CONVERGENCE OF RANDOM ZEROS ON COMPLEX MANIFOLDS

BERNARD SHIFFMAN

Abstract. We show that the zeros of random sequences of Gaussian systems of polynomials of increasing degree almost surely converge to the expected limit distribution under very general hypotheses. In particular, the normalized distribution of zeros of systems of $m$ polynomials of degree $N$, orthonormalized on a regular compact set $K \subset \mathbb{C}^m$, almost surely converge to the equilibrium measure on $K$ as $N \to \infty$.

1. Introduction

The central theme of this paper is the almost sure convergence to an equilibrium distribution of zeros of random sequences of holomorphic zero sets. We work with simultaneous zeros of random polynomials on $\mathbb{C}^m$ or, more generally, zeros of random sections of powers of a holomorphic line bundle $L \to M$ over a compact Kähler manifold. To review some history, the asymptotic properties of the zeros of random real polynomials were studied by Kac [Kac] in 1949; a few years later, Hammersley [Ham] investigated the zeros of the complexification of the Kac ensembles. While the zeros of the Hammersley ensembles tend to accumulate on the unit circle in $\mathbb{C}$, the distribution of zeros is uniform (with respect to the Fubini-Study measure on $\mathbb{CP}^1$) for the “SU(2) polynomials” studied in the physics literature (e.g., [BBL, FH, Han, NV]).

There has been a recent interest in the statistical properties of zeros and simultaneous zeros of random functions of several variables. For example, statistics on zeros and simultaneous zeros of random polynomials of several real variables were given in [EK, Ro, SS, Ws]. Results on zeros of random polynomials of several complex variables as well as of random holomorphic sections of line bundles can be found in [Be1, BSZ1, BSZ2, BS, EK, SZ1, SZ3, SZ4, Zr] and elsewhere.

In joint work with Zelditch [SZ1] in 1999, we showed that if $L$ is a positive Hermitian line bundle, the normalized zero currents $\frac{1}{N}Z_{s_N}$ of a random sequence $s_N \in H^0(M, L^N)$ of holomorphic sections of increasing powers of $L$ almost surely converge to the curvature form of $L$. This result was derived as a consequence of an asymptotic expansion for the expected values $E(\frac{1}{N}Z_{s_N})$ of these zero currents together with an elementary variance estimate. Furthermore, as a consequence of the sharp variance asymptotics in a recent paper with Zelditch [SZ4], the normalized expected zero currents $\frac{1}{N^k}Z_{s_{N_1},\ldots, s_{N_k}}$ of $k$ independent random sections almost surely converge to a uniform distribution (for $1 \leq k \leq \dim M$). On the other hand, Bloom showed in [Bl1] that for random sequences of polynomials of increasing degree, orthonormalized with respect to certain measures on a compact set $K \subset \mathbb{C}$, the normalized zero distributions converge almost surely to the equilibrium measure on $K$.

In this paper, we show that for all sequences of ensembles of random sections of increasing powers of line bundles (e.g., random polynomials of increasing degree), whenever the expected normalized zero currents converge to a limit current, the convergence holds almost surely to the limit distribution.

Research partially supported by NSF grant DMS-0600982.
surely for random sequences. The only condition imposed on the sequence of ensembles is that the probability measures are (complex) Gaussian.

Our results are stated in terms of the currents of integration over zero sets, which we call zero currents. For a system $S_N = (s_N^1, \ldots, s_N^k)$ of $k$ holomorphic sections $s_N^j \in H^0(M, L^N)$, $j = 1, \ldots, k$ (where $1 \leq k \leq m = \dim M$), we let

$$
|Z_{S_N}| := \{z \in M : s_N^1(z) = \cdots = s_N^k(z) = 0\}
$$

denote its zero set, and we consider the current of integration $Z_{S_N} \in D^{k,k}(M)$ defined by

$$(Z_{S_N}, \varphi) = \int_{|Z_{S_N}|} \varphi, \quad \varphi \in D^{m-k, m-k}(M),$$

whenever the zero set of $S_N$ is a codimension $k$ subvariety without multiplicity. (For $L^N$ base point free, $|Z_{S_N}|$ is almost surely a smooth codimension $k$ subvariety without multiplicity.) We recall that $D^{j,j}(M)$ denotes the space of real $C\infty$ forms of bidegree $(j, j)$ on $M$.

Our convergence result (Corollary 1.3) is a consequence of the following variance estimate:

**Theorem 1.1.** Let $L \to (M, \omega)$ be an ample holomorphic line bundle over a compact Kähler manifold of dimension $m$, and let $S$ be a linear subspace of the space $H^0(M, L)$ of holomorphic sections of $L$. Suppose that $S$ has a Gaussian probability measure. Let $1 \leq k \leq m$ and let $\varphi \in D^{m-k, m-k}(M \setminus B)$, where $B$ is the base point set of $S$. Then the standard deviation of the zero statistics of $k$ independent random sections $s_1, \ldots, s_k$ of $S$ satisfies the bound

$$\sqrt{\text{Var}(Z_{s_1, \ldots, s_k}, \varphi)} \leq C_m \|\partial \bar{\partial} \varphi\|_\infty \int_M \omega^{m-k+1} \wedge c_1(L)^{k-1},$$

where the constant $C_m$ depends only on the dimension $m$ of $M$.

The base point set of $S$ is the set of points $z \in M$ where $s(z) = 0$ for all $s \in S$. In [8] we prove a slightly more general variance bound (Theorem 3.1).

The key point is that the variance bound involves the $(k-1)$-th power of $c_1(L)$ instead of the $k$-th power. It follows that the standard deviations of simultaneous zeros of random sections of the $N$-th tensor powers $L^N$ of $L$ grow at a lower rate than the expected values. To be precise, given $k$ sections $s_N^1, \ldots, s_N^k$ of $L^N$, we define the normalized zero current

$$
\tilde{Z}_{s_N^1, \ldots, s_N^k} := \frac{1}{N^k}Z_{s_N^1, \ldots, s_N^k}.
$$

We then have the following asymptotic variance bound:

**Theorem 1.2.** Let $L \to M$ be a holomorphic line bundle over a projective algebraic manifold of dimension $m$ and let $\varphi \in D^{m-k, m-k}(U)$, where $U$ is an open subset of $M$. Suppose that we are given subspaces $S_N^j \subset H^0(M, L^N)$ endowed with arbitrary complex Gaussian probability measures $\gamma^j_N$, such that $S_N^j$ has no base points in $U$, for $1 \leq j \leq k$, $N \geq 1$. Then for independent random sections $s_N^j \in S_N^j$,

$$\text{Var}(\tilde{Z}_{s_N^1, \ldots, s_N^k}, \varphi) \leq O\left(\frac{1}{N^2}\right).$$
If $L$ is ample, the conclusion of Theorem 1.2 follows immediately from Theorem 1.1. For the general case, the result follows from a modified version (Proposition 4.1) of Theorem 1.1.

