Set of Holonomic and Protected Gates on Topological Qubits for Realistic Quantum Computer

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Abstract. In recent years qubit designs such as transmons approached the fidelities of up to 0.999. However, even these devices are still insufficient for realizing quantum error correction requiring better than 0.9999 fidelity. Topologically protected superconducting qubits are arguably most prospective for building a realistic quantum computer as they are intrinsically protected from noise and leakage errors that occur in transmons. We propose a topologically protected qubit design based on a $\pi$-periodic Josephson element and a universal set of gates: protected Clifford group and highly robust (with infidelity $\sim 10^{-4}$) non-discrete holonomic phase gate. The qubit is controlled via charge($Q$) and flux($\Phi$)-biases. The holonomic gate is realized by quickly, but adiabatically, going along a particular closed path in the two-dimensional $\{\Phi, Q\}$-space – a path where computational states are always degenerate, but Berry curvature is localized inside the path. This gate is robust against currently achievable noise levels. This qubit architecture allows building a realistic scalable superconducting quantum computer with leakage and noise-induced errors as low as $10^{-4}$, which allows performing realistic error correction codes with currently available fabrication techniques.

Overview. The main challenge in building a realistic quantum computer is building a qubit and developing logical operations that can be used for efficient error corrections[1]. The problem with currently existing qubits is two fold. First, the best fidelity of the existing qubits is either insufficient for error correction or requires an impractically large hardware overhead. Second, the transmons displaying the best fidelity achieve it by reducing non-linearity, increasing leakage out of computational space that is very difficult to correct within surface code[2]. With increasing complexity of the quantum algorithm, surface codes require rapidly increasing number of physical qubits to perform error correction. As an alternative, it is desirable to create qubits that are protected against noise on the hardware level. Arguably the most prospective design involves using a $\pi$-periodic element[3, 4, 5, 6, 7] – effectively a Josephson element that only allows tunneling of even number of Cooper pairs and has phase($\varphi$)-energy($E$) relation $E = -E_2 \cos 2\varphi$ with $E_2$ being the Josephson energy for double Cooper pair tunneling. Such an element coupled to a large capacitor $C$ with charging energy $E_C \ll E_2$ (here $E_C = (2e)^2 / 2C$) forms the qubit (Fig. 1b) in which two logical states are characterized by the charge parity
Figure 1: Schematics of the protected \(\pi\)--periodic qubit (a) and its modification that allows full set of operations (b).

(i.e. parity of number of Cooper pairs) on the superconducting island: “0” and “1” logical states are encoded by even and odd charge states respectively. The dephasing rate for such qubit is exponentially suppressed with increasing value of \(\sqrt{E_2/E_C}\), which is a square of the characteristic width of the wavefunction \(\psi(n)\) in the charge \((n)\) space. In this respect, the \(\pi\)--periodic qubit is similar to the transmon where protection is achieved due to the exponential suppression of the energy dispersion as a function of the charge offset \((Q)\). However, unlike transmons, \(\pi\)--periodic qubits are strongly anharmonic and have nearly degenerate computational states well separated from excited ones, preventing leakage outside of the computational space.

Ideally, a protected qubit should allow a universal set of fault tolerant operations during which the qubit remains protected. Here the term “protected” implies exponential suppression of any noise and term “robust” – suppression of linear noise. In this paper we show that a relatively minor modification of the \(\pi\)--periodic qubit gives an almost ideal protected qubit. Namely, it allows fault-tolerant (i.e. with exponentially small error) \(Z(\pi/2)\) discrete phase gate and robust non-discrete holonomic phase gate \(Z(\Theta)\) along with previously proposed\([7]\) \(X(\pi/2)\)- and \(X \otimes X(\pi/2)\)-gates. Altogether these gates allow universal qubit control\([8]\). During all these operations the qubit states remain degenerate. Whilst this degeneracy is exponentially protected during discrete gates, it is only insensitive in the linear order to the charge noise for holonomic phase gate. Furthermore, due to degeneracy of the computational states the holonomic operation is not sensitive to a precise form of the pulse shape in time domain. Because holonomic operations are robust but not exponentially protected the resulting qubit is almost ideal.

