Note on ETH of descendant states in 2D CFT

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Abstract

We investigate the eigenstate thermalization hypothesis (ETH) of highly excited descendant states in two-dimensional large central charge $c$ conformal field theory. We use operator product expansion of twist operators to calculate the short interval expansions of entanglement entropy and relative entropy for an interval of length $\ell$ up to order $\ell^{12}$. Using these results to ensure ETH of a heavy state when compared with the canonical ensemble thermal state up to various orders of $c$, we get the constraints on the expectation values of the first few quasiprimary operators of vacuum conformal family at the corresponding order of $c$. Similarly, we also obtain the constraints from the first few Korteweg-de Vries charges. We check these constraints for the primary and various descendant states. We find that at most only the leading order ones of these constraints can be satisfied for the descendant states that are close to their primary states, otherwise, they are violated for the descendant states that are far away from their primary states.
1 Introduction

By eigenstate thermalization hypothesis (ETH) [1, 2], a typical highly excited energy eigenstate in a quantum chaotic system behaves like a thermal state. It explains how various statistical ensembles emerge in quantum many-body systems. It was originally formulated in terms local operators in basis of energy eigenstates and can be called local ETH. The two-dimensional (2D) conformal field theory (CFT) with a large central charge $c$ is dual to quantum gravity in three-dimensional (3D) anti-de Sitter (AdS) space with a small Newton constant [3], and it is the precursor of AdS/CFT correspondence [4–6]. ETH in a 2D large $c$ CFT is interesting and is related to the information lost paradox of black hole in 3D AdS space, i.e. the Bañados-Teitelboim-Zanelli (BTZ) black hole [7]. In fact, both the diagonal and off-diagonal parts of the local ETH in 2D CFT are consistent with the coarse-grained results obtained from modular covariance of the one-point and multi-points functions on a torus [8–11].

As a generalization of local ETH, subsystem ETH was proposed in [12, 13] and one directly compares the reduced density matrix (RDM) of the excited energy eigenstate with the RDM of the thermal state. By investigating the two-point functions of light operators and entanglement entropy (EE), it was found that ETH is satisfied for the heavy primary states at the leading order of large $c$ expansion, and thus the heavy primary states behave like the canonical ensemble thermal states [14, 17]. With $1/c$ corrections, it was found that the RDM’s of the primary excited state and canonical ensemble
thermal state are in fact different by comparing their Rényi entropy, entanglement entropy, relative entropy, trace square distance, and other quantities \cite{12,18,22}. A possible resolution is to replace the canonical thermal state with the generalized Gibbs ensemble (GGE) thermal state \cite{23} by including an infinite number of Korteweg-de Vries (KdV) hierarchy conserved charges \cite{24,26} and their corresponding chemical potentials in the ensemble \cite{13,22}. We will briefly discuss about ETH of GGE in the conclusion part of this part.

Previous studies of ETH in 2D CFT focus only on the primary excited states. For each primary state, there exists an infinite tower of descendant states. Though the properties of descendant states are algebraically determined by those of primary states, it does not necessarily imply that they also satisfy ETH as the primary states do even at the leading order of large $c$ expansion. In this paper we will investigate the subsystem ETH for a few of special descendant states, and find that some of them behave similarly to the primary states while some others behave differently.

The remaining part of the paper is organized as follows. In section 2 we obtain the constraints on the expectations values of the first few quasiprimary operators in the vacuum conformal family for a general state to satisfy ETH in terms of EE, relative entropy, and KdV charges at different orders of $c$. In section 3 we check these constraints for the primary states and various descendant states. We conclude with discussion in section 4. We collect calculation details in the appendices. In appendix A we give some useful details of the quasiprimaries and their correlation functions in 2D CFTs, including both reviews and new calculations. In appendix B we first review the short interval expansions of the EE and relative entropy of an interval with length $\ell$ with the details of enumerating the quasiprimaries, and then calculate the results up to order $\ell^{12}$, which is higher than the order $\ell^8$ in literature. In appendix C we give the proof of a statement used in the previous appendix.

2 Constraints of expectation values from ETH

We consider a 2D large $c$ CFT on a cylinder with spatial period $L$, and choose the subregion $A$ as a short interval with length $\ell \ll L$. In appendix B we calculate the EE and relative entropy for general translationally invariant states up to order $\ell^{12}$. See (B.21) for EE and (B.27) for relative entropy. Without loss of generality, we only include the contributions from the holomorphic sector. The first few KdV charges written in terms of quasiprimary operators are listed in (A.6). One may define ETH in a 2D large $c$ CFT in terms of different quantities and at different orders of $c$. Thus, in terms of EE, relative entropy, and KdV charges at different orders of $c$, we extract the constraints on the expectations values of the first few quasiprimary operators in vacuum conformal family for a general state to satisfy ETH.

To define ETH, we also need to take the thermodynamic limit \cite{12}, i.e., taking the energy $E$ and the total length $L$ to infinity but keeping the energy density $\varepsilon = E/L$ to be finite. In the thermodynamic limit, the inverse temperature $\beta$ and the interval length $\ell$ satisfy $\beta/L \to 0$, $\ell/L \to 0$, but there is no requirement for $\ell/\beta$. All the constraints in this section should be understood as under the thermodynamic limit. To do short interval expansions of the EE and relative entropy we further
require $\ell \ll \beta$. In summary we need $\ell \ll \beta \ll L$, and the constraints should satisfy for all orders of the expansion of $\ell/\beta$.

### 2.1 Constraints for all orders of large $c$

For two general states $\rho$, $\sigma$, requiring $S_{A,\rho} = S_{A,\sigma}$ we get the constraints

$$
\langle T \rangle_\rho = \langle T \rangle_\sigma, \quad \langle A \rangle_\rho = \langle A \rangle_\sigma \quad \text{or} \quad \langle A \rangle_\rho + \langle A \rangle_\sigma = \frac{2(5c + 22)}{5c} \langle T \rangle_\sigma^2, \quad \cdots.
$$

(2.1)

From $S(\rho_A \| \sigma_A) = 0$, we get

$$
\langle T \rangle_\rho = \langle T \rangle_\sigma, \quad \langle A \rangle_\rho = \langle A \rangle_\sigma, \quad \langle D \rangle_\rho = \langle D \rangle_\sigma, \quad \cdots.
$$

(2.2)

In fact $S(\rho_A \| \sigma_A) = 0$ is equivalent to $\rho_A = \sigma_A$, and so all local operators $\{X\}$ satisfy $\langle X \rangle_\rho = \langle X \rangle_\sigma$.

For the KdV charges $Q_{2k-1}^{\rho} = Q_{2k-1}^{\beta}$, $k = 1, 2, 3, \cdots$, we get

$$
\langle T \rangle_\rho = \langle T \rangle_\beta, \quad \langle A \rangle_\rho = \langle A \rangle_\beta, \quad \langle D \rangle_\rho = \langle D \rangle_\beta, \quad \cdots
$$

(2.3)

Generally, the three sets of constraints are not equivalent, and their relations are summarized in figure 1. In fact, two states having the same KdV charges, i.e. satisfying (2.3), lead to non-vanishing relative entropy

$$
S(\rho_A \| \sigma_A) = \frac{25(5c^2 + 203c + 791)\ell^2}{105080976c(2c - 1)(5c + 22)}(\langle B \rangle_\rho - \langle B \rangle_\sigma)^2 + O(\ell^4).
$$

(2.4)

![Figure 1: Relations of the three sets of constraints from the EE, relative entropy and KdV charges.](image_url)

More specifically for requiring ETH, we choose the state $\sigma$ to be the canonical thermal state $\rho_\beta$ and compare its RDM $\rho_{A,\beta}$ with the RDM $\rho_A$ of the state $\rho$. Because the modular Hamiltonian of RDM $\rho_{A,\beta}$ is a local integral of the stress tensor [27], requiring $S(\rho_A \| \rho_{A,\beta}) = 0$ or $S(\rho_{A,\beta} \| \rho_A) = 0$ is equivalent to requiring $S_A = S_{A,\beta}$ [12], which then yields for all orders of $c$,

$$
\langle T \rangle_\rho = \langle T \rangle_\beta, \quad \langle A \rangle_\rho = \langle A \rangle_\beta, \quad \langle B \rangle_\rho = \langle B \rangle_\beta, \quad \langle D \rangle_\rho = \langle D \rangle_\beta, \quad \cdots
$$

(2.5)

with the expectation values in thermal state as given (A.9). One can then see the KdV charges of the thermal state as given in (A.10). On the other hand, By requiring that KdV charges $Q_{2k-1}^\rho = Q_{2k-1}^\beta$, $k = 1, 2, 3, \cdots$, we get

$$
\langle T \rangle_\rho = \langle T \rangle_\beta, \quad \langle A \rangle_\rho = \langle A \rangle_\beta, \quad \langle D \rangle_\rho = \langle D \rangle_\beta, \quad \cdots
$$

(2.6)

Again, constraints (2.6) do not lead to (2.5). A state $\rho$ satisfying (2.6) leads to possibly non-vanishing relative entropy.
2.2 Constraints for fixed orders of large \( c \)

In thermodynamical limit, the exact form of the thermal state EE is known \(^{28}\)

\[
S_{A,\beta} = \frac{c}{6} \log\left(\frac{\beta}{\pi \epsilon} \sinh \frac{\pi \ell}{\beta}\right). \tag{2.7}
\]

We may relax the ETH condition up to different orders of \( c \), i.e., by requiring \( S_A - S_{A,\beta} = O(c^0) \), then we get the leading order constraints of EE

\[
\langle T \rangle_\rho = -\frac{\pi^2 c}{6 \beta^2} + t_0 + O(1/c), \quad \langle A \rangle_\rho = \frac{\pi^4 c^2}{36 \beta^4} + a_1 c + O(c^0),
\]

\[
\langle B \rangle_\rho = O(c), \quad \langle D \rangle_\rho = -\frac{\pi^6 c^3}{216 \beta^6} - \frac{c^2(\pi^4 t_0 + 6 \pi^2 \beta^2 a_1)}{12 \beta^4} + O(c), \quad \ldots. \tag{2.8}
\]

with \( t_0, a_1 \) being arbitrary order \( c^0 \) constants. On the other hand, if we ask for more stringent ETH condition by requiring \( S_A - S_{A,\beta} = O(1/c) \), we get

\[
\langle T \rangle_\rho = -\frac{\pi^2 c}{6 \beta^2} + O(1/c), \quad \langle A \rangle_\rho = \frac{\pi^4 c^2}{36 \beta^4} + \frac{11 \pi^4 c}{90 \beta^4} + O(c^0),
\]

\[
\langle B \rangle_\rho = \frac{62 \pi^6 c}{525 \beta^6} + O(c^0), \quad \langle D \rangle_\rho = -\frac{\pi^6 c^3}{216 \beta^6} - \frac{11 \pi^6 c^2}{180 \beta^6} + O(c), \quad \ldots. \tag{2.9}
\]

We call them the next-to-leading order constraints of EE.

