QUANTUM GEOMETRY OF BOOLEAN ALGEBRAS AND DE MORGAN DUALITY

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Abstract. We take a fresh look at the geometrization of logic using the recently developed tools of ‘quantum Riemannian geometry’ applied in the digital case over the field \( F_2 = \{0, 1\} \), extending de Morgan duality to this context of differential forms and connections. We look in detail at the Boolean algebra on a set of 3,4 points. The triangle case in particular is known to have curved quantum geometries with conserved Einstein tensor as part of a classification of all geometries over \( F_2 \) to dimension 3 in recent work.

1. Introduction

Boolean algebra essentially encodes the algebra of subsets with intersection and union. It is also the model for propositional logic where \( a \subseteq b \) appears as \( a \Rightarrow b \) for entailment. Here we demonstrate the robustness of the recently developed theory of quantum Riemannian geometry\cite{14, 5} by showing that it reduces in the case of a Boolean algebra to a reasonable theory. As de Morgan duality – interchanging a set with its complement and \( \cap \) with \( \cup \) or \( a \Rightarrow b \) to \( \overline{b} \Rightarrow \overline{a} \) in propositional terms – is one of the most famous features of Boolean algebra, it becomes an interesting question if this extends to the quantum geometry.

We will see that it does. Although the present work will be mathematics, this question can also be loosely motivated from physics as follows. Indeed, some 30 years ago in \cite{12} I proposed that if one regarded a Boolean algebra as the simplest ‘theory of physics’ then one could ask what became of de Morgan duality -- in more advanced theories. I argued that while clearly broken by quantum theory and gravity alone (for example, apples curve space but the presence of not-apples, meaning the absence of applies, does not) such a duality but might re-emerge as a symmetry of quantum gravity. This is meant to be thought-provoking speculation rather than something understood, but the idea is that we might say that a region of space is ‘as full of apples’ as GR allows (forming a black hole and expanding if we put more apples in) while someone else using the dual picture might say that this same region of space was as empty of not-apples as their quantum field theory allows (where in QFT space is never completely empty in some sense due to vacuum fluctuations). One can go further \cite{13} and introduce the dual to Schroedinger’s cat. Just as the latter is in a mixed state that is neither dead or alive, I proposed co-Schroedinger’s cat as a cat falling into a black hole. This is both dead in finite proper...
time and alive forever in the frame of the observer at infinity. In other words, while quantum theory is intuitionistic as in a Heyting algebra, where we relax the rule that \( a \cup \bar{a} = \text{everything} \), gravity might be expected to be co-intuitionistic in character in the de Morgan dual sense, as in a coHeyting algebra where we relax the rule that \( a \cap \bar{a} = \emptyset \). The latter has also been proposed for other reasons in \[10\] as geometric in nature with \( \partial a = a \cap \bar{a} \) a kind of boundary of \( a \). This then requires both effects or quantum gravity for the symmetry to be maintained. These ideas are not to be taken too literally but they suggest the glimmer of an idea that could be worth exploring and which meanwhile is one of our motivations.

It should be mentioned that at the time of \[12\], such duality ideas motivated the view that quantum gravity needs geometry that is at the same time quantum or noncommutative, with the duality realised slightly differently in concrete ‘toy models’ \[11\] as observer-observed, representation theoretic and Hopf algebra duality. The bicrossproduct quantum groups associated to Lie group factorisations emerging from this as well as the Drinfeld-Jimbo one q-deforming complex simple Lie groups contributed to a concrete ‘constructive’ approach to such quantum Riemannian geometry and included the first convincing model \[20\] of quantum spacetime with quantum symmetry. This is somewhat different from Connes’ approach to ‘noncommutative geometry’ \[6\] founded in cyclic cohomology and spectral triples or ‘Dirac operators’ but not incompatible with it \[4\]. In recent years it was developed particularly (but not only) with bimodule connections \[8\] in a series of works with Beggs as covered in the book \[5\] and lecture notes \[14\]. See also some of the recent literature such as \[3, 15, 17, 16, 19, 1\]. It is this approach which we will use.

Particularly, \[2, 18, 19\] already showed that quantum Riemannian geometry in this form specialises nontrivially over the field \( \mathbb{F}_2 = \{0, 1\} \). Here \[19\] classified such ‘digital quantum geometries’ for algebras up to dimension 3 while \[2\] constructed some first quantum geometries of algebra dimension 4. We therefore are in position to revisit our old idea about de Morgan duality back in its original setting of Boolean algebras. While colourfully motivated as above, this article will be limited to some self-contained mathematics but which will include elements of gravity in the loose sense of a curved metric and an element of quantum theory in the minimal sense that differential forms on Boolean algebras do not commute with algebra elements. \( \mathbb{F}_2 \) geometry is also interesting in its own right \[2\] and could have other applications, such as to the transfer of geometric ideas to digital electronics \[18, 19\], providing another reason to consider the Boolean algebra case and within it de Morgan duality. We will also put the latter into a wider context beyond the Boolean case.

In this paper \( \bar{\cdot} \) will always denote complementation or its generalisation (not complex conjugation) and indeed we work exclusively over \( \mathbb{F}_2 \) so that all algebra elements are their own additive inverse. We will recall the quantum Riemannian geometry formalism as we progress, but for orientation purposes suffice it to say here that if \( A \) plays the role of coordinate algebra then its extension to an exterior algebra of differential forms \( (\Omega, d) \) plays the role of a differential structure. A metric is an element \( g \in \Omega^1 \otimes_A \Omega^1 \) and a linear connection is a map \( \nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1 \) (evaluating the first factor of the output against a vector field would give a covariant derivative along the vector field). The simplest setting is to require the connection to respect the fact that one can multiply \( \Omega^1 \) by \( A \) from either side (it
is a ‘bimodule’). Such a ‘bimodule connection’ involves a ‘generalised braiding’ \( \sigma : \Omega^1 \otimes_A \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1 \) as in [8]. In this case a quantum Levi-Civita connection or QLC is a bimodule connection which is torsion free and metric compatible. The discrete nature of atomic Boolean algebras means that their differential structure at the level of \( \Omega^1 \) is described by a graph[15]. Our first task is to translate these quantum geometry ideas back to the language of subsets.

2. Boolean algebras and de Morgan duality for differentials

There are several ways to generalise Boolean algebras, one of which is to view them as unital algebras \( A \) over \( \mathbb{F}_2 \). The latter just means a unital ring such that every element is its own additive inverse. Among these, Boolean algebras are characterised as having all elements square to themselves. We limit ourselves to the atomic case of power sets.

2.1. Differential Venn diagrams. If \( X \) is a set then \( A = P(X) \) its power set of subsets with product given by \( \cap \) and addition given by exclusive OR \( a \oplus b = (a \cup b) \cap \overline{a} \cap \overline{b} \) can be identified with \( \mathbb{F}_2(X) \) (the algebra of \( \mathbb{F}_2 \)-valued functions) via the characteristic function \( \chi \). Thus, as functions,

\[
\begin{align*}
\chi_\emptyset &= 0, & \chi_X &= 1, & \chi_a \chi_b &= \chi_{a \cap b}, & \chi_a + \chi_b &= \chi_{a \oplus b}
\end{align*}
\]

where \( a, b \in X \). In this way we view Boolean algebras as function algebras over \( \mathbb{F}_2 \) and can do geometry on them using the tools of quantum differential geometry, which apply also to any algebra over any field in the approach of [5].

For any algebra we define a differential structure on \( A \) as a bimodule \( \Omega^1 \) of ‘1-forms’ and a map \( d : A \rightarrow \Omega^1 \) obeying the Leibniz rule and such that \( AdA = \Omega^1 \). There is also a reasonable notion of diffeomorphism \( A \rightarrow B \) between algebras with differentials namely algebra maps that extend to bimodule maps between the corresponding \( \Omega^1 \), forming a commuting square with \( d \).

In the case of functions on a discrete space \( X \), the possible \( \Omega^1 \) are classified by the possible directed graphs with vertex set \( X \). So from now on we fix both a set \( X \) and a choice of set Arr = \( \{ x \rightarrow y \} \) of arrows between some distinct elements of \( X \). Here \( \Omega^1 \) has basis labelled by the arrows and over \( \mathbb{F}_2 \) each basis element appears or doesn’t appear in an element \( \omega \in \Omega^1 \), so we can identify \( \Omega^1(P(X)) = P(\text{Arr}) \) as the set of subsets of the arrow set of the graph with its \( \oplus \) addition law as a Boolean algebra in its own right. Translating the usual finite-difference formulae in [15] back to \( P(X) \) we then find the following noncommutative bimodule and differential structure:

\[
\begin{align*}
(a \cap \omega) &= \{ \text{arrows in } \omega \text{ with tail in } a \}, & (\omega \cap a) &= \{ \text{arrows in } \omega \text{ with tip in } a \} \\
da &= \{ \text{arrows with one end in } a \text{ and other end in } \overline{a} \}
\end{align*}
\]

where \( a \subseteq X \) and \( \overline{a} \) is its complement. We extend the usual meaning of \( \cap \) to apply between subsets of \( X \) and subsets of Arr as indicated (but note that this is not commutative) and we use these extensions for the bimodule product, so \( a \omega = a \cap \omega \) and \( \omega a = \omega \cap a \). Thus \( da \) is the set of arrows that cross the boundary of \( a \) in a
Here \( da \cap b \) are arrows in \( da \) with tips in \( b \) and \( a \cap db \) are arrows in \( db \) with tails in \( a \). From their union we exclude those in their intersection, leaving the boxed arrows with one end in \( a \cap b \) and one end out, i.e. \( d(a \cap b) \).

Venn diagram. It’s a nice check using Venn diagrams that \( d \) is indeed a derivation, see Figure 1. This property in terms of \( \cap, \cup \) on \( P(\text{Arr}) \) is

\[
d(a \cap b) = (da \cap b) \oplus (a \cap db) = ((da \cap b) \cup (a \cap db)) \cap (da \cap b) \cap (a \cap db)
\]

where \((da) \cap b \cap (a\cap db) \) means arrows that cross both \( a \) and \( b \) boundaries and have tip in \( b \) and tail in \( a \), i.e. the two show that connect \( a \cap b \to \bar{a} \cap b \) in the figure; we exclude these.

