Hyperbolic Complex Numbers in Two Dimensions

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Abstract

A system of commutative hyperbolic complex numbers in 2 dimensions is studied in this paper. Exponential and trigonometric forms are obtained for the hyperbolic two-complex numbers. Expressions are given for the elementary functions of hyperbolic two-complex variable. The functions of a hyperbolic two-complex variable which are defined by power series are analytic. Relations of equality exist between partial derivatives of the real components a function of a hyperbolic two-complex variable. The integral of a two-complex function between two points is independent of the path connecting the points. A hyperbolic two-complex polynomial can be written as a product of linear or quadratic factors, although the factorization may not be unique.

1 Introduction

A regular, two-dimensional complex number $x + iy$ can be represented geometrically by the modulus $\rho = (x^2 + y^2)^{1/2}$ and by the polar angle $\theta = \arctan(y/x)$. The modulus $\rho$ is multiplicative and the polar angle $\theta$ is additive upon the multiplication of ordinary complex numbers.

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The quaternions of Hamilton are a system of hypercomplex numbers defined in four dimensions, the multiplication being a noncommutative operation, \([1]\) and many other hypercomplex systems are possible, \([2]-[4]\) but these interesting hypercomplex systems do not have all the required properties of regular, two-dimensional complex numbers which rendered possible the development of the theory of functions of a complex variable.

A system of hypercomplex numbers in 2 dimensions is described in this work, for which the multiplication is associative and commutative, and for which an exponential form and the concepts of analytic twocomplex function and contour integration can be defined. The twocomplex numbers introduced in this work have the form \(u = x + \delta y\), the variables \(x, y\) being real numbers. The multiplication rules for the complex units \(1, \delta\) are \(1 \cdot \delta = \delta, \delta^2 = 1\).

In a geometric representation, the twocomplex number \(u\) is represented by the point \(A\) of coordinates \((x, y)\). The product of two twocomplex numbers is equal to zero if both numbers are equal to zero, or if one of the twocomplex numbers lies on the line \(x = y\) and the other on the line \(x = -y\).

The exponential form of a twocomplex number, defined for \(x + y > 0, x - y > 0\), is \(u = \rho \exp(\delta \lambda/2)\), where the amplitude is \(\rho = (x^2 - y^2)^{1/2}\) and the argument is \(\lambda = \ln \tan \theta\), \(\tan \theta = (x + y)/(x - y), 0 < \theta < \pi/2\). The trigonometric form of a twocomplex number is \(u = d \sqrt{\sin 2\theta} \exp\{(1/2)\delta \ln \tan \theta\}\), where \(d^2 = x^2 + y^2\). The amplitude \(\rho\) is equal to zero on the lines \(x = \pm y\). The division \(1/(x + \delta y)\) is possible provided that \(\rho \neq 0\). If \(u_1 = x_1 + \delta y_1, u_2 = x_2 + \delta y_2\) are twocomplex numbers of amplitudes and arguments \(\rho_1, \lambda_1\) and respectively \(\rho_2, \lambda_2\), then the amplitude and the argument \(\rho, \lambda\) of the product twocomplex number \(u_1 u_2 = x_1 x_2 + y_1 y_2 + \delta(x_1 y_2 + y_1 x_2)\) are \(\rho = \rho_1 \rho_2, \lambda = \lambda_1 + \lambda_2\). Thus, the amplitude \(\rho\) is a multiplicative quantity and the argument \(\lambda\) is an additive quantity upon the multiplication of twocomplex numbers, which reminds the properties of ordinary, two-dimensional complex numbers.

Expressions are given for the elementary functions of twocomplex variable. Moreover, it is shown that the region of convergence of series of powers of twocomplex variables is a rectangle having the sides parallel to the bisectors \(x = \pm y\).

A function \(f(u)\) of the twocomplex variable \(u = x + \delta y\) can be defined by a corresponding power series. It will be shown that the function \(f\) has a derivative \(\lim_{u \to u_0}[f(u) - f(u_0)]/(u-u_0)\).
u_0) independent of the direction of approach of u to u_0. If the twocomplex function f(u) of the twocomplex variable u is written in terms of the real functions P(x, y), Q(x, y) of real variables x, y as f(u) = P(x, y) + δQ(x, y), then relations of equality exist between partial derivatives of the functions P, Q, and the functions P, Q are solutions of the two-dimensional wave equation.

