Fractional Coloring (Orthogonal Access) achieves All-unicast Capacity (DoF) Region of Index Coding (TIM) if and only if Network Topology is Chordal

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Abstract—The main result of this work is that fractional coloring (orthogonal access) achieves the all-unicast capacity (degrees of freedom (DoF)) region of the index coding (topological interference management (TIM)) problem if and only if the bipartite network topology graph (with sources on one side and destinations on the other, and edges identifying presence of non-trivial channels whose communication capacity is not zero or infinity) is chordal, i.e., every cycle that can contain a chord, does contain a chord. The all-unicast capacity (DoF) region includes the capacity (DoF) region for any arbitrary choice of a unicast message set, so e.g., the results of Maleki and Jafar on the optimality of orthogonal access for the sum-capacity (DoF) of one-dimensional convex networks are recovered as a special case.

I. INTRODUCTION

The index coding problem is one of the most intriguing open problems in network information theory due to its rich connections to various other prominent problems ranging from distributed storage [1], [2], caching [3] and general instances of network coding [4], [5] to topological interference management [6] in wireless networks and hat-guessing problems [7] in recreational mathematics. On the surface it seems to be a very simple problem — any instance of the index coding problem can be described as a wired network with only one non-trivial link. However, in spite of various graph-theoretic [8], random coding [9], rate-distortion [10], as well as interference alignment [11] approaches that have been brought to bear upon it, the index coding problem remains open. The difficulty is evident in recent results that prove the necessity, in general, of non-linear coding schemes for achievability arguments [4], [11], [12], and of non-Shannon information inequalities for converses [13]–[15], neither of which are well understood. As such much of the recent progress on index coding has come about from a divide and conquer approach aimed at identifying broad classes of solvable instances of the problem. For example, [8], [16] identify all instances where it is optimal to serve only one user at a time. All half-rate feasible instances are identified in [11], [13]. All instances for which the alignment graph has no cycles or no forks are solved in [6] and this class is further generalized to include instances for which the alignment graph has no overlapping cycles in [14]. All instances with 5 or fewer messages are solved in [9], all single uniprior instances are solved in [17], and all instances where each source has non-trivial communication capacity to at most two of the destinations are solved in [10].

Continuing with the divide and conquer strategy, a problem of great interest from both theoretical and practical perspectives, is to identify those instances where the simplest achievable schemes are also optimal. Such an approach has been quite successful recently in wireless interference networks where much progress has been made in identifying settings where the simple scheme of treating interference as noise is (in a generalized DoF sense) optimal [18]. For index coding, perhaps the simplest scheme is graph coloring, i.e., simultaneous scheduling of only those messages that do not interfere with each other. Progress along this direction is reported in [19], where coloring is shown to achieve the sum-capacity of a fairly broad class of index coding problems, known as one-dimensional convex networks. This result is our starting point.

A. One-Dimensional Convex Networks

The topological interference management problem (TIM), introduced in [6], studies the degrees of freedom (DoF) of partially connected one-hop wireless networks with no channel state information at the transmitters except the network topology. For the TIM problem, [19] shows that orthogonal access achieves the sum-DoF of a one-dimensional cellular network (all nodes placed on a straight line) that satisfies (i) source convexity, (ii) destination convexity, and (iii) message convexity. An example of such a network (with 19 messages)
is shown in Fig. 1. The convexity assumptions are motivated
by the observation that signals are stronger and communication
is more likely to occur between nodes that are physically
closer to each other than between nodes that are farther
apart. For example, since source $S_1$ is heard by destination
$D_4$, it must also be heard by destinations that are closer,
e.g., destinations $D_1, D_2, D_3$. This is referred to as source
convexity. Destination convexity is similarly defined. Further,
if source $S_1$ has a desired message for destination $D_2$, it
must have a desired message for destination $D_1$ because $D_1$
is closer to $S_1$ than $D_2$. This is denoted as the message
convexity assumption. Under these three assumptions, [19]
shows that orthogonal access is optimal in terms of sum-
DoF. For example, since the network in Fig. 1 satisfies these
assumptions, an orthogonal scheduling scheme that schedules
messages only between non-interfering source-destination pairs
$S_1 \rightarrow D_4, S_3 \rightarrow D_5, S_5 \rightarrow D_7, S_6 \rightarrow D_{10}$ and $S_4 \rightarrow D_{11}$,
achieves the optimal sum-DoF value (5 in this case).

