NORMAL NUMBERS FROM STEINHAUS’ VIEWPOINT

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Abstract. In this paper we recall a non-standard construction of the Borel sigma-algebra $\mathcal{B}$ in $[0,1]$ and construct a family of measures (in particular, Lebesgue measure) in $\mathcal{B}$ by a completely non-topological method. This approach, that goes back to Steinhaus, in 1923, is now used to introduce natural generalizations of the concept of normal numbers and explore their intrinsic probabilistic properties. We show that, in virtually all the cases, almost all real number in $[0,1]$ is normal (with respect to this generalized concept). This procedure highlights some apparently hidden but interesting features of the Borel sigma-algebra and Lebesgue measure in $[0,1]$.

1. Introduction

The Borel sigma-algebra in $[0,1]$ is, in general, defined as the sigma-algebra generated by the (open) intervals in $[0,1]$. So, we have a natural “topological” component in the Borel sigma-algebra (and Lebesgue measure) in $[0,1]$. However, as it will be shown, a completely different (non-topological) approach will be extremely useful to deal with an extension of the concept of normal numbers and their probabilistic aspects. The proposed characterization of the Borel sigma-algebra in $[0,1]$, which goes back to 1923, with Steinhaus [5], in the beginning of the conception of the modern probability theory, shows a natural way to consider “weighted” measures in $[0,1]$, and to define and discuss normality of numbers with respect to these measures.

The paper is organized as follows: In Section 2 we recall some background results concerning product measures, in Section 3 we characterize the Borel sigma-algebra in $[0,1]$ as a product sigma-algebra and introduce a family of “weighted” measures in $[0,1]$ and, in the last section, we apply the previous results to generalize the concept of normal numbers and explore their probabilistic behavior. The main goal of this note is to translate, to the modern notation and terminology, Steinhaus’ striking ideas concerning normal numbers and also to call attention to his contributions to the birth of modern probability theory.

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Throughout, \( \mathbb{R} \) denotes the set of all real numbers, \( \mathbb{N} \) represents the set of all natural numbers \( \{1, 2, \ldots\} \) and \( \mathcal{B} \) is the Borel sigma-algebra in the closed interval \([0, 1]\). If \( \Omega \) is any set, \( 2^\Omega \) denotes the set of all subsets of \( \Omega \). If \( A \subset 2^\Omega \), then \( \sigma(A) \) represents the sigma-algebra generated by \( A \), \( \#A \) denotes the cardinality of \( A \) and \( A^\mathbb{N} \) represents \( A \times A \times \cdots \).

2. Preliminary results

Let \( \Omega \neq \emptyset \) be a denumerable (or finite) set, \( A = 2^\Omega \) and let \( \rho \) be a probability measure in \( A \).

Let \( (\Omega^\mathbb{N}, \otimes \mathcal{A}, \otimes \rho) \) be the product space and denote \( \mu = \mu_\rho = \otimes \rho \) and \( \mathcal{D} = \otimes \mathcal{A} \).

**Definition 1.** \( w = (a_i)_{i=1}^\infty \in \Omega^\mathbb{N} \) is simply normal, with respect to \( \mu \), if
\[
\lim_{n \to \infty} \frac{S(w, r, n)}{n} = \rho(\{r\}) \quad \forall r \in \Omega,
\]
where \( S(w, r, n) \) denotes the total of indexes \( i, 1 \leq i \leq n \), such that \( a_i = r \).

**Definition 2.** \( w = (a_i)_{i=1}^\infty \in \Omega^\mathbb{N} \) is normal, with respect to \( \mu \), if
\[
\lim_{n \to \infty} \frac{S(w, B_k, n)}{n} = \prod_{j=1}^k \rho(\{b_j\}) \quad \forall k \in \mathbb{N},
\]
for each word \( B_k = b_1 \ldots b_k \) of \( k \) elements from \( \Omega \), where
\[
S(w, B_k, n) = \# \{ i \in \{1, \ldots, n\}; a_{i+j-1} = b_j \text{ for every } j = 1, \ldots, k \}.
\]

The next two propositions are standard applications of the Strong Law of Large Numbers. We sketch the proofs for the sake of completeness.

