A FAST COMPUTATION OF DENSITY OF EXPONENTIALLY S-NUMBERS

VLADIMIR SHEVELEV

Abstract. The author [4] proved that, for every set \( S \) of positive integers containing 1 (finite or infinite) there exists the density \( h = h(E(S)) \) of the set \( E(S) \) of numbers whose prime factorizations contain exponents only from \( S \), and gave an explicit formula for \( h(E(S)) \). In this paper we give an equivalent polynomial formula for \( \log h(E(S)) \) which allows to get a fast calculation of \( h(E(S)) \).

1. Introduction

Let \( \mathbf{G} \) be the set of all finite or infinite increasing sequences of positive integers beginning with 1. For a sequence \( S = \{s(n)\}, n \geq 1, \) from \( \mathbf{G} \), a positive number \( N \) is called an exponentially \( S \)-number \( (N \in E(S)) \), if all exponents in its prime power factorization are in \( S \). The author [4] proved that, for every sequence \( S \in \mathbf{G} \), the sequence of exponentially \( S \)-numbers has a density \( h = h(E(S)) \in [\frac{6}{\pi^2}, 1] \). More exactly, the following theorem was proved in [4]:

Theorem 1. For every sequence \( S \in \mathbf{G} \) the sequence of exponentially \( S \)-numbers has a density \( h = h(E(S)) \) such that

\[
1 = h(E(S))x + O \left( \sqrt{x} \log x e^{\frac{\sqrt{x}}{\log \log x}} \right),
\]

with \( c = 4\sqrt{\frac{2.4}{\log 2}} = 7.443083... \) and

\[
h(E(S)) = \prod_p \left( 1 + \sum_{i \geq 2} \frac{u(i) - u(i-1)}{p^i} \right),
\]

where the product is over all primes, \( u(n) \) is the characteristic function of sequence \( S : u(n) = 1, \) if \( n \in S \) and \( u(n) = 0 \) otherwise.

In case when \( S \) is the sequence of square-free numbers (see Toth [6]) Arias de Reyna [5, A262276], using the Wrench method of fast calculation [7], did the calculation of \( h \) with a very high degree of accuracy. In this paper, using Wrench's method for formula [2], we find a general representation of \( h(E(S)) \) based on a special polynomial over partitions of \( n \) which allows to get a fast calculation of \( h(E(S)) \) for every \( S \in \mathbf{G} \). Note also that Wrench's
method was successfully realized in a special case by Arias de Reyna, Brent and van de Lune in [2].

Everywhere below we write \( \{ h(E(S)) \} \), understanding \( \{ h(E(S)) \}_{S \in \mathbb{G}} \).

2. A computing idea in Wrench’s style

Consider function given by power series

\[
F_S(x) = 1 + \sum_{i \geq 2} (u(i) - u(i - 1))x^i, \quad x \in (0, \frac{1}{2}).
\]

Since \( u(n) - u(n - 1) \geq -1 \), then \( F_S(x) \geq 1 - \frac{x^2}{1-x} > 0 \). By [2], we have

\[
h(E(S)) = \prod_{p} F_S \left( \frac{1}{p} \right).
\]

and

\[
\log h(E(S)) = \sum_{p} \log F_S(x)|_{x=\frac{1}{p}}.
\]

Let

\[
\log F_S(x) = \sum_{i \geq 2} \frac{f^{(S)}_i}{i} x^i.
\]

Since \( |u(n) - u(n - 1)| \leq 1 \), then by [3], \( F_S(x) \leq 1 + \frac{x^2}{1-x} \) and \( 0 < \log F_S(x) \leq 2x^2 \), \( x \in (0, \frac{1}{2}] \). Thus the series (5) is absolutely convergent. Now, according to [3] - (6), we have

\[
\log h(E(S)) = \sum_{n=2}^{\infty} \frac{f^{(S)}_n}{n} P(n),
\]

where \( P(n) = \sum_p \frac{1}{p^n} \) is the prime zeta function. The series (7) is fast convergent and very suitable for the calculation of \( h(E(S)) \).