The modification that gives almost ideal qubit is the ability to vary the value of the Josephson energy, \(E_2\) together with the offset charge, \(Q\). Thus, we control two parameters, and a closed path in the 2D parameter space produces Berry phase of the qubit. Variation of the effective \(E_2\) can be achieved by replacing a \(\pi\)-periodic element by a dc-SQUID-like loop of two similar \(\pi\)-periodic elements connected in parallel (Fig. 1b). We refer to this circuit as \(\pi\)-SQUID with effective Josephson energy \(E_2^{\text{eff}}\) depending on flux \(\Phi\) through the loop. For \(\Phi = 0\), effective \(E_2^{\text{eff}}\) is the largest and the qubit behaves like a regular protected
Figure 2: Berry curvature and phase. a) Berry curvature in parameter space \{\Phi, Q\}. The protected region is shown by red color, unprotected by green and blue. The holonomic transformation is achieved by changing the parameters along dashed orange line that starts at the point (1) in the protected regime. b) Cartoons of the wave function at a few characteristic points in parameter space. Points 1, 4, 7 correspond to protected regime, points 2, 3, 5, 6 to unprotected one, see text.

0 − π qubit. Increasing \( \Phi \) decreases \( E_2^{\text{eff}} \). When \( \sqrt{E_2^{\text{eff}}/E_C} \lesssim 1 \) a relatively slow variation of the parameters (the estimate will be given below) squeezes the qubit wavefunction to only one or two charge states and protection is lifted. This temporary removal of protection creates a strong charge\((Q)\)-dispersion and allows to perform different phase gates.

We show below that flux and charge bias variables \{\Phi, Q\} form a 2D parameter space, in which the qubit possesses a Berry curvature shown in Fig. 2, that is obtained analytically. The Berry curvature has a strong peak at \( (\Phi = \Phi_0/4, Q = 0) \). The non-discrete phase gate is performed by adiabatically going in a loop around this peak and gaining different Berry phases for the two logical states. It is crucial that one can chose the path so that at every point the computational states remain degenerate and Berry curvature is zero. The preservation of degeneracy implies that the gate is holonomic: lifting of protection does not cause a gain of unwanted time-dependent dynamic phase. Furthermore, it provides exponential protection against flux noise, while the symmetric nature of the \( Q = 1/2 \) point implies the absence of the linear response to charge noise. Throughout the paper we use units of \( \hbar = 2e = 1 \) and \( \Phi_0 = 2\pi \).

The discrete protected \( Z(\frac{\pi}{2}) \) phase gate is performed by turning the \( \pi \)-SQUID off thereby allowing the qubit to evolve only under the quadratic capacitor Hamiltonian \( H_C = E_C Q^2 \). Analogously to the \( X(\frac{\pi}{2}) \)-gate[7], by choosing the proper gate timing we can make even states to gain 0 dynamic phase and all odd states – dynamic phase of \( -\pi/2 \). Similarly to the \( X(\frac{\pi}{2}) \)-gate this transformation is protected and the errors are flagged by the qubit excitation.
to a high energy state.

**Qualitative description of the holonomic gate.** Figure 2a shows the path (dashed orange loop) along which the holonomic gate is performed. Red areas of the figure depict protected regions where $\Phi$ is close to 0 (or $\pi$) and $\sqrt{E_{\text{eff}}^{2}(\Phi)/E_{C}} \gg 1$. Outside that region ($\sqrt{E_{\text{eff}}^{2}/E_{C}} \lesssim 1$) the qubit is in an unprotected regime.

