A review on recent results of the $\zeta$-function regularization procedure in curved spacetime

Valter Moretti

Department of Mathematics, Trento University and Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Trento, I-38050 Povo (TN), Italy.

E-mail: moretti@science.unitn.it

February 1999

Abstract: Some recent (1997-1998) theoretical results concerning the $\zeta$-function regularization procedure used to renormalize, at one-loop, the effective Lagrangian, the field fluctuations and the stress-tensor and some applications are reviewed.

1 Quasifree QFT in curved static manifolds, Euclidean approach $\zeta$-function technique.

1.1 Preliminaries. Let $(L, g_{\mu\nu}^L)$ be a Lorentzian manifold which is supposed to be globally hyperbolic. This manifold is said (locally) static whenever it admits a (local) time-like Killing vector $\partial_t$ normal to a Cauchy surface $\Sigma$. In other words, the manifold admits a (local) coordinate frame $(x^0, x^1, x^2, x^3) \equiv (t, \vec{x})$, where $g_{0i} = 0$ ($i = 1, 2, 3$) and $\partial_t g_{\mu\nu} = 0$. Let us consider a real scalar field $\phi$ propagating in $L$, its evolution equation can be written down

$$A'_{L,\phi} = 0,$$

(1)

where $A'_{L} := -\nabla^\mu \nabla^\mu + V$, $V$ being a smooth scalar field of the form

$$V(x) := \xi R + m^2 + V'(x).$$

(2)

Above, $V'$ is another smooth scalar field satisfying $\partial_t V = 0$, moreover $\xi$ is a constant $\xi \in \mathbb{R}$, $R$ is the scalar curvature and $m^2$ the squared mass of the particles associated to the field. The operator $A'_{L}$ works on a space of real-valued $C^\infty$ functions. A straightforward way to build up the QFT is the definition of opportune Green functions of the operator $A'_{L}$ \[FR87\], in particular the Feynman propagator $G_F(x, x')$ or, equivalently, the Wightman functions $W_{\pm}(x, x')$. Then, generalizations of GNS theorems \[KW91\] in curved spacetime allow one to build up a corresponding Fock space and a quasifree QFT, $G_F$ corresponding to a (not necessarily pure) vacuum state. In a globally hyperbolic region where the static coordinates above are defined, it must be possible a choice of Green functions which are invariant under translations of the local static Killing time $x^0 = t$. These Green functions should determine a static vacuum which may
represent a thermal state. Moreover these coordinates allow one to perform the Wick rotation to get the Euclidean formulation of the QFT. This means that (locally) one can pass from the Lorentzian manifold \((L, g^L_{\mu\nu})\) to a Riemannian manifold \((M, g^R_{ab})\) by the analytic continuation \(t \to i\tau\) where \(t, \tau \in \mathbb{R}\). This defines a (local) Killing vector \(\partial_\tau\) in the Riemannian manifold and a corresponding (local) “static” coordinate frame \(\langle \tau, \vec{x} \rangle\) therein. As is well-known [FR87], in the case the Riemannian manifold has been made compact along the Euclidean time \(\tau\) with a period \(\beta\), \(T = 1/\beta\) has to be interpreted as the temperature of the quantum state. The corresponding Feynman propagator, whenever continued in the Riemannian manifold, admits \(\beta\) as Euclidean temporal period. In this approach, the Feynman propagator \(G_F(t-t', \vec{x}, \vec{x}')\) determines and, generally speaking [FR87], itself is completely determined by, a proper Green function (in the spectral theory sense) \(S_\beta(\tau-\tau', \vec{x}, \vec{x}')\) of a self-adjoint extension \(A\) of the operator

\[
A' := -\nabla_a \nabla^a + V(\vec{x}) : C^\infty_0(M) \to L^2(M, d\mu_g). \tag{3}
\]

(Above, \(M\) can be restricted to an opportune region where the metric is static.) \(S_\beta(\tau-\tau', \vec{x}, \vec{x}')\), said the Schwinger function, is the integral kernel of \(A^{-1}\), supposing \(A > 0\).

The partition function of the quantum state can be computed as the functional integral evaluated over the field configurations periodic with period \(\beta\) in the Euclidean time

\[
Z_\beta = \int \mathcal{D}\phi \ e^{-S_E[\phi]}, \tag{4}
\]

the Euclidean action \(S_E\) being \((d\mu_g := \sqrt{g} d^4 x)\)

\[
S_E[\phi] = \frac{1}{2} \int_M d\mu_g(x) \phi(x) A_x \phi(x). \tag{5}
\]

Thus, formally speaking, one has

\[
Z_\beta = \left\{ \det \left( \frac{A}{\mu^2} \right) \right\}^{-1/2}, \tag{6}
\]

where \(\mu\) is a mass scale which is necessary for dimensional reasons.