It will also be shown that the integral $\int_A^B f(u)du$ of a twocomplex function between two points A, B is independent of the path connecting the points A, B.

A polynomial $u^n + a_1u^{n-1} + \cdots + a_{n-1}u + a_n$ can be written as a product of linear or quadratic factors, although the factorization may not be unique.

This paper belongs to a series of studies on commutative complex numbers in n dimensions.\[5\] The twocomplex numbers described in this work are a particular case for n = 2 of the polar hypercomplex numbers in n dimensions.\[5, 6\]

2 Operations with hyperbolic twocomplex numbers

A hyperbolic complex number in two dimensions is determined by its two components (x, y). The sum of the hyperbolic twocomplex numbers (x, y) and (x', y') is the hyperbolic twocomplex number (x + x', y + y'). The product of the hyperbolic twocomplex numbers (x, y) and (x', y') is defined in this work to be the hyperbolic twocomplex number (xx' + yy', xy' + yx').

Twocomplex numbers and their operations can be represented by writing the twocomplex number (x, y) as $u = x + \delta y$, where $\delta$ is a basis for which the multiplication rules are

\begin{equation}
1 \cdot \delta = \delta, \ \delta^2 = 1.
\end{equation}

Two twocomplex numbers $u = x + \delta y, u' = x' + \delta y'$ are equal, $u = u'$, if and only if $x = x', y = y'$. If $u = x + \delta y, u' = x' + \delta y'$ are twocomplex numbers, the sum $u + u'$ and the product $uu'$ defined above can be obtained by applying the usual algebraic rules to the sum $(x + \delta y) + (x' + \delta y')$ and to the product $(x + \delta y)(x' + \delta y')$, and grouping of the resulting terms,

\begin{equation}
u + u' = x + x' + \delta(y + y'),
\end{equation}

\begin{equation}uu' = xx' + yy' + \delta(xy' + yx').\end{equation}
If \( u, u', u'' \) are twocomplex numbers, the multiplication is associative

\[(uu')u'' = u(u'u'')\] (4)

and commutative

\[uu' = u'u,\] (5)

as can be checked through direct calculation. The twocomplex zero is \( 0 + \delta \cdot 0 \), denoted simply 0, and the twocomplex unity is \( 1 + \delta \cdot 0 \), denoted simply 1.

The inverse of the twocomplex number \( u = x + \delta y \) is a twocomplex number \( u' = x' + \delta y' \) having the property that

\[uu' = 1.\] (6)

Written on components, the condition, Eq. (6), is

\[xx' + yy' = 1,\]
\[yx' + xy' = 0.\] (7)

The system (6) has the solution

\[x' = \frac{x}{\nu},\] (8)
\[y' = -\frac{y}{\nu},\] (9)

provided that \( \nu \neq 0 \), where

\[\nu = x^2 - y^2.\] (10)

The quantity \( \nu \) can be written as

\[\nu = v_+ v_,\] (11)

where

\[v_+ = x + y, v_- = x - y.\] (12)

The variables \( v_+, v_- \) will be called canonical hyperbolic twocomplex variables. Then a twocomplex number \( u = x + \delta y \) has an inverse, unless

\[v_+ = 0, \text{ or } v_- = 0.\] (13)
For arbitrary values of the variables $x, y$, the quantity $\nu$ can be positive or negative. If $\nu \geq 0$, the quantity
\[
\rho = \nu^{1/2}, \quad \nu > 0,
\] (14)
will be called amplitude of the twocomplex number $x + \delta y$. The normals of the lines in Eq. (13) are orthogonal to each other. Because of conditions (13) these lines will be also called the nodal lines. It can be shown that if $uu' = 0$ then either $u = 0$, or $u' = 0$, or one of the twocomplex numbers $u, u'$ is of the form $x + \delta x$ and the other is of the form $x - \delta x$.

### 3 Geometric representation of hyperbolic twocomplex numbers

The twocomplex number $x + \delta y$ can be represented by the point $A$ of coordinates $(x, y)$. If $O$ is the origin of the two-dimensional space $x, y$, the distance from $A$ to the origin $O$ can be taken as
\[
d^2 = x^2 + y^2.
\] (15)
The distance $d$ will be called modulus of the twocomplex number $x + \delta y$.

