co}(\Sigma)$ and its dual $H^{-1/2}_{co}(\Sigma)$.

Note that, by (2.3), it is easily verified that $\Lambda^\Sigma_{\sigma}$ is selfadjoint. We shall denote by $\| \cdot \|_*$ the norm on the Banach space of bounded linear operators between $H^{1/2}_{co}(\Sigma)$ and $H^{-1/2}_{co}(\Sigma)$.
2.2. Our assumptions

We give here the precise assumptions for the domain $\Omega$ under investigation and its conductivity $\sigma$. The dimension of the space for $\Omega$ is denoted by $n$ and for sake of simplicity we only consider $n \geq 3$.

2.2.1. Assumptions about the domain $\Omega$.

(1) We assume that $\Omega$ is a domain in $\mathbb{R}^n$ satisfying
\[ |\Omega| \leq Nr_0^2, \]  
where $|\Omega|$ denotes the Lebesgue measure of $\Omega$.

(2) We assume that $\partial \Omega$ is of Lipschitz class with constants $r_0, L$.

(3) We fix an open non-empty subset $\Sigma$ of $\partial \Omega$ (where the measurements in terms of the local D–N map are taken).

(4) $\bar{\Omega} = \bigcup_{j=1}^N D_j$, where $D_j, j = 1, \ldots, N$ are known open sets of $\mathbb{R}^n$, satisfying the conditions below.
(a) $D_j, j = 1, \ldots, N$ are connected and pairwise nonoverlapping.
(b) $\partial D_j, j = 1, \ldots, N$ are of Lipschitz class with constants $r_0, L$.
(c) There exists one region, say $D_1$, such that $\partial D_1 \cap \Sigma$ contains a $C^{1,\alpha}$ portion $\Sigma_1$ with constants $r_0, M$.
(d) For every $i \in \{2, \ldots, N\}$ there exists $j_{i-1}, \ldots, j_k \in \{1, \ldots, N\}$ such that
$$D_{j_i} = D_1, \quad D_{j_k} = D_i.$$  
In addition we assume that, for every $k = 1, \ldots, K$, $\partial D_k \cap \partial D_{k-1}$ contains a $C^{1,\alpha}$ portion $\Sigma_k$ (here we agree that $D_{j_0} = \mathbb{R}^n \setminus \Omega$), such that
$$\Sigma_1 \subset \Sigma,$$
and, for every $k = 1, \ldots, K$, there exists $P_k \in \Sigma_k$ and a rigid transformation of coordinates under which we have $P_k = 0$ and
$$\Sigma_k \cap Q_{\eta/3} = \left\{ x \in Q_{\eta/3} \mid x_n = \phi_k(x') \right\},$$
$$D_{j_k} \cap Q_{\eta/3} = \left\{ x \in Q_{\eta/3} \mid x_n > \phi_k(x') \right\},$$
$$D_{j_{k-1}} \cap Q_{\eta/3} = \left\{ x \in Q_{\eta/3} \mid x_n < \phi_k(x') \right\},$$  
where $\phi_k$ is a $C^{1,\alpha}$ function on $B_{\eta/3}$ satisfying
$$\phi_k(0) = \left| \nabla \phi_k(0) \right| = 0$$
and
$$\| \phi_k \|_{C^{1,\alpha}(B_\eta)} \leq M_{\eta_0}.$$
2.2.2. A priori information on the conductivity $\gamma$: the class $C$

**Definition 2.4.** We shall say that $\sigma \in C$ if $\sigma$ is of type

$$\sigma(x) = \sigma_A(x) = \sum_{j=1}^{N} \gamma_j A(x)_{\chi D_j}(x), \quad x \in \Omega,$$

(2.7)

where $\gamma_j$ are unknown real numbers, $D_j, j = 1, \ldots, N$ are the given subdomains introduced in section 2.2.1 and

$$\tilde{\gamma} \leq \gamma_j \leq \tilde{\gamma}^{-1}, \quad \text{for any } j = 1 \ldots N. \quad (2.8)$$

$A(x)$ is a known Lipschitz matrix valued function satisfying

$$\|A\|_{C^0(\Omega)} \leq \tilde{A}, \quad (2.9)$$

where $\tilde{A} > 0$ is a constant and

$$\lambda^{-1} |\xi|^2 \leq A(x) \xi \cdot \xi \leq \lambda |\xi|^2, \quad \text{for almost every } x \in \Omega,$n

$$\text{for every } \xi \in \mathbb{R}^n. \quad (2.10)$$

**Definition 2.5.** Let $N, \alpha_0, L, M, \alpha, \lambda, \tilde{\gamma}, \tilde{A}$ be given positive numbers with $N \in \mathbb{N}$ and $\alpha \in (0, 1]$. We will refer to this set of numbers, along with the space dimension $n$, as to the a priori data. Our main result is the following.

**Theorem 2.1.** Let $\Omega, D_j, j = 1, \ldots, N$ and $\Sigma$ be a domain, $N$ subdomains of $\Omega$ and a portion of $\partial \Omega$ as in section 2.2.1 respectively. If $\sigma_A^{(i)} \in C, i = 1, 2$ are two conductivities of type

$$\sigma_A^{(i)}(x) = \sum_{j=1}^{N} \gamma_j^{(i)} A(x)_{\chi D_j}(x), \quad x \in \Omega, \quad i = 1, 2, \quad (2.11)$$

then we have

$$\|\sigma_A^{(1)} - \sigma_A^{(2)}\|_{L^\infty(\Omega)} \leq C \left\|A_{\sigma_A^{(1)}} - A_{\sigma_A^{(2)}}\right\|_{s}, \quad (2.12)$$

where $C$ is a positive constant that depends on the a priori data only.

3. Proof of the main result

The proof of our main result (theorem 2.1) is based on an argument that combines asymptotic type of estimates for the Green’s function of the operator

$$L = \text{div}(\sigma(x) \nabla) \quad \text{in} \quad \Omega, \quad (3.1)$$

(theorem 3.4), with $\sigma \in C$, together with a result of unique continuation (proposition 3.5) for solutions to

$$Lu = 0, \quad \text{in} \quad \Omega.$$

We shall give the precise formulation of these results in what follows.
3.1. Measurable conductivity $\sigma$

We shall start with some general considerations about the Green’s function $G(x, y)$ associated to the operator (3.1), where $\sigma$ is merely a measurable matrix valued function satisfying the ellipticity condition (2.1).

3.1.1. Green's function. If $L$ is the operator given in (3.1), then for every $y \in \Omega$, the Green’s function $G(\cdot, y)$ is the weak solution to the Dirichlet problem

\[
\begin{cases}
\text{div} (\sigma V G(\cdot, y)) = -\delta(\cdot - y), & \text{in } \Omega, \\
G(\cdot, y) = 0, & \text{on } \partial\Omega,
\end{cases}
\]  

(3.1)

where $\delta(\cdot - y)$ is the Dirac measure at $y$. We recall that $G$ satisfies the properties ([Lit-St-W])

\[
G(x, y) = G(y, x), \quad \text{for every } x, y \in \Omega, \quad x \neq y, \quad \text{(3.2)}
\]

\[
0 < G(x, y) < |x - y|^{2-n}, \quad \text{for every } x, y \in \Omega, \quad x \neq y. \quad \text{(3.3)}
\]

Moreover, the following result holds true.

**Proposition 3.1.** For any $y \in \Omega$ and every $r > 0$ we have that

\[
\int_{\Omega \setminus B_r(y)} |\nabla G(\cdot, y)|^2 \leq C r^{2-n},
\]

(3.4)

where $C > 0$ depends on $\lambda$ and $n$ only.