Similarly, we can obtain the constraints by requiring the two relative entropies \( S(\rho_A || \rho_{A,\beta}) \) and \( S(\rho_{A,\beta} || \rho_A) \) to be of different orders. If we require \( S(\rho_A || \rho_{A,\beta}) = O(c^0) \) or \( S(\rho_{A,\beta} || \rho_A) = O(c^0) \), we get exactly the same results as the leading order constraints of EE \(^{28}\). However, by requiring \( S(\rho_A || \rho_{A,\beta}) = O(1/c) \) or \( S(\rho_{A,\beta} || \rho_A) = O(1/c) \), we get the next-to-leading order constraints of relative entropy

\[
\langle T \rangle_\rho = -\frac{\pi^2 c}{6 \beta^2} + t_0 + O(1/c), \quad \langle A \rangle_\rho = \frac{\pi^4 c^2}{36 \beta^4} + \frac{(11 \pi^4 - 30 \pi^2 \beta^2 t_0) c}{90 \beta^4} + O(c^0),
\]

\[
\langle B \rangle_\rho = \frac{62 \pi^6 c}{525 \beta^6} + O(c^0), \quad \langle D \rangle_\rho = -\frac{\pi^6 c^3}{216 \beta^6} - \frac{(11 \pi^6 - 15 \pi^4 \beta^2 t_0) c^2}{180 \beta^6} + O(c), \quad \ldots. \tag{2.10}
\]

which are different from \(^{2,9}\) for the next-to-leading order constraints of EE.

As for the KdV charges, we require the following quantities to be small,

\[
\frac{Q^\rho_{2k-1}}{Q^\beta_{2k-1}}, \quad k = 1, 2, 3, \ldots. \tag{2.11}
\]

Note that these quantities do not depend on normalization convention of the KdV charges. Requiring \(^{2,11}\) to be at order \( 1/c \), we get the leading order constraints of KdV charges

\[
\langle T \rangle_\rho = -\frac{\pi^2 c}{6 \beta^2} + O(1/c), \quad \langle A \rangle_\rho = \frac{\pi^4 c^2}{36 \beta^4} + O(c^0),
\]

\[
\langle B \rangle_\rho = b_2 c^2 + O(c), \quad \langle D \rangle_\rho = -\frac{(\pi^6 - 10 \beta^6 b_2) c^3}{216 \beta^6} + O(c^2), \quad \ldots. \tag{2.12}
\]

Requiring \(^{2,11}\) to be at order \( 1/c^2 \), we get the next-to-leading order constraints of KdV charges

\[
\langle T \rangle_\rho = -\frac{\pi^2 c}{6 \beta^2} + O(1/c), \quad \langle A \rangle_\rho = \frac{\pi^4 c^2}{36 \beta^4} + \frac{11 \pi^4 c}{90 \beta^4} + O(c^0), \quad \langle B \rangle_\rho = b_2 c^2 + b_1 c + O(c^0),
\]

\[
\langle D \rangle_\rho = -\frac{(\pi^6 - 10 \beta^6 b_2) c^3}{216 \beta^6} - \frac{(151 \pi^6 - 105 \beta^6 b_1 - 1344 \beta^6 b_2) c^2}{2268 \beta^6} + O(c), \quad \ldots. \tag{2.13}
\]
3 Checks for primary and various descendant eigenstates

We check the constraints obtained in the previous section for various highly excited energy eigenstates. The states to be considered include the excited state $|\phi\rangle$ with $\phi$ a primary operator of conformal weight $h_\phi$, and descendant states of the same conformal family $|\tilde{\phi}\rangle$, $|\partial^m \phi\rangle$, $|\partial^m \tilde{\phi}\rangle$. We have the definition $\tilde{\phi} = \langle T \phi \rangle - \frac{3}{2(2\kappa+1)} \partial^2 \phi$. We also consider the vacuum conformal family descendant states $|\partial^m T\rangle$, $|\partial^m A\rangle$. Note that $|\partial^m T\rangle$ is just a special case of $|\partial^m \tilde{\phi}\rangle$ with $h_\phi = 0$. One can see details of these states in appendix A.

Note that in this section we only check the aforementioned constraints up to level 6. If the constraints are violated, the results are conclusive. If the constraints are satisfied to level 6, we do not know whether the constraints will be violated at higher levels.

We take the thermodynamic limit for the CFT [12]. This requires that the level of the state is of order $L^2$ in $L \to \infty$ and is of order $c$ in large $c$ limit. In the thermodynamic limit, the energy eigenstates we consider fall into three types depending on the values of the parameters.

The type I states include $|\phi\rangle$ and $|\tilde{\phi}\rangle$ with $h_\phi = HL^2 + o(L^2)$, as well as $|\partial^m \phi\rangle$ and $|\partial^m \tilde{\phi}\rangle$ with $h_\phi = HL^2 + o(L^2)$ and $m = o(L^2)$. Note that $o(L^2)$ denoting terms satisfying $\lim_{L \to \infty} \frac{o(L^2)}{L^2} = 0$. Using results in appendix A we get the expectation values

\[
\langle T \rangle_\rho = -4\pi^2 H, \quad \langle A \rangle_\rho = 16\pi^4 H^2, \quad \langle B \rangle_\rho = 0, \quad \langle D \rangle_\rho = -64\pi^6 H^3, \quad \ldots. \tag{3.1}
\]

Neither the all-order constraints (2.5) of EE/relative entropy nor (2.6) of the KdV charges are satisfied by the above expectation values for any $H$. However, the leading constraints of EE and relative entropy (2.8) are satisfied with the following identification of the parameter

\[
H = \frac{c}{24\beta^2} + H_0 + O(1/c), \tag{3.2}
\]

and $t_0 = -4\pi^2 H_0$, $a_1 = \frac{4\pi^4}{3\beta^2} H_0$. Similarly, the leading order constraints of KdV charges (2.12) are satisfied with the following identification of the parameter,

\[
H = \frac{c}{24\beta^2} + O(c^0), \tag{3.3}
\]

and $b_2 = 0$. Despite that, none of the next-to-leading order constraints of EE (2.9), relative entropy (2.10), or KdV charges (2.13) can be satisfied.

The type II states include $|\partial^m \phi\rangle$ and $|\partial^m \tilde{\phi}\rangle$ with $h_\phi = HL^2 + o(L^2)$, $m = ML^2 + o(L^2)$.

\[
\langle T \rangle_\rho = -4\pi^2 (H + M), \quad \langle A \rangle_\rho = \frac{8\pi^4 (H + M) (2H^2 + 10HM + 5M^2)}{H}, \quad \langle B \rangle_\rho = 0,
\]

\[
\langle D \rangle_\rho = -\frac{8\pi^6 (H + M) (8H^4 + 112H^3 M + 308H^2 M^2 + 252HM^3 + 63M^4)}{H^2}, \quad \ldots. \tag{3.4}
\]

These expectation values of quasiprimaries do not satisfy the all-order constraints (2.5) or (2.6). However, they satisfy (2.8) by the parameters

\[
H = \frac{c}{24\beta^2} + H_0 + O(1/c), \quad M = M_0 + O(1/c), \tag{3.5}
\]
with \( t_0 = -4\pi^2(H_0 + M_0) \), \( a_1 = \frac{4\pi^4}{3\beta^2}(H_0 + 3M_0) \). They also satisfy \([2.12]\) by the parameters

\[
H = \frac{c}{24\beta^2} + O(c^0), \quad M = O(c^0). \tag{3.6}
\]

Otherwise, they do not satisfy any of the next-to-leading order constraints \([2.9]\), \([2.10]\), or \([2.13]\). For \( M = O(c) \), all the constraints are violated.

The type III states include states \( |\partial^m\phi\rangle \) and \( |\partial^m\tilde{\phi}\rangle \) with \( h_\phi = o(L^2) \), \( m = ML^2 + o(L^2) \), as well as \( |\partial^mT\rangle \) and \( |\partial^mA\rangle \) with \( m = ML^2 + o(L^2) \). In thermodynamic limit, such states have divergent expectations values, and would never be close the canonical thermal state.

We summarize the results of this section in table \( \text{II} \). The states considered can at most match the thermodynamic state at leading order of large \( c \). For the descendant states considered to match the thermal state at the leading order of large \( c \), it requires that (1) the corresponding primary state matches the thermal state at the leading order of large \( c \) and (2) the level difference of the primary and descendant states is at most of order \( L^2 \) in large \( L \) limit and the order \( L^2 \) part, if non-vanishing, is at most of order \( c^0 \) in large \( c \) limit. We do not know if the results and requirements apply to general descendant states. Roughly, we conjecture that the descendant states that are close to their primary states satisfy ETH at the leading order of large \( c \), and descendant states that are far away from their primary states do not satisfy ETH.

| type | state | constraints to level 6 |
|------|-------|------------------------|
|      |       | leading | next-to-leading | all |
| I    | \(|\phi\rangle\) with \( h_\phi = HL^2 + o(L^2) \) | ✓ | × | × |
|      | \(|\tilde{\phi}\rangle\) with \( h_\phi = HL^2 + o(L^2) \) | ✓ | × | × |
|      | \(|\partial^m\phi\rangle\) with \( h_\phi = HL^2 + o(L^2) \) \( m = o(L^2) \) | ✓ | × | × |
|      | \(|\partial^m\tilde{\phi}\rangle\) with \( h_\phi = HL^2 + o(L^2) \) \( m = o(L^2) \) | ✓ | × | × |
| II   | | | |
|      | \(|\partial^m\phi\rangle\) with \( h_\phi = HL^2 + o(L^2) \) \( m = ML^2 + o(L^2) \) | ✓ | × | × |
|      | \(|\partial^m\tilde{\phi}\rangle\) with \( h_\phi = HL^2 + o(L^2) \) \( m = ML^2 + o(L^2) \) | ✓ | × | × |
|      | \(|\partial^mT\rangle\) with \( m = ML^2 + o(L^2) \) | × | × | × |
|      | \(|\partial^mA\rangle\) with \( m = ML^2 + o(L^2) \) | × | × | × |

Table 1: The three types primary and descendant states we consider in this section, and whether they satisfy the ETH constraints derived from EE, relative entropy, and KdV charges up to level 6. In the 2nd column we have definitions \( H = \frac{c}{24\beta^2} + O(c^0) \), \( M = O(c^0) \). Note that for \( M = O(c) \) all the constraints are violated. The 3rd, 4th, and 5th columns all apply to the constraints derived from EE, relative entropy, and KdV charges. We mark ✓ for constraints that are satisfied and mark × otherwise.

4 Conclusion and discussion

We calculated the short interval expansions of EE and relative entropy to order \( \ell^{12} \). Using the results to require ETH for a highly excited state, we got the leading order and next-to-leading order constraints...
in the large \( c \) expansion on the expectation values of the first few quasiprimary operators in the vacuum conformal family with respect to the state considered. We also obtained the constraints from the first few KdV charges.

We checked the constraints for the primary and various descendant states. We found that these constraints can only be satisfied for at most the leading order of large \( c \), and even the leading order constraints are violated for descendant states that are far away from their primary states. Note that when we say a descendant state is close to or far away from its primary state, we refer to just the primary state in the same conformal family of the descendant state.