The loop can be split into two branches: top ($Q>0$) and bottom ($Q<0$). First we go along the $Q > 0$-branch applying a positive charge bias on the superconducting island while keeping the qubit protected (Fig. 2a, (1)). Then, increasing the flux through the loop, we lift the protection against dephasing and squeeze the even wavefunction into only one charge state of $n = 0$ and odd wavefunction – to $n = 1$ (2,3). Importantly, the flux through the $\pi$-SQUID loop also creates a gauge transformation that rotates each charge state $n$ on the island by a different phase factor $n\tilde{\Phi}$, where $\tilde{\Phi}$ is some gauge-related rotation angle that will be discussed and derived further. As a result, on the $Q > 0$-branch the $n = 0$-state remains unaffected, but the strongest odd state (Fig. 2b. (2-3)) $n = +1$ gains the phase of $(+1)\tilde{\Phi}$. Then we return to our initial state through the $Q < 0$-branch (Fig. 2b. (4-7)). On this branch the gauge transform is performed in the opposite direction ($\tilde{\Phi} \rightarrow -\tilde{\Phi}$). The dominant even state $n = 0$ is again unaffected, but the odd state, now represented by the $n = -1$ charge state, gains the phase of $(-1)(-\tilde{\Phi})$ which has the same sign as on the $Q > 0$-branch. Thus, on both halves of the path the odd state is rotated in the same direction causing a non-discrete rotation that is smaller than $\pi$ by a value proportional to the asymmetry of the $\pi$-SQUID.

**Quantitative description of the holonomic gate.** First, let us discuss the properties of the Hamiltonian and its eigenfunctions. We focus on the designs of $\pi$-periodic elements where computational energy levels are well separated from excited ones[3, 4]. Energy of the $\pi$-squid is $E(\varphi) = -E_{2}\cos(\varphi - \Phi/2) - E'_{2}\cos(\varphi + \Phi/2)$. In this case the relevant low energy degrees of freedom of the qubit are described by the Hamiltonian:

$$\mathcal{H} = E_{C} (n - Q)^{2} - E_{2}^{\text{eff}}(\Phi) \cos 2 \left( \varphi - \tilde{\Phi}(\Phi) \right).$$

Here

$$E_{2}^{\text{eff}}(\Phi) = \sqrt{E_{2}^{2} + E'_{2}^{2} + 2E_{2}E'_{2}\cos 2\Phi};$$

$$\tilde{\Phi}(\Phi) = \frac{1}{2} \arctan \left( \frac{\sin \Phi}{E_{2} + E'_{2} \frac{\cos \Phi}{E'_{2} - E_{2}}} \right),$$

with Josephson energies $E_{2}$ and $E'_{2}$ for the two $\pi$-periodic elements. We choose parameters so that $E_{2} + E'_{2} \gg E_{C} \gg |E_{2} - E'_{2}|$. This allows to have both protected ($E_{2}^{\text{eff}}(0)/E_{C} \gg 1$) and unprotected ($E_{2}^{\text{eff}}(\pi/2)/E_{C} \ll 1$) regimes. The eigenfunctions $\psi(n)$ of such Hamiltonian are represented by either even or odd charge states enclosed by the envelope function of width $\sim (E_{2}^{\text{eff}}/E_{C})^{1/4}$ that
is centered around \( n = Q \) in the charge space (Fig. 2b.1-2). Note that for half-integer \( Q \) even and odd eigenstates have a mirror symmetry, which means that for half-integer \( Q \) computational states are degenerate regardless of \( \Phi \). Thus, offset charge \( Q \) changes the balance between even and odd states. Flux bias, in turn, changes the width of the wavefunction by modifying \( E_{2}^{\text{eff}} \). However, flux bias has another crucial effect: although it is tempting to disregard the phase offset \( \tilde{\Phi} \) in (1), it should not be done because in our gate \( \Phi \) is a function of \( Φ \), which is not constant. In fact, this term plays a key role in realizing the gate: since the potential in (1) is shifted by \( \tilde{\Phi} \) in the \( ϕ \)-space, the wavefunction \( ψ(ϕ) \) is also transformed as \( ψ(ϕ) \rightarrow ψ(ϕ − \tilde{Φ}) \). In the charge space this results in a gauge transformation \( ψ(n) \rightarrow \exp(in\tilde{Φ})ψ(n) \), mentioned in the previous section.

Calculating Berry curvature. The phase accrued by the computational states when going along the loop is calculated as an integral of the Berry curvature over the area enclosed by the loop. In this section we derive the expression for the Berry curvature.