**Proposition 1.** The measure of the set
\[
\{ w = (a_i)_{i=1}^\infty \in \Omega^\mathbb{N}; w \text{ is simply normal} \}
\]
is 1.

**Proof.** Let \( \mathcal{B}_\mathbb{R} \) denote the Borel sigma-algebra on \( \mathbb{R} \) and \( r \in \Omega \). Define, for every \( n \in \mathbb{N} \),
\[
(2.1) \quad X^r_n : (\Omega^\mathbb{N}, \mathcal{D}, \mu) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})
\]
\[
w = (a_i)_{i=1}^\infty \mapsto \begin{cases} 0, & \text{if } a_n \neq r \\ 1, & \text{if } a_n = r \end{cases}
\]

It is easy to see that each \( X^r_n \) is measurable and hence \( (X^r_n)_{n=1}^\infty \) is a sequence of real random variables. Moreover, \( (X^r_n)_{n=1}^\infty \) is an independent, integrable and identically distributed sequence.
Note that
\[ \int_{\Omega^N} X_n^r d\mu = \mu((X_n^r)^{-1}(\{1\})) = \rho(\{r\}). \]
Hence, a well-known result due to Kolmogorov asserts that the sequence
\( (X_n^r)_{n=1}^{\infty} \) satisfies the Strong Law of Large Numbers, and so
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_n^r(w) = \rho(\{r\}) \text{ (\( \mu \)-a.e)} \]
Therefore
\[ \lim_{n \to \infty} \frac{S(w,r,n)}{n} = \rho(\{r\}) \text{ (\( \mu \)-a.e)} \]
Denoting
\[ M^r = \left\{ w \in \Omega^N; \lim_{n \to \infty} \frac{S(w,r,n)}{n} = \rho(\{r\}) \right\}, \]
we have that the set composed by the simply normal sequences on \( \Omega^N \) is precisely
\[ \bigcap_{r \in \Omega} M^r. \]
Since \( \mu(\Omega^N \setminus M^r) = 0 \) and \( \Omega \) is, at most, denumerable, we have
\[ \mu \left( \bigcap_{r \in \Omega} M^r \right) = 1. \]
\[ \square \]
**Proposition 2.** The measure of the set \( \{w = (a_i)_{i=1}^{\infty} \in \Omega^N; \ w \text{ is normal}\} \) is 1.

Proof. Let \( r_1,...,r_k \) be a word with \( k \) elements from \( \Omega \), with \( k \in \mathbb{N} \). Using the notation from (2.1), define, for every \( n \in \mathbb{N} \),
\[
\begin{cases}
Y_{r_1,...,r_k, n}^{(1)}(w) = X_{kn-(k-1)}^{r_1}(w)X_{kn-(k-2)}^{r_2}(w)...X_{kn}^{r_k}(w) \\
Y_{r_1,...,r_k, n}^{(2)}(w) = X_{kn-(k-2)}^{r_1}(w)X_{kn-(k-3)}^{r_2}(w)...X_{kn+1}^{r_k}(w) \\
\vdots \\
Y_{r_1,...,r_k, n}^{(k)}(w) = X_{kn}^{r_1}(w)X_{kn+1}^{r_2}(w)...X_{kn+(k-1)}^{r_k}(w).
\end{cases}
\]
It is plain that, for every \( j = 1, ..., k \), \( (Y_{r_1,...,r_k, n}^{(j)})_{n=1}^{\infty} \) are integrable, independent and identically distributed sequences. We also have, for every \( j = 1, ..., k \),
\[ \int_{\Omega^N} Y_{r_1,...,r_k, n}^{(j)} d\mu = \mu \left( \left( Y_{r_1,...,r_k, n}^{(j)} \right)^{-1}(\{1\}) \right) = \prod_{j=1}^{k} \rho(\{r_j\}). \]
Hence, for every \( j = 1, \ldots, k \), \( (Y_{(j),n}^{r_1 \ldots r_k})_{j=1}^{\infty} \) satisfy the Strong Law of Large Numbers and

