CLASSIFYING SEVERAL CLASSES OF LEIBNIZ ALGEBRAS

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ABSTRACT

We extend results related to maximal subalgebras and ideals from Lie to Leibniz algebras. In particular, we classify minimal non-elementary Leibniz algebras and Leibniz algebras with a unique maximal ideal. In both cases, there are types of these algebras with no Lie algebra analogue. We also give a classification of E-Leibniz algebras which is very similar to its Lie algebra counterpart. Note that a classification of elementary Leibniz algebras has been shown in [3].

I. PRELIMINARIES

Loday introduced Leibniz algebras as a noncommutative generalization of Lie algebras. Lie algebra results have been extended to these new algebras. Properties of the Frattini subalgebra and ideal are studied in [3] for Leibniz algebras. In particular, elementary Leibniz algebras, those algebras with the property that the Frattini ideal is 0 for every subalgebra, were classified over algebraically closed fields, extending Lie algebra results of Towers [12], [13] and Towers and Varea [14]. Algebras closely related to elementary Lie algebras are E-Lie algebras and minimal non-elementary Lie algebras, which have been investigated in [13]. We extend these results to Leibniz algebras. In the minimal non-elementary case, the classification contains many new algebras. Similar in definition to the Frattini subalgebra is the Jacobson radical, which is the intersection of all maximal ideals. This concept appears in [5] and [9] for Lie algebras. In particular, Lie algebras with a unique maximal ideal are classified in [5]. We investigate these ideas in Leibniz algebras, again getting new cases in the classification result.

Let A be an algebra. The intersection of all maximal subalgebras, \( F(A) \), is called the Frattini subalgebra and the largest ideal of A contained in \( F(A) \), \( \Phi(A) \), is called the Frattini ideal. For Lie algebras these concepts have been widely studied, see [13]. If A is a solvable Lie algebra or is over a field of characteristic 0, then \( F(A) \) is an ideal, but not generally. For Leibniz algebras \( F(A) \) is an ideal at characteristic 0, but not necessarily when A is solvable [3]. The intersection of all maximal ideals, \( J(A) \), is called the Jacobson radical. The nilradical, \( \text{Nil}(A) \), is the maximal nilpotent ideal of A; it exists by [6]. The sum of all minimal abelian ideals is denoted by \( \text{Asoc}(A) \). An algebra is elementary if \( \Phi(B)=0 \) for all
subalgebras $B$ of $A$ and is minimal non-elementary if $\Phi(B) = 0$ for all proper subalgebras of $A$ but $\Phi(A) \neq 0$. $A$ is called an E-algebra if $\Phi(B) \subseteq \Phi(A)$ for all subalgebras of $B$ of $A$.

Following Barnes [1], we call an algebra $L$ (left) Leibniz if left multiplication by each $x$ in $L$ is a derivation. Many authors consider right Leibniz algebras instead. Thus an algebra $L$ is Leibniz if $x(yz) = (xy)z + y(xz)$ is an identity on $L$. The product need not be antisymmetric.

II. E-LEIBNIZ ALGEBRAS

$L$ is an E-algebra if $\Phi(B) \subseteq \Phi(L)$ for all subalgebras $B$ of $L$. The following result is an extension of Proposition 2 of [11] to Leibniz algebra. The proof is the same as the original, hence we omit it.

Theorem 1. Let $L$ be a Leibniz algebra. Then $L$ is an E-Leibniz algebra if and only if $L/\Phi(L)$ is elementary.

Proposition 2. If $L$ is solvable over a field of characteristic 0, then $L$ is an E-Leibniz algebra.

Proof. Since $L$ is solvable, $L^2$ is nilpotent. By Theorem 3.5 of [3] the result holds.

Lemma 3. If $L=B\oplus C$ and $B$ and $C$ are elementary, then $L$ is elementary.

Proof. Let $S$ be a subalgebra of $B\oplus C$. We show that $\Phi(S)=0$. For $x\in B+S$, $x=b+c$, where $b\in B$ and $c\in C$ and $c=x-b \in C\cap (B+S)$ The projection mapping from $B+S$ onto $(B+S)\cap C$, where $x$ goes to $c$, has kernel $B$. Hence $S/(B\cap S)\cong (B+S)/B\cong (B+S)\cap C$. Since $C$ is elementary, $\Phi((B+S)\cap C)=0$. Hence $\Phi(S)\subseteq B\cap S$. Similarly $\Phi(S)\subseteq C\cap S$. Hence $\Phi(S)\subseteq B\cap C=0$.

