Notes on Lax Ends

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October 5, 2022

Abstract

In enriched category theory, the notion of extranatural transformations is more fundamental than that of ordinary natural transformations, and the ends, the universal extranatural transformations, play a critical role. On the other hand, 2-category theory makes use of several other natural transformations, such as lax and pseudo transformations. For these weak transformations, it is known that we can define the corresponding extranatural transformations or ends. However, there is little literature describing such results in detail. We provide a detailed calculation of the lax end, including its relation to the lax limits. We prove the bicategorical coYoneda lemma as the dual of the bicategorical Yoneda lemma, and also show that the weight of any lax end is a PIE weight, but it might not be a weight for a lax limit.

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1 Introduction

1.1 Background

While 2-categories can be defined as \textbf{Cat}-enriched categories, there are various types of weak notions of functors and weak transformations. These weak gadgets cannot be defined in general enriched categories, but it is known that for many statements in enriched category theory, it is also possible to prove statements that are replaced by weak gadgets in 2-category theory. However,

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while the proofs of those statements in enriched category theory can be written concisely using ends, the corresponding theorems with weak gadgets in 2-category theories are written using jumbled diagrams and are not treated in a very unified manner.

Bozapalides introduced lax end as a universal extraordinary lax natural transformation, just as an end is a universal extraordinary natural transformation [3, 4]. However, the notion of lax ends has not been well studied, and the only reference containing a survey on lax ends is [10]. In these notes, we give detailed calculations and proofs of basic results on lax ends in 2-category theory. In particular, we treat some calculations on weighted lax limits in Section 4 and the bicategorical (co)Yoneda lemma in Section 7. In Section 6, we also examine the class of limits where lax ends live.

For simplicity, in these notes, we restrict our attention to 2-categories and 2-functors, although Bozapalide defined lax ends for lax (or quasi) functors between 2-categories in [3, 4] and Corner proved coYoneda lemma for bicategories and pseudo functors in [5].

1.2 Notations

Each term 2-categories, 2-functors, and 2-transformations implies Cat-enriched categories, functors, and transformations. Other morphisms of 2-categories include (op)lax and pseudo functors and the same for transformations. Out of these kinds of functors, we only deal with 2-functors in these notes. The functor categories which appear in these notes will be denoted as follows.

- **Lax[A, B]**: 2-functors, lax transformations, and modifications
- **Ps[A, B]**: 2-functors, pseudo transformations, and modifications
- **[A, B]**: 2-functors, 2-transformations, and modifications

If \( F: A \to \text{Cat} \) and \( G: A \to K \) are 2-functors, the \( F \) weighted limit of the diagram \( G \) is denoted as \( \text{lim}^F G \). Similarly, \( \text{colim}^F G \) is the \( F \) weighted colimit of the diagram \( G \). Weighted lax limits, which will be defined in Section 4, will be denoted as \( \text{lax lim}^F G \).

As special weighted limits, we have ends and powers. We write the end of \( T: A \times B \to K \) as \( \int_{A \in A} T(A, A) \), and the coend as \( \int^{A \in A} T(A, A) \). If \( K \in K \) and \( A \in \text{Cat} \), we write the power as \( A \downarrow K \) and the copower as \( A \cdot K \).

To denote 2-categories, we tend to use the calligraphic font \( A, B, K, \ldots \), and for 1-categories, the blackboard bold fonts \( \mathbb{A}, \mathbb{1}, \ldots \). For the terminal 2-category, we use \( \mathbb{1} \). The enriched yoneda embedding functor is denoted as \( Y: A \to [A^{\text{op}}, \text{Cat}] \).

Let \( G: A \to B \) a 2-functor. We define two 2-functors \( \tilde{G}: B^{\text{op}} \to [A, \text{Cat}] \) and \( \bar{G}: B \to [A^{\text{op}}, \text{Cat}] \) by \( \tilde{G} = B(-, G?) \) and \( \bar{G} = B(G?, -) \).

2 Lax naturalities

**Definition 2.1.** Let \( T: A^{\text{op}} \times A \to K \) be a 2-functor and \( K \) an object of \( K \). A lax wedge (or extraordinary lax natural transformation) \( \sigma: K \to T \) consists of:

- a 1-cell \( \sigma_A: K \to T(A, A) \) for each object \( A \) in \( A \);
• a 2-cell \( \sigma_f : T(1_A, f) \sigma_A \to T(f, 1_B) \sigma_B \) for each 1-cell \( f : A \to B \) as shown in the diagram below:

\[
\begin{array}{ccc}
K & \xrightarrow{\sigma_A} & T(A, A) \\
\downarrow \sigma_B & \searrow & \downarrow \sigma_f \searrow (1_A, f) \\
T(B, B) & \xrightarrow{T(f, 1_B)} & T(A, B)
\end{array}
\]

satisfying the following three equalities:

\[
\begin{array}{ccc}
K \xrightarrow{\sigma_A} & T(A, A) \\
\downarrow \sigma_B & \searrow 1 \\
T(A, A) & \xrightarrow{1} & T(A, A)
\end{array}
\] (2.1)

\[
\begin{array}{ccc}
K \xrightarrow{\sigma_A} & T(A, A) \\
\downarrow \sigma_B & \searrow \sigma_f \searrow T(1_A, f) \\
T(A, B) & \xrightarrow{T(1, f)} & T(A, B)
\end{array}
\] (2.2)

\[
\begin{array}{ccc}
K \xrightarrow{\sigma_A} & T(A, A) \\
\downarrow \sigma_B & \searrow 1 \\
T(A, A) & \xrightarrow{1} & T(A, A)
\end{array}
\] (2.3)

Dually, a **lax cowedge** is a lax wedge in \( K^{op} \).

Of course, an **oplax wedge** \( K \to T \) can also be defined as a lax wedge in \( K^{co} \), but this is the same as a lax wedge \( K \to T' \) with \( T'(A, B) = T(B, A) \). Also, we will briefly discuss pseudo wedges later in Section 7.

**Definition 2.2.** Let \( \sigma, \tau : K \to T \) be a pair of lax wedges. A modification \( \Gamma \) from \( \sigma \) to \( \tau \) is a family
Proposition 2.4. Let \( \sigma_A : \tau_A \to \sigma_A \) be checked that the coherence required for lax wedges and for lax transformations are equivalent. Then, \( \sigma_{AB} \) by checking the lax naturality in \( A \to \sigma_B \) of \( 2 \)-cells \( \gamma \tau \sigma \)_{AB}.

Proof. The "if" part is trivial and the other part can be shown by pasting some diagrams.

Proposition 2.3. Let \( F, G : \mathcal{A}^{op} \to \mathcal{K} \) be 2-functors. Lax transformations \( \alpha : F \to G \) are bijective to lax wedges of type \( 1 \to \mathcal{K} (F?, G?) \). And also, modifications between lax transformations are bijective to modifications of corresponding lax wedges.

