Stability for $t$-intersecting families of permutations

David Ellis

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Abstract

A family of permutations $A \subset S_n$ is said to be $t$-intersecting if any two permutations in $A$ agree on at least $t$ points, i.e. for any $\sigma, \pi \in A$, $|\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t$. It was proved by Friedgut, Pilpel and the author in [6] that for $n$ sufficiently large depending on $t$, a $t$-intersecting family $A \subset S_n$ has size at most $(n - t)!$, with equality only if $A$ is a coset of the stabilizer of $t$ points (or ‘$t$-coset’ for short), proving a conjecture of Deza and Frankl. Here, we first obtain a rough stability result for $t$-intersecting families of permutations, namely that for any $t \in \mathbb{N}$ and any positive constant $c$, if $A \subset S_n$ is a $t$-intersecting family of permutations of size at least $c(n - t)!$, then there exists a $t$-coset containing all but at most a $O(1/n)$-fraction of $A$. We use this to prove an exact stability result: for $n$ sufficiently large depending on $t$, if $A \subset S_n$ is a $t$-intersecting family which is not contained within a $t$-coset, then $A$ is at most as large as the family $D = \{\sigma \in S_n : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = j \text{ for some } j > t + 1\} \cup\{(1 t + 1), (2 t + 1), \ldots, (t t + 1)\}$ which has size $(1 - 1/e + o(1))(n - t)!$. Moreover, if $A$ is the same size as $D$ then it must be a ‘double translate’ of $D$, meaning that there exist $\pi, \tau \in S_n$ such that $A = \pi D \tau$. The $t = 1$ case of this was a conjecture of Cameron and Ku and was proved by the author in [5]. We build on the methods of [5], but the representation theory of $S_n$ and the combinatorial arguments are more involved. We also obtain an analogous result for $t$-intersecting families in the alternating group $A_n$.

1 Introduction

We work first on the symmetric group $S_n$, the group of all permutations of $\{1, 2, \ldots, n\} = [n]$. A family of permutations $A \subset S_n$ is said to be $t$-intersecting if any two permutations in $A$ agree on at least $t$ points, i.e. for
any \( \sigma, \pi \in \mathcal{A}, |\{i \in [n] : \sigma(i) = \pi(i)\}| \geq t \). Deza and Frankl \cite{Deza} conjectured that for \( n \) sufficiently large depending on \( t \), a \( t \)-intersecting family \( \mathcal{A} \subset S_n \) has size at most \((n-t)!\); this became known as the Deza-Frankl conjecture. It was proved in 2008 by Friedgut, Pilpel and the author in \cite{Friedgut2008} using eigenvalue techniques and the representation theory of the symmetric group; it was also shown in \cite{Friedgut2008} that equality holds only if \( \mathcal{A} \) is a coset of the stabilizer of \( t \) points (or ‘\( t \)-coset’ for short). In this paper, we will first prove a rough stability result for \( t \)-intersecting families of permutations. Namely, we show that for any fixed \( t \in \mathbb{N} \) and \( c > 0 \), if \( \mathcal{A} \subset S_n \) is a \( t \)-intersecting family of size at least \( cn > n! \), then there exists a \( t \)-coset \( \mathcal{C} \) such that \( |\mathcal{A} \setminus \mathcal{C}| \leq \Theta((n-t-1)!)) \), i.e. \( \mathcal{C} \) contains all but at most a \( O(1/n) \)-fraction of \( \mathcal{A} \).

We then use some additional combinatorial arguments to prove an exact stability result: for \( n \) sufficiently large depending on \( t \), if \( \mathcal{A} \subset S_n \) is a \( t \)-intersecting family which is not contained within a \( t \)-coset, then \( \mathcal{A} \) is at most as large as the family

\[
\mathcal{D} = \{ \sigma \in S_n : \sigma(i) = i \forall i \leq t, \sigma(j) = j \text{ for some } j > t + 1 \}
\]

\[
\cup \{(t+1), (2t+1), \ldots, (tt+1)\}
\]

which has size \((1-1/e+o(1))(n-t)!\). Moreover, if \( \mathcal{A} \) is the same size as \( \mathcal{D} \), then it must be a ‘double translate’ of \( \mathcal{D} \), meaning that there exist \( \pi, \tau \in S_n \) such that \( \mathcal{A} = \pi \mathcal{D} \tau \). Note that if \( \mathcal{F} \subset S_n \), any double translate of \( \mathcal{F} \) has the same size as \( \mathcal{F} \), is \( t \)-intersecting iff \( \mathcal{F} \) is and is contained within a \( t \)-coset of \( S_n \) iff \( \mathcal{F} \) is; this will be our notion of ‘isomorphism’.

In other words, if we demand that our \( t \)-intersecting family \( \mathcal{A} \subset S_n \) is not contained within a \( t \)-coset of \( S_n \), then it is best to take \( \mathcal{A} \) such that all but \( t \) of its permutations are contained within some \( t \)-coset.

One may compare this with the situation for \( t \)-intersecting families of \( r \)-sets. We say a family \( \mathcal{A} \subset [n]^{(r)} \) of \( r \)-element subsets of \( [n] \) is \( t \)-intersecting if any two of its sets contain at least \( t \) elements in common, i.e. \( |x \cap y| \geq t \) for any \( x, y \in \mathcal{A} \). Wilson \cite{Wilson} proved using an eigenvalue technique that provided \( n \geq (t+1)(r-t+1) \), a \( t \)-intersecting family \( \mathcal{A} \subset [n]^{(r)} \) has size at most \((r-1)^{\binom{n-t}{r-t}} \), and that for \( n > (t+1)(r-t+1) \), equality holds only if \( \mathcal{A} \) consists of all \( r \)-sets containing some fixed \( t \)-set. Later, Ahlswede and Khachatrian \cite{Ahlswede} characterized the \( t \)-intersecting families of maximum size in \([n]^{(r)} \) for all values of \( t, r \) and \( n \) using entirely combinatorial methods based on left-compression. They also proved that for \( n > (t+1)(r-t+1) \), if \( \mathcal{A} \subset [n]^{(r)} \) is \( t \)-intersecting and non-trivial, meaning that there is no \( t \)-set contained in all of its members, then \( \mathcal{A} \) is at most as large as the family

\[
\{x \in [n]^{(r)} : \{t \} \subset x, x \cap \{t+1, \ldots, r+1\} \neq \emptyset \} \cup \{r+1\} \setminus \{i : i \in [t]\}
\]
if $r > 2t + 1$, and at most as large as the family

$$\{ x \in [n]^r : |x \cap [t + 2]| \geq t + 1 \}$$

if $r \leq 2t + 1$. This had been proved under the assumption $n \geq n_1(r, t)$ by Frankl [7] in 1978. Note that the first family above is ‘almost trivial’, and is the natural analogue of our family $D$.

The $t = 1$ case of our result was a conjecture of Cameron and Ku and was proved by the author in [5]. We build on the methods of [5], but the representation theory of $S_n$ and the combinatorial arguments required are more involved.

We also obtain analogous results for $t$-intersecting families of permutations in the alternating group $A_n$. We use the methods of [6] to show that for $n$ sufficiently large depending on $t$, if $\mathcal{A} \subset A_n$ is $t$-intersecting, then $|\mathcal{A}| \leq (n-t)!/2$. Interestingly, it does not seem possible to use the methods of [6] to show that equality holds only if $\mathcal{A}$ is a coset of the stabilizer of $t$ points. Instead, we deduce this from a stability result. Using the same techniques as for $S_n$, we prove that if $\mathcal{A} \subset A_n$ is $t$-intersecting but not contained within a $t$-coset, then it is at most as large as the family

$$\mathcal{E} = \{ \sigma \in A_n : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = (n-1)n(j) \text{ for some } j > t+1 \}$$

$$\cup \{(1 \ t+1)(n-1)n, (2 \ t+1)(n-1)n, \ldots, (t \ t+1)(n-1)n\}$$

which has size $(1 - 1/e + o(1))(n-t)!/2$; if $\mathcal{A}$ is the same size as $\mathcal{E}$, then it must be a double translate of $\mathcal{E}$, meaning that $\mathcal{A} = \pi \mathcal{E} \tau$ for some $\pi, \tau \in A_n$.