3. A recursion for coefficients

Denoting

\[
v_n = u(n) - u(n - 1), \quad n \geq 2,
\]

by [3] and (6), we have

\[
F_S(x) = 1 + \sum_{n \geq 2} v_n x^n,
\]

\[
\log(1 + \sum_{n \geq 2} v_n x^n) = \sum_{i \geq 2} \frac{f^{(S)}_i}{i} x^i.
\]

Lemma 1. Coefficients \( \{ f^{(S)}_n \} \) satisfy the recurrence
(11) \( f_{n+1}^{(S)} = (n + 1)v_{n+1} - \sum_{i=1}^{n-2} v_{n-i} f_{i+1}^{(S)}, \quad n \geq 1. \)

Proof. Differentiating (10), we have
\[
\frac{\sum_{n \geq 2} n v_n x^{n-1}}{F_S(x)} = \sum_{j=1}^{n+1} f_j^{(S)} x^j.
\]
Hence,
\[
\sum_{n \geq 2} n v_n x^{n-1} = (1 + \sum_{n \geq 2} v_n x^n)(\sum_{j=1}^{n+1} f_j^{(S)} x^j).
\]
Equating the coefficients of \( x^n \) in both sides, we get
\[
(n + 1)v_{n+1} = f_{n+1}^{(S)} + \sum_{j=1}^{n-2} v_{n-j} f_{j+1}^{(S)}
\]
and the lemma follows. \( \square \)

Corollary 1. All \( \{ f_n^{(S)} \} \) are integers.

Proof. For \( n=1,2,3 \), by the recurrence (11), we have
\[
f_2^{(S)} = 2v_2, \quad f_3^{(S)} = 3v_3, \quad f_4^{(S)} = 4v_4 - 2v_2^2, \]
now the corollary follows by induction. \( \square \)

4. Explicit polynomial formula

To apply (10) we need a fast way to generate the coefficients \( f_i^{(S)} \). Since, for \( x \in (0, \frac{1}{2}) \), \( \sum_{n \geq 2} v_n x^n \leq \frac{x^2}{1-x} \leq \frac{1}{2} \), then
\[
(12) \quad \log(1 + \sum_{n \geq 2} v_n x^n) = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} (\sum_{n \geq 2} v_n x^n)^m.
\]
Expanding these powers, we get a great sum of terms of type
\[
(13) \quad t_{\lambda_1, s_1} (v_{\lambda_1} x^{\lambda_1})^{s_1} \ldots t_{\lambda_r, s_r} (v_{\lambda_r} x^{\lambda_r})^{s_r}, \quad s_i \geq 1, \quad \lambda_i \geq 2.
\]
When we collect all the terms with a fixed sum of exponents of \( x \), say, \( n \), we get a sum of terms (13) with \( \lambda_1 s_1 + \ldots + \lambda_r s_r = n \), i.e., we have \( s_i \) parts \( \lambda_i \) in partition of \( n \). Therefore, the considered expansion has the form
\[
\log(1 + \sum_{n \geq 2} v_n x^n) = \sum_{n \geq 2} \left( \sum_{\sigma \in \Sigma_n} t_\sigma v_\sigma \right) \frac{x^n}{n} = \sum_{n \geq 2} \frac{f_n^{(S)}}{n} x^n,
\]
where \( \Sigma_n \) is the set of the partitions \( \{ \sigma \} \) of \( n \) with parts \( \lambda_i \geq 2 \) and \( t_\sigma, v_\sigma \) are functions of partitions \( \sigma \) defined by (13) such that with every partition \( \sigma \) of \( n \) we associate the monomial
(14) \[ v_{\sigma} = \prod_{i=1}^{r} v_{\lambda_i}^{s_i} \ (\lambda_1 s_1 + ... + \lambda_r s_r = n, \ \lambda_i \geq 2). \]

So

(15) \[ f_n^{(S)} = \sum_{\sigma \in \Sigma_n} t_{\sigma} v_{\sigma}. \]

Substituting (15) in equation (11), we get

\[
\sum_{\sigma \in \Sigma_{n+1}} t_{\sigma} v_{\sigma} = (n + 1)v_{n+1} - \sum_{i=1}^{n-2} v_{n-i} \sum_{\sigma \in \Sigma_{i+1}} t_{\sigma} v_{\sigma} =
\]

(16) \[ (n + 1)v_{n+1} - \sum_{j=2}^{n-1} v_j \sum_{\sigma \in \Sigma_{n+1-j}} t_{\sigma} v_{\sigma}. \]

Note that, using (16), one can proved that all coefficients \( t_{\sigma} \) are integer numbers. Let partition \( \sigma = (b_2, ..., b_{n+1}) \in \Sigma_{n+1} \) contains \( b_2 \) elements 2, ..., \( b_{n+1} \) elements \( n + 1 \) such that \( 2b_2 + ... + (n + 1)b_{n+1} = n + 1, \ b_i \geq 0 \). In particular, evidently, \( b_{n+1} = 0 \) or 1 and in the latter case all other \( b_i = 0 \).

