Research Article
Dynamics of COVID-19 Using SEIQR Epidemic Model

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Received 19 May 2022; Revised 9 August 2022; Accepted 20 August 2022; Published 26 September 2022

Academic Editor: Ali Sajid

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The major goal of this study is to create an optimal technique for managing COVID-19 spread by transforming the SEIQR model into a dynamic (multistage) programming problem with continuous and discrete time-varying transmission rates as optimizing variables. We have developed an optimal control problem for a discrete-time, deterministic susceptible class (S), exposed class (E), infected class (I), quarantined class (Q), and recovered class (R) epidemic with a finite time horizon. The problem involves finding the minimum objective function of a controlled process subject to the constraints of limited resources. For our model, we present a new technique based on dynamic programming problem solutions that can be used to minimize infection rate and maximize recovery rate. We developed suitable conditions for obtaining monotonic solutions and proposed a dynamic programming model to obtain optimal transmission rate sequences. We explored the positivity and unique solvability nature of these implicit and explicit time-discrete models. According to our findings, isolating the affected humans can limit the danger of COVID-19 spreading in the future.

1. Introduction

Mathematical models are useful for determining how an infection behaves when it enters a population and determining whether it will be eradicated or continue under different settings. COVID-19 is currently causing tremendous concern among researchers, governments, and the general public due to its rapid spread and a high number of deaths [1]. The transmission of this disease is caused by the tiny particles or droplets called aerosols that carry the virus into the atmosphere caused by a contaminated person while sneezing, coughing, or exhaling. Many researchers and scientists are continuously working to reduce the transmission of this vicious disease throughout the world. Infectious diseases are the disciplines that focus on the study of the dynamics of infectious diseases as well as the relationship between these diseases and the various factors involved in their appearance and evolution, in order to implement a fight against this spread. Despite its youth, mathematical modeling is a valuable tool for understanding disease transmission mechanisms, is playing an increasingly important role in epidemiology, and has already contributed to significant successes. The most influential work in the field of mathematics epidemiology was first introduced by Kermack and McKendrick as the SIR model in the year 1927 [2]. Cao et al.[3] discussed a modified model of the SIR (susceptible, infected, recovered) epidemic introduced in order to detect the confirmed number of infected cases and consecutive burdens on isolation wards and ICUs. Also, Nesteruk [4] developed the variables used in the proposed model by introducing a SIR epidemic model and explaining how to dominate the spread of the disease. To restore the pandemic with the involvement of social distancing and lockdown, Gerberry and Milner presented a data-driven susceptible, exposed, infected, quarantined, and recovered (SEIQR) model in [5]. From the publication of Zeb et al. [6], epidemiological’s SEIQR model with isolation class in 2020 and their mathematical epidemiology has expanded in numerous directions, involving biology and computer science by Zima et al.[7], Zhou et al. [8], and Kermack and McKendrick.
Some recent studies have focused on this area of research by He et al. [11], Rahimi et al. [12], Hussain et al. [13], Youssef et al. [14], Prabakaran et al. [15], and Youssef et al. [16]. In this current paper, we implement the discrete type of SEIQR model and discuss the solvability of both continuous and discrete type SEIQR model. We examined the behaviour of the time-continuous model. We have developed two time-discrete models: time-implicit and time-explicit. We looked at the theory and methods for solving the time-implicit model. Then, to control and anticipate the dynamics of COVID dissemination, we establish appropriate transmission rate limits. To do this, we devised a dynamic programming problem to optimize transmission rate sequences under arbitrary beginning conditions. We propose safety guidelines and essential precautionary measures based on the optimized rate sequences to control COVID spread. The article gives the technique for optimizing the transmission rate sequences. The epidemic models and their time-discrete variations have been studied by Allen [17] and Ghosh et al. [18]. Several approaches towards fractional-order mathematical models of COVID-19 were studied by the authors Alghatani et al. [19], Valliammal and Ravichandran [20], Nisar et al. [21], Vijayakumar et al. [22], and Alderremy et al. [23]. However, the aforementioned studies and references mostly contain explicit approaches with respect to time-discrete epidemic models.

1.1. Review Literature. In 2020, COVID-19 is a worldwide emergency. The first cases occurred in December 2019, and as of 6:34 pm CEST, 28 July 2022, there have been 571,198,904 confirmed cases of COVID-19, including 6,387,863 deaths, reported to WHO. As of 25 July 2022, a total of 12,248,795,623 vaccine doses have been administered. The rapid spread of COVID-19 has already caused great public attention and many heated discussions, and the Chinese mass media have been reporting relevant information about the virus and the outbreak.

Ming et al. [24] show that effective public health measures are required to be implemented in time to avoid the breakdown of the health system, and the media can certainly play a crucial role in conveying updated policies and regulations from authorities to the citizens. The finding that SARS-2-S exploits ACE2 for entry, which was also reported by Kermack and McKendrick [9] while the present manuscript was in revision, suggests that the virus might target a similar spectrum of cells as SARS-CoV. However, upon its outbreak, various research, including but not limited to Okhuese [25], began to predict the scale that the virus would hit the world; the ratio of the death to recovery rate has seemingly been a positive proportion. Allen [17] studied about time-discrete SI, SIR, and SIS epidemic models, and its properties. Kermack and McKendrick [10] analyzed an outbreak such as the one in Hubei is captured by SIR dynamics where the population is divided into three compartments that differentiate the state of individuals with respect to the contagion process: infected (I), susceptible (S) to infection, and removed (R) (i.e., not taking part in the transmission process).
discussed about the complex systems and network science approaches, along with technological advances and data availability, are becoming instrumental for the design of effective containment strategies. In a non-essential region, Hilfer’s neutral fractional derivative provided controllability results using Monch’s method, Banach’s contraction principle, fractional calculus, and semigroup property was studied by the authors Nisar et al. [21]. Some recent updates regarding the modeling of the coronavirus, the authors Alderremy et al. [23] constructed a mathematical model based on the fuzzy fractional derivative and obtained the results. The authors Valliammal and Ravichandran [20] are discussed in detail the fractional integro-differential equation with different conditions in various spaces. In [30], the authors Awal et al. proposed a framework that uses Bayesian optimization to optimize the hyperparameters of the classifier and adaptive synthetic (ADASYN) algorithm to balance the COVID and non-COVID classes of the dataset. In 2022, the authors Youssef et al. considered the transmission of COVID-19 based upon data of epidemiological parameters. In this article, we have considered the transmission of COVID and non-COVID classes of the dataset. In [6,30], we have extended the idea of the article [1] to the considered model. As a result, we can optimize COVID-19 outbreaks by taking the real cases from Saudi Arabia [16]. Alqhtani et al. [19] analyzed about discrete-time epidemic model for the analysis of transmission of COVID-19 based upon data of epidemiological parameters. In this article, we have considered the Caputo fractional operator. Ghosh et al. [18] have studied the principle, fractional calculus, and semigroup property was effective containment strategies. In a nonsense region, we have investigated the properties of its time-continuous formulation, and design an implicit numerical solution approach that preserves the time-continuous variant’s primary properties. In our investigations, COVID-19 outbreaks might be caused by human-to-human interaction. As a result, isolation of the infected humans can reduce the COVID-19 spread in the future. Literature review and comparison of various of these models are presented in Table 1.

More precisely, our main contributions can be summarized as follows:

(i) First, we suggest a time-continuous SEIQR model modification with time-varying transmission and recovery rates.
(ii) Second, we draw the conclusion that the formulation of our time-continuous problem is well-posed. This comprises continuous reliance on initial conditions and time-varying rates, global existence in time, and global uniqueness in time, all of which are based on an inductive application of Banach’s fixed point theorem.

(iii) In the case of the time-discrete implicit model, we provide unique solvability, monotonicity properties, and an upper error bound between the solution of the implicit time-discrete problem formulation and the solution of the time-continuous problem formulation.

(iv) In order to maximize transmission rate sequences under arbitrary beginning conditions, we have developed a dynamic programming problem. Based on the optimal rate sequences, we suggest safety guidelines and important safety precautions to control COVID spread.

The paper is arranged as follows: Section 1 is dedicated to the introduction. In section 2, we present the time-continuous and time-discrete SEIQR model. In section 3, we give the monotonicity properties and long-time behaviour. An error analysis is given in section 4. The conclusion of our research work is implemented in the last section 5.

2. Time-Continuous SEIQR Model

The time-continuous SEIQR model is formulated, and its behaviours are described using the Lipschitz condition and Bellman’s inequality in this section.

2.1. Mathematical Background Material. Here, we revisit the Lipschitz continuity of a function on Euclidean spaces, the local Lipschitz condition, Banach’s fixed point theorem, and the method of variation of the parameter, which will be used in the subsequent sections.

Definition 1 (see [46]). Let \(q_1\) and \(q_2\) be two positive integers and \(D \subset \mathbb{R}^{q_1}\). A function \(H: D \rightarrow \mathbb{R}^{q_2}\) is said to be Lipschitz continuous on \(D\) if there exists a nonnegative constant \(L \geq 0\) such that \(\|H(x) - H(y)\|_{\mathbb{R}^{q_2}} \leq L \cdot \|x - y\|_{\mathbb{R}^{q_1}}\) holds for \(x, y \in D\).

(i) Let \(U \subset \mathbb{R}^{q_1}\) be an open set and \(H: U \rightarrow \mathbb{R}^{q_2}\). Then, \(H\) is called as locally Lipschitz continuous if for every element \(y_0 \in U\) there exists a neighborhood \(V\) of \(y_0\) such that the restrictions of \(H\) to \(V\) are Lipschitz continuous on \(V\).

