Expect at most one billionth of a new Fermat Prime!

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Abstract: We provide compelling evidence that all Fermat primes were already known to Fermat.

Prologue

What are the known Fermat primes? Hardy and Wright [HW] say that only four such primes are known, but this is incorrect since taking $F_n = 2^{2^n} + 1$, as they did, $F_0, F_1, F_2, F_3$, and $F_4$ are prime. However, it’s not clear that this is the definition that Fermat preferred. Taking “Fermat prime” to mean “prime of the form $2^n + 1$”, there are six known Fermat primes, namely those for $n = 0, 1, 2, 4, 8, 16$. We shall pronounce the last letter of Fermat’s name, as he did, when we include 2 among the Fermat primes, as he did. In this paper, we indicate this by italicizing that last letter, as we already did.

Whichever definition we use, all the known such primes were already known to Fermat. Euler showed in 1732 that $2^{32} + 1 = 4294967297 = 641 \times 6700417$ is composite. In the endnotes to [HW], Hardy and Wright give a list of known composite Fermat numbers $F_n$, which is extended to $5 \leq n \leq 32$ with many other known composite values of $F_n$ (some with known factors, others merely known not prime); this is still the state of things. They go on to suggest that “the number of primes $F_n$ is finite”.

They then say “This is what is suggested by considerations of probability ... The probability that a number $n$ is prime is at most $A/\log n$ and therefore the expected number of new Fermat primes is at most [a formula equivalent to]”

$$A \sum \frac{1}{\log F_n} < \frac{A}{\log 2} \sum 2^{-n} < 3A.$$ 

In this paper we produce compelling evidence for our thesis (why should only Church have a thesis?):

Thesis: The only Fermat primes are 2 (according to taste), 3, 5, 17, 257 and 65537.

As Hardy and Wright also say, their argument (notwithstanding its general lack of precision) assumes that there are no special reasons why a Fermat number should likely be a prime, while there are some. The most compelling ones are the result of Euler that every prime divisor of $F_n$ is congruent to 1 modulo $2^n+1$ (we say “$F_n$ is $2^n+1$-full”), and Lucas’ 1891 strengthening of this to “$F_n$ is $2^{n+2}$-full”. A second reason is that the Fermat numbers are coprime, since

$$F_{n+1} = F_0 F_1 F_2 \ldots F_n + 2.$$

The Fermat number $F_n$ is either prime or not prime: the question of how to approximate the probability of primality for a general $n$ is delicate. $F_n$ is not a generic odd but, if it were, according to the Prime Number Theorem (PNT), as there are about $n/\log n$ primes up to $n$, we would have $2/\log n$ as a naive first estimate for the primality of $F_n$. But the $F_n$ do have very special properties - and specific forms for prime factors.

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Since $F_n$ has no prime divisors that are $o(2^n)$, a simple (but loose) conditional probability estimate based on a lack of small divisors looks like
\[ \frac{2}{\log n} \prod_{p \leq B} \left(1 - \frac{1}{p}\right)^{-1} \]
and by a classical result of Mertens, this is
\[ e^\gamma \frac{\log B}{\log n}. \]
A tighter view is by way of the conditional probability that $F_n$ is prime given that all of its prime factors are congruent to 1 (mod $2^{n+2}$); this is a basis for our discourse.

Prime divisors of the special form (as given by Lucas) seem to divide Fermat numbers with a greater likelihood than a generic prime might divide a generic number of the same size. Dubner and Keller [DK] comment that it “appears that the probability of each prime [of the form] $k2^m + 1$ [$k$ odd] dividing a standard Fermat number is $1/k$”. If this were true, it may be contrasted with a prime $q$ of size $k2^m$ dividing $X$ of size $F_m$ as
\[ \frac{1}{q} = \frac{1}{k2^m} \approx \frac{1}{k \log X}, \]
in turn suggesting that the special form of the potential prime divisors of a Fermat number make them more likely to divide a Fermat number than a generic prime divisor. Our main result (11) uses the most powerful conjectures about the distribution of primes in order to understand the limits of the method and shows that even with the special form of possible prime factors, the probability that $F_n$ is prime is only at most a constant multiple larger than the naive estimate $2/\log F_n$, which demonstrates that the restriction of the possible divisors is nearly balanced out by the increased likelihood of each of these contender factors dividing $F_n$.

