USING STRICTLY ISOSPECTRAL UNBROKEN NONRELATIVISTIC SUPERSYMMETRY AS A TOY MODEL

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Summary: I employ heuristically the strictly isospectral double Darboux method based on the general superpotential of unbroken nonrelativistic supersymmetry suggesting a few small steps of principle for extending its range of applications toward relativistic (gauge) physics. The application of the method to minisuperspace quantum cosmology is also briefly presented.

PACS: 11.30.Pb - Supersymmetry.
PACS: 04.60 - Quantum gravity.

1. Introduction

As recently emphasized by Poppitz and Trivedi [1], from the point of view of the concept of supersymmetry breaking the difference between supersymmetric quantum mechanics (SUSYQM) and the 3+1 dimensional renormalizable supersymmetric field theory with spin-0 and spin-1/2 fields is only minimal in the sense that the “spin-orbit” term in the first case corresponds to the Yukawa interaction between the bosons and fermions in the supermultiplet in the second case. This can also be seen by noting that the N=1 four-dimensional SUSY algebra

\[
\{Q_{\alpha}, \bar{Q}_{\dot{\alpha}}\} = -2\sigma^\mu_{\alpha,\dot{\alpha}}P_\mu
\]

reduces, in the rest frame of the system, \(P_0 = H, \bar{P} = 0\), after appropriate scaling, to the nonrelativistic SUSYQM algebra. Thus, it might be possible that lessons from the simple (toy) SUSYQM be useful when extrapolated in an unambiguous way to the complicated field theories. As a matter of fact, at the conceptual level, one may well make use of toy models for better pointing out the main ideas. Here, the toy (simple) idea is a double Darboux, strictly isospectral scheme of unbroken SUSYQM first discussed in the 80’s by Mielnik [3] but not so well known even to people working in SUSYQM (for recent applications see [4]). In this work, the acronym DDGR (double Darboux general Riccati) will be used for this mathematical scheme. The key feature of the DDGR method is to introduce a sort of free parameters labeling the DDGR-modulated degenerate vacua.

Usually, in quantum field theories the full unbroken supersymmetry is considered as an uninteresting case. The favored idea is that of supersymmetry unbroken at the tree level, but broken due to instanton tunneling, because the instanton calculus can generate small scales therefore explaining the hierarchy of scales in nature. Moreover, the Euclidean instanton calculus is also basic in the fundamental topic of the quantum creation of the universe. However, within the full unbroken nonrelativistic SUSY, the DDGR construction can produce degenerate true zero modes, that may be considered as nonrelativistic counterparts of quantum field degenerate vacua. This important fact is practically unknown to field theorists.

The organization of the paper is the following. In the next section, I provide a short but detailed sketch of the DDGR scheme and present its main ingredients. In section 3, I comment on the application to minisuperspace quantum gravity/cosmology where the appropriate DDGR parameter can be determined from a stationary phase approximation and I end up with conclusion.
2. The DDGR construction

Witten’s SUSYQM looks so simple that apparently is well known to every layman and will not be repeated here. There is however more than that as first shown by Mielnik for the harmonic oscillator case. It is possible to construct families of potentials strictly isospectral with respect to the initial (bosonic) one, if one asks for the most general superpotential (i.e., the general Riccati solution) such that \( V_+(x) = w_+^2 + \frac{dw_+}{dx} \), where \( V_+ \) is the fermionic partner potential. It is easy to see that one particular solution to this equation is \( w_p = w(x) \), where \( w(x) \) is the common Witten superpotential. One is led to consider the following Riccati equation

\[
\frac{dw(x)}{dx} = w(x)^2 + 2w(x)\frac{dw_0(x)}{dx},
\]

where \( w(x) \) is an unknown function. Using this ansatz, one obtains for the function \( w(x) \) the following Bernoulli equation

\[
\frac{dv(x)}{dx} - 2v(x)w_p(x) = 1,
\]

that has the solution

\[
v(x) = I_0(x) + \lambda u_0^2(x),
\]

where \( I_0(x) = \int x u_0^2(y) dy \), and \( \lambda \) is an integration constant thereby considered as a free DDGR parameter. Thus, \( w_p(x) \) can be written as follows

\[
w_p(x; \lambda) = w_p(x) + \frac{d}{dx} \ln \left( I_0(x) + \lambda \right)
\]