In a more general framework, we consider a nonlinear initial value problem (IVP) \(z'(t) = H(t, z(t))\); \(z(0) = z_0\), where \(z(t) = (y_1(t), y_2(t), \ldots, y_{q_1}(t))^T\) is solution vector, and \(H(t, z(t)) = (h_1(t, z(t)), h_2(t, z(t)), \ldots, h_{q_2}(t, z(t)))^T\) is vectorial function with initial point \(z_0 \in \mathbb{R}^{q_1}\). The following theorem, which is a direct consequence of Gronwall’s lemma, can be used to prove global existence in time.

Theorem 1 (see [17]). Let \(G: \mathbb{R}^{q_1} \rightarrow \mathbb{R}^{q_2}\) be locally Lipschitz conditions. If there exist nonnegative real constants \(B\) and \(k\) such that \(\|G(t, z(t))\|_{\mathbb{R}^{q_2}} \leq k |z(t)| + B\) holds for all \(z(t) \in \mathbb{R}^{q_2}\), then the solution of the initial value problem \(z'(t) = G(t, z(t));\ z(0) = z_0\), exists for all time \(t \in \mathbb{R}\), and moreover, \(z(t)\) satisfies \(\|z(t)\|_{\mathbb{R}^{q_2}} \leq \|z_0\|_{\mathbb{R}^{q_1}} \cdot \exp(k \cdot |t|) + (B/k) (\exp(k \cdot |t|) - 1), \forall t \in \mathbb{R}\).

We give the following: Banach’s fixed point theorem, which will be used to preserve the global uniqueness in time [47, 47].
Table 1: Overview of models studied.

| Models     | Compartments                                                                 | Model type     | Study area                                       |
|------------|------------------------------------------------------------------------------|----------------|-------------------------------------------------|
| SEIR/SLIR  |Susceptible (S), exposed/latent (E/L), infectious (I), removed (R)           |Deterministic   |Europe and North America, New York, Mexico, Zhejiang, Guangdong, Japan, India [31–36]|
| SEIQR      |Susceptible (S), exposed (E), hospitalized infected (I), quarantine (Q), recovered or removed (R) |Deterministic   |India [37]                                        |
| SIR-X      |Infected (I), susceptible (S), removed (R), quarantined (X)                  |Deterministic   |China [38]                                        |
| SIRD       |Susceptible (S), infected (I), recovered (R), dead (D)                        |Deterministic   |China, Italy, and France [39]                     |
| SEIHARD    |Susceptible (S), exposed (E), asymptomatic infectious (I), hospitalized (H), asymptomatic infectious (A), recovered (R), deaths (D) |Deterministic   |Washington, New York [40]                         |
| SIRU       |Susceptible (S), asymptomatic infectious (I), reported symptomatic infectious (R), unreported symptomatic infectious (U) |Deterministic   |China, Hubei, Wuhan [41]                          |
| SEIPAHRF   |Susceptible (S), exposed (E), symptomatic (I), super-spreaders class (P), asymptomatic infectious (A), hospitalization (H), recovery (R), death (D) |Deterministic   |Wuhan [42]                                        |
| SEIRU      |Susceptible (S), asymptomatic noninfectious (E), asymptomatic infectious (I), reported symptomatic infectious (R), unreported symptomatic infectious (U) |Deterministic   |China [43]                                        |
| SEISS      |Susceptible (S), exposed (E), infectious (I), removed (R), pathogens (P)     |Deterministic   |South Korea [44]                                  |
| SEIRP      |Susceptible (S), exposed (E), infectious (I), removed (R)                     |Deterministic   |Pakistan [45]                                     |

Theorem 2 (see [48]). Let \((X, \mu)\) be a complete metric space. Let \(T: X \rightarrow X\) be a strict contraction, that is, there exists a constant \(K \in [0, 1)\) such that \(\mu(T(x), T(y)) \leq K \cdot \mu(x, y)\) holds for all \(x, y \in X\). Then, the mapping \(T\) has a unique fixed point.

In the following theorem, we present the Gronwall and Bellman inequality, which will be used in the subsequent theorems related to the continuous functions.

Theorem 3 (see [49]). Let \(I: = [a, b], u, f: I \rightarrow [0, \infty)\) be two continuous and nonnegative functions and \(g: I \rightarrow (0, \infty)\) be a continuous, positive, and nondecreasing function. If the inequality

\[
u(t) \leq g(t) + \int_a^t f(s) \cdot u(s) ds,
\]

holds for all \(t \in I\), we have

\[
u(t) \leq g(t) \cdot \exp\left(\int_a^t f(s) \cdot u(s) ds\right).
\]

Theorem 4. (Method of variation of parameter) For a first-order nonhomogeneous linear differential equation, \(y'(t) + p(t)y(t) = f(t)\) has the general solution \(y(t) = v(t)e^{\int_A^t} + Ae^{\int_A^t}\), where \(v'(t) = e^{-\int_A^t} f(t)\), \(P(t) = -\int p(t)dt\) and \(A\) is an arbitrary constant.

2.2. Continuous Problem Formulation. At first, let us assume the following assumptions [50, 51] for the upcoming calculations.

(i) Let the population size varies over time be \(N\) is varying over time (i.e., population size = \(\mu N(t)\) for all \(t \in [0, \infty)\)).

(ii) We divide the population into five homogeneous subgroups, namely susceptible people (S), exposed (E), infectious (I), quarantined (Q), and recovered (R). We can clearly assign every individual to exactly one subgroup. Hence, we obtain S, E, I, Q, R model satisfying the condition \(\mu N = S(t) + E(t) + I(t) + Q(t) + R(t)\) for all \(t \in [0, \infty)\).

(iii) Each time-varying transmission rate \(\beta: [0, \infty) \rightarrow (0, \infty)\) is Lipschitz continuous and continuously differentiable, and there exist constants \(x_{\min}\) and \(x_{\max}\) such that \(0 < x_{\min} \leq x(t) \leq x_{\max}\) for all \(t \geq 0\) and \(x \in \{\pi, \beta, \gamma, \sigma, \theta, \mu\}\).

The choice of time-dependent transmission rates is possible because the countermeasures such as lockdowns, social distancing, or other political actions like curfews and different medical treatments reduce possible contact between susceptible and infectious people.

Our equations of the time-continuous SEIQR model read as follows:

\[
\begin{align*}
\frac{dS(t)}{dt} &= \mu N - \mu S(t) - \beta(N)S(t)[E(t) + I(t)], \\
\frac{dE(t)}{dt} &= \beta(N)S(t)[E(t) + I(t)] - \pi E(t) - (\mu + \gamma)E(t) \\
\frac{dI(t)}{dt} &= \pi E(t) - \sigma I(t) - \mu I(t), \\
\frac{dQ(t)}{dt} &= \gamma E(t) + \sigma I(t) - \theta Q(t) - \mu Q(t) \\
\frac{dR(t)}{dt} &= \theta Q(t) - \mu R(t),
\end{align*}
\]
with initial conditions $S(0) = S_0 > 0$, $E(0) = E_0 > 0$, $I(0) = I_0 > 0$, $Q(0) = Q_0 > 0$ and $R(0) = R_0 \geq 0$. The detailed parameters and description are given in Table 2.

2.3. Nonnegativity and Boundedness of Solutions. Now, we prove the boundedness of the solution to (3). For this purpose, we modify ideas given in [51, 52] deriving the following lemmas, so consider the bounded, time-varying transmission rates given above.

Lemma 1. Each solution of system (3) is bounded below by zero.

Proof. Consider, the first relation of (3),
\[
\frac{dS(t)}{dt} = \mu N - \mu S(t) - \beta(N)S(t)[E(t) + I(t)].
\] (4)

By taking $(dS(t)/dt) = S'(t)$, equation (4) can be expressed as a first-order nonhomogeneous linear differential equation in $S(t)$ as
\[
S'(t) + [\mu + \beta(N)E(t) + \beta(N)I(t)]S(t) = \mu N.
\] (5)

Applying Theorem 4, and by applying the same procedure to the first-order nonhomogenous linear equation in $E(t)$,
\[
\frac{dE(t)}{dt} - [\beta(N)S(t) - \pi - \gamma]E(t) = \beta(N)S(t)I(t).
\] (6)

We can easily show that $E(t) \geq 0$ for all $t \in [0, \infty)$. Proceeding like this, we can show that $I(t)$, $Q(t)$, and $R(t) \geq 0$ for all $t \in [0, \infty)$.

Since $\mu N(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$, and total population is finite, $S(t)$, $E(t)$, $I(t)$, $Q(t)$, and $R(t)$ are bounded above, and hence, the proof is complete.

Theorem 5. For all solution functions of (3), we have $0 \leq X \leq \mu N$, where $X \in [S(t), E(t), I(t), Q(t), R(t)]$.

Proof. The proof follows from $\mu N(t) = S(t) + E(t) + I(t) + Q(t) + R(t)$ and Lemma 1.

2.4. Global Existence in Time. We arrive at a theorem regarding global existence in the time of (3) based on Theorem 1. For abbreviation, we use the supremum norm $\|f(t)\|_{\infty} = \sup_{t \in [a,b]} |f(t)|$ for an arbitrary continuous function $f: [a, b] \longrightarrow \mathbb{R}$. A similar definition holds for vector-valued bounded functions. In our case, using the boundedness of $S, E, I, Q, R$ on $[0, \infty)$, obtain the following global existence theorem.

Theorem 6. The system of nonlinear first-order ODE (3) has at least one solution which exists for all $t \geq 0$.