### 1.2 The \(\zeta\)-function technique

The most intriguing problem related to the applications of (6) concerns the interpretation of the determinant of an operator. An interesting suggestion is given by the \(\zeta\)-function procedure. Suppose \(A\) is a \(n \times n\) positive-definite Hermitian matrix with eigenvalues \(0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\). Then one can define the complex-valued function

\[
\zeta(s|A) = \sum_{j=1}^{n} \lambda_j^{-s}, \tag{7}
\]

where \(s \in \mathbb{C}\). (Notice that \(\lambda_j^{-s}\) is well-defined since \(\lambda_j > 0\).) It is now a trivial task to prove that

\[
\det A = e^{-\frac{d\zeta(s|A)}{ds}|_{s=0}}. \tag{8}
\]
In the case $M$ is a $D$-dimensional Riemannian compact manifold and $A'$ is bounded below by some constant $b \geq 0$, this procedure can be generalized to operators. In this case, $A'$ admits the Friedrichs self-adjoint extension $A$ which is also bounded below by the same bound of $A'$, moreover the spectrum of $A$ is discrete and each eigenspace has a finite dimension. Then one can consider the series with $s \in \mathcal{C}$ (the prime on the sum means that any possible null eigenvalues is omitted)

$$\zeta(s|A/\mu^2) := \sum_j' \left( \frac{\lambda_j}{\mu^2} \right)^{-s}.$$  

(9)

As is well-known, the series above converges provided $\text{Re} \ s > D/2$, where $D$ is the dimension of $M$. Moreover, it is possible to continue the right-hand side above into a meromorphic function of $s$ which is regular at $s = 0$ (see [Mo98] for a short review of the principal properties of the convergence of the series above and for the corresponding bibliography). Following (6) and (8), the idea [Ha77, EORBZ] is to define

$$Z_\beta := e^{\frac{1}{2} \int_0^1 ds \zeta(s|A/\mu^2)},$$  

(10)

where the function $\zeta$ on the right-hand side is the analytic continuation of that defined in (9). It is possible the define the $\zeta$ function in terms of the heat kernel of the operator $A$, $K(t, x, y|A)$. One has, for $\text{Re} \ s > D/2$,

$$\zeta(s|A/\mu^2) = \int_M d\mu_g(x) \int_0^\infty dt \frac{\mu^{2s}t^{s-1}}{\Gamma(s)} [K(t, x, x|A) - P_0(x, x|A)],$$  

(11)

$P(x, y|A)$ is the integral kernel of the projector on the null-eigenspace of $A$. If the manifold $M$ is not compact, $A$ has also a continuous-spectrum part, however, it is still possible to generalize the definitions and the results above considering opportune integrals on the spectrum of the operator $A$ provided $A$ is strictly positive (e.g., see [Wa79]).

Another very useful tool is the local $\zeta$ function [Wa79], which can be defined in two different but equivalent ways:

$$\zeta(s, x|A/\mu^2) = \int_0^\infty dt \frac{\mu^{2s}t^{s-1}}{\Gamma(s)} [K(t, x, x|A) - P_0(x, x|A)],$$  

(12)

and, $\phi_j$ being the smooth eigenvector of the eigenvalue $\lambda_j$,

$$\zeta(s, x|A/\mu^2) = \sum_j' \left( \frac{\lambda_j}{\mu^2} \right)^{-s} \phi_j(x) \phi_j^*(x).$$  

(13)

Both the integral and the series converges for $\text{Re} \ s > D/2$. The local zeta function enjoys the same analyticity properties of the integrated $\zeta$ function ([Wa79, Mo98]). For future convenience it is also useful to define, in the sense of the analytic continuation,

$$\zeta(s, x, y|A/\mu^2) = \int_0^\infty dt \frac{\mu^{2s}t^{s-1}}{\Gamma(s)} [K(t, x, y|A) - P_0(x, y|A)],$$  

