We propose a relativistic generalization of integrable systems describing $M$ interacting elliptic $gl(N)$ Euler–Arnold tops. The obtained models are elliptic integrable systems that reproduce the spin elliptic $GL(M)$ Ruijsenaars–Schneider model with $N = 1$ and relativistic integrable $GL(N)$ elliptic tops with $M = 1$. We construct the Lax pairs with a spectral parameter on the elliptic curve.

Keywords: elliptic integrable system, spin Ruijsenaars–Schneider model, integrable top

DOI: 10.1134/S0040577919110035

1. Introduction

In [1], Krichever and Zabrodin proposed an ansatz for the Lax pair with a spectral parameter of the spin elliptic $GL(M)$ Ruijsenaars–Schneider model:

$$L_{ij}(z) = S_{ij} \phi(z, q_{ij} + \eta), \quad q_{ij} = q_i - q_j, \quad \text{Res}_{z=0} L(z) = S \in \text{Mat}(M, \mathbb{C}),$$

$$M_{ij}(z) = -\delta_{ij}(E_1(z) + E_1(\eta))S_{ii} - (1 - \delta_{ij})S_{ij} \phi(z, q_{ij}) \quad i, j = 1, \ldots, M \quad (1.2)$$

The definitions of the Kronecker function $\phi$ and the elliptic functions $E_1$ and $E_2$ are given in the appendix. Under the conditions

$$S_{ii} = \dot{q}_i, \quad i = 1, \ldots, M, \quad (1.3)$$

the Lax equation

$$\dot{L}(z) = [L(z), M(z)] \quad (1.4)$$

yields the equations of motion for the diagonal part of the matrix $S$,

$$\dot{S}_{ii} = \ddot{q}_i = - \sum_{k: k \neq i} S_{ik} S_{ki} (E_1(q_{ik} + \eta) + E_1(q_{ik} - \eta) - 2E_1(q_{ik})), \quad (1.5)$$

and for its nondiagonal part (for $i \neq j$), it yields

$$\dot{S}_{ij} = \sum_{k: k \neq j} S_{ik} S_{kj} (E_1(q_{kj} + \eta) - E_1(q_{kj})) - \sum_{k: k \neq i} S_{ik} S_{kj} (E_1(q_{ik} + \eta) - E_1(q_{ik})). \quad (1.6)$$
or, equivalently,

\[
\dot{S}_{ij} = S_{ij}(S_{ii} - S_{jj})(E_1(q_{ij} + \eta) - E_1(q_{ij})) + \\
+ \sum_{k: k \neq i, j}^M S_{ik}S_{kj}(E_1(q_{kj}) - E_1(q_{kj} + \eta)) - (1.7)
\]

These equations can be viewed as a relativistic deformation \[2\] (the deformation parameter is \(\eta \in \mathbb{C}\)) of the spin elliptic Calogero–Moser model \[3\].

\[
\ddot{q}_i = \sum_{k: k \neq i}^M S_{ik}S_{ki}E_2(q_{ik}), \quad (1.8)
\]

\[
\dot{S}_{ii} = 0, \quad \dot{S}_{ij} = \sum_{k: k \neq i, j}^M S_{ik}S_{kj}(E_2(q_{ik}) - E_2(q_{kj})). \quad (1.10)
\]

System (1.8) admits an anisotropic \(gl(NM)\) generalization, where the spin variables \(S_{ij}\) are replaced with matrix-valued variables \(S^{ij} \in \text{Mat}(N, \mathbb{C})\). The equations of motion have the forms

\[
\dot{S}^{ii} = [S^{ii}, J(S^{ii})] + \sum_{k: k \neq i}^M (S^{ik}J^{q_{ik}}(S^{kk}) - J^{q_{ik}}(S^{ik})S^{ki}), \quad (1.9)
\]

\[
\dot{S}^{ij} = S^{ij}J(S^{ij}) - J(S^{ii})S^{ij} + \sum_{k: k \neq j}^M S^{ik}J^{q_{kj}}(S^{kj}) - \sum_{k: k \neq i}^M J^{q_{ik}}(S^{ik})S^{kj}, \quad (1.10)
\]

\[
\ddot{q}_i = -\frac{1}{N} \sum_{k: k \neq i}^M \partial_{q_i} \text{tr}(J^{q_{ik}}(S^{ik})S^{ki}). \quad (1.11)
\]

The anisotropy means that the linear operators \(J\) and \(\dot{J}^{q_{ij}}\) acting in the matrix space \(\text{Mat}(N, \mathbb{C})\) appear in the equations of motion. In the case \(N = 1\), Eqs. (1.9)–(1.11) reproduce (1.8), and in the case \(M = 1\), Eq. (1.9) is simplified to the Euler–Arnold equation for the integrable elliptic top \[4\]:

\[
\dot{S} = [S, J(S)], \quad S \in \text{Mat}(N, \mathbb{C}). \quad (1.12)
\]

Systems of type (1.9)–(1.11) appeared in \[5\] in studies of matrix models and were later described as examples of the Hitchin systems on \(SL(NM, \mathbb{C})\) bundles (over an elliptic curve) with nontrivial characteristic classes \[6\], \[7\]. Equations (1.9)–(1.11) for a more general class of the operators \(J\) were obtained in \[8\].

If the matrix of spin variables \(S = \sum_{ij} E_{ij} \otimes S^{ij} \in \text{Mat}(NM, \mathbb{C})\) has rank 1, then the right-hand sides of (1.9) and (1.11) are represented in terms of only diagonal blocks of the matrix \(S\) (i.e., in terms of matrices \(S^{ii}\))

\[
\dot{S}^{ii} = [S^{ii}, J(S^{ii})] + \sum_{k: k \neq i}^M [S^{ii}, \dot{J}^{q_{ik}}(S^{kk})],
\]

\[
\ddot{q}_i = -\frac{1}{N} \sum_{k: k \neq i}^M \partial_{q_i} \text{tr}(S^{ii} \dot{J}^{q_{ik}}(S^{kk})). \quad (1.13)
\]

This allows interpreting the equations as the dynamics of \(M\) interacting tops with the positions \(q_i\). Written in such a form, the model resembles the initial formulation of the (quantum) spin Calogero–Moser model \[9\].
Our purpose here is to construct a relativistic deformation of models (1.9)–(1.11) and (1.13). We show that such a generalization exists and has the form

\[
\dot{S}^{ii} = [S^{ii}, J^{0}(S^{ii})] + \sum_{k: k \neq i}^{M} (S^{ik} J^{q_{ik}}(S^{ki}) - J^{q_{ik}}(S^{ik}) S^{ki}),
\]

(1.14)

\[
\dot{S}^{ij} = S^{ij} J^{0}(S^{jj}) - J^{0}(S^{ij}) S^{ij} + \sum_{k: k \neq j}^{M} S^{ik} J^{q_{ik}}(S^{kj}) - \sum_{k: k \neq i}^{M} J^{q_{ik}}(S^{ik}) S^{kj},
\]

(1.15)

\[
\ddot{q}_{i} = \frac{1}{N} \text{tr}(\dot{S}^{ii}) = \frac{1}{N} \sum_{k: k \neq i}^{M} \text{tr}(S^{ik} J^{q_{ik}}(S^{ki}) - J^{q_{ik}}(S^{ik}) S^{ki}).
\]

(1.16)

