UNIVALENCE OF THE AVERAGE OF TWO ANALYTIC FUNCTIONS

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Abstract. Let $A$ denote the set of all analytic functions $f$ in the unit disk $D = \{ z : |z| < 1 \}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Let $U$ denote the set of all $f \in A$, $f(z)/z \neq 0$ and satisfying the condition

$$\left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| < 1 \text{ for } z \in D.$$ 

Functions in $U$ are known to be univalent in $D$. For $\alpha \in [0,1]$, let

$$N(\alpha) = \left\{ f_\alpha : f_\alpha(z) = (1 - \alpha)f(z) + \alpha \int_0^z \frac{f(t)}{t} \, dt, \ f \in A \text{ with } |a_n| \leq n \text{ for } n \geq 2 \right\}.$$ 

In this paper, we first show that the condition $\Sigma_{n=2}^{\infty} n|a_n| \leq 1$ is sufficient for $f$ to be in $U$ and the same condition is necessary for $f \in U$ in case all $a_n$'s are negative. Next, we obtain the radius of univalence of functions in the class $N(\alpha)$. Also, for $f, g \in U$ with $f(z) + g(z) \neq 0$ in $D$, $F(z) = (f(z) + g(z))/2$, and $G(z) = r^{-1}F(rz)$, we determine a range of $r$ such that $G \in U$. As a consequence of these results, several special cases are presented.

1. Introduction

Let $A$ denote the set of all normalized analytic functions $f$ in the unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For $f \in A$ and $n \in \mathbb{N}$, we write $s_n(f)(z) = z + \sum_{k=2}^{n} a_k z^k$, for the $n$-th partial sums or sections of $f$. Also, we denote by $S$ the class of all univalent functions in $A$. It is well-known that if $f \in S$ then $|a_n| \leq n$ for $n \geq 2$. The class of convex and starlike functions are two important geometric subclasses of $S$, denoted by $C$ and $S^*$, respectively (see [3, 6]). For $f \in C$, one has $|a_n| \leq 1$ for all $n \geq 2$. Two other subclasses of $S$ that are studied extensively are

$$R_1 = \{ f \in A : |f'(z) - 1| < 1 \text{ for } z \in D \}$$

$$S_1^* = \{ f \in S : |(zf'(z)/f(z)) - 1| < 1 \text{ for } z \in D \}.$$ 

Finally, we define ([1, 8, 10])

$$U = \{ f \in A : f(z)/z \neq 0 \text{ and } |U_f(z) - 1| < 1 \text{ for } z \in D \}.$$ 

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where
\[ U_f(z) = f'(z) \left( \frac{z}{f(z)} \right)^2. \]

In the recent years, the class \( U \) and its association with a number of subclasses of \( S \) together with certain integral transformations have been studied in details (see [4, 8, 10, 11]). It is well-known that \( U \subseteq S \) (see [1, 12]). It is interesting to observe that the Koebe function \( k(z) = z/(1-z)^2 \) belongs to \( U \). The following result is well-known.

**Theorem A.** [2, Theorem 3] If \( \sum_{n=2}^{\infty} n|a_n| \leq 1 \), then \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) belongs to \( S^* \cap R_1 \).

Let \( F \) and \( G \) be two subclasses of \( A \). If for every \( f \in F \), \( G_r(z) = r^{-1}f(rz) \in G \) for \( r \leq r_0 \), and \( r_0 \) is the maximum value for which this holds, then we say that \( r_0 \) is the \( G \)-radius in \( F \). That is,
\[ r_0 = r_G(F) := \sup \{ r > 0 : G_r \in G \} \text{ for every } f \in F. \]

There are many results of this type in the theory of univalent functions, see [6, Chapter 13, Vol. 2]. For instance, the \( C \)-radius in \( S \) is \( 2 - \sqrt{3} \) which is referred to as the radius of convexity of \( S \). Similarly, the radius of starlikeness for the class \( S \) is \( \tanh(\pi/4) \approx 0.65579 \) (see [3, p. 44] and [6, Chapter 8, Vol. 1]). We now recall the following result due to Gavrilov [5, Theorem 1] which does not seem to be known for many readers.

**Theorem B.** Let \( F = \{ f \in A : |a_n| \leq n \text{ for } n \geq 2 \} \). Then \( f \) and each of its \( n \)-th partial sum \( s_n(f) \) is univalent for \( |z| < r_S \), where \( r_S \approx 0.164878 \) is the root of the equation \( 2(1-r)^3 - r - 1 = 0 \) in \((0,1)\). Here \( r_S \) is the radius of univalence of \( g(z) = 2z - \frac{z}{(1-z)^2}, \ z \in \mathbb{D}. \)

Equivalently, \( r_S \) is the \( S \)-radius in \( F \).

