A study of Nonlinear Galerkin Finite Element for time-dependent incompressible Navier-Stokes equations

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Abstract

In this article, we discuss a couple of nonlinear Galerkin methods (NLGM) in finite element set up for time dependent incompressible Navier-Stokes equations. We show the crucial role played by the non-linear term in determining the rate of convergence of the methods. We have obtained improved error estimate in $L^2$ norm, which is optimal in nature, for linear finite element approximation, in view of the error estimate available in literature, in $H^1$ norm.

Key Words. nonlinear Galerkin method, Navier-Stokes equations, optimal error estimates.

1 Introduction

In the study of dynamical systems, generated by evolution partial differential equations, we look into long time behavior of the solutions. In certain cases, solutions converge asymptotically to a compact set called global attractor. Owing to complicated structures of such sets, notion of inertial manifold (IM) was introduced, which is a smooth finite dimensional manifold, attracting all the solution trajectories exponentially. Once we can prove the existence of an IM for a evolution equation, it is far easier to study it than to study the global attractor. Unfortunately, the existence of an IM for Navier-Stokes equations could not be established, even for a simpler case like 2D spatially periodic flow. Hence came the concept of artificial inertial manifold (AIM), a sequence of smooth and finite dimensional manifolds of increasing dimensions and that the global attractor lies in a small neighborhood of each such manifold with the distance vanishing exponentially as the dimension increases. AIM has been shown to exist for the Navier-Stokes equations in 2D. A numerical technique based on AIM was introduced by Marion and Temam in [12] ans was coined non-linear Galerkin method (NLGM). The method involves splitting the solution of a dissipative system into large and small scales, simplifying the small scale equation, thereby obtaining small scale in terms of large scale, relatively easily. In other words, small scales are made slaves to large scales. This technique and its modifications are studied in depth in early nineties. For details and a history of developments, we refer to [2-15].

NLGM was originally developed in the context of spectral Galerkin approximations. The method in [12] was based on the eigenvectors of the underlying linear elliptic operator. Later in [13], it was expanded to more general bases and more specifically to finite elements. But for non-spectral Galerkin discretizations, very few results are available. The problem was that the extension to such cases is not natural, since the splitting of discrete solution space into large and small scales (or low and high frequency modes) is not obvious any longer. Marion and Temam [13] developed various NLGMs based on finite element and later on, Marion and Xu [14] and, Ammi
and Martin [1] worked on general semi-linear reaction diffusion and the Navier-Stokes equations with higher convergence rate in $H^1$-norm for a NLGM type two-level finite element method. There, the strong orthogonality properties of spectral Galerkin approximation was substituted by weaker $L^2$-orthogonality of finite element approximation. In [14], Marion and Xu considered a semi-linear evolution equation and showed that the $H^1$-error estimate to be of $O(H^3)$ (here $H$ is a space discretizing parameter). They expected $L^2$-error estimate to be of $O(H^4)$ and remarked that this is an open problem. We prove this higher order estimate in a couple of cases and believe that it can not be extended to other possible cases.

The advent of NLGMs created a lot of interest, since it outperformed the standard Galerkin method, which is restricted to the large scales only. Also, it was thought of as well-suited for turbulence modeling. But Heywood and Rannacher [10] argued that it is not turbulence modeling that is responsible for the better performance of NLGM over Galerkin method. On the other hand, it is the ability of NLGM to handle better, the Gibb phenomenon induced by higher order boundary incompatibilities induced by no-slip boundary condition. They substantiated it by showing that, in periodic domain, both perform identically. Later, Guermou and Prudhomme, in [2], revisited NLGM and showed that, in case of arbitrary smoothness, NLGM will always outperform Galerkin method. And it has nothing to do with turbulence modeling, but due to the well-known fact of superconvergence of elliptic projection in $H^1$-norm. We have also exploited this simple fact to improve the existing $L^2$-error estimate.

The incompressible time dependent Navier-Stokes equations are given by
\begin{equation}
\frac{\partial u}{\partial t} + u \cdot \nabla u - \nu \Delta u + \nabla p = f(x,t), \quad x \in \Omega, \quad t > 0\end{equation}
with incompressibility condition
\begin{equation}\nabla \cdot u = 0, \quad x \in \Omega, \quad t > 0,\end{equation}
and initial and boundary conditions
\begin{equation}u(x,0) = u_0 \quad \text{in} \quad \Omega, \quad u = 0, \quad \text{on} \quad \partial \Omega, \quad t \geq 0.\end{equation}
Here, $\Omega$ is a bounded domain in $\mathbb{R}^2$ with boundary $\partial \Omega$ and $\nu$ is the inverse of the Reynolds number. $u$ and $p$ stand for the velocity and the pressure of a fluid occupying $\Omega$, respectively. Initial velocity $u_0$ is a solenoidal vector field and $f$ is the forcing term.

Error estimations of NLGM in mixed finite element set up for Navier-Stokes’ equations are carried out in [1]. The small scales equations carry both nonlinearity and time dependence and the following estimates were established.
\begin{equation}
\| (u^h - u_h)(t) \|_{H^1(\Omega)} \leq c(t)H^2, \quad \| (p^h - p_h)(t) \|_{L^2(\Omega)} \leq c(t)H^2,
\end{equation}
where $(u_h, p_h)$ is the Galerkin approximation and $(u^h, p^h)$ is the nonlinear Galerkin approximation. These results were improved to $O(H^3)$ and similar result was obtained in $L^2$-norm for velocity approximation, but for semilinear parabolic problems, by Marion and Xu, in [14]. We feel that it is not straight forward to carry forward these results to Navier-Stokes equations. The proof of [14] depends on one important estimate involving the nonlinear term $f(u)$, namely
\begin{equation}
\| f(y_h + z_h) - f(y_h), \chi \| \leq \| z_h \|_{L^2(\Omega)} \| \chi \|_{L^2(\Omega)},
\end{equation}
where $y_h$ and $z_h$ are the large and small scales of $u^h$, and $\chi$ is any element from the appropriate space containing small scales, see [14, pp. 1176]. But it may not be possible to establish similar estimate for the nonlinear term $(u \cdot \nabla)u$. On the other hand, it is observed that the improvement
in the order of convergence is due to the fact that the splitting in space is done on the basis of $L^2$ projections, unlike previous approaches, where the splitting is based on hierarchical basis. Similar approach is adopted in this article.

No further improvement is observed for (piecewise) linear finite element discretization and with forcing term $f \in L^2(\Omega)$. Results, similar to the ones mentioned above, are observed in [3, 4], but for $f \in H^1(\Omega)$ and for Navier-Stokes equations. In [15], various nonlinear Galerkin finite elements are studied in depth for one-dimensional problems and similar results are obtained, although termed as optimal in nature. For example, the method is studied for piecewise polynomials of degree $2n - 1$.

And the error estimates obtained in energy-norm and $L^2$-norm are as follows

$$\| (u_h - u^h)(t) \|_{H^n(\Omega)} \leq c(t) H^{3n-1},$$

$$\| (u_h - u^h)(t) \|_{L^2(\Omega)} \leq c(t) \min\{ t^{-1/2} H^{4n-1}, H^{7n/2-1} \},$$

respectively. For $n = 1$, we have the piecewise linear finite element approximation and error estimates are of $O(H^2)$ and $O(H^{5/2})$, in energy-norm and $L^2$-norm, respectively. Later on He et. al have studied NLGM and modified NLGM in both spectral and finite element set ups, [5, 6, 8, 7] to name a few. In these articles, fully discrete NLGMs were considered and were shown to exhibit better convergence rate than Galerkin method. But none were optimal.

Recently in [11], a new projection is employed for a two-level finite element for Navier-Stokes equations. Error estimates of $O((\log h)^{1/2} H^3)$ and $O((\log h)^{1/2} H^4)$, in energy-norm and $L^2$-norm, respectively, are obtained. Logarithmic term appears due to the use of the finite dimensional case of the Brezis-Gallouet inequality. Note that here the forcing term is taken in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$.

In this article, we have established optimal $L^2$-error estimate for a couple of NLGMs, while the forcing term is in $L^2$. The approach here is an heuristic one and we basically highlight the importance of nonlinearity of the small scales equations as the key to the higher or lower order error estimate. Since these methods can be considered as two-level methods, this suggests ways of constructions or limitations of two or multi-level methods for lower-order approximations.

The article is organized as follows. In section 2, we briefly recall the notion of a suitable weak solution of the Navier-Stokes equations. Section 3 deals with the Galerkin approximation and state the error estimates. In Section 4, we introduce the NLGMs. Error estimates for the methods are discussed in Section 5. Finally, in Chapter 6, we summarize the results obtained in this article.

