ON THE LOCAL QUOTIENT STRUCTURE OF ARTIN STACKS

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ABSTRACT. We show that near closed points with linearly reductive stabilizer, Artin stacks are formally locally quotient stacks by the stabilizer. We conjecture that the statement holds étale locally and we provide some evidence for this conjecture. In particular, we prove that if the stabilizer of a point is linearly reductive, the stabilizer acts algebraically on a miniversal deformation space generalizing results of Pinkham and Rim.

1. INTRODUCTION

This paper was motivated by the question of whether an Artin stack is “locally” a quotient stack by the stabilizer. While this question may appear quite technical and stacky in nature, we hope that a positive answer would lead to intrinsic constructions of moduli schemes parameterizing objects with infinite automorphisms (eg. vector bundles on a curve) without the use of classical geometric invariant theory.

We restrict ourselves to studying Artin stacks $\mathcal{X}$ over a base $S$ near closed points $\xi \in |\mathcal{X}|$ with linearly reductive stabilizer.

We conjecture that this question has an affirmative answer in the étale topology. Precisely,

Conjecture 1. If $\mathcal{X}$ is an Artin stack finitely presented over an algebraic space $S$ and $\xi \in |\mathcal{X}|$ is a closed point with linearly reductive stabilizer with image $s \in S$, then there exists an étale neighborhood $S' \to S$, $s' \mapsto s$ and an étale, representable morphism $f : [\mathcal{X}/G] \to \mathcal{X}$ where $G \to S'$ is a flat and finitely presented group algebraic space acting on an algebraic space $X \to S'$ such that the groups $\text{Aut}_{\mathcal{X}(k)}(x)$ and $G \times S' k$ are isomorphic for any geometric point $x \in \mathcal{X}(k)$ representing $\xi$. There is a lift of $\xi$ to $x : \text{Spec } k \to X$ such that $f$ induces an isomorphism $G_x \to \text{Aut}_{\mathcal{X}(k)}(f(x))$.

For example, if $S = \text{Spec } k$ and $x \in \mathcal{X}(k)$, Conjecture 1 implies that the stabilizer $G_x$ acts on an algebraic space $X$ fixing some point $\bar{x}$ and there exists an étale, representable morphism $f : [X/G_x] \to \mathcal{X}$ mapping $\bar{x}$ to $x$ and inducing an isomorphism on stabilizer groups.

There are natural variants of Conjecture 1 that one might hope are true. One might desire to find a presentation $[X/G_x] \to \mathcal{X}$ with $X$ affine over $S$; in this case, one would have that étale locally on $\mathcal{X}$, there exists a good moduli space. One might also like to relax the condition that $G_x$ is linearly reductive to geometrically reductive. However, some reductivity assumption on the stabilizer seems necessary (see Example 3.10).

Conjecture 1 is known for Artin stacks with quasi-finite diagonal (see Section 3.1). By a combination of an application of Sumihiro’s theorem and a Luna’s slice
argument, this conjecture is true over an algebraically closed field \(k\) for global quotient stacks \([X/G]\) where \(X\) is a regular scheme separated and finite type over \(k\) and \(G\) is a connected algebraic group (see Section 3.3). This conjecture is also known by Luna’s étale slice theorem ([Lun73]) over \(\text{Spec } k\) with \(k\) algebraically closed for global quotient stacks \([\text{Spec } A/G]\) where \(G\) is linearly reductive.

However, the conjecture appears to be considerably more difficult for general Artin stacks with non-finite stabilizer group schemes (eg. \(G_m^n, \text{PGL}_n, \text{GL}_n,...\)). For starters, there is not in general a coarse moduli scheme on which to work étale locally. Second, if \(G \to \text{Spec } k\) is not finite, an action of \(G\) on \(\text{Spf } A\) for a complete local noetherian \(k\)-algebra may not lift to an action of \(G\) and \(\text{Spec } A\) (consider \(G_m = \text{Spec } k[t]_t\) on \(\text{Spf } k[[x]]\) by \(x \mapsto tx\)) so that for certain deformation functors where one may desire to apply Artin’s approximation/algebraization theorems (such as in the proof of [AOV08 Prop 3.6]), formal deformations may not be effective.

While we cannot establish a general étale local quotient structure theorem, we establish the conjecture formally locally:

**Theorem 1.** Let \(X\) be a locally noetherian Artin stack over a scheme \(S\) and \(\xi \in |X|\) be a closed point with linearly reductive stabilizer. Let \(G_\xi \hookrightarrow X\) be the induced closed immersion and \(\mathcal{X}_n\) be its nilpotent thickenings.

(i) If \(S = \text{Spec } k\) and there exists a representative \(x : \text{Spec } k \to X\) of \(\xi\), then there exists affine schemes \(U_i\) and actions of \(G_x\) on \(U_i\) such that \(\mathcal{X}_i \cong [U_i/G_x]\). If \(G_x \to \text{Spec } k\) is smooth, the schemes \(U_i\) are unique up to \(G_x\) equivariant isomorphism.

(ii) Suppose \(x : \text{Spec } k \to X\) is a representative of \(\xi\) with image \(s \in S\) such that \(k(s) \to k\) is a finite, separable extension and \(G_x \to \text{Spec } k\) a smooth, affine group scheme. Fix an étale morphism \(S' \to S\) and a point \(s' \in S'\) with residue field \(k\). Then there exists affine schemes \(U_i\) and linearly reductive smooth groups schemes \(G_i\) over \(S'_n = \text{Spec } O_{S',s'}/m_{s'}^{n+1}\) with \(G_n = G_x\) such that \(\mathcal{X}_n \times_S S' \cong [U_n/G_n]\). The group schemes \(G_n \to S'_n\) are unique and the affine schemes \(U_n\) are unique up to \(G_n\-equivariant isomorphism.

This theorem implies that the stabilizer acts algebraically on a miniversal deformation space of \(\xi\) and this action is unique up to \(G_x\)-equivariant isomorphism.

After this paper was written, the author was made aware of similar results by Pinkham and Rim. In [Pin74], Pinkham shows that if \(G_m\) acts on an affine variety \(X\) over an algebraically closed field \(k\) with an isolated singular point, then the deformation space of \(X\) inherits a \(G_m\)-action. In [Rim80], Rim showed that for arbitrary homogeneous category fibered in groupoids, if the stabilizer is a linearly reductive algebraic group, then the stabilizer acts on a miniversal deformation.

Both Pinkham and Rim follow Schlessinger’s approach of building a versal deformation and show inductively that choices can be made equivariantly. We use an entirely different method. Following the techniques of [AOV08], we use a simple (although technical) deformation theory argument to give a quick proof recovering Rim’s result when then category fibered in groupoids is an Artin stack. Our result is more general in that (1) when the base is a field, we allow for non-reduced stabilizer groups and (2) we can work over any base scheme. Additionally, Pinkham and Rim appear to give actions on the tangent space and
deformation space only by the abstract group of \( k \)-valued points. Our methods show immediately that these actions are algebraic.

