Reconstruction for the Asymmetric Ising Model on Regular Trees

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Abstract
It is known that the Kesten-Stigum reconstruction bound is tight for roughly symmetric binary channels. In this paper, we will adopt a refined analysis of moment recursion on a weighted version of the asymmetric Potts model, and establish the critical condition of the asymmetric Ising model to make Kesten-Stigum bound the reconstruction threshold on regular \(d\)-ary trees.

I. Introduction

I.1. Basic definitions
We start with the following broadcasting process that stands as a discrete, irreducible, aperiodic, and reversible Markov chain. Let \(T = (V, E, \rho)\) be a tree with nodes \(V\), edges \(E\) and root \(\rho \in V\). Each edge of the tree acts as a channel on a finite characters set \(C\), whose elements are configurations on \(T\), denoted by \(\sigma\). Next set a probability transition matrix \(M = (M_{ij})\) as the noisy communication channel on each edge. The state of the root \(\rho\), denoted by \(\sigma_{\rho}\), is chosen according to an initial distribution \(\pi\) on \(C\). This symbol is then propagated in the tree as follows. For each vertex \(v\) having as a parent \(u\), the spin at \(v\) is defined according to the probabilities

\[
P(\sigma_v = j \mid \sigma_u = i) = M_{ij} \tag{1}
\]

with \(i, j \in C\). Roughly speaking, the problem of reconstruction is to investigate whether the symbols received at the vertices of the \(n\)th generation contain a non-vanishing information transmitted by the root as \(n\) goes to \(\infty\). The following is the formal definition of the reconstruction.

Definition 1 The reconstruction problem for the infinite tree \(T\) is solvable if for some \(i, j \in C\),

\[
\limsup_{n \to \infty} d_{TV}(\sigma^i(n), \sigma^j(n)) > 0 \tag{2}
\]

where \(d_{TV}\) is the total variation distance. When the \(\limsup\) is 0 we will say the model has non-reconstruction on \(T\).

This paper will restrict to regular \(d\)-ary trees, i.e., the infinite rooted tree where every vertex has exactly \(d\) offspring. Let \(\sigma(n)\) denote the spins at distance \(n\) from the root and let \(\sigma^i(n)\) denote \(\sigma(n)\) conditioned on \(\sigma_0 = i\). The objective model taken into account is the asymmetric binary channel with the configuration set \(C = \{1, 2\}\), whose transition matrix is of the form

\[
M = \frac{1}{2} \begin{pmatrix}
1 + \theta & 1 - \theta \\
1 - \theta & 1 + \theta
\end{pmatrix} + \Delta \begin{pmatrix}
-1 & 1 \\
-1 & 1
\end{pmatrix}, \tag{3}
\]

where \(\Delta\) is used to describe the deviation of \(M\) from the symmetric channel and obviously there is a restriction of \(|\theta| + |\Delta| \leq 1\). Actually the process of broadcasting on a tree with the channels \(M\) corresponds to the ferromagnetic Ising model with external field on the tree. Furthermore it is apparent that the second eigenvalue of the channel \(M\) is \(\theta\) which plays a crucial role in the reconstruction problem.

I.2. Background
Determining the reconstruction threshold of a Markov random field in probability, as the interdisciplinary subject, has attracted more and more attention from probabilists, statistical physicists, biologists, etc. In fact, the investigation of the reconstruction problem originated from spin systems in statistical physics by establishing that the reconstruction threshold happens to be the threshold for extremality of the infinite-volume Gibbs measure with free boundary conditions [6]. It is shown that the reconstruction bound determines the efficiency of the Glauber dynamics on trees and random graphs [1] [8] [14], for example, the mixing time for the Glauber dynamics undergoes a phase transition...
at the reconstruction threshold. The reconstruction threshold is also believed to play an important role in a variety of other contexts, such as the efficiency of reconstructing phylogenetic ancestors in evolutionary biology (11), communication theory in the study of noisy computation (5), network tomography (2) (derive the link delays in the interior from end-to-end delays in a computer network), etc.

It is well known that the reconstruction solvability result when $d(\theta) > 1$ for any channel (7). Specially for the binary symmetric channel, it was shown in (3) that $d(\theta) > 1$ is not only the sufficient but necessary condition for the reconstruction solvability, which we refer to as the Kesten-Stigum bound. As for other channels, proving non-reconstructibility turned out to be harder. Although coupling arguments easily yield nonreconstruction, these arguments are typically not tight. Mossel (12) showed that the Kesten-Stigum bound is not the bound for reconstruction in the binary-asymmetric model with sufficiently large asymmetry or in the Potts model with sufficiently many characters, opening a window to exploit the tightness of the Kesten-Stigum bound.