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} Y_{(j),n}^{r_1 \ldots r_k}(w) = \prod_{j=1}^{k} \rho(\{r_j\}) \quad (\mu\text{-a.e}).
\]

We thus have

\[
\lim_{n \to \infty} \frac{S(w, r_1 \ldots r_k, n)}{n} = \prod_{j=1}^{k} \rho(\{r_j\}) \quad (\mu\text{-a.e}).
\]

Denoting

\[
M_{r_1 \ldots r_k} = \left\{ w \in \Omega^N; \lim_{n \to \infty} \frac{S(w, r_1 \ldots r_k, n)}{n} = \prod_{j=1}^{k} \rho(\{r_j\}) \right\},
\]

the set composed by the normal sequences on \( \Omega^N \) is precisely

\[
\bigcap_{k \in \mathbb{N}} \left( \bigcap_{r_1 \ldots r_k \in \Omega^k} M_{r_1 \ldots r_k} \right).
\]

Since \( \mu(\Omega^N \setminus M_{r_1 \ldots r_k}) = 0 \) and \( \Omega \) is, at most, denumerable, we have

\[
\mu\left( \bigcap_{k \in \mathbb{N}} \left( \bigcap_{r_1 \ldots r_k \in \Omega^k} M_{r_1 \ldots r_k} \right) \right) = 1.
\]

\( \square \)

3. A FAMILY OF MEASURES IN \([0, 1]\)

The present section can be regarded, in some sense, as a translation of Steinhaus’ ideas from the roots of the modern probability theory (see [5]).

Henceforth, \( \Omega = \{0, \ldots, 9\}, A = 2^\Omega \) and \( \rho \) is a probability measure on \( A \).

If \( (\Omega^N, D, \mu_\rho) \) is the product space, then \( D = \sigma(C) \), with

\[
C = \left\{ \Omega \times \cdots \times \Omega \times \{a\} \times \Omega \times \cdots; a \in \Omega \text{ and } n \in \mathbb{N} \right\}.
\]

Consider the mapping

\[
\Psi_{\mu_\rho}: (\Omega^N, D, \mu_\rho) \to ([0, 1], \sigma(\Psi_{\mu_\rho}(C)))
\]

\[
w = (a_j)_{j=1}^{\infty} \mapsto \sum_{j=1}^{\infty} a_j 10^{-j}.
\]
Note that $\Psi_{\mu}$ is “almost injective”, in the sense that, for every $(a_j)_{j=1}^{\infty} \in \Omega^\mathbb{N}$,
\[
\# \left( \{ (b_j)_{j=1}^{\infty} \in \Omega^\mathbb{N}; (b_j)_{j=1}^{\infty} \neq (a_j)_{j=1}^{\infty} \text{ and } \Psi_{\mu}( (b_j)_{j=1}^{\infty}) = \Psi_{\mu}((a_j)_{j=1}^{\infty}) \} \right) = 0 \text{ or } 1.
\]
Besides,
\[
\# \left( \{ (a_j)_{j=1}^{\infty} \in \Omega^\mathbb{N}; \exists (b_j)_{j=1}^{\infty} \neq (a_j)_{j=1}^{\infty} \text{ with } \Psi_{\mu}( (b_j)_{j=1}^{\infty}) = \Psi_{\mu}((a_j)_{j=1}^{\infty}) \} \right) = \#(\mathbb{N}).
\]
From now on, sometimes we will write $\mu$ in the place of $\mu_{\rho}$ and $\Psi_{\mu_{\rho}}$ will be simply denoted by $\Psi_{\mu}$.

Note that $\Psi_{\mu}$ is a random variable. In fact, if $A \in \Psi_{\mu}(C)$, then
\[
A = \Psi_{\mu}(\Omega \times \cdots \times \Omega \times \{b\} \times \Omega \times \cdots)
\]
for some $b \in \Omega$ and thus
\[
(\Psi_{\mu})^{-1}(A) = (\Omega \times \cdots \times \Omega \times \{b\} \times \Omega \times \cdots) \cup D,
\]
where $D \in \mathcal{D}$ is a denumerable set. So,
\[
(\Psi_{\mu})^{-1}(A) \in \mathcal{D}
\]
and we can conclude that $\Psi_{\mu}$ is a random variable.