(14)
(see [Mo98, Mo98b] for the properties of this off-diagonal \( \zeta \)-function). In the framework of the \( \zeta \)-function regularization framework, the effective Lagrangian is defined as
\[
\mathcal{L}(x|A)_{\mu^2} := \frac{1}{2} \left. \frac{d}{ds} \right|_{s=0} \zeta(s, x|A/\mu^2),
\]
and thus, in a thermal theory, \( Z_\beta = e^{-S_\beta} \) where \( S_\beta = \int d\mu g \mathcal{L}_\beta \mu^2 \). A first recent result which, in compact manifolds at least, generalizes to any dimension an earlier results by Wald [Wa79], has been obtained in [Mo98]. This result proves how the effective Lagrangian can be obtained by a point-splitting procedure (see below); for \( D \) even it reads
\[
\mathcal{L}(y|A)_{\mu^2} = \lim_{x \to y} \left\{ -\int_0^\infty \frac{dt}{2t} K(t, x, y|A) - \frac{a_{D/2}(x, y)}{2(4\pi)^{D/2}} \ln \frac{\mu^2 \sigma(x, y)}{2} \right\}
+ \sum_{j=0}^{D/2-1} \frac{D}{2j-1} \frac{a_j(x, y|A)}{2(4\pi)^{D/2}} \left( \frac{2}{\sigma(x, y)} \right)^{D/2-j},
\]
and for \( D \) odd (notice that \( \mu \) disappears from the final result)
\[
\mathcal{L}(y|A)_{\mu^2} = \lim_{x \to y} \left\{ -\int_0^\infty \frac{dt}{2t} K(t, x, y|A) - \sqrt{\frac{2}{\sigma(x, y)}} \frac{a_{(D-1)/2}(x, y)}{2(4\pi)^{D/2}} \right\}
+ \sum_{j=0}^{(D-3)/2} \frac{(D-2j-2)!!}{2^{(D+1)/2-j}} \frac{a_j(x, y|A)}{2(4\pi)^{D/2}} \left( \frac{2}{\sigma(x, y)} \right)^{D/2-j}.
\]
Above, \( \sigma(x, y) \) is one half the square of the geodesical distance of \( x \) from \( y \) and the coefficients \( a_j \) are the well-known off-diagonal coefficients of the small-\( t \) expansion of the heat-kernel (see [Mo98] for a short review on the properties of these coefficients).

2 Improvements of the local \( \zeta \)-function technique.

2.1 Generalizations of the local \( \zeta \) function technique. Besides the effective Lagrangian and the effective action, further important one-loop quantities are the (quantum) field fluctuation and the averaged (quantum) stress tensor. These quantities are given, in terms of the Euclidean path integral, by
\[
< \phi^2(x) > = \frac{\delta}{\delta J(x)} |_{J=0} \ln \int \mathcal{D}\phi \ e^{-S_E} + \int d\mu \phi^2 J,
\]
\[
< T_{ab}(x) > = \frac{2}{\sqrt{g(x)}} \frac{\delta}{\delta g^{ab}(x)} \ln \int \mathcal{D}\phi \ e^{-S_E [g]}. \]

A very popular method to compute the quantities above which in the practice diverge, is the so-called point-splitting procedure [BD82, Fu91, Wa94], anyhow, it is possible to generalize the \( \zeta \)-function method in order to build up opportune \( \zeta \) functions which regularize the quantities above.
directly, similarly to the procedure for the effective Lagrangian \[Mo97\], \([M98\], Mo98, Mo98b\]. Let us consider the stress tensor, the way to get a direct \(\zeta\)-function regularization procedure is based on the following chain of formal identities \[Mo97\]

\[
\sqrt{g(x)} < T_{ab}(x) > = \frac{2\delta}{\delta g^{ab}(x)} \ln Z_{\beta} = \frac{\delta}{\delta g^{ab}(x)} \frac{d}{ds}|_{s=0} \zeta(s|A/\mu^2)
\]

\[
= \frac{\delta}{\delta g^{ab}(x)} \frac{d}{ds}|_{s=0} \sum' \left( \frac{\lambda_j}{\mu^2} \right)^{-s} = \frac{d}{ds}|_{s=0}\mu^{-2s} \sum' \frac{\delta \lambda_j^{-s}}{\delta g^{ab}(x)}.
\]

(20)

Thus, one define the \(\zeta\)-regularized (or renormalized) stress tensor as

\[
<T_{ab}(x|A) >_{\mu^2} := \frac{1}{2} \frac{d}{ds}|_{s=0} Z_{ab}(s, x|A/\mu^2),
\]

where, in the sense of the analytic continuation of the left-hand side

\[
Z_{ab}(s, x|A/\mu^2) := 2 \sum' \mu^{-2s} \frac{\delta \lambda_j^{-s}}{\delta g^{ab}(x)}.
\]

(22)

The mathematical problem is whether the right-hand side above can be computed in the practice and whether it defines an analytic function of \(s\) in a neighborhood of \(s = 0\). We have the result \[Mo97, Mo98\]

**T1.** If \(M\) is compact, \(A \geq 0\) and \(\mu^2 > 0\), then \(Z_{ab}(s, x|A/\mu^2)\) is well-defined and is a \(C^\infty\) function of \(x\) which is also meromorphic in \(s \in \mathbb{C}\). In particular, it is analytic in a neighborhood of \(s = 0\).

