A GENERALIZED WINTGEN INEQUALITY FOR LEGENDRIAN SUBMANIFOLDS IN ALMOST KENMOTSU MANIFOLDS

R. GÖRÜNÜŞ, I. KüPELI ERKEN, AZIZ YAZLA, AND C. MURATHAN

Abstract. Main interest of the present paper is to obtain the generalized Wintgen inequality for Legendrian submanifolds in almost Kenmotsu manifolds.

1. Introduction

One of the most fundamental problems in a Riemannian submanifold theory is to establish a simple sharp relationships between intrinsic and extrinsic invariants. The main extrinsic invariants are the extrinsic normal curvature, the squared mean curvature and the main intrinsic invariants include the Ricci curvature and the scalar curvature. In 1979, Wintgen [33] obtained a basic inequality involving Gauss curvature $K$, normal curvature $K^\perp$ and the squared mean curvature $\|H\|^2$ of an oriented surface $M^2$ in $E^4$, that is,

$$K \leq \|H\|^2 - |K^\perp|$$

with the equality holding if and only if the ellipse of curvature of $M^2$ in $E^4$ is a circle. The inequality (1.1), now called Wintgen inequality, attracted the attention of several authors.

Over time P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken [11] gave a conjecture for Wintgen inequality in an $n$-dimensional Riemannian submanifold $M^n$ of a real space form $R^{n+p}(c)$, namely,

$$\rho \leq \|H\|^2 - \rho^\perp + c,$$

where

$$\rho = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \langle R(e_i,e_j)e_j,e_i \rangle,$$

is the normalized scalar curvature of $M^n$

$$\rho^\perp = \frac{2}{n(n-1)} \sqrt{\sum_{1 \leq i < j \leq n} \sum_{1 \leq \alpha < \beta \leq m} \langle R^\perp(e_i,e_j)u_\alpha,u_\beta \rangle^2},$$

where $\{e_1,...,e_n\}$ and $\{u_1,...,u_p\}$ respectively orthonormal frames of tangent bundle $TM$ and normal bundle $T^\perp M$ and they also proved that this conjecture holds.

1991 Mathematics Subject Classification. 53C25, 53C40, 53D10.

Key words and phrases. Statistical Manifolds, Cosymplectic Manifolds, Kenmotsu Manifolds, Wintgen inequality, Legendrian submanifold.
for codimension \( p = 2 \). This type of inequality known later came to be known as the DDVV conjecture. A special version of the DDVV conjecture,
\[
\rho \leq \|H\|^2 + c,
\]
was proved by B.Y. Chen in [7]. F. Dillen, J. Fastenakels and J. Van Der Vekens [14] proved that DDVV conjecture is equivalent to an algebraic conjecture. Recently DDVV-conjecture was proven by Z. Lu [22] and by Ge and Z. Tang [17] independently.

In recent years, I. Mihai [25] proved DDVV conjecture for Lagrangian submanifolds in complex space forms and obtained Wintgen inequality for Legendrian submanifolds in Sasakian space forms (see [26]). On the other hand, the product spaces \( S^n(c) \times \mathbb{R} \) and \( \mathbb{R} \times H^n(c) \) are studied to obtain generalized Wintgen inequality by Q. Chen and Q. Cui [9]. Then J. Roth [30] extended DDVV inequality to submanifolds of warped product manifolds \( \mathbb{R} \times fM^n(c) \). Furuhata et al. [16] studied on Kenmotsu statistical manifolds and warped product. It is well known that a Kenmotsu manifold is locally considered as the warped product of a Kaehler manifold and a line.

Nowadays Wintgen inequality of statistical submanifolds in statistical manifolds of constant curvature has been studied in [3], [4] and [5]. The generalized Wintgen inequality for statistical submanifolds of statistical warped product manifolds is proved in [27].

Motivated by the studies of the above author, in this study, we consider generalized Wintgen inequality for Legendrian submanifolds in almost Kenmotsu manifolds.

2. Preliminaries

An almost Hermitian manifold \((N^{2n}, g, J)\) is a smooth manifold endowed with an almost complex structure \(J\) and a Riemannian metric \(g\) compatible in the sense
\[
J^2X = -X, \quad g(JX, Y) = -g(X, JY)
\]
for any \(X, Y \in \Gamma(TN)\). The fundamental 2-form \(\Omega\) of an almost Hermitian manifold is defined by
\[
\Omega(X, Y) = g(JX, Y)
\]
for any vector fields \(X, Y\) on \(N\). For an almost Hermitian manifold \((N^{2n}, g, J)\) with Riemannian connection \(\nabla\), the fundamental 2-form \(\Omega\) and the Nijenhuis torsion of \(J, N_J\) satisfy
\[
2g((\nabla_X J)Y, Z) = g(JX, N_J(Y, Z)) + 3d\Omega(X, JY, JZ) - 3d\Omega(X, Y, Z)
\]
where \(N_J(X, Y) = [X, Y] - [JX, JY] + J[X, JY] + J[JX, Y]\) (see [35]). An almost Hermitian manifold is said to be an almost Kaehler manifold if its fundamental form \(\Omega\) is closed, that is, \(d\Omega = 0\). If \(d\Omega = 0\) and \(N_J = 0\), the structure is called Kaehler. Thus by (2.1), an almost Hermitian manifold \((N, J, g)\) is Kaehler if and only if its almost complex structure \(J\) is parallel with respect to the Levi-Civita connection \(\nabla^0\), that is, \(\nabla^0 J = 0\) (35).

It is known that a Kaehlerian manifold \(N\) is of constant holomorphic sectional curvature \(c\) if and only if
\[
R(X, Y)Z = \frac{c}{4}(g(X, Z)Y - g(Y, Z)X + g(JX, Z)Y - g(JY, Z)JX + 2g(JX, Y)JZ,
\]
and is denoted by $N(c)$ (see [35]).

Let $M$ be a $(2n + 1)$-dimensional differentiable manifold and $\phi$ is a $(1, 1)$ tensor field, $\xi$ is a vector field and $\eta$ is a one-form on $M$. If $\phi^2 = -Id + \eta \otimes \xi$, $\eta(\xi) = 1$ then $(\phi, \xi, \eta)$ is called an almost contact structure on $M$. The manifold $M$ is said to be an almost contact manifold if it is endowed with an almost contact structure [6].

Let $M$ be an almost contact manifold. $M$ will be called an almost contact metric manifold if it is additionally endowed with a Riemannian metric $g$, i.e.

\begin{equation}
\eta(X) = g(X, \xi), \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0.
\end{equation}

Furthermore, a 2-form $\Phi$ is defined

\begin{equation}
\Phi(X, Y) = g(\phi X, Y),
\end{equation}

and usually is called fundamental form.

On an almost contact manifold, the $(1, 2)$-tensor field $N^{(1)}$ is defined by

\begin{equation}
N^{(1)}(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi,
\end{equation}

where $[\phi, \phi]$ is the Nijenhuis torsion of $\phi$

\[ [\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]. \]

If $N^{(1)}$ vanishes identically, then the almost contact manifold (structure) is said to be normal [6]. The normality condition says that the almost complex structure $J$ defined on $M \times \mathbb{R}$

\[ J(X, \lambda \frac{d}{dt}) = (\phi X + \lambda \xi, \eta(X)\frac{d}{dt}) \]

is integrable.

An almost contact metric manifold $M^{2n+1}$, with a structure $(\phi, \xi, \eta, g)$ is said to be an almost cosymplectic manifold, if

\begin{equation}
d\eta = 0, \quad d\Phi = 0.
\end{equation}

If additionally normality condition is fulfilled, then manifold is called cosymplectic.

