Properly colored Hamilton cycles in Dirac-type hypergraphs

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July 1, 2020

Abstract

We consider a robust variant of Dirac-type problems in $k$-uniform hypergraphs. For instance, we prove that if $H$ is a $k$-uniform hypergraph with minimum codegree at least $(\frac{1}{2} + \gamma)n$, $\gamma > 0$, and $n$ sufficiently large, then any edge coloring $\phi$ satisfying appropriate local constraints yields a properly colored tight Hamilton cycle in $H$. Similar results for loose cycles are also shown.

1 Introduction

Our study lies at the intersection of three classical areas of research – we will extend several Dirac-type theorems for hypergraphs to an edge-colored setting. The famous theorem of Dirac states that any $n$-vertex graph with minimum degree $n/2$ has a Hamilton cycle. Among the numerous research lines stemming from the Theorem are so-called Dirac-type problems, where one aims to embed a spanning subgraph into a graph of given minimum degree.

Dirac-type problems in hypergraphs, first considered by Katona and Kierstead [21] in 1999, now constitute a fruitful and dynamic theory. One reason for this multitude of results is that there is a number of different notions of hypergraph degrees and cycles, and consequently several extremal constructions. Given $\ell \in [k-1]$, a $(k, \ell)$-overlapping cycle $C_n^{(k)}(\ell)$, or shortly an $(k, \ell)$-cycle is an $n$-vertex $k$-graph whose vertices can be ordered cyclically such that each of its edges consists of $k$ consecutive vertices and every two consecutive edges share exactly $\ell$ vertices. Note that $C_n^{(k)}(\ell)$ exists whenever $k - \ell$

\footnote{Research supported by ARC Discovery Project DP180103684.}
\footnote{Research supported by Polish NSC grant 2018/29/B/ST1/00426.}

Keywords and phrases: hypergraph, Hamilton cycle, Dirac’s theorem, absorbing method.
Mathematics Subject Classification: 05C65, 05C45, 05D40.
divides $n$ (we will further use standard notation $(k - \ell)|n$ to indicate this fact). For $k = 2$ and $\ell = 1$, this reduces to the graph cycle. The two extreme cases, $\ell = 1$ and $\ell = k - 1$, are usually referred to, respectively, as loose and tight cycles. We say that a $k$-graph contains a Hamilton $(k, \ell)$-cycle, or is $\ell$-Hamiltonian if it contains an $(k, \ell)$-cycle as a spanning subhypergraph.

The degree of a set $S \subset V(H)$ in a hypergraph $H$, denoted by $\deg(S)$, is the number of edges of $H$ containing $S$. For sets of order 1 or $k - 1$, the terms vertex degree and codegree (respectively) are often used. The minimum and maximum $s$-degree of $H$ are defined as

$$
\delta_s(H) = \min\{\deg(S) : S \subset V(H), |S| = s\}, \quad \Delta_s(H) = \max\{\deg(S) : S \subset V(H), |S| = s\}.
$$

The following three results are directly relevant to our topic.

**Theorem 1.** Let $1 \leq \ell < k/2$ and $\gamma > 0$. Let $H$ be a $k$-graph on $n$-vertices for sufficiently large $n$.

(i) [28] If $\delta_{k-1}(H) \geq \left(\frac{1}{2} + \gamma\right)n$, then $H$ contains a tight Hamilton cycle.

(ii) [24, 14] If $\delta_{k-1}(H) \geq \left(\frac{1}{2(k-\ell)} + \gamma\right)n$, then $H$ contains a Hamilton $(k, \ell)$-cycle.

(iii) [4] If $k = 3$ and $\delta_1(H) \geq \left(\frac{7}{16} + \gamma\right)n^2/2$, then $H$ contains a loose Hamilton cycle.

The minimum-degree conditions in all three statements are asymptotically tight. In some cases, the corresponding exact thresholds are known [29, 16, 15]. The most recent survey covering hypergraph Dirac-type problems is [32]. The area has witnessed the development of several ubiquitous techniques such as the absorbing method and tools in hypergraph regularity, including the hypergraph blow-up lemma [19]. As we shall see in our proofs, hypergraph Turán problems also play an important role.

In our setting, the host graph will be edge-colored and, given appropriate restrictions on the coloring, we will be embedding properly colored Hamilton cycles. A hypergraph is properly colored if every two intersecting edges are assigned different colors. Our results can be seen as robust versions of the above-mentioned Dirac-type theorems.

Sufficient conditions for finding properly colored Hamilton cycles in edge-colorings of the complete graph $K_n$ were first proposed by Daykin [9] in 1976. Let us call an edge colouring of a hypergraph $H$ locally $t$-bounded if any colour appears on at most $t$ edges incident to any given vertex. Daykin conjectured that there is a positive constant $\mu$ such that for any $n$, a locally $\mu n$-bounded colouring of $K_n$ contains a properly colored Hamilton cycle. The conjecture was resolved by Chen and Daykin [6], as well as Bollobás and Erdős [2]. After a number of improvements and extensions (see, e.g., [30, 1, 3]), Lo [25] showed that Daykin’s conjecture holds for any $\mu < \frac{1}{2}$ and large $n$, which is asymptotically optimal. Analogous ‘global’ sufficient conditions for rainbow embeddings (in which no two edges have the same colour) have often been proved in parallel, also motivated by famous problems which can be encoded in terms of rainbow subgraphs, such as Latin transversals. However, we will streamline most of our discussion towards the properly colored setting.
The question of Daykin also has natural hypergraph analogues. Dudek, Frieze and Ruciński [11] showed that for \( \mu > 0 \), \( n \) sufficiently large and \( \ell \in [k-1] \), locally \( \mu n^{k-\ell} \)-bounded colourings of the complete \( k \)-uniform hypergraph \( K_n^{(k)} \) contain a properly coloured \( \ell \)-overlapping cycle. However, this is only known to be tight for the loose cycle \( C_n^{(k)}(1) \), and the problem remains wide open for tight cycles. In [10] and [18], the boundedness condition was weakened to codegrees – in particular, if the subhypergraph induced by any color has maximum codegree at most \( \mu n \), then one can find a tight Hamilton cycle.

Several authors have considered replacing the host (hyper)graph by an incomplete (hyper)graph [23, 22, 12, 5, 8, 7]. Krivelevich, Lee and Sudakov have shown that Daykin’s conjecture holds even when \( K_n \) is replaced by an arbitrary graph of minimum degree at least \( n/2 \) (often called a Dirac graph), thus confirming a conjecture of Häggvist. They placed their result in the context of a number of recent studies on robustness of extremal and probabilistic results [31].

We will show that a similar phenomenon occurs in \( k \)-uniform hypergraphs. A colored hypergraph is a pair \((H, \phi)\), where \( H \) is a \( k \)-graph and \( \phi : H \to N \) is a coloring of the edges of \( H \). For each \( i \in N \), we denote by \( H_i = \{ e \in H : \phi(e) = i \} \) the subhypergraph of \( H \) consisting of the edges of color \( i \). We now state our three new results which correspond to the three parts of Theorem 1 above. In each of them we make a suitable assumption on the coloring in terms of \( \Delta(H_i) \).

**Theorem 2.** For every \( k \geq 3 \) and \( \gamma > 0 \) there exist \( c > 0 \) and \( n_0 > 0 \) such that if \((H, \phi)\) is an \( n \)-vertex colored \( k \)-graph with \( n \geq n_0 \), \( \delta_{k-1}(H) \geq (1/2 + \gamma)n \) and \( \Delta_{k-1}(H_i) \leq cn \) for every \( i \in N \), then \((H, \phi)\) contains a properly colored tight Hamilton cycle \( C_n^{(k)}(k-1) \).

**Theorem 3.** For every \( k \geq 3 \), \( 1 \leq \ell < k/2 \) and \( \gamma > 0 \) there exist \( c > 0 \) and \( n_0 > 0 \) such that if \((H, \phi)\) is an \( n \)-vertex colored \( k \)-graph with \((k - \ell) \mid n \) \( \geq n_0 \), \( \delta_{k-1}(H) \geq (\frac{1}{2(k-\ell)} + \gamma)n \) and \( \Delta_{\ell}(H_i) \leq cn^{k-\ell} \) for every \( i \in N \), then \((H, \phi)\) contains a properly colored Hamilton \((k, \ell)\)-cycle \( C_n^{(k)}(\ell) \).

**Theorem 4.** For every \( \gamma > 0 \) there exist \( c > 0 \) and \( n_0 > 0 \) such that if \((H, \phi)\) is an \( n \)-vertex colored \( 3 \)-graph with \( 2 \mid n \geq n_0 \), \( \delta_1(H) \geq (7/16 + \gamma)n^2/2 \) and \( \Delta_1(H_i) \leq cn^2 \) for every \( i \in N \), then \((H, \phi)\) contains a properly colored loose Hamilton cycle \( C_n^{(3)}(1) \).

As, trivially, \( \Delta_{\ell}(H_i) \leq (\frac{n-\ell}{n-k}) \), the assumptions on \( \Delta_{\ell}(H_i) \) in our theorems are optimal up to a constant factor.

In our proofs we adapt the absorbing method to the properly colored setting, which may turn out to be a fruitful direction given that alternative approaches do not extend straightforwardly. For instance, a uniformly random vertex ordering analysed using the Local Lemma (used, for instance, in [3, 18]), immediately fails for host hypergraphs which are not ‘almost’ complete. Recently, the blow-up lemma for graphs has been adapted and fruitfully applied to the rainbow setting [12], but the hypergraph blow-up lemma used in [20] is significantly more intricate.

Finally, Theorems 2-4 provide new evidence in support of a meta-conjecture formulated by Coulson, Keevash, Perarnau and Yepremyan [7] in the context of rainbow problems. Namely, for a large class of Dirac-type problems for \( k \)-graphs, the rainbow
counterparts for bounded colorings should have asymptotically the same degree threshold as the original problem. Our results show that the conjecture holds for Hamilton cycles if the rainbow colorings are replaced by less restrictive proper colorings. To our knowledge, these are the first results of this type on embedding spanning hypergraphs.

The paper is structured as follows. Section 2 contains preliminary definitions and tools. Theorems 2, 3 and 4 are proved in Sections 3, 4 and 5 respectively.

2 Preliminaries

In this preliminary section we give some definitions and results which are relevant throughout the paper.

We begin with two simple existential results whose proofs employ the standard probabilistic method. The first of them is cited from [27] (see Lemma 3.10 therein). We will use it three times when proving respective ‘reservoir lemmas’. However, part (c) will be needed only in Section 5.1.

Proposition 1 ([27]). For every \( p \), \( 0 < p < 1 \), there is \( n_0 \) such that the following holds. Let \( U_1, \ldots, U_s \) be subsets of an \( n \)-element set \( V \), \( n \geq n_0 \), and let \( G_1, \ldots, G_t \) be graphs on \( V \), where \( s \) and \( t \) are both polynomials in \( n \). Further, let \( |U_i| \geq \alpha_i n \), \( i = 1, \ldots, s \), and \( |G_j| \geq \beta_j \binom{n}{2} \), \( j = 1, \ldots, t \), for some constants \( 0 < \alpha_i, \beta_j < 1 \). Then there exists a subset \( R \subset V \) such that

(a) \( |R| - pn \leq pn^{2/3} \),

(b) for all \( i = 1, \ldots, s \), we have \( |U_i \cap R| \geq (\alpha_i - 2n^{-1/3}) |R| \), and

(c) for all \( i = 1, \ldots, t \), we have \( |G_j[R]| \geq (\beta_j - 3n^{-1/3}) \left( \frac{|R|}{2} \right)^2 \).

Proof. We use the probabilistic method. Let \( R \subset [n] \) be a random family to which each \( t \)-tuple is sampled independently with probability \( p = \frac{\alpha n^{-1/3}}{4} \). Using Markov’s inequality, we infer that

\[ P(|R| \geq \alpha n) \leq \frac{1}{4} \]

and, likewise, the probability that the number of intersecting pairs of \( t \)-tuples in \( R \) exceeds \( \alpha^2 t^2 n/4 \) is no more than \( \frac{1}{4} \). To see the latter, note that the expected number of intersecting pairs of \( t \)-tuples in \( R \) is at most

\[ t^2 n^{2t-1} p^2 = \frac{1}{16} \alpha^2 t^2 n. \]
Now consider \( i \in [m] \). The random variable \(|R \cap A_i|\) is a binomially distributed random variable with expectation

\[ |A_i|p \geq 4n^2\alpha t^2n'p = \alpha^2 t^2n. \]

Therefore, using the well-known Chernoff bound (see, e.g., [17]), followed by the union bound over all \( i \in [m] \), we infer that a.a.s.

\[ |R \cap A_i| \geq \frac{1}{2} \alpha^2 t^2n \quad \text{for all } i \in [m]. \]

Thus, the probability that \( R \) satisfies all three above properties is at least \( \frac{1}{2} - o(1) > 0 \), so the event is nonempty. Let \( F' \) be an instance of \( R \) satisfying them. Further, let \( F \) be obtained from \( F' \) by removing one \( t \)-tuple from each intersecting pair and disregarding all \( t \)-tuples of \( F' \) which do not belong to \( \bigcup_{i=1}^{m} A_i \). Clearly, we still have \(|F| \leq \alpha n\) and, moreover, for all \( i \in [m] \),

\[ |F \cap A_i| \geq \frac{1}{2} \alpha^2 t^2n - \frac{1}{4} \alpha^2 t^2n = \frac{1}{4} \alpha^2 t^2n, \]

as required. \( \square \)

A \((k, \ell)\)-overlapping path, or shortly \((k, \ell)\)-path, on \( s \) vertices is defined as an \( s \)-vertex \( k \)-graph whose vertices can be ordered linearly such that each of its edges consists of \( k \) consecutive vertices and every two consecutive edges share exactly \( \ell \) vertices. The length of \( P \) is defined as \(|V(P)| = s\). Note that many vertex orderings yield the same path. We will typically fix one of them.