In the practice, the result above has been checked also in several noncompact manifolds also containing singularities \[Mo97\]. In these cases the summation on the right hand side of (22) has to be changed into a spectral integration. The form of the series on the right-hand side of (22) is \[Mo97, Mo98\],

\[
s \sum' \left\lbrace \frac{2}{\mu^2} \left( \frac{\lambda_j}{\mu^2} \right)^{-s-1} T_{ab}[\phi_j, \phi_j^*] (x) + g_{ab}(x) \left( \frac{\lambda_j}{\mu^2} \right)^{-s} \right\rbrace.
\]

Above \(T_{ab}[\phi_j, \phi_j^*] (x)\) is the classical stress tensor evaluated on the modes of the Euclidean-motion operator \(\hat{A} = -\Delta + \xi R + m^2 + V(x)'\) (see \[Mo97, Mo98\] for details). The series converges for \(Re\ s > 3D/2 + 2\).

Similarly, it is possible to built up a \(\zeta\) function for the field fluctuation \[M98, Mo98\]. One has

\[
< \phi^2(x|A) >_{\mu^2} := \frac{d}{ds}|_{s=0} \Phi(s, x|A/\mu^2),
\]

where

\[
\Phi(s, x|A/\mu^2) := \frac{s}{\mu^2} \zeta(s + 1, x|A/\mu^2).
\]

(23)
The properties of these functions have been studied in [IM98, Mo98] and several applications on concrete cases are considered (e.g. cosmic-string spacetime and homogeneous spacetimes). In particular, in [Mo98], the problem of the change of the parameter $m^2$ in the field fluctuations has been studied.

2.2 Physically correctness of the given regularization procedures. We are now concerned with the physical interest of the found regularization techniques. To this end, the following quite general results are relevant and prove that the proposed technique are physically good candidates [Mo97, Mo98a]

T2 If $M$ is compact, $A \geq 0$ and $\mu^2 > 0$, and the averaged quantities above are those defined above in terms of local $\zeta$-function regularization, then

(a) $< T_{ab}(x|A) >_{\mu^2}$ defines a $C^\infty$ symmetric tensorial field.

(b) Similarly to the classical result,

$$\nabla^b < T_{bc}(x|A) >_{\mu^2} = -\frac{1}{2} < \phi^2(x|A) >_{\mu^2} \nabla_c V'(x).$$

(24)

(c) Concerning the trace of the stress tensor, it is naturally decomposed in the classical and the (correct) anomalous part

$$g^{ab}(x) < T_{ab}(x|A) >_{\mu^2} = \left( \frac{\xi_D - \xi}{4\xi_D - 1} - m^2 - V'(x) \right) < \phi^2(x|A) >_{\mu^2} + \delta_D \frac{a_D(x,x|A)}{(4\pi)^{D/2}} - P_0(x,x|A),$$

(25)

where $\delta_D = 0$ if $D$ is odd and $\delta_D = 1$ if $D$ is even, $\xi_D = (D - 2)/[4(D - 1)],$

(d) for any $\alpha > 0$

$$< T_{ab}(x|A) >_{\alpha\mu^2} = < T_{ab}(x|A) >_{\mu^2} + t_{ab}(x) \ln \alpha,$$

(26)

where, the form of $t_{ab}(x)$ which depends on the geometry only and is in agreement with Wald’s axioms [Wa94], has been given in [Mo98a].

(e) In the case $\partial_0 = \partial_\tau$ is a global Killing vector, the manifold admits periodicity $\beta$ along the lines tangent to $\partial_0$ and $M$ remains smooth (near any fixed points of the Killing orbits) fixing arbitrarily $\beta$ in a neighborhood and, finally, $\Sigma$ is a global surface everywhere normal to $\partial_0$, then

$$\frac{\partial}{\partial \beta} \ln Z(\beta)_{\mu^2} = \int_{\Sigma} d^D\vec{x} \sqrt{g(\vec{x})} < T_0^0(x,\beta|x|A) >_{\mu^2}.$$  

(27)

Another general result, concerning the possibility to get a Lorentzian theory from an Euclidean one, is the following one [Mo98b].