**Proof.** The proof can be obtained by combining Caccioppoli inequality with (3.3) ([A-V], proposition 3.1).

3.1.2. Integral solutions of $L$. Let $\sigma^{(i)}$, $i = 1, 2$ be two measurable matrix valued functions satisfying the ellipticity condition (2.1) and let $G_i(x, y)$ be the Green’s functions associated to the operators

\[
L_i = \text{div}(\sigma^{(i)}(x) V) \quad \text{in } \Omega, \quad i = 1, 2.
\]

(3.5)

Let $U'$ be an open subset of $\Omega$ and $W = \Omega \setminus U'$. For any $y, z \in W$ we define

\[
S_{U'}(y, z) = \int_{U'} (\sigma^{(1)}(x) - \sigma^{(2)}(x)) V_i G_i(x, y) \cdot V_i G_2(z, x) \, dx.
\]

(3.6)

**Remark 3.2.**

\[
|S_{U'}(y, z)| \leq C \|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} (d(y) d(z))^{1-\frac{n}{2}}, \quad \text{for every } y, z \in W,
\]

(3.7)

where $d(y) = \text{dist}(y, U')$ and $C$ is a positive constant depending on $\lambda$ and $n$ only.

Observe that (3.7) is a straightforward consequence of Hölder inequality and proposition 3.1. We constructed in this way an integral function $S_{U'}(\cdot, \cdot)$ on $W \times W$, which is written in terms of the two Green’s functions $G_1(\cdot, y), G_2(\cdot, z)$ of $L_1, L_2$ respectively; $S_{U'}(\cdot, z)$, $S_{U'}(y, \cdot)$ are in turn solutions for $L_1, L_2$ respectively on the complement part of $U'$ in $\Omega$. More precisely we have
Proposition 3.3. For every \( y, z \in \mathbb{W} \) we have that \( S(y, \cdot) \), \( S\varepsilon \varepsilon \) \( H_{w_{\cdot}}(\mathbb{W}) \) are weak solutions to
\[
\text{div}(\sigma^{(1)}(\cdot) V S(y, \cdot)) = 0, \quad \text{div}(\sigma^{(2)}(\cdot) V S_\varepsilon(y, \cdot)) = 0, \quad \text{in } \mathbb{W}. \tag{3.8}
\]

Proof. The proof relies on differentiation under the integral sign arguments and the symmetry of \( G_i, 1, 2 \).

3.2. Conductivity \( \sigma \in \mathbb{C} \)

We shall denote with
\[
\Gamma(x, y) = \frac{1}{(n-2) \omega_n} |x - y|^{2-n}, \tag{3.9}
\]
the fundamental solution of the Laplace operator (here \( \omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \)). If \( D_i, i = 1, \ldots, N \) are the domains introduced in section 2.2.1 and \( L \) is the operator given by (3.1), with \( \sigma \in \mathbb{C} \), we shall give asymptotic estimates for the Green’s function of \( L \), with respect to (3.9) at the interfaces between the domains \( D_i, i = 1, \ldots, N \). These estimates are given below. In what follows let \( G \) be the Green’s function associated to the operator \( L \) in \( \Omega \).

3.2.1. Green’s function

Theorem 3.4. (Asymptotic estimates) For every \( l \in \{1, \ldots, K - 1\} \), let \( \nu(P_{l+1}) \) denote the unit exterior normal to \( D_{j_0+1} \) at the point \( P_{l+1} \). There exist constants \( \beta \in (0, \alpha) \) and \( \bar{C} > 1 \) depending on \( \Gamma, \lambda, M, \alpha \) and \( n \) only such that the following inequalities hold true for every \( x \in B_{r_{l+1}}(P_{l+1}) \cap D_{j_0+1} \) and every \( y = P_{l+1} + r \nu(P_{l+1}) \), where \( r \in (0, \frac{1}{2}) \)
\[
\begin{align*}
G(x, y) &= \frac{2}{\gamma_{l+1} + \gamma_{j_0+1}} \Gamma(J(x), J(y)) \leq \frac{\bar{C}}{r_{l+1}^2} |x - y|^{\beta+2-n}, \tag{3.10} \\
V_x G(x, y) &= \frac{2}{\gamma_{l+1} + \gamma_{j_0+1}} V_x \Gamma(J(x), J(y)) \leq \frac{\bar{C}}{r_{l+1}^2} |x - y|^{\beta+1-n}. \tag{3.11}
\end{align*}
\]
where \( J \) is the positive definite matrix such that \( J = \sqrt{A(P_{l+1})^{-1}} \).

3.2.2. Integral solutions of \( L \): unique continuation. We recall that up to a rigid transformation of coordinates we can assume that
\[
R = 0; \quad (\mathbb{R}^n \setminus \Omega) \cap B_{\phi} = \left\{ (x', x_n) \in B_{\phi} \mid x_n < \phi(x') \right\},
\]
where \( \phi \) is a Lipschitz function such that
\[
\phi(0) = 0 \quad \text{and} \quad \|\phi\|_{C^\alpha(B_{\phi})} \leq L_0.
\]
Denoting by
\[
D_0 = \left\{ x \in \left( \mathbb{R}^n \setminus \Omega \right) \cap B_{r_0} \left| \begin{array}{l}
  x_i < \frac{2}{3} r_0, \ i = 1, \ldots, n - 1, \\
  x_n - \frac{r_0}{6} < \frac{5}{6} r_0
\end{array} \right. \right\},
\]
it turns out that the augmented domain \( \Omega_0 = \Omega \cup D_0 \) is of Lipschitz class with constants \( \frac{r_0}{3} \) and \( \tilde{L} \), where \( \tilde{L} \) depends on \( L \) only. We consider the operator \( L_i \) given by (3.5) and extend \( \sigma^{(i)} \in C \) to \( \tilde{\sigma}^{(i)} \tilde{A} \) on \( \Omega_0 \), by setting \( \tilde{\sigma}^{(i)}|_{D_0} = 1 \), and extending \( A \) to \( \tilde{A} \in C^{0,1}(\Omega_0) \) with Lipschitz constant \( L \), for \( i = 1, 2 \). We denote by \( \tilde{G}_i \) the Green function associated to \( \tilde{\sigma}^{(i)} \tilde{A} \) in \( \Omega_0 \), for \( i = 1, 2 \). For any number \( r \in \left( 0, \frac{2}{3} r_0 \right) \) we also denote
\[
(D_0)_r = \{ x \in D_0 \left| \text{dist}(x, \Omega) > r \right. \}.
\]

Let us fix \( k \in \{ 2, \ldots, N \} \) and recall that there exist \( j_1, j_2, \ldots, j_K \in \{ 1, \ldots, N \} \) such that
\[
D_{j_i} = D_{j_1} \ldots D_{j_K} = D_k.
\]

We denote
\[
W_K = \bigcup_{i=0}^{K} D_{j_i} \quad \text{and} \quad U_{\tilde{K}} = \Omega_0 \setminus \overline{W_K}, \quad \text{when} \ k \geq 0.
\]

\((D_{j_i} = D_0)\) and for any \( y, z \in W_K \)
\[
\tilde{S}_{U_{\tilde{K}}}(y, z) = \int_{U_{\tilde{K}}} \left( \sigma_A^{(1)}(\cdot) - \sigma_A^{(2)}(\cdot) \right) \tilde{V} \tilde{G}_1(\cdot, y) \cdot \tilde{V} \tilde{G}_2(\cdot, z), \quad \text{when} \ k \geq 0.
\]

We introduce for any number \( b > 0 \) as in [A-V], the concave non-decreasing function \( \omega_b(t) \), defined on \((0, +\infty)\)
\[
\omega_b(t) = \begin{cases} 
2^b e^{-2t} \log(t)^{-b}, & t \in \left( 0, e^{-2} \right), \\
e^{-t}, & t \in \left[ e^{-2}, +\infty \right)
\end{cases}
\]
and denote
\[
\omega^{(1)}_b = \omega, \quad \omega^{(j)}_b = \omega_b \circ \omega^{(j-1)}_b.
\]

The following parameters shall also be introduced
\[
\beta = \arctan \left( \frac{1}{L} \right), \quad \beta_1 = \arctan \left( \frac{\sin \beta}{4} \right), \quad \lambda_1 = \frac{r_0}{1 + \sin \beta_1}.
\]
\[
\rho_1 = \lambda_1 \sin \beta_1, \quad \rho_2 = \frac{1 - \sin \beta_1}{1 + \sin \beta_1}, \quad \lambda_k = a \lambda_{k-1}, \quad \rho_k = a \rho_{k-1}, \quad \text{for every} \ k \geq 2,
\]
\[
d_k = \lambda_k - \rho_k, \quad k \geq 1.
\]