The fundamental case considered in [SZ1, SZ4] is where $L$ is an ample line bundle and $S_N = H^0(M, L^N)$. Then $L$ has a Hermitian metric $h$ with positive curvature, and we give $M$ the Kähler form $\omega = \pi c_1(L, h)$, where $c_1(L, h)$ is the Chern form (see (18)). The Hermitian metric $h$ on $L$ and the Kähler form $\omega$ induce Hermitian inner products on the spaces $H^0(M, L^N)$:

$$\langle s_N, s'_N \rangle = \int_M h^N(s_N, s'_N) \frac{1}{m!} \omega^m, \quad s_N, s'_N \in H^0(M, L^N),$$

where $h^N$ denotes the induced metric on $L^N$. These inner products in turn induce Gaussian probability measures on the corresponding spaces (see (16)). It was shown in [SZ1] that in this case,

$$E(\tilde{Z}_{s_1^N, \ldots, s_k^N, \varphi}) = \int_M \omega^k \wedge \varphi + O\left(\frac{1}{N}\right),$$

where $E(Y)$ denotes the expected value of a random variable $Y$. For the fundamental case, we further have the sharp variance bound from [SZ4a]:

$$\text{Var}(\tilde{Z}_{s_1^N, \ldots, s_k^N}, \varphi) \leq O\left(\frac{1}{N^{m+2}}\right).$$

(4)

In fact when $k = 1$, we have the precise formula

$$\text{Var}(\tilde{Z}_{s^N}, \varphi) = N^{-m-2} \left[\frac{\pi^{m-2} \zeta(m+2)}{4} \|\partial \overline{\partial} \varphi\|_{L^2}^2 + O(N^{-\frac{1}{2}+\varepsilon})\right].$$

(5)

The variance formula (5) was previously obtained for zeros of polynomials in one variable (the SU(2) ensemble) by Sodin and Tsirelson [ST].

We point out here that the weaker bound of Theorem 1.2 holds for an arbitrary sequence of Hermitian inner products on arbitrary subspaces $S^j_N \subset H^0(M, L^N)$, whereas the sharp bound (4) is a consequence of the off-diagonal asymptotics of the Szegő kernels for the full spaces $H^0(M, L^N)$ with the inner products (2). (See [BSZ1, SZ4] for discussions of the Szegő kernel asymptotics.)

Theorem 1.2 implies the following general result on almost sure convergence to the average of sequences of zeros of i.i.d. $k$-tuples of sections in $S_N$:

**Corollary 1.3.** Let $L \rightarrow M$ be a holomorphic line bundle over a projective algebraic manifold. Suppose that $\gamma_N$ is a Gaussian probability measure on a subspace $S_N$ of $H^0(M, L^N)$, for $N = 1, 2, 3, \ldots$, and let $\gamma = \prod_{N=1}^\infty \gamma_N^k$ denote the product measure on $S_\infty := \prod_{N=1}^\infty (S_N)^k$. Let $U$ be an open subset of $M$ such that $S_N$ has no base points on $U$, for all $N$.

Suppose that the expected normalized zero currents

$$E_{\gamma_N^k}(\tilde{Z}_{s_1^N, \ldots, s_k^N}|U)$$

converge weakly in $\mathcal{D}^{k,k}(U)$ to a current $\Psi \in \mathcal{D}^{k,k}(U)$. Then for $\gamma$-almost all sequences $\{(s_1^N, \ldots, s_k^N)\}_{N=1}^\infty \in S_\infty$,

$$\tilde{Z}_{s_1^N, \ldots, s_k^N}|U \rightarrow \Psi \quad \text{weak}^*$$
(in the sense of measures); i.e., for almost all sequences,
\[
\lim_{N \to \infty} \left( \frac{1}{N^k} Z_{s_N^1, \ldots, s_N^k}, \varphi \right) = \int_M \Psi \wedge \varphi
\]
for all continuous \((\dim M - k, \dim M - k)\) forms \(\varphi\) with support contained in \(U\).

The proof of Corollary 1.3 is given in [4]. Applying Corollary 1.3 to the full ensembles \(S_N^j = H^0(M, L^N)\) with the inner product \([2]\), we conclude from \([3]\) that the simultaneous zeros of random sequences \(\{(s_N^1, \ldots, s_N^k)\}\) are almost always asymptotically uniform; i.e., \(\tilde{Z}_{s_N^1, \ldots, s_N^k} \to \omega^k\) almost surely, as noted in [SZ4] using the sharp variance bound \([4]\) (and in [SZ1] for the case \(k = 1\)).

We now mention some new applications of Corollary 1.3. The first application is to the result given in joint work with Bloom [BS] on zeros of random polynomial systems orthonormalized on compact sets in \(\mathbb{C}^m\):

**Theorem 1.4.** Let \(\mu\) be a Borel probability measure on a regular compact set \(K \subset \mathbb{C}^m\), and suppose that \((K, \mu)\) satisfies a Bernstein-Markov inequality. Let \(1 \leq k \leq m\), and let \((P_N^k, \gamma_N^k)\) denote the ensemble of \(k\)-tuples of i.i.d. Gaussian random polynomials of degree \(\leq N\) with the Gaussian measure \(d\gamma_N\) induced by \(L^2(\mu)\). Then for almost all sequences of \(k\)-tuples of polynomials \(\{(f_N^1, \ldots, f_N^k)\} \in \prod_{N=1}^{\infty} P_N^k\),
\[
\tilde{Z}_{f_N^1, \ldots, f_N^k} \to \left( \frac{i}{\pi} \partial \bar{\partial} V_K \right)^k \text{ weak* ,}
\]
where \(V_K\) is the pluricomplex Green function of \(K\) with pole at infinity. In particular, for \(k = m\),
\[
\tilde{Z}_{f_N^1, \ldots, f_N^m} \to \mu_{eq}(K) := \left( \frac{i}{\pi} \partial \bar{\partial} V_K \right)^m \text{ weak* a.s. .}
\]

The one-variable case of Theorem 1.4 was given in [Bl1], generalizing a result in [SZ2]. The pluricomplex Green function in the theorem is given by
\[
V_K(z) := \sup \{ u(z) \in \mathcal{L} : u \leq 0 \text{ on } K \} ,
\]
where
\[
\mathcal{L} := \{ u \in \text{PSH}(\mathbb{C}^m) : u(z) \leq \log^+ ||z|| + O(1) \} .
\]
If \(\mu\) is a probability measure on a compact set \(K \subset \mathbb{C}^m\), one says that \((K, \mu)\) satisfies a Bernstein-Markov inequality if for all \(\varepsilon > 0\), there is a positive constant \(C = C(\varepsilon)\) such that
\[
\|p\|_{K} \leq C e^{\varepsilon \deg(p)} \|p\|_{L^2(\mu)} ,
\]
for all polynomials \(p\). The measure \(\mu_{eq}(K)\) is called the equilibrium measure of \(K\); it is supported on the Silov boundary of \(K\), and \((K, \mu_{eq}(K))\) satisfies a Bernstein-Markov inequality (for \(K\) regular).

