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Abstract. Here we consider the influence on the electron spin in the magnetohydrodynamic (MHD) regime. Recently developed models that include spin–velocity correlations are taken as the starting point. A theoretical argument is presented, suggesting that in the MHD regime a single-fluid electron model with spin correlations is equivalent to a model with spin-up and spin-down electrons constituting different fluids, but where the spin–velocity correlations are omitted. Three-wave interaction of two shear Alfvén waves and a compressional Alfvén wave is then taken as a model problem to evaluate the asserted equivalence. The theoretical argument turns out to be supported, because the predictions of the two models agree completely. Furthermore, the three-wave coupling coefficients obey the Manley–Rowe relations, which further support the soundness of the models and the validity of the assumptions made in the derivation. Finally, we point out that the proposed two-fluid model can be incorporated in standard particle-in-cell schemes with only minor modifications.

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1. Introduction

Considerable interest has recently been devoted to the study of quantum plasmas; see e.g. [1–8]. Much of the research has been motivated by applications to quantum wells [9], plasmonics [10], spintronics [11], astrophysics [12] and ultra-cold plasmas [13]. Two of the most basic and much studied quantum effects are those of Fermi pressure and particle dispersive effects (directly associated with the Bohm–de Broglie potential); see e.g. [1–8]. Other studies [3, 14–21] focus on the electron spin properties that result in a magnetic dipole force and a magnetization current, in addition to some more complex aspects of the spin dynamics. Although most quantum effects have a tendency to be more important in plasmas of high density and low temperature, the relevant regimes differ to some extent for the various quantum effects; see [19] for a discussion of this issue. A consequence is that it is possible to focus on certain quantum effects and ignore the others. In this paper, we make use of this fact and concentrate on the physics associated with the spin coupling in the Pauli Hamiltonian. Our starting point is a recently presented spin-fluid model [16], derived from kinetic theory [15], which in addition to the most basic spin precession dynamics includes the effects of spin–velocity correlations. Evaluating this model in the magnetohydrodynamic (MHD) regime, we make a conjecture based on certain theoretical arguments: that there are two equivalent ways of modelling spin-MHD dynamics, either by a one-fluid model including spin–velocity correlations, or by a two-fluid model without spin–velocity correlations. In the latter, the spin-up and spin-down states relative to the magnetic field are regarded as different fluids [14]. The conjectured equivalence of these models in the MHD regime is tested by considering a specific problem of three-wave interaction. For this purpose, we calculate the coupling coefficients between two shear Alfvén waves and one compressional Alfvén wave (fast magnetosonic (MS) wave) in a magnetized plasma. The coupling coefficients indeed turn out to be identical in the two cases, and the coefficients are also seen to obey the Manley–Rowe relations, which further support the soundness of the models used. The applicability of the two-fluid model without spin–velocity correlations in the MHD regime, which is strongly supported by our findings, is a very useful result. This is because the model can be easily adopted into standard particle-in-cell (PIC) schemes with only small modifications, as will be discussed in the final section.
2. Model equations

2.1. Preliminaries

In the treatment that follows we will include certain quantum effects due to the spin, while simultaneously we neglect others. As a prerequisite, let us briefly review the foundations for this procedure. The starting point for a nonrelativistic quantum theory of electrons is the Pauli Hamiltonian. Using the density matrix, combined with the Wigner and Q-transform \[15\], a scalar kinetic equation for a spin-1/2 particle can be derived. The only approximation made in the derivation is the omission of higher-order particle correlations, which is a good approximation for plasmas that are not strongly coupled. Thus, equation (41) of \[15\] includes all quantum effects of a plasma that is not strongly coupled, including wave particle dispersive effects (related to the Bohm–de Broglie potential). In the standard MHD regime, where the scale lengths are longer than the ion Larmor radius, typically the scale lengths are also longer than the characteristic de Broglie wavelength\(^2\). Limiting ourselves to scale lengths longer than the de Broglie wavelength, particle dispersive effects are negligible, in which case the kinetic evolution equation is reduced to equation (64) of \[15\]. This evolution equation still includes all quantum effects associated with the spin. In particular within this kinetic model, quantum effects associated with the Fermi pressure may be significant, depending on the temperature and density. However, we will not focus much on pressure perturbations in the treatment below, and thus the difference between a classical equation of state and a Fermi pressure (which becomes the issue when fluid models are to be deduced from kinetic theory) will be of relatively minor concern to us.

Although particle dispersive effects are dropped, we stress that the magnetic dipole force associated with the electron spin may still be significant. We note that the magnetic dipole force density on a strongly spin-polarized fluid element is of the same order as the magnetic force density provided:

\[
j \times B \sim n_e \nabla (\mu^e B),
\]

where \(j\) is the (free) current density, \(B\) is the magnetic field, \(B = |B|\), \(n_e\) is the electron number density and \(\mu^e\) is the magnetic dipole moment of electrons. The comparison (1) reduces to

\[
c_A^2 \sim \frac{\mu_B B}{m_e}
\]

in the MHD regime, where \(c_A\) is the Alfvén velocity, and we have approximated the magnitude of \(\mu^e\) with the Bohr magneton \(\mu_B\). We will henceforth assume that the ordering displayed in (2) is adequate (which for a reasonably weak magnetic field can be fulfilled for a rather modest plasma density), such that the magnetic dipole force density on a spin-polarized fluid element is not small compared to the classic force density. However, we note that the significance of this scaling potentially can be reduced because fluid elements contain (approximately) equal numbers of spin-up and spin-down states, such that there are approximate cancelations of the magnetic dipole force. In fact, for a thermodynamic equilibrium state, the relative overweight of the low-energy spin over the high-energy state is proportional to \(\tanh(\mu_B B / k_B T)\) \[15\], which is