Proof. One can easily check that a family \( \{ \sigma_A : 1 \to \mathcal{K} (FA, GA) \}_{A \in \mathcal{A}} \) corresponds to a family \( \{ \bar{\sigma}_A : FA \to GA \}_{A \in \mathcal{A}} \), and \( \{ \bar{\sigma}_f : \mathcal{K} (f, 1) \sigma_A \to \mathcal{K} (f, 1) \sigma_B \} \) to \( \{ \bar{\sigma}_f : G f \circ \sigma_A \to \bar{\sigma}_B \circ f \} \). Also, it can be checked that the coherence required for lax wedges and for lax transformations are equivalent. \( \square \)

Unlike general enriched categories, lax naturality in \( (A, B) \in \mathcal{A} \times \mathcal{B} \) can not be simply verified by checking the lax naturality in \( A \) for each fixed \( B \) and vice versa. It requires additional equalities.

Proposition 2.4. Let \( T : \mathcal{A}^{op} \times \mathcal{B}^{op} \times \mathcal{A} \times \mathcal{B} \to \mathcal{K} \) be a 2-functor, and assume we have

- a family of 1-cells \( \{ \sigma_{AB} : K \to T(A, B, A, B) \} \) in \( \mathcal{K} \),
- families of 2-cells \( \{ \sigma_{fB} : T(1, 1, f, 1) \sigma_{AB} \to T(f, 1, 1) \sigma_{A'B} \}_{f} \) for each \( B \), and
- families of 2-cells \( \{ \sigma_{gA} : T(1, 1, g) \sigma_{AB} \to T(1, g, 1) \sigma_{A'B} \}_{g} \) for each \( A \).

Then, \( \sigma_{-B} \) and \( \sigma_{A'} \) are wedges for each \( B \) and \( A \) satisfying the following equality,

\[
\begin{array}{ccc}
T(11f) & \xrightarrow{T(11g)} & T(11g) \\
\sigma_{AB} & \xleftarrow{\sigma_{fB}} & \sigma_{gA} \\
T(11f) & \xrightarrow{T(11g)} & T(11g) \\
K & \xleftarrow{\sigma_{A'B}} & \sigma_{gA'} \\
T(f11) & \xrightarrow{T(11g)} & T(11g) \\
\end{array}
\]

if and only if \( \sigma \) is a wedge for \( T : (\mathcal{A} \times \mathcal{B})^{op} \times (\mathcal{A} \times \mathcal{B}) \to \mathcal{K} \) with \( \sigma_{fg} \) defined by this 2-cell.

Proof. The "if" part is trivial and the other part can be shown by pasting some diagrams. \( \square \)
3 Lax ends in Cat

As in enriched category theory, we wish to define the lax end as a universal lax wedge. In this section, we define a lax end for a $\textbf{Cat}$-valued 2-functor $\mathcal{A}^{\text{op}} \times \mathcal{A} \to \textbf{Cat}$, and examine its properties in this and the following two sections. Of course, we can also define a lax end for general 2-category $\mathcal{K}$, which will be discussed later in Section 5.

**Definition 3.1.** A lax end of a 2-functor $T: \mathcal{A}^{\text{op}} \times \mathcal{A} \to \textbf{Cat}$ consists of a category $\int_A T(A, A)$ and a lax wedge $\lambda: \int_A T(A, A) \to T$, with the following universal properties:

1-dimensional For each lax wedge $\sigma: X \to T$, there is a unique functor $u: X \to \int_A T(A, A)$ satisfying $\sigma = \lambda u$, that is, $\sigma_A = \lambda_A u$ for each object in $\mathcal{A}$, and $\sigma_f = \lambda_f u$ for each 1-cell in $\mathcal{A}$.

2-dimensional For each modification $\Gamma: \lambda u \to \lambda v: X \to T$, There is a unique 2-cell $\gamma: u \to v$ such that $\Gamma = \lambda \ast \gamma$, that is, $\Gamma_A = \lambda_A \gamma$ for each $A$ in $\mathcal{A}$.

Dually, we also define a lax coend $T \to \int_A^T T(A, A)$ as a universal cowedge in the same way. ■

We show that there is a lax (co)end for every $T: \mathcal{A}^{\text{op}} \times \mathcal{A} \to \textbf{Cat}$ later in Proposition 3.7, but before we prove it, let us now examine some of the properties that hold when it exists.

Let $T$ be a 2-functor $T: \mathcal{A}^{\text{op}} \times \mathcal{A} \to \textbf{Cat}$, and assume that its lax end $\int_A T(A, A)$ exists. Then, since the objects of $\int_A T(A, A)$ correspond to lax wedges of type $1 \to T$, its data can explicitly be written down; an object of $\int_A T(A, A)$ consists of a pair of families $\{x_A \in T(A, A)\}_{A \in \mathcal{A}}$ and $\{x_f: T(1, f)x_A \to T(f, 1)x_B \in T(A, B)\}_{f: A \to B}$, satisfying the three axioms:

- $x_{1A} = \text{id}$,
- for each composable pair $(g, f)$ in $\mathcal{A}$, $T(f, 1)x_g \circ T(1, g)x_f = x_{gf}$,
- for each 2-cell $\alpha: f \to g$,

\[
\begin{array}{ccc}
T(1, f)x_A & \xrightarrow{\alpha} & T(1, g)x_A \\
\downarrow x_f & & \downarrow x_g \\
T(f, 1)x_B & \xrightarrow{\alpha} & T(g, 1)x_B
\end{array}
\]

A morphism from $x$ to $y$ in $\int_A T(A, A)$ is a family $\{\gamma_A: x_A \to y_A\}_{A}$ satisfying $T(f, 1)\gamma_B \circ x_f = y_f \circ T(1, f)\gamma_A$ for each $f: A \to B$ in $\mathcal{A}$.

If $\Delta: \mathcal{A}^{\text{op}} \times \mathcal{A} \to \textbf{Cat}$ is a constant functor which returns a category $\mathcal{B}$, then the lax end $\int_A \Delta \mathcal{B}$ is the functor category $[\mathcal{A}, \mathcal{B}]$.

Although we gave an explicit presentation of a lax end as above, we will not use this presentation below, and all the following discussions will be based on its universality.

First, we show that the two basic propositions – analogous to the enriched case – hold in our lax setting.

**Proposition 3.2.** There is an isomorphism of categories

\[
\text{Lax}([\mathcal{A}, \mathcal{K}](F, G)) \cong \int_{A \in \mathcal{A}} \mathcal{K}(FA, GA).
\] (3.1)
Proof. Trivial from Proposition 2.3.

**Proposition 3.3.** A lax end commutes with representables

\[
\left[ X, \underleftarrow{\int}_{A \in A} T(A, A) \right] \cong \underleftarrow{\int}_{A \in A} \left[ X, T(A, A) \right].
\]  

(3.2)

Proof. From the universal property of lax ends, the objects in left-hand side are bijective to lax wedges of form \( X \to T \), and the morphisms corresponds to those modifications. On the other hand, objects in the right-hand side are lax wedges of form \( \mathbb{1} \to [X, T(?,-)] \), and the morphisms are those modifications. One can easily show these two categories are isomorphic.

In order to show the existence of lax ends, we would like to represent a lax end with a weighted limit in \( \textbf{Cat} \) and deduce the existence by the completeness of \( \textbf{Cat} \). To this end, we introduce lax descent objects.

