Non-perturbative renormalization group approach for a scalar theory in higher-derivative gravity

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Abstract

A renormalization group study of a scalar theory coupled to gravity through a general functional dependence on the Ricci scalar in the action is discussed. A set of non-perturbative flow equations governing the evolution of the new interaction terms generated in both local potential and wavefunction renormalization is derived. It is shown for a specific model that these new terms play an important role in determining the scaling behavior of the system above the mass of the inflaton field.

04.62+v, 11.10.Hi, 05.40+j
I. INTRODUCTION

It is well known that modifications of Einstein’s theory of gravity are required in order to include quantum effects. Although a consistent fundamental theory of the spacetime near the Planck scale is still not available, one can consider General Relativity as a quantum effective field theory \[1,2\]. From this point of view one gives up the requirement of (perturbative) renormalizability to consider a more general action that is a functional of any geometrical invariant \(\mathcal{R}\) which can be constructed from the principle of general covariance, 

\[
\mathcal{R} = \{ R, R_{\alpha\beta} R^{\alpha\beta}, C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}, \ldots \},
\]

and of the matter field \(\phi\) and their derivatives

\[
S[\phi, g] = \int d^4x \sqrt{-g} \mathcal{L}(\mathcal{R}, \phi, \nabla^2 \mathcal{R}, \nabla^2 \phi, \nabla^2 \mathcal{R}, \nabla^2 \phi, \ldots)
\]

(1)

This new theory is analogous to the Euler-Heisenberg lagrangian, low-energy effective theory of the more fundamental microscopic QED. Due to the smallness of the ratio between the Newton constant and the Fermi constant, this modification of Einstein’s theory is experimentally indistinguishable from the standard theory at ordinary energy scales \[3\]. However at higher energy scales \[4\] it has significant physical implications, mainly for the inflationary scenarios \[5\], nowadays tested by measuring the temperature fluctuations of the cosmic microwave background radiation (CMBR) over large angular scales.

Higher-derivative gravity theories also arise through the coupling between a quantized field and the classical background geometry \[6\]. In this case one adds new terms to the standard Einstein-Hilbert lagrangian. Those terms are general functions of \(\mathcal{R}\) and are needed to cancel the ultraviolet (UV) divergences at any given order in perturbation theory. This approach is useful to study the short-distance behavior of the model, but it loses its validity if we are interested in the low energy behavior.

In fact, the standard perturbative approach based on the scaling property of the Green’s functions under a rescaling of the metric \[7\] handles only a finite number of operators, those which are important for the UV fixed point. In this way one cannot follow the evolution of the irrelevant, i.e. non-renormalizable operators. Although the standard classification
of the interactions is given accordingly to the behavior of the renormalization group (RG) transformation near a fixed point, we use a more general definition, calling a coupling irrelevant, marginal or relevant if, in cut-off units, either it gets smaller, or it does not run, or it grows under a RG transformation. The irrelevant operators, even if they are not present in the bare lagrangian, mix their evolution with the renormalized trajectory of the relevant couplings. Although near the gaussian UV fixed point their running is suppressed, they can behave in a quite different manner in other scaling regions thus deviating the renormalized flow.

It is therefore important to trace the evolution of all the couplings constants generated by the renormalization procedure in order to decide what can be neglected and what cannot. If for example a new scale other than the cut-off is present in the problem, the scaling laws change near that scale and the standard UV relevant interactions might not be sufficient to describe the physics at the crossover. This happens in the $O(N)$ model where, in the non-symmetric phase, the presence of singularity in the beta functions for the longitudinal components indicates that new infrared (IR) relevant interactions, due to the Goldstone modes, appear after the condensate has formed [8].

Actually we deal with a similar situation when we move up in energy, going from an effective theory defined at some low energy scale, towards the high energy region. In fact integrating out the heavy quarks contribution in the Standard Model generates non-renormalizable vertices that are suppressed at the weak scale [9] but that can show up when the heavy mass threshold is reached [10].

Although there is no way to localize the energy stored in the gravitational field, a local definition of a “mass” scale can always be given, even if no matter field acts as a source in Einstein’s equations, i.e. $T_{\mu\nu} = 0$. A possibility is to consider $m_G = \sqrt{|\Psi^2|}$ where $\Psi^2$ is the coulombian component of the Weyl tensor. For instance, in spherical symmetry it coincides with the square of the Weyl tensor, and for a Schwarzschild spacetime $\Psi^2 = m/r^3$. Since $m_G^2 \sim |R_{\alpha\beta\gamma\delta}|$ one can think of $m_G$ as the local effective mass producing the curvature in that point. In this respect the gravitational field acts as an external field that influences the
quantum dynamics within scales $l_G \sim m_G^{-1}$.

On the other hand, during the slowroll phase of the inflationary era, the only relevant “mass” scale is provided by $H = \dot{a}/a$, $a(t) \sim a_0 e^{Ht}$ is the scale factor, and the quantum evolution of the fluctuations is stopped after crossing the horizon. The subsequent classical behavior is therefore dependent on the hierarchy between these two scales, the mass of the inflaton field, $m_\phi^2$, and the scalar curvature $R$.

An important issue is related to the running of the conformal coupling constant $\xi$ which couples directly the inflaton with gravity through the $\xi R \phi^2$ term in the action. The value of this coupling is determined in principle by the spectral index of the CMBR temperature fluctuations $n_s = 1.17 \pm 0.31$, depending on the model of inflation considered. Although for some models negative values of $\xi$ are preferred, in perturbation theory with the minimal subtraction scheme one finds $\xi = 1/6$ to be an infrared fixed point for the renormalized coupling.

The renormalization group approach used in Statistical Mechanics is the best tool to study problems where many scales are coupled together. In this paper we shall use the differential form of the RG transformation formulated by Wegner and Houghton. Starting from a bare action $S_k$ at the cut-off $k$ one first calculates $S_{k-\Delta k}$ in

$$e^{-S_{k-\Delta k} [\phi]} = \int D[\psi] e^{-S_k [\phi + \psi]}$$

by using the loop expansion, where $\psi$ and $\phi$ respectively have non-zero Fourier components only in the momentum shells $k - \Delta k < p \leq k$ and $p \leq k - \Delta k$. The differential RG transformation is obtained by taking the limit of an infinitesimal shell $\Delta k/k \to \delta k/k$. The higher loop contributions in Eq. (2) are suppressed as powers of $\Delta k/k$ in the limit $\Delta k \to 0$ for finite $k$ and an exact, non-perturbative, one-loop RG equation is obtained

$$k \frac{dS_k[\phi]}{dk} = -\frac{1}{2} \langle \ln \delta^2 S[\phi] \rangle + \langle \frac{\delta S[\phi]}{\delta \phi} \left( \frac{\delta^2 S[\phi]}{\delta \phi^2} \right)^{-1} \frac{\delta S[\phi]}{\delta \phi} \rangle$$

where the brackets indicates sum over the Fourier components within the shell. This functional equation rules the evolution of all the interaction terms that are generated in the
renormalization procedure. Despite the solution is not know in the general case one can derive approximate, non-perturbative evolution equations in terms of the gradient expansion \[16\], by writing

\[ S[\phi] = \int d^D x \sum_n U_n(\phi, \partial^{2n} \phi) \]  

where \( U_n \) is an homogeneous function of the field and of its derivatives. In this way Eq. (3) decouples in a set of infinite non-linear partial differential equation for the \( U_n \). The explicit construction of the RG transformation can be achieved by the blocking transformation or “coarse-graining” of the fields by means of a smearing function that introduces a sharp cut-off in the high momentum modes \[17\]. A perturbative construction of the coarse-graining approach has been firstly discussed in a gravitational context in \[18\] and proposed in the formalism of the “average action” in \[19,20\]. In particular in \[19\], by considering the contribution of the blocked potential \( U_0(\phi) \) in Eq. (4), it has been shown that \( \xi = 1/6 \) is not necessarily an infrared fixed point for the massive theory coupled with gravity.

