Infinite Symmetry in the Fractional Quantum Hall Effect

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Abstract

We have generalized recent results of Cappelli, Trugenberger and Zemba on the integer quantum Hall effect constructing explicitly a $\mathcal{W}_{1+\infty}$ for the fractional quantum Hall effect such that the negative modes annihilate the Laughlin wave functions. This generalization has a nice interpretation in Jain’s composite fermion theory. Furthermore, for these models we have calculated the wave functions of the edge excitations viewing them as area preserving deformations of an incompressible quantum droplet, and have shown that the $\mathcal{W}_{1+\infty}$ is the underlying symmetry of the edge excitations in the fractional quantum Hall effect. Finally, we have applied this method to more general wave functions.

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1 Introduction

The experimental discoveries of the integer quantum Hall effect (IQHE) \[1\] and of the fractional quantum Hall effect (FQHE) \[2\] are one of the most interesting physical phenomena in solid state physics in recent years. The conductance of a two-dimensional electron gas in a high magnetic field at low temperature exhibits quantized plateau values of the form \(\sigma_{xy} = \frac{e^2}{h}\nu\), where the filling factor \(\nu\) is an integer or fractional number. In many respects, both the integer as well as the fractional effect share very similar underlying physical characteristics and concepts, for instance the two-dimensionality of the system, the quantization of the Hall conductance with simultaneous vanishing of the longitudinal resistance, and the interplay between disorder and the magnetic field giving rise to the existence of extended states. In other respects, they encompass entirely different physical principles and ideas. In particular, while the IQHE is thought of essentially as a noninteracting electron phenomenon \[3\], the FQHE is believed to arise from a condensation of the two-dimensional electrons into a new incompressible state of matter as a result of interelectron interaction \[4\], see also \[5, 6\].

An important step was taken by Laughlin writing down the wave functions for the fundamental fractions \(\nu = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\) which played a special role in a hierarchical scheme in which a daughter state was obtained at each step from a condensation of quasiparticles of the parent state into a correlated low-energy state \[7, 8\]. Extensive calculations have proven these wave functions to be extremely close to the numerical exact solutions \[5\].

In the last years J.K. Jain developed the composite fermion theory which could describe IQHE and FQHE by a common principle attaching to each electron an even number of magnetic flux quanta which gives an easy explanation of the experimentally observed fractional fillings as well as a new derivation of Laughlin’s wave functions starting from the well understood IQHE \[3\]. New experiments are in good agreement with this theory \[9, 10, 11, 12\].

The incompressibility of these quantum fluids is explained by an finite energy gap above the ground state. Recently, for the IQHE \((\nu = 1)\) it was shown that incompressibility also results in an infinite symmetry which describes the area preserving nonsingular deformations of the quantum droplet and commutes with the Hamiltonian \[13\]. The quantization of this symmetry is well known in physics as the nonsingular part of a \(\mathcal{W}_{1+\infty}\) and arises e.g. in string theories or two dimensional gravity \[14, 15, 16\]. These deformations are directly related to edge excitations which should live on the one dimensional boundaries and were studied by a number of authors \[17, 18, 19, 21, 22\]. The dynamics of these edge states is mainly based on the relation of Chern-Simons gauge theories and conformal field theory \[23\].