Denote the distribution of $\Psi_{\mu}$ by $\lambda_{\mu}$. The measures $\lambda_{\mu}$ in $\sigma(\Psi_{\mu}(\mathcal{C}))$ present an interesting behavior, since, in general, these measures are “weighted”, i.e., they “protect” some digits. For example, it is not hard to see that if
\[
\rho(\{9\}) = \frac{3}{10},
\]
then
\[
\lambda_{\mu}(\frac{9}{10}, 1) = \frac{3}{10}.
\]

Next, we will prove the following results, that are probably folkloric, but, as far as we know, are (at least) very difficult to be found in the literature:

- $\Psi_{\mu}(\mathcal{D}) = \sigma(\Psi_{\mu}(\mathcal{C})) = \mathcal{B}$.
- If $\rho(\{r\}) = 1/10$ for every $r \in \Omega$, then $\lambda_{\mu}$ is precisely the Lebesgue measure, i.e., when the measure $\rho$ is “non-weighted”, $\lambda_{\mu}$ coincides with the Lebesgue measure.

**Theorem 1.** $\sigma(\Psi_{\mu}(\mathcal{C})) = \mathcal{B}$

Proof. Recall that
\[
\mathcal{B} = \sigma (\{[a, b]; a < b \text{ and } a, b \in [0, 1]\}).
\]
The following notation will be convenient:

{ > a } = \{ x \in \Omega; x > a \}
{ < a } = \{ x \in \Omega; x < a \}
{ \geq a } = \{ x \in \Omega; x \geq a \}
{ \leq a } = \{ x \in \Omega; x \leq a \}
{ > a, < b } = \{ x \in \Omega; x > a \text{ and } x < b \}.

We also define \{ > a, \leq b \}, \{ \geq a, < b \} and \{ \geq a, \leq b \} in a similar way.

If \( a, b \in [0, 1[ \), we can always write (uniquely)
\[
a = \sum_{j=1}^{\infty} a_j 10^{-j} \quad \text{and} \quad b = \sum_{j=1}^{\infty} b_j 10^{-j}
\]
with \( \#\{ j; a_j < 9 \} = \#\{ j; b_j < 9 \} = \#(\mathbb{N}) \). Note that
\[
[a, b] = \bigcap_{n=n_0+1}^{\infty} \left[ \sum_{j=1}^{n} a_j 10^{-j}, \sum_{j=1}^{n} b_j 10^{-j} + \sum_{j=n+1}^{\infty} 9.10^{-j} \right],
\]
where \( n_0 \) is the smallest index for which \( a_{n_0} \neq b_{n_0} \).

Since, for each \( n \geq n_0 \),
\[
\left[ \sum_{j=1}^{n} a_j 10^{-j}, \sum_{j=1}^{n} b_j 10^{-j} + \sum_{j=n+1}^{\infty} 9.10^{-j} \right] = \Psi \left( \{ a_1 \} \times \{ a_2 \} \times \cdots \times \{ a_{n-1} \} \times \{ \geq a_n \}, \leq b_{n_0} \} \times \Omega \times \Omega \times \cdots \right)
\]
\[
\bigcup_{k=n_0+1}^{n} \Psi \left( \bigcup_{k=n_0+1}^{n} \{ a_1 \} \times \{ a_2 \} \times \cdots \times \{ a_{k-1} \} \times \{ > a_k \} \times \Omega \times \cdots \right)
\]
\[
\bigcup_{k=n_0+1}^{n} \Psi \left( \bigcup_{k=n_0+1}^{n} \{ b_1 \} \times \{ b_2 \} \times \cdots \times \{ b_{k-1} \} \times \{ < b_k \} \times \Omega \times \cdots \right)
\]
we can easily conclude that
\[
\left[ \sum_{j=1}^{n} a_j 10^{-j}, \sum_{j=1}^{n} b_j 10^{-j} + \sum_{j=n+1}^{\infty} 9.10^{-j} \right] \in \sigma(\Psi_{\mu}(\mathcal{C}))
\]
and hence
\[
\mathcal{B} \subset \sigma(\Psi_{\mu}(\mathcal{C})).
\]

