Algebraic proof and application of Lumley’s realizability triangle

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Lumley [Lumley J.L.: Adv. Appl. Mech. 18 (1978) 123–176] provided a geometrical proof that any Reynolds-stress tensor $u'_i u'_j$ (indeed any tensor whose eigenvalues are invariably nonnegative) should remain inside the so-called Lumley’s realizability triangle. An alternative formal algebraic proof is given that the anisotropy invariants of any positive-definite symmetric Cartesian rank-2 tensor in the 3-D Euclidean space $E^3$ define a point which lies within the realizability triangle. This general result applies therefore not only to $u'_i u'_j$ but also to many other tensors that appear in the analysis and modeling of turbulent flows. Typical examples are presented based on DNS data for plane channel flow.

1 Introduction

The introduction in [10] of Lumley’s [9] realizability triangle is without doubt one of the most important contributions to statistical turbulence theory. The Reynolds-stress tensor property that serves to prove that every possible (realizable) Reynolds-stress tensor should lie within Lumley’s [9] realizability triangle, is the positivity of the diagonal components of the covariance of velocity-fluctuations

$$r_{ij} := u'_i u'_j$$

in every reference-frame, and hence also in the frame of its principal axes [9], implying that the tensor $u'_i u'_j$ is positive-definite [15] Theorem 2.3, p. 186], is exactly the same as that behind Schumann’s [11] realizability conditions. Throughout the paper, $u_i \in \{u, v, w\}$ are the velocity components in a Cartesian coordinates system $x_i \in \{x, y, z\}$, $\nu$ is the kinematic viscosity, $(\cdot)'$ denotes Reynolds (ensemble) fluctuations and $(\cdot)$ denotes Reynolds (ensemble) averaging.

Lee and Reynolds [8] further argued that Lumley’s [9] realizability triangle also applies to the dissipation tensor

$$\varepsilon_{ij} := 2\nu \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}$$

and to the covariance of the fluctuating vorticity components

$$\zeta_{ij} := \omega'_i \omega'_j$$

where $\omega'_i$ are the fluctuating vorticity components. Obviously the diagonal components of both these tensors are positive for every orientation of the axes of the Cartesian coordinates system.

Realizability constraints are essential not only in theory and modelling [9, 10] but also in computational implementations of second-moment closures [2, 3]. The same positivity of the diagonal components for every orientation of the axes of coordinates, which is equivalent to the positive-definiteness of the symmetric real tensor [15] Theorem 2.2, p. 186], and implies Lumley’s [9] realizability triangle, can also be of interest to the unresolved stresses [14] in partially-resolved approaches [6].

Lumley’s [9] proof of the realizability triangle is geometric, based on representing the behaviour of 2 of the principal values of the traceless anisotropy tensor, and taking into account the corresponding behaviour of the invariants. An alternative easy-to-follow algebraic proof is possible, based on just 2 requirements

1. the symmetric Reynolds-stress tensor has 3 real eigenvalues [12] Theorem 2, p. 55] which are nonnegative [9, 11] with nonzero trace (positive kinetic energy)

which also apply to any symmetric real positive-definite rank-2 tensor in $E^3$.

2 Anisotropy, principal axes and invariants

Before giving the proof, we summarize for completeness some basic definitions and properties [9, 12]. The tensor of the 2-moments of fluctuating velocities $r_{ij}$ is real and symmetric, and is therefore diagonalizable in the frame of its principal axes [12] Theorem 5, p. 59], where its diagonal components (principal values) are its real [12] Theorem 2, p. 55] eigenvalues [12] Theorem 4, p. 58]. This implies that the eigenvalues of $r$, being its diagonal components in
the frame of its principal axes, are nonnegative. Since the
eigenvalues of the symmetric tensor \( r \) are nonnegative, \( r \) is
polynomial semi-positive definite [15] Theorem 2.3, p. 186]. Inversely,
the diagonal components of every positive-semi-definite ten-
sor are nonnegative [15] p. 186]. The halfrace of \( r \) is the tur-
bulent kinetic energy and is therefore nonzero (trr = 2k > 0),
implies that at least one of its eigenvalues is nonzero (there-
fore \( r \) is positive-definite, which inversely implies nonzero
trace). Let \( A_r \) and \( A_{br} \)

\[
A_r := \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad ; \quad A_{br} := \begin{bmatrix} \lambda_{br,1} & 0 & 0 \\ 0 & \lambda_{br,2} & 0 \\ 0 & 0 & \lambda_{br,3} \end{bmatrix}
\]

be the diagonal matrices of the eigenvalues of \( r \) and \( br \), re-
respectively, where

