ON DISTINCT UNIT GENERATED FIELDS THAT ARE TOTALLY COMPLEX

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Abstract. We consider the problem of characterizing all number fields \( K \) such that all algebraic integers \( \alpha \in K \) can be written as the sum of distinct units of \( K \). We extend a method due to Thuswaldner and Ziegler [9] that previously did not work for totally complex fields and apply our results to the case of totally complex quartic number fields.

1. Introduction

Jacobson [6] observed in the 1960’s that the two number fields \( \mathbb{Q}(\sqrt{2}) \) and \( \mathbb{Q}(\sqrt{5}) \) share the property that every algebraic integer is the sum of distinct units. Moreover, he conjectured that these two quadratic number fields are the only quadratic number fields with this property. Let us call a field with this property a distinct unit generated field or DUG-field for short.

In the 1970’s Śliwa [8] solved this problem for quadratic number fields and showed that even no pure cubic number field is DUG. These results have been extended to cubic and quartic fields by Belcher [1, 2]. In particular, Belcher solved the case of imaginary cubic number fields completely [2].

The problem of characterizing all number fields in which every algebraic integer is a sum of distinct units is still unsolved. Let us note that this problem is contained in Narkiewicz’ list of open problems in his famous book [7, see page 539, Problem 18].

Recently methods originating from the theory of number systems and enumeration Thuswaldner and Ziegler [9] obtained a new approach to the problem and introduced the following definition in order to measure how far is a number field away from being a DUG-field.

Definition 1. Let \( \mathfrak{o} \) be some order in a number field \( K \) and \( \alpha \in \mathfrak{o} \). Suppose \( \alpha \) can be written as a linear combination of units

\[
\alpha = a_1\epsilon_1 + \cdots + a_\ell\epsilon_\ell,
\]

such that \( \epsilon_1, \ldots, \epsilon_\ell \in \mathfrak{o}^* \) are all distinct and \( a_1 \geq \cdots \geq a_\ell > 0 \) are positive integers. Choose a representation with \( a_1 \) minimal, then we call \( \omega(\alpha) = a_1 \) the unit sum height of \( \alpha \). Moreover we define \( \omega(0) = 0 \) and \( \omega(\alpha) = \infty \) if \( \alpha \) is not the sum of units.

We define

\[
\omega(\mathfrak{o}) = \max\{\omega(\alpha) : \alpha \in \mathfrak{o}\}
\]

if the maximum exists. If it does not exist we write \( \omega(\mathfrak{o}) = \omega \) in case of \( \mathfrak{o} \) is generated by units and \( \omega(\mathfrak{o}) = \infty \) otherwise.

In case of \( \mathfrak{o} \) is the maximal order of \( K \) we also write \( \omega(K) = \omega(\mathfrak{o}) \).

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Unfortunately the method of Thuswaldner and Ziegler [9] only works for number fields which are not totally complex. However, Hajdu and Ziegler [5] proved the following theorem concerning totally complex quartic fields:

**Theorem 1** (Hajdu, Ziegler [5]). Let $\zeta_\mu$ be a primitive $\mu$-th root of unity. If $K$ is a totally complex quartic field with $\omega(K) = 1$, then it is of the form

- $\mathbb{Q}(\zeta_\mu)$ where $\mu = 5, 8, 12$ or,
- $\mathbb{Q}(\gamma)$ where $\gamma$ is the root of one of the polynomials $X^4 - X + 1$, $X^4 + X^2 - X + 1$, $X^4 + 2X^2 - 2X + 1$, $X^4 - X^3 + X + 1$, $X^4 - X^3 + X^2 + X + 1$, $X^4 - X^3 + 2X^2 - X + 2$ or,
- $\mathbb{Q}(\sqrt{a + b\zeta_4})$, with $(a, b) = (1, 1), (1, 2), (1, 4), (7, 4)$ or,
- $\mathbb{Q}(\sqrt{a + b\zeta_3})$, with $(a, b) = (2, 1), (4, 1), (8, 1), (3, 2), (4, 3), (7, 3), (11, 3), (5, 4), (9, 4), (13, 4), (12, 5), (11, 7), (9, 8), (15, 11), (19, 11), (17, 12), (17, 16)$ or,
- $\mathbb{Q}(\zeta_4, \sqrt{5})$ or $\mathbb{Q}(\zeta_3, \sqrt{d})$, with $d = 5, 6, 21$ or,
- $\mathbb{Q}\left(\sqrt{-1 - \sqrt{2}}\right)$ or $\mathbb{Q}\left(\sqrt{-1 + \sqrt{5}}\right)$.

Note that Hajdu and Ziegler [5] could not prove that all these fields listed above are DUG-fields. They only succeeded to do so for the fields $K = \mathbb{Q}(\gamma)$, where

$$\gamma \in \left\{ \alpha, \zeta_5, \zeta_8, \zeta_{12}, \sqrt{-1 - \sqrt{2}}, \sqrt{-\frac{1 + \sqrt{5}}{2}}, \zeta_3 + \sqrt{5}, \zeta_4 + \sqrt{5} \right\}$$

and $\alpha$ is a root of the polynomial $X^4 + X^2 - X + 1$ using similar techniques as Belcher [2]. Based on a large computer search Hajdu and Ziegler [5] conjecture that all the fields in Theorem 1 are indeed DUG. However, for all the remaining fields Hajdu and Ziegler could not even provide a bound for $\omega(K)$.

