THE BCS FUNCTIONAL FOR GENERAL PAIR INTERACTIONS

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Abstract. The Bardeen-Cooper-Schrieffer (BCS) functional has recently received renewed attention as a description of fermionic gases interacting with local pairwise interactions. We present here a rigorous analysis of the BCS functional for general pair interaction potentials. For both zero and positive temperature, we show that the existence of a non-trivial solution of the non-linear BCS gap equation is equivalent to the existence of a negative eigenvalue of a certain linear operator. From this we conclude the existence of a critical temperature below which the BCS pairing wave function does not vanish identically. For attractive potentials, we prove that the critical temperature is non-zero and exponentially small in the strength of the potential.

1. Introduction

Over the last few years, significant attention has been devoted to ultra-cold gases of fermionic atoms, both from the experimental and the theoretical point of view. Because of the ability to magnetically tune Feshbach resonances it has become possible to vary interatomic interactions via external magnetic fields, and hence to control the strength of the interactions among the atoms. This method was used in remarkable experiments to study the crossover from the regime of Bose-Einstein condensation (BEC) of tightly bound diatomic molecules to the Bardeen-Cooper-Schrieffer (BCS) regime of weakly bound Cooper pairs in ultra-cold gases of fermionic atoms. For a recent review on this topic we refer to [5, 7].

The main purpose of this paper is to give a mathematically precise study of the BCS regime. This regime is usually described by the BCS functional, derived by Leggett [10] based on the original work of Bardeen-Cooper-Schrieffer [3]. We note that while in the original BCS model simple phonon-induced non-local interaction potentials were considered, the interaction potentials are local for atomic Fermi gases. In order to allow for a wide range of applications, we will consider here the BCS functional for general pair interaction potentials $V$.

One purpose of our paper is to obtain necessary and sufficient conditions on $V$ for the system to display superfluid behavior. Recall that in the BCS model superfluidity (or, rather, superconductivity in the original BCS case, where electrons in a crystal structure have been considered) is related to the non-vanishing of the pairing wave function, describing particle pairs with opposite spin and zero total momentum. That is, superfluidity occurs if the paired state is energetically favored over the normal state described by the Fermi-Dirac distribution.

More precisely, for arbitrary temperature $T = 1/\beta \geq 0$ a system is in a superfluid state if there exists a non-trivial solution $\Delta$ to the BCS gap equation

$$\Delta(p) = -\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} V(p - q) \frac{\Delta(q)}{E(q)} \tanh \frac{E(q)}{2T} dq , \quad (1.1)$$

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with \( E(p) = \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2} \). Here, \( \mu \) denotes the chemical potential. Notice that the gap equation is highly \textit{nonlinear}. In Theorem\( \text{I} \) below we will show that the existence of a non-trivial solution to the gap equation is equivalent to the existence of a negative eigenvalue of the corresponding \textit{linear} operator

\[
K_{\beta,\mu} + V, \quad K_{\beta,\mu} = (p^2 - \mu)\left(\frac{e^{\beta p^2} - 1}{e^{\beta (p^2 - \mu)} - 1}\right).
\]

The operator \( K_{\beta,\mu} \) is understood as a multiplication operator in momentum space, i.e., \( p^2 \) equals \( -\Delta \) in configuration space. Observe that in the limit \( T \to 0 \) this operator reduces to the Schrödinger type operator

\[
|p^2 - \mu| + V.
\]

This \textit{linear} characterization of the existence of solutions to the nonlinear BCS equation represents a considerable simplification of the analysis. In particular, it proves the existence of solutions to \( (1.1) \) for a wide range of interaction potentials \( V \), and hence generalizes previous results \cite{4, 13, 18, 19} valid only for non-local \( V \)'s under suitable assumptions.

Moreover, the monotonicity of \( K_{\beta,\mu} \) in \( \beta \) guarantees the existence of a \textit{critical temperature} \( T_c \), with \( 0 \leq T_c < \infty \), such that superfluidity occurs whenever \( T < T_c \) and normal behavior for \( T \geq T_c \). In particular, there is a phase transition at \( T_c \). For positive values of \( \mu \), we shall show that \( T_c \) is strictly positive whenever the negative part of \( V \), denoted by \( V_- \), is non-vanishing and the positive part \( V_+ \) is small enough in a suitable sense. Moreover, we prove that \( T_c \) is exponentially small in the strength of the interaction potential. More precisely, if the potential is given by \( \lambda V \) for some fixed \( V \in L^1 \cap L^{3/2} \), then \( T_c \lesssim e^{-1/\lambda} \) as \( \lambda \to 0 \). This generalizes the corresponding result in \cite{3, Eq. (3.29)} to a very large class of interaction potentials. For negative potentials, also a corresponding lower bound will be shown to hold. In a forthcoming work \cite0{9}, the precise asymptotics of \( T_c \) for \( \lambda \to 0 \) will be investigated.

At zero temperature, the existence of a non-trivial solution of the BCS gap equation is also related to the non-vanishing of the energy gap between the ground state and the first excited state in the effective quadratic BCS Hamiltonian on Fock space. For very specific attractive interaction potentials, it has been argued that this gap is non-vanishing, see e.g. \cite{10, 14, 15}. We note, however, that the existence of such an energy gap is not necessarily implied by the non-vanishing of the solution to the BCS equation. In fact, the presence of a gap is not necessary for our analysis. We consider general potentials where such an energy gap need not be present, a priori. We do prove that the gap is non-vanishing for a certain class of interaction potentials, however.

We note that our results are relevant for actual experiments on cold atomic gases as long as the description via the BCS functional is applicable. There is an extensive literature on this subject, see e.g. \cite{1, 5, 6, 7, 14, 15, 16, 17}. According to Leggett\cite{10} this is the case for weak interactions as long as the range of the potential \( V \) is much smaller than \( \mu^{-1/2} \) (in appropriate units), i.e., for weakly coupled dilute gases. In particular, “weakly coupled” means that the corresponding Schrödinger operator \( p^2 + V \) typically does not allow for a negative energy bound state since otherwise the system is in the BEC regime of tightly bound bosonic molecules. We refer to \cite{10} for a detailed discussion on this question.

\section{Model and Main Results}

The precise definition of the BCS functional considered in this paper is given as follows. For the convenience of reader, we describe in the appendix the motivation and physical background; the discussion there follows Leggett’s derivation in \cite{10}. See also \cite{2} for a detailed study of the BCS approximation to the Hubbard model.
We use the standard convention \( \hat{\alpha}(p) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \alpha(x)e^{-ipx}dx \) for the Fourier transform.

**Definition 1.** Let \( \mathcal{D} \) denote the set of pairs of functions \((\gamma, \alpha)\), with \( \gamma \in L^1(\mathbb{R}^3, (1 + p^2)dp), 0 \leq \gamma(p) \leq 1, \) and \( \alpha \in H^1(\mathbb{R}^3, dx) \), satisfying \( |\hat{\alpha}(p)|^2 \leq \gamma(p)(1 - \gamma(p)) \). Let \( V \in L^{3/2}(\mathbb{R}^3, dx) \) be real-valued and \( \mu \in \mathbb{R} \). For \( T = 1/\beta \geq 0 \) and \((\gamma, \alpha) \in \mathcal{D}\), the energy functional \( \mathcal{F}_{\beta} \) is defined as

\[
\mathcal{F}_{\beta}(\gamma, \alpha) = \int (p^2 - \mu)\gamma(p)dp + \int |\alpha(x)|^2V(x)dx - \frac{1}{\beta}S(\gamma, \alpha),
\]

where

\[
S(\gamma, \alpha) = -\int [s(p)\ln s(p) + (1 - s(p))\ln (1 - s(p))] dp,
\]

with \( s(p) \) determined by \( s(1 - s) = \gamma(1 - \gamma) - |\hat{\alpha}|^2 \).

As explained in the Appendix, \( 2(\pi)^{3/2} \) has the physical interpretation of \( -\frac{1}{2} \) times the pressure of a system of spin \( 1/2 \) fermions at temperature \( T \) interacting via a pair potential given by \( 2V \).