2 Background

In [6], in order to prove the Deza-Frankl conjecture, we constructed (for $n$ sufficiently large depending on $t$) a weighted graph $Y$ which was a real linear combination of Cayley graphs on $S_n$ generated by conjugacy-classes of permutations with less than $t$ fixed points, such that the matrix $A$ of weights of $Y$ had maximum eigenvalue 1 and minimum eigenvalue

$$\omega_{n,t} = -\frac{1}{n(n-1)\ldots(n-t+1) - 1}$$

The 1-eigenspace was the subspace of $\mathbb{C}[S_n]$ consisting of the constant functions. The direct sum of the 1-eigenspace and the $\omega_{n,t}$-eigenspace was the subspace $V_t$ of $\mathbb{C}[S_n]$ spanned by the characteristic vectors of the $t$-cosets of $S_n$. All other eigenvalues were $O(|\omega_{n,t}|/n^{1/6})$; this can in fact be improved to
$O(|\omega_{n,t}|/n)$, but any bound of the form $o(|\omega_{n,t}|)$ will suffice for our purposes. We then appealed to a weighted version of Hoffman’s bound (Theorem 11 in [6]):

**Theorem 1.** Let $A$ be a real, symmetric, $N \times N$ matrix with real eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ (where $\lambda_1 > 0$), such that the all-1’s vector $f$ is an eigenvector of $A$ with eigenvalue $\lambda_1$, i.e. all row and column sums of $A$ equal $\lambda_1$. Let $X \subset [N]$ such that $A_{x,y} = 0$ for any $x, y \in X$. Let $U$ be the direct sum of the subspace of constant vectors and the $\lambda_N$-eigenspace. Then

$$|X| \leq \frac{|\lambda_N|}{\lambda_1 + |\lambda_N|} N$$

and equality holds only if the characteristic vector $v_X$ lies in the subspace $U$.

Applying this to our weighted graph $Y$ proved the Deza-Frankl conjecture:

**Theorem 2.** For $n$ sufficiently large depending on $t$, a $t$-intersecting family $A \subset S_n$ has size $|A| \leq (n-t)!$.

Note that equality holds only if the characteristic vector $v_A$ of $A$ lies in the subspace $V_t$ spanned by the characteristic vectors of the $t$-cosets of $S_n$. It was proved in [3] that the Boolean functions in $V_t$ are precisely the disjoint unions of $t$-cosets of $S_n$, implying that equality holds only if $A$ is a $t$-coset of $S_n$.

We also appealed to the following cross-independent weighted version of Hoffman’s bound:

**Theorem 3.** Let $A$ be as in Theorem 1, and let $\nu = \max(|\lambda_2|, |\lambda_N|)$. Let $X, Y \subset [N]$ such that $A_{x,y} = 0$ for any $x \in X$ and $y \in Y$. Let $U$ be the direct sum of the subspace of constant vectors and the $\pm \nu$-eigenspaces. Then

$$|X||Y| \leq \left(\frac{\nu}{\lambda_1 + \nu} N\right)^2$$

and equality holds only if $|X| = |Y|$ and the characteristic vectors $v_X$ and $v_Y$ lie in the subspace $U$.

Applying this to our weighted graph $Y$ yielded:

**Theorem 4.** For $n$ sufficiently large depending on $t$, if $A, B \subset S_n$ are $t$-cross-intersecting, then $|A||B| \leq ((n-t)!)^2$. 

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This will be a crucial tool in our stability analysis. Note that if equality holds in Theorem 4, then the characteristic vectors \( v_A \) and \( v_B \) lie in the subspace \( V_t \) spanned by the characteristic vectors of the \( t \)-cosets of \( S_n \), so by the same argument as before, \( A \) and \( B \) must both be equal to the same \( t \)-coset of \( S_n \).

We will need the following ‘stability’ version of Theorem 1:

**Lemma 5.** Let \( A, X \) and \( U \) be as in Theorem 4. Let \( \alpha = |X|/N \). Let \( \lambda_M \) be the negative eigenvalue of second largest modulus. Equip \( \mathbb{C}^N \) with the inner product:

\[
\langle x, y \rangle = \frac{1}{N} \sum_{i=1}^{N} \bar{x}_i y_i
\]

and let

\[
||x|| = \sqrt{\frac{1}{N} \sum_{i=1}^{N} |x_i|^2}
\]

be the induced norm. Let \( D \) be the Euclidean distance from the characteristic vector \( v_X \) of \( X \) to the subspace \( U \), i.e. the norm \( ||P_{U^\perp}(v_X)|| \) of the projection of \( v_X \) onto \( U^\perp \). Then

\[
D^2 \leq \frac{(1 - \alpha)|\lambda_N| - \lambda_1 \alpha}{|\lambda_N| - |\lambda_M|} \alpha
\]

For completeness, we include a proof:

**Proof.** Let \( u_1 = f, u_2, \ldots, u_N \) be an orthonormal basis of real eigenvectors of \( A \) corresponding to the eigenvalues \( \lambda_1, \ldots, \lambda_N \). Write

\[
v_X = \sum_{i=1}^{N} \xi_i u_i
\]

as a linear combination of the eigenvectors of \( A \); we have \( \xi_1 = \alpha \) and

\[
\sum_{i=1}^{N} \xi_i^2 = ||v_X||^2 = |X|/N = \alpha
\]

Then we have the crucial property:

\[
0 = \sum_{x,y \in X} A_{x,y} = v_X^\top A v_X = \sum_{i=1}^{N} \lambda_i \xi_i^2 \geq \lambda_1 \xi_1^2 + \lambda_N \sum_{i: \lambda_i = \lambda_N} \xi_i^2 + \lambda_M \sum_{i>1: \lambda_i \neq \lambda_N} \xi_i^2
\]
Note that
\[ \sum_{i > 1: \lambda_i \neq \lambda_N} \xi_i^2 = D^2 \]
and
\[ \sum_{i: \lambda_i = \lambda_N} \xi_i^2 = \alpha - \alpha^2 - D^2 \]
so we have
\[ 0 \geq \lambda_1 \alpha^2 + \lambda_N (\alpha - \alpha^2 - D^2) + \lambda_M D^2 \]
Rearranging, we obtain:
\[ D^2 \leq \frac{(1 - \alpha)|\lambda_N| - \lambda_1 \alpha}{|\lambda_N| - |\lambda_M|} \alpha \]
as required. \( \Box \)

Our weighted graph \( Y \) has \( \lambda_N = \omega_{n,t} \) and \( |\lambda_M| = O(|\omega_{n,t}|/n^{1/6}) \), so applying the above result to a \( t \)-intersecting family \( A \subset S_n \) gives:
\[ \|P_{V_t^\perp}(v_A)\|^2 \leq (1 - |A|/(n-t)!)(1 + O(n^{1/6}))|A|/n! \tag{1} \]
Next, we find a formula for the projection \( P_{V_t}(v_A) \) of the characteristic vector of \( A \) onto the subspace \( V_t \) spanned by the characteristic vectors of the \( t \)-cosets of \( S_n \). But first, we need some background on non-Abelian Fourier analysis and the representation theory of the symmetric group.

**Background from non-Abelian Fourier analysis**

We now recall some information we need from [6]. [Notes for algebraists are included in square brackets and may be ignored without prejudicing the reader’s understanding.]

If \( G \) is a finite group, a representation of \( G \) is a vector space \( W \) together with a group homomorphism \( \rho : G \to \text{GL}(W) \) from \( G \) to the group of all automorphisms of \( W \), or equivalently a linear action of \( G \) on \( W \). If \( W = \mathbb{C}^m \), then \( \text{GL}(W) \) can be identified with the group of all complex invertible \( m \times m \) matrices; we call \( \rho \) a complex matrix representation of degree (or dimension) \( m \). [Note that \( \rho \) makes \( \mathbb{C}^m \) into a \( \mathbb{C}G \)-module of dimension \( m \).]

We say a representation \((\rho,W)\) is irreducible if it has no proper sub-representation, i.e. no proper subspace of \( W \) is fixed by \( \rho(g) \) for every \( g \in G \). We say that two (complex) representations \((\rho,W)\) and \((\rho',W')\) are equivalent if there exists a linear isomorphism \( \phi : W \to W' \) such that \( \rho'(g) \circ \phi = \phi \circ \rho(g) \ \forall g \in G \).
For any finite group $G$, there are only finitely many equivalence classes of irreducible complex representations of $G$. Let $(\rho_1, \rho_2, \ldots, \rho_k)$ be a complete set of pairwise non-equivalent complex irreducible matrix representations of $G$ (i.e. containing one from each equivalence class of complex irreducible representations).

**Definition 1.** The (non-Abelian) Fourier transform of a function $f : G \to \mathbb{C}$ at the irreducible representation $\rho_i$ is the matrix

$$\hat{f}(\rho_i) = \frac{1}{|G|} \sum_{g \in G} f(g) \rho_i(g)$$

Let $V_{\rho_i}$ be the subspace of functions whose Fourier transform is concentrated on $\rho_i$, i.e. with $\hat{f}(\rho_j) = 0$ for each $j \neq i$. [Identifying the space $\mathbb{C}[G]$ of all complex-valued functions on $G$ with the group module $\mathbb{C}G$, $V_{\rho_i}$ is the sum of all submodules of the group module isomorphic to the module defined by $\rho_i$; it has dimension $\dim(V_{\rho_i}) = (\dim(\rho_i))^2$. The group module decomposes as

$$\mathbb{C}G = \bigoplus_{i=1}^{k} V_{\rho_i}$$

Write $\text{Id} = \sum_{i=1}^{k} e_i$, where $e_i \in V_{\rho_i}$ for each $i \in [k]$. The $e_i$’s are called the primitive central idempotents of $\mathbb{C}G$; they are given by the following formula:

$$e_i = \frac{\dim(\rho_i)}{|G|} \sum_{g \in G} \chi_{\rho_i}(g^{-1}) g$$

They are in the centre $Z(\mathbb{C}G)$ of the group module, and satisfy $e_i e_j = \delta_{i,j}$. Note that $V_{\rho_i}$ is the two-sided ideal of $\mathbb{C}G$ generated by $e_i$. For any $x \in \mathbb{C}G$, the unique decomposition of $x$ into elements of the $V_{\rho_i}$’s is given by $x = \sum_{i=1}^{k} e_i x$.]

