COMPACTNESS OF THE SPACE OF MINIMAL HYPERSURFACES WITH BOUNDED VOLUME AND $p$–TH JACOBI EIGENVALUE

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Abstract. Given a closed Riemannian manifold of dimension less than eight, we prove a compactness result for the space of closed, embedded minimal hypersurfaces satisfying a volume bound and a uniform lower bound on the first eigenvalue of the stability operator. When the latter assumption is replaced by a uniform lower bound on the $p$–th Jacobi eigenvalue for $p \geq 2$ one gains strong convergence to a smooth limit submanifold away from at most $p - 1$ points.

1. Introduction

Let $(N^{n+1}, g)$ be a closed Riemannian manifold and let us denote by $\mathfrak{M}^\kappa(N)$ the class of closed, smooth and embedded minimal hypersurfaces $M \subset N$. By the seminal work of Almgren-Pitts [7] (and Schoen-Simon [8]) we know that such a set $\mathfrak{M}^\kappa(N)$ is not empty whenever $2 \leq n \leq 6$, the higher-dimensional counterpart of their method being obstructed by the occurrence of singularities of mass-minimizing currents. Over the last three decades, several existence results have been proven by means of equivariant constructions, desingularization, gluing and more recently, high-dimensional min-max techniques that ensure that in many cases of natural geometric interest the set $\mathfrak{M}^\kappa(N)$ contains plenty of elements. Most remarkably, it was proven by Marques and Neves in [6] that when $2 \leq n \leq 6$ and the Ricci curvature of $g$ is positive, then $N$ contains at least countably many closed, embedded minimal hypersurfaces. Thus, one is led to investigate the global structure of the class $\mathfrak{M}^\kappa(N)$ and the most basic question in this sense is perhaps that of finding geometrically natural and meaningful conditions that ensure the compactness of subsets of this space. In the three-dimensional scenario, namely for $n = 2$, and under an assumption on the positivity of the ambient Ricci curvature a prototypical statement was obtained, in 1985, by Choi and Schoen:

Theorem 1.1. [3] Let $N$ be a compact 3-dimensional manifold with positive Ricci curvature. Then the space of compact embedded minimal surfaces of fixed topological type in $N$ is compact in the $C^k$ topology for any $k \geq 2$. Furthermore, if $N$ is real analytic, then this space is a compact finite-dimensional real analytic variety.

Roughly speaking, the idea behind this result is that a uniform bound on the genus suffices for controlling both the area (as had already been observed in [4]) and the second fundamental form of the minimal surfaces in question, in a uniform fashion.

When $n \geq 3$ new phenomena appear and a statement of this type cannot possibly be expected. Indeed, it was shown by Hsiang [5] that in $S^{n+1}, n = 3, 4, 5$ there exists a sequence of embedded minimal hyperspheres $M^n$ that have uniformly bounded volume and converge, in the sense of varifolds, to a singular minimal subvariety with two conical singularities located at antipodal points of the ambient manifold. Based on the seminal works of

1Throughout this paper, we shall always tacitly assume all hypersurfaces to be connected.
Schoen-Simon-Yau [9] and Schoen-Simon [8] concerning stable minimal hypersurfaces, one is naturally led to conjecture that some sort of control on the spectrum of the Jacobi operator (together with a volume bound) should indeed suffice to obtain compactness. A first result of this flavour, that holds true up to (and including) ambient dimension seven, was recently proven by the third-named author.

**Theorem 1.2.** Let $N^{n+1}$ be a smooth, closed Riemannian manifold with $\text{Ric}_N > 0$ and $2 \leq n \leq 6$. Then given any $0 < \Lambda < \infty$ and $I \in \mathbb{N}$ the class

$$\mathcal{I}(\Lambda, I) := \{M \in \mathcal{M}^n(N) : \mathcal{H}^n(M) \leq \Lambda, \text{ index}(M) \leq I\}$$

is compact in the $C^k$ topology for all $k \geq 2$.

Here and above the word compactness is understood as single-sheeted graphical convergence to some limit $M \in \mathcal{I}(\Lambda, I)$. As the reader can see, in analogy with Theorem 1.1, here one does also need to assume positivity of the ambient Ricci curvature to derive a compactness theorem, for otherwise smooth, graphical convergence can only be ensured away from at most $I$ points (cmp. Theorem 2.3 in [11]).