Proof. By denoting $z(t) = (S(t), E(t), I(t), Q(t), R(t))$, we can set $G: [0, \infty) \times \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ by
\[
(t, z(t)) \rightarrow \begin{pmatrix}
A - \mu S(t) - \beta(N)S(t)(E(t) + I(t)) \\
\beta(N)S(t)(E(t) + I(t)) - \pi E(t) - (\mu + \gamma)E(t) \\
\pi E(t) - \sigma I(t) - \mu I(t) \\
\gamma E(t) + \sigma I(t) - \theta Q(t) - \mu Q(t) \\
\theta Q(t) - \mu R(t)
\end{pmatrix}.
\] (7)

Clearly, $G$ is Lipchitz continuous, due to the continuity of each components.

Assuming the supremum norm on our Euclidean space, and with the help of triangle inequality, we arrive
We present the global uniqueness theorem for (3) by utilizing the inductive approach. Assume that the requirements of Theorem 1 are fulfilled, and our proof is complete. □

2.5. Global Uniqueness in Time. We present the global uniqueness theorem for (3) by utilizing the inductive application of Banach’s fixed point theorem.

**Theorem 7.** The nonlinear ODE system (3) has a unique solution that exists for all $t \geq 0$.

**Proof.** Consider the system of equations given in (3):

1. Consider the time interval $[0, r]$ is applicable to Banach’s fixed point theorem.
2. For $x_1, x_2, y_1, y_2 \in \mathbb{R}$, by triangle inequality, we have $|x_1 - y_1| \leq |x_1, y_1| + |y_2| - |x_1, y_2|$
3. We assume that $S, E, I, Q, R, \bar{S}, \bar{E}, \bar{I}, \bar{Q}, \bar{R} : [0, \infty) \rightarrow [0, \infty)$ are two solutions of (3).

Beginning the proof by letting

\[
\|G(t, z(t))\|_{\infty} \leq k\|z(t)\|_{\infty},
\]

where \( k = \mu_{\text{max}} + \beta_{\text{max}} + \pi_{\text{max}} + \gamma_{\text{max}} + \sigma_{\text{max}} + \theta_{\text{max}}. \)

From the boundedness of our solution functions and the boundedness of our time-varying transmission rates, all requirements of Theorem 1 are fulfilled, and our proof is complete. □

2.6. Time-Discrete Implicit SEIQR Model. Assume that our time interval $[0, r]$ can be divided by strictly increasing sequence $\{t_j\}_{j=0}^M$ for $M \in \mathbb{N}$ with $t_0 = 0$ and $t_M = T$. We write $f(t_j) = f_j$, $j \in \{1, 2, 3, \ldots, M\}$ for an arbitrary time-dependent function $f$.

2.7. Discussion and Formulation. Here, we are transforming the continuous system (3) to the fully explicit discrete scheme (14) as given as follows:
\[
\frac{\Delta S_j}{\Delta t_j} = \mu N - \mu_{j+1} S_j(t) - \beta_{j+1} (N) S_j(t) \left[ E_j(t) + I_j(t) \right], \\
\frac{\Delta E_j}{\Delta t_j} = \beta_{j+1} (N) S_j(t) [E_j + I_j] - \pi_{j+1} E_j \left( \mu_{j+1} + \gamma_{j+1} \right) E_j, \\
\frac{\Delta I_j}{\Delta t_j} = \pi_{j+1} E_j - \sigma_{j+1} I_j - \mu_{j+1} I_j, \\
\frac{\Delta Q_j}{\Delta t_j} = \gamma_{j+1} E_j + \sigma_{j+1} I_j - \theta_{j+1} Q_j - \mu_{j+1} Q_j, \\
\frac{\Delta R_j}{\Delta t_j} = \theta_{j+1} Q_j - \mu_{j+1} R_j,
\]

(14)

and a fully implicit scheme (15) as

\[
\frac{\Delta S_j}{\Delta t_j} = \mu_{j+1} N - \mu_{j+1} S_{j+1} - \beta_{j+1} (N) S_j \left[ E_{j+1} + I_{j+1}(t) \right], \\
\frac{\Delta E_j}{\Delta t_j} = \beta_{j+1} N S_{j+1} [E_{j+1} + I_{j+1}] - \left[ \pi_{j+1} + \mu_{j+1} + \gamma_{j+1} \right] E_{j+1}, \\
\frac{\Delta I_j}{\Delta t_j} = \pi_{j+1} E_{j+1} - \sigma_{j+1} I_{j+1} - \mu_{j+1} I_{j+1}, \\
\frac{\Delta Q_j}{\Delta t_j} = \gamma_{j+1} E_{j+1} + \sigma_{j+1} I_{j+1} - \theta_{j+1} Q_{j+1} - \mu_{j+1} Q_{j+1}, \\
\frac{\Delta R_j}{\Delta t_j} = \theta_{j+1} Q_{j+1} - \mu_{j+1} R_{j+1},
\]

(15)

where \((\Delta X_j/\Delta t_j) = (X_{j+1} - X_j)/(t_{j+1} - t_j)\) for \(j \in \{1, 2, 3, \ldots, M - 1\}\).

Observe that \(\mu_{j+1} N = S_{j+1} + E_{j+1} + I_{j+1} + Q_{j+1} + R_{j+1}\) for all \(j \in \{1, 2, 3, \ldots, M - 1\}\). Since the fully explicit scheme (14) simply reduces to a linear system, our main interest is in a fully implicit discrete scheme because it preserves the nonlinear structure of the continuous problem.

2.8. Implicit Time-Discrete Problem Formulation. In this section and subsequent sections, we derive the recurrence type of solutions for the implicit scheme (15). For that, we assume that \(0 < x_{\text{min}} \leq x_i \leq x_{\text{max}} \in [\mu, \beta, \pi, \gamma, \sigma, \theta]\) for \(j \in \{1, 2, 3, \ldots, M\}\) and that \(0 < t_{j+1} - t_j \leq 1\) for \(j \in \{1, 2, 3, \ldots, M\}\) and that \(S_0 > 0, E_0 > 0, I_0 > 0, Q_0 > 0, R_0 \geq 0\). Now, (15) can be expressed as

\[
\begin{align*}
S_{j+1} &= \frac{S_j + \mu_{j+1} N(t_{j+1} - t_j)}{1 + [\mu_{j+1} + \beta_{j+1} N E_{j+1} + \beta_{j+1} N I_{j+1}](t_{j+1} - t_j)}, \\
E_{j+1} &= \frac{E_j + \beta_{j+1} N S_{j+1} I_{j+1}(t_{j+1} - t_j)}{1 + [\pi_{j+1} + \mu_{j+1} + \gamma_{j+1} + \beta N S_{j+1}](t_{j+1} - t_j)}, \\
I_{j+1} &= \frac{I_j + \pi_{j+1} E_{j+1} (t_{j+1} - t_j)}{1 + [\sigma_{j+1} + \mu_{j+1}](t_{j+1} - t_j)}, \\
Q_{j+1} &= \frac{Q_j + [\gamma_{j+1} E_{j+1} + \sigma_{j+1} I_{j+1}](t_{j+1} - t_j)}{1 + [\theta_{j+1} + \mu_{j+1}](t_{j+1} - t_j)}, \\
R_{j+1} &= \frac{R_j + \theta_{j+1} Q_{j+1} (t_{j+1} - t_j)}{1 + \mu_{j+1}(t_{j+1} - t_j)}.
\end{align*}
\]

(16)

2.9. Unique Solvability. In this section, we provide the method for finding the solution of (16). Letting \(t_{j+1} - t_j = \Delta t_j\) in \(S_{j+1}\), we obtain

\[
S_{j+1} = \frac{S_j + \mu_{j+1} N \Delta t_j}{1 + [\mu_{j+1} + \beta_{j+1} N E_{j+1} + \beta_{j+1} N I_{j+1} \Delta t_j]}. 
\]

(17)

where \(j = 0, S_1 = (S_0 + \mu_1 N \Delta t_0)/1 [\mu_1 + \beta_1 N E_1 + \beta_1 N I_1 \Delta t_0] \)

when \(j = 1, S_2 = (S_1 + \mu_2 N \Delta t_1)/1 [\mu_2 + \beta_2 N E_2 + \beta_2 N I_2 \Delta t_1] \).