Let us denote here and in the sequel
\[
E = \| \sigma_A^{(1)} - \sigma_A^{(2)} \|_{L^1(\Omega)}.
\]

The following estimate for \( \tilde{S}_{U_{\tilde{K}}}(y, z) \) holds true.
Proposition 3.5. (Estimates of unique continuation) If, for a positive number \( \varepsilon_0 \), we have

\[
\left| \tilde{S}_{U_k}(y, z) \right| \leq \varepsilon_0^{2-n} \varepsilon_0, \quad \text{for every } (y, z) \in (D_0)_0 \times (D_0)_0,
\]

then the following inequality holds true for every \( r \in (0, d_1] \)

\[
\left| \tilde{S}_{U_k}\left( w_h(P_{K+1}), w_h(P_{K+1}) \right) \right| \leq \varepsilon_0^{2-n} C^6 (E + \varepsilon_0) \left( \alpha_{1/2}^2(K) \left( \frac{\varepsilon_0}{E + \varepsilon_0} \right) \right)^{(1/C)^r},
\]

where \( P_{K+1} \in \Sigma_{K+1}, \) \( \tilde{h} = \min \{ k \in \mathbb{N} | d_k \leq r \}, \) \( w_h(P_{K+1}) = P_{K+1} - \lambda_h \nu (P_{K+1}). \) \( \nu \) is the exterior unit normal to \( \partial D_K \) and \( C \geq 1 \) depends on the a priori data only.

3.3. Proof of theorem 2.1

Proof. We denote by \( \Lambda_i \) the map \( \Lambda_i(\sigma_i) \), for \( i = 1, 2 \) and, for every \( k \in \{0, \ldots, K \} \), the subscript \( j_k \) will be replaced by \( k \). This should simplify the notation. Let us point out that

\[
\left\| \left( \sigma^{(1)}_{j_k} - \sigma^{(2)}_{j_k} \right) \right\|_{L^{\infty}(\Omega)} \lesssim \tilde{A} \left\| \gamma^{(1)} - \gamma^{(2)} \right\|_{L^{\infty}(\Omega)},
\]

where

\[
\gamma^{(i)} = \sum_{j=1}^{N} \gamma^{(i)}_{j_k} \chi_{D_j}(x), \quad i = 1, 2,
\]

therefore (2.12) trivially follows from

\[
\left\| \gamma^{(1)} - \gamma^{(2)} \right\|_{L^{\infty}(\Omega)} \leq C \left\| A_1 - A_2 \right\|_b,
\]

which we shall prove. Moreover we shall denote

\[
\varepsilon = \left\| A_1 - A_2 \right\|_b, \quad \delta_k = \left\| \gamma^{(1)} - \gamma^{(2)} \right\|_{L^{\infty}(W_k)}.
\]

We start by recalling that for every \( y, z \in D_0 \) we have

\[
\left\langle (A_1 - A_2) G_1 (\cdot, y), \tilde{G}_2 (\cdot, z) \right\rangle = \int_{\Omega} \left( \gamma^{(1)} - \gamma^{(2)} \right) \hat{A} (x) \nabla \tilde{G}_1 (\cdot, y) \cdot \nabla \tilde{G}_2 (\cdot, z)
\]

and that, for every \( k \in \{1, \ldots, K \} \),

\[
\tilde{S}_{U_k}(y, z) = \int_{U_{k-1}} \left( \gamma^{(1)} - \gamma^{(2)} \right) \hat{A} (x) \nabla \tilde{G}_1 (\cdot, y) \cdot \nabla \tilde{G}_2 (\cdot, z),
\]

therefore

\[
\left| \tilde{S}_{U_k}(y, z) \right| \leq \varepsilon \left\| \tilde{G}_1 (\cdot, y) \right\|_{H^{1/2}(\Sigma)} \left\| \tilde{G}_2 (\cdot, z) \right\|_{H^{1/2}(\Sigma)} + \delta_{k-1} \varepsilon \left\| \nabla \tilde{G}_1 (\cdot, y) \right\|_{L^2(W_{k-1})} \left\| \nabla \tilde{G}_2 (\cdot, z) \right\|_{L^2(W_{k-1})}
\]

\[
\leq C (\varepsilon + \delta_{k-1}) r_0^{2-n}, \quad \text{for every } y, z \in (D_0)_{\rho_0 \Omega},
\]

where \( C \) depends on \( A, L, \lambda, \tilde{A} \) and \( n \). Let \( \rho_0 = \frac{a}{C} \), where \( C \) is the constant introduced in theorem 3.4, let \( r \in (0, d_2) \) and denote