The aim of this paper is to extend the method of Thuswaldner and Ziegler [9] to totally complex number fields and to apply this method to extend the list of fields in Theorem 1 where $\omega(K) = 1$ is confirmed. Unfortunately we failed in proving that all fields listed in Theorem 1 are distinct unit generated, but at least we can provide upper bounds for the unit sum height.

**Theorem 2.** Let $\zeta_\mu$ be a primitive $\mu$-th root of unity. If $K$ is a totally complex quartic field and if it is of the form

- $\mathbb{Q}(\zeta_\mu)$ where $\mu = 5, 8, 12$ or,
- $\mathbb{Q}(\gamma)$ where $\gamma$ is the root of one of the polynomials $X^4 - X + 1$, $X^4 + X^2 - X + 1$, $X^4 + 2X^2 - 2X + 1$, $X^4 - X^3 + X + 1$, $X^4 - X^3 + X^2 + X + 1$, $X^4 - X^3 + 2X^2 - X + 2$ or,
- $\mathbb{Q}(\sqrt{a + b\zeta_4})$, with $(a, b) = (1, 1), (1, 2), (1, 4), (7, 4)$ or,
- $\mathbb{Q}(\sqrt{a + b\zeta_3})$, with $(a, b) = (2, 1), (4, 1), (8, 1), (3, 2), (4, 3), (7, 3), (11, 3), (5, 4), (9, 4), (13, 4), (12, 5), (11, 7), (9, 8), (15, 11), (19, 11), (17, 12), (17, 16)$ or,
- $\mathbb{Q}(\zeta_4, \sqrt{5})$ or $\mathbb{Q}(\zeta_3, \sqrt{d})$, with $d = 5, 6, 21$ or,
- $\mathbb{Q}\left(\sqrt{-1 - \sqrt{2}}\right)$ or $\mathbb{Q}\left(\sqrt{-1 + \sqrt{5}}\right)$.

then $\omega(K) \leq 3$. Moreover all the fields listed above are DUG except those marked with $\dagger$ or $\ddagger$. Those fields marked with $\dagger$ satisfy at least $\omega(K) \leq 2$ and those marked with $\ddagger$ satisfy only $\omega(K) \leq 3$. 

In the next section we generalize a theorem due to Thuswaldner and Ziegler [9, Theorem 2.1] to the case of totally complex number fields. With this generalization at hand we consider the case that a totally complex number field \( K \) contains a primitive \( \mu \)-th root of unity with \( \mu > 2 \). This enables us to prove Theorem 2 up to the second item in the list of fields given there. In Section 4 we apply a variant of our method to the remaining fields and prove Theorem 2 up to the case that \( K = \mathbb{Q}(\gamma) \), where \( \gamma \) is a root of \( X^4 - X + 1 \). This special case is solved in the last section of the paper by a combinatorial approach.

2. Computing upper bounds for the unit sum height

Before we state and prove our main tool (see Theorem 3 below) to compute upper bounds for the unit sum height we have to introduce some notations.

Let \( A \subset \mathbb{C} \) be a set and \( \alpha \in \mathbb{C} \) an element. As commonly used we write \( \alpha + A = \{ \alpha + a : a \in A \} \) and \( \alpha A = \{ \alpha a : a \in A \} \).

Let \( K \) be a totally complex number field of degree 2, signature \((0, s)\) and let \( \mathfrak{o} \) be an order of \( K \). Also let us fix the complex embeddings \( \sigma_1 = \overline{\sigma}_{s+1}, \ldots, \sigma_s = \overline{\sigma}_{2s} \) of \( K \). For \( \alpha \in K \) we denote by \( \alpha^{(i)} = \sigma_i \alpha \) the Galois conjugates of \( \alpha \) and by convention we write \( \alpha = \alpha^{(1)} \).

Let \( \epsilon \in \mathfrak{o}^* \) be such that \( |\epsilon| > 1 \) and \( |\epsilon^{(i)}| < 1 \) for all \( i = 2, \ldots, s \). Let \( \Sigma \subset \mathfrak{o} \) be finite and denote
\[
C_i := \max\{|c^{(i)}| : c \in \Sigma\}, \quad \text{for } i = 2, \ldots, s.
\]

Consider a compact set \( P \subset \mathbb{C} \), containing at least a neighborhood of 0 and denote by \( B(\epsilon, \Sigma, P) \) the cylinder defined by
\[
B(\epsilon, \Sigma, P) := \left\{ \alpha \in \mathfrak{o} : \alpha \in P \text{ and } |\alpha^{(i)}| \leq \frac{C_i}{1 - |\epsilon^{(i)}|} \text{ for } i = 2, \ldots, s \right\}.
\]

Note that since the lattice \( \Lambda_{\mathfrak{o}} := \{(\alpha^{(i)})_{1 \leq i \leq s} : \alpha \in \mathfrak{o}\} \subset \mathbb{C}^s \) is discrete, the set \( B(\epsilon, \Sigma, P) \) is finite.

Now we have all the notations to state the main result of this section:

**Theorem 3.** Let us assume that
\[
\epsilon P \subset \bigcup_{s \in \Sigma} (s + P).
\]

Then for each \( \alpha \in \mathfrak{o} \) there exist \( N, n \in \mathbb{N} \) such that
\[
\alpha \epsilon^N = \beta + \sum_{i=0}^{n} c_i \epsilon^i,
\]

with \( c_i \in \Sigma \) and \( \beta \) is contained in the finite set \( B(\epsilon, \Sigma, P) \). The elements of \( B(\epsilon, \Sigma, P) \setminus \{0\} \) will be called critical points.