We also define \( \theta := \text{Arr} \) as the identity element of \( P(\text{Arr}) \) and then each subset \( a \) partitions the set of all arrows as

\[
\theta = \text{Arr} = (a \cap \theta \cap a) \oplus da \oplus (\bar{a} \cap \theta \cap \bar{a})
\]

into subsets of arrows that, respectively, lie entirely within \( a \) (i.e. the restricted graph on \( a \cap X \)), or cross the boundary, or arrows that lie entirely outside \( a \) (i.e. the restricted graph on \( \bar{a} \cap X \)). Moreover,

\[
da = (\theta \cap a) \oplus (a \cap \theta) = \theta.a + a.\theta
\]

in a more algebraic language for the bimodule products and addition, i.e. the calculus is inner via \( \theta \) in the sense of [5].

We will also need a full exterior algebra \( \Omega \). These can be obtained by the ‘maximal prolongation’ of \( \Omega^1 \) (basically, products of 1-forms modulo some minimal set of relations) followed by further quotients of our choice. The latter requires further data as follows. Firstly

\[
\Omega^1 \otimes_{P(X)} \Omega^1 = P(\text{Arr}^{(2)}), \quad \omega \otimes_{P(X)} \eta = \{2 \text{ - steps starting in } \omega \text{ and ending in } \eta\}
\]

where \( \text{Arr}^{(2)} = \{x \to y \to z\} \) denotes the set of 2-step arrows in \( X \), i.e. \( \otimes_{P(X)} \) is the concatenation of compatible arrows. This is a \( P(X) \)-bimodule with \( an \) and \( na \)
defined as in (2.1) with ‘tail’ and ‘tip’ now referring to the initial tail or the final tip. Let \( p(\text{Arr}^{(2)}_q) \) denote the set of 2-step arrows between fixed \( p, q \) and consider the collections of subsets

\[
(2.2) \quad N_{\text{max}} = \{ p(\text{Arr}_q^{(2)}) | p \neq q \} \subseteq N_{\text{med}} = \{ p(\text{Arr}_q^{(2)}) | p \neq q \} \subseteq N_{\text{min}} = \{ p(\text{Arr}_q^{(2)}) \}
\]

where the first collection runs over \( p \neq q \) for which there is no arrow \( p \rightarrow q \). We then define \( \Omega^2_{\text{max}}, \Omega^2_{\text{med}}, \Omega^2_{\text{min}} \) by a quotient of \( P(\text{Arr}^{(2)}) \) by an equivalence relation where \( \omega \sim \eta \) if \( \omega \oplus \eta \) is the union of a subset of the relevant collections \( N \). One can extend this to all forms but we will need only \( \Omega^2 \). According to (2.3) the max one is the maximal prolongation and the other two are successive quotients. The latter two are inner with the same \( \theta \) as above. Once we have specified the 2-forms we set

\[
(2.3) \quad d\omega = \{ 2 - \text{steps where one step is in } \omega \text{ and the other step is not} \}
\]

but with the output viewed up to the chosen equivalence. One can check for example that

\[
ddaa = \emptyset
\]

for all \( a \in X \). Here the left hand side consists of all 2-steps where one step crosses the boundary of \( a \) and the other does not cross the boundary of \( a \). If we fix \( p \in a \) and \( q \in a \), for example and if there is such a 2-step \( p \rightarrow x \rightarrow q \) then all \( x \) meet the criterion so all of \( p(\text{Arr}_q^{(2)}) \) is included. Similarly for \( p \in a \) and \( q \in a \).

2.2. De Morgan duality for differential forms. The classical de Morgan’s theorem is that in any equality in Boolean algebra we can swap

\[
a \leftrightarrow \bar{a}, \quad \cap \leftrightarrow \cup, \quad 1 = X \leftrightarrow 0 = \emptyset
\]

and still have a valid equality. This translates into propositional logic for example as \( a \Rightarrow b \) if and only if \( \bar{b} \Rightarrow \bar{a} \). In this section we want to see how this duality extends to differential forms.

The first thing to note is that complementation does not respect addition by \( \oplus \) on \( P(X) \) so it cannot be expressed as any kind of operator on this as a vector space over \( \mathbb{F}_2 \). Rather, we define \( \bar{P}(X) \) as again the power set of subsets of \( X \) but now with product given by \( \cup \) and addition given by the de Morgan dual exclusive OR (built using \( \cap, \cup \) swapped), namely what we call inclusive AND,

\[
a \odot b := a \cup b, \quad a \oplus b := (a \cap b) \cup \bar{a} \cap \bar{b} = (a \cap b) \cup (\bar{a} \cap \bar{b}) = (a \cup b) \cap \bar{a} \cap \bar{b} = \bar{a} \oplus b.
\]

One can check that this again makes the power set of \( X \) into an algebra over \( \mathbb{F}_2 \) (as it must by de Morgan’s theorem) and that we now have an isomorphism of algebras

\[
\sim: P(X) \rightarrow \bar{P}(X).
\]

Here \( \bar{a} \oplus \bar{b} = a \oplus b \) so that \( \bar{a} \oplus \bar{b} = \bar{a} \oplus \bar{b} = a \oplus b \) as required for linearity over \( \mathbb{F}_2 \).

Next, define the 1-forms \( \Omega^1 := \Omega^1(\bar{P}(X)) := \bar{P}(\text{Arr}) \) meaning its addition law is by \( \oplus \) of subsets of arrows, with bimodule structure and exterior derivative

\[
a \odot \omega = \{ \text{arrows in } \omega \text{ or with tail in } a \}, \quad \omega \odot a = \{ \text{arrows in } \omega \text{ or with tip in } a \}
\]

(2.4)

\[
d\omega = \{ \text{arrows wholly in } a \text{ or wholly in } \bar{a} \} = \bar{d}a
\]

where complementation of a subset of arrows is in \( \text{Arr} \). We extended \( \cup \) to apply between subsets of \( X \) and subsets of \( \text{Arr} \) as stated and a little thought shows that

\[
(2.5) \quad \bar{a} \cup \bar{\omega} = \bar{a} \cap \bar{\omega}, \quad \bar{\omega} \cup \bar{a} = \bar{\omega} \cap \bar{a}.
\]
Figure 2. Venn diagram to check that \( \tilde{\partial}(a \cup b) = (\tilde{\partial}a \cup b) \oplus (a \cup \tilde{\partial}b) \).

Here solid \( \partial a \cup b \) are arrows wholly in or out of \( a \) or have tips in \( b \), while dotted \( a \cup \tilde{\partial}b \) are arrows wholly in or out of \( b \) or have tails in \( a \). The parallel arrows are in both subsets and we see that they are almost all the arrows wholly in or out of \( a \cup b \), i.e. almost \( \tilde{\partial}(a \cup b) \).

We need to add the missing type of arrow shown dashed which is not in the union of all the other arrows shown.

We use this extended \( \cup \) for the bimodule structure of \( \tilde{P}(X) \), so \( \bar{a} \omega = a \cup \omega \) and \( \omega \bar{a} = \omega \cup a \). Figure 2 checks that this indeed obeys the derivation rule for a differential calculus on \( \tilde{P}(X) \). However, this must be the case by de Morgan’s theorem in view of the symmetry between \( \cup \) and \( \cap \). In terms of \( \cup, \cap \) this is

\[
\tilde{\partial}(a \cup b) = (\tilde{\partial}a \cup b) \oplus (a \cup \tilde{\partial}b) = ((\tilde{\partial}a \cup b) \cap (a \cup \tilde{\partial}b)) \cup (\tilde{\partial}a \cup b) \cup (a \cup \tilde{\partial}b).
\]

Here \( (\tilde{\partial}a \cup b) \cup (a \cup \tilde{\partial}b) \) means arrows wholly on our out of \( a \) or with tip in \( b \) or wholly in or out of \( b \) or with tail in \( a \).

One can also check that this calculus is inner with \( \bar{\theta} = \emptyset \) of arrows. Thus

\[
(\emptyset \cup a) \oplus (a \cup \emptyset) = \{ \text{arrows with tip in } a \} \oplus \{ \text{arrows with tail in } a \} = \{ \text{arrows wholly in } a \text{ or } \bar{a} \} = \bar{a} \delta a
\]

using the above definition of \( \oplus \).

**Proposition 2.1.** Let \( X \) be a graph. The algebra isomorphism \( \gamma : P(X) \rightarrow \tilde{P}(X) \) with their respective differential structures is a diffeomorphism.

**Proof.** The key observation is that \( \partial, \tilde{\partial} \) as defined are symmetric between \( a, \bar{a} \), so in particular we have \( \bar{\partial}a = \partial a = \bar{a} \delta a \) so that complementation of arrows forms a commutative diagram

\[
P(\text{Arr}) \xrightarrow{\tilde{\partial}} \tilde{P}(\text{Arr})
\]

\[
\text{d} \uparrow \uparrow \tilde{\partial}
\]

\[
P(X) \rightarrow \tilde{P}(X).
\]

The top map is a bimodule map in the sense \( \bar{\partial} \omega = \bar{a} \cdot \omega \) and similarly on the other side, by the observation \( (2.5) \) already given. \( \square \)
Next, whereas \( \omega \otimes_{P(X)} \eta \) is the set of possible concatenations or a kind of intersection of a tip in \( \omega \) and a tail in \( \eta \), we define the dual \textit{coconcatenation} of subsets of arrows

\[
\omega \otimes_{P(X)} \eta = \{ 2 - \text{steps starting in } \omega \text{ or ending in } \eta \} = \overline{\omega \otimes_{P(X)} \eta}
\]

and one can check that

\[
(\omega \cup a) \otimes_{P(X)} \eta = \omega \otimes_{P(X)} \eta (a \cup \eta)
\]

as both sides are arrows that start in \( \omega \) or end in \( \eta \) or have middle vertex in \( a \). For the addition law we use \( \oplus \) and we define a bimodule structure on 2-steps by extending \( \omega \oplus \eta \) and \( \omega \oplus a \) in (2.2) to 2-steps with ‘tail’ and ‘tip’ referring to the initial tail or the final tip. We still have (2.5) with this extension. In this way we identify

\[
\overline{\Omega^1} \otimes_{P(X)} \overline{\Omega^1} = \overline{\Omega^1} \otimes_{P(X)} \overline{\Omega^1}
\]

and by construction the complementation map

\[
\bar{\cdot} : P(\text{Arr}^{(2)}) = \overline{\Omega^1} \otimes_{P(X)} \overline{\Omega^1} \to \overline{\Omega^1} = \overline{\Omega^1} \otimes_{P(X)} \overline{\Omega^1}
\]

intertwines the bimodule structures in same way as for \( \bar{\cdot} : \Omega^1 \to \overline{\Omega^1} \) in Proposition 2.1, namely \( \bar{\cdot} a \cap \omega \otimes_{P(X)} \eta = \bar{\cdot} a \cup \omega \otimes_{P(X)} \eta \) and similarly on the other side.

