String Bit Description of Antiperiodic
Fermion Worldsheet Fields

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Abstract

We study a string bit Hamiltonian whose continuum limit describes antiperiodic (AP) anticommuting worldsheet fields. We calculate the amplitude for transitions between an AP spin chain and a periodic (P) one in the continuum limit, \( M \to \infty \) where \( M \) is the bit number of either chain. We also numerically evaluate the corresponding amplitudes at increasing finite \( M \) to assess the convergence rate to the continuum. We then give the overlap equations for the transition AP+AP→AP, and numerically solve them for increasing \( M \) values at a fixed value of \( x = K/M \), where \( M \) is the bit number of the large chain and \( K \) is the bit number of one of the smaller chains. For this case, in contrast to the situation with an even number of AP chains, there is an obstacle to directly finding the continuum limit analytically. We suggest an indirect analytic approach to this problem: using the AP→P transition followed by a P→AP+AP transition, each of which has a relatively simple analytic continuum limit. We also show how bosonization of the fermion fields enables an analytic recursive evaluation of the AP+AP→AP amplitude.

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1 Introduction

The string bit concept [1] provides a vehicle for the holographic emergence [2] of space from a quantum mechanical system of bits, where only a finite number of states are available to each bit [3, 4]. Each bit creation operator is an \( N \times N \) matrix \((\phi^\dagger a_1 \cdots a_n)\). The Hamiltonian for the underlying quantum system is assumed to be a single trace operator which commutes with bit number \( M = \text{Tr} \sum \phi^\dagger \phi \). Then the 't Hooft limit \( N \to \infty \) [5] implies that the energy eigenstates are a collection of noninteracting string bit chains, which for very large bit number behave as continuous strings. In the \( 1/N \) expansion one can represent a closed string as a linear combination of single trace states of the form [6]

\[
\text{Tr} \phi^\dagger_1 \cdots \phi^\dagger_n |0\rangle.
\tag{1}
\]

where \( \phi|0\rangle = 0 \) for all \( \phi \). Then the string bit Hamiltonian applied to such a linear combination behaves at large \( N \) as

\[
H|\Psi_0\rangle = h|\Psi_0\rangle + O(N^{-1}) \times \text{multi trace states}
\tag{2}
\]

where \( h \), which maps single trace states to single-trace states, acts as the effective Hamiltonian that describes the closed chain at zeroth order in the \( 1/N \) expansion \(^2\).

In the original formulation of string bit models, [1], in addition to discrete labels, \( \phi \) also depended on \( d = D - 2 \) continuous coordinates \( x_i \), which label points in the transverse space in lightcone quantization of a string moving in \( D - 1 \) dimensional space [7–9]. The proposal of [3, 4] is that each transverse coordinate can emerge from a spin system. In [10] we explored this mechanism in great detail for the Heisenberg spin chain

\[
h = \frac{1}{4} \sum_{k=1}^{M} (\sigma^x_k \sigma^x_{k+1} + \sigma^y_k \sigma^y_{k+1} + \Delta \sigma^z_k \sigma^z_{k+1})
\tag{3}
\]

at the free fermion point \( \Delta = 0 \). The Jordan-Wigner trick converts \( \sigma^x,y_k \) into spin variables \( S^x,y_k \) that anticommute at different sites. However then the resulting Fermi fields are periodic or antiperiodic depending on the values of \( M \) and \( Q = \sum_k \sigma^z \), which commute with the Hamiltonian. That is, some sectors are described by periodic fields while in other sectors by antiperiodic fields. It turned out that the overlap amplitude between a large chain and two smaller chains was nonzero in given sectors only if the three chains involved included an even number of periodic sectors and an odd number of antiperiodic sectors—a pattern that matches the three string vertices of the RNS (Ramond-Neveu-Schwarz) [11] string theory and the corresponding formulation of the superstring [12].

Oddly this pattern is the reverse of the GS (Green-Schwarz) superstring [13]. In the GS case, closed form formulas for the interaction vertices can be found by various methods [14].

\(^2\)This \( h \) is analogous, in quantum field theory, to the “first quantized” Hamiltonian \( h = \sqrt{p^2 + m^2} \) of a free single particle at zero coupling. In this article we shall use upper case \( H \) to denote the full ("second-quantized") Hamiltonian and lower case \( h \) the corresponding first quantized Hamiltonian obtained, for string bit models when \( N \to \infty \).
In contrast the RNS pattern of vertices frustrates the known direct methods for obtaining the vertices in closed form. An indirect method, based on factorizing RNS dual resonance amplitudes on onshell DDF states in the critical dimension, was used by Hornfeck [15, 16] to obtain an expression for the vertex as a double sum over expressions similar to the vertices for the bosonic string or GS superstring. This procedure gives the full onshell RNS vertex in the critical dimension, including the necessary operator prefactor. To dig out the contribution of the overlap discussed in this article would require further work.

In the present article we study the two string transition between a periodic sector and an antiperiodic sector which can be evaluated explicitly. With this transition amplitude in hand one can hope to shed light on the relation of the two formulations of superstring theory to each other. Specifically we have in mind another indirect construction of the vertex for 3 antiperiodic chains as illustrated in Fig. 1. The desired vertex would then be obtained by taking the $T \to 0$ limit after the continuum limit. The two vertices in this diagram will be obtained in this article, but we shall not attempt the implied sum over $P$ states.

In section 2 we give our choice for the “first quantized” string bit Hamiltonian $h_{AP}$ which implements antiperiodic boundary conditions on the worldsheet effective fermion fields. We obtain its eigenvalue spectrum and evaluate some simple expectation values. In section 3 we formulate the $AP \to P$ transition amplitude at finite $M$ and then obtain its continuum limit $M \to \infty$. In section 4 we formulate the finite $M$ three chain transition $AP \to AP + AP$, and discuss the obstacles to finding its continuum limit analytically. Numerical calculations of selected transitions for finite $M$ values ranging from $M = 14$ to $M = 1792$ show convergence to a limiting value as $M$ gets large. We also describe in a few cases how this limiting value can be analytically obtained using the hypothesis of bosonization. In Appendices we collect some known results for the $P \to P + P$ transition amplitude and calculate the continuum limit of the $P \to AP + AP$ transitions which could be converted to the $AP \to AP + AP$ by applying the $AP \to P$ transition obtained in this article to the $P$ leg.

Figure 1: Construction of vertex for 3 antiperiodic (AP) chains from the vertices for two antiperiodic chains to one periodic (P) chain and 1 periodic chain to one antiperiodic chain. The desired vertex is the $T \to 0$ limit.
2 String bits for antiperiodic fermion worldsheet fields

We use the same notation established in [6, 17], which dealt with spinor fields \( S^a, \tilde{S}^a \) which obeyed periodic boundary conditions. The index \( a \) labels several independent spinor fields. This \( P \) boundary condition was implemented at the discrete level by substituting \( S^a_{M+1} \rightarrow S^a_1, \tilde{S}^a_{M+1} \rightarrow \tilde{S}^a_1 \) in the expression for the Hamiltonian \( h \). This choice was actually dictated by our chosen string bit Hamiltonian; which was designed to describe the superstring. Introducing a minus sign in this substitution \( S^a_{M+1} \rightarrow -S^a_1, \tilde{S}^a_{M+1} \rightarrow -\tilde{S}^a_1 \) leads to antiperiodic (AP) boundary conditions a la Neveu-Schwarz. The resulting contribution to \( h \) is then

\[
h_{AP} = \sum_{k=1}^{M-1} \left[ -iS^a_k \tilde{S}^a_{k-1} + i\tilde{S}^a_k \tilde{S}^a_{k+1} - iS^a_k \tilde{S}^a_{k+1} + i\tilde{S}^a_k \tilde{S}^a_{k+1} + 2iS^a_k \tilde{S}^a_{k} \right] \\
- \left[ -iS^a_M S^a_1 + i\tilde{S}^a_M \tilde{S}^a_1 - iS^a_M \tilde{S}^a_1 + i\tilde{S}^a_M S^a_1 - 2iS^a_M \tilde{S}^a_M \right].
\]

(4)

Since \( h \) is bilinear in the spin variables, its eigenvalues can be found by finding energy raising and lowering operators. To find them we compute, for \( k = 2, \ldots, M - 1, \)

\[
[h_{AP}, S^a_k] = 2i(S^a_{k+1} - S^a_{k-1} + \tilde{S}^a_{k+1} + \tilde{S}^a_{k-1} - 2S^a_k) \]

(5)

\[
[h_{AP}, \tilde{S}^a_k] = -2i(\tilde{S}^a_{k+1} - \tilde{S}^a_{k-1} + S^a_{k+1} + S^a_{k-1} - 2\tilde{S}^a_{k}) \]

(6)

We then handle \( k = 1, M \) separately

\[
[h_{AP}, S^a_1] = 2i(S^a_2 + S^a_M + \tilde{S}^a_2 - \tilde{S}^a_M - 2S^a_1) \]

(7)

\[
[h_{AP}, \tilde{S}^a_1] = -2i(\tilde{S}^a_2 + \tilde{S}^a_M + S^a_2 - S^a_M - 2S^a_1) \]

(8)

\[
[h_{AP}, S^a_M] = 2i(-S^a_1 - S^a_{M-1} + \tilde{S}^a_1 + \tilde{S}^a_{M-1} - 2S^a_M) \]

(9)

\[
[h_{AP}, \tilde{S}^a_M] = -2i(-\tilde{S}^a_1 - \tilde{S}^a_{M-1} - S^a_1 + S^a_{M-1} - 2\tilde{S}^a_M) \]

(10)

In the rest of this section we suppress the label \( a \) to reduce clutter. We see that these special cases can be incorporated into the general \( k \) equations with the identifications, suppressing the \( a \) label, \( S_0 = -S_M, S_{M+1} = -S_1, \tilde{S}_0 = -\tilde{S}_M, \tilde{S}_{M+1} = -\tilde{S}_1 \). This means that in solving the eigenoperator equations by Fourier transforms, we have to use half odd integer modes:

\[
S_k = \frac{1}{\sqrt{M}} \sum_{r=1/2}^{M-1/2} e^{2\pi ikr/M} D_r, \quad \tilde{S}_k = \frac{1}{\sqrt{M}} \sum_{r=1/2}^{M-1/2} e^{2\pi ikr/M} \tilde{D}_r
\]

(11)

\[
D_r = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} e^{-2\pi ikr/M} S_k, \quad \tilde{D}_r = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} e^{-2\pi ikr/M} \tilde{S}_k
\]

(12)

Then

\[
[h_{AP}, D_r] = 2i(e^{2\pi ir/M} - e^{-2\pi ir/M})D_r + 2i(e^{2\pi ir/M} + e^{-2\pi ir/M} - 2)\tilde{D}_r
\]

\[
= -8 \sin \frac{\pi r}{M} \left[ D_r \cos \frac{\pi r}{M} + i\tilde{D}_r \sin \frac{\pi r}{M} \right]
\]

(13)

\[
[h_{AP}, \tilde{D}_r] = 8 \sin \frac{\pi r}{M} \left[ \tilde{D}_r \cos \frac{\pi r}{M} + iD_r \sin \frac{\pi r}{M} \right]
\]

(14)
Then we look for eigenoperators

\[ [h_{AP}, D_r + \xi \tilde{D}_r] = 8 \sin \frac{\pi r}{M} \left[ \tilde{D}_r \left( \xi \cos \frac{\pi r}{M} - i \sin \frac{\pi r}{M} \right) + D_r \left( i \xi \sin \frac{\pi r}{M} - \cos \frac{\pi r}{M} \right) \right] \]
\[ \equiv \Delta (D_r + \xi \tilde{D}_r) \quad (15) \]

The analysis is identical to the periodic case:

\[ \xi_\pm = \begin{cases} 
  i \tan \frac{\pi r}{2M} \\
  -i \cot \frac{\pi r}{2M} 
\end{cases} \]
\[ \Delta_\pm = \mp 8 \sin \frac{\pi r}{M}. \quad (16) \]

The energy lowering operators are

\[ G_r = D_r \cos \frac{\pi r}{2M} + i \tilde{D}_r \sin \frac{\pi r}{2M}, \quad (17) \]

and the raising operators are

\[ \bar{G}_r = D_r \sin \frac{\pi r}{2M} - i \tilde{D}_r \cos \frac{\pi r}{2M}, \quad (18) \]

which can be inverted

\[ D_r = G_r \cos \frac{\pi r}{2M} + G_r \sin \frac{\pi r}{2M}, \quad i \tilde{D}_r = G_r \sin \frac{\pi r}{2M} - G_r \cos \frac{\pi r}{2M}. \quad (19) \]

We notice that

\[ G_r^\dagger = D_{M-r} \cos \frac{\pi r}{2M} - i \tilde{D}_{M-r} \sin \frac{\pi r}{2M} = \bar{G}_{M-r} \quad (20) \]

\[ \{ G_r, \bar{G}_s \} = 2 \sin \frac{\pi r}{2M} \cos \frac{\pi s}{2M} \delta_{r+s,M} + 2 \cos \frac{\pi r}{2M} \sin \frac{\pi s}{2M} \delta_{r,s,M} = 2 \delta_{r,s} \quad (21) \]

\[ \{ G_r, G_s^\dagger \} = 2 \delta_{r,s} \quad (22) \]