The TIM result translates directly into an index coding result
as a remarkable association between the two problems
identified in [6], such that there is a TIM instance associated
with each index coding instance and vice versa (see Fig. 2).
Corresponding instances of both problems are jointly described
by a bipartite topology graph with sources on one side,
destinations on the other, and edges representing the presence
of a non-trivial communication channel whose communication
capacity is not zero (TIM) or infinity (index coding). Reference
[6] shows that (expressed in normalized units) the capacity
region of any instance of the index coding problem acts
as an outer bound on the DoF region of the corresponding
instance of the TIM problem, and furthermore, the two are
equivalent when restricted to linear schemes over the same
field. Therefore, the optimality of orthogonal access for sum-
DoF of one-dimensional convex networks, established in [19],
automatically implies the optimality of coloring for the sum-
capacity of the corresponding class of index coding instances.

![Fig. 2: (a) An instance of the TIM problem (edges constitute the network topology graph) and (b) the corresponding instance of the index coding problem. Red links (solid and dashed) have infinite capacity. Solid red links form the antidote graph, which is the complement of the network topology graph.](image)

### B. Beyond One-Dimensional Convex Networks

On the one hand, the optimality of orthogonal access for
the physically motivated and fairly broad class of topologies
represented by one-dimensional convex networks is surprising
because it is known that simple schemes such as (fractional)
coloring or (partition) multicast (cf. TDMA and CDMA) can
be severely sub-optimal in general. For example, there exist
instances of TIM and index coding with $K$ messages where
optimal schemes (involving interference alignment) achieve
a sum-capacity (sum-DoF) value that is a factor of $(1/3 +
\alpha/1)$ higher than the best achievable through fractional
coloring or partition multicast [13].

On the other hand, however, the result is limited by the
assumption of a one-dimensional placement of nodes and the
convexity constraints. For instance, even for physically moti-
vated TIM topologies that satisfy all the convexity constraints,
it is shown in [19] that going from one-dimensional settings
to the much more realistic two-dimensional placements of
sources and destinations, one immediately runs into examples
where orthogonal access is no longer optimal and interference
alignment solutions significantly outperform conventional base-
lines. The convexity assumptions are also not applicable to
heterogeneous networks, where a user may hear a distant high
power base station, but still not be able to hear a closer but
lower power base station. Moreover, beyond the TIM context,
for the index coding problem in general, the one-dimensional
node placements or convexity constraints are of little physical
significance. Last but not the least, the focus on sum-capacity
(DoF) is restrictive as well.

This brings us to the motivation of this work, which is
to go beyond these limitations, to answer the question —
what is the fundamental topological structure that determines
the optimality of fractional coloring (orthogonal access) for
the index coding (TIM) problem, making other sophisticated
schemes redundant?

### C. Summary of Contribution

The main contribution of this work, as highlighted in the
title, is to show that fractional coloring (orthogonal access)
achieves the all-unicast capacity (DoF) region of the index
coding (TIM) problem if and only if the network topology
graph is chordal, i.e., any cycle of length 6 or more must
contain a chord. Since the topology graph is bipartite, this
simply means that any cycle that can contain a chord, must
contain a chord. Whether or not a network is chordal is easy
to check (polynomial time). The all-unicast setting involves an
independent message from each source to each destination, i.e.,
all possible unicast messages. Characterizing the all-unicast
capacity (DoF) region automatically characterizes the capacity
(DoF) region for any arbitrary subset of unicast messages, as
well as other traditional metrics such as symmetric or sum
capacity (DoF). So, when fractional coloring achieves the all-
unicast capacity region, it achieves the capacity region of any
arbitrary subset of messages as well.

Note that chords are only defined for cycles, so e.g., tree topologies are
also chordal graphs. 

Footnote 3: Fig. 2 is such an example. It corresponds to a 2 dimensional convex
network (see [2]) with optimal sum-DoF value of 8/3, achieved by interference
alignment, whereas orthogonal schemes cannot achieve more than 2 DoF.
While not restricted to the one-dimensional convex topologies studied in [19], chordal networks do include them as a special case. Since convex message sets are included in all-unicasts, and the capacity region includes sum-capacity, the results of [19] are recovered as a special case of our result. Within the context of one-dimensional networks, our result generalizes the results of [19] — it shows that for one-dimensional placement of sources and destinations, if the network satisfies either source convexity or destination convexity (without requiring both, turns out either assumption is sufficient (not necessary) to imply a chordal topology, then for any arbitrary unicast message set (without requiring message convexity), the entire capacity region (without restriction to only sum-capacity) is achieved by fractional coloring. For the setting illustrated in Fig. 1 (note that the topology is chordal), the result shows that the entire capacity (DoF) region for the 19 messages (as well as any other set of unicast messages possible in this setting) is achieved by fractional coloring (orthogonal access). Other examples that further highlight the generality of this result are presented in Fig. 8 (does not satisfy destination convexity) and Fig. 4 (does not satisfy source or destination convexity) where also the topology graph is chordal and fractional coloring (orthogonal access) is optimal for the capacity (DoF) region under all possible unicast message sets. On the other hand, Fig. 2 is an example where the topology is not chordal, and so fractional coloring (orthogonal access) schemes cannot achieve the capacity (DoF) region (as indeed is shown in [6]).

