Speed of the biased random walk on a Galton–Watson tree

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Summary. We give an expression of the speed of the biased random walk on a Galton–Watson tree. In the particular case of the simple random walk, we recover the result of Lyons, Pemantle and Peres [7]. The proof uses a description of the invariant distribution of the environment seen from the particle.

Keywords. Random walk, Galton–Watson tree, speed, invariant measure.

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1 Introduction

Let $T$ be a Galton–Watson tree with root $e$, and $\nu$ be its offspring distribution with values in $\mathbb{N}$. We suppose that $m := \mathbb{E}[\nu] > 1$, so that the tree is super-critical. In particular, the event $\mathcal{S}$ that $T$ is infinite has a positive probability, and we let $q := 1 - \mathbb{P}(\mathcal{S}) < 1$ be the extinction probability. We call $\nu(x)$ the number of children of the vertex $x$ in $T$. For $x \in T \setminus \{e\}$, we denote by $x_*$ the parent of $x$, that is the neighbour of $x$ which lies on the path from $x$ to the root $e$, and by $x_i, 1 \leq i \leq \nu(x)$ the children of $x$. We call $T_*$ the tree $T$ on which we add an artificial parent $e_*$ to the root $e$.

For any $\lambda > 0$, and conditionally on $T_*$, we introduce the $\lambda$-biased random walk $(X_n)_{n \geq 0}$ which is the Markov chain such that, for $x \neq e_*$,

\begin{align}
(1.1) \quad P(X_{n+1} = x_* \mid X_n = x) &= \frac{\lambda}{\lambda + \nu(x)}, \\
(1.2) \quad P(X_{n+1} = x_i \mid X_n = x) &= \frac{1}{\lambda + \nu(x)} \text{ for any } 1 \leq i \leq \nu(x),
\end{align}

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and which is reflected at $e_\ast$. It is easily seen that this Markov chain is reversible. We denote by $P_x$ the quenched probability associated to the Markov chain $(X_n)_n$ starting from $x$ and by $\mathbb{P}_x$ the annealed probability obtained by averaging $P_x$ over the Galton–Watson measure. They are respectively associated to the expectations $E_x$ and $\mathbb{E}_x$.

When $\lambda < m$, we know from Lyons [6] that the walk is almost surely transient on the event $S$. Moreover, if we denote by $|x|$ the generation of $x$, Lyons, Pemantle and Peres [8] showed that, conditionally on $S$, the limit $\ell_\lambda := \lim_{n \to \infty} \frac{|X_n|}{n}$ exists almost surely, is deterministic and is positive if and only if $\lambda \in (\lambda_c, m)$ with $\lambda_c := \mathbb{E}[\nu q^{\nu - 1}]$. This is the regime we are interested in.

For any vertex $x \in T$, let

$$\tau_x := \min\{n \geq 1 : X_n = x\}$$

be the hitting time of the vertex $x$ by the biased random walk, with the notation that $\min\emptyset := \infty$, and, for $x \neq e_\ast$,

$$\beta(x) := P_x(\tau_{x_\ast} = \infty)$$

be the quenched probability of never reaching the parent of $x$ when starting from $x$. Notice that we have $\beta(x) > 0$ if and only if the subtree rooted at $x$ is infinite. Then, let $(\beta_i, i \geq 0)$ be, under $\mathbb{P}$, generic i.i.d. random variables distributed as $\beta(e)$, and independent of $\nu$.

**Theorem 1.1.** Suppose that $m \in (1, \infty)$ and $\lambda \in (\lambda_c, m)$. Then,

$$\ell_\lambda = \mathbb{E} \left[ \frac{(\nu - \lambda)\beta_0}{\lambda - 1 + \sum_{i=0}^{\nu} \beta_i} \right] / \mathbb{E} \left[ \frac{(\nu + \lambda)\beta_0}{\lambda - 1 + \sum_{i=0}^{\nu} \beta_i} \right].$$

The speed in the case $\lambda = 1$ was already obtained by Lyons, Pemantle and Peres [7], who found that $\ell_1 = \mathbb{E}[\frac{\nu}{\nu + 1}]$, whereas Ben Arous, Hu, Olla and Zeitouni [2] computed the derivative of $\ell_\lambda$ at the point $\lambda = m$. In this paper, the authors give another representation of the speed $\ell_\lambda$, at least when $\lambda$ is close enough to $m$. In the zero speed regime $\lambda \leq \lambda_c$, Ben Arous, Fribergh, Gantert and Hammond [1] showed tightness of the properly rescaled random walk, though a limit law fails. A central limit theorem was obtained by Peres and Zeitouni [10], by means, in the case $\lambda = m$, of a construction of the invariant distribution on the space of trees. The invariant distribution in the case $\lambda > m$ was given in [2]. We mention that, so far, the only case in the transient regime $\lambda < m$ for which such an invariant distribution was known was the simple random walk case $\lambda = 1$ studied in [7]. Theorem 4.1 in Section 4 gives a description of the invariant measure for
all $\lambda \in (\lambda_c, m)$. In the setting of random walks on Galton–Watson trees with random conductances, Gantert, Müller, Popov and Vachkovskaia \cite{Gantertetal} obtained a similar formula for the speed via the construction of the invariant measure in terms of effective conductances.

The paper is organized as follows. Section 2 introduces some notation and the concept of backward tree seen from a vertex. Section 3 investigates the law of the tree seen from a vertex that we visit for the first time. Using a time reversal argument, we are able to describe the distribution of this tree in Proposition 3.2. Then, we obtain in Section 4 the invariant measure of the tree seen from the particle. Theorem 1.1 follows in Section 5.

2 Preliminaries

2.1 The space of words $\mathcal{U}$

We let $\mathcal{U} := \{e\} \cup \bigcup_{n \geq 1} (\mathbb{N}^*)^n$ be the set of words, and $|u|$ be the length of the word $u$, where we set $|e| := 0$. We equip $\mathcal{U}$ with the lexicographical order. For any word $u \in \mathcal{U}$ with label $u = i_1 \ldots i_n$, we denote by $\overline{u} \in \mathcal{U}$ the word with letters in reversed order $\overline{u} := i_n \ldots i_1$ (and $\overline{e} := e$). If $u \neq e$, we denote by $u_*$ the parent of $u$, that is the word $i_1 \ldots i_{n-1}$, and by $u_{*k}$ the word $i_1 \ldots i_{n-k}$, which stands for the ancestor of $u$ at generation $|u| - k$. We have $u_{*k} := e$ if $k = |u|$ and $u_{*k} := u$ if $k = 0$. Finally, for $u, v \in \mathcal{U}$, we denote by $uv$ the concatenation of $u$ and $v$. We add to the set of words the element $e_*$, which stands for the parent of the root and we write $\mathcal{U}_* := \mathcal{U} \cup \{e_*\}$. We set $|e_*| = -1$, hence $u_{*k} = e_*$ for $k = |u| + 1$ for any $u \in \mathcal{U}$. We denote by $S_x := \{x_{*k}, 1 \leq k \leq |x| + 1\}$ the set of strict ancestors of $x$.

2.2 The space of trees $\mathcal{T}$

Following Neveu \cite{Neveu}, a tree $T$ is defined as a subset of $\mathcal{U}$ such that

- $e \in T$,
- if $x \in T \setminus \{e\}$, then $x_* \in T$,
- if $x = i_1 \ldots i_n \in T \setminus \{e\}$, then any word $i_1 \ldots i_{n-1}j$ with $j \leq i_n$ belongs to $T$.

We call $\mathcal{T}$ the space of all trees $T$. For any tree $T$, we define $T_*$ as the tree on which we add the parent $e_*$ to the root $e$. Then, let $\mathcal{T}_* := \{T_*, T \in \mathcal{T}\}$. For a tree $T \in \mathcal{T}$, and a vertex $u \in T_*$, we denote by $\nu_T(u) = \nu_{T_*}(u)$ the number of children of $u$ in $T_*$, and we
notice that $\nu_T(e_{\ast}) = \nu_{T_{r}}(e_{\ast}) = 1$. We will write only $\nu(u)$ when there is no doubt about which tree we are dealing with.

