A DYNAMICAL SYSTEM APPROACH TO THE INVERSE SPECTRAL PROBLEM FOR HANKEL OPERATORS: A MODEL CASE

ZHEHUI LIANG AND SERGEI TREIL

Abstract. We present an alternative proof of the result by P. Gérard and S. Grellier [5], stating that given two real sequences $(\lambda_n)_{n=1}^{\infty}$, $(\mu_n)_{n=1}^{\infty}$ satisfying the intertwining relations

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > ... > |\lambda_n| > |\mu_n| > ... > 0, \quad \lambda_n \to 0,$$

there exists a unique compact Hankel operator $\Gamma$ such that $\lambda_n$ are the (simple) eigenvalues of $\Gamma$ and $\mu_n$ are the simple eigenvalues of its truncation $\Gamma_1$ obtained from $\Gamma$ by removing the first column.

We use the dynamical systems approach originated in [8], and the proof is split into three independent parts. The first one, which is a slight modification of a result in [8] is an abstract operator-theoretic statement reducing the problem to the asymptotic stability of some operator. The second one is the proof of the asymptotic stability, which is usually the hardest part, but in our case of compact operators it is almost trivial. And the third part is an abstract version of the Borg’s two spectra theorem, which is essentially a simple exercise in graduate complex analysis.

0. Notation

All operators act on or between Hilbert spaces, and we consider only separable Hilbert spaces.

$S$ the forward shift in $\ell^2$, $S(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots)$;

$S^*$ adjoint of $S$, $S^*(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$;

$|A|$ modulus of the operator $A$, $|A| := (A^*A)^{1/2}$;

$P_E$ the orthogonal projection onto a subspace $E$.

$a^*$ for $a \in \mathcal{H}$, the symbol $a^*$ denotes the linear functional on $\mathcal{H}$, given by $a^* x = (x, a)_{\mathcal{H}}$

1. Introduction and main results

A Hankel operator is a bounded linear operator in $\ell^2 = \ell^2(\mathbb{Z}_+)$ with matrix whose entries depend on the sum of indices,

$$\Gamma = (\gamma_{j+k})_{j,k \geq 0}.$$

Denoting by $S$ the shift operator in $\ell^2$,

$$S(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots),$$

and by $S^*$ its adjoint (the backward shift),

$$S^*(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$$

Work of S. Treil is Supported in part by the National Science Foundation under the grants DMS-1856719, DMS-2154321.
we can see that an operator \( \Gamma \) on \( \ell^2 \) is a Hankel operator if and only if
\[
\Gamma S = S^* \Gamma;
\]
sometimes this formula is used for the definition of a Hankel operator.

In [8] a complete description of self-adjoint operators unitarily equivalent to a Hankel operator was obtained. It was clear from the construction that such Hankel operator is not unique, and the question of finding additional spectral conditions providing uniqueness looks like a natural question.

In a breakthrough series of papers [3, 4, 5, 6] P. Gérard and S. Grellier investigated the inverse problem for Hankel operators; their motivation come the study of the so-called cubic Szegö equation, which is a completely integrable Hamiltonian system.

One of their discoveries was that the spectral invariants of the pair of Hankel operator \( \Gamma \) and \( \Gamma_1 = \Gamma S = S^* \Gamma \), completely determine the operator \( \Gamma \).

A simple illustration of that principle, is Theorem 1.1 below, proved in [5].

In this paper we present a simple proof of this result.

**Theorem 1.1.** Given sequences \( (\lambda_n)_{n=1}^\infty \) and \( (\mu_n)_{n=1}^\infty \) of non-zero real numbers, satisfying intertwining relations
\[
|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > \ldots > |\lambda_n| > |\mu_n| > \ldots > 0,
\]
and such that \( \lim_{n \to \infty} \lambda_n = 0, \lim_{n \to \infty} \mu_n = 0 \), there exists a unique self-adjoint compact Hankel operator \( \Gamma \) such that non-zero eigenvalues of \( \Gamma \) and \( \Gamma S = S^* \Gamma \), completely determine the operator \( \Gamma \).

Moreover, \( \ker \Gamma = \{0\} \) if and only if:
\[
\sum_{j=1}^\infty \left( 1 - \frac{\mu_j^2}{\lambda_j^2} \right) = \infty \tag{1.2}
\]
\[
\sum_{j=1}^\infty \left( \frac{\mu_j^2}{\lambda_{j+1}^2} - 1 \right) = \infty \tag{1.3}
\]

The proof presented in [5] was based on a deep analysis of Hankel operators.

In this paper we present a simple proof consisting of three essentially separate parts. The first one is a simple operator-theoretic statement (which was essentially proved in [8]); it is presented in Section 2 below. The second part, which was the hardest part in [8], is proving the asymptotic stability of some operator. However in our case of compact operator it can be obtained essentially for free. This part is discussed in Section 3.

The third part is an abstract version of the Borg’s two spectra theorem in [2], which is just an exercise in graduate complex analysis.

**Remark 1.2.** The above theorem holds for the finite rank case as well. In this case we have finite sequences \( (\lambda_n)_{n=1}^N \) and \( (\mu_n)_{n=1}^N \) satisfying the intertwining relations (1.1), \( \lambda_N \neq 0 \) (but \( \mu_N \) can be 0).

Then there exists a unique finite rank Hankel operator \( \Gamma \) such that non-zero eigenvalues of \( \Gamma \) are simple and coincide with \( (\lambda_n)_{n=1}^N \), and the non-zero eigenvalues of \( \Gamma S \) are also simple and coincide with non-zero members of \( (\mu_n)_{n=1}^N \).

The kernel of \( \Gamma \) is always non-trivial, which agrees with Theorem 1.1, because for finite sequences conditions (1.2), (1.3) always fail.

The proof of the finite rank case is simpler, since the asymptotic stability and the abstract Borg’s theorem are trivial in this case.
2. An abstract inverse problem theorem

2.1. Plan of the game. Let $\Gamma$ be a self-adjoint Hankel operator. Define $\Gamma_1 = \Gamma S = S^* \Gamma$ (which is also a self-adjoint Hankel operator). Let $(e_n)_{n=0}^{\infty}$ be the standard basis in $\ell^2$. Using the fact that $SS^* = I - (\cdot, e_0)e_0 = I - e_0e_0^*$ we can write

$$\Gamma_1^2 = \Gamma S^* S \Gamma = \Gamma (I - e_0e_0^*) \Gamma = \Gamma^2 - uu^*, \quad (2.1)$$

where $u := \Gamma e_0$.

Now let us try to go the opposite direction. Suppose that we are given two self-adjoint operators $R, R_1$ on a Hilbert space $H$, and we want to find a Hankel operator $\Gamma$, such that the Hankel operators $\Gamma$ and $\Gamma_1 = \Gamma S = S^* \Gamma$ are unitarily equivalent to $R$ and $R_1$ respectively. We can see from (2.1) that $R$ and $R_1$ should satisfy the relations

$$R^2 - R_1^2 = pp^*, \quad (2.2)$$

for some $p \in H$.

To solve the inverse problem we want to find an operator $\Sigma^*$, (hopefully unitarily equivalent to the backward shift $S^*$) such that $R_1 = \Sigma^* R$. The tool to find $\Sigma^*$ is the following simple lemma.

Lemma 2.1 (Douglas Lemma). Let $A$ and $B$ be bounded operators in a Hilbert space $\mathcal{H}$ such that

$$\|Bh\| \leq \|Ah\| \quad \forall h \in \mathcal{H},$$

or, equivalently, $B^* B \leq A^* A$.

Then there exists a contraction $T$ (i.e. $\|T\| \leq 1$) such that $B = TA$.

Moreover, if $A$ has dense range, the operator $T$ is unique.

Remark 2.2. If $\ker A = \{0\}$ and $A$ has dense range, then the (unbounded) operator $A^{-1}$ is defined on the dense set $\text{Ran} A$, and the operator $T$ is given on this dense set by $T = BA^{-1}$. The adjoint $T^*$ is given by $T^* = (A^{-1})^* B^*$; it is not hard to show that $\text{Ran} B^*$ is in the domain of $(A^{-1})^*$, so it is well defined on all space.

