Abstract

We investigate whether fairness is compatible with efficiency in economies with multi-self agents, who may not be able to integrate their multiple objectives into a single complete and transitive ranking. We adapt envy-freeness, egalitarian-equivalence and the fair-share guarantee in two different ways. An allocation is unambiguously-fair if it satisfies the chosen criterion of fairness according to every objective of any agent; it is aggregate-fair if it satisfies the criterion for some aggregation of each agent’s objectives.

While efficiency is always compatible with the unambiguous fair-share guarantee, it is incompatible with unambiguous envy-freeness in economies with at least three agents. Two agents are enough for efficiency and unambiguous egalitarian-equivalence to clash. Efficiency and the unambiguous fair-share guarantee can be attained together with aggregate envy-freeness, or aggregate egalitarian-equivalence.

1 Introduction

We consider multi-self agents who pursue a variety of potentially conflicting objectives. An agent may, for example, base her decisions on her career prospects, her family life and her immediate enjoyment. A fully rational agent is able to aggregate all her objectives into a single complete and transitive ranking; a boundedly rational agent may fail to do so. In an economy with such boundedly rational agents, we consider three criteria of fairness.
An allocation has the fair-share guarantee (FS) if each agent prefers their bundle to the average bundle; it is envy free (EF) if each agent prefers their bundle to any other agent’s; it is egalitarian-equivalent (EE) if each agent is indifferent between their bundle and some fixed reference bundle.

For each fairness criterion, we call an allocation unambiguously fair if it satisfies the fairness criterion according to each objective of each agent. For an illustration, consider agents that are driven by two selves; one with a cool-headed long-run view and another that greedily searches immediate gratification. An allocation among such “tempted” agents is unambiguously envy-free if neither self of any agent envies any other agent. In this case no agent envies any other agent according to any aggregation of her two objectives. The agents may then avoid the mental load of weighing their long-term goals against their immediate greed.

With rational agents, each of the three concepts of fairness is — under generic conditions — compatible with efficiency. We then ask whether unambiguous fairness is compatible with efficiency when agents are boundedly rational. The assumption of bounded rationality has two countervailing effects: On the one hand, the set of Pareto optima can be large. Unambiguous fairness is, on the other hand, hard to satisfy. A-priori, it is not clear which of these two effects dominates. We find in Theorem 2.1 that efficiency is always compatible with the unambiguous fair-share guarantee. Theorem 3.1 shows that any economy with just two agents, all whose selves have convex preferences, has an unambiguously envy-free Pareto optimum. Unambiguous no-envy, however, conflicts with efficiency in economies with three or more agents (Proposition 3.2). Two agents suffice for unambiguous egalitarian-equivalence to conflict with efficiency (Proposition 4.1).

Faced with the non-existence of unambiguously envy-free or egalitarian-equivalent Pareto optima, we investigate these fairness criteria according to some aggregation of all agents’ objectives into rational preferences. Theorems 3.3 and 4.2 show that some Pareto optima with the unambiguous fair-share guarantee respectively satisfy aggregate no-envy and aggregate egalitarian-equivalence.

1 Also known as proportionality.

2 Kreps (1979) and Gul and Pesendorfer (2001, 2004) introduced such agents together with proposals for their complete and transitive rankings over choice sets.
equivalence, when all selves have strictly convex preferences. The aggregators used in the two results do not vary with the economies under consideration. To understand these aggregators, interpret the agents and their selves as families and their members. If the Rawlsian criterion of justice is used to resolve intra-family conflict, the lexicmin aggregator used in Theorem 3.3 arises. If the family instead aggregates its members’ objectives using Nash bargaining (maximizing the product of members’ utilities), we obtain the aggregator used in Theorem 4.2.

When agents are rational, market equilibria from equal endowments have the fair-share guarantee and are envy-free (Foley 1967). So when agents are rational, envy-free Pareto optima with the fair share guarantee exist wherever there are market equilibria. With boundedly rational agents this strong nexus between no envy and market equilibrium disappears: Proposition 3.4 shows that, even if some unambiguously envy free Pareto optima arise as market equilibria, it may be necessary to give less rational agents larger endowments to obtain such allocations in market equilibrium.

2 Preliminaries

2.1 Goods and Agents

There are $G$ different homogeneous divisible goods and a finite set of agents $I$. The set of different consumption bundles is $\mathbb{R}_+^G$; the total endowment is $e \in \mathbb{R}_+^G$ with $e \gg 0$. An allocation is a vector $x = \{x_i\}_{i \in I}$ of consumption bundles $x_i \in \mathbb{R}_+^G$ whose sum does not exceed the total endowment: $\sum_{i \in I} x_i \leq e$. $X$ is the set of all allocations. The average bundle $\bar{e} := \frac{1}{|I|}e$ defines each agent’s fair-share of the total endowment, and each agent gets the fair share in the equal split $\bar{e} = (\bar{e}, \ldots, \bar{e})$.

3For any two vectors $x, x' \in \mathbb{R}^m$ for some integer $m$, say $x \geq x'$ if $x_i \geq x'_i$ for all $i$, $x > x'$ if $x \geq x'$ but not $x = x'$ and $x \gg x'$ if $x_i > x'_i$ for all $i$. 

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2.2 Selves and Aggregators

When evaluating allocations, agents only consider their own bundles. Each agent $i$ has a set of selves $S_i$. The preferences of each such self $s \in S_i$ are represented by a continuous utility function $u_s : \mathbb{R}^G_+ \to \mathbb{R}$, normalized such that $u_s(\mathbf{0}) = 0$. We assume throughout that $u_s$ is strictly increasing in all components, so all selves have strictly monotone preferences. The preferences represented by the utility $u_s$ are convex if $u_s(x) > u_s(x')$ implies $u_s((1 - \alpha)x + \alpha x') > u_s(x')$ for all $x, x' \in \mathbb{R}^G_+$ and $\alpha \in (0, 1)$. They are strictly convex if $u_s((1 - \alpha)x + \alpha x') > u_s(x')$ also holds for any two different bundles $x$ and $x'$ with $u_s(x) = u_s(x')$. We say that an agent $i$ is more rational than an agent $j$ if $S_i \subset S_j$.

Agent $i$ unambiguously prefers an option if all his selves prefer this option. Formally, agent $i$’s unambiguous preference $\succsim^U_i$ is a — typically incomplete — transitive relation, where $x \succsim^U_i x'$ holds if and only if $u_s(x) \geq u_s(x')$ for all $s \in S_i$ and where $x \succ^U_i x'$ holds if in addition $u_s(x) > u_s(x')$ holds for some $s' \in S_i$. A transitive and complete preference $\succsim^{agg}_i$ is an aggregator of agent $i$’s selves if for all bundles $x, x'$, $x \succsim^{agg}_i x'$ implies $x \succ^{agg}_i x'$ and $x \succ^U_i x'$ implies $x \succ^{agg}_i x'$. Szpilrajn (1930)’s extension theorem guarantees the existence of such aggregators. An agent unambiguously prefers some bundle $x$ to a different bundle $x'$ ($x \succsim^U_i x'$) if and only if he prefers $x$ to $x'$ ($x \succ^{agg}_i x'$) according to every aggregator $\succsim^{agg}_i$. We also call a vector of aggregators $\succsim^{agg} : = (\succsim^{agg}_i)_{i \in I}$ an aggregator.