We introduce double trees. For any $u \in U$, let $u^- := (u, -1)$ and $u^+ := (u, 1)$. Given two trees $T, T^+ \in \mathcal{T}$, we define the double tree $T \bullet T^+$ as the tree obtained by drawing an edge between the roots of $T$ and $T^+$. Formally, $T \bullet T^+$ is the set $\{u^-, u \in T\} \cup \{u^+, u \in T^+\}$. We root the double tree at $e^+$. Given $r$ an element of $T$, we say that $X$ is the $r$-parent of $Y$ in $T \bullet T^+$ if either

- $Y = y^+$ and $X = y^+_s$,
- $Y = e^+$ and $X = e^-$,
- $Y = y^-$ with $y \notin S_r \cup \{u \in U : u \geq r\}$ and $X = y^-_s$,
- $Y = r^-_{s_k}$ and $X = r^-_{s_{k-1}}$ for some $k \in [1, |r|]$.

In words, the $r$-parent of a vertex $x$ is the vertex which would be the parent of $x$ if we were 'hanging' the tree at $r$. Notice that we defined the $r$-parent only for the vertices which do not belong to $\{u^- : u \in U, u \geq r\}$.

![Figure 1: A double tree](image-url)
2.3 The backward tree $B_x(T_*)$

Let $\delta$ be some cemetery tree. For a tree $T_* \in \mathcal{T}_*$ and a word $x \in \mathcal{U}$, we define the tree $T_* \leq x \in \mathcal{T}_* \cup \{\delta\}$ cut at $x$ by

$$T_* \leq x := \begin{cases} 
\delta & \text{if } x \notin T_*, \\
T_* \setminus \{u \in \mathcal{U} : x < u\} & \text{if } x \in T_*. 
\end{cases}$$

![Diagram](image)

Figure 2: The backward tree at $x$

In other words, if $x \in T_*$, then $T_* \leq x$ is the tree $T_*$ in which you remove the strict descendants of $x$. We call $\mathcal{U}_* \leq x$ the set of words $\mathcal{U}_* \setminus \{u \in \mathcal{U} : x < u\}$. We now introduce the 'backward' tree at $x$. For any word $x \in \mathcal{U}$, let $\Psi_x : \mathcal{U}_* \leq x \to \mathcal{U}_* \leq x$ such that:

- for any $k \in [0, |x| + 1]$, $\Psi_x(x_k) = \overline{x}_{|x| - k + 1}$,

- for any $k \in [1, |x|]$ and $v \in \mathcal{U}$ such that $x_k v$ is not a descendant of $x_{k+1}$, $\Psi_x(x_k v) = \Psi_x(x_k) v$.

The application $\Psi_x$ is a bijection, with inverse map $\Psi_x$. For any tree $T_* \in \mathcal{T}_*$, we call backward tree at $x$ the tree

$$B_x(T_*) := \Psi_x(T_* \leq x),$$

image of $T_* \leq x$ by $\Psi_x$, with the notation that $\Psi_x(\delta) := \delta$. This is the tree obtained by cutting the descendants of $x$ and then 'hanging' the tree $T_*$ at $x$. We observe that,
\[ \nu_{B_x(T_*)}(e) = 1, \]
\[ \nu_{B_x(T_*)}(\bar{x}) = 0, \]
\[ \text{for any other } u \in B_x(T_*), \text{ we have } \nu_{B_x(T_*)}(u) = \nu_{T_*}(\Psi(x)). \]

Recall that \( T \) is a Galton–Watson tree with offspring distribution \( \nu \).

**Lemma 2.1.** Let \( x \in U \). The distributions of the trees \( B_x(T_*) \) and \( T_*^{\leq \bar{x}} \) are the same.

**Proof.** For any sequence \((k_u, u \in U) \in \mathbb{N}^U\), denote by \( M(k_u, u \in U) \in T_* \) the unique tree such that for any \( u \in M(k_u, u \in U) \) the number of children of \( u \) is 1 if \( u = e \), and \( k_u \) otherwise. Take \((\kappa(u), u \in U)\) i.i.d. random variables distributed as \( \nu \). Then notice that the tree \( M(\kappa(u), u \in U) \) is distributed as \( T_* \). Therefore, we set in this proof

\[ T_* := M(\kappa(u), u \in U). \]

We check that we can extend the map \( \Psi \) to a bijection on \( U \) by letting \( \Psi(\bar{x}v) := xv \) for any strict descendant \( \bar{x}v \) of \( \bar{x} \). Suppose that \( x \in T_* \). We know that if \( u \in B_x(T_*), \) then the number of children of \( u \) is 1 if \( u = e \), 0 if \( u = \bar{x} \) and \( \kappa(\Psi(u)) \) otherwise. By definition, this yields that

\[ B_x(T_*) = M(\kappa(\Psi(u))1_{\{u \neq \bar{x}\}}, u \in U). \]

Let \( \bar{T}_* := M(\kappa(\Psi(u)), u \in U) \). We notice that \( M(\kappa(\Psi(u))1_{\{u \neq \bar{x}\}}, u \in U) = \bar{T}_*^{\leq \bar{x}}. \)

Therefore, if \( x \in T_* \), then

\[ B_x(T_*) = \bar{T}_*^{\leq \bar{x}}. \]

We check that the equality holds also when \( x \notin T_* \). Observe that \( \bar{T}_* \) is distributed as \( T_* \) to complete the proof. \( \square \)

3 The environment seen from the particle at first-visit times

For any tree \( T_* \in T_* \), we denote by \( P^{T_*} \) a probability measure under which \( (X_n)_{n \geq 0} \) is a Markov chain on \( T_* \) with transition probabilities given by (1.1) and (1.2). For any vertex \( x \in T_* \), we denote by \( P^{T_*} \) the probability \( P^{T_*}(\cdot \mid X_0 = x) \). We will just write \( P_x \) if the tree \( T_* \) is clear from the context.
Lemma 3.1. Suppose that \( \lambda > 0 \). Let \( T_x \) be a tree in \( T_x \), \( x \) be a vertex in \( T_x \), and \((e_\ast = u_0, u_1, \ldots, u_n = x)\) be a nearest-neighbour trajectory in \( T_x \) such that \( u_j \notin \{e_\ast, x\} \) for any \( j \in (0, n) \). Then,

\[
P_{e_\ast}^T (X_j = u_j, \forall j \leq n) = P_{e_\ast}^{B_e^c(T_x)} (X_j = \Psi_x(u_{n-j}), \forall j \leq n).
\]

Proof. We decompose the trajectory \((u_j, j \leq n)\) along the ancestral path \( S_x \). Let \( j_0 := 0 \). Supposing that we know \( j_i \), we define \( j_{i+1} \) as the smallest integer \( j_{i+1} > j_i \) such that \( u_{j_{i+1}} \) is an ancestor of \( u_{j_i} \). Let \( m \) be the integer such that \( u_{j_m+1} = x \). We see that necessarily \( j_1 = 1 \), \((u_{j_0}, u_{j_1}) = (e_\ast, e)\) and \((u_{j_m}, u_{j_{m+1}}) = (x_\ast, x)\). For \( i \in [1, m] \), let \( c_i \) be the cycle \((u_{j_i}, u_{j_{i+1}}, \ldots, u_{j_{i+1}-1})\). Notice that in this cycle, the vertex \( u_{j_i} \) is the unique element of \( S_x \) visited, at least twice at times \( j_i \) and \( j_{i+1} - 1 \). We set for any cycle \( c = (z_0, z_1, \ldots, z_k) \),

\[
P_{e_\ast}^T (c) := \prod_{\ell = 0}^{k-1} P_{z_\ell}^T (X_1 = z_{\ell+1})
\]

with the notation that \( \prod_{\emptyset} := 1 \). Using the Markov property, we see that

\[
P_{e_\ast}^T (X_j = u_j, \forall j \leq n) = \prod_{i=1}^{m} P_{e_\ast}^T (c_i) \prod_{i=1}^{m} P_{u_{j_i}}^T (X_1 = u_{j_{i+1}}).
\]

