BOLTZMANN DIFFUSIVE LIMIT WITH MAXWELL BOUNDARY CONDITION

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ABSTRACT. Based on a recent $L^6 - L^\infty$ approach, validity of diffusive limit is established for both steady and unsteady Boltzmann equation in the presence of the classical Maxwell boundary condition for a full range of the accommodation coefficient $0 \leq \alpha \leq 1$. A general stretching method is developed to control bouncing trajectories for the specular reflection with $\alpha = 0$ in the hydrodynamic limit, and refined estimates uniform with respect to $0 \leq \alpha \leq 1$ for the macroscopic distribution $Pf$ are derived.

Contents

1. Introduction 1
1.1. Background 1
1.2. Main results 4
1.3. Difficulties and illustrations 8
1.4. Organization of the paper 14
2. Steady Limit 14
2.1. $L^2$ coercivity estimate and $L^6$ bound 14
2.2. $L^\infty$ estimate 25
2.3. Validity of the steady problem 36
3. Unsteady Limit 42
3.1. $L^2$ coercivity estimate and $L^2L^3_t$ bound 42
3.2. $L^\infty$ estimate 55
3.3. Validity of unsteady equation 59
Appendix A. Elliptic estimates 66
References 70

1. Introduction

1.1. Background.

The purpose of this paper is to establish the incompressible Navier-Stokes-Fourier limit from the steady and unsteady Boltzmann equation to general Maxwell boundary condition in bounded domain. The rescaled Boltzmann equation, under the action of the external field $\Phi$, is given in the diffuse regime with small Knudsen and Mach numbers

$$
\begin{align*}
\varepsilon \partial_t F + v \cdot \nabla_x F + \varepsilon^2 \Phi \cdot \nabla_v F &= \varepsilon^{-1} Q(F, F) \quad \text{in } \mathbb{R}_+ \times \Omega \times \mathbb{R}^3, \\
F|_{\gamma_-} &= (1 - \alpha) \mathcal{L}F + \alpha \mathcal{P}_w F \quad \text{on } \mathbb{R}_+ \times \gamma_-, \\
F(t, x, v)|_{t=0} &= F_0(x, v) \geq 0 \quad \text{on } \Omega \times \mathbb{R}^3,
\end{align*}
$$

(1.1)

(1.2)

(1.3)
where \( F(t, x, v) \) is the distribution for particles at time \( t \geq 0 \), position \( x \in \Omega \) and velocity \( v \in \mathbb{R}^3 \). The Boltzmann collision operator \( Q \) takes the form

\[
Q(F_1, F_2)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \omega) [F_1(v')F_2(u') - F_1(v)F_2(u)] d\omega du
\]

where \( v' = v - [(v-u) \cdot \omega] \omega \), \( u' = u + [(v-u) \cdot \omega] \omega \) and \( B \) is chosen as the hard spheres cross section

\[
B(v-u, \omega) = |(v-u) \cdot \omega|.
\]

Throughout this paper, \( \Omega = \{ x : \xi(x) < 0 \} \) is a general bounded domain in \( \mathbb{R}^3 \), not necessarily convex. Denote its boundary by \( \partial \Omega = \{ x : \xi(x) = 0 \} \), where \( \xi(x) \) is a \( C^2 \) function. We assume \( \nabla \xi(x) \neq 0 \) at \( \partial \Omega \). The outward normal vector at \( \partial \Omega \) is given by

\[
n(x) = \frac{\nabla \xi(x)}{|\nabla \xi(x)|},
\]

and it can be extended smoothly near \( \partial \Omega \). We denote the phase boundary in the space \( \Omega \times \mathbb{R}^3 \) as \( \gamma = \partial \Omega \times \mathbb{R}^3 \), and split it into an outgoing boundary \( \gamma^+ \), an incoming boundary \( \gamma^- \) and a singular boundary \( \gamma_0 \) for grazing velocities

\[
\gamma^+ := \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v > 0 \}, \quad \gamma^- := \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v < 0 \}, \quad \gamma_0 := \{(x, v) \in \partial \Omega \times \mathbb{R}^3 : n(x) \cdot v = 0 \}.
\]

The physically relevant boundary condition \( \text{(1.2)} \), called the Maxwell boundary condition, was proposed by Maxwell [13] in 1879 to describe the interaction between particles and the wall. The non-dimensional parameter \( \alpha \in [0, 1] \), called the accommodation coefficient, is used to measure the roughness of the boundary. If the boundary is perfectly smooth, the reflection is specular, which corresponds to \( \alpha = 0 \) in \( \text{(1.2)} \) and is called the specular reflection boundary condition. While if the boundary is rough, the reflection is diffuse, which corresponds to \( \alpha = 1 \) in \( \text{(1.2)} \) and is called the diffuse reflection boundary condition. The specular reflection operator \( \mathcal{L} \) is given by

\[
\mathcal{L} F(t, x, v) = F(t, x, R_x v),
\]

where \( R_x v = v - 2[n(x) \cdot v]n(x) \) is the velocity before the collision against the wall. The diffuse reflection operator \( \mathcal{P} \) is given by

\[
\mathcal{P} F(t, x, v) = M_w(x, v) \int_{n(x) \cdot u > 0} F(t, x, u)[n(x) \cdot u] du,
\]

where \( M_w \) is the wall Maxwellian defined for a prescribed wall temperature \( T_w \) on \( \partial \Omega \)

\[
M_w = \sqrt{\frac{2\pi}{T_w}} M_{1,0,T_w},
\]

and \( M_{\rho,u,T} \) is the local Maxwellian with the density \( \rho \), bulk velocity \( u \) and temperature \( T \)

\[
M_{\rho,u,T} = \frac{\rho}{(2\pi T)^{3/2}} \exp \left( -\frac{|v-u|^2}{2T} \right).
\]

The prescribed wall temperature is given by \( T_w = 1 + \varepsilon \vartheta_w \), where \( \vartheta_w \in W^{1,\infty}(\partial \Omega) \) is a given function defined on \( \partial \Omega \).

The incompressible Navier-Stokes-Fourier system (INSF) can be derived from the rescaled Boltzmann equation \( \text{(1.1)} \) through a scaling

\[
F = \mu + \varepsilon \sqrt{\mu} f,
\]

where \( f \) is the fluctuation and \( \mu \) is the global Maxwellian

\[
\mu(v) = M_{1,u,1} = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}.
\]
Indeed, we will show that
\[ f \rightarrow f_1 = \left[ \rho + u \cdot v + \theta \left\| v \right\|^2 - 3 \right] \sqrt{\mu} \text{ weakly in } L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3) \] as \( \varepsilon \rightarrow 0 \),

where \((\rho, u, \theta)\) represents the mass density, bulk velocity and temperature fluctuations physically and satisfies INSF

\[
\begin{align*}
\partial_t u + u \cdot \nabla_x u + \nabla_x p &= \sigma \Delta u + \Phi, \quad \nabla_x \cdot u = 0 \quad \text{in } \mathbb{R}_+ \times \Omega, \\
\partial_t \theta + u \cdot \nabla_x \theta &= \kappa \Delta \theta, \quad \nabla_x (\rho + \theta) = 0 \quad \text{in } \mathbb{R}_+ \times \Omega, \\
u|_{t=0} = u_0, \quad \theta|_{t=0} = \theta_0 \quad \text{on } \Omega
\end{align*}
\]

supplemented by the Dirichlet boundary condition

\[ u|_{\partial \Omega} = 0, \quad \theta|_{\partial \Omega} = \vartheta \quad \text{on } \mathbb{R}_+ \times \partial \Omega \]

if \( \lim_{\varepsilon \rightarrow 0} \frac{\sqrt{\varepsilon}}{\varepsilon} = \infty \), and by the Navier boundary condition

\[
\begin{align*}
\alpha (\nabla_x u + (\nabla_x u)^T) \cdot n + \lambda u \right\| \tan &= 0, \quad u \cdot n = 0 \quad \text{on } \mathbb{R}_+ \times \partial \Omega, \\
\kappa \partial_n \theta + \frac{4}{9} \lambda (\theta - \vartheta_w) &= 0 \quad \text{on } \mathbb{R}_+ \times \partial \Omega
\end{align*}
\]

if \( \lim_{\varepsilon \rightarrow 0} \frac{\sqrt{\varepsilon}}{\varepsilon} = \sqrt{2\pi} \lambda \in [0, \infty) \). Here \( \sigma > 0 \) stands for the kinematic viscosity, \( \kappa > 0 \) denotes the heat thermal conductivity. In the Navier boundary condition \([1.14]\), proposed by Navier \([15]\) in 1823, \( \lambda \) denotes the reciprocal of the slip length, reflecting the interaction between the fluid and the boundary. In particular, \( \lambda = 0 \) corresponds to \textit{perfect slipping}. Moreover, we can show that the limit point of the steady Boltzmann equation leads to the corresponding steady INSF.

The subject of derivation of fluid dynamical equations from kinetic theory of gases goes back to the founding work of Maxwell \([42]\) and Boltzmann \([9]\). At a time when the existence of atoms was controversial, kinetic theory could explain how to estimate the size of a gas molecule from macroscopic data such as the viscosity of the gas. Much later, in his famous sixth problem, Hilbert proposed the question of describing the transition between atomistic and continuous models for gas dynamics by rigorous mathematical convergence results \([33]\). Hilbert himself investigated the connection between the Boltzmann equation and hydrodynamics at formal level in the pioneering work \([34]\). Since then, the hydrodynamic limits from the Boltzmann equation have been gotten a lot of interest and a huge amount of literatures have been devoted to this field.

It is known that INSF can be derived formally from the Boltzmann equation in a regime of small, slowly varying fluctuations of number density about a uniform Maxwellian state. Bardos, Golse and Levermore \([3, 13]\) initiated the program of showing that the DiPerna-Lions renormalized solutions \([14]\) of the Boltzmann equation converge to the Leray-Hopf weak solutions of INSF with some a priori assumptions. Since then a series of contributions \([6, 7, 22, 30, 17]\) have been made to remove some of these assumptions. A complete proof for such a limit was established by Golse and Saint-Raymond for bounded collision kernel \([23]\). This breakthrough stimulated further investigations with more general collision kernels \([24, 39]\). As for bounded spatial domain, Masmoudi and Saint-Raymond justified linear Stokes-Fourier limit \([14]\), and Saint-Raymond considered Navier-Stokes limit \([18]\). Recently, Jiang and Masmoudi established the Navier-Stokes-Fourier limit for DiPerna-Lions-Mischler renormalized solutions \([13]\) for both cases of \( \lim_{\varepsilon \rightarrow 0} \frac{\sqrt{\varepsilon}}{\varepsilon} < \infty \) and \( \alpha \sim \sqrt{\varepsilon} \) by constructing boundary layers \([30]\).

In the context of classical solutions, Nishida and Caflisch studied the compressible Euler limit \([40, 11]\), and Yu derived the Euler limit near a shock \([54]\). De Masi, Esposito and Lebowitz employed the idea of \([11]\) to investigate the Navier-Stokes limit \([13]\). Later on, Guo justified the Navier-Stokes limit on high order corrections with hard, soft and Landau collision kernels \([25]\). We also mention the related works \([3, 8, 33, 37, 49]\). See review literatures \([48, 52]\) for more complete references in this direction.

Much less is known for hydrodynamic limits of the steady Boltzmann equation, where the analog of DiPerna-Lions renormalized solutions is not available due to lacks of \( L^1 \) and entropy estimates. In fact, as Golse pointed in \([21]\), the derivation of the steady INSF from the steady Boltzmann equation is important and has been an outstanding open problem. Several investigations have been
made to this thesis for special cases \[2, 16, 19, 25\]. Esposito, Guo, Kim and Marra partially solved this problem by deriving INSF to the Dirichlet boundary condition from the steady Boltzmann equation to the diffuse reflection boundary condition in bounded domain \[17\]. More recently, Esposito, Guo and Marra extended such efforts to exterior domains \[18\].

In this paper, we derive INSF limit from steady and unsteady Boltzmann equation to the general Maxwell boundary condition with complete accommodation coefficient \(0 \leq \alpha \leq 1\). We can treat the specular reflection boundary condition \(\alpha = 0\), the most concerned and singular case where all of the boundary energy is reflected. Compared to the diffuse reflection boundary condition with accommodation coefficient \(\alpha = 1\), the main difficulty comes from the pointwise estimate and singularity in boundary terms, induced by the specular reflection effect.

1.2. Main results.

Recall the definition of the linearized collision operator
\[
Lf = -\frac{1}{\sqrt{\mu}}[Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)],
\]
and the nonlinear collision operator
\[
\Gamma(f, g) = \frac{1}{2\sqrt{\mu}}[Q(\sqrt{\mu}f, \sqrt{\mu}g) + Q(\sqrt{\mu}g, \sqrt{\mu}f)].
\]
The null space of \(L\), which we denote by \(N(L)\), is a five-dimensional subspace of \(L^2(\mathbb{R}^3)\)
\[
N(L) = \text{span}\{\sqrt{\mu}, v\sqrt{\mu}, |v|^2 - \frac{3}{2}\sqrt{\mu}\}.
\]
We denote the orthogonal projection of \(f\) onto \(N(L)\) as
\[
Pf = \left[a + b \cdot v + c\frac{|v|^2 - 3}{2}\right]\sqrt{\mu},
\]
and \((I - P)f\) the projection on the orthogonal complement of \(N(L)\). It is known that
\[
Lf = \nu f - Kf,
\]
where \(\nu = \nu(v)\) is collision frequency defined by
\[
\nu(v) = \int_{\mathbb{R}^3 \times S^2} |(v - u) \cdot \omega|\sqrt{\mu(u)}d\omega du,
\]
and \(K\) is a compact operator on \(L^2(\mathbb{R}^3)\) defined by
\[
Kf = \frac{1}{\sqrt{\mu}}[Q_+(\mu, \sqrt{\mu}f) + Q_+(\sqrt{\mu}f, \mu) - Q_-(\mu, \sqrt{\mu}f)] = \int_{\mathbb{R}^3} k(v, u)f(u)du.
\]
For hard sphere cross section, there are positive constants \(C_1, C_2\) such that
\[
C_1 \langle v \rangle \leq \nu(v) \leq C_2 \langle v \rangle
\]
where \(\langle v \rangle := \sqrt{1 + |v|^2}\).

For Maxwell boundary condition \((1.2)\), mass is conserved for \((1.1)\). Without loss of generality, we may assume that the mass conservation law holds for \(t \geq 0\), in terms of the perturbation \(f\) in \((1.10)\):
\[
\int_{\Omega \times \mathbb{R}^3} f(\sqrt{\mu}dxdv = 0,
\]
so that \(\int_{\Omega \times \mathbb{R}^3} Fdxdv = \int_{\Omega \times \mathbb{R}^3} \mu dxdv = |\Omega|\).

We first drive limit of the steady Boltzmann equation. We look for a steady solution \(F_s\) of \((1.1)\) and \((1.2)\) in the form
\[
F_s = \mu + \varepsilon\sqrt{\mu}[f_w + f_s],
\]
where \(f_w\) is a prescribed function defined by
\[
f_w := \sqrt{\mu}[\Theta_w \frac{|v|^2 - 3}{2} + \rho_w], \quad \rho_w = -\Theta_w + \frac{|\Omega|}{1} \int_{\Omega} \Theta_w,
\]
where the average of $\Theta_w$ is added so that $\int_{\Omega} F_w = 0$, and $\Theta_w$ is a fixed smooth function on $\Omega$ such that $\Theta_w |_{\partial \Omega} = \vartheta_w$ and

$$\|\vartheta_w \|_{W^{1,\infty}(\Omega)} \leq C \|\vartheta_w \|_{W^{1,\infty}(\partial \Omega)}.$$  \hfill (1.19)

Then in terms of $f_s$, (1.1) reads

$$v \cdot \nabla f_s + \varepsilon^2 \frac{1}{\sqrt{\mu}} \Phi \cdot \nabla_v (\sqrt{\mu} f_s) + \varepsilon^{-1} L f_s = \Gamma(f_s, f_s) + 2 \Gamma(f_s, f_w) + R_s,$$  \hfill (1.20)

where

$$R_s := \varepsilon \Phi \cdot v \sqrt{\mu} - v \cdot \nabla f_w - \varepsilon^2 \frac{1}{\sqrt{\mu}} \Phi \cdot \nabla_v [\sqrt{\mu} f_w] + \Gamma(f_w, f_w).$$  \hfill (1.21)

Elementary calculation shows that

$$\mathbf{P} R_s = \varepsilon \Phi \cdot v \sqrt{\mu} - \varepsilon^2 \rho w \Phi \cdot v \sqrt{\mu}.$$  \hfill (1.22)

To derive the boundary condition that $f_s$ satisfies, we see from the definition of $\Theta_w$ that

$$M_{1+\varepsilon\rho w, 0, 1+\varepsilon \Theta_w} |_{\gamma_-} = (1 - \alpha)\mathcal{L}(M_{1+\varepsilon\rho w, 0, 1+\varepsilon \Theta_w}) + \alpha \mathcal{P}_w^w(M_{1+\varepsilon\rho w, 0, 1+\varepsilon \Theta_w}).$$

By expanding $M_{1+\varepsilon\rho w, 0, 1+\varepsilon \Theta_w}$ in $\varepsilon$, we get

$$M_{1+\varepsilon\rho w, 0, 1+\varepsilon \Theta_w} = \mu + \varepsilon f_w \sqrt{\mu} + \varepsilon^2 \varphi \varepsilon \sqrt{\mu},$$

where $\varphi \in L^\infty(\Omega)$ satisfies

$$\varphi(x, v) = \varphi_v(x, |v|), \quad |\varphi_v| \leq O(\|\varphi_w\|_{L^1(\partial \Omega)})^4 \sqrt{\mu}(v).$$

Therefore, on $\gamma_-$

$$\mu + \varepsilon f_w \sqrt{\mu} + \varepsilon^2 \varphi \varepsilon \sqrt{\mu} = (1 - \alpha)\mathcal{L}(\mu + \varepsilon f_w \sqrt{\mu} + \varepsilon^2 \varphi \varepsilon \sqrt{\mu}) + \alpha \mathcal{P}_w^w(\mu + \varepsilon f_w \sqrt{\mu} + \varepsilon^2 \varphi \varepsilon \sqrt{\mu}).$$

On the other hand, from (1.17) and (1.2), on $\gamma_-$,

$$\mu + \varepsilon (f_w + f_s) \sqrt{\mu} = (1 - \alpha)\mathcal{L}(\mu + \varepsilon (f_w + f_s) \sqrt{\mu}) + \alpha \mathcal{P}_w^w(\mu + \varepsilon (f_w + f_s) \sqrt{\mu}).$$

Subtracting the above two equations, we obtain the boundary condition for $f_s$

$$f_s |_{\gamma_-} = (1 - \alpha)\mathcal{L} f_s + \alpha P_\gamma f_s + \alpha r_s,$$  \hfill (1.23)

where

$$r_s := \varepsilon \mathcal{D}_1(f_s) + \varepsilon \mathcal{D}_2(\varphi_v),$$  \hfill (1.24)

$$\mathcal{D}_1(f) := \varepsilon^{-1} [\mu^{-1} \mathcal{P}_w^w(\sqrt{\mu} f) - P_\gamma f],$$  \hfill (1.25)

$$\mathcal{D}_2(\varphi_v) := \varepsilon [\varphi_v - \mu^{-1} \mathcal{P}_w^w(\sqrt{\mu} \varphi_v)],$$  \hfill (1.26)

$$P_\gamma f(x, v) := \sqrt{2\pi} \int_0^{\infty} f(x, u) \sqrt{\mu(u)} |n(x) \cdot u| du,$$  \hfill (1.27)

and we have used the fact that $\mathcal{L}(\mu) = \mu$ and $\mathcal{L}(\varphi_v) = \varphi_v$. From

$$\sqrt{2\pi} \int_{v < 0} \mu |n \cdot v| dv = -1, \quad \int_{v < 0} M^w |n \cdot v| dv = -1$$

and (1.25) and (1.26), we have

$$\int_{n(x) \cdot v < 0} \sqrt{\mu} \mathcal{D}_1(f_s)[n(x) \cdot v] dv = 0, \quad \int_{n(x) \cdot v < 0} \sqrt{\mu} \mathcal{D}_2(\varphi_v)[n(x) \cdot v] dv = 0 \quad \text{for } x \in \partial \Omega,$$

so that

$$\int_{n(x) \cdot v < 0} \sqrt{\mu} r_s[n(x) \cdot v] dv = 0 \quad \text{for } x \in \partial \Omega.$$  \hfill (1.29)

Our first main result is on the hydrodynamic limit of the steady Boltzmann equation.
Theorem 1.1. Assume that $\Phi \in C^1(\bar{\Omega})$, $\vartheta_w \in W^{1,\infty}(\Omega)$ and
\[
\|\Phi\|_{L^2(\Omega)} + |\vartheta_w|_{H^{1/2}(\partial\Omega)} \ll 1.
\] (1.30)

Then, for $0 < \varepsilon \ll 1$, there is a unique positive solution $F_0 \geq 0$, given by (1.17) with $f_0$ satisfying (1.20), (1.23) and
\[
\|f_0\|_2 + \|Pf_0\|_6 + \frac{1}{\varepsilon} \|\Phi f_0\|_6 + \varepsilon^{3/2} \|w f_0\|_\infty + \sqrt{\frac{\alpha}{\varepsilon}} (1 - P_\gamma) f_0 |_{2+, \infty} + \sqrt{\frac{\alpha}{\varepsilon}} |f_0|_{2, \infty} \ll 1,
\] (1.31)

where $w(\nu) = e^{\beta|\nu|^2}$ with $0 < \beta' \ll 1$. Besides,
\[
f_0 \to f^*_0 = \left[ \rho_s + u_s \cdot v + \theta_s \frac{|v|^2 - 3}{2} \right] \sqrt{\mu} \text{ weakly in } L^2(\Omega \times \mathbb{R}^3) \text{ as } \varepsilon \to 0,
\]

where $(\rho_s, u_s, \theta_s)$ solves the steady INSF with the external field $\Phi$
\[
\begin{align*}
\rho_s \cdot \nabla u_s + \nabla p_s &= \sigma \Delta u_s + \Phi, \quad \nabla \cdot u_s = 0 \quad \text{in } \Omega, \\
\rho_s \cdot \nabla (\theta_s + \Theta_w) &= \kappa \Delta (\theta_s + \Theta_w), \quad \nabla \rho_s + \theta_s = 0 \quad \text{in } \Omega.
\end{align*}
\] (1.32)

Moreover, if $\lim_{\varepsilon \to 0} \frac{\varepsilon}{\rho} = \infty$ then (1.32) is supplemented by the Dirichlet boundary condition
\[
u_s = 0, \quad \theta_s = 0 \quad \text{on } \partial\Omega,
\] (1.33)

and if $\lim_{\varepsilon \to 0} \frac{\varepsilon}{\rho} = \sqrt{2\pi \lambda} \in [0, \infty)$ then (1.32) is supplemented by the Navier boundary condition
\[
\begin{align*}
\left[ \sigma (\nabla u_s + (\nabla u_s)^T) \cdot n + \lambda u_s \right]^\text{tan} &= 0, \quad u_s \cdot n = 0 \quad \text{on } \partial\Omega, \\
\kappa \partial_n (\theta_s + \Theta_w) + \frac{4}{5} \lambda \theta_s &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\] (1.34)

Note that in order to have a stationary solution with non-vanishing velocity field for (1.32), we may assume that $\Phi$ is not a potential field, such that $\text{div}\Phi = 0$. Otherwise, if $\Phi = -\nabla p_s$ is a potential field, then $\nu \equiv 0$, $p \equiv p_s$ is a steady solution to (1.32).

Then we derive limit of the unsteady Boltzmann equation. For this, let
\[
F = \mu + \varepsilon \sqrt{\mu} (f_0 + f_0 + \tilde{f}),
\] (1.35)

where $f_0$ is the solution of (1.20) and (1.23). Then in terms of $\tilde{f}$, (1.11)–(1.13) read as
\[
\begin{align*}
\varepsilon \partial_t \tilde{f} + v \cdot \nabla \tilde{f} + \varepsilon^2 \frac{1}{\sqrt{\mu}} \Phi \cdot \nabla \sqrt{\mu} \tilde{f} + \varepsilon^{-1} L \tilde{f} &= \Gamma(\tilde{f}) + 2 \Gamma(\tilde{f}, f_0 + f_0), \\
\tilde{f}|_{\gamma^-} &= (1 - \alpha) L \tilde{f} + \alpha P_\gamma \tilde{f} + \alpha \tilde{r}, \\
\tilde{f}|_{\gamma^-} &= 0,
\end{align*}
\] (1.36)

where we used (1.10), (1.20), (1.23) and the notation $\tilde{r} := \varepsilon \mathcal{Q}_1(\tilde{f})$. By (1.28), there holds
\[
\int_{n(x) \cdot v < 0} \sqrt{\mu} n(x) \cdot v)dv = 0 \quad \text{for } x \in \partial\Omega.
\] (1.37)

We define the energy as
\[
\mathcal{E}_\lambda[f](t) := \sup_{0 \leq s \leq t} \|e^{\lambda s} f(s)\|_2^2 + \sup_{0 \leq s \leq t} \|e^{\lambda s} f_0(s)\|_2^2,
\] (1.38)

and the dissipation as
\[
\mathcal{D}_\lambda[f](t) := \int_0^t \|e^{\lambda s} Pf(t)\|_2^2 + \int_0^t \|e^{\lambda s} Pf(t)\|_2^2 + \frac{1}{\varepsilon^2} \int_0^t \|e^{\lambda s} (I - P)f(t)\|_2^2 + \alpha \int_0^t \|e^{\lambda s} (1 - P_\gamma)f(t)\|_2^2 + \alpha \int_0^t \|e^{\lambda s} f(t)\|_2^2 + \alpha \int_0^t \|e^{\lambda s} f_0(t)\|_2^2.
\] (1.39)

The second main result is on the hydrodynamic limit of the unsteady Boltzmann equation.
Theorem 1.2. Assume the hypothesis of Theorem 1.1 holds. Let $F_s$ be the steady solution given by (1.14). Let the initial datum take the form of $F_0 = F_s + \varepsilon \sqrt{P} f_0 \geq 0$ satisfying
\begin{equation}
\delta \lambda f(0) + \varepsilon \psi \|w f_0\|_{L^2_{\nu,v}} + \varepsilon \phi \|w \partial_t f_0\|_{L^2_{\nu,v}} + \frac{1}{\varepsilon} \|\nu f_0 (I - P) f_0\|_{L^2_{\nu,v}} + \frac{\alpha}{\varepsilon} \|(1 - P) f_0\|_{L^2_{\nu,v}}
+ \| [v \cdot \nabla x + \varepsilon^2 \Phi \cdot \nabla \nu] f_0\|_{L^2_{\nu,v}} + \| [v \cdot \nabla x + \varepsilon^2 \Phi \cdot \nabla \nu] \partial_t f(0)\|_{L^2_{\nu,v}} \leq 1,
\end{equation}
where $w(v) = \varepsilon |v|^2$ with $0 < \beta \leq 1$.

Then, for $0 < \varepsilon \ll 1$, there is a unique global solution $F \geq 0$, given by (1.15) with $\tilde{f}$ satisfying (1.32) and the estimate
\begin{equation}
\delta \lambda \tilde{f}(\infty) + \mathcal{D} \lambda \tilde{f}(\infty) + \varepsilon \phi \|we^{\lambda t} \tilde{f}\|_{L^1_{\nu,v}} + \varepsilon \phi \|we^{\lambda t} \tilde{f}\|_{L^1_{\nu,v}} \leq 1.
\end{equation}

Besides,
\begin{equation}
\tilde{f} \to \tilde{f}^* = \left[ \rho + \tilde{u} \cdot v + \tilde{\theta} \right] \frac{|v|^2 - 3}{2} \sqrt{\mu} \text{ weakly in } L^2(\mathbb{R}^3 \times \Omega) \text{ as } \varepsilon \to 0,
\end{equation}
where $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ satisfies the unsteady INSF
\begin{align}
\partial_t \tilde{u} + \tilde{u} \cdot \nabla x (\tilde{u} + u_s) + u_s \cdot \nabla x \tilde{u} + \nabla x \tilde{u} &= \sigma \Delta \tilde{u}, \quad \nabla x \cdot \tilde{u} = 0 \quad \text{in } \mathbb{R}^3 \times \Omega,

\partial_t \tilde{\theta} + \tilde{u} \cdot \nabla x (\tilde{\theta} + \theta_s + \theta_w) + u_s \cdot \nabla x \tilde{\theta} &= \kappa \Delta \tilde{\theta}, \quad \nabla x (\tilde{\rho} + \tilde{\theta}) = 0 \quad \text{in } \mathbb{R}^3 \times \Omega,

\tilde{u}|_{t=0} = \tilde{u}_0, \quad \tilde{\theta}|_{t=0} = \tilde{\theta}_0 \quad \text{on } \Omega.
\end{align}
Moreover, (1.42) is supplemented by the Dirichlet boundary condition
\begin{equation}
\tilde{u} = 0, \quad \tilde{\theta} = 0 \quad \text{on } \mathbb{R}^3 \times \partial \Omega
\end{equation}
if $\lim_{r \to 0} \frac{\rho}{\varepsilon} = \infty$, and by the Navier boundary condition
\begin{equation}
\left[ \sigma (\nabla x \tilde{u} + (\nabla x \tilde{u})^T) \cdot n + \lambda \tilde{u} \right]_{\text{tan}} = 0, \quad \tilde{u} \cdot n = 0 \quad \text{on } \mathbb{R}^3 \times \partial \Omega,

\kappa \partial_n \tilde{\theta} + \frac{4}{5} \lambda \tilde{\theta} = 0 \quad \text{on } \mathbb{R}^3 \times \partial \Omega
\end{equation}
if $\lim_{r \to 0} \frac{\rho}{\varepsilon} = \sqrt{2\pi} \lambda \in [0, \infty)$.

Remark 1.3. (i) Theorem 1.1 and Theorem 1.2 cover complete accommodation coefficient $0 \leq \alpha \leq 1$ and general domain (not necessarily convex). In particular, they hold for the specular boundary condition $F|_{\gamma_e} = \mathcal{L} F$. It is well-known that global well-posedness for Boltzmann equation (with $\varepsilon = 1$) with specular boundary condition ($\alpha = 0$) remains an outstanding question for a 3D non-convex domain. Remarkably, our results have settled such an open question in the regime of hydrodynamic limit with $\varepsilon \ll 1$. The key is to employ a stretching method (see detailed illustration below) to reduce the analysis of complex bouncing trajectories to the case of trajectories with at most one bounce. Such a stretching method is robust, which is expected to play a crucial role in future study for other hydrodynamic limit problems. As a consequence, our method would lead to the first global well-posedness for Boltzmann equation (with $\varepsilon = 1$) for a ‘large stretched’ non-convex 3D domain. Moreover, due to lack of boundary control for $\alpha < 1$, it is a well-known difficulty to obtain energy estimate uniform for $\frac{\rho}{\varepsilon} \to \infty$ (the case for perfect slipping for the Navier boundary condition). We are able to obtain refined delicate estimate for $\|P f\|_{L^1_{\nu,v}}$ and $\|f\|_{L^1_{\nu,v}}$ for the full range of $0 \leq \alpha \leq 1$ (see detailed illustration below).

(ii) Let $f_s, \tilde{f}$ be the solutions of (1.20) and (1.32), respectively. Write
\begin{equation}
g^s = f_w + f_s, \quad g^t = f_w + f_s + \tilde{f}.
\end{equation}
Then Theorem 1.1 and Theorem 1.2 indicate that
\begin{equation}
g^s \to g^* = \left[ \rho_g + u_g \cdot v + \theta_g \frac{|v|^2 - 3}{2} \right] \sqrt{\mu}, \quad f^t \to f^* = \left[ \rho_f + u_f \cdot v + \theta_f \frac{|v|^2 - 3}{2} \right] \sqrt{\mu}
\end{equation}
weakly as $\varepsilon \to 0$, where $(\rho_f, u_f, \theta_f)$ solves the standard INSF (1.13, 1.14) and $(\rho_g, u_g, \theta_g)$ satisfies the corresponding steady form, cf. the proofs of Theorems 1.1 and 1.2.
(iii) The initial data $\partial_t f(0)$ in (1.40), expressed through the equation (1.39), is a sharp condition. One can compensate the corresponding diverging factors in its expression by choosing $f_0$ properly, cf. Remark 1.4 of [17].

(iv) Theorem 1.9 and Theorem 1.12 hold true for 2-dimensional case $\Omega \subset \mathbb{R}^2$, where $L^4$-norm should be employed instead of $L^p$-norm and the proof is simpler.

1.3. Difficulties and illustrations.

1.3.1. $L^p - L^\infty$ framework and stretching method.

In order to treat the quadratic term $\Gamma(f,f)$, the $L^\infty$-norm has to be used to control solutions of the nonlinear equations (1.20) and (1.39). To illustrate the main idea concisely, let consider a simplified problem

$$v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_v f + \varepsilon^{-1} \nu_0 f = \varepsilon^{-1} \int_{|\nu'| \leq N} f(\nu') d\nu' \quad \text{in } \Omega \times \mathbb{R}^3,$$

$$f|_{\gamma^t} = (1 - \alpha)\mathcal{L} f + \alpha P_2 f \quad \text{on } \partial \Omega \times \mathbb{R}^3,$$

where the integral in (1.45) is the truncation of $Kf$ and $\nu_0$ is the lower bound of $\nu(v)$. The trajectory (or the characteristics) for (1.45) is

$$X(s; t, x, v) = x + v(s - t) + \varepsilon^2 \int_t^s \int_t^\tau \Phi(X(\tau'; t, x, v)) d\tau' d\tau,$$

$$V(s; t, x, v) = v + \varepsilon^2 \int_t^s \Phi(X(\tau; t, x, v)) d\tau.$$

Define

$$t_b(x, v) := \inf \{ t \geq 0 : X(-t; 0, x, v) \notin \Omega \},$$

$$y_b(x, v) := X(-t_b(x, v); 0, x, v),$$

$$v_b(x, v) := V(-t_b(x, v); 0, x, v).$$

Clearly $(y_b(x, v), v_b(x, v)) \in \gamma^-_b$.

Taken into account of the Maxwell boundary condition (1.40), after multiple collisions against the boundary $\partial \Omega$, the trajectory $[X(s; t, x, v), V(s; t, x, v)]$ becomes very complicated. In fact, at each collision against $\partial \Omega$, the trajectory splits as the backward specular reflection and backward diffuse reflection. Let $(t_0, x_0, v_0) := (t, x, v)$ be the initial of the backward trajectory. Define the first backward bouncing time, position and velocity along the trajectory

$$t_1 := t_0 - t_b(x_0, v_0), \quad x_1 := x_b(x_0, v_0), \quad v_1 = \begin{cases} R_x f_1(V(t_1; t_0, x_0, v_0)) & \text{if specular,} \\ v_1^f & \text{if diffuse,} \end{cases}$$

where $v_1^f$ stands for an independent variable. The second backward bouncing time, position and velocity along the trajectory is more complicated, since it may propagate from either the specular reflection $(t_1, x_1, R_x f_1(V(t_1; t_0, x_0, v_0)))$ or the diffuse reflection $(t_1, x_1, v_1^f)$. We may employ the expression given by [10]. For $m \geq 1$ and $i \in \{ 1, 2, \ldots, m \}$, define $\theta_m(i)$ the set of strictly increasing functions from $\{ 1, 2, \ldots, i \}$ into $\{ 1, 2, \ldots, m \}$. It embodies every possible combination of specular reflections among all the collisions against the boundary. For $\ell \in \theta_m(i)$ and free variables $v_k^* \in \mathbb{R}^3$, define the sequence $(t_k^\ell, x_k^\ell, v_k^\ell)_{1 \leq k \leq m}$ by induction

$$t_k^\ell = t_{k-1}^\ell - t_b(x_{k-1}^\ell, v_{k-1}^\ell), \quad x_k^\ell = x_b(x_{k-1}^\ell, v_{k-1}^\ell) = X(x_k^\ell; t_{k-1}^\ell, x_{k-1}^\ell, v_{k-1}^\ell),$$

$$v_k^\ell = \begin{cases} R_x f_i(V(t_k^\ell; t_{k-1}^\ell, x_{k-1}^\ell, v_{k-1}^\ell)) & \text{if } k \in \ell \{ 1, 2, \ldots, i \}, \\ v_k^f & \text{otherwise.} \end{cases}$$

Define the back-time cycles as

$$X_{cl}(s; t, x, v) = \sum_k 1_{[t_{k+1}^\ell, t_k^\ell]}(s) X(s; t_k^\ell, x_k^\ell, v_k^\ell), \quad V_{cl}(s; t, x, v) = \sum_k 1_{[v_{k+1}^\ell, v_k^\ell]}(s) V(s; t_k^\ell, x_k^\ell, v_k^\ell).$$
Along the back-time cycles, solutions of (1.45) can be expressed by

\[ f(x, v) = \int_0^t e^{-\frac{\nu_0}{\varepsilon}(t-s)} \int_0^s e^{-\frac{\nu_0}{\varepsilon}(s-\tau)} \int_0^\tau f(X_{cl}(s; t, x, v), v') dv' d\tau ds + e^{-\frac{\nu_0}{\varepsilon} t} f(X_{cl}(0; t, x, v), V_d(0; t, x, v)). \]

(1.48)

Note that this formula is an abstract form and there are many terms, brought by the repeated usage of the Maxwell boundary condition (1.46) accompanied by the multiple collisions against the boundary. Then plug the same form into the integrand \( f(X_{cl}(s; t, x, v), v') \)

\[ f(x, v) = \int_0^t e^{-\frac{\nu_0}{\varepsilon}(t-s)} \int_0^s e^{-\frac{\nu_0}{\varepsilon}(s-\tau)} \int_0^\tau f(X_{cl}(s; t, x, v), v', v'') dv'' d\tau ds + \text{other terms}. \]

(1.49)

The spirit of \( L^p - L^\infty \) approach, first introduced in [27] and then employed by [15] [16] [17] [28] [29] [30] [31], is to generate one \( L^p \)-norm from the \( v' \)-integration through the change of variable \( v' \rightarrow z := X_{cl}(\tau; s, X_{cl}(s; t, x, v), v') \).

The key point is to ensure that the Jacobian has a positive lower bound, that is,

\[ |J| := \left| \text{det} \left[ \frac{dX_{cl}(\tau; s, X_{cl}(s; t, x, v), v')}{d\tau} \right] \right| \gtrsim \delta > 0 \]

away from a small set of \( s \). It follows that

\[ \|f\|_{L^p_{\tau,v}} \lesssim \left( \int_\Omega \int_{|v'|\leq N} |f(z, v'')|^p \frac{1}{\delta} dv' dz \right)^{\frac{1}{p}} + \text{other terms}, \]

which controls the bulk of the \( L^\infty \) norm through the \( L^p \)-norm.

Once specular reflection is involved in boundary condition, the specular backward cycles in \( X_{cl}(s; t, x, v) \) reflect repeatedly against the boundary, and \( \frac{dX_{cl}(\tau; s, X_{cl}(s; t, x, v), v')}{dv'} \) is very complicated to compute and (1.49) is quite difficult to verify. This is in part due to the fact that there is no apparent way to analyze \( \frac{dX_{cl}(\tau; s, X_{cl}(s; t, x, v), v')}{dv'} \) inductively with finite bounces. As for non-limit problems, for example, for \( \varepsilon = 1 \) and \( \alpha = 0 \) in (1.45) and (1.46), [27] used the fact that, for convex domain \( \Omega \), \( \frac{dX_{cl}(\tau; s, X_{cl}(s; t, x, v), v')}{dv'} \) can be computed asymptotically in a delicate iterative fashion for special cycles almost tangential to the boundary. It is then combined with the analyticity assumption of \( \partial \Omega \) to conclude that (1.50) is valid after cutting off a small set of \( J = 0 \). Later on, [35] removed this analyticity restriction on \( \partial \Omega \) by applying triple Duhamel formula

\[ f(x, v) = \int_0^t e^{-\nu_0(t-s)} \int_0^s e^{-\nu_0(s-\tau)} \int_0^\tau f(X_{cl}(\tau'; \tau, X_{cl}(\tau; s, X_{cl}(s; t, x, v), v'), v''), v''') dv'''' dv'' dv' d\tau ds + \cdots. \]

Note that in both of [27] [35], the back-time cycles may collide many times against the boundary and the most inner time integration must be cut off away from a small set, say

\[ \int_0^t = \int_0^s + \int_0^{s-\delta}, \quad \int_0^t \int_0^s = \int_0^s \int_0^{s-\delta}, \quad \int_0^t \int_0^s \int_0^\tau = \int_0^\tau \int_0^{\tau-\delta} + \int_0^\tau \int_0^{s-\delta}, \quad \text{to guarantee (1.50) for the second integration which carries out a change of variable in each formula.} \]

As for limit problems, one has to calculate accurately how the lower bound of \( J \) depends on \( \varepsilon \), which differs greatly from non-limit problems like [27] [35] where ensuring positive lower bound of \( J \) is enough. For this, we have to cut off the inner time integration with an \( \varepsilon \)-order interval \( o(1) \varepsilon \) to offset the factor \( \varepsilon^{-1} \) in front of the velocity integration in (1.49), that is,

\[ \text{(1.49)} = \int_0^t \int_0^s + \int_0^t \int_0^{s-\alpha(1) \varepsilon} + \text{others terms}, \]

(1.51)
where the first integration leads to

$$\int_0^t e^{-\frac{a_1}{\epsilon}(t-s)}\epsilon^{-1}ds \times o(1)\epsilon \times \frac{1}{\epsilon^3}\|f\|_{L^\infty_{x,v}} \lesssim o(1)\|f\|_{L^\infty_{x,v}}.$$  

For the second integration \(\int_0^t \int_0^s o(1)\epsilon\) in (1.51), the trajectory \(X_{cl}(\tau; s, X_{cl}(s; t, x, v), v')\), experienced the truncation \(s = o(1)\epsilon\), continues to propagate. And after multiple specular reflections against the boundary, the relation between \(X_{cl}(\tau; s, X_{cl}(s; t, x, v), v')\) and \(\epsilon\) becomes extremely ambiguous and (1.50) is completely unverifiable! Thus, the approaches employed in [27] and [38] fail to apply to the current limit problem.