For some time every number-theorist has believed that the number of Fermat primes is finite. However, the Wikipedia page for Fermat Numbers currently provides, but does not endorse, an argument in the reverse direction demonstrating the infinitude of the number of Fermat primes. (We invite the reader to find the flaw in the implied conditional probability argument that makes the estimate too large.)

In the following section with details of our argument (the precision of which is the significant difference between ours and previous heuristics), we take account of the main properties of Fermat numbers, concluding that the probability that $F_n$ is prime is approximately no larger than $4^{2^n}$, which summed over $n \geq 33$ yields $\frac{4^{32}}{2^{30}}$ as an upper bound for the expected number of new Fermat primes, indeed less than one billionth.

An Argument for our Thesis

Our objective is to estimate the probability that a number of size $x$ is a prime given only that it satisfies the various conditions that Fermat numbers are known to satisfy.

Let $\infty(x)$ denote a function that tends to infinity with $x$ and $\epsilon$ a fixed, arbitrarily small positive number (that may vary in value from one use to the next). We write $\pi(I)$ for the number of primes in the interval $I$ and $\pi_{a(q)}(I)$ for the number of primes congruent to $a$ (mod $q$) therein. For our argument, $I$ will be the interval $[x-r, x+r]$. We further set $\pi_{cond}(I)$ to be the number of primes in $I$ that satisfy a condition, “$\text{cond}$”. We write $\#_{\text{cond}}(I)$ for the number of integers in the interval $I$ satisfying the condition $\text{cond}$. And by $f(x) \sim g(x)$ we mean $f(x) = (1 + o(1))g(x)$.

We are concerned with the number of primes in the interval $[x-r, x+r]$ that satisfy various conditions. It is well-known that the number of primes in such an interval is approximately $2r/(\log x)$ provided $r$ is sufficiently large compared to $x$. If the primes are restricted to be congruent to $a$ modulo $q$, then we expect a proportion $1/\phi(q)$ of this number if $r$ is large compared to $q$. 

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We recall that the Fermat number $F_k = 2^{2^k} + 1$ is $2^{k+2}$-full, that is all its prime divisors are congruent to 1 (mod $2^{k+2}$). We desire to estimate the probability that a number $x$ (that satisfies some cond) in a certain interval $I$ is prime if we know that the number $x$ is $K$-full, in other words what we model as:

$$\frac{\pi_{\text{cond}}(I)}{\#K-\text{full}(I)}$$

where the length of the interval $I$ is taken as small as possible such that the expression in (1) is meaningful.

I. Dealing with the $K$-full numbers

If $x$ is large and $K$ is at least as large as $\log \log x$, it may be shown by a standard application of the Linear Sieve (see, for example, [HR]) that a first estimate for $\#K-\text{full}[0, x]$ is $c\pi_{1(K)}[0, x]$ where $c$ is a constant smaller than 2. The evaluation of the constant $c$ depends upon Mertens-type estimates for primes in arithmetic progressions. (See [LZ].) This first estimate, applied to the interval $[x - r, x + r]$ where $x$ is large and fixed, still holds as the value of $r$ decreases from near $x$ to $(\log x)^{\delta}$, for some fixed $\delta$, and, meanwhile, the value of $c$ decreases to its limit of 1. Numerical calculations kindly performed for us by Alex Ryba amply confirm this and indeed suggest that the ratio

$$\frac{\#K-\text{full}[x - r, x + r]}{\pi_{1(K)}[x - r, x + r]}$$

is much nearer to 1 than 2 for small $r$. So at the cost of at most a factor of 2, and very likely less than 1.1, we can replace “$K$-full number” by “prime congruent to 1(mod $K$) in the denominator of (1) and the resulting quotient is an upper bound for the probability in (1).

II. Equidistribution if $r$ is large compared to $q$

So we are now faced with the question regarding primes in short intervals, which has been studied deeply. With no a priori information, the probability that a number of size $x$ be a prime is $(1/\log x)$ and this implies that if $y$ is sufficiently large, an interval of length $y$ should contain about $y/(\log x)$ primes. Selberg [SE] showed, assuming the Riemann Hypthesis, that for almost all $x > 0$,

$$\pi(x + y) - \pi(x) \sim \frac{y}{\log x}$$

holds provided that

$$y > \infty(x) \log^2 x$$

(2)

(where by almost all we mean in the sense of Lebesgue measure).