**Lemma 2.2.** \( \bar{\cdot} \) \( \text{descends to the relevant max, med, min prolongations in a way that commutes with } d, \bar{d} \text{ on degree } 1. \)

**Proof.** We let \( N \) be one of the collections (2.2). Its elements are the \( \oplus \) of any subset \( Y \) of the allowed \( (p, q) \) in the relevant collection and such an element maps to \( \oplus_{(p, q) \in Y_p \text{Arr}^{(2)}_q} = \oplus_{(p, q) \in Y_p \text{Arr}^{(2)}_q} \) as the corresponding element of \( N \). The latter is defined by the same collections as \( N \) but with elements constructed in \( P(\text{Arr}^{(2)}) \) using \( \oplus \). Then by construction \( \bar{\cdot} \) descends to \( \Omega^2 \) where we quotient by \( N \) on mapping to \( \Omega^2 \) where we quotient by \( \bar{N} \). That the differentiability diagram for \( d, \bar{d} \) commutes follows similarly to the proof for \( P(X) \) given the form of \( d \) in (2.3) and the dual version

\[
(2.6) \quad \bar{d} \omega = \{ 2 - \text{steps wholly in } \omega \text{ or wholly out} \}.
\]

\( \square \)

3. **Elements of Quantum Riemannian Geometry on \( P(X) \)**

We continue in the case of \( X \) a graph with \( A = P(X) \) and \( \Omega^1 = P(\text{Arr}) \). Now we suppose the graph is bidirected. Then the unique quantum metric is

\[
(3.1) \quad g = \oplus_{p \text{Arr}^{(2)}_p} \in \Omega^1 \otimes_{P(X)} \Omega^1
\]

i.e. all 2-steps that go to another point and come back. Hence \( \wedge g = 0 \) provided we use \( \Omega^2_{\text{min}} \), which we henceforth do.

A connection has to map subsets of \( \text{Arr} \) to subsets of \( \text{Arr}^{(2)} \) subject to certain properties. It is shown in [15], which we now specialise to the \( \mathbb{F}_2 \) case, that bimodule connections are in fact determined here by two bimodule maps \( \alpha : P(\text{Arr}) \to P(\text{Arr}^{(2)}) \) and \( \sigma : P(\text{Arr}^{(2)}) \to P(\text{Arr}^{(2)}) \) as

\[
(3.2) \quad \nabla \omega = \theta \otimes \omega + \sigma(\omega \otimes \theta) + \alpha(\omega).
\]
To be torsion free we need the image of $\alpha, \sigma + \text{id}$ to land in $N_{min}$, i.e. each element in the image should be the union of a subset of the $p\text{Arr}^{(2)}_q$. To be metric compatible, we need
g \sigma_1 \sigma_2 g + (\alpha \otimes \text{id}) g + \sigma_1 (\text{id} \otimes \alpha) g = 0.

### 3.1. Boolean Riemannian geometries on $n$-gons

Here we describe some curved Boolean Riemannian geometries. The ones on $n \leq 3$ points are from [19] but we convert them to our $P(X)$ form where $\nabla: P(\text{Arr}) \to P(\text{Arr}^{(2)})$. The polygon has recently been solved for all $n$ over generic fields in [1]; although not the Boolean case over $\mathbb{F}_2$, it will inform some of our observations for $n = 4$ points.

Let $X$ be a discrete set of $n$ elements. For $\Omega^1$ we fix the $n$-gon graph which means we number the vertices $0, \ldots, n - 1$ and we have $2n$ arrows $i \to i + 1$ and $i \to i - 1$ understood with labels mod $n$. This is also a Cayley graph for the group $\mathbb{Z}_n$ with its generators $\pm 1$, so there are two left-invariant 1-forms when $n > 2$ by a standard construction [5] [14], which we write in subset form

\[ e^+ = \bigoplus_i \{i \to i + 1\} = \{i \to i + 1 \mid i = 0, \ldots, n-1\}; \quad e^- = \bigoplus_i \{i \to i - 1\} = \{i \to i - 1 \mid i = 0, \ldots, n-1\} \]

We regard the 1-forms here as $e^\pm$ as subsets of arrows and a general subset of arrows can be expressed in the form $\omega = (a_+ \cap e^+) \oplus (a_- \cap e^-)$ for some $a_\pm \subseteq P(X)$ which can be recovered from $\omega$ by

\[ a_\pm = \{\text{tails of } \omega \cap e^\pm\} \subseteq X. \]

Next, for $n > 2$ the canonical Cayley graph exterior algebra has $(e^+)^2 = 0$ and $\text{Vol} = e^+ e^- = e^- e^+$ in our case over $\mathbb{F}_2$ is our basis of $\Omega^2$ over the algebra. This agrees with the general graph $\Omega^2_{min}$ construction when $n \neq 4$. In this case the $4n$ 2-step arrows $\text{Arr}^{(2)}$ are partitioned into subsets

\[ \text{Arr}^{(2)}_1 = \{i \to i + 1 \to i, i \to i - 1 \to i\}, \quad \text{Arr}^{(2)}_{1+2} = \{i \to i + 1 \to i + 2\}, \quad \text{Arr}^{(2)}_{1-2} = \{i \to i - 1 \to i - 2\}. \]

The canonical $N_{min}$ sets all of these subsets of arrows as well as all their unions to zero (in the sense of an equivalence relation on $P(\text{Arr}^{(2)})$). Then $\Omega^2$ is $n$-dimensional over $\mathbb{F}_2$ with every element represented as $a \cap \text{Vol}$ where

\[ \text{Vol} = e^+ e^- = e^- e^+ = \bigoplus_i \{i \to i + 1 \to i\} \]

in the quotient and in agreement with the Cayley graph construction. The metric as a subset is

\[ g = e^+ \otimes e^- + e^- \otimes e^+ = \bigoplus_i \{i \to i \pm 1 \to i\} \]

and we see that $\cap (g) = 0$ in the quotient as the two entries for each $i$ are equivalent with respect to $N_{min}$. The $\bigoplus$ components here are mutually disjoint so we could have written $\cup$ in place of $\bigoplus$. Later on, for the Ricci tensor, we will need a lift $\Omega^2 \to \Omega^2 \otimes P(X) \Omega^1$ and we have two natural group-invariant ones

\[ i_+ (\text{Vol}) = e^+ \otimes e^- = \bigoplus_i \{i \to i + 1 \to i\}, \quad i_- (\text{Vol}) = e^- \otimes e^+ = \bigoplus_i \{i \to i - 1 \to i\} \]

amounting to two halves of the metric.

There is an obvious trivial QLC given by $\sigma = \text{flip on the generators and } \alpha = 0$, resulting in $\nabla_0 e^\pm = 0$ with zero curvature. In terms of $P(X)$, $\sigma : P(\text{Arr}^{(2)}) \rightarrow$
\( P(\text{Arr}(2)) \) is given elementwise on subsets by the maps

\[
\sigma_{\text{Arr}(2)}^{(3.8)} = \text{swap}, \quad \sigma_{\text{Arr}(2)}^{(3.9)} = \text{id}, \quad \sigma_{\text{Arr}(2)}^{(3.10)} = \text{id}
\]

where swap gives the other element of the 2-element set. Clearly \( \land \) in the quotient the swap in the first map has no effect. So \( \nabla_0 \) defined by the above \( \sigma \) and \( \alpha = 0 \) in (3.2) is torsion free. For metric compatibility we have

\[
g \otimes \theta = \oplus_p \{ p \to p \pm 1 \to p \to p \pm' 1 \to p \}
\]

where \( \pm' \) are independent so each set has 4 elements. Applying \( \sigma \) as in (3.8) one readily sees that as sets \( \sigma_{\text{Arr}(2)}^{(3.3)}(g \otimes \theta) = \theta \otimes g \) so that \( \nabla_0 \) is metric compatible by (3.3) and hence a QLC.

To see what \( \nabla_0 \) looks like we compute for example

\[
\nabla_0\{i \to i + 1\} = \{i + 1 \to i \to i + 1\} \oplus \sigma(\{i \to i + 1 \to i + 2, i \to i + 1 \to i\})
\]

\[
= \{i \to i + 1 \to i, i - 1 \to i \to i + 1, i + 1 \to i \to i + 1, i \to i + 1 \to i + 2\}
\]

\[
= e^+_i \oplus e^+_{i+1}
\]

where we define

\[
e_i^+ = \{i \to i - 1 \to i, i - 1 \to i \to i + 1\}, \quad e_i^- = \{i \to i + 1 \to i, i + 1 \to i \to i - 1\}
\]

and use \( e_i^\pm \) for a similar result \( \nabla_0\{i \to i\} = e_{i-1}^- \oplus e_i^- \). For the general case let \( \omega \in \text{Arr} \) and

\[
\partial_+ \omega = \{i \in \text{tails of } \omega \mid i - 1 \notin \text{tails of } \omega\} \cup \{i \in \text{heads of } \omega \mid i + 1 \notin \text{heads of } \omega\}
\]

\[
\partial_- \omega = \{i \in \text{heads of } \omega \mid i + 1 \notin \text{heads of } \omega\} \cup \{i \in \text{heads of } \omega \mid i + 1 \notin \text{heads of } \omega\}
\]

where \( \omega = \omega \cap e^\pm \) are the increasing/decreasing arrows of \( \omega \). Then

\[
\nabla_0 : P(\text{Arr}) \to P(\text{Arr}(2)), \quad \nabla_0 \omega = (\oplus_{i \in \partial_+ \omega} e_i^+) \oplus (\oplus_{i \in \partial_- \omega} e_i^-)
\]

as depicted in Figure 3. All components here are disjoint so we could have written \( \cup \) in place of \( \oplus \).

A more general connection is specified by \( \nabla e^\pm = \Gamma^\pm \) and then has the form

\[
\nabla \omega = \nabla_0 \omega \oplus a_+ \cap \Gamma^+ \oplus a_- \cap \Gamma^-
\]

due to \( \omega = a_+ \cap e^+ \oplus a_- \cap e^- \) with \( a_+ \) defined in (3.4). We now describe the known QLCs for small numbers of points.