In order to prove Theorem 3 the following lemma is useful.

**Lemma 1.** Let \( x \in \mathbb{C} \) and assume that (1) holds. Then there exists \( n \in \mathbb{N} \) and \( c_0, \ldots, c_n \in \Sigma \) such that
\[
x - \sum_{j=0}^{n} c_j \epsilon^j \in P.
\]
Proof. Let \( n \geq -1 \) be an integer such that \( x \in \epsilon^{n+1}P \). Such an integer \( n \) exists since \( P \) contains a neighborhood of 0. We prove the Lemma by induction on \( n \). The case \( n = -1 \) is trivial.

Now let us assume the lemma is proved for all integers \( M \leq n \) and assume that \( x \in \epsilon^{n+1}P \). Since by assumption \( x \in \epsilon^{n+1}P \subset \bigcup_{s \in \Sigma} (se^n + \epsilon^n P) \), there exists some \( s = c_n \in \Sigma \) with \( x - se^n \in \epsilon^n P \). Since we assume by induction that the lemma is true for all \( M \leq n \) we know that there exist \( c_0, \ldots, c_{n-1} \in \Sigma \) such that

\[
\begin{align*}
x - sc^n = -\sum_{j=0}^{n-1} c_j \epsilon^j &= x - \sum_{j=0}^{n} c_j \epsilon^j \in P.
\end{align*}
\]

With Lemma \( \blacksquare \) at hand, we are ready to prove Theorem 3.

Proof of Theorem 3. Let \( \alpha \in \mathfrak{o} \) be arbitrary. Since \( |\epsilon| > 1 \) and for all other conjugates we have \( |\epsilon(i)| < 1 \), there exists for each \( \delta > 0 \) a non-negative integer \( N \) such that

\[
\left|\alpha(i)(\epsilon(i))^N - \sum_{j=0}^{n} c_j \epsilon^j\right| < \delta \quad \text{for} \quad 2 \leq i \leq s.
\]

Since \( \Lambda_o \) is a discrete set and since we assume that \( P \) is compact there exists a \( \delta_0 \) such that for every \( 0 < \delta < \delta_0 \) the conditions (3) and (4) imply \( \beta \in \mathcal{B}(\epsilon, \Sigma, P) \). 

3. APPLICATION TO FIELDS WITH A FOURTH OR SIXTH ROOT OF UNITY

Assume that \( K \) contains a \( \mu \)-th root of unity with \( \mu > 2 \) and denote by \( \zeta_\mu \) some primitive \( \mu \)-th root of unity. We may assume that \( \mu \) is even. Indeed if \( \mu \) is odd then with \( \zeta_\mu \) also \( -\zeta_\mu = \zeta_\mu^2 \) is an element of \( K \). The role of the digit set \( \Sigma \) will be taken by the set of all possible sums of roots of unity with bounded coefficients. Therefore we write

\[
\Sigma = \Sigma_\mu(w) := \left\{ \sum_{i=1}^{\mu} d_i \zeta_\mu^i : 0 \leq d_i \leq w \quad \text{for} \quad 1 \leq i \leq \mu \right\}.
\]

First, let us assume that \( K \) is a totally complex (not necessarily quartic) field that contains a fourth root of unity. We apply Theorem 3 to the case where \( P \subset \mathbb{C} \) is the square with vertices \( \pm \frac{1+\sqrt{3}}{2} \) and obtain a simple criterion such that the covering property (1) holds:
Lemma 2. Let \( P \subset \mathbb{C} \) be the square with vertices \( \pm \frac{1+i}{2} \). Let \( \eta = \epsilon \frac{1+i}{2} \) then \((1)\) is satisfied, provided
\[
\max\{|\Re(\eta)|, |\Im(\eta)|\} \leq \frac{1 + 2w}{2}.
\]
Proof. Note that
\[
\bigcup_{s \in \Sigma_w(\epsilon)} s + P = (1 + 2w)P.
\]
Since \( P \) is convex it suffices to prove that all the vertices of \( \epsilon P \) lie within the square \((1 + 2w)P\), i.e.
\[
\max\{|\Re(\eta)|, |\Im(\eta)|\} \leq \frac{1 + 2w}{2}.
\]
\(\Box\)

In view of Theorem 1 we want to apply Lemma 2 together with Theorem 3 to the fields \( \mathbb{Q}(\sqrt{1 + \zeta_4}), \mathbb{Q}(\sqrt{1 + 2\zeta_4}), \mathbb{Q}(\sqrt{1 + 4\zeta_4}) \) and \( \mathbb{Q}(\sqrt{7 + 4\zeta_4}) \). Since the computations in all cases are similar we only give details for the case \( K = \mathbb{Q}(\sqrt{1 + \zeta_4}) \). For some details in the other cases see Table 1 below.

Let us discuss the case that \( K = \mathbb{Q}(\sqrt{1 + \zeta_4}) \) and \( \sigma \) is the maximal order of \( K \). Let us write \( \gamma = \sqrt{1 + \zeta_4} \) and choose a branch of the logarithm such that \( \Re(\gamma) > 0 \). The fundamental unit is \( \epsilon = 1 + \gamma \) and
\[
\epsilon \frac{1+i}{2} = \eta \simeq 0.822 + 1.277i.
\]
Due to Lemma 2 we may apply Theorem 3 with \( w = 1 \) and we obtain the critical points
\[
B(\epsilon, \Sigma, P) = \left\{ \alpha \in \sigma : |\Re(\alpha)|, |\Im(\alpha)| \leq \frac{1}{2}, |\alpha^{(2)}| \leq 2.647 \right\}
\]
which are exactly \( \zeta_4^k(1 - \gamma) \), with \( k = 0, 1, 2, 3 \). But, these critical points are \( \zeta_4^k \epsilon^{-1} \) with \( k = 0, 1, 2, 3 \) written in terms of \( \epsilon^{-1} \). Since all the critical points can be written as the sum of distinct units such that the exponent of \( \epsilon \) is negative we deduce from Theorem 3 that each algebraic integer of \( K \) is the sum of distinct units, i.e. \( K \) is DUG.