The last equation is obtained by taking the trace of both sides of (1.14) under the conditions

\[
\dot{q}_{i} = \frac{1}{N} \text{tr}(S^{ii}), \quad i = 1, \ldots, M.
\]

(1.17)

In the particular case where the matrix of spin variables has rank 1, we obtain a system of M interacting tops associated with the \( GL(N, \mathbb{C}) \) group. It is similar to the nonrelativistic case:

\[
\dot{S}^{ii} = [S^{ii}, J^{0}(S^{ii})] + \sum_{k: k \neq i}^{M} (S^{ik} \tilde{J}^{q_{ik}}(S^{kk}) - \tilde{J}^{q_{ik}}(S^{kk}) S^{ii}),
\]

(1.18)

\[
\ddot{q}_{i} = \frac{1}{N} \sum_{k: k \neq i}^{M} \text{tr}(S^{ik} \tilde{J}^{q_{ik}}(S^{kk}) - \tilde{J}^{q_{ik}}(S^{kk}) S^{ii}).
\]

(1.19)

The obtained models can be viewed as an anisotropic matrix generalization of the spin elliptic \( GL(M) \) Ruijsenaars–Schneider model, which is reproduced in the case \( N = 1 \). If \( M = 1 \), then we obtain the relativistic deformation of elliptic top (1.12), previously known [10].

In the respective Secs. 2 and 3, we describe the spin elliptic Ruijsenaars–Schneider model and the relativistic elliptic top in detail. In Sec. 4, we describe model (1.14)–(1.16) and give a Mat\((NM, \mathbb{C})\)-valued Lax representation with a spectral parameter on the elliptic curve. In Sec. 5, we study the case \( \text{rk}(S) = 1 \) and obtain Eqs. (1.18) and (1.19). We derive the explicit form of the linear operators \( \tilde{J}^{q_{ik}} \) and \( \tilde{J}^{q_{ij}} \) in (1.18) at the end of Sec. 5 using the finite-dimensional Fourier transform of elliptic functions. We also describe the nonrelativistic limit and show that the limit reproduces the previously obtained results [8] in the elliptic case.

2. Spin Ruijsenaars–Schneider model

In this section, we derive the equations of motion. In what follows, this simplifies the proof of a more complicated statement about interacting tops. More precisely, we prove the following proposition.

Proposition 2.1. The equations of motion

\[
\dot{S}^{ii} = -\sum_{k: k \neq i}^{M} S^{ik} S^{ki} (E_{1}(q_{ik} + \eta) + E_{1}(q_{ik} - \eta) - 2E_{1}(q_{ik}))
\]

(2.1)
and (1.6) are equivalent to the Lax equations with an additional term\(^1\)

\[
\dot{L}(z) = [L(z), M(z)] + \sum_{i,j=1}^M E_{ij}(\mu_i - \mu_j)S_{ij}f(z, q_{ij} + \eta) \tag{2.2}
\]

for the pair of matrices (1.1) and (1.2) and the set of variables \(\mu_i = \dot{q}_i - S_{ii}, \ i = 1, \ldots, M\). Under the on-shell constraints \(\mu_i = 0\), matrices (1.1) and (1.2) satisfy Lax equation (1.4) and yield equations of motion (1.5) and (1.6).

**Proof.** The additional term is absent in the diagonal part of (2.2). We consider the \(i\)th diagonal element. We have \(\dot{S}_{ii}\phi(z, \eta)\) in the left-hand side of (2.2) and the expression

\[
\sum_{k: k \neq i} L_{ik}M_{ki} - M_{ik}L_{ki} = \sum_{k: k \neq i} S_{ik}S_{k}(\phi(z, q_{ik})\phi(z, q_{ki}) - \phi(z, q_{ik} + \eta)\phi(z, q_{ki})) \tag{2.3}
\]

in the right-hand side. Using relation (A.6), we obtain Eq. (2.1). In the off-diagonal part of the Eq. (2.2) for the \(ij\)th matrix element (with \(i \neq j\)), we have

\[
\dot{S}_{ij}\phi(z, q_{ij} + \eta) + S_{ij}(\dot{q}_i - \dot{q}_j)f(z, q_{ij} + \eta) \tag{2.4}
\]

in the left-hand side and

\[
(M_{jj} - M_{ii})L_{ij} + (L_{ii} - L_{jj})M_{ij} + \sum_{k: k \neq i,j} (L_{ik}M_{kj} - M_{ik}L_{kj}) + (\mu_i - \mu_j)S_{ij}f(z, q_{ij} + \eta) =
\]

\[
= (S_{ii} - S_{jj})S_{ij}(E_1(z) + E_1(\eta))\phi(z, q_{ij} + \eta) - (S_{ii} - S_{jj})S_{ij}\phi(z, q_{ij} + \eta) - \sum_{k: k \neq i,j} S_{ik}S_{kj}(\phi(z, q_{ik} + \eta)\phi(z, q_{kj}) - \phi(z, q_{ik})\phi(z, q_{kj} + \eta)) + (\mu_i - \mu_j)S_{ij}f(z, q_{ij} + \eta) \tag{2.5}
\]

in the right-hand side. We transpose the second term in (2.4) from the left-hand side of Eq. (2.2) to its right-hand side. The terms proportional to \((\dot{q}_i - \dot{q}_j)S_{ij}\) then cancel. For the terms proportional to \((S_{ii} - S_{jj})S_{ij}\), we obtain a common factor:

\[
(E_1(z) + E_1(\eta))\phi(z, q_{ij} + \eta) - \phi(z, q_{ij})\phi(z, \eta) - f(z, q_{ij} + \eta) \overset{(A.6)}{=} \overset{(A.7)}{=} \phi(z, q_{ij} + \eta)(E_1(q_{ij} + \eta) - E_1(q_{ij})). \tag{2.6}
\]

Also using relation (A.6) for the expression in the sum in (2.5), we finally obtain the off-diagonal part of the equations of motion in form (1.7). It is easy to see that the latter is equivalent to (1.6). The proof is finished.

The nonrelativistic limit appears as follows. We redefine the time variable

\[
t \mapsto \frac{t}{\eta}. \tag{2.7}
\]

\(^1\)The function \(f\) in (2.2) is defined in (A.7), and \(\{E_{ij}\}\) is the standard basis in Mat\((M, \mathbb{C})\).
In particular, this means that \( \dot{q}_i \rightarrow \eta \dot{q}_i \) and \( \ddot{q}_i \rightarrow \eta^2 \ddot{q}_i \). From definition (A.3) in the vicinity of \( \eta = 0 \), we obtain

\[
E_1(q + \eta) = E_1(q) - \eta E_2(q) - \frac{1}{2} \eta^2 E_2'(q) + O(\eta^3).
\] (2.8)

In the limit \( \eta \rightarrow 0 \), constraints (1.3) become the set of conditions

\[
S_{ii} = 0, \quad i = 1, \ldots, M,
\] (2.9)

and equations of motion (1.5)–(1.7) in view of (2.9) become

\[
\ddot{q}_i = \sum_{k: k \neq i}^M S_{ik} S_{ki} E_2'(q_{ik}).
\] (2.10)

The equations of motion for the diagonal part of the spin variables are \( \dot{S}_{ii} = 0 \), and for \( i \neq j \), we have

\[
\dot{S}_{ij} = \sum_{k: k \neq i, j}^M S_{ik} S_{kj} (E_2(q_{ik}) - E_2(q_{kj})).
\] (2.11)

Thus, we obtain the equations of motion of the classical spin Calogero–Moser model [3]. We note that the choice of the constraints \( \mu_i = 0 \) is unnecessary. All derivations also hold for the constraints \( \mu_i = \nu \) = const for all \( i = 1, \ldots, M \). This is a set of the first-class constraints in the Calogero–Moser model. They should be supplied with \( M \) conditions of gauge fixation with respect to the coadjoint action of the Cartan subgroup of \( GL(M, \mathbb{C}) \), i.e., with respect to conjugation with diagonal matrices. The total set of \( 2M \) conditions thus forms the second-class constraints, and the Poisson reduction with respect to these constraints should be performed. The reduction procedure changes the equations of motion because the number of independent variables is reduced and Dirac terms appear in the reduced Poisson brackets.