**Definition 3.4.** Coherence data in a 2-category \( K \) consists of three objects \( X_1, X_2, X_3 \), six 1-cells

\[
X_3 \xleftrightarrow{s} X_2 \xleftrightarrow{r} X_1 \xrightarrow{v} X_1 \xleftarrow{i} X_2 \xrightarrow{w} X_2
\]

and five equalities

\[
\begin{align*}
\delta : iv &= 1, \\
\gamma : 1 &= iw, \\
\kappa : rv &= sv, \\
\lambda : tw &= sw, \\
\rho : rw &= tv.
\end{align*}
\]

If we regard the simplex category as a discrete 2-category, this coherence data is a full sub 2-category of the simplex category.

An example of coherence data (with non-trivial 2-cells) can be found in 2-monad theory [7].

**Definition 3.5.** A lax descent object of the coherence data consists of an object \( X \), a 1-cell \( x : X \to X_1 \), and a 2-cell \( \xi : vx \Rightarrow wx \) satisfying the following diagram equalities,

\[
\begin{align*}
\begin{array}{c}
X \xrightarrow{x} X_1 \\
\downarrow x & \quad \downarrow v & \quad \downarrow i \\
X_1 & \xrightarrow{w} X_2 & \xrightarrow{i} X_1 \\
\end{array}
= 
\begin{array}{c}
X \xrightarrow{x} X_1 \\
\downarrow w & \quad \downarrow 1 \\
X_2 & \xrightarrow{i} X_1
\end{array}
\end{align*}
\]

(3.3)

\[
\begin{align*}
\begin{array}{c}
X \xrightarrow{x} X_1 \\
\downarrow x & \quad \downarrow v & \quad \downarrow w \\
X_1 & \xrightarrow{r} X_2 & \xrightarrow{w} X_1 \\
\end{array}
= 
\begin{array}{c}
X \xrightarrow{x} X_1 \\
\downarrow x & \quad \downarrow v \\
X_2 & \xrightarrow{r} X_2 \\
\end{array}
\end{align*}
\]

(3.4)
with the one and two-dimensional universal property. The one dimensional universal property is that, if there is another triple \((Y, y: Y \to X_1, \eta: vy \Rightarrow wy)\) satisfying the same equality of 2-cells, there uniquely exists a 1-cell \(u: Y \to X\) such that \(xu = y\) and \(\xi u = \eta\). And the two-dimensional universal property is that, if there is a pair of 1-cells \(u, u': Y \to X\) and a 2-cell \(\alpha: xu \Rightarrow xu'\) satisfying \(\xi u' \circ v\alpha = w\alpha \circ \xi u\), there uniquely exists a 2-cell \(\beta: u \Rightarrow u'\) such that \(\alpha = x\beta\).

This lax descent object can easily be obtained by taking several limits.

**Lemma 3.6.** Let \(\mathcal{K}\) be a 2-category. If \(\mathcal{K}\) admits inserters and equifiers, then \(\mathcal{K}\) also admits lax descent objects.

**Proof.** Take an inserter of \(v\) and \(w\),

\[
\begin{array}{ccc}
I & \xrightarrow{a} & X_1 \\
\downarrow a & & \downarrow v \\
X_1 & \xrightarrow{\zeta} & X_2 \\
\end{array}
\]

and the lax descent object is obtained by taking equifiers twice for the diagrams in (3.3) and (3.4) with \(\xi\) properly substituted using \(\zeta\).

Finally, we show that a lax end is a kind of lax descent object, and prove its existence.

**Proposition 3.7.** For a small \(\mathcal{A}\), any 2-functor \(T: \mathcal{A}^{\text{op}} \times \mathcal{A} \to \text{Cat}\) has a lax end and a lax coend.

**Proof.** Let \(X_1, X_2, X_3\) be as follows.

\[
\begin{align*}
X_1 &= \prod_{A \in \mathcal{A}} T(A, A) \\
X_2 &= \prod_{A, B \in \mathcal{A}} [\mathcal{A}(A, B), T(A, B)] \\
X_3 &= \prod_{A, B, C \in \mathcal{A}} [\mathcal{A}(B, C) \times \mathcal{A}(A, B), T(A, C)]
\end{align*}
\]

Then we need to define six functors

\[
\begin{array}{ccc}
X_3 & \xleftarrow{s} & X_2 \\
\downarrow i & & \downarrow w \\
X_2 & \xleftarrow{r} & X_1
\end{array}
\]

Since this is very complicated and redundant, we omit the detail and give an outline. We define six functors as follows.

\[
\begin{align*}
v_{AB} &= T(A, -): \mathcal{A}(A, B) \to [T(A, A), T(A, B)] \\
w_{AB} &= T(-, B): \mathcal{A}(A, B) \to [T(B, B), T(A, B)] \\
i_A &= 1_A: \mathcal{A}(A, A) \\
r_{ABC} &= T(A, -): \mathcal{A}(B, C) \to [T(A, C), T(A, C)] \\
s_{ABC} &= c_{ABC}: \mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C) \\
t_{ABC} &= T(-, C): \mathcal{A}(A, B) \to [T(B, C), T(A, C)]
\end{align*}
\]
Functors $v, w, i, r, s, t$ are those canonically constructed from these data. For example, $v$ is defined by the product of the functors

$$X_1 \xrightarrow{\pi_A} T(A, A) \xrightarrow{v_{AB}} [A(A, B), T(A, B)]$$

where $v_{AB}$ is the transpose of $u_{AB}$. One can check that these six functors actually constitute coherence data, whose five 2-cells are all identities.

Since $\textbf{Cat}$ is complete, there is a descent object $(X, x, \xi)$ for this coherence data. Let $\lambda_A$ be the composite

$$X \xleftarrow{x} X_1 \xrightarrow{\pi_A} T(A, A)$$

and $\lambda_{AB}$ be the 2-cell

$$\begin{array}{c}
X_1 \xrightarrow{v} T(A, A) \\
X_2 \xrightarrow{v_{AB}} [A(A, B), T(A, B)] \\
X_1 \xrightarrow{w} T(B, B)
\end{array}$$

One can show that, for each $A$ and $B$, this 2-cell $\lambda_{AB}$ corresponds to a family $\{\lambda_f\}_{f: A \to B}$

$$\begin{array}{c}
X \xrightarrow{\lambda_A} T(A, A) \\
\lambda_B \downarrow \quad \lambda_f \Downarrow \xi \\
T(B, B) \xrightarrow{T(f, 1_B)} T(A, B)
\end{array}$$

which is natural in $f$ in the sense of $(2.1)$. This family $\lambda_f$ also satisfies each of $(2.2)$ and $(2.3)$ because the equality of diagrams $(3.3)$ and $(3.4)$ in the definition of descent objects corresponds respectively. Also, the universality of a lax end follows from that of limits.

From this construction of lax ends with weighted limits, we deduce the following two corollaries.

**Corollary 3.8.** Let $T$ be a 2-functor $\mathcal{A}^{op} \times A \times B \to \textbf{Cat}$. There is a canonical 2-functor $\int_A T(A, A, -)$, sending $B$ to $\int_B T(A, A, B)$.

**Proof.** It follows from Proposition 3.7 and the functoriality of weighted limits.

**Corollary 3.9.** Lax ends commutes with weighted limits

$$\lim^F \left( \int_A T(A, A, -) \right) \cong \int_A \lim^F T(A, A, -).$$

As a special case, for $T: \mathcal{A}^{op} \times \mathcal{B}^{op} \times A \times B \to \textbf{Cat}$,

$$\int_B \int_A T(A, B, A, B) \cong \int_A \int_B T(A, B, A, B).$$
Proof. It follows from Proposition 3.7 and the commutativity of weighted limits. 