\[
\begin{align*}
b_r &:= \frac{r}{trr} - \frac{1}{3} I_3 \quad \Rightarrow \\
I_{br} &= trb_r = 0 \\
I_{br} &= -\frac{1}{2} \delta_{ij} b_{r ij} \\
I_{br} &= -\frac{1}{2} \lambda_{br 1} \lambda_{br i} < 0 \\
I_{br} &= detb_r = \lambda_{br 1} \lambda_{br 2} \lambda_{br 3}
\end{align*}
\]

is the traceless anisotropy tensor corresponding to \( r \) and \( I_3 \)
is the 3 \times 3 identity tensor, with the usual definition of the in-
variants [12] (6), p. 51], simplified [9] by the relation \( I_{br} = trb_r = 0 \) [25],
and the corresponding expressions in terms of the eigenvalues in the frame of principal axes [9].
By definition \( I_{br} \) is real symmetric, and has therefore
real eigenvalues [12] Theorem 2, p. 55]. It is straightforward
to show that the eigenvectors of \( r \) are also eigenvectors of \( br \).
Let \( X_r \) and \( X_{br} \) denote the orthonormal matrices [12] Theo-
rem 5, p. 59] whose columns are the right eigenvectors of \( r \)
and \( br \), respectively, and therefore satisfy

\[
\begin{align*}
r \cdot X_r &= A_r \cdot X_r \\
\quad ; \quad b_r \cdot X_{br} &= A_{br} \cdot X_{br}
\end{align*}
\]

By straightforward computation using \( I_{br} \) [3a]

\[
\begin{align*}
b_r \cdot X_r V(\frac{r}{trr} - \frac{1}{3} I_3) \cdot X_r &= \frac{1}{trr} \cdot X_r - \frac{1}{3} X_r \\
\quad \Rightarrow \quad \frac{1}{trr} \cdot X_r = \frac{1}{trr} \cdot X_r \quad (3b)
\end{align*}
\]

implying by \( 3a \)

\[
\begin{align*}
X_{br} &= V(\frac{1}{trr} \cdot X_r \cdot \frac{1}{trr}) \\
\quad \Rightarrow \quad \frac{1}{trr} \cdot A_r - \frac{1}{3} I_3 \quad \Rightarrow \quad \lambda_{br i} = (trr) (\lambda_{br 1} + \frac{1}{3})
\end{align*}
\]

ie \( r \) and \( br \) have the same system of principal axes and their
eigenvalues are related by \( 3d \).

3 Proof

As stated in the introduction the algebraic proof of Lum-
ley’s [9] realizability triangle can be easily obtained from 2
well-known conditions, also discussed in [2]. The eigen-
values of \( br \) satisfy the characteristic polynomial [12] (5), p. 51]

\[
\lambda_{br 1}^3 - \frac{1}{3} \lambda_{br 2} \lambda_{br 3} + \frac{1}{2} \lambda_{br 2} - \frac{1}{2} \lambda_{br 3} = 0
\]

The roots of the cubic equation \( 4a \) are real iff [7] pp. 44–45] its
determinant is negative, \( \Delta_3 = [\frac{1}{3} (3b - a^2)]^3 + \frac{1}{2} (c + \frac{2}{3} a^2 -
\frac{2}{27} ab)^2 \leq 0 \). If \( \Delta_3 = 0 \) then 2 roots are identical.
Fig. 1. Lumley’s realizability triangle (6) for a positive-definite symmetric real rank-2 Cartesian tensor $\mathbf{r}$ in the 3-D Euclidean space, plotted in the $(\text{III}_b, -\text{II}_b)$-plane of the invariants (2b) of the corresponding anisotropy tensor $\mathbf{b}_i$ (2b), and representation of the inequalities (4b) whose intersection defines the region of realizable states.

4 Applications

Obviously the property applies not only to the Reynolds-stresses $r_{ij}$ (1a), their dissipation $\varepsilon_{ij}$ (1b) or the vorticity covariance $\zeta_{ij}$ (1c), but also to any tensor with nonnegative diagonal values. Typical examples are the destruction-of-dissipation tensor $\varepsilon_{ij}$ and the destruction-of-vorticity-co-variance tensor $\zeta_{ij}$, which represent the destruction of $\varepsilon_{ij}$ by the action of molecular viscosity (5 (3.3), p. 17] or the destruction-of-vorticity-co-variance tensor $\zeta_{ij}$, which represents the destruction of $\zeta_{ij}$ by the action of molecular viscosity (1 (20), p. 458).

