Generalized Solutions to Nonlinear First Order Cauchy Problems

Jan Harm van der Walt

Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria 0002, South Africa
E-mail: janharm.vanderwalt@up.ac.za

Received September 20, 2007; Accepted in Revised Form Month *, ****

Abstract
The recent significant enrichment, see [14] through [15], of the Order Completion Method for nonlinear systems of PDEs [12] resulted in the global existence of generalized solutions to a large class of such equations. In this paper we consider the existence and regularity of the generalized solutions to a family of nonlinear first order Cauchy problems. The spaces of generalized solutions are obtained as the completion of suitably constructed uniform convergence spaces.

"...provided also if need be that the notion of a solutions shall be suitably extended"

–cited form Hilbert’s 20th problem

1 Introduction
It is widely held misconception that there can be no general, type independent theory for the existence and regularity of solutions to nonlinear PDEs. Arnold [4] ascribes this to the more complicated geometry of \( \mathbb{R}^n \), as opposed to \( \mathbb{R} \), which is relevant to ODEs alone. Evans [8], on the other hand, cites the wide variety of physical and probabilistic phenomena that are modelled with PDEs.

There are, however, two general, type independent theories for the solutions of nonlinear PDEs. The Central Theory for PDEs, as developed by Neuberger [10], is based on a generalized method of steepest descent in suitably constructed Hilbert spaces. It delivers generalized solutions to nonlinear PDEs in a type independent way, although the method is not universally applicable. However, it does yield spectacular numerical results. The Order Completion Method [12], on the other hand, yields the generalized solutions to arbitrary, continuous nonlinear PDEs of the form

\[
T(x, D) u(x) = f(x)
\] (1.1)
where $\Omega \subseteq \mathbb{R}^n$ is open and nonempty, $f$ is continuous, and the PDE operator $T(x, D)$ is defined through some jointly continuous mapping
\[
F : \Omega \times \mathbb{R}^K \rightarrow \mathbb{R}
\]
by
\[
T(x, D) : u(x) \mapsto F(x, u(x), ..., D^\alpha u(x), ...)
\]
(1.3)
The generalized solutions are obtained as elements of the Dedekind completion of certain spaces of functions, and may be assimilated with usual Hausdorff continuous, interval valued functions on the domain of the PDE operator $\mathcal{L}$.

Recently, see [14] through [16], the Order Completion Method [12] was reformulated and enriched by introducing suitable uniform convergence spaces, in the sense of [6]. In this new setting it is possible, for instance, to treat PDEs with addition smoothness, over and above the mere continuity of the PDE operator, in a way that allows for a significantly higher degree of regularity of the solutions [15].

The aim of this paper is to show how the ideas developed in [14] through [16] may be applied to initial and/or boundary value problems. In this regard, we consider a family of nonlinear first order Cauchy problems. The generalized solutions are obtained as elements of the completion of a suitably constructed uniform convergence space. We note the relative ease and simplicity of the method presented here, compared to the usual linear function analytic methods. In this way we come to note another of the advantages in solving initial and/or boundary value problems for linear and nonlinear PDEs in this way. Namely, initial and/or boundary value problems are solved by precisely the same kind of constructions as the free problems. On the other hand, as is well known, this is not so when function analytic methods - in particular, involving distributions, their restrictions to lower dimensional manifolds, or the associated trace operators - are used for the solution of such problems.

The paper is organized as follows. In Section 2 we introduce some definitions and results as are required in what follows. We omit the proofs, which can be found in [14] through [16]. Section 3 is concerned with the solutions of a class of nonlinear first order Cauchy problems. In Section 4 we discuss the possible interpretation of the generalized solutions obtained.