We next express \( h_{AP} \) in terms of the raising and lowering operators

\[ h_{AP} = 2 \sum_{r=1/2}^{M-1/2} \sin \frac{r \pi}{M} \left[ G_r^\dagger G_r - G_r G_r^\dagger \right] = 2 \sum_{r=1/2}^{M-1/2} \sin \frac{r \pi}{M} \left[ 2G_r^\dagger G_r - 2 \right] \quad (23) \]

From which we read off the ground state energy, for \( s \) independent spinor fields

\[ E_{G}^{AP} = -4s \sum_{r=1/2}^{M-1/2} \sin \frac{r \pi}{M} = -\frac{4s}{\sin(\pi/(2M))} \sim -\frac{8s}{\pi} M - \frac{\pi s}{3M} + O(M^{-3}) \quad (24) \]
2.1 Expectation values

A nice exercise is to calculate the expectation value of \( h_{AP} \) in the ground state of \( h_P \) or the expectation of \( h_P \) in the ground state of \( h_{AP} \). To do this we use

\[
h_{AP} = h_P - 2 \left[ -i S_M S_1 + i \bar{S}_M \bar{S}_1 - i S_M \bar{S}_1 + i \bar{S}_M S_1 \right]
\]

(25)

The terms in square brackets can be expanded either in terms of periodic modes to calculate \( \langle G_P | h_{AP} | G_P \rangle \) or in terms of antiperiodic modes to calculate \( \langle G_{AP} | h_P | G_{AP} \rangle \). Starting with the first one,

\[
\langle G_P | h_{AP} | G_P \rangle = E_G^P - \frac{2}{M} \sum_{m,n=0}^{M-1} e^{2\pi i n M} i \left( \langle G_P \left| \left[ -B_mB_n + \bar{B}_m \bar{B}_n - B_m \bar{B}_n + \bar{B}_m B_n \right] \right| G_P \right)
\]

(26)

So we need

\[
\langle G_P \left| \left[ \bar{B}_0 - B_0 \right] (B_0 + \bar{B}_0) \right| G_P \rangle = \langle G_P \left| 2\bar{B}_0 B_0 \right| G_P \rangle
\]

(27)

for zero modes, and for nonzero modes

\[
\langle G_P \left| \left[ \bar{B}_m - B_m \right] (B_n + \bar{B}_n) \right| G_P \rangle = - \langle G_P \left| (F_m - i \bar{F}_m)(F_n + i \bar{F}_n) \right| G_P \rangle e^{i(m-n)\pi/2M}
\]

\[
= - \langle G_P \left| iF_m \bar{F}_n \right| G_P \rangle e^{i(m-n)\pi/2M}
\]

\[
= 2\delta_{m+n,M} e^{-i\pi n M}
\]

(28)

Then

\[
\langle G_P | h_{AP} | G_P \rangle = E_G^S - \frac{4}{M} \langle G_P \left| i \bar{B}_0 B_0 \right| G_P \rangle - \frac{4i}{M} \sum_{n=1}^{M-1} e^{i\pi n M}
\]

\[
= E_G^S - \frac{4}{M} \langle G_P \left| i \bar{B}_0 B_0 \right| G_P \rangle + \frac{4}{M} \cot \frac{\pi}{2M}
\]

(29)

The square of the operator \( i \bar{B}_0 B_0 \) is 1 implying that its eigenvalues are \( \pm 1 \). It is noteworthy that \( \langle G_P | h_{AP} | G_P \rangle - E_G^S \to 8/\pi = O(1) \) at large bit number, which is much larger than the excitation energies \( = O(1/M) \) in the same limit.

One can repeat the exercise for the expectation value of \( h_P \) in the ground state of \( h_{AP} \).

In this case there are no zero modes and we have

\[
\langle G_{AP} | h_P | G_{AP} \rangle = E_G^{AP} - \frac{4i}{M} \sum_{r=1/2}^{M-1/2} e^{i\pi r M}
\]

\[
= E_G^{AP} + \frac{4}{M} \csc \frac{\pi}{2M}
\]

(30)

In each case the expectation of the “wrong” \( h \) is much higher than the ground energy of the “right” \( h \), consistently with the variational principle.
3 Periodic/Antiperiodic Transition Amplitude

A very simple overlap problem is the relationship of eigenstates of a Hamiltonian with periodic boundary conditions to those of a Hamiltonian with antiperiodic boundary conditions. The same set of variables $S_k$ are used in both Hamiltonians. To address this problem we can expand the eigenoperators of one Hamiltonian in terms of those of the other one. So for example we can write

\[ D_r = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} e^{-2\pi i r k / M} S_k = \frac{1}{M} \sum_{k=1}^{M} e^{-2\pi i r k / M} \sum_{n=0}^{M-1} B_n e^{2\pi i n / M} \]

\[ = \frac{1}{M} \sum_{n=0}^{M-1} B_n e^{2\pi i (n-r) / M} \frac{1}{1 - e^{2\pi i (n-r) / M}} = \frac{2}{M} \sum_{n=0}^{M-1} B_n \frac{e^{2\pi i (n-r) / M}}{1 - e^{2\pi i (n-r) / M}} \] (31)

\[ \tilde{D}_r = \frac{2}{M} \sum_{n=0}^{M-1} \tilde{B}_n \frac{e^{2\pi i (n-r) / M}}{1 - e^{2\pi i (n-r) / M}} \] (32)

\[ G_r = \frac{2}{M} \sum_{n=0}^{M-1} \left( B_n \cos \frac{\pi r}{2M} + i \tilde{B}_n \sin \frac{\pi r}{2M} \right) \frac{e^{2\pi i (n-r) / M}}{1 - e^{2\pi i (n-r) / M}} \]

\[ = \frac{2}{M} \sum_{n=0}^{M-1} \left( F_n \cos \frac{\pi (n-r)}{2M} + \tilde{F}_n \sin \frac{\pi (n-r)}{2M} \right) \frac{e^{2\pi i (n-r) / M}}{1 - e^{2\pi i (n-r) / M}} \] (33)

For nonzero modes $\tilde{F}_n = F_{M-n}^{\dagger}$ so their contribution to $G_r$ may be written

\[ \sum_{n=1}^{M-1} \left( F_n C_{rn} + F_n^{\dagger} S_{rn} \right). \] (34)

However the zero mode contribution is

\[ \frac{2}{M} \left( B_0 \cos \frac{\pi r}{2M} + i \tilde{B}_0 \sin \frac{\pi r}{2M} \right) \frac{e^{-2\pi i r / M}}{1 - e^{-2\pi i r / M}} \] (35)

where $B_0, \tilde{B}_0$ are anticommuting hermitian operators. To treat them more like the nonzero modes we define $f_0 = (B_0 + i \tilde{B}_0) / 2$, so that $\{f_0, f_0^{\dagger}\} = 1$ in terms of which the zero mode contribution can be written

\[ \frac{2}{M} \left( f_0 \left( \cos \frac{\pi r}{2M} + \sin \frac{\pi r}{2M} \right) + f_0^{\dagger} \left( \cos \frac{\pi r}{2M} - \sin \frac{\pi r}{2M} \right) \right) \frac{e^{-2\pi i r / M}}{1 - e^{-2\pi i r / M}} \]

\[ = \sqrt{2} \frac{2}{M} \left( f_0 \cos \left( \frac{\pi r}{2M} - \frac{\pi}{4} \right) + f_0^{\dagger} \cos \left( \frac{\pi r}{2M} + \frac{\pi}{4} \right) \right) \frac{e^{-2\pi i r / M}}{1 - e^{-2\pi i r / M}} \] (36)

Then we rescale $F_n = f_n \sqrt{2}$ so that $G_r$ becomes

\[ G_r = \sqrt{2} \sum_{n=0}^{M-1} \left( f_n C_{rn} + f_n^{\dagger} S_{rn} \right). \] (37)
where the matrices $C$ and $S$ are defined by this equation:

$$
C_{rn} = \frac{2}{M} \cos \left( \frac{\pi (n-r)}{2M} \right) \frac{e^{2\pi i (n-r)/M}}{1 - e^{2\pi i (n-r)/M}}, \quad n \neq 0
$$

$$
S_{rn} = \frac{2}{M} \cos \left( \frac{\pi (n+r)}{2M} \right) \frac{e^{-2\pi i (n+r)/M}}{1 - e^{-2\pi i (n+r)/M}}, \quad n \neq 0
$$

$$
C_{r0} = \frac{2}{M} \cos \left( \frac{\pi}{4} - \frac{\pi r}{2M} \right) \frac{e^{-2\pi i r/M}}{1 - e^{-2\pi i r/M}}
$$

$$
S_{r0} = \frac{2}{M} \cos \left( \frac{\pi}{4} + \frac{\pi r}{2M} \right) \frac{e^{-2\pi i r/M}}{1 - e^{-2\pi i r/M}}
$$

To discuss the continuum limit we need to consider $M \to \infty$ in four situations: $r, n$ fixed, $r, n' \equiv M - n$ fixed, $r' \equiv M - r, n$ fixed, and $r', n'$ fixed.

$r, n$ fixed:

$$
C_{rn} \to \frac{2}{-2\pi i (n-r)}, \quad n \neq 0
$$

$$
S_{rn} \to \frac{2}{2\pi i (n+r)}, \quad n \neq 0
$$

$$
C_{r0} \to \frac{\sqrt{2}}{2\pi i r}
$$

$$
S_{r0} \to \frac{\sqrt{2}}{2\pi i r}
$$

$r', n'$ fixed:

$$
C_{rn} \to \frac{2}{2\pi i (n'-r')}, \quad n' \neq 0
$$

$$
S_{rn} \to \frac{2}{2\pi i (n'+r')}, \quad n' \neq 0
$$

$$
C_{r0} \to -\frac{\sqrt{2}}{2\pi i r'}
$$

$$
S_{r0} \to \frac{\sqrt{2}}{2\pi i r'}
$$

$r, n'$ fixed:

$$
C_{rn} \to \frac{1}{2iM} \to 0, \quad n' \neq 0
$$

$$
S_{rn} \to -\frac{1}{2iM} \to 0, \quad n' \neq 0
$$

$$
C_{r0} \to \frac{\sqrt{2}}{2\pi i r}
$$

$$
S_{r0} \to \frac{\sqrt{2}}{2\pi i r}
$$
\[ r', n \text{ fixed:} \]

\[
C_{rn} \rightarrow -\frac{1}{2iM} \rightarrow 0, \quad n \neq 0 \tag{54}
\]

\[
S_{rn} \rightarrow -\frac{1}{2\pi iM} \rightarrow 0, \quad n \neq 0 \tag{55}
\]

\[
C_{r0} \rightarrow -\sqrt{\frac{2}{\pi ir'}} \tag{56}
\]

\[
S_{r0} \rightarrow \sqrt{\frac{2}{\pi ir'}} \tag{57}
\]

### 3.1 AP Ground state

Let us first construct the ground state in the AP sector in terms of P sector states. Then we have to solve \( G_r |G\rangle = 0 \) for all \( r \). We make the ansatz

\[
|G\rangle \propto \exp \left\{ \frac{1}{2} f_{m}^{\dagger} D_{mn} f_{n}^{\dagger} \right\} |0\rangle \tag{58}
\]

where \( f_{n}|0\rangle = 0 \) for \( n = 0, 1, 2, \ldots \). Then, if we require \( D \) to be antisymmetric, \( G_r |G\rangle = 0 \) implies

\[
S_{rm} + C_{rn} D_{nm} = 0. \tag{59}
\]

We attempt to solve for \( D \) in the continuum limit.

