STRATIFYING ENDSMORPHISM ALGEBRAS USING EXACT CATEGORIES

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We dedicate this paper to the memory of J.A. Green

ABSTRACT. This paper constructs enlargements of Hecke algebras over \( \mathbb{Z}[t, t^{-1}] \) to certain standardly stratified algebras. The latter are obtained as endomorphism algebras of modules with dual left cell module filtrations in the sense of Kazhdan-Lusztig. A novel feature of the proofs is the use of suitably chosen exact categories to avoid difficult \( \text{Ext}^1 \)-vanishing conditions.

1. INTRODUCTION

This paper is the second in a series aimed at proving versions of a conjecture made by the authors in 1996. The conjecture concerns the enlargement, in a framework involving Kazhdan-Lusztig cell theory, of those Hecke endomorphism algebras which occur naturally in the cross characteristic representation theory of finite groups of Lie type. See [DPS98] for the original version of the conjecture, and [DPS15] for a reformulation.

The [DPS98] conjecture is set in the context of a Hecke algebra \( \mathcal{H} \) for a finite Weyl group, using the dual left cell modules \( S_\omega, \omega \in \Omega \), in the sense of [Lu03]. (Thus, each \( S_\omega \) is a right \( \mathcal{H} \)-module.) The base ring (in [DPS15]) is \( \mathbb{Z}[t, t^{-1}] \), where \( t \) is an indeterminate. One of the conjecture’s implications is that there is a faithful right \( \mathcal{H} \)-module \( T^\dagger \), filtered by various \( S_\omega \), such that the modules \( \Delta(\omega) := \text{Hom}_\mathcal{H}(S_\omega, T^\dagger) \), with \( \omega \in \Omega \), form a stratifying system (in the sense of [DPS98]) for the endomorphism algebra \( A^\dagger := \text{End}_\mathcal{H}(T^\dagger) \). Using exact category methods, we are able to prove this statement. See Theorem 4.9 below.

A strength of the “stratifying system” construction is that it is well-behaved under base change, so that the resulting algebra \( A^\dagger \otimes k \) inherits a stratification from that of \( A^\dagger \) over any Noetherian commutative ring or field \( k \) in which \( t \) is specialized to an invertible element.

The endomorphism algebras \( A^\dagger \) constructed here have other good properties. In particular, based changed versions \( \tilde{A}^\dagger, \tilde{T}^\dagger \) can be shown to satisfy the particular “cyclotomic” local versions of the conjecture which were treated in [DPS15, Theorem 5.6], using results of [GGOR03] on the module categories \( \mathcal{O} \) for rational Cherednik algebras. The present paper raises the possibility that the [DPS98] conjecture can be proved directly within the global framework of \( \mathbb{Z}[t, t^{-1}] \)-algebras and modules, perhaps with the present \( A^\dagger \), or a close variation.

The authors began developing a general theory in [DPS98] for constructing the required enlarged algebras, centered around a set of requirements contained in what we call the “stratification hypothesis.” The most difficult condition to verify in this hypothesis is an \( \text{Ext}^1 \)-vanishing requirement for some of the modules involved. The present paper takes a novel approach to this problem by building new exact categories containing the relevant modules, effectively making the \( \text{Ext}^1 \)-groups
involved smaller and better behaved. While there are Specht modules and analogues for all finite Weyl groups, there are no troublesome self-extensions, or extensions in the “wrong order,” because of the exact structure we construct. As a result, many issues of “bad characteristic” do not arise.

The present paper also contains new results on exact category constructions. In particular, the main Lemma 3.1 gives a very general construction in an abstract setting. It very quickly leads to new exact module categories \((\mathcal{A}, E)\) for algebras \(B\) over Noetherian domains \(\mathcal{K}\), when the \(K\)-algebra \(B_K\) obtained by base change from \(\mathcal{K}\) to its quotient field \(K\) is semisimple. The underlying additive category \(\mathcal{A}\) is the full subcategory of \(B\)-mod consisting of all modules which are finitely generated and torsion-free over \(\mathcal{K}\). The “exact sequences” in \(E\) are required to be exact on certain filtrations; see Construction 3.5. Both this construction and that of Lemma 3.1 apply to all standard axiom systems for exact categories. We use the Quillen axiom system \([Q73]\), \([K90]\) which, in particular, does not require that “idempotents split”. This generality is especially useful when using cell modules, whose direct summands may not be cell modules, and a further Construction 3.8 exploiting this flexibility, leads to the main theorem.

2. Stratifying algebras and exact categories

Throughout this section, let \(\mathcal{K}\) be a fixed Noetherian commutative ring. Often \(\mathcal{K}\) will also be a domain. Later, in the main application to Hecke algebras, \(\mathcal{K}\) will be the ring \(\mathbb{Z}[t, t^{-1}]\) of Laurent polynomials in a variable \(t\). A \(\mathcal{K}\)-module \(V\) is called finite if it is finitely generated as a \(\mathcal{K}\)-module.

By a quasi-poset, we mean a (usually finite) set \(\Lambda\) with a transitive and reflexive relation \(\leq\). An equivalence relation \(\sim\) is defined on \(\Lambda\) by putting \(\lambda \sim \mu\) if and only if \(\lambda \leq \mu\) and \(\mu \leq \lambda\). Let \(\bar{\Lambda}\) be the equivalence class containing \(\lambda \in \Lambda\). Of course, \(\bar{\Lambda}\) inherits a poset structure.

2.1 Stratifying systems. We will briefly review the notion of a (strict) stratifying system\(^1\) for a finite \(\mathcal{K}\)-algebra \(A\) and a quasi-poset \(\Lambda\). Assume that \(A\) is projective over \(\mathcal{K}\). For \(\lambda \in \Lambda\), we require a finitely generated \(A\)-module \(\Delta(\lambda)\), projective as a \(\mathcal{K}\)-module\(^2\) and a finitely generated, projective \(A\)-module \(P(\lambda)\), together with an epimorphism \(P(\lambda) \to \Delta(\lambda)\). The following conditions are assumed to hold:

(SS1) For \(\lambda, \mu \in \Lambda\),

\[\text{Hom}_A(P(\lambda), \Delta(\mu)) \neq 0 \implies \lambda \leq \mu.\]

(SS2) Every irreducible \(A\)-module \(L\) is a homomorphic image of some \(\Delta(\lambda)\).

\(^1\)Contrary to popular beliefs, the notion of an “exact category” is not exactly well-defined. There are at least three axiom systems, all quite useful. The weakest set of axioms is that of Quillen \([Q73]\), as reduced to a smaller set by Keller \([K90]\). See our Appendix A. Then there is the axiom system of Gabriel-Roiter \([GR97]\). Keller shows in the appendix to \([DRSS99]\) that this set is equivalent to that of Quillen after adding the additional condition that retraction have kernels. This axiom set is generally easier to use for producing new exact sequences from others, but the retraction axiom may be hard to verify in integral settings, or simply is not true. It is implied by the stronger, yet simpler requirement, that all idempotents split. (An idempotent \(e : A \to A\) in an additive category is called split, if \(e\) can be factored as \(e = \alpha \beta\), \(\alpha : B \to A\) and \(\beta : A \to B\), where \(\beta \alpha = 1_B\), i.e., \(\beta\) is a retraction.) The latter has several applications, including a six term “long exact sequence” for Hom and Ext\(^1\) in \([DRSS99]\), and it is used by Neeman \([Ne90]\) to build derived categories. But in the context of the proof of Theorem 4.9 below, our main result, idempotents do not split. For a discussion of derived categories in the Quillen framework, see \([K90]\).

\(^2\)In \([DPS98]\), these systems were called strict stratifying systems. In this paper, we drop the word “strict” and do not consider more general systems. (The more general stratifying systems in \([DPS98]\) allowed \(\bar{\mu} \geq \lambda\) in condition (SS3).)

This condition was inadvertently omitted in the discussion \([DPS15]\) p. 231 but explicitly assumed in \([DPS15]\) Thm. 1.1] and in the original definition of a stratifying system \([DPS98]\) Defn. 1.2.4].
(SS3) For $\lambda \in \Lambda$, the $A$-module $P(\lambda)$ has a finite filtration by $A$-submodules with top section $\Delta(\lambda)$ and other sections of the form $\Delta(\mu)$ with $\mu > \lambda$.

When these conditions all hold, the data consisting of the $\Delta(\lambda)$, $P(\lambda)$, etc. form (by definition) a stratifying system for the category $A$-mod of finitely generated $A$-modules. It is also clear that $\Delta(\lambda),P(\lambda),\ldots$ is a stratifying system for $A_{K'}$-mod for any base change $K' \rightarrow K$, provided $K'$ is a Noetherian commutative ring. (Notice that condition (SS2) is redundant, if it is known that the direct sum of the projective modules in (SS3) is a progenerator—a property preserved by base change.)

An ideal $J$ in the $K$-algebra $A$ above is called a stratifying ideal provided that the inclusion $J \rightarrow A$ is $K$-split (or, equivalently, $A/J$ is $K$-projective) and, for $M,N \in A/J$-mod, inflation from $A/J$ to $A$ defines an isomorphism
\[
\text{Ext}_{A/J}^n(M,N) \cong \text{Ext}_A^n(M,N), \quad \forall n \geq 0
\]
of Ext-groups.