In this article, we derive the rather surprising conclusion that no assumption on the ambient manifold is needed in proving a strong convergence theorem provided an upper bound on the Morse index is replaced by a lower bound on the first eigenvalue of the Jacobi operator. To state our result, we need to recall a definition: in the setting described above, we will say that $M_k \to M$ in the sense of smooth graphs at $p \in M$ if there exists $\rho > 0, \eta > 0$ such that in normal coordinates centered at $p$ the intersection of $M_k$ with $B^p_\rho(0) \times B^0_\eta(0)$ consists, for $k$ large enough, of the collection of the graphs of smooth defining functions $u^1_k, \ldots, u^l_k$ with $u^j_k \to 0$ in $C^m$ for all $m \geq 2$ and $1 \leq j \leq l$. We remark that if $M_k \to M$ in the sense of smooth graphs away from a finite set $\mathcal{Y}$ and $M$ is connected (and embedded) then the number of leaves of the convergence is constant.

**Theorem 1.3.** Let $2 \leq n \leq 6$ and $N^{n+1}$ a smooth, closed Riemannian manifold. Denote by $\mathcal{M}^n(N)$ the class of closed, smooth and embedded minimal hypersurfaces $M \subset N$. Let $\lambda_p(M)$ denote the $p$-th eigenvalue of the Jacobi operator for $M \in \mathcal{M}^n(N)$. Given any $0 < \Lambda < \infty$ and $0 \leq \mu < \infty$, define the class

$$\mathcal{M}_p(\Lambda, \mu) := \{M \in \mathcal{M}^n(N) : \mathcal{H}^n(M) \leq \Lambda, \lambda_p(M) \geq -\mu\}.$$ 

Given a sequence $\{M_k\} \subset \mathcal{M}_p(\Lambda, \mu)$ there exists $M \in \mathcal{M}_p(\Lambda, \mu)$ such that $M_k \to M$ in the varifold sense and furthermore:

1. if $p = 1$ then $M_k \to M$ locally in the sense of smooth graphs;
2. if $p \geq 2$ then there exists a finite set $\mathcal{Y} = \{y_i\}_{i=1}^P$ with $P \leq p - 1$ such that the convergence $M_k \to M$ is smooth and graphical for all $x \in M \setminus \mathcal{Y}$; if the number of leaves of the convergence is one then $\mathcal{Y} = \emptyset$.

From such general assertion we can derive a strong compactness result under a purely topological assumption on the ambient manifold $N$.

**Corollary 1.4.** Let $2 \leq n \leq 6$ and $N^{n+1}$ a smooth, closed Riemannian manifold not containing any one-sided minimal hypersurface (which holds true, for instance, if $N$ is simply connected). Then given any $0 < \Lambda < \infty$ and $0 \leq \mu < \infty$ the class $\mathcal{M}_1(\Lambda, \mu)$ is compact in the $C^k$ topology for all $k \geq 2$. 


Indeed, in such a scenario it is of course the case that the limit hypersurface $M$ (whose existence is ensured by Theorem 1.3) is itself two-sided and thus for $k$ large enough $M_k$ is a finite covering thereof, hence the conclusion comes via an elementary topological argument due to the connectedness assumption on $M_k$.

In particular, we deduce from this statement that (in presence of a volume bound) the Morse index of an element in $M_1(\Lambda, \mu)$ is uniformly bounded from above, which seems a rather unexpected conclusion from a purely analytic viewpoint, as we are considering an infinite family of elliptic operators parametrized by minimal hypersurfaces in $M_1(\Lambda, \mu)$. On the other hand, we know that a bound on index does not give a bound on $\lambda_1$ (since even with bounded index we could have non-smooth convergence: to see this one only needs to consider a sequence of catenoids $M_k = \frac{1}{k} M$ in $\mathbb{R}^n$, all centred at the origin, and blowing down to a double plane. One gets that eventually the catenoid has index one in the unit ball (when it is scaled down sufficiently far, and only considering compactly supported variations), moreover that $\lambda_1(M_k \cap B(0)) \to -\infty$.) In particular, given a sequence with only volume and index bounds, we must have that $\lambda_1 \to -\infty$ in general - moreover the gaps between the eigenvalues must also diverge (since the index is bounded).