\(^2\) For an exception to the rule that length scales longer than the Larmor radius imply scale lengths longer than the characteristic de Broglie wavelength Larmor radius, we need ultra-strong magnetic field strengths, comparable to magnetar field strengths or higher.
normally a very small factor, with the possible exception of strongly magnetized astrophysical plasmas [22] or ultra-cold plasmas [23]. Excluding these latter regimes, we will focus on the more common case \( \tanh(\mu_B B / k_B T) \approx \mu_B B / k_B T \ll 1 \), such that the spin is isotropically distributed to a good approximation and the magnetization of the unperturbed plasma state is small enough to be neglected. However, we stress that this does not imply that the spin force is negligible in general. For a weakly collisional plasma, wave perturbations will drive the spin state from thermodynamic equilibrium to induce a spin polarization and, as a consequence, a finite magnetic dipole force on the fluid elements. The remainder of this manuscript will focus on the modeling associated with wave-induced spin polarization in the MHD regime.

2.2. Basic equations

Starting from a scalar kinetic equation for a spin-1/2 particle \([15]\), spin-fluid equations can be derived \([16]\). These are given by the continuity equation

\[
\partial_t n^{(s)} + \nabla \cdot (n^{(s)} \mathbf{v}^{(s)}) = 0
\]

and the fluid momentum equation

\[
m^{(s)} \frac{D\mathbf{v}^{(s)}}{Dt} = q^{(s)} \left( \mathbf{E} + \epsilon_{ijk} v^{(s)}_j B_k \right) + \mu^{(s)} S^{(s)}_j \frac{\partial B_j}{\partial x_i} - \frac{1}{n^{(s)}} \frac{\partial P^{(s)}_i}{\partial x_j},
\]

where the superscript \( s = e, i \) denotes the species (electrons or ions), \( D/Dt \equiv \partial_t + \mathbf{v}^{(s)} \cdot \nabla \), and summation over repeated indices \( i, j, k = x, y, z \) is implied. Here \( m^{(s)} \) is the mass, \( q^{(s)} \) is the charge, \( \mu^{(s)} \) is the magnetic dipole moment, \( n^{(s)} \) is the number density and \( v^{(s)} \) is the fluid velocity of species \( s \). Furthermore, \( S \) is the spin vector normalized to unity, \( P_{ij} \) is the pressure tensor and \( \epsilon_{ijk} \) is the Levi–Civita symbol. Since the ions normally have a much smaller magnetic moment than electrons\(^3\), the spin contribution due to the ions can be neglected compared to the electron contribution, i.e. we may let \( \mu^{(i)} \approx 0 \). The pressure may here be of a classical type, or a Fermi pressure, depending on temperature and density. The philosophy behind neglecting particle dispersive effects (the so-called Bohm–de Broglie potential) has been described in the preliminaries. However, for a further discussion of the importance of various quantum effects in different regimes, see e.g. \([19, 24]\). Provided particle dispersive effects can be neglected, the pressure moment satisfies the evolution equation

\[
\frac{DP_{ij}^{(s)}}{Dt} = -P_{ik}^{(s)} \frac{\partial v_j^{(s)}}{\partial x_k} - P_{jk}^{(s)} \frac{\partial v_i^{(s)}}{\partial x_k} - P_{ij}^{(s)} \frac{\partial v_k^{(s)}}{\partial x_k} + q^{(s)} m^{(s)} \epsilon_{imn} P_{jm}^{(s)} B_n + q^{(s)} \epsilon_{jmn} P_{im}^{(s)} B_n
\]

\[+ \mu^{(s)} \sum_k \frac{\partial B_k}{\partial x_j} + \mu^{(s)} \sum_k \frac{\partial B_k}{\partial x_i},
\]

where again the last two terms can be dropped for ion species.

Furthermore, to describe the spin dynamics we need the electron spin evolution equation, which is given by

\[
\frac{DS_{ij}^{(s)}}{Dt} = \frac{2 \mu^{(e)}}{h} \epsilon_{ijk} S^{(e)}_j B_k - \frac{1}{m^{(e)} n^{(e)}} \frac{\partial \Sigma_{ij}^{(e)}}{\partial x_j},
\]

\(^3\) Taking protons as a typical example of ions, we may note that the He proton magnetic moment is of the order of the nuclear magneton, which is a factor \( m_e/m_p \) smaller than the Bohr magneton.
where $\Sigma_{ij}^{(e)}$ is the spin–velocity correlation tensor. Finally, the evolution of the spin–velocity moment is described by

$$
\frac{D \Sigma_{ij}^{(e)}}{Dt} = -\Sigma_{ij}^{(e)} \frac{\partial v_j^{(e)}}{\partial x_k} - \Sigma_{ik}^{(e)} \frac{\partial v_j^{(e)}}{\partial x_k} + \frac{q^{(e)}}{m^{(e)}} \epsilon_{ijkl} \Sigma_{kl}^{(e)} B_l + \frac{2\mu^{(e)}}{\hbar} \epsilon_{ikl} \Sigma_{lj}^{(e)} B_l + \mu^{(e)} n^{(e)} \frac{\partial B_i}{\partial x_j} - \mu^{(e)} n^{(e)} S_i^{(e)} S_j^{(e)} \frac{\partial B_k}{\partial x_j}. $$

(7)

In equation (5), we have neglected the heat flux tensor $Q_{ijk}$ to obtain a closed set of equations. Similarly, we have neglected the higher-order tensor $\Lambda_{ijk}$ in the evolution equation for the spin–velocity tensor equation (7). The validity of the truncation has been investigated in [15, 25], and the truncation seems to be an acceptable approximation in the low-temperature limit. The equations above together with Maxwell’s equations constitute a closed system, where a magnetization current density $j_M = \nabla \times M$, due to the spin, should be added to the free current density, and where naturally all species contribute in the latter term. The set of equations (3)–(7) has been studied by [16, 25], but without inclusion of the ion dynamics. The aim of the current paper is to apply the above set of equations to the MHD regime where the ion dynamics is essential, at the same time carefully evaluating the electron spin magnetization.