A standard stratification of length $n$ of $A$ is a sequence $0 = J_0 \subset J_1 \subset \cdots \subset J_n = A$ of stratifying ideals of $A$ such that each $J_i/J_{i-1}$ is a projective $A/J_{i-1}$-module. If $A$-mod has a stratifying system with quasi-poset $\Lambda$, then it has a standard stratification of length $n = |\Lambda|$; see [DPS98, Thm. 1.2.8].

**Lemma 2.1.** Suppose that $A$ has a stratifying system as above. Let $\lambda,\mu \in \Lambda$. Then
\[
\text{Ext}_A^n(\Delta(\lambda),\Delta(\mu)) \neq 0 \iff \lambda < \mu.
\]

**Proof.** Assume that $\lambda \neq \mu$, and let $Q(\lambda)$ be the kernel of the given epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$. Then $\text{Ext}_A^n(\Delta(\lambda),\Delta(\mu))$ is homomorphic image of $	ext{Hom}_A(Q(\lambda),\Delta(\mu))$. But $Q(\lambda)$ has a filtration with sections of the form $\Delta(\tau)$ for $\tau \geq \lambda$, so that $\text{Hom}_A(\Delta(\tau),\Delta(\mu))) = 0$ since $\tau \leq \mu$. \hfill $\square$

Given a finite quasi-poset $\Lambda$, a height function on $\Lambda$ is a mapping $ht : \Lambda \rightarrow \mathbb{Z}$ with the properties that $\lambda < \mu \implies ht(\lambda) < ht(\mu)$ and $\lambda = \bar{\mu} \implies ht(\lambda) = ht(\mu)$. Given $\lambda \in \Lambda$, a sequence $\lambda = \lambda_0 > \lambda_1 > \cdots > \lambda_0$ is called a chain of length $n$ starting at $\lambda = \lambda_n$. Then the standard height function $ht : \Lambda \rightarrow \mathbb{N}$ is defined by setting $ht(\lambda)$ to be the maximal length of a chain beginning at $\lambda$.

Given $A$-modules $X,Y$, recall that the trace module $\text{trace}_X(Y)$ of $Y$ in $X$ is the submodule of $X$ generated by the images of all morphisms $Y \rightarrow X$.

**Proposition 2.2.** Suppose that $A$ has a stratifying system as above, and let $ht : \Lambda \rightarrow \mathbb{Z}$ be a height function. Let $\lambda \in \Lambda$. Then the $\Delta$-sections arising from the filtration (SS3) of $P(\lambda)$ can be reordered (constructively, as in the proof below) so that, if we set
\[
P(\lambda)_j = \text{trace}_{P(\lambda)}\left( \bigoplus_{ht(\mu) \geq j} P(\mu) \right),
\]
then $P(\lambda)_{j+1} \subset P(\lambda)_j$, for $j \in \mathbb{Z}$, and
\[
P(\lambda)_j / P(\lambda)_{j+1}
\]
is a direct sum of modules $\Delta(\mu)$ satisfying $ht(\mu) = j$.

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4In particular, the $n = 1$ case implies that $J^2 = J$, see [CPS90]. If an ideal $J$ is known to be projective as an $A$-module, then $J^2 = J$ implies (1.1); see Appendix B.

5The word “stratifying” may be replaced by “idempotent”, given the projectivity assumption on $J_i/J_{i-1}$. See fn. 4. This is the more usual definition of a standard stratification [DPS98], but not our focus here.
Proof. First, fix \( j \) maximal with a section \( \Delta(\mu) \) appearing in \( P(\lambda) \) such that \( ht(\mu) = j \). Lemma \( 2.1 \) implies that, whenever \( M \) is a module with a submodule \( D \cong \Delta(\nu) \) and \( M/D \cong \Delta(\mu) \), with \( \mu, \nu \in \Lambda \) and \( ht(\nu) \leq ht(\mu) \), then \( M \) is the direct sum of \( D \) and a submodule \( E \cong \Delta(\mu) \). Of course the quotient \( M/E \) is isomorphic to \( D \). This interchange of \( E \) with \( D \) can be repeatedly applied to adjacent \( \Delta \)-sections in a filtration (SS3) of \( P(\lambda) \) to construct a submodule \( P(\lambda)(j) \), a term in a modified filtration, which is filtered by modules \( \Delta(\nu) \) with \( ht(\nu) = j \), and \( P(\lambda)/P(\lambda)(j) \) filtered by modules \( \Delta(\nu) \) with \( ht(\nu) < j \). Axiom (SS1) clearly gives \( P(\lambda)(j) = P(\lambda)_j \), and \( P(\lambda)_j + 1 = 0 \). Clearly, \( P(\lambda)_j / P(\lambda)_{j+1} \) is a direct sum as required by the proposition. We have not used projectivity of \( P(\lambda) \), only its filtration properties. Induction applied to the quotient module \( P(\lambda)/P(\lambda)(j) \) completes the proof. \( \square \)

Remark 2.3. The proposition above shows that the projective modules have a canonically described filtration, given any height function \( ht \). This suggests that, if \( A \) is to be realized as an endomorphism algebra of a given module, that module might also reflect that filtration in a canonical way. In §§3,4, this is successfully approached using semisimple base change and exact categories. The latter also builds in a height function version of the vanishing condition in Lemma \( 2.1 \).

The proposition can also be used, in conjunction with Lemma \( 2.4 \) below, to build stratifying ideals in an algebra Morita equivalent to \( A \), and then in \( A \). See [DPS98, Lem. 1.2.7, Thm. 1.2.8]. We will not need to return to this in this paper.

Lemma 2.4. Suppose \( A \) has a stratifying system as above. Then

\[
P := \bigoplus_{\lambda \in \Lambda} P(\lambda)
\]

is a projective generator for \( A\)-mod.

Proof. Obvious from (SS2) and (SS3). \( \square \)

2.2 Exact categories and the stratification hypothesis. This section provides a way to construct stratifying systems in a setting involving exact categories. Previously, the construction was based on assuming a “stratification hypothesis” in [DPS98, Hyp. 1.2.9, Thm. 1.2.10]. The method required a difficult \( \text{Ext}^1 \)-vanishing condition (see [DPS98, Thms. 2.3.9, 2.4.4]). The advantage of the exact category approach is that the relevant \( \text{Ext}^1 \)-vanishing conditions involve smaller spaces (and so are hopefully easier to make vanish).

Let \( (\mathcal{A}, \mathcal{E}) \) be exact category in the sense of Quillen [Q73], as discussed in Appendix A using axioms of Keller [K90]. In particular, \( \mathcal{A} \) is an additive category and \( \mathcal{E} \) is a class of sequences \( X \rightarrow Y \rightarrow Z \) satisfying certain properties. In the hypotheses below we will assume the more explicit setup in which \( \mathcal{A} \) is an additive full subcategory of \( \text{mod} - B \) where \( B \) is a finite and projective \( K \)-algebra. The sequences \( X \rightarrow Y \rightarrow Z \in \mathcal{E} \) are among the short exact sequences \( 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \) in \( \text{mod} - B \). Thus, \( \mathcal{A} \) is an “exact subcategory” of \( \text{mod} - B \). Note, however, we do not assume that all exact sequences in \( \text{mod} - B \) whose object terms lie in \( \mathcal{A} \) necessarily belong to \( \mathcal{E} \).

Next, we discuss the variation of the stratification hypothesis based on the notion of an exact category. First, there are several preliminary assumptions.
Assume there is given a collection of objects $S_\lambda \in \mathcal{A}$ indexed by the elements $\lambda$ of a finite quasi-poset $\Lambda$. For each $\lambda \in \Lambda$, $S_\lambda$ is a subobject of $T_\lambda \in \mathcal{A}$. Write

$$T := \bigoplus_{\lambda \in \Lambda} T_\lambda \in \mathcal{A}.$$  

With this notation, the following statements make up a straightforward version of the “stratification hypothesis” in an exact categorical context.

**Hypothesis 2.5.** The stratification hypothesis holds in $(\mathcal{A}, \mathcal{E})$ provided the following statements hold.

1. For $\lambda \in \Lambda$, there is a fixed sequence $\nu_{\lambda,0}, \cdots, \nu_{\lambda,l(\lambda)}$ where $l(\lambda) \geq 0$, $\nu_{\lambda,0} = \lambda$, and $\nu_{\lambda,i} > \lambda$ for each $i > 0$. Also, there is an increasing filtration

$$0 = F_{\lambda}^{-1} \subseteq F_{\lambda}^0 \subseteq \cdots \subseteq F_{\lambda}^{l(\lambda)} = T_\lambda$$

such that each inclusion $F_{\lambda}^{i-1} \subseteq F_{\lambda}^i$ is an inflation such that

$$F_{\lambda}^i / F_{\lambda}^{i-1} \cong S_{\nu_{\lambda,i}}$$

for $0 \leq i \leq l(\lambda)$.

2. For $\lambda, \mu \in \Lambda$, $\text{Hom}_\mathcal{A}(S_\mu, T_\lambda) \neq 0 \implies \lambda \leq \mu$.

3. For all $\lambda \in \Lambda$, $\text{Ext}^1_E(T_\lambda / F_{\lambda}^i, T) = 0$, $\forall i \geq 0$. (See Appendix A for a definition of $\text{Ext}^1_E$.)

The proof of the following result parallels the analogous result in [DPS98, Thm. 1.2.10], using Proposition A.2(a).