The equation breaks into 6 equations since \( r \) is either near 0 or near \( M \) and \( m \) equals 0, is near 0, or is near \( M \). We first write out the equation for \( r \) near 0 and \( m = 0 \):

\[
S_{r0} + C_{r0} D_{00} + C_{rn} D_{n0} + C_{rn'} D_{n'0} = 0 \tag{60}
\]

where the prime indicates that the index is near \( M \) and the zero index is explicitly singled out. Putting in the limiting forms and cancelling a factor of \( 2\pi i \) leads to

\[
\frac{\sqrt{2}}{r} = \sum_{n=1}^{\infty} \frac{2}{n-r} D_{n0} \tag{61}
\]

where we used antisymmetry, which implies \( D_{00} = 0 \). For \( m \neq 0 \) but near 0, the equation reads

\[
S_{rm} + C_{r0} D_{om} + C_{rn} D_{nm} + C_{rn'} D_{n'm} = 0 \tag{62}
\]

and inserting the limiting forms

\[
\frac{2}{r + m} = -\frac{\sqrt{2}}{r} D_{om} + \sum_{n=1}^{\infty} \frac{2}{n-r} D_{nm}, \quad m \neq 0. \tag{63}
\]
Keeping \( r \) near 0 but letting \( m \) be near \( M \), we write out
\[
S_{rm'} + C_{r0}D_{0m'} + C_{rn}D_{nm'} + C_{rn'}D_{n'm'} = 0 \tag{64}
\]
Putting in the limiting forms these equations reduce to
\[
0 = \frac{\sqrt{2}}{r} D_{0m'} - \sum_{n=1}^{\infty} \frac{2}{n - r} D_{nm'} \tag{65}
\]
The remaining 3 equations come from taking \( r \) near \( M \). The limiting forms of the \( C \)'s all change sign when \( r' \) is substituted for \( r \):
\[
\frac{\sqrt{2}}{r'} = -\sum_{n'=1}^{\infty} \frac{2}{n' - r'} D_{n'0} \tag{66}
\]
\[
\frac{2}{r' + m'} = \frac{\sqrt{2}}{r'} D_{0m'} + \sum_{n'=1}^{\infty} \frac{2}{n' - r'} D_{n'm'} \tag{67}
\]
\[
0 = \frac{\sqrt{2}}{r'} D_{0m} - \sum_{n'=1}^{\infty} \frac{2}{n' - r'} D_{n'm} \tag{68}
\]
We next derive a pair of useful identities following a method developed by J. Goldstone [9].

Introduce a meromorphic function \( g(z) = \Gamma(z)/\Gamma(z+1/2) \): it has poles at \( z = 0, -1, -2, \ldots \) and zeroes at \( z = -1/2, -3/2, \ldots \). Near \( z = -m \),
\[
g(z) \sim \frac{1}{z + m} \frac{1}{\pi mg(m)}, \quad m = 0, 1, 2, \ldots \tag{69}
\]
Here, for the case \( m = 0 \), we replace \( \pi mg(m) \rightarrow \sqrt{\pi} \). The function \( g(z) \) behaves as \( Cz^{-1/2} \) as \( z \rightarrow \infty \). Because of the zeroes, \( g(z)/(z + r) \) has the same poles as \( g(z) \) as long as \( r \) is a positive half odd integer. We therefore can expand
\[
\frac{g(z)}{z + r} = \sum_{n=0}^{\infty} \frac{1}{z + n} \frac{1}{\pi ng(n)} \frac{1}{r - n} \tag{70}
\]
where it is important that the left side vanishes as \( z \rightarrow \infty \). Putting \( z = m > 0 \) this becomes
\[
\frac{1}{m + r} = \sum_{n=0}^{\infty} \frac{1}{m + n} \frac{1}{\pi ng(n)g(m)} \frac{1}{r - n}, \quad m = 1, 2, \ldots \tag{71}
\]
which resembles the equations we wish to solve.

We can get another identity by expanding \( zg(z)/(z + r) \).
\[
\frac{zg(z)}{z + r} = \sum_{n=1}^{\infty} \frac{-n}{z + n} \frac{1}{\pi ng(n)} \frac{1}{r - n} \tag{72}
\]
\[
\frac{1}{m + r} = \sum_{n=1}^{\infty} \frac{-n}{m + n} \frac{1}{\pi ng(n)mg(m)} \frac{1}{r - n}, \quad m = 0, 1, 2, \ldots \tag{73}
\]
In the second equation we put \( z = m \). In this case, \( m \) is allowed to be 0, because \( zg(z) \to 1/\sqrt{\pi} \) is finite as \( z \to 0 \). The difference of the two identities cancels the inhomogeneous term. This is useful in solving for the mixed matrix elements \( D_{n'm} \) and \( D_{nm'} \):

\[
0 = \sum_{n=0}^{\infty} \frac{1}{\pi n g(n) m g(m)} \frac{1}{r - n} = \frac{1}{\sqrt{\pi mg(m)}} \frac{1}{r} + \sum_{n=1}^{\infty} \frac{1}{\pi n g(n) m g(m)} \frac{1}{r - n}, \quad m = 1, 2, \ldots \quad (74)
\]

The zero index components aren’t determined from these equations since the equations are homogeneous. They are fixed by the two inhomogeneous equations.

By comparing the identities to the inhomogeneous equations to be solved, it is straightforward to infer that

\[
\frac{D_{nm'}}{D_{0m'}} = \frac{1}{\sqrt{2 \pi n g(n)}}, \quad n > 0 \quad (75)
\]

\[
\frac{D_{n'm}}{D_{0m}} = \frac{1}{\sqrt{2 \pi n' g(n')}}; \quad n' > 0 \quad (76)
\]

It is also useful in analyzing ambiguities in the solution of the equations. However, in the case of \( D_{mn} \), the ambiguities in the determination of \( D_{mn} \) are fixed by simply requiring antisymmetry \( D^T = -D \).

Comparing to the equations involving these mixed matrix elements, we learn that

\[
D_{nm} = -D_{mn} = -\frac{m - n}{m + n} \frac{1}{2 \pi n g(n) m g(m)}. \quad m, n > 0 \quad (77)
\]

\[
D_{n0} = -D_{0n} = \frac{1}{\sqrt{2 \pi n g(n)}}. \quad n > 0. \quad (78)
\]

where the antisymmetry of \( D \) is assured by taking the sum of the two identities for \( m > 0 \),

\[
\frac{2}{m + r} = \frac{1}{\sqrt{\pi mg(m)}} \frac{1}{r} + \sum_{n=1}^{\infty} \frac{m - n}{m + n} \frac{1}{\pi n g(n) m g(m)} \frac{1}{r - n}, \quad m = 1, 2, \ldots \quad (79)
\]

and by using only the second identity for \( m = 0 \),

\[
\frac{1}{r} = -\sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n g(n)}} \frac{1}{r - n}. \quad (80)
\]

The components \( D_{n'm'} \) and \( D_{0n'} \) are the negatives of \( D_{nm}, D_{0n} \), with \( n', m' \) substituted for \( n, m \).

These results are for the continuum limit \( M \to \infty \). To assess convergence rates we use MATLAB to numerically evaluate the first few matrix elements of \( D = -C^{-1} S \), which we collect in table 1.
The anticommutation relations of the operators:

\[ \{ P_{\alpha} \} = 0 \]

or in components \( C \alpha \beta \) is antisymmetric as another calculation quickly shows.

Then we can generate all AP energy eigenstates from the ground state via the coherent states.

\[ g_r^\dagger \equiv \frac{G_r^\dagger}{\sqrt{2}} = \sum_{n=0}^{M-1} (f_n^\dagger C^*_{rn} + f_n S^*_{rn}) \quad (81) \]

Then we can generate all AP energy eigenstates from the ground state via the coherent states.

\[ e^{\sum_r \beta_r g_r^\dagger} |G\rangle = \exp \left\{ \frac{1}{2} f_n^\dagger D_n f_l^\dagger + \beta_r K_{rn} f_n^\dagger + \frac{1}{2} \beta_{r} K_{rn} S^{*}_{sn} \beta_{s} \right\} |0\rangle [\det (I + \hat{D}^\dagger)]^{-1/4} \]

\[ = \exp \left\{ \frac{1}{2} f^\dagger T D f^\dagger + \beta T K f^\dagger + \frac{1}{2} \beta^T K S^* T \beta \right\} |0\rangle [\det (I + \hat{D}^\dagger)]^{-1/4} \quad (82) \]

\[ K_{rn} \equiv C^*_{rn} + S^*_{rl} D_{ln} \quad (83) \]

The anticommutation relations of the \( G \)'s imply the conditions

\[ SS^\dagger + CC^\dagger = I, \quad SC^T + CS^T = 0. \quad (84) \]

A short calculation shows that

\[ CK^T = I \quad (85) \]

or in components \( C_{rn} K_{ns} = \delta_{rs} \). That is \( K^T \) is a right inverse of \( C \). The matrix \( \hat{D} \equiv KS^\dagger \) is antisymmetric as another calculation quickly shows.

The same matrices figure in the inverse transition relating S eigenoperators to AP eigenoperators:

\[ f_n = \sum_r (g_r \hat{C}_{nr} + g^*_r \hat{S}_{nr}) \quad (86) \]

\[ \hat{C} = C^\dagger, \quad \hat{S} = S^T \quad (87) \]
The relation between $\hat{C}, \hat{S}$ and $C, S$ follows from direct comparison of the mode expansions. More generally, one easily sees that consistency of the equations $g = Cf + Sf^\dagger$ and $f = \hat{C}g + \hat{S}g^\dagger$ leads to the conditions

$$C\hat{C} + S\hat{S}^* = I, \quad C\hat{S} + S\hat{C}^* = 0,$$

which are solved by $\hat{C} = C^\dagger$ and $\hat{S} = S^T$, although this argument doesn’t assure uniqueness.

Preservation of anticommutation relations then imposes the further constraints

$$C^\dagger C + S^T S^* = I, \quad S^T C^* + C^\dagger S = 0.$$

(88)

One can then show that $K^T$ is also the left inverse of $C$ (i.e. $K^TC = I$) as follows:

$$K^TC = [(C^TK)^T = [(C^\dagger C)^* + C^\dagger S^* D]^T = [(C^\dagger C)^* - S^T CD]^T = [(C^\dagger C)^* + S^T S]^T = I$$

(90)

3.2.1 Continuum limit of $K$

As before taking the continuum limit requires breaking the equation into cases: $rs, rs', r's,$ and $r's'$:

$$C_{r0}K_{0s} + C_{rn}K_{ns} + C_{rn'}K_{n's} = \delta_{rs}$$

(91)

$$C_{r'0}K_{0s} + C_{r'n}K_{ns} + C_{r'n'}K_{n's} = 0$$

(92)

$$C_{r0}K_{0s'} + C_{rn}K_{ns'} + C_{rn'}K_{n's'} = 0$$

(93)

$$C_{r'0}K_{0s'} + C_{r'n}K_{ns'} + C_{r'n'}K_{n's'} = \delta_{r's'}$$

(94)

Putting in the limiting forms, these become

$$\sqrt{2} \frac{r}{r'} K_{0s} - \sum_{n=1}^{\infty} \frac{2}{n - r} K_{ns} = 2\pi i \delta_{rs}$$

(95)

$$-\sqrt{2} \frac{r'}{r} K_{0s} + \sum_{n'=1}^{\infty} \frac{2}{n' - r'} K_{n's} = 0$$

(96)

$$\sqrt{2} \frac{r}{r'} K_{0s'} - \sum_{n=1}^{\infty} \frac{2}{n - r} K_{ns'} = 0$$

(97)

$$-\sqrt{2} \frac{r'}{r} K_{0s'} + \sum_{n'=1}^{\infty} \frac{2}{n' - r'} K_{n's'} = 2\pi i \delta_{r's'}$$

(98)

We shall need some more identities to handle the continuum limit with excited states. If we set $z = -s$ with $s$ a positive half odd integer in (70) or (72), the left side goes to zero.

3For invertible finite dimensional square matrices this is automatic, but these considerations show that we should require this even in the continuum limit when the matrices become infinite dimensional.
by virtue of the zeroes of \( g(z) \), unless \( s = r \). If \( s = r \), the zero in \( g(z) \) is cancelled by the explicit pole:

\[
\frac{g(z)}{z + r} \rightarrow -\delta_{rs} \frac{\pi}{g(r)}, \quad \text{as} \quad z \rightarrow -s
\]  

(99)

So we get

\[
-\delta_{rs} \frac{\pi}{sg(s)} = \sum_{n=0}^{\infty} \frac{1}{-s + n} \frac{1}{\pi ng(n)} \frac{1}{r - n}
\]  

(100)

from (70), and

\[
\delta_{rs} \frac{\pi}{g(s)} = \sum_{n=1}^{\infty} \frac{-n}{-s + n} \frac{1}{\pi ng(n)} \frac{1}{r - n}
\]  

(101)

from (72). Adding (101) to \( s \times (100) \), gives

\[
0 = -\frac{1}{\sqrt{\pi}} \frac{1}{r} - \sum_{n=1}^{\infty} \frac{1}{\pi ng(n)} \frac{1}{r - n} = -\sum_{n=0}^{\infty} \frac{1}{\pi ng(n)} \frac{1}{r - n}
\]  

(102)

where in the last form \( ng(n) \) is understood to be \( 1/\sqrt{\pi} \) when \( n = 0 \). This last identity can be used to find a solution of the middle two equations relating \( K_{n's} \) to \( K_{0s} \) and \( K_{ns'} \) to \( K_{0s'} \):

\[
K_{n's} = K_{0s} \cdot \frac{1}{\sqrt{2\pi n'g(n')}}
\]  

(103)

\[
K_{ns'} = K_{0s'} \cdot \frac{1}{\sqrt{2\pi ng(n)}}
\]  

(104)

We can also add an \( s \) dependent factor times the right side of (102) to the right side of (100) to rearrange it:

\[
2\pi i \delta_{rs} = \sum_{n=0}^{\infty} 2isg(s) + \xi_s(s - n) \frac{1}{s - n} \frac{1}{\pi ng(n)} \frac{1}{r - n}
\]  