Note that, for coends and colimits, Corollaries 3.8 and 3.9 have duals. Also, Fubini’s rule for lax ends is proved as follows.

Proposition 3.10. Let \( T : \mathcal{A}^{\text{op}} \times \mathcal{B}^{\text{op}} \times \mathcal{A} \times \mathcal{B} \to \text{Cat} \). Fubini’s theorem holds for both lax ends and coends.

\[
\begin{align*}
\int_{A,B} T(A, B, A, B) & \cong \int_{A} \int_{B} T(A, B, A, B) \cong \int_{B} \int_{A} T(A, B, A, B) \\
\int_{A} T(A, B, A, B) & \cong \int_{B} \int_{A} T(A, B, A, B) \cong \int_{A} \int_{B} T(A, B, A, B)
\end{align*}
\tag{3.7}
\]

\[
\begin{align*}
\int_{A,B} T(A, B, A, B) & \cong \int_{A} \int_{B} T(A, B, A, B) \cong \int_{B} \int_{A} T(A, B, A, B) \\
\int_{A} T(A, B, A, B) & \cong \int_{B} \int_{A} T(A, B, A, B) \cong \int_{A} \int_{B} T(A, B, A, B)
\end{align*}
\tag{3.8}
\]

Proof. It suffices to prove the left isomorphism of (3.7). The right isomorphism follows from the left, and (3.8) is the dual.

From the universality of \( \int_{A,B} \), a functor \( X \to \int_{A,B} T(A, B, A, B) \) corresponds to a lax wedge \( X \to \int_{A} T(A, B, A, B) \), which is a pair \( \{ \tau_A \} \) and \( \{ \tau_f \} \) as

\[
\begin{array}{ccc}
X & \xrightarrow{\tau_A} & \int_{A} T(A, B, A, B) \\
\downarrow \tau_A & & \downarrow \int_{A} T(1,1,f,1) \\
\int_{A} T(A, B, A, B) & \xrightarrow{\int_{A} T(f,1,1,1)} & \int_{A} T(A, B, A', B)
\end{array}
\]

Again, from the universality of \( \int_{B} \), each \( \tau_A \) corresponds to a wedge from \( X \) to \( T(A, -, A, -) \). We now have a family of 1-cells \( \{ \sigma_{AB} : X \to T(A,B,A,B) \} \), families of 2-cells \( \{ \sigma_{Ag} \} \) making \( \sigma_{A-} \) a wedge, and the family \( \{ \tau_f \} \).

By defining \( \sigma_{fB} \) as below,

\[
\begin{array}{ccc}
\int_{B} T(A, B, A, B) & \longrightarrow & T(A, B, A, B) \\
\sigma_A & \downarrow & \int_{B} T(1,1,f,1) \\
X & \downarrow \tau_f & \longrightarrow \int_{B} T(A, B, A', B) \\
\downarrow \sigma_C & & \downarrow \int_{B} T(f,1,1,1) \\
\int_{B} T(A', B, A', B) & \longrightarrow & T(A', B, A', B)
\end{array}
\]

\( \sigma_{-B} \) also becomes a wedge. From the universality of \( \int_{B} T(A, B, A', B) \), \( \sigma_{fB} \) defines a modification of wedges. Writing down the of requirements for \( \sigma_{fB} \) to be a modification, it turns out that \( \sigma_{Ag} \) and \( \sigma_{fB} \) are compatible in the sense of the condition in Proposition 2.4. Thus, we deduce that \( \sigma \) is a wedge \( X \to T \). Since we have proven that the universality for \( \int_{A} \int_{B} T(A, B, A, B) \) coincides with that for \( \int_{A,B} T(A, B, A, B) \), these are isomorphic. 

\[\square\]
4 Lax limits, lax end calculus

In the previous section, we established some basic isomorphisms for lax ends. Now, we will see more advanced results for lax ends, including the relation with lax limits.

The first important theorem is Theorem 4.2, which establishes two adjunctions between a presheaf 2-category \([A^{op}, \text{Cat}]\) and lax presheaf 2-category \(\text{Lax}[A^{op}, \text{Cat}]\), which are lax morphism classifier and coclassifier.

\[
\begin{aligned}
\text{Lax}[A^{op}, \text{Cat}] & \quad \text{Lax}[A^{op}, \text{Cat}] \\
\downarrow & \quad \downarrow \\
[A^{op}, \text{Cat}] & \quad [A^{op}, \text{Cat}]
\end{aligned}
\]

To this end, we first define this \((-)^\sharp\) and \((-)^\flat\).

**Definition 4.1.** Let \(F: A^{op} \to \text{Cat}\) be a 2-functor. We define 2-functors \(F^\sharp\) and \(F^\flat\) from \(A^{op}\) to \(\text{Cat}\) as follows:

\[
\begin{aligned}
F^\sharp & : \hat{A^{op}} \leftarrow \hat{A^{op}} \times FA \quad (4.1) \\
F^\flat & : \hat{A^{op}} \leftarrow [A(A, -), FA] \quad (4.2)
\end{aligned}
\]

Now that we have already proven some useful isomorphisms, it just suffices to combine them.

**Theorem 4.2.** The inclusion \([A^{op}, \text{Cat}] \to \text{Lax}[A^{op}, \text{Cat}]\) has both a left adjoint \((-)^\sharp\) and a right adjoint \((-)^\flat\).

\[
\begin{aligned}
\text{Lax}[A^{op}, \text{Cat}](F, H) & \cong [A^{op}, \text{Cat}](F^\sharp, H) \quad (4.3) \\
\text{Lax}[A^{op}, \text{Cat}](H, F) & \cong [A^{op}, \text{Cat}](H, F^\flat) \quad (4.4)
\end{aligned}
\]

**Proof.** For the right adjoint,

\[
\begin{aligned}
[A^{op}, \text{Cat}](H, F^\flat) & \cong \int_C [HC, F^\flat C] \\
& \cong \int_C [HC, \int_A [A(A, C), FA]] \quad \text{by (4.2)} \\
& \cong \int_A \int_C [A(A, C), FA] \quad \text{by (3.2)} \\
& \cong \int_A \int_C [A(A, C) \times HC, FA] \quad \text{by (3.6)} \\
& \cong \int_A [HA, FA] \quad \text{by Yoneda Lemma} \\
& \cong \text{Lax}[A^{op}, \text{Cat}](H, F) \quad \text{by (3.1)}
\end{aligned}
\]
And the left adjoint is the dual.

The left adjoint \((-\)\) we established above is known to give a representation of lax limits \(\text{lax lim}^F G\) with usual weighted limits \(\text{lim}^F G\). We start by recalling the definition of lax limits and then show this statement.