\[
w = P_1 + \sigma v (P_1), \quad \text{where } \sigma = a^{k-1} \lambda_1,
\]
where \( \nu(R) \) denotes the exterior unit normal to \( \partial D_k \) in \( P_k \), then
\[
\tilde{S}_{U_k}(w, w) = I_1(w) + I_2(w), \tag{3.16}
\]
where
\[
I_1(w) = \int_{B_\rho(R) \cap D_k} \left( \gamma^{(1)} - \gamma^{(2)} \right) \mathcal{A}(\cdot) \mathcal{V}\tilde{G}_1(\cdot, w) \cdot \mathcal{V}\tilde{G}_1(\cdot, w),
\]
\[
I_2(w) = \int_{U_k \setminus (B_\rho(R) \cap D_k)} \left( \gamma^{(1)} - \gamma^{(2)} \right) \mathcal{A}(\cdot) \mathcal{V}\tilde{G}_1(\cdot, w) \cdot \mathcal{V}\tilde{G}_1(\cdot, w)
\]
and (see [A-V])
\[
|I_2(w)| \leq C E \rho_0^{2-n}, \tag{3.17}
\]
where \( C \) depends on \( \lambda, A, \alpha \) and \( n \) only. To estimate \( I_1(w) \) we recall theorem 3.4 which leads to
\[
|I_1(w)| \geq C \left\{ |\gamma^{(1)} - \gamma^{(2)}| \int_{B_\rho(R) \cap D_k} \left[ |\nabla \mathcal{J}(x, Jw)| \right]^2 \right. \\
- \int_{B_\rho(R) \cap D_k} \left[ A(x) \| \nabla \mathcal{J}(x, Jw) \| |k - w|^{1-n+\beta} \right] \left. \right\}.
\]
where \( C \) is a constant that depends on \( M, \lambda, \alpha, \bar{A} \) and \( n \) only. Therefore, by combining (3.15) together with (3.16) and (3.17), we obtain
\[
|I_1(w)| \geq C \left\{ |\gamma^{(1)} - \gamma^{(2)}| \int_{B_\rho(R) \cap D_k} \left[ \frac{|J^2(x - w)|^2}{|J(x - w)|^\rho} \right] \right. \\
- \frac{E}{\rho_0^\beta} \int_{B_\rho(R) \cap D_k} \left[ \frac{|J^2(x - w)|}{|J(x - w)|^\rho} \right] |k - w|^{1-n+\beta} \right. \\
- \frac{E}{\rho_0^\beta} \int_{B_\rho(R) \cap D_k} \left[ |k - w|^{2(1-n)+\beta} \right] \right\}.
\]
Therefore
\[
|I_1(w)| \geq C \left\{ |\gamma^{(1)} - \gamma^{(2)}| \int_{B_\rho(R) \cap D_k} |k - w|^{2(1-n)} \right. \\
- \frac{E}{\rho_0^\beta} \int_{B_\rho(R) \cap D_k} |k - w|^{2(1-n)+\beta} \right. \\
- \frac{E}{\rho_0^\beta} \int_{B_\rho(R) \cap D_k} |k - w|^{2(1-n)+\beta} \right\}, \tag{3.18}
\]
which leads to
\[ |I_1(w)| \geq C_1 \left| \gamma_k^{(1)} - \gamma_k^{(2)} \right| \sigma^{2-n} - C_2 E \frac{\sigma^{2-n+\beta}}{\rho_0^\beta}, \]  
(3.19)
where $\beta$ is the number introduced in theorem 3.4 and $C_1, C_2$ are constants depending on $M, \lambda, \alpha, \bar{A}$ and $n$ only. By combining (3.19) together with (3.16) and (3.17) we obtain
\[ C_1 \left| \gamma_k^{(1)} - \gamma_k^{(2)} \right| \sigma^{2-n} \lesssim \left| \tilde{S}_{\mathcal{U}_\gamma}(w, w) \right| + C_2 E \frac{\sigma^{2-n+\beta}}{\rho_0^\beta} \]  
(3.20)
and by proposition 3.5 and (3.15) we obtain
\[ \left| \tilde{S}_{\mathcal{U}_\gamma}(w, w) \right| \lesssim \sigma^{2-n} C \tilde{h}(E + \epsilon + \delta_{k-1}) \left( \omega_{\frac{1}{E + \epsilon + \delta_{k-1}}} \left( E + \epsilon + \delta_{k-1} \right) \right)^{\frac{1}{2}}, \]
where $C \geq 1$ is a constant depending on $A, L, \tilde{A}, M, N, \alpha, \lambda$ and $n$ only, therefore
\[ \left| \gamma_k^{(1)} - \gamma_k^{(2)} \right| \lesssim C \tilde{h}(E + \epsilon + \delta_{k-1} + E) \left( \omega_{\frac{1}{E + \epsilon + \delta_{k-1} + E}} \left( E + \epsilon + \delta_{k-1} \right) \right)^{\frac{1}{2}} \]  
(3.21)
We need to estimate $C \tilde{h}$ and $\left( \frac{1}{C} \right)^{\frac{1}{2}}$, where $C > 1$. It turns out that
\[ C \tilde{h} \lesssim C^2 \left( \frac{d_1}{r} \right)^{\frac{1}{2}}, \]
\[ \left( \frac{1}{C} \right)^{\frac{1}{2}} \lesssim \left( \frac{r}{d_1} \right)^{\frac{1}{2}}, \]  
(3.22)
therefore
\[ \left| \gamma_k^{(1)} - \gamma_k^{(2)} \right| \lesssim C \left( E + \epsilon + \delta_{k-1} + E \right) \left( \frac{d_1}{r} \right)^{C} \left( \omega_{\frac{1}{E + \epsilon + \delta_{k-1} + E}} \left( E + \epsilon + \delta_{k-1} \right) \right)^{\frac{1}{2}} \]  
(3.23)
By (3.23) we obtain for every $k \in \{1, \ldots, K\}$
\[ \delta_k \leq \delta_{k-1} + C \left( E + \epsilon + \delta_{k-1} + E \right) \left( \omega_{\frac{1}{E + \epsilon + \delta_{k-1} + E}} \left( E + \epsilon + \delta_{k-1} \right) \right)^{\frac{1}{2}}, \]
which leads to
\[ \| \gamma^{(1)} - \gamma^{(2)} \|_{L^\infty(I)} \leq C \left( E + E \right) \left( \omega_{\frac{1}{E + E}} \left( E + \frac{E}{E + E} \right) \right)^{\frac{1}{2}}, \]
therefore
\[ E \leq C \left( E + E \right) \left( \omega_{\frac{1}{E + E}} \left( E + E \right) \right)^{\frac{1}{2}}. \]  
(3.24)
Assuming that $E > \varepsilon e^2$ (if this is not the case then the theorem is proven) we obtain
\[ E \leq C \left( \frac{E}{e^2} + E \left( \omega_{+}^{(N)} \left( \frac{\varepsilon}{E} \right) \right)^* \right), \]
which leads to
\[ \frac{1}{C} \leq \omega_{+}^{(N)} \left( \frac{\varepsilon}{E} \right), \]
therefore
\[ E \leq \frac{1}{\omega_{+}^{(-N)} \left( \frac{1}{\varepsilon} \right) e}, \]
which concludes the proof. \[ \square \]

4. Proof of technical propositions

4.1. Proof of the asymptotic estimates

Whenever $\varphi$ is a Lipschitz continuous function on $\mathbb{R}^{n-1}$, we shall denote by $Q_{\varphi,r}^+$ and $Q_{\varphi,r}^-$ the following sets
\[ Q_{\varphi,r}^+ = \left\{ (x', x_n) \in Q_r \left| x_n > \varphi(x') \right. \right\}, \quad (4.1) \]
\[ Q_{\varphi,r}^- = \left\{ (x', x_n) \in Q_r \left| x_n < \varphi(x') \right. \right\}. \quad (4.2) \]

Let $0 < \mu < 1$ and $B^+ \in C^\mu(Q_{\varphi,r}^+), B^- \in C^\mu(Q_{\varphi,r}^-)$ be symmetric, positive definite matrix valued functions and define
\[ B(x) = \begin{cases} B^+(x), & x \in Q_{\varphi,r}^+, \\ B^-(x), & x \in Q_{\varphi,r}^-. \end{cases} \]
such that $B$ satisfies the uniform ellipticity condition
\[ \lambda_0^{-1} |\xi|^2 \leq B(x) \xi \cdot \xi \leq \lambda_0 |\xi|^2, \quad \text{for almost every } x \in Q_r, \]
\[ \text{for every } \xi \in \mathbb{R}^n, \quad (4.3) \]
where $\lambda_0 > 0$ is a constant.

Theorem 4.1. Let $k > 0$, $r > 0$ and $0 < \alpha < 1$ be fixed numbers. Moreover, let $B$ be a matrix as above. Assume that $\varphi \in C^{1,\alpha}(B_r')$ and let $U \in H^1(Q_r)$ be a solution to
\[ \text{div} \left( \left( 1 + (k - 1)\chi_{Q_{\varphi,r}^+}^\ast \right) B V U \right) = 0. \quad (4.4) \]

Suppose $\alpha'$ satisfies at the same time $0 < \alpha' \leq \mu$ and $\alpha' < \frac{\alpha}{(\alpha + 1)n}$. Then, there exists a positive constant $C$ such that for any $\rho \leq \frac{r}{2}$ and for any $x \in Q_{r-2\rho}$ the following estimate holds true

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\[ \| \nabla U \|_{L^p(Q_r(\alpha \lambda \phi \rho \phi \rho))} \leq C \frac{1}{\rho^{1+n/2}} \| U \|_{L^p(Q_r(\alpha \lambda \phi \rho \phi \rho))}, \]  
where \( C \) depends on \( \| \phi \|_{C^{\alpha/2}(\mathbb{R}^n)} \), \( k \), \( \alpha \), \( n \), \( \lambda_0 \), \( \| B^+ \|_{C^{-\alpha/2}(\mathbb{R}^n)} \) and \( \| B^- \|_{C^{-\alpha/2}(\mathbb{R}^n)} \) only.

**Proof.** For the proof we refer to [Li-Vo theorem 1.1], where the authors, among various results, obtain piecewise \( C^{1,\alpha/2} \) estimates for solutions to divergence form elliptic equations with piecewise Hölder continuous coefficients (see also [Li-Ni]).

We fix \( l \in \{1, \ldots, K - 1\} \). There exists a rigid transformation of coordinates under which \( P_{l+1} = 0 \) and

\[ \Sigma_l \cap Q_{r_1} = \left\{ x \in Q_{r_1} : x_n = \varphi(x') \right\}, \]  
\[ D_{j_1} \cap Q_{r_1} = \left\{ x \in Q_{r_1} : x_n > \varphi(x') \right\}, \]  
\[ D_{j_2} \cap Q_{r_1} = \left\{ x \in Q_{r_1} : x_n < \varphi(x') \right\}, \]

where \( \varphi \) is a \( C^{1,\alpha/2} \) function on \( B_{r_0}^2 \) satisfying

\[ \varphi(0) = |\varphi(0)| = 0, \quad \| \varphi \|_{C^{\alpha/2}} \left( B_{r_0}^2 \right) \leq M_0. \]  

Moreover, up to a possible replacement of \( \gamma \) with \( \gamma \frac{1}{r_0} \), we can assume that \( \gamma |_{D_{j_1}} = 1 \) and \( \gamma |_{D_{j_2}} = k \), where \( k \) is a real number which satisfies

\[ \gamma \leq k \leq \gamma. \]  

Let \( \tau \) be a \( C^\infty \) function on \( \mathbb{R} \) such that \( 0 \leq \tau \leq 1 \), \( \tau(s) = 1 \) for every \( s \in (-1, 1) \), \( \tau(s) = 0 \) for every \( s \in \mathbb{R} \setminus (-2, 2) \) and \( |\tau'(s)| \leq 2 \) for every \( s \in \mathbb{R} \).

