A percolation system with extremely long range connections and node dilution

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Abstract. - We study the very long-range bond-percolation problem on a linear chain with both sites and bonds dilution. Very long range means that the probability \( p_{ij} \) for a connection between two occupied sites \( i,j \) at a distance \( r_{ij} \) decays as a power law, i.e., \( p_{ij} = \rho/\left[ r_{ij}^{\alpha} N^{1-\alpha} \right] \) when \( 0 \leq \alpha < 1 \), and \( p_{ij} = \rho/\left[ r_{ij} \ln(N) \right] \) when \( \alpha = 1 \). Site dilution means that the occupancy probability of a site is \( 0 < p_s \leq 1 \). The behavior of this model results from the competition between long-range connectivity, which enhances the percolation, and site dilution, which weakens percolation. The case \( \alpha = 0 \) with \( p_s = 1 \) is well-known, being the exactly solvable mean-field model. The percolation order parameter \( P_\infty \) is investigated numerically for different values of \( \alpha \), \( p_s \) and \( \rho \). We show that in the ranges \( 0 \leq \alpha \leq 1 \) and \( 0 < p_s \leq 1 \) the percolation order parameter \( P_\infty \) depends only on the average connectivity \( \gamma \) of sites, which can be explicitly computed in terms of the three parameters \( \alpha \), \( p_s \), and \( \rho \).

1 - Introduction. – During the last fifty years, percolation theory has brought new understanding and methods to a broad range of topics in physics like materials science, complex networks, surface roughening, epidemiology, geography, and fire propagation, to cite just a few [1,2]. This theory was first considered for the optimization of masks supplied to the miners in the coal pits needing a protection which could block poisoning materials, while permitting the passage of air. In other words, it was needed an appropriate dosage of porosity of the material which composed the masks in order to have connected path for air and unconnected path for poisoning materials. After that, the theory was applied to the study of movement and filtering of fluids through porous materials (the most familiar phenomena probably being coffee percolation), and its scope has been progressively extended to all other domains [3–5].

Nowadays, percolation is still a very active field of research and applied to an always increasing number of phenomena in physics as, for example, fluid flow in random media [6], dielectric breakdown [7], and reaction-diffusion processes in two-dimensional percolating structures [8].

In the context of percolation theory, a percolation transition is characterized by a set of universal critical exponents, which describe the fractal properties of the percolating medium at large scales and sufficiently close to the transition. The exponents are universal in the sense that they only depend on the type of percolation model and on the space dimension. They are expected not to depend on microscopic details like the lattice structure or whether site or bond percolation is considered [9,11].

Percolation models have also been increasingly adopted to many systems in Nature to understand important features of many chemical, biological, and social phenomena. These systems differ mainly in their topology: many of them form complex networks, whose vertices are the elements of the system and whose edges represent the interactions between them. For example, living systems form a huge genetic network, whose vertices are proteins, while the edges represent the chemical interactions between them [12]. Equally complex networks occur also in social science, where the vertices are individuals, organizations or countries, and the edges characterize the social interaction between them [13]. Due to their large size and the complexity of the interactions, the topology of these networks is largely unknown or unexplored.

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The first percolation model was the well-known Bernoulli (or bond) percolation model, whose generalization was introduced as the Fortuin-Kasteleyn random cluster model, which has many connections with the Ising and Potts models in the language of percolation theory. Furthermore, the effect of connectivity on biodiversity can be supported by percolation theory hypothesizing the existence of a connectivity threshold, whose measurement is considered as a practical means to facilitate biological fluxes and optimize species colonization possibilities in designing a policy of conservation biology. Recent advances in this field points to universal laws and offer a new conceptual framework that could potentially revolutionize our view of biology and disease pathologies in the twenty-first century. Applications of complex networks, based largely on graph theory, have been rapidly translated to characterize brain network organization as well.

The effect of long-range connections on percolation is of fundamental interest, since they give rise to a variety of new interesting dynamical and thermodynamical phenomena. In view of that, long-range models have been intensively studied in recent times in different context. The phenomenology becomes very interesting when long range connections appears together with site dilution. In this case, the presence of competitions between long-range connectivity enhance percolation and site dilution, leading to a weaker percolation effect.

In this work, we want to investigate the very long-range bond-percolation problem on a linear chain with both sites and bonds dilution. Very long range means that the probability $p_{ij}$ for a connection between two occupied sites $i, j$ at a distance $r_{ij}$ decays as a power law, i.e. $p_{ij} = p/[r_{ij}^{\alpha}N^{1-\alpha}]$ when $0 \leq \alpha < 1$, and $p_{ij} = p/[r_{ij}\ln(N)]$ when $\alpha = 1$. Site dilution means that the occupancy probability of a site is $0 < p_{s} \leq 1$. Notice that for this very long range models it is necessary to assume that the probability of connection decays as $1/N^{1-\alpha}$ (or $1/\ln(N)$) in order to obtain the correct thermodynamic limit.

