A STABLE COHOMOTOPY REFINEMENT OF
SEIBERG-WITTEN INVARIANTS: I

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Abstract. The monopole map defines an element in an equivariant stable cohomotopy
group refining the Seiberg-Witten invariant. Part I discusses the definition of this stable
homotopy invariant and its relation to the integer valued Seiberg-Witten invariants. A
gluing theorem for these invariants, proved in part II, gives new results on diffeomorphism
types of decomposable four-manifolds.

1. Introduction

In the year 1994, Nathan Seiberg and Edward Witten found new differential invariants [8]
of four-manifolds. The basic geometric ingredient in their construction was a space well
known to physicists: The moduli space of monopoles. This space is defined for a closed
Riemannian four-manifold \((X, g)\), which is \(K\)-theory oriented (or equivalently equipped
with both an orientation and a \(spin^c\)-structure). Under certain genericity assumptions, this
moduli space is a smooth, orientable finite dimensional manifold and comes with a natural
complex line bundle. The Seiberg-Witten invariant is the characteristic number obtained
from these data by specifying an orientation of the moduli space.

In [3], a new point of view was introduced to monopolized four-manifold theory: Instead of
the moduli space of monopoles, consider the monopole map! The monopole map \(\mu = \mu_g\) is
an \(\mathbb{T}\)-equivariant map between certain affine Hilbert spaces. The moduli space of monopoles
is obtained as the quotient of the zero-set of \(\mu\) by the action of the group \(\mathbb{T}\) of complex
numbers of unit length.

The present article carries this point of view a little further. In a finite dimensional analogue,
the map \(\mu\) as a proper map between finite dimensional Hilbert spaces would extend to
the one-point compactifications, thus defining an element in some equivariant version of
the stable homotopy groups of spheres. This stable homotopy element by the Pontrijagin-
Thom construction would encode the moduli space as an element in some sort of equivariant
framed bordism group.

With a grain of salt, this concept actually carries over to the infinite dimensional setting.
Salt is provided by a theorem of A. S. Schwarz [5], which seems to have fallen into oblivion
during the past decades. Applying his ideas to the monopole map leads to the following
theorem.
Theorem 1.1. The monopole map $\mu$ defines an element in an equivariant stable cohomotopy group

$$\pi_{b,H}^{b}(\text{Pic}^{0}(X); \text{ind}(D)),$$

which is independent of the chosen Riemannian metric. For $b > \dim(\text{Pic}^{0}(X)) + 1$, a homology orientation determines a homomorphism of this stable cohomotopy group to $\mathbb{Z}$, which maps $[\mu]$ to the integer valued Seiberg-Witten invariant.

Here $\text{Pic}^{0}(X)$ denotes the Picard torus $H^{1}(X; \mathbb{R})/H^{1}(X; \mathbb{Z})$. The Dirac operator associated to the chosen $\text{spin}^c$-structure defines a virtual complex index bundle $\text{ind}(D)$ over the Picard torus, and $b = b_{+}(X)$ denotes the dimension of a maximal linear subspace $P$ of the second de Rham group of $X$, on which the cup product pairing is positive definite. An orientation of $P \times \text{Pic}^{0}(X)$ is called a homology orientation. It determines an orientation of the monopole moduli space. The suffix $H$, finally, stands for a universe for the $T$-action, that is a Hilbert space with an orthogonal $T$-action. Its finite dimensional invariant linear subspaces provide the suspension coordinates in the construction of equivariant cohomotopy groups.

The stable cohomotopy invariant as formulated in the theorem does not capture all the features of Seiberg-Witten theory. In the case $b = 1$, $\text{Pic}^{0}(X) = 0$, for example, the above groups are vanishing. The Seiberg-Witten invariants, however, are nonvanishing in general, but depend in a well understood manner on the Riemannian metric and on an additional perturbation parameter.

Indeed it is possible to recapture this phenomenon in the stable homotopy setting at the expense of losing convenient algebraic structure: One may consider $\mu$ up to equivariant homotopy relative to the fixed point set. In case $b > \dim(\text{Pic}^{0}(X)) + 1$ the resulting relative homotopy classes are one-to-one with the homotopy classes which form the stable cohomotopy groups above. In general the sets of such stable relative homotopy classes don’t admit a natural group structure. The relative homotopy class of $\mu$ as an element in such a set turns out to be independent of the parameters used in the definition of the Seiberg-Witten number. The reason is that the restriction of the monopole map to the $T$-fixed point set is quite special: It is affine linear.

So, how can a discrete invariant (as the Seiberg-Witten number) of any sort of homotopy class jump along a continuous path of parameters? To find an answer, one has to carefully distinguish between parameters: One parameter, the metric, will move $\mu$ within its relative homotopy class. The second, problematic parameter is used in the definition of a map from the set of relative homotopy classes to the integers, which associates to the class of $\mu$ a number. Seiberg-Witten theory defines such maps by performing an oriented count of orbits in the preimage of generic fixed points in the target space of $\mu$. However, there is no natural choice of a generic fixed point. The affine linear map $\mu^{T}$ on the fixed point set maps onto a real hyperplane which divides the $T$-fixed point set in the target space.
COHOMOTOPY INVARIANTS: I

into two “chambers”. The oriented count has different results for generic points in different chambers.

Let’s illustrate this phenomenon in a characteristic example: View the spinning globe as a two-sphere with an $\mathbb{T}$-action and choose the north pole as base point. As a target space, take a one-sphere with trivial action and choose two points on this one-sphere as “poles”, the north pole again as base point. Based equivariant maps from the spinning globe to the one-sphere are determined by their restriction to a latitude, which as an arc is a contractible space. So there is only the trivial homotopy class of equivariant such maps.

