NEW GENERALIZED GEOMETRIC DIFFERENCE SEQUENCE SPACES

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ABSTRACT. In this current study, the most apparent aspect is to submit a new geometric sequence space. We investigate its topological properties, inclusion relations, geometric statistical convergence and geometric property of Orlicz function and. Moreover, we also check its dual properties.

1. Introduction and preliminaries

Scientists were able to approach problems in science and engineering from a different angle, non-Newtonian calculus, which was developed between 1967 and 1970. Michael Grossman and Robert Katz explored non-Newtonian calculus, that covers many areas such as geometric, anageometric, and bigeometric calculus in [12], which, by depending on multiplication rather than addition, is an opposite to the traditional calculus of Newton and Leibnitz. Geometric analysis was named as multiplicative calculus by D.Stanley [27] and further it was generalised by D.Campell [5], Bashirov et al. [2] and several other researchers [4, 11, 1].

The traditional difference sequence spaces were first introduced by Kizmaz in 1981 \( X(\Delta) \), for \( X = c, c_0, l_\infty \) which are the class of convergent, null and bounded sequences respectively, defined as

\[
X(\Delta) = \{ x = (x_i) : (\Delta x_i) \in X \}
\]

where \( \Delta x_i = x_i - x_{i+1} \).

In 1995 Et. and Çolak [7] generalized \( X(\Delta) \) to \( m \)th order difference sequence spaces by defining \( X(\Delta^m) = \{ x = (x_i) : (\Delta^m x_i) \in X \} \), where \( \Delta^m x_i = \Delta^m (x_i - x_{i+1}) \).

Türkmen and Başar [28] studied the sets of geometric real numbers and geometric complex numbers by \( \mathbb{R}(G) \) and \( \mathbb{C}(G) \) respectively as

\[
\mathbb{R}(G) = \{ e^x : x \in \mathbb{R} \} = \mathbb{R}^+ / 0, \\
\mathbb{C}(G) = \{ e^x : x \in \mathbb{C} \} = \mathbb{C} / 0.
\]

Geometric addition \((\oplus)\), geometric subtraction \((\ominus)\), geometric multiplication \((\odot)\) and geometric limit \( G \lim_{n \to \infty} \) defined in [23, 24, 25, 26].

In fact they introduced the geometric sequence spaces \( l_\infty(G), c(G), c_0(G), l_p(G) \).

[23, 14]
Borua et al. [14, 15, 16, 17, 18, 19] have studied various properties of Geometric sequence spaces. The $m^{th}$ order geometric difference sequence spaces as

\[ l_G^\infty ( \Delta_G^m ) = \{ x = (x_i) : \Delta_G^m x \in l_G^\infty \} , \]
\[ c_G^\infty ( \Delta_G^m ) = \{ x = (x_i) : \Delta_G^m x \in c_G^\infty \} , \]
\[ c_0^m ( \Delta_G^m ) = \{ x = (x_i) : \Delta_G^m x \in c_0^m \} , \]

respectively where $m \in \mathbb{N}$

\[
\Delta_G^0 = (x_i) \\
\Delta_G^1 = (\Delta_G x_i) = (x_i \oplus x_{i+1}) \\
\Delta_G^2 = (\Delta_G^2 x_i) = (\Delta_G x_i \oplus \Delta_G x_{i+1}) \\
= (x_i \oplus x_{i+1} \oplus x_{i+1} \oplus x_{k+2}) \\
= (x_i \oplus e^2 \oplus x_{i+1} \oplus x_{i+1}) \\
\Delta_G^3 = (\Delta_G^3 x_i) = (\Delta_G^2 x_i \oplus \Delta_G^2 x_{i+1}) \\
= (x_i \oplus e^2 \oplus x_{i+1} \oplus e^3 \oplus x_{i+1} \oplus x_{k+3})
\]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \]

\[
\Delta_G^m = (\Delta_G^m x_i) = (\Delta_G^{m-1} x_i \oplus \Delta_G^{m-1} x_{i+1}) \\
= G \sum_{\nu=0}^{m} (\oplus e)^{\nu} \odot e^{(\nu)} \odot x_{k+\nu}, \text{ with } (\oplus e)^{0_G} = e. \tag{1.1}
\]

In 1968 Cesàro sequence spaces were brought to lime light by Dutch Mathematical Society which were of the form

\[ [C, 1] = \left\{ x = (x_i) \in \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |x_i - \ell| = 0 \text{, for some } \ell \right\}. \]
\[ (C, 1) = \left\{ x = (x_i) \in \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (x_i - \ell) = 0 \text{, for some } \ell \right\}. \]

Then after, in 1970, Shiue [21, 22] researched the Cesàro function and sequence spaces. Since then, other academics have become interested in Cesàro sequence spaces.

Leindler [29] first developed the strongly$(V, \lambda)$– summable sequence spaces

\[ [V, \lambda] = \left\{ x = (x_i) \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{i \in I_n} |x_i - \ell| = 0 \text{, for some } \ell \right\}, \]
\[ (V, \lambda) = \left\{ x = (x_i) \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{i \in I_n} (x_i - \ell) = 0 \text{, for some } \ell \right\}. \]
Where, the de la Vallee-Pousin meaning is $t_n(x) = \frac{1}{\lambda_n} \sum_{i \in I_n} x_i$, in which the non-decreasing sequence of positive reals $(\lambda_n)$ generally tends to infinity with $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$ and $I_n = [n - \lambda_n + 1, n], n = 1, 2, 3, \ldots$

The reduction of the $(V, \lambda)$-summable sequence spaces to the Cesàro sequences space is noticed when $\lambda_n = n$.

The new geometric sequence spaces is now being introduced as follows:

$$(C, 1)G(\Delta^m_G) = \left\{ x = (x_i) \in \omega(G) : \lim_{\eta \to \infty} (e \otimes e^\eta) \odot G \sum_{i=1}^{n} (\Delta^m_G x_i \otimes L)^G = 1, \text{ for some } L \right\},$$

$$(C, 1)_G(\Delta^m_G) = \left\{ x = (x_i) \in \omega(G) : \lim_{\eta \to \infty} (e \otimes e^\eta) \odot G \sum_{i=1}^{n} |\Delta^m_G x_i \otimes L|^G = 1, \text{ for some } L \right\},$$

$$(V, \lambda)G(\Delta^m_G) = \left\{ x = (x_i) \in \omega(G) : \lim_{\eta \to \infty} (e \otimes e^{\lambda \eta}) \odot G \sum_{i=1}^{n} (\Delta^m_G x_i \otimes L)^G = 1, \text{ for some } L \right\},$$

$$(V, \lambda)_G(\Delta^m_G) = \left\{ x = (x_i) \in \omega(G) : \lim_{\eta \to \infty} (e \otimes e^{\lambda \eta}) \odot G \sum_{i=1}^{n} |\Delta^m_G x_i \otimes L|^G = 1, \text{ for some } L \right\}.$$  

2. **Main Results**

2.1. **Topological Structure.** The spaces $(C, 1)_G(\Delta^m_G)$, $[C, 1]_G(\Delta^m_G)$, $(V, \lambda)_G(\Delta^m_G)$ and $[V, \lambda]_G(\Delta^m_G)$ are Banach spaces have the given norms

$$
||x||^G_{\Delta_G} = G \sum_{i=1}^{n} |x_i|^G \oplus \sup_{\eta \in \mathbb{N}} (e \otimes e^\eta) \odot G \sum_{i=1}^{n} \Delta^m_G x_i^G, \quad (2.1)
$$

$$
||x||^G_{\Delta_G} = G \sum_{i=1}^{n} |x_i|^G \oplus \sup_{\eta \in \mathbb{N}} (e \otimes e^\eta) \odot G \sum_{i=1}^{n} |\Delta^m_G x_i|^G, \quad (2.2)
$$

$$
||x||^G_{\Delta_G} = G \sum_{i=1}^{n} |x_i|^G \oplus \sup_{\eta \in \mathbb{N}} (e \otimes e^{\lambda \eta}) \odot G \sum_{i \in I_n} \Delta^m_G x_i^G, \quad (2.3)
$$

and

$$
\text{ and } ||x||^G_{\Delta_G} = G \sum_{i=1}^{n} |x_i|^G \oplus \sup_{\eta \in \mathbb{N}} (e \otimes e^{\lambda \eta}) \odot G \sum_{i \in I_n} |\Delta^m_G x_i|^G. \quad (2.4)
$$

respectively.