In [BS], we showed that that the expected values of the normalized zero currents of Theorem 1.4 satisfy the asymptotics:
\[
E \left( \tilde{Z}_{f_N^1, \ldots, f_N^k} \right) \to \left( \frac{i}{\pi} \partial \bar{\partial} V_K \right)^k \text{ weak* .}
\]

Theorem 1.4 follows from Corollary 1.3 and \([9]\).
We remark that a generalization of (9) with weights was recently given by Bloom [Bl2], answering a question posed in [SZ2]. To state Bloom’s result, we let \( w : K \to [0, +\infty) \) be a continuous weight (such that the set \( \{w > 0\} \) is non-pluripolar), and we give each space \( \mathcal{P}_N \) of polynomials of degree \( \leq N \) the Gaussian measure induced by \( L^2(w^{2N}d\mu) \). We let \( \varphi = -\log w \) and define the “weighted pluricomplex Green function”

\[
V_{K,\varphi}(z) := \sup\{u(z) \in \mathcal{L} : u \leq \varphi \text{ on } K\}.
\]

If \((K, \mu)\) satisfies a “weighted Bernstein-Markov inequality” (replace \( p \) with \( w^N \) in (8)), one then has the asymptotics

\[
E_{w^N}(\tilde{f}_{N,1},...,\tilde{f}_{N,k}) \to \left(\frac{i}{\pi} \partial \bar{\partial} V_{K,\varphi}\right)^k \text{ weak},
\]

(10)

where \( E_{w^N} \) denotes the expected value for the weighted ensemble [Bl2, Th. 2.1]. It then follows as before from Corollary 1.3 that

\[
\tilde{f}_{N,1},...,\tilde{f}_{N,k} \to \left(\frac{i}{\pi} \partial \bar{\partial} V_{K,\varphi}\right)^k \text{ weak}, \quad \text{a.s.}.
\]

Our next application is to systems of random polynomials with fixed Newton polytopes as discussed in [SZ3]. Given a convex integral polytope \( P \subset \mathbb{C}^m \), we denote by \( \text{Poly}(P) \) the space of polynomials

\[
f(z_1, \ldots, z_m) = \sum_{\alpha \in P \cap \mathbb{Z}^m} c_\alpha z_1^{\alpha_1} \cdots z_m^{\alpha_m}
\]

with Newton polytope contained in \( P \). It is a subspace of \( H^0(\mathbb{CP}^m, \mathcal{O}(p)) \), the space of all homogeneous polynomials of degree \( p \), where \( p \) is the maximal degree of polynomials in \( \text{Poly}(P) \). The \( SU(m+1) \)-invariant inner product on \( H^0(\mathbb{CP}^m, \mathcal{O}(p)) \) then restricts to \( \text{Poly}(P) \) to define an inner product and Gaussian measure there. It is the conditional Gaussian measure on polynomials with the condition of having Newton polytope \( P \). In joint work with Zelditch [SZ2], we studied the asymptotic statistical patterns of zeros of polynomials in \( \text{Poly}(NP) \), where \( NP \) denotes the dilate of \( P \) by \( N \).

We apply Corollary 1.3 with \( L = O(1) \to M = \mathbb{CP}^m \) and \( S_N = \text{Poly}(NP) \) with the conditional Gaussian measure described above. With this choice of ensembles, the expected zero current is not uniformly distributed over \( \mathbb{CP}^m \). Instead, it was shown in [SZ3] that for each integral polytope \( P \), there is associated a (discontinuous, piecewise smooth) \((1,1)\)-form \( \psi_P \) on \((\mathbb{C}^*)^m \) (where \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \)) so that

\[
E(\tilde{f}_{N,k}|(\mathbb{C}^*)^m) \to \psi_P \quad \text{and hence} \quad E(\tilde{f}_{N,1},...,\tilde{f}_{N,k}|(\mathbb{C}^*)^m) \to \psi_P^k, \quad \text{for } 1 \leq k \leq m.
\]

(11)

By Corollary 1.3, we then have:

**Theorem 1.5.** For almost all sequences \( \{(f_{N,1}, ..., f_{N,k})\} \in \text{Poly}(NP)^k, N = 1, 2, 3, \ldots, \)

\[
\tilde{f}_{N,1},...,\tilde{f}_{N,k}|(\mathbb{C}^*)^m \to \psi_P^k \text{ weak*}.
\]

In fact, to each polytope \( P \) there is associated an allowed region \( \mathcal{A}_P \subset (\mathbb{C}^*)^m \) where \( \psi_P = p_\omega \psi_{FS} \) (where \( \psi_{FS} \) denotes the Fubini-Study form on \( \mathbb{CP}^m \)), and hence the zeros of random sections of \( \text{Poly}(NP) \) tend to be equidistributed on \( \mathcal{A}_P \), for \( N \) large. On the complementary forbidden region, \( \psi_P^m = 0 \) and hence a random system of \( m \) polynomials with
Newton polytope \( NP \) has, on average, few zeros in the forbidden region, for \( N \) large. It follows from Theorem 1.5 that sequences of simultaneous zeros of systems of random polynomials \( f_1^N, \ldots, f_m^N \) in \( \text{Poly}(NP) \) will almost surely become concentrated in the allowed region \( \mathcal{A}_P \) and be uniformly distributed there as \( N \to \infty \).

R. Berman [Be2] recently gave an extension to non-positively curved line bundles of the Szegö kernel asymptotics of \([\text{Ca}, \text{Ti}, \text{Ze}]\) on which (3) is based. These asymptotics lead to similar convergence results for random zeros. To state Berman’s result, we let \((L,h) \to (M,\omega)\) be an ample Hermitian line bundle over a compact Kähler manifold. Although \( L \) is assumed to be ample, we do not assume that the metric \( h \) has positive curvature. We give \( H^0(M,L^N) \) the inner product (2) and the induced Szegö kernel \( \Pi_N \) and Gaussian probability measure (see (16)–(17)). We let \( L_{(X,L)} \) denote the class of all (possibly singular) metrics on \( L \) with positive curvature form, and we define the equilibrium metric \( h_e \) on \( L \) by

\[
h_e := \inf \{ \tilde{h} \in L_{(X,L)} : \tilde{h} \geq h \} .
\]

Choosing a local nonvanishing section \( e_L \) of \( L \), we write \( \varphi_e = -\log |e_L|^2_h \), which is plurisubharmonic. Berman showed [Be2, Th. 2.3] that \( \varphi_e \) is \( C^{1,1} \) and that the “equilibrium measure”

\[
\mu_{eq}(h) := \left( \frac{i}{2\pi} \partial \bar{\partial} \varphi_e \right)^m = c_1(L,h_e)^m
\]

is absolutely continuous, i.e., is given by pointwise multiplication of the Chern forms. Berman then showed [Be2, Th. 3.6] that

\[
\frac{1}{N} \log \Pi_N(z,z) \to \log \frac{h(z)}{h_e(z)} \text{ uniformly,}
\]

and hence

\[
E(\tilde{Z}_{s_1^N, \ldots, s_m^N}) \to \mu_{eq}(h) \text{ weak*}.
\]

Thus it follows from (14) and Corollary 1.3 that

\[
\tilde{Z}_{s_1^N, \ldots, s_m^N} \to \mu_{eq}(h) \text{ weak* a.s.} .
\]

Similar results hold for equilibrium measures on pseudoconcave domains in compact Kähler manifolds (see [Be1]).

### 2. Expected distribution of zeros and Szegö kernels

In this section, we review the formulas from [SZ1, SZ4] for the expected current of integration over the zero set of \( k \leq m \) i.i.d. Gaussian random sections of a holomorphic line bundle.