For any vertex \( z \), let \( a(z) := (\lambda + \nu_{T_x}(z))^{-1} \). Notice that the term corresponding to \( i = m \) in the second product is

\[
P_{x_\ast}^T (X_1 = x) = a(x_\ast).
\]

For any \( z \neq e_\ast \), let \( N_u(z) \) be the number of times the oriented edge \((z, z_\ast)\) is crossed by the trajectory \((u_j, j \leq n)\). Notice that the oriented edge \((z_\ast, z)\) is crossed \( 1 + N_u(z) \) times when \( z \in S_x \). Using the transition probabilities (1.1) and (1.2), we deduce that

\[
\prod_{i=1}^{m} P_{u_{j_i}}^T (X_1 = u_{j_{i+1}}) = \prod_{k=1}^{x-1} \left( \lambda a(x_k) a(x_{k+1}) \right)^{N_u(x_k)} a(x_{k+1}).
\]

Therefore, we can rewrite (5.6) as

\[
P_{e_\ast}^T (X_j = u_j, \forall j \leq n) = \prod_1 \prod_2
\]

where

\[
\prod_1 := \prod_{i=1}^{m} P_{e_\ast}^T (c_i),
\]

\[
\prod_2 := a(x_\ast) \prod_{k=1}^{x-1} \left( \lambda a(x_k) a(x_{k+1}) \right)^{N_u(x_k)} a(x_{k+1}).
\]
We look now at the probability $P_{B_x(T_x)}^X(X_j = v_j, \forall j \leq n)$, where $v_j := \Psi_x(u_{n-j})$. We decompose the trajectory $(v_j, j \leq n)$ along $S_x$. Observe that $(v_j, j \leq n)$ is the time-reversed trajectory of $(u_j, j \leq n)$ looked in the backward tree. Therefore, the cycles of $(v_j, j \leq n)$ are the image by $\Psi_x$ of the time-reversed cycles of $(u_j, j \leq n)$. We need some notation. Let $c_i$ be the path $c_i$ time-reversed, and $\Psi_x\left(c_i^{-}\right)$ be its image by $\Psi_x$, that is

$$\Psi_x\left(c_i^{-}\right) = (\Psi_x(u_{j_i+1}), \Psi_x(u_{j_i+2}), \ldots, \Psi_x(u_{j_i})).$$

Let

$$p_{B_x(T_x)}^x\left(\Psi_x\left(c_i^{-}\right)\right) := \prod_{\ell = j_i}^{j_i+1-2} p_{B_x(T_x)}^x(X_1 = \Psi_x(u_{\ell}) | X_0 = \Psi_x(u_{\ell+1})).$$

We introduce for any vertex $z \in B_x(T_x)$,

$$a_B(z) := (\lambda + \nu_{B_x(T_x)}(z))^{-1}$$

and, for $z \neq e_x$, $N_v(z)$ the number of times the trajectory $(v_j, j \leq n)$ crosses the directed edge $(z, z^*)$. Equation (3.6) reads for the trajectory $(v_j, j \leq n)$,

$$P_{B_x(T_x)}^X(X_j = v_j, \forall j \leq n) = \Pi_{B,1} \Pi_{B,2}$$

where

$$\Pi_{B,1} := \prod_{i=1}^m p_{B_x(T_x)}^x\left(\Psi_x\left(c_i^{-}\right)\right),$$

$$\Pi_{B,2} := a_B(x) \prod_{k=1}^{\vert x \vert - 1} (\lambda a_B(x^{*k}) a_B(x^{*k+1}))^{N_v(x^{*k})} a_B(x^{*k+1}).$$

Going from $T_x$ to $B_x(T_x)$, we did not change the configuration of the subtrees located outside the ancestral path $S_x$ of $x$. This yields that $p_{B_x(T_x)}^x\left(\Psi_x\left(c_i^{-}\right)\right) = p_{T_x}^x\left(c_i^{-}\right)$ which is $p_{T_x}^x(c_i)$ since the Markov chain $(X_n)_{n \geq 0}$ is reversible. By definition of $\Pi_1$ in (3.7), we get

$$\Pi_{B,1} = \Pi_1.$$

We observe that $a_B(z) = a(\Psi_x(z))$ whenever $z \notin \{e_x, x\}$, and $\Psi_x(x^{*k}) = x_{\vert x \vert - k+1}$ by definition. Moreover, for any $k \in [1, \vert x \vert - 1]$, we have $N_v(x^{*k}) = N_u(x_{\vert x \vert - k})$. This gives
Proposition 3.2. Suppose that \( \lambda > 0 \). Let \( k \geq 1 \). Under \( P_{\epsilon^*} (\cdot \mid \theta_k < \tau_{e^*}) \), we have

\[
(\mathcal{B}_{\xi_k}(T_*), (\Psi_{\xi_k}(X_{\theta_k-j}))_{j \leq \theta_k}) \xrightarrow{(d)} (T_{\theta_k}, (X_j)_{j \leq \theta_k}) .
\]

Proof. For any relevant bounded measurable map \( F \) and any word \( x \in \mathcal{U} \), we have

\[
\mathbb{E}_{\epsilon^*} \left[ F \left( \mathcal{B}_{\xi_k}(T_*), (\Psi_{\xi_k}(X_{\theta_k-j}))_{j \leq \theta_k} \right) 1_{\{\xi_k = x, \theta_k < \tau_{e^*}\}} \right] = \mathbb{E}_{\epsilon^*} \left[ F \left( \mathcal{B}_{\xi_k}(T_*), (\Psi_{\xi_k}(X_{\theta_k-j}))_{j \leq \theta_k} \right) 1_{\{\xi_k = x, \theta_k < \tau_{e^*}\}} \right] = \mathbb{E}_{\epsilon^*} \left[ F \left( \tilde{T}_{\theta_k}, (\tilde{X}_j)_{j \leq \theta_k} \right) 1_{\{\tilde{X}_k = x, \tilde{\theta}_k < \tilde{\tau}_{e^*}\}} \right]
\]

by Lemma 3.1, where \( (\tilde{X}_n)_{n \geq 0} \) is the \( \lambda \)-biased random walk on the tree \( \mathcal{B}_x(T_*) \), and the variables \( \tilde{\theta}_k, \tilde{\xi}_k \) and \( \tilde{\tau}_{e^*} \) are the analogues of \( \theta_k, \xi_k \) and \( \tau_{e^*} \) for the Markov chain \( (\tilde{X}_n)_{n \geq 0} \). By Lemma 2.1, this yields that

\[
\mathbb{E}_{\epsilon^*} \left[ F \left( \mathcal{B}_{\xi_k}(T_*), (\Psi_{\xi_k}(X_{\theta_k-j}))_{j \leq \theta_k} \right) 1_{\{\xi_k = x, \theta_k < \tau_{e^*}\}} \right] = \mathbb{E}_{\epsilon^*} \left[ F \left( T_{\theta_k}, (X_j)_{j \leq \theta_k} \right) 1_{\{\xi_k = x, \theta_k < \tau_{e^*}\}} \right] = \mathbb{E}_{\epsilon^*} \left[ F \left( T_{\theta_k}, (X_j)_{j \leq \theta_k} \right) 1_{\{\xi_k = x, \theta_k < \tau_{e^*}\}} \right] .
\]