In (13), the Potts model was completely investigated by means of the recursive structure of the tree, and more importantly, the author engaged the refined recursive equations of vector-valued distributions and concentration analyses to confirm much of the picture predicted by Mézard and Montanari (9).

But exact thresholds for non-solvability of the asymmetric Ising model had not been known until (14), in which Borgs et al displayed a delicate analysis of the moment recursion on a weighted version of the magnetization, and thus achieved a breakthrough result. However this conclusion has just established the existence of the sufficiently small $\Delta$ without estimating the range of the symmetry bias to keep Kesten-Stigum bound tight.

**I.3. Main results**

Inspired by Sly’s work (13)’s, we are able to present the critical relationship between $\Delta$ and $\theta$ to preserve tightness of the Kesten-Stigum bound. Since $d(\theta)^2 > 1$ always guarantees the reconstruction, it suffices to consider $1/2 \leq d(\theta)^2 \leq 1$ in the following context.

**Theorem I.1** When $\Delta^2 > (1 - \theta)^2/3$, for every $d$ the Kesten-Stigum bound is not tight. In other words, the reconstruction problem is solvable for some $\theta$ even if $d(\theta)^2 < 1$.

Furthermore with the assistance of the central limit theorem and gaussian approximation, we figure out the precise condition to keep the tightness of the Kesten-Stigum bound for fixed $\pi$ and large $d$.

**Theorem I.2** When $\Delta^2 < (1 - \theta)^2/3$, there exists a $D = D(\pi) > 0$ such that for $d > D$ the Kesten-Stigum bound is sharp. Furthermore there is non-reconstruction at the Kesten-Stigum bound, when $d(\theta)^2 = 1$.

**II. Main ideas of the proof**

**II.1. Notations**

Note first that the stationary distribution $\pi = (\pi_1, \pi_2)$ of $M$ is given by

$$\pi_1 = \frac{1}{2} + \frac{\Delta}{2(1 - \theta)} \text{ and } \pi_2 = \frac{1}{2} - \frac{\Delta}{2(1 - \theta)},$$

and without loss of generality, it is convenient to assume $\pi_1 \geq \pi_2$. Let $u_1, \ldots, u_d$ be the children of $v$ and $T_v$ be the subtree of descendants of $v \in T$. Furthermore, if we set $d(v, \cdot)$ as the graph-metric distance on $T$, denote the $n$th level of the tree by $L(n) = \{ v \in V : d(\rho, v) = n \}$. With the notation above, let $\sigma(n)$ and $\sigma_j(n)$ denote the spins on $L_n$ and $L(n) \cap T_{u_j}$, respectively. For a configuration $A$ on $L(n)$ define the posterior function by

$$f_n(i, A) = P(\sigma_\rho = i \mid \sigma(n) = A).$$

By the recursive nature of the tree for a configuration $A$ on $L(n+1) \cap T_{u_j}$ we can give the equivalent form of the previous one

$$f_n(i, A) = P(\sigma_{u_j} = i \mid \sigma_j(n+1) = A).$$

Now for $i = 1, 2$ and $1 \leq j \leq d$ define

$$X_i = X_i(n) = f_n(i, \sigma(n)); \quad Y_j = Y_j(n) = f_n(1, \sigma^j(n+1))$$

and

$$X^+ = X^+(n) = f_n(1, \sigma^1(n)) ; \quad X^- = X^-(n) = f_n(2, \sigma^2(n)).$$

And it is clear that the random variables $\{Y_j\}_{1 \leq j \leq d}$ are independent and identical in distribution. Last introduce the objective quantities in this paper:

$$x_n = E(X^+(n) - \pi_1) = E_f(n, 1, \sigma^1(n)) - \pi_1$$

and

$$z_n = E(X^+(n) - \pi_1)^2 = E_f(n, 1, \sigma^1(n)) - \pi_1)^2.$$  

Referring to (10), Proposition 14, it suffices to investigate the asymptotic behavior of $x_n$ as $n$ goes to infinity. Then we can establish the equivalent condition for non-reconstruction.