Theorem 4. Let $L$ be a Leibniz algebra over $K$, an algebraically closed field of characteristic 0. Then $L$ is an E-Leibniz algebra if and only if:

1. $L$ is solvable, or
2. $L\cong sl_2(K)\oplus \cdots \oplus sl_2(K)$, or
3. $L=R+S$, where $R$ is a solvable ideal, $S\cong sl_2(K)\oplus \cdots \oplus sl_2(K)$, and $RS+SR$ is contained in $\Phi(L)$.

Proof. If $L$ is solvable, then $L$ is an E-Leibniz algebra by Proposition 2. If $L$ is as in (2), then $L$ is a Lie algebra and $L$ is elementary by Theorem 3.2 of [13]. Let $L$ be as in (3). $S$ is elementary as in (2). $\Phi(L)$ is a nilpotent ideal in $L$, hence $\Phi(L) \subseteq R$. Thus, $L/(\Phi(L))\cong R/\Phi(L)+S$. Since $RS+SR\subseteq \Phi(L)$, the previous sum is a direct sum. Thus, by Theorem 4.8 of [12], $0=\Phi(L/\Phi(L))=\Phi(S)\oplus \Phi(R/\Phi(L))$. This shows that $R/\Phi(L)$ is
elementary using Proposition 2. Then \( L/\Phi(L) \) is elementary by Lemma 3 and \( L \) is an
E-algebra by Theorem 1.

Conversely, suppose that \( L \) is an E-algebra. Then \( L/\Phi(L) \) is elementary from Theorem 1. \( L/\Phi(L) \) is the direct sum of its radical, \( R \), and a semisimple ideal \( S \cong \text{sl}_2(K) \oplus \ldots \oplus \text{sl}_2(K) \) by Theorem 4.3 of [3], either of which may be 0. If \( S=0 \), then \( L \) is solvable and (1) holds. If \( R=0 \), the \( L \) is the direct sum of copies of \( \text{sl}_2(K) \) as in (2). If neither \( R \) nor \( S \) is 0, then \( RS+SR \subseteq \Phi(L) \) as a consequence of Theorem 4.3 of [3].

The following result addresses the special case in which \( L \) is a perfect Leibniz algebra.

Corollary 5. Let \( L \) be a perfect Leibniz algebra \((L^2=L)\) over \( K \), an algebraically closed
field of characteristic 0. Then \( L \) is an E-Leibniz algebra if and only if \( L \cong \text{sl}_2(K) \oplus \ldots \oplus \text{sl}_2(K) \).

Proof. Let \( L=R+S \) be the Levi decomposition for \( L \) as in [2], where \( R \) is the radical and \( S \)
is a semisimple subalgebra. Then \( L^2=S^2+RS+SR+R^2 \). \( R=RS+SR+R^2 \) since \( L \) is perfect.
Since \( L \) is an E-algebra, \( RS+SR \subseteq \Phi(L) \). Hence \( R^2+\Phi(L)=R \) and \( R^2+\Phi(L)+S=L \). Hence \( R^2+S=L \) since no proper subalgebra can supplement \( \Phi(L) \). Thus \( R^2=R \), which implies that \( R=0 \) since \( R \) is solvable. Hence \( L=S \) and \( L \) is Lie. Then, by Corollary 4.5 of [13], \( L \cong \text{sl}_2(K) \oplus \ldots \oplus \text{sl}_2(K) \).

III. MINIMAL NON-ELEMENTARY LEIBNIZ ALGEBRAS

An algebra, \( L \), is called minimal non-elementary if \( L \) is not elementary but all proper
subalgebras of \( L \) are elementary. In [5], conditions for a Lie algebra to be minimal non-
elementary are found when \( L^2 \) is nilpotent. The next result is the Leibniz algebra version.
Note that cases 3 and 4 have no Lie algebra counterpart and case 1 has a case not found
in Lie algebras. When \( L^2 \) is nilpotent, \( \Phi(L)=0 \) if and only if \( L^2 \subseteq \text{Asoc}(L) \) and \( L^2 \)
is complemented in \( L \) by Proposition 3.1 of [3]. We often use this fact in the following proof.