Since
\[
(x + y)^2 + (x - y)^2 = 2d^2,
\] (16)
$x + y$ and $x - y$ can be written as
\[
x + y = \sqrt{2}d \sin \theta, \quad x - y = \sqrt{2}d \cos \theta,
\] (17)
so that
\[
x = d \sin(\theta + \pi/4), \quad y = -d \cos(\theta + \pi/4).
\] (18)
If $u = x + \delta y, u_1 = x_1 + \delta y_1, u_2 = x_2 + \delta y_2$, and $u = u_1u_2$, and if
\[
v_{j+} = x_j + y_j, \quad v_{j-} = x_j - y_j, \quad 2d_j^2 = v_{j+}^2 + v_{j-}^2, \quad x_j + y_j = \sqrt{2}d_j \sin \theta_j, \quad x_j - y_j = d_j \sqrt{2} \cos \theta_j,
\] (19)
for $j = 1, 2$, it can be shown that
\[
v_+ = v_{1+}v_{2+}, \quad v_- = v_{1-}v_{2-}, \quad \tan \theta = \tan \theta_1 \tan \theta_2.
\] (20)
The relations (20) are a consequence of the identities

\[(x_1 x_2 + y_1 y_2) + (x_1 y_2 + y_1 x_2) = (x_1 + y_1)(x_2 + y_2),\]  
\[(x_1 x_2 + y_1 y_2) - (x_1 y_2 + y_1 x_2) = (x_1 - y_1)(x_2 - y_2).\]

A consequence of Eqs. (20) is that if \(u = u_1 u_2\), then

\[\nu = \nu_1 \nu_2,\]  
where

\[\nu_j = v_j^+ v_j^-,\]

for \(j = 1, 2\). If \(\nu > 0, \nu_1 > 0, \nu_2 > 0\), then

\[\rho = \rho_1 \rho_2,\]  
where

\[\rho_j = \nu_j^{1/2},\]

for \(j = 1, 2\).

The twocomplex numbers

\[e_+ = \frac{1 + \delta}{2}, \quad e_- = \frac{1 - \delta}{2},\]

are orthogonal,

\[e_+ e_- = 0,\]

and have also the property that

\[e_+^2 = e_+, \quad e_-^2 = e_.\]

The ensemble \(e_+, e_-\) will be called the canonical hyperbolic twocomplex base. The twocomplex number \(u = x + \delta y\) can be written as

\[x + \delta y = (x + y)e_+ + (x - y)e_-,\]

or, by using Eq. (12),

\[u = v_+ e_+ + v_- e_-,\]
which will be called the canonical form of the hyperbolic two complex number. Thus, if
\[ u_j = v_{j+} e_+ + v_{j-} e_-, \; j = 1, 2, \] and \( u = u_1 u_2, \) then the multiplication of the hyperbolic
two complex numbers is expressed by the relations \((20)\).

The relation \((23)\) for the product of two complex numbers can be demonstrated also by
using a representation of the multiplication of the two complex numbers by matrices, in which
the two complex number \( u = x + \delta y \) is represented by the matrix
\[
\begin{pmatrix}
x & y \\
y & x \\
\end{pmatrix},
\]
\((32)\)
The product \( u = x + \delta y \) of the two complex numbers \( u_1 = x_1 + \delta y_1, u_2 = x_2 + \delta y_2, \) can be
represented by the matrix multiplication
\[
\begin{pmatrix}
x & y \\
y & x \\
\end{pmatrix} = \begin{pmatrix}
x_1 & y_1 \\
y_1 & x_1 \\
\end{pmatrix} \begin{pmatrix}
x_2 & y_2 \\
y_2 & x_2 \\
\end{pmatrix},
\]
\((33)\)
It can be checked that
\[
\det \begin{pmatrix}
x & y \\
y & x \\
\end{pmatrix} = \nu.
\]
\((34)\)
The identity \((23)\) is then a consequence of the fact the determinant of the product of matrices
is equal to the product of the determinants of the factor matrices.