**Lemma II.1** The non-reconstruction is equivalent to

$$\lim_{n \to \infty} x_n = 0.$$

**Proof.** The maximum-likelihood algorithm, which is the optimal reconstruction algorithm of $\sigma_\rho$ given $\sigma(n)$, is successful with probability

$$\Delta_n = E \max \{ X_1(n), X_2(n) \}.$$  

Therefore it follows immediately the inequality of $x_n + \pi_1 \leq \Delta_n$. On the other side, recalling the
assumption of $\pi_1 \geq \pi_2$, we could apply Cauchy-Schwartz inequality, in tandem with the identity (13) to conclude

$$\Delta_n = \pi_1 + \mathbb{E} \max \{X_1(n) - \pi_1, X_2(n) - \pi_1\}$$

$$\leq \pi_1 + \mathbb{E} \max \{X_1(n) - \pi_1, X_2(n) - \pi_2\}$$

$$= \pi_1 + \mathbb{E}[X_1(n) - \pi_1]$$

$$\leq \pi_1 + \left(\mathbb{E}(X_1(n) - \pi_1)^2\right)^{1/2}$$

$$\leq \pi_1 + \pi_1^{1/2} \pi_1^{1/2}.$$  \hfill (10)

To sum up, we come up with the inequalities

$$x_n \leq \Delta_n - \pi_1 \leq \pi_1^{1/2} x_n^{1/2},$$

implying that $\lim_{n \to \infty} x_n = 0$ is equivalent to $\lim_{n \to \infty} \Delta_n = \pi_1$, which is in turn equivalent to non-reconstruction [10].

II.2. Preparations

Before giving the the outline of the proof, it is convenient to derive some basic identities concerning $x_n$. First we reveal the relation between the first and second moments of $X^+$.

**Lemma II.2** \ For any $n \in \mathbb{N} \cup \{0\}$, we have

$$x_n = \frac{1}{\pi_1} \mathbb{E}(X_1(n) - \pi_1)^2$$

$$= \mathbb{E}(X^+(n) - \pi_1)^2 + \frac{\pi_2}{\pi_1} \mathbb{E}(X^-(n) - \pi_2)^2$$

$$\geq z_n \geq 0.$$  \hfill Proof. By Bayes’ rule, we have

$$\mathbb{E}X^+(n) = \mathbb{E}f_n(1, \sigma^1(n))$$

$$= \sum_A \int_A \mathbb{P}(\sigma(n) = A | \sigma_\rho = 1)$$

$$= \frac{1}{\pi_1} \sum_A \int_A \mathbb{P}(\sigma(n) = A)$$

$$= \frac{1}{\pi_1} \mathbb{E}(X_1^2)$$  \hfill (11)

and similarly,

$$\mathbb{E}X^- = \mathbb{E}f_n(2, \sigma^2(n)) = \frac{1}{\pi_2} \mathbb{E}(X_2^2).$$  \hfill (12)

Then it follows from the fact of $\mathbb{E}(X_1) = \pi_1$ that

$$x_n = \frac{1}{\pi_1} \mathbb{E}(X_1^2) - \pi_1^2 = \frac{1}{\pi_1} \mathbb{E}(X_1(n) - \pi_1)^2.$$  \hfill (13)

Next referring to the identity $X_1(n) + X_2(n) = 1$ we obtain

$$x_n = \frac{1}{\pi_1} \mathbb{E}(X_2 - \pi_2)^2 = \frac{\pi_2}{\pi_1} \mathbb{E}(X^- - \pi_2).$$  \hfill (14)

Last from (13) we present the quantitative relation between $x_n$ and $z_n$:

$$x_n = \frac{1}{\pi_1} \left[\mathbb{P}(\sigma_\rho = 1) \mathbb{E}((X_1 - \pi_1)^2 | \sigma_\rho = 1)\right]$$

$$+ \frac{1}{\pi_1} \left[\mathbb{P}(\sigma_\rho = 2) \mathbb{E}((X_2 - \pi_2)^2 | \sigma_\rho = 2)\right]$$

$$= \frac{1}{\pi_1} \left[\pi_1 \mathbb{E}(X^+(n) - \pi_1)^2 + \pi_2 \mathbb{E}(X^-(n) - \pi_2)^2\right]$$

$$= \mathbb{E}(X^+(n) - \pi_1)^2 + \frac{\pi_2}{\pi_1} \mathbb{E}(X^-(n) - \pi_2)^2$$

$$\geq z_n \geq 0.$$  \hfill $\Box$

Next with the preceding results, we could evaluate the means and variances of $Y_j$.

**Lemma II.3** \ For each $1 \leq j \leq d$, we have

$$\mathbb{E}(Y_j - \pi_1) = \theta x_n$$

and

$$\mathbb{E}(Y_j - \pi_1)^2 = \theta z_n + \pi_1(1 - \theta)x_n.$$  \hfill Proof. If $\sigma^1_{u_j} = 1, Y_j$ is distributed according to $X^+(n)$, while as $1 - X^-(n)$ given $\sigma^1_{u_j} = 2$. Therefore display our discussion in virtue of the total probability formula as

$$\mathbb{E}(Y_j - \pi_1) = \mathbb{P}(\sigma^1_{u_j} = 1) \mathbb{E}(X^+(n) - \pi_1)$$