**Theorem 2.6.** Let $\mathcal{A}, \mathcal{E}, B, T$ be as above. (In particular, $\mathcal{A}$ is an additive full subcategory of $\text{mod–}B$.) Assume that Hypothesis 2.5 holds in $(\mathcal{A}, \mathcal{E})$. Put

$$A^+ = \text{End}_B(T)$$

and, for $\lambda \in \Lambda$, define $\Delta(\lambda) := \text{Hom}_B(S_\lambda, T) \in A^+\text{-mod}$. Assume that each $\Delta(\lambda)$ is $\mathcal{K}$-projective. Then $\{\Delta(\lambda)\}_{\lambda \in \Lambda}$ is a stratifying system for $A^+\text{-mod}$. 

**Remark 2.7.** The main function of condition (3) in Hypothesis 2.5 in proving Theorem 2.6 is to ensure the existence of various exact sequences when $\text{Hom}_\mathcal{A}(-, T)$ is applied. This exactness still works and Theorem 2.6 still holds if $S_\lambda$ is used in place of $T_\lambda / F_{\lambda}^i$, at least for the exact categories we use. For one precise formulation, see Lemma 3.10 below. This discussion is necessary when using the Quillen axiom system. In the idempotent split context studied in [DRSS99], the functor $\text{Ext}^1_E$ is half-exact in each variable; see [DRSS99, Thm. 1.3], who quote arguments of [BH61, Thm. 1.1]. In this case, the original version of condition (3) holds as written when all the $\text{Ext}^1_E(S_\lambda, T)$ vanish. Finally, another useful modification of Hypothesis 2.5 (1) is obtained by replacing $S_{\nu_{\lambda,i}}, i \geq 1$, by the direct sum of such objects, all with $i \geq 1$. Again, Theorem 2.6 holds with essentially the same arguments.

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*In an (abstract) exact category setting, $F_{\lambda}^{i-1} \subseteq F_{\lambda}^i$ might be taken as a notation for a monomorphism $F_{\lambda}^{i-1} \rightarrow F_{\lambda}^i$. In the case above, we intended that the sequence $F_{\lambda}^{i-1} \rightarrow F_{\lambda}^i \rightarrow F_{\lambda}^i / F_{\lambda}^{i-1}$ belongs to $\mathcal{E}$. 

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3. SOME CONSTRUCTIONS OF EXACT CATEGORIES

Let \((\mathcal{A}, \mathcal{E})\) be an exact category in the sense of Quillen \([Q73]\); see Appendix A. Suppose that \(\mathcal{E}\) is a given abelian category, and let \(F : \mathcal{A} \to \mathcal{E}\) be an additive functor. Then \(F\) is called \(\mathcal{E}\)-exact (resp., left \(\mathcal{E}\)-exact) if given any \((X \to Y \to Z) \in \mathcal{E}\), the sequence \(0 \to F(X) \to F(Y) \to F(Z) \to 0\) (resp., \(0 \to F(X) \to F(Y)\)) is exact in \(\mathcal{E}\).

**Lemma 3.1.** Let \(\mathcal{E}\) be an abelian category.\(^7\) Also, let \((\mathcal{A}, \mathcal{E}')\) be an exact category and let \(F : \mathcal{A} \to \mathcal{E}\) be a left \(\mathcal{E}'\)-exact, additive functor. Define \(\mathcal{E}\) to be the class of those \((X \to Y \to Z) \in \mathcal{E}'\) such that \(0 \to F(X) \to F(Y) \to F(Z) \to 0\) is exact in \(\mathcal{E}\). Then \((\mathcal{A}, \mathcal{E})\) is an exact category.

**Proof.** First, since \(F\) is left \(\mathcal{E}'\)-exact, \(\mathcal{E}\) can also be described as the class of all \((X \to Y \to Z) \in \mathcal{E}'\) such that \(F(Y) \to F(Z)\) an epimorphism in \(\mathcal{E}\). Axioms 0, 1 in Appendix A are immediate. Consider Axiom 2 and the following commutative diagram in \(\mathcal{A}\)

\[
\begin{array}{ccc}
X & \longrightarrow & Y' \\
\downarrow & & \downarrow_{f'} \\
X & \longrightarrow & Y
\end{array}
\]

\[
\begin{array}{ccc}
& & \longrightarrow \downarrow_{f} \\
& & Z' \\
Y' & \longrightarrow & Z' \oplus Y
\end{array}
\]

in which the bottom row belongs to \(\mathcal{E}\), so that the sequence is \(\mathcal{E}'\)-exact (in \(\mathcal{E}'\) and \(F(d) : F(Y) \to F(Z)\) is an epimorphism), and the top row is the pullback of the bottom row (through the map \(f\)).

The object \(Y'\) is identified as the kernel of the epimorphism \((-f, d) : Z' \oplus Y \to Z\) in the bottom row of the commutative diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & Z \\
\downarrow_{0} & & \downarrow_{1_{Y}} \\
Y' & \longrightarrow & Z' \oplus Y
\end{array}
\]

The bottom row \(Y' \to Z' \oplus Y \xrightarrow{(-f, d)} Z\) is isomorphic to \(Y' \xrightarrow{(-f', d)} Z' \oplus Y \xrightarrow{(f, d)} Z\), which is shown in \([K90]\) p. 406 to belong to \(\mathcal{E}'\). (See also Remark \([A.1]\) in Appendix A below for an alternate argument.) Now apply the functor \(F\), and use the natural isomorphism \(F(Z' \oplus Y) \cong F(Z') \oplus F(Y)\) to obtain the following commutative diagram

\[
\begin{array}{ccc}
F(Y) & \longrightarrow & F(Z) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & F(Y') \longrightarrow F(Z') \oplus F(Y)
\end{array}
\]

\[
\begin{array}{ccc}
& & \longrightarrow \downarrow_{F(f), F(d)} \\
& & F(Z) \longrightarrow 0
\end{array}
\]

\(^7\)After an earlier posting of this paper, T. Bühler drew our attention to Exercise 5.5 in \([B010]\), which he credits to M. Künzer. This exercise is similar to Lemma 3.1. However, while it uses a general exact category as functor target, it does require an apparently stronger “admissible kernel preserving” hypothesis. Indeed, in the case of an abelian category functor target, the hypothesis of Exercise 5.5 implies the hypotheses of Lemma 3.1. However, we do not know if there is a converse implication. The conclusions are the same for both assertions.
As noted above, the morphism $F(d)$ is an epimorphism. Thus, since $F$ is left exact, the bottom row is exact, and it identifies $F(Y')$ as the pullback in the abelian category $\mathcal{C}$ of $F(f)$ and $F(d)$. Since $F(d)$ is an epimorphism, so is its pullback $F(d')$. This verifies Axiom 2.

Finally, we must check that Axiom $2^c$ holds. Consider a commutative pushout diagram

$$
\begin{array}{c}
0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{d} Z \longrightarrow 0
\end{array}
$$

(3.0.1)

$$
\begin{array}{c}
0 \longrightarrow X' \xrightarrow{i'} Y' \xrightarrow{d'} Z \longrightarrow 0
\end{array}
$$

in which the top row belongs to $\mathcal{E}$. We must prove that $X' \rightarrow Y' \rightarrow Z$ also belongs to $\mathcal{E}$. But the diagram (3.0.1) gives the following commutative diagram

$$
\begin{array}{c}
Y' \xrightarrow{\sim} Y
\end{array}
$$

(3.0.2)

$$
\begin{array}{c}
h \downarrow \quad d \downarrow
\end{array}
$$

After applying $F$, we get the following commutative diagram

$$
\begin{array}{c}
F(Y) \xrightarrow{\sim} F(Y)
\end{array}
$$

$$
\begin{array}{c}
F(h) \downarrow \quad F(d) \downarrow
\end{array}
$$

in which $F(d)$ is an epimorphism, since the top row of (3.0.1) belongs to $\mathcal{E}$. This implies that $F(d')$ is an epimorphism, and, hence, the bottom row of (3.0.2) is exact in $\mathcal{C}$. Thus, the bottom row of (3.0.1) belongs to $\mathcal{E}$, and Axiom $2^c$ holds, completing the proof of the lemma. □

We now make some assumptions which will often be in force for the rest of this paper.

**Assumptions 3.2.** Let $\mathcal{K}$ be a fixed Noetherian integral domain with fraction field $K$. Let $H$ be $\mathcal{K}$-algebra which is finite and torsion-free as a $\mathcal{K}$-module. Assume that $H_K$ is semisimple. The isomorphism classes of irreducible right $H_K$-modules are indexed by a finite set $\Lambda$. Given $\lambda \in \Lambda$, let $E\lambda$ denote a representative from the corresponding irreducible class. Fix a function $ht : \Lambda \rightarrow \mathbb{Z}$, taking, for convenience, non-negative values. (We call $ht$ a height function, though there is no immediate assumption that $\Lambda$ is a quasi-poset.)

Let mod-$H$ be the category of $\mathcal{K}$-finite right $H$-modules, and let mod–$H_K$ be category of finite dimensional right $H_K$-modules. Let $\mathcal{A}$ be the full subcategory of mod–$H$ which consists of $\mathcal{K}$-torsion-free $H$-modules.