We complete the proof by summing over \( x \in \mathcal{U} \). \( \square \)
The last lemma gives the asymptotic probability that \( n \) is a first-visit time. To state it, we introduce the regeneration epochs \( (\Gamma_k, k \geq 1) \) defined by \( \Gamma_1 := \inf\{\ell \in \{0\} \cup \{\theta_k, k \geq 1\} : X_j \neq (X_\ell)_\ast \forall j \geq \ell, X_\ell \neq e\_\ast\} \) and for any \( k \geq 2, \)
\[
(3.12) \quad \Gamma_k := \inf\{\ell > \Gamma_{k-1} : \ell \in \{\theta_k, k \geq 1\}, X_j \neq (X_\ell)_\ast \forall j \geq \ell\},
\]
where \((X_\ell)_\ast\) stands for the parent of the vertex \( X_\ell \). For any \( k \geq 1 \), it is well-known that, under \( P \), the random walk after time \( \Gamma_k \) is independent of its past. Moreover, the walk \( (X_\ell, \ell \geq \Gamma_k) \) seen in the subtree rooted at \( X_{\Gamma_k} \) is distributed as \((X_\ell, \ell \geq 0)\) under \( P_e(\cdot | \tau_{e\_\ast} = \infty) \). We refer to Section 3 of [8] for the proof of such facts. We have that \( \Gamma_k < \infty \) for any \( k \geq 1 \) almost surely on the event \( S \) when \( \lambda < m \), and \( E_e[\Gamma_2 | \tau_{e\_\ast} = \infty] < \infty \) if and only if \( \lambda \in (\lambda_c, m) \).

**Lemma 3.3.** Suppose that \( m > 1 \) and \( \lambda \in (0, m) \). We have
\[
\lim_{n \to \infty} P_{e\_\ast}(n \in \{\theta_k, k \geq 1\}, \tau_{e\_\ast} > n) = \frac{1}{E_e[\Gamma_2 | \tau_{e\_\ast} = \infty]}.
\]

**Proof.** By the Markov property at time \( n \) and the branching property at vertex \( X_n \), we observe that
\[
P_{e\_\ast}(n \in \{\theta_k, k \geq 1\}, \tau_{e\_\ast} > n)P_e(\tau_{e\_\ast} = \infty) = P_{e\_\ast}(n \in \{\Gamma_k, k \geq 1\}, \tau_{e\_\ast} = \infty)
\]
and
\[
P_{e\_\ast}(n \in \{\theta_k, k \geq 1\}, \tau_{e\_\ast} > n) = P_{e\_\ast}(n \in \{\Gamma_k, k \geq 1\}) | \tau_{e\_\ast} = \infty).
\]
We notice that \( P_{e\_\ast}(n \in \{\Gamma_k, k \geq 1\}) | \tau_{e\_\ast} = \infty) = P_e(n - 1 \in \{\Gamma_k, k \geq 1\}) | \tau_{e\_\ast} = \infty). \)
We mention that \( \Gamma_1 = 0 \) on the event that \( \tau_{e\_\ast} = \infty \), when starting from \( e \). Since \((\Gamma_{k+1} - \Gamma_k, k \geq 1)\) are a sequence of i.i.d random variables under \( P_e(\cdot | \tau_{e\_\ast} = \infty) \) with mean \( E_e[\Gamma_2 | \tau_{e\_\ast} = \infty] \), the lemma follows from the renewal theorem pp. 360, XI.1 [4]. \( \square \)

### 4 Asymptotic distribution of the environment seen from the particle

We equip \( T \) and \( T_\ast \) with the topology generated by finite subtrees. For any tree \( T \in T \) and any \( x \in T_\ast \), let
\[
T_x := \{u \in T : u \geq x\}
\]
be the subtree composed of \( x \) and its descendants. We recall that \( B_x(T_\ast) \) defined in (2.4) is the tree backward. This section is devoted to the asymptotic distribution of the tree
seen from the particle. In other words, we want to know the joint distribution of the couple \((B_{X_n}(T_\ast), T_{X_n}) \in T_\ast \times T\) as \(n\) becomes large. Let \(T\) and \(T^+\) be two independent Galton–Watson trees. We recall that we labelled our trees with the space of words \(\mathcal{U}\). For any tree \(T_\ast \in T_\ast\) and any vertex \(x \neq e_\ast\), we can define \(\beta_{T_\ast}(x)\) as the probability that the biased random walk on \(T_\ast\) never hits \(x_\ast\) starting from \(x\). We write only \(\beta(x)\) when the tree \(T_\ast\) is clear from the context. We write in the following theorem \(\nu^+(e) := \nu_{T^+}(e)\), \(\beta(e) := \beta_{T_\ast}(e)\), \(\beta^+(i) := \beta_{T^+}(i)\).

**Theorem 4.1.** Suppose that \(m \in (1, \infty)\) and \(\lambda \in (\lambda_c, m)\). Under \(\mathbb{P}_{e_\ast}(\cdot | S)\), the couple \((B_{X_n}(T_\ast), T_{X_n})\) converges in distribution as \(n \to \infty\). The limit distribution has density

\[
C^{-1} \frac{(\lambda + \nu^+(e))\beta(e)}{\lambda - 1 + \beta(e) + \sum_{i=1}^{\nu^+(e)} \beta^+(i)}
\]

with respect to \((T_\ast, T^+)\), where \(C\) is the renormalising constant.

### 4.1 On the conductance \(\beta\)

In this section, let \(T_\ast \in T_\ast\) be a fixed tree, and write \(\beta(x), \nu(x)\) for \(\beta_{T_\ast}(x), \nu_{T_\ast}(x)\). The quantity \(\beta(e)\) is also called conductance of the tree, because of the link between reversible Markov chains and electrical networks, see [3]. It satisfies the recurrence equation

\[
(4.13) \quad \beta(e) = \frac{\sum_{i=1}^{\nu(e)} \beta(i)}{\lambda + \sum_{i=1}^{\nu(e)} \beta(i)}.
\]

Letting \(\beta_n(x)\) be the probability to hit level \(n\) before \(x_\ast\), we have actually, for \(n \geq 1\),

\[
(4.14) \quad \beta_n(e) = \frac{\sum_{i=1}^{\nu(e)} \beta_n(i)}{\lambda + \sum_{i=1}^{\nu(e)} \beta_n(i)}.
\]

This is easily seen from the Markov property. Indeed, notice that

\[
\beta_n(e) = \sum_{k \geq 0} P^T_{e_\ast}(\tau_e < \tau_e \land \tau_n)^k P^T_{e_\ast}(\tau_n < \tau_e)
\]

where \(\tau_n\) is the hitting time of level \(n\). Since

\[
P^T_{e_\ast}(\tau_e < \tau_e \land \tau_n) = \sum_{i=1}^{\nu(e)} \frac{1}{\lambda + \nu(e)}(1 - \beta_n(i))
\]

and

\[
P^T_{e_\ast}(\tau_n < \tau_e) = \sum_{i=1}^{\nu(e)} \frac{1}{\lambda + \nu(e)} \beta_n(i),
\]

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equation (4.14) follows. Let \( n \to \infty \) to get (4.13). The next lemma implies that the renormalizing constant in Theorem 4.1 is finite indeed.