Let \( P \) be a \((k, \ell)\)-overlapping path in \( H \) on the vertices \( v_1, \ldots, v_s \) (in this order). For \( 1 \leq p < q \leq s \), we say that \( P \) connects, or lies between, the segments \( v_1, \ldots, v_p \) and \( v_q, \ldots, v_s \). When \( p = k \) and \( q = s - k + 1 \), these two segments span edges which we call the end-edges of \( P \).

For \( \ell < k/2 \), we will also need the notion of \( \ell \)-ends of \( P \), defined as the \( \ell \)-sets \( \{v_1, \ldots, v_\ell\} \) and \( \{v_{s-\ell+1}, \ldots, v_s\} \).

Our final ‘meta-theorem’ will be used six times (twice in the proof of each of our three main results). In all these applications either the set \( Q \) or its complement will be very small, but linear in \( n \).

**Proposition 3** (Connecting Meta-Statement). Let \((H, \phi)\) be an \( n \)-vertex colored \( k \)-graph, \( 1 \leq \ell < k \), and \( Q \subset V := V(H) \). For integers \( g \) and \( m \), where \( m = m(n) \), consider two statements.

\[ \begin{align*}
I. & \quad \text{For all } Q' \subset Q, \ |Q'| \leq mg, \text{ and all pairs of disjoint, properly colored } (k, \ell)\text{-paths } P_1, P_2 \text{ in } (H - Q, \phi), \text{ each having at least } \left\lceil \ell/(k - \ell) \right\rceil \text{ edges, there exist vertices } v_1, \ldots, v_{g'} \in Q \setminus Q', \ g' \leq g, \text{ such that } P_1 v_1 \cdots v_{g'} P_2 \text{ is also a properly colored } (k, \ell)\text{-path in } (H, \phi). \\
II. & \quad \text{For every collection of } m \text{ disjoint, properly colored } (k, \ell)\text{-paths } P_1, \ldots, P_m \text{ in } (H - Q, \phi), \text{ each having at least } \left\lceil \ell/(k - \ell) \right\rceil \text{ edges, there exist a properly colored } (k, \ell)\text{-cycle } C, \text{ and a properly colored } (k, \ell)\text{-path } P \text{ in } (H, \phi) \text{ which contain all paths } P_1, \ldots, P_m \text{ and have at most } mg \text{ vertices in } R. \text{ Moreover } P \text{ connects } P_1 \text{ with } P_m \text{ and } P \subset C.\
\end{align*} \]
Then, Statement I implies Statement II.

Proof. Fixing \( b \), in order to show the existence of a path \( P \), we perform induction on \( m \).
Trivially, it is true for \( m = 2 \) (as it then follows from Statement I). Assume there is a
properly colored path \( P' \) which contains all paths \( P_1, \ldots, P_{m-1} \) and connects \( P_1 \) with
\( P_{m-1} \). Set \( Q' = V(P') \cap Q \) and note that \( |Q'| \leq (m - 2)g \). Thus, we are in position
to apply Statement I to \( Q', P_{m-1}, P_m \) obtaining a desired path \( P \) which completes the
induction. In order to obtain a cycle \( C \), we apply Statement I to the pair \( (P_m, P_1) \) with
\( Q' = V(P) \cap Q \).

We finish this preliminary section with a couple of definitions related to the regularity
method used in the proofs of ‘covering lemmas’. Let \( V_1, \ldots, V_k \subset V \) be mutually disjoint
non-empty vertex sets of a \( k \)-graph \( \mathcal{H} \) on \( V \). We define the density \( d_{\mathcal{H}}(V_1, \ldots, V_k) \) of \( \mathcal{H} \)
with respect to \( (V_1, \ldots, V_k) \) as the ratio of the number of edges in \( \mathcal{H} \) with one vertex
in each \( V_i \) to \( |V_1||V_2| \cdots |V_k| \). We call the \( k \)-tuple \( (V_1, V_2, \ldots, V_k) \) \((\varepsilon, d)\)-regular
(for the hypergraph \( \mathcal{H} \)) if whenever \( (A_1, \ldots, A_k) \) is a \( k \)-tuple of subsets \( A_i \subset V_i \)
satisfying \( |A_i| \geq \varepsilon|V_i| \) for \( i \in [k] \), we have

\[
|d_{\mathcal{H}}(A_1, \ldots, A_k) - d| \leq \varepsilon.
\]

For \( t \in \mathbb{N} \), we call sets \( V_1, \ldots, V_t \) equitable if \( ||V_i| - |V_j|| \leq 1 \) for all \( i, j \in [t] \). A \( k \)-partite
\( k \)-graph with equitable parts is also called equitable.

3 Tight Hamilton cycles with co-degree conditions

This section is devoted to the proof of Theorem 2, so we will be constructing tight paths
and cycles and the attribute ‘tight’ will sometimes be omitted. We adapt the proof
in [28] to the context of colored hypergraphs.

Throughout the section, given an integer \( k \geq 3 \) and a sufficiently small \( \gamma \in (0, 1), \)
\((\mathcal{H}, \phi)\) is an \( n \)-vertex colored \( k \)-graph on vertex set \( V \) with

\[
\delta_{k-1}(\mathcal{H}) \geq (1/2 + \gamma)n \quad \text{and} \quad \Delta_{k-1}(\mathcal{H}_i) \leq cn \quad \text{for all} \quad i \in \mathbb{N},
\]

where \( c > 0 \) is sufficiently small with respect to \( \gamma \), while \( n \) is sufficiently large. (In fact,
in the actual proof of Theorem 2, Lemmas 5 and 6 will be applied to large induced
sub-hypergraphs \( \mathcal{H}' \) and \( \mathcal{H}'' \) of \( \mathcal{H} \).

We define the end-paths of a tight path \( P = v_1, \ldots, v_s \) of length \( s \geq 2k - 2 \) as the
sequences \( (v_1, \ldots, v_{2k-2}) \) and \( (v_{s-2k+3}, \ldots, v_s) \), each spanning \( k - 1 \) edges. This non-
standard definition reflects our frequent need to preserve the property of being properly
colored for the union of two properly colored paths that share one of their end-paths.

We first state three lemmas from which the proof of Theorem 2 follows in a standard
way. The first of them establishes the existence of an absorbing path.

**Lemma 4** (Absorbing Lemma). Let \( c = c(\gamma) > 0 \) be sufficiently small. Then \((\mathcal{H}, \phi)\)
contains a properly colored tight path \( A \) with \( |V(A)| \leq \frac{1}{4}(\gamma/2)^{k-2}n \) and such that for
any \( U \subset V \), \( |U| \leq \frac{1}{4}(\gamma/2)^{2k}n \), there is a properly colored tight path \( A_U \) on the vertex set
\( V(A) \cup U \) having the same end-paths as \( A \). (We will say that \( A \) can absorb \( U \).)
The next lemma sets aside a small set \( R \subset V \), called reservoir, which retains the properties of \((\mathcal{H}, \phi)\) and is, hence, useful in connecting properly colored paths into one almost hamiltonian tight cycle, without ‘interfering’ with the previously built part. We remark that we could have requested the maximum degree in each color within \( R \) to ‘scale’ appropriately, but this is not necessary, since our proof requires \( c \ll \gamma \) anyway. The role of the set \( Z \) is to prevent \( R \) from overlapping with \( V(A) \). The vertices of \( P \) which do not belong to its ends are called inner.

**Lemma 5** (Reservoir Lemma). Let \( \rho = \rho(\gamma) > 0 \) be sufficiently small and \( 0 < c \leq \frac{\rho \gamma}{6(2k-1)} \). Then there is a set of vertices \( R \subset V \) of size \( |R| \leq \rho n \) such that for any \( R' \subset R \) with \( |R'| \leq \frac{\gamma}{10} |R| \) and any two disjoint \((2k-2)\)-tuples \( \vec{v}, \vec{w} \in (V \setminus R)^{2k-2} \), both inducing properly colored tight paths in \((\mathcal{H}, \phi)\), there is in \((\mathcal{H}, \phi)\) a properly colored tight path of length at most \( 64k\gamma^{-2} \) with end-paths \( \vec{v} \) and \( \vec{w} \), whose all inner vertices belong to \( R \setminus R' \).

The third lemma allows us, given \( A \) and \( R \), to cover almost all vertices of the set \( V \setminus (V(A) \cup R) \) by a bounded number of disjoint, properly colored paths.

**Lemma 6** (Covering Lemma). For any \( \delta > 0 \), let \( c = c(\gamma, \delta) > 0 \) be sufficiently small and \( q = q(\gamma, \delta) \) be sufficiently large. Then there is a family \( \mathcal{P} \) of at most \( q \) vertex-disjoint properly colored tight paths in \((\mathcal{H}, \phi)\) covering all but at most \( 6n \) vertices of \( \mathcal{H} \).

The proof of Theorem 2 follows the classical outline of the absorbing method (see [28] or [26]).

**Proof of Theorem 2.** Set

\[
\lambda = \frac{1}{4}(\gamma/2)^{2k}, \quad \rho = \min\{\rho_5(7\gamma/8), \lambda/2\}, \quad \delta = \min\{\delta_0(\gamma/2), \lambda/2\}, \quad (2)
\]

where \( \rho_5(7\gamma/8) \) and \( \delta_0(\gamma/2) \) are given, respectively, by Lemma 5 and Lemma 6, with, respectively, \( 7\gamma/8 \) and \( \gamma/2 \) instead of \( \gamma \). Let \( c_i, i = 2, 3, 4, \) be a constant \( c \) yielded by, Lemma 4, Lemma 5, and Lemma 6 respectively, with appropriately altered \( \gamma \) – see above. Finally, let \( c = \min\{c_4, c_5/2, c_6/2\} \) and \( q = q(\gamma/2, \delta) \) be as in Lemma 6, and let \( n \) be sufficiently large.

Let \((\mathcal{H}, \phi)\) satisfy (1) and let \( A \) be a path provided by Lemma 4, that is, a path of length

\[
|V(A)| \leq \frac{1}{4}(\gamma/2)^{k-2}n \leq \frac{1}{8}\gamma n
\]

which can absorb any set \( U \) of up to \( \lambda n \) vertices.

Apply Lemma 5 to \( \mathcal{H}' = \mathcal{H} - V(A) \) with \( \gamma := 7\gamma/8 \) and \( \rho \) and \( c \) selected above. It is feasible as, setting \( n' = |V(\mathcal{H})| = n - |V(A)| \geq (1 - \gamma/8)n, \)

\[
\delta_{k-1}(\mathcal{H}') \geq \delta_{k-1}(\mathcal{H}) - |V(A)| \geq \left( \frac{1}{2} + \frac{7}{8}\gamma \right)n \geq \left( \frac{1}{2} + \frac{7}{8}\gamma \right)n'
\]

and

\[
\Delta_{k-1}(\mathcal{H}') \leq \Delta_{k-1}(\mathcal{H}_i) \leq cn \leq \frac{c_5}{2(1-\gamma/8)}n' \leq c_5n'.
\]

Let \( R \) be the resulting set of vertices of size \( |R| \leq \rho n \leq \lambda n'/2. \)
Set $\mathcal{H}'' = \mathcal{H}' - R$ and note that, since $\lambda \leq \frac{3}{4}\gamma$ and $n'' := |V(\mathcal{H}'')| \geq (1 - \gamma/2)n$,
\[\delta_{k-1}(\mathcal{H}'') \geq \left(\frac{1}{2} + \frac{7}{8}\gamma\right)n' - |R| \geq \left(\frac{1}{2} + \frac{7}{8}\gamma - \frac{1}{2}\lambda\right)n' \geq \left(\frac{1}{2} + \frac{\gamma}{2}\right)n''\]
and
\[\Delta_{k-1}(\mathcal{H}'') \leq \Delta_{k-1}(\mathcal{H}_i) \leq cn \leq \frac{c_0}{2(1 - \gamma/2)^2} n'' \leq c_0 n''.\]

Apply Lemma 6 to $(\mathcal{H}'', \phi)$ with $\delta = \delta(\gamma/2)$ defined as in (2), to obtain a family $\mathcal{P}$ of at most $q$ properly colored paths which cover all but at most $\delta n'' \leq \lambda n/2$ vertices of $\mathcal{H}''$. Let $W$ be the set of all vertices of $\mathcal{H}''$ not covered by the paths in $\mathcal{P}$. Set $\mathcal{P}^A = \mathcal{P} \cup \{A\}$.

To connect the paths from $\mathcal{P}^A$ into one cycle $C$, we apply Proposition 3 with $Q = R$ and $m = |\mathcal{P}^A| = q + 1$. To verify Statement I therein with $g = 64k\gamma^{-2} - (4k - 4)$, let $Q'$, $P_1$ and $P_2$ in $(\mathcal{H} - R, \phi)$ be as in the statement. We invoke Lemma 5 with $R' = Q'$ since, for $n$ large enough,
\[|Q'| \leq mg \leq (q + 1) \cdot 64k\gamma^{-2} \leq \frac{\gamma}{10}|R|.
\]
Indeed, the left-hand side of the second inequality above is a constant, while the right-hand side grows linearly with $n$. By Lemma 5, applied to one end-path $\vec{v}$ of $P_1$ and one end-path $\vec{w}$ of $P_2$, we see that Statement I of Proposition 3 holds and thus so does Statement II.

Let $U$ be the set of vertices of $\mathcal{H}$ not covered by $C$. Since $U \subset R \cup W$, we have
\[|U| \leq |R| + |W| \leq \rho n + \delta n \leq \lambda n.
\]
Therefore, a tight Hamilton cycle can be constructed in $\mathcal{H}$ by absorbing $U$ into the absorbing path $A$, which is a sub-path of the cycle $C$. Since the new path $A_U$ which is replacing $A$ in $C$ has the same end-paths as $A$, the obtained Hamilton cycle is properly colored as well. 

### 3.1 Connecting and Reservoir Lemmas

The following connecting lemma is a key tool in constructing both the absorbing path and the reservoir.