For $N \in \text{mod–}H_K$, the height function $ht$ induces a natural increasing (finite) filtration

$$
0 = N^{i-1} \subseteq N^0 \subseteq \cdots \subseteq N^i \subseteq N^{i+1} \subseteq \cdots \subseteq N,
$$

defining $N^i$ to be the sum of all irreducible right $H_K$-submodules isomorphic to $E\lambda$ with $ht(\lambda) \leq i$. (Thus, $N^j = N$ for all $j \geq |\Lambda|$.) Then, if $M \in \mathcal{A}$, there is an induced filtration

$$
0 = M^{-1} \subseteq M^0 \subseteq \cdots \subseteq M^i \subseteq M^{i+1} \subseteq \cdots \subseteq M
$$
on $M$ defined by setting

$$
M^i = M \cap (M_K)^i, \quad i \geq 0.
$$
Observe that each $M^i \in \mathcal{A}$, as are the modules $M/M^i$ and $M^i/M^{i-1}$. Also, $(M^i/M^{i-1})_K$ is a direct sum of $H_K$-modules $E_\lambda$ with $\text{ht}(\lambda) = i$.

Our goal is to show that the above data define the structure of an exact category on the additive category $\mathcal{A}$ of $\mathcal{K}$-torsion-free right $H$-modules, once an appropriate family $\mathcal{E}$ of conflations $X \to Y \to Z$ has been defined.

First, we require more preliminaries, including the proposition below. Note that if $X \xrightarrow{f} Y$ is a map in $\mathcal{A}$, then $f$ induces a map $f_i : X^i \to Y^i$ and a map $\overline{f}_i : X^i/X^{i-1} \to Y^i/Y^{i-1}$ for each integer $i$. In addition, if $g : Y \to Z$ is another morphism in $\mathcal{A}$, then $(gf)_i = g_if_i$ and $\overline{gf}_i = g_i\overline{f}_i$ for each $i$. Finally, if $f : X \to Y$ is an inclusion $X \subseteq Y$, then

$$X \cap (Y_K)^i = X \cap (X_K)^i = X^i, \quad \forall i.$$  

In the following proposition, we continue to assume that Assumptions\textsuperscript{3.2} are in force.

**Proposition 3.3.** Suppose $X, Y, Z \in \mathcal{A}$ and $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an exact sequence in mod--$H$. Then, for each $i \in \mathbb{Z}$, the following statements hold.

(a) The sequence $0 \to X^i \to Y^i \to Z^i$ is exact in mod--$H$.

(b) The sequence $0 \to X^h \to Y^h \to Z^h \to 0$ is a short exact sequence in mod--$H$, for each $h \leq i$, if and only if

$$0 \to X^j/X^{j-1} \to Y^j/Y^{j-1} \to Z^j/Z^{j-1} \to 0$$

is exact for each $j \leq i$.

(c) The sequence $0 \to X^j/X^{j-1} \to Y^j/Y^{j-1} \to Z^j/Z^{j-1} \to 0$ is a short exact sequence for all $j \leq i$ if and only if $Y^g/Y^{g-1} \to Z^g/Z^{g-1}$ is an epimorphism for all $g \leq i$.

**Proof.** Throughout this proof, the word “exact” means exact in the usual sense in the category of right $H$-- (or possibly $H_K$--) modules.

We first prove (a). Without loss of generality, we can assume that the map $f : X \to Y$ is an inclusion of a submodule. Clearly, each $f_i$ is an inclusion. Also, $g_if_i = (gf)_i = 0$, so that the image of $f_i$ is contained in the kernel of $g_i$. To prove the reverse inclusion, let $y \in \ker g_i$. Thus, $y \in \ker g$, so $y \in X$. But also $y \in Y^i \subseteq (Y_K)^i$. So $y \in X \cap (Y_K)^i = X^i$, as per (3.0.3). This proves (a).

We next prove (b). For every integer $j$, we have a $3 \times 3$ diagram

\[
\begin{array}{ccc}
X^i & \longrightarrow & Y^j \\
\downarrow & & \downarrow \\
X^j & \longrightarrow & Y^j \\
\downarrow & & \downarrow \\
X^j/X^{j-1} & \longrightarrow & Y^j/Y^{j-1} \\
\end{array}
\]

in which the columns are short exact sequences. Then assume that each $0 \to X^h \to Y^h \to Z^h \to 0$ is exact for each $h \leq j$. Then the $3 \times 3$ Lemma [Mac94, p. 49] implies that $0 \to X^j/X^{j-1} \to Y^j/Y^{j-1} \to Z^j/Z^{j-1} \to 0$ is exact for all $j \leq i$.

Conversely, assume that, for any $j \leq i$, the sequence $0 \to X^j/X^{j-1} \to Y^j/Y^{j+1} \to Z^j/Z^{j-1} \to 0$ is exact. By induction, we can assume that $0 \to X^i-1 \to Y^i-1 \to Z^i-1 \to 0$ is exact. In addition, the composition map $X^i \to Y^i \to Z^i$ is zero. Since the top and bottom rows of (3.0.4) are short
exact sequences, [Mac94, Ex. 2, p. 51] implies the middle horizontal line is a short exact sequence, as required.

As for (c), the \( \implies \) direction is obvious. Conversely, it is easy to see that if the maps \( Y^g/Y^g-1 \to Z^h/Z^h-1 \) are epimorphisms for all \( g \leq i \), then each map \( Y^h \to Z^h, h \leq i \), is an epimorphism. Now apply (a) and (b). \( \square \)

In the context of Proposition \( \ref{thm:exactness} \) (b), it is easy to give examples where \( 0 \to X^h \to Y^h \to Z^h \to 0 \) is not a short exact sequence.

**Example 3.4.** Let \( \mathcal{K} = \mathbb{Z} \), and let \( H = \mathbb{Z}C_2 \), where \( C_2 = \{1, s\} \) is the cyclic group of order 2. Let \( S_2 \) be the trivial module for \( H \). It is free of rank 1 over \( \mathbb{Z} \) with basis vector 1. Let \( S_1 \) be the sign module for \( H \), also free of rank 1 with basis vector denoted \( \epsilon \) (so that \( s \cdot \epsilon := -\epsilon \)). Consider the short exact sequence \( 0 \to X^0 \to Y^0 \to Z^0 \to 0 \) of torsion-free \( H \)-modules where \( X = S_2 \), \( Y = H \), and \( Z = S_1 \). Here \( \alpha(1) = 1 + s \), and \( \beta(1) = \beta(s) = -\epsilon \). Assign \( S_{2,\mathbb{Q}} \) height 2 and \( S_{1,\mathbb{Q}} \) height 1, then

\[
\begin{aligned}
X^1 &= S^1_2 = 0; \\
Y^1 &= H^1 = \mathbb{Z}(1 - s); \\
Z^1 &= Z.
\end{aligned}
\]

Then \( \beta(Y^1) = 2\mathbb{Z} \epsilon \), so that \( Y^1 \to Z^1 \) is not surjective. Thus, taking \( h = 1 \), the sequence \( 0 \to X^h \to Y^h \to Z^h \to 0 \) is a short exact sequence. However, with the same height function, but interchanging the roles of \( X \) and \( Z \), the short exact sequence \( 0 \to Z \to Y \to X \to 0 \) (where \( \phi(\epsilon) = 1 - s \), and \( \psi(1) = \psi(s) = 1 \)) has the property that \( 0 \to Z^h \to Y^h \to X^h \to 0 \) is exact for all \( h \). If the height function assignment is reversed, then the sequence \( 0 \to X^h \to Y^h \to Z^h \to 0 \) is also exact for all \( h \).

**Construction 3.5.** Keep Assumptions \( \ref{assum:modlat} \) with \( H, \mathcal{A} \) and \( \text{ht} \) as described there. Now define \( \mathcal{E} \) as follows. A pair \( (\iota, \delta) \) of morphisms \( X \xrightarrow{\iota} Y \) and \( Y \xrightarrow{\delta} Z \) in \( \mathcal{A} \) belongs to \( \mathcal{E} \) if and only if the sequence \( 0 \to X \xrightarrow{\iota} Y \xrightarrow{\delta} Z \to 0 \) and the induced sequences \( 0 \to X^i/X^{i-1} \to Y^i/Y^{i-1} \to Z^i/Z^{i-1} \to 0 \) (\( i \in \mathbb{N} \)) are exact in mod-\( H \).

We note that, by Proposition \( \ref{thm:exactness} \) each sequence \( 0 \to X^i \to Y^i \to Z^i \to 0 \) is also exact. The Example \( \ref{ex:example} \) shows that the height function determines which sequences are exact (i.e., belongs to \( \mathcal{E} \)).

**Theorem 3.6.** The pair \( (\mathcal{A}, \mathcal{E}) \) is an exact category.

**Proof.** First observe that there is the standard exact category \( (\mathcal{A}, \mathcal{E}') \). Here \( \mathcal{E}' \) consists of all exact triples \( X \to Y \to Z \) in \( \text{mod-}H \) with \( X, Y, Z \) objects in \( \mathcal{A} \) (i.e., \( X, Y, Z \) are \( \mathcal{K} \)-torsion-free). Let \( \mathcal{C} \) be the abelian category of right \( H \)-modules (not necessarily finitely generated), and \( F : \mathcal{A} \to \mathcal{C} \) the functor \( FX = \bigoplus_{i \geq 0} X^i \). Then \( F \) is left \( \mathcal{E}' \)-exact, and \( \mathcal{E} \) (as defined in Construction \( \ref{const:construction} \)) consists of precisely those \( (X \to Y \to Z) \in \mathcal{E}' \) for which \( 0 \to F(X) \to F(Y) \to F(Z) \to 0 \) is a short exact sequence in \( \mathcal{C} \). (Apply Proposition \( \ref{thm:exactness} \) (b).) Thus, \( (\mathcal{A}, \mathcal{E}) \) is an exact category by Lemma \( \ref{lem:exactness} \). \( \square \)

**Remark 3.7.** Though the construction of \( (\mathcal{A}, \mathcal{E}) \) requires the tools of exact category theory, they can all be interpreted here in the larger (and more familiar) category mod-\( H \). Similar remarks apply to the second construction below.
Construction 3.8. Keep Assumptions \[\ref{assumptions} \] For each integer \( i \), let \( \mathcal{S}_i \) be a full, additive subcategory of \( \mathcal{A} \) such that if \( S \in \mathcal{S}_i \), then \( S_K \) is a direct sum of irreducible right \( H_K \)-modules having height \( \ell(S_K) \). (If \( i \) is not in the image of the height function, then put \( \mathcal{S}_i := [0] \).) Let \( \mathcal{J} \) be the set-theoretic union of the \( \mathcal{S}_i \). Let \( \mathcal{A}(\mathcal{J}) \) be the full subcategory of \( \mathcal{A} \) above having objects \( M \) satisfying \( M^j/M^{j-1} \in \mathcal{J}_j \) for all \( j \) (or, equivalently, \( M^j/M^{j-1} \) is in \( \mathcal{J} \) for all integers \( j \)).