To generalize Selberg’s result to arithmetic progressions (mod $q$), as we need to do, we first require that the primes in $I$ be uniformly distributed among the $\phi(q)$ residue classes provided only that there are sufficiently many of them. That is, provided $r$ is large enough compared to $q$. This is solved by

**The Equidistribution Lemma:** If $B$ balls are independently distributed into $C$ cups then provided that $B > \infty(C)C\log C$ one expects that for any fixed $\epsilon > 0$, the number of balls contained in any one cup is between $(1 - \epsilon)(B/C)$ and $(1 + \epsilon)(B/C)$.

(We thank Noam D. Elkies [NDE] for an argument establishing this Lemma - a result we have not been able to locate in the literature although, as Elkies remarks and we agree, it must be “well-known”.)

† It is worth noting that a non-trivial lower bound on $K$ is needed, for if $K$ is bounded, $\#K-\text{full}[0, x]$ is genuinely of size $x$.  

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This Lemma shows that the primes in $I = [x - r, x + r]$ will be uniformly distributed (in the above sense) provided that

$$\frac{r}{\log x} > \infty(x)\phi(q)\log \phi(q) .$$

We now set $q = (\log x)^{\delta}$, with $\delta \geq 1$, so that the Lemma’s requirement - and our requirement on $I$ - becomes

$$r > (\log x)^{1+\delta+\epsilon} .$$

(3)

(Note that this implies that $r > \infty(x)\log^2 x$, so Selberg’s condition (2) is automatically satisfied and our application of the Lemma with $r/(\log x)$ “balls” is justified.)

II. Uniformity if $r$ is large compared to $x$

In order to establish a lower bound condition on $r$ (half the length of $I$) for the expected result on primes in arithmetic progression

$$\pi_1(q)(I) \sim \frac{2r}{\phi(q)\log x}$$

(4)

to hold we first set

$$I(x, r, q, a) = \int_x^{2x} \left( \psi(y+h, q, a) - \psi(y, q, a) - \frac{h}{\phi(q)} \right)^2 dy$$

where

$$\psi(y, q, a) = \sum_{\substack{n \leq x \\ n \equiv a (q)}} \Lambda(n) ,$$

and $\Lambda(n)$ is the von Mangoldt function (defined by $\Lambda(n) = \log p$ if $n$ is a power of the prime $p$ and 0 otherwise). Then we define

$$I(x, h, q) = \sum_{a (mod q)} I(x, h, q, a)$$

where “$a (mod q)$” indicates that the sum is taken over all residue classes $a (mod q)$ with $(a, q) = 1$. On the Generalized Riemann Hypothesis (GRH), Prachar [PR] showed that\[†\]

$$I(x, h, q) = O \left( h x \log^2(q x) \right) .$$

(5)

We wish in some way to identify the $a (mod q)$ with $(a, q) = 1$ for which for which the expected asymptotic relation

$$\psi(y+h, q, a) - \psi(y, q, a) \sim \frac{h}{\phi(q)}$$

fails to hold (for a positive proportion of $y$ in $[x, 2x]$). Using $\sum^*$ to indicate summation over such $a$, note that

$$I(x, h, q) \geq \sum^* \int_x^{2x} \left( \psi(y+h, q, a) - \psi(y, q, a) - \frac{h}{\phi(q)} \right)^2 dy$$

\[†\] In the other direction, assuming the GRH, for $x \geq 2$, $1 \leq q \leq h \leq x$ and $q \leq h \leq (x q)^{1/3-\epsilon}$, Goldston and Yildirim [GY] have shown that

$$I(x, h, q, a) \geq \frac{1}{2} \frac{x h}{\phi(q)} \log \left( \frac{x q}{h^3} \right) (1 - o(1)) .$$

Combining this with the Equidistribution Lemma, Prachar’s result (5) is at most $1 - \epsilon$ logarithms away from being best possible.
and so, from (5),
\[ \sum 1 = O \left( \frac{\phi^2(q) \log^2 x}{h} \right). \tag{7} \]

It follows immediately that if \( h > \infty(x) \phi(q) \log^2 x \), then the asymptotic estimate equivalent to (6), namely
\[ \pi(y + h, q, a) - \pi(y, q, a) \sim \frac{h}{\phi(q) \log y} \tag{8} \]
holds for almost all \( a \) mod \( q \) for fixed \( q \) for almost all \( y \) in \([x, 2x]\).