The case $\alpha = 0$, with $p_{s} = 1$, is well-known, being the exactly solvable mean-field model, while the case $\alpha = 0$ with $p_{s} < 1$ is its almost trivial extension. In the other regions, the percolation order parameter $P_{\infty}$ is investigated numerically for different values of $\alpha, p_{s}$, and $\rho$. Intuitively, one expects the percolation order parameter $P_{\infty}$ be reduced by the inclusion of diluted sites. Indeed, we will show not only that this is true, but we also show that in all range $0 \leq \alpha \leq 1$, $0 < p_{s} \leq 1$ the percolation order parameter $P_{\infty}$ depends only on the average connectivity $\gamma$ of sites, which can be explicitly computed in terms of the three parameters $\alpha, p_{s}$, and $\rho$. Besides, the connectivity reduces when dilution increases.

From a technical point of view, we will show that for values of $\alpha$ either between 0.2 and 0.8, or for $\alpha = 1$, considering different values of $p_{s}$, all curves collapse.

The paper is organized as follows: In section 2 we consider the simple case $\alpha = 0$ in the absence and in the presence of dilution. Sections 3 and 4 discuss the cases $0 < \alpha < 1$ and $\alpha = 1$ respectively. Finally, our conclusions are depicted in section 5.

2 - Mean-field ($\alpha=0$). – The percolation order parameter $P_{\infty}$ is defined as the fraction of sites of the system that belong to the infinite cluster. Obviously, $P_{\infty}$ attains its maximum value ($P_{\infty} = 1$) when all the sites of the system appear inside the infinite cluster, whereas $P_{\infty} = 0$ below a certain threshold, when it is not possible to produce an infinite cluster.

A particularly simple model is the mean-field, which corresponds to $\alpha = 0$. We describe below this almost trivial case, first when only bonds are diluted, and afterwards considering dilution for both sites and bonds.

![Fig. 1: Order parameter $P_{\infty}$ versus the control parameter $\gamma(\rho, \alpha, p_{s})$. For a given choice of the parameters $\alpha$ and $p_{s}$, $\gamma$ only depends on $\rho$. The figure shows a complete collapse, i.e. the shape is the same for any choice of $\alpha$ and $p_{s}$, and coincides with the well-known mean-field result.

a - Mean-field (bond dilated). In mean-field bond diluted model ($\alpha = 0$, $p_{s} = 1$), one assumes that there are $N$ nodes. Any pair of nodes is connected (closed bond) with probability $\rho/N$ and unconnected (open bond) with probability $1 - \rho/N$.

The average connectivity $\gamma$ of a given node (the average number of connections of a node to the remaining $N - 1$ nodes) is given by

$$\gamma = \frac{\rho}{N} (N - 1) \simeq \rho.$$ (1)

This number is simply obtained by multiplying the number $N - 1$ of remaining nodes with the probability that a bond is closed.

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Let us call \( P_\infty \) the fraction of nodes in the giant component (number of nodes in the giant component divided by the total number of nodes \( N \)), which can also be seen as the probability that a node belongs to the giant component itself. The order parameter \( P_\infty \) satisfies the self-consistency equation (see, for example, [27][28])

\[
\exp(-\gamma P_\infty) = 1 - P_\infty, \tag{2}
\]

which gives \( P_\infty(\gamma) \). The critical value of the control parameter is \( \gamma_c = 1 \), as it is depicted in Fig. 1.

b - Mean-field (bond and node diluted). This model has a distribution of bonds as the previous model but node dilution is introduced. It is assumed that a node has a distribution of bonds as the previous model but which gives a parameter \( \gamma \) obeys a power-law, i.e.

\[
P(\gamma) \propto \gamma^{-\alpha}, \tag{3}
\]

The average connectivity \( \gamma \) of a given active node is

\[
\gamma = \frac{\rho}{N} (N_s - 1) \simeq \rho p_s. \tag{3}
\]

This number is simply obtained by multiplying the number \( N_s - 1 \simeq p_s N \) of remaining active nodes with the probability \( \rho/N \) that a bond is closed.