4 Exponential and trigonometric forms of a two complex number

The exponential function of the two complex variable \( u \) can be defined by the series
\[
\exp u = 1 + u + u^2/2! + u^3/3! + \cdots.
\]
\((35)\)
It can be checked by direct multiplication of the series that
\[
\exp(u + u') = \exp u \cdot \exp u'.
\]
\((36)\)
If \( u = x + \delta y, \) then \( \exp u \) can be calculated as \( \exp u = \exp x \cdot \exp(\delta y). \) According to Eq. \((1)\),
\[
\delta^{2m} = 1, \; \delta^{2m+1} = \delta,
\]
\((37)\)
where $m$ is a natural number, so that $\exp(\delta y)$ can be written as

$$
\exp(\delta y) = \cosh y + \delta \sinh y.
$$

(38)

From Eq. (38) it can be inferred that

$$(\cosh t + \delta \sinh t)^m = \cosh mt + \delta \sinh mt.$$

(39)

The two-complex numbers $u = x + \delta y$ for which $v_+ = x + y > 0$, $v_- = x - y > 0$ can be written in the form

$$
x + \delta y = e^{x_1 + \delta y_1}.
$$

(40)

The expressions of $x_1, y_1$ as functions of $x, y$ can be obtained by developing $e^{\delta y_1}$ with the aid of Eq. (38) and separating the hypercomplex components,

$$
x = e^{x_1} \cosh y_1,
$$

(41)

$$
y = e^{x_1} \sinh y_1,
$$

(42)

It can be shown from Eqs. (41)-(42) that

$$
x_1 = \frac{1}{2} \ln(v_+v_-), \quad y_1 = \frac{1}{2} \ln \frac{v_+}{v_-}.
$$

(43)

The two-complex number $u$ can thus be written as

$$
u = \rho \exp(\delta \lambda),
$$

(44)

where the amplitude is $\rho = (x^2 - y^2)^{1/2}$ and the argument is $\lambda = (1/2) \ln \{(x + y)/(x - y)\}$, for $x + y > 0$, $x - y > 0$. The expression (44) can be written with the aid of the variables $d, \theta$, Eq. (17), as

$$
u = \rho \exp \left(\frac{1}{2} \delta \ln \tan \theta\right),
$$

(45)

which is the exponential form of the two-complex number $u$, where $0 < \theta < \pi/2$.

The relation between the amplitude $\rho$ and the distance $d$ is

$$
\rho = d \sin^{1/2} \theta.
$$

(46)

Substituting this form of $\rho$ in Eq. (45) yields

$$
u = d \sin^{1/2} \theta \exp \left(\frac{1}{2} \delta \ln \tan \theta\right),
$$

(47)

which is the trigonometric form of the two-complex number $u$. 

8
5 Elementary functions of a twocomplex variable

The logarithm $u_1$ of the twocomplex number $u$, $u_1 = \ln u$, can be defined for $v_+ > 0, v_- > 0$ as the solution of the equation

$$u = e^{u_1},$$

(48)

for $u_1$ as a function of $u$. From Eq. (45) it results that

$$\ln u = \ln \rho + \frac{1}{2} \delta \ln \tan \theta$$

(49)

It can be inferred from Eqs. (49) and (20) that

$$\ln(u_1 u_2) = \ln u_1 + \ln u_2.$$  

(50)

The explicit form of Eq. (49) is

$$\ln(x + \delta y) = \frac{1}{2} (1 + \delta) \ln(x + y) + \frac{1}{2} (1 - \delta) \ln(x - y),$$

(51)

so that the relation (49) can be written with the aid of Eq. (27) as

$$\ln u = e_+ \ln v_+ + e_- \ln v_-.$$  

(52)

The power function $u^n$ can be defined for $v_+ > 0, v_- > 0$ and real values of $n$ as

$$u^n = e^{n \ln u}.$$  

(53)

It can be inferred from Eqs. (53) and (50) that

$$(u_1 u_2)^n = u_1^n u_2^n.$$  

(54)

Using the expression (52) for $\ln u$ and the relations (28) and (29) it can be shown that

$$(x + \delta y)^n = \frac{1}{2} (1 + \delta) (x + y)^n + \frac{1}{2} (1 - \delta) (x - y)^n.$$  

(55)

For integer $n$, the relation (55) is valid for any $x, y$. The relation (55) for $n = -1$ is

$$\frac{1}{x + \delta y} = \frac{1}{2} \left( \frac{1 + \delta}{x + y} + \frac{1 - \delta}{x - y} \right).$$  

(56)

The trigonometric functions $\cos u$ and $\sin u$ of a twocomplex variable are defined by the series

$$\cos u = 1 - u^2/2! + u^4/4! + \cdots,$$  

(57)
\[ \sin u = u - u^3/3! + u^5/5! + \cdots. \]

(58)