In [29], we have investigated the conditions for a state to have classical gravity dual, and we call a state satisfying these conditions a geometric state. Based on the holographic EE and Rényi entropy [30,32], the geometric state conditions are obtained by requiring the Rényi entropy to be at most of order \( c \) in large \( c \) limit. There are some differences between the geometric state conditions and the ETH state constraints in this paper as listed below. (1) For the geometric state conditions, the states does not necessarily have translational symmetry and so the one-point functions are not necessarily constants. For the ETH state constraints, we consider the globally excited energy eigenstate, and the one-point functions are constants. (2) In the geometric state conditions we do not require the state to be at high energy, and so we focus on the order of \( c \) and do not care about the order of \( L \). In the ETH state constraints we need to firstly take thermodynamic limit \( L \to \infty \) and then do expansion of large \( c \).

Although there are differences mentioned above, still it is interesting to compare the results. We stress that the comparison in this paragraph is under the thermodynamic limit. For translationally invariant states, the geometric state conditions in [29] can be recast as

\[
\langle T \rangle_\rho = ct_1 + t_0 + O(1/c), \quad \langle A \rangle_\rho = c^2 t_1^2 + ca_0 + O(c^0),
\]

\[
\langle B \rangle_\rho = O(c), \quad \langle D \rangle_\rho = c^3 t_1^3 + 3c^2 t_1(a_1 - t_1 t_0) + O(c), \quad \cdots . \quad (4.1)
\]

The states satisfying the ETH state constraints of EE and relative entropy (2.8), (2.9), (2.10) also satisfy the geometric state constraints (4.1). However, the states satisfying the ETH constraints of KdV charges (2.12), (2.13) do not necessarily satisfy the geometric state constraints. In fact the geometric state conditions (4.1) are not only consistent with but also equivalent to the leading order constraints of EE (2.8). Requiring the energy density be nonnegative in the thermodynamic limit, we get \( t_1 \) in (4.1) is nonpositive \( t_1 \leq 0 \). We obtain that (4.1) and (2.8) are equivalent with the identification \( t_1 = -\frac{x^2}{6\beta^2} \).

The equivalence of the geometric conditions and the leading order constraints is remarkable. The geometric conditions come from \( S_A^{(n)} = O(c) \), which leads to \( S_A = O(c) \). The leading order ETH constraints of EE come from \( S_A = S_{A,\beta} + O(c^0) \) with (2.7). A possible way to understand this is that a translationally invariant geometric state is dual to a Bañados metric [33] with constant stress tensor and in thermodynamic limit the classical Bañados geometry metric is exactly the same as the BTZ black hole metric with some identification of parameters. By the holographic EE [30,31], it ensures that the translationally invariant geometric state satisfies the leading order ETH constraints of EE (2.8).
The ETH state constraints we obtained are about how close a state $\rho$ and the canonical ensemble thermal state $\rho_\beta = \frac{1}{Z(\beta)} \sum_i e^{-\beta E_i} |i\rangle\langle i|$ with $Z(\beta) = \sum_i e^{-\beta E_i}$ can be in the thermodynamic limit. Although the RDM of a primary state is the same as RDM of the thermal state $\rho_\beta$ in the leading order of $c$, the RDM’s of the considered descendant states that are far away from the primary state are very different from the thermal state RDM. Recently we have proved that the RDM of the canonical ensemble thermal state $\rho_\beta$ and RDM of the microcanonical ensemble thermal state $\rho_E = \frac{1}{\Omega(E)} \sum_i \delta(E - E_i) |i\rangle\langle i|$ with $\Omega(E) = \sum_i \delta(E - E_i)$ are the same in thermodynamic limit \cite{34}. In both of the canonical and microcanonical ensembles, one needs to average over all the states, including both the primary and descendant states. In fact, there are far more descendant states than primary states at high energy. The density of all the states is given by Cardy formula \cite{35}

$$\Omega(E) \sim e^{\sqrt{2 \pi c \varepsilon L}} / \varepsilon^{3}, \quad (4.2)$$

and the density of primary states is \cite{8}

$$\Omega_p(E) \sim e^{\sqrt{2 \pi (c-1) L} \left(E \frac{\varepsilon}{\pi} \right) / \varepsilon}. \quad (4.3)$$

Note that in the thermodynamic limit one has $E \to \infty$, $L \to \infty$, and $\varepsilon = E/L$ is finite. We get the ratio of the number of primary states and the number of all states is

$$\frac{\Omega_p(E)}{\Omega(E)} \sim e^{-L \left[ \sqrt{\frac{2 \pi \varepsilon}{3}} - \sqrt{\frac{2 \pi (c-1)}{3} \left( \frac{\varepsilon}{\pi L} \right) } \right]} \to 0 \text{ as } L \to \infty. \quad (4.4)$$

It is exponentially suppressed in the thermodynamic limit. So at high temperature or energy the average in the canonical or microcanonical ensemble is dominated by descendant states. It is an intriguing puzzle how the average over descendant states behaves like the primary states in large $c$ limit. Note that we have only consider a few very special descendant states in this paper, it is possible most of the descendant states at a high level behave like the primary states at the same level. Another possibility is that each descendant state is different from the primary state at the same level, while the average erase the differences. It would be nice if the issue can be addressed by investigations of general descendant states.

The RDM’s of the primary excited state and the canonical ensemble thermal state are the same at the leading order of large $c$ \cite{14,17}, but they are in fact different with subleading corrections of $1/c$ \cite{19,22}. In this paper we obtained the same results for some descendant states that are close to their primary states. A possible resolution to the mismatch of the excited states and the thermal state was proposed in \cite{19,22}, and it is to replace the canonical ensemble thermal state by the GGE \cite{23}. In 2D CFT there are an infinite number of conserved charges that commute with the Hamiltonian \cite{24,26}, which are just the KdV charges. A strong form of ETH for GGE is that the RDM a typical energy eigenstate is the same of the RDM of GGE thermal state in the thermodynamic limit. A weak form is to require the state to be an eigenstate all KdV charges. The chemical potentials of the GGE thermal state are determined by requiring that the thermal state has the same KdV charges as the excited state. As can be seen in (2.4) and figure \cite{1} two general states have the same KdV charges do not necessarily have the same RDM’s. It is an open question whether ETH of GGE is correct for the 2D large $c$ CFT.
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A Quasiprimaries and their correlation functions

In this appendix, we give some useful details of 2D CFT we need in this paper, including both reviews and new calculations. The basics of 2D CFT can be found in [36–38].

A.1 Vacuum conformal family

Many details of the quasiprimary operators in the vacuum conformal family can be found in the papers [19,39,40]. We only consider the holomorphic sector, and the anti-holomorphic sector is similar. We count the number of independent holomorphic operators at each level in the vacuum conformal family as

$$\text{tr} x^{L_0} = \prod_{k=2}^{\infty} \frac{1}{1 - x^k},$$  \hspace{1cm} (A.1)

among which the holomorphic quasiprimary operators are counted as

$$x + (1 - x)\text{tr} x^{L_0} = 1 + x^2 + x^4 + 2x^6 + 3x^8 + x^9 + 4x^{10} + 2x^{11} + 7x^{12} + 3x^{13} + O(x^{14}).$$  \hspace{1cm} (A.2)

We list these quasiprimary operators in table 2 and to level 9 their explicit forms can be found in [19,39,40].

| level | 0 | 2 | 4 | 6 | 8 | 9 | 10 | 11 | 12 | 13 | ... |
|-------|---|---|---|---|---|---|----|----|----|----|-----|
| quasiprimary | 1 | T | A | B, D | E, H, I | A^{(9)} | A^{(10,m)} | A^{(11,m)} | A^{(12,m)} | A^{(13,m)} | ... |

Table 2: Holomorphic quasiprimary operators in the vacuum conformal family. The ranges in which the m’s take values can be read in (A.2). For examples, at level 10 we have $m = 1, 2, 3, 4$, and at level 11 we have $m = 1, 2$.

At level 0, it is trivially the identity operator 1. At level 2, it is the stress tensor $T$ with the usual normalization $\alpha_T = \frac{c}{2}$. At level 4 we have the quasiprimary operator and its normalization factor

$$\mathcal{A} = (TT) - \frac{3}{10} \partial^2 T, \quad \alpha_{\mathcal{A}} = \frac{c(5c + 22)}{10}.$$  \hspace{1cm} (A.3)
At level 6, we have the orthogonalized quasiprimary operators

\[
B = (\partial T \partial T) - \frac{4}{5} (\partial^2 TT) - \frac{1}{42} \partial^4 T,
\]
\[
D = (T(\partial T)) - \frac{9}{10} (\partial^2 TT) - \frac{1}{28} \partial^4 T + \frac{93}{70c + 29} B,
\]

and their normalization factors are

\[
\alpha_B = \frac{36c(70c + 29)}{175}, \quad \alpha_D = \frac{3c(2c - 1)(5c + 22)(7c + 68)}{4(70c + 29)}.
\]

We do not need the explicit forms of other quasiprimary operators in this paper. The conformal transformation rules of the quasiprimary operators to level 8 can be found in \[19\].

In the vacuum conformal family, one can define an infinite number of mutually commuting conserved KdV charges \(Q_{2k-1}, k = 1, 2, \cdots \). In terms of the quasiprimary operators, we can write the first four KdV charges on a cylinder as \[29\].

\[
Q_1 = \frac{1}{(2\pi i)^2} \int_0^L \frac{dw}{L} T(w), \quad Q_3 = \frac{1}{(2\pi i)^4} \int_0^L \frac{dw}{L} \left[ A(w) + \frac{3}{10} \partial^2 T(w) \right],
\]
\[
Q_5 = \frac{1}{(2\pi i)^6} \int_0^L \frac{dw}{L} \left[ D - \frac{25(2c + 7)(7c + 68)}{108(70c + 29)} B - \frac{2c - 23}{108} \partial^4 A - \frac{c - 14}{280} \partial^4 T \right].
\]

For the CFT on a circle with period \(L\) in the ground state, which is just a vertical cylinder with spatial period \(L\), we have the one point functions

\[
\langle T \rangle_L = \frac{\pi^2 c}{6L^2}, \quad \langle A \rangle_L = \frac{\pi^4 c(5c + 22)}{180L^4},
\]
\[
\langle B \rangle_L = -\frac{62\pi^6 c}{525L^6}, \quad \langle D \rangle_L = \frac{\pi^6 c(2c - 1)(5c + 22)(7c + 68)}{216(70c + 29)L^6}.
\]

For the CFT on a circle with period \(L\) in thermal state with inverse temperature \(\beta \gg L\), which is a torus with module \(\tau = i\beta / L\), we use the results in \[41\] and get the one point functions expanded by \(q = e^{-2\pi \beta / L}\)

\[
\langle T \rangle_{L,q} = \frac{\pi^2}{6L^2} \left[(c - 48q^2 - 72q^3 - 144q^4 + O(q^5))\right],
\]
\[
\langle A \rangle_{L,q} = \frac{\pi^4 c(5c + 22)}{180L^4} \left[(5c + 22) + 480(5c + 22)q^2 + 2160(5c + 22)q^3 + 30240(c + 6)q^4 + O(q^5)\right],
\]
\[
\langle B \rangle_{L,q} = -\frac{2\pi^6}{525L^6} \left[31c - 1008(120c + 1)q^2 - 1512(720c + 161)q^3 - 3024(1640c + 841)q^4 + O(q^5)\right],
\]
\[
\langle D \rangle_{L,q} = \frac{\pi^6 (2c - 1)(7c + 68)}{216(70c + 29)L^6} \left[(5c + 22) + 1584(5c + 22)q^2 + 6696(5c + 22)q^3 + 432(215c - 638)q^4 + O(q^5)\right].
\]

Note that, as stated in \[41\], the above one point functions can be expanded to an arbitrary order of \(q\).