To overcome this difficulty, our resolution is based on the following principle: prevent the truncation of time integration from the involvement of \(\epsilon\), and avoid repeated bounces against the boundary. We introduce the so-called stretching method: for given \(0 < \epsilon \ll 1\), stretch the domain \(\Omega\) via the change of variable

$$\Omega \rightarrow \Omega_\epsilon := \epsilon^{-1}\Omega; \quad x \mapsto y := \epsilon^{-1}x,$$  

where \(\Omega_\epsilon\) is very large due to smallness of \(\epsilon\) and maintains unit normal invariant

$$n(y) = \frac{\nabla_y[\xi(\epsilon y)]}{|\nabla_y[\xi(\epsilon y)]|} = \frac{\nabla_x[\xi(\epsilon y)]}{|\nabla_x[\xi(\epsilon y)]|} = \frac{\nabla_x[\xi(x)]}{|\nabla_x[\xi(x)]|} = n(x) \quad \text{for } x \in \partial\Omega, y = \epsilon^{-1}x \in \partial\Omega_\epsilon,$$  

and then transform (1.45)–(1.46) into an equivalent problem

$$v \cdot \nabla_y \tilde{f} + \epsilon^3 \tilde{\Phi} \cdot \nabla_v \tilde{f} + \nu_0 \tilde{f} = \int_{|v'| \leq N} \tilde{f}(y, v')dv' \quad \text{in } \Omega_\epsilon \times \mathbb{R}^3,$$  

$$\tilde{f}|_{\gamma_\epsilon} = (1 - \alpha)\mathcal{L}\tilde{f} + \alpha P_\gamma \tilde{f} \quad \text{on } \partial\Omega_\epsilon \times \mathbb{R}^3$$  

by defining

$$\tilde{f}(y, v) := f(x, v), \quad \tilde{\Phi}(y) := \Phi(x).$$  

The trajectory of (1.55) reads as

$$Y(s; t, y, v) = y + v(s - t) + \epsilon^3 \int_t^s \int_t^{\tau} \tilde{\Phi}(Y(\tau'; t, y, v))d\tau'd\tau,$$  

$$V(s; t, y, v) = v + \epsilon^3 \int_t^s \tilde{\Phi}(Y(\tau; t, y, v))d\tau.$$  

Denote the first and the second bounces (if they do happen) against the boundary along the backward specular trajectory by

$$(t_1, y_1) = (t - t_b(y, v), y_b(y, v)), \quad (t_2, y_2) = (t_1 - t_b(y_1, v_1), y_b(y_1, v_1)).$$

From (1.58) we know that

$$|y_2 - y_1| \approx |v(t_2 - t_1)|.$$  

For given \((t, y, v) \in [0, T_0] \times \Omega_\epsilon \times \mathbb{R}^3\) satisfying \((y, v) \notin \gamma_0\), the left hand side of (1.59) \(|y_2 - y_1| \approx O(\frac{1}{\epsilon})\) due to the stretch (1.53), while the right hand side \(|v(t_2 - t_1)|\) is bounded. This indicates that the specular backward trajectory starting from \((t, y, v)\) undergoes “at most one bounce” against the boundary for small \(\epsilon \ll 1\), cf. Lemma 2.5. With this crucial observation, solutions of (1.53)
can be expressed by (note that the Maxwell boundary condition in (1.50) is used at most once)
\[
\begin{align*}
  \tilde{f}(y,v) &= e^{-\nu_0 t} \tilde{f}(Y(0, t, y, v), V(0, t, y, v)) + \int_0^t e^{-\nu_0(t-s)} \int_{|v'| \leq N} \tilde{f}(Y(s; t, y, v), v') dv' ds \\
  &+ (1 - \alpha)e^{-\nu_0(t-0)} \tilde{f}(Y_{cl}^a(0; t, y, v), V_{cl}^a(0; t, y, v)) \\
  &+ (1 - \alpha) \int_0^{\tilde{t}_1} e^{-\nu_0(t-s)} \int_{|v'| \leq N} \tilde{f}(Y_{cl}^a(s; t, y, v), v') dv' ds \\
  &+ \alpha \int_{|v'| \leq N} \int_0^{\tilde{t}_1} e^{-\nu_0(t-s)} \int_{|v'| \leq N} \tilde{f}(Y_{cl}^d(s; t, y, v), v') dv' dv' ds d\tilde{t}_1,
\end{align*}
\]
where \(Y_{cl}^a(s; t, y, v)\) and \(Y_{cl}^d(s; t, y, v)\) stand for the trajectory undergoing one specular reflection and one diffuse reflection, respectively. Then plug the same form into the integrands in (1.60) and one diffuse reflection, respectively. Then plug the same form into the integrands in (1.60).

where \(Y_{cl}^a(s; t, y, v)\) and \(Y_{cl}^d(s; t, y, v)\) stand for the trajectory undergoing one specular reflection and one diffuse reflection, respectively. Then plug the same form into the integrands in (1.60). In (1.61), we have cut off the inner time integration with small constant \(\delta\) (independent of \(\varepsilon\)) and used the notation \(t'_1 := s - t_1(Y_{cl}^a(s; t, y, v), v')\). By careful calculations, we show that, for the underbraced term in (1.61), where \(\tau < t'_1 - \delta < t'_1 < s\), the Jacobian of the change of variable \([v' \mapsto Y_{cl}^a(\tau; s, Y_{cl}^a(s; t, y, v), v')]\) has a positive lower bound independent of \(\varepsilon\)

\[
\left| \det \left[ \frac{dY_{cl}^a(\tau; s, Y_{cl}^a(s; t, y, v), v')}{dv'} \right] \right| \geq |s - \tau|^3 + O(\varepsilon) \geq \delta^3,
\]

cf. Lemma 2.7. This first settles the uniform (in \(\varepsilon\)) \(L^\infty\) estimate for limit problems containing specular reflection effect in boundary conditions.

For the unsteady limit, we consider the evolutionary version of (1.63)

\[
\varepsilon \partial_t f + v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_v f + \varepsilon^{-1} \nu_0 f = \varepsilon^{-1} \int_{|v'| \leq N} f(v') dv' \quad \text{in} \quad \Omega \times \mathbb{R}^3.
\]

The above stretching method leads to

\[
\varepsilon^2 \partial_{\tilde{t}} \tilde{f} + v \cdot \nabla_y \tilde{f} + \varepsilon^3 \Phi \cdot \nabla_v \tilde{f} + \nu_0 \tilde{f} = \int_{|v'| \leq N} \tilde{f}(v') dv' \quad \text{in} \quad \Omega \times \mathbb{R}^3.
\]

To express solutions of (1.64), one needs to go through the above steps for the steady case again by defining new characteristics \(Z(s; t, y, v) = Y(t - \frac{s}{\varepsilon^2}; t, y, v)\). However, we can skip these lengthy steps by stretching the time variable at the same time as (1.53)

\[
[0, \infty] \rightarrow [0, \infty]; \quad t \mapsto \tilde{t} := \varepsilon^{-2} t.
\]

Then (1.63) is transformed into

\[
\partial_{\tilde{t}} \tilde{f} + v \cdot \nabla_y \tilde{f} + \varepsilon^3 \Phi \cdot \nabla_v \tilde{f} + \nu_0 \tilde{f} = \int_{|v'| \leq N} \tilde{f}(v') dv',
\]

where

\[
\tilde{f}(\tilde{t}, y, v) := f(t, x, v), \quad \tilde{\Phi}(y) := \Phi(x).
\]

One can find that (1.66) and its steady version (1.55) enjoy the same characteristic (1.58). Thus the above “at most one bounce” argument still holds true for given finite time \(\tilde{t} \in [0, T_0]\), and the \(L^\infty\) estimate follows readily from the steady case. Finally, returning back to the original problem (1.63), we get the estimate on \([0, \varepsilon^2 T_0] \times \Omega \times \mathbb{R}^3\) and then extend to \([\varepsilon^2 T_0, 2\varepsilon^2 T_0], \ldots, [n\varepsilon^2 T_0, (n +\]
1) $\epsilon^2 T_0, \cdots$. Thus in the $L^\infty$ estimate of the unsteady problem, we can simplify the procedure greatly by stretching the time and space variables simultaneously.

1.3.2. $L^2$ coercivity estimate with $\frac{a}{\epsilon}$ small or vanishing.

It is well-known that $L$ is semi-positive \[12\]
\[
\int_{\mathbb{R}^3} f L f \, dv \geq \sigma_L \|(I - P)f\|^2_{L^2(\mathbb{R}^3)}.
\]
Thus standard $L^2$ energy estimate of \[1.36\] leads to
\[
\|\tilde{f}(t)\|^2_2 + \frac{1}{\epsilon^2} \int_0^t \|\tilde{(I - P)f}\|^2_\nu + \frac{\alpha}{\epsilon} \int_0^t \|(1 - P_\gamma)\tilde{f}\|^2_{2, +} \lesssim \frac{1}{c} \int_0^t \|\Gamma(\tilde{f}, \tilde{\phi})\|^2_2 + 1,
\]
(1.68)
The missing $\int_0^t \|\tilde{P}f\|^2_2$ is estimated by the coercivity estimates with proper test function in the weak formulation together with local conservation law, first employed by \[10\]. Note that in \[1.67\], only $\frac{a}{\epsilon} \int_0^t \|(1 - P_\gamma)\tilde{f}\|^2_{2, +}$ can be controlled and it may be very small if $\frac{a}{\epsilon} \to 0$, induced by the Maxwell boundary condition. In particular, for the specular reflection boundary condition $\alpha = 0$, there is no boundary integration on the left hand side of \[1.68\]. Thus in order to close the $L^2$ energy, the boundary integration in the upper bound of $P \tilde{f}$ should not exceed $\frac{a}{\epsilon} \int_0^t \|(1 - P_\gamma)\tilde{f}\|^2_{2, +}$.

To demonstrate the treatments concisely, we consider a simplified version of \[1.36\]
\[
\epsilon \partial_t f + v \cdot \nabla_x f + \epsilon^{-1} L f = g \in L^2(\Omega \times \mathbb{R}^3), \quad \int_{\Omega \times \mathbb{R}^3} f \, dv \, dx = 0,
\]
(1.69)
and seek for the estimates of $a$, $b$ and $c$ in \[1.15\]. Weak formulation of \[1.69\] reads as
\[
\int_0^t \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi)(P f + (1 - P) f) = \int_0^t \int_{\partial \Omega \times \mathbb{R}^3} f \psi [n \cdot v] + \text{other terms.}
\]
(1.71)
A test function $\psi$ needs to be chosen such that $L^2$ control of $a$, $b$, $c$ comes from $\int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi) P f$ and the boundary integration reads as (we omit the time integral for simplicity)
\[
\int_{\partial \Omega \times \mathbb{R}^3} \psi f [n \cdot v] = \int_{\gamma_+} \psi f \, d\gamma - \int_{\gamma_+} \psi (R_x v) [(1 - \alpha) f(v) + \alpha P_\gamma f] \, d\gamma,
\]
(1.72)
where we used $d\gamma := |n(x) \cdot v| dS(x) \, dv$ and the change of variable $v \mapsto R_x v$ on $\gamma_-$.

For $|a|_2$, we choose a test function $\psi_a = (|v|^2 - 10) \sqrt{\mu} (v \cdot \nabla_x \phi_a)$ as in \[10\], where $\phi_a$ solves $-\Delta_x \phi_a = a$ with $\partial_n \phi_a |_{\partial \Omega} = 0$, due to $\int_{\Omega \times \mathbb{R}^3} f = 0$. By elliptic estimate and trace theorem, \[1.67\] is bounded by $a^2 \|(1 - P_\gamma) f\|^2_{2, +} + o(1) ||a||_2^2$. For $||b||_2$, we choose $\psi_b = \sum_{i,j=1}^{3} \partial_j \theta_i (v) v_j \sqrt{\mu} - \sum_{i=1}^{3} \partial_i \phi^b |v|^2 - a \frac{1}{2} \|a\|^2_2$ introduced by \[53\], where $\phi^b = (\phi_1^b, \phi_2^b, \phi_3^b)$ solves the elliptic system
\[
-\Delta_x \phi^b = b \quad \text{in} \quad \Omega, \quad \phi^b \cdot n = 0 \quad \text{on} \quad \partial \Omega, \quad \partial_n \phi^b = (\phi^b \cdot n) n \quad \text{on} \quad \partial \Omega.
\]
(1.73)
This system satisfies complementing boundary conditions in the sense of Agmon-Douglis-Nirenberg and thus leads to elliptic estimate $\|\phi^b\|^2_{H^2} \lesssim ||b||_2$, cf. Lemma \[A.1\] and Lemma \[A.2\] in the Appendix. Then \[1.72\] is controlled by $a^2 \|(1 - P_\gamma) f\|^2_{2, +} + o(1) ||b||_2^2$.

The estimate of $||c||_2$ is rather delicate. On one hand, test function $\psi_c$ should be odd on $v$ such that the integral $\int_{\Omega \times \mathbb{R}^3} \psi_c (v \cdot \nabla_x f)$ leads to $\int_{\Omega} \Delta_x \phi_c$ and then to $\int_{\Omega} c^2$. For this, let $\psi_c = \chi(|v|)(v \cdot \nabla_x \phi_c)$, with $\chi$ and $\phi_c$ to be determined. Then \[1.72\] reads as
\[
\alpha \int_{\gamma_+} \chi(|v|) (v \cdot \nabla_x \phi_c) (1 - P_\gamma) f \, d\gamma + 2 \int_{\gamma_+} \chi(|v|) (v \cdot \nabla_x \phi_c) (1 - \alpha) f(v) + \alpha P_\gamma f \, d\gamma,
\]
(1.74)
where the second integration can not be controlled by $\frac{a}{\epsilon} \int_0^t \|(1 - P_\gamma)\tilde{f}\|^2_{2, +}$, unless $\partial_n \phi_c \lesssim \frac{a}{\epsilon}$. On the other hand, the information of $\int_{\Omega} f \, dv \, dx$ is unknown at all (note that there is no energy conservation law for \[1.37\], due to Maxwell boundary condition), so that we must handle the goal $\partial_n \phi_c \lesssim \alpha$ very carefully to avoid compatible conditions for elliptic equation satisfied by $\phi_c$. We balance these
factors and construct a test function \( \psi_e = (|v|^2 - 5)\sqrt{\mu}(v \cdot \nabla_x \phi_e) \), where \( \phi_e \) solves the following elliptic equation with convection term

\[- \Delta_x \phi_e + E \cdot \nabla_x \phi_e = c, \quad \partial_t \phi_e|_{\partial \Omega} = 0, \quad \int_\Omega \phi_e \, dx = 0, \quad (1.75)\]

where \( E \in [L^\infty(\Omega)]^3 \) is given with small \( \|E\|_\infty \ll 1 \). Lemma A.3 in the Appendix proves that there exists such a vector \( E \) to guarantee unique solvability of (1.75) and uniform elliptic estimate \( \|\phi_e\|_{H^2} \lesssim \|c\|_2 \). With this carefully chosen \( \phi_e \), we have the bulk lower bound for the integral

\[ \left| \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_e) Pf \right| = \left| 5 \int_\Omega c \Delta \psi_e \right| = \left| 5 \int_\Omega c^2 - 5 \int_\Omega c(\mathbf{E} \cdot \nabla_x \phi_e) \right| \geq 4\|c\|_2^2 \]

and upper bound \( \alpha^2 \| (1 - \mathbf{P}) f \|_{L^2_{L^2_x}}^2 + o(1) \|c\|_2^2 \) for (1.72) or (1.74). Combining these estimates we finally get the refined estimate

\[ \int_0^t \| Pf \|_{L^\infty_t L^1_x}^2 \lesssim \int_0^t \| (I - Pf) \|_{L^2_t}^2 + \alpha \int_0^t \| (1 - P\mathbf{P}) \|_{L^2_{L^2_x}}^2, \quad (1.76) \]

Another obstacle arises from the treatment of the collision operator

\[ \int_0^t \| \Gamma(Pf, Pf) \|_{L^\infty_t L^1_x}^2 \lesssim \| Pf \|_{L^\infty_t L^1_x} \| Pf \|_{L^2_{L^2_x}} \]

in order to close (1.68). \( \| Pf \|_{L^\infty_t L^1_x} \) is estimated like the steady case and \( \| Pf \|_{L^2_{L^2_x}} \) can be bounded through the gain of integrability by extending \( f \), cf. [1.7]. However, owing to (1.68) again, the boundary integration in the upper bound of \( \| Pf \|_{L^2_{L^2_x}} \) should not exceed \( \frac{\alpha}{\epsilon} \int_0^t \| (1 - P\mathbf{P}) \|_{L^2_{L^2_x}} \), especially when \( \epsilon \rightarrow 0 \). We overcome this difficulty by first cutting off the whole grazing set \( \gamma \) and then extending \( f \) to \( f_\delta \) satisfying \( f_\delta \|_{\gamma \gamma_-} = 0 \) and \( \| f_\delta \|_{L^2(\mathbb{R} \times \gamma \gamma_-)} \sim \| f_{1\gamma_-} \|_{L^2(\mathbb{R} \times \gamma \gamma_-)} \). Here \( \gamma \) is the non-grazing set

\[ \gamma := \{ (x, v) \in \gamma : |n(x) \cdot v| \geq \delta, \quad \delta \leq |v| \leq \frac{1}{\delta} \}. \quad (1.76) \]

Then we follow the same argument as in [1.7] to derive

\[ \| Pf \|_{L^\infty_t L^1_x} \lesssim \| f_{1\gamma_-} \|_{L^2(\mathbb{R} \times \gamma \gamma_-)} + \text{other terms.} \]

For the outgoing non-grazing set \( \gamma \), we can use the trace lemma to bound \( \int_{\gamma_-} \int_{\gamma_-} \| Pf \|_{\partial \gamma}^2 d\gamma \). While for the incoming non-grazing set \( \gamma \) where the trace lemma does not hold true anymore, we apply the Maxwell boundary condition and convert it to the outgoing part by the change of variable \( v \mapsto R_x v \) on \( \gamma_- \)

\[ \int_0^t \int_{\gamma_-} |f_{1\gamma_-}|^2 d\gamma \lesssim (1 - \alpha)^2 \int_0^t \int_{\gamma_-} |f|^2 d\gamma + \alpha^2 \int_0^t \int_{\gamma_-} \| P\mathbf{P} f \|^2 d\gamma + 1, \quad (1.76) \]

and then use the trace lemma again, cf. Lemma 3.4.

Finally, recall from (1.41) that only \( \frac{\alpha}{\epsilon} \int_{\gamma_-} |f|^2 d\gamma \) and \( \frac{\alpha}{\epsilon} \int_0^t \| (1 - P\mathbf{P}) f \^2_{L^2_x} d\gamma \) are controlled (rather than \( \int_{\gamma_-} |f|^2 d\gamma \) and \( \int_0^t \| (1 - P\mathbf{P}) f \^2_{L^2_x} d\gamma \), which would be very small if \( \epsilon \rightarrow 0 \). When taking limits of the boundary conditions, we have to transfer the integral \( \frac{1}{\epsilon} \int_{\gamma_-} [(1 - \alpha) \mathcal{L} f + \alpha P\mathbf{P} f] \) onto \( \gamma \) to offset the bulk \( \frac{1}{\epsilon} \int_{\gamma_-} (1 - \alpha) f \) and remain the small contribution \( \frac{2}{\epsilon} \int_{\gamma_-} (1 - P\mathbf{P}) f \). In the case of the Dirichlet boundary condition, we can take limit directly in Maxwell boundary condition and show that the convergence is strong. While for derivation of the Navier boundary condition, we have to take limits of (1.20) and (1.23) in the weak formulation and that the moments satisfy the weak formulation of INSF.
1.4. Organization of the paper.

This paper is organized as follows. In Section 2 we study the steady limit of \[ \text{Problems used in previous sections.} \]
and give the proof of Theorem 1.2. Finally, in the Appendix, we give proofs on some elliptic boundary problems used in previous sections.

**Notation.** For notational simplicity, we use \( C \) to denote some generic positive constant and use \( X \lesssim Y \) to denote \( X \leq DY \), where \( D \) is a constant not depending on \( X \) and \( Y \). We subscript this to denote dependence on parameters, thus \( X \lesssim_\beta Y \) means \( X \leq D\beta Y \). Let \( o(1) \) stand for a small constant. For \( 1 \leq p \leq \infty \), we use \( \| \cdot \|_p \) and \( \| \cdot \|_{L^p} \) for the \( L^p(\Omega \times \mathbb{R}^3) \) norm or the \( L^p(\Omega) \) norm, depending on the context. We subscript this to denote the variables, thus \( \| \cdot \|_{L^p} \) means \( L^p(\{ y \in Y \}) \). We also denote \( \| \cdot \|_{L^p_{L^p}} := \| \cdot \|_{L^p(\Omega \times \mathbb{R}^3)} : = \| \| \cdot \|_{L^p} \|_{L^p} \). We use \( (\cdot, \cdot) \) for the \( L^2(\Omega \times \mathbb{R}^3) \) inner product or \( L^2(\Omega) \) inner product, and \( (\cdot, \cdot) \) for the \( L^2(\mathbb{R}^3) \) inner product. Write \( \| \cdot \|_{\nu} \equiv \| \cdot \|^{1/2}_{L^2} \| \cdot \|_2 \). For the phase boundary integration, we use \( | \cdot | \) norm. More precisely, define \( d\gamma = |a(x)| \cdot v|dS(x)| \) where \( dS(x) \) is the surface measure, use \( |f|^p_p = \int_{\Omega} |f(x, v)|^p d\gamma \) or \( |f|^p_p = \int_{\partial \Omega} |f(x)|^p dS(x) \) for \( 1 \leq p < \infty \), with the corresponding spaces as \( L^p(\partial \Omega \times \mathbb{R}^3; d\gamma) = L^p(\partial \Omega \times \mathbb{R}^3) \) or \( L^p(\partial \Omega; dS(x)) = L^p(\partial \Omega) \). Similarly, let \( |f|_{\infty} \) represent \( \sup_{x \in \partial \Omega} |f(x)| \) or \( \sup_{(x,v) \in \gamma} |f(x,v)| \).

We also denote \( |f|_{p, \pm} = |f|_{1_{\gamma \pm}} \).

2. Steady Limit

2.1. \( L^2 \) coercivity estimate and \( L^6 \) bound.

In this section, we construct the \( L^2 \) coercivity estimate of the steady linear problem. We define

\[
\langle f \rangle := \left( \int_{\Omega \times \mathbb{R}^3} f \sqrt{\mu} d\gamma \right) / \left( \int_{\Omega \times \mathbb{R}^3} \mu d\gamma \right).
\]

Denote

\[
\hat{a}(x) := a(x) - \langle f \rangle, \quad \hat{f} := f - \langle f \rangle \sqrt{\mu}.
\]

It follows that

\[
\int_{\Omega} \hat{a}(x) dx = 0 \quad \text{and} \quad \mathbf{P} \hat{f} = \left[ \hat{a}(x) + b(x) \cdot v + c(x) \frac{|v|^2 - 3}{2} \right] \sqrt{\mu}.
\]

We have the following estimates for \( \mathbf{P} \hat{f} \).

**Lemma 2.1.** Let \( 0 < \varepsilon \ll 1 \) and \( \lambda > 0 \) be given. Suppose that \( \Phi \in L^\infty(\Omega), g \in L^2(\Omega \times \mathbb{R}^3), r \in L^2(\gamma_-) \) and satisfy

\[
\int_{\Omega \times \mathbb{R}^3} r \sqrt{\mu} |n| d\gamma = 0 \quad \text{for all} \ x \in \partial \Omega. \tag{2.1}
\]

Assume that \( f \) satisfies

\[
[\lambda + (1 - \theta)\varepsilon^{-1} \nu - \frac{1}{2} \varepsilon^2 \Phi \cdot v] f + v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_v f + \theta \varepsilon^{-1} Lf = g,
\]

\[
f|_{\gamma_-} = (1 - \alpha) \mathcal{L} f + \alpha \mathcal{P} r, \tag{2.2}
\]

where \( \theta \in [0, 1] \).

Then for \( \theta \) close to \( 1^- \),

\[
\lambda \langle f \rangle \lesssim (1 - \theta) \varepsilon^{-1} \| f \|_2. \tag{2.3}
\]

Moreover,

\[
\| \mathbf{P} \hat{f} \|_2^2 \lesssim \varepsilon^{-2} \| (1 - \mathbf{P}) f \|_p^2 + \alpha^2 |(1 - \mathcal{P} r)|_{2,+}^2 + \alpha^2 |r|_{2,-}^2 + \| \nu^{-\frac{1}{2}} g \|_2^2 + \lambda \langle f \rangle^2. \tag{2.4}
\]
and 
\[
\| \mathbf{P} \dot{f} \|_6^6 \lesssim \left( \varepsilon^{-1} \| (\mathbf{I} - \mathbf{P}) f \|_\infty \right)^6 + \left[ \alpha \varepsilon^{3/2} \| (1 - P_\gamma) f \|_{2, +} \right]^6 + \left[ \alpha \varepsilon^{3/2} \| r_{2, +} \| \right]^6 \\
+ \left[ \alpha \varepsilon^{3/2} \| \nu r \|_\infty \right]^6 + \| \nu - \frac{3}{2} g \|_2^2 + \rho_1 \left[ \alpha \varepsilon \| w f \|_\infty \right]^6 + \lambda(\| f \|_6^6),
\]
where \( \rho_1 > 0 \) is a small constant.

**Proof.** Step 1. Estimate of (2.3).

Multiply a test function \( \psi(x, v) \), which will be determined later, to the equation (2.2) and integrate on \( \Omega \times \mathbb{R}^3 \),
\[
\iint_{\Omega \times \mathbb{R}^3} \left[ \lambda + (1 - \theta) \varepsilon^{-1} \nu f + \frac{1}{2} \varepsilon^2 \Phi \cdot v f \right] \psi - \iint_{\partial \Omega \times \mathbb{R}^3} (v \cdot n) f \psi - \iint_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi) f \\
- \iint_{\partial \Omega \times \mathbb{R}^3} \varepsilon^2 (\Phi \cdot \nabla_v \psi) f + \varepsilon^{-1} \theta \iint_{\Omega \times \mathbb{R}^3} L(\mathbf{I} - \mathbf{P}) f \psi = \iint_{\Omega \times \mathbb{R}^3} g \psi.
\]
In (2.6), we choose \( \psi = \sqrt{\mu} \). By the change of variable \( v \mapsto R_x v \) on \( \gamma_- \), we have
\[
\iint_{\partial \Omega \times \mathbb{R}^3} f \sqrt{\mu} |n \cdot v| dv \mathrm{d}x = \int_{\gamma_+} f \sqrt{\mu} d\gamma - \int_{\gamma_-} \left[ (1 - \alpha) \mathcal{L} f + \alpha P_\gamma f + \alpha r \right] \sqrt{\mu} d\gamma \\
= \int_{\gamma_+} f \sqrt{\mu} d\gamma - \int_{\gamma_+} \left[ (1 - \alpha) f + \alpha P_\gamma f \right] \sqrt{\mu} d\gamma \\
= \alpha \int_{\gamma_+} (f - P_\gamma f) \sqrt{\mu} d\gamma = 0,
\]
where we used (2.1) and \( \sqrt{2\pi} \int_{n \cdot v > 0} \mu |n \cdot v| dv = 1 \). Combining this with (2.1), we get
\[
\int_{\Omega \times \mathbb{R}^3} [\lambda + (1 - \theta) \varepsilon^{-1} \nu] f \sqrt{\mu} = 0.
\]
From the definition of \( \dot{f} \) we have \( \iint_{\Omega \times \mathbb{R}^3} \dot{f} \sqrt{\mu} = 0 \). Noticing \( f = \dot{f} + (f) \sqrt{\mu} \) and
\[
\iint_{\Omega \times \mathbb{R}^3} (1 - \theta) \varepsilon^{-1} \nu f \sqrt{\mu} \lesssim \Omega (1 - \theta) \varepsilon^{-1} \| f \|_2 \lesssim (1 - \theta) \varepsilon^{-1} \| f \|_2,
\]
we prove (2.8) for \( \theta \in [0, 1] \) close to 1.

In the following we prove (2.4) and (2.5). It follows from (2.6) that
\[
\iint_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_m^i) \mathbf{P} f = \sum_{j = 1}^6 \Psi_m^{i,j} \quad \text{for} \ m \in \{2, 6\}, i \in \{a, b, c\},
\]
where \( \psi_m^i(x, v) \) are test functions and
\[
\Psi_m^{i,1} := - \iint_{\Omega \times \mathbb{R}^3} [\lambda + (1 - \theta) \varepsilon^{-1} \nu] \psi_m^i, \quad \Psi_m^{i,2} := - \iint_{\partial \Omega \times \mathbb{R}^3} (v \cdot n) \psi_m^i, \\
\Psi_m^{i,3} := \iint_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_m^i)(\mathbf{I} - \mathbf{P}) f, \quad \Psi_m^{i,4} := \varepsilon^2 \iint_{\Omega \times \mathbb{R}^3} \left[ \Phi \cdot \nabla_v \psi_m^i \sqrt{\mu} \right] f, \\
\Psi_m^{i,5} := - \varepsilon^{-1} \theta \iint_{\Omega \times \mathbb{R}^3} \psi_m^i L(\mathbf{I} - \mathbf{P}) f, \quad \Psi_m^{i,6} := \iint_{\Omega \times \mathbb{R}^3} g \psi_m^i.
\]

Step 2. Estimates of \( \| \hat{a} \|_2 \) and \( \| \hat{a} \|_6 \).

Choose a test function
\[
\psi_m^a := \left| \phi_m^a \right|^2 - 10 \sqrt{\mu} \cdot \nabla_x \phi_m^a(x), \quad m = 2, 6
\]
in (2.9), where \( \phi_m^a \) is defined through
\[
- \Delta_x \phi_m^a = a^{m-1} - \int \hat{a}^{m-1} dx, \quad \partial_n \phi_m^a \big|_{\partial \Omega} = 0, \quad \int_{\Omega} \phi_m^a dx = 0.
\]
\]
\]
\]
\]
\]
\]
\]
Here we used the notation $\int g(x) := |\Omega|^{-1} \int_{\Omega} g(x)$. Elliptic estimates lead to
\begin{align}
\|\phi_0\|_{H^2} &\lesssim \|\hat{a}\|_2, \\
\|\phi_0\|_{W^{1,\infty}_{x\psi}} &\lesssim \|\hat{a}\|_2 - \int \phi_0^5 \lesssim \|\hat{a}\|_2 + \|\Omega^{-\frac{1}{2}} \| \hat{a}\|_{L^6} \lesssim \|\hat{a}\|_6. \tag{12.21}
\end{align}

The left-hand side of (2.21) equals
\[- \int_{\Omega \times \mathbb{R}^3} (\nu \cdot \nabla_x \psi_m) P f = 5 \int \Delta \phi_m^a [\hat{a} + \langle f \rangle],
\]
where we used oddness of $v_k v_i v_j (|v|^2 - 10)\mu$ and
\[\int_{\mathbb{R}^3} v_k v_i v_j (|v|^2 - 10)\mu = -5\delta_{ik}, \quad \int_{\mathbb{R}^3} v_i^2 (|v|^2 - 10)\left| \frac{|v|^2 - 3}{2} \right| \mu = 0, \quad i, k = 1, 2, 3. \tag{12.13}\]
By $\int \hat{a} = 0$ and (2.11) we have
\[- \int_{\Omega \times \mathbb{R}^3} (\nu \cdot \nabla_x \psi_m^a) P f = -5 \int \Delta \phi_m^a [\hat{a} + \langle f \rangle] = -5\|\hat{a}\|_2, \tag{14.14}\]
\[- \int_{\Omega \times \mathbb{R}^3} (\nu \cdot \nabla_x \psi_m) P f = -5 \int \Delta \phi_m [\hat{a} + \langle f \rangle] = -5\|\hat{a}\|_6. \tag{15.15}\]

To estimate $\Psi_m^{a,1}$, we calculate
\[\Psi_m^{a,1} \lesssim [\lambda + (1 - \theta)\varepsilon^{-1}] \|f\|_2 \|\nabla \phi_m^a\|_2. \tag{16.16}\]

For $m = 2$ we employ (2.12), $\|\hat{a}\|_2 \lesssim \|P f\|_2$ and $\theta = 1 + o(1)\varepsilon\lambda$ to get
\[|\Psi_m^{a,1}| \lesssim [\lambda + (1 - \theta)\varepsilon^{-1}] \|f\|_2 \|\hat{a}\|_2 \lesssim \lambda \|P f\|_2 + \|I - P\|f\|_2. \tag{17.17}\]

For $m = 6$, we use the embedding $W^{1,\frac{5}{6}}(\Omega) \hookrightarrow L^2(\Omega)$ and Sobolev–Gagliardo–Nirenberg inequality
\[\|\nabla \phi_m^a\|_2 \lesssim \|\nabla \phi_m^a\|_{W^{1,\frac{5}{6}}} \lesssim \|\hat{a}\|_6. \tag{18.18}\]

It follows from Young inequality that
\[|\Psi_m^{a,1}| \lesssim [\lambda + (1 - \theta)\varepsilon^{-1}] \|f\|_2 \|\hat{a}\|_6 \lesssim \lambda \|P f\|_6 + \|I - P\|f\|_6. \tag{19.19}\]

To estimate $\Psi_m^{a,2}$, noticing the expression
\[P_+ f = \sqrt{\mu} z_\gamma(x) \text{ with } z_\gamma(x) := \sqrt{2\pi} \int_{n \cdot v > 0} f \sqrt{\mu} [n \cdot v] \, dv, \tag{20.20}\]
and the decomposition $v = v_n \cdot n + v_\perp$, we have
\[\int_{\gamma_+} \psi_m^a P_+ f \, d\gamma = \int_{\gamma_+} (v_0^2 - 10) \sqrt{\mu} (v_n \partial_n \phi_m^a + v_\perp \nabla_x \phi_m^a) \sqrt{\mu} z_\gamma(x) \, d\gamma = 0, \tag{21.21}\]
where we used $\partial_n \phi_m^a = 0$ for the first term and oddness in $v_\perp$ for the second one. Besides, by $\partial_n \phi_m^a = 0$ and the change of variable $u = R_x v$ we have
\[v \cdot \nabla_x \phi_m^a = [u - 2(u \cdot n)n] \cdot \nabla_x \phi_m^a = u \cdot \nabla_x \phi_m^a - 2(u \cdot n) \partial_n \phi_m^a = u \cdot \nabla_x \phi_m^a, \tag{22.22}\]
which further indicates
\[\int_{\gamma_-} \psi_m^a L f \, d\gamma = \int_{\gamma_+} (|u|^2 - 10) \sqrt{\mu} (u \cdot \nabla_x \phi_m^a) f \, d\gamma = \int_{\gamma_+} \psi_m^a f \, d\gamma. \tag{23.23}\]
It follows that
\[-\Psi_m^{a,2} = \int_{\gamma_+} \psi_m^a f \, d\gamma - \int_{\gamma_-} \psi_m^a [(1 - \alpha) L f + \alpha P_+ f + \alpha r] \, d\gamma \tag{24.24}\]
\[= \alpha \int_{\gamma_+} \psi_m^a (1 - P_+) f \, d\gamma - \alpha \int_{\gamma_-} \psi_m^a r \, d\gamma,
\]
where we used (2.19). For $m = 2$, we use (2.12) and the trace theorem $|\nabla_x \phi_m^a|_2 \lesssim \|\phi_m^a\|_{H^2}$,
\[|\Psi_m^{a,2}| \lesssim \alpha \|(1 - P_+) f\|_{2,+} + |r|_{2,-} \|\nabla_x \phi_m^a\|_2 \lesssim \alpha^2 (1 - P_+) f^2_{2,+} + \alpha^2 |r|_{2,-} + o(1) \|\hat{a}\|_2. \tag{25.25}\]
For \( m = 6 \), we first use the trace theorem \( W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega) \) and Sobolev embedding \( W^{1-\frac{1}{p},p}(\partial\Omega) \subset L^{\frac{p(N-1)}{N-p}}(\partial\Omega) \) for \( p = \frac{3}{2} \) and \( N = 3 \) to get
\[
|\nabla_x \phi^6_0|_\frac{1}{4} \lesssim \| \phi^6_0 \|_{W^{1,\frac{3}{2}}} \lesssim \| \tilde{a} \|_6^3.
\] (2.25)

It follows from interpolation and Young inequality that
\[
\alpha \int_{\gamma^+_\nu} \nu^6_0 f d\gamma | \lesssim \alpha |\mu|^{\frac{1}{2}} (1 - P_\gamma) f|_{|2, +}| |\nabla_x \phi^6_0|_\frac{1}{4}
\lesssim \left[ \alpha \epsilon^{-\frac{1}{2}} |(1 - P_\gamma) f|_{|2, +}| \right] \frac{1}{2} \left[ \alpha \epsilon^\frac{1}{2} \| w f \|_{\infty} \right] \frac{1}{2} \| \tilde{a} \|_6^5
\lesssim \left[ \alpha \epsilon^{-\frac{1}{2}} |(1 - P_\gamma) f|_{|2, +}| \right]^6 + o(1) \left[ \alpha \epsilon^\frac{1}{2} \| w f \|_{\infty} \right]^6 + o(1) \| \tilde{a} \|_6^5.
\] (2.26)

Similarly, we have
\[
\alpha \int_{\gamma^+_\nu} \nu^6_0 r d\gamma | \lesssim \left[ \alpha \epsilon^{-\frac{1}{2}} |r|_{|2, +}| \right]^6 + \left[ \alpha \epsilon^\frac{1}{2} |wr|_{\infty} \right]^6 + o(1) \| \tilde{a} \|_6^5.
\]

Combining this with (2.23) and (2.26), we get
\[
|\Psi^a_6| \lesssim \left[ \alpha \epsilon^{-\frac{1}{2}} |(1 - P_\gamma) f|_{|2, +}| \right]^6 + o(1) \left[ \alpha \epsilon^\frac{1}{2} \| w f \|_{\infty} \right]^6 + \left[ \alpha \epsilon^{-\frac{1}{2}} |r|_{|2, +}| \right]^6 + \left[ \alpha \epsilon^\frac{1}{2} |wr|_{\infty} \right]^6 + o(1) \| \tilde{a} \|_6^5.
\] (2.27)

\( \Psi^a_{m,3} \sim \Psi^a_{m,6} \) are estimated directly. For \( m = 2 \), we have
\[
|\Psi^a_2| \lesssim \| (I - P) f \|_6 + \| \nabla^2 \phi^6_0 \|_\frac{1}{2} \lesssim \| (I - P) f \|_6 \| \tilde{a} \|_6^5 \lesssim \| (I - P) f \|_6 + o(1) \| \tilde{a} \|_6^6,
\]
\[
|\Psi^a_2| \lesssim \epsilon^2 \| \Phi \|_{\infty} \| \nabla^2 \phi^6_0 \|_\frac{1}{2} \left[ \| a \|_6 + \| c \|_6 + \| (I - P) f \|_6 \right]
\lesssim \epsilon^2 \| \Phi \|_{\infty} \| \tilde{a} \|_6^5 + \| c \|_6 + \| (I - P) f \|_6 \| \tilde{a} \|_6^5.
\] (2.28)

where in \( \Psi^a_2 \) the \( b \) contribution vanishes by oddness. For \( m = 6 \), similar estimates lead to
\[
|\Psi^a_6| \lesssim \| (I - P) f \|_6 \| \nabla^2 \phi^6_0 \|_\frac{1}{2} \lesssim \| (I - P) f \|_6 \| \tilde{a} \|_6^5 \lesssim \| (I - P) f \|_6 + o(1) \| \tilde{a} \|_6^6,
\]
\[
|\Psi^a_6| \lesssim \epsilon^2 \| \Phi \|_{\infty} \| \nabla^2 \phi^6_0 \|_\frac{1}{2} \left[ \| a \|_6 + \| c \|_6 + \| (I - P) f \|_6 \right]
\lesssim \epsilon^2 \| \Phi \|_{\infty} \| \tilde{a} \|_6^5 + \| c \|_6 + \| (I - P) f \|_6 \| \tilde{a} \|_6^5.
\] (2.29)

where we have used (2.17).

Collecting (2.14), (2.15), (2.16), (2.18), (2.24), (2.27), (2.28) and (2.29) and absorbing small contributions concerning \( \| \tilde{a} \|_2 \), we get
\[
\| \tilde{a} \|_2^3 \lesssim \lambda \| P f \|_6^2 + \left[ \epsilon^{-1} |(I - P) f|_{\nu} \right]^2 + \alpha^2 |(1 - P_\gamma) f|_{|2, +}|^2 + \alpha^2 |r|_{|2, +}|^2
\]
\[
+ \| \nu^{-\frac{1}{2}} g \|_2^2 + \epsilon \| \Phi \|_{\infty} \| |c|_6^2 + |(f)|_2^2 \| \tilde{a} \|_6^2.
\] (2.30)

and
\[
\| \tilde{a} \|_6^6 \lesssim \lambda \| P f \|_6^6 + \left[ \epsilon^{-1} |(I - P) f|_{\nu} \right]^6 + \| (I - P) f \|_6^6 + o(1) \left[ \alpha \epsilon^\frac{1}{2} \| w f \|_{\infty} \right]^6
\]
\[
+ \left[ \alpha \epsilon^{-\frac{1}{2}} |r|_{|2, +}| \right]^6 + \| \nu^{-\frac{1}{2}} g \|_2^6 + \epsilon^2 \| \Phi \|_{\infty} \| |c|_6^2 + |(f)|_2^6 \| \tilde{a} \|_6^5
\] (2.31)

\textbf{Step 3. Estimates of} \( \| c \|_2 \) \textbf{and} \( \| c \|_6 \).

Choose a test function
\[
\psi^m_\nu := (|v|^2 - 5) \sqrt{\mu} v \cdot \nabla_x \phi^m_\nu(x), \quad m = 2, 6
\]
in \(2.39\). Here \(\phi_m^c\) is defined through
\[
- \Delta_x \phi_m^c + E_m \cdot \nabla_x \phi_m^c = c^{m-1}, \quad \partial_n \phi_m^c |_{\partial \Omega} = 0, \quad \int \phi_m^c \, dx = 0, \quad (2.32)
\]
where \(E_m \in [L^\infty(\Omega)]^3\) is a chosen function with small \(\|E_m\|_\infty \ll 1\), guaranteed by Lemma \[A.3\] in the Appendix. Moreover, Lemma \[A.3\] gives
\[
\|\phi_m^c\|_{H^2} \leq C\|c\|_2, \quad \|\phi_m^c\|_{W^{2,\frac{4}{3}}} \leq C\|c\|_6^5, \quad (2.33)
\]
provided \(\|E_m\|_\infty\) is small enough, see \(A.17\).

The left-hand side of \(2.39\) is treated as \(2.14\) and \(2.15\),
\[
- \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_m^c) P f = -5 \int_{\Omega} c \Delta \phi_m^c = 5\|c\|_m^m - 5 \int_{\Omega} c (E_m \cdot \nabla_x \phi_m^c), \quad (2.34)
\]
where the \(b\) contribution vanishes by oddness,
\[
\int_{\mathbb{R}^3} (|v|^2 - 5)v_i^2 \mu = 0, \quad \int_{\mathbb{R}^3} (|v|^2 - 5)\frac{|v|^2 - 3}{2}v_i^2 \mu = 5, \quad i = 1, 2, 3, \quad (2.35)
\]
and we used \(2.32\) in the last equality. By \(2.33\) and the embedding \(2.17\) we have
\[
\left| \int_{\Omega} c (E_m \cdot \nabla_x \phi_m^c) \right| \leq \|E_m\|_\infty \|\nabla_x \phi_m^c\|_2 \leq C\|E_m\|_\infty \|c\|_2, \quad (2.36)
\]
\[
\int_{\Omega} c (E_m \cdot \nabla_x \phi_m^c) \|c\|_6 \leq C\|E_m\|_\infty \|c\|_6. \quad (2.37)
\]
It follows that
\[
\int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_m^c) P f \geq 4\|c\|_m^m, \quad (2.38)
\]
provided \(\|E_m\|_\infty\) is sufficiently small.

Similarly as \(2.10\) and \(2.13\), we estimate \(\Psi_m^c\) by
\[
|\Psi_m^c| \lesssim \lambda \|P f\|_m^m + \|\Pi - P\|_m^m, \quad m = 2, 6. \quad (2.39)
\]
To estimate \(\Psi_m^c\), by the change of variable \(v \mapsto R_v v\) like \(2.21\) and \(2.22\), we have
\[
-\Psi_m^c = \int_{\gamma_1} (|v|^2 - 5)\sqrt{\mu} (v \cdot \nabla_x \phi_m^c) f d\gamma - \alpha \int_{\gamma_1} \psi_m^c r d\gamma
\]
\[
- \int_{\gamma_2} (|v|^2 - 5)\sqrt{\mu} [(v \cdot \nabla_x \phi_m^c) - 2(n \cdot v) \partial_n \phi_m^c] [(1 - \alpha) f(v) + \alpha P_v f] d\gamma
\]
\[
= \alpha \int_{\gamma_2} (|v|^2 - 5)\sqrt{\mu} (v \cdot \nabla_x \phi_m^c)(1 - P_v) f d\gamma - \alpha \int_{\gamma_2} \psi_m^c r d\gamma,
\]
where the \(P_v f\) contribution vanishes due to
\[
(n \cdot v)^2 = \sum_{i,j=1}^3 v_i v_j n_i n_j \quad \text{and} \quad \int_{n \cdot v > 0} (|v|^2 - 5)v_i^2 \mu = 0, \quad k = 1, 2, 3. \quad (2.40)
\]
Then, similarly as \(2.24\) and \(2.27\), we get
\[
|\Psi_m^c| \lesssim \alpha^2 [(1 - P_v) f]_{2,+}^2 + \alpha^2 |r|_{2,-}^2 + o(1)\|c\|_2, \quad (2.41)
\]
and
\[
|\Psi_m^c| \lesssim [\alpha \varepsilon^{-\frac{1}{2}} (1 - P_v) f]_{2,\pm}^6 + o(1)\|\nabla f\|_\infty^6 + \|v\|_\infty^6 + o(1)\|c\|_6^6.
\]
\[
|\Psi_m^c| \lesssim \|\Pi - P\|_m^m + o(1)\|c\|_m^m, \quad |\Psi_m^c| \lesssim \varepsilon^2 \|\Phi\|_\infty [\|c\|_m^m + \|\Pi - P\|_m^m],
\]
\[
|\Psi_m^c| \lesssim \varepsilon^{-\frac{5}{2}} \|\Pi - P\|_m^m + o(1)\|c\|_m^m, \quad |\Psi_m^c| \lesssim \|\nabla g\|_2^2 + o(1)\|c\|_m^m.
\]

where $a, b$ contributions in $\Psi_{\alpha}^{m, b}$ vanish due to oddness and $\int_{\Omega} [2v_i^2 + (|v|^2 - 5)] \mu dv = 0$.

Collect (2.37), (2.38), (2.39)–(2.41) and absorb small contributions concerning $|c|_2$,

$$
\|c\|^2_2 \lesssim \lambda \|P f\|^2_2 + \epsilon^{-2} \|(I - P) f\|^2_2 + \alpha^2 |(1 - P\gamma) f|_{2,+}^2 + \alpha^2 |r|_{2,-}^2 + \|v^{-\frac{1}{2}} g\|^2_2
$$

and

$$
\|c\|^6_6 \lesssim \lambda \|P f\|^6_6 + [\epsilon^{-1} \|(I - P) f\|^6_2] + \|P f\|^6_6 + \|v^{-\frac{1}{2}} g\|^6_2 + \left[\alpha \epsilon^{-\frac{1}{2}} |(1 - P\gamma) f|_{2,+}^6 + \alpha \epsilon^{-\frac{1}{2}} |r|_{2,-}^6 + \alpha \epsilon^{-\frac{1}{2}} |wr|_{\infty}^6 \right].
$$

Step 4. Estimates of $\|b\|_2$ and $\|b\|_6$.