Let \((L,h)\) be a Hermitian holomorphic line bundle over a complex manifold \( M \) and let \( S \) be a finite-dimensional subspace of \( H^0(M,L) \) with a Hermitian inner product. The inner product on \( S \) induces the complex Gaussian probability measure

\[
d\gamma(s) = \frac{1}{\pi^m} e^{-|c|^2} dc , \quad s = \sum_{j=1}^n c_j S_j ,
\]

on \( S \), where \( \{S_j\} \) is an orthonormal basis for \( S \) and \( dc \) is \( 2n \)-dimensional Lebesgue measure. This Gaussian is characterized by the property that the \( 2n \) real variables \( \text{Re} c_j, \text{Im} c_j \)

CONVERGENCE OF RANDOM ZEROS ON COMPLEX MANIFOLDS

$(j = 1, \ldots, n)$ are independent random variables with mean 0 and variance $\frac{1}{2}$; equivalently,

\[ \mathbb{E}c_j = 0, \quad \mathbb{E}c_j c_k = 0, \quad \mathbb{E}c_j \bar{c}_k = \delta_{jk}. \]

To state the explicit formula for the expected distribution of zero divisors, we let

\[ \Pi_S(z, z) = \sum_{j=1}^{n} |S_j(z)|^2_h, \quad z \in M, \]

(17)
denote the Szegő kernel for $S$ on the diagonal.

Remark: The Szegő kernel for the fundamental case $S = H^0(M, L)$ (with the inner product (2)) is given as follows: we let $X \overset{\pi}{\to} M$ denote the circle bundle of unit vectors in the dual bundle $L^{-1} \to M$, and we identify sections $s \in \mathcal{S}$ with functions $\hat{s}$ in the space $\hat{\mathcal{S}}^\infty$ functions on $X$ such that $\bar{\partial}_b \hat{s} = 0$ and $\hat{s}(e^{i\theta}x) = e^{i\theta} \hat{s}(x)$. The Szegő projector is the orthogonal projector $\Pi : L^2(X) \to \hat{\mathcal{S}}$, which is given by the Szegő kernel

\[ \Pi(x, y) = \sum_{j=1}^{n} \overline{\hat{S}_j(x)} \hat{S}_j(y) \quad (x, y \in X). \]

On the diagonal, we may write $\Pi(z, z) = \Pi(x, x)$, where $\pi(x) = z$; then $\Pi(z, z) = \Pi_S(z, z)$ as defined by (17). For details, see [SZ1].

We now consider a local holomorphic frame $e_L$ over a trivializing chart $U$, and we write $S_j = f_j e_L$ over $U$. Any section $s \in \mathcal{S}$ may then be written as

\[ s = \langle c, F \rangle e_L, \quad \text{where} \quad F = (f_1, \ldots, f_n), \quad \langle c, F \rangle = \sum_{j=1}^{n} c_j f_j. \]

If $s = f e_L$, its Hermitian norm is given by $|s(z)|_h = a(z)^{-\frac{1}{2}} |f(z)|$ where $a(z) = |e_L(z)|^{-2}$. The Chern form $c_1(L, h)$ of $L$ is given locally by

\[ c_1(L, h) = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log a. \]

(18)

The current of integration over the zeros of $s = \langle c, F \rangle e_L$ is then given locally by the Poincaré-Lelong formula:

\[ Z_s = \frac{\sqrt{-1}}{\pi} \bar{\partial} \partial \log |\langle c, F \rangle|. \]

(19)

We now recall the formula for the expected zero divisor for the general case where $S$ has base points.

**Proposition 2.1.** ([SZ1 Prop. 3.1], [SZ4 Prop. 2.1]) Let $(L, h)$ be a Hermitian holomorphic line bundle on a complex manifold $M$, and let $\mathcal{S}$ be a finite-dimensional subspace of $H^0(M, L)$. We give $\mathcal{S}$ an inner product and we let $\gamma$ be the induced Gaussian probability measure on $\mathcal{S}$. Then the expected zero current of a random section $s \in \mathcal{S}$ is given by

\[ \mathbb{E}_\gamma(Z_s) = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log \Pi_S(z, z) + c_1(L, h). \]
We note that the expected zero current \( E_T(Z_s) \) is a smooth form outside the base point set of \( S \). Proposition 2.1 also holds for infinite dimensional spaces \( S \); see [SZ4a].

We next state our general result on simultaneous expected zeros:

**Proposition 2.2.** Let \( M \) be a projective algebraic manifold, and let \((L_1, h_1), \ldots, (L_k, h_k)\) be Hermitian holomorphic line bundles on \( M \) (\( 1 \leq k \leq \dim M \)). Suppose we are given subspaces \( S_j \subset H^0(M, L_j) \) with inner products \( \langle \cdot, \cdot \rangle_j \) and let \( \gamma_j \) denote the associated Gaussian probability measure on \( S_j \) (for \( 1 \leq j \leq k \)). Let \( U \) be an open subset of \( M \) on which \( S_j \) has no base points for all \( j \). Then the expected simultaneous zero current of independent random sections \( s_1 \in S_1, \ldots, s_k \in S_k \) is given over \( U \) by

\[
E_{\gamma_1 \times \cdots \times \gamma_k}(Z_{s_1}, \ldots, Z_{s_k}) = \bigwedge_{j=1}^k E_{\gamma_j}(Z_{s_j}) = \bigwedge_{j=1}^k \left[ \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \Pi_{j}(z, z) + c_1(L_j, h_j) \right].
\]

Proposition 2.2 is a generalization of Proposition 2.2 in [SZ4], where the formula is proved under the assumption that the \( S_j \) are identical subspaces of the same line bundle and are base point free on all of \( M \). To use the argument in [SZ4] to prove the above form of the proposition, we must first show that, for each fixed test form \( \varphi \in \mathcal{D}_K^{m-k,m-k}(U) \), the map

\[
(s_1, \ldots, s_k) \mapsto (Z_{s_1}, \ldots, Z_{s_k}, \varphi)
\]

(20)
is \( L^\infty \).

To verify this assertion, we let \( A \) be a very ample line bundle of the form \( A = L \otimes L' \), where \( L' \) is also very ample. Suppose that the \( Z_{s_j} \) are smooth divisors intersecting transversely in \( U \), which is the case almost surely (by Bertini’s theorem), since the \( S_j \) have no base points in \( U \). Let \( \tilde{s}_j = s_j \otimes t_j \in H^0(M, A) \), where the sections \( t_j \in H^0(M, L') \) are chosen so that the zero divisors \( Z_{\tilde{s}_j} \) are smooth and intersect transversely in \( U \). Next deform the sections \( \tilde{s}_j \) to sections \( \sigma_j^\nu \in H^0(M, A) \), with \( \sigma_j^\nu \to \tilde{s}_j \) as \( \nu \to \infty \), such that the zero divisors \( Z_{\sigma_j^\nu} \) are smooth and intersect transversely on all of \( M \). Then

\[
(Z_{\sigma_j^\nu}, \ldots, \sigma_k^\nu, \omega^{m-k}) = \int_M c_1(A, h)^k \wedge \omega^{m-k}.
\]

Letting \( \nu \to \infty \), we conclude that

\[
\int_{Z_{s_1} \cap U} \omega^{m-k} \leq \int_{Z_{\tilde{s}_1} \cap U} \omega^{m-k} = \lim_{\nu \to \infty} \int_{Z_{\sigma_1^\nu} \cap U} \omega^{m-k} \leq \int_M c_1(A, h)^k \wedge \omega^{m-k},
\]

(21)

and hence

\[
\left\| (Z_{s_1}, \ldots, s_k, \varphi) \right\| \leq \frac{\| \varphi \|_\infty}{(m-k)!} \int_{Z_{s_1} \cap U} \omega^{m-k} \leq \frac{\| \varphi \|_\infty}{(m-k)!} \int_M c_1(A, h)^k \wedge \omega^{m-k},
\]

verifying that the map (20) is bounded.

