General(ized) Hartman effect

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Abstract – In this letter we prove explicitly that if the Hartman effect exists for an arbitrary “unit cell” potential, then it also exists for a periodic system constructed using the same “unit cell” potential repeatedly. We further show that if the Hartman effect exists, the tunneling time in the limiting case of a sufficiently thick “unit cell” potential is the same as that of its periodic system. This is true for any arbitrary value of the intervening gap between the consecutive “unit cell” of the periodic system. Thus, the generalized Hartman effect always occurs for a general potential constructed using multiple copies of single potential which shows Hartman effect.

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Introduction. – The tunneling of a particle from a classically forbidden region is one of the most fundamental and earliest studied problems of quantum mechanics [1–5]. However, how much time a particle takes to tunnel through a potential barrier is still an open problem. In the year 1962, Hartman studied the time taken by a wave packet to cross the classically forbidden region imposed by a metal-insulator-metal sandwich [6]. He applied the stationary phase (SPM) method to calculate the tunneling time. It was found that for large thickness of the classically forbidden region, the tunneling time becomes independent of the thickness. In an independent study, this phenomenon was also confirmed by Fletcher in later years [7]. To present, this paradox is unresolved and is known as Hartman effect. According to SPM, if the transmission coefficient through a potential barrier V(x), 0 ≤ x ≤ b is t(E) = |t(E)|e^(iδ(E)), then the tunneling time \( \tau \) through the potential barrier is

\[
\tau = \delta' + \frac{b}{2k},
\]

where \( \delta' = \frac{d\delta}{dx} \) and the wave vector \( k = \sqrt{E} \) (we have chosen the unit 2m = 1, \( \hbar = 1, c = 1 \)). For the case of a rectangular barrier \( V(x) = V \) for \( 0 ≤ x ≤ b \), the tunneling time \( \tau \) according to SPM yields [8]

\[
\tau = \frac{d}{dE} \tan^{-1} \left( \frac{k^2 - q^2}{2kq} \right),
\]

In the above equation, \( q = \sqrt{V - E} \). We observe that \( \tau \to 0 \) as \( b \to 0 \) as expected. However, when \( b \to \infty \) we note

\[
\lim_{b \to \infty} \tau = \frac{1}{qk},
\]

i.e., tunneling time is independent of the width of the barrier \( b \) for a sufficiently opaque barrier. This is the famous Hartman effect.

Several studies have been conducted by different authors to understand this effect. The numerical monitoring of time evolution of the tunneling wave packet has shown that the tunneling time agrees with that obtained by the stationary phase method [9]. In the case of two opaque barriers separated by a finite distance, it was found that the tunneling time is independent of the separation of the barrier in the limit when the thickness of the barrier is large [10]. This has given rise to the notion of generalized Hartman effect. The phenomenon of the tunneling time being insensitive to the intervening gap of the thick barriers for double- or multi-barrier tunneling is known as generalized Hartman effect. For multi-barrier tunneling it has been shown that the total tunneling time is independent of barrier thickness and inter-barrier separation [11]. For the case of complex potential associated with elastic and inelastic channels, the Hartman effect has been found to occur for the case of weak absorption [12]. For the array of potentials associated with elastic and inelastic channels, the tunneling time saturates with the number of barriers [13].
Several experiments have been conducted to test the finding of the theoretical results on tunneling time [14–20]. The tunneling time in all such experiments was not found to depend upon the thickness of the tunneling region. Experiments were also conducted with double-barrier photonic band gaps [21] and double-barrier optical gratings [19]. The tunneling time was found to be independent of the gap between the two barriers and therefore also to favor the generalized Hartman effect. For critical comments on the generalized Hartman effect we refer to the articles [22–24].