$$+ \mathbb{P}(\sigma^1_{u_j} = 2) \mathbb{E}(1 - X^-(n) - \pi_1)$$

$$= M_{11} x_n - M_{12} \frac{\pi_1}{\pi_2} x_n$$

$$= \theta x_n$$

and similarly,

$$\mathbb{E}(Y_j - \pi_1)^2 = \mathbb{P}(\sigma^1_{u_j} = 1) \mathbb{E}(X^+(n) - \pi_1)^2$$

$$+ \mathbb{P}(\sigma^1_{u_j} = 2) \mathbb{E}(1 - X^-(n) - \pi_1)^2$$

$$= M_{11} x_n - M_{12} \frac{\pi_1}{\pi_2} (x_n - z_n)$$

$$= \theta z_n + \pi_1(1 - \theta)x_n.$$  \hfill $\Box$

III. Moment recursion

III.1. Distributional recursion

It is known that the asymptotic behavior of $x_n$ plays a crucial role in determining the reconstruction, however, it is still too difficult and not necessary to get the explicit expression for $x_n$. In fact we only need to investigate the recursive formula of $x_n$, from which it is possible to illustrate the trend of $x_n$ as $n$ goes to infinity. Thus the key method is to analyze the recursive relation between $X^+(n)$ and $X^+(n + 1)$ by the structure of the tree. Suppose $A$ is a configuration on $L(n + 1)$ and $A_j$ denotes the restriction to
With all preliminary results, we are ready to figure approximations of means and variances of Markov random field property that

\[
f_{n+1}(1, A) = \frac{N_1}{N_1 + N_2},
\]

where

\[
N_1 = \pi_1 \prod_{j=1}^{d} \left[ \frac{M_{11}}{\pi_1} f_n(1, A_j) + \frac{M_{12}}{\pi_2} f_n(2, A_j) \right] - \pi_1 \prod_{j=1}^{d} \left[ 1 + \theta \frac{\tau_1}{\pi_1} (f_n(1, A_j) - \pi_1) \right].
\]

and

\[
N_2 = \pi_2 \prod_{j=1}^{d} \left[ \frac{M_{21}}{\pi_1} f_n(1, A_j) + \frac{M_{22}}{\pi_2} f_n(2, A_j) \right] - \pi_2 \prod_{j=1}^{d} \left[ 1 - \theta \frac{\tau_2}{\pi_2} (f_n(1, A_j) - \pi_1) \right]
\]

Next conditioning the root to be 1 and setting \( A = c^1(n+1) \) in (15) give the recursive formula of the random variable

\[
X^+(n+1) = \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2},
\]

where

\[
Z_1 = \prod_{j=1}^{d} \left[ 1 + \theta \frac{\tau_1}{\pi_1} (f_n(1, A_j) - \pi_1) \right]
\]

\[
= \prod_{j=1}^{d} \left[ 1 + \theta \frac{\tau_1}{\pi_1} (Y_j(n) - \pi_1) \right];
\]

\[
Z_2 = \prod_{j=1}^{d} \left[ 1 - \theta \frac{\tau_2}{\pi_2} (f_n(1, A_j) - \pi_1) \right]
\]

\[
= \prod_{j=1}^{d} \left[ 1 - \theta \frac{\tau_2}{\pi_2} (Y_j(n) - \pi_1) \right].
\]

III.2. Main expansion of \( x_{n+1} \)

With all preliminary results, we are ready to figure out the recursion relation of \( x_{n+1}, \) say, its major expansions, which would play a crucial role in the further discussion. As regards \( x_{n+1}, \) we could expand it out by virtue of the identity

\[
\frac{a}{s + r} = \frac{a}{s} - \frac{a r^2}{s^2 (s + r)}.
\]

and specifically plugging \( a = \pi_1 Z_1, \ r = \pi_1 Z_1 + \pi_2 Z_2 - 1 \) and \( s = 1 \) in (17) yields

\[
x_{n+1} = E X^+(n + 1) - E Z_1
\]

\[
= E[\pi_1 Z_1] - E[\pi_1 Z_1 (\pi_1 Z_1 + \pi_2 Z_2 - 1)]
\]

\[
+ E \left[ (\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1. \right]
\]

In order to estimate terms in (18), we adapt Lemma 2.6 in [13] to our model, and then obtain Taylor series approximations of means and variances of \( Z_1 \)’s.