It is easy to show that the size \( P_\infty(\gamma) \) of the giant component is still the function of the average connectivity given by eq. (2). In fact, it is sufficient to remark that the size of the giant component is, by definition, the number of active nodes in the giant component divided by the total number of active nodes. Therefore, it is enough to consider a system composed only by active nodes (whose number \( N_s \) is about \( p_s N \)) which are connected (active bond) with a probability \( \rho/N \simeq \rho p_s/N_s \). In this way, we are re-conducted to the previous model with the difference that \( \rho \) and \( N \) are replaced by \( \rho p_s \) and \( N_s \) in eq. (1). Observe that the average connectivity was \( \gamma = \rho \) in previous model, while now \( \gamma = \rho p_s \). So, eq. (2) must hold also for the present model with \( \gamma \) given by (3).

Notice that \( \gamma \) increases linearly with \( p_s \), while \( P_\infty \) is a non-decreasing function of \( \gamma \). Therefore, dilution decreases the value of \( P_\infty \) as expected.

3 - Power law model \((0 < \alpha < 1)\).

a - Definitions. Here we consider a one-dimensional (periodic chain) problem where nodes are active according to a given rate \( p_s \). Two nodes are connected (closed bonds), at a power law probability depending on their distance.

Assuming that \( i \) is the position of a node on the chain, periodic boundary conditions implies that the node \( i \) coincides with node \( i + N \). Furthermore, given the periodic boundary conditions, the distance \( r_{ij} \) between two nodes \( i \) and \( j \) is \( r_{ij} = |i-j| \) if \( 1 \leq |i-j| \leq N/2 \) and \( r_{ij} = N-|i-j| \) if \( N/2 < |i-j| < N \).

Therefore, we assume that two active nodes \( i \) and \( j \) of the chain are connected by a closed bond depending upon their distance \( r_{ij} \) according to the probability \( P(r_{ij}) \) which obeys a power-law, i.e.

\[
p_{ij} = \frac{\rho}{(r_{ij})^\alpha N^{1-\alpha}}, \tag{4}
\]

with \( 0 \leq \alpha < 1 \). According to the above prescription, nearest neighbors nodes (when both are active) are connected with probability \( \rho/N^{1-\alpha} \).

The exponent \( 0 \leq \alpha < 1 \) controls the range of network interaction. When \( \alpha = 0 \), the model reduces to the model described in subsection 2b of section 2.

b - Solution. The probability that a given active node at position \( i \) is connected to another active node at position \( j \) at distance \( r \) is

\[
\frac{p_s}{r^\alpha N^{1-\alpha}}. \tag{5}
\]

Given the periodic boundary conditions, there are two nodes at any given distance \( 1 \leq r \leq N/2 \). Therefore, the average connectivity \( \gamma = \gamma(\rho, \alpha, p_s) \) is obtained by the sum

\[
\gamma = \sum_{r=1}^{N/2} \frac{p_s}{r^\alpha N^{1-\alpha}} \simeq \frac{2^\alpha}{1-\alpha} \rho p_s, \tag{6}
\]

where we have neglected terms which vanish in the thermodynamical limit. Remark that (6) reduces to (3) when \( \alpha \to 0 \), and that the average distance \( d \) between two active connected nodes of the system has the the size of a finite fraction of the the system size. In fact:

\[
d = \frac{1}{\gamma} \sum_{r=1}^{N/2} 2r \frac{p_s}{r^\alpha N^{1-\alpha}} \simeq \frac{1 - \alpha}{2(2 - \alpha)} N. \tag{7}
\]
This fact should imply mean-field properties for the system. Given that $P_\infty$ is the ratio between the number of active nodes in the giant component and the total number of active nodes, $P_\infty(\gamma)$ should be still given by eq. (2), provided that $\gamma$ is given by (6), which is the aim of the numerical work.

In practice, for any value of $\alpha$, $\rho$ and $p_s$ one should compute numerically $P_\infty$ and plot against $\gamma = \gamma(\alpha, \rho, p_s)$ given by (6). Once obtained a curve, one should compare it with the mean-field curve given by eq. (1). In Fig. 1 we show that, indeed, the shape of $P_\infty(\lambda)$ is the same independently on the numerical parameters, and coincides with the well known mean-field result.

For a given choice of the parameters $\alpha$ and $p_s$, $\gamma$ only depends on $\rho$. Therefore, we have considered various values of $p_s$ and $\alpha$ and plotted $P_\infty(\lambda)$ with respect to $\lambda$. For all cases we have considered a system of $N = 10,000$ sites, and we have obtained $P_\infty$ as an average over 500 different independent realizations of the network.