In the other cases the critical points are less obvious. But by a computer search we were able to confirm that all critical points can be written in the form \( \sum_{k=-1}^B s_i \epsilon^i \) with \( s_i \in \Sigma_4(w) \). Further, let us denote by \( C \) the number of critical points.

| \( K \) | \( w \) | \( C \) | \( B \) | \( K \) | \( w \) | \( C \) | \( B \) |
|---|---|---|---|---|---|---|---|
| \( \mathbb{Q}(\sqrt{1 + \zeta_4}) \) | 1 | 4 | 1 | \( \mathbb{Q}(\sqrt{1 + 4\zeta_4}) \) | 1 | 16 | 2 |
| \( \mathbb{Q}(\sqrt{1 + 2\zeta_4}) \) | 1 | 8 | 2 | \( \mathbb{Q}(\sqrt{7 + 4\zeta_4}) \) | 2 | 8 | 2 |

Remark 1. It is rather plausible that another choice of \( P \) might yield \( w = 1 \) in the case that \( K = \mathbb{Q}(\sqrt{1 + 4\zeta_4}) \). The best choice for \( P \) seems to be the unique compact set \( P \) which satisfies
\[
P = \bigcup_{s \in \Sigma} f_s(P),
\]
where \( f_s(x) = s + \frac{x}{\sqrt{3}} \) for all \( s \in \Sigma_4(1) \) (e.g. see [3, Theorem 9.1]). Obviously this set \( P \) is compact and satisfies condition (1) of Theorem 3. Unfortunately this iterated function system does not fulfill the so called “open set condition” (see e.g. [3, page 118]) and therefore we are unable to show that \( P \) contains a neighborhood of 0, which is essential in the proof of Theorem 3.

Now let us assume that \( K \) is a totally complex (not necessarily quartic) field that contains sixth roots of unity. In this case we choose \( P \) to be a regular hexagon.

**Lemma 3.** Let \( P \subset \mathbb{C} \) be the hexagon with vertices \( v_k = \frac{1}{\sqrt{3}} \exp \left( \frac{2\pi i}{12} \right) \) and \( k = 0, \ldots , 5 \). Let \( \eta_k = \epsilon v_k \) then (1) is satisfied, provided

\[
\max_{0 \leq k \leq 5} \{| \Im(\eta_k) | \} \leq \frac{5w + 2}{2\sqrt{3}}.
\]

**Proof.** Note that

\[
\bigcup_{s \in \Sigma_6(w)} s + P \supset \exp(\pi i/6) \frac{5w + 2}{\sqrt{3}} P.
\]

Therefore it suffices to prove that all the vertices of \( \epsilon P \) lie within the hexagon \( \frac{5w + 2}{2\sqrt{3}} \exp(\pi i/6) P \), i.e.

\[
\max_k \{| \Im(\eta_k) | \} \leq \frac{5w + 2}{2\sqrt{3}}.
\]

For a better illustration see Figure 1, where the case that \( w = 1 \) is shown. The black hexagons are translations of \( P \) by all possible \( s \in \Sigma_6(1) \). The gray hexagon is the hexagon \( \exp(\pi i/6) \frac{5w + 2}{\sqrt{3}} P = \exp(\pi i/6) \frac{7\sqrt{3}}{2} P \).

We proceed as described in the case that \( K \) contains a fourth root of unity. Since the computations are similar to those made in the case that \( \mu = 4 \) we only give a few details (see Table 2 below).

**Table 2.** Details to the computations in case that \( K \) contains sixth roots of unity

| \( K \)       | \( w \) | \( \mathcal{C} \) | \( B \) | \( K \)       | \( w \) | \( \mathcal{C} \) | \( B \) |
|---------------|---------|------------------|-------|---------------|---------|------------------|-------|
| \( \mathbb{Q}(\sqrt{2} + \zeta_3) \) | 1       | 6                | 1     | \( \mathbb{Q}(\sqrt{12} + 5\zeta_3) \) | 1       | 6                | 1     |
| \( \mathbb{Q}(\sqrt{4} + \zeta_3) \) | 1       | 67               | 3     | \( \mathbb{Q}(\sqrt{11} + 7\zeta_3) \) | 1       | 0                |       |
| \( \mathbb{Q}(\sqrt{8} + \zeta_3) \) | 1       | 6                | 1     | \( \mathbb{Q}(\sqrt{9} + 8\zeta_3) \) | 1       | 6                | 1     |
| \( \mathbb{Q}(\sqrt{3} + 2\zeta_3) \) | 1       | 0                |       | \( \mathbb{Q}(\sqrt{15} + 11\zeta_3) \) | 1       | 0                |       |
| \( \mathbb{Q}(\sqrt{4} + 3\zeta_3) \) | 1       | 0                |       | \( \mathbb{Q}(\sqrt{19} + 11\zeta_3) \) | 2       | 6                | 1     |
| \( \mathbb{Q}(\sqrt{7} + 3\zeta_3) \) | 1       | 6                | 1     | \( \mathbb{Q}(\sqrt{17} + 12\zeta_3) \) | 2       | 6                | 1     |
| \( \mathbb{Q}(\sqrt{11} + 3\zeta_3) \) | 1       | 0                |       | \( \mathbb{Q}(\sqrt{17} + 16\zeta_3) \) | 2       | 6                | 1     |
| \( \mathbb{Q}(\sqrt{5} + 4\zeta_3) \) | 1       | 24               | 2     | \( \mathbb{Q}(\sqrt{5}, \sqrt{6}) \) | 1       | 0                |       |
| \( \mathbb{Q}(\sqrt{9} + 4\zeta_3) \) | 1       | 6                | 1     | \( \mathbb{Q}(\zeta_3, \sqrt{21}) \) | 1       | 6                | 1     |
| \( \mathbb{Q}(\sqrt{13} + 4\zeta_3) \) | 1       | 0                |       |               |         |                  |       |