Choose a test function

$$
\psi_m^{b} := \sum_{i,j=1}^{3} \partial_{i} \phi_{i}^{m,b} v_{i} v_{j} \sqrt{\mu} - \sum_{i=1}^{3} \partial_{i} \phi_{i}^{m,b} \frac{|v|^2 - 1}{2} \sqrt{\mu}, \quad m = 2, 6
$$
in (2.44), first introduced in [53]. Here $\phi^{m,b} = (\phi_{1}^{m,b}, \phi_{2}^{m,b}, \phi_{3}^{m,b})$ satisfies the elliptic system

$$
\Delta_x \phi^{m,b} = b^{m-1} \text{ in } \Omega, \quad \phi^{m,b} \cdot n = 0 \text{ on } \partial \Omega, \quad \partial_n \phi^{m,b} = (\phi^{m,b} \cdot n) n \text{ on } \partial \Omega,
$$

where we used the notation $b^{m-1} := (b_{1}^{m-1}, b_{2}^{m-1}, b_{3}^{m-1})$. Existence of $\phi^{m,b}$ is proved in Lemma A.2 in the Appendix. Moreover, Lemma A.2 gives the elliptic estimates

$$
\|\phi_{2}^{b}\|_{H^2} \leq C\|b\|_2, \quad \|\phi_{6}^{b}\|_{W^{2,4}} \leq C\|b\|_6.
$$

Direct computation shows

$$
-v \cdot \nabla_x \psi_m^{b} = -\sum_{i,j,k=1}^{3} \partial_{jk}^{2} \phi_i^{m,b}(I - P)(v_i v_j v_k \sqrt{\mu}) - \sum_{i,j,k=1}^{3} \partial_{jk}^{2} \phi_i^{m,b} P(v_i v_j v_k \sqrt{\mu})
$$

$$
+ \sum_{i,k=1}^{3} \partial_{ik}^{2} \phi_i^{m,b} v_k \frac{|v|^2 - 1}{2} \sqrt{\mu}.
$$

By oddness of $v_i v_j v_k$, we have

$$
\sum_{i,j,k=1}^{3} \partial_{jk}^{2} \phi_i^{m,b} P(v_i v_j v_k \sqrt{\mu}) = \sum_{i,j,k=1}^{3} v_i \sqrt{\mu} \sum_{l=1}^{3} \partial_{jk}^{2} \phi_i^{m,b} \int_{\mathbb{R}^3} v_i v_j v_k v_l \mu dv
$$

$$
= \sum_{i=1}^{3} v_i \sqrt{\mu} \left( \sum_{l=i,k,l}^{3} + \sum_{i\neq j,i:k,l}^{3} + \sum_{i\neq j,i:l,k,j}^{3} \right) \partial_{jk}^{2} \phi_i^{m,b} \int_{\mathbb{R}^3} v_i v_j v_k v_l \mu dv.
$$

For fixed $l = 1, 2, 3$,

$$
\sum_{i,j,k=1}^{3} \partial_{jk}^{2} \phi_i^{m,b} \int_{\mathbb{R}^3} v_i v_j v_k v_l \mu dv = (\sum_{i=l}^{3} + \sum_{i\neq l}^{3}) \partial_{ll}^{2} \phi_i^{m,b} \int_{\mathbb{R}^3} v_i^2 v_l^2 \mu dv = 3 \partial_{ll}^{2} \phi_i^{m,b} + \sum_{i\neq l}^{3} \partial_{ll}^{2} \phi_i^{m,b},
$$

$$
\sum_{i,j,k=1, k=l}^{3} \partial_{jk}^{2} \phi_i^{m,b} \int_{\mathbb{R}^3} v_i v_j v_k v_l \mu dv = \sum_{i\neq l}^{3} \partial_{ll}^{2} \phi_i^{m,b} \int_{\mathbb{R}^3} v_i^2 v_l^2 \mu dv = \sum_{i\neq l}^{3} \partial_{ll}^{2} \phi_i^{m,b},
$$

$$
\sum_{i,j,k=1, i:k, l}^{3} \partial_{jk}^{2} \phi_i^{m,b} \int_{\mathbb{R}^3} v_i v_j v_k v_l \mu dv = \sum_{j\neq l}^{3} \partial_{ll}^{2} \phi_i^{m,b} \int_{\mathbb{R}^3} v_i^2 v_l^2 \mu dv = \sum_{j\neq l}^{3} \partial_{ll}^{2} \phi_i^{m,b},
$$

where we used

$$
\int_{\mathbb{R}^3} v_i^2 v_j^2 \mu dv = \begin{cases} 
3, & \text{if } i = j, \\
1, & \text{if } i \neq j.
\end{cases}
$$
Thus the second term in (2.46) leads to
\[
- \int_{\Omega \times \mathbb{R}^3} P f \sum_{i,j,k=1}^{3} \partial_{j k} \phi_m^{m,b} \mathbf{P}(v_i v_j v_k \sqrt{\mu})
\]
\[= - \sum_{i=1}^{3} \int_{\Omega} b_i (3 \partial_{i}^{2} \phi_m^{m,b} + \sum_{i \neq j} \partial_{i}^{2} \phi_i^{m,b} + \sum_{j \neq k} \partial_{j k} \phi_j^{m,b} + \sum_{j \neq k} \partial_{j k} \phi_j^{m,b}) dx. \quad (2.47)
\]
The third term in (2.46) leads to
\[
\int_{\Omega \times \mathbb{R}^3} P f \sum_{i,k=1}^{3} \partial_{i k} \phi_i^{m,b} v_k \frac{|v|^2 - 1}{2} \sqrt{\mu} = \int_{\Omega} \left( \sum_{i=1}^{3} b_i \partial_{i}^{2} \phi_i^{m,b} + \sum_{i \neq j} b_k \partial_{i k} \phi_j^{m,b} \right) dx, \quad (2.48)
\]
by \( \int_{\mathbb{R}^3} \frac{|v|^2 - 1}{2} \mu dv = 2 \). Combining (2.46)–(2.48), we have
\[
\int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_m^{b}) P f = - \sum_{i=1}^{3} \int_{\Omega} b_i (\partial_{i}^{2} \phi_i^{m,b} + \sum_{i \neq j} \partial_{i j} \phi_j^{m,b}) dx
\]
\[= - \sum_{i=1}^{3} \int_{\Omega} b_i \Delta \phi_i^{m,b} dx = \|b\|_m^m, \quad (2.49)
\]
where we used (2.44) in the last step.
Similarly as (2.19) and (2.18) we estimate \( \psi_{m}^{b,1} \) by
\[
|\psi_{m}^{b,1}| \lesssim \lambda \|P f\|_m^m + \|\mathbf{I} - P\|_2^m. \quad (2.50)
\]
To estimate \( \psi_{m}^{b,2} \), we have
\[
\psi_{m}^{b,2} = \int_{\partial \Omega \times \mathbb{R}^3} (v \cdot n) \left( \sum_{i,j=1}^{3} \partial_{j \phi_i^{m,b}} v_i v_j \sqrt{\mu} - \sum_{i=1}^{3} \partial_{i \phi_i^{m,b}} \frac{|v|^2 - 1}{2} \sqrt{\mu} \right) f dv d\gamma =: I_1 - I_2. \quad (2.51)
\]
Noticing \( \frac{|v|^2 - 1}{2} \sqrt{\mu} \) is invariant under the change of variable \( v \mapsto R_x v \), we deduce similarly as (2.22) to get
\[
I_2 = \alpha \int_{\gamma} (1 - P_{\gamma}) \sum_{i=1}^{3} \partial_{i \phi_i^{m,b}} \frac{|v|^2 - 1}{2} \sqrt{\mu} dv d\gamma = - \alpha \int_{\gamma} \sum_{i=1}^{3} \partial_{i \phi_i^{m,b}} \frac{|v|^2 - 1}{2} \sqrt{\mu} dv d\gamma. \quad (2.52)
\]
Then we estimate \( I_1 \). Through a coordinate rotation, we may assume \( n = (0, 0, 1) \). From the first boundary condition \( \phi_m^{m,b} \cdot n = 0 \) in (2.44), we get \( \phi_3^{m,b} = 0 \) on \( \partial \Omega \), which indicates that the tangential derivative is also zero, that is,
\[
\partial_3 \phi_3^{m,b} = 0, \quad \partial_2 \phi_3^{m,b} = 0 \quad \text{on} \ \partial \Omega.
\]
The second boundary condition \( \partial_n \phi_m^{m,b} = (\partial_n \phi_2^{m,b} \cdot n)n \) in (2.44) implies
\[
\partial_3 \phi_1^{m,b} = 0, \quad \partial_3 \phi_1^{m,b} = 0 \quad \text{on} \ \partial \Omega.
\]
It follows that
\[
I_1 = \int_{\partial \Omega \times \mathbb{R}^3} \left( \partial_{3 \phi_1^{m,b}} v_3 v_1 + \partial_{3 \phi_2^{m,b}} v_3 v_2 + \partial_1 \phi_3^{m,b} v_1 v_3 + \partial_2 \phi_3^{m,b} v_2 v_3 + \partial_3 \phi_3^{m,b} v_3^2 \right)
\]
\[+ \sum_{i,j=1}^{2} \partial_{j \phi_i^{m,b}} v_j v_i (v \cdot n) \sqrt{\mu} f dv d\gamma \quad (2.53)
\]
\[= \int_{\partial \Omega \times \mathbb{R}^3} \left( \partial_{3 \phi_1^{m,b}} v_3^2 + \sum_{i,j=1}^{2} \partial_{j \phi_i^{m,b}} v_j v_i \right) (v \cdot n) \sqrt{\mu} f dv d\gamma.
\]
Observe that $\partial_3 \phi_i^{m,b} v_3^2 + \sum_{i,j=1}^2 \partial_j \phi_i^{m,b} v_j v_i$ is even in $v_3$. By the change of variable $u = R x v$, $(u_1, u_2, u_3) = (v_1, v_2, v_3) - 2 v_3 (0, 0, 1) = (v_1, v_2, -v_3)$ and $v' = -u' \cdot n$. Thus we have

$$\int_{\partial \Omega \times \mathbb{R}^3} \partial_3 \phi_i^{m,b} v_3^2 (v' \cdot n) \sqrt{\rho} f \, dv d\Omega = \int_{\gamma^+} \partial_3 \phi_i^{m,b} v_3^2 (v' \cdot n) \sqrt{\rho} f \, dv d\Omega - \int_{\gamma^-} \partial_3 \phi_i^{m,b} v_3^2 (v' \cdot n) \sqrt{\rho} f \, dv d\Omega$$

$$= \int_{\gamma^+} \partial_3 \phi_i^{m,b} v_3^2 (v' \cdot n) \sqrt{\rho} f \, dv d\Omega - \int_{\gamma^-} \partial_3 \phi_i^{m,b} v_3^2 (v' \cdot n) \sqrt{\rho} f \, dv d\Omega - \int_{\gamma^-} \partial_3 \phi_i^{m,b} (v_3) \frac{\partial}{\partial u_3} \left[ (1 - \alpha) f \right] (u_1, u_2, u_3) \, du_1 du_2 d\rho$$

$$= \alpha \int_{\gamma^+} \partial_3 \phi_i^{m,b} v_3^2 (v' \cdot n) \sqrt{\rho} (1 - P) f \, dv d\Omega - \alpha \int_{\gamma^-} \partial_3 \phi_i^{m,b} v_3^2 (v' \cdot n) \sqrt{\rho} f \, dv d\Omega.$$

The computation of the term concerning $\sum_{i,j=1}^2 \partial_j \phi_i^{m,b} v_j v_i$ is similarly. We conclude

$$I^b_1 = \alpha \int_{\gamma^+} \left( \partial_3 \phi_i^{m,b} v_3^2 + \sum_{i,j=1}^2 \partial_j \phi_i^{m,b} v_j v_i \right) \sqrt{\rho} (1 - P) f \, dv d\gamma$$

$$- \alpha \int_{\gamma^-} \left( \partial_3 \phi_i^{m,b} v_3^2 + \sum_{i,j=1}^2 \partial_j \phi_i^{m,b} v_j v_i \right) \sqrt{\rho} f \, dv d\gamma.$$

Collecting (2.51), (2.52) and (2.55) and estimating similarly as (2.24) and (2.27), we get

$$|\Psi_2^{b,2}| \lesssim \alpha^2 \left| (1 - P) f \right|_2^2 + \alpha^2 \left| r \right|_{2,-}^2 + o(1) \| b \|_2^2,$$

$$|\Psi_6^{b,2}| \lesssim \left[ \alpha \left| \left( 1 - P \right) f \right|_{2,+} \right]^6 + o(1) \left[ \alpha \left| w f \right|_{\infty} \right]^6 + \left[ \alpha \left| w r \right|_{\infty} \right]^6 + o(1) \| b \|_6^6,$$

where we used (2.54).

We estimate $|\Psi_3^{m,3}| \approx |\Psi_6^{b,6}|$ directly as (2.28) and (2.29),

$$|\Psi_5^{b,3}| \lesssim \| (I - P) f \|_{m}^m + o(1) \| b \|_m^m,$$

$$|\Psi_6^{b,4}| \lesssim \left[ \alpha \varepsilon \frac{1}{2} \right] \| (I - P) f \|_{m}^m + o(1) \| b \|_m^m,$$

where in $\Psi_6^{b,4}$ the $a, c$ contributions vanish due to oddness.

Collecting the estimates from (2.30), (2.31), (2.32) and (2.33), (2.34), (2.35) and (2.36), we get

$$\| b \|_2^2 \lesssim \lambda \| (I - P) f \|_{m}^m + \varepsilon^2 \| (I - P) f \|_{m}^m + \alpha^2 \| (1 - P) \|_{2,+}^2 + \alpha^2 \left| r \right|_{2,-}^2 + \nu^{-\frac{1}{2}} 2^2,$$

and

$$\| b \|_6^6 \lesssim \lambda \| (I - P) f \|_{m}^m + \left[ \varepsilon^{-\frac{1}{2}} \left| (I - P) f \right|_v \right]^6 + \nu^{-\frac{1}{2}} 2^6 + \left[ \alpha \varepsilon \frac{1}{2} \right] \left| (I - P) f \right|_{2,+}^6 + \nu^{-\frac{1}{2}} 6^6 + \left[ \alpha \varepsilon \frac{1}{2} \right] \left| w f \right|_{\infty}^6 + \left[ \alpha \varepsilon \frac{1}{2} \right] \left| w r \right|_{\infty}^6.$$

Finally, combining (2.30), (2.31) and (2.32) and employing (2.34) and (2.35) we get, for small $\lambda > 0$,

$$\| (I - P) f \|_{m}^m \lesssim \varepsilon^{-2} \| (I - P) f \|_{m}^m + \alpha^2 \| (1 - P) \|_{2,+}^2 + \alpha^2 \left| r \right|_{2,-}^2 + \nu^{-\frac{1}{2}} 2^2,$$

and

$$\| (I - P) f \|_{m}^m \lesssim \| (I - P) f \|_{m}^m + \left[ \varepsilon^{-\frac{1}{2}} \left| (I - P) f \right|_v \right]^6 + \left[ \varepsilon^{-\frac{1}{2}} \left| (I - P) f \right|_v \right]^6 + \nu^{-\frac{1}{2}} 2^6 + \left[ \alpha \varepsilon \frac{1}{2} \right] \left| w f \right|_{\infty}^6 + \left[ \alpha \varepsilon \frac{1}{2} \right] \left| w r \right|_{\infty}^6.$$
we get
\[
\|Pf\|_6^6 \leq \left[ \varepsilon^{-1}\| (I-P)f \|_\nu \right]^6 + \left[ \alpha \varepsilon^{-\frac{1}{2}} |(1 - P_\gamma)f_{2,+}| \right]^6 + \left[ \alpha \varepsilon^{-\frac{1}{2}} |r|_{2,+} \right]^6 + \left[ \alpha \varepsilon^{-\frac{1}{2}} \| wr \|_\infty \right]^6 \\
+ \| \nu^{-\frac{1}{2}} g \|_2^6 + \rho_1 \left[ \alpha \varepsilon^{-\frac{1}{2}} \| w f \|_\infty \right]^6 + (\lambda + \varepsilon^2 \| \Phi \|_\infty ) |f|^6,
\] (2.63)
where \( \rho_1 \) is a small constant. This proves (2.4) and (2.5). \( \square \)

We need the following trace lemma for steady transport equations with small external fields given by \([17]\).

**Lemma 2.2.** Suppose that \( f \in L^1(\Omega \times \mathbb{R}^3) \) and \( v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_v f \in L^1(\Omega \times \mathbb{R}^3) \). Then \( f \) has a trace on \( \gamma_\pm \) and
\[
|f|_{1,\gamma} \lesssim_{\delta, \Omega} \| f \|_1 + \| v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_v f \|_1.
\]

We construct the \( L^2 \) coercivity estimate for the steady linear equation.

**Theorem 2.3.** Let \( \Phi, g \) and \( r \) satisfy the assumption of Lemma \([17]\) Then
\[
v \cdot \nabla_x f + \varepsilon^2 \frac{1}{\sqrt{\nu}} \Phi \cdot \nabla_v (\sqrt{\nu} f) + \varepsilon^{-1} L f = g,
\]
\[
f \big|_{\gamma} = (1 - \alpha) L f + \alpha P_\gamma f + \alpha r
\]
has a unique weak solution \( f \) satisfying
\[
\int_{\Omega \times \mathbb{R}^3} f(x, v) \sqrt{\nu} dv dx = 0
\] (2.65)
and
\[
\| Pf \|_2 + \frac{1}{\varepsilon} \| (I-P)f \|_\nu + \frac{\alpha}{\sqrt{\varepsilon}} |(1 - P_\gamma)f_{2,+}| + \frac{\alpha}{\sqrt{\varepsilon}} |f|_2
\leq \frac{\sqrt{\alpha}}{\varepsilon} |r|_{2,-} + \frac{1}{\varepsilon} \| Pg \|_2 + \| \nu^{-\frac{1}{2}} (I-P)g \|_2.
\] (2.66)
Moreover,
\[
\| Pf \|_6 \leq \frac{1}{\varepsilon} \| (I-P)f \|_\nu + \frac{\alpha}{\sqrt{\varepsilon}} |(1 - P_\gamma)f_{2,+}| + \alpha \varepsilon^{-\frac{1}{2}} |wr|_\infty + o(1) \alpha \varepsilon^{-\frac{1}{2}} \| w f \|_\infty
\] + \frac{\alpha}{\sqrt{\varepsilon}} |r|_{2,-} + \frac{1}{\varepsilon} \| Pg \|_2 + \| \nu^{-\frac{1}{2}} (I-P)g \|_2.
\] (2.67)

**Proof.** Step 1. For fixed \( \lambda > 0 \), we consider the approximate problem
\[
[(\lambda + (1 - \theta)\varepsilon^{-1} \nu - \frac{1}{2} \varepsilon^2 \Phi \cdot v) + v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla_v + \varepsilon^{-1} \theta L] f_\theta = g,
\]
\[
f_\theta \big|_{\gamma} = (1 - \alpha) L f_\theta + \alpha P_\gamma f_\theta + \alpha r, \quad \theta \in \left[ \frac{3}{4}, 1 \right]
\] (2.68)
and show that \( \| f_\theta \|_2 \) is bounded uniformly in \( \theta \).

Multiply (2.68) with \( f_\theta \) and integrate on \( \Omega \times \mathbb{R}^3 \),
\[
\lambda \| f_\theta \|^2_2 + \varepsilon^{-1} (1 - \theta) \| f_\theta \|^2_\nu + \varepsilon^{-1} \theta |L| \| (I-P)f_\theta \|^2_\nu + \int_{\gamma_+} |f_\theta|^2 d\gamma
\leq \int_{\gamma_-} |f_\theta|^2 d\gamma + 2 \int_{\Omega \times \mathbb{R}^3} |gf_\theta| + \varepsilon^2 \| \Phi \|_\infty \| f_\theta \|_\nu^2.
\] (2.69)
By the decomposition
\[
|f_\theta|^2_{2,+} = |(1 - P_\gamma)f_\theta|^2_{2,+} + |P_\gamma f_\theta|^2_{2,+}
\] (2.70)
and the change of variable $v \mapsto R_x v$, we have
\[
\int_{\gamma_-} \left[ (1 - \alpha)\mathcal{L} f_\theta + \alpha P_v f_\theta \right]^2 d\gamma = \int_{\gamma_+} \left[ (1 - \alpha)f_\theta + \alpha P_v f_\theta \right]^2 d\gamma.
\]
\[= (1 - \alpha)^2 (1 - P_\gamma) f_\theta^2_{2, +} + |P_v f_\theta|_{2, +}^2. \tag{2.71}\]

It follows from (2.29) and (2.13) that \( \int_{\gamma_-} r P_v f_\theta d\gamma = 0 \). Then by the change of variable $v \mapsto R_x v$ and (2.71), we get
\[
|2 \int_{\gamma_-} \left[ (1 - \alpha)\mathcal{L} f_\theta + \alpha P_v f_\theta \right] \alpha r d\gamma | \leq \eta_1 \alpha (1 - \alpha) \mathcal{L} f_\theta^2_{2, -} + \alpha C_\gamma |r|_{2, -}^2
\leq \eta_1 \alpha (1 - \alpha)^2 |f_\theta|_{2, +}^2 + \alpha C_\gamma |r|_{2, -}^2 \tag{2.72}
for some small constant $\eta_1 < 1$. It follows from (2.71) and (2.72) that
\[
\int_{\gamma_-} |f_\theta|^2 d\gamma \leq (1 - \alpha)^2 (1 + \eta_1 \alpha)((1 - P_\gamma) f_\theta^2_{2, +} + (1 + \eta_1 \alpha)|P_v f_\theta|_{2, +}^2 + \alpha C_\gamma |r|_{2, -}^2. \tag{2.73}
\]

We split \( \int_{\Omega \times \mathbb{R}^3} |g f_\theta| \) as
\[
\int_{\Omega \times \mathbb{R}^3} |g f_\theta| \leq \eta_2 \epsilon^{-1} \| (I - P) f_\theta \|_2^2 + C_\eta \epsilon \| \nu^{-\frac{1}{2}} (I - P) g \|_2^2 + \eta_3 \epsilon \| P f_\theta \|_2^2 + C_\eta \epsilon^{-1} \| P g \|_2^2. \tag{2.74}
\]

Combining (2.69), (2.73) and (2.74) and absorbing small contributions, we get
\[
\lambda \| f_\theta \|_2^2 + \epsilon^{-1} (1 - \theta) \| f_\theta \|_2^2 + \epsilon^{-1} \theta \sigma_L \| (I - P) f_\theta \|_2^2 + \alpha |(1 - P_\gamma) f_\theta|_{2, +}^2
\leq C_\eta [\alpha |r|_{2, -}^2 + \epsilon^{-1} \| P g \|_2^2 + \epsilon \| \nu^{-\frac{1}{2}} (I - P) g \|_2^2] + + \epsilon^2 \| \Phi \|_\infty \| f_\theta \|_2^2
+ + \eta \alpha |P_v f_\theta|_{2, +}^2 + \eta \| P f_\theta \|_2^2 \tag{2.75}
\]

for some new small constant $0 < \eta < 1$. Trace Lemma 2.2 implies
\[
|P_\gamma f_\theta|_{2, +}^2 \lesssim \int_{\gamma_-} |f_\theta^2 \mathbf{1}_{\gamma_+}^L| d\gamma + |(1 - P_\gamma) f_\theta|_{2, +}^2
\leq \delta \| f_\theta \|_1^2 + |(v \cdot \nabla_x + \epsilon^2 \Phi \cdot \nabla_v) f_\theta^2 \|_1 + |(1 - P_\gamma) f_\theta|_{2, +}^2
\leq \delta \| f_\theta \|_2^2 + \| g f_\theta - \epsilon^{-1} \theta f_\theta L f_\theta - [\lambda + (1 - \theta) \epsilon^{-1} \nu - \frac{1}{2} \epsilon^2 \Phi \cdot v] f_\theta^2 \|_1 + \| P_\gamma f_\theta |_{2, +}^2
\leq \delta \| f_\theta \|_2^2 + \| P f_\theta \|_2^2 + \| P_\gamma f_\theta |_{2, +}^2 + \| \nu^{-\frac{1}{2}} (I - P) g \|_2^2 + (1 + \eta) \epsilon^2 \| \Phi \|_\infty \| f_\theta \|_2^2
\leq \delta \| f_\theta \|_2^2 + \| P f_\theta \|_2^2 + \| P_\gamma f_\theta |_{2, +}^2 + \| \nu^{-\frac{1}{2}} (I - P) g \|_2^2 + (1 + \eta) \epsilon^2 \| \Phi \|_\infty \| f_\theta \|_2^2. \tag{2.76}\]

Multiplying (2.70) with $\eta \alpha$ and adding to (2.75), we get
\[
\lambda - \eta \alpha (1 + \lambda) \| f_\theta \|_2^2 + \epsilon^{-1} (1 - \theta)(1 - \eta) \| f_\theta \|_2^2 + \alpha (1 - \eta) |(1 - P_\gamma) f_\theta|_{2, +}^2 + \epsilon^{-1} [\sigma_L \theta - \eta \alpha (1 + \theta)] \| (I - P) f_\theta \|_2^2
\leq C_\eta [\alpha |r|_{2, -}^2 + \epsilon^{-1} \| P g \|_2^2 + \epsilon \| \nu^{-\frac{1}{2}} (I - P) g \|_2^2] + (1 + \eta) \epsilon^2 \| \Phi \|_\infty \| f_\theta \|_2^2
\leq \delta \| f_\theta \|_2^2 + \| P f_\theta \|_2^2 + \| P_\gamma f_\theta |_{2, +}^2 + \| \nu^{-\frac{1}{2}} (I - P) g \|_2^2. \tag{2.77}\]

For given $0 < \lambda < 1, 0 < \epsilon \ll 1, 0 \leq \alpha \leq 1$ and $\frac{4}{7} \leq \theta \leq 1$, we choose $\eta$ small such that
\[
\lambda - \eta \alpha (1 + \lambda) - \eta \epsilon (1 + \alpha) - (1 + \eta) \epsilon^2 \| \Phi \|_\infty \geq \frac{1}{2} \lambda,
1 - \eta \geq \frac{1}{2} \sigma_L \theta - \eta \alpha (1 + \theta) - (1 + \eta) \epsilon^3 \| \Phi \|_\infty \geq \frac{1}{2} \sigma_L.
\]

It follows from (2.77) and the fact $\| P f_\theta \|_2 \sim \| P f_\theta \|_\nu$, that
\[
\lambda \| f_\theta \|_2^2 + \epsilon^{-1} (1 - \theta) \| f_\theta \|_2^2 + \epsilon^{-1} \| (I - P) f_\theta \|_2^2 + \alpha |(1 - P_\gamma) f_\theta|_{2, +}^2
\leq \alpha |r|_{2, -}^2 + \epsilon^{-1} \| P g \|_2^2 + \epsilon \| \nu^{-\frac{1}{2}} (I - P) g \|_2^2. \tag{2.78}\]

This proves the uniform boundedness of $f_\theta |_{2, +}$.
Step 2. In this step, we consider the penalization problem of (2.64)

\[ \lambda + v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla v - \frac{1}{2} \varepsilon^2 \Phi \cdot v + \varepsilon^{-1} L \mathbf{f}_\lambda = g, \quad (2.79) \]

with parameter \( \lambda > 0 \), and give the uniform estimates.

Solution of (2.79) can be seen as a fixed point of the map

\[ f_\lambda \mapsto \mathcal{S}^{-1}[\varepsilon^{-1} K f_\lambda + g], \quad (2.80) \]

where \( \mathcal{S}^{-1} \) is the solution operator of the linear problem

\[ \mathcal{S} u_\lambda := \left[ (\lambda + \varepsilon^{-1} \nu - \frac{1}{2} \varepsilon^2 \Phi \cdot v) + v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla v \right] u_\lambda = g, \]

\[ u_\lambda|_{\gamma_-} = (1 - \alpha) \mathcal{L} u_\lambda + \alpha P\gamma u_\lambda + \alpha r. \quad (2.81) \]

In fact, solvability of (2.81) can be obtained via iteration argument together with similar uniform estimates given in Step I, cf. proof of Lemma 2.10 of [17]. We omit the details for brevity.

Define \( f_0 = \mathcal{S}^{-1} h_\theta \). Then \( h_\theta \) is bounded uniformly in \( \theta \), by boundedness of \( \|f_\theta\|_2 \). Consider the fixed point problem

\[ h_\theta \mapsto \theta \varepsilon^{-1} K \mathcal{S}^{-1} h_\theta + g, \quad \theta \in \left[ \frac{3}{4}, 1 \right]. \quad (2.82) \]

It follows from uniform boundedness of \( h_\theta \), compactness of \( K \mathcal{S}^{-1} \) which was proved by Lemma 2.11 of [17], and Schaefer’s fixed point theorem that

\[ \text{the map } h_\lambda \mapsto \varepsilon^{-1} K \mathcal{S}^{-1} h_\lambda + g \text{ has a fixed point } h_\lambda. \quad (2.83) \]

By writing \( f_\lambda = \mathcal{S}^{-1} h_\lambda \), then (2.80) is equivalent to (2.83) and thus has a fixed point. This proves that the penalization problem (2.79) has a solution \( f_\lambda \).

Make difference between (2.68) and its limit (2.79),

\[ \left[ \lambda + v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla v - \frac{1}{2} \varepsilon^2 \Phi \cdot v + \varepsilon^{-1} \theta L \right] (f_0 - f_\lambda) = (1 - \theta)\varepsilon^{-1}(L f_\lambda + \nu f_\theta), \quad (2.84) \]

Standard \( L^2 \) estimate leads to

\[ \lambda \| f_0 - f_\lambda \|_2^2 + \varepsilon^{-1} \theta \| (I - P)(f_0 - f_\lambda) \|_2^2 + \alpha \|(1 - P\gamma)(f_0 - f_\lambda)\|_2^2, \]

\[ \lesssim \varepsilon^2 \| \Phi \|_\infty \| f_0 - f_\lambda \|_2^2 + (1 - \theta)\varepsilon^{-1}(\| f_\theta \|_2 + \| f_\lambda \|_2) \| f_0 - f_\lambda \|_2. \]

Recall that we have proved the uniform boundedness of \( \| f_\theta \|_\nu \) and \( \| f_\lambda \|_\nu \) in (2.78). Letting \( \theta \to 1^- \), we obtain

\[ \| f_0 - f_\lambda \|_2 \to 0, \quad \| (1 - P\gamma)(f_0 - f_\lambda) \|_2 \to 0 \quad \text{as } \theta \to 1. \]

It follows from the trace Lemma (2.22) that \( \| f_0 - f_\lambda \|_2 \to 0 \), which combining with boundary condition of \( f_0 - f_\lambda \) imply \( \| f_0 - f_\lambda \|_2 \to 0 \). Thus, for fixed \( \lambda \), we have

\[ \| (f_\lambda) \| = \lim_{\theta \to 1^-} \| (f_\theta) \| \lesssim \lambda \lim_{\theta \to 1^-} (1 - \theta)\varepsilon^{-1}\| f_\theta \|_2 = 0. \quad (2.85) \]

Now we apply (2.3) and (2.4) to (2.79),

\[ \| P f_\lambda \|_2^2 \lesssim \| P f_\lambda \|_2^2 + \| (f_\lambda) \|^2 \]

\[ \lesssim \varepsilon^2 \| (I - P) f_\lambda \|_2^2 + \alpha^2 (1 - P\gamma) \| f_\lambda \|_2^2 + \alpha^2 \| f_\lambda \|_2^2 \lesssim \| P g \|_2^2 + \varepsilon \| (I - P) g \|_2^2. \quad (2.86) \]

Taking limit of (2.78) we get

\[ \| f_\lambda \|_2^2 + \varepsilon^{-1}\| (I - P) f_\lambda \|_2^2 + \| (1 - P\gamma) f_\lambda \|_2^2 \lesssim \alpha \| f_\lambda \|_2^2 + \varepsilon \| (I - P) g \|_2^2. \quad (2.87) \]
Multiplying (2.86) with $\delta\varepsilon$ for small constant $\delta > 0$ and adding to (2.87), we get
\[
\| Pf \|_2 + \frac{1}{\varepsilon} \| (I - P) f \|_2 + \sqrt{\frac{\alpha}{\varepsilon}} (1 - P) f \|_2 + \frac{\alpha}{\varepsilon} \| v \|_2 + \frac{1}{\varepsilon} \| Pf \|_2 + \| \nu^{-\frac{1}{2}} (I - P) g \|_2.
\] (2.88)

**Step 3.** Now we consider weak solutions of (2.64) and prove (2.66) and (2.67).

It follows from (2.88) that $\| f \|_2$ is bounded uniformly in $\lambda$. Thus there exists $f \in L^2(\Omega \times \mathbb{R}^3)$ such that
\[
f_{\lambda} \to f \quad \text{weakly in } L^2(\Omega \times \mathbb{R}^3) \text{ as } \lambda \to 0.
\]

Besides, $\langle f \rangle = 0$ leads to $\langle f \rangle = 0$.

To prove strong convergence, we make difference between (2.79) and its limit
\[
[\lambda + v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla_v - \frac{1}{2} \varepsilon^2 \Phi \cdot v + \varepsilon^{-1} L] (f - f_{\lambda}) = \lambda f,
\]
\[
(f - f_{\lambda})|_{\gamma} = (1 - \alpha) \mathcal{L} (f - f_{\lambda}) + \alpha P_{\gamma} (f - f_{\lambda}).
\] (2.89)

Similarly as (2.90), we get
\[
\lambda \| f - f_{\lambda} \|_2^2 + \varepsilon^{-1} \| (I - P) (f - f_{\lambda}) \|_2^2 + \alpha (1 - P_{\gamma}) (f - f_{\lambda}) \|_2^2 + \lambda \| \nu^{-\frac{1}{2}} f \|_2^2.
\]
(2.90)

Moreover, apply (2.21) and (2.23) to (2.89),
\[
\| Pf \|_2 + \frac{1}{\varepsilon} \| (I - P) f \|_2 + \sqrt{\frac{\alpha}{\varepsilon}} (1 - P_{\gamma}) f \|_2 + \frac{\alpha}{\varepsilon} \| P f \|_2 + \| \nu^{-\frac{1}{2}} (I - P) g \|_2.
\] (2.91)

Combining (2.90) and (2.91) and using $\| f - f_{\lambda} \|_2^2 \lesssim \| f \|_2^2 + \| (I - P) (f - f_{\lambda}) \|_2^2$, we get $\| f - f_{\lambda} \|_2 \to 0$ as $\lambda \to 0$. Uniqueness of $f$ follows from standard argument.

Furthermore, take limit $\lambda \to 0$ in (2.89),
\[
\| Pf \|_2 + \frac{1}{\varepsilon} \| (I - P) f \|_2 + \sqrt{\frac{\alpha}{\varepsilon}} (1 - P_{\gamma}) f \|_2 + \frac{\alpha}{\varepsilon} \| g \|_2 + \| \nu^{-\frac{1}{2}} (I - P) g \|_2.
\] (2.92)

By the trace lemma and (2.92) we get
\[
\alpha \varepsilon \| P f \|_2^2 + \frac{\alpha}{\varepsilon} \| g \|_2^2 \lesssim \frac{\alpha}{\varepsilon} \| P g \|_2^2 + \| \nu^{-\frac{1}{2}} (I - P) g \|_2^2 + \frac{\alpha}{\varepsilon} (1 - P_{\gamma}) f \|_2 + \frac{\alpha}{\varepsilon} (1 - P_{\gamma}) f \|_2.
\] (2.93)

From boundary condition of $f$ and (2.92) and (2.93), we have
\[
\frac{\alpha}{\varepsilon} \| f \|_2^2 \lesssim \frac{\alpha}{\varepsilon} \| g \|_2^2 + \| \nu^{-\frac{1}{2}} (I - P) g \|_2^2 + \frac{\alpha}{\varepsilon} (1 - P_{\gamma}) f \|_2 + \frac{\alpha}{\varepsilon} (1 - P_{\gamma}) f \|_2.
\] (2.94)

Then (2.66) follows from (2.92), (2.93), (2.94).

Finally, (2.67) follows readily from (2.65), (2.92) and the fact $\langle f \rangle = 0$. This completes the proof.

### 2.2. $L^\infty$ estimate.

In this section we give the $L^\infty$ estimate for the steady linear equation.

Define
\[
K_\beta := \int_{\mathbb{R}^3} k_\beta(v, u)g(u) du,
\]
where $0 < \beta < \frac{1}{4}$ and
\[
k_\beta(v, u) := (|v - u| + |v - u|^{-1}) \exp \left\{ - \beta |v - u|^{-2} - \beta \frac{|v|^{-2} - |u|^{-2}}{|v - u|^{-2}} \right\}.
\]

Introduce the weight $w(v) = e^{\beta'|v|^2}$ for $0 < \beta' \leq \beta$.

The main result of this subsection is as follows.
Theorem 2.4. Suppose that $f$ satisfies
\[
[v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla_v + \varepsilon^{-1} \tilde{C}(v) + \lambda] |f| \leq \varepsilon^{-1} K_{\beta} |f| + |g| \quad \text{in } \Omega \times \mathbb{R}^3,
\]
\[
|f|_{\gamma_\cdot} \leq (1 - \alpha) \mathcal{L}^t |f| + \alpha P \bar{g} |f| + \alpha |r| \quad \text{on } \partial \Omega \times \mathbb{R}^3,
\]
for some constant $\lambda \geq 0$. Then
\[
\|wf\|_\infty \lesssim \varepsilon^{-\frac{1}{2}} \|Pf\|_6 + \varepsilon^{-\frac{1}{2}} \|I - P\|_2 + \alpha |wr|_\infty + \varepsilon \|v\|^{-1} \|wq\|_\infty.
\]
To prove Theorem 2.4, we employ the so-called stretching method through (1.58) and (1.57), see Subsection 1.3. Recall (1.51) and (1.56). Then (2.95) is transformed into the equivalent problem
\[
[v \cdot \nabla_y + \varepsilon^3 \Phi \cdot \nabla_v + \tilde{C}(v) + \lambda \varepsilon] |\tilde{f}| \leq K_{\beta} |\tilde{f}| + \varepsilon \tilde{g} \quad \text{in } \Omega \times \mathbb{R}^3,
\]
\[
|\tilde{f}|_{\gamma_\cdot} \leq (1 - \alpha) \mathcal{L}^t |\tilde{f}| + \alpha P \bar{g} |\tilde{f}| + \alpha |r| \quad \text{on } \partial \Omega \times \mathbb{R}^3,
\]
where $\tilde{g}(y,v) := g(x,v)$ and $\tilde{r}(y,v) := r(x,v)$.

For given $(y,v) \in \bar{\Omega} \times \mathbb{R}^3$, define
\[
S_y(v) := \{ v \in \mathbb{R}^3 : n \left( Y(t - t_b(y,v); t, y, v) \right) \cdot V \left( t - t_b(y,v); t, y, v \right) = 0 \}. \tag{2.98}
\]
Following similar argument as the proof of Lemma 17 in [27], one can show that the Lebesgue measure of $S_y(v)$ is zero.

We need the following lemma concerning the specular backward trajectory.

Lemma 2.5. Let $(t,y,v) \in [0, T_0] \times \bar{\Omega}_\varepsilon \times \mathbb{R}^3$ be given and satisfy $(y,v) \notin \gamma_0$ or $v \notin S_y(v)$. Then there is at most one bounce against the boundary $\partial \Omega_\varepsilon$ along the specular backward trajectory (1.58) starting from $(t,y,v)$ for small $0 < \varepsilon \ll 1$.

Proof. To prove this assertion, it is suffice to show that for given $(t,y,v)$ with $t \in [0, T_0]$ and $(y,v) \notin \gamma_0$ or $v \notin S_y(v)$, there is no more bounce against the boundary along the specular backward trajectory after the first bounce (if it does happens). Let $x_1 \in \partial \Omega$. From page 725 of [27] we know that
\[
\lim_{x' \in \partial \Omega, x' \to x_1} \frac{(x_1 - x') \cdot n(x_1)}{|x_1 - x'|} = 0,
\]
and then
\[
|(x_1 - x') \cdot n(x_1)| \leq C_{\xi} |x_1 - x'|^2.
\]
Denote $y_1 = \varepsilon^{-1} x_1$, $y' = \varepsilon^{-1} x'$. Then $y_1, y' \in \partial \Omega_\varepsilon$ and
\[
\varepsilon |(y_1 - y') \cdot n(x_1)| \leq C_\xi |y_1 - y'|^2 \varepsilon^2. \tag{2.99}
\]
Let $t' = t - t_b(y_1, v_1)$, $y' = y_b(y_1, v_1)$. Then
\[
y' = Y(t'; t_1, y_1, v_1) = y_1 + v_1 (t' - t_1) + \varepsilon^3 \int_{t_1}^{t'} \int_{t_1}^{t'} \Phi (Y(t'; t_1, y_1, v_1)) \, dt' \, d\tau,
\]
where $t_1 = t - t_b(y,v)$, and $(t_1, y_1, v_1)$ is the first backward bouncing time, position and velocity. Taking inner product with $n(y_1)$, we get
\[
(y' - y_1) \cdot n(y_1) = (t' - t_1) [v_1 \cdot n(y_1)] + \varepsilon^3 \int_{t_1}^{t'} \int_{t_1}^{t'} \left[ \Phi (Y(t'; t_1, y_1, v_1)) \cdot n(y_1) \right] \, dt' \, d\tau.
\]
Recall that $\|\Phi\|_\infty$ is bounded, $0 < \varepsilon \ll 1$ and $0 \leq t' < t_1 \leq t \leq T_0$. It follows that
\[
|y' - y_1| \leq 2 |t' - t_1| |v_1|, \quad |(y' - y_1) \cdot n(y_1)| \geq \frac{1}{2} |t' - t_1| |v_1 \cdot n(y_1)|. \tag{2.100}
\]
Moreover, along the specular backward trajectory,
\[
v_1 = R_{y_1} (V(t_1; t, y, v)), \quad V(t_1; t, y, v) = v + \varepsilon^3 \int_{t_1}^{t} \Phi (\tau; t, y, v) \, d\tau.
\]
Similarly as (2.100), we have
\[
\frac{1}{2} |v| \leq |v_1| \leq 2 |v|, \quad \frac{1}{2} |v \cdot n(y_1)| \leq |v_1 \cdot n(y_1)| \leq 2 |v \cdot n(y_1)|,
\]
where we used the fact that \( |v_1| = |V(t_1; t, y, v)| \) and \( v_1 \cdot n(y_1) = -V(t_1; t, y, v) \cdot n(y_1) \). Combining (2.100) and (2.101), we have
\[
|y' - y_1| \leq 4 |t' - t_1| |v|, \quad |(y' - y_1) \cdot n(y_1)| \geq \frac{1}{4} |t' - t_1| |v \cdot n(y_1)|.
\]
Substitute (2.102) into (2.99),
\[
\frac{1}{4} |t' - t_1| |v \cdot n(x_1)| \leq 16 \varepsilon^2 C_\xi |t' - t_1|^2 |v|^2,
\]
where we used (1.54). It follows that
\[
t_b(y, v) = |t' - t_1| \geq \varepsilon^{-1} \frac{|v \cdot n(x_1)|}{64 C_\xi |v|^2}.
\]
This indicates that for given \( t \in [0, T_0] \) and \((y, v) \notin \gamma_0 \) or \( v \notin S_y(v) \), there holds \(|t' - t_1| \geq T_0\) provided \( \varepsilon \ll 1 \) is sufficiently small. Thus there is no more bounce against the boundary after the first bounce. This completes the proof.

We need the following lemma concerning the backward trajectory starting from \( \partial \Omega_\varepsilon \).

**Lemma 2.6.** Let \((t_1, y_1, v_1) \in [0, T_0] \times \partial \Omega_\varepsilon \times \{ |v_1| \leq N, |n(y_1) \cdot v_1| > \eta \}\) be given with constants \( N, \eta > 0 \). Then there is no collision against the boundary along the backward trajectory (1.58) starting from \((t_1, y_1, v_1)\) for small \( 0 < \varepsilon \ll 1 \).

**Proof.** It is sufficient to show that for given \((t_1, y_1, v_1) \in [0, T_0] \times \partial \Omega_\varepsilon \times \{ |v_1| \leq N, |n(y_1) \cdot v_1| > \eta \}\), if the backward trajectory collides against the boundary, then the backward exist time \( t_b(y_1, v_1) \) is larger than \( T_0 \).

