Global Solution for Gas-Liquid Flow of 1-D van der Waals Equation of State with Large Initial Data

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Abstract. This paper is concerned with a diffuse interface model for the gas-liquid phase transition. The model consists the compressible Navier-Stokes equations with van der Waals equation of state and a modified Allen-Cahn equation. The global existence and uniqueness of strong solution with the periodic boundary condition (or the mixed boundary condition) in one dimensional space is proved for large initial data. Furthermore, the phase variable and the density of the gas-liquid mixture are proved to stay in the physical reasonable interval. The proofs are based on the elementary energy method and the maximum principle, but with new development, where some techniques are introduced to establish the uniform bounds of the density and to treat the non-convexity of the pressure function.

Keywords: global solution, Navier-Stokes equations, Allen-Cahn equation, gas-liquid flow, van der Waals equation of state
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1 Introduction and Main Result

In the last few decades, there have been many progresses on modelling and analysis of the multiphase and phase transition problems, in particular on the phase field models of the phenomena, see [2]-[4], [10] [20] [30] [34] [36] and the references therein. In this paper, we investigate Navier-Stokes-Allen-Cahn system proposed by Blesgen [3] which describes the compressible two-phase flow with diffusive interface. The system consists of the compressible Navier-Stokes equations and a modified Allen-Cahn equation, and it is especially useful for analyzing the phase transition properties of gas-liquid flow. It allows phases to shrink or grow due to changes of density in the fluid and incorporates their transport with the current.

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The Navier-Stokes-Allen-Cahn system is commonly expressed as follows (see [3], [8], [7], [6], [35] etc.)

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho \mathbf{u}) &= 0, \\
\partial_t (\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - (\nu \Delta \mathbf{u} + \eta \nabla \text{div} \mathbf{u}) &= -\epsilon \text{div}(\nabla \chi \otimes \nabla \chi - \frac{1}{2} |\nabla \chi|^2 I), \\
\partial_t (\rho \chi) + \text{div}(\rho \chi \mathbf{u}) &= -\frac{1}{\epsilon} \frac{\partial f(\rho, \chi)}{\partial \chi} + \frac{\epsilon}{\rho} \Delta \chi,
\end{align*}
\]  

(1.1)

where \( \rho = \rho(x,t), \mathbf{u} = \mathbf{u}(x,t) \) and \( \chi = \chi(x,t) \) are the density, the velocity and the concentration difference of the gas-liquid mixture. The constants \( \nu > 0, \eta \geq 0 \) are viscosity coefficients, and the constant \( \epsilon > 0 \) is defined as the thickness of the diffuse interface of the gas-liquid mixture. The potential energy density \( f = f(\rho, \chi) \), satisfying the Ginzburg-Landau double-well potential model (see [11], [7], [6] and the references therein), follows that:

\[
f(\rho, \chi) = -3 \rho + \frac{8 \Theta}{3} \ln \frac{\rho}{3 - \rho} + \frac{1}{4} (\chi^2 - 1)^2,
\]

(1.2)

with \( 0 < \Theta \) is the positive constant related to the ratio of the actual temperature to the critical temperature. The pressure \( p \) is given by the following van der Waals equation of state (see [30], [17], [11], [26], [27], [15], [9] and the references therein)

\[
p(\rho) = \begin{cases} 
\rho^2 \frac{\partial f}{\partial \rho}, & \text{if } 0 \leq \rho < 3, \\
+\infty, & \text{if } \rho \geq 3.
\end{cases}
\]

(1.3)

We have the following properties of the pressure \( p \):

(i) \( p(\rho) > 0 \) for \( \rho > 0 \), \( p(0) = 0 \);

(ii) When \( \Theta \geq 1 \), \( p(\rho) \) is a monotone increasing function. When \( 0 < \Theta < 1 \), there exist two positive densities \( 3 > \beta > \alpha > 0 \) such that \( p(\rho) \) is increasing on \([0, \alpha]\) and on \([\beta, 3]\), \( p(\rho) \) is decreasing on \((\alpha, \beta)\);

(iii) \( p'(\rho) = -\frac{6(\rho^3 - 6\rho^2 + 9\rho - 4\Theta)}{(3 - \rho)^2} \). When \( 0 < \Theta < 1 \), there exist a positive density \( \gamma \), such that, \( p(\gamma) = p(\beta), \) and \( p(\rho) > p(\gamma) \) for \( \rho > \gamma \), \( p \) is increasing on \([0, \gamma]\).

Remark 1.1. The van der Waals state equation (1.3) is proposed by the Dutch physicist J. D. van der Waals [30]. It is a thermodynamic equation of state which is based on the theory that fluids are composed of particles with non-zero volumes, and subject to an inter-particle attractive force. Over the critical temperature (i.e. \( \Theta \geq 1 \) in (1.3)), this equation of state is an improvement over the ideal gas law. And what’s more, below the critical temperature (i.e. \( 0 < \Theta < 1 \) in (1.3)), this equation is also qualitatively reasonable for the low-pressure gas-liquid states.
Remark 1.2. The concentration difference $\chi$ of the gas-liquid mixture can be understood as $\chi = \chi_1 - \chi_2$, where $\chi_i = \frac{M_i}{V}$ is the mass concentration of the fluid $i$ ($i = 1, 2$), $M_i$ is the mass of the components in the representative material volume $V$. The item of $\epsilon \left( \nabla \chi \otimes \nabla \chi - \frac{\nabla \chi^2}{2} \right)$ in the momentum equation (1.1) can be seen as an additional stress contribution in the stress tensor. This describes the capillary effect associated with free energy $E_{\text{free}}(\rho, \chi) = \int_\Omega \left( \frac{\rho}{\epsilon} f(\rho, \chi) + \frac{\epsilon}{2} |\nabla \chi|^2 \right) dx$, (see [1], [8], [7], [6], [5] and the references therein).