In this paper we give a generalization of this infinite symmetry to the FQHE \((\nu = \frac{1}{2p+1})\) showing that the Laughlin wave functions are annihilated by the negative, nonsingular generators of the \(\mathcal{W}_{1+\infty}\). The very interesting point is that,
constructing the $W_{1+\infty}$ for the FQHE, interelectron interaction effects enter which automatically cancel out for the IQHE and agree with the result of reference \[13\]. Furthermore, we can show that these interactions can be interpreted as arising from an even number of magnetic flux quanta which are attached to the electrons as in the composite fermion picture. It turns out that the interaction is hidden in a nontrivial measure which is the $N$-point function of $N$ localized flux quanta vortices and should be described by an abelian Chern-Simons theory. Viewing the QHE-states as a droplet of an incompressible quantum fluid, the gapless edge excitations can be interpreted coming from surface waves or area preserving deformations of the droplet. We have calculated the wave functions for edge excitations with $\nu = \frac{1}{2p+1}$ using the fact that they are generated by the positive modes of the $W_{1+\infty}$. Here our result agrees with former ones of M. Stone for $\nu = 1$ and X.G. Wen for the FQHE \[24, 18\]. Finally, we apply the previous method to more general wave functions describing multi layer systems or systems of interacting Landau levels for every fractional filling and show that the $W_{1+\infty}$ is indeed the fundamental symmetry of the edge excitations.

The paper is organized as follows: First we give an introduction to the basics of the QHE. Next we show how to generalize the construction of the $W_{1+\infty}$ from the IQHE to the FQHE and interprete this generalization by Jain’s composite fermion theory. Then we calculate the wave functions of the edge excitations using the $W_{1+\infty}$. Finally, we consider the case of more general wave functions.

2 Preliminaries

Let us start by reviewing some elementary facts about a two dimensional electron in an uniform, transverse magnetic field $B$. The Schrödinger equation for such an electron is given by

$$H \psi = \frac{1}{2m}(p - \frac{e}{c} A)^2 \psi = E \psi,$$  

(2.1)

where the momentum $p = -i\hbar \nabla$ and the gauge potential $A$ live in the plane. This problem can be solved exactly. Let us choose the symmetric gauge $A = \frac{B}{2}(-y, x)$ and introduce complex variables: $z = x + iy$, $\bar{z} = x - iy$ and $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. Defining all lengths in units of the magnetic length,

$$l = \left(\frac{2\hbar c}{eB}\right)^{\frac{1}{2}},$$  

(2.2)

and the energies in units of the Landau level spacing,

$$\omega_c = \frac{eB}{mc},$$  

(2.3)
the Hamiltonian can be reexpressed as:

$$H = 2\hbar \omega_c l^2 \left( -\partial \bar{\partial} + \frac{1}{2l^2} (\bar{z} \partial - z \bar{\partial}) + \frac{1}{4l^4} z \bar{z} \right).$$  
(2.4)

Letting $\hbar = m = l = 1$ the hamiltonian and the angular momentum $J$ can be written in terms of a pair of independent harmonic oscillators:

$$H = a^\dagger a + a a^\dagger,$$  
(2.5)

$$J = b^\dagger b - a^\dagger a,$$  
(2.6)

where these operators are

$$a = \frac{z}{2} + \partial, \quad a^\dagger = \frac{\bar{z}}{2} - \partial,$$  
(2.7)

$$b = \frac{\bar{z}}{2} + \partial, \quad b^\dagger = \frac{z}{2} - \bar{\partial},$$  
(2.8)

and satisfy the commutation relations

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1,$$  
(2.9)

with all other commutators vanishing. The vacuum is determined by the condition $a\psi_{0,0} = b\psi_{0,0} = 0$ and given as

$$\psi_{0,0} = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{1}{2} |z|^2\right).$$  
(2.10)

In terms of the operators $a^\dagger$ and $b^\dagger$ the solutions can finally be written as

$$\psi_{n,l} = \frac{(b^\dagger)^l (a^\dagger)^n}{\sqrt{l! n!}} \psi_{0,0}$$  
(2.11)

with energy $E_n = 2n + 1$, which determines the Landau level. These energy states are infinitely degenerate due to the rotational invariance around the z-axis. It is useful to note that in the lowest Landau level the polynomial part of the wave function is holomorphic, and involves in the second Landau level at most one power of $\bar{z}$. In general the highest power of $\bar{z}$ in the $n^{th}$-Landau level is $n - 1$.