It is natural to require \( \mathcal{F}_{\beta} \) is minimized on \( \mathcal{D} \) by the choice \( \alpha \equiv 0 \) and \( \gamma(p) = \gamma_0(p) = [e^{\beta(p^2 - \mu)} + 1]^{-1} \). We refer to this state as the *normal state* of the system. We will be concerned with the question of whether a non-vanishing \( \alpha \) can lower the energy for given potential \( V \).

Note that at \( T = 0 \) it is natural to formulate the model in terms of a functional that depends only on \( \alpha \) and not on \( \gamma \). Namely, for fixed \( |\hat{\alpha}(p)|^2 \), the optimal choice of \( \gamma(p) \) is

\[
\gamma(p) = \begin{cases} 
\frac{1}{2}(1 + \sqrt{1 - 4|\hat{\alpha}(p)|^2}) & \text{for } p^2 \leq \mu \\
\frac{1}{2}(1 - \sqrt{1 - 4|\hat{\alpha}(p)|^2}) & \text{for } p^2 > \mu
\end{cases}
\]

(2.2)
in this case. Similarly, also at \( T > 0 \) it is enough to minimize \( \mathcal{F}_{\beta} \) only over pairs \((\gamma, \alpha)\) satisfying a certain relation independent of \( V \), which is displayed in Eq. (3.4) below.

According to the usual interpretation of the BCS theory, the normal state is unstable if the energy can be lowered by the formation of Cooper pairs, i.e., if \( \mathcal{F}_{\beta} \) can be lowered by choosing \( \alpha \neq 0 \). The following Theorem, which is the main result of this paper, shows that this property is equivalent to the existence of a negative eigenvalue of a certain linear operator.

**Theorem 1.** Let \( V \in L^{3/2}(\mathbb{R}^3), \mu \in \mathbb{R}, \) and \( \infty > T = 1/\beta \geq 0 \). Then the following statements are equivalent:

(i) The normal state \((\gamma_0, 0)\) is unstable under pair formation, i.e.,

\[
\inf_{(\gamma, \alpha) \in \mathcal{D}} \mathcal{F}_{\beta}(\gamma, \alpha) < \mathcal{F}_{\beta}(\gamma_0, 0).
\]

(ii) There exists a pair \((\gamma, \alpha) \in \mathcal{D}\), with \( \alpha \neq 0 \), such that

\[
\Delta(p) = -\frac{p^2 - \mu}{\gamma(p) - \frac{1}{2}} \hat{\alpha}(p)
\]

(2.3)
satisfies the BCS gap equation

\[
\Delta = -\hat{V} \ast \left( \frac{\Delta}{E} \tanh \frac{\beta E}{2} \right), \text{ with } E(p) = \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}.
\]

(2.4)
The linear operator

\[ K_{\beta,\mu} + V, \quad K_{\beta,\mu} = (p^2 - \mu)e^{\beta(p^2 - \mu)} + 1, \] (2.5)

has at least one negative eigenvalue.

Here, * denotes convolution in the BCS gap equation (2.4). In order to avoid having to write factors of \(2\pi\), we find it convenient to define convolution with a factor \((2\pi)^{-3/2}\) in front, i.e., \((f \ast g)(p) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} f(p-q)g(q) dq\).

**Remark 1.** The operator (2.5) is obtained by taking the second derivative of the functional \(\mathcal{F}_\beta\) with respect to \(\alpha\) at the normal state \(\gamma_0, 0\) (see Eq. (3.10) below). Since the normal state is a critical point of \(\mathcal{F}_\beta\), it is then easy to prove that (iii) implies (i). The opposite direction is not immediate, however. It says that if \(\gamma_0, 0\) is not a minimizer, it can also not be locally stable.

**Remark 2.** As mentioned above, a minimizer of \(\mathcal{F}_\beta\) satisfies the \(V\)-independent relation (2.2) at \(T = 0\), and (3.4) at \(T > 0\). It is not necessary to assume that these equations hold in part (ii) of Theorem 1, however.

Note that in the zero temperature case \(T = 0\) the gap equation (2.4) takes the form

\[ \Delta = -\hat{V} + \frac{\Delta}{E}, \] (compare with [3, 8, 10, 12]) and the operator in (2.5) is given by \(|p^2 - \mu| + V\).

For any potential \(V\) we shall prove the existence of a critical temperature \(0 \leq T_c = \frac{1}{\beta_c} < \infty\) at which a phase transition from a normal to superfluid state takes place. More precisely, we shall show that the gap equation (2.4) has a non-trivial solution for all \(T < T_c\), whereas it has no non-trivial solution if \(T \geq T_c\). This follows from the next theorem, in combination with Theorem 1.

**Theorem 2.** For any \(V \in L^{3/2}(\mathbb{R}^3)\) there exists a critical temperature \(0 \leq T_c = \frac{1}{\beta_c} < \infty\) such that

\[
\inf \text{spec}(K_{\beta,\mu} + V) < 0 \quad \text{if} \quad T < T_c,
\]

\[
K_{\beta,\mu} + V \geq 0 \quad \text{if} \quad T \geq T_c.
\]

With the aid of the Birman-Schwinger principle, it is possible to characterize the value of the critical temperature \(T_c\), at least if \(T_c > 0\). In order to do this, it is convenient to decompose \(V\) into its positive and negative parts, i.e., \(V = V_+ - V_-\), with \(V_+ \geq 0, V_- \geq 0\) and \(V_+ V_- = 0\). We shall show in Proposition 4 that \(T_c > 0\) satisfies the equation

\[ 1 = \left\| V_+^{1/2} \frac{1}{K_{\beta,\mu} + V_+} V_+^{1/2} \right\|. \] (2.6)

Note that since \(K_{\beta,\mu} \geq 2/\beta\), this leads immediately to the general rough upper bound

\[ T_c \leq \frac{1}{\beta} \| V_- \|_\infty. \] (2.7)

For negative potentials \(V\) and a positive chemical potential \(\mu\), we shall show that the critical temperature is always positive, i.e., that \(|p^2 - \mu| + V\) has a negative energy bound state. We can even say more, namely we can guarantee superfluidity for small temperature whenever the positive part \(V_+\) is not too large in a suitable sense.

**Theorem 3.** Let \(V \in L^{3/2}(\mathbb{R}^3)\) be not identically zero, and let \(\mu > 0\).

(i) If \(V \leq 0\), then the corresponding critical temperature is positive, \(T_c = \frac{1}{\beta_c} > 0\).
(ii) Let \( V = V_+ - V_- \), and let \( \beta > \beta_c(-V_-) \), where \( \beta_c(-V_-) \) is the critical inverse temperature for the potential \(-V_-\). Then the system at temperature \( 1/\beta \) is still in a superfluid state if \( \|v^{1/2}K_{\beta,\mu}v^{1/2}\| \) is small enough.

In particular, this theorem states that for any negative potential one can add a sufficiently small positive part such that the critical temperature \( T_c \) is still strictly positive.

The next question concerns the magnitude of \( T_c \). The rough upper bound given in (2.8) turns out to be much too rough, in general. In fact, the following Theorem states the bound precisely, let \( a := \inf_{t>0} t(e^t + 1)/((t + 2)(e^t - 1)) \). Numerically, \( a \approx 0.654 \). Let further

\[
f(t) := \frac{1}{2\pi^2} \int_0^\infty dp p^2 \left( \frac{1}{|p^2 - 1| + t} - \frac{1}{p^2 + 1 + t} \right). \tag{2.8}
\]

Note that \( f \) is a positive and monotone decreasing function, with \( f(t) \sim \ln(1/t) \) as \( t \to 0 \). We obtain the following upper bound on the critical temperature \( T_c \).

**THEOREM 4.** Let \( f^{-1} \) be the inverse of the function \( f \) in (2.8). Let \( \mu > 0 \), and let \( V = V_+ - V_- \), with \( V \in L^{3/2} \), \( V_- \in L^1 \cap L^{3/2} \), and with \( \frac{1}{3} (\frac{2}{3})^{4/3} \|V_-\|_{3/2} < a \approx 0.654 \). Then the critical temperature satisfies the upper bound

\[
T_c \leq \frac{\mu}{2} f^{-1}\left( a - \frac{4}{3} (\frac{2}{3})^{4/3} \|V_-\|_{3/2} \right). \tag{2.9}
\]

Note that, in particular, \( f^{-1}(t) \sim e^{-t} \) for large \( t \). Hence the critical temperature is exponentially small in the potential for \( V_- \in L^1 \cap L^{3/2} \).