Combining Theorem 1.3 with Remark 3.2 we deduce the following:

**Corollary 1.5.** Let $N^{n+1}$ be a smooth, closed Riemannian manifold with $\text{Ric}_N > 0$ and $2 \leq n \leq 6$. Then given any $0 < \Lambda < \infty$ and $0 \leq \mu < \infty$ and $p \geq 1$ the class $M_p(\Lambda, \mu)$ is compact in the $C^k$ topology for all $k \geq 2$.

From a different perspective, Theorem 1.3 gives us an interesting description of what goes wrong when absence of a smooth limit occurs for a family $\{M_k\}$ satisfying a uniform volume bound: necessarily, every eigenvalue of the Jacobi operator has to diverge to $-\infty$, which somehow captures the well-known picture of $M_k$ exhibiting some neck-pinching around finitely many points.

**Corollary 1.6.** Let $\{M_k\} \subset M^n(N)$ be a sequence satisfying a uniform volume bound, so that possibly by extracting a subsequence we know [10] that $M_k \to M$ for some stationary, integral varifold $M$ in $N$.

1. If $M$ is not smooth, then $\lambda_p(M_k) \to -\infty$ as $k \to \infty$ for every $p \geq 1$.
2. If $M$ has multiplicity greater than one, then $\lambda_p(M_k) \to -\infty$ as $k \to \infty$ for every $p \geq 1$ provided $\text{Ric}_N > 0$.
3. If we denote by $\mathcal{Y}$ the set of points of $M$ where the convergence $M_k \to M$ is not smooth and graphical, then $\lambda_p(M_k) \to -\infty$ for all $1 \leq p \leq |\mathcal{Y}|$. In particular, if $|\mathcal{Y}| = \infty$ then $\lambda_p(M_k) \to -\infty$ for all $p \geq 1$.

For instance: this applies to the aforementioned Hsiang minimal hyperspheres [5] (since they have a non-smooth limit) or more generally to the hyperspheres produced in [2]. We further remark that the positivity assumption on the Ricci curvature of $N$ in item (2) above is essential because of the example described in Remark 3.3.

When $n = 2$, the scenario we obtain by combining Theorem 1.1 with Theorem 1.2 and Theorem 1.3 is rather enlightening.

**Corollary 1.7.** Let $\mathcal{C}^n \subset M^n$ be a subclass of closed minimal hypersurfaces inside some smooth closed Riemannian manifold $N^{n+1}$ of dimension $2 \leq n \leq 6$ satisfying $\text{Ric}_N > 0$. Then a uniform bound on any one of the following quantities for every $M \in \mathcal{C}^n$ leads to a bound on the rest of them for every $M \in \mathcal{C}^n$:
- the genus of $M$ (when $n = 2$)
- index$(M) + \mathcal{H}^n(M)$
- $\lambda_p(M) + \mathcal{H}^n(M)$
- $\sup_M |A| + \mathcal{H}^n(M)$
- $\int_M |A|^n + \mathcal{H}^n(M)$.

Lastly, our main theorem extends (with minor variations in the proof) to the case when $N$ is a complete (not necessarily compact) Riemannian manifold, provided one replaces, in the statement, the set $\mathcal{M}_p(\Lambda, \mu)$ by the set

$$\mathcal{M}_p^\Omega(\Lambda, \mu) := \{ M \in \mathcal{M}^n(N) : M \subset \Omega, \mathcal{H}^n(M) \leq \Lambda, \lambda_p(M) \geq -\mu \}$$

for some open, bounded domain $\Omega$. In various situations of great geometric interest one can in fact drop the requirement that the minimal hypersurfaces in question are contained in a given, bounded domain.

**Remark 1.8.** The conclusions of [13] also hold true when

1. $(N^{n+1}, g)$ is a compact Riemannian manifold with mean-convex boundary;
2. $(N^{n+1}, g)$ is a complete Riemannian manifold such that for some compact set $K$ each component of $M \setminus K$ is foliated by closed, mean-convex leaves (in particular: asymptotically flat, asymptotically cylindrical and asymptotically hyperbolic manifolds).

