On generalized Erdős–Ginzburg–Ziv constants for $\mathbb{Z}_d^2$

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Abstract

Let $G$ be a finite abelian group, and $r$ be a multiple of its exponent. The generalized Erdős–Ginzburg–Ziv constant $s_r(G)$ is the smallest integer $s$ such that every sequence of length $s$ over $G$ has a zero-sum subsequence of length $r$. We find exact values of $s_{2m}(\mathbb{Z}_d^2)$ for $d \leq 2m + 1$. Connections to linear binary codes of maximal length and codes without a forbidden weight are discussed.

Keywords: Erdős–Ginzburg–Ziv constant, zero-sum subsequence

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1. Introduction

Let $G$ be a finite abelian group written additively. We denote by $\text{exp}(G)$ the exponent of $G$ that is the least common multiple of the orders of its elements. Let $r$ be a multiple of $\text{exp}(G)$. The generalized Erdős–Ginzburg–Ziv constant $s_r(G)$ is the smallest integer $s$ such that every sequence of length $s$ over $G$ has a zero-sum subsequence of length $r$. If $r = \text{exp}(G)$, then $s(G) = s_{\text{exp}(G)}(G)$ is the classical Erdős–Ginzburg–Ziv constant.

In the case when $k$ is a power of a prime, Gao [12] proved $s_{km}(\mathbb{Z}_k^d) = km + (k - 1)d$ for $m \geq k^{d-1}$ and conjectured that the same equality holds when $km > (k - 1)d$.

In this paper, we consider the case $G = \mathbb{Z}_d^2$. We show that the problem of determining $s_{2m}(\mathbb{Z}_d^2)$ is essentially equivalent to finding the lowest redundancy of a linear binary code of given length which does not contain words of Hamming weight $2m$. When $m = 2$, this problem is also equivalent to finding the maximal length of a linear binary code of redundancy $d$ and distance 5 or higher.

We prove that $s_{2m}(\mathbb{Z}_2^d) = 2m + d$ for $d < 2m$, validating the Gao’s conjecture for $k = 2$. We also prove $s_{2m}(\mathbb{Z}_2^{2m}) = 4m + 1$, $s_{2m}(\mathbb{Z}_2^{2m+1}) = 4m + 2$ for even $m$, and $s_{2m}(\mathbb{Z}_2^{2m+1}) = 4m + 5$ for odd $m$. Our results provide counterexamples to Conjectures 4.4 and 4.6 from [10].

This paper is organized as follows. We discuss maximal length linear binary codes in Section 2, linear codes without a forbidden weight in Section 3, and generalized Erdős–Ginzburg–Ziv constants in Section 4. We present our results for $s_{2m}(\mathbb{Z}_2^d)$ in Section 5. Section 6 contains the proofs.
2. Linear binary codes of maximal length

In this section, we will provide basic definitions and some results from coding theory (for details, see [26]).

Let $F_2$ be the binary field and $F_2^n$ be the $n$-dimensional vector space over $F_2$. The Hamming weight of vector $x \in F_2^n$ is the number of its entries equal to 1. The dot product of vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is defined as $x \cdot y = x_1y_1 + x_2y_2 + \ldots + x_ny_n$. A linear binary code of length $n$ is a subspace in $F_2^n$. Its elements are called words. The distance of a linear code is the smallest Hamming weight of its nonzero word. A trivial code of dimension 0 has distance $\infty$. A linear binary code $C$ is called an $(n, k, d)$ code when it has length $n$, dimension $k$ and distance $d$. The dual code $C^\perp$ is the subspace of vectors in $F_2^n$ orthogonal to $C$. The redundancy of $C$ is the dimension of its dual code, $r = n - k$, which may be interpreted as the number of parity check bits. If $y^{(1)}, y^{(2)}, \ldots, y^{(r)}$ form a basis of $C^\perp$, then $C$ consists of vectors $x$ such that $x \cdot y^{(i)} = 0$ for every $i = 1, 2, \ldots, r$. If $y^{(i)} = (y_{i1}, y_{i2}, \ldots, y_{in})$, the $(r \times n)$-matrix $\begin{bmatrix} y_{ij} \end{bmatrix}$ is called a parity-check matrix of $C$. In fact, any binary $r \times n$ matrix of rank $r$ is a parity-check matrix of some linear code of length $n$ and redundancy $r$.

An $(n, k, 2t + 1)$ code is capable of correcting up to $t$ errors in a word of length $n$ that carries $k$ bits of information. It is natural to seek codes of maximal possible length with prescribed error-correction capabilities. We denote by $N(r, d)$ the largest length of a linear code with redundancy $r$ and distance $d$ or higher, that is the largest $n$ such that an $(n, n - r, \geq d)$ code exists.

It follows from the well known Hamming bound (see [26]) that

$$\sum_{i=0}^{t} \binom{N(r, 2t + 1)}{i} \leq 2^r .$$

The primitive binary BCH code (see [3, 19]) is a $(2^m - 1, 2^m - 1 - mt, 2t + 1)$ code. It gives the lower bound

$$N(mt, 2t + 1) \geq 2^m - 1 .$$

It is easy to see that $N(m, 3) = 2^m - 1$. When $t \geq 2$, the bound (2.2) is not sharp: some codes of slightly larger length are known. Goppa [13] constructed $(2^m, 2^m - mt, 2t + 1)$ codes. Chen [4] found $(2^m + 1, 2^m + 1 - 2m, 5)$ codes for even $m$. Sloane, Reddy, and Chen [4, 25] obtained $(2^m + 2^{\lceil m/2 \rceil} - 1, 2^m + 2^{\lceil m/2 \rceil} - 1 - (2m + 1), 5)$ codes. Hence,

$$N(4s, 5) \geq 2^{2s + 1}, \quad N(4s + 2, 5) \geq 2^{2s+1} ,$$

$$N(4s + 1, 5) \geq 2^{2s} + 2^s - 1, \quad N(4s + 3, 5) \geq 2^{2s+1} + 2^{s+1} - 1 .$$

The values of $N(r, d)$ for small $r$ and $d$ can be derived from tables in [14]. We list these values for $4 \leq r \leq 14$, $d = 5$:

| $r$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|-----|---|---|---|---|---|---|----|----|----|----|----|
| $N(r, 5)$ | 5 | 6 | 8 | 11 | 17 | 23 | 33 | 47–57 | 65–88 | 81–124 | 128–179 |
It follows from (2.1) that $N(r, 5) \leq 2^{(r+1)/2}$. When $r$ is large, the best known lower and upper bounds for $N(r, 5)$ differ by a factor of $\sqrt{2}$ if $r$ is even, and by a factor of 2 if $r$ is odd. For $a > 1$, not a single $(2^m + a, 2^m + a - 2m, 5)$ code is known.

For future use we need

**Theorem 2.1** (MacWilliams identities [22]). Let $C$ be a $k$-dimensional linear binary code of length $n$. Let $A_j$ denote the number of words of Hamming weight $j$ in $C$, and $B_j$ denote the number of words of Hamming weight $j$ in the dual code $C^\perp$. Then for every $\lambda = 0, 1, \ldots, n$,

$$2^{n-k} \sum_{j=0}^{\lambda} \binom{n-j}{\lambda-j} A_j = 2^\lambda \sum_{j=0}^{n-\lambda} \binom{n-j}{\lambda} B_j.$$  \hfill (2.3)

**3. Codes without a forbidden weight**

Let $R_{2m}(n)$ be the smallest redundancy of a linear code of length $n$ which has no words of Hamming weight $2m$. The problem of determining $R_{2m}(n)$ was studied in [1, 7] in notation $l(n, 2^m) = n - R_{2m}(n)$. It follows from Theorem 1.1 of [7] that

$$R_{2m}(n) = n - 2m + 1 \quad \text{for} \quad 2m - 1 \leq n \leq 4m - 1,$$

$$R_{2m}(4m) = 2m. \quad \hfill (3.1)$$

We will solve some new cases with $n > 4m$ in Corollary 5.18.

It follows from Theorem 6 of [1] that

$$R_{2m}(n)/m = \log_2 n + O(1), \quad \hfill (3.2)$$

when $m$ is fixed and $n \to \infty$.

The proof of (3.1) and (3.2) in [7] uses the notion of binormal form of a binary matrix. Following [7], we say that a $k \times n$ binary matrix $M = [a_{ij}]$ is in binormal form if $n \geq 2k$, $a_{i,2j-1} = a_{i,2j}$ for $i \neq j$, and $a_{i,2i-1} \neq a_{i,2i}$ ($i, j = 1, 2, \ldots, k$).

**Lemma 3.1** (Proposition 2.1 [7]). If $k \times n$ binary matrix $M$ is in binormal form, then for any $k$-dimensional binary vector $x$, there is a unique choice of $k$ indices $j_i \in \{2i - 1, 2i\}$ ($i = 1, 2, \ldots, k$) such that the sum of columns $j_1, j_2, \ldots, j_k$ of $M$ is equal to $x$. In particular, one can pick up $k$ columns in $M$ whose sum is the $k$-dimensional zero vector.

**Lemma 3.2** (Lemma 2.2 [7]). Let $n$ be odd, $2k < n$, and $M$ be a $k \times n$ binary matrix of rank $k$. If the sum of entries in each row is 0, then $M$ can be brought to binormal form by such operations as permutations of the columns and additions of one row to another.