**Theorem 2.1.** The spaces $(C, 1)_G(\Delta^m_G)$, $[C, 1]_G(\Delta^m_G)$, $(V, \lambda)_G(\Delta^m_G)$ and $[V, \lambda]_G(\Delta^m_G)$ are linear over $\mathbb{C}(G)$. 

Proof. Let \( x = (x_i) \) and \( y = (y_i) \) be any two elements of \( [C, 1]_G (\Delta^m_G) \) then there exist \( L \) & \( L' \) s.t.

\[
G \lim_{\eta \to \infty} (e \otimes e^{\eta}) \odot G \sum_{i=1}^n |\Delta^m_G x_i \oplus L|_G = 1,
\]

\[
G \lim_{\eta \to \infty} (e \otimes e^{\eta}) \odot G \sum_{i=1}^n |\Delta^m_G y_i \oplus L'|_G = 1.
\]

For any two scalars \( \alpha, \beta \in \mathbb{C}(G) \), by setting \( L'' = \alpha L \oplus \beta L' \), we have

\[
G \lim_{\eta \to \infty} (e \otimes e^{\eta}) \odot G \sum_{i=1}^n |\Delta^m_G (\alpha x_i \oplus \beta y_i) \oplus L''|_G
\]

\[
= G \lim_{\eta \to \infty} (e \otimes e^{\eta}) \odot G \sum_{i=1}^n |\Delta^m_G (\alpha x_i \oplus \beta y_i) \oplus (\alpha L \oplus \beta L')|_G
\]

\[
\leq |\alpha| G \lim_{\eta \to \infty} (e \otimes e^{\eta}) \odot G \sum_{i=1}^n |\Delta^m_G x_i \oplus L|_G
\]

\[
\oplus |\beta| G \lim_{\eta \to \infty} (e \otimes e^{\eta}) \odot G \sum_{i=1}^n |\Delta^m_G y_i \oplus L'|_G
\]

\[
\to 1 \text{ as } \eta \to \infty
\]

\[
\Rightarrow \alpha x_i \oplus \beta y_i \in [C, 1]^G (\Delta^m_G).
\]

For other spaces the results follow using similar techniques.

\[\square\]

**Theorem 2.2.** The space \((C, 1)_G (\Delta^m_G)\) is a normed linear space w.r.t the norm

\[
\|x\|_{\Delta_G}^G = G \sum_{i=1}^n |x_i|^G \oplus \sup_{\eta \in \mathbb{N}} (e \otimes e^{\eta}) \odot G \sum_{i=1}^n \Delta^m_G x_i
\]

Proof. Consider \( x = (x_i), y = (y_k) \in (C, 1)_G (\Delta^m_G), \) for normed linear spaces we will prove the following four conditions:

(i) \( \|x\|_{\Delta_G}^G \geq 1 \)

(ii) \( \|x\|_{\Delta_G}^G = 1 \Leftrightarrow x = 0_G. \)

(iii) \( \|\lambda \odot x\|_{\Delta_G}^G = |\lambda| \odot \|x\|_{\Delta_G}^G, \) for \( \lambda \in \mathbb{C}(G) \) and \( x \in X, \)

(iv) \( \|x \oplus y\|_{\Delta_G}^G \leq \|x\|_{\Delta_G}^G \oplus \|y\|_{\Delta_G}^G, \) for all \( x, y \in X, \) (Triangle inequality)
(i) Here

\[ |x_i|^G \geq 1 \text{ and } G \sum_{i=1}^{n} |\Delta_G^m x_i|^G \geq 1. \]

Consider \[ \|x\|_{\Delta_G}^G = G \sum_{i=1}^{n} |x_i|^G \| \sup_{\eta \in N} (e \circ e^\eta) \| G \sum_{i=1}^{n} \Delta_G^m x_i \|_{\Delta_G}^G \]

\[ = G \sum_{i=1}^{n} |x_i|^G \| \sup_{\eta \in N} (e \circ e^\eta) \| G \sum_{i=1}^{n} \Delta_G^m x_i \|_{\Delta_G}^G \geq 1. \]

(ii)

\[ \|x\|_{\Delta_G}^G = 1 \iff G \sum_{i=1}^{n} |x_i|^G \| \sup_{\eta \in N} (e \circ e^\eta) \| G \sum_{i=1}^{n} \Delta_G^m x_i \|_{\Delta_G}^G = 1 \]

\[ \iff |x_1|^G \sup_{\eta \in N} |x_i \ominus x_{i+1}|^G = 1, \ \forall i \]

\[ \iff |x_1|^G = 1 \text{ and } |x_i \ominus x_{i+1}|^G = 1 \]

\[ \iff x_1 = 1 \text{ and } x_i \ominus x_{i+1} = 1, \ \forall i \]

\[ \iff x_1 = 1 \text{ and } x_i/x_{i+1} = 1, \ \forall i \]

\[ \iff x_1 = 1 \text{ and } x_i = x_{i+1}, \ \forall i \]

\[ \iff x_i = 1, \ \forall i \]

\[ \iff x = (1, 1, 1, 1, \ldots) = 1_G. \]

(iii) For any scalar \( \lambda \), consider

\[ \| \lambda \circ x \|_{\Delta_G}^G = G \sum_{i=1}^{n} |\lambda \circ x_i|^G \| \sup_{m,n} (e \circ e^\eta) \| G \sum_{i=1}^{n} \Delta_G^m (\lambda \circ x) \|_{\Delta_G}^G \]

\[ \leq |\lambda| \circ G \sum_{i=1}^{n} |x_i|^G \| \sup_{m,n} (e \circ e^\eta) \| G \sum_{i=1}^{n} \Delta_G^m x_i \|_{\Delta_G}^G \]

\[ = |\lambda| \circ \left[ G \sum_{i=1}^{n} |x_i|^G \| \sup_{\eta \in N} (e \circ e^\eta) \| G \sum_{i=1}^{n} \Delta_G^m x_i \|_{\Delta_G}^G \right] \]

\[ = |\lambda| \circ \|x\|_{\Delta_G}^G. \]

(iv) Suppose

\[ \|x\|_{\Delta_G}^G = G \sum_{i=1}^{n} |x_1|^G \| \sup_{\eta \in N} (e \circ e^\eta) \| G \sum_{i=1}^{n} \Delta_G^m x_i \|_{\Delta_G}^G \]  \hspace{1cm} (2.5)
and

\[ \|y\|_{\Delta_G}^G = \sum_{i=1}^n |y_i|^G \oplus \sup_{\eta \in \mathbb{N}} \left( (e \otimes e^\eta) \odot G \sum_{i=1}^n \Delta^m_G x_i \right)^G \]  

(2.6)

Consider

\[ \|x \oplus y\|_{\Delta_G}^G = \sum_{i=1}^n \left| x_i \oplus y_i \right|^G \oplus \sup_{\eta \in \mathbb{N}} \left( (e \otimes e^\eta) \odot G \sum_{i=1}^n \Delta^m_G (x \oplus y) \right)^G \]

\[ \leq \left( G \sum_{i=1}^n \left| x_i \right|^G \oplus \sup_{\eta \in \mathbb{N}} \left( (e \otimes e^\eta) \odot G \sum_{i=1}^n \Delta^m_G x_i \right)^G \right) \oplus \left( G \sum_{i=1}^n \left| y_i \right|^G \oplus \sup_{\eta \in \mathbb{N}} \left( (e \otimes e^\eta) \odot G \sum_{i=1}^n \Delta^m_G (y) \right)^G \right) \]

\[ = \|x\|_{\Delta_G}^G \oplus \|y\|_{\Delta_G}^G. \]

\[ \square \]