A function $f : G \to \mathbb{C}$ may be recovered from its Fourier transform using the Fourier Inversion Formula:

$$f(g) = \sum_{i=1}^{k} \dim(\rho_i) \text{Tr} \left( \hat{f}(\rho_i) \rho_i(g^{-1}) \right)$$

where $\text{Tr}(M)$ denotes the trace of the matrix $M$. It follows from this that the projection of $f$ onto $V_{\rho_i}$ has $g$-coordinate

$$P_{V_{\rho_i}}(f)_g = \frac{\dim(\rho_i)}{|G|} \sum_{h \in G} f(h) \text{Tr}(\rho_i(h g^{-1})) = \frac{\dim(\rho_i)}{|G|} \sum_{h \in G} f(h) \chi_{\rho_i}(h g)$$

where $\chi_{\rho_i}(g) = \text{Tr}(\rho_i(g))$ denotes the character of the representation $\rho_i$. 

7
Background on the representation theory of $S_n$

A partition of $n$ is a non-increasing sequence of positive integers summing to $n$, i.e., a sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_l \geq 1$ and $\sum_{i=1}^{l} \alpha_i = n$; we write $\alpha \vdash n$. For example, $(3, 2, 2) \vdash 7$; we sometimes use the shorthand $(3, 2, 2) = (3, 2^2)$.

The cycle-type of a permutation $\sigma \in S_n$ is the partition of $n$ obtained by expressing $\sigma$ as a product of disjoint cycles and listing its cycle-lengths in non-increasing order. The conjugacy-classes of $S_n$ are precisely

$$\{\sigma \in S_n : \text{cycle-type}(\sigma) = \alpha\}_{\alpha \vdash n}.$$  

Moreover, there is an explicit 1-1 correspondence between irreducible representations of $S_n$ (up to isomorphism) and partitions of $n$, which we now describe.

Let $\alpha = (\alpha_1, \ldots, \alpha_l)$ be a partition of $n$. The Young diagram of $\alpha$ is an array of $n$ dots, or cells, having $l$ left-justified rows where row $i$ contains $\alpha_i$ dots. For example, the Young diagram of the partition $(3, 2^2)$ is

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If the array contains the numbers $\{1, 2, \ldots, n\}$ in some order in place of the dots, we call it an $\alpha$-tableau; for example,

$\begin{array}{c}
6 & 1 & 7 \\
5 & 4 \\
3 & 2
\end{array}$

is a $(3, 2^2)$-tableau. Two $\alpha$-tableaux are said to be row-equivalent if for each row, they have the same numbers in that row. If an $\alpha$-tableau $s$ has rows $R_1, \ldots, R_l \subset [n]$ and columns $C_1, \ldots, C_k \subset [n]$, we let $R_s = S_{R_1} \times S_{R_2} \times \ldots \times S_{R_l}$ be the row-stabilizer of $s$ and $C_s = S_{C_1} \times S_{C_2} \times \ldots \times S_{C_k}$ be the column-stabilizer.

An $\alpha$-tabloid is an $\alpha$-tableau with unordered row entries (or formally, a row-equivalence class of $\alpha$-tableaux); given a tableau $s$, we write $[s]$ for the tabloid it produces. For example, the $(3, 2^2)$-tableau above produces the following $(3, 2^2)$-tabloid
Consider the natural left action of $S_n$ on the set $X^\alpha$ of all $\alpha$-tabloids; let $M^\alpha = \mathbb{C}[X^\alpha]$ be the corresponding permutation module, i.e. the complex vector space with basis $X^\alpha$ and $S_n$ action given by extending this action linearly. Given an $\alpha$-tableau $s$, we define the corresponding $\alpha$-polytabloid

$$e_s := \sum_{\pi \in C_s} \epsilon(\pi) \pi[s]$$

We define the Specht module $S^\alpha$ to be the submodule of $M^\alpha$ spanned by the $\alpha$-polytabloids:

$$S^\alpha = \text{Span}\{e_s : s \text{ is an } \alpha\text{-tableau}\}.$$ 

A central observation in the representation theory of $S_n$ is that the Specht modules are a complete set of pairwise non-isomorphic, irreducible representations of $S_n$. Hence, any irreducible representation $\rho$ of $S_n$ is isomorphic to some $S^\alpha$. For example, $S^{(n)} = M^{(n)}$ is the trivial representation; $M^{(1^n)}$ is the left-regular representation, and $S^{(1^n)}$ is the sign representation $S$.

We say that a tableau is standard if the numbers strictly increase along each row and down each column. It turns out that for any partition $\alpha$ of $n$,

$$\{e_t : t \text{ is a standard } \alpha\text{-tableau}\}$$

is a basis for the Specht module $S^\alpha$.

Given a partition $\alpha$ of $n$, for each cell $(i, j)$ in its Young diagram, we define the ‘hook-length’ $(h^\alpha_{i,j})$ to be the number of cells in its ‘hook’ (the set of cells in the same row to the right of it or in the same column below it, including itself) — for example, the hook-lengths of $(3, 2^2)$ are as follows:

$${5 \quad 4 \quad 1}
{3 \quad 2}
{2 \quad 1}$$

The dimension $f^\alpha$ of the Specht module $S^\alpha$ is given by the following formula

$$f^\alpha = n! / \prod (\text{hook lengths of } [\alpha])$$

From now on we will write $[\alpha]$ for the equivalence class of the irreducible representation $S^\alpha$, $\chi_\alpha$ for the irreducible character $\chi_{S^\alpha}$, and $\xi_\alpha$ for
the character of the permutation representation $M^\alpha$. Notice that the set of $\alpha$-tabloids form a basis for $M^\alpha$, and therefore $\xi_\alpha(\sigma)$, the trace of the corresponding permutation representation at $\sigma$, is precisely the number of $\alpha$-tabloids fixed by $\sigma$.

We now explain how the permutation modules $M^\beta$ decompose into irreducibles.

**Definition 2.** Let $\alpha, \beta$ be partitions of $n$. A generalized $\alpha$-tableau is produced by replacing each dot in the Young diagram of $\alpha$ with a number between 1 and $n$; if a generalized $\alpha$-tableau has $\beta_i$’s ($1 \leq i \leq n$) it is said to have content $\beta$. A generalized $\alpha$-tableau is said to be semistandard if the numbers are non-decreasing along each row and strictly increasing down each column.

**Definition 3.** Let $\alpha, \beta$ be partitions of $n$. The Kostka number $K_{\alpha, \beta}$ is the number of semistandard generalized $\alpha$-tableaux with content $\beta$.

Young’s Rule states that for any partition $\beta$ of $n$, the permutation module $M^\beta$ decomposes into irreducibles as follows:

$$M^\beta \cong \bigoplus_{\alpha \vdash n} K_{\alpha, \beta} S^\alpha$$

For example, $M^{(n-1,1)}$, which corresponds to the natural permutation action of $S_n$ on $[n]$, decomposes as

$$M^{(n-1,1)} \cong S^{(n-1,1)} \oplus S^{(n)}$$

and therefore

$$\xi^{(n-1,1)} = \chi^{(n-1,1)} + 1$$

Let $V_\alpha$ be the subspace of $\mathbb{C}[S_n]$ consisting of functions whose Fourier transform is concentrated on $[\alpha]$; equivalently, $V_\alpha$ is the sum of all submodules of $\mathbb{C}S_n$ isomorphic to the Specht module $S^\alpha$.

We call a partition of $n$ (or an irreducible representation of $S_n$) ‘fat’ if its Young diagram has first row of length at least $n - t$. Let $F_{n,t}$ denote the set of all fat partitions of $n$; note that for $n \geq 2t$,

$$|F_{n,t}| = \sum_{s=0}^{t} p(s)$$

where $p(s)$ denotes the number of partitions of $s$. This grows very rapidly with $t$, but (as will be crucial for our stability analysis) it is independent of $n$ for $n \geq 2t$. Note that $\{[\alpha] : \alpha \text{ is fat}\}$ are precisely the irreducible constituents of the permutation module $M^{(n-t,1^t)}$ corresponding to the action
of $S_n$ on $t$-tuples of distinct numbers, since $K_{\alpha,(n-t,1^t)} \geq 1$ iff there exists a semistandard generalized $\alpha$-tableau of content $(n-t,1^t)$, i.e. iff $\alpha_1 \geq n-t$.