So to answer the question that motivates this work — chordal network topology is the fundamental topological structure that determines the optimality of fractional coloring (orthogonal access), making all other sophisticated schemes unnecessary.

II. SYSTEM MODEL

In this section we define the TIM and index coding problems and the relationship between them, and then present some of the key definitions. More common terms such as capacity region and DoF region are used only in the standard Shannon theoretic sense and their definitions are omitted.

A. Topological Interference Management Problem (TIM)

An arbitrary instance of the TIM problem [6] is represented as a partially connected network with M sources, labeled $S_1$, $S_2$, ..., $S_M$, and N destinations, labeled $D_1$, $D_2$, ..., $D_N$. All sources and destinations are equipped with a single antenna each. The received signal for destination $D_j$ at time instant $t$ is:

$$ Y_j(t) = \sum_{i=1}^{M} t_{ji} h_{ji}(t) X_i(t) + Z_j(t) $$

(1)

where $X_i(t)$ is the transmitted signal from source $S_i$. All transmitted signals are subject to a power constraint $P$. $Z_j(t)$ is the Gaussian noise with zero-mean and unit-variance at destination $D_j$, $h_{ji}(t)$ is the channel coefficient between source $S_i$ and destination $D_j$, and $t_{ji} \in \{0, 1\}$ is equal to 1 if and only if source $S_i$ is connected to destination $D_j$. The actual channel realizations are not available at the sources, yet the topology matrix (i.e., $T = [t_{ji}]_{N\times M}$) is known by all sources and destinations. Only desired channel coefficients are known by destinations. The channel coefficients and network topology are assumed to be fixed throughout the duration of communication. Time-varying channels will be discussed in Section V-F.

B. Index Coding Problem (IC)

The index coding problem consists of $M$ sources, labeled $S_1, S_2, ..., S_M$. $N$ destinations, labeled $D_1, D_2, ..., D_N$ and two additional nodes $B_1, B_2$, that are connected by a unit capacity edge from $B_1$ to $B_2$. There is an infinite capacity link from every source to $B_1$, and from $B_2$ to every destination. We also have infinite capacity links between sources and destinations identified by the antidote matrix $A = [a_{ji}]_{N\times M}$, where $a_{ji} = 1$ means that source $S_i$ is connected to destination $D_j$ through an infinite capacity link, and $a_{ji} = 0$ otherwise.

C. Relationship between TIM and IC

As shown in [6], for every TIM problem described by topology matrix $T$, there is an associated IC problem (and vice versa) described by antidote matrix $A = 1 - T$. Here 1 denotes the matrix with all entries equal to 1. For corresponding instances of IC and TIM, the IC capacity region is an outer bound on the TIM DoF region, and the two are equivalent under linear solutions. Fig. 8 shows an example.

D. Key Definitions

**Definition 1** (All-Unicast). The all-unicast setting means that from each source $S_i$ there is an independent message $W_{ji}$ to each destination $D_j$ if $t_{ji} = 1$ ($a_{ji} = 0$), i.e., if there exists a non-trivial channel between them. Note that this includes arbitrary message sets as special cases, by setting the rates of some messages to zero.

**Definition 2** (Topology Graph). The topology graph is a bipartite graph with sources on one side, destinations on the other, and an edge between $S_i$ and $D_j$ whenever $t_{ji} = 1$ ($a_{ji} = 0$). Note that the TIM representation directly shows the topology graph, whereas for the IC representation, it is the complement of the antidote graph.

**Definition 3** (Cycle). A cycle is a set of vertices and edges that form a closed loop. The length of a cycle is the number of vertices in this cycle. A cycle of length 6 or more is called a long cycle.

**Definition 4** (Chord). A chord is an edge that connects two non-adjacent vertices of a cycle.

**Definition 5** (Chordal Network). An index coding (TIM) instance is chordal if its topology graph does not contain a chordless long cycle.

Note that because a source (destination) is not connected to any other sources (destinations), it is not possible for a length-4 cycle to contain a chord. Also a bipartite graph cannot contain odd length cycles. Thus the chordal condition can
be interpreted as – any cycle that can contain a chord, must contain a chord.

Definition 6 (Fractional Coloring (Orthogonal Access) [19]). Fractional coloring (orthogonal access) refers to a scheme that schedules messages \( W_\nu \) for transmission simultaneously only if they are non-interfering (orthogonal), i.e., \( a_{ji} = a_{j'i'} = 1 \) for IC \( (t_{j'j} = t_{ji} = 0 \) for TIM), \( \forall W_{ji},W_{j'i'} \in W_o, i \neq i',j \neq j' \).