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### 2. Background

We will assume schemes and algebraic spaces to be quasi-separated. An Artin stack, in this paper, will have a quasi-compact and separated diagonal. We will work over a fixed base scheme \( S \).

#### 2.1. Stabilizer preserving morphisms.

The following definition generalizes the notion of fixed-point reflecting morphisms was introduced by Deligne, Kollár ([Kol97, Definition 2.12]) and by Keel and Mori ([KM97, Definition 2.2]). When translated to the language of stacks, the term stabilizer preserving seems more appropriate and we will distinguish between related notions.

**Definition 2.1.** Let \( f : \mathcal{X} \rightarrow \mathcal{Y} \) be a morphism of Artin stacks. We define:

1. \( f \) is **stabilizer preserving** if the induced \( \mathcal{X} \)-morphism \( \psi : I_{\mathcal{X}} \rightarrow I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X} \) is an isomorphism.
2. For \( \xi \in |\mathcal{X}| \), \( f \) is **stabilizer preserving at** \( \xi \) if for a (equivalently any) geometric point \( x : \text{Spec} \ k \rightarrow \mathcal{X} \) representing \( \xi \), the fiber \( \psi_x : \text{Aut}_{\mathcal{X}(k)}(x) \rightarrow \text{Aut}_{\mathcal{Y}(k)}(f(x)) \) is an isomorphism of group schemes over \( k \).
3. \( f \) is **pointwise stabilizer preserving** if \( f \) is stabilizer preserving at \( \xi \) for all \( \xi \in |\mathcal{X}| \).

**Remark 2.2.** One could also consider in (ii) the weaker notion where the morphism \( \psi_x \) is only required to be isomorphisms of groups on \( k \)-valued points. This property would be equivalent if \( \mathcal{X} \) and \( \mathcal{Y} \) are Deligne-Mumford stacks over an algebraically closed field \( k \).

**Remark 2.3.** Any morphism of algebraic spaces is stabilizer preserving. Both properties are stable under composition and base change. While a stabilizer preserving morphism is clearly pointwise stabilizer preserving, the converse is not true. For example, consider the action of \( \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \sigma, \tau \rangle \) on the affine line with a double origin \( X \) over a field \( k \) where \( \sigma \) acts by inverting the line but keeping both origins fixed and \( \tau \) acts by switching the origins. Then the stabilizer group scheme \( S_X \hookrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times X \rightarrow X \) has a fibers \( (1, \tau) \) everywhere except over the origins where fibers are \( (1, \sigma) \). The subgroup \( H = \langle 1, \tau \sigma \rangle \) acts freely on \( X \) and there is an induced trivial action of \( \mathbb{Z}_2 \) on the non-locally separated line \( Y = X/H \). There is \( \mathbb{Z}_2 \)-equivariant morphism \( Y \rightarrow \mathbb{A}^1 \) (with the trivial \( \mathbb{Z}_2 \) action on \( \mathbb{A}^1 \)) which induces a morphism \( [Y/\mathbb{Z}_2] \rightarrow [\mathbb{A}^1/\mathbb{Z}_2] \) which is pointwise stabilizer preserving but not stabilizer preserving. We note that the induced map \( [Y/\mathbb{Z}_2] \rightarrow \mathbb{A}^1 \) is not a \( \mathbb{Z}_2 \)-gerbe even though the fibers are isomorphic to \( B \mathbb{Z}_2 \). This example arose in discussions with Andrew Kresch.

It is natural to ask when the property of being pointwise stabilizer preserving is an open condition and what additional hypotheses are necessary to insure that a pointwise stabilizer preserving morphism is stabilizer preserving. First, we have:
Proposition 2.4. ([Ryd07] Prop. 3.5) Let $f : X \to Y$ be a representable and unramified morphism of Artin stack with $I_Y \to Y$ proper. The locus $U \subseteq |X|$ over which $f$ is pointwise stabilizer preserving is open and $f|_U$ is stabilizer preserving.

Proof. The cartesian square

\[
\begin{array}{ccc}
I_{X'} & \xrightarrow{\psi} & I_Y \times_Y X' \\
\downarrow & & \downarrow \\
X' & \xrightarrow{\Delta_{X'/Y} \times_Y X} & X \times_Y X'
\end{array}
\]

implies that $\psi$ is an open immersion and since the projection $p_2 : I_Y \times_Y X' \to X'$ is proper, the locus $U = X \setminus p_2(I_Y \times_Y X' \setminus I_X)$ is open.

\[
\square
\]

Remark 2.5. The proposition is not true if $f$ is ramified: if $f : [\mathbb{A}^1/\mathbb{Z}_2] \to [\mathbb{A}^1/\mathbb{Z}_2]$ where $\mathbb{Z}_2$ is acting by the non-trivial involution and trivially, respectively, then $\psi$ is only an isomorphism over the origin. The proposition also fails without the properness hypothesis: if $f : [\mathbb{A}^2/G_m] \to [\mathbb{A}^1/G_m]$ where $G_m$ is acting by vertical scaling on $\mathbb{A}^2$ and trivially on $\mathbb{A}^1$, then $\psi_x$ is only an isomorphism over the $x$-axis.

The following proposition gives a criterion for a pointwise stabilizer preserving morphism to be stabilizer preserving. We note that it is obvious that if $f$ is unramified and pointwise stabilizer preserving, then $f$ is stabilizer preserving.

Proposition 2.6. If $f : X \to Y$ is a representable pointwise stabilizer preserving morphism of Artin stacks and $I_X \to X$ finite, then $f$ is stabilizer preserving.

Proof. Since $I_X \to X$ is proper, it follows that $\psi : I_X \to I_Y \times_Y X$ is proper. As the hypotheses imply that $\psi$ is also quasi-finite, $\psi$ is a finite morphism. Since $\psi$ is the pull back of the monomorphism $\Delta_{X'/Y} : X' \to X \times_Y X'$, $\psi$ is also a monomorphism. Since any finite monomorphism is a closed immersion, $\psi$ is a closed immersion. But for all $x : \text{Spec} k \to X$, $\phi_x$ is an isomorphism and since $I_X \to X$ is finite, it follows that $\psi$ is an isomorphism.

\[
\square
\]

3. Evidence for Conjecture 1

3.1. Conjecture 1 is known for stacks with quasi-finite diagonal. An essential ingredient in the proof of the Keel-Mori theorem (see [KM97] Section 4) is the existence of étale, stabilizer preserving neighborhoods admitting finite, flat covers by schemes. We note that the existence of flat, quasi-finite presentations was known to Grothendieck (see [SGA3] Exp V, 7.2). We find the language of [Con05] more appealing:

Proposition 3.1. ([Con05] Lemma 2.1 and 2.2) Let $X$ be an Artin stack locally of finite presentation over a scheme $S$ with quasi-finite diagonal $\Delta_{X/S}$. For any point $\xi \in |X|$, there exists a representable, étale morphism $f : W \to X$ from an Artin stack $W$ admitting a finite fppf cover by a separated scheme and point $\omega \in |W|$ such that $f$ is stabilizer preserving at $\omega$. In particular, $W$ has finite diagonal over $S$.