Let \( \mathcal{E} \) be as in Construction \[\ref{construction} \] Define \( \mathcal{E}(\mathcal{J}) \) to be the class of those conflations \( X \to Y \to Z \) in \( \mathcal{E} \) such that \( X, Y, Z \in \mathcal{A}(\mathcal{J}) \) and with the additional property that, for each integer \( i \),

\[ 0 \to X^i/X^{i-1} \to Y^i/Y^{i-1} \to Z^i/Z^{i-1} \to 0 \]

is a split short exact sequence in \( \text{mod–}H \). (Thus, by definition, \( \mathcal{E}(\mathcal{J}) \subseteq \mathcal{E} \).)

Theorem 3.9. The pair \( (\mathcal{A}(\mathcal{J}), \mathcal{E}(\mathcal{J})) \) is an exact category.

Proof. The first two axioms in Appendix A are easily verified. (Note again that \( \mathcal{E}(\mathcal{J}) \subseteq \mathcal{E} \).) To check Axiom 2, consider the diagram

\[
\begin{array}{ccc}
X & \longrightarrow & Y' \longrightarrow Z' \\
\downarrow & & \downarrow f \\
X & \longrightarrow & Y \longrightarrow Z
\end{array}
\]

where the bottom row is in \( \mathcal{E}(\mathcal{J}) \) and the top row is a pullback (in \( \text{mod–}H \)) with \( Z' \in \mathcal{A}(\mathcal{J}) \). However, since the bottom row lies in \( \mathcal{E} \), we have that \( X' \to Y' \to Z' \) also belongs to \( \mathcal{E} \). The issue is whether it splits section by section (which, in particular, would imply that \( Y' \in \mathcal{A}(\mathcal{J}) \)). This splitting at the section level follows easily from the fact that the pullback of a split short exact sequence is split. A similar argument gives Axiom 2°. \( \square \)

The following lemma shows a common vanishing condition leads to expected exact sequences.

Lemma 3.10. Suppose that \( X \in \mathcal{A}(\mathcal{J}) \) satisfies \( \text{Ext}^1_{\mathcal{E}(\mathcal{J})}(S, X) = 0 \), for all \( S \in \mathcal{J} \). Let \( E \to F \to G \) belong to \( \mathcal{E}(\mathcal{J}) \). Then

\[ 0 \to \text{Hom}_{\mathcal{E}(\mathcal{J})}(G, X) \to \text{Hom}_{\mathcal{E}(\mathcal{J})}(F, X) \to \text{Hom}_{\mathcal{E}(\mathcal{J})}(E, X) \to 0 \]

is a short exact sequence of \( \mathcal{J} \)-modules.

Proof. The lemma is obvious, from Proposition \[\ref{proposition} \]a, when \( F = F^h \) for some \( h \in \mathbb{Z} \) and \( E = F^{h-1} \), since \( G = F^h/F^{h-1} \in \mathcal{J}_h \).

This special case applies to all columns of the commutative diagram, upon applying the functor \( \text{Hom}_{\mathcal{E}(\mathcal{J})}(-, X) \) to the diagram

\[
\begin{array}{ccc}
E^{h-1} & \longrightarrow & F^{h-1} \longrightarrow G^{h-1} \\
\downarrow & & \downarrow & \downarrow \\
E & \longrightarrow & F \longrightarrow G \\
\downarrow & & \downarrow & \downarrow \\
E^{h}/E^{h-1} & \longrightarrow & F^{h}/F^{h-1} \longrightarrow G^{h}/G^{h-1}.
\end{array}
\]

\[8\] We think of \( \mathcal{J}_i \) as a special class of objects in \( \mathcal{A} \); the stated condition on \( S_K \) is necessary, but not always sufficient for membership in \( \mathcal{J}_i \).
Here, $h$ is chosen so that $F = F^h$, and it follows that $E = E^h$ and $G = G^h$. Moreover, we can assume the top row of the resulting diagram is exact by induction (on, say, the number of indices $j$ for which $F_j^j/F_{j-1}^j \neq 0$). Finally, the bottom split row, of course, remains split exact in the new $3 \times 3$ diagram. Since the middle row of the latter satisfies the hypothesis of [Mac94, Ex. 2, p. 51], it defines a short exact sequence. This proves the lemma. □

4. SOME FURTHER RESULTS FOR $(\mathcal{A}(\mathcal{S}), \mathcal{E}(\mathcal{S}))$ AND CONSTRUCTION OF $T^<$

In this section, we consider further the exact category $(\mathcal{A}(\mathcal{S}), \mathcal{E}(\mathcal{S}))$ introduced in Construction 3.8. In particular, Assumptions 3.2 are in force.

**Proposition 4.1.** Let $M, N \in \mathcal{A}(\mathcal{S})$, and let $h$ be any integer.
(a) There is a natural isomorphism $\operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}(N^h, M) \cong \operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}(N^h, M^h)$.
(b) In particular, if $S \in \mathcal{S}_h$, we have $\operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}(S, M) \cong \operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}(S, M^h)$.
(c) Assume that $S \in \mathcal{S}_h$. Suppose that $M = M^h$ and $M^{h-1} = 0$. Then $\operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}(S, M) = 0$.

**Proof.** Without loss of generality, take $N = N^h$ in (a). Obviously, there is a natural transformation 
$$\eta(N, M) : \operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}(N, M) \to \operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}(N, M^h)$$
which sends $(M \to Y \to N) \in \mathcal{E}(\mathcal{S})$ to $(M^h \to Y^h \to N^h) \in \mathcal{E}(\mathcal{S})$. The inverse is obtained by pushout. This proves (a), and (b) follows. Finally, (c) follows immediately from the definition of $\mathcal{E}(\mathcal{S})$.

We also have the following result. It is immediate from the definitions.

**Lemma 4.2.** Let $M \in \mathcal{A}(\mathcal{S})$. If $S \in \mathcal{S}_h$, then $\operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}(S, M^{h-1}) \cong \operatorname{Ext}^1_H(S, M^{h-1})$.

**Proposition 4.3.** Let $S \in \mathcal{S}_h$, let $M \in \mathcal{A}(\mathcal{S})$, and let $j$ be a non-negative integer. There is a 6-term exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}(\mathcal{S})}(S, M^j) \to \operatorname{Hom}_{\mathcal{A}(\mathcal{S})}(S, M) \to \operatorname{Hom}_{\mathcal{A}(\mathcal{S})}(S, M/M^j) \to \operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}(S, M^j) \to \operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}(S, M) \to \operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}(S, M/M^j).$$

It is compatible with the first 6 terms of the long exact sequence for the functor $\operatorname{Hom}_{\mathcal{A}(\mathcal{S})}(S, -) = \operatorname{Hom}_H(S, -)$ applied to the short exact sequence $0 \to M^j \to M \to M/M^j \to 0$ of $H$-modules. This sequence belongs to $\mathcal{E}(\mathcal{S})$.

**Proof.** The last assertion that the sequence $M^j \to M \to M/M^j$ belongs to $\mathcal{E}(\mathcal{S})$ follows from the hypothesis that $M \in \mathcal{A}(\mathcal{S})$ and the definition of $(\mathcal{A}(\mathcal{S}), \mathcal{E}(\mathcal{S}))$. All the maps are standard: the connecting map $f$ uses pullbacks, and the other $\operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}$-maps arise from functoriality (and use pushouts). The composition of any two consecutive maps is zero. Note that $\operatorname{Hom}_{\mathcal{A}(\mathcal{S})}(S, -) = \operatorname{Hom}_H(S, -)$, when applied to $\mathcal{A}(\mathcal{S})$. All $\operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}$-groups are contained in (and are compatible with) their $\operatorname{Ext}^1_{\mathcal{E}(\mathcal{S})}$ counterparts. Now an element in the kernel of $g$ is also in the kernel of its classical counterpart, so lies in the image of $f$, since the first three terms of the “long exact sequence” are identical to those in the classical case (i.e., mod-$H$).
Now consider exactness at the 5th term. By Proposition 4.1(b), we may assume \( M = M^h \). If \( j \geq h \), then \( g \) is clearly an isomorphism and exactness at the 5th term follows. If \( j < h \), then \((M^j)^{h-1} = M^j \) and so, by Lemma 4.2, \( \text{Ext}^{1}_{\mathcal{E}(\mathcal{S})}(S, M^j) = \text{Ext}_{H}^{1}(S, M^j) \). Thus, the first four terms of the “long exact sequence” are identical to those in mod-\( H \). Now, exactness at the next term follows as before.