Definition 7 (Message Conflict Graph). It is a graph where each message is a vertex, and edges exist between two vertices if the two messages conflict with each other. Two messages \( W_{ji},W_{j'i'} \) conflict if they originate from the same source \( (i = i') \), or are intended for the same destination \( (j = j') \), or if one source interferes with the other destination, i.e., \( (a_{ji},a_{j'i'}) \neq (1,1) \) for IC \( ((t_{ji},t_{j'i'}) = (0,0) \) for TIM).

Definition 8 (Set of Cliques of Conflict Graph \( \mathcal{S} \)). A clique \( \mathcal{C} \) of the message conflict graph is a set of nodes (messages) such that any two are adjacent. The set of all cliques of the message conflict graph is denoted as \( \mathcal{S} \).

### III. Main Result

The main result is stated in the following theorem.

**Theorem 1.** Fractional coloring (orthogonal access) achieves the all-unicast capacity (DoF) region of the index coding (TIM) problem if and only if the network is chordal. For chordal networks, the capacity (DoF) region of the index coding (TIM) problem is the set of rate (DoF) tuples comprised of \( R_{ji} (d_{ji}) \) in \( \mathbb{R}^+ \) that satisfy the following constraints.

\[
\forall \mathcal{C} \in \mathcal{S}, \sum_{W_{ji} \in \mathcal{C}} R_{ji} \leq 1 \quad \text{and} \quad \sum_{W_{ji} \in \mathcal{C}} d_{ji} \leq 1 \text{ for TIM}
\]

where \( R_{ji} (d_{ji}) \) is the rate (DoF) of the message \( W_{ji} \).

Next we illustrate this result through some examples. While we will use the TIM representation to present these examples because the topology graph is directly visible in the TIM formulation, note that the examples apply to corresponding instances of the index coding problem as well.

**Example 1.** Consider the one-dimensional TIM network instance with message \( W_{ji} \) from \( S_i \) to \( D_i \) as shown in Fig. 3 where convexity applies to the sources only. Because the topology graph is chordal, Theorem 1 applies and we have the DoF region

\[
\mathcal{D} = \left\{ (d_{11}, \ldots, d_{55}) \in \mathbb{R}^5_+ \middle| \begin{array}{l}
d_{11} + d_{22} + d_{33} \leq 1 \\
d_{11} + d_{44} \leq 1 \\
d_{33} + d_{55} \leq 1 \\
d_{44} + d_{55} \leq 1
\end{array} \right\}
\]

**Example 2.** Consider the TIM instance in Fig. 4 with message set \( \mathcal{M} = \{W_{13},W_{24},W_{31},W_{45},W_{52},W_{55},W_{73}\} \) (which is a subset of the all-unicast message set). Since the network is chordal, Theorem 1 applies and we have the DoF region

\[
\mathcal{D} = \left\{ (d_{11}, \ldots, d_{55}) \in \mathbb{R}^5_+ \middle| \begin{array}{l}
d_{42} + d_{13} \leq 1 \\
d_{31} + d_{13} + d_{37} \leq 1 \\
d_{31} + d_{54} + d_{37} \leq 1 \\
d_{54} + d_{55} + d_{37} \leq 1 \\
d_{25} + d_{54} + d_{55} \leq 1
\end{array} \right\}
\]

Additionally, we are able to show (proof relegated to Section V-E) that one-dimensional TIM instances with either source convexity or destination convexity (not necessarily both) are chordal. Thus, the optimality of orthogonal access shown in [19] for one-dimensional convex networks, applies with either source or destination convexity, to any arbitrary unicast message set, and to the entire DoF region.

**IV. Conclusion**

We show that the necessary and sufficient condition for fractional coloring (orthogonal access) to achieve the all-unicast capacity (DoF) region of index coding (TIM) is that the network topology should be chordal. The absence/presence of chordless cycles prevents/creates opportunities for more sophisticated achievable schemes. Among other interesting observations, we note that for chordal networks while fractional coloring is needed to achieve the capacity region, regular coloring suffices to achieve all the corner points. Finally, in the TIM context, the main result holds regardless of channel coherence time. The proof is presented in Section V-F.

**V. Proofs**

**A. Graph Theoretic Preliminaries**

Consider the simple undirected topology graph \( G \). A line graph of \( G \) is another graph, denoted by \( G_L \), such that each
vertex of \( G_e \) represents an edge of \( G \), and any two vertices in \( G_e \) are adjacent if and only if their corresponding edges in \( G \) have an endpoint in common. The square of a graph \( G \), denoted by \( G^2 \), is another graph that has the same set of vertices, but in which two vertices are adjacent when their distance in \( G \) is at most 2. Interestingly, \( G_{e_2}^2 \) is the message conflict graph corresponding to the topology graph \( G \). We state this as the following lemma. The proof is presented in Section V-C.