On the other hand, Kenmotsu studied in [19] another class of almost contact manifolds, defined by the following conditions on the associated almost contact structure

\begin{equation}
d\eta = 0, \quad d\Phi = 2\eta \wedge \Phi.
\end{equation}

A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold.

3. Statistical Manifolds

Let $(M, g)$ be a Riemannian manifold and $\nabla$ an affine connection on $M$. An affine connection $\nabla^*$ is said to be dual connection of $\nabla$ if

\begin{equation}
Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y)
\end{equation}

for any $X, Y, Z \in \Gamma(M)$. The notion of “conjugate connection” is given an excellent survey by Simon ([?], [?]). In the triple $(g, \nabla, \nabla^*)$ is called a dualistic structure on $M$. It appears that $(\nabla^*)^\ast = \nabla$. The statistical model often forms a geometrical
manifold so that the geometry of a manifold plays an important role in statistics. The manifold structure of statistical distributions was first introduced by Amari [2] and used by Lauritzen [20].

A statistical manifold \((M, \nabla, g)\) is a Riemannian manifold \((M, g)\) endowed torsion free connection \(\nabla\) such that the Codazzi equation

\[
(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)
\]

holds for any \(X, Y, Z \in \Gamma(TM)\) (see [2]). If \((M, \nabla, g)\) is a statistical manifold, so is \((M, \nabla^*, g)\). For a statistical manifold \((M, g, \nabla, \nabla^*)\) the difference \((1, 2)\) tensor \(K\) of a torsion free affine connection \(\nabla\) and Levi-Civita connection \(\nabla^0\) is defined as

\[
K_X Y = \mathcal{K}(X, Y) = \nabla_X Y - \nabla^0_X Y.
\]

Because of \(\nabla\) and \(\nabla^0\) are torsion free, we have

\[
K_X Y = K_Y X, \quad g(K_X Y, Z) = g(Y, K_X Z)
\]

for any \(X, Y, Z \in \Gamma(TM)\). By (3.1) and (3.3), one can obtain

\[
K_X Y = \nabla^0_X Y - \nabla^*_X Y.
\]

Using (3.3) and (3.5), we find

\[
2K_X Y = \nabla_X Y - \nabla^*_X Y.
\]

By (3.3), we have

\[
g(\nabla_X Y, Z) = g(K_X Y, Z) + g(\nabla^0_X Y, Z).
\]

It can be also shown that any torsion-free affine connection \(\nabla\) has a dual connection given by

\[
\nabla^0 = \frac{1}{2}(\nabla + \nabla^*),
\]

where \(\nabla^0\) is Levi-Civita connection of the Riemannian manifold \((M, g)\). If \(\nabla = \nabla^*\) then \((M, \nabla, g)\) is called trivial statistical manifold.

Denote by \(R\) and \(R^*\) the curvature tensors on \(M\) with respect to the affine connection \(\nabla\) and its conjugate \(\nabla^*\), respectively. Then the relation between \(R\) and \(R^*\) can be given as following

\[
g(R(X, Y) Z, W) = -g(Z, R^*(X, Y)W)
\]

for any \(X, Y, Z, W \in \Gamma(TM)\).

By using (3.3) and (3.5), we have

\[
R(X, Y) Z + R^*(X, Y) Z = 2R^0(X, Y) Z + 2[\mathcal{K}, \mathcal{K}](X, Y) Z,
\]

where \([\mathcal{K}, \mathcal{K}](X, Y) Z = [\mathcal{K}_X, \mathcal{K}_Y] Z = \mathcal{K}_X \mathcal{K}_Y Z - \mathcal{K}_Y \mathcal{K}_X Z\) (see [29]). The \([\mathcal{K}, \mathcal{K}]\) is a curvature-like tensor.

In [36], L.Todjihounde gave a method how to establish a dualistic structure on the warped product manifold. If we adapt this method for \(I \times f N\), we have

**Proposition 1 (36).** Let \((\mathbb{R}, dt, \nabla)\) be a trivial statistical manifold and \((N, g_N, N, \nabla^*)\) be a statistical manifold. If the connections \(\nabla\) and \(\nabla^*\) satisfy the following relations on \(\mathbb{R} \times N\)

\[
\begin{align*}
(a) \quad \nabla_{\partial_t} \partial_t &= \nabla_{\partial_t} \partial_t = 0 \\
(b) \quad \nabla_{\partial_t} X &= \nabla_X \partial_t = \frac{f'(t)}{f(t)} X
\end{align*}
\]
A GENERALIZED WINTGEN INEQUALITY FOR LEGENDRIAN SUBMANIFOLDS IN ALMOST KENMOTSU MANIFOLDS

(3.10) \( \bar{\nabla}_X Y = N \nabla_X Y - \frac{<X,Y>}{f} f'(t) \partial_t \)
and
(i) \( \bar{\nabla}_X^a \partial_t = \nabla_X^a \partial_t = 0 \),
(ii) \( \bar{\nabla}_X^a \partial_t = \nabla_X^a \partial_t = \frac{f''(t)}{f(t)} X \)
(iii) \( \bar{\nabla}_X^* Y = N \nabla_X^* Y - \frac{<X,Y>}{f} f'(t) \partial_t \)

then \( \mathbb{R} \times_f N, <, >, \bar{\nabla}, \bar{\nabla}^* \) is a statistical manifold, where \( \bar{X}, \bar{Y} \) are vertical lifts of \( X, Y \in \Gamma(TM) \) and \( \partial_t = \frac{\partial}{\partial t} \) is horizontal lift of \( \partial_t \) and the notation is simplified by writing \( f \) for \( f \circ \pi \) and \( \text{grad } f \) for \( \text{grad}(f \circ \pi) \).

Assuming \( (\mathbb{R}, dt, \nabla) \) is trivial statistical manifold and denoting \( R \) and \( R^* \) are curvature tensors respect to the dualistic structure \( (\langle, >, \nabla, \nabla^* \rangle) \) on \( \mathbb{R} \times N \) then we can give the following lemma by Proposition II. In practice, \( (\cdot) \) is omitted from lifts.

**Lemma 1 (30).** \( (\tilde{M} = \mathbb{R} \times_f N, <, >, \bar{\nabla}, \bar{\nabla}^* \) is a statistical manifold, be a statistical warped product. If \( U, V, W \in \Gamma(N) \), then:
(a) \( R(V, \partial_t) \partial_t = -\frac{f''(t)}{f(t)} V \),
(b) \( R(V, U) \partial_t = 0 \),
(c) \( R(\partial_t, V) W = -\frac{f''(t)}{f(t)} <V, W > \partial_t \),
(d) \( R(V, W) U = R^N(V, W) U - \frac{f''(t)}{f(t)} \frac{(f'(t))^2}{(f(t))^2} [<V, U > V - <V, U > W] \)
and
(a') \( R^*(V, \partial_t) \partial_t = -\frac{f''(t)}{f(t)} V \),
(b') \( R^*(V, U) \partial_t = 0 \),
(c') \( R^*(\partial_t, V) W = -\frac{f''(t)}{f(t)} <V, W > \partial_t \),
(d') \( R^*(V, W) U = R^{*N}(V, W) U - \frac{f''(t)}{f(t)} \frac{(f'(t))^2}{(f(t))^2} [<V, U > V - <V, U > W] \)
where \( R^N, R^{*N} \) are curvature tensors of \( N \) with respect to the connections \( N \nabla, N^* \nabla \).

3.1. **Statistical submanifolds.** In this section, we will give some basic notations, formulas, definitions taken from reference [32].