**Definition 4.3.** Let \(F: \mathcal{A} \to \text{Cat}\) and \(G: \mathcal{A} \to \mathcal{K}\) be 2-functors. A **lax limit** is a representing object \(\text{lax lim}^F G\) of \(\text{Lax}[\mathcal{A}, \text{Cat}](F, \text{K})\).

\[
\mathcal{K}(\text{K}, \text{lax lim}^F G) \cong \text{Lax}[\mathcal{A}, \text{Cat}](F, \mathcal{K}(\text{K}, G-)) \quad (4.5)
\]

Dually, for \(F: \mathcal{A}^{\text{op}} \to \text{Cat}\) and \(G: \mathcal{A} \to \mathcal{K}\), a **lax colimit** is a representing object \(\text{lax colim}^F G\) of \(\text{Lax}[\mathcal{A}^{\text{op}}, \text{Cat}](F, \text{G})\).

\[
\mathcal{K}(\text{lax colim}^F G, \text{K}) \cong \text{Lax}[\mathcal{A}^{\text{op}}, \text{Cat}](F, \mathcal{K}(G-, \text{K})) \quad (4.6)
\]

If \(\mathcal{K} = \text{Cat}\), we can calculate the right-hand side of (4.5) as

\[
\text{Lax}[\mathcal{A}, \text{Cat}](F, [X, G-]) \cong \int_A [FA, [X, GA]] \quad \text{by (3.1)}
\]

\[
\cong \int_A [X, [FA, GA]]
\]

\[
\cong [X, \int_A [FA, GA]] \quad \text{by (3.2)}
\]

\[
\cong [X, \text{Lax}[\mathcal{A}, \text{Cat}](F, G)]. \quad \text{by (3.1)}
\]

Therefore, by Yoneda lemma, we have

\[
\text{lax lim}^F G \cong \int_A [FA, GA] \cong \text{Lax}[\mathcal{A}, \text{Cat}](F, G) \quad (4.7)
\]

for any \(F, G \in [\mathcal{A}, \text{Cat}]\). And, with Theorem 4.2,

\[
\text{lax lim}^F G \cong [\mathcal{A}, \text{Cat}] (F^\sharp, G) \cong \text{lim}^F G, \quad (4.8)
\]

or conversely,

\[
\text{lax lim}^F G \cong [\mathcal{A}, \text{Cat}] (F, G^\circ) \cong \text{lim}^F G. \quad (4.9)
\]

Isomorphisms (4.8) and (4.9) show that every lax limits in \(\text{Cat}\) can be represented by weighted limits with the same weights or diagrams. We show (4.8) can be generalized in general 2-categories.

**Proposition 4.4.** Let \(G: \mathcal{A} \to \mathcal{K}\) be a 2-functor. Then

\[
\text{lax lim}^F G \cong \text{lim}^F G, \quad (4.10)
\]

\[
\text{lax colim}^F G \cong \text{colim}^F G, \quad (4.11)
\]

whenever they exist.
Proof. (4.10) follows from
\[ \mathcal{K}(K, \text{lax lim}^F G) \cong \text{Lax}[A, \textbf{Cat}](F, \widehat{G}K) \]
by (4.5)
\[ \cong [A, \textbf{Cat}](F^\sharp, \widehat{G}K) \]
by (4.3)
\[ \cong \mathcal{K}(K, \text{lax lim}^{F^\sharp} G). \]

(4.11) is similar.

Note that, by similar arguments as in (4.7) and proposition 4.4, the lax colimits in \textbf{Cat} can be presented in the following several other ways:
\[ \text{lax colim}^F G \cong \int^A F A \times GA \]
\[ \cong \int^A GA \times FA \]
\[ \cong \text{lax colim}^G F \]
\[ \cong \text{colim}^{G^\sharp} F \]

We can directly deduce the commutativity of lax ends from the commutativity of weighted limits, which is the right isomorphism in Fubini’s theorem 3.10. And from Fubini’s theorem and the representation of lax limits in \textbf{Cat} with lax ends (4.7), we deduce that lax limits commute.

**Corollary 4.5.** Let \( F: A \to \textbf{Cat}, F': B \to \textbf{Cat}, \) and \( G: A \times B \to \textbf{Cat}. \) Then,
\[ \text{lax lim}^F \text{lax lim}^{F'} G \cong \text{lax lim}^{F'} \text{lax lim}^F G \]
\[ \text{lax colim}^F \text{lax colim}^{F'} G \cong \text{lax colim}^{F'} \text{lax colim}^F G. \]

**Example 4.6.** One of the simple lax limits is the conical lax limit, that is, a lax limit weighted by the constant functor \( \Delta_1: A \to \textbf{Cat}. \) By Proposition 4.4, this is \( \text{lax lim}^{\Delta_1} G \cong \text{lim}^{(\Delta_1)^\sharp} G. \) Let us examine the weight \( (\Delta_1)^\sharp \) of the right-hand side. By the definition of \( (\cdot)^\sharp \), \( (\Delta_1)^\sharp \) is a 2-functor \( \int^A \mathcal{A}(A, \cdot), \) sending \( C \in A \) to \( \int^A \mathcal{A}(A, C) \cong \text{lax colim}^{\Delta_1} \mathcal{A}(-, C), \) In fact, \( (\Delta_1)^\sharp C \) turns out to be the lax slice category \( A/C, \) whose object is a 1-cell into \( C, \) and whose morphism from \( p \) to \( q \) is a pair \((f, \bar{f})\) as follows.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
\downarrow{p} & \Rightarrow & \downarrow{q} \\
C & & \\
\end{array}
\]

To show this actually is a lax colimit, we then construct the unit lax cocone \( \{\lambda_A: \mathcal{A}(A, C) \to A/C\} \) with \( \{\lambda_f: \lambda_A \circ \mathcal{A}(f, C) \to \lambda_{A'}\}_f: A \to A', \) by
\[
\lambda_A(p: A \to C) = p, \quad \lambda_A(\alpha: p \to p') = (1_A, \alpha), \\
(\lambda_f)_q: A' \to C = (f, 1): qf \to q.
\]
We need to show its universality. To this end, let us assume there is another lax cocone \( \{ \sigma_A : \mathcal{A}(A, C) \to X \} \) with \( \{ \sigma_f : \sigma_A \circ \mathcal{A}(f, C) \to \sigma_{A'} \} \), and a morphism of cocone \( u : \mathcal{A}/C \to X \). Then, these data must satisfy the following:

\[
\begin{align*}
\mathcal{A}(A, C) & \xrightarrow{\lambda_A} \mathcal{A}/C \xrightarrow{u} X, \\
\mathcal{A}(A', C) & \xrightarrow{\lambda_{A'}} \mathcal{A}/C \xrightarrow{u} X.
\end{align*}
\]

Therefore, we need to have

- for each \( p : A \to C \), \( u(p) = \lambda_A(p) = \sigma_A(p) \),
- for each \( \alpha : p \to q \) in \( \mathcal{A}(A, C) \), \( u((1, \alpha)) = \lambda_A(\alpha) = \sigma_A(\alpha) \),
- for each \( f : A \to A' \) and \( q : A' \to C \), \( u((f, 1)) = u(\lambda_f)_q = (\sigma_f)_q \).

Since an arrow \( (f, \bar{f}) : p \to q \) in \( \mathcal{A}/C \) is the composite \( p \xrightarrow{(1, \bar{f})} qf \xrightarrow{(f, 1)} q \), the data of \( u \) is fully determined by \( \sigma \). This proves the uniqueness of \( u \), and the existence of \( u \) is checked by some diagram chasing.