**Lemma 1.** For any topology graph \( G \), the square of its line graph, \( G^2_e \), is its message conflict graph.

Next, we carry over the chordal property of \( G \) to the perfect-graph property of \( G^2_e \). \( G \) is a chordal bipartite graph (see Chapter 12.4 in [20]), thus weakly chordal [21]. It is proved in [21] that if \( G \) is weakly chordal, then \( G^2_e \) is also weakly chordal. As weakly chordal graphs are subclass of perfect graphs (see Chapter 66.5d in [22]), \( G^2_e \) is perfect, a crucial result that we state as the following lemma.

**Lemma 2.** If \( G \) is chordal, then \( G^2_e \) is perfect.

Let us also recall the definition of a Demand Graph.

**Definition 9** (Demand Graph). The demand graph is a directed bipartite graph with messages on one side and destinations on the other, with a directed edge from a message to a destination if this message is intended for this destination, and with a directed edge from a destination to a message if the source from which this message originates is not connected to the destination in the topology graph.

**Lemma 3** (From [6], [8], [10]). The sum rate (DoF) of a message set that forms an acyclic demand graph is bounded by 1.

For a chordal network topology, an acyclic demand graph connects to a clique in \( G^2_e \), as stated in the following lemma. The proof is presented in Section V-D.

**Lemma 4.** If \( G \) is chordal, then for each clique in \( G^2_e \), the associated messages form an acyclic demand graph.

**B. Proof of the Main Theorem**

**Sufficiency:** We prove that if a network is chordal, then orthogonal access achieves the capacity (DoF) region.

First consider the outer bound. As Lemma 4 shows that each clique in \( G^2_e \) corresponds to an acyclic demand graph, and by Lemma 3 the sum rate (DoF) value of associated messages in an acyclic demand graph is bounded by 1, we obtain the following outer bounds.

\[
\forall e \in E, \sum_{j \in e} R_{ji} \leq 1 \quad (\sum_{j \in e} d_{ji} \leq 1 \text{ for TIM}).
\]

The above inequalities are called the clique inequalities [22].

Next we proceed to the achievability and show that the above outer bound region is achievable by fractional coloring (orthogonal access). To this end, we show that the outer bound region has integral vertices, meaning that each coordinate of the rate (DoF) tuples of the vertices is either 0 or 1. Note that an integral vertex of the polytope defined by the clique inequalities corresponds to a set of orthogonal messages that do not conflict, such that they can be scheduled over one time slot. As each vertex of the outer bound region can be achieved by one shot scheduling, time sharing between these vertices can achieve the whole region and the overall scheme is fractional coloring (orthogonal access).

We are left to prove the outer bound region has integral vertices. This follows from the fact that \( G^2_e \) is a perfect graph and it is known that the polytope defined by the clique inequalities of a perfect graph has integral vertices (see [23] and Chapter 65 in [22]).

The sufficiency proof is complete.

**Necessity:** We want to show that if a network is not chordal (contains chordless cycles with length \( 2n, \ n = 3, 4, \ldots \)), then fractional coloring (orthogonal access) does not achieve the capacity (DoF) region.

Suppose now the topology graph contains a chordless cycle with \( 2n \) vertices, which consist of \( n \) distinct source nodes, labeled \( S_1, S_2, \ldots, S_n \) and \( n \) distinct destination nodes, labeled \( D_1, D_2, \ldots, D_n \). We consider the sub-network induced by these nodes. As the cycle is chordless, the sub-network topology is cyclic where each source \( S_i, i \in \{1, \ldots, n\} \) is connected to two destinations \( D_{i-1}, D_i \) and each destination \( D_i \) is connected to two sources \( S_i, S_{i+1} \) (source/destination indices are interpreted modulo-\( n \), i.e., \( n + 1 = 1, 0 = n \)). An example with \( n = 4 \) is shown in Fig. 2. To prove the desired claim, it suffices to find a rate (DoF) tuple that is achievable, thus inside the capacity (DoF) region, but cannot be achieved by fractional coloring (orthogonal access).

We consider the cases where \( n \) is odd or even separately.

When \( n \) is odd, we consider the interference channel message setting, i.e., there are \( n \) desired messages in the sub-network, one each from source \( S_i \) to destination \( D_i \). As each destination only suffers interference from one non-desired source, multicast (CDMA) can achieve rate (DoF) tuple \( (1/2, 1/2, \ldots, 1/2) \) with \( n \) elements [6]. However, it is easily seen that fractional coloring (orthogonal access) cannot schedule more than \( (n - 1)/2 < n/2 \) messages over one time slot, and is therefore unable to achieve this rate (DoF) tuple.