**Theorem 2.3.** The space \((C, 1)_G(\Delta^m_G)\) is a Banach space w.r.t. the norm

\[ \|x\|_{\Delta_G}^G = G \sum_{i=1}^n |x_i|^G \oplus \sup_{\eta \in \mathbb{N}} \left( (e \otimes e^\eta) \odot G \sum_{i=1}^n |\Delta^m_G x_i|^G \right). \]

**Proof.** Let \((x^n)\) is a Cauchy sequence in \((C, 1)_G(\Delta^m_G)\), where \(x^n = (x^{(n)}_i) = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots)\) for \(\eta \in \mathbb{N}\) and \(x_j^{(n)}\) is the \(j\)th co-ordinate of \(x^n\). Then

\[ \|x^n \ominus x^m\|_{\Delta_G}^G \xrightarrow{G} 1 \text{ as } m, \eta \to \infty, \]

**i.e.**

\[ G \sum_{i=1}^n \left| x^n_i \ominus x^m_i \right|^G \oplus \sup_{\eta \in \mathbb{N}} \left( (e \otimes e^\eta) \odot G \sum_{i=1}^n \Delta^m_G (x_i^n \ominus x_i^m) \right)^G \xrightarrow{G} 1 \]

**i.e.**

\[ G \sum_{i=1}^n \left| x^n_i \ominus x^m_i \right|^G \cdot \sup_{\eta \in \mathbb{N}} \left( (e \otimes e^\eta) \odot G \sum_{i=1}^n \Delta^m_G (x_i^n \ominus x_i^m) \right)^G \xrightarrow{G} 1 \]

**i.e.**

\[ G \sum_{i=1}^n \left| x^n_i \ominus x^m_i \right|^G \xrightarrow{G} 1 \text{ and sup} \sup_{\eta \in \mathbb{N}} \left( (e \otimes e^\eta) \odot G \sum_{i=1}^n \Delta^m_G (x_i^n \ominus x_i^m) \right)^G \xrightarrow{G} 1. \]

Hence we get

\[ |x^n \ominus x^m|^G \xrightarrow{G} 1 \text{ as } m, n \to \infty. \]

Since \(\mathbb{C}(G)\) is complete, \((x_i^{(n)}) = (x_i^{(1)}, x_i^{(2)}, x_i^{(3)}, \ldots)\) is a Cauchy sequence. Hence by completeness of \(\mathbb{C}(G)\), \((x_i^{(n)})\) converges to \(x_i\). Since \((x^n)\) is a Cauchy sequence so for each \(\varepsilon > 1, \exists N = N(\varepsilon)\) s.t. \(\|x^n \ominus x^m\|_{\Delta_G}^G < \varepsilon \forall m, n \geq N.\)

Hence we get
$G \sum_{i=1}^{n} |x_i^n \otimes x_i^m|^G < \varepsilon$ and $\sup_{\eta \in N} |(e \otimes e^n) \circ G \sum_{i=1}^{n} \Delta^m_G (x_i^n \otimes x_i^m)|^G < \varepsilon$

for all $k \in \mathbb{N}$ and $m, n, \geq N$. So we have

$$\|x^n \otimes x^m\|_{\Delta G} = G \lim_{m \to \infty G} \sum_{i=1}^{n} |x_i^n \otimes x_i^m|^G \oplus G \lim_{m \to \infty \eta \in N} \sup_{\eta \in N} |(e \otimes e^n) \circ G \sum_{i=1}^{n} \Delta^m_G (x_i^n \otimes x_i^m)|^G$$

$$= \varepsilon \oplus \varepsilon = \varepsilon^2.$$

Hence the sequence $(x^n)$ converges to the sequence $x = (x_i)$.

Here only we show that the sequence $x = (x_i) \in (C, 1)_G (\Delta^m_G)$. So we consider

$$\left| (e \otimes e^n) \circ G \sum_{i=1}^{n} \Delta^m_G x_i \right|^G$$

$$= \left| (e \otimes e^n) \circ G \sum_{i=1}^{n} \Delta^m_G (x_i \oplus x_i^N \otimes x_i^N) \right|^G$$

$$\leq \left| (e \otimes e^n) \circ G \sum_{i=1}^{n} \Delta^m_G x_i \right|^G \oplus \left| (e \otimes e^n) \circ G \sum_{i=1}^{n} \Delta^m_G (x_i^N \otimes x_i) \right|^G.$$

From the inequalities

$G \sum_{i=1}^{n} |x_i^n \otimes x_i^m|^G < \varepsilon$ and $\sup_{\eta \in N} |(e \otimes e^n) \circ G \sum_{i=1}^{n} \Delta^m_G (x_i^n \otimes x_i^m)|^G < \varepsilon$, hence the sequence $(x_i^N) \in (C, 1)_G (\Delta^m_G)$, hence $(x_i) \in (C, 1)_G (\Delta^m_G)$. Therefore, the space $(C, 1)_G (\Delta^m_G)$ is a Banach space.

Hence Proved. □

**Remark 2.4.** The spaces $[C, 1]_G(\Delta^m_G), (V, \lambda)_G(\Delta^m_G)$ and $[V, \lambda]_G(\Delta^m_G)$ are Banach spaces with their corresponding norms (2.1) using the similar techniques as in theorem 2.3.

**Theorem 2.5.**

(i) $[C, 1]_G(\Delta^m_G) \subset (C, 1)_G(\Delta^m_G)$

(ii) $[V, \lambda]_G(\Delta^m_G) \subset (V, \lambda)_G(\Delta^m_G)$

and the inclusions are strict.

**Proof.** (i) Suppose $x \in [C, 1]_G(\Delta^m_G)$ then there exists a $L \in C(G)$ s.t.

$$G \lim_{\eta \to \infty} (e \otimes e^n) \circ G \sum_{i=1}^{n} \Delta^m_G (x_i) \oplus L^G = 1$$

$$\Rightarrow G \lim_{\eta \to \infty} (e \otimes e^n) \circ G \sum_{i=1}^{n} (\Delta^m_G (x_i) \oplus L)^G = 1.$$

So it means $x \in (C, 1)_G(\Delta^m_G)$.

Hence $[C, 1]_G(\Delta^m_G) \subset (C, 1)_G(\Delta^m_G)$.

Similarly we can prove (ii). □
3. GEOMETRIC STATISTICAL CONVERGENCE

In this section, we define a $\lambda-$ geometric statistical convergence $S_\lambda (\Delta^m_G)$ and then establish the relationship of $S_\lambda (\Delta^m_G)$ with $[V, \lambda]_G (\Delta^m_G)$ and $[C, 1]_G (\Delta^m_G)$. The idea of statistical convergence was introduced by Fast [10] and studied subsequently in different spaces by several authors including [6, 9, 8].

Now we define geometric statistical convergence separately for the spaces $[C, 1]_G (\Delta^m_G)$ and $[V, \lambda]_G (\Delta^m_G)$.

**Definition 3.1.** Now we define a sequence $x = (x_i)$ is said to be $\Delta^m_G-$ geometric statistically convergent $\Delta^m_G-$ statistically convergent if there is a complex number $L$ such that

$$G\lim_{\eta \to \infty} (e \odot e^n) \odot \left| \{k \leq n : |\Delta^m_Gx_i \ominus L| \geq \varepsilon \} \right| = 1$$

for every $\varepsilon > 1$. In this case we write $x_i \rightarrow_G LS(\Delta^m_G)$. The set $\Delta^m_G-$ statistically convergent sequences will be denoted by $S(\Delta^m_G)$.

**Definition 3.2.** A sequence $x = (x_i)$ is said to be $\lambda-$ geometric statistically convergent or $S_\lambda-$ geometric convergent to $L$, if for every $\varepsilon > 1$,

$$G\lim_{\eta \to \infty} (e \odot e^{\lambda^n}) \odot \left| \{k \in I_n : |\Delta^m_Gx_i \ominus L|^G \geq \varepsilon \} \right|^G = 1.$$

We write

$$S_\lambda \ominus G\lim x = L \text{ or } x_i \rightarrow_G L (S_\lambda(\Delta^m_G))$$

that is $S_\lambda(\Delta^m_G) = \{x \in \omega(G) : S_\lambda \ominus G\lim x = L \text{ for some } L\}$.