### 3.2. All QLCs for the triangle with \( n = 3 \) points.

By the classification results in [19] (for the algebra \( \mathcal{B} \) there) there are just four QLCs for this algebra including the above trivial one. They all have the same \( \sigma \) as above, so

\[
\sigma\{0 \to 1 \to 2\} = \{0 \to 1 \to 2\}, \quad \sigma\{0 \to 2 \to 1\} = \{0 \to 2 \to 1\}
\]

\[
\sigma\{0 \to 1 \to 0\} = \{0 \to 2 \to 0\}, \quad \sigma\{0 \to 2 \to 0\} = \{0 \to 1 \to 0\}
\]

and cyclic rotations of these.

(0) We have the above trivial flat connection with \( \alpha = 0 \) and \( \nabla_0 e^\pm = 0 \) or in our terms for example

\[
\nabla_0\{0 \to 1\} = \{0 \to 2 \to 0, 2 \to 0 \to 1, 1 \to 0 \to 1, 0 \to 1 \to 2\}
\]

\[
\nabla_0\{0 \to 1, 1 \to 2\} = \{0 \to 2 \to 0, 2 \to 0 \to 1, 2 \to 1 \to 2, 1 \to 2 \to 0\}
\]

\[
\nabla_0\{0 \to 2\} = \{0 \to 1 \to 0, 1 \to 0 \to 2, 2 \to 0 \to 2, 0 \to 2 \to 1\}
\]
Figure 3. Half of the trivial connection $\nabla_0 \omega$. Here the boxes are $\omega \cap e^+$ and the open circles are the boundary points $\partial \omega$. Each of these contribute two 2-steps $e^+_i$ to $\nabla_0 \omega$ as shown dashed. The other half of $\nabla_0$ is the same construction applied to $\omega \cap e^-$ with boundary points $\partial - \omega$ contributing $e^-_i$. and similarly for a mix of increasing and decreasing arrows.

(1) $\nabla e^+ = e^- \otimes e^-$, $\nabla e^- = 0$ again with zero curvature and in our terms
\[
\nabla \omega = \nabla_0 \omega \oplus \{i \to i - 1 \to i - 2 \mid i \in a_+\}; \quad \nabla \{0 \to 1\} = \nabla_0 \{0 \to 1\} \oplus \{0 \to 2 \to 1\}
\]

etc. where the additional term is $\alpha(\omega)$. For decreasing arrow there is no change.

(2) $\nabla e^- = e^+ \otimes e^+$, $\nabla e^+ = 0$ again with zero curvature and in our terms
\[
\nabla \omega = \nabla_0 \omega \oplus \{i \to i + 1 \to i + 2 \mid i \in a_-\}; \quad \nabla \{0 \to 2\} = \nabla_0 \{0 \to 2\} \oplus \{0 \to 1 \to 2\}
\]

etc. where the additional term is $\alpha(\omega)$. There is no change for increasing arrows.

(3) $\nabla e^- = e^+ \otimes e^+$, $\nabla e^+ = e^- \otimes e^-$ with curvature $R_{\nabla} e^\pm = \text{Vol} \otimes e^\pm$. The connection has both the additional $\alpha$ terms as above so
\[
\nabla \omega = \nabla_0 \omega \oplus \{i \to i - 1 \to i - 2 \mid i \in a_+\} \oplus \{i \to i + 1 \to i + 2 \mid i \in a_-\}
\]
\[
R_{\nabla} \omega = \text{Vol} \otimes \omega
\]
on subsets. For example
\[
\nabla \theta = \oplus_i \{i \to i + 1 \to i - 1, i \to i - 1 \to i + 1\}, \quad R_{\nabla} \theta = \oplus_i \{i \to i + 1 \to i \to i + 1, i \to i + 1 \to i \to i - 1\}
\]
\[
\nabla \{0 \to 1, 2 \to 0\} = \{0 \to 2 \to 1, 2 \to 1, 1 \to 0 \to 0 \to 1, 2 \to 0 \to 1, 0 \to 1 \to 0\}
\]
\[
R_{\nabla} \{0 \to 1, 2 \to 0\} = \{0 \to 1 \to 0, 2 \to 1, 1 \to 0 \to 2, 0 \to 2 \to 1, 0 \to 1 \to 0\}
\]
where the first two steps in the 3-step items are up to equivalence, since
\[
\text{Vol} = \{0 \to 1 \to 0, 1 \to 2 \to 1, 2 \to 0 \to 2\}
\]
up to equivalence. From the Riemann curvature and the choice of lift $i$ we define the Ricci tensor and Ricci scalar by [3, 5]
\[
(3.11) \quad \text{Ricci} = ((, ) \otimes \text{id})(\text{id} \otimes (i \otimes \text{id})R_{\nabla})g, \quad S = (, )\text{Ricci}
\]
where \((e^\pm, e^\mp) = 1\) and \((e^\pm, e^\pm) = 0\) is the inverse metric in our case. The two lifts \(i_\pm\) immediately give us two Ricci tensors with the same Ricci scalar

\[
\text{Ricci}_+ = e^+ \otimes e^+ = \bigoplus \{ i \to i - 1 \to i \}, \quad \text{Ricci}_- = e^- \otimes e^- = \bigoplus \{ i \to i + 1 \to i \}, \quad S = 1
\]

meaning \(S = X\) as an element of \(P(X)\). General lifting maps \(i\) are considered in [19] but the above are the two natural ones in this example. Classically we would average them to to give the canonical lift of \(\text{Vol}\) to an antisymmetric cotensor, but we do not have that luxury here. One approach is to think of \(i_2 = i_+ + i_-\) as twice the classical lift. If we use this in (3.11) then we would have twice the Ricci tensor that we would otherwise have, namely \(\text{Ricci}_2 = \text{Ricci}_+ + \text{Ricci}_- = g\), so in this sense the Boolean algebra with this connection is ‘Einstein’. It is also a good question as to what should be the Einstein tensor and a tentative proposal in [19] for the digital case is

\[
\text{Eins} = \text{Ricci}_+ S g.
\]

It was found in [19] for this model to be non-zero for all lifts \(i\), but with exactly two lifts, which turn out to be \(i_\pm\), for which Eins is conserved in the sense \(\nabla \cdot \text{Eins} = 0\). Clearly, these give

\[
\text{Eins}_\pm = e^\pm \otimes e^\mp = i_\pm(\text{Vol})
\]

and it is easy to verify that they are indeed conserved. For example

\[
\nabla (\text{Eins}_+) = \nabla (e^+ \otimes e^-) = \nabla e^+ \otimes e^- + \sigma_{12}(e^+ \otimes \nabla e^-) = e^- \otimes e^- \otimes e^- + e^+ \otimes e^+ \otimes e^+,
\]

which then vanishes when we contract the first two factors with the inverse metric \((\ , \ )\). Similarly for conservation of Eins..

### 3.3. Some QLCs for \(n = 2, 4\) points.

These are a little exceptional and we consider them briefly.

For 2 points there is just one left-invariant 1-form \(e = \{0 \to 1\} \oplus \{1 \to 0\}\). The possible subsets of arrows are expressed equivalently as subsets of \(X\) as namely

\[
\{0\} \cap e = \{0 \to 1\}, \quad \{1\} \cap e = \{1 \to 0\}, \quad \{0, 1\} \cap e = \{0 \to 1, 0 \to 1\}
\]

and the empty set for the empty set of arrows. The canonical exterior algebra has \(e^2 = 0\) so \(\Omega^2 = 0\). This is also the canonical choice by our arrow construction as \(\text{Arr}_0^{(2)} = \{0 \to 1 \to 0\}\) and \(\text{Arr}_1^{(2)} = \{1 \to 0 \to 1\}\) are both singleton sets and partition \(\text{Arr}^{(2)}\).

The metric is \(g = e \otimes e = \{0 \to 1 \to 0, 1 \to 0 \to 1\} = \text{Arr}^{(2)}\) as a subset of arrows. From the classification in [19], the only QLC is the trivial one \(\nabla_0 e = 0\) and \(R_\nabla = 0\), or in our arrow terms

\[
\nabla \{0 \to 1\} = \nabla \{1 \to 0\} = g, \quad \nabla \text{Arr} = \emptyset.
\]

and \(\sigma = \text{id}\) on \(\text{Arr}^{(2)}\).

For 4 points we have \(i + 2 = i - 2\) with the result that \(\Omega^{2}_{\text{min}}\) has 4 fewer relations namely \(\text{Arr}_{i+2}^{(2)} \sim \emptyset\) have two elements for the two ways to go around the square and their sum is zero (or over \(F_2\) we identify them) rather than separately setting them to zero. This is 2D over \(\mathbb{C}(Z_2)\) and has the relations \(e^2 + e^{-2} = 0\) rather than setting these separately to zero. When we do the latter then we have a quotient
of $\Omega^2$ of $\Omega^2_{\min}$ with $\text{Vol} = e^+e^-$ as central basis. Explicitly, we have a collection $\mathcal{N}$ which partitions $\text{Arr}^{(2)}$ into

\[
\{0 \to 1 \to 0, 0 \to 3 \to 0\}, \quad \{1 \to 2 \to 1, 1 \to 0 \to 1\},
\]
\[
\{2 \to 3 \to 2, 2 \to 1 \to 2\}, \quad \{3 \to 0 \to 3, 3 \to 2 \to 3\},
\]
\[
\{0 \to 1 \to 2\}, \quad \{1 \to 2 \to 3\}, \quad \{2 \to 3 \to 0\}, \quad \{3 \to 0 \to 1\},
\]
\[
\{2 \to 1 \to 1\}, \quad \{3 \to 2 \to 1\}, \quad \{0 \to 3 \to 2\}, \quad \{1 \to 0 \to 3\}.
\]

The quantum metric $g$ is quantum symmetric as it must be since we are in a quotient of $\Omega^2_{\min}$. For $n = 4$ we have not got a complete classification of QLCs but in the inner construction $\theta = e^+ + e^-$ and we must have $\alpha = 0$ for a bimodule map. Then for $\sigma$ we have at least the following five:

(0) The trivial $\nabla$ as above.

(1) From the $\mathbb{F}_2$ limit of generic field calculations for the 4-gon in [1], we have

\[
\sigma(e^+ \otimes e^+) = e^+ \otimes e^+, \quad \sigma(e^+ \otimes e^-) = e^- \otimes e^+
\]
\[
\sigma(e^- \otimes e^+) = e^+ \otimes e^-, \quad \sigma(e^- \otimes e^-) = e^+ \otimes e^+ + e^- \otimes e^-
\]
\[
\nabla e^+ = 0, \quad \nabla e^- = e^+ \otimes e^+
\]

with zero curvature and the same form of $\nabla$ as for $n = 3$ case (1).