(105)

Comparing this identity to the first and last of the four equations for \( K \), we can read off

\[
K_{0s} = \frac{2ig(s) + \xi_s}{\sqrt{2\pi}}
\]  

(106)

\[
K_{ns} = \frac{1}{2\pi ng(n)} \left[ \frac{2isg(s) + \xi_s(s - n)}{s - n} \right]
\]  

(107)

\[
K_{0s'} = -\frac{2ig(s') + \xi_s'}{\sqrt{2\pi}}
\]  

(108)

\[
K_{n's'} = -\frac{1}{2\pi n'g(n')} \left[ \frac{2is'g(s') + \xi_s'(s' - n')}{s' - n'} \right]
\]  

(109)
In these formulas ξ, ξ′ are undetermined sets of numbers. It will be convenient to replace
ξ = −ig(s)(1 − αs) and ξ′ = −ig(s′)(1 + αs′)

\[ K_{0s} = \frac{ig(s)(1 + \alpha_s)}{\sqrt{2\pi}} \] (110)
\[ K_{ns} = \frac{ig(s)}{2\pi n g(n)} \left[ \frac{s + n}{s - n} + \alpha_s \right] \] (111)
\[ K_{0s'} = \frac{ig(s')(-1 + \alpha_{s'})}{\sqrt{2\pi}} \] (112)
\[ K_{n's'} = \frac{ig(s')}{2\pi n' g(n')} \left[ \frac{s' + n'}{s' - n'} + \alpha_{s'} \right] \] (113)
\[ K_{n's} = \frac{ig(s)}{2\pi n' g(n')}(1 + \alpha_s) \] (114)
\[ K_{n's'} = \frac{ig(s')}{2\pi n' g(n')}(-1 + \alpha_{s'}) \] (115)

We have noted that \( K^T \) is a right inverse of \( C \). At the level of string bits, since \( C \) is a finite dimensional square matrix, if the inverse exists, i.e. if \( \det C \neq 0 \), its inverse is unique and is both a left and right inverse. However, in the continuum limit \( C \) becomes an infinite dimensional matrix and these facts need not hold. We seek to determine the \( \xi \)'s by insisting that \( K^T \) is also a left inverse of \( C \).

We start by writing out \( K^T C = I \) in components (in the continuum limit);

\[ 2\pi i \delta_{mn} = \sum_r K_{mr} \frac{2}{r - n}, \quad m, n \neq 0 \] (116)
\[ 2\pi i \delta_{0n} = 0 = \sum_r K_{0r} \frac{2}{r - n}, \quad n \neq 0 \] (117)
\[ 2\pi i \delta_{m0} = 0 = \sum_r K_{mr} \frac{\sqrt{2}}{r} - \sum_{r'} K_{mr'} \frac{\sqrt{2}}{r'}, \quad m \neq 0 \] (118)
\[ 2\pi i \delta_{00} = 2\pi i = \sum_r K_{0r} \frac{\sqrt{2}}{r} - \sum_{r'} K_{0r'} \frac{\sqrt{2}}{r'}, \] (119)
\[ 2\pi i \delta_{m'n'} = \sum_{r'} K_{m'r'} \frac{-2}{r' - n'} \] (120)
\[
0 = \sum_{r'} K_{m'r'} \frac{-2}{r'-n'}, \quad m \neq 0 \quad (121)
\]
\[
0 = \sum_{r'} K_{0'r'} \frac{-2}{r'-n'} \quad (122)
\]
\[
0 = \sum_{r} K_{m'r} \frac{2}{r-n}, \quad n \neq 0 \quad (123)
\]
\[
0 = \sum_{r} K_{m'r} \frac{\sqrt{2}}{r} - \sum_{r'} K_{m'r'} \frac{\sqrt{2}}{r'} \quad (124)
\]

where we have separated the components with \(m=0\) or \(n=0\) or both. Recall that when an index is primed it is always nonzero.

In order to solve these equations by Goldstone’s method, we employ \(1/g(z)\) instead of \(g(z)\). The poles of \(1/g\) are at \(z = -r\):

\[
\frac{1}{g(z)} \sim -\frac{rg(r)}{\pi(z+r)}, \quad r = 1/2, 3/2, \ldots \quad (125)
\]

and we have the expansion

\[
\frac{1}{g(z)} \left( \frac{1}{z+n} \right) = \sum_{r=1/2}^\infty \frac{-rg(r)}{\pi(z+r)} \frac{1}{n-r}, \quad n = 0, 1, 2, \ldots \quad (126)
\]

Setting \(z = -m\) the left side vanishes for \(m \neq n\). For \(m = n\), \(g(z)(z+n) \to 1/[\pi ng(n)]\), so we get

\[
\delta_{mn}\pi mg(m) = \sum_{r=1/2}^\infty \frac{-rg(r)}{\pi(-m+r)} \frac{1}{n-r}, \quad m, n = 0, 1, 2, \ldots \quad (127)
\]

Using \(1/[zg(z)]\) instead gives the expansion

\[
\frac{1}{zg(z)} \left( \frac{1}{z+n} \right) = \sum_{r=1/2}^\infty \frac{g(r)}{\pi(z+r)} \frac{1}{n-r}, \quad n = 1, 2, \ldots \quad (128)
\]

\(n = 0\) is excluded because in that case the pole at \(z = 0\) is not cancelled. Letting \(z \to -m \neq 0\), the left side goes to \(-\delta_{mn}\pi g(m)\), producing the identity

\[
-\delta_{mn}\pi g(m) = \sum_{r=1/2}^\infty \frac{g(r)}{\pi(r-m)} \frac{1}{n-r}, \quad m, n = 1, 2, \ldots \quad (129)
\]

Again it is convenient to choose linear combinations of (127) and (129). For both \(m, n\) nonzero we can add \(m\) times (129) to (127) so that the left sides cancel

\[
0 = -\sum_{r=1/2}^\infty \frac{g(r)}{\pi mg(m)} \frac{1}{n-r}, \quad m, n = 1, 2, \ldots \quad (130)
\]
Then one can add an \( m \) dependent multiple of this last equation to (127) to put the latter in the form

\[
2\pi i \delta_{mn} = \sum_{r=1/2}^{\infty} \frac{2ig(r)}{\pi g(m)} \left[ -\frac{r}{m} + \frac{1}{n-r} \right], \quad m, n = 1, 2, \ldots
\]  

Comparing this last equation to (116) and (120) allows us to read off the nonzero unmixed index components of \( K^T \)

\[
K_{mr} = \frac{2ig(r)}{2\pi mg(m)} \left[ -\frac{r}{m} - \eta_m \right], \quad m \neq 0 \tag{132}
\]

\[
K_{m'r'} = \frac{2ig(r')}{2\pi m'g(m')} \left[ -\frac{r'}{m'} - \eta_{m'} \right], \quad m' \neq 0 \tag{133}
\]

The mixed components \( K_{mr} \) and \( K_{m'r'} \) satisfy (121) or (123). Referring to (130), we see that these equations can be solved as \( m \) or \( m' \) dependent factors times \( g(r') \) or \( g(r) \). Information about the \( m \) or \( m' \) dependence is contained in (118) or (124). We compute

\[
\sum_r K_{mr} \frac{1}{r} = \sum_r \frac{ig(r)}{\pi mg(m)} \left[ 1 - \frac{\eta_m}{r} \right] = -\frac{i\eta_m}{\pi mg(m)} \sum_r \frac{g(r)}{r}. \tag{134}
\]

Inserting this into (118) gives

\[
K_{mr'} = -\frac{i\eta_m g(r')}{\pi mg(m)} \tag{135}
\]

Similarly (124) can be used to determine

\[
K_{m'r} = -\frac{i\eta_{m'} g(r)}{\pi m'g(m')} \tag{136}
\]

It is again convenient to replace \( \eta_m = (1 + \beta_m) / 2 \) and \( \eta_{m'} = (-1 + \beta_{m'}) / 2 \).

\[
K_{mr} = \frac{i(1 + \beta_m)g(r)}{2\pi mg(m)} \left[ \frac{r + m}{r - m} - \beta_m \right], \quad m \neq 0 \tag{137}
\]

\[
K_{m'r'} = \frac{i(1 + \beta_{m'})g(r')}{2\pi m'g(m')} \left[ \frac{r' + m'}{r' - m'} - \beta_{m'} \right], \quad m' \neq 0 \tag{138}
\]

\[
K_{mr'} = -\frac{i(-1 + \beta_m)g(r')}{2\pi mg(m)} \tag{139}
\]

\[
K_{m'r} = -\frac{i(-1 + \beta_{m'})g(r)}{2\pi m'g(m')} \tag{140}
\]

We finally consider \( K_{0r} \) and \( K_{0'r'} \), which satisfy (117) and (122) respectively. These equations are solved by \( K_{0r} = C_0 g(r) \) and \( K_{0'r'} = C'_0 g(r') \). Then the final equation (119) reads

\[
2\pi i = \sqrt{2}(C_0 - C'_0) \sum_r \frac{g(r)}{r}. \tag{141}
\]
The $m = n = 0$ case of (127) reads
\[ \sqrt{\pi} = \frac{1}{\pi} \sum_r \frac{g(r)}{r} \]  \hspace{1cm} (142)
so that
\[ C_0 - C'_0 = i \sqrt{\frac{2}{\pi}} \]  \hspace{1cm} (143)
We write the solutions as $C_0 = i(1 - \beta_0)/\sqrt{2\pi}$ and $C'_0 = i(-1 - \beta_0)/\sqrt{2\pi}$, completing the solution of the left inverse equations:
\[ K_{0r} = \frac{ig(r)(1 - \beta_0)}{\sqrt{2\pi}} \]  \hspace{1cm} (144)
\[ K'_{0r} = \frac{ig(r')(-1 - \beta_0)}{\sqrt{2\pi}} \]  \hspace{1cm} (145)
Equating these solutions to the corresponding results of the right inverse calculation, leads to the conclusion that all $\beta$’s have the common value $\beta$, all the $\alpha$’s have the same value $\alpha$, and $\beta = -\alpha$.

By insisting that $K^T$ be both a left and right inverse of $C$ we have reduced the ambiguities in the solution to the single parameter $\alpha$, which is not determined by the the full set of left and right inverse equations. However, since interchanging primed and unprimed indices reverses the sign of the elements of $C$, we can require that this applies also for the elements of $K$. This then would determine $\alpha = 0$.

\[ K_{mr} = \frac{ig(r)}{2\pi mg(m)} \left[ \frac{r + m}{r - m} \right], \quad m \neq 0 \]
\[ K_{m'r'} = -\frac{ig(r')}{2\pi m'g(m')} \left[ \frac{r' + m'}{r' - m'} \right] \]
\[ K_{0r} = \frac{ig(r)}{\sqrt{2\pi}} \]
\[ K'_{0r} = -\frac{ig(r')}{\sqrt{2\pi}} \]
\[ K_{mr'} = -i \frac{g(r')}{2\pi mg(m)}, \quad m \neq 0 \]
\[ K_{m'r} = i \frac{g(r)}{2\pi m'g(m')} \]  \hspace{1cm} (146)

\[ 3.2.2 \text{ Continuum limit of } \hat{D} \]

From its definition $\hat{D} \equiv KS^\dagger$ we can quickly show that $\hat{D}$ is antisymmetric:
\[ \hat{D}^T = S^* K^t = S^* (C^t + D^T S^\dagger) S^\dagger = -(C^* + S^8 D) S^\dagger = -KS^\dagger = -\hat{D} \]  \hspace{1cm} (147)
Furthermore we can obtain an equation for $\hat{D}$, similar to the one satisfied by $D$:

$$\hat{D}_C = KS^\dagger C = -KC^T S^* = -(CK^T)^T S^* = -S^*$$  \hspace{1cm} (148)$$

Then we can search for solutions in the continuum limit.

$$2\pi i (-S^*_{rn}) \rightarrow \frac{2}{r + n} = \sum_s \hat{D}_{rs} \frac{2}{s - n}, \quad n \neq 0$$  \hspace{1cm} (149)$$

$$\frac{\sqrt{2}}{r} = \sum_s \hat{D}_{rs} \frac{\sqrt{2}}{s} - \sum_{s'} \hat{D}_{rs'} \frac{\sqrt{2}}{s'},$$  \hspace{1cm} (150)$$

$$0 = \sum_{s'} \hat{D}_{rs'} \frac{2}{s - n'}, \quad n \neq 0$$  \hspace{1cm} (151)$$

$$\frac{\sqrt{2}}{r'} = \sum_s \hat{D}_{r's} \frac{\sqrt{2}}{s} - \sum_{s'} \hat{D}_{r's'} \frac{\sqrt{2}}{s'}$$  \hspace{1cm} (153)$$

$$\frac{2}{r' + n'} = \sum_{s'} \hat{D}_{r's'} \frac{2}{s' - n'},$$  \hspace{1cm} (154)$$

In (126) change the summation index to $s$ and then set $z = r$ to get

$$\frac{1}{r + n} = \sum_{s=1/2}^{\infty} \frac{-sg(s)g(r)}{\pi(r + s)} \frac{1}{n - s}, \quad n = 0, 1, 2, \cdots$$  \hspace{1cm} (155)$$

Doing the same in (126) gives a second identity

$$\frac{1}{r + n} = \sum_{s=1/2}^{\infty} \frac{rg(r)g(s)}{\pi(r + s)} \frac{1}{n - s}, \quad n = 1, 2, \cdots$$  \hspace{1cm} (156)$$

For $n > 0$ adding (155) and (156) shows that

$$\hat{D}_{rs} = \frac{(s - r)g(s)g(r)}{2\pi(r + s)}$$  \hspace{1cm} (157)$$

satisfies (149) while

$$\hat{D}_{r's'} = \frac{-(s' - r')g(s')g(r')}{2\pi(r' + s')}$$  \hspace{1cm} (158)$$

satisfies (154).

