Relativistic Covariance and Quark-Diquark Wave Functions for Baryons *

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Abstract

We derive covariant wave functions for hadrons composed of two constituents for arbitrary Lorentz boosts. Focussing explicitly on baryons as quark-diquark systems, we reduce their manifestly covariant Bethe-Salpeter equation to covariant 3-dimensional forms by projecting on the relative quark-diquark energy. Guided by a phenomenological multi gluon exchange representation of covariant confining kernels, we derive explicit solutions for harmonic confinement and for the MIT Bag Model. We briefly sketch implications of breaking the spherical symmetry of the ground state and the transition from the instant form to the light cone via the infinite momentum frame.

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Modern electron and hadron accelerators investigate the internal structure and dynamics of hadrons and implications for the non perturbative regime of QCD: detailed information is extracted from scattering experiments at large momentum transfers of typically 1 GeV and beyond. The corresponding form factors map out the various internal (generalized) charge distributions and provide stringent information on the underlying quark and gluon degrees of freedom. Presently various experiments are ongoing with electron and proton beams at various labs (1,2).

With energy and momentum transfers increasing beyond a typical scale of 1 GeV, at least approximate covariance is an important ingredient in any microscopic parametrization; however, its quantitative importance and implementation is still under investigations for practical calculations and quantitatively not fully understood. Traditionally performed in the instant (equal time) form, observables, such as form factors, are sensitive to the pertinent problem of center-of-mass (CM) corrections for the many-body problem and from substantial effects from Lorentz contraction at increasing momentum transfers. Both for the CM corrections (3-6) and the formulation of covariant hadronic wave functions, various recipes have been developed and applied in practical calculations (7-28). A possible alternative, the evaluation of form factors on the light cone, where Lorentz boosts are kinematical (29), has so far entered only selectively in applications at low and intermediate scattering energies; beyond that, such an approach suffers from other deceases, such as the loss of strict rotational invariance (30). As in general the construction of boosted, Lorentz contracted wave functions in the instant form is nearly as complicated as the solution of the full problem (as here the boost are dynamical and depend explicitly on the interaction kernel employed in the problem), in most practical applications simple kinematical prescriptions for the rescaling of the coordinate along the direction of the momentum transfer are applied (examples are given ref. (31,32)); however, specific questions, as the dependence of Lorentz corrections on the confining kernel in quark models, are not addressed.

In this note we formulate an economical model for hadron baryon wave functions, which leads to results suitable for practical applications. As it is our main goal to end up with analytical formulae, we model the baryon - in the following we focus on the proton, though our approach is fairly general - as a quark-diquark system (33) and restrict ourselves, for technical simplicity and without any loss of generality, to spin-isospin scalar diquarks; including axial diquarks or modelling mesons as \( q\bar{q} \) systems basically only introduces a more angular momentum structure of the corresponding amplitudes (see our comment below).

Our starting point is the manifestly covariant 4-dimensional Bethe-Salpeter equation (34)

\[
\Gamma = K G \Gamma \quad \text{and} \quad \Psi = G \Gamma
\]  

(1)
with the vertex function and the Bethe-Salpeter amplitude \( \Gamma \) and \( \Psi \), respectively, and the interaction kernel \( K \). In the two-body Greens function for the quark and diquark masses \( m \) and \( m^* \), we fix the relative energy dependence from the covariant projection on the diquark \( 35 \)

\[
G(P, q) = \frac{\hat{q} + m}{\nu(q^2 - m^2) + (1 - \nu)((Q - q)^2 - m^2) - i\epsilon} \delta_+(\nu(q^2 - m^2) - (1 - \nu)((Q - q)^2 - m^2)).
\]  \( \nu \to \infty \) which results up to \( 0 \left( \frac{q^2}{2M^2} \right) \) in the single particle Dirac equation for the quark for systems with arbitrary overall 4-momentum \( Q = (E(P) = \sqrt{P^2 + M^2}, 0, 0, P) \)

\[
((\frac{M^2}{2E(P)} + \frac{P}{E(P)}q_z) - (\alpha q + \beta m))\phi(P, q) = \frac{1}{E(P)} \int K(p, q, k)\phi(P, k)dk
\]  with \( M \) being the mass of the baryon (the external momentum \( P \) is chosen along the \( z \) axis).