Lemmas 3.1 and 3.2 yield

**Corollary 3.3.** Let $n$ be odd and $2k < n$. Let $M$ be a $k \times n$ binary matrix of rank $k$ where the sum of entries in each row is 0. One can pick up $k$ columns in $M$ whose sum is the $k$-dimensional zero vector.
4. Generalized Erdős–Ginzburg–Ziv constant

Let $G$ be a finite abelian group written additively. The classical Erdős–Ginzburg–Ziv constant $s(G)$ is the smallest integer $s$ such that every sequence of length $s$ over $G$ has a zero-sum subsequence of length $\exp(G)$ (see [5, 6, 9, 12, 17, 20, 23]). In 1961, Erdős, Ginzburg, and Ziv [8] proved $s(\mathbb{Z}_k) = 2k - 1$. Kemnitz’ conjecture, $s(\mathbb{Z}_2^k) = 4k - 3$ (see [20]), was open for more than twenty years and finally was proved by Reiher [23] in 2007.

The following generalization of the classical Erdős–Ginzburg–Ziv constant was introduced by Gao [12]. If $r$ is a multiple of $\exp(G)$, then $s_r(G)$ denotes the smallest integer $s$ such that every sequence of length $s$ over $G$ has a zero-sum subsequence of length $r$. (Notice that if $r$ is not a multiple of $\exp(G)$, then there is an element $x \in G$ whose order is not a divisor of $r$, and the infinite sequence $x, x, x, \ldots$ contains no zero-sum subsequence of length $r$.) Obviously, $s_{\exp(G)}(G) = s(G)$. Constants $s_r(G)$ were studied in [2, 10–12, 15, 16, 18, 21].

A sequence that consists of $(km - 1)$ copies of the zero vector and $(k - 1)$ copies of each of the basis vectors demonstrates that

$$s_{km}(\mathbb{Z}_k^d) \geq km + (k - 1)d.$$  \hspace{1cm} (4.1)

If $km \leq (k - 1)d$, we can add $(1, 1, \ldots, 1)$ to this sequence. Hence,

$$s_{km}(\mathbb{Z}_k^d) \geq km + (k - 1)d + 1 \quad \text{when} \quad km \leq (k - 1)d.$$  \hspace{1cm} (4.2)

It is easy to see that

$$s_{km}(\mathbb{Z}_k^d) + (k - 1) \leq s_{km}(\mathbb{Z}_k^{d+1}).$$  \hspace{1cm} (4.3)

Indeed, consider a sequence $S$ over $\mathbb{Z}_k^d$ that does not have zero-sum subsequences of size $km$. Attach 0 to each vector in $S$ as the $(d + 1)$th entry and add to the sequence $(k - 1)$ copies of a vector whose $(d + 1)$th entry is 1. The resulting sequence over $\mathbb{Z}_k^{d+1}$ will not contain a zero-sum subsequence of length $km$, either.

In the case when $k$ is a power of a prime, Gao [12] proved the equality in (4.1) for $m \geq k^{d-1}$ and conjectured

$$s_{km}(\mathbb{Z}_k^d) = km + (k - 1)d \quad \text{for} \quad km > (k - 1)d.$$  \hspace{1cm} (4.4)

The connection between generalized Erdős–Ginzburg–Ziv constants of $\mathbb{Z}_2^d$ and linear binary codes is evident from the following observation. Let $S$ be a sequence of length $n$ over $\mathbb{Z}_2^d$. Write its $n$ vectors column-wise to get a $d \times n$ binary matrix $M$. Obviously, $S$ has a zero-sum subsequence of length $r$ if and only if $M$ has $r$ columns that sum up to a zero vector. Let $C$ be the subspace in $\mathbb{Z}_2^n$ generated by the rows of $M$. If $M$ has $r$ columns that sum up to a zero vector, then the same will be true for any basis of $C$ written row-wise. The $n$-dimensional vector, whose non-zero entries are positioned in these $r$ columns, will be orthogonal to any word of $C$. It means that the dual code $C^\perp$ has a word of weight $r$. The same arguments work in the opposite way, too. If a
linear binary code has a word of weight $r$ then any its parity check matrix has $r$ columns that sum up to a zero vector.

When $k > 2$ is a power of a prime, a similar connection exists between the generalized Erdős–Ginzburg–Ziv constants of $\mathbb{Z}_k^d$ and linear $k$-ary codes (which are subspaces of vector spaces over field $\mathbb{F}_k$), but unfortunately, it works only one way. If a sequence over $\mathbb{F}_k^d$ has a zero-sum subsequence of length $r$, then being written column-wise, it serves as a parity check matrix of a $k$-ary code which has a word whose $r$ entries are equal to 1 and the rest are equal to 0. However, the fact that a $k$-ary code has a word with $r$ nonzero entries does not guarantee that its parity-check matrix has $r$ columns that sum up to a zero vector.

5. Summary of results

In this section, we consider the case $G = \mathbb{Z}_2^d$. We will show (see Theorem 5.9) that Gao’s conjecture (4.4) holds for $k = 2$. We will also show that constants $N(d, 5)$ from Section 2 and $s_4(\mathbb{Z}_2^d)$ are equivalent (namely, $s_4(\mathbb{Z}_2^d) = N(d, 5) + 4$) while constants $R_{2m}(n)$ from Section 3 and $s_{2m}(\mathbb{Z}_2^d)$ are closely related.

To simplify notation, we will write $s_{2m}(d)$ instead of $s_{2m}(\mathbb{Z}_2^d)$. Let $W$ be a set of positive integers which contains at least one even number. We denote by $\beta_W(d)$ the largest size of a set in $\mathbb{Z}_2^d$ which has no zero-sum subsets of size $w \in W$. We will use the following shortcuts:

$$\beta_{2m}(d) = \beta_{(2m)}(d), \quad \beta_{2[k,m]}(d) = \beta_{(2k,2k+2,\ldots,2m)}(d),$$

$$\beta_{[1,2m]}(d) = \beta_{(1,2,\ldots,2m)}(d).$$

As it should be expected, $s_{2m}(d)$ and $\beta_{2m}(d)$ are close:

$$\beta_{2m}(d) + 1 \leq s_{2m}(d) \leq \beta_{2m}(d) + 2m - 1. \quad (5.1)$$

The lower bound in (5.1) is trivial. The upper bound was proved in [24, Theorem 4.1]. The set of $d$ basis vectors in $\mathbb{Z}_2^d$ with addition of vector $(1,1,\ldots,1)$ demonstrates that

$$\beta_{[1,2m]}(d) \geq d + 1 \quad \text{for} \quad d \geq 2m. \quad (5.2)$$

If $d \geq 3m$, there exist vectors $x, y \in \mathbb{Z}_2^d$ such that each of the three vectors $x, y, x + y$ has Hamming weight $2m$. Then $x, y$ and the $d$ basis vectors demonstrate that

$$\beta_{[1,2m]}(d) \geq d + 2 \quad \text{for} \quad d \geq 3m. \quad (5.3)$$

Theorem 5.1. $\beta_{[1,2m]}(d) = N(d, 2m + 1)$.

Theorem 5.2. $\beta_{2[1,m]}(d) = \beta_{[1,2m]}(d) + 1$.

It is easy to see that $\beta_{2m}(d) = 2^d$ if $2m > 2^d$. 

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Theorem 5.3. $\beta_{2m}(d) = \begin{cases} 2m + 2, & \text{if } 2^{d-1} \leq 2m < 2^d, \quad m \neq 2^{d-1} - 2, \\ 2m, & \text{if } m = 2^{d-1} - 2. \end{cases}$

Theorem 5.4. Let $m$ be odd. Then $\beta_{2m}(d) \geq 2\beta_{2[1,m]}(d - 1)$. If $b$ is even and $m < b \leq \beta_{2[1,m]}(d - 1)$, then $\beta_{2b-2m}(d) \geq 2b$.

Theorem 5.5. $s_{2m}(d) = 1 + \max_{1 \leq j \leq m} \{\beta_{2j,m}(d) + (2m - 2j)\}$.

Theorem 5.6. $s_{4}(d) = \beta_{4}(d) + 3 = N(d,5) + 4$.

Theorem 5.7. $s_{6}(d) = 1 + \max_{1 \leq j \leq m} \{\beta_{j,m}(d) + (2m - 2j)\}$.

Theorem 5.8. $R_{2m}(n) = d$ if $n < s_{2m}(d)$.

Theorem 5.9. $s_{2m}(d) = 2m + d$ for $d < 2m$.

Theorem 5.10. $s_{2m}(2m) = 4m + 1$.

The theorems 5.2 and 5.4 together with (5.2) and (5.3) yield

Corollary 5.11. If $m$ is odd, then $\beta_{2m}(d) \geq 2d + 2$ for $d \geq 2m + 1$, and $\beta_{2m}(d) \geq 2d + 4$ for $d \geq 3m + 1$.

Theorem 5.12. $s_{2m}(2m + 1) = 4m + 5$ for odd $m$.

Theorem 5.13. If $m \geq 3$ is odd and $2m - 3 \leq d \leq 2m + 1$, then $\beta_{2m}(d) = s_{2m}(d) - 1$.

In line with Corollary 5.11 and Theorem 5.12, we propose

Conjecture 5.14. If $m$ is odd, $s_{2m}(d) = 2d + 3$ for $2m + 1 \leq d \leq 3m$.