Remark 3.2. The stack $W$ is constructed as the étale locus of the relative Hilbert stack $\text{Hilb}_{V/X} \to X$ where $V \to X$ is a quasi-finite, fppf scheme cover. In fact, the morphism $W \to X$ is stabilizer preserving at points $\text{Spec} k \to W$ corresponding
to the entire closed substack of $V \times_X \text{Spec} \, k$ so that every point $x \in |X|$ has some preimage at which $f$ is stabilizer preserving. If $X$ has finite inertia, it follows from Proposition 2.4 that $f$ is stabilizer preserving. In fact, as shown in [Con05, Remark 2.3], the converse is true: for $X$ as above with a representable, quasi-compact, étale, pointwise stabilizer preserving cover $W \to X$ such that $W$ is separated over $S$ and admits a finite fppf scheme cover, then $X$ has finite inertia.

We now restate one of the main results from [AOV08].

**Proposition 3.3.** ([AOV08, Prop. 3.6]) Let $X$ be an Artin stack locally of finite presentation over a scheme $S$ with finite inertia. Let $\phi : X \to Y$ be its coarse moduli space and let $\xi \in |X|$ be a point with linearly reductive stabilizer with image $y \in Y$. Then there exists an étale morphism $U \to Y$, a point $u$ mapping to $y$, a finite linearly reductive group scheme $G \to U$ acting on a finite, finitely presented scheme $V \to U$ and an isomorphism $[V/G] \simto U \times_X Y$ of Artin stacks over $S$. Moreover, it can be arranged that there is a representative of $\xi$ by $x : \text{Spec} \, k(u) \to X$ such that $G \times_U k(u)$ and $\text{Aut}_{X(k(u))}(x)$ are isomorphic as group schemes over $\text{Spec} \, k(u)$.

**Proof.** Strictly speaking, the last statement is not in [AOV08] although their construction yields the statement. □

**Remark 3.4.** In particular, this proposition implies that given any Artin stack $X$ locally of finite presentation with finite inertia, the locus of points with linearly reductive stabilizer is open.

**Corollary 3.5.** Conjecture [1] is true for Artin stacks $X$ locally of finite presentation over $S$ with quasi-finite diagonal. In fact, étale presentations $[X/G]$ can be chosen so that $X$ is affine.

**Proof.** Given $\xi \in |X|$, by Proposition 3.1 there exists an étale neighborhood $f : W \to X$ stabilizer preserving at some $\omega \in |X|$ above $\xi$ such that $W$ has finite inertia. Apply Proposition 3.3 to $W$ achieves the result. □

**Remark 3.6.** In fact, the conjecture is even true for Deligne-Mumford stacks with finite inertia which are not necessarily tame (ie. have points with non-linearly reductive stabilizer). This follows easily from (see [AV02, Lemma 2.2.3] and [Ols06b, Thm 2.12]). We wonder if any Artin stack with finite inertia can étale locally be written as a quotient stack by the stabilizer. We note that non-reduced, non-linearly reductive finite fppf group schemes are still geometrically reductive.

### 3.2. Examples

Here we list three examples of non-separated Deligne-Mumford stacks and give étale presentations by quotient stacks by the stabilizer verifying Conjecture [1]. In these examples, good moduli spaces do not exist Zariski-locally. We will work over an algebraically closed field with $\text{char} \, k \neq 2$.

**Example 3.7.** Let $G \to \mathbb{A}^1$ be the group scheme which has fibers isomorphic to $\mathbb{Z}_n$ everywhere except over the origin where it is trivial. The group scheme $G \to \mathbb{A}^1$ is not linearly reductive. The classifying stack $BG$ does not admit a good moduli space Zariski-locally around the origin although there does exist a coarse moduli space. The cover $f : \mathbb{A}^1 \to BG$ satisfies the conclusion of Conjecture [1]. The morphism $f$ is stabilizer preserving at the origin but nowhere else. This example
stresses that one cannot hope to find étale charts \([X/G] \to \mathcal{X}\) of quotient stacks of linearly reductive group schemes which are pointwise stabilizer preserving everywhere.

**Example 3.8.** \((4 \text{ unordered points in } \mathbb{P}^1 \text{ up to } \text{Aut}(\mathbb{P}^1))\)

Consider the quotient stack \(\mathcal{X} = \mathbb{P}(V)/\text{PGL}_2\) where \(V\) is the vector space of degree 4 homogeneous polynomials in \(x\) and \(y\). Let \(\mathcal{U} \subseteq \mathcal{X}\) be the open substack consisting of points with finite automorphism group. Any point in \(p \in \mathcal{U}\) can be written as \(xy(x - y)(x - \lambda y)\) for \(\lambda \in \mathbb{P}^1\). If \(\lambda \neq 0, 1\) or \(\infty\), the stabilizer is

\[
G_p = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ 1 & 1 - \lambda \end{pmatrix}, \begin{pmatrix} 1 & -\lambda \\ 1 & 1 \end{pmatrix} \right\}
\]

As \(\lambda \to 1\) (resp. \(\lambda \to 0, \lambda \to \infty\)), only the identity and \(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), (resp. \(\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\)) survive. Therefore, the stabilizer group scheme of the morphism \(\mathbb{P}^1 \to \mathcal{U}\) is a non-finite group scheme which is \(\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{P}_1\) but with two elements removed over each of the fibers over 0,1 and \(\infty\) (so that the generic fiber is \(\mathbb{Z}_2 \times \mathbb{Z}_2\) and the fiber over 0,1 or \(\infty\) is \(\mathbb{Z}_2\).

We give an étale presentation around 1. Let \(\mathbb{Z}_2\) act on \(X = \mathbb{A}^1 \setminus \{0\}\) via \(\lambda \mapsto 1/\lambda\). The morphism \(f : X \to \mathbb{P}^4, \lambda \mapsto [xy(x - y)(x - \lambda y)]\) is \(\mathbb{Z}_2\) invariant where \(\mathbb{Z}_2\) acts on \(\mathbb{P}^4\) via the inclusion \(\mathbb{Z}_2 \hookrightarrow \text{PGL}_2, -1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). The induced morphism \([X/\mathbb{Z}_2] \to \mathcal{X}\) is étale and stabilizer preserving at 1. However, it is not pointwise stabilizer preserving in a neighborhood of \(1\). The \(j\)-invariant \(j : \mathcal{U} \to \mathbb{P}^1, [xy(x - y)(x - \lambda y)] \mapsto [(\lambda^2 - \lambda + 1)^2, \lambda^4 - 1]\) gives a coarse moduli space which is not a good moduli space.

The following example due to Rydh shows that coarse moduli spaces (or even categorical quotients) may not exist for non-separated Deligne-Mumford stacks.