## 2 Preliminaries

For our subsequent use, we denote by bold face letters the $\mathbb{R}^2$-valued function space such as

$$H^1_0 = [H^1_0(\Omega)]^2, \quad L^2 = [L^2(\Omega)]^2.$$

Note that $H^1_0$ is equipped with a norm

$$\| \nabla v \| = \left( \sum_{i,j=1}^2 (\partial_j v_i, \partial_j v_i) \right)^{1/2} = \left( \sum_{i=1}^2 (\nabla v_i, \nabla v_i) \right)^{1/2}.$$

Further, we introduce divergence free function spaces:

$$J_1 = \{ \phi \in H^1_0 : \nabla \cdot \phi = 0 \},$$

$$J = \{ \phi \in L^2 : \nabla \cdot \phi = 0 \text{ in } \Omega, \phi \cdot n|_{\partial \Omega} = 0 \text{ holds weakly} \}. $$
where $\mathbf{n}$ is the outward normal to the boundary $\partial \Omega$ and $\phi \cdot \mathbf{n}|_{\partial \Omega} = 0$ should be understood in the sense of trace in $H^{-1/2}(\partial \Omega)$, see [16]. For any Banach space $X$, let $L^p(0, T; X)$ denote the space of measurable $X$-valued functions $\phi$ on $(0, T)$ such that

$$\int_0^T \|\phi(t)\|_X^p \, dt < \infty \quad \text{if} \quad 1 \leq p < \infty,$$

and for $p = \infty$

$$\text{ess sup}_{0 < t < T} \|\phi(t)\|_X < \infty \quad \text{if} \quad p = \infty.$$ 

Throughout this paper, we make the following assumptions:

(A1). For $g \in L^2$, let the unique pair of solutions $\{v \in J_1, q \in L^2/R\}$ for the steady state Stokes problem

$$-\Delta v + \nabla q = g, \quad \nabla \cdot v = 0 \quad \text{in} \quad \Omega, \quad v|_{\partial \Omega} = 0,$$

satisfy the following regularity result

$$\|v\|_2 + \|q\|_{H^1/R} \leq C\|g\|.$$ 

Note that for a $C^2$-domain or for a two-dimensional convex polygon, assumption (A1) holds. (A2). The initial velocity $u_0$ and the external force $f$ satisfy for positive constant $M_0$, and for $T$ with $0 < T \leq \infty$

$$u_0 \in J_1, \quad f, f_t \in L^\infty(0, T; L^2) \quad \text{with} \quad \|u_0\|_1 \leq M_0, \quad \sup_{0 < t < T} \{\|f\|, \|f_t\|\} \leq M_0.$$ 

We now present a weak solution of (1.1)-(1.3). Find a pair of functions $\{u(t), p(t)\}$, $t > 0$, such that

$$\begin{cases}
(u_t, \phi) + \nu(\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi) = (p, \nabla \cdot \phi) + (f, \phi) & \forall \phi \in H^1_0, \\
(\nabla \cdot u, \chi) = 0 & \forall \chi \in L^2.
\end{cases} \quad (2.1)$$

Equivalently, find $u(t) \in J_1$, $t > 0$ such that

$$\begin{cases}
(u_t, \phi) + \nu(\nabla u, \nabla \phi) + (u \cdot \nabla u, \phi) = (f, \phi) & \forall \phi \in J_1.
\end{cases} \quad (2.2)$$

The existence, uniqueness and regularity of the solution of the 2D time dependent Navier-Stokes equations can be found in [16].

### 3 Galerkin Method

From now on, we denote $h$ with $0 < h < 1$ to be a real positive discretization parameter tending to zero. Let $H_h$ and $L_h$, $0 < h < 1$ be two family of finite dimensional subspaces of $H^1_0$ and $L^2/\mathbb{R}$, respectively, approximating velocity vector and the pressure. Assume that the following approximation properties are satisfied for the spaces $H_h$ and $L_h$:

(B1) For each $w \in H^1_0 \cap H^2$ and $q \in H^1/\mathbb{R}$ there exist approximations $i_h w \in H_h$ and $j_h q \in L_h$ such that

$$\|w - i_h w\| + h\|\nabla (w - i_h w)\| \leq K_0 h^2\|w\|_2, \quad \|q - j_h q\| \leq K_0 h\|q\|_1.$$ 

Further, suppose that the following inverse hypothesis holds for $w_h \in H_h$:

$$\|\nabla w_h\| \leq K_0 h^{-1}\|w_h\|. \quad (3.1)$$
For defining the Galerkin approximations, set for \( v, w, \phi \in H^1_0 \),
\[
a(v, \phi) = (\nabla v, \nabla \phi), \quad b(v, w, \phi) = \frac{1}{2}(v \cdot \nabla w, \phi) - \frac{1}{2}(v \cdot \nabla \phi, w).
\]

Note that the operator \( b(\cdot, \cdot, \cdot) \) preserves the antisymmetric property of the original nonlinear term, that is,
\[
b(v_h, w_h, w_h) = 0 \quad \forall v_h, w_h \in H_h.
\]
The discrete analogue of the weak formulation (2.1) now reads as: Find \( u_h(t) \in H_h \) and \( p_h(t) \in L_h \) such that \( u_h(0) = u_{0h} \) and for \( t > 0 \)
\[
(3.2) \quad \begin{cases}
(u_{ht}, \phi_h) + \nu a(u_h, \phi_h) + b(u_h, u_h, \phi_h) - (p_h, \nabla \cdot \phi_h) = (f, \phi_h), & \phi_h \in H_h \\
(\nabla \cdot u_h, \chi_h) = 0, & \chi_h \in L_h.
\end{cases}
\]
Here \( u_{0h} \in H_h \) is a suitable approximation of \( u_0 \in J_1 \). For continuous dependence of the discrete pressure \( p_h(t) \in L_h \) on the discrete velocity \( u_h(t) \in J_h \), we assume the following discrete inf-sup (LBB) condition for the finite dimensional spaces \( H_h \) and \( L_h \):

(B2') For every \( q_h \in L_h \), there exists a non-trivial function \( \phi_h \in H_h \) and a positive constant \( K_0 \), independent of \( h \), such that
\[
|(q_h, \nabla \cdot \phi_h)| \geq K_0 \| \nabla \phi_h \| \| q_h \|.
\]
In order to consider a discrete space, analogous to \( J_1 \), we impose the discrete incompressibility condition on \( H_h \) and call it as \( J_h \). Thus, we define \( J_h \), as
\[
J_h = \{ v_h \in H_h : (\chi_h, \nabla \cdot v_h) = 0 \quad \forall \chi_h \in L_h \}.
\]
Note that \( J_h \) is not a subspace of \( J_1 \). With \( J_h \) as above, we now introduce an equivalent Galerkin formulation as: Find \( u_h(t) \in J_h \) such that \( u_h(0) = u_{0h} \) and for \( \phi_h \in J_h \), \( t > 0 \)
\[
(3.3) \quad (u_{ht}, \phi_h) + \nu a(u_h, \phi_h) = -b(u_h, u_h, \phi_h) + (f, \phi_h).
\]
Since \( J_h \) is finite dimensional, the problem (3.3) leads to a system of nonlinear differential equations. For global existence of a unique solution of (3.3) (or unique solution pair of (3.2)), we refer to [9].

We further assume that the following approximation property holds true for \( J_h \).

(B2) For every \( w \in J_1 \cap H^2 \), there exists an approximation \( r_h w \in J_h \) such that
\[
\| w - r_h w \| + h \| \nabla (w - r_h w) \| \leq K_5 h^2 \| w \|_2.
\]
The \( L^2 \) projection \( P_h : L^2 \rightarrow J_h \) satisfies the following properties: for \( \phi \in J_h \),
\[
(3.4) \quad \| \phi - P_h \phi \| + h \| \nabla P_h \phi \| \leq C h \| \nabla \phi \|,
\]
and for \( \phi \in J_1 \cap H^2 \),
\[
(3.5) \quad \| \phi - P_h \phi \| + h \| \nabla (\phi - P_h \phi) \| \leq C h^2 \| \Delta \phi \|.
\]
With the definition of the discrete operator \( \Delta_h : H_h \rightarrow H_h \) through the bilinear form \( a(\cdot, \cdot) \),
\[
(3.6) \quad a(v_h, \phi_h) = (-\Delta_h v_h, \phi) \quad \forall v_h, \phi_h \in H_h,
\]
we set the discrete analogue of the Stokes operator \( \tilde{\Delta} = P(-\Delta) \) as \( \tilde{\Delta}_h = P_h(-\Delta_h) \). Using Sobolev imbedding and Sobolev inequality, it is easy to prove the following Lemma.
Lemma 3.1. Suppose conditions (A1), (B1) and (B2) are satisfied. Then there exists a positive constant $K$ such that for $v, w, \phi \in H_h$, the following holds:

$$
(3.7) \quad \| (v \cdot \nabla w, \phi) \| \leq K \begin{cases} 
\|v\|^{1/2}\|\nabla v\|^{1/2}\|\nabla w\|^{1/2}\|\Delta_h w\|^{1/2}\|\phi\|, \\
\|v\|\|\Delta_h v\|\|\nabla w\|\|\phi\|, \\
\|v\|\|\nabla w\|\|\phi\|^{1/2}\|\Delta_h \phi\|^{1/2}, \\
\|v\|\|\nabla w\|\|\phi\|^{1/2}\|\nabla \phi\|^{1/2}.
\end{cases}
$$

The following lemma and theorem present a priori estimates of the semi-discrete solution and optimal error estimate, respectively.