We shall write \( v_{\sigma} = v_{b_2}^{b_2} ... v_{b_{n+1}}^{b_{n+1}} \) and \( t_{\sigma} = t(v_{b_2}^{b_2} ... v_{b_{n+1}}^{b_{n+1}}) \). According to (16), the coefficient of the monomial \( v_{n}^{0} ... v_{n+1}^{0} v_{n+1}^{1} \) equals \( n + 1 \), i.e., for partition of \( n + 1 \) with only part we have \( t(\sigma) = n + 1 \). We agree that \( 0^0 = 1 \).

Denote by \( \Sigma'_{n+1} \) the set of partitions of \( n + 1 \) with parts \( \geq 2 \) and \( \leq n \).

Then, by (16), we have

(17) \[ \sum_{\sigma \in \Sigma_{n+1}} t_{\sigma} v_{\sigma} = - \sum_{j=2}^{n-1} v_j \sum_{\sigma \in \Sigma'_{n+1-j}} t_{\sigma} v_{\sigma}. \]

For every partition \( (b_2, ..., b_{n+1}) \in \Sigma'_{n+1} \) we have \( b_{n+1} = 0 \) and \( b_n = 0 \) (the latter since all parts \( \geq 2 \)). Then (17) leads to the formula:

\[
t(v_{b_2}^{b_2} ... v_{b_{n-1}}^{b_{n-1}} v_{b_{n+1}}^{0} v_{n+1}^{0}) = -t(v_{b_2}^{b_2-1} v_{b_3}^{b_3} ... v_{b_{n-1}}^{b_{n-1}} v_{n}^{0} v_{n+1}^{0}) -
\]

(18) \[ t(v_{b_2}^{b_2} v_{b_3}^{b_3} ... v_{n-1}^{b_{n-1}} v_{n+1}^{0} v_{n+1}^{0}) - ... - t(v_{b_2}^{b_2} v_{b_3}^{b_3} ... v_{n-1}^{b_{n-1}-1} v_{n}^{0} v_{n+1}^{0}). \]

Using (18), we find an explicit formula for \( f_n^{(S)} \).

**Lemma 2.** Let, for \( n \geq 3 \), \( (b_2, ..., b_{n-1}, 0, 0) \in \Sigma'_{n+1} \). Then

(19) \[ t(v_{b_2}^{b_2} ... v_{b_{n-1}}^{b_{n-1}} v_{n}^{0} v_{n+1}^{0}) = (-1)^{b_{n-1}-1} \frac{(B_{n-1}-1)!}{b_2! ... b_{n-1}!} (n + 1), \]

where \( B_{n-1} = b_2 + ... + b_{n-1} \).

**Proof.** Let \( n = 3 \). We saw that \( f_4^{(S)} = 4v_4 - 2v_2^2 \). So, \( t(v_2^{b_2}) = -2 \) with
$b_2 = 2$ and, by \eqref{eq:20}, we also obtain $t(v_2^{b_2}) = -2$. Let the lemma holds for $t(v_2^{b_2} \ldots v_{n-1}^{b_{n-1}})$, $n \geq 3$, where all $c_i \leq b_i$ such that not all equalities hold. Then, by the relation \eqref{eq:18} and the induction supposition, we have

$$t(v_2^{b_2} \ldots v_{n-1}^{b_{n-1}}) = -(-1)^{B_{n-1}}(B_{n-1} - 2)!/(b_2 - 1)!b_3! \ldots b_{n-1}!(n + 1 - 2) +$$

$$\frac{(B_{n-1} - 2)!}{b_2!(b_3 - 1)! \ldots b_{n-1}!}(n + 1 - 3) + \ldots + \frac{(B_{n-1} - 1)!}{b_2!b_3! \ldots (b_{n-1} - 1)!}(n + 1 - (n - 1)) =$$

$$(-1)^{B_{n-1}-1}(B_{n-1} - 2)!/b_2! \ldots b_{n-1}!(b_2(n + 1 - 2) + b_3(n + 1 - 3) + \ldots +$$

$$b_{n-1}(n + 1 - (n - 1)) = (-1)^{B_{n-1}-1}(B_{n-1} - 2)!/b_2! \ldots b_{n-1}!(B_{n-1}(n + 1) -$$

$$(2b_2 + 3b_3 + \ldots + (n - 1)b_{n-1})$$

and, since $2b_2 + 3b_3 + \ldots + (n - 1)b_{n-1} = n + 1$, the lemma follows. \hfill \Box