There are a lot of works on the well-posedness of the solutions to compressible Navier-Stokes system. We refer to the work of Matsumura-Nishida [21], Matsumura-Nishihara [22]-[23], Lions [19], Huang-Li-Xin [16], Mei [24]-[25], Huang-Li-Matsumura [12], Huang-Matsumura-Xin [13], Huang-Wang-Wang-Yang [14], Shi-Yong-Zhang [29] and the references therein.

The study of interfacial phase changing in mixed fluids can be traced back to the work by van der Waals (1894). van der Waals described the interface between two immiscible fluids as a layer in the pioneer paper [30]. His idea was successfully applied by Cahn-Hilliard [4] and Allen-Cahn [2] to describe the complicated phase separation and coarsening phenomena, the motion of anti-phase boundaries in the mixture respectively. Lowengrub-Truskinovsky [20] added the effect of the motion of the particles and the interaction with the diffusion into the Cahn-Hilliard equation, and the Navier-Stokes-Cahn-Hilliard system was put forward. Blesgen [3] then combined the compressible Navier-Stokes system with the modified Allen-Cahn equation to describe the behavior of cavitation in a flowing liquid, which was known as Navier-Stokes-Allen-Cahn system. The difference between Navier-Stokes-Allen-Cahn system and Navier-Stokes-Cahn-Hilliard system is that, for the former, the diffusion fluxes are neglected and the development of the constitutive equation for mass conversion of any of the considered phases is focused. This leads to that the latter conserves the volume fractions while the former does not.

Nowadays, Navier-Stokes-Allen-Cahn system and Navier-Stokes-Cahn-Hilliard system are widely used in the interfacial diffusion problems of fluid mechanics and material science. Comparatively speaking, the numerical treatment to the former is simpler than that of the latter which involves fourth-order differential operators. However, because the concentration difference $\chi$ in (1.1) does not preserve overall volume fraction, a Lagrange multiplier is usually introduced in (1.1)_3 as a constraint to conserve the volume, see Yang-Feng-Liu-Shen [32], Zhang-Wang-Mi [33] and the references therein. Feireisl-Petzeltová-Rocca-Schimperna [8] obtained the global existence of weak solutions for the isentropic case, where the method they used is the framework introduced by Lions [19]. Along the way proposed by Feireisl et al., Ding-Li-Luo [7] proved the global existence of one-dimensional strong solution in the bounded domain for initial density without vacuum states. Chen-Guo [6] generalized Ding-Li-Luo’s result to the case that the initial vacuum is allowed.

However, all the results above are for the ideal fluid. In order to study the gas-liquid phase transition, we need to consider the non-ideal viscous fluid in which, there is an interval of the density $\rho$ where the pressure $p$ decreases as $\rho$ increases, and the phase transition takes place.
The equations of state (1.3) proposed by van der Waals is quite satisfactory in describing this phenomena. Hsieh-Wang [11] solved the isentropic compressible Navier-Stokes system model by the van der Waals state equation numerically by a pseudo-spectral method with a form of artificial viscosity. They showed that the phase transition depends on the selection of the initial density. He-Liu-Shi [9] investigated the large time behavior for van der Waals fluid in 1-D by using a second order TVD Runge-Kutta splitting scheme combined with Jin-Xin relaxation scheme. Mei-Liu-Wong [26, 27] studied Navier-Stokes system with additional artificial viscosity and 

\[
p(\rho) = \rho^{-3} - \rho^{-1}.
\]

By using the Liapunov functional method, they proved the existence, uniqueness, regularity and uniform boundedness of the periodic solution in 1-D. Hoff and Khodia [17] considered the dynamic stability of certain steady-state weak solutions of system (1.1) for compressible van der Waals fluids in 1-D whole space with the small initial disturbance.

In this paper, we study the global existence of the solution for the system (1.1) with the van der Waals state equation (1.3) in one dimension. More precisely, for general initial conditions without vacuum state, our purpose is to study the existence and uniqueness of global strong solution for the isentropic Navier-Stokes-Allen-Cahn systems (1.1) even with large initial data. Moreover we show that the phase variable \( \chi \) belongs to the physical interval \([-1, 1]\). Some new techniques are developed to establish the up and low bounds of the density \( \rho \), and to treat the non-convexity of the pressure \( p(\rho) \), both are crucial steps in the proof.

We now present our main result. The 1-D isentropic Navier-Stokes-Allen-Cahn system in the Euler coordinates is expressed in the following

\[
\begin{aligned}
&\rho_t + (\rho u)_x = 0, \quad x \in \mathbb{R}, t > 0, \\
&\rho u_t + \rho uu_x + p_x = \nu u_{xx} - \frac{\epsilon}{2}(\chi_x^2)_x, \quad x \in \mathbb{R}, t > 0, \\
&\rho \chi_t + \rho u \chi_x = -\frac{1}{\epsilon}(\chi^3 - \chi) + \frac{\epsilon}{\rho} \chi_{xx}, \quad x \in \mathbb{R}, t > 0,
\end{aligned}
\]

(1.4)

with the \( L \)-periodic boundary value condition:

\[
\begin{aligned}
&\left. (\rho, u, \chi) \right|_{x=t} = (\rho, u, \chi)(x + L, t), \quad x \in \mathbb{R}, t > 0, \\
&\left. (\rho, u, \chi) \right|_{t=0} = (\rho_0, u_0, \chi_0), \quad x \in \mathbb{R}.
\end{aligned}
\]

(1.5)

We introduce the Hilbert space \( L^2_{\text{per}} \) of square integrable functions with the period \( L \):

\[
L^2_{\text{per}} = \left\{ g(x) \mid g(x + L) = g(x) \text{ for all } x \in \mathbb{R}, \text{ and } g(x) \in L^2(0, L) \right\},
\]