Recall from [6] that $V_t$ is the subspace of functions whose Fourier transform is concentrated on the ‘fat’ irreducible representations of $S_n$; equivalently,

$$V_t = \bigoplus_{\text{fat } \alpha} V_\alpha \tag{3}$$

The projection of $u \in \mathbb{C}[S_n]$ onto $V_\alpha$ has $\sigma$-coordinate

$$P_{V_\alpha}(u)_\sigma = \frac{f_\alpha}{n!} \sum_{\pi \in S_n} u(\pi) \chi_\alpha(\pi \sigma^{-1})$$

and therefore the projection of $u$ onto $V_t$ has $\sigma$-coordinate

$$P_{V_t}(u)_\sigma = \frac{1}{n!} \sum_{\text{fat } \alpha} f_\alpha \sum_{\pi \in S_n} u(\pi) \chi_\alpha(\pi \sigma^{-1}) \tag{4}$$

3 Stability

We are now in a position to prove our rough stability result:

**Theorem 6.** Let $t \in \mathbb{N}, c > 0$ be fixed. If $A \subset S_n$ is a $t$-intersecting family with $|A| \geq c(n-t)!$, then there exists a $t$-coset $C$ such that $|A \setminus C| \leq O((n-t-1)!)$.

In other words, if $A \subset S_n$ is a $t$-intersecting family of size at least a constant proportion of the maximum possible size $(n-t)!$, then there is some $t$-coset containing all but at most a $O(1/n)$-fraction of $A$.

To prove this, we will first prove the following weaker statement:

**Lemma 7.** Let $t \in \mathbb{N}, c > 0$ be fixed. If $A \subset S_n$ is a $t$-intersecting family of size at least $c(n-t)!$, then there exist $i$ and $j$ such that all but at most $O((n-t-1)!)$ permutations in $A$ map $i$ to $j$.

In other words, a large $t$-intersecting family is almost contained within a 1-coset. Theorem 6 will follow easily from this by an inductive argument.

Given distinct $i_1, \ldots, i_t$ and distinct $j_1, \ldots, j_t$, we will write

$$\mathcal{A}_{i_1 \mapsto j_1, i_2 \mapsto j_2, \ldots, i_t \mapsto j_t} := \{ \sigma \in A : \sigma(i_k) = j_k \ \forall k \in [t] \}$$

To prove Lemma 7 we will first observe from (1) that if $A \subset S_n$ is a $t$-intersecting family of size at least $c(n-t)!$ then the characteristic vector
\( v_A \) of \( A \) is close to the subspace \( V_t \) spanned by the characteristic vectors of the \( t \)-cosets. We will use this, combined with representation-theoretic arguments, to show that there exists some \( t \)-coset \( C_0 \) such that

\[
|A \cap C_0| \geq \omega((n - 2t)!) \]

—without loss of generality, \( C_0 = \{ \sigma \in S_n : \sigma(1) = 1, \ldots, \sigma(t) = t \} \), so

\[
|A_{1 \rightarrow t, 2 \rightarrow 2, \ldots, t \rightarrow t}| \geq \omega((n - 2t)!) \]

Note that the average size of the intersection of \( A \) with a \( t \)-coset is

\[
|A| / n(n - 1) \ldots (n - t + 1) = \Theta((n - t)!/n!) \]

We only know that \( A \cap C_0 \) has size \( \omega \) of the average size. This statement would at first seem to weak to help us. However, for any distinct \( j_1 \neq 1, j_2 \neq 2, \ldots, \) and \( j_t \neq t \), the pair of families

\[
A_{1 \rightarrow j_1, 2 \rightarrow j_2, \ldots, t \rightarrow j_t}
\]

is \( t \)-cross-intersecting, so we may compare their sizes. In detail, we will deduce from Theorem 3 that

\[
|A_{1 \rightarrow j_1, 2 \rightarrow j_2, \ldots, t \rightarrow j_t}| \leq ((n - 2t)!)^2
\]

giving \( |A_{1 \rightarrow j_1, \ldots, t \rightarrow j_t}| \leq o((n - 2t)!). \) Summing over all choices of \( j_1, \ldots, j_t \) will show that all but at most \( o((n - t)! \) permutations in \( A \) fix some point of \([t]\), enabling us to complete the proof.

**Proof of Lemma 7:**
Let \( A \subset S_n \) be a \( t \)-intersecting family of size at least \( c(n - t)! \); write \( \delta = 1 - c < 1 \). From (1), we know that the Euclidean distance from \( v_A \) to \( V_t \) is small:

\[
||P_{V_t}(v_A)||^2 \leq \delta(1 + O(n^{1/6}))|A|/n!
\]

From (1), the projection of \( v_A \) onto \( V_t \) has \( \sigma \)-coordinate:

\[
P_{V_t}(v_A)_{\sigma} = \frac{1}{n!} \sum_{\alpha} f_{\alpha} \sum_{\pi \in A} \chi_{\alpha}(\pi \sigma^{-1})
\]

Write \( P_\sigma = P_{V_t}(v_A)_{\sigma} \); then

\[
\frac{1}{n!} \left( \sum_{\sigma \in A} (1 - P_\sigma)^2 + \sum_{\sigma \notin A} P_\sigma^2 \right) \leq \delta(1 + O(1/n^{1/6}))|A|/n!
\]

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i.e.
\[
\sum_{\sigma \in A} (1 - P_\sigma)^2 + \sum_{\sigma \notin A} P_\sigma^2 \leq \delta (1 + O(1/n^{1/6})) |A|
\]
Choose \(C > 0 : |A|(1 - 1/n)\delta(1 + C/n^{1/6}) \geq \text{RHS}\); then the subset
\[
S := \{\sigma \in A : (1 - P_\sigma)^2 < \delta(1 + C/n^{1/6})\}
\]
has size at least \(|A|/n\). Similarly, \(P_\sigma^2 < 2\delta/n\) for all but at most
\[
n|A|(1 + O(1/n)) / 2
\]
permutations \(\sigma \notin A\). Provided \(n\) is sufficiently large, \(|A| \leq (n - t)!\), and therefore the subset \(T = \{\sigma \notin A : P_\sigma^2 < 2\delta/n\}\) has size
\[
|T| \geq n! - (n - t)! - n(n-t)!(1 + O(1/n)) / 2
\]
The permutations \(\sigma \in S\) have \(P_\sigma\) close to 1; the permutations \(\pi \in T\) have \(P_\pi\) close to 0. Using only our lower bounds on the sizes of \(S\) and \(T\), we may prove the following:

**Claim:** There exist permutations \(\sigma \in S\), \(\pi \in T\) such that \(\sigma^{-1}\pi\) is a product of at most \(h = h(n)\) transpositions, where \(h = \sqrt{2(t+2)(n-1)\log n}\).

**Proof of Claim:** Define the transposition graph \(H\) to be the Cayley graph on \(S_n\) generated by the transpositions, i.e. \(V(H) = S_n\) and \(\sigma \pi \in E(H)\) iff \(\sigma^{-1}\pi\) is a transposition. We use the following isoperimetric inequality for \(H\), essentially the martingale inequality of Maurey:

**Theorem 8.** Let \(X \subset V(H)\) with \(|X| \geq \gamma n!\) where \(0 < \gamma < 1\). Then for any \(h \geq h_0 := \sqrt{1/2(n-1)\log \frac{1}{\gamma}}\),
\[
|N_h(X)| \geq \left(1 - e^{-2(h-h_0)^2/n-1}\right) n!
\]

\(\square\)

For a proof, see for example [10]. Applying this to the set \(S\), which has \(|S| \geq (1 - \delta)(n - t)!/n \geq \frac{n!}{n^{t+2}}\) (provided \(n\) is sufficiently large), with \(\gamma = 1/n^{t+2}\), \(h = 2h_0\), gives \(|N_h(S)| \geq (1-n^{-(t+2)})n!\), so certainly \(N_h(S) \cap T \neq \emptyset\), proving the claim.
We now have two permutations \( \sigma \in A, \pi \notin A \) which are ‘close’ to one another in \( H \) (differing in only \( O(\sqrt{n \log n}) \) transpositions) such that

\[
P_\sigma > 1 - \sqrt{\delta (1 + C/n^{1/6})}, \quad P_\pi < \sqrt{2 \delta / n}
\]

and therefore

\[
P_\sigma - P_\pi > 1 - \sqrt{\delta} - O(1/n^{1/12})
\]

Hence, by averaging, there exist two permutations \( \rho, \tau \) that differ by just one transposition and satisfy

\[
P_\rho - P_\tau > (1 - \sqrt{\delta} - O(1/n^{1/12}))/h \geq 1 - \sqrt{\delta} - O(1/n^{1/12})
\]

i.e.

\[
\sum_{\alpha \in \mathcal{F}_{n,t}} \frac{f^\alpha}{n!} \left( \sum_{\pi \in A} \chi_\alpha(\pi \rho^{-1}) - \sum_{\pi \in A} \chi_\alpha(\pi \tau^{-1}) \right) \geq \frac{1 - \sqrt{\delta} - O(1/n^{1/12})}{\sqrt{2(t + 2)n \log n}}
\]