**Example 3.9.** The Keel-Mori theorem states that any Artin stack \(\mathcal{X} \to S\) where the inertia stack \(I_{\mathcal{X}/S} \to \mathcal{X}\) is finite admits a coarse moduli space. The finiteness of inertia hypothesis cannot be weakened to requiring that the diagonal is quasi-finite. Let \(X\) be the non-separated plane attained by gluing two planes \(\mathbb{A}^2 = \text{Spec } k[x, y]\) along the open set \(\{x \neq 0\}\). The action of \(\mathbb{Z}_2\) on \(\text{Spec } k[x, y]_x\) given by \((x, y) \mapsto (x, -y)\) extends to an action of \(\mathbb{Z}_2\) on \(X\) by swapping and flipping the axis (explicitly, if \(X = U_1 \cup U_2\), the multiplicity is defined by \(\mathbb{Z}_2 \times U_1 \to U_2, (x_2, y_2) \mapsto (x_1, -y_1)\) and \(\mathbb{Z}_2 \times U_2 \to U_1, (x_1, y_1) \mapsto (x_2, -y_2)\)). Then \(\mathcal{X} = [X/\mathbb{Z}_2]\) is a non-separated Deligne-Mumford stack. There is an isomorphism \(\mathcal{X} = [\mathbb{A}^2/G]\) where \(G = \mathbb{A}^1 \cup \{0\} \to \mathbb{A}^1\) is the group scheme over \(\mathbb{A}^1\) whose fibers are \(\mathbb{Z}_2\) over the origin where it is trivial and \(G\) acts on \(\mathbb{A}^2 = \text{Spec } k[x, y]\) over \(\mathbb{A}^1 = \text{Spec } k[x]\) by the non-trivial involution \(y \mapsto -y\) away from the origin.

David Rydh shows in [Ryd07, Example 7.15] that this stack does not admit a good coarse moduli space. In fact, there does not even exist an algebraic stack \(Z\) and a morphism \(\phi : \mathcal{X} \to Z\) which is universal for maps to schemes. The above statements are also true for any open neighborhood of the origin.

The following is a counterexample for Conjecture \([1]\) if the stabilizer is not linearly reductive.
Counterexample 3.10. Over a field $k$, let $G \to \mathbb{A}^1$ be a group scheme with generic fiber $G_m$ and with a $G_a$ fiber over the origin. Explicitly, we can write $G = \text{Spec } k[x, y]_{xy+1} \to \text{Spec } k[x]$ with the multiplication $G \times_{\mathbb{A}^1} G \to G$ defined by $y \mapsto xyy' + y + y'$. Let $\mathcal{X} = [\mathbb{A}^1/G]$ be the quotient stack over $\text{Spec } k$ and $x : \text{Spec } k \to \mathcal{X}$ be the origin. The stabilizer $G_x = G_a$ acts trivially on the tangent space $T_x(k[\epsilon])$. The nilpotent thickening $\mathcal{X}_1$ cannot be a quotient stack by $G_a$ giving a counterexample to Conjecture [1] and Theorem [1] in the case that the stabilizer is not linearly reductive.

3.3. Conjecture [1] is known for certain quotients stacks. 

Theorem. Let $\mathcal{X}$ be an Artin stack over an algebraically closed field $k$. Suppose $\mathcal{X} = [X/G]$ is a quotient stack and $x \in \mathcal{X}(k)$ has smooth linearly reductive stabilizer. Suppose that one of the following hold:

1. $G$ is a connected algebraic group acting on a regular scheme $X$ separated and finite type over $\text{Spec } k$.
2. $G$ is a linearly reductive algebraic group acting on an affine scheme $X$.

Then there exists a locally closed $G_x$-invariant affine $W \hookrightarrow X$ with $w \in W$ such that

$$[W/G_x] \to [X/G]$$

is affine and étale.

Proof. Part (2) follows directly from Luna’s étale slice theorem ([Lun73]). We note that although the results in [Lun73] are stated over fields of characteristic 0, the same arguments apply in characteristic $p$ to smooth linearly reductive group schemes.

For part (1), by applying [Sum74] Theorem 1 and Lemma 8], there exists an open $G$-invariant affine $U_1$ containing $x$ and an $G$-equivariant immersion $U_1 \hookrightarrow X = \mathbb{P}(V)$ where $V$ is a $G$-representation. Since the action of $G_x$ on $\text{Spec } \text{Sym}^* V^\vee$ fixes the line spanned by $x$, there exists a $G_x$-invariant homogeneous polynomial $f$ with $f(x) \neq 0$. It follows that $V = X_f \cap U_1$ is a $G_x$-invariant quasi-affine neighborhood of $x$ with $i : V \hookrightarrow V := (U_1)_f$ is an open immersion and $V$ is affine. If we let $\pi : V \to V/G_x$ be the GIT quotient. Since $V \setminus V$ and $x \in V$ are disjoint $G_x$-invariant closed subschemes, $\pi(V \setminus V)$ and $\pi(x)$ are closed and disjoint. Let $Y \subseteq (\overline{V}/G_x) \setminus (\pi(V \setminus V))$ be an affine containing $x$. Then $U = \pi^{-1}(Y)$ is a $G_x$-invariant affine containing $x$.

The stabilizer acts naturally on $T_x X$ and there exists a $G_x$-invariant morphism $U \to T_x X$ which is étale since $x \in X$ is regular. Since $G_x$ is linearly reductive, we may write $T_x X = T_x o(x) \oplus W_1$ for a $G_x$-representation $W_1$. Define the $G_x$-invariant affine $W \subseteq U$ by the cartesian diagram

$$
\begin{array}{ccc}
W & \longrightarrow & W_1 \\
\downarrow & & \downarrow \\
U & \longrightarrow & T_x X = T_x o(x) \oplus W_1
\end{array}
$$

and let $w \in W$ be the point corresponding to $x$.

The stabilizer $G_x$ acts on $G \times W$ via $h \cdot (g, w) = (gh^{-1}, h \cdot w)$ for $h \in G_x$ and $(g, w) \in G \times W$. The quotient $G \times_{G_x} W := (G \times W)/G_x$ is affine. Since the
quotient morphism \( G \times W \to G \times_{G_x} W \) is a \( G_x \)-torsor it follows that \( T_{(e,e)}G \times_{G_x} W = (T_eG \oplus T_wW)/T_xG \) where \( T_eG_x \subseteq T_eG \oplus T_wW \) is induced via the inclusion \( G_x \to G \times W ; h \mapsto (h^{-1}, h \cdot w) \). Therefore, \( G \times_{G_x} W \to X \) is étale at \((e, w)\). Furthermore, \( G \times_{G_x} W \to X \) is affine. It follows that the induced morphism of stacks \( f : W/G_x \to [X/G] \) is affine and étale at \( w \).