6
Let $M$ be compact, $A \geq 0$, $\mu^2 > 0$, let $M$ be also globally static with global Killing time $\partial_\tau$ and (orthogonal) global spatial section $\Sigma$ and finally, $\partial_\tau V' \equiv 0$. Then

(a) $\partial_\tau < \phi^2(x|A) >_{\mu^2} \equiv 0$;
(b) $\partial_\tau L(x|A)_{\mu^2} \equiv 0$;
(c) $\partial_\tau < T_{ab}(x|A) >_{\mu^2} \equiv 0$;
(d) $< T_{0i}(x|A) >_{\mu^2} \equiv 0$ for $i = 1, 2, 3, \ldots, D - 1$

where the averaged quantities above are those defined above in terms of local $\zeta$-function regularization and coordinates $\tau \equiv x^0, \vec{x} \in \Sigma$ are employed.

These properties allow one to continue the Euclidean considered quantities into imaginary values of the coordinate $\tau \rightarrow it$ obtaining real functions of the Lorentzian time $t$.

Some of the properties above (concerning $T_1, T_2, T_3$) have been checked also in noncompact and symmetric manifolds (Rindler spacetime, cosmic string spacetime, Einstein’s open spacetime, $H^N$ spaces, Gödel spacetime, BTZ spacetime) \cite{Mo97, IM98, Ca98, Ra98, BMVZ98}. In particular, the theory of the regularization of the stress tensor and field fluctuations via local $\zeta$ function has been successfully employed to compute the back reaction on the three-dimensional BTZ metric \cite{BMVZ98} in the case of the singular ground state containing a naked singularity and a semiclassical implementation of the cosmic censorship conjecture has been found.

2.3. The relation with the point-splitting technique. The procedure of the point-splitting to renormalize the field fluctuation as well as the stress tensor we are concerned with \cite{BD82, Wa94} can be summarized as

\begin{align}
< \phi^2(y) >_{ps} &= \lim_{x \rightarrow y} \{G(x,y) - H(x,y)\} + g_{ab}(y)Q(y), \\
< T_{ab}(y) >_{ps} &= \lim_{x \rightarrow y} D_{ab}(x,y) \{G(x,y) - H(x,y)\} + g_{ab}(y)Q(y),
\end{align}

where, $G(x,y)$ is one half the Hadamard function (i.e., one half the sum of the two Wightman function) of the considered quantum state or, in Euclidean approach, the corresponding Green (Schwinger) function. $H(x,y)$ is the Hadamard local fundamental solution, a parametrix for the Green function which depends on the local geometry only and takes the short-distance singularity into account. $H(x,y)$ is represented in terms of a truncated series of functions of $\sigma(x,y)$. The operator $D_{ab}(x,y)$ is a bi-tensorial operator obtained by “splitting” the argument of the classical expression of the stress tensor (see \cite{Mo98b} for a quite general expression of this operator). Finally $Q(y)$ is a scalar obtained by imposing several physical conditions (essentially, the appearance of the conformal anomaly, the conservation of the stress tensor and the triviality of the Minkowskian limit) \cite{Wa94} on the left-hand side of (29) (see \cite{BD82, Fu91, Wa94, Mo98b} for details). The expression of $H(x,y)$ is the following one, in a geodesically convex neighborhood containing both $x$ and $y$,

\begin{align}
H(x,y) &= \frac{\sum_{j=0}^L u_j(x,y)\sigma(x,y)^j}{(4\pi)^{D/2}(\sigma(x,y)/2)^{D/2-1}} + \delta_D \left[ \sum_{j=0}^M v_j(x,y)\sigma(x,y)^j \ln \left( \frac{\sigma(x,y)}{2} \right) \right]
\end{align}
\[ + \delta_D \sum_{j=0}^{N} w_j(x, y) \sigma(x, y)^j. \quad (30) \]

Above, \( L, M, N \) are fixed integers (see \[Mo98b\] for details), \( \delta_D = 0 \) if \( D \) is odd and \( \delta_D = 1 \) otherwise. The coefficients \( u_j \) and \( v_j \) are smooth functions of \((x, y)\) which are completely determined by the local geometry. Conversely, the coefficients \( w_j \) are determined once one has fixed \( w_0 \). Dealing with Euclidean approaches, it is possible to explicit \( u_j \) and \( v_j \) in terms of heat-kernel coefficients \[Mo98, Mo98b\]. The problem is the determination of the coefficient \( w_0 \). This coefficient should determine the form of the term \( Q \) above. However, there is no guarantee that any choice of \( w_0 \) determine such a \( Q \) that the obtained stress tensor fulfills the requested physical conditions. (Anyhow, as well-know \[Wa94\] a possible choice is \( w_0 \equiv 0 \)).