Let \( (M^n, g) \) be a statistical submanifold of \( (\tilde{M}^{n+d}, <, >) \). Then the Gauss and Weingarten formulas are given respectively by

(3.10) \( \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X \xi = -A^*_\xi X + D_X \xi, \)

(3.11) \( \tilde{\nabla}_X^* Y = \nabla_X^* Y + h^*(X, Y), \quad \tilde{\nabla}_X^* \xi = -A^*_\xi X + D_X^* \xi, \)

for \( X, Y \in \Gamma(TM) \) and \( \xi \in \Gamma(T^* M) \), respectively. Furthermore, the following hold for \( X, Y, Z \in \Gamma(TM) \) and \( \xi, \eta \in \Gamma(T^* M) \):

\[ X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z), \]
\[ < h(X, Y), \xi > = g(A^*_\xi X, Y), \quad < h^*(X, Y), \xi > = g(A^*_\xi X, Y) \]
and
\[ X < \xi, \eta > = < D_X \xi, \eta > + < \xi, D_X^* \eta >. \]

The mean curvature vector fields of \( M \) are defined with respect to \( \tilde{\nabla} \) and \( \tilde{\nabla}^* \) by

\[ H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) \quad \text{and} \quad H^* = \frac{1}{n} \sum_{i=1}^{n} h^*(e_i, e_i) \]
where $\{e_1, ..., e_n\}$ is a local orthonormal frame of the tangent bundle $TM$ of $M$. By (3.10) and (3.11), we have $2h^0 = h + h^*$ and $2H^0 = H + H^*$, where $h^0$ and $H^0$ are second fundamental form and mean curvature with respect to Levi-Civita connection $\hat{\nabla}^0$.

**Proposition 2** ([32]). Let $(M^m, g, \nabla, \nabla^*)$ be statistical submanifold of $(\tilde{M}^{m+n}, <, >, \nabla, \nabla^*)$. Denote $\tilde{R}$ and $\tilde{R}^*$ the curvature tensors on $\tilde{M}^{m+n}$ with respect to connections $\nabla$ and $\nabla^*$. Then

\[
<\tilde{R}(X, Y)Z, W> = g_M(R(X, Y)Z, W) + <h(X, Z), h^*(Y, W)> - <h^*(X, W), h(Y, Z)>,
\]

\[
(3.12)
\]

\[
<\tilde{R}^*(X, Y)Z, W> = g_M(R^*(X, Y)Z, W) + <h^*(X, Z), h(Y, W)> - <h(X, W), h^*(Y, Z)>,
\]

\[
(3.13)
\]

\[
<\tilde{R}^\perp(X, Y)\xi, \eta> = <\tilde{R}(X, Y)\xi, \eta> + g_M([A_\xi, A_\eta]X, Y),
\]

\[
(3.14)
\]

\[
<\tilde{R}^{\ast\perp}(X, Y)\xi, \eta> = <\tilde{R}^*(X, Y)\xi, \eta> + g_M([A_\xi, A_\eta]X, Y),
\]

\[
(3.15)
\]

where $R^\perp$ and $R^{\ast\perp}$ are curvature tensors with respect to $D$ and $D^*$ and

\[
[A_\xi, A_\eta] = A_\xi A_\eta^* - A_\eta A_\xi^*,
\]

\[
[A_\xi^*, A_\eta] = A_\xi^* A_\eta - A_\eta^* A_\xi^*
\]

for $X, Y, Z, W \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^{\perp}M)$.

### 4. Almost Kenmotsu and cosymplectic statistical manifolds

**Definition 1** ([15]). Let $(M, g, \nabla)$ be statistical manifold with almost complex structure $J \in \Gamma(TM^{(1, 1)})$. Denote by $\Omega$ the fundamental form with respect to $J$ and $g$, that is, $\Omega(X, Y) = g(X, JY)$. The triplet $(\nabla, g, J)$ is called a holomorphic statistical structure on $M$ if $\Omega$ is a $\nabla$-parallel 2-form.

**Lemma 2** ([37]). For an almost Hermitian statistical manifold we have

\[
(\nabla_X \Omega)(Y, Z) = g(\langle \nabla_X J \rangle Y, Z) - 2g(K_X J Y, Z),
\]

\[
(4.1)
\]

and

\[
(\nabla^*_X \Omega)(Y, Z) = g(\langle \nabla^*_X J \rangle Y, Z) + 2g(K_X J Y, Z)
\]

\[
(4.2)
\]

for any $X, Y, Z \in \Gamma(TM)$.

**Corollary 1** ([37]). For an almost Hermitian statistical manifold we have

\[
(\nabla_X \Omega)(Y, Z) = (\nabla^*_X \Omega)(Y, Z) - g(K_X J Y + JK_X Y, Z)
\]

\[
(4.3)
\]

and

\[
(\nabla^*_X \Omega)(Y, Z) = (\nabla^*_X \Omega)(Y, Z) + g(K_X J Y + JK_X Y, Z)
\]

\[
(4.4)
\]

for any $X, Y, Z \in \Gamma(TM)$.

By Lemma 2 and Corollary 1 we have following.

**Proposition 3** ([15], [37]). Let $(M, g, \nabla, J)$ be a holomorphic statistical manifold and $K_X J Y + JK_X Y = 0$ for any $X, Y \in \Gamma(TM)$. Then following staments are equivalent.
• \((M, g, \nabla, J)\) is a holomorphic statistical manifold,
• \((M, g, \nabla^*, J)\) is a holomorphic statistical manifold,
• \((M, g, \nabla^0, J)\) is a Kaehler manifold.

**Definition 2.** Let \((M^{2n+1}, g, \nabla, \nabla^*)\) be a statistical manifold. If \(M^{2n+1}\) is an almost contact metric manifold then \(M^{2n+1}\) is called almost contact metric statistical manifold.

**Corollary 2 \([37]\).** For an almost contact metric statistical manifold we have
\[
(4.5) \quad (\nabla_X \Phi)(Y, Z) = (\nabla_X^0 \Phi)(Y, Z) - g(\nabla_X \phi Y + \phi K_X Y, Z)
\]
and
\[
(4.6) \quad (\nabla_X \Phi)(Y, Z) = (\nabla_X^0 \Phi)(Y, Z) + g(\nabla_X \phi Y + \phi K_X Y, Z)
\]
for any \(X, Y, Z \in \Gamma(TM)\).

**Proposition 4 \([37]\).** Let \((M^n, g, \nabla, \nabla^*)\) be a statistical manifold and \(\psi\) be a skew symmetric \((1,1)\) tensor field on \(M\). Then we have
\[
(4.7) \quad g(\nabla_X \psi Y + \phi K_X Y, Z) + g(\nabla_Z \psi X + \phi K_Z X, Y) + g(\nabla_Y \psi Z + \phi K_Y Z, X) = 0
\]
for any \(X, Y, Z \in \Gamma(TM)\).

If we resort to the relation \((4.5)\) and \((4.7)\), we have
\[
(\nabla_X \Phi)(Y, Z) = (\nabla_X^0 \Phi)(Y, Z) + (\nabla_X^0 \Phi)(X, Y)
\]
\[
\quad + (\nabla_X^0 \Phi)(Z, X) - 2g(\nabla_X \phi Y + \phi K_X Y, Z)
\]
\[
\quad + g(\nabla_Z \phi X + \phi K_Z X, Y) + g(\nabla_Y \phi Z + \phi K_Y Z, X)
\]
\[
\quad = (\nabla_X^0 \Phi)(Y, Z) + (\nabla_X^0 \Phi)(X, Y) + (\nabla_X^0 \Phi)(Z, X),
\]
where \(U, V, W \in \Gamma(TM)\).