For negative potentials, one can show that the bound given in Theorem 4 is indeed optimal. More precisely, if the potential is given by \( \lambda V \) for some fixed negative \( V \in L^{3/2} \), we shall show in Proposition 5 the lower bound

\[
T_c \geq g_{\mu,V}(\lambda),
\]

where \( g_{\mu,V} \) is some positive function depending only on \( \mu \) and \( V \) satisfying \( g_{\mu,V}(t) \sim e^{-1/t} \) for small \( t \). Combining both bounds then implies that for negative potentials \( V \) the critical temperature is exponentially small in the coupling parameter. This generalizes well-known calculations in the physics literature, see e.g. [3, Eq. (3.29)], to a very large class of interaction potentials. Note that, in particular, this result implies that it is not possible to calculate the critical temperature via a perturbative expansion in \( V \).

Finally, we comment on the continuity properties of minimizers of \( F_\beta \). In fact, for positive temperature \( T > 0 \) the minimizing \( \gamma \) and \( \hat{\alpha} \) can be shown to be continuous functions. (We show this in Prop. 6 below.) This may not be surprising, since already the minimizer without potential, \( \gamma_0(p) = 1/(e^{\beta(p^2 - \mu)} + 1) \), is continuous. But in the case of \( T = 0 \) the situation is not so clear, since the normal state is a step function with jump at \( p^2 = \mu \) for \( \mu > 0 \).

The continuity of a minimizer \( \gamma \) at \( T = 0 \) is closely related to the existence of an energy gap, given by

\[
\Xi := \inf_p E(p), \tag{2.10}
\]

with \( E \) defined in (2.4). As explained in the Appendix, \( E(p) \) has the interpretation of the dispersion relation for quasi-particle excitations. The following shows that in case \( V \) has a strictly negative Fourier transform, \( \gamma \) is continuous even at \( T = 0 \) and the gap \( \Xi \) is non-vanishing.
Proposition 1. Assume that \( \inf \text{spec}(p^2 - \mu + V) < 0 \), and that \( \hat{V} \) is strictly negative. Let \( (\gamma, \alpha) \in \mathcal{D} \) be a minimizer of \( F_{\infty} \). Then \( \Xi > 0 \) and \( \gamma \) is a continuous function.

This result implies, in particular, that \( \Xi > 0 \) for negative \( V \) with strictly negative Fourier transform. We note that this property need not necessarily be true for all potentials \( V \), however.

We conclude this section with two remarks.

Remark 3. Our results can be generalized to non-local potentials given by an integral kernel \( V(x, y) \). Exactly such a non-local potential was used in the original paper of BCS [3] who considered \( V \) of the form \( V(x, y) = -V_0 \phi(x) \phi(y) \), with \( \phi(p) = 1 \) if \( |p^2 - \mu| \leq h \) and 0 otherwise. This models an effective potential arising from electron-phonon interactions in crystals. In this case, the minimizer of \( F_{\beta} \) can be evaluated explicitly (see, e.g., [12]).

In particular, our analysis can be used to generalize previous results [4, 13, 18, 19] on the existence of solutions of the gap equation, as already mentioned in the Introduction. None of this previous work allows for an interaction kernel \( \hat{V}(k, k') \) of the form \( \hat{V}(k - k') \), however, since such a kernel does not satisfy appropriate \( L^p \) conditions. Such conditions were necessary for utilization of fixed point arguments in the previous investigations of this problem.

Remark 4. In [6, 15] the BCS-BEC crossover is studied numerically for certain potentials \( V \) that are negative and have negative Fourier transform. These potentials satisfy all the assumptions of our results stated above.

In the following Sections 3 and 4 the proofs of the claims of this section will be given.

3. Properties of the Minimizer of \( F_{\beta} \)

In this section we shall give the proof of Theorem 1 and Proposition 1 stated above. We start by showing the existence of a minimizer of \( F_{\beta} \).

Proposition 2. There exists a minimizer of \( F_{\beta} \) in \( \mathcal{D} \).

Proof. We first show that \( F_{\beta} \) dominates both the \( L^1(\mathbb{R}^3, (1 + p^2)dp) \) norm of \( \gamma \) and the \( H^1(\mathbb{R}^3, dx) \) norm of \( \alpha \). Hence any minimizing sequence will be bounded in these norms. We have

\[
F_{\beta}(\gamma, \alpha) \geq C_1 + \frac{3}{4} \int (p^2 - \mu) \gamma(p) dp + \int |\alpha(x)|^2 V(x) dx,
\]

where

\[
C_1 = \inf_{(\gamma, \alpha) \in \mathcal{D}} \left( \frac{1}{4} \int (p^2 - \mu) \gamma(p) dp - \frac{1}{\beta} S(\gamma, \alpha) \right) = -\frac{1}{\beta} \int \ln(1 + e^{-\frac{2}{\beta} (p^2 - \mu)}) dp.
\]

Since \( V \in L^{3/2} \) by assumption, it is relatively bounded with respect to \( -\Delta \) (in the sense of quadratic forms), and hence \( C_2 = \inf \text{spec} (p^2/4 + V) \) is finite. Using \( |\hat{\alpha}(p)|^2 \leq \gamma(p) \), we thus have that

\[
\frac{1}{4} \int p^2 \gamma(p) dp + \int V(x) |\alpha(x)|^2 dx \geq C_2 \int \gamma(p) dp.
\]

Using again that \( |\hat{\alpha}(p)|^2 \leq \gamma(p) \leq 1 \), it follows that

\[
F_{\beta}(\gamma, \alpha) \geq -A + \frac{1}{8} \|\alpha\|^2_{H^1(\mathbb{R}^3, dx)} + \frac{1}{8} \|\gamma\|_{L^1(\mathbb{R}^3, (1 + p^2)dp)},
\]

where

\[
A = -C_1 - \int \left[ p^2/4 - 3\mu/4 - 1/4 + C_2 \right]_+ dp,
\]
with $[\cdot]_- = \min\{\cdot,0\}$ denoting the negative part.

To show that a minimizer of $F_\beta$ exists in $\mathcal{D}$, we pick a minimizing sequence $(\gamma_n, \alpha_n) \in \mathcal{D}$, with $F_\beta(\gamma_n, \alpha_n) \leq 0$. From \textit{Lemma 1.4}, we conclude that $\|\alpha_n\|_{H^1} \leq 8A$, and hence we can find a subsequence that converges weakly to some $\tilde{\alpha} \in H^1$. Since $V \in L^{3/2}(\mathbb{R}^3)$, this implies that

$$
\lim_{n \to \infty} \int_{\mathbb{R}^3} V(x)|\alpha_n(x)|^2 \, dx = \int_{\mathbb{R}^3} V(x)|\tilde{\alpha}(x)|^2 \, dx
$$

\text{[11 Thm. 11.4].}

It remains to show that the remaining part of the functional,

$$
F_\beta^0(\gamma, \alpha) = \int \left[ (p^2 - \mu)\gamma(p)dp - \frac{1}{\beta} s(p)\ln s(p) + \frac{1}{\beta} (1 - s(p))\ln (1 - s(p)) \right] \, dp,
$$

is weakly lower semicontinuous. Note that $F_\beta^0$ is \textit{jointly convex} in $(\gamma, \alpha)$, and that its domain $\mathcal{D}$ is a convex set. We already know that $\alpha_n \rightharpoonup \tilde{\alpha}$ weakly in $H^1(\mathbb{R}^3)$. Moreover, since $\gamma_n$ is uniformly bounded in $L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we can find a subsequence such that $\gamma_n \rightharpoonup \tilde{\gamma}$ weakly in $L^p(\mathbb{R}^3)$ for some $1 < p < \infty$. We can then apply Mazur’s theorem \textit{[11 Theorem 2.13]} to construct a new sequence as convex combinations of the old one, which now converges \textit{strongly} to $(\tilde{\gamma}, \tilde{\alpha})$ in $L^p(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

By going to a subsequence, we can also assume that $\gamma_n \rightharpoonup \gamma$ and $\alpha_n \rightharpoonup \tilde{\alpha}$ pointwise \textit{[11 Theorem 2.7]}. Because of convexity of $F_\beta^0$, this new sequence is again a minimizing sequence.