Let \( \phi : [W/G_x] \to Y = W//G_x \) be the good moduli space corresponding to the GIT quotient \( \pi : W \to W//G_x \). If \( Z \subseteq [W/G_x] \) is the closed locus where \( f \) is not étale, then \( Z \) is disjoint to \( \{ w \} \) and it follows that \( \phi(Z) \) and \( \phi(w) \) are closed and disjoint. Let \( Y' \subseteq Y \) is an open affine containing in \( Y \) containing \( \phi(Z) \) containing \( \phi(w) \) so that \( \phi^{-1}(Y') = [W'/G_x] \) where \( W' = \pi^{-1}(Y') \) is a \( G_x \)-invariant affine. The morphism \( f : [W'/G_x] \to [X/G] \) satisfies the desired properties.

\[ \square \]

**Remark 3.11.** In a future paper by the author, it will be shown that the same statement holds if \( X = [X/G] \) is quotient stack over an algebraically closed field admitting a good moduli space where \( X \) is an arbitrary algebraic space finite type over \( k \) and \( G \) is an algebraic group.

### 4. Actions on deformations

#### 4.1. Setup

Let \( \mathcal{X} \) be a category fibered in groupoids over \( \text{Sch}/S \) with \( S = \text{Spec} \, R \).

For an \( R \)-algebra \( A \), an object \( a \in \mathcal{X}(A) \), and a morphism \( A' \to A \) of \( R \)-algebras, denote by \( F_{\mathcal{X},a}(A') \) the category of arrows \( a \to a' \) over \( \text{Spec} \, A \to \text{Spec} \, A' \) where a morphism \( (a \to a_1) \to (a \to a_2) \) is an arrow \( a_1 \to a_2 \) over the identity inducing a commutative diagram

\[
\begin{array}{ccc}
 a_1 & \to & a_2 \\
 \downarrow & & \downarrow \\
 a & \to & a
\end{array}
\]

Let \( \overline{F}_{\mathcal{X},a}(A') \) be the set of isomorphism classes of \( F_{\mathcal{X},a}(A') \). When there is no risk of confusion, we will denote \( F_a(A') := F_{\mathcal{X},a}(A') \) and \( \overline{F}_a(A') = \overline{F}_{\mathcal{X},a}(A') \).

For an \( A \)-module \( M \), denote by \( A[M] \) the \( R \)-algebra \( R \oplus M \) with \( M^2 = 0 \).

**Definition 4.1.** We say that \( \mathcal{X} \) is \( S1(b) \) (resp. \emph{strongly} \( S1(b) \)) if for every surjection \( B \to A \) (resp. any morphism \( B \to A \)), finite \( A \)-module \( M \), and arrow \( a \to b \) over \( \text{Spec} \, A \to \text{Spec} \, B \), the canonical map

\[
\overline{F}_b(B[M]) \longrightarrow \overline{F}_a(A[M])
\]

is bijective. (Note that we are not assuming that \( A \) is reduced as in \([\text{Art74}]\).)

**Remark 4.2.** We are using the notation from \([\text{Art74}]\). Recall that there is another condition \( S1(a) \) such that when both \( S1(a) \) and \( S1(b) \) are satisfied (called \emph{semi-homogeneity} by Rim), then there exists a miniversal deformation space (or a hull) by \([\text{Sch68}]\) and \([\text{Rim80}]\). We are isolating the condition \( S1(b) \) and strongly \( S1(b) \) to indicate precisely what is necessary for algebraicity of the action of the stabilizer on the tangent space.
Remark 4.3. Any Artin stack $\mathcal{X}$ over $S$ satisfies the following homogeneity property: for any surjection of $R$-algebras $C' \to B'$ with nilpotent kernel, $B \to B'$ any morphism of $R$-algebras, and $b' \in \mathcal{X}(B')$, the natural functor

$$\mathcal{X}_{b'}(C' \times_{B'} B) \to \mathcal{X}_{b'}(C') \times \mathcal{X}_{b'}(B)$$

is an equivalence of categories (see [Ols, Lemma 1.4.4]). In particular, any Artin stack $\mathcal{X}$ over $S$ is strongly $S1(b)$.

It is easy to see that if $\mathcal{X}$ satisfies $S1(b)$, then for any $R$-algebra $A$, object $a \in \mathcal{X}(A)$ and finite $A$-module $M$, the set $\mathcal{T}_a(A[M])$ inherits an $A$-module structure. In particular, for $x \in \mathcal{X}(k)$, the tangent space $\mathcal{T}_x(k[\varepsilon])$ is naturally a $k$-vector space. For any $k$-vector space, the natural identification $\text{Hom}(k[\varepsilon], k[V]) \cong V$ induces a morphism

$$\mathcal{T}_x(k[\varepsilon]) \otimes_k V \to \mathcal{T}_x(k[V])$$

which is an isomorphism for finite dimensional vector spaces $V$.

Remark 4.4. If $\mathcal{X}$ is also locally of finite presentation, then this is an isomorphism for any vector space $V$ since if we write $V = \lim V_i$ with $V_i$ finite dimensional then $\lim \mathcal{T}_x(k[V_i]) \to \mathcal{T}_x(k[V])$ is bijective.

4.2. Actions on tangent spaces. For $a \in \mathcal{X}(A)$, the abstract group $\text{Aut}_{\mathcal{X}(A)}(a)$ acts on the $R$-module $\mathcal{T}_a(A[\varepsilon])$ via $A$-module isomorphisms: $g \in \text{Aut}_{\mathcal{X}(A)}(a)$ and $(\alpha : a \to a') \in \mathcal{T}_a(A[\varepsilon])$, then $g \cdot (a \to a') = (a \xrightarrow{g^{-1}} a \xrightarrow{\alpha} a')$.

Remark 4.5. For example, suppose $\mathcal{X}$ is parameterizing schemes with $X_0 \to \text{Spec} A$ is an object in $\mathcal{X}(A)$. An element $g \in \text{Aut}(X_0)$ acts on infinitesimal deformations via

$$\begin{pmatrix} X_0 \to X \to \text{Spec} A \to \text{Spec} A[\varepsilon] \\ i \downarrow \downarrow \downarrow \downarrow \\ \text{Spec} A \end{pmatrix} \mapsto \begin{pmatrix} X_0 \to X \to \text{Spec} A \to \text{Spec} A[\varepsilon] \\ g^{-1} \circ i \downarrow \downarrow \downarrow \downarrow \\ \text{Spec} A \end{pmatrix}$$

If $x \in \mathcal{X}(k)$ with stabilizer $G_x$, we have shown that there is a homomorphism of abstract groups

$$G_x(k) \to \text{GL}(\mathcal{T}_x(k[\varepsilon]))(k)$$

We are interested in determining when this is algebraic (ie. arising from a morphism of group schemes $G_x \to \text{GL}(\mathcal{T}_x(k[\varepsilon]))$). For any $k$-algebra $A$, let $a \in \mathcal{X}(A)$ be a pullback of $x$. Note that there is a canonical identification $\text{Aut}_{\mathcal{X}(A)}(a) \cong G_x(A)$ which induces a homomorphism

$$G_x(A) \to \text{GL}(\mathcal{T}_a(A[\varepsilon]))(A)$$

If $\mathcal{X}$ is strongly $S1(b)$, then using the isomorphism $A[\varepsilon] \times_A k \to k[A]$, we have a bijection $\mathcal{T}_x(k[A]) \to \mathcal{T}_a(A[\varepsilon])$. The natural maps induce a commutative diagram
of $A$-modules
\[
\begin{array}{c}
F_x(k[e]) \otimes_k A \\
\downarrow \\
F_a(A[e])
\end{array}
\]
If $\mathcal{X}$ is locally of finite presentation over $S$, by Remark 4.4, the top arrow is bijective so that the diagonal arrow is as well. Therefore, we have a natural homomorphism of groups
\[
G_x(A) \longrightarrow \text{GL}(F_x(k[e]) \otimes_k A) = \text{GL}(F_x(k[e]))(A)
\]
for any $k$-algebra $A$ which induces a morphism of group schemes $G_x \to \text{GL}(F_x(k[e]))$.