Theorem 6. Let \( L \) be a finite dimensional Leibniz algebra over an algebraically closed
field. Suppose that \( L^2 \) is nilpotent. \( L \) is minimal non-elementary if and only if:

1. \( L \) is three dimensional non-nilpotent with basis \( x,y,z \) and non-zero multiplication as:
   (a) \( xz=cz, \ xy=cy+z, \ zx=0 \) and \( yx=0 \), where \( c \) is a non-zero scalar, or
   (b) \( xz=cz, \ xy=cy+z, \ zx=-cz, \ yx=-cy-z \) where \( c \) is a non-zero scalar, or

2. \( L \) is Heisenberg, or

3. \( L \) is generated by \( a \), where \( a^2 \neq 0 \), \( L \) is nilpotent and \( \dim L \geq 2 \), or

4. \( L \) is the four dimensional non-nilpotent algebra generated by \( a,b,x \) and \( y \) with multiplication:
\[ ax = \alpha x, \quad ay = \alpha y, \quad bx = \beta x, \quad by = \beta y, \quad xa = -\alpha x, \quad ya = 0, \quad xb = y - \beta x \quad \text{and} \quad yb = 0 \] where \( \alpha, \beta \) are non-zero scalars.

Proof. Let \( L \) be minimal non-elementary. \( L \) is solvable by the conditions. Suppose that \( L \) is not nilpotent. Then \( \Phi(L) \neq L^2 \) and there exists a maximal subalgebra, \( M \), of \( L \) such that \( L = L^2 + M \). Let \( B \) be an algebra of minimum dimension such that \( L = -L^2 + B \). By Lemma 7.1 of [12], \( L^2 \cap B \subseteq \Phi(B) = 0 \). \( L^2 \) is nilpotent and elementary, hence it is abelian. Clearly \( B \) is also abelian.

Suppose that \( \dim B > 1 \). For any \( a \in B \), let \( H(a) = L^2 + (a) \). Then \( \Phi(H(a)) = 0 \) since all proper subalgebras are elementary. Then \( L^2 \subseteq \text{Nil}(H(a)) = \text{Asoc}(H(a)) \). \( L^2 \) is abelian since it is elementary and nilpotent, and it is completely reducible under the action of \( a \). On each minimal ideal, either \( R_a = -L_a \) or \( R_a = 0 \) [1]. Hence the minimal ideals are one dimensional eigenspaces for \( L_a \) and \( R_a \) acting on \( L^2 \) and \( L_a \) and \( R_a \) are simultaneously diagonalizable on \( L^2 \). This holds for all \( a \) in \( B \), and since the left multiplications, \( L_a \), commute, they are simultaneously diagonalizable.

Consider \( B \) acting on \( W = L^2 \) by left multiplication and decompose \( L^2 \) as the direct sum of weight modules, \( \{ x \in W : ax = \alpha x \text{ for all } a \in B \} \). Each of these weight modules is invariant under \( R_b, b \in B \). If there is more than one weight module, then, for each weight \( \alpha \), let \( B_\alpha = W_\alpha + B \). Since \( \Phi(B_\alpha) = 0, W_\alpha \subseteq \text{Asoc}(B_\alpha) = \text{Nil}(B_\alpha) \) and \( W_\alpha \) is completely reducible as a \( B \)-bimodule. Hence \( W \subseteq \text{Asoc}(L) \). Since \( B \) complements \( W \), \( \text{Asoc}(L) \) is complemented and \( \Phi(L) = 0 \), a contradiction.