When $d > 2m$, the cases of even and odd $m$ differ significantly. For even $m$ and $2m < d < 3m$, the best lower bound we know follows from (4.3) and Theorem 5.10: $s_{2m}(d) \geq s_{2m}(2m) + (d - 2m) = d + 2m + 1$.

Theorem 5.15. $s_{2m}(2m + 1) = 4m + 2$ for even $m$.

Conjecture 5.16. If $m$ is even, $s_{2m}(d) = d + 2m + 1$ for $2m + 1 \leq d \leq 3m - 1$.

To prove Conjecture 5.16, it would be sufficient to show that $s_{4k}(6k - 1) \leq 10k$. This is equivalent to the statement that every $(4k + 1)$-dimensional code of length $10k$ has a word of weight $4k$.

Computer search confirmed that Conjectures 5.14 and 5.16 hold for $m = 3$ and $m = 4$, respectively.
We can improve this bound for odd $s$.

**Corollary 5.18.** Theorems 5.8, 5.10, 5.12, and 5.15 yield notation of [10], $\eta_m$, our Theorem 5.6), and is likely to hold for even $s$. Future work can be easily derived from Theorem 5.6 and (3.3):

**Conjecture 4.4 of [10] claims $s_{2m}(d) = \beta_{[1,2m]}(d) + 2m$ for every $m$. (In notation of [10], $\eta_{2m}(Z^d) = \beta_{[1,2m]}(d) + 1$.) The equality holds for $m = 2$ (see our Theorem 5.6), and is likely to hold for even $m$ in general. Our Theorem 5.12 disproves this conjecture for odd $m \geq 3$ and $d = 2m + 1$. Indeed, it is easy to see that a two-dimensional binary code of length $n \geq 7$ and distance $n - 2$ or higher can not exist. Therefore, $\beta_{[1,2m]}(2m+1) = N(2m+1,2m+1) < 2m+3$. By applying (5.2), we get $\beta_{[1,2m]}(2m+1) = 2m+2$ while $s_{2m}(2m+1) = 4m+5$.

**Observation 5.17.** If $s_{2m}(d) - s_{2m-2}(d) \geq 3$ then $\beta_{2m}(d) = s_{2m}(d) - 1$. Indeed, consider a sequence $S$ of length $s_{2m}(d) - 1$ over $\mathbb{Z}_d^2$ that does not contain a zero-sum subsequence of length $2m$. If $\beta_{2m}(d) < s_{2m}(d) - 1$, there is $z \in \mathbb{Z}_d^2$ which appears in $S$ at least twice. Remove two copies of $z$ to obtain a sequence $S'$ of length $s_{2m}(d) - 3$. As $s_{2m}(d) - 3 \geq s_{2m-2}(d)$, $S'$ must contain a zero-sum subsequence of length $2m - 2$. By adding back two copies of $z$ we get a zero-sum subsequence of length $2m$ in $S$.

In light of Conjectures 5.12, 5.12, and Observation 5.17, we expect $\beta_{2m}(d) = s_{2m}(d) - 1$ to hold for all odd $m$ and $2m - 3 \leq d \leq 3m$.

It follows from Theorem 5.15 and (4.3) that $s_{2m}(2m+2) \geq 4m+3$ for even $m$. By Corollary 5.11 and (5.1), $s_{2m}(2m+2) \geq 4m+7$ for odd $m$. Hence, Theorems 5.8, 5.10, 5.12, and 5.15 yield

**Corollary 5.18.**

$$R_{2m}(n) = \begin{cases} 2m + 1, & \text{if } n = 4m + 1, \\ 2m + 1, & \text{if } 4m + 2 \leq n \leq 4m + 4, \text{ m is odd}, \\ 2m + 2, & \text{if } 4m + 5 \leq n \leq 4m + 6, \text{ m is odd}, \\ 2m + 2, & \text{if } n = 4m + 2, \text{ m is even}. \end{cases}$$

The next statement (which was also proved in [24]) can be easily derived from Theorem 5.8 and (3.3):

**Corollary 5.19.** For any fixed $m$, $s_{2m}(d) = \Theta(2^{d/m})$ as $d \to \infty$.

It follows from (2.2) and Theorem 5.8 that $\limsup_{d \to \infty} s_{2m}(d) 2^{-d/m} \geq 1$. We can improve this bound for odd $m$. Namely, (5.1) and Theorems 5.1, 5.2, 5.4 yield $s_{2m}(d) \geq 2N(d-1,2m+1)+2$. Coupled with (2.2), it leads to

**Corollary 5.20.** If $m$ is odd, $\limsup_{d \to \infty} s_{2m}(d) 2^{-d/m} \geq 2^{1-1/m}$.

By Theorem 5.6, bounds on $N(d,5)$ from Section 2 translate into bounds on $s_{4}(d)$. For $d \leq 10$, we get the exact values:
6. Proofs of theorems

For $A \subseteq \mathbb{Z}_2^d$, we denote the sum of elements of $A$ by $\sum A$.

**Proof of Theorem 5.1.** First, we will prove $\beta_{[1,2m]}(d) \geq N(d,2m+1))$. Consider a linear code $C$ of length $n = N(d,2m+1)$, redundancy $d$ and distance at least $2m+1$. Its parity check matrix $M$ has size $d \times n$. Since $C$ has no words of weights $1,2,\ldots,2m$, the sum of any $k \in \{1,2,\ldots,2m\}$ columns of $M$ is not a zero vector. It means that the columns of $M$, being interpreted as $n$ vectors in $\mathbb{Z}_2^d$, form a set without zero-sum subsets of sizes $2m$ and less.

To prove $\beta_{[1,2m]}(d) \leq N(d,2m+1)$, consider a set $A$ of size $\beta_{[1,2m]}(d)$ in $\mathbb{Z}_2^d$ which has no zero-sum subsets of sizes $2m$ and less. If $A$ does not contain a basis in $\mathbb{Z}_2^d$, then there exists a vector $x \in \mathbb{Z}_2^d$ which cannot be represented as a sum of some vectors from $A$. Then $A \cup \{x\}$ would have no zero-sum subsets of sizes $2m$ and less which contradicts with the maximality of $|A|$. Hence, $A$ contains a basis. Then a $d \times |A|$ matrix, whose columns are the vectors in $A$, has rank $d$ and is a parity check matrix of a code of length $n$, redundancy $d$ and distance at least $2m+1$.

**Proof of Theorem 5.2.** Consider a set $A$ of size $\beta_{[1,2m]}(d)$ in $\mathbb{Z}_2^d$ which has no zero-sum subsets of sizes $1,2,\ldots,2m$. In particular, $0 \notin A$. It is obvious that $A \cup \{0\}$ does not have zero-sum subsets of sizes $2,4,\ldots,2m$. Hence, $\beta_{[1,2m]}(d) \geq \beta_{[1,2m]}(d) + 1$.

To prove the opposite inequality, consider a set $B$ of size $\beta_{[2,1m]}(d)$ in $\mathbb{Z}_2^d$ which has no zero-sum subsets of sizes $2,4,\ldots,2m$. Select $y \in B$ and define $B_y = \{x + y \mid x \in B\}$. Notice that $B_y$ does not have zero-sum subsets of sizes $2,4,\ldots,2m$ and contains the zero vector. Then $B_y \setminus \{0\}$ will have no zero-sum subsets of sizes $1,2,\ldots,2m$. Therefore, $\beta_{[1,2m]}(d) \geq |B_y \setminus \{0\}| = \beta_{[2,1m]}(d) - 1$.

**Lemma 6.1.** If $A \subseteq \mathbb{Z}_2^d$, $|A| > 2d-1$ and $\sum A \neq 0$, then there exists $B \subset A$ such that $|B| = |A| - 2$ and $\sum B = 0$.

**Proof.** There are exactly $2^d$ solutions $(x,y)$ of the equation $x + y = \sum A$ where $x,y \in \mathbb{Z}_2^d$. Let $\overline{A} = \mathbb{Z}_2^d \setminus A$. The number of solutions with $x \in \overline{A}$ or $y \in \overline{A}$ is at most $2|\overline{A}| = 2(2^d - |A|) < 2^d$. Hence, there exists a solution $(x,y)$ where $x,y \in A$. As $x + y = \sum A \neq 0$, we get $x \neq y$. Set $B = A \setminus \{x,y\}$. Then $|B| = |A| - 2$ and $\sum B = 0$.

**Lemma 6.2.** If $A \subseteq \mathbb{Z}_2^d$, $|A| \geq 2d - 1 + 2$, then there exists $B \subset A$ such that $|B| = |A| - 3$ and $\sum B = 0$.

**Proof.** As $|A| \geq 2$, there exists $x \in A$ such that $x \notin \sum A$. Set $A_1 = A \setminus \{x\}$, then $\sum A_1 \neq 0$. By Lemma 6.1, there exists $B \subset A_1$ such that $|B| = |A| - 3$ and $\sum B = 0$.

\begin{table}
\begin{tabular}{cccccccccccc}
\hline
$d$ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
$s_4(d)$ & 5 & 6 & 7 & 9 & 10 & 12 & 15 & 21 & 27 & 37 \\
\hline
\end{tabular}
\end{table}
Proof of Theorem 5.3. We will prove the lower bound first. In the case \( m \neq 2^{d-1} - 2 \), select \( A \subseteq \mathbb{Z}_2^d \) such that \( |A| = 2m + 2 \) and \( \sum A = 0 \). If \( B \subseteq A \), \( |B| = 2m \), and \( A \setminus B = \{x, y\} \), then \( \sum B = \sum A - (x + y) = x + y \neq 0 \). Hence, \( \beta_{2m}(d) \geq 2m + 2 \). In the case \( m = 2^{d-1} - 2 \), the bound \( \beta_{2m}(d) \geq 2m \) is trivial.