This relation shows clearly that
\[
(4.8) \quad 3d\Phi(X, Y, Z) = (\nabla_X^0 \Phi)(Y, Z) + (\nabla_X^0 \Phi)(X, Y) + (\nabla_X^0 \Phi)(Z, X)
\]
\[
\quad = (\nabla_X \Phi)(Y, Z) + (\nabla_Z \Phi)(X, Y) + (\nabla_Y \Phi)(Z, X).
\]

Let \((N, \tilde{\nabla}, g, J)\) be an almost Hermitian manifold statistical manifold and \((\mathbb{R}, dt, f, \tilde{\nabla})\) be trivial statistical manifold. Let us consider the warped product \(\tilde{M} = \mathbb{R} \times f N\), with warping function \(f > 0\), endowed with the Riemannian metric
\[
<,> = dt^2 + f^2 g
\]
Denoting by \(\xi = \frac{\partial}{\partial t}\) the structure vector field on \(\tilde{M}\), one can define arbitrary any vector field on \(\tilde{M}\) by \(\tilde{X} = \eta(\tilde{X})\xi + X\), where \(X\) is any vector field on \(N\) and \(dt = \eta\).

By the help of tensor field \(J\), a new tensor field \(\phi\) of type \((1,1)\) on \(\tilde{M}\) can be given by
\[
(4.9) \quad \phi \tilde{X} = JX, \quad X \in \Gamma(TN),
\]
for \(\tilde{X} \in \Gamma(T\tilde{M})\). So we get \(\phi \xi = 0\), \(\eta \circ \phi = 0\), \(\phi^2 \tilde{X} = -\tilde{X} + \eta(\tilde{X})\xi\) and
\[
< \phi \tilde{X}, \tilde{Y} >= - < \tilde{X}, \phi \tilde{Y} > \text{ for } \tilde{X}, \tilde{Y} \in \Gamma(T\tilde{M}).
\]
Furthermore, we have \(< \phi \tilde{X}, \phi \tilde{Y} >= < \tilde{X}, \tilde{Y} > - \eta(\tilde{X}) \eta(\tilde{Y})\). Thus \((\tilde{M}, <,>, \phi, \xi, \eta)\) is an almost contact metric manifold. By Proposition \([4]\) and similar argument as in \([4]\) we have
\[
(4.10) \quad (\tilde{\nabla}_X \phi)(\tilde{Y}) = (\nabla_X J)Y - \frac{f'(t)}{f(t)} < \tilde{X}, \phi \tilde{Y} > - \frac{f'(t)}{f(t)} \eta(\tilde{Y}) \phi \tilde{X}
\]
Using Proposition \[1\] we get
\[
\tilde{\mathcal{K}}_X Y = \mathcal{K}_X Y, \tilde{\mathcal{K}}_X \xi = \tilde{\mathcal{K}}_\xi X = 0, \tilde{\mathcal{K}}_\xi \xi = 0,
\]
where \(\mathcal{K}_X Y = \nabla_X Y - \nabla^0_X Y\) and \(\tilde{\mathcal{K}}_X Y = \tilde{\nabla}_X Y - \tilde{\nabla}^0_X Y\).

By \[4.10\] and \[4.8\] it is readily found that the relation
\[
(\tilde{\nabla}_X \Phi)(\tilde{Y}, \tilde{Z}) = f^2(\nabla_X \Omega)(Y, Z) - \frac{f'(t)}{f(t)} \langle \tilde{X}, \phi \tilde{Y} \rangle \eta(\tilde{Z})
\]
\[
- \frac{f'(t)}{f(t)} \eta(\tilde{Y}) \Phi(\tilde{X}, \tilde{Z})
\]
We thus conclude that
\[\tag{4.11} d\Phi = f^2 d\Omega + 2\left(-\frac{f'(t)}{f(t)}\right) \eta \wedge \Phi\]
and Proposition \[1\] leading to the following theorem.

**Theorem 1.** Let \((\mathbb{R}, dt, \nabla)\) be trivial statistical manifold. Then the warped product \(\tilde{M} = \mathbb{R} \times f N\) is an almost \((\mathcal{L}(t) - \text{Kenmotsu})\) statistical manifold if and only if \((N, \nabla, g, J)\) is an almost Kaehler statistical manifold. Moreover \(\tilde{\mathcal{K}}_X Y = \mathcal{K}_X Y, \tilde{\mathcal{K}}_X \xi = \tilde{\mathcal{K}}_\xi X = 0, \tilde{\mathcal{K}}_\xi \xi = 0,\) where \(\mathcal{K} = \nabla - \nabla^g,\) and \(\tilde{\mathcal{K}} = \tilde{\nabla} - \tilde{\nabla}^{g, >}\).

Choosing \(f = \text{const} \neq 0\) we have following corollary.

**Corollary 3.** Let \((\mathbb{R}, dt, \nabla)\) be trivial statistical manifold. Then the product manifold \(M = \mathbb{R} \times N\) is an almost coisometric statistical manifold if and only if \((N, \nabla, g, J)\) is an almost Kaehler statistical manifold.

Using same methods as in [21], we get following proposition.

**Proposition 5.** Let \(\tilde{M} = I \times f N(c)\) be statistical warped product manifold and \(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \in \Gamma(\tilde{M}),\) where \(I \subset \mathbb{R}\) is trivial statistical manifold and \(N(c)\) is statistical complex space form. Then the curvature tensor \(\tilde{R}\) and \(\tilde{R}^*\) are given by
\[
\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = \tilde{R}^*(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = \left[ \frac{c}{4f^2} - \frac{(f')^2}{f^2} \right] \langle \tilde{Y}, \tilde{Z} > < \tilde{X}, \tilde{W} > - < \tilde{X}, \tilde{Z} > < \tilde{Y}, \tilde{W} >
\]
\[
+ \left[ \frac{c}{4f^2} - \frac{(f')^2}{f^2} \right] \langle \tilde{X}, \tilde{Z} > < \tilde{Y}, \partial_t > < \tilde{W}, \partial_t >
\]
\[
- < \tilde{X}, \tilde{Z} > < \tilde{Y}, \partial_t > < \tilde{W}, \partial_t > + < \tilde{Y}, \tilde{W} > < \tilde{X}, \partial_t > < \tilde{Z}, \partial_t >
\]
\[
- < \tilde{X}, \tilde{W} > < \tilde{Y}, \partial_t > < \tilde{Z}, \partial_t >
\]
\[
+ \left[ \frac{c}{4f^2} \right] < \tilde{X}, \phi \tilde{Z} > < \phi \tilde{Y}, \tilde{W} > - < \tilde{Y}, \phi \tilde{Z} > < \phi \tilde{X}, \tilde{W} >
\]
\[
+ 2 < \tilde{X}, \phi \tilde{Y} > < \phi \tilde{Z}, \tilde{W} >
\]
and \([\mathcal{K}, \mathcal{K}] = 0\).