Hence there is one weight module and each left multiplication is a scalar. Pick \( a \in B \) and suppose that \( ax = \alpha x \) on \( W \). Let \( W_0 = \{ x \in W : xa = 0 \} \) and \( W_1 = \{ x \in W : xa = -\alpha x \} \). Since \( W \) is the direct sum of one dimensional \( a \)-invariant submodules, \( W = W_0 + W_1 \). If, for each \( a \in B \), \( W = W_0 \) or \( W = W_1 \), then each right multiplication is a scalar on \( L^2 \), and \( L^2 \subseteq \text{Asoc}(L) \). Again \( \text{Asoc}(L) \) is complemented in \( L \) and \( \Phi(L) = 0 \), a contradiction. Thus assume there is an \( a \in B \) such that neither \( W_0 \) nor \( W_1 \) is 0. \( W_0 \) is a submodule. If \( W_1 \) is a submodule, then by induction both components are completely reducible under \( B \), and \( W \subseteq \text{Asoc}(L) \). Thus \( \text{Asoc}(L) \) is complemented and \( \Phi(L) = 0 \), a contradiction. Hence assume there is an \( x \) in \( W_1 \) and \( b \) in \( B \) such that \( xb \) is not in \( W_1 \). Let \( N = (x, xb) \), \( bx = \beta x \), and set \( y = \beta x + xb \). The following multiplications hold: \( ax = \alpha x, \quad ay = \alpha y, \quad bx = \beta x, \quad by = \beta y, \quad xa = -\alpha x, \quad ya = \beta x + xb \). \( \alpha = -\alpha x + \alpha \beta x + \alpha \beta x = 0, \quad \beta y = (\beta x + xb) \beta x + xb \beta x = 0 \). Let \( C = (a, b) \). Hence \( N \) is a \( C \)-bimodule and \( y \) is a submodule which is not complemented in \( N \), for suppose that \( \sigma x + \tau y \) is \( C \)-invariant where \( \sigma \neq 0 \). Then \( (\sigma x + \tau y) a = -\sigma ax \) which yields that \( \tau = 0 \). Then \( ax b = \sigma y - \sigma \beta x \) and \( \sigma = 0 \) since \( xb \) is not in \( W_1 \). Thus no complement exists. Hence \( \text{Asoc}(N + C) \) is not complemented in \( N + C \) and \( \Phi(N + C) \neq 0 \). Thus \( B = C, N = W = L^2, \dim B = 2 \) and \( \dim L^2 = 2 \) and the multiplication for \( B \) acting on \( L^2 \) is the one given in this paragraph.
Let $B=(a,b)$ and $L^2=(x,y)$. From the foregoing $ax=\alpha x$, $ay=\alpha y$, $bx=\beta x$, $by=\beta y$, $xa=-\alpha x$, $ya=0$, $yb=0$, and $xb=y-\beta x$. $N=(y)$ is a one dimensional submodule of $L$, and $L^2$ is not completely reducible under the action of $L$. This is the algebra in (4).

Suppose that $\dim B = 1$. Hence $L=L^2+B$ and $L^2$ is abelian. We now show that there exists a chain of ideals $0 \subseteq L_1 \subseteq \ldots \subseteq L_{n-1} \subseteq L^2 \subseteq L_n=\text{Nil}(L)$. If $P \subseteq Q \subseteq L^2$ are ideals of $L$ with $Q/P$ irreducible under the action of $L$, then $\dim Q/P=1$ since the action of $x$ on $Q/P$ determines the action of $L$ on $Q/P$, and either $R_x=-L_x$ or $R_x=0$. Then any eigenvector of $L_x$ in $Q/P$ must span $Q/P$ and $\dim Q/P=1$. Hence there exists a flag from 0 to $L$ as claimed. Now $M=L_{n-2}+B$ is a maximal subalgebra of $L$, hence $\Phi(M)=0$ by assumption. Hence $L_{n-2} \subseteq \text{Asoc}(L)=\text{Nil}(M)$. Since $L^2$ is abelian and $B$ is one dimensional, $\text{Asoc}(M) \subseteq \text{Asoc}(L)$. Thus $L_{n-2} \subseteq \text{Asoc}(L)$. If $L_{n-2} \neq \text{Asoc}(L)$, then $\Phi(L)=0$ from Theorem 3.1 of [3], and $L$ is elementary. Otherwise, $\text{Asoc}(L)=L_{n-2}$ and $\text{Asoc}(L)$ has co-dimension two in $L$. If $\text{Asoc}(L)$ is not contained in $\Phi(L)$, then $L$ splits over $\text{Asoc}(L)$ by Theorem 7.1 of [12]. Again $\Phi(L)=0$. and $L$ is elementary. Thus assume that $\text{Asoc}(L)=\Phi(L)$. Since all minimal ideals are one dimensional, they are eigenspaces for $L_x$ where $B=\langle x \rangle$. Note that $R_x=0$ or $R_x=-L_x$ on each of these minimal ideals. Since $L^2/\text{Asoc}(L)$ is one-dimensional, there exists a scalar $\alpha$ with $(L_x-\alpha I)^2=0$ and $L_x-\alpha I \neq 0$. Hence there exist $y,z \in L^2$ with the following possible multiplications.