In a finite sample of area $A$ one can show, that the degeneracy of each Landau level is determined by the number of the magnetic flux quanta

$$N_A = \frac{\Phi_{\text{mag}}}{\Phi_0},$$  
(2.12)

where $\Phi_{\text{mag}} = BA$ is the magnetic flux through the area $A$ and $\Phi_0 = \hbar/e$ is a single flux quantum.

Let us now consider the case of $N$ such electrons. If there is no interaction between them, the many particle problem splits into $N$ copies of the single particle
problem. Therefore, we just get $N$ operators, identical to the single particle operators $a, b$, but now labelled by an index $i$ referring to the coordinate of the $i$th electron: $a_i, b_i$. Since the magnetic field $B$ controls the number of states and thus the density of electrons per state, its action can be considered as an external pressure. Actually, the electron density per state is the correct quantum measure of the electron density, the filling fraction $\nu$

$$\nu = \frac{N}{N_A}. \quad (2.13)$$

The IQHE is well understood by a gauge argument of Laughlin. Later it was shown that the conductivity can be interpreted as the Chern character of a $U(1)$-fibre bundle over a torus [25, 26, 27] or as an element in the cyclic cohomology of a $C^*$-algebra [28].

For the FQHE with filling fraction $\nu = \frac{n}{2p+1}$ by numerical experiments Laughlin found the groundstates given by the following wave functions:

$$\psi_p = \prod_{i<j} (z_i - z_j)^{2p+1} \exp\left(-\frac{1}{2} \sum_i |z_i|^2\right), \quad (2.14)$$

where $p$ should be an integer to respect the Pauli principle. In the composite fermion theory this wave function was reinterpreted by J.K. Jain as a wave function not of bare single electrons, but of electrons bound to an even (here $2p$) number of vortices or flux quanta. Starting with the wave function $\phi_n$ of the IQHE with filling fraction $\nu = n$ one attaches $2p$ flux quanta to each electron, which is given by multiplying $\phi_n$ with $D^{2p}$,

$$\psi_\nu = D^{2p}\phi_n \quad \text{with} \quad D = \prod_{i<j} (z_i - z_j). \quad (2.15)$$

Using mean field arguments, this leads to an electron state in which $n^{-1} \pm 2p$ flux quanta are available to each electron. Thus this composite fermion state has filling fraction [3]

$$\nu = \frac{n}{2pn \pm 1}. \quad (2.16)$$

Thus, the Laughlin wave functions are given for $n = 1$. When calculating some expectation values via path integrals, only closed paths contribute to the partition function, because it is the trace of $\exp(-\beta H)$. Closed paths are given by exchanging electrons or by moving them around each other. The phase associated with each path has two contributions. One is the statistical phase due to the Fermi statistics of the electrons, and the other one is the Aharonov-Bohm phase due to the flux enclosed in the loop. But adding to a fermion an even number of flux quanta again gives a fermion and also the Aharonov-Bohm phase factor is the same because a flux quantum produces a phase factor of unity. Thus, adding an even number of flux quanta to each electron does not change the expectation values. This argument is due to A. Lopez and E. Fradkin [29].
The wave functions of the last section should describe the condensation of the electrons to new states of matter, to incompressible quantum superfluids. Normally, the incompressibility is explained by a finite energy gap above the ground state. Recently, A. Cappelli, C.A. Trugenberger and G.R. Zemba have given another explanation of this incompressibility for the $\nu = 1$ case; they have found a $W_{1+\infty}$ symmetry which is the algebra of the area preserving nonsingular diffeomorphisms commuting with the Hamiltonian of the system, defining an incompressible state now to be a highest weight vector of the $W_{1+\infty}$ [13]. They constructed the generators of the $W_{1+\infty}$ in the following way:

$$L_{m,n} = \sum_{i=1}^{N} (b_i^\dagger)^{m+1}(b_i)^{n+1} \quad \text{for} \quad n, m \geq -1.$$ \hspace{1cm} (3.1)