We now can apply the proof in [SZ4]: The case \( k = 1 \) follows from Proposition 2.1 with \( M = U \), and the inductive step follows by the proof of Proposition 2.2 in [SZ4] with \( M \) replaced by \( U \).
3. The variance estimate

In this section, we prove the following variance estimate, which is a slight generalization of Theorem 3.1.

**Theorem 3.1.** Let $L_1, \ldots, L_k$ be stably base point free line bundles on a projective Kähler manifold $(M, \omega)$, where $1 \leq k \leq m = \dim M$. Suppose we are given subspaces $S_j \subset H^0(M, L_j)$ endowed with Gaussian probability measures $d\gamma_j$ (for $1 \leq j \leq k$). Let $\varphi \in \mathcal{D}^{m-k,m-k}_R(U)$, where $U$ is an open subset of $M$ on which $S_j$ has no base points for $1 \leq j \leq k$.

Then for a system $S = (s_1, \ldots, s_k)$ of sections of $S_1, \ldots, S_k$ chosen independently and at random, we have:

$$\sqrt{\text{Var}(Z_S, \varphi)} \leq C_m \|\partial \bar{\partial} \varphi\|_\infty \int_M \omega^{m-k+1} \wedge \sum_{\lambda=1}^k \prod_{1 \leq j \leq k, j \neq \lambda} c_1(L_j),$$

where $C_m$ is a universal constant depending only on the dimension $m$.

A line bundle $L$ is said to be stably base point free if the base point set of $H^0(M, L^N)$ is empty for $N$ sufficiently large. In particular, ample line bundles are stably base point free as a consequence of the Kodaira embedding theorem.

**Remark:** We remark that the hypothesis that $L$ is stably base point free is essential for the estimate of Theorem 3.1, indeed, the stated upper bound of the theorem might be negative. For example, let $m = k = 3$ and let $M = Y \times \mathbb{CP}^1$, where $Y$ is the blow-up of a point in $\mathbb{CP}^2$. To construct the line bundles, we let $E$ be the exceptional divisor in $Y$ and $\hat{H}$ the pull-back to $Y$ of a line $H \subset \mathbb{CP}^2$, and we let $\pi_1 : M \to Y$, $\pi_2 : M \to \mathbb{CP}^1$ denote the projections. We then let $L_j = L_D$ ($j = 1, 2, 3$), where $D = \pi_1^*(\hat{H} + 4E) + \pi_2^*\{p\}$, and let $[\omega] = \pi_1^*(2\hat{H} - E) + \pi_2^*\{p\}$. Then

$$\int_M \omega \wedge c_1(L_D)^2 = (\hat{H} + 4E)^2 + 2(\hat{H} + 4E) \cdot (2\hat{H} - E) = -15 + 12 = -3.$$

We shall prove Theorem 3.1 by induction on $k$. The case $k = 1$ is essentially Lemma 3.3 in [SZ1]. To go from $k = 1$ to $k = 2$ (and subsequently to higher $k$), we shall use the fact that $Z_{s_1,s_2} = Z_{s_1} \cap Z_{s_2}$ is the current of integration over the intersection $Z_{s_1} \cap Z_{s_2}$ and hence $(Z_{s_1,s_2}, \varphi)$ reduces to the integration of $\varphi|_{Z_{s_1}}$ against $Z_{s_2}|_{Z_{s_1} \cap U}$, which is almost surely smooth.

We begin with the $k = 1$ step, which is based on a result from [SZ1].

**Lemma 3.2.** Under the hypotheses and notation of Proposition 2.1, we have

$$\text{Var}(Z_s, \varphi) \leq C \|\partial \bar{\partial} \varphi\|_1^2, \quad \varphi \in \mathcal{D}^{m-1,m-1}_R(M),$$

where $C$ is a universal constant. (The main point is that the constant $C$ is independent of $\dim S$ as well as $M$ and $L$.)

**Proof.** For completeness, we include a modified version of the argument of [SZ1], Lemma 3.3]. As in [2], we let $\{S_j\}$ be an orthonormal basis for $S$ and we write sections locally as

$$s = \sum_{j=1}^n c_j S_j = \langle c, S \rangle = \langle c, F \rangle e_L.$$
where $c = (c_1, \ldots, c_n)$, $S = (S_1, \ldots, S_n)$, $F = (f_1, \ldots, f_n)$. By (18)–(19), we have
\[
Z_s = \frac{1}{\pi} \partial \bar{\partial} \log |\langle c, F \rangle| = \frac{1}{\pi} \partial \bar{\partial} \log |\langle c, S \rangle|_h + c_1(L, h) .
\]

Let $\varphi \in D^{m-1,m-1}_0(M)$ and consider the random variable $Y : S \to \mathbb{C}$ given by
\[
Y(s) = (Z_s, \varphi) - \int_M c_1(L, h) \wedge \varphi = \left( \frac{1}{\pi} \partial \bar{\partial} \log |\langle c, S \rangle|_h, \varphi \right) = \frac{1}{\pi} \int_M \log |\langle c, S \rangle|_h \partial \bar{\partial} \varphi . \tag{22}
\]