When \( \mathcal{A} \) is a category \( \mathbf{2} \) for example, the resulting weight \( (\Delta 1)^\sharp \) sends the diagram \( 0 \to 1 \) to \( 0 \xrightarrow{1} 2 \).

In enriched category theory, \( \int_A T(A, A) \) could be rewritten as \( \lim^{\Hom_A} T \). Correspondingly, we wish to rewrite lax ends using limits or lax limits. The reader might naturally think of \( \lim^{\Hom_A} T \) as a candidate for the rewriting. However, it turns out to be incorrect. In fact, calculating \( [\mathcal{A} \to \mathcal{X} \lim^{\Hom_A} T] \) yields \( \int_A \int_C [\mathcal{A}(C, A), [\mathcal{X}, T(C, A)]] \), which is different from \( \int_A [\mathcal{X}, \int_A \mathcal{A}(C, A), [\mathcal{X}, T(C, A)]] \).

The next theorem gives the correct answer.

**Theorem 4.7.** Let \( T : \mathcal{A}^{\op} \times \mathcal{A} \to \mathbf{Cat} \) be a \( 2 \)-functor. Then

\[
\begin{align*}
\int_A T(A, A) & \cong \lim^{(\mathcal{Y}^\circ)^\sharp} T(?, -) \quad (4.12) \\
\int^A T(A, A) & \cong \colim^{(\mathcal{Y}^\circ)^\sharp} T(?, -) \quad (4.13)
\end{align*}
\]

where \( (\mathcal{Y}^\circ)^\sharp \) is a profunctor sending \( (A, A') \in \mathcal{A}^{\op} \times \mathcal{A} \) to \( (Y A')^\sharp A \).
Proof. (4.12) follows from the following isomorphisms. (4.13) is similar.

\[
\begin{align*}
[ X, \lim_{(Y^-)^{\circ}, T(?, -)} ] \\
\cong \int_{C,C'} [(Y'C')^\circ C, [X, T(C, C')] ] \\
= \int_{C,C'} \left[ \int^A \mathcal{A}(C, A) \times \mathcal{A}(A, C'), [X, T(C, C')] \right] \\
\cong \int_{C,C'} \int^A \left[ \mathcal{A}(C, A) \times \mathcal{A}(A, C'), [X, T(C, C')] \right] \\
\cong \left[ \int^A \mathcal{A}(C, A), [X, T(C, A)] \right] \\
\cong \left[ X, \int^A T(A, A) \right].
\end{align*}
\]

by (4.1)

by the dual of (3.2)

by (3.6)

by Yoneda lemma

by Yoneda lemma

by (3.2)

\[\square\]

5 Lax ends in general 2-Categories

In the previous two sections, we examined lax ends in \textbf{Cat}, but of course, lax ends can be defined in general 2-categories, which we discuss in this section. The problem is that, while there are several characterizations of the lax ends in \textbf{Cat}, it is not obvious which one should be used for the generalization. Here, we adopt its commutativity with representables as the definition for the general case and show that several other characterizations coincide.

**Definition 5.1.** Let \( T \) be a 2-functor \( T : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{K} \). We define lax [co]ends in \( \mathcal{K} \) as representing objects

\[\begin{align*}
\mathcal{K} \left( B, \int^A T(A, A) \right) &\cong \int^A \mathcal{K}(B, T(A, A)), \\
\mathcal{K} \left( \int^A T(A, A), B \right) &\cong \int^A \mathcal{K}(T(A, A), B).
\end{align*}\] (5.1)

Firstly, we check the universality of unit in (5.1).

**Proposition 5.2.** Let \( T \) be a 2-functor \( T : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{K} \). Then, \( T \) has a lax end in \( \mathcal{K} \) if and only if there is a lax wedge that has the same universality as lax ends in \textbf{Cat}.
Proof. As we observed in Proposition 3.3, there is also a bijection between the set of lax wedges $K \rightarrow T$ in $\mathcal{K}$ and the set of lax wedges $\mathbb{1} \rightarrow \mathcal{K}(B,T(?))$ in $\mathbf{Cat}$. Moreover, those modifications precisely coincide.

Let $T: A^{\text{op}} \times A \times B \rightarrow \mathcal{K}$ be a 2-functor. To verify the notation $\mathcal{L}_{A} T(A, A, ?)$ defined in the case $\mathcal{K} = \mathbf{Cat}$ is compatible with the definition of the general lax end, we need to check that the lax end in a functor category is computed pointwise. This is checked by the following isomorphism for an arbitrary $F$: $B \rightarrow \mathcal{K}$.

$$\mathcal{L}_{A} \left[ B, \mathcal{K} \right] (F, T(A, A, ?)) \cong \mathcal{L}_{A} \int_{B} \mathcal{K}(FB, T(A, A, B))$$

$$\cong \int_{B} \mathcal{L}_{A} \mathcal{K}(FB, T(A, A, B)) \quad \text{by (3.6)}$$

$$\cong [B, \mathcal{K}] \left( F \mathcal{L}_{A}, \mathcal{L}_{A} T(A, A, ?) \right).$$

Therefore, we can use the notation $\mathcal{L}_{A} T(A, A, ?)$ freely.

The other characterization of the lax limits in $\mathbf{Cat}$ was a weighted limit $\lim^{(Y,-)?} T(?,-)$, proved in Theorem 4.7. Its generalization to arbitrary 2-categories again can be shown to agree with the definition above by similar calculation: for general $T: A^{\text{op}} \times A \rightarrow \mathcal{K}$,

$$\mathcal{L}_{A} T(A, A) \cong \lim^{(Y,-)?} T(?,-)$$

$$\mathcal{L}_{A} T(A, A) \cong \text{colim}^{(Y,-)?} T(?,-).$$

As a corollary, it is proved that, if $\mathcal{K}$ is [co]complete, then $\mathcal{K}$ admits lax [co]ends.

By the definition of lax ends in $\mathcal{K}$ and Corollary 3.9, lax [co]ends in general $\mathcal{K}$ also commute with weighted [co]limits.

$$\lim^{F} \left( \mathcal{L}_{A} T(A, A, ?) \right) \cong \mathcal{L}_{A} \lim^{F} T(A, A, ?) \quad (5.2)$$

Also, the Fubini’s rule follows from the $\mathbf{Cat}$ case (Proposition 3.10).

Next, we would like to show the adjunctions between functor categories with 2/lax transformations in general $\mathcal{K}$ again. As in the $\mathbf{Cat}$ case, we define $(-)^{\sharp}$ and $(-)^{\flat}$.

**Definition 5.3.** Let $F: A \rightarrow \mathcal{K}$ be a 2-functor. We define $F^{\sharp}, F^{\flat}: A \rightarrow \mathcal{K}$ as follows.

$$F^{\sharp}: \mathcal{L}_{A} \mathcal{K}(A(-), FA)$$

$$F^{\flat}: \mathcal{L}_{A} \mathcal{K}(A(\mathcal{L}_{A}), FA) \quad (5.3)$$

Note that a general $\mathcal{K}$ do not admit $(-)^{\flat}$[or $(-)^{\sharp}$], since $\mathcal{K}$ might not have [co]limits. And by the same calculation as in the $\mathbf{Cat}$ case, the next theorem holds.
Theorem 5.4. If $\mathcal{K}$ is complete, then the inclusion $[\mathcal{A}, \mathcal{K}] \to \text{Lax}[\mathcal{A}, \mathcal{K}]$ has a right adjoint $(-)^\flat$. Dually, if $\mathcal{K}$ is cocomplete, then the inclusion $[\mathcal{A}, \mathcal{K}] \to \text{Lax}[\mathcal{A}, \mathcal{K}]$ has a left adjoint $(-)^\sharp$.