**Theorem 3.3.** Let $\lambda = (\lambda_k)$ be a non-decreasing sequence of positive reals which tend to infinity with $\lambda_1 = 1$ and $\lambda_{k+1} \leq \lambda_k + 1$ and $I_k = [k - \lambda_k + 1, k], k = 1, 2, 3, \ldots$.

Then

(i) If $x_i \rightarrow_G L [V, \lambda]_G (\Delta^m_G)$, then $x_i \rightarrow_G LS(\Delta^m_G)$.

(ii) If $x \in l^G_G (\Delta^m_G)$ and $x_i \rightarrow_G L (S_\lambda(\Delta^m_G))$, then $x_i \rightarrow_G L [V, \lambda]_G (\Delta^m_G)$ and hence $x_i \rightarrow_G L [C, 1]_G (\Delta^m_G)$ provided $x = (x_i)$ is not eventually constant.

(iii) $S_\lambda (\Delta^m_G) \cap l^G_G (\Delta^m_G) = [V, \lambda]_G (\Delta^m_G) \cap l^G_G (\Delta^m_G)$, where $l^G_G (\Delta^m_G) = \{x = (x_i) : \Delta^m_Gx \in l^G_G\}$.

**Proof.** (i) Suppose $\varepsilon > 1$ and $x_i \rightarrow_G L [V, \lambda]_G (\Delta^m_G)$, then we have

$$G\sum_{k \in I_n} |\Delta^m_Gx_i \ominus L|^G \geq G\sum_{k \in I_n} |\Delta^m_Gx_i \ominus L|^G \geq \varepsilon \left| \left\{k \in I_n : |\Delta^m_Gx_i \ominus L|^G \geq \varepsilon \right\} \right|^G.$$

Therefore $x_i \rightarrow_G LS(\Delta^m_G)$.

(ii) Suppose $x_i \rightarrow_G LS(\Delta^m_G)$ and $x \in l^G_G (\Delta^m_G)$,

$|\Delta^m_Gx_i \ominus L|^G \leq M$ for all $k$. Given $\varepsilon > 1$, we have
Theorem 3.4. If

(iii) From (i) and (ii) it is obvious. □

we obtain

\[ x \]

Proof.

Given \( x \), we have

\[ \{ k \in I_n : | \Delta_G^m x_i \oslash L | \geq \varepsilon \} \subset \{ k \leq n : | \Delta_G^m x_i \oslash L | \geq \varepsilon \}. \]

As \( n \to \infty \), the right hand side goes to one, this gives \( x_i \to L[V, \lambda]_G(\Delta_G^m) \).

If \( x_i \in L[V, \lambda]_G(\Delta_G^m) \), then to show \( x_i \in L(C,1)_G(\Delta_G^m) \).

Consider

\[ x_i \in L[V, \lambda]_G^m(\Delta_G^m) \]

\[ \Rightarrow (e \otimes e^\lambda) \otimes G \sum_{k \in I_n} | \Delta_G^m x_i \oslash L |^G \leq \varepsilon, \text{ for some } L. \]

To show

\[ (e \otimes e^\eta) \otimes G \sum_{k=1}^{n} (\Delta_G^m x_i \oslash L)^G \leq \varepsilon, \text{ for some } L. \]

Consider

\[ (e \otimes e^\eta) \otimes G \sum_{k=1}^{n} (\Delta_G^m x_i \oslash L)^G \]

\[ \Rightarrow (e \otimes e^\eta) \otimes G \sum_{k=1}^{n-\lambda} (\Delta_G^m x_i \oslash L) \oplus (e \otimes e^\eta) \otimes G \sum_{k=n-\lambda+1}^{n} (\Delta_G^m x_i \oslash L)^G \]

\[ \leq \left( (e \otimes e^\lambda) \otimes G \sum_{k \in I_n} | \Delta_G^m x_i \oslash L |^G \right) \oplus \left( (e \otimes e^\lambda) \otimes G \sum_{k \in I_n} | \Delta_G^m x_i \oslash L |^G \right) \]

\[ \leq e^2 \otimes (e \otimes e^\lambda) \otimes G \sum_{k \in I_n} | \Delta_G^m x_i \oslash L |^G \]

we obtain \( x_i \to L(C,1)_G(\Delta_G^m) \).

(iii) From (i) and (ii) it is obvious. □

Theorem 3.4. If \( \lim_{n \to \infty} \inf (e^\lambda \oslash e^\eta) > 1 \), then \( S(\Delta_G^m) \subset S_\lambda(\Delta_G^m) \).

Proof. Given \( \varepsilon > 1 \), we have

\[ \{ k \in I_n : | \Delta_G^m x_i \oslash L |^G \geq \varepsilon \} \subset \{ k \leq n : | \Delta_G^m x_i \oslash L |^G \geq \varepsilon \}. \]
Therefore,

\[
(e \otimes e^n) \left\{ k \leq n : |\Delta_G^m x_i \ominus L|^G \geq \varepsilon \right\}^G \\
\geq (e \otimes e^n) \left\{ k \in I_n : |\Delta_G^m x_i \ominus L|^G \geq \varepsilon \right\}^G \\
= (e^{\lambda_n} \otimes e^n) \otimes (e \otimes e^{\lambda_n}) \left\{ k \in I_n : |\Delta_G^m x_i \ominus L|^G \geq \varepsilon \right\}^G.
\]

Taking the limit as \( n \to \infty \) and by hypothesis \( \lim_{\eta \to \infty} e^{\lambda_n} \otimes e^n > 1 \), we get \( x_i \overset{G}{\to} LS(\Delta_G^m) \Rightarrow x_i \overset{G}{\to} LS(\lambda_m) \).

\[\square\]

4. **NEW GEOMETRIC SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS**

This part deals with some new types of geometric difference sequence spaces which includes Orlicz functions. Also here we examine their topological properties.

Lindenstrauss and Tzafriri [29] constructed the sequence space \( l_M \) by using the Orlicz function \( M \) as follows:

\[
l_M = \left\{ x = (x_i) \in \omega : \sum_{k=1}^{\infty} M \left( \frac{|x_i|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.
\]

With norm

\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_i|}{\rho} \right) \leq 1 \right\}
\]

this is a Banach space.
For $M$ an Orlicz function, $m$ a positive integer also $p(G) = (p_k)_G$ any sequence of strictly positive real numbers now we define the spaces,

$$[V, \lambda, M, p]_G^G(\Delta^m_G) = \left\{ x = (x_i) : \begin{array}{l}
g \lim_{\eta \to \infty} (e \otimes e^{\lambda \eta}) G \sum_{k \in I_n} \left[ M \otimes \left( \frac{\Delta^m_G x_i \otimes L^G_G}{\rho} \right) \right]^{(p_k)_G} \\
= 1, \text{ for some } L \text{ and } \rho > 1 \end{array} \right\},$$

$$[V, \lambda, M, p]_0^G(\Delta^m_G) = \left\{ x = (x_i) : G \lim_{\eta \to \infty} (e \otimes e^{\lambda \eta}) G \sum_{k \in I_n} \left[ M \otimes \left( \frac{\Delta^m_G x_i}{\rho G} \right) \right]^{(p_k)_G} \\
= 1, \text{ for some } \rho > 1 \right\},$$

$$[V, \lambda, M, p]_\infty^G(\Delta^m_G) = \left\{ x = (x_i) : \sup_{n \in \mathbb{N}} (e \otimes e^{\lambda \eta}) G \sum_{k \in I_n} \left[ M \otimes \left( \frac{\Delta^m_G x_i}{\rho G} \right) \right]^{(p_k)_G} < \infty, \text{ for some } \rho > 1 \right\}.$$

For $p(G) = (p_k)_G = 1, \forall i$, we denote the above spaces by $[V, \lambda, M]_G^G(\Delta^m_G)$, $[V, \lambda, M]_0^G(\Delta^m_G)$ and $[V, \lambda, M]_\infty^G(\Delta^m_G)$ respectively. Then we have certain topological properties of the spaces.