We begin with the region of \( Φ \approx \pi/2 \) where \( E_{2}^{\text{eff}} \ll E_{C} \) and the computational eigenfunctions are squeezed to only one or two charge states (Fig. 2 (2)). For any value of \( Q \) it is convenient to write the effective Hamiltonian in terms of the two charge states \( n \) nearest to \( Q \). For example, for \( 0 < Q < 1 \) the odd state can be written in the basis of two wavefunctions \( ψ_{±1}(ϕ) = (2π)^{-1/2} \exp(±iϕ) \), corresponding to \( n = ±1 \) charge states, while the relevant even states are \( ψ_{0}(ϕ) = (2π)^{-1/2} \) and \( ψ_{2}(ϕ) = (2π)^{-1/2} \exp(2iϕ) \):

\[
\mathcal{H}_{\text{eff}}^{\text{odd}} \approx -\frac{1}{2} E_{2}^{\text{eff}} \left( σ^{x} \cos 2\tilde{Φ} + σ^{y} \sin 2\tilde{Φ} \right) + QE_{C}σ^{z}, \quad (4)
\]

\[
\mathcal{H}_{\text{eff}}^{\text{even}} \approx -\frac{1}{2} E_{2}^{\text{eff}} \left( σ^{x} \cos 2\tilde{Φ} + σ^{y} \sin 2\tilde{Φ} \right) + (1 − Q)E_{C}σ^{z}. \quad (5)
\]

Here \( σ^{x,y,z} \) are Pauli matrices in the space of \( |+1⟩, |−1⟩ \) charge state vectors in (4), and in the basis of \( |0⟩, |2⟩ \) in (5). Ground states of \( \mathcal{H}_{\text{eff}}^{\text{even}} \) and \( \mathcal{H}_{\text{eff}}^{\text{odd}} \) represent even and odd computational states respectively. Higher eigenstates of \( \mathcal{H}_{\text{eff}}^{\text{odd(even)}} \) lie outside of the computational space. For the path shown in Fig. 2b, the excited states always remain separated from the computational states by a large energy gap, so that effect of these excited states can be ignored. For \( −\frac{1}{2} \lesssim Q \lesssim \frac{1}{2} \) the accumulated phase is only due to odd eigenstates. To avoid the excitations of the higher energy states of \( \mathcal{H}_{\text{eff}}^{\text{odd(even)}} \) the gate speed needs to be much slower than the smallest energy gap between the eigenvalues of \( \mathcal{H}_{\text{eff}}^{\text{odd(even)}} \) on the path, i.e.

\[
τ_{\text{gate}}^{-1} \ll E_{C} \quad (6)
\]

where \( τ_{\text{gate}} \) is time of the gate operation. Analytically, probability of Landau-Zener tunneling from even(odd) computational state to the even(odd) excited state can be evaluated from the Hamiltonian (4,5) as [9,10]

\[
P = \exp[-τ_{\text{gate}}E_{C}^{2}/(E_{2}^{2} + E_{2}′^{2})]. \quad (7)
\]
Figure 3: Coordinate transformations. a) Mapping of \( \Phi \) onto \( \tilde{\Phi} \) for \( \delta = 0.01, 0.05 \). b) Coordinate transformation that relates the loop in the qubit parameter space \( \Phi \) and \( Q \) to the effective field acting on the spin, characterized by Euler angles \( \xi \) and \( \theta \): mapping of our loop onto the Bloch sphere.

The ground state of (4) is described by a spinor \( |\text{spinor}\rangle = (e^{-i\xi/2} \cos \theta/2, e^{+i\xi/2} \sin \theta/2)^T \).