**Lemma 4.2.** Suppose that \( m > 1 \) and \( \lambda \in (\lambda_c, m) \). We have

\[
\mathbb{E} \left[ \frac{1_S}{\lambda - 1 + \beta(e)} \right] < \infty.
\]

**Proof.** The statement is trivial if \( \lambda > 1 \). Suppose first that \( \lambda < 1 \). By coupling with a one-dimensional random walk, we see that on the event \( S \), we have \( \beta(e) \geq 1 - \lambda \). In particular, \( \beta_n(e) \geq 1 - \lambda \) for any \( n \geq 1 \). Use the recurrence equation (4.14) to get that

\[
(4.15) \quad \frac{\beta_n(e)}{\lambda - 1 + \beta_n(e)} = \frac{1}{\lambda} \frac{\sum_{i=1}^{\nu(e)} \beta_n(i)}{\lambda - 1 + \sum_{i=1}^{\nu(e)} \beta_n(i)}.
\]

On the event \( S \), there exists an index \( I \leq \nu(e) \) such that the tree rooted at \( I \) is infinite. Since \( \beta_n(I) \geq 1 - \lambda \), we see that

\[
\sum_{i \leq \nu(e), i \neq I} \beta_n(i) \leq 1.
\]

On the event that there exists \( J \neq I \) such that the tree rooted at \( J \) is also infinite, we have

\[
\frac{\beta_n(I)}{\lambda - 1 + \sum_{i=1}^{\nu(e)} \beta_n(i)} \leq \frac{\beta_n(J)}{\beta_n(I)} \leq \frac{1}{1 - \lambda}.
\]

We get that

\[
\mathbb{E} \left[ \frac{\sum_{i=1}^{\nu(e)} \beta_n(i)}{\lambda - 1 + \sum_{i=1}^{\nu(e)} \beta_n(i)} \right] \leq 1 + \frac{1}{1 - \lambda} + \mathbb{E} \left[ \frac{\beta_n(I)1_{\{\beta(j)=0 \forall j \neq I\}}}{\lambda - 1 + \sum_{i=1}^{\nu(e)} \beta_n(i)} \right]
\]

\[
= \frac{\lambda}{1 - \lambda} + \mathbb{E} \left[ \frac{\beta_n(I)1_{\{\beta(j)=0 \forall j \neq I\}}}{\lambda - 1 + \beta_n(I)} \right]
\]

\[
= \frac{\lambda}{1 - \lambda} + \mathbb{E} \left[ \nu q^{\nu-1} \right] \mathbb{E} \left[ \frac{\beta_{n-1}(e)}{\lambda - 1 + \beta_{n-1}(e)} \right].
\]

Recall that \( \lambda_c := \mathbb{E} [\nu q^{\nu-1}] \). In view of (4.15), we end up with, for any \( n \geq 1 \),

\[
\mathbb{E} \left[ \frac{\beta_n(e)}{\lambda - 1 + \beta_n(e)} \right] \leq 1 + \frac{\lambda_c}{1 - \lambda} \mathbb{E} \left[ \frac{\beta_{n-1}(e)}{\lambda - 1 + \beta_{n-1}(e)} \right].
\]

Applying the above inequality for \( n, n-1, \ldots, 1 \), we obtain that, for any \( \lambda \in (\lambda_c, 1) \) and any \( n \geq 1 \),

\[
\mathbb{E} \left[ \frac{\beta_n(e)}{\lambda - 1 + \beta_n(e)} \right] \leq \frac{1}{1 - \lambda} + \frac{1}{1 - \lambda} + \left( \frac{\lambda_c}{\lambda} \right)^n \frac{1}{\lambda}.
\]
Fatou’s lemma yields that
\[ E \left[ \frac{\beta(e)}{\lambda - 1 + \beta(e)} \right] \leq \frac{1}{1 - \lambda} \frac{1}{1 - (\lambda_e/\lambda)}. \]
Observe that \( E \left[ \frac{\beta(e)}{\lambda - 1 + \beta(e)} \right] \geq (1 - \lambda) E \left[ \frac{\lambda_e}{\lambda - 1 + \beta(e)} \right] \) to complete the proof in the case \( \lambda < 1 \).
In the case \( \lambda = 1 \), we have to show that
\[ \frac{1}{\beta_n(e)} = 1 + \frac{1}{\sum_{i=1}^{\nu(e)} \beta_n(i)}. \]
Let \( \varepsilon > 0 \). With \( I \) being defined as before, we check that, on the event \( S \),
\[ \frac{1}{\sum_{i=1}^{\nu(e)} \beta_n(i)} \leq \frac{1}{\beta_n(I)} \frac{1}{1} + \frac{1}{\varepsilon}. \]
Hence,
\[ E \left[ \frac{1}{\beta_n(e)} \right] \leq 1 + \varepsilon^{-1} E \left[ \frac{1}{\beta_n(I)} \frac{1}{1} + \frac{1}{\varepsilon} \right] \]
with \( q_\varepsilon := P(\beta(e) < \varepsilon) \). Notice that \( q_\varepsilon \to q \) as \( \varepsilon \to 0 \). Taking \( \varepsilon > 0 \) small enough such that \( \lambda_\varepsilon := E[\nu q_\varepsilon^{-1}] < 1 \), we have that
\[ E \left[ \frac{1}{\beta_n(e)} \right] \leq (1 + \varepsilon^{-1}) \frac{1}{1 - \lambda_\varepsilon} + \lambda_\varepsilon (1 - q). \]
Use Fatou’s lemma to complete the proof.

4.2 Random walks on double trees

Recall that we introduced the concepts of double trees and of \( r \)-parents in Section 2.2. For two trees \( T, T^+ \in \mathcal{T} \), and under some probability \( P_{\ast e^\bullet T^+} \), we introduce two Markov chains on the double tree \( T^\bullet T^+ \).
For any \( r \in T \), we define the biased random walk \( (Y_n^{(r)})_{n \geq 0} \) on \( T^\bullet T^+ \) with respect to \( r \) as the Markov chain, starting from \( e^+ \) which moves with weight \( \lambda \) to the \( r \)-parent of the current vertex, with weight 1 to the other neighbors and which is reflected at the vertex \( r^- \). In particular, \( Y_n^{(r)} \) never visits the subtree \( \{u^-, u > r\} \).
On the other hand, we define \( (Y_n)_{n \geq 0} \) the Markov chain on \( T^\bullet T^+ \) which has the transition probabilities of the biased random walk in \( T \) and in \( T^+ \). More precisely, if we set \((e_*, -1) := e^+\) and \((e_*, 1) := e^-\), the Markov chain \( (Y_n)_{n \geq 0} \), while being at \((u, \eta) \in U \times \{-1, 1\} \), goes to \((u_*, \eta)\) with weight \( \lambda \) and to \((u, \eta)\) with weight 1, this for every child \( u_\eta \) of \( u \) in \( T \) if \( \eta = -1 \) and every child \( u_\eta \) of \( u \) in \( T^+ \) if \( \eta = 1 \).

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Lemma 4.3. Let $T^- \bullet T^+$ be a double tree. Let $(e^+ = u_0, u_1, \ldots, u_n = e^+)$ be a sequence of vertices in $T^- \bullet T^+$ such that $u_k \notin \{u^-, u \geq r\}$ for any $k \leq n$. Denoting by $N_u(y, z)$ the number of crosses of the directed edge $(y, z)$ by the trajectory $(u_k)_{k \leq n}$, we have

$$P_{e^+T^+}^{T^-T^+} \left( Y^{(r)}_k = u_k, \forall k \leq n \right) = \lambda^{-N_u(e^+, e^-)} P_{e^+T^+}^{T^-T^+} (Y_k = u_k, \forall k \leq n).$$

Proof. Let $p^{(r)}(x, y)$, resp. $p(x, y)$, denote the transition probability of the walk $Y^{(r)}$, resp. the walk $Y$, from $x$ to $y$. We have

$$P_{e^+T^+}^{T^-T^+} \left( Y^{(r)}_k = u_k, \forall k \leq n \right) = \prod_{k=0}^{n-1} p^{(r)}(u_k, u_{k+1}).$$

Similarly,

$$P_{e^+T^+}^{T^-T^+} (Y_k = u_k, \forall k \leq n) = \prod_{k=0}^{n-1} p(u_k, u_{k+1}).$$

We notice that $p^{(r)}(u_k, u_{k+1}) = p(u_k, u_{k+1})$ if $u_k$ or $u_{k+1}$ does not belong to $\{r^{-}_{s}, \ell \in
We define $\lambda$. This gives that any $k \geq 1$.