### 2.3. MHD limit

As concluded in the previous section, we will primarily consider wave dynamics in the MHD regime where the frequencies are smaller than the ion–cyclotron frequency and the wavelengths are longer than the Larmor radius. Under these assumptions, the system will be described by the magnetohydrodynamic equation [20]

$$
\rho \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) u = -\nabla \left( \frac{B^2}{2\mu_0} - M \cdot B \right) + (B \cdot \nabla)M - \nabla P,
$$

(8)

where $P$ is the sum of the electron and ion pressure, together with the equations

$$
\frac{\partial}{\partial t} \mathbf{J} = \nabla \times (\mathbf{u} \times \mathbf{B})
$$

(9)

and

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.
$$

(10)

Here we have neglected the electron contribution to the fluid mass density by setting $\rho \approx m^{(e)} n^{(e)}$, and the fluid velocity can be written as $u = (n^{(e)} m^{(e)} \mathbf{u}^{(e)} + m^{(i)} m^{(i)} \mathbf{u}^{(i)})/\rho \approx \mathbf{u}^{(i)}$. We have neglected the magnetic moment of the ions such that the magnetization $\mathbf{M} = \mu^{(e)} n^{(e)} \mathbf{S}^{(e)}$ is purely due to the electron spin. The derivation of equation (8) made in [20] was done starting from a somewhat less elaborate set of equations, not including the spin–velocity correlations. However, the derivation does not involve equations (6) and (7) describing the spin dynamics, and hence we may adopt this result within the current model.

Without the magnetization $\mathbf{M}$, we obtain standard ideal MHD equations, and thus (8)–(10) constitute a closed system. The magnetization can be easily included with the spin determined by equation (6) together with (7), where terms containing derivatives of the velocity turn out to be negligible. The aim is to solve for the magnetization in terms of the magnetic field, in which case equations (8)–(10) are sufficient to produce a closed spin-MHD theory. This can be achieved in two different ways: either by considering the electrons in spin-up and spin-down
states relative to the magnetic field as two separate fluids, or by treating them as a single fluid with a macroscopic spin that is proportional to the difference in population density of the two spin states. We will now discuss this in further detail.

2.4. One-fluid versus two-fluid

In this sub-section, we will demonstrate how one can determine magnetization, in order to use equations (8)–(10). In particular, we will show the equivalence in the MHD regime of a one-fluid model with spin–velocity correlations and of a two-fluid model where spin–velocity correlations are omitted. The derivation below is based on the following premises:

1. That the fluid model equations (3)–(7) are applicable. Firstly, this means that kinetic effects such as e.g. wave–particle interaction can be neglected. Furthermore, another important assumption implied by using equations (3)–(7) is that truncation of the higher-moment hierarchy is applicable. The validity of this particular assumption has been discussed in some detail by Stefan et al [25].

2. That the magnitude of various terms in equations (6) and (7) can be compared assuming the same type of scaling as in ideal MHD theory.

3. That we may adopt the fluid hierarchy for the low-energy and high-energy spin states separately, to obtain separate fluid theories for the two spin states. That this procedure is indeed applicable has been clarified by Zamanian et al [16].

To find the magnetization, we first need to solve (6) to determine the spin. The first term on the right-hand side of equation (6) is the basic spin precession. If the spin–velocity correlations in (6) can be neglected (which as we will see below is valid under certain circumstances), the solutions for $S$ are particularly simple in the MHD regime. This is because the left-hand-side term of equation (6) is smaller than the spin precession term by a factor of the order of $O(\omega_{\text{ch}}/\omega_{\text{cg}})$, where $\omega_{\text{ch}} \sim \partial t$ is the characteristic frequency scale of the problem and $\omega_{\text{cg}} \sim 2\mu eB/\hbar$ is the characteristic spin precession frequency (which is close to the characteristic cyclotron frequency $\omega_{\text{c}} \sim qB/m$). Assuming tentatively that spin–velocity correlations can be omitted, we note that the spin evolution equation in the MHD regime reduces to

$$\varepsilon_{ijk}S^{(e)}_j B_k = 0. \quad (11)$$

This has two solutions, where $S$ is either parallel or antiparallel to $B$, that is $S_i = \pm b_i$, where $b_i = B_i/B$ is a unit vector in the direction of $B$. The physical reason behind this relation is simply that the rapid spin precessions average out on the low-frequency MHD scale. A comparatively general way of dealing with spins obeying $S_i = \pm b_i$ is to consider a two-fluid model of electrons, where for one of the species the electron spin state is parallel to $B$, and for the other species antiparallel. Equation (11) then implies that these spin states are conserved. However, as seen from the above discussion, this is only an adequate approximation if the spin–velocity correlations give a small contribution in equation (6). Thus our next step is to outline the solutions of equation (7) using MHD approximations, in order to determine the contribution from $\Sigma^{(e)}_{ij}$ in (6). Firstly, we note that the first three terms in (7) are at most of order $\omega_{\text{c}}^{(ch)} \Sigma_{ij}$, whereas the fifth and sixth terms are of order $\omega_{\text{cg}}^{(ch)} \Sigma_{ij} \sim \omega_{\text{cg}}^{(ch)} \Sigma_{ij}$. Thus, neglecting the first three terms we can write equation (7) in the form

