Derivation of constitutive equation based on Miura-homogeneous equivalent model

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Abstract. As a new type of sandwich structure, the Miura-folded sandwich structure shows excellent mechanical properties, and its internal transparent design is also conducive to the realization of multi-function. This kind of structure can solve the problem of the condensation of water vapor which will cause structural failure in the honeycomb sandwich structure. Based on the Miura origami structure, a homogenization mechanical analysis model of this folded sandwich is established. Through the model, we derived the theoretical expressions of structural modulus of sandwich panel under compression and shear loads, and the simulation results are verified in ABAQUS.

1. Introduction
With the development of modern aerospace technology, the demand for lightweight and multifunctional panel structure is increasingly strong [1]. In recent years, based on the inspiration of origami technology and the improvement of preparation technology [2], the folded sandwich structure has been highly valued by scholars. Therefore, it is of great significance to study the equivalent modulus of sandwich panel for the optimal design of folded sandwich structure [3]. In this paper, we chosen the unit cell of Miura-folded sandwich structure as the research object, and simplified the equivalent mechanical analysis model. From the standard beam theory and homogenization method [4-6], the equivalent modulus of Miura unit cell under in-plane compression and shear load is given, which provides a theoretical basis for further study of the mechanical properties of Miura sandwich plate [7-8]. ABAQUS is used to complete the corresponding simulation, which is compared with the theoretical results to verify the universality of the theoretical expression.

2. Design Method
Based on the construction of folded sandwich structure, this paper chosen Miura origami structure to construct sandwich panel. The cell structure is shown in Figure 1-a).
The whole cell is composed of parallelogram as shown in Figure 1-b), and the position of the cell can be completely determined by geometric parameters \( \alpha, \theta, a \) and \( b \). Where \( \alpha \) is the angle between the side with side length \( b \) and the bottom, \( \theta \) is half of the angle between the two sides with side length \( a \), \( a \) and \( b \) are the side lengths of the parallelogram. The angle \( \varphi \) between the two sides of a parallelogram has the following relationship with the angle \( \alpha \) and \( \theta \):

\[
\cos \varphi = \cos \alpha \cos \theta \\
\sin^2 \varphi = \sin^2 \alpha + \cos^2 \alpha \sin^2 \theta
\]  

(1)

The whole sandwich panel structure can be regarded as a unit cell array, and it can be analyzed from two scales in the derivation: (I) In the macro-scale, homogenize the cell into a homogeneous hexahedron, as shown in Figure 2-a). (II) In micro-scale, it is considered as the stress deformation of beam, as shown in Figure 2-b).

By completing the simplest beam analysis, the vertical plate is further deduced, and then the theoretical formula of adding inclination angle is obtained.

2.1. Basic assumptions

Following basic assumptions are defined to simplify the equations and calculations:

1. The homogenization assumption is that the unit cell is equivalent to a continuous medium of equal volume;
2. Only consider the in-plane stress component of the wall;
3. The large deformation of the wall is not considered, that is, the wall material has small linear elastic deformation.

In the calculation of the integral sandwich panel, it is considered as a small deflection plate, ignoring the influence of \( z \) direction, and the corresponding stress-strain relationship is

\[
\left\{ \begin{array}{l}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{array} \right\} = \left[ Q_{11} \quad Q_{12} \quad 0 \\
Q_{21} \quad Q_{22} \quad 0 \\
0 \quad 0 \quad Q_{66} \end{array} \right] \left\{ \begin{array}{l}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_{xy}
\end{array} \right\}
\]  

(2)

Where

\[
Q_{11} = \frac{E_1}{1 - \nu_{12} \nu_{21}}, \quad Q_{12} = Q_{21} = \frac{\nu_{12} E_1}{1 - \nu_{12} \nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12} \nu_{21}}, \quad Q_{66} = G_{12} = \frac{E_1}{2(1 + \nu)}, \quad E_1, E_2, v_{12}, v_{21}, G_{12}
\]

is the in-plane modulus in each direction.
Based on the above assumption, the whole cell is compressed and the shear load is applied, as shown in Figure 3.