To avoid non-uniqueness in finding $\Sigma^*$ it is convenient to deal only with the core of the operator $\Gamma$, i.e. with the operator $\Gamma^c := \Gamma |_{(\ker \Gamma)^\perp}$. So, given self-adjoint operators $R$ and $R_1$ satisfying (2.2), $\ker R = \{0\}$, we want to find Hankel operator $\Gamma$ such that the pair $R, R_1$ is unitarily equivalent to the pair $\Gamma^c = \Gamma |_{(\ker \Gamma)^\perp}$, $\Gamma_1^c = S^* \Gamma |_{(\ker \Gamma)^\perp}$. In this case, by the above Lemma 2.1 there exists a unique contraction $\Sigma^*$ such that $R_1 = \Sigma^* R$ (note that $\Sigma^* = R_1 R^{-1}$, see Remark 2.2). Clearly, if the pair $R, R_1$ is unitarily equivalent to the pair $\Gamma^c, \Gamma_1^c$, then $\Sigma^*$ should be unitarily equivalent to $S^*$ restricted to $(\ker \Gamma)^\perp$, meaning that

$$\Sigma^* = V^* S^* |_{(\ker \Gamma)^\perp} V,$$

for some unitary operator $V$.

In this situation the vector $p$ from (2.2) should be $p = Rq$, where $\|q\| \leq 1$; think of $q$ as of being

$$q = V^* P_{(\ker \Gamma)^\perp} e_0.$$

Using the fact that $R_1 = \Sigma^* R = RS \Sigma$ we can rewrite (2.2) as

$$R(I - \Sigma \Sigma^*) R = Rqq^* R$$

which implies that

$$I - \Sigma \Sigma^* = qq^* = (\cdot, q)q \quad (2.3)$$
So, we arrive at the following setup: $R$ and $R_1$ are self-adjoint operators, $\ker R = \{0\}$,
\[ R^2 - R_1^2 = pp^*, \quad p = Rq, \quad \|q\| \leq 1, \tag{2.4} \]
and $\Sigma^*$ is the unique contraction satisfying $R_1 = \Sigma^* R$; note that in this case $\Sigma^*$ satisfies (2.3).

**Remark 2.3.** The operator $\Sigma^*$ is defined on a dense set as $\Sigma^* = R_1 R^{-1}$, see Remark 2.2.

**Definition 2.4.** We say an operator $T$ is asymptotically stable iff $T^n \to 0$ in the strong operator topology as $n \to \infty$, i.e. iff for for all $x \in \mathcal{H}$
\[ \lim_{n \to \infty} \|T^n x\| \to 0. \]

**Proposition 2.5.** If the operator $\Sigma^*$ introduced above is asymptotically stable, then there exists a unique Hankel operator $\Gamma$ such that the pair $\Gamma|_{\ker \Gamma}^\perp$, $\Gamma S|_{\ker \Gamma}^\perp$ is unitary equivalent to $R$, $R_1$, i.e. that there exists a unitary operator $V : \mathcal{H} \to (\ker \Gamma)^\perp$ such that
\begin{align*}
\Gamma|_{\ker \Gamma}^\perp &= VRV^*, \\
\Gamma S|_{\ker \Gamma}^\perp &= VR_1V^*. \tag{2.5}
\end{align*}
Moreover, multiplying $V$ by a unimodular constant one can always get that
\begin{align*}
P|_{\ker \Gamma}^\perp e_0 &= Vq, \\
\Gamma e_0 &= Vp. \tag{2.6}
\end{align*}
Finally, $\ker \Gamma = \{0\}$ if and only if $\|q\| = 1$ and $q \notin \text{Ran} R$.

**Remark 2.6.** The unitary operator $V$ is clearly not unique, since replacing $V$ by $\alpha V$, $|\alpha| = 1$ does not change (2.5) and (2.6). However, the multiplication by a unimodular constant is the only degree of freedom for $V$; we will see in the proof of Proposition 2.5 that a unitary operator $V$ satisfying (2.5), (2.6) and one of the identities (2.7), (2.8) is unique.

**Proof of Proposition 2.5.** Treating (2.3) as an identity for quadratic forms and substituting $x \in \mathcal{H}$ into it we get
\[ \|x\|^2 - \|\Sigma^* x\|^2 = |(x, q)|^2. \]
Applying this identity to $(\Sigma^*)^k x$ we get
\[ \|(\Sigma^*)^k x\|^2 - \|(\Sigma^*)^{k+1} x\|^2 = |(\Sigma^*)^k x, q|^2, \]
so taking the sum we get
\[ \|x\|^2 - \|(\Sigma^*)^{n+1} x\|^2 = \sum_{k=0}^{n} |(\Sigma^*)^k x, q|^2. \]
Taking the limit as $n \to \infty$ and using the asymptotic stability of $\Sigma^*$ we see that
\[ \|x\|^2 = \sum_{k=0}^\infty |(\Sigma^*)^k x, q|^2, \]
which means that the operator $\mathcal{V} : \mathcal{H} \to \ell^2$,
\[ \mathcal{V} x := ((x, q), (\Sigma^* x, q), ((\Sigma^*)^2 x, q), \ldots) = \left( (\Sigma^*)^k x, q \right)_{k=0}^\infty \tag{2.9} \]
is an isometry.
We can see that
\[ \mathcal{V} \Sigma^* x = ((\Sigma^* x, q), ((\Sigma^*)^2 x, q), ((\Sigma^*)^3 x, q), \ldots) = S^* V x, \]
\[ V \Sigma^* = S^* V, \] so \( \Sigma^* \) is unitarily equivalent to either \( S^* \) (if \( \text{Ran} V = \ell^2 \)) or to the restriction of \( S^* \) to \( S^\perp \)-invariant subspace \( \text{Ran} V \subset \ell^2 \) (if \( \text{Ran} V \neq \ell^2 \)).

Denoting by \( \Sigma \) the adjoint of \( \Sigma^* \) and taking the adjoint of \( \mathcal{V} \Sigma^* = S^* \mathcal{V} \) we get \( \Sigma \mathcal{V} = \mathcal{V} S \).

Define \( \Gamma := \mathcal{V} R \mathcal{V}^* \). Then

\[
\Gamma S = \mathcal{V} R \mathcal{V}^* S = \mathcal{V} R \Sigma \mathcal{V} = \mathcal{V} \Sigma^* R \mathcal{V} = S^* \mathcal{V} R \mathcal{V}^* = S^* \Gamma; \tag{2.10}
\]

(here in the second equality we used \( \mathcal{V} \Sigma^* = S^* \mathcal{V} \), and in the forth one \( \mathcal{V} \Sigma^* = S^* \mathcal{V} \)), so \( \Gamma \) is a Hankel operator.

We can also see from (2.10) that \( \Gamma S = \mathcal{V} \Sigma^* R \mathcal{V}^* = \mathcal{V} R_1 \mathcal{V}^* \).

Let \( V \) be the operator \( \mathcal{V} \) with the target space restricted to \( \text{Ran} \mathcal{V} \), so \( V : \mathcal{H} \to \text{Ran} \mathcal{V} \) is a unitary operator. Since \( \text{Ker} R = \{0\} \), the identity \( \Gamma = \mathcal{V} R \mathcal{V}^* \) implies that \( \text{ker} \Gamma = \{0\} \), so identities \( \Gamma = \mathcal{V} R \mathcal{V}^* \), \( \Gamma_1 = \mathcal{V} R_1 \mathcal{V}^* \) translate to (2.5), (2.6).

Let us now discuss identities (2.7), (2.8). We can see from the definition of \( \mathcal{V} \) that

\[
(x, V^* e_0)_{\mathcal{H}} = (\mathcal{V} x, e_0)_{\ell^2} = (x, q)_{\mathcal{H}},
\]

so \( V^* e_0 = q \), which is equivalent to (2.7). Next,

\[
u = \Gamma e_0 = V R \mathcal{V}^* e_0 = V R q = \nu p,
\]

which is exactly (2.8). Thus for the operator \( V \) constructed above, identities (2.7), (2.8) are satisfied.

Let us discuss uniqueness. Suppose the identities (2.5), (2.6) hold for some unitary operator \( V : \mathcal{H} \to \text{clos} \text{Ran} \Gamma = (\text{Ker} \Gamma)^\perp \subset \ell^2 \), not necessarily the one constructed above.

