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ON STABILITY AND STABILIZATION OF ELASTIC SYSTEMS BY TIME-VARIANT FEEDBACK

Abstract. We study a class of elastic systems described by a (hyperbolic) partial differential equation. Our working example is the equation of a vibrating string subject to linear disturbance. The main goal is to establish conditions for stabilization and asymptotic stabilization by applying a fast oscillating control to the string. In the first situation studied we assume that system is subject to a damping force; next we consider the system without damping. We extend the tools of high-order averaging and of chronological calculus for studying stability of this distributed parameter system.

1. Introduction

Stability and stabilization by imposing fast oscillations onto a system have been extensively studied in literature. Basic example is stabilization of equilibrium of inverted pendulum by means of fast harmonic oscillation of its suspension point, \[1\]. Methods of high-order averaging developed in \[2\] are applicable to study stability and asymptotic stability for a wider class of (non-harmonically) time-variant systems. As an illustration, a stabilization condition for pendulum under non-harmonic (fast) oscillation of its suspension point has been established. An extension of this work onto the case of planar and spherical double inverted pendulum was done in \[3\].

In the present work we apply the latter approach to a class of distributed parameter systems: we study a possibility to stabilize an elastic system (a string) destabilized by a linear disturbance. A condition for asymptotic stability under application of a state feedback control to the damped system is presented. We also obtain a stability condition for the undamped vibrating string under an output feedback control.

Our main results are Theorem 5.1 in Section 5 and Theorem 6.1 in Section 6. Our problem setting in Section 5 is close to the one of \[4\] but the class of stabilizers and the results differ; we provide some comments on it in Section 7.

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2. Notation

Let \( \Omega \subset \mathbb{R} \) be a closed interval. As usual \( L_2(\Omega) \) is the space of all measurable (in the sense of Lebesgue) functions on \( \Omega \) having finite norm

\[
\|u\| = \left( \int_\Omega u^2 \, dx \right)^{1/2}.
\]

The inner product in \( L_2(\Omega) \) is denoted by \( \langle \cdot, \cdot \rangle \). The Hilbert space consisting of the elements \( L_2^m(\Omega) \) which possess generalized derivatives up to order \( m \) also in \( L_2(\Omega) \), is denoted by \( W_2^m(\Omega) \). The norm in this space is denoted by \( \| \cdot \|_m \).

Bellow, \( W_2^1(\Omega) \) stands for a subspace of the space \( W_2^1(\Omega) \) which coincides with the closure of the set of infinitely differentiable functions with compact supports in \( \Omega \).

Let \( \{\phi_n(x)\} \) be a orthonormal basis of \( L_2(\Omega) \). We denote by \( G_N \) be the subspace of \( L_2(\Omega) \) spanned by \( \{\phi_n(x)\}_{n=1}^N \) and by \( G_N^\perp \) the orthogonal complement of \( G_N \). For an element \( v \in L_2(\Omega) \) we denote by \( v_N \) its projection onto \( G_N \) and by \( v^N \) the projection onto \( G_N^\perp \).

3. Problem setting

We consider forced elastic system described by a (hyperbolic) partial differential equation. In particular, we consider the equation for an elastic string

\[
u_{\tau \tau} = a^2 u_{xx}, \quad x \in \Omega
\]

where \( u_\xi \) denotes the partial derivative of \( u \) with respect to a variable \( \xi \) and the domain \( \Omega = [0, 2\pi] \). We fix boundary conditions

\[
u(x, \tau) = 0, \quad x \in \partial \Omega.
\]

We introduce into (4) a disturbance \( \gamma^2 u \) and the equation becomes

\[
u_{\tau \tau} = a^2 u_{xx} + \gamma^2 u
\]

whose zero solution is unstable if \( \gamma^2 > a^2/4 \).

If in addition, we assume that there is a damping, then (4) turns into

\[
u_{\tau \tau} = a^2 u_{xx} - \alpha u_\tau + \gamma^2 u, \quad x \in \Omega
\]

with damping coefficient \( \alpha > 0 \).

We would like to achieve stabilization (asymptotic stabilization) of the null solution of (3) (4) by applying time-variant high-gain linear feedback control term

\[
h(u, \kappa \tau) = \delta k^2 g(\kappa \tau) u.
\]
The resulting “controlled” equations is

\[ u_{\tau \tau} = a^2 u_{xx} + \gamma^2 u + h(u, k \tau). \]  

and

\[ u_{\tau \tau} = a^2 u_{xx} - \alpha u_{\tau} + \gamma^2 u + h(u, k \tau). \]

The function \( g(\tau) \) is assumed to be bounded, continuous and 1-periodic, \( \delta > 0 \) is a small parameter and \( k \geq 1 \) can be chosen arbitrarily. Besides, we will assume for \( g \):

**Assumption 1.** \( \int_0^1 g(\xi) \, d\xi = 0; \)

**Assumption 2.** For \( G(t) = \int_0^t g(\xi) \, d\xi \) there holds

\[ \int_0^1 G(\xi) \, d\xi = 0. \]

Observe that, under these assumptions, the time-average of the vibrational term vanishes, in a way that the averaged equations (6) and (7), coincide with the equations (3) and (4), respectively. Therefore these averages are unstable.