In the foregoing discussion, we say that \( f \) belongs to \( U \) in the disk \( |z| < r \) if the inequality in the above definition of \( U \) holds for \( |z| < r \) instead of the whole unit disk \( \mathbb{D} \). In other words, this is equivalent to saying that \( G \) defined by \( G(z) = r^{-1}f(rz) \) belongs to \( U \), when \( f \) belongs to \( U \) in the disk \( |z| < r \).

In order to indicate another example of the type as in Theorem B, we may recall the following result which is indeed a corollary to a general result (see [9]).

**Theorem C.** \( r_U(S) = 1/\sqrt{2} \).

The paper is organized as follows. The main results are stated in Section 2 and their proofs and some of their consequences are presented in Section 3. First we present a sufficient coefficient condition for a function \( f \) to be in \( U \). The condition is also shown to be necessary if the coefficients of \( f \in A \) are negative. Next, we present several simple observations concerning the radius of univalence (and of starlikeness) of certain class of analytic functions in the unit disk. Finally, we obtain a radius of univalence of average of two functions from \( U \).
2. Main Results

We now state our first result.

**Theorem 1.** If \( \sum_{n=2}^{\infty} n|a_n| \leq 1 \), then \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) belongs to \( U \). The result is sharp.

At this place, it is worth recalling that the class \( U \) is neither included in \( S^* \) nor includes the class \( S^* \). Also, the class \( U \) is neither contained in \( R_1 \) nor contains the class \( R_1 \) (see for example [10]). For instance, the function \( f \) defined by

\[
f(z) = \frac{z}{1 + \frac{1}{2}z + \frac{1}{2}z^3}
\]

is in \( U \setminus S^* \) (see also [4]). Indeed for this function

\[
\frac{zf'(z)}{f(z)} = \frac{1 - z^3}{1 + \frac{1}{2}z + \frac{1}{2}z^3}
\]

we have at \( z_0 = (-1 + i)/\sqrt{2} \),

\[
\frac{z_0 f'(z_0)}{f(z_0)} = \frac{2 - 2\sqrt{2}}{3} + \frac{1 - 2\sqrt{2}}{3}i
\]

which gives that Re \( \{z_0 f'(z_0)/f(z_0)\} < 0 \) showing that \( f \) is not starlike in \( \mathbb{D} \). Moreover, although the Koebe function \( k(z) \) belongs to \( U \cap S^* \), it is known that \( S \not\subset U \).

In view of these observations, Theorem 1 refines Theorem A and hence, Theorem 1 will be useful to generate nice class of examples of functions in \( U \) that are also starlike in \( \mathbb{D} \).

For functions with negative coefficients, the converse of Theorem 1 is also true.

**Theorem 2.** A function \( f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \) is in the class \( U \) if and only if \( \sum_{n=2}^{\infty} n|a_n| \leq 1 \).

From the result of Silverman [15], we may now formulate the above discussion as

**Corollary 1.** Suppose that \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \) belongs to \( A \). Then we have the following equivalent statements:

\[
f \in U \iff f \in S^* \iff f \in R_1 \iff \sum_{n=2}^{\infty} n|a_n| \leq 1.
\]

In connection with a problem due to [7], Ruscheweyh and Wirths [14] discussed the univalency of functions in the set of convex linear combinations of the form

\[
\mu f(z) + (1 - \mu)g(z), \ \mu \in [0, 1],
\]

when \( f, g \) belonging to suitable subsets of \( S \). We shall consider a similar problem which is indeed a generalization of Theorem B. For \( \alpha \in [0, 1] \), let

\[
\mathcal{N}(\alpha) = \left\{ f_\alpha : f_\alpha(z) = (1 - \alpha)f(z) + \alpha \int_0^z \frac{f(t)}{t} dt, \ f \in A \text{ with } |a_n| \leq n \text{ for } n \geq 2 \right\}
\]

and for convenience, we set \( r_S(\alpha) := r_S(\mathcal{N}(\alpha)) \).
\[\begin{array}{|c|c|}
\hline
\alpha & r_S(\alpha) \\
\hline
0 & 0.164878 \\
1/5 & 0.178866 \\
1/4 & 0.182817 \\
1/3 & 0.189894 \\
1/2 & 0.206299 \\
3/4 & 0.23931 \\
4/5 & 0.247756 \\
1 & 0.292893 \\
\hline
\end{array}\]

Table 1. The radius \(r_S(\alpha)\) of univalence for \(f_\alpha \in \mathcal{N}(\alpha)\)

**Theorem 3.** The number \(r_S(\alpha)\) is the root in \((0,1)\) of the equation

\[
2(1 - r)^3 + (2\alpha - 1)r - 1 = 0.
\]

The extremal function is

\[
g(z) = 2z - \frac{\alpha z}{1 - z} - \frac{(1 - \alpha)z}{(1 - z)^2}, \quad z \in \mathbb{D}.
\]

For a ready reference, the values of \(r_S(\alpha)\) for certain values of \(\alpha \in [0,1]\) are listed in Table 1.