Let \( y'' = y_b(y_1, v_1), t'' = t_b(y_1, v_1) \). To prove this assertion, it is sufficient to show that \( t_b(y_1, v_1) \) is larger than \( T_0 \). Following the same argument as Lemma 2.5, we have
\[
\varepsilon \frac{|(y'' - y_1) \cdot n(x_1)|}{(2.104)} \leq C_\xi |y'' - y_1|^2 \varepsilon^2,
\]
where \( x_1 \in \partial \Omega, y_1 = \varepsilon^{-1} x_1 \in \partial \Omega_\varepsilon \) and \( n(x_1) = n(y_1) \). We also have
\[
y'' - y_1 = (t'' - t_1)v_1 + \varepsilon^3 \int_{t_1}^{t''} \int_{t_1}^{t'} \Phi(Y(t'; t_1, y_1, v_1)) d\tau' dt',
\]
\[
(y'' - y_1) \cdot n(y_1) = (t'' - t_1)|v_1 \cdot n(y_1)| + \varepsilon^3 \int_{t_1}^{t''} \int_{t_1}^{t'} \Phi(Y(t'; t_1, y_1, v_1)) \cdot n(y_1) d\tau' dt'.
\]
It follows from \( \|\Phi\|_\infty \ll 1 \) and \( \varepsilon \ll 1 \) and \( 0 \leq t'' < t_1 \leq T_0 \) that
\[
|y'' - y_1| \leq 2 |t'' - t_1| |v_1|, \quad |(y'' - y_1) \cdot n(y_1)| \geq \frac{1}{2} |t'' - t_1| |v_1 \cdot n(y_1)|.
\]
Combining this with (2.104), we have
\[
|t_b(y_1, v_1)| = |t'' - t_1| \geq \varepsilon^{-1} \frac{|v_1 \cdot n(x_1)|}{8 C_\xi |v_1|^2} \geq T_0,
\]
provided \( \varepsilon \ll 1 \) is sufficiently small. This means that the backward trajectory would not collide anymore against the boundary after departure from \((t_1, y_1, v_1)\).

**Lemma 2.7.** Suppose that \( \tilde{f} \) satisfies (2.97). Then
\[
\|w\tilde{f}\|_{L^\infty_{\varepsilon,v}(\Omega_{\varepsilon} \times \mathbb{R}^3)} \lesssim \|P\|_{L^0_{\varepsilon,v}(\Omega_{\varepsilon} \times \mathbb{R}^3)} + \| (I - P) \tilde{f}\|_{L^2_{\varepsilon,v}(\Omega_{\varepsilon} \times \mathbb{R}^3)} + \alpha \| w\tilde{F}\|_{L^\infty_{\varepsilon,v}(\Omega_{\varepsilon} \times \mathbb{R}^3)}
\]
\[
+ \varepsilon \|(v)^{-1}w\tilde{g}\|_{L^\infty_{\varepsilon,v}(\Omega_{\varepsilon} \times \mathbb{R}^3)}.
\]

(2.105)
Proof. Define
\[ h(y, v) := \frac{1}{w(v)\sqrt{\mu(v)}}. \]  
(2.106)

By Lemma 3 of [27], there exists \( \tilde{\beta} = \tilde{\beta}(\beta, \beta') > 0 \) such that \( k_{\tilde{\beta}}(v, u) \frac{w(v)}{w(u)} \lesssim k_{\beta}(v, u) \). Then we have
\[
\begin{align*}
|v \cdot \nabla_y + \varepsilon^3 \Phi \cdot \nabla_y + \tilde{C}(v)| h | & \leq \int_{\mathbb{R}^3} k_{\tilde{\beta}}(v, u) h(u) du + \varepsilon |w\tilde{g}|, \\
|h(y, v)|_{J} | & \lesssim (1 - \alpha)|h(y, R_y(v))| + \alpha \frac{1}{w(v)} \int_{n(y) \cdot u > 0} |h(y, u)| w(u) d\sigma + \alpha |w\tilde{r}|,
\end{align*}
\]
(2.107)

where we used \( \tilde{C}(v) + \lambda \varepsilon \sim \tilde{C}(v) \), the symmetry of \( w(v) \) and
\[
\tilde{w}(v) := \frac{1}{w(v)\sqrt{\mu(v)}}, \quad d\sigma := C_{\mu}(u)[n(y) \cdot u] du.
\]
(2.108)

We claim that, for \( t = T_0 \) and any given \((y, v) \in \Omega \times \mathbb{R}^3 \) satisfying \((y, v) \notin \gamma_0 \) or \( v \notin S_y(v) \),
\[
|h(y, v)| \lesssim o(1) ||h||_{L^\infty} + ||Pf||_{L^\infty_y} + ||(1 - P)f||_{L^\infty_y} + ||w\tilde{r}||_{L^\infty_y} + \varepsilon ||\tilde{v}||_1 |w\tilde{g}| ||L^\infty_y \),
\]
(2.109)

where \( o(1) \) is a small constant independent of \( \varepsilon \) and \( T_0 \) is sufficiently large.

Once (2.109) is proved. By taking \( L^\infty_y \) on (2.109), using (2.106) and absorbing the small contribution \( o(1) ||h||_{L^\infty} \), we obtain (2.109) readily. Note that the exclusion of zero measurable sets \( \gamma_0 \) and \( S_y(v) \) in (2.109) does not affect this uniform \( L^\infty \) estimate.

In the following, we focus on the proof of the claim (2.109). From (1.58) and (2.107), for \( t_1 < s \leq t \),
\[
\frac{d}{ds} \left[ e^{-t_1^s \tilde{C}(V(t, s, y, v))} d\tau \right] h(Y(s; t, y, v), V(s; t, y, v)) \]
\[
\leq e^{-t_1^s \tilde{C}(V(t, s, y, v))} d\tau \int_{\mathbb{R}^3} k_{\tilde{\beta}}(V(s; t, y, v), u) |h(Y(s; t, y, v), u)| du
\]
\[
+ e^{-t_1^s \tilde{C}(V(t, s, y, v))} d\tau \varepsilon |w\tilde{g}| (Y(s; t, y, v), V(s; t, y, v)).
\]
(2.110)

Along the backward trajectory, we have
\[
|h(y, v)| \leq J_0(t, y, v) + J_k(t, y, v) + J_g(t, y, v) + J_r(t, y, v) + J_{sp}(t, y, v) + J_{di}(t, y, v),
\]
(2.111)

where
\[
J_0(t, y, v) = 1_{\{t_1 \leq 0\}} e^{-t_1^0 \tilde{C}(V(t, y, v))} d\tau |h(Y(0; t, y, v), V(0; t, y, v))|
\]
\[
J_k(t, y, v) = \int_{\max\{0, t_1\}}^t ds e^{-t_1^s \tilde{C}(V(t, y, v))} d\tau \int_{\mathbb{R}^3} k_{\tilde{\beta}}(V(s; t, y, v), u)
\]
\[
\times |h(Y(s; t, y, v), u)|,
\]
\[
J_g(t, y, v) = \int_{\max\{0, t_1\}}^t ds e^{-t_1^s \tilde{C}(V(t, y, v))} d\tau \varepsilon |w\tilde{g}| (Y(s; t, y, v), V(s; t, y, v))|
\]
\[
J_r(t, y, v) = 1_{\{t_1 > 0\}} e^{-t_1^0 \tilde{C}(V(t, y, v))} d\tau \alpha |w\tilde{r}| (y_1, V(t_1; y, v))|
\]
\[
J_{sp}(t, y, v) = 1_{\{t_1 > 0\}} e^{-t_1^0 \tilde{C}(V(t, y, v))} d\tau (1 - \alpha) |h(y_1, v_1)|,
\]
\[
J_{di}(t, y, v) = 1_{\{t_1 > 0\}} e^{-t_1^0 \tilde{C}(V(t, y, v))} d\tau \varepsilon \frac{1}{w(V(t, y, v))} \int_{n(y_1) \cdot u_1 > 0} |h(y_1, u_1)| d\sigma_1^*,
\]
(2.112)

where we used the notations
\[
t_1 := t - t_b(y, v), \quad y_1 := Y(t_1; t, y, v) = y_b(y, v),
\]
\[
v_1 := R_{y_1}(V(t_1; t, y, v)), \quad d\sigma_1^* := C_{\mu}(v_1^*) [n(y_1) \cdot u_1^*] d\sigma_1^*.
\]

It is not difficult to see from \( \tilde{C}(V(t, y, v)) \gtrsim y_0 \) that
\[
J_0(t, y, v) \lesssim T_0 \varepsilon^{-v_0}, \quad J_g(t, y, v) \lesssim \varepsilon^{-1} |w\tilde{g}| \lesssim L^\infty_y, \quad J_r(t, y, v) \lesssim \alpha |w\tilde{r}| L^\infty_y,
\]
(2.113)
Similarly as (2.113), we have

\[ J_{sp}(t, y, v) \le J_{sp,0}(t, y, v) + J_{sp,g}(t, y, v) + J_{sp,k}(t, y, v), \]

where

\[ J_{sp,0}(t, y, v) = 1\{t_1 > 0\} e^{-\int_{t_1} \bar{C}(V(t, y, v)) \, dt} (1 - \alpha) e^{-\int_{0}^{t_1} \bar{C}(V(t, y, v)) \, dt} \times |h(Y(0; t_1, y_1, v_1), V(0; t_1, y_1, v_1))|, \]

\[ J_{sp,g}(t, y, v) = 1\{t_1 > 0\} e^{-\int_{t_1} \bar{C}(V(t, y, v)) \, dt} (1 - \alpha) \int_0^{t_1} ds e^{-\int_s^{t_1} \bar{C}(V(t, y, v)) \, dt} \times |\varepsilon(w\tilde{g}) (Y(s; t_1, y_1, v_1), V(s; t_1, y_1, v_1))|, \]

\[ J_{sp,k}(t, y, v) = 1\{t_1 > 0\} e^{-\int_{t_1} \bar{C}(V(t, y, v)) \, dt} (1 - \alpha) \int_0^{t_1} ds e^{-\int_s^{t_1} \bar{C}(V(t, y, v)) \, dt} \times \int_{\mathbb{R}^3} dv' \ k_\beta(V(s; t_1, y_1, v_1), v') |h(Y(s; t_1, y_1, v_1), v')|. \]

Similarly as (2.113), we have

\[ J_{sp,0}(t, y, v) \lesssim e^{-\varepsilon}\|h\|_{L^\infty_{y,v}}, \quad J_{sp,g}(t, y, v) \lesssim \|\varepsilon w\tilde{g}\|_{L^\infty_{y,v}}. \]

To estimate \( J_{sp,k}(t, y, v) \), for \( N > 1 \), we can choose \( m = m(N) \gg 1 \) such that

\[ k_m(V, v') := 1_{|V - v'| \ge \frac{1}{\pi^2}} 1_{|v'| \le m} k_\beta(V, v'), \]

\[ \sup_V \int_{\mathbb{R}^3} |k_m(V, v') - k_\beta(V, v')| dv' \le \frac{1}{N}. \]

We split

\[ k_\beta(V, v') = \left[ k_\beta(V, v') - k_m(V, v') \right] + k_m(V, v'), \]

where the first difference would lead to a small contribution \( o(1)\|h\|_{L^\infty_{y,v}} \) for \( N \gg T_0 \). We write

\[ Y_1(s) := Y(s; t_1, y_1, v_1), \quad V_1(s) := V(s; t_1, y_1, v_1). \]

Then the second term in (2.116) leads to

\[ C_m \int_{|v'| \le m, |v' \cdot n(Y_1(s))| < \eta} + C_m \int_{|v'| \le m, |v' \cdot n(Y_1(s))| \ge \eta} \]

for small \( \eta > 0 \), which are further bounded by

\[ o(1)\|h\|_{L^\infty_{y,v}} + C_m J_{sp,k*}(t, y, v), \]

where

\[ J_{sp,k*}(t, y, v) := 1\{t_1 > 0\} \int_0^{t_1} ds e^{-\varepsilon\|s\|} \int_{|v'| \le m, |v' \cdot n(Y_1(s))| \ge \eta} |h(Y_1(s), v')|. \]

Note that the small coefficient in (2.118) is independent of \( \varepsilon \), though \( Y_1(s) \) depends on \( \varepsilon \). In fact, by the change of variable \( v'_\parallel = [v' \cdot n(Y_1(s))] n(Y_1(s)) \) and \( v'_\perp = v' - v'_\parallel \) for \( |v' \cdot n(Y_1(s))| < \eta \), there holds

\[ \int_{|v'| \le m, |v' \cdot n(Y_1(s))| < \eta} k_\beta(V_1(s), v') |h(Y_1(s), v')| dv' \le C_m\|h\|_{L^\infty_{y,v}} \int_{-\eta}^{\eta} dv' \int_{|v'_\perp| \le m} dv'_\perp \le \eta C_m\|h\|_{L^\infty_{y,v}} = o(1)\|h\|_{L^\infty_{y,v}}, \]

provided \( \eta \) is small.
Then we use Duhamal principle again to the underbraced term in (2.149),

\[
J_{sp,k*}(t, y, v) \lesssim J_{sp,k*}^0(t, y, v) + J_{sp,k*}^k(t, y, v) + J_{sp,k*}^g(t, y, v) + J_{sp,k*}^{sp}(t, y, v) + J_{sp,k*}^{dp}(t, y, v),
\]

where

\[
J_{sp,k*}^0(t, y, v) = \mathbf{1}_{\{t_1 > 0\}} \int_0^{t_1} ds e^{-\nu(t-s)} \int_{|v'| \leq m, |v' - n(Y(s))| \geq \eta} du' \mathbf{1}_{\{t_1' < 0\}}
\]

\[
	imes e^{-J^\nu_{t_1'} C(V(\tau; s, Y_1(s), v'))} \int h(Y(0; s, Y_1(s), v'), V(0; s, Y_1(s), v')) |\Delta h|,\\
\]

\[
J_{sp,k*}^k(t, y, v) = \mathbf{1}_{\{t_1 > 0\}} \int_0^{t_1} ds e^{-\nu(t-s)} \int_{|v'| \leq m, |v' - n(Y(s))| \geq \eta} du' \int_{\max\{0, t_1'\}}^\infty d\tau
\]

\[
	imes e^{-J^\nu_{t_1'} C(V(\tau; s, Y_1(s), v'))} \int_{\mathbb{R}^3} du \mathbf{k}_\beta \left(V(\tau; s, Y_1(s), v'), u\right) \times |h(Y(\tau; s, Y_1(s), v'), v)|,
\]

\[
J_{sp,k*}^g(t, y, v) = \mathbf{1}_{\{t_1 > 0\}} \int_0^{t_1} ds e^{-\nu(t-s)} \int_{|v'| \leq m, |v' - n(Y(s))| \geq \eta} du' \int_{\max\{0, t_1'\}}^\infty d\tau
\]

\[
	imes e^{-J^\nu_{t_1'} C(V(\tau; s, Y_1(s), v'))} \int |\omega| \left(Y(\tau; s, Y_1(s), v'), V(\tau; s, Y_1(s), v')\right),
\]

\[
J_{sp,k*}^p(t, y, v) = \mathbf{1}_{\{t_1 > 0\}} (1 - \alpha) \int_0^{t_1} ds e^{-\nu(t-s)} \int_{|v'| \leq m, |v' - n(Y(s))| \geq \eta} du' \mathbf{1}_{\{t_1' > 0\}}
\]

\[
	imes e^{-J^\nu_{t_1'} C(V(\tau; s, Y_1(s), v'))} |\omega| \left(y'_1, V(t'_1; s, Y_1(s), v')\right),
\]

\[
J_{sp,k*}^{dp}(t, y, v) = \mathbf{1}_{\{t_1 > 0\}} \int_0^{t_1} ds e^{-\nu(t-s)} \int_{|v'| \leq m, |v' - n(Y(s))| \geq \eta} du' \mathbf{1}_{\{t_1' > 0\}}
\]

\[
	imes e^{-J^\nu_{t_1'} C(V(\tau; s, Y_1(s), v'))} |\omega| \left(y'_1, v'_1\right) \times |\Delta h|,
\]

where we used notations

\[
t_1' := s - t_b(s, Y_1(s), v'), \quad y'_1 := Y(t'_1; s, Y_1(s), v') = y_b(Y_1(s), v'),
\]

\[
v'_1 := R_{y'_1} \left(V(t'_1; s, Y_1(s), v')\right), \quad da'_1 := C_\mu \mu(u'_1) \left[n(Y(t'_1)) \cdot u'_1\right] du'_1.
\]

Firstly, for \( t = T_0 \) large such that \( T_0e^{-\nu(T_0)} \ll 1 \), we have

\[
J_{sp,k*}^0(t, y, v) \lesssim T_0 e^{-\nu(T_0)} ||h||_\infty \lesssim o(1)||h||_\infty.
\]

\[
J_{sp,k*}^g(t, y, v) \quad \text{and} \quad J_{sp,k*}^r(t, y, v) \quad \text{are estimated as (2.133)},
\]

\[
J_{sp,k*}^0(t, y, v) \lesssim \epsilon ||\omega||^{-1} ||\omega||_\infty, \quad J_{sp,k*}^r(t, y, v) \lesssim o(1)||\omega||_\infty.
\]

We estimate \( J_{sp,k*}^k(t, y, v), J_{sp,k*}^{sp}(t, y, v) \) and \( J_{sp,k*}^{dp}(t, y, v) \) in the following Step 1.1 ~ Step 1.3.

**Step 1.1.** Estimate of \( J_{sp,k*}^k(t, y, v) \).

Similarly as the treatment of (2.116), for \( m = m(N) \gg 1 \) we split \( k_\beta(v, u) = [k_\beta(v, u) - k_m(v, u)] + k_m(v, u) \), where the first difference is bounded by \( o(1)||h||_{L_y^\infty} \). We further split the time integration of the second term:

\[
\int_{s-\delta}^s \int_{\max\{0, t_1'\}}^{s-\delta} d\tau.
\]

(2.126)
where the first integration is bounded by
\[ \delta \sup_v \int_{|u| \leq m} k_m(v, u) du \| h \|_\infty \lesssim \delta \| h \|_\infty, \]
due to the small-in-time truncation. The underbraced integration in (2.120) is bounded by
\[ J_{sp,ks}^*(t, y, v) := \int_0^t ds \int_{|v'| \leq m} \int_{|v'| \leq m} \int_{|v'| \leq m} \int_{|v'| \leq m} \int_{|v'| \leq m} |h(Y(t; s, Y_1(s), v), u)| du' \]
(2.127)

We consider the change of variable \( v' \mapsto y := Y(t; s, Y_1(s), v') \).

Recall that
\[ Y(t; s, Y_1(s), v') = Y_1(s) + (t - s)v' + \varepsilon^3 \int_s^t \int_s^{t'} \Phi(Y(t'; s, Y_1(s), v')) dt' ds', \]
where \( Y_1(s) \) is defined in (2.117). Simple computation shows, for \( 0 \leq \tau \leq s - \delta < s \),
\[ |\det \nabla v' Y(t; s, Y_1(s), v')| = |s - \tau|^3 |\det (\delta_{ij} + O(\varepsilon^3))| \geq \frac{1}{2} \delta^3. \]

Integrating over time first and using \( |h(u)| = w(u)|f(u)| \lesssim_m |f(u)| \) for \( |u| \leq m \), we have
\[ J_{sp,ks}^*(t, y, v) \lesssim \sup_{0 \leq \tau \leq s - \delta < s \leq t_1} \int_{|v'| \leq m} \int_{|u| \leq m} |h(Y(t; s, Y_1(s), v'), u)| du' \]
(2.128)
\[ + \sup_{0 \leq \tau \leq s - \delta < s \leq t_1} \int_{|v'| \leq m} \int_{|u| \leq m} \left( |I - P| f(Y(t; s, Y_1(s), v'), u) \right) du'. \]

For \( P f \) contribution,
\[ \sup_{0 \leq \tau \leq s - \delta < s \leq t_1} \int_{|v'| \leq m} \int_{|u| \leq m} |P f(Y(t; s, Y_1(s), v'), u)| du' \]
(2.129)
\[ \lesssim_m \left[ \int_{\Omega_s} \| P f(y) \|_{L^6(\mathbb{R}^3)}^6 \frac{2}{\delta^3} dy \right]^{1/6} \]
\[ \lesssim_m \| P f \|_{L^6(\Omega_s \times \mathbb{R}^3)}. \]

For \( (I - P) f \) contribution,
\[ \sup_{0 \leq \tau \leq s - \delta < s \leq t_1} \int_{|v'| \leq m} \int_{|u| \leq m} \left( |I - P| f(Y(t; s, Y_1(s), v'), u) \right) du' \]
(2.130)
\[ \lesssim_m \left[ \iint_{\Omega_s \times \mathbb{R}^3} \| (I - P) f(y, u) \|_{L^2(\mathbb{R}^3)}^2 \frac{2}{\delta^3} dy da \right]^{1/2} \]
\[ \lesssim_m \| (I - P) f \|_{L^2(\Omega_s \times \mathbb{R}^3)}. \]

Collecting (2.126) - (2.130) we obtain
\[ J_{sp,ks}^*(t, y, v) \lesssim o(1) \| h \|_{L^\infty_{v,u}} + \| P f \|_{L^6_{v,u}} + \| (I - P) f \|_{L^2_{v,u}}. \]

Step 1.2. Estimate of \( J_{sp,ks}^*(t, y, v) \).
Recall the formula of $J_{\text{sp,}k_s}(t, y, v)$. For given

$$(Y_t(s), v') \in \left\{ \Omega \times \mathbb{R}^3 : |v'| \leq m, |v' \cdot n(Y_t(s))| \geq \eta \right\},$$

(2.132)

similarly as [2.124], Lemma 2.25 guarantees that the specular backward trajectory starting from $(Y_t(s), v')$ continues to bounce back to initial plane $t = 0$, after its first collision at $(y'_1, v'_1)$. Thus

$$J_{\text{sp,}k_s}(t, y, v) \leq J_{\text{sp,}0}^{p, 0}(t, y, v) + J_{\text{sp,}k_s}^{p, g}(t, y, v) + J_{\text{sp,}k_s}^{p, k}(t, y, v),$$

(2.133)

where

$$J_{\text{sp,}k_s}^{p, 0}(t, y, v) = \mathbf{1}_{\{t_1 > 0\}}(1 - \alpha) \int_0^{t_1} ds \ e^{-\nu_0(t-s)} \int_{|v'| \leq m, |v' \cdot n(Y_t(s))| \geq \eta} \ dv' \mathbf{1}_{\{t'_1 > 0\}}$$

$$\times \ e^{-\int_0^s \tilde{C}(V(\tau, s, y'_1, v'_1))d\tau} \ e^{-\int_s^{t_1} \tilde{C}(V(\tau; t'_1, y'_1, v'_1))d\tau}$$

$$\times |h(Y(0; t'_1, y'_1, v'_1), V(0; t'_1, y'_1, v'_1))|,$$

$$J_{\text{sp,}k_s}^{p, g}(t, y, v) = \mathbf{1}_{\{t_1 > 0\}}(1 - \alpha) \int_0^{t_1} ds \ e^{-\nu_0(t-s)} \int_{|v'| \leq m, |v' \cdot n(Y_t(s))| \geq \eta} \ dv' \mathbf{1}_{\{t'_1 > 0\}}$$

$$\times \ e^{-\int_0^s \tilde{C}(V(\tau, s, y'_1, v'_1))d\tau} \ e^{-\int_s^{t_1} \tilde{C}(V(\tau; t'_1, y'_1, v'_1))d\tau}$$

$$\times |h(Y(0; t'_1, y'_1, v'_1), V(0; t'_1, y'_1, v'_1))|,$$

$$J_{\text{sp,}k_s}^{p, k}(t, y, v) = \mathbf{1}_{\{t_1 > 0\}}(1 - \alpha) \int_0^{t_1} ds \ e^{-\nu_0(t-s)} \int_{|v'| \leq m, |v' \cdot n(Y_t(s))| \geq \eta} \ dv' \mathbf{1}_{\{t'_1 > 0\}}$$

$$\times \ e^{-\int_0^s \tilde{C}(V(\tau, s, y'_1, v'_1))d\tau} \ e^{-\int_s^{t_1} \tilde{C}(V(\tau; t'_1, y'_1, v'_1))d\tau}$$

$$\times \int \ d\hat{u} \ \bar{K}_3(V(\tau; t'_1, y'_1, v'_1), \hat{u}) \ |h(Y(\tau; t'_1, y'_1, v'_1), \hat{u})|.$$
Here we used the fact that $\nabla x \xi(\varepsilon Y(s)) \neq 0$ near the boundary. Thus we have
\[
n(y'_1) = n(\varepsilon y'_1) = n(\varepsilon Y_1(s) + \varepsilon(t'_1 - s)v' + O(\varepsilon^4)) = n(\varepsilon Y_1(s)) + O(\varepsilon),
\]
\[
V(t'_1; s, Y_1(s), v') = v' + \varepsilon^3 \int_{t'_1}^{t} \bar{\Phi}(\tau; s, Y_1(s), v') d\tau = v' + O(\varepsilon),
\]
where we used (2.137). Combining this with (2.138), we have
\[
v'_1 = V(t'_1; s, Y_1(s), v') - 2\left[ n(y'_1) \cdot V(t'_1; s, Y_1(s), v') \right] n(y'_1)
= v' - 2\left[ n(\varepsilon Y_1(s)) \cdot v' \right] n(\varepsilon Y_1(s)) + O(\varepsilon).
\]
It follows from (2.137) and (2.138) that
\[
Y(\tau; t'_1, y'_1, v'_1)
= y'_1 + (\tau - t'_1)v'_1 + \varepsilon^3 \int_{t'_1}^{\tau} \int_{t'_1}^{\tau} \bar{\Phi}(\tilde{\tau}; t'_1, y'_1, y'_1) d\tilde{\tau} d\tau,
= Y_1(s) + (t'_1 - s)v' + \left[ (\tau - s) - (t'_1 - s) \right] \left[ v' - 2\left[ n(\varepsilon Y_1(s)) \cdot v' \right] n(\varepsilon Y_1(s)) \right] + O(\varepsilon)
= Y_1(s) + (\tau - s)v' + 2\left[ (t'_1 - s) - (\tau - s) \right] \left[ n(\varepsilon Y_1(s)) \cdot v' \right] n(\varepsilon Y_1(s)) + O(\varepsilon)
= Y_1(s) + (\tau - s)v' + 2\left[ (t'_1 - s) - (\tau - s) \right] \frac{\nabla x \xi(\varepsilon Y_1(s)) \cdot v'}{\nabla x \xi(\varepsilon Y_1(s))^2} \nabla x \xi(\varepsilon Y_1(s)) + O(\varepsilon).
\]
It is not difficult to see that
\[
\frac{\partial Y(\tau; t'_1, y'_1, v'_1)}{\partial v'_j} = (\tau - s)\delta_{ij} + 2\left[ (t'_1 - s) - (\tau - s) \right] \frac{\partial \xi(\varepsilon Y_1(s))}{\nabla x \xi(\varepsilon Y_1(s))^2} \frac{\partial \xi(\varepsilon Y_1(s))}{\partial v'_j} + O(\varepsilon)
= (\tau - s)\delta_{ij} + a_{ij} + O(\varepsilon),
\]
where $\partial \xi = \frac{\partial \xi}{\partial x}$ stands for spatial derivative and we used the notations
\[
a_{ij} := b_i c_j, \quad b_i := \partial_i \xi(\varepsilon Y_1(s)),
\]
\[
c_j := 2\left[ (t'_1 - s) - (\tau - s) \right] \frac{\partial \xi(\varepsilon Y_1(s))}{\nabla x \xi(\varepsilon Y_1(s))^2} + 2\left[ \nabla x \xi(\varepsilon Y_1(s)) \cdot v' \right] \frac{\partial (t'_1 - s)}{\partial v'_j}.
\]
Elementary computation shows that
\[
\sum_{k=1}^{3} a_{kk} = 2\left[ (t'_1 - s) - (\tau - s) \right] + 2\left[ \nabla x \xi(\varepsilon Y_1(s)) \cdot v' \right] \frac{\nabla v(\varepsilon Y_1(s)) \cdot \nabla x \xi(\varepsilon Y_1(s))}{\nabla x \xi(\varepsilon Y_1(s))^2},
\]
\[
\det B_{ij} := \det \begin{pmatrix} a_{ii} & a_{ij} & a_{ij} \\ a_{ji} & a_{jj} & a_{ij} \end{pmatrix} = \det \begin{pmatrix} b_i c_i & b_i c_j \\ b_j c_i & b_j c_j \end{pmatrix} = 0 \quad \text{for } i \neq j,
\]
\[
\det C := \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \det \begin{pmatrix} b_1 c_1 & b_1 c_2 & b_1 c_3 \\ b_2 c_1 & b_2 c_2 & b_2 c_3 \\ b_3 c_1 & b_3 c_2 & b_3 c_3 \end{pmatrix} = 0.
\]
It follows from these relations and (2.140) that
\[
\det \nabla v_s Y(\tau; t'_1, y'_1, v'_1)
= (\tau - s)^3 + (\tau - s)^2 \sum_{k=1}^{3} a_{kk} + (\tau - s)^3 \sum_{1 \leq i < j \leq 3} \det B_{ij} + \det C + O(\varepsilon),
= -(\tau - s)^3 + 2(\tau - s)^2 \left\{ (t'_1 - s) + \frac{\nabla x \xi(\varepsilon Y_1(s)) \cdot v'}{\nabla x \xi(\varepsilon Y_1(s))^2} \left[ \nabla v(\varepsilon Y_1(s)) \cdot \nabla x \xi(\varepsilon Y_1(s)) \right] \right\}
+ O(\varepsilon).
\]
Recall that $y'_1 \in \partial \Omega$, $\varepsilon y'_1 \in \partial \Omega$ and $\varepsilon Y_1(s)$ is near the boundary $\partial \Omega$. From the condition \( \nabla_x \xi(x) \neq 0 \) for $x$ near the boundary $\partial \Omega$, we have \( |\nabla_x \xi(\varepsilon Y_1(s))| \neq 0 \). It follows that
\[
[|v' \cdot \nabla_x \xi(\varepsilon Y_1(s))|] = |\nabla_x \xi(\varepsilon Y_1(s))| \cdot [|v' \cdot n(\varepsilon Y_1(s))|] \neq 0,
\]
where we used \(2.132\) and the fact $n(\varepsilon Y_1(s)) = n(Y_1(s))$. From the expansion
\[
0 = \xi(\varepsilon y'_1) = \xi(\varepsilon Y_1(s) + \varepsilon(t'_1 - s)v' + O(\varepsilon^3))
\]
\[
= \xi(\varepsilon Y_1(s)) + \varepsilon(t'_1 - s) \left[ \nabla_x \xi(\varepsilon Y_1(s)) \cdot v' \right] + O(\varepsilon^2),
\]
we take partial derivative $\partial_{v'_j}$ and get
\[
(t'_1 - s)\partial_{j} \xi(\varepsilon Y_1(s)) + \frac{\partial(t'_1 - s)}{\partial v'_j} \left[ \nabla_x \xi(\varepsilon Y_1(s)) \cdot v' \right] + O(\varepsilon) = 0.
\]
Then
\[
\left[ \nabla_x \xi(\varepsilon Y_1(s)) \cdot v' \right] \left[ \nabla_{v'}(t'_1 - s) \cdot \nabla_x \xi(\varepsilon Y_1(s)) \right] = -(t'_1 - s)|\nabla_x \xi(\varepsilon Y_1(s))|^2 + O(\varepsilon).
\]
It follows that
\[
(t'_1 - s) + \frac{\left[ \nabla_x \xi(\varepsilon Y_1(s)) \cdot v' \right]}{|\nabla_x \xi(\varepsilon Y_1(s))|^2} \left[ \nabla_{v'}(t'_1 - s) \cdot \nabla_x \xi(\varepsilon Y_1(s)) \right] = O(\varepsilon).
\]
(2.142)

Since $0 < \tau < t'_1 - \delta < t'_1 < s < t < T_0$, we have $s - \tau > t'_1 - \tau - \delta$. Combining \(2.141\) and \(2.132\), we get the lower bound of the Jacobian
\[
|\det \nabla_w Y(\tau, t'_1, y'_1, v'_1)| \gtrsim |s - \tau|^3 + O(\varepsilon) \geq \frac{1}{2} \delta^3.
\]
Note that this lower bound is independent of $\varepsilon$.

Now integrating over time $\tau$ and deducing similarly as \(2.128\)-\(2.130\), we get
\[
J^{sp, k}_{sp, k^*}(t, y, v) \lesssim \|P \hat{f}\|_{L^6_{v,w}} + \|(I - P) \hat{f}\|_{L^2_{v,w}}.
\]
(2.143)

Collecting \(2.134\) and \(2.143\), we have
\[
J^{sp, k}_{sp, k^*}(t, y, v) \lesssim o(1) ||h||_{L^\infty_{v,w}} + \varepsilon \|v\|^{-1} w g||_{L^\infty_{v,w}} + \|P \hat{f}\|_{L^6_{v,w}} + \|(I - P) \hat{f}\|_{L^2_{v,w}}.
\]
(2.144)

**Step 1.3. Estimate of $J^{dis}_{sp, k^*}(t, y, v)$.**
To estimate $J^{dis}_{sp, k^*}(t, y, v)$, we split the integration domain \{\(n(y'_1) \cdot u'_1 > 0\)\} into
\[
\{\|u'_1\| > m, \; n(y'_1) \cdot u'_1 > 0\} \cup \{\|u'_1\| \leq m, \; n(y'_1) \cdot u'_1 \leq \eta\} \cup \{\|u'_1\| \leq m, \; n(y'_1) \cdot u'_1 \geq \eta\}.
\]
The first two sets lead to small contribution $o(1) \|h\|_{L^\infty_{v,w}}$, which is independent of $\varepsilon$, as stated in \(2.120\). Thus
\[
J^{dis}_{sp, k^*}(t, y, v) \lesssim o(1) \|h\|_{L^\infty_{v,w}} + J^{dis}_{sp, k^*}(t, y, v),
\]
(2.145)

where the remaining bulk is denoted as
\[
J^{dis}_{sp, k^*}(t, y, v) := 1_{\{t_1 > 0\}} 1_{\{t'_1 > 0\}} \int_0^{t_1} ds \int_{|v'| \leq m} \int_{|u'_1| \leq m, \; |n(y'_1) \cdot u'_1| \geq \eta} \right| h(y'_1, u'_1) \left| d\sigma'_1.
\]
Let \((y'_1, u'_1) \in \partial \Omega \times \{\|u'_1\| \leq m, \; |n(y'_1) \cdot u'_1| \geq \eta\}\) be given. By virtue of Lemma \(2.6\), there is no more collision against the boundary along the backward trajectory starting from \((y'_1, u'_1)\). Thus $J^{dis}_{sp, k^*}$ bounces back to initial plane $t = 0$, and then
\[
J^{dis}_{sp, k^*}(t, y, v) \leq J^{dis, 0}_{sp, k^*}(t, y, v) + J^{dis, 2}_{sp, k^*}(t, y, v) + J^{dis, k}_{sp, k^*}(t, y, v),
\]
(2.146)
To treat the integration on \(J_k\), we collect (2.115), (2.118), (2.121), (2.124), (2.125), (2.131), (2.144) and (2.148). We follow the same argument as the estimate of \(\det t,y,v(0,0;\bar{Y};u^s_t)\) in (2.121). Finally we get
\[
1 \preceq |t| \preceq \tau_1 \times \max \{ |\hat{f}(t'_1, v'_1, u'_1)|, |\hat{h}(Y(t'; t'_1, y'_1, u'_1), \bar{u})| \}.
\]

We estimate \(J_{sp,k}^{div,0}(t, y, v)\) and \(J_{sp,k}^{div,g}(t, y, v)\) similarly as (2.123) and (2.125). For \(J_{sp,k}^{div,k}(t, y, v)\), similarly as the treatment of \(J_k^{sp,k}(t, y, v)\), we split \(k^\beta(V(\tau), \bar{u})\) and \([0, t'_1] = [0, t'_1 - \delta] \cup [t'_1 - \delta, t'_1]\). To treat the integration on \([0, t'_1 - \delta]\), we consider the change of variable
\[
u^s_t \mapsto Y(\tau; t'_1, y'_1, u'_1).
\]
Recall that
\[
Y(\tau; t'_1, y'_1, u'_1) = y'_1 + (\tau - t'_1)u^s_t + \varepsilon \int_0^{t'_1} \int_{t'_1}^\tau \Phi(Y(t'; t'_1, y'_1, u'_1)) dt' dt''.
\]
Simple computation shows, for \(0 \leq \tau \leq t'_1 - \delta,
\[
|\det \nabla u^s_t Y(\tau; t'_1, y'_1, u'_1)| = |t'_1 - \tau|^3 \left| \det (\delta_{ij} + O(\varepsilon^3)) \right| \geq \frac{1}{2} \delta^3.
\]
Following the same argument as (2.123) and (2.125), we get
\[
J_{sp,k}^{div,k} \lesssim o(1) ||h||_{L^{\infty}_{y,v}} + ||P\hat{f}||_{L^2_{y,v}} + ||(I - P)\hat{f}||_{L^2_{y,v}}.
\] (2.147)
Combining (2.145), (2.146) and (2.147), we obtain
\[
J_{sp,k}^{div}(t, y, v) \lesssim o(1) ||h||_{L^{\infty}_{y,v}} + \varepsilon ||(v)^{-1} w\tilde{g}||_{L^{\infty}_{y,v}} + ||P\hat{f}||_{L^2_{y,v}} + ||(I - P)\hat{f}||_{L^2_{y,v}}.
\] (2.148)
Now we collect (2.115), (2.118), (2.121), (2.124), (2.125), (2.131), (2.144) and (2.148)
\[
J_{sp}(t, y, v) \lesssim o(1) ||h||_{L^{\infty}_{y,v}} + \varepsilon ||(v)^{-1} w\tilde{g}||_{L^{\infty}_{y,v}} + \alpha ||w\tilde{r}||_{L^{\infty}_{y,v}} + ||P\hat{f}||_{L^2_{y,v}} + ||(I - P)\hat{f}||_{L^2_{y,v}}.
\] (2.149)

**Step 2.** Estimate of \(J_k(t, y, v)\).
Recall \(J_k(t, y, v)\) in (2.142). We follow the same argument as the estimate of \(J_{sp,k}(t, y, v)\), that is, first cut off the integration domain of \(u\), then use Duhamal principle again to the integrand \(h(s, Y(s; t, y, v), u)\) in \(J_k(t, y, v)\) and get similar formula as (2.121). Finally we get
\[
J_k(t, y, v) \lesssim o(1) ||h||_{L^{\infty}_{y,v}} + \varepsilon ||(v)^{-1} w\tilde{g}||_{L^{\infty}_{y,v}} + \alpha ||w\tilde{r}||_{L^{\infty}_{y,v}} + ||P\hat{f}||_{L^2_{y,v}} + ||(I - P)\hat{f}||_{L^2_{y,v}}.
\] (2.150)

**Step 3.** Estimate for \(J_{di}(t, y, v)\).
Recall $J_{di}(t, y, v)$ in (2.112). We split the integration domain \(\{n(y_1) \cdot v_1^* > 0\}\) similarly as in the estimate of $J_{sp,k}^{di}(t, y, v)$, and estimate $J_{di}$ by

\[
J_{di}(t, y, v) \lesssim o(1) \|h\|_{L_{p,v}^\infty} + J_{dis}(t, y, v),
\]

where

\[
J_{dis}(t, y, v) := 1_{\{t_1 > 0\}} e^{-\nu_0(t-t_1)} \int_{|n(y_1) \cdot v_1^*| \leq m, |n(y_1) \cdot v_1^*| \geq \eta} |h(y_1, v_1^*)| \, dv_1^*.
\]

For given \((y_1, v_1^*) \in \partial \Omega_c \times \{ |v_1^*| \leq m, |n(y_1) \cdot v_1^*| \geq \eta\}\), by Lemma 2.6 the backward trajectory starting from \((y_1, v_1^*)\) bounces back to initial plane \(t = 0\). Thus

\[
J_{dis}(t, y, v) \leq J_{dis}^0(t, y, v) + J_{dis}^1(t, y, v) + J_{dis}^k(t, y, v),
\]

where

\[
J_{dis}^0(t, y, v) = 1_{\{t_1 > 0\}} e^{-\nu_0(t-t_1)} \int_{|n(y_1) \cdot v_1^*| \leq m, |n(y_1) \cdot v_1^*| \geq \eta} |h(Y(0; t_1, y_1, v_1^*))| \, dv_1^*
\]

\[
\times \left| h(Y(0; t_1, y_1, v_1^*), V(0; t_1, y_1, v_1^*)) \right| \, ds_1,
\]

\[
J_{dis}^1(t, y, v) = 1_{\{t_1 > 0\}} e^{-\nu_0(t-t_1)} \int_{|n(y_1) \cdot v_1^*| \leq m, |n(y_1) \cdot v_1^*| \geq \eta} e^{-\nu_0(t_1)} C(V(t_1, y_1, v_1^*)) \, dr
\]

\[
\times \left| h(Y(s; t_1, y_1, v_1^*), V(s; t_1, y_1, v_1^*)) \right| \, ds,
\]

\[
J_{dis}^k(t, y, v) = 1_{\{t_1 > 0\}} e^{-\nu_0(t-t_1)} \int_{|n(y_1) \cdot v_1^*| \leq m, |n(y_1) \cdot v_1^*| \geq \eta} e^{-\nu_0(t_1)} C(V(t_1, y_1, v_1^*)) \, dr
\]

\[
\times \int_{\mathbb{R}^3} dv' k_\beta(V(s; t_1, y_1, v_1^*), v') \left| h(Y(s; t_1, y_1, v_1^*), v') \right|.
\]

We treat $J_{dis}^0(t, y, v)$ and $J_{dis}^1(t, y, v)$ similarly as (2.124) and (2.125). $J_{dis}^k(t, y, v)$ is estimated by the change of variable $v_1^* \mapsto Y(s; t_1, y_1, v_1^*)$ and bounded by (2.147), similarly as the estimate of $J_{sp,k}^{dis}(t, y, v)$. We conclude

\[
J_{di}(t, y, v) \lesssim o(1) \|h\|_{L_{p,v}^\infty} + \varepsilon \|w\|_{L_{p,v}^\infty}^{-1} \|w\|_{L_{p,v}^\infty} + \|P \bar{f}\|_{L_{p,v}^\infty} + \|(I - P) \bar{f}\|_{L_{p,v}^\infty}.
\]

Finally, combining (2.111), (2.113), (2.145), (2.151) and (2.154), we verify the claim (2.109). This completes the proof.

With the help of Lemma 2.7, we can prove Theorem 2.4 readily.

**Proof of Theorem 2.4**

Recall that (1.67). It follows that

\[
\|w f\|_{L_{p,v}^\infty(\Omega \times \mathbb{R}^3)} = \|w f\|_{L_{p,v}^\infty(\Omega \times \mathbb{R}^3)}, \quad \|P f\|_{L_{p,v}^\infty(\Omega \times \mathbb{R}^3)} = \varepsilon \frac{2}{\varepsilon} \|P \bar{f}\|_{L_{p,v}^\infty(\Omega \times \mathbb{R}^3)},
\]

\[
\|(I - P) f\|_{L_{p,v}^\infty(\Omega \times \mathbb{R}^3)} = \varepsilon \frac{2}{\varepsilon} \|(I - P) \bar{f}\|_{L_{p,v}^\infty(\Omega \times \mathbb{R}^3)}.
\]

Then (2.56) follows readily from these relations and Lemma 2.7. This completes the proof.

2.3. Validity of the steady problem.

In this section we give the proof of Theorem 1.1.

Define the norm

\[
[[f]] := \|P f\|_6 + \frac{1}{\varepsilon} \|(I - P) f\|_{\nu} + \varepsilon \frac{2}{\varepsilon} \|w f\|_{\infty} + \sqrt{\frac{c}{\varepsilon}} \|f\|_{L_{\nu}^1(\Omega \times \mathbb{R}^3)} + \frac{\alpha}{\varepsilon} |r|_{2, +} + \sqrt{\frac{c}{\varepsilon}} |r|_{2, -} + \alpha \varepsilon \frac{2}{\varepsilon} |wr|_{\infty}.
\]

Then we have the following result.

**Theorem 2.8.** Suppose that all the assumptions of Theorem 2.4 hold. Then for $0 < \varepsilon \ll 1$,

\[
[[f]] \lesssim \varepsilon^{-1} \|P g\|_2 + \|\nu^{-\frac{2}{\nu}} (I - P) g\|_2 + \varepsilon \frac{2}{\varepsilon} \|(v)^{-1} w g\|_{\infty} + \sqrt{\frac{c}{\varepsilon}} |r|_{2, +} + \alpha \varepsilon \frac{2}{\varepsilon} |wr|_{\infty}.
\]
Proof. Recall that $Lf = \nu f - Kf$, $\nu(v) \sim \langle v \rangle$ and $|k(v, u)| \lesssim k_0(v, u)$. Moreover,
$$
\varepsilon^{-1} \nu(v) - \varepsilon \frac{\Phi \cdot v}{2} \gtrsim \varepsilon^{-1} \tilde{C}(v) - \varepsilon |\Phi|_\infty |v| \gtrsim \varepsilon^{-1} \tilde{C}(v).
$$
Therefore, the conditions of Theorem 2.23 are satisfied. Substitute (2.90) into (2.67),
$$
\|Pf\|_6 \lesssim \varepsilon \|f\|_6 + \frac{\alpha}{\varepsilon} |(1 - P_\gamma)f|_{2, +} + \frac{\alpha}{\varepsilon} |r|_{2, -} + \alpha \varepsilon^{\frac{1}{2}} |wr|_\infty + \varepsilon^2 |\Phi|_\infty |v| \lesssim \varepsilon^{-1} \tilde{C}(v).
$$
(2.154)
Multiplying (2.154) with a small constant and adding to (2.100), we get (2.155).