Now, substituting \(S_2\) value in \(S_3\), we obtain

\[
S_2 = \frac{N \mu_2 \Delta t_1 \left[ S_1 + (\mu_2 + N \beta_2 E_2 + N \beta_2 I_2) \Delta t_0 \right] + N \mu_1 \Delta t_0 + S_0}{\prod_{m=1}^2 \left[ 1 + (\mu_m + N \beta_m E_m + N \beta_m I_m) \Delta t_m - 1 \right]},
\]

(18)

Similarly, we can find the value of \(S_3\) as

\[
S_3 = \frac{S_2 + \mu_3 N \Delta t_2}{1 + [\mu_3 + \beta_3 N E_{j+1} + \beta_3 N I_{j+1} \Delta t_2]}. 
\]

(19)

Finally, we obtain the general solution as

\[
S_k = \frac{S_0}{\prod_{m=1}^2 \left[ 1 + (\mu_m + N \beta_m E_m + N \beta_m I_m) \Delta t_m - 1 \right]} + \sum_{z=1}^k N \mu_z (\Delta t_{z+1}) \prod_{m=0}^{z-1} \left[ 1 + (\mu_m + N \beta_m E_m + N \beta_m I_m) \Delta t_m - 1 \right], 
\]

(20)

where \(I_x = E_x = \mu_x = \beta_x = t_x = 0\) for \(x \leq 0\).
Theorem 8. Assume \(0 < \mu_{\text{min}} \leq \mu \leq \mu_{\text{max}} < 1, 0 < \beta_{\text{min}} \leq \beta \leq \beta_{\text{max}} < 1, 0 < \gamma_{\text{min}} \leq \gamma \leq \gamma_{\text{max}} < 1, 0 < \sigma_{\text{min}} \leq \sigma \leq \sigma_{\text{max}} < 1, \) and \(0 < t_{j+1} - t_j \leq 1\) holds for all \(j \in [1, 2, \ldots, M - 1]\), and \(S_1 > 0, E_j > 0, I_j > 0, Q_j > 0\) and \(R_j > 0\). The implicit solution scheme (16) uniquely solvable for all \(j \in [1, 2, \ldots, M - 1]\), and we have

\[
E_j = \frac{E_j + E_j XT + T[E_j N\beta T + \beta NT(N\mu T + S_j)]}{1 + XT + \phi T + \phi T^2 X - N\beta[N\mu T + S_j]T + T[N\beta T + \phi T^2 N\beta]},
\]

where \(\pi + \mu + \gamma = \phi, \mu + N\beta E = X, \) and \(\Delta T = T\).

Again, substituting the \(T\) value above, we arrive at

\[
\tilde{E} = \frac{\text{Numerator}(\tilde{N})}{\text{Denominator}(\overline{D})},
\]

\[
\tilde{N} = (E_j + E_j \mu T)\delta + (E_j N\beta T + \beta NT(N\mu T + S_j))I_j + E[N\beta E, T\delta + (E_j N\beta T + \beta NT(N\mu T + S_j))\pi T],
\]

\[
\overline{D} = (1 + \mu T + \phi T + \phi T^2 \mu - N\beta(N\mu T + S_j)T)\delta + I_j(N\beta T + \phi T^2 N\beta) + E[N\beta T + \phi T^2 N\beta]\delta + \pi T(N\beta T + \phi T^2 N\beta),
\]

where \(\Delta T = T, 1 + (\sigma + \mu)T = \delta\).

We get the quadratic equation in the form when we solve equation (24)

\[
AE_j^2 + BE_j + C = 0 \quad (i.e.) AE_{j+1} + BE_{j+1} + C = 0,
\]

where

\[
E_{j+1} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, \quad j \in \{1, 2, \ldots, M - 1\},
\]

where \(A, B, \) and \(C\) are given in upcoming equations in (27)–(29).

Proof. Substituting \(S_{j+1} \) in \(E_{j+1}\) and taking \(S_{j+1} = 0\), \(E_{j+1} = E, I_{j+1} = I, Q_{j+1} = Q\) and \(R_{j+1} = R\), we get

\[
E_{j+1} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, \quad j \in \{1, 2, \ldots, M - 1\},
\]

where \(A, B, \) and \(C\) are given in upcoming equations in (27)–(29).

The proof is completed by taking the roots of equation (26).

Similarly, by substituting \(I_{j+1}\) in \(Q_{j+1}\) in the system, the equation \(Q_{j+1}\) is in the function of \(E_{j+1}\). Then, replacing \(E_{j+1}\)
by equation (24), we will get an explicit formula for computing $Q_j$.

Since $S_{j+1} + E_{j+1} + I_{j+1} + Q_{j+1} + R_{j+1} = \mu_{j+1}N$, we can easily get an explicit formula for $R_{j+1}$ for $j \in \{1,2,\ldots,M-1\}$.

3. Monotonicity Properties and Long-Time Behaviour

In this section, we develop a suitable atmosphere in which our implicit scheme obeys the monotonic properties as in the continuous case. For this, we give the following lemmas and finally provide a nonlinear programming problem to optimize the transmission sequences.

Lemma 2. If $S_j[\beta_j + \mu_j] N \Delta t_j$, then $S_j \leq S_j$, i.e., $(S_n)_{n=1}^\infty$ becomes a decreasing sequence.

Proof. Taking $x = \mu_j N \Delta t_j$ and $y = 1 + [\mu_j + \beta_j N (E_j + I_j)]$ in $S_{j+1}$ of (16). We arrive the relation

$$S_{j+1} = \frac{y + S_j}{1 + x}. \quad (30)$$

From (30), we obtain $(1+x)S_{j+1} = y + S_j$ and which implies $S_{j+1} - S_j = y - xS_{j+1}$. Clearly, $y - xS_{j+1} \leq 0$ if $S_{j+1}[\beta_j + \mu_j + \beta_j N (E_j + I_j)] \leq y \leq 1$. Thus,

$$S_{j+1} - S_j \leq 0, \quad \text{for } j \in \{1,2,\ldots,m\}, \quad (31)$$

and the proof is complete.

Lemma 3. If $\beta_j NS_j [1 + I_j] E_j^{-1} \leq \pi_j + \mu_j + \gamma_j$, then $(E_j)_{n=1}^\infty$ is a decreasing sequence.

Proof. Taking $x = \beta_j NS_j [1 + I_j] E_j^{-1} \leq \pi_j + \mu_j + \gamma_j$, then $(E_j)_{n=1}^\infty$ is a decreasing sequence.

3.1. Formulation and Discussion. Due to the ongoing nature of the COVID-19 pandemic, it was impossible to fully comprehend the short- or long-term implications of this global disruption. In our study, when there is no quarantine, a single infected individual can spread the infection to about two other people; however, when quarantine is imposed, there is a chance of preventing further transmission of infection. However, some of the exposed individuals may avoid quarantine due to fear of stigma and death. In other words, this does not achieve zero infection in the population, implying that additional interventions are required to eradicate the virus. If there are no adequate interventions in place, the virus will remain in the population for a long time, but it will eventually drop over time. But, still, there will be a small number of sick people who have the ability to start another outbreak even after measures like quarantine and public health education/awareness raise the number of exposed and infected people dramatically but not to zero. According to this, COVID-19 will not be completely eradicated even with prompt development of measures. In order to study the effects of isolation, quarantine, and the percentage of exposed people who will be quarantined, we did numerical simulations. Many authors have developed numerous mathematical models to limit the spread of viruses. Here, we have developed the optimization technique to control the spread of the virus. This approach enables us to control a viruses future spread and predict how it will spread in the future. After summarizing all of the prior principles, the problem is transformed into a dynamic
programming problem model with constraints imposed by
the previous lemmas by keeping $S_k, E_k, I_k, Q_k$, and $R_k$ are
constants and $\beta_k, \pi_k, \mu_k, \gamma_k, \sigma_k$, and $\theta_k$ are variables at each
level of the optimization. Note that the following dynamic
programming problem preserves the monotonic properties.

The dynamic programming problem is given by

$$
\text{Min}(\beta_k + \pi_k + \mu_k) - (\gamma_k + \sigma_k + \theta_k) \quad \text{subject to the constraints:}
$$

$$
S_k[\beta_k(E_k + I_k) + \mu_k N^{-1}] - \mu_k \geq 0,
$$

$$
(\pi_k + \mu_k + \gamma_k)E_k - \beta_k NS_k(E_k + I_k) \geq 0,
$$

$$
I_k(\sigma_k + \mu_k) - \pi_k E_k \geq 0,
$$

$$
\gamma_k I_k + \sigma_k I_k - Q_k(\theta_k + \mu_k) \geq 0,
$$

$$
\theta_k Q_k - R_k \mu_k \geq 0,
$$

$$
S_k + E_k + I_k + Q_k + R_k - \mu_k N = 0,
$$

$$
S_k, E_k, I_k, Q_k, R_k \geq 0.
$$

(38)

Since $S_0, E_0, I_0, Q_0$, and $R_0$ are known initial conditions,
we get a dynamic programming problem for level-0 time if we keep $k = 0$ in the above model. We shall obtain an optimal (feasible) solution $(\beta_0, \pi_0, \mu_0, \gamma_0, \sigma_0, \theta_0)$ by employing the optimization technique in operation research.

Then, these values are assigned as $\beta_1 = \beta_0^*, \pi_1 = \pi_0^*, \mu_1 = \mu_0^*, \gamma_1 = \gamma_0^*, \sigma_1 = \sigma_0^*, \theta_1 = \theta_0^*$. We will receive the values for finding $S_1, E_1, I_1, Q_1$, and $R_1$. We will receive a dynamic programming problem for level-1 time if we keep $k = 1$ in this model. We will achieve the ideal solution as $(\beta_1^*, \pi_1^*, \mu_1^*, \gamma_1^*, \sigma_1^*, \theta_1^*)$ using the optimization technique. These values are assigned as $\beta_2 = \beta_1^*, \pi_2 = \pi_1^*, \mu_2 = \mu_1^*, \gamma_2 = \gamma_1^*, \sigma_2 = \sigma_1^*, \theta_2 = \theta_1^*$. We will receive the values for $S_2, E_2, I_2, Q_2$, and $R_2$. If we keep going in this direction, we will end up with transmission rate sequences $(\beta_k)_{k=1}^{\infty}, (\pi_k)_{k=1}^{\infty}, (\mu_k)_{k=1}^{\infty}, (\gamma_k)_{k=1}^{\infty}, (\sigma_k)_{k=1}^{\infty}$, and $(\theta_k)_{k=1}^{\infty}$ which provide a sufficiently viable stable solution for the situation, correspond to $S_0, E_0, I_0, Q_0$, and $R_0$. After a certain stage, each transmission rate becomes constant. We must follow the appropriate standard operating procedures and safety precautions in order to acquire these sequences in practice. The isolation class, it appears, plays a significant role in achieving this possible solution.