Now we will prove the upper bound. If \( m \neq 2^{d-1} - 2 \), consider \( A \subseteq \mathbb{Z}_2^d \) where \( |A| = 2m + 3 \geq 2^{d-1} + 3 \). By Lemma 6.2, there exists \( B \subseteq A \) such that \( |B| = 2m \) and \( \sum B = 0 \). Therefore, \( \beta_{2m}(d) \leq 2m + 2 \). If \( m = 2^{d-1} - 2 \), consider \( A \subseteq \mathbb{Z}_2^d \) where \( |A| = 2m + 1 = 2^d - 3 \). Let \( x = \sum A \), and \( \mathbb{Z}_2^d \setminus A = \{a, b, c\} \). Since \( x = \sum A = a + b + c = a + (b + c) \neq a \) (and similarly, \( x \neq b, c \)), we conclude that \( x \in A \). Set \( B = A \setminus \{x\} \). Then \( |B| = 2m \) and \( \sum B = 0 \), so \( \beta_{2m}(d) \leq 2m \).

Proof of Theorem 5.4. Consider a set \( A \) of size \( b \leq \beta_{2\{1,m\}}(d-1) \) in \( \mathbb{Z}_2^d \), which does not have zero-sum subsets of sizes \( 2, 4, \ldots, 2m \). For \( i = 0, 1 \), obtain \( A_{i} \subseteq \mathbb{Z}_2^d \) by attaching \( i \) to each vector of \( A \) as the \( d \)th entry. We claim that \( A_{0} \cup A_{1} \) does not have zero-sum subsets of size \( 2m \). Indeed, suppose \( X \subseteq (A_{0} \cup A_{1}) \), \( |X| = 2m \). Let \( X_{i} = \{(z_{1}, z_{2}, \ldots, z_{d-1}) \mid (z_{1}, z_{2}, \ldots, z_{d-1}, i) \in X\} \) and \( Y = (X_{0} \cup X_{1}) \setminus (X_{0} \cap X_{1}) \). Then \( X_{0}, X_{1}, Y \) are subsets of \( A \). If \( \sum X = 0 \) then \( |X_{0}|, |X_{1}| \) are even and \( \sum X_{0} + \sum X_{1} = 0 \). Notice that \( |X_{0} \cap X_{1}| = m \) is impossible, as it would imply \( |X_{0}| = |X_{1}| = m \), but \( m \) is odd. Hence, \( \sum Y = \sum X_{0} + \sum X_{1} - 2 \sum (X_{0} \cap X_{1}) = 0 \) while \( |Y| = 2m - 2 |X_{0} \cap X_{1}| \) is even and not equal to zero. Selecting \( b = \beta_{2\{1,m\}}(d-1) \) in \( \mathbb{Z}_2^d \), we get \( \beta_{2m}(d) \geq |A_{0} \cup A_{1}| = 2 \beta_{2\{1,m\}}(d-1) \).

When \( b \) is even, \( \sum (A_{0} \cup A_{1}) = 0 \), so \( A_{0} \cup A_{1} \) does not have zero-sum subsets of size \( 2b - 2m \) either. Hence, \( \beta_{2b-2m}(d) \geq |A_{0} \cup A_{1}| = 2b \).

Proof of Theorem 5.5. We will show first that \( s_{2m}(d) \geq 1 + \beta_{2\{1,m\}}(d) + (2m - 2j) \). Indeed, consider a set \( A \) of size \( \beta_{2\{1,m\}}(d) \) in \( \mathbb{Z}_2^d \) which does not have zero-sum subsets of sizes \( 2j, 2j + 2, \ldots, 2m \). Select \( x \in A \) and form a sequence from all the elements of \( A \) plus \( (2m - 2j) \) extra copies of \( x \). It is easy to see that this sequence does not have zero-sum subsequences of size \( 2m \). Hence, \( s_{2m}(d) \geq 1 + \max_{1 \leq j \leq m} \{ \beta_{2\{1,m\}}(d) + (2m - 2j) \} \). Now we need to prove the opposite inequality. Let \( s \) be an integer such that \( s > \beta_{2\{1,m\}}(d) + (2m - 2j) \) for every \( j = 1, 2, \ldots, m \). We define the product of an integer \( n \) and \( z \in \mathbb{Z}_2^d \) as \( z \) if \( n \) is odd, and 0 if \( n \) is even. Let \( f(z) \) be the number of appearances of \( z \in \mathbb{Z}_2^d \) in a sequence of length \( s \) over \( \mathbb{Z}_2^d \). We are going to show that there exists a nonnegative integer function \( g \) on \( \mathbb{Z}_2^d \) such that \( g \leq f \), \( \sum_{z \in \mathbb{Z}_2^d} g(z) = 2m \), and \( \sum_{z \in \mathbb{Z}_2^d} g(z)z = 0 \). Indeed, set \( f_{1}(z) = 1 \) if \( f(z) \) is odd, and \( f_{1}(z) = 0 \) if \( f(z) \) is even. Set \( f_{2}(z) = f(z) - f_{1}(z) \). All values of \( f_{2}(z) \) are even. If \( \sum_{z \in \mathbb{Z}_2^d} f_{2}(z) \geq 2m \), then there exists \( g \leq f_{2} \) such that all values of \( g \) are even and \( \sum_{z \in \mathbb{Z}_2^d} g(z) = 2m \). Hence, we can assume that \( \sum_{z \in \mathbb{Z}_2^d} f_{2}(z) = 2(m - j) \) where \( j \in \{1, 2, \ldots, m\} \). Let \( A = \{z \in \mathbb{Z}_2^d : f_{1}(z) = 1\} \). Since \( |A| = \sum_{z \in \mathbb{Z}_2^d} (f(z) - f_{2}(z)) = s - 2(m - j) > \beta_{2\{1,m\}}(d) \), there exists \( B \subseteq A \) such that \( |B| = k \in \{2j, 2j + 2, \ldots, 2m\} \) and \( \sum_{z \in B} z = 0 \). Set \( f_{3}(z) = 1 \) if \( z \in B \), and \( f_{3}(z) = 0 \) otherwise. Choose a function \( f_{3} \leq f_{2} \) such that all values of \( f_{3} \) are even and \( \sum_{z \in \mathbb{Z}_2^d} f_{3}(z) = 2m - k \). Then \( f_{3} + f_{B} \) is the required function \( g \).
Proof of Theorem 5.6. By definition, \( \beta_4(d) = \beta_{(2,4)}(d) \). Thus, Theorems 5.1 and 5.2 imply \( \beta_4(d) = N(d, 5) + 1 \), while Theorem 5.5 implies \( s_4(d) = \beta_4(d) + 3 \).

Proof of Theorem 5.8. We will show first that \( n < s_{2m}(d) \) implies \( R_{2m}(n) \leq d \), and then show that \( n \geq s_{2m}(d - 1) \) implies \( R_{2m}(n) \geq d \). As \( s_{2m}(d - 1) < s_{2m}(d) \) (see (4.3)), these two statements establish the theorem.

Let \( n < s_{2m}(d) \). Then there exists a sequence \( S \) of length \( n \) over \( \mathbb{Z}_2^d \) which does not have zero-sum subsequences of size \( 2m \). Write the vectors of \( S \) as \( d \times n \) binary matrix \( M \). This matrix does not have \( 2m \) columns that sum up to a zero vector. Its rank is \( r \leq d \). Take \( r \) independent rows of \( M \) to get an \( r \times n \) matrix \( M_1 \) of rank \( r \). We claim that \( M_1 \) does not have \( 2m \) columns that sum up to a zero vector. Indeed, any row of \( M \) that is not in \( M_1 \) is the sum of some rows of \( M_1 \). If \( M_1 \) had a set of \( 2m \) columns which sum up to a zero vector, then the same columns in \( M \) would also sum up to a zero vector. Let \( C \) be the linear code whose parity check matrix is \( M_1 \). This code has length \( n \), redundancy \( r \) and does not have a word of weight \( 2m \). Hence, \( R_{2m}(n) \leq r \).

Let \( n \geq s_{2m}(d - 1) \). Consider a linear code \( C \) of length \( n \) and redundancy \( r \) which does not have words of weight \( 2m \). Let \( M \) be an \( r \times n \) parity check matrix of \( C \). If \( r \leq d - 1 \), then \( n \geq s_{2m}(r) \), and there exists a set of \( 2m \) columns in \( M \) that sum up to a zero vector. It contradicts with the assumption that \( C \) does not have words of weight \( 2m \). Hence, \( r \geq d \). We have proved that \( R_{2m}(n) \geq d \).

Proof of Theorem 5.9. It follows from (3.1) that \( R_{2m}(2m + d) = d + 1 \) when \( d < 2m \). Hence, by Theorem 5.8, \( s_{2m}(d) \leq 2m + d \). The opposite inequality is provided by (4.1).