The proof follows along the same lines of Theorem 1.3 modulo exploiting a geometric maximum principle (see, for instance, [13]) in order to reduce our compactness analysis to a bounded domain of the manifold $N$.

## 2. Preliminaries

We shall recall here the definition of the Morse index and the Jacobi eigenvalues $\lambda_p$ for general smooth minimal hypersurfaces $M \hookrightarrow N$. First of all, if $M$ is orientable then the second variation of the area functional can be written down purely in terms of sections of the normal bundle $v \in \Gamma(NM)$ by

$$Q(v, v) := \int_M |\nabla v|^2 - |A|^2|v|^2 - \text{Ric}_N(v, v).$$

Standard results on the spectra of compact self-adjoint operators on separable Hilbert spaces tell us that there is an orthonormal basis $\{v_i\}_{i=1}^\infty$ of $L^2(\Gamma(TN))$ consisting of eigenfunctions for the operator

$$L^\perp v := \Delta^\perp v + |A|^2v + \text{Ric}_N^\perp(v)$$

with associated eigenvalues $\{\lambda_i\}_{i=1}^\infty$ of $Q$. Moreover, we have the following Rayleigh characterization of the eigenvalues due to R. Courant:

$$\lambda_k := \inf_{\dim(V) = k} \max_{v \in V} \frac{Q(v, v)}{\int_M |v|^2}$$

where of course $V$ is a linear subspace of $\Gamma(NM)$.

Now, if $M$ is non-orientable then we simply lift the problem to its orientable double cover $\tilde{M}$ via $\pi : \tilde{M} \to M$. Consider the linear subspace of smooth sections $v \in \Gamma(\pi^*NM)$ such that $v \circ \tau = v$ where $\tau : \tilde{M} \to \tilde{M}$ is the unique deck transformation of $\pi$ which reverses orientation. Denote this subspace by $\tilde{\Gamma}(\pi^*NM)$. We can also pull back the quantities
|A(x)|^2 := |A(\pi(x))|^2 and \( Ric_N(a, b) := Ric_N(\pi_\ast a, \pi_\ast b) \). Thus consider the quadratic form
\[ \tilde{Q}(v, v) := \int_M |\nabla^2 v|^2 - |A|^2 |v|^2 - Ric_N(v, v) \]
over \(\tilde{\Gamma}(\pi^\ast NM)\). As before we can define the spectrum, and therefore index of \(M\) to be that of \(\tilde{M}\) with respect to \(\tilde{Q}\) and \(\tilde{\Gamma}(\pi^\ast NM)\).

If the ambient manifold \(N\) is orientable, a closed hypersurface is orientable if and only if it is two-sided, namely if there exists a global section \(\nu = \nu_M\) of its normal bundle inside \(TN\). If this is the case (namely if \(M \hookrightarrow N\) is minimal and two-sided) the spectrum defined above patently coincides with the spectrum of the \emph{scalar} Jacobi or stability operator of \(M\), namely
\[ Lu := \Delta_M u + (Ric_N(\nu, \nu) + |A|^2)u \]
when we regard \(L : W^{1,2}(M) \to W^{-1,2}(M)\).

Furthermore, we shall introduce the following notation: given a minimal hypersurface \(M\) in the Riemannian manifold \((N, g)\) and a bounded open domain \(\Omega \subset M\) we shall set
\[ \lambda_1^M(\Omega) = \inf \left\{ - \int_M vL^1 v \mid v \in C^\infty_0(\Omega; NM) \text{ and } \int_M |v|^2 = 1 \right\} \]
where \(C^\infty_0(\Omega; NM)\) denotes the (smooth) sections of the normal bundle whose support, projected on the base \(M\) is relatively compact in \(\Omega\).

3. Proofs

For the sake of conceptual clarity, we will separate the proof of Theorem 1.3 in the case \(p = 1\) and \(p \geq 2\), the former being a building block for the latter.