In contrast, consider equivariant maps, which take north and south pole to north and south pole, respectively. Such a map basically wraps a latitude $n + \frac{1}{2}$ times around the one-sphere. This wrapping number classifies the “relative homotopy” class of such a map. Choosing a generic point in the one-sphere, the oriented count of preimages in a fixed latitude defines in a natural way a map of the set of relative homotopy classes to the integers. This oriented count, however, depends on the choice of the generic point. It changes by $\pm 1$, if the generic point is chosen in the “other half” of the one-sphere.

In the case of a spin-manifold, the monopole map is $Pin(2)$-equivariant. It thus defines an element in a $Pin(2)$-equivariant cohomotopy group. One can recover the $\frac{10}{8}$-theorem of [9] in this setup: In the case of vanishing first Betti number, there is a fixed point homomorphism from this $Pin(2)$-equivariant group to the 0-th stable stem, i.e. to $\mathbb{Z}$. It is shown in [14] that surjectivity of this fixed point homomorphism is equivalent to the existence of equivariant maps as used in the proof of [9].

Donaldson’s theorem on the diagonalizability of the quadratic form on manifolds with negative definite intersection form follows from number theory and from a well known result on $\mathbb{T}$-equivariant maps between spheres.

Addendum: The stable homotopy invariant in the theorem was found independently by both authors. For various reasons, publication of the corresponding preprints [2] and [10] was unduly delayed. As a consequence, we finally opted for a joint publication.

2. Fredholm maps and stable homotopy

A Fredholm map $f : H' \to H$ in this paper will be a compact perturbation of a linear Fredholm operator between separable Hilbert spaces. This means that $f$ is of the form $f = l + c$, where $l$ is linear Fredholm and the continuous map $c$ maps bounded sets to subsets of compact sets.

A theorem, which goes back to A. S. Schwarz [15], associates to certain such Fredholm maps stable homotopy classes of maps between finite dimensional spheres: Let $D' \subset H'$ be a disc with boundary $\partial D'$. Two continuous maps $f_i : \partial D' \to H \setminus \{0\}$ are called compactly homotopic relative to $l$, if there is a continuous and compact map $c : \partial D' \times [0,1] \to H$ with
\[
fi = l + ci \text{ for } i \in \{0,1\} \quad \text{and} \quad \left[f_i(h') \right] = l(h') + ci(h') \neq 0
\]
for all \(t \in [0,1]\) and \(h' \in \partial D'\).

**Theorem 2.1.** The compact homotopy classes of continuous Fredholm maps relative to \(l\) are in one-to-one correspondence with elements of the stable homotopy group \(\pi_{\text{ind}}^{st}(S^0)\) of the sphere.

This correspondence can be described as follows: Any compact map \(c\) on the bounded set \(\partial D'\) can be uniformly approximated by maps \(c_n\) mapping to finite dimensional linear subspaces \(V_n \subset H\) containing \(\text{im}(l)^\perp\). The correspondence associates to \(l + c\) the maps \(f_n/||f_n||\) with \(f_n\) being the restriction of \(l + c_n\) to \(l^{-1}(V_n) \cap \partial D'\). A detailed proof of this theorem can be found in [3], p.257f.

In this paper, this concept will be used with a few modifications: Firstly, we will consider equivariant maps, which are furthermore parametrized over some space. Secondly, we will consider Fredholm maps which extend continuously to maps

\[
f^+ : H'^+ \to H^+
\]

between the one-point completed Hilbert spheres. Equivalently, we suppose \(f\) to satisfy a boundedness condition: The preimages of bounded sets are bounded.

The second condition is convenient and avoids clumsy notation: The construction of [15] considers maps between pairs \((D', \partial D') \to (H, H \setminus \{0\})\) of spaces. The boundedness condition makes it possible to instead consider maps \(H'^+ \to H^+\) between spheres equipped with a natural base point. As in finite dimensions, both points of view are equivalent for maps which satisfy the boundedness condition. This is due to the fact that starting from the pairs \((H'^+, +)\) and \((H^+, +)\) one gets the pairs \((D', \partial D')\) and \((H, H \setminus \{0\})\) via homotopy equivalence and excision with intermediary pairs \((H'^+, H'^+ \setminus \bar{D'})\) and \((H^+, H^+ \setminus \{0\})\). In the monopole setup the boundedness condition is satisfied.

First a short discussion of the boundedness condition: In finite dimensions this condition is equivalent to \(f\) being proper, i.e. \(f\) is closed and the preimage of any point in the target space is compact. Here is a proof that in the setting of Fredholm maps the boundedness condition implies properness:

**Lemma 2.2.** Let \(l : H' \to H\) be a continuous linear Fredholm map between Hilbert spaces and let \(c : H' \to H\) be a compact map. Then the restriction of the map \(f = l + c\) to any closed and bounded subset \(A' \subset H'\) is proper.

In particular, if preimages of bounded sets in \(H\) under the map \(f\) are bounded, then \(f\) is proper and extends to a proper map \(f^+ : H'^+ \to H^+\) between the one point completions.
Lemma 2.3. There are finite dimensional linear subspaces $V \subset H$, such that the following statements hold:
at least 1 − f the homotopy. At starting time, it is mapped to ε to 1, and furthermore, in an in the direct system of finite dimensional subspaces in H

\[ \text{Let} \]

Indeed, if \( H \) is separable, then the subspaces \( V \) satisfying these three conditions are cofinal in the direct system of finite dimensional subspaces in \( H \).