## 2 Preliminaries

Let $\Omega$ be some open and nonempty subset of $\mathbb{R}^n$, and let $\overline{\mathbb{R}}$ denote the extended real line
\[
\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}
\]
A function $u : \Omega \rightarrow \overline{\mathbb{R}}$ belongs to $\mathcal{ML}_o^m(\Omega)$, for some integer $m$, whenever $u$ is normal lower semi-continuous, in the sense of Dilworths [7], and
\[
\exists \quad \Gamma_u \subset \Omega \text{ closed nowhere dense}:
\begin{align*}
&1) \quad \text{mes}(\Gamma_u) = 0 \\
&2) \quad u \in C^m(\Omega \setminus \Gamma_u)
\end{align*}
\]
(2.1)
Here \( \text{mes}(\Gamma_u) \) denotes the Lebesgue measure of the set \( \Gamma_u \). Recall \( [1] \) that a function \( u : \Omega \rightarrow \mathbb{R} \) is normal lower semi-continuous whenever
\[
\forall \ x \in \Omega :
I (S(u))(x) = u(x)
\tag{2.2}
\]
where
\[
\forall \ u : \Omega \rightarrow \mathbb{R}:
1) \ I(u) : \Omega \ni x \mapsto \sup \{ \inf \{ u(y) : y \in B_\delta(x) \} : \delta > 0 \} \in \mathbb{R}
2) \ S(u) : \Omega \ni x \mapsto \inf \{ \sup \{ u(y) : y \in B_\delta(x) \} : \delta > 0 \} \in \mathbb{R}
\]
are the lower- and upper- Baire Operators, respectively, see \([1]\) and \([5]\). Note that each function \( u \in \mathcal{ML}_0^m(\Omega) \) is measurable and nearly finite with respect to Lebesgue measure. In particular, the space \( \mathcal{ML}_0^m(\Omega) \) contains \( C_0^m(\Omega) \). In this regard, we note that the partial differential operators
\[
D^\alpha : C_0^m(\Omega) \rightarrow C_0^0(\Omega), \ |\alpha| \leq m
\]
extend to mappings
\[
\mathcal{D}^\alpha : \mathcal{ML}_0^m(\Omega) \ni u \rightarrow (I \circ S)(D^\alpha u) \in \mathcal{ML}_0^0(\Omega)
\tag{2.3}
\]
A convergence structure \( \lambda_a \), in the sense of \([6]\), may be defined on \( \mathcal{ML}_0^0(\Omega) \) as follows.

**Definition 1.** For any \( u \in \mathcal{ML}_0^0(\Omega) \), and any filter \( \mathcal{F} \) on \( \mathcal{ML}_0^0(\Omega) \),
\[
\mathcal{F} \in \lambda_a(u) \iff \left( \exists \ E \subset \Omega : \begin{array}{ll}
a) & \text{mes}(E) = 0 \\
b) & x \in \Omega \setminus E \Rightarrow \mathcal{F}(x) \text{ converges to } u(x) \end{array} \right)
\]
Here \( \mathcal{F}(x) \) denotes the filter of real numbers given by
\[
\mathcal{F}(x) = \{ \{v(x) : v \in F\} : F \in \mathcal{F}\} \tag{2.4}
\]
That \( \lambda_a \) does in fact constitute a uniform convergence structure on \( \mathcal{ML}_0^0(\Omega) \) follows by \([6]\) Example 1.1.2 (iii)]. Indeed, \( \lambda_a \) is the almost everywhere convergence structure, which is Hausdorff. One may now introduce a complete uniform convergence structure \( \mathcal{J}_a \), in the sense of \([6]\), on \( \mathcal{ML}_0^0(\Omega) \) in such a way that the induced convergence structure \([6]\) Definition 2.1.3] is \( \lambda_a \).