**Lemma 7** (Connecting Lemma). Let $(\mathcal{H}, \phi)$ satisfy (1). For any $c \leq \frac{\gamma}{4(2k - 2)}$, a subset $V' \subset V$ with $|V'| \leq \gamma n/10$, and any two disjoint $(2k - 2)$-tuples $\vec{v}, \vec{w} \in (V \setminus V')^{2k-2}$, both inducing properly colored tight paths in $(\mathcal{H}, \phi)$, there is a properly colored tight path $P$ in $(\mathcal{H}, \phi)$ of length at most $8(2k - 1)\gamma^{-2}$ with end-paths $\vec{v}$ and $\vec{w}$, whose all vertices belong to $V \setminus V'$.

Lemma 7 will be deduced from an adaptation of Lemma 2.4 in [28]. To state this adaptation, consider a directed $k$-graph $\overrightarrow{\mathcal{H}}$ where each edge is a $k$-tuple (a sequence of vertices), rather than a set. For a sequence of vertices $\overrightarrow{v}$ and a vertex $u$ disjoint from $\overrightarrow{v}$, the two concatenations of $u$ and $\overrightarrow{v}$ will be denoted by $\overrightarrow{uv}$ and $\overrightarrow{vu}$. Given a directed $k$-graph $\overrightarrow{\mathcal{H}}$ and a $(k - 1)$-tuple $\overrightarrow{v}$ of its vertices, set
\[d^+(\overrightarrow{v}) = |\{u \in V : \overrightarrow{vu} \in \overrightarrow{\mathcal{H}}\}| \quad \text{and} \quad d^-(\overrightarrow{v}) = |\{u \in V : \overrightarrow{vu} \in \overrightarrow{\mathcal{H}}\}|.
\]
Furthermore, define $d^+(v) = \min(d^+(v), d^-(v))$ and call $v$ extendable if $d^+(v) > 0$ or $d^-(v) > 0$. A brief glance at the proof of Lemma 2.4 in [28] reveals that it goes through for any directed $k$-graph with high values of $d^+(v)$ for every extendable $(k-1)$-tuple. Consequently, the following version of that lemma is also true.

**Lemma 8** ([28]). Let $\gamma > 0$ and $\overrightarrow{H}$ be a directed $n$-vertex $k$-graph. Assume that for any extendable $(k-1)$-tuple $v$ of vertices of $\overrightarrow{H}$, $d^+(v) \geq (\frac{1}{2} + \gamma)n$. Then, for every pair of disjoint, extendable $(k-1)$-tuples $v$ and $w$, there is a tight path in $\overrightarrow{H}$ of length at most $2k/\gamma^2$ between them.

**Proof of Lemma 7.** Given $(H, \phi)$, define an auxiliary directed, $(2k-1)$-uniform hypergraph $\overrightarrow{H} = \overrightarrow{H}(\phi, V')$ with vertex set $V \setminus V'$ and the edge set consisting of sequences of distinct vertices $(v_1, \ldots, v_{2k-1}) \in (V \setminus V')^{2k-1}$ such that the sets

$$\{v_1, \ldots, v_k\}, \ldots, \{v_k, \ldots, v_{2k-1}\}$$

are edges of $H$ with distinct colors. Equivalently, the edges of $\overrightarrow{H}$ correspond to the properly colored tight paths of length $2k - 1$ in $(H, \phi)$ with a fixed direction.

By this construction, one cannot exclude the possibility that a $(2k-2)$-tuple $v$ is not a segment of an edge of $\overrightarrow{H}$, and thus $\max(d^+(v), d^-(v)) = 0$ (for instance, if $v$ does not span a path in $H$). However, it turns out that $\overrightarrow{H}$ satisfies the assumptions of Lemma 8 with $k := 2k - 1$ and $\gamma := \gamma/2$.

Indeed, let $v$ be a $(2k-2)$-tuple in $\overrightarrow{H}$ with $\max(d^+(v), d^-(v)) > 0$. This means, in particular, that $\overrightarrow{v}$ spans in $(H, \phi)$ a properly colored path. Let $U$ be the set of vertices $u \in V \setminus V'$ such that $\{u, v_k, \ldots, v_{2k-2}\} \in H$ and $\phi(u, v_k, \ldots, v_{2k-2})$ does not appear on the consecutive $k$-tuples of $v$. Clearly, $\overrightarrow{vu}$ is an edge of $\overrightarrow{H}$ for any $u \in U$, so $d^-(v) = |U|$. Using the assumed bounds $\delta_k(H) \geq (1/2 + \gamma)n$ and $\Delta_{k-1}(H) \leq cn$, we infer that

$$d^-(v) = |U| \geq \left(\frac{1}{2} + \gamma - \frac{\gamma}{10} - (k-1)c\right)n \geq \left(\frac{1}{2} + \frac{\gamma}{2}\right)n,$$

since $c \leq \frac{\gamma}{3(k-1)}$ and $|V'| \leq \gamma n/10$. Analogously, $d^+(v) \geq \left(\frac{1}{2} + \frac{\gamma}{10}\right)n$.

To deduce Lemma 7, consider vertex-disjoint $(2k-2)$-tuples $\overrightarrow{v}, \overrightarrow{w} \in (V \setminus V')^{2k-2}$ spanning properly colored paths in $(H, \phi)$. Applying Lemma 8 to $\overrightarrow{H}$ with $k := 2k - 1$ and $\gamma = \gamma/2$, we get a path $P$ in $\overrightarrow{H}$ of length at most $8(2k-1)(\gamma)^{-2}$ between $\overrightarrow{v}$ and $\overrightarrow{w}$. It remains to check that $P$ corresponds to a properly colored tight path in $H$. Indeed, any two intersecting $k$-tuples contained in $P$ are also contained in some edge of $\overrightarrow{H}$, so they carry distinct colors by definition of $\overrightarrow{H}$.

**Proof of Lemma 5.** Let $R$ be a set of vertices guaranteed by Proposition 1 with $p = \frac{2}{3}\rho$, $U_S = \mathcal{N}_S(S)$ for $S \in \binom{V}{k-1}$ and $\alpha_S = \frac{1}{2} + \gamma$. In particular, for large $n$, $\frac{1}{n} \rho n \leq |R| \leq \rho n$ and, for each $S \in \binom{V}{k-1}$,

$$|\mathcal{N}_S(S) \cap R| \geq \left(\frac{1}{2} + \frac{2\gamma}{3}\right)|R|.$$
We claim that $R$ fulfills the conclusion of Lemma 5. To see this, consider a set $R' \subset R$, $|R'| \leq \gamma |R|/10$, and two disjoint $(2k - 2)$-tuples of vertices $\vec{v}, \vec{w} \in (V \setminus R)^{2k-2}$ inducing properly colored tight paths in $(H, \phi)$. Let
\[ R = H[(R \cup \{\vec{v}, \vec{w}\})], \]
where $\vec{v}, \vec{w}$ are viewed as sets. Set $r = |V(R)|$ and note that $\frac{1}{2} \rho n \leq |R| \leq r \leq |R| + 4k$. Thus,
\[ \delta_{k-1}(R) \geq \delta_{k-1}(H[R]) \geq \left(\frac{1}{2} + \frac{2}{3}\gamma\right) |R| \geq \left(\frac{1}{2} + \frac{\gamma}{2}\right) r \quad \text{and} \quad \Delta_{k-1}(R_i) \leq cn \leq \frac{2c}{\rho}r. \]
Using Lemma 7 with $H := R$, $\phi := \phi|_R$, $\gamma := \gamma$, $c := \frac{2c}{\rho} \leq \gamma/(3k)$, and $V' = R'$, we get the desired path of length at most $64k\gamma^{-2}$ between $\vec{v}$ and $\vec{w}$. \[ \square \]

### 3.2 Absorbing path

For a vertex $v \in V$, a $v$-absorber is a $(4k - 4)$-tuple of vertices $\vec{w} = (w_1, \ldots, w_{4k-4})$ spanning a tight path $T$ in $H$ such that the sequence $(w_1, \ldots, w_{2k-2}, v, w_{2k-1}, \ldots, w_{4k-4})$ spans another tight path $T_v$ in $H$. If both $T$ and $T_v$ are properly colored paths, we call the $(4k - 4)$-tuple $\vec{w}$ a properly colored $v$-absorber. To construct an absorbing path, we first need to show that for every $v$ there are many, that is, $\Theta(n^{4k-4})$, properly colored $v$-absorbers.

**Lemma 9.** Let $c \leq 4(\gamma/2)^k(3k)^{-2}$. For every $v \in V(H)$ there are at least $4(\gamma/2)^k n^{4k-4}$ properly colored $v$-absorbers in $(H, \phi)$.

**Proof.** We begin by counting $v$-absorbers. A $v$-absorber $\vec{w} = (w_1, w_2, \ldots, w_{4k-4})$ can be constructed by sequentially selecting the vertices $w_i$, $i = 1, \ldots, 4k - 4$, and each time counting the number of ways to do it. Initially, we will be allowing repetitions, $w_i = w_j$, as well as choices $w_i = v$. Those will be discarded at the end.

- For $i = 1, \ldots, k - 1$, there are no constraints on $w_i$, so the number of choices of $w_i$ is $n$.
- For $i = k, \ldots, 2k - 3$, the $k$-tuple $\{w_{i-k+1}, \ldots, w_i\}$ must form an edge in $H$, so the number of choices of $w_i$ is at least $\delta_{k-1}(H) \geq (1/2 + \gamma)n$.
- For $i = 2k - 2, \ldots, 3k - 3$, vertex $w_i$ must belong to two edges, $\{w_{i-k+1}, \ldots, w_i\}$ and $\{v, w_{i-k+2}, \ldots, w_i\}$, so the number of choices of $w_i$ is at least $2(1/2 + \gamma)n - n = 2\gamma n$.
- For $i = 3k - 2, \ldots, 4k - 4$, the $k$-tuple $\{w_{i-k+1}, \ldots, w_i\}$ must form an edge, so the number of choices of $w_i$ is, again, at least $(1/2 + \gamma)n > n/2$.

Altogether, as the number of choices with vertex repetitions is $O(n^{4k-5})$, for sufficiently large $n$, the total number of $v$-absorbers in $H$ is at least
\[ n^{k-1} ((1/2 + \gamma)n)^{2k-3} (2\gamma n)^k - O(n^{4k-5}) \geq 8 \left(\frac{\gamma}{2}\right)^k n^{4k-4}. \]
Now we have to subtract the number of \( v \)-absorbers with a color conflict. Let \( T \) and \( T_v \) be the paths as in the definition of a \( v \)-absorber \( \bar{w} \). Our task is to count the \( v \)-absorbers \( \bar{w} \) in which either \( T \) or \( T_v \) contains two intersecting edges with the same color.

Let us estimate from above the number of \( v \)-absorbers in which we have \( \phi(e) = \phi(f) \) for a given pair \( (e,f) \) of edges in \( T \). Let \( w_{ij} \in f \setminus e \). There are no more than \( n^{4k-5} \) choices of the vertices \( w_{ij}, i = 1, \ldots, 4k-4, i \neq j \). However, since \( \Delta_{k-1}(\mathcal{H}_{\phi(e)}) \leq cn \), and we want \( \phi(e) = \phi(f) \), vertex \( w_{ij} \) can be chosen in at most \( cn \) ways. Altogether, this gives us at most \( cn^{4k-4} \) \( v \)-absorbers. By the same token, the number of \( v \)-absorbers in which we have \( \phi(e) = \phi(f) \) for a given pair \( (e,f) \) of edges in \( T_v \) is also at most \( cn^{4k-4} \).

By the union bound over all possible pairwise edge intersections in \( T \) and in \( T_v \) and using the assumption on \( c \), the total number of properly colored \( v \)-absorbers is at least

\[
8 \left( \frac{\gamma}{2} \right)^k n^{4k-4} - 2 \cdot \binom{3k-2}{2} cn^{4k-4} \geq 4 \left( \frac{\gamma}{2} \right)^k n^{4k-4}.
\]

**Proof of Lemma 4.** For each \( v \in V \), let \( A_v \) be the family of all properly colored \( v \)-absorbers in \((\mathcal{H}, \phi)\). Based on Lemma 9, we apply Proposition 2 to the family \( \{A_v : v \in V\} \) with \( m = n, t = 4k - 4 \), and

\[
\alpha = \frac{(\gamma/2)^k}{(4k-4)^2}.
\]

As an outcome, we obtain a family \( \mathcal{F} \) of vertex disjoint properly colored paths of length \((4k-4)\) with \( |\mathcal{F}| \leq an \) and such that for each vertex \( v \) there are at least

\[
\alpha^2(4k-4)^2n/4 = \frac{1}{4} \left( \frac{\gamma}{2} \right)^{2k} n
\]

properly colored \( v \)-absorbers in \( \mathcal{F} \).

To connect the paths from \( \mathcal{F} \) into one path \( A \), we apply Proposition 3 with \( Q = V \setminus \bigcup_{F \in \mathcal{F}} V(F) \) and \( m = |\mathcal{F}| \). To verify Statement I therein, we invoke Lemma 7 with \( Z = Q' \cup \bigcup_{F \in \mathcal{F}} V(F) \), so we set \( g = 8(2k-11)\gamma^{-2} - (4k-4) \). Note that

\[
|Z| \leq |Q'| + |\mathcal{F}|(4k-4) \leq \alpha n \cdot \frac{8(2k-1)}{\gamma^2} \leq \frac{16k}{\gamma^2} \alpha n = \frac{\gamma^{k-2}k}{2k(k-1)^2} n < 10n.\gamma
\]

By Lemma 7, applied to one end-path \( \bar{v} \) of \( P_1 \) and one end-path of \( \bar{w} \) \( P_2 \), we see that Statement I of Proposition 3 holds and thus Statement II follows. The obtained path \( A \) has length

\[
|V(A)| \leq \alpha n \cdot \frac{8(2k-1)}{\gamma^2} \leq \frac{\gamma^{k-2}k}{2k(k-1)^2} n < \frac{1}{4} \left( \frac{\gamma}{2} \right)^{k-2} n,
\]

as required.