Regarding acceleration fluctuations $(Du_i)'$, most authors generally study the variances of its components (17, 18) and its splitting, based on the momentum equation, in a pressure part $\rho^{-1}\partial_k p'$ (also called inviscid) and a viscous part $\nabla^2 u_i'$ (also called solenoidal because, by the fluctuating continuity equation (5 (3.2a), p. 17), it is divergence-free). As for the fluctuating vorticity correlations, we may define the symmetric positive-definite tensor of fluctuating acceleration correlations $\alpha_{ij}$ and the corresponding inviscid and solenoidal parts

\[
\alpha_{ij} := \frac{D u_i}{D t} \left( \frac{D u_j}{D t} \right) \quad (7c)
\]

\[
\alpha_{ij}^{(p)} := \frac{1}{\rho^2} \partial_k \partial_j \frac{\partial u_i}{\partial x_k} \quad (7d)
\]

\[
\alpha_{ij}^{(v)} := \nabla^2 u_i' \frac{\partial^2 u_j'}{\partial x_i \partial x_j} \quad (7e)
\]

We consider DNS results of fully developed (streamwise invariant in the mean) turbulent plane channel flow (4 [5], in a streamwise $\times$ wall-normal $\times$ spanwise $L_x \times L_y \times L_z = 4\pi \delta \times 2\delta \times \frac{4}{3}\pi \delta$ computational box, and use standard definitions (5 [3.2, p. 18] of computational parameters (Figs. 2 [3]).
Regarding $r_{ij}$ (1a), its dissipation-rate $\varepsilon_{ij}$ (1b) and the destruction of that dissipation $\varepsilon_{ij}$ (7a), notice that the shear component $(\cdot)_{xy}$ is generally of the order-of-magnitude of the wall-normal component $(\cdot)_{yy}$ (Fig. 2). Sufficiently far from the wall [8] $r_{ij}$ is expected to reflect the anisotropy of the large turbulent scales (typical size $\ell_T$), whereas $\varepsilon_{ij}$ is expected to reflect the anisotropy of the smaller scales (of the order of the Taylor microscale $\lambda$). The scaling arguments of

$$
Re_{x_w} \quad N_x \times N_y \times N_z \quad L_x \times L_y \times L_z \quad \Delta x^+ \quad \Delta y^+ \quad \Delta t^+ \quad \Delta t_{L}\quad N_{y_1} \quad \Delta y_L \quad \Delta z^+ \quad \Delta t^+ \quad r_{obs} \quad \Delta t^+
$$

$180 \quad 401 \times 201 \times 201 \quad 4.7 \times 25 \times 1.7 \quad 5.7 \quad 0.2 \quad 0.1 \quad 3 \quad 1.9 \quad 0.0060 \quad 2 \quad 0.0060$

Fig. 2. Components, in wall-units [5] (A3), p. 28, of the positive-definite symmetric tensors (1, 7), plotted against the inner-scaled wall-distance $y^+$ (logscale and linear wall-zoom), from DNS computations of turbulent plane channel flow [4, 5].
Tennekes and Lumley [16, pp. 88–92] suggest that, again sufficiently far from the wall, \( \epsilon_{ij} \) reflects the anisotropy of scales between \( \lambda \) and the Kolmogorov scale \( \ell_K \). It is therefore noteworthy that they appear to share a seemingly similar anisotropy \( (\cdot)_{xx} > (\cdot)_{zz} > (\cdot)_{yy} \) \( \forall y^+ \geq 1 \) (Fig. 2). Nonetheless, very near the wall \( (y^+) \ll 1; \) Fig. 2), where all these length-scales collapse to 0, \( \epsilon_{zz} \) becomes larger than \( \epsilon_{xx} \). Vorticity covariance \( \zeta_{ij} \) [16] is expected [16, pp. 88–92] to reflect the
anisotropy of the same scales as \( \varepsilon_{ij} \), and its destruction \( \varepsilon_{ij}^{(p)} \) corresponding to the same scales as \( \varepsilon_{ij}^{(v)} \). Both \( \zeta_{ij} \) and \( \varepsilon_{ij}^{(p)} \) have a very weak shear component \( \langle \cdot \rangle_{xy} \) (Fig. 2). Their componentality obviously differs from that of \( \{ r_{ij}, \varepsilon_{ij}, \varepsilon_{ij}^{(v)} \} \), because in the major part of the channel \( \langle \cdot \rangle_{xx} \approx \langle \cdot \rangle_{yy} \approx \langle \cdot \rangle_{zz} \) \((y^+ \approx 10; \text{Fig. 2})\). Nonetheless, \( \zeta_{yy} \rightarrow 0 \) (2-C at the wall), contrary to \( \varepsilon_{xy} \) (Fig. 2), and in the sublayer \( \varepsilon_{xx} \) and \( \varepsilon_{zz} \) cross each other \((y^+ \approx 1; \text{Fig. 2})\), in analogy with the observed behaviour of \( \varepsilon_{ij} \).