Lemma 3.2. Let $0 < \alpha < \nu \lambda_1$, where $\lambda_1 > 0$ is the smallest eigenvalue of the Stokes’ operator. Let the assumptions (A1), (A2), (B1) and (B2) hold. Then the semi-discrete Galerkin approximation $u_h$ of the velocity $u$ satisfies, for $t > 0$,

$$
(3.8) \quad \|u_h(t)\| + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\nabla u_h(s)\|^2 \, ds \leq K,
$$

$$
(3.9) \quad \|\nabla u_h(t)\| + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\Delta_h u_h(s)\|^2 \, ds \leq K,
$$

$$
(3.10) \quad (\tau^*(t))^{1/2}\|\Delta_h u_h(t)\| \leq K,
$$

where $\tau^*(t) = \min\{t, 1\}$. The positive constant $K$ depends only on the given data. In particular, $K$ is independent of $h$ and $t$.

Theorem 3.1. Let $\Omega$ be a convex polygon and let the conditions (A1)-(A2) and (B1)-(B2) be satisfied. Further, let the discrete initial velocity $u_{0h} \in J_h$ with $u_{0h} = P_h u_0$, where $u_0 \in J_1$. Then, there exists a positive constant $C$, that depends only on the given data and the domain $\Omega$, such that for $0 < T < \infty$ with $t \in (0, T]$

$$
\|(u - u_h)(t)\| + h\|\nabla (u - u_h)(t)\| \leq C e^{\frac{C}{2}h^2 t^{-1/2}}.
$$

A detail account of the finite element spaces, estimates of the semi-discrete solutions and error estimates can be found in [9].

Remark 3.1. From numerical point of view, it is a standard practice to work with mixed formulation (3.2) rather than (3.3). But following [9], we prefer to base our analysis on the divergence free formulation (3.3). The same analysis can easily be carried forward to mixed formulation. It is just a matter of taste.

4 Nonlinear Galerkin Method

In this section, we introduce another space discretizing parameter $H$ such that $0 < h < H$ and both $h, H$ tend to 0. And based on that, we split finite element space $J_h$ into two.

$$
(4.1) \quad J_h = J_H + J^H_h, \quad \text{with} \quad J^H_h = (I - P_H)J_h
$$

Note that, by definition, the spaces $J_H$ and $J^H_h$ are orthogonal with respect to the $L^2$-inner product $\langle \cdot, \cdot \rangle$. In practice, $J_H$ corresponds to a coarse grid and $J^H_h$ corresponds to a fine grid. In case of mixed method, we only split the velocity space and pressure space remains the same.
The following properties are crucial for our error estimates. For a proof, we refer to [1].

\[(4.2)\quad \|\chi\| \leq cH\|\chi\|_1, \quad \chi \in J_H^H.\]

And there exists \(0 < \rho < 1\) independent of \(h\) and \(H\) such that

\[(4.3)\quad |a(\phi, \chi)| \leq (1 - \rho)\|\phi\|_1\|\chi_h\|_1, \quad \phi \in J_H, \chi \in J_h^H.\]

From (4.3), we can easily deduce that

\[(4.4)\quad \rho(\|\phi\|_1^2 + \|\chi\|_1^2) \leq \|\phi + \chi_h\|_1^2, \quad \phi \in J_H, \chi \in J_h^H.\]

In nonlinear Galerkin methods, we look for a solution \(u_h \in J_h\) in terms of its components \(y^H\) (coarse grid) and \(z^h\) (fine grid).

\[u_h = y^H + z^h \in J_H + J_h^H.\]

In the first of the two methods (NLGM I), for \(t > t_0 > 0\), we look for a pair \((y^H, z^h)\) satisfying

\[(4.5)\quad \begin{cases} y^H_t(\phi) + va(u^h, \phi) + b(u^h, u^h, \phi) = (f, \phi), & \phi \in J_H, \\ va(u^h, \chi) + b(u^h, u^h, \chi) = (f, \chi), & \chi \in J_h^H. \end{cases}\]

In the second one (NLGM II), we again look for a pair \((y^H, z^h)\) satisfying, for \(t > t_0 > 0\)

\[(4.6)\quad \begin{cases} y^H_t(\phi) + va(u^h, \phi) + b(u^h, y^H, \phi) = (f, \phi), & \phi \in J_H, \\ va(u^h, \chi) + b(u^h, y^H, \chi) + b(y^H, z^h, \chi) = (f, \chi), & \chi \in J_h^H. \end{cases}\]

We set \(y^H(t_0) = P_H u(t_0)\). Note that the two systems differ only by the term \(b(z^h, z^h, \chi)\), which has been dropped from the second system, assuming \(z^h\) is small.

**Remark 4.1.** Here and henceforth, subscript means the classical Galerkin method and superscript means the nonlinear Galerkin method.

**Remark 4.2.** The nonlinear Galerkin approximations are carried out away from \(t = 0\). We can employ Galerkin approximation to obtain \((u_h, p_h)\) on the interval \((0, t_0]\). This is done to avoid nonlocal compatibility conditions. In [9], Heywood and Rannacher showed that to assume higher regularity for the solution demands some nonlocal compatibility conditions to be satisfied by initial velocity and initial pressure. These conditions are very difficult to verify and do not arise in physical context. In order to avoid them, we must admit singularity of velocity field in a higher order norm at \(t = 0\), like (1.6) from [9]. Since the error analysis of NLGM demands higher regularity of the velocity and this means higher singularity at \(t = 0\), the idea is to avoid these kinds of singularity by staying away from \(t = 0\).

**Remark 4.3.** Both the NLGs can be heuristically derived from (3.3) as follows. We split the Galerkin approximation \(u_h\) with the help of the \(L^2\) projection \(P_H\).

\[(4.7)\quad u_h = P_H u_h + (I - P_H) u_h = y^H + z_h,\]

And we project the system (3.3) on \(J_H, J_h^H\) to obtain the coupled system:

\[(4.8)\quad \begin{cases} y^H_t(\phi) + va(u^h, \phi) = -b(u_h, u_h, \phi) + (f, \phi), \\ z^H_t(\chi) + va(u^h, \chi) = -b(u^h, u^h, \chi) + (f, \chi), \end{cases}\]

for \(\phi \in J_H, \chi \in J_h^H\). Assuming the time derivative and higher space derivatives of \(z_h\) are small, various (modified) NLG methods are defined. In case, time derivative of \(z_h\) is retained in the equation, different time-steps can be employed for the two equations; (much) smaller time-step for the equation involving \(y^H\), thereby keeping \(z_h\) remains steady.
The well-posedness of both the NLGMs (4.5) and (4.6) follows easily from the works of Marion et al. [1, 13, 14]. For the sake of completeness, we present below, the \textit{a priori} estimates for the approximate solution pair \( \{ y^H, z^h \} \). And for the sake of brevity, we only sketch a proof.

**Lemma 4.1.** Under the assumptions of Lemma 3.2, the solution pair \( (y^H, z^h) \) of (4.8) or (4.6) satisfy the following estimates, for \( t > t_0 \)

\[
\| y^H \|^2 + e^{-2\alpha t} \int_{t_0}^{t} e^{2\alpha s} \| \nabla (y^H + z^h) \|^2 ds \leq K. 
\]

And if \( H \) is small enough to satisfy

\[
\nu - cL_H \| y^H \| > 0,
\]

where \( L_H \sim |\log H|^{1/2} \), then the following estimate holds

\[
\| z^h \| \leq K.
\]

The constant \( K > 0 \) depends only on the given data. In particular, \( K \) is independent of \( h, H \) and \( t \).