Polar and azimuthal angles of the spinor are related to the qubit parameters as

\[
\xi = 2\tilde{\Phi}(\Phi); \\
\theta = \arctan \left(-\frac{E'_{\text{eff}}(\Phi)}{E'_{\text{C}}}, \frac{E'_{\text{C}}}{E'_{\text{C}}}\right).
\]

In order to obtain the Berry curvature we perform a coordinate transform \( \Phi = \tilde{\Phi}(\xi), Q = Q(\theta, \xi) \) and map the well-known Berry curvature of a spin \( \Omega_{\xi\theta}^{\text{spin}} = (1/2) \sin \theta \) onto variables \( (\Phi, Q) \) as \( \Omega_{\Phi Q}^{\text{eff}} = \frac{\partial \xi}{\partial \Phi} \frac{\partial \theta}{\partial Q} \Omega_{\xi\theta}^{\text{spin}} \). Since \( \frac{\partial \xi}{\partial \Phi} \) and \( \frac{\partial \theta}{\partial \Phi} \) has a maximum at \( \Phi = \pi/2 \) (Fig.3a), most of the Berry curvature is concentrated in the vicinity of half-integer values of \( \Phi/\pi \). This allows us write equations in the vicinity of \( \Phi = \Phi_0/4 = \pi/2 \). Let us introduce new notations \( \alpha \overset{\text{def}}{=} \Phi - \pi/2, \ E_\Sigma \overset{\text{def}}{=} E_2 + E'_2, \ \delta \overset{\text{def}}{=} |E'_2 - E_2|/E_\Sigma \) and \( \epsilon_C \overset{\text{def}}{=} 2E_C/E_\Sigma \). For large \( \alpha \gg \epsilon_C \) equations (4, 5) break down but this regime gives little contribution to the Berry phase because in it the Berry curvature, along with the qubit charge dispersion is exponentially suppressed with \( (\alpha/\epsilon_C)^{1/2} \).

The dimensionless parameter that controls the Berry phase accumulated in the adiabatic evolution is

\[
\eta = \delta/\epsilon_C = |E'_2 - E_2|/2E_C.
\]

In the following we assume that \( \eta \ll 1 \). The protection is removed when \( \alpha \lesssim \epsilon_C \ll 1 \), and restored when \( \alpha \gg \epsilon_C \). In the former regime the adiabatic evolution leads to accumulation of significant Berry phase. From (2) we
approximate $E_2^{\text{eff}}(\alpha) \approx E_2 \sqrt{\delta^2 + \alpha^2}$ and Berry curvature for the Hamiltonian \[ (9) \] reduces to a simple form

$$\Omega_{\Phi Q}^{\text{peak}} \left( \frac{\pi}{2} + \alpha, Q \right) = \frac{e_C}{2} \frac{\delta}{\left( \delta^2 + \alpha^2 + (e_C Q)^2 \right)^{3/2}} \left\{ 1 + \mathcal{O} \left( \alpha^2 \right) \right\}. \quad (9)$$

The total Berry curvature (difference between $\Omega_{\Phi Q}^{\text{peak}}$ for even and odd states) that determines the phase difference gained between odd and even states can be evaluated as

$$\Omega_{\Phi Q} (\Phi, Q) = \begin{cases} 
\Omega_{\Phi Q}^{\text{peak}} (\Phi, Q) - \Omega_{\Phi Q}^{\text{peak}} (\Phi, Q - 1) & , Q > 0 \\
\Omega_{\Phi Q}^{\text{peak}} (\Phi, Q) - \Omega_{\Phi Q}^{\text{peak}} (\Phi, Q + 1) & , Q < 0 
\end{cases} \quad (10)$$

Since $\Omega_{\Phi Q}$ is an odd function of $Q - 1/2$ it is equal to zero at half-integer values of $Q$. This expression holds for $|Q| \leq 1/2$ and $0 \leq \Phi \leq \pi$, but can be generalized to the entire $\{\Phi, Q\}$-space by keeping in mind that $\Omega_{\Phi Q}$ has a period of $\pi$ in $\Phi$ and period of 2 in $Q$. Also, at half-integer values of $Q$ the computational states are degenerate. Additionally, as mentioned above, the Berry curvature and energy splitting between the two lowest states is exponentially suppressed for $\Phi \approx 0$ and $\Phi \approx \pi$. Thus, we choose our holonomic adiabatic path to go through these regions of $\Phi = 0, \pi$ and $Q = \pm 1/2$.