**Remark 1.** If \(f \in \mathcal{S}\), then the Biernacki integral \((Bf)(z) = \int_0^z (f(t)/t) \, dt\) is not necessarily univalent in \(\mathbb{D}\).

The case \(\alpha = 0\) leads to Theorem B. Set \(\mathcal{N} = \mathcal{N}(1)\) so that

\[
\mathcal{N} = \{f \in \mathcal{A} : |a_n| \leq 1 \text{ for } n \geq 2\}
\]

and therefore, the case \(\alpha = 1\) of Theorem 3 may be reformulated as

**Corollary 2.** Then \(f \in \mathcal{N}\) and each of its \(n\)-th partial sum \(s_n(f)\) is univalent for \(|z| < r\), where \(r \approx 1 - \frac{\sqrt{2}}{2} \approx 0.292893\) is the root of the equation \(2(1 - r)^3 + r - 1 = 0\) in \((0,1)\). The extremal function is

\[
g(z) = 2z - \frac{z}{1 - z}, \quad z \in \mathbb{D}.
\]

We remark that

\[
\mathcal{N} \supset \{f \in \mathcal{A} : \text{Re} \left(\frac{f(z)}{z}\right) > 1/2, \quad z \in \mathbb{D}\} \supset \{f : f \in \mathcal{C}\}.
\]

In the case of \(\alpha = 1/2\), we see that if

\[
\mathcal{N}(1/2) = \left\{f_{1/2}(z) = \frac{1}{2} \left( f(z) + \int_0^z \frac{f(t)}{t} \, dt \right) : f \in \mathcal{A} \text{ and } |a_n| \leq n \text{ for } n \geq 2 \right\},
\]

then functions \(f_{1/2}\) in \(\mathcal{N}(1/2)\) are univalent in \(|z| < 1 - 2^{-1/3} \approx 0.206299\), but not necessarily in any larger disk as the function

\[
g(z) = 2z - \frac{z}{2(1 - z)} - \frac{z}{2(1 - z)^2}, \quad z \in \mathbb{D},
\]
In order to motivate our next theorem, let us consider the sum of two univalent functions \( f, g \in S \). Then the average function \( F \) defined by

\[
F(z) = \frac{f(z) + g(z)}{2}, \quad z \in \mathbb{D},
\]

belongs to the class \( \mathcal{N}(0) \), and hence, by Theorem B, we conclude that the radius of univalence of the function \( F \) is not smaller than that of the number given in Theorem B.

Similarly, if \( f, g \in C \) then the necessary coefficient estimates on the Taylor coefficients of \( f, g \) show that the average function \( F \) defined by

\[
F(z) = \frac{f(z) + g(z)}{2}, \quad z \in \mathbb{D},
\]

belongs to the class \( \mathcal{N} \) defined in Corollary 2. Again, the radius of univalence of the function \( F \) is not smaller than that given in Corollary 2. Thus, any result which gives radius bigger than that given in these two corollaries will provide us results with an improved bound for the radius. Our next result fills this idea if we consider the subclass \( \mathcal{U} \) of \( S \) for our investigation.

In the following theorem, we use certain well-known basic facts. Each \( f \in \mathcal{U} \) with \( a_2 = f''(0)/2 \) can be written in the form

\[
(2) \quad f'(z) \left( \frac{z}{f(z)} \right)^2 = -z \left( \frac{z}{f(z)} \right)' + \frac{z}{f(z)} = 1 + w(z), \quad z \in \mathbb{D},
\]

where \( w: \mathbb{D} \to \mathbb{D} \) is analytic with \( w(0) = w'(0) = 0 \). Consequently, the classical Schwarz lemma gives

\[
|U_f(z) - 1| \leq |z|^2 \quad \text{for} \quad z \in \mathbb{D}
\]

and, by (2), we easily have

\[
(3) \quad \frac{z}{f(z)} = 1 - a_2 z - \int_0^1 \frac{w(tz)}{t^2} dt, \quad z \in \mathbb{D}
\]

so that

\[
(4) \quad \left| \frac{z}{f(z)} - 1 - a_2 z \right| \leq |z|^2, \quad z \in \mathbb{D}.
\]