**Proof.** Choose \( \phi = y^H, \chi = z^h \) in (4.3) or (4.6) and add the resulting equations. Note that the nonlinear terms sum up to 0. Multiply by \( e^{2\alpha t} \). Use Poincaré inequality and then kickback argument. Finally, integrate from \( t_0 \) to \( t \) and multiply by \( e^{-2\alpha t} \) to complete the first estimate. For the second estimate, we put \( \chi = z^h \) in the second equation of (4.8) or (4.6).

\[
\nu \| \nabla z^h \|^2 \leq \| f \| \| z^h \| + \nu \| \nabla y^H \| \| \nabla z^h \| + |b(y^H + z^h, y^H, z^h)|. 
\]

Using (3.1) and (4.2), we find

\[
b(y^H, y^H, z^h) \leq 2^{1/2} \| y^H \|^{1/2} \| \nabla y^H \|^{3/2} \| z^h \|^{1/2} \| \nabla z^h \|^{1/2} \leq c \| y^H \| \| \nabla y^H \| \| \nabla z^h \| \| \nabla z^h \|
\]

\[
b(z^h, y^H, z^h) \leq \| z^h \| \| \nabla z^h \| \| y^H \|_\infty \leq cL_H \| z^h \| \| \nabla z^h \| \| \nabla y^H \|
\]

We have used the finite dimensional case of Brezis-Gallouet inequality (see [11, (3.12)])

\[
\| u_h \|_\infty \leq cL_h \| \nabla u_h \|; \quad L_h \sim |\log h|^{1/2}. 
\]

Incorporate these estimates in (4.12). With the re-use of (4.2), we have

\[
\nu \| \nabla z^h \| \leq cH \| f \| + \nu \| \nabla y^H \| + c \| y^H \| \| \nabla y^H \| + cL_H \| z^h \| \| \nabla y^H \|.
\]

Again use (3.1), we find

\[
\nu \| z^h \| \leq cH \| \nabla z^h \| \leq cH^2 \| f \| + c \| y^H \| + c \| y^H \|^2 + cL_H \| z^h \| \| y^H \|
\]

Apply (4.10) and (4.9), we conclude the proof.

**Remark 4.4.** Under the smallness assumption on \( H \), that is, (4.10), we could similarly prove higher order estimates of \( y^H \) and \( z^h \). For details, we refer to [1, 13, 14].
5 Error Estimate

In this section, we work out the error between classical Galerkin approximation and nonlinear Galerkin approximation of velocity.

Before actually working out the error estimates, we present below the Lemma involving the estimate of \( z_h \). For a proof, we refer to [1].

**Lemma 5.1.** Under the assumptions Lemma 3.2 and for the solution \( u_h \) of (3.3), the following estimates are satisfied for \( z_h = (I - P_H)u_h \) and for \( t > t_0 \)

\[
\begin{aligned}
\|z_h\| + H\|z_h\|_1 &\leq K(t)H^2, \\
\|z_{ht}\| + H\|z_{ht}\|_1 &\leq K(t)H^2, \\
\|z_{htt}\| + H\|z_{htt}\|_1 &\leq K(t)H^2.
\end{aligned}
\]

In order to separate the effect of the nonlinearity in the error, we introduce

\[
\tilde{u}(\in J_h) = P_H\tilde{u} + (I - P_H)\tilde{u} = \tilde{y} + \tilde{z} \in J_H + J^H
\]

satisfying the following linearized system \((t > t_0)\)

\[
\begin{aligned}
(\tilde{y}_t, \phi) + \nu a(\tilde{u}, \phi) + \int_{t_0}^t \beta(t-s)a(\tilde{u}(s), \phi)\, ds = -b(u_h, u_h, \phi) + (f, \phi) \quad \phi \in J_H \\
\nu a(\tilde{u}, \chi) + \int_{t_0}^t \beta(t-s)a(\tilde{u}(s), \chi)\, ds = -b(u_h, u_h, \chi) + (f, \chi) \quad \chi \in J^H,
\end{aligned}
\]

and \( \tilde{y}(t_0) = y(t_0) \). Being linear it is easy to establish the well-posedness of the above system and the following estimates.

**Lemma 5.2.** Under the assumptions of Lemma 3.2, we have

\[
\|\nabla \tilde{u}^h\|^2 + e^{-2\alpha t} \int_{t_0}^t e^{2\alpha t}\|\tilde{\Delta}_h \tilde{u}^h\|^2\, ds \leq K, \quad \|\tilde{\Delta}_h \tilde{u}^h\| \leq K,
\]

where the constant depends on \( u_0 \) and \( f \).

5.1 NLGM I

We define

\[
e := u_h - u^h = (y_H - y^H) + (z_h - z^h) =: e_1 + e_2.
\]

We further split the errors as follows:

\[
\begin{aligned}
e_1 &= y_H - y^H = (y_H - \tilde{y}) - (y^H - \tilde{y}) = \xi_1 - \eta_1 \in J_H \\
e_2 &= z_h - z^h = (z_h - \tilde{z}) - (z^h - \tilde{z}) = \xi_2 - \eta_2 \in J^H.
\end{aligned}
\]

For the sake of simplicity, we write

\[
\xi = \xi_1 + \xi_2, \quad \eta = \eta_1 + \eta_2.
\]

And the equations in \( \xi \) and \( \eta \): subtract (5.2) from (4.5) and subtract (5.2) from (4.5) to obtain

\[
\begin{aligned}
(\xi_{1,t}, \phi) + \nu a(\xi, \phi) &= 0 \\
\nu a(\xi, \chi) &= -(z_{ht}, \chi),
\end{aligned}
\]

\[
\begin{aligned}
(\eta_{1,t}, \phi) + \nu a(\eta, \phi) &= b(u_h, u_h, \phi) - b(u^h, u^h, \phi) \\
\nu a(\eta, \chi) &= b(u_h, u_h, \chi) - b(u^h, u^h, \chi),
\end{aligned}
\]

for \( \phi \in J_H \) and \( \chi \in J^H \).
Lemma 5.3. Under the assumptions of Lemma 3.2, the following holds.

\[(5.6)\]
\[e^{-2\alpha t} \int_{t_0}^{t} e^{2\alpha \tau} \|\xi(\tau)\|^2 d\tau \leq K(t) H^8.\]

Proof. Choose \(\phi = e^{2\alpha t} \xi_1, \ \chi = e^{2\alpha t} \xi_2\) in (5.4), add the two resulting equations and with the notation \(\hat{\xi} = e^{\alpha t} \xi\), we get

\[(5.7)\]
\[\frac{1}{2} \frac{d}{dt} \|\hat{\xi}_1\|^2 - \alpha \|\hat{\xi}_1\|^2 + \nu \|\hat{\xi}\|^2 \leq e^{\alpha t} \|z_{nt}\| \|\hat{\xi}_2\|.\]

Using (4.2) and (5.1), we can bound the right-hand side as:

\[\leq e^{\alpha t} K(t) H^2 cH \|\hat{\xi}_2\| \leq \frac{\nu \rho}{2} \|\hat{\xi}_1\|^2 + K(t) H^6 e^{2\alpha t}.\]

And using (4.4), we have

\[-\alpha \|\hat{\xi}_1\|^2 + \nu \|\hat{\xi}\|^2 \geq (\nu \rho - \frac{\alpha}{\lambda_1}) \|\hat{\xi}_1\|^2 + \nu \rho \|\hat{\xi}_2\|^2.\]

As a result, we obtain from (5.7)

\[(5.8)\]
\[\frac{d}{dt} \|\hat{\xi}_1\|^2 + 2(\nu \rho - \frac{\alpha}{\lambda_1}) \|\hat{\xi}_1\|^2 + \nu \rho \|\hat{\xi}_2\|^2 \leq K(t) H^6 e^{2\alpha t}.\]

Integrate from \(t_0\) to \(t\) and multiply the resulting inequality by \(e^{-2\alpha t}\).

\[(5.9)\]
\[\|\xi_1\|^2 + e^{-2\alpha t} \int_{t_0}^{t} (\|\hat{\xi}_1\|^2 + \|\hat{\xi}_2\|^2) \, ds \leq K(t) H^6.\]

To obtain \(L^2(\mathbb{L}^2)\)-norm estimate, we consider the following discrete backward problem: for fixed \(t_0\), let \(w(\tau) \in J_h, \ w = w_1 + w_2\) such that \(w_1 \in J_H, \ w_2 \in J^H_h\) be the unique solution of \((t_0 \leq \tau < t)\)

\[(5.10)\]
\[\left\{ \begin{array}{l}
(\phi, w_1) - \nu a(\phi, w) = e^{2\alpha \tau} (\phi, \xi_1) \\
(\chi, w) = e^{2\alpha \tau} (\chi, \xi_2) \\
w_1(t_0) = 0.
\end{array} \right.\]

With change of variable, we can make it a forward problem and it turns out to be a linearized version of (4.5) and hence is well-posed. The following regularity result holds.

\[(5.11)\]
\[\int_{t_0}^{t} e^{-2\alpha \tau} \|w\|^2 d\tau \leq C \int_{t_0}^{t} \|\hat{\xi}\|^2 d\tau.\]

Now, choose \(\phi = \xi_1, \ \chi = \xi_2\) in (5.10) and use (5.4) with \(\phi = w_1, \ \chi = w_2\) to find that

\[\|\hat{\xi}(\tau)\|^2 = (\xi_1, w_1) - \nu a(\xi, w) \leq \frac{d}{dt} (\xi_1, w_1) + (z_{ht}, w_2).\]

Integrate from \(t_0\) to \(t\).