These generators commute with the Hamiltonian of the system and fulfill the following commutation relations:

$$[L_{n,m}, L_{k,l}] = \sum_{s=0}^{\min(m,k)} \frac{(m+1)!(k+1)!}{(m-s)!(k-s)!(s+1)!} L_{n+k-s,m+l-s} - (m \leftrightarrow l, n \leftrightarrow k).$$ \hspace{1cm} (3.2)

Then they have shown that

$$L_{m,n}\psi_0 = 0 \quad \text{for} \quad n > m \geq -1,$$ \hspace{1cm} (3.3)

which means that $\psi_0$ is a highest weight vector of the algebra of area preserving nonsingular diffeomorphisms.

The aim of our paper is to generalize this result to the FQHE. We are doing this by changing the definition of the $b_i$ introducing an interaction term in the following way: ($b_i^\dagger$ remains unchanged)

$$b_i = \partial_i + \frac{\bar{z}_i}{2} - 2p \sum_{i \neq j} \frac{1}{\bar{z}_i - z_j}.$$ \hspace{1cm} (3.4)

For $p = 0$ one recovers the original definition for the $b_i$ and $b_i^\dagger$ as before, so we are not changing our notation. The commutators of the $b_i$ and $b_i^\dagger$ change in the following way

$$[b_i, b_j^\dagger] = 1 + 2p\pi \sum_{i \neq j} \delta(z_i - z_j),$$ \hspace{1cm} (3.5)

$$[b_i, b_j^\dagger] = -2p\pi\delta(z_i - z_j) \quad \text{for} \quad i \neq j.$$ \hspace{1cm} (3.6)

Defining the $L_{mn}$ as above but with the new $b_i$ they fulfill the commutation relations of the same $W_{1+\infty}$ up to terms involving delta functions. In the case
of fermions, which have to respect the Pauli principle, the delta functions do not
contribute, since the wave function has to approach zero for \( z_i \to z_j, i \neq j \).
For the first Landau level, where the wave functions are holomorphic up to the
exponential term, one can rewrite the operators \( b_i \) and \( b_i^\dagger \) such that they only act
on the holomorphic part,

\[
b_i = \partial_i - 2p \sum_{i \neq j} \frac{1}{z_i - z_j}, \tag{3.7}
\]

\[
b_i^\dagger = z_i. \tag{3.8}
\]

Note, that \( b_i^\dagger \) just acts by multiplication. Thus, in the case of the first Landau
level no delta functions will occur. In the standard notation of \( \mathcal{W}_{1+\infty} \) we set

\[
W_n^{(s)} \sim \mathcal{L}_{n+s-2,s-2}, \quad s \geq 1, \quad n \geq -s+1, \tag{3.9}
\]

where \( W_n^{(s)} \) is the \( n^{th} \)-Fourier mode of a spin \( s \) field. After some calculations
which can be found in the appendix one obtains the action of the modes \( W_n^{(s)} \) on
the Laughlin wave function \( \psi_p \):

\[
W_n^{(s)} \psi_p = (s - 1)! \sum_{1 \leq j_0 < j_1 < \ldots < j_{s-1} \leq N} \psi_p
\]

\[
\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
\zeta_{j_0} & \zeta_{j_1} & \cdots & \zeta_{j_{s-2}} & \zeta_{j_{s-1}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\zeta_{j_0}^{s-2} & \zeta_{j_1}^{s-2} & \cdots & \zeta_{j_{s-2}}^{s-2} & \zeta_{j_{s-1}}^{s-2} \\
\zeta_{j_0}^{s-1+n} & \zeta_{j_1}^{s-1+n} & \cdots & \zeta_{j_{s-2}}^{s-1+n} & \zeta_{j_{s-1}}^{s-1+n}
\end{array}
\]

(3.10)

from which immediately follows that acting on \( \psi_p \) the negative modes vanish,

\[
W_n^{(s)} \psi_p = 0 \quad \text{for} \quad -s < n \leq -1. \tag{3.11}
\]