It follows from all that has been said that Eqs. (2.10) and (2.11) should be regarded as an intermediate stage of the Poisson reduction corresponding to simple restriction of the unreduced system (with linear Poisson–Lie brackets) to the imposed constraints \( \mu_i = \nu \), but the reduction procedure has not yet been performed. Equations of motion (1.5)–(1.7) in the relativistic case should be understood in the same manner on constraints (1.3). We note that the Poisson structure (and the classical \( r \)-matrix structure) for the spin elliptic Ruijsenaars–Schneider model is still unknown. At the same time, the Poisson structures and the group theory description are known for the trigonometric and rational models [11]–[15].

3. Relativistic integrable top

A special basis (the sine-algebra basis) is used to describe elliptic tops in the space \( \text{Mat}(N, \mathbb{C}) \). The basis has the form

\[
T_\alpha = T_{\alpha_1 \alpha_2} = \exp\left(\frac{\pi i}{N} \alpha_1 \alpha_2\right) Q^{\alpha_1} \Lambda^{\alpha_2}, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_N \times \mathbb{Z}_N,
\] (3.1)

where \( Q \) and \( \Lambda \) are a pair of matrices (for which \( Q^N = \Lambda^N = 1_{N \times N} \)) with the elements

\[
Q_{kl} = \delta_{kl} \exp\left(\frac{2\pi i}{N} k\right), \quad \Lambda_{kl} = \delta_{k-l+1=0 \mod N}, \quad k, l = 1, \ldots, N.
\] (3.2)

It is the finite-dimensional representation of the Heisenberg group:

\[
\Lambda^{\alpha_2} Q^{\alpha_1} = \exp\left(\frac{2\pi i}{N} \alpha_1 \alpha_2\right) Q^{\alpha_1} \Lambda^{\alpha_2}.
\] (3.3)
It is easy to see that the product of the basis matrices can be written as

$$T_\alpha T_\beta = \kappa_{\alpha,\beta} T_{\alpha + \beta}, \quad \kappa_{\alpha,\beta} = \exp\left(\frac{\pi i}{N} (\beta_1 \alpha_2 - \beta_2 \alpha_1)\right),$$

(3.4)

where $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$. In particular, it follows from the latter that

$$\text{tr}(T_\alpha T_\beta) = N \delta_{\alpha,-\beta}.$$

(3.5)

The term “sine-algebra” comes from the form of the structure constants of the $gl(N)$ Lie algebra, which are

$$[T_\alpha, T_\beta] = C_{\alpha\beta} T_{\alpha + \beta}, \quad C_{\alpha\beta} = \kappa_{\alpha,\beta} - \kappa_{\beta,\alpha} = 2i \sin\left(\frac{\pi i}{N} (\beta_1 \alpha_2 - \beta_2 \alpha_1)\right).$$

(3.6)

For brevity in what follows, we let a single zero denote the index $(0,0)$:

$$T_{(0,0)} = 1_N = T_0.$$

(3.7)

**The relativistic top** in the GL(2) case was essentially described by Sklyanin in [16] as a Hamiltonian system with the quadratic Poisson bracket (the classical Sklyanin algebra) in terms of the semiclassical limit of the RLL relations. Here, we use the description of the GL(N)-top proposed in [10], [17].

The dynamical variables are the components $S_\alpha$ of the matrix $S \in \text{Mat}(N, \mathbb{C})$ in basis (3.1). We define the action of the linear operator (multidimensional analogue of the inverse inertia tensor in principal axes):

$$J^n(S) = \sum_{\alpha \neq 0} T_\alpha S_\alpha \left(E_1(\omega_\alpha + \eta) - E_1(\omega_\alpha)\right), \quad S = \sum_\alpha T_\alpha S_\alpha \in \text{Mat}(N, \mathbb{C}),$$

(3.8)

where $\omega_\alpha$ is defined as in (A.8). The equations of motion are the Euler–Arnold equations for the dynamics of a rigid body in a multidimensional space:

$$\dot{S} = [S, J^n(S)].$$

(3.9)

The Lax pair has the form

$$L(z) = \sum_\alpha T_\alpha S_\alpha \varphi_\alpha(z, \eta + \omega_\alpha), \quad M(z) = -\sum_{\alpha \neq 0} T_\alpha S_\alpha \varphi_\alpha(z, \omega_\alpha).$$

(3.10)

**Proposition 3.1.** Lax equation (1.4) for the pair of matrices (3.10) is equivalent to equation of motion (3.9).

**Proof.** The left-hand side of the Lax equation is equal to $\sum_\alpha T_\alpha \dot{S}_\alpha \varphi_\alpha(z, \eta + \omega_\alpha)$. For the right-hand side, we have

$$-\sum_{\beta, \gamma \neq 0} [T_\beta, T_\gamma] S_\beta S_\gamma \varphi_\beta(z, \omega_\beta + \eta) \varphi_\gamma(z, \omega_\gamma),$$

(3.11)

where the term with $\beta = 0$ is absent because of (3.7) (it is proportional to the identity matrix). Antisymmetrizing this expression with respect to the indices $\beta$ and $\gamma$, we obtain

$$-\frac{1}{2} \sum_{\beta, \gamma \neq 0} [T_\beta, T_\gamma] S_\beta S_\gamma (\varphi_\beta(z, \omega_\beta + \eta) \varphi_\gamma(z, \omega_\gamma) - \varphi_\gamma(z, \omega_\gamma + \eta) \varphi_\beta(z, \omega_\beta)) \overset{(A.10)}{=} (3.6)$$

$$\overset{(A.10)}{=} \overset{(3.6)}{-\frac{1}{2} \sum_{\beta, \gamma \neq 0} C_{\beta, \gamma} T_{\beta + \gamma} S_\beta S_\gamma \varphi_{\beta + \gamma}(z, \eta + \omega_{\beta + \gamma}) (J^n_\beta - J^n_\gamma) = -\frac{1}{2} \sum_{\beta, \gamma \neq 0} C_{\beta, \gamma} T_{\beta + \gamma} S_\beta S_\gamma \varphi_{\beta + \gamma}(z, \eta + \omega_{\beta + \gamma}) J^n_\gamma.}$$

(3.12)
In the components of basis (3.1), the equations of motion become
\[
\dot{S}_0 = 0, \quad \dot{S}_\alpha = \sum_{\beta \neq 0} C_{\beta,\alpha-\beta} S_\beta S_{\alpha-\beta} J_\alpha^\beta, \quad \alpha \neq 0, \quad (3.13)
\]
which means that the Lax equation is equivalent to (3.9).