Note that the integrand in (3.2) is bounded from below independently of $\gamma$ and $\alpha$ by $-\beta^{-1}\ln(1 + \epsilon/\beta(p^2 - \mu))$. Since this function is integrable, we can apply Fatou’s Lemma \textit{[11 Lemma 1.7]}, together with the pointwise convergence, to conclude that

$$
\liminf_{n \to \infty} F_\beta^0(\gamma_n, \alpha_n) \geq F_\beta^0(\tilde{\gamma}, \tilde{\alpha}).
$$

We have thus shown that

$$
F_\beta(\tilde{\gamma}, \tilde{\alpha}) \leq \liminf_{n \to \infty} F_\beta(\gamma_n, \alpha_n).
$$

It is easy to see that $(\tilde{\gamma}, \tilde{\alpha}) \in \mathcal{D}$, hence it is a minimizer. This proves the claim.}

Next, we establish a few auxiliary results that will be needed below in the proof of Theorem \textit{[11]}

\textbf{Lemma 1.} \textit{Let $(\gamma, \alpha) \in \mathcal{D}$ be a minimizer of $F_\beta$ for $0 < \beta < \infty$. Then, for a.e. $p \in \mathbb{R}^3$,

$$
(\mathring{V} * \tilde{\alpha})(p) = (p^2 - \mu) \frac{\tilde{\alpha}(p)}{2\gamma(p) - 1},
$$

(3.3)

$$
(p^2 - \mu) = -\frac{2\gamma(p) - 1}{\beta} f\left(2\sqrt{(\gamma(p) - \frac{1}{2})^2 + |\tilde{\alpha}(p)|^2}\right),
$$

(3.4)

where $f(a) := \frac{1}{2} \ln \frac{1 + a}{1 - a}$ for $0 \leq a \leq 1$.}

\textit{Proof.} For $\epsilon > 0$, let $A_\epsilon := \{p \in \mathbb{R}^3 : |\tilde{\alpha}(p)|^2 + (\gamma(p) - 1/2)^2 \leq 1/4 - \epsilon\}$. Let $g \in H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, with Fourier transform supported in $A_\epsilon$. Then $(\gamma, \alpha + \epsilon g) \in \mathcal{D}$ for small enough $|\epsilon|$. By the Lebesgue dominated convergence theorem, it follows that

$$
\frac{d}{dt} F_\beta(\gamma, \alpha + \epsilon g) \bigg|_{t=0} = 2 \text{Re} \int V(x)\overline{g(x)}\alpha(x) \, dx
$$

$$
+ \frac{2}{\beta} \text{Re} \int g(p)\tilde{\alpha}(p)f\left(2\sqrt{(\gamma(p) - \frac{1}{2})^2 + |\tilde{\alpha}(p)|^2}\right) \, dp.
$$

(3.5)
Similarly, for \( \tilde{g} \) real, \((\gamma + t\tilde{g}, \alpha) \in \mathcal{D}\) for small \(|t|\), and we get
\[
\frac{d}{dt} \mathcal{F}_\beta(\gamma + t\tilde{g}, \alpha)\bigg|_{t=0} = \int (p^2 - \mu)\tilde{g}(p)dp \tag{3.6}
\]
\[
+ \frac{1}{\beta} \int \tilde{g}(p)(2\gamma(p) - 1)f\left(2\sqrt{(\gamma(p) - \frac{1}{2})^2 + |\tilde{\alpha}(p)|^2}\right)dp.
\]
Since \((\gamma, \alpha)\) minimize \(\mathcal{F}_\beta\) by assumption, the expressions in (3.5) and (3.6) vanish. It follows that for a.e. \( p \in A\), we have
\[
(\hat{V} \ast \tilde{\alpha})(p) = -\frac{1}{\beta} \tilde{\alpha}(p)f\left(2\sqrt{(\gamma(p) - \frac{1}{2})^2 + |\tilde{\alpha}(p)|^2}\right),
\]
\[
(p^2 - \mu) = -\frac{1}{\beta} (2\gamma(p) - 1)f\left(2\sqrt{(\gamma(p) - \frac{1}{2})^2 + |\tilde{\alpha}(p)|^2}\right). \tag{3.7}
\]
To conclude that (3.7) holds a.e. in \(\mathbb{R}^3\), it remains to show that \(B := \mathbb{R}^3 \setminus \cup_{t>0}A_t\) has zero measure. By definition, if \( p \in B \) we have \(\gamma(p)(1 - \gamma(p)) = |\tilde{\alpha}(p)|^2\), i.e. \(s(p)(s(p) - 1) = 0\). It is then easy to see that if \(B\) has non-zero measure, \(\mathcal{F}_\beta(\gamma, \alpha)\) can be lowered by modifying \(\gamma\) and \(\tilde{\alpha}\) on \(B\) (since \(t \mapsto t \ln t\) has a diverging derivative at zero). This contradicts the fact the \((\gamma, \alpha)\) is a minimizer, and hence \(B\) has zero measure.

**Remark 5.** A similar strategy as in the proof of Lemma 1 leads to the conclusion that Eq. (3.3) is also valid at \(T = 0\). The relation between \(\gamma\) and \(\alpha\), given in Eq. (3.3), for \(T > 0\), has to be replaced by (2.2) at \(T = 0\), however. We omit the details.

Note that Eq. (3.4) and the positivity of \(f\) immediately imply that \(\gamma(p) \geq 1/2\) for \(p^2 < \mu\), and \(\gamma(p) \leq 1/2\) for \(p^2 > \mu\). Of greater importance below will be the following monotonicity property, which bounds \(\gamma\) in the opposite direction.

**Lemma 2.** Assume that \((\gamma, \alpha) \in \mathcal{D}\) satisfies Eq. (3.4), and let \(\gamma_0(p) = [e^{\beta(p^2 - \mu)} + 1]^{-1}\). For a.e. \( p \in \mathbb{R}^3\), we have \(\gamma(p) \leq \gamma_0(p)\) if \(p^2 < \mu\), while for \(p^2 > \mu\) we have \(\gamma(p) \geq \gamma_0(p)\). Moreover, these inequalities are strict if \(\tilde{\alpha}(p) \neq 0\). In particular,
\[
\frac{p^2 - \mu}{1 - 2\gamma(p)} \leq \frac{p^2 - \mu}{1 - 2\gamma_0(p)} \text{ on the support of } \tilde{\alpha}.
\]

**Proof.** Let \(p^2 > \mu\) such that \(\tilde{\alpha}(p) \neq 0\) and assume by way of contradiction that \(\gamma(p) \leq \gamma_0(p)\). We then have
\[
\frac{1}{2} - \gamma_0(p) \leq \frac{1}{2} - \gamma(p) < \sqrt{|\tilde{\alpha}(p)|^2 + \left(\frac{1}{2} - \gamma(p)\right)^2}.
\]
Since \(f\) is a strictly monotone increasing function, we obtain
\[
f(1 - 2\gamma_0(p)) \leq f\left(2\sqrt{|\tilde{\alpha}(p)|^2 + \left(\frac{1}{2} - \gamma(p)\right)^2}\right)
\]
and hence
\[
(1 - 2\gamma_0(p))f(1 - 2\gamma_0(p)) < (1 - 2\gamma(p))f\left(2\sqrt{|\tilde{\alpha}(p)|^2 + \left(\frac{1}{2} - \gamma(p)\right)^2}\right). \tag{3.8}
\]
This, however, contradicts the fact that both \((\gamma_0, 0)\) and \((\gamma, \alpha)\) are solutions of (3.4), and hence both sides of (3.8) are equal to \(\beta(p^2 - \mu)\).

The case \(p^2 < \mu\) is treated similarly.