For the CFT on an infinite straight line in thermal state with inverse temperature \(\beta\), which is just a horizontal cylinder with temporal period \(\beta\), we have \[39\]

\[
\langle T \rangle_\beta = -\frac{\pi^2 c}{6\beta^2}, \quad \langle A \rangle_\beta = \frac{\pi^4 c(5c + 22)}{180\beta^4},
\]
\[
\langle B \rangle_\beta = \frac{62\pi^6 c}{525\beta^6}, \quad \langle D \rangle_\beta = -\frac{\pi^6 c(2c - 1)(5c + 22)(7c + 68)}{216(70c + 29)\beta^6}.
\]
Then we use (A.6), and get the KdV charges of the thermal state
\[
Q_1^2 = \frac{c}{24\beta^2}, \quad Q_3^3 = \frac{c(5c + 22)}{2880\beta^4}, \quad Q_5^5 = \frac{c(3c + 14)(7c + 68)}{290304\beta^6}.
\] (A.10)

We have orthogonalized the operators, such that the correlation function of two holomorphic quasiprimary operators \(X_{1,2}\) on a complex plane \(C\) with coordinate \(f\) takes the form
\[
\langle X_1(f_1)X_2(f_2) \rangle_C = \frac{\delta_{X_1 X_2 \alpha X_1}}{f_{12}^{2h_{X_1}}},
\] (A.11)
where we have defined \(f_{12} = f_1 - f_2\). The correlation function of three holomorphic quasiprimary operators \(X_{1,2,3}\) takes the form
\[
\langle X_1(f_1)X_2(f_2)X_3(f_3) \rangle_C = \frac{C_{X_1 X_2 X_3}}{f_{12}^{h_{X_1}+h_{X_2}-h_{X_3}} f_{13}^{h_{X_1}+h_{X_3}-h_{X_2}} f_{23}^{h_{X_2}+h_{X_3}-h_{X_1}}},
\] (A.12)
with \(C_{X_1 X_2 X_3}\) being the structure constant. In this paper, we need the structure constants
\[
C_{TTT} = c, \quad C_{TTA} = \frac{c(5c + 22)}{10}, \quad C_{TTB} = -\frac{2c(70c + 29)}{35},
\]
\[
C_{TTD} = 0, \quad C_{TTA} = \frac{2c(5c + 22)}{5}, \quad C_{TAB} = -\frac{24c(5c + 22)}{25},
\]
\[
C_{TAD} = \frac{3c(2c - 1)(5c + 22)(7c + 68)}{4(70c + 29)}, \quad C_{AAA} = \frac{c(5c + 22)(5c + 64)}{25},
\]
\[
C_{AAB} = -\frac{4c(5c + 22)(14c + 73)}{35}, \quad C_{AAD} = \frac{6c(2c - 1)(5c + 22)(7c + 68)}{70c + 29}.\] (A.13)

We also need the four-point functions
\[
\langle T(f_1)T(f_2)T(f_3)T(f_4) \rangle_C = \frac{c^2}{4} \left( \frac{1}{f_{12} f_{34}} + \cdots \right)_3 + \frac{c}{f_{12} f_{23} f_{34} f_{41}} + \cdots \right)_3, \]
\[
\langle T(f_1)T(f_2)T(f_3)A(f_4) \rangle_C = \frac{c(5c + 22)}{20} \left[ \left( \frac{f_{12} f_{34}}{f_{13} f_{24} f_{14} f_{23}} + \cdots \right)_3 + 2 \left( \frac{1}{f_{12} f_{14} f_{23} f_{34}} + \cdots \right)_3 \right],
\]
\[
\langle T(f_1)T(f_2)T(f_3)B(f_4) \rangle_C = -\frac{2c(70c + 29)}{35} \left[ \frac{f_{12} f_{13} f_{24} f_{14} f_{23} f_{34}}{f_{12} f_{13} f_{24} f_{14} f_{23} f_{34}} + \cdots \right)_3 + \frac{36c(4c - 1)}{5} \frac{1}{f_{12} f_{24} f_{34}}.
\]
\[
\langle T(f_1)T(f_2)T(f_3)D(f_4) \rangle_C = \frac{3c(2c - 1)(5c + 22)(7c + 68)}{4(70c + 29)} \frac{1}{f_{12} f_{24} f_{34}}
\]
\[
\langle T(f_1)T(f_2)A(f_3)A(f_4) \rangle_C = \frac{c^2(5c + 22)}{20} \left[ \frac{1}{f_{12} f_{34}} + \frac{2}{f_{12} f_{24} f_{34}} + \cdots \right)_2
\]
\[
+ \frac{c(5c + 22)}{25} \left[ \frac{1}{f_{12} f_{14} f_{23} f_{24} f_{34} f_{14} f_{23} f_{24} f_{34}} - \frac{28}{f_{12} f_{14} f_{23} f_{24} f_{34} f_{14} f_{23} f_{24} f_{34}}
\]
\[
- \frac{8}{f_{12} f_{14} f_{23} f_{24} f_{34} f_{14} f_{23} f_{24} f_{34}} + \cdots \right)_2 + 10 \left( \frac{1}{f_{12} f_{24} f_{34} f_{24} f_{34}} + \cdots \right)_2,\] (A.14)

the five-point functions
\[
\langle T(f_1)T(f_2)T(f_3)T(f_4)T(f_5) \rangle_C = \frac{c^2}{2} \left( \frac{1}{f_{12} f_{23} f_{34} f_{45}} + \cdots \right)_1 + \frac{c}{f_{12} f_{23} f_{34} f_{45} f_{35} f_{51}} + \cdots \right)_2,
\]
\[
\langle T(f_1)T(f_2)T(f_3)T(f_4)A(f_5) \rangle_C = \frac{c(5c + 22)}{20} \left[ \frac{1}{f_{12} f_{23} f_{34} f_{45}} + \cdots \right)_1 + 4 \left( \frac{1}{f_{12} f_{23} f_{15} f_{23} f_{35} f_{45}} + \cdots \right)_2
\]
\[
+ \frac{1}{f_{12} f_{23} f_{34} f_{45} f_{23} f_{35} f_{45}} + \cdots \right)_3 + 2 \left( \frac{f_{12} f_{23} f_{15} f_{23} f_{35} f_{45}}{f_{12} f_{23} f_{34} f_{45} f_{23} f_{35} f_{45}} + \cdots \right)_3,\] (A.15)
and the six-point function
\[
\langle T(f_1)T(f_2)T(f_3)T(f_4)T(f_5)T(f_6)\rangle_C = C^3 \left( \frac{1}{f_{12}^2 f_{13}^2 f_{14}^2 + \cdots} \right)_{15} + C^2 \left( \frac{1}{f_{12}^2 f_{13}^2 f_{14}^2 + \cdots} \right)_{45} + c^2 \left( \frac{1}{f_{12}^2 f_{13}^2 f_{14}^2 + \cdots} \right)_{60}.
\] (A.16)

In the above multi-point functions, we have used the notation in the form
\[
\langle \cdots \rangle_#, \tag{A.17}
\]
where \( \cdots \) denotes permutation terms of \(*\) and \# is the total number of terms in the parentheses. For example, the five-point function \(\langle T(f_1)T(f_2)T(f_3)T(f_4)A(f_5)\rangle_C\) is invariant under the permutations of \(f_1, f_2, f_3, f_4,\) and for it we have
\[
\left( \frac{1}{f_{12}^2 f_{13}^2 f_{14}^2 + \cdots} \right)_{6} = \frac{1}{f_{12}^2 f_{13}^2 f_{14}^2} + \frac{1}{f_{13}^2 f_{14}^2 f_{15}^2} + \frac{1}{f_{14}^2 f_{15}^2 f_{16}^2} + \frac{1}{f_{15}^2 f_{16}^2 f_{17}^2} + \frac{1}{f_{16}^2 f_{17}^2 f_{18}^2} + \frac{1}{f_{17}^2 f_{18}^2 f_{19}^2}. \tag{A.18}
\]

The above multi-point functions are derived from the two-point and three-point functions [A.11], [A.12], the conformal Ward identity, and the operator product expansions (OPE's)
\[
T(f)T(0) = \frac{c}{2f} + \frac{2T(0)}{f} + \frac{\partial T(0)}{f} + \cdots,
\]
\[
T(f)A(0) = \frac{(5c + 22)T(0)}{5f^4} + \frac{4A(0)}{f^2} + \frac{\partial A(0)}{f} + \cdots,
\]
\[
T(f)B(0) = \frac{4(70c + 29)}{35 f^6} \left( T(0) - \frac{f \partial T(0)}{2} + \frac{f^2 \partial^2 T(0)}{20} - \frac{48A(0)}{5f^4} - \frac{6B(0)}{f^2} + \frac{\partial B(0)}{f} + \cdots \right),
\]
\[
T(f)D(0) = \frac{15(2c - 1)(7c + 68)A(0)}{2(70c + 29)f^4} + \frac{6D(0)}{f^2} + \frac{\partial D(0)}{f} + \cdots. \tag{A.19}
\]

A.2 Non-vacuum conformal family

We consider the non-vacuum conformal family of the primary operator \(\phi\), with conformal weight \(h_\phi > 0\) and normalization \(\alpha_\phi\). Without loss of generality and for simplicity, we choose it to be holomorphic, and the generalization to a non-chiral primary operator is straightforward. The next quasiprimary operator in the conformal family of \(\phi\) is
\[
\tilde{\phi} = (T\phi) - \frac{3}{2(2h_\phi + 1)} \partial^2 \phi, \tag{A.20}
\]
with conformal weight \(h_{\tilde{\phi}} = h_\phi + 2\), and normalization factor
\[
\alpha_{\tilde{\phi}} = \frac{\alpha_\phi [16h_\phi^2 + 2(c - 5)h_\phi + c]}{2(2h_\phi + 1)}. \tag{A.21}
\]

Using state operator correspondence in 2D CFT, on a cylinder with spatial period \(L\) we construct the ket and bra states
\[
|\phi\rangle = \phi(0)|0\rangle, \quad \langle \phi| = \langle 0|\phi(\infty) = \lim_{z \to 0} \langle 0|z^{-2h_\phi}\phi(z^{-1}),
\]
\[
|\partial^m \phi\rangle = \partial^m \phi(0)|0\rangle, \quad \langle \partial^m \phi| = \langle 0|\partial^m \phi(\infty) = \lim_{z \to 0} \langle 0|\partial^m_z [z^{-2h_\phi}\phi(z^{-1})]. \tag{A.22}
\]
Similarly we define $|\tilde{\phi}\rangle$, $\langle \tilde{\phi}|$, $|\partial^m \tilde{\phi}\rangle$, $\langle \partial^m \tilde{\phi}|$. The state $|\phi\rangle$ is primary, and states $|\tilde{\phi}\rangle$, $|\partial^m \tilde{\phi}\rangle$, $|\partial^m \tilde{\phi}|$ are descendant. We have the normalization factors

$$\alpha_{\partial^m \phi} = \langle\partial^m \phi|\partial^m \tilde{\phi}\rangle = \frac{m!(2h_\phi + m - 1)!}{(2h_\phi - 1)!} \alpha_\phi,$$

$$\alpha_{\partial^m \tilde{\phi}} = \langle\partial^m \tilde{\phi}|\partial^m \tilde{\phi}\rangle = \frac{m!(2h_\phi + m + 3)!}{(2h_\phi + 3)!} \alpha_\tilde{\phi}. \quad (A.23)$$