For given initial data

\[ u(x, 0) = \varphi(x), \quad u_\tau(x, 0) = \psi(x), \]

where \( \varphi \in W^2_2(\Omega) \cap W^1_2(\Omega) \) and \( \psi \in W^1_2(\Omega) \) the corresponding (classical) solution for (6), (2) and (7), (2) exists and is unique in \( W^2_2(Q_T) \), \( Q_T = \Omega \times (0, T) \). By a (classical) solution we mean an element of \( W^2_2(\Omega) \) which satisfies the equation for all \( t \) and for almost all \( x \in \Omega \). Next we precis the meaning of stability and asymptotic stability of the null solution.

**Definition 1 (Stability).** The null solution \( \tilde{u}(x, t) \equiv 0 \) of (6), (2) is stable if for every \( \eta > 0 \) there exists \( 0 < \xi < \eta \) such that for any solution \( u(x, t) \) with \( \|u(x, 0)\|^2_1 + \|u_t(x, 0)\|^2 < \xi \) implies

\[ \|u(x, t)\|^2_1 + \|u_t(x, t)\|^2 < \eta \]

for all \( t \geq 0 \).

**Definition 2 (Asymptotic stability).** The null solution \( \tilde{u}(x, t) \equiv 0 \) of (7), (2) is exponentially asymptotically stable if it is stable and in addition there exist \( \sigma, C > 0 \) such that

\[ \|u(\cdot, t)\|^2_1 + \|u_t(\cdot, t)\|^2 \leq C e^{-\sigma t}(\|u(x, 0)\|^2_1 + \|u_t(x, 0)\|^2), \]

for all \( t \geq 0 \).
For convenience we perform the change of the time variable $t = k \tau$. The transformed equation (9) is transformed into
\[
\frac{\partial^2 u}{\partial t^2} = k^{-2} a^2 \frac{\partial^2 u}{\partial x^2} + k^{-2} \gamma^2 u + k^{-2} h(u, t)
\]
\[
u(0, t) = u(2 \pi, t) = 0,
\]
while for the damped equation we obtain
\[
\frac{\partial^2 u}{\partial t^2} = k^{-2} a^2 \frac{\partial^2 u}{\partial x^2} - k^{-1} \alpha \frac{\partial u}{\partial t} + k^{-2} \gamma^2 u + k^{-2} h(u, t),
\]
\[
u(0, t) = u(2 \pi, t) = 0.
\]
The initial data (8) is converted into
\[
u(x, 0) = \varphi(x), \quad u_t(x, 0) = k^{-1} \psi(x).
\]

4. Chronological calculus: variational formula, logarithm and stability

Chronological calculus has been developed by Agrachev and Gamkrelidze in [6] for studying nonlinear time-variant systems. The systems, to be considered in this work, are linear and we present some reformulations for the (time-variant) linear case.

Consider the time-variant linear system of ordinary differential equations
\[
\dot{z}(t) = A(t) z(t), \quad z(0) = z_0
\]
where $z(t) \in \mathbb{R}^n$ and $t \mapsto A(t) \in \mathbb{R}^{n \times n}$ is a matrix-valued function depending continuously on $t$. Considering its solutions $z(t; z_0)$ we can introduce a flow of linear maps $P^t : z_0 \mapsto z(t, z_0)$, or a fundamental matrix solution. Obviously, $P^t$ is the unique solution to the matrix ordinary differential equation with initial condition
\[
\dot{P}^t = A(t) P^t, \quad P^0 = \text{Id}.
\]

Following [6], we call the flow $P^t$ left chronological exponential of $A(t)$ and denote it
\[
P^t = \exp \int_0^t A(\tau) \, d\tau.
\]
We can also define right chronological exponential $\exp^\text{r} \int_0^t A(\tau) \, d\tau$ as solution of
\[
\dot{P}^t = P^t A(t), \quad P^0 = \text{Id}.
\]

Using this notation, the solution of (12) is represented as
\[
z(t) = \exp \int_0^t A(\tau) \, d\tau z_0 = P^t z_0.
\]
The flow \( \exp \int_0^t A(\tau) \, d\tau \) admits a Volterra series expansion of the form

\[
\exp \int_0^t A(\tau) \, d\tau = \text{Id} + \int_0^t A(\tau_1) \, d\tau_1 + \sum_{n=1}^{\infty} \int_0^t \int_0^{\tau_1} \ldots \int_0^{\tau_n} A(\tau_1) \ldots A(\tau_n) \, d\tau_n \ldots d\tau_1.
\]

4.1. Variational formula

Assume now that, in (12), we take \( A(t) = B(t) + C(t) \), considering \( B(t) \) as a reference matrix of coefficients and \( C(t) \) as its perturbation. Then the following chronological calculus variational formula holds, \[6\],

\[
\exp \int_0^t [B(\tau) + C(\tau)] \, d\tau = \exp \int_0^t B(\tau) \, d\tau \circ \exp \int_0^t \left( \exp \int_0^\tau \text{ad} \, B(\theta) \, d\theta \right) C(\tau) \, d\tau.
\]