We then would like to return to the topic of lax limits. Let $\mathcal{K}$ be a complete 2-category, and $F: \mathcal{A} \to \mathcal{K}$, $G: \mathcal{A} \to \mathcal{K}$ be 2-functors. By similar isomorphisms just before (4.7), we have an isomorphism,

$$\text{Lax}[\mathcal{A}, \text{Cat}](F, \mathcal{K}(K, G-)) \cong \int_{A}[FA, \mathcal{K}(K, GA)] \quad \text{by (3.1)}$$

$$\cong \int_{A} \mathcal{K}(K, FA \downarrow GA) \quad \cong \mathcal{K}(K, \int_{A} FA \downarrow GA). \quad \text{by (5.1)}$$

Therefore, a lax limit can also be presented as,

$$\text{lax lim}^F G \cong \int_{A} FA \downarrow GA. \quad (5.5)$$

Proposition 4.4 showed that this could be presented as the limit $\text{lim}^F G$. However, since we did not define $G^\flat$ for general $\mathcal{K}$ in the previous section, the isomorphism

$$\text{lax lim}^F G \cong \text{lim}^F G^\flat \quad (5.6)$$

in (4.9) did not make sense at that time. But now, since we defined $G^\flat$ in (5.4), one can check (5.6) by

$$\mathcal{K}(K, \text{lim}^F G^\flat) \cong [\mathcal{A}, \text{Cat}](F, \mathcal{K}(K, G^\flat-))$$

$$\cong [\mathcal{A}, \text{Cat}](F, \mathcal{K}(K, G-)^\flat)$$

$$\cong \text{Lax}[\mathcal{A}, \text{Cat}](F, \mathcal{K}(K, G-)),$$

where the second isomorphism is from the fact that representables preserves lax ends and powers.

6 The class of limits where lax ends live

The classification of 2-categorical limits is an interesting and complicated problem. To classify them, several classes of limits have been invented, such as the flexible limits [1] and the PIE limits [11]. In this section, we show that a lax end is a PIE limit, but not a lax limit. We first show the former part: a lax end is a PIE limit.

Theorem 6.1. If a 2-category $\mathcal{K}$ admits products, inserters, and equifiers, Then it admits all lax ends and lax limits.
Proof. In the same way as we did in Proposition 3.7, we can define $X_1$, $X_2$, $X_3$ as follows,

$$X_1 = \prod_{A \in A} T(A, A)$$
$$X_2 = \prod_{A, B \in A} A(A, B) \cdot T(A, B)$$
$$X_3 = \prod_{A, B, C \in A} A(B, C) \times A(A, B) \cdot T(A, C)$$

with six 1-cells in coherence data in the same manner, and check the five identities. Since $\mathcal{K}$ has inserters and equifiers, a descent object of this coherence data does exist in $\mathcal{K}$, which is the desired lax end in $\mathcal{K}$.

The isomorphism (5.5) shows that, since $\mathcal{K}$ admits lax ends and powers, it also admits lax limits.

Then, we show that there is a 2-category with all lax limits but not all lax ends.

Lemma 6.2. A 2-category $\mathcal{K}$ with lax ends admits powers, all lax limits and oplax limits.

Proof. Let $\mathcal{K} \in \mathcal{K}$ and $\mathbb{A}$ be a category. The lax end of the constant functor $\Delta \mathcal{K} \colon \mathbb{A}^{op} \times \mathbb{A} \to \mathcal{K}$ determines the power $\mathbb{A} \sqcap \mathcal{K}$ since we have an isomorphism

$$\mathcal{K} \left( X, \int_{A \in \mathbb{A}} \Delta \mathcal{K} \right) \cong \int_{A \in \mathbb{A}} \mathcal{K}(X, K) \cong [\mathbb{A}, \mathcal{K}(X, K)] \cong \mathcal{K}(X, \mathbb{A} \sqcap \mathcal{K}).$$

Therefore, from (5.5), $\mathcal{K}$ has all lax limits.

As we mentioned in Section 2, an oplax wedge and end of $T$ can be defined as a lax wedge and end of another $T'$ with $T'(A, B) = T(A, B)$. And therefore, writing oplax ends with $\int$, $\mathcal{K}$ also admits oplax limits $\int_{\mathbb{A}} F(A, B) \sqcap G(A) \cong \text{oplax lim}_{\mathcal{K}} F$.

Theorem 6.3. There is a 2-category with all lax limits, but does not have lax ends.

Proof. Let $\mathcal{K}$ be a full sub 2-category of $\textbf{Cat}$ whose objects are categories with finite products. Since there is a 2-monad $T$ on $\textbf{Cat}$ which has $\mathcal{K}$ as the 2-category of algebras and oplax morphisms, $\mathcal{K}$ has all lax limits, which was shown in [8, 9].

However, this $\mathcal{K}$ lacks oplax limits of an arrow. To see this, first observe that $\mathcal{K}$ is dense in $\textbf{Cat}$, which is because $\mathbb{1}$ is dense in $\textbf{Cat}$ as a $\textbf{Cat}$-enriched category. And therefore all the limits in $\mathcal{K}$ are those in $\textbf{Cat}$. Let $\mathbb{B}$ a category with finite products and $b \in \mathbb{B}$. Then, the oplax limit of the arrow $\mathbb{1} \rightarrow \mathbb{B}$ in $\textbf{Cat}$ is the slice category $\mathbb{B}/b$. Since the product in slice category is the pullback, the oplax limit is not included in $\mathcal{K}$ in general.

7 Pseudo ends and bicategorical (co)Yoneda lemma

As is widely known, the theory of bicategories has its version of the Yoneda lemma, the bicategorical Yoneda lemma. For simplicity, we restrict to 2-categories and 2-functors here.
Theorem 7.1 (bicategorical Yoneda lemma). Let $F : \mathcal{A}^{\text{op}} \to \textbf{Cat}$ be a 2-functor. Then there is a following equivalence of categories, which exists the pseudo-natural in $A \in \mathcal{A}$ and $F \in \text{Ps}[\mathcal{A}^{\text{op}}, \textbf{Cat}]$.

$$\text{Ps}[\mathcal{A}^{\text{op}}, \textbf{Cat}](Y_A, F) \simeq FA \quad (7.1)$$

For the proof, consult other literature such as [6].

Clearly, since everything proved in Sections 2 to 4 has its counterpart in pseudo case with pseudo transformations/ends/limits, etc. We denote pseudo ends by integral with tilde $\int_A$, and adjunctions corresponding to Theorem 4.2 by double sharp $\sharp$ and double flat $\flat$.

$$\text{Ps}[\mathcal{A}^{\text{op}}, \textbf{Cat}](F, H) \simeq [\mathcal{A}^{\text{op}}, \textbf{Cat}](F^\ast, H) \quad (7.2)$$

$$\text{Ps}[\mathcal{A}^{\text{op}}, \textbf{Cat}](H, F) \simeq [\mathcal{A}^{\text{op}}, \textbf{Cat}](H, F^\flat)$$

Since the hom-category of $\text{Ps}[\mathcal{A}^{\text{op}}, \textbf{Cat}]$ can be represented with pseudo ends, the bicategorical Yoneda lemma is equivalent to saying

$$\int_{C \in \mathcal{A}} [\mathcal{A}(C, A), FC] \simeq FA.$$ 

Since the left-hand side is the definition of $F^\beta A$, there is an equivalence $F^\beta A \simeq FA$.