We need the following collision estimate on $\Gamma(f, g)$ given by (17).

Lemma 2.9. For $w = \varepsilon^\beta |v|^2$ with $0 < \beta' < 1$, there holds
$$
\|\nu^{\frac{1}{2}} \Gamma_{\pm}(f, g)\|_{L^2_{x, w}} \lesssim \varepsilon \|\nu^{\frac{1}{2}} \Gamma_{\pm}(f, g)\|_{L^2_{x, w}} + \|Pf\|_{L^2_{x, w}} \|Pf\|_{L^2_{x, w}} + \varepsilon^2 \|\nu^{\frac{1}{2}} \Gamma_{\pm}(f, g)\|_{L^2_{x, w}}.
$$
(2.155)

Lemma 2.10. Let $f_w, R_s, Q_1, Q_2$ be as in (1.18), (1.21), (1.23), (1.26). Then we have
$$
\|P R_s\|_2 \lesssim \varepsilon \|\Phi\|_2 (1 + \varepsilon |\partial_w|_\infty),
$$
$$
\|\nu^{\frac{1}{2}} (I - P) R_s\|_2 \lesssim |\partial_w|_{H^{\frac{1}{2}}(\partial\Omega)} + \varepsilon^2 \|\Phi\|_2 |\partial_w|_\infty + |\partial_w|^2_{H^{\frac{1}{2}}(\partial\Omega)}
$$
$$
\|w R_s\|_2 \lesssim \|w\|_{W^{1, \infty}(\partial\Omega)} + |\partial_w|^2_{H^{\frac{1}{2}}(\partial\Omega)}
$$
$$
\|f_w\|_{L^2_{x, w}} + \|w f_w\|_\infty + |\partial_2 \varphi_{x, 2, -} + w \partial \varphi_{x, 2, -} \lesssim |\partial_w|_\infty,
$$
$$
\|Q_1 f|_{2, -} + |w Q_2 f|_\infty \lesssim |\partial_w|_\infty (1 + \varepsilon |\partial_w|_\infty) (|\partial w|_{2, +} + |\mu f|_\infty).
$$
Proof. The first estimate follows from (1.22). The second and third estimates follow Lemma 2.9 and the trace theorem $\|\Theta\|_{H^{\frac{1}{2}}(\Omega)} \lesssim |\partial_w|_{H^{\frac{1}{2}}(\Omega)}$ from (1.23) and (1.26) and the expansion of $M^w$
$$
M^w(x, v) = \sqrt{2 \pi \mu(v) + \varepsilon \partial_w} \sqrt{2 |v|^2 - 2 \mu(v)} + \varepsilon^2 O(|\partial_w|^2) \mu(v) + \varepsilon^2 O(|\partial_w|^2) \mu(v),
$$
(2.156)
we have
$$
Q_2 f = \left[ \frac{|v|^2 - 4}{2} t_w + \varepsilon O(|\partial_w|^2) \langle v \rangle^4 \right] \sqrt{2 \pi \mu} \int_{\mathbb{R}^N} f \sqrt{\mu} (n \cdot u) du,
$$
which proves the last estimate.

Now we are ready to give the proof of Theorem 1.1. Note that the positivity of $F_s$ is proved in the end of Section 3, with the help of the positivity and asymptotic behavior of $F$.

Proof of Theorem 1.1. Step 1. We set the iteration for $k \in \mathbb{N}$
$$
v \cdot \nabla v f^{k+1} + \varepsilon^2 \frac{1}{\sqrt{R}} \Phi \cdot \nabla \phi (\sqrt{R} f^{k+1}) + \varepsilon^{-1} L f^{k+1} = \Gamma(f^k, f^k) + \Gamma(f^k, f_w) + R_s,
$$
(2.157)
$$
f^{k+1} = (1 - \alpha) f^k + \alpha P_\gamma f^{k+1} + \alpha \varepsilon \partial_2 f^k + \varepsilon \partial_2 \varphi_{x, 2, -},
$$
where $f^0 \equiv 0$, $R_s$ is defined at (1.23), and $Q_1$ and $Q_2$ are defined at (1.25) and (1.26). Note that Theorem 2.23 with (1.22) and (1.29), guarantees the solvability of the linear problem (2.157).

For $0 < M_0 \ll 1$, we assume the induction hypothesis
$$
\sup_{0 \leq t \leq \varepsilon^2 k} ||f^k|| \leq M_0.
$$
(2.158)
In the following, we will verify the same bound for the iteration sequence $||f^{k+1}||$. For this, we assume that
$$
|\partial_w|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\Phi\|_2 + \varepsilon^{\frac{1}{2}} |\partial_w|_{W^{1, \infty}(\partial\Omega)} + \varepsilon^{\frac{1}{2}} \|\Phi\|_\infty \leq c_0 M_0
$$
(2.159)
for $0 < c_0 \ll 1$.

Applying Theorem 2.8 with $f = f^{k+1}$, $g = \Gamma(f^k, f^k) + 2\Gamma(f^k, f_w) + R_s$, $r = \varepsilon \mathcal{D}_1 f^k + \varepsilon \mathcal{D}_2 \varphi$, 

\[
[[f^{k+1}]] \lesssim \frac{1}{\varepsilon} \|Pg\|_2 + \|v^{-\frac{2}{3}}(I - P)g\|_2^3 + \sqrt{\frac{\alpha}{\varepsilon}} |r|_2, - + \alpha \varepsilon \frac{2}{3} |w|_\infty + \varepsilon \frac{2}{3} \|v\|^{-1} w g\|_\infty. \tag{2.160}
\]

It follows from Lemma 2.9 that 

\[
\begin{align*}
\|v^{-\frac{2}{3}} \Gamma(f^k, f^k)\|_2 & \lesssim \|P f^{k}\|_6 + \varepsilon \frac{2}{3} \|w f^{k}\|_\infty \|e^{-1} (I - P) f^k\|_{\nu} \lesssim \|f^k\|^2, \\
\|v^{-\frac{2}{3}} \Gamma(f^k, f_w)\|_2 & \lesssim \|\Theta_w\|_3 \|P f^k\|_6 + \varepsilon \|\Theta_w\|_\infty \|e^{-1} (I - P) f^k\|_{\nu} \\
& \lesssim (\|\Theta_w\|_{H^\frac{1}{2} (\partial \Omega)} + \|\Theta_w\|_\infty) \|f^k\|,
\end{align*}
\]

where we used $(I - P) f_w = 0$, Sobolev embedding and trace theorem. Applying Lemma 2.10 

\[
\begin{align*}
\varepsilon^{-1} \|Pg\|_2 & = \varepsilon^{-1} \|P R_s\|_2 \lesssim \|\Phi\|_2 (1 + \varepsilon |\vartheta|_\infty), \\
\|v^{-\frac{2}{3}} (I - P) R_s\|_2 & \lesssim |\vartheta|_{H^\frac{1}{2} (\partial \Omega)} + \varepsilon \|\Phi\|_2 |\vartheta|_\infty + |\vartheta|_{H^\frac{1}{2} (\partial \Omega)}^2, \\
\varepsilon \frac{2}{3} |w|_\infty & \leq \varepsilon |\vartheta|_\infty [(1 + \varepsilon |\vartheta|_\infty) |f^k| + 1], \\
\varepsilon \frac{2}{3} |\vartheta|_\infty & \leq \varepsilon |\vartheta|_\infty [(1 + \varepsilon |\vartheta|_\infty) |f^k| + \varepsilon |\Phi|_\infty],
\end{align*}
\]

and 

\[
\varepsilon \frac{2}{3} \|\Theta_w\|_3 |f^k| + \varepsilon \|\Theta_w\|_\infty |f^k| + \varepsilon \|\Phi\|_\infty + |\vartheta|_{H^\frac{1}{2} (\partial \Omega)}^2
\]

where we used the inequality 

\[
\begin{align*}
\|w \Gamma^\dagger (g_1, g_2)\|_{L^\infty_{x,v}} & \lesssim \|w g_1\|_{L^\infty_{x,v}} \|w g_2\|_{L^\infty_{x,v}} |\nu^{-1} w \Gamma (w^{-1}, w^{-1})| \\
& \lesssim \|w g_1\|_{L^\infty_{x,v}} \|w g_2\|_{L^\infty_{x,v}}.
\end{align*}
\tag{2.161}
\]

due to $|w \Gamma^\dagger (w^{-1}, w^{-1})| \lesssim |\nu|$. 

Substituting the above estimates into (2.160), we get 

\[
[[f^{k+1}]] \lesssim c_0 M_0 + \left( |\vartheta|_{H^\frac{1}{2}} + \varepsilon \frac{2}{3} |\vartheta|_\infty (1 + \varepsilon |\vartheta|_\infty) + \|f^k\| \right) \|f^k\| \leq M_0, \tag{2.162}
\]

provided $c_0$ and $M_0$ are small enough. This proved the uniform boundedness of $[[f^{k+1}]]$.

**Step 2.** To prove strong convergence of $f^k$ in $L^\infty \cap L^2$, we repeat Step 1 for $f^{k+1} - f^k$. It is standard to conclude that the weak limit $f^\varepsilon(x, v) := \lim_{k \to \infty} f^k(x, v)$ solves the steady Boltzmann equation (1.20) and (1.23). The uniqueness follows from standard argument, cf. [16].

**Step 3.** In this step, we show that the weak limit of $f^\varepsilon_s$, denoted by $f_s$, leads to steady INSF.

Let $g^\varepsilon_s = f_w + f^\varepsilon_s$. Then $g^\varepsilon_s$ satisfies 

\[
\begin{align*}
\nu \cdot \nabla_x g^\varepsilon_s + \varepsilon^2 \frac{2}{\sqrt{\mu}} \Phi \cdot \nabla_v (g^\varepsilon_s \sqrt{\mu}) + \varepsilon^{-1} L g^\varepsilon_s & = \Gamma (g^\varepsilon_s, g^\varepsilon_s) + \varepsilon \Phi \cdot v \sqrt{\mu}, \\
g^\varepsilon_s |_{x_\gamma} & = (1 - \alpha \varepsilon) \mathcal{L} g^\varepsilon_s + \alpha \varepsilon P g^\varepsilon_s + \alpha \varepsilon R g^\varepsilon_s, \tag{2.163}
\end{align*}
\]

where 

\[
\mathcal{R} g^\varepsilon_s = \frac{M_w - \sqrt{2\pi \mu}}{\varepsilon \sqrt{2\pi \mu}} + \frac{M_w - \sqrt{2\pi \mu}}{\sqrt{2\pi \mu}} P g^\varepsilon_s = \vartheta_w \left[ |v|^2 - 4 \mu + \varepsilon P g^\varepsilon_s + O(\varepsilon^2) \right], \tag{2.164}
\]

by the expansion of $M_w(x, v)$ given in (2.156).

It follows from the uniform boundedness of $[[f^\varepsilon]]$ that 

\[
\|P f^\varepsilon_s\|_6 \text{ is bounded, and } \|P (I - P) f^\varepsilon_s\|_{\nu} \to 0 \text{ as } \varepsilon \to 0. \tag{2.165}
\]

Noticing $\|Pg^\varepsilon_s\|_2 \sim \|Pg^\varepsilon_s\|_6$, we see that $g^\varepsilon_s$ is bounded in $L^2(\Omega \times \mathbb{R}^3)$, and there exists $g_s \in L^2(\Omega \times \mathbb{R}^3)$ such that, up to extraction of a subsequence, 

\[
g^\varepsilon_s \to g_s \text{ weakly in } L^2(\Omega \times \mathbb{R}^3). \tag{2.166}
\]
From Lemma 3.9 and the proof of (2.162), we know that $\nu^{-\frac{1}{2}}\Gamma(g^\varepsilon_s, g^\varepsilon_s), \varepsilon^{-1}Lg^\varepsilon$ and $\varepsilon\Phi \cdot v\sqrt{\mu}$ are all bounded in $L^2(\Omega \times \mathbb{R}^3)$. It follows that
\[
\nu^{-\frac{1}{2}}v \cdot \nabla_x g^\varepsilon_s + \varepsilon^{-1} \nu^{-\frac{1}{2}} \frac{1}{\sqrt{\mu}} \Phi \cdot \nabla_v (g^\varepsilon_s \sqrt{\mu}) \text{ is bounded in } L^2(\Omega \times \mathbb{R}^3),
\]
and thus has a weak limit in $L^2(\Omega \times \mathbb{R}^3)$, up to extraction of a subsequence. On the other hand, it is easy to verify that
\[
\nu^{-\frac{1}{2}}v \cdot \nabla_x g^\varepsilon_s \to \nu^{-\frac{1}{2}}v \cdot \nabla_x g_s, \quad \varepsilon^{-1} \nu^{-\frac{1}{2}} \frac{1}{\sqrt{\mu}} \Phi \cdot \nabla_v (g^\varepsilon_s \sqrt{\mu}) \to 0 \text{ in the sense of distributions.}
\]

It follows from the uniqueness of distribution limit that $\nu^{-\frac{1}{2}}v \cdot \nabla_x g_s \in L^2(\Omega \times \mathbb{R}^3)$ and
\[
\nu^{-\frac{1}{2}}v \cdot \nabla_x g^\varepsilon_s + \varepsilon^{-1} \nu^{-\frac{1}{2}} \frac{1}{\sqrt{\mu}} \Phi \cdot \nabla_v (g^\varepsilon_s \sqrt{\mu}) \to 0 \text{ weakly in } L^2(\Omega \times \mathbb{R}^3). \tag{2.168}
\]

Multiplying (2.163) by $\varepsilon \nu^{-\frac{1}{2}}$ and taking weak limit in $L^2(\Omega \times \mathbb{R}^3)$, we get
\[
Lg^\varepsilon_s \to 0 \text{ weakly in } L^2(\Omega \times \mathbb{R}^3). \tag{2.169}
\]

Thus we have
\[
Lg_s = 0 \quad \text{and} \quad g_s = \left[ \rho_g + u_g \cdot v + \theta_g \right] \left[ \frac{|v|^2 - 3}{2} \right] \sqrt{\mu}. \tag{2.170}
\]

From the linear independence of $\nu^{-\frac{1}{2}}v \{1, v, v \times v, |v|^2, |v|^2 \} \sqrt{\mu}$, we deduce that $\rho_g, u_g, \theta_g \in H^1(\Omega)$.

Then we show that $(\rho_g, u_g, \theta_g)$ solves steady INSF. Firstly, applying $P$ to (2.163) and taking weak limit in $L^2(\Omega \times \mathbb{R}^3)$, we get $P(\nu \cdot \nabla_x g_s) = 0$, which further yields
\[
\nabla \cdot u_g = 0, \quad \nabla (\rho_g + \theta_g) = 0. \tag{2.171}
\]

To take limit in (2.163), we first claim that
\[
P g^\varepsilon_s \to P g_s \text{ strongly in } L^2(\Omega \times \mathbb{R}^3). \tag{2.172}
\]

In fact, by using the cut-off argument in definition (3.61) and extension lemma 3.6 in [17], we can employ the averaging lemma to show that $\int_{\mathbb{R}^3} \nu^{-\frac{1}{2}}g^\varepsilon_s \psi dv \in H^2(\mathbb{R}^3)$, where $\psi \in \mathcal{D}(\mathbb{R}^3)$ are test functions. It follows that, up to extraction of a subsequence,
\[
\int_{\mathbb{R}^3} \nu^{-\frac{1}{2}}g^\varepsilon_s \psi dv \to \int_{\mathbb{R}^3} \nu^{-\frac{1}{2}}g_s \psi dv \text{ strongly in } L^2(\Omega).
\]

On the other hand, (2.160) and the fact $(1 - P)f_w = 0$ imply that $\|(1 - P)g^\varepsilon_s\|_\nu$ converges to 0. Thus, the coefficients of $P g^\varepsilon_s$ converge strongly in $L^2(\Omega)$ and the claim (2.172) is proved.

Then we can take weak limit directly from (2.163) in $L^2(\Omega \times \mathbb{R}^3)$ and get
\[
\lim_{\varepsilon \to 0} \varepsilon^{-1}Lg^\varepsilon_s = \Gamma(g_s, g_s) - v \cdot \nabla_x g_s. \tag{2.173}
\]

We multiply (2.163) by $\varepsilon^{-1} \frac{|v|^2 - 5}{2} \sqrt{\mu}$ and $\varepsilon^{-1} \nu \sqrt{\mu}$, respectively, integrate in $\mathbb{R}^3$ and take weak limit in $L^2(\Omega \times \mathbb{R}^3)$. With the help of (2.173), we calculate as in [1]
\[
0 = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left( \frac{|v|^2 - 5}{2} \sqrt{\mu} \cdot v \cdot \nabla_x g^\varepsilon_s \right) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \nabla \cdot \left( L^{-1} \left[ \frac{|v|^2 - 5}{2} \sqrt{\mu} \right], Lg^\varepsilon_s \right) = \nabla \cdot \left( L^{-1} \left[ \frac{|v|^2 - 5}{2} \sqrt{\mu} \right], \Gamma(g_s, g_s) - v \cdot \nabla_x g_s \right) = \nabla \cdot \left[ \frac{5}{2} \nabla \cdot \nabla \theta_g - \frac{5}{2} u_g \theta_g \right], \tag{2.174}
\]
and
\[
\Phi = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left( \sqrt{\mu} v \cdot \nabla_x g^\varepsilon_s \right)
= \lim_{\varepsilon \to 0} \varepsilon^{-1} \nabla \cdot \left( L^{-1} \left[ (v \cdot v - \frac{|v|^2}{3}) \sqrt{\mu} \right], Lg^\varepsilon_s \right) + \lim_{\varepsilon \to 0} \varepsilon^{-1} \nabla \cdot \left( \frac{|v|^2}{3} \sqrt{\mu}, g^\varepsilon_s \right)
= \nabla \cdot \left( L^{-1} \left[ (v \cdot v - \frac{|v|^2}{3}) \sqrt{\mu} \right], \Gamma(g_s, g_s) - v \cdot \nabla_x g_s \right) + \nabla_p g
= \nabla \cdot \left[ 2u_g \otimes u_g - \frac{2}{3} |u_g|^2 I - \sigma (\nabla u_g + (\nabla u_g)^T) \right] + \nabla_p g, \tag{2.175}
\]
where we used the notation \( p_g := \lim_{\varepsilon \to 0} \varepsilon^{-1} \left( |\nabla|^2 \sqrt{\mu} \right) \). Hence \((\rho_g, u_g, \theta_g)\) forms a weak solution to the steady INSF

\[
\begin{align*}
    u_g \cdot \nabla_x u_g + \nabla_x p_g &= \sigma \Delta u_g + \Phi \quad \text{in } \Omega, \\
    u_g \cdot \nabla_x \theta_g &= \kappa \Delta \theta_g \quad \text{in } \Omega,
\end{align*}
\]

supplemented by divergence free condition and Boussinesq relation (2.171).

**Step 4.** In this step, we derive the limiting boundary conditions.

We firstly set up a priori estimate coming from the inside. Observing the relation

\[
\int_{\partial \Omega} \nu^\frac{1}{2} \phi g_s^\varepsilon \, d\gamma = \int_{\Omega \times \mathbb{R}^3} \nu^\frac{1}{2} \phi \cdot \nabla \phi g_s^\varepsilon + \int_{\Omega \times \mathbb{R}^3} \nu^\frac{1}{2} \phi \cdot \nabla \phi g_s^\varepsilon, \quad \text{where } \phi(x, v) \text{ are test functions satisfying } \phi(\cdot, v) \in C^\infty(\bar{\Omega}) \text{ and } \phi(x, \cdot) \in C^\infty_0(\mathbb{R}^3),
\]

we employ the weak convergence of \( g_s^\varepsilon \) and \( \nu^\frac{1}{2} \phi \cdot \nabla \phi g_s^\varepsilon \) to get

\[
\nu^\frac{1}{2} g_s^\varepsilon \big|_{\partial \Omega} \to \nu^\frac{1}{2} g_s \big|_{\partial \Omega} \quad \text{in the sense of distributions. (2.177)}
\]

Then we set up a priori estimate coming from the boundary. From the boundedness of \(|f_s^\varepsilon|\)
we know that

\[
\sqrt{\frac{\alpha}{\varepsilon}} |f_s^\varepsilon|_2 \text{ is uniformly bounded. (2.178)}
\]

Define \( \langle g \rangle_{\partial \Omega} := \sqrt{2\pi} \int_{x \cdot \nu > 0} g \, d\gamma \sqrt{\mu}(\nu \cdot \nu) \, dv \). It follows that \( P_\gamma g = \sqrt{\mu}(g)_{\partial \Omega} \) and \( g_s^\varepsilon |_{\partial \Omega} - \sqrt{\mu}(g_s^\varepsilon)_{\partial \Omega} = (1 - P_\gamma) g_s^\varepsilon \).

If \( \lim_{\varepsilon \to 0} \frac{\alpha}{\varepsilon} = \sqrt{2\pi} \lambda \in (0, +\infty) \), then from (2.178) we know that \(|f_s^\varepsilon|_2\) is uniformly bounded, so that \(|g_s^\varepsilon|_2\) is bounded in \( L^2(d\gamma) \) and thus has a weak limit in \( L^2(d\gamma) \), up to a subsequence. It follows from (2.177) and the uniqueness of distribution limit that

\[
\nu^\frac{1}{2} g_s |_{\partial \Omega} \in L^2(d\gamma), \quad \nu^\frac{1}{2} g_s^\varepsilon |_{\partial \Omega} \to \nu^\frac{1}{2} g_s |_{\partial \Omega} \text{ weakly in } L^2(d\gamma) \text{ as } \varepsilon \to 0. \tag{2.179}
\]

By (2.179) and the fact that \( \langle g_s^\varepsilon \rangle_{\partial \Omega} \) is independent of \( v \), we have

\[
\langle g_s^\varepsilon \rangle_{\partial \Omega} \to \langle g_s \rangle_{\partial \Omega} \text{ weakly in } L^2(\partial \Omega) \text{ as } \varepsilon \to 0.
\]

Combining this with (2.179), we get

\[
\nu^\frac{1}{2} (g_s^\varepsilon |_{\partial \Omega} - \sqrt{\mu}(g_s^\varepsilon)_{\partial \Omega}) \to \nu^\frac{1}{2} (g_s |_{\partial \Omega} - \sqrt{\mu}(g_s)_{\partial \Omega}) \quad \text{weakly in } L^2_{\gamma^+}(d\gamma) \text{ as } \varepsilon \to 0. \tag{2.180}
\]

If \( \lim_{\varepsilon \to 0} \frac{\alpha}{\varepsilon} = \sqrt{2\pi} \lambda = 0 \), then it follows from (2.178) that \( \sqrt{\frac{\alpha}{\varepsilon}} g_s^\varepsilon |_2 \) is uniformly bounded and thus has a weak limit in \( L^2(d\gamma) \), up to a subsequence. It follows from (2.177) and the uniqueness of distribution limit that

\[
\nu^\frac{1}{2} g_s^\varepsilon |_{\partial \Omega} \to 0 \text{ weakly in } L^2(d\gamma) \text{ as } \varepsilon \to 0. \tag{2.177}
\]

Similarly as (2.180), we have

\[
\sqrt{\frac{\alpha}{\varepsilon}} \nu^\frac{1}{2} (g_s^\varepsilon |_{\partial \Omega} - \sqrt{\mu}(g_s^\varepsilon)_{\partial \Omega}) \to 0 \text{ weakly in } L^2_{\gamma^+}(d\gamma) \text{ as } \varepsilon \to 0. \tag{2.181}
\]

Now we check the boundary conditions that \((u_g, \theta_g)\) satisfies. We first consider the case \( \lim_{\varepsilon \to 0} \frac{\alpha}{\varepsilon} = \infty \). In this case, we can take limit directly in Maxwell boundary condition and show that the convergence is strong. From the boundedness of \( \sqrt{\frac{\alpha}{\varepsilon}}(1 - P_\gamma)f_s^\varepsilon |_{\partial \Omega^+} \) we know

\[
f_s^\varepsilon |_{\partial \Omega} - \sqrt{\mu}(f_s^\varepsilon)_{\partial \Omega} = (1 - P_\gamma)f_s^\varepsilon \to 0 \quad \text{strongly in } L^2_{\gamma^+}(d\gamma) \text{ as } \varepsilon \to 0.
\]

It follows that

\[
g_s^\varepsilon |_{\partial \Omega} - \sqrt{\mu}(g_s^\varepsilon)_{\partial \Omega} = f_w |_{\partial \Omega} - \sqrt{\mu}(f_w)_{\partial \Omega} \quad \text{strongly in } L^2_{\gamma^+}(d\gamma) \text{ as } \varepsilon \to 0. \tag{2.182}
\]

Combining (2.180) and (2.182), we get

\[

\nu^\frac{1}{2} [g_s |_{\partial \Omega} - \sqrt{\mu}(g_s)_{\partial \Omega}] = \nu^\frac{1}{2} [f_w |_{\partial \Omega} - \sqrt{\mu}(f_w)_{\partial \Omega}].
\]

\[
\nu^\frac{1}{2} [g_s |_{\partial \Omega} - \sqrt{\mu}(g_s)_{\partial \Omega}] = \nu^\frac{1}{2} [f_w |_{\partial \Omega} - \sqrt{\mu}(f_w)_{\partial \Omega}].
\]
which gives the Dirichlet boundary condition
\[ u_g|_{\partial \Omega} = 0, \quad \theta_g|_{\partial \Omega} = \theta_w. \quad (2.183) \]

Then we focus on the case \( \lim_{\varepsilon \to 0} \frac{\varepsilon}{\mu} = \sqrt{2\pi} \lambda \in [0, +\infty) \). Noticing \( (2.165) \) and the boundedness of \( \sqrt{\alpha_{\varepsilon}} P_{3} g_{s}^{\varepsilon}|_{\Omega} \) and \( \sqrt{\alpha_{\varepsilon}} g_{s}^{\varepsilon}|_{\Omega} \), we take weak limit directly from \( (2.164) \),
\[ g_{s}|_{\gamma} = \mathcal{L}(g_{s}|_{\gamma}). \]

This, combined with \( (2.170) \), imply the condition of zero mass flux
\[ n \cdot u_g|_{\partial \Omega} = 0. \quad (2.184) \]

To verify the Navier boundary condition, we have to take limits of \( (2.163) \) in the weak formulation and that the moments \( u_g \) and \( \theta_g \) satisfy the weak formulation of INSF. For this, we take a test function \( \phi \in C^\infty(\Omega) \) and a test vector field \( w \in C^\infty(\Omega) \) satisfying \( \nabla_{x} \cdot w = 0 \) and \( n \cdot w|_{\partial \Omega} = 0 \). Multiply equation \( (2.163) \) by \( \varepsilon^{-1} \frac{|v|^2}{2} \sqrt{\mu} \phi \) and \( \varepsilon^{-1} v \sqrt{\mu} w \), respectively, integrate in \( \Omega \times \mathbb{R}^3 \) and take weak limit in \( L^2(\Omega \times \mathbb{R}^3) \),
\[ 0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega \times \mathbb{R}^3} \left( v \cdot \nabla_{x} g_{s}^{\varepsilon} \right) \left( \frac{|v|^2}{2} - 5 \right) \sqrt{\mu} \phi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\partial \Omega} \langle v \rangle \left( \frac{|v|^2}{2} - 5 \right) \sqrt{\mu} \phi, \quad (2.185) \]

and
\[ \int_{\Omega} \Phi \cdot w = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega \times \mathbb{R}^3} \left( v \cdot \nabla_{x} g_{s}^{\varepsilon} \right) \sqrt{\mu} w = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\partial \Omega \times \mathbb{R}^3} \left\{ \nabla_{x} \cdot \left( v \otimes v - \frac{|v|^2}{3} \mathbb{I} \right) \sqrt{\mu} g_{s}^{\varepsilon} \right\} \cdot w + \nabla_{x} \cdot \left( \frac{|v|^2}{3} \mathbb{I} \sqrt{\mu} g_{s}^{\varepsilon} \right) \cdot w = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\partial \Omega \times \mathbb{R}^3} \langle v \otimes v - \frac{|v|^2}{3} \mathbb{I} \rangle \sqrt{\mu} g_{s}^{\varepsilon} \cdot \nabla_{x} w. \quad (2.186) \]

It follows from \( (2.114) \) and \( (2.116) \) that
\[ \lim_{\varepsilon \to 0} \varepsilon^{-1} \langle v \rangle \left( \frac{|v|^2}{2} - 5 \right) \sqrt{\mu} g_{s}^{\varepsilon} = \frac{5}{2} \kappa \nabla_{x} \theta_g - \frac{5}{2} u_g \theta_g, \]
\[ \lim_{\varepsilon \to 0} \varepsilon^{-1} \langle v \otimes v - \frac{|v|^2}{3} \mathbb{I} \rangle \sqrt{\mu} g_{s}^{\varepsilon} = 2 u_g \otimes u_g - \frac{2}{3} |u_g|^2 \mathbb{I} - \nu \left[ \nabla_{x} u_g + (\nabla_{x} u_g)^T \right]. \quad (2.187) \]

For the boundary integration in \( (2.185) \), we use \( (2.164) \), \( (2.165) \), \( (2.170) \), \( (2.180) \) and \( (2.181) \) and the change of variable \( v \mapsto R_{x} v \) on \( \gamma_{-} \)
\[ \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{\partial \Omega \times \mathbb{R}^3} g_{s}^{\varepsilon} \left( \frac{|v|^2}{2} - 5 \right) \sqrt{\mu} \phi \langle n \cdot v \rangle dvS_{x} = \lim_{\varepsilon \to 0} \frac{\alpha_{\varepsilon}}{\varepsilon} \int_{\gamma_{+}} \left( \frac{|v|^2}{2} - 5 \right) \sqrt{\mu} \phi \left[ g_{s}^{\varepsilon}|_{\partial \Omega} - \sqrt{\mu}(g_{s}^{\varepsilon})|_{\partial \Omega} \right] \langle n \cdot v \rangle dvS_{x} \]
\[ + \lim_{\varepsilon \to 0} \frac{\alpha_{\varepsilon}}{\varepsilon} \int_{\gamma_{-}} \left( \frac{|v|^2}{2} - 5 \right) \sqrt{\mu} \phi \left[ g_{s}^{\varepsilon}|_{\partial \Omega} - \sqrt{\mu}(g_{s}^{\varepsilon})|_{\partial \Omega} \right] \langle n \cdot v \rangle dvS_{x} = \lambda \sqrt{2\pi} \int_{\gamma_{+}} \left( \frac{|v|^2}{2} - 5 \right) \sqrt{\mu} \phi \left[ g_{s}^{\varepsilon}|_{\partial \Omega} - \sqrt{\mu}(g_{s}^{\varepsilon})|_{\partial \Omega} \right] \langle n \cdot v \rangle dvS_{x} \]
\[ + \lambda \sqrt{2\pi} \int_{\gamma_{-}} \left( \frac{|v|^2}{2} - 5 \right) \sqrt{\mu} \phi \langle \nabla_{x} \theta_g \rangle \langle n \cdot v \rangle dvS_{x} = 2\lambda \int_{\partial \Omega} (\theta_g - \theta_w) dS_{x}. \quad (2.188) \]
For the boundary term in (2.180), noticing $n \cdot w|_{\partial \Omega} = 0$ and derive similarly as above, we get
\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{\partial \Omega \times \mathbb{R}^3} (v \cdot w) \sqrt{\mu} g^s_\varepsilon [n \cdot v] dv \, ds_x \\
= \lim_{\varepsilon \to 0} \frac{\alpha_\varepsilon}{\xi} \int_{\gamma^+} (v \cdot w) \sqrt{\mu} [g^s_\varepsilon |_{\partial \Omega} - \sqrt{\mu} (g^s_\varepsilon |_{\partial \Omega})] [n \cdot v] dv \, ds_x \\
+ \lim_{\varepsilon \to 0} \frac{\alpha_\varepsilon}{\xi} \int_{\gamma^-} (v \cdot w) \sqrt{\mu} [g^s_\varepsilon [n \cdot v] dv \, ds_x \\
= \lambda \sqrt{2\pi} \int_{\gamma^+} (v \cdot w) \sqrt{\mu} [g_\varepsilon |_{\partial \Omega} - \sqrt{\mu} (g_\varepsilon |_{\partial \Omega})] [n \cdot v] dv \, ds_x \\
+ \lambda \sqrt{2\pi} \int_{\gamma^-} (v \cdot w) \sqrt{\mu} \theta_w \left( \frac{|v|^2 - 4}{2} \right) \sqrt{\mu} [n \cdot v] dv \, ds_x \\
= \lambda \int_{\partial \Omega} w \cdot u_g dv \, ds_x.
\]
Combining (2.183)–(2.189), we see that $(\rho_g, u_g, \theta_g)$ satisfies the following weak formulation of steady INSF with Navier boundary condition
\[
- \int_{\Omega} \left[ u_g \otimes u_g - \nu (\nabla u_g + (\nabla u_g)^T) \right] : \nabla w + \lambda \int_{\partial \Omega} u_g \cdot w - \int_{\Omega} \Phi \cdot w = 0, \\
- \int_{\Omega} \left[ u_g \theta_g - \kappa \nabla \theta_g \right] \cdot \nabla \phi + \frac{4}{5} \int_{\partial \Omega} \phi (\theta_g - \theta_w) = 0.
\tag{2.190}
\]
Note that if $\alpha_\varepsilon = 0$ then (2.188) and (2.189) vanish, and there is no boundary integration in (2.185), (2.186) and (2.190).

Returning back to $f^s_\varepsilon$, we have
\[
f^s_\varepsilon \to f_s := [\rho_g + u_g \cdot v + \theta_s \frac{|v|^2 - 3}{2}] \sqrt{\mu} \quad \text{weakly in } L^2(\Omega \times \mathbb{R}^3),
\]
where we used $Lf_w = 0$ and $g^s_\varepsilon = f_w + f^s_\varepsilon$. By the relation $g_s = f_w + f_s$ we have
\[
\rho_s = \rho_g, \quad \theta_s = \theta_g - \Theta_w, \quad \rho_s = \rho_g - \rho_w.
\]
Then the divergence free condition and Boussinesq relation in (1.32) and the Dirichlet boundary condition (1.33) follow readily. Moreover, (2.191) leads to
\[
- \int_{\Omega} \left[ u_s \otimes u_s - \nu (\nabla u_s + (\nabla u_s)^T) \right] : \nabla w + \lambda \int_{\partial \Omega} u_s \cdot w - \int_{\Omega} \Phi \cdot w = 0, \\
- \int_{\Omega} \left[ u_s (\theta_s + \Theta_w) - \kappa \nabla (\theta_s + \Theta_w) \right] \cdot \nabla \phi + \frac{4}{5} \lambda \int_{\partial \Omega} \phi \theta_s = 0.
\tag{2.191}
\]
This is the weak form of the steady INSF (1.32) subject to the Navier boundary condition (1.33). This completes the proof. \qed

3. Unsteady Limit

3.1. $L^2$ coercivity estimate and $L^2_L L^3_T$ bound.

In this section, we construct the $L^2$ coercivity estimate for the unsteady linear equation.

We first estimate the macroscopic part $f^s_0 \| Pf \|_2$ and have the following result.

**Lemma 3.1.** Let $\Phi \in C^1(\Omega)$, $g \in L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3)$, $r \in L^2(\mathbb{R}_+ \times \gamma_-)$ such that for all $t > 0$
\[
\int_{\Omega \times \mathbb{R}^3} g(t,x,v) \sqrt{\mu} dv \, dr = 0, \quad \int_{\Omega \times \mathbb{R}^3} r(t,x,v) \sqrt{\mu} [n \cdot v] dv = 0 \quad \text{for all } x \in \partial \Omega.
\tag{3.1}
\]
Suppose that $f$ solves
\[\varepsilon \partial_t f + v \cdot \nabla_x f + \varepsilon^2 \frac{1}{\sqrt{\mu}} \Phi \cdot \nabla_v (\sqrt{\mu} f) + \varepsilon^{-1} Lf = g,\]
\[f|_{t=0} = f_0,\]
and satisfies
\[\int_{\Omega \times \mathbb{R}^3} f(t, x, v) \sqrt{\mu} dv = 0 \quad \text{for all } t > 0. \tag{3.3}\]

Then there exists a function $G(t)$ such that $G(t) \lesssim \varepsilon \|f(t)\|_2$ and
\[
\int_0^t \|Pf(s)\|_2^2 \lesssim \int_0^t \left( \varepsilon^{-2} \|(I - P)f(s)\|_2^2 + \alpha^2 \|(1 - P_\gamma)f(s)\|_{2,-} + \alpha^2 |r(s)|_{2,-} \right) + G(t) - G(0). \tag{3.4}\]

**Proof.** Note that (3.1), (3.2) and (3.3) are all invariant under a standard $t$-mollification for all $t > 0$. The estimates in Step 1 to Step 3 below are obtained via a $t$-mollification so that all the functions are smooth in $t$. For the notational simplicity we do not write explicitly the parameter of the regularization.

Multiply a test function $\psi(t, x, v)$, which will be determined later, to the equation in (3.2)
\[
\int_0^t \int_{\Omega \times \mathbb{R}^3} \left[ - \varepsilon f \partial_t \psi - (v \cdot \nabla_x \psi)f - \varepsilon^2 (\Phi \cdot \nabla_v (\sqrt{\mu} f) + \varepsilon^{-1} L(I - P)f \psi \right] + \varepsilon \int_{\Omega \times \mathbb{R}^3} [(f \psi)(t) - (f \psi)(0)] + \int_0^t \int_{\partial \Omega \times \mathbb{R}^3} (v \cdot n) f \psi = \int_0^t \int_{\Omega \times \mathbb{R}^3} g \psi. \tag{3.5}\]

It follows from (3.5) that
\[
- \int_0^t \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_k) Pf = \sum_{j=1}^7 \Psi_j k, \tag{3.6}\]
where $k \in \{a, b, c\}$ and
\[
\begin{align*}
\Psi_k^1 &:= -\varepsilon \int_{\Omega \times \mathbb{R}^3} [(f \psi_k)(t) - (f \psi_k)(0)], \\
\Psi_k^2 &:= -\int_0^t \int_{\partial \Omega \times \mathbb{R}^3} (v \cdot n) f \psi_k, \\
\Psi_k^3 &:= \int_0^t \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_k)(I - P)f, \\
\Psi_k^4 &:= \varepsilon^2 \int_0^t \int_{\Omega \times \mathbb{R}^3} (\Phi \cdot \nabla_v (\sqrt{\mu} f), \\
\Psi_k^7 &:= -\varepsilon^{-1} \int_0^t \int_{\Omega \times \mathbb{R}^3} L(I - P)f \psi_k, \\
\Psi_k^8 &:= \int_0^t \int_{\Omega \times \mathbb{R}^3} g \psi_k, \\
\Psi_k^9 &:= \varepsilon \int_0^t \int_{\Omega \times \mathbb{R}^3} f \partial_t \psi_k.
\end{align*}
\]

In the following, we prove (3.4) by using the same choice of test functions as in Lemma 2.1.

**Step 1.** Estimate of $\int_0^t \|a\|_2^2$.

Choose test function $\psi_a := (|v|^2 - 10)\sqrt{\mu} v \cdot \nabla_x \phi_a(t, x)$, where $\phi_a(t, x)$ satisfies
\[-\Delta_x \phi_a(t, x) = a(t, x), \quad \partial_n \phi_a |_{\partial \Omega} = 0, \quad \int_\Omega \phi_a(t, x) dx = 0.\]

Note that $\int_\Omega a(t, x) = \int_{\Omega \times \mathbb{R}^3} f(t, x, v) \sqrt{\mu} = 0$, so that the compatible condition is satisfied for the equation.

By virtue of (2.14), the left-hand side of (3.6) equals
\[
- \int_0^t \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \Psi_a) Pf = -5 \int_0^t \|a\|_2^2. \tag{3.7}\]
Firstly, $\Psi_a^1$ contributes to $G(t) - G(0)$. Similarly as (2.24) and (2.28), we can show that

$$|\Psi_a^2| + |\Psi_a^3| + |\Psi_a^4| + |\Psi_a^5| + |\Psi_a^6|$$

$$\leq \int_0^t \left( \alpha^2 (1 - P_t) f_s^2 + \alpha^2 |r|^2_{2,-} + \varepsilon^2 ||c||^2_2 + \varepsilon^2 \| (I - P_t) f_s \|^2_2 + \| \nu^{-\frac{3}{2}} g \|^2_2 + o(1) ||a||^2_2 \right).$$

The new contribution $\Psi_a^5$ is bounded by

$$|\Psi_a^5| \leq \varepsilon \int_0^t \left( ||b||_2 + ||(I - P_t) f_s||_2 \right) \| \partial_t \nabla_x \phi_a \|_2,$$

where we used (2.13) and oddness of $v_i ||v||^2 - 10 \sqrt{a}$.

To estimate $\| \partial_t \nabla_x \phi_a \|_2$, we use the decomposition of the weak formulation (3.5) between $t$ and $t + \delta$ (instead of between $0$ and $t$) with $\psi(t, x, v) = \phi(x) \sqrt{a}$, where $\phi \in H^1(\Omega)$ is independent of $t$ and satisfies $\int_\Omega \phi \, dx = 0$. From (3.5) we have

$$- \int_t^{t+\delta} \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \phi) \sqrt{a} \, f = -\varepsilon \int_\Omega \phi [a(t+\delta) - a(t)] + \int_t^{t+\delta} \int_{\Omega \times \mathbb{R}^3} g \phi \sqrt{a},$$

where the boundary integration vanishes due to (3.7). Besides, direct computation shows

$$- \int_t^{t+\delta} \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \phi) \sqrt{a} \, f = - \int_t^{t+\delta} \int_{\Omega} (b \cdot \nabla_x \phi).$$

Combining the above two equalities, we get

$$\varepsilon \int_\Omega \phi [a(t+\delta) - a(t)] = \int_t^{t+\delta} \int_{\Omega \times \mathbb{R}^3} g \phi \sqrt{a} + \int_t^{t+\delta} \int_{\Omega} b \cdot \nabla_x \phi. \quad (3.10)$$

It follows by Poincaré inequality that

$$\left| \varepsilon \int_\Omega \phi \partial_t a \right| \leq ||\nu^{-\frac{3}{2}} g||_2 \| \phi \|_2 + ||b||_2 \| \nabla_x \phi \|_2 \leq (||\nu^{-\frac{3}{2}} g||_2 + ||b||_2) \| \nabla_x \phi \|_2. \quad (3.11)$$

Besides, by taking $\phi = 1$ in (3.10) and using (2.7) and (3.3), we have $\int_\Omega \partial_t a dx = 0$. Now we define

$$W_a := \{ \phi \in H^1(\Omega) : \int_\Omega \phi \, dx = 0 \}, \quad \| \phi \|_{W^*_a} := \| \nabla_x \phi \|_2.$$

The dual space is denoted by $W^*_a$. Then by (3.11) we have

$$\| \partial_t a \|_{W^*_a} \leq \varepsilon^{-1} (||\nu^{-\frac{3}{2}} g||_2 + ||b||_2). \quad (3.12)$$

In fact, $\partial_t a$ is the solution of

$$- \Delta_x \partial_t a = \partial_t a, \quad \partial_n (\partial_t a) \big|_{\partial \Omega} = 0, \quad \int_\Omega \partial_t a \, dx = 0.$$

It follows from standard variational theory that

$$\| \partial_t a \|_{W^*_a} \leq \| \partial_t a \|_{W^*_a},$$

where the compatible condition $\int_\Omega \partial_t a dx = 0$ was used. Combining this with (3.12), we have

$$\| \partial_t \nabla_x \phi_a \|_2 = \| \nabla_x \partial_t \phi_a \|_2 = \| \partial_t \phi_a \|_{W^*_a} \leq \| \partial_t a \|_{W^*_a} \leq \varepsilon^{-1} (||\nu^{-\frac{3}{2}} g||_2 + ||b||_2). \quad (3.13)$$

Substituting (3.12) into (3.9), we get

$$|\Psi_a^5| \leq \int_0^t \left( ||(I - P_t) f_s ||^2_2 + ||\nu^{-\frac{3}{2}} g||^2_2 + ||b||_2^2 \right).$$

Collecting the estimates from $\Psi_a^4$ to $\Psi_a^7$ and absorbing small contributions, we get

$$\int_0^t ||a||_2^2 \leq G(t) - G(0) + \varepsilon^{-2} \int_0^t ||(I - P_t) f_s ||^2_2 + \alpha^2 \int_0^t ||(1 - P_t) f_s ||^2_2 +$$

$$+ \alpha^2 \int_0^t ||r||^2_{2,-} + \int_0^t \| \nu^{-\frac{3}{2}} g \|^2_2 + \int_0^t ||b||^2_2 + \varepsilon^2 \| \Phi \|_\infty \int_0^t ||c||^2_2. \quad (3.14)$$
Step 2. Estimate of $\int_0^t ||c||^2$. 