It can be checked by series multiplication that the usual addition theorems hold also for the twocomplex numbers \( u_1, u_2, \)

\[ \cos(u_1 + u_2) = \cos u_1 \cos u_2 - \sin u_1 \sin u_2, \]

(59)

\[ \sin(u_1 + u_2) = \sin u_1 \cos u_2 + \cos u_1 \sin u_2. \]

(60)

The cosine and sine functions of the hypercomplex variables \( \delta y \) can be expressed as

\[ \cos \delta y = \cos y, \quad \sin \delta y = \delta \sin y. \]

(61)

The cosine and sine functions of a twocomplex number \( x + \delta y \) can then be expressed in terms of elementary functions with the aid of the addition theorems Eqs. (59), (60) and of the expressions in Eq. (61).

The hyperbolic functions \( \cosh u \) and \( \sinh u \) of the twocomplex variable \( u \) are defined by the series

\[ \cosh u = 1 + u^2/2! + u^4/4! + \cdots, \]

(62)

\[ \sinh u = u + u^3/3! + u^5/5! + \cdots. \]

(63)

It can be checked by series multiplication that the usual addition theorems hold also for the twocomplex numbers \( u_1, u_2, \)

\[ \cosh(u_1 + u_2) = \cosh u_1 \cosh u_2 + \sinh u_1 \sinh u_2, \]

(64)

\[ \sinh(u_1 + u_2) = \sinh u_1 \cosh u_2 + \cosh u_1 \sinh u_2. \]

(65)

The \( \cosh \) and \( \sinh \) functions of the hypercomplex variables \( \delta y \) can be expressed as

\[ \cosh \delta y = \cosh y, \quad \sinh \delta y = \delta \sinh y. \]

(66)

The hyperbolic cosine and sine functions of a twocomplex number \( x + \delta y \) can then be expressed in terms of elementary functions with the aid of the addition theorems Eqs. (64), (65) and of the expressions in Eq. (66).
6 Twocomplex power series

A twocomplex series is an infinite sum of the form

\[ a_0 + a_1 + a_2 + \cdots + a_n + \cdots, \quad (67) \]

where the coefficients \( a_n \) are twocomplex numbers. The convergence of the series (67) can be defined in terms of the convergence of its 2 real components. The convergence of a twocomplex series can however be studied using twocomplex variables. The main criterion for absolute convergence remains the comparison theorem, but this requires a number of inequalities which will be discussed further.

The modulus of a twocomplex number \( u = x + \delta y \) can be defined as

\[ |u| = (x^2 + y^2)^{1/2}, \quad (68) \]

so that according to Eq. (63) \( d = |u| \). Since \( |x| \leq |u|, |y| \leq |u| \), a property of absolute convergence established via a comparison theorem based on the modulus of the series (67) will ensure the absolute convergence of each real component of that series.

The modulus of the sum \( u_1 + u_2 \) of the twocomplex numbers \( u_1, u_2 \) fulfills the inequality

\[ ||u_1| - |u_2|| \leq |u_1 + u_2| \leq |u_1| + |u_2|. \quad (69) \]

For the product the relation is

\[ |u_1u_2| \leq \sqrt{2}|u_1||u_2|, \quad (70) \]

which replaces the relation of equality extant for regular complex numbers. The equality in Eq. (70) takes place for \( x_1 = y_1, x_2 = y_2 \) or \( x_1 = -y_1, x_2 = -y_2 \). In particular

\[ |u^2| \leq \sqrt{2}|u|^2. \quad (71) \]

The inequality in Eq. (70) implies that

\[ |u^m| \leq 2^{(m-1)/2}|u|^m. \quad (72) \]

From Eqs. (70) and (72) it results that

\[ |au^m| \leq 2^{m/2}|a||u|^m. \quad (73) \]
A power series of the two complex variable \( u \) is a series of the form

\[
a_0 + a_1 u + a_2 u^2 + \cdots + a_l u^l + \cdots.
\] (74)

Since

\[
\left| \sum_{l=0}^{\infty} a_l u^l \right| \leq \sum_{l=0}^{\infty} 2^{l/2} |a_l||u|^l,
\] (75)

a sufficient condition for the absolute convergence of this series is that

\[
\lim_{l \to \infty} \frac{\sqrt{2}|a_{l+1}||u|}{|a_l|} < 1.
\] (76)

Thus the series is absolutely convergent for

\[
|u| < c_0,
\] (77)

where

\[
c_0 = \lim_{l \to \infty} \frac{|a_l|}{\sqrt{2}|a_{l+1}|}.
\] (78)