**Definition 2.** A filter \( \mathcal{U} \) on \( \mathcal{ML}_0^0(\Omega) \times \mathcal{ML}_0^0(\Omega) \) belongs to \( \mathcal{J}_a \) whenever there exists \( k \in \mathbb{N} \) such that
\[
\forall \ i = 1, \ldots, k :
\exists \ u_i \in \mathcal{ML}_0^0(\Omega) :
\exists \ \mathcal{F}_i \text{ a filter on } \mathcal{ML}_0^0(\Omega) :
\begin{array}{ll}
a) & \mathcal{F}_i \in \lambda_a(u_i) \\
b) & (\mathcal{F}_1 \times \mathcal{F}_1) \cap \ldots \cap (\mathcal{F}_k \times \mathcal{F}_k) \subseteq \mathcal{U}
\end{array}
\]
The uniform convergence structure $\mathcal{J}_a$ is referred to as the uniform convergence structure associated with the convergence structure $\lambda_a$, see [6, Proposition 2.1.7].

We note that the concept of a convergence structure on a set $X$ is a generalization of that of topology on $X$. With every topology $\tau$ on $X$ one may associate a convergence structure $\lambda_\tau$ on $X$ through

$$\forall x \in X :$$
$$\forall F \text{ a filter on } X :$$
$$F \in \lambda_\tau(x) \iff V_\tau(x) \subseteq F$$

where $V_\tau(x)$ denotes the filter of $\tau$-neighborhoods at $x$. However, not every convergence structure $\lambda$ on $X$ is induced by a topology in this way. Indeed, the convergence structure $\lambda_a$ specified above is one such an example. A uniform convergence space is the generalization of a uniform space in the context of convergence spaces. The reader is referred to [6] for details concerning convergence spaces.

3 First Order Cauchy Problems

Let $\Omega = (-a,a) \times (-b,b) \subset \mathbb{R}^2$, for some $a, b > 0$, be the domain of the independent variables $(x,y)$. We are given

$$F : \overline{\Omega} \times \mathbb{R}^4 \rightarrow \mathbb{R}$$

and

$$f : [-a,a] \rightarrow \mathbb{R}$$

Here $F$ is jointly continuous in all of its variables, and $f$ is in $C^1[-a,a]$. We consider the Cauchy problem

$$D_y u(x,y) + F(x,y,u(x,y),u(x,y),D_x u(x,y)) = 0, \ (x,y) \in \Omega$$

(3.3)

$$u(x,0) = f(x), \ x \in (-a,a)$$

(3.4)

Denote by $T : C^1(\Omega) \rightarrow C^0(\Omega)$ the nonlinear partial differential operator given by

$$\forall u \in C^1(\Omega) :$$
$$\forall (x,y) \in \Omega :$$
$$Tu : (x,y) \mapsto D_y u(x,y) + F(x,y,u(x,y),u(x,y),D_x u(x,y))$$

(3.5)

Note that the equation (3.3) may have several classical solutions. Indeed, in the particular case when the operator $T$ is linear and homogeneous, there is at least one classical solution to (3.3) which is the function which is everywhere equal to 0. However, the presence of the initial condition (3.4) may rule out some or all of the possible classical solutions. What is more, there is a well known physical interest in nonclassical or generalized solutions to (3.3) through
First Order Cauchy Problems

(3.4) Such solution may, for instance, model shocks waves in fluids. In this regard, it is convenient to extend the PDE operator \( T \) to \( ML^1_0(\Omega) \) through

\[
\forall \ u \in ML^1_0(\Omega) : \\
\forall \ (x, y) \in \Omega : \\
Tu : (x, y) \mapsto (I \circ S)(D_yu + F(\cdot, Dxu, u))(x, y)
\]

As mentioned, the solution method for the Cauchy problem (3.3) through (3.4) uses exactly the same techniques that apply to the free problem [15]. However, in order to incorporate the additional condition (3.4), we must adapt the method slightly. In this regard, we consider the space

\[
ML^1_{0,0}(\Omega) = \{ u \in ML^1_0(\Omega) : u(\cdot, 0) \in C^1[-a, a] \}
\]

and the mapping

\[
T_0 : ML^1_{0,0}(\Omega) \ni u \mapsto (Tu, R_0u) \in ML^0_0(\Omega) \times C^1[-a, a]
\]

where

\[
\forall \ u \in ML^1_{0,0}(\Omega) : \\
\forall \ x \in [-a, a] : \\
R_0u : x \mapsto u(x, 0)
\]

That is, \( R^0 \) assigns to each \( u \in ML^1_{0,0}(\Omega) \) its restriction to \( \{(x, y) \in \Omega : y = 0\} \). This amounts to a separation of the initial value problem (3.4) form the problem of satisfying the PDE (3.3).