Therefore, if $\mathcal{X} \to S$ is locally of finite presentation and is strongly $S$-1(b), then for $x \in \mathcal{X}(k)$, the stabilizer $G_x$ acts algebraically on $F_x(k[e])$. In particular,

**Proposition 4.6.** If $\mathcal{X}$ is an Artin stack locally of finite presentation over a scheme $S$ and $x \in \mathcal{X}(k)$, then the stabilizer $G_x$ acts algebraically on the tangent space $F_x(k[e])$.

**Remark 4.7.** The above proposition is certainly well known, but we are unaware of a rigorous proof in the literature. We thank Angelo Vistoli for pointing out the simple argument above.

In [Pin74, Prop. 2.2], Pinkham states that if $\mathcal{X}$ is the deformation functor over an algebraically closed field of an affine variety with an isolated singular point with $\mathbb{G}_m$-action, then the tangent space $T^1$ inherits an algebraic $\mathbb{G}_m$-action. However, it appears that he only gives a homomorphism of algebraic groups $\mathbb{G}_m(k) \to \text{GL}(T^1(k))$. There can be group homomorphisms $k^* \to \text{GL}_n(k)$ which are not algebraic.

In [Rim80, p. 220-1], Rim states that if $\mathcal{X}$ is category fibered in groupoids over the category $B$ of local Artin $k$-algebras with residue field $k$ with $\mathcal{X}(k) = \{x\}$ which is homogeneous in the sense that (1) is an equivalence for a surjection $C' \to B'$ and any morphism $B \to B'$ in $B$, then $F_x(k[e])$ inherits a linear representation. However, he only shows that there is a homomorphism of algebraic groups $G_x(k) \to \text{GL}(F_x(k[e]))(k)$. While it is clear that there are morphisms of groups $G_x(A) \to \text{GL}(F_x(k[e]))(A)$ for local Artin $k$-algebras with residue field $k$, it is not clear to us that this gives a morphism of group schemes $G_x \to \text{GL}(F_x(k[e]))$ without assuming a stronger homogeneity property.

### 4.3. Actions on deformations.

Let $\mathcal{X}$ is an Artin stack over $S$ and suppose $G \to S$ is a group scheme with multiplication $\mu : G \times_S G \to G$ acting on a scheme $U \to S$ via $\sigma : G \times_S U \to U$. To give a morphism
\[
[U/G] \longrightarrow \mathcal{X}
\]
is equivalent to giving an object $a \in \mathcal{X}(U)$ and an arrow $\phi : \sigma^*a \to p_2^*a$ over the identity satisfying the cocycle $p_{23}^*\phi \circ (\text{id} \times \sigma)^*\phi = (\mu \times \text{id})^*\phi$. We say that $G$ acts on $a \in \mathcal{X}(U)$ if such data exists. (In fact, there is an equivalence of categories between $\mathcal{X}([U/G])$ and the category parameterizing the above data.)
Remark 4.8. Suppose $X$ parameterizes families of schemes and we are given a deformation of an object $x \in X(k)$ corresponding to a scheme $X_0 \to \text{Spec } k$.

$$
\begin{array}{c}
X_0 \xrightarrow{i} X \\
\downarrow \\
\text{Spec } k \xrightarrow{p} U
\end{array}
$$

If $G$ acts on a scheme $U$ over $k$, then giving a morphism $[\text{Spec } A/G] \to X$ is equivalent to giving an action of $G$ on $X$ compatible with the action on $U$.

4.4. Action of formal deformations. Let $\mathcal{U}$ be a noetherian formal scheme over $S$ with ideal of definition $\mathfrak{I}$. Set $U_n$ to be the scheme $[|\mathcal{U}|, \mathcal{O}_{\mathcal{U}}/\mathfrak{I}^{n+1}]$. If $\mathcal{X}$ is an category fibered in groupoids over $\text{Sch}/S$, one defines $\mathcal{X}(\mathcal{U})$ to be the category where the objects are a sequence of arrows $a_0 \to a_1 \to a_2 \to \cdots$ over the nilpotent thickenings $U_0 \hookrightarrow U_1 \hookrightarrow \cdots$ and a morphism $(a_0 \to a_1 \to \cdots) \to (a'_0 \to a'_1 \cdots)$ is compatible sequence of arrows $a_i \to a_i$ over the identity. One checks that if $\mathcal{I}$ is replaced with a different ideal of definition, then one obtains an equivalent category. Given a morphism of formal schemes $p : \mathcal{U}' \to \mathcal{U}$, one obtains a functor $p^* : \mathcal{X}(\mathcal{U}) \to \mathcal{X}(\mathcal{U}')$.

If $G \to S$ is a group scheme over $S$ with multiplication $\mu$ acting on the formal scheme $\mathcal{U}$ via $\sigma : G \times_S \mathcal{U} \to \mathcal{U}$ such that $\mathfrak{I}$ is an invariant ideal of definition, we say that $G$ acts on a deformation $\tilde{\mathfrak{a}} = (a_0 \to a_1 \cdots) \in \mathcal{X}(\mathcal{U})$, if above there is an arrow $\phi : \sigma^*\tilde{\mathfrak{a}} \to p_2^*\tilde{\mathfrak{a}}$ in $\mathcal{X}(G \times_S \mathcal{U})$ satisfying the cocycle $p_{23}^*\phi \circ (\text{id} \times \sigma)^*\phi = (\mu \times \text{id})^*\phi$. This is equivalent to giving compatible morphisms $[U_\lambda/G] \to \mathcal{X}$. (Given an appropriate definition of a formal stack $[\mathcal{U}/G]$, this should be equivalent to giving a morphism $[\mathcal{U}/G] \to \mathcal{X}$.)

5. LOCAL QUOTIENT STRUCTURE

We show that for closed points with linearly reductive stabilizer, the stabilizer acts algebraically on the deformation space. In other words, Artin stacks are “formally locally” quotient stacks around such points which gives a formally local answer to Conjecture [1]. We will use the same method as in [AOV08] to deduce that all nilpotent thickenings are quotient stacks.

5.1. Deformation theory of $G$-torsors. We will need to know the deformation theory of $G$-torsors over Artin stacks. We recall for the reader the necessary results of the deformation theory of $G$-torsors from [Ols06a] and [AOV08].