**Theorem 4.** Let \( f, g \in \mathcal{U} \) with \( \frac{f(z) + g(z)}{z} \neq 0 \) in the unit disk \( \mathbb{D} \), and \( F \) be defined by

\[
(5) \quad F(z) = \frac{f(z) + g(z)}{2}, \quad z \in \mathbb{D}.
\]

Then \( F \in \mathcal{U} \) for \( |z| < r_0 \), where \( r_0 \approx 0.262453 \) is the root of the equation

\[
(6) \quad r^2 + (1 + r^2) \frac{(2r + r^2)^2}{\left(1 - 2r - \frac{\pi}{\sqrt{3}} \frac{r^2}{\sqrt{1-r^2}}\right)^2} = 1.
\]

Using Theorem C and Theorem 4, one can quickly deduce the following result.
Theorem 5. Let \( f, g \in S \) with \( \frac{f(z) + g(z)}{z} \neq 0 \) in the unit disk \( \mathbb{D} \). Then the function \( F \) defined by (5) belongs to \( \mathcal{U} \) for \( |z| < \frac{1}{\sqrt{2}} r_0 \), where \( r_0 \approx 0.262453 \) is the root of the equation (6).

At this place it is worth recalling that \( S \subset \mathcal{N}(0) \).

3. A Lemma and an Example

For the proof of Theorem 4 and the discussion in Example 1 below, we need the following lemma.

Lemma 1. Let \( \phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \) be a non-vanishing analytic function on \( \mathbb{D} \) and let \( f \) be of the form \( f(z) = z \phi(z) \).

Then, we have the following:

(a) If \( \sum_{n=2}^{\infty} (n - 1) |b_n| \leq 1 \), then \( f \in \mathcal{U} \).
(b) If \( \sum_{n=2}^{\infty} (n - 1) |b_n| \leq 1 - |b_1| \), then \( f \in \mathcal{S}^* \).
(c) If \( f \in \mathcal{U} \), then \( \sum_{n=2}^{\infty} (n - 1)^2 |b_n|^2 \leq 1 \).

The conclusion (a) in Lemma 1 is from [8, 11] whereas the (b) is due to Reade et al. [13, Theorem 1]. Finally, as \( f \in \mathcal{U} \), we have

\[
\left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| = \left| -z \left( \frac{z}{f(z)} \right)' + \frac{z}{f(z)} - 1 \right| = \sum_{n=2}^{\infty} (n - 1) b_n z^n \leq 1
\]

and so (c) is an immediate consequence of Gronwall’s area theorem, and can be obtained also from Parseval’s relation.

Example 1. Consider

\[
f(z) = \frac{z}{(1 - iz)^2} \quad \text{and} \quad g(z) = \frac{z}{(1 + iz)^2}, \quad z \in \mathbb{D}.
\]

Then \( f, g \in S \), and for \( \alpha \in [0, 1] \), we obtain

\[
f_\alpha(z) = \frac{(1 - \alpha)z}{(1 - iz)^2} + \frac{\alpha z}{1 - iz} \quad \text{and} \quad g_\alpha(z) = \frac{(1 - \alpha)z}{(1 + iz)^2} + \frac{\alpha z}{1 + iz}, \quad z \in \mathbb{D}.
\]

We observe that \( f_\alpha, g_\alpha \in \mathcal{N}(\alpha) \). Now, we introduce

\[
F_\alpha(z) = \frac{1}{2} (f_\alpha(z) + g_\alpha(z)), \quad z \in \mathbb{D}.
\]

Simple calculation shows that

\[
F_\alpha(z) = \frac{z[1 - (1 - 2\alpha)z^2]}{(1 + z^2)^2}
\]

and

\[
\frac{z}{F_\alpha(z)} = \frac{(1 + z^2)^2}{1 - (1 - 2\alpha)z^2} = 1 + (3 - 2\alpha)z^2 + \frac{4(1 - \alpha)^2 z^4}{1 - (1 - 2\alpha)z^2}, \quad z \in \mathbb{D}.
\]
For $0 < r \leq 1$, define $G_\alpha$ by $G_{\alpha,r}(z) = r^{-1}F_\alpha(rz)$ so that

$$\frac{z}{G_{\alpha,r}(z)} = 1 + \sum_{n=2}^{\infty} B_{2n}(r)z^{2n}$$

where

$$B_{2n}(r) = \begin{cases} (3 - 2\alpha)r^2 & \text{if } n = 1 \\ 4(1 - \alpha)^2(1 - 2\alpha)^{n-2}r^{2n} & \text{if } n \geq 2. \end{cases}$$