At this point we introduce what we call the

**Uniformity Conceit**: If an assertion holds for almost all values of \( a \) it is very probable that it holds for any particular \( a \) unless there is a good reason why it should not.

We apply the Conceit with \( h > \infty(x) \phi(q) \log^2 x \) to (8) with \( a = 1 \) to deduce that very probably
\[ \pi(x + h, q, 1) - \pi(x, q, 1) \sim \frac{h}{\phi(q) \log x} \]
holds. Setting \( h = 2r \) and shifting \( x \) to \( x - r \) we find that very probably (4) holds so long as the following condition (stricter than (3) which was needed for equidistribution across residue classes) is satisfied:
\[ r > (\log x)^{2+\delta+\epsilon}. \tag{9} \]

**IV. Towards the probability that \( F_n \) is prime**

So now we return to the probability that a number \( N \), satisfying condition \( cond \), in \( I = [x - r, x + r] \) is a prime given that it is \( K\)-full. From the probability given in (1), we take \( x \) to be the Fermat number \( F_n \) with \( K = 2^{n+2} \). Then using our replacement of "\( K\)-full" by "prime congruent to 1 (mod \( K \))", an upper bound for this probability is
\[ \frac{\pi_{\text{cond}}(I)}{\pi_{1(2^{n+2})}(I)} \tag{10} \]
for \( r \) as small as possible such that this is meaningful.

The Fermat number \( F_n \) is \( 2^{n+2}\)-full so \( F_n \) is in the arithmetic progression 1 (mod \( 2^{n+2} \)). Of course, we know that \( F_n \equiv 1 \) (mod \( 2^{2n} \)) but we cannot use a value of \( q = 2^{2n} \) because The Equidistribution Lemma requires that
\[ \phi(q) = o \left( \frac{x}{\log^2 x} \right). \]
However, we do need and can use something larger than \( 2^{n+2} \) and can in fact use \( q = 2^{2n} \) so \( F_n \) lies in the progression 1 (mod \( 2^{2n} \)); we take this as condition \( cond \) in the numerator of (10).†

On all of our reasonable assumptions (including the GRH), we may evaluate this probability provided \( r \) satisfies (9)‡. For \( q = (\log x)^\delta \sim 2^{2n} \) we have \( \delta = 2 \) so, from (9), the interval \( I \) must be at least as large as

† Selecting a larger value of \( q \) such as \( 2^\alpha n \) for \( \alpha > 2 \) leads to a smaller upper bound for the probability we seek, but increasing the value of \( \alpha \) requires a larger \( r \) which is discordant with our model, making the interval around \( x \) as small as possible.

‡ The observant reader will have noticed that we have not mentioned the second of Hardy and Wright’s reasons that the Fermat numbers are more likely to be prime than other numbers, namely that they are mutually coprime; but that observant reader will also have noticed that the Fermat numbers increase so rapidly that at no time in this Argument has this made any difference.
\[2(\log x)^{4+\epsilon};\] we select \( r \) to be \((\log x)^{4+\epsilon}\). Thus, from (4), in accord with our model, the probability that \( F_n = x \) is prime is at most

\[
\frac{\pi_1(2^{2n})[x-r,x+r]}{\pi_1(2^{n+2})[x-r,x+r]} \sim \frac{1}{\varphi(2^{2n}) \log x} \frac{2^{2n}}{2^{n+2}} \cdot \frac{2^{2n}}{2^{n+2}} = \frac{4}{2^n}. \tag{11}
\]

For a new Fermat prime \( F_n \), \( n \) must be at least 33, so the expected number of new Fermat primes is at most

\[
\frac{1}{2^{33}} + \frac{1}{2^{34}} + \frac{1}{2^{35}} + \ldots = \frac{4}{2^{32}}
\]

and since \( 2^{30} \) exceeds one billion, this is indeed less than one-billionth, justifying the title of our paper.

V. Improvements?

We - your authors - are firmly convinced that there will be no significant improvement on our paper throughout all of future time. This seems an extremely audacious statement, but the evidence we now present should make it extremely plausible.