Case 1: Let $xz=cz$, $xy=cy+z$, $zx=0$, $yx=-dcy+ez$, where $d=0$ or $1$ and $c$ is a non-zero scalar since $L$ is not nilpotent. From $x(yx)=(xy)x+y(xx)$ we obtain that $-cd+ce=ce$. Hence $d=0$. From $y(xx)=(yx)x+x(yx)$, we obtain that $-cde+cd+ce=0$. Hence $e=0$, and $yx=0$.

Case 2: Let $xz=cz$, $xy=cy+z$, $zx=-cz$, $yx=-dcy+ez$ where $d=0$ or $1$ and $c \neq 0$. Using $x(yx)=(xy)x+y(xx)$, we obtain that $-dc=-c$. Hence $d=1$. From $y(xx)=(yx)x+x(yx)$, we obtain that $cde=-cd$. Hence $e=-1$, and $yx=-cy-z$.

Hence $H=\langle x,y,z \rangle$ has $\Phi(H)=\langle z \rangle$ and $L=H$, which is the algebra in (1).

Let $L$ be nilpotent with all proper subalgebras elementary. If there exists an $a \in L$ with $a^2 \neq 0$, then the subalgebra $B$ with basis $a$, $a^2$, $\ldots$, $a^n$ and $aa^n=0$ has $\Phi(B)=B^2$. Since $a^2 \in \Phi(B)$, $B=L$, which is the algebra in (3). If no such $a$ exists, then $L$ is Lie and hence Heisenberg by Theorem 4.7 of [8], yielding case (2).

Conversely, in cases (1) and (2) and (4), clearly all proper subalgebras are elementary. Suppose that $L$ is as in case (3). Then $L=\langle a,a^2,\ldots,a^n \rangle$ with $aa^n=0$. Since $L$ is nilpotent, $L^2=\Phi(L)$. Now $b=c_1a+c_2a^2+\ldots+c_na^n$ is in $\Phi(L)$ if and only if $c_1=0$. If the subalgebra $b$ contains an element that is not in $\Phi(L)$, then $B+\Phi(L)=L$ since $L/\Phi(L)$ is one dimensional.
Hence $B=L$. Therefore all proper subalgebras of $L$ are contained in $L^2$, which is abelian, and hence are elementary. Thus $L$ satisfies the conditions of the theorem.

We turn to the Leibniz algebra version of a Lie algebra result of Towers [13].

Theorem 7. Let $L$ be a Leibniz algebra over $K$, an algebraically closed field of characteristic 0. $L$ is minimal non-elementary if and only if:

1. $L$ is three dimensional non-nilpotent with basis $x,y,z$ and non-zero multiplication as:
   
   (a) $xz=cz$, $xy=cy+z$, $zx=0$ and $yx=0$, where $c$ is a non-zero scalar, or
   
   (b) $xz=cz$, $xy=cy+z$, $zx=-cz$, $yx=-cy-z$ where $c$ is a non-zero scalar, or

2. $L$ is Heisenberg, or

3. $L$ is generated by $a$, where $a^2 \neq 0$, $L$ is nilpotent and $\text{dim } L \geq 2$, or

4. $L$ is the four dimensional non-nilpotent algebra generated by $a,b,x$ and $y$ with multiplication. $ax=\alpha x$, $ay=\beta y$, $bx=\beta x$, $by=\beta y$, $xa=-\alpha x$, $ya=0$, $xb=y-\beta x$ and $yb=0$ where $\alpha$ and $\beta$ are non zero scalars.

Proof: Suppose that $L$ is minimal non-elementary. Then $L$ is an E-algebra.

Suppose that $L$ is not solvable. Hence there exists a $k$ such that $L^{(k)}=L^{(k+1)}$. If $k=1$, then $L$ is a perfect Lie algebra and $L$ is elementary by Corollary 5, a contradiction. If $k \geq 2$, then $L^k$ is perfect and $L^{(k)} \cong \text{sl}_2(K) \oplus \cdots \oplus \text{sl}_2(K)$. If $R$ is the radical of $L$, then, then $L=R \oplus L^{(k)}$. Since both summands are elementary, $L$ is elementary by Lemma 3, a contradiction. Hence $L$ must be solvable. The result now follows from Theorem 6.