Here the operator “\( \text{ad} \)” corresponds to the matrix commutator \( \text{ad} \, A(t) \, B(t) = -[A(t), B(t)] = B(t) \, A(t) - A(t) \, B(t) \) and \( Q^t = \exp \int_0^t \text{ad} \, B(\theta) \, d\theta \) is the solution of the operator differential equation

\[
\frac{d}{dt} Q^t = Q^t \circ \text{ad} \, B(t), \quad Q^0 = \text{Id}.
\]

4.2. Formal expansion for \( \ln \exp \int_0^t A(\tau) \, d\tau \)

Let \( A(t) \overset{\text{def}}{=} A_t \) be a matrix-valued function with time-variant entries. We are interested in a formal expansion for the logarithm

\[
\Lambda_{0, t}(A_t) = \ln \exp \int_0^t A_t \, d\tau.
\]

It has been shown in \[6\] that \( \Lambda_{0, t}(A_t) \) admits a representation

\[
(14) \quad \Lambda_{0, t}(A_t) = \sum_{m=1}^{\infty} \Lambda^{(m)}
\]

where

\[
(15) \quad \Lambda^{(m)} = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{m-1}} d\tau_m \, p_m(A_{\tau_1}, \ldots, A_{\tau_m}),
\]

and, for each \( m \geq 2, \, p_m(A_{\tau_1}, \ldots, A_{\tau_m}) \) is a homogeneous polynomial of first degree in each \( A_{\tau_i} \). Moreover, it is a commutator polynomial in \( A_{\tau_1}, \ldots, A_{\tau_m} \).
i.e., $p_m$ can be expressed as a linear combination of $A_\tau, \ldots, A_{\tau_m}$ and of their iterated commutators.

Expressions for the first $\Lambda^{(m)}$ can be founded e.g. in [2]; in particular $\Lambda^{(1)}$ coincides with the averaging of $A_\tau$:

$$\Lambda^{(1)} = \int_0^t A_\tau \, d\tau.$$  

(16)

The series (14) is known to be absolutely convergent if $\int_0^t \|A_\tau\| \, d\tau \leq 0.44$, [6].

4.3. The logarithm and stability of time-variant systems

Consider linear fast-oscillating system

$$\dot{z}(t) = A(k t) z(t)$$  

(17)

where $x \in \mathbb{R}^n$, $A(\tau)$, $\tau \geq 0$, is matrix-valued function, which is continuous and 1-periodic with respect to $\tau$; $k > 0$ is a large parameter.

A condition for stability of time-periodic system (17) uses the monodromy matrix $P^n$. If all the eigenvalues of $P^n$ are located in the interior of the unit circle, then the system is asymptotically stable; system is unstable if at least one eigenvalue lies outside unit the circle. In general it is difficult to compute spectrum of $P^n$.

Due to the asymptotic expansion (14) it is more convenient to deal with the logarithm $\ln P^n$. One can arrive to a conclusion on the system’s stability by analyzing the truncations of the logarithm series if one gets, somehow, an estimate for the rest term of such truncation.

In terms of logarithm, stability conditions can be formulated as follows: system (17) is asymptotically stable if all the eigenvalues of $\ln P^n$ are located in the open left complex half-plane and is unstable if at least one eigenvalue lies in the open right complex half-plane. System (17) is stable if all the eigenvalues of $\ln P^n$ have non positive real part and purely imaginary eigenvalues are distinct.

5. Vibrating string with damping

We consider the problem of asymptotically stabilizing the null solution of (10) by means of a state feedback control

$$h(u, t) = \delta k^2 g(t) u.$$  

In other words we study asymptotic stability of the null solution of the equation

$$u_{ttt} = k^{-2} a^2 u_{xx} - k^{-1} \alpha u_t + [k^{-2} \gamma^2 + \delta g(t)] u,$$

$$u(0, t) = u(2 \pi, t) = 0.$$  

(18)
5.1. Infinite dimensional system of ODE

Following the Fourier method, we look for a solution of (18) of the form

\[ u(x, t) = T(t) X(x) \]

We will use \textit{dot} and \textit{prime}, respectively, to denote the derivative with respect to \( t \) and to \( x \). Differentiating \( u \) and replacing it and its derivatives in the partial differential equation we obtain, for \( X \neq 0, \dot{T} \neq 0 \),

\[ \ddot{T} + \frac{k-1}{k-2} \alpha \dot{T} + \frac{b(t)}{k-2 a^2 T} X'' = \lambda, \]

where

\[ b(t) = -[k^{-2} \gamma^2 + \delta g(t)]. \]

From (19) it is clear that \( \lambda \) must be constant, that is \( X'' = \lambda X, \lambda \in \mathbb{R} \) and

\[ \ddot{T} + \frac{k-1}{k-2} \alpha \dot{T} + \left[ b(t) - \lambda k^{-2} a^2 \right] T = 0. \]

The problem

\[ X'' = \lambda X, \quad X(0) = X(2\pi) = 0, \]

is a “spectral problem for the elliptic operator”

\[ \mathcal{L}_0 = \frac{\partial^2}{\partial x^2} \]

with boundary conditions derived from the original ones in (18). We conclude that in (22)

\[ \lambda_n = -\mu_n^2 = -\frac{n^2}{4}, \quad \tilde{\phi}_n(x) = \sin(\mu_n x), \quad n = 1, 2, \ldots \]

are the eigenvalues and eigenfunctions. Instead of \( \tilde{\phi}_n(x) \) we take

\[ \phi_n(x) = \pi^{-1/2} \tilde{\phi}_n(x); \]

now \( \{\phi_n\} \) forms an orthonormal basis in \( L_2(\Omega) \).