**Remark 4.4.** Observe that exactness of the first 5 terms of the proposition holds for any \( S \in A(\mathcal{S}) \), not just in \( \mathcal{S} \). Also, as noted in Proposition A.2(b), the \( \text{Ext}^{1}_{\mathcal{E}(\mathcal{S})} \)-groups above are all naturally \( \mathcal{H} \)-modules. The proof of that proposition shows they are \( \mathcal{H} \)-submodules of the corresponding \( \mathcal{H} \)-modules \( \text{Ext}^{1}_{H} \). All maps in the above proposition are \( \mathcal{H} \)-module maps.

When “idempotents split”, there is a general 6 term exact sequence; see fn. 1. For any exact category satisfying the Quillen axioms, there is always a general 4 term exact sequence; see Proposition A.2(a).

**Corollary 4.5.** Let \( S \in \mathcal{S} \), and let \( M \in A(\mathcal{S}) \).

(a) The map \( \text{Ext}^{1}_{\mathcal{E}(\mathcal{S})}(S, M^{h-1}) \to \text{Ext}^{1}_{\mathcal{E}(\mathcal{S})}(S, M^{h}) \cong \text{Ext}^{1}_{\mathcal{E}(\mathcal{S})}(S, M) \) is surjective.

(b) We have \( \text{Ext}^{1}_{\mathcal{E}(\mathcal{S})}(S, M) = 0 \) if and only if the map

\[
\text{Hom}_{A}(S, M^{h}/M^{h-1}) \to \text{Ext}^{1}_{\mathcal{E}(\mathcal{S})}(S, M^{h-1})
\]

is surjective.

(c) Suppose that \( \text{Ext}^{1}_{\mathcal{E}(\mathcal{S})}(S, M^{h-1}) \) is generated as a \( \mathcal{H} \)-module by \( \epsilon_{1}, \cdots, \epsilon_{n} \). Let \( M^{h-1} \to N \to S^{\oplus n} \) represent the element of

\[
\text{Ext}^{1}_{\mathcal{E}(\mathcal{S})}(S^{\oplus n}, M^{h-1}) \cong \text{Ext}_{H}^{1}(S^{\oplus n}, M^{h-1})
\]

(see Lemma 4.2) corresponding to \( \chi := \epsilon_{1} \oplus \cdots \oplus \epsilon_{n} \). Finally, suppose there is a commutative diagram

\[
\begin{array}{ccc}
M^{h-1} & \longrightarrow & N \\
\downarrow & & \downarrow f \\
M^{h-1} & \longrightarrow & M^{h} \longrightarrow M^{h}/M^{h-1}
\end{array}
\]

where \( f \) is a morphism in \( A(\mathcal{S}) \). Then \( \text{Ext}^{1}_{\mathcal{E}(\mathcal{S})}(S, M) = 0 \).

**Proof.** Assertion (a) follows from the 6-term exact sequence of Proposition 4.3 and the fact that \( \text{Ext}^{1}_{\mathcal{E}(\mathcal{S})}(S, M^{h}/M^{h-1}) = 0 \). (The equality follows from Proposition 4.1(b).) The proof of (b) is similar. Next, if \( M^{h-1} \to N_{i} \to S \) corresponds to \( \epsilon_{i} \), there is a pullback (of the top row in the display above) with top row \( \epsilon_{i} \). Thus, \( \epsilon_{i} \) is a pullback of \( M^{h-1} \to M^{h} \to M^{h}/M^{h-1} \) under the evident composite \( g_{i} : S \to S^{\oplus n} \to M^{h}/M^{h-1} \). Consequently, the image of \( g_{i} \in \text{Hom}_{A}(S, M^{h}/M^{h-1}) \) under the connecting homomorphism to \( \text{Ext}^{1}_{\mathcal{E}(\mathcal{S})}(S, M^{h-1}) \) is \( \epsilon_{i} \). Since \( i \) was arbitrary, the connecting homomorphism in (b) is surjective. Hence, \( \text{Ext}^{1}_{\mathcal{E}(\mathcal{S})}(S, M) = 0 \).

**Remark 4.6.** The argument above has already appeared in a module theoretic form in [DPS15]. However, the argument given there required stronger hypotheses, e.g., that \( \text{Ext}^{1}_{H}(S, S) = 0 \).

**Theorem 4.7.** Assume that each \( \mathcal{S}_{i} \) is strictly generated as an additive category by finitely many objects, i.e., every object in \( \mathcal{S}_{i} \) is isomorphic to a finite direct sum of a given finite set of objects in \( \mathcal{S}_{i} \). Let \( M \in A(\mathcal{S}) \). Then there exists an object \( X \) in \( A(\mathcal{S}) \) and an inflation \( M \overset{i}{\rightarrow} X \) such that
\[ \text{Ext}_{\mathcal{E}(\mathcal{S})}^1(S, X) = 0 \text{ for all } S \in \mathcal{S}. \] In addition if \( h \) is chosen minimal such that \( M^{h-1} \neq 0 \), it may be assumed that the inflation induces an isomorphism \( M^{h-1} \cong X^{h-1} \).

**Proof.** Without loss of generality, we can assume that \( M \neq 0 \), and also that \( \text{Ext}_{\mathcal{E}(\mathcal{S})}^1(S, M) \neq 0 \) for some \( S \in \mathcal{S} \). Choose an integer \( h \) minimal with such a non-vanishing occurring for some \( S \in \mathcal{S}_h \). Note that \( M^{h-1} \neq 0 \) by Proposition \[4.1\](c). We will next enlarge \( M \) to an object \( X \), closer to the \( X \) required in the theorem.

Let \( S_1, \ldots, S_m \) be generators for \( \mathcal{S}_h \). For each index \( i \), let \( \epsilon_{i,1}, \ldots, \epsilon_{i,n_i} \) be a finite set of generators for \( \text{Ext}_H^1(S_i, M^{h-1}) \cong \text{Ext}_{\mathcal{E}(\mathcal{S})}^1(S_i, M^{h-1}). \) Form an extension \( 0 \to M^{h-1} \to Y_i \to S_i^{\oplus n_i} \to 0 \) corresponding to \( \chi_i := \epsilon_{i,1} \oplus \cdots \oplus \epsilon_{i,n_i} \in \text{Ext}_H^1(S_i^{\oplus n_i}, M^{h-1}). \) Put \( \chi := \chi_1 \oplus \cdots \oplus \chi_m \), and let \( \chi' \in \text{Ext}_H^1(M^h/M^{h-1}, M^{h-1}) \) correspond to the extension \( 0 \to M^{h-1} \to M^h \to M^h/M^{h-1} \to 0. \) Put \( Z := \oplus_i S_i^{\oplus n_i} \oplus M^h/M^{h-1} \), and let \( M^{h-1} \to X^h \to Z \) correspond to \( \chi \oplus \chi' \). Observe there is a commutative diagram

\[
\begin{array}{ccc}
M^{h-1} & \longrightarrow & Y_i \\
\downarrow & & \downarrow \\
M^{h-1} & \longrightarrow & X^h \\
& & \downarrow \\
& & Z,
\end{array}
\]

in which the top row corresponds to \( \chi_i \) and the bottom row to \( \chi \oplus \chi' \). Comparison with Corollary \[4.5\](c), allowing for the differences in notation, shows \( \text{Ext}_{\mathcal{E}(\mathcal{S})}^1(S_i, X^h) = 0 \) for all \( i \). Thus, \( \text{Ext}_{\mathcal{E}(\mathcal{S})}^1(S, X^h) = 0 \), for all \( S \in \mathcal{S}_h \). Note we have the same vanishing for \( S \in \mathcal{S}_j \) with \( j < h \), by our choice of \( h \). In all cases, we can replace \( X^h \) with any \( X' \) containing it with \( (X')^h = X^h \).

So far, we have not constructed an object \( X \), only \( X^h \). However, the latter may be viewed as the middle term of an exact sequence of right \( H \)-modules \( 0 \to M^h \to X^h \to S' \to 0 \), where \( S' := \bigoplus_i S_i^{\oplus n_i} \in \mathcal{S}_h \). This sequence clearly corresponds to a conflation in \( \mathcal{E}(\mathcal{S}) \). (Note how \( Z \) above is split.) Applying a pushout construction using \( M^h \to M \) (see Proposition \[4.1\]b) and its proof), we obtain an object \( X \) in \( \mathcal{S}(\mathcal{S}) \) which contains a copy of \( M \) under an inflation, and has our constructed \( X^h \) as its image under the functor \((-)^h \). In addition \( X^j = M^j \) for \( j \leq h - 1 \).

Applying Proposition \[4.1\]b again, we find that \( \text{Ext}_{\mathcal{E}(\mathcal{S})}^1(-, X) \) vanishes on all objects in \( \mathcal{S}_j \) with \( j \leq h - 1 \) (and, thus, \( j \leq h \)). Now repeat the argument with \( X \) in the role of \( M \). This requires a bigger \( h \), unless \( \text{Ext}_{\mathcal{E}(\mathcal{S})}^1(S, X) \) already vanishes for all \( S \in \mathcal{S} \). Eventually the process stops, at which point \( X \) satisfies all requirements of the theorem.