Finally, one easily gets the \( V_-(x; \lambda) \) family of potentials

\[
V_-(x; \lambda) = w_+^2(x; \lambda) - \frac{dw_+(x; \lambda)}{dx}
\]

All \( V_-(x; \lambda) \) have the same supersymmetric partner potential \( V_+(x) \) obtained by deleting the ground state. They are asymmetric double-well potentials that may be considered as a sort of intermediates between the bosonic potential \( V_-(x) \) and the fermionic partner \( V_+(x) = V_-(x) - 2\sigma_{0,x}(x) \), where \( \sigma_{0,x}(x) = \frac{d^2}{dx^2} \ln u_0 \), is the notation for logarithmic derivatives in the book of Matveev and Salle. From Eq. (4c) one can infer the ground state wave functions for the potentials \( V_-(x; \lambda) \) as follows

\[
u_0(x; \lambda) = f(\lambda) \frac{u_0(x)}{I_0(x) + \lambda},
\]

where \( f(\lambda) \) is a normalization factor that can be shown to be of the form \( f(\lambda) = \sqrt{\lambda(\lambda + 1)} \). One can now understand the double Darboux feature of the strictly isospectral construction by writing the parametric family in terms of their unique “fermionic” partner

\[
V_-(x; \lambda) = V_+(x) - 2\frac{d^2}{dx^2} \ln \left( \frac{1}{u_0(x; \lambda)} \right),
\]

which shows that the DDGR transformation is of the inverse Darboux type, allowing at the same time a two-step (double Darboux) interpretation, namely, in the first step one goes to the fermionic system and in the second step one returns to a deformed bosonic system.
From the normalization factor one can see that in the \( \lambda \)-parameter space the interval \([-1,0]\) is forbidden. A connection with other isospectral methods has been found, by noticing that the limiting values \(-1\) and \(0\) for the parameter \(\lambda\) lead to the Abraham-Moses procedure \([8]\) and Pursey’s one \([9]\), respectively. Actually, the discussion is more involved because of the singularities that may appear both in the strictly isospectral solutions and potentials, see \([10]\). If the normalization factor \(f(\lambda)\) is included all \(u_0(x; \lambda)\) are true Schroedinger zero modes labeled by \(\lambda\). According to our experience \([4]\), the denominator of these DDGR zero modes acts as a modulational factor, introducing some additional structure in the shape of the zero mode as a result of the form of the DDGR double-well potentials. More details on the construction such as the connection with the general zero-energy Schroedinger solution and an intertwining operator approach can be found in my recent work in collaboration \([10]\). The DDGR method can be applied to any one-dimensional system, whose dynamics is dictated by a Schroedinger(-like) equation. Moreover, one can employ combinations of any pairs of Abraham-Moses procedure, Pursey’s one, and the Darboux one. However, only the DDGR method leads to reflection and transmission amplitudes identical to those of the original potential, showing the complete degeneracy produced by such a construction. Moreover, the scheme can be used iteratively leading to multiple-parameter families of solutions \([11, 10]\).

Now, let us present some small steps toward extrapolating DDGR to much more complicated theories. In my view, if this will prove possible, the following philosophy is suggested by the DDGR method.

(i) The fermionic system, as it stands in the Schroedinger-Riccati “entanglement” \([12]\), is merely an intermediate step of the mathematical procedure.

(ii) If one assigns physical meaning to the DDGR vacua, since they are parametrically defined, one may think of fixing the parameter (and thus the bosonic vacuum) through some mathematical procedure. In the next section, for the example of minisuperspace cosmology, it is briefly shown how one can fix the parameter through a stationary phase approximation.

(iii) One can also claim that an extrapolation of the DDGR construction to gauge theories might offer a different perspective on high energy physics, allowing the freedom of selecting the true bosonic vacuum from a DDGR parametric family of vacua.