When \( n \) is even, we consider the all-unicast message setting and the rate (DoF) tuple \( (1/3, 1/3, \ldots, 1/3) \) with \( 2n \) elements, which is achievable by interference alignment [6], but not by fractional coloring (orthogonal access). Orthogonal access cannot schedule more than \( n/2 < 2n/3 \) messages at the same time, e.g., if we schedule the message from \( S_i \) to \( D_i \), then messages from \( S_i \) to \( D_{i-1} \), from \( S_{i+1} \) to \( D_i \) and \( S_{i-1} \) to \( D_{i-1} \) cannot be scheduled. Thus, orthogonal access is again sub-optimal.

This completes the necessity proof.

**Remark 1.** From the capacity (DoF) region, it is not hard to verify that symmetric capacity (DoF) is given by \( 1/\chi(G_e^2) \), where \( \chi(G_e^2) \) is the chromatic number of \( G_e^2 \), and sum capacity (DoF) is given by the independence set number of \( G_e^2 \).
C. Proof of Lemma 1

First, $G^2_e$ and the message conflict graph of $G$ have the same vertex set. In the message conflict graph, there is a vertex for each message in $G$. In $G^2_e$, there is a vertex for each edge in $G$ and in the all-unicast message setting, each edge in $G$ corresponds to a message. Thus the claim follows.

Second, we prove $G^2_e$ and the message conflict graph of $G$ have the same edge set. In the message conflict graph, two messages (vertices) $W_{ji}, W_{j'i'}$ are connected if and only if they originate from the same source ($i = i'$), or are intended for the same destination ($j = j'$), or one source interferes with the other destination ($a_{ji} = 0(a_{j'i'}) = 1$). When $i = i'$ or $j = j'$, the two edges representing $W_{ji}, W_{j'i'}$ in $G$ share a common vertex such that these two messages (vertices) are connected in $G_e$ (have distance 1, thus connected in $G^2_e$). When $a_{ji} = 0(a_{j'i'}) = 1$, the two edges representing messages $W_{ji}, W_{j'i'}$ in $G$ both connect to the source that emits the message in the topology graph, means that a destination wants a message and is not connected to only two sources, labeled $S_1, S_2, S_3$ and 3 destinations, labeled $D'_1, D'_2, D'_3$. In the topology graph, $S'_1$ is connected to $D'_3, D'_1, S'_2$ is connected to $D'_1, D'_2$, and $S'_3$ is connected to $D'_2, D'_3$. The topology graph is not chordal as all nodes form a length-6 chordless cycle. Consider the messages $W_{11}, W_{22}, W_{33}$. They mutually conflict and form a clique in $G^2_e$. But the demand graph formed by them is not acyclic as multicast (CDMA) can achieve rate (DoF) 1/2 per message and the sum rate (DoF) is not bounded up by 1.

D. Proof of Lemma 2

We show that if a set of messages forms a clique in $G^2_e$ (they mutually conflict), the demand graph formed by these messages and their desired destinations must be acyclic, given $G$ is chordal. To set up a proof by contradiction, suppose a set of messages forms a clique in $G^2_e$ and there exist cycles in the demand graph. Let us consider such set of messages with minimum cardinality (denoted by $k$) such that the cycle in the demand graph is chordless, the sources from which the message originate are distinct and the destinations that want the messages are also distinct.

First, as the demand graph is bipartite, $k$ must be even such that $k = 2n, n = 1, 2, \ldots$.

Second, $n \neq 1$ as such a length-2 cycle in the demand graph means that a destination wants a message and is not connected to the source that emits the message in the topology graph, which is not possible.

Third, $n = 2$, as otherwise the two messages in the cyclic demand graph do not conflict as the two desired destinations do not hear the interfering source. This contradicts the assumption that these two messages form a clique in $G^2_e$.

Finally, we consider $n = 3, 4, \ldots$. As the sources from which the messages originate are distinct, we can replace each message in the demand graph by a source. As such, a chordless cycle in the demand graph is translated to a chordless cycle with length $2n$ in $G$, which contradicts the assumption that $G$ does not contain chordless cycles with length 6 or more.

Therefore, for chordal $G$, whenever we have a clique in $G^2_e$, the associated messages form an acyclic demand graph. The proof is complete.

Remark 2. Note that a clique in $G^2_e$ may not correspond to an acyclic demand graph if $G$ is not chordal. For a counter example, consider a cyclic network with 3 sources, labeled $S_1, S_2, S_3$ and 3 destinations, labeled $D'_1, D'_2, D'_3$. In the topology graph, $S'_1$ is connected to $D'_3, D'_1, S'_2$ is connected to $D'_1, D'_2$, and $S'_3$ is connected to $D'_2, D'_3$. The topology graph is not chordal as all nodes form a length-6 chordless cycle. Consider the messages $W_{11}, W_{22}, W_{33}$. They mutually conflict and form a clique in $G^2_e$. But the demand graph formed by them is not acyclic as multicast (CDMA) can achieve rate (DoF) 1/2 per message and the sum rate (DoF) is not bounded up by 1.