An equivalent construction of the exact RG equations in the Wilsonian approach is presented in \[21\]. In particular in \[22\] the infrared behavior of the Einstein-Hilbert action is studied in the case of pure gravity.

In this paper we explicitly construct the RG transformation for the blocked potential \( U_k \) and the wavefunction renormalization \( Z_k \) which are general functions of both \( \phi \) and the curvature scalar \( R \). In particular we construct the RG transformation by means of an \( O(D) \) symmetric smearing function (\( D \) is the dimension of the spacetime). This will be achieved by introducing locally a momentum space \[23\] and by working up to first non-trivial order in the expansion of the metric. The set of equations we derive generates the running of any interaction term of the form \( g_{ij} R^i \phi^j \) in both \( U_k \) and \( Z_k \).

We study the RG flow in the two different scaling regions of the ultraviolet and infrared domain. In particular the crossover in correspondence of the mass gap is analyzed in a specific model by means of a numerical investigation of the equations.

The organization of the paper is the following. In Sec.II the flow equations for \( Z_k \) and \( U_k \)
are derived. In Sec. III we analitically study the behavior in the UV and in the IR region. In sec. IV we apply the method to a model by numerically integrating the flow equations. The results of the numerical analysis are then displayed. Sec. V is devoted to the conclusions.

II. RENORMALIZATION GROUP TRANSFORMATION

In the following we suppose that the spacetime has a well defined Euclidean section $\Omega$. The general form of the Euclidean bare action, quadratic in the derivatives of the scalar field $\phi$ and of the Ricci scalar $R$ is

$$S_\Lambda[\phi, g] = \int d^Dx \sqrt{g} \left\{ \frac{1}{2} Z(\phi, R) \partial^\mu \phi \partial_\mu \phi + V(\phi, R) + \frac{1}{2} W(\phi, R) \partial_\mu R \partial^\mu R + Y(\phi, R) \partial_\mu \phi \partial^\mu R \right\}$$

where $\Lambda$ is the ultraviolet cut-off of the theory, $V$ is the local potential, while $Z, Y$ are general functions of $\phi$ and $R$. The action (5) is equivalent to the following one

$$S_\Lambda[\phi, g] = \int d^Dx \sqrt{g} \left\{ -\frac{1}{2} Z(\phi, R) \phi \nabla^2 \phi + V(\phi, R) - \frac{1}{2} W(\phi, R) R \nabla^2 R + Y(\phi, R) \partial_\mu \phi \partial^\mu R \right\}$$

provided

$$Z = (Z\phi)', \quad W = \frac{\partial(WR)}{\partial R}, \quad Y = \mathcal{Y} + \frac{1}{2} RW' + \frac{1}{2} \phi \frac{\partial Z}{\partial R}$$

where $\nabla^2 = \nabla^\mu \nabla_\mu$ is the Laplace-Beltrami operator, and the prime indicates derivation with respects to $\phi$.

In order to construct the renormalization group transformation in the Wilson-Kadanoff approach one first defines the average blocked field by coarse-graining the original field

$$\phi_k(x) = \int d^Dx' \sqrt{g} \rho_k(x, x') \phi(x')$$

where $\rho_k(x, x')$ is a smearing function which is constant in a given volume where the field is averaged, and it is zero outside. The blocked action is by definition
\[ e^{-\tilde{S}_k[\Phi,g]} = \int D[\phi] \prod_x \delta(\phi_k(x) - \Phi(x)) e^{-S_\Lambda[\phi,g]}. \]  

(9)

In this way one “averages out” the high energy degrees of freedom that are relevant for the short-distance behavior of the theory.

Let us consider the decomposition of a generic function \( f(x) \) of \( L^2(\Omega) \), in eigenfunctions of the Laplace-Beltrami operator \( \psi_i(x) \), \( f(x) = \sum_i f_i \psi_i(x) \), where the \( \psi_i(x) \) represent an orthonormal base in this space (\( i \) is an integer if the manifold is compact). Once the spectrum of the operator \( \nabla^2 \) is known, the sharp momentum smearing function defined in [19] can then be used to explicitly construct the projector \( \rho_k(x,x') \) in Eq. (8). In particular, for a general spacetime one can introduce locally a momentum space and use the \( O(D) \) sharp-cutoff smearing function as in [17]. This is a good approximation as long as the cutoff of the theory is much above the scale of energy where the curvature becomes dynamically relevant and, at the same time, it gives a precise meaning to the UV cutoff \( \Lambda \) introduced above. The field is then split

\[ \phi(x) = \vartheta(x) + \xi(x) = \sum_M \vartheta_m \psi_m(x) + \sum_N \xi_n \psi_n(x) \]  

(10)

where \( M \) and \( N \) respectively indicate the slow and the fast variables; more precisely \( M \) is the set of index values \( m \) labelling components with low momentum: \( p < k \) and \( N \) is the set of index values \( n \) and \( n' \) corresponding to high momentum \( k < p < \Lambda \), \( k \) being a fixed momentum scale below the cutoff \( \Lambda \). The functional integration in Eq. (9) yields

\[ \tilde{S}_k[\Phi] = -\ln \int \mathcal{D}[\vartheta] \mathcal{D}[\xi] \prod_x \delta(\vartheta_k(x) - \Phi(x)) e^{-S[\vartheta+\xi]} = \]

\[ -\ln \int \mathcal{D}[\vartheta] \prod_M \delta(\vartheta_m - \Phi_m) \int \mathcal{D}[\xi] \exp\left\{ -S[\vartheta] - \frac{1}{2} \sum_n F_n \xi_n - \frac{1}{2} \sum_{n,n'} \xi_n K_{n,n'} \xi_{n'} \right\} = \]

\[ -\ln \int \mathcal{D}[\vartheta] \prod_M \delta(\vartheta_m - \Phi_m) \exp\left\{ -S[\vartheta] + \frac{1}{2} \sum_n F_n K_{n,n}^{-1} F_{n'} - \frac{1}{2} \sum_n (\ln K)_{n,n} \right\} = \]

\[ S[\Phi] - \sum_n F_n K_{n,n}^{-1} F_{n'} \big|_{\Phi} + \frac{1}{2} \text{Tr} \ln K[\Phi] \]  