The function $g((u_s(\cdot))_{s \in S_i}) : X \to \mathbb{R}$ represents an aggregator for agent $i$ if $g : \mathbb{R}^{|S_i|} \to \mathbb{R}$ is strictly increasing in all its components. If $g$ is the

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4 Alternatively, one could think of the agent’s selves as his frames in the sense of Salant and Rubinstein (2008).

5 The relation “more rational than-” is incomplete. For example, a fully rational agent $i$ with a single self $S_i = \{u_s\}$ is not more-rational than an agent $j$ with two selves if $u_s \notin S_j$.

6 Mandler (2014, 2020) calls $\succsim^U_i$ the behavioral preference. Bernheim and Rangel (2009) define a similar relation based on revealed choices, denote it by $R'$ and call it weak unambiguous choice preference.

7 The proof is simple and we omit it. It is available upon request.
sum of its components, then $g$ represents the utilitarian aggregator. The leximin aggregator is not representable by any function. To define it, fix any two bundles $x$ and $x'$, and consider the two minimal utilities of any selves of the agent: if $\min_{s \in S_i} u_s(x) > \min_{s \in S_i} u_s(x')$ then $x \succ^i_{lex} x'$. If these two minimal utilities are identical, then the leximin aggregator uses the two second-lowest utilities to rank $x$ and $x'$. Proceeding inductively $\succ^i_{lex}$ ranks $x$ and $x'$ as indifferent if and only if both are mapped to the same vector of utilities in increasing order of utility (Dubins and Spanier 1961; Moulin 2004). To illustrate such aggregators at the hand of a specific decision theoretic model, consider the Bewley (2002) model of Knightian uncertainty, where the different selves of an agent use different priors to evaluate uncertain events. Bewley (2002) axiomatized the unambiguous preference $\succ^i_U$ where agent $i$ prefers an option to a different one if the former yields a higher expected utility than the latter according to every prior of the agent. If such an agent uses the utilitarian aggregator, he is an expected utility maximizer. If he instead evaluates every choice according to the minimal expected utility over the set of all priors, he is represented by a maximin expected utility, following Gilboa and Schmeidler (1989).

### 2.3 Notions of Fairness

An allocation satisfies the fair-share guarantee if each agent weakly prefers his bundle to the fair-share; it is envy-free if no agent strictly prefers the bundle of another agent; it is egalitarian equivalent if each agent is indifferent between his bundle and some fixed reference bundle $r$. For a given aggregator $\succ_{agg}$ and a given fairness notion, we call an allocation $\succ_{agg}$-aggregate-fair if each agent $i$ considers it fair according to their aggregator $\succ^i_{agg}$. Aggregate fairness coincides with standard fairness for rational agents whose preferences are represented by the aggregators. So existing tools suffice to study aggregate fairness. But there is a drawback: we have to commit to particu-

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8The definitions of the utilitarian and leximin aggregators depend on the representations $u_s$ of the selves’ preferences. The income-equivalence principle (Decancq et al. 2015; Fleurbaey and Schokkaert 2013) provides a specific normalization with $u_s(\pi) = 0$.

9The proof of Theorem 3.3 addresses the difference between the leximin aggregator and the function that uses the minimal utility of any self to evaluate options.
lar aggregators. To avoid such a dependence, we study unambiguously-fair allocations, which are fair according to every aggregator.

Specifically, an allocation \( x \) is unambiguously envy-free (EF) if there are no two agents \( i, i' \) and self \( s \in S_i \) such that \( u_s(x_{i'}) > u_s(x_i) \). It is unambiguously egalitarian equivalent (EE), if there exists a reference bundle \( r \) such that for each agent \( i \) \( u_s(r) = u_s(x_i) \) holds for all \( s \in S_i \). The allocation \( x \) satisfies the unambiguous fair-share guarantee (FS) if for each agent \( i \) \( u_s(x_i) \geq u_s(r) \) holds for all \( s \in S_i \), and \( E \) is the set of all unambiguous-FS allocations.\(^{10}\)

Unambiguous fairness relies on the transitive but incomplete preferences \( \succeq^U_i \) of agents. For a different approach to fairness for boundedly rational agents, we instead consider complete but intransitive aggregators in Section 5.

2.4 Pareto Optimality

An allocation \( x \) Pareto dominates a different allocation \( x' \) if \( u_s(x_i) \geq u_s(x'_{i}) \) holds for every self \( s \in S_i \) of every agent \( i \), while \( u_{s'}(x_j) > u_{s'}(x_j) \) holds for at least one agent \( j \) and self \( s' \in S_j \). An allocation is Pareto-optimal if it is not Pareto-dominated by any other allocation.

To formalize the two countervailing effects of bounded rationality on the set of fair and efficient allocations, consider a more-rational and a less-rational economy. Say that some allocation \( x' \) Pareto dominates a different allocation \( x \) in the less-rational economy, and say that no selves are indifferent between the two allocations. Then, clearly, \( x' \) also dominates \( x \) according to the smaller sets of selves in the more-rational economy.\(^{11}\) So, abstracting away from the issue of indifferences, the set of Pareto-optima increases when the economy becomes less rational. On the other hand, the set of unam-

\(^{10}\)For each of the fairness notions, an allocation is unambiguously-fair if and only if it is aggregate-fair according to every vector of aggregators. The proof is available upon request.

\(^{11}\)To see that the clause of indifference matters, consider a fully rational economy in which all selves are indifferent between all bundles. Now add one self to one agent’s set of selves and assume that this self strictly ranks all allocations. While no allocation dominates any other in the rational economy, this does not hold in the boundedly rational economy.
ously fair allocations in the more-rational economy is larger, since the chosen fairness notion then has to hold for fewer selves. Our first result shows that — no matter the level of irrationality — the set of Pareto optima contains some unambiguously-fair allocations.

**Theorem 2.1** Any economy has unambiguous-FS Pareto optima.

**Proof** Since $u_s$ is for each $s \in \bigcup_{i \in N} S_i$ continuous, the set $E$ of unambiguous-FS allocations is compact. Define a function $F : X \to \mathbb{R}$ as the sum of all utilities of all agents’ selves: $F(x) = \sum_{i \in N} \sum_{s \in S_i} u_s(x_i)$ for all $x \in X$. By the compactness of $E$ and the continuity of $F$, some allocation $x^*$ maximizes $F(x)$ over all $x \in E$, so that $x^*$ is Pareto optimal in $E$. Since no allocation outside $E$ Pareto dominates any allocation in $E$, $x^*$ is Pareto optimal. Since $x^* \in E$, $x^*$ is unambiguous-FS. 

The above proof continues to hold if we use a different method to find a Pareto optimum in $E$ and if we drop the assumption of monotonic preferences.

### 2.5 Market Equilibria

We borrow our notions of market equilibrium from behavioral welfare economics (Fon and Otani 1979; Mandler 2014). A triplet $(p, x^*, x^0)$ with $x^*, x \in X$ and $p \in \mathbb{R}^G$ is a *market equilibrium from (endowments) $x^0 \in X$, if the following two conditions hold for each agent $i \in I$:

1. $px_i^* \leq px_i^0$, that is, each agent $i$ can afford bundle $x_i^*$; and—

2. For any bundle $x' \in \mathbb{R}_+^G$, if $px' \leq px_i^0$ then either $u_s(x') = u_s(x_i^*)$ for all selves $s \in S_i$, or $u_s(x') < u_s(x_i^*)$ for at least one self $s \in S_i$.