\[(5.12)\]
\[\int_{t_0}^{t} \|\hat{\xi}(\tau)\|^2 d\tau = ((\xi_1(t), w_1(t)) - (\xi_1(t_0), w_1(t_0))) + \int_{t_0}^{t} (z_{ht}, w_2) d\tau.\]
But \( w_1(t) = 0 \) and \( \xi_1(t_0) = y_H(t_0) - \hat{y}(t_0) = 0 \). Next, we observe that

\[
\begin{align*}
  w_1 \in J_H & \implies P_H w_1 = w_1, \text{ i.e., } w_1 - P_H w_1 = 0 \\
  w_2 \in J^H_h & \implies P_H w_2 = 0, \text{ i.e., } w_2 - P_H w_2 = w_2 \end{align*}
\]

therefore,

\[
(\cdot_{ht}, w_2) = (\cdot_{ht}, w - P_H w)
\]

(5.13)

From (5.12), we get

\[
\int_{t_0}^t \|\mathbf{\hat{v}}(\tau)\|^2 d\tau \leq K(t) e^{2\gamma t} H^4 \left( \int_{t_0}^t e^{-\alpha \tau} \|w\|^2 d\tau \right)^{1/2}.
\]

Use (5.11) to conclude. \( \Box \)

In order to obtain optimal \( L^\infty(L^2) \) estimate, we would like to introduce Stokes-type projections \((S_H, S^H_h)\) for \( t > t_0 \) defined as below:

\[
S_H : J_h \to J_H, \quad S^H_h : J_h \to J^H_h,
\]

and with the notations

\[
\zeta_1 := y_H - S_H u_h \in J_H, \quad \zeta_2 := z_{ht} - S^H_h u_h \in J^H_h
\]

the following system is satisfied.

(5.14)

\[
\begin{cases}
  \nu a(\zeta, \phi) = 0, \quad \phi \in J_H, \\
  \nu a(\zeta, \chi) = - (\cdot_{ht}, \chi), \quad \chi \in J^H_h.
\end{cases}
\]

For the sake of convenience, we have written \( \zeta = \zeta_1 + \zeta_2 \). Note that, given a semi-discrete Galerkin approximation \( u_h \) of NSE with \textit{a priori} estimates, the system (5.14) is a Stokes system, with Stokes problem in \( J_h \) projected to subspaces \( J_H \) and \( J^H_h \), and hence is well-posed.

**Lemma 5.4.** Under the assumptions of Lemma 3.2, we have

(5.15)

\[
||\zeta|| + ||\zeta_t|| \leq K(t) H^4.
\]

**Proof.** Choose \( \phi = e^{2\gamma t} \zeta_1, \chi = e^{2\gamma t} \zeta_2 \) in (5.14) to obtain

(5.16)

\[
\nu \|\mathbf{\hat{v}}\|_{L^2}^2 \leq e^{2\gamma t} ||z_{ht}|| \|\mathbf{\hat{v}}\|_{L^2}.
\]

As in (5.7), we establish

(5.17)

\[
\|\zeta\|_{L^6}^2 \leq \|\zeta_1\|_{L^6}^2 + \|\zeta_2\|_{L^6}^2 \leq K(t) H^6.
\]

In order to obtain optimal \( L^\infty(L^2) \)-norm estimate, we would use Aubin-Nitsche duality argument. For that purpose, we consider the following Galerkin approximation of steady Stoke problem: let \( w_h \in J_h \) be the solution of

\[
\nu a(\mathbf{v}, w_h) = (\mathbf{v}, \hat{\zeta}_1 + \hat{\zeta}_2), \quad \mathbf{v} \in J_h.
\]

Writing \( w_{h1} = P_H w_h, \quad w_{h2} = (I - P_H) w_h \), we split the above equation as

(5.18)

\[
\begin{cases}
  \nu a(\phi, w_{h1}) = (\phi, \hat{\zeta}_1), \quad \phi \in J_H, \\
  \nu a(\chi, w_{h2}) = (\chi, \hat{\zeta}_2), \quad \chi \in J^H_h.
\end{cases}
\]
Standard elliptic regularity leads us to the following result.

\[(5.19) \quad \|w_h = w_{h1} + w_{h2}\|_2 \leq c\|\dot{\zeta}_1 + \dot{\zeta}_2\|.
\]

Now, put \(\phi = \dot{\zeta}_1, \chi = \dot{\zeta}_2\) in (5.18) and use (5.14) with \(\phi = w_{h1}, \chi = w_{h2}\) to find that

\[\|\dot{\zeta}\|^2 = \nu a(\dot{\zeta}, w_h) = -e^{2\alpha t}(z_{ht}, w_{h2}).\]

As in (5.13) along with (5.19), we find

\[\|\zeta\| \leq K(t)H^4.\]

For the remaining part, we differentiate (5.14) and proceed as above to complete the rest of the proof.

Now we are in a position to estimate \(L^\infty(L^2)\)-norm of \(\xi\), that is, of \(\xi_1\) and \(\xi_2\). Using the definitions of \(\xi_i, \zeta_i, i = 1, 2\), we write

\[
\begin{align*}
\xi_1 &= y_H - \dot{y} = (y_H - S_Hu_h) - (\dot{y} - S_Hu_h) =: \zeta_1 - \theta_1, \\
\xi_2 &= z_h - \dot{z} = (z_h - S_hu_h) - (\dot{z} - S_hu_h) =: \zeta_2 - \theta_2.
\end{align*}
\]

From (5.14) and (5.14), we have

\[(5.20) \quad \begin{cases} 
(\theta_{1,t}, \phi) + \nu a(\theta, \phi) = (\zeta_{1,t}, \phi), & \phi \in J_H, \\
\nu a(\theta, \chi) = 0, & \chi \in J^H.
\end{cases}
\]

**Lemma 5.5.** Under the assumptions Lemma 3.2, we have

\[\|\xi\| \leq K(t)H^4.\]

**Proof.** Put \(\phi = e^{2\alpha t}\theta_1, \chi = e^{2\alpha t}\theta_2\) in (5.20) to find

\[(5.21) \quad \frac{1}{2} \frac{d}{dt}\|\dot{\theta}_1\|^2 - \alpha\|\dot{\theta}_1\|^2 + \nu\|\dot{\theta}\|^2 \leq e^{\alpha t}\|\zeta_{1,t}\|\|\dot{\theta}_1\|.
\]

We recall that the spaces \(J_H\) and \(J^H\) are orthogonal in \(L^2\)-inner product. That is,

\[\text{for } \phi \in J_H, \chi \in J^H, (\phi, \chi) = 0.
\]

Hence

\[\|\theta_1\|^2 \leq \|\dot{\theta}_1\|^2 + \|\dot{\theta}_2\|^2 = \|\dot{\theta}\|^2 \leq \|\dot{\xi}\|^2 + \|\dot{\zeta}\|^2, \quad \|\zeta_{1,t}\|^2 \leq \|\zeta_t\|^2.
\]

And

\[-\alpha\|\dot{\theta}_1\|^2 + \nu\|\dot{\theta}\|^2 = (\nu - \alpha\lambda)\|\dot{\theta}_1\|^2 + \nu\|\dot{\theta}_2\|^2.
\]

As a result, after integrating (5.21) with respect to time from \(t_0\) to \(t\), we obtain

\[(5.22) \quad \|\theta_1\|^2 + \int_{t_0}^t (\|\dot{\theta}_1\|^2 + \|\dot{\theta}_2\|^2) \, ds \leq \left( \int_{t_0}^t e^{2\alpha t}\|\zeta_t\|^2 \, ds \right)^{1/2} \left( \int_{t_0}^t (\|\dot{\xi}\|^2 + \|\dot{\zeta}\|^2) \, ds \right)^{1/2}.
\]

We now use Lemmas 5.3 and 5.4 to conclude from (5.22) that

\[(5.23) \quad \|\theta_1\|^2 + e^{-2\alpha t} \int_{t_0}^t e^{2\alpha s} (\|\theta_1\|^2 + \|\theta_2\|^2) \, ds \leq K(t)H^8.
\]
We again choose \( \chi = e^{2\alpha t} \theta_2 \) in (5.20) to find

\[
\nu ||\hat{\theta}_2||^2 = -\nu a(\hat{\theta}_1, \hat{\theta}_2) \implies ||\theta_2||_1 \leq ||\theta_1||_1.
\]

Since \( \theta_1 \in J_H \), we use inverse inequality (3.1) and (5.23) to note that

\[
||\theta_1||_1 \leq cH^{-1}||\theta|| \leq K(t)H^3.
\]

Hence, we conclude that

\[
||\theta_2||_1 \leq K(t)H^3.
\]

Now use (4.2) to see that

\[
(5.24) \quad ||\theta_2|| \leq K(t)H^4.
\]

Combining (5.23) with (5.24), we establish

\[
||\theta|| \leq K(t)H^4.
\]

Use triangle inequality and the estimates of \( \zeta \) and \( \theta \) to complete the proof.

We are now left with the estimate of \( \eta \), the error due to the nonlinearity.