$$\vec{O} \cdot \vec{\Sigma} = \vec{\sigma}, \quad (12)$$
where $\vec{O}$ is a $9 \times 9$-matrix where all coefficients are $\pm \omega_{ca}$ or $\pm \omega_{cg}$. Here, $\Sigma$ is a nine-component vector containing all elements of $\Sigma_{ij}$ and $\vec{\sigma}$ is a nine-component vector containing the source terms, i.e. $P_{jk}(\partial S_i/\partial x_k), \mu n(\partial B_i/\partial x_j)$ and $\mu n S_i S_k (\partial B_k/\partial x_j)$. Moreover, $\omega_{ca} = q B_a/m$ and $\omega_{cg} = 2 \mu e B_a/\hbar$ with $\alpha = x, y, z$. Note that the full field strength is used and not the linearized field when defining $\omega_{ca}$ and $\omega_{cg}$. Since $\vec{O}$ contains no operators, we can perform simple matrix inversion to find $\tilde{\Sigma} = \vec{O}^{-1} \vec{\sigma}$. This turns out to be sufficient to determine all components of $\Sigma_{ij}$, except for a component directed as $b \otimes b$. Thus this approximation scheme allows us to compute $\Sigma_{ij}$ apart from a contribution $\Sigma_{ij}^{(d)}$ yet to be determined, which can be expressed as

$$\Sigma_{ij}^{(d)} = \Phi b_i b_j,$$

where $\Phi$ is a scalar field to be found. Using the solution $\tilde{\Sigma} = \vec{O}^{-1} \vec{\sigma}$, we can easily check that the determined components of $\Sigma_{ij}$ are of order $\Sigma_{ij} \sim \mu_B n k B/\omega_c^{(ch)}$. This means that the contributions from $\Sigma_{ij}$ are sufficiently small to be neglected in (6). However, in general we must also account for the contribution $\Sigma_{ij}^{(d)}$ whose magnitude is unknown. Since the scalar field $\Phi$ cannot be determined if the first three terms of (7) are omitted, we must extend our model to solve the full case of equation (7). We will do so within a one-fluid model in section 3.1, and it turns out that $\Phi$ becomes sufficiently large for the component $\Sigma_{ij}^{(d)}$ to significantly influence the solutions to (6), also within the MHD regime. However, it also turns out that in order to obtain a large value of $\Phi$, we must have a spin vector that is different from $\pm b$. This is the normal case in a one-fluid theory, where the macroscopic spin results from averaging over all spin states. However, within a two-fluid MHD model (treating spin-up and -down states as different species) without spin–velocity correlations, equation (11) can nevertheless be applied leading to $S_i = \pm b_i$, and the situation would then again be modified.

The key difference between the one-fluid and two-fluid cases can be simply deduced as follows. Contracting equation (7) with $b_i b_j$ to compute the source terms for $\Phi$, we find that all the source terms for $\Sigma_{ij}^{(d)}$ vanish if $S_i = \pm b_i$, as the fourth term in equation (7) becomes $b_i b_j P_{jk}(\partial S_i/\partial x_j)$, which is zero as $b_i (\partial S_i/\partial x_j) = \pm (1/2) \partial (b_i b_i)/\partial x_j$, whereas terms (7) and (8) together become

$$b_i b_j \frac{\partial B_i}{\partial x_j} - b_i b_j S_i S_k \frac{\partial B_k}{\partial x_j} = 0,$$

where $S_i = \pm b_i$ was used in the last step. This result provides the theoretical basis for adopting a two-fluid model of electrons in the MHD regime and omitting spin–velocity correlations in (6), leading to $S_i = \pm b_i$. The division into two fluids leaves the rest of the basic equations structurally unaffected, but we now obtain two contributions such that the magnetization is calculated as $M = \mu n_i s_i + \mu n_j s_j$ due to the difference in density perturbations of the two spin states. We will consider this in more detail within perturbation theory in our model problem below. The conclusion of this section is then confirmed, since the one-fluid models that keep the spin–velocity correlations in equation (6) indeed give an identical expression for magnetization as the two-fluid model with up and down spins $S_i = \pm b_i$. The allowance for independent density variations of the two species in the latter model provides the physical mechanism that reproduces the effects of spin–velocity correlations in the one-fluid model. It should, however, be stressed that this conclusion regarding the equivalence of models is limited to the MHD regime.
3. Three-wave interaction—a model problem

We will now consider a model problem with the purpose of testing our conclusion about the equivalence between the one-fluid and two-fluid models presented in the previous section. Specifically, we consider three-wave interaction between two shear Alfvén waves \((A, A')\) and one compressional Alfvén wave \((\text{MS})\): \(\text{MS} \rightarrow A + A'.\) Using three-wave interaction as a model problem has the advantage that an unphysical assumption (or an incorrect calculation) is likely to result in a broken Manley–Rowe symmetry \([26]\), in which case one gets a clear indication that something needs to be revised.