The second condition is weak: agent $i$’s selves may not unanimously prefer a different affordable bundle to the choice $x_i^*$. Equivalently, any agent may buy any bundle that is optimal according to some aggregation of her selves. So the definition remains agnostic as to the aggregator an agent uses when
she shops. Our result on the impossibility of attaining fairness in market equilibrium (Proposition 3.4) is strengthened by this agnosticism.

2.6 Examples with two goods

In economies with two goods, these are called $y$ and $z$, and we normalise the total endowment of each good to $|I|$, so that $\bar{e} = (1, 1)$. For a differentiable utility $u_s : \mathbb{R}_+^2 \to \mathbb{R}$, the marginal rate of substitution between $y$ and $z$ is:

$$MRS_s(y, z) := \frac{d(u_s(y, z))}{dy} / \frac{d(u_s(y, z))}{dz}.$$ 

Two different utilities $u_s$ and $u_{s'}$ have the single crossing property if any two indifference-curves defined by $u_s(y, z) = \alpha$ and $u_{s'}(y, z) = \beta$ for some $\alpha, \beta \in \mathbb{R}$ share at most one point. So $u_s$ and $u_{s'}$ have the single-crossing property if for all bundles $(y, z)$ and $(y', z')$ and either $u_s = u$, $u_{s'} = u'$ or $u_s = u'$, $u_{s'} = u$:

If $y' > y$ and $z' < z$ then $u(y', z') \geq u(y, z) \Rightarrow u'(y', z') > u'(y, z)$

If $y' < y$ and $z' > z$ then $u'(y', z') \geq u'(y, z) \Rightarrow u'(y', z') > u(y, z)$.

Two differentiable utilities have the single-crossing property if the marginal rate of substitution of one is higher than the other’s at each bundle $(y, z)$. For example, any two Cobb-Douglas utilities $u_s(y, z) = y^\alpha z^{1-\alpha}$ with different $\alpha$ have the single-crossing property.

Our proofs and examples use the characterization of all interior Pareto-optimal allocations as the set of allocations at which the intersection of all agents’ MRS-ranges, the intervals between the smallest and the largest MRS of their selves, is non-empty.

**Theorem 2.2** Fix a two-good economy with convex preferences representable by differentiable utilities and an allocation $x$ in the interior of $X$. Then $x$ is

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12If we would instead require that any self of agent $i$ prefers $x^*_i$ to all affordable bundles, nonexistence of market equilibria would be immediate, as an agent’s selves need not agree on optimal choices. Using aggregate preferences in the second condition would tie us down to a particular rational theory on how agents weigh their different selves.
Pareto optimal if and only if:

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\bigcap_{i \in I} \left[ \min_{s \in S_i} MRS_s(y_i, z_i), \max_{s \in S_i} MRS_s(y_i, z_i) \right] \neq \emptyset.
\]

Theorem 2.2 has been shown to hold in much more general environments by Fon and Otani (1979); Mandler (2014) and we omit its proof here. Simple modifications of the arguments in Mas-Colell (1974) also prove Theorem 2.2.

3 Envy-freeness

Theorem 3.1 shows that two-agent economies with convex preferences always have unambiguous-EF Pareto optima. Economies with more agents, in contrast, need not have such allocations (Proposition 3.2). Even if unambiguous-EF Pareto optima exist, they may not arise as market equilibria from equal endowments. The market approach, however, turns out to be useful to find unambiguous-FS Pareto optima that are aggregate-EF. The impossibility results hold even if all but one agent are fully rational.

3.1 Unambiguous envy-freeness

Theorem 3.1 Any two-agent economy, where all selves have convex preferences, has an unambiguous-EF and unambiguous-FS Pareto optimum.

Proof By Theorem 2.1 there exists an unambiguous-FS Pareto optimum \(x\). To see that \(x\) is unambiguous-EF, suppose by contradiction that some self \(s\) of some agent \(i\) envies agent \(j\), so that \(u_s(x_i) < u_s(x_j) = u_s(e - x_i)\). By convexity, \(u_s(x_i)\) must also be smaller than \(u_s(\frac{1}{2}x_i + \frac{1}{2}(e - x_i))\). But the bundle \(\frac{1}{2}x_i + \frac{1}{2}(e - x_i)\) is exactly the fair-share \(e\) — a contradiction to the assumption that \(x\) is unambiguous-FS. \(\square\)

The preceding proof only requires continuous and convex preferences, the assumption of monotonicity can be dropped. To show that Theorem 3.1 does not extend to economies with more agents, we construct a two-goods economy with two fully rational and one boundedly rational agent. All selves’ utilities have the single-crossing property and the preferences of the two fully rational
agents 1 and 2 are intermediate between the two selves of the boundedly rational agent 3. To avoid envy between agent 3 and the two fully rational agents, agent 3 must consume the same bundle as agents 1 and agent 2. A contradiction then arises since agents 1 and 2 have different marginal rates of substitution at any bundle, and therefore must consume different bundles in any interior Pareto optimum. Figure 1 illustrates.

**Proposition 3.2** Economics with three agents and two goods need not have any unambiguous-EF Pareto optima.

**Proof** Fix a two-good economy with two fully rational agents 1 and 2 represented by $u_1$ and $u_2$ and a boundedly rational agent 3 whose two selves are represented by $u_h$ and $u_w$. All selves’ utilities are differentiable and convex, and satisfy a single-crossing property so that for any two bundles
$(y, z), (y', z')$:

If $y' > y$ and $z' < z$ then:

\[ u_h(y', z') \geq u_h(y, z) \Rightarrow u_2(y', z') > u_2(y, z), \]
\[ u_2(y', z') \geq u_2(y, z) \Rightarrow u_1(y', z') > u_1(y, z), \]
\[ u_1(y', z') \geq u_1(y, z) \Rightarrow u_w(y', z') > u_w(y, z). \]

If $y' < y$ and $z' > z$ then:

\[ u_w(y', z') \geq u_w(y, z) \Rightarrow u_1(y', z') > u_1(y, z), \]
\[ u_1(y', z') \geq u_1(y, z) \Rightarrow u_2(y', z') > u_2(y, z), \]
\[ u_2(y', z') \geq u_2(y, z) \Rightarrow u_h(y', z') > u_h(y, z). \]

Fix an unambiguous-EF allocation $x = ((y_1, z_1), (y_2, z_2), (y_3, z_3))$. Since all preferences are strictly monotonic, either $(y_1 - y_3)(z_1 - z_3) < 0$ or $(y_1, z_1) = (y_3, z_3)$ must hold for agents 1 and 3 not to envy each other:

- If $y_1 > y_3$ and $z_1 < z_3$, then $u_1(y_1, z_1) \geq u_1(y_3, z_3)$ and single-crossing imply $u_w(y_1, z_1) > u_w(y_3, z_3)$, so self $w$ of agent 3 envies agent 1.

- If $y_1 < y_3$ and $z_1 > z_3$, then $u_1(y_1, z_1) \geq u_1(y_3, z_3)$ and single-crossing imply $u_h(y_1, z_1) > u_h(y_3, z_3)$, so self $h$ of agent 3 envies agent 1.