Now, we must show that \( \sigma(\Psi_{\mu}(\mathcal{C})) \subset \mathcal{B} \).
It suffices to show that $\Psi(\Omega \times \cdots \Omega \times \{b\} \times \Omega \times \cdots)$

$$A = \bigcup_{\substack{a_j \in \{0,\ldots,9\} \quad j=1,\ldots,n-1 \quad \text{position } n}} \left[ \sum_{j=1}^{n-1} a_j 10^{-j} + \frac{b}{10^n}, \sum_{j=1}^{n-1} a_j 10^{-j} + \frac{b}{10^n} + \sum_{j=n+1}^{\infty} 9 10^{-j} \right] \in \mathcal{B}.$$ 

$\Box$

**Theorem 2.** $\Psi(\mathcal{D}) = \mathcal{B}$.

Proof. Using the previous result, all we need to show is that $\sigma(\Psi(\mathcal{C})) = \Psi(\mathcal{D})$.

The proof that $\Psi(\mathcal{D})$ is a sigma-algebra needs a little bit of hardwork, but is standard and we omit.

It is plain that $\sigma(\Psi(\mathcal{C})) \subset \Psi(\mathcal{D})$.

We will prove the converse inclusion.

It is not difficult to show that

$$\mathcal{R} = \{ A \in \mathcal{D}; \Psi(\mathcal{A}) \in \sigma(\Psi(\mathcal{C})) \}$$

is a sigma-algebra and, since $\mathcal{C} \subset \mathcal{R}$, we have

$$\mathcal{D} = \sigma(\mathcal{C}) \subset \mathcal{R}.$$ 

From the definition of $\mathcal{R}$ we conclude that

$$\Psi(\mathcal{D}) \subset \sigma(\Psi(\mathcal{C}))$$

and the proof is done. $\Box$

Finally we have:

**Theorem 3.** If $\rho(\{a\}) = 1/10$ for every $a \in \Omega$, the distribution of the random variable $\Psi_{\mu}$ is the Lebesgue measure.

Proof. Let $\pi$ be the distribution of $\Psi_{\mu}$ and $\lambda$ be the Lebesgue measure on the Borel sigma-algebra on $[0,1]$. In order to show that $\pi$ and $\lambda$ coincide, it suffices to show that they coincide over the intervals of $[0,1]$. In fact, in this case they will coincide in the algebra $\mathcal{U}$ composed by the finite union of disjoint intervals and, by invoking Carathéodory Extension Theorem, the measures $\pi$ and $\lambda$ will coincide in $\mathcal{B} = \sigma(\mathcal{U})$.

Note that the set

$$J = \left\{ x \in [0,1]; \exists m \in \mathbb{N} \text{ so that } x = \sum_{j=1}^{m} x_j 10^{-j}, \ 0 \leq x_j \leq 9 \ (\forall j = 1,\ldots,m) \right\}$$
is dense in \([0, 1]\). So, we just need to show that \(\mu\) and \(\lambda\) coincide over the intervals \([a, b] \subset [0, 1]\), with \(a, b \in J\).