Lemma III.1 For each positive integer \( k, \) there exists a \( C = C(\pi, k) \) only depending on \( \pi \) and \( k \) such that for each \( 0 \leq k_1, k_2 \leq k, \)

\[
E Z_1^{k_1} Z_2^{k_2} \leq C,
\]

Then it can be concluded from the

\[
E Z_1^{k_1} Z_2^{k_2} \leq C,
\]

and

\[
E Z_1^{k_1} Z_2^{k_2} \leq C.
\]

Taken together, plugging all the previous results in (18) yields

\[
x_{n+1} = E(\pi_1 Z_1) - E[\pi_1 Z_1 (\pi_1 Z_1 + \pi_2 Z_2 - 1)]
\]

\[
+ \pi_1 E(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 - \pi_1 + S
\]

\[
= \delta^2 x_n + \frac{1 - 6 \pi_1 \pi_2 d (d - 1)}{\pi_1 \pi_2^2} \theta^4 x_n + R + S + T
\]

where \( |T| \leq C_T(\pi) x_n^3 \) and

\[
|R| \leq C_R(\pi) \frac{d (d - 1)}{2} \theta^5 \left| \frac{z_n}{x_n} - \pi_1 \right| x_n^2 = O(\theta) \left( \frac{z_n}{x_n} - \pi_1 \right) x_n^2
\]

with \( C_T(\pi), C_R(\pi) \) constants depending only on \( \pi, \)

\[
S = E(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \left( \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1 \right)
\]

will be handled in the following concentration analysis.

IV. Sufficient condition for the non-tightness of the Kesten-Stigum bound

IV.1. Estimates of \( R \) and \( S \)

The purpose of the following lemma is to describe how close the linear term in the recursive expansion approaches to \( x_{n+1}. \)

Lemma IV.1 For any \( \epsilon > 0, \) there exists a constant \( \delta = \delta(\pi, \epsilon) \) such that for all \( n, \) if \( x_n < \delta \) then

\[
| x_n + 1 - \delta^2 x_n | \leq \epsilon x_n.
\]
Proof. First note that $Z_1, Z_2 \geq 0$, and thus $0 < \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} \leq 1$. Then it is concluded from (18) and (II.1) that
\[
|x_{n+1} - d \theta^2 x_n| \leq C \varepsilon^2 \|x_n\|
\]
where $C = C(\delta)$ depends only on $\pi$, the first inequality follows from the fact of $0 \leq z_n \leq x_n$, and the last holds if $x_n < \delta$ for $\delta = \delta(\pi, \varepsilon)$ small enough.

Before investigating the concentration, we would like to introduce a significant lemma showing that $x_n$ does not drop from a very large value to a very small one.

**Lemma IV.2.** For any fixed $\varrho > 0$, assume $|\theta| > \varrho$. Then there exists a constant $\gamma = \gamma(\pi, \varrho) > 0$ such that for all $n$, $x_{n+1} \geq \gamma x_n$.

Proof. For a configuration $A = (A_1, \ldots, A_d)$ on $L(n+1)$ with $A_j$ on $T_{\gamma_1} \cap L(n+1)$ define
\[
f_{\pi,n+1}^z(1, A) = P(\sigma_j = 1 | \sigma_{n+1} = A) = \frac{\pi_1}{\pi_1 + \pi_2} P(\sigma_{n+1} = A | \sigma_n = 1)
\]

By IV.3, for any $\varrho > 0$, assume $|\theta| > \varrho$ for some $\varrho > 0$. Then there exist $N = N(\pi, \varepsilon, \varrho)$ such that whenever $n \geq N$, $P \left( \left| \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1 \right| > \varepsilon \right) \leq C \varepsilon^n$. With the preceding concentration results, it is feasible to bound $S$ in (22).

**Corollary IV.4.** Assume $|\theta| > \varrho$ for some $\varrho > 0$. For any $\varepsilon > 0$, there exist $N = N(\pi, \varepsilon, \varrho)$ such that $n \geq N$ and $x_n \leq \delta$ then $|S| \leq C \varepsilon^n$.

Proof. For any $\gamma > 0$, combining Cauchy-Schwartz inequality and Lemma IV.3 gives
\[
|S| = \left| \text{E}((\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \left( \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1 \right) \right|
\]
}\leq \text{E} \left( (\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \left( \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1 \right) \right; \left| \pi_1 Z_1 + \pi_2 Z_2 - \pi_1 \right| \leq \eta
\]
\+
\text{E} \left( (\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \left( \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1 \right) ; \left| \pi_1 Z_1 + \pi_2 Z_2 - \pi_1 \right| > \eta \right)
\leq \eta \left( \text{E}((\pi_1 Z_1 + \pi_2 Z_2 - 1)^2) \right)^{1/2}
\times \left( \text{P} \left( \left| \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1 \right| > \eta \right) \right)^{1/2}.
\]

Besides it follows from Lemma III.1 that
\[
\text{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^2 \leq C_1(\pi) x_n^2
\]
and
\[
(\text{E}(\pi_1 Z_1 + \pi_2 Z_2 - 1)^4)^{1/2} \leq C_2(\pi).
\]