(11)

where

\[ F = \left. \frac{\delta S}{\delta \phi} \right|_{\phi=\Phi}, \quad K = \left. \frac{\delta^2 S}{\delta \phi \delta \phi} \right|_{\phi=\Phi} \]  

(12)
In coordinate representation we get

\[ F(x) = \frac{\delta S}{\delta \phi(x)} \bigg|_\Phi = -\frac{1}{2} Z' \partial_\mu \Phi \partial^\mu \Phi - Z \nabla^2 \Phi + V' \]  

(13)

and

\[ K(x, x') = \frac{\delta^2 S}{\delta \phi(x) \delta \phi(x')} \bigg|_\Phi = \{ -\frac{1}{2} Z'' \partial_\mu \Phi \partial^\mu \Phi - Z' (\partial_\mu \Phi \partial^\mu + \nabla^2 \Phi) - Z \nabla^2 + V'' \} \delta(x, x') = \{ -\frac{1}{2} Z' \nabla^2 \Phi - Z \nabla^2 + V'' \} \delta(x, x') \]  

(14)

where

\[ \delta(x, x') = g^{-1/4}(x) \delta(x - x') g^{-1/4}(x') = \delta(x - x') / \sqrt{g(x)} \]  

(15)

In deriving Eqs. (13,14), we have supposed that, for our purposes the fluctuations of \( R \) are negligible or, in other words, we have suppressed in the action the terms containing \( \partial_\mu R \); as a consequence \( W \) and \( Y \), although in principle could be not vanishing, do not appear in Eqs. (13,14) and in the following calculations.

The last equality in Eq. (14) follows after some manipulations, but the easiest way to verify it, is to compute the second functional derivative from the action (6), and use the relation for \( Z \) in Eq. (7).

In order to explicitly carry out the blocking procedure, the functional form of the blocked action in Eq. (11) is needed, and the usual assumption is to maintain the same form of the starting action; thus, according to the restriction to constant \( R \), the blocked action reads

\[ \tilde{S}_k[\Phi] = \int d^D x \sqrt{g} \left( \frac{1}{2} \tilde{Z}_k(\Phi, R) \partial_\mu \Phi \partial^\mu \Phi + U_k(\Phi, R) + O(\nabla^4) \right) \]  

(16)

The function \( \tilde{Z} \) is generated by performing the path integral in Eq. (11) for non constant field configurations

\[ \Phi(x) = \Phi_0 + \tilde{\phi}(x) \]  

(17)
By expanding the action in Eq. (16) up to second order in $\tilde{\phi}$, one can single out the contribution to the blocked potential $U$ and to the wavefunction renormalization $\tilde{Z}$, as it has been shown in [13] and in [8]. In a general spacetime this procedure might be problematic because the effective potential is in general ill defined when the spacetime dynamics becomes relevant. For this purpose it is convenient to consider energy scales much higher than the characteristic curvature scale $k^2 \ll R$, and as anticipated above, to introduce locally a momentum space [23] and to use an $O(D)$ symmetric smearing function in Eq. (8) to evaluate Eq. (11). In fact, although several techniques are available in the literature [24] to compute the functional determinant in Eq. (11), they are often combined with dimensional regularization and minimal subtraction prescription. Instead, the explicit use of a sharp cut-off regulator allows a clear separation between the UV and the IR domain of the theory.

Let us briefly outline the computation of the blocked action. It is convenient at this point to introduce a covariant notation: for any operator $A$ the Lorentz invariant operator $\overline{A}$ is defined by $\overline{A}(x, x') \equiv g^{1/4}(x)A(x, x')g^{1/4}(x')$. In particular for the trace in Eq. (11), properly written in coordinate representation, we get

$$\text{Tr} \ln K = \int d^Dx \sqrt{g(x)} \ln K(x, x) = \int d^Dx \ln \overline{K}(x, x) \equiv \text{Tr} \overline{K}$$

where the definition of the logarithm as power series of operators has been used.

Let us examine the various pieces entering $\overline{K}$. According to the above definition $g^{1/4}(x)\nabla_x^2 \delta(x, x')g^{1/4}(x') = \nabla_x^2 \delta(x - x')$ where Eq. (15) has been used and

$$\nabla^2 = g^{\mu\nu}(x)\partial_{\mu\nu}^2 + b^\mu(x)\partial_\mu + a(x)$$

$$b^\mu(x) = g^{-1/2}(\partial^\mu \sqrt{g}) + 2g^{1/4}(\partial^\mu g^{-1/4}) + (\partial_\nu g^\nu\mu)$$

$$a(x) = g^{-1/4}(\partial_\mu \sqrt{g})(\partial^\mu g^{-1/4}) + g^{1/4} \partial_\mu \partial^\mu g^{-1/4}.$$  

(19)

Analogously

$$g^{1/4}(x)Z'(\nabla_x^2 \Phi)\delta(x, x')g^{1/4}(x') = g^{-1/4}(x)Z' \nabla_x^2 (g^{1/4} \Phi)\delta(x - x')$$

(20)

Thus, making the replacements
\[ \Phi(x)g^{1/4}(x) \to \Phi(x), \quad V(x)g^{1/2}(x) \to V(x) \quad (21) \]

we finally get
\[ K(x, x') = \{-\frac{1}{2}Z' \nabla^2(\Phi) - Z\nabla^2 + V''\} \delta(x - x') \quad (22) \]

where the prime now indicates the derivative with respect to the new scalar field of Eq. (21).

Note that we could have obtained Eq. (22), making the replacements of Eq. (21) in the original action and taking second functional derivatives with respect to the new scalar field. Since the following developments are naturally expressed in terms of the replaced variables in Eq. (21), we shall keep on using the new field and potential, but for simplicity we shall not change the notation.

We can now introduce normal coordinates \( \{ y^\alpha \} \) around the point \( x' \) corresponding to \( y^\alpha = 0 \), expand the expressions in Eqs. (19) and (22) up to some order and invert the propagator as shown in [23] and [24] where they fixed \( Z = 1 \) at the bare level. However for our purposes it is more convenient to work out separately the expansions for the metric and the field \( \Phi \) since in principle they involve different scales. We expand the metric tensor up to second order in its derivatives therefore writing
\[
g^{\mu\nu}(y) = \delta^{\mu\nu} + \frac{1}{3} R^\mu \alpha \beta y^\alpha y^\beta + O(\partial^3 g) 
\]
\[
g = 1 - \frac{1}{3} R_{\alpha\beta} y^\alpha y^\beta + O(\partial^3 g) \quad (23)
\]

This truncation is consistent with the choice of neglecting \( \partial_\mu R \), that would appear starting from the third order in the metric expansion.