Using the states, we have the normalized density matrices

$$\rho_{L,\phi} = \frac{|\phi\rangle\langle\phi|}{\alpha_\phi}, \quad \rho_{L,\partial^m \phi} = \frac{|\partial^m \phi\rangle\langle\partial^m \phi|}{\alpha_{\partial^m \phi}},$$

$$\rho_{L,\tilde{\phi}} = \frac{|\tilde{\phi}\rangle\langle\tilde{\phi}|}{\alpha_{\tilde{\phi}}}, \quad \rho_{L,\partial^m \tilde{\phi}} = \frac{|\partial^m \tilde{\phi}\rangle\langle\partial^m \tilde{\phi}|}{\alpha_{\partial^m \tilde{\phi}}}. \quad (A.24)$$

As in [19], we get the structure constants

$$C_{\phi T} = \alpha_\phi h_\phi, \quad C_{\phi A} = \frac{\alpha_\phi h_\phi(5h_\phi + 1)}{5}, \quad C_{\phi B} = -\frac{2\alpha_\phi h_\phi(14h_\phi + 1)}{35},$$

$$C_{\phi D} = \frac{\alpha_\phi h_\phi[(70c + 29)h_\phi^2 + (42c - 57)h_\phi + 2(4c - 1)]}{70c + 29}. \quad (A.25)$$

From the structure constants and the conformal transformation rules of $T$, $A$, $B$, $D$, we get the expectation values

$$\langle T \rangle_\phi = -\frac{\pi^2(24h_\phi - c)}{6L^2}, \quad \langle A \rangle_\phi = \frac{\pi^4[2880h_\phi^2 - 240(c + 2)h_\phi + c(5c + 22)]}{180L^4},$$

$$\langle B \rangle_\phi = \frac{2\pi^6(504h_\phi - 31c)}{525L^6},$$

$$\langle D \rangle_\phi = -\frac{\pi^6}{216(70c + 29)L^6} [13824(70c + 29)h_\phi^3 - 1728(c + 4)(70c + 29)h_\phi^2 + 72(70c^3 + 617c^2 + 938c - 248)h_\phi - c(2c - 1)(5c + 22)(7c + 68)]. \quad (A.26)$$

Similarly, we get the structure constants

$$C_{\tilde{\phi} T} = \alpha_{\tilde{\phi}}(h_\phi + 2), \quad C_{\tilde{\phi} A} = \frac{\alpha_{\tilde{\phi}}[10h_\phi^3 + 127h_\phi^2 + 5(2c + 3)h_\phi + 5c + 22]}{5(2h_\phi + 1)},$$

$$C_{\tilde{\phi} B} = -\frac{2\alpha_{\tilde{\phi}}[26h_\phi^3 + 2368h_\phi^2 + (280c - 1227)h_\phi + 2(70c + 29)]}{35(2h_\phi + 1)},$$

$$C_{\tilde{\phi} D} = \frac{\alpha_{\tilde{\phi}} h_\phi}{(70c + 29)(2h_\phi + 1)} [2(70c + 29)h_\phi^3 + (4354c + 1655)h_\phi^2 + (420c^2 + 568c - 12421)h_\phi + 210c^2 + 1103c + 7558]. \quad (A.27)$$

Then we get the expectations values

$$\langle T \rangle_\tilde{\phi} = -\frac{\pi^2(24h_\tilde{\phi} - c + 48)}{6L^2},$$

$$\langle A \rangle_\tilde{\phi} = \frac{\pi^4[5760h_\tilde{\phi}^3 - 480(c - 148)h_\tilde{\phi}^2 + 2(5c^2 + 2302c + 1680)h_\tilde{\phi} + (c + 480)(5c + 22)]}{180(2h_\tilde{\phi} + 1)L^4},$$

$$\langle B \rangle_\tilde{\phi} = \frac{2\pi^6[1936368h_\tilde{\phi}^2 + 2(120929c - 603540)h_\tilde{\phi} + 120929c + 1008]}{525(2h_\tilde{\phi} + 1)L^6}. \quad (A.28)$$
\[\langle D \phi \rangle = - \frac{\pi^6}{216(70c + 29)(2h_\phi + 1)L^6} [27648(70c + 29)h_\phi^4 - 3456(c - 240)(70c + 29)h_\phi^3 \\
+ 144(70c^3 + 19937c^2 - 139570c - 1090280)h_\phi^2 - 2(70c^4 + 109313c^3 - 324562c^2 \\
- 6990752c - 52852896)h_\phi - (c + 1584)(2c - 1)(5c + 22)(7c + 68)]. \]  
(A.28)

For general quasiprimary operators \( X, Y \), we have the three point function on a complex plane

\[\langle \frac{\partial}{\partial m} Y(\infty) \frac{\partial}{\partial p} X(f) \frac{\partial}{\partial m} Y(0) \rangle_C = C_{YXY} \frac{1}{p^{h_X + p}} (-p)^{h_Y} \sum_{i=0}^{m} (C_{h_{\delta} + i - 1}^i)^2 C_{2h_{\delta} - h_X + m - i - 1}, \]  
(A.29)

with the binomial coefficient \( C_{x}^y = \Gamma(x+1)/\Gamma(y+1)\Gamma(x-y+1) \). Using the three point function we get the expectations values \( \langle X \rangle_{\frac{\partial}{\partial m} \phi} \), \( \langle X \rangle_{\frac{\partial}{\partial m} \tilde{\phi}} \) with \( X = T, A, B, D \). Similarly, we obtain the expectations values \( \langle X \rangle_{\frac{\partial}{\partial m} T} \), \( \langle X \rangle_{\frac{\partial}{\partial m} A} \) with \( X = T, A, B, D \). We will not give the explicit forms of these expectations values.

B Short interval expansion of EE and relative entropy

We review the short interval expansion of EE and relative entropy from OPE of twist operators. We also obtain the EE and relative entropy to higher orders than the ones in literature. The method twist operators was proposed in [28] to calculate Rényi entropy in 2D CFT. The OPE of twist operators was formulated in [42–44]. Without loss of generality, we only include the contributions from the holomorphic sector, and the contributions from the anti-holomorphic sector can be added easily.

B.1 OPE of twist operators and enumerating quasiprimaries

We consider one short interval \( A = [0, \ell] \) on a general Riemann surface \( \mathcal{R} \) with translational symmetry, and the constant time slice is in a state with density matrix \( \rho \). Tracing the degrees of freedom of the complement of \( A \) that we call \( \bar{A} \), we get the RDM \( \rho_A = \text{tr}_{\bar{A}} \rho \). To get the EE

\[S_A = -\text{tr}_A (\rho_A \log \rho_A), \]  
(B.1)

we use the replica trick and first calculate the Rényi entropy

\[S^{(n)}_A = -\log \frac{\text{tr}_A \rho_A^n}{n - 1}, \]  
(B.2)

and then take the \( n \to 1 \) limit. We calculate the partition function \( \text{tr}_A \rho_A^n \) of the CFT on an \( n \)-fold Riemann surface \( \mathcal{R}^n \), and it equals the two-point function of twist operators \( \sigma, \tilde{\sigma} \) in the \( n \)-fold CFT on single copy of the Riemann surface \( \mathcal{R} \) [28]

\[\text{tr}_A \rho_A^n = \langle \sigma(\ell) \tilde{\sigma}(0) \rangle_\mathcal{R}. \]  
(B.3)

Note that in this paper we just focus on the holomorphic sector of the twist operators.

The twist operators are primary operators with conformal weights [28]

\[h_\sigma = h_{\tilde{\sigma}} = \frac{c(n^2 - 1)}{24n}. \]  
(B.4)
We can write the OPE of twist operators as \( \sigma(z) \tilde{\sigma}(w) = \frac{c_n}{(z-w)^{2h_\sigma}} \sum_K d_K \sum_{p=0}^{\infty} \frac{\epsilon_K^p}{p!} (z-w)^{h_K+p} \partial^p \Phi_K(w), \) \(^{(B.5)}\)

with \( c_n \) being the normalization factor, the summation \( K \) being over all the orthogonalized holomorphic quasiprimary operators \( \Phi_K \) in CFT\(^n\), and \( h_K \) being the conformal weight of \( \Phi_K \). We have definition

\[ c_K^p \equiv \frac{C^p_{h_K+p-1}}{C^2_{2h_K+p-1}}, \]

with \( C^n_p \) denoting the binomial coefficient. The OPE coefficient \( d_K \) can be calculated as \( \alpha_K \)

\[ d_K = \frac{1}{\alpha_K \ell^{h_K}} \lim_{z \to \infty} z^{2h_K} \langle \Phi_K(z) \rangle_{\mathcal{R}_{n,1}}, \] \(^{(B.7)}\)

with \( \alpha_K \) being the normalization of \( \Phi_K \). We have used \( \mathcal{R}_{n,1} \) to denote the \( n \)-fold Riemann surface \( \mathcal{R} \) that results from the replica trick for one interval \( A = [0, \ell] \) on the complex plane \( \mathbb{C} \). To calculate the right-hand side of \( \langle \Phi_K(z) \rangle_{\mathcal{R}_{n,1}} \), we map \( \mathcal{R}_{n,1} \) with coordinate \( z \) to a complex plane with coordinate \( f \) by the transformation \( f(z) = \left( \frac{z-\ell}{z} \right)^{1/n} \). \(^{(B.8)}\)

Because of the translational symmetry, all the one-point functions on \( \mathcal{R} \) are constants. Then we use \( \langle \Phi_K(z) \rangle_{\mathcal{R}_{n,1}} \) and write \( \langle \Phi_K(z) \rangle_{\mathcal{R}_{n,1}} \) as

\[ \langle \Phi_K(z) \rangle_{\mathcal{R}_{n,1}} = \langle \Phi_1(z) \rangle_{\mathcal{R}} \cdots \langle \Phi_k(z) \rangle_{\mathcal{R}}, \] \(^{(B.9)}\)

with the summation of \( K \) being over all the CFT\(^n\) holomorphic quasiprimary operators \( \Phi_K \) that are direct products of the holomorphic quasiprimary operators in each copy of the original CFT. In this paper we only consider the contributions of the holomorphic part of the vacuum conformal family, and the relevant CFT\(^n\) quasiprimary operators are counted as

\[
[x + (1-x)\text{tr} x^L] = 1 + n x^2 + \frac{n(n+1)}{2} x^4 + \frac{n(n^2 + 3n + 8)}{6} x^6 + \frac{n(n+1)(n^2 + 5n + 30)}{24} x^8 \\
+ n x^9 + \frac{n(n+1)(n+2)(n^2 + 7n + 72)}{120} x^{10} + n(n+1) x^{11} \\
+ \frac{n(n+3)(n^4 + 12n^3 + 169n^2 + 438n + 640)}{720} x^{12} \\
+ \frac{n(n+1)(n+2)}{2} x^{13} + O(x^{14}).
\]

Note the definition of \( \text{tr} x^L \) in \( \langle A_{11} \rangle \). The direct product quasiprimary operators take the form

\[ \Phi_K^{j_1 \cdots j_k} = \Lambda_1^{j_1} \cdots \Lambda_k^{j_k}, \] \(^{(B.11)}\)

and we list these operators to level 13 in table 3.

