Description of the measuring device mathematical model by the methods of quasinormed spaces

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Abstract. The article contains a description of the measuring device mathematical model by the methods of quasi-normalized spaces. This model serves as an application to the theory of positive degenerate holomorphic groups of operators in Quasi-Sobolev spaces of sequences. This approach is relevant due to the possibility of finding positive solutions or values of the measuring device, which means that it is more consistent with the physical meaning. Indeed, as a rule, we talk about its absence when we get a negative or imaginary value of a quantity that is positive or real in meaning.

Introduction

Recently, there has been an increased interest in mathematical models that characterize physical processes and have positive (i.e. non-negative) solutions. The adequacy of these models can be questioned under the found characteristics, the qualitative and quantitative aspects of which are in continuous dependence on the initially set values of the model. Our task is to describe such a mathematical model, the non-negative initial data of which will give non-negative solutions.

Note that this model is taken from the theory of optimal measurements, the foundations of which were laid in [1].

Let us consider the model of the measuring device (MD)

\[ \dot{x} = Ax + Du, \quad y = Cx, \] (1)

The model has the following interpretation: \( x = x(t) \) is state vector function of MD, \( u = u(t) \), \( y = y(t) \) are input and output vector functions, respectively, \( u = (u_1, u_2, ..., u_m) \) and \( y = (y_1, y_2, ..., y_l) \), moreover matrices of MD \( A \), of sensor \( D \) and of output \( C \) have corresponding dimensions \( n \times n \), \( n \times m \) and \( l \times n \).

The mathematical model of the measuring device is designed to process linear signals that are continuous in time. As a rule, the values of such signals are distorted and contain gross deviations from their models. The classical formulation of tasks for measuring signal parameters is based on the use of normalized spaces.

Work [1] shows that the model of MD [1] is reduced to a linear inhomogeneous Leontief-type system

\[ L \dot{z} = Mz + f. \] (2)
Here $L$ and $M$ are square matrices of order $n$, and besides the matrix $M$ is $L$-regular [2]. Note that system (2) is degenerate, $\det L = 0$.

Due to the fact that positive solutions are more "physical", the question of the existence of positive solutions to system (2) is topical. To study this issue, the theory of positive degenerate holomorphic groups of operators [3] is used, which corresponds to the adequacy of the model and is based on complete normed Banach spaces, called Banach structures. As is known, any Banach space is quasi-Banach, and the reverse is not true.

A quasinormed space $(\mathcal{U}, \| \cdot \|_\mu)$ is a linear space $\mathcal{U}$ over a field $\mathbb{R}$ with a quasinorm $\| \cdot \|_\mu$, that differs from the norm only by the axiom "triangle inequality"

$$\forall u, v \in \mathcal{U}, \| u + v \| \leq C (\| u \| + \| v \|),$$

where constant $C \geq 1$. A complete quasi-normed space $\mathcal{U}$ is called quasi-Banach space. Quasi-normed spaces have already been used earlier both in solving abstract problems [5] and in solving applied problems [6]. Speaking about our description of the measuring device model, it is worth mentioning once again that the quasinorm $\| \cdot \|_\mu$ naturally defines the topology on $\mathcal{U}$. The basis of the neighborhood is the collection of all sets of the form $\{ u \in \mathcal{U} : \| u - v \| < \varepsilon \}$, where $\varepsilon \in \mathbb{R}_+$. We choose the quasisobolean spaces of sequences as the spaces in which we carry out the reasoning.

The main purpose of this article is to describe the positive solutions to system (1). The presence of such solutions in quasi-normed spaces will make it possible to continue the study of positive optimal dynamic measurements in quasi-normed spaces.

1. Preliminary information
Consider the Leontief-type system (2) mentioned in the introduction

$$Lu = Mu,$$

(3)

Well, as the second component of the mathematical model of the measuring device, we indicate the initial Showalter - Sidorov condition

$$P(u(0) - u_0) = 0,$$

(4)

where $P$ is a projector in space $\mathbb{R}^n$, built in accordance with matrices $L$ and $M$. The degenerate system of ordinary differential equations is taken as a finite-dimensional analogue of Sobolev-type equations, so that further when using the theory of positive holomorphic groups of operators there would be no contradictions with the methods of the theory of degenerate semigroups of operators [7]. Note also that the initial condition (3) is more convenient in the algorithms of numerical calculations that are planned to be introduced in the future.