**Remark 2.** Figure 1 shows that Lemma 3 is not optimal and it may happen that \( \epsilon P \subset \bigcup_{s \in \Sigma_6(w)} s + P \) holds but condition (5) in Lemma 3 fails. Although this seems to be rather unlikely, we checked in case that \( K \) is one of the fields \( \mathbb{Q}(\sqrt{19} + 11\zeta_3) \),
Remark 3. In case that $\mu > 6$ it is not hard to get criteria which are similar to the criteria in Lemmas 2 and 3 such that the covering property (1) holds. As we can already see in the case that $\mu = 6$ such results are either not best possible or not very simple. So in view of Theorem 2 we abandon to discuss criteria for $\mu > 6$.

Remark 4. We want to note that in case of $\mu = 8$ and $K = \mathbb{Q}(\zeta_8)$ Theorem 2 yields a new proof that $\mathbb{Q}(\zeta_8)$ is indeed DUG. Indeed choose $P$ to be the square with vertices $\frac{\pm 1 \pm i}{2}$. Then it is easy to show that

$$\epsilon P = (1 + \sqrt{2})P \subset \bigcup_{s \in \Sigma_8(1)} (s + P).$$

Since the critical points in this case are $\frac{\pm 2^{k+1}}{e}$ for $k = 0, 1, 2, 3$ we see that $K$ is indeed DUG.
4. Five special cases

Now we consider the remaining five number fields in Theorem 2 namely those which do not contain any roots of unity \( \zeta_{\mu} \) for \( \mu > 2 \). The same approach as in the previous section will not lead to success, since the alphabet \( \Sigma_{\mu}(w) = \{ -w, \ldots, 0, \ldots, w \} \) is contained in the real line. Instead, we take the digit set \( \Sigma = \{ d_0 + d_1 \epsilon : -w \leq d_0, d_1 \leq w \} \), and expand the number \( \alpha \in \mathbf{e} \) in base \( \epsilon = \epsilon^2 \), where \( \epsilon \) is a fundamental unit with \( |\epsilon| > 1 \). The compact set \( P \subseteq \mathbb{C} \) is taken to be the parallelogram with vertices \( \frac{\pm 1 + \epsilon}{2} \).

**Lemma 4.** Let \( P \subseteq \mathbb{C} \) be the parallelogram with vertices \( \frac{\pm 1 + \epsilon}{2} \). Let

\[
A = \begin{pmatrix} 1 \text{Re}(\epsilon) & 0 \\ \text{Re}(\epsilon) & \text{Im}(\epsilon) \end{pmatrix}, \quad a_1 = \text{Re}(\frac{\epsilon^2}{2}(1 + \epsilon)), \quad b_1 = \text{Im}(\frac{\epsilon^2}{2}(1 + \epsilon)), \\
a_2 = \text{Re}(\frac{\epsilon^2}{2}(1 - \epsilon)), \quad b_2 = \text{Im}(\frac{\epsilon^2}{2}(1 - \epsilon)).
\]

Then (1) is satisfied, provided

\[
\max_{k=1,2} \left| (1,0)A^{-1}(a_k) \right|, \left| (0,1)A^{-1}(a_k) \right| \leq \frac{1 + 2w}{2}.
\]

**Proof.** Note that

\[
\bigcup_{s \in \Sigma} (s + P) = (1 + 2w)P.
\]

Therefore it suffices to prove that all the vertices of \( \epsilon P \), namely \( \frac{\epsilon^2}{2}(1 \pm \epsilon) \), lie within the parallelogram \( (1 + 2w)P \). In order to check this, it is convenient to consider 1, \( \epsilon \) as a basis of \( \mathbb{C} \) over \( \mathbb{R} \) instead of 1, \( i \). If \( A \) is as above, we have \( z = a + bi = c + d\epsilon \), where \((\begin{array}{c} a \\ b \end{array}) = A^{-1}(a_k)\). Hence \( z \) lies within \( (1 + 2w)P \), if \( |c|, |d| \leq \frac{1 + 2w}{2} \). \( \square \)

Let us note that the bound for \( w \) obtained in Lemma 3 depends on which embedding \( K \hookrightarrow \mathbb{C} \) we chose. For instance in the case that \( K \) is the number field with minimal polynomial \( X^4 + 2X^2 - 2X + 1 \) one obtains either \( w = 2 \) or \( w = 4 \) depending on the choice of the actual embedding \( K \hookrightarrow \mathbb{C} \). However since the unit sum height \( \omega(K) \) does not depend on the embedding we can choose \( \epsilon \) such that in view of Lemma 4 the quantity \( w \) is minimal.