The one-point function of \( \Lambda_1^{j_1} \cdots \Lambda_k^{j_k} \) on \( \mathcal{R} \) is independent of the replica indices

\[ \langle \Lambda_1^{j_1} \cdots \Lambda_k^{j_k} \rangle_{\mathcal{R}} = \langle \Lambda_1 \rangle_{\mathcal{R}} \cdots \langle \Lambda_k \rangle_{\mathcal{R}}, \] \(^{(B.12)}\)
| level | quasiprimary | ? | # | # |
|-------|-------------|---|---|---|
| 0     | 1           | - | 1 | 1 |
| 2     | $T$         | ✓ | $n$ | $n$ |
| 4     | $A$         | $\times n$ | $\frac{n(n+1)}{2}$ |
|       | $TT$        | $\checkmark$ | $\frac{n^2}{2}$ |
| 6     | $B, D$      | $\times 2n$ | $\frac{n(n+3)(n+8)}{6}$ |
|       | $TA$        | $\times n_2$ | $\frac{n(n+1)}{2}$ |
|       | $TTT$       | $\checkmark$ | $\frac{n^3}{6}$ |
| 8     | $A$         | $\times n_2$ | $\frac{n(n+1)(n^2+5n+10)}{24}$ |
|       | $TTA$       | $\checkmark$ | $\frac{n^3}{2}$ |
|       | $TTTT$      | $\checkmark$ | $\frac{n^4}{24}$ |
| 9     | $A^{(9)}$   | $\times n$ | $n$ |
|       | $A^{(10,m)}$| $\times 4n$ | $\frac{n(n+1)(n^2+7n+12)}{120}$ |
| 10    | $TE, TH, TT$| $\times 5n_2$ | $\frac{n(n+1)(n^2+7n+22)}{120}$ |
|       | $AB, AD$    | $\times n_2$ | $\frac{n^2}{2}$ |
|       | $TTA$       | $\checkmark$ | $\frac{n^3}{6}$ |
|       | $TTTT$      | $\checkmark$ | $\frac{n^4}{120}$ |
| 11    | $A^{(11,m)}$| $\times 2n$ | $n(n+1)$ |
|       | $TA^{(9)}$  | $\times n_2$ | $\frac{n(n+1)(n^2+2n+3)}{6}$ |
| 12    | $A^{(12,m)}$| $\times 7n$ | $\frac{n(n+1)(n^2+3n+6)}{6}$ |
|       | $TA^{(10,m)}$| $\times 8n_2$ | $\frac{n(n+1)(n^2+2n+3)(n^2+3n+6)}{6}$ |
|       | $AE, AH, AL$| $\times BD$ | $\frac{n(n+1)(n^2+2n+3)(n^2+3n+6)}{6}$ |
|       | $BB, DD$    | $\checkmark$ | $n_2$ |
| 13    | $TTTE, TTTH$| $\times \frac{3n_3}{2}$ | $\frac{n(n+1)(n^2+2n+3)(n^2+3n+6)}{6}$ |
|       | $TTTI$      | $\checkmark$ | $\frac{13n_3}{6}$ |
|       | $TABA, TADA$| $\checkmark$ | $\frac{7n_4}{12}$ |
|       | $TTTB, TTDD$| $\checkmark$ | $\frac{n_5}{24}$ |
|       | $TTTT$      | $\checkmark$ | $\frac{n_6}{720}$ |
| 14    | $A^{(13,m)}$| $\times 3n$ | $\frac{n(n+1)(n^2+2n+3)(n^2+3n+6)(n^2+4n+6)}{6}$ |
|       | $TA^{(11,m)}$| $\times 3n_2$ | $\frac{n(n+1)(n^2+2n+3)(n^2+3n+6)(n^2+4n+6)}{6}$ |
|       | $AA^{(9)}$  | $\times n_2$ | $\frac{n^3}{24}$ |
|       | $TTAA$      | $\checkmark$ | $\frac{n^4}{720}$ |
|       | $TTTTTT$    | $\checkmark$ | $\frac{n^5}{1440}$ |

Table 3: The CFT\textsuperscript{n} holomorphic quasiprimary operators that are direct products of the vacuum conformal family holomorphic quasiprimary operators in each copy of the original CFT. We have omitted the replica indices for these operators. For example, at level 8, $TTA$ denotes $T_{j_1}T_{j_2}A_{j_3}$ with $0 \leq j_1, j_2, j_3 \leq n-1$, $j_1 < j_2$, $j_1 \neq j_3$, $j_2 \neq j_3$. In the third column, we mark ✓ for nonidentity operators with generally non-vanishing contributions to the single interval EE in a translationally invariant state, i.e., with non-vanishing coefficients $a_K$ defined in (B.15), and we mark × for operators with vanishing coefficients $a_{X_1\ldots X_k}$. Note that for $k = 0$, i.e. the identity operator, we do not need to calculate the coefficient $a_{X_1\ldots X_k}$. We count the number of operators in the fourth and fifth columns, with the notation $n_k = n(n-1)\cdots(n-k+1)$. The counting is consistent with (B.10).
We sum the replica indices of the OPE coefficient $d_{X_1 \cdots X_k}^{j_1 \cdots j_k}$ and define \[^{[41]}\]

$$b_{X_1 \cdots X_k} \equiv \sum_{j_1, \cdots, j_k} d_{X_1 \cdots X_k}^{j_1 \cdots j_k} \text{ with some constraints for } 0 \leq j_1, \cdots, j_k \leq n - 1. \quad \tag{B.13}$$

The constraints are to avoid overcounting of the quasiprimary operators. For example, the constraints for $d_{TT,A}^{ij:j}$ are $j_1 < j_2$, $j_1 \neq j_3$, $j_2 \neq j_3$. Except the identity operator, all the coefficients $b_{X_1 \cdots X_k}$ are vanishing in the limit $n \to 1$. We get the Rényi entropy

$$S_A^{(n)} = \frac{c(n + 1)}{12 n} \log \frac{\ell}{\epsilon} - \frac{1}{n - 1} \log \left( 1 + \sum_{k=1}^{n} \sum_{\{X_1, \cdots, X_k\}} \ell^h X_1 + \cdots + h X_k b_{X_1 \cdots X_k} \langle X_1 \rangle_R \cdots \langle X_k \rangle_R \right). \quad \tag{B.14}$$

From the coefficient $b_{X_1 \cdots X_k}$, we further define \[^{[21,45]}\]

$$a_{X_1 \cdots X_k} \equiv \lim_{n \to 1} \frac{b_{X_1 \cdots X_k}}{n - 1}, \quad \tag{B.15}$$

and get the EE written as

$$S_A = \frac{c}{6} \log \frac{\ell}{\epsilon} + \sum_{k=1}^{\infty} \sum_{\{X_1, \cdots, X_k\}} \ell^h X_1 + \cdots + h X_k a_{X_1 \cdots X_k} \langle X_1 \rangle_R \cdots \langle X_k \rangle_R. \quad \tag{B.16}$$

In this paper we focus on the entanglement entropy, instead of the Rényi entropy. To calculate the coefficients $a_{X_1 \cdots X_k}$ we do not need the full forms of $b_{X_1 \cdots X_k}$ or $d_{X_1 \cdots X_k}^{j_1 \cdots j_k}$. A general holomorphic quasiprimary operator $X$ transform under a general map $z \rightarrow f(z)$ as

$$X(z) = f^{h_X} X(f) + \cdots, \quad \tag{B.17}$$

with $\cdots$ denoting terms with the Schwarzian derivative

$$s(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2. \quad \tag{B.18}$$

For the transformation \[^{(B.8)}\] we have

$$s(z) = \frac{(n^2 - 1) \ell^2}{2n^2 z^2 (z - \ell)^2}. \quad \tag{B.19}$$

When we calculate $d_{X_1 \cdots X_k}^{j_1 \cdots j_k}$ using \[^{(B.7)}\], the contributions from $\cdots$ terms in \[^{(B.17)}\] would be of order $O(n - 1)$ or of higher orders. For $k \geq 2$, when we compute $b_{X_1 \cdots X_k}$ using \[^{(B.7)}\], the summation of the replica indices would lead to another order $O(n - 1)$ or higher order factor for each term. For $k \geq 2$, the $\cdots$ terms in \[^{(B.17)}\] only contribute order $O(n - 1)^2$ or higher order terms to $b_{X_1 \cdots X_k}$, and so would not contribute to $a_{X_1 \cdots X_k}$ defined in \[^{(B.15)}\]. For $k = 1$, $a_X = 0$ for $h_X > 2$. We will prove it the next appendix.

To level 13 the nonidentity CFT\(^n\) quasiprimary operators with non-vanishing $a_{X_1 \cdots X_k}$ are marked with $\checkmark$ in the 3rd column of table \[^{[3]}\]. For $k = 1$ we only need to calculate $a_T$. For $k \geq 2$, we use the various multi-point functions in appendix \[^{[A]}\] and get $d_{X_1 \cdots X_k}^{j_1 \cdots j_k}$ with some irrelevant $O(n - 1)$ terms.
Summing the replica indices, we get $b_{X_1 \ldots X_k}$ with some irrelevant $O(n-1)^2$ terms. Then we get $a_{X_1 \ldots X_k}$.

The non-vanishing factors $a_{X_1 \ldots X_k}$ are listed as follows

$$a_T = -\frac{1}{6}, \quad a_{TT} = -\frac{1}{30c}, \quad a_{TTT} = -\frac{4}{315c^2},$$
$$a_{AAA} = -\frac{1}{126c(5c + 22)}, \quad a_{TTA} = -\frac{32}{3465c^2}, \quad a_{TTTT} = -\frac{4}{630c^4},$$
$$a_{TAA} = -\frac{16}{693c^2(5c + 22)}, \quad a_{TTTA} = \frac{1}{3465c^2}, \quad a_{TTTTT} = -\frac{16(c + 5)}{3465c^4},$$
$$a_{BB} = -\frac{10}{25}, \quad a_{DD} = -\frac{70c + 29}{123552c(70c + 29)},$$
$$a_{TAB} = -\frac{10}{1287c^2(70c + 29)}, \quad a_{TAD} = \frac{5}{3003c^2(5c + 22)}, \quad a_{AAA} = \frac{4(5c + 64)}{3003c^2(5c + 22)^2},$$
$$a_{TTTT} = \frac{5(14c + 43)}{9009c^3(70c + 29)}, \quad a_{TTTD} = -\frac{2}{9009c^3}, \quad a_{TTTA} = -\frac{585c + 10804}{90090c^3(5c + 22)},$$
$$a_{TTTTA} = \frac{2(33c + 784)}{45045c^4}, \quad a_{TTTTTT} = \frac{2(11c^2 + 380c + 1480)}{45045c^5}. \quad (B.20)$$

To level 8, the coefficients $a_{X_1 \ldots X_k}$ have been calculated in [21] using the results in [39,41], and at level 10 and level 12 the results here are new. We have used the coefficients to level 10 to calculate the Holo information in [34].