**Proof.** The preimage \( f^{-1}(D) \) of the unit disc \( D \) in \( H \) is bounded in \( H' \). So the closure \( C \) of its image under the compact map \( c \) is compact in \( H \). Cover \( C \) by finitely many balls with radius \( \varepsilon \leq \frac{1}{4} \), centered at points \( v_i \). Together with the orthogonal complement to the image of the linear Fredholm map \( l \), these points \( v_i \) span a finite dimensional linear subspace \( V \) of \( H \). Let’s check the second condition: Suppose \( w \in S(W^+) \) is in the image of \( f|_{W^+} \). Then \( f^{-1}(w) \cap W^+ \subset f^{-1}(D_1(H)) \) will be mapped by \( f|_{W'} = (l + c)|_{W'} \) to a subspace of \( W + C \). So \( w \) will be contained both in \( S(W^+) \) and \( W + C \). However, these two subsets of \( H \) are at least \( 1 - \varepsilon \geq \frac{\sqrt{3}}{4} \) apart.

We will identify \( W' \) with the orthogonal sum \( U \perp V' \) via the map

\[ w' \mapsto (l \circ (1 - pr_{V'})(w'), pr_{V'}(w')). \]

To prove the last claim, it suffices to show that \( \text{id}_{U'} \land \rho_{V'} f|_{W'} \) and \( f|_{W'} \) are homotopic as maps \( W'^+ \to H^+ \setminus S(W^+) \). Let \( D' \subset H' \) be a disk, centred at the origin, which contains the preimage \( f^{-1}(D) \) of the unit disk in \( H \). Consider the homotopy \( h : D' \times [0, 3] \to H^+ \setminus S(W^+) \), defined by:

\[
h_t = \begin{cases} 
  l + ((1 - t)id_H + t \cdot pr_V) \circ c & \text{for } 0 \leq t \leq 1, \\
  l + pr_V \circ c \circ ((2 - t)id_{V'} + (t - 1)pr_{V'}) & \text{for } 1 \leq t \leq 2, \\
  pr_{V'} \circ l + ((3 - t)pr_V + (t - 2)\rho_V) \circ (l + c) \circ pr_{V'} & \text{for } 2 \leq t \leq 3.
\end{cases}
\]

Note that the image during the homotopy stays within an \( \varepsilon \)-neighbourhood of \( W \). The homotopy is chosen in such a way that the image of the sphere \( S' = \partial D' \cap W' \) during the homotopy stays away not only from the unit sphere \( S(W^+) \) in \( W^+ \), but from the whole of \( W^+ \). Before we check this, let’s consider the consequences: Since \( H^+ \setminus (D \cap W^+) \) is contractible, the homotopy \( h_t \) can be extended to the complement of \( D' \cap W' \) in \( W'^+ \), thus defining a homotopy as claimed.

Let \( s' \) be an element in the sphere \( S' \). We will track the path its image will take during the homotopy. At starting time, it is mapped to \( f(s') \), which is of norm greater or equal to 1, and furthermore, in an \( \varepsilon \)-neighbourhood of \( W \). In particular, its distance from \( W^+ \) is at least \( 1 - \varepsilon \geq \frac{\sqrt{3}}{4} \). During the first part of the homotopy, the image will move at most a distance of \( \varepsilon \), so it will definitely stay away from \( W^+ \).
From time \( t = 1 \) on one has \( pr_U \circ h_t(s') = pr_U \circ l(s') \). Since \( pr_U(W^\perp) = 0 \) by definition, we are reduced to checking the case \( pr_U \circ l(s') = 0 \), that is for \( s' \in S' \cap V' \). But for such an element, the image during the second part of the homotopy stays fixed and during the third part moves on a straight line between \( pr_V(f(s')) \) and \( pr_V(f(s')) \), which are nonzero vectors in \( V \), differing by a positive real factor. This concludes the proof of 2.3.

In particular, the restrictions \( f|_{l^{-1}(V)} \) to finite dimensional linear subspaces \( V \subset H \) as in 2.3 together define an element in the colimit of pointed homotopy classes

\[
[f] = \colim_{V \subset H}[(f|_{l^{-1}(V)})^+] \in \colim_{V \subset H}[(l^{-1}(V))^+, H^+ \setminus S(V^\perp)].
\]

The homotopy equivalences \( V^+ \subset (H^+ \setminus S(V^\perp)) \) combine to an isomorphism

\[
\pi_{\text{ind}l}^S(S^0) = \colim_{V \subset H} [(l^{-1}(V))^+, V^+] \xrightarrow{\sim} \colim_{V \subset H} [(l^{-1}(V))^+, H^+ \setminus S(V^\perp)].
\]

In this way \([f]\) can be identified as an element in the stable homotopy group \( \pi_{\text{ind}l}^S(S^0) \):

**Corollary 2.4.** Let \( f = l + c : H' \to H \) be a compact perturbation of the linear Fredholm map \( l \) such that the preimages of bounded sets under the map \( f \) are bounded. Then \( f \) defines an element \([f] \in \pi_{\text{ind}l}^S(S^0)\). □

In the construction above the linear map \( l \) seems to play an essential rôle. In fact it will turn out that the homotopy class \([f]\) basically is independent of the choice of decomposition of \( f \) as a sum \( f = l + c \). In order to show this, we will have to consider a parametrized version of the above situation and reach back some way:

Let \( Y \) be a finite CW-complex. The group \( KO^0(Y) \) can be described as follows (cf. [16]):

A (real) Hilbert bundle over \( Y \) is a locally trivial fiber bundle with fiber a separable Hilbert space \( H \), whose structure group is the group of linear isometric bijections, equipped with the norm topology. A cocycle \( \lambda = (E', l, E) \) over \( Y \) consists of two Hilbert bundles over \( Y \) and a Fredholm morphism \( l : E' \to H \) between them. Here a Fredholm morphism is a continuous map which is fiber preserving and fiberwise linear Fredholm over \( Y \). Two cocycles \( \lambda_i \) over \( Y \) for \( i \in \{0, 1\} \) are homotopic, if there is a cocycle \( \lambda \) over \( Y \times [0, 1] \) such that the restriction \( \lambda|_{Y \times \{i\}} \) is isomorphic to \( \lambda_i \). A cocycle \( (E', l, E) \) is trivial, if \( l \) is invertible. Two cocycles \( \lambda_0 \) and \( \lambda_1 \) are equivalent, if there is a trivial cocycle \( \tau \) such that \( \lambda_0 \oplus \tau \) and \( \lambda_1 \oplus \tau \) are homotopic. The group \( KO^0(Y) \) is the set of equivalence classes of cocycles with addition given by the Whitney sum of cocycles.