Choose a test function $\psi_c := (|v|^2 - 5)\sqrt{\mu}v \cdot \nabla_x \phi_c(t, x)$, where

$$-\Delta_x \phi_c(t, x) + E \cdot \nabla_x \phi_c = c(t, x), \quad \partial_n \phi_c|_{\partial \Omega} = 0, \quad \int_\Omega \phi_c \, dx = 0,$$

where $E \in [L^\infty(\Omega)]^3$ is defined as in (2.32) and Lemma A.3.

It follows from (2.34) and (2.35) that

$$\int_0^t \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_c) P f = 5 \int_0^t \int_{\Omega} ||c||^2 - 5 \int_0^t \int_{\Omega} c(E \cdot \nabla_x \phi_c) \geq 4 \int_0^t ||c||^2,$$

provided that $||E||_\infty$ is sufficiently small. We estimate $\Psi^1_c$ like $\Psi^1_a$. Similarly as (2.39) and (2.41) we can prove that

$$|\Psi^1_c| + |\Psi^2_c| + |\Psi^3_c| + |\Psi^5_c| + |\Psi^6_c| \lesssim \int_0^t \alpha^2(1 - P_\gamma)f^2_{2, +} + \alpha^2|r|^2_{2, -} + \varepsilon^{-2}|||I - P||f||^2_2 + ||\nu^{-\frac{1}{2}}g||^2_2 + o(1)||c||^2_2.$$

The new contribution $\Psi^7_c$ is estimated by

$$|\Psi^7_c| \lesssim \varepsilon \int_0^t ||I - P||f||_2|| \partial_t \nabla_x \phi_c||_2,$$

where $a, c$ contributions vanish due to oddness, and the $b$ contribution vanishes by (2.35).

To estimate $||\partial_t \nabla_x \phi_c||_2$, we use (3.15) with $\psi(t, x, v) = \phi(x)\frac{|v|^2}{2}\sqrt{\mu}$, where $\phi \in H^1(\Omega)$ is independent of $t$ and satisfies $\int_\Omega \phi \, dx = 0$. From (3.15) we have

$$-\int_t^{t+\delta} \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \phi) \frac{|v|^2 - 3}{2} \sqrt{\mu} P f$$

$$= -\frac{3}{2} \int_{\Omega} \phi(c(t + \delta) - c(t)) - \alpha \int_{\gamma_+} \phi \frac{|v|^2 - 3}{2} \sqrt{\mu}(1 - P_\gamma)f \, d\gamma + \alpha \int_{\gamma_-} \phi \frac{|v|^2 - 3}{2} \sqrt{\mu} r \, d\gamma + \int_t^{t+\delta} \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \phi)(I - P)f + g \phi \frac{|v|^2 - 3}{2} \sqrt{\mu} \left(1 - \frac{1}{2} \sqrt{\mu} \right)(1 - \alpha)f + \alpha P_\gamma f \right) \, d\gamma$$

$$= \alpha \int_{\gamma_+} \phi \frac{|v|^2 - 3}{2} \sqrt{\mu}(1 - P_\gamma)f \, d\gamma - \alpha \int_{\gamma_-} \phi \frac{|v|^2 - 3}{2} \sqrt{\mu} r \, d\gamma.$$

One the other hand, direct computation leads to

$$-\int_t^{t+\delta} \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \phi) \frac{|v|^2 - 3}{2} \sqrt{\mu} P f = -\int_t^{t+\delta} \int_{\Omega} (b \cdot \nabla_x \phi),$$

by $\int_{\mathbb{R}^3} v^2 |v|^2 - 3 \sqrt{\mu} = 1$. Combining this with (3.17), we get

$$\varepsilon \int_{\Omega} \phi \partial_t c \lesssim ||b||_2 ||\nabla_x \phi||_2 + ||(I - P)f||_2 ||\nabla_x \phi||_2 + \varepsilon^2 ||\Phi||_\infty ||b||_2 ||\phi||_2 + ||\nu^{-\frac{1}{2}} g||_2 ||\phi||_2$$

$$+ \alpha (1 - P_\gamma)f|_{2, +} ||\phi||_2 - \alpha |r|_2, - ||\phi||_2$$

$$\lesssim \left(||(I - P)f||_2 + \alpha (1 - P_\gamma)f|_{2, +} + \alpha |r|_2, - + ||\nu^{-\frac{1}{2}} g||_2 + ||b||_2 \right)||\nabla_x \phi||_2,$$

where we used trace theorem $||\phi||_2 \lesssim ||\phi||_{H^1}$ and Poincaré inequality.
As in Step 1, we define
\[ W_c := \{ \phi \in H^1(\Omega) : \int_\Omega \phi \, dx = 0 \}, \quad \|\phi\|_{W_c} := \|\nabla_x \phi\|_2. \]

Its dual space is denoted by \( W_c^* \). It follows from (3.15) that
\[ \varepsilon \|\partial_t c\|_{W_c^*} \lesssim \| (I - P) f \|_2 \quad \text{with} \quad \|\phi\|_{W_c} := \|\nabla_x \phi\|_2. \] (3.19)
We know that \( \partial_t \phi_c \) is the solution of
\[ -\Delta_x \partial_t \phi_c + E \cdot \nabla_x (\partial_t \phi_c) = \partial_t c, \quad \partial_n (\partial_t \phi_c)|_{\partial \Omega} = 0, \quad \int_\Omega \partial_t \phi_c \, dx = 0, \]
where \( E = E(x) \) is as before. Standard variational theory and Poincaré inequality imply that
\[ \|\partial_t \phi_c\|_{W_c} \lesssim \|\partial_t c\|_{W_c^*}, \]
provided that \( \|E\|_\infty \) is small. Combining this with (3.19), we have
\[ \|\partial_t \nabla_x \phi_c\|_2 = \|\nabla_x \partial_t \phi_c\|_2 = \|\partial_t \phi_c\|_{W_c} \lesssim \|\partial_t c\|_{W_c^*} \lesssim \varepsilon^{-1} \left( \| (I - P) f \|_2 + \alpha \| (1 - P) f \|_2 + \alpha \| r \|_2 + \| \nu^{-\frac{1}{2}} g \|_2 + \| b \|_2 \right). \] (3.20)

Substituting (3.20) into (3.16),
\[ |\Psi_0^2| \lesssim \int_0^t \left( \| (I - P) f \|_2^2 + \alpha^2 \| (1 - P) f \|_2^2 + \alpha^2 \| r \|_2^2 + \| \nu^{-\frac{1}{2}} g \|_2^2 + \eta_1 \| b \|_2^2 \right), \]
where \( \eta_1 > 0 \) is a small constant.

Collecting (3.15) and the estimates from \( \Psi_0^1 \) to \( \Psi_0^7 \) and absorbing small terms, we get
\[ \int_0^t \| c \|_2^2 \lesssim G(t) - G(0) + \varepsilon^2 \int_0^t \| (I - P) f \|_2^2 + 2 \int_0^t \| r \|_2^2 + 2 \int_0^t \| \nu^{-\frac{1}{2}} g \|_2^2 + \eta \int_0^t \| b \|_2^2. \] (3.21)

Step 3. Estimate of \( \int_0^t \| b \|_2^2 \).

Choose a test function \( \psi_b := \sum_{i,j=1}^3 \partial_j \phi_i \psi_i \sqrt{\nu} - \sum_{i=1}^3 \partial_i \phi_i \sqrt{\nu} \) where \( \phi^b = (\phi_1^b, \phi_2^b, \phi_3^b) \) is defined through the elliptic system
\[ -\Delta_x \phi^b(t, x) = b(t, x) \quad \text{in} \quad \Omega, \quad \phi^b \cdot n = 0 \quad \text{on} \quad \partial \Omega, \quad \partial_n \phi^b = [\phi^b \cdot n] n \quad \text{on} \quad \partial \Omega, \]
(cf. Lemma A.1 and Lemma A.2) in the Appendix. It follows from (2.49) (with \( m = 2 \)) that
\[ -\int_0^t \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_b) Pf = \int_0^t \| b \|_2^2. \] (3.22)

\( |\Psi_0^1| \) is estimated as \( |\Psi_0^4| \). From (2.40) and (2.45) (with \( m = 2 \)) we have
\[ |\Psi_0^2| + |\Psi_0^3| + |\Psi_0^4| + |\Psi_0^5| + |\Psi_0^6| \lesssim \int_0^t \left( \alpha^2 \| (1 - P) f \|_2^2 + \alpha^2 \| r \|_2^2 + \varepsilon^{-2} \| (I - P) f \|_2^2 + \| \nu^{-\frac{1}{2}} g \|_2^2 + o(1) \| b \|_2^2 \right). \]

For the new contribution \( \Psi_0^7 \), we have
\[ \Psi_0^7 \lesssim \varepsilon \int_0^t \left( \| c \|_2 + \| (I - P) f \|_2 \right) \left( \| \nabla_x \phi^b \|_2 \right), \] (3.23)
where the $b$ contribution vanishes by oddness and the $a$ contribution also vanishes due to

$$
\int_{\Omega \times \mathbb{R}^3} a \sqrt{\mu} \left( \sum_{i,j=1}^{3} \partial_i \partial_j \phi_i^b v_i v_j \sqrt{\mu} - \sum_{i=1}^{3} \partial_i \partial_i \phi_i^b \frac{|v|^2 - 1}{2 \sqrt{\mu}} \right)
$$

$$\begin{align*}
&= \sum_{i,j=1}^{3} \int_{\Omega} a \partial_i \partial_j \phi_i^b \int_{\mathbb{R}^3} \mu v_i v_j - \sum_{i=1}^{3} \int_{\Omega} a \partial_i \partial_i \phi_i^b \int_{\mathbb{R}^3} \mu \frac{|v|^2 - 1}{2} \\
&= \int_{\Omega} a \partial_i \text{div} \phi^b - \int_{\Omega} a \partial_i \text{div} \phi^b = 0.
\end{align*}
$$

To estimate $\| \partial_t \nabla_x \phi^b \|_2$, we choose $\psi(t, x, v) = \sqrt{\mu} v \cdot \phi(x)$ in (3.23), where $\phi \in [H^1(\Omega)]^3$ is independent of $t$ and satisfies $\phi \cdot n|_{\partial \Omega} = 0$. From (3.5) we have

$$\begin{align*}
&- \int_{\Omega}^{t+\delta} \int_{\Omega \times \mathbb{R}^3} \left[ v \cdot \nabla_x (v \cdot \phi) \right] \sqrt{\mu} P f \\
&= - \varepsilon \int_{\Omega} \phi \cdot \left[ b(t + \delta) - b(t) \right] - \alpha \int_{\gamma_+} \sqrt{\mu} (v \cdot \phi) (1 - P) f d\gamma + \alpha \int_{\gamma_-} \sqrt{\mu} (v \cdot \phi) r d\gamma + \frac{\varepsilon}{3} \int_{\Omega}^{t+\delta} \int_{\Omega \times \mathbb{R}^3} \left( [v \cdot \nabla_x (v \cdot \phi)] \sqrt{\mu} (I - P) f + g(v \cdot \phi) \sqrt{\mu} \right) + \varepsilon^2 \int_{\Omega}^{t+\delta} \int_{\Omega} (\Phi \cdot \phi) a.
\end{align*}
$$

Here we treated the boundary integration by the change of variable $v \mapsto R_x v$

$$\begin{align*}
&\int_{\Omega \times \mathbb{R}^3} (v \cdot n) f \sqrt{\mu} (v \cdot \phi) \\
&= \int_{\gamma_+} \sqrt{\mu} f (v \cdot \phi) a \gamma - \int_{\gamma_+} \sqrt{\mu} (v \cdot \phi) \left( [1 - \alpha] f + \alpha P \gamma f \right) d\gamma - \int_{\gamma_-} \sqrt{\mu} (v \cdot \phi) r d\gamma \\
&= \int_{\gamma_+} \sqrt{\mu} (v \cdot \phi) (1 - P) f d\gamma - \int_{\gamma_-} \sqrt{\mu} (v \cdot \phi) r d\gamma,
\end{align*}
$$

where we used the boundary condition $\phi \cdot n|_{\partial \Omega} = 0$. On the other hand, direct calculation indicates

$$\begin{align*}
- \int_{\Omega}^{t+\delta} \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi) P f &= - \int_{\Omega}^{t} \int_{\Omega} (a + c) \text{ div} \phi,
\end{align*}
$$

where we used $\int_{\mathbb{R}^3} v_i^2 \sqrt{\mu} = 1$ and $\int_{\mathbb{R}^3} v_i^2 \frac{|v|^2 - 1}{2 \sqrt{\mu}} \sqrt{\mu} = 1$. Combining this with (3.24),

$$\begin{align*}
\varepsilon \int_{\Omega} \phi \partial_t b &\lesssim \left( \|a\|_2 + \|c\|_2 + \|[I - P] f\|_2 + \|\nu^{-\frac{1}{2}} g\|_2 \right) \|\nabla_x \phi\|_2 + \|\frac{g}{\sqrt{\mu}}\|_2 \|\phi\|_2 \\
&\quad + \alpha \|(1 - P) f\|_{2+,} + |\phi|_2 - \alpha |r|_{2,-} |\phi|_2 \\
&\lesssim \left( \|\nabla_x \phi\|_2 + \alpha \|(1 - P) f\|_{2,+} + \alpha |r|_{2,-} + \|\nu^{-\frac{1}{2}} g\|_2 + \|a\|_2 + \|c\|_2 \right) \|\phi\|_{H^1},
\end{align*}
$$

where we used the trace theorem $|\phi|_2 \lesssim \|\phi\|_{H^1}$.

We define

$$V^{1,2}(\Omega) := \{ \phi \in [H^1(\Omega)]^3 : \phi \cdot n|_{\partial \Omega} = 0 \}, \quad \|\phi\|_{V^{1,2}} := \|\phi\|_{H^1}.
$$

The dual space is denoted by $[V^{1,2}(\Omega)]^*$. Then we have

$$\|\partial_t b\|_{V^{1,2}, 1} \lesssim \varepsilon^{-1} \left( \|\nabla_x (I - P) f\|_2 + \alpha \|(1 - P) f\|_{2,+} + \alpha |r|_{2,-} + \|\nu^{-\frac{1}{2}} g\|_2 + \|a\|_2 + \|c\|_2 \right). \tag{3.26}
$$

In fact, $\partial_t b$ solves the elliptic system

$$-\Delta_x \partial_t \phi^b(t, x) = \partial_t b(t, x) \quad \text{in} \ \Omega, \quad (\partial_t \phi^b) \cdot n = 0 \ \text{on} \ \partial \Omega, \quad \partial_n (\partial_t \phi^b) = [(\partial_t \phi^b) \cdot n] n \ \text{on} \ \partial \Omega,
$$

It was proved in Appendix of [83] that for given $f \in [V^{1,2}(\Omega)]^*$, this system has a unique weak solution $\phi^b(t, \cdot) \in V^{1,2}(\Omega)$ and the solution operator $\partial_t b \mapsto \partial_t \phi^b$ is bounded from $[V^{1,2}(\Omega)]^*$ to $V^{1,2}(\Omega)$, that is,

$$\|\partial_t \phi^b\|_{V^{1,2}} \lesssim \|\partial_t b\|_{V^{1,2}, 1}.$$
Combining this inequality with (3.26), we get
\[
\| \partial_t \nabla_x \phi^b \|_{L^2} \leq \| \nabla_x \partial_t \phi^b \|_{V_1,2} \leq \| \partial_t b \|_{V_1,2},
\]
\[
\lesssim \varepsilon^{-1} \left( \|(I - P)f\|_{L^2} + \alpha |(1 - P_\gamma)f|_{L^2} + \| \nu^{-\frac{1}{2}} g \|_{L^2} + \| a \|_{L^2} + \| c \|_{L^2} \right) .
\]
(3.27)
Substituting (3.27) into (3.23) we have
\[
\| \hat{\Psi}_T \| \lesssim \int_0^T \left( \|(I - P)f\|_{L^2} + \alpha^2 |(1 - P_\gamma)f|_{L^2} + \alpha^2 |r|_{L^2} + \| \nu^{-\frac{1}{2}} g \|_{L^2} + \| c \|_{L^2} + \eta_2 \| a \|_{L^2} \right),
\]
where \( \eta_2 > 0 \) is a small constant.
Combining (3.22) and the estimates from \( \Psi_T^b \) to \( \Psi_T^7 \) and absorbing small contributions,
\[
\int_0^t \| b \|_{L^2}^2 \lesssim G(t) - G(0) + \varepsilon^{-2} \int_0^t \|(I - P)f\|_{L^2}^2 + \alpha^2 \int_0^t |(1 - P_\gamma)f|_{L^2}^2 + 
\alpha^2 \int_0^t |r|_{L^2}^2 + \int_0^t \| \nu^{-\frac{1}{2}} g \|_{L^2}^2 + \int_0^t \| c \|_{L^2}^2 + \eta_2 \| a \|_{L^2}^2 .
\]
(3.28)
Finally, we multiply small constants \( \delta_1 \) to (3.14) and \( \delta_2 \) to (3.28), then add to (3.21),
\[
(\delta_1 - \delta_2 \eta_2) \int_0^t \| a \|_{L^2}^2 + (\delta_2 - \delta_1 - \eta_1) \int_0^t \| b \|_{L^2}^2 + (1 - \delta_2) \int_0^t \| c \|_{L^2}^2
\]
\[
\lesssim G(t) - G(0) + \int_0^t \varepsilon^{-2} \|(I - P)f\|_{L^2}^2 + \alpha^2 \|(1 - P_\gamma)f\|_{L^2}^2 + \alpha^2 |r|_{L^2}^2 + \| \nu^{-\frac{1}{2}} g \|_{L^2}^2 .
\]
(3.29)
We first choose \( \delta_2 = 4\delta_1 \) small such that \( 1 - \delta_2 \geq \frac{1}{2} \), then choose \( \eta_1 \) small such that \( \delta_2 - \delta_1 - \eta_1 \geq \frac{\delta_1}{2} \),
and finally find \( \eta_2 \) small such that \( \delta_2 - \delta_1 - \eta_2 \geq \frac{\delta_1}{2} \). Finally, (3.4) follows readily from (3.29), where
\( G(s) = -\varepsilon \int_{\Omega \times \mathbb{R}^3} |f\psi|(s) \lesssim \varepsilon \| f \|_{L^2}^2 \) by the selection of \( \psi \). This completes the proof. \( \square \)

Recall the definition of \( \gamma^4 \) in (1.40). We need the following trace lemma proved by [17] for unsteady transport equations on the outgoing non-grazing set \( \gamma^4 \).

**Lemma 3.2.** For \( f \in L^1([0, T] \times \Omega \times \mathbb{R}^3) \), there holds
\[
\int_0^T \int_{\gamma^4} |f(t, x, v)| \, dv \, dt \lesssim \varepsilon \int_{\Omega \times \mathbb{R}^3} |f(0, x, v)| \, dv \, dx + \varepsilon \int_0^T \int_{\Omega \times \mathbb{R}^3} |f(t, x, v)| \, dv \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega \times \mathbb{R}^3} \left| \varepsilon \partial_t f + v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_v f \right| (t, x, v) \, dv \, dx \, dt .
\]
(3.30)
We have the following \( L^2 \) existence and decay estimate for the unsteady equation (3.2).

**Theorem 3.3.** Let \( \Phi, g, r \) be as in Lemma 3.1 and satisfy (3.1). Then for \( 0 < \varepsilon \ll 1 \), the problem (3.2) has a unique weak solution \( f \in L^2(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3) \) satisfying (3.28) and the following decay estimate
\[
\| e^{\lambda t} f(t) \|_{L^2}^2 + \frac{1}{\varepsilon^2} \int_0^t \| (I - P) |e^{\lambda s} f(s)| \|_{L^2}^2 + \int_0^t \| P |e^{\lambda s} f(s)| \|_{L^2}^2
\]
\[
+ \frac{\alpha}{\varepsilon} \int_0^t |(1 - P_\gamma)|e^{\lambda s} f(s)| \|_{L^2}^2 + \frac{\alpha}{\varepsilon} \int_0^t |e^{\lambda s} f(s)|^2
\]
\[
\lesssim \| f_0 \|_{L^2}^2 + \int_0^t \left( \frac{\alpha}{\varepsilon} \| e^{\lambda s} r(s) \|_{L^2}^2 + \frac{1}{\varepsilon^2} \| P |e^{\lambda s} g(s)| \|_{L^2}^2 + \| (I - P) |e^{\lambda s} g(s)| \|_{L^2}^2 \right) ,
\]
where \( 0 < \lambda \ll 1 \) is a small constant.
Proof. Step 1. Let \( j \in \mathbb{N} \) be fixed and consider the iteration problem for \( t \in \mathbb{N} \)
\[
\partial_t f^{l+1} + \epsilon^{-1} v \cdot \nabla_x f^{l+1} + \epsilon \Phi \cdot \nabla_x f^{l+1} + \epsilon^{-2} \bar{\nu} f^{l+1} = \epsilon^{-2} K f^l + \epsilon^{-1} g,
\]
\[
f^{l+1}|_{\gamma} = (1 - \frac{1}{j})[(1 - \alpha)\mathcal{L} f^l + \alpha P f^l] + \alpha \eta,
\]
(3.32)
\[
f^{l+1}|_{t=0} = f_0,
\]
where \( f^0 \equiv f_0, \bar{\nu} = \nu - \frac{\epsilon}{2}\epsilon \Phi \cdot v \).

Standard \( L^2 \) energy estimate leads to
\[
\|f^{l+1}(t)\|_2^2 + \epsilon^{-2} \int_0^t \|f^{l+1}\|_2^2 + 2\epsilon^{-1} \int_0^t \int_{\gamma} |f^{l+1}|^2 d\gamma
\leq \|f_0\|_2^2 + C \int_0^t \left( \epsilon^{-2} \|f^l\|_2^2 + \epsilon^{-2} \|\mathbf{P} g\|_2^2 + \|\nu^{-\frac{3}{2}}(I - \mathbf{P}) g\|_2^2 \right) + 2\epsilon^{-1} \int_0^t \int_{\gamma} |f^{l+1}|^2 d\gamma,
\]
(3.33)
where we used \((\bar{\nu}, f, f) \geq \frac{1}{2}\|f\|_2^2 \) and
\[
(K f^l, f^{l+1}) \leq \left[ \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} k(v, u)|f^l(v)|^2 du dv \right]^\frac{3}{2} \left[ \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} k(v, u)|f^{l+1}(v)|^2 dv du \right]^\frac{1}{2}
\leq \left[ \sup_v \int_{\mathbb{R}^3} |k(v, u)| du \int_{\mathbb{R}^3} |f^l(v)|^2 dv \right]^\frac{3}{2} \left[ \sup_u \int_{\mathbb{R}^3} |k(v, u)| du \int_{\mathbb{R}^3} |f^{l+1}(v)|^2 dv \right]^\frac{1}{2}
\leq \eta_1 \|f^l\|_2^2 + C_n \|f^{l+1}\|_2^2.
\]

By the change of variable \( v \mapsto R_x v, \{2.71\}\) and \{2.71\}, we have
\[
\int_{\gamma} \left( (1 - \alpha)\mathcal{L} f^l + \alpha P f^l \right) d\gamma = (1 - \alpha)^2 |f^l|_{2, +}^2 + |\langle 1 - (1 - \alpha)^2 \rangle | P f^l |_{2, +}^2 \leq |f^l|_{2, +}^2.
\]
(3.34)
Similarly as \{2.72\}, we have
\[
2 \int_{\gamma} \left( (1 - \alpha)\mathcal{L} f^l + \alpha P f^l \right) d\gamma \leq (1 - \alpha)^2 |f^l|_{2, +}^2 + \alpha C_j |r|_{2, -}^2.
\]
(3.35)
It follows from \{3.31\} and \{3.35\} that
\[
\int_{\gamma} |f^{l+1}|^2 d\gamma \leq \left( 1 - \frac{1}{j} \right)^2 |f^l|_{2, +}^2 + (1 - \frac{1}{j}) |f^l|_{2, +}^2 + \alpha C_j |r|_{2, -}^2 \leq (1 - \frac{1}{j}) |f^l|_{2, +}^2 + \alpha C_j |r|_{2, -}^2.
\]
(3.36)
Combining \{3.33\} and \{3.36\} and absorbing small contributions, we get
\[
\|f^{l+1}(t)\|_2^2 + \epsilon^{-2} \int_0^t \|f^{l+1}\|_2^2 + 2\epsilon^{-1} \int_0^t |f^{l+1}|_{2, +}^2
\leq \left[ \|f_0\|_2^2 + C_j \int_0^t \left( (1 - \epsilon)^2 \|f^l\|_2^2 + \epsilon^{-2} \|\mathbf{P} g\|_2^2 + \|\nu^{-\frac{3}{2}}(I - \mathbf{P}) g\|_2^2 + \epsilon^{-1} |r|_{2, -}^2 \right) \right]
+ 2\epsilon^{-1} \int_0^t \int_{\gamma} |f^{l+1}|^2 d\gamma
\leq \sum_{k=0}^l \eta^k \left[ \|f_0\|_2^2 + C_j \int_0^t \left( (1 - \epsilon)^2 \|f^l\|_2^2 + \epsilon^{-2} \|\mathbf{P} g\|_2^2 + \|\nu^{-\frac{3}{2}}(I - \mathbf{P}) g\|_2^2 + \epsilon^{-1} |r|_{2, -}^2 \right) \right]
+ 2\epsilon^{-1} \int_0^t \int_{\gamma} |f^{l+1}|^2 d\gamma
\]
\[
\leq \frac{1 - \eta^{l+1}}{1 - \eta} \left[ \|f_0\|_2^2 + C_j \int_0^t \left( (1 - \epsilon)^2 \|f^l\|_2^2 + \epsilon^{-2} \|\mathbf{P} g\|_2^2 + \|\nu^{-\frac{3}{2}}(I - \mathbf{P}) g\|_2^2 + \epsilon^{-1} |r|_{2, -}^2 \right) \right]
+ 2\epsilon^{l+1} - \epsilon^l |f_0|_{2, +}^2 t.
\]
where $\eta := (1 - \frac{1}{t}) < 1$ and we iterated on $2\varepsilon^{-1} \int_0^t |f^l|^2_2$. It follow that

$$
\max_{1 \leq i \leq l+1} \|f^i(t)\|_2^2 \leq C_j \varepsilon^{-2} \int_0^t \max_{1 \leq i \leq l+1} |f^i|_2^2 + C_j ((\varepsilon^{-2} t + 1)\|f_0\|_2^2 + \varepsilon^{-1} |f_0|_2^2 t)
+ C_j \int_0^t (\varepsilon^{-2} \|P g\|_2^2 + \|\nu^{-\frac{1}{2}}(I - P) g\|_2^2 + \varepsilon^{-1} |r|_2^2) \mathrm{d}t.
$$

For fixed $\varepsilon \ll 1$, $j \in \mathbb{N}$ and $t > 0$, Gronwall inequality implies the uniform boundedness of $f^i$:

$$
\max_{1 \leq i \leq l+1} \|f^i(t)\|_2^2 \leq C_{j, \varepsilon, t} \left[ \int_0^t (\varepsilon^{-2} \|P g\|_2^2 + \|\nu^{-\frac{1}{2}}(I - P) g\|_2^2 + \varepsilon^{-1} |r|_2^2) \mathrm{d}t \right] + (\varepsilon^{-2} t + 1)\|f_0\|_2^2 + \varepsilon^{-1} |f_0|_2^2.
$$

(3.39)

Returning back to (3.37) we get

$$
\max_{l \geq 1} \left\{ \|f^i(t)\|_2^2 + \varepsilon^{-2} \int_0^t \|f^i(s)\|_2^2 + 2 \varepsilon^{-1} \int_0^t |f^i(s)|_2^2 \right\}
\leq C_{j, \varepsilon, t} \left[ \int_0^t (\varepsilon^{-2} \|P g\|_2^2 + \|\nu^{-\frac{1}{2}}(I - P) g\|_2^2 + \varepsilon^{-1} |r|_2^2) \mathrm{d}t + (\varepsilon^{-2} t + 1)\|f_0\|_2^2 \right].
$$

(3.38)

This indicates that there exists $f^l \in L^2((0, t) \times \Omega \times \mathbb{R}^3)$ such that, up a subsequence,

$$
f^l \to f_j \quad \text{weakly in } L^2((0, t) \times \Omega \times \mathbb{R}^3) \text{ as } l \to \infty.
$$

To prove strong convergence, we make difference between (3.32) for $l + 1$ and $l$

$$
[\partial_t + \varepsilon^{-1} \nu \cdot \nabla x + \varepsilon \Phi \cdot \nabla v + \varepsilon^{-2} P] (f^{l+1} - f^l) = \varepsilon^{-2} K (f^l - f^{l-1}),
$$

(3.40)

$$
(f^{l+1} - f^l)|_{t=0} = 0.
$$

Similarly as the first inequality of (3.37), we have

$$
\left\|f^{l+1} - f^l\right\|_2^2 + \varepsilon^{-2} \int_0^t \left\|f^{l+1} - f^l\right\|_2^2 + \varepsilon^{-1} \int_0^t |f^{l+1} - f^l|_2^2
\leq C_{\eta} \varepsilon^{-2} \int_0^t \left\|f^l - f^{l-1}\right\|_2^2 + \eta \varepsilon^{-1} \int_0^t |f^l - f^{l-1}|_2^2 + \eta
\leq \varepsilon^{-2} C_{\eta} T_* \sup_{0 \leq s \leq T_*} \|f^l(s) - f^{l-1}(s)\|_2^2 + \eta \varepsilon^{-1} \int_0^t |f^l - f^{l-1}|_2^2,
$$

(3.41)

for $t \in [0, T_*]$ and $\eta = 1 - \frac{1}{t}$. For fixed $\varepsilon \ll 1$ and $j \in \mathbb{N}$, we choose $T_*$ sufficiently small such that $C_{\eta} \varepsilon^{-2} T_* < 1$. Define $k_* := \max\{C_{\eta} \varepsilon^{-2} T_*, \eta\} < 1$. It follows that

$$
\sup_{0 \leq s \leq T_*} \left\|f^{l+1}(s) - f^l(s)\right\|_2^2 + \varepsilon^{-1} \int_0^t |f^{l+1} - f^l|_2^2
\leq k_* \left[ \sup_{0 \leq s \leq T_*} \|f^l(s) - f^{l-1}(s)\|_2^2 + \varepsilon^{-1} \int_0^t |f^l - f^{l-1}|_2^2 \right].
$$

(3.42)

This means that

$$
\left\{ \sup_{0 \leq s \leq T_*} \left\|f^{l+1}(s) - f^l(s)\right\|_2^2 + \varepsilon^{-1} \int_0^t |f^{l+1} - f^l|_2^2 \right\}_{t=0}^\infty
$$
forms a contraction sequence on local interval $[0, T_*]$. Repeating the argument for $[T_*, 2T_*], [2T_*, 3T_*], \cdots$, we deduce that for finite time $t$,

$$
f^l \to f^i \quad \text{strongly in } L^2((0, t) \times \Omega \times \mathbb{R}^3) \text{ as } l \to \infty.
$$

Step 2. Consider the process $j \to \infty$. 

It follows from Step 1 that the limit of (3.32) (as \( t \to \infty \)) reads as

\[
\partial_t f^j + \varepsilon^{-1} v \cdot \nabla_x f^j + \varepsilon \Phi \cdot \nabla_x f^j \frac{1}{2} \varepsilon (\Phi \cdot v) f^j + \varepsilon^{-2} L f^j = \varepsilon^{-1} g,
\]

\[
f^j|_{\gamma^-} = (1 - \frac{1}{j}) [(1 - \alpha) \mathcal{L} f^j + \alpha P_j f^j] + \alpha r,
\]

(3.43)

\[
f^j|_{t=0} = f_0.
\]

Similarly as (3.33), \( L^2 \) energy estimate leads to

\[
\| f^j(t) \|_2^2 + \varepsilon^{-2} \int_0^t \| (I - P) f^j \|_2^2 + \alpha \varepsilon^{-1} \int_0^t \| (1 - P_j) f^j \|_{2,+}^2 + \alpha \varepsilon^{-1} \int_0^t \| P f^j \|_2^2 + \alpha \varepsilon^{-1} \int_0^t \| P f^j \|_{2,+}^2 + \alpha \varepsilon^{-1} \int_0^t \| P f^j \|_{2,+}^2
\]

\[
\leq \| f_0 \|_2^2 + \varepsilon^2 \| \Phi \|_\infty \int_0^t \| f^j \|_2^2 + \beta \int_0^t \left( \| P f^j \|_2^2 + \alpha \varepsilon^{-1} \| P f^j \|_{2,+}^2 \right) + C_\beta \int_0^t \left( \alpha \varepsilon^{-1} \| v \|_2^2 + \varepsilon^{-2} \| g \|_2^2 + \| \nu^{-\frac{1}{2}} (I - P) g \|_2^2 \right),
\]

(3.44)

where \( \beta > 0 \) is a small constant and we used (2.71) and (2.72) to control the boundary integration. Trace Lemma (3.2) implies

\[
\int_0^t \| P_J f^j \|_{2,+}^2 \lesssim \int_0^t \int_\gamma \| (f^j)^2 \|_{1,\gamma} \mathrm{d} \gamma + \int_0^t \| (1 - P_j) f^j \|_{2,+}^2 + \varepsilon \int_0^t \| (f^j)^2 \|_2^2 + \varepsilon \int_0^t \| \varepsilon \partial_x + v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla_x \| (f^j)^2 \|_2^2 + \int_0^t \| (1 - P_j) f^j \|_{2,+}^2
\]

\[
\lesssim \varepsilon \| f_0 \|_2^2 + \varepsilon \int_0^t \| f^j \|_2^2 + \varepsilon \int_0^t \| (I - P_j) f^j \|_2^2 + \varepsilon \int_0^t \| g \|_2^2 + \varepsilon \int_0^t \| P f^j \|_2^2 + \varepsilon \int_0^t \| (1 - P_j) f^j \|_{2,+}^2 + \varepsilon \int_0^t \| P f^j \|_{2,+}^2 + \varepsilon \int_0^t \| P f^j \|_{2,+}^2 + \varepsilon \int_0^t \| P g \|_2^2 + \varepsilon \int_0^t \| \nu^{-\frac{1}{2}} (I - P) g \|_2^2,
\]

(3.45)

where we used (3.43). Employing the macroscopic estimate (3.41) on (3.33), we get

\[
\int_0^t \| P f^j \|_2^2 \lesssim \int_0^t \left( \varepsilon^{-2} \| (I - P) f^j \|_2^2 + \alpha^2 \| (1 - P_j) f^j \|_{2,+}^2 + \alpha^2 \| v \|_{2,-}^2 + \| \nu^{-\frac{1}{2}} g \|_{2}^2 \right) + \varepsilon \| f^j(t) \|_2^2 - \varepsilon \| f_0 \|_2^2.
\]

(3.46)

Multiply (3.45) with \( \beta \alpha \varepsilon^{-1} \) and (3.46) with \( \rho \), then add to (3.45),

\[
(1 - \rho \varepsilon) \| f^j(t) \|_2^2 + (1 - \beta \alpha - \rho) \varepsilon^{-2} \int_0^t \| (I - P) f^j \|_2^2 + (\rho - \beta - \beta \alpha) \int_0^t \| P f^j \|_2^2
\]

\[
+ (1 - \beta - \rho \varepsilon) \alpha \varepsilon^{-1} \int_0^t \| (1 - P_j) f^j \|_{2,+}^2 + C_{\rho, \beta} \left( \| f_0 \|_2^2 + \int_0^t \left( \alpha \varepsilon^{-1} \| v \|_{2,-}^2 + \varepsilon^{-2} \| P g \|_2^2 + \| \nu^{-\frac{1}{2}} (I - P) g \|_2^2 \right) + (\beta \alpha + \varepsilon^2 \| \Phi \|_\infty) \int_0^t \| f^j \|_{2,+}^2 \right)
\]

(3.47)

Choose \( \rho = 8 \beta > 0 \) small enough (independent of \( \varepsilon \)) such that

\[
1 - \rho \varepsilon \geq \frac{1}{2}, \quad 1 - \beta \alpha - \rho (\beta \alpha + \varepsilon^2 \| \Phi \|_\infty), \quad \frac{1}{2}, \quad 1 - \beta - \rho \varepsilon \geq \frac{1}{2}, \quad \rho - \beta - \beta \alpha - (\beta \alpha + \varepsilon^2 \| \Phi \|_\infty) \geq \frac{\rho}{2}.
\]
Then we get the $L^2$ energy estimate
\[
\|f^j(t)\|_{L^2}^2 + \int_0^t (\varepsilon^{-2} \|f^j\|_{L^2}^2 + \|P f^j\|_{L^2}^2 + \alpha \varepsilon^{-1} |(1 - P_\gamma) f^j|_{H^2}^2) dt \\
\lesssim \|f_0\|_{L^2}^2 + \int_0^t (\varepsilon^{-2} \|P g\|_{L^2}^2 + \|\nu^{-\frac{1}{2}} (I - P) g\|_{L^2}^2 + \alpha \varepsilon^{-1} |r|_{H^2}^2). \tag{3.48}
\]
This uniform boundedness of $f^j$ indicates that there exists $f \in L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3)$ such that, up a subsequence,
\[ f^j \to f \quad \text{weakly in } L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3) \quad \text{as } j \to \infty. \]
Furthermore, we claim that for $0 < \alpha \leq 1$,
\[ f^j \to f \quad \text{strongly in } L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3) \quad \text{as } j \to \infty. \tag{3.49} \]
In fact, by replacing the factor $1 - \frac{\alpha}{2}$ with $1 - \frac{\alpha}{2}$ in (3.52) and deducing Step 1 again, we can still obtain (3.48). With this factor $1 - \frac{\alpha}{2}$, we make difference between (3.48) and its limit
\[
\left[ \partial_t + \varepsilon^{-1} v \cdot \nabla_x + \varepsilon \Phi \cdot \nabla_v - \frac{1}{2} \Phi (v) - \varepsilon^{-2} L \right] (f^j - f) = 0,
\]
\[ (f^j - f) \mid_{\gamma_+} = (1 - \frac{\alpha}{2}) \left[ (1 - \alpha) \mathcal{L} (f^j - f) + \alpha P_\gamma (f^j - f) \right] - \frac{\alpha}{2} \left[ (1 - \alpha) \mathcal{L} f + \alpha P_\gamma f \right], \tag{3.50} \]
\[ (f^j - f) \mid_{t=0} = 0. \]
By using the mean value inequality for the cross term and (2.71), we get
\[
\int_{\gamma_+} \left| f^j - f \right|^2 d\gamma \
\leq \left(1 - \frac{\alpha}{2}\right) \int_{\gamma_+} \left[ (1 - \alpha) \mathcal{L} (f^j - f) + \alpha P_\gamma (f^j - f) \right]^2 d\gamma + \frac{\alpha}{2} \int_{\gamma_+} \left[ (1 - \alpha) \mathcal{L} f + \alpha P_\gamma f \right]^2 d\gamma 
\leq \left[ (1 - \alpha)^2 (1 - P_\gamma) (f^j - f)^2 \right]_{L^2} + \left[ |P_\gamma f|^2 \right]_{L^2}.
\]
Combining this with the $L^2$ estimate of (3.50), which can be derived similarly as (3.44), we get
\[
\|f^j - f\|_{L^2}^2 + \varepsilon^{-2} \int_0^t \| (I - P) (f^j - f) \|_{L^2}^2 + \alpha \varepsilon^{-1} \int_0^t |(1 - P_\gamma) (f^j - f)|_{H^2}^2 dt \\
\lesssim \varepsilon^2 \Phi \|f\|_{L^\infty} \int_0^t \|f^j - f\|_{L^2}^2 + \frac{\alpha}{2} \int_0^t \left| (1 - P_\gamma) f^j \right|_{L^2}^2 + |P_\gamma f|_{L^2}^2. \tag{3.51}
\]
Employ (3.4) on (3.50),
\[
\int_0^t \|P (f^j - f)\|_{L^2}^2 \lesssim \varepsilon^{-2} \int_0^t \| (I - P) (f^j - f) \|_{L^2}^2 + \alpha \varepsilon^{-1} \int_0^t |(1 - P_\gamma) (f^j - f)|_{H^2}^2 dt \\
+ \frac{\alpha}{2} \int_0^t \left| (1 - P_\gamma) f^j \right|_{L^2}^2 + |P_\gamma f|_{L^2}^2 + \varepsilon \|f^j - f\|_{L^2}^2.
\]
Besides, from (3.48) and Lemma 3.2 we can show the uniform boundedness of $\frac{\alpha}{2} \int_0^t \left| (1 - P_\gamma) f^j \right|_{L^2}^2 + |P_\gamma f|_{L^2}^2$. Thus (3.49) follows readily from the above two estimates.

If $\alpha = 0$ then the boundary integration vanishes on the left hand side of (3.1). Note that this lack of trace $f_\gamma$ may not guarantee uniqueness of solutions within the $L^2$ framework. But this can be remedied by showing $\int_0^t |f^j|^2 d\gamma < \infty$ with the help of $L^\infty$ estimate given in Theorem 5.6 with property $w f \in L^\infty(\Omega) \cap L^\infty(\gamma)$, cf. the corresponding statement in page 758 of [27].

**Step 3.** Proof of the decay estimate (3.31).
Let $h = e^M f$ for $0 < \lambda \ll 1$. Then
\[
\varepsilon \partial_t h + v \cdot \nabla_x h + \varepsilon^2 \Phi \cdot \nabla_v h - \frac{1}{2} \varepsilon^2 (\Phi \cdot v) h + \varepsilon^{-1} L h = g e^M + \lambda \varepsilon h,
\]
\[
h|_{\gamma_-} = (1 - \alpha) \mathcal{L} h + \alpha P h + \alpha v e^M,
\]
\[
h|_{t=0} = f_0.
\]

It follows from standard energy estimate that
\[
\|h(t)\|_{L^2}^2 + \varepsilon^2 \int_0^t \| (I - P) h \|_{L^2}^2 + \varepsilon^{-1} \int_0^t | (1 - P_{\gamma}) h |_{L^2}^2 + \lambda \varepsilon \int_0^t \| h \|_{L^2}^2 + \varepsilon^2 \int_0^t \| (I - P) h \|_{L^2}^2
\]
\[
\leq \| f_0 \|_{L^2}^2 + \lambda \int_0^t \| h \|_{L^2}^2 + \varepsilon^2 \| \Phi \|_{L^\infty} \int_0^t \| h \|_{L^2}^2 + \varepsilon \int_0^t \| (P h) \|_{L^2}^2 + \varepsilon^{-1} \| P_{\gamma} h \|_{L^2}^2 + C_\eta \int_0^t (\varepsilon^2 - |P| e^{\lambda s} g|_{L^2}^2 + \| \nu^{-\frac{1}{2}} (I - P) | e^{\lambda s} g|_{L^2}^2).
\]

From (3.1) and (3.3) we know that
\[
\begin{align*}
\int_{\Omega \times \mathbb{R}^3} (g e^M + \lambda \varepsilon h) \sqrt{\mu} \text{div} \tilde{x} = 0, & \quad \int_{\gamma_+} \mu e^{\lambda t} \sqrt{\mu} \text{d} \gamma = 0.
\end{align*}
\]

Employing (3.2) on (3.51), we get
\[
\int_0^t \| P h \|_{L^2}^2 \lesssim \varepsilon \| h \|_{L^2}^2 + \varepsilon \int_0^t \| h \|_{L^2}^2 + \varepsilon \int_0^t (\varepsilon^2 - |I - P| h|_{L^2}^2 + \varepsilon \| (I - P_{\gamma}) h \|_{L^2}^2 + \lambda \varepsilon \int_0^t \| h \|_{L^2}^2 + \varepsilon^2 \int_0^t \| (I - P) h \|_{L^2}^2)
\]
\[
\hspace{1cm} + \varepsilon \int_0^t \| (I - P) h \|_{L^2}^2 + \varepsilon \| (I - P) h \|_{L^2}^2 + \varepsilon^2 \| (I - P) h \|_{L^2}^2 + \varepsilon \| (I - P) h \|_{L^2}^2.
\]

Similarly as (3.45), we use the trace Lemma 3.2 and get
\[
\int_0^t \| P_{\gamma} h \|_{L^2}^2 \lesssim \varepsilon \| f_0 \|_{L^2}^2 + \varepsilon \int_0^t \| h \|_{L^2}^2 + \varepsilon \int_0^t \| h \|_{L^2}^2 + \varepsilon \int_0^t (\varepsilon^{-1} - |I - P| h|_{L^2}^2 + \varepsilon \| (I - P_{\gamma}) h \|_{L^2}^2 + \lambda \varepsilon \int_0^t \| h \|_{L^2}^2 + \varepsilon^2 \int_0^t \| (I - P) h \|_{L^2}^2)
\]
\[
\hspace{1cm} + \varepsilon \int_0^t \| (I - P) h \|_{L^2}^2 + \varepsilon \| (I - P) h \|_{L^2}^2 + \varepsilon^2 \| (I - P) h \|_{L^2}^2 + \varepsilon \| (I - P) h \|_{L^2}^2.
\]

Combine (3.42), (3.43) and (3.51) and derive similarly as (3.45)
\[
\|h(t)\|_{L^2}^2 + \int_0^t (\varepsilon^2 - |I - P| h|_{L^2}^2 + \varepsilon^2 \| (I - P) h \|_{L^2}^2 + \varepsilon \| \nu^{-\frac{1}{2}} (I - P) | e^{\lambda s} g|_{L^2}^2)
\]
\[
\lesssim \| f_0 \|_{L^2}^2 + \int_0^t (\varepsilon \| e^{\lambda s} g|_{L^2}^2 + \varepsilon^{-2} \| (I - P) | e^{\lambda s} g|_{L^2}^2 + \| \nu^{-\frac{1}{2}} (I - P) | e^{\lambda s} g|_{L^2}^2)
\]
for small $0 < \lambda \ll 1$.