(2) Again from the $\mathbb{F}_2$ limit of generic field calculations for the 4-gon in [1],

\[
\sigma(e^+ \otimes e^+) = e^+ \otimes e^+ + e^- \otimes e^-, \quad \sigma(e^+ \otimes e^-) = e^- \otimes e^+
\]
\[
\sigma(e^- \otimes e^+) = e^+ \otimes e^-, \quad \sigma(e^- \otimes e^-) = e^- \otimes e^- 
\]
\[
\nabla e^+ = e^- \otimes e^-, \quad \nabla e^- = 0
\]

with zero curvature and the same form of $\nabla$ as for $n = 3$ case (2).

(3) Again from the $\mathbb{F}_2$ limit of generic field calculations for the 4-gon in [1],

\[
\sigma(e^+ \otimes e^+) = e^- \otimes e^-, \quad \sigma(e^+ \otimes e^-) = e^+ \otimes e^-
\]
\[
\sigma(e^- \otimes e^+) = e^- \otimes e^+, \quad \sigma(e^- \otimes e^-) = e^+ \otimes e^+
\]
\[
\nabla e^+ = \nabla e^- = \theta \otimes \theta
\]

again with zero curvature. In set terms for this connection, $\Gamma^+ = \Gamma^- = \text{Arr}^{(2)}$ as the set of all possible 2-steps. Hence

\[
\nabla \omega = \nabla_{\theta^2} \omega \oplus a_+ \cap \text{Arr}^{(2)} \oplus a_- \cap \text{Arr}^{(2)} = \nabla_{\theta^2} \omega \oplus (a_+ \oplus a_-) \cap \text{Arr}^{(2)}
\]

$a_+ \oplus a_- = \{\text{points which are tails of an increasing or decreasing arrow in } \omega \text{ but not both}\}$. For example, $\nabla \theta = \emptyset$ as $a_+ = a_- = \emptyset$ since all points are tails of one increasing and one decreasing arrow.

(4) In addition, based on the pattern for $n = 3$, we also consider

\[
\sigma(e^+ \otimes e^+) = e^+ \otimes e^+ + e^- \otimes e^-, \quad \sigma(e^+ \otimes e^-) = e^+ \otimes e^+
\]
\[
\sigma(e^- \otimes e^+) = e^+ \otimes e^-, \quad \sigma(e^- \otimes e^-) = e^+ \otimes e^+ + e^- \otimes e^-
\]
\[
\nabla e^+ = e^- \otimes e^-, \quad \nabla e^- = e^+ \otimes e^+
\]

with curvature $R_{\nabla}e^\pm = \text{Vol} \otimes e^\pm$ and Ricci as for $n = 3$ case (3) and the same form of $\nabla$ as there. However, due to the different form of $\sigma$, this is not a QLC but has obeys a weaker metric compatibility

$$\text{coT}_\nabla = (d \otimes \text{id} - \text{id} \wedge \nabla)g = e^+ \wedge \nabla e^- + e^- \wedge \nabla e^+ = (e^+)^2 \otimes e^- + (e^-)^2 \otimes e^+ = 0$$

When this ‘cotsion’ and the torsion both vanish then one has a WQLC [5, 11] and we see that this arises naturally for $n = 4$. As for $n = 3$ one can make this explicit our set theory terms.

An alternate $\Omega^2$ for $n = 4$ is also possible as in [10], namely the same square can be viewed as a Cayley graph for $\mathbb{Z}_2 \times \mathbb{Z}_2$ leading to different invariant 1-forms

$$e^1 = \{00 \rightarrow 10, 10 \rightarrow 00, 01 \rightarrow 11, 11 \rightarrow 01\}, \quad e^2 = \{00 \rightarrow 01, 01 \rightarrow 00, 10 \rightarrow 11, 11 \rightarrow 10\}.$$ 

If we identify the vertices by $00 = 0, 01 = 1, 11 = 2, 10 = 3$ then this is

$$e^1 = \{0 \rightarrow 3, 3 \rightarrow 0, 1 \rightarrow 2, 2 \rightarrow 1\}, \quad e^2 = \{0 \rightarrow 1, 1 \rightarrow 0, 3 \rightarrow 2, 2 \rightarrow 3\}.$$

The quotient of the maximal prolongation this time requiring $e^i$ to anticommute means a collection $\mathcal{N}$ consisting of the partition of $\text{Arr}^{(2)}$ into

$$\{0 \rightarrow 3 \rightarrow 0\}, \quad \{3 \rightarrow 0 \rightarrow 3\}, \quad \{1 \rightarrow 2 \rightarrow 1\}, \quad \{2 \rightarrow 1 \rightarrow 2\},$$

$$\{0 \rightarrow 1 \rightarrow 0\}, \quad \{1 \rightarrow 0 \rightarrow 1\}, \quad \{2 \rightarrow 3 \rightarrow 2\}, \quad \{3 \rightarrow 2 \rightarrow 3\},$$

$$\{0 \rightarrow 3 \rightarrow 2, 0 \rightarrow 1 \rightarrow 2\}, \quad \{1 \rightarrow 2 \rightarrow 3, 1 \rightarrow 0 \rightarrow 3\},$$

$$\{2 \rightarrow 1 \rightarrow 0, 2 \rightarrow 3 \rightarrow 0\}, \quad \{3 \rightarrow 0 \rightarrow 1, 3 \rightarrow 2 \rightarrow 1\}$$

with unique but not central volume form

$$\text{Vol} = e^1 e^2 = \{0 \rightarrow 3 \rightarrow 2, 3 \rightarrow 0 \rightarrow 1, 1 \rightarrow 2 \rightarrow 3, 2 \rightarrow 1 \rightarrow 0\}.$$ 

There is again a trivial flat QLC with $\alpha = 0$ and $\sigma = \text{flip}$ on the invariant 1-forms, so that $\nabla_\alpha e^i = 0$. This is the only $\mathbb{F}_2$ limit of the generic field analysis in [10], but one could expect at least another flat one.

Similarly for polygons with $n > 5$, the generic field analysis in [11] does not yield another QLC beyond $\nabla_0$ when restricted to $\mathbb{F}_2$, but one could expect at least another flat one and likely others. A full classification as for $n = 3$ in [10] is currently beyond reach by the methods there due to computer limitations.

### 3.4. De Morgan dual connections on the polygon.

For de Morgan duality we note for general $n$ that $\nabla_0(\omega \oplus \bar{\omega}) = \nabla_0\theta = \emptyset$, hence $\nabla_0\bar{\omega} = \nabla_0\omega$ and

$$\nabla_0\omega = \nabla_0\bar{\omega} = \nabla_0\omega \oplus \text{Arr}^{(2)}.$$ 

One also has $\nabla\bar{\omega} = \nabla\omega$ and hence the same conclusion $\nabla\bar{\omega} = \nabla\omega \oplus \text{Arr}^{(2)}$ for the flat QLC case (3) for $n = 4$ case. We now focus on the curved QLC connection (3) for $n = 3$. Here

$$\nabla\bar{\omega} = \nabla\omega \oplus \{i \rightarrow i + 1 \rightarrow i - 1, i \rightarrow i - 1 \rightarrow i + 1 \mid i = 0, 1, 2\}$$

which in turn tells us that the corresponding connection on $\tilde{P}(X)$ is

$$\nabla\omega := \nabla\bar{\omega} = \nabla\omega \oplus \text{Arr}^{(2)} = \nabla\omega \oplus g$$

where $g$ is the metric and the result is viewed in $\tilde{P}(\text{Arr}^{(2)})$. 

For the curvature we first note that \( \text{Vol} = \{ i \rightarrow i + 1 \rightarrow i \mid i = 0, 1, 2 \} \oplus \text{Arr}^{(2)} \) modulo the union of subsets of the collections

\[
\mathcal{N}_{\text{min}} = \{ \{ 0 \rightarrow 1 \rightarrow 0, 0 \rightarrow 2 \rightarrow 0 \}, \{ 1 \rightarrow 2 \rightarrow 1, 1 \rightarrow 0 \rightarrow 1 \}, \{ 2 \rightarrow 0 \rightarrow 2, 2 \rightarrow 1 \rightarrow 2 \}, \{ 0 \rightarrow 1 \rightarrow 2 \}, \{ 1 \rightarrow 2 \rightarrow 0 \}, \{ 2 \rightarrow 0 \rightarrow 1 \}, \{ 2 \rightarrow 1 \rightarrow 0 \}, \{ 0 \rightarrow 2 \rightarrow 1 \}, \{ 1 \rightarrow 0 \rightarrow 2 \} \).
\]

It follows that

\[
\overline{\text{Vol}} = \{ i \rightarrow i + 1 \rightarrow i \mid i = 0, 1, 2 \} = \text{Vol}
\]

but taken modulo \( \mathcal{N}_{\text{min}} \), which is defined in the same way as \( \mathcal{N}_{\text{min}} \) but with \( \oplus \) of the \( p \text{Arr}^{(2)} \) subsets. For example, if we took \( \text{Vol} = \{ i \rightarrow i+1 \rightarrow i \mid i = 0, 1, 2 \} \oplus \{ 1 \rightarrow 2 \rightarrow 0 \} \) then

\[
\overline{\text{Vol}} = \{ i \rightarrow i + 1 \rightarrow i \mid i = 0, 1, 2 \} \oplus \{ 1 \rightarrow 2 \rightarrow 0 \} = \{ i \rightarrow i+1 \rightarrow i \mid i = 0, 1, 2 \} \oplus \{ 1 \rightarrow 2 \rightarrow 0 \}.
\]