The waves are assumed to be small perturbations on a homogeneous background, and we write \(\mathbf{B} = B_0 \hat{z} + \mathbf{B}_1, \) \(\rho = \rho_0 + \rho_1,\) etc, but omit index 1 on variables whose background values are zero. Furthermore, we omit index 1 whenever the Cartesian components are specified for notational convenience, i.e. we write \(\mathbf{B}_1 = B_y \hat{x} + B_z \hat{y} + B \hat{z}.\) Moreover, we assume that there is no drift so that \(\mathbf{u}_0 = 0\) and also that there is no spin–velocity correlation in the background distribution, i.e. \(\Sigma_0 = 0.\) For simplicity, we further assume that the temperature is sufficiently low so that the equilibrium pressure can be neglected\(^4\), \(P_0 = 0\) (i.e. that we have a low-beta plasma with the ion-acoustic velocity much smaller than the Alfvén velocity). Furthermore, assuming that \(\mu_B B_0 / (k_B T) \ll 1\) we can make the approximation that the equilibrium spin-up and spin-down populations are equal \([19]\) such that \(n_{0\uparrow} = n_{0\downarrow}\) in the two-fluid model, which implies that the total zeroth-order magnetization vanishes\(^5\). For consistency between the one-fluid and two-fluid models, we should pick \(\mathbf{M}_0 = 0\) also in the latter case. The difference in the model equations between the one-fluid and two-fluid approach is then primarily that in the two-fluid model we have \(\mathbf{S}_0 = \pm \hat{z}\) for the two spin states (in which case we obtain a finite zeroth-order magnetization if only one of the electron fluids is counted) whereas in the one-fluid model \(\mathbf{M}_0 = 0\) and \(\mathbf{S}_0 = 0.\)

Next we make a harmonic decomposition \(\partial_t \rightarrow -i\omega\) and \(\partial_k \rightarrow ik\) for each wave, where the frequencies and wave vectors satisfy the conditions

\[
\omega^{\text{MS}} = \omega^A + \omega^{A'}, \quad (14)
\]

\[
\mathbf{k}^{\text{MS}} = \mathbf{k}^A + \mathbf{k}^{A'}, \quad (15)
\]

with the index MS denoting the compressional Alfvén (or fast MS) wave, and \(A\) and \(A'\) denoting the shear Alfvén waves. The coordinate system is defined so that the \(z\)-direction points in the direction of the unperturbed magnetic field, \(\mathbf{B} = B_0 \hat{z},\) and for simplicity we assume all wave vectors to lie in the \(xz\)-plane.

Throughout the calculation we will use \(\omega / \omega_c\) and \(k C_A / \omega_c\) as small expansion parameters (where \(\omega_c = q B_0 / m\) is the cyclotron frequency), in accordance with standard MHD theory. Here \(C_A = (B_0^2 / \mu_0 m n_0)^{1/2} = (B_0^2 / \mu_0 \rho_0)^{1/2}\) is the Alfvén velocity, and \(\omega\) and \(k\) represent any of the wave frequencies or wave vector components. We also note that \(\omega_{\text{cg}} \simeq \omega_c,\) where

\(^4\) We also note that a nonzero Fermi pressure could very well be significant in a plasma of comparatively high density in the low-temperature limit. However, since it is well known how one can modify the equation of state to account for this effect, we ignore such a contribution and concentrate on determining the magnetization instead.

\(^5\) For the purpose of calculating a coupling coefficient corresponding to a thermodynamic equilibrium state, such an approximation may not be adequate, as we know that there are slightly more particles in the lower-energy spin state, corresponding to a nonzero total magnetization. However, our chosen background state is nevertheless a true dynamic equilibrium, which suffices for the main purpose of this paper, which is to validate the inferred equivalence of the two models.
ωcg = 2μeB0/h is the spin precession frequency. Furthermore, ωcg − ωc is of the same order as the ion–cyclotron frequency, and is therefore much larger than wave frequencies within the MHD regime. We will therefore drop terms proportional to (ωcg − ωc)−1 compared to ω−1 in our final results.

It should be pointed out that unlike the classical case, the pressure tensor Pij does not necessarily vanish in the limit of zero temperature (see footnote 3). However, we note that we need not be concerned about the contribution from the pressure term in this particular case. This is because Pij vanishes linearly in the MHD limit and thereby enters as a cubic nonlinearity (which does not affect the three-wave interaction) in the evolution equation for Σij. The pressure tensor, however, also gives a contribution in the MHD equation (8), but it turns out that this is a nonlinear contribution proportional to (ωcg − ωc)−1, which is small compared with leading terms proportional to ω−1. We may therefore neglect the contribution from Pij altogether.

3.1. One-fluid calculation

We start by considering the one-fluid model for which, as mentioned above, the unperturbed spin-density is zero, S0 = 0. Our first aim is to find the linear dispersion relation as well as the linear eigenvectors (polarizations) of the shear Alfvén wave and the compressional Alfvén wave. We note that in the MHD equation (8) we need an expression for the magnetization. We therefore begin by solving the spin–velocity evolution equation to find the Σij-tensor. Linearly, this is straightforward and we find

\[
Σ_{ij} = i n_0 μ \begin{pmatrix}
\frac{k_x B_x ω_{cg}}{ω^2 - ω_{cg}^2} & -\frac{k_x B_x ω_c}{ω^2 - ω_{cg}^2} & \frac{k_x B_y}{ω_{cg}} \\
-\frac{k_x B_x ω_{cg}}{ω^2 - ω_{cg}^2} & \frac{k_x B_y ω_c}{ω^2 - ω_{cg}^2} & -\frac{k_x B_y}{ω_{cg}} \\
0 & -\frac{k_x B_z}{ω_c} & \frac{k_x B_z}{ω}
\end{pmatrix},
\]

(16)

As can be seen, the components in (16) have different magnitudes, but we keep all of them at this stage in the calculation. Next we use the linear Σ-tensor in the spin evolution equation (6) to find an expression for the linear spin S and thereby the linear magnetization M = μn0S:

\[
M = \frac{n_0 μ^2}{m} \begin{pmatrix}
k_x^2 B_x \\
k_x^2 B_y \\
k_x^2 B_z
\end{pmatrix}
\]

(17)

Here we have dropped contributions to components of M that are smaller by factors |ω^2 − ω_{cg}^2|/ω_{cg}^2 and/or ω/ω_{cg}. Substituting (17) into (8), the linear dispersion relations are obtained