3. DDGR methods in quantum cosmology

It is well known that any system invariant under spacetime reparametrizations has a vanishing Hamiltonian; for a compact discussion the reader is directed to a paper by Gamboa and Zanelli \([15]\). Such a situation is common in quantum gravity and cosmology \([16]\). Quantum cosmology is an area dominated by the Wheeler-DeWitt (WDW) equation. In some simple, minisuperspace models the WDW equation can be reduced to a stationary Schroedinger-like equation at zero energy. In papers with Socorro, we have already applied DDGR to a couple of minisuperspace models \([13]\). Generically, a WDW equation of DDGR type can be written down as follows

\[
-\frac{d^2\Psi(\Omega; \lambda)}{d\Omega^2} + V(\Omega, \lambda)\Psi(\Omega; \lambda) = 0,
\]

(8)

where, as an example, \(\Omega\) is Misner’s minisuperspace variable \([14]\), and \(\Psi(\Omega; \lambda)\) are solutions of the type given by Eq. (6), either with or without the normalization factor. Here I pose the problem of what would be an appropriate \(\lambda\) in quantum cosmology. To answer this question, I recall that Salopek \([17]\) discussed an interesting (semiclassical) principle of superposition for Hamilton-Jacobi (HJ) theory that applies to Schroedinger solutions which depend on a continuous parameter. For Schroedinger solutions \(\psi(x; \lambda)\) depending on a continuous parameter \(\lambda\) any linear superposition is also a solution

\[
\psi(x) = \int d\lambda p(\lambda)\psi(x; \lambda),
\]

(9)

where the weighting function \(p(\lambda)\) is arbitrary. If we work in the semiclassical limit, \(\hbar \to 0\), then \(\psi(x; \lambda)\) and \(p(\lambda)\) may be approximated by phase factors

\[
\psi(x; \lambda) \approx e^{iS(x; \lambda)/\hbar}, \quad p(\lambda) = e^{ig(\lambda)/\hbar}.
\]

(10)

\(S(x; \lambda)\) is then a solution of the HJ equation which depends on the parameter \(\lambda\). As remarked by Salopek, if \(S\) is real, then one deals with classical phenomena, whereas if \(S\) is complex, one may describe quantum
phenomena such as tunneling or the initial wavefunction of the universe. The superposition integral may be approximated using the stationary phase approximation

\[ \psi(x) = \exp\left[i(S(x; \lambda_{st}) + g(\lambda_{st}))/\hbar\right], \]  

(11)

where \( \lambda_{st} = \lambda(x) \) is now chosen so that the phase of the integrand has a maximum or minimum, i.e.,

\[ \frac{\partial}{\partial \lambda} \left[S(x; \lambda) + g(\lambda)\right] = 0, \]  

(12)

for \( \lambda = \lambda_{st} \).

To implement these ideas within the DDGR solutions one should consider the superposition

\[ \Phi_0(\Omega) = \int d\lambda p(\lambda) \Psi_0(\Omega; \lambda), \]  

(13)

as the most appropriate WDW cosmological solution and apply a stationary phase approximation as above. Since the DDGR solutions are of the type \( \Psi_0(\Omega; \lambda) \propto e^{-\int_{\Omega} w_g dy} \), one gets \( S \propto -i\hbar \int_{\Omega} w_g dy \), i.e, \( S \propto -i\hbar \Psi_0(\Omega; \lambda) \) and the condition of stationary phase approximation reads

\[ i \frac{\partial}{\partial \lambda} \left[- \Psi_0(\Omega; \lambda) + g(\lambda)\right] = 0, \]  

(14)

from which one can get \( \lambda_{st}(\Omega) \).

### 4. Conclusion

I have tackled the toy idea of DDGR supersymmetric scheme, presenting its main ingredients. It might be that, as suggested in this work, an unambiguous extrapolation of this construction to relativistic (gauge) physics, if possible, may show up interesting consequences, such as parametric families of DDGR bosonic field vacua. As commented, for the case of minisuperspace quantum cosmology, one can determine the appropriate DDGR cosmological parameter from a stationary phase approximation.

This work was supported in part by the CONACyT project 458100-5-25844E. The author wishes to thank the referee for his comments.

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