Also for $n > 0$ taking the difference (155) minus (156) gives

$$0 = \sum_{s=1/2}^{\infty} \frac{g(s)g(r)}{\pi} \frac{1}{s - n},$$  \hspace{1cm} (159)$$
which shows that
\[ \hat{D}_{\alpha \beta} = \delta_{\alpha} \frac{g(s')g(r)}{\pi} \]  
\[ \hat{D}_{\alpha \beta} = \delta_{\alpha} \frac{g(s)g(r')}{\pi} \]

satisfy (151) (152) respectively.

Finally we consider (155) for \( n = 0 \):

\[ \frac{1}{r} = \sum_{s=1/2}^{\infty} \frac{g(s)g(r)}{\pi(r + s)} \]

which we can use to evaluate

\[ \sum_{s} \hat{D}_{\alpha s} \sqrt{\frac{2}{s}} = \sum_{s} \frac{(s - r)g(s)g(r)}{2\pi(r + s)} \frac{\sqrt{2}}{s} = \sum_{s} \frac{\sqrt{2}g(s)g(r)}{\pi(r + s)} - \sum_{s} \frac{\sqrt{2}g(s)g(r)}{2\pi s} = \frac{\sqrt{2}}{r} - g(r) \sum_{s} \frac{\sqrt{2}g(s)}{2\pi s} \]  

Then (150) reads

\[ 0 = -g(r) \sum_{s} \frac{\sqrt{2}g(s)}{2\pi s} - \delta_{\alpha} g(r) \sum_{s} \frac{\sqrt{2}g(s')}{\pi s'} \]

determining \( \delta_{\alpha} = -1/2 \). Similarly applying the same analysis to (153) determines \( \delta_{\alpha'} = +1/2 \). Thus

\[ \hat{D}_{\alpha s'} = -\frac{g(s')g(r)}{2\pi} \]  
\[ \hat{D}_{\alpha s} = \frac{g(s)g(r')}{2\pi} \]

This completes the determination of all the matrix elements of \( \hat{D} \) in the continuum limit. The ambiguities are resolved because we imposed antisymmetry \( \hat{D}^T = -\hat{D} \).
4 Vertex for 3 antiperiodic strings.

We next consider the transition of two smaller chains into one larger chain. If all three strings are antiperiodic, it is easy enough to evaluate the overlap matrices with $M$ finite.

\[
C_{sr_1} = -\frac{1}{\sqrt{MK}} \left( 1 + e^{-2\pi isK/M} \right) \cos \left( \frac{s\pi}{2M} - \frac{r\pi}{2K} \right)
\]

\[
C_{sr_2} = -\frac{1}{\sqrt{ML}} \left( 1 + e^{-2\pi isK/M} \right) \cos \left( \frac{s\pi}{2M} - \frac{r\pi}{2L} \right)
\]  

\[
S_{sr_1} = -\frac{1}{\sqrt{MK}} \left( 1 + e^{-2\pi isK/M} \right) \cos \left( \frac{s\pi}{2M} + \frac{r\pi}{2K} \right)
\]

\[
S_{sr_2} = -\frac{1}{\sqrt{ML}} \left( 1 + e^{-2\pi isK/M} \right) \cos \left( \frac{s\pi}{2M} + \frac{r\pi}{2L} \right).
\]  

The continuum limit of these matrices is $K, L \to \infty$ with $K/M = x$ with $0 < x < 1$ fixed and $s$ or $M - s$, $r_1$ or $K - r_1$, and $r_2$ or $L - r_2$ finite. Unlike the cases where the number of antiperiodic strings is even, when the $C$’s and $S$’s can be reduced to reciprocals of linear combinations of integers, there is no common overall factor that can be removed from $C, S$ when the number of antiperiodic strings is odd and the dependence on exponentials of the indices cannot be factored off. We have not been able to modify Goldstone’s method to handle this more complicated situation.

However if the number of bits $M$ stays finite there is nothing to prevent a numerical solution of the equations for $D, K,$ and $\hat{D}$. To illustrate this (and indicate the accuracy of the numerics), we used MATLAB to calculate the first few elements of $D$, with increasing $M$ at fixed $x = K/M$, in Tables 2 and 3. For an analytic treatment of the continuum limit, in the absence of an adaptation of Goldstone’s method to this situation, we have the method described in the introduction and illustrated in Fig. 1, and a less direct approach exploiting bosonization explained in the following subsection.

| M   | Re 100D_{4,1}  | Re 100D_{5,1}  | Re 100D_{5,3}  | Re 100D_{7,1}  | Re 100D_{7,3}  | Re 100D_{7,5}  |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 14  | 2.2447648      | 1.3055708      | 0              | 0              | -.51953822     | -.36957678     |
| 28  | 2.8152666      | 2.5958446      | .54056077      | 2.0870933      | .58406856      | .13751509      |
| 56  | 2.9583407      | 2.9233054      | .67876263      | 2.6263502      | .87204834      | .27324926      |
| 112 | 2.9941314      | 3.0054722      | .71347885      | 2.7623182      | .94472719      | .30761115      |
| 224 | 3.0038000      | 3.0260318      | .72216757      | 2.7963820      | .96293806      | .31622643      |
| 448 | 3.0053171      | 3.0311728      | .72434031      | 2.8049023      | .96749326      | .31838175      |
| 896 | 3.0058764      | 3.0324580      | .72488353      | 2.8070326      | .96863220      | .31892066      |
| 1792| 3.0060162      | 3.0327794      | .72501933      | 2.8075652      | .96891695      | .31905540      |

Table 2: The real parts of the first few elements of $100D$, with both indices on short string 1, for the vertex of 3 antiperiodic strings for the case $x = K/M = 4/7$ and increasing values of $M$. 

\[\text{Table 2: The real parts of the first few elements of } 100D, \text{ with both indices on short string 1, for the vertex of 3 antiperiodic strings for the case } x = K/M = 4/7 \text{ and increasing values of } M.\]
Table 3: The imaginary parts of the first few elements of $100 \times D$, with both indices on short string 1, for the vertex of 3 antiperiodic strings for the case $x = K/M = 4/7$. Notice that the imaginary parts tend to 0 as $1/M$.

| M  | $\text{Im}-100D_{\pm \pm}$ | $\text{Im}-100D_{\pm \mp}$ | $\text{Im}-100D_{\mp \mp}$ | $\text{Im}-100D_{\mp \pm}$ | $\text{Im}-100D_{\pm \pm}$ |
|-----|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 14  | 2.2447648                 | 3.1519267                | .89896526                 | 3.4549332               | 1.2542762               |
| 28  | 1.1661216                 | 1.7344879                | .54056077                 | 2.0870933               | .87412037               |
| 56  | .58845055                 | .88677499                | 1.0878699                 | .46611955               | .18257932               |
| 112 | .29489620                 | .44519168                | .54945926                 | .23664184               | .09331282               |
| 224 | .14753186                 | .22321352                | .07112730                 | .04690776               | .02348525               |
| 448 | .07377639                 | .1164470                 | .03558456                 | .01174654               | .01174654               |
| 896 | .03688950                 | .05582709                | .01779489                 | .00587376               | .00587376               |
|1792 | .01844492                 | .02791414                | .00889777                 | .00587376               | .00587376               |

4.1 Bosonization

Another tool to evaluate vertex overlaps, in the continuum limit, is to double the number of real fermion fields and describe the resulting system in the language of bosonization. Since the boson fields are free, one can then borrow Mandelstam’s results for the overlap for bosonic coordinates and then express them in terms of the fermion fields using the explicit formulas of bosonization. It is convenient to combine the pair of real fermion fields into a single complex fermion field using $\psi = (S^1 + iS^2)/\sqrt{2}$, or in terms of modes $b = (b^1 + ib^2)/\sqrt{2}$, $d = (b^1 - ib^2)/\sqrt{2}$, where $b$ destroys 1 unit of positive charge and $d$ destroys one unit of negative charge. Then for the three vertex the ground state of the long string has the structure

$$|G\rangle \propto \exp \left\{ \frac{1}{2} \sum_{m,n>0} \alpha_m N_{mn} a_{-m} a_{-n} + \sum_{m>0} N_m a_{-m} - \tau_0 \frac{p^2}{2\alpha_1 \alpha_2 \alpha_3} \right\} |0\rangle$$

where the indices $r, s$ run over the (half integer) modes of the two smaller strings. The structure in bosonized language is [8]

$$|G\rangle \propto \exp \left\{ \frac{1}{2} \sum_{m,n>0} \alpha_m N_{mn} \tilde{a}_{-m} \tilde{a}_{-n} + \sum_{m>0} N_m \tilde{a}_{-m} - \tau_0 \frac{\tilde{p}^2}{2\alpha_1 \alpha_2 \alpha_3} \right\} |0\rangle$$

where $m, n$ run over the bosonic modes of the two shorter strings. $\tau_0$ depends only on the $\alpha$’s:

$$\tau_0 = \alpha_1 \ln \left| \frac{\alpha_1}{\alpha_3} \right| + \alpha_2 \ln \left| \frac{\alpha_2}{\alpha_3} \right| = |\alpha_3||x \ln x + (1 - x) \ln (1 - x)|.$$  \hspace{1cm} (171)

The symbol $P$ contains the zero mode information

$$P = \alpha_1 a_0^2 - \alpha_2 a_0^1, \quad \tilde{P} = \alpha_1 \tilde{a}_0^2 - \alpha_2 \tilde{a}_0^1.$$ \hspace{1cm} (172)
where \( \alpha_i = 2P_i^+ > 0 \) for \( i = 1, 2 \) where \( i \) labels the two short strings. The long string is labeled by \( i = 3 \) and \( \alpha_3 = -\alpha_1 - \alpha_2 \) so \( P^+ \) conservation means \( \sum_{r=1}^{3} \alpha_r = 0 \). While an uncompactified string coordinate requires \( a_0 = \tilde{a}_0 \), the bosonic coordinate, describing a spin system, is compactified and this equality need not hold. We shall first assume that \( P = \tilde{P} = 0 \). This corresponds in fermion language to restricting the states to be neutral in both the charge and the chiral charge. In the boson language charge corresponds to Kaluza-Klein (KK) momenta and chiral charge to winding number. The dependence on them will be determined later. The \( N_{mn}, N_m \) are all known from Mandelstam’s work on interacting strings [8]. we quote them using Goldstone’s \( g(z) \) function

\[
g(z) = \frac{\Gamma(1+xz)\Gamma(1+(1-x)z)e^{xz}}{z\Gamma(1+z)\sqrt{x(1-x)}}
\]

Making the appropriate modifications to apply to our choice of parameterization, The \( N \)’s occurring in our bosonic formula are

\[
N^1_m = -\frac{1}{\alpha_1 m} \frac{\sqrt{x}}{m^2 g(m/x)\sqrt{1-x}}
\]

\[
N^2_m = \frac{1}{\alpha_2 m} \frac{(1-x)^{3/2}}{m^2 g(m/(1-x))\sqrt{x}}
\]

\[
N^{11}_{mn} = \frac{x}{(m+n)mg(m/x)ng(n/x)}
\]

\[
N^{22}_{mn} = \frac{1-x}{(m+n)mg(m/(1-x))ng(n/(1-x))}
\]

\[
N^{12}_{mn} = -\frac{x(1-x)}{(m(1-x)+nx)mg(m/x)ng(n/(1-x))}
\]

Mandelstam’s \( N \)’s have extra sign factors \((-)^m\) for \( N^1_m \) and \( N^{12}_{mn} \) and \((-)^{m+n}\) for \( N^{11}_{mn} \).