Let \( f : E' \to E \) be a continuous map between Hilbert bundles of the form \( f = l + c \), where \( \lambda = (E', l, E) \) is a cocycle over \( Y \) and \( c \) is fiber preserving and compact, i.e. maps bounded disk bundles in \( E' \) to subspaces in \( E \), which are proper over \( Y \). Let’s call such a map \( f \) a *Fredholm map over \( Y \).* The boundedness condition in this parametrized situation reads: The preimages of bounded disk bundles are contained in bounded disk bundles. An equivalent
condition is: The Fredholm map over $Y$ extends to the fiberwise one-point completions of $E'$ and $E$.

Every Hilbert bundle over the compact space $Y$ is trivial, i.e. $E \cong Y \times H$ by the theorem of Kuiper [13]. The boundedness condition on $f$ thus translates to the condition that the composed map $pr_H \circ f : E' \to H$ extends to the one-point completions, defining a continuous map

$$(pr_H \circ f)^+ : T(E') \to H^+$$

from the Thom space of the Hilbert bundle $E'$ to the Hilbert sphere $H^+$.

The stage is now set for the definition of stable cohomotopy groups with coefficients: Let $\lambda$ be a finite dimensional virtual vector bundle over $Y$. Suppose we are given a presentation $\lambda = F_0 - F_1$ with vector bundles $F_i$ such that $F_1 \cong Y \times V$ is a trivial vector bundle with $V$ a finite dimensional linear subspace of a Hilbert space $H$. With $TF_0$ denoting the Thom space of the bundle $F_0$, stable cohomotopy groups are defined as the colimits

$$\pi^n_H(Y; \lambda) = \colim_{U \subset \mathbb{V}^+} [U^+ \wedge TF_0 , U^+ \wedge V^+ \wedge S^n]$$

of pointed homotopy classes of maps, where the colimits are over the finite dimensional linear subspaces $U \subset V^+ \subset H$ and $W = U + V \subset H$, respectively. Here the connecting morphism for $W \subset W_1$ with $U_1 = W^\perp \cap W_1$ is the suspension map $(\text{id}_{U_1^+} \wedge \cdot)$. The symbol $T \lambda$ stands not anymore for a space, but for a spectrum.

The reason for keeping the Hilbert space $H$ in the notation lies in the equivariant version: For a compact Lie group $G$ we fix a $G$-universe $H$, i.e. a real Hilbert space $H$ equipped with an orthogonal $G$-action such that $H$ contains the trivial representation and, furthermore, the space of equivariant morphisms $\text{Hom}_G(V, H)$ for a real $G$-module $V$ either is zero or infinite dimensional. Let $\lambda = F_0 - F_1$ be a virtual equivariant vector bundle over a finite $G$-CW complex $Y$ such that $F_1 \cong Y \times V$ is a trivial bundle with $V \subset H$ a finite dimensional $G$-representation. Stable equivariant cohomotopy groups are the colimits

$$\pi^n_{G,H}(Y; \lambda) = \colim_{U \subset \mathbb{V}^G} [U^+ \wedge TF_0 , U^+ \wedge V^+ \wedge S^n]^G$$

of pointed equivariant homotopy classes of maps, where the colimit now is over the finite dimensional subrepresentations $U \subset V^+ \subset H$ and $W = U + V \subset H$, respectively. This definition of stable equivariant cohomotopy groups differs a little from the usual one as we allow for coefficients $\lambda$ in the equivariant $KO$-group $KO_G^0(Y)$ and our universe $H$ need not contain all irreducible representations.

Let $f : E' \to E$ be a $G$-equivariant Fredholm map between $G$-Hilbert space bundles over the finite $G$-CW complex $Y$ such that $E \cong Y \times H$ is a trivialised bundle. Let $f = l + c$
be a presentation of \( f \) as a sum of a linear Fredholm morphism and a compact map. For sufficiently large linear \( G \)-subspaces \( V \subset H \), the cocycle \( \lambda = (E', l, E) \) admits a presentation as virtual index bundle
\[
\lambda = F_0(V) - F_1(V)
\]
with equivariant vector bundles \( F_0(V) = (\text{pr}_H \circ l)^{-1}(V) \subset E' \) and \( F_1(V) = Y \times V \). The following lemma parallels 2.3. Its proof is omitted, as it is almost verbatim the same.

**Lemma 2.5.** There exist finite dimensional linear \( G \)-subspaces \( V \subset H \) such that the following hold:

1. For every \( y \in Y \), the subspace \( V \) is mapped onto \( \text{coker} \ (l_y : E'_y \to H) \). In particular, \( F_0(V) \) is a bundle over \( Y \) and \( \lambda = F_0(V) - F_1(V) \) represents the virtual index bundle \( \text{ind}(l) \).
2. For any \( G \)-linear \( W = W' + V \) with \( W' \subset V^\perp \), the restricted map \( f(W)^+ = (\text{pr}_H \circ f)|_{F_0(W)} : TF_0(W) \to H^+ \) misses the unit sphere \( S(W^\perp) \).
3. The maps \( \rho_W f(W)^+ \) and \( \text{id}_{W'^+} \wedge \rho_V f(V)^+ \) are \( G \)-homotopic as pointed maps
\[
F_0(W)^+ \cong W'^+ \wedge F_0(V)^+ \to W'^+ \wedge V^+ = W^+.
\]

**Theorem 2.6.** An equivariant Fredholm map \( f = l + c : E' \to E \) between \( G \)-Hilbert space bundles over \( Y \) with \( E \cong Y \times H \), which extends continuously to the fiberwise one-point completions, defines a stable cohomotopy Euler class
\[
[f] \in \pi^0_{G,H}(Y; \text{ind}
\]
This Euler class is independent of the presentation of \( f \) as a sum.