4. Error Analysis

Now, we will set an upper limit for error propagation. We need to construct certain assumptions for our convergence analysis before proving the required statements. The following is a list of them:

(i) Let $[0, T]$ be the time interval under consideration with $t_1 = 0 < t_2 < \cdots < t_{M-1} < t_M = T$

(ii) Allow the time-continuous and time-discrete models beginning circumstances to coincide

(iii) Let $S, E, I, Q, R: [0, T] \rightarrow [0, \infty)$ be twice continuously differentiable solution functions

(iv) Allow the time-varying transmission rates $\mu, \beta, \pi, \sigma, \theta: [0, T] \rightarrow [0, \infty)$ be continuously differentiable just once

(v) Allow the time-varying transmission and recovery rates to be bounded by 0 and 1

(vi) Choose $\Delta_p < \min\{1/12, (\mu_{\max} + \beta_{\max}) + 4\pi_{\max} + \gamma_{\max} + \sigma_{\max} + \theta_{\max}\} \leq 1$ for all $p \in N$ and $\Delta_p = \max_{p \in N} \Delta_p$

We get the following theorem under these conditions, in which we adopt notions from the error analysis of an explicit-implicit solution algorithm.

**Theorem 9.** The difference between the solutions of the time-continuous system formulation (3) and the time-discrete system (16) fulfills if the aforementioned assumptions are met, then

$$
\|z_{p+1} - z(t_{p+1})\| \leq C_{loc} \cdot \Delta \left(1 - \left[6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \sigma_{\max} + \theta_{\max})\right]^p - 1\right).
$$

(39)

**Proof.** Since this is technical proof, we will start with a brief description of our technique. The first step is the estimation of local errors between time-continuous and time-discrete solutions. After that, we look at error propagation over time. Finally, we look into the accumulation of these errors. At the same time, time-discrete solutions are expressed as $S_p$ at time $t_p$, whereas time-continuous solutions are written as $S(t_p)$.

1) For the purpose of examining local errors, we assume that

$$
(t_p, S_p)^T = (t_p, S(t_p))^T,
$$

$$
(t_p, E_p)^T = (t_p, E(t_p))^T,
$$

$$
(t_p, I_p)^T = (t_p, I(t_p))^T,
$$

$$
(t_p, Q_p)^T = (t_p, Q(t_p))^T,
$$

$$
(t_p, R_p)^T = (t_p, R(t_p))^T,
$$

(40)
hold for arbitrary \( p \in \{1, 2, \ldots, M - 1\} \) on the time interval \([t_p, t_{p+1}]\). Here, we consider one time step and denote corresponding time-discrete solutions by \( \overline{S}_{p+1}, \overline{E}_{p+1}, \overline{I}_{p+1}, \overline{Q}_{p+1}, \) and \( \overline{R}_{p+1} \).

a) For the time-discrete solution \( S_p \) at time \( t_p \), we take

\[
\overline{S}_{p+1} = S(t_p) - \frac{(\mu_{p+1} + \beta_{p+1}N\overline{E}_{p+1} + \beta_{p+1}N\overline{I}_{p+1})\Delta t_{p+1}S(t_p)}{1 + (\mu_{p+1} + \beta_{p+1}N\overline{E}_{p+1} + \beta_{p+1}N\overline{I}_{p+1})\Delta t_{p+1}},
\]

which yields

\[
\left| S(t_{p+1}) - \overline{S}_{p+1} \right| = \left| S(t_p) - S(t_p) + \frac{(\mu_{p+1} + \beta_{p+1}N\overline{E}_{p+1} + \beta_{p+1}N\overline{I}_{p+1})\Delta t_{p+1}S(t_p) - \mu_{p+1}N\Delta t_{p+1}}{1 + (\mu_{p+1} + \beta_{p+1}N\overline{E}_{p+1} + \beta_{p+1}N\overline{I}_{p+1})\Delta t_{p+1}} \right|.
\]

which implies

\[
\int_{t_p}^{t_{p+1}} S(t)\, dt + \frac{\mu_{p+1} + \beta_{p+1}N\overline{E}_{p+1} + \beta_{p+1}N\overline{I}_{p+1})\Delta t_{p+1}S(t) - \mu_{p+1}N\Delta t_{p+1}}{1 + (\mu_{p+1} + \beta_{p+1}N\overline{E}_{p+1} + \beta_{p+1}N\overline{I}_{p+1})\Delta t_{p+1}}.
\]

Now, add and subtract \( \Delta t_{p+1} \cdot S(t_p) \). And then, applying the triangle inequality in the above relation, we get

\[
\left| S(t_{p+1}) - \overline{S}_{p+1} \right| \leq \int_{t_p}^{t_{p+1}} S(t)\, dt - \Delta t_{p+1} \cdot S(t_p) + \frac{(\mu_{p+1} + \beta_{p+1}N\overline{E}_{p+1} + \beta_{p+1}N\overline{I}_{p+1})\Delta t_{p+1}S(t) - \mu_{p+1}N\Delta t_{p+1}}{1 + (\mu_{p+1} + \beta_{p+1}N\overline{E}_{p+1} + \beta_{p+1}N\overline{I}_{p+1})\Delta t_{p+1}}
\]

\[
I_{S_1} := \int_{t_p}^{t_{p+1}} \left| S(t) - S(t_p) \right|\, dt = \int_{t_p}^{t_{p+1}} \left| S(t) - S(t_p) \right|\, dt.
\]

By applying the mean value theorem, there exists \( \xi_{S_1} \in (t_p, t_{p+1}) \) such that

\[
S''(\xi_{S_1}) = \frac{S(t) - S(t_p)}{t - t_p} \leq \|S''(t)\|_{\infty}.
\]

This yields

\[
I_{S_1} \leq \|S''(t)\|_{\infty} \cdot \int_{t_p}^{t_{p+1}} (t - t_p)\, dt = \frac{\Delta t_{p+1}^2}{2} \cdot \|S''(t)\|_{\infty}.
\]

Also,

\[
II_{S_1} := \left| \Delta t_{p+1} \cdot S(t_p) + \frac{(\mu_{p+1} + \beta_{p+1}N\overline{E}_{p+1} + \beta_{p+1}N\overline{I}_{p+1})\Delta t_{p+1}S(t_p) - \mu_{p+1}N\Delta t_{p+1}}{1 + (\mu_{p+1} + \beta_{p+1}N\overline{E}_{p+1} + \beta_{p+1}N\overline{I}_{p+1})\Delta t_{p+1}} \right|,
\]

\[
III_{S_1} := \left| \left( N_1 - \mu_{p-1} \right) - S(t_p) \left( \mu_{p+1} - \mu_{p-1} \right) \right| + \left| \left( \mu_{p+1} - \mu_{p-1} \right) \left( N_1 - \mu_{p-1} \right) \right|,
\]

\[
IV_{S_1} := \left| \mu_{p+1}N - S(t_p) \left( \mu_{p+1} + \beta_{p+1}N \right) \right| \cdot \left| \mu_{p+1}N - S(t_p) \left( \mu_{p+1} + \beta_{p+1}N \right) \right| \Delta t_{p+1}.
\]

Now, applying the triangle inequality in \( III_{S_1} \), we obtain

\[
\leq N_{\mu_{p-1} - \mu_{p+1}} \cdot |S(t_p)| \cdot V_{S_1},
\]

where

\[
V_{S_1} := \left| N_1 - \mu_{p-1} \right| + \left| \left( \mu_{p+1} - \mu_{p-1} \right) \left( N_1 - \mu_{p-1} \right) \right| + \left| \mu_{p+1}N - S(t_p) \left( \mu_{p+1} + \beta_{p+1}N \right) \right| \cdot \left| \mu_{p+1}N - S(t_p) \left( \mu_{p+1} + \beta_{p+1}N \right) \right| \Delta t_{p+1}.
\]
\[ V_{S,1} := \left| \mu_p - \mu_{p+1} \right| + N \left[ \beta_p E(t_p) + \Gamma_1(t_p) \right] \beta_{p+1} E(t_{p+1}) \left| \frac{E(t_{p+1}) - E(t_p)}{E(t_{p+1})} \right|, \]

let \( N|\mu_{p+1} - \mu_p| = N(t_{p+1} - t_p) \left| \frac{\mu_{p+1} - \mu_p}{t_{p+1} - t_p} \right|, \)

\[ N|\mu_{p+1} - \mu_p| = N \Delta t_{p+1} \left| \frac{\mu_{p+1} - \mu_p}{t_{p+1} - t_p} \right|. \]  

(50)

By the mean value theorem, there exists \( \xi \in [t_p, t_{p+1}] \) such that

\[ \left| \mu(\xi) \right| = \left| \frac{\mu_p - \mu_p}{t_{p+1} - t_p} \right| \leq \| \mu'(t) \|_\infty, \]  

(51)

Applying equation (57) and (58) in (55), we get

\[ N|\mu_{p+1} - \mu_p| \leq N \cdot \Delta t_{p+1} \cdot \| \mu'(t) \|_\infty. \]  

(52)

Using the triangle inequality again in \( V_{S,1} \) yields

\[ V_{S,1} \leq |\mu_{p+1} - \mu_p| + N|\beta_p E(t_p) - \beta_{p+1} E(t_{p+1})| \]

\[ + N|\beta_p E(t_p) - \beta_{p+1} E(t_{p+1})| \left| \frac{E(t_{p+1}) - E(t_p)}{E(t_{p+1})} \right|. \]  