The explicit bosonization formula is

\[
a_{-m} = a^\dagger_m = \sum_{r=1/2}^{\infty} \left[ b^\dagger_{r+m} b_r - d^\dagger_{r+m} d_r \right] + \sum_{r=1/2}^{m-1/2} b^\dagger_{m-r} d^\dagger_r
\]

with a corresponding formula relating \( \tilde{a}_m \) to \( \tilde{b}_r, \tilde{d}_r \). One finds, with due care with operator ordering,

\[
[a_m, a_n] = m\delta_{m+n,0}, \quad [a_m, \tilde{a}_n] = 0, \quad [\tilde{a}_m, \tilde{a}_n] = m\delta_{m+n,0}
\]

\[4\]Mandelstam’s original \( N \)’s were obtained for open strings. For closed strings, the phases of the \( N \)’s depend on the \( \sigma \)’s chosen for the interaction points. The following \( N \)’s correspond to the \( \sigma \) for string 1 identified with the interval \( 0 < \sigma < P^+_1 \) on the long string, with the \( \sigma \) for string 2 identified with the interval \( P^+_1 < \sigma < P^+ \) on the long string. The open string \( N \)’s are appropriate for the closed string parameterization chosen in [14]
We see that the zero mode
\[
a_0 = \sum_{r=1/2}^{\infty} [b_r^\dagger b_r - d_r^\dagger d_r]
\]
is just the charge carried by the \(b, d\) operators and \(\tilde{a}_0\) is the charge carried by the \(\tilde{b}, \tilde{d}\) operators, and there are certainly states for which these charges are different. The KK momentum is \(a_0 + \tilde{a}_0\) and the winding number is \(a_0 - \tilde{a}_0\).

One can now consider the vertex for the particular states
\[
a_m^\dagger a_n^\dagger |0\rangle = \sum_{s=1/2}^{n-1/2} (b_{m+n-s}^\dagger d_s^\dagger - b_{s}^\dagger d_{m+n-s}^\dagger) |0\rangle + \left(\sum_{s=1/2}^{n-1/2} b_{m-s}^\dagger d_s^\dagger\right) \left(\sum_{r=1/2}^{m-1/2} b_{m-r}^\dagger d_r^\dagger\right) |0\rangle
\]
Plugging the left side into Mandelstam’s formula yields \(mnN_{mn}\). On the other hand plugging the right side into the assumed structure for the fermion language yields
\[
mnN_{mn} = \sum_{s=1/2}^{n-1/2} (D_{m+n-s,s} - D_{s,m+n-s}) + 2 \sum_{s=1/2}^{n-1/2} \sum_{r=1/2}^{m-1/2} (D_{n-s,s} D_{m-r,r} - D_{n-s,r} D_{m-r,s})
\]
This formula determines recursively the \(D\)’s in terms of the \(N\)’s. In particular, for \(x = 4/7\) (the case tabulated in the tables)
\[
D_{3/2,1/2}^{11} = \frac{1}{2} N_{11}^{11} = \frac{1}{4} \left[\frac{3}{7}\right]^{5/2} \approx 0.0300606
\]
\[
D_{5/2,1/2}^{11} = N_{21}^{11} = \frac{10}{21} \left[\frac{3}{7}\right]^{13/4} \approx 0.0303289
\]
which are very close to the last entries in Table 2, for \(M = 1792\). Here the superscripts refer to which of the two shorter strings the mode numbers \(r, s\) belong to. The table is compiled assuming they both refer to string 1.

The quadratic terms of the recursion formula do not contribute in the determination of \(D_{3/2,1/2}^{11}\) and \(D_{5/2,1/2}^{11}\). That changes for larger values of the indices. Indeed, by studying the state \(a_{-3} a_{-1} |0\rangle\) one learns that
\[
D_{7/2,1/2} = \frac{3}{2} N_{31}^{11} - \frac{1}{2} (D_{3/2,1/2}^{11})^2 = \frac{3 \cdot 13 \cdot 17}{4 \cdot 49} \left(\frac{3}{7}\right)^{3/2} D_{3/2,1/2} - \frac{1}{2} (D_{3/2,1/2}^{11})^2 \approx 0.0280774
\]
again very close to the tabulated result for \(M = 1792\).

So far we have considered only states with \(P = \tilde{P} = 0\). To check on the zero mode structure of the bosonic formalism we need to study short string states with nonzero \(a_0\) and/or nonzero \(\tilde{a}_0\). For example the states
\[
b_{1/2}^\dagger b_{3/2}^\dagger b_{7/2}^\dagger \cdots b_{n-1/2}^\dagger |0\rangle
\]
have \( a_0 = n > 0 \) and the corresponding states with \( b \to d \) have \( a_0 = -n < 0 \), with similar constructions for the tilde operators. Since there are no gaps in the string of operators, it is easy to show that all these states are annihilated by \( a_n, \tilde{a}_n \) with \( n > 0 \). This means that if the short string states are selected from these, the only state dependence of the bosonic vertex is given by the zero mode factors

\[
\exp \left\{ -\tau_0 \frac{P^2}{2\alpha_1\alpha_2\alpha_3} - \tau_0 \frac{\tilde{P}^2}{2\alpha_1\alpha_2\alpha_3} \right\}
\]

(188)

Since the ground state of the long string has \( a_0 = \tilde{a}_0 = 0 \) the two short strings must have opposite values of these charges.

| \( M \) | \( \text{Re}D_{1/2,1/2}^{12} \) | \( \text{Re}D_{3/2,1/2}^{12} \) |
|---|---|---|
| 14 | -0.11078489 | -0.10214029 |
| 28 | -0.056457379 | -0.054671329 |
| 56 | -0.028362255 | -0.027791394 |
| 112 | -0.014197838 | -0.013952835 |
| 224 | -0.0071010083 | -0.0069835650 |
| 448 | -0.0035507653 | -0.0034926761 |
| 896 | -0.0017754153 | -0.0017464497 |
| 1792 | -0.00088771173 | -0.00087323883 |

Table 4: Two elements of the matrix \( \text{Re}D \) with the first index = 1/2 or = 3/2 on short string 1 and the second = 1/2 on short string 2, for the vertex of 3 antiperiodic strings for the case \( x = K/M = 4/7 \). Notice that the real parts vanish as \( 1/M \).

Let’s take, as a simple example, the two short string state \( |0; 1, -1\rangle = b_1^{\dagger}d_1^{\dagger}e^{i\eta} |0\rangle \), which has \( a_0^1 = -a_0^2 = 1 \) and \( \tilde{a}_0 = 0 \). We included a phase factor, which is not fixed by specifying only the charges of the two strings. For this state \( \tilde{P} = 0 \) and \( P = -(\alpha_1 + \alpha_2) = \alpha_3 \). Since this is the lowest energy two string state with these charges, \( a_n|0; 1, -1\rangle = 0 \) for all \( n > 0 \). Then

\[
\exp \left\{ -\tau_0 \frac{P^2}{\alpha_1\alpha_2\alpha_3} \right\} = x^{1/(1-x)/2}(1-x)^{1/x/2} \to (4/7)^{7/6}(3/7)^{7/8} \approx 0.24801384
\]

(189)

for the value \( x = 4/7 \) which was used in the tabulations. Tables 4, 5 show the values for the matrix elements \( D_{1/2,1/2}^{12} \) and \( D_{3/2,1/2}^{12} \) for increasing values of \( M \) in the string bit model. The bosonic calculation is quite close to \( \text{Im}D_{1/2,1/2}^{12} \) in the last entry of Table 5. The corresponding real part shown in Table 4 is also very small. In other words there is excellent agreement if the phase of the state is chosen to be \( e^{i\eta} = e^{-i\pi/2} = -i \). Moreover, from the equations satisfied by \( D \) and the explicit forms for \( C \) and \( S \), one can argue that, in the continuum
Table 5: Two elements of the matrix $\text{Im}D$ with the first index $= 1/2$ or $= 3/2$ on short string 1 and the second $= 1/2$ on short string 2, for the vertex of 3 antiperiodic strings for the case $x = K/M = 4/7$. Here the imaginary parts converge to a nonzero limit for large $M$.

| M  | $\text{Im}D_{1/2,1/2}$ | $\text{Im}D_{3/2,1/2}$ |
|-----|------------------------|------------------------|
| 14  | 0.22464955             | 0.089574631            |
| 28  | 0.24213249             | 0.12066219             |
| 56  | 0.24654097             | 0.12867619             |
| 112 | 0.24764546             | 0.13069489             |
| 224 | 0.24792173             | 0.13120051             |
| 448 | 0.24799081             | 0.13132697             |
| 896 | 0.24800808             | 0.13158559             |
| 1792| 0.24801240             | 0.13136650             |

Thus we can anticipate that we should identify the states

$$|0; n, -n\rangle = \pm b_{1/2}^{1\dagger} d_{1/2}^{2\dagger} b_{3/2}^{1\dagger} d_{3/2}^{2\dagger} \cdots b_{n-1/2}^{1\dagger} d_{n-1/2}^{2\dagger} |0\rangle e^{-in\pi/2}$$

where the overall sign can be settled by comparing the analytic bosonic calculation to the numerical fermionic calculation for large $M$. Since the bosonization formulas don’t fix the relative phase of states of different strings, this is satisfactory.

We have also tabulated the element $D_{3/2,1/2}$ which is determined by the state $b_{3/2}^{1\dagger} d_{1/2}^{2\dagger} |0\rangle$. To use bosonization to calculate the $M \to \infty$ limit, we first note that

$$a_{-1}^{\dagger} |0; 1, -1\rangle = -ia_{-1}^{\dagger} b_{1/2}^{1\dagger} d_{1/2}^{2\dagger} |0\rangle$$

$$= -ib_{3/2}^{1\dagger} d_{1/2}^{2\dagger} |0\rangle$$

The contribution of the last line to the long string ground state is just $-iD_{3/2,1/2}$. In bosonized language the left side of the first line shows that it is $PN_1^1 (-iD_{1/2,1/2}^{12})$. Thus we conclude that

$$D_{3/2,1/2}^{12} = PN_1^1 D_{1/2,1/2}^{12}.$$  

For the value of $x = 4/7$ chosen for the tables, $P = \alpha_3 = - (\alpha_1 + \alpha_2)$ and

$$PN_1^1 = (1 - x)^{(1-x)/x} \to \left(\frac{3}{7}\right)^{3/4} \approx 0.52968468$$

which is very close to the ratio $\text{Im}D_{3/2,1/2}^{12}/\text{Im}D_{1/2,1/2}^{12} \approx 0.52967778$ evaluated using the last line of the tables ($M = 1792$).
These comparisons of the bosonic overlap to the string bit model for large finite $M$ give strong support to the use of bosonization (with due care with phase conventions) to calculate the continuum limit of spin chain models, and, in particular, of string bit models. A more systematic analysis of the use of bosonization to calculate the 3 AP string vertex will be the subject of another paper.

Appendices

A Matrix elements

We quote a generally useful formula for matrix elements of the sort required in the overlap calculations:

$$\langle 0 | \exp \left\{ \frac{1}{2} f^T A f + \alpha^T f \right\} \exp \left\{ \frac{1}{2} f^{\dagger T} B f^{\dagger} + \beta^T f^{\dagger} \right\} | 0 \rangle = \det^{1/2}(I + BA)$$

$$\times \exp \left\{ \frac{1}{2} \alpha^T B (I + AB)^{-1} \alpha + \frac{1}{2} \beta^T (I + AB)^{-1} A \beta + \beta^T (I + AB)^{-1} \alpha \right\}$$

(194)

For example, this formula enables the normalization of the ground state of $h_{AP}$ when it is expressed in terms of the eigenstates of $h_P$, or vice versa.