**Theorem 4.1.** For a bounded sequence $p^G = (p_k)_G$ of strictly positive geometric real numbers, the space $[V, \lambda, M, p]_G^G(\Delta^m_G)$, $[V, \lambda, M, p]_0^G(\Delta^m_G)$ and $[V, \lambda, M, p]_\infty^G(\Delta^m_G)$ are linear over $\mathbb{C}(G)$, the field of geometric complex numbers.

**Proof.** Let $x, y \in [V, \lambda, M, p]_G^G(\Delta^m_G)$ and $\alpha, \beta \in \mathbb{C}(G)$. Then there exist positive numbers $\rho_1, \rho_2$ such that

$$G \lim_{\eta \to \infty} (e \otimes e^{\lambda \eta}) G \sum_{k \in I_n} \left[ M \otimes \left( \frac{\Delta^m_G x_i}{\rho_1 G} \right) \right]^{(p_k)_G} = 1,$$

and

$$G \lim_{\eta \to \infty} (e \otimes e^{\lambda \eta}) G \sum_{k \in I_n} \left[ M \otimes \left( \frac{\Delta^m_G y_i}{\rho_2 G} \right) \right]^{(p_k)_G} = 1.$$

Define $\rho_3 = \max \left\{ 2 \left| \alpha \right| \rho_1, 2 \left| \alpha \right| \rho_2 \right\}$. 
Since $\Delta_m^G$ is linear and $M$ is non-decreasing

$$(e \odot e^{\lambda y}) \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\Delta_m^G (\alpha x_i \oplus \beta y_i)|^G}{\rho_3} \right) \right]^{(p_k)^G}$$

$$= (e \odot e^{\lambda y}) \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\alpha \Delta_m^G x_i \oplus \beta \Delta_m^G y_i|^G}{\rho_3} \right) \right]^{(p_k)^G}$$

$$\leq (e \odot e^{\lambda y}) \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\alpha \Delta_m^G x_i|^G}{\rho_1} \right) \oplus M \odot \left( \frac{|\beta \Delta_m^G y_i|^G}{\rho_2} \right) \right]^{(p_k)^G}$$

$$\leq (e \odot e^{\lambda y}) \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\Delta_m^G x_i|^G}{\rho_1} \right) \oplus M \odot \left( \frac{|\Delta_m^G y_i|^G}{\rho_2} \right) \right]^{(p_k)^G}$$

$$\leq C (e \odot e^{\lambda y}) \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\Delta_m^G x_i|^G}{\rho_1} \right) \right]^{(p_k)^G}$$

$$\oplus C \odot (e \odot e^{\lambda y}) \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\Delta_m^G y_i|^G}{\rho_2} \right) \right]^{(p_k)^G}$$

$\rightarrow 1,$

where $C = \max \{1, 2^{H-1}\}, H = \sup_{i \in \mathbb{N}} (p_k)^G$.

So that $\alpha x_i \oplus \beta y_i \in [V, \lambda, M, p]_0^G \left( \Delta_m^G \right)$.

Hence $[V, \lambda, M, p]_0^G \left( \Delta_m^G \right)$ is a paranormed space (not necessarily totally paranormed) with a paranorm

$g(x) =
\inf \left\{ \frac{(p_n)^G}{\rho} : \left( e \odot e^{\lambda y} \right) \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\Delta_m^G x_i|^G}{\rho} \right) \right]^{(p_k)^G} \right\} \leq 1, n = 1, 2, 3, ...$

where $H = \max \{1, \sup_{i \in \mathbb{N}} p_k^G\}$. For $m$ be a positive integer, $M$ any Orlicz function.

**Theorem 4.2.** A bounded sequence $p^G = (p_k)^G$ of strictly positive real numbers, $[V, \lambda, M, p]_0^G \left( \Delta_m^G \right)$ is a paranormed space (not necessarily totally paranormed) with a paranorm

$g(x) =
\inf \left\{ \frac{(p_n)^G}{\rho} : \left( e \odot e^{\lambda y} \right) \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\Delta_m^G x_i|^G}{\rho} \right) \right]^{(p_k)^G} \right\} \leq 1, n = 1, 2, 3, ...$

where $H = \max \{1, \sup_{i \in \mathbb{N}} p_k^G\}$. For $m$ be a positive integer, $M$ any Orlicz function.

**Proof.** It is obvious that $g(x) = g(\odot x)$. The subadditivity of $g$ follows from the proof of theorem (4.1) taking $\alpha = 1, \beta = 1$. It is trivial that $\Delta_m^G x = 1$ for $x = 1$. Since $M(1) = 1$, we get $\inf \frac{(p_n)^G}{\rho} = 1$ for $x = 1$. 
Now, here we have to prove scalar multiplication is continuous. Let us take $\alpha$ be any complex number. For the continuity of scalar multiplication let $\alpha$ be fixed and $x \to 1$ in $[V, \lambda, M, p]_0^G(\Delta^n_G)$.

Consider

$$g(\alpha \odot x) = \inf \left\{ \rho (\frac{\rho n}{H}) G : \left( e \odot e^{\lambda y} \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\Delta^n_G(\alpha x_i)|^G}{\rho} \right) \right] (p_{k})^G \right) \leq 1, 
\quad n = 1, 2, 3, ..., \right\}$$

$$= \inf \left\{ \rho (\frac{\rho n}{H}) G : \left( e \odot e^{\lambda y} \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\alpha \Delta^n_G(x_i)|^G}{\rho} \right) \right] (p_{k})^G \right) \leq 1, 
\quad n = 1, 2, 3, ..., \right\}$$

$$(\because \text{ the linearity of } \Delta^n_G).$$

Then

$$g(\alpha \odot x) = \inf \left\{ (|\alpha| r) (\frac{\rho n}{H}) G : \left( e \odot e^{\lambda y} \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\alpha \Delta^n_G(x_i)|^G}{r} \right) \right] (p_{k})^G \right) \leq 1, 
\quad n = 1, 2, 3, ..., \right\}$$

where $r = \rho/|\alpha|$. Since $|\alpha|^{(p_{n})^G} \leq \max \{1, |\alpha|^{\sup(p_{n})^G}\}$ we have

$$g(\alpha \odot x) \leq \max \{1, |\alpha|^{\sup(p_{n})^G}\} \left( \frac{1}{\pi} \right) \odot$$

$$\inf \left\{ \left( r \frac{\rho n}{H} \right) G : \left( e \odot e^{\lambda y} \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\alpha \Delta^n_G(x_i)|^G}{r} \right) \right] (p_{k})^G \right) \leq 1, n = 1, 2, 3, .... \right\},$$

(4.1)

Therefore $\alpha \odot x \to 1$ in $[V, \lambda, M, p]_0^G(\Delta^n_G)$. Let $x$ be fixed and $\alpha \to 1$, so equation (4.1) converges to one.

Let $\alpha_i \to 1$ as $i \to \infty$. Let $x$ be a fixed sequence in $[V, \lambda, M, p]_0^G(\Delta^n_G)$. For arbitrary $\varepsilon > 1$, let $N$ be a positive integer such that

$$(e \odot e^{\lambda y} \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\Delta^n_G(x_i)|^G}{\rho} \right) \right] (p_{k})^G \leq \left( \frac{\varepsilon}{2} G \right)^H.$$

This implies

$$\left( e \odot e^{\lambda y} \odot G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\Delta^n_G(x_i)|^G}{\rho} \right) \right] (p_{k})^G \right)^{\frac{1}{\pi}} \leq \frac{\varepsilon}{2} G.$$
for some $\rho > 1$ and all $n > N$. Using convexity of $M$, let $1 < |\alpha| < e$, we get

\[
(e \otimes e^{\lambda \eta}) \odot_G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\alpha \Delta^m x_i|^G}{\rho} \right) \right]_{(p_k)^G} \\
\leq (e \otimes e^{\lambda \eta}) \odot_G \sum_{k \in I_n} \left[ |\alpha| \odot M \odot \left( \frac{|\Delta^m x_i|^G}{\rho} \right) \right]_{(p_k)^G} \\
\leq \left( \frac{\varepsilon}{2} G \right)^H.
\]

Since $\alpha \to 1$, corresponding to $\varepsilon > 1$ chosen earlier, we can take one $\delta > 1$ depending upon $\varepsilon$ s.t. $|\alpha| < \delta$, which implies

\[
\left( (e \otimes e^{\lambda \eta}) \odot_G \sum_{k \in I_n} \left[ M \odot \left( \frac{|s_i \Delta^m x_i|^G}{\rho} \right) \right]_{(p_k)^G} \right)^{\frac{1}{H}} \leq \varepsilon \frac{2}{G}, \text{ for } n \leq N.
\]

Hence

\[
\left( (e \otimes e^{\lambda \eta}) \odot_G \sum_{k \in I_n} \left[ M \odot \left( \frac{|s_i \Delta^m x_i|^G}{\rho} \right) \right]_{(p_k)^G} \right)^{\frac{1}{H}} \leq \left( e^{\frac{\varepsilon}{2}} \oplus \varepsilon \frac{2}{2} \right) = \varepsilon
\]

for $|\alpha| < \min(1, \delta)$. Therefore $\alpha \odot x \rightarrow 1$ in $[V, \lambda, M, p]_0^G (\Delta^m_G)$.