The solution $u = u(t)$ to equations (3) will be called the classical solution to problem (3), (4).

A matrix $M$ will be called $(L, p)$-regular $(p \in \{0\} \cup \mathbb{N})$, if we add the condition $p = 0$ (i.e. $\det L \neq 0$) to the case $p = \max \{ n_1, n_2, \ldots, n_k \} \in \mathbb{N}$, where it is the order of the pole at the point of $\infty$ of $L$ -resolvent $(\mu L - M)^{-1}$ of the matrix $M$.

Lemma 1. [8] Let the matrix $M$ be $(L, p)$-regular. Then matrices

$$P = \frac{1}{2\pi i} \int_\gamma (\mu L - M)^{-1} Ld\mu, Q = \frac{1}{2\pi i} \int_\gamma L(\mu L - M)^{-1} d\mu$$

are projectors.
Theorem 1. [8] Let the matrix $M$ be $(L, p)$-regular, where $p \in \{0\} \cup \mathbb{N}$. Then the unique maximal holomorphic resolving group of equations (3) exists and can be represented in the form

$$U' = \frac{1}{2\pi i} \int_{\gamma} R^L_\mu(M) e^{\mu t} d\mu,$$

where $t \in \mathbb{R}$, $\gamma \subset \mathbb{C}$ is (closed) contour bounding the domain containing $L$ - spectrum $\sigma^L(M)$ of the operator $M$.

2. The main method for describing the positive decisions of the mathematical model of the measuring device. Homogeneous case

Let $L$ and $M$ be square matrices of order $n$. The matrix $(L, p)$ is regular. In accordance with [2], we call the $(L, p)$ - resolvent set, and $\sigma^L(M) = \{\mu_1, \mu_2, \ldots, \mu_s\}, s < n$ the $L$ - spectrum of the matrix $M$, which coincides with the spectrum of both the matrix $L^{-1}M$ and the matrix $ML^{-1}$, provided $\det L \neq 0$. Let's define a contour $\gamma = \{\mu \in \mathbb{C} : |\mu| = r\}, \ r > \max \{|\mu_1|, |\mu_2|, \ldots, |\mu_s|\}$, construct matrix projectors

$$P = \frac{1}{2\pi i} \int_{\gamma} (\mu L - M)^{-1} Ld\mu, \quad Q = \frac{1}{2\pi i} \int_{\gamma} (\mu L - M)^{-1} d\mu.$$

Let $(\mu L - M)^{-1} = \tilde{L}^L_\mu(M)$ be right and left $L$ - resolvents of the matrix $M$ respectively.

The following holomorphic matrix groups

$$U' = \frac{1}{2\pi i} \int_{\gamma} R^L_\mu(M) e^{\mu t} d\mu, \quad \tilde{F}' = \frac{1}{2\pi i} \int_{\gamma} \tilde{L}^L_\mu(M) e^{\mu t} d\mu$$

also exist, where $t \in \mathbb{R}$, $\gamma \subset \mathbb{C}$ [8].

The units of these groups are projectors $U^0 = P$ and $\tilde{F}^0 = Q$. In the theory of positive degenerate holomorphic groups of operators in quasisobolens spaces, a Banach structure $\mathcal{B} = (\mathcal{B}, \mathcal{C})$ is a complete normed linear Riesz space with a given order relation $\geq$ and $\mathcal{C}$ is a proper generating cone [3]. A one-parameter family of matrices $V' = \{V' : t \in \mathbb{R}\}$ is acting on the space $\mathcal{B}$, which will be called the resolving positive group of system (2), provided that $V'u \geq 0$ for $u \geq 0, \ u \in \mathbb{R}^n$.

The solution $u(t) = U' u_0$ of the system of equations (3) will be the solution to the problem (3), (4), if $Pu_0 = u_0$. By the phase space of system (3) we mean a set $\mathcal{B} \subset \mathbb{R}^n$ such that any solution to system (2) lies in pointwise, i.e. $u(t) \in \mathcal{B}$ for all $t \in \mathbb{R}$. For any $u_0 \in \mathcal{B}$ there is a unique solution to problem (3), (4). Thus, a sufficient condition $u_0 \in \text{im} P$ is necessary.

Condition (4) is considered appropriate in the case of $(L, p)$ - regular matrix. And in this case, this condition (4) is equivalent to the condition $R^L_\mu(M)^{p+1}(u(0) - u_0) = 0$ for any $\alpha \in \rho^L(M)$[11].