Moreover, the states \( \psi_p \) are eigenstates for the zero modes,

\[
W_0^{(s)} \psi_p = (s - 1)! \left( \frac{N}{s} \right) \psi_p. \tag{3.12}
\]

Let us emphasize this result: We have shown, that the Laughlin wave functions
are highest weight states of the quantized algebra of non singular area preserving
diffeomorphisms which means that all surface waves on the droplet move in the
same direction. The singular deformations cannot be included in the algebra in
that way, since they would change the topology of the droplet. Now, we have a common formulation for $\nu = 1$ and $\nu = \frac{1}{2p+1} \text{QHE}$, where automatically the IQHE is described by an single electron theory, but the FQHE needs interelectron interaction in the first Landau level.

At that point the reader may worry that $b_i$ and $b_i^\dagger$ are not hermitian conjugate. However, we can take an inner product of the form

$$< \Psi_1 | \Psi_2 > = \int \Psi_1^\dagger \mu \Psi_2,$$  
(3.13)

where $\mu$ is given as:

$$\mu(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) = \prod_{i<j} | z_i - z_j |^{-4p}.$$  
(3.14)

Using this inner product $b_i$ and $b_i^\dagger$ become hermitian conjugate to each other. One will see that this measure is very important in the following, especially for the interpretation of the new interaction term in the $b_i$s. Namely introducing the nontrivial measure the hamiltonian would be non hermitian. Thus we have to change the definition of the $a_i^\dagger$ in the following way: ($a_i$ remains also unchanged)

$$a_i^\dagger = -\partial_i + \frac{\bar{z}_i}{2} + 2p \sum_{i \neq j} \frac{1}{z_i - z_j}.$$  
(3.15)

The commutation relations are now given as:

$$[a_i, a_i^\dagger] = 1 - 2p \pi \sum_{i \neq j} \delta(z_i - z_j),$$  
(3.16)

$$[a_i, a_j^\dagger] = 2p \pi \delta(z_i - z_j) \text{ for } i \neq j.$$  
(3.17)

The hamiltonian is defined as before

$$H = \sum_{i=1}^{N} (a_i a_i^\dagger + a_i^\dagger a_i)$$  
(3.18)

and commutes with the $W_{1+\infty}$ without occurrence of any delta functions. The Landau level structure is not destroyed and the Laughlin wave function for $\nu = \frac{1}{2p+1}$ is an eigenfunction in the lowest Landau level.

The configuration space for distinguishable particles is given by

$$C_N = \{ (z_1, \ldots, z_N) \in \mathbb{C}^N; \ z_i \neq z_j \text{ for } i \neq j \}.$$  
(3.19)

The $(a_i, a_i^\dagger)$ can be considered as covariant derivatives on a $U(1) \otimes \ldots \otimes U(1)$ bundle over $C_N$ as in the paper of E. Verlinde on the non-abelian Aharanov-Bohm effect [30]. Thus, the curvature is given by (3.14) which describes a constant magnetic field plus $2p$ flux quanta added to each electron. This is exactly the
FQHE interpretation of J.K. Jain by the composite fermion theory mentioned in the previous section. These flux quanta can be described in an abelian Chern-Simons theory by localized Wilson loops. Considering the N-point function of these flux quanta localized at the positions $z_i$ of the electrons one sees that it is proportional to the measure $\mu$ (3.14) using that those Wilson loop operators can be expressed by vertex operators [31]. This explains the former observation on the relation between vertex operator correlators and the Laughlin wave function [22, 24, 33, 34].

This picture is in good agreement with the argument of Lopez and Fradkin stated previously that adding an even number of flux quanta to each electron leaves all expectation values invariant. Calculating the expectation values of the Laughlin wave function one also has to introduce the measure $\mu$ (3.14):

$$\int \psi_p^\dagger \mu \psi_p \, dz^N.$$  (3.20)

It is easy to see that this expression is independent of $p$, thus, adding flux quanta does not change the expectation value.