Observation 6.3. The following operations do not change the fact whether a binary matrix has a set of \( 2m \) columns that sum up to a zero vector: permutations of columns (rows), additions of one row to another, additions of the same vector to each column. In particular, bringing a matrix to binormal form by Lemma 3.2 does not change the fact whether such a set of \( 2m \) columns exists.

Lemma 6.4. Let \( M \) be an \( d \times n \) binary matrix whose rank is less than \( d \). If \( n \geq s_{2m}(d - 1) \), then \( M \) contains \( 2m \) columns that sum up to a zero vector.

Proof. Use additions of one row to another to make the last row entirely zero. By Observation 6.3, it does not change the fact whether there is a set of \( 2m \) columns that sum up to a zero vector. As \( n > s_{2m}(d - 1) \), such a set of columns must exist. By Observation 6.3, the same is true for the original matrix.

Proof of Theorem 5.10. When \( k = 2 \) and \( d = 2m \), (4.2) yields \( s_{2m}(2m) \geq 4m + 1 \). To prove \( s_{2m}(2m) \leq 4m + 1 \), consider a sequence \( S \) of size \( 4m + 1 \) over \( \mathbb{Z}_2^m \). We need to prove that \( S \) has a zero-sum subsequence of size \( 2m \). Let \( x \) be the sum of all elements of \( S \). Add \( x \) to each element of \( S \) and denote the resulting sequence by \( S_0 \). If \( S_0 \) has a zero-sum subsequence of size \( 2m \), then \( S \) has it, too.
Consider a $(2m) \times (4m+1)$ binary matrix $M$ whose columns represent the vectors from $S_0$. By Theorem 5.9, $4m + 1 \geq s_{2m}(2m-1)$. If the rank of $M$ is less than $2m$, then by Lemma 6.4, it has a set of $2m$ columns that sum up to a zero vector. Suppose that the rank of $M$ is $2m$. Since the sum of all vectors of $S_0$ is a zero vector, $M$ satisfies the conditions of Corollary 3.3. Thus, there are $2m$ columns in $M$ whose sum is a zero vector.

**Lemma 6.5.** Let $C = [c_{ij}]$ be a $t \times t$ binary matrix. If $C$ does not have a row where all off-diagonal entries are equal, then it has three distinct rows $i, j, k$ such that $c_{ij} + c_{ik} = 1$ and $c_{ij} + c_{jk} = 1$.

**Proof.** Let $a_i$ be the number of off-diagonal nonzero entries in row $i$, and $b_i = (t-1)-a_i$ be the number of off-diagonal zero entries. Let $a = \min\{a_1, a_2, \ldots, a_t\}$, $b = \min\{b_1, b_2, \ldots, b_t\}$. Since the statement of the lemma holds for matrix $[c_{ij}]$ if and only if it holds for matrix $[1 - c_{ij}]$, we may assume that $a \leq b$. Let index $i$ be such that $a_i = a$. Since $a_i > 0$, there exists index $j \neq i$ such that $c_{ij} = 1$. Consider two cases for $c_{ji}$.

- $0)$. $c_{ji} = 0$. As $a_j > a_i - 1$, there exists $k \neq i, j$ such that $c_{ik} = 0$ and $c_{jk} = 1$.
- $1)$. $c_{ji} = 1$. As $a_i + a_j = a_i + (t - 1 - b_j) \leq (t - 1) + a_i - b < t$, there exists $k \neq i, j$ such that $c_{ik} = c_{jk} = 0$.

Let $M$ be an $r \times c$ matrix, and $1 \leq r' \leq r'' \leq r$, $1 \leq c' \leq c'' \leq c$. We denote by $M[r' : r'', c' : c'']$ the submatrix of $M$ formed by rows $r', r' + 1, \ldots, r''$ and columns $c', c' + 1, \ldots, c''$.

**Lemma 6.6.** Let $k$ be even, and $M$ be a $(k+1) \times (2k+5)$ matrix in binormal form. One can pick up $k$ columns in $M$ whose sum is the $(k+1)$-dimensional zero vector.

**Proof.** Let $M = [a_{ij}]$. Let $C = [c_{ij}]$ be a $(k+1) \times (k+1)$ binary matrix where $c_{ij} = a_{i,2j-1} = a_{i,2j}$ for $i \neq j$. The values of diagonal entries $c_{ii}$ are not important and may be set arbitrarily. Suppose, there is a row in $C$ where the sum of off-diagonal entries is 0. Without loss of generality, we may assume that it is the last row, so $\sum_{i=1}^{k} c_{k+1,i} = 0$. Since $M[1 : k, 1 : 2k]$ is in binormal form, by Lemma 3.1, there is a set of $k$ indices $j(i) \in \{2i-1, 2i\}$ ($i = 1, 2, \ldots, k$) such that the sum of columns $j(1), j(2), \ldots, j(k)$ in $M[1 : k, 1 : 2k]$ is the $k$-dimensional zero vector. Since $a_{k+1, j(i)} = c_{k+1,i}$, the condition $\sum_{i=1}^{k} c_{k+1,i} = 0$ guarantees that the sum of columns $j(1), j(2), \ldots, j(k)$ in $M$ is the $(k+1)$-dimensional zero vector. Thus, we may assume that the sum of off-diagonal entries in each row of $C$ is equal to 1. It means that $C$ satisfies the conditions of Lemma 6.5. Hence, without loss of generality, we may assume that $c_{k-1,k} + c_{k-1,k+1} = 1$ and $c_{k,k-1} + c_{k,k+1} = 1$. Since the sum of off-diagonal entries in any row of $C$ is 1, we get $\sum_{j=1}^{k-2} c_{ij} = 0$ for $i = k-1, k$. Let $x = \sum_{j=1}^{k-2} c_{k+1,j}$. Then the sum of all columns in the $3 \times (k-2)$ matrix $C[k-1 : k+1, 1 : k-2]$ is equal to $(0, 0, x)^T$. We claim that the $3 \times 9$ matrix $M[k-1 : k+1, 2k-3 : 2k+5]$ has two columns whose sum is $(0, 0, x)^T$. Indeed, among 9 columns there must
be two equal, their sum is \((0, 0, 0)^T\). Since \(M\) is in binormal form, its columns 
\((2k + 1)\) and \((2k + 2)\) differ only in the last entry. Hence, \(M[k - 1 : k + 1, 2k - 3 : 2k + 5]\) contains two rows whose sum is \((0, 0, 1)^T\). Now we select two columns in 
\(M[k - 1 : k + 1, 2k - 3 : 2k + 5]\) whose sum is \((0, 0, x)^T\) and call the columns of 
\(M\) which contain them special. Let \((y_1, y_2, \ldots, y_{k+1})^T\) be the sum of the two 
special columns. We already know that \(y_{k-1} = y_k = 0\) and \(y_{k+1} = x\). Since 
\(M[1 : k - 2, 1 : 2k - 4]\) is in binormal form, by Lemma 3.1, there is a set of \(k - 2\) 
indices \(l(i) \in \{2i - 1, 2i\} \ (i = 1, 2, \ldots, k - 2)\) such that the sum of columns 
l(1), l(2), \ldots, l(k - 2) in \(M[1 : k - 2, 1 : 2k - 4]\) is equal to \((y_1, y_2, \ldots, y_{k-2})^T\). 
Then the sum of these \(k - 2\) columns plus the two special columns in \(M\) is the 
\((k + 1)\)-dimensional zero vector. □

**Proof of Theorem 5.12.** By (5.1) and Corollary 5.11, 
\[s_{2m}(2m + 1) \geq \beta_{2m}(2m + 1) + 1 \geq 4m + 5.\] 
To prove \(s_{2m}(2m + 1) \leq 4m + 5\), consider a sequence \(S\) of size \(4m + 5\) over \(\mathbb{Z}^{2m+1}_2\). We need to prove that \(S\) has a zero-sum 
subsequence of size \(2m\). Let \(x\) be the sum of all elements of \(S\). Add \(x\) to each 
element of \(S\) and denote the resulting sequence by \(S_0\). If \(S_0\) has a zero-sum 
subsequence of size \(2m\), then \(S\) has it, too.

Consider a \((2m + 1) \times (4m + 5)\) binary matrix \(M\) whose columns represent the 
vectors from \(S_0\). By Theorem 5.10, \(4m + 5 \geq s_{2m}(2m)\). If the rank of \(M\) is 
less than \(2m + 1\), then by Lemma 6.4, it has a set of \(2m\) columns that sum up to 
a zero vector.