B Overlap Matrices

The transition amplitude between eigenstates of two different Hamiltonians can be determined by the relation between eigenoperators of the corresponding Hamiltonian. Call the lowering operators for $H_1$ and $H_2$ $f^{\dagger}_k$ and $f^{\dagger}_k$ respectively. Then define the matrices $C$ and $S$ by

$$f^{\dagger}_k = C_{kl} f^{\dagger}_l + S_{kl} f^{\dagger}_l$$

(195)

To determine the ground state of $H_2$ in terms of the eigenstates of $H_1$, we need to solve the equation

$$CD = -S$$

(196)

where $D$ enters the ground state ket of $H_2$ as

$$|G2\rangle = \exp \left\{ \frac{1}{2} f^{\dagger}_k D_{kl} f^{\dagger}_l \right\} |G1\rangle$$

(197)
We first quote the $C$ and $S$ matrices for two periodic strings ($H_1$) transitioning to one periodic string ($H_2$) obtained in [6].

$$C_{m0} = \frac{1}{\sqrt{LK}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2\pi m/M}} \cos \left( \frac{m\pi}{2M} - \frac{\pi}{4} \right)$$

$$C_{mn} = \frac{1}{\sqrt{ML}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2\pi n/L - m/M}} \cos \left( \frac{n\pi}{2L} - \frac{m\pi}{2M} \right)$$

$$C_{mr} = \frac{1}{\sqrt{MK}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2\pi (n/K - m/M)}} \cos \left( \frac{n\pi}{2K} - \frac{m\pi}{2M} \right)$$

$$S_{m0} = -\frac{1}{\sqrt{LK}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2\pi m/M}} \cos \left( \frac{m\pi}{2M} + \frac{\pi}{4} \right)$$

$$S_{mn} = -\frac{1}{\sqrt{ML}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2\pi (n/L + m/M)}} \cos \left( \frac{n\pi}{2L} + \frac{m\pi}{2M} \right)$$

$$S_{mr} = \frac{1}{\sqrt{MK}} \frac{1 - e^{-2\pi imL/M}}{1 - e^{2\pi (n/K + m/M)}} \cos \left( \frac{n\pi}{2K} + \frac{m\pi}{2M} \right).$$

The continuum limit of these matrices was given in [6]. Replacing the two periodic strings with two antiperiodic strings leads to the modifications

$$C_{0r_1} = -\frac{1}{\sqrt{MK}} \frac{2}{1 - e^{-2\pi i r/K}} \cos \left( \frac{r\pi}{2K} - \frac{\pi}{4} \right)$$

$$C_{mr_1} = -\frac{1}{\sqrt{MK}} \frac{2}{1 + e^{-2\pi i mK/M}} \cos \left( \frac{m\pi}{2M} - \frac{r\pi}{2K} \right)$$

$$C_{0r_2} = -\frac{1}{\sqrt{ML}} \frac{2}{1 - e^{-2\pi i r/L}} \cos \left( \frac{r\pi}{2L} - \frac{\pi}{4} \right)$$

$$C_{mr_2} = -\frac{1}{\sqrt{MK}} \frac{2}{1 + e^{-2\pi i mK/M}} \cos \left( \frac{m\pi}{2M} - \frac{r\pi}{2K} \right)$$

$$S_{0r_1} = -\frac{1}{\sqrt{MK}} \frac{2}{1 - e^{2\pi i r/K}} \cos \left( \frac{r\pi}{2K} + \frac{\pi}{4} \right)$$

$$S_{mr_1} = -\frac{1}{\sqrt{MK}} \frac{2}{1 + e^{2\pi i mK/M}} \cos \left( \frac{m\pi}{2M} + \frac{r\pi}{2K} \right)$$

$$S_{0r_2} = -\frac{1}{\sqrt{ML}} \frac{2}{1 - e^{2\pi i r/L}} \cos \left( \frac{r\pi}{2L} + \frac{\pi}{4} \right)$$

$$S_{mr_2} = -\frac{1}{\sqrt{ML}} \frac{2}{1 + e^{2\pi i mK/M}} \cos \left( \frac{m\pi}{2M} + \frac{r\pi}{2L} \right).$$

The continuum limit of these matrices is $K, L \to \infty$ with $K/M = x$ with $0 < x < 1$ fixed and $m$ or $M - m$, $r1$ or $K - r1$, and $r2$ or $L - r2$ finite. It is convenient to remove some overall factors from $C$ and $S$. Define

$$C_{mr} = \frac{1 + e^{-2\pi i m}}{2\pi i} c_{mr}, \quad S_{mr} = \frac{1 + e^{-2\pi i m}}{2\pi i} s_{mr}$$

(202)
We quote the continuum limit in terms of the $c$ and $s$ matrices. There are eight distinct cases:

1. $m, r_1, r_2$ finite

\[
\begin{align*}
  c_{0r1} &\to \frac{-\sqrt{x}}{r \sqrt{2}} \\
  c_{0r2} &\to \frac{-\sqrt{1-x}}{r \sqrt{2}} \\
  c_{mr1} &\to \frac{1}{m \sqrt{x} - r / \sqrt{x}} \\
  c_{mr2} &\to \frac{1}{m \sqrt{1-x} - r / \sqrt{1-x}} \\
  s_{0r1} &\to \frac{\sqrt{x}}{r \sqrt{2}} \\
  s_{0r2} &\to \frac{\sqrt{1-x}}{r \sqrt{2}} \\
  s_{mr1} &\to \frac{1}{m \sqrt{x} + r / \sqrt{x}} \\
  s_{mr2} &\to \frac{1}{m \sqrt{1-x} + r / \sqrt{1-x}} \\
\end{align*}
\]

(203)

2. $m' = M - m \neq 0, r_1, r_2$ finite

\[
\begin{align*}
  c_{mr1} &\sim O(M^{-1}) \to 0, & c_{mr2} &\sim O(M^{-1}) \to 0 \\
  s_{mr1} &\sim O(M^{-1}) \to 0, & s_{mr2} &\sim O(K^{-1}) \to 0 \\
\end{align*}
\]

(205) (206)

3. $m, r'_1 = K - r_1, r_2$ finite

\[
\begin{align*}
  c_{0r1} &\to \frac{\sqrt{x}}{r' \sqrt{2}} \\
  c_{0r2} &\to \frac{-\sqrt{1-x}}{r' \sqrt{2}} \\
  c_{mr1} &\to 0, & c_{mr2} &\to \frac{1}{m \sqrt{1-x} - r / \sqrt{1-x}} \\
  s_{0r1} &\to \frac{\sqrt{x}}{r' \sqrt{2}} \\
  s_{0r2} &\to \frac{\sqrt{1-x}}{r' \sqrt{2}} \\
  s_{mr1} &\to 0, & s_{mr2} &\to \frac{1}{m \sqrt{1-x} + r / \sqrt{1-x}} \\
\end{align*}
\]

(207) (208)

4. $m', r', r_2$ finite

\[
\begin{align*}
  c_{mr1} &\to -\frac{1}{m' \sqrt{x} - r' / \sqrt{x}}, & c_{mr2} &\to 0 \\
  s_{mr1} &\to \frac{1}{m' \sqrt{x} + r' / \sqrt{x}}, & s_{mr2} &\to 0 \\
\end{align*}
\]

(209) (210)

5. $m, r_1, r_2'$ finite

\[
\begin{align*}
  c_{0r1} &\to \frac{-\sqrt{x}}{r \sqrt{2}} \\
  c_{0r2} &\to \frac{\sqrt{1-x}}{r' \sqrt{2}} \\
  c_{mr1} &\to \frac{1}{m \sqrt{x} - r / \sqrt{x}} \\
  c_{mr2} &\to 0 \\
  s_{0r1} &\to \frac{\sqrt{x}}{r \sqrt{2}} \\
  s_{0r2} &\to -\frac{\sqrt{1-x}}{r' \sqrt{2}} \\
  s_{mr1} &\to \frac{1}{m \sqrt{x} + r / \sqrt{x}} \\
  s_{mr2} &\to 0 \\
\end{align*}
\]

(211) (212)
6. \( m', r_1, r'_2 \) finite

\[
\begin{align*}
    c_{mr_1} & \to 0, \quad c_{mr_2} \to -\frac{1}{m' \sqrt{1 - x - r'/\sqrt{1 - x}}} \\
    s_{mr_1} & \to 0, \quad s_{mr_2} \to \frac{1}{m' \sqrt{1 - x + r'/\sqrt{1 - x}}}
\end{align*}
\]  

(213)

(214)

7. \( m, r'_1, r'_2 \) finite

\[
\begin{align*}
    c_{0r_1} & \to \frac{\sqrt{x}}{r'\sqrt{2}}, \quad c_{0r_2} \to \frac{\sqrt{1 - x}}{r\sqrt{2}} \\
    c_{mr_1} & \to 0, \quad c_{mr_2} \to 0 \\
    s_{0r_1} & \to \frac{\sqrt{x}}{r'\sqrt{2}}, \quad s_{0r_2} \to -\frac{\sqrt{1 - x}}{r\sqrt{2}} \\
    s_{mr_1} & \to 0, \quad s_{mr_2} \to 0
\end{align*}
\]  

(215)

(216)

8. \( m', r'_1, r'_2 \) finite

\[
\begin{align*}
    c_{mr_1} & \to -\frac{1}{m' \sqrt{x - r'/\sqrt{x}}} \quad c_{mr_2} \to -\frac{1}{m' \sqrt{1 - x - r'/\sqrt{1 - x}}} \\
    s_{mr_1} & \to \frac{1}{m' \sqrt{x + r'/\sqrt{x}}} \quad s_{mr_2} \to \frac{1}{m' \sqrt{1 - x + r'/\sqrt{1 - x}}}
\end{align*}
\]  

(217)

(218)

C. The continuum limit of \( D \) for 2 antiperiodic strings \( \to 1 \) periodic string

In [6] we obtained the continuum limit of \( D \) for the case that all strings are periodic using Goldstone’s method [9]. To do the analogous analysis for this case, we work with a slightly modified meromorphic function

\[
g(z) = \frac{\Gamma(1/2 + xz)\Gamma(1/2 + (1 - x)z)}{\Gamma(1 + z)\sqrt{x(1 - x)}} e^{\xi z}
\]

\[
\xi = -x \ln x - (1 - x) \ln(1 - x).
\]  

(220)

It has zeroes at negative integers and poles at \( z = -(1/2 + n)/x \) and at \( z = -(1/2 + n)/(1 - x) \) with \( n = 0, 1, 2, \cdots \):

\[
g(z) \sim \frac{1}{z + r/x \arg(r/x)}, \quad g(z) \sim \frac{1}{z + r/(1 - x) \arg(r/(1 - x))}
\]  

(220)

respectively. Its large \( z \) behavior is

\[g(z) \sim \sqrt{2\pi} z^{-1/2}, \quad |\arg(z)| < \pi\]  

(221)
For \( m \) a positive integer, \( g(z)/(z + m) \) has the same poles as \( g(z) \), since \( g(-m) = 0 \). Thus we can expand

\[
\frac{g(z)}{z + m} = \sum_r \left[ \frac{1}{z + r/x} \frac{1}{rg(r/x)} \frac{1}{m - r/x} \right. \\
+ \left. \frac{1}{z + r/(1 - x)} \frac{1}{rg(r/(1 - x))} \frac{1}{m - r/(1 - x)} \right], \quad m = 1, 2, \cdots. (222)
\]

Another identity follows by expanding

\[
\frac{zg(z)}{z + m} = \sum_r \left[ \frac{-r/x}{z + r/x} \frac{1}{rg(r/x)} \frac{1}{m - r/x} \right. \\
+ \left. \frac{-r/(1 - x)}{z + r/(1 - x)} \frac{1}{rg(r/(1 - x))} \frac{1}{m - r/(1 - x)} \right], \quad m = 0, 1, 2, \cdots. (223)
\]

Note that this second identity is valid for \( m = 0 \), which will be important for a complete determination of \( D \). Also note that the right sides of both identities are symmetric under \( x \to 1 - x \).

Setting \( z = s/x \) or \( z = s/(1 - x) \) turns these identities into equations similar to those to be solved. And comparing them in various cases, allows one to read off explicit expressions for the matrix elements of \( D \):

\[
\frac{1}{m + s/x} = \sum_r \left[ \frac{1}{s/x + r/x} \frac{1}{rg(r/x)g(s/x)} \frac{1}{m - r/x} \right. \\
+ \left. \frac{1}{s/x + r/(1 - x)} \frac{1}{rg(r/(1 - x))g(s/x)} \frac{1}{m - r/(1 - x)} \right], \quad (224)
\]

\[
\frac{1}{m + s/x} = \sum_r \left[ \frac{-r/x}{s/x + r/x} \frac{1}{rg(r/x)g(s/x)} \frac{1}{m - r/x} \right. \\
+ \left. \frac{-r/(1 - x)}{s/x + r/(1 - x)} \frac{1}{rg(r/(1 - x))g(s/x)} \frac{1}{m - r/(1 - x)} \right], \quad (225)
\]

\[
\frac{1}{m + s/(1 - x)} = \sum_r \left[ \frac{1}{s/(1 - x) + r/x} \frac{1}{rg(r/x)g(s/(1 - x))} \frac{1}{m - r/x} \right. \\
+ \left. \frac{1}{s/(1 - x) + r/(1 - x)} \frac{1}{rg(r/(1 - x))g(s/(1 - x))} \frac{1}{m - r/(1 - x)} \right], \quad (226)
\]

\[
\frac{1}{m + s/(1 - x)} = \sum_r \left[ \frac{-r/x}{s/(1 - x) + r/x} \frac{1}{rg(r/x)g(s/(1 - x))} \frac{1}{m - r/x} \right. \\
+ \left. \frac{-r/(1 - x)}{s/(1 - x) + r/(1 - x)} \frac{1}{rg(r/(1 - x))g(s/(1 - x))} \frac{1}{m - r/(1 - x)} \right], \quad (227)
\]

Again the second pair of equations is the first pair with \( x \to 1 - x \). The sum of the two identities enables the construction of antisymmetric solutions for \( D_{rs} \) or \( D_{r's'} \). (As before we
use a prime to distinguish indices close to their upper limit.)

\[
\frac{1}{m + s/x} = \frac{1}{2} \sum_r \left[ \frac{s - r}{s/x + r/x \, r g(r/x) s g(s/x)} \frac{1}{m - r/x} \right.
\]

\[
+ \frac{s - x r/(1 - x)}{s/x + r/(1 - x) \, r g(r/(1 - x)) s g(s/x)} \frac{1}{m - r/(1 - x)} \right]
\]

\[
\frac{1}{m + s/(1 - x)} = \frac{1}{2} \sum_r \left[ \frac{s - (1 - x) r/x}{s/(1 - x) + r/x \, r g(r/x) s g(s/(1 - x))} \frac{1}{m - r/x} \right.
\]

\[
+ \frac{s - r}{s/(1 - x) + r/(1 - x) \, r g(r/(1 - x)) s g(s/(1 - x))} \frac{1}{m - r/(1 - x)} \right]
\]