For the sake of a more compact exposition, we will denote by \( X \) the space \( ML^1_{0,0}(\Omega) \) and by \( Y \) the space \( ML^0_0(\Omega) \times C^1[-a, a] \). On \( C^1[-a, a] \) we consider the convergence structure \( \lambda_0 \), and with it the associated u.c.s. \( J_0 \).

**Definition 3.** For any \( f \in C^1[-a, a] \), and any filter \( \mathcal{F} \) on \( C^1[-a, a] \),

\[
\mathcal{F} \in \lambda_0(f) \iff [f] \subseteq \mathcal{F}
\]

Here \([x]\) denotes the filter generated by \( x \). That is,

\[
[x] = \{ F \subseteq C^1[-a, a] : x \in F \}
\]

The associated u.c.s. \( J_0 \) on \( C^1[-a, a] \) consists of all filters \( \mathcal{U} \) on \( C^1[-a, a] \times C^1[-a, a] \) that satisfies

\[
\exists \ k \in \mathbb{N} : \\
\left( \forall \ i = 1, \ldots, k : \\
\exists \ f_i \in C^1[-a, a] : \\
\exists \ \mathcal{F}_i \ a \ filter \ on \ C^1[-a, a] : \\
\quad a) \ \mathcal{F}_i \in \lambda_0(u_i) \\
\quad b) \ (\mathcal{F}_1 \times \mathcal{F}_1) \cap \ldots \cap (\mathcal{F}_k \times \mathcal{F}_k) \subseteq \mathcal{U} \right)
\]

(3.8)

This u.c.s. is uniformly Hausdorff and complete. The space \( Y \) carries the product u.c.s. \( J_Y \) with respect to the u.c.s.'s \( J_a \) on \( ML^0_0(\Omega) \) and \( J_0 \) on \( C^1[-a, a] \). That is, for any filter \( \mathcal{V} \) on \( Y \times Y \)

\[
\mathcal{V} \in J_Y \iff \left( \begin{array}{c} a) \ (\pi_0 \times \pi_0) (\mathcal{V}) \in J_a \\
\quad b) \ (\pi_1 \times \pi_1) (\mathcal{V}) \in J_0 \end{array} \right)
\]

(3.9)
Here $\pi_0$ denotes the projection on $\mathcal{ML}_0^0(\Omega)$, and $\pi_1$ is the projection on $\mathcal{C}^1[-a,a]$. With this u.c.s., the space $Y$ is uniformly Hausdorff and complete \[\text{[6, Proposition 2.3.3 (iii)].}\]

On the space $X$ we introduce an equivalence relation $\sim_{T_0}$ through

$$\forall \ u, v \in X : \ u \sim_{T_0} v \iff T_0 u = T_0 v$$ \hspace{1cm} (3.10)

The quotient space $X/\sim_{T_0}$ is denoted $X_{T_0}$. With the mapping $T_0 : X \rightarrow Y$ we may now associate an injective mapping $\hat{T}_0 : X_{T_0} \rightarrow Y$ so that the diagram

\[
\begin{array}{ccc}
X & \overset{T_0}{\longrightarrow} & Y \\
q_{T_0} \downarrow & & \downarrow \ i_Y \\
X_{T_0} & \overset{\hat{T}_0}{\longrightarrow} & Y
\end{array}
\]

commutes. Here $q_{T_0}$ denotes the quotient mapping associated with the equivalence relation $\sim_{T_0}$, and $i_Y$ is the identity mapping on $Y$. We now define a u.c.s. $\mathcal{J}_{T_0}$ on $X_{T_0}$ as the initial u.c.s. \[\text{[6, Proposition 2.1.1 (i)] on } X_{T_0} \text{ with respect to the mapping } \hat{T}_0 : X_{T_0} \rightarrow Y. \text{ That is,}\]

$$\forall \ \mathcal{U} \text{ a filter on } X_{T_0} :$$

$$\mathcal{U} \in \mathcal{J}_{T_0} \iff \left( \hat{T}_0 \times \hat{T}_0 \right) (\mathcal{U}) \in \mathcal{J}_Y$$ \hspace{1cm} (3.11)