**Proof.** The only statement left to prove is the final one. Note that the restriction maps
\[
\pi^0_{G,H}(Y \times [0, 1], \lambda) \to \pi^n_{G,H}(Y \times \{i\}, \lambda|_{Y \times \{i\}})
\]
are isomorphisms. Thus a homotopy of cocycles naturally induces an isomorphism of the corresponding cohomotopy groups. (An extension of this statement to equivalences of cocycles needs further discussion of universes; it seems unnecessary in the present context.) If \( f = l + c = l' + c' \) are two different presentations as a sum, then the constant homotopy \( f = f_t = (1-t)(l+c) + t(l'+c') \) defines an Euler class in the cohomotopy group of \( Y \times [0, 1] \), which restricts for \( i \in \{0, 1\} \) to the Euler classes defined via the respective presentations of \( f \). \( \square \)

**2.7. Remarks.**

- Indeed any element in \( \pi^0_{G,H}(Y; \text{ind}
\]
where the preimage of the basepoint consists only of the base point. Now take the
smash product with the identity on an infinite dimensional Hilbert sphere and remove the base point.

- The stable cohomotopy Euler class has been defined and investigated by Crabb and Knapp [1]. It is related to the standard Euler class the following way: A section of an oriented vector bundle $\xi$ over $Y$ can be regarded as a map $\sigma : Y \times \mathbb{R}^0 \to \xi$. Choosing an bundle isomorphism $\xi \oplus \eta \cong Y \times \mathbb{R}^n$, this section and the projection to the fibers of a trivialized bundle together define a map $(\sigma + id_{\eta})^+ : \eta^+ \to (Y \times \mathbb{R}^n)^+ \to S^n$ and thus an element of $\pi^0(Y; -\xi)$. The choice of a Thom class $u \in H^r(Y; \xi) = H^r(D\xi, S\xi)$ corresponds to choosing an orientation of $\xi$. The standard Euler class is defined by $e(\xi) = \sigma^*(u) \in H^r(Y)$. A generator $1 \in H^0(S^0)$ gives rise to the Hurewicz map $\pi^0(Y; -\xi) \to H^0(Y; -\xi)$, which associates to a stable pointed map $\sigma : T(-\xi) \to S^0$ the image $\sigma^*(1)$. Using the cup product pairing $H^r(Y; -\xi) \times H^s(Y; \xi) \to H^{r+s}(Y)$, the singular cohomology Euler class and the stable cohomotopy one are related by $e(\xi) = \sigma^*(1) \cdot u$.

- The approach of [2] and the one outlined above obviously are closely related: If $f = l + k : H' \to H$ admits a priori estimates and $D' \setminus \partial D' \subset H'$ contains $f^{-1}(0)$, then its compact homotopy class in $\mathcal{C}^0(\partial D', H \setminus \{0\})$ corresponds to $[f] \in \pi_H^0(pt; \text{ind} \ l) = \pi_{\text{ind}}^0(S^0)$.

3. The monopole map

Let $S^+$ and $S^-$ denote the Hermitian rank-2 bundles associated to the given Spin$^c$ structure on $X$ and let $L$ denote their determinant line bundle. Clifford multiplication $T^*X \times S^\pm \to S^\pm$ defines a linear map $\rho : \Lambda^2 \to \text{End}_C(S^\pm)$ from the bundle of 2-forms to the endomorphism bundle of the positive spinor bundle. The kernel of this homomorphism is the subbundle $\Lambda^-$ of anti-selfdual 2-forms. Its image is the subbundle of trace-free Hermitian endomorphisms. For a spin$^c$-connection $A$, denote by $D_A : \Gamma(S^+) \to \Gamma(S^-)$ its associated Dirac operator. The monopole map $\tilde{\mu}$ is defined for triples $(A, \phi, a)$ of a spin$^c$-connection $A$, a positive spinor $\phi$ and a 1-form $a$ on $X$ by

$$\tilde{\mu} : \text{Conn} \times (\Gamma(S^+) \oplus \Omega^1(X)) \to \text{Conn} \times (\Gamma(S^-) \oplus \Omega^+(X) \oplus H^1(X; \mathbb{R}) \oplus \Omega^0(X)/\mathbb{R})$$

$$(A, \phi, a) \mapsto (A, D_{A+a}\phi, F_{A+a}^+, -\sigma(\phi), a_{\text{harm}}, d^*a).$$

Here $\sigma(\phi)$ denotes the trace free endomorphism $\phi \otimes \phi^* - \frac{1}{2}||\phi||^2 \cdot \text{id}$ of $S^+$, considered via the map $\rho$ as a selfdual 2-form on $X$. As a map over the space $\text{Conn}$ of spin$^c$-connections, the monopole map is equivariant with respect to the action of the gauge group $\mathcal{G} = \text{map}(X, \mathbb{T})$. This group acts on spinors via multiplication with $u : X \to \mathbb{T}$, on connections via addition of $iud^{-1}$ and trivially on forms. Fixing a base point $* \in X$, the based gauge group $\mathcal{G}_0$ is obtained as the kernel of the evaluation homomorphism $\text{map}(X, \mathbb{T}) \to \mathbb{T}$ at $*$. 