By double translation, we may assume without loss of generality that \( \rho = \text{Id}, \tau = (1 2) \). So we have:

\[
\sum_{\alpha \in \mathcal{F}_{n,t}} \frac{f^\alpha}{n!} \left( \sum_{\pi \in A} \chi_\alpha(\pi) - \sum_{\pi \in A} \chi_\alpha(\pi(1 2)) \right) \geq \frac{1 - \sqrt{\delta} - O(1/n^{1/12})}{\sqrt{2(t + 2)n \log n}}
\]

The above sum is over \( |\mathcal{F}_{n,t}| = \sum_{s=0}^{t} p(s) \) partitions \( \alpha \) of \( n \); this grows very rapidly with \( t \), but is independent of \( n \) for \( n \geq 2t \). By averaging, there exists some \( \alpha \in \mathcal{F}_{n,t} \) such that

\[
\frac{f^\alpha}{n!} \left( \sum_{\pi \in A} \chi_\alpha(\pi) - \sum_{\pi \in A} \chi_\alpha(\pi(1 2)) \right) \geq \frac{1 - \sqrt{\delta} - O(1/n^{1/12})}{\sqrt{2(t + 2)n \log n} \sum_{s=0}^{t} p(s)} = \Omega(1/\sqrt{n \log n})
\]

Recall that the ‘fat’ irreducible representations \([\alpha] : \alpha \in \mathcal{F}_{n,t}\) are precisely the irreducible constituents of \( M^{(n-t,1^t)} \), so very crudely, for each fat \( \alpha \),

\[
f^\alpha \leq \dim(M^{(n-t,1^t)}) = n(n - 1) \ldots (n - t + 1)
\]

Hence,

\[
\sum_{\pi \in A} \chi_\alpha(\pi) - \sum_{\pi \in A} \chi_\alpha(\pi(1 2)) \geq \Omega(1/\sqrt{n \log n})(n - t)!
\]
But for any \( \alpha \in F_{n,t} \), we may express the irreducible character \( \chi_\alpha \) as a linear combination of permutation characters \( \xi_\beta : \beta \in F_{n,t} \) using the following ‘determinantal formula’ (see [8]). For any partition \( \alpha \) of \( n \),

\[
\chi_\alpha = \sum_{\pi \in S_n} \epsilon(\pi) \xi_{\alpha - \text{id} + \pi}
\]

Here, for \( \alpha = (\alpha_1, \ldots, \alpha_l) \vdash n \), we set \( \alpha_i = 0 \) (\( l < i \leq n \)), we think of \( \alpha \), \( \text{id} \) and \( \pi \) as sequences of length \( n \), and we define addition and subtraction of these sequences pointwise. In general,

\[
\alpha - \text{id} + \pi = (\alpha_1 - 1 + \pi(1), \alpha_2 - 2 + \pi(2), \ldots, \alpha_n - n + \pi(n))
\]

will be a sequence of \( n \) integers with sum \( n \), i.e. a composition of \( n \). If \( \lambda \) is a composition of \( n \) with all its terms non-negative, then let \( \lambda \) be the partition of \( n \) produced by ordering the terms of \( \lambda \) in non-increasing order, and define \( \xi_\lambda = \xi_\lambda' \); if \( \lambda \) has a negative term, we define \( \xi_\lambda = 0 \). If \( \alpha \in F_{n,t} \), then as \( \alpha_1 \geq n-t \), any composition occurring in the above sum has first term at least \( n-t \), and therefore \( \xi_\beta \) can only occur in the above sum if \( \beta \in F_{n,t} \).

Observe further that since \( \alpha \) has at most \( t+1 \) non-zero parts, \( \alpha_i = 0 \) for every \( i > t+1 \), and therefore any permutation \( \pi \in S_n \) with \( \xi_{\alpha - \text{id} + \pi} \neq 0 \) must have \( \pi(i) \geq i \) for every \( i > t+1 \), so must fix \( t+2, t+3, \ldots, \) and \( n \). Therefore, the above sum is only over \( \pi \in S_{\{1, \ldots, t+1\}} \), i.e.

\[
\chi_\alpha = \sum_{\pi \in S_{t+1}} \epsilon(\pi) \xi_{\alpha - \text{id} + \pi} \forall \alpha \in F_{n,t}
\]

Therefore, \( \chi_\alpha \) is a \((\pm 1)\)-linear combination of at most \((t+1)!\) permutation characters \( \xi_\beta \) (\( \beta \in F_{n,t} \)), possibly with repeats. Hence, by averaging, there exists some \( \beta \in F_{n,t} \) such that

\[
\left| \sum_{\pi \in A} \xi_\beta(\pi) - \sum_{\pi \in A} \xi_\beta(\pi(1 2)) \right| \geq \Omega(1/\sqrt{n \log n})(n-t)!/(t+1)! = \Omega(1/\sqrt{n \log n})(n-t)!
\]

Without loss of generality, we may assume that the above quantity is positive, i.e.

\[
\sum_{\pi \in A} \xi_\beta(\pi) - \sum_{\pi \in A} \xi_\beta(\pi(1 2)) \geq \Omega(1/\sqrt{n \log n})(n-t)!
\]
Let \( T_\beta \) be the set of \( \beta \)-tabloids; the LHS is then
\[
\#\{ (T, \pi) : T \in T_\beta, \pi \in \mathcal{A}, \pi(T) = T \} - \#\{ (T, \pi) : T \in T_\beta, \pi \in \mathcal{A}, \pi(1 \ 2)(T) = T \}
\]
Interchanging the order of summation, this equals
\[
\sum_{T \in T_\beta} (\#\{ \pi \in \mathcal{A} : \pi(T) = T \} - \#\{ \pi \in \mathcal{A} : \pi(1 \ 2)(T) = T \})
\]
The above summand is zero for all \( \beta \)-tabloids \( T \) with 1 and 2 in the first row of \( T \) (as then \((1 \ 2)T = T\)). Write \( \beta = (n-s, \beta_2, \ldots, \beta_l) \), where \( 0 \leq s \leq t \).

The number of \( \beta \)-tabloids with 1 not in the first row is
\[
s(n-1)(n-2) \cdots (n-s+1) / \prod_{i=2}^t \beta_i!
\]
and therefore the number of \( \beta \)-tabloids with 1 or 2 below the first row is at most
\[
2s(n-1)(n-2) \cdots (n-s+1) / \prod_{i=2}^t \beta_i! \leq 2t(n-1)(n-2) \cdots (n-s+1)
\]
\[
= 2t(n-1)! / (n-s)!
\]

Hence by averaging, for one such \( \beta \)-tabloid \( T \),
\[
\#\{ \pi \in \mathcal{A} : \pi(T) = T \} - \#\{ \pi \in \mathcal{A} : \pi(1 \ 2)(T) = T \} \geq \Omega(1 / \sqrt{n \log n}) \frac{(n-s)!}{2t(n-1)!} (n-t)!
\]
and therefore the number of permutations in \( \mathcal{A} \) fixing \( T \) satisfies
\[
\#\{ \pi \in \mathcal{A} : \pi(T) = T \} \geq \Omega(1 / \sqrt{n \log n}) \frac{(n-s)!}{2t(n-1)!} (n-t)!
\]

Without loss of generality, we may assume that the first row of \( T \) consists of the numbers \{ \( s+1, \ldots, n \} \). There are \( \beta_2! \beta_3! \cdots \beta_l! \leq s! \leq t! \) permutations of \( [s] \) fixing the \( 2^{nd}, 3^{rd}, \ldots, \) and \( t^{th} \) rows of \( T \); any permutation fixing \( T \) must agree with one of these permutations on \( [s] \). Hence, there exists a permutation \( \rho \) of \( [s] \) such that at least
\[
\Omega(1 / \sqrt{n \log n}) \frac{(n-s)!}{2t(n-1)! t!} (n-t)!
\]

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permutations in $A$ agree with $\rho$ on $[s]$. Without loss of generality, we may assume that $\rho = \text{Id}_{[s]}$, so the number of permutations in $A$ fixing $[s]$ pointwise satisfies

$$|A_{1 \mapsto 1, \ldots, s \mapsto s}| \geq \Omega(1/\sqrt{n \log n}) \frac{(n-s)!/(n-t)!}{2t(n-1)!}$$

$$= \Omega(1/\sqrt{n \log n}) \frac{(n-s)!/(n-t)!}{(n-1)!}$$

We may write $A_{1 \mapsto 1, \ldots, s \mapsto s}$ as a disjoint union

$$A_{1 \mapsto 1, \ldots, s \mapsto s} = \bigcup_{j_{s+1}, \ldots, j_t > s \text{ distinct}} A_{1 \mapsto 1, \ldots, s \mapsto s+1 \mapsto j_{s+1}, \ldots, t \mapsto j_t}$$

and there are $(n-s)(n-s-1) \ldots (n-t+1)$ choices of $j_{s+1}, \ldots, j_t$, so by averaging, there exists a choice such that

$$|A_{1 \mapsto 1, \ldots, s \mapsto s+1 \mapsto j_{s+1}, \ldots, t \mapsto j_t}| \geq \Omega(1/\sqrt{n \log n}) \frac{(n-t)!}{(n-1)!}$$