**Remark 1.** In [16] Furuhata, Hasegawa, Okuyama an Sato introduced Kenmotsu statistical manifolds. They proved that if \(M\) has a holomorphic statistical structure, \((N = \mathbb{R} \times C, \langle ., . \rangle, \phi, \xi)\) is Kenmotsu manifold satisfying property \(\tilde{\mathcal{K}}_X Y = \mathcal{K}_X Y, \tilde{\mathcal{K}}_X \xi = \mathcal{K}_\xi X = 0, \tilde{\mathcal{K}}_\xi \xi = \lambda \xi\) then \(N\) has a holomorphic statistical structure, where \(\lambda \in C^\infty(N)\) now give a new example of a statistical warped product manifold.
Example 1 ([27]). We consider \((\mathbb{R}^2, \tilde{g} = dx^2 + dy^2)\) Euclidian space and define the affine connection by

\[
\begin{align*}
\nabla^2 \frac{\partial}{\partial x} &= \frac{\partial}{\partial y}, \\
\nabla^2 \frac{\partial}{\partial y} &= 0.
\end{align*}
\]

Then its conjugate \(\tilde{\nabla}^{2*}\) is given as follows;

\[
\begin{align*}
\tilde{\nabla}^{2*} \frac{\partial}{\partial x} &= -\frac{\partial}{\partial y}, \\
\tilde{\nabla}^{2*} \frac{\partial}{\partial y} &= 0,
\end{align*}
\]

Thus we can verify that \((\mathbb{R}^2, \tilde{\nabla}, \tilde{g})\) is a statistical manifold of constant curvature \(-1\). An affine connection and its conjugate connection are defined on \((\mathbb{R}, dt^2)\) Euclidean space as following

\[
\begin{align*}
\tilde{\nabla}^1 \frac{\partial}{\partial t} &= 0, \\
\tilde{\nabla}^1 \frac{\partial}{\partial x} &= -\frac{\partial}{\partial y}.
\end{align*}
\]

On the other hand, \(\mathbb{R} \times \mathbb{R}^2, \langle, \rangle = dt^2 + e^{2t}(dx^2 + dy^2)\) is a warped product model of hyperbolic space \((\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}, \tilde{g}_{\mathbb{H}^3} = dx^2 + dy^2 + dz^2)\) and it has natural Kenmotsu structure. We also have \((\mathbb{R} \times \mathbb{R}^2, \langle, \rangle)\) is a statistic manifold with following affine connection \(\tilde{\nabla}\);

\[
\begin{align*}
\tilde{\nabla} \frac{\partial}{\partial t} &= 0, \\
\tilde{\nabla} \frac{\partial}{\partial x} &= \frac{\partial}{\partial y}, \\
\tilde{\nabla} \frac{\partial}{\partial y} &= \frac{\partial}{\partial x} - e^{2t} \frac{\partial}{\partial t}, \\
\tilde{\nabla} \frac{\partial}{\partial t} &= -\frac{\partial}{\partial y}.
\end{align*}
\]

5. Generalized Wintgen Inequality for Almost \((-\frac{f'(t)}{f(t)})\)-Kenmotsu Statistical Manifold

Let \(\tilde{M}^n\) be a complex \(m\)-dimensional (real \(2m\) dimensional) almost Hermitian manifold with Hermitian metric \(g_{\tilde{M}}\) and almost complex structure \(J\) and \(N^n\) be a Riemannian manifold with Riemannian metric \(g_N\). If \(J(T_pN) \subset T^\perp_pN\), at any point \(p \in N\), then is called totally real submanifold. In particular, a totally real submanifold of maximum dimension is called a Lagrangian submanifold.

Let \(M^n\) be submanifold of \(\tilde{M}^{2m+1}\). \(\phi\) maps any tangent space of \(M^n\) into the normal space, that is, \(\phi(T_pM^n) \subset T^\perp_pM^{2m+1}\), for every \(p \in M^n\) then \(M^n\) is called anti invariant submanifold. If \(\text{dim}(\tilde{M}) = 2\ \text{dim}(M) + 1\) and \(\xi_p\) is orthogonal to \(T_pM\) for all \(p \in M^n\) then \(M^n\) is called Legendre submanifold.

I. Mihai, [23, 20] obtained the DDVV inequality, also known as generalized Wintgen inequality for Lagrangian submanifold of a complex space form \(\tilde{M}^{m+1}(4c)\) and Legendrian submanifolds in Sasakian space forms,

\[
(\rho^2) \leq (\|H\|^2 - \rho + c)^2 + \frac{4}{n(n-1)}(\rho - c) + \frac{2c^2}{n(n-1)},
\]
We have
\[(\rho^1)^2 \leq (\|H\|^2 - \rho + c)^2 \quad \text{rested, respectively.}
\]

In \[8\], the following theorem is proved

**Theorem 2 (8).** Let \(M^n\) be a Lagrangian submanifold of a holomorphic statistical space form \(\tilde{M}^m(c)\). Then
\[(\rho^1)^2 \geq \frac{c}{n(n-1)}(\rho - \frac{c}{4^n}) + \frac{c}{n(n-1)}g(H^*, H) - \|H\|^2 \|H^*\|.
\]

Now, we will prove Generalized Wintgen Inequality for almost \((-\frac{f'(t)}{f(t)})\)-Kenmotsu statistical manifold.

**Theorem 3.** Let \((\mathbb{R}, dt, \mathbb{R} \nabla)\) be trivial statistical manifold and \(N(c)\) be a holomorphic statistical space form. If \(M^n\) is a Legendrian submanifold of the statistical warped product manifold \(\tilde{M} = \mathbb{R} \times f N(c)\) then we have
\[\rho^1 \nabla \nabla^* \leq 2\rho^0 \nabla^* - 8\rho^0 + \frac{1}{4f^2}(2f \|c\| - c + 4(f')^2)
\]
\[+ 4 \|H^0\|^2 + \|H\|^2 + \|H^*\|^2
\]

**Proof.** Let \(M^n\) be an \(n\)-dimensional Legendrian real submanifold of a \(2n + 1\)-dimensional almost \((-\frac{f'(t)}{f(t)})\)-Kenmotsu statistical manifold \(\tilde{M} = \mathbb{R} \times f N(c)\) and \(\{e_1, e_2, \ldots, e_n\}\) an orthonormal frame on \(M^n\) and \(\{e_{n+1} = \phi e_1, e_{n+2} = \phi e_2, \ldots, e_{2n} = \phi e_n, e_{2n+1} = \xi\}\) an orthonormal frame in normal bundle \(T^\perp M^n\), respectively. By Proposition 5 and (3.12), we have
\[g_M(R(X,Y)Z,W) = \langle \tilde{R}(X,Y)Z,W \rangle + \langle h^*(X,W), h(Y,Z) \rangle - \langle h(X,Z), h^*(Y,W) \rangle
\]
\[\quad \quad \text{(5.1)} = \left[\frac{c}{4f^2} - \frac{(f')^2}{f^2}\right] \langle X, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle
\]
\[\quad \quad \quad + \langle h^*(X,W), h(Y,Z) \rangle - \langle h(X,Z), h^*(Y,W) \rangle.
\]
and
\[g_M(R^*(X,Y)Z,W) = \left[\frac{c}{4f^2} - \frac{(f')^2}{f^2}\right] \langle X, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle
\]
\[\quad \quad \text{(5.2)} + \langle h^*(X,W), h^*(Y,Z) \rangle - \langle h^*(X,Z), h^*(Y,W) \rangle.
\]
for \(X, Y, Z, W \in \Gamma(TM)\). Setting \(X = e_i = W, Y = e_j = Z\) in (5.1) and (5.2), we have
\[g_M(R(e_i,e_j)e_i,e_j) = \left(\frac{c}{4f^2} - \frac{(f')^2}{f^2}\right) \langle e_j, e_j \rangle \langle e_i, e_j \rangle - \langle e_i, e_j \rangle \langle e_i, e_j \rangle
\]
\[\quad \quad \text{(5.3)} + \langle h^*(e_i,e_j), h(e_j,e_j) \rangle - \langle h(e_i,e_j), h^*(e_i,e_j) \rangle
\]
and
\[g_M(R^*(e_i,e_j)e_i,e_j) = \left(\frac{c}{4f^2} - \frac{(f')^2}{f^2}\right) \langle e_j, e_j \rangle \langle e_i, e_j \rangle - \langle e_i, e_j \rangle \langle e_i, e_j \rangle
\]
\[\quad \quad \text{(5.4)} + \langle h(e_i,e_j), h^*(e_j,e_j) \rangle - \langle h^*(e_i,e_j), h(e_i,e_j) \rangle
\]
Using (5.1) in (5.4), we have
\[\langle R^\perp(X,Y)U,V \rangle = \frac{c}{4f^2} \langle - \phi X, U \rangle \langle \phi Y, V \rangle
\]
\[\quad \quad \quad + \langle \phi X, V \rangle \langle \phi Y, U \rangle + g_M([A^*_U, A^*_V]X,Y),
\]
If we make use of the equality (5.2) in (3.14)I, we obtain

\[(5.6) \quad <R^\perp(X,Y)\xi,\eta> = \frac{c}{4f^2}(-<\phi X,U><\phi Y,V> + <\phi X,V><\phi Y,U>) + g_M([A_\xi,A_\eta]X,Y).\]