Since \( \overline{\cdot} \) is a diffeomorphism, the connection \( \overline{\nabla} \) must have

\[
R_{\overline{\nabla}}(\omega) = \overline{\text{Vol}} \odot P(X) \overline{\omega} = \overline{\text{Vol}} \odot \overline{P(X)} \omega
\]

with the left factor now understood modulo \( \mathcal{N}_{\text{min}} \). As a check on our entire dual formalism, we verify this directly on \( \omega = \{ 0, 1 \} \) as follows. During calculations, we will adopt a shorthand where \( ij \) means \( i \rightarrow j \), \( ijk \) means \( i \rightarrow j \rightarrow k \) etc., and \( \odot \) means \( \odot P(X) \) while \( \overline{\odot} = \odot \overline{P(X)} \). We start with

\[
\overline{\nabla} \{ 0 \rightarrow 1 \} = \{ 020, 201 \} \oplus g = \{ 010, 121, 012, 202, 212, 201 \} = \{ 020, 101, 120, 102, 210 \}
\]

\[
= \{ 02, 10, 12, 21 \} \odot \{ 20, 01, 02, 10 \} = \{ 02, 10, 12, 21 \} \odot \{ 20, 01, 02, 10 \}
\]

\[
= \{ 020, 010, 202, 101, 120, 012 \} \odot \{ 20, 01, 02, 10 \}
\]

\[
= \{ 020, 010, 202, 101, 120, 012 \} \odot \{ 20, 01, 02, 10 \}
\]

\[
= \{ 020, 0201, 2020, 0101, 0102, 1010, 1202, 1201, 0120 \}
\]

where \( d \) on a subset of arrows is all 2-steps ‘between’ arrows wholly in our out of the subset of arrows, for example \( 20 \rightarrow 01 \) (since they form a 2-step) contributes the 2-step 201. Its result should be understood modulo \( \mathcal{N}_{\text{min}} \). On the other factor of the output of \( \overline{\nabla} \) we compute

\[
\{ 02, 10, 12, 21 \} \odot \overline{\nabla} \{ 20, 01, 02, 10 \} = \{ 02, 10, 12, 21 \} \odot \overline{\nabla} \{ 20, 01, 02, 10 \}
\]

\[
= \{ 02, 10, 12, 21 \} \odot \overline{\{ 101, 121, 202, 212 \}} = \{ 02, 10, 12, 21 \} \odot \overline{\{ 101, 121, 202, 212 \}}
\]

\[
= \{ 0202, 0212, 1202, 1212, 2101, 2121 \}
\]

again with the first 2 steps modulo \( \mathcal{N}_{\text{min}} \) (which we denote \( \overline{\cdot} \)). The curvature computed in \( \overline{P(X)} \) is the \( \overline{\odot} \) of these two results:

\[
R_{\overline{\nabla}} \{ 0 \rightarrow 1 \} = (\overline{d} \oplus \text{id} \odot \text{id} \odot \overline{\nabla}) \overline{\nabla} \{ 0 \rightarrow 1 \}
\]

\[
= \{ 0202, 0201, 2020, 0101, 0102, 2010, 1202, 1201, 0120 \} \odot \{ 0202, 0212, 1202, 1212, 2101, 2121 \}
\]

\[
= \{ 0101, 0102, 0201, 0212, 1010, 1201, 1212, 2020, 2101, 2121 \}
\]

\[
= \{ 0102, 1210, 1212, 2020, 2121 \} = \{ 010, 121, 202 \} \oplus \{ 10, 02, 20, 12, 21 \} = \text{Vol} \oplus \{ 0 \rightarrow 1 \}
\]

when in the 4th equality we use the relations of \( \mathcal{N}_{\text{min}} \) inside the overline to simplify.
4. Generalised de Morgan duality over $\mathbb{F}_2$

We now generalise the de Morgan duality ideas above to any unital algebra $A$ over $\mathbb{F}_2$. We define ‘complementation’ as the vector space bijection

$$A \rightarrow A, \quad a \mapsto \bar{a} = 1 + a$$

which we view as an isomorphism of $A$ with a new algebra structure on $A$, denoted $\bar{A}$, with new product and addition

$$a\bar{b} = ab + b, \quad a\bar{b} = a + b + 1.$$

**Lemma 4.1.** $\bar{A}$ with the above product and addition is again a unital algebra over $\mathbb{F}_2$ with $1 = 0$ and $0 = 1$. Moreover, it obeys $\bar{a}\bar{a} = a^2$ so the new algebra is Boolean if and only if the initial one is.

**Proof.** This involves checking all the axioms of an algebra. For example,

$$(a\bar{b})\bar{(a\bar{c})} = 1 + a\bar{b} + a\bar{c} = 1 + a + b + ab + a + c + ac$$

$$= a + (1 + b + c) + a + ab + ac = a\bar{(1 + b + c)} = a\bar{(b\bar{c})}$$

$$a\bar{(b\bar{c})} = a + (b\bar{c}) + a(b\bar{c}) = a + b + c + bc + ab + ac = (2a\bar{b})\bar{c}$$

where the last step is similar to the preceding ones but in reverse. We also have $1\bar{a} = 1 + 1 = a$ over $\mathbb{F}_2$ while $0\bar{a} = 0 + 0 = a$ which agrees with $1 = 0$ and $0 = 1$. As a check, we then have $1\bar{a} = 1 + a + 1 = a$ over $\mathbb{F}_2$ which is $0\bar{a} = 0$.  

In the example of $\mathbb{F}_2(X)$ we have $\bar{\chi}_a = \chi_\bar{a}$ for $a \subseteq X$ and

$$\chi_a\bar{\chi}_b = \chi_{a\oplus b}, \quad \chi_a\bar{\chi}_b = \chi_{a\ominus b}$$

so our algebra $\bar{\cdot}$ operation reduces in this case to de Morgan duality as an algebra isomorphism $P(X) \rightarrow \bar{P}(X)$ on the power set of subsets of $X$. Or directly on $P(X)$,

$$a\bar{b} = a \oplus b \oplus (a \cap b) = ((a \cup b) \cap a \bar{\cap} b) \cup (a \cap b) = a \cup b \cup (a \cap b) = a \cup b$$

$$a\bar{b} = X \oplus (a \oplus b) = \bar{a} \oplus \bar{b} = a\bar{b}$$

is the algebra structure of $\bar{P}(X)$. In a more algebraic language $A = \mathbb{F}_2[\delta_i]/(\delta_i\delta_j - \delta_j\delta_i)$ where $i \in X$, while in $\bar{A}$ their product obeys

$$\delta_i\bar{\delta}_j + \delta_j\bar{\delta}_i + \bar{\delta}_i\delta_j(1 + \delta_i) = \delta_i + \delta_j + \delta_i\delta_j + 1 + \delta_i + 1 + \delta_j + 1 + \delta_j(1 + 0 + \delta_i) = 1 = 0.$$  

We used here that if $\mu = 0$, then $+\mu a$ in $\bar{A}$ means $+\bar{0} = +0$ in $A$ if $\mu = 0$ and $+1 + a$ in $A$ if $\mu = 1$, i.e. $+\mu a = +\mu(1 + a)$ in $A$. We obtained just the relations in $A$ of the complementary projectors $\epsilon_i := 1 + \delta_i$ obeying $\epsilon_i\epsilon_j = 1 + \epsilon_i + \epsilon_j + \delta_i\delta_j(1 + \epsilon_i) = 0$. Thus one can also view de Morgan duality as a change of variables within $A$. The same applies more generally, for example as follows.

**Lemma 4.2.** Let $A = \mathbb{F}_2[x]/(f(x))$ for some relation $f(x) = 0$. Then we can identify $A \cong \mathbb{F}_2[y]/(f(1 + y)) = A$ as a change of variables $y = 1 + x$.

**Proof.** We check this for $f(x) = f_3x^3 + f_2x^2 + f_1x + f_0$. Then $g(y) := f(1 + y) = f_3(1 + y + y^2 + y^3) + f_2(1 + y^2) + f_1(1 + y) + f_0 = f_3y + f_2y^2 + f_1y + f_0$.
Hence starting in \( \tilde{A} \) and using that \( \tilde{\mu}a = \mu(1 + a) \) in \( A \),
\[
g_{\tilde{A}}(x) = \tilde{\partial} f_3 x^2 + (f_3 + f_2) x^2 + (f_3 + f_1) x^2 (f_3 + f_2 + f_1 + f_0) = 1 + f_3 x^2 + (f_3 + f_2) x^2 + (f_3 + f_1) x + (f_3 + f_2 + f_1 + f_0) 0
+ f_3 + f_3 + f_2 + f_1 + f_3 + f_2 + f_1 + f_0
= 1 + f_3(x^3 + x + x^2) + (f_3 + f_2)x^2 + (f_3 + f_1)x + f_0
= 1 + f_3x^3 + f_2x^2 + f_1x + f_0 = 1 + f(x)
\]
as an element of \( A \). Hence \( g_{\tilde{A}}(x) = \tilde{0} \) in \( \tilde{A} \) is equivalent to \( f(x) = 0 \) in \( A \) which in turn is equivalent to a new variable \( y = 1 + x \) with \( g(y) = 0 \) in \( A \). \( \qed \)

Also for any algebra over \( \mathbb{F}_2 \) we define a ‘generalised derivation’
\[
\partial a = a\tilde{a} = a + a^2; \quad \partial(ab) = \partial a + \partial b + (\partial a)(\partial b) = (\partial a)(\partial b), \quad \partial(a + b) = \partial a + \partial b.
\]
In fact this just the ‘infinitesimal part’ of the canonical Frobenius automorphism in the sense that the latter is \( F = \text{id} + \partial \). For a Boolean algebra this is zero but for a more general algebra it wont be everywhere zero and we think of it in the spirit of \( \Omega \) as a kind of ‘boundary’ of \( a \) (the intersection of a subset and its complement which in the Venn diagram would be the boundary). This not the same as our exterior derivative but is a little similar, without needing a graph.

Now let \( (\Omega^1, d) \) be a differential calculus on \( A \) and \( \theta \in \Omega^1 \). We define \( \tilde{\Omega}^1 \) as the same set as \( \Omega^1 \).

**Proposition 4.3.** Let \( (\Omega^1, d) \) be a differential calculus on \( A \) and \( \theta \in \Omega^1 \). Then \( \tilde{\Omega}^1 \) defined as the same set as \( \Omega^1 \) but with a new addition, bimodule structure and differential
\[
\omega \tilde{+} \eta = \theta + \omega + \eta, \quad \tilde{\omega} = a\theta + (a + 1)\omega, \quad \omega a = \theta a + \omega(a + 1), \quad \tilde{\partial}a = \theta + da
\]
is a differential calculus on \( \tilde{A} \). Moreover,

(i) \( \tilde{\omega} : \Omega^1 \rightarrow \tilde{\Omega}^1 \) defined by \( \omega = \theta + \omega \) makes \( \tilde{\omega} : A \rightarrow \tilde{A} \) a diffeomorphism.

(ii) \( \theta \) is the zero element of \( \tilde{\Omega}^1 \).

(iii) \( \theta \) makes \( \Omega^1 \) inner if and only if the zero element of \( \Omega^1 \) makes \( \Omega^1 \) inner.