\[ M = \frac{F l \sin \theta}{2} - \frac{M l^2 \sin \theta}{12 E I}, \quad A_1 A_2 = \frac{\sigma}{E} \frac{l}{I} \]

2.2. Simplified derivation of \( E_1, E_2 \)
Take the beam with thickness \( t \), depth \( h \), length \( l = b \) and modulus \( E \). The stress analysis of beam \( ABC \) is carried out, as shown in Figure 4-a). When the extrusion is applied in the X direction, the beam \( AB \) changed into \( A_2 B \). \( AA_1 \) is the deflection perpendicular to the beam direction and \( A_1 A_2 \) is the axial compression parallel to the beam direction. As shown in Figure 4-b), when the extrusion is applied in the Y direction, the beam \( AB \) changed into \( A_3 B \). \( AA_3 \) is the deflection perpendicular to the beam direction and \( A_3 A_4 \) is the axial compression parallel to the beam direction.

The moment \( M \) tending to bend the beam of length \( l \) is:

\[ M_1 = F_1 l \sin \theta / 2, \quad M_2 = F_2 l \cos \theta / 2 \] (3)

where \( F_1 = \sigma h l \sin \theta, \quad F_2 = \sigma h l \cos \theta \).

From the standard beam theory, the beam deflects by

\[ A_1 A_2 = \frac{F_1 l^3 \sin \theta}{3 E I} - \frac{M l^2 \sin \theta}{2 E I} = \frac{F_1 l^3 \sin \theta}{12 E I}, \quad A_1 A_2 = \frac{\sigma l}{E I} \frac{F_1 \cos \theta l}{E h t} \]

\[ A_3 A_4 = \frac{F_2 l^3 \cos \theta}{3 E I} - \frac{M l^2 \cos \theta}{2 E I} = \frac{F_2 l^3 \cos \theta}{12 E I}, \quad A_3 A_4 = \frac{\sigma l}{E I} \frac{F_2 \sin \theta l}{E h t} \] (4)

With this deformation, the strain \( \varepsilon_i (i = 1, 2) \) in the corresponding direction can be calculated, and the modulus \( E_i = \sigma_i / \varepsilon_i (i = 1, 2) \) in the corresponding direction can be obtained and \( I = h t^3 / 12 \).
\[ \varepsilon_{11} = \frac{AA \sin \theta + A_A \cos \theta}{l \cos \theta} = \frac{F_1 l^2 \sin^2 \theta}{E h t^3 \cos \theta} \left(1 + \cot^2 \theta \frac{t^2}{l^2}\right) \]

\[ \varepsilon_{12} = \frac{A_A \cos \theta - A_A \sin \theta}{l \sin \theta} = \frac{F_2 l^3 \sin \theta}{E h t^3} \left(1 - \frac{t^2}{l^2}\right) \]

\[ \varepsilon_{21} = -\frac{A_A \sin \theta - A_A \cos \theta}{l \cos \theta} = -\frac{F_2 l^3 \sin \theta}{E h t^3} \left(1 - \frac{t^2}{l^2}\right) \]

\[ \varepsilon_{22} = \frac{A_A \cos \theta + A_A \sin \theta}{l \sin \theta} = \frac{F_2 l^3 \cos^2 \theta}{E_s h t^3 \sin \theta} \left(1 + \tan^2 \theta \frac{t^2}{l^2}\right) \]

Through equation (5), the modulus is written as:

\[ \nu_{12} = -\frac{\varepsilon_{12}}{\varepsilon_{11}} = \frac{\cos^2 \theta}{\sin \theta} \left(1 - \frac{t^2}{l^2}\right), \quad E_i = \frac{\sigma_i}{\varepsilon_i} = \frac{F_i / h l \sin \theta}{l^3 \sin^3 \theta} = \frac{E_s}{1 + \cot^2 \theta t^2 / l^2} \]

\[ \nu_{21} = -\frac{\varepsilon_{21}}{\varepsilon_{22}} = \tan^2 \theta \left(1 - \frac{t^2}{l^2}\right), \quad E_2 = \frac{\sigma_2}{\varepsilon_2} = \frac{F_i / h l \cos \theta}{l^3 \sin^3 \theta} = \frac{E_s}{1 + \tan^2 \theta t^2 / l^2} \]