We introduce

\[ n_1 = \frac{n_0}{3} \min \left\{ \frac{1}{2}, \frac{1}{8}, \frac{1}{4} \right\} \]  

and we consider the following change of variable \( \xi = \Phi(x) \) given by

\[ \begin{cases} 
\xi' = x', \\
\xi_n = x_n - \varphi(x') \tau \left( \frac{|x'|}{n_1} \right) \tau \left( \frac{x_n}{n_1} \right).
\end{cases} \]

It can be verified that the map \( \Phi \) is a \( C^{1,\alpha}(\mathbb{R}^n, \mathbb{R}^n) \) and it satisfies the following properties

\[ \Phi \left( \Sigma_l \cap Q_{n_1} \right) = \left\{ x \in Q_{n_1} : x_n = 0 \right\}, \]  
\[ \Phi(x) = x, \quad \text{for every } x \in \mathbb{R}^n \setminus Q_{2n_1}, \]  
\[ C^{-1} \left| x_1 - x_2 \right| \leq \left| \Phi(x_1) - \Phi(x_2) \right| \leq C \left| x_1 - x_2 \right|, \quad \text{for every } x_1, x_2 \in \mathbb{R}^n, \]
\[
|\Phi(x) - x| \leq \frac{C}{r_0^\alpha} |x|^{1+\alpha}, \quad \text{and} \quad (4.16)
\]

\[
|D\Phi(x) - I| \leq \frac{C}{r_0^\alpha} |x|^\alpha, \quad \text{for every } x \in \mathbb{R}^n, \quad (4.17)
\]

where \( C > 1 \) is a constant depending on \( M \) and \( \alpha \) only and \( I \) denotes the identity matrix.

Let \( y_n \in (-\frac{\alpha}{2}, 0) \) and \( y = ye_n \). We set
\[
\eta = \Phi(y), \quad (4.18)
\]
\[
G_B(\xi, \eta) = G\left(\Phi^{-1}(\xi), \Phi^{-1}(\eta)\right), \quad (4.19)
\]
\[
J(\xi) = \langle D\Phi \rangle \left(\Phi^{-1}(\xi)\right), \quad (4.20)
\]
\[
\sigma_B(\xi) = \frac{1}{\det J(\xi)} J(\xi) y \left(\Phi^{-1}(\xi)\right) A \left(\Phi^{-1}(\xi)\right) \left(J(\xi)\right)^T, \quad (4.21)
\]
we have that \( G_B(\cdot, \eta) \) is a solution to
\[
\begin{cases}
\text{div} \left( \sigma \nabla \gamma B(\cdot, \eta) \right) = -\delta(\cdot - \eta), & \text{in } \Omega, \\
G_B(\cdot, \eta) = 0, & \text{on } \partial\Omega.
\end{cases} \quad (4.22)
\]

We have
\[
\sigma_B(\xi) = \left(1 + (k - 1)\chi^+(\xi)\right) B(\xi), \quad \text{for any } \xi \in Q_n, \quad (4.23)
\]
where \( \chi^+ \) is the characteristic function of \( \mathbb{R}^n_+ \) and
\[
B(\xi) = \frac{1}{\det J(\xi)} J(\xi) A \left(\Phi^{-1}(\xi)\right) \left(J(\xi)\right)^T. \quad (4.24)
\]
Furthermore, we have that \( B \) is of class \( C^\alpha \) and
\[
\|B\|_{C^\alpha(\Omega)} \leq C, \quad (4.25)
\]
where \( C > 0 \) is a constant depending on \( M, \alpha, \lambda, \bar{A} \) only. We also have that \( B(0) = A(0) \).

We denote
\[
\sigma_{A(0)}(\xi) = \left(1 + (k - 1)\chi^+(\xi)\right) A(0) \quad (4.26)
\]
and we refer to \( G_0 \) as the Green’s function solution to
\[
\begin{cases}
\text{div} \left( \sigma_{A(0)}(\cdot) V G_0(\cdot, y) \right) = -\delta(\cdot - y), & \text{in } \Omega, \\
G_0(\cdot, y) = 0, & \text{on } \partial\Omega.
\end{cases} \quad (4.27)
\]

We then define
\[
R(\xi, \eta) = G_B(\xi, \eta) - G_0(\xi, \eta). \quad (4.28)
\]
Lemma 4.2. For every $\xi \in B_{\tau}^{\pm}$ and $\eta_n \in (-\frac{\xi}{\tau}, 0)$ we have that
\[
\left| R(\xi, \epsilon_n \eta_n) \right| + \left| \xi - \epsilon_n \eta_n \right| \| V_{\xi} R(\xi, \eta) \| \leq \frac{C}{\tau^d} \left| \xi - \epsilon_n \eta_n \right|^{\beta + 2/n},
\tag{4.29}
\]
where $\beta \in (0, \alpha^2]$ depends on $\alpha$ and $n$ only and $C$ depends on $M$, $\bar{\gamma}$, $\lambda$, $\bar{A}$ only.

Proof. It is easy to check that $R$ in (4.28) satisfies
\[
\begin{cases}
\text{div}_\xi \left( \sigma_B(\cdot) V_{\xi} R(\cdot, \eta) \right) = -\text{div}_\xi \left( \left( \sigma_B(\cdot) - \sigma_{A(0)}(\cdot) \right) V_{\xi} G_0(\cdot, \eta) \right), & \text{in } \Omega, \\
R(\cdot, \eta) = 0, & \text{on } \partial \Omega.
\end{cases}
\tag{4.30}
\]
By the representation formula over $\Omega$ we have that $R$ in (4.28) satisfies
\[
R(\xi, \eta) = \int_{\Omega} \left( \sigma_B(\zeta) - \sigma_{A(0)}(\zeta) \right) V_{\xi} G_0(\zeta, \eta) \cdot V \tilde{G}(\zeta, \xi) d\zeta.
\tag{4.31}
\]
We consider $\xi \in Q_{\frac{\xi}{\tau}}$ and $\eta = \epsilon_n \eta_n$ and we split $R$ as the sum of the following integrals
\[
R_1(\xi, \eta) = \int_{\Omega \setminus Q_{\eta_1}} \left( \sigma_B(\zeta) - \sigma_{A(0)}(\zeta) \right) V_{\xi} G_0(\zeta, \eta) \cdot V \tilde{G}(\zeta, \xi) d\zeta,
\tag{4.32}
\]
\[
R_2(\xi, \eta) = \int_{Q_{\eta_1}} \left( \sigma_B(\zeta) - \sigma_{A(0)}(\zeta) \right) V_{\xi} G_0(\zeta, \eta) \cdot V \tilde{G}(\zeta, \xi) d\zeta.
\tag{4.33}
\]
By the bounds (2.8)–(2.10) and by combining the Schwartz inequality with the Caccioppoli inequality we get
\[
\left| R_1(\xi, \eta) \right| \leq \frac{C}{\tau^d} \left\| G_0(\cdot, \eta) \right\|_{L^2(\Omega \setminus Q_{\eta_1})} \left\| G_{B}(\cdot, \eta) \right\|_{L^2(\Omega \setminus Q_{\eta_1})},
\tag{4.34}
\]
where $C > 0$ depends on $M$, $\alpha$, $\bar{\gamma}$, $\lambda$ and $\bar{A}$ only. By the standard behaviour (3.3) of the Green’s functions at hand, it follows that
\[
\left| R_1(\xi, \eta) \right| \leq C\xi^{2-n},
\tag{4.35}
\]
where $C > 0$ depends on $M$, $\alpha$, $\bar{\gamma}$, $\lambda$ and $\bar{A}$ only. Moreover being $B(0) = A(0)$, it follows that (2.9) and (4.25) lead to
\[
\left| \sigma_B(\xi) - \sigma_{A(0)}(\xi) \right| \leq \max\{1, k\} \left| B(\xi) - A(0) \right| \leq \frac{C}{\tau^d} |\xi|^n,
\tag{4.36}
\]
for any $\xi \in Q_{\eta_1}$, where $C$ depends on $M$, $\alpha$, $\bar{A}$ and $\bar{\gamma}$ only. By (3.3) and by theorem 4.1 we have that
\[
\left| V_{\xi} G_0(\zeta, \xi) \right| \leq C|\zeta - \xi|^n, \quad \text{for every } \zeta, \xi \in Q_{\eta_1},
\tag{4.37}
\]
where $C$ depends on $M$, $\alpha$, $\bar{A}$ and $\bar{\gamma}$ only. By (4.15) and the same arguments used above, we infer that
\[
\left| V_{\xi} G_B(\zeta, \xi) \right| \leq C|\zeta - \xi|^n, \quad \text{for every } \zeta, \xi \in Q_{\eta_1},
\tag{4.38}
\]
where $C$ depends on $M$, $\alpha$, $\bar{A}$ and $\bar{\gamma}$ only. We denote
\[
I_1 = \int_{R_{\eta_1}} |\eta|^n |\xi - \eta|^n |\xi - \eta|^{-n} d\eta
\tag{4.39}
\]
and