**Theorem 4.3.** Let $X$ stand for the spaces $[V, \lambda, M]^G, [V, \lambda, M]^G_0$ or $[V, \lambda, M]^G_\infty$ and $m \geq e$, the geometric identity. The inclusion $X(\Delta^{m-1}) \subset X(\Delta^m)$ is strict in general $X(\Delta^i) \subset X(\Delta^m)$ for all $i = 1, 2, \ldots m - 1$.

**Proof.** We give only proof for the space $X = [V, \lambda, M]^G_\infty$, and the proof for other geometric spaces follows in a similar way.

Let $x \in [V, \lambda, M]^G_\infty(\Delta^m_G)$. Then we have

\[
\sup_{\eta \in N} (e \otimes e^{\lambda \eta}) \odot_G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\Delta^m G x_i|^G}{\rho} \right) \right] < \infty, \quad (4.2)
\]
for some $\rho > 1$.
As $M$ is convex function and non-decreasing, we have
\[
(e \otimes e^{\lambda_n}) \odot_G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\Delta_G^{m}x_i|^G}{2\rho} \right) \right]
\]
\[
= (e \otimes e^{\lambda_n}) \odot_G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\Delta_G^{m-1}x_i \ominus \Delta_G^{m-1}x_{i+1}|^G}{2\rho} \right) \right]
\]
\[
= (e \otimes e^{\lambda_n}) \odot_G \sum_{k \in I_n} \left[ \frac{1}{2G} M \odot \left( \frac{|\Delta_G^{m-1}x_i|^G}{\rho} \right) \right] \oplus
\]
\[
(e \otimes e^{\lambda_n}) \odot_G \sum_{k \in I_n} \left[ \frac{1}{2G} M \odot \left( \frac{|\Delta_G^{m-1}x_{i+1}|^G}{\rho} \right) \right] < \infty \text{ by (4.2).}
\]
Thus $[V, \lambda, M]^G_{\infty}(\Delta_G^{m-1}) \subset [V, \lambda, M]^G_{\infty}(\Delta_G^m)$. Hence $[V, \lambda, M]^G_{\infty}(\Delta_G^i) \subset [V, \lambda, M]^G_{\infty}(\Delta_G^m)$ for $i = 1, 2, ..., m - 1$.

Now consider the following example to show the inclusion is strict.

**Example 4.4.** The sequence $x = (k^m) \in [V, \lambda, M]^G_{\infty}(\Delta_G^m)$, $M(x) = x \hat{p}_k = 1 \forall k \in \mathbb{N}$ and $\lambda_n = n$ for all $n \in \mathbb{N}$.

(If $x = (k^m)$, then $\Delta_G^m x_i = (\ominus e)^m_G \odot m!_G$ and $\Delta_G^{m-1} x_i = (\ominus e)^{m+1}_G \odot m!_G (k \ominus (\frac{m}{2})_G)$ for all $k \in \mathbb{N}$.) by equation (1.1) Hence $x \notin [V, \lambda, M]^G_{\infty}(\Delta_G^{m-1})$. This implies the above inclusion (4.3) is strict.

**Theorem 4.5.** The sequence spaces $[V, \lambda, M, p]_0^G$ and $[V, \lambda, M, p]_{\infty}^G$ are solid.

**Proof.** We give the proof for $[V, \lambda, M, p]_0^G$ . Let $(x_i) \in [V, \lambda, M, p]_0^G$ and $\alpha_k$ be any sequence of scalars such that $|\alpha_k|^G \leq 1$ for all $k \in \mathbb{N}$. Then we have
\[
(e \otimes e^{\lambda_n}) \odot_G \sum_{k \in I_n} \left[ M \odot \left( \frac{|\alpha_k \odot x_i|^G}{\rho} \right) \right]
\]
\[
\leq (e \otimes e^{\lambda_n}) \odot_G \sum_{k \in I_n} \left[ M \odot \left( \frac{|x_i|^G}{\rho} \right) \right] \to 1
\]
as $n \to \infty$.

Hence $(\alpha_k \odot x_i) \in [V, \lambda, M, p]^G_0$ for all sequences of scalars $(\alpha_k)$ with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, whenever $(x_i) \in [V, \lambda, M, p]^G_0$. 

**Theorem 4.6.** For any Orlicz function $M$ which satisfies $\Delta_2$– condition hence we have $[V, \lambda](\Delta^m_G) \subset [V, \lambda](\Delta^n_G)$.

**Proof.** Let $x \in [V, \lambda](\Delta^m_G)$ so that
\[
A_n \equiv (e \otimes e^{\lambda_n}) \odot_G \sum_{k \in I_n} |\Delta_G^m x_i \ominus L|^G \to 1,
\]

$n \to \infty$, for some $L$.  

Let $\varepsilon > 0$ and choose $\delta$ with $1 < \delta < e$ such that $M(t) < \varepsilon$ for $1 \leq t \leq \delta$. We can write

$$
(e \otimes e^{\lambda_n}) \odot G \sum_{k \in I_n} M \odot (|\Delta_{G}^m x_i \ominus L|)^G
$$

$$
= (e \otimes e^{\lambda_n}) \odot G \sum_{k \in I_n} M \odot (|\Delta_{G}^m (x_i) \ominus L|^G)
$$

$$
\oplus (e \otimes e^{\lambda_n}) \odot G \sum_{k \in I_n} M \odot (|\Delta_{G}^m x_i \ominus L|^G)
$$

$$
< [K \odot \delta^{-1} \odot M(t) \odot A_n] \oplus [(e \otimes e^{\lambda_n}) \odot (\lambda_n^\varepsilon)]
$$

By taking $\eta \to \infty$, it follows that $x \in [V, \lambda, M](\Delta_{G}^m)$. \(\square\)

**Theorem 4.7.** For $m$ be a positive integer, $M$ be any Orlicz function, $[V, \lambda, M](\Delta_{G}^m) \subset S_\lambda(\Delta_{G}^m)$.

**Proof.** Let $x \in [V, \lambda, M](\Delta_{G}^m)$ and $\varepsilon > 1$ be given. Then

$$
(e \otimes e^{\lambda_n}) \odot G \sum_{k \in I_n} M \odot \left(\frac{|\Delta_{G}^m x_i \ominus L|^G}{\rho}\right)
$$

$$
\geq (e \otimes e^{\lambda_n}) \odot G \sum_{k \in I_n} M \odot \left(\frac{|\Delta_{G}^m x_i \ominus L|^G}{\rho}\right)
$$

$$
> (e \otimes e^{\lambda_n}) \odot M(\varepsilon / \rho) \{k \in I_n : |\Delta_{G}^m x_i \ominus L| \geq \varepsilon\}^G.
$$

Hence $x \in S_\lambda(\Delta_{G}^m)$. \(\square\)

## 5. Dual Spaces

Geometric $\alpha-$ dual and $\beta-$ dual of the sequence space $X$ in geometric calculus are denoted by $X_\alpha^G$ and $X_\beta^G$ and defined as follows:

$$
X_\alpha^G = \{ a = (a_k) \in \omega(G) : \sum_{k=1}^{\infty} |a_k \odot x_i| < \infty \},
$$

$$
X_\beta^G = \{ a = (a_k) \in \omega(G) : \sum_{k=1}^{\infty} a_k x_i \text{ is convergent} \}
$$

for each $x \in X$ respectively.