The nonrelativistic limit \( \eta \to 0 \) is taken together with rescaling the time variable \( t \to t/\eta \). Equation of motion (3.9) becomes the Euler–Arnold equation of the nonrelativistic elliptic top [4]:
\[
\dot{S} = [S, J(S)], \quad J(S) = -\sum_{\alpha \neq 0} T_\alpha S_\alpha E_2(\omega_\alpha), \quad S = \sum_\alpha T_\alpha S_\alpha \in \text{Mat}(N, \mathbb{C}), \quad (3.14)
\]
where we use the function \( E_2 \) given by (A.3).

4. \textit{GL}(NM) generalization of the spin M-body Ruijsenaars–Schneider model

In this section, we simultaneously define a generalization of both the spin Ruijsenaars–Schneider model in Sec. 2 and the relativistic top in Sec. 3. The Lax representation for the new model has the size \( NM \times NM \) and has a natural block-matrix structure:
\[
\mathcal{L}(z) = \begin{pmatrix}
\mathcal{L}^{11}(z) & \mathcal{L}^{12}(z) & \cdots & \mathcal{L}^{1M}(z) \\
\mathcal{L}^{21}(z) & \mathcal{L}^{22}(z) & \cdots & \mathcal{L}^{2M}(z) \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{L}^{M1}(z) & \mathcal{L}^{M2}(z) & \cdots & \mathcal{L}^{MM}(z)
\end{pmatrix}, \quad (4.1)
\]

where each column consists of \( M \) blocks of size \( N \times N \). In other words,
\[
\mathcal{L}(z) = \sum_{i,j=1}^M E_{ij} \otimes \mathcal{L}^{ij}(z) \in \text{Mat}(NM, \mathbb{C}), \quad \mathcal{L}^{ij}(z) \in \text{Mat}(N, \mathbb{C}). \quad (4.2)
\]

Similarly to how the residue of Lax matrix (1.1) was equal to the matrix \( S \), the residue of Lax matrix (4.1) is here equal to \( \text{Res}_{z=0} \mathcal{L}(z) = S \in \text{Mat}(NM, \mathbb{C}) \), and for each block, we have
\[
\mathcal{L}^{ij}(z) = \mathcal{L}^{ij}(S^{ij}, z), \quad S^{ij} = \text{Res}_{z=0} \mathcal{L}^{ij}(z) \in \text{Mat}(N, \mathbb{C}). \quad (4.3)
\]
The explicit expression for the \( ij \)th block of the Lax matrix becomes
\[
\mathcal{L}^{ij}(z) = \sum_{\alpha} T_\alpha S^{ij}_\alpha \varphi_\alpha(z, \omega_\alpha + q_{ij} + \eta), \quad q_{ij} = q_i - q_j, \quad \omega_\alpha = \frac{\alpha_1 + \alpha_2 \tau}{N}, \quad (4.4)
\]
For the \( ij \)th block of the \( M \)-matrix, we obtain
\[
\mathcal{M}^{ij}(z) = -T_0 S^{ij}_0 (E_1(z) + E_1(\eta)) - \sum_{\alpha \neq 0} T_\alpha S^{ij}_\alpha \varphi_\alpha(z, \omega_\alpha), \quad (4.5)
\]
\[
\mathcal{M}^{ij}(z) = -\sum_{\alpha} T_\alpha S^{ij}_\alpha \varphi_\alpha(z, \omega_\alpha + q_{ij}) \quad \text{for } i \neq j.
\]
In the case \( N = 1 \), we have only the scalar parts of the above matrices, i.e., those proportional to \( T_0 = 1_N \). Lax pair (1.1), (1.2) is thus reproduced. If \( M = 1 \), then we have a single (diagonal) block. In this case, we obtain Lax pair (3.10) up to the term \(-E_1(q)T_0S_0^{11}\) in the \( M \)-matrix. But for \( M = 1 \), this does not affect the equations of motion, because it is proportional to the identity matrix.

For a compact form of the equations of motion, we define the linear operator

\[
J^{\eta,q_{ij}}(S^{ij}) = \sum_{\alpha} T_\alpha S_{ij}^{\alpha} \left( E_1 (\omega_\alpha + q_{ij} + \eta) - E_1 (\omega_\alpha + q_{ij}) \right)
\]  

(4.6)

acting on the \( ij \)th block (\( i \neq j \)) of the matrix \( S \). At the same time, we use the linear operator \( J^\eta \) given by (3.8) for the diagonal blocks. As is shown below, the equations of motion coming from the Lax equation with Lax pair (1.1), (1.2) become

\[
\dot{S}_{ii} = [S_{ii}, J^\eta(S^{ii})] + \sum_{k: k \neq i} (S_{ik} J^{\eta,q_{ki}}(S^{kj}) - J^{\eta,q_{ik}}(S^{ik}) S_{kj})
\]  

(4.7)

for the diagonal blocks of the matrix \( S \) and

\[
\dot{S}_{ij} = S_{ij} J^\eta(S^{ij}) - J^\eta(S^{ii}) S_{ij} + S_{ij} J^{\eta,q_{ij}}(S^{ij}) - J^{\eta,q_{ij}}(S^{ij}) S_{jj} +
\]  

\[ + \sum_{k: k \neq i,j} (S_{ik} J^{\eta,q_{kj}}(S^{kj}) - J^{\eta,q_{ik}}(S^{ik}) S^{kj})
\]  

(4.8)

or, equivalently,

\[
\dot{S}_{ij} = S_{ij} J^\eta(S^{ij}) - J^\eta(S^{ii}) S_{ij} + \sum_{k: k \neq j} S_{ik} J^{\eta,q_{kj}}(S^{kj}) - \sum_{k: k \neq i} J^{\eta,q_{ik}}(S^{ik}) S^{kj}
\]  

(4.9)

for the non-diagonal blocks.

For the positions of particles, we have the equation

\[
\ddot{q}_i = \frac{1}{N} \text{tr}(\dot{S}^{ii}) = \frac{1}{N} \sum_{k: k \neq i} \text{tr}(S_{ik} J^{\eta,q_{ki}}(S^{kj}) - J^{\eta,q_{ik}}(S^{ik}) S^{kj})
\]  

(4.10)

which is derived from Eq. (4.7) by taking the trace of both parts together with the conditions

\[
\dot{q}_i = S_0^{ii} = \frac{1}{N} \text{tr}(S^{ii}), \quad i = 1, \ldots, M,
\]  

(4.11)

which are analogues of relations (1.3) for the generalized model.

We note that for \( N = 1 \), the linear operators \( J^{\eta,q_{ij}}(S^{ij}) \) given by (4.6) and \( J^\eta(S^{ii}) \) given by (3.8) become

\[
J^{\eta,q_{ij}}(S_{ij}) = (E_1(q_{ij} + \eta) - E_1(q_{ij})) S_{ij}, \quad J^\eta(S_{ii}) = 0.
\]  

(4.12)

The operator \( J^\eta \) is equal to zero in the case \( N = 1 \) because the scalar term (with \( \alpha = 0 \)) in the right-hand side of Eq. (3.8) is absent. Using (4.12), we can easily see that for \( N = 1 \), equations of motion (4.7)–(4.9) and (4.10) become (1.5)–(1.7). We prove a statement similar to Proposition 2.1.
Proposition 4.1. Equations of motion (4.7) and (4.8), (4.9) for diagonal and off-diagonal blocks of the matrix \( S \) are equivalent to the Lax equation with an additional term

\[
\dot{\mathcal{L}}(z) = [\mathcal{L}(z), \mathcal{M}(z)] + \sum_{i,j=1}^{M} E_{ij} \otimes T_{\alpha}(\mu_{0}^{i} - \mu_{0}^{j})S^{ij}_{\alpha} f_{\alpha}(z, \omega_{\alpha} + q_{ij} + \eta) \tag{4.13}
\]

for Lax pair (4.4), (4.5) and the set of variables

\[
\mu_{0}^{i} = \dot{q}_{i} - S^{ii}_{0} = \dot{q}_{i} - \frac{1}{N} \text{tr}(S^{ii}), \quad i = 1, \ldots, M. \tag{4.14}
\]

Under the on-shell constraints \( \mu_{0}^{i} = 0 \), matrices (4.4) and (4.5) satisfy the Lax equation \( \dot{\mathcal{L}}(z) = [\mathcal{L}(z), \mathcal{M}(z)] \) and yield equations of motion (4.7)–(4.9) and (4.10).