Thus, in our formulation of the FQHE we consider a hamiltonian without explicit interelectron interaction as in the IQHE, but describing the interaction with the help of a nontrivial measure coming from the N-point correlation function of the flux quanta in an abelian Chern-Simons theory.

## 4 Edge excitations

Halperin was the first who pointed out that the IQHE states contain gapless edge excitations, which are responsible for nontrivial transport properties [17]. Using gauge arguments, one can easily show that FQHE states also support gapless edge excitations. X.G. Wen has shown that these states span a representation of a Kac-Moody current algebra [18] and M. Stone has described them using Schur functions or homogenous symmetric polynomials [24]. In this section we derive these results with the help of the $\mathcal{W}_{1+\infty}$.

Viewing the QHE-states as a droplet of an incompressible quantum fluid we consider the edge excitations as area preserving deformations of the droplet, which are described by the $\mathcal{W}_{1+\infty}$. Thus, the highest weight representation on the QHE wave function should give the spectrum of these edge states:

$$W_{n_1}^{(s_1)} W_{n_2}^{(s_2)} \ldots W_{n_k}^{(s_k)} \psi_p , \ s_i \geq s_{i+1} . \ n_i \geq n_{i+1} \text{ if } s_i = s_{i+1} .$$  (4.1)

In fact, equation (3.10) shows, that applying one mode of a $\mathcal{W}_{1+\infty}$ generator to $\psi_p$ yields $\psi_p$ multiplied by a symmetric function, since the fraction of the determinants is equal to the Schur function $S^{(0,0,\ldots,0,n)}$. Actually, every Schur function
can be written as a fraction of a certain determinant and the Vandermonde determinant. If we use the notation

\[
D\{m_1, m_2, \ldots, m_n\} = \begin{vmatrix}
  z_1^{m_1} & z_2^{m_1} & \cdots & z_n^{m_1} \\
  z_1^{1+m_2} & z_2^{1+m_2} & \cdots & z_n^{1+m_2} \\
  \vdots & \vdots & \ddots & \vdots \\
  z_1^{n-1+m_n} & z_2^{n-1+m_n} & \cdots & z_n^{n-1+m_n}
\end{vmatrix},
\]

(4.2)

then the Schur functions can be expressed as

\[
S\{m_1, m_2, \ldots, m_n\} = \frac{D\{m_1, m_2, \ldots, m_n\}}{D\{0, 0, \ldots, 0\}}.
\]

(4.3)

By induction it follows that monomials as in (4.1) also yield polynomial symmetric functions multiplied with \(\psi_p\). The reason is that any such state must be totally antisymmetric due to the Pauli principle. But since \(\psi_p\) is always reproduced, the only way to get a totally antisymmetric polynomial function is to multiply \(\psi_p\) by a totally symmetric one. Moreover, the current \(j \equiv W^{(1)}\) already yields a complete set of symmetric functions, namely the products of power sums \(s_k = \sum_{i=1}^N z_i^k\),

\[
\hat{j}_{n_1} \hat{j}_{n_2} \cdots \hat{j}_{n_k} \psi_p = s_{n_1} s_{n_2} \cdots s_{n_k} \psi_p.
\]

(4.4)

This is a basis of all symmetric functions provided \(n_i \geq n_{i+1}\). To see this, one just has to note that the action of

\[
W_k^{(1)} = \sum_{i=1}^N (b_i^\dagger)^k
\]

(4.5)
on \(f(z_1, \ldots, z_N) \exp(-\frac{1}{2} \sum_{i=1}^N |z_i|^2)\) with \(f(z_1, \ldots, z_N)\) any holomorphic function on \(\mathbb{C}^N\) is just given by the multiplication with \(s_k\) which is a holomorphic function for \(k \geq 0\).