- So we must have $(y_1, z_1) = (y_3, z_3)$.

Mutatis mutandis $(y_2, z_2) = (y_3, z_3)$ also holds and all three agents consume the same bundle. So $x = \bar{e}$. But $\bar{e}$ cannot be Pareto optimal since the two fully rational agents consume different bundles in any interior Pareto-optimum (Theorem 2.2).

The economy in the proof of Proposition 3.2 illustrates the two countervailing forces of bounded rationality on the set of unambiguously fair Pareto optima. By standard results, a fully rational version of this economy, where agent 3 has only one self, has an unambiguous-EF Pareto optimum. With the addition of agent 3’s second self, the set of Pareto-optima increases while the set of unambiguous-EF allocations decreases. The second effect dominates: the example in the proof has no unambiguous-EF Pareto optimum. But with the further addition of $\frac{1}{2}u_w + \frac{1}{2}u_h$ to agent 1’s and agent 2’s sets of selves, the set of Pareto optima increases enough for $\bar{e}$ to be an unambiguous-EF Pareto optimum in this less-rational economy, by Theorem 2.2.
The example used to prove Proposition 3.2 can be extended to any number of agents by adding fully rational agents with preferences that are intermediate between the two selves of agent 3. To extend it to any number of goods, replace the single good $z$ with $G - 1$ perfect substitutes $z_1, \ldots, z^{G-1}$, and define self $s$'s utility for bundle $y, z_1, \ldots, z^{G-1}$ as $u_s(y, z_1 + \ldots + z^{G-1})$, where $u_s$ is the utility of $s$ in the two-goods economy.

### 3.2 Aggregate envy-freeness

While Theorem 3.2 shows that unambiguous-EF Pareto optima need not exist, standard results show that mild conditions suffice for existence of aggregate-EF Pareto optima. Here we investigate conditions under which some aggregate-EF Pareto optima have the unambiguous fair-share guarantee. When all selves’ preferences are strictly convex, the set of Pareto optima that guarantee the fair-share to each self contains some allocations that are envy free according to a particular aggregator—the leximin aggregator.

**Theorem 3.3** If all selves’ preferences are strictly convex there exist unambiguous-FS and $\succeq_{lex}$-aggregate-EF Pareto optima.

**Proof** For each agent $i$ the function $U_i^{\min}(\cdot) := \min_{s \in S_i} u_s(\cdot)$ inherits the continuity, monotonicity and strict convexity of all selves’ preferences.\(^{14}\) Say $(p, x^*, \bar{e})$ is a market equilibrium from equal endowments in the economy where each agent $i$’s preferences are represented by $U_i^{\min}$.

**Claim 1:** $x^*$ is unambiguous-FS. Fix an arbitrary agent $i$. Thanks to the normalization $u_s(\bar{e}) = 0$, and since $\bar{e}$ is in each agent’s budget set,

\(^{13}\)Since $U_i^{\min}$ is indifferent between any two bundles associated with the same minimal utility over all selves, $U_i^{\min}$ does not represent an aggregator for agent $i$’s selves. We can therefore not directly use $U_i^{\min}$ as the aggregator in Theorem 3.3.

\(^{14}\)To see the strict convexity, fix any two bundles $x \neq x'$, an $\alpha \in (0, 1)$, and an agent $i \in I$. Suppose that $U_i^{\min}(x) \geq U_i^{\min}(x')$. Let $s^* \in S_i$ be the self for which the latter minimum is attained, that is, $U_i^{\min}(x') = u_{s^*}(x')$. So we have for all $s \in S_i$

$$u_s(x') \geq u_{s^*}(x') \quad \text{and} \quad u_s(x) \geq U_i^{\min}(x) \geq U_i^{\min}(x') = u_{s^*}(x').$$

Since each $u_s$ represents a strictly convex preference we have $u_s((1 - \alpha)x + \alpha x') > u_{s^*}(x')$ for all $s \in S_i$, which implies $U_i^{\min}((1 - \alpha)x + \alpha x') > u_{s^*}(x') = U_i^{\min}(x')$. 

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\( \min_{s \in S_i} u_s(x_i^*) = U_i^\min(x_i^*) \geq U_i^\min(\bar{e}) = \min_{s \in S_i} u_s(\bar{e}) = 0. \) So \( u_s(x_i^*) \geq 0 \)

holds for each \( s \in S_i \), and \( x^* \) satisfies unambiguous-FS.

**Claim 2:** \( x^* \) is \( \succeq_i^{\text{lex}} \)-aggregate-\( f \). Fix an arbitrary agent \( i \). Since \( x^* \) is an equilibrium allocation, \( x_i^* \) is \( U_i^\min \)-maximal in agent \( i \)'s budget set. Since budget sets are convex and since \( U_i^\min \) represents strictly convex preferences \( x_i^* \) is the unique \( U_i^\min \)-maximum. By the definition of \( \succeq_i^{\text{lex}} \), the set of \( \succeq_i^{\text{lex}} \)-maxima is a subset of the set of \( U_i^\min \)-maxima, so \( x_i^* \) is also the unique \( \succeq_i^{\text{lex}} \)-maximum in the budget set. Since all agents have identical budgets, \( x_i^* \) is \( \succeq_i^{\text{lex}} \) preferred to any \( x_j^* \), and the allocation \( x^* \) is \( \succeq_i^{\text{lex}} \)-aggregate-\( f \).

**Claim 3:** \( x^* \) is Pareto optimal. Suppose that some allocation \( x' \) Pareto-improves on \( x^* \), so that \( u_s(x'_i) \geq u_s(x_i^*) \) for all \( s \in S_i \) and \( i \in I \), and \( u_{s'}(x'_i) > u_{s'}(x_i^*) \) for some \( s' \in S_j \) and \( j \in I \). The definition of \( \succeq_i^{\text{lex}} \) then yields \( x'_i \succeq_i^{\text{lex}} x_i^* \) for all \( i \in I \), and \( x'_j \succeq_j^{\text{lex}} x_j^* \) for \( j \in I \). By the arguments in Claim 2, \( x_i^* \) is for each agent \( i \) the unique \( \succeq_i^{\text{lex}} \)-maximal choice in their budget set. So \( p^* \cdot x'_i > p^* \cdot x_j^* \) holds for any agent \( j \) with \( x'_j \succeq_j^{\text{lex}} x_j^* \) while \( x_i^* = x_i^\prime \) holds for all other agents \( i \). The monotonicity of agents’ preferences implies \( p^* \cdot x_i^* = p^* \cdot \bar{e} \) for any agent \( i \). Summing over all agents, yields \( \sum_{i \in I} p^* \cdot x_i^* > p^* \cdot \bar{e} \) for any agent \( i \). A contradiction to the feasibility of \( x^* \) which requires \( \sum_{i \in I} x_i^* \leq \bar{e} \). \( \square \)

The requirement that allocations satisfy no-envy according to at least one self of each agent would appear to be weaker than \( \succeq_i^{\text{lex}} \)-aggregate no envy. To see that this is not the case, we construct a two-goods economy where no unambiguous-FS Pareto optimum is one-self envy-free. Assume two rational agents with different Cobb-Douglas utilities and a boundedly rational agent each of whose two selves only cares about one good. Say \( x^* \) was an unambiguous-FS Pareto optimum satisfying no-envy according to at least one self of each agent. By unambiguous-FS, agent 3’s selves prefer \( (y_3^*, z_3^*) \) to the fair-share \( \bar{e} = (1, 1) \), so \( y_3^* \geq 1 \) and \( z_3^* \geq 1 \). W.l.o.g., say that no envy holds for the self of agent 3 that only cares about \( y \), so that \( y_3^* \geq y_2^*, z_3^* \). For agents 1 and 2 not to envy agent 3, we then must have \( z_1^*, z_2^* \geq z_3^* \). By the single crossing property agents 1 and 2 must consume different bundles. The agents, in sum, consume more than three units of \( z \) — a contradiction.