(1.6)

with the norm denoted also by \( \| \cdot \| \) (without confusion) which is given by \( \| g \| = \left( \int_0^L |g(x)|^2 \, dx \right)^{\frac{1}{2}} \). \( H^l_{\text{per}} \) \((l \geq 0)\) denotes the \( L^2_{\text{per}} \)-functions \( g \) on \( \mathbb{R} \) whose derivatives \( \partial^j_x g, j = 1, \ldots, l \) are \( L^2_{\text{per}} \) functions, with the norm \( \| g \|_l = \left( \sum_{j=0}^l \| \partial^j_x g \|_{L^2} \right)^{\frac{1}{2}} \). The initial and boundary data for the density, velocity and concentration difference of two components are assumed to be:

\[
(\rho_0, u_0) \in H^1_{\text{per}}, \quad \chi_0 \in H^2_{\text{per}}; \quad 0 < \rho_0 < 3, \quad -1 \leq \chi_0 \leq 1;
\]

(1.7)
\[
\chi_t(x, 0) = -u_0 \chi_0 x + \frac{\epsilon}{\rho_0^2} \chi_0 xx - \frac{1}{\epsilon \rho_0} \left( \chi_0^3 - \chi_0 \right). \tag{1.8}
\]

**Theorem 1.1.** Assume that \((\rho_0, u_0, \chi_0)\) satisfies (1.7) - (1.8), then there exists a unique global strong solution \((\rho, u, \chi)\) of the system (1.4) - (1.5) such that for any \(T > 0\),

\[
\begin{align*}
\rho &\in L^{\infty}(0, T; H^1_{\text{per}}) \cap L^2(0, T; H^1_{\text{per}}), \\
u &\in L^{\infty}(0, T; H^2_{\text{per}}) \cap L^2(0, T; H^2_{\text{per}}), \\
\chi &\in L^{\infty}(0, T; H^3_{\text{per}}) \cap L^2(0, T; H^3_{\text{per}}), \\
-1 &\leq \chi \leq 1, \quad 0 < \rho < 3, \quad \text{for all } (x, t) \in \mathbb{R} \times [0, T],
\end{align*}
\]

and

\[
\sup_{t \in [0,T]} \{ \| \rho(t) \|_1^2 + \| u \|_2^2 + \| \chi \|_3^2 \} + \int_0^T (\| \rho \|_1^2 + \| u \|_2^2 + \| \chi \|_3^2) dt \leq C, \tag{1.10}
\]

where \(C\) is a positive constant depending only on the initial data and \(T\).

**Remark 1.3.** There are two difficulties to overcome in proving Theorem 1.1. One is the upper and lower bounds of the density \(\rho\), the other is the non-convexity of the pressure. For the former, we use the singularity of pressure and the energy estimation of \(\| \frac{\rho}{\rho_0} \|_{L^\infty([0,L] \times [0,T])}\). For the latter, we decompose the pressure according to its convexity. The results of the Theorem 1.1 are valid even for large initial data. They also match well with the existing numerical studies in [11] and [9].

Moreover, we consider the following mixed boundary value problem:

\[
\begin{cases}
\rho_t + (\rho u)_x = 0, \\
\rho u_t + \rho uu_x + p_x = \nu u xx - \frac{\epsilon}{2} \left( \chi_x^2 \right)_x, \\
\rho \chi_t + \rho u \chi_x = -\frac{1}{\epsilon} (\chi^3 - \chi) + \frac{\epsilon}{\rho} \chi xx, \\
(u, \chi)|_{x=0,L} = (0,0), \\
(p, u, \chi)|_{t=0} = (\rho_0, u_0, \chi_0).
\end{cases} \tag{1.11}
\]

Similarly, we have the following existence theorem for the mixed boundary problem (1.11). The proof will be omitted.

**Theorem 1.2.** Assume that \((\rho_0, u_0, \chi_0)\) satisfies

\[
(\rho_0, u_0) \in H^1, \quad \chi_0 \in H^2, \quad 0 < \rho_0 < 3, \quad -1 \leq \chi_0 \leq 1, \tag{1.12}
\]

\[
\chi_t(x, 0) = -u_0 \chi_0 x + \frac{\epsilon}{\rho_0^2} \chi_0 xx - \frac{1}{\epsilon \rho_0} \left( \chi_0^3 - \chi_0 \right). \tag{1.13}
\]
then there exists a unique global strong solution \((\rho, u, \chi)\) of the system \((1.11)\), such that for any \(T > 0\),

\[
\begin{align*}
\rho &\in L^\infty(0, T; H^1) \cap L^2(0, T; H^1), \\
u &\in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \\
\chi &\in L^\infty(0, T; H^2) \cap L^2(0, T; H^3),
\end{align*}
\]

(1.14)

and

\[
\sup_{t \in [0, T]} \{\|(\rho, u)(t)\|_2^2 + \|\chi\|_2^2\} + \int_0^T (\|\rho\|_2^2 + \|u\|_2^2 + \|\chi\|_3^2)\,dt \leq C,
\]

(1.15)

where \(C\) is a positive constant depending only on the initial data and \(T\).

The outline of this paper is as follows. In Section 2, we first give the local existence of the solution for the system \((1.4)-(1.5)\). Then, we give a series of lemmas which lead us the desired a priori estimates. Finally, Theorem 1.1 is proved by the well-known alternative result and the maximum principle for parabolic equation.