We denote by \( T_n \) the solution of (21) with \( \lambda \) replaced by \( \lambda_n = -\mu_n^2 \). The series

\[ u(x, t) = \sum_{n=1}^{\infty} T_n(t) \phi_n(x) \]

satisfies, formally, (18). For given initial data (11), we obtain

\[ \phi(x) = \sum_{n=1}^{\infty} T_n(0) \phi_n(x), \quad \psi(x) = \sum_{n=1}^{\infty} k \dot{T}_n(0) \phi_n(x). \]
Since \( \{ \phi_n \} \) is an orthonormal basis in \( L_2(\Omega) \), there holds
\[
T_n(0) = (\varphi, \phi_n) \quad \text{and} \quad \dot{T}_n(0) = k^{-1} (\psi, \phi_n).
\]
Therefore, if \( T_n(0) \) and \( \dot{T}_n(0) \) are given by (26), then series (26) formally satisfies (18) with given initial data.

Substituting \( u \), given by (25) into the partial differential equation (18) and equalizing the terms which contain \( \sin(\mu_n x) \) we obtain an infinite system of second order linear ordinary differential equations
\[
\ddot{T}_n(t) + k^{-1} \alpha \dot{T}_n(t) + [b(t) + \mu_n^2 k^{-2} a^2] T_n(t) = 0,
\]
for \( n = 1, 2, \ldots \). Formally speaking, (18) is equivalent to infinite-dimensional system of equations (27).

In the next section we illustrate the essence of our method by studying asymptotic stability of an equation (27).

5.2. Stabilization of (27)

If we take time-averaging of the coefficients of (27), then, due to Assumption II, the average of \( b(t) \) equals \(-k^{-2} \gamma^2\) and we arrive to the equation
\[
\ddot{T}_n(t) + k^{-1} \alpha \dot{T}_n(t) + k^{-2}(\mu_n^2 a^2 - \gamma^2) T_n(t) = 0.
\]
If destabilizing linear disturbance is sufficiently large, e.g.,
\[
\gamma^2 > a^2 \mu_n^2
\]
then the averaged equation (28) is unstable. We prove though that, even in this case, one can achieve stability by choosing sufficiently large “frequency” \( k \) of (non-harmonic) oscillation (5).

To derive an asymptotic stability condition for a single equation in (27) we first transform it into a two-dimensional first-order linear system by introducing a new variable \( S_n = k \dot{T}_n \) (c.f. (8)). We get for \( z_n^T = (T_n, S_n) \)
\[
\dot{z}_n(t) = A_n(t) z_n(t)
\]
where
\[
A_n(t) = \begin{pmatrix} 0 & k^{-1} \\ k^{-1} (\gamma^2 - \mu_n^2 a^2) + k \delta g(t) & -k^{-1} \alpha \end{pmatrix}.
\]
We represent \( A_n(t) \) as \( B_n(t) + C_n \), where
\[
B_n(t) = \delta k g(t) B, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]
and
\[
C_n = k^{-1} \begin{pmatrix} 0 & \mu_n^2 \\ \gamma^2 - a^2 \mu_n^2 & -\alpha \end{pmatrix}.
\]
According to the chronological calculus variational formula, the monodromy matrix of the system can be represented as

\[ P_n = \exp \int_0^1 A_n(\xi) \, d\xi = \exp \int_0^1 [B_n(\xi) + C_n] \, d\xi = \exp \int_0^1 B_n(\tau) \, d\tau \circ \exp \int_0^1 \left( \exp \int_0^\tau \text{ad} \, B_n(\theta) \, d\theta \, C_n \right) \, d\tau. \]

From the Assumption it follows that

\[ \int_0^1 B_n(\tau) \, d\tau = 0 \quad \text{and, hence} \quad \exp \int_0^1 B_n(\tau) \, d\tau = \text{Id}_2, \]

where 0 and Id_2 are, respectively, the two-dimensional null and identity matrices. Besides

\[ \exp \left( \int_0^t \text{ad} \, B_n(\theta) \, d\theta \right) C_n = e^{-\delta k G(t) \text{ad} \, B \, C_n} \]

where \[ \int_0^t g(\theta) \, d\theta = G(t) \] (c.f. Assumption).