If $p = (p_n)$ is bounded, then

$$
V(\lambda, p) = \left\{ x \in \omega : \sum_{n=0}^{\infty} \left( \frac{1}{\lambda_n} \sum_{k \in I_n} |x_i| \right)^{p_n} < \infty \right\}
$$
with the norm \[3\]
\[\|x\| = \inf\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\}\]

Now the above space in geometric difference sequence spaces in mth order is given by

\[V_p^\lambda(\Delta^m_G) = \left\{ x = (x_i) \in \omega(G) : G \sum_{\eta=1}^{\infty} \left( (e \otimes e^{\lambda \eta}) \circ_G \sum_{k \in I_n} \Delta^m_G x_i \right)^{\eta} < \infty \right\}
\]

\[V_\infty^\lambda(\Delta^m_G) = \left\{ x = (x_i) \in \omega(G) : \sup_{\eta \in \mathbb{N}} \left( (e \otimes e^{\lambda \eta}) \circ_G \sum_{k \in I_n} \Delta^m_G x_i \right)^{\eta} < \infty \right\}
\]

Now for an operator \(u : X \to X\) by \(u(x) = (1, \ldots, 1, x_{m+1}, x_{m+2}, \ldots)\) for all \(x = (x_i) \in X\). Consider the sets \(uV_p^\lambda(\Delta^m_G)\) and \(uV_\infty^\lambda(\Delta^m_G)\) as

\[uV_p^\lambda(\Delta^m_G) = \left\{ x = (x_i) : x \in V_p^\lambda(\Delta^m_G) \text{ and } x_1 = x_2 = \cdots = x_m = 1 \right\}
\]

and

\[uV_\infty^\lambda(\Delta^m_G) = \left\{ x = (x_i) : x \in V_\infty^\lambda(\Delta^m_G) \text{ and } x_1 = x_2 = \cdots = x_m = 1 \right\}.
\]

**Lemma 5.1.** If \(x = (x_i) \in uV_\infty^\lambda(\Delta^m_G)\), then \(\sup_{\eta \in \mathbb{N}} (e \otimes e^{\lambda \eta}) \circ |\Delta^{-1}_G x_1|^G < \infty\).

**Proof.** Let \(x = (x_i) \in uV_\infty^\lambda(\Delta^m_G)\), then

\[\sup_{\eta \in \mathbb{N}} \left| (e \otimes e^{\lambda \eta}) \circ_G \sum_{k \in I_n} \Delta^m_G x_i \right|^G < \infty. \quad (5.1)
\]

\(I_n = [n - \lambda_n + 1, n], \ n = 1, 2, 3, \ldots\) Consider

\[G \sum_{k \in I_n} \Delta^m_G x_i \leq G \sum_{i=1}^{n} \Delta^m_G (x_i \ominus x_{i+1})
\]

\[= G \sum_{i=1}^{n} (\Delta^{-1}_G x_i \ominus \Delta^{-1}_G x_{i+1})
\]

\[= \Delta^{-1}_G x_1 \ominus \Delta^{-1}_G x_{n+1}
\]

As \(x_1 = x_2 = \cdots = x_m = 1\), we get \(\Delta^{-1}_G x_1 = 1\). Therefore,

\[G \sum_{k \in I_n} \Delta^m_G x_i = 1 \ominus \Delta^{-1}_G x_{n+1}.
\]

Since \(0_G\) is 1 hence we omit it and write

\[G \sum_{k \in I_n} \Delta^m_G x_i = \ominus \Delta^{-1}_G x_{n+1}.
\]

Now substituting this value in (5.1),

\[\sup_{\eta \in \mathbb{N}} \left| (e \otimes e^{\lambda \eta}) \circ (\ominus \Delta^{-1}_G x_{n+1}) \right|^G < \infty.
\]
By general geometric arithmetic property
\[ \leq \sup_{\eta \in \mathbb{N}} \left| (e \otimes e^\lambda) \right|^G \circ \left| \left( \otimes \Delta_G^{m-1} x_{n+1} \right) \right|^G < \infty \]
\[ = \sup_{\eta \in \mathbb{N}} \left| (e \otimes e^\lambda) \right|^G \circ \left| \Delta_G^{m-1} x_{n+1} \right|^G < \infty. \]

Since \((e \otimes e^\lambda) = e^{(\lambda_n)^{-1}} > 1\), so
\[ \sup_{\eta \in \mathbb{N}} e^{(\lambda_n)^{-1}} \circ \left| \Delta_G^{m-1} x_{n+1} \right|^G < \infty. \] (5.2)

Now,
\[ \sup_{\eta \in \mathbb{N}} e^{(\lambda_n)^{-1}} \circ \left| \Delta_G^{m-1} x_n \right|^G \]
\[ = \sup_{\eta \in \mathbb{N}} e^{(1-\frac{1}{\lambda_n})^{-1}} \circ \left| \Delta_G^{m-1} x_n \right|^G \]
\[ \leq \sup_{\eta \in \mathbb{N}} e^{(\lambda_n-1)^{-1}} \circ \left| \Delta_G^{m-1} x_n \right|^G. \]

Replacing the R.H.S. \(n = n + 1\) and comparing it with equation (5.2) we get
\[ \sup_{\eta \in \mathbb{N}} e^{(\lambda_n)^{-1}} \circ \left| \Delta_G^{m-1} x_n \right|^G < \infty. \]

Replacing \(n\) as \(k\) we have
\[ \sup_k e^{(\lambda_k)^{-1}} \circ \left| \Delta_G^{m-1} x_k \right|^G < \infty. \]

\[ \square \]

**Corollary 5.2.** If
\[ \sup_k e^{(\lambda_k)^{-1}} \circ \left| \Delta_G^{m-1} x_k \right|^G < \infty, \]
then
\[ \sup_k e^{(k)^{-m}} \circ |x_k|^G < \infty. \]

**Lemma 5.3.** A sequence \(x = (x_i) \in uV_\infty^\lambda(\Delta_G^m)\), then it gives
\[ \sup_k e^{(\lambda_k)^{-m}} \circ |x_i|^G < \infty. \]

**Proof.** Let us consider \(x = (x_i) \in uV_\infty^\lambda(\Delta_G^m)\), then from Lemma(5.1)
\[ \sup_k e^{(\lambda_k)^{-1}} \circ \left| \Delta_G^{m-1} x_k \right|^G < \infty, \]
which then by corollary (5.2) gives that
\[ \sup_k e^{(\lambda_k)^{-m}} \circ |x_k|^G < \infty. \]

\[ \square \]

**Theorem 5.4.**
\[ [uV_\infty^\lambda(\Delta_G^m)]^\alpha = \left\{ a = (a_k) : \ G \sum_{k=1}^{\infty} e^{(\lambda_k)^m} \circ |a_k|^G < \infty \right\}. \]
Proof. We consider

\[ W = \left\{ a = (a_k) : G \sum_{k=1}^{\infty} e^{(\lambda_k)^m} \odot |a_k|^G < \infty \right\}. \]

Let \( a = (a_k) \in W \); then for any \( x = (x_i) \in uV_\lambda^G(\Delta_G^m) \), we have

\[
G \sum_{k=1}^{\infty} |a_k \odot x_i|^G = G \sum_{k=1}^{\infty} e^{(\lambda_k)^m} \odot |a_k|^G \odot (e^{(\lambda_k)^m} \odot |x_i|^G)
\leq \sup_{k \in \mathbb{N}} (e^{(\lambda_k)^m} |x_i|^G) \odot G \sum_{k=1}^{\infty} e^{(\lambda_k)^m} \odot |a_k|^G
\]

By Lemma (5.1) this is finite. Also \( x = (x_i) \) is arbitrary, we have \( a = (a_k) \in [uV_\lambda^G(\Delta_G^m)]^\alpha \).