We have ω_{cg} − ωc = (g/2 − 1)ωc ≈ 0.0016ωc, which is of the order of the ion–cyclotron frequency. Thus, we have ω/(ω_{cg} − ωc) ≪ 1 within the standard MHD regime. Accordingly, ω/(ω_{cg} − ωc) has been used as an expansion parameter in sections 2 and 3, although in some cases we have kept first-order corrections in this parameter for illustrative purposes.
from (8)–(10). Similar to the classical ideal MHD case, the modes decouple into the shear Alfvén wave described by

$$D_A(\omega, k) \equiv \omega^2 - k_z^2 C_A^2 \left(1 - \frac{n_0 \mu_0 k^2}{m} \frac{k_x^2}{\omega_c^2 - \omega_c^2} \right) = 0,$$  \hspace{1cm} (18)

and the compressional Alfvén (fast MS) wave with the dispersion relation

$$D_{MS}(\omega, k) \equiv \omega^2 - k_z^2 C_A^2 \left(1 + \frac{n_0 \mu_0 k^2}{m} \frac{k_z^2}{\omega_c^2} \right) - k_z^2 C_A^2 \left(1 - \frac{n_0 \mu_0 k^2}{m} \frac{k_x^2}{\omega_c^2 - \omega_c^2} \right) = 0.$$ \hspace{1cm} (19)

Note that the last terms in (18) and (19) are smaller than the first spin-modification in (19) as \(\omega^2 \ll |\omega_c^2 - \omega_c^2|\). Furthermore, the linear eigenvector components for the shear Alfvén wave are

$$u_A^x = u_A^z = 0, \quad B_A^x = B_A^z = 0, \quad \rho_1^A = 0 \quad \text{and} \quad B_A^y = -\frac{k_A^0 B_0}{\omega_A^x} u_A^y.$$ \hspace{1cm} (20)

For the compressional Alfvén wave (index MS) we instead obtain

$$u_{MS}^y = u_{MS}^z = 0, \quad B_{MS}^y = 0, \quad \text{and} \quad B_{MS}^z = -\frac{k_{MS}^0}{\omega_{MS}^x} u_{MS}^x.$$ \hspace{1cm} (21)

Next we aim to calculate the three-wave coupling coefficients due to quadratic nonlinearities. We have calculated the nonlinear contribution to the coupling coefficients including all terms proportional to \((\omega_c g - \omega_c)^{-1}\). However, our results show that these terms only give rise to small corrections to the leading terms proportional to \(\omega^{-1}\). Since the full analysis is rather tedious, we will therefore only write out the leading terms in the nonlinear contribution to the \(\Sigma\)-tensor as well as to the magnetization \(M\). Under the given approximations, keeping the resonant terms, we find that the components with a nonzero nonlinear contribution to the \(\Sigma\)-tensor are

$$\Sigma_{yz}^A = -\frac{k_A^0 B_A^x}{\omega_{cg}} + \frac{2i \mu k_{MS}^0 B_{MS}^y B_A^x}{\omega_{cg} \omega_{MS}},$$ \hspace{1cm} (24)

and

$$\Sigma_{zy}^A = -\frac{k_A^0 B_A^z}{\omega_{cg}} + \frac{i q k_{MS}^0 B_{MS}^y B_A^z}{\omega_{cg} \omega_{MS}}.$$ \hspace{1cm} (25)

for the Alfvén wave. Here * denotes complex conjugation. For the MS wave, the component with a nonzero nonlinear contribution is

$$\Sigma_{zz}^{MS} = \frac{k_{MS}^0 B_{MS}^z}{\omega} + \frac{2i \mu}{\hbar} \frac{1}{\omega_{cg} \omega_{MS}} (k_A^0 + k_z^0) B_A^z B_y^A.$$ \hspace{1cm} (26)
Solving the spin evolution equation with the sources from $\Sigma$ given above, we find a nonlinear contribution to the $z$-component of the magnetization,

$$M^z_{\text{MS}} = -\frac{n_0 \mu^2}{m} \left( \frac{k_z^2 B^z_{\text{MS}}}{\omega^2} + \frac{2 \mu k_z^{\text{MS}} (k_z^{A} + k_z^{A'})}{\hbar} B^A_{y} B^A_{y} \right),$$

(27)

for the MS wave, and a nonlinear contribution to the $y$-component of the magnetization,

$$M^y_{A'} = \frac{n_0 \mu^2}{m} \left( \frac{k_x^2 B_y}{\omega^2} - \frac{2 \mu k_x^{2(\text{MS})}}{\hbar} B_{z}^{\text{MS}} B_{y}^{A'} \right),$$

(28)

for the Alfvén wave. Now that we have expressed the magnetization in terms of the magnetic field, correcting to second order in the amplitude, we may substitute these results into (8), and perform the rest of the calculations using (8)–(10) as in standard MHD theory. Accounting for time-dependent amplitudes with the substitution $D_A(\omega, k) \rightarrow [\partial D_A/\partial \omega]i\partial/\partial t$ and $D^{}_{\text{MS}}(\omega, k) \rightarrow [\partial D_{\text{MS}}/\partial \omega]i\partial/\partial t$, doing successive elimination keeping the velocity variables as the wave amplitudes, we find the following coupled equations for the different wave modes:

$$\frac{\partial u^y_{A'}}{\partial t} = -i \frac{\omega^{2(A')}}{\partial D^{}_{A'}} C u^A_{y} u^{}_{x}^{\text{MS}}$$

(29)

and

$$\frac{\partial u^x_{\text{MS}}}{\partial t} = -i \frac{\omega^{2(\text{MS})}}{\partial D^{}_{\text{MS}}} C u^{A y} u^{A'}_{y}$$