We now have all the necessary prerequisites at our disposal to give the proof of Theorem 1.
Proof of Theorem \[1\] \([i] \Rightarrow [ii]\): Consider first the case \(T > 0\). According to Lemma 1, a minimizing pair \((\gamma, \alpha) \in \mathcal{D}\) with \(\alpha \neq 0\) satisfies Eqs. \((3.3)\) and \((3.4)\). Using the definitions of \(\Delta\) and \(E\) in \((2.3)\) and \((2.4)\) we see that \(\sqrt{(\gamma(p) - 1/2)^2 + |\alpha(p)|^2} = \frac{1}{p^2 - \mu} E(p)\). Plugging this into Eq. \((3.4)\) leads to
\[
\frac{p^2 - \mu}{2\gamma(p) - 1} = -\frac{1}{\beta} f \left( \frac{1 - 2\gamma(p)}{p^2 - \mu} E(p) \right) = \frac{2}{\beta E(p)} \frac{p^2 - \mu}{2\gamma(p) - 1} \tanh^{-1} \left( \frac{1 - 2\gamma(p)}{p^2 - \mu} E(p) \right),
\]
or
\[
\tanh \frac{\beta E(p)}{2} = \frac{1 - 2\gamma(p)}{p^2 - \mu} E(p). \tag{3.9}
\]
Therefore, we can rewrite the relation between \(\hat{\alpha}\) and \(\Delta\) as \(\hat{\alpha}(p) = \frac{\Delta(p)}{2E(p)} \tanh \frac{\beta E(p)}{2}\).

Together with Eq. \((3.3)\) this gives the required form of the gap equation.

The case \(T = 0\) is treated similarly (cf. Remark \[5\]).

\([ii] \Rightarrow [iii]\): Given \(T \geq 0\) and a \(\Delta\) satisfying \((2.3)\) for some \((\gamma, \alpha) \in \mathcal{D}\), we first show that there exists a \((\tilde{\gamma}, \tilde{\alpha}) \in \mathcal{D}\) yielding the same \(\Delta\) but satisfying in addition Eq. \((2.2)\) at \(T = 0\) or Eq. \((3.4)\) at \(T > 0\), respectively. Namely, we can let \(\hat{\alpha}(p) = m(p)\hat{\alpha}(p)\) and \(\gamma(p) = 1/2 + m(p)\gamma(p) - 1/2\) without changing \(\Delta\). If we choose
\[
m(p) = \text{sgn} \left( \frac{p^2 - \mu}{1 - 2\gamma(p)} \right) \frac{\tanh(\beta E(p)/2)}{2\sqrt{(\gamma(p) - 1/2)^2 + |\hat{\alpha}(p)|^2}},
\]
with \(E(p)\) as in \((2.4)\), it is easy to see that \((3.9)\) is satisfied. Using in addition the gap equation, the same reasoning as in the previous part of the proof leads to the conclusion that Eqs. \((3.3)\) and \((2.2)\) (respectively \((3.4)\)) are satisfied for this \((\tilde{\gamma}, \tilde{\alpha})\).

Hence there exists a \((\gamma, \alpha) \in \mathcal{D}\) satisfying \((3.3)\) as well as \((2.2)\) (at \(T = 0\)) or \((3.4)\) (at \(T > 0\)). Using \((3.3)\) and the definition of \(K_{\beta, \mu}\), we have that
\[
\langle \alpha, (K_{\beta, \mu} + V)\alpha \rangle = \alpha, \left( \frac{1}{1 - 2\gamma_0(p)} - \frac{1}{1 - 2\gamma(p)} \right) \left( p^2 - \mu \right) \alpha \right). \tag{3.10}
\]
It follows from Lemma \[2\] that this expression is negative whenever \(\alpha\) is not identically zero.

\([iii] \Rightarrow [i]\): Consider first the case \(T > 0\). Showing that the existence of a negative eigenvalue of the operator \((2.5)\) implies that the minimizing \(\alpha\) is not identically zero is equivalent to proving that \(\alpha \equiv 0\) implies that \(\inf \text{spec } (K_{\beta, \mu} + V) \geq 0\). First, we note that, for any \(\delta \in C_0^\infty\), \(\| (t\delta)^2 + (\gamma_0 - \frac{1}{2})^2 \|_\infty < \frac{1}{2}\) for all small \(|t|\). With the aid of the Lebesgue dominated convergence theorem it is easy to see that \(\frac{d^2}{dt^2} \mathcal{F}_{\beta}(\gamma_0, t\delta)\) exists for small \(t\), and is given by
\[
\frac{d^2}{dt^2} \mathcal{F}_{\beta}(\gamma_0, t\delta) = 2 \int V(x) |\delta(x)|^2 dx + \frac{2}{\beta} \int |\delta(p)|^2 \frac{(\gamma_0(p) - \frac{1}{2})^2}{|t\delta(p)|^2 + (\gamma_0(p) - \frac{1}{2})^2} \left( 2 |t\delta(p)|^2 + (\gamma_0(p) - \frac{1}{2})^2 \right) dp + \frac{1}{\beta} \int \frac{|t\delta(p)|^4}{|t\delta(p)|^2 + (\gamma_0(p) - \frac{1}{2})^2} \left| (\gamma_0(p) - 1 - (\gamma_0(p) - \frac{1}{2}) \right|^2 dp.
\]
Assuming that the minimizing \(\alpha\) is identically zero, it follows that the corresponding \(\gamma = \gamma_0\) and hence \(\frac{d^2}{dt^2} \mathcal{F}_{\beta}(\gamma_0, t\delta)|_{t=0} \geq 0\). But
\[
\frac{d^2}{dt^2} \mathcal{F}_{\beta}(\gamma_0, t\delta) \bigg|_{t=0} = 2 \int V(x) |\delta(x)|^2 dx + 2 \int |\delta(p)|^2 K_{\beta, \mu}(p) dp, \tag{3.10}
\]
and hence \(\langle \delta, (K_{\beta, \mu} + V)\delta \rangle \geq 0\) for all \(\delta \in C_0^\infty\). This proves the statement.
In the case $T = 0$, it is sufficient to minimize $F_\beta$ over $(\gamma, \alpha) \in D$ with $\gamma$ of the form (2.2), as remarked earlier. For simplicity, we abuse the notation slightly and denote the resulting functional by $F_\infty(\alpha)$. Subtracting its value for $\alpha = 0$, it is given by
\[
F_\infty(\alpha) = F_\infty(0) + \frac{1}{2} \int |p^2 - \mu| (1 - \sqrt{1 - 4|\hat{\alpha}(p)|^2}) dp + \int V(x)|\alpha(x)|^2 dx.
\] (3.11)

Assume that $|p^2 - \mu| + V$ has a negative eigenvalue. We can then find an $\alpha \in C_0^\infty(\mathbb{R}^3)$ such that $\langle \alpha, (|p^2 - \mu| + V)\alpha \rangle < 0$. For $0 < \epsilon \ll 1$ small enough, we then have
\[
F_\infty(\epsilon \alpha) - F_\infty(0) = \epsilon^2 \langle \alpha, (|p^2 - \mu| + V)\alpha \rangle + O(\epsilon^3) < 0.
\] (3.12)

This implies that a minimizer $\alpha$ does not vanish identically.

In the following, we investigate the continuity properties of a minimizing pair $(\gamma, \alpha) \in D$.

**Proposition 3.** If $(\gamma, \alpha) \in D$ is a minimizer of $F_\beta$ for $0 < \beta < \infty$, then both $\gamma$ and $\hat{\alpha}$ are continuous functions.