Since $\hat{T}_0$ is injective, the u.c.s. $\mathcal{J}_{T_0}$ is uniformly Hausdorff, and $\hat{T}_0$ is actually a uniformly continuous embedding. Moreover, if $X_{T_0}^\sharp$ denotes the completion of $X_{T_0}$, then there exists a unique uniformly continuous embedding

$$\hat{T}_0^\sharp : X_{T_0}^\sharp \rightarrow Y$$

such that the diagram
commutes. Here \( \iota_{X_{T_0}} \) denotes the uniformly continuous embedding associated with the completion \( X^\sharp_{T_0} \) of \( X_{T_0} \). A generalized solution to (3.3) through (3.4) is a solution to the equation

\[
\hat{T}_0^\sharp U^\sharp = (0, f)
\]  

The existence of generalized solutions is based on the following basic approximation result [12, Section 8].

**Lemma 1.** We have

\[\forall \varepsilon > 0 :\]
\[\exists \delta > 0 :\]
\[\forall (x_0, y_0) \in \Omega :\]
\[\exists u = u_{\varepsilon, x_0, y_0} \in C^1(\overline{\Omega}) :\]
\[\forall (x, y) :\]
\[\left( \frac{|x - x_0|}{\delta} < \delta \right) \Rightarrow -\varepsilon \leq Tu(x, y) \leq 0 \]  

Furthermore, we also have

\[\forall \varepsilon > 0 :\]
\[\exists \delta > 0 :\]
\[\forall x_0 \in [-a, a] :\]
\[\exists u = u_{\varepsilon, x_0} \in C^1(\overline{\Omega}) :\]
\[a) \forall (x, y) \in \Omega :\]
\[\left( \frac{|x - x_0|}{\delta} < \delta \right) \Rightarrow -\varepsilon \leq Tu(x, y) \leq 0 \]
\[b) u(x, 0) = f(x), x \in [-a, a], |x - x_0| < \delta \]  

As a consequence of the approximation result above, we now obtain the existence and uniqueness of generalized solutions to (3.3) through (3.4). In this regard, we introduce the concept of a finite initial adaptive \( \delta \)-tiling. A finite initial adaptive \( \delta \)-tiling of \( \Omega \) is any finite collection \( \mathcal{K}_\delta = \{K_1, ..., K_\nu\} \) of perfect, compact subsets of \( \mathbb{R}^2 \) with pairwise disjoint interiors such that

\[\forall K_i \in \mathcal{K}_\delta :\]
\[ (x, y), (x_0, y_0) \in K_i \Rightarrow \left( \frac{|x - x_0|}{\delta} < \delta \right) \]  

(3.15)
and
\[ \{ (x, 0) : -a \leq x \leq a \} \cap (\cup_{K_i \in \mathcal{K}_\delta} \partial K_i) \text{ at most finite} \]  
(3.16)

where \( \partial K_i \) denotes the boundary of \( K_i \). For any \( \delta > 0 \) there exists at least one finite initial adaptive \( \delta \)-tiling of \( \Omega \), see for instance [12, Section 8].

**Theorem 1.** For any \( f \in C^1[-a, a] \), there exists a unique \( U^* \in X^2_{\mathcal{K}_0} \) that satisfies (3.12).