By Lemmas 1(a) and (b), the function $G_{\alpha,r} \in \mathcal{U} \cap \mathcal{S}^*$ if $r$ satisfies the condition

$$S(r) = \sum_{n=2}^{\infty} (2n - 1)|B_{2n}(r)| \leq 1.$$

A computation gives

$$S(r) = (3 - 2\alpha)r^2 + 4(1 - \alpha)^2r^4 \sum_{n=2}^{\infty} (2n - 1)|1 - 2\alpha|^{n-2}r^{2(n-2)}$$

$$= (3 - 2\alpha)r^2 + 4(1 - \alpha)^2r^4 \left\{ \frac{3 - |1 - 2\alpha|r^2}{(1 - |1 - 2\alpha|r^2)^2} \right\}.$$

Thus, $S(r) \leq 1$ if and only if $r$ satisfies the inequality

$$(3 - 2\alpha)r^2(1 - |1 - 2\alpha|r^2)^2 + 4(1 - \alpha)^2r^4(3 - |1 - 2\alpha|r^2) - (1 - |1 - 2\alpha|r^2)^2 \leq 0.$$

If $1 - 2\alpha > 0$, then the last condition is equivalent to

$$(1 + r^2)\left( (1 - 2\alpha)r^4 - 6(1 - \alpha)r^2 + 1 \right) \geq 0$$

which is true if $r^2 \leq K_\alpha^2$, where

$$K_\alpha = \sqrt{\frac{3(1 - \alpha) - \sqrt{9\alpha^2 - 16\alpha + 8}}{1 - 2\alpha}} \quad \text{for } 0 \leq \alpha < 1/2.$$

Thus, for $0 \leq \alpha < 1/2$, the inequality $S(r) \leq 1$ holds if $r \leq K_\alpha$. Moreover, for $\alpha = 1/2$, we obtain that $S(r) \leq 1$ whenever

$$r \leq \lim_{\alpha \to (1/2)^-} K_\alpha = \frac{1}{\sqrt{3}}.$$

Finally, if $1 - 2\alpha < 0$ then the inequality $S(r) \leq 1$ holds if and only if

$$(3 - 2\alpha)r^2(1 + (1 - 2\alpha)r^2)^2 + 4(1 - \alpha)^2r^4(3 + (1 - 2\alpha)r^2) - (1 + (1 - 2\alpha)r^2)^2 \leq 0,$$

or equivalently $r \leq K'_\alpha$, where $K'_\alpha$ for $1/2 < \alpha \leq 1$ is the root of the equation

$$-1 + (1 + 2\alpha)r^2 + (17 - 36\alpha + 16\alpha^2)r^4 + (7 - 16\alpha + 8\alpha^2)(1 - 2\alpha)r^6 = 0$$

in the unit interval $(0, 1)$. Setting

$$r(\alpha) = \begin{cases} K_\alpha & \text{if } 0 \leq \alpha < 1/2 \\ \frac{1}{\sqrt{3}} & \text{if } \alpha = 1/2 \\ K'_\alpha & \text{if } 1/2 < \alpha \leq 1, \end{cases}$$

we see that the function $G_{\alpha,r}(z)$ is univalent in $\mathbb{D}$ if $0 < r \leq r(\alpha)$. Equivalently,
it means that the function \( F_\alpha \) is univalent in the disk \(|z| < r(\alpha)\) and the result is sharp. In Table 2, we list the values of \( r(\alpha) \) for certain choices of \( \alpha \).

The above discussion suggests the following

**Problem 1.** Suppose that for \( \alpha \in [0,1] \),

\[
S(\alpha) = \left\{ f_\alpha(z) = (1 - \alpha)f(z) + \alpha \int_0^z \frac{f(t)}{t} dt : f \in S \right\}.
\]

and

\[
F(\alpha) = \left\{ F_\alpha(z) = \frac{1}{2}(f_\alpha(z) + g_\alpha(z)) : f_\alpha, g_\alpha \in S(\alpha) \right\}.
\]

Determine the radius of univalence of functions in \( F_\alpha \in F(\alpha) \).

### 4. Proofs and some of their consequences

**Proof of Theorem 1.** Suppose that \( \sum_{n=2}^{\infty} n|a_n| \leq 1 \) and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \).