The convergence of the series (74) can be also studied with the aid of the formula (55)

which, for integer values of \( l \), is valid for any \( x, y, z, t \). If \( a_l = a_l x + \delta a_l y \), and

\[
A_{l+} = a_l x + a_l y, A_{l-} = a_l x - a_l y,
\] (79)

it can be shown with the aid of relations (28) and (29) that

\[
a_l e_+ = A_{l+} e_+, a_l e_- = A_{l-} e_-,
\] (80)

so that the expression of the series (74) becomes

\[
\sum_{l=0}^{\infty} \left( A_{l+} v^l_+ e_+ + A_{l-} v^l_- e_- \right),
\] (81)

where the quantities \( v_+, v_- \) have been defined in Eq. (12). The sufficient conditions for the absolute convergence of the series in Eq. (81) are that

\[
\lim_{l \to \infty} \frac{|A_{l+1,+}||v_+|}{|A_{l+}|} < 1, \lim_{l \to \infty} \frac{|A_{l+1,-}||v_-|}{|A_{l-}|} < 1.
\] (82)

Thus the series in Eq. (81) is absolutely convergent for

\[
|x + y| < c_+, |x - y| < c_-,
\] (83)
where
\[ c_+ = \lim_{l \to \infty} \frac{|A_{l+}|}{|A_{l+1,+}|}, \quad c_- = \lim_{l \to \infty} \frac{|A_{l-}|}{|A_{l+1,-}|}. \quad (84) \]

The relations (83) show that the region of convergence of the series (81) is a rectangle having the sides parallel to the bisectors \( x = \pm y \). It can be shown that \( c_0 = (1/\sqrt{2})\min(c, c') \), where \( \min(c, c') \) designates the smallest of the numbers \( c, c' \). Since \( |u|^2 = (v_+^2 + v_-^2)/2 \), it can be seen that the circular region of convergence defined in Eqs. (77), (78) is included in the parallelogram defined in Eqs. (83) and (84).

7 Analytic functions of twocomplex variables

The derivative of a function \( f(u) \) of the twocomplex variables \( u \) is defined as a function \( f'(u) \) having the property that
\[ |f(u) - f(u_0) - f'(u_0)(u - u_0)| \to 0 \quad \text{as} \quad |u - u_0| \to 0. \quad (85) \]

If the difference \( u - u_0 \) is not parallel to one of the nodal hypersurfaces, the definition in Eq. (85) can also be written as
\[ f'(u_0) = \lim_{u \to u_0} \frac{f(u) - f(u_0)}{u - u_0}. \quad (86) \]

The derivative of the function \( f(u) = u^m \), with \( m \) an integer, is \( f'(u) = mu^{m-1} \), as can be seen by developing \( u^m = [u_0 + (u - u_0)]^m \) as
\[ u^m = \sum_{p=0}^{m} \frac{m!}{p!(m-p)!} u_0^{m-p}(u_0 - u)^p, \quad (87) \]
and using the definition (85).

If the function \( f'(u) \) defined in Eq. (85) is independent of the direction in space along which \( u \) is approaching \( u_0 \), the function \( f(u) \) is said to be analytic, analogously to the case of functions of regular complex variables. The function \( u^m \), with \( m \) an integer, of the twocomplex variable \( u \) is analytic, because the difference \( u^m - u_0^m \) is always proportional to \( u - u_0 \), as can be seen from Eq. (87). Then series of integer powers of \( u \) will also be analytic functions of the twocomplex variable \( u \), and this result holds in fact for any commutative algebra.
If an analytic function is defined by a series around a certain point, for example $u = 0$, as

$$ f(u) = \sum_{k=0}^{\infty} a_k u^k, \quad (88) $$

an expansion of $f(u)$ around a different point $u_0$,

$$ f(u) = \sum_{k=0}^{\infty} c_k (u - u_0)^k, \quad (89) $$
can be obtained by substituting in Eq. (88) the expression of $u^k$ according to Eq. (87).