We note that $\text{Var}(Z_s, \varphi) = \text{Var}(Y)$. By Proposition 2.1, we have
\[
\mathbb{E}(Y) = \frac{1}{2\pi} \int_M \log \Pi_S(z, z) \partial \bar{\partial} \varphi(z) = \frac{1}{\pi} \int_M \log |S|_h \partial \bar{\partial} \varphi . \tag{23}
\]
Furthermore, by (22) we have
\[
\mathbb{E}(Y^2) = \frac{-1}{\pi^2} \int_M \int_M \partial \bar{\partial} \varphi(z) \partial \bar{\partial} \varphi(w) \int_{\mathbb{C}^n} \log |\langle c, S(z) \rangle|_h \log |\langle c, S(w) \rangle|_h d\gamma(c) . \tag{24}
\]
We let $u(z) = |S(z)|_h^{-1}S(z)$ so that $|u(z)|_h \equiv 1$, and we have
\[
\log |\langle c, S(z) \rangle|_h \log |\langle c, S(w) \rangle|_h = \log |S(z)|_h \log |S(w)|_h + \log |S(z)|_h \log |\langle c, u(w) \rangle|_h + \log |\langle c, u(z) \rangle|_h \log |\langle c, u(w) \rangle|_h ,
\]
which decomposes (24) into four terms. By (23), the first term contributes
\[
\frac{-1}{\pi^2} \int_M \int_M \partial \bar{\partial} \varphi(z) \partial \bar{\partial} \varphi(w) \log |S(z)|_h \log |S(w)|_h = (\mathbb{E}Y)^2 . \tag{25}
\]
The $c$-integral of the second term is independent of $w$ and hence the second term in the expansion of (24) vanishes. The third term likewise vanishes. Therefore,
\[
\text{Var}(Z_s, \varphi) = \frac{-1}{\pi^2} \int_M \int_M \partial \bar{\partial} \varphi(z) \partial \bar{\partial} \varphi(w) \int_{\mathbb{C}^n} \log |\langle c, u(z) \rangle|_h \log |\langle c, u(w) \rangle|_h d\gamma(c) . \tag{26}
\]
By Cauchy-Schwartz,
\[
\left| \int_{\mathbb{C}^n} \log |\langle c, u(z) \rangle| \log |\langle c, u(w) \rangle| d\gamma(c) \right| \leq \left( \int_{\mathbb{C}^n} (\log |\langle c, u(z) \rangle|)^2 d\gamma(c) \right)^{1/2} \left( \int_{\mathbb{C}^n} (\log |\langle c, u(w) \rangle|)^2 d\gamma(c) \right)^{1/2} = \int_{\mathbb{C}^n} (\log |c_1|)^2 d\gamma(c) = \frac{1}{\pi} \int_{\mathbb{C}} (\log |c_1|)^2 e^{-|c_1|^2} dc_1 . \tag{27}
\]
The conclusion follows immediately from (26)–(27).
Proof of Theorem 3.7: We shall prove by induction on $k$ that the variance bound holds when $\omega$ is an arbitrary closed semi-positive $(1, 1)$-form on $M$ that is strictly positive on $U$. Let $\omega$ be such a form, and let $\Omega := \frac{1}{m!} \omega^m |U$ denote the induced volume form on $U$. Let $\eta \in D^{2m}(U)$ be a compactly supported, top degree form on $U$, and write $\eta = f \Omega$. We define the sup norm $||\eta||_\infty := ||f||_\infty$. The $L^1$ norm is given by

$$||\eta||_1 = \int_U |f| \Omega \leq ||\eta||_\infty \int_U \Omega = ||\eta||_\infty \text{Vol}(U).$$  \hfill (28)

We note that while the $L^\infty$ norm depends on $\omega$, the $L^1$ norm of $\eta$ is independent of the choice of the Kähler form on $U$.

The case $k = 1$ is an immediate consequence of Lemma 3.2 (with $M$ replaced by $U$) and (28). Now let $2 \leq k \leq m$ and assume the inequality has been proven for $k - 1$ sections. We let $S = (s_1, \ldots, s_k) \in \prod_{j=1}^k S_j$ be a random $k$-tuple of sections. We write $S = (S', s_k)$, where $S' = (s_1, \ldots, s_{k-1})$.

By Bertini’s Theorem, the hypersurfaces $|Z_{s_j}|$ ($1 \leq j \leq k$) are smooth in $U$ and intersect transversely in $U$ for almost all $S$, so that we may write $Z_S = Z_{S'} \backslash Z_{s_k}$. Let $\varphi \in D^{m-k, m-k}(U)$ be a test form. By Proposition 2.2 $E Z_S = E Z_{S'} \wedge E Z_{s_k}$ and hence

$$\text{Var}(Z_S, \varphi) = E (Z_S \wedge Z_{s_k}, \varphi)^2 - (E Z_{S'} \wedge E Z_{s_k}, \varphi)^2. \hfill (29)$$

We write

$$(Z_{S'} \wedge Z_{s_k}, \varphi)^2 - (E Z_{S'} \wedge E Z_{s_k}, \varphi)^2 = G_1 + G_2, \quad \text{where}$$  \hfill (30)

$$G_1 = G_1(S', s_k) = (Z_{S'} \wedge Z_{s_k}, \varphi)^2 - (Z_{S'} \wedge E Z_{s_k}, \varphi)^2 \text{ a.e.}, \hfill (31)$$

$$G_2 = G_2(S') = (Z_{S'} \wedge E Z_{s_k}, \varphi)^2 - (E Z_{S'} \wedge E Z_{s_k}, \varphi)^2 \text{ a.e.} \hfill (32)$$

Hence

$$\text{Var}(Z_S, \varphi) = EG_1 + EG_2. \hfill (33)$$

We now let $V = |Z_{S'}| \cap U$. (Recall that $|Z|$ denotes the support of a zero current $Z$.) The idea of the proof is to notice that $(Z_{S'} \wedge Z_{s_k}, \varphi) = (Z_{s_k}|_V, \varphi|_V)$ and then to apply Lemma 3.2 with $M$ replaced by $|Z_{S'}|$ and with $Z_s = Z_{s_k}$ in order to obtain the desired bound for $EG_1$. We then reverse the roles of $Z_{S'}$ and $Z_{s_k}$ and use a similar argument to obtain the bound for $EG_2$.

To obtain a bound for $EG_1$, we first integrate over $S_k$:

$$\int_{S_k} G_1(S', s_k) \, d\gamma_k(s_k) = \int_{S_k} [(Z_{s_k}|_V, \varphi|_V)^2 - (E Z_{s_k}|_V, \varphi|_V)^2] \, d\gamma_k(s_k)$$

$$= \text{Var}(Z_{s_k}|_V, \varphi|_V) \leq C \left( \int_V |\bar{\partial} \varphi| \right)^2$$

$$\leq C \left[ ||\partial \bar{\partial} \varphi||^2_{\infty} \int_V \omega^{m-k+1} \right]_V, \hfill (34)$$

where the first inequality is by Lemma 3.2 (with $M$ replaced by $V$).

We claim that

$$\int_V \omega^{m-k+1} \leq \int_M \omega^{m-k+1} \wedge \prod_{j=1}^{k-1} c_1(L_j) \text{ for a.a. } S'. \hfill (35)$$
To verify (35), choose a positive integer \( N \) so that the line bundles \( L_j^N \) are base point free, and then (by applying Bertini’s theorem, as before) deform the sections \( s_j^\nu \) to sections \( \sigma_j^\nu \in H^0(M, L_j^N) \), with \( \sigma_j^\nu \to s_j^\nu \) as \( \nu \to \infty \), such that the zero sets \( Z_{\sigma_1^\nu, \ldots, \sigma_{k-1}^\nu} \) are smooth reduced varieties of dimension \( m - k + 1 \) (in all of \( M \)). We then have

\[
\int_{Z_{\sigma_1^\nu, \ldots, \sigma_{k-1}^\nu} \cap \bar{U}} \omega^{m-k+1} \leq \int_{Z_{\sigma_1^\nu, \ldots, \sigma_{k-1}^\nu}} \omega^{m-k+1} = N^{k-1} \int_M \omega^{m-k+1} \wedge \prod_{j=1}^{k-1} c_1(L_j). \tag{36}
\]

Letting \( \nu \to \infty \) and noting that

\[
\int_{Z_{\sigma_1^\nu, \ldots, \sigma_{k-1}^\nu} \cap \bar{U}} \omega^{m-k+1} \to N^{k-1} \int_V \omega^{m-k+1},
\]

we obtain (35). Hence by (34)–(35), we have

\[
\text{EG}_1 = \int_{S'} \int_{S_k} G_1(S', s_k) d\gamma_k(s_k) d\gamma'(S') \leq C \| \overline{\partial \partial} \varphi \|_\infty^2 \left( \int_M \omega^{m-k+1} \wedge \prod_{j=1}^{k-1} c_1(L_j) \right)^2. \tag{37}
\]

where \( S' = S_1 \times \cdots \times S_{k-1}, \gamma' = \gamma_1 \times \cdots \times \gamma_{k-1}, \) and \( C \) denotes a constant depending only on \( m \).