Once we have chosen the optimal embedding \( K \hookrightarrow \mathbb{C} \) we can proceed as before and we only give a few details on the applications of Lemma 4 and Theorem 3. In particular, see Table 3 below for details.

**Table 3.** Details to the computations in case \( K \) has the following minimal polynomial.

| minimal polynomial | \( w \) | \( \epsilon \) | \( \mathcal{C} \) | \( B \) |
|-------------------|------|--------|-------|-----|
| \( X^4 - X + 1 \) | 2    | 0.727 + 0.934i | 112   | 7   |
| \( X^4 - X^3 + X^2 + X + 1 \) | 3    | -0.933 + 1.132i | 42    | 4   |
| \( X^4 - X^3 + X + 1 \) | 3    | -1.066 + 0.864i | 56    | 4   |
| \( X^4 + 2X^2 - 2X + 1 \) | 2    | 0.475 + 1.509i | 18    | 4   |
| \( X^4 - X^3 + 2X^2 - X + 2 \) | 2    | 0.204 + 1.664i | 12    | 3   |
5. A combinatorial approach

The aim of this section is to prove that \( K = \mathbb{Q}(\gamma) \) is DUG, where \( \gamma \) is a root of the polynomial \( X^4 - X + 1 \). Although we already proved in the previous section that \( \omega(K) \leq 2 \) we do not assume this result in this section. Independently from the rest of the paper we prove:

**Proposition 1.** The field \( K = \mathbb{Q}(\gamma) \) with \( \gamma \) being a root of the polynomial \( X^4 - X + 1 \) is DUG.

Since the maximal order \( \mathfrak{o} \) of \( K \) is of the form \( \mathfrak{o} = \mathbb{Z}[\gamma] = \mathbb{Z}[\gamma, \gamma^{-1}] \) we can write every element \( \alpha \in \mathfrak{o} \) in the form

\[
\alpha = \sum_{n=-\infty}^{\infty} v_n \gamma^n
\]

with \( v_n \in \mathbb{Z} \) and \( v_n \neq 0 \) for at most finitely many indices. Such a \( \gamma \)-representation of \( \alpha \) is sometimes written

\[
\alpha = \cdots v_2 v_1 v_0 \bullet v_{-1} v_{-2} \cdots ,
\]

where the fractional point \( \bullet \) separates between the coefficients at negative and non-negative powers of the base \( \gamma \). We are only interested in the fact that non-vanishing coefficients in the \( \gamma \)-representation are finitely many. Thus we will abbreviate representation (6) by the finite word

\[
v_k v_{k-1} \cdots v_{\ell+1} v_{\ell}, \quad \text{where the indices } k \text{ and } \ell \text{ are such that } v_n = 0 \text{ for all } n > k \text{ and all } n < \ell, \text{ without marking the fractional point. Note that the } \gamma \text{-representation is not unique. Since } \gamma^n (\gamma^4 - \gamma + 1) = 0 \text{ for all } n, \text{ position-wise addition or subtraction of } 100\overline{11}, \text{ with } \overline{1} = -1, \text{ at any position does not change the value of } \alpha \text{ but only its } \gamma \text{-representation, i.e. the words } v_k \cdots v_n v_{n-1} v_{n-2} v_{n-3} v_{n+4} \cdots v_{\ell} \text{ and } v_k \cdots (v_n+1) v_{n-1} v_{n-2} (v_{n-3} - 1)(v_{n+4} + 1) \cdots v_{\ell} \text{ represent the same element } \alpha \in \mathfrak{o}.

From this point of view any element \( \alpha \in \mathfrak{o} \) has some \( \gamma \)-representation

\[
x_{3}x_{2}x_{1}x_{0}, \quad x_i \in \mathbb{Z}
\]

and if \( \alpha \) is also a sum of distinct units, there exists another \( \gamma \)-representation of the form

\[
v_k v_{k-1} \cdots v_{0} v_{-1} \cdots v_{\ell}, \quad v_i \in \{1, 0, 1\}, \ell, k \leq \mathbb{Z}.
\]

Hence, if we want to prove that the field \( K \) is DUG, we have to show that any representation of the form (7) can be rewritten into (8) without changing the value of the represented number.

**Definition 2.** Let \( A \subseteq \mathbb{Z} \) be an alphabet and let \( w \in A^* \) be a finite word. We say that the word \( u \in A^* \) can be rewritten by \( w \) to \( v \in A^* \), if it is possible to obtain \( v \) from \( u \) by finitely many position-wise additions or subtractions of shifts of \( w \). We denote this by \( u \leftrightarrow_w v \) or just \( u \leftrightarrow v \), if \( w \) is understood. In this context we call \( w \) the rewriting rule.

Moreover let \( v = v_k \cdots v_{\ell} \in A^* \) be a finite word, then we denote by

\[
W(v) = \sum_{n=\ell}^{k} |v_n|
\]

the weight of \( v \).
Let us note that the symbol $\mathbb{Z}^*$ bears some ambiguity. It may be the set of finite words with alphabet $\mathbb{Z}$ or it may denote the set $\{\pm 1\}$, which is the group of units of $\mathbb{Z}$. Since from the context the meaning of $\mathbb{Z}^*$ is always clear in this paper we allow this ambiguity.