**B.2 EE**

Using the above coefficients, we get EE of a short interval $A$ in state $\rho$ on a Riemann surface $\mathcal{R}$ [21]

$$S_A = \frac{c}{6} \log \frac{\ell}{\epsilon} + \ell^2 a_T \langle T \rangle_\rho + \ell^4 a_{TT} \langle T \rangle_\rho^2 + \ell^6 a_{TTT} \langle T \rangle_\rho^3$$
$$\quad + \ell^8 (a_{AAA} \langle A \rangle_\rho^2 + a_{TTA} \langle T \rangle_\rho ^2 \langle A \rangle_\rho + a_{TTTT} \langle T \rangle_\rho ^4)$$
$$\quad + \ell^{10} (a_{TAA} \langle T \rangle_\rho \langle A \rangle_\rho^2 + a_{TTTA} \langle T \rangle_\rho^3 \langle A \rangle_\rho + a_{TTTTT} \langle T \rangle_\rho^5)$$
$$\quad + \ell^{12} (a_{BB} \langle B \rangle_\rho^2 + a_{DD} \langle D \rangle_\rho^2 + a_{TAB} \langle T \rangle_\rho \langle B \rangle_\rho + a_{TAD} \langle T \rangle_\rho \langle A \rangle_\rho \langle D \rangle_\rho$$
$$\quad + a_{AAA} \langle A \rangle_\rho^3 + a_{TTT} \langle T \rangle_\rho^3 \langle B \rangle_\rho + a_{TTT} \langle T \rangle_\rho^3 \langle D \rangle_\rho + a_{TTAA} \langle T \rangle_\rho \langle A \rangle_\rho^2$$
$$\quad + a_{TTTTA} \langle T \rangle_\rho^4 \langle A \rangle_\rho + a_{TTTTTT} \langle T \rangle_\rho^6) + O(\ell^{14}). \quad (B.21)$$

To check the EE formula and the coefficients $a_{X_1 \ldots X_k}$ in (B.20), we consider several examples. The first case is that $\mathcal{R}$ is a vertical cylinder with spatial period $L$. We denote the state density matrix as $\rho_L$, we have the expectation values (A.7). We get the entanglement entropy

$$S_{A,L} = \frac{c}{6} \log \frac{\ell}{\epsilon} - \frac{\pi^2 \ell^2}{36L^2} - \frac{\pi^4 \ell^4}{1080L^4} - \frac{\pi^6 \ell^6}{17010L^6} - \frac{\pi^8 \ell^8}{22680L^8} - \frac{\pi^{10} \ell^{10}}{2806650L^{10}} - \frac{691\pi^{12} \ell^{12}}{22986463500L^{12}} + O(\ell^{14}), \quad (B.22)$$

which matches the exact result [28]

$$S_{A,L} = \frac{c}{6} \log \left(\frac{L}{\pi\ell} \sin \frac{\pi \ell}{L}\right). \quad (B.23)$$

It is similar for the CFT on an infinite straight line in thermal state with inverse temperature $\beta$, which is just the 2D CFT on a horizontal cylinder with temporal period $\beta$. We use the expectation values (A.9) and get a result that matches the EE (2.7).
On a torus with low temperature, we have spatial and temporal periods $L$ and $\beta$ that satisfy $L \ll \beta$. We denote the density matrix as $\rho_{L,q}$ with $q = e^{-2\pi \beta/L} \ll 1$. On the low temperature torus one has the one-point functions (A.8), putting which in (B.21) we get the EE

$$S_{A,L,q} = \frac{c}{6} \log \frac{\ell}{\epsilon} + \left[ -\frac{1}{36} + \frac{4q^2}{3} + 2q^3 + 4q^4 + O(q^5) \right] \left( \frac{\pi \ell}{L} \right)^2 + \left[ -\frac{c}{1080} + \frac{4q^2}{45} + \frac{2q^3}{15} \right] + \frac{4(c-8)q^4}{15c} + O(q^5) \left( \frac{\pi \ell}{L} \right)^4 + \left[ -\frac{c}{17010} + \frac{8q^2}{945} + \frac{4q^3}{315} + \frac{8(c-16)q^4}{315c} \right] + O(q^5) \left( \frac{\pi \ell}{L} \right)^6 + \left[ -\frac{c}{226800} + \frac{4q^2}{4725} + \frac{2q^3}{1575} - \frac{4(159c + 728)q^4}{1575c} \right] + O(q^5) \left( \frac{\pi \ell}{L} \right)^8 + \left[ -\frac{c}{2806650} + \frac{8q^2}{93555} + \frac{4q^3}{31185} - \frac{104(295c + 1312)q^4}{155925c} \right] + O(q^5) \left( \frac{\pi \ell}{L} \right)^{10} + O(q^5) \left( \frac{\pi \ell}{L} \right)^{12} + O(\ell^{14}),$$

and it is consistent with

$$S_{A,L,q} = \frac{c}{6} \log \left( \frac{L}{\pi \epsilon} \sin \frac{\pi \ell}{L} \right) + \left( 1 - \frac{\pi \ell}{L} \cot \frac{\pi \ell}{L} \right) 2(2q^2 + 3q^3 + 6q^4) + \frac{64}{315} \sin^8 \frac{\pi \ell}{2L} - \frac{1}{192 \cos^8 \frac{\pi \ell}{L}} \left( 76 + 87 \cos \frac{2\pi \ell}{L} + 44 \cos \frac{4\pi \ell}{L} + 3 \cos \frac{6\pi \ell}{L} \right) + \frac{1}{32 \ell} \cot \frac{\pi \ell}{L} \frac{1}{\cos^8 \frac{\pi \ell}{L}} \left( 9 + 18 \cos \frac{2\pi \ell}{L} + 6 \cos \frac{4\pi \ell}{L} + 2 \cos \frac{6\pi \ell}{L} \right) - \frac{1}{15c \cos^2 \frac{\pi \ell}{L}} \left( 97 + 59 \cos \frac{2\pi \ell}{L} - 7 \cos \frac{4\pi \ell}{L} + \cos \frac{6\pi \ell}{L} \right) + \frac{2}{c \ell} \cot \frac{\pi \ell}{L} \frac{1}{\cos^4 \frac{\pi \ell}{L}} \left( 2 + 3 \cos \frac{2\pi \ell}{L} \right) \right] q^4 + O(q^5),$$

which is valid as long as the interval length $\ell$ is not comparable with total length $L$. The order $c^0$ part of (B.25) was calculated to order $q^2$ in [46], to order $q^3$ in [47], and to order $q^4$ in [48]. The order $1/c$ part of (B.25) is new, and we calculate it using the method in [46–48].

For the CFT on a cylinder with spatial period $L$ in the primary state $|\phi\rangle$, we denote the density matrix as $\rho_{L,\phi}$. We have the expectation values (A.26), from which we get the EE

$$S_{A,L,\phi} = \frac{c}{6} \log \frac{\ell}{\epsilon} + \frac{\pi^2 \epsilon^2 (24h_\phi - c)}{36L^2} - \frac{\pi^4 \ell^4 (24h_\phi - c)^2}{1080cL^4} + \frac{\pi^6 \ell^6 (24h_\phi - c)^3}{17010c^2L^6} - \frac{226800c^3 (5c + 22)L^8}{184320(9c + 88)h_\phi^4 - 921600(3c + 22)ch_\phi^3 + 1152(15c + 82)c^2h_\phi^2 - 96(5c + 22)c^3h_\phi + (5c + 22)c^4} + \frac{\pi^{10} \ell^{10} (24h_\phi - c)}{2806650c^3 (5c + 22)L^{10}} [552960(3c + 110)h_\phi^3 - 276480(c + 22)c^3h_\phi + 3456(5c + 54)c^2h_\phi^2 - 96(5c + 22)c^3h_\phi + (5c + 22)c^4] + \frac{\pi^{12} \ell^{12} c^3 (2c - 1)(5c + 22)^2(7c + 68)L^{12}}{16721510400(2764c^4 + 430763c^3 + 6713346c^2 + 20890232c - 12177440)c^4h_\phi^4 - 1393459200(8292c^4 + 917833c^3 + 13434350c^2 + 40315616c - 23630816)c^4h_\phi^4}. $$

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The result to order $\ell^8$ has been calculated in [19]. Setting $\hbar = e^{\pi/(\sqrt{27} + 1)}$ in (B.26), we get a result that matches (2.7) to order $c$ in large $c$ limit, and this is consistent with the result in [16,17].

B.3 Relative entropy

Similarly, for two translationally invariant states $\rho$ and $\sigma$, which correspond to the Riemann surface $\mathcal{R}$ and $\mathcal{S}$ respectively, we have the relative entropy [21]

$$S(\rho_{A,L} || \sigma_{A}) = -\ell^4 a_{TTT}(\langle T \rho \rangle - \langle T \sigma \rangle)^2 - \ell^6 a_{TTTT}(\langle T \rho \rangle - \langle T \sigma \rangle)^2(\langle T \rho \rangle + 2\langle T \sigma \rangle)$$

$$- \ell^8 \left[ a_{TTTT}(\langle T \rho \rangle - \langle T \sigma \rangle)^2 + a_{TTTT}(\langle T \rho \rangle - \langle T \sigma \rangle)(\langle T \rho \rangle(\langle T \rho \rangle + \langle T \sigma \rangle - 2\langle T \sigma \rangle(\langle T \sigma \rangle) \right)$$

$$+ a_{TTTT}(\langle T \rho \rangle - \langle T \sigma \rangle)^2(\langle T \rho \rangle + 2\langle T \rho \rangle(\langle T \sigma \rangle + 3\langle T \sigma \rangle)]$$

$$- \ell^{10}\left[ a_{TTTT}(\langle T \rho \rangle(\langle T \rho \rangle + \langle T \sigma \rangle)(\langle T \rho \rangle(\langle T \rho \rangle + \langle T \sigma \rangle) - 2\langle T \sigma \rangle(\langle T \sigma \rangle) \right)$$

$$+ a_{TTTT}(\langle T \rho \rangle(\langle T \rho \rangle + \langle T \sigma \rangle)(\langle T \rho \rangle(\langle T \rho \rangle + \langle T \sigma \rangle + 3\langle T \sigma \rangle]$$(B.26)

$$- \ell^{12}\left[ a_{BB}(\langle B \rangle - \langle B \rangle)^2 + a_{DD}(\langle D \rangle - \langle D \rangle)^2$$

$$- a_{TTAB}(\langle T \rho \rangle(\langle T \rho \rangle(\langle B \rangle(\langle B \rangle + \langle T \sigma \rangle)(\langle T \rho \rangle(\langle B \rangle + \langle T \sigma \rangle(\langle B \rangle - 2\langle T \rho \rangle(\langle T \sigma \rangle(\langle B \rangle)$$

$$- a_{TTAB}(\langle T \rho \rangle(\langle T \rho \rangle(\langle T \rho \rangle(\langle D \rangle + \langle T \sigma \rangle(\langle T \rho \rangle(\langle D \rangle + \langle T \sigma \rangle(\langle D \rangle - 2\langle T \rho \rangle(\langle T \sigma \rangle(\langle D \rangle$$

$$+ a_{TTTT}(\langle T \rho \rangle - \langle T \sigma \rangle)^2(\langle T \rho \rangle^2 + 2\langle T \rho \rangle(\langle T \sigma \rangle + 4\langle T \sigma \rangle)$$(B.27)

$$+ a_{TTTT}(\langle T \rho \rangle(\langle T \rho \rangle + \langle T \sigma \rangle(\langle T \rho \rangle + \langle T \sigma \rangle(\langle T \rho \rangle + \langle T \rho \rangle(\langle T \sigma \rangle + \langle T \rho \rangle(\langle T \sigma \rangle - 4\langle T \sigma \rangle(\langle T \sigma \rangle)$$

$$+ a_{TTTT}(\langle T \rho \rangle(\langle T \rho \rangle + \langle T \sigma \rangle(\langle T \rho \rangle + \langle T \sigma \rangle(\langle T \rho \rangle + 5\langle T \sigma \rangle)] + O(\ell^{14}).$$