**Proof.** For every \( n \in \mathbb{N} \), set \( \epsilon_n = 1/n \). Applying Lemma 1 we find \( \delta_n > 0 \) such that
\[
\forall (x_0, y_0) \in \Omega : \\
\exists u = u_{n,x_0,y_0} \in C^1(\Omega) : \\
\forall (x, y) : \\
\left( \frac{|x - x_0|}{\delta_n} < \epsilon_n \right) \Rightarrow -\frac{\epsilon_n^2}{2} \leq T u(x, y) \leq 0
\]
and
\[
\forall x_0 \in [-a, a] : \\
\exists u = u_{n,x_0} \in C^1(\Omega) : \\
a) \forall (x, y) \in \Omega : \\
\left( \frac{|x - x_0|}{\delta} < \epsilon_n \right) \Rightarrow -\frac{\epsilon_n^2}{2} \leq T u(x, y) \leq 0 \\
b) u(x, 0) = f(x), x \in [-a, a], |x - x_0| < \delta
\]

Let \( \mathcal{K}_{\delta_n} = \{ K_1, ..., K_{\nu_n} \} \) be a finite initial adaptive \( \delta_n \)-tiling. If \( K_i \in \mathcal{K}_{\delta_n} \), and
\[ K_i \cap \{(x, 0) : |x| \leq a\} = \emptyset \]  
(3.19)
then take any \( (x_0, y_0) \in \text{int}(K_i) \) and set
\[ u_n^i = u_{n,x_0,y_0} \]
Otherwise, select \( (x_0, 0) \in \{[-a, a] \times \{0\}\} \cap K_i \) and set
\[ u_n^i = u_{n,x_0} \]
Consider the function \( u_n \in \mathcal{M}\mathcal{L}^1_{0,0}(\Omega) \) defined as
\[ u_n = (I \circ S) \left( \sum_{i=1}^{\nu} \chi_i u_n^i \right) \]
where, for each \( i, \chi_i \) is the indicator function of \( \text{int}(K_i) \). It is clear that
\[
\forall (x, y) \in \Omega : \\
-\epsilon_n < T u_n(x, y) \leq 0
\]
and
\[
\forall x \in [-a, a] : \\
\mathcal{R}_0 u_n(x) = 0
\]
Let $U_n$ denote the $\sim_{T_0}$ equivalence class associated with $u_n$. Then the sequence $(\hat{T}_0 U_n)$ converges to $(0, f) \in Y$. Since $\hat{T}_0$ is uniformly continuous embedding, the sequence $(U_n)$ is a Cauchy sequence in $X_{T_0}$ so that there exists $U^\sharp \in X^\sharp_{T_0}$ that satisfies (3.12). Moreover, this solution is unique since $\hat{T}_0^\sharp$ is injective.

4 The Meaning of Generalized Solutions

Regarding the meaning of the existence and uniqueness of the generalized solution $U^\sharp \in X^\sharp_{T_0}$ to (3.3) through (3.4), we recall the abstract construction of the completion of a uniform convergence space. Let $(Z, J)$ be a Hausdorff uniform convergence space. A filter $\mathcal{F}$ on $Z$ is a $J$-Cauchy filter whenever $\mathcal{F} \times \mathcal{F} \in J$. If $C(Z)$ denotes the collection of all $J$-Cauchy filters on $Z$, one may introduce an equivalence relation $\sim_c$ on $C(Z)$ through

$$\forall \mathcal{F}, \mathcal{G} \in C(Z) : \mathcal{F} \sim_c \mathcal{G} \iff \exists \mathcal{H} \in C(Z) : \mathcal{H} \subseteq \mathcal{G} \cap \mathcal{F}$$

(4.1)

The quotient space $Z^\sharp = C(Z) / \sim_c$ serves as the underlying set of the completion of $Z$. Note that, since $Z$ is Hausdorff, if the filters $\mathcal{F}, \mathcal{G} \in C(Z)$ converge to $x \in Z$ and $z \in Z$, respectively, then

$$\mathcal{F} \sim_c \mathcal{G} \iff x = z$$

so that one obtains an injective mapping

$$\iota_Z : Z \ni z \mapsto [\lambda(z)] \in Z^\sharp$$

where $[\lambda(z)]$ denotes the equivalence class generated by the filters which converge to $z \in Z$. One may now equip $Z^\sharp$ with a u.c.s. $J^\sharp$ in such a way that the mapping $\iota_Z$ is a uniformly continuous embedding, $Z^\sharp$ is complete, and $\iota_Z(Z)$ is dense in $Z^\sharp$.