Furthermore, we first apply Lemma 3.2 to estimate $\alpha \varepsilon^{-1} \int_0^t |P_{\gamma} h|_{L^2}^2$, like (3.45), then use boundary condition to treat $\alpha \varepsilon^{-1} \int_0^t |h|_{L^2}^2$. Finally we get (3.51). This completes the proof. \qed

Lemma 3.4. Let $g \in L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3)$, $r \in L^2(\mathbb{R}_+ \times \gamma_-)$, $f_0 \in L^2(\Omega \times \mathbb{R}^3)$. Let $f \in L^\infty(\mathbb{R}_+ ; L^2(\Omega \times \mathbb{R}^3)) \cap L^2(\mathbb{R}_+ ; L^2(\Omega \times \mathbb{R}^3))$ and solve
\[
\varepsilon \partial_t f + v \cdot \nabla_x f + \varepsilon^2 \Phi \cdot \nabla_v f = g
\]
in the sense of distribution and satisfy $f(0, x, v) = f_0(x, v)$ and
\[
f|_{\gamma_-} = (1 - \alpha) \mathcal{L} f + \alpha P f + \alpha v.
\]

Then there exist $S_1 f(t, x), S_2 f(t, x)$ and $S_3 f(t, x)$ satisfying
\[
|a(t, x)| + |b(t, x)| + |c(t, x)| \leq S_1 f(t, x) + S_2 f(t, x) + S_3 f(t, x),
\]
and
\[
\|S_3 f\|_{L^2}^2 \lesssim \| (I - P) f\|_{L^2}^2.
\]
\begin{align}
\|S_1 f\|_{L_t^2 L_x^2} + \varepsilon^{\frac{1}{4}} \|S_2 f\|_{L_t^2 L_x^2} & \lesssim \|\nu^\frac{1}{2} f\|_{L_t^2 L_x^2} + \alpha (1 - P_\gamma) f|_{L_t^2 L_x^2} + \|\nu^{-\frac{1}{4}} g\|_{L_t^2 L_x^2} \\
& \quad + \|f_0\|_{L_t^2 L_x^2} + \|\nu \cdot \nabla_x f_0 + \varepsilon^2 \Phi \cdot \nabla_x f_0\|_{L_t^2 L_x^2} + \alpha |r|_{L_t^2 L_x^2}.
\end{align}

(3.60)

**Proof.** For \((t, x, v) \in \mathbb{R} \times \tilde{\Omega} \times \mathbb{R}^3\) and \(0 < \delta \ll 1\), we define

\[
f_\delta(t, x, v) := [1 - \chi(\frac{n(x) \cdot v}{\delta}) \chi \left(\frac{|v|}{\delta} \right)] [1 - \chi \left(\frac{|\delta|}{\delta} \right)] \chi(\delta |v|) \times \left[1_{t \in [0, \infty)} f(t, x, v) + 1_{t \in (-\infty, 0]} \chi(t) f_0(x, v)\right],
\]

(3.61)

where \(n(x)\) is given in (1.53) and \(\chi \in C_0^\infty(\mathbb{R})\) is defined by

\[
0 \leq \chi \leq 1, \quad \chi'(x) \geq -4 \times 1_{\frac{1}{2} \leq |x| \leq 1} \quad \text{and} \quad \chi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{4}, \\
0 & \text{if } |x| \geq 1. \end{cases}
\]

We extended \(f_\delta\) to the negative time so that we are able to take the time derivative. Note that the definition of \(f_\delta\) includes the factor \(1 - \chi(\frac{|v|}{\delta})\). Thus, \(f_\delta(t, x, v)\) vanishes on the whole near-grazing set

\[
f_\delta(t, x, v) \equiv 0 \quad \text{for} \quad (x, v) \in \gamma_{\delta} \backslash \gamma_{\delta}^4, \quad (3.62)
\]

where \(\gamma_{\delta}^4\) is defined in (1.76). Clearly,

\[
\|f_\delta\|_{L^2(\mathbb{R} \times \tilde{\Omega} \times \mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R} \times \tilde{\Omega} \times \mathbb{R}^3)} + \|f_0\|_{L^2(\tilde{\Omega} \times \mathbb{R}^3)} ,
\]

\[
\|f_\delta\|_{L^2(\mathbb{R} \times \gamma)} \lesssim \|f_1\gamma_{\delta}^4\|_{L^2(\mathbb{R} \times \gamma)} + \|f_0\gamma_{\delta}^4\|_{L^2(\gamma)} .
\]

(3.63)

Existence of \(S_t f\) and estimate (3.59) follow from Proposition 3.4 in [17]. Moreover, similarly as the proof of Lemma 3.6 and Proposition 3.4 in [17], we have

\[
\|S_1 f\|_{L_t^2 L_x^2} + \varepsilon^{\frac{1}{4}} \|S_2 f\|_{L_t^2 L_x^2} \lesssim \|w^{-1} g\|_{L_t^2 L_x^2} + \|f_0\|_{L_t^2 L_x^2} + \|v \cdot \nabla_x f_0 + \varepsilon^2 \Phi \cdot \nabla v f_0\|_{L_t^2 L_x^2} \\
+ \|f_1\gamma_{\delta}^4\|_{L^2(\mathbb{R} \times \gamma)} + \|f_0\gamma_{\delta}^4\|_{L^2(\gamma)}. \quad (3.64)
\]

Note that the boundary term \(|f_1\gamma_{\delta}^4\|_{L^2(\mathbb{R} \times \gamma)}\) comes from the definition of \(f_\delta\), (3.62) and (3.63).

In the following, we estimate \(\|f_0\gamma_{\delta}^4\|_{L^2(\gamma)}\) and \(\|f_1\gamma_{\delta}^4\|_{L^2(\mathbb{R} \times \gamma)}\) furthermore.

To bound \(|f_0\gamma_{\delta}^4\|_{L^2(\gamma)}\), we use Lemma 2.2 on both of the out-going and in-coming non-grazing set

\[
|f_0\gamma_{\delta}^4|_{L^2(\gamma)}^2 \lesssim \|f_0\|_{L_t^2 L_x^2} + \|v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla v\|_{L_t^1 L_x^2} \|f_0\|_{L_t^2 L_x^2} \\
\lesssim \|f_0\|_{L_t^2 L_x^2} + \|v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla v\| f_0 \|_{L_t^2 L_x^2} . \quad (3.65)
\]

To estimate \(|f_1\gamma_{\delta}^4|_{L^2(\mathbb{R} \times \gamma)}\), we use Lemma 3.2 on the out-going non-grazing set \(\gamma_{\delta}^4\)

\[
\int_0^t \int_{\gamma_{\delta}^4} |f_1\gamma_{\delta}^4|^2 d\gamma \lesssim \|f_0\|_1 + \varepsilon \int_0^t \|f_2\|_1 + \int_0^t \|\varepsilon \partial_t + v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla v\| (f_2^2) \|_1 \\
\lesssim \|f_0\|_2 + \varepsilon \int_0^t \|f_2\|_2 + \int_0^t \|f_0\|_2 + \int_0^t \|\nu^{-\frac{1}{2}} g\|_2 . \quad (3.66)
\]
For \( f, 1_{\delta} \in L^2(\mathbb{R}^3) \), where trace lemma does not hold true anymore on \( \gamma_\delta \), we use boundary condition \( \mathcal{B} \) and the change of variable \( v \mapsto R_x v \) on \( \gamma_\delta \\
\int_0^t \int_{\gamma_\delta} |f 1_{\delta}|^2 \gamma \\leq \int_0^t \int_{\gamma_\delta} |L f|^2 \gamma + \alpha^2 \int_0^t \int_{\gamma_\delta} |P_\gamma f|^2 \gamma + \alpha^2 \int_0^t \int_{\gamma_\delta} |r|^2 \gamma \\
\leq \int_0^t \int_{\gamma_\delta} |f|^2 \gamma + \alpha^2 \int_0^t \int_{\gamma_\delta} |(1 - P_\gamma f)^2 \gamma \gamma + \alpha^2 \int_0^t \int_{\gamma_\delta} |\gamma|^2 \gamma \rfloor_2 + \bar{\epsilon} \begin{cases} \leq 2 + \int_0^t \parallel f \parallel_2^2 \quad + \int_0^t \parallel \gamma \parallel_2^2 + \int_0^t \parallel \gamma \parallel_2^2 \end{cases} \\
\leq \delta, \text{ we get } (3.60). \text{ This completes the proof.} \quad \Box \n
3.2. \textbf{L}_\infty \textbf{ estimate.}

In this subsection, we give the \( L^\infty \) estimate of unsteady linear problem.

\textbf{Theorem 3.5.} Suppose that \( f \) satisfies
\[
\left[ \varepsilon \partial_t + v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla_v + \varepsilon^{-1} \hat{C}(v) \right] \parallel f \parallel \leq \varepsilon^{-1} K_\beta \parallel f \parallel + |g| \quad \text{in } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3, \\
| f |_{\gamma_\delta} \leq (1 - \alpha) |L f | + \alpha P_\gamma |f | + \alpha |r | \quad \text{in } \mathbb{R}^+ \times \partial \Omega \times \mathbb{R}^3, \\
| f |_{t=0} \leq | f_0 | \quad \text{in } \Omega \times \mathbb{R}^3.
\]
Then
\[
\parallel w f(t) \parallel_\infty \leq \parallel w f_0 \parallel_\infty + \varepsilon^{-\frac{1}{2}} \sup_{0 \leq s \leq t} \parallel Pf(s) \parallel_6 + \varepsilon^{-\frac{3}{2}} \sup_{0 \leq s \leq t} \parallel (I - P)f(s) \parallel_2 \leq \alpha \sup_{0 \leq s \leq t} \parallel w r(s) \parallel_\infty + \varepsilon \sup_{0 \leq s \leq t} \parallel \langle v \rangle^{-1} w g(s) \parallel_\infty, \quad (3.69)
\]
and
\[
\parallel w f(t) \parallel_\infty \leq \parallel w f_0 \parallel_\infty + \varepsilon^{-\frac{3}{2}} \sup_{0 \leq s \leq t} \parallel f(s) \parallel_2 + \alpha \sup_{0 \leq s \leq t} \parallel w r(s) \parallel_\infty + \varepsilon \sup_{0 \leq s \leq t} \parallel \langle v \rangle^{-1} w g(s) \parallel_\infty. \quad (3.70)
\]
To prove Theorem 3.5 similarly as the steady case, we stretch the space and time variables simultaneously through \((1.63)\) and \((1.65)\). Then \((3.68)\) is transformed into the equivalent problem
\[
\left[ \partial_t + v \cdot \nabla_v + \varepsilon^2 \hat{\Phi} \cdot \nabla_v + \hat{C}(v) \right] \parallel f \parallel \leq K_\beta \parallel f \parallel + \varepsilon \hat{g} \quad \text{in } \mathbb{R}^+ \times \Omega_\varepsilon \times \mathbb{R}^3, \\
| f |_{\gamma_\varepsilon} \leq (1 - \alpha) |L f | + \alpha P_\varepsilon |f | + \alpha |r | \quad \text{in } \mathbb{R}^+ \times \partial \Omega_\varepsilon \times \mathbb{R}^3, \\
| f |_{t=0} \leq | f_0 | \quad \text{in } \Omega_\varepsilon \times \mathbb{R}^3, \quad (3.71)
\]
where we used \((1.67)\) and \( f_0(y, v) := f_0(x, v), \hat{g}(\bar{t}, y, v) := g(t, x, v) \) and \( \hat{r}(\bar{t}, y, v) := r(t, x, v) \). We have the following estimates for \( \hat{f} \).

\textbf{Lemma 3.6.} Suppose that \( \hat{f} \) satisfies \((3.71)\). Then for \( \bar{t} \in [0, T_0] \) with some large \( T_0 \geq 1 \),
\[
\parallel \hat{w} \hat{f}(\bar{t}) \parallel_{L^\infty_{\sqrt{\gamma}}} \leq \varepsilon^{-\frac{3}{2}} \parallel w f_0 \parallel_{L^\infty_{\sqrt{\gamma}}} + o(1) \sup_{0 \leq s \leq t} \parallel \hat{w} \hat{f}(s) \parallel_{L^\infty_{\sqrt{\gamma}}} \\
+ \sup_{0 \leq s \leq t} \parallel \hat{P} \hat{f}(s) \parallel_{L^\infty_{\sqrt{\gamma}}} + \sup_{0 \leq s \leq t} \parallel (I - P) \hat{f}(s) \parallel_{L^\infty_{\sqrt{\gamma}}} \leq \alpha \sup_{0 \leq s \leq t} \parallel \hat{w} \hat{r}(s) \parallel_{L^\infty_{\sqrt{\gamma}}} + \varepsilon \sup_{0 \leq s \leq t} \parallel \hat{r}(s) \parallel_{L^\infty_{\sqrt{\gamma}}}. \quad (3.72)
\]

\[\]
and
\[ \| w\hat{f}(\bar{t}) \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})} \lesssim e^{-\frac{\epsilon}{2} \tau} \| w\hat{f}_{0} \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})} + o(1) \sup_{0 \leq s \leq \bar{t}} \| w\hat{f}(\bar{s}) \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})} + \alpha \sup_{0 \leq s \leq \bar{t}} \| w\hat{r}(\bar{s}) \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})} + \varepsilon \sup_{0 \leq s \leq \bar{t}} \| (v)^{-1}w\hat{g}(\bar{s}) \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})}. \] (3.73)

Proof. Define
\[ h(\bar{t}, y, v) := w(v)\hat{f}(\bar{t}, y, v). \] (3.74)

Deriving similarly as (2.107), we get
\[ \[ \partial_{\bar{t}} + v \cdot \nabla_{y} + \varepsilon^{2} \Phi \cdot \nabla_{v} + \tilde{C}(\bar{v}) \] |h| \leq \int_{\mathbb{R}^{3}} k_{\bar{y}}(v, u) h(u) du + \varepsilon |w\hat{g}|, \]
\[ |h(\bar{t}, y, v)|_{\gamma_{\epsilon}} \lesssim (1 - \alpha)|h(\bar{t}, y, R_{y}(v))| + \alpha \frac{1}{\hat{w}(v)} \int_{n(y) > 0} |h(\bar{t}, y, u)| \hat{w}(u) d\sigma + \alpha |w\hat{r}|, \] (3.75)
\[ |h|_{\bar{t}=0} \leq |h_{0}|. \]

where \( \hat{w}(v) \) and \( d\sigma \) are the same as (2.108).

We claim that, for any given \((\bar{t}, y, v) \in [0, T_{0}] \times \Omega_{\epsilon} \times \mathbb{R}^{3}\) satisfying \( y \notin \gamma_{0} \) or \( v \notin S_{y}(v) \),
\[ |h(\bar{t}, y, v)| \lesssim e^{-\frac{\epsilon}{2} \tau} \| h_{0} \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})} + o(1) \sup_{0 \leq s \leq \bar{t}} \| h(\bar{s}) \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})} + \alpha \sup_{0 \leq s \leq \bar{t}} \| \tilde{C}(\bar{v}) \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})} \] (3.76)
\[ + \alpha \sup_{0 \leq s \leq \bar{t}} \| (v)^{-1}w\hat{g}(\bar{s}) \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})} \]
and
\[ |h(\bar{t}, y, v)| \lesssim e^{-\frac{\epsilon}{2} \tau} \| h_{0} \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})} + o(1) \sup_{0 \leq s \leq \bar{t}} \| h(\bar{s}) \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})} + \alpha \sup_{0 \leq s \leq \bar{t}} \| \tilde{C}(\bar{v}) \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})} \] (3.77)
\[ + \alpha \sup_{0 \leq s \leq \bar{t}} \| (v)^{-1}w\hat{g}(\bar{s}) \|_{L_{\infty}^{m}(\Omega_{\epsilon} \times \mathbb{R}^{3})}. \]

Once (3.76) and (3.77) are proved, using (3.74) and taking \( L_{\infty}^{m} \), on both sides of (3.76) and (3.77), we get (3.72) and (3.73).

It remains to prove the claims (3.76) and (3.77). Note that (3.76) and (2.107) enjoy the same trajectory (1.58). Thus, (2.111) still holds for (3.76), and along the backward trajectory,
\[ |h(\bar{t}, y, v)| \leq \tilde{J}_{0}(\bar{t}, y, v) + \tilde{J}_{k}(\bar{t}, y, v) + \tilde{J}_{g}(\bar{t}, y, v) + \tilde{J}_{r}(\bar{t}, y, v) + \tilde{J}_{sp}(\bar{t}, y, v) + \tilde{J}_{d}(\bar{t}, y, v), \] (3.78)

where
\[ \tilde{J}_{0}(\bar{t}, y, v) = \int_{\{ t_{1} \leq 0 \}} e^{-\int_{0}^{t_{1}} \tilde{C}(V(\tau; \bar{t}, y, v)) d\tau} |h_{0}(0, Y(0; \bar{t}, y, v), V(0; \bar{t}, y, v))|, \]
\[ \tilde{J}_{k}(\bar{t}, y, v) = \int_{\max(0, t_{1})}^{\bar{t}} d\bar{s} e^{-\int_{\bar{s}}^{\bar{t}} \tilde{C}(V(\tau; \bar{t}, y, v)) d\tau} \int_{\mathbb{R}^{3}} \hat{f}_{\bar{y}}(V(\bar{s}; \bar{t}, y, v), u) \]
\[ \times |h(\bar{s}, Y(\bar{s}; \bar{t}, y, v), u)|, \]
\[ \tilde{J}_{g}(\bar{t}, y, v) = \int_{\max(0, t_{1})}^{\bar{t}} d\bar{s} e^{-\int_{\bar{s}}^{\bar{t}} \tilde{C}(V(\tau; \bar{t}, y, v)) d\tau} \int_{\mathbb{R}^{3}} \hat{g}(\bar{s}, Y(\bar{s}; \bar{t}, y, v), V(\bar{s}; \bar{t}, y, v)) \]
\[ \times |\varepsilon(w\hat{g})(\bar{s}, Y(\bar{s}; \bar{t}, y, v), V(\bar{s}; \bar{t}, y, v))|, \] (3.79)
\[ \tilde{J}_{r}(\bar{t}, y, v) = \int_{\{ t_{1} > 0 \}} e^{-\int_{t_{1}}^{t_{1}} \tilde{C}(V(\tau; y, v)) d\tau} \alpha |(w\hat{r})(t_{1}, y_{1}, V(t_{1}; \bar{t}, y, v))|, \]
\[ \tilde{J}_{sp}(\bar{t}, y, v) = \int_{\{ t_{1} > 0 \}} e^{-\int_{t_{1}}^{t_{1}} \tilde{C}(V(\tau; y, v)) d\tau} (1 - \alpha) |h(t_{1}, y_{1}, v_{1})|, \]
\[ \tilde{J}_{d}(\bar{t}, y, v) = \int_{\{ t_{1} > 0 \}} e^{-\int_{t_{1}}^{t_{1}} \tilde{C}(V(\tau; y, v)) d\tau} \alpha |(w\hat{r})(t_{1}, y_{1}, v_{1})| \int_{n(y_{1}) > 0} |h(t_{1}, y_{1}, v_{1})| d\sigma_{1}. \]
where we used
\[ t_1 := \tilde{t} - t_b(y, v), \quad y_1 := Y(t_1; \tilde{t}, y, v) = y_b(y, v), \]
\[ v_1 := R_{y_1}(V(t_1; \tilde{t}, y, v), \quad \partial^*_{v_1} := C_{\mu}(v_1') \left[ n(y_1) \cdot v_1' \right] \, dv_1'. \]

It is not difficult to see that (3.78) and (3.79) have the same expressions as (2.111) and (2.112). Thus we can estimate \( J_0(\tilde{t}, y, v) \sim J_{d_1}(\tilde{t}, y, v) \) in the same way as \( J_0(t, y, v) \sim J_{d_1}(t, y, v) \) in Lemma 2.7 and finally get (3.76). (3.77) is obtained like (3.76), except that we use \( \| \hat{f}(s) \|_{L^2_{x,v}(\Omega, \times \mathbb{R}^3)} \) directly, rather than splitting \( \hat{f} \) into \( P\hat{f} \) and \( (I-P)\hat{f} \), when implementing the change of variable, cf. (2.128–2.130). This completes the proof. \( \square \)

**Proof of Theorem 3.5** Recall (1.63), (1.65) and (1.67). Then for \( \Omega \subseteq \mathbb{R}^3 \) and \( 0 \leq t \leq \varepsilon T_0 \), we have
\[
\sup_{0 \leq s \leq t} \| Pf(s) \|_{L^1_{x,v}(\Omega \times \mathbb{R}^3)} = \| Pf(0) \|_{L^1_{x,v}(\Omega \times \mathbb{R}^3)},
\]
\[
= \varepsilon^{\frac{1}{2}} \| Pf(0) \|_{L^1_{x,v}(\Omega \times \mathbb{R}^3)} + \varepsilon^{\frac{1}{2}} \| (I-P)f(0) \|_{L^1_{x,v}(\Omega \times \mathbb{R}^3)}.
\]

It follows from these relations and Lemma 3.6 that, for \( 0 \leq t \leq \varepsilon T_0 \),
\[
\| w f(t) \|_{L^1_{x,v}(\Omega \times \mathbb{R}^3)} \leq \varepsilon^{\frac{1}{2}} \| w f(0) \|_{L^1_{x,v}(\Omega \times \mathbb{R}^3)} + \varepsilon \sup_{0 \leq s \leq t} \| Pf(s) \|_{L^1_{x,v}(\Omega \times \mathbb{R}^3)} + \varepsilon \sup_{0 \leq s \leq t} \| (I-P)f(s) \|_{L^1_{x,v}(\Omega \times \mathbb{R}^3)} \quad (3.80)
\]
and
\[
\| w f(t) \|_{L^1_{x,v}(\Omega \times \mathbb{R}^3)} \leq \varepsilon^{\frac{1}{2}} \| w f(0) \|_{L^1_{x,v}(\Omega \times \mathbb{R}^3)} + \varepsilon \sup_{0 \leq s \leq t} \| f(s) \|_{L^1_{x,v}(\Omega \times \mathbb{R}^3)} + \varepsilon \sup_{0 \leq s \leq t} \| w r(s) \|_{L^1_{x,v}(\Omega \times \mathbb{R}^3)} \quad (3.81)
\]

Define
\[
D(s) := \varepsilon \sup_{0 \leq s \leq t} \| w f(s) \|_{L^1_{x,v}} + \varepsilon \sup_{0 \leq s \leq t} \| f(s) \|_{L^1_{x,v}} + \varepsilon \sup_{0 \leq s \leq t} \| w r(s) \|_{L^1_{x,v}}
\]

Applying (3.80) successively, we get
\[
\| h(n \varepsilon T_0) \|_{L^1_{x,v}} \leq \varepsilon^{\frac{1}{2}} \| h((n-1) \varepsilon T_0) \|_{L^1_{x,v}} + \sum_{(n-1) \varepsilon T_0 \leq s \leq n \varepsilon T_0} D(s)
\]
\[
\leq \left( e^{-\frac{C_T_0}{n+1}} \right)^2 \| h((n-2) \varepsilon T_0) \|_{L^1_{x,v}} + \sum_{(n-2) \varepsilon T_0 \leq s \leq n \varepsilon T_0} D(s)
\]
\[
\vdots
\]
\[
\leq \left( e^{-\frac{C_T_0}{n+1}} \right)^n \| h_0 \|_{L^1_{x,v}} + \sum_{j=0}^{n-1} \left( e^{-\frac{C_T_0}{n+j}} \right)^j \sup_{0 \leq s \leq n \varepsilon T_0} D(s)
\]
\[
\leq C_1 \| h_0 \|_{L^1_{x,v}} + C_1 \sup_{0 \leq s \leq n \varepsilon T_0} D(s)
\]
for some constant \( C_1 > 0 \), where in the last step we used
\[
e^{-\frac{C_T_0}{n+1}} < \infty, \quad \sum_{j=0}^{n-1} e^{-\frac{C_T_0}{n+j}} < \infty \quad \text{for large } T_0.
\]
Let (3.69). (3.70) can be proved similarly with the help of (3.81). This completes the proof.

We have the following estimate with Theorem 3.3. Then, for \( \lambda \) sufficiently small,

\[
\sup_{0 \leq s \leq t} \| P[e^{\lambda s} f] \|_{L^6_{x,v}} \lesssim \frac{1}{\varepsilon} \| (I - P) f_0 \|_\nu + \frac{\alpha}{\sqrt{\varepsilon}} \| (1 - P_\gamma) f_0 \|_{L^2_x} + \sqrt{\mathcal{D}_\lambda[f](t)} + \varepsilon \sqrt{\delta_\lambda[f](t)}
\]

\[
+ \frac{\alpha}{\sqrt{\varepsilon}} e^\lambda \| s \|_{L^2_x} + \alpha \varepsilon^{\frac{1}{2}} \| \nu^{-1} e^\lambda \|_{L^6_{x,v}}
\]

(3.82)

\[
\sup_{0 \leq s \leq t} \| P[e^{\lambda s} f] \|_{L^6_{x,v}} \lesssim \frac{\varepsilon}{\varepsilon} \| w f(t) \|_{L^\infty_{x,v}} + \frac{1}{\varepsilon} \| (I - P) f_0 \|_\nu + \frac{\alpha}{\sqrt{\varepsilon}} \| (1 - P_\gamma) f_0 \|_{L^2_x}
\]

\[
+ \sqrt{\mathcal{D}_\lambda[f](t)} + \varepsilon \sqrt{\delta_\lambda[f](t)} + \frac{\alpha}{\sqrt{\varepsilon}} e^\lambda \| s \|_{L^2_x} + \alpha \varepsilon^{\frac{1}{2}} \| \nu^{-1} e^\lambda \|_{L^6_{x,v}}
\]

(3.83)

\[
+ \| \nu^{-1} e^\lambda \|_{L^6_{x,v}}^2 + \varepsilon^{\frac{1}{2}} \| (v)^{-1} \|_{L^2_{x,v}}^2 + \varepsilon^{\frac{1}{2}} \| (v)^{-1} \|_{L^2_{x,v}}^2
\]

\[
+ \varepsilon \| (v)^{-1} \|_{L^2_{x,v}}^2 + \varepsilon \| (v)^{-1} \|_{L^2_{x,v}}^2
\]

and employ (2.154),

\[
\| P[e^{\lambda s} f] \|_{L^6_{x,v}} \lesssim \frac{1}{\varepsilon} \| (I - P) f_0 \|_\nu + \frac{\alpha}{\sqrt{\varepsilon}} \| (1 - P_\gamma) f_0 \|_{L^2_x} + \alpha \varepsilon^{\frac{1}{2}} \| e^\lambda \|_{L^6_{x,v}}
\]

(3.84)

It is not difficult to see that

\[
\sup_{s \in [0,t]} \frac{\alpha^2}{\varepsilon} \| (1 - P_\gamma) e^\lambda f \|_{L^2_x}^2 \lesssim \frac{2}{\varepsilon^2} \int_0^t \| (I - P) [e^{\lambda s} f] \| \| (I - P) [e^{\lambda s} f_\nu] + \lambda \| (I - P) [e^{\lambda s} f] \|_\nu \| + \frac{1}{\varepsilon^2} \| (I - P) f_0 \|_\nu^2
\]

(3.85)

\[
\leq (2\lambda + 1) \mathcal{D}_\lambda[f](t) + \frac{1}{\varepsilon^2} \| (I - P) f_0 \|_\nu^2
\]

and

\[
\sup_{s \in [0,t]} \frac{\alpha^2}{\varepsilon} \| (1 - P_\gamma) e^\lambda f \|_{L^2_x}^2 \lesssim \frac{2}{\varepsilon^2} \int_0^t \| (1 - P_\gamma) [e^{\lambda s} f] \|_{L^2_x}^2 + \| (1 - P_\gamma) [e^{\lambda s} f_\nu] + \lambda \| (1 - P_\gamma) [e^{\lambda s} f] \|_{L^2_x}^2 \]

(3.86)
The first two terms on the right hand side of (3.83) are bounded by
\[ \varepsilon^{-1} \|(I - P)f_0\|_\nu + \varepsilon^{-1}|(1 - P_\gamma)f_0|_{2,v} + \sqrt{\mathcal{D}_\lambda}\|f\| \]
The last two terms in (3.83) are bounded by
\[ \varepsilon \sqrt{\mathcal{D}_\lambda}\|f\| + \varepsilon^3 \|e^{\lambda t}\nu^{-\frac{1}{2}} g\|_{L^r_{l,v}} + \varepsilon^2 \|e^{\lambda t}\nu^{-\frac{1}{2}} g\|_{L^r_{l,v}} \]
Collecting these estimates, we get (3.82).
To show (3.83), we estimate the \( f_t \) term in (3.82) by (3.70)
\[ \varepsilon^2 \|e^{\lambda t}\nu^{-\frac{1}{2}} w f_t\|_{L^r_{l,v}} \]
\[ \leq \varepsilon^2 \|w f_t(0)\|_{\infty} + \alpha \varepsilon^2 \|\nu t \nu^{-\frac{1}{2}} w f_t\|_{L^r_{l,v}} + \varepsilon^2 \|e^{\lambda t}\nu^{-\frac{1}{2}} w g\|_{L^r_{l,v}} + \varepsilon \|e^{\lambda t} f_t\|_{L^r_{l,v}} \]
where the last term is bounded by \( \varepsilon \sqrt{\mathcal{D}_\lambda}(t) \). Then (3.83) follows readily.

### 3.3. Validity of unsteady equation.

In this section, we give the proof of Theorem 1.2. We first need the following preliminary lemma on collision operator \( \Gamma(f, g) \) proved in [17].

**Lemma 3.8.** Assume that \( f, g \in L^2(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3) \) and satisfy
\[ |a(f)| + |b(f)| + |c(f)| \leq \sum_{k=1}^3 S_k f(t, x), \quad |a(g)| + |b(g)| + |c(g)| \leq \sum_{k=1}^3 S_k g(t, x) \quad \text{for} \ t > 0, \]
where \( a, b, c \) are defined in [11]. Then
\[ \|\nu^{-\frac{1}{2}} \Gamma(f, g)\|_{L^2_{l,v}} \leq \varepsilon^2 \left[ |\nu^{-\frac{1}{2}}(I - P)^2 f\|_{L^2_{l,v}} + \varepsilon^{-1}\|S_1 f\|_{L^2_{l,v}} \right] \]
\[ + \varepsilon^2 \left[ |\nu^{-\frac{1}{2}} |\nu^{-\frac{1}{2}} g\|_{L^r_{l,v}} + \varepsilon^{-1}\|S_2 f\|_{L^r_{l,v}} \right] \]
\[ + \varepsilon^2 \left[ \|e^{\lambda t}\nu^{-\frac{1}{2}} g\|_{L^r_{l,v}} \right] \left[ |\nu^{-\frac{1}{2}}(I - P)\|_{L^2_{l,v}} + \varepsilon^{-1}\|S_3 f\|_{L^2_{l,v}} \right] \]
and
\[ \|\nu^{-\frac{1}{2}} \Gamma(f, g)\|_{L^2_{l,v}} \leq \varepsilon^2 \left[ \|e^{\lambda t}\nu^{-\frac{1}{2}} g\|_{L^r_{l,v}} \right] \left[ |\nu^{-\frac{1}{2}}(I - P)\|_{L^2_{l,v}} + \varepsilon^{-1}\|S_1 f\|_{L^2_{l,v}} \right] \]
\[ + \varepsilon^2 \left[ |\nu^{-\frac{1}{2}} |\nu^{-\frac{1}{2}} g\|_{L^r_{l,v}} + \varepsilon^{-1}\|S_2 f\|_{L^r_{l,v}} \right] \]
\[ + \varepsilon^2 \left[ \|e^{\lambda t}\nu^{-\frac{1}{2}} g\|_{L^r_{l,v}} \right] \left[ \varepsilon^{-1}\|S_3 f\|_{L^2_{l,v}} \right] \]
and
\[ \|\nu^{-\frac{1}{2}} \Gamma(f, g)\|_{L^2_{l,v}} \leq \varepsilon^2 \left[ |\nu^{-\frac{1}{2}}(I - P)\|_{L^2_{l,v}} + \varepsilon^{-1}\|S_1 f\|_{L^2_{l,v}} \right] \]
\[ + \varepsilon^2 \left[ |\nu^{-\frac{1}{2}} |\nu^{-\frac{1}{2}} g\|_{L^r_{l,v}} + \varepsilon^{-1}\|S_2 f\|_{L^r_{l,v}} \right] \]
\[ + \varepsilon^2 \left[ \|e^{\lambda t}\nu^{-\frac{1}{2}} g\|_{L^r_{l,v}} \right] \left[ \varepsilon^{-1}\|S_3 f\|_{L^2_{l,v}} \right] \]
Now we are ready to give the proof of Theorem 1.2.
**Proof of Theorem 3.2.** We set the following iteration for \( k \in \mathbb{N} \)

\[
\varepsilon \partial_t [e^{\lambda t} \hat{f}^{k+1}] + v \cdot \nabla_x [e^{\lambda t} \hat{f}^{k+1}] + \varepsilon^2 \Phi \cdot \nabla_x [e^{\lambda t} \hat{f}^{k+1}] + \varepsilon^{-1} L [e^{\lambda t} \hat{f}^{k+1}] \\
= \lambda [e^{\lambda t} \hat{f}^{k+1}] + \frac{1}{2} \varepsilon^2 \Phi \cdot \varepsilon [e^{\lambda t} \hat{f}^{k+1}] + 2 \Gamma (e^{\lambda t} \hat{f}_s, f_s + f_w) + e^{-\lambda t} \Gamma (e^{\lambda t} \hat{f}, e^{\lambda t} \hat{f}), \tag{3.91}
\]

\[
e^{\lambda t} \hat{f}^{k+1}_{|_{t=0}} = \bar{f}_0,
\]

where \( \bar{f}(t, x, v) = 0 \). Furthermore, \( \hat{f}^{k+1} \) satisfies

\[
\varepsilon \partial_t [e^{\lambda t} \hat{f}^{k+1}] + v \cdot \nabla_x [e^{\lambda t} \hat{f}^{k+1}] + \varepsilon^2 \Phi \cdot \nabla_x [e^{\lambda t} \hat{f}^{k+1}] + \varepsilon^{-1} L [e^{\lambda t} \hat{f}^{k+1}] \\
= \lambda [e^{\lambda t} \hat{f}^{k+1}] + \frac{1}{2} \varepsilon^2 \Phi \cdot \varepsilon [e^{\lambda t} \hat{f}^{k+1}] + 2 \Gamma (e^{\lambda t} \hat{f}_s, f_s + f_w) + 2e^{-\lambda t} \Gamma (e^{\lambda t} \hat{f}, e^{\lambda t} \hat{f}), \tag{3.92}
\]

\[
e^{\lambda t} \hat{f}^{k+1}_{|_{t=0}} = (1 - \alpha) e^{\lambda t} \hat{f}_s + \alpha P_\gamma \hat{f}^{k+1} + \alpha \varepsilon \mathcal{D}_1 [e^{\lambda t} \hat{f}],
\]

where \( \hat{f}_0(t, x, v) = 0 \). Furthermore, \( \hat{f}^{k+1} \) satisfies

\[
\varepsilon \partial_t [e^{\lambda t} \hat{f}^{k+1}] + v \cdot \nabla_x [e^{\lambda t} \hat{f}^{k+1}] + \varepsilon^2 \Phi \cdot \nabla_x [e^{\lambda t} \hat{f}^{k+1}] + \varepsilon^{-1} L [e^{\lambda t} \hat{f}^{k+1}] \\
= \lambda [e^{\lambda t} \hat{f}^{k+1}] + \frac{1}{2} \varepsilon^2 \Phi \cdot \varepsilon [e^{\lambda t} \hat{f}^{k+1}] + 2 \Gamma (e^{\lambda t} \hat{f}_s, f_s + f_w) + 2e^{-\lambda t} \Gamma (e^{\lambda t} \hat{f}, e^{\lambda t} \hat{f}), \tag{3.93}
\]

Note that Theorem 3.3 with (1.37), guarantees the solvability of linear problems (3.91) and (3.92).

Define a norm

\[
\|[[f]]\| := \varepsilon \partial_x [f] + \|D_x [f] + \|S_1 [e^{\lambda t} f]\|_{L^2_t L^2_x} + \frac{1}{\varepsilon} \|S_2 [e^{\lambda t} f]\|_{L^2_t L^2_x} + \|P [e^{\lambda t} f]\|_{L^2_t L^2_v} + \frac{1}{\varepsilon} \|P [e^{\lambda t} f]\|_{L^2_t L^2_v}, \tag{3.94}
\]

where \( S_1 f, S_2 f \) are given in Lemma 3.4.

**Step 1.** We show the uniform boundedness of the iteration sequence \([[[\hat{f}^k]]]\) for all \( k \in \mathbb{N} \).

For \( 0 < M_1 \ll 1 \), we assume the induction hypothesis

\[
\sup_{0 \leq k \leq k} \|[[f^k]]\| \leq M_1. \tag{3.95}
\]

For small \( 0 < c_1 \ll 1 \), assume

\[
\|\hat{f}_0\|_{L^2_v} + \|\partial_t \hat{f}(0)\|_{L^2_v} + \frac{1}{\varepsilon} \|\hat{w} \hat{f}_0\|_{L^\infty_v} + \frac{1}{\varepsilon^2} \|\omega \partial_x \hat{f}_0\|_{L^\infty_v} \\
+ \|\left[ v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla_x \right] \hat{f}_0\|_{L^2_v} + \|\left[ v \cdot \nabla_x + \varepsilon^2 \Phi \cdot \nabla_x \right] \partial_t \hat{f}(0)\|_{L^2_v} + \frac{1}{\varepsilon} \|\hat{w} (I - P) \hat{f}_0\|_{L^2_v} < c_1 M_1
\]

and

\[
\|P f_s\|_{L^2_v} + \frac{1}{\varepsilon} \|\hat{w} (I - P) f_s\|_{L^2_v} + \frac{1}{\varepsilon^2} \|\hat{w} f_s\|_{L^\infty_v} < c_1. \tag{3.96}
\]

Note that smallness of \( f_s \) can be obtained through Theorem 3.4 by letting (1.39) further small.

**Step 1.1.** Estimates of collision terms.

Applying (3.88) with \( f = e^{\lambda t} \hat{f}_s, g = e^{\lambda t} \hat{f}_s \), we have

\[
\|\nu^{-\frac{1}{2}} \Gamma (e^{\lambda t} \hat{f}_s, e^{\lambda t} \hat{f}_s)\|_{L^2_v} \leq \varepsilon \frac{1}{\varepsilon} \left[ \frac{1}{\varepsilon} \|\hat{w} e^{\lambda t} \hat{f}_s\|_{L^\infty_v} \right] \frac{1}{\varepsilon} \|\nu^{-\frac{1}{2}} (I - P) (e^{\lambda t} \hat{f}_s)\|_{L^2_v} + \varepsilon^{-1} \|S_3 (e^{\lambda t} \hat{f}_s)\|_{L^2_v} \\
+ \frac{1}{\varepsilon} \left[ \frac{1}{\varepsilon^2} \|\hat{w} e^{\lambda t} \hat{f}_s\|_{L^\infty_v} \right] \frac{1}{\varepsilon^2} \left[ \frac{1}{\varepsilon} \|\hat{w} (I - P) e^{\lambda t} \hat{f}_s\|_{L^2_v} \right] \frac{1}{\varepsilon^2} \|\hat{w} e^{\lambda t} \hat{f}_s\|_{L^\infty_v} \leq \frac{1}{\varepsilon} \left[ \frac{1}{\varepsilon^2} \|\hat{w} e^{\lambda t} \hat{f}_s\|_{L^\infty_v} \right] \frac{1}{\varepsilon^2} \|\hat{w} e^{\lambda t} \hat{f}_s\|_{L^\infty_v}, \tag{3.97}
\]

\[
\leq (1 + \varepsilon^{-\frac{1}{2}} + \varepsilon^{-\frac{1}{2}}) \|[[\hat{f}^k]]\|^2.
\]
Applying (3.90) with \( f = e^{\lambda t} \tilde{f}^k, g = e^{\lambda t} \tilde{f}^k \),

\[
\| \nu^{-\frac{1}{2}} \Gamma(e^{\lambda t} \tilde{f}^k, e^{\lambda t} \tilde{f}^k) \|_{L^2_t L^\infty_x L^2_v} \\
\lesssim e^\frac{\nu}{2} \| w e^{\lambda t} \tilde{f}^k \|_{L^\infty_t L^\infty_x L^2_v} [e^{-1} \| \nu^{\frac{1}{2}} (I - P) e^{\lambda t} \tilde{f}^k \|_{L^2_t L^\infty_x L^2_v} + e^{-1} \| S_3 e^{\lambda t} \partial_t \tilde{f}^k \|_{L^2_t L^\infty_x L^2_v}] \\
+ e^\frac{\nu}{2} \| w e^{\lambda t} \tilde{f}^k \|_{L^\infty_t L^\infty_x L^2_v} \left[ \| \partial_\lambda \tilde{f}^k \|_{L^\infty} + e^{-1} \| \nu^{\frac{1}{2}} (I - P) \tilde{f}_0 \|_{L^2_x} \right] \| S_1 e^{\lambda t} \tilde{f}^k \|_{L^2_t L^\infty_x L^2_v} \\
+ \| P [e^{\lambda t} \tilde{f}^k] \|_{L^\infty_t L^2_x L^2_v} \| S_1 [e^{\lambda t} \partial_t \tilde{f}^k] \|_{L^2_t L^\infty_x L^2_v} \\
+ \| e^{\lambda t} \| w e^{\lambda t} \tilde{f}^k \|_{L^\infty_t L^\infty_x L^2_v} \left[ \| \partial_\lambda \tilde{f}^k \|_{L^\infty} + e^{-1} \| \nu^{\frac{1}{2}} (I - P) \tilde{f}_0 \|_{L^2_x} \right] \| S_1 e^{\lambda t} \tilde{f}^k \|_{L^2_t L^\infty_x L^2_v} \\
\lesssim (1 + e^\frac{\nu}{2} + e^\frac{\nu}{2}) \| [\tilde{f}^k] \| + e^{-1} \| \nu^{\frac{1}{2}} (I - P) \tilde{f}_0 \|_{L^2_x} \| [\tilde{f}^k] \| \\
\lesssim (1 + e^\frac{\nu}{2} + e^\frac{\nu}{2}) \| [\tilde{f}^k] \| \\
\]

where we used (3.99) with \( f = e^{\lambda t} \tilde{f}^k, g = f_s + f_w \),

\[
\| \nu^{-\frac{1}{2}} \Gamma(e^{\lambda t} \tilde{f}^k, f_s + f_w) \|_{L^2_t L^\infty_x L^2_v} \\
\lesssim e^\frac{\nu}{2} \| w (f_s + f_w) \|_{L^\infty_t L^\infty_x L^2_v} [e^{-1} \| \nu^{\frac{1}{2}} (I - P) e^{\lambda t} \tilde{f}^k \|_{L^2_t L^\infty_x L^2_v} + e^{-1} \| S_3 e^{\lambda t} \tilde{f}^k \|_{L^2_t L^\infty_x L^2_v}] \\
+ e^\frac{\nu}{2} \| w (f_s + f_w) \|_{L^\infty_t L^\infty_x L^2_v} \left[ \| P [f_s + f_w] \|_{L^\infty_t L^2_x L^2_v} \right] \| e^{\lambda t} \tilde{f}^k \|_{L^2_t L^\infty_x L^2_v} \\
+ \| P [f_s + f_w] \|_{L^\infty_t L^2_x L^2_v} \| S_1 [e^{\lambda t} \tilde{f}^k] \|_{L^2_t L^\infty_x L^2_v} \\
+ \| e^{\lambda t} \| w (f_s + f_w) \|_{L^\infty_t L^\infty_x L^2_v} \left[ \| \nu^{\frac{1}{2}} (I - P) f_s + f_w \|_{L^2_x} \right] \| P [e^{\lambda t} \tilde{f}^k] \|_{L^2_t L^\infty_x L^2_v} \\
\lesssim \left[ (e^\frac{\nu}{2} + e^\frac{\nu}{2}) e^\frac{\nu}{2} \right] \| w (f_s + f_w) \|_{L^\infty_t L^\infty_x L^2_v} \| P [f_s + f_w] \|_{L^\infty_t L^2_x L^2_v} // [[[\tilde{f}^k]]] \\
+ \| e^{-1} \| \nu^{\frac{1}{2}} (I - P) f_s + f_w \|_{L^\infty_t L^\infty_x L^2_v} \| P [e^{\lambda t} \tilde{f}^k] \|_{L^2_t L^\infty_x L^2_v} \\
\lesssim (c_1 + e^\frac{\nu}{2} + e^\frac{\nu}{2}) \| [\tilde{f}^k] \| \\
\]

Applying (3.99) with \( f = e^{\lambda t} \tilde{f}^k, g = f_s + f_w \) and deriving similarly as (3.99),

\[
\| \nu^{-\frac{1}{2}} \Gamma(e^{\lambda t} \tilde{f}^k, f_s + f_w) \|_{L^2_t L^\infty_x L^2_v} \\
\lesssim \left[ (e^\frac{\nu}{2} + e^\frac{\nu}{2}) e^\frac{\nu}{2} \right] \| w (f_s + f_w) \|_{L^\infty_t L^\infty_x L^2_v} \| P [f_s + f_w] \|_{L^\infty_t L^2_x L^2_v} // [[[\tilde{f}^k]]] \\
+ \| e^{-1} \| \nu^{\frac{1}{2}} (I - P) f_s + f_w \|_{L^\infty_t L^\infty_x L^2_v} \| e^\frac{\nu}{2} \| w (f_s + f_w) \|_{L^\infty_t L^\infty_x L^2_v} \| P [e^{\lambda t} \tilde{f}^k] \|_{L^2_t L^\infty_x L^2_v} \\
\lesssim \left[ (1 + e^\frac{\nu}{2} + e^\frac{\nu}{2} + e^\frac{\nu}{2} + e^\frac{\nu}{2}) \right] \| [\tilde{f}^k] \| \\
\]

Here \( \| P [e^{\lambda t} \tilde{f}^k] \|_{L^2_t L^\infty_x L^2_v} \) is bounded by (3.98),

\[
\| P [e^{\lambda t} \tilde{f}^k] \| \lesssim \| P [e^{\lambda t} \tilde{f}^k] \| \| P [e^{\lambda t} \tilde{f}^k] \|_L^2_t L^\infty_x L^2_v \lesssim \sqrt{\| \partial_\lambda \tilde{f}^k \|_{L^\infty} + \| [\tilde{f}^k] \|} \\
\]

where the first term is bounded by

\[
\| \nu^\frac{1}{2} \sqrt{\| S_1 e^{\lambda t} \tilde{f}^k \|_{L^2_t L^\infty_x L^2_v}} \lesssim \| S_1 e^{\lambda t} \tilde{f}^k \|_{L^2_t L^\infty_x L^2_v} \\
\]

and the last two terms are estimated by interpolation and (3.59),

\[
\| \nu^\frac{1}{2} \sqrt{\| S_2 e^{\lambda t} \tilde{f}^k \|_{L^2_t L^\infty_x L^2_v}} \lesssim \| e^{-\frac{1}{2}} \| e^\frac{\nu}{2} \| \sqrt{\| S_2 e^{\lambda t} \tilde{f}^k \|_{L^2_t L^\infty_x L^2_v}} \lesssim \| \| [\tilde{f}^k] \| \\
\]


where we used Lemma 2.10 to bound \( \epsilon^2 \| e^{-1} \nu^2 \sqrt{\mathcal{M}_3} e^{\lambda f_t^k} \|_{L^2_t L^3_x, v} \leq \epsilon^2 \| e^{-1} \nu^2 \sqrt{\mathcal{M}_3} e^{\lambda f_t^k} \|_{L^2_t L^2_{x,v}} \). 

\[
\| e^\lambda f_t^k \|_2 + \int_0^t \left( \frac{1}{\tau^2} \| (I - P)[e^{\lambda s} f_t^k] \|_2^2 + \| P[e^{\lambda s} f_t^k] \|_2^2 \right) \mathrm{d} s \\
\lesssim \| f_0 \|_2 + \int_0^t \left( \| e^\lambda f_t^k \|_2^2 + \| e^\lambda f_t^k \|_2^2 + \| e^\lambda f_t^k \|_2^2 \right) \mathrm{d} s \\
\lesssim \left( c_1 M_1 + (1 + \epsilon^{1/2} + \epsilon^{1/4})[[[f_t^k]]] + (1 + \epsilon^{1/2} + \epsilon^{1/4})[[[f_t^k]]]^2 \right) \tag{3.101}
\]

where we used Lemma 2.10 to bound \( \alpha \| \mathcal{D}_1 (e^\lambda f_t^k) \|_{L^2_v} \). 