## 2 Proofs of the main theorem

In this section, we will present the global existence on strong solution for the periodic problem \((1.4)-(1.5)\). Firstly, for \(\forall m > 0, \ M > 0, \ T > 0\), we define the periodic solution space:

\[
X_{\text{per}, m, M}([0, T]) \equiv \left\{ (\rho, u, \chi) \mid (\rho, u) \in C^0([0, T]; H^1_{\text{per}}), \chi \in C^0([0, T]; H^2_{\text{per}}), \right. \\
\left. \quad \rho \in L^2([0, T]; H^1_{\text{per}}), u \in L^2([0, T]; H^2_{\text{per}}), \chi \in L^2([0, T]; H^3_{\text{per}}), \quad (2.1) \right. \\
\left. \inf_{x \in \mathbb{R}, t \in [0, T]} \rho(x, t) \geq m, \sup_{t \in [0, T]} \{\|(\rho, u)\|_2^2, \|\chi\|_2^2\} \leq M \right\}.
\]

**Proposition 2.1** (Local existence). For \(\forall m > 0, \ M > 0, \) if \(\inf_{x \in \mathbb{R}} \rho(x, t) \geq m, \|\rho(0, 0, \rho)\|_2^2, \|\chi(0, 0, \chi)\|_2^2 \leq M\), then there exists a small time \(T_0 = T_0(\rho_0, u_0, \chi_0) > 0\) such that the periodic boundary problem \((1.4)-(1.5)\) admits a unique solution \((\rho, u, \chi)\) satisfying that \((\rho, u, \chi) \in X_{\text{per}, \rho, \chi, 2M}([0, T_s]).\)

**Proof.** Taking \(0 < T < +\infty\), for \(\forall m > 0, \ M > 0\), we construct an iterative sequence \((\rho^{(n)}, u^{(n)}, \chi^{(n)}), n = 1, 2, \ldots\), satisfying \((\rho^{(0)}, u^{(0)}), \chi^{(0)}) = (v_0, u_0, \chi_0)\), and the following iterative scheme

\[
\begin{align*}
\rho_t^{(n)} + (\rho^{(n)}u^{(n-1)})_x &= 0, \\
\rho u_t^{(n)} + (\rho^{(n)}u^{(n-1)})_x + (p(\rho^{(n)}))_x &= \nu u_{xx} - \frac{\epsilon}{2}((\chi^{(n)})_x^2)_x, \\
\rho\chi_t^{(n)} + (\rho^{(n)}\chi^{(n-1)})_x &= -\frac{1}{\epsilon}((\chi^{(n-1)})^3 - \chi^{(n-1)}) + \frac{\epsilon}{\rho^{(n)}}\chi_{xx}^{(n)}, \\
(\rho^{(n)}, u^{(n)}, \chi^{(n)})(x, t) &= (\rho^{(n)}, u^{(n)}, \chi^{(n)})(x + L, t), \\
(\rho^{(n)}, u^{(n)}, \chi^{(n)})(x, 0) &= (\rho_0, u_0, \chi_0)(x),
\end{align*}
\]

(2.2)
By using the usual iterative approach (c.f. [5]), we can obtained the local existence of the solution for the periodic boundary problem (1.4)-(1.5), the details are omitted.

Now we will prove the global existence and uniqueness of the solution for the periodic boundary problem (1.4)-(1.5). Setting
\[ \mu = \frac{1}{c}(\chi^3 - \chi) - \frac{c}{\rho} \chi_{xx}. \]  
(2.3)

From the physical point of view, the functional \( \mu \) in (2.3) can be understood as the chemical potential. The basic energy equality is presented below. From the definition of the pressure \( p \) in (1.3), we fix a positive reference density \( \tilde{\rho} \) satisfying (see the properties of \( p \))
\[ 0 < \tilde{\rho} < \gamma < 3, \]  
(2.4)
and define
\[ \Phi(\rho) = \rho \int_{\tilde{\rho}}^{\rho} \frac{p(s) - p(\tilde{\rho})}{s^2} ds. \]  
(2.5)

Noting that
\[ \Phi'(\rho) = \frac{\Phi(\rho) + p(\rho) - p(\tilde{\rho})}{\rho}, \quad \text{and} \quad \Phi''(\rho) = \frac{p'(\rho)}{\rho}, \]
then \( \Phi(\tilde{\rho}) = \Phi'(\tilde{\rho}) = 0 \), and so that, there exist positive constants \( c_1, c_2 > 0 \) such that
\[ c_1(\rho - \tilde{\rho})^2 \leq \Phi(\rho) \leq c_2(\rho - \tilde{\rho})^2. \]  
(2.6)

Moreover, combining with the mass conservation equation (1.4), one gets
\[ \Phi(\rho)_t + (\Phi(\rho)u)_x + (p(\rho) - p(\tilde{\rho}))u_x = 0. \]  
(2.7)

Taking advantage of the local existence result Proposition 2.1, we know that there exists a unique strong solution of the system (1.4)-(1.5) for \( T > 0 \) small enough. By using the well-known alternative result, and the maximum principle for parabolic equation (see [28]), it suffices to show the following a priori estimate.

**Proposition 2.2** (A priori estimate). Assume that \((v_0, u_0, \chi_0)\) satisfies (1.7)-(1.8), let \((\rho, u, \chi) \in X_{\text{per}, m, M}([0, T])\) be a local solution for a given \( T > 0 \), then there exists a positive constant \( C \), such that
\[ \sup_{t \in [0, T]} \{ (||\rho, u(t)||_1^2 + ||\chi||_2^2) + \int_0^T (||\rho||_1^2 + ||u||_2^2 + ||\chi||_2^3) dt \} \leq C. \]  
(2.8)

Proposition 2.2 can be obtained by the following series of lemmas.