In order to compute \( \tilde{Z} \) we first expand Eq. (22) around \( \Phi(y) = \Phi_0 + \tilde{\phi}(y) \) up to second order in \( \tilde{\phi} \) obtaining \( K = K_0 + K_{\tilde{\phi}} + K_{\tilde{\phi}^2} \) where \( (Z_0 \text{ and } V_0 \text{ are used for constant scalar field } \Phi_0 \text{ and } \Box = \delta^{\mu\nu} \partial_\mu \partial_\nu ) \)
\[
K_0(y, 0) = \{-Z_0 \Box + V_0'' - \frac{R}{12}(2Z_0 + \Phi_0 Z_0')\} \delta(y) 
\]
\[
K_{\tilde{\phi}}(y, 0) = \{-\frac{1}{2} Z_0' (\Box \tilde{\phi} + 2 \tilde{\phi} \Box) + V_0''' \tilde{\phi} - \frac{R}{12} (3Z_0' + \Phi_0 Z_0'') \tilde{\phi} \} \delta(y) 
\]
\[
K_{\tilde{\phi}^2}(y, 0) = \{-\frac{1}{2} Z_0'' (\tilde{\phi} \Box \tilde{\phi} + \tilde{\phi}^2 \Box) + \frac{1}{2} V_0''' \tilde{\phi}^2 - \frac{R}{24} (4Z_0'' + \Phi_0 Z_0''') \tilde{\phi}^2 \} \delta(y) \quad (24)
\]
In deriving these expressions we have taken into account that $\nabla^2$, when applied to a scalar quantity, becomes $\nabla^2 = (\Box + (1/3)R^\alpha_\alpha^\mu \nu y^\alpha y^\beta \partial^2_{\mu\nu} - (1/3)R^\alpha_\alpha^\mu y^\beta \partial_\mu) = \Box + R/6$. We rewrite Eq. (18) using

$$\text{Tr} \ln K = \text{Tr} \ln K_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{Tr}\{K_0^{-n}(K_0^{-1}K_0^{-1}K_0^{-1})^n\}$$

(25)

One therefore also expands the expression (16) up to second order in $\tilde{\phi}$ and compare it with Eq. (11), obtaining

$$\tilde{S}_k[\Phi] = \tilde{S}_k^0 + \tilde{S}_k^1 + \tilde{S}_k^2 = S_\Lambda^0 + S_\Lambda^1 + S_\Lambda^2 +$$

$$\frac{1}{2} \text{Tr} \ln K_0 + \frac{1}{2} \text{Tr} K_0^{-1}K_0^{-1}K_0^{-1}K_0^{-1}K_0^{-1}K_0^{-1}$$

(26)

where $S^0, S^1, S^2$ are, respectively, the zeroth, the first and the second order terms in the functional expansion of the action in $\tilde{\phi}$. The zeroth order gives the blocked potential

$$\tilde{S}_k^0 = S_\Lambda^0[\Phi_0] + \frac{1}{2} \text{Tr} \ln K_0$$

(27)

wherefrom (in the following we use the notation $\hat{f}_p = \int_{|p|<\Lambda} \frac{d^3p}{(2\pi)^3}$)

$$U_k = V_0 + \frac{1}{2} \hat{f}_p \ln (Z_0p^2 + V''_0) - \frac{R}{12} (2Z_0 + \Phi_0Z'_0).$$

(28)

In deriving the last relation we have evaluated the trace in Eq. (27) in the Riemann frame. Since the integrand is calculated in $\Phi_0$ and $K_0$ is diagonal in the local momentum representation, a volume term can be factored out, yielding a covariant equality between constant quantities. The blocked potential $U(\Phi)$ is obtained by replacing the constant field $\Phi_0$ with $\Phi(x)$. It should also be observed that Eq. (28) for $k = 0$ gives the effective potential, obtained by taking the coincidence limit in the fluctuation determinant and by retaining only the first order terms in the Schwinger-De Witt proper time expansion.

A similar procedure must be used for selecting the contribution of $\hat{Z}_k$ in the other terms in the expansion (24). In this case, for computing the traces, according to (16) we retain at the same time operators in the momentum and in the coordinates representation, as long as the operators in one representation are on the right side of the operators in the other.
representation. In the terms where the required ordering is not fulfilled, we employ commutation rules for \( p \)-dependent and \( x \)-dependent terms in order to move the \( p \)-dependence on the left of the \( x \)-dependence and disentangle the volume integration from the \( p \) integration. The main commutation rule, from which all the others can be deduced with no ambiguity, is

\[
[\Phi(x),p_\mu] = -i\partial_\mu \Phi(x) \tag{29}
\]

Thus we get

\[
\begin{align*}
\mathbf{Tr} K^{-1}_{0} \tilde{\phi} &= \int d^D y \int_p \frac{1}{K_p} \left( Z_0 p^2 + V_0'' - \frac{R}{12} (3Z_0' + \Phi_0 Z_0'') \right) \tilde{\phi}(y) \\
\mathbf{Tr} K^{-1}_{0} \tilde{\phi} &= \int d^D y \int_p \frac{1}{K_p} \left( \frac{1}{2} Z_0'' (\partial_\mu \tilde{\phi}(y) \partial^\mu \tilde{\phi}(y)) + p^2 \tilde{\phi}^2(y) \right) + \frac{1}{2} V_0''' \tilde{\phi}^2(y) - \frac{R}{24} (4Z_0'' + \Phi_0 Z_0'''') \right)
\end{align*}
\tag{30}
\]

where

\[
K_p = Z_0 p^2 + V_0'' - (R/12)(2Z_0 + \Phi_0 Z_0')
\tag{31}
\]

and, after having used the commutation relation (29) for computing \( \mathbf{Tr} \ln(K^{-1}_{0} K^{-1}_{\phi} K^{-1}_{0} K^{-1}_{\phi}) \) we eventually obtain the following renormalization group equations

\[
\begin{align*}
U_k &= V + \frac{1}{2} \int_p \ln(K_p) \\
Z_k &= Z + \frac{1}{2} \int_p \left( \frac{Z''}{K_p} - \frac{2Z' K'_p}{K_p^2} + \frac{Z K''}{K_p^3} + \frac{2 K'_p Z p^2 K'_p}{K_p^3} - \frac{Z^2 p^2 (K'_p)^2}{K_p^4} \right)
\end{align*}
\tag{32}
\]

Note that there is no contribution from the term \( FK^{-1}F \) in Eq. (11) in the gradient expansion as it can be proved by direct calculation. The reason is that in this term no volume integration occurs and one obtains the Fourier transform of \( \tilde{\phi} \) which is constrained to have \( p < k \). Since the momentum integration is performed for \( p > k \) this term identically vanishes.

These equations have been obtained in the independent mode approximation where one computes the blocked action by an independent integration of the degrees of freedom between
the UV cut-off $\Lambda$ and $k$. In the limit $k \to 0$ one recovers the effective action. In particular if we neglect the field dependence in $Z$, therefore reducing it to a running coupling $\eta$, the previous equations read

$$U_k = V + \frac{1}{2} \int p \ln \frac{\eta p^2 + V'' - (R/6)\eta}{(\eta p^2 + V'\Phi_{=R=0})}$$

$$\hat{Z}_k = \eta + \frac{1}{2} \int p \left( \frac{\eta (K_p^\prime)^2}{K_p^3} - \frac{\eta^2 p^2 (K_p^\prime)^2}{K_p^4} \right).$$

(33)

where now $K_p = \eta p^2 + V'' - (R/6)\eta$ and the cosmological constant that corresponds to the constant part of $V$ has been modified in order to include the field independent part in the logarithm in Eq. (33).