Since for a Hankel operator \( \text{Ker} \Gamma \) is always \( S^\perp \)-invariant, the subspace \( (\text{Ker} \Gamma)^\perp \) is \( S^\perp \)-invariant, so the restriction \( S^* |_{(\text{Ker} \Gamma)^\perp} \) is well defined. The identities (2.5), (2.6) with \( V = \tilde{V} \) and the definition of \( \Sigma^* \) then imply that

\[
S^* |_{(\text{Ker} \Gamma)^\perp} = V \Sigma^* V^*; \tag{2.11}
\]

As it was discussed in Section 2, see (2.1),

\[
(\Gamma S)^2 = \Gamma^2 - uu^*,
\]

where

\[
\nu := \Gamma e_0 = \Gamma P_{(\text{Ker} \Gamma)^\perp} e_0.
\]

Then identities (2.5), (2.6) imply that

\[
R_1^2 = R^2 - \tilde{p} \tilde{p}^*, \quad \text{where} \quad \tilde{p} = R \tilde{q}, \quad \tilde{q} = V^* P_{(\text{Ker} \Gamma)^\perp} e_0.
\]

Comparing this with \( R_1^2 = R^2 - pp^* \) we conclude that \( \tilde{p} = \alpha p, \tilde{q} = \alpha q, |\alpha| = 1 \), so multiplying \( V \) by \( \alpha \) we get the identities (2.7), (2.8) (without spoiling (2.5), (2.6)).

Computing the coefficients \( \gamma_k \) of the operator \( \Gamma \) we write

\[
\gamma_k = (\Gamma e_0, S^k e_0) = ((S^*)^k \Gamma e_0, e_0) = ((S^*)^k |_{(\text{Ker} \Gamma)^\perp}) P_{(\text{Ker} \Gamma)^\perp} e_0, P_{(\text{Ker} \Gamma)^\perp} e_0
\]

\[
= ((S^*)^k R \tilde{q}, \tilde{q}) = ((S^*)^k \tilde{p}, \tilde{q}) = ((S^*)^k p, q),
\]

meaning that the coefficients \( \alpha_k \) do not depend on \( V \). So, uniqueness of the Hankel operator \( \Gamma \) is proved.

Finally, let us discuss the kernel of \( \Gamma \). As we discussed above, \( \text{Ker} \Gamma \) is trivial if and only if the operator \( V \) defined by (2.9) satisfies \( \text{Ran} V = \ell^2 \). In this case \( \Sigma^* \) is unitarily equivalent to \( S^* \), or, equivalently, \( \Sigma \) is unitarily equivalent to \( S \).
So, let $\text{Ker} \Gamma = \{0\}$, so $\Sigma^*$ is unitarily equivalent to $S^*$. We know that 

$$I - SS^* = e_0 e_0^*, \quad I - \Sigma \Sigma^* = qq^*,$$

so by the unitary equivalence $\|q\| = \|e_0\| = 1$, and $\Sigma^* q = 0$. If $q \in \text{Ran} R$, i.e. $q = Rf$, then 

$$R \Sigma f = \Sigma^* Rf = \Sigma^* q = 0,$$

and since $\text{Ker} R = \{0\}$, we conclude that $\Sigma^* f = 0$. But if $\text{Ker} \Gamma = \{0\}$, then $\Sigma$ is an isometry, which contradicts $\Sigma^* f = 0$. So $q \notin \text{Ran} R$.

Let now $\|q\| = 1$ and $q \notin \text{Ran} R$. We know that 

$$\Sigma^* R^2 \Sigma = R_1^2 = R \Sigma \Sigma^* R = R(I - q q^*) R,$$

and that $\text{Ker}(I - q q^*) = \text{span}\{q\}$. Since $q \notin \text{Ran} R$, we see that $\text{Ker} R(I - q q^*) R = \{0\}$, so $\text{Ker} \Sigma = \{0\}$.

Applying (2.3) to vector $q$ we get that 

$$q - \Sigma \Sigma^* q = q,$$

and since $\text{Ker} \Sigma = \{0\}$ we see that $\Sigma^* q = 0$.

Left and right multiplying (2.3) by $\Sigma^*$ and $\Sigma$ respectively, we get that 

$$\Sigma^* \Sigma - \Sigma^* \Sigma \Sigma^* \Sigma = \Sigma^* q q^* \Sigma,$$

and since $\Sigma^* q = 0$, we have $\Sigma \Sigma^* = (\Sigma \Sigma^*)^2$, which means $\Sigma \Sigma^*$ is an orthogonal projection.

Since $\text{Ker} \Sigma = \{0\}$, we conclude that $\Sigma^* \Sigma = I$, i.e. that $\Sigma$ is an isometry. Since $\Sigma^*$ is asymptotically stable, $\Sigma^*$ (and so $\Sigma$) has no unitary part. The identity (2.3) implies that 

$$\text{rank}(I - \Sigma \Sigma^*) = 1,$$

so $\Sigma$ is unitarily equivalent to the shift operator $S$, i.e. that 

$$S = V \Sigma V^*$$

for some unitary operator $V : \mathcal{H} \to \ell^2$. Defining $\Gamma = VRV^*$, $\Gamma_1 = VR_1 V^*$, we can see that 

$$\Gamma_1 = \Gamma S = S^* \Gamma,$$

so $\Gamma$ is indeed the Hankel operator with trivial kernel, satisfying the conditions (2.5), (2.6) (we already proved that such Hankel operator is unique).

\[\square\]

3. Asymptotic Stability

The hard part of solving the inverse problem for Hankel operator is usually the proof of asymptotic stability of operator $T$. However, under the compact operator case we will get asymptotic stability for free.

Let us recall the setup. We had compact self-adjoint operators $R$ and $R_1$, $\text{ker} R = \{0\}$, satisfying $R_1^2 = R^2 - pp^*$, where $\|R^{-1} p\| \leq 1$, with $\{\lambda_k\}_{k \geq 1}$ and $\{\mu_k\}_{k \geq 1}$ being the non-zero eigenvalues of $R$ and $R_1$ respectively. We also assume that the eigenvalues satisfy the intertwining relations (1.1).

**Proposition 3.1.** Under the above assumptions the vector $p$ is cyclic for $R^2$ and the operator $\Sigma^*$ is asymptotically stable.

The cyclicity of $p$ is easy. Indeed, if $p$ is not cyclic for $R^2$, then projection of $p$ onto some eigenspace $\text{ker}(R^2 - \lambda_k I)$ is zero, so operators $R^2$ and $R_1^2$ coincide on their common eigenspace $\text{ker}(R^2 - \lambda_k I)$, which means that $|\lambda_k| = |\mu_k|$.
3.1. **Preparation.** To prove the asymptotic stability of the contraction $\Sigma^*$ we will use the following simple lemma, which is a slight modification of [8, lemma 3.2].

**Lemma 3.2.** Let $\|T\| \leq 1$, and let $K$ be a compact operator with a dense range. Assume that an operator $A$ satisfies

$$TK = KA \tag{3.1}$$

If $A$ is weakly asymptotically stable, meaning that $A_n \to 0$ in the weak operator topology (W.O.T) as $n \to \infty$, then $T$ is asymptotically stable.

**Proof.** Iterating (3.1) we get that $T^y K = KA^y, n \geq 1$. Take $x \in H$. Since $A^n \to 0$ in W.O.T. and $K$ is compact, we have that $\|KA^y x\| \to 0$.

So $\lim_{n \to \infty} \|T^n y\| = 0$ for all $y \in \text{Ran } K$. Thus, we have strong convergence on a dense set, and since $\|T^n\| \leq 1$, we conclude (by $\varepsilon/3$-Theorem) that $T^n \to 0$ in the strong operator topology. □

Recall, that for an operator $R$ (in a Hilbert space) its modulus $|R|$ is defined as $|R| := (R^* R)^{1/2}$

**Lemma 3.3.** For the operators $R$ and $\Sigma^*$ from Section 2 there exists a unique contraction $A$, such that:

$$\Sigma^* |R|^{1/2} = |R|^{1/2} A$$

**Proof.** The inequality $\Sigma \Sigma^* \leq I$ implies that

$$|R|^2 = R^2 \geq R \Sigma \Sigma^* R = \Sigma \Sigma^* |R|^2 \Sigma \geq \Sigma^* |R| \Sigma \Sigma^* |R| \Sigma = (\Sigma^* |R| \Sigma)^2.$$

Recall, that the Löwner–Heinz inequality states that for self-adjoint operators $W, W_1$ the inequality $W \geq W_1 \geq 0$ imply that $W^{\alpha} \geq W_1^{\alpha}$ for all $\alpha \in (0, 1)$. Applying this inequality with $\alpha = 1/2$ to operators $|R|^2$ and $(\Sigma^* |R| \Sigma)^2$, we conclude that

$$|R| \geq \Sigma^* |R| \Sigma,$$

or, equivalently,

$$\| |R|^{1/2} x \| \geq \| |R|^{1/2} \Sigma x \| \quad \forall x \in H.$$

By Lemma 2.1 there exists a unique contraction, denote it to be $A^*$, satisfies:

$$A^* |R|^{1/2} = |R|^{1/2} \Sigma.$$

Taking the adjoint, we get the conclusion of the lemma. □

The operator $A$ constructed in the above Lemma 3.3 satisfies the identity (3.1) with $K = |R|^{1/2}$. Since $|R|^{1/2}$ is compact, Lemma 3.2 says that the weak asymptotic stability of $A$ implies the asymptotic stability of $T$.