We now estimate \( \text{EG}_2 \). First we note that \( \text{E}(G_2) \) is the variance of the random variable \( X \) on \( S' \) given a.e. by

\[
X(S') = (Z_{S'} \cap E Z_{S_k}, \varphi).
\]

Hence

\[
\text{EG}_2 = \text{Var}(X) = \text{EG}_2',
\]

where

\[
G_2'(S') := [X(S') - E X]^2 = ((Z_{S'} - E Z_{S'}) \wedge E Z_{S_k}, \varphi)^2. \tag{38}
\]

By Cauchy-Schwartz, we have the upper bound

\[
G_2'(S') = \left[ \int_{S_k} ((Z_{S'} - E Z_{S'}) \wedge Z_{S_k}, \varphi) d\gamma_k(s_k) \right]^2 \leq \int_{S_k} ((Z_{S'} - E Z_{S'}) \wedge Z_{S_k}, \varphi)^2 d\gamma_k(s_k),
\]

for all \( S' \) such that \( Z_{S'} \cap U \) is a smooth submanifold of codimension \( k - 1 \).

Writing \( W_{S_k} = |Z_{S_k}| \cap U \), we then have

\[
\text{EG}_2 = \text{EG}_2' \leq \int_{S'} d\gamma'(S') \int_{S_k} d\gamma_k(s_k) \left( (Z_{S'} - E Z_{S'}) \wedge Z_{S_k}, \varphi \right)^2 \leq \int_{S_k} d\gamma_k(s_k) \int_{S'} d\gamma'(S') \left( (Z_{S'} - E Z_{S'})|_{W_{S_k}}, \varphi|_{W_{S_k}} \right)^2 \leq \int_{S_k} d\gamma_k(s_k) \text{Var}(Z_{S'}|_{W_{S_k}}, \varphi|_{W_{S_k}}). \tag{39}
\]
where the variance is with respect to \( S' \). If \(|Z_{sk}|\) is a smooth submanifold of \( M \), then we can apply the inductive hypothesis to \(|Z_{sk}|\) to conclude that

\[
\sqrt{\text{Var}(Z_{sk}|w_{sk}, \varphi|w_{sk})} \leq C_{m-1} \| \bar{\partial} \partial \varphi \|_{\infty} \int_{|Z_{sk}|} \omega^{m-k+1} \wedge \sum_{\lambda=1}^{k-1} \left[ \prod_{1 \leq j \leq k-1, j \neq \lambda} c_1(L_j) \right] \\
= C_{m-1} \| \bar{\partial} \partial \varphi \|_{\infty} \int_{M} \omega^{m-k+1} \wedge \sum_{\lambda=1}^{k-1} \left[ \prod_{1 \leq j \leq k-1, j \neq \lambda} c_1(L_j) \right]. \tag{40}
\]

If \( L_k \) is base point free on \( M \), then \(|Z_{sk}|\) will almost surely be smooth and hence (40) will hold almost surely. For the general case, we use the following argument: Since \( S_k \) has no base points in \( U \), \( Z_{sk} \) will almost surely be smooth in \( U \). Now suppose \( Z_{sk} \) is smooth in \( U \), but has singularities in \( M \). Let \( \pi: \bar{M} \to M \) be a resolution of the singularities of \( Z_{sk} \); i.e., \( \pi \) is a modification of \( M \) that is biholomorphic outside the singular locus of \( Z_{sk} \) such that the proper transform \( \bar{Z}_{sk} \subset \bar{M} \) of \( Z_{sk} \) is smooth. Since \( Z_{sk} \) is smooth in \( U \), \( \pi \) does not blow up points of \( U \). Applying the inductive assumption to the linear systems \( \bar{S}_j := \pi^* S_j \bar{Z}_{sk} \) \((1 \leq j \leq k-1)\) and semi-positive form \( \bar{\omega} := \pi^* \omega \bar{Z}_{sk} \) (which is strictly positive on \( U = U \)), we obtain (40) for almost all \( s_k \in S_k \).

Hence it follows from (39)–(40) that

\[
\text{EG}_2 \leq C_{m-1}^2 \| \bar{\partial} \partial \varphi \|_{\infty}^2 \left( \int_{M} \omega^{m-k+1} \wedge \sum_{\lambda=1}^{k-1} \left[ \prod_{1 \leq j \leq k-1, j \neq \lambda} c_1(L_j) \right] \right)^2. \tag{41}
\]

The inductive step follows from (33), (37) and (41).

\[\square\]

### 4. Almost sure convergence of zeros

We complete this paper by verifying Theorem 1.2 and Corollary 1.3. Theorem 1.2 is a consequence of the following variant of Theorem 1.1:

**Proposition 4.1.** Let \( L \to (M, \omega) \) be a holomorphic line bundle over a compact Kähler manifold of dimension \( m \), and let \( S_j \subset H^0(M, L) \), \( j = 1, \ldots, k \), be linear spaces endowed with Gaussian probability measures. Let \( \varphi \in D^{m-k, m-k}_R(U) \), where \( U \) is an open subset of \( M \) on which \( S_j \) has no base points for \( j = 1, \ldots, k \).

Suppose that \( A \) is a very ample line bundle on \( M \) of the form \( A = L \otimes L' \), where \( L' \) is also very ample. Then for independent random line bundles \( s_j \in S_j \),

\[
\sqrt{\text{Var}(Z_{s_1, \ldots, s_k}, \varphi)} \leq C_m \| \bar{\partial} \partial \varphi \|_{\infty} \int_{M} \omega^{m-k+1} \wedge c_1(A)^{k-1},
\]

where the constant \( C_m \) depends only on the dimension \( m \) of \( M \).

**Proof.** The result follows by repeating the proof of Theorem 3.1 with \( c_1(L_j) \) replaced by \( c_1(A) \). Instead of (35), we use the inequality

\[
\int_{|Z_{s_j}| \cap U} \omega^{m-k+1} \leq \int_{M} \omega^{m-k+1} \wedge c_1(A)^{k-1}. \tag{42}
\]
where $S' = (s_1, \ldots, s_{k-1}) \in \prod_{j=1}^{k-1} S_j$ is chosen as before so that $|Z_{S'}| \cap U$ is a smooth reduced submanifold of dimension $m - k + 1$. The inequality (42) is the same as the inequality (21) (with $k$ replaced by $k - 1$), which was verified in the proof of Proposition 2.2. In place of (40), we have

$$\sqrt{\text{Var}(Z_{S'}|W_{sk}, \varphi|W_{sk})} \leq C_{m-1} \|\partial \bar{\partial} \varphi\|_{\infty} \int_{|Z_{sk}|} \omega^{m-k+1} \wedge c_1(A)^{k-1}$$

and hence

$$\sqrt{\text{Var}(Z_{s_1, \ldots, s_k}|W_{sk}, \varphi|W_{sk})} = O(N^{-1}) .$$

The proof of (43) is exactly the same as that of (40).