(228) (229)

On the other hand the difference of the two identities is useful for the construction of solutions for \( D_{rs'} \) or \( D_{r's} \):

\[
0 = \sum_r \left[ \frac{x}{r g(r/x) s g(s/x)} \frac{1}{m - r/x} \right.
\]

\[
+ \frac{x}{r g(r/(1 - x)) s g(s/x)} \frac{1}{m - r/(1 - x)} \right]
\]

(230)

\[
0 = \sum_r \left[ \frac{1 - x}{r g(r/x) s g(s/(1 - x))} \frac{1}{m - r/x} \right.
\]

\[
+ \frac{1 - x}{r g(r/(1 - x)) s g(s/(1 - x))} \frac{1}{m - r/(1 - x)} \right]
\]

(231)

Notice that the \( r \) and \( s \) dependence factorizes, which shows that the second equation is an \( s \) dependent factor times the first. The content of either is therefore

\[
0 = \sum_r \left[ \frac{1}{r g(r/x) m - r/x} + \frac{1}{r g(r/(1 - x)) m - r/(1 - x)} \right], \quad m = 1, 2, \cdots
\]

(232)

Finally we recall that the \( m = 0 \) case of the second identity is also valid and can be rewritten:

\[
g(s/x) = \sum_r \left[ \frac{1}{s/x + r/x \, r g(r/x)} \frac{1}{s/x + r/(1 - x) \, r g(r/(1 - x))} \right]
\]

(233)

\[
g(s/(1 - x)) = \sum_r \left[ \frac{1}{s/(1 - x) + r/x \, r g(r/x)} \frac{1}{s/(1 - x) + r/(1 - x) \, r g(r/(1 - x))} \right]
\]

(234)

Once again, the second equation is simply the first with the substitution \( x \to 1 - x \).

We next write out the equation \( cD = -s \) in the continuum limit, employing our convention about primed and unprimed indices:. There are 12 types of matrix elements of \( c \) and
the first index can be 0, \( m \neq 0, m' \) and the second can be \( s_1, s_1', s_2, s_2' \). Matrix elements of \( c, s \) that link primed indices and nonzero unprimed indices are zero. Let’s begin with the \( ms_1 \) matrix element

\[
-s_{ms_1} = -\frac{1/\sqrt{x}}{m + r/x} = c_{mr_1}D_{r_1s_1} + c_{mr_2}D_{r_2s_1}
\]

\[
= \frac{1/\sqrt{x}}{m - r/x}D_{r_1s_1} + \frac{1/\sqrt{1 - x}}{m - r/(1 - x)}D_{r_3s_1}
\]

(235)

Comparing to the first identity we read off

\[
D_{r_1s_1} = -\frac{1}{2} \frac{s - r}{s/x + r/x} \frac{1}{rg(r/x)sg(s/x)}
\]

(236)

\[
D_{r_2s_1} = -\frac{1}{2} \sqrt{\frac{1 - x}{x}} \frac{s - xr/(1 - x)}{s/x + r/(1 - x)} \frac{1}{rg(r/(1 - x))sg(s/x)}
\]

(237)

A parallel analysis of the \( ms_2 \) matrix element

\[
-s_{ms_2} = -\frac{1}{m\sqrt{1 - x + r/\sqrt{1 - x}}} = c_{mr_1}D_{r_1s_2} + c_{mr_2}D_{r_2s_2}
\]

\[
= \frac{1/\sqrt{x}}{m - r/x}D_{r_1s_2} + \frac{1/\sqrt{1 - x}}{m - r/(1 - x)}D_{r_3s_2}
\]

(238)

yields

\[
D_{r_1s_2} = -\frac{1}{2} \sqrt{\frac{x}{1 - x}} \frac{s - (1 - x)r/x}{s/x + r/x} \frac{1}{rg(r/x)sg(s/(1 - x))}
\]

(239)

\[
D_{r_2s_2} = -\frac{1}{2} \frac{s - r}{s/(1 - x) + r/(1 - x)} \frac{1}{rg(r/(1 - x))sg(s/(1 - x))}
\]

(240)

Note that \( D_{r_1s_1} \) and \( D_{r_2s_2} \) are explicitly antisymmetric, and they both go into each other on the substitution \( x \to 1 - x \). For the mixed indices \( D_{r_1s_2} = -D_{s_2r_1} \) by inspecting the results of the two calculations.

The elements so far obtained have all indices a finite distance from 0. When all indices are a finite distance from their upper limits, we can obtain the \( D \) elements by noting that in the latter case \( c \) and \( s \) can be obtained from the former case by priming all indices and multiplying the \( c \) matrix elements by \(-1\). It follows that corresponding \( D \) elements are the negatives of those in the unprimed case.
\[ D_{r's1} = \frac{1}{2} s' - r' \frac{1}{s'/x + r'/x r'g(r'/x) s'g(s'/x)} \]  
\[ D_{r's2} = \frac{1}{2} \sqrt{\frac{1 - x}{x}} s' - x r'/(1 - x) \frac{1}{s'/x + r'/x (1 - x) r'g(r'/x) s'g(s'/x)} \]  
\[ D_{r's1} = \frac{1}{2} \sqrt{\frac{1 - x}{x}} s' - (1 - x) r'/x \frac{1}{s'/x + r'/x r'g(r'/x) s'g(s'(1 - x))} \]  
\[ D_{r's2} = \frac{1}{2} \frac{s' - r'}{s'/x + r'/x (1 - x) r'g(r'/x) s'g(s'(1 - x))} \]  

It remains to obtain the mixed \( D \) elements with one index unprimed and the other primed. We begin with the equation (and the one with \( s'2 \) in place of \( s'1 \).)

\[ s_{ms'1} = 0 = c_{mr1} D_{r's1} + c_{mr2} D_{r's2} \]  
\[ = \sum_r \left[ \frac{1/\sqrt{x}}{m - r/x} D_{r's1} + \frac{1/\sqrt{1-x}}{m - r/(1-x)} D_{r's2} \right] \]  

Referring to the homogeneous identity we read off

\[ D_{r's1} = \kappa_1(s') \frac{\sqrt{x}}{r g(r/x)} \]  
\[ D_{r's2} = \kappa_1(s') \frac{\sqrt{1-x}}{r g(r/(1-x))} \]  

\( \kappa(s') \) is undetermined because the equation is homogeneous. Substituting \( s'2 \) for \( s'1 \) in the equation doesn’t change the \( r \) dependence but the \( \kappa \) may be different:

\[ D_{r's2} = \kappa_2(s') \frac{\sqrt{x}}{r g(r/x)} \]  
\[ D_{r's2} = \kappa_2(s') \frac{\sqrt{1-x}}{r g(r/(1-x))} \]  

The equation

\[ s_{ms'1} = 0 = c_{mr's1} D_{r's1} + c_{mr's2} D_{r's2} \]  
\[ = \sum_r \left[ \frac{1/\sqrt{x}}{m - r/x} D_{r's1} + \frac{1/\sqrt{1-x}}{m - r/(1-x)} D_{r's2} \right] \]  

33
and the one with $s_1 \rightarrow s_2$ constrain the mixed $D$ elements with the first index primed:

$$D_{\nu s_1} = \kappa'_1(s) \frac{\sqrt{r}}{r' g(r'/x)}$$  \hfill (251)

$$D_{\nu s_2} = \kappa'_1(s) \frac{\sqrt{r - x}}{r' g((r'/(1 - x))}$$  \hfill (252)

$$D_{\nu s_2} = \kappa'_2(s) \frac{\sqrt{r}}{r' g(r'/x)}$$  \hfill (253)

$$D_{\nu s_2} = \kappa'_2(s) + \frac{\sqrt{r - x}}{r' g(r'/((1 - x))}$$  \hfill (254)

Again the $\kappa$'s can be different in each case. To determine them we turn to the $m = 0$ equations.

$$-s_{0s_1} = c_{0r_1} D_{r s_1} + c_{0r_2} D_{r s_2} + c_{0r_1} D_{r' s_1} + c_{0r_2} D_{r' s_2}$$  \hfill (255)

$$-s_{0s_2} = c_{0r_1} D_{r s_2} + c_{0r_2} D_{r s_2} + c_{0r_1} D_{r' s_2} + c_{0r_2} D_{r' s_2}$$  \hfill (256)

together with the two equations with $s$ primed. The first, in the continuum limit, reads

$$-\frac{\sqrt{x}}{s\sqrt{2}} = -\frac{\sqrt{x}}{r\sqrt{2}} D_{r s_1} - \frac{\sqrt{r - x}}{r \sqrt{2}} D_{r s_2} + \frac{\sqrt{x}}{r' \sqrt{2}} D_{r' s_1} + \frac{\sqrt{r - x}}{r' \sqrt{2}} D_{r' s_2}$$  \hfill (257)

The sums in the first two terms can be simplified by writing the numerator in the $D$’s as a linear combination of $s$ and the denominator in the $D$: $s - r = s + r - 2r$ for $D_{r s_1}$ and $s + x r/(1 - x) - 2 x r/(1 - x)$. The contribution of the first term cancels the denominator and the second enters as a term in the zero mode identity.

$$c_{0r_1} D_{r s_1} + c_{0r_2} D_{r s_2} = \frac{\sqrt{x}}{2 \sqrt{2 s g(s/x)}} \sum_r \left[ \frac{x}{r^2 g(r/x)} + \frac{(1 - x)}{r^2 g(r/(1 - x))} \right]$$

$$-\frac{\sqrt{x}}{\sqrt{2 s g(s/x)}} \sum_r \left[ \frac{s + r}{s + r g(r/x)} + \frac{1}{s + r g(r/(1 - x))} \right]$$

$$= \frac{\sqrt{x}}{2 \sqrt{2 s g(s/x)}} \sum_r \left[ \frac{x}{r^2 g(r/x)} + \frac{(1 - x)}{r^2 g(r/(1 - x))} \right] - \frac{\sqrt{x}}{\sqrt{2 s}}$$  \hfill (258)

where we used the zero mode identity to get the last line. Plugging this in, the equation becomes

$$0 = \frac{\sqrt{x}}{2 \sqrt{2 s g(s/x)}} \sum_r \left[ \frac{x}{r^2 g(r/x)} + \frac{(1 - x)}{r^2 g(r/(1 - x))} \right]$$

$$+ \kappa'_1(s) \left[ \frac{x}{r^2 g(r'/x) \sqrt{2}} + \frac{1 - x}{r^2 g(r'/((1 - x) \sqrt{2})} \right]$$

$$\kappa'_1(s) = -\frac{\sqrt{x}}{2 s g(s/x)}$$  \hfill (259)

34
Applying the same procedure to the remaining 3 zero mode equations determines the remaining $\kappa$’s:

\begin{align*}
\kappa_2'(s) &= -\frac{\sqrt{1-x}}{2\text{sg}(s/(1-x))} \\
\kappa_1(s') &= \frac{\sqrt{x}}{2\text{sg}(s'/x)} \\
\kappa_2(s') &= \frac{\sqrt{1-x}}{2\text{sg}(s'/(1-x))}
\end{align*}

We have thus fully determined the mixed matrix elements of $D$:

\begin{align*}
D_{r'1s1} &= -\frac{x}{2r'g(r'/x)\text{sg}(s/x)} \quad (261) \\
D_{r'2s1} &= -\frac{\sqrt{x(1-x)}}{2r'g(r'/(1-x))\text{sg}(s/x)} \quad (262) \\
D_{r'1s2} &= -\frac{\sqrt{x(1-x)}}{2r'g(r'/x)\text{sg}(s/(1-x))} \quad (263) \\
D_{r'2s2} &= -\frac{1-x}{2r'g(r'/(1-x))\text{sg}(s/(1-x))} \quad (264) \\
D_{r1's1} &= \frac{2rg(r/x)s'g(s'/x)}{x} \quad (265) \\
D_{r2's1} &= \frac{\sqrt{x(1-x)}}{2rg(r/(1-x))s'g(s'/x)} \quad (266) \\
D_{r1's2} &= \frac{\sqrt{x(1-x)}}{2rg(r/x)s'g(s'/x)} \quad (267) \\
D_{r2's2} &= \frac{1-x}{2rg(r/(1-x))s'g(s'/(1-x))} \quad (268)
\end{align*}

One can easily verify the antisymmetry of these matrix elements

\begin{align*}
D_{r'1s1} &= -D_{s1r'1}, \quad D_{r'2s1} = -D_{s1r'2} \\
D_{r'1s2} &= -D_{s2r'1}, \quad D_{r'2s2} = -D_{s2r'2}
\end{align*}

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