**Proof.** Clearly \( \omega \tilde{+} (\eta \tilde{+} \zeta) = \omega + \eta + \zeta \) is associative. Moreover
\[
a^\ast(\omega \tilde{+} \eta) = a^\ast(\theta + \omega + \eta) = a\theta + (1 + a)\theta + (1 + a)(\omega + \eta)
+ \theta + (\theta + (1 + a)\omega) + (\theta + (1 + a)\eta) = \tilde{a}\omega + a^\ast\eta
\]
\[
a^\ast(b \tilde{\omega}) = a\theta + (a + 1)(b\omega) = a\theta + (a + 1)b\theta + (a + 1)(b + 1)\omega
= (a\tilde{b})\theta + (1 + (a\tilde{b}))\omega = (a\tilde{b})\omega
\]
\[
a^\ast(\omega \tilde{b}) = a\theta + (a + 1)\omega b = a\theta + (a + 1)b\theta + (a + 1)\omega b + 1
\]
where we make the same steps in reverse. So \( \tilde{\Omega}^1 \) is a bimodule. We also have
\[
(d\omega)\tilde{b} + a^\ast db = \theta + (\theta b + (\theta + da)(b + 1)) + (a\theta + (a + 1)(\theta + db))
\]
\[
= \theta + da + b\tilde{d} + da^\ast db + a\tilde{d} = \tilde{d}(a + b + ab) = \tilde{d}(a + b).
\]
One can check that the surjectivity condition for a differential calculus holds as it does for \( \Omega^1 \).
For the additional facts: (i) Clearly $\bar{a} \cdot \bar{\omega} = (1 + a) \cdot (\theta + \omega) = (1 + a)\theta + a(\theta + \omega) = \theta + \omega = \bar{\omega}$ and similarly on the other side, so $\bar{\cdot} : \Omega^1 \to \bar{\Omega}^1$ is a bimodule map in the required sense. The diagram with $d, d$ also clearly commutes. (ii) $\theta \cdot \omega = \theta + \theta + \omega = \omega$ so $\theta$ is the zero element of $\bar{\Omega}^1$ (iii) $a \cdot 0 = a\theta + (a + 1)0 = a\theta$ and $0 \cdot a = a\theta$ similarly. Thus $a \cdot 0 + 0 \cdot a = \theta + [\theta, a] = \bar{0}a$.

In the example of $\bar{P}(X)$ we can take $\theta = \text{Arr}$ to have $\bar{\Omega}^1 = \bar{P}(\text{Arr})$ as a canonical choice, i.e., we recover the procedure in Section 2.2. The other canonical choice is $\theta = 0$ in which case $\bar{\Omega}^1$ has an unchanged addition law but a modified product

$$\bar{a} \cdot \bar{\omega} = \omega \oplus a \cap \omega = \bar{a} \cap \omega,$$

the set of arrows in $\omega$ with tip not in $a$. This is not as natural as our previous choice, so we will stick with that. In this case we also have $\theta \otimes_A \theta = \text{Arr}^{(2)}$ for $\bar{P}(X)$ which motivates us to similarly general to define ‘complementation’ on tensor products. With $\omega, \eta \in \bar{\Omega}^1$ viewed in $\bar{\Omega}^1$, we define

$$\omega \otimes_A \eta := \omega \otimes_A \eta + \theta \otimes \eta + \omega \otimes_A \theta \in \bar{\Omega}^1 \otimes_A \bar{\Omega}^1$$

and one can check that $\omega \cdot a \otimes_A \eta = \omega \otimes_A a \cdot \eta$. Here $\bar{\Omega}^1 \otimes_A \bar{\Omega}^1$ is the same vector space as $\Omega^1 \otimes_A \Omega^1$ and similarly to our treatment of $\bar{\Omega}^1$ is a bimodule with

$$(4.1) \quad \bar{\omega} \cdot \eta = \theta \otimes \omega + \omega + \eta, \quad a \cdot \omega = a\theta \otimes_A \theta + (a + 1)\omega, \quad \omega \cdot a = \theta \otimes \theta a + \omega(a + 1),$$

where now $\omega, \eta \in \Omega^1 \otimes_A \Omega^1$. One can check that $\otimes_A$ is bilinear with respect to $\bar{\cdot}$. By construction we can now define

$$\bar{\cdot} : \bar{\Omega}^1 \otimes_A \bar{\Omega}^1 \to \bar{\Omega}^1 \otimes_A \bar{\Omega}^1, \quad \bar{\omega} \otimes_A \bar{\eta} = \theta \otimes_A \theta + \omega \otimes_A \eta$$

and check that it connects the bimodule structures on the two sides compatibly with $\bar{\cdot}$ on $A$, e.g. on one side this is

$$a \cdot \bar{\omega} = (a + 1)(\theta \otimes_A \theta + \omega) = (a + 1)\theta \otimes_A \theta + a(\theta \otimes_A \theta + \omega) = \theta \otimes_A \theta + a\omega = \bar{a} \cdot \bar{\omega}.$$ 

We now ask when this descends to the wedge product.

Lemma 4.4. Let $\Omega^1$ extend to an exterior algebra over $A$ at least to $\Omega^2$ and $d\theta = 0$.

(i) $\bar{\Omega}^2$ defined as the same vector space as $\Omega^2$ with bimodule structure as in (4.1) but now using $\theta^2$ and with $d\omega = \theta^2 + d\omega$ for $\omega \in \Omega^1$ forms the degree 2 part of an exterior algebra over $A$.

(ii) $\bar{\Omega}^2 \to \bar{\Omega}^2$ defined by $\bar{\omega} = \bar{\theta}^2 + \bar{\omega}$ is a map of DGA's to degree 2.

Proof. The structure of $\bar{\Omega}^2$ follows the same structure and proofs as $\bar{\Omega}^1 \otimes_A \bar{\Omega}^1$, namely

$$\omega \cdot \eta = \theta^2 + \omega + \eta, \quad a \cdot \omega = a\theta^2 + (a + 1)\omega, \quad \omega \cdot a = \theta^2 a + \omega(a + 1)$$

for $\omega, \eta \in \Omega^2$ and we also have $\omega \cdot \eta = \omega \eta + \theta \eta + \omega \theta$ for $\omega, \eta \in \Omega^1$. We check the Leibniz rule

$$d(\bar{\omega} \cdot \eta) = \theta^2 + d(\theta a + (a + 1)\omega) + (da)\eta + a\eta + (da)\omega = (a + 1)(\theta^2 + d\omega)$$

$$d(a \cdot \bar{\omega}) = \theta^2 + (\theta + da)\omega + \theta a + (\theta + da)\theta + a\theta^2 + (a + 1)(\theta^2 + d\omega)$$

which agree provided $d\theta = 0$. This is also needed for $d\bar{\omega} = \theta^2 + d(\theta + da) = \theta^2 + \theta + d\bar{\omega} = \theta^2$ which is the zero element of $\Omega^2$. That $\bar{\cdot}$ extends our previous map $\bar{\Omega}^1 \to \bar{\Omega}^1$ compatibly with $d$ is also immediate provided $d\theta = 0$. We also have $\bar{\omega} \cdot \bar{\eta} = (\theta + \omega)(\theta + \eta) + \theta(\theta + \eta) + (\theta + \omega)\theta = \theta^2 + d\omega = \bar{\omega} \bar{\eta}$ by construction. □
Here \(d\theta = \{\theta, \theta\} = 2\theta = 0\) is automatic over \(F_2\) if the calculus in inner by \(\theta\). This is the case for \(P(X)\) with \(\Omega^2_{min}\) where indeed \(\theta \otimes P(X) \theta = \text{Arr}(2)\) is the union of all the elements of \(N_{min}\). We also note that the lemma works similarly for forms of all degrees; we have focussed on the degree 2 case as this is all that is needed for Riemannian geometry.

4.1. Example of group algebra \(A = F_2Z_3\). To illustrate the above, we now turn to the Hopf algebra dual model to the Boolean algebra \(F_2Z_3\) studied before. This is \(A = F_2Z_3\) or the D model from \([19]\) except that we change \(z, x\) there to \(x, y\) and use left-invariant 1-forms. Then \(A\) has basis \(1, x, x^2\), the relation \(x^3 = 1\) and the universal \(\Omega^1\) calculus. We define a left invariant basis \(e^+ = x^2dx\) and \(e^- = xdx^2\) and have the relations, volume form and inner element

\[
e^+x = x(e^+ + e^-), \quad e^+x^2 = x^2e^-; \quad e^-x = xe^+, \quad e^-x^2 = x^2(e^+ + e^-)
\]

\[
(e^+)^2 = (e^-)^2 = e^+e^- + e^-e^+ = 0, \quad \text{Vol} = e^+e^-, \quad \theta = e^+ + e^-.
\]

There are three quantum metrics

\[
g_i = x^i(e^+ \otimes e^- + e^- \otimes e^+), \quad i = 0, 1, 2
\]

(denoted \(g_{D,3}, g_{D,1}, g_{D,2}\) in \([19]\) and they each have a flat QLC

\[
g_0: \quad \nabla e^+ = 0, \quad g_1: \quad \nabla e^+ = e^+ \otimes e^+ + g_0, \quad \nabla e^- = e^+ \otimes e^- + e^- \otimes e^-
\]

\[
g_2: \quad \nabla e^+ = e^+ \otimes e^+ + e^- \otimes e^-, \quad \nabla e^- = e^- \otimes e^- + g_0.
\]

Next, each metric has three nonflat equal-curvature connections, with joint curvatures respectively

\[
g_0: \quad R\nabla e^+ = \text{Vol} \otimes e^+, \quad g_1: \quad R\nabla e^+ = x\text{Vol} \otimes (e^+ + e^-), \quad R\nabla e^- = x\text{Vol} \otimes e^+
\]

\[
g_2: \quad R\nabla e^+ = x^2\text{Vol} \otimes e^-, \quad R\nabla e^- = x^2\text{Vol} \otimes (e^+ + e^-)
\]

(notating that \(\text{Vol}\) in \([19]\) is \(x^2\text{Vol}\) now). Similarly to the \(F_2Z_3\) model, \([19]\) tells us that there are two natural lifts that result in an Einstein tensor \(Eins = \text{Ricci + g}\) that is conserved in the sense \(\nabla \cdot \text{Eins} = 0\). When converted to our left-invariant basis, these are

\[
i_\pm(\text{Vol}) = e^+ \otimes e^+ + e^- \otimes e^- + e^\pm \otimes e^\mp,
\]

\[
g_0: \quad \Eins_+ = i_+(\text{Vol}), \quad g_1: \quad \Eins_+ = \begin{cases} x_i_+(\text{Vol}) \\ 0 \end{cases}, \quad g_2: \quad \Eins_+ = \begin{cases} 0 \\ x_2i_+(\text{Vol}) \end{cases}
\]