As an example let us consider the standard $\lambda \phi^4$ bare scalar self-interacting theory in curved spacetime

$$\mathcal{L} = \frac{1}{2} \eta \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} m^2 \Phi^2 + \frac{1}{2} \xi R \Phi^2 + \frac{1}{4!} \lambda \Phi^4$$

(34)

the RG equations (33) for $\hat{Z}$ can be integrated up to $k = 0$ and one obtains in $D = 4$ the finite contribution to the wavefunction renormalization function

$$\hat{Z}_{k=0} = \eta + \frac{1}{192 \pi^2} \frac{\lambda^2 \Phi^2}{m^2 + \frac{1}{3} \Phi^2 + (\xi - \frac{1}{6})R}$$

(35)

which coincides with the result in [20].

We want to remark that, due to the replacements in Eq. (21), the couplings and the field appearing in the bare lagrangian do not coincide with the ones appearing in the evolution equations, although, in case of small curvature the difference becomes practically negligible. However it is formally correct to expand in terms of running couplings the modified potential defined in Eq.(21) and not the bare lagrangian as in Eq. (34).

**III. NON-PERTURBATIVE FLOW EQUATIONS**

The previous equations have been obtained by eliminating the degrees of freedom between $\Lambda$ and $k$. The non-perturbative RG equations in Wegner-Houghton’s formulation
follow if one infinitesimally lowers the running cut-off \( k \to k - \Delta k \) thus retaining the contribution of the modes which have been previously integrated in the infinitesimal momentum shell. In practice the dependent mode improved approximation is obtained by taking the derivatives with respect to \( k \) of Eq. (32) rewritten for a general potential and wavefunction renormalization. One obtains the following set of non-linear partial differential equations for the evolution of \( \hat{Z} \) and \( U \) (for simplicity the subscript \( k \) is omitted)

\[
\begin{align*}
\frac{dU}{dk} &= -a_D k^D \ln \left( \frac{\hat{Z}k^2 + U'' - (R/12)(2\hat{Z} + \Phi \hat{Z}')}{\hat{Z}k^2 + U'' - (R/12)(2\hat{Z} + \Phi \hat{Z}')|_{\Phi=0}} \right) \\
\frac{d\hat{Z}}{dk} &= -a_D k^D \frac{\hat{Z}''}{K} \left( \hat{Z}'' - \frac{2\hat{Z}'K'}{K} + \frac{\hat{Z}(K')^2}{K^2} + \frac{\hat{Z}\hat{Z}'K^2}{K^2} - \frac{\hat{Z}^2k^2(K')^2}{K^3} \right)
\end{align*}
\]

(36)

where \( a_D = 1/2^D \pi^{D/2} \Gamma(D/2) \) and

\[
K = \hat{Z}k^2 + U'' - (R/12)(2\hat{Z} + \Phi \hat{Z}')
\]

(37)

These equations rule the evolution of all coupling constants generated by the renormalization procedure, and have been derived without any assumption on the functional form of the potential and the wavefunction renormalization. For instance, if \( U \) and \( \hat{Z} \) are analytic functionals we write

\[
\hat{Z} = \sum_{i,j=0}^{\infty} \frac{1}{i!j!} h_{ij}(k) \Phi^i R^j, \quad U = \sum_{i,j=0}^{\infty} \frac{1}{i!j!} g_{ij}(k) \Phi^i R^j.
\]

(38)

The beta functions for the couplings in Eq. (38) are

\[
\begin{align*}
\frac{d g_{ij}}{dk} &= \frac{\partial^i+j U}{\partial^i \Phi \partial^j R} |_{\text{vac}} \\
\frac{d h_{ij}}{dk} &= \frac{\partial^i+j Z}{\partial^i \Phi \partial^j R} |_{\text{vac}}
\end{align*}
\]

(39)

and one obtains an infinite dynamical system that can be conveniently truncated to some order to obtain an approximate solution of Eq. (36). Let us for instance neglect the field dependence in \( \hat{Z} \) using the improved version of Eqs. (33) in \( D = 4 \) (the notation \( \hat{Z} = \eta_k \) is adopted analogously to Eqs. (33) )
\[ k \frac{dU}{dk} = -\frac{k^4}{16\pi^2} \ln \frac{\eta_k k^2 + U'' - (R/6)\eta_k}{(\eta_k k^2 + U'')|_{\Phi=R=0}} \]

\[ \frac{d\ln \eta_k}{d\ln k} = -\frac{k^4}{16\pi^2} \frac{(U''')^2(U'' - (R/6)\eta_k)}{(\eta_k k^2 + U'' - (R/6)\eta_k)^4} \]  

(40)

It is interesting to employ these equations in order to study the behavior of the couplings in the model (34) which corresponds to a specific truncation in Eq. (38).

We define the running parameters \( m_k^2 = g_{20}, \lambda_k = g_{40}, \xi_k = g_{21}, \eta_k = h_{00} \) and, as usual, the renormalized parameters \( m_R, \lambda_R, \xi_R, \eta_R \) are obtained in the limit \( k \to 0 \) and, in the broken phase where \( \langle \phi(x) \rangle = v \), \( m_R^2 = g_{20}(k=0) + \lambda_R v^2/2 \).

By fixing the vacuum in Eq. (39) at the values \( \langle \phi(x) \rangle = v \) and the Ricci scalar at the value \( R = \mathcal{R} \) we obtain the following \( \beta \)-functions

\[
\begin{align*}
    k \frac{d m_k^2}{dk} &= -\frac{k^4}{16\pi^2} f_k (\lambda_k - f_k \lambda_k v) \\
    k \frac{d \lambda_k}{dk} &= \frac{k^4}{16\pi^2} f_k^2 (3\lambda_k^2 + 6 f_k \lambda_k^4 v^4 + 12 \lambda_k^3 v^2) \\
    k \frac{d \xi_k}{dk} &= \frac{k^4}{16\pi^2} f_k^2 \lambda_k (\xi_k - \eta_k/6)(1 - f_k 2 \lambda_k v^2) \\
    k \frac{d \tau_k}{dk} &= \frac{k^4}{16\pi^2} v \lambda_k^2 f_k^2 (3 - 2 f_k \lambda_k v^2) \\
    k \frac{d \sigma_k}{dk} &= \frac{k^4}{16\pi^2} v \lambda_k^2 f_k^2 (\xi_k - \eta_k/6) \\
    \gamma_k &= \frac{d \ln \eta_k}{d \ln k} = -\frac{k^4}{16\pi^2} f_k^4 \lambda_k^2 v^2 (m_k^2 + (\xi_k - \eta_k/6)\mathcal{R} + \lambda_k v^2/2) \\
\end{align*}
\]

(41)

where

\[
f_k^{-1} = \eta_k k^2 + m_k^2 + \lambda_k v^2/2 + (\xi_k - \eta_k/6)\mathcal{R} \]

(42)

From the above equations it is evident that the presence of a nonvanishing expectation value of the scalar field \( v \) has the consequence of generating new couplings \( \tau_k \) for the operator \( \Phi^3 \) and \( \sigma_k \) for \( \Phi \mathcal{R} \). In principle the location of the vacuum should be determined after solving the equation of the minimum for the effective potential, in the improved scheme. This amounts to solve the system (41) coupled to the equation of the minimum for the running
potential. Here things are more complicated since in higher-derivative gravitational theories the vacuum does not necessarily coincide with the $R = 0$ configuration \cite{3}. In the following we just consider $v$ and $R$ as adjustable parameters.