**Lemma 5.6.** Under the assumptions of Lemma 3.2 and that \( H \) is small enough to satisfy (4.10) and

\[
\mu \rho - 2H||\hat{u}||_2 \geq 0, \quad \mu - H(||\hat{u}||_2 + ||y_H||_2) \geq 0,
\]

we have

\[
||\mathbf{u}_h - \mathbf{u}_h^0(t)|| \leq K(t)H^4.
\]

**Proof.** We choose \( \phi = e^{2\alpha t} \eta_1 \), \( \chi = e^{2\alpha t} \eta_2 \) in (5.5).

\[
(5.25) \quad \frac{1}{2} \frac{d}{dt} ||\hat{\eta}_1||^2 + \nu ||\hat{\eta}||_1^2 = e^{2\alpha t} \Lambda_h(\eta_1, \eta_2),
\]

where

\[
\Lambda_h(\eta_1, \eta_2) = \Lambda_{h,1}(\eta_1) + \Lambda_{h,2}(\eta_2),
\]

and

\[
\Lambda_{h,1}(\eta_1) = b(\mathbf{u}_h, \mathbf{u}_h, \eta_1) - b(\mathbf{u}_h^0, \mathbf{u}_h, \eta_1)
\]

\[
= b(\xi - \eta, \mathbf{u}_h, \eta_1) + b(\mathbf{u}_h, \xi - \eta, \eta_1) - b(\xi - \eta, \xi - \eta, \eta_1)
\]

\[
\Lambda_{h,2}(\eta_2) = b(\mathbf{u}_h, \mathbf{u}_h, \eta_2) - b(\mathbf{u}_h^0, \mathbf{u}_h, \eta_2)
\]

\[
= b(\xi - \eta, \mathbf{u}_h, \eta_2) + b(\mathbf{u}_h, \xi - \eta, \eta_2) - b(\xi - \eta, \xi - \eta, \eta_2).
\]

Therefore

\[
(5.26) \quad \Lambda_h(\eta_1, \eta_2) = b(\xi - \eta, \hat{\mathbf{u}}, \eta) + b(\mathbf{u}_h, \xi, \eta).
\]

We estimate the nonlinear terms as follows:

\[
b(\mathbf{u}_h, \xi, \eta) + b(\xi - \eta_1, \hat{\mathbf{u}}, \eta) \leq \{ ||\xi||_1 ||\mathbf{u}_h||_2 + (||\xi||_1 + ||\eta_1||_1) ||\hat{\mathbf{u}}||_2 \} ||\eta||_1,
\]

\[
b(\eta_2, \hat{\mathbf{u}}, \eta) \leq ||\eta_2||_1 ||\hat{\mathbf{u}}||_2 (||\eta_1||_1 + ||\eta_2||_1) \leq ||\eta_2||_1 ||\mathbf{u}_h||_2 ||\eta_1||_1 + H ||\hat{\mathbf{u}}||_2 ||\eta_2||^2_1.
\]
Therefore, for \( \epsilon, \epsilon_1 > 0 \),
\[
\Lambda_h(\eta_1, \eta_2) \leq \epsilon \| \eta \|_1^2 + \epsilon_1 \| \eta_2 \|_1^2 + c(\epsilon)(\| u_h \|_2^2 + \| \bar{u} \|_2^2)\| \xi \|_1^2 + c(\epsilon, \epsilon_1)\| \bar{u} \|_2 \| \eta_1 \|_1^2 + H \| \bar{u} \|_2 \| \eta_2 \|_2^2.
\]

Now, from [5.23], we find that
\[
\frac{d}{dt}\| \dot{\eta}_1 \|_2^2 + 2\nu \rho (\| \dot{\eta}_1 \|_1^2 + \| \dot{\eta}_2 \|_1^2) \leq 2\epsilon \| \dot{\eta} \|_1^2 + 2\epsilon_1 \| \dot{\eta}_2 \|_1^2 + c(\epsilon)(\| u_h \|_2^2 + \| \bar{u} \|_2^2)\| \xi \|_1^2 + c(\epsilon, \epsilon_1)\| \bar{u} \|_2 \| \dot{\eta}_1 \|_1^2 + 2H \| \bar{u} \|_2 \| \dot{\eta}_2 \|_2^2.
\]
(5.27)

We choose \( \epsilon = \epsilon_1 = \nu \rho \) and assume that \( H \) small enough such that
\[
\nu \rho - 2H \| \bar{u} \|_2 \geq 0
\]
to obtain after integration
\[
\| \eta_1 \|_2^2 + cH^{-1} \| \eta_1 \| \leq K(t)H^3.
\]

For \( \eta_2 \), we again put \( \chi = e^{2\alpha t} \eta_2 \) in \([5.25]\).

Recall that
\[
\Lambda_h,2(\eta_2) = b(\xi - \eta_1, \bar{u}, \eta_2) + b(u_h, \xi - \eta_1, \eta_2) + b(\xi - \eta_1 - \eta_2, \eta_1).
\]

And
\[
b(\xi - \eta_1, \bar{u}, \eta_2) + b(u_h, \xi - \eta_1, \eta_2) \leq (\| \xi \|_1 + \| \eta_1 \|_1)(\| \bar{u} \|_2 + \| u_h \|_2)\| \eta_2 \|_1
\]
\[
b(\xi - \eta_1, \xi - \eta_1, \eta_2) \leq (\| \xi \|_1 + \| \eta_1 \|_1)\| \eta_1 \|_1 \| \eta_2 \|_1
\]
\[
b(-\eta_2, \bar{u}, \eta_2) + b(-\eta_2, \eta_1, \eta_2) \leq H(\| \bar{u} \|_2 + \| \eta_1 \|_2)\| \eta_2 \|_1.
\]

Note that \( \| \eta_1 \| \leq \| y^H \| + \| \bar{y} \| \). And under the assumption
\[
\nu - H(\| \bar{u} \|_2 + \| y^H \|_2) \geq 0
\]
we easily obtain that
\[
\| \eta_2 \|_1 \leq K(t)H^3
\]
and hence
\[
\| \eta_2 \| \leq cH\| \eta_2 \|_1 \leq K(t)H^4.
\]

Now, triangle inequality completes the proof.
5.2 NLGM II

In this subsection, we deal with the error estimate for NLGM II. As earlier, we split the error in two, that is, \( e = u_h - u^h = \xi - \eta \). The equations and hence the estimates regarding \( \xi \) remain same and are optimal in nature. The equation in \( \eta \) reads as follows:

\[
(\ref{5.29}) \quad \left\{ \begin{array}{l}
(\eta_{1,t}, \phi) + \nu a(\eta, \phi) = b(u_h, u_h, \phi) - b(u^h, u^h, \phi), \quad \phi \in J^H_H \\
(\eta, \chi) = b(u_h, u_h, \chi) - b(u^h, u^h, \chi) + b(z^h, z^h, \chi), \quad \chi \in J^H_H
\end{array} \right.
\]

**Lemma 5.7.** Under the assumptions of Lemma \ref{5.6}, we have

\[ \| (u_h - u^h)(t) \| \leq K(t)H^3. \]

**Proof.** We choose \( \phi = e^{2\alpha t}\eta_1, \quad \chi = e^{2\alpha t}\eta_2 \) in \((\ref{5.29})\).

\[
(\ref{5.30}) \quad \frac{1}{2} \frac{d}{dt} \| \eta_1 \|^2 + \nu \| \eta_1 \|^2 = e^{2\alpha t} \{ A_h(\eta_1, \eta_2) + b(z^h, z^h, \eta_2) \},
\]

Since \( z^h = z_h + \eta_2 - \xi_2 \), we have

\[ b(z^h, z^h, \eta_2) = b(z_h + \eta_2 - \xi_2, z_h - \xi_2, \eta_2). \]

Now

\[
b(\xi_2, z_h - \xi_2, \eta_2) \leq \| \xi_2 \|_1 (\| z_h \|_1 + \| \xi_2 \|_1) \| \eta_2 \|_1 \\
= b(\xi_2, z_h - \xi_2, \eta_2) \leq cH \| z_h \|_1 \| \eta_2 \|_1^2 \\
b(z_h, z_h - \xi_2, \eta_2) \leq \| z_h \|_1^{1/2} \| z_h \|_1^{1/2} (\| z_h \|_1 + \| \xi_2 \|_1) \| \eta_2 \|_1^{1/2} \| \eta_2 \|_1^{1/2} \\
+ \| z_h \|_1^{1/2} \| \xi_2 \|_1 \| \eta_2 \|_1^{1/2} \| \eta_2 \|_1 + \| \xi_2 \|_1 \\
\leq cH \| z_h \|_1 (\| z_h \|_1 + \| \xi_2 \|_1) \| \eta_2 \|_1 + cH^{1/2} \| z_h \|_1 \| \eta_2 \|_1 \| \eta_2 \|_1 \| \xi_2 \|_1 \\
\leq cH \| z_h \|_1 \| \eta_2 \|_1 + cH \| z_h \|_1 \| \xi_2 \|_1 \| \eta_2 \|_1.
\]