The definition and interpretation of tunneling time is not unique. Some notable definitions of tunneling times are Keldysh time, Buttiker-Landauer time, Eisenbud-Wigner time, Pollack-Miller time, Larmor time, Bohmian time (see [25] for definition and brief review). With the advancement of technologies to probe light-matter interaction at much shorter time scales and with more stable and finer control on experimental parameters, various experiments have been attempted to validate different models of the tunneling times. However the tunneling time remains a matter of debate among the physics community. More precise experiments have been conducted recently such as [26,27]. While [26] appears to rule out all the known definition of tunneling times, the experimental outcome of [27] is in good agreement with the Buttiker-Landauer time. However the experimental work in [27] probes the tunneling macroscopically and therefore at much shorter scale of tunneling problem, the universality of Buttiker-Landauer time is yet to be established. To the best of our knowledge, the current debate on tunneling time is unable to establish a universal definition of tunneling time for quantum regime. Also to the best of our knowledge, the Hartman effect is also not universally ruled out experimentally. As the saturation of tunneling time with barrier thickness $b$ (when calculated by SPM) happens quickly with $b$ and $b$ need not be close to infinity (as shown in fig. 1), the Hartman effect may exist for a finite range of $b$.

The studies of tunneling time from multiple-barrier systems [10,11] which has led to the concept of the generalized Hartman effect are conducted by taking a “unit cell” system as rectangular barrier which is known to show Hartman effect. Our motivation for the present work aims to understand whether this is a general phenomenon or not. In other words, if an arbitrary “unit cell” system shows Hartman effect in the limit of increasing thickness, then whether the periodic system constructed using multiple copies of the “unit cell” will also show Hartman effect in the same limit. This leads to the question: does the occurrence of the Hartman effect from a potential also imply the occurrence of the generalized Hartman effect? We found that this is always the case. Our result will be useful to study the Hartman effect for many other complicated potentials.

We organize this letter as follows: in the next section, we establish the main result of this work that the generalized Hartman effect always occurs for a potential that displays Hartman effect. In the third section we discuss the implications of this result.

**Generalized Hartman effect.** – In this section we prove that the generalized Hartman effect always exists for a potential that displays Hartman effect. Consider that the transmission coefficient for an arbitrary potential $V(x)$ confined over the length $b$ is expressed as

$$t_1 = \frac{1}{M_1},$$

where $M_1 = \sqrt{v e^{-i\delta}}$, $v \in R^+$ and $\delta \in R$. $\frac{1}{v}$ is the transmission amplitude and $\delta$ is the phase of transmission coefficient. If this “unit cell” potential $V(x)$ displays Hartman effect, then the following quantity will be independent of $b$ in the limit $b \to \infty$:

$$\lim_{b \to \infty} \left( \delta' + \frac{b}{2k} \right) = \tau_0.$$  

$\tau_0$ is the tunneling time from the potential $V(x)$ in the limit $b \to \infty$. From the knowledge of $M_1$, the transmission coefficient for a periodic system made by $N$ repetitions of $V(x)$ is given by [28]

$$t_N = \frac{e^{-ikNs}}{U_N^{-1}(\chi) - U_N^{-2}(\chi)},$$  

where

$$\chi = \sqrt{v} \cos(\delta + ks).$$

In the above $s = b + L$, where $L$ is the intervening gap between the two consecutive “unit cell” potential of the periodic system. From eq. (6), the phase of $t_N$ is given by

$$\Phi = \phi_N - kNs,$$

where

$$\phi_N = \tan^{-1} \left[ \frac{\sqrt{v} \sin(\delta + ks) U_N^{-1}(\chi)}{U_N^{-2}(\chi)} \right].$$
The tunneling time according to SPM to traverse the length $(N-1)s + b$ of the periodic system is

$$\tau_N = \left( \phi_N' - \frac{Ns}{2k} \right) + \frac{(N-1)s + b}{2k}. \quad (10)$$

In the above $\phi_N' = \frac{d\phi_N}{dE}$ and the term in the parenthesis is the phase delay time. Equation (10) is simplified to

$$\tau_N = \phi_N' - \frac{s}{2k} + \frac{b}{2k}. \quad (11)$$

To investigate the generalized Hartman effect, we calculate the limiting value of $\phi_N'$ when $b \to \infty$. As $b$ and $E$ are two independent quantities related to the tunneling problem, we write

$$\lim_{b \to \infty} \phi_N' = \frac{d}{dE} \left( \lim_{b \to \infty} \phi_N \right). \quad (12)$$