We now discuss the scaling behavior of the couplings in the two extreme scaling limits: ultraviolet and infrared. In the UV domain we neglect all scales with respect to $k$, obtaining

\begin{align}
\frac{d m_k^2}{dk} &= -\frac{k^2 \lambda_k}{\eta_k 16 \pi^2} \\
\frac{d \lambda_k}{dk} &= \frac{3 \lambda_k^2}{\eta_k^2 16 \pi^2} \\
\frac{d \xi_k}{dk} &= \frac{\lambda_k (\xi_k - \eta_k/6)}{\eta_k^2 16 \pi^2} \\
\frac{d \tau_k}{dk} &= \frac{3 \lambda_k^2 v}{\eta_k^2 16 \pi^2} \\
\frac{d \sigma_k}{dk} &= \frac{v \lambda_k}{\eta_k^2 16 \pi^2} (\xi_k - \frac{\eta_k}{6})
\end{align}

while the anomalous dimension vanishes as

$$\gamma_k \sim O\left(\frac{1}{k^4}\right)$$

as it should be. This result coincides with the findings in \cite{7,13} for the $\beta$-functions in the UV domain. However once $k$ becomes much smaller than $m_k$, the scaling laws change. Suppose first that the following hierarchy holds

$$m_{R}^2 \gg R$$

then, below the mass gap the renormalized flow is arrested and in $k = 0$ the flow always reaches a (completely trivial) fixed point. In particular this shows that the findings in \cite{19} that $\xi_{R} = 1/6$ is not necessarily an IR attractor, is still valid when the contribution of $\eta_k$ is considered. At this level of the approximation we note that $\xi_k = 1/6$ is a fixed point only if $\eta_k = 1$, statement which is not true in the IR domain for the broken phase. In the symmetric phase, instead if we fix $\eta_k$ at the bare level to one it does not change and $\xi_k = 1/6$ is a fixed point for the system \cite{11}. However, as we shall see in the next section, when the contribution
of the irrelevant operators is included, this fixed point disappears. These conclusions should apply for any $D$ as one can see by direct calculation of the $\beta$-functions derived from Eq. (40). We also note that the gaussian fixed point is always present in any dimension.

When Eq. (45) is not satisfied the structure of the vacuum can be different. In fact in general the curvature competes to determine the sign of the gravitational running mass $\partial^2 U$ and it can lead the sistem to a new phase. In fact, by looking at Eq. (42) we note that around

$$\eta_k k^2 \sim (\eta_k/6 - \xi_k)R - m_R^2$$

(46)

the equations in (41) become unstable because of the presence of poles in the $\beta$-functions signaling that a new phase sets and that new IR relevant interactions are needed to describe the renormalized system.

One might worry that around $k^2 \sim R$ the local momentum expansion is not reliable. This is not the case. In fact for the Einstein Universe we have the following exact finite-difference RG equation for the local potential

$$U_{n-\Delta n, k-\Delta k} - U_{n, k} = -\frac{k}{4\pi^3 a^3} n^2 \ln(a^2 k^2 + a^2 \partial^2 U(\Phi, a^2) + n^2 - 1) \Delta n \Delta k$$

(47)

where $n$ is the quantum number associated with the smearing on the spatial sections with topology $S^3$, $k$ is a cut-off in the Euclidean “time” direction and $a^2 = 6/R$ is the radius of $S^3$. We find also in this case the appearance of a singular behavior at some scale $n$ if the “restoring force” $\partial^2 U(\Phi, a^2)$ becomes negative.

The physical reason of this instability is that the saddle-point expansion in Eq. (11) has been performed by considering perturbations around homogeneus configurations which do not dominate the path-integral in Eq. (11) on scales smaller than the symmetry-breaking scale. In this case a more refined computation is needed, and the naive gradient expansion cannot be applied.
IV. THE CROSSOVER

The study of the crossover is instead more complicated and one has to integrate numerically the flow equations by keeping the contribution of the “irrelevant” operators as well. In order to deal with a more tractable problem, we employ a truncation in the expressions of the local potential and the wavefunction renormalization function, by keeping up to dimension-6 operators. We thus consider the following action defined at the scale $k$

$$
S = \int d^Dx \sqrt{g} \left\{ \epsilon_0 + \epsilon_1 R + \frac{\epsilon_2}{2} R^2 + \frac{\epsilon_3}{3!} R^3 + \frac{1}{2} \eta_k \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{4} \alpha_k \Phi^2 \partial_\mu \Phi \partial^\mu \Phi \\
+ \frac{1}{2} \beta_k R \partial^2 \Phi \partial_\mu \Phi + \frac{1}{2} m^2_k \Phi^2 + \frac{1}{4} \xi_k R \Phi^2 + \frac{1}{4!} \lambda_k \Phi^4 + \frac{1}{2!2!} \chi_k R^2 \Phi^2 + \frac{\zeta_k}{4!} \Phi^4 + \frac{1}{6!} \omega_k \Phi^6 \right\}
$$

(48)

where $\epsilon_0 = m^2_{PL} \Lambda_{CS} / 8\pi$, $\epsilon_1 = -m^2_{PL} / 16\pi$, $m_{PL}$ is the Planck mass, $\Lambda_{CS}$ is the cosmological constant. Note that the remark at the end of sect. II also applies for the more general action (48).

In order to avoid problems with triviality we keep the UV cut-off $\Lambda$ fixed. It is convenient to work in running cut-off units, by introducing the dimensionless variables

$$
x = \Phi k^{-(D-2)/2}, \quad y = R k^{2-D}
$$

(49)

and calling $u, z$ respectively the dimensionless potential and wave function renormalization, we have the expansions

$$
z = \sum_{i,j} \frac{1}{i!j!} z_{ij}(k) x^i y^j, \quad u = \sum_{i,j} \frac{1}{i!j!} u_{ij}(k) x^i y^j
$$

(50)

where one can easily find the proper correspondence among $z_{ij}(k)$, $u_{ij}(k)$ and the various couplings appearing in Eq. (48). If we define

$$
A_i(t, x, y) = z(t, x, y) + \partial_x^2 u(t, x, y) - y(2z + x \partial_x z)/12
$$

(51)

the renormalization group equations for $u$ and $z$ can be rewritten in terms of the new variables in the following way
\[
\frac{du}{dt} = \frac{(D-2)}{2} (x \partial_x u + 2y \partial_y u) - Du - a_D \ln \left( \frac{A_t}{A_t[x=y=0]} \right)
\]
\[
\frac{dz}{dt} = \frac{(D-2)}{2} (x \partial_x z + 2y \partial_y z) - a_D A^{-1}_t \left( \partial_x^2 z - 2A^{-1}_t \partial_x z \partial_x A_t + A^{-2}_t z [(\partial_x A_t)^2 + \partial_x z \partial_x A_t] - A^{-3}_t z^2 (\partial_x A_t)^2 \right). \tag{52}
\]

We have rescaled the momentum variable, by defining \( t = \ln(k/\Lambda) \) so that we generate the renormalized flow by lowering the cut-off \( \Lambda \to \Lambda e^t \), with \( t \leq 0 \).