**Berry phase.** Let us evaluate the the Berry phase that is given by integral of the Berry curvature \[ (9) \] over $\alpha$ and $Q$ in the leading approximation in $\eta \ll 1$. The integral is dominated by the region $|\alpha| \lesssim \delta, |e_C Q| \lesssim \delta$. For $\delta \ll e_C\phi$, this implies that $\alpha \ll e_C, Q \ll 1$ which justifies the use of \[ (9) \]. If we were to integrate the curvature \[ (9) \] in the infinite limits of $\alpha$ and $Q$ we would get $\Theta_0 = \pi$ for the Berry phase. However, exponential suppression of curvature for $\alpha > e_C\phi$ and finite size of the loop limited by $Q \approx \pm 1/2$ implies that actual Berry phase is given by the integral that is cut off in these directions:

$$\Theta \approx \int \sim_{-\infty}^{\alpha = e_C} \int \approx_{-\infty}^{\alpha^{e_C/2}} \approx_{-\infty}^{\alpha^{e_C/2}} \frac{\delta/2}{\left( \delta^2 + \alpha^2 + (e_C Q)^2 \right)^{3/2}} \sim \pi - A \frac{\delta}{e_C}, \quad (11)$$

giving us leading approximation in $\eta = \delta/e_C$. This simple analytical computation does not give the value of the constant $A \sim 1$ which we determined numerically. Our numerical calculations done by diagonalizing the Hamiltonian \[ (1) \] in the basis of 201 charge-state wavevector $\{|-100\rangle, |\ldots|, |\pm100\rangle\}$ and going along the contour with step of $0.001\pi$ in $\Phi$ and $0.01$ in $Q$ determine the numerical constant $A = 2.97 \pm 0.02$.

Most importantly, the resulting phase is not discrete: it deviates from a discrete rotation of $\pi$ by a value that can be controlled by tuning the qubit design. For a reasonably achievable values of $\delta \sim 10^{-2}$ and $e_C \sim 10^{-1}$, the non-discrete rotation is $A \eta \sim 0.3\text{rad}$. Non-discreteness of this gate can be understood by mapping the rectangular path $\gamma$ in the $\{\Phi, Q\}$-space onto a Bloch
Figure 4: Phase gates. a) Phase $\Theta$ as a function of the $\pi$-SQUID asymmetry $\delta$ assuming $\epsilon_C = 0.1$. b) Sketch of the Berry curvature map with depiction of paths that realize different gates. Orange dashed line: rectangular adiabatic loop to realize the holonomic phase gate. Black solid line: quick diabatic $X(\pi/2)$-gate realized by quick frustration of the $\pi$-SQUID. Dark-blue dashed line: example of an idle gate that can be used in CPMG or multiple echo sequences. During the idle gate only accrual of the positive or negative noise-induced dynamic phase occurs. This can be used to create a multiple charge-echo effect to partially compensate the random dynamic phase accrued during the holonomic gate.

sphere $\{\xi, \theta\}$ using equations (8). As shown in figure 3b, the path covers an area, which is somewhat less than half of the Bloch sphere. Moreover, our numerical modeling shows that further increasing of $\eta$ can yield any non-discrete phase (Fig.4a) from $\Theta \approx \pi$ for $\eta \ll 1$ to $\Theta \rightarrow 0$ for $\eta \gtrsim 1$ – regime when qubit does not leave the protected state ($E_{2,}\Phi > E_C$) and $\Omega_{\Phi Q}$ remains exponentially small.

The relative gate error and timing. Since the part of the path sensitive to the flux noise ($\Phi \approx 0, \pi; -\frac{1}{2} \leq Q \leq \frac{1}{2}$) is located in the protected regime of $E_{2,}\Phi > E_C$, effects of the flux noise are exponentially suppressed. In order to estimate the effect of the charge noise, assume that horizontal (i.e. in $\Phi$-direction) part of the path is shifted vertically by a small value $\epsilon_Q$, so that $Q = 1/2 + \epsilon_Q$ instead of $Q = 1/2$. This path would give Berry phase that differs from (11) by

$$\Delta \Theta \sim d\epsilon_Q \Omega_{\Phi Q} (\frac{\pi}{2} + \alpha, \frac{1}{2} + \epsilon_Q) \sim \eta \epsilon_Q^2.$$