Some small progress will undoubtedly be made. We divide the numbers \( F_n (n \geq 33) \) into two lists

(A) 36,37,38,39,42,43,...

the numbers for which a prime factor of \( F_n \) has been found and

(B) 33,34,35,40,41,...

for which nothing is known. The list (A) has been formed by taking a prime \( p \) of the shape \( k2^m+1 \) where \( k \) is sufficiently small and \( m \) sufficiently large and finding that the sequence \( 2, 2^2, 2^4, 2^8 \ldots \) formed by repeated squaring (mod \( p \)). If the number \( 2^2 \) is congruent to \(-1 \mod p \) then one has shown that \( F_r \) is divisible by \( p \) so that \( r \) should belong to list (A).

This procedure will obviously be continued; so that numbers will gradually be promoted to list (A) from list (B). Your authors, who have not memorized the two lists, will not regard this as significant progress.

There is a necessary and sufficient test for primality of \( F_n \) but it involves so much computation that the largest \( F_n \) that has been proved composite this way is \( F_{24} \) [CMP]. For this, the computation took the better part of a year of computer time in 1999. Now \( F_{33} \) is \( 2^{30} \) times as long as \( F_{24} \), it will may well fall to this method at least when quantum computation fulfills the promises that have been made for it. If so, some of our readers may live to see a paper that updates our title to "Expect at most one trillionth ....". Will you regard this as a substantial improvement?

Epilogue

The famous question of the infinitude of the Mersenne primes falls in the orbit of our approach; setting \( M_p = 2^p - 1 = x \) it is easy to find that \( M_p \) is \( p - full \). When it comes to Mersenne primes, naive (PNT) estimates suggest there are infinitely many because \( \sum 1/p \) diverges; we do not quibble with this expectation. But the approach in the Argument is not directly amenable to the Mersenne prime question since we lack a second condition. Adding in such a condition into our model leads to our

**Conjecture:** There are finitely many Mersenne primes \( M_p \) whose index \( p \) is a twin prime.

The evidence is equally strong for the

**Conjecture:** There are finitely many Mersenne primes \( M_p \) whose index \( p \) is a Sophie Germain prime (i.e. a prime \( p \) where \( 2p + 1 \) is also prime).
More generally, if we fix integers $a$ and $b$ where $(b,p) = 1$ and consider primes of the form $M_p$ where both $p$ and $ap + b$ are primes, the Selberg Sieve (see [HR]) establishes an upper bound on the count of such primes up to $x$,

$$
\sum_{p<x} 1 = O\left(\frac{x}{\log^2 x}\right).
$$

We may apply the technique in the Argument, with $M_p$ taking on $a$ distinct residues modulo $ap + b$ so the Chinese Remainder Theorem determining $a$ values of $M_p$ modulo $p(ap + b)$, to produce an upper bound probability of size $1/(ap + b)$ for the probability that such an $M_p$ is prime. Summing over the special primes using partial summation yields a total expectation of $O(1)$ of these special Mersenne primes. So we are led to formulate the general

**Conjecture:** There are only finitely many Mersenne primes $2^p - 1$ where $p$ is a prime and $ap + b$ is also prime (for some fixed integers $a$ and $b$ where $(b,p) = 1$).

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**References**

[CMP] Crandall, R., Mayer, E. and Papadopoulos, J.: The Twenty-Fourth Fermat Number is Composite. *Math. of Computation* 72 (2002), 1555-1572

[DK] Dubner, H. and Keller, W.: Factors of generalized Fermat numbers. *Math. of Computation* 64 (1995), 397-405

[GY] Goldston, D. and Yildirim, C.: On the Second Moment for primes in an arithmetic progression. *Acta Arith.* 100 (2001), 85-104

[HR] Halberstam, H. and Richert, H.-E.: *Sieve Methods*. New York, Academic Press, 1974

[HW] Hardy, G.H. and Wright, E.M.: *An Introduction to the Theory of Numbers* [6th ed. (2008)] Oxford University Press

[LZ] Languasco, A. and Zaccagnini, A.: On the constant in the Mertens product for arithmetic progressions II: Numerical Values. *Math. of Computation* 78 (2009), 315 - 326

[NDE] Private email communication to the first author, January 3, 2014

[PR] Prachar, K.: Generalisation of a theorem of A. Selberg on primes in short intervals. In: *Topics in Number Theory, Colloquia Mathematica Societatis Janós Bolyai 13*. Debrecen, 1974, 267-280

[SE] Selberg, A.: On the normal density of primes in small intervals, and the difference between consecutive primes. *Archiv for Mathematik og Naturvidenskap* 47 (1943), 87-105