IV. THE JACOBSON RADICAL

The Jacobson radical $J(L)$ is the intersection of maximal ideals of $L$. This concept was considered in [5] and [9] when $L$ is a Lie algebra. If $L$ is nilpotent, then $J(L)=\Phi(L)$, since then all maximal subalgebras are ideals [1]. Clearly $J(L) \subseteq L^2$, since if $x$ is not in $L^2$, then we can find a complementary subspace, $M$, of $x$ in $L$ that contains $L^2$ and, since $L^2 \subseteq M$, $M$ is a maximal ideal of $L$ and $x$ is not in $M$.

If $L$ is a linear Lie algebra, let $R=\text{Rad}(L)$, and let $\text{Rad}(L^*)$ be the radical of the associative envelope, $L^*$ of $L$. Then, by corollary 2 p. 45 of [7], $L \cap \text{Rad}(L^*) = \text{all nilpotent elements of } R$ and $[R,L] \subseteq \text{Rad}(L^*)$.

Theorem 8. Let $L$ be a Leibniz algebra and $R=\text{Rad}(L)$ be the radical of $L$. Then $LR+RL \subseteq N=\text{Nil}(L)$.
Proof: Let $\mathcal{L}(L)=\{L_x: x \in \mathbb{N}\}$. The map $\pi: L \to \mathcal{L}(L)$ is a homomorphism and $\mathcal{L}(L)$ is a Lie algebra under commutation. Also $\pi : R \to R(\mathcal{L}(L))$, the radical of $\mathcal{L}(L)$. By the result in the first paragraph, $[\mathcal{L}(L), R(\mathcal{L}(L))] \subseteq R(\mathcal{L}(L)^*)$. Hence there exists an $n$ such that $[L_{s_n}t_n]...[L_{s_2}t_2][L_{s_1}t_1]=0$, where $s_i \in L$, $t_i \in R$ or $s_i \in R$ and $t_i \in L$. Hence $L_{s_n}t_n...L_{s_1}t_1 = 0$. Hence $s_n t_n(...(s_1 t_1(x))...) = 0$ for all $x \in L$. Hence $s_n t_n(...(s_1 t_1(s_0 t_0))...) = 0$. Therefore $(LR+RL)^n+1=0$. Since $LR+RL$ is an ideal in $L$, $LR+RL \subseteq N$.

Proposition 9. If $L$ is of characteristic 0, then $L^2 \cap R=LR+RL \subseteq N$.

Proof: $L=R+S$ as in the Levi decomposition for Leibniz algebras ([2]). Then $L^2 = S^2 + LR+RL$ and $L^2 \cap R = S^2 \cap R + (LR+RL) \cap R = LR+RL$ since $S \cap R = 0$ and $LR+RL \subseteq R$.

Lemma 10. If $L$ is solvable, then $J(L)=L^2$.

Proof: If $M$ is a maximal ideal of $L$, then $L/M$ is abelian and, in fact, one-dimensional since $L$ is solvable. Hence $L^2 \subseteq M$ for all $M$ and $L^2 \subseteq J(L)$. Since $J(L) \subseteq L^2$ always holds, $L^2=J(L)$.

Proposition 11. Let $L$ be a Leibniz algebra over a field of characteristic 0. Let $R=R(L)$. Then $J(L)=LR+RL$.

Proof. Let $S$ be a Levi factor of $L$. Then $S$ is Lie and $S=S_1 \oplus ... \oplus S_t$ where each $S_i$ is simple. Let $M_i$ be the sum of $R$ and all of the $S_j$ except $S_i$. Then $M_i$ is a maximal ideal of $L$. Then $J(L) \subseteq \bigcap M_i=R$ and $J(L) \subseteq R \cap L^2$.

If $M$ is a maximal ideal of $L$, then $L/M$ is abelian or simple. In the first case, $L^2 \subseteq M$ and in the second case, $R \subseteq M$. The intersection of all of the first type of $M$ contains $L^2$, and all of the second type contains $R$. Therefore, $R \cap L^2 \subseteq J(L)$ and the result follows.

Corollary 12. $J(L)$ is nilpotent.

Proof; $J(L)=LR+RL \subseteq Nil(L)$ by Propositions 9 and 11.

Corollary 13. $\Phi(L) \subseteq J(L)$ when $L$ has characteristic 0.

Proof: $L=R+S$ as in the Levi decomposition and $S$ is Lie. Hence $\Phi(S)=0$. Thus $\Phi(L) \subseteq R$. Since $\Phi(L) \subseteq L^2$, it follows that $\Phi(L) \subseteq R \cap L^2 = LR+RL = J(L)$ using Proposition 9 and 11.