Hence

\[ W \subseteq [uV_\lambda^G(\Delta_G^m)]^\alpha. \tag{5.3} \]

Conversely, let \( a \in [uV_\lambda^G(\Delta_G^m)]^\alpha \).

\[ G \sum_{k=1}^{\infty} |a_k \odot x_i|^G \]

is less than infinite for every \( x = (x_i) \in uV_\lambda^G(\Delta_G^m) \). Now, look about the sequence \( x = (x_i) \) that is defined by

\[ x_i = \begin{cases} 1, & (k \leq m), \\ e^{(\lambda_k)^m}, & (k > m), \end{cases} \tag{5.4} \]

This is \( \in uV_\lambda^G(\Delta_G^m) \). Hence,

\[ G \sum_{k=m+1}^{\infty} |a_k \odot (e^{(\lambda_k)^m}|^G < \infty. \]

Now,

\[
G \sum_{k=1}^{\infty} |a_k \odot (e^{(\lambda_k)^m}|^G
= G \sum_{k=1}^{m} |a_k \odot (e^{(\lambda_k)^m}|^G \oplus G \sum_{k=m+1}^{\infty} |a_k \odot (e^{(\lambda_k)^m}|^G < \infty
\]

Hence \( a \in W \). Thus,

\[ [uV_\lambda^G(\Delta_G^m)]^\alpha \subseteq W. \tag{5.5} \]

From (5.3) and (5.5)

\[ W = [uV_\lambda^G(\Delta_G^m)]^\alpha \]

\[ \square \]

Theorem 5.5. \([V_\lambda^G(\Delta_G^m)]^\alpha = [uV_\lambda^G(\Delta_G^m)]^\alpha\)
Proof. \([uV^\lambda_\infty(\Delta^m_G)]^\alpha \subset [V^\lambda_\infty(\Delta^m_G)]^\alpha\), so we have
\[ [V^\lambda_\infty(\Delta^m_G)]^\alpha \subseteq [uV^\lambda_\infty(\Delta^m_G)]^\alpha \] (5.6)
Conversely let \(a = (a_k) \in [uV^\lambda_\infty(\Delta^m_G)]^\alpha\),
\[ \mathcal{G} \sum_{k=1}^{\infty} |a_k \odot x_i|^G \text{is less than finite} \]
for each \(x = (x_i) \in uV^\lambda_\infty(\Delta^m_G)\).
Now take any sequence \(x' = (x'_i) \in V^\lambda_\infty(\Delta^m_G)\), the following sequence \((1, 1, 1, \ldots, 1, x'_{m+1}, x'_{m+2}, \ldots) \in [uV^\lambda_\infty(\Delta^m_G)]^\alpha\) and
\[ \mathcal{G} \sum_{k=m+1}^{\infty} |a_k \odot x'_i|^G < \infty. \]
Now,
\[ \mathcal{G} \sum_{k=1}^{\infty} |a_k \odot x'_i|^G \]
\[ = \mathcal{G} \sum_{k=1}^{m} |a_k \odot x'_i|^G + \mathcal{G} \sum_{k=m+1}^{\infty} |a_k \odot x'_i|^G < \infty \]
\(\forall \ x = (x'_i) \in V^\lambda_\infty(\Delta^m_G)\). Therefore, the sequence \(a = (a_k) \in [V^\lambda_\infty(\Delta^m_G)]^\alpha\) and hence
\[ [uV^\lambda_\infty(\Delta^m_G)]^\alpha \subseteq [V^\lambda_\infty(\Delta^m_G)]^\alpha \] (5.7)
Hence from equations (5.6) and (5.7) we obtain our required result. \(\blacksquare\)

References
[1] D., Aniszewska, (2007): Multiplicative Runge Kutta methods, Nonlinear Dyn., 50, 265-272.
[2] Bashirov, A. E., Kurpınar E. M. and Ozyapıcı, A., (2008): Multiplicative Calculus and its applications, J. Math. Anal. Appl., 337, 36-48.
[3] Bakery, A.A. (2013): Mappings of type generalized de La Vallée Poussin’s mean, J. Inequal. Appl., 2013 (1).
[4] Bashirov, A. E. and Rıza,M., (2011): On Complex multiplicative differentiation, TWMS J. App. Eng. Math., 1(1), 75-85.
[5] Campbell, D. , (1999) : Multiplicative calculus and student projects, Primus 9 (4).
[6] Connor J., (1988): The statistical and strong p-Cesàro convergence of sequences, Analysis, 8, 47-63.
[7] Et, M. and Çolak, R. (1995): On some generalized difference sequence spaces, Soochow J. Math., 21 (4), 377-386.
[8] Et, M. and Nuray, F. (2001): \(\Delta^m\)-statistical convergence,Indian J. Pure Appl. Math.,32, 961-969.
[9] Et, M., Altin, Y. and Altınok, H., (2003): On some generalized difference sequence spaces defined by a Modulus function, Filomat, 17, 23-33.
[10] Fast H.(1951): Sur la convergence statistique, Colloq. Math., 2, 241-244.
[11] Florack, L. and Assen,H. (2012): Multiplicative calculus in biomedical image analysis, J. Math. Imaging Vis. 42 64–75.
[12] Grossman, M. ,Katz, R. (1972): Non-Newtonian Calculus, Lee Press, City place Piegov Cove, State Massachusetts.
[13] Kızmaz H. (1981): On certain sequence spaces, Canad. Math. Bull., 24, 169-176.
[14] Boruah, K., Hazarika, B., (2017) Application of geometric calculus in numerical analysis and difference sequence spaces, Journal of Mathematical Analysis and Applications, 449, 1265-1285.
[15] Boruah, K., Hazarika, B., (2017) On some generalized geometric difference sequence spaces, Proyecciones Journal of Mathematics, 36(3), 373-395.
[16] Boruah, K., Hazarika, B., (2018) G-CALCULUS, TWMS J. App. Eng. Math., 8(1), 94-105.
[17] Boruah, K., Hazarika, B., (2018) BIGEOMETRIC INTEGRAL CALCULUS, TWMS J. App. Eng. Math., 8(2), 374-385.
[18] Boruah, K., Hazarika, B., (2020) Some basic properties of bigeometric calculus and its applications in numerical analysis, Afrika Matematika, https://doi.org/10.1007/s13370-020-00821-1.
[19] Boruah, K., Hazarika, B., (2021) Solvability of Bigeometric Differential Equations by Numerical Methods, Bol. Soc. Paran. Mat., 39(2), 203-222.
[20] Mursaleen, M. (2000): λ-statistical convergence, Math. Slovaca, 50, 111-115.
[21] Shiue, J.S., (1970): On the Césaro sequence spaces, Tamkang J. Math., 1, 19-25.
[22] Shiue, J.S., (1970): A note on Césaro function spaces, Tamkang J. Math., 1, 91-95.
[23] Singh, S. and Dutta, S. (2019): Geometric zweier convergent lacunary sequence spaces, iJET, 10(2b), 126-129.
[24] Singh, S. and Dutta, S. (2020): Classes of new geometric difference sequence spaces, Test Engineering and Management, 83 (January - February 2020), 14358-14364.
[25] Singh, S. and Dutta, S. (2020): On some fractional order Binomial sequence spaces with infinite matrices, arXiv:2012.07364 [math.FA].
[26] Singh, S., Dutta, S., Dash, D. and Sharma, R., (2021): Strongly summable Fibonacci Difference Geometric Sequences defined by Orlicz functions, GANITA, 71(2), 99-109.
[27] Stanley, D. (1999): A multiplicative calculus, Primus, IX(4), 310-326.
[28] Türkmen, C. and Başar, F., (2012): Some Basic Results on the sets of Sequences with Geometric Calculus, Commun. Fac. Sci. Univ. Ank. Series A1, 81(2), 17-34.
[29] Lindenstrauss, J. and Tzafriri, L. (1971): On Orlicz sequence spaces, Israel J. Math., 10(3), 379-390.

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