Let $A$ be a fixed connection. The subspace $A + \ker(d) \subset \text{Conn}$ is invariant under the free action of the based gauge group with quotient space isomorphic to

$$\text{Pic}^0(X) = H^1(X; \mathbb{R})/H^1(X; \mathbb{Z}).$$

Let $\mathcal{A}$ and $\mathcal{C}$ denote the quotients

$$\mathcal{A} = (A + \ker d) \times (\Gamma(S^+) \oplus \Omega^1(X))/G_0$$

$$\mathcal{C} = (A + \ker d) \times (\Gamma(S^-) \oplus \Omega^+(X) \oplus H^1(X; \mathbb{R}) \oplus \Omega^0(X)/\mathbb{R})/G_0$$

by the pointed gauge group. Both spaces are bundles over $\text{Pic}^0(X)$ and the quotient $\mu = \tilde{\mu}/G_0 : \mathcal{A} \rightarrow \mathcal{C}$ of the monopole map is a fiber preserving, $\mathbb{T}$-equivariant map over $\text{Pic}^0(X)$.

For a fixed $k > 4$, consider the fiberwise $L^2_k$ Sobolev completion $\mathcal{A}_k$ and the fiberwise $L^2_{k-1}$ Sobolev completion $\mathcal{C}_{k-1}$ of $\mathcal{A}$ and $\mathcal{C}$. The monopole map extends to a continuous map $\mu = \mu_k : \mathcal{A}_k \rightarrow \mathcal{C}_{k-1}$ over $\text{Pic}^0(X)$. It is the sum $\mu = l + c$ of the linear Fredholm map $l = D_A \oplus d^+ \oplus \text{pr}_{\text{harm}} \oplus d^*$ and a term $c : (\phi, a) \mapsto (0, F^+_A, 0, 0) + (a \cdot \phi, -\sigma(\phi), 0, 0)$. This map $c$ is compact as the sum of the constant map $F^+_A$ and the composition of a multiplication map $\mathcal{A}_k \times \mathcal{A}_k \rightarrow \mathcal{C}_k$, which is continuous for $k > 2$, and a compact restriction map $\mathcal{C}_k \rightarrow \mathcal{C}_{k-1}$. The following statement and its proof are only slight variations of standard ones in Seiberg-Witten theory, compare e.g. [12]:

**Proposition 3.1.** Preimages $\mu^{-1}(B) \subset \mathcal{A}_k$ of bounded disk bundles $B \subset \mathcal{C}_{k-1}$ are contained in bounded disk bundles.

**Proof.** It is sufficient to prove this fiberwise for the Sobolev completions of the restriction of the monopole map to the space $\{A\} \times (\Gamma(S^+) \oplus \ker(d^*))$, which maps to $\{A\} \times (\Gamma(S^-) \oplus \Omega^+(X) \oplus H^1(X; \mathbb{R}) \oplus \Omega^0(X)/\mathbb{R})$. (Note that the monopole map is a linear isomorphism on the additional factor; in the stable homotopy picture the restricted map and the full monopole map thus differ only by infinite dimensional, but otherwise irrelevant suspension.) Using the elliptic operator $D = D_A + d^+$ and its adjoint, define the $L^2_k$-norm via the scalar product on the respective function spaces through

$$(\cdot, \cdot)_i = (\cdot, \cdot)_0 + (D \cdot, D \cdot)_{i-1}, \quad (\cdot, \cdot)_0 = (\cdot, \cdot) = \int_X <\cdot, \cdot >$$

The norms for the $L^p_k$-spaces are defined correspondingly. Let $\mu(A, \phi, a) = (A, \varphi, b, a_{\text{harm}}) \in \mathcal{C}_{k-1}$ be bounded by some constant $R$. The Weitzenböck formula for the Dirac operator associated to the connection $A + a = A'$ reads

$$D_{A'}^* D_A = \nabla_{A'}^* \nabla_{A'} + \frac{1}{4} s - \frac{1}{2} F^+_{A'}.$$
with $s$ denoting the scalar curvature of $X$. As a consequence, there is a pointwise estimate:

$$\Delta|\phi|^2 = 2 < \nabla_A^* \nabla_A \phi, \phi > - 2 < \nabla_A \phi, \nabla_A^* \phi >$$

$$\leq 2 < \nabla_A^* \nabla_A \phi, \phi >$$

$$= 2 < D_A^* \nabla_A \phi - \frac{s}{4} \phi + \frac{1}{2} F^+_A \phi, \phi >$$

$$= < 2 D_A^* \phi - \frac{s}{2} \phi + (b + \sigma(\phi)) \phi, \phi >$$

In particular, there are inequalities

$$\Delta|\phi|^2 + \frac{s}{2} |\phi|^2 + \frac{1}{2} |\phi|^4 \leq < 2 D_A^* \phi, \phi > + < 2 a \cdot \phi, \phi > + < b \phi, \phi >$$

$$\leq 2 (||D_A^* \phi||_{L^1} + ||a||_{L^1} ||\varphi||_{L^\infty}) \cdot |\phi| + ||b||_{L^\infty} \cdot |\phi|^2$$

$$\leq c_1 ((1 + ||a||_{L^1}) ||\varphi||_{L^2_{-1}} \cdot |\phi| + ||b||_{L^2_{-1}} \cdot |\phi|^2),$$

with a constant $c_1$ by applying the Sobolev embedding theorems. To get a bound for the remaining term, use a Sobolev estimate $||a||_{L^\infty} \leq c_2 ||a||_{L^p}$ for some $p > 4$ and the elliptic estimate $||a||_{L^p} \leq c_3 (||d^+ a||_{L^p} + ||a_{harm}||)$. Combination with the equality $d^+ a = b - F^+_A + \sigma(\phi)$ then leads to an estimate

$$||a||_{L^\infty} \leq c_4 (||a_{harm}|| + ||b||_{L^0} + ||F^+_A||_{L^0} + ||\sigma(\phi)||_{L^0})$$

$$\leq c_5 (||a_{harm}|| + ||b||_{L^2_{-1}} + ||F^+_A||_{L^2} + ||\phi||_{L^\infty}).$$