Recall that, by (3), for each vertex \( v \in V \), the path \( A \) contains at least \( \frac{1}{4} \left( \frac{\gamma}{2} \right)^{2k} n \) vertex-disjoint properly colored \( v \)-absorbers. Therefore, one can absorb into \( A \) any set of vertices \( U \) of size \(|U| \leq \frac{1}{4} \left( \frac{\gamma}{2} \right)^{2k} n \), one by one, obtaining a new properly colored path.
follows that in [28], Section 4. The proof in [28] relied on five
below is an analogue of Claim 4.1 in [28].

\[ A_U. \] After absorbing a vertex \( v \) into \( A \) via a \( v \)-absorber \( \overline{w} \), only the edges containing \( w_k, \ldots, w_{3k-3} \) are reconfigured, so the new path is properly colored, has the same end- paths, and all the other absorbers remain unaffected. In particular, the final path \( A_U \) has the same end-paths as \( A \).

\[ \square \]

3.3 Covering by long paths

Our proof of Lemma 6 follows that in [28], Section 4. The proof in [28] relied on five technical claims and only the first two of them require, due to the coloring, certain modifications. Claim 1 below is an analogue of Claim 4.1 in [28].

Claim 1. Let \((H, \phi)\) be a colored \( k \)-partite graph with at most \( m \) vertices in each part and at least \( dm^k \) edges. If \( \Delta_{k-1}(H_i) \leq \frac{d}{2}m \) for all \( i \in \mathbb{N} \), then \( H \) contains a properly colored tight path on at least \( \frac{m}{2} \) vertices.

Proof. Denote the parts of \( H \) by \( V_1, \ldots, V_k \), and call a set \( S \subset V(H) \) relevant if \( |S| = k-1 \) and \( |S \cap V_i| \leq 1 \) for \( i \in [k] \). Note that there are at most \( km^{k-1} \) relevant sets. We start by preprocessing \( H \) as follows. If there is a relevant set \( S \) whose degree in the current hypergraph is smaller than \( \frac{dm}{k^2} \), delete all edges containing \( S \). Repeat this step until all relevant sets have degree either 0 or at least \( \frac{dm}{k} \). Denote the resulting hypergraph by \( H' \). Observe that the number of deleted edges is strictly smaller than \( dm \cdot km^{k-1} = dm^k \) since the edges containing each relevant set were removed at most once. Hence \( H' \) is non-empty.

Let \( P \) be the longest properly colored tight path in \( H' \). Denote its vertices by \( v_1, \ldots, v_{\ell} \) in that order, and observe that since \( P \) is tight, any part \( V_i \) contains precisely every \( k \)-th vertex of \( P \). Let \( S = \{v_1, \ldots, v_{k-1}\} \), and let \( C \) be the set of colors appearing on the \( k-1 \) edges of \( P \) intersecting \( S \). Let \( U \) be the set of vertices \( v \) for which \( e_v := S \cup \{v\} \) is an edge of \( H' \) with \( \phi(e_v) \notin C \). By construction of \( H' \) and the assumption on \( \phi \),

\[ |U| \geq \frac{dm}{k} - \frac{(k-1)dm}{2k^2} \geq \frac{dm}{2k}. \]

On the other hand, by maximality of \( P \), \( U \subset V(P) \). Notice that there is \( j \in [k] \) such that \( U \subset V_j \), and thus, each vertex \( v \in U \) has \( k-1 \) predecessors on \( P \) not belonging to \( U \). Therefore \( |V(P)| \geq k|U| \geq \frac{dm}{2} \), as required. \[ \square \]

Our next result is a suitably modified version of Claim 4.2 in [28].

Claim 2. For all \( 0 < \varepsilon < d < 1 \), every \( \varepsilon \)-regular, equitable \( k \)-partite graph \( H \) on \( n \) vertices with density \( d_H \geq d \) and with \( \Delta_{k-1}(H_i) \leq \frac{dm}{2k}n \) for all \( i \in \mathbb{N} \), contains a family \( \mathcal{P} \) of vertex-disjoint properly colored paths such that

\[ \begin{align*}
&\text{for each } P \in \mathcal{P} \text{ we have } |V(P)| \geq \frac{\varepsilon(d-\varepsilon)n}{2k} \quad \text{and} \quad \sum_{P \in \mathcal{P}} |V(P)| \geq (1-2\varepsilon)n.
\end{align*} \]

Proof. Let \( \mathcal{P} \) be a largest family of vertex-disjoint properly colored paths with \( |V(P)| \geq \varepsilon(d-\varepsilon)n/(2k) \) for each \( P \in \mathcal{P} \). Suppose that \( \sum_{P \in \mathcal{P}} |V(P)| < (1-2\varepsilon)n \). The proof goes along the lines of the one in [28] (with \( \alpha := d \) and \( Q := \mathcal{P} \), except for the very
end, where we focus on a sub-$k$-graph $\hat{\mathcal{H}}$ with $m \geq \varepsilon n/k$ vertices in each partition class which is vertex disjoint from all paths in $\mathcal{P}$. Here we have to note that

$$\Delta_{k-1}(\hat{\mathcal{H}}_i) \leq \Delta_{k-1}(\mathcal{H}_i) \leq \frac{\varepsilon d}{2k^3} n = \frac{d}{2k^2} (\varepsilon n/k) \leq \frac{d}{2k^2} m.$$ 

Moreover, by the $\varepsilon$-regularity of $\mathcal{H}$, we have $|\hat{\mathcal{H}}| \geq (d - \varepsilon)m^k$, so Claim 1 can be applied to $\hat{\mathcal{H}}$, producing a properly colored path in $\mathcal{H}$ of length at least $\varepsilon(d - \varepsilon)n/(2k)$, vertex disjoint from $\mathcal{P}$. This yields a contradiction with the maximality of $\mathcal{P}$ (for details see [28].)

The remaining three claims from the proof in [28], combined together, imply the existence of a vertex-decomposition of our hypergraph into $\varepsilon$-regular $k$-tuples (plus some leftover vertices). This was achieved by using the Weak Hypergraph Regularity Lemma. Notice that this statement concerns hypergraphs only, making no mention of the coloring. Given an $\varepsilon$-regular partition $(V_1, \ldots, V_t)$ of a $k$-graph $\mathcal{H}$ and a real $d > 0$, we denote by $\mathcal{K}(V_1, \ldots, V_t; d, \varepsilon)$ the $k$-graph on vertex set $[t]$ where edges correspond to $\varepsilon$-regular $k$-tuples $(V_{i_1}, \ldots, V_{i_k})$ with density at least $d$. Our last claim corresponds to Claims 4.3-4.5 in [28].

**Claim 3.** Given $\gamma > 0$ and sufficiently small $\varepsilon = \varepsilon(\gamma) > 0$, there exist integers $T_0$ and $n_0$ such that the following holds. Any $n$-vertex $k$-graph $\mathcal{H}$, $n \geq n_0$, with $\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2} + \gamma) n$ contains an equitable partition $V = V_1 \cup \cdots \cup V_t$, $t \leq T_0$, such that the corresponding $k$-graph $\mathcal{K} = \mathcal{K}(V_1, \ldots, V_t; \frac{1}{k}, \varepsilon)$ possesses a matching covering at least $(1 - 2k\eta) t$ vertices, where $\eta = k(\sqrt{\varepsilon})^{1/(k-1)}$. $\square$

**Proof of Lemma 6.** For any $\gamma > 0$ and $\delta > 0$, let $\varepsilon$ satisfy $2\varepsilon + 4k\eta \leq \delta$. Further, let $T_0 = T_0(\varepsilon)$ and $n_0 = n_0(\varepsilon)$ be as in Claim 3 and set

$$c = \frac{\varepsilon}{16k^2 T_0} \quad \text{and} \quad q = \frac{16T_0}{\varepsilon}.$$ 

Let $\mathcal{H}$ be an $n$-vertex $k$-graph, $n \geq n_0$, with

$$\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2} + \gamma) n \quad \text{and} \quad \Delta_{k-1}(\mathcal{H}_i) \leq cn.$$ 

By Claim 3, there is an equitable partition $V = V_1 \cup \cdots \cup V_t$, $t \leq T_0$, such that the corresponding $k$-graph $\mathcal{K} = \mathcal{K}(V_1, \ldots, V_t; \frac{1}{k}, \varepsilon)$ possesses a matching $\mathcal{M}$ covering at least $(1 - 2k\eta) t$ vertices of $\mathcal{K}$. Note that

$$n/2T_0 \leq |n/t| \leq |V_i| \leq \lceil n/t \rceil.$$ 

For each $e \in \mathcal{M}$, let $\mathcal{H}_e$ be the sub-$k$-graph of $\mathcal{H}$ induced by the partition sets constituting the edge $e$ of $\mathcal{K}$. We have

$$\Delta_{k-1}((\mathcal{H}_e)_i) \leq \Delta_{k-1}(\mathcal{H}_i) \leq \frac{\varepsilon}{8k^2} \left( \frac{n}{2T_0} \right) \leq \frac{\varepsilon}{8k^3} (k|n/t|) \leq \frac{\varepsilon}{8k^3} |V(\mathcal{H}_e)|.$$ 

13
Thus, by Claim 2 applied to $\mathcal{H} := \mathcal{H}_e$ with $d = 1/4$, there is a family $\mathcal{P}_e$ of vertex-disjoint properly colored paths such that

for each $P \in \mathcal{P}_e$ we have $|V(P)| \geq \frac{\varepsilon(1/4 - \varepsilon)n}{2T_0}$ and $\sum_{P \in \mathcal{P}_e} |V(P)| \geq (1 - 2\varepsilon)|V(\mathcal{H}_e)|$.

Applying the same argument to each $e \in M$ gives a collection of paths $\mathcal{P}$ which cover all but at most $(2\varepsilon + 4k\eta)n \leq \delta n$ vertices of $\mathcal{H}$. Clearly,

$$|\mathcal{P}| \leq \frac{2T_0}{\varepsilon(1/4 - \varepsilon)} \leq \frac{16T_0}{\varepsilon}.$$

\[\square\]

4 Hamilton $(k, \ell)$-cycles with co-degree conditions

This section is devoted to the proof of Theorem 3. Recall that given a $k$-graph $\mathcal{H}$ and a coloring $\phi$ of its edges, $\mathcal{H}_i = \{e \in \mathcal{H} : \phi(e) = i\}$. Throughout this section, given integers $1 \leq \ell < k/2$, $(\mathcal{H}, \phi)$ is an $n$-vertex colored $k$-graph on vertex set $V$ with $n$ sufficiently large and divisible by $k - \ell$, and such that

$$\Delta_\ell(\mathcal{H}_i) \leq cn^{k-\ell} \text{ for all } i \in \mathbb{N},$$

where $c > 0$ is sufficiently small. As all our statements below are about $(k, \ell)$-paths and $(k, \ell)$-cycles, we will often call them just paths and cycles. As in previous section, we will first state three lemmas from which Theorem 3 follows.

Observe that, since $2\ell < k$, the degree of any vertex on a path or a cycle is either one or two. Recall that for $\ell < k/2$, the $\ell$-ends of a $(k, \ell)$-path are defined as the sets of the first and the last $\ell$ vertices of the path.

Lemma 10 (Absorbing Lemma). There exist constants $a = a(k, \ell) < 1$ and $b = b(k, \ell) < 1$ such that for every $\lambda > 0$ and $c \leq b\lambda^5$, if $(k - \ell)n$, $\delta_{k-1}(\mathcal{H}) \geq \lambda n$ and (4) holds, then there is a properly colored $(k, \ell)$-path $A$ in $\mathcal{H}$ with $|V(A)| \leq \lambda^5n$ such that for every $U \subset V$ with $|U| \leq a(k - \ell)\lambda^{10}n$ and $(k - \ell)||U|$, there is a properly colored $(k, \ell)$-path $P_U$ in $\mathcal{H}$ with $V(P_U) = V(A) \cup U$ and with the same end-edges and $\ell$-ends as $A$. (We will say that $A$ can absorb $U$.)

Lemma 11 (Reservoir Lemma). For every $0 < \rho < 1$, $d > 0$, and $c \leq \frac{(d/2)^3}{2\text{ln}(\rho)}(\rho/2)^{k-\ell}$, if $\delta_{k-1}(\mathcal{H}) \geq dn$ and (4) holds, then there is a set $R \subset V$ with $|R| \leq \rho n$ such that for any two disjoint $\ell$-sets $X, Y \subset V \setminus R$, any two colors $c_X, c_Y$, and any subset $R' \subset R$, $|R'| \leq d|R|/20$, there exists a $3$-edge properly colored $(k, \ell)$-path $P = X, v_1 \ldots v_{3k-4\ell}, Y$ connecting $X$ and $Y$, with $\{v_1 \ldots v_{3k-4\ell}\} \subset R \setminus R'$ and such that $\phi(X, v_1 \ldots v_{k-\ell}) \neq c_X$ and $\phi(v_{2k-3\ell+1} \ldots v_{3k-4\ell}, Y) \neq c_Y$.

Lemma 12 (Covering Lemma). For every $\gamma > 0$ and $\delta > 0$, let $c = c(\gamma, \delta) > 0$ be sufficiently small and $q = q(\gamma, \delta)$ be sufficiently large. If $\delta_{k-1}(\mathcal{H}) \geq \left(\frac{1}{2(k-\ell)} + \gamma\right)n$ and (4) holds, then there is a family $\mathcal{P}$ of at most $q$ vertex-disjoint properly colored $(k, \ell)$-paths in $(\mathcal{H}, \phi)$ covering all but at most $\delta n$ vertices of $\mathcal{H}$.
Proof of Theorem 3. Let $\gamma > 0$ be given and let $a$ and $b$ be the constants provided by Lemma 10. Set
\[
\lambda = (\gamma/4)^{1/5}, \quad \rho = \delta = \frac{1}{2}a\lambda^{10} = \frac{a\gamma^2}{32}, \quad d = \frac{1}{2(k-\ell)}, \quad q = q_{12}(\gamma/2, \delta),
\]
and
\[
c = \min \left( b\lambda^5, \frac{1}{217k!}, \left( \frac{d}{2} \right)^3 \left( \frac{\rho}{2} \right)^{k-\ell}, \frac{1}{2}c_{12}(\gamma/2, \delta) \right),
\]
where the subscript 12 means that the constant comes from Lemma 12. Note that the second and third ingredients of the minimum above incorporate an extra margin to accommodate the forthcoming estimates.