In order to show that \( f \in \mathcal{U} \), we need to show that \(|U_f(z) - 1| < 1\) for \( z \in \mathbb{D} \). As

\[
\left|f'(z) - \left(\frac{f(z)}{z}\right)^2\right| = \left|1 + \sum_{n=2}^{\infty} n a_n z^{n-1} - \left(1 + \sum_{n=2}^{\infty} a_n z^{n-1}\right)^2\right| \\
= \left|\sum_{n=2}^{\infty} (n - 2) a_n z^{n-1} - \left(\sum_{n=2}^{\infty} a_n z^{n-1}\right)^2\right| \\
= \left|z\right|^2 \left|\sum_{n=3}^{\infty} (n - 2) a_n z^{n-3} - \left(\sum_{n=2}^{\infty} a_n z^{n-2}\right)^2\right|,
\]
it follows that
\[
\left| f'(z) - \left( \frac{f(z)}{z} \right)^2 \right| < \sum_{n=2}^{\infty} (n-2)|a_n| + \left( \sum_{n=2}^{\infty} |a_n| \right)^2
\]
\[
\leq 1 - 2 \sum_{n=2}^{\infty} |a_n| + \left( \sum_{n=2}^{\infty} |a_n| \right)^2
\]
\[
\leq \left( 1 - \sum_{n=2}^{\infty} |a_n| \right)^2 \leq \left| \frac{f(z)}{z} \right|^2
\]
which implies that
\[
|U_f(z) - 1| = \left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| < 1, \quad z \in \mathbb{D}.
\]
and hence, \( f \in \mathcal{U} \).

To see that the upper bound 1 in the coefficient condition cannot be replaced by \( 1 + \epsilon, \epsilon > 0 \), we consider the function
\[
f_\epsilon(z) = z + \frac{1+\epsilon}{\epsilon} z^n \quad (n \geq 2).
\]
We obtain that \( f_\epsilon'(z) = 1 + (1+\epsilon)z^{n-1} \) has a zero in \( \mathbb{D} \) as \( \epsilon > 0 \) and hence, \( f_\epsilon \) is not univalent in \( \mathbb{D} \). In particular, \( f_\epsilon \notin \mathcal{U} \). Thus, the result is sharp. \( \Box \)

**Remark 2.** Theorem A in particular gives condition on the Taylor coefficients of \( f \) so that \( f \) and its \( n \)-th partial sum \( s_n(f) \) is not only univalent but is starlike in \( \mathbb{D} \). In view of this observation, from the proof of Theorem 3, it follows that the quantity \( r_S(\alpha) \) in Theorem 3 is indeed the \( S \)-radius in \( \mathcal{N}(\alpha) \) as well as the radius of starlikeness of functions in the class \( \mathcal{N}(\alpha) \). These observations and the proof of Theorem 1 show that Theorem 1 may be stated in an improved form.

**Proof of Theorem 2.** In view of Theorem 1, it suffices to show the only if part. Assume that \( |U_f(z) - 1| < 1 \) in \( \mathbb{D} \). That is,
\[
\left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| = \left| \frac{1 - \sum_{n=2}^{\infty} n|a_n| z^{n-1}}{\left( 1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \right)^2} - 1 \right|
\]
\[
= \left| \frac{- \sum_{n=3}^{\infty} (n-2)|a_n| z^{n-3} - \left( \sum_{n=2}^{\infty} |a_n| z^{n-2} \right)^2}{\left( 1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \right)^2} \right| < 1
\]
for $z \in \mathbb{D}$. Choose values of $z$ on the real axis so that $U_f(z) - 1$ is real. Upon clearing the denominator in the last expression and letting $z \to 1^-$ through real values, we obtain

$$
\sum_{n=3}^{\infty} (n-2)|a_n| + \left( \sum_{n=2}^{\infty} |a_n| \right)^2 \leq \left( 1 - \sum_{n=2}^{\infty} |a_n| \right)^2.
$$

Thus, $\sum_{n=2}^{\infty} n|a_n| \leq 1$, and the proof is complete. \hfill \square

**Proof of Theorem 3.** Let $f_\alpha \in \mathcal{N}(\alpha)$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then $f'_\alpha$ takes the form

$$
f'_\alpha(z) = (1 - \alpha)f'(z) + \alpha \frac{f(z)}{z} = 1 + \sum_{n=2}^{\infty} ((1 - \alpha)n + \alpha)a_n z^{n-1}.
$$

As $|a_n| \leq n$ for $n \geq 2$, it follows that for $|z| = r$

$$
|f'_\alpha(z) - 1| \leq \sum_{n=2}^{\infty} ((1 - \alpha)n + \alpha)nr^{n-1}
$$

$$
= (1 - \alpha) \left( \frac{1+r}{(1-r)^3} - 1 \right) + \alpha \left( \frac{1}{(1-r)^2} - 1 \right)
$$

so that $|f'_\alpha(z) - 1| < 1$ whenever $R_\alpha(r) > 0$, where

$$
R_\alpha(r) = 2(1-r)^3 + (2\alpha - 1)r - 1.
$$

Consequently, by (1), $\Re f'_\alpha(z) > 0$ for $|z| < r_0$ where $r_0 \geq r_S(\alpha)$. Next we show that $r_0 = r_S(\alpha)$. For this, we consider the function

$$
g(z) = z - \sum_{n=2}^{\infty} ((1 - \alpha)n + \alpha)z^n \in \mathcal{N}(\alpha).
$$