Proof of Theorem 1.3 case \(p = 1\). Let \(\{M_k\} \subset \mathcal{M}_1(\Lambda, \mu)\) be a sequence of closed, embedded minimal hypersurfaces satisfying our bounds on the volume and first Jacobi eigenvalue: we claim the existence of a constant \(C = C(N, \Lambda, \mu) > 0\) such that
\[ \sup_{k \geq 1} \sup_{z \in M_k} |A_k(z)| \leq C \]
where \(A_k\) denotes the second fundamental form of \(M_k\) in \((N, g)\). For the sake of a contradiction, let us assume instead that such a uniform curvature bound does not hold. Then, we could find (for every \(k \geq 1\)) a sequence of points \(\{z_k\} \subset N\) such that \(A_k\) attains its maximum value at \(z_k \in M_k\) and, furthermore, \(\lim_{k \to \infty} |A_k(z_k)| = +\infty\). Thanks to the compactness of the ambient manifold \(N\), possibly by extracting a subsequence (which we shall not rename) we can assume that \(z_k \to y\) for some point \(y \in N\). Let us pick, once and for all, a small radius \(r_0 > 0\) (less than the injectivity radius of \((N, g)\) at \(y\)) and let us denote by \(\{x\}\) a system of geodesic normal coordinates centered at \(y\) and by \(g_{ij}(x)\) the corresponding components of the Riemannian metric \(g\). For \(k\) large enough we know that \(M_k \cap B_{r_0/2}(z_k) \subset B_{r_0}(y)\) and we can assume, without loss of generality, that \(M_k \cap B_{r_0}(y)\) is two-sided. We can then consider the blown-up hypersurfaces defined by
\[ \hat{M}_k := |A_k(z_k)|(M_k - z_k) \]
and the appropriately rescaled Riemannian metrics on \(\hat{B}_k := B_{r_0|A_k(z_k)|/2}(0) \subset \mathbb{R}^{n+1}\)
\[ \hat{g}_k(x) := g \left( z_k + \frac{x}{|A_k(z_k)|} \right). \]
(For the sake of clarity we have identified, in the equation above, the hypersurface $M_k$ with its portion in $B_{r_0/2}(z_k)$). Now, the hypersurface $\hat{M}_k$ is minimal in metric $\hat{g}_k$ and patently satisfies volume and curvature bounds, for $M_k \in M_1(\Lambda, \mu)$ implies
\[
\frac{H^n(M_k \cap B_r(0))}{r^n} \leq \Lambda, \text{ for any } r < \frac{r_0|A_k(z_k)|}{2}
\]
and by scaling
\[
\sup_{x \in B_k} |\hat{A}_k(x)| \leq 1, \quad \hat{A}_k(0) = 1
\]
where $\hat{A}_k$ denotes the second fundamental form of $\hat{M}_k$ in metric $\hat{g}_k$. It follows that the sequence $\{\hat{M}_k\}$ converges (for any $m$ in $C^m_{\text{loc}}$, in the sense of smooth graphs) to a complete, embedded minimal hypersurface $\hat{M}_\infty \subset \mathbb{R}^{n+1}$ (in flat Euclidean metric). We further claim that $\hat{M}_\infty$ has to be stable. If not, we could find a smooth, compactly supported vector field $u$ such that the second variation
\[
\left[\frac{d^2}{dt^2}\right]_{t=0} H^n((\varphi_t)_\# \hat{M}_\infty) < 0
\]
where $\{\varphi_t\}$ is the flow of diffeomorphisms generated by $u$ (which coincides with the identity outside of a compact set). As a result, thanks to the locally strong convergence $\hat{M}_k \to \hat{M}_\infty$ we would have
\[
\left[\frac{d^2}{dt^2}\right]_{t=0} H^n((\varphi_t)_\# \hat{M}_k) < 0
\]
for all indices $k$ that are large enough and, more specifically, for a fixed open, bounded set $\Omega \subset \mathbb{R}^{n+1}$ we would have $\lambda^M_1(\Omega) \leq -\varepsilon$ for some $\varepsilon > 0$. Therefore, scaling back and keeping in mind the Rayleigh characterization of the eigenvalues of an elliptic operator we must conclude that
\[
-\mu \leq \lambda^M_1(\Omega) \leq -\varepsilon|A_k(z_k)|,
\]
which is impossible when $k$ attains sufficiently large values. It follows that $\hat{M}_\infty$ is a stable minimal hypersurface in $\mathbb{R}^{n+1}$ (with polynomial volume growth), hence an affine hyperplane by the work of Schoen-Simon [8] and thus on the one hand $\hat{A}_\infty$ vanishes identically, while on the other $\hat{A}_\infty(0) = 1$ and this contradiction completes the proof of our initial claim. Once those uniform curvature estimates are gained, the strong convergence of $M_k \to M$ follows from a geometric counterpart of the Arzelà-Ascoli compactness theorem, and the fact that in this case the volume and first Jacobi eigenvalue of $M$ are also controlled, namely $M \in M_1(\Lambda, \mu)$ is also clear. \hfill \Box