**Proof.** Since $V \in L^{3/2}$ and $\alpha \in H^1$, we know that $\hat{V} \in L^3$ and $\hat{\alpha} \in L^{6/5+\epsilon}$ for all $\epsilon > 0$. Hence $\hat{V} * \hat{\alpha}$ is continuous [11 Thm. 2.20]. From Eq. (3.7) and the definition of $\Delta$ and $E$ given in (2.3) and (2.4), we observe that $\Delta * \hat{\alpha} = -\frac{1}{2} \hat{\alpha}$ and hence both $\Delta$ and $E$ are continuous functions. As in Eq. (3.9) in the proof of Thm. 1, Eq. (3.4) can be rewritten in the form
\[
\frac{p^2 - \mu}{2\gamma(p)} = -\frac{e^{\beta E(p)} + 1}{e^{\beta E(p)} - 1} E(p).
\] (3.13)

The right side of this equation is continuous and does not vanish, not even when $E(p) = 0$ (which could happen, in principle, for $p^2 = \mu$). Hence $\gamma$ is continuous. Using again Eq. (3.3), this also implies continuity of $\hat{\alpha}$.

The situation is different at zero temperature, however. In the limit $\beta \to \infty$, the right side of Eq. (3.13) vanishes in case $E(p) = 0$. Hence $\gamma$ could possibly be discontinuous if $p^2 = \mu$. In fact, this is what happens in the case of vanishing potential. On the other hand, a non-vanishing gap $\Xi = \inf_p E(p) > 0$ implies continuity of $\gamma$. At $T = 0$, the gap equation (2.4) takes the form
\[
E(p)\hat{\alpha}(p) = -(\hat{V} * \hat{\alpha})(p),
\] (3.14)

with $E(p) = \frac{|p^2 - \mu|}{\sqrt{1 - 4|\hat{\alpha}(p)|^2}}$. Hence the gap $\Xi$ is guaranteed to be non-zero if the right side of (3.14) vanishes nowhere on the sphere $p^2 = \mu$. This can be easily shown in case $\hat{V}$ is strictly negative, which is the content of our Proposition.

**Proof of Proposition**

Since $\hat{V} \leq 0$,
\[
\int \hat{\alpha}(\hat{V} * \hat{\alpha}) \geq \int |\hat{\alpha}|(\hat{V} * |\hat{\alpha}|).
\] (3.15)

Moreover, since $\hat{V}$ is assumed to be strictly negative, there can be equality in (3.15) only if $\hat{\alpha}(p) = e^{\kappa|\hat{\alpha}(p)|}$ for some constant $\kappa \in \mathbb{R}$. This implies that this is the case for a minimizer of $F_\infty$, and hence $\hat{V} * \hat{\alpha}$ vanishes nowhere. The statement then follows from Eq. (3.14).
4. The Critical Temperature

In this section we investigate the relation between properties of the interaction potential $V$ and the critical temperature $T_c$ at which the phase transition takes place. In particular, we shall prove Theorems 2–4.

Proof of Theorem 2. Let $\beta_c = \inf \{ \beta > 0 : \inf \text{spec}(K_{\beta,\mu} + V) < 0 \}$. It is easy to see that $\beta_c > 0$, since $V$ is relatively form-bounded with respect to $|p^2 - \mu|$ and $K_{\beta,\mu} \geq c(|p^2 - \mu| + 1/\beta)$ for some $c > 0$ (cf. Lemma 3 below).

From the definition of $K_{\beta,\mu}$ in (2.5) one easily sees that

$$K_{\beta_1,\mu} \leq K_{\beta_2,\mu} \quad \text{if} \quad \beta_1 \geq \beta_2. \quad (4.1)$$

It follows that $\inf \text{spec}(K_{\beta,\mu} + V) < 0$ for all $\beta > \beta_c$. Moreover, since $K_{\beta,\mu}$ depends continuously on $\beta$ (in a norm resolvent sense), we have that $K_{\beta,\mu} + V \geq 0$. Setting $\beta_c = \infty$ in the case $\inf \text{spec}(K_{\beta,\mu} + V) \geq 0$ for all $\beta$ and letting $T_c = 1/\beta_c$ finishes the proof. \hfill \Box

Next, we prove the first assertion of Theorem 3, namely that for any negative potential the critical temperature is strictly positive.

Proof of Theorem 3(i). In light of Theorem 2 it suffices to prove that $\inf \text{spec}(|p^2 - \mu| + V) < 0$. According to the Birman-Schwinger principle, this is equivalent to the fact that for some $\varepsilon > 0$, the Birman-Schwinger operator

$$B_{\varepsilon} := |V|^{1/2}(|p^2 - \mu| + \varepsilon)^{-1}|V|^{1/2}$$

has an eigenvalue 1. Notice that $\lim_{\varepsilon \to 0} \|B_{\varepsilon}\| = 0$ since $V \in L^{3/2}$ is relatively bounded with respect to $p^2$ (in the sense of quadratic forms). Because of continuity in $\varepsilon$, it thus suffices to find an $\varepsilon > 0$ such that $\|B_{\varepsilon}\| > 1$.

Hence we have to show that there exists a $\phi \in L^2(\mathbb{R}^3)$ with $\|\phi\|_2 = 1$ such that $\langle \phi, B_{\varepsilon} \phi \rangle \geq 1$ for an appropriate $\varepsilon$. For this purpose, we choose a normalized function $\phi$ in such a way that $\phi|V|^{1/2}$ does not vanish in a neighborhood of some point $p$ with $p^2 = \mu$. We can, for instance, choose $\phi$ to be rapidly decaying, in which case $\phi|V|^{1/2}$ is continuous and hence strictly positive on some small ball. This ball can then be made to overlap with the sphere $p^2 = \mu$ by simply multiplying $\phi$ by a factor $e^{iqx}$ for appropriate $q$, which translates the function $\phi|V|^{1/2}$ by $q$. In this way,

$$\langle \phi, B_{\varepsilon} \phi \rangle = \int \frac{|\phi|^2}{|p^2 - \mu| + \varepsilon} dp \to \infty \quad \text{as} \quad \varepsilon \to 0.$$ 

Thus there exists $\varepsilon > 0$ such that $\langle \phi, B_{\varepsilon} \phi \rangle > 1$. \hfill \Box

Before proceeding with the proof of the second part of Theorem 3, we introduce an alternative characterization for the critical temperature $T_c$ in the case $T_c > 0$, which we already referred to in Eq. (2.6).

Proposition 4. For all $V \in L^{3/2}(\mathbb{R}^3)$, such that $V = V_+ - V_-$, with $V_+ \geq 0$, $V_- \geq 0$ and $V_- - V_+ = 0$, the critical temperature $T_c = 1/\beta_c$ is characterized by

$$\left\| V_{1/2}^{1/2} \frac{1}{K_{\beta,\mu} + V_+} V_{1/2}^{1/2} \right\| = 1,$$

whenever $T_c \neq 0$.

Recall that $K_{\beta,\mu} \geq 2/\beta$, hence $K_{\beta,\mu} + V_+$ has a bounded inverse.
Proof. Let the operator $B_\beta^3$ be defined as
\[ B_\beta^3 = V^{-1/2} \frac{1}{K_{\beta,\mu} + V_+ + e} V^{1/2}. \]
This operator is compact (in fact, Hilbert-Schmidt) and depends continuously on $\beta$ and $e$. Moreover, $\|B_\beta^3\|$ is strictly decreasing in $e$ and $1/\beta$.

Since $T_c > 0$ by assumption, we know from Theorem 2 that $K_{\beta,\mu} + V$ has a negative eigenvalue for $T < T_c$. Hence the Birman-Schwinger principle implies that $\|B_\beta^3\| > 1$ for $1/\beta$ and $e$ small enough. On the other hand, $\|B_0^3\| \to 0$ as $\beta \to 0$. Thus there exists a unique $\beta$ such that $\|B_\beta^3\| = 1$.

For $\beta > \beta_c$, $\|B_\beta^3\| > 1$ for $e$ small enough, and hence $K_{\beta,\mu} + V$ has a negative energy bound state. For $\beta < \beta_c$, however, $\|B_\beta^3\| < 1$ for all $e \geq 0$, and hence $K_{\beta,\mu} + V \geq 0$. This shows that $\beta = \beta_c$.

From Prop. 4 it is then easy to conclude that the critical temperature stays strictly positive under small positive perturbations to negative potentials.