Suppose $G \to S$ is a fppf group scheme and $p : \mathcal{P} \to \mathcal{X}$ is a $G$-torsor. Let $i : \mathcal{X} \to \mathcal{X}'$ be a closed immersion of stacks defined by a square-zero ideal $I \subseteq \mathcal{O}_{\mathcal{X}'}$. Then the collection of 2-cartesian diagrams

$$
\begin{array}{ccc}
\mathcal{P} \xrightarrow{\mathfrak{u}} \mathcal{P}' \\
\downarrow p \\
\mathcal{X} \xrightarrow{i} \mathcal{X}'
\end{array}
$$

with $p' : \mathcal{P}' \to \mathcal{X}'$ a $G$-torsor form in a natural way a category.
Proposition 5.1. Let $L_{BG/S}$ denote the cotangent complex of $BG \to S$ and $f: X \to BG$ be the morphism corresponding to the $G$-torsor $p: \mathcal{P} \to X$.

(i) There is a canonical class $o(x, i) \in \text{Ext}^1(Lf^*L_{BG/S}, I)$ whose vanishing is necessary and sufficient for the existence of an extension $(i', p')$ filling in the diagram

(ii) If $o(x, i) = 0$, then the set of isomorphism of extensions filling in the diagram is naturally a torsor under $\text{Ext}^0(Lf^*L_{BG/S}, I)$.

(iii) For any extension $(i', p')$, the group of automorphisms of $(i', p')$ (as a deformation of $\mathcal{P} \to X$) is canonically isomorphic to $\text{Ext}^{-1}(Lf^*L_{BG/S}, I)$.

Proof. This is a special case of [Ols06a, Theorem 1.5] with $\mathcal{Y} = \mathcal{Y}' = BG$ and $Z = Z' = S$. □

Proposition 5.2. Let $G \to S$ be an fppf group scheme. Then

(i) $L_{BG/S} \in D_{\text{coh}}^{[0,1]}(O_BG)$.

(ii) If $G \to S$ is smooth, $L_{BG/S} \in D_{\text{coh}}^{[1]}(O_BG)$.

If $G \to \text{Spec} k$ is linearly reductive and $\mathcal{F}$ is a coherent sheaf on $BG$, then

(iii) $\text{Ext}^i(L_{BG/k}, \mathcal{F}) = 0$ for $i \neq -1, 0$.

(iv) If $G \to \text{Spec} k$ is smooth, $\text{Ext}^i(L_{BG/k}, \mathcal{F}) = 0$ for $i \neq -1$.

Proof. Part (i) and (ii) follow from the distinguished triangle induced by the composition $S \to BG \to S$ as in [AOV08] Lemma 2.18 with the observation that $G \to S$ is a local complete intersection. Part (iii) is given in the proof of [AOV08] Lemma 2.17 and (iv) is clear from (ii). □

5.2. Proof of Theorem 1.

Theorem 1. Let $X$ be a locally noetherian Artin stack over a scheme $S$ and $\xi \in |X|$ be a closed point with linearly reductive stabilizer. Let $\mathcal{G}_\xi \hookrightarrow X$ be the induced closed immersion and $X_\xi$ be its nilpotent thickenings.

(i) If $S = \text{Spec} k$ and there exists a representative $x: \text{Spec} k \to X$ of $\xi$, then there exists affine schemes $U_i$ and actions of $G_x$ on $U_i$ such that $X_i \cong [U_i/G_x]$. If $G_x \to \text{Spec} k$ is smooth, the schemes $U_i$ are unique up to $G_x$ equivariant isomorphism.

(ii) Suppose $x: \text{Spec} k \to X$ is a representative of $\xi$ with image $s \in S$ such that $k(s) \hookrightarrow k$ is a finite, separable extension and $G_x \to \text{Spec} k$ a smooth, affine group scheme. Fix an étale morphism $S' \to S$ and a point $s' \in S'$ with residue field $k$. Then there exists affine schemes $U_i$ and linearly reductive smooth groups schemes $G_i$ over $S'_n = \text{Spec} O_{S'/s'}/m_{s'}^{n+1}$ with $G_n = G_x$ such that $X_n \times_S S' \cong [U_n/G_n]$. The group schemes $G_n \to S'_n$ are unique and the affine schemes $U_n$ are unique up to $G_n$-equivariant isomorphism.

Proof. We prove inductively that each $X_i$ is a quotient stack by $G_x$ by deformation theory. For (i), let $p_0: U_0 = \text{Spec} k \to X_0$ be the canonical $G_x$-torsor. Suppose we have a compatible family of $G_x$-torsors $p_i: U_i \to X_i$ with $U_i$ affine. This gives a
2-cartesian diagram

\[
\begin{array}{ccccccc}
U_0 & \rightarrow & \cdots & \rightarrow & U_{n-1} & \rightarrow & j_n \rightarrow U_n \\
| & & & | & & & | \\
p_0 & & p_{n-1} & & j_n & & p_n \\
\mathcal{X}_0 & \rightarrow & \cdots & \rightarrow & \mathcal{X}_{n-1} & \rightarrow & i_n \rightarrow \mathcal{X}_n
\end{array}
\]

By Corollary 5.1, the obstruction to the existence of a \( G_x \)-torsor \( p_n : U_n \rightarrow \mathcal{X}_n \) restricting to \( p_{n-1} : U_{n-1} \rightarrow \mathcal{X}_{n-1} \) is an element

\[ o \in \text{Ext}^1(Lf^*L_{BG_x/k}(\mathcal{I}^n), \mathcal{I}^n/\mathcal{I}^{n+1}) = 0 \]

where \( f : \mathcal{X}_{n-1} \rightarrow BG_x \) is the morphism defined by \( U_{n-1} \rightarrow \mathcal{X}_{n-1} \) and \( \mathcal{I} \) denotes the sheaf of ideals defining \( \mathcal{X}_0 \). The vanishing is implied by Proposition 5.2(iii). Therefore, there exists a \( G_x \)-torsor \( U_n \rightarrow \mathcal{X}_n \) extending \( U_{n-1} \rightarrow \mathcal{X}_{n-1} \). Since \( U_0 \) is affine, so is \( U_n \) and the \( G_x \)-torsor \( p_n \) gives an isomorphism \( \mathcal{X}_n \cong [U_n/G_x] \). Furthermore, if \( G_x \) is smooth, this extension is unique by Proposition 5.2(iv).