Thus, the \(W_{1+\infty}\) algebra yields all possible edge excitations which respect the Pauli principle. The resulting spectrum is given by the set of all symmetric polynomial functions with the partition function being nothing but

\[
Z(q = e^{2\pi i \tau}) = \sum_{n=0}^\infty p(n) q^n = \prod_{n=1}^\infty \frac{1}{1 - q^n},
\]

(4.6)

where \(p(n)\) denotes the number of partitions of \(n\) in positive integers. Thus, the positive modes of the current \(j \equiv W^{(1)}\) alone generate all edge excitations which means that these excitations can be interpreted as surface waves moving in the same direction and moving with the same velocity. Therefore the spectrum is equivalent with that of the \(U(1)\)-Kac-Moody algebra at level 1. In this way the results of X.G. Wen and M. Stone reappear in an unified way [18, 24].
These considerations show that the conformal field theory which corresponds to the Chern Simons theory describing the attachment of flux quanta to the electrons and which is defined on the boundary of the system (the Laughlin droplet) must be generated by a \( U(1) \)-Kac-Moody current. Thus, it follows that the conformal theory must have the effective central charge \( c_{\text{eff}} = 1 \). This agrees with the fact that the non-trivial measure introduced in the third section, where it arose from the Knizhnik-Zamolodchikov connection describing the effect of the attached flux quanta, is given by a correlation function of a \( c_{\text{eff}} = 1 \) conformal field theory.

5 Generalizations

There exist a lot of other examples of trial wavefunctions not only for filling fraction \( \nu = \frac{1}{2p+1} \). Most of these wavefunctions have the following structure \([18, 21, 35, 36]\):

\[
\psi_K = \prod_{i<j} (z_i^I - z_j^J)^{K_{i,j}} \prod_I \prod_{i<j} (z_i^I - z_j^J)^{K_{i,I}} \exp\left(-\frac{1}{2} \sum_{i,I} |z_i^I|^2 \right),
\]

(5.1)

where \( K \) is a symmetric, integer valued \( m \times m \) Matrix with odd integers on the main diagonal. Then, the filling fraction is given by

\[
\nu = \sum_{I,J} (K^{-1})_{I,J}.
\]

(5.2)

Thus, one can get different wave functions for the same filling fraction \( \nu \). The physical picture behind this ansatz is to couple different independent Hall fluids (i.e., sets of eventually interacting Landau levels or different layers). Viewing the filling fraction \( \nu \) to be proportional to the Hall conductivity \( \sigma \) one sees, that the total Hall conductivity is determined by the Hall conductivities of the several Hall fluids (or Landau levels) according to the Kirchhoff rules for coupling them in parallel or in series.

Now, it is easy to see that the \( W_{1+\infty} \) can be constructed in the same way as before defining \( b_i^I \) and \( b_i^{I\dagger} \) as

\[
b_i^I = \partial_i^I + \frac{z_i^I}{2} + \sum_{j \neq i} A_{I,j} \sum_{i<j} \frac{1}{z_i^I - z_j},
\]

\[
b_i^{I\dagger} = -\bar{\partial}_i^I + \frac{z_i^I}{2},
\]

(5.3)

(5.4)

and

\[
\mathcal{L}_{m,n} = \sum_{l,i} (b_i^{I\dagger})^{m+1} (b_i^I)^{n+1}.
\]

(5.5)

The heighest weight condition can be fulfilled, if

\[
K = I - A,
\]

(5.6)
where \( I \) is the \( m \times m \) identity matrix. For example, the \( \nu = \frac{m}{2p+1} \) FQHE can be obtained if \( K \) is given by the following \( m \times m \) matrix [21, 35],

\[
K = \begin{pmatrix}
2p+1 & 2p & \ldots & 2p \\
2p & 2p+1 & 2p & \\
\vdots & \ddots & \ddots & \vdots \\
2p & \ldots & 2p & 2p+1
\end{pmatrix}
\]

(5.7)

Thus, the matrix \( A \) is nothing but

\[
The edge excitations are generated by the action of the \( \mathcal{W}_{1+\infty} \) in a completely analogous manner. But now, if \( m > 1 \), the current \( j \equiv W^{(1)} \) contained in the \( \mathcal{W}_{1+\infty} \) is not longer sufficient to generate all the edge excitations. The partition function (4.6) has to be replaced by its \( m \)th power, i.e. the edge excitations are generated by \( m \) currents [18]. In the same way it is possible to reproduce the hierarchy picture of F.D.M. Haldane and B.I. Halperin [21, 35].