Since \(<R(X,Y)Z,W>\) is not skew-symmetric relative to \(Z\) and \(W\). Then the sectional curvature on \(\tilde{M}\) can not be defined. But \(<R(X,Y)Z,W> + <R^*(X,Y)Z,W>)\) is skew-symmetric relative to \(Z\) and \(W\). So the sectional curvature \(K^{\nabla,\nabla^*}\) is defined by

\[K^{\nabla,\nabla^*}(X \wedge Y) = \frac{1}{2}[<R(X,Y)Y,X> + <R^*(X,Y)Y,X>],\]

for any orthonormal vectors \(X,Y, \in T_pM, p \in M\) (see [4]).

In [4], the normalized scalar curvature \(\rho^{\nabla,\nabla^*}\) and the normalized normal scalar curvature \(\rho^{\perp,\nabla^*}\) are respectively defined by

\[\rho^{\nabla,\nabla^*} = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} K^{\nabla,\nabla^*}(e_i \wedge e_j)\]

\[= \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} <R(e_i,e_j)e_j,e_i> + <R^*(e_i,e_j)e_j,e_i>\]

and

\[\rho^{\perp,\nabla^*} = \frac{1}{n(n-1)} \left\{ \sum_{n+1 \leq r < s \leq 2n} \sum_{1 \leq i < j \leq n} \left[ <R^\perp(e_i,e_j)e_r,e_s> + <R^{\perp*}(e_i,e_j)e_r,e_s> \right]^2 \right\}^{1/2},\]

where \(\{e_1,\ldots,e_n\}\) and \(\{e_{n+1} = \phi e_1,\ldots,e_{2n} = \phi e_n, e_{2n+1} = \xi\}\) are respectively orthonormal basis of \(T_pM\) and \(T_p^\perp M\) for \(p \in M\). Due to the equations (5.2) and (5.3), we obtain

\[\rho^{\nabla,\nabla^*} = \frac{1}{2n(n-1)} \sum_{i \neq j} \left( \frac{c}{4f^2} - \frac{(f')^2}{f^2} \right) + <h^*(e_i,e_i),h(e_j,e_j)> - <h(e_i,e_j),h^*(e_i,e_j)> + \frac{1}{2n(n-1)} \sum_{i \neq j} <h^*(e_i,e_i),h(e_j,e_j)> + <h(e_i,e_i),h^*(e_j,e_j)> - 2 <h(e_i,e_j),h^*(e_i,e_j)>\]

\[= \frac{c}{4f^2} - \frac{(f')^2}{f^2} + \frac{1}{2n(n-1)} \sum_{i \neq j} <h(e_i,e_i),h^*(e_i,e_i),h^*(e_j,e_j),h(e_j,e_j)> - <h(e_i,e_j),h(e_j,e_j)> - <h(e_i,e_j),h^*(e_i,e_j),h^*(e_j,e_j)> - (h(e_i,e_j) + h^*(e_i,e_j)h(e_i,e_j) + h^*(e_i,e_i) + h(e_i,e_i) - h(e_i,e_j))>\]
Because of \(2h^0 = h + h^*\) and \(2H^0 = H + H^*\), we thus get

\[
\rho \nabla, \nabla^* = \left(\frac{c}{4f^2} - \frac{(f')^2}{f^2}\right) + \frac{1}{2n(n-1)} \sum_{i \neq j} [4 < h^0(e_i, e_i), h^0(e_j, e_j)] >
- < h(e_i, e_i), h(e_j, e_j) > - < h^*(e_i, e_i), h^*(e_j, e_j) >
- (4 < h^0(e_i, e_j), h^0(e_j, e_i) > - < h(e_i, e_j), h(e_i, e_j) >
- < h^*(e_i, e_j), h^*(e_i, e_j) >)
\]

\[
\rho \nabla, \nabla^* = \left(\frac{c}{4f^2} - \frac{(f')^2}{f^2}\right) + \frac{1}{2n(n-1)} [4n^2 \parallel H^0 \parallel^2 - n^2 \parallel H \parallel^2 - n^2 \parallel H^* \parallel^2
- (4 \parallel H^0 \parallel^2 - \parallel H \parallel^2 - \parallel H^* \parallel^2].
\]

Denote \(\tau^0 = h^0 - H^0 g\), \(\tau = h - H g\) and \(\tau^* = h^* - H^* g\) the traceless part of second fundamental forms. Then we find \(\parallel \tau^0 \parallel^2 = \parallel h^0 \parallel^2 - n^2 \parallel H^0 \parallel^2, \parallel \tau \parallel^2 = \parallel h^2 - n^2 \parallel \parallel H \parallel^2\) and \(\parallel \tau^* \parallel^2 = \parallel h^* - n^2 \parallel \parallel H^* \parallel^2\). Thus, we get

\[
\rho \nabla, \nabla^* = \left(\frac{c}{4f^2} - \frac{(f')^2}{f^2}\right) + \frac{1}{2n(n-1)} [4n^2 \parallel H^0 \parallel^2 - n^2 \parallel H \parallel^2 - n^2 \parallel H^* \parallel^2
- (4 \parallel \tau^0 \parallel^2 + 4n \parallel H^0 \parallel^2 - \parallel \tau \parallel^2 - n \parallel H \parallel^2 - \parallel \tau^* \parallel^2 - n \parallel H^* \parallel^2].
\]

This relation gives rise to

\[
\rho \nabla, \nabla^* = \left(\frac{c}{4f^2} - \frac{(f')^2}{f^2}\right)
+ 2 \parallel H^0 \parallel^2 - \frac{2}{n(n-1)} \parallel \tau^0 \parallel^2
\]

(5.7)

\[
\left\{ \begin{array}{l}
- \frac{1}{2} \parallel H \parallel^2 + \frac{1}{2n(n-1)} \parallel \tau \parallel^2 \\
- \frac{1}{2} \parallel H^* \parallel^2 + \frac{1}{2n(n-1)} \parallel \tau^* \parallel^2
\end{array} \right. 
\]

From (5.5) and (5.6), the normalized normal scalar curvature satisfies

\[
\rho \nabla, \nabla^* = \frac{1}{n(n-1)} \left\{ \sum_{1 \leq r < s \leq n+1} \sum_{1 \leq i < j \leq n} \left[ g([A_{e_{n+r}}^* A_{e_{n+s}}^*], e_i, e_j) + g([A_{e_{n+r}} A_{e_{n+s}}], e_i, e_j) \right] \right\}^{1/2}
\]

(5.8)

\[
\left\{ \begin{array}{l}
\sum_{1 \leq r < s \leq n+1} \sum_{1 \leq i < j \leq n} \left[ g([A_{e_{n+r}}^* A_{e_{n+s}}^*], e_i, e_j) + g([A_{e_{n+r}} A_{e_{n+s}}], e_i, e_j) \right] \right\}^{1/2}
\]

By Proposition 4 and the equations 5.10, 5.11 we have

\[
(5.9) \quad A_{e}X = A_{e}^* X = \frac{f'(t)}{f(t)} X.
\]