Without any details we add a brief comment on the CM corrections in our model: evidently there is a direct coupling between the internal and external momenta \( q \) and \( P \), or equivalently, between boosts and the CM motion. In the rest system the leading center-of-mass corrections are absorbed for \( \epsilon = m + \epsilon_b \), where \( \epsilon_b \) is the binding energy of the quark in

\[
\left( \frac{q^2}{2\mu} + \epsilon_b - V_n(r) \right)\varphi(r) = 0
\]  with the reduced mass \( 1/\mu \cong 1/m + 1/(m + m^*) \) for an arbitrary quark potential \( V_n(r) \).

The important step for a practical model is the formulation of a covariant interaction kernel in eq. \( 4 \). As the dynamics of the quark - quark interactions, particularly the microscopic nature of the confinement, lacks an understanding on the fundamental level of QCD, all current models in practical calculations rely on phenomenological formulations of the interaction kernel. Here we proceed here along similar lines: we parametrize the interaction kernel as a superposition of appropriately weighted gluon exchange contributions; quantitative parameters can be extracted in comparison with studies to baryon spectroscopy, decay rates or form factors \( 36 \). Thus we start from the general kernel

\[
K(P, q, k) = \sum_n \frac{k_n(P)}{((q - k)^2 - \mu^2 - i\epsilon)^{n+1}}
\]
for arbitrary powers of \( n \) (which reflect different parametrizations of the confining kernel; for \( n=1 \) eq.(5) contains the linear confinement in the Cornell potential (37) and as confirmed from lattice calculations). Upon projecting out the relative energy dependence this yields the covariant, 3-dimensional kernel

\[
K_n(P, q, k) \propto \lim_{\mu \to 0} \left( \frac{d}{d\mu^2} \right)^n \frac{1}{\lambda^2(P)q_z^2 + q_{\perp}^2 + \mu^2 - i\epsilon}
\]

with the "quenching parameter"

\[
\lambda(P) = M/\sqrt{M^2 + P^2} = M/E(P)
\]

where we introduced the mass scale \( \mu \) (to regularize the Fourier transform to coordinate space). Already simple power counting signals, that a kernel with the power \( n \geq 1 \) leads to confinement with \( \sim r^{2n-1} \). Upon performing the corresponding Fourier transform to coordinate space and taking the limit \( \mu \to 0 \) we find

\[
K_n(P, q) \to (1 + \beta)/2 V_n(\sqrt{(z/\lambda(P))^2 + \rho^2})
\]

(with \( \rho \) as the perpendicular component to \( z \)), where we introduced for convenience the particular Dirac structure of the kernel to facilitate the evaluation of the resulting Dirac equation (more general Dirac invariants modify in addition the structure of the small Dirac component; compare our remark below). Eliminating the small component in eq. (3) with the kernel from eq. (6) and upon dropping CM corrections and \( \epsilon_b^2 \) terms for compactness, we end up with the Schroedinger type equation for the large component of the Dirac equation

\[
(2m\epsilon_b - \lambda^2(q_z - \frac{m}{M}P)^2) - q_{\perp}^2 - \nabla^2\left(\sqrt{(z/\lambda(P))^2 + \rho^2}/R\right) u(z, \rho) = 0
\]

(with the typical length scale \( R \); in the rest system the equation above reduces to the standard spherical Schrödinger type equation for a particle with mass \( m \)). The final equation defines with its connection to the small component by a simple differentiation the full relativistic covariant quark - diquark wave function for arbitrary Lorentz systems. In the equation above we see the shortcoming from the phenomenological nature of the interaction kernel: we absorb the explicit \( \epsilon \) and \( P \) dependence of the kernel in the definition of the energy scale \( V_n \) for the confining force; including an explicit \( P \) dependence in \( V_n \) would require a detailed knowledge of its microscopic origin.

Approximate or numerical solutions for eq. (9) can be obtained for different confining scenarios. Here we discuss briefly two examples, which allow a rigorous analytical solution: i.e. harmonic confinement and bag models (in the limit \( n \to \infty \) in eq. (6)).
- Harmonic confinement:

With the harmonic kernel defined as (39)

\[
K(P, q, k) = -\frac{12}{\pi} \lim_{(\mu \to 0)} \left( (d/d\mu)^2(\mu/2 + (d/d\mu^2)\mu^3/3) \frac{1}{q^2 - \mu^2 - i\epsilon} \right)
\]

the solutions for arbitrary excitations of the baryon are easily obtained in momentum space. After a redefinition of the longitudinal momentum and upon separating the longitudinal and the perpendicular component, the general solution is given by a product of confluent hypergeometric functions (40). Here we focus only on the nucleon as the quark - diquark ground state and obtain explicitly

\[
u(q_z, q_\perp) = N e^{-\frac{a^2}{2} \left( \lambda^2(q_z - \frac{m}{M^2}P)^2 + q_\perp^2 \right)}
\]

with the oscillator parameter \(a^2 = \frac{2}{\sqrt{V_c}}\) and \(V_c \propto 1/R^4\) being the confinement strength. As expected the standard solution for the spherical harmonic oscillator is recovered in the rest system, i.e. for \(P=0\) and \(\lambda(0)=1\). As the characteristic result we find a quenching of the effective \(P\)-dependent size parameter

\[
a^2(P) = (\lambda(P)a)^2 = \frac{M^2}{P^2 + M^2} a^2
\]

which leads to Lorentz quenching in coordinate space along the z - axis and thus to a significant increase of the longitudinal momentum components with increasing \(P\) (Fig.1(a,b));

- Bag Model:

As mentioned above we generate the Bag from the transition \(n \to \infty\) in the power of the gluon-exchange kernel. As we are unable to present an analytical solution for arbitrary \(n\) (a closed solution for the \(z\)-component exists only in the limit of vanishing binding (40)), we first perform the limit \(n \to \infty\) and then solve the equation

\[
\left( \epsilon_b + \frac{P}{M} q_z + m - (\alpha q + \beta m) \right) u(z, \rho) = 0
\]

with the boundary condition for the large and small components at \(z = \lambda(P)R\) for the bag radius \(R\). For the large component the ground state solution can be represented as

\[
u(z, \rho) = N \cos \left( \frac{k_z}{\lambda} z \right) J_0(k_\perp \rho)
\]

where the \(\lambda\) dependence of the \(z\)-component again reflects the quenching of the bag. The corresponding boundary condition \(z = \lambda R\) for \(\rho = 0\) for the deformed bag,
which reduces to the standard boundary condition for the spherical bag, can be solved only numerically (41). Characteristic results for 3 different boost momenta are presented for the large and small component of the bag ground state solution in Fig. 2.

Comparing our findings with simple recipes we find that a general and simple extension of the parametrization of the spherical wave functions and momentum distributions in the rest system to a boosted system, by rescaling the size parameter of the system, but keeping otherwise the spherical character of the solutions, is certainly very unsatisfactory and breaks down completely for boost momenta of typically $P/M \geq 1$. Only for very small boost momenta $P$ simple approximations, such as

$$u(z, \rho, a) \approx u \left( r, a/\left(\sqrt{3} \lambda(P)\right)\right) \quad \text{and} \quad (16a)$$

$$u(z, \rho, R) \approx \exp(-r/(\sqrt{3} \lambda(P)R)) u(r, R) \quad (16b)$$

simulate very qualitatively Lorentz quenching of slowly moving systems.

The breaking of the spherical symmetry for moving systems leads to significant modifications of the structure of composite objects with increasing boost momenta. Without entering into details we list just a few consequences.

One novel feature for moving particles is the modification of their spin-orbit interaction for interaction kernels involving different Dirac invariants. As an example, for a purely scalar confining potential the elimination of the lower component in the Dirac equation leads to

$$V_{ls} \sim (\sigma_z q_z + c \sigma q) V(z, \rho) (\sigma_z q_z + c' \sigma q) \quad (17)$$

which gives rise to terms $\sim 1_{\perp} s_{\perp}$ and thus to a spin-orbit interaction even for baryons initially in a relative s-state .

Similarly, boosting a spherical kernel leads even for baryons with quarks in relative s-states - due to the breaking of spherical symmetry - to the admixture of additional angular momenta, which significantly enhance the momentum spectrum of the ground state with increasing $q$ due to their radial structure (in an harmonic basis, dropping normalization constants)

$$\phi(P, r) \sim \frac{1}{aa^{1/2}(P)} \exp(-\frac{r^2}{2a^2}(1 + \frac{a^2 - a^2(P)}{a^2(P)} \frac{4\pi}{3} Y_{10}^2(\hat{r}))), \quad (18)$$
which yields for the spectrum in momentum space from the corresponding Fourier transform

\[ \phi(P, q) \sim \int_0^1 dt \frac{2p^2(t) - (qt)^2}{p^5(t)} \exp\left(-\frac{(qt)^2}{4p^2(t)}\right) \]

with \( p^2(t) = (1 + a^2 - a^2(P)a^2(P)t^2)/(2a^2) \).