**Proof of Theorem 1.2:** Let $L \to M$, $U$, $(S_j^N, \gamma_j^N)$, $\varphi$ be as in the statement of the theorem. Let $L'$ be an ample line bundle on $M$ such that the line bundle $A := L \otimes L'$ is ample. Applying Proposition 4.1 with $L, L', A$ replaced with $L^N, L'^N, A^N$, respectively, we conclude that

$$\sqrt{\text{Var}(Z_{s_1, \ldots, s_k}|W_{sk}, \varphi|W_{sk})} \leq C_m \|\partial \bar{\partial} \varphi\|_{\infty} N^{k-1} \int_{M} \omega^{m-k+1} \wedge c_1(A)^{k-1} = O(N^{k-1}) ,$$

and hence

$$\sqrt{\text{Var}(\tilde{Z}_{s_1, \ldots, s_k}, \varphi)} = O(N^{-1}) .$$

**Proof of Corollary 1.3:** The proof follows from the elementary argument in §3.3 of [SZ1], which we include here for completeness. Consider a random sequence $s = \{S_N\}$ in $S_\infty$, where $S_N = (s_1^N, \ldots, s_k^N) \in (S_N)^k$. Since the masses of $\tilde{Z}_{S_N}$ are bounded independent of $N$, we may assume that $\varphi$ is a smooth form in $D_{\mathbb{R}}^{m-k,m-k}(U)$. Now consider the random variables

$$Y_N(s) := (\tilde{Z}_{S_N} - E\tilde{Z}_{S_N}, \varphi)^2 \geq 0 .$$

By Theorem 1.2 we have

$$\int_S Y_N(s) d\gamma(s) = \text{Var}(\tilde{Z}_{S_N}, \varphi) = O \left( \frac{1}{N^2} \right) .$$

Therefore

$$\int_S \sum_{N=1}^{\infty} Y_N d\gamma = \sum_{N=1}^{\infty} \int_S Y_N d\gamma < +\infty ,$$

and hence $Y_N \to 0$ almost surely, i.e.

$$(\tilde{Z}_{S_N}, \varphi) - (E\tilde{Z}_{S_N}, \varphi) \to 0 \quad a.s. \quad (45)$$

By hypothesis,

$$(E\tilde{Z}_{S_N}, \varphi) \to \int_U \Psi \wedge \varphi ,$$

(46)
and therefore by (45)–(46),

$$\langle Z_{S_N}, \varphi \rangle \to \int_U \Psi \wedge \varphi \quad \text{a.s.},$$

completing the proof of Corollary 1.3.

References

[Be1] R. Berman, Bergman kernels, random zeroes and equilibrium measures for polarized pseudoconcave domains, arXiv:math/0608226v2.
[Be2] R. Berman, Bergman kernels and equilibrium measures for ample line bundles, arXiv:0704.1640v1.
[BSZ1] P. Bleher, B. Shiffman and S. Zelditch, Universality and scaling of correlations between zeroes on complex manifolds, Invent. Math. 142 (2000), 351–395.
[BSZ2] P. Bleher, B. Shiffman, and S. Zelditch, Correlations between zeros and supersymmetry, Comm. Math. Phys. 224 (2001), 255–269.
[Bli] T. Bloom, Random polynomials and Green functions, Int. Math. Res. Not. 2005 (2005), 1689–1708.
[Bli2] T. Bloom, Random polynomials and (pluri)potential theory, Ann. Polon. Math. 91 (2007), 131-141.
[BS] T. Bloom and B. Shiffman, Zeros of random polynomials on $\mathbb{C}^n$, Math. Res. Lett. 14 (2007), 469-479.
[BBL] E. Bogomolny, O. Bohigas, and P. Lebeuf, Quantum chaotic dynamics and random polynomials, J. Statist. Phys. 85 (1996), 639–679.
[Ca] D. Catlin, The Bergman kernel and a theorem of Tian, Analysis and Geometry in Several Complex Variables, G. Komatsu and M. Kuranishi, eds., Birkhäuser, Boston, MA, 1999.
[EK] A. Edelman and E. Kostlan, How many zeros of a random polynomial are real? Bull. Amer. Math. Soc. 32 (1995), 1–37.
[FH] P. J. Forrester and G. Honner, Exact statistical properties of the zeros of complex random polynomials, J. Phys. A 32 (1999), 2961–2981.
[Ham] J. M. Hammersley, The zeros of a random polynomial, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, 89–111, University of California Press, Berkeley and Los Angeles, 1956.
[Ham] J. H. Hannay, Chaotic analytic zero points: exact statistics for those of a random spin state, J. Phys. A 29 (1996), L101–L105.
[Kac] M. Kac, On the average number of real roots of a random algebraic equation, II, Proc. London Math. Soc. 50 (1949), 390–408.
[NV] S. Nonnenmacher and A. Voros, Chaotic eigenfunctions in phase space, J. Statist. Phys. 92 (1998), 431–518.
[Ro] J. M. Rojas, On the average number of real roots of certain random sparse polynomial systems, The mathematics of numerical analysis (Park City, UT, 1995), 689–699, Lectures in Appl. Math. 32, Amer. Math. Soc., Providence, RI, 1996.
[SZ1] B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles, Comm. Math. Phys. 200 (1999), 661–683.
[SZ2] B. Shiffman and S. Zelditch, Equilibrium distribution of zeros of random polynomials, Int. Math. Res. Not. 2003 (2003), 25–49.
[SZ3] B. Shiffman and S. Zelditch, Random polynomials with prescribed Newton polytope, J. Amer. Math. Soc. 17 (2004), 49–108.
[SZ4] B. Shiffman and S. Zelditch, Number variance of random zeros on complex manifolds, Geom. Funct. Anal., to appear.
[SZ4a] B. Shiffman and S. Zelditch, Number variance of random zeros on complex manifolds, arXiv:math/0608743v1. (This early version of [SZ4] contains additional results to be published elsewhere.)
[SS] M. Shub and S. Smale, Complexity of Bezout’s theorem. II. Volumes and probabilities, Computational algebraic geometry (Nice, 1992), 267–285, Progr. Math. 109, Birkhäuser, Boston, MA, 1993.
[ST] M. Sodin and B. Tsirelson, Random complex zeros, I. Asymptotic normality, *Israel J. Math.* 144 (2004), 125–149.

[Ti] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, *J. Diff. Geometry* 32 (1990), 99–130.

[Ws] M. Wschebor, On the Kostlan-Shub-Smale model for random polynomial systems. Variance of the number of roots, *J. Complexity* 21 (2005), 773–789.

[Ze] S. Zelditch, Szegő kernels and a theorem of Tian, *Int. Math. Res. Not.* 1998 (1998), 317–331.

[Zr] S. Zrebiec, The order of the decay of the hole probability for Gaussian random SU$(m + 1)$ polynomials, arXiv:0704.2733v1.

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA

E-mail address: shiffman@math.jhu.edu