We now show that each $A_{1 \mapsto 1, \ldots, t \mapsto t}$ is small using Theorem 4. Let $J = \{j_1, \ldots, j_t\}$. Notice that $\mathcal{E} := A_{1 \mapsto 1, \ldots, t \mapsto t} \cup \ldots \cup A_{t \mapsto t}$ and $\mathcal{F} := A_{1 \mapsto j_1, \ldots, t \mapsto j_t}$ is a $t$-cross-intersecting pair of families, so for any $\sigma \in \mathcal{E}$ and $\pi \in \mathcal{F}$, there are $t$ distinct points $i_1, i_2, \ldots, i_t > t$ such that $\sigma(i_k) = \pi(i_k) \notin [t] \cup J$ for each $k \in [t]$. But then

$$(1 \ j_1)(2 \ j_2) \ldots (t \ j_t)\pi(i_k) = \sigma(i_k) \quad \text{for each} \ k \in [t]$$
so letting $\mathcal{G} := (1 \ j_1)(2 \ j_2) \ldots (t \ j_t) \mathcal{F}$, the pair of families $\mathcal{E}, \mathcal{G}$ fix $[t]$ pointwise and $t$-cross-intersect on $\{t + 1, t + 2, \ldots, n\}$. Deleting $1, \ldots, t$ we obtain a $t$-cross-intersecting pair $\mathcal{E}', \mathcal{G}'$ of subsets of $S_{\{t+1,...,n\}}$. By Theorem 4,

$$|A_{1-1,...,t-t}||A_{1-j_1,...,t-j_t}| = |\mathcal{E}||\mathcal{G}| = |\mathcal{E}'||\mathcal{G}'| \leq ((n - 2t)!)^2$$

Since

$$|A_{1-1,...,t-t}| \geq \omega((n - 2t)!))$$

we have

$$|A_{1-j_1,...,t-j_t}| \leq o((n - 2t)!)$$

There are $\leq n(n - 1)(n - 2) \ldots (n - t + 1)$ possible choices of $j_1, \ldots, j_t$, and therefore the number of permutations in $\mathcal{A}$ with no fixed point in $[t]$ satisfies

$$|A \setminus (A_{1-1} \cup A_{2-2} \cup \ldots \cup A_{t-t})| \leq o((n - 2t)!n(n - 1) \ldots (n - t + 1) = o((n - t)!$$

Since $|\mathcal{A}| \geq c(n - t)!$, we have

$$|A_{1-1} \cup A_{2-2} \cup \ldots \cup A_{t-t}| \geq (c - o(1))(n - t)!$$

By averaging, there exists some $i \in [t]$ such that

$$|A_{i-i}| \geq (c - o(1))(n - t)!/t$$

We may assume that $i = 1$, so $|A_{1-1}| \geq (c - o(1))(n - t)!/t$. Now, using the same trick as before, we may use Theorem 4 to show that $|A \setminus A_{1-1}| \leq O((n - t - 1)!$. Indeed, write $A \setminus A_{1-1}$ as a disjoint union

$$A \setminus A_{1-1} = \bigcup_{j \neq 1} A_{1-j}$$

We will show that each $A_{1-j}$ is small. Notice as before that the pair of families $A_{1-1}$, $(1 \ j)A_{1-j}$ fixes 1 and $t$-cross-intersects on the domain $\{2, \ldots, n\}$, so Theorem 4 gives

$$|A_{1-1}||A_{1-j}| \leq ((n - t - 1)!)^2$$

Since $|A_{1-1}| \geq \Omega((n - t)!)$, we obtain $|A_{1-j}| \leq O((n - t - 2)!$, and therefore

$$|A \setminus A_{1-1}| = \sum_{j \neq 1} |A_{1-j}| \leq O((n - t - 1)!$$

proving Lemma 7.
Proof of Theorem 6.
By induction on $t$. The $t = 1$ case is the same as that of Lemma 7. Assume the theorem is true for $t - 1$; we will prove it for $t$. Let $A \subseteq S_n$ be a $t$-intersecting family of size at least $c(n - t)!$. By Lemma 7 there exist $i$ and $j$ such that $|A \setminus A_{i \mapsto j}| \leq O((n - t - 1)!)$.

Without loss of generality we may assume that $i = j = 1$, so $|A \setminus A_{1 \mapsto 1}| \leq O((n - t - 1)!)$.

Hence, $|A_{1 \mapsto 1}| \geq |A| - O((n - t - 1)!)$. Deleting 1 from each permutation in $A_{1 \mapsto 1}$, we obtain a $(t - 1)$-intersecting family $A' \subseteq S_{\{2, 3, \ldots, n\}}$ of size $\geq c(n - t)!$. Choose any positive constant $c' < c$; then provided $n$ is sufficiently large, we have $|A'| \geq c'(n - t)!$.

By the induction hypothesis, there exists a $(t - 1)$-coset $C'$ of $S_{2, 3, \ldots, n}$ such that $|A' \setminus C'| \leq O((n - t - 1)!)$. Then if $C$ is the $t$-coset obtained from $C'$ by adjoining $1 \mapsto 1$, we have $|A \setminus C| \leq O((n - t - 1)!)$.

This completes the induction and proves Theorem 6.

□

We now use our rough stability result to prove an exact stability result. First, we need some more definitions.

Let $d_n$ be the number of derangements of $[n]$ (permutations of $[n]$ without fixed points). It is well known that $d_n = (1/e + o(1))n!$.

Following Cameron and Ku [3], given a permutation $\rho \in S_n$ and $i \in [n]$, we define the $i$-fix of $\rho$ to be the permutation $\rho_i$ which fixes $i$, maps the preimage of $i$ to the image of $i$, and agrees with $\rho$ at all other points of $[n]$, i.e.

$$
\rho_i(i) = i; \quad \rho_i(\rho^{-1}(i)) = \rho(i); \quad \rho_i(k) = \rho(k) \forall k \neq i, \rho^{-1}(i)
$$

In other words, $\rho_i = \rho(\rho^{-1}(i))$. We inductively define

$$
\rho_{i_1, \ldots, i_t} = (\rho_{i_1, \ldots, i_{t-1}})_{i_t}
$$

Notice that if $\sigma$ fixes $j$, then $\sigma$ agrees with $\rho_j$ wherever it agrees with $\rho$.

Theorem 9. For $n$ sufficiently large depending on $t$, if $A \subseteq S_n$ is a $t$-intersecting family which is not contained within a $t$-coset, then $A$ is no larger than the family

$$
D = \{ \sigma \in S_n : \sigma(i) = i \forall i \leq t, \sigma(j) = j \text{ for some } j > t + 1 \}
\cup \{(1 t + 1), (2 t + 1), \ldots, (t t + 1)\}
$$

which has size $(n - t)! - d_{n-t} - d_{n-t-1} + t = (1 - 1/e + o(1))(n - t)!$. If $A$ is the same size as $D$, then $A$ is a double translate of $D$, i.e. $A = \pi D \tau$ for some $\pi, \tau \in S_n$. 

\]
**Proof.** Suppose \( \mathcal{A} \subset S_n \) is a \( t \)-intersecting family which is not contained within a \( t \)-coset, and has size

\[
|\mathcal{A}| \geq (n-t)! - d_{n-t} - d_{n-t-1} + t = (1 - 1/e + o(1))(n-t)!
\]

Applying Theorem 6 with any constant \( c \) such that \( 0 < c < 1 - 1/e \), we see that (provided \( n \) is sufficiently large) there exists a \( t \)-coset \( \mathcal{C} \) such that

\[
|A \setminus C| \leq O(1/n)(n-t)!
\]

By double translation, without loss of generality we may assume that \( C = \{ \sigma \in S_n : \sigma(1) = 1, \ldots, \sigma(t) = t \} \). We have:

\[
|A \cap C| \geq (n-t)! - d_{n-t} - d_{n-t-1} + t - O(1/n)(n-t)!
= (1 - 1/e + o(1))(n-t)!
\]  

(5)

We now claim that every permutation in \( A \setminus C \) fixes exactly \( t-1 \) points of \([t]\). Suppose for a contradiction that \( A \) contains a permutation \( \tau \) fixing at most \( t-2 \) points of \([t]\). Then every permutation in \( A \cap C \) must agree with \( \tau \) on at least 2 points of \([t+1, \ldots, n]\), so

\[
|A \cap C| \leq \binom{n-t}{2}(n-t-2)! = \frac{1}{2}(n-t)!
\]

contradicting (5), provided \( n \) is sufficiently large.

Since we are assuming that \( \mathcal{A} \) is not contained within a \( t \)-coset, \( A \setminus C \) contains some permutation \( \tau \); \( \tau \) must fix all points of \([t]\) except for one. By double translation, we may assume that \( \tau = (1 \ t + 1) \). We will show that under these hypotheses, \( A = D \).