Proof of Theorem 1.3 case $p \geq 2$. We shall start by stating the following important:

**Lemma 3.1.** Let $\Omega_1, \Omega_2, \ldots, \Omega_p \subset N$ be $p$ pairwise disjoint, bounded open sets. If we assume $M \in M_p(\Lambda, \mu)$ and $M \cap \Omega_i \neq \emptyset$ for $i = 1, \ldots, p$ then there exists an index $i_0$ such that $\lambda^M_1(\Omega_{i_0}) \geq -\mu$.

Indeed, suppose that were not the case: then we could find, for each index $i = 1, \ldots, p$ a section $\phi_i \in C_0^\infty(\Omega_i; NM)$ that ensures $\lambda^M_1(\Omega_i) < -\mu$ (that is to say $\int_M |\phi_i|^2 = 1$ and $Q(\phi_i, \phi_i) < -\mu$) and thus
\[
Q(\phi, \phi) < -\mu \text{ for all } \phi \in W = (\phi_1, \ldots, \phi_p)_R \text{ such that } \int_M |\phi|^2 = 1
\]
which contradicts the assumption that $\lambda_\mu(M) \geq -\mu$ due to the well-known min-max characterization of the eigenvalues of an elliptic operator. (In case $M$ is not orientable, one
needs to consider the space \( \hat{W} = \langle \hat{\phi}_1, \ldots, \hat{\phi}_p \rangle \) where each \( \hat{\phi}_i \) is the lift of \( \phi_i \) to \( \hat{M} \) and the quadratic form \( Q \) is evaluated on \( \hat{M} \) as explained in Section 2.

Now, let a sequence \( \{M_k\} \subset M_p(\Lambda, \mu) \) be given: thanks to the volume bound, we know that possibly by taking a subsequence (which we shall not rename) \( M_k \to V \) in the sense of varifolds, for some integral varifold \( V \). Furthermore, given \( \varepsilon > 0 \) the Lemma 3.1 we have just stated and a standard covering argument ensure that there exists a set \( \mathcal{Y} = \{y_1\}_{i=1}^p \subset \text{spt}(V) \) consisting of at most \( p - 1 \) points and a subsequence \( \{M_{l(k)}\} \) such that in \( N \setminus \mathcal{Y} \) the hypersurfaces \( M_{l(k)} \) locally converge to \( V \) strongly in the sense of smooth graphs. This descends from the fact that for any \( z \in N \setminus \mathcal{Y} \) and \( \varepsilon > 0 \) the sequence \( M_{l(k)} \) satisfies a uniform bound on the first Jacobi eigenvalue, which in turn implies

\[
\sup_{k \geq 1} \sup_{x \in B_{\varepsilon}(z)} |A_{l(k)}(x)| \leq C = C(N, \Lambda, \mu)
\]

by following the argument that has been used to prove Theorem 1.3 in the case \( p = 1 \). Here \( A_{l(k)} \) stands for the second fundamental form of \( M_{l(k)} \) in the ambient manifold \( (N, g) \). In particular, this implies that the varifold \( V \) is supported on a smooth submanifold \( M \) away from finitely many points, namely those points belonging to the set \( \mathcal{Y} \).