In view of Proposition 1 let us fix $w = 100\overline{1}1$. If $u$ and $v$ are $\gamma$-representations with $u \leftrightarrow_w v$, then $u$ and $v$ represent the same element $\alpha \in \mathfrak{a}$, as explained above. Hence Proposition 1 is equivalent to the following:

**Proposition 2.** For every word $x_3x_2x_1x_0 \in \mathbb{Z}^*$ there exists a word $v \in \{\overline{1}, 0, 1\}^*$ such that

$$x_3x_2x_1x_0 \leftrightarrow_w v.$$

Note that for two digits $a$ and $b$, we denote by $ab$ their concatenation and by $a \cdot b$ standard multiplication. In order to prove Proposition 2 we need the following lemma.

**Lemma 5.** Let $u \in \mathbb{Z}^*$ be a finite word. Then $u \leftrightarrow_w v = v_kv_{k-1}\cdots v_{\ell+1}v_\ell$, where $v$ fulfills the following conditions:

1. $v_i \in \{0, 1, 2\}$ for all $i \in \{k, \ldots, \ell\}$
2. $v_{i+m} \cdot v_i > 0 \Rightarrow m \notin \{1, 2, 4\}$
3. $v_{i+m} \cdot v_i < 0 \Rightarrow m \notin \{1, 3\}$
4. $v_{i+2} \cdot v_i < 0 \Rightarrow v_{i+4} = v_{i+5} = 0$
5. $v_{i+3} \cdot v_i > 0 \Rightarrow v_{i+6} = 0$

**Proof.** Multiple application of the original rewriting rule $w = w_1 = 100\overline{1}1$ gives rise to other useful ones, in particular $w_2 = 1000000030000001$, $w_3 = 10010010\overline{1}1$ and $w_4 = 111000001$. We prove the lemma by showing that these rewriting rules can be used in such a manner, that they decrease the weight $W(u)$ of the word $u$ in every rewriting step until $u$ satisfies the conditions of the lemma. We apply the rewriting rules $w_1, w_2, w_3$ or $w_4$ in the following situations (the underlined digits indicate which digits we want to “reduce” in order to obtain a smaller weight):

(a) If $|v_i| \geq 3$, then

$$v_{i+7}v_{i+6}\cdots v_{i+5}v_{i-6} \leftrightarrow_w (v_{i+7} \pm 1)v_{i+6}\cdots (v_i \pm 3)v_{i-5}(v_{i-6} \pm 1).$$

(b) If $v_{i+1} \cdot v_i < 0$, then

$$v_{i+4}v_{i+3}v_{i+2}v_{i+1}v_i \leftrightarrow_w (v_{i+4} \mp 1)v_{i+3}v_{i+2}(v_{i+1} \pm 1)(v_i \mp 1).$$

(c) If $v_{i+3} \cdot v_i < 0$, then

$$v_{i+3}v_{i+2}v_{i+1}v_i \leftrightarrow_w (v_{i+3} \pm 1)v_{i+2}v_{i+1}(v_i \mp 1)(v_{i-1} \pm 1).$$

(d) If $v_{i+4} \cdot v_i > 0$, then

$$v_{i+4}v_{i+3}v_{i+2}v_{i+1}v_i \leftrightarrow_w (v_{i+4} \pm 1)v_{i+3}v_{i+2}(v_{i+1} \mp 1)(v_i \pm 1).$$

(e) If $v_{i+2} \cdot v_i < 0$ and $v_{i+5} \cdot v_i < 0$, then

$$v_{i+9}\cdots v_{i+5}v_{i+4}v_{i+3}v_{i+2}v_{i+1}v_i \leftrightarrow_w (v_{i+9} \mp 1)\cdots (v_{i+5} \pm 1)v_{i+4}v_{i+3}(v_{i+2} \pm 1)v_{i+1}(v_i \mp 1)(v_{i-1} \pm 1).$$
Moreover the digits of $v$ are contained in a factor $\alpha$. If an application of (g) does not decrease the weight of the word, then also the occurrences of digits $\pm 2$ in a neighborhood around $v$ will be achieved by reading the word $v$ from left to right and rewriting some of its occurrences.

Claim 1. Let $v = v_k \ldots v_1$ be a word satisfying the conditions of Lemma\cite{5}. Then $v \leftrightarrow_{10011} v'$ with $v' \in \{1, 0, \overline{1}\}^*$ and the factor 020, resp. 010, which is at the right most position $i$, is rewritten into 011, 110 or 010, (0\overline{1}1, 110 or 0\overline{1}0 respectively). Moreover the digits of $v$ and $v'$ are equal for all indices $< i - 1$.

We prove this claim by induction on the number $N$ of appearances of the digits $\pm 2$. Of course the case $N = 0$ is trivial. For each $i$, with $v_i = \pm 2$ we define $\Delta_i = +\infty$ if it is the left most occurrence of the digits $\pm 2$ in $v$ and

$$\Delta_i = \min\{j > 0 : v_{i+j} = \pm 2\}$$
otherwise. Without loss of generality, assume that \( v_i = 2 \). First, consider \( \Delta_i \geq 6 \). In this case we derive from (i)–(v) that we only have four cases which can be rewritten as indicated below.

| \( u_{i+4}v_{i+3}v_{i+2} \) | rewriting \( u_2 \) | output |
|-------------------------|----------------|--------|
| 001020                  | 101110         |        |
| 010020                  | 110110         |        |
| 000020                  | 1100110        |        |
| \( \mathbf{T}00020 \)  | 1100011        |        |

(9)

Note that this also settles the case that \( N = 1 \).