Every permutation in \( A \cap C \) must \( t \)-intersect \((1 \ t + 1)\) and must therefore have at least one fixed point \( > t+1 \), i.e. \( A \cap C \) is a subset of the family

\[
\mathcal{E} := \{ \sigma \in S_n : \sigma(i) = i \ \forall i \in [t], \ \sigma(j) = j \text{ for some } j > t+1 \}
\]

which has size

\[
(n-t)! - d_{n-t} - d_{n-t-1}
\]

We now make the following observation:

**Claim:** \( A \setminus C \) may only contain the transpositions \( \{(i \ t + 1) : i \in [t]\} \).

**Proof of Claim:**
Suppose for a contradiction that \( A \setminus C \) contains a permutation \( \rho \) not of this
form. Then $\rho(j) \neq j$ for some $j \geq t + 2$. We will show that there are at least $d_{n-t-1}$ permutations in $E$ which fix $j$ and disagree with $\rho$ at every point of $\{t+1, t+2, \ldots, n\}$, and therefore cannot $t$-intersect $\rho$. Let $l$ be the unique point of $[t]$ not fixed by $\rho$. If $\sigma$ fixes both $l$ and $j$, then $\sigma$ agrees with $\rho_{j,l} = (\rho_j)_l$ wherever it agrees with $\rho$. Notice that $\rho_{j,l}$ fixes $1, 2, \ldots, t$ and $j$.

There are exactly $d_{n-t}$ permutations in $E$ which fix $j$ and disagree with $\rho_{j,l}$ at every point of $\{t+1, t+2, \ldots, n\}$; each disagrees with $\rho$ at every point of $\{t+1, t+2, \ldots, n\}$. So none $t$-intersect $\rho$, so none are in $A$, and therefore $|A \cap C| \leq |E| - d_{n-t-1} = (n-t)! - d_{n-t} - 2d_{n-t-1}$.

Since we are assuming that $|A| \geq (n-t)! - d_{n-t} - d_{n-t-1} + t$, this means that $|A \setminus C| \geq d_{n-t-1} + t = (1/e + o(1))(n-t-1)!$.

Notice that for any $m \leq n$ we have the following trivial upper bound on the size of an $m$-intersecting family $H \subset S_n$:

$$|H| \leq \binom{n}{m} (n-m)! = n!/m!$$

since every permutation in $H$ must agree with a fixed permutation in $H$ in at least $m$ places.

Hence, $A \setminus C$ cannot be $(\log n)$-intersecting and therefore contains two permutations $\pi, \tau$ agreeing on at most $\log n$ points. The number of permutations fixing $[t]$ pointwise and agreeing with both $\pi$ and $\tau$ at one of these log $n$ points is therefore at most $(\log n)(n-t-1)!$. All other permutations in $A \cap C$ agree with $\pi$ and $\tau$ at two separate points of $\{t+1, \ldots, n\}$, and by the above argument, the same holds for $\pi_p$ and $\tau_q$, where $p$ and $q$ are the points of $[t]$ shifted by $\pi$ and $\tau$ respectively. The number of permutations in $C$ that agree with $\pi_p$ and $\tau_q$ at two separate points of $\{t+1, \ldots, n\}$ is at most $((1-1/e)^2 + o(1))(n-t)!$ (it is easily checked that given two fixed permutations, the probability that a uniform random permutation agrees with them at separate points is at most $(1-1/e)^2 + o(1)$, which implies that

$$|A \cap C| \leq ((1-1/e)^2 + o(1))(n-t)! + (\log n)(n-t-1)! = ((1-1/e)^2 + o(1))(n-t)!$$

contradicting (5), provided $n$ is sufficiently large. This proves the claim.

Since we are assuming $|A| \geq |E| + t$, we must have equality, so $A = D$, proving Theorem 9. □

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Similar arguments give the following stability results for \( t \)-cross-intersecting families. Say two pairs of families \((A, B), (C, D)\) in \( S_n\) are isomorphic if there exist permutations \( \pi, \rho \in S_n\) such that \( A = \pi C \rho \) and \( B = \pi D \rho \). We have:

**Theorem 10.** If \( n \) sufficiently large depending on \( t \), if \( A, B \subset S_n\) are \( t \)-cross-intersecting but not both contained within the same \( t \)-coset, then

\[
\min(|A|, |B|) \leq (n-t)! - d_{n-t} - d_{n-t-1} + t
\]

with equality iff \((A, B)\) is isomorphic to the pair of families

\[
\{\sigma : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = \tau(j) \ \text{for some} \ j > t + 1\} \cup \{(i \ t + 1) : i \in [t]\}
\]

\[
\{\sigma : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = j \ \text{for some} \ j > t + 1\} \cup \{(1 i \tau(1)) : i \in [t]\}
\]

where \( \tau(1) \neq 1 \) and if \( t \geq 2 \), \( \tau \) fixes \( 2, 3, \ldots, t \) and at least two points \( J > t + 1 \), whereas if \( t = 1 \), \( \tau \) intersects \((1 2)\).

**Theorem 11.** For \( n \) sufficiently large depending on \( t \), if \( A, B \subset S_n\) are \( t \)-cross-intersecting but not both contained within the same \( t \)-coset, then

\[
|A||B| \leq ((n-t)! - d_{n-t} - d_{n-t-1})(n-t)! + t
\]

with equality iff \((A, B)\) is isomorphic to the pair of families

\[
\{\sigma \in S_n : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = j \ \text{for some} \ j > t + 1\}
\]

\[
\{\sigma \in S_n : \sigma(i) = i \ \forall i \leq t\} \cup \{(1 t + 1), (2 t + 1), \ldots, (t t + 1)\}
\]

The proofs are very similar to the proof of Theorem 9 and we omit them.

### 4 The Alternating Group

We now turn our attention to the alternating group \( A_n\), the index-2 subgroup of \( S_n\) consisting of the even permutations of \( \{1, 2, \ldots, n\}\). The following may be deduced from the proof of the Deza-Frankl conjecture in [6]:

**Theorem 12.** For \( n \) sufficiently large depending on \( t \), if \( A \subset A_n\) is \( t \)-intersecting, then \(|A| \leq (n-t)!/2\).

**Remark:** This implies the Deza-Frankl conjecture. To see this, let \( A \subset S_n\) be \( t \)-intersecting; then \( A \cap A_n \) and \( (A \setminus A_n)(1 2) \) are both \( t \)-intersecting families of permutations in \( A_n\), so by Theorem 12 both have size at most \((n-t)!)/2\). Hence,

\[
|A| = |A \cap A_n| + |A \setminus A_n| \leq (n-t)!
\]
Proof. Recall that in [6], we constructed a weighted graph $Y_{\text{even}}$ which was a real linear combination of Cayley graphs on $S_n$ generated by conjugacy-classes of \textit{even} permutations with less than $t$ fixed points, and whose matrix of weights had maximum eigenvalue 1 and minimum eigenvalue

$$\omega_{n,t} = -\frac{1}{n(n-1)\ldots(n-t+1)-1}.$$ 

Clearly, $Y_{\text{even}}$ has no (non-zero) edges between $A_n$ and $S_n \setminus A_n$. Let $Y_1$ be the weighted subgraph of $Y_{\text{even}}$ induced on $A_n$, and $Y_2$ the weighted subgraph induced on $S_n \setminus A_n$. Notice that the map

$$\phi : A_n \to S_n \setminus A_n; \quad \sigma \mapsto (1 \, 2)\sigma$$

is a graph isomorphism from $Y_1$ to $Y_2$. To see this, note that

$$\phi(\sigma)(\phi(\pi))^{-1} = ((1 \, 2)\sigma)((1 \, 2)\pi)^{-1} = (1 \, 2)\sigma\pi^{-1}(1 \, 2)$$

which is conjugate to $\sigma\pi^{-1}$. Since $Y_{\text{even}}$ is a linear combination of Cayley graphs generated by conjugacy-classes of $S_n$, the edge $\phi(\sigma)\phi(\pi)$ has the same weight in $Y_{\text{even}}$ as the edge $\sigma\pi$. Hence, $Y_{\text{even}}$ is a disjoint union of the two isomorphic subgraphs $Y_1$ and $Y_2$, so the eigenvalues of $Y_{\text{even}}$ are the same as those of $Y_1$ (with double the multiplicities). Applying Theorem 1 to $Y_1$ proves Theorem 12.

Our next aim is to show that equality holds in Theorem 12 only if $A$ is a coset of the stabilizer of $t$ points. As for $S_n$, we will call these families the ‘$t$-cosets of $A_n$’.

Let $W_t$ be the subspace of $\mathbb{C}[A_n]$ spanned by the characteristic vectors of the $t$-cosets of $A_n$. It is easily checked that $W_t$ is the direct sum of the 1 and $\omega_{n,t}$-eigenspaces of $Y_1$. Hence, by Theorem 1 if equality holds in Theorem 12 then the characteristic vector $v_A$ of $A$ lies in the subspace $W_t$.

We would like to show that the Boolean functions which are linear combinations of the characteristic functions of the $t$-cosets of $A_n$ are precisely the characteristic functions of the disjoint unions of $t$-cosets of $A_n$. To do this for $S_n$ in [6], it was first proved that if a non-negative function $f : S_n \to \mathbb{R}_{\geq 0}$ is a linear combination of the characteristic functions of the $t$-cosets of $S_n$, then it can be expressed as a linear combination of them with non-negative coefficients. However, this is not true in the case of $A_n$, even for $t = 1$: 

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Claim: There exists a non-negative function in $W_1$ which cannot be written as a non-negative linear combination of the characteristic functions of the 1-cosets of $A_n$.