(30)

with the coupling coefficient

$$C = \frac{k_x^{\text{MS}}}{\omega^{\text{MS}}} \left( 1 + \frac{n_0 \mu_0 \mu^2}{m} \frac{k_z^{2(\text{MS})}}{\omega^{2(\text{MS})}} \right).$$

(31)

Due to the symmetry between the two shear Alfvén waves, the equation for $\partial u^A_{y}/\partial t$ is obtained by exchanging $A$ and $A'$ in equation (29). The appearance of the common factor $C$ in the three coupled equations is a reflection of the Manley–Rowe symmetry [26]. The first term of $C$ is a purely classical contribution, which agrees with [27, 28] in the cold limit. For the spin contribution in equation (31) to be important as compared to the classical one, a rather dense plasma is required. In contrast, other MHD phenomena exist that require less extreme parameters for the electron spin to be important [24]. Nevertheless, as will be discussed in the final section, the results derived here have a number of interesting theoretical consequences. It should be stressed that the contribution to the magnetization in this one-fluid model stems from the $\Sigma$-tensor. This is in contrast to the two-fluid model, as we will see below.

### 3.2. Two-fluid calculation

We now consider the problem of three-wave coupling using the two-fluid model. The spin is then determined from (6) with the contribution from $\Sigma$ omitted, as described in section 2, but we now have two species of electrons, which have the unperturbed spin $S_{0\uparrow} = \hat{z}$ and $S_{0\downarrow} = -\hat{z}$,
respectively. The total magnetization is then written as

$$M = \mu \left( n^{(i)} S^{(i)} + n^{(i)} S^{(i)} \right),$$

(32)

which gives $M_0 = 0$ in agreement with the previous section, provided we let $n_{0\uparrow} = n_{0\downarrow} = n_0/2$, which will be used henceforth. Next we find the linear spin-vector to be

$$S_1 = \begin{pmatrix} \frac{2\mu}{\hbar \omega_{cg}} S_0 B_x + \frac{\mu}{m} \frac{k_x^2 B_z}{\omega_c^2 - \omega_{cg}^2} \\ \frac{2\mu}{\hbar \omega_{cg}} S_0 B_y + \frac{\mu}{m} \frac{k_y^2 B_z}{\omega_c^2 - \omega_{cg}^2} \\ 0 \end{pmatrix}. \quad (33)$$

Note here that although the terms $\propto S_0$ in (33) are larger than the terms $\propto (\omega_c^2 - \omega_{cg}^2)^{-1}$, the former has opposite signs for the up and down species, and hence give no contribution to the linear magnetization. It turns out that the terms in (33) $\propto (\omega_c^2 - \omega_{cg}^2)^{-1}$ are needed to get agreement with the linear magnetization obtained with the one-fluid model (17). However, it can be noted that these terms have been dropped in the expression for the coupling coefficient (31) where, in the end, only the leading term is kept. Next we need to find an expression for the fluid densities of the electron spin fluids. From the continuity equation (3), we have

$$n^{(s)} = n_0^{(s)} + n_0^{(s)} \frac{k \cdot v^{(s)}}{\omega} + n^{(s)NL}, \quad (34)$$

where $n^{(s)NL}$ is a nonlinear contribution that can be shown not to contribute to the magnetization after summation of the spin states $s = \uparrow, \downarrow$. An expression for the electron velocities is obtained by solving the fluid momentum equation (4) together with $-\partial_t \mathbf{B} = \nabla \times \mathbf{E}$. To close this system, we may in general need to use a full fluid description. However, within the MHD regime and for this specific problem, it suffices to determine the magnetization. For our case with $n_{0\uparrow} = n_{0\downarrow} = n_0/2$, several terms vanish in equation (32) after the summation over up and down species. In MHD, we make the approximation so that we may write the electric field as $\mathbf{E} = -v^{(i)} \times \mathbf{B} \simeq -\mathbf{u} \times \mathbf{B}$. This allows us to again make use of equations (20)–(23). Under these assumptions we note that $E_z$ vanishes linearly, and also that $E^{NL}_z$ is only proportional to quadratic combinations of $B$-field components and will therefore not contribute to the magnetization. Thus, $E_z$ may therefore be set to zero from now on. Under these assumptions, it is easy to show that the fluid velocities may be written as

$$v_x = \frac{q}{m} \frac{\omega}{\omega_c} B_x + \frac{\mu}{m} \frac{k_x}{\omega_c} \frac{1}{\omega} (-v_x B_c + v_x B_x), \quad (35a)$$

$$v_y = -\frac{q}{m} \frac{\omega}{\omega_c} B_y + i \frac{\mu}{m \omega_c} k_y S_0 - \frac{q}{m} \frac{1}{\omega} (v_y B_x - v_x B_y) - i \frac{\mu}{m \omega_c} k_y (B_x S_x - B_y S_y), \quad (35b)$$

$$v_z = -\frac{\mu}{m \omega} k_z B_z S_0 + \frac{q}{m} \frac{1}{\omega} (v_z B_y - v_y B_z) - \frac{\mu}{m \omega} k_z (B_x S_x - B_y S_y). \quad (35c)$$

From equations (32)–(35), we find the linear magnetization, which agrees identically with the expression obtained from the one-fluid model. Furthermore, an extended analysis gives agreement for the nonlinear terms of the magnetization as well. Consequently, the coupling
coefficients remain the same regardless of whether the one-fluid or two-fluid model is used to determine the magnetization. This corroborates the usefulness of the two-fluid model in the MHD regime.