$$I_2 = \int_{\mathbb{R}^n \setminus B_{2h}} |\nabla^\alpha \zeta| \, d\eta. \quad (4.40)$$

By (4.36)–(4.38) we have that

$$|R_2(\xi, \eta)| \leq \frac{C}{\eta^2} (I_1 + I_2). \quad (4.41)$$

Let us denote now \( h = |\xi - \eta| \) and consider the following change of variables \( \xi = hw \); we set \( t = \frac{\xi}{h} \) and \( s = \frac{\eta}{h} \); it follows that for any \( t, s \in \mathbb{R}^n \) we have that \( |t - s| = 1 \). We obtain

$$I_1 \leq 4^3 h^{n+2-n} \int_{B_1} |t - w|^{1-n} |s - w|^{1-n} d\sigma. \quad (4.42)$$

Let us now set

$$F(t, s) = \int_{B_1} |t - w|^{1-n} |s - w|^{1-n} d\sigma. \quad (4.43)$$

From standard bounds (see for instance, [Mi chapter 2]) we have that

$$F(t, s) \leq C, \quad (4.44)$$

where \( C \) depends on \( n \) only. Hence

$$I_1 \leq C h^{n+2-n}, \quad (4.45)$$

where \( C \) depends on \( n \) only. We consider now integral \( I_2 \) and we recall that \( \eta = e_n \eta_n \), where \( \eta_n \in (-\frac{n}{2}, 0) \) and \( \xi \in Q_2^\perp \); hence we have

$$|\eta| = -\eta_n \leq -\eta_n + \xi_n \leq |\xi - \eta| = h, \quad (4.46)$$

which leads to

$$|\xi| \leq |\xi - \eta| + |\eta| \leq 2h. \quad (4.47)$$

On the other hand, we have that for any \( \zeta \in \mathbb{R}^n \setminus B_{4h} \)

$$|\zeta| \leq |\zeta - \eta| + |\eta| \leq |\zeta - \eta| + \frac{1}{4} |\zeta|, \quad (4.48)$$

hence we get

$$\frac{3}{4} |\zeta| \leq |\zeta - \eta|. \quad (4.49)$$

and

$$\frac{1}{2} |\zeta| \leq |\zeta - \xi|, \quad \text{for any } \zeta \in \mathbb{R}^n \setminus B_{4h}. \quad (4.50)$$

By combining (4.49) together with (4.50), we obtain

$$I_2 \leq \left( \frac{8}{3} \right)^{1-n} \int_{\mathbb{R}^n \setminus B_{4h}} |\nabla^{n+2-2n} \zeta| \, d\zeta \leq C h^{n+2-n}, \quad (4.51)$$

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where $C$ depends on $\alpha$ and $n$ only. By combining (4.35), (4.41), (4.45) and (4.51) we obtain

$$|R(\xi, \eta)| \leq \frac{C}{r^{\frac{1}{1 + \beta}}} h^{\alpha + 2 - n},$$

(4.52)

where $C$ depends on $M$, $\alpha$, $\bar{A}$, $n$ and $\bar{\gamma}$ only. Let us fix $\xi \in B^+_\bar{\tau}$, $\eta_n \in (-\eta/4, 0)$ and consider the cylinder

$$Q = B^+_\bar{\tau}(\xi') \times \left( \frac{\xi_n}{\eta_n} + \frac{h}{8} \right).$$

(4.53)

Observing that $h = |\xi - (0, \eta, \bar{\eta})| \leq \frac{\eta}{2}$ we deduce that $Q \subset Q^+_\bar{\tau}$. Moreover $Q \subset Q_{\bar{\tau}^2}$ and $\xi \in \partial Q$, then by choosing for instance $\alpha' = \frac{1}{2} \min \left\{ \alpha, \frac{a}{(a + 1)n} \right\}$ in the statement of theorem 4.1 and observing that $(0, \eta_n, \bar{\eta}) \notin Q_{\bar{\tau}^2}(\xi)$, by (4.5) we obtain the following bound for the seminorm

$$\left\| V_\xi G(\cdot, e_n \eta_n) \right\|_{\alpha', Q} \leq \left\| V_\xi G(\cdot, e_n \eta_n) \right\|_{\alpha', Q_{\bar{\tau}^2} \cap Q_{\bar{\tau}^2} \bar{\tau}} \leq Ch^{\alpha' - n + 1/2} \left\| V_\xi G(\cdot, e_n \eta_n) \right\|_{L^2(\{\bar{\tau}^2\})},$$

(4.54)

where $C$ depends on $M$, $\alpha$, $\bar{A}$, $n$ and $\bar{\gamma}$ only. Furthermore by observing that for any $\xi \in Q_{\bar{\tau}^2}(\xi)$ we have that $|\xi - (0, e_n \eta_n)| \geq \frac{h}{2}$ and by (3.3) we have

$$\left\| V_\xi G(\cdot, e_n \eta_n) \right\|_{\alpha', Q} \leq Ch^{\alpha' + 1 - n'},$$

(4.55)

where $C$ depends on $M$, $\alpha$, $\bar{A}$, $n$ and $\bar{\gamma}$ only. By analogous argument we may also infer that

$$\left\| V_\xi G(\cdot, e_n \eta_n) \right\|_{\alpha', Q} \leq Ch^{\alpha' + 1 - n'},$$

(4.56)

where $C$ depends on $M$, $\alpha$, $\bar{A}$, $n$ and $\bar{\gamma}$ only. Hence by (4.28), (4.55) and (4.56) we obtain

$$\left\| V_\xi R(\cdot, e_n \eta_n) \right\|_{\alpha', Q} \leq Ch^{\alpha' + 1 - n'},$$

(4.57)

where $C$ depends on $M$, $\alpha$, $\bar{A}$, $n$ and $\bar{\gamma}$ only. We recall the following interpolation inequality (see for instance [A-S], proposition 8.3)

$$\left\| V_\xi R(\cdot, e_n \eta_n) \right\|_{L^2(Q)} \leq \left\| R(\cdot, e_n \eta_n) \right\|_{L^2(Q)} \left\| V_\xi R(\cdot, e_n \eta_n) \right\|_{L^2_{\bar{\tau}^2}(Q)},$$

(4.58)

where $C$ depends on $M$, $\alpha$, $\bar{A}$, $n$ and $\bar{\gamma}$ only. By the above estimate and (4.52) we get

$$\left\| V_\xi R(\xi, e_n \eta_n) \right\| \leq \frac{C}{r_0^{1 + \beta}} h^{\beta + 1 - n}, \text{ for every } \xi \in B^+_\bar{\tau} \text{ and } \eta \in \left( -\frac{\eta}{4}, 0 \right),$$

(4.59)

where $C$ depends on $M$, $\alpha$, $\bar{A}$, $n$ and $\bar{\gamma}$ only. The thesis follows with $\beta = \frac{a^2}{1 + a}$. \hfill \square