(4.16) \[
\prod_{\ell=1}^{\lfloor r \rfloor} \left( p^{(r)}(r_{s_{t_{e}}}, r_{s_{t_{e+1}}}) \right)^{N_{u}(r_{s_{t_{e}}}, r_{s_{t_{e+1}}})} = \lambda^{-N_{u}(e^+, e^-)} \prod_{\ell=1}^{\lfloor r \rfloor} \left( p^{(r)}(r_{s_{t_{e}}}, r_{s_{t_{e+1}}}) \right)^{N_{u}(r_{s_{t_{e}}}, r_{s_{t_{e+1}}})}.
\]

This comes from the following observations: for any $\ell \in [1, |r| - 1]$, $p^{(r)}(r_{s_{t_{e}}}, r_{s_{t_{e+1}}}) = \lambda^{-1}p(r_{s_{t_{e+1}}}, r_{s_{t_{e}}})$ and $p^{(r)}(r_{s_{t_{e+1}}}, r_{s_{t_{e}}}) = \lambda p(r_{s_{t_{e+1}}}, r_{s_{t_{e}}})$. For $\ell = |r|$, we have $p^{(r)}(r_{s_{t_{e}}}, r_{s_{t_{e+1}}}) = \lambda^{-1}p(r_{s_{t_{e+1}}}, r_{s_{t_{e}}})$ and $p^{(r)}(r_{s_{t_{e+1}}}, r_{s_{t_{e}}}) = p^{(r)}(r_{s_{t_{e+1}}}, r_{s_{t_{e}}})$. Furthermore, $N_{u}(r_{s_{t_{e}}}, r_{s_{t_{e+1}}}) = N_{u}(r_{s_{t_{e+1}}}, r_{s_{t_{e}}})$ for any $\ell \in [1, |r|]$. A straightforward computation yields (4.16), and completes the proof.

For any $\ell \geq 0$, let $N_{\ell}^{T}(e^+, e^-) := \sum_{k=0}^{\ell-1} 1\{Y_{k}=e^+, Y_{k+1}=e^-\}$ with $\sum_{\emptyset} := 0$. We call $E^{T\bullet T^+}_{e^+}$ the expectation associated to the probability $P^{T\bullet T^+}_{e^+}$. In the next lemma, we write $\beta(x) = \beta_{T^+}(x)$, $\beta^+(x) = \beta_{T^+}(x)$ and $\nu^+(e) = \nu_{T^+}(e)$.

Lemma 4.4. Let $T \bullet T^+$ be an infinite double tree. We have

(4.17) \[
E^{T\bullet T^+}_{e^+} \left[ \sum_{\ell \geq 0} \lambda^{-N_{\ell}^{T}(e^+, e^-)} 1\{Y_{\ell}=e^+\} \right] = \frac{\lambda + \nu^+(e)}{\lambda - 1 + (e) + \sum_{i=1}^{\nu^+(e)} \beta^+(i)}.
\]

Proof. We compute the left-hand side. We observe that

\[
\sum_{\ell \geq 0} \lambda^{-N_{\ell}^{T}(e^+, e^-)} 1\{Y_{\ell}=e^+\} = \sum_{k \geq 0} \lambda^{-k} \sum_{\ell \geq 0} 1\{N_{\ell}^{T}(e^+, e^-)=k, Y_{\ell}=e^+\}.
\]

Let $(s_{k}, k \geq 0)$ be stopping times defined by

\[
s_{k} := \inf \{ \ell \geq 0 : N_{\ell}^{T}(e^+, e^-) = k \}.
\]

We define $t_{k} := \inf \{ \ell \geq s_{k} : X_{\ell} = e^+ \}$, and we have that $t_{0} = s_{0} = 0$. Notice that, for any $k \geq 0$,

\[
\sum_{\ell \geq 0} 1\{N_{\ell}^{T}(e^+, e^-)=k, Y_{\ell}=e^+\} = 1\{t_{k}<\infty\} \sum_{\ell=t_{k}}^{s_{k+1}} 1\{Y_{\ell}=e^+\}.
\]

This gives that

\[
E^{T\bullet T^+}_{e^+} \left[ \sum_{\ell \geq 0} \lambda^{-N_{\ell}^{T}(e^+, e^-)} 1\{Y_{\ell}=e^+\} \right] = \sum_{k \geq 0} \lambda^{-k} E^{T\bullet T^+}_{e^+} \left[ 1\{t_{k}<\infty\} \sum_{\ell=t_{k}}^{s_{k+1}} 1\{Y_{\ell}=e^+\} \right].
\]
By the strong Markov property at time \( t_k \), we have, for any \( k \geq 0 \),
\[
E_{e^+}^{T^*T^+} \left[ \sum_{\ell=t_k}^{s_k+1} 1\{Y_{\ell}=e^+\} \right] = P_{e^+}^{T^*T^+}(t_k < \infty) E_{e^+}^{T^*T^+} \left[ \sum_{\ell=0}^{s_1} 1\{Y_{\ell}=e^+\} \right].
\]
We see that \( P_{e^+}^{T^*T^+}(t_k < \infty) = [(1 - \beta^+(e))(1 - \beta^+(e))]^k \). Moreover, for \( \tau_{e^+}^Y := \inf\{n \geq 1 : Y_n = e^+\} \), we have \( P_{e^+}^{T^*T^+}(\tau_{e^+}^Y < s_1) = \frac{1}{1 + \nu^+(e)} \sum_{i=1}^{\nu^+(e)} (1 - \beta^+(i)) \). This yields that
\[
E_{e^+}^{T^*T^+} \left[ \sum_{\ell=0}^{s_1} 1\{Y_{\ell}=e^+\} \right] = \frac{1}{1 + \nu^+(e) \sum_{i=1}^{\nu^+(e)} \beta^+(i)}.
\]
Since \( T^*T^+ \) is infinite, we have by coupling with a one-dimensional random walk, \( \beta(e) > 1 - \lambda \) or \( \beta^+(e) > 1 - \lambda \). Hence \( \lambda^{-1}(1 - \beta^+(e))(1 - \beta(e)) < 1 \). We end up with
\[
E_{e^+}^{T^*T^+} \left[ \sum_{\ell=0}^{s_1} \lambda^{-\nu^+(e)\nu^+(e^-)} 1\{Y_{\ell}=e^+\} \right] = \frac{1}{1 + \lambda^{-1}(1 - \beta(e))(1 - \beta^+(e)) \sum_{i=1}^{\nu^+(e)} \beta^+(i)}.
\]
Apply the recurrence equation (4.13) to \( \beta^+(e) \) to complete the proof. \( \square \)

### 4.3 Proof of Theorem 4.1

**Proof of Theorem 4.1.** Let \( F_1 \) and \( F_2 \) be two bounded measurable functions respectively on \( T_* \) and \( T \) which depend only on a finite subtree. Recall the definition of the regeneration epochs \( (\Gamma_k, k \geq 1) \) in (3.12). We will show that

\[
\lim_{n \to \infty} E_{e_*} \left[ F_1(B_{X_n}(T_*)) F_2(T_{X_n}) 1_S \right] = \frac{\mathbb{P}(S)}{\mathbb{E}[\Gamma_2 1\{\tau_* = \infty\}]} \mathbb{E}_{e_2}^{\Gamma_2} \left[ F_1(T_*^*) F_2(T_2^*) \lambda - 1 + \beta(e) + \sum_{i=1}^{\nu^+(e)} \beta^+(i) \right],
\]

which proves the theorem. Let us prove (4.19). We first show that

\[
\lim_{n \to \infty} E_{e_*} \left[ F_1(B_{X_n}(T_*)) F_2(T_{X_n}) 1\{\tau_* > n\} \right] = \frac{1}{\mathbb{E}[\Gamma_2 1\{\tau_* = \infty\}]} \mathbb{E}^{\Gamma_2} \left[ F_1(T_*^*) F_2(T_2^*) \lambda - 1 + \beta(e) + \sum_{i=1}^{\nu^+(e)} \beta^+(i) \right].
\]