3.2. **Weak asymptotic stability of $A$.** We will show below in Section 3.3 that under our assumptions the operator $A$ is a strict contraction, meaning that $\|Ax\| < \|x\|$ for all $x \neq 0$.

**Lemma 3.4.** Let $A : H \to H$ be a strict contraction. Then $A^n \to 0$ in the weak operator topology (WOT) as $n \to \infty$.

**Proof.** First we notice that the assumption that $A$ is a strict contraction implies that $A$ is completely non unitary, meaning that there is no reducing subspace of $A$ on which $A$ acts unitarily. But every completely non-unitary contraction admits the functional model, i.e. it is unitarily equivalent to the model operator $\mathcal{M}_\theta$ on the model space $\mathcal{K}_\theta$, where $\theta$ is the so-called characteristic function of $A$. Without going into details, which are not important for
our purposes, we just mention that the model space $\mathcal{K}_\theta$ is a subspace of a vector-valued space $L^2(E) = L^2(\mathbb{T}, m; E)$ of square integrable (with respect to the normalized Lebesgue measure $m$ on $\mathbb{T}$) functions with values in an auxiliary Hilbert space $E$. The model operator $\mathcal{M}_\theta$, to which $A$ is unitarily equivalent, is just the compression of the multiplication operator $M_z$ by the independent variable $z$

$$\mathcal{M}_\theta f = P_{\mathcal{K}_\theta} M_z f, \quad f \in \mathcal{K}_\theta;$$

recall that the multiplication operator $M_z$ is defined by $M_z f(z) = zf(z)$, $z \in \mathbb{T}$.

What is also essential for our purposes, is that the multiplication operator $M_z$ is the dilation of the model operator $\mathcal{M}_\theta$, i.e. that for all $n \geq 1$

$$\mathcal{M}_\theta^n f = P_{\mathcal{K}_\theta} M_z^n f, \quad f \in \mathcal{K}_\theta.$$

Since trivially $M_z^n \to 0$ in the weak operator topology of $B(L^2(\mathbb{T}, m; E))$ as $n \to +\infty$, we conclude that $\mathcal{M}_\theta^n \to 0$ as $n \to +\infty$ in the weak operator topology of $B(\mathcal{K}_\theta)$, and so $A^n \to 0$ in the weak operator topology as well. \hfill \Box

3.3. $A$ is a strict contraction. To prove the weak asymptotic stability of $A$ we need to investigate its structure in more detail.

We know that $|R_1|^2 = |R|^2 - pp^* \leq |R|^2$. By the Löwner–Heinz inequality with $\alpha = 1/2$ we have that $|R_1| \leq |R|$, so by Lemma 2.1 there exists a unique contraction $Q$ such that:

$$|R_1|^{1/2} = Q|R|^{1/2} \quad (3.2)$$

Let us write the polar decompositions of $R$ and $R_1$

$$R = J|R| \quad R_1 = J_1|R_1|.$$

Since $R$ and $R_1$ are self-adjoint, the unitary operators $J$ and $J_1$ are also self-adjoint and commute with $|R|$ and $|R_1|$ respectively.

The following simple Lemma (see [8, Lemma 3.5]) gives the structure of the operator $A$.

**Lemma 3.5.**

$$A = Q^* J_1 Q J$$

**Proof.** Note (see Remark 2.2) that $Q = |R_1|^{1/2} |R|^{-1/2}$. Using Remark 2.2 again we can write

$$A = |R|^{-1/2} \Sigma^* |R|^{1/2} = |R|^{-1/2} R_1 R^{-1} |R|^{1/2} = |R|^{-1/2} |R_1| J_1 |R|^{-1/2}$$

$$= |R|^{-1/2} |R_1|^{1/2} J_1 |R|^{1/2} |R|^{-1/2} J = Q^* J_1 Q J;$$

here we used the fact that $J$ and $J_1$ commute with $|R|$ and $|R_1|$ respectively. \hfill \Box

In addition, the following lemma, see [8, Lemma 3.6] gives the structure of $Q$

**Lemma 3.6.** Let $\mathcal{H}_0$ be the smallest invariant subspace of $|R|$ that contains $|R|^{-1} p = Jq$ (recall that $q$ is defined by $p = Rq$ with $\|q\| \leq 1$). Then $Q$ has the following block structure in the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^1$:

$$Q = \begin{pmatrix} Q_0 & 0 \\ 0 & I \end{pmatrix} \quad (3.3)$$

where $Q_0$ is a pure contraction (i.e. $\|Q_0 h\| < \|h\|$ for all $h \neq 0$).
Applying both sides to $x$ relations ($R_p H$), hence set $|R| = 0$ then Ker($|R|$) = Ker($|R|$), which is a reducing subspace for both operators, so $Q$ has the form (3.3), and only need to show that $Q_0$ is a strict contraction.

Using (3.4) and the identity $|R|^1/2 = Q|R|1/2 = |R|1/2Q^*Q$, we can write

$$R^2 - pp^* = R^2 = |R|1/2Q^*Q|R|1/2.$$ 

Recalling that $p = Rq = J|R|q = |R|Jq$, we can rewrite the above identity as

$$|R|^{1/2} \left( |R| - (|R|^{1/2} Jq)(|R|^{1/2} Jq)^* \right) |R|^{1/2} = |R|^{1/2}Q^*Q|R|Q^*Q|R|^{1/2}$$

which, because Ker $R = \{0\}$, implies that

$$Q^*Q|R|Q^*Q = |R| - (|R|^{1/2} Jq)(|R|^{1/2} Jq)^*.$$ 

(3.5)

Applying both sides to $x$, and taking the inner product with $x$, we get

$$\langle |R|Q^*Qx, Q^*Qx \rangle = \langle |R|x, x \rangle - \langle x, |R|^{1/2} Jq \rangle^2.$$ 

(3.6)

Now, take $x$ such that $\|Qx\| = \|x\|$. Since $\|Q\| \leq 1$, this happens if and only if $x = Q^*Qx$. The equation (3.6) can be rewritten in this case as

$$\langle |R|x, x \rangle = \langle |R|x, x \rangle - \langle x, |R|^{1/2} Jq \rangle^2,$$

which implies that $x \perp |R|^{1/2} Jq$. Applying equation (3.5) to such $x$, and using again the fact that $Q^*Qx = x$, we get that

$$Q^*Q|R|x = |R|x.$$ 

Hence set $\mathcal{H}_1 := \{h \in \mathcal{H} : h \in \mathcal{H}, \|Qh\| = \|h\|\} = \text{Ker}(I - Q^*Q)$ is an invariant subspace for $|R|$ (and so for $|R|^{1/2}$), which is orthogonal to $J|R|^{1/2} q$. Therefore

$$\mathcal{H}_1 \perp \text{span}\{|R|^{n/2}|R|^{1/2} Jq : n \geq 0\} = \text{span}\{|R|^{n/2}p : n \geq 2\} \supset \text{span}\{|R|^{n/2}p : n \geq 1\} = \mathcal{H}_0;$$

in the last equality we used the fact that $|R|p$ is also cyclic for $|R|\|_{\mathcal{H}_0}$. Thus $Q_0 = Q_{|\mathcal{H}_0}$ is a strict contraction, and the lemma is proved.