So we have

\[
g([A_{e}^* A_{e_{n+s}}, e_i, e_j]) = g(A_{e} A_{e_{n+s}}, e_i, e_j) - g(A_{e_{n+s}}, A_{e}^* e_i, e_j)
\]

(5.10)

\[
= \frac{f'(t)}{f(t)} g(A_{e_{n+s}}, e_i, e_j) + \frac{f'(t)}{f(t)} g(A_{e_{n+s}}, e_i, e_j)
\]

\[
= 0
\]
and by using similar calculation we obtain

\[(5.11) \quad ([A_{e_{n+r}}, A^*_{e_{n+s}}]e_i, e_j) = 0.\]

On the other hand we recall

\[(5.12) \quad \langle \phi X, \xi \rangle = 0.\]

Using the equations (5.10), (5.11) and (5.12) in (5.8) we find that

\[
\rho_{\perp \nabla, \nabla^*} = \frac{1}{n(n-1)} \left\{ \sum_{1 \leq r < s \leq n+1} \sum_{1 \leq i < j \leq n} \left[ g([A^*_{e_{n+r}}, A_{e_{n+s}}]e_i, e_j) + g([A_{e_{n+r}}, A^*_{e_{n+s}}]e_i, e_j) \right] \right\}^{1/2}
\]

which is equivalent

\[(5.13) \quad \rho_{\perp \nabla, \nabla^*} = \frac{1}{n(n-1)} \left\{ \sum_{1 \leq r < s \leq n+1} \sum_{1 \leq i < j \leq n} \left[ 4g([A^0_{e_{n+r}}, A^0_{e_{n+s}}]e_i, e_j) + g([A_{e_{n+r}}, A^*_{e_{n+s}}]e_i, e_j) \right] \right\}^{1/2},
\]

where \(2A^0_{\xi_r} = A_{\xi_r} + A^*_{\xi_r}\). By the Cauchy–Schwarz inequality, we have the algebraic inequality

\[(5.14) \quad (\lambda + \mu + \nu + w)^2 \leq 4(\lambda^2 + \mu^2 + \nu^2 + w^2), \forall \lambda, \mu, \nu \in \mathbb{R}.\]

We obtain from (5.14) that

\[
\rho_{\perp \nabla, \nabla^*} \leq \frac{2}{n(n-1)} \left\{ \sum_{1 \leq r < s \leq n+1} \sum_{1 \leq i < j \leq n} \left[ 16g([A^0_{e_{n+r}}, A^0_{e_{n+s}}]e_i, e_j)^2 + g([A_{e_{n+r}}, A_{e_{n+s}}]e_i, e_j)^2 \right] \right\}^{1/2}
\]

\[
\leq \frac{2}{n(n-1)} \left\{ \frac{2^2}{4f^2} n^2(n-1)^2 + \frac{1}{4} \sum_{r,s=1}^n \sum_{i,j=1}^m \left[ 16g([A^0_{\xi_r}, A^0_{\xi_s}]e_i, e_j)^2 + g([A_{\xi_r}, A_{\xi_s}]e_i, e_j)^2 \right] \right\}^{1/2}
\]

Now we define sets \(\{S^0_1, ..., S^0_n\}, \{S_1, ..., S_n\}, \{S^*_1, ..., S^*_n\}\) of symmetric with trace zero operators on \(T_pM\) by

\[
\langle S^0_\alpha X, Y \rangle = \langle \tau^0(X, Y), \xi_\alpha \rangle,
\]

\[
\langle S_\alpha X, Y \rangle = \langle \tau(X, Y), \xi_\alpha \rangle,
\]

\[
\langle S^*_\alpha X, Y \rangle = \langle \tau^*(X, Y), \xi_\alpha \rangle
\]

for all \(X, Y \in T_pM, p \in M\). Clearly, we obtain

\[
S^0_\alpha = A^0_{\xi_\alpha} = \langle H^0, \xi_\alpha \rangle > I,
\]

\[
S_\alpha = A_{\xi_\alpha} = \langle H, \xi_\alpha \rangle > I,
\]

\[
S^*_\alpha = A^*_{\xi_\alpha} = \langle H^*, \xi_\alpha \rangle > I
\]
and

\[ [S^0_\alpha, S^0_\beta] = [A^0_\xi, A^0_\xi], \]
\[ [S^\alpha, S^\beta] = [A_\xi, A_\xi], \]
\[ [S^\alpha, S^\beta] = [A^\xi, A^\xi]. \]

Therefore, it is clear that

\[
\rho_{\nabla^*, \nabla^*} \leq \frac{2}{n(n-1)} \left\{ \frac{c^2}{4f^2}n^2(n-1)^2 + \frac{1}{4} \sum_{r,s=1}^{n} (16\|[S^0_\alpha, S^0_\beta]\|^2 + \|[S_\alpha, S_\beta]\|^2 + \|[S^\alpha, S^\beta]\|^2) \right\}^{1/2}.
\]

In [22] Lu proved the following theorem

**Theorem 4** ([22]). Let \( M^m \) be Riemannian submanifold of Riemannian space form \( \tilde{M}^{m+n}(c) \). For every set \( \{B_1, \ldots, B_n\} \) of symmetric \((n \times n)\)-matrices with trace zero the following inequality holds:

\[
\sum_{\alpha, \beta=1}^{n} \|[B_\alpha, B_\beta]\|^2 \leq (\sum_{\alpha=1}^{n} \|B_\alpha\|^2)^2.
\]

By Theorem 4 ([22]), can be written as

\[
\rho_{\nabla^*, \nabla^*} \leq \frac{|c|}{2f} + \frac{4}{n(n-1)} \sum_{r=1}^{n} \|S^0_r\|^2 + \frac{1}{n(n-1)} \sum_{r=1}^{n} \|[S_r, S_r]\|^2 + \frac{1}{n(n-1)} \sum_{r=1}^{n} \|[S^\alpha, S^\alpha]\|^2
\]

\[
(5.16) \quad \leq \frac{|c|}{2f} + \frac{8}{n(n-1)} \|\tau^0\|^2 + \frac{2}{n(n-1)} \|\tau\|^2 + \frac{1}{n(n-1)} \|\tau^*\|^2.
\]

Using (5.17) in (5.16), we get

\[
(5.17) \quad \rho_{\nabla^*, \nabla^*} \leq \frac{|c|}{2f} + \frac{8}{n(n-1)} \|\tau^0\|^2 + 2\rho_{\nabla^*, \nabla^*} - \frac{2c}{4f^2} + \frac{2(f')^2}{f^2}
\]

\[-4 \|H^0\|^2 + \|H\|^2 + \|H^*\|^2.
\]

The other hand normalized scalar curvature \( \rho^0 \) of \( M^m \) with respect to Levi-civita connection \( \nabla^0 \) can be obtained as

\[
(5.18) \quad \rho^0 = \left( \frac{c}{4f^2} - \frac{(f')^2}{f^2} \right) + \frac{1}{n(n-1)} \|H^0\|^2 - \|h^0\|^2.
\]

(see [30]).