Suppose that the rank of \(M\) is \(2m + 1\). Since the sum of all vectors of \(S_0\) is 
a zero vector, \(M\) satisfies the conditions of Lemma 3.2 and can be brought to 
matrix \(M'\) in binormal form by permutations of columns and additions of one 
row to another. By Lemma 6.6, \(M'\) has a set of \(2m\) columns that sum up to a 
zero vector. Then by Observation 6.3, \(M\) has such a set, too. □

**Proof of Theorem 5.13.** By (5.1), \[\beta_{2m}(d) \leq s_{2m}(d) - 1.\] It remains to prove 
\[\beta_{2m}(d) \geq s_{2m}(d) - 1.\] The values of \(s_{2m}(d)\) for \(d \leq 2m + 1\) were determined in 
Theorems 5.9, 5.10, and 5.12. Theorems 5.2 and 5.4 yield 
\[\beta_{2m}(d) \geq 2\beta_{1,2m}(d - 1) + 2.\] The set of \(d - 1\) basis vectors in \(\mathbb{Z}^{d-1}_2\) demonstrates that 
\[\beta_{1,2m}(d - 1) \geq d - 1,\] so we get 
\[\beta_{2m}(2m - 1) \geq 2(2m - 2) + 2 = 4m - 2 = s_{2m}(2m - 1) - 1,\]
\[\beta_{2m}(2m) \geq 2(2m - 1) + 2 = 4m = s_{2m}(2m) - 1.\]

By (5.2), \[\beta_{1,2m}(2m) \geq 2m + 1,\] and hence,
\[\beta_{2m}(2m + 1) \geq 2(2m + 1) + 2 = 4m + 4 = s_{2m}(2m + 1) - 1.\]

We use Theorem 5.4 with odd \(m \geq 1\) and \(b = 2m + 2 \leq \beta_{2[1,m]}(2m)\) to get 
\[\beta_{2m+1}(2m + 1) = \beta_{2b-2m}(2m + 1) \geq 2b = 4m + 4\] for \(m \geq 1\). Substituting 
\(m - 2 \geq 1\) instead of \(m\), we get 
\[\beta_{2m}(2m - 3) \geq 4m - 4 = s_{2m}(2m - 3) - 1 \text{ for } m \geq 3.\]
Lemma 6.7. Let $C$ be a linear binary code of length $n \geq 10$. If $C$ does not have words of Hamming weight 2 and 4, then its dual code $C^\perp$ has a nonzero word of weight $l$ where $|l-n/2| \geq 2$.

Proof. Suppose, to the contrary, that the weights of nonzero words of $C^\perp$ lay in the interval $[(n-3)/2, (n+3)/2]$. Let $D_n = \{0, (n-3)/2, (n-1)/2, (n+1)/2, (n+3)/2\}$ if $n$ is odd, and $D_n = \{0, (n-2)/2, (n+2)/2\}$ if $n$ is even. Let $r$ be the dimension of $C^\perp$, so the dimension of $C$ is $k = n - r$. Let $A_j$ ($B_j$) denote the number of words of weight $j$ in $C$ ($C^\perp$). Then $A_0 = 1$, $A_2 = A_4 = 0$, $B_0 = 1$, $B_j = 0$ for $j \notin D_n$, and $\sum_{j \in D_n} B_j = 2^r$. Consider a linear combination of MacWilliams identities (2.3) with $\lambda = 1, 2, 3, 4$, where coefficients $c_\lambda$ will be selected later:

$$
\sum_{\lambda=1}^4 c_\lambda 2^r \sum_{j=0}^{\lambda} \binom{n-j}{\lambda-j} A_j = \sum_{\lambda=1}^4 c_\lambda 2^r \sum_{j=0}^{n-\lambda} \binom{n-j}{\lambda} B_j.
$$

As $D_n \subseteq [0, n-\lambda]$ for $n \geq 10$, $\lambda \leq 4$, and $B_j = 0$ for $j \notin D_n$, we can rewrite it as

$$
2^r \sum_{\lambda=1}^4 c_\lambda \sum_{j=0}^{\lambda} \binom{n-j}{\lambda-j} A_j = \sum_{j \in D_n} f_j B_j,
$$

where

$$
f_j = \sum_{\lambda=1}^4 c_\lambda 2^r \sum_{j=0}^{n-\lambda} \binom{n-j}{\lambda}.
$$

As $A_2 = A_4 = 0$, the left hand side of (6.1) is a linear combination of $A_0, A_1, A_3$. We are going to choose coefficients $c_1, c_2, c_3, c_4$ in such a way that $A_1$ and $A_3$ are eliminated while the values of $f_j$ are equal for all $j \in D_n \setminus \{0\}$.

If $n$ is even, set $c_1 = 4n(n-1)(n-2)$, $c_2 = -12(n-2)^2$, $c_3 = 24(n-3)$, $c_4 = -24$. In this case, we get $f_0 = 0$, $f_j = n^2(n+2)(n-2)$ for $j \in D_n \setminus \{0\}$, and (6.1) is reduced to

$$
2^r n(n-1)(n-2)(n+3) A_0 = n^2(n+2)(n-2) \sum_{j \in D_n \setminus \{0\}} B_j.
$$

Since $A_0 = 1$ and $\sum_{j \in D_n \setminus \{0\}} B_j = 2^r - 1$, we can simplify it further to

$$
n(n-2)(3 \cdot 2^r - n(n+2)) = 0,
$$

which has no integer solutions for $n > 6$. 

Finally,

$$
\beta_{2m}(2m-2) \geq \beta_{2m}(2m-3) + 1 \geq 4m - 3 = s_{2m}(2m-2) - 1 \text{ for } m \geq 3.
$$
If $n$ is odd, set $c_1 = 4(n + 1)(n - 1)(n - 3)$, $c_2 = -12(n - 1)(n - 3)$, $c_3 = 24(n - 3)$, $c_4 = -24$. In this case, we get $f_0 = 0$, $f_j = (n + 3)(n + 1)(n - 1)(n - 3)$ for $j \in D_n \setminus \{0\}$, and (6.1) is reduced to

$$2' n(n - 1)(n - 3)(n + 4)A_0 = (n + 3)(n + 1)(n - 1)(n - 3) \sum_{j \in D_n \setminus \{0\}} B_j.$$ 

Since $A_0 = 1$ and $\sum_{j \in D_n \setminus \{0\}} B_j = 2'^{n - 1}$, we can simplify it further to

$$(n - 1)(n - 3)(3 \cdot 2' - (n + 1)(n + 3)) = 0,$$

which has no integer solutions for $n > 3$. □

**Lemma 6.8.** $\beta_{2[1,m]}(d) \leq \max\{9, 2\beta_{2[1,m]}(d - 1) - 4\}$.

**Proof.** Suppose, $\beta_{2[1,m]}(d) \geq 10$. We need to prove $\beta_{2[1,m]}(d) \leq 2\beta_{2[1,m]}(d - 1) - 4$. Consider a set $A$ of size $n = \beta_{2[1,m]}(d)$ in $\mathbb{Z}_2^d$ which does not have zero-sum subsets of sizes $2, 4, \ldots, 2m$. Write $n$ vectors of $A$ column-wise as a $d \times n$ binary matrix $M$. Similarly to the proof of Theorem 5.1, the maximality of $|A|$ ensures that $M$ has rank $d$. Let $C$ be the linear code of length $n$ whose parity check matrix is $M$. This code does not have words of weight $2, 4, \ldots, 2m$. As $n \geq 10$, by Lemma 6.7, the dual code $C^\perp$ has a word of weight $l$ where $l - n/2 \geq 2$. Then there exists a parity check matrix $M_1$ of code $C$ such that this word is the first row of $M_1$. Notice that the sum of any $2k$ columns of $M_1$ is not a zero vector ($k = 1, 2, \ldots, m$). If $l \geq (n + 4)/2$, remove from $M_1$ all columns which contain 0 in the first row. If $l \leq (n - 4)/2$, remove from $M_1$ all columns which contain 1 in the first row. The resulting matrix $M_2$ is of size $d \times t$ where $t \geq (n + 4)/2$. All entries in the first row of $M_2$ are equal. Remove the first row to get matrix $M_3$ of size $(d - 1) \times t$. As $M_2$ does not have sets of columns of size $2k$ ($k = 1, 2, \ldots, m$) which sum up to a zero vector, the same is true for $M_3$. Therefore, $\beta_{2[1,m]}(d - 1) \geq t \geq (n + 4)/2 = (\beta_{2[1,m]}(d) + 4)/2$. □

**Proof of Theorem 5.7.** For $3 \leq d \leq 7$, the statement of the theorem follows from Theorem 5.13. If $d \geq 8$, (5.2) and Theorem 5.2 yield $\beta_{2[4,6]}(d) \geq 10$. We may apply Lemma 6.8 to get $\beta_{2[4,6]}(d) \leq 2\beta_{2[4,6]}(d - 1) - 4$. By Theorem 5.4, $\beta_6(d) \geq 2\beta_{2[4,6]}(d - 1) \geq \beta_{2[4,6]}(d) + 4$. By definition, $\beta_{4,6}(d) = \beta_{2[4,6]}(d)$, hence, Theorem 5.5 yields $s_6(d) = 1 + \max\{\beta_{2[4,6]}(d) + 4, \beta_6(d)\} = 1 + \beta_6(d)$. □

**Lemma 6.9.** Let $D$ be a digraph with $n \equiv 1 \pmod{4}$ vertices where every vertex has odd out-degree. Then one can find in $D$ either 3 vertices that span a subgraph with 2 vertices of odd out-degree, or 5 vertices that span a subgraph with 3 vertices of odd out-degree.

**Proof.** Let the vertex set of $D$ be $\{1, 2, \ldots, n\}$. Let $C = [c_{ij}]$ be the adjacency matrix: $c_{ij} = 1$ if the arc $(i, j)$ is present in $D$, otherwise, $c_{ij} = 0$ ($i \neq j$). The diagonal entries $c_{ii}$ are zeros.

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For a subset $A$ of vertices we define the type of $A$ as the number of vertices of odd out-degree in the subgraph of $D$ spanned by $A$. We need to show that there exists either a triple of type 2 or a quintuple of type 3. For a triple $\{i, j, k\}$, let $t(i, j, k)$ denote its type. Similarly, for a quintuple $\{i, j, k, l, m\}$, its type is denoted by $t(i, j, k, l, m)$. Suppose, $D$ contains no triple of type 2, so every triple of even type must have type 0. We are going to show that there is a quintuple of type 3.