Let $(\mathcal{H}, \phi)$ be an $n$-vertex colored $k$-graph on vertex set $V$ with $n$ sufficiently large and divisible by $k-\ell$. Assume that $\delta_{k-1}(\mathcal{H}) \geq \left( \frac{1}{2(k-\ell)} + \gamma \right) n$. Since, for sufficiently small $\gamma = \gamma(k, \ell)$ we have $\frac{1}{2(k-\ell)} + \gamma \geq \frac{1}{2(k-\ell)} \geq \lambda$, we may apply Lemma 10 to $(\mathcal{H}, \phi)$ to find an absorbing path $A$ with $|V(A)| \leq \lambda^5n = \gamma n/4$ which can absorb any set $U$ of up to $a(k-\ell)\lambda^{10} n$ vertices.

Next, let $R \subset V \setminus V(A)$ be the set given by Lemma 11 applied to $\mathcal{H}' = \mathcal{H} - V(A)$ and to the coloring $\phi'$ induced in $\mathcal{H}'$ by $\phi$. Such an application is feasible since, setting $n' = |V(\mathcal{H}')| = n - |V(A)| \geq (1-\gamma/4)n$, we have
\[
\delta_{k-1}(\mathcal{H}') \geq \delta_{k-1}(\mathcal{H}) - |V(A)|n \geq \frac{n}{2(k-\ell)} \geq dn'
\]
and
\[
\Delta_{\ell}(\mathcal{H}_i') \leq \Delta_{\ell}(\mathcal{H}_i) \leq c n^{k-\ell} \leq \frac{c}{(1-\gamma/4)^{k-\ell}(n')^{k-\ell}} \leq \frac{1}{216k!^3} \left( \frac{d}{2} \right)^3 \left( \frac{\rho}{2} \right)^{k-\ell} (n')^{k-\ell}.
\]

Further, let $\mathcal{H}'' = \mathcal{H}' - R$ and let $\phi''$ be the coloring induced in $\mathcal{H}''$ by $\phi$. Set $n'' := |V(\mathcal{H}'')|$. Since $\gamma/4 + a\gamma^2/32 \leq \gamma/2$,
\[
\delta_{k-1}(\mathcal{H}'') \geq \delta_{k-1}(\mathcal{H}) - (|V(A)| + |R|)n \geq \left( \frac{1}{2(k-\ell)} + \gamma/2 \right) n''
\]
and
\[
\Delta_{\ell}(\mathcal{H}_i'') \leq \Delta_{\ell}(\mathcal{H}_i) \leq cn^{k-\ell} \leq \frac{c_{12}(\gamma/2, \delta)}{2(1-\gamma/2)^{k-\ell}} (n'')^{k-\ell} \leq c_{12}(\gamma/2, \delta)(n'')^{k-\ell}.
\]

Hence, we can apply Lemma 12 to the colored hypergraph $(\mathcal{H}'', \phi'')$ with $\gamma := \gamma/2$ to obtain a family $\mathcal{P}$ of at most $q$ properly colored paths which cover all but at most $\frac{1}{2}a\lambda^{10} n$ vertices of $\mathcal{H}''$. Let $W$ be the set of vertices of $\mathcal{H}''$ not covered by any of the paths in $\mathcal{P}$. Include the absorbing path $A$ into $\mathcal{P}$ to get a family of paths $\mathcal{P}^A = \mathcal{P} \cup \{A\}$.

To connect the paths from $\mathcal{P}^A$ into one cycle $C$, we apply Proposition 3 with $Q = R$ and $m = |\mathcal{P}^A| = q + 1$. To verify Statement I therein with $g = 3k - 4\ell$, let $Q'$, $P_1$ and $P_2$ in be as in the statement. We invoke Lemma 11 with $R' = Q'$ since, for $n$ large enough,
\[
|Q'| \leq mg \leq (q + 1)(3k - 4\ell) \leq \frac{d}{20}|R|.
\]

15
By Lemma 11, applied to one \( \ell \)-end \( X \) of \( P_1 \) and one \( \ell \)-end \( Y \) of \( P_2 \), we see that Statement I of Proposition 3 holds and thus so does Statement II.

Let \( U \) be the set of vertices of \( V \) not covered by \( C \). As \(|V(C)|\) is divisible by \( k - \ell \), so is \(|U|\). Since

\[
|U| \leq |W| + |R| \leq \delta n + \rho n \leq a\lambda^{10}n \leq a(k - \ell)\lambda^{10}n,
\]

\( U \) can be absorbed into \( A \) by replacing \( A \) with a properly colored path \( P_U \). Since \( P_U \) and \( A \) have the same \( \ell \)-ends, the absorption extends \( C \) to a Hamilton cycle in \( H \). Since, in addition, \( P_U \) and \( A \) have the same end-edges, the obtained Hamilton cycle remains properly colored. \( \square \)

### 4.1 Connecting and Reservoir Lemmas

In this subsection we prove first a simple connecting lemma which is used in the proofs of both the Absorbing and the Reservoir Lemma. It says, roughly, that any two disjoint \( \ell \)-sets can be connected via a short, properly colored path which avoids a given relatively small set of vertices. In fact, although we do not need it here, we show that there are many such paths.

**Lemma 13** (Connecting Lemma). For every \( \kappa > 0 \) and \( c \leq 10^{-3}k^3!^{-3} \), let a colored \( k \)-graph \( (H, \phi) \) be given with \( \delta_{k-1}(H) \geq \kappa n \) and \( \Delta_{\ell}(H_i) \leq cn^{k-\ell} \). Further, let \( V' \subseteq V \), \(|V'| \leq \kappa n/10 \). Then, for sufficiently large \( n \), for any pair of disjoint \( \ell \)-sets \( X, Y \subseteq V \setminus V' \) and any two colors \( c_X, c_Y \), there exist at least \( \kappa n^3n^{3k-4\ell}/(54k!) \) \( 3 \)-edge properly colored paths \( Xv_1 \cdots v_{3k-4\ell}Y \), where \( v_1, \ldots, v_{3k-4\ell} \notin V' \), such that, in addition, \( \phi(X, v_1, \ldots, v_{k-\ell}) \neq c_X \) and \( \phi(v_{2k-3\ell+1}, \ldots, v_{3k-4\ell}, Y) \neq c_Y \).

**Proof.** We will first estimate from below the number of paths \( Xv_1 \cdots v_{3k-4\ell}Y \) with no regard to coloring. Then we will subtract an upper bound on the number of those among them which have a color conflict. Throughout we assume that \( n \) is large enough for all the estimates to hold.

Note that by the assumption \( \delta_{k-1}(H) \geq \kappa n \), every set \( Z \subseteq V \) of size \(|Z| = s < k \) is contained in at least

\[
\binom{n-s}{k-1-s} \frac{\kappa n}{k-s} \geq \frac{\kappa n^{k-s}}{2k!}.
\]

edges of \( H \). Now, fix two disjoint \( \ell \)-sets \( X, Y \subseteq V \setminus V' \). By (5), there are at least \( \frac{\kappa n^{k-\ell}}{3k!} \) edges \( X' \) of \( H \) such that \( X \subseteq X' \) and \( X' \cap (V' \cup Y) = \emptyset \). Similarly, there are at least \( \frac{\kappa n^{k-\ell}}{3k!} \) edges \( Y' \) of \( H \) such that \( Y \subseteq Y' \) and \( Y' \cap (V' \cup X') = \emptyset \). For a fixed pair \((X', Y')\), select arbitrarily subsets \( Z_X \subseteq X' \setminus X \) and \( Z_Y \subseteq Y' \setminus Y \) of size \(|Z_X| = |Z_Y| = \ell \). Set \( Z = Z_X \cup Z_Y \) and notice that \(|Z| = 2\ell < k \). Thus, there are at least \( \frac{\kappa n^{k-2\ell}}{3k!} \) edges \( T \) of \( H \) such that \( Z \subseteq T \) and \( T \cap (X' \cup Y') = \emptyset \). Altogether, there are at least \( \frac{\kappa n^{3k-4\ell}}{27k!} \) choices of \((X', Y', T)\) and each of them corresponds to at least one \( 3 \)-edge path between \( X \) and \( Y \).

Now we count the paths with color conflicts. In addition to \( \phi(X, v_1, \ldots, v_{k-\ell}) = c_X \) and \( \phi(v_{2k-3\ell+1}, \ldots, v_{3k-4\ell}, Y) = c_Y \), there are two more potential conflicts, corresponding to the two pairs of edges intersecting, respectively, at \( Z_X \) and at \( Z_Y \). Using the
assumption $\Delta_\ell(H_i) \leq cn^{k-\ell}$ each one of them disqualifies at most $cn^{3k-4\ell}$ paths. We conclude that the number of required paths is at least
\[
\frac{r^3n^{3k-4\ell}}{27k^4} - 4cn^{3k-4\ell} \geq \frac{k^3n^{3k-4\ell}}{54k^4},
\]
by our assumption on $c$.

Proof of Lemma 11. Let $R$ be a set of vertices guaranteed by Proposition 1 with $p = \frac{2}{3}\rho$, $U_S = N_H(S)$ for $S \in \binom{V}{k-1}$ and $\alpha_S = d$. In particular, for large $n$, $\frac{1}{2}\rho n \leq |R| \leq \rho n$ and, for each $S \in \binom{V}{k-1}$,
\[
|N_H(S) \cap R| \geq \frac{2}{3}d|R|.
\]
We claim that $R$ fulfils the conclusion of Lemma 11. To see this, consider a set $R' \subset R$, $|R'| \leq d|R|/20$, two disjoint $\ell$-tuples of vertices $X, Y \subset V \setminus V'$ and colors $c_X, c_Y$. Let $R = H[(R \cup X \cup Y)$, set $r = |V(R)|$ and note that $\frac{1}{2}\rho n \leq |R| \leq r \leq |R| + 2\ell$. Thus, for large $n$,
\[
\delta_{k-1}(R) \geq \frac{2}{3}d|R| \geq \frac{1}{2}dr \quad \text{and} \quad \Delta_{k-\ell}(R_i) \leq cn^{k-\ell} \leq c \left(\frac{2}{\rho}\right)^{k-\ell} r^{k-\ell}.
\]
Using Lemma 13 with $H := R$, $\phi := \phi|_R$, $\kappa := \frac{d}{2}$,
\[
c := c \left(\frac{2}{\rho}\right)^{k-\ell} \leq \left(\frac{d}{2}\right)^3 \cdot \frac{1}{216k^4} \left(\frac{p}{2}\right)^{k-\ell},
\]
and $V' = R'$, we get the desired path of length three between $X$ and $Y$.

4.2 Absorbing path

In this subsection we prove Lemma 10. As usual, the absorbing path will be built from small pieces called absorbers. For convenience, paths are here represented by sequences of edges (with an implicit ordering of the vertices). Given a set $S \in \binom{V}{k-\ell}$, an $S$-absorber is a $3$-edge path $P = (E_1, G, E_2)$ along with its $\ell$-ends $F_1$ and $F_2$ for which there exists a $4$-edge path $Q = (E_1, G_1, G_2, E_2)$ with $V(Q) = V(P) \cup S$ whose $\ell$-ends are $F_1$ and $F_2$.

Proposition 9 in [14] and the initial part of the proof of Lemma 5 in [14] together imply the following lower bound on the number of $S$-absorbers. (When citing that result, we use $\lambda$ in place of $\varepsilon$.)

**Proposition 14** ([14]). For all $\lambda > 0$, if $\delta_{k-1}(H) \geq \lambda n$, then for every $S \in \binom{V}{k-\ell}$ there are at least
\[
\frac{\lambda^5(3k - 4\ell)!}{2^{6+3k}k^4(3k - 2\ell)!} \left(\frac{n}{3k - 2\ell}\right)^3 \geq \zeta \left(\frac{n}{3k - 2\ell}\right)^3 \geq \frac{\zeta}{2(3k - 2\ell)!} n^{3k-2\ell}\]
$S$-absorbers in $H$. 

17
An S-absorber is called \textit{properly colored} if both paths, \(E_1, G, E_2\) and \(E_1, G_1, G_2, E_2\), are properly colored. Note that there are two intersecting pairs of edges in \(E_1, G, E_2\) and three in \(E_1, G_1, G_2, E_2\). Thus, arguing as in the proof of Lemma 13, there are no more than \(5cn^{3k-2\ell}\) S-absorbers which are not properly colored.

\textbf{Corollary 15.} For \(\zeta\) as in (6) and \(c \leq \frac{\zeta}{4(3k-2\ell)}\), for every \(S \in \binom{V}{k-\ell}\) there are at least 

\[
\frac{\zeta}{4(3k-2\ell)} n^{3k-2\ell}
\]

properly colored \(S\)-absorbers.

\textit{Proof of Lemma 10.} We are going to prove Lemma 10 with

\[
a = a(k, \ell) = \frac{(3k - 4\ell)!!}{2^{2k+6k} k!(3k - 2\ell)!} \quad \text{and} \quad b(k, \ell) = \frac{(3k - 4\ell)!}{5 \cdot 2^8 + 3k \cdot (3k - 2\ell)!}.
\]

Note that, with this \(b\), the bound on \(c\) in Corollary 15 coincides with that in Lemma 10.