2.3. Simplified derivation of \( G_{12} \)

When the shear force is applied in the direction of X and Y direction in the same time, the deformation of the beam can be regarded as the combination of shear deformation in two directions, as shown in Figure 5:

![Figure 5 Shear force](image)

The shear force in x direction forms the deformation of \( AA_A \), and the shear force in y directions forms the deformation of \( A_A A_6 \). In the process of calculating the deformation angle, the axial tension and compression deformation of the beam is ignored, and the deformation of the whole beam can be regarded as rotation. The angle change caused by shear deformation is obtained by simplifying to isosceles triangle, as shown in Figure 6.
Figure 6  Simplification of shear modulus analysis

Where \( M_3 = F_3 l \sin \theta /2 , \quad M_4 = F_4 l \cos \theta /2 , \quad F_3 = \tau_{12} h l \sin \theta , \quad F_4 = \tau_{21} h l \cos \theta . \)

The deformations are expressed:

\[
AA_5 = \frac{F_3 l \sin \theta}{12E_s I} = \frac{F_1 \sin \theta l^3}{E_s b t^3} \quad AA_6 = \frac{F_4 l \cos \theta}{12E_s I} = \frac{F_2 \cos \theta l^3}{E_s b t^3}
\]

Through equation (7), the angle is written as:

\[
\gamma = 2 \left( \frac{AA_5 - 2AA_6}{l} \right) = 2 \left( \frac{F_3 \sin \theta l^3}{E_s b t^3} - 2 \frac{F_4 \cos \theta l^3}{E_s b t^3} \right) / l ,
\]

and the shear modulus are written as:

\[
G_{xy} = \tau / \gamma = E_s l^3 / 2 \left( \sin^2 \theta - 2 \cos^2 \theta \right) l^3
\]

3. Result

The modulus of each direction is derived by the homogenization simplified model, and FEA is carried out by ABAQUS/Explicit. The elements of the core were linear quad shell element which adopts S4R shell element. The core is connected with the upper and lower panels by tie constraint. The core models were developed using SolidWorks. The specific geometric parameters and material parameters are shown in Table 1.

| Geometric parameters | b=l | h | t | \( \beta \) |
|----------------------|-----|---|---|-----|
| a 30mm               | 40mm| 1mm| 60° |

| Material parameters | \( E_s \) | \( \nu \) | 0.34 |
|---------------------|--------|--------|------|

In the process of establishing the model, one side of the cell is fixed, the other side is coupled to the reference point, and the force or displacement load is applied through the reference point to complete the force simulation of the whole cell, as shown in Figure 7-a).
Through the FEA model, the stress-strain curve can be obtained by coupling the force displacement curve with the reference point, and its slope is the modulus of the corresponding sandwich panel cell. The above geometric parameters and material parameters are substituted into equation (6), (8) and compared, as shown in Table 2.

| Screw-number | Modulu   | Analytical | FEA     | error   |
|--------------|----------|------------|---------|---------|
| θ=45°        | $E_1$    | 499Mpa     | 455Mpa  | 8.8%    |
|              | $E_2$    | 499Mpa     | 463Mpa  | 7.2%    |
|              | $G$      | 500Mpa     | 449Mpa  | 10.2%   |

4. Conclusion

In this paper, a homogenization equivalent model is proposed based on Miura's Sandwich cell. With the simplified model, the approximate analytical expressions of the equivalent elastic modulus of the sandwich cell under compression and shear loads are given. ABAQUS is used to complete the comparative verification. From the error results, the simplified model is universal. Based on the equivalent elastic modulus obtained above, the energy absorption and vibration reduction characteristics of the Miura-folded sandwich can be further studied.

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