Proof of Claim: Let $w_{i \rightarrow j}$ be the characteristic function of the 1-coset \{\(\sigma \in A_n : \sigma(i) = j\)\}. We say a real $n \times n$ matrix $B$ represents a function $f \in W_1$ if $f$ can be written as a linear combination of $w_{i \rightarrow j}$’s with coefficients given by the matrix $B$, i.e.

$$f = \sum_{i,j=1}^{n} b_{i,j} w_{i \rightarrow j}$$

or equivalently,

$$f(\sigma) = \sum_{i=1}^{n} b_{i,\sigma(i)} \quad \forall \sigma \in A_n$$

It is easy to see that, provided $n \geq 4$, any function $f \in W_1$ has a unique extension to a function $\tilde{f} \in V_1$. Hence, if $B$ and $C$ are two matrices both representing $f$, they must both represent the same function $\tilde{f} : S_n \rightarrow \mathbb{R}$, and therefore

$$\sum_{i=1}^{n} b_{i,\sigma(i)} = \sum_{i=1}^{n} c_{i,\sigma(i)} \quad \forall \sigma \in S_n$$

Now let $f$ be the function represented by the matrix

$$B = \begin{pmatrix}
1 & -1/2 & 1 & 1 & \ldots & 1 \\
-1/2 & 1 & 1 & 1 & \ldots & 1 \\
1 & 1 & 0 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots & \\
1 & 1 & \ldots & 0
\end{pmatrix}$$

This takes only non-negative values on $A_n$, since

$$\sum_{i=1}^{n} b_{i,\sigma(i)} \geq 0 \quad \forall \sigma \in A_n$$

but if $\tau$ is the transposition $(1 2)$, then

$$\sum_{i=1}^{n} b_{i,\tau(i)} = -1$$
Hence, any matrix $C$ representing the same function as $B$ must also have

$$\sum_{i=1}^{n} c_{i,\tau(i)} = -1$$

and therefore cannot have non-negative entries. Therefore, $f$ is a non-negative function in $W_1$ that cannot be written as a non-negative linear combination of the $w_{i,j}$’s, proving the claim.

Instead, we obtain our desired characterization of equality in Theorem 12 from a stability result for $t$-intersecting families in $A_n$.

Let $e_n, o_n$ denote the number of respectively even/odd derangements of $[n]$. It is well known that $e_n - o_n = (-1)^{n-1}(n-1) \forall n \in \mathbb{N}$; combining this with the fact that $d_n = (1/e + o(1))n!$ gives $e_n = (1/(2e) + o(1))n!$, $o_n = (1/(2e) + o(1))n!$.

We now prove the following analogue of Theorem 9:

**Theorem 13.** For $n$ sufficiently large depending on $t$, if $A \subset A_n$ is a $t$-intersecting family which is not contained within a $t$-coset of $A_n$, then $A$ cannot be larger than the family

$$B = \{ \sigma \in A_n : \sigma(i) = i \ \forall i \leq t, \ \sigma(j) = (n-1)n(j) \text{ for some } j > t+1 \}$$

$$\cup \{(1 \ t + 1)(n-1) \ n, (2 \ t + 1)(n-1) \ n, \ldots, (t \ t + 1)(n-1) \ n \}$$

which has size $(n-t)!/2 - o_{n-t} - o_{n-t-1} + t = (1 - 1/e + o(1))(n-t)!/2$.

If $A$ is the same size as $B$, then $A$ is a double translate of $B$, meaning that $A = \pi B \tau$ for some $\pi, \tau \in A_n$.

**Proof.** Let $A \subset A_n$ be a $t$-intersecting family which is not contained within a $t$-coset of $A_n$ and has size

$$|A| \geq (n-t)!/2 - o_{n-t} - o_{n-t-1} + t = (1 - 1/e + o(1))(n-t)!/2.$$ 

Applying Theorem 6 with any constant $c$ such that $0 < c < (1 - 1/e)/2$, we see that (provided $n$ is sufficiently large) there exists a $t$-coset $C$ such that

$$|A \setminus C| \leq O(1/n)(n-t)!$$

By double translation, without loss of generality we may assume that $C = \{ \sigma \in A_n : \sigma(1) = 1, \ldots, \sigma(t) = t \}$. We have:

$$|A \cap C| \geq (n-t)!/2 - o_{n-t} - o_{n-t-1} + t - O(1/n)(n-t)!$$

$$= (1 - 1/e + o(1))(n-t)!/2$$

$$= (1 - 1/e + o(1))(n-t)!/2$$  \hspace{1cm} (6)
We now claim that every permutation in $A \setminus C$ fixes exactly $t - 1$ points of $[t]$. Suppose for a contradiction that $A$ contains a permutation $\tau$ fixing at most $t - 2$ points of $[t]$. Then every permutation in $A \cap C$ must agree with $\tau$ on at least 2 points of $\{t+1, \ldots, n\}$, so

$$|A \cap C| \leq \binom{n-t}{2} (n-t-2)!/2 = \frac{1}{2} (n-t)!/2$$

contradicting (6), provided $n$ is sufficiently large.

Since we are assuming that $A$ is not contained within a $t$-coset, $A \setminus C$ contains some permutation $\tau$; $\tau$ must fix all points of $[t]$ except for one. By double translation, we may assume that $\tau = (1 \ t+1)(n-1 \ n)$. We will show that under these hypotheses, $A = B$. Every permutation in $A \cap C$ must agree with $(n-1 \ n)$ at some point $\geq t+2$, i.e. $A \cap C$ is a subset of the family

$$\mathcal{E} := \{\sigma \in A_n : \sigma(i) = i \ \forall i \in [t], \ \sigma(j) = (n-1 \ n)(j) \ \text{for some } j \geq t+2\}$$

which has size

$$(n-t)!/2 - o_{n-t} - o_{n-t-1}$$

We now make the following observation:

**Claim:** $A \setminus C$ may only contain the permutations $\{(i+1)(n-1) : i \in [t]\}$.

**Proof of Claim:**

Suppose for a contradiction that $A \setminus C$ contains a permutation $\rho$ not of this form. Then $\rho(j) \neq (n-1 \ n)(j)$ for some $j \geq t+2$, so by a very similar argument to in the proof of Theorem 6 there are at least $\min(e_{n-t-1}, o_{n-t-1})$ even permutations which fix $1, 2, \ldots, t$ and agree with $(n-1 \ n)$ at $j$ (and are therefore in $\mathcal{E}$) and also disagree with $\rho$ at all points of $\{t+1, t+2, \ldots, n\} \setminus \{j\}$. Since $\rho$ has exactly $t - 1$ fixed points in $[t]$, none of these permutations can $t$-intersect $\rho$, and therefore

$$|A \cap C| \leq |\mathcal{E}| - \min(e_{n-t-1}, o_{n-t-1})$$

$$= (n-t)! - o_{n-t} - o_{n-t-1} - \min(e_{n-t-1}, o_{n-t-1})$$

Since we are assuming that $|A| \geq (n-t)! - o_{n-t} - o_{n-t-1} + t$, this means that

$$|A \setminus C| \geq \min(e_{n-t-1}, o_{n-t-1}) + t = (1/e + o(1))(n-t-1)!/2$$

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Notice that for any $m < n$ we have the following trivial upper bound on the size of an $m$-intersecting family $H \subset A_n$:

$$|H| \leq \binom{n}{m} (n-m)!/2 = n!/(2m!)$$

since every permutation in $H$ must agree with a fixed permutation in $H$ in at least $m$ places.

Hence, $A \setminus C$ cannot be $(\log n)$-intersecting and therefore contains two permutations $\pi, \tau$ agreeing on at most $\log n$ points. The number of permutations in $C$ which agree with $\pi$ and $\tau$ at one of these $\log n$ points is clearly at most $(\log n)(n - t - 1)!/2$. All other permutations in $A \cap C$ agree with $\pi$ and $\tau$ at two separate points of $\{t+1, \ldots, n\}$, and therefore the same holds for $\pi_p$ and $\tau_q$, where $p$ and $q$ are the unique points of $[t]$ shifted by $\pi$ and $\tau$ respectively. The number of permutations in $C$ that agree with $\pi_p$ and $\tau_q$ at two separate points of $\{t+1, \ldots, n\}$ is at most $((1 - 1/e)^2 + o(1))(n - t)!/2$ (it is easily checked that given two fixed permutations, the probability that a uniform random even permutation agrees with them at separate points is at most $(1 - 1/e)^2 + o(1)$), which implies that

$$|A \cap C| \leq ((1 - 1/e)^2 + o(1))(n - t)!/2 + (\log n)(n - t - 1)!/2$$

contradicting (6), provided $n$ is sufficiently large. This proves the claim.

Since we are assuming $|A| \geq |E| + t$, we must have equality, so $A = B$, proving Theorem 13.

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