We represent the monodromy matrix as

\[ P_n = \exp \int_0^1 D_n(t) \, dt, \]

where

\[ D_n(t) = e^{-\delta k G(t) \text{ad} \, B \, C_n} = [\text{Id}_2 - \delta k G(t) \text{ad} \, B + \frac{(\delta k)^2}{2} G^2(t) \text{ad}^2 B - \frac{(\delta k)^3}{3!} G(t) \text{ad}^3 B + \ldots] C_n. \]

As far as \( \text{ad}^j \, B \, C_n = 0_2, \) \( j \geq 3, \) the series for \( e^{-\delta k G(t) \text{ad} \, B \, C_n} \) ends at the third term and therefore

\[ D_n(t) = \left( k^{-1}(\gamma^2 - a^2 \mu_n^2) - \tilde{G}(t) - k^{-1} \alpha - \delta G(t) \right) \]

where \( \tilde{G}(t) = \alpha \delta G(t) + k \delta^2 G^2(t). \)

Let now

\[ \Lambda = \ln P_n = \ln \left( \exp \int_0^1 D_n(t) \, dt \right) \]

be the logarithm of the monodromy matrix.

Consider the expansion for the logarithm, with \( \Lambda^{(m)} \) defined by.

We will show that, under some conditions, the stability properties of are determined by \( \Lambda^{(1)}. \)
As we said in Section 4.2, $\Lambda^{(1)}$ coincides with the averaging of $D_n(t)$. According to Assumption 2

$$
\Lambda^{(1)} = \begin{pmatrix}
0 & k^{-1} \\
k^{-1}(\gamma^2 - a^2 \mu_n^2) - \delta^2 k \Gamma & k^{-1} - k^{-1} \alpha
\end{pmatrix}
$$

where

$$
\Gamma = \int_0^1 G^2(t) \, dt > 0.
$$

The eigenvalues of $\Lambda^{(1)}$ can be either real or complex conjugated; they are equal to

$$
\xi_i^{(1)} = \frac{\Theta \pm \sqrt{\Theta^2 - 4\Delta}}{2}
$$

where $\Theta = -k^{-1} \alpha$ and $\Delta = \delta^2 \Gamma - k^{-2}(\gamma^2 - a^2 \mu_n^2)$ are the trace and the determinant of $\Lambda^{(1)}$, respectively. Obviously, $\Theta < 0$. $\Lambda^{(1)}$ possess real negative eigenvalues if

$$
k^{-2} \frac{4\gamma^2 - a^2 n^2}{4} < \delta^2 \int_0^1 G^2(t) \, dt \leq k^{-2} \frac{\alpha^2 + 4\gamma^2 - a^2 n^2}{4}.
$$

For large $n$ the latter inequality is never satisfied and then for

$$
\delta^2 \int_0^1 G^2(t) \, dt > k^{-2} \frac{\alpha^2 + 4\gamma^2 - a^2 n^2}{4}
$$

we obtain a pair of conjugate complex numbers with negative real part.

What for the rest term $\Lambda - \Lambda^{(1)}$, then the following estimate is available.

**Proposition 1 (3).** For $n^2 < 4\frac{k^2 + \gamma^2}{a}$ there exists a positive constant $c$ and a $2 \times 2$ matrix $R$ such that

$$
\Lambda - \Lambda^{(1)} = \delta^2 R, \quad \|R\|_M < c
$$

where neither $c$ nor $R$ depend on $n$. ■

Here $\| \cdot \|_M$ stands for some matrix norm. The upper bound on $n$ is irrelevant; we can assume it to be fulfilled for $k > k_0$ as stated in the next proposition. The estimate can be derived from convergence results for the chronological exponential presented in [6]; in [3] the detailed exposition for the particular case of stability of inverse double pendulum can be found.

From (34), (35) and (36), we arrive to the condition for the asymptotic stability of the two-dimensional system (29).
Proposition 2. For each $\epsilon > 0$ there exist $\delta_0 > 0$, $k_0 > 1$ such that the null solution $z_n(t) \equiv 0$ of (29) is exponentially asymptotically stable if $0 < \delta < \delta_0$, $k > k_0$ and

$$\delta^2 \int_0^1 G^2(t) \, dt > k^{-2} \frac{4\gamma^2 - a^2 n^2}{4} + \epsilon,$$

and unstable if $0 < \delta < \delta_0$, $k > k_0$ and

$$\delta^2 \int_0^1 G^2(t) \, dt < k^{-2} \frac{4\gamma^2 - a^2 n^2}{4} - \epsilon. \blacksquare$$

Observe that if the condition (37) is fulfilled for $n = N$ then it is fulfilled for all $n \geq N$.

5.3. Asymptotic stability of Galerkin’s approximation

The Galerkin approximation for (18) is a system of (decoupled) equations (27) for $n = 1, \ldots, N$.

This system is equivalent to a $2N$ block diagonal system

$$\dot{z}_N(t) = H_N(t) z_N(t)$$

where $z_N^T(t) = (\tilde{z}_1(t), \ldots, \tilde{z}_N(t))$ and

$$H_N(t) = \text{diag}\{D_1(t), \ldots, D_N(t)\}.$$

Due to the decoupling, we may conclude that the system (39) is asymptotically stable whenever each of the systems

$$\dot{z}_n(t) = D_n(t) z_n(t), \quad n = 1, \ldots, N,$$

is asymptotically stable; it is unstable if at least one of them is unstable. Instability occurs if (38) holds for some $n = n_1 \leq N$ (then it holds for all $n \leq n_1$). According to Proposition 2 system (39) is exponentially asymptotically stable if

$$\delta^2 \int_0^1 G^2(t) \, dt > k^{-2} \frac{4\gamma^2 - a^2 n^2}{4} + \epsilon.$$

Observe that the latter condition does not depend on $N$.