(53)

From equation (52), we can easily find that

\[ |\mu_{p+1} - \mu_p| \leq \Delta t_{p+1} \cdot \| \mu'(t) \|_\infty. \]  

(54)

Now, substituting \( E(t_{p+1}) \) value (53), we arrive

\[ N|\beta_p E(t_p) - \beta_{p+1} E(t_{p+1})| = \left| N \beta_p E(t_p) - \beta_{p+1} \left( \frac{E(t_p) + \beta_{p+1} N S_{p+1} E(t_{p+1})}{1 + (\beta_{p+1} N S_{p+1}) \Delta t_{p+1}} \right) \right|, \]  

\[ \leq N \cdot I_{S,2} + N \cdot II_{S,2}, \]

where

\[ I_{S,2} := |\beta_p E(t_p) - \beta_{p+1} E(t_{p+1})| \]

\[ \left| \frac{E(t_p) + \beta_{p+1} N S_{p+1} E(t_{p+1})}{1 + (\beta_{p+1} N S_{p+1}) \Delta t_{p+1}} \right|, \]  

(56)

\[ II_{S,2} := \left| \frac{\beta_p E(t_p) - \beta_{p+1} \left( \frac{E(t_p) + \beta_{p+1} N S_{p+1} E(t_{p+1})}{1 + (\beta_{p+1} N S_{p+1}) \Delta t_{p+1}} \right)}{1 + (\beta_{p+1} N S_{p+1}) \Delta t_{p+1}} \right|. \]

Now, \( I_{S,2} \leq |E(t_p)|: \)

\[ \left| \beta_p + \beta_{p+1} (\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1} N S_{p+1}) \Delta t_{p+1} \right| \]

\[ \leq \frac{N}{\beta_{\max}} \left| \beta_{p+1} \right| \leq \frac{N}{1 - \beta_{\max}} \left| \beta_{p+1} \right| \leq \frac{N}{1 - \beta_{\max}} \left\{ \| \beta'(t) \|_\infty + \beta_{\max} \left( \pi_{\max} + \mu_{\max} + \gamma_{\max} + \beta_{\max} \right) \right\}. \]  

(57)

Similarly, for \( II_{S,2} \), it is easy to arrive

\[ II_{S,2} = \left| \frac{N \beta_{p+1} E(t_{p+1})}{1 + (\beta_{p+1} N S_{p+1}) \Delta t_{p+1}} \right| \]

\[ \leq \frac{N \beta_{p+1} E(t_{p+1})}{1 - \beta_{\max}} \]  

(58)

Applying equation (57) and (58) in (55), we get

\[ N|\beta_p E(t_p) - \beta_{p+1} E(t_{p+1})| \leq \frac{N^2 \Delta t_{p+1}}{1 - \beta_{\max}} \left\{ \| \beta'(t) \|_\infty + \beta_{\max} \left( \pi_{\max} + \mu_{\max} + \gamma_{\max} + \beta_{\max} \right) \right\}. \]  

(59)

Proceeding through the similar steps, we can find
Hence, applying equations (54), (59), and (60) in $V_{S,I}$, we obtain

$$V_{S,I} \leq \Delta t_{p+1} \| \mu'(t) \|_{\infty} + \frac{N^2 \Delta t_{p+1}}{1 - \beta_{\max}} \left\{ \| \beta'(t) \|_{\infty} + \beta_{\max} \left( \sigma_{\max} + \mu_{\max} + \gamma_{\max} + \beta_{\max} \right) + \beta_{\max}^2 \right\}$$

Using equations (52) and (61), $III_{S,I}$ will be

$$III_{S,I} \leq 2N\Delta t_{p+1} \cdot \| \mu'(t) \|_{\infty} + \frac{N^2 \Delta t_{p+1}}{1 - \beta_{\max}} \left\{ \| \beta'(t) \|_{\infty} + \frac{N \Delta t_{p+1}}{1 - \beta_{\max}} \left[ \| \beta'(t) \|_{\infty} + \beta_{\max} \left( \sigma_{\max} + \mu_{\max} \right) \right] \right\}$$

Now, the term $IV_{S,I}$ satisfies the inequality

$$IV_{S,I} \leq I_{S,3} + \| S(t_p) \| \cdot I_{S,3}, \quad (63)$$

where

$$I_{S,3} := \mu_p N \left[ \mu_{p+1} + \beta_{p+1} N \left( E_{p+1} - I_{p+1} \right) \right] \Delta t_{p+1},$$

$$II_{S,3} := \mu_p + \beta_{p+1} N \left( E(t_p) + I(t_p) \right) \left[ \mu_{p+1} + \beta_{p+1} N \left( E_{p+1} + tI_{p+1} \right) \right] \Delta t_{p+1},$$

which yields

$$I_{S,3} = N \Delta t_{p+1} \| \mu_p \| \left[ \mu_{p+1} + \beta_{p+1} N \left( E_{p+1} + tI_{p+1} \right) \right],$$

$$\leq N \Delta t_{p+1} \| \mu_{\max} \| \left[ \mu_{\max} + \beta_{\max} \right], \quad (65)$$

Hence,

$$\| S(t_{p+1}) - \overline{S_{p+1}} \| \leq \frac{(\Delta t_{p+1})^2}{2} \cdot \| S(t) \|_{\infty} + 2N \left( \Delta t_{p+1} \right)^2 \cdot \| \mu'(t) \|_{\infty}$$

$$+ \frac{N^3 (\Delta t_{p+1})^2}{1 - \beta_{\max}} \left\{ \| \beta'(t) \|_{\infty} + \beta_{\max} \left( \sigma_{\max} + \mu_{\max} + \gamma_{\max} + \beta_{\max} \right) + \beta_{\max}^2 \right\}$$

$$+ \frac{N^2 (\Delta t_{p+1})^2}{1 - \beta_{\max}} \| \beta'(t) \|_{\infty} + \beta_{\max} \left( \sigma_{\max} + \mu_{\max} \right)$$

From (65) and (66), we get

$$IV_{S,I} \leq N \Delta t_{p+1} \left[ \mu_{\max} + \beta_{\max} \right]$$

$$+ N^2 \Delta t_{p+1} \left( \mu_{\max} + \beta_{\max} \right)^2. \quad (67)$$

Similarly, from (62) and (66),

$$II_{S,3} = N \Delta t_{p+1} \left( \mu_{\max} + \beta_{\max} \right)^2. \quad (66)$$
b) For the time-discrete solution for $E_p$ at time $t_p$, we take as well as for time-continuous solutions $E(t_p)$, we find

$$
\overline{E_{p+1}} = \frac{E_p + \beta_{p+1}NS_{p+1}p_{p+1}\Delta t_{p+1}}{1 + (\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1}NS_{p+1})\Delta t_{p+1}},
$$

(70)

Now, applying the procedure similar to the case (a), we can easily find

$$
|E(t_{p+1}) - \overline{E_{p+1}}| \leq \left| \int_{t_p}^{t_{p+1}} E(t)\,dt - \Delta t_{p+1}E(t_p) \right| + \left| \Delta t_{p+1}E(t_p) + \frac{(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1}NS_{p+1})\Delta t_{p+1}E(t_p)}{1 + (\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1}NS_{p+1})\Delta t_{p+1}} \right|
$$

(71)

$$
\text{where}
$$

$$
|E(t_{p+1}) - \overline{E_{p+1}}| \leq \frac{(\Delta t_{p+1})^2}{2} \| E'(t) \|_\infty + \Delta t_{p+1} \cdot I_{E,1},
$$

(73)

$$
I_{E,1} := \left| \frac{(\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1}NS_{p+1})\Delta t_{p+1}E(t_p)}{1 + (\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} - \beta_{p+1}NS_{p+1})\Delta t_{p+1}} \right|
$$

(74)

which can be written as

$$
E(t_p) \left[ 1 + (\phi_{p+1} - \beta_{p+1}NS_{p+1})\Delta t_{p+1} \right] + (\phi_{p+1} - \beta_{p+1}NS_{p+1})E(t_p) - \beta_{p+1}NS_{p+1}t_{p+1}N
$$

(75)

and $\pi_{p+1} + \mu_{p+1} + \gamma_{p+1} = \phi_{p+1}$. Thus,

$$
I_{E,1} \leq \frac{1}{1 - \beta_{\text{max}}} \left| (\beta_{p+1}NS(t_p)I(t_p) - \beta_{p+1}NS_{p+1}t_{p+1}) \right|
$$

(76)

where
\[ H_{E2} := \{ \beta_p N S(t_p) I(t_p) - \beta_{p+1} N S_{p+1} I_{p+1} \}, \]
\[ III_{E2} := E(t_p) \{ [\beta_p N S(t_p) - \beta_{p+1} N S_{p+1}] + \phi_{p+1} - \phi_p \} \}
\[ IV_{E2} := [\beta_p N S(t_p) I(t_p) + E(t_p) [\beta_p N S(t_p) - \phi_{p+1}] \} \]
\[ H_{E2} := N \{ \beta_p S(t_p) I(t_p) - \beta_p N S(t_p) \} \]
\[ \leq N \{ \beta_p S(t_p) I(t_p) - \beta_p N S(t_p) \} \]

where
\[ V_{E2} := \beta_p S(t_p) I(t_p) - \beta_p N S(t_p) \]
\[ VI_{E2} := \beta_p S(t_p) I(t_p) - \beta_p N S(t_p) \]