We further claim that for any \( y \in \mathcal{Y} \) there exists \( \varepsilon_0 > 0 \) such that

\[
\lambda_1^M(B_{\varepsilon_0}(y) \setminus \{y\}) \geq -\mu \quad (*)
\]

If this claim were false, then we could find a smooth, normal vector field \( u_0 \), compactly supported in \( B_{\varepsilon_0}(y) \setminus \{y\} \) and hence (say) supported in \( B_{\varepsilon_0}(y) \setminus B_{\varepsilon_1}(y) \) for some \( 0 < \varepsilon_1 < \varepsilon_0 \) such that

\[
\frac{Q(u_0, u_0)}{\int_M |u_0|^2} < -\mu.
\]

At that stage, we shall observe that it cannot be \( \lambda_1^M(B_{\varepsilon_1}(y) \setminus \{y\}) \geq -\mu \) either (for otherwise we would have gained property \((*)\) with \( \varepsilon_1 \) in lieu of \( \varepsilon_0 \)) and hence, again, there is a smooth vector field \( u_1 \) that is compactly supported in \( B_{\varepsilon_1}(y) \setminus \{y\} \) and hence (say) supported in \( B_{\varepsilon_1}(y) \setminus B_{\varepsilon_2}(y) \) for some \( 0 < \varepsilon_2 < \varepsilon_1 \) such that

\[
\frac{Q(u_1, u_1)}{\int_M |u_1|^2} < -\mu
\]

with the same notation as above. Of course, we can repeat this argument \( p \) times, hereby getting sections \( u_0, \ldots, u_{p-1} \) supported on smaller and smaller annuli, specifically \( u_j \) shall be supported on \( B_{\varepsilon_j}(y) \setminus B_{\varepsilon_{j+1}}(y) \) for \( 0 < \varepsilon_p < \ldots < \varepsilon_0 \). But we already know that \( M_{l(k)} \to M \) on (the closure of) \( B_{\varepsilon_j}(y) \setminus B_{\varepsilon_{j+1}}(y) \) for each \( j \leq p - 1 \) and thus we derive (for \( k \) large enough) the conclusion

\[
\lambda_1^{M_{l(k)}}(B_{\varepsilon_j}(y) \setminus B_{\varepsilon_{j+1}}(y)) < -\mu, \text{ for } j = 0, 1, \ldots, p - 1
\]

which contradicts our preliminary Lemma 3.1. This ensures the validity of \((*)\) for some suitable choice of \( \varepsilon_0 > 0 \). At that stage, let \( V^{(i)} \) for each fixed \( y_i \in \mathcal{Y} \) be a tangent cone to the varifold \( V \) at \( y_i \); necessarily \( V^{(i)} \) has to be stable, for otherwise we could scale back and argue as in the proof of Theorem 1.3 to show that the bound \((*)\) cannot possibly hold. As a result, \( V^{(i)} \) is a stable minimal hypercone in \( \mathbb{R}^{n+1} \) and hence an hyperplane for \( 2 \leq n \leq 6 \) due to the classic work of J. Simons [12]. Therefore \( V \) is regular (in fact, smooth) in a neighborhood of each point \( y_i \) thanks to Allard’s regularity theorem [1], and
we can conclude that its support $M$ is a smooth, minimal hypersurface in $(N, g)$, as we had to prove.

Varifold convergence directly implies that $\mathcal{H}^n(M) \leq \Lambda$, while the fact that $\lambda_p(M) \geq -\mu$ is more delicate as the set $\mathcal{Y}$ may not be empty. To that aim, we argue as follows. For small $r > 0$ let $\eta^{(j)}$ be a smooth non-negative function on $M$ that vanishes on the geodesic ball of radius $r$ centered at $y_j$ and equals one outside of the ball of radius $2r$. For the sake of contradiction, suppose there exists $p$ linearly independent (and orthonormal) sections in $W^{1,2}(M; NM)$ (in fact $W^{1,2}(M; \pi^*NM)$ when $M$ is not orientable), say $\phi_1, \ldots, \phi_p$ such that

$$\frac{Q(\phi, \phi)}{\int_M |\phi|^2} < -\mu \text{ on } V = \langle \phi_1, \ldots, \phi_p \rangle_\mathbb{R}$$