Let \( \varepsilon \in (0, 1) \) and, for any random tree \( T, S_T \) be the event that \( T \) is infinite. We deduce from dominated convergence that

\[
\mathbb{E}_{e_*} \left[ F_1(B_{X_n}(T_*)) F_2(T_{X_n}) 1\{\tau_* > n\} \right] = \mathbb{E}_{e_*} \left[ F_1(B_{X_n}(X_n)) F_2(T_{X_n}) 1\{\tau_* > n, |X_n| \geq n^\varepsilon\} 1_{S_{B_{X_n}(T_*)}} \right] + o_n(1).
\]
Recall the definition of $\theta_k$ and $\xi_k$ in (3.10) and (3.11). We have for any $n \geq 1$,

$$\mathbb{E}_{e^*} \left[ F_1(\mathcal{B}_{X_n}(\tau_*)) F_2(\tau_{X_n}) \mathbf{1}_{\{\tau_* > n, |X_n| \geq n^\varepsilon\}} 1_{S_{B_{X_n}(\tau_*)}} \right]$$

$$= \sum_{k \geq 1} \mathbb{E}_{e^*} \left[ F_1(\mathcal{B}_{\xi_k}(\tau_*)) F_2(\tau_{\xi_k}) \mathbf{1}_{\{X_n = \xi_k, \tau_* > n, |\xi_k| \geq n^\varepsilon\}} 1_{S_{B_{X_n}(\tau_*)}} \right].$$

We want to reroot the tree at $\xi_k$. Notice that $\tau_{\xi_k}$ is a Galton-Watson tree independent of $\mathcal{B}_{\xi_k}(\tau_*)$. By the strong Markov property at time $\theta_k$ and Proposition 3.2, we have that for any $k \geq 1$,

$$\mathbb{E}_{e^*} \left[ F_1(\mathcal{B}_{\xi_k}(\tau_*)) F_2(\tau_{\xi_k}) \mathbf{1}_{\{X_n = \xi_k, \tau_* > n, |\xi_k| \geq n^\varepsilon\}} 1_{S_{B_{X_n}(\tau_*)}} \right]$$

$$= \mathbb{E}_{e^*} \left[ F_1(\mathcal{T}_{\xi_k}^{\leq n^\varepsilon}) F_2(\tau_{\xi_k}^{\leq n^\varepsilon}) \mathbf{1}_{\{Y_n = e^+, \tau_{\xi_k}^{(e^+)} > \theta_k\}} \mathbf{1}_{\{\tau_* > \theta_k, |\xi_k| \geq n^\varepsilon\}} 1_{S_{\tau_{\xi_k}^{\leq n^\varepsilon}}} \right].$$

In the last expectation, the Markov chain $(X_n)_{n \geq 0}$ being the biased random walk on $\tau_{\xi_k}$ starting at $e^*$, the variables $\theta_k$, $\xi_k$ and $\tau_x$ are given by (3.10), (3.11) and (1.3). Moreover, conditionally on $\mathcal{T}$, $\tau^+$ and $\{X_{\tau^+}, \ell \leq \theta_k\}$, we take $(Y_n^{(e^+), \xi_k})_{n \geq 0}$ a biased random walk starting at $e^+$ with respect to $\xi_k$ on the double tree $\mathcal{T} \cdot T^+$ as defined in Section 4.2, and $\tau_{\xi_k}^{(e^+)} := \inf\{\ell \geq 1 : Y_{\tau^+}^{(e^+)} = (\xi_k, -1)\}$. Since $F_1$ depends only on a finite subtree, we get that for $n$ large enough

$$\mathbf{1}_{\{\tau_* > n, |X_n| \geq n^\varepsilon\}} 1_{S_{B_{X_n}(\tau_*)}}$$

(4.21) $$= \sum_{k \geq 1} \mathbb{E}_{e^*} \left[ F_1(\mathcal{T}_{\xi_k}^{\leq n^\varepsilon}) F_2(\tau_{\xi_k}^{\leq n^\varepsilon}) \mathbf{1}_{\{Y_n = e^+, \tau_{\xi_k}^{(e^+)} > \theta_k\}} \mathbf{1}_{\{\tau_* > \theta_k, |\xi_k| \geq n^\varepsilon\}} 1_{S_{\tau_{\xi_k}^{\leq n^\varepsilon}}} \right].$$

Lemma 4.3 implies that

$$\mathbf{1}_{\{\tau_* > n, |X_n| \geq n^\varepsilon\}} 1_{S_{B_{X_n}(\tau_*)}}$$

(4.22) $$= \sum_{k \geq 1} \mathbb{E}_{e^*} \left[ F_1(\mathcal{T}_{\xi_k}^{\leq n^\varepsilon}) F_2(\tau_{\xi_k}^{\leq n^\varepsilon}) \mathbf{1}_{\{Y_n = e^+, \tau_{\xi_k}^{(e^+)} > \theta_k\}} \mathbf{1}_{\{\tau_* > \theta_k, |\xi_k| \geq n^\varepsilon\}} 1_{S_{\tau_{\xi_k}^{\leq n^\varepsilon}}} \right].$$

where, conditionally on $\mathcal{T}$, $\tau^+$, the Markov chain $(Y_n)_{n \geq 0}$ is the biased random walk on the double tree $\mathcal{T} \cdot T^+$ as defined in Section 4.2, taken independent of $(X_n)_{n \geq 0}$, and $\tau_{\xi_k}^{(e^+)} := \inf\{\ell \geq 1 : Y_{\tau^+} = (\xi_k, -1)\}$. In view of (4.20), (4.21) and (4.22), we see that, as $n \rightarrow \infty$,

$$\mathbb{E}_{e^*} \left[ F_1(\mathcal{B}_{X_n}(\tau_*)) F_2(\tau_{X_n}) \mathbf{1}_{\{\tau_* > n\}} \right]$$

$$= \mathbb{E}_{e^*} \left[ F_1(\mathcal{T}_*) F_2(\tau^+) \sum_{k \geq 1} \lambda^{-N_{n-\theta_k}(e^+, e^-)} \mathbf{1}_{\{Y_n = e^+, \tau_{\xi_k}^{(e^+)} > \theta_k\}} \mathbf{1}_{\{\tau_* > \theta_k, |\xi_k| \geq n^\varepsilon\}} 1_{S_{\tau_{\xi_k}^{(e^+)}}} \right] + o_n(1).$$
Lemma 4.2 shows that

\[ \mathbb{E}_e \left[ F_1(T_s) F_2(T^+) \sum_{k \geq 1} \lambda^{-N_{n-\theta_k}(e^+,e^-)} 1_{\{ Y_{n-\theta_k} = e^+, \tau_{\xi_k} > n-\theta_k \}} 1_{\{ \tau_{\xi_k} > \theta_k, |\xi_k| \geq n^\varepsilon \}} 1_{S_{\tau_{\xi_k}} \leq \ell_k} \right] \]

\[ \mathbb{E}_e \left[ F_1(T_s) F_2(T^+) \sum_{\ell=0}^{n-1} \lambda^{-N_{Y}\ell}(e^+,e^-) 1_{\{ Y_{\ell} = e^+ \}} 1_{\{ \tau_{\xi_k} > n-\ell, |X_{n-\ell}| \geq n^\varepsilon, \tau_{X_{n-\ell}} > \ell, n-\ell \in \{ \theta_k, k \geq 1 \} \}} 1_{S_{\tau_{X_{n-\ell}}}} \right] . \]

We choose \( u \) deterministic integer such that

\[ |u| \geq K - 1 \].