**Lemma 2.1.** Under the assumption of Proposition 2.2, for \( \forall T > 0 \), it holds that
\[ \int_0^L \left( \rho u^2 + \Phi(\rho) + \chi_x^2 + \rho(\chi^2 - 1)^2 \right) dx + \int_0^T \int_0^L (\mu^2 + u_x^2) dx dt \leq C, \]  
(2.9)
where \( \mu \) is defined in (2.3).
Proof. Multiplying Eq. (1.4) 2 by $u$ and Eq. (1.4) 3 by $\mu$, integrating the resultant equations over $[0, L]$ and adding them up, one has

$$
\frac{d}{dt} \int_0^L \left( \frac{\rho u^2}{2} + \frac{\epsilon \chi^2}{2} + \frac{\rho(\chi^2 - 1)^2}{4\epsilon} \right) dx + \int_0^L \left( \mu^2 + \nu u_x^2 + up_x(\rho) \right) dt = 0. \quad (2.10)
$$

Integrating (2.7) and adding the result to (2.10), one then gets

$$
\frac{d}{dt} \int_0^L \left( \rho u^2 + \epsilon \chi^2 + \Phi(\rho) + \rho(\chi^2 - 1)^2/4\epsilon \right) dx + \int_0^L \left( \mu^2 + \nu u_x^2 \right) dx dt = 0. \quad (2.11)
$$

Integrating (2.11) over $[0, T]$, one has

$$
\sup_{t \in [0, T]} \int_0^L \left( \frac{\rho u^2}{2} + \frac{\epsilon \chi^2}{2} + \Phi(\rho) + \rho(\chi^2 - 1)^2/4\epsilon \right) dx + \int_0^T \int_0^L \left( \mu^2 + \nu u_x^2 \right) dx d\tau = E_0, \quad (2.12)
$$

where $E_0 = \int_0^L \left( \frac{1}{2} \rho_0 u_0^2 + \frac{\epsilon}{2} \chi_0^2 + \Phi(\rho_0) + \rho_0(\chi_0^2 - 1)^2/4\epsilon \right) dx$. The proof is obtained.

Lemma 2.2. Under the assumption of Proposition 2.2, for $\forall T > 0$, it holds that

$$
\| \chi \|_{L^\infty_{per}} \leq C. \quad (2.13)
$$

Proof. Integrating the mass equation (1.4) 1 over $[0, L] \times [0, t]$, one has

$$
\int_0^L \rho(x, t) dx = \int_0^L \rho_0(x) dx. \quad (2.14)
$$

By Lemma 2.1, we then have

$$
\int_0^L \rho \chi^4 dx \leq 2 \int_0^L \rho \chi^2 dx - \int_0^L \rho dx + C_1 \leq \frac{1}{2} \int_0^L \rho \chi^4 dx + C. \quad (2.15)
$$

Therefore

$$
\int_0^L \rho \chi^4 dx \leq C, \quad \int_0^L \rho \chi dx \leq \int_0^L \rho \chi^4 dx + \int_0^L \rho dx \leq C. \quad (2.16)
$$

From (2.9), one has

$$
|\chi(x, t)| = \frac{1}{\int_0^L \rho_0 dx} \left| \int_0^L \rho(y, t) dy \right| \leq \frac{1}{\int_0^L \rho_0 dx} \left( \left| \int_0^L (\chi(x, t) - \chi(y, t)) \rho(y, t) dy \right| + \left| \int_0^L \chi(y, t) \rho(y, t) dy \right| \right) \leq \frac{1}{\int_0^L \rho_0 dx} \left( \left| \int_0^L \rho(y, t) \left( \int_y^L \chi(s, t) ds \right) dy \right| + \left| \int_0^L \chi(y, t) \rho(y, t) dy \right| \right) \leq \frac{1}{\int_0^L \rho_0 dx} \int_0^L |\chi_x| dx \int_0^L \rho(y, t) dy + C_1 \leq C. \quad (2.17)
$$

The proof is completed. \( \square \)
Lemma 2.3. Under the assumption of Proposition 2.2, for $\forall T > 0$, it holds that
\[
\|\rho\|_{L^\infty_{\text{per}}([0,L]\times[0,T])} < 3, \quad \int_0^T \int_0^L \chi_{xx}^2 dx \leq C. \tag{2.18}
\]

Proof. Observing Lemma 2.1, one has
\[
\sup_{t\in[0,T]} \int_0^L \Phi(\rho) dx \leq E_0 = \int_0^L \left( \frac{1}{2} \rho_0 u_0^2 + \frac{\epsilon}{2} \chi_0^2 + \Phi(\rho_0) + \frac{\rho_0}{4\epsilon} (\chi_0^2 - 1)^2 \right) dx. \tag{2.19}
\]
From the definitions of (2.5) and (1.3), one gets
\[
\lim_{\delta \to 0} \text{mes}\{ (x,t) \in [0,L] \times [0,T] \mid \rho(x,t) \geq 3 - \delta \} = 0, \tag{2.20}
\]
thus
\[
\|\rho(x,t)\|_{L^\infty([0,L]\times[0,T])} < 3. \tag{2.21}
\]
Moreover, from the equation (2.3) and the energy inequalities (2.16), (2.17), one obtains
\[
\int_0^T \int_0^L \chi_{xx}^2 dx = \int_0^T \int_0^L \left( \rho(\chi^3 - \chi) - \rho \mu \right)^2 dx \leq C.
\]
The proof is completed. 