Incorporate these in \((\ref{5.30})\). Integrate and as earlier, for small \( H \), we obtain

\[
\| \eta_1 \|^2 + e^{-2\alpha t} \int_{t_0}^t (\| \eta_1 \|^2 + \| \eta_2 \|^2) \, ds \leq K(t)H^8 + K(t)H^2 \| z_h \|_1^2 + K \int_{t_0}^t \| \eta_1(s) \|^2 \, ds,
\]

which results in

\[
\| \eta_1 \|^2 + e^{-2\alpha t} \int_{t_0}^t (\| \eta_1 \|^2 + \| \eta_2 \|^2) \, ds \leq K(t)H^6.
\]

That is

\[
\| \eta_1 \| \leq K(t)H^3, \quad \| \eta_1 \|_1 \leq K(t)H^2.
\]

As in the previous section, using only the second equation of \((\ref{5.29})\), we can easily conclude that

\[
\| \eta_2 \| \leq K(t)H^3, \quad \| \eta_2 \|_1 \leq K(t)H^2.
\]

\[ \square \]

**Remark 5.1.** The analysis reveals that the decrease in the order of convergence is due to the presence of \( b(z_h, z_h, \chi) \) in the error equation. So, whereas in NLGI, we keep the nonlinearity in both the equations, in NLGM II, the second equation is made linear in \( z^h \) by dropping the term \( b(z_h, z_h, \chi) \) and which in turn appears in the error equation and is responsible for bringing down the rate of convergence in the above analysis.
5.3 Improved Error Estimate

In this section, we try to improve the rate of convergence, using the technique of Marion & Xu [14]. But this is not straightforward, as the estimate of the function \( f(u) \) in their semi-linear problem does not hold for our \( f(u) \) and we have to be careful in order to obtain similar results.

First, we note that the second equation of (4.8) can be written as

\[
(5.31) \quad \nu \Phi(h) = \Phi(y),
\]

where \( \Phi : J \to J^H \). Using this, we can write the equation in \( \Phi(y) \), for \( \chi \in J^H \)

\[
(5.32) \quad \nu \Phi(y) + \Phi(y, \chi) + b(y, \Phi(y), \chi) + b(y, \Phi(y), \chi) = (f, \chi)
\]

Lemma 5.8. Under the assumptions of Lemma 3.2 and that \( H \) is small enough to satisfy

\[
\frac{\nu}{2} - cH\|u_h\|_1 \geq 0,
\]

we have

\[
(5.33) \quad \|z_h - \Phi(y)\| + H\|z_h - \Phi(y)\|_1 \leq K(\nu)H^4.
\]

Proof. With the notation \( \Phi_e := z_h - \Phi(y) \in J^H \), we have, by deducting (5.32) from the second equation of (4.8)

\[
(5.34) \quad \nu \Phi_e(\chi) = -(z_{h,t}, \chi) - b(u_h, u_h, \chi) + b(y, \Phi(y), \chi) + b(y, \Phi(y), y, \chi).
\]

Put \( \chi = \Phi_e \) in (5.34) to obtain

\[
(5.35) \quad \nu \|\Phi_e\|_1^2 = -(z_{h,t}, \Phi_e) - b(\Phi_e, u_h, \Phi_e) - b(\Phi(y), \Phi(y), \Phi_e).
\]

Note that

\[
-(z_{h,t}, \Phi_e) \leq \|z_{h,t}\|\|\Phi_e\| \leq K(\nu)H^3\|\Phi_e\|_1
\]

\[
-b(\Phi_e, u_h, \Phi_e) \leq c\|u_h\|_1\|\Phi_e\|\|\Phi_e\|_1 \leq cH\|u_h\|_1\|\Phi_e\|_1^2
\]

\[
-b(\Phi(y), \Phi(y), \Phi_e) = -(z_h - \Phi_e, z_h, \Phi_e)
\]

\[
\leq (\|z_h\|_1/2\|z_h\|_1/2 + \|\Phi_e\|_1/2\|\Phi_e\|_1/2)\|z_h\|_1\|\Phi_e\|_1/2\|\Phi_e\|_1/2
\]

\[
\leq K(\nu)H^3\|\Phi_e\|_1 + cH\|z_h\|_1\|\Phi_e\|_1^2.
\]

Put these estimates in (5.35) to find

\[
(5.36) \quad \frac{\nu}{2}\|\Phi_e\|_1^2 \leq K(\nu)H^6 + cH\|u_h\|_1\|\Phi_e\|_1^2.
\]

We have used the fact that \( \|z_h\| \leq \|u_h\| + \|y\| \leq c\|u_h\| \). And assuming \( H \) to be small enough to satisfy

\[
\frac{\nu}{2} - cH\|u_h\|_1 \geq 0
\]

we establish from (5.36)

\[
\|\Phi_e\|_1 \leq K(\nu)H^3.
\]

And hence

\[
\|\Phi_e\| \leq cH\|\Phi_e\|_1 \leq K(\nu)H^4.
\]

This completes the proof.
\textbf{Lemma 5.9.} Under the assumptions of Lemma 5.8, we have

\begin{align}
|e_2|^2 & \leq K(t)H^6 + K|e_1|^2, \\
|e_2|^2 & \leq K(t)H^8 + KH^2|e_1|^2.
\end{align}

\textbf{Proof.} Recall that \(e_2 = z_h - z^h = (z_h - \Phi(y_H)) - (z^h - \Phi(y_H)).\) With the notation \(\Phi = z^h - \Phi(y_H),\) we have \(e_2 = \Phi - \Phi.\) The equation in \(\Phi\) can be obtained by deducting (5.32) from the second equation of (5.6).

\begin{align}
\nu a(\Phi - e_1, \chi) = -b(u^h, y, \chi) - b(y^H, z, \chi) + b(\Phi(y_H), \chi) + b(y_H + \Phi(y_H), y_H, \chi).
\end{align}

Put \(\chi = \Phi\) to obtain

\begin{align}
\nu \|\Phi\|^2_1 = \nu a(e_1, \Phi) + b(u^h, e_1, \Phi) + b(e_1 - \Phi, y_H, \Phi) + b(e_1, \Phi(y_H), \Phi)
\end{align}

Note that

\begin{align}
b(u^h, e_1, \Phi) & = b(u_h - e_1 - \Phi, e_1, \Phi) + b(\Phi, e_1, \Phi) \\
nb & \leq |u_h||e_1||\Phi|| + |e_1||\Phi|| + |\Phi||e_1|| \Phi|| \\
nb & \leq cH(1 + L_H)|e_1||\Phi||.
\end{align}

Incorporate these estimates in (5.40). Whenever it suits us, we bound \(|e_1||\Phi(y_H)||y^H||_1 \leq K.\) And therefore, we have, after kickback argument

\begin{align}
\frac{\nu}{2} \|\Phi\|^2_1 \leq K|e_1|^2 + K(t)H^6 + cH(1 + L_H)(|y_H||1 + |y^H||_1)||\Phi||^2.
\end{align}

Assuming \(H\) small enough to satisfy

\begin{align}
\frac{\nu}{2} - cH(1 + L_H)(|y_H||1 + |y^H||_1) > 0,
\end{align}

we obtain

\begin{align}
|\Phi|^2 \leq K(t)H^6 + K|e_1|^2.
\end{align}

And so

\begin{align}
|\Phi|^2 \leq K(t)H^8 + KH^2|e_1|^2.
\end{align}

Using triangle inequality, we complete the proof.

\textbf{Remark 5.2.} Recall that \(e_i = \xi_i - \eta_i, i = 1, 2\) and since the linearized error \(\xi\) (that is, \(\xi_1, \xi_2\)) is optimal in nature, we have from (5.37)-(5.38)

\begin{align}
|\eta_2|^2 & \leq K(t)H^6 + c|\eta_1|^2 \\
|\eta_2|^2 & \leq K(t)H^8 + cH^2|\eta_1|^2.
\end{align}
Following Marion & Xu [14], we introduce the operator $R_h^H : J_h \rightarrow J_h^H$ satisfying
\begin{equation}
(5.44) \quad a(v - R_h^H v, \chi) = 0, \quad \forall \chi \in J_h^H.
\end{equation}

With the notations
\begin{equation}
\|v\|_R = \|(I - R_h^H)v\|_1, \quad (v, w)_R = a((I - R_h^H)v, (I - R_h^H)w),
\end{equation}
we have, from Lemma 4.1 of [13],
\begin{equation}
(5.45) \quad c_1\|v\|_1 \leq \|v\|_R \leq c_2\|v\|_1,
\end{equation}
where $c_1, c_2$ are positive constants independent of $h, H$. And similar to Lemmas 4.5 and 4.6 of [14], we find for $\phi \in J_H$
\begin{align}
(5.46) \quad (y_h^H, \phi) + \nu(y_h^H, \phi)_R &= (f, (I - R_h^H)\phi) - b(u^h, u^h, (I - R_h^H)\phi) - b(z^h, z^h, R_h^H \phi) \\
(5.47) \quad (y_{h,t}, \phi) + \nu(y_h^H, \phi)_R &= (\Gamma, (I - R_h^H)\phi) - b(u_h, u_h, (I - R_h^H)\phi) + (u_{h,t}, R_h^H \phi).
\end{align}