Let $\mathcal{G}_N$ be the subspace of $L_2(\Omega)$ spanned by $\{\phi_n(x)\}_{n=1}^N$ defined in (21) and let $\mathcal{G}_N^\perp$ be the orthogonal complement of $\mathcal{G}_N$ in $L_2(\Omega)$.

We denote the projection of the (unique) solution of (18) on $\mathcal{G}_N$ by $u_N$ and the projection of $u$ onto $\mathcal{G}_N^\perp$ by $u^N$.

Then

$$u_N(x, t) = \sum_{n=1}^N (u, \phi_n) \phi_n(x) = \sum_{n=1}^N T_n(t) \phi_n(x)$$

is the solution of the $N$th Galerkin’s approximation of (18) and the following proposition holds.
Proposition 3. For each \( \epsilon > 0 \) there exist \( \delta_0 > 0, k_0 > 0 \) if \( 0 < \delta < \delta_0, k > k_0 \) and

\[
\delta^2 \int_0^1 G^2(t) \, dt > k^{-2} \frac{4\gamma^2 - a^2}{4} + \epsilon
\]

then the null solution \( z_n(t) \equiv 0 \) of the Galerkin’s approximation of (13) is exponentially asymptotically stable. 

Therefore, if (41) holds then

\[
\|u_N(\cdot,t)\|_1^2 + \|(u_N)_{t}(\cdot,t)\|_2^2 \to 0 \quad \text{as} \quad t \to +\infty.
\]

To establish asymptotic stability of the problem (7), (2) it suffices to prove, that for sufficiently large \( N \), the projection of \( u \) onto \( G^\perp_N \) tends to zero as \( t \to +\infty \)

\[
\|u_N(\cdot,t)\|_1^2 + \|u^N_N(\cdot,t)\|_2^2 \to 0 \quad \text{as} \quad t \to +\infty.
\]

5.4. Convergence of \( \|u^N(\cdot,t)\|_1^2 + \|u^N_N(\cdot,t)\|_2^2 \)

As far as \( u \) is an element of \( W^2_0(Q_T) \), where \( Q_T = \Omega \times (0,T) \), \( u^N(\cdot,t) \) is in \( W^2_0(\Omega) \) for all \( t \in (0,T) \). We prove now

Proposition 4. For the projection \( u^N \) there exists \( c > 0 \) such that

\[
\|u^N(\cdot,t)\|_1^2 + \|u^N_N(\cdot,t)\|_2^2 \leq e^{-ct}(\|u^N(\cdot,0)\|_1^2 + \|u^N_N(\cdot,0)\|_2^2),
\]

for all \( t \geq 0 \).

Proof. Introduce a function \( V \) defined, for any \( u \in W^2_0(Q_T) \), by

\[
V = \frac{k^{-2}a^2}{4}\|u\|^2 + \frac{k^{-2}a^2}{2}\|u_x\|^2 + \frac{1}{2}\|u_t\|^2 + \frac{k^{-1}\alpha}{2} < u, u_t > .
\]

We recall that \( < \cdot, \cdot > \) and \( \| \cdot \| \) stand for the inner product and for the norm in \( L^2_2(\Omega) \), respectively (c.f. Section 2). Let \( V_N \) denote the restriction of the function \( V \) onto \( G^\perp_N \).

We first note that there is a positive constant \( C_1 \) such that

\[
V_N \geq C_1(\|u^N\|^2_1 + \|u^N_N\|^2_2).
\]

Indeed, \( V \) can be represented as

\[
V = \frac{k^{-2}a^2}{2}\|u_x\|^2 + \frac{1}{4}\|u_t\|^2 + \frac{1}{4}\|k^{-1}\alpha u + u_t\|^2.
\]

As long as the inequality \( \|u^N_N\|^2 \geq N^2\|u^N\|^2 \) holds, then

\[
V_N \geq \frac{k^{-2}a^2}{4}N^2\|u^N\|^2 + \frac{k^{-2}a^2}{4}\|u^N_x\|^2 + \frac{1}{4}\|u^N_t\|^2 + \frac{1}{4}\|k^{-1}\alpha u^N + u^N_t\|^2.
\]
and (43) follows with $C_1 = C_1(k) = \min\left(\frac{k-2}{4}, \frac{1}{4}\right)$.