Taking $\alpha = 6$ in Lemma IV.3, there exist $C_3 = C_3(\pi, \eta, \varrho)$ and $N = N(\pi, \eta)$ such that if $n \geq N$ then
\[
\text{P} \left( \left| \frac{\pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1 \right| > \eta \right) \leq C_3 x_n^6.
\]
Finally take \( \eta = \epsilon/(2C_1) \) and \( \delta = \epsilon/(2C_2C_3) \) and thus if \( n \geq N \) and \( x_n \leq \delta \) then
\[
|S| \leq \eta C_1 x_n^2 + C_2 C_3 x_n^3 \leq \epsilon x_n^2.
\]

IV.2. Proof of Theorem 1.1

To accomplish the proof, it suffices to show that when \( d^2 \theta^2 \) is close enough to 1, \( x_n \) does not converge to 0. For any fixed \( d \) and \( \pi_1 \), by the assumption of \( d^2 \theta^2 \geq 1/2 \), say, \( |\theta| \geq (2d)^{-1/2} \) take \( \epsilon = (2d)^{-1/2} \) in Lemma IV.2 and then get \( \gamma = \gamma(\pi,d) > 0 \). When \( \Delta^2 > (1-\theta)^2/3 \), namely, \( 1 - 6\pi_1 \pi_2 > 0 \), by Lemma IV.3 and Corollary IV.5 there exist \( N = N(\pi) \) and \( \delta = \delta(\pi,d) > 0 \) such that if \( n \geq N \) and \( x_n \leq \delta \) then the remainders in (19) could be bounded respectively by
\[
|R| \leq \frac{1}{6} \frac{1 - 6\pi_1 \pi_2}{\pi_1 \pi_2^2} \frac{d(d - 1)}{2} \theta^4 x_n^2 \tag{24}
\]
\[
|S| \leq \frac{1}{6} \frac{1 - 6\pi_1 \pi_2}{\pi_1 \pi_2^2} \frac{d(d - 1)}{2} \theta^4 x_n^2 \tag{25}
\]
\[
|T| \leq \frac{1}{6} \frac{1 - 6\pi_1 \pi_2}{\pi_1 \pi_2^2} \frac{d(d - 1)}{2} \theta^4 x_n^2 \tag{26}
\]
Consequently combining (24), (25) and (26) together gives
\[
x_{n+1} \geq d\theta^2 x_n + \frac{1}{2} \frac{1 - 6\pi_1 \pi_2}{\pi_1 \pi_2^2} \frac{d(d - 1)}{2} \theta^4 x_n^2. \tag{27}
\]
Furthermore in light of \( x_0 = 1 - \pi_1 = \pi_2 \) and Lemma IV.2 for all \( n \) we have
\[
x_n \geq \pi_2 \gamma^N. \tag{28}
\]
Thus define \( \epsilon = (\pi,d) = \min\{\pi_2 \gamma^N, \delta \gamma\} > 0 \), and then (28) implies that \( x_n \geq \epsilon \) when \( n \leq N \). Next by choosing suitable \( |\theta| < d^{-1/2} \), it is feasible to achieve
\[
d\theta^2 + \frac{1}{2} \frac{1 - 6\pi_1 \pi_2}{\pi_1 \pi_2^2} \frac{d(d - 1)}{2} \theta^4 \epsilon \geq 1, \tag{29}
\]
since \( \epsilon \) is independent of \( \theta \). Therefore, suppose \( x_n \geq \epsilon \) for some \( n \geq N \). If \( x_n \geq \gamma^{-1} \epsilon \), then Lemma IV.2 gives \( x_{n+1} \geq \gamma x_n \geq \epsilon \). If \( \epsilon \leq x_n \leq \gamma^{-1} \epsilon \leq \delta \) then by (27) and (29),
\[
x_{n+1} \geq d\theta^2 x_n + \frac{1}{2} \frac{1 - 6\pi_1 \pi_2}{\pi_1 \pi_2^2} \frac{d(d - 1)}{2} \theta^4 x_n^2 \geq x_n \left[ d\theta^2 + \frac{1}{2} \frac{1 - 6\pi_1 \pi_2}{\pi_1 \pi_2^2} \frac{d(d - 1)}{2} \theta^4 \epsilon \right] \geq x_n \geq \epsilon.
\]
Finally show by induction that \( x_n \geq \epsilon \) for all \( n \), namely, the Kesten-Stigum bound is not tight.