E. One-dimensional Convex Networks

We show that one-dimensional TIM instances with either source convexity or destination convexity (not necessarily both) are chordal.

We first define convexity. In a one-dimensional network, the source and destination nodes are placed along a straight line. We define the relation $a < b$ between two nodes to indicate that node $a$ is “to the left of” node $b$.

Definition 10 (Source Convexity). Source convexity refers to the property that if a source (say $S_i$) can be heard by two destination nodes (say $D_j, D_k$), then it must also be heard by all other destination nodes that are in between, i.e., $(D_j < D_i < D_k)$ AND $(t_{ki} = t_{ji} + 1) \Rightarrow t_{ki} = 1$.

Definition 11 (Destination Convexity). Destination convexity refers to the property that if a destination (say $D_j$) can hear two source nodes (say $S_j, S_k$), then it must also hear all other source nodes that are in between, i.e., $(S_j < S_i < S_k)$ AND $(t_{ij} = t_{ik} + 1) \Rightarrow t_{ij} = 1$.

We next proceed to the proof. Since the name of source or destination is entirely cosmetic, we consider a one-dimension TIM instance with only source convexity, without loss of generality. Similar proof applies to only destination convexity as well. To set up a proof by contradiction, suppose the TIM instance is not chordal, i.e., its topology graph contains a chordless cycle with length $2n, n = 3, 4, \ldots$, which corresponds to a cyclic sub-network with $n$ distinct sources, labeled $S'_1, \ldots, S'_n$ and $n$ distinct destinations, labeled $D'_1, \ldots, D'_n$.

Source $S'_i, i \in \{1, \ldots, n\}$ is connected to two destinations $D'_{i-1}, D'_{i+1}$ and destination $D'_i$ is connected to two sources $S'_{i-1}, S'_{i+1}$ (source/destination indices are interpreted modulo-$n$, i.e., $n + 1 = 1, 0 = n$). As the cycle is chordless, each source is connected to only two destinations. Further, because of source convexity, the destinations that are connected to the same source must be consecutive. For example, as $S'_2$ is connected to only $D'_1$ and $D'_2$, there can not be any destination in the interval between $D'_1$ and $D'_2$. Similarly, there is no destination in between $D'_3$ and $D'_{i+1}$, such that the order of the destinations in one straight line must appear as $D'_1 < D'_2 < \cdots < D'_n$ or $D'_1 > D'_2 > \cdots > D'_n$. In both cases, $D'_1$ and $D'_n$ are not consecutive. We arrive at a contradiction.
F. Coherence Time

We show that in the TIM context, Theorem 1 holds regardless of channel coherence time, i.e., orthogonal access achieves the all-unicast DoF region of the TIM problem if and only if the network is chordal, regardless of channel coherence time.

We now do not require the channel coefficients $h_{ij}(t)$ to be constant. Instead, $h_{ij}(t)$ can vary in an arbitrary manner as long as the values are bounded away from zero and infinity, i.e., there is no requirement on the channel coherence time.

**Sufficiency:** We prove that if a TIM network is chordal, then orthogonal access achieves the DoF region, regardless of channel coherence time.

The outer bounds provided in Section V-B hold regardless of channel coherence time. Also, orthogonal access scheme applies to both constant and time-varying channel setting, such that the achievability proof in Section V-B is not affected. Hence sufficiency is proved.

**Necessity:** We prove that if a TIM network is not chordal, then orthogonal access does not achieve the DoF region, regardless of channel coherence time.

We follow the proof in Section V-B. Suppose the topology graph is not chordal, such that it contains a chordless $2n$ cycle, $n = 3, 4, \ldots$. Such a chordless cycle corresponds to a cyclic sub-network. We show that orthogonal access can not achieve the DoF region for such a cyclic sub-network.

We also consider the cases where $n$ is odd or even separately. When $n$ is odd, we use the same interference channel message setting and consider DoF tuple $(1/2, 1/2, \ldots, 1/2)$. As CDMA can be applied to time-varying channels, the DoF tuple is still achievable, regardless of channel coherence time. However, orthogonal access cannot schedule more than $(n - 1)/2$ messages over one time slot, and is therefore unable to achieve this DoF tuple.

When $n$ is even, we still consider the all-unicast message setting. In this case, the DoF tuple $(1/3, 1/3, \ldots, 1/3)$ does not work as the interference alignment scheme used to achieve this tuple requires the channels to be constant for 3 symbol periods [6]. Instead, we consider the sum-DoF value. As shown in Section V-B, orthogonal access can not achieve more than $n/2$ sum-DoF. However, the scheme in [24] can achieve $(n + 1)/2$ sum-DoF, regardless of channel coherence time. See Fig. 9 in [24] for a pictorial illustration for $n = 4$ case. Thus, orthogonal access is again sub-optimal, regardless of channel coherence time. This completes the necessity proof.