Similarly, substituting the \( I_{p+1} \) value in \( V_{E2} \) and then solving the \( V_{E2} \), we get
\[ V_{E2(a)} = \frac{1}{1 - \beta_{max}} \left[ \beta_p I(t_p) - \beta_p N S(t_p) \right] \]
\[ + \frac{1}{1 - \beta_{max}} \left[ \beta_p I(t_p) \right] \left[ \sigma_{p+1} + \mu_{p+1} \right] \Delta t_{p+1} + \left[ \mu_{p+1} + \beta_{p+1} N E_{p+1} + \beta_{p+1} N \Delta t_{p+1} \right] \]
\[ + \left[ \sigma_{p+1} \right] \left[ \mu_{p+1} + \beta_{p+1} N E_{p+1} + \beta_{p+1} N \Delta t_{p+1} \right] \Delta t_{p+1} \]
\[ V_{E2(a)} = \frac{N \Delta t_{p+1}}{1 - \beta_{max}} \left[ \beta_p I(t_p) \right] \left[ \sigma_{max} + \mu_{max} \right] \Delta t_{p+1} \]

We can also easily find
\[ V_{E2(b)} = \frac{\pi_{max} \Delta t_{p+1}}{1 - \beta_{max}} \]

Finally,
\[ V_{E2} \leq \frac{N^2 \Delta t_{p+1}}{1 - \beta_{max}} \left[ \beta_p I(t_p) \right] \left[ \sigma_{max} + \mu_{max} \right] \Delta t_{p+1} \]
\[ + \frac{N^2 \Delta t_{p+1}}{1 - \beta_{max}} \left[ \beta_p I(t_p) \right] \left[ \sigma_{max} + \mu_{max} \right] \Delta t_{p+1} \]
\[ VI_{E2} \leq \frac{N^2 \Delta t_{p+1}}{1 - \beta_{max}} \left[ \beta_p I(t_p) \right] \left[ \sigma_{max} + \mu_{max} \right] \Delta t_{p+1} \]

and
\[ II_{E2} \leq \frac{N^3 \Delta t_{p+1}}{1 - \beta_{max}} \left[ \beta_p I(t_p) \right] \left[ \phi_{max} + 2 \left( \mu_{max} + \beta_{max} \right) \right] \]

Using the same procedure given above, we get
\[ III_{E2} \leq \frac{N^3 \Delta t_{p+1}}{1 - \beta_{max}} \left[ \beta_p I(t_p) \right] \left[ \phi_{max} + 2 \left( \mu_{max} + \beta_{max} \right) \right] \]
and

$$IV_{E,2} \leq \frac{N^3 \Delta t_{p+1}}{1 - \beta_{\text{max}}} \left[ \max \left( \pi_{\text{max}} + \mu_{\text{max}} + \gamma_{\text{max}} + \beta_{\text{max}} \right) \right]$$

$$+ \frac{N^3 \Delta t_{p+1}}{1 - \beta_{\text{max}}} \left( \beta_{\text{max}} + \pi_{\text{max}} + \mu_{\text{max}} + \gamma_{\text{max}} \right)^2;$$

(85)

Substituting equations (83)–(85) in $I_{E,1}$, we arrive

$$I_{E,1} \leq \frac{N^3 \Delta t_{p+1}}{1 - \beta_{\text{max}}} \left[ 2\left\| \beta'(t) \right\|_{\infty} \right]$$

$$+ \frac{N^3 \Delta t_{p+1}}{1 - \beta_{\text{max}}} \left\{ \left( \sigma_{\text{max}} + 2 \left( \beta_{\text{max}} + \beta_{\text{max}} \right) \right) \left( \mu_{\text{max}} + 2 \beta_{\text{max}} \right) \left( \pi_{\text{max}} + \mu_{\text{max}} + \gamma_{\text{max}} + \beta_{\text{max}} \right) \right\}$$

$$+ \frac{N^3 \Delta t_{p+1}}{1 - \beta_{\text{max}}} \left\{ \beta_{\text{max}} + \mu_{\text{max}} + \beta_{\text{max}} \right\}$$

$$+ \frac{N^3 \Delta t_{p+1}}{1 - \beta_{\text{max}}} \left\{ \left( \beta_{\text{max}} + \pi_{\text{max}} + \mu_{\text{max}} + \gamma_{\text{max}} \right)^2 + \Delta t_{p+1} \right\} \left\{ \left\| \pi(t) \right\|_{\infty} + \left\| \gamma(t) \right\|_{\infty} \right\}.$$  

(86)

Substituting equation (86) in (73), we obtain the case.

c) For the time-discrete solution $I_p$ at time $t_{p+1}$, let

$$\overline{I}_{p+1} = \frac{I_p + \pi_{p+1} \overline{E}_{p+1} \Delta t_{p+1}}{1 + (\sigma_{p+1} + \mu_{p+1}) \Delta t_{p+1}},$$

(87)

and for the time-continuous solutions $I(t_p)$, let

$$\left| I(t_{p+1}) - \overline{I}_{p+1} \right| \leq \frac{(\Delta t_{p+1})^2}{2} \left\| I_n(t) \right\|_{\infty} + \left\| \Delta t_{p+1} I(t_p) \right\| + \left( \frac{\sigma_{p+1} + \mu_{p+1}}{1 + (\sigma_{p+1} + \mu_{p+1}) \Delta t_{p+1}} \right)$$

$$\left( \Delta t_{p+1} I(t_p) - \pi_{p+1} \overline{E}_{p+1} \Delta t_{p+1} \right),$$

(89)

where

$$I_{f,1} := \frac{\Delta t_{p+1} I(t_p) + (\sigma_{p+1} + \mu_{p+1}) \Delta t_{p+1} I(t_p) - \pi_{p+1} \overline{E}_{p+1} \Delta t_{p+1}}{1 + (\sigma_{p+1} + \mu_{p+1}) \Delta t_{p+1}}.$$  

(90)

Applying $I(t_p)$ value in (89), we obtain

$$I_{f,1} = \Delta t_{p+1} \left| \pi_p E(t_p) - (\sigma_p + \mu_p) I(t_p) \right| + \left( \frac{\sigma_{p+1} + \mu_{p+1}}{1 + (\sigma_{p+1} + \mu_{p+1}) \Delta t_{p+1}} \right)$$

$$\left( \pi_p E(t_p) - I(t_p) \right) \left( \sigma_p + \mu_p \right) \left( \sigma_{p+1} + \mu_{p+1} \right) \Delta t_{p+1},$$

(91)

Solving the above equations, we get

$$I_{f,1} \leq \Delta t_{p+1} \left[ \left| \pi_p E(t_p) - \pi_{p+1} \overline{E}_{p+1} \right| + \left| I(t_p) \right| \left[ \left| \sigma_p - \sigma_p \right| + \left| \mu_p - \mu_p \right| + \left| \sigma_p E(t_p) - I(t_p) \right( \sigma_p + \mu_p \right) \left( \sigma_{p+1} + \mu_{p+1} \right) \Delta t_{p+1} \right],$$

(92)

which yields

$$I_{f,1} \leq \frac{N(\Delta t_{p+1})^2}{1 - \beta_{\text{max}}} \left\{ \left\| \pi(t) \right\|_{\infty} + \pi_{\text{max}} \left( \pi_{\text{max}} + \mu_{\text{max}} + \gamma_{\text{max}} + 2 \beta_{\text{max}} \right) \right\}$$

$$+ \frac{N(\Delta t_{p+1})^2}{1 - \beta_{\text{max}}} \left\{ \left\| \mu(t) \right\|_{\infty} + \mu_{\text{max}} \left( \pi_{\text{max}} + \sigma_{\text{max}} + \mu_{\text{max}} \right) \right\}.$$  

(93)
The case (c) is verified by substituting equation (93) in (89).

d) Let the time-discrete solution for \( Q_p \) at time \( t_p \) be

\[
\tilde{Q}_{p+1} = Q_p + \left( \frac{\theta_{p+1} + \mu_{p+1}}{\mu_p} \right) \Delta t_{p+1} \tilde{Q}_{p+1},
\]

where

\[
Q_{p+1} = Q(t_p) - \left( \frac{\theta_{p+1} + \mu_{p+1}}{\mu_p} \right) \Delta t_{p+1} Q(t_p) + \left( \frac{\theta_{p+1} + \mu_{p+1}}{\mu_p} \right) \Delta t_{p+1} \tilde{Q}_{p+1},
\]

which implies

\[
|Q(t_{p+1}) - \tilde{Q}_{p+1}| \leq \frac{(\Delta t_{p+1})^2}{2} \| \theta(t) \|_\infty + \| \mu(t) \|_\infty,
\]

where

\[
I_{Q,1} = |Q(t_p)\Delta t_{p+1} + \left( \frac{\theta_{p+1} + \mu_{p+1}}{\mu_p} \right) Q(t_p)\Delta t_{p+1} - (\theta_{p+1} + \mu_{p+1}) Q(t_p)\Delta t_{p+1} | + \left( \frac{\theta_{p+1} + \mu_{p+1}}{\mu_p} \right) \| \theta(t) \|_\infty + \| \mu(t) \|_\infty,
\]

Simplifying the above equation, we arrive

\[
I_{Q,1} \leq \frac{N(\Delta t_{p+1})^2}{1 - \beta_{\max}} \left\{ \| \theta(t) \|_\infty + \| \mu(t) \|_\infty \right\} + \frac{N(\Delta t_{p+1})^2}{1 - \beta_{\max}} \left\{ \| Q(t) \|_\infty + \| \mu(t) \|_\infty \right\}
\]

Substituting equation (98) in (96), we get the quarantine case.