Proof. The proof is similar to the proof of Proposition 2.1. We consider the left-hand side of (4.13):

\[
\dot{\mathcal{L}}^{ii}(z) = \sum_{\alpha} T_{\alpha} \dot{S}^{ii}_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha} + \eta). \tag{4.15}
\]

For \( i \neq j \), we have

\[
\dot{\mathcal{L}}^{ij}(z) = \sum_{\alpha} T_{\alpha} (\dot{S}^{ij}_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha} + q_{ij} + \eta) + (\dot{q}_{i} - \dot{q}_{j})S^{ij}_{\alpha} f_{\alpha}(z, \omega_{\alpha} + q_{ij} + \eta)) =
\]

\[
= \sum_{\alpha} T_{\alpha} \dot{S}^{ij}_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha} + q_{ij} + \eta) + \sum_{\alpha} T_{\alpha} (\mu_{0}^{i} - \mu_{0}^{j})S^{ij}_{\alpha} f_{\alpha}(z, \omega_{\alpha} + q_{ij} + \eta) +
\]

\[
+ \sum_{\alpha} T_{\alpha} (S^{ii}_{0} - S^{ij}_{0})S^{ij}_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha} + q_{ij} + \eta)(E_{1}(z + \omega_{\alpha} + q_{ij} + \eta) - E_{1}(\omega_{\alpha} + q_{ij} + \eta)). \tag{4.16}
\]

The expression (function) in the last summation in the right-hand side is just the function \( f_{\alpha}(z, \omega_{\alpha} + q_{ij} + \eta) \) in (A.9) transformed using (A.7). The second summation in the right-hand side of (4.16) exactly cancels with the additional term in the right-hand side of Eq. (4.13). The rest of the terms in the right-hand side (i.e., those coming from the commutator) have the form

\[
[\mathcal{L}^{ii}, \mathcal{M}^{ij}] + \sum_{k=k\neq i} (\mathcal{L}^{ik} \mathcal{M}^{ki} - \mathcal{M}^{ik} \mathcal{L}^{ki}) \tag{4.17}
\]

for the diagonal block and the form

\[
\mathcal{L}^{ii} \mathcal{M}^{ij} - \mathcal{M}^{ii} \mathcal{L}^{ij} + \mathcal{L}^{ij} \mathcal{M}^{ij} - \mathcal{M}^{ij} \mathcal{L}^{ij} + \sum_{k=k\neq i,j} (\mathcal{L}^{ik} \mathcal{M}^{kj} - \mathcal{M}^{ik} \mathcal{L}^{kj}) \tag{4.18}
\]

for the off-diagonal \( ij \)th block \( (i \neq j) \).

The computations for the diagonal blocks are very similar to those in Secs. 2 and 3. The first term (commutator) in (4.17) provides the first term (also a commutator) in the right-hand side of equation of motion (4.7) in the same way as in deriving Eq. (3.9) for the relativistic top. The sum in (4.17) is simplified similarly to (2.3):

\[
\mathcal{L}^{ik} \mathcal{M}^{ki} - \mathcal{M}^{ik} \mathcal{L}^{ki} = \sum_{\beta, \gamma} T_{\beta} T_{\gamma} S^{ik}_{\beta} S^{ki}_{\gamma} (\varphi_{\beta}(z, \omega_{\beta} + q_{ik})\varphi_{\gamma}(z, \omega_{\gamma} + q_{ki} + \eta) -
\]

\[
- \varphi_{\beta}(z, \omega_{\beta} + q_{ik} + \eta)\varphi_{\gamma}(z, \omega_{\gamma} + q_{ki})). \tag{4.19}
\]
Applying (A.10) to the expression in parentheses, we obtain

\[ \sum_{\beta, \gamma} T_\beta T_\gamma S^i_\beta S^i_\gamma \varphi_{\beta+\gamma}(z, \omega_{\beta+\gamma} + q_{ii} + \eta) \times \]

\[ \times (E_1(\omega_\gamma + q_{ki} + \eta) - E_1(\omega_\gamma + q_{ki}) - E_1(\omega_{\beta} + q_{ik} + \eta) + E_1(\omega_{\beta} + q_{ik})) \]  

\[ (4.20) \]

which yields the sum in the right-hand side of equations of motion (4.7). The scalar (i.e., zero) component, which provides (4.11) because of relations (3.4) and (3.5), corresponds to the terms with \( \beta = -\gamma \).

We consider the off-diagonal block \( ij \). For the first four terms in (4.18), we separately write all the terms containing the scalar (zero) components of the diagonal blocks of matrix \( S \) (i.e., the terms containing \( S^i_0 \)):

\[ (S^i_0 - S^i_0) \sum_\alpha T_\alpha S^i_\alpha \varphi_\alpha(z, \omega_\alpha + q_{ij} + \eta) - \varphi_\alpha(z, \omega_\alpha + q_{ij})\phi(z, \eta) \]  

\[ (A.10) \Rightarrow (4.10) \]

\[ (4.21) \]

There are the same type of terms (containing \( S^i_0 \)) in the left-hand side of Eq. (4.13). They are in the last summation in (4.16). We transpose them to the right-hand side and sum with result (4.21). We then obtain

\[ (S^i_0 - S^i_0) \sum_\alpha T_\alpha S^i_\alpha \varphi_\alpha(z, \omega_\alpha + q_{ij} + \eta)(E_1(\omega_\alpha + q_{ij} + \eta) - E_1(\omega_\alpha + q_{ij})). \]  

\[ (4.22) \]