The size of $C$ is odd, and the sum of the entries in each row is odd. Hence, the sum of all entries of $C$ is odd. In the sum of expressions $f(i, j, k) = c_{ij} + c_{ji} + c_{ik} + c_{ki} + c_{jk} + c_{kj}$ over all triples $\{i, j, k\}$, each off-diagonal entry of $C$ appears $n - 2 \equiv 1 \pmod{2}$ times. Notice that $t(i, j, k)$ is odd (even) when $f(i, j, k)$ is odd (even). Hence, the sum of the types of all triples is odd. As $n \equiv 1 \pmod{4}$, the number of all triples, $\binom{n}{3}$, is even. It means that there exists at least one triple of even type, and at least one triple of odd type. Then we can find a triple of even type and a triple of odd type that share a pair. Notice that the expression $f(i, j, k) + f(i, j, l) + f(i, j, l) + f(j, k, l)$ is always even, because each arc of the subgraph spanned by $\{i, j, k, l\}$ is counted there twice. Thus, the sum of types of the 4 triples within one quadruple is even. In particular, a quadruple which contains a triple of even type and a triple of odd type must contain two triples of even type and two triples of odd type. Without limiting generality, we can assume that triples $\{1, 2, 3\}$, $\{2, 3, 4\}$ are of type 0, and $\{1, 2, 4\}$, $\{1, 3, 4\}$ are of odd type. Hence,

$$c_{12} = c_{13}, \quad c_{24} = c_{43}, \quad c_{21} = c_{23} = c_{24}, \quad c_{31} = c_{32} = c_{34}, \quad c_{14} \neq c_{41}.\quad (6.2)$$

Notice that the following operation on $D$ preserves the types of all subsets of odd sizes (in particular of sizes 3, 5, $n$): for a given vertex $i$, remove all arcs $(i, j)$ that were present in $D$, and add all arcs $(i, j)$ that were not present. It is equivalent to replacing each off-diagonal entry $c_{ij}$ in row $i$ of $C$ with $1 - c_{ij}$. We apply this switching operation, if necessary, to some of vertices 1, 2, 3, 4 to ensure

$$c_{12} = c_{13} = c_{24} = c_{43} = c_{21} = c_{23} = c_{24} = c_{31} = c_{32} = c_{34} = 0.\quad (6.2)$$

As indices 1 and 4 appear in (6.2) symmetrically, and $c_{14} \neq c_{41}$, we can assume, without limiting generality, that $c_{14} = 1$ and $c_{41} = 0$. Define

$$V_1 = \{i \in \{4, 5, \ldots, n\} : t(i, 2, 3) = 0, \quad t(i, 1, 2) \equiv 1 \pmod{2}, \quad t(i, 1, 3) \equiv 1 \pmod{2}\},$$

$$V_2 = \{i \in \{4, 5, \ldots, n\} : t(i, 1, 3) = 0, \quad t(i, 2, 3) \equiv 1 \pmod{2}, \quad t(i, 1, 2) \equiv 1 \pmod{2}\},$$

$$V_3 = \{i \in \{4, 5, \ldots, n\} : t(i, 1, 2) = 0, \quad t(i, 1, 3) \equiv 1 \pmod{2}, \quad t(i, 2, 3) \equiv 1 \pmod{2}\},$$

$$V_4 = \{i \in \{4, 5, \ldots, n\} : t(i, 2, 3) = 0, \quad t(i, 1, 2) \equiv 1 \pmod{2}, \quad t(i, 1, 3) \equiv 1 \pmod{2}\},$$

$$V_5 = \{i \in \{4, 5, \ldots, n\} : t(i, 1, 2) = 0, \quad t(i, 2, 3) \equiv 1 \pmod{2}, \quad t(i, 1, 3) \equiv 1 \pmod{2}\},$$
W_0 = \{ i \in \{4, 5, \ldots, n\} : t(i, 1, 2) = t(i, 1, 3) = t(i, 2, 3) = 0 \},
\quad W_1 = V_1 \cup \{1\}, \quad W_2 = V_2 \cup \{2\}, \quad W_3 = V_3 \cup \{3\}.

As the sum of types of all triples within one quadruple is even, \{W_0, W_1, W_2, W_3\}
form a partition of \{1, 2, \ldots, n\}. Notice that 4 \in V_j.

We claim that t(i, j, 3) = 0 for any i \in V_1 and j \in V_2 (if V_2 is empty, this
statement becomes trivial). Indeed, by the definition of V_1 and V_2, t(1, 2, 3) = t(1, 3, 3) = t(1, j, 3) = 0 while t(i, 1, 2), t(i, 1, 3), t(j, 2, 3), t(j, 1, 2) are odd.
Suppose, t(i, j, 3) is odd. Then t(i, j, 1) \equiv t(i, j, 3) + t(i, 1, 3) + t(j, 1, 3) \equiv 0
(mod 2) and t(i, j, 2) \equiv t(i, j, 3) + t(i, 2, 3) + t(j, 2, 3) \equiv 0 (mod 2), which
means that there are 5 triples of type 0 in cyclic pattern:

\[ t(i, j, 1) = t(j, i, 3) = t(1, 3, 2) = t(3, 2, i) = t(2, i, j) = 0. \]

If so, the principal minor of C, formed by rows and columns i, j, 1, 2, 3, must
carry equal off-diagonal entries within each row. But then we would have
\[ t(i, j, 3) = 0 \]
which contradicts with the initial assumption \[ t(i, j, 3) \equiv 1 \] (mod 2).
Therefore, \[ t(i, j, 3) = 0. \]

Similarly, \[ t(i, 2, k) = 0 \] for any \[ i \in V_1, k \in V_3. \]

For every \[ i \in (W_0 \cup W_1 \setminus \{1, 4\}) \], we have \[ t(i, 2, 3) = 0 \], hence \[ c_{2i} = c_{23} = 0, \]
\[ c_{3i} = c_{32} = 0, \] and \[ c_{12} = c_{13}. \] If \[ c_{12} = 1 \], apply switching operation to vertex\[ i \] to
ensure \[ c_{2i} = c_{i3} = 0. \] Now we have \[ c_{2i} = c_{23} = c_{3i} = c_{32} = c_{12} = c_{13} = 0 \] for all
\[ i \in W_0 \cup W_1. \]

For any \[ i \in V_1 \] and \[ j \in V_2 \], we have \[ t(i, j, 3) = 0 \], hence \[ c_{ij} = c_{i3} = 0. \] The
same is true when \[ i = 1 \] or \[ j = 2 \]. Therefore, \[ c_{ij} = 0 \] for all \[ i \in W_1 \] and \[ j \in W_2 \],
and similarly, \[ c_{ik} = 0 \] for all \[ i \in W_1 \] and \[ k \in W_3. \]

Denote by \[ F \] the subgraph of \[ D \] spanned by \[ W_1 \]. If \[ i, j \in W_1, i \neq j, \]
and \[ c_{ij} = c_{ji} = 1 \], then \[ t(i, j, 2) = 2 \] (since \[ c_{12} = c_{2i} = c_{j2} = c_{2j} = 0 \]). Hence, \[ F \] does
not have a pair of opposite arcs.

We know that \[ c_{14} = 1, c_{41} = 0 \], so \[ F \] contains a transitive tournament
of size 2 spanned by vertices 1 and 4, and 1 is the vertex of zero in-degree in
this tournament. Let \[ T \] be a transitive tournament of the largest possible size
contained in \[ F \] such that 1 is its vertex of zero in-degree. Let \[ i \] denote the vertex
of zero out-degree in \[ T \]. As the out-degree of \[ i \] in the whole \[ D \] is odd, there
exists vertex \[ j \], distinct from \[ i \] and 1, such that \[ c_{ij} = 1. \] As \[ c_{ij} = 1, \] \[ j \] cannot
belong to \[ W_2 \] or \[ W_3 \]. The two remaining cases are \[ j \in W_1 \] and \[ j \in W_0. \]

If \[ j \in W_1 \], then \[ c_{ji} = 0 \]. By the maximality of \[ T \], there exists a vertex \[ k \] in
\[ T \] such that \[ c_{kj} = 0. \] As \[ i \] is the zero out-degree vertex in tournament \[ T \], we
get \[ c_{ki} = 1 \] and \[ c_{ik} = 0. \] If \[ c_{jk} = 0 \], we get \[ t(k, i, j) = 2 \]. If \[ c_{jk} = 1 \], we get
\[ t(k, i, j, 2, 3) = 3. \]

If \[ j \in W_0 \], then \[ t(1, 2, j) = 0 \]. Hence, \[ c_{1j} = c_{12} = 0 \] and \[ c_{j1} = c_{j2} = 0. \] As
1 is the zero in-degree vertex in tournament \[ T \], we get \[ c_{1i} = 1 \] and \[ c_{11} = 0. \] If
\[ c_{ji} = 0 \], we get \[ t(1, i, j) = 2 \]. If \[ c_{ji} = 1 \], we get \[ t(1, i, j, 2, 3) = 3. \]

\[ \square \]

**Lemma 6.10.** Let \( m \) be even, and \( M \) be a \((2m+1) \times (4m+2)\) matrix in binormal
form. One can pick up \( 2m \) columns in \( M \) whose sum is the \((2m+1)\)-dimensional
zero vector.
Proof. Let $n = 2m + 1$, $M = [a_{ij}]$. Let $C = [c_{ij}]$ be an $n \times n$ binary matrix where $c_{ij} = a_{i,2j-1} = a_{i,2j}$ for $i \neq j$. The values of diagonal entries $c_{ii}$ are not important and may be set to zero. For a subset $I \subseteq \{1,2,\ldots,n\}$ and $i \in I$, we denote $\sigma_i(I) = \sum_{j \in I \setminus \{i\}} c_{ij}$. We say that $I$ is of type $t$ if among $|I|$ values $\sigma_i(I)$ with $i \in I$ there are exactly $t$ that are equal to 1. Let $t(I)$ denote the type of $I$.