In Fig. 3 a characteristic result for the admixture of higher angular momenta to the \( L=0 \) ground state of a baryon (with all quarks in relative s states), is shown for different boosts upon projecting out the \( L=2 \) orbital angular momentum: evidently, higher angular momenta dominate form factors of hadrons with increasing momentum transfers (for an explicit inclusion of a d-state in the nucleon the present quark-scalar diquark representation has to include in addition axial diquarks with \( S = 1 \), which allow the coupling of the orbital angular momentum \( L = 2 \) to the total spin \( 1/2 \) of the nucleon).

A test for the consistency of the simple approach above is the transition to the infinite momentum frame for boost momenta \( P \to \infty \): as in this limit the physics becomes effectively \((1+1)\)-dimensional, our approach has to merge into the corresponding dynamical equations on the light cone. Without entering into details of such a limiting process, we, for simplicity, only sketch here our reasoning for two spin less bosons with equal masses. With a projection of the BSE for the two particles symmetrically off their mass shell (following Blankenbecler and Sugar (35); a similar result is obtained in the Gross limit (42), we obtain with the momentum fraction \( x \) and the total longitudinal light cone momentum \( P_+ \), from

\[ G_{IF} \sim \delta_+(q_0 - \frac{P}{E(P)}q_z), \quad G_{LC} \sim \delta(q_- + M^2x); \quad x = \frac{q_+}{P_+} - \frac{1}{2}, \]

for \( P \to \infty \) the equation

\[ \left(\frac{M^2}{4} - m^2 - y^2 - q_{\perp}\right)\Phi(y, q_{\perp}, M^2) = \int K(y - y', q_{\perp} - q'_{\perp}), \Phi(y', q'_{\perp}, M^2) dy' dq'_{\perp} \]

with

\[ y \equiv y_{IF} = M \frac{q_z}{P}, \quad y \equiv y_{LC} = Mx, \]

respectively. Similarly, the kernels also become identical in the IF and on the LC in the limit \( M \to \infty \).

Summarizing the main findings in this note, we derive covariant wave functions and their transformation properties in an analytical quark - diquark model for the baryon. As
expected, we find characteristic modifications from the baryon rest system to moving Lorentz systems for different confining kernels already for moderate boost momenta.

Our findings suggest possible extensions and basic shortcomings of the model. Evidently its application to mesons as $q\bar{q}$ systems, towards a more realistic quark-diquark description of baryons or to genuine 3-quark systems (together with a systematic inclusion of CM corrections) imposes only technical problems and is clearly feasible. Here we only mention that the quenching factor from eq.(7) is recovered in leading order for all different projections of the BS equation.

A more serious problem is the detailed formulation of confining kernels with a finite power in the inter quark distance $r$, in particular of the Lorentz quenching for the kernels itself (in Bag models the dependence is absorbed in the boundary condition). Here the main problem it the poor phenomenological understanding the confining mechanism: it is not clear how the full $P$ dependence enters into the kernel (for a simple extension compare ref. (43)). We feel that a more realistic formulation of present phenomenological quark models requires a deeper analytical understanding of confinement, beyond the numerical results from lattice calculations. Here significant progress in various directions has been achieved recently, to mention only the modelling of confinement of QCD in the Coulomb gauge (44) or the extension of concept of instantons to merons as solutions of the classical QCD equations (45). Given such a more detailed understanding may eventually provide some answer to our question at the beginning, i. e. on the importance of covariance and the appropriate incorporation of boosts in exclusive processes at typical energy and momentum transfers of 1 GeV. Valuating both the approximations and the practicability of calculations on the one side, and recent progress in light cone physics, such as restoring rotational invariance (29,46), on the other side, we expect that systematic studies of both electromagnetic and hadronic exclusive two-body reactions will finally map out the most adequate parametrization of the underlying short ranged physics of exclusive processes at the low GeV scale, advocating either the instant form or the front form.

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Figure 1: Longitudinal dependence of the large component of the harmonic oscillator ground state in coordinate (a) and momentum space (b) for different boost momenta $P$. 
Figure 2: Quenching and boundary conditions for the bag along the boost momenta $P=0$, 1 and 2 GeV/c. The functions $f(z)$ and $g(z)$ denote the large and small Dirac components of the ground state wave function for a (static) bag radius $R = 0.7$ fm.
Figure 3: D-state admixture to a boosted harmonic oscillator ground state. Compared are the (spherical L=0) s-wave momentum distribution for $P=0$ with the L=2 component for $P = 1$ and 2 GeV/c.