Notice that necessarily, \( |X_{\Gamma_K}| \geq K - 1 \). In particular, \( F_1(T_s) \) is independent
of the subtree rooted at $X_{\Gamma_K}$. Recall that $T^+$ is independent of $T_\ast$, hence of $(X_n)_n$ as well. Using the regenerative structure of the walk $(X_n)_n$ at time $\Gamma_K$, we get that

$$
\mathbb{E}_{e_\ast} \left[ F_1(T_\ast) F_2(T^+) \sum_{\ell \geq 0} \lambda^{-N^Y_\ell(\ell^+,-\ell^-)} 1_{\{\ell_\ast = \infty, n - \ell \in \{\theta_k, k \geq 1\}, n - \ell \geq \Gamma_K}\} \right]
$$

$$
= \mathbb{E}_{e_\ast} \left[ F_1(T_\ast) F_2(T^+) \sum_{\ell \geq 0} \lambda^{-N^Y_\ell(\ell^+,-\ell^-)} 1_{\{\ell_\ast = \infty, n - \ell \in \{\theta_k, k \geq 1\}, n - \ell \geq \Gamma_K}\} b_{n - \ell - \Gamma_K} \right]
$$

with, for any integer $i \geq 0$, $b_i := \mathbb{P}_e(i \in \{0\} \cup \{\theta_k, k \geq 1\} | \ell_\ast = \infty)$. Lemma 3.3 says that $b_i \rightarrow \mathbb{E}_{[\Gamma_2 | \ell_\ast = \infty]}$ as $i \rightarrow \infty$, hence

$$
\lim_{n \rightarrow \infty} \mathbb{E}_{e_\ast} \left[ F_1(T_\ast) F_2(T^+) \sum_{\ell \geq 0} \lambda^{-N^Y_\ell(\ell^+,-\ell^-)} 1_{\{\ell_\ast = \infty, n - \ell \in \{\theta_k, k \geq 1\}, n - \ell \geq \Gamma_K}\} \right]
$$

$$
= \frac{1}{\mathbb{E}_{[\Gamma_2 | \ell_\ast = \infty]} \mathbb{E}_{e_\ast} \left[ F_1(T_\ast) F_2(T^+) \sum_{\ell \geq 0} \lambda^{-N^Y_\ell(\ell^+,-\ell^-)} 1_{\{\ell_\ast = \infty\}} \right]} \mathbb{E}_{e_\ast} \left[ F_1(T_\ast) F_2(T^+) \sum_{\ell \geq 0} \lambda^{-N^Y_\ell(\ell^+,-\ell^-)} 1_{\{\ell_\ast = \infty\}} \right].
$$

Consequently,

$$
\lim_{n \rightarrow \infty} \mathbb{E}_{e_\ast} \left[ F_1(B_{X_n}(T_\ast)) F_2(T_n) 1_{\{\ell_\ast > n\}} \right]
$$

$$
= \frac{1}{\mathbb{E}_{[\Gamma_2 | \ell_\ast = \infty]} \mathbb{E}_{e_\ast} \left[ F_1(T_\ast) F_2(T^+) \sum_{\ell \geq 0} \lambda^{-N^Y_\ell(\ell^+,-\ell^-)} 1_{\{\ell_\ast = \infty\}} \right]} \mathbb{E}_{e_\ast} \left[ F_1(T_\ast) F_2(T^+) \sum_{\ell \geq 0} \lambda^{-N^Y_\ell(\ell^+,-\ell^-)} 1_{\{\ell_\ast = \infty\}} \right].
$$

Recall that $\beta(e) = \mathbb{P}^T_{e_\ast}(\ell_\ast = \infty)$ by definition. Then apply Lemma 4.4 to complete the proof of (4.19). It remains to remove the conditioning on $\{\ell_\ast > n\}$ on the left-hand side. Fix $\ell \geq 1$. For $n \geq \ell$, we have by the Markov property,

$$
\mathbb{E}_{e_\ast} \left[ F_1(B_{X_n}(T_\ast)) F_2(T_n) 1_{\{\ell_\ast = \ell\}} \right] = \mathbb{E}_{e_\ast} \left[ 1_{E_\ell} \phi(X_\ell, n - \ell) \right]
$$

where, for any $k \geq 0$ and $x \in T_\ast$,

$$
\phi(x, k) := \mathbb{E}_x \left[ F_1(B_{X_k}(T_\ast)) F_2(T_k) 1_{\{\ell_\ast = \infty\}} \right]
$$

and, for any $\ell \geq 0$, $E_\ell$ is the event that $X_\ell \neq e_\ast$ and that at time $\ell$, every (non-directed) edge that has been visited has been visited at least twice, except the edge between $X_\ell$ and its parent. Since $F_1$ depends on a finite subtree, we can use, when $|X_{n-\ell}|$ is big enough (actually greater than $K - 1$), the branching property for the Galton–Watson tree at the vertex $X_\ell$ to obtain that

$$
\mathbb{E}_{e_\ast} \left[ 1_{E_\ell} \phi(X_\ell, n - \ell) \right] = \mathbb{P}_{e_\ast} (E_\ell) \mathbb{E}_e \left[ F_1(B_{X_{n-\ell}}(T_\ast)) F_2(T_{n-\ell}) 1_{\{\ell_\ast = \infty\}} \right] + o_n(1).
$$
Notice that, for any \( n - \ell \geq 0 \),
\[
E_e \left[ F_1(\mathcal{B}_{X_{n-\ell}}(T_*))F_2(T_{X_{n-\ell}})1_{\{\tau_* = \infty\}} \right] = E_e \left[ F_1(\mathcal{B}_{X_{n-\ell+1}}(T_*))F_2(T_{X_{n-\ell+1}})1_{\{\tau_* = \infty\}} \right].
\]

Equation (4.19) implies that
\[
\lim_{n \to \infty} E_{\tau_*} \left[ F_1(\mathcal{B}_{X_n}(T_*))F_2(T_{X_n})1_{\{\Gamma_1 = \ell\}} \right] = \frac{1}{E_{\tau_*}[\Gamma_2 | \tau_* = \infty]} \mathbb{E} \left[ F_1(T_*)F_2(T^+) \frac{(\lambda + \nu^+(e))\beta(e)}{\lambda - 1 + \beta(e) + \sum_{i=1}^{\nu^+(e)} \beta^+(i)} \right].
\]

Since \( \{\Gamma_1 < \infty\} = S \), we deduce that
\[
\lim_{n \to \infty} E_{\tau_*} \left[ F_1(\mathcal{B}_{X_n}(T_*))F_2(T_{X_n})1_S \right] = \sum_{\ell \geq 1} \mathbb{P}_{\tau_*}(E_{\ell}) \mathbb{E}_{\tau_*} \left[ F_1(T_*)F_2(T^+) \frac{\lambda + \nu^+(e)}{\lambda - 1 + \beta(e) + \sum_{i=1}^{\nu^+(e)} \beta^+(i)} \right].
\]

We notice that \( \mathbb{P}_{\tau_*}(E_{\ell})\mathbb{P}_{\tau_*}(\tau_* = \infty) = \mathbb{P}_{\tau_*}(\Gamma_1 = \ell) \), hence
\[
\sum_{\ell \geq 1} \mathbb{P}_{\tau_*}(E_{\ell}) = \frac{\mathbb{P}(S)}{\mathbb{P}_{\tau_*}(\tau_* = \infty)}.
\]

This proves (4.18), hence the theorem. \( \square \)

5 Proof of Theorem 1.1

Proof. By dominated convergence, we have \( \ell_\lambda = \lim_{n \to \infty} E_{\tau_*} \left[ \frac{|X_n|}{n} \right] \). We observe that
\[
E_{\tau_*} \left[ |X_n| \right] = \sum_{k=0}^{n-1} E_{\tau_*} \left[ |X_{k+1} - |X_k| \right] = \sum_{k=0}^{n-1} E_{\tau_*} \left[ \frac{\nu(X_k) - \lambda}{\nu(X_k) + \lambda} \right].
\]

Use Theorem 4.1 to complete the proof. \( \square \)

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