While the two selves of agent 3 violate our standard monotonicity assump-
tion, nearby economies with strictly monotonic preferences will, by continuity, also lack unambiguous-FS Pareto optima that are one-self envy-free. To construct such a nearby economy, represent the preferences agent 3’s two selves by $y_3 + \epsilon \sqrt{z_3}$ and $z_3 + \epsilon \sqrt{y_3}$ for some small $\epsilon > 0$.15

3.3 Market equilibrium and unambiguous envy-freeness

The classic proof that envy-free Pareto optima exist (Foley, 1967) argues that market equilibria from equal endowments yield such allocations. Since the fair-share is each agent’s endowment, each agent must weakly prefer the bundle she buys in equilibrium to the fair-share. Since each agent faces the same budget set, each agent could choose any other agent’s bundle. So no agent envies any other. This technique does not fare as well in economies with boundedly rational agents. Proposition 3.2 already shows that unambiguous-EF Pareto optima need not exist. But even if such allocations exist, they may not arise as market equilibria from equal endowments.

**Proposition 3.4** (a) Some economies have unambiguous-EF and unambiguous-FS Pareto optima, even though no such allocation arises as a market equilibrium from equal endowments.

(b) In any economy, if an unambiguous-EF allocation $x$ can be sustained as a market equilibrium, and if agent $i$ is more rational than agent $j$, then agent $j$’s equilibrium budget must be at least as large as agent $i$’s.

**Proof** (a) Fix a three-agent two-good economy. Say the two fully rational agents 1 and 2 are represented by Cobb-Douglas utilities with respectively $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{2}{3}$. These utilities also represent the two selves of the bound-

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15 A similar example shows that we cannot replace the leximin aggregator in Theorem 3.3 with an arbitrary aggregator: there is no unambiguous-FS aggregate-EF Pareto optimum, according to the aggregator where agent 3 ranks any bundle with more $y$ above any bundle with less $y$ and considers $z$ only to compare two bundles with equally much $y$. Moreover, for every real constant $\kappa$, suppose that the preferences of agent $i$ are aggregated using the function $\sum_{s \in S_i} u_s^\kappa / \kappa$ (this is equivalent to the utilitarian aggregator for $\kappa = 1$, approaches the Nash aggregator for $\kappa = 0$, and approaches the leximin aggregator for $\kappa \to -\infty$; see Moulin (2004)). Then, a similar example shows that there may be no unambiguous-FS aggregate-EF Pareto optimum for any $\kappa > -\infty$. 

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14
Figure 2: Proof of Prop. 3.4. The blue discs denote the allocations of the agents in the symmetric unambiguous-EF Pareto-optimum. The solid green line is an indifference curve of agent 1 and self h of agent 3; the dashed red line is an indifference curve of agent 2 and self w of agent 3. The black dotted line represents the budget constraint with endowment \((0.981, 0.981) < \mathbf{z}\).

edly rational agent 3. We first show that this economy has a unambiguous-EF Pareto-optimum \(\mathbf{x}^* = ((y_1, z_1), (y_2, z_2), (y_3, z_3))\). For agents 1 and 2 not to envy agent 3 and vice versa, \(\frac{y_1^3}{z_1^3} = \frac{y_2^3}{z_2^3}\) and \(\frac{y_2^3}{z_2^3} = \frac{y_3^3}{z_3^3}\) must hold. If \(\mathbf{x}^*\) is on the boundary of \(X\) then at least one self has zero utility. No envy then implies that all selves have utility zero at \(\mathbf{x}^*\) which can therefore not be Pareto optimal. So \(\mathbf{x}^*\) must be in the interior of \(X\) and \(\frac{1}{2} y_1 = MRS_1(y_1, z_1) = MRS_2(y_2, z_2) = 2 \frac{y_2}{y_2}\) must hold. Combining the preceding equations with the resource constraints \(y_1 + y_1 + y_3 = 3\) and \(z_1 + z_2 + z_2 = 3\) we obtain a system of 5 equations in 6 unknowns. To obtain a concrete solution \(\mathbf{x}^*\) (illustrated in Figure 2), we additionally impose the symmetry condition \(y_3 = z_3\). The symmetric solution is:

\[
\begin{align*}
y_1^* &\approx 0.654 & z_1^* &\approx 1.308 \\
y_2^* &\approx 1.308 & z_2^* &\approx 0.654 \\
y_3^* &\approx 1.038 & z_3^* &\approx 1.038
\end{align*}
\]

Since agent 3’s bundle in \(\mathbf{x}^*\) contains strictly more than the average share from each good, \(\mathbf{x}^*\) cannot be obtained as a market equilibrium from equal endowments. So suppose some other unambiguous-EF Pareto optimum
\((y_1, z_1), (y_2, z_2), (y_3, z_3)\) could be obtained as a market equilibrium from equal endowments. By Pareto-optimality and the single-crossing property we have \((y_1, z_1) \neq (y_2, z_2)\), so that \((y_3, z_3) \neq (y_1, z_1)\) or \((y_3, z_3) \neq (y_2, z_2)\) (or both). W.l.o.g. say \((y_3, z_3) \neq (y_1, z_1)\). Since all selves have strictly convex preferences, the fully rational agent 1 strictly prefers \((y_1, z_1)\) to all other bundles in the budget set, in particular \(u_1(y_1, z_1) > u_1(y_3, z_3)\). But then agent 3 envies agent 1 according to the \(u_1\) self — a contradiction.

(b) Say \(x\) is unambiguous-EF and \((p, x, x^0)\) is a market equilibrium from some endowment \(x^0\). Suppose agent \(i\)'s equilibrium budget was strictly larger than agent \(j\)'s (so \(p x_i^0 > p x_j^0\)). By definition of market equilibrium, agent \(i\)'s bundle \(x_i\) is optimal in agent \(i\)'s budget set according to some aggregator \(\succsim_i\). Denote by \(x_i^*\) the \(\succsim_i\)-optimal choice from agent \(j\)'s budget set, so that \(x_i^* \succsim_i x_j\). Since agent \(i\)'s budget set strictly contains agent \(j\)'s, and since all selves preferences (and therefore also \(\succsim_i\)) are strictly monotonic, \(x_i \succsim_i x_i^*\). Since \(x\) is unambiguous-EF, \(u_s(x_j) \geq u_s(x_i)\) holds for all \(s \in S_j\). Since agent \(i\) is more rational than agent \(j\), we have \(S_i \subset S_j\). Since \(\succsim_i\) is an aggregator of agent \(i\)'s selves, we in sum get the contradiction \(x_j \succsim_i x_i \succsim_i x_i^* \succsim_i x_j\).