We next write the rest of the first four terms in (4.18), i.e., the terms not containing \( S^i_0 \):

\[ \sum T_\beta T_\gamma S^i_\beta S^i_\gamma \varphi_\beta(z, \omega_\beta) \varphi_\gamma(z, \omega_\gamma + q_{ij} + \eta) - \varphi_\beta(z, \omega_\beta + \eta)\varphi_\gamma(z, \omega_\gamma + q_{ij}) \]

\[ - \sum T_\gamma T_\beta S^i_\gamma S^i_\beta \varphi_\beta(z, \omega_\beta) \varphi_\gamma(z, \omega_\gamma + q_{ij} + \eta) - \varphi_\beta(z, \omega_\beta + \eta)\varphi_\gamma(z, \omega_\gamma + q_{ij}) \]

\[ (4.23) \]

where the primed sum is the sum over the two indices \( \beta, \gamma \in \mathbb{Z}_N^2 \) with condition \( \beta \neq 0 \) (the terms with \( \beta = 0 \) were already included in (4.21)). Applying (A.10), we obtain

\[ \sum T_\beta T_\gamma S^i_\beta S^i_\gamma \varphi_{\beta+\gamma}(z, \omega_{\beta+\gamma} + q_{ij} + \eta)(E_1(\omega_{\beta} + \eta) - E_1(\omega_\beta)) \]

\[ - \sum T_\beta T_\gamma S^i_\beta S^i_\gamma \varphi_{\beta+\gamma}(z, \omega_{\beta+\gamma} + q_{ij} + \eta)(E_1(\omega_\gamma + q_{ij} + \eta) - E_1(\omega_\gamma + q_{ij})) \]

\[ + \sum T_\beta T_\gamma S^i_\beta S^i_\gamma \varphi_{\beta+\gamma}(z, \omega_{\beta+\gamma} + q_{ij} + \eta)(E_1(\omega_{\gamma} + q_{ij} + \eta) - E_1(\omega_{\gamma} + q_{ij})) \]

\[ - \sum T_\beta T_\gamma S^i_\beta S^i_\gamma \varphi_{\beta+\gamma}(z, \omega_{\beta+\gamma} + q_{ij} + \eta)(E_1(\omega_{\beta} + q_{ij} + \eta) - E_1(\omega_{\beta} + q_{ij})). \]  

\[ (4.24) \]

Hence, in the right-hand side of the Lax equation, we have only terms (4.24) and (4.22). We note that adding (4.22) to (4.24) is equivalent to removing the primes in the last two sums in (4.24) because terms (4.22) correspond to the index value \( \beta = 0 \) in these sums. Therefore, the sum of (4.22) and (4.24) reproduce the first four terms in the right-hand side of equation of motion (4.8).

It remains to determine the contribution to the equation of motion from the last sum in (4.18). For this, we consider the difference

\[ L^{ik} M^{kj} - M^{ik} L^{kj} = \sum_{\beta, \gamma} T_\beta T_\gamma S^i_\beta S^i_\gamma \varphi_\beta(z, \omega_\beta + q_{ik}) \varphi_\gamma(z, \omega_\gamma + q_{kj} + \eta) \]

\[ - \varphi_\beta(z, \omega_\beta + q_{ik} + \eta) \varphi_\gamma(z, \omega_\gamma + q_{kj}) \]  

\[ (4.25) \]

\[ 1574 \]
Again applying (A.10) to the expression in the parentheses, we obtain

\[
\sum_{\beta, \gamma} T_{\beta} T_{\gamma} S_{\beta} S_{\gamma} \phi_{\beta + \gamma}(z, \omega_{\beta + \gamma} + q_{ij} + \eta) \times
\]
\[
\times \left( E_1(\omega_{\gamma} + q_{kj} + \eta) - E_1(\omega_{\gamma} + q_{kj}) - E_1(\omega_{\beta} + q_{lk} + \eta) + E_1(\omega_{\beta} + q_{lk}) \right),
\]

which reproduces the sum in the right-hand side of equation of motion (4.8). The proof is finished.

Similarly to (2.7)–(2.11) and (3.14) in the nonrelativistic limit \( \eta \to 0 \), we have

\[
\dot{S}_{ii} = [S_{ii}, J(S_{ii})] + \sum_{k : k \neq i}^M \left( S_{ik} J q_{ki} (S_{ki}) - J q_{ik} (S_{ik}) S_{ki} \right),
\]

(4.27)

\[
\dot{S}_{ij} = S_{ij} J(S_{ij}) - J(S_{ij}) S_{ij} + \sum_{k : k \neq j}^M S_{ik} J q_{kj} (S_{kj}) - \sum_{k : k \neq i}^M J q_{ik} (S_{ik}) S_{kj}
\]

(4.28)

and

\[
\dot{q}_{i} = -\frac{1}{N} \sum_{k : k \neq i}^M \partial q_{i} \text{tr}(J q_{ik} (S_{ik})),
\]

(4.29)

where \( J(S_{ij}) \) are as in (3.14) and

\[
J q_{ij} (S_{ij}) = -\sum_{\alpha} T_{\alpha} S_{ij} E_2(\omega_{\alpha} + q_{ij}).
\]

(4.30)

The last equation of motion (4.29) is obtained similarly to (2.8), i.e., for the operator \( J^{q_{ij}} (S_{ij}) \) given by (4.6), we use the expansion in the vicinity of \( \eta = 0 \)

\[
J^{q_{ij}} (S_{ij}) = \eta J^{q_{ij}} (S_{ij}) + \frac{1}{2} \eta^2 \partial_{\eta} J^{q_{ij}} (S_{ij}) + O(\eta^3).
\]

(4.31)

For \( N = 1 \), Eqs. (4.27)–(4.29) become Eqs. (2.10) and (2.11) of the spin Calogero-Moser model.

Equations (4.27)–(4.30) were recently derived in [8] for more general case. Constraints (4.14) in the nonrelativistic limit become

\[
\frac{1}{N} \text{tr}(S_{ii}) = 0, \quad i = 1, \ldots, M.
\]

(4.32)

The comment at the end of Sec. 2 is also applicable to these constraints. In this case, the gauge is fixed with respect to the coadjoint action of a \( M \)-dimensional subgroup in the Cartan subgroup of \( GL(NM, \mathbb{C}) \), i.e., with respect to conjugation by matrices of the form \( D = \sum_k d_k E_{kk} \otimes T_0 \).

5. Interacting relativistic tops

In this section, we consider the special case of the \( GL(NM, \mathbb{C}) \) model in the preceding section where the matrix \( S \) has rank 1, i.e.,

\[
S_{ij} = \xi_i \otimes \rho^j \in \text{Mat}(N, \mathbb{C}), \quad i, j = 1, \ldots, M,
\]

(5.1)

or

\[
S_{ij} = \xi_i \rho_j, \quad i, j = 1, \ldots, M, \quad a, b = 1, \ldots, N.
\]

(5.2)

\footnote{Slightly different normalization factors were used in the elliptic case in [8]. In particular, the positions of particles were divided by \( N \) everywhere.}
where $\xi^i$ is a set of $M$ vector-columns each of height $N$ and $\rho^i$ is a set of $M$ vector-rows each of length $N$.

Our purpose here is to rewrite the right-hand side of equations of motion (4.7) for the diagonal blocks in terms of only the diagonal blocks. For $M$ diagonal blocks, we then obtain a closed system of $M$-matrix equations of motion. A problem in using conditions (5.1) and (5.2) is that they are written in the standard basis, while the basis $T_\alpha$ given by (3.1) is used for the operators $J^{q_i,q_j}$ (4.6) in Eqs. (4.7). We use the tensor notation to overcome this difficulty.