Here we used the fact that $\Omega_{\Phi Q} \sim \eta \epsilon_Q$ is linear with $\epsilon_Q$ near $Q = \pm 1/2$. Thus, from (11), relative error of the nondiscrete part ($\Theta \ mod \pi$) of the phase rotation $\Theta$ is $\epsilon_{\Theta}^\text{rel} \sim \Delta \Theta/\eta \sim \epsilon_Q^2$. Assuming a high value of the charge noise $\epsilon_Q \sim 10^{-2}$ on few-second timescales, we arrive at a relative gate error as low as $\epsilon_{\Theta}^\text{rel} \sim 10^{-4}$.
In order for the gate to be considered adiabatic to a reasonable degree, we want to have the previously estimated probability of Landau-Zener tunneling \[ \frac{\tau_0}{F} \]
out of the computational space to be \[ P = \exp[-\tau_0 e_C^2 E_0^{\Sigma}/4] \approx 10^{-4} \].
For reasonable values \( e_C \sim 10^{-1} \) and \( E_0^{\Sigma} \sim 2\pi \times 40\text{GHz} \), we get the gate timing \( \tau_0 \gtrsim 15\text{ns} \).

Finally, we consider an error due to accumulation of unwanted dynamic phase. We consider this to be the bottleneck problem for any protected qubit design because in order to perform a non-discrete rotation one needs to temporarily remove the protection by either (a) lifting the degeneracy of the computational states, which leads to error linear with error in gate timing or (b) keep the degeneracy of the computational states (like in our case) at a cost of gaining linear dispersion of the computational states which leads to error linear in noise amplitude. In our gate, for example, the qubit remains in unprotected regime \( \{ \Phi \approx \frac{\pi}{2} \pm e_C, Q = \pm \frac{\pi}{2} \} \) during the time \( \tau_0 \sim e_C \tau_{\text{gate}} \), when it gains dynamic phase \( \gamma \). There the computational states are degenerate but their charge dispersion \( \Delta E_1,0(e_Q) \) is linear with deviation \( e_Q \) of charge offset from \( Q = \pm 0.5 \).

At \( \Phi = \pi/2 \) the dispersion is \( \Delta E_1,0(e_Q) = \text{sign}(Q) E_0^{\Sigma} e_C e_Q \). Notably, for \( Q > 0 \) and \( Q < 0 \) parts of the path the dynamic phase has opposite sign. Such accumulation of dynamic phase is identical to charge echo experiments\[15,12,10\] which are sensitive only to high-frequency \( f \gtrsim \tau_u^{-1} \) noise. Approximating the “on” function for the unprotected regime as a square pulse we can characterize the dynamic phase by its mean square using a well-known expression\[15,15]:

\[
\overline{\gamma^2} \sim \int d\omega \Delta E^2_{1,0} \frac{A}{\omega} \left( \frac{\sin(\omega \tau_u/2)}{\omega/2} \right)^2 \sim A \left( \Delta E_{1,0} \tau_u \right)^2.
\]

Here \( A/\omega \) is the spectral density of 1/f-noise. We can now estimate the infidelity of the phase gate. Assume that the ideal gate acting on initial qubit state \(|\text{initial}\rangle \) gives the state \(|\text{ideal}\rangle = Z(\Theta)|\text{initial}\rangle \). The physical gate gives instead a state \(|\text{real}\rangle = Z(\Theta + \gamma)|\text{initial}\rangle \) with \( \gamma \ll 1 \). Define the mean infidelity as \( 1 - F = 1 - |\langle \text{ideal}|\text{real}\rangle| \approx \gamma^2/2 \). Assuming same parameters as above, \( E_0^{\Sigma} \sim 2\pi \times 40\text{GHz} \); \( e_C = 0.1 \), \( \tau_{\text{gate}} = 15\text{ns} \) and high-frequency 1/f charge noise with amplitude \( e_Q = A^{1/2} \) between \( 1.5 \times 10^{-4} \times (2e) \) and \( 6.5 \times 10^{-4} \times (2e) \)[13,12,17,18] we get \( 1 - F \) between \( \sim 10^{-4} \) and \( \sim 10^{-5} \). This estimate relies on the assumption that charge noise follows 1/f dependence up to GHz frequencies, similar to the charge sensitive devices studied in works by Astafiev et al.[12] and expected theoretically by Faoro et al.[19]. Notice that the dephasing can be decreased even further by controlling the qubit using GRAPE[20] pulses to further minimize gate timing and by using CPMG-like[21] or multiple-echo sequences to filter out 1/f noise (Fig.4b). This would give us further improvement in fidelity. To the best of our knowledge, this gate performance is much better than in any currently existing qubit[22,23,24], especially considering a complexity of non-discrete gates.