In the context of PDEs, and in particular the first order Cauchy problem (3.3) through (3.4), the generalized solution $U^\sharp \in X_{T_0}$ to (3.3) through (3.4) may be expressed as

$$U^\sharp = \left\{ \mathcal{F} \in C(X_{T_0}) \left| \begin{array}{l} a) \quad \pi_0 \left( \hat{T}_0 (\mathcal{F}) \right) \in \lambda_0 (0) \\ b) \quad \pi_1 \left( \hat{T}_0 (\mathcal{F}) \right) \in \lambda_0 (f) \end{array} \right. \right\}$$

(4.2)

Any classical solution $u \in C^1 (\Omega)$, and more generally any shockwave solution $u \in ML^1_{0,0} (\Omega) = X$ to (3.3) through (3.4), is mapped to the generalized solution $U^\sharp$, as may be seen from the following commutative diagram.
Hence there is a consistency between the generalized solutions $U^♯ \in X^♯_{T_0}$ and any classical and shockwave solutions that may exists.

5 Conclusion

We have shown how the ideas developed in [14] through [16] may be applied to solve initial and / or boundary value problems for nonlinear systems of PDEs. In this regard, we have established the existence and uniqueness of generalized solutions to a family of nonlinear first order Cauchy problems. The generalized solutions are seen to be consistent with the usual classical solutions, if the latter exists, as well as with shock wave solutions. It should be noted that the above method applies equally well to arbitrary nonlinear systems of equations.

References

[1] Anguelov R, Dedekind order completion of C(X) by Hausdorff continuous functions Quaestiones Mathematicae, 27(2004) 153-170.

[2] Anguelov R and Rosinger E E, Solving large classes of nonlinear systems of PDE’s, Computers and Mathematics with Applications 53 (2007) 491-507

[3] Anguelov R and van der Walt J H, Order convergence structure on $C(X)$, Quaestiones Mathematicae 28 No. 4 (2005), 425–457.

[4] Arnold V I, Lectures on PDEs, Springer Universitext, 2004.
[5] Baire R, Lecons sur les fonctions discontinues, *Collection Borel*, Paris, 1905.

[6] Beattie R and Butzmann H-P, Convergence structures and applications to functional analysis, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[7] Dilworth R P, The normal completion of the lattice of continuous functions, *Trans. AMS* (1950), 427-438.

[8] Evans L C, Partial differential equations, AMS Graduate Studies in Mathematics 19, AMS, 1998.

[9] Luxemburg W A and Zaanen A C, Riesz Spaces I, North-Holland, Amsterdam, London, 1971.

[10] Neuberger J W, Sobolev gradients and differential equations, Springer Lecture Notes in Mathematics vol. 1670, 1997.

[11] Neuberger J W, Prospects of a central theory of partial differential equations, 27 no. 3 (2005), 47-55.

[12] Oberguggenberger M B and Rosinger E E, Solution of continuous nonlinear PDEs through order completion, North-Holland, Amsterdam, London, New York, Tokyo, 1994.

[13] Ordman E T, Convergence almost everywhere is not topological, *A. Math. Mon.* 73, (1966), 182-183.

[14] van der Walt J H, The order completion method for systems of nonlinear PDEs: Pseudo-topological Perspectives, Technical Report UPWT 2007/07, University of Pretoria.

[15] van der Walt J H, On the completion of uniform convergence spaces and an application to nonlinear PDEs, Technical Report UPWT 2007/14, University of Pretoria.

[16] van der Walt J H, Generalized functions and nonlinear PDEs, To Appear.