One has the following result \[Mo98, Mo98b\].

**T4.** If \( M \) is compact, \( A \geq 0 \) and \( \mu^2 > 0 \), and the averaged quantities above are those defined above in terms of local \( \zeta \)-function regularization, then

\[ < \phi^2 (y|A) > \mu^2 = \lim_{x \to y} \{ G(x, y) - H(x, y) \} , \quad (31) \]

\[ < T_{ab} (y|A) > \mu^2 = \lim_{x \to y} \mathcal{D}_{ab}(x, y) \{ G(x, y) - H_{\mu^2}(x, y) \} + g_{ab}(y) Q(y) , \quad (32) \]

where \( G(x, y) = \zeta(1, x, y|A/\mu^2) \) given in \[14\], \( H_{\mu^2} \) is completely determined by posing

\[ w_0(x, y) := -\frac{a_{D/2-1}(x, y|A)}{(4\pi)^{D/2}} \frac{[2\gamma + \ln \mu^2]}{2}, \quad (33) \]

and the term \( Q \) is found to be

\[ DQ(y) = -P_0(y, y|A) + \delta_D \frac{a_{D/2}(y, y|A)}{(4\pi)^{D/2}} . \quad (34) \]

All the requests on the stress tensor necessary to determine \( Q \) in the point-splitting approach are now fulfilled by the left-hand side of \((32)\). Moreover, not depending on the \( \zeta \)-function approach, the choice above for \( w_0 \) and \( Q \) should work also in Lorentzian and noncompact manifolds as pointed-out in \[Mo98b\], where this conjecture has been checked for Minkowski spacetime.

It is worthwhile stressing that, in 1997-1998, several results have been obtained concerning the so-called *multiplicative anomaly* via \( \zeta \)-function techniques \[ECFVZ98\]. Anyhow, we shall not review these results here because these are quite far from the main arguments of this talk.

**References**

[BD82] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).
[BMVZ98] D. Binosi, V. Moretti, L. Vanzo, S. Zerbini Quantum scalar field on the massless (2 + 1)-dimensional black-hole background gr-qc/9809041, Phys. Rev. D (in press).

[Ca98] M. Caldarelli, Quantum scalar fields on anti-deSitter spacetime hep-th/9809144.

[ECFVZ98] E. Elizalde, G. Cognola, S. Zerbini Nucl. Phys. B 532, 407 (1998); E. Elizalde, A. Filippi, L. Vanzo, S. Zerbini, Phys. Rev. D57, 7430, (1998).

[EORBZ] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko, S. Zerbini, Zeta Regularization techniques with Applications (World Scientific, Singapore, 1994); E. Elizalde, Ten Physical Applications of Spectral Zeta Functions (Springer, Berlin, 1995).

[FR87] S.A. Fulling and S.N.M. Ruijsenaars, Phys. Rep. 152, 135 (1987).

[Fu91] S.A. Fulling, Aspects of Quantum Field Theory in Curved Space-Time (Cambridge University Press, Cambridge, 1991).

[Ha77] S. W. Hawking, Commun. Math. Phys. 55, 133 (1977).

[IM98] D. Iellici and V. Moretti, Phys. Lett. B 425, 33 (1998).

[KW91] B.S. Kay and R.M. Wald, Phys. Rep. 207, 49 (1991).

[Mo97] V. Moretti, Phys. Rev. D 56, 7797 (1997).

[Mo98] V. Moretti, Local ζ-function techniques vs point-splitting procedures: a few rigorous results UTM 535, gr-qc/9805091. Commun. Math. Phys. in press.

[Mo98b] V. Moretti, One-loop stress-tensor renormalization in curved background: the relation between ζ-function and point-splitting approaches, and an improved point-splitting procedure. UTM 540, gr-qc/9809006, submitted.

[Ra98] E. Radu, Phys. Lett. A 247, 207 (1998).

[Wa79] R. M. Wald, Commun. Math. Phys. 70, 226 (1979).

[Wa94] R.M. Wald, Quantum Field theory and Black Hole Thermodynamics in Curved Spacetime (The University of Chicago Press, Chicago, 1994).