**Remark 4.8.** We do not have here any canonical choice for \( X \). In the more local context of \[DPS15\], we did obtain some useful uniqueness results, effectively characterizing analogs of \( X \) as injective hulls in a suitable category; see \[DPS15\] Props. 6.1&6.2.

For the main result, we let \( H \) be the Hecke algebra \( \mathcal{H} \) over \( \mathcal{K} = \mathbb{Z}[t, t^{-1}] \) associated to a finite Coxeter system \( (W, S) \). See \[Lu03\] Ch. 8 for a very general “unequal parameter” version of \( \mathcal{H} \), and a corresponding Kazhdan-Lusztig cell theory. We use dual left cell modules \( S_\omega \) as generators for the various additive categories \( \mathcal{S}_i \). Here \( \omega \) is a left cell in \( W \). There are also right cells, and two-sided cells. These are all defined as equivalence classes associated to certain quasi-posets in \( W \). We shall make use of the opposite \( \leq_{LR}^{op} \) of the quasi-poset order \( \leq_{LR} \), defined in \[Lu03\] Ch. 8. However, we view it as an order on the set \( \Omega \) of left cells (rather than on \( W \)). Using \( \leq_{LR}^{op} \) for \( \leq \) in the discussion above Proposition \[2.2\] earlier in this paper, choose a height function \( h : \Omega \to \mathbb{Z} \) on the
quasi-poset \((\Omega, \leq^\text{op}_{LR})\). Thus, \(ht\) takes a constant value for left cells occurring in the same two-sided cell. Observe that, given two left cells \(\omega, \omega'\), if \(S(\omega)_K\) and \(S(\omega')_K\) have a common composition factor then \(\omega\) and \(\omega'\) are contained in a two-sided cell and so \(ht(\omega) = ht(\omega')\). It follows that \(ht\) defines a height function, still denoted \(ht\), on the set \(\Lambda\) of irreducible \(\mathcal{H}_K\)-modules, equipped with the same quasi-poset structure. Now, for each integer \(i\), define \(\mathcal{H}_i\) as the additive category generated by all dual left cell modules \(S_i\) for which \(ht(\omega) = i\).

For \(\omega, \omega' \in \Omega\), define a preorder
\[
\omega \preceq \omega' \iff ht(\omega) < ht(\omega')\text{, or } ht(\omega) = ht(\omega')\text{ and } \omega \sim_{LR} \omega'
\]
(compare the order \(\leq_f\) given on [DPST15, p.236]). Then \((\Omega, \preceq)\) becomes a quasi-poset and \(ht\) remains a height function with respect to \(\preceq\). We remark that the preorder \(\preceq\) is “strictly compatible” with the partition into two-sided cells, in the sense of [DPST15 Conj. 1.2].

For each \(\omega \in \Omega\), construct \(X = X^\omega\) as in the above theorem, with \(M = S^\omega\). Choose positive integers \(m^\omega, \omega \in \Omega\), and let
\[
T^\dagger = \bigoplus X^\omega \oplus m^\omega.
\]
The use of chosen positive integers \(m^\omega\) is a useful flexibility—all choices of \(m^\omega > 0\) lead to Morita equivalent endomorphism algebras \(A^\dagger\) in the statement below.

**Theorem 4.9.** The \(\mathbb{Z}[t,t^{-1}]\)-algebra \(A^\dagger := \text{End}_{\mathcal{H}}(T^\dagger)\) is standardly stratified. In fact, it has stratifying system consisting of all \(\Delta(\omega) := \text{Hom}_{\mathcal{H}}(S^\omega, T^\dagger)\), with \(S^\omega\) ranging over the dual left cell modules and relative to the quasi-poset \((\Omega, \preceq)\).

**Proof.** The result follows by applying Theorem 2.6 as modified by Remark 2.7 by using Lemma 3.10, taking \(\mathcal{H} = \mathbb{Z}[t,t^{-1}]\). The projectivity of \(\Delta(\lambda)\) over \(\mathcal{H}\) can be proved by [DDPW08 Cor. C.19] and the argument after its proof, which uses the Auslander-Goldman Lemma (see [DDPW08 Lem. C.17]). Alternatively, see [DPS98 Cor. 1.2.12]. The projective \(A^\dagger\)-modules for (SS1) and (SS3) in (2.1) may be taken as the various \(\text{Hom}_{\mathcal{H}}(X^\omega, T^\dagger)\). We leave the straightforward details to the reader. □

**Remark 4.10.** (a) We mention, with only a brief indication of the proof, that \(T^\dagger\) can be chosen with the regular module \(\mathcal{H}\) as a direct summand. We do not yet know if it is possible to do the same with other permutation module analogs. In the case of the regular module \(\mathcal{H}\) itself, one constructs a \(\mathcal{H}\)-split injective composite
\[
\mathcal{H} \to \bigoplus_j (\mathcal{H}/\mathcal{H}^j) \to T^\dagger
\]
and uses the well-known fact that \(\mathcal{H}\) is self-dual as a left (\(\mathcal{H}\)-torsion free) \(\mathcal{H}\)-module (thus, “injective relative to \(\mathcal{H}^\dagger\”)).

(b) The referee asked if, in the more general context of Theorem 4.7, one always gets a standardly stratified algebra from the construction above Theorem 4.9 letting \(\mathcal{H}\) be generated by the single module \(H^i/H^{i-1}\). This is true when \(\mathcal{H}\) is regular of Krull dimension at most 2 (e.g., \(\mathcal{H} = \mathbb{Z}[t, t^{-1}]\)) if each \(H^i/H^{i-1}\) is projective over \(\mathcal{H}\) (a property of dual left cell modules implicitly used in the proof of Theorem 4.9 to ensure \(\Delta(\lambda)\) is projective over \(\mathcal{H}\)). Another positive answer occurs if \(\mathcal{H}\) is a DVR or Dedekind domain, without restriction on \(H^i/H^{i-1}\). The latter module is always torsion-free, but, in the generality of Theorem 4.7, not much more is known about it.
(c) In the general Lusztig setup discussed above, after Remark 4.8 one knows $\mathcal{H}^i/\mathcal{H}^{i-1}$ is a direct sum of dual left cell modules with a largely explicit action of $\mathcal{H}$ available using a (generalised) Kazhdan–Lusztig basis. The height function, which determines $\mathcal{H}^i/\mathcal{H}^{i-1}$, is also important. Though not needed for our argument above, an explicit choice may usually be given using Lusztig’s $a$-function. For instance, the $a$-function may be used in the “split” or “quasi-split case” (in the terminology of Lusztig [Lu03]). These cases include all unknown instances of our conjecture [DPS98, DPS15] mentioned in the introduction.

Appendix A. A Summary of Exact Categories

This brief appendix summarizes, for the convenience of the reader, some basic material concerning exact categories. We closely follow Keller’s treatment in the appendix to [DRSS99]. (See also Keller’s paper [K90].)

Let $\mathcal{A}$ be an additive category. We do not repeat the standard definition, but refer to [Mac94, Chp. 9, §1] for a precise discussion. A pair $(i, d)$ of composable morphisms $i : X \to Y$ and $d : Y \to Z$ in $\mathcal{A}$ is called exact if $i : X \to Y$ is the kernel of $d : Y \to Z$ and $d$ is the cokernel of $i$. Let $\mathcal{E}$ be a class of exact pairs, which is closed under isomorphisms. If $(i, d) \in \mathcal{E}$, then $i$ (resp., $d$) is called an inflation (resp., deflation), and the pair $(i, d)$ itself can be called a conflation. We often just write $X \xrightarrow{i} Y \xleftarrow{f} Z$ or merely $X \xrightarrow{i} Y \xleftarrow{f} Z$ to denote elements (i.e., conflations) in $\mathcal{E}$.

The pair $(\mathcal{A}, \mathcal{E})$ is called an exact category provided the following axioms hold:

0. $1_0 \in \text{Hom}(0, 0)$ is a deflation, where $0$ is the zero object in $\mathcal{A}$.
1. The composition of two deflations is a deflation.
2. Morphisms $Y \xrightarrow{d} Z \xleftarrow{f} Z'$ in $\mathcal{A}$ in which $d$ is a deflation can be completed to a pullback diagram

$$
\begin{array}{ccc}
Y' & \xrightarrow{d'} & Z' \\
\downarrow f' & & \downarrow f \\
Y & \xrightarrow{d} & Z
\end{array}
$$

in which $d'$ is a deflation.
3. Morphisms $X' \xleftarrow{i'} X \xrightarrow{i} Y$ in $\mathcal{A}$ in which $i$ is an inflation can be completed to a pushout diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow f & & \downarrow f' \\
X' & \xrightarrow{i'} & Y'
\end{array}
$$

in $\mathcal{A}$ in which $i'$ is an inflation.

Remarks A.1. (a) The axioms above are part of Quillen’s axioms [Q73] for an exact category, and they are shown in [K90] to be equivalent to the full set of axioms. Since the Quillen axioms are self-dual, it follows that any exact category in the sense of the above conditions also satisfies each corresponding dual condition. For example, the composition of any two inflations is an inflation.
(b) Continuing the above remark, note that the opposite category $\mathcal{A}^{\text{op}}$ inherits an exact category structure from that of $\mathcal{A}$. Now assume that $\mathcal{A}$ is small. (If one believes in the set-theoretic philosophy of universes, every $\mathcal{A}$ can be regarded as small in an appropriate set-theoretic universe.) Applying [K90] Prop. A2] to the opposite category $\mathcal{A}^{\text{op}}$, we find that there is an abelian category $\mathcal{B}$ and faithful full embedding $G : \mathcal{A} \to \mathcal{B}$, such that an exact pair $(i, d)$ belongs to $\mathcal{E}$ if and only if $0 \to G(X) \xrightarrow{G(i)} G(Y) \xrightarrow{G(d)} G(Z) \to 0$ is a short exact sequence in $\mathcal{B}$. Moreover, we can assume that the strict image $\mathcal{M}$ of $G$ (which is equivalent to $\mathcal{A}$) is closed under extensions in $\mathcal{B}$.