It is a simple exercise to see that

$$
g(z) = \frac{z(1 - (4 - \alpha)z + 2z^2)}{(1 - z)^2} = 2z - \frac{\alpha z}{1 - z} - \frac{(1 - \alpha)z}{(1 - z)^2}
$$

so that

$$
g'(z) = \frac{2(1-z)^3 - 1 + (2\alpha - 1)z}{(1-z)^3}
$$

and therefore, we have that $g'(r_S(\alpha)) = 0$. This observation shows that $g$ cannot be univalent in $|z| < r$ if $r > r_S(\alpha)$. Thus, $r_0 = r_S(\alpha)$. \hfill \square

**Proof of Theorem 4.** Assume that $f, g \in \mathcal{U}$ with $\frac{f(z) + g(z)}{z} \not= 0$ in the unit disk $\mathbb{D}$. Then

$$
|U_f(z) - 1| \leq |z|^2 \quad \text{and} \quad |U_g(z) - 1| \leq |z|^2.
$$

Further, let

$$
\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots \quad \text{and} \quad \frac{z}{g(z)} = 1 + c_1 z + c_2 z^2 + \cdots,
$$

where $b_1 = -f''(0)/2$ and $c_1 = -g''(0)/2$ so that $|b_1| \leq 2$ and $|c_1| \leq 2.$
Then, (3) gives that
\[ \frac{z}{f(z)} = 1 + b_1 z - \int_0^1 \frac{w_1(tz)}{t^2} \, dt \quad \text{and} \quad \frac{z}{g(z)} = 1 + c_1 z - \int_0^1 \frac{w_2(tz)}{t^2} \, dt, \quad z \in \mathbb{D} \]
for some analytic functions \( w_j : \mathbb{D} \to \mathbb{D} \) such that \( w_j(0) = w'_j(0) = 0 \) for \( j = 1, 2 \).
Thus, for \( |z| = r \), we have
\[ S_1 = \frac{1}{2} \left| \frac{z}{f(z)} - \frac{z}{g(z)} \right| \leq \frac{1}{2} |b_1 - c_1| r + r^2 \leq 2r + r^2 \]
since \( |b_1 \pm c_1| \leq |b_1| + |c_1| \leq 4 \). Also, for \( |z| = r \), we have that
\[ S_2 = \frac{1}{2} \left| \frac{z}{f(z)} + \frac{z}{g(z)} \right| \geq 1 - \frac{|b_1 + c_1|}{2} r - \sum_{n=2}^{\infty} \frac{|b_n| + |c_n|}{2} r^n. \]
By the Cauchy-Schwarz inequality and Lemma 1(c), it follows that
\[ \sum_{n=2}^{\infty} |b_n|r^n \leq \left( \sum_{n=2}^{\infty} (n-1)^2 |b_n|^2 \right)^{1/2} \left( \sum_{n=2}^{\infty} \frac{r^{2n}}{(n-1)^2} \right)^{1/2} \leq r \sqrt{A(r)} \]
where
\[ A(r) = \sum_{n=1}^{\infty} \frac{r^{2n}}{n^2} \leq \left( \sum_{n=1}^{\infty} \frac{1}{n^4} \right)^{1/2} \left( \sum_{n=1}^{\infty} r^{4n} \right)^{1/2} = \frac{\pi^2}{\sqrt{90}} \frac{r^2}{\sqrt{1 - r^4}}. \]
Therefore, we have
\[ \sum_{n=2}^{\infty} |b_n|r^n \leq r \sqrt{A(r)} \leq \frac{\pi r^2}{\sqrt{90} \sqrt{1 - r^4}} \]
and similarly,
\[ \sum_{n=2}^{\infty} |c_n|r^n \leq r \sqrt{A(r)} \leq \frac{\pi r^2}{\sqrt{90} \sqrt{1 - r^4}}. \]
Using these two inequalities, we deduce that
\[ S_2 \geq 1 - 2r - \frac{\pi r^2}{\sqrt{90} \sqrt{1 - r^4}}. \]
Next we consider the function \( F \) defined by (5) so that
\[ U_F(z) - 1 = \left( \frac{z}{F(z)} \right)^2 F'(z) - 1 = \left( \frac{2z}{f(z) + g(z)} \right)^2 \frac{f'(z) + g'(z)}{2} - 1. \]
We may now rewrite the right hand expression and obtain