Lemma 2.4. Under the assumption of Proposition 2.2, for $\forall T > 0$, it holds that
\[
\sup_{t\in[0,T]} \|\rho_x\|_{L^2_{\text{per}}} \leq C, \quad \|\frac{1}{\rho}\|_{L^\infty_{\text{per}}([0,L]\times[0,T])} \leq C. \tag{2.22}
\]

Proof. From the mass conservation equation (1.4)\textsuperscript{1}, one has
\[
u = -\frac{1}{\rho} \left( \rho_t + \rho_x u \right)_x = \left( -\ln \rho \right)_t + \rho u (\frac{1}{\rho})_x = \left[ \rho_x (\frac{1}{\rho})_t + \rho u (\frac{1}{\rho})_x \right]_x = \rho_1(\frac{1}{\rho})_t + \rho u (\frac{1}{\rho})_x + \rho_x (\frac{1}{\rho})_t + (\rho u)_x (\frac{1}{\rho})_x.
\]
Substituting (2.23) into the momentum equation (1.4)\textsuperscript{2}, one gets
\[
(\rho u)_t + (\rho u^2)_x + \rho' \rho_x = \nu \left[ \rho \frac{d}{dt} \left( \frac{1}{\rho} \right)_x + \rho u (\frac{1}{\rho})_{xx} \right] - \frac{\epsilon}{2} \chi_x^2.
\]
Multiplying \((2.24)\) by \((\frac{1}{\rho})_x\), and integrating over \([0, L]\), further

\[
\frac{d}{dt} \int_0^L \left(\nu \frac{1}{\rho} \right)_x |^2 - \rho u \frac{1}{\rho}_x \right) dx + \int_0^L \frac{p'(\rho)}{\rho^2} \rho^2_x dx \\
= - \int_0^L \rho u \frac{1}{\rho}_x dx + \int_0^L (\rho u^2)_x \frac{1}{\rho}_x dx + \frac{\epsilon}{2} \int_0^L (\chi_x^2)_x \frac{1}{\rho}_x dx \\
= \int_0^L (\rho u)_x (-\frac{\rho u}{\rho^2}) + (\rho u^2)_x (-\frac{\rho u}{\rho^2}) dx + \epsilon \int_0^L \chi_x \chi_{xx} \frac{1}{\rho}_x dx \\
= \int_0^L u^2_x dx + \epsilon \int_0^L \chi_x \chi_{xx} \frac{1}{\rho}_x dx \\
\leq \int_0^L u^2_x dx + \epsilon \left( \left\| \frac{1}{\rho} \right\|_{L^\infty_{\text{per}}} + \int_0^L \rho (\frac{1}{\rho}_x)^2 dx \right) \| \chi_{xx} \|_{L^2_{\text{per}}}^2.
\]

In view of the mean value theorem, there exists \(a(t) \in [0, L]\) satisfying \(\rho(a(t), t) = \frac{1}{L} \int_0^L \rho_0 dx\), so that

\[
\frac{1}{\rho(x, t)} = \frac{1}{\rho(x, t)} - \frac{1}{\rho(a(t), t)} + \frac{1}{\rho(a(t), t)} \\
= \int_{a(t)}^x \frac{1}{\rho(y, t)} dy + \frac{L}{\int_0^L \rho_0 dx} \\
\leq \int_0^L \frac{\rho_x(x, t)}{\rho^2(x, t)} dx + \frac{L}{\int_0^L \rho_0 dx} \\
\leq \left( \int_0^L \frac{1}{\rho} dx \right)^{\frac{1}{2}} \left( \int_0^L \frac{\rho_x^2(x, t)}{\rho^3(x, t)} dx \right)^{\frac{1}{2}} + \frac{L}{\int_0^L \rho_0 dx} \\
\leq \frac{1}{2} \left\| \frac{1}{\rho} \right\|_{L^\infty_{\text{per}}} + \frac{L}{2} \int_0^L \rho (\frac{1}{\rho}_x)^2 dx + \frac{L}{\int_0^L \rho_0 dx},
\]

then one has the Sobolev inequality about \(\frac{1}{\rho}\),

\[
\left\| \frac{1}{\rho} \right\|_{L^\infty_{\text{per}}} \leq L \int_0^L \rho (\frac{1}{\rho}_x)^2 dx + \frac{2L}{\int_0^L \rho_0 dx}.
\]

Substituting the above expression into the inequality \((2.25)\), one gets

\[
\frac{d}{dt} \int_0^L \left(\nu \frac{1}{\rho} \right)_x |^2 - \rho u \frac{1}{\rho}_x \right) dx + \int_0^L \frac{p'(\rho)}{\rho^2} \rho^2_x dx \\
\leq \int_0^L u^2_x dx + \epsilon \left( \left\| \frac{1}{\rho} \right\|_{L^\infty_{\text{per}}} + \int_0^L \rho (\frac{1}{\rho}_x)^2 dx + \frac{2L}{\int_0^L \rho_0 dx} \right) \| \chi_{xx} \|_{L^2_{\text{per}}}^2.
\]

\[10\]
Setting

\[ A_{\text{increase}}(t) = \{ x \in [0, L] | 0 \leq \rho(x, t) < \alpha \} \cup \{ x \in [0, L] | \beta < \rho \leq M \}, \quad (2.29) \]
\[ A_{\text{decrease}}(t) = \{ x \in [0, L] | \alpha \leq \rho(x, t) \leq \beta \}, \quad (2.30) \]
then multiplying (2.28) by \( \frac{d}{2} \), and adding up (2.11), one gets

\[
\frac{d}{dt} \int_0^L \left( \frac{\mu^2}{4} \rho \left( \frac{1}{\rho} \right)_x^2 - \frac{\mu_0 u_0}{2} \left( \frac{1}{\rho} \right)_x + \frac{\rho u^2}{2} + \Phi(\rho) + \frac{(\chi^2 - 1)^2}{4e} + \frac{\epsilon \chi_x^2}{2} \right) dx \\
+ \int_{A_{\text{increase}}(t)} pp'(\rho) \left( \frac{1}{\rho} \right)_x^2 dx + \int_0^L (\mu^2 + \nu u_x^2) dx \\
\leq \frac{\epsilon \nu}{2} \left( (L + 1) \int_0^L \rho \left( \frac{1}{\rho} \right)_x^2 dx + \frac{2L}{\int_0^L \rho \rho_0 dx} \right) \| \chi_{xx} \|_{L_{\text{per}}}^2 - \int_{A_{\text{decrease}}(t)} pp'(\rho) \left( \frac{1}{\rho} \right)_x^2 dx \\
\leq \frac{\epsilon \nu}{2} \left( (L + 1) \int_0^L \rho \left( \frac{1}{\rho} \right)_x^2 dx + \frac{2L}{\int_0^L \rho \rho_0 dx} \right) \| \chi_{xx} \|_{L_{\text{per}}}^2 + \frac{6(27 - 4\Theta)}{(3 - \beta)^2} \int_0^L \rho \left( \frac{1}{\rho} \right)_x^2 dx.
\]