We insert Eq. (50) in Eq. (52) and we evaluate the derivatives in the symmetric vacuum \( \Phi = 0 \) at non-zero curvature \( R = \mathcal{R} \) (the corresponding dimensionless value of \( y \) is indicated with \( r: r = \mathcal{R}k^{2-D} \)), obtaining in \( D=4 \) the system of ordinary differential equations displayed in the appendix. For simplicity we shall not discuss here the symmetry broken phase with \( v \neq 0 \).

The results of the numerical analysis is shown in figures 1 - 5, for some values (including zero) of the curvature \( r \). We set all bare non-gaussian couplings to zero. The bare value of \( \lambda_k \) and \( \eta_k \) are kept fixed respectively to \( u_{40}(t = 0) = 0.1 \) and \( z_{00}(t = 0) = 1 \), while we have considered several bare values of the conformal coupling. We locate the critical line for \( r = 0 \) near \( u_{20}(t = 0) = -0.000325 \) and this value does not depend on the value of \( u_{21} \).

When the theory is away from the critical line, one sees that below a certain value of \( t \), the evolution of all couplings stops. In fact below that scale \( f_t \), defined in the appendix, exponentially vanishes and the tree level scaling of the couplings is recovered. In particular for the mass

\[ u_{20} k^2 = m_k^2 \simeq m_R^2. \tag{53} \]

The specific constant value of the renormalized mass \( m_R \) depends on how the other parameters are fixed. For instance, for the above initial conditions we see that the mass scale corresponds to \( t \sim 6 \).

Let us consider the scaling above the mass threshold. The running of some non-gaussian couplings is characterized by the presence of a “plateau” as it is shown in fig.1 for the running
of $u_{03} = \epsilon_3 k^2$ and we also observe a similar behavior for $u_{41} = \zeta_k k^2$. This is interesting, we believe, because it contrasts with the usual notion of “irrelevance” since those interactions behave like marginals for almost three orders of magnitude below the cutoff.

A striking result concerns the running of the $u_{22} = \chi_k k^2$ interaction term. In fig.2 we see that across and above the mass scale its value is not negligible, since for instance $\chi_k k^2 \sim 0.2$ around $t = -4$. Note the strong dependence of $\chi_k$ from the bare value of $\xi_k$.

The conformal coupling $\xi_k = u_{21}$ deserves a more detailed discussion. Its evolution is shown in fig.3 for the minimally coupled bare theory and in fig.4 for the conformally coupled bare theory. It is worth to remark the small increase of $u_{21}$ in fig.3, even for non vanishing non-renormalizable couplings (see curve 4), within the three orders of magnitude range between $t = 0$ and the mass threshold, below which the curves become flat. This means that starting at $t = 0$ with a bare $\xi$ far from the conformal value $1/6$, it is very likely that the coupling does not reach that value, being stopped before by the mass threshold.

Actually, looking at the evolution equation for $u_{21}$ in the appendix, we see that in the infrared region, where the various scales cannot be neglected when compared to $k$, it is extremely difficult to establish the existence of a non trivial IR fixed point for this coupling.

In fig.4 the behavior of $u_{21}$ in the neighbourhood of $1/6$ is explored. Here we see that there is no deviation from $1/6$ only for zero non-renormalizable couplings at $t = 0$. However, turning on these couplings, we observe small deviations, of the same order of magnitude of the changes observed in fig.3. Again no strong attraction toward $u_{21} = 1/6$ is observed.

Finally in fig.5 the influence of the curvature on the phase diagram for the $\Phi$ field is investigated. For a small value of the curvature $r = 0.01$, we see that for sufficiently high values of the conformal coupling (greater than about 2 in our example) the running adimensional mass $u_{20}(t)$ grows negative in the IR region while choosing smaller or negative $u_{21}(t = 0)$, we see that $u_{20}(t)$ is driven away from the critical line toward positive values, at earlier “times” $t$. In fact, for such curvature, the mass in fig.5 practically represents the second derivative of the potential with respect to the scalar field, since the additional term $(\xi_k - \eta_k/6)r$ is practically $t$-independent and negligible for large values of $t$. Thus we are
led to a result which is opposite to the naive statement, obtained by looking at the not RG-improved propagator, that the phase transition, when \( r \) is positive, is approached by decreasing the conformal coupling.

V. SUMMARY AND CONCLUSIONS

In this paper we have discussed the renormalization of a scalar theory coupled to the gravitational field by means of the blocking procedure. We have obtained non-perturbative flow equations for the blocked potential and the wavefunction renormalization that rule the flow of all coupling constants generated when the cut-off is lowered from the UV region towards the infrared domain. We have seen that the gravitational field influences the scaling laws of the field in the crossover region between the mass gap and the cut-off. In fact our equations reproduce the standard perturbative \( \beta \)-functions only in the UV region when the contribution of the irrelevant couplings is neglected. However we show that any coupling of the kind \( R^i \Phi^j \) is generated in both potential and wavefunction renormalization even if it is not present at the bare level. We have seen that they can be important in determining the scaling law across the mass gap. In particular, due to these non renormalizable couplings one cannot claim that \( \xi = 1/6 \) is a fixed point for the conformal coupling constant whereas, independently on the bare value of \( \xi_k \), only very small changes of the value of this coupling are observed in the flow from the UV to the IR region.

Another interesting issue addressed with our formalism concerns the structure of the phase diagram for the quantum field. In the model analyzed in the previous section the onset of criticality is determined by taking into account the coupling with the gravitational field. It turns out that the approach to the critical line strongly depends on the strenght of the conformal coupling.

It would be important to understand the consequences of our findings in the context of the inflationary models. For instance it has been recently pointed out that the “fine tuning” problem in models with the symmetry-breaking scale near the Planck mass, can be
avoided in potentials dominated by non-quadratic “irrelevant” interactions terms like $\phi^m$ with $m > 2$.

In fact in the standard slow-rolling approximation it is assumed that the inflaton field evolves as a free field leading to a gaussian spectrum for the large scale angular anisotropy. Since this happens above the mass scale of the inflaton field, one may think that the presence of non-quadratic interaction terms whose fluctuations are not suppressed above the mass scale generates non-gaussian fluctuations in the CMBR. At the present state of the experimental data there is no secure bound on non-gaussian fluctuations on large angular scales, therefore this possibility cannot be ruled out. It would be worth to pursue these investigations in a specific model in order to answer these questions in detail.

ACKNOWLEDGMENTS

The authors are grateful to Janos Polonyi for his constant advice and many important suggestions. A.B. has also benefited from discussions with M. Reuter and J. Bartlett. A.B. gratefully acknowledges Fondazione Angelo Della Riccia and I.N.F.N for financial support.