For (ii), first choose a scheme \( S' \) and an étale morphism \( S' \rightarrow S \) such that \( S' \times_S k(s) = k \). Let \( s' \in S' \) denote the preimage of \( s \) and \( S'_n = \text{Spec} \mathcal{O}_{S', s'}/m_{s'+1}^n \). The group scheme \( G_0 = G_x \rightarrow \text{Spec} k \) extends uniquely to smooth group schemes \( G_i \rightarrow S_n' \) (SGA3 Expose III, Thm. 3.5) which by [Alp08, Prop. 3.9(iii)] are linearly reductive. If \( \mathcal{X}' = \mathcal{X} \times_S S' \), then \( BG_x \hookrightarrow \mathcal{X}' \) is a closed immersion with nilpotent thickenings \( \mathcal{X}'_n \) isomorphic to \( \mathcal{X}_n \times_S S' \). Let \( p_0 : U_0 = \text{Spec} k \rightarrow \mathcal{X}_0 \) be the canonical \( G_x \)-torsor which we may also view as a torsor over \( G_n \rightarrow S_n' \). Suppose we have a compatible family of \( G_n \rightarrow S_n \) torsors \( p_i : U_i \rightarrow \mathcal{X}_i \) with \( U_i \) affine. This gives a 2-cartesian diagram

\[
\begin{array}{ccccccc}
U_0 & \rightarrow & \cdots & \rightarrow & U_{n-1} & \rightarrow & j_n \rightarrow U_n \\
| & & & | & & & | \\
p_0 & & p_{n-1} & & j_n & & p_n \\
\mathcal{X}_0' & \rightarrow & \cdots & \rightarrow & \mathcal{X}_{n-1}' & \rightarrow & i_n \rightarrow \mathcal{X}_n'
\end{array}
\]

of Artin stacks over \( S_n' \). By Corollary 5.1, the obstruction to the existence of a \( G_x \)-torsor \( p_n : U_n \rightarrow \mathcal{X}'_n \) restricting to \( p_{n-1} : U_{n-1} \rightarrow \mathcal{X}'_{n-1} \) is an element

\[ o \in \text{Ext}^1(Lf^*L_{BG_n/S_n}(\mathcal{I}^n), \mathcal{I}^n/\mathcal{I}^{n+1}) = 0 \]

where \( f : \mathcal{X}'_{n-1} \rightarrow BG_x \) is the morphism defined by \( U_{n-1} \rightarrow \mathcal{X}_{n-1} \). Since the set of extensions is \( H^1(BG_x, \mathcal{I}^n/\mathcal{I}^{n+1}) = 0 \), there is a unique extension \( p_n : U_n \rightarrow \mathcal{X}'_n \).

**Corollary 5.3.** Let \( \mathcal{X} \) be a locally noetherian Artin stack over \( \text{Spec} k \) and \( \xi \in |\mathcal{X}| \) be a closed point with linearly reductive stabilizer with representative \( x : \text{Spec} k \rightarrow \mathcal{X} \) of \( \xi \). Then there exists a miniversal deformation \( (A, \xi) \) of \( x \) with \( G_x \)-action. If \( G_x \) is smooth, then this is unique up to \( G_x \)-invariant isomorphism.

**Proof.** The first statement follows directly from the above theorem with the observation that \( \varprojlim U_i \rightarrow \mathcal{X} \) is a miniversal deformation. □
Remark 5.4. The action of $G_x$ on $\text{Spf} \, A$ fixes the maximal ideal so we get an induced algebraic action of $G_x$ on $(m/m)^\vee$. The miniversality of $\xi$ gives an identification of $k$-vector spaces $\Psi : (m/m^2)^\vee \cong F_x(k[\epsilon])$ which we claim is $G_x$-equivariant.

The map $\Psi$ is defined as follows: if $\tau : \text{Spec} \, k[\epsilon] \to \text{Spec} \, A/m^2$ there is an induced diagram

\[
\begin{array}{ccc}
x & \longrightarrow & \tau^* \xi_1 \\
\downarrow & & \downarrow \\
\xi_1 & \rightarrow & \text{Spec} \, k[\epsilon] \quad \text{over} \quad \text{Spec} \, A/m^2
\end{array}
\]

then $\Psi(\tau) = (x \to \tau^* \xi_1)$. The action of $G_x$ on $F_x(k[\epsilon])$ is given in Section 4.2 Under the identification $(m/m^2)^\vee \cong F_{U,u}(k[\epsilon])$ where $U = \text{Spec} \, A/m^2$ and $u : \text{Spec} \, k \to U$ is the closed point, then $G_x$-action on $F_{U,u}(k[\epsilon])$ can be given explicitly: If $p : \text{Spec} \, B \to \text{Spec} \, k$, then an element $g \in G_x(R)$ gives a $B$-algebra isomorphism $\alpha_g : A/m^2 \otimes_k B \to A/m^2 \otimes_k B$ and an element $\sigma \in F_{U,u}(k[\epsilon])$ corresponds to a $B$-module homomorphism $A/m^2 \otimes_k B$ and $g \cdot \sigma \in F_{U,u}(k[\epsilon])$ is the $B$-module homomorphism corresponding to the composition $A/m^2 \otimes_k B \xrightarrow{\alpha_g^{-1}} A/m^2 \otimes_k B \to B$.

We also note that if $p : \text{Spec} \, B \to \text{Spec} \, k$, then under the isomorphisms given in Section 4.2 we have a commutative diagram

\[
\begin{array}{ccc}
\overline{F}_{U,u}(k[\epsilon]) \otimes_k B & \xrightarrow{\sim} & \overline{F}_{U,p^* u}(B[\epsilon]) \\
\downarrow \Psi \otimes_k B & & \downarrow \Psi_B \\
\overline{F}_{X,x}(k[\epsilon]) \otimes_k B & \xrightarrow{\sim} & \overline{F}_{X,p^* x}(B[\epsilon])
\end{array}
\]

where for $(\tau : \text{Spec} \, B[\epsilon] \to U) \in \overline{F}_{U,p^* u}(B[\epsilon])$, $\Psi_B(\tau) = (p^* x \to \tau^* \xi_1)$.

For $g \in G_x(B)$ and $(\tau : \text{Spec} \, B[\epsilon] \to U) \in \overline{F}_{U,u}(B[\epsilon])$, the pullback of the cocycle $\phi : \sigma^* \xi_1 \to p^*_2 \sigma$ (defining the $G_x$-action on $\xi_1$) under the morphism $(g, \text{id}) : \text{Spec} \, B \times_k U \to G_x \times_k U$ gives an arrow $\beta$ making a commutative diagram

\[
\begin{array}{ccc}
p^* x & \xrightarrow{g} & p^* x \\
\downarrow \beta & & \downarrow \\
p^*_2 \xi_1 & \rightarrow & p^*_2 \xi_1 \quad \text{over} \quad \text{Spec} \, B \times_k U \xrightarrow{\alpha_g} \text{Spec} \, B \times_k U
\end{array}
\]

We have a commutative diagram

\[
\begin{array}{ccc}
p^* x & \xrightarrow{g} & p^* x \\
\downarrow \beta & & \downarrow \\
\tau^* \xi_1 & \rightarrow & (\alpha_g \circ \tau)^* \xi_1 \quad \text{over} \quad \text{Spec} \, B \times_k U \xrightarrow{\alpha_g} \text{Spec} \, B \times_k U
\end{array}
\]
where $\gamma : \tau^*\xi_1 \to (g \circ \tau)^*\xi_1$ is the unique arrow making the bottom square commute. The arrow $\gamma$ identifies $g \cdot \Psi(\tau) = (p^*x \xrightarrow{\alpha_g^{-1}} p^*x \to \tau^*\xi_1)$ and $\Psi(g \cdot \tau) = (x \to (\alpha_g \circ \tau)^*\xi_1)$.

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