Applying (3.31) with \( f = f_t^k \), \( g = 2 \Gamma \left( f_t^k, f_s + f_w \right) + \Gamma \left( f_t^k, f_t^k \right) \), \( r = \epsilon \mathcal{D}_1 f_t^k \), 

\[
\| e^\lambda f_t^k \|_2 + \int_0^t \left( \frac{1}{\tau^2} \| (I - P)[e^{\lambda s} f_t^k] \|_2^2 + \| P[e^{\lambda s} f_t^k] \|_2^2 \right) \mathrm{d} s \\
\lesssim \| f_0 \|_2 + \int_0^t \left( \| e^\lambda f_t^k \|_2^2 + \| e^\lambda f_t^k \|_2^2 + \| e^\lambda f_t^k \|_2^2 \right) \mathrm{d} s \\
\lesssim \left( c_1 M_1 + (1 + \epsilon^{1/2} + \epsilon^{1/4})[[[f_t^k]]] + (1 + \epsilon^{1/2} + \epsilon^{1/4})[[[f_t^k]]]^2 \right) \tag{3.102}
\]

Combining (3.101) and (3.102) we get 

\[
\sqrt{\mathcal{E}_3[f_t^k]} + \sqrt{\mathcal{D}_3[f_t^k]} \lesssim c_1 M_1 + [\epsilon^{1/2} + \epsilon^{1/4} + (1 + \epsilon^{1/2} + \epsilon^{1/4} + \epsilon^{1/4})\epsilon_0 c_1] [[[[f_t^k]]]] + (1 + \epsilon^{1/2} + \epsilon^{1/4})[[[f_t^k]]]^2. \tag{3.103}
\]
Applying Lemma 3.4 to (3.91) with
$$\nu e^{\lambda t} f^{k+1} + 2\Gamma e^{\lambda t} f^{k}, f_s + f_w) + e^{-\lambda t} \Gamma(e^{\lambda t} f^{k}, e^{\lambda t} f^{k}), r = \epsilon \mathcal{O}_2 \hat{f}^k,$$

$$\|S_1[e^{\lambda t} f^{k+1}]\|_{L^2_t L^2_{x,v}} + \epsilon \|S_2[e^{\lambda t} f^{k+1}]\|_{L^2_t L^2_{x,v}}$$

$$\lesssim \|\nu \epsilon e^{\lambda t} f^{k+1}\|_{L^2_t L^2_{x,v}} + \epsilon^{-1} \|L[e^{\lambda t} f^{k+1}]\|_{L^2_t L^2_{x,v}} + \epsilon e^{\lambda t} \|v[e^{\lambda t} f^{k+1}]\|_{L^2_t L^2_{x,v}} + \epsilon^2 \|\Phi \cdot v[e^{\lambda t} f^{k+1}]\|_{L^2_t L^2_{x,v}}$$

$$+ \|\nu^{-1} \Gamma e^{\lambda t} f^{k}, f_s + f_w)\|_{L^2_t L^2_{x,v}} + \epsilon e^{-\lambda t} \|\Gamma(e^{\lambda t} f^{k}, e^{\lambda t} f^{k})\|_{L^2_t L^2_{x,v}} + \epsilon \|\Phi \|_{L^2_t L^2_{x,v}}$$

$$\lesssim c_1 M_1 + (1 + \epsilon + \lambda \epsilon + \epsilon^2 \|\Phi\|_{\infty}) \left(\sqrt{\Theta_\lambda[f^{k+1}]_{\infty}} + \sqrt{\Theta_\lambda[f^{k+1}]_{\infty}}\right)$$

$$+ (1 + \epsilon + \lambda \epsilon + \epsilon^2 \|\Phi\|_{\infty}) (1 - P_\gamma) \lambda e^{\lambda t} f^{k+1}|_{L^2_{x,v} + \alpha\|v\|_{L^2_t L^2_{x,v}}}$$

(3.104)

Applying Lemma 3.4 to (3.91) with
$$f = e^{\lambda t} f^{k+1}, g = \epsilon^{-1} L[e^{\lambda t} f^{k+1}] + \epsilon e^{\lambda t} f^{k+1} + \frac{1}{2} \epsilon^2 \Phi \cdot v[e^{\lambda t} f^{k+1}] + 2\Gamma(e^{\lambda t} f^{k}, f_s + f_w) + e^{-\lambda t} \Gamma(e^{\lambda t} f^{k}, e^{\lambda t} f^{k}) r = \epsilon \mathcal{O}_1 \hat{f}^k,$$

$$\|S_1[e^{\lambda t} f^{k+1}]\|_{L^2_t L^2_{x,v}} + \epsilon \|S_2[e^{\lambda t} f^{k+1}]\|_{L^2_t L^2_{x,v}}$$

$$\lesssim c_1 M_1 + (1 + \epsilon + \lambda \epsilon + \epsilon^2 \|\Phi\|_{\infty}) \left(\sqrt{\Theta_\lambda[f^{k+1}]_{\infty}} + \sqrt{\Theta_\lambda[f^{k+1}]_{\infty}}\right)$$

$$+ (1 + \epsilon + \lambda \epsilon + \epsilon^2 \|\Phi\|_{\infty}) \|\Phi\|_{\infty} + (1 + \epsilon + \lambda \epsilon + \epsilon^2 \|\Phi\|_{\infty}) (1 - P_\gamma) \lambda e^{\lambda t} f^{k+1}|_{L^2_{x,v} + \alpha\|v\|_{L^2_t L^2_{x,v}}}$$

(3.105)

**Step 1.4.** Estimates of $\|P[e^{\lambda t} f^{k+1}]\|_{L^\infty_t L^2_{x,v}}, \epsilon \|v e^{\lambda t} f^{k+1}\|_{L^\infty_t L^2_{x,v}}$ and $\epsilon \|v e^{\lambda t} f^{k+1}\|_{L^\infty_t L^2_{x,v}}$.

Applying (3.83) to (3.91) with $f = \hat{f}^{k+1}, g = 2\Gamma(\hat{f}^k, f_s + f_w) + \Gamma(\hat{f}^k, \hat{f}^k), r = \epsilon \mathcal{O}_1(\hat{f}^k)$,

$$\|P[e^{\lambda t} f^{k+1}]\|_{L^\infty_t L^2_{x,v}}$$

$$\lesssim \epsilon \|\hat{f}_0(0)\|_{L^\infty_t L^2_{x,v}} + \epsilon \|\hat{f}_0(0)\|_{L^2_t L^2_{x,v}}$$

$$+ \epsilon \|\hat{f}_0(0)\|_{L^2_t L^2_{x,v}} + \epsilon \|\hat{f}_0(0)\|_{L^2_t L^2_{x,v}}$$

$$+ \epsilon \|\hat{f}_0(0)\|_{L^2_t L^2_{x,v}} + \epsilon \|\hat{f}_0(0)\|_{L^2_t L^2_{x,v}}$$

$$+ \epsilon \|\hat{f}_0(0)\|_{L^2_t L^2_{x,v}} + \epsilon \|\hat{f}_0(0)\|_{L^2_t L^2_{x,v}}$$

(3.106)

where we have used Lemma 2.10 and an inequality of (3.86) to form to bound

$$\|\hat{f}_0(0)\|_{L^\infty_t L^2_{x,v}} \lesssim \epsilon \|\hat{f}_0(0)\|_{L^\infty_t L^2_{x,v}}$$

$$\lesssim \epsilon \|\hat{f}_0(0)\|_{L^\infty_t L^2_{x,v}} + \epsilon \|\hat{f}_0(0)\|_{L^\infty_t L^2_{x,v}}$$

Besides, we have used Lemma 2.9 and (3.85) to estimate

$$\|\nu^{-1} \Gamma(e^{\lambda t} f^{k}, f_s + f_w)\|_{L^\infty_t L^2_{x,v}}$$

$$\lesssim \epsilon \|\hat{f}_0(0)\|_{L^\infty_t L^2_{x,v}} + \epsilon \|\hat{f}_0(0)\|_{L^\infty_t L^2_{x,v}}$$

$$+ \epsilon \|\hat{f}_0(0)\|_{L^\infty_t L^2_{x,v}} + \epsilon \|\hat{f}_0(0)\|_{L^\infty_t L^2_{x,v}}$$

$$+ \epsilon \|\hat{f}_0(0)\|_{L^\infty_t L^2_{x,v}} + \epsilon \|\hat{f}_0(0)\|_{L^\infty_t L^2_{x,v}}$$

(3.107)
and
\[ \|\nu^{-\frac{1}{2}} \Gamma(e^M \hat{f}_k, e^M \hat{f}_k)\|_{L^\infty L^2_y} \leq \varepsilon^\frac{1}{2} \left[ \varepsilon^\frac{1}{2} \|w e^M \hat{f}_k\|_{L^\infty L^2_y} + \sqrt{D_x[\hat{f}_k]}(\infty) \right] + \|P[e^M \hat{f}_k]\|_{L^\infty L^2_y} + \|P[e^M \hat{f}_k]\|_{L^\infty L^2_y} \varepsilon^{-1} \|\nu^{-\frac{1}{2}} (I - P)f_0\|_{L^2_y} + \frac{\sqrt{D_x[\hat{f}_k]}(\infty)}{\varepsilon}. \]

Finally, collecting (3.103)–(3.108), we get
\[ \varepsilon \mathcal{D}_1 (e^M \hat{f}_k), \]
\[ r = \varepsilon \mathcal{D}_1 (e^M \hat{f}_k), \]
\[ \varepsilon^\frac{1}{2} \|we^M \hat{f}_k\|_{L^\infty L^2_y} \leq \varepsilon^\frac{1}{2} \|w \hat{f}_0\|_{L^\infty L^2_y} + \|P[e^M \hat{f}_k]\|_{L^\infty L^2_y} + \|P[e^M \hat{f}_k]\|_{L^\infty L^2_y} + \varepsilon^{-1} \|\nu^{-\frac{1}{2}} (I - P)e^M \hat{f}_k\|_{L^2_y} \]
\[ + \alpha \varepsilon^\frac{\lambda}{2} \|we^M \hat{f}_k\|_{L^\infty L^2_y} + \varepsilon^\frac{1}{2} \|\nu^{-\frac{1}{2}} (I - P)e^M \hat{f}_k\|_{L^2_y} \]
\[ + \varepsilon \|P[e^M \hat{f}_k]\|_{L^\infty L^2_y} + \|P[e^M \hat{f}_k]\|_{L^\infty L^2_y} \]
\[ \leq c_1 M_1 + \|P[e^M \hat{f}_k]\|_{L^\infty L^2_y} + \sqrt{D_x[\hat{f}_k]}(\infty) + (\varepsilon + \varepsilon^\frac{1}{2} c_1)\|[(\hat{f}_k)]\| + \varepsilon^\frac{1}{2} \|[(\hat{f}_k)]\|^2, \]
where we used (2.161) to bound the collision terms.

Applying (3.70) to (3.92) with \( f = e^M \hat{f}_k, \)
\[ r = \varepsilon \mathcal{D}_1 (e^M \hat{f}_k), \]
\[ \varepsilon^\frac{1}{2} \|we^M \hat{f}_k\|_{L^\infty L^2_y} \leq \varepsilon^\frac{1}{2} \|w \hat{f}_0\|_{L^\infty L^2_y} + \|P[e^M \hat{f}_k]\|_{L^\infty L^2_y} + \|P[e^M \hat{f}_k]\|_{L^\infty L^2_y} + \varepsilon^{-1} \|\nu^{-\frac{1}{2}} (I - P)e^M \hat{f}_k\|_{L^2_y} \]
\[ + \alpha \varepsilon^\frac{\lambda}{2} \|we^M \hat{f}_k\|_{L^\infty L^2_y} + \varepsilon \|P[e^M \hat{f}_k]\|_{L^\infty L^2_y} + \|P[e^M \hat{f}_k]\|_{L^\infty L^2_y} \]
\[ \leq c_1 M_1 + \sqrt{D_x[\hat{f}_k]}(\infty) + (\varepsilon + \varepsilon^\frac{1}{2} c_1)\|[(\hat{f}_k)]\| + \varepsilon^\frac{1}{2} \|[(\hat{f}_k)]\|^2, \]
where we have used (2.161) again.

Finally, collecting (3.103)–(3.108), we get
\[ \|[(\hat{f}_k + \varepsilon^\frac{1}{2})]\| \leq c_1 M_1 + (c_1 + \varepsilon^\frac{1}{2})\|[(\hat{f}_k)]\| + \|[(\hat{f}_k)]\|^2 \leq M_1, \]
provided \( c_1 \) and \( M_1 \) are small enough. This proves the uniform boundedness of \( \|[(\hat{f}_k)]\| \) for all \( k \in \mathbb{N} \).

**Step 2.** Strong convergence of \( \hat{f}_k \) in \( L^\infty \cap L^2 \) can be proved through repeating Step 1 for \( \hat{f}_k+1 - \hat{f}_k \).

We conclude that the weak limit \( \hat{f}(t, x, v) := \lim_{k \to \infty} \hat{f}_k(t, x, v) \) solves the unsteady Boltzmann equation (1.30). Uniqueness follows from standard argument.

**Step 3.** In this step we prove the weak convergence of \( \hat{f} \) and show its limit leads to the unsteady INSF (1.12) to the boundary conditions (1.13) and (1.14).

Let \( \hat{f}^\varepsilon = f_\varepsilon + f_s + \hat{f}_\varepsilon \). Then \( \hat{f} \) satisfies
\[ \varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \varepsilon^\frac{1}{2} \Phi \cdot \nabla_v (\sqrt{\mu} f^\varepsilon) + \varepsilon^{-1} L f^\varepsilon = \Gamma(f^\varepsilon, f^\varepsilon) + \varepsilon \sqrt{\mu} \Phi \cdot v, \]
\[ f^\varepsilon|_{\gamma_-} = (1 - \alpha_\varepsilon) \mathcal{L} f + \alpha_\varepsilon P_\gamma f^\varepsilon + \alpha_\varepsilon \mathcal{R} f^\varepsilon, \]
\[ f^\varepsilon|_{t=0} = f_0^\varepsilon, \]
where \( f_0^\varepsilon = f_\varepsilon + f_s + \hat{f}_\varepsilon \) and the operator \( \mathcal{R} \) is defined in (2.165).

We follow the same argument as in the proof of Theorem (1.1) and have
\[ f^\varepsilon \to f_1 := \left[ \rho_1 + u f \cdot v + \theta f \frac{|v|^2 - 3}{2} \right] \sqrt{\mu} \text{ weakly in } L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3), \]
and \( (\rho_1, u_1, \theta_1) \) satisfies the following INSF
\[ \partial_t u_1 + u_1 \cdot \nabla_x u_1 + \nabla_x P_1 = \sigma \Delta u_1 + \Phi, \quad \nabla_x \cdot u_1 = 0 \quad \text{in } \Omega, \]
\[ \partial_t \theta_1 + u_1 \cdot \nabla_x \theta = \kappa \Delta \theta_1, \quad \nabla_x (\rho_1 + \theta_1) = 0 \quad \text{in } \Omega. \]
The limiting boundary conditions follow similarly as in the steady case. Specially, for the limiting Navier boundary condition when \( \lim_{\epsilon \to 0} \frac{\partial f}{\partial n} = \sqrt{2\pi} \lambda \in [0, +\infty) \), we multiply equation \( 3.110 \) by \( \frac{\epsilon^{-1} |v|^2 - 5}{2} \sqrt{\mu} \phi \) and \( \epsilon^{-1} v \sqrt{\mu} \), integrate on \([t_1, t_2] \times \Omega \times \mathbb{R}^3\),

\[
0 = \lim_{\epsilon \to 0} \int_{t_1}^{t_2} \int_{\Omega \times \mathbb{R}^3} \partial_t f^\varepsilon \frac{|v|^2 - 5}{2} \sqrt{\mu} \phi - \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_1}^{t_2} \int_{\Omega} \left( |v|^2 - 5 \right) \sqrt{\mu} \cdot f^\varepsilon \right) \cdot \nabla_x \phi \\
+ \lim_{\epsilon \to 0} \int_{t_1}^{t_2} \int_{\partial \Omega \times \mathbb{R}^3} f^\varepsilon \frac{|v|^2 - 5}{2} \sqrt{\mu} \phi [n \cdot v] (3.113)
\]

\[
= \frac{5}{2} \int_{\Omega} [\theta f(t_2) - \theta f(t_1)] \phi - \int_{t_1}^{t_2} \int_{\Omega} (u_f \theta_f - \kappa \nabla \theta_f) \cdot \nabla \phi + 2\lambda \int_{t_1}^{t_2} \int_{\partial \Omega} (\theta_f - \partial_w) \phi
\]

and

\[
\int_{t_1}^{t_2} \int_{\Omega} \Phi \cdot w
\]

\[
= \lim_{\epsilon \to 0} \int_{t_1}^{t_2} \int_{\Omega \times \mathbb{R}^3} \partial_t f^\varepsilon \sqrt{\mu} v \cdot w - \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_1}^{t_2} \int_{\Omega} \left( |v| \otimes v - \frac{|v|^2}{3} \right) \sqrt{\mu} \cdot f^\varepsilon \right) \cdot \nabla_x w \\
+ \lim_{\epsilon \to 0} \int_{t_1}^{t_2} \int_{\partial \Omega \times \mathbb{R}^3} (v \cdot w) \sqrt{\mu} f^\varepsilon [n \cdot v] (3.114)
\]

\[
= \int_{\Omega} [u_f(t_2) - u_f(t_1)] \cdot w - \int_{t_1}^{t_2} \int_{\Omega} \left[ u_f \otimes u_f - \sigma (\nabla u_f + (\nabla u_f)^T) \right] : \nabla w \\
+ \lambda \int_{t_1}^{t_2} \int_{\partial \Omega} u_f \cdot w
\]

where \( \phi \in C^\infty(\Omega) \) and \( w \in C^\infty(\Omega) \) are test functions satisfying \( \nabla_x \cdot w = 0 \) and \( n \cdot w|_{\partial \Omega} = 0 \). This is the weak formulation of unsteady INSF with to the Navier boundary condition satisfied by \( (\rho_f, u_f, \theta_f) \).

Returning back to \( \tilde{f} \), we have

\[
\tilde{f} \to \tilde{f}_1 := [\tilde{\rho} + \tilde{u} \cdot v + \tilde{\theta} |v|^2 - 3] \sqrt{\mu} \text{ weakly in } L^2(\mathbb{R}_+ \times \Omega \times \mathbb{R}^3),
\]

where we used \( L(f_w + f_s) = 0 \). By the relation \( f = f_w + f_s + \tilde{f} \) we have

\[
\tilde{u} = u_f - u_s, \quad \tilde{\theta} = \theta_f - \theta_s - \Theta_w, \quad \tilde{\rho} = \rho_f + \rho_s - \rho_w.
\]

Then the divergence free condition and Boussinesq relation in \( 1.42 \) and the Dirichlet boundary condition \( 1.43 \) follow readily. Moreover, make difference between \( 3.113 \), \( 3.114 \) and \( 2.191 \),

\[
\int_{\Omega} [\tilde{u}(t_2) - \tilde{u}(t_1)] \cdot w - \int_{t_1}^{t_2} \int_{\Omega} [\tilde{u} \otimes (\tilde{u} + u_s) + u_s \otimes \tilde{u} - \sigma (\nabla \tilde{u} + (\nabla \tilde{u})^T)] : \nabla w \\
+ \lambda \int_{t_1}^{t_2} \int_{\partial \Omega} \tilde{u} \cdot w = 0, \quad (3.116)
\]

\[
\int_{\Omega} [\tilde{\theta}(t_2) - \tilde{\theta}(t_1)] \phi - \int_{t_1}^{t_2} \int_{\Omega} [\tilde{u}(\tilde{\theta} + \theta_s + \Theta_w) + u_s \tilde{\theta} - \kappa \nabla \tilde{\theta}] \cdot \nabla \phi + \frac{4}{5} \lambda \int_{t_1}^{t_2} \int_{\partial \Omega} \tilde{\theta} \phi = 0.
\]

This is the weak formulation of the unsteady INSF \( 1.42 \) subject to the Navier boundary condition \( 1.44 \), satisfied by \( (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \). This completes the proof of Theorem \( 1.2 \) \( \square \)

Finally, we show the non-negativity of \( F(t, x, v) \) in Theorem \( 1.2 \) and \( F_s(x, v) \) in Theorem \( 1.4 \).
Proof of the non-negativity of $F$ and $F_*$. We set the following positivity-preserving iteration introduced in [10, 27]

$$
\varepsilon \partial_t F^{k+1} + v \cdot \nabla_x F^{k+1} + \varepsilon^2 \Phi \cdot \nabla_x F^{k+1} + \varepsilon^{-1} \nu(F^k) F^{k+1} = \varepsilon^{-1} Q_+(F^k, F^k),
$$

$$
F^{k+1}(x, v)|_{\gamma} = (1 - \alpha) L F^{k+1} + \alpha P f_k^{k+1} + \alpha \mathcal{R}(F^{k+1}),
$$

where $F^0(t, x, v) := F_0(x, v) \geq 0$ and $\nu(F^k)(v) = \int_{\mathbb{R}^2 \times \mathbb{R}^3} B(v - u, \omega) F^k(u) \omega \, du$. Furthermore, by setting $F_k = \mu + \varepsilon \sqrt{\mu} f$ and $F_0 = \mu + \varepsilon \sqrt{\mu} f_0$, we derive from (3.117) that

$$
\begin{align*}
\varepsilon \partial_t f_k + v \cdot \nabla_x f_k + \varepsilon^{-1} \nu(f_k) f_k &= \varepsilon^{-1} K f_k + \Gamma_+(f_k, f_k) + \varepsilon \Phi \cdot v \sqrt{\mu}, \\
f^{k+1}_{\gamma} &= (1 - \alpha) L f^{k+1} + \alpha P f^{k+1} + \alpha \mathcal{R}(f^{k+1}),
\end{align*}
$$

where $\mathcal{R}$ is defined in (2.105).

To show the non-negativity of $F_k$ and uniform boundedness of $\|wf\|_{L^\infty_{t,x,v}}$, we employ the stretching method $(t, x) \mapsto (\varepsilon^{-2} t, \varepsilon^{-1} x)$ to (3.117) and (3.118), as we did in the $L^\infty$ estimate of (3.68), cf. proof of Theorem 3.3 and Lemma 3.6. Thus, we can use the argument given in Section 3.8 of [17] to show that $F \geq 0$ a.e. on $\mathbb{R}^+ \times \Omega$ as $\varepsilon$-weak limit of $F_k$ (up to a subsequence). Furthermore, the non-negativity of the steady solution $F_*$ follows from the non-negativity of $F$ and asymptotic stability of $F_*$ in $L^2$ profile. This completes the proof. \qed

APPENDIX A. Elliptic estimates

In this part, we give solvability and elliptic estimates of the elliptic system (1.73) and the elliptic equation (1.75).

Lemma A.1. The elliptic system

$$
-\Delta u = f \quad \text{in } \Omega, \quad u \cdot n = 0 \quad \text{on } \partial \Omega, \quad \partial_n u - (\partial_n u \cdot n)n = 0 \quad \text{on } \partial \Omega
$$

(A.1)

satisfies complementing boundary conditions in the sense of Agmon-Douglas-Nirenberg [1].

Proof. To verify that the boundary conditions satisfy complementing boundary conditions, we need to describe the equations and boundary conditions in equivalent algebraic characterizations.

In terms of the notations in [1], the equations in (A.1) reads as

$$
\sum_{j=1}^3 l_{ij}(\partial) u_j = f_i, \quad i = 1, 2, 3,
$$

where $l_{ij}(\partial) = -\delta_{ij} \sum_{k=1}^3 \partial^2_{kk} = -\delta_{ij} \Delta$. Denote, for real $\xi \in \mathbb{R}^3$ and $\xi \neq 0$,

$$
L(\xi) := \det (l_{ij}(\xi)) = -|\xi|^6.
$$

Let $n$ denote the unit normal and $\Xi \neq 0$ any tangent to $\partial \Omega$. Obviously, $L(\Xi + \tau n)$ is of degree 6 with respect to $\tau$ and has exactly 3 roots (in $\tau$) with positive imaginary part:

$$
\tau^+_1(\Xi) = \tau^+_2(\Xi) = \tau^+_3(\Xi) = i|\Xi|.
$$

Denote

$$
M^+(\Xi, \tau) := \prod_{h=1}^3 (\tau - \tau^+_h(\Xi)) = (\tau - i|\Xi|)^3.
$$

Let $(L^k(\Xi + \tau n))$ denote the matrix adjoint to $(l_{ij}(\Xi + \tau n))$. Then

$$
(l_{ij}(\Xi + \tau n)) = -(|\Xi|^2 + \tau^2) I_{3 \times 3}, \quad (L^k(\Xi + \tau n)) = -(|\Xi|^2 + \tau^2)^2 I_{3 \times 3}.
$$

where $I_{3 \times 3}$ is unit matrix.
The boundary condition \( \partial_n u - (\partial_n u \cdot n)n = 0 \) and \( u \cdot n = 0 \) are rewritten as

\[
\sum_{j=1}^{3} B_{hj}(\partial) u_j = 0, \quad h = 1, 2, 3, 4.
\]

Then the equivalent algebraic characterizations read as

\[
(B_{hj}(\Xi + \tau n)) = \tau \begin{pmatrix}
1 - n_1^2 & -n_1 n_2 & -n_1 n_3 \\
-n_2 n_1 & 1 - n_2^2 & -n_2 n_3 \\
-n_3 n_1 & -n_3 n_2 & 1 - n_3^2
\end{pmatrix} := \tau \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix} \quad \text{for } h = 1, 2, 3, \quad (A.2)
\]

\[
B_{h4}(\Xi + \tau n) = (n_1, n_2, n_3) \quad \text{for } h = 4.
\]

Note that the rank of the \( 3 \times 3 \) coefficient matrix in (A.2) is 2. Without loss of generality, we assume that \( e_1 = (1-n_1^2, -n_1 n_2, -n_1 n_3) \) and \( e_2 = (-n_2 n_1, 1-n_2^2, -n_2 n_3) \) are linearly independent and form a maximal linearly independent system of this matrix. We claim that \( n_3 \neq 0 \). In fact, if \( n_3 = 0 \), then the coefficient matrix in (A.2) reduces to

\[
\begin{pmatrix}
1 - n_1^2 & -n_1 n_2 & 0 \\
-n_2 n_1 & 1 - n_2^2 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Noticing the fact \( n_1^2 + n_2^2 = 1 \), one can find that the first two rows \( (1-n_1^2, -n_1 n_2, 0) \) and \( (-n_2 n_1, 1-n_2^2, 0) \) in (A.3) are linearly independent, which contradicts with the selection of \( e_1 \) and \( e_2 \). This proves the claim \( n_3 \neq 0 \).

Thus the boundary operators are reformulated as

\[
(\hat{B}_{hj}(\Xi + \tau n)) := \begin{pmatrix}
\tau (1-n_1^2) & -\tau n_1 n_2 & -\tau n_1 n_3 \\
-\tau n_2 n_1 & \tau (1-n_2^2) & -\tau n_2 n_3 \\
n_1 & n_2 & n_3
\end{pmatrix}.
\]

We calculate

\[
\sum_{j=1}^{3} \hat{B}_{hj}(\Xi + \tau n)L^{jk}(\Xi + \tau n) = -(|\Xi|^2 + \tau^2)^2 \begin{pmatrix}
\tau (1-n_1^2) & -\tau n_1 n_2 & -\tau n_1 n_3 \\
-\tau n_2 n_1 & \tau (1-n_2^2) & -\tau n_2 n_3 \\
n_1 & n_2 & n_3
\end{pmatrix}.
\]

Note that the elements of this matrix are regarded as polynomials in the indeterminate \( \tau \). By modulo \( M^+(\Xi, \tau) \) we get

\[
\sum_{j=1}^{3} \hat{B}_{hj}(\Xi + \tau n)L^{jk}(\Xi + \tau n) \quad (\text{mod } M^+(\Xi, \tau))
\]

\[
= 4|\Xi|^2(\tau - i|\Xi|)^2 \begin{pmatrix}
i|\Xi|(1-n_1^2) & -i|\Xi|n_1 n_2 & -i|\Xi|n_1 n_3 \\
-i|\Xi|n_2 n_1 & i|\Xi|(1-n_2^2) & -i|\Xi|n_2 n_3 \\
n_1 & n_2 & n_3
\end{pmatrix}.
\]

Simple computation shows that

\[
\det \begin{pmatrix}
i|\Xi|(1-n_1^2) & -i|\Xi|n_1 n_2 & -i|\Xi|n_1 n_3 \\
-i|\Xi|n_2 n_1 & i|\Xi|(1-n_2^2) & -i|\Xi|n_2 n_3 \\
n_1 & n_2 & n_3
\end{pmatrix} = -|\Xi|^2 n_3 \neq 0, \quad (A.5)
\]

due to \( \Xi \neq 0 \) and \( n_3 \neq 0 \). It follows from (A.4) and (A.5) that the rows of the matrix \( \sum_{j=1}^{3} \hat{B}_{hj}(\Xi + \tau n)L^{jk}(\Xi + \tau n) \) are linearly independent modulo \( M^+(\Xi, \tau) \). This verifies that the boundary conditions in (A.1) satisfy complementing boundary conditions in the sense of Agmon-Douglis-Nirenberg [1]. This completes the proof. \( \square \)

**Lemma A.2.** Let \( f \in L^p(\Omega)^3 \) and \( 1 < p < \infty \). Then the elliptic system (A.1) has a unique strong solution \( u \in W^{2,p}(\Omega)^3 \) satisfying

\[
\|u\|_{W^{2,p}} \leq C\|f\|_p.
\]

(A.6)
Proof. Firstly, by Lemma A.1, we can employ the $L^p$ estimate for elliptic system with complementing boundary given in [1].

$$
\|u\|_{W^{2,p}} \leq C(\|f\|_p + \|u\|_p) \quad \text{for} \quad u \in [W^{2,p}(\Omega)]^3.
$$

(A.7)

With the help of this estimate, the result follows from standard approach of elliptic theory. We give the sketch of the proof and omit the details for the sake of brevity.

Define the Banach spaces

$$
V^{k,p}(\Omega) := \{ v \in [W^{k,p}(\Omega)]^3 : v \cdot n = 0 \text{ on } \partial \Omega \}, \quad \|v\|_{V^{k,p}(\Omega)} := \|v\|_{W^{k,p}(\Omega)}
$$

for $k = 1, 2$. The dual space is denoted by $[V^{k,p}(\Omega)]^*$. It follows that

$$
[H_0^1(\Omega)]^3 \subset V^{1,2}(\Omega) \subset [H^1(\Omega)]^3, \quad [H_0^{-1}(\Omega)]^3 \subset [V^{1,2}(\Omega)]^* \subset [H^{-1}(\Omega)]^3.
$$

Existence and uniqueness of weak solutions of (A.1) have been established in the Appendix of [53]: for given $f \in [V^{1,2}(\Omega)]^*$, (A.1) has a unique weak solution $u \in V^{1,2}(\Omega)$ and the solution operator $\Sigma^{-1} : f \mapsto u$ is bounded from $[V^{1,2}(\Omega)]^*$ to $V^{1,2}(\Omega)$. Thus there exists a constant $C$ such that

$$
\|u\|_{H^1} \leq C\|f\|_{[V^{k,p}]^*} \leq C\|f\|_2.
$$

(A.8)

For any $\phi \in V^{1,2}(\Omega)$, it follows from the boundary condition $\partial_n u - (\partial_n u \cdot n)n = 0$ that

$$
\int_{\partial \Omega} \partial_n u \cdot \phi dS_x = \int_{\partial \Omega} (\partial_n u \cdot n)n \cdot \phi dS_x = 0,
$$

(A.9)

which, combined with Green’s formula, yield

$$
\sum_{i=1}^3 \int_{\Omega} \nabla u_i \cdot \nabla \phi_i dx = \int_{\Omega} f \cdot \phi dx.
$$

(A.10)

With this variational form, we can employ the standard method in Chapter 5 of [50] or Chapter 2 of [51] to show that this unique weak solution $u$ belongs to $[H^2(\Omega)]^3$ and satisfies

$$
\|u\|_{H^2} \leq C(\|f\|_2 + \|u\|_2) \leq C\|f\|_2,
$$

(A.11)

where (A.8) was used in the last inequality. This proves existence and uniqueness of strong solutions in $[H^2(\Omega)]^3$ and estimate (A.0) for $p = 2$.

Then we consider the case $1 < p < \infty$ and $p \neq 2$. In fact, existence and uniqueness of strong solutions in $W^{2,p}(\Omega)$ for $p \neq 2$ is based on the above $H^2$ theory. Using the method of Chapter 9 of [20] we can show the following regularity property

If $f \in L^p(\Omega)$ and $u \in [W^{2,r}(\Omega)]^3 \cap V^{1,r}(\Omega) \quad (1 < r < q < \infty)$ is a strong solution of (A.1), then $u \in [W^{2,q}(\Omega)]^3 \cap V^{1,q}(\Omega)$.

(A.12)

For $2 < p < \infty$, since $f \in L^p(\Omega) \subset L^2(\Omega)$, it follows from the above $H^2$ theory that there exists a unique strong solution $u \in [H^2(\Omega)]^3 \cap V^{1,2}(\Omega)$. Then the regularity property (A.12) indicates that $u \in W^{2,p}(\Omega) \cap V^{1,p}(\Omega)$. (A.10) follows from this unique solvability in $W^{2,p}(\Omega) \cap V^{1,p}(\Omega)$ and global $L^p$ estimate (A.7).

For $1 < p < 2$, choose approximate sequence $\{f_k\}_{k=1}^\infty \subset L^p(\Omega) \cap L^2(\Omega)$ such that $\lim_{k \to \infty} f_k = f$ in $L^p(\Omega)$. Then $\Sigma^{-1}$ follows from the above $H^2$ theory that there exists a unique strong solution

$$
u_k \in H^2(\Omega) \cap V^{1,2}(\Omega) \subset W^{2,p}(\Omega) \cap V^{1,p}(\Omega), \quad k = 1, 2, \ldots
$$

of the approximate elliptic system

$$
-\Delta u_k = f_k \quad \text{in} \quad \Omega, \quad u_k \cdot n = 0 \quad \text{on} \quad \partial \Omega, \quad \partial_n u_k - (\partial_n u_k \cdot n)n = 0 \quad \text{on} \quad \partial \Omega.
$$

(A.13)

Note that solutions of (A.1) is unique in $W^{2,p}(\Omega) \cap V^{1,p}(\Omega)$. In fact, let $f = 0$ in (A.1) and $\tilde{u}$ be the solution of (A.1) in $W^{2,p}(\Omega) \cap V^{1,p}(\Omega)$. Then the regularity property (A.12) implies $\tilde{u} \in H^2(\Omega) \cap V^{1,2}(\Omega)$, which combined with uniqueness in $H^2(\Omega) \cap V^{1,2}(\Omega)$, indicate $\tilde{u} = 0$. Combining
this uniqueness of solutions in $W^{2,p}(\Omega) \cap V^{1,p}(\Omega)$, global $L^p$ estimate (A.7) and contradiction argument as the proof of Lemma 9.17 in [20], we get
\[ \|u\|_{W^{2,p}} \leq C\|f\|_p \quad \text{for all} \quad u \in W^{2,p}(\Omega) \cap V^{1,p}(\Omega). \] (A.14)
Applying this estimate on $u_k - u_j$, we have
\[ \|u_k - u_j\|_{W^{2,p}} \leq C\|f_k - f_j\|_p \to 0 \quad \text{as} \quad k, j \to \infty. \] (A.15)
This implies that $\{u_k\}_{k=1}^\infty$ is a Cauchy sequence and thus has a limit $u \in W^{2,p}(\Omega) \cap V^{1,p}(\Omega)$. Taking limit in approximate problem (A.13) in $L^p(\Omega)$, we see that this limit $u$ is the solution of (A.1) in $W^{2,p}(\Omega) \cap V^{1,p}(\Omega)$. This completes the proof. \[ \square \]

**Lemma A.3.** Let $g \in L^p(\Omega)$ be given with $1 < p < \infty$. There exists a function $E \in [L^\infty(\Omega)]^3$ with $\|E\|_{L^\infty(\Omega)} \ll 1$, such that the elliptic boundary value problem
\[ -\Delta \phi + E \cdot \nabla \phi = g, \quad \partial_n \phi |_{\partial \Omega} = 0, \quad \int_{\Omega} \phi dx = 0 \] (A.16)
has a unique strong solution $\phi \in W^{2,p}(\Omega)$ satisfying
\[ \|\phi\|_{W^{2,p}} \leq C\|g\|_p. \] (A.17)

**Proof.** We remark that since we do not know whether $\int_{\Omega} g dx$ is zero or not, we have to choose $E$ carefully to avoid compatible conditions on $g$. If $\int_{\Omega} g dx = 0$ then we let $E \equiv 0$, which leads to the well-known Poisson equation with homogeneous Neumann boundary condition. If $\int_{\Omega} g dx \neq 0$, then we choose a non-zero potential field $E = \nabla U \in L^\infty(\Omega)$ satisfying $\|E\|_{\infty} \ll 1$. Note that this selection excludes the case that $\text{div} E = 0$ on $\Omega$ and $E \cdot n = 0$ on $\partial \Omega$, which would leads to $\int_{\Omega} E \cdot \nabla x \phi dx = 0$ and compatible condition of $g$. In the following we show that this selection of $E$ could guarantee unique solvability of (A.16).

Firstly, we establish existence and uniqueness of weak solutions of (A.16) in $H^1(\Omega)$ by Lax-Milgram Theorem. Define Banach space
\[ \tilde{H}^1(\Omega) := \{ v \in H^1(\Omega) : \int_{\Omega} v(x) dx = 0 \}, \quad \|v\|_{\tilde{H}^1(\Omega)} := \|\nabla v\|_{L^2(\Omega)}, \]
thanks to Poincaré inequality. Introduce the bilinear form
\[ B(\phi, v) := \int_{\Omega} \left[ \nabla \phi \cdot \nabla v + (E \cdot \nabla \phi) v \right] dx, \quad \forall v \in \tilde{H}^1(\Omega) \]
and the linear functional
\[ F(v) := \int_{\Omega} g v dx, \quad \forall v \in \tilde{H}^1(\Omega). \]
By Poincaré inequality we have, for all $v \in \tilde{H}^1(\Omega)$,
\[ \left| \int_{\Omega} (E \cdot \nabla \phi) v dx \right| \leq \|E\|_{\infty} \|\nabla \phi\|_2 \|v\|_2 \leq C_p \|E\|_{\infty} \|\nabla \phi\|_2 \|\nabla v\|_2, \]
\[ \left| \int_{\Omega} g v dx \right| \leq \|g\|_2 \|v\|_2 \leq C_p \|g\|_2 \|\nabla v\|_2, \]
where $C_p$ is the Poincaré constant. Then we can see that $F$ is continuous in $\tilde{H}^1(\Omega)$ with
\[ \|F\|_{[\tilde{H}^1(\Omega)]^*} \leq C_p \|g\|_{L^2(\Omega)}, \]
and $B$ is continuous and coercivity in $\tilde{H}^1(\Omega)$
\[ |B(\phi, v)| \leq (1 + C_p \|E\|_{\infty}) \|\nabla \phi\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}, \]
\[ |B(\phi, \phi)| \geq (1 - C_p \|E\|_{\infty}) \|\nabla \phi\|_{L^2(\Omega)}^2, \]
provided $\|E\|_{\infty} \ll 1$ is small enough. Lax-Milgram Theorem indicates that for given $g \in [\tilde{H}^1(\Omega)]^*$, (A.16) has a unique weak solution $\phi \in \tilde{H}^1(\Omega)$ satisfying (A.17) for $p = 2$. 

Then we may employ the standard method of elliptic theory, as shown in the proof of Lemma A.2, to establish solvability of strong solutions in $W^{2,p}(\Omega)$ and uniform estimate (A.17). We omit the details for brevity. □

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