We consider an operator $A$ of the form

$$A(S) = \sum_\alpha T_\alpha S_\alpha A_\alpha \in \text{Mat}(N, \mathbb{C}), \quad S = \sum_\alpha T_\alpha S_\alpha \in \text{Mat}(N, \mathbb{C}),$$

and introduce the notation

$$A_{12} = \sum_\alpha A_\alpha T_\alpha \otimes T_{-\alpha}, \quad \breve{A}_{12} = \sum_\alpha \breve{A}_\alpha T_\alpha \otimes T_{-\alpha}, \quad A_{12}, \breve{A}_{12} \in \text{Mat}(N, \mathbb{C})^\otimes 2,$$

and

$$\breve{A}_{21} = \sum_\alpha \breve{A}_\alpha T_{-\alpha} \otimes T_\alpha = \sum_\alpha \breve{A}_{-\alpha} T_\alpha \otimes T_{-\alpha} = P_{12} \breve{A}_{12} P_{12} = P_{12} A_{12},$$

where $P_{12}$ is the permutation operator

$$P_{12} = \sum_{a,b=1}^N e_{ab} \otimes e_{ba} = \frac{1}{N} \sum_\alpha T_\alpha \otimes T_{-\alpha}$$

(the set of matrices $\{e_{ab}\}$ forms the standard basis of the space $\text{Mat}(N, \mathbb{C})$). We write a few main properties of the permutation operator: $(P_{12})^2 = 1_N \otimes 1_N$ and $(B \otimes C)P_{12} = P_{12} (C \otimes B)$ for matrices $B, C \in \text{Mat}(N, \mathbb{C})$. Also, using the standard notation $S_1 = S \otimes 1_N$ and $S_2 = 1_N \otimes S$ (see, e.g., [16]), we have $S_2 P_{12} = P_{12} S_1$ and $\text{tr}_2(P_{12} S_2) = S_1$, where $\text{tr}_2$ is a trace over the second tensor component.

With the above notation, operator (5.3) becomes

$$A(S) = \left(5.5\right) \frac{1}{N} \text{tr}_2(A_{12} S_2) = \frac{1}{N} \text{tr}_2(\breve{A}_{12} P_{12} S_2) = \frac{1}{N} \text{tr}_2(S_2 \breve{A}_{12} P_{12}) =$$

$$= \frac{1}{N} \text{tr}_2(S_2 P_{12} \breve{A}_{21}) = \frac{1}{N} \text{tr}_2(P_{12} S_1 \breve{A}_{21}),$$

where we use a cyclic permutation of matrices in the second tensor component in the last equality in the first line.

We consider the expression

$$S^{ik} A(S^{kj}) = \frac{1}{N} \text{tr}_2(S_1^{ik} P_{12} S_1^{kj} \breve{A}_{21}),$$

For matrix (5.1), we obtain

$$S_1^{ik} P_{12} S_1^{kj} = \sum_{a,b=1}^N ((\xi^i \otimes \rho^k) e_{ab}(\xi^k \otimes \rho^j)) \otimes e_{ba} = \sum_{a,b=1}^N (\xi^i \otimes \rho^j) (\rho^k e_{ab} \xi^k) \otimes e_{ba} =$$

$$= \sum_{a,b=1}^N S^{i^j} \text{tr}(e_{ab} S^{kk}) \otimes e_{ba} = S_1^{i^i} S_2^{kk},$$

(5.9)
where we use the fact that \((\rho^k e_a e^k)\) is a scalar. For (5.8), we then have

\[
S^{ik} A(S^{ki}) = \frac{1}{N} S^{ii} \text{tr}_2(\tilde{A}_{21}) S^{kk}.
\]

(5.10)

Similarly,

\[
A(S^{ik}) S^{ki} = \frac{1}{N} \text{tr}_2(\tilde{A}_{12} S^{ik} P_{12} S^{ki}) = \frac{1}{N} \text{tr}_2(\tilde{A}_{12} S^{kk}) S^{ii}.
\]

(5.11)

Using (5.10) and (5.11) for Eq. (4.10), we obtain the following statement.

**Proposition 5.1.** In case (5.1), if the matrix of spin variables has rank 1, then equations of motion (4.7) become

\[
\dot{S}^{ii} = [S^{ii}, J^{\eta}(S^{ii})] + \sum_{k: k \neq i}^{M} (S^{ii} \tilde{J}^{\eta, q_{ii}} (S^{kk}) - \tilde{J}^{\eta, q_{ii}} (S^{kk}) S^{ii}),
\]

(5.12)

\[
\ddot{q}_i = \frac{1}{N} \text{tr} (\dot{S}^{ii}) = \frac{1}{N} \sum_{k: k \neq i}^{M} \text{tr} (S^{ii} \tilde{J}^{\eta, q_{ii}} (S^{kk}) - \tilde{J}^{\eta, q_{ii}} (S^{kk}) S^{ii}),
\]

(5.13)

where (we obtain explicit expressions for \(\tilde{J}^{\eta, q_{ij}}\) and \(\tilde{J}^{\eta, q_{ij}}\) in (5.20) below)

\[
\tilde{J}^{\eta, q_{ij}} (S^{kk}) = \frac{1}{N} \text{tr}_2(\tilde{J}^{\eta, q_{ij}} S^{kk}), \quad \tilde{J}^{\eta, q_{ij}} (S^{kk}) = \frac{1}{N} \text{tr}_2(\tilde{J}^{\eta, q_{ij}} S^{kk}),
\]

(5.14)

and

\[
J^{\eta, q_{ij}}_{12} = \sum_{\alpha} T_\alpha \otimes T_{-\alpha} (E_1(\omega_\alpha + q_{ij} + \eta) - E_1(\omega_\alpha + q_{ij})).
\]

(5.15)

We derive explicit expression for \(\tilde{J}^{\eta, q_{ij}}_{12} = J^{\eta, q_{ij}}_{12} P_{12}\) from (5.14):

\[
J^{\eta, q_{ij}}_{12} P_{12} = \frac{1}{N} \sum_{\alpha, \gamma} T_\alpha T_\gamma \otimes T_{-\alpha} T_{-\gamma} (E_1(\omega_\alpha + q_{ij} + \eta) - E_1(\omega_\alpha + q_{ij})) = \frac{1}{N} \sum_{\alpha, \gamma} \kappa^{2}_{\alpha, \gamma} T_\alpha T_{\gamma} (E_1(\omega_\alpha + q_{ij} + \eta) - E_1(\omega_\alpha + q_{ij})).
\]

(3.4)

\[
= \frac{1}{N} \sum_{\alpha, \gamma} \kappa^{2}_{\alpha, \gamma} T_\alpha T_{\gamma} (E_1(\omega_\alpha + q_{ij} + \eta) - E_1(\omega_\alpha + q_{ij})).
\]

(5.16)

The summation over the index \(\alpha\) is the finite-dimensional Fourier transform of the expression in the parentheses. We use the formulas [18]

\[
\frac{1}{N} \sum_{\alpha} (E_1(\omega_\alpha + \eta) + 2\pi i \partial_\alpha \omega_\alpha) = E_1(N\eta),
\]

\[
\frac{1}{N} \sum_{\alpha} \kappa^{2}_{\alpha, \gamma} (E_1(\omega_\alpha + \eta) + 2\pi i \partial_\alpha \omega_\alpha) = \varphi_\gamma(N\eta, \omega_\gamma) \text{ for } \gamma \neq 0.
\]

(5.17)

Substituting this in (5.16), we obtain

\[
\tilde{J}^{\eta, q_{ij}}_{12} = J^{\eta, q_{ij}}_{12} P_{12} = \sum_{\alpha} I^{\eta, q_{ij}}_{\alpha} T_\alpha \otimes T_{-\alpha},
\]

(5.18)
where

\[ I_{0}^{q,i} = E_{1}(Nq_{i} + N\eta) - E_{1}(Nq_{i}), \]

\[ I_{\alpha}^{q,i} = \varphi_{\alpha}(Nq_{i} + N\eta, \omega_{\alpha}) - \varphi_{\alpha}(Nq_{i}, \omega_{\alpha}) \quad \text{for} \quad \alpha \neq 0. \]