V. High degree discussion

V.1. Gaussian approximation

For \( 1 \leq j \leq d \), define
\[
U_j = \log \left[ 1 + \frac{\theta}{\pi_1}(Y_j - \pi_1) \right] \tag{30}
\]
and
\[
V_j = \log \left[ 1 - \frac{\theta}{\pi_2}(Y_j - \pi_1) \right] \tag{31}
\]
Lemma V.1 There exist positive constants \( C = C(\pi) \) and \( D = D(\pi) \) such that when \( d > D \),
\[
\begin{align*}
|d\text{E}U_j - \frac{d\theta^2}{2\pi_1} x_n| & \leq C d^{-1/2}; \\
|d\text{E}V_j + \frac{1 + \pi_2}{2\pi_2} d\theta^2 x_n| & \leq C d^{-1/2}; \\
|d\text{Var}(U_j) - \frac{\pi_1}{\pi_2} d\theta^2 x_n| & \leq C d^{-1/2}; \\
|d\text{Var}(V_j) - \frac{\pi_1}{\pi_2} d\theta^2 x_n| & \leq C d^{-1/2}; \\
|d\text{Cov}(U_j, V_j) + \frac{d\theta^2}{\pi_2} x_n| & \leq C d^{-1/2}.
\end{align*}
\]

Proof. Starting with the Taylor series expansion of \( \log(1 + w) \), there exists a constant \( W > 0 \) such that when \( |w| < W \),
\[
\left| \log(1 + w) - w - \frac{w^2}{2} \right| \leq |w|^3. \tag{32}
\]
If we take \( D = D(\pi) \) sufficiently large, when \( d > D \), \( |\theta| \leq d^{-1/2} \) is small enough to guarantee (32) for \( w = \theta(Y_j - \pi_1)/\pi_1 \) and then
\[
\begin{align*}
|d\text{E}U_j - \frac{\theta^2}{2\pi_1} x_n| & \leq \mathbb{E} \left| \frac{\theta^3}{\pi_1^3} |Y_j - \pi_1|^3 + \frac{\theta^3}{2\pi_1^2} |z_n - \pi_1 x_n| \right| \\
& \leq \frac{\theta^3}{\pi_1^3} + \frac{\theta^3}{2\pi_1^2} \\
& \leq C(\pi) d^{-3/2}
\end{align*}
\]
for some constant \( C = C(\pi) \), where the third inequality follows from \( 0 \leq z_n \leq x_n \leq 1 \). The rest estimates would follow similarly.

In view of the complexity of (15), it is convenient to come up with the "better" recursive approximation under results of Lemma V.1. Define a 2-dimensional vector
\[
\mu = (\mu_1, \mu_2) = \left( \frac{1}{2\pi_1}, -\frac{1 + \pi_2}{2\pi_2^2} \right) \tag{33}
\]
and a \( 2 \times 2 \)-covariance matrix
\[
\Sigma = \begin{pmatrix}
\frac{1}{\pi_1} & -\frac{1}{\pi_2^2} \\
-\frac{1}{\pi_2} & \frac{1}{\pi_2^2}
\end{pmatrix}. \tag{34}
\]
Suppose \((W_1, W_2)\) has a Gaussian distribution \(N(0, \Sigma)\), and then \((s\mu_1 + \sqrt{s}W_1, s\mu_2 + \sqrt{s}W_2)\) is distributed according to \(N(s\mu, s\Sigma)\). According to

\[
x_{n+1} = \frac{x_n + \pi_1 Z_1}{\pi_1 Z_1 + \pi_2 Z_2} - \pi_1,
\]

we could construct a differentiable function

\[
f(s) = \frac{\pi_1 \exp(s\mu_1 + \sqrt{s}W_1)}{\pi_1 \exp(s\mu_1 + \sqrt{s}W_1) + \pi_2 \exp(s\mu_2 + \sqrt{s}W_2)} - \pi_1.
\]

From the fact that \[\left| \frac{2n^d}{t} \right| \left( 1 + \frac{2n^d}{t} \right)^2 \leq 1/4\] holds for any \(t \in \mathbb{R}\). Then we establish the differentiability of \(f(s)\).

Now we can reinvestigate the recursive approximation with the assistance of \(f(s)\) and the following lemma established immediately by using Central Limit Theorem, Gaussian approximation and Portmanteau Theorem.