6 Conclusion

In this paper we have shown that the \( \mathcal{W}_{1+\infty} \) is the underlying symmetry in the IQHE as well as in the FQHE which generates all edge excitations. This \( \mathcal{W}_{1+\infty} \) was first introduced in the case \( \nu = 1 \) IQHE by A. Cappelli, C.A. Trugenberger and G.R. Zemba, describing the incompressibility of the quantum droplet. We have shown that the Laughlin wave functions for \( \nu = \frac{1}{2p+1} \) can be interpreted as highest weight vectors of a \( \mathcal{W}_{1+\infty} \) which describes the quantized algebra of the area preserving diffeomorphisms. For this generalization we have introduced an electron-electron interaction term which can be considered as adding flux quanta to each electron as in J.K. Jain’s composite fermion theory. Further, we calculated all edge excitations of this quantum droplet interpreting them as area preserving surface deformations and we could show that these are surface waves which are moving with the same velocity and in the same direction. There exist a lot of other examples of trial wave functions not only for filling fraction \( \nu = \frac{1}{2p+1} \). We have applied our methods to wave functions for multi layer systems and systems of interacting Landau levels. An open question still is how the Coulomb interaction in the solid breaks this symmetry.
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7 Appendix

In this appendix we sketch a derivation of equation (3.10). First, one shows inductively that

$$(b_i)^n \psi_p = \sum_{1 \leq j_k \neq j_l \leq N}^{1 \leq k \neq l \leq n} \prod_{k=1}^{n} \frac{1}{z_{j_k} - z_{j_l}} \psi_p,$$

(7.1)

Thus, the action of $W_n^{(s)}$ on $\psi_p$ is given by

$$W_n^{(s)} \psi_p = \sum_{1 \leq j_k \neq j_l \leq N}^{0 \leq k \neq l \leq s} s_{s-1} \prod_{k=1}^{n} \frac{z^n_{j_0} - z_{j_k}}{z_{j_0} - z_{j_l}} \psi_p.$$

(7.2)

Note, that this expression does not explicitly depend on $p$. Now, we can rewrite the sums in terms of determinants in the following way:

$$\sum_{1 \leq j_k \neq j_l \leq N}^{0 \leq k \neq l \leq s} s_{s} \prod_{k=1}^{n} \frac{z^n_{j_0} - z_{j_k}}{z_{j_0} - z_{j_l}}$$

(7.3)

$$= \sum_{1 \leq j_0 < j_1 < ... < j_s \leq N} s \prod_{l=0}^{s} \left( - \right)^{l+1} z_{j_l}^{s} \prod_{k=1}^{l-1} (z_{j_k} - z_{j_l})^{-1} \prod_{k=l+1}^{s} (z_{j_l} - z_{j_k})^{-1}$$

(7.4)

$$= s! \sum_{1 \leq j_0 < j_1 < ... < j_s \leq N} \prod_{m,n \neq l}^{m<n} \left( z_{j_m} - z_{j_n} \right) \prod_{i<k} \left( z_{j_i} - z_{j_k} \right)$$

(7.5)

where we sum over all fixed point free permutations of $S_{s+1}$ and where the sign comes from the asymmetry of the factors $(z_i - z_j)$. The last expression is nothing but the expansion of a determinant divided by a Vandermonde determinant, hence we arrive at eqn. (3.10).
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