This equivalence concludes the equivalence of functors $F \simeq F^\beta$. However, it should be noted that this is one that in $\text{Ps}[\mathcal{A}^{\text{op}}, \textbf{Cat}]$ and not in $[\mathcal{A}^{\text{op}}, \textbf{Cat}]$, since the bicategorical Yoneda lemma is only pseudo-natural in $A$.

In the remaining part of this section, we show the bicategorical coYoneda lemma $F^\ast A \simeq FA$ as the dual for the bicategorical Yoneda lemma $F^\beta A \simeq FA$, which is also shown in [5].

Theorem 7.2 (bicategorical coYoneda lemma). There exists the following equivalence of categories which is pseudo natural for $A \in \mathcal{A}$ and 2-natural for $F \in \text{Ps}[\mathcal{A}^{\text{op}}, \textbf{Cat}]$.

$$F^\ast A = \int_{C \in \mathcal{A}} [\mathcal{A}(A, C) \times FC] \simeq FA.$$ 

Proof. Let the following be the universal cowedges for the end or the pseudo end.

$$\lambda_{CA} : \mathcal{A}(C, A) \times FC \to \int^C \mathcal{A}(C, A) \times FC$$

$$\tilde{\lambda}_{CA} : \mathcal{A}(C, A) \times FC \to \int^C \mathcal{A}(C, A) \times FC$$

Note that these are both 2-natural for $A$.

Since 2-wedge $\lambda_{-A}$ is also a pseudo wedge, from the universality of pseudo ends, there is a unique functor $\varepsilon_A : F^\ast A \to FA$ satisfying $\varepsilon_A \tilde{\lambda}_{CA} = \lambda_{CA}$.

$$\mathcal{A}(A, C) \times FC \xrightarrow{\varepsilon_A} \int_{C \in \mathcal{A}} [\mathcal{A}(A, C) \times FC] \simeq FA$$
This $\varepsilon_A$ is the counit for the left adjoint $(-)^{\ast}$. Since $\tilde{\lambda}_{AD}$ is 2-natural for $D$, the transpose $\alpha_{AD}: FA \rightarrow [A(A, D), F^{\ast}D]$ is also 2-natural for $D$. So, $\alpha_{AD}$ induces $\eta_A: FA \rightarrow \int_D [A(A, D), F^{\ast}D] \cong F^{\ast}A$, which is the unit, pseudo natural for $A$.

The one side of triangular identity for the left adjoint $(-)^{\ast}$ tells $\varepsilon_A\eta_A = id$. Therefore, it suffices to show $\eta_A\varepsilon_A \cong id$. To show this, we precompose $\tilde{\lambda}_{CA}$ and postcompose $F^{\ast}A \cong \int_D [A(D, A), F^{\ast}D] \xrightarrow{\rho_DA} [A(D, A), F^{\ast}D]$ to both $\eta_A\varepsilon_A$ and id, where $\rho_DA$ is the universal wedge. By showing these are natural isomorphic, $\eta_A\varepsilon_A \cong id$ is deduced from the universality of the pseudo coend and the end.

Precomposition of $\tilde{\lambda}_{CA}$ to $\varepsilon_A$ is $\overline{F_{AC}}$, where $\overline{F_{AC}}$ is the transpose of $F_{AC}: A(A, C) \rightarrow [FC, FA]$, and postcomposition of $\rho_DA$ to $\eta_A$ is $\alpha_{AD}$. Thus, the left-hand side is $\alpha_{AD}\overline{F_{AC}}$. On the other hand, the right-hand side – the composition of $\tilde{\lambda}_{CA}$ and $\rho_DA$ – is the transpose of $A(D, A) \times A(C, A) \times FC \xrightarrow{\varepsilon_{DAC}\times 1} A(D, C) \times FC \xrightarrow{\tilde{\lambda}_{CD}} F^{\ast}D$. (7.3)

Here, we used the 2-naturality of $\tilde{\lambda}_{CA}$ for $A$. The pseudo naturality of $\tilde{\lambda}_{CA}$ for $A$ produces the natural isomorphism,

$$
\begin{array}{ccc}
A(A, C) & \xrightarrow{A(D, A) \times F^-} & [A(D, A) \times FC, A(D, A) \times FA] \\
A(D, -) \times FC & \cong & [1, \tilde{\lambda}_{CA}] \\
[A(D, A) \times FC, A(D, C) \times FC] & \xrightarrow{[1, \tilde{\lambda}_{CD}]} & [A(D, A) \times FC, F^{\ast}D].
\end{array}
$$

Carefully checked, it can be proved that $\alpha_{AD}\overline{F_{AC}}$ is the transpose of the upper right of the diagram, and (7.3) is the transpose of the down left.

Showing the naturality of $A$ and $F$ requires a bit more work. That is because, as we showed in Proposition 2.4, the naturality for each variable does not show the whole naturality. The compatibility of naturality with $C$ or $D$ needs to be checked, but we omit the proof here for redundancy. One can also check this naturality from 2-monad theory in the following section.

In the same way, one can prove bicategorical Yoneda lemma $F^{\ast}A \simeq FA$ dually.

## 8 From the point of view of 2-monad theory

The adjunction we showed at (7.2)

$$
\text{Ps}[A^{op}, \text{Cat}](F, H) \cong [A^{op}, \text{Cat}](F^{\ast}, H) \quad (8.1)
$$

$$
\text{Ps}[A^{op}, \text{Cat}](H, F) \cong [A^{op}, \text{Cat}](H, F^{\ast}) \quad (8.2)
$$

can also be derived from 2-monad theory [2]. There is a 2-monad $T$ over the 2-category $[\text{ob}(A^{op}), \text{Cat}]$ whose strict algebras are 2-functors and whose strict/pseudo/lax morphisms are 2-/pseudo/lax transformations. This 2-category $[\text{ob}(A^{op}), \text{Cat}]$ is complete and cocomplete, and 2-monad $T$ has a right adjoint [9]. Therefore, $T$ satisfies the coherence condition [7], that is,

- The inclusion $T^{\ast}\text{-Alg}_{\ast} = [A^{op}, \text{Cat}] \hookrightarrow T^{\ast}\text{-Alg} = \text{Ps}[A^{op}, \text{Cat}]$ has a left adjoint $(-)^{\ast}$ and a right adjoint $(-)^{\ast}$,
• whose units or counits in $\mathbf{Ps}[\mathcal{A}^{\text{op}}, \mathbf{Cat}]$ are (pseudo) equivalences $F^\ast \simeq F \simeq F^{\sharp \flat}$.

On the other hand, for lax natural transformations, there are two adjunctions in $\mathbf{Lax}[\mathcal{A}^{\text{op}}, \mathbf{Cat}]$ between $F$ and $F^\sharp$, and between $F$ and $F^\flat$.

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