We apply Proposition 2, with parameters \(t = 3k - 2\ell\) and \(\alpha = \zeta(2t!t^{-1})\), where \(\alpha\) is as in (6), to the families \(A_S\) of properly colored \(S\)-absorbers viewed as vertex sequences. Note that, by Corollary 15, we have, indeed, \(|A_S| \geq 4\alpha t^2 n^t\). As an outcome, we obtain a family \(F\) of disjoint absorbers, i.e. members of \(\bigcup_S A_S\), of size \(|F| \leq \alpha n\) and such that for all \(S\) we have

\[
|F \cap A_S| \geq \alpha^2 t^2 n/4 = \frac{\zeta^2}{16t!} n = a\lambda^{10} n.
\]

To connect the paths from \(F\) into one path \(A\), we apply Proposition 3 with \(Q = V \setminus \bigcup_{F \in F} V(F)\) and \(m = |F| \leq \alpha n\). To verify Statement I therein with \(g = 3k - 4\ell\), let \(Q' \subset Q\) and \(P_1, P_2\) in \((H - Q, \phi)\) be as in the statement. We invoke Lemma 13 with \(\kappa = \lambda\) and \(V' = Q' \cup \bigcup_{F \in F} V(F)\) satisfying

\[
|V'| \leq |Q'| + |F|(3k - 2\ell) \leq mg + \alpha n(3k - 2\ell) \leq 6(k - \ell)an \leq \zeta \leq \lambda^5 \leq \lambda n/10.
\]

Moreover, the upper bound \(c \leq b\lambda^5\) is stronger than that required for Lemma 13. Applying the lemma to one \(\ell\)-end \(X\) of \(P_1\) and one \(\ell\)-end \(Y\) of \(P_2\), we see that Statement I of Proposition 3 holds and thus Statement II follows. Note that the obtained path \(A\) has length

\[
|V(A)| \leq |F|(6k) \leq \zeta n \leq \lambda^5 n,
\]
as required.

Finally, for every \(U \subset V\) of size \(|U| \leq a(k - \ell)\lambda^{10} n\) where \((k - \ell)||U\), we may partition it into sets of size \(k - \ell\), say \(U = S_1 \cup \ldots \cup S_u\), where \(u = |U|/(k - \ell) \leq a\lambda^{10} n\). Since for each \(i = 1, \ldots, u\), there are at least \(a\lambda^{10} n\) disjoint properly colored \(S_i\)-absorbers on \(A\), we can greedily absorb each set \(S_i\) onto \(A\), obtaining a new \((k, \ell)\)-path \(P_U\) with the same end-edges and the same \(\ell\)-ends as \(A\). Note that in each step, we replace a 3-edge sub-path of \(A\) by a 4-edge path with the same end-edges, so the resulting path remains properly colored. \(\square\)
4.3 Covering by long paths

We emphasize that for the proof of Lemma 12 the argument from [14] goes through practically verbatim. All we have to do is to incorporate the coloring constraints into Propositions 19 and Lemma 20 in [14]. We begin with a colored version of Proposition 19.

For disjoint subsets $V_1, V_2, \ldots, V_k \subset V$ of the vertex set $\mathcal{J}$, a $(k, \ell)$-path $P = v_1 v_2 \ldots v_s$ is called canonical if for each $j = i, \ldots, s$,

$$v_i \in (V_1 \cup \cdots \cup V_{i-1}) \cup (V_{i+1} \cup \cdots \cup V_k) \quad \text{if and only if} \quad \deg_P(v_i) = 2.$$ 

The following result is an analog of Claim 1. Note that, trivially, $\Delta_{k-1}(\mathcal{J}) \leq dm$ implies $\Delta_{i}(\mathcal{J}) \leq dm^{k-\ell}$.

**Claim 4.** Let $(\mathcal{J}, \phi)$ be a colored $k$-partite $k$-graph with partition classes $W_1, \ldots, W_k$, $|W_i| \leq m$ for all $i \in [k]$, and at least $dm^k$ edges. If $\Delta_{i}(\mathcal{J}_i) \leq dm^{k-\ell}/4$ for all $i$, then $\mathcal{J}$ contains a properly colored canonical path on at least $dm/4$ vertices.

**Proof.** As in the proof of Proposition 19 in [14], by deleting iteratively some edges of $\mathcal{J}$, we construct a (non-empty) sub-hypergraph $\mathcal{J}'$ of $\mathcal{J}$ in which all $\ell$-element sets $L$ with exactly one vertex in each $W_i$, $i = 1, \ldots, \ell$, as well as, all $\ell$-element sets $L$ with exactly one vertex in each $W_i$, $i = k-\ell+1, \ldots, k$ satisfy: $\deg_{\mathcal{J}'}(L) = 0$ or $\deg_{\mathcal{J}'}(L) \geq dm^{k-\ell}/2$.

Let $P = v_1 v_2 \ldots v_s$ be the longest properly colored canonical path in $\mathcal{J}'$. The set $L = \{v_{s-\ell+1}, \ldots, v_s\}$ has, clearly, a nonzero degree in $\mathcal{J}'$, and so it is contained in at least $dm^{k-\ell}/2$ edges of $\mathcal{J}'$. Moreover, by our assumption, at most $dm^{k-\ell}/4$ of them have the same color as the last edge of $P$. Hence, there are at least $dm^{k-\ell}/2 - dm^{k-\ell}/4 = dm^{k-\ell}/4$ edges in $\mathcal{J}'$ which contain $L$ and have a different color than the last edge of $P$. On the other hand, by the maximality of $P$, each one of these edges must intersect $V(P) \setminus L$ and so there cannot be more than $sm^{k-\ell+1}$ such edges. Consequently, $dm^{k-\ell}/4 \leq sm^{k-\ell+1}$ which implies that $s \geq dm/4$, as required. \qed

The outcome of the proof of Lemma 7 in [14], excluding the last two sentences, is summarized in the following lemma.

**Lemma 16.** For sufficiently small $\varepsilon := \varepsilon(\gamma) > 0$, there exists an integer $T_0$ such that the following holds. Any $n$-vertex $k$-graph $\mathcal{H}$ with $\delta_{k-1}(\mathcal{H}) \geq (1/2(k-\ell) + \gamma)n$ contains a collection $\mathcal{C}$ of at most $T_0$ vertex-disjoint $(\varepsilon, \gamma/6)$-regular $k$-tuples $(U_1, \ldots, U_{k-1}, U_k)$, $1 \leq j \leq |\mathcal{C}| \leq T_0$, of sets of sizes $|U_1| = \cdots = |U_{k-1}| = (2k - 2\ell - 1)m$ and $|U_k| = (k-1)m$, for some $m$, which cover all but $\varepsilon n$ vertices of $\mathcal{H}$. \[\square\]

In fact, in [14] the number of such collections is at most $\frac{2k-2\ell-1}{2k-2\ell-2}t$, where $t \leq T_{14}$ and $T_{14}$ is a constant delivered by the Weak Regularity Lemma (Lemma 14 in [14]).

As a final ingredient of the proof of Lemma 12, we now prove a colored modification of Lemma 20 in [14].

**Lemma 17.** For all $\beta > 0$ and $d > 0$ there exist $c', \varepsilon > 0$ and $q \in \mathbb{N}$ such that the following holds for sufficiently large $m$. Let $(\mathcal{J}, \phi)$ be a colored $k$-partite $k$-graph with the partition classes forming an $(\varepsilon, d)$-regular $k$-tuple $(U_1, \ldots, U_{k-1}, U_k)$ with $|U_1| = \cdots = |U_{k-1}| = (2k - 2\ell - 1)m$ and $|U_k| = (k-1)m$. If $\Delta_{i}(\mathcal{J}_i) \leq c'm^{k-\ell}$, then there is a family of at most $q$ vertex-disjoint, properly colored paths which cover all but at most $\beta m$ vertices of $\mathcal{J}$.
Proof. A brief analysis of the proof of Lemma 20 in [14] reveals that the only alteration is an application of our Claim 4 instead of their Proposition 19. To do so, we need to set \( c' \leq \varepsilon (2k\varepsilon)^{k-\ell}/4 \). Indeed, then for the sets \( W_i \subset U_i \) defined in the proof of Lemma 20 in [14], setting \( m' = 2k\varepsilon m \) and assuming \( \varepsilon \leq d/2 \), we have \(|W_i| = m', i = 1, \ldots, k\), and
\[
e(W_1, \ldots, W_k) \geq (d - \varepsilon)(m')^k \geq \varepsilon (m')^k,
\]
while
\[
\Delta_e(\mathcal{J}[W_1, \ldots, W_k]) \leq c'm^{k-\ell} = \frac{c'(2k\varepsilon)^{k-\ell}(m')^{k-\ell}}{4} \leq \frac{\varepsilon}{4}(m')^{k-\ell}.
\]
Thus, by Claim 4 with \( d := \varepsilon, \mathcal{J}[W_1, \ldots, W_k] \) contains a properly colored canonical path on at least \( \varepsilon m'/4 \) vertices, which leads to a contradiction in the proof of Lemma 20 in [14] (note that in [14] the length of a path is measured by the number of edges rather than vertices). The rest of that proof carries on unchanged.

Proof of Lemma 20. For any \( \gamma > 0 \) and \( \delta > 0 \), let \( \varepsilon_{16} \) and \( T_0 \) be as in Lemma 16 and let \( \beta \leq \delta k^2/(2T_0) \). Further, set \( d = \gamma/6 \) and let \( \varepsilon_{17}, \varepsilon, q \) be as in Lemma 17. Finally, set \( \varepsilon = \min\{\varepsilon_{16}, \varepsilon_{17}, \delta/2, \gamma/12\} \). We are going to prove Lemma 20 with \( c = c'/(4T_0 k^2)^{k-\ell} \) and \( Q = T_0q \).

By Lemma 16, \( \mathcal{H} \) contains a collection of vertex-disjoint \((\varepsilon, \gamma/6)\)-regular \( k \)-tuples \( \mathcal{C} = \{(U^j_1, U^j_2, \ldots, U^j_{k-1}, U^j_k) : 1 \leq j \leq |\mathcal{C}|, \text{ such that } |\mathcal{C}| \leq T_0, \text{ the sets } U^j_i \text{ cover all but } \varepsilon n \text{ vertices of } \mathcal{H} \text{ and satisfy } |U^j_1| = \cdots = |U^j_{k-1}| = (2k-2\ell-1)m, |U^j_k| = (k-1)m \} \) for an integer \( m \). Note that
\[
(1-\varepsilon)n \leq |\mathcal{C}|[(k-1)(2k-2\ell-1) + (k-1)]m \leq 2T_0k^2m.
\]
Since \( \varepsilon \leq 1/2 \), the above estimate implies that \( n \leq (4T_0 k^2)m \). On the other hand, we also have
\[
|\mathcal{C}|[(k-1)(2k-2\ell-1) + (k-1)]m \leq n,
\]
and so \( m \leq n/k^2 \).

We now verify the hypothesis of Lemma 17. Setting \( \mathcal{J}^j = \mathcal{H}[U^j_1, \ldots, U^j_k] \), we have
\[
\Delta_e(\mathcal{J}^j) \leq \Delta_e(\mathcal{H}_i) \leq cn^{k-\ell} \leq c(4T_0 k^2)^{k-\ell}m^{k-\ell} = c'm^{k-\ell}.
\]
Thus, by Lemma 17, for each \( j \) there is a family \( \mathcal{P}^j \) of at most \( q \) vertex-disjoint, properly colored paths in \( \mathcal{J}^j \) which cover all but at most \( \beta m \) vertices of \( (U^j_1, \ldots, U^j_{k-1}, U^j_k) \).

Consider the family \( \bigcup_{j=1}^{|\mathcal{C}|} \mathcal{P}^j \). It consists of at most \( |\mathcal{C}|q \leq T_0q = Q \) vertex-disjoint, properly colored paths. Moreover, the number of vertices of \( V \) not covered by these paths, by our estimates on \( \beta, m, \) and \( \varepsilon \), is at most
\[
T_0(\beta m) + \varepsilon n \leq \frac{\delta}{2} k^2 \frac{n}{k^2} + \frac{\delta}{2} n = \delta n.
\]
This completes the proof of Lemma 20. \( \square \)
5 Loose Hamilton cycles with degree conditions

This section is devoted to the proof of Theorem 4. As all our statements below are about
loose paths and cycles, the attribute ‘loose’ will sometimes be dropped. Recall that given
a 3-graph \( \mathcal{H} \) and a coloring \( \phi \) of its edges, \( \mathcal{H}_i = \{ e \in \mathcal{H} : \phi(e) = i \} \). Throughout this
section, given a sufficiently small \( \gamma \in (0, 1) \), \((\mathcal{H}, \phi)\) is an \( n \)-vertex colored 3-graph on
vertex set \( V \) with \( n \) sufficiently large,

\[
\delta_1(\mathcal{H}) \geq (7/16 + \gamma)n^2/2 \quad \text{and} \quad \Delta_1(\mathcal{H}_i) \leq cn^2 \quad \text{for all} \quad i \in \mathbb{N},
\]

where \( c > 0 \) is sufficiently small with respect to \( \gamma \) and some other constants introduced
later.

Our proof follows, again, the standard absorbing method. However, there is a serious
obstacle related to the act of absorption which affects two of the three crucial lemmas.
Due to the structure of loose paths, one cannot absorb vertices into \( P \) one by one,
but rather in pairs. Moreover, the proper-coloring constraint results in not every pair of
vertices being absorbable. Thus, we introduce an auxiliary graph \( G \) on \( V \) whose edges
represent the pairs of vertices which can be absorbed into \( P \). Recall that the 1-ends of
loose path \( P = v_1 \cdots v_s \) are just \( v_1 \) and \( v_s \).

**Lemma 18** (Absorbing Lemma). For every \( 0 < \lambda \leq 10^{-14} \) and \( c \leq 30^{-9} \), if \((?)\) holds
(even with \( \gamma = 0 \)), then there is a properly colored loose path \( A \) in \( \mathcal{H} \) of order at most
\( 12\lambda n \) and a graph \( G = G(\mathcal{H}, \phi) \) on \( V \) with \( \delta(G) \geq \frac{2}{3} n \) such that the following holds.