Proposition 14. Let $B$ be a nilpotent ideal in a Leibniz algebra, $L$. Then $J(B) \subseteq J(L)$. 
Proof: Since $B$ is nilpotent, $J(B)=B^2$, which is an ideal in $L$. Suppose that $x \in J(B)$, $x \notin J(L)$. Let $M$ be a maximal ideal of $L$ such that $x$ is not in $M$. Then $L=J(B)+M$ and $B=J(B)+(M \cap B)$. $M \cap B$ is a proper ideal of $B$ that supplements $J(B)$, a contradiction.

V. LEIBNIZ ALGEBRAS WITH A UNIQUE MAXIMAL IDEAL.

Lie algebras with a unique maximal ideal were classified in [5] when the field of scalars has characteristic 0. We extend this result to Leibniz algebras.

Lemma 15. Suppose that $L=N+\langle x \rangle$ where $\langle x \rangle$ is a one dimensional vector space, $x^2 \neq 0$, and $N$ is an abelian ideal. Then $xL$ is an ideal in $L$.

Proof. Since $xL \subseteq N$, $N(xL)=(xL)N=0$. $(xL)x=(x(x+N))x=(x^2+xN)x=(xN)x=x(Nx)-N(x^2) \subseteq xL$ since $x^2 \in N$ and $N$ is abelian. Furthermore $x(xL) \subseteq xL$. Hence the result holds.

Lemma 16. Suppose that $L=N+\langle x \rangle$ where $\langle x \rangle$ is a one dimensional vector space, $x^2 \neq 0$ and $N$ is an abelian ideal. If $xL+Lx=N$, then $xL=N$.

Proof. If $xL \neq N$, then $xL$ is contained in an ideal $M$ of $L$, $M \subseteq N$, and $N/M$ is a minimal ideal of $L/M$. Note that $x^2 \in M$. We may take $M=0$. Then $xN=0$, which yields $Nx=0$ by [1]. Then $Lx=(N+x)x=0$. Therefore $xL+Lx=0$, a contradiction.

We now show

Theorem 17. Let $L$ be a Leibniz algebra over a field of characteristic 0. $L$ has a unique maximal ideal if and only if one of the following holds.

1) $L$ is nilpotent and cyclic, or

2) $L=\text{Nil}(L)+S$ where $S$ is simple and $N/N^2=S(N/N^2)+(N/N^2)S$, or

3) $L=\text{Nil}(L)+\langle x \rangle$ where $\langle x \rangle$ is a one dimensional vector space, $x^2 \neq 0$ and $N/(N^2)=x(L/N^2)$, or

4) $L=\text{Nil}(L)+\langle x \rangle$ where $x^2=0$ and $N/(N^2)=x(N/N^2)+(N/N^2)x$.

Proof. Let $L$ have only one maximal ideal which is then $J(L)=J$. Then $L/J$ is simple or one-dimensional. Since $J \subseteq \text{Nil}(L)=N$, if $L/J$ is simple, then $J=N$ and $L=J+S=N+S$. To show the second part of (2), assume that $N^2=0$ and let $R=\text{Rad}(L)$. Then $N=R$ and $J=LR+RL=SN+NS$. 
Suppose that \( \dim(L/J)=1 \). Since \( J \subseteq N \subseteq L \), either \( J=N \) or \( N=L \). If \( N=L \), then \( L \) is nilpotent and \( J=\Phi(L)=L^2 \), so \( L \) is cyclic generated by some \( a \) and \( L \) is nilpotent and (1) holds.

Suppose that \( N=J \). Then \( L=N+\langle x \rangle \) where \( \langle x \rangle \) is a one dimensional subspace. Suppose that \( x^2 \neq 0 \). By Theorem 3.1 of [4], \( L/N^2 \) is not nilpotent, hence \( \text{Nil}(L/N^2)=N/N^2 \). Suppose that \( N=J=0 \) and \( T \) is the algebra generated by \( x \). Then \( \text{Nil}(L)=J(L)=xL+Lx=(L/N^2)x=x(N/T^2) \).

We now show that each of the algebras in 1-4 have a unique maximal ideal. For the algebras in (3), we have the following lemma.