Moreover, there is no loss of generality in dealing with intervals of the form

\[ I = \left[ \sum_{j=1}^{n} a_j 10^{-j}, \sum_{j=1}^{n} b_j 10^{-j} \right]. \]

Let \(n_0\) be the smallest index such that \(a_{n_0} \neq b_{n_0}\). Hence

\[ \lambda(I) = \sum_{j=n_0}^{n} (b_j - a_j) 10^{-j}. \]

Consider \(A \in \mathcal{D}\) given by

\[ A = \left( \{a_1\} \times \{a_2\} \times \cdots \times \{a_{n_0-1}\} \times \{a_{n_0}, < b_{n_0}\} \times \Omega \times \cdots \right) \]
\[ \bigcup \left( \bigcup_{k=n_0+1}^{n} \{a_1\} \times \{a_2\} \times \cdots \times \{a_k-1\} \times \{a_k\} \times \Omega \times \cdots \right) \]
\[ \bigcup (\{a_1\} \times \{a_2\} \times \cdots \times \{a_{n}\} \times \Omega \times \Omega \times \cdots) \]
\[ \bigcup \left( \bigcup_{k=n_0+1}^{n} \{b_1\} \times \{b_2\} \times \cdots \times \{b_{k-1}\} \times \{b_k\} \times \Omega \times \cdots \right) \]
\[ \bigcup (\{b_1\} \times \{b_2\} \times \cdots \times \{b_{n}\} \times \{0\} \times \{0\} \times \cdots) . \]

Hence \((\Psi_{\mu_{\rho}})^{-1}(I) = A \cup D\) with \(\mu_{\rho}(D) = 0\) and

\[ \mu_{\rho}(A) = \frac{1}{10^{n_0}}(b_{n_0} - a_{n_0} - 1) + \sum_{k=n_0+1}^{n} \frac{1}{10^k}(9 - a_k) + \frac{1}{10^n} + \sum_{k=n_0+1}^{n} \frac{b_k}{10^k} + 0, \]

and straightforward calculations show that

\[ \mu_{\rho}(A) = \lambda(I). \]

We thus have

\[ \overline{\mu}(I) = \mu_{\rho}\left((\Psi_{\mu_{\rho}})^{-1}(I)\right) = \mu_{\rho}(A \cup D) = \mu_{\rho}(A) = \lambda(I). \]

\(\square\)

4. A MORE GENERAL APPROACH TO NORMAL NUMBERS

The notion of normal numbers (with respect to Lebesgue measure) was introduced by E. Borel [11], in 1909, and, since then, several interesting questions on normal numbers have been investigated and various intriguing problems remain open (for example, the normality of \(\sqrt{2}\)). The results of the previous sections turns natural to consider the concept of normal numbers
with respect to other measures than the Lebesgue measure on $[0,1]$. In this section, as an application of the previous results, we generalize the concept of normal numbers and obtain the measure of the sets of normal numbers (with this generalized concept). In particular, we give an alternative simple proof (essentially due to Steinhaus [5]) for the fact that almost all real numbers in $[0,1]$ are normal, with respect to Lebesgue measure (different proofs of this result can be found, for example, in [1], [2], [3], [4]).

If $\Omega = \{0,\ldots,9\}$, $\rho$ is a probability measure on $A = 2^{\Omega}$ and $\lambda_{\mu_\rho}$ is the distribution of the random variable $\Psi_{\mu_\rho}$ defined in (3.1), a number $\eta \in [0,\infty[$, represented in the decimal scale by

\begin{equation}
\eta = [\eta] + \sum_{j=1}^{\infty} a_j 10^{-j}, \ a_j \in \{0,\ldots,9\}, \ \forall j \in \mathbb{N},
\end{equation}

with $[\eta] = \sup\{r \in \mathbb{N}; r \leq \eta\}$ and $\#\{n; a_n < 9\} = \#(\mathbb{N})$, is said to be simply normal (with respect to $\lambda_{\mu_\rho}$) when

\begin{equation}
\lim_{n \to \infty} \frac{S(\eta, r, n)}{n} = \rho(\{r\}) \ \forall r \in \{0,\ldots,9\},
\end{equation}

where $S(\eta, r, n)$ denotes the total of indexes $i, 1 \leq i \leq n$ such that $a_i = r$. A number $\eta$, as in (4.1), is said to be normal (with respect to $\lambda_{\mu_\rho}$) if

\begin{equation}
\lim_{n \to \infty} \frac{S(\eta, B_k, n)}{n} = \prod_{j=1}^{k} \rho(\{b_k\}) \ \forall k \in \mathbb{N}.
\end{equation}

for each word $B_k = b_1\ldots b_k$ of $k$ digits, where

$S(\eta, B_k, n) = \# \{i \in \{1,\ldots,n\}; a_{i+j-1} = b_j \text{ for every } j = 1,\ldots,k\}.$

In particular, if $\rho(\{a\}) = \frac{1}{10}$ for every $a \in \Omega$, this concept is precisely Borel’s original concept of normal numbers, with respect to Lebesgue measure.