VI. APPENDIX

In this appendix the complete set of equations ruling the couplings of the action (18), is displayed in terms of dimensionless quantities introduced in Eqs. (19), (50) for the vacuum characterized by the values $\langle \phi(x) \rangle = 0$ and $R = R$ or, in terms of dimensionless quantities, $x = 0$ and $y = r$. Defining $f_t$ as

$$f_t^{-1} = u_{22}t^2/8 + u_{20} + (u_{21} - z_{00}/6)r + z_{00} + z_{01}r(1 - r/6)$$

we get

$$\frac{dz_{00}}{dt} = 2rz_{01} - \frac{1}{16\pi^2} f_t z_{20}$$

$$\frac{dz_{20}}{dt} = 2z_{20} + \frac{5}{16\pi^2} z_{20} f_t^2 (u_{40} + u_{41}r + z_{20}(1 - r/3)) + \frac{1}{8\pi^2} f_t^3 (u_{40} + u_{41}r + z_{20}(1 - r/3)).$$
\[
(z_0 + rz_{01})[f_t(z_0 + rz_{01})(u_{40} + u_{41}r + z_{20}(1 - r/3)) - ru_{41} - u_{40} - z_{20}(2 - r/3)]
\]

\[
\frac{dz_{01}}{dt} = 2z_{01} + \frac{1}{16\pi^2}f_t^2z_{20}(u_{21} - \frac{z_{00}}{6} + \frac{u_{22}r}{4} + z_{01}(1 - r/3))
\]

\[
\frac{du_{20}}{dt} = -2u_{20} + u_{22}r^2 + \frac{f_t}{16\pi^2}(u_{40} + u_{41}r + z_{20}(1 - r/3))
\]

\[
\frac{du_{21}}{dt} = u_{22}r^2 + \frac{f_t}{16\pi^2}(z_{20}/3 - u_{41}) + \frac{f_t^2}{16\pi^2}(u_{40} + u_{41}r + z_{20}(1 - r/3))
\]

\[
z_{20}(1 - r/3)(u_{21} - \frac{z_{00}}{6} + \frac{u_{22}r}{4} + z_{01}(1 - r/3))
\]

\[
\frac{du_{40}}{dt} = 2ru_{41} - \frac{f_t}{16\pi^2}u_{60} + \frac{3f_t}{16\pi^2}(u_{40} + u_{41}r + z_{20}(1 - r/3))^2
\]

\[
\frac{du_{60}}{dt} = 2u_{60} + \frac{15}{16\pi^2}f_t^2u_{60}(u_{40} + u_{41}r + z_{20}(1 - r/3)) - \frac{30}{16\pi^2}f_t^3(u_{40} + u_{41}r + z_{20}(1 - r/3))^3
\]

\[
\frac{du_{22}}{dt} = u_{22}/2 + \frac{f_t^2}{16\pi^2}(u_{40} + u_{41}r + z_{20}(1 - r/3))(u_{22}/4 - z_{10}/3) + \frac{f_t^2}{8\pi^2}(u_{21} - \frac{z_{00}}{6} + \frac{u_{22}r}{4} + z_{01}(1 - r/3))[(u_{41} - z_{20}/3) - f_t(u_{40} + u_{41}r + z_{20}(1 - r/3))(u_{21} - \frac{z_{00}}{6} + \frac{u_{22}r}{4} + z_{01}(1 - r/3))]
\]

\[
\frac{du_{41}}{dt} = 2u_{41} + \frac{3f_t^2}{8\pi^2}(u_{40} + u_{41}r + z_{20}(1 - r/3))(u_{41} - z_{20}/3) + \frac{f_t^2}{16\pi^2}(z_{21} - \frac{z_{00}}{6} + \frac{z_{22}r}{4} + z_{01}(1 - r/3))[(u_{60} - 6f_t(u_{40} + u_{41}r + z_{20}(1 - r/3))^2)]
\]

\[
\frac{du_{01}}{dt} = -2u_{10} - \frac{f_t}{16\pi^2}(u_{21} - \frac{z_{00}}{6} + \frac{u_{22}r}{4} + z_{01}(1 - r/3))
\]

\[
\frac{du_{02}}{dt} = 2u_{03}r + \frac{f_t}{16\pi^2}(\frac{z_{10}}{3} - \frac{u_{22}}{4}) + \frac{f_t^2}{16\pi^2}(u_{21} - \frac{z_{00}}{6} + \frac{u_{22}r}{4} + z_{01}(1 - r/3))^2
\]

\[
\frac{du_{03}}{dt} = 4u_{30} - \frac{4f_t^3}{16\pi^2}(u_{21} - \frac{z_{00}}{6} + \frac{u_{22}r}{4} + z_{01}(1 - r/3))^3 - \frac{6f_t^2}{16\pi^2}(\frac{z_{10}}{3} - \frac{u_{22}}{4})(u_{2,1} - \frac{z_{00}}{6} + \frac{u_{22}r}{4} + z_{01}(1 - r/3))
\]

\[
\frac{du_{00}}{dt} = -4u_{00} - 2u_{01}r + \frac{u_{03}}{3} + \frac{1}{16\pi^2}ln \frac{f_t}{f_t(x = 0, y = r)}.
\]
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FIGURE CAPTIONS

Figure 1. \( u_{03}(t) \) with \( u_{21}(t = 0) = 1 \), (1), and \( u_{21}(t = 0) = -1 \), (2). For both curves the non renormalizable couplings are fixed to zero at \( t = 0 \) and \( r = 0.01 \).

Figure 2. \( u_{22}(t) \) with \( u_{21}(t = 0) = -8 \) and \( r = 0 \) (1), \( u_{21}(t = 0) = -8 \) and \( r = 0.01 \) (2), \( u_{21}(t = 0) = -4 \) and \( r = 0.1 \) (3). The non renormalizable couplings are fixed to zero at \( t = 0 \) for all curves.

Figure 3. Flow of \( u_{21}(t) \) starting at \( u_{21}(t = 0) = 0 \) for zero non-renormalizable couplings at \( t = 0 \) and \( r = 0 \) (1), \( r = 0.01 \) (2), \( r = 0.1 \) (3); with the various non-renormalizable couplings set to 0.01 at \( t = 0 \) and \( r = 0.01 \) (4).

Figure 4. Flow of \( u_{21}(t) \) starting at \( u_{21}(t = 0) = 1/6 \) with the various non-renormalizable couplings set to 0.01 at \( t = 0 \) and and \( r = 0 \) (1), \( r = 0.01 \) (3), \( r = 0.1 \) (4). The flat curve (2) corresponds to zero non-renormalizable couplings at \( t = 0 \) and it is not sensitive to \( r \).

Figure 5. \( u_{20}(t) \) plotted for three different values of the conformal coupling: \( u_{21}(t = 0) = 0 \) (1), \( u_{21}(t = 0) = 1 \) (2), \( u_{21}(t = 0) = 2.5 \) (3); \( r \) is kept fixed to \( r = 0.01 \) and the the bare non-renormalizable couplings are zero.
fig. 4

- $u_21$ versus $t$

- Lines labeled (1), (2), (3), (4)