Returning to our situation, recall that the operator $R^2_1$ is a rank one perturbation of $R^2$,

$$R^2 = R^2 - pp^*,$$

and that $R^2$ and $R^2_1$ have simple eigenvalues $\lambda^2_k$ and $\mu^2_k$ respectively. The strict intertwining relations (1.1) imply that for all $k$

$$P_{\text{Ker}(R^2 - \lambda^2_k I)} p \neq 0$$

(because if $P_{\text{Ker}(R^2 - \lambda^2_k I)} p = 0$ then Ker($R^2 - \lambda^2_k I$) = Ker($R^2_1 - \lambda^2_k I$), so $\mu_k = \lambda_k$), so $p$ is a cyclic vector for $|R|$. Therefore in out case $\mathcal{H}_0 = \mathcal{H}$, so $Q$ is a pure contraction. Since $A = Q^*JQJ$, the operator $A$ is also a pure contraction.
4. Abstract Borg’s theorem

Let us introduce some terminology. Consider a triple \( R, R_1, p \), where \( R, R_1 \) are operators on a Hilbert space \( \mathcal{H} \) and \( p \in \mathcal{H} \) (we call this a triple on \( \mathcal{H} \)). We say that two triples \( \tilde{R}, \tilde{R}_1, \tilde{p} \) on \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) respectively are unitary equivalent if there exists a unitary operator \( U : \mathcal{H} \to \tilde{\mathcal{H}} \) such that

\[
\tilde{R} = U R U^*, \quad \tilde{R}_1 = U R_1 U^*, \quad \tilde{p} = U p.\]

The main result, Theorem 1.1 easily follows, see Section 4.1 from the theorem below, applied to the operators \( W = R^2 \) and \( W_1 = R^2_1 = W - p p^* \).

**Theorem 4.1** (Abstract Borg’s Theorem). Given two sequences \((\lambda_k^2)_{k \geq 1}\) and \((\mu_k^2)_{k \geq 1}\) satisfying intertwining relations (1.1) and such that \( \lambda_k^2 \to 0 \) as \( k \to \infty \), there exist a unique (up to unitary equivalence) triple \( W, W_1, p \), such that

(i) \( W = W^* \geq 0 \), \( \text{Ker} W = \{0\} \) is a compact operator with simple eigenvalues \( (\lambda_k^2)_{k = 1}^{\infty} \);

(ii) \( p \in \mathcal{H} \) and \( W_1 = W - p p^* \) is a compact operator with non-zero eigenvalues \( (\mu_k^2)_{k = 1}^{\infty} \) (\( W_1 \) can also have a simple eigenvalue at 0, and it is not hard to show that all the eigenvalues are simple).

Moreover, \( ||W^{-1/2}p|| = 1 \) if and only if

\[
\sum_{j=1}^{\infty} \left( 1 - \frac{\mu^2_j}{\lambda^2_j} \right) = \infty; \tag{4.1}
\]

in addition, if (4.1) holds, then \( ||W^{-1}p|| = \infty \) (i.e. \( p \notin \text{Ran} W \)) if and only if

\[
\sum_{j=1}^{\infty} \left( \frac{\mu^2_j}{\lambda^2_{j+1}} - 1 \right) = \infty. \tag{4.2}
\]

**Remark 4.2.** The original Borg’s theorem [2] states that the potential \( q \) of a Schrödinger operator \( L \), \( Ly = y'' + q(x)y \) on an interval is uniquely defined by the two sets of eigenvalues, corresponding to two specific boundary conditions. Later Levinson [7] extended this result by showing that essentially any non-degenerate pair of self-adjoint boundary conditions would work.

Changing boundary conditions for a Schrödinger operator is essentially a rank one perturbation (by an unbounded operator). Namely, if \( L_1 \) and \( L_2 \) are Schrödinger operators on an interval with the same potential, but with two different self-adjoint boundary conditions, then for any \( \lambda \notin \sigma(L_1) \cup \sigma(L_2) \) the difference \( (L_1 - \lambda I)^{-1} - (L_2 - \lambda I)^{-1} \) is a rank one operator (and the operators \( (L_1 - \lambda I)^{-1}, (L_2 - \lambda I)^{-1} \) are compact). Thus, by picking a real \( \lambda \) the problem can be reduced to rank one perturbations of compact self-adjoint operators.

Our Theorem 4.1 deals with rank one perturbations of (abstract) compact self-adjoint operators, hence the name. We do not assume that our operators came from Schrödinger operators, so we only reconstructing the spectral measure, and are not concerned with the reconstruction of the potential. However, it is well known how to reconstruct the potential from the spectral measure, or, more precisely, from the Titchmarsh–Weyl m-function, so it should be possible to get the Borg’s result from our abstract theorem.

Note also, that Theorem 4.1 give not only uniqueness, but the existence as well.

4.1. Abstract Borg’s Theorem implies the main result. Let us now explain how the above Theorem 4.1 implies Theorem 1.1. First of all, if we know the triple \( W = R^2, W_1 = R^2_1, p \), and know the eigenvalues of \( R \) and \( R_1 \) (it is sufficient to know only their signs), we can
reconstruct the unique triple $R$, $R_1$, $p$ just by taking appropriate square roots of $W$ and $W_1$. Namely, if $u_k$ and $v_k$ are eigenvectors of $W$ and $W_1$,

$$W u_k = \lambda_k^2 u_k, \quad W_1 v_k = \mu_k^2 v_k,$$

we put

$$R u_k = \lambda_k u_k, \quad R_1 v_k = \mu_k v_k$$

(and of course, we put $R_1 x = 0$ for $x \in \ker W_1$). And it is easy to see that a self-adjoint square root with prescribed signs of eigenvalues is unique and is given by the above formulas.

So, if we are given the eigenvalues $(\lambda_k)_{k \geq 1}$, $(\mu_k)_{k \geq 1}$, we first construct the (unique up to unitary equivalence) triple $W$, $W_1$, $p$ using the eigenvalues $(\lambda_k^2)_{k \geq 1}$, $(\mu_k^2)_{k \geq 1}$, and then take the (unique) square roots with prescribed signs of eigenvalues to get the operators $R$ and $R_1$. Note, that the triple $R$, $R_1$, $p$ is unique up to unitary equivalence.

By Proposition 3.1 the operator $\Sigma^* = R_1 R^{-1}$ is asymptotically stable, so by Proposition 2.5 there exists a unique Hankel operator $\Gamma$ such that the triple $\Gamma_{(\ker \Gamma)^+}$, $(\Gamma S)_{(\ker \Gamma)^+}$, $u := \Gamma e_0 = \Gamma P_{(\ker \Gamma)^+} e_0$ is unitarily equivalent to the triple $R$, $R_1$, $p$. This implies, in particular, that the non-zero eigenvalues of $\Gamma$ and of $\Gamma S$ are simple and coincide with $(\lambda_k)_{k \geq 1}$ and $(\mu_k)_{k \geq 1}$ respectively. So, the existence and uniqueness part of Theorem 1.1 is proved.

As for the conditions for $\ker \Gamma = \{0\}$, we just note that the conditions $\|q\| = 1$ and $q \notin \operatorname{Ran} R$ from Proposition 2.5 translate to $\|R^{-1} p\| = 1$ and $\|R^{-2} p\| = \infty$. But $\|R^{-1} p\| = \|W^{-1/2} p\|$, $\|R^{-2} p\| = \|W^{-1} p\|$, and thus the statement about triviality of the kernel of $\Gamma$ follows.

\[ \square \]

4.2. Proof of the abstract Borg’s Theorem: existence and uniqueness part. First of all notice that the intertwining condition (1.1) implies that the vector $p$ must be cyclic for $W$. Since everything is defined up to unitary equivalence, we than can assume without loss of generality that $W$ is the multiplication $M_s$ by the independent variable $s$ in the weighted space $L^2(\rho)$, where $\rho$ is the spectral measure, corresponding to the vector $p$.

Recall, that this spectral measure can be defined as the unique (finite, Borel, compactly supported) measure on $\mathbb{R}$ such that

$$((W - zI)^{-1} p, p) = \int_{\mathbb{R}} \frac{d\rho(s)}{s - z} \quad \forall z \in \mathbb{C} \setminus \mathbb{R}. $$

Since $W$ is a compact operator with eigenvalues $(\lambda_k^2)_{k \geq 1}$, the measure $\rho$ is purely atomic,

$$\rho = \sum_{k \geq 1} a_k \delta_{\lambda_k^2}, \quad a_k > 0. \quad (4.3)$$

Note also that in this representation the vector $p$ is represented by the function 1 in $L^2(\rho)$.

Since everything is considered up to unitary equivalence, we can always assume that $W$ is the multiplication operator $M_s$ by the independent variable $s$ in the weighted space $L^2(\rho)$, where the spectral measure $\rho$ is given by (4.3), with $p \equiv 1$. Note that in this representation position $\lambda_k^2$ of delta functions are fixed (because they must correspond to the eigenvalues of $W$), but the weights $a_k$ are at the moment unknown.