Assuming that the series are absolutely convergent so that the order of the terms can be modified and ordering the terms in the resulting expression according to the increasing powers of $u - u_0$ yields

$$ f(u) = \sum_{k,l=0}^{\infty} \frac{(k + l)!}{k! l!} a_{k+l} u_0^l (u - u_0)^k \quad (90) $$

Since the derivative of order $k$ at $u = u_0$ of the function $f(u)$, Eq. (88), is

$$ f^{(k)}(u_0) = \sum_{l=0}^{\infty} \frac{(k + l)!}{l!} a_{k+l} u_0^l \quad (91) $$

the expansion of $f(u)$ around $u = u_0$, Eq. (90), becomes

$$ f(u) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(u_0) (u - u_0)^k \quad (92) $$

which has the same form as the series expansion of 2-dimensional complex functions. The relation (92) shows that the coefficients in the series expansion, Eq. (89), are

$$ c_k = \frac{1}{k!} f^{(k)}(u_0). \quad (93) $$

The rules for obtaining the derivatives and the integrals of the basic functions can be obtained from the series of definitions and, as long as these series expansions have the same form as the corresponding series for the 2-dimensional complex functions, the rules of derivation and integration remain unchanged.

If the twocomplex function $f(u)$ of the twocomplex variable $u$ is written in terms of the real functions $P(x, y), Q(x, y)$ of real variables $x, y$ as

$$ f(u) = P(x, y) + \delta Q(x, y), \quad (94) $$
then relations of equality exist between partial derivatives of the functions \( P, Q \). These relations can be obtained by writing the derivative of the function \( f \) as

\[
\lim_{\Delta x, \Delta y \to 0} \frac{1}{\Delta x + \delta \Delta y} \left[ \frac{\partial P}{\partial x} \Delta x + \frac{\partial P}{\partial y} \Delta y + \delta \left( \frac{\partial Q}{\partial x} \Delta x + \frac{\partial Q}{\partial y} \Delta y \right) \right],
\]

(95)

where the difference \( u - u_0 \) in Eq. (86) is \( u - u_0 = \Delta x + \delta \Delta y \). The relations between the partials derivatives of the functions \( P, Q \) are obtained by setting successively in Eq. (95) \( \Delta x \to 0, \Delta y = 0 \); then \( \Delta x = 0, \Delta y \to 0 \). The relations are

\[
\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y},
\]

(96)

\[
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.
\]

(97)

The relations (96)-(97) are analogous to the Riemann relations for the real and imaginary components of a complex function. It can be shown from Eqs. (96)-(97) that the components \( P, Q \) are solutions of the equations

\[
\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial y^2} = 0,
\]

(98)

\[
\frac{\partial^2 Q}{\partial x^2} - \frac{\partial^2 Q}{\partial y^2} = 0,
\]

(99)

As can be seen from Eqs. (98)-(99), the components \( P, Q \) of an analytic function of twocomplex variable are solutions of the wave equation with respect to the variables \( x, y \).

### 8 Integrals of twocomplex functions

The singularities of twocomplex functions arise from terms of the form \( 1/(u - u_0)^m \), with \( m > 0 \). Functions containing such terms are singular not only at \( u = u_0 \), but also at all points of the lines passing through \( u_0 \) and which are parallel to the nodal lines.

The integral of a twocomplex function between two points \( A, B \) along a path situated in a region free of singularities is independent of path, which means that the integral of an analytic function along a loop situated in a region free from singularities is zero,

\[
\oint_{\Gamma} f(u)du = 0.
\]

(100)
Using the expression, Eq. (94) for \( f(u) \) and the fact that \( du = dx + \delta dy \), the explicit form of the integral in Eq. (100) is

\[
\oint_{\Gamma} f(u) \, du = \oint_{\Gamma} [(P \, dx + Q \, dy) + \delta (Q \, dx + P \, dy)]
\] (101)

If the functions \( P, Q \) are regular on the surface \( \Sigma \) enclosed by the loop \( \Gamma \), the integral along the loop \( \Gamma \) can be transformed with the aid of the theorem of Stokes in an integral over the surface \( \Sigma \) of terms of the form \( \partial P/\partial y - \partial Q/\partial x \) and \( \partial P/\partial x - \partial Q/\partial y \) which are equal to zero by Eqs. (96)-(97), and this proves Eq. (100).

The exponential form of the two-complex numbers, Eq. (44), contains no cyclic variable, and therefore the concept of residue is not applicable to the two-complex numbers defined in Eqs. (1).