For completeness we also give the three underlying equally curved QLCs from \([19]\) for each metric but converted in terms of our left-invariant forms.

\[
g_0: \quad \begin{aligned}
(i) \quad \nabla e^+ &= e^- \otimes e^- + g_0, \quad \nabla e^- = e^+ \otimes e^+ + e^- \otimes e^-

(ii) \quad \nabla e^+ &= e^+ \otimes e^+ + e^- \otimes e^- + g_0

(iii) \quad \nabla e^+ &= e^+ \otimes e^+ + g_0, \quad \nabla e^- = e^- \otimes e^- + g_0
\end{aligned}
\]

\[
g_1: \quad \begin{aligned}
(i) \quad \nabla e^+ &= (1 + x^2)e^+ \otimes e^+ + x^2e^- \otimes e^- + g_0, \quad \nabla e^- = (1 + x^2)e^+ \otimes e^+ + e^- \otimes e^- + x^2g_0

(ii) \quad \nabla e^+ = (1 + x^2)(e^+ \otimes e^+ + g_0), \quad \nabla e^- = e^+ \otimes e^+ + (1 + x^2)e^- \otimes e^- + x^2g_0

(iii) \quad \nabla e^+ = e^+ \otimes e^+ + x^2e^- \otimes e^- + (1 + x^2)g_0, \quad \nabla e^- = (1 + x^2)(e^\mp \otimes e^\pm)
\end{aligned}
\]

\[
g_2: \quad \begin{aligned}
(i) \quad \nabla e^+ &= e^+ \otimes e^+ + (1 + x)e^- \otimes e^- + xg_0, \quad \nabla e^- = xe^+ \otimes e^+ + (1 + x)e^- \otimes e^- + g_0

(ii) \quad \nabla e^+ = (1 + x)(e^+ \otimes e^+ + e^- \otimes e^-) + xg_0, \quad \nabla e^- = (1 + x)(e^\mp \otimes e^\pm) + g_0

(iii) \quad \nabla e^+ = (1 + x)(e^+ \otimes e^+ + e^- \otimes e^-), \quad \nabla e^- = xe^+ \otimes e^+ + e^- \otimes e^- + (1 + x)g_0.
\end{aligned}
\]
Swapping $e^\pm$ and $x, x^{-1}$ interchanges the $g_1$ and $g_2$ solutions while the $g_0$ solutions transform among themselves with (iii) invariant. In summary, the quantum geometry for the base metric $g_0$ is very similar to that for the Boolean algebra on 3 points in Section 3.2 except that now we have one flat and 3 curved QLCs rather than the other way around, and we also have the possibility of conformally scaled metrics $g_1, g_2$ with slightly different curvatures.

The de Morgan dual algebra is isomorphic but with dual generator $y = 1 + x$ with a new product $x \bar{x} = x^2, x \bar{x}^2 = 1 + x + x^2 = x + x^2$. So $x \bar{x}^2 + \bar{x}^2 + x = 0$ in $\bar{A}$. We also have $(1 + x)^3 + (1 + x)^2 + (1 + x) = x + x^2 + 1 + x^2 + 1 + x = 0$ so that de Morgan duality is equivalent to a change of variable to $y$ in the same algebra. The associated ‘derivation’ is $\partial x = x + x^2 = \partial x^2$ so that $\partial \partial x = 0$. This isomorphism extends to $\Omega_1 \cong \bar{\Omega}_1$, for example

$$(\bar{d}x) x + \bar{x} \bar{d}x = (\theta + (1 + x)dx) + (\theta + (dx)(1 + x)) = \theta + d(x^2) = \bar{d}(x^2) = \bar{d}(x^2)$$

and this is also necessarily true for $y$ in $\Omega_1$ as $(dy)y + y(dy) = d(y^2)$. At degree 2 we have the same vector space for $\Omega^2$ as $\theta^2 = 0$ and for example

$$(\bar{d}x \bar{d}(x^2)) = (\theta + dx)(\theta + dx^2) + (\theta + dx^2)(\theta + dx)\theta = (dx)dx^2 = xe^+x^2e^- = (e^-)^2 = 0$$

is parallel to $(dy)dy^2 = 0$ in $\Omega$. We can also write

$$e^- = e^+ = \theta + x^2dx = \theta + (dx^2)x = (\bar{d}x^2)(1 + x)$$

$$e^+ = e^- = \theta + xdx = \theta + (dx)x = \theta(1 + x^2) + (\theta + dx^2) x^2 = (\bar{d}x)(1 + x^2)$$

as elements of $\bar{\Omega}_1$, obeying $e^+e^- = 0 = \theta$ and

$$e^+e^+ = \theta e^+ = e^+\theta = 0, \quad e^+e^+ = e^+e^+ + (e^+ + e^-)e^+ + e^+(e^+ + e^-) = e^+e^- = e^-\bar{e} = \bar{e}^-\bar{e}^+.$$  

We have $\bar{d} = d$ acting on degree 1 and $\bar{\text{Vol}} = \text{Vol}$ as $\theta^2 = 0$, and one can check that the zero element of $\bar{\Omega}_1$ makes $\bar{\Omega}$ inner. In short, $\bar{\Omega}$ looks different but can also be viewed within $\Omega$ as a change of variables, i.e. de Morgan duality invariance is ultimately part of diffeomorphism invariance.

5. Concluding remarks

We specialised the formalism of quantum Riemannian geometry, as in [5] and references therein, to the atomic Boolean case and extended de Morgan duality to this. To this end we introduced the noncommutative $\cap, \cup$ of subsets with arrows, where the differential forms on the powerset $P(X)$ correspond to arrows in a Venn diagram picture. We saw some aspects of the duality ideas in the introduction, for example the element $\theta = \text{Arr}$ consisting of all arrows mapping to the zero differential form in the de Morgan dual differential calculus. We did not realise the full ‘physical motivation’ relating quantum theory and gravity but this is not surprising as the quantum aspects were limited to noncommutative differentials and gravity was limited to some models with Riemann curvature. We did see that ultimately the de Morgan duality extends as a diffeomorphism between $P(X)$ and the dual model on $\bar{P}(X)$ where $\cup, \cap$ are swapped. More generally, that quantum Riemannian geometry produces reasonable answers in such a special case itself speaks to the coherence of the formalism.

Looking beyond, it is certainly possible to consider quantum geometric structures on Boolean algebras more generally [9] as well as to take a more formal propositional
logic point of view. Given the importance of Heyting algebras and topos theory in physics, see e.g. [7], one might try to explore quantum Riemannian geometry in such contexts. It is also possible to consider duality ideas in more general logic, such as the notion of a bi-Heyting algebra [22]. In Section 4 we went in a different direction and extended the de Morgan duality to unital algebras $A$ over $\mathbb{F}_2$, where we can already define quantum geometry and where $A$ could even be noncommutative. Here we could see more clearly that the duality is equivalent to a change of variables or ‘coordinate transformation’ within $A$ itself, bringing the proposal of de Morgan duality for physics in [12] in line with the equivalence principle of General Relativity (provided we use quantum geometry).

We illustrated this for $\mathbb{F}_2 \mathbb{Z}_3$ the Hopf-algebra dual model to the Boolean algebra on 3 points (with $\mathbb{Z}_3$-invariant geometry) studied in Section 3.2. The representation-theoretic duality it a little different from de Morgan duality and the two models, both with curvature, have different moduli of metrics and connections. This latter duality between geometries on $\mathbb{C}(S_3)$ and $\mathbb{C}S_3$ and potentially other finite groups was introduced in [21]. In the first model the possible differential structures are labelled by conjugacy classes and the eigenvectors of the resulting Laplacians ‘waves’ provided by matrix elements of irreducible representations, in the dual model the possible differential structures are labelled by representations and eigenvectors of the Laplacians by conjugacy classes. Over $\mathbb{C}$ the $\mathbb{Z}_3$ models would be isomorphic by Hopf algebra self-duality, but this is not the case over $\mathbb{F}_2$. Nevertheless we saw that the Boolean $\mathbb{F}_2(\mathbb{Z}_3)$ and the $\mathbb{F}_2 \mathbb{Z}_3$ quantum Riemannian geometries are strikingly similar. Both have 4 QLC’s for each metric just in one case 3 are flat and one is curved and in the other vice versa. Moreover, in both models there are two natural lifts maps $i$ such that the Einstein tensor is conserved in the sense $\nabla \cdot \text{Eins} := ((\cdot, \cdot) \otimes \text{id}) \nabla (\text{Eins}) = 0$. The quantum geometries from both models were taken from the classification in [19] but simplified in terms of left-invariant 1-forms. That work also finds several interesting quantum Riemannian geometries on $\mathbb{F}_8$ as a 3-dimensional algebra over $\mathbb{F}_2$.

Returning to de Morgan duality, this could potentially be extended further to quivers (where there can be self-arrows and multiple arrows between vertices). One still has a differential calculus in the sense of part of a DGA but not all 1-forms need be sums of elements of the form $a dB$ [21]. One can still do quantum Riemannian geometry but now a metric is not a number on each edge but a matrix [21], which could still be interesting over $\mathbb{F}_2$ where it would have a more combinatoric flavour (for example the invertible elements of $M_2(\mathbb{F}_2)$ form the group $S_3$).

Another comment is that for a quantum metric on a graph, we need the graph to be bidirected for $g$ to be invertible. There could be some mileage and applications to one-way processes where arrows might exist only in one direction. The set of $P(X)$ and propositional logic can be viewed as coming with a tautological graph, in the former case according to inclusion. This is very different from the graph on $X$ explored in the present paper and would be more relevant to field theory if one were doing geometry on the set of $P(X)$ itself. This could be explored from the point of view of developing QFT where we quantise the ‘function space’ itself. (To this end it could be reasonable to work over $\mathbb{C}$ for the second quantisation even if the initial geometry was over $\mathbb{F}_2$.) Finally, the kind of noncommutative geometry in [5] works over any field, for example over $\mathbb{F}_p$ or indeed over $\mathbb{R}$ or $\mathbb{C}$. In the real case a metric
on a graph would assign a (say positive) real number more in line with Riemannian geometry. We would no longer have the kind of duality transformation $A$ to $\bar{A}$ in Section[4] as this was specific to characteristic 2, but possibly the de Morgan duality and its generalisations could re-emerge in a probabilistic interpretation. These are some ongoing directions for further work.

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