**Lemma 18.** Suppose that \( L \) is Leibniz, and \( L=\text{Nil}(L)+\langle x \rangle \) where \( x^2 \neq 0 \). Suppose that \( xL=N \) and \( N=\text{Nil}(L) \). Then \( \text{Nil}(L)=J(L) \).

**Proof.** Let \( J=J(L) \). Since \( xL=N \), \( N=L^2 \). Let \( T \) be the subalgebra generated by \( x \). Since \( \dim(\text{Ker} \ L_x|_L)=1 \) and \( 0 \neq \text{Ker} \ L_x|_T \subseteq \text{Ker} \ L_x|_L \), it follows that \( \text{Ker} \ L_x|_T=\text{Ker} \ L_x|_L \). Since \( \dim(\text{Ker} \ L_x|_T)=1 \), the Fitting null component of \( L_x|_T \) is the union \( T_0 \subseteq T_1 \subseteq ... \subseteq T_k \) where \( \dim(T_j)=j \) and \( xT_j \subseteq T_{j-1} \). The same holds for the Fitting null component of \( L_x \) on \( N \) where the invariant subspaces end with \( L_m \). Clearly \( T_i=L_i \) for \( i \leq k \). We claim that \( m=k \).

There is a \( T_i \) that is not contained in \( N \) since the Fitting one component of \( L_x \) on \( N \) is contained in \( N \). Then \( \alpha x+n \in T_i, \alpha \neq 0, n \in N \). If \( i \neq k \), then there exists \( \beta x+p \) such that \( x(\beta x+p)=\alpha x+n \), which is impossible since \( x(\beta x+p) \in N=L^2 \). Thus \( T_{k-1} \subseteq N \) while \( T_k \) is not contained in \( N \). Since \( T_k=L_k \) is not contained in \( N \), the chain of \( L_i \)'s must stop at \( L_k \) since only the final term in the string is not in \( N \). Hence the Fitting null component for \( L_x \) acting on \( L \) is the same as \( L_x \) acting on \( T \) and the Fitting null component of \( L_x \) on \( N \) and \( T^2 \) are the same, both equal to \( T_{k-1} \subseteq T^2 \).

Let \( J=J(L) \). Since \( xL=N \), \( N=L^2 \). Let \( N=N_0+N_1 \) be the Fitting decomposition of \( L_x \) on \( N \). Let \( K \) be a maximal ideal of \( L \) and suppose that \( K \) is not contained in \( L^2 \). There exists \( y \in K \) such that \( y=\alpha x+n_0+n_1 \) where \( \alpha \neq 0 \), \( n_0 \in N_0 \) and \( n_1 \in N_1 \). For any \( t \in N_1 \), there exists \( s \in N_1 \) such that \( xs=t \). Then \( ys=(\alpha x+n_0+n_1)s=\alpha t \) since \( N^2=0 \). Thus \( \alpha t \in K \) and \( N_1 \subseteq K \). Also \( n_0 \in N_0=0 \) \subseteq \( T_{k-1} \subseteq T^2 \) by the last paragraph. Hence \( n_0=\alpha_2 x^2+...+\alpha_1 x^4 \) and \( x=\alpha x+\alpha_2 x^2+...+\alpha_1 x^4+n_1 x=\alpha x^2+n_1 x \in K \). Since \( n_1 x \in K, \alpha x^2 \in K \). Therefore \( x^2 \in K \) and \( T^2 \subseteq K \). Since \( N_0 \subseteq T^2 \) and \( N \subseteq K \), \( N \subseteq K \). Since \( J(L)=N \) has codimension 1, \( J(L)=N=K \) for all maximal ideals \( K \) of \( L \). Hence algebras as in (3) have a unique maximal ideal.
We now consider the algebras in (1), (2), and (4). Suppose that \( L \) is as in (2). We may assume that \( N^2 = 0 \) since \( N/N^2 = \text{Nil}(L/N^2) \). Thus \( J = LR + RL = SN + NS = \text{Nil}(L) \) and \( J(L) \) is the only maximal ideal of \( L \). If \( L \) is as in (4), then assume that \( N^2 = 0 \) since \( N/N^2 = \text{Nil}(L/N^2) \). Then \( J(L) = LR + RL = xN + Nx \). Since \( \dim(L/N) = 1 \), \( N \) is the only maximal ideal of \( L \).

If \( L \) is as in (1), then \( L^2 = J(L) \) since \( L \) is nilpotent, and \( J(L) \) is the unique maximal ideal.

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