When Nonlocal Coupling Between Oscillators Becomes Stronger: Patched Synchrony or Multi-Chimera States

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This paper discusses the tendency of coupled systems to exhibit both coherent and incoherent parts, which we all know describes a *chimera*. Specifically, the authors intend to present evidence that chimera behavior extends to neuronal models.

It is believed that chimeras can account for synchrony in neural activity, such as eye movement.
The authors consider a system of \( N \) nonlocally coupled *FitzHugh-Nagumo* oscillators, arranged on a ring.

- Model has relevance not only to neuronal models, but also electronic oscillators and nonlinear electronic circuits.

- Described by the following equations:

\[
\epsilon \frac{du_k}{dt} = u_k - \frac{u_k^3}{3} - v_k + \frac{\sigma}{2R} \sum_{j=k-R}^{j=k+R} \left[ b_{uu}(u_j - u_k) + b_{uv}(v_j - v_k) \right]
\]

\[
\epsilon \frac{dv_k}{dt} = u_k + a_k + \frac{\sigma}{2R} \sum_{j=k-R}^{j=k+R} \left[ b_{vu}(u_j - u_k) + b_{vv}(v_j - v_k) \right]
\]

- Where \( u_k \) and \( v_k \) are the activator and inhibitor variables, respectively.

- \( \epsilon > 0 \) is a small parameter characterizing a timescale separation.

- Fixed at \( \epsilon = 0.05 \) throughout paper.
Parameter $a_k$

- In the paper, $a_k$ is called the threshold variable. Model is either oscillator or excitable depending on its magnitude.
- In other words, $a$ is the bifurcation parameter.
- Authors assume $a_k$ is in oscillatory regime for each oscillator. ($a \in (-1, 1)$)

\[ |a_k| < 1 \quad \text{and} \quad |a_k| > 1 \]
Coupling of the System

- Coupling scheme inspired by experimental neuroscience.
- Neuronal networks IRL are often structured such that strong interconnections exist with a range $R$ and vanish at longer distances.
- Authors approximate this feature by incorporating constant coupling of strength $\sigma$ for the $R$ nearest neighbors on each side of a given oscillator.
- Another important feature of the model is that it not only has direct $u - u$ and $v - v$ coupling, but also cross coupling between variables.
- Modeled by (familiar) rotational matrix:

$$B = \begin{pmatrix} b_{uu} & b_{uv} \\ b_{vu} & b_{vv} \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$$

- Four parameters to vary: $a, \sigma, R,$ and $\phi$. 
Phase Reduction

In the absence of coupling, the oscillators follow a limit cycle with $u = U_a$ and $v = V_a$, with approximate period

$$T = 3 + (1 - a^2)\ln\frac{1-a^2}{4-a^2}$$

The authors perform a phase reduction of the network model, inspired by the results of Izhikevich in *Phase Equations for Relaxation Oscillators*, transforming the model into the following form:

$$\frac{d\theta_K}{dt} = -\frac{1}{R} \sum_{j=k-R}^{j=k+R} \left[ H(\theta_k - \theta_j) - H(0) \right]$$

With $T$-periodic phase interaction curve, $H$

$$H(\psi) = \frac{1}{T} \int_0^T \frac{p(t-\psi) - (1-U^2_a(t))q(t-\psi)}{(1-U^2_a(t))(U_a(t)+a)} dt - \frac{1}{T} \sum_{m=1}^{2} 2c_m p(t_m - \psi)$$
...and

\[ p(t) = U_a(t) \cos \phi + V_a(t) \sin \phi \]

\[ q(t) = -U_a(t) \sin \phi + V_a(t) \cos \phi \]

In *Chimera states as chaotic spatiotemporal patterns*, authors Oleh E. Omel’chenko, Matthias Wolfrum, and Yuri L. Maistrenko showed that chimera states can be found in systems of the form of this phase reduction, with \( H(\psi) = \sin(\psi + \alpha) \) if phase lag parameter \( \alpha \) is close to, but not equal to, \( \pi/2 \)