For $\rho_{A,L_1}$ and $\rho_{A,L_2}$ we use (A.7) and get

$$S(\rho_{A,L_1} || \rho_{A,L_2}) = \frac{\pi^4 c(L_1^2 - L_2^2)^2 \ell^4}{1080 L_1^2 L_2^4} + \frac{\pi^6 c(2L_1^4 - 3L_2^2 L_1^4 + L_2^6) \ell^6}{17010 L_1^4 L_2^8} + \frac{\pi^8 c(3L_1^8 - 4L_2^2 L_1^8 + L_2^8) \ell^8}{226800 L_1^6 L_2^{12}} + \frac{\pi^{10} c(4L_1^{10} - 5L_2^2 L_1^{10} + L_2^{12}) \ell^{10}}{2806650 L_1^{10} L_2^{12}} + \frac{691 \pi^{12} c(5L_1^{12} - 6L_2^2 L_1^{12} + L_2^{14}) \ell^{12}}{22986463500 L_1^{12} L_2^{18}} + O(\ell^{14}),$$

(B.28)

which is consistent with the exact result [49,50]

$$S(\rho_{A,L_1} || \rho_{A,L_2}) = c \log \frac{L_2 \sin \frac{\pi \ell}{2L_1}}{L_1 \sin \frac{\pi \ell}{2L_2}} + \frac{c}{12} \left(1 - \frac{L_2}{L_1} \right) \left(1 - \frac{\pi \ell}{L_2} \right) \left(1 - \frac{\pi \ell}{L_2} \right).$$

(B.29)
C Proof of $a_X = 0$ for $h_X > 2$

In this appendix, we give a proof of $a_X = 0$ with $X$ being a quasiprimary operator in the holomorphic sector of the vacuum conformal family and $h_X > 2$. Note that $h_X > 2$ is equivalent to $h_X \geq 4$. General $a_{X_1 \cdots X_k}$ is defined in (B.15), and $a_X$ is the special $k = 1$ case.

Under a general conformal transformation $z \to f(z)$, the operator $X$ transforms formally as

$$X(z) = \sum_Y \sum_p F_{X, \partial^p Y}[f(z)] \partial^p Y(f(z)), \quad (C.1)$$

with the coefficients $F_{X, \partial^p Y}[f(z)]$ being composed by derivatives of $f(z)$ and the summation of $Y$ being over all holomorphic quasiprimary operators including the identity operator 1. For example, the stress tensor $T$, it is

$$T(z) = f'(z)^2 T(f(z)) + \frac{c}{12} s(z), \quad (C.2)$$

with the Schwarzian derivative (B.18). We focus on the coefficient with $Y = 1$

$$F_X(z) = F_{X,1}[f(z)]. \quad (C.3)$$

For examples

$$F_T(z) = \frac{c}{12} s(z), \quad F_A(z) = \frac{c(5c + 22)}{720} s(z)^2. \quad (C.4)$$

We have

$$a_X = -\frac{1}{a_X e^{h_X}} \lim_{n \to 1, z \to \infty} \frac{z^{2h_X} F_X(z)}{n - 1}, \quad (C.5)$$

for the conformal transformation (B.8). Note that (B.19), $s(z) = O(n - 1)$. To prove $a_X = 0$ for $h_X > 2$, we only need to show $F_X = O(s^2)$ for a small $s$.

All the operators in holomorphic sector of the vacuum conformal family can be constructed from $T$ by derivatives, normal orderings, and linear combinations. We can recursively organize all general holomorphic quasiprimary operators $\{X\}$ as linear combinations of operators in the forms $(\partial^p T X)$, $\partial^q X$ with integers $p = 0, 1, 2, \ldots$, $q = 1, 2, 3, \ldots$. Note the relation $(\partial^p T \partial^q X) = \partial((\partial^p T \partial^{q-1} X) - (\partial^{p+1} T \partial^{q-1} X))$, we do not need the include $(\partial^p T \partial^r X)$ with $r \geq 1$. For examples, at level 2 we have $T$, at level 4 we have $(TT)$, $\partial^2 T$ and get $A$, and at level 6 we have $(\partial^2 TT)$, $(T A)$, $\partial^4 T$, $\partial^2 A$ and get $B$, $D$. Explicitly, we can recursively write a quasiprimary operator $X$ with $h_X > 2$ as

$$X = \sum_Y \sum_{XY}[u_{XY}(\partial^{h_X-h_Y-2} T Y) + v_{XY} \partial^{h_X-h_Y} Y], \quad (C.6)$$

with the summation of all nonidentity quasiprimary operators $Y$ with $h_Y \leq h_X - 1$. In fact the constants $u_{XY} = 0$, for $h_Y \geq h_X - 1$, $v_{XY} = 0$ for $h_Y \geq h_X$. Generally for a fixed $X$, the decomposition (C.6) may not be unique. Writing in terms of states, we have

$$|X\rangle = \sum_{Y} [u_{XY}(h_X - h_Y - 2)! L_{-h_X+h_Y} + v_{XY} L_{-1}^{h_X-h_Y}]|Y\rangle \quad (C.7)$$

with $L_k$ being modes of the stress tensor $T$. We multiply it with the bra state

$$\langle \partial^{h_X-2} T | = (h_X - 2)! (0) L_{h_X}. \quad (C.8)$$
Using the orthogonality of the quasiprimary operators and the Virasoro algebra we get

$$v_{\mathcal{Y}T} = -\frac{12u_{\mathcal{Y}T}}{(h_{\mathcal{Y}} - 3)(h_{\mathcal{Y}} - 2)h_{\mathcal{Y}}(h_{\mathcal{Y}} + 1)}.$$  \hfill (C.9)

We write $\mathcal{X}$ as

$$\mathcal{X} = \sum_{\mathcal{Y} \neq T} [u_{\mathcal{X}\mathcal{Y}}(\partial^{h_{\mathcal{X}}-h_{\mathcal{Y}}-2}T\mathcal{Y}) + v_{\mathcal{X}\mathcal{Y}}\partial^{h_{\mathcal{X}}-h_{\mathcal{Y}}}\mathcal{Y}]$$  \hfill (C.10)

$$+ u_{\mathcal{X}T}[\partial^{h_{\mathcal{X}}-2}T] - \frac{12}{(h_{\mathcal{X}} - 3)(h_{\mathcal{X}} - 2)h_{\mathcal{X}}(h_{\mathcal{X}} + 1)}\partial^{h_{\mathcal{X}}-2}T].$$

Note that for $\mathcal{Y} \neq T$, we also have $h_{\mathcal{Y}} \geq 4$.

The normal ordering operator can be written as

$$(\partial^p T\mathcal{Y})(w) = \frac{1}{2\pi i} \oint_{\gamma} dz \partial^p T(z)\mathcal{Y}(w).$$ \hfill (C.11)

Note that $\mathcal{Y}$ is a quasiprimary operator, and at least $F_{\mathcal{Y}} = O(s)$. From the conformal transformations of $T$ and $\mathcal{Y}$ we get

$$F_{(\partial^p T\mathcal{Y})}(w) = \frac{c}{12} \sum_{q} \frac{p!(q + 3)!}{(p + q + 4)!} F_{\mathcal{Y},\partial^p T}[f(w)] \partial^{p+q+4}f \left(\frac{(z - w)^{q+4}f'(z)^2}{[f(z) - f(w)]^{q+4}}\right)_{z=w} + O(s^2).$$ \hfill (C.12)

For an SL(2, C) conformal transformation $f(z) = \frac{az + \beta}{cz + \delta}$ with constants $\alpha, \beta, \gamma, \delta$ satisfying $\alpha\delta - \beta\gamma = 1$\footnote{One should not confuse the constant $\beta$ here with the inverse temperature used in other places of the paper.}, we have

$$\partial^{p+q+4}z \left(\frac{(z - w)^{q+4}f'(z)^2}{[f(z) - f(w)]^{q+4}}\right)_{z=w} = O(s^4).$$ \hfill (C.14)

For quasiprimary operator $\mathcal{Y} \neq T$, we have at least $F_{\mathcal{Y},\partial^p T} = O(s)$, and so we get

$$F_{(\partial^p T\mathcal{Y})} = O(s^2).$$ \hfill (C.15)

For $\mathcal{Y} = T$, we get

$$F_{(\partial^p TT)}(w) = \frac{cp!}{2(p + 4)!} \partial^{p+4}z \left[\frac{(z - w)^{p+4}f'(z)^2f'(w)^2}{(f(z) - f(w))^{p+4}}\right]_{z=w} + O(s^2).$$ \hfill (C.16)

To evaluate it we need the Aharonov invariants $\psi_p$ that are defined as

$$\frac{(z - w)^2f'(z)f'(w)}{[f(z) - f(w)]^2} = 1 + \sum_{p=2}^{+\infty}(p-1)(z - w)^p\psi_p(w).$$ \hfill (C.17)

Note that $\psi_2 = \frac{z}{6}$. For $p \geq 3$, there is the nonlinear recursive formula

$$\psi_p = \frac{1}{p + 1}\left(\psi'_p - \sum_{q=2}^{p-2}\psi_q\psi_{p-q}\right).$$ \hfill (C.18)
We get for $p \geq 2$

$$\psi_p = \frac{1}{(p+1)!} s^{(p-2)} + O(s^2). \quad (C.19)$$

Then we obtain

$$F(\partial^p TT) = \frac{c}{(p+1)(p+2)(p+4)(p+5)} s^{(p+2)} + O(s^2). \quad (C.20)$$

Note that

$$F_{\partial^p T} = \frac{c}{12} s^{(p)}. \quad (C.21)$$

From (C.15), (C.20), (C.21) and (C.10), we get for $h_X \geq 4$

$$F_X = \sum_{Y \neq T} v_{X,Y} F_Y^{(h_X-h_Y)} + O(s^2). \quad (C.22)$$

From $F_A = O(s^2)$, we get by induction $F_X = O(s^2)$ for all holomorphic quasiprimary operators in the vacuum conformal family with $h_X \geq 4$. Thus we prove that $a_X = 0$ for $h_X > 2$.

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