We thus also obtain an explicit answer for (5.14): 

\[ \tilde{j}_{\eta,i}(S^{kk}) = \frac{1}{N} \text{tr}_{2}(\tilde{j}_{12}^{\eta,i} S_{2}^{kk}) = \sum_{\alpha} T_{\alpha} S_{\alpha}^{kk} I_{\alpha}^{q,i}, \]

\[ \tilde{j}^{q,i}(S^{kk}) = \frac{1}{N} \text{tr}_{2}(\tilde{j}_{21}^{q,i} S_{2}^{kk}) = \sum_{\alpha} T_{\alpha} S_{\alpha}^{kk} I_{\alpha}^{q,i}. \]  

(A.19)

In the nonrelativistic limit \( \eta \to 0 \) (see (2.7)), we have

\[ \tilde{\dot{s}}^{ii} = [S^{ii}, J(S^{ii})] + \sum_{k: k \neq i}^{M} [S^{ii}, \tilde{j}^{q,k}(S^{kk})], \]

\[ \tilde{\dot{q}}_{i} = -\frac{1}{N} \sum_{k: k \neq i}^{M} \partial_{q_{i}} \text{tr}(S^{ii} \tilde{j}^{q,k}(S^{kk})), \]

where

\[ \tilde{j}^{q,i}(S^{kk}) = \partial_{q_{i}} \tilde{j}_{\eta,i} q_{i}(S^{kk})|_{\eta=0} = \partial_{q_{i}} \sum_{\alpha} T_{\alpha} S_{\alpha}^{kk} F_{\alpha}^{q,i}, \]

\[ F_{0}^{q,i} = E_{1}(Nq_{i}), \quad F_{\alpha}^{q,i} = \varphi_{\alpha}(Nq_{i}, \omega_{\alpha}). \]

Such an answer follows because the function \( E_{2}(z) \) is even and consequently

\[ \partial_{q_{i}} \tilde{j}^{\eta,q}_{21}|_{\eta=0} = \partial_{q_{i}} \tilde{j}^{\eta,-q}_{12}|_{\eta=0} = \partial_{q_{i}} \tilde{j}^{\eta,-q}_{12}|_{\eta=0} = \partial_{q_{i}} \tilde{j}^{\eta,q}_{21}|_{\eta=0}. \]  

(A.20)

The classical spin variables in the models of Calogero and Ruijsenaars types are often described in terms of quiver parameterization [3, 11, 5, 14]. For the \( GL(M, \mathbb{C}) \) model, this means introducing \( 2NM \) variables \( \xi_{i}^{a}, \rho_{i}^{a}, i = 1, \ldots, M, a = 1, \ldots, N \), such that the spin variables (in the \( GL(M, \mathbb{C}) \) case) have the form \( S_{ij} = \sum_{a} \xi_{a}^{i} \rho_{a}^{j} \). In the trigonometric and rational cases, the Poisson structure is known, and equations of motion can be written for the set of Mat\((N, \mathbb{C})\)-valued variables \( S^{i} \) and \( S_{ab}^{i} = \xi_{a}^{i} \rho_{b}^{i} \) (see, e.g., [3, 11]). Such equations can be viewed as isotropic analogues of Eqs. (4.27)–(4.29). In our approach, we have a Mat\((NM, \mathbb{C})\)-valued variable \( S \), and the \( 2NM \)-dimensional parameterization is not given by a pair of rectangular matrices of size \( N \times M \) but arises as a particular case (of rank 1) for the matrix of size \( NM \times NM \), as in (5.2).

**Appendix: Elliptic functions**

The main tool for constructing the Lax pairs with a spectral parameter on the elliptic curve \( \Sigma_{\tau} \) with moduli \( \tau \) (\( \text{Im} \tau > 0 \)) is the Kronecker function

\[ \phi(z, q) = \frac{\theta'(0)\vartheta(z + q)}{\vartheta(z)\vartheta(q)}, \]

(A.1)

defined in terms of the Riemann theta function

\[ \theta(z) = \sum_{k \in \mathbb{Z}} \exp(\pi i \tau(k + \frac{1}{2})^{2} + 2\pi i \left( z + \frac{1}{2} \right)(k + \frac{1}{2})), \]

(A.2)
which has a simple zero at \( z = 0 \) because it is odd. We also use the first and second Eisenstein functions

\[
E_1(z) = \frac{\vartheta'(z)}{\vartheta(z)}, \quad E_2(z) = -\partial_z E_1(z) = \varphi(z) - \frac{1}{3} \frac{\vartheta'''(0)}{\vartheta'(0)}, \tag{A.3}
\]

where \( \varphi(z) \) is the Weierstrass \( \varphi \)-function. The function \( E_2(z) \) is double-periodic on the lattice \( \mathbb{Z} + \tau \mathbb{Z} \) and has a second-order pole at \( z = 0 \). The first Eisenstein function and the Kronecker function have a simple pole at zero with the residue equal to one. They transform on the lattice as

\[
E_1(z + 1) = E_1(z), \quad E_1(z + \tau) = E_1(z) - 2\pi i, \tag{A.4}
\]

\[
\varphi(z + 1, q) = \varphi(z, q), \quad \varphi(z + \tau, q) = e^{-2\pi i q} \varphi(z, q).
\]

The main relation for function (A.1) is the Fay identity of genus one:

\[
\varphi(z_1, q_1) \varphi(z_2, q_2) = \varphi(z_1 - z_2, q_1) \varphi(z_2, q_1 + q_2) + \varphi(z_2 - z_1, q_2) \varphi(z_1, q_1 + q_2). \tag{A.5}
\]

We use its degeneration to derive the Lax equation:

\[
\varphi(z, q_1) \varphi(z, q_2) = \varphi(z, q_1 + q_2) \left( E_1(z) + E_1(q_1) + E_1(q_2) - E_1(q_1 + q_2 + z) \right). \tag{A.6}
\]

We use the notation

\[
f(z, q) = \partial_q \varphi(z, q) \quad \text{(A.3)} \quad \varphi(z, q)(E_1(z + q) - E_1(q)) \tag{A.7}
\]

to derive the Kronecker function with respect to the second argument. To describe the models of elliptic tops, we also define the set of functions labeled by the index \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_N \times \mathbb{Z}_N \) in accordance with the labeling of elements of matrix basis (3.1):

\[
\varphi_\alpha(z, \omega_\alpha + \eta) = \exp \left( 2\pi i \frac{\alpha_2}{N} z \right) \varphi(z, \omega_\alpha + \eta), \quad \omega_\alpha = \frac{\alpha_1 + \alpha_2 \tau}{N}. \tag{A.8}
\]

Similarly,

\[
f_\alpha(z, \omega_\alpha + \eta) = \exp \left( 2\pi i \frac{\alpha_2}{N} z \right) f(z, \omega_\alpha + \eta). \tag{A.9}
\]

By virtue of (A.6), the set of functions (A.8) satisfies the relations

\[
\varphi_\alpha(z, \omega_\alpha + q_1) \varphi_\beta(z, \omega_\beta + q_2) = \\
= \varphi_{\alpha + \beta}(z, \omega_{\alpha + \beta} + q_1 + q_2) \left( E_1(z) + E_1(\omega_\alpha + q_1) + E_1(\omega_\beta + q_2) - E_1(z + \omega_{\alpha + \beta} + q_1 + q_2) \right). \tag{A.10}
\]

**Conflicts of interest.** The author declares no conflicts of interest.

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