Note, that for $p \equiv 1$ a choice of the weights $a_k$ completely defines the pair $W$, $p$ up to unitary equivalence.\(^1\) Moreover, the choice of the weights $a_k$ is completely defines the triple $W$, $W_1$, $p$ (up to unitary equivalence). Indeed if the weights $a_k$ are known (recall

\(^1\)Of course, any choice of non-vanishing weights $a_k > 0$ gives unitary equivalent operators $W$, but the vectors $p \equiv 1$ are not transformed according to the unitary equivalence.
that in our representation $p \equiv 1$, the operator $W_1$ is uniquely defined in the above spectral representation of $W$ as $W_1 = W - pp^*$.

4.2.1. Cauchy transforms of spectral measures. Let us recall some standard facts in perturbation theory. Let $W = W^*$ be a (bounded, for simplicity) self-adjoint operator, and let $p$ be its cyclic vector. The spectral measure $\rho$ corresponding to the vector $p$ is defined as the unique Borel measure on $\mathbb{R}$ such that

$$ F(z) := ((W - zI)^{-1}p, p) = \int_{\mathbb{R}} \frac{d\rho(s)}{s - z} \quad \forall z \in \mathbb{C} \setminus \mathbb{R}. \quad (4.4) $$

As it is customary in perturbation theory, define a family of rank one perturbations $W^{(\alpha)} := W + \alpha pp^*$, $\alpha \in \mathbb{R}$.

The spectral measures $\rho^{(\alpha)}$ are defined as the unique Borel measures such that

$$ F^{(\alpha)}(z) := ((W^{(\alpha)} - zI)^{-1}p, p) = \int_{\mathbb{R}} \frac{d\rho^{(\alpha)}(s)}{s - z} \quad \forall z \in \mathbb{C} \setminus \mathbb{R}; \quad (4.5) $$

the functions $F^{(\alpha)}$ are the Cauchy transforms of $\rho^{(\alpha)}$.

The relation between the Cauchy transforms $F$ and $F^{(\alpha)}$ is given by the famous Aronszajn–Krein formula,

$$ F^{(\alpha)} = \frac{F}{1 + \alpha F}, \quad (4.6) $$

which is an easy corollary of the standard resolvent identities.

Recall that our operator $W_1$ is exactly the operator $W^{(-1)}$, so let us for the consistency denote $F_1 := F^{(-1)}$. Identity (4.6) can then be rewritten as

$$ F_1 = \frac{F}{1 - F} \quad (4.7) $$

so

$$ \frac{F}{F_1} = 1 - F. \quad (4.8) $$

So, to prove the first part of Theorem 4.1 (existence and uniqueness of the triple $W$, $W_1$, $p$), it is sufficient to show that there exists a unique measure $\rho$ of form (4.3) such that if $F$ is the Cauchy transform of $\rho$

$$ F(z) = \int_{\mathbb{R}} \frac{d\rho(s)}{s - z} \quad \forall z \in \mathbb{C} \setminus \mathbb{R} \quad (4.9) $$

then the function $F_1$ defined by (4.7) has the poles exactly at points $\mu_k^2$. Here the points $\lambda_k^2$ and $\mu_k^2$ are given, and the weights $a_k > 0$ are to be found.

4.2.2. Guessing the function $F$. We want to reconstruct the spectral measure $\rho$ from the sequences $\{\lambda_k\}_{k \geq 1}$ and $\{\mu_k\}_{k \geq 1}$.

Denote

$$ \sigma := \sigma(W) = \{\lambda_k^2\}_{k \geq 1} \cup \{0\}, \quad \sigma_1 := \sigma(W_1) = \{\mu_k^2\}_{k \geq 1} \cup \{0\}. $$

It trivial that $F$ and $F_1$ are analytic in $\mathbb{C} \setminus \sigma$ and $\mathbb{C} \setminus \sigma_1$ respectively, and having simple poles at points $\lambda_k$ and $\mu_k$, $k \geq 1$ respectively (note that while the measure $\rho_1 = \rho^{(-1)}$ can have a mass at 0, the point 0 is not an isolated singularity).
Identity (4.8) (which holds on $\mathbb{C} \setminus (\sigma \cup \sigma_1)$) implies that $F/F_1$ has simple poles at the points $\lambda_k^2$, $k \geq 1$, and that it can be analytically extended to $\mathbb{C} \setminus \sigma$. The (isolated) zeroes of $1 - F$ must be at points where $F(z) = 0$, so they must be only at the poles of $F_1$, i.e. at the points $\mu_k^2$, $k \geq 1$.

So the function $F/F_1 = 1 - F$ must be a function analytic on $\mathbb{C} \setminus \sigma$ with simple poles at the points $\lambda_k^2$, $k \geq 1$ and simple zeroes at the points $\mu_k^2$.

One can try to write such a function, namely one could guess that

$$1 - F(z) = \prod_{k \geq 1} \left( \frac{z - \mu_k^2}{z - \lambda_k^2} \right).$$

We will show that this is indeed the case; from there we will trivially get the formula for $F$, and then compute the weights $a_k$.

### 4.2.3. Existence and uniqueness of our guess for $F$.

Denote

$$\Phi(z) := \prod_{k \geq 0} \left( \frac{z - \mu_k^2}{z - \lambda_k^2} \right). \quad (4.10)$$

We have guessed that $1 - F(z) = \Phi(z)$. But first we need to show that the product (4.10) is well defined.

**Lemma 4.3.** The product (4.10) converges uniformly on compact subsets of $\mathbb{C} \setminus \sigma$, and moreover

$$\Phi(\infty) := \lim_{z \to \infty} \Phi(z) = 1 \quad (4.11)$$

**Proof.** Fix a compact $K \subset \mathbb{C} \setminus \sigma$ Trivially

$$\left| 1 - \frac{z - \mu_k^2}{z - \lambda_k^2} \right| = \left| \frac{\lambda_k^2 - \mu_k^2}{z - \lambda_k^2} \right| \leq C(K)(\lambda_k^2 - \mu_k^2) \quad \forall z \in K. \quad (4.12)$$

Since

$$\sum_{k \geq 1} (\lambda_k^2 - \mu_k^2) \leq \sum_{k \geq 1} (\lambda_k^2 - \lambda_{k+1}^2) \leq \lambda_1^2 < \infty \quad (4.13)$$

we have

$$\sum_{k \geq 1} \left| 1 - \frac{z - \mu_k^2}{z - \lambda_k^2} \right| \leq C(K)\lambda_1^2 < \infty \quad \forall z \in \mathbb{C} \setminus \sigma.$$

But this implies that the product (4.10) converges uniformly on compact subsets of $\mathbb{C} \setminus \sigma$.

To prove the second statement, take any $R > 2\lambda_1^2$. Then, see (4.12)

$$\left| 1 - \frac{z - \mu_k^2}{z - \lambda_k^2} \right| = \left| \frac{\lambda_k^2 - \mu_k^2}{z - \lambda_k^2} \right| \leq \frac{2}{R}(\lambda_k^2 - \mu_k^2) \quad \forall z : |z| > R.$$ 

Using (4.13) we get that for all $|z| > R$

$$\sum_{k \geq 1} \left| 1 - \frac{z - \mu_k^2}{z - \lambda_k^2} \right| \leq \frac{2\lambda_1^2}{R},$$

which immediately implies (4.11). \qed

We will need the following definition
Definition 4.4. Recall that an analytic in the upper half-place \( C_+ \) function \( f \) is called Herglotz (or Nevanlinna) if \( \text{Im} f(z) \geq 0 \) for all \( z \in C_+ \).

We should mention, that any non-constant Herglotz function satisfies the strict inequality \( f(z) > 0 \) for all \( z \in C_+ \).

Lemma 4.5. The function \( \Phi \) defined by (4.10) satisfies the following properties

(i) \( \Phi(z) = \overline{\Phi(z)} \); in particular, \( \Phi(x) \) is real for all \( x \in \mathbb{R} \setminus \sigma \).

(ii) The function \( \Phi \) has simple poles at points \( \lambda_k^2 \), and its only zeroes are the simple zeroes at points \( \mu_k^2 \).

(iii) \( \lim_{z \to \infty} \Phi(z) = 1 \).

(iv) The function \(-\Phi\) is a Herglotz (Nevanlinna) function in \( C_+ \), i.e. \( \text{Im} \Phi(z) < 0 \) for all \( z \in C_+ \).

Proof. Statements (i), (ii) are trivial, statement (iii) was stated and proved in Lemma 4.3 above.