Proof of Theorem 3(ii). According to Prop. 4 it suffices to prove that for $\beta > \beta_c(-V_0)$ and for $\lambda > 0$ small enough, $\|V^{1/2}(K_{\beta,\mu} + \lambda V_+)^{-1}V^{1/2}\| > 1$. In order to simplify the notation let $B_\lambda = V^{1/2}(K_{\beta,\mu} + \lambda V_+)^{-1}V^{1/2}$. It follows that
\begin{align*}
B_\lambda &= V^{-1/2} \frac{1}{K_{\beta,\mu}^{1/2}} \left( \frac{1}{1 + \lambda K_{\beta,\mu}^{-1/2} V_+ K_{\beta,\mu}^{-1/2}} \right) \frac{1}{K_{\beta,\mu}^{1/2}} V^{-1/2} \\
&\geq B_0 \left( 1 + \lambda \|K_{\beta,\mu}^{-1/2} V_+ K_{\beta,\mu}^{-1/2}\|^{-1} \right).
\end{align*}

Since $V_+ \in L^{3/2}$ by assumption, it is relatively form-bounded with respect to $K_{\beta,\mu}$, and hence $\|K_{\beta,\mu}^{-1/2} V_+ K_{\beta,\mu}^{-1/2}\| = \|V_+ K_{\beta,\mu}^{-1} V_+\|$ is finite. Therefore $\|B_\lambda\| > 1$ if $\|B_0\| > 1$ and $\lambda$ is small enough.

Next, we shall derive upper and lower bounds on the critical temperature. These bounds are similar to the ones found in the literature, but generalize these results from simple rank 1 potentials as in 3 to very general pair interactions.

We start with the upper bound given in Theorem 4. To this aim we first prove two auxiliary lemmas.

Lemma 3. Let $K_{\beta,\mu}(p) = (p^2 - \mu)(e^{2(p^2 - \mu)} + 1)(e^{2(p^2 - \mu)} - 1)^{-1}$. Then there is a constant $a > 0$ such that
\[ a \left( |p^2 - \mu| + \frac{2}{\beta} \right) \leq K_{\beta,\mu}(p) \leq \left( |p^2 - \mu| + \frac{2}{\beta} \right). \]

Proof. Let $g(t) = t(e^t + 1)/(t + 2)(e^t - 1)$ for $t \geq 0$. It is easy to see that $g(t) \leq 1$. Moreover, $g(0) = 1$ and $\lim_{t \to \infty} g(t) = 1$, hence $a := \inf_{t > 0} g(t) > 0$. In fact, $a \approx 0.654$ numerically. Since $K_{\beta,\mu}(p) = (|p^2 - \mu| + 2/\beta)g(|p^2 - \mu|)$, this implies the statement.

Lemma 4. Assume that $V \in L^1 \cap L^{3/2}$ and that $\mu > 0$. Then
\[ \left\| V^{1/2} \frac{1}{|p^2 - \mu| + e} V^{1/2} \right\| \leq \mu^{1/2} f(\mu) + \frac{1}{3} \left( \frac{2}{\pi} \right)^{4/3} \left\| V \right\|_{3/2}, \]
where $f$ is defined in (2.8).

Proof. Define $k_{e,\mu}(p) \geq 0$ by
\[ \frac{1}{|p^2 - \mu| + e} = \frac{1}{p^2 + \mu + e} + k_{e,\mu}(p). \]
Then
\[
\left\| V^{1/2} \frac{1}{p^2 - \mu + \epsilon} V^{1/2} \right\| \leq \left\| V^{1/2} \frac{1}{p^2 + \mu + \epsilon} V^{1/2} \right\| + \left\| V^{1/2} k_{\epsilon, \mu}(p) V^{1/2} \right\|. 
\]
(4.5)
The last term is bounded by
\[
\left\| V^{1/2} k_{\epsilon, \mu}(p) V^{1/2} \right\| = \sup_{\|\psi\|=1} \left\langle \psi | V^{1/2}, \hat{k}_{\epsilon, \mu} * (|V|^{1/2}) \right\rangle \leq \left(2\pi\right)^{-3/2} \|\hat{k}_{\epsilon, \mu}\|_{\infty} \|V\|_1 \leq \left(2\pi\right)^{-3} \|k_{\epsilon, \mu}\|_1 \|V\|_1.
\]
Moreover, \(\|k_{\epsilon, \mu}\|_1 = (2\pi)^{3/2} \mu^{1/2} f(e/\mu)\), with \(f\) given in (4.3).

For the first term on the right side of (4.5), the use of the Hardy-Littlewood-Sobolev inequality and Hölder’s inequality yields
\[
\left\| V^{1/2} \frac{1}{p^2 + \mu + \epsilon} V^{1/2} \right\| \leq \sup_{\|\psi\|=1} \left\langle \psi | V^{1/2}, \frac{1}{p^2} \psi | V^{1/2} \right\rangle \leq \frac{1}{3} \left(\frac{2}{\pi}\right)^{4/3} \|V\|_{3/2}.
\]

Equipped with these two lemmas we are now able to prove the upper bound on the critical temperature stated in Theorem 4.

**Proof of Theorem 4.** Assume that \(T_c > 0\), for otherwise there is nothing to prove. According to Proposition 4 the critical temperature \(T_c = 1/\beta_c\) is determined by the equation
\[
\left\| V^{1/2} \frac{1}{K_{\beta_c, \mu} + V} V^{1/2} \right\| = 1.
\]
Using Lemmas 3 and 4 as well as \(V_c \geq 0\), we thus have
\[
1 \leq \frac{1}{a} \left\| V^{1/2} \frac{1}{p^2 - \mu + 2/\beta_c} V^{1/2} \right\| \leq \frac{\mu^{1/2} \|V_c\|_1}{a} f(2/(\beta_c \mu)) + \frac{1}{3a} \left(\frac{2}{\pi}\right)^{4/3} \|V\|_{3/2}.
\]
Since \(f\) is monotone decreasing, this implies the statement. \(\square\)

Finally, we turn to a lower bound for the critical temperature. We assume that the interaction potential is given by \(\lambda V\), with \(V \leq 0\) not identically zero, and \(\lambda > 0\) a coupling parameter.

**Proposition 5.** Let \(V \in L^{3/2}\), with \(V \leq 0\) not identically zero, and let \(\mu > 0\). Then there exists a monotone increasing strictly positive function \(g_{\mu, V}\), with \(g_{\mu, V}(t) \sim e^{-t/\mu}\) as \(t \to 0\), such that the critical temperature for the potential \(\lambda V\) satisfies the lower bound
\[
T_c \geq g_{\mu, V}(\lambda)
\]
(4.6) for \(\lambda > 0\).

**Proof.** We have already shown in Thm. 3 that \(T_c > 0\) for positive \(\lambda\). By Prop. 3 the fact that \(V\) is non-positive and (1.2) we have that
\[
1/\lambda = \left\| V^{1/2} \frac{1}{K_{\beta_c, \mu}} V^{1/2} \right\| \geq \left\| V^{1/2} \frac{1}{p^2 - \mu + 2/\beta_c} V^{1/2} \right\|.
\]
For any function \(\psi \in L^2\) with \(\|\psi\|_2 = 1\) we therefore obtain
\[
1/\lambda \geq \int |\psi|^{3/2}(p) \left\{ \frac{1}{p^2 - \mu} + \frac{1}{2/\beta_c} \right\} dp.
\]
The latter integral in monotone increasing in \(\beta_c\) and diverges logarithmically as \(\beta_c \to \infty\) for appropriate \(\psi\) such that \(\psi|V|^{1/2}(p)\) does not vanish identically if
\[ p^2 = \mu. \] (See the proof of Thm. 3(i) for an argument concerning the existence of such a \( \psi \).) This implies the statement. \[ \square \]

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**Appendix A. Motivation and Physical Background**

In this Appendix, we shall briefly explain the derivation of the BCS functional defined in Definition 1. Our presentation follows the one of Leggett in [10]. Similar derivations can be found in [3, 12]. The BCS functional is obtained from several approximations and assumptions on the underlying many-body problem. We assume the reader to be familiar with the usual Fock space formalism in quantum statistical mechanics.