**Proof of theorem 3.4.** We first assume that the auxiliary hypothesis that $A(0) = I$ is fulfilled and denote with $H(\xi, \eta)$ the half space fundamental solution of the operator $\text{div}_\xi ((1 + (k - 1))\chi^+(\xi)) I(\xi) V_\xi$ which has the following explicit form
where $\Gamma$ is the distribution introduced in (3.9) and for any $\xi = (\xi', \xi_n)$ we denote $\xi^* = (\xi', -\xi_n)$. Let $\eta_n \in (-\frac{n}{2}, 0)$, then we have

$$
\begin{aligned}
\text{div}_{\xi} & \left( (1 + (k - 1))I(\xi) V_\xi \left( G_0(\xi, e_n, \eta_n) - H(\xi, e_n, \eta_n) \right) \right) = 0, \quad \text{in } Q_2^n, \\
\left| G_0(\xi, e_n, \eta_n) - H(\xi, e_n, \eta_n) \right| & \leq C\rho^n - 2, \quad \text{on } \partial Q_2^n.
\end{aligned}
$$

Hence by the maximum principle we can infer that

$$
\| G_0(\cdot, e_n, \eta_n) - H(\cdot, e_n, \eta_n) \|_L^2\left( Q_2^n \right) \leq C\rho^n - 2
$$

and by theorem 4.1 we deduce that

$$
\| V_\xi G_0(\cdot, e_n, \eta_n) - V_\xi H(\cdot, e_n, \eta_n) \|_L^2\left( Q_2^n \right) \leq C\rho^n - 1.
$$

We now consider $x \in \Phi^{-1}(B^+_1)$ and $\eta_n \in (-\frac{n}{2}, 0)$, then we observe that being $\Phi(y) = y$ we have that

$$
|\Phi(y)| = |\Phi(y) - \Phi(0)| \leq |\Phi(y) - \Phi(x)|.
$$

Moreover, by (4.15) and the above estimate we have

$$
C^{-1}|x| \leq |\Phi(x)| \leq |\Phi(x) - \Phi(y)| + |\Phi(y)| \leq C|x - y|.
$$

By combining the above estimate with (4.16), we infer that

$$
|\Phi(x) - x| \leq \frac{C}{\rho^n} |x|^{\alpha + \epsilon} \leq \frac{C}{\rho^n} \left| x - e_n\gamma \right|^{1+\alpha},
$$

where $C$ depends on $M$ and $\alpha$ only. Let $\{ A_k \}_{k \geq 1}$ be a regularizing sequence for $A$ obtained by convolution with a sequence of mollifiers, then we have that

$$
\| A_k \|_{C^{0}(\Omega)} \leq 2A, \quad \text{for any } k \in \mathbb{N}
$$

and $A_k$ satisfies (2.10), with $A = A_k, k \in \mathbb{N}$. Let us introduce the following function

$$
F_k: B_n \setminus \{ e_n\gamma \} \rightarrow \mathbb{R}
$$

$$
z \mapsto < A_k(z)(z - e_n\gamma), (z - e_n\gamma) >^{\frac{\alpha}{2}},
$$

where $< \cdot, \cdot >$ denotes the Euclidean scalar product of vectors in $\mathbb{R}^n$. Given $z_1, z_2 \in B_n \setminus \{ e_n\gamma \}$ by the mean-value theorem, there exists $t_k, 0 < t_k < 1$ such that
where \(a_1 = z_1 + t_k(z_2 - z_1)\) and where \(C\) depends on depends on \(M, \alpha, \bar{A}\) and \(n\) only. Let us denote with \(G_k\) the fundamental solution introduced in (3.9) associated to the matrix \(A_k\).

We choose \(z_1 = \Phi(x)\) and \(z_2 = x\), then we have

\[
\left| G_k \left( \Phi(x), e_n y_n \right) - I_k \left( x, e_n y_n \right) \right| \leq C \left| \Phi(x) - x \right| \| x - e_n y_n + t_k (\Phi(x) - x) \|^{1-n},
\]

where \(C\) depends on \(M, \alpha, \bar{A}, \lambda\) and \(n\) only. By (4.65) and the triangle inequality we deduce that for any \(x \in D_{\bar{A}+} \cap B_{\frac{\mu}{\alpha + n}}\)

\[
\left| x - e_n y_n - t_k (\Phi(x) - x) \right| \geq \left| x - e_n y_n \right| - \left| t_k \right| \left| \Phi(x) - x \right| \\
\geq \left| x - e_n y_n \right| - \left| x - e_n y_n \right|^{1+\alpha} \geq \frac{1}{2} \left| x - e_n y_n \right|,
\]

Finally combining the above estimates and (4.65) we obtain

\[
\left| G_k \left( \Phi(x), e_n y_n \right) - I_k \left( x, e_n y_n \right) \right| \leq C \left| x - e_n y_n \right|^{2-n+\alpha},
\]

where \(C\) depends on \(M, \alpha, \lambda, \bar{A}\) and \(n\) only. Now since \(A_k\) converges uniformly to \(A\) in \(\bar{D}\) we can infer that

\[
\left| I_k \left( \Phi(x), e_n y_n \right) - I_k \left( x, e_n y_n \right) \right| \leq C \left| x - e_n y_n \right|^{2-n+\alpha},
\]

for \(x \in \Phi^{-1}(B_{\bar{A}+}^\iota)\), where \(C\) depends on \(M, \alpha, \lambda, \bar{A}\) and \(n\) only. By (4.61), (4.62) and (4.72) we have

\[
\left| G_0 \left( \Phi(x), e_n y_n \right) - H \left( x, e_n y_n \right) \right| \leq \left| G_0 \left( \Phi(x), e_n y_n \right) - H \left( \Phi(x), e_n y_n \right) \right| \\
+ \left| H \left( \Phi(x), e_n y_n \right) - H \left( x, e_n y_n \right) \right| \\
\leq \frac{C}{r_0^\alpha} \left| x - e_n y_n \right|^{2n+\alpha},
\]

and

\[
\left| V G_0 \left( \Phi(x), e_n y_n \right) - V H \left( x, e_n y_n \right) \right| \leq \frac{C}{r_0^\alpha} \left| x - e_n y_n \right|^{2n+\alpha},
\]

for \(x \in \Phi^{-1}(B_{\bar{A}+}^\iota)\), where \(C\) depends on \(M, \lambda, \alpha\) and \(n\) only. Moreover, by lemma 4.2 and (4.15) and recalling that \(\Phi(y) = y\), we get

\[
\left| R \left( \Phi(x), e_n y_n \right) \right| + \left| x - e_n y_n \right| \| V \xi R \left( \Phi(x), \eta \right) \| \leq \frac{C}{r_0^\alpha} \left| \xi - e_n y_n \right|^{\beta+2-n},
\]

for \(x \in \Phi^{-1}(B_{\bar{A}+}^\iota)\), \(C\) depends on \(M, \lambda, \alpha\) and \(n\) only. Gathering (4.73)–(4.75) and recalling that
we first find that

\[
G\left(\bar{x}, e_n y_n\right) = G_0\left(\Phi(\bar{x}), e_n y_n\right) + R\left(\Phi(\bar{x}), e_n y_n\right)
\]

(4.76)

for a.e. \(\bar{x} \in D_{\beta,\alpha} \cap B_{\frac{1}{\max(\lambda,\gamma)}}\) and \(y_n \in (-\eta/(4C)^{1/\alpha}, 0)\), where \(C\) depends on \(M, \lambda, \tilde{\gamma}, \tilde{A}, \alpha\) and \(n\) only. The thesis then follows for the case \(A(0) = I\).