For every \( U \subset V \) with \( |U| \leq 2\lambda^2 n \), if \( G[U] \) has a perfect matching, then there is a
properly colored loose path \( P_U \) in \( \mathcal{H} \) with \( V(P_U) = V(A) \cup U \) with the same end-edges
and \( \ell \)-ends as \( A \). (We will say that \( A \) can absorb \( U \).)

Part (b) of the next lemma ensures that the vertices left outside the long cycle at the
end of the proof induce a perfect matching in the graph \( G \) guaranteed by Lemma 18.

**Lemma 19** (Reservoir Lemma). Let \( G \) be a graph on \( V \) with \( \delta(G) \geq 0.71n \). For every
\( \rho > 0 \) and \( c \leq \frac{\rho^2}{410} \), if \((?)\) holds (even with \( \gamma = 0 \)), then there is a set \( R \subset V \) with
\( |R| \leq \rho n \), which has the following properties:

(i) For any two vertices \( x, y \notin R \), any two colors \( c_x, c_y \), and any subset \( R' \subset R \),
\( |R'| \leq |R|/100 \), there exist \( v_1, v_2, v_3, v_4, v_5 \in R \setminus R' \) such that \( xv_1v_2v_3v_4v_5y \)
is a properly colored loose path in \( \mathcal{H} \) with \( \phi(xv_1v_2) \neq c_x \) and \( \phi(v_4v_5y) \neq c_y \).

(ii) If \( U \subset V \), \( |U| \) even, \( |R \setminus U| \leq |R|/100 \), and \( |U \setminus R| \leq |R|/100 \), then \( G[U] \) has a
perfect matching.

Only the covering lemma is not affected by the above-mentioned problem with absor-
sion.

**Lemma 20** (Covering Lemma). For every \( \gamma > 0 \) and \( \delta > 0 \), let \( c = c(\gamma, \delta) > 0 \) be
sufficiently small and \( q = q(\gamma, \delta) \) be sufficiently large. If \((?)\) holds, then there is a family
\( \mathcal{P} \) of at most \( q \) vertex-disjoint properly colored loose paths in \((\mathcal{H}, \phi)\) covering all but at
most \( \delta n \) vertices of \( \mathcal{H} \).
Proof of Theorem 4. Set
\[\lambda = \min\left(\frac{10^{-14}}{c}, \gamma/25\right), \quad \rho = \lambda^2, \quad \delta = \lambda^2/200, \quad q = q_{20}(\gamma/2, \delta),\]
and
\[c = \min\left(30^{-9}, \rho^2/(5 \cdot 10^5), \frac{1}{2}c_{20}(\gamma/2, \delta)\right),\]
where the subscript 20 means that the constant comes from Lemma 20.

Apply Lemma 18 to find a graph \(G\) with \(\delta(G) \geq \frac{\gamma}{2} n\) and an absorbing path \(A\) with \(|V(A)| \leq 12\lambda n\) which can absorb any set \(U\) of up to \(2\lambda^2 n\) vertices such that \(G[U]\) contains a perfect matching.

Next, let \(R \subset V \setminus V(A)\) be the set given by Lemma 19 applied to \(H' = H - V(A)\), \(G' = G - V(A)\) and to the coloring \(\phi'\) induced in \(H'\) by \(\phi\). Such an application is feasible, because, setting \(n' = |V(H')| = n - |V(A)|\), we have
\[\delta_1(H') \geq \delta_1(H) - |V(A)| n \geq \left(\frac{7}{16} + \gamma\right) \frac{n^2}{2} - 12\lambda n^2 \geq \frac{7}{32}(n')^2,\]
\[\Delta_1(H') \leq \Delta_1(H) \leq cn^2 \leq \frac{\rho^2}{5 \cdot 10^5 (1 - 10\lambda)^2} (n')^2 \leq \frac{\rho^2}{4 \cdot 10^5} (n')^2,\]
and
\[\delta_1(G') \geq \delta(G) - |V(A)| \geq \frac{3}{4} n - 12\lambda n \geq 0.71n \geq 0.71n'.\]

Further, let \(H'' = H' - R\) and let \(\phi''\) be the coloring induced in \(H''\) by \(\phi\). Set \(n'' := |V(H'')|\). Since, for \(\lambda \leq \frac{\gamma}{25}\), we have \(\lambda^2 + 12\lambda \leq \gamma/2\),
\[\delta_1(H'') \geq \delta_1(H) - (|V(A)| + |R|) n \geq \left(\frac{7}{16} + \frac{\gamma}{2}\right) (n'')^2/2\]
and
\[\Delta_1(H'') \leq \Delta_1(H) \leq cn^2 \leq \frac{c_{20}}{2(1 - \lambda/2)^2} (n')^2 \leq c_{20}(n'')^2.\]

Hence, we can apply Lemma 20 to the colored hypergraph \((H'', \phi'')\) with \(\gamma := \gamma/2\) to obtain a family \(\mathcal{P}\) of at most \(q\) properly colored paths which cover all but at most \(\lambda^2 n''/200\) vertices of \(H''\). Let \(W\) be the set of vertices of \(H''\) not covered by any of the paths in \(\mathcal{P}\). Include the absorbing path \(A\) into \(\mathcal{P}\) to get a family of paths \(\mathcal{P}^A = \mathcal{P} \cup \{A\}\).

To connect the paths from \(\mathcal{P}^A\) into one cycle \(C\), we apply Proposition 3 with \(Q = R\) and \(m = |\mathcal{P}^A| = q + 1\). To verify Statement I therein, we invoke Lemma 5 with \(R' = Q'\), so we set \(g = 5\). For \(n\) large enough, we have
\[|Q'| \leq mg \leq 5(q + 1) \leq \frac{1}{100} |R|.\]
By Lemma 5(i), applied to one 1-end \(x\) of \(P_1\) and one 1-end \(y\) of \(P_2\), we see that Statement I of Proposition 3 holds and thus Statement II follows.

Let \(U\) be the set of vertices of \(V\) not covered by \(C\). Since \(|V(C)|\) is even, so is \(|U|\). We have
\[|U \setminus R| \leq |W| \leq \frac{\lambda^2 n''}{200} \leq \frac{|R|}{100}\quad\text{and}\quad |R \setminus U| \leq |R'| \leq \frac{|R|}{100}.\]
Thus, by Lemma \ref{lem:20}(ii), the graph $G[U]$ has a perfect matching. Moreover, since
\[ |U| \leq |W| + |R| \leq \delta n + \rho n < 2\lambda^2 n, \]
$U$ can be absorbed into $A$ (by replacing $A$ with a properly colored path $P_U$), and thus into $C$, forming a Hamilton cycle in $\mathcal{H}$. Since $P_U$ and $A$ have the same end-edges, the obtained Hamilton cycle remains properly colored. \hfill \Box

5.1 Connecting and Reservoir Lemmas

The following lemma states that if $x$ and $y$ are vertices already present in edges of given colors $c_x$ and $c_y$, respectively, then one can connect them via a short properly colored path which avoids a given relatively small set of vertices without creating a color conflict. In fact, although we do not need it here, we show that there are many such paths.

**Lemma 21** (Connecting Lemma). Let a colored 3-graph $(\mathcal{H}, \phi)$ be given with $\delta_1(\mathcal{H}) \geq n^2/5$ and $\Delta_1(\mathcal{H}_i) \leq 10^{-5} n^2$. Further, let $Z \subset V(\mathcal{H})$, $|Z| \leq 0.01 n$. Then, for any pair of vertices $x, y \in V(\mathcal{H}) \setminus Z$, and any two colors $c_x, c_y$, there exist at least $4(n/10)^5$ properly colored paths $xv_1v_2v_3v_4v_5y$, where $v_1, v_2, v_3, v_4, v_5 \notin Z$, such that, in addition, $\phi(xv_1v_2) \neq c_x$ and $\phi(v_2v_3y) \neq c_y$.

**Proof.** We will first estimate from below the number of paths $xv_1v_2v_3v_4v_5y$ with no regard to coloring. Then we will subtract an upper bound on the number of those among them which have a color conflict. Given a vertex $u \in V(\mathcal{H})$, we define its neighborhood in $\mathcal{H}$ as $N_\mathcal{H}(u) = \{S \subset V(\mathcal{H}) : S \cup \{v\} \in \mathcal{H}\}$ and denote $L_u = N_\mathcal{H}(u)$. Note that under our assumptions, $L_u$ is a graph with $|L_u| \geq n^2/5$.

Choose $\{v_4, v_5\} \subset L_y$, $v_4, v_5 \neq x$, arbitrarily. There are at least $n^2/5 - n$ ways to do so. Turning, for a moment, to the other end of the to-be-path, let $X = \{v \in V : \deg_{L_x}(v) \leq n/20\}$ and set $|X| = \tau n$. Then
\[
\frac{1}{5} n^2 - n \leq |L_x| \leq |X|n/20 + \left(\frac{|V \setminus X|}{2}\right) \leq \left(\frac{1}{10} \tau + (1 - \tau)^2\right) n^2/2,
\]
which implies (since $n$ is arbitrarily large) that
\[
f(\tau) := \frac{1}{10} \tau + (1 - \tau)^2 \geq \frac{2}{5}.
\]
Note that $f$ has the unique minimum at $\frac{20}{21}$, while $f(1) = 1/10 < 2/5$ and $f(2/5) = 2/5$. It follows that $\tau \leq 2/5$. As the same is true for $L_{v_4}$, we infer that there are at least $n/5$ vertices $v \in V$ with both $\deg_{L_x}(v) \geq n/20$ and $\deg_{L_{v_4}}(v) \geq n/20$. Thus, avoiding $Z \cup \{x, y, v_4, v_5\}$, we have at least
\[
(n/5 - |Z| - 4)(n/20 - |Z| - 4)(n/20 - |Z| - 5) \geq \frac{3}{10^{11}} n^3
\]
triples $(v_1, v_2, v_3)$ with $v_1, v_2, v_3 \notin Z$. $\{v_1, v_2\} \in L_x$ and $\{v_2, v_3\} \in L_{v_4}$. Consequently, factoring in the number of ordered pairs $(v_4, v_5)$, there are at least
\[
2 \left(\frac{n^2}{5} - n\right) \frac{3}{10^4} n^3 \geq \frac{8}{10^6} n^5
\]
paths $xv_1v_2v_3v_4v_5y$ with $v_1, v_2, v_3, v_4, v_5 \notin Z$.

Now, we count the paths with color conflicts. On top of the forbidden conflicts $\phi(xv_1v_2) = c_x$ and $\phi(v_2v_3y) = c_y$, there are two more corresponding to the two pairs of edges intersecting, respectively, at $v_2$ and at $v_4$. Using the assumption $\Delta_1(H_i) \leq 10^{-5}n^2$, each one of them disqualifies at most $(n/10)^5$. Altogether, the number of required paths is at least

$$8 \left( \frac{n}{10} \right)^5 - 4 \left( \frac{n}{10} \right)^5 \geq 4 \left( \frac{n}{10} \right)^5.$$

\[ \square \]

Proof of Lemma 19. Let $p = \frac{2}{3} \rho$. Applying Proposition 1 with $U_v = N_G(v)$, $G_v = N_H(v)$, $\alpha_v = 0.71$, $\beta_v = \frac{7}{16}$, for all $v \in V$, we obtain a subset $R \subset V$ such that

\[ \frac{1}{7} \rho n \leq |R| \leq \rho n, \quad \text{and, for all } v \in V, \]

1. $|N_G(v) \cap R| \geq (0.71 - 2n^{-1/3})|R| \geq \frac{2}{3} |R|$, and

2. $|N_H(v)[R]| \geq (\frac{7}{16} - 3n^{-1/3}) \left( \frac{|R|}{2} \right) \geq \frac{13}{32} \left( \frac{|R|}{2} \right)$.

We claim that for any set $R$ satisfying properties (1) and (2) above, conditions (i) and (ii) of Lemma 19 hold.

To prove (i) consider a pair of vertices $x, y \notin R$, two colors $c_x, c_y$, and a subset $R' \subset R$ with $|R'| \leq |R|/100$. Let $\mathcal{R} = H[R \cup \{x, y\}]$ and set $r = |V(\mathcal{R})| = |R| + 2$. Since $r \geq \rho n/2$ and $n$ is sufficiently large, by (2),

$$\delta_1(\mathcal{R}) \geq \frac{13}{32} \left( \frac{|R|}{2} \right) \geq \frac{r^2}{5},$$

and, by our assumption on $c$, for each $i$,

$$\Delta_1(\mathcal{R}_i) \leq cn^2 \leq \frac{4c}{\rho^2} |R|^2 \leq 10^{-5}r^2.$$

Thus, we are in position to apply Lemma 21 with $\mathcal{H} := \mathcal{R}$, $\phi := \phi|_{\mathcal{R}}$, $Z := R'$, and $c := 4c/\rho^2$, obtaining the desired $v_1, v_2, v_3, v_4, v_5 \in R \setminus R'$.

To prove (ii), it is enough to show that $\delta(G[U]) \geq \frac{1}{7}|U|$, as this, recalling that $|U|$ is even, implies the existence of a perfect matching in $G[U]$. To this end, note that, since $|U \setminus R| \leq |R|/100$, we have $|U| \leq |R| + |R|/100$, or equivalently, $|R| \geq \frac{100}{101} |U|$. Thus, for any $u \in U$, using (1) and the inequality $|R \setminus U| \leq |R|/100$,

$$|N_G(u) \cap U| \geq |N_G(u) \cap R| - |R \setminus U| \geq \frac{2}{3} |R| - \frac{1}{100} |R| = \frac{197}{300} |R| \geq \frac{197}{303} |U| > \frac{1}{2} |U|,$$

which completes the proof of the lemma. \[ \square \]
5.2 Absorbing path

In this subsection we prove Lemma 18. As usual, the absorbing path will be built from small pieces called *absorbers*. For a pair of distinct vertices \( x, y \in V \), an \((x, y)\)-*absorber* is a 7-tuple \( \vec{v} = (v_1, v_2, \ldots, v_7) \) of vertices of \( \mathcal{H} \) such that:

(i) \( v_1v_2v_3, v_3v_4v_5, v_5v_6v_7 \in \mathcal{H} \),

(ii) \( v_2xv_4, v_4yv_6 \in \mathcal{H} \).