**Discrete fault-tolerant \( Z(\pi/2)\)-gate.** The proposed qubit can also be used to perform a protected discrete \( \exp(-i\pi \sigma^2/4)\)-gate. The idea is similar to the gate proposed in work by Brooks et al.[7]. In contrast to the nondiscrete
gate, this gate is not performed adiabatically, but by quick modification of the Hamiltonian. While the adiabatic change of $\Phi$ leads to squeezing of the wavefunction in the charge space, in the case of the quick modification of the Hamiltonian, the wavefunction does not have time to squeeze, but it remains as it was, delocalized in the charge space. In more detail, we start with the qubit in the protected state with $\Phi = 0$ and $E_{\text{eff}}^2 = E_C$. Then we quickly change the flux to $\Phi = \pi/2$. This effectively turns off the $\pi$-SQUID leaving only the capacitive part of the Hamiltonian $H(\pi/2) = n^2/2C + O(\delta E_C \cos 2\varphi)$ and hence the qubit evolves under the operator $U_{\pi/2}(t) \approx \exp\{-i n^2/(2C) t\}$. After time $T = \pi C$ qubit is brought back into the initial state. With this gate timing $U_{\pi/2}(T) \approx \exp\{-i\pi n^2/2\}$. As a result, all even charge states are multiplied by a factor of $\exp\{-i\pi n_{\text{even}}^2/2\} = 1$ and odd states – by $\exp\{-i\pi n_{\text{odd}}^2/2\} = -i$. Hence, we realize a $\exp(-i\pi \sigma^z/4)$-gate. This gate is dual to the gate proposed in the work by Brooks et al. and therefore has similar exponential stability against gate timing error $T \to \pi C + \Delta T$ (see section VI of [7]) and perturbation stability (e.g. against small perturbation $O(\delta E_C \cos 2\varphi)$ in the Hamiltonian: see section XI of work by Brooks et al. [7]). In both cases errors and perturbations only result into excitations of high-energy levels that are outside of our computational space.

Since in a protected state ($\Phi = 0$) our qubit is identical to a standard 0–$\pi$ qubit[6, 7] it is also possible to implement a $\exp(i\pi \sigma^x/4)$-gate described by Brooks et al.[7]. With these two discrete gates it is possible to realize the topologically protected Clifford group $C_1$ and a two-qubit gate[7] $X \otimes X(\pi/2)$. In combination with the semiprotected holonomic gate described above it results in a universal qubit control[8] with high fidelity.

**Conclusion.** We showed that by adding one more degree of freedom to the protected qubit architecture based on double periodic Josephson junctions it is possible to realize two more types of gates: a discrete protected gate and a robust continuous holonomic gate that is not sensitive to the flux noise and to charge noise in the linear order. Together with previously existing one- and two-qubit flip-gate[7] it is possible to build a first realistic scalable quantum computer with universal qubit control and infidelity of the order of $10^{-4}$/gate (importantly, with potential for further improvement). For the holonomic gate, in principle, one can also choose a different, more complicated path in the $\{\Phi, Q\}$-space and achieve different phase gates with different noise sensitivity. This diversity arises due to non-trivial Berry curvature landscape of an essentially two-dimensional system that is controlled by two bias channels. We expect that by creating more complex circuits with more degrees of freedom one can create systems with more complex Berry curvature landscapes and gauge fields. In prospective it will be interesting to generalize this approach to other types of protected or robust qubits such as fluxonium[25, 26, 27]. We hope that further development of similar holonomic qubit architectures will allow achieving higher degrees of protection for continuous gates.

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