To treat the general case when \(A(0) \neq I\), we introduce the fundamental solution \(H_{A(0)}\) of the operator \(\nabla (1 + (k - 1)) \chi^+(\bar{x}) A(0) V_{\bar{x}}\). We set \(\sigma_I(\bar{x}) = (1 + (k - 1)) \chi^+(\bar{x}) I\) and recall that \(\sigma_{A(0)}(\bar{x}) = (1 + (k - 1)) \chi^+(\bar{x}) A(0)\). Let us introduce the linear change of variable

\[
L: \mathbb{R}^n \rightarrow \mathbb{R}^n
\]

(4.79)

\[
\xi \mapsto L\xi = R\sqrt{A^{-1}(0)} \xi,
\]

(4.80)

where \(R\) is the planar rotation in \(\mathbb{R}^n\) that rotates the unit vector \(\frac{v}{\|v\|}\), where \(v = \sqrt{A(0)} e_n\), to the \(n\)th standard unit vector \(e_n\) and such that

\[
R\big|_{\langle\xi\rangle^+} \equiv Ld\big|_{\langle\xi\rangle^+},
\]

where \(\pi\) is the plane in \(\mathbb{R}^n\) generated by \(v\) and \(e_n\) and \((\pi)^\perp\) denotes the orthogonal complement of \(\pi\) in \(\mathbb{R}^n\). For this choice of \(L\) we have

(i) \(A(0) = L^{-1} \cdot (L^{-1})^T\),

(ii) \((L\xi) \cdot e_n = \frac{1}{\|v\|} \xi \cdot e_n\),

which leads to

\[
\sigma_{A(0)}(\xi) = L^{-1} \sigma_I(L\xi)(L^{-1})^T.
\]

(i.e. \(L^{-1}: x \mapsto \xi\) is the linear change of variables that maps \(\sigma_I(x)\) into \(\sigma_{A(0)}(\xi)\). Therefore the fundamental solution for the operator \(\nabla (1 + (k - 1)) \chi^+(\bar{x}) A(0) V_{\bar{x}}\) turns out to be

\[
H_{A(0)}(\xi, \eta) = \begin{cases} 
\left|J\right| \left(\frac{1}{k} \Gamma(L\xi, L\eta) + \frac{k - 1}{k(k + 1)} \Gamma\left(L\xi, L^\eta\right)\right), & \text{if } \xi_n, \eta_n > 0, \\
\left|J\right| \left(\frac{2}{k + 1} \Gamma(L\xi, L\eta)\right), & \text{if } \xi_n \eta_n < 0,
\end{cases}
\]

(4.81)

where \(|J|\) denotes the determinant of matrix \(J = \sqrt{A^{-1}(0)}\), matrix \(L^\eta = \left(l^\eta_{i,j}\right)_{i,j=1}^n\) is such that \(l^\eta_{i,j} = l_{i,j}\) for \(i = 1, \ldots, n - 1, j = 1, \ldots, n\) and \(l^\eta_{i,j} = -l_{n,i,j}\) for \(j = 1, \ldots, n\). In particular we have that when \(\xi_n \eta_n < 0\)

\[
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\]

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\[ H_{A(0)}(\xi, \eta) = \frac{2 |I|}{k + 1} < A^{-1}(0)(\xi - \eta, \xi - \eta) > \frac{2}{\pi}. \]

Hence for the case \( A(0) \neq I \) (4.77) and (4.78) shall be replaced by
\[
\left| G(x, e_n y_n) - \frac{2}{1 + k} < A^{-1}(0)(x - e_n y_n), (x - e_n y_n) > \right| \leq \frac{C}{r_0^2} |x - e_n y_n|^{2-n+\beta},
\]
\[
\left| \nabla_x G(x, e_n y_n) - \frac{2}{1 + k} \nabla_x < A^{-1}(0)(x - e_n y_n), (x - e_n y_n) > \right| \leq \frac{C}{r_0^2} |x - e_n y_n|^{\beta+n-1},
\]
for \( x \in D_{i+1} \cap R_{-\varepsilon}(\varepsilon(a)/4, 0) \) and \( y_n \in (-\eta/(4C)^{1/(\alpha)}, 0) \), where \( C \) depends on \( M, \lambda, \tilde{r}, \tilde{A}, \alpha \) and \( n \) only. Hence the thesis follows also for the general case.

4.2. Proof of unique continuation estimates

Let \( P_1, D_0, \Omega_0, (D_0) \), and \( \tilde{G}_j \), for \( i = 1, 2 \) be as in subsection 3.2.1. Let us fix \( k \in \{2, \ldots, N\} \) and recall that there exist \( j_1, \ldots, j_k \in \{2, \ldots, N\} \) such that
\[
D_{\tilde{j}} = D_1, \ldots, D_{j_k} = D_k.
\]
We recall that
\[
W_K = \bigcup_{i=1}^{K} D_{j_i}, \quad U_K = \Omega_0 \setminus W_K, \quad \text{when } k \geq 0
\]
\( (D_k = D_0) \) and for any \( y, z \in W_k \)
\[
\tilde{S}_{U_k}(y, z) = \int_{U_k} \left( \tilde{s}_{A(1)} - \tilde{s}_{A(2)} \right) \nabla G_1(\cdot, y) \cdot \nabla G_2(\cdot, z), \quad \text{when } k \geq 0.
\]

The proof of proposition 3.5 is a straightforward consequence of the following result (see [A-V], proof of proposition 4.6).

**Proposition 4.3.** Let \( v \) be a weak solution to
\[
\text{div}(\sigma V_y) = 0, \quad \text{in } W_K,
\]
where \( \sigma \) is either equal to \( \tilde{\sigma}_{A(1)} \) or to \( \tilde{\sigma}_{A(2)} \). Assume that, for given positive numbers \( \varepsilon_0 \) and \( E_0 \), \( v \) satisfies
\[
|v(x)| \leq \varepsilon_0 r_0^{-\frac{n}{2}}, \quad \text{for every } x \in (D_0)^2, \quad \text{(4.1)}
\]
and
\[
|v(x)| \leq E_0 \left( \eta_0 d(x) \right)^{-\frac{n}{2}}, \quad \text{for every } x \in W_k, \quad \text{(4.2)}
\]
where \( d(x) = \text{dist}(x, \Sigma_{k+1}) \). Then the following inequality holds true for every \( r \in (0, d_1] \)
\[
\left| v(w_{\tilde{K}}(P_{k+1})) \right| \leq \varepsilon_0^{-\frac{n}{2}} C^\delta \left( E_0 + \varepsilon_0 \right) \left( \frac{E_0}{E_0 + \varepsilon_0} \right)^{\frac{1}{1+C}}. \quad \text{(4.3)}
\]

**Proof.** We observe that the proof of this result follows the same line of the argument used in [A-V proof of proposition 4.4] which is independent from the presence of isotropy/anisotropy in \( \tilde{\sigma} \). In fact their proof is based on an argument of unique continuation which requires \( \tilde{\sigma} \) to be
Lipschitz continuous and the interfaces between each domain $D_j$ to contain a $C^{1,\alpha}$ portion, therefore we simply recall [A-V], proof of proposition 4.4 for a complete proof of this proposition. Here we simply recall for sake of completeness the main fact proven in [A-V], proof of proposition 4.4. By defining the quantities
\[
\eta = \frac{\eta_0}{4}, \quad \tilde{\rho} = \frac{\eta_0}{128\sqrt{1 + L^2}}.
\]
let $\gamma_m \in D_m$ be a point ‘near the portion’ $\Sigma_{m+1}$ of the interface between $D_m$ and $D_{m+1}$ defined by
\[
\gamma_m = P_{m+1} - \frac{\eta_0}{32} \nu (P_{m+1}),
\]
where $P_{m+1} \in \Sigma_{m+1}$. Their main point is the proof of the following fact
\[
\|v\|_{L^2(B_\rho(\gamma_m))} \leq L \tilde{\rho} (E_0 + \epsilon_0)^{\frac{m+1}{\tau}} \left( \frac{\epsilon_0}{E_0 + \epsilon_0} \right), \tag{4.4}
\]
where $\tilde{\rho}$ has been chosen above so that $B_\rho(\gamma_m) \subset D_m$. The proof of the above inequality is done by induction. The so-called argument of \textit{global propagation of smallness} is used there to prove (4.4) for $m = 0$. We refer to [A-R-R-V], theorem 5.3 for a complete treatment of this topic. The rest of the proof is based on the \textit{three sphere inequality}, therefore we simply refer to [A-V], proof of proposition 4.4 for this.

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