9 Factorization of two-complex polynomials

A polynomial of degree \( m \) of the two-complex variable \( u = x + \delta y \) has the form

\[
P_m(u) = u^m + a_1 u^{m-1} + \cdots + a_{m-1} u + a_m,
\] (102)

where the constants are in general two-complex numbers. If \( a_m = a_{mx} + \delta a_{my} \), and with the notations of Eqs. (12) and (79) applied for \( 0, 1, \cdots, m \), the polynomial \( P_m(u) \) can be written as

\[
P_m = \left[ v_+^m + A_{1+} v_+^{m-1} + \cdots + A_{m-1,+} v_+ + A_{m+} \right] e_+ + \left[ v_-^m + A_{1-} v_-^{m-1} + \cdots + A_{m-1,-} v_- + A_{m-} \right] e_-.
\] (103)

Each of the polynomials of degree \( m \) with real coefficients in Eq. (103) can be written as a product of linear or quadratic factors with real coefficients, or as a product of linear factors which, if imaginary, appear always in complex conjugate pairs. Using the latter form for the simplicity of notations, the polynomial \( P_m \) can be written as

\[
P_m = \prod_{l=1}^{m} (v_+ - v_{l+}) e_+ + \prod_{l=1}^{m} (v_- - v_{l-}) e_-.
\] (104)
where the quantities \(v_l^+\) appear always in complex conjugate pairs, and the same is true for the quantities \(v_l^-\). Due to the properties in Eqs. (28) and (29), the polynomial \(P_m(u)\) can be written as a product of factors of the form

\[
P_m(u) = \prod_{l=1}^{m} \left[ (v_l^+ - v_l^-)e_+ + (v_- - v_l^-)e_- \right].
\]

(105)

This relation can be written with the aid of Eqs. (31) as

\[
P_m(u) = \prod_{l=1}^{m} (u - u_l),
\]

(106)

where

\[
u_l = e_+ v_l^+ + e_- v_l^-,
\]

(107)

for \(l = 1, \ldots, m\). The roots \(v_l^+\) and the roots \(v_l^-\) defined in Eq. (104) may be ordered arbitrarily, which means that Eq. (107) gives sets of \(m\) roots \(u_1, \ldots, u_m\) of the polynomial \(P_m(u)\), corresponding to the various ways in which the roots \(v_l^+, v_l^-\) are ordered according to \(l\) in each group. Thus, while the hypercomplex components in Eq. (104) taken separately have unique factorizations, the polynomial \(P_m(u)\) can be written in many different ways as a product of linear factors.

If \(P(u) = u^2 - 1\), the degree is \(m = 2\), the coefficients of the polynomial are \(a_1 = 0, a_2 = -1\), the twocomplex components of \(a_2\) are \(a_{2x} = -1, a_{2y} = 0\), the components \(A_{2+}, A_{2-}\) are \(A_{2+} = -1, A_{2-} = -1\). The expression, Eq. (103), of \(P(u)\) is \(P(u) = e_+ (v_2^2 - 1) + e_- (v_-^2 - 1)\), and the factorization in Eq. (106) is \(u^2 - 1 = (u - u_1)(u - u_2)\), where \(u_1 = \pm e_+ \pm e_-\), \(u_2 = -u_1\). The factorizations are thus \(u^2 - 1 = (u + 1)(u - 1)\) and \(u^2 - 1 = (u + \delta)(u - \delta)\). It can be checked that \((\pm e_+ \pm e_-)^2 = e_+ + e_- = 1\).

### 10 Representation of hyperbolic two-complex complex numbers by irreducible matrices

If the matrix in Eq. (32) representing the two-complex number \(u\) is called \(U\), and

\[
T = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix},
\]

(108)
it can be checked that

\[ TUT^{-1} = \begin{pmatrix} x + y & 0 \\ 0 & x - y \end{pmatrix}. \]  

(109)

The relations for the variables \( v_+ = x + y, v_- = x - y \) for the multiplication of twocomplex numbers have been written in Eq. (20). The matrix \( TUT^{-1} \) provides an irreducible representation \( \mathbb{S} \) of the twocomplex numbers \( u = x + \delta y \), in terms of matrices with real coefficients.

11 Conclusions

An exponential form exists for the twocomplex numbers, involving the amplitude \( \rho \) and the argument \( \lambda \). The twocomplex functions defined by series of powers are analytic, and the partial derivatives of the components of the twocomplex functions are closely related. The integrals of twocomplex functions are independent of path in regions where the functions are regular. The polynomials of tricomplex variables can be written as products of linear or quadratic factors.

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