\[
|U_F(z) - 1| = \left| 2U_f(z) \frac{f^2(z)}{(f(z) + g(z))^2} + 2U_g(z) \frac{g^2(z)}{(f(z) + g(z))^2} + \frac{(f(z) - g(z))^2}{(f(z) + g(z))^2} \right|
\]

\[
\leq 2|z|^2 \left( |f(z)|^2 + |g(z)|^2 + |f(z) - g(z)|^2 \right) \frac{1}{|f(z) + g(z)|^2}
\]

\[
= |z|^2 + (1 + |z|^2) \frac{|f(z) - g(z)|^2}{|f(z) + g(z)|^2}
\]

\[
= |z|^2 + (1 + |z|^2) \frac{S_2^2}{S_1^2}
\]

Using the estimates for \(S_1\) and \(S_2\) from (7) and (8), we find that for \(|z| = r\),

\[
|U_F(z) - 1| \leq r^2 + (1 + r^2) \frac{(2r + r^2)^2}{\left(1 - 2r - \frac{\pi}{\sqrt{90}} \frac{r^2}{\sqrt{1 - r^2}}\right)^2}.
\]

Thus, \(|U_F(z) - 1| < 1\) for \(|z| = r < r_0\) (and hence, by the maximum modulus theorem, \(F \in \mathcal{U}\) for \(|z| < r_0\)) if \(0 < r < r_0\), where \(r_0 \approx 0.262453\) is the root of the equation (6).

**Corollary 3.** Let \(f, g \in \mathcal{U}\) with \(f(z) + g(z) \neq 0\) in the unit disk \(\mathbb{D}\), \(f''(0) + g''(0) = 0\), and \(F\) be defined by (5). Then \(F \in \mathcal{U}\) for \(|z| < r_0\), where \(r_0 \approx 0.3512\) is the root of the equation

\[
r^2 + (1 + r^2) \frac{(2r + r^2)^2}{\left(1 - \frac{\pi}{\sqrt{90}} \frac{r^2}{\sqrt{1 - r^2}}\right)^2} = 1.
\]

**Proof.** The hypothesis that \(f''(0) + g''(0) = 0\) gives that \(b_1 + c_1 = 0\) and therefore, it suffices to observe that the estimate for \(S_2\) in the proof of Theorem 4 takes the form

\[
S_2 \geq 1 - \frac{\pi r^2}{\sqrt{90} \sqrt{1 - r^4}}
\]

and the rest of the proof is similar.  \(\square\)

Similarly, by the obvious observation in the proof of Theorem 4, we easily have the following.

**Corollary 4.** Let \(f, g \in \mathcal{U}\) with \(f(z) + g(z) \neq 0\) in the unit disk \(\mathbb{D}\), either \(f''(0) = 0\) or \(g''(0) = 0\), and \(F\) be defined by (5). Then \(F \in \mathcal{U}\) for \(|z| < r_0\), where \(r_0 \approx 0.400502\) is the root of the equation

\[
r^2 + (1 + r^2) \frac{(r + r^2)^2}{\left(1 - r - \frac{\pi}{\sqrt{90}} \frac{r^2}{\sqrt{1 - r^4}}\right)^2} = 1.
\]
Corollary 5. Let \( f, g \in \mathcal{U} \) with \( f(z) + g(z) \neq 0 \) in the unit disk \( \mathbb{D} \), \( f''(0) = g''(0) = 0 \), and \( F \) be defined by (5). Then \( F \in \mathcal{U} \) for \( |z| < r_0 \), where \( r_0 \approx 0.667827 \) is the root of the equation
\[
 r^2 + (1 + r^2) \left( 1 - \frac{r^4}{1 - r^4} \right)^2 = 1.
\]

As in Corollaries 3-5, Theorem 5 may be stated with an improved form in the cases where either \( f''(0) + g''(0) = 0 \), or \( f''(0) = 0 \) or \( g''(0) = 0 \), or \( f''(0) = g''(0) = 0 \), respectively. The sharpness of radii \( r_0 \) in Theorem 4 and related corollaries are open. Moreover, if
\[
 \mathcal{U}_2 := \{ f \in \mathcal{U} : f''(0) = 0 \},
\]
then the inequality (4) shows that each \( f \in \mathcal{U}_2 \) satisfies the condition \( \text{Re} \left( \frac{f(z)}{z} \right) > 1/2 \) in \( \mathbb{D} \) and hence, \( \mathcal{U}_2 \subset \mathcal{N} \). Therefore, it is natural to ask the analog of Corollary 5 when \( f, g \in \mathcal{N} \).

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