Let $\dot{V}_N$ denote the time derivative of $V_N(u^N(\cdot, t))$. After some simple computation we conclude

$$-\dot{V}_N \geq C_2(N, k, t)\|u^N\|^2 + \frac{k^{-3}}{4} \alpha^2\|u^N_x\|^2 + \frac{k^{-1}}{4} \alpha\|u^N_t\|$$

where

$$C_2(N, k, t) = \frac{k^{-3}}{4} \alpha^2 N^2 - \left[\frac{1}{2} k^{-1} \alpha b(t) + \frac{1}{k^{-1} \alpha} b^2(t)\right].$$

As long as $g(t)$ and $b(t)$ are periodic and bounded, there exists $\beta > 0$ such that $|b(t)| \leq \beta$ for all $t \geq 0$, (c.f. (20)). Then, for all $t \geq 0$

$$C_2(N, k, t) \geq \hat{C}_2(N, k) = \frac{k^{-3}}{4} \alpha^2 N^2 - \left[\frac{1}{2} k^{-1} \alpha \beta + \frac{\beta^2}{k^{-1} \alpha}\right].$$

We assume $N$ to be sufficiently large to guarantee $\hat{C}_2(N, k) > 0$ (for all $\delta < \delta_0$, $k > k_0$). Then, for those $N$, $\hat{C}_2$ can be chosen independently on $N$

$$-\dot{V}_N \geq \hat{C}_2(N, k)\|u^N\|^2 + \frac{k^{-3}}{4} \alpha^2\|u^N_x\|^2 + \frac{k^{-1}}{4} \alpha\|u^N_t\| > 0.$$

We can also show that

$$V_N \leq \frac{3}{8} k^{-2} \alpha^2\|u^N\|^2 + \frac{1}{2} k^{-2} \alpha^2\|u^N_x\|^2 + \frac{k^{-1}}{4} \alpha\|u^N_t\|.$$

Therefore, there exists a positive constant $C(k)$ such that $\dot{V}_N \leq -C(k) V_N$, hence

(44) \hspace{1cm} V_N(t) \leq V_N(0) e^{-C(k) t}$.

From (43) and (44) we conclude

$$\|u^N(\cdot, t)\|^2 + \|u^N_t(\cdot, t)\|^2 \to 0 \quad \text{as} \quad t \to +\infty. \blacksquare$$

From the Propositions 3 and 4 we conclude the asymptotic stability of the null solution of infinite dimensional system (39) and the consequent asymptotic stability of the null solution of the problem (7), (2).

5.5. Main result for damped string

Sumarizing the reasoning of the previous sections we state

**Theorem 5.1.** Consider the equations (7), (2) for damped elastic string subject to linear disturbance and controlled by a time-variant (high-gain fast-oscillating non-harmonic) linear feedback control (5).

For each $\epsilon > 0$ there exist $\delta_0 > 0$, $k_0 > 0$ such that if $0 < \delta < \delta_0$, $k > k_0$ and the inequality (11) holds, then the null solution $u(x, t) \equiv 0$ of problem (7), (2) is exponentially asymptotically stable $\blacksquare$.
6. Undamped vibrating string

In this section, we consider the problem of stabilization of the null solution of undamped equation \( \text{(3)} \). We try to stabilize it by a forcing term, which results in equation

\[
\begin{align*}
\ddot{u} & = k^{-2} a^2 u_{xx} + k^{-2} \gamma^2 u + k^{-2} h(u, t) \\
u(t) & = u(2\pi, t) = 0.
\end{align*}
\]

(45)

Our choice is a time-variant “output” feedback, \( \text{(46)} \),

\[
h(u, t) = \delta k^2 g(k\tau) \Pi(N) u,
\]

c.f. (5), where \( \Pi(N) \) stands for the orthogonal projection of the “state” \( u \) onto the subspace of \( L^2(\Omega) \) generated by the first \( N \) harmonics.

For the control \( h \) defined by (46), the equation (45) is formally equivalent to an infinite dimensional system of second order linear ordinary differential equations

\[
\begin{align*}
\dddot{T}_n(t) + \left( k^{-2} \mu_n^2 a^2 - \gamma^2 \right) T_n(t) & = 0, \quad n \leq N \\
\dddot{T}_n(t) + k^{-2} \mu_n^2 a^2 - \gamma^2 T_n(t) & = 0, \quad n > N.
\end{align*}
\]

(47)

This system is obtained by introducing the (formal) expansion (25) of \( u \).

The analysis of the stability of \( u_N = \Pi_N u \) goes along with what was done in Sections 5.2 and 5.3 for the damped case.

As before, we introduce a new variable \( S_n = k \dot{T}_n \) and reduce (47) to a two-dimensional first-order linear system

\[
\dot{z}_n(t) = \tilde{A}_n(t) z_n(t)
\]

(48)

where

\[
\tilde{A}_n(t) = \begin{pmatrix}
0 & k^{-1} \\
k^{-1} (\gamma^2 - \mu_n^2 a^2) + k \delta g(t) & 0
\end{pmatrix}.
\]

Using chronological calculus variational formula (13), we compute the monodromy matrix for (48)

\[
Q^1_n = \exp \int_0^1 \tilde{D}_n(t) \, dt,
\]

where

\[
\tilde{D}_n(t) = \begin{pmatrix}
\delta G(t) & -k^{-1} G(t) \\
-\delta^2 G^2(t) & -k \delta G(t)
\end{pmatrix}
\]

and \( G(t) = \int_0^t g(\xi) \, d\xi \). Next we consider the logarithm \( \Lambda = \ln Q^1_n \) given by the asymptotic expansion (14) and compute \( \Lambda^{(1)} \)

\[
\Lambda^{(1)} = \begin{pmatrix}
0 & k^{-1} \\
k^{-1} (\gamma^2 - \mu_n^2 a^2) - k \delta^2 k \Gamma & 0
\end{pmatrix}.
\]
where \( \Gamma = \int_0^1 G^2(t) \, dt > 0 \). It is clear that the main difference between damped and undamped systems is that now \( \Lambda^{(1)} \) is a traceless matrix while \( \Lambda^{(1)} \) had negative trace. Hence, to achieve stability we have to ensure that its eigenvalues are two imaginary (conjugate) numbers. Stability condition for (each of) the two-dimensional system is analogous to the one of the Proposition.