Integrating the inequality (2.31) over \([0, T]\), applying Lemma 2.2-2.3 and combining with Gronwall’s inequality, one obtains

\[
\int_0^T \int_0^L \left( \rho \left( \frac{1}{\rho} \right)_x^2 + \rho u^2 + (\rho - \bar{\rho})^2 + \rho (\chi^2 - 1)^2 + \chi_x^2 \right) dx dt + \int_0^T \int_0^L (\mu^2 + u_x^2) dx dt \leq C.
\]

In view of (2.27), combining with \( \int_0^L \rho \left( \frac{1}{\rho} \right)_x^2 dx \geq \frac{1}{\| \rho \|_{L_{\text{per}}}^2} \int_0^L \rho_x^2 dx \), the proof of Lemma 2.4 is completed. \( \Box \)

The estimate of the higher order derivatives for the phase parameter \( \chi \) and the velocity \( u \) can be obtained in a simpler way then in Lemma 2.1-Lemma 2.4.

**Lemma 2.5.** Under the assumption of Proposition 2.3, for \( \forall T > 0 \), it holds that

\[
\sup_{t \in [0, T]} \left( \| \chi_t \|_{L_{\text{per}}^2}^2 + \| \chi_{xx} \|_{L_{\text{per}}^2}^2 \right) + \int_0^T \int_0^L \left( \chi_{xt}^2 + \chi_t^2 + \chi_{xx}^2 \right) dx dt \leq C,
\]

\[
\sup_{t \in [0, T]} \| u_x \|_{L_{\text{per}}^2}^2 + \int_0^T \int_0^L \left( u_t^2 + u_{xx}^2 \right) dx dt \leq C.
\]

**Proof.** For the sake of convenience, we introduce the Lagrange coordinate system below:

\[
y = \int_0^x \rho(s, t) ds, \quad t = t; \quad v = \frac{1}{\rho}.
\]

Integrating (1.3) over \([0, R] \times [0, t]\) and using the boundary condition (1.5), we have

\[
\frac{1}{L} \int_0^L \rho dx = \frac{1}{L} \int_0^L \rho_0 dx := \bar{\rho}.
\]

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Setting
\[ \tilde{L} := \bar{\rho}L, \]  
then the system (1.4) can be reduced into
\[
\begin{cases}
  v_t - u_y = 0, & y \in \mathbb{R}, t > 0, \\
  u_t + p_y = \nu \left( \frac{u_y}{v} \right)_y - \epsilon \left( \frac{\chi^2}{v^2} \right)_y, & y \in \mathbb{R}, t > 0, \\
  \chi_t = -\frac{v}{\epsilon} (\chi^3 - \chi) + \nu v \left( \frac{\chi_y}{v} \right)_y, & y \in \mathbb{R}, t > 0, \\
  (v, u, \chi)(y, t) = (v, u, \chi)(y + \tilde{L}, t), & y \in \mathbb{R}, t > 0, \\
  (v, u, \chi)|_{t=0} = (v_0, u_0, \chi_0), & y \in \mathbb{R}.
\end{cases}
\]  
(2.37)

From (2.37), one has
\[
\chi_t = -\frac{1}{\epsilon} v (\chi^3 - \chi) + \frac{\chi_{yy}}{v} - \frac{2 \chi_y v_y}{v^2},
\]  
(2.38)
then (2.38) and Lemma 2.1-2.4 implies that
\[
\int_0^{\tilde{L}} \left( u^2 + v^2 + u^2 + (\chi^2 - 1)^2 \right) dy + \int_0^{T} \int_0^{\tilde{L}} \left( \mu^2 + u_y^2 + \chi_t^2 + \chi_{yy}^2 \right) dy \leq C,
\]  
(2.39)
and
\[
\int_0^{\tilde{L}} \chi_{yy}^2 dy \leq C \left( \int_0^{\tilde{L}} \chi_t^2 dy + 1 \right).
\]  
(2.40)

Differentiating (2.38) with respect to \( t \), one gets
\[
\chi_{tt} = -\frac{v_t}{\epsilon} (\chi^3 - \chi) - \frac{v}{\epsilon} (3 \chi^2 - 1) \chi_t + \nu v \left( \frac{\chi_y}{v} \right)_y + \nu \left( \frac{\chi_y}{v^2} \right) y_t.
\]  
(2.42)

Multiplying (2.42) by \( \chi_t \), and integrating it over \([0, \tilde{L}]\) with respect of \( y \), one obtains
\[
\frac{1}{2} \frac{d}{dt} \int_0^{\tilde{L}} \chi_t^2 dy + \epsilon \int_0^{\tilde{L}} \frac{\chi_{yy}}{v} dy
= -\frac{1}{\epsilon} \int_0^{\tilde{L}} \left( (\chi^3 - \chi) u_y \chi_t + v(3 \chi^2 - 1) \chi_t^2 \right) dy + \epsilon \int_0^{\tilde{L}} u_y \left( \frac{\chi_y}{v^2} \right)_y \chi_t dy
- \epsilon \int_0^{\tilde{L}} v_y \left( \frac{\chi_y}{v^2} \right)_t \chi_t dy + \epsilon \int_0^{\tilde{L}} \frac{2}{v^2} \chi_y u_y \chi_{yt} dy
= I_1 + I_2 + I_3 + I_4.
\]  
(2.43)