**Remark 1.** This generalized concept arises some interesting situations. For example, for “degenerate” cases, in which $\rho(\{a\}) = 1$ for some $a \in \{0,1,\ldots,9\}$, normal numbers are very special numbers, with a strong preference to the digit $\{a\}$. For example, if $\rho(\{a\}) = 1$ for some $a \in \{0,1,\ldots,9\}$, then

$0, a0aa0aaa0aaaa0aaaaa0…$

is normal with respect to $\lambda_{\mu_\rho}$.

The next result shows that, in virtually all cases, the sets $N_{\mu_\rho}$, of normal numbers in $[0,1]$, with respect to $\lambda_{\mu_\rho}$, are so that $\lambda_{\mu_\rho}(N_{\mu_\rho}) = 1$, but there is one situation in which $\lambda_{\mu_\rho}(N_{\mu_\rho}) = 0$. 


Theorem 4. The measure of the set of normal numbers in \([0, 1]\), with respect to \(\lambda_{\mu_{\rho}}\) is:

(a) 0, if \(\rho(\{9\}) = 1\).
(b) 1, if \(\rho(\{9\}) < 1\).

Proof. In the following, \(N_{\mu_{\rho}}\) denotes the set of all normal numbers in \([0, 1]\), with respect to \(\lambda_{\mu_{\rho}}\) and \(M_{\mu_{\rho}}\) represents the set of all normal sequences in \(\Omega^N\), with respect to \(\mu_{\rho}\).

(a) If \(\rho(\{9\}) = 1\), then \(\mu_{\rho}(\{9\}^N) = 1\). We have

\[
(\Psi_{\mu_{\rho}})^{-1}(N_{\mu_{\rho}}) = D_1,
\]

with

\[
D_1 = M_{\mu_{\rho}} \setminus \{(a_j)_{j=1}^\infty; \exists N \in \mathbb{N} \text{ such that } a_n = 9 \text{ for every } n \geq N\}.
\]

Hence \(N_{\mu_{\rho}} \in \mathcal{B}\) and, since

\[
D_1 \cap \{9\}^N = \emptyset,
\]

we have

\(\lambda_{\mu_{\rho}}(N_{\mu_{\rho}}) = \mu_{\rho}(D_1) = 0\).

(b) If \(\rho(\{9\}) < 1\), note that

\[
(\Psi_{\mu_{\rho}})^{-1}(N_{\mu_{\rho}}) = M_{\mu_{\rho}} \cup D_2,
\]

with \(\mu_{\rho}(D_2) = 0\). Hence

\[
N_{\mu_{\rho}} \in (\Psi_{\mu_{\rho}}(D) = \mathcal{B}
\]

and by invoking Proposition 2 we conclude that

\(\lambda_{\mu_{\rho}}(N_{\mu_{\rho}}) = \mu_{\rho}(M_{\mu_{\rho}}) = 1\).

\qed

Corollary 1. (Borel [1]) The set of normal numbers in \([0, 1]\), with respect to Lebesgue measure, has measure 1.

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References

[1] M. E. Borel, Les probabilités denombreables et leurs applications arithmetiques, Suppl. Rend. Circ. Mat. Palermo 27 (1909), 247-271.
[2] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Fourth Edition, Oxford University Press, London (1975).
[3] I. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, John Wiley & Sons, (1974).
[4] R. Nielsen, Normal numbers without measure theory, Amer. Math. Monthly 107 (2000), no. 7, 639-644.

[5] H. Steinhaus, Les probabilités denomérables et leur rapport à la théorie de la mesure, Fundamenta Math. 4 (1923), 286-310.

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