Regarding the acceleration correlations, \( \alpha_{ij} \) \((7c)\), \( \alpha_{ij}^{(p)} \) \((7d)\), and \( \alpha_{ij}^{(v)} \) \((7e)\), again the shear component is substantially smaller than the diagonal components (Fig. 2). Recall that the fluctuating momentum equation \( [5, (3.2b), \text{p. 17}] \)

\[
\frac{D u'_i}{D t} = -\frac{1}{\rho} \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u'_i}{\partial x_m \partial x_m} \tag{8a}
\]

readily implies

\[
(7c) \implies \alpha_{ij} = \alpha_{ij}^{(p)} + \alpha_{ij}^{(v)} = -\frac{\nu}{\rho} \frac{\partial p'}{\partial x_i} \frac{\partial^2 u'_j}{\partial x_m \partial x_m} + \nu \frac{\partial^2 u'_i}{\partial x_m \partial x_m} \tag{8b}
\]

where the last cross-correlation tensor is symmetric but indefinite. The componentality of the acceleration correlations \( \alpha_{ij} \) \((7c)\) is quite different from that of its pressure \( \alpha_{ij}^{(p)} \) \((7d)\) and viscous \( \alpha_{ij}^{(v)} \) \((7e)\) parts, as these correlations are the footprint of different mechanisms occurring mainly at different scales. In the buffer layer \((10 \lesssim y^+ \lesssim 100; \text{Fig. 3})\) viscous acceleration is mainly in the streamwise direction, but in the sublayer \( \alpha_{i}^{(v)} \) increases and crosses with \( \alpha_{i}^{(v)} \) at \( y^+ \approx 1 \) (Fig. 2), in analogy with the other diagonal components of the fluctuating velocity Hessian, \( \varepsilon_{ij} \) \((7a)\) and \( \xi_{ij} \) \((1b)\). The wall normal component \( \alpha_{xy} \) becomes comparable to the other diagonal components only sufficiently away from the wall \((y^+ \gtrsim 30; \text{Fig. 2})\). On the other hand, acceleration induced by fluctuating pressure forces \( \alpha_{ij}^{(p)} \) \((7d)\) exhibits a \( \alpha_{i}^{(p)} > \alpha_{xy}^{(p)} > \alpha_{i}^{(v)} \), anisotropy in the buffer layer \((10 \lesssim y^+ \lesssim 100; \text{Fig. 2})\), whereas near the wall \( \alpha_{xy}^{(p)} \rightarrow 0 \) \((y^+ \lesssim 10; \text{Fig. 2})\). Finally, the acceleration correlations behave quite differently from the 2 parts in the fluctuating momentum equation \( (8a) \), implying that the cross-term in \( (8b) \) is important, and especially so near the wall where scale-separation tends to disappear, and is directly responsible for the differences in limiting behavior (Fig. 2).

\[
\lim_{y^+ \to 0} \alpha_{ij} = 0 \tag{8c}
\]
\[
\lim_{y^+ \to 0} \alpha_{ij}^{(p)} \neq 0 \tag{8d}
\]
\[
\lim_{y^+ \to 0} \alpha_{ij}^{(v)} \neq 0 \tag{8e}
\]

5 Conclusion

The simple algebraic proof presented above, can be summarized in the following mathematical proposition:

**Theorem** (Lumley’s realizability triangle). Let \( r \) be a real rank-2 Cartesian tensor in the 3-D Euclidean space \( \mathbb{E}^3 \). Assume \( r \) symmetric and positive definite. Then the locus of the invariants \( (2b) \) of the corresponding anisotropy tensor \( b \) \((2b)\), in the \((III_{b_{r}}, -II_{b_{r}})-\text{plane}\), lies within Lumley’s realizability triangle \( (6) \).

The proof \((5)\) follows directly from the well-known fact that the principal values of \( r \) are real and nonnegative. By its algebraic nature it leads directly to the inequality \( (6) \), which defines Lumley’s realizability triangle. It is obtained in the actual \((III_{b_{r}}, -II_{b_{r}})-\text{plane}\), with no need of transformation of the invariants or explicit analysis of the limit states at the boundaries of the realizability triangle. The novel algebraic proof reported in the paper helps to better grasp the classic geometric proof given in Lumley \( [9] \).

Many symmetric tensors with nonnegative diagonal values are encountered in the analysis of turbulent flows. Several, by no means exhaustive, examples are studied, using DNS data for plane channel flow, illustrating how anisotropy invariant mapping (AIM) within the realizability triangle can improve our understanding of their componentality behavior.

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