Our permissive definition of market equilibrium (Sub. 2.5) strengthens the above result. No matter which aggregators the agents happen to use while shopping, no market equilibrium from equal endowments is unambiguous-EF in the above example. The result continues to hold for any more restrictive notion of market equilibrium.

To illustrate Proposition 3.4, consider insurance policies against earthquakes and flooding. Two fully rational agents with the same income but different priors may buy different insurance policies. Now consider a boundedly rational third agent, whose selves use the priors of the two preceding agents, as in the Bewley (2002) model of Knightian uncertainty. The third agent then needs a higher income to avoid envy of the other two agents. The more rational agents can better target their income to achieve their goals. Indeed, the symmetric allocation in part a) of the above proof, can arises as a market equilibrium when we endow the boundedly rational agent 3 with a higher income than the fully rational agents.
4 Egalitarian Equivalence

4.1 Unambiguous egalitarian-equivalence

Proposition 4.1 shows that unambiguous egalitarian-equivalence is even more restrictive than unambiguous envy-freeness, in the sense that even two-agent economies may lack unambiguous-EE Pareto optima. The fact that an agent with two selves with the single-crossing property is only indifferent between a reference bundle \( r \) and some bundle \( x \) if \( x = r \) is the driving force behind the upcoming non-existence proof. This proof specifies an economy with two such agents who must both consume the reference bundle \( r \) in any unambiguous-EE allocation. At the same time, we differentiate the agents enough for them to consume different bundles in any Pareto optimum.

**Proposition 4.1** When there are at least two agents and at least two goods, a unambiguous-EE Pareto optimum might not exist.

**Proof** Fix a two-agents-two-goods economy. Define \( u_\gamma(y, z) := y^\gamma + z \). Represent the two selves of agents 1 and 2 by \( u_{1/5} \) and \( u_{1/4} \), and respectively \( u_{1/3} \) and \( u_{1/2} \). Fix an unambiguous-EE Pareto optimum \( x \) with reference bundle \( r \). For \( x \) to be an unambiguous-EE Pareto optimum, each agent must consume a positive amount of \( y \). Assuming large enough total endowment of \( z \), each agent must also consume a positive quantity of \( z \) in \( x \). The single-crossing property together with \( u_{1/4}(y_1, z_1) = u_1(r) \) and \( u_{1/5}(y_1, z_1) = u_1(r) \) imply \( (y_1, z_1) = r \). By the same token, agent 2 must also consume \( r \), so that \( x = (r, r) \). Since \( \max_{s \in S_1} MRS_s(r) > \min_{s \in S_2} MRS_s(r) \) the allocation is by Theorem 2.2 not Pareto optimal. Figure 3 illustrates.

The analysis of the above example remains unchanged if we add further agents to the above economy. As long as such agents have multiple selves with the single-crossing property, they must consume the reference bundle \( r \) to be unambiguously indifferent between their consumption and \( r \). With enough differentiation between the agents they cannot all consume the same bundle in a Pareto optimum. For an example with more than two goods, replace the single good \( z \) with some \( G - 1 \) perfect substitutes \( z^1, \ldots, z^{G-1} \), as

\(^{16}\)This holds since the marginal utility of \( y \) at 0 is for each \( u_\gamma \) infinite.
Figure 3: Proof of Proposition 4.1. Agent 1 and 2’s two selves are respectively illustrated by the solid and dotted indifference curves. To be indifferent between their bundle and some $r$, they must consume exactly $r$. But this allocation is Pareto dominated by giving the solid agent $r + (\epsilon, -\epsilon)$ and giving the dotted agent $r + (-\epsilon, \epsilon)$, for some small $\epsilon$.

in comment after Proposition 3.2. As is the case for unambiguous-EF Pareto-optima, the existence of unambiguous-EE Pareto-optima is not “monotone” in the variability of selves: if we increase each agent’s set of selves in the above example to include each other’s selves, then the equal split is Pareto-optimal and unambiguous-EE by Theorem 2.2.

4.2 Aggregate egalitarian-equivalence

In parallel to Subsection 3.2 on envy freeness, there exist aggregators for which efficiency, the unanimous fair-share guarantee and aggregate egalitarian equivalence are compatible. Theorem 4.2 shows that such compatibility is achieved for the Nash aggregator. To define this aggregator, interpret agents and their selves as families and their members. If these families use Nash-bargaining with the fair-share as the outside option, they choose the bundle $x_i^*$ that maximizes the product of their members’ utilities.

The monotonicity and continuity of all selves’ preferences allows us to normalize any $u_s$ such that $u_s(x_i) = t - 1$ if $u_s(x_i) = u_s(t \cdot \overline{e})$, where $u_s(\overline{e}) = 0$ continues to hold. Define $U_i^{Nash}(x) = |S_i| \sqrt{\prod_{s \in S_i} u_s(x_i)}$ to represent agent $i$’s Nash-bargaining aggregator over bundles in the interior of $E$. For any
\[ t \geq 1 \text{ we have } U_i^{Nash}(t \cdot e) = |S| \sqrt{\prod_{s \in S_i} u_s(t \cdot e)} = |S| \sqrt{(t - 1)|S|} = t - 1. \] Since \( U_i^{Nash} \) only represents an aggregator for bundles that all selves strictly prefer to \( e \), we use the leximin aggregator for all remaining comparisons to define an aggregator \( \succsim_i^{Nash} \). Formally, the value of \( x \succsim_i^{Nash} x' \) equals —

\[
U_i^{Nash}(x) \geq U_i^{Nash}(x') \quad \text{if } u_s(x), u_s(x') > 0 \text{ for all } s \in S_i; \\
x \succsim_i^{lex} x' \quad \text{otherwise.}
\]

**Theorem 4.2** If all selves have strictly convex preferences, there exist unambiguous-FS and \( \succsim_i^{Nash} \)-aggregate-EE Pareto optima.

**Proof** Define \( x^* \) as the allocation in \( E \) that maximizes the minimal Nash bargaining utility:

\[
x^* \in \arg \max_{x \in E} \min_{i \in I} U_i^{Nash}(x_i)
\]

Such an allocation exists since each \( U_i^{Nash} \) is continuous on the compact and non-empty set \( E \). Define \( t^* := 1 + \min_{i \in I} U_i^{Nash}(x_i^*) \). Since \( x^* \in E, t^* - 1 \geq 0. \)

**Claim 1:** There is no allocation \( x \in E \) for which \( U_i^{Nash}(x_i) \geq t^* - 1 \) for all \( i \) and \( U_j^{Nash}(x_j) > t^* - 1 \) for some \( j \in I \). Suppose \( E \) contained such an \( x \). Since \( U_j^{Nash} \) is continuous, there exists a small bundle \( \epsilon > 0 \) such that \( U_j^{Nash}(x_j - \epsilon) > t^* - 1 \) and \( u_s(x_j - \epsilon) > 0 \) for each \( s \in S_j \). Evenly redistribute \( \epsilon \) from agent \( j \) to all others to obtain the allocation \( x^0 \). The strict monotonicity of all selves’ preferences yields \( U_i^{Nash}(x_i^0) > t^* - 1 \) for all \( i \neq j \). In sum we get \( \min_{i \in I} U_i^{Nash}(x_i^0) > t^* - 1 \) — a contradiction to the definition of \( x^* \).