At the maximum of $|\phi|^2$, its Laplacian is non-negative. So, putting everything together, one obtains a polynomial estimate of the form

$$||\phi||_{L^1} \leq c R \left( (1 + R) ||\phi||_{L^\infty} + ||\phi||_{L^\infty} + ||\phi||_{L^\infty}^2 + ||s||_{L^\infty} ||\phi||_{L^\infty}^2 \right)$$

Combining the last two estimates, one obtain bounds for the $L^\infty$-norm and a fortiori for the $L^p$-norm of $(\phi, a)$ for every $p \geq 1$.

Now comes bootstrapping: For $i \leq k$, assume inductively $L^p_{i-1}$-bounds on $(\phi, a)$ with $p = 2^{k-i}$. To obtain $L^p_i$-bounds, compute:

$$||((\phi, a))||_{L^p_i}^p = ||(D_A \phi, d^+ a)||_{L^p_i}^p$$

$$= ||(\varphi, b, a_{harm})||_{L^p_{i-1}}^p + ||(a \phi, - F^+_A - \sigma(\phi))||_{L^p_{i-1}}^p.$$ 

The latter equality holds as $D_A' = D_A + a$. The summands in the last expression are bounded by the assumed $L^p_{i-1}$-bounds on $(\phi, a)$.

The proposition in particular implies that the assumptions of 2.6 are satisfied for the monopole map $\mu$. The conclusion is spelled out in the following

**Corollary 3.2.** The monopole map defines an element $[\mu]$ in the stable cohomotopy group

$$\pi^0_\mathcal{T}(Pic^0(X); \lambda) = \pi^0_\mathcal{T}(Pic^0(X); ind(D)).$$
where $H$ is a Sobolev completion of the sum $\Gamma(S^- \oplus \Lambda^2_+(T^*X))$ of the vector spaces of negative spinors and selfdual two-forms on $X$. The virtual index bundle $\lambda = \text{ind}(D) \otimes H_+$ is the difference of the complex virtual index bundle of the Dirac operator over $\text{Pic}^0(X)$ and the trivial bundle $H_+$ with fiber $H^2_+(X; \mathbb{R})$, which for a chosen metric on $X$ may be viewed as the space of selfdual harmonic two-forms. The $\mathbb{T}$-action on $\text{ind}(D)$ is by multiplication with complex numbers and on $H_+$ is trivial.

There is a comparison map from the stable equivariant cohomotopy group above to the integers, which relates the element defined by the monopole map with the integer valued Seiberg-Witten invariant associated to it:

**Proposition 3.3.** Let $X$ be a closed 4-manifold with $b = b_+ > b_1 + 1$. The choice of a homology orientation (i.e. an orientation of $H^1(X; \mathbb{R}) \oplus H^2_+(X; \mathbb{R})$) then determines a homomorphism $t : \pi_{T,H}^b(\text{Pic}^0(X); \text{ind}(D)) \to \mathbb{Z}$, which maps the class of the monopole map to the integer valued Seiberg-Witten invariant.

**Proof.** Any element in $\pi_{T,H}^b(\text{Pic}^0(X); \text{ind}(D))$ is represented by a pointed equivariant map $\mu : TF \to V^+$ from the Thom space of a bundle $F$ over $\text{Pic}^0(X)$ to a sphere $V^+ = (V' \oplus H^2_+(X; \mathbb{R}))^+$, where $F - \text{Pic}^0(X) \times V'$ represents the equivariant virtual index bundle of the Dirac operator. The integer Seiberg-Witten invariant is constructed as follows. After possible perturbation of the map $\mu$, the $\mathbb{T}$-fixed point set $TF^\mathbb{T}$ is mapped to a subspace of $V^{T^+}$ of codimension at least $b - b_1 \geq 2$. After perturbing further, the preimage of a generic point in the complement is a manifold $M$ with a free $\mathbb{T}$-action. The homology orientation, together with a standard orientation of complex vector spaces, defines a relative orientation for the pair $F, V$ and thus an orientation of $M$ and on the manifold $M/\mathbb{T}$. The dimension of $M$ is $\text{ind}_R(D) + b_1 - b_2$. Now suppose $M$ is a manifold of odd dimension $2k + 1$ (otherwise the SW-number is zero). Then the Seiberg-Witten number is the evaluation of the Euler class of the complex vector bundle $(M \times \mathbb{C}^k)/\mathbb{T}$ over $M/\mathbb{T}$ at the fundamental class.

Equivalently, one could start with the map $\gamma(\mathbb{C})^k \circ \mu$, where $\gamma(\mathbb{C})^k : V^+ \to (V \oplus \mathbb{C}^k)^+$ is the one-point compactified inclusion of vector spaces. After perturbing this map equivariantly as before, the preimage of a generic fixed point would be a finite number of oriented $\mathbb{T}$-orbits. The oriented count of it is the Seiberg-Witten number again. This is basically the same construction as above, as the Euler class of a vector bundle over a manifold is the Poincaré dual of a generic section.