4. Discussion

In this paper, we have studied a recently presented fluid model accounting for the electron spin [16], and adopted it to the MHD regime. The main feature of the original model is that in addition to basic spin effects such as magnetic dipole force, spin precession and magnetization current, it incorporates spin–velocity correlations. The spin–velocity correlations have been shown to be important for a number of spin plasma phenomena [16, 25]. Introducing the approximations appropriate for the MHD regime, it turns out that essentially the ordinary MHD equations are recovered, but with a magnetization that needs to be determined. This can be done in different ways: firstly from a single-fluid model of electrons that besides the basic spin precession contains spin–velocity correlations; secondly from a two-fluid model where spin-up and spin-down electrons constitute different species. A theoretical argument is presented in section 2.2, suggesting that these two models are equivalent in the MHD regime. However, the equivalence argument depends on some assumptions that are difficult to justify rigorously, and thus practical tests of the equivalence are valuable. For this purpose, we have evaluated the different models using a nonlinear three-wave interaction as the test problem. Specifically, we have computed the coupling coefficients between two shear Alfvén waves and a compressional Alfvén wave. Classical as well as quantum mechanical (spin) contributions to the coupling coefficients are found, and the coupling coefficients are indeed identical in the two models. Furthermore, the coupling coefficients obey the Manley–Rowe symmetries [26]. The Manley–Rowe relations are a reflection of the underlying Hamiltonian structure [29] of the model. The fact that the coefficients preserve these relations strongly suggests that the approximations made when deriving the models are sound, as otherwise it is highly likely that the Manley–Rowe symmetries would be broken.

Since we have put forward two somewhat different models in this paper, one may ask which one is most easy to use. For the analytical calculations made here, the degree of complexity is found to be roughly the same. However, the two-fluid model has a great advantage in case one would like to carry out PIC simulations. In that case, the only modification of a standard code would be to have two species of electrons, and add a force proportional to $\pm \mu_e \nabla B$ in the momentum equation, as well as to compute $\mathbf{M} = \mu_e \mathbf{b} (n_\uparrow - n_\downarrow)$ to find the contribution from the magnetization current in Ampere’s law. Because the variables for the density and magnetic field are monitored throughout the PIC simulations anyway, no new equations and only little extra complexity is added to the general concept. Developing a PIC scheme incorporating spin–velocity correlations, however, is a much more cumbersome project, as the evolution equation for this object is not present in present PIC schemes, and also such equations are considerably more complex. Furthermore, it is not clear whether it is at all possible to model spin–velocity correlations as a single-particle property, which makes the adoption of this model in the PIC scheme conceptually questionable. The importance of being able to use a PIC scheme is that the spin effects are likely to be most significant in the strongly nonlinear regime, where the effect of the nonlinearly induced spin polarization is fully developed. Unfortunately, this regime is very difficult to treat by analytical means. Finally, we conclude that the two-fluid model is
valuable for the purpose of incorporating electron spin effects in PIC simulations, although we stress that the applicability of such an approach will be limited to the MHD regime.

References

[1] Shukla P K and Eliasson B 2010 Phys.—Usp. 53 51
[2] Manfredi G 2005 Fields Inst. Commun. 46 263
[3] Shukla P K 2009 Nat. Phys. 5 92
[4] Garcia L G, Haas F, de Oliveira L P L and Goedert J 2005 Phys. Plasmas 12 012302
[5] Misra A P and Chowdhury A R 2006 Phys. Plasmas 13 072305
[6] Haas F, Eliasson B, Shukla P K and Manfredi G 2008 Phys. Rev. E 78 056407
[7] Crouseilles N, Hervieux P-A and Manfredi G 2008 Phys. Rev. B 78 155412
[8] Haas F 2005 arXiv:physics/0503021v1 [physics.plasm-ph]
[9] Manfredi G and Hervieux P-A 2007 Appl. Phys. Lett. 91 061108
[10] Atwater H A 2007 Sci. Am. 296 56
[11] Wolf S A et al 2001 Science 294 1488
[12] Kouveliotou C et al 1998 Nature 393 235
   Palmer D M, Barthelmy S and Gehrels N 2008 Nature 434 1107
   Harding A K and Lai D 2006 Rep. Prog. Phys. 69 2631
[13] Robinson M P et al 2000 Phys. Rev. Lett. 85 4466
[14] Brodin G, Marklund M, Zamanian J, Ericsson A and Mana L P 2008 Phys. Rev. Lett. 101 245002
[15] Zamanian J, Marklund M and Brodin G 2010 New J. Phys. 12 043019
[16] Zamanian J, Stefan M, Marklund M and Brodin G 2010 Phys. Plasmas 17 10
[17] Asenjo F A 2009 Phys. Lett. A 373 4460
[18] Mushtaq A and Vladimirov S V 2010 Phys. Plasmas 17 102310
[19] Lundin J and Brodin G 2010 Phys. Rev. E 82 056407
[20] Brodin G and Marklund M 2007 New J. Phys. 9 277
[21] Brodin G, Misra A P and Marklund M 2010 Phys. Rev. Lett. 105 105004
[22] Harding A K and Lai D 2006 Rep. Prog. Phys. 69 2631
[23] Robinson M P et al 2000 Phys. Rev. Lett. 85 4466
[24] Brodin G, Marklund M and Manfredi G 2008 Phys. Rev. Lett. 100 175001
[25] Stefan M, Zamanian J, Brodin G and Marklund M 2011 Phys. Rev. E 83 036410
[26] Weiland J and Wilhelmsson H 1976 Coherent Nonlinear Interaction of Waves in Plasmas (Oxford: Pergamon)
[27] Chin Y C and Wentzel D G 1972 Astrophys. Space Sci. 16 465
[28] Brodin G and Stenflo L 1988 J. Plasma Phys. 39 277
[29] Larsson J and Wiklund K 1999 Phys. Scr. T82 71