Let us prove statement (iv). Let \( \text{Arg} \) denote the principal value of the argument, taking values in the interval \( (-\pi, \pi] \). It is easy to see that for \( z \in C_+ \)

\[- \text{Arg} \left( \frac{z - \mu_k^2}{z - \lambda_k^2} \right) = - \text{Arg} \left( \frac{\mu_k^2 - z}{\lambda_k^2 - z} \right) = \alpha_k(z)\]

where \( \alpha_k(z) > 0 \) is the aperture of the angle at which an observer at the point \( z \in C_+ \) sees the interval \([\mu_k^2, \lambda_k^2]\). Since for the whole real line \( \mathbb{R} \) the aperture is \( \pi \) and the intervals \([\mu_k^2, \lambda_k^2]\) do not intersect \( \mathbb{R}_- = (-\infty, 0) \), we can conclude that

\[0 < - \sum_{k \geq 1} \text{Arg} \left( \frac{\mu_k^2 - z}{\lambda_k^2 - z} \right) < \pi,\]

so \(-\text{Im} \Phi(z) > 0 \) for all \( z \in C_+ \).

Lemma 4.6. The function \( \Phi \) defined by (4.10) is the only analytic in \( C \setminus \sigma \) function satisfying properties (i)–(iv) of Lemma 4.5.

Proof. Let \( \Phi_1 \) be another such function. Both functions have simple poles at \( \lambda_k^2 \) and simple zeroes at \( \mu_k^2 \) (and these are the only isolated singularities and zeroes for both functions), hence \( \Psi(z) := \Phi(z) / \Phi_1(z) \) is analytic and zero-free in \( C \setminus \{0\} \).

Moreover, for \( x \in \mathbb{R} \setminus \{0\} \) we have \( \Psi(x) > 0 \). Indeed, on \( \mathbb{R} \setminus \sigma \setminus \{1\} \) the functions \( \Phi_{1,2} \) are real and have the same sign, so \( \Psi(x) > 0 \) on \( \mathbb{R} \setminus \sigma \setminus \{1, \sigma\} \). Since \( \Psi \) is continuous and zero-free on \( \mathbb{R} \setminus \{0\} \), this tells us that \( \Psi \) is positive on \( \mathbb{R} \setminus \{0\} \).

Next, let us notice that \( \Psi(z) \) does not take negative real values. If \( \text{Im} z > 0 \), then \( \text{Im} \Phi(z) > 0 \), \( \text{Im} \Phi_1(z) > 0 \), so \( \Psi(z) = \Phi(z) / \Phi_1(z) \) cannot be negative real. If \( \text{Im} z < 0 \), the symmetry \( \Psi(z) = \overline{\Psi(z)} \) implies the same conclusion. And, as we just discussed above, on the real line \( \Psi \) takes positive real values.

So, \( \Psi \) omits infinitely many points, therefore by the Picard’s Theorem the point 0 is not an essential singularity for \( \Psi \). Trivial analysis shows that 0 cannot be a pole, otherwise \( 1/\Psi \) is analytic at 0, which also contradicts to the fact that \( \Psi \) can’t take negative real values. Hence the point 0 is a removable singularity for function \( \Psi \), so \( \Psi \) is an entire function. By Liouville’s Theorem, condition \( \Psi(\infty) = 1 \) implies that \( \Psi \equiv 1 \) for all \( z \in \mathbb{C} \), so \( \Psi \equiv \Psi_1 \).

We can avoid using Picard’s theorem by considering the square root \( \Psi^{1/2} \), where we take the principal branch of the square root (cut along the negative half-axis). Since \( \Psi \) does not take negative real values, the function \( \Psi^{1/2} \) is well defined (and analytic) on \( \mathbb{C} \setminus \{0\} \). Trivially
Re $\Psi(z)^{1/2} \geq 0$, so by the Casorati–Weierstrass Theorem $0$ cannot be the essential singularity for $\Psi^{1/2}$. Again, trivial reasoning shows that $0$ cannot be a pole, so again, $\Psi^{1/2}$ is an entire function. The condition $\Psi^{1/2}(\infty) = 1$ then implies that $\Psi^{1/2}(z) \equiv 1$. □

4.2.4. Computing $F$ and the spectral measure $\rho$. Now the proof of the first part of Theorem 4.1 (existence and uniqueness of the triple $W$, $W_1$, $p$) is almost completed. Namely, we define $F := 1 - \Phi$, where $\Phi$ is defined by (4.10). The function $F_1$ defined by (4.6) has poles exactly at the points $\mu_k^2$. Therefore, if we show that $F$ is the Cauchy transform (4.9) of the measure $\rho$ of form (4.3), then according to the discussion at the end of Section 4.2.1 we get the existence of the triple $W$, $W_1$, $p$.

The uniqueness of the triple $W$, $W_1$, $p$, i.e. the uniqueness of the measure $\rho$ follows from the uniqueness of the function $\Phi$, see Lemma 4.6. Namely, if $F$ is the Cauchy transform (4.9) of the measure $\rho$ of form (4.3), and $F_1$ given by (4.6) has poles exactly at the points $\mu_k^2$, then the function $\Phi := 1 - F$ satisfies the properties (i)–(iv) of Lemma 4.5. But by Lemma 4.6 such function $\Phi$ is unique, and so are the function $F$ and so the measure $\rho$.

The fact $F$ is indeed the Cauchy transform of an appropriate measure $\rho$ can be obtained from the general theory of Herglotz functions. However, we will not use a general theory, but present an elementary proof, see the lemma below.

$$F(z) = \int \frac{d\rho(s)}{s - z},$$

and that the measure $\rho$ is supported on $\sigma$.

**Lemma 4.7.** The function $\Phi$ defined by (4.10) can be decomposed as

$$\Phi(z) = 1 - \sum_{n \geq 1} \frac{a_n}{\lambda_n^2 - z},$$

where

$$a_n = (\lambda_n^2 - \mu_n^2) \prod_{k \neq n} \left( \frac{\lambda_n^2 - \mu_k^2}{\lambda_n^2 - \lambda_k^2} \right)$$

(4.15)

**Proof.** Consider functions $\Phi_N$

$$\Phi_N(z) = \prod_{k=1}^{N} \left( \frac{z - \mu_k^2}{z - \lambda_k^2} \right)$$

Trivially

$$\Phi_N(z) = 1 - \sum_{n \geq 1} a_n^N \frac{z - \lambda_n^2}{\lambda_n^2 - z}$$

(4.16)

where

$$a_n^N = (\lambda_n^2 - \mu_n^2) \prod_{k=1}^{N} \left( \frac{\lambda_n^2 - \mu_k^2}{\lambda_n^2 - \lambda_k^2} \right) \quad \text{if } n \leq N,$$

and $a_n^N = 0$ if $n > N$.

---

2a computation will be needed to show that $\rho$ does not have a mass at 0.
We know, see Lemma 4.3 that $\Phi_N(z)$ converges to $\Phi(z)$ uniformly on compact subsets of $\mathbb{C} \setminus \sigma$. Hence, to prove the lemma it remains to show that

$$\sum_{n=1}^{N} \frac{a_n^N}{\lambda_n^2 - z} \to \sum_{n \geq 1} \frac{a_n}{\lambda_n^2 - z} \quad \text{as } N \to \infty$$

uniformly on compact subsets of $\mathbb{C} \setminus \sigma$.

Take $z = 0$ in (4.16). Then

$$1 - \sum_{n \geq 1} \frac{a_n^N}{\lambda_n^2} = \prod_{k=1}^{\infty} \left( \frac{\mu_k^2}{\lambda_k^2} \right) > 0,$$

so $\sum_{n \geq 1} \frac{a_n^N}{\lambda_n^2} \leq 1$.

Notice that for any fixed $n$ the sequence $a_n^N \nearrow a_n$ as $N \to \infty$, so $\sum_{n \geq 1} \frac{a_n}{\lambda_n^2} \leq 1$.

Take an arbitrary compact $K \subset \mathbb{C} \setminus \sigma$. Clearly for any $z \in K$

$$\left| \frac{a_n^N}{\lambda_n^2 - z} \right| \leq \frac{a_n^N}{\text{dist}(K, \sigma)} \leq \frac{a_n}{\text{dist}(K, \sigma)} \cdot \frac{\lambda_n}{\lambda_n^2}$$

so the condition $\sum_{n \geq 1} \frac{a_n}{\lambda_n^2} \leq 1$ implies that the series $\sum_{n \geq 1} \frac{a_n^N}{\lambda_n^2 - z}$ converges uniformly on the compact $K$. □

4.3. **Proof of the abstract Borg’s theorem: the trivial kernel condition.** To complete the proof of Theorem 4.1 is remains to show that the condition $\|W^{-1/2}p\| = 1$ is equivalent to (4.1), and that if (4.1) holds, then the condition $\|W^{-1}p\| = \infty$ is equivalent to (4.2).