Similarly to the proof of Lemma 6.6, we can assume that the sum of off-diagonal entries in each row of $C$ is equal to 1. Hence, $t(\{1,2,\ldots,n\}) = n$.

We claim that if there exists $I \subseteq \{1,2,\ldots,n\}$ such that $|I| = 2t(I) - 1$, then one can pick up $2m$ columns in $M$ with zero sum. Indeed, as $t(1,2,\ldots,n) = n$, we have $|I| \neq n$, so $|I| \leq n - 2$. Without limiting generality, we may assume that $I = \{n - 2t + 2, n - 2t + 3, \ldots, n\}$ where $t = t(I)$, $2t \leq n - 1$, $\sigma_i(I) = 1$ for $n - 2t + 2 \leq i \leq n - t + 1$, and $\sigma_i(I) = 0$ for $n - t + 2 \leq i \leq n$. As $M[1 : n - 2t + 1, 1 : 2(n - 2t + 1)]$ is in binormal form, by Lemma 3.1, there is a set of indices $j(i) \in \{2i - 1, 2i\}$ ($i = 1,2,\ldots,n - 2t + 1$) such that the sum of columns $j(1), j(2), \ldots, j(n - 2t + 1)$ in $M[1 : n - 2t + 1, 1 : 2(n - 2t + 1)]$ is equal to the $(n - 2t + 1)$-dimensional zero vector. As

$$\sum_{j=1}^{n-2t+1} c_{ij} = 1 - \sigma_i(I)$$

for $n - 2t + 2 \leq i \leq n$, the sum of columns $j(1), j(2), \ldots, j(n - 2t + 1)$ in $M$ has the last $t - 1$ entries equal to 1, and the rest equal to 0. Columns $2r - 1$ and $2r$ in $M$ differ only in the $r$'th row, so the sum of columns $2n - 2t + 3, 2n - 2t + 4, \ldots, 2n$ in $M$ also has the last $t - 1$ entries equal to 1, and the rest equal to 0. Set $j(i) = n + 1 + i$ for $n - 2t + 2 \leq i \leq n - 1$. Then the sum of columns $j(1), j(2), \ldots, j(n - 1)$ in $M$ is the $n$-dimensional zero vector.

Let $D$ be a digraph with vertices $1,2,\ldots,n$ where an arc $(i,j)$ is present if and only if $c_{ij} = 1$. As $n \equiv 1$ (mod 4), and every vertex has odd out-degree, $D$ satisfies conditions of Lemma 6.9. Thus, there is a subset $I \subset \{1,2,\ldots,n\}$ (a triple or a quintuple) such that $|I| = 2t(I) - 1$. As we just have shown, it guarantees the existence of $2m$ columns in $M$ with zero sum.

Lemma 6.11. Let $m$ be even, and $M$ be a $(2m+1) \times (4m+3)$ matrix in binormal form where the sum of all columns is a zero vector. If $M$ has two identical columns, then it also has a set of $2m$ columns with zero sum that includes at most one of the two identical columns.

Proof. Set $n = 2m + 1$. Let $x_i = (a_{1i}, a_{2i}, \ldots, a_{ni})^T$ be the $i$th column of $M$ ($i = 1,2,\ldots,2n+1$). As $M$ is in binormal form, we can define $n \times n$ matrix $C = [c_{ij}]$ where $c_{ij} = a_{i,2j-1} = a_{i,2j}$ for $i \neq j$. The values of diagonal entries $c_{ii}$ are not important and may be set arbitrarily.

If one of the two columns that are identical is the last column, then the statement of the lemma follows from Lemma 6.10. Hence, without limiting generality, we may assume that the two identical columns are $2n - 2$ and $2n$.

Suppose, $c_{n,n-1} = 1$. Then $a_{n,2n-2} = 1$. As $x_{2n} = x_{2n-2}$, we get $a_{n,2n} = 1$ and $a_{n,2n-1} = 0$. As $M$ is in binormal form, $x_1 + x_2 + \ldots + x_{2n} = (1,1,\ldots,1)^T$. As the sum of all columns of $M$ is a zero vector, $x_{2n+1} = (1,1,\ldots,1)^T$. Add row $n$ to each row $i \in \{1,2,\ldots,n-1\}$ where $a_{i,2n-1} \neq a_{i,2n+1}$. After this is done, columns $2n - 1$ and $2n + 1$ differ only in the last entry. Now swap columns
2n and 2n + 1. The resulting matrix \( M' \) is in binormal form. By Lemma 6.10, there is a set of 2m = n − 1 columns in \( M' \) with zero sum that does not include column 2n + 1 of \( M' \) (which originated from column 2n of \( M \)). In this case, \( M \) has a set of 2m columns with zero sum that does not include column 2n. Hence, we may assume \( c_{n,n-1} = 0 \), and similarly, \( c_{n-1,n} = 0 \).

The \((n - 1) \times (2m - 2)\) submatrix \( M[1 : n - 1, 1 : 2m - 2] \) is in binormal form. By Lemma 3.1, there is a set of indices \( j(i) \in \{2i - 1, 2i\} \) \( (i = 1, 2, \ldots, n - 1) \) such that the sum of columns \( j(1), j(2), \ldots, j(n - 1) \) in \( M[1 : n - 1, 1 : 2m - 2] \) is the \((n - 1)\)-dimensional zero vector. We recall that \( c_{n,n-1} = 0 \). If \( \sum_{j=1}^{n-2} c_{n,j} = 0 \), then the sum of columns \( j(1), j(2), \ldots, j(n - 1) \) in \( M \) is the \( n \)-dimensional zero vector. Hence, we may assume \( \sum_{j=1}^{n-2} c_{n,j} = 1 \), and similarly, \( \sum_{j=1}^{n-2} c_{n-1,j} = 1 \).

The \((n - 2) \times (2m - 4)\) submatrix \( M[1 : n - 2, 1 : 2m - 4] \) is in binormal form. By Lemma 3.1, there is a set of indices \( j(i) \in \{2i - 1, 2i\} \) \( (i = 1, 2, \ldots, n - 2) \) such that \( x_{j(1)} + x_{j(2)} + \ldots + x_{j(n-2)} \) has the first \( n - 2 \) entries equal to 1. Since \( \sum_{j=1}^{n-2} c_{n-1,j} = 1 \) and \( \sum_{j=1}^{n-2} c_{n,j} = 1 \), the last two entries of \( x_{j(1)} + x_{j(2)} + \ldots + x_{j(n-2)} \) are also equal to 1. As \( x_{2n+1} = (1, 1, \ldots, 1)^T \), we get \( x_{j(1)} + x_{j(2)} + \ldots + x_{j(n-2)} + x_{2n+1} = 0 \).

**Proof of Theorem 5.15.** By (4.3) and Theorem 5.10, \( s_{2m}(2m + 1) \geq s_{2m}(2m + 1) = 4m + 2 \). To prove \( s_{2m}(2m + 1) \leq 4m + 2 \), consider a sequence \( S \) of size \( 4m + 2 \) over \( \mathbb{Z}_2^{2m+1} \). Let \( M \) be a \((2m + 1) \times (4m + 2)\) binary matrix whose columns represent the vectors from \( S \). We need to prove that \( M \) has a set of \( 2m \) columns whose sum is a zero vector.

By Theorem 5.10, \( 4m + 2 \geq s_{2m}(2m) \). If the rank of \( M \) is less than \( 2m + 1 \), then by Lemma 6.4, there is a set of \( 2m \) columns that sum up to a zero vector. Suppose that the rank of \( M \) is \( 2m + 1 \). Let \( x \in \mathbb{Z}_2^{2m+1} \) be the sum of all columns of \( M \), and \( x_{4m+2} \in \mathbb{Z}_2^{2m+1} \) be the last column. Add \( x + x_{4m+2} \) to each column of \( M \), and expand the matrix by a new column equal to \( x \). In the resulting \((2m + 1) \times (4m + 3)\) matrix \( M' \), the sum of all columns is a zero vector, and the last two columns are equal to \( x \). \( M' \) satisfies conditions of Lemma 3.2 and can be brought to binormal form \( M'' \) by permutations of the columns and additions of one row to another. There are two identical columns in \( M'' \) that originated from the last two columns of \( M' \). By Lemma 6.11, there is a set of \( 2m \) columns in \( M'' \) with zero sum which includes at most one of these two columns. By Observation 6.3, \( M' \) has a set of \( 2m \) columns with zero sum that does not include the last column. Therefore, \( M \) has a set of \( 2m \) columns with zero sum.

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