**Proposition 5.** For each \( \epsilon > 0 \) there exist \( \delta_0 > 0, k_0 > 1 \) such that the null solution \( z_n(t) \equiv 0 \) of \( \text{(48)} \) is exponentially asymptotically stable if \( 0 < \delta < \delta_0, k > k_0 \) and

\[
\delta^2 \int_0^1 G^2(t) \, dt > k^{-2} \frac{4 \gamma^2 - a^2 n^2}{4} + \epsilon
\]

and unstable if \( 0 < \delta < \delta_0, k > k_0 \) and

\[
\delta^2 \int_0^1 G^2(t) \, dt < k^{-2} \frac{4 \gamma^2 - a^2 n^2}{4} - \epsilon. \]

Recall that Galerkin’s approximation for the equation \( \text{(45)} \) is a system of \( N \) decoupled second order equations \( \text{(48)}, \) where \( n = 1, \ldots, N \). As in Section \( \text{5.3} \) this system is stable whenever all the equations are stable, that is, when the Proposition holds for all \( n \leq N \).

To be able to conclude stability of the null solution of \( \text{(45)}, \text{(46)} \) we have to analyze stability of the projection of \( \text{(45)} \) onto \( \mathcal{G}_N^\perp \)

\[
u_{tt}^N = k^{-2} a^2 u_{xx}^N + k^{-2} \gamma^2 u^N.
\]

Let us consider a function

\[
V = -k^{-2} \frac{a^2}{2} \|u\|^2 + k^{-2} \frac{a^2}{2} \|u_x\|^2 + \frac{1}{2} \|u_t\|^2.
\]

This function is not necessarily positive but its restriction \( V_N \) onto \( \mathcal{G}_N^\perp \) is positive provided that \( N \) is sufficiently large, say, \( N^2 > 2 \gamma^2/a^2 \). Indeed, since the inequality \( \|u^N\|^2 \geq N^2 \|u^N\|^2 \) holds, then in this case

\[
V_N \geq \frac{k^{-2}}{2} [a^2 N^2 - \gamma^2] \|u^N\|^2 + k^{-2} \frac{a^2}{4} \|u_x\|^2 + \frac{1}{2} \|u_t^N\|^2 \geq 0.
\]

Let \( \dot{V}_N \) denote the time derivative of \( V_N(u^N(\cdot, t)) \). Then

\[
\dot{V}_N = -k^{-2} \gamma^2 < u^N, \dot{u}^N > + k^{-2} a^2 < u_x^N, \dot{u}_{x}^N > + k^{-2} a^2 < \dot{u}_t^N, \dot{u}_{tt}^N > + k^{-2} a^2 < u^N, \dot{u}_t^N > + k^{-2} \gamma^2 < u_t^N, u^N >
\]

\[
= 0.
\]
The equality \( < u^N_{x^2}, u^N_{xt} > = - < u^N_{x^2}, u^N_t > \), which holds under the boundary conditions (2) is concluded by integration by parts with respect to \( x \).

Therefore \( V_N(u^N(\cdot, t)) \equiv \text{const} \) wherefrom we obtain an upper estimate for \( u^N \) and conclude stability of the null solution of of the projection of (35) onto \( G^\perp_N \). Completing this by the result of the Proposition 5 we conclude stability of (35), (2).

**Theorem 6.1.** Consider the equations (6), (2) for an undamped elastic string which is subject to linear disturbance and is controlled by a time-variant (high-gain fast-oscillating non-harmonic) linear output feedback control (46). Then for each \( \epsilon > 0 \) there exist \( \delta_0 > 0 \), \( k_0 > 0 \) such that if \( 0 < \delta < \delta_0 \), \( k > k_0 \) and the inequality

\[
\delta^2 \int_0^1 G^2(t) \, dt > k^{-2} \frac{4\gamma^2 - a^2}{4} + \epsilon
\]

holds, then the null solution of problem (6), (2) is stable. ■

7. Final remarks

From (33), it is clear that the degree of asymptotic stability increases with \( \alpha \). An interesting but not unexpected phenomena arises when \( \Theta^2 - 4\Delta \) equals zero: multiple negative eigenvalue bifurcates to a conjugate complex pair.

If (38) holds for some \( n = n_1 \leq N \), then finite dimensional approximation (39) is unstable implying that the null solution of the problem (7), (2) is unstable.

The output feedback control constructed in the Section 6 is suitable for the asymptotic stabilization of the damped string (system (7), (2)).

The approach to the problem presented in this work is more general than the one of [4]. The main difference is that we do not select a particular type of vibrational control, in contrast to the harmonic vibration and rather restrictive assumption for frequencies, which appeared in [4]. Besides here we were able to deal with stabilization of an undamped string by means of output feedback.

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