Now let us assume that \( N \geq 2 \) and that the claim is true for all words with strictly less than \( N \) appearances of the digits \( \pm 2 \). Let us assume that \( i \) is the lowest index such that \( |v_i| = 2 \).

If \( \Delta_i \geq 6 \), we split \( v = u_2u_1 \) into the two words \( u_2 = v_k \ldots v_{i+6}v_{i+5} \) and \( u_1 = v_{i+4}v_{i+3} \ldots v_{k} \). Since \( u_2 \) has \( N - 1 \) appearances of the digits \( \pm 2 \) we have by induction \( v = u_2u_1 \leftrightarrow u'_2u_1 \) with \( u'_2 \in \{1, 0, 1\}^* \). Now applying (9) we obtain \( u_1 \leftrightarrow u'_1 \) with \( u'_1 \in \{1, 0, 1\}^* \), hence \( v = u_2u_1 \leftrightarrow u'_2u'_1 = v' \) with \( v' \in \{1, 0, 1\}^* \).

If \( \Delta_i = 5 \) we split up \( v = u_2u_1 \) into the two words \( u_2 = v_k \ldots v_{i+5}v_{i+4} \) and \( u_1 = v_{i+3}v_{i+2} \ldots v_{k} \). By induction \( u_2 \leftrightarrow u'_2 = v'_k \ldots v'_{i+4} \) and \( u'_2 \in \{1, 0, 1\}^* \) is a word ending with \( \pm (011), \pm (110) \) or \( \pm (010) \). Now the following computations settle the case:

| \( (v_{i+6})v_{i+5}v_{i+4}v_{i+3}v_{i+2} \) | rewriting \( u_2 \) | final output |
|-------------------------------|----------------|----------------|
| 2000020                        | 1000020        | 1100011        |
| 1100020                        | 1100011        | 1100011        |
| \( \mathbf{T}000020 \)        | 1100011        | 1100011        |
| \( (0)2010020 \)              | 1101020        | 1110011        |
| \( 0)1010020 \)               | 1110011        | 1110011        |

(10)

The case that \( \Delta_i = 4 \) runs analogously. We split up \( v = u_2u_1 \) into two words \( u_2 = v_k \ldots v_{i+4}v_{i+3} \) and \( u_1 = v_{i+2}v_{i+1} \ldots v_{k} \) and compute

| \( v_{i+4}v_{i+3}v_{i+2} \) | rewriting \( u_2 \) | final output |
|----------------------------|----------------|----------------|
| 2000020                     | 1000020        | 1100011        |
| \( \mathbf{T}000020 \)      | 1100011        | 1100011        |

(11)

Now let us examine the case that \( \Delta_i = 3 \). We split up \( v = u_2u_1 \) into two words \( u_2 = v_k \ldots v_{i+3}v_{i+2} \) and \( u_1 = v_{i+1}v_{i} \ldots v_{k} \) and get

| \( v_{i+4}v_{i+3}v_{i+2} \) | rewriting \( u_2 \) | final output |
|----------------------------|----------------|----------------|
| 020020                      | 011020         | 111110         |
| 110020                      | 0100110        | 1100110        |
| 010020                      | 110110         | 110110         |

(12)

We are left with the case \( \Delta_i = 2 \). In this case we split up \( v = u_2u_1 \) into \( u_2 = v_k \ldots v_{i+2}v_{i+1} \) and \( u_1 = v_{i+1}v_{i} \ldots v_{k} \). Further, this case implies that \( \Delta_{i+2} \geq 4 \) and by (iv) also \( v_{i+4} = v_{i+5} = 0 \), i.e. \( u_2 \) ends in 00020. Looking at the possible rewritings of such words from (9), (10) and (11) we obtain the following cases:

| \( v_{i+4}v_{i+3}v_{i+2} \) | rewriting \( u_2 \) | final output |
|----------------------------|----------------|----------------|
| 002020                      | 001120         | 101010         |
| 001020                      | 110110         | 110110         |
| 011020                      | 111110         | 111110         |
| \( \mathbf{T}01020 \)      | 111110         | 111110         |

(13)
Therefore the proof of the claim and hence the proof of Proposition 2 is complete.

\[\square\]

Remark 5. The method used in the proof of Proposition 1 is very particular for the field \(K = \mathbb{Q}(\gamma)\), where \(\gamma\) is a root of the polynomial \(X^4 - X + 1\), which provided us rewriting rules \(w\) with low weight but large support. We failed in proving an analogous result to Lemma 5 for the remaining cases of Theorem 2, since the corresponding fields seem not to provide such rewriting rules.

Let us mention that the possibility to rewrite any finite word with integer digits into the alphabet \(\mathcal{A} = \{-1, 0, 1\}\) is closely connected to the so-called finiteness property of numeration systems. In particular, we study whether

\[
\mathbb{Z}[\gamma, \gamma^{-1}] = \text{Fin}_{\mathcal{A}}(\gamma) := \left\{ \sum_{i=1}^{k} a_i \gamma^i : k, l \in \mathbb{Z}, a_i \in \mathcal{A} \right\},
\]

which will be true, if \(\text{Fin}_{\mathcal{A}}(\gamma)\) is closed under addition. This is in general a very hard question and only few results are known. For example, in [4], it was shown that for any algebraic integer \(\gamma\) without conjugates on the unit circle there exists an alphabet \(\mathcal{A}\) of consecutive integers, such that \(\text{Fin}_{\mathcal{A}}(\gamma)\) is closed under addition. This is however very far from stating that \(\mathcal{A} = \{-1, 0, 1\}\) is sufficient.

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