In other words, both \( \vec{v} \) and \( \vec{v}_{x,y} := (v_1, v_3, v_2, x, v_4, y, v_5, v_7) \) induce loose paths in \( \mathcal{H} \) in that ordering. If both \( \vec{v} \) and \( \vec{v}_{x,y} \) are properly colored in \( \mathcal{H} \), then \( \vec{v} \) is called a *properly colored* \((x, y)\)-*absorber*. Note that \( \vec{v} \) and \( \vec{v}_{x,y} \) have the same end-edges and 1-ends, though the order of vertices in the end-edges does change.

In ([4], Proposition 8) it was shown that under some milder degree assumptions, for any pair of vertices \( x, y \in V(\mathcal{H}) \), the number of \((x, y)\)-absorbers is \( \Theta(n^7) \).

**Proposition 22** ([4]). For every \( \xi \in (0, 3/8) \) there exists \( n_0 \) such that the following holds. Suppose that \( \mathcal{H} \) is a 3-graph on \( n > n_0 \) vertices with \( \delta_1(\mathcal{H}) \geq (\frac{8}{3} + \xi)^2 \binom{n}{2} \). Then for every pair of vertices \( x, y \in V(\mathcal{H}) \) the number of \((x, y)\)-absorbers is at least \((\xi n)^7 / 8\).

Under our stronger assumption on \( \delta_1(\mathcal{H}) \), we may take \( \xi = 1/30 \) and deduce the following.

**Corollary 23.** There exists \( n_0 \) such that if \( \mathcal{H} \) be a 3-graph on \( n > n_0 \) vertices with \( \delta_1(\mathcal{H}) \geq \frac{1}{100} \binom{n}{2} \), then for every pair of vertices \( x, y \in V(\mathcal{H}) \) the number of \((x, y)\)-absorbers is at least \( \frac{1}{8}(n/30)^7 \).

Consider a pair of vertices \( x, y \in V(\mathcal{H}) \). Although there are many \((x, y)\)-absorbers, there is no guarantee that any of them are properly colored. Indeed, it may be the case that for every choice of \( v_2, v_4, v_6 \in V(\mathcal{H}) \), we have \( \phi(v_2xv_4) = \phi(v_4yv_6) \). The following proposition states that for every vertex \( x \) there are many vertices \( y \) such that the number of properly colored \((x, y)\)-absorbers is of the order of magnitude as large as possible. A pair of distinct vertices \( x, y \in V(\mathcal{H}) \) is called *pc-absorbable* if there exists at least \( \frac{1}{16}(n/30)^7 \) properly colored \((x, y)\)-absorbers.

**Proposition 24.** Let \( c \leq 30^{-9} \). For every \( x \in V \), there are at least \( \frac{3}{4}n \) vertices \( y \in V \) such that the pair \((x, y)\) is pc-absorbable.

**Proof.** Given \( x \in V \) it is enough to show that for each \( Y \subset V \), \( |Y| = n/4 \), there is a vertex \( y \in Y \) such that the pair \((x, y)\) is pc-absorbable.

Set \( \beta = \frac{1}{16}(n/30)^7 \) for convenience. By Corollary 23, the number of pairs \((y, \vec{v})\), where \( y \in Y \) and \( \vec{v} = (v_1, \ldots, v_7) \) is an \((x, y)\)-absorber is at least \( \frac{1}{8} \beta n^8 \). We wish to count for how many of these pairs \( \vec{v} \) is not a properly colored \((x, y)\)-absorber, i.e. at least two intersecting edges in \( \vec{v} \) or in \( \vec{v}_{x,y} \) have the same color. As there are in total five intersecting pairs among the edges of the two paths (three in \( \vec{v} \) and two in \( \vec{v}_{x,y} \)), this number can be crudely bounded from above by \( 5cn^8 \).

Indeed, fixing two intersecting edges, say, \( v_1v_2v_3 \) and \( v_3v_4v_5 \), the number of pairs \((y, \vec{v})\) in question can be bounded from above by first bounding crudely the number of choices of \( y, v_1, v_2, v_3, v_6, v_7 \) by \( n^6 \) and then, setting \( i := \phi(v_1v_2v_3) \), bounding the number
of choices of $v_4, v_5$ by $\Delta(\mathcal{H}_i) \leq cn^2$, as $v_3v_4v_5$ must be of the same color as $v_1v_2v_3$. If one of the intersecting edges involves $x$, as in the pair $v_2xv_4, v_4yv_5$, then the bound $cn^2$ is applied to the other edge in the pair (in the given example, we would use this bound to the number of choices of $y, v_5$).

Consequently, the number of pairs $(y, \bar{v})$, where $y \in Y$ and $\bar{v}$ is a properly colored $(x, y)$-absorber is at least
\[
\left(\frac{1}{2}\beta - 5c\right)n^8.
\]
By averaging, there exists $y \in Y$ for which the number of properly colored $(x, y)$-absorbers is at least
\[
(2\beta - 20c)n^7 \geq \frac{1}{16}(n/30)^7,
\]
where we also used the inequality $20c < \beta$. Thus, the pair $(x, y)$ is pc-absorbable. \hfill \Box

**Proof of Lemma 18.** By definition, for any pc-absorbable pair $(x, y)$, the family $\mathcal{A}_{(x,y)}$ of properly colored $(x, y)$-absorbers has size
\[
|\mathcal{A}_{(x,y)}| \geq 2^{-4}(n/30)^7 > 4 \cdot 7^2 \lambda n,
\]
where the last inequality follows by our assumption on $\lambda$. The elements of $\bigcup_{(x,y)} \mathcal{A}(x,y)$ will be called *absorbers*. Applying Proposition 2 to the families $\mathcal{A}_{(x,y)}$ with $t = 7$ and $\alpha = \lambda$, one can find a family $\mathcal{F}$ of absorbers satisfying $|\mathcal{F}| \leq \lambda n$ and $|\mathcal{F} \cap \mathcal{A}_{(x,y)}| \geq \frac{49}{4} \lambda^2 n$ for all pc-absorbable pairs $(x, y)$.

To connect the paths from $\mathcal{F}$ into one path $A$, we apply Proposition 3 with $Q = V \setminus \bigcup_{F \in \mathcal{F}} V(F)$ and $m = |\mathcal{F}| \leq \lambda n$. To verify Statement I therein with $g = 5$, let $Q', P_1$ and $P_2$ in $(\mathcal{H} - R, \phi)$ be as in the statement. We invoke Lemma 7 with $Z = Q' \cup \bigcup_{F \in \mathcal{F}} V(F)$, so we set $b = 5$. Note that, for $n$ large enough,
\[
|Z| \leq |Q'| + 7|\mathcal{F}| \leq mg + 7\lambda n \leq 12\lambda n \leq 0.01n.
\]
By Lemma 7, applied to one 1-end $x$ of $P_1$ and one 1-end $y$ of $P_2$, we see that Statement I of Proposition 3 holds and thus Statement II follows. Note that the obtained path $A$ has length
\[
|V(A)| \leq 12|\mathcal{F}| \leq 12\lambda n,
\]
as required.

Let $G$ be an auxiliary graph on vertex set $V$ whose edges correspond to pc-absorbable pairs. Observe that there is no ambiguity in this definition, since, by reversing the 7-tuples, $(x, y)$ is a pc-absorbable pair if and only if $(y, x)$ is. Proposition 24 implies that $\delta(G) \geq \frac{3}{4} n$. Consider a set $U \subset V$ of size $|U| \leq 2\lambda^2 n$ which induces a matching $M$ in $G$. For each edge $xy$ of $M$, the path $A$ contains
\[
|\mathcal{A}_{(x,y)}| \geq \frac{49}{4} \lambda^2 n \geq \lambda^2 n
\]
properly colored $(x, y)$-absorbers. Hence, one can absorb the pairs of $M$ greedily to obtain a properly colored path $P_U$ on the vertex set $V(A) \cup U$. Note that, by construction of absorbers, in each step the 1-ends and end-edges of the resulting path stay intact. Thus, the final path $P_U$ has the same 1-ends and end-edges as the path $A$. \hfill \Box
5.3 Covering by long paths

The proof of Lemma 20 follows closely the proof of Lemma 10 in [4] which, in turn, is based partly on Lemma 20 from [14]. Since we already have a colored version of that lemma, namely Lemma 17 in Section 4, all we need is a decomposition result analogous to Lemma 16. Such a result can be deduced from the results in [4]. It is based on the Weak Hypergraph Regularity Lemma (Proposition 15 in [4]) and an extremal result developed in [4] (Lemma 11 therein); this deduction is shown in the proof of Lemma 10 in [4]. (In our version, we apply Lemma 11 with \(\alpha := \varepsilon\).) Notice that this statement makes no reference to edge coloring.

**Lemma 25** ([4]). For a sufficiently small \(\varepsilon = \varepsilon(\gamma) > 0\), there exists an integer \(T_0\) such that the following holds. Any \(n\)-vertex 3-graph \(H\) with \(\delta_1(H) \geq \left(\frac{7}{10} + \gamma\right) \binom{n}{3}\) contains a collection \(C\) of at most \(T_0\) vertex-disjoint \((\varepsilon, \gamma/3)\)-regular triples \((U_1^j, U_2^j, U_3^j)\), \(j = 1, \ldots, |C| \leq T_0\), which cover all but \(\varepsilon n\) vertices of \(H\). Moreover, for some \(m\), half of the triples satisfy the identity \(|U_1^j| = |U_2^j| = |U_3^j| = 3m\), while for the other half \(|U_1^j| = |U_2^j| = |U_3^j| = 6m\). \(\square\)

**Proof of Lemma 20.** In this proof, which is very similar to that of Lemma 12, we utilize Lemma 25 combined with Lemma 17 for \(k = 3\) and \(\ell = 1\). For any \(\gamma > 0\) and \(\delta > 0\), let \(\varepsilon_{25}\) and \(T_0\) be as in Lemma 25 and let \(\beta \leq 6\delta\). Further, set \(d = \gamma/3\) and let \(\varepsilon_{17}, c', q\) be as in Lemma 17 with \(k = 3\) and \(\ell = 1\). Finally, set \(\varepsilon = \min\{\varepsilon_{25}, \varepsilon_{17}, \delta/2, \gamma/6\}\). We are going to prove Lemma 12 with \(c = c'/(24T_0)^2\) and \(Q = T_0q\).

By Lemma 25, \(H\) contains a collection \(C\) of size \(|C| \leq T_0\) of vertex-disjoint \((\varepsilon, \gamma/3)\)-regular triples of prescribed sizes which cover all but \(\varepsilon n\) vertices of \(H\). Note that

\[(1 - \varepsilon)n \leq 12|C|m \leq 12T_0m.\]

Since \(\varepsilon \leq 1/2\), the above estimate implies that \(n \leq 24T_0m\). On the other hand, we also have \(12|C|m \leq n\).

In order to apply Lemma 17, we need to verify its assumption (with \(k = 3\) and \(\ell = 1\)). Setting, \(J = H[U_1^j, U_2^j, U_3^j]\), we have

\[\Delta_1(J^j) \leq \Delta_1(H) \leq cn^2 \leq c(24T_0)^2m^2 = c'm^2.\]

Thus, by Lemma 17, for each \(j\) there is a family \(P^j\) of at most \(q\) vertex-disjoint, properly colored paths in \(J^j\) which cover all but at most \(bm\) vertices of \((U_1^j, U_2^j, U_3^j)\).

Consider the family \(\bigcup_{j=1}^{\lfloor C\rfloor} P^j\). It consists of at most \(|C|q \leq T_0q = Q\) vertex-disjoint, properly colored paths. Moreover, the number of vertices of \(V\) not covered by these paths, by our estimates on \(b, m\), and \(\varepsilon\), is at most

\[|C|(bm) + \varepsilon n \leq \beta n + \delta n \leq 2n.\]

This completes the proof of Lemma 20. \(\square\)

6 Concluding remarks

We point out an extension and some directions for further research.
Compatibility systems: Theorems 2-4 actually hold in the setting of ‘compatibility systems’ which generalise edge colorings. Compatible cycles were considered for instance in [23, 13].

Dirac hypergraphs: Do our theorems hold for Dirac hypergraphs, that is, with the constant $\gamma = 0$? Analogous results are presently only known for graphs [23, 12].

The rainbow setting: It is likely that the rainbow analogues of Theorems 2-4 hold, where an additional assumption on the number of occurrences of each colour is needed. This was already asked by [7], but let us spell out the conjecture for tight cycles. For loose cycles, the conjecture may even hold without the local-boundedness assumption, as in [11, 18].

**Conjecture 1.** For every $k \geq 3$ and $\gamma > 0$ there exist $c > 0$ and $n_0 > 0$ such that if $(\mathcal{H}, \phi)$ is an $n$-vertex colored $k$-graph with $n \geq n_0$, $\delta_{k-1}(\mathcal{H}) \geq (1/2 + \gamma)n$, $\Delta_0(\mathcal{H}_i) \leq cn^{k-1}$ and $\Delta_{k-1}(\mathcal{H}_i) \leq cn$ for every $i \in \mathbb{N}$, then $(\mathcal{H}, \phi)$ contains a properly colored tight Hamilton cycle $C_n^{(k)}(k - 1)$.

Existing rainbow results for incomplete hypergraphs [7] are achieved using the method of switchings, which is suitable for embedding $H$-factors because it exploits their symmetry. In other words, there are many ways to locally modify an $H$ factor and obtain another $H$-factor. It is likely that new ideas are needed for connected spanning hypergraphs.

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