Now, for $\phi \in J_H$, we write the equation in $e_1 = y_H - y^H$ as
\begin{align}
(5.48) \quad (e_{1,t}, \phi) + \nu(e_1, \phi)_R &= + (u_{h,t}, R_h^H \phi) + b(z^h, z^h, R_h^H \phi) - b(e_1 + e_2, u_h, (I - R_h^H)\phi) \\
&\quad - b(u^h, e_1 + e_2, (I - R_h^H)\phi).
\end{align}

**Lemma 5.10.** Under the assumptions Lemma 5.8 we have
\begin{equation}
\|e_1\|^2 + \int_{t_0}^t \|e_{1,t}\|^2 ds \leq K(t)H^6.
\end{equation}

**Proof.** Put $\phi = e_{1,t}$ in (5.48) and observe that
\begin{align*}
(u_{h,t}, R_h^H e_{1,t}) &= \frac{d}{dt}(u_{h,t}, R_h^H e_1) - (u_{h,t}, R_h^H e_1) \\
&= \frac{d}{dt}(u_{h,t}, R_h^H e_1) - ((I - P_H)u_{tt}, R_h^H e_1) - ((u_h - u)_{tt}, R_h^H e_1) \\
&\leq \frac{d}{dt}(u_{h,t}, R_h^H e_1) + K(t)H^3\|e_1\|_1,
\end{align*}
\begin{align*}
b(e_1 + e_2, u_h, (I - R_h^H)\phi_{1,t}) - b(u^h, e_1 + e_2, (I - R_h^H)\phi_{1,t}) \\
&\leq c(\|\phi\|_1 + \|e_2\|_1)(\|u_h\|_2 + \|u^h\|_2)\|e_{1,t}\|.
\end{align*}
\begin{align*}
b(z^h, z^h, R_h^H e_{1,t}) &= b(z_h - e_2, z_h - e_2, R_h^H e_{1,t}) = \frac{d}{dt}b(z_h, z_h, R_h^H e_1) - b(z_{h,t}, z_h, R_h^H e_1) \\
&\quad - b(z_h, z_{h,t}, R_h^H e_1) + b(z^h, -e_2, R_h^H e_{1,t}) + b(e_2, z_h, R_h^H e_{1,t}) \\
&\quad - b(z_{h,t}, z_h, R_h^H e_1) - b(z_h, z_{h,t}, R_h^H e_1) \\
&\leq cH\|z_h\|_1\|z_{h,t}\|_1\|e_1\|_1
\end{align*}
\begin{align*}
b(e_2, z_h, R_h^H e_{1,t}) &\leq c\|e_2\|^2\|e_1\|_1\|z_{h,t}\|_1\|e_{1,t}\|_1 \\
&\leq cH\|e_2\|_1\|z_h\|_1\|e_{1,t}\|_1 + c\|e_2\|_1\|z_{h,t}\|_1\|e_{1,t}\|_1.
\end{align*}

Here, we have used that $\|(I - R_h^H)\phi_{1,t}\| \leq \|\phi_{1,t}\| + cH|\phi_{1,t}| \leq c|\phi_{1,t}|$. And now we find
\begin{equation}
(5.49) \quad \|e_{1,t}\|^2 + \nu \frac{d}{dt}\|e_1\|_R^2 \leq K(t)H^3\|e_1\|_1 + \frac{d}{dt}\left\{ (u_{h,t}, R_h^H e_1) + b(z_h, z_h, R_h^H e_1) \right\} \\
+ K(\|\phi\|_1 + \|e_2\|_1)\|e_{1,t}\| + cH\|z_{h,t}\|_1\|e_{1,t}\|_1 + c\|e_2\|_1(\|z_h\|_1 + \|z^h\|_1)\|e_{1,t}\|_1.
\end{equation}
Integrate (5.49), use (5.45) and the fact that \( e_1(t_0) = 0 \) to find
\[
\|e_1\|^2 + \int_{t_0}^{t} \|e_{1,t}\|^2 ds \leq K(t)H^6 + c \int_{t_0}^{t} (\|e_1\|^2 + \|e_2\|^2) ds + (u_{h,t}, R^{H}_h e_1) + b(z_h, z_h, R^{H}_h e_1).
\]

As earlier, we estimate the last three terms to obtain
\[
\|e_1\|^2 + \int_{t_0}^{t} \|e_{1,t}\|^2 ds \leq K(t)H^6 + c \int_{t_0}^{t} (\|e_1\|^2 + \|e_2\|^2) ds.
\]

We note from (5.42) and triangle inequality that
\[
\|e_2\|^2 \leq K(t)H^6 + \|\eta_1\|^2 \leq K(t)H^6 + \|e_1\|^2.
\]

Therefore
\[
\|e_1\|^2 + \int_{t_0}^{t} \|e_{1,t}\|^2 ds \leq K(t)H^6 + c \int_{t_0}^{t} \|e_1\|^2 ds.
\]

Use Gronwall’s lemma to establish
\[
\|e_1\|^2 + \int_{t_0}^{t} \|e_{1,t}\|^2 ds \leq K(t)H^6.
\]

\[\square\]

**Remark 5.3.** This tells us that
\[
\|\eta_1\|_1 \leq K(t)H^3,
\]
and as a result, from Remark 5.2, we have
\[
\|\eta_2\| + \|\eta_1\|_1 \leq K(t)H^4.
\]

Another application of triangle inequality results in
\[
\|e_2\| + \|e_1\|_1 \leq K(t)H^4.
\]

For the final estimate, we note down the equations in terms of \( e_i, i = 1, 2 \).

(5.51) \[
\begin{align*}
(e_{1,t}, \phi) + \nu a(e, \phi) &= -b(u_h, u_h, \phi) + b(u_h^h, u_h^h, \phi),
\nu a(e, \chi) &= -(z_{ht}, \chi) - b(u_h, u_h, \chi) + b(u_h^h, u_h^h, \chi) - b(z_h^h, z_h^h, \chi),
\end{align*}
\]

**Lemma 5.11.** Under the assumptions of Lemma 5.8, we have
\[
\|(u_h - u^h)(t)\| \leq K(t)H^4, \quad t > t_0.
\]

**Proof.** With the notation \( \tilde{\Delta}_H = P_H(-\Delta_h) \), we choose \( \phi = \tilde{\Delta}_H^{-1}e_{1,t} \) in the first equation of (5.51) to find
\[
\|e_{1,t}\|^2_{-1} + \frac{\nu}{2} \frac{d}{dt} \|e_1\|^2 = b(e_1 + e_2, u_h, \tilde{\Delta}_H^{-1}e_{1,t}) + b(u_h^h, e_1 + e_2, \tilde{\Delta}_H^{-1}e_{1,t}).
\]

As earlier, we have
\[
b(e_1 + e_2, u_h, \tilde{\Delta}_H^{-1}e_{1,t}) + b(u_h^h, e_1 + e_2, \tilde{\Delta}_H^{-1}e_{1,t}) \leq c(\|e_1\| + \|e_2\|)(\|u_h\|_2 + \|u_h^h\|_2)\|e_{1,t}\|_{-1}.
\]

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Integrate (5.52) and use the above estimate to find

\[ \|e_1\|^2 + \int_{t_0}^{t} \|e_{1,t}\|_{L^2}^2 \leq c \int_{t_0}^{t} (\|e_1\|^2 + \|e_2\|^2) \, ds \leq K(t)H^8 + c \int_{t_0}^{t} \|e_1\|^2 \, ds. \]

Apply Gronwall’s lemma to conclude

\[ \|e_1\|^2 + \int_{t_0}^{t} \|e_{1,t}\|_{L^2}^2 \leq K(t)H^8. \]

**Remark 5.4.** It is clear from our above analysis is that the linearized error between NLG approximation and Galerkin approximation is of order $H^4$ in $L^2$-norm. However, non-linearized part of the error may not always be of same order. For example, if the equation in $z^h$ contains only $b(y^H, y^H, \chi)$, then the non-linearized part of the error (i.e. the equation in $\eta$) will contain additional terms like $b(y^H, z^h, \chi)$ and $b(z^h, y^H, \chi)$ apart from the non-linear terms of the second equation of (5.29). And with one of these terms, we believe, we can only manage $H^3$ order of convergence in $L^2$-norm.

6 Summary

In this work, our main focus is in obtaining optimal $L^2$-error estimate, that is, $O(H^4)$ for nonlinear Galerkin finite element approximations. For that purpose, we have discussed two NLGMs. In the first one, small scales are assumed stationary. In the second one, we have an additional assumption that interactions between small scales are negligible. And in both these cases, we have managed to show optimal $L^2$-error estimate. But any further simplification of the small scales equations will lead to sub-optimal error estimate, say $O(H^3)$, as has been observed in the remark 5.4.

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