We note that for $b \to \infty$ the transmission from the potential $V(x)$ is expected to vanish, i.e., $t_1 \to 0$. This implies $v \to \infty$ when $b \to \infty$. Thus, for $b \to \infty$ we have $|\chi| \to \infty$ from eq. (7) provided $(\delta + ks) \neq (2p+1)\frac{\pi}{2}, p \in \{0, 1^\ast \}$. For $b \to \infty$, the condition of $(\delta + ks)$ being an odd integer multiple of $\frac{\pi}{2}$ never satisfies “for those potentials that display Hartman effect”. Therefore, our initial assumption on the “unit cell” displaying Hartman effect, we always have $\lim_{b \to \infty} |\chi| \to \infty$. Therefore, by our initial assumption on the “unit cell” displaying Hartman effect, we always have $\lim_{b \to \infty} |\chi| \to \infty$.

Using the above result in the expression of $\phi_N$ (eq. (9)) we get

$$\lim_{\chi \to \infty} \frac{U_{N-1}(\chi)}{T_N(\chi)} = \lim_{\chi \to \infty} \frac{U_{N-1}(\chi)}{T_N(\chi)} \cdot \frac{1}{\chi}. \quad (13)$$

Equation (15) proves the generalized Hartman effect for the periodic potential system. The tunneling from the periodic system is independent of $L$ and is equal to the tunneling time of the “unit cell” system in the limit $b \to \infty$.

Our derived result are due to the limiting value of the ratio $U_{N-1}(\chi)/T_N(\chi) = \chi^{-1}$ when $\chi \to \infty$. The limit $\lim_{b \to \infty} \chi(b, L) \to \infty$ is due to $v \to \infty$ when $b \to \infty$. As $\chi(b, L)$ has only trigonometric dependence on $L$ (as $v$ is independent of $L$), $\chi(b \to \infty, L) \to \infty$ for any fixed value of $L$, finite or infinite. This is also true even if $L$ is variable with $b$, i.e., $L = f(b)$ for an arbitrary function $f$ which can be finite or infinite as $b \to \infty$. Therefore, eq. (15) is always true for any value of $L$ in the limit $b \to \infty$.

This is to be noted that the results represented by eq. (15) have been found in [10,11] for real “unit cell” potential and in [29,30] for $PT$-symmetric “unit cell” potential. In all these cases, the “unit cell” potentials showed Hartman effect. Thus, our result, that is if a “unit cell” potential shows Hartman effect, then its periodic system will also show Hartman effect, was already found by different authors for specific cases of “unit cell” potential. Moreover, its quantitative representation by eq. (15) was also noted by these authors. In the present letter, we have shown this as a general result.

**Results and discussions.** – We have given the proof that if a “unit cell” potential system displays Hartman effect, then the generalized Hartman effect always occurs for the periodic system made using the multiple copies of the same “unit cell” potential repeatedly. We have given the proof for an arbitrary $N$ repetitons of the “unit cell” system. This result has several implications in understanding the tunneling time by stationary phase method. As a special case we take the example of a rectangular barrier which is known to display Hartman effect. If the rectangular system is arranged in a super-periodic fashion (in which a periodic system repeats periodically and the new generated system further repeats periodically and this operation continues for arbitrary times [31]), the generated arrangement will show generalized Hartman effect. Similarly, the case of fractal potential such as Cantor-1/3, general Cantor, Smith-Volterra-Cantor (SVC) potential, general SVC potential, etc., will also show generalized Hartman effect.

We have shown that for arbitrary $N$ repetitions of the “unit cell” potential, the tunneling time for the periodic potential is the same as that of its “unit cell” potential when the “unit cell” potential is sufficiently thick. This quantitative result has a simple qualitative interpretation. When the thickness of the “unit cell” potential is much larger as compared to the intervening gap between the two “unit cell” potential, the incident wave packet begins to see the periodic potential system as a single potential system. As the tunneling time from the “considered” single potential system (unit cell) is independent of thickness for large thickness, the net tunneling time is the same as that of the single potential system. Therefore, the generalized Hartman effect is not a distinct feature of a specific periodic system and can be considered as the Hartman effect. However, why a potential system displays Hartman effect is a paradoxical result and needs more investigations.