**Corollary 2.** Let, for $n \geq 3$, $(b_2, \ldots, b_{n+1}) \in \Sigma_{n+1}$. Then

$$t(v_2^{b_2} \ldots v_{n+1}^{b_{n+1}}) = (\delta(b_{n+1}, 1) + (-1)^{B_{n+1}}(B_{n+1} - 1)!/b_2! \ldots b_{n+1}!)(n + 1),$$

where $B_{n+1} = b_2 + \ldots + b_{n+1}$.

**Proof.** The statement follows from Lemma \eqref{Lemma2} and addition of the coefficient $n + 1$ of $v_{n+1}$ in equation \eqref{eq:16} in case when $\delta(b_{n+1}, 1) = 1$. \hfill \Box

Now, using \eqref{eq:7}, \eqref{eq:15}, Corollary \eqref{Corollary2} and the initial values of the coefficients $f_2^{(S)} = 2v_2$, $f_3^{(S)} = 3v_3$, and changing $n$ by $n - 1$, we get a suitable formula to compute $\log h(E(S))$.

**Theorem 2.** We have

$$\log h(E(S)) = P(2)v_2 + P(3)v_3 + \sum_{n=4}^{\infty} P(n)(v_n + M(v_2, \ldots, v_{n-2})),$$

where $P(n)$ is the prime zeta function, $M$ is the polynomial defined as

$$M(v_2, \ldots, v_{n-2}) = \sum_{2b_2 + \ldots + (n-2)b_{n-2} = n} (-1)^{B_{n-2}}(B_{n-2} - 1)!/b_2! \ldots b_{n-2}! v_2^{b_2} \ldots v_{n-2}^{b_{n-2}},$$

where $B_{n-2} = b_2 + \ldots + b_{n-2}$, $b_i \geq 0$, $i = 2, \ldots, n - 2$, $n \geq 4$.

In particular, for $n = 4, 5, 6, \ldots$, we have

$$M(v_2) = -\frac{v_2^2}{2}, M(v_2, v_3) = -v_2v_3, M(v_2, v_3, v_4) = -v_2v_4 - \frac{v_3^2}{2} + \frac{v_4^2}{3}, \ldots$$

For example, in case $n = 6$ the diophantine equation $2b_2 + 3b_3 + 4b_4 = 6$ has 3 solutions.
a) $b_2 = 1, b_3 = 0, b_4 = 1$ with $B_4 = 2$;

b) $b_2 = 0, b_3 = 2, b_4 = 0$ with $B_4 = 2$;

c) $b_2 = 3, b_3 = 0, b_4 = 0$ with $B_4 = 3$.

Besides, using (11), for $M_n = M_n(v_2, ..., v_{n-2})$ we have the recursion

\[
M_2 = 0, M_3 = 0, M_n = -\frac{1}{n} \sum_{j=2}^{n-2} j v_{n-j}(v_j + M_j), \ n \geq 4
\]

which, possibly, more suitable for fast calculations by Theorem 2.

5. Examples

1) As we already mentioned, in case when $S$ is the sequence of square-free numbers, Arias de Reyna [5,A262276] obtained

\[
h = \prod_p \left( 1 + \sum_{i \geq 4} \frac{\mu(i)^2 - \mu(i-1)^2}{p^i} \right) = 0.95592301586190237688...
\]

By the results of [1], the coefficients $f_n^{(S)}$ [15] in this case (see A262400 [5]) have very interesting congruence properties.

2) The case of $S = 2^n$ was essentially considered by the author [3]. He found that $h = 0.872497...$ The author asked Arias de Reyna to get more digits. Using Theorem 2, he obtained

\[
h = 0.87249717935391281355...
\]

3) Among the other several calculations by Arias de Reyna, we give the following one. Let $S$ be 1 and the primes (A008578 [5]). Then

\[
h = 0.9467193375527801046...
\]

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DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY OF THE NEGEV, BEER-SHEVA 84105, ISRAEL. E-MAIL:shevelev@bgu.ac.il