Consider a system of spin \( \frac{1}{2} \) fermions confined to a cubic box \( \Omega \subset \mathbb{R}^3 \), with periodic boundary conditions, interacting via a two-body potential \( V \). Assume, for simplicity, that \( V \in L^1(\mathbb{R}^3) \) so that it has a bounded Fourier transform, denoted by \( \hat{V}(p) = (2\pi)^{-3/2} \int V(x)e^{-ipx}dx \). The many-body Hamiltonian on Fock space is then given as

\[
\mathcal{H} = \sum_{\sigma, k} k^2 a_{k,\sigma}^\dagger a_{k,\sigma} + \frac{(2\pi)^{3/2}}{2|\Omega|} \sum_{k, p, q, \sigma, \nu} \hat{V}(k)a_{p-k,\sigma}^\dagger a_{q+k,\nu}^\dagger a_{q,\nu} a_{p,\sigma} . \tag{A.1}
\]

Here, \( a_{k,\sigma} \) and \( a_{k,\sigma}^\dagger \) denote the usual creation and annihilation operators for a particle of momentum \( k \) and spin \( \sigma \), satisfying the canonical anticommutation relations. The momentum sums run over \( k \in (2\pi/L)\mathbb{Z}^3 \), where \( L \) is the side length of the cube \( \Omega \), whereas the spin sums are over \( \sigma \in \{\uparrow, \downarrow\} \). Units are chosen such that \( \hbar = 2m = 1 \), \( m \) denoting the particle mass.

Following [3, 10] we restrict \( \mathcal{H} \) to BCS type states. These are quasi-free states that do not have a fixed number of particles. For details on the formalism, we refer the reader to [2]. Quasi-free states are linear maps \( \rho \) from bounded operators on the Fock space \( \mathcal{F} \) to the complex numbers, satisfying \( \rho(A^\dagger A) \geq 0 \), \( \rho(I) = 1 \), and

\[
\rho(e_1e_2e_3e_4) = \rho(e_1e_2)\rho(e_3e_4) - \rho(e_1e_3)\rho(e_2e_4) + \rho(e_1e_4)\rho(e_2e_3),
\]

whenever \( e_i \) is either \( a_{k,\sigma} \) or \( a_{k,\sigma}^\dagger \).

Assuming both translation invariance and \( SU(2) \) invariance for rotations of the spins, the state \( \rho \) is completely determined by the quantities

\[
\gamma(k) = \rho(a_{k,\uparrow}^\dagger a_{k,\uparrow}) = \rho(a_{k,\downarrow}^\dagger a_{k,\downarrow}), \quad \varphi(k) = \rho(a_{-k,\uparrow} a_{k,\downarrow}) = -\rho(a_{-k,\downarrow} a_{k,\uparrow}).
\]

All other expectation values of quadratic expressions are zero under these symmetry assumptions. Note that \( 0 \leq \gamma(p) \leq 1 \) and \( |\varphi(p)|^2 \leq \gamma(p)(1 - \gamma(-p)) \) by Schwarz’s inequality and the canonical anticommutation relations. Since also \( \varphi(p) = \varphi(-p) \), this implies that \( |\varphi(p)|^2 \leq \gamma(p)(1 - \gamma(p)) \).
The energy expectation value can be conveniently expressed in terms of these quantities:

\[
\rho(H) = 2 \sum_k k^2 \gamma(k) + \frac{(2\pi)^{3/2}}{|\Omega|} \sum_{k,p} \hat{V}(k)[\gamma(p-k)\varphi(p) + \gamma(k)\varphi(p-k)] \\
+ \frac{2(2\pi)^{3/2}}{|\Omega|} \hat{V}(0) \left( \sum_p \gamma(p) \right)^2 - \frac{(2\pi)^{3/2}}{|\Omega|} \sum_{k,p} \hat{V}(k)\gamma(p-k)\gamma(p). \tag{A.2}
\]

The first term in the second line is just a constant times the square of the number of particles. The last term is the exchange terms. Both these terms will be dropped in the sequel. This truncations can be justified on physical grounds for interaction potentials that are of short range such that s-wave scattering is dominant [10].

The von-Neumann entropy of the quasi-free state \( \rho \) under consideration here, \( S(\rho) = -\text{Tr}_x \rho \ln \rho \), can be conveniently expressed in terms of the \( 2 \times 2 \) matrix

\[
\Gamma(p) = \begin{pmatrix} \gamma(p) & \varphi(p) \\ \varphi(p) & 1 - \gamma(p) \end{pmatrix}.
\tag{A.3}
\]

Note that \( 0 \leq \Gamma(p) \leq 1 \) as an operator on \( \mathbb{C}^2 \). The entropy is then given by (compare with [2, Eq. (2.c.10)])

\[
S(\rho) = -2 \sum_p \text{Tr}_{\mathbb{C}^2} [\Gamma(p) \ln \Gamma(p)] .
\]

The factor 2 results from the 2 spin components.

The thermodynamic pressure, \( P \), of a state at a fixed temperature \( T = \frac{1}{\beta} \geq 0 \) and chemical potential \( \mu \) is defined as

\[
P(\mu) = \frac{1}{|\Omega|} [TS(\rho) - \rho(H - \mu N)].
\]

Here, \( N = \sum_{\sigma,k} \hat{a}_{k,\sigma}^+ \hat{a}_{k,\sigma} \) denotes the number operator. Dropping the last two terms in (A.2) and taking a formal infinite volume limit, one thus obtains

\[
P(\gamma, \alpha) := \lim_{\Omega \to \mathbb{R}^3} P(\rho) = \frac{2}{(2\pi)^3} \int_{\mathbb{R}^3} (T \text{Tr}_{\mathbb{C}^2} \Gamma(p) \ln \Gamma(p) + (\mu^2 - \mu \gamma(p)) dp) \\
- \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |\alpha(x)|^2 V(x) dx .
\]

Here, \( \alpha \) denotes the inverse Fourier transform of \( \varphi \), \( \alpha(x) = (2\pi)^{-3/2} \int \varphi(p)e^{ipx} dp \). If we set \( F_\beta = -\frac{1}{2} (2\pi)^3 P \) and replace \( V \) by \( 2V \) to slightly simplify the notation, we are thus led to the definition of the BCS model given in Definition [11].

Finally, we will briefly explain in which sense \( E(\rho) \) is an effective dispersion relation and how it is related to an energy gap between the ground state and the first excited state of the system.

With \( \varphi = \hat{\varphi} \) the Fourier transform of a maximizer of the pressure functional at \( T = 0 \), we introduce the following quadratic Hamiltonian as an approximation to the full Hamiltonian \( H \):

\[
H_{\text{qu}} = \sum_{\sigma,k} k^2 \hat{a}_{k,\sigma}^+ \hat{a}_{k,\sigma} + \frac{(2\pi)^{3/2}}{|\Omega|} \sum_{k,p} \hat{V}(k) \left( \varphi(k-p)a_{-p,\uparrow}a_{p,\downarrow} + \varphi(p)a_{p,\uparrow}^+ a_{-p,\downarrow}^+ \right). \tag{A.4}
\]

The Hamiltonian \( H_{\text{qu}} - \mu N \) can be diagonalized in a standard way (see, e.g., [12]) via a Bogoliubov transformation. Using the fact that \( \varphi \) satisfies the gap equation
\[ E(k)\phi(k) = -\frac{1}{2}(2\pi)^{3/2} \sum_{k'} \hat{V}(k-k')\phi(k'), \]

where \( E(k) = \sqrt{(k^2 - \mu)^2 + |\Delta(k)|^2} \) and \( \Delta \) is determined from \( \phi \) via (2.2)–(2.4), this leads to

\[ \sum_{k} E(k) \left( \hat{c}_k^\dagger \hat{c}_k + \hat{d}_k^\dagger \hat{d}_k \right). \]

The operators \( \hat{c}_k, \hat{c}_k^\dagger, \hat{d}_k \) and \( \hat{d}_k^\dagger \) satisfy the canonical anticommutation relations, and both \( \hat{c}_k \) and \( \hat{d}_k \) annihilate the BCS ground state. Equation (A.5) thus explains why \( E(k) \) is interpreted as the quasi-particle dispersion relation and \( \Xi = \inf_{k} E(k) \) as the energy gap of the system.

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