\( H(\psi) \) can be qualitatively approximated by their Fourier series: (cont. on next slide)
With Fourier coefficients $h_0, h_1^c, h_1^s$ given by

$$h_0 = \frac{2}{T} \int_0^T H(\psi) d\psi$$

$$h_1^c = \frac{2}{T} \int_0^T H(\psi) \cos \left(\frac{2\pi}{T} \psi + \alpha\right) d\psi$$

$$h_1^s = \frac{2}{T} \int_0^T H(\psi) \sin \left(\frac{2\pi}{T} \psi\right) d\psi$$
All this yields approximate equations for phase-lag parameter $\alpha$:

\[
\cos \alpha = \frac{h_1^s}{\sqrt{(h_1^c)^2 + (h_1^s)^2}}
\]

\[
\sin \alpha = \frac{h_1^c}{\sqrt{(h_1^c)^2 + (h_1^s)^2}}
\]
As previously stated, it has been shown that chimera states can be generated in systems of the form

$$H(\psi) = \sin(\psi + \alpha)$$

If $\alpha$ is close to, but not equal to $\pi/2$.

Looking at the equations for $\alpha$ shown on the previous slide, we can see that $\alpha$ has a dependence on the parameters $a$ and $\phi$.

Therefore, we can use these equations for $\alpha$ to pinpoint regions in the parameter space that favor chimeras.
Pinpointing Chimeras

- In this manner, the authors show that chimeras are expected for a pronounced off diagonal coupling \( \phi \approx \frac{\pi}{2} \), but not for diagonal coupling \( \phi \approx 0 \) or \( \phi \approx \pi \).
- The authors confirm this via numerical simulation.
- To right, the hatched area shows parameter values for which the authors observe chimera states in their simulations.
  - *initial conds. randomly distributed on circle \( u^2 + v^2 = 4 \)
• In the figure to the right, we can clearly distinguish between coherent and incoherent regions.

• Fig 2(c) shows mean phase velocities of all the oscillators, calculated as

\[ \omega_k = \frac{2\pi M_k}{\Delta T} \]

• Where \( M_k \) is the number of complete rotations over an interval \( \Delta T \).
Local order parameter

- The spatial coherence of the oscillators can also be expressed in terms of a local order parameter:

  \[ Z_k = \left| \frac{1}{2\delta} \sum_{|j-k| \leq \delta} e^{i\Theta_j} \right|, \quad k = 1, \ldots, N, \]

- Where \( \Theta_j = \arctan\left( \frac{v_j}{u_j} \right) \) is geometric phase of the \( j^{th} \) oscillator.

- Authors also use a spatial average, with window size of \( \delta = 25 \) elements.

- A value \( Z_k = 1 \) indicates that the \( k^{th} \) unit belongs to the coherent part, and \( Z_k < 1 \) for incoherent parts.

- Figure 2(d) on the previous slide depicts the local order parameter where bright yellow color denotes coherence.
Varying $r$ and $\sigma$

- Having determined suitable values for $\phi$ and $a$ to yield chimeras, the authors then begin looking at the dependence of chimeras on $r$, the coupling radius, and $\sigma$, the coupling strength.
- To the right we can parameter space for $(\sigma, r)$. Different colored regions denote different types of chimera states.
- Near the borders of each of the regimes (hatching), multistability of states is found; the type of chimera depends upon initial conditions.
More Chimeras

• Here we can see spatial patterns and mean phase velocities for various pairs \((r, \sigma)\).

Image included for reference.
Conclusions

- The authors identified non-local off-diagonal coupling as a crucial ingredient for generating chimeras.
- Their findings also strengthen the hypothesized ‘universality’ of chimera states.
- Furthermore, the authors found a new type of chimera consisting of multiple domains of coherence.
  - This behavior is the result of strong coupling interaction, as thus does not exist in simple phase models.
Thank You!
Bibliography

- Omelchenko, Iryna & Omel'chenko, Oleh & Hövel, Philipp & Schöll, Eckehard. (2013). When Nonlocal Coupling Between Oscillators Becomes Stronger: Patched Synchrony or Multi-Chimera States. Physical review letters. 110. 224101. 10.1103/PhysRevLett.110.224101.

- Izhikevich, Eugene M. “Phase Equations for Relaxation Oscillators.” SIAM Journal on Applied Mathematics, vol. 60, no. 5, 2000, pp. 1789-1804. JSTOR, www.jstor.org/stable/3061710. Accessed 16 June 2021.

- Omel'chenko, Oleh & Wolfrum, Matthias & Maistrenko, Yuri. (2010). Chimera states as chaotic spatiotemporal patterns. Physical review. E, Statistical, nonlinear, and soft matter physics. 81. 065201. 10.1103/PhysRevE.81.065201.