and set $\phi_\mu^r = \phi_1 \prod_{j=1}^p \eta^{(j)}_\mu$. The fact that points have zero capacity in $\mathbb{R}^n$ for any $n \geq 2$ implies that on the subspace $V_r = \langle \phi_1^r, \ldots, \phi_p^r \rangle_\mathbb{R}$ the inequality above must also hold for $r$ small enough, and hence (replacing each $\phi_i$ by $\phi_i^r$ and then applying the Gram-Schmidt process to the latter family, without further renaming) we can assume that the sections in question vanish on small geodesic neighborhoods of the points in the set $\mathcal{Y}$. Now, such vector fields can be extended to a tubular neighborhood of $M \hookrightarrow N$ (without renaming) and since each $M_k$ has to be contained in that neighborhood for $k$ large enough we can define sections $\phi_i^{r,k} \in \Gamma(M_k; NM_k)$ by projecting those extended vector fields onto the normal bundle of $M_k \hookrightarrow N$. The strong convergence of $M_k$ to $M$ away from the points in $\mathcal{Y}$ together with the assumption $M_k \in \mathcal{M}_p(\Lambda, \mu)$ implies that we can find real coefficients $\alpha_1^k, \ldots, \alpha_p^k$ such that

$$\sum_{i=1}^p \alpha_i^k \phi_i^{r,k} = 0.$$

Possibly by dividing the coefficients by $\max_i |\alpha_i^k|$ and renaming we can assume that $|\alpha_i^k| \leq 1$ for each index $i$ and $|\alpha_i^k| = 1$ for some index $i_0$. Hence, squaring the previous equation and integrating over $M_k$ we get

$$0 = \int_{M_k} \left| \sum_{i=1}^p \alpha_i^k \phi_i^{r,k} \right|^2 = \sum_{i,j} \alpha_i^k \alpha_j^k \int_{M_k} g(\phi_i^{r,k}, \phi_j^{r,k})$$

and by letting $k \to \infty$ the orhogonality of the family $\{\phi_1^r, \ldots, \phi_p^r\}$ implies that

$$\sum_{i=1}^p (\alpha_i)^2 = 0.$$

where (possibly by extracting a subsequence, which we shall not rename) $\alpha_i^k \to \alpha_i$ as $k \to \infty$ for each $i = 1, \ldots, p$. Thus $\alpha_i = 0$ for each $i$ but on the other hand (by construction) $\sum_{i=1}^p (\alpha_i)^2 \geq 1$, a contradiction. This proves that $M \in \mathcal{M}_p(\Lambda, \mu)$. Lastly, the fact that single-sheeted convergence implies $\mathcal{Y} = \emptyset$ is a direct consequence of Allard’s interior regularity theorem. \(\square\) Thereby, the proof is complete.

Remark 3.2. When the number of leaves in the convergence of $M_k$ to $M$ is known, then one can deduce further information about the limit hypersurface $M$. Specifically:

- if the number of sheets in the convergence is one
  - if $M$ is two-sided and $M_k \cap M = \emptyset$ eventually then $M$ is stable
  - if $M$ is two-sided and $M_k \cap M \neq \emptyset$ eventually then $\text{index}(M) \geq 1$
- if the number of sheets in the convergence is at least two
  - if $N$ has $\text{Ric}_N > 0$ then $M$ cannot be one-sided
– if $M$ is two-sided then $M$ is stable.

All of these statements follow from variations on the same argument, which consists of constructing a global section in the kernel of the Jacobi operator $L$ of $M$ (or a suitable lift, in the case of the third assertion) by appropriately renormalizing the distance function between $M$ and $M_k$ (first and second assertion) and two adjacent leaves of $M_k$ (third and fourth assertion). The reader can consult pages 10-13 of [11] for detailed arguments.

**Remark 3.3.** The assertion given in part (1) of Theorem 1.3 is sharp at that level of generality in the sense that one can provide explicit examples of Riemannian manifolds $(N, g)$ and sequences $\{M_k\} \subset M_1(\Lambda, \mu)$ such that $M_k \to M$ in the sense of smooth graphs, but with multiple leaves. For instance: let $(N^3, g)$ be gotten by taking the quotient of the product manifold $(S^2 \times \mathbb{R}, g_{\text{round}} \times dt^2)$ modulo the equivalence relation $(x, t) \sim (-x, -t)$. If we consider $\pi$ the associated Riemannian projection, then $\pi(S^2 \times \{t\})$ is a totally geodesic, stable minimal sphere for any $t \neq 0$ while $\pi(S^2 \times \{0\})$ is a stable minimal $\mathbb{RP}^2$ and for any sequence $t_k \downarrow 0$ we have that $\pi(S^2 \times \{t_k\}) \to \pi(S^2 \times \{0\})$ strongly in the sense of graphical, two-sheeted convergence.

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