(c) Assume the setting of Axiom 2. Let $i : X \to Y$ (resp. $i' : X' \to Y'$) be inflations with $(i, d)$ and $(i', d')$ in $\mathcal{E}$. Then there is a commutative diagram

$$
\begin{array}{c}
\begin{array}{ccc}
X' & \xrightarrow{i'} & Y' & \xrightarrow{d'} & Z' \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{i} & Y & \xrightarrow{d} & Z.
\end{array}
\end{array}
$$

The morphism $X' \to X$, induced by the zero composition $X' \to Y' \to Y \to Z$, is an isomorphism, with inverse given by the map $X \to X'$ induced from the evident zero morphism $X \to Y' \to Z$ (where $X \to Y'$ is obtained by pull-back from the morphism $i : X \to Y$, and the zero morphism $X \to Z'$). This is all in $(\mathcal{A}, \mathcal{E})$, but a similar construction may be made with a pullback of $G(Z') \to G(Z)$ and $G(d) : G(Y) \to G(Z)$. It follows easily that $G(Y')$ is a pullback in $\mathcal{B}$ of these two maps. Similar constructions apply for any exact embedding of $\mathcal{A}$ into an abelian category, or even an exact category; see [Bu10, Prop. 5.2]. A dual discussion applies for Axiom 2$^\circ$.

(d) The embedding in (b) can be used to prove “with elements” that useful exact sequences belong to $\mathcal{E}$. For example, if we put $f' : Y' \to Y$ and $f : Z' \to Z$, then there is a sequence $Y' \xrightarrow{i} Z' \oplus Y \xrightarrow{\epsilon} Z$ of maps in $\mathcal{A}$, where $i$ is defined as the “product map” associated with $d', f'$ and $\epsilon$ the coproduct map of $-d, f$. We claim $(i, \epsilon)$ belongs to $\mathcal{E}$. The analogous assertion for an abelian module category is easy to prove. (Arguing with the given notation, a pullback of $f, d$ may be constructed in $Z' \oplus Y$ as all elements $z' \oplus y$ with $f(z') = d(y)$; this is precisely the kernel of $\epsilon$, and is naturally isomorphic to any other pullback, such as $Y'$.v) We may assume $\mathcal{B}$ has been replaced by such a category. Applying $G$ gives a sequence $G(Y') \xrightarrow{G(i)} G(Z') \oplus G(Y) \xrightarrow{G(\epsilon)} G(Z)$ in $\mathcal{B}$. One can see this sequence is exact by the facts that (1) $G(i)$ is the product map of $G(f')$ and $G(d')$ and $G(\epsilon)$ is the coproduct map of $G(-d)$ and $G(f)$; (2) applying $G$ to the right square of the digram above yields a pullback diagram by (c). Thus, $(i, \epsilon)$ is exact in $\mathcal{A}$, that is, $(i, \epsilon)$ belongs to $\mathcal{E}$.

(e) The abelian category $\mathcal{B}$ can also be used to extend the exact sequence in Proposition [A.2] below to the right by one term, as in the argument for Proposition 4.3. As previously mentioned, [DRSS99] effectively gives a general 6 term version, using the “split idempotent” hypothesis, which we cannot assume.

Let $(\mathcal{A}, \mathcal{E})$ be an exact category. For $X, Z \in \mathcal{A}$, let $\mathcal{E}(Z, X)$ be the set of sequences $X \to Y \to Z$ in $\mathcal{E}$. Define the usual equivalence relation $\sim$ on $\mathcal{E}(Z, X)$ by putting

$$(X \to Y \to Z) \sim (X \to Y' \to Z)$$
provided there is a morphism $Y \to Y'$ giving a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y & \longrightarrow & Z \\
\| & & \| & & \\
X & \longrightarrow & Y' & \longrightarrow & Z
\end{array}
$$

The morphism $Y \to Y'$ is necessarily an isomorphism (as follows from Remark A.1(b) and a diagram chase, for example). Let $\Ext^1_{\mathfrak{A}}(Z,X) = \mathfrak{A}(Z,X)/\sim$.

**Proposition A.2.** (a) $\Ext^1_{\mathfrak{A}}(Z,X)$ has a natural abelian group structure such that given any $A \to B \to C$ in $\mathfrak{A}$ and object $Z \in \mathfrak{A}$, there are exact sequences

$$
0 \to \Hom_{\mathfrak{A}}(Z,A) \longrightarrow \Hom_{\mathfrak{A}}(Z,B) \longrightarrow \Hom_{\mathfrak{A}}(Z,C) \longrightarrow \Ext^1_{\mathfrak{A}}(Z,A)
$$

where $f$ is defined by pullback as in Axiom 2. A dual contravariant version holds, using the contravariant functor $\Hom_{\mathfrak{A}}(-,Z)$ and pullback as in Axiom 2°.

(b) Let $\mathcal{K}$ be a fixed commutative, Noetherian ring. If $\mathfrak{A}$ is a $\mathcal{K}$-category, then $\Ext^1_{\mathfrak{A}}(Z,A)$ is naturally a $\mathcal{K}$-module.

**Proof.** The usual argument involving the Baer sum $(\alpha + \beta = \nabla_X (\alpha \oplus \beta) \Delta_Z)$ proves (a); see [Mac94] p. 85, (5.4)]. We next prove (b). Using standard embedding theorems, we can reduce to the case where $\mathfrak{A}$ is a $\mathcal{K}$-category of $\mathcal{K}$-modules (We remark that this is the only case to which we make applications in this paper.). Assuming this, we have what appears to be two actions of $\mathcal{K}$ on $\Ext^1_{\mathfrak{A}}(Z,X)$, one through the action of $\mathcal{K}$ on $Z$, and one through its action on $X$. The first of the two actions uses a pullback of multiplication by any given element $b$ in $\mathcal{K}$ on $Z$, and the second uses a pushout of the $X \to Y \to Z$ action of $b$ on $X$. We take part (b) as asserting, in this context, that the actions are the same, and that is (all of) what we will prove.

Suppose we are given an element of $\Ext^1_{\mathfrak{A}}(Z,X)$ represented by $X \xrightarrow{i} Y \xrightarrow{d} Z$, and let $b \in \mathcal{K}$. Form the pullback and pushout objects as above, denoting the pullback by $Y'$ and the pushout by $Y^\#$. The pullback object is formed by all pairs $(y,z)$ with $dy = bz (y \in Y, z \in Z)$. It is an object in $\mathfrak{A}$ which is a subobject of $Y \oplus Z$. There is an evident sequence $X \to Y' \to Z$, which we also call a pullback. The pushout object $Y^\#$ is formed as a quotient of $X \oplus Y$ by the subobject $W$ consisting of all pairs $(-bx, ix)$, with $x \in X$. We represent an element of this quotient as a bracketed pair $[x,y]$, with the representative pair $(x,y)$ well-defined only up to addition of an element of $W$. There is a corresponding pushout sequence $X \to Y^\# \to Z$. We claim this sequence represents the same element of $\Ext^1_{\mathfrak{A}}(Z,X)$ as the pullback sequence with $Y'$. To prove this, all we have to do is exhibit a map $Y^\# \to Y'$ in the $\mathcal{K}$-category $\mathfrak{A}$ giving the expected commutative diagram. Such a map may be defined by sending a pair $x, y \in X \oplus Y$ to $(by + ix, dy) \in Y \oplus Z$, a pair which is actually in $Y'$, since $d(by + ix) = b(dy)$. Moreover, the map has $W$ in its kernel since, if $x \in X, (b(ix) + (-bx), d(ix)) = (0,0)$. Thus, induces a map to $Y^\# \to Y'$. We leave it to the reader to check the required commutativites. This proves the claim and completes the proof of part (b).

For a relatively recent survey of exact categories, starting from the Quillen axioms (though without any explicit discussion of $\Ext^1_{\mathfrak{A}}$), see [Bu10].
Appendix B. Idempotent Ideals

The following result is proved in [CPS90]. For convenience, we indicate a short proof.

**Proposition B.1.** Let $J$ be an idempotent ideal in a ring $A$. Assume that $AJ$ is projective. Let $M, N$ be $A/J$-modules. For any integer $n \geq 0$, inflation provides an isomorphism

$$\text{Ext}^n_{A/J}(M, N) \simto \text{Ext}^n_A(M, N)$$

of abelian groups. (On the right hand side, $M, N$ are regarded as $A$-modules through the morphism $A \to A/J$.)

**Proof.** Using the short exact sequence $0 \to J \to A \to A/J \to 0$ of left $A$-modules, the projectivity of $AJ$ implies that $\text{Ext}^n_A(A/J, N) = 0$ for $n > 1$. Since $J^2 = J$, $\text{Hom}_A(J, N) = 0$. Thus, any projective $A/J$-module is acyclic for the functor $\text{Hom}_A(-, N)$. The proposition follows. □

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