Now, If we set \( \|\tau^0\|^2 = \|h^0\|^2 - n \|H^0\|^2 \) in (5.18) then we get

\[
(5.19) \quad \rho^0 = \left( \frac{c}{4f^2} - \frac{(f')^2}{f^2} \right) + \|H^0\|^2 - \frac{1}{n(n-1)} \|\tau^0\|^2.
\]

In view of the equations (5.17) and (5.18) we have

\[
\rho_{\nabla^*, \nabla^*} \leq 2\rho_{\nabla^*, \nabla^*} - 8\rho^0 + \frac{1}{4f^2}(2f \cdot c \cdot c - 4(f')^2)
\]

\[
+ 4 \|H^0\|^2 + \|H\|^2 + \|H^*\|^2
\]

which completes proof. \( \square \)
Corollary 4. Let \((\mathbb{R}, dt, \nabla)\) be trivial statistical manifold and \(N(c)\) be a holomorphic statistical space form. If \(M^n\) be a Legendrian submanifold of the statistical Kenmotsu manifold \(\mathbb{R} \times e^t \mathbb{C}^n\), then we get
\[
\rho^1 \nabla, \nabla^* \leq 2\rho \nabla, \nabla^* - 8\rho^0 + 4 \| H^0 \|^2 + \| H \|^2 + \| H^* \|^2 + 1.
\]
In this case \(\mathbb{R} \times e^t \mathbb{C}^n\) is locally isometric to the hyperbolic space \(H^{2n+1}(-1)\).

Corollary 5. Let \((\mathbb{R}, dt, \nabla)\) be trivial statistical manifold and \(N(c)\) be a holomorphic statistical space form. If \(M^n\) be a Legendrian submanifold of the statistical cosymplectic manifold \(\mathbb{R} \times N(c)\), then we have
\[
\rho^1 \nabla, \nabla^* \leq 2\rho \nabla, \nabla^* - 8\rho^0 + 4 \| H^0 \|^2 + \| H \|^2 + \| H^* \|^2 + \frac{1}{4}(2 |c| - c)
\]

REFERENCES

[1] A. Carriazo, A., M.J., Perez-Garcia, Slant submanifolds in neutral almost contact pseudo-metric manifolds. Differ. Geom. Appl. 54, 71–80 (2017). https://doi.org/10.1016/j.difgeo.2017.02.002
[2] S. Amari,: Differential-Geometrical Methods in Statistics, Lecture Notes in Statistics, vol. 28. Springer, Berlin (1985)
[3] M. E. Aydin, A. Mihai, I. Mihai, Some inequalities on submanifolds in statistical manifolds of constant curvature, Filomat 29 (3) (2015), 465-477.
[4] M. E. Aydin, A. Mihai, I. Mihai, Generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature, Bull. Math. Sci., Vol. 7, Issue 1, (2017) 155-166.
[5] M. E. Aydin, I. Mihai, Wintgen inequality for statistical surfaces, Wintgen inequality for statistical surfaces (2015). arXiv:1511.04987 [math.DG]
[6] D. E., Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, (2002).
[7] B. Y. Chen, Mean curvature and shape operator of isometric immersions in real-space forms, Glasgow Math. J. 38 (1996), 87-97.
[8] M. N. Boyom, M. Aquib, M. H. Shahid and M. Jamali, Generalized Wintgen Type Inequality for Lagrangian Submanifolds in Holomorphic Statistical Space Forms, Frank Nielsen • Frédéric Barbaresco (Eds.) Geometric Science of Information Third International Conference, GSI 2017 Paris, France, November 7–9, 2017 Proceedings.
[9] Q. Chen and Q. Cui, Normal scalar curvature and a pinching theorem in \(S^m \times \mathbb{R}\) and \(H^m \times \mathbb{R}\), Science China Math. 54(9) (2011),1977-1984
[10] T. Choi and Z. Lu, On the DDVV conjecture and the comass in calibrated geometry I. Math. Z. 260 (2008), 409-429.
[11] P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, A pointwise inequality in submanifold theory, Arch. Math. (Brno) 35 (1999), 115-128.
[12] G. Dileo, A.M. Pastore, Almost Kenmotsu manifolds and local symmetry. Bull. Belg. Math. Soc. Simon Stevin 14 (2007), no. 2, 343-354.
[13] F. Dillen, J. Fastenakels and J. Van Der Veken, A pinching theorem for the normal scalar curvature of invariant submanifolds, J. Geom. Phys. 57 (2007), 833-840.
[14] F. Dillen, J. Fastenakels and J. Van Der Veken, Remarks on an inequality involving the normal scalar curvature, Pure and Applied Differential Geometry-PADGE 2007, 83-92, Ber. Math., Shaker Verlag, Aachen, 2007.
[15] H. Furuhata, Hypersurfaces in statistical manifolds, Diff. Geom. Appl. 27 (2009) 420-429.
[16] H. Furuhata, I. Hasegawa.Y. Okuyama, and K. Sato, Kenmotsu statistical manifolds and warped product, J. Geom. 108 (2017), 1175–1191.
[17] J. Ge and Z. Tang, A proof of the DDVV conjecture and its equality case, Pacific J. Math. 237 (2008), 87-95.
[18] I. V. Guadalupe, L. Rodriguez, Normal curvature of surfaces in space forms, Pacific J. Math. 106 (1983) 95-103.
[19] K. Kenmotsu, A class of contact Riemannian manifold, Tohoku Math. Journal, 24 (1972), 93-103.
[20] Lauritzen, S.: In statistical manifolds. In: Amari, S., Barndorff-Nielsen, O., Kass, R., Lauritzen, S., Rao, C.R. (eds.) Differential Geometry in Statistical Inference, vol. 10, pp. 163–216. IMS Lecture Notes-Institute of Mathematical Statistics, Hayward (1987)

[21] Lawn, M, Ortega M. A fundamental theorem for hypersurfaces in semi-Riemannian warped products, Journal of Geometry and Physics. 90 (2015), 55-70.

[22] Z. Lu, Normal scalar curvature conjecture and its applications, J. Funct. Analysis 261 (2011), 1284-1308.

[23] G. D., Ludden, Submanifolds of cosymplectic manifolds, J. Differential Geometry, 4 (1970), 237-244.

[24] A. Mihai, Scalar normal curvature of Lagrangian 3-dimensional submanifolds in complex space forms, Pure and Applied Differential Geometry PADGE 2007, 171-177, Ber. Math., Shaker Verlag, Aachen, 2007.

[25] I. Mihai, On the generalized Wintgen inequality for Lagrangian submanifolds in complex space forms, Non-linear Anal. 95 (2014), 714-720.

[26] I. Mihai, On the generalized Wintgen inequality for Legendrian submanifolds in Sasakian space forms, Tohoku Math. J., 69 (2017), 43-53. 87-95

[27] C. Murathan, B. Şahin, A study of Wintgen like inequality for submanifolds in statistical warped product manifolds. J. Geom. 109 (2018), no. 2, Art. 30, 18 pp.

[28] B. O'Neill, Semi-Riemannian Geometry. Academic Press, New York (1983)

[29] B. Opozda, A sectional curvature for statistical structures. Linear Algebra Appl. 2016, 497, 134–161

[30] J. Roth, A DDVV Inequality for submanifolds of warped products, Bull. Aust. Math. Soc. 95 (2017), 495-499.

[31] H. Shima, The Geometry of Hessian Structures; World Scientific: Singapore, 2007.

[32] Vos, P.W.: Fundamental equations for statistical submanifolds with applications to the Bartlett correction. Ann. Inst. Stat. Math. 41(3), 429–450 (1989).

[33] P. Wintgen, Sur l’inégalité de Chen-Willmore, C. R. Acad. Sci., Paris Sér. A-B 288 (1979) A993-A995.

[34] K. Yano, M. Kon, Anti-Invariant Submanifolds, M. Dekker, New York, 1976.

[35] K. Yano, M. Kon, Structures on manifolds, Series in Pure Mathematics, 3. World Scientific Publishing Co., Singapore, 1984.

[36] L. Todjihounde, Dualistic structures on warped product manifolds, Diff.Geom.-Dyn. Syst. 8, (2006), 278-284.

[37] A. Yazla, İ. Küpeli Erken, C. Murathan, Almost cosymplectic statistical manifolds,(2018). arXiv:1801.08900 [math.DG].