**Lemma V.3** For arbitrary \(\varepsilon > 0\) there exists a \(D = D(\pi, \varepsilon) > 0\) such that whenever \(d > D\),

\[
x_{n+1} = f(d^2 x_n) \leq \varepsilon.
\]

**V.2. Proof of Theorem [L.2]**

First referring to Mathematica, it is possible to establish

**Lemma V.4** When \(\Delta^2 < (1 - \theta)^2 / 3\), for any \(0 < s \leq \pi_2\) we have

\[
f(s) < s.
\]

When \(\Delta^2 < (1 - \theta)^2 / 3\), namely, \(1 - 6\pi_1 \pi_2 < 0\), the proof of Theorem [L.2] would resemble Theorem [L.1] to establish the analogous recursive inequality of (27) under the condition of \(x_n < \delta\) and \(n \geq N\) for suitable \(\delta = \delta(\pi, d)\) and \(N = N(\pi)\). However, there still exists a crucial discrepancy between two proofs, that is, Theorem [L.2] relies tightly on large \(d\), while the case of small degree emerges as the more intractable issue. Thus in order to establish an analogue of (27), it is necessary to eliminate the dependency of \(\delta\) on \(d\) as follows.

**Proof of Theorem [L.2]** Resembling the proof of Theorem [L.1] we evaluate \(R, S\) and \(T\) in (19) respectively, under \(1 - 6\pi_1 \pi_2 < 0\). First take \(D = D(\pi) = (6C(\pi) \pi\pi_2^2)/(6\pi\pi_2 - 1)^2\) such that if \(d > D\) that implies \(|\theta| < d^{-1/2} < D^{-1/2}\), then it is concluded by recalling (20) and \(|z_n/x_n - \pi_1| \leq 1\) that

\[
|R| \leq C_R(\pi) \frac{d(1 - d)}{2} |\theta|^5 \left| \frac{z_n}{x_n} - \pi_1 \right| x_n^2
\]

Next applying Lemma [V.3] to deal with \(S\), there exist \(N = N(\pi)\) and \(\delta = \delta(\pi) > 0\) independent of \(d\) such that if \(n \geq N\) and \(x_n < \delta\) then the analogues of (25) and (26) still hold as

\[
|S| \leq \frac{11 - 6\pi_1 \pi_2 d(1 - d)}{6 \pi\pi_2} \theta^4 x_n^2
\]

and

\[
|T| \leq \frac{11 - 6\pi_1 \pi_2 d(1 - d)}{6 \pi\pi_2} \theta^4 x_n^2.
\]
Finally taken together, if \( d > D, n \geq N \) and \( x_n < \delta \), then (39), (40) and (41) give
\[
x_{n+1} \leq d\sigma^2 x_n + \frac{1}{2} \left( 1 - 6\pi_1 \pi_2 \right) d\left( d - 1 \right) \frac{1}{\pi_1 \pi_2^2} \theta^d x_n^2 \leq x_n.
\]
Therefore it follows that \( L = \lim_{n \to \infty} x_n \) exists, since the sequence \( \{x_n\}_{n \geq N} \) is bounded and decreasing. Thus taking limits to both sides of (42) yields
\[
L \leq d\sigma^2 L - \frac{1}{2} \left( 6\pi_1 \pi_2 - 1 \right) d\left( d - 1 \right) \frac{1}{\pi_1 \pi_2^2} \theta^d L^2,
\]
(43)
which implies \( L = \lim_{n \to \infty} x_n = 0 \), and hence non-reconstruction.

So here it suffices to find some \( m \geq N \) such that \( x_m < \delta \). Define \( \epsilon = \epsilon(\pi, \delta) = \epsilon(\pi) = \frac{1}{2} \min_{s \geq \delta} (s - f(s)) \). Since the function \( s - f(s) \) is continuous and positive on \([\delta, \pi_2]\), it follows by Lemma [V4] that \( \epsilon > 0 \). Then by Lemma [V3] there exists a \( D = D(\pi, \epsilon) = D(\pi) > 0 \) such that when \( d > D \), if \( x_n \geq \delta \) and \( n \geq N \), we have
\[
x_{n+1} < f(d\sigma^2 x_n) + \epsilon \leq f(x_n) + \epsilon \leq x_n - \epsilon,
\]
where the second inequality is from Lemma [V2] say, \( f(s) \) is increasing on \([0, \pi_2] \). Therefore there must exist \( m \geq N \) such that \( x_m < \delta \), as desired.

VI. Conclusion

In this paper, we have studied the asymmetric Ising model on regular trees and figured out the critical conditions for the reconstruction by means of the recursive structure of the tree. The key idea is to analyze the relation between the distributions \( P(\sigma_0 = 1 \mid \sigma^i(n)) \) and \( P(\sigma_0 = 1 \mid \sigma^i(n + 1)) \). Our result not only establishes the existence of the symmetry bias to keep Kesten-Stigum bound tight, but determines the exact thresholds for non-solvability of the asymmetric Ising model.

More importantly, together with some results of non-linear dynamic system, our skills could also be applied to explore the reconstruction of the \( d \)-ary tree on continuous state space, and even the general phylogenetic reconstruction.

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