**Claim 2:** \( x^* \) is unambiguous-FS. Follows from \( x^* \in E \).

**Claim 3:** If \( x^* \neq \bar{e} \) then \( x^* \) is in the interior of \( E \), that is, \( x^* \) yields strictly positive utilities to all selves. Suppose for contradiction that \( x^* \neq \bar{e} \) but \( u_s(x_i^*) = 0 \) for some agent \( i \) and \( s \in S_i \). This implies that \( U_i^{Nash}(x_i^*) = 0 \), and \( t^* - 1 = \min_{i \in I} U_i^{Nash}(x_i^*) = 0 \). Define \( x' := \frac{1}{2} x^* + \frac{1}{2} \bar{e} \). Since \( x^* \neq \bar{e} \), there exists an agent \( j \) with \( x_j^* \neq \bar{e} \). By the strict convexity of all selves’ preferences, \( U_i^{Nash}(x'_i) \geq U_i^{Nash}(\bar{e}) = \min_{i \in I} U_i^{Nash}(x_i^*) = 0 \) holds for all \( i \in I \) and \( U_j^{Nash}(x'_j) > 0 \) — a contradiction to Claim 1.
Claim 4: \( x^* \) is Pareto-optimal. If some allocation \( x \) Pareto-dominates \( x^* \), then \( U_i^{\text{Nash}}(x_i) \geq U_i^{\text{Nash}}(x_i^*) \geq \min_{i \in I} U_i^{\text{Nash}}(x_i^*) = t^* - 1 \) for all \( i \) and \( U_j^{\text{Nash}}(x_j) \geq U_j^{\text{Nash}}(x_j^*) \geq \min_{i \in I} U_i^{\text{Nash}}(x_i^*) = t^* - 1 \) for some \( j \in I \), contradicting Claim 1.

Claim 5: \( x^* \) is \( \succsim_{\text{Nash}} \)-aggregate-EE. If \( x^* = \bar{e} \), then each agent gets exactly \( \bar{e} \), and \( x^* \) is even unambiguous-EE. So assume that \( x^* \neq \bar{e} \). By Claim 3, \( x^* \) is in the interior of \( E \). By Claim 1, \( U_i^{\text{Nash}}(x_i^*) = t^* - 1 = U_i^{\text{Nash}}(t^* \bar{e}) \) holds for all agents \( i \in I \). Since \( U_i^{\text{Nash}} \) represents the aggregator \( \succsim_i^{\text{Nash}} \) in the interior of \( E \), \( x^* \) is \( \succsim_{\text{Nash}} \)-aggregate-EE with the reference bundle \( t^* \bar{e} \). \( \square \)

The proof of Theorem 4.2 uses two properties of the Nash aggregator: Firstly, the continuity of \( U_i^{\text{Nash}} \) on \( E \) is required for \( x^* \) to be well-defined. Secondly, \( U_i^{\text{Nash}}(x_i^*) > 0 \) holds if and only if \( u_s(x_i^*) > 0 \) for all \( s \in S_i \), so the allocation \( x^* \) is either \( (\bar{e}, \ldots, \bar{e}) \) or in the interior of \( E \) (Claim 3).

The proof holds unchanged for any aggregator with these two properties. For example, \( U_i(x_i) := \int_{t_1=0}^{u_1(x_i)} \ldots \int_{t_k=0}^{u_k(x_i)} f_i(t_1, \ldots, t_k) \) for any positive measurable function \( f_i \) of \( k \) variables, where \( S_i = \{u_1, \ldots, u_k\} \). The function \( U_i^{\text{Nash}} \) corresponds to the special case \( f_i \equiv 1 \). Other natural aggregators, such as the leximin or the utilitarian aggregator, violate one of these properties. Similarly, if we consider egalitarian equivalence according to one self per agent, then the second property is violated. We do not know for which alternative aggregators there always exist unambiguous-FS aggregate-EE Pareto-optima.

4.3 Weakly convex preferences

Proposition 4.3 below shows that economies where selves have merely convex (not strictly convex) preferences need not have any unambiguous-FS and aggregate-EE Pareto optima. We can however guarantee any two out of these three properties:

Proposition 4.3 When all selves have (weakly) convex preferences: (a) Some three-agent economies have no unambiguous-FS and aggregate-EE optimum. (b) There always exist aggregate-EE Pareto optima, unambiguous-FS Pareto optima, and aggregate-EE unambiguous-FS allocations.

\[ \text{We are grateful to Magma and Martin R. for these insights.} \]
Figure 4: Proof of Proposition 4.3. The solid green line is $y + z = 2$; in any unambiguous-FS allocation, all bundles must be on that line.

**Proof** (a) Say $u_3(y, z) = y + z$ represents the preferences of the fully rational agent 3. Agents 1 and 2 have two selves each. The preference of their first selves coincide with agent 3’s, and we have $u_1(y, z) = u_2(y, z) = u_3(y, z) = y + z$. The two other selves of agents 1 and 2 are represented by different Cobb-Douglas utilities: $v_1(y, z) = y^{1/4}z^{3/4}$ and $v_2(y, z) = y^{3/4}z^{1/4}$.

For the fair-share guarantee to hold according to the three selves with identical linear preferences, $y_1 + z_1 = y_2 + z_2 = y_3 + z_3 = 2$ must hold. The only Pareto-optimal allocation satisfying these equations gives agent 1 $(1/2, 3/2)$ and agent 2 $(3/2, 1/2)$; see Figure 4.

Now suppose the allocation was aggregate-EE for some aggregator and reference bundle $r := (y_r, z_r)$. For agent 3 to be indifferent between his bundle $(1, 1)$ and $r$, $y_r + z_r$ must equal 2, so $r$ must lie on the solid green line in Figure 4. This implies that the two other agents are according to their linear selves indifferent between their bundles and $r$. But, for any $r$ with $y_r + z_r = 2$, at least one agent prefers her bundle to $r$ according to her Cobb-Douglas self. This agent must according any aggregator strictly prefer
her bundle to $r$.\footnote{To see where the proof of Theorem 4.2 fails on this economy, note that $E$ contains only allocations $x$ with $y_i + z_i = 2$ for all agents $i$ (the set represented by the solid green line). So $U^N_{i,x}(x_i) = 0$ holds for all $i \in I$ and all $x \in E$, and any $x \in E$ satisfies the definition of $x^*$. In particular, Claim 3 fails.}

(b) Theorem 2.1 proved the existence of unambiguous-FS Pareto optima. The equal allocation is aggregate-EE and unambiguous-FS. To find an aggregate-EE Pareto optimum without unambiguous-FS, replace the Nash aggregator in the proof of Theorem 4.2 with the utilitarian aggregator. As this aggregator is represented by a continuous function on the entire set $X$ (not only in the interior of $E$), the proof of aggregate-EE is now valid without the need for Claim 3.

\hfill $\Box$

5 Intransitive aggregators

An alternative approach towards fairness in the context of behavioral economics would explicitly consider some intransitive aggregations of the agents’ selves. Agents might aggregate their objectives intransitively, using voting rules or by sequential procedures, as suggested by respectively Green and Hojman (2007) and Apesteguia and Ballester (2013). For a concrete example, say an allocation is majority-fair if at least half of the selves of any given agent deem it to be fair.\footnote{Majority fairness was studied in the context of cake-cutting (Segal-Halevi and Nitzan, 2019) and indivisible item allocation (Segal-Halevi and Suksompong, 2019).} So an allocation is majority envy free (majority-EF) if no more than half of any agent’s selves envy a different agent, it is majority egalitarian equivalent (majority-EE) if there exists a references bundle $r$ such that at least half of the selves of any given agent are indifferent between $r$ and their bundle.