Theorem 2. Let the matrix $M$ be $(L, p)$-regular, where $p \in \{0\} \cup \mathbb{N}$, $\det M \neq 0$. Then the following conditions are equivalent:

1. $(\mu R^L_\mu(M))^{p+1}, (\mu L^L_\mu(M))^{p+1}$ are positive for all large enough $\mu \in \mathbb{R}_+$;

2. Degenerate holomorphic groups $U'$, $\tilde{F}'$ are positive.
As noted earlier, the phase space (3) is a subspace in $\text{im} \; P$.
Then there is a unique positive solution to problem (3), (4), which, moreover, has the form $u(t) = U^t u_0$, $\forall u_0 \in U^n : u_0 \geq 0$.

3. Example

Let the spaces $\mathfrak{U}$ and $\mathfrak{F}$ be such that $\mathfrak{U} = \mathfrak{F} = \mathbb{R}^2$ and the operators $L, M \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ be given by the matrices:

$$L = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

Taking into account the above, we construct the $L$-resolvent $\left(\mu L - M\right)^{-1}$ of the operator $M$:

$$\left(\mu L - M\right)^{-1} = \left(2 \mu - 1\right)^{-1} \begin{pmatrix} \mu & 1 - \mu \\ 1 - \mu & \mu \end{pmatrix}.$$  

This means that the operator $M$ is $(L, 0)$-bounded and its $L$-spectrum $\sigma^L(M) = \left\{ \frac{1}{2} \right\}$. And besides, if the right $L$-resolvent $R^T_\mu(L) = \left(\mu L - M\right)^{-1} L$ has the form $R^T_\mu(M) = \left(2 \mu - 1\right)^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$,

then the degenerate holomorphic group of operators $U^* = \left\{ U^t : t \in \mathbb{R} \right\}$ has the form

$$U^t = \frac{1}{2 \pi i} \int_{\mathfrak{F}^{-1}} R^T_\mu(M) e^{\mu t} d\mu = \frac{e^{\mu t}}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$  

and its unit is the projector $P = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

An equal affirmation may be said about the degenerate holomorphic group $F^* = \left\{ F^t : t \in \mathbb{R} \right\}$.

After that, we find the phase space $\mathfrak{U}^1$ of equation (3) using the relation $Pu = u$ and we find the kernel $\mathfrak{U}^0$ of the group $U^*$ by the relation $Pu = 0$.

$$\mathfrak{U}^1 = \left\{ u \in \mathbb{R}^2 : u_2 - u_1 = 0 \right\}, \mathfrak{U}^0 = \left\{ u \in \mathbb{R}^2 : u_2 + u_1 = 0 \right\}.$$  

Further by the formula $(u = \text{col}(u_1, u_2) \geq v = \text{col}(v_1, v_2) \Leftrightarrow (u_1 \geq v_1) \land (u_2 \geq v_2))$ we define the order relation $\geq$ on the space $\mathfrak{U}$ that defines the canonical cone $\mathcal{C}_\mathfrak{U} = \mathbb{R}_+^2 \times \mathbb{R}_-^2$. Note that the operator $R^T_\mu(M)$ is positive when $\mu \in \left( \frac{1}{2} ; +\infty \right)$, and the group is also positive. Consequently the order $\geq$ generated by the cone $\mathcal{C}_\mathfrak{U} = \mathfrak{U}^1 \cap \mathcal{C}_\mathfrak{U} = \left\{ u \in \mathbb{R}^2 : u_2 = u_1 \geq 0 \right\}$ exists on the space $\mathfrak{U}^1$.

The cone $\mathcal{C}_\mathfrak{U}^0 = \left\{ u \in \mathbb{R}^2 : u_2 = -u_1 \geq 0 \right\}$ can also be defined on a subspace $\mathfrak{U}^0$ . The direct sum $\mathcal{C}_\mathfrak{U}^0 \oplus \mathcal{C}_\mathfrak{U}^1$ is its own generating cone $\mathcal{C}_\mathfrak{U}$, which generates the order $\succ$ on $\mathbb{R}^2$ such that...
\( (u > 0) \Leftrightarrow ((u_2 - u_1 \geq 0) \land (u_1 + u_2 \geq 0)) \). Note that both the right \( L \)-resolvent \( R^L_M \) and the group \( U' \) are positive in the sense of the order \( \succ \).

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