Following from Sobolev inequality and Lemma 2.1-2.4 and (2.39)-(2.40), one deduces
\[
|I_1| \leq C_1 (\|u_y\|^2 + \|\chi_t\|^2),
\]  
(2.44)
\begin{align*}
\left| I_2 \right| & \leq C \left( \int_0^{\bar{L}} \left| u_y x_y y_t \right| dy + \int_0^{\bar{L}} \left| u_y x_y v_y x_t \right| dy \right) \\
& \leq C \left( \| x_t \|_{L^\infty} \| u_y \| \| x_y \| + \| x_t \|_{L^\infty} \| x_y \|_{L^\infty} \| u_y \| \| v_y \| \right) \\
& \leq C \left( \| x_t \|^2 \| u_y \| + \| x_t \|^2 \| u_y \|^\frac{4}{3} + \| x_t \|^2 \| u_y \|^\frac{2}{3} \| u_y \|^\frac{1}{3} + \| x_t \| \| u_y \| + \| x_t \|^2 \| u_y \|^2 \right) \\
& \quad \quad + \frac{\epsilon}{4} \| x_y \|^2, \\
\text{(2.45)}
\end{align*}

and
\begin{align*}
\left| I_3 \right| + \left| I_4 \right| & \leq C \int_0^{\bar{L}} \left( \left| v_y x_y y_t x_t \right| + \left| v_y x_y u_y x_t \right| + \left| x_y u_y x_y x_t \right| \right) dy \\
& \leq C \left( \| x_t \| \| u_y \|^2 + \| x_t \|^2 \right) + \frac{\epsilon}{4} \| x_y \|^2. \\
\text{(2.46)}
\end{align*}

Substituting (2.44)–(2.46) into (2.43), applying the Gronwall’s inequality, one drives
\begin{align*}
\int_0^{\bar{L}} \chi_t^2 dy + \int_0^T \int_0^{\bar{L}} \chi_{yy}^2 dy \leq C. \\
\text{(2.47)}
\end{align*}

Combining with (2.41), one gets
\begin{align*}
\int_0^{\bar{L}} \chi_{yy}^2 dy \leq C. \\
\text{(2.48)}
\end{align*}

It holds that
\begin{align*}
\int_0^{\bar{L}} (\chi_t^2 + \chi_{yy}) dy + \int_0^T \int_0^{\bar{L}} (\chi_{yy}^2 + \chi_t^2 + \chi_{yy}) dy \leq C. \\
\text{(2.49)}
\end{align*}

Multiplying (2.31) by \(-u_{yy}\), integrating over \([0, \bar{L}]\) by parts, by using Sobolev inequality, Lemma 2.1-4 and (2.32), one obtains
\begin{align*}
& \left( \frac{1}{2} \int_0^{\bar{L}} u_y^2 dy \right)_t + \nu \int_0^{\bar{L}} \frac{u_y^2}{v} dy \\
& = \int_0^{\bar{L}} u_{yy} (p_y)' v_y dy + \int_0^{\bar{L}} u_{yy} u_y v_y dy + \int_0^{\bar{L}} \frac{2 \epsilon \chi_y x_y u_{yy}}{v^2} dy - \int_0^{\bar{L}} \frac{3 \epsilon \chi_y^2 v_y u_{yy}}{v^4} dy \\
& \leq C (\| u_y \|^2 + 1) + \frac{\nu}{2} \int_0^{\bar{L}} u_{yy}^2 dy. \\
\text{(2.50)}
\end{align*}

Thus it holds that
\begin{align*}
\int_0^{\bar{L}} u_y^2 dy + \int_0^T \int_0^{\bar{L}} u_{yy}^2 dy \leq C. \\
\text{(2.51)}
\end{align*}

Let’s go back to the Euler coordinates, by using (2.49), (2.51), combining with \(\chi_{xxx} = 2 \rho \rho_x x_t + \rho^2 \chi_{xt} + 2 \rho \rho_x u_x x_t + \rho^2 u_x \chi_x + \rho^2 u_x \chi_{xx} + \rho_x (\chi^3 - \chi) + \rho (\chi^2 - 1) \chi_x\), one has
\begin{align*}
\sup_{t \in [0, T]} \left( \| \chi_t \|^2_{L^2_{\text{per}}} + \| \chi_{xx} \|^2_{L^2_{\text{per}}} \right) + \int_0^T \int_0^{\bar{L}} (\chi_{xt}^2 + \chi_t^2) dx dt \leq C, \\
\text{(2.52)}
\end{align*}
\[
\sup_{t \in [0,T]} \|u_x\|_{L^2_{per}}^2 + \int_0^T \int_0^L u_{xx}^2 \, dx \, dt \leq C, \tag{2.53}
\]
and
\[
\int_0^T \int_0^L \chi_{xxx}^2 \, dx \, dt \leq C. \tag{2.54}
\]
Furthermore, by using \(u_t = -(p\delta)_y + \nu \left( \frac{u_x}{v} \right)_y - \epsilon \left( \frac{\chi^2}{v^2} \right)_y \), one obtains
\[
\int_0^T \int_0^L u_t^2 \, dx \, dt \leq C. \tag{2.55}
\]
Then proof of Lemma 3.5 is achieved.

From Lemma 2.1-Lemma 2.5, Proposition 2.2 is obtained, and the proof of Theorem 1.1 is completed.

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