Let us first investigate the condition $\|W^{-1/2}p\| = 1$. The operator $W$ is the multiplication by the independent variable $s$ in the weighted space $L^2(\rho)$, where $\rho = \sum_{k \geq 1} a_n \delta_{\lambda_n^2}$ with $a_n$ given by (4.15). Recall, see Lemma 4.7, that

$$\prod_{k=1}^{\infty} \left( \frac{z - \mu_k^2}{z - \lambda_k^2} \right) = 1 - \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2} \quad (4.17)$$

Plugging the real $x < 0$ into (4.17) and taking the limit of both sides as $x \to 0^-$ we get that

$$\prod_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^2} = 1 - \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2},$$

the interchange of limit and sum (product) can be justified, for example, by the monotone convergence theorem.

Therefore $\sum_{k \geq 1} \frac{a_k}{\lambda_k^2} = 1$ if and only if $\prod_{k \geq 0} (\mu_k^2/\lambda_k^2) = 0$. The latter condition is equivalent to

$$\sum_{k \geq 1} \left( 1 - \frac{\mu_k^2}{\lambda_k^2} \right) = \infty,$$

which is exactly the condition (4.1).

Now, let us investigate the condition $\|W^{-1}p\| = \infty$. It can be rewritten as

$$\sum_{k \geq 1} \frac{a_k}{\lambda_k^2} = \infty. \quad (4.18)$$
Rewriting identity (4.17) as
\[
\prod_{k=1}^{\infty} \left( \frac{z - \mu_k^2}{z - \lambda_k^2} \right) = 1 - \sum_{k=1}^{\infty} a_k \frac{(\lambda_k^2 - z) + z}{\lambda_k^2(\lambda_k^2 - z)}
\]
\[
= 1 - \sum_{k \geq 1} a_k \frac{1}{\lambda_k^2} - \sum_{k \geq 1} \frac{a_k z}{\lambda_k^2(\lambda_k^2 - z)},
\]
we see that if the condition (4.1) holds, then
\[
- \frac{1}{z} \prod_{k=1}^{\infty} \left( \frac{z - \mu_k^2}{z - \lambda_k^2} \right) = \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2(\lambda_k^2 - z)}.
\]
Substituting \( z = -\lambda_N^2 \) we get that
\[
\frac{1}{\lambda_N^2} \prod_{k=1}^{\infty} \left( \frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2} \right) = \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2(\lambda_N^2 + \lambda_k^2)}.
\]
By the monotone convergence theorem
\[
\lim_{N \to \infty} \sum_{k \geq 1} \frac{a_k}{\lambda_k^2(\lambda_N^2 + \lambda_k^2)} = \sum_{k \geq 1} \frac{a_k}{\lambda_k^4},
\]
so the condition (4.18) can be rewritten as
\[
\lim_{N \to \infty} \frac{1}{\lambda_N^2} \prod_{k=1}^{\infty} \left( \frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2} \right) = \infty \quad (4.19)
\]
The following simple lemma completes the proof.

**Lemma 4.8.** The condition (4.19) holds if and only if
\[
\sum_{k \geq 1} \left( \frac{\mu_k^2}{\lambda_{k+1}^2} - 1 \right) = \infty.
\]

**Proof.** First of all notice that
\[
0 < C \leq \prod_{k=N}^{\infty} \left( \frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2} \right) \leq 1 \quad (4.20)
\]
with \( C > 0 \) independent of \( N \).

Indeed, for all \( k \geq N \) we trivially have
\[
\frac{1}{2} \leq \frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2} \leq 1.
\]
The upper bound in (4.21) trivially implies the upper bound in (4.20).

To get the lower bound in (4.20) we use the estimate (4.21) and the inequality
\[
\ln x \geq (\ln 2)(x - 1), \quad \forall x \in [1/2, 1].
\]
Thus
\[
\sum_{k=N}^{\infty} \ln 2 \left( \frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2} - 1 \right) = \sum_{k=N}^{\infty} -\ln 2 \frac{\lambda_N^2 - \mu_k^2}{\lambda_N^2 + \lambda_k^2} \geq -\ln 2 \sum_{k=N}^{\infty} \frac{\lambda_N^2 - \mu_k^2}{\lambda_N^2} \geq -\ln 2
\]
we see that the lower bound in (4.20) holds with \( C_1 = \frac{1}{2} \).
The estimate (4.20) implies that the condition (4.19) is equivalent to
\[
\lim_{N \to \infty} \frac{1}{\lambda_N^2} \prod_{k=1}^{N-1} \left( \frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2} \right) = \infty.
\]
Since \( \lambda_N^2 \leq \lambda_N^2 + \lambda_k^2 \leq 2\lambda_N^2 \) for \( k \geq N \), the above condition is equivalent to
\[
\lim_{N \to \infty} \prod_{k=1}^{N-1} \left( \frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2 + 1} \right) = \infty, \tag{4.22}
\]
Denote
\[
G_N(z) := \prod_{k=1}^{N-1} \left( \frac{\mu_k^2 - z}{\lambda_k^2 + 1 - z} \right)
\]
The function \( G_N \) is analytic in the half-plane \( \text{Re } z < \lambda_N^2 \) and satisfies the inequality \( |G_N(z)| \geq 1 \) there. Therefore the function \( \ln |G_N| \) is harmonic and non-negative in the disc \( D_N \) of radius \( 2\lambda_N^2 \) centered at \( -\lambda_N^2 \). So by the Harnack inequality
\[
\frac{1}{3} \ln |G_N(-\lambda_N^2)| \leq \ln |G_N(0)| \leq 3 \ln |G_N(-\lambda_N^2)|.
\]
Note that \( G_N(0) = \prod_{k=1}^{N-1} \mu_k^2 / \lambda_k^2 + 1 \), so the condition (4.22) (which translates to the condition \( \lim_{N \to \infty} G_N(-\lambda_N^2) = \infty \)) is equivalent to
\[
\lim_{N \to \infty} G_N(0) = \lim_{N \to \infty} \prod_{k=1}^{N-1} \frac{\mu_k^2}{\lambda_k^2 + 1} = \infty.
\]
The latter condition is equivalent to the condition
\[
\prod_{k \geq 1} \frac{\mu_k^2}{\lambda_k^2 + 1} = \infty,
\]
which in turn is equivalent to (4.2).

\[\Box\]

**References**

[1] M. S. Birman, M. Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert space*, translated from the Russian by S. Khrushchev and V. Peller, Springer-Verlag, Dordrecht Netherlands, 1987.

[2] G. Borg, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe : Bestimmung der Differentialgleichung durch die Eigenwerte*, Acta Math. 78(1946) 1-96.

[3] P. Gerard, S. Grellier, *The cubic Szegő equation*, Ann. Scient. Éc. Norm. Sup. 43(2010), 761-810.

[4] P. Gerard, S. Grellier, *Invariant Tori for the cubic Szegő equation*, Invent. Math. 187(2012), 707–754.

[5] P. Gérard, S. Grellier, *Inverse spectral problems for compact Hankel Operators*, J. Inst. Math. Jessieu 13(2014), no. 2, 273–301.

[6] P. Gérard, S. Grellier, *The cubic Szegő equation and Hankel operators*, Astérisque 389 (2017).

[7] N. Levinson, *The inverse Sturm-Liouville problem*, Matematisk Tidsskrift. B(1949), pp.25–30.

[8] A. V. Megretski, V. V. Peller, and S. R. Treil’, *The inverse spectral problem for self-adjoint Hankel operators*, Acta Math. 174 (1995), no. 2, 241–309.

[9] N. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 1, Hardy, Hankel, and Toeplitz*, Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002, Translated from the French by A. Hartmann and revised by the author.

[10] N. K. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 2, Model operators and systems*, Mathematical Surveys and Monographs, vol. 93, American Mathematical Society, Providence, RI, 2002, Translated from the French by A. Hartmann and revised by the author.
[11] B. Sz.-Nagy, C. Foias, H. Bercivici, and L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space. Second Edition*

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02912, USA
*Email address: zhehui_liang@brown.edu*

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02912, USA
*Email address: treil@math.brown.edu*