**References**

[1] A. Mazumdar, “Storage capacity of repairable networks,” arXiv:1408.4862, Aug. 2014.
[2] K. Shanmugam and A. G. Dimakis, “Bounding multiple unicasts through index coding and locally repairable codes,” arXiv:1402.3895, Feb. 2014.
[3] M. Maddah-Ali and U. Niesen, “Fundamental limits of caching,” IEEE Trans. Inf. Theory, vol. 60, no. 5, pp. 2856–2867, May 2014.
[4] S. Rouayheb, A. Sprintson, and C. Georghiades, “On the index coding problem and its relation to network coding and Matroid theory,” IEEE Trans. Inf. Theory, vol. 56, no. 7, pp. 3187–3195, July 2010.
[5] M. Effros, S. El Rouayheb, and M. Langberg, “An equivalence between network coding and index coding,” ArXiv:1211.6660, Nov. 2012.
[6] S. A. Jafar, “Topological interference management through index coding,” IEEE Trans. Inf. Theory, vol. 60, no. 1, pp. 529–568, Jan. 2014.
[7] S. Riis, “Information flows, graphs and their guessing numbers,” The Electronic Journal of Combinatorics, vol. 14, no. 1, p. R44, 2007.
[8] Z. Bar-Yossef and Y. Birk and T. S. Jayram and T. Kol, “Index coding with side information,” IEEE Trans. Inf. Theory, vol. 57, no. 3, pp. 1479–1494, March 2011.
[9] F. Arbabjolfaei, B. Bandemer, Y.-H. Kim, E. Sasoglu, and L. Wang, “On the capacity region for index coding,” in ISIT 2013.
[10] S. Unal and A. B. Wagner. “A rate-distortion approach to index coding,” in Inf. Theory and Applications Workshop (ITA), 2014, pp. 1–5.
[11] H. Maleki, V. Cadambe, and S. Jafar, “Index coding – an interference alignment perspective,” IEEE Trans. Inf. Theory, vol. 60, no. 9, pp. 5402–5432, Sep. 2014.
[12] A. Blasiak, R. Kleinberg, and E. Lubetzky, “Lexicographic products and the power of non-linear network coding,” ArXiv:1108.2489, Aug. 2011.
[13] ——, “Broadcasting with side information: Bounding and approximating the broadcast rate,” IEEE Trans. Inf. Theory, vol. 59, no. 9, pp. 5811–5823, 2013.
[14] H. Sun and S. A. Jafar, “Index coding capacity: How far can one go with only Shannon inequalities?” arXiv:1303.7000, Mar. 2013.
[15] R. Baber, D. Christofides, A. N. Dang, S. Riis, and E. R. Vaughan, “Multiple unicasts, graph guessing games, and non-Shannon inequalities,” in International Symposium on Network Coding (NetCod), 2013, pp. 1–6.
[16] M. J. Neely, A. S. Tehrani, and Z. Zhang, “Dynamic index coding for wireless broadcast networks,” IEEE Trans. Inf. Theory, vol. 59, no. 11, pp. 7525–7540, Nov. 2013.
[17] L. Ong, C. K. Ho, and F. Lim, “The single-uniprior index-coding problem: The single-sender case and the multi-sender extension,” arXiv:1412.1520, Dec. 2014.
[18] C. Geng, N. Naderializadeh, S. Avestimehr, and S. Jafar, “On the optimality of treating interference as noise,” ArXiv:1305.4610, May 2013.
[19] H. Maleki and S. A. Jafar, “Optimality of orthogonal access for one-dimensional convex cellular networks,” IEEE communications letters, vol. 17, no. 9, pp. 1770–1773, Sept. 2013.
[20] M. C. Golumbic, Algorithmic graph theory and perfect graphs. Elsevier, 2004, vol. 57.
[21] K. Cameron, R. Sridharan, and Y. Tang, “Finding a maximum induced matching in weakly chordal graphs,” Discrete Mathematics, vol. 266, no. 1, pp. 133–142, 2003.
[22] A. Schrijver, Combinatorial optimization: polyhedra and efficiency. Springer, 2003, vol. 24.
[23] V. Chvátal, “On certain polytopes associated with graphs,” Journal of Combinatorial Theory, Series B, vol. 18, no. 2, pp. 138–154, 1975.
[24] S. A. Jafar, “Elements of cellular blind interference alignment — aligned frequency reuse, wireless index coding and interference diversity,” ArXiv:1203.2384, March 2012.
[25] S. A. Jafar, “Blind Interference Alignment,” IEEE Journal of Selected Topics in Signal Processing, vol. 6, no. 3, pp. 216–227, June 2012.