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**References**

[1] Nordheim L., *Proc. R. Soc. A*, 119 (1928) 173.
[2] Gurney R. W. and Condon E. U., *Nature*, 122 (1928) 439.
[3] Condon E. U., *Rev. Mod. Phys.*, 3 (1931) 43.
[4] Wigner E. P., *Phys. Rev.*, 98 (1955) 145.

20001-p3
[5] Bohm D., *Quantum Theory* (Prentice-Hall, New York) 1951.

[6] Hartman T. E., *J. Appl. Phys.*, **33** (1962) 3427.

[7] Fletcher J. R., *J. Phys. C*, **18** (1985) L55.

[8] Dutta Roy B., *Elements of Quantum Mechanics* (New Age Science Ltd.) 2009.

[9] Aquino V. M., Navarro A., Goto M. and Iwamoto H., *Phys. Rev. A.*, **58** (1998) 4359.

[10] Dutta Roy B., *Elements of Quantum Mechanics* (New Age Science Ltd.) 2009.

[11] Olkhovsky V. S., Recami E. and Salesi G., *Europhys. Lett.*, **57** (2002) 879.

[12] Steinberg A. M., Kwiat P. G. and Chiao R. Y., *Phys. Rev. E.*, **67** (2003) 016609.

[13] Ghatak A., Hasan M. and Mandal B. P., Hartman-Fletcher effect for array of complex barriers, arXiv:1505.03163.

[14] Nimtz G., Spieker H. and Brodowsky H. M., *Phys. Lett. A*, **222** (1996) 125.

[15] Balcou Ph. and Dutriaux L., *Phys. Lett. A*, **78** (1997) 851.

[16] Grifiths D. J. and Steinkauf A. C., *Am. J. Phys.*, **69** (2001) 137.

[17] Ragni L., *Phys. Rev. E*, **79** (2009) 046609.

[18] Sattari F. and Faizabadi E., *AIP Adv.*, **2** (2012) 12123.

[19] Longhi S., Marano M., Laporta P. and Belmonte M., *Phys. Rev. E*, **64** (2001) 055602.

[20] Olindo C., Sagioro M. A., Matinaga F. M., Delgado A., Monken C. H. and Padua S., *Opt. Commun.*, **272** (2007) 161.

[21] Longhi S., Laporta P., Belmonte M. and Recami E., *Phys. Rev. E*, **65** (2002) 046610.

[22] Kudaka S. and Matsumoto S., *Phys. Lett. A*, **375** (2011) 3259.

[23] Milanovic V. and Ranovacic J., *Phys. Lett. A.*, **376** (2012) 1401.

[24] Kudaka S. and Matsumoto S., *Phys. Lett. A*, **376** (2012) 1403.

[25] Sainadh U. S., Sang R. T. and Litvinyuk I. V., *J. Phys. Photon.*, **2** (2020) 042002.

[26] Sainadh U. S., Xu H., Wang X., Noor A. A., Wallace W. C., Douguet N., Bray A., Ivanov I., Bartasch K., Kheifets A., Sang R. T. and Litvinyuk I. V., *Nature*, **568** (2019) 75.

[27] Ramos R., Spierings D., Racicot I. and Steinberg A. M., *Nature*, **583** (2020) 529.

[28] Ghiaouthakis D. J. and Steinkauf A. C., *Am. J. Phys.*, **69** (2001) 137.

[29] Hasen S. and Mandal B. P., *Eur. Phys. J. Plus*, **135** (2020) 84.

[30] Hasen S. and Mandal B. P., *Eur. Phys. J. Plus*, **135** (2020) 640.

[31] Hasen S. and Mandal B. P., *Ann. Phys.*, **391** (2018) 240.