Fig. 1 only shows results up to $\alpha = 0.6$ because for larger values of $\alpha$, a size $N = 10,000$ is not enough to reach the thermodynamical limit. Nevertheless, we are able to show, by a scaling analysis, that the mean-field value is anyway reached in the $N \to \infty$ limit. This can be seen in Fig. 2 where the difference $P_\infty(\lambda) - P_\infty(\lambda)$ is plotted against $N$ in a log-log scale for the case $\alpha = 0.8$. The function $P_\infty(\lambda)$ is the value calculated analytically, while $P_N(\lambda)$ is the value obtained by a simulation of a network of size $N$. We observe that the difference $P_\infty(\lambda) - P_\infty(\lambda)$ converges as a power law to zero, confirming our data collapse for various values of the parameters $\rho$ and $p_s$. In particular, the upper (lower) data of Fig. 2 correspond to two different choices of the parameters, both considering $\gamma = 1.5$ (3.5). In all cases, the values of $P_N(\lambda)$ are obtained as an average over 500 different independent realizations of the network, with $N$ ranging from 1,000 to 64,000.

4 - The case $\alpha = 1$. – When $\alpha = 1$ everything goes as before but scaling $1/\ln(N)$ replaces scaling $1/N^{1-\alpha}$ in eq. (4), i.e.:

$$p_{ij} = \frac{\rho}{r_{ij} \ln(N)}.$$  

Then, the average connectivity is

$$\gamma = \sum_{r=1}^{N/2} 2 \frac{\rho p_s}{r \ln(N)} \simeq 2 \rho p_s.$$  

We remark that now, at variance with the case $0 \leq \alpha < 1$, the average distance $d$ between two active connected nodes of the system is not of the order of the size of the system. In fact:

$$d = \frac{1}{\gamma} \sum_{r=1}^{N/2} 2 \frac{\rho p_s}{\ln(N)} \simeq \frac{1}{2} \frac{N}{\ln(N)}.$$  

Nevertheless, the system still has mean-field properties.

We have checked numerically that $P_\infty(\gamma)$ given by eq. (2) should still be valid, provided $\gamma$ is given by (9), as it is shown in Fig. 3, where $P_\infty(\gamma)$ is plotted against $\gamma$. Considering $\alpha = 1$, for a given choice of the parameters $p_s$, $\gamma$ only depends on $\rho$. We have taken into account various values of $p_s$ and, for all cases, a system with $N = 10,000$ sites. $P_\infty$ is obtained as an average over 500 different independent realizations of the network.

5 - Discussion. – We have shown that very long-range bond-percolation ($0 \leq \alpha \leq 1$), with both sites and bonds dilution on a linear chain behaves as in the mean-field theory if $P_\infty$ is expressed in terms of the average connectivity $\gamma$ of a site. In other words, all data collapse on the same universal curve if $P_\infty$ is plotted against $\gamma(\alpha, \rho, p_s)$.

Noticeably, collapse is absent when the parameter $\alpha$ is larger than unity [19, 24, 25]. In this case, in fact, connectivity is not an exhaustive description of the topology of the system, as it can be easily understood. If $\alpha > 1$ one can assume $p_{ij} = \rho/r_{ij}^{\alpha}$ and then compute the average connectivity

$$\gamma = \sum_{r=1}^{N/2} 2 \rho p_s/r^{\alpha} \simeq 2 \rho p_s \zeta(\alpha),$$

where $\zeta(\alpha) = \sum_{r=1}^{\infty} r^{-\alpha}$ is the zeta Riemann function. In this $\alpha > 1$ case, at variance with the case $0 \leq \alpha \leq 1$, the probability of connections to sites at a distance of order unity remains finite in the thermodynamic limit.
This implies that fluctuations in connections with few close sites and bonds, may prevent the emergence of a giant component even when $\gamma > 1$.

To clarify this point consider the simple problem of percolation on a linear chain occupancy probability $p_s$ and with connections only between nearest-neighbor sites. This case corresponds to the limit $\alpha \rightarrow \infty$ where $p_{ij} = \rho / r_{ij}^\alpha$ equals 1 if $r_{ij} = 1$, vanishing otherwise. In this case, whenever $\rho < 1$ or $p_s < 1$ percolations is forbidden. In other words $P_\infty = 1$ for $p_s, \rho = 1$ and $P_\infty = 0$ for $p_s, \rho < 1$. Since the average connectivity is $\gamma = 2 \rho p_s$, since $\lim_{\alpha \rightarrow \infty} \zeta(\alpha) = 1$, one can also state $P_\infty = 1$ for $\gamma = 2$ (which is the maximum possible value for $\gamma$) and $P_\infty = 0$ for $\gamma < 2$. This behavior is at very variance with those described in this paper for the very long range model.

We would like to finally stress that while we are always able to explicitly compute $\zeta$ in terms of $\alpha, p_s$ and $\rho$ for all possible range of these three parameters, the case $\alpha = 0$ is the only one which we are able to completely treat analytically. Our conclusions concerning the region $0 < \alpha < 1$ are very precise and hopefully correct, although a rigorous mathematical proof of our conclusions still remains open.

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