e) Let the time-discrete solution for \( R_p \) at time \( t_p \) be

\[
\tilde{R}_{p+1} = R_p + \left( \frac{\theta_{p+1} + \mu_{p+1}}{\mu_p} \right) \Delta t_{p+1} \tilde{R}_{p+1},
\]

and the time-continuous solutions \( R(t_p) \), let

\[
\tilde{R}_{p+1} = R(t_p) - \left( \frac{\mu_{p+1} \Delta t_{p+1} R(t_p)}{1 + \mu_{p+1} \Delta t_{p+1}} \right),
\]

which gives

\[
|R(t_{p+1} - \tilde{R}_{p+1})| \leq \frac{(\Delta t_{p+1})^2}{2} \| R(t) \|_\infty + I_{R,1},
\]

where

\[
I_{R,1} = |\Delta t_{p+1} R(t_p) - \left( \frac{\mu_{p+1} \Delta t_{p+1} R(t_p)}{1 + \mu_{p+1} \Delta t_{p+1}} \right) |.
\]
(69) can be rewritten in the following form:
\[
\|s(t_{p+1}) - s_{p+1}\| \leq \Delta^2 t_{p+1} C_{loc,S}.
\]  
\[
\tag{104}
\]
Similarly, we can get the other cases as
\[
E(t_{p+1}) - E_{p+1} \leq \Delta^2 t_{p+1} C_{loc,E},
\]
\[
I(t_{p+1}) - I_{p+1} \leq \Delta^2 t_{p+1} C_{loc,I},
\]
\[
Q(t_{p+1}) - Q_{p+1} \leq \Delta^2 t_{p+1} C_{loc,Q},
\]
\[
R(t_{p+1}) - R_{p+1} \leq \Delta^2 t_{p+1} C_{loc,R}.
\]
\[
\tag{105}
\]

**Definition 2.** Define 
\[
C_{loc} := \max \{ C_{loc,S}, C_{loc,E}, C_{loc,I}, C_{loc,Q}, C_{loc,R} \}
\]
which holds
\[
\|z_{p+1} - z(t_{p+1})\| \leq \|z_p - z(t_p)\| + \Delta t_{p+1} \|G(t_{p+1}, z_{p+1}) - G(t_{p+1}, z(t_{p+1})\|_{\infty}.
\]
\[
\tag{108}
\]
Here,
\[
\begin{aligned}
G(t_{p+1}, z_{p+1}) - G(t_{p+1}, z(t_{p+1})) & = \\
& = \begin{pmatrix}
A - \mu_{p+1}(S(t_{p+1}) - S_{p+1}) - \beta_{p+1} N\{S(t_{p+1})(E(t_{p+1}) + (t_{p+1})) - S_{p+1}(E_{p+1} + I_{p+1})\} \\
\beta_{p+1} N\{S(t_{p+1})(E_{p+1} + I_{p+1}) - S_{p+1}E_{p+1} + I_{p+1}\} - (\sigma_{p+1} + \mu_{p+1} + \gamma_{p+1})\{E_{p+1}(t_{p+1} - I_{p+1})\} \\
\sigma_{p+1}(I(t_{p+1}) - I_{p+1}) \\
\theta_{p+1}(Q(t_{p+1}) - Q_{p+1}) - \mu_{p+1}(R(t_{p+1}) - R_{p+1})
\end{pmatrix}_{\infty},
\end{aligned}
\]
\[
\tag{109}
\]
which is less than equal to
\[
\begin{aligned}
& \leq A + \mu_{p+1}(S_{p+1} - S(t_{p+1})) + \beta_{p+1} N\{2\|S_{p+1} - S(t_{p+1})\|_{\infty} + \|E_{p+1} - E(t_{p+1})\|_{\infty} + \|I_{p+1} - I(t_{p+1})\|_{\infty}\} \\
& + \beta_{p+1} N\{2\|S_{p+1} - S(t_{p+1})\|_{\infty} + \|E_{p+1} - E(t_{p+1})\|_{\infty} + \|I_{p+1} - I(t_{p+1})\|_{\infty}\} \cdot (\sigma_{p+1} + \mu_{p+1} + \gamma_{p+1})\|E_{p+1} - E(t_{p+1})\|_{\infty} \\
& + \sigma_{p+1}\|I_{p+1} - I(t_{p+1})\|_{\infty} \cdot (\sigma_{p+1} + \mu_{p+1})\|Q_{p+1} - Q(t_{p+1})\|_{\infty} \\
& + \theta_{p+1}\|Q_{p+1} - Q(t_{p+1})\|_{\infty} - \mu_{p+1}\|R_{p+1} - R(t_{p+1})\|_{\infty}
\end{aligned}
\]
\[
\leq 6(\mu_{max} + \beta_{max}) + 2(\sigma_{max} + \gamma_{max} + \theta_{max}) \cdot \|z_{p+1} - z(t_{p+1})\|_{\infty},
\]
\[
\tag{110}
\]
Thus,
\[
\|z_{p+1} - z(t_{p+1})\|_{\infty} \leq \|z_p - z(t_p)\|_{\infty} + 6(\mu_{max} + \beta_{max}) + 2(\sigma_{max} + \gamma_{max} + \theta_{max}) \cdot \|z_{p+1} - z(t_{p+1})\|_{\infty},
\]
\[
\tag{111}
\]
Hence, we conclude
\[
\|z_{p+1} - z(t_{p+1})\|_{\infty} \leq \|z_p - z(t_p)\|_{\infty} + 6(\mu_{max} + \beta_{max}) + 2(\sigma_{max} + \gamma_{max} + \theta_{max}) \cdot \|z_{p+1} - z(t_{p+1})\|_{\infty},
\]
\[
\tag{112}
\]
with
\[
\Delta =: \max_{p=1,2,3,\ldots,M-1} \Delta t_{p+1}
\]
\[
\tag{113}
\]
for local errors on time interval \([t_p, t_{p+1}]\).

2) In reality, \((t_p, S_p)^T, (t_p, E_p)^T, (t_p, I_p)^T, (t_p, Q_p)^T, \) and \((t_p, R_p)^T\) do not exactly lie on the graph of the time-continuous solution. Thus, we should examine how procedural errors such as \(S_p - S(t_p), E_p - E(t_p), I_p - I(t_p), Q_p - Q(t_p), \) and \(R_p - R(t_p)\) propagate to the \((p + 1)\)th time step. That is,
\[
\|z_{p+1} - z(t_{p+1})\|_{\infty} \leq \|z_p - z(t_p)\|_{\infty} + \Delta t_{p+1}\{G(t_{p+1}, z_{p+1}) - G(t_{p+1}, z(t_{p+1}))\},
\]
\[
\tag{107}
\]
3) The upper bound between the time-discrete solution and the time-continuous solution is verified below.

For \( p = 0 \) in (112), it becomes \( \|z_1 - z(t_1)\|_\infty = 0 \).

For \( p = 1 \), we get \( \|z_2 - z(t_2)\|_\infty \leq \|z_2 - z(t_2)\|_\infty + \|z_2 - z(t_2)\|_\infty \). By applying equations (106) and (112), we get

\[
\|z_2 - z(t_2)\|_\infty \leq \frac{1}{1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})] \Delta} + C_{\text{loc}} \cdot \Delta^2,
\]

which becomes

\[
\|z_2 - z(t_2)\|_\infty \leq C_{\text{loc}} \cdot \Delta^2.
\]

For \( p = 2 \), we get

\[
\|z_4 - z(t_4)\|_\infty \leq \|z_4 - z(t_2)\|_\infty + \|z_4 - z(t_2)\|_\infty \leq \frac{1}{1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})] \Delta} + C_{\text{loc}} \cdot \Delta^2,
\]

which yields \( \|z_4 - z(t_4)\|_\infty \)

\[
\leq \sum_{j=0}^{2} C_{\text{loc}} \cdot \Delta^2 (1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})] \Delta)^j.
\]

Proceeding like this, we can write \( \|z_{p+1} - z(t_{p+1})\|_\infty \)

\[
\leq \sum_{j=0}^{p-1} C_{\text{loc}} \cdot \Delta^2 (1 - [6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})] \Delta)^j.
\]

Assuming \( \Delta < 1/(12(\mu_{\max} + \beta_{\max}) + 4(\pi_{\max} + \gamma_{\max} + \theta_{\max})) \), we conclude \((\Delta/1 - (6(\mu_{\max} + \beta_{\max}) + 2(\pi_{\max} + \gamma_{\max} + \theta_{\max})] \Delta) - 1 \leq 1.

Hence, we get (39).

5. Conclusion

The present paper is devoted to the analysis and optimization of the SEIQR epidemic model containing an isolation class. We derived both continuous and discrete schemes of the SEIQR model as well as the global existence of solutions and nonnegativity bounded properties for both schemes. Along with this, we have illustrated the solving technique for the discrete scheme and proposed a new optimization technique with the help of a dynamic programming problem. In addition, we have analyzed the error between continuous and discrete schemes in the last section. Finally, we conclude that the isolation class plays a vital role in controlling the COVID-19 pandemic situation. More interestingly, the results also reveal that COVID-19 can exhibit oscillatory behaviour in the future. On the other hand, social distancing methods, quarantine efficiency, and isolation can be used to keep it under control. Future research could look into the effects of current coronavirus mutations like Delta and Omicron on the COVID-19 pandemic’s dynamics. We also suggested an alternative dynamic model of the SEIQR class, which can theoretically be generalized to generate continuous and discrete-time models such as SEIR, SEIRS,
SIRS, SEI, SEIS, SI, SIS, SEIQR, SIDARTHE, and others in future work.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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