Our negative results can be extended to show that majority-fairness is incompatible with efficiency. For a parallel to the our result on unambiguous-EF results Theorem 3.2, consider an economy with four rational agents $(1, 2, 3, 4)$ and one boundedly rational agent 5 with three selves $(5a, 5b, 5c)$. Say that the utilities of all selves have the single-crossing property. The
indifference-curves of the rational agents 1 and 2 are between the indifference curves of the selves 5a and 5b; the indifference curves of the remaining two rational agents 3 and 4 are between the indifference curves of selves 5b and 5c. Suppose a majority-EF efficient allocation $x^*$ exists and that all agents consume positive quantities of each good in this allocation. Due to the single crossing property, no two rational agents can consume the same quantity in the allocation $x^*$. In any majority-EF allocation, at most one self of agent 5 may envy any other agent. So say without loss of generality that self 5a as well as self $s \in \{5b, 5c\}$ do not envy any other agent. Now we have a situation similar to the example used to prove Proposition 3.2: No envy, together with the fact that agent 1’s indifference curves lie between the indifference curves of selves 5a and $s$, imply that agents 1 and 5 must consume the same bundle. Mutatis mutandis, we see that also agent 2 and agent 5 have to consume the same bundle — a contradiction to agents 1 and 2 consuming different bundles in any interior Pareto optimum.

Clearly, the dearth of unambiguous-EF allocations is owed to the fact that the unambiguous ranking according to which each agent must prefer their own bundle to any other agent’s might be highly incomplete. Our notion of majority-EF then presents an interesting contrast: even though majority preference is a complete ranking, the example used to prove the non-existence result for unambiguous-EF transfers with some modification to the case of majority-EF allocations. So we see that both - completeness and transitivity - play their role in existence of envy free Pareto optima in rational environments.

Similarly, our example in the proof of result that unambiguous-EE need not exist (Theorem 4.1) can be extended to majority egalitarian equivalence: Consider an economy with 2 agents, each of whom has 3 selves. Say all utilities in the model satisfy the single-crossing property. In any majority-EE allocation, at least two selves of each agent must be indifferent between their bundle and the reference bundle $r$, so the situation is similar to the one analyzed in Proposition 4.1.
6 Related work

6.1 Behavioral Welfare Economics

While there is a large literature on boundedly rational choices by single agents, our paper belongs to the much smaller literature on welfare economics with two or more boundedly rational agents. [Bernheim and Rangel (2009)] define several notions of Pareto-optimality and of competitive equilibrium for economies with multi-self agents, and prove that competitive equilibria always satisfy one of these notions of Pareto-optimality. [Mandler (2014)] proves that, with multi-self agents, the set of Pareto-optima might be very large, in the sense that it has the same dimension as the set of all allocations. Since almost all Pareto-optima lie in full-dimensional neighborhoods of other Pareto optima, he argues that the Pareto criterion is of limited use for policy decisions. To close this gap of indecisiveness, [Mandler (2020)] suggests to use utilitarian-optimality. [Danan et al. (2015)] characterize a utilitarian social-choice rule when agents have incomplete preferences, and [Fleurbaey and Schokkaert (2013)] characterizes an egalitarian social-choice rule — based on the principle of income-equivalence — with incomplete preferences.

6.2 Fairness criteria

The modern study of fairness in economics was initiated by [Steinhaus (1948)], who proved the existence of fair-share allocations of a heterogeneous good ("cake"). Since then, fairness has been extensively studied in economics (Young, 1995; Moulin, 2004; Thomson, 2011) as well as in other disciplines such as mathematics and computer science.

The existence of envy-free Pareto-optima was initially proved as a consequence of the existence of market equilibrium (Foley, 1967). Varian (1974) showed that no-envy Pareto-optima may exist even when a competitive equilibrium does not. Varian (1974) replaces assumption of convex preferences with a condition on the set of allocations associated with any weakly-Pareto-optimal vector of utilities. If any such set is a singleton an envy free Pareto optimum exists. Svensson (1983) and Diamantaras (1992) then respectively showed that Varian (1974)'s “singleton”-requirement can be replaced by con-
vexity and contractibility. Diamantaras (1992)’s result applies to economies with public goods. Svensson (1994) and Bogomolnaia et al. (2017) showed that envy free Pareto optima exist under yet further relaxed conditions.

The egalitarian equivalence criterion was introduced by Pazner and Schmeidler (1978). They argue that the equal split \( \bar{e} \) where each agent gets the same bundle is, from an egalitarian perspective, an ideal division. Since \( \bar{e} \) is usually not efficient, Pazner and Schmeidler (1978) propose to regain efficiency by considering all allocations for which there exists an (infeasible) allocation where all agents consume the same bundle, such that each agent is indifferent between that bundle and their consumption in the allocation. Pazner and Schmeidler (1978) proved that egalitarian-equivalent Pareto optima exist in economies with production, which may not have any envy-free Pareto-optima (Vohra, 1992). Thomson (1990) showed that egalitarian equivalent Pareto optima are generically not envy-free when there are at least three agents.

### 6.3 Fair division among families

Several papers study fair division with multi-self agents under the interpretation of agents and their selves as families and their members. Segal-Halevi and Nitzan (2019) study fair division in cake-cutting problems, where the challenge is to allocate connected pieces, or at least pieces made of a small number of connected components. Manurangsi and Suksompong (2017); Suksompong (2018); Segal-Halevi and Suksompong (2019); Kyropoulou et al. (2019) study fair division in problems with indivisible goods. Since there may be no fair allocation of indivisible goods, the focus in these studies is on finding approximately-fair allocations. In contrast, our model of fully-divisible goods always allows for fair allocations, and the challenge is to find allocations that are both fair and efficient. Ghodsi et al. (2018) studies fair division of rooms and rent among families of tenants, using three notions of fairness termed strong, aggregate and weak. Their strong-fairness is our unambiguous no-envy; their weak-fairness holds if at least one self per agent does not envy; their aggregate-fairness corresponds to our aggregate fairness with the utilitarian aggregator.
6.4 Public and club goods

The existence of fair allocations with public and private goods has been studied e.g. by Diamantaras (1992); Diamantaras and Wilkie (1994, 1996); Guth and Kliemt (2002). With multi-self agents, our fairness notions correspond to club goods (Buchanan, 1965), that is, goods that are public among all selves of the agent — but private outside. As far as we know, fair allocation of goods among different clubs has not yet been considered. Under the interpretation of multi-self agents as clubs, questions such as the optimal number of members in a club, optimal number of clubs, optimal quantity of club-good provision, pricing policies and exclusion mechanisms have been studied (Sandler and Tschirhart, 1980; Hillman, 1993; Sandler and Tschirhart, 1997; Loertscher and Marx, 2017; Mackenzie and Trudeau, 2019). A modern example of a club good is information: information is partly excludable (via intellectual property law), but once it is given to a group, it is not rival. Therefore our work may have implications on fair division of information, for example, dividing training samples for machine learning among groups of researchers.

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