Here is another equivalent description of this map in purely algebraic topological terms. In the long exact sequence of stable homotopy groups of equivariant maps associated to the pair $(TF, TF^\mathbb{T})$

$$\{\Sigma TF^\mathbb{T}, V^+\}_{T,H} \to \{(TF, TF^\mathbb{T}), (V^+, \emptyset^+)\}_{T,H} \to \{TF, V^+\}_{T,H} \to \{TF^\mathbb{T}, V^+\}_{T,H}$$
the first and last term are vanishing because of the dimension assumption \( b > b_1 + 1 \). So the map \( \mu \) can be described by a stable map of pairs. Now apply equivariant cohomology to this map of pairs. Since the \( \mathbb{T} \)-action on \( (TF, TF^\mathbb{T}) \) is relatively free, the equivariant cohomology group \( H^*_\mathbb{T}(TF, TF^\mathbb{T}) \) identifies with the nonequivariant cohomology \( H^*(TF/\mathbb{T}, TF) \) of the quotient, which after replacing \( TF^\mathbb{T} \) by a tubular neighbourhood, is a connected manifold relative to its boundary. An orientation of \( H^1(X; \mathbb{R}) \) and thus of \( \text{Pic}^0(X) \) together with the standard orientation of complex vector bundles defines an orientation class \([TF/\mathbb{T}]\) of this manifold. Similarly, the chosen homology orientation of \( X \) and the orientation of \( \text{Pic}^0(X) \) determine the orientation of \( V \) and thus a generator \([V^+]\) in reduced equivariant cohomology as a free \( H_*^\mathbb{T}(\ast) \sim \mathbb{Z}[x] \)-module of rank one. The homomorphism \( t \) associates to \( \mu \) the degree zero part of \( \mu^*([\sum_0^\infty x^i[V^+]] \cap [TF/\mathbb{T}] \). Using the alternate description above, the same integer is obtained as \( (\gamma(C)^{C \circ \mu}([V \oplus C^{k}]) \cap [TF/\mathbb{T}] \).

If the first Betti number of \( X \) vanishes, the group \( \pi^{i-1}_\mathbb{T,H}(\ast; C^d) \) simplifies: The index of the Dirac operator is a complex vector space of complex dimension

\[
d = \frac{c(s)^2 - \text{signature}(X)}{8},
\]

where \( c(s) \) is the first Chern class of the spinor bundles \( S^{\pm} \) associated to the spin\(^c\)-structure \( s \).

**Proposition 3.4.** For \( i > 1 \), the stable equivariant cohomotopy groups \( \pi^i_{\mathbb{T,H}}(\ast; C^d) \) are isomorphic to the nonequivariant stable cohomotopy groups \( \pi^{i-1}(CP^{d-1}) \) of complex projective \((d-1)\)-space. In particular, if \( X \) is a closed 4-manifold with \( b_1 = 0 \) and \( b_+ > 1 \), then the monopole map determines an element in \( \pi^{b-1}(CP^{d-1}) \).

**Proof.** The long exact stable cohomotopy sequence for the pair \((D(C^d), S(C^d))\) consisting of the unit disk and sphere in the complex vector space \( C^d \) allows to identify for \( i > 1 \) the groups \( \pi^i_{\mathbb{T,H}}(\ast; C^d) \) with \( \pi^{i-1}_{\mathbb{T,H}}(S(C^d)^+) \). But for the free \( \mathbb{T} \)-space \( S(C^d) \) equivariant cohomotopy is isomorphic to the nonequivariant cohomotopy of its quotient \( S \). To analyse this cohomotopy of projective spaces a little further, consider the Hurewicz map

\[
\pi^i(Y) \rightarrow H^i(Y) \\
[f] \mapsto f^*(1),
\]

with \( 1 \in H^1(S^i) \cong \tilde{H}^0(S^0) \) defined by the orientation. Rationally it is an isomorphism, as rationally the sphere spectrum is an Eilenberg MacLane spectrum by Serre’s theorem. However, nonrationally, the Hurewicz map has both kernel and cokernel, as displayed in low dimensions below.

In the following lemma the results are ordered according to \( k \), which can be interpreted as the ”expected dimension of the moduli space”, i.e. the dimension of the preimage of
a generic point in the sphere. Thanks go to N. Minami for pointing out a mistake in the computation for $k = 3$ in an earlier version.

**Lemma 3.5.** Let $d > 1$ be an integer. The Hurewicz map of reduced cohomology groups $h^{2d-2-k}: \pi^{2d-2-k}(C P^d-1) \to \tilde{H}^{2d-2-k}(C P^d-1)$

0. for $k = 0$ is an isomorphism.
1. for $k = 1$ has kernel isomorphic to $\mathbb{Z}/\gcd(2,d)$.
2. for $k = 2$ has kernel isomorphic to $\mathbb{Z}/\gcd(2,d)$ and cokernel to $\mathbb{Z}/\gcd(2,d-1)$.
3. for $k = 3$ has kernel isomorphic to $\mathbb{Z}/l$ with $l = \gcd(24,d)$, if $d$ is even, and $l = \gcd(24,d-3)/2$ otherwise.
4. for $k = 4$ has trivial kernel and, for $d > 2$, cokernel isomorphic to $\mathbb{Z}/m$ with $lm = 48$, if $d$ is even, and $lm = 12$ otherwise.

**Proof.** The proof employs the Atiyah-Hirzebruch spectral sequence with $E_2$-term

$$H^*(Y; \pi^*(pt)) \Rightarrow \pi^*(Y)$$

and uses the following facts:

1. The attaching map of the 4-cell in $C P^2$ is the Hopf map, which is the generator $\eta$ of $\pi^{-1}(pt) \cong \mathbb{Z}/2$.
2. The group $\pi^{-2}(pt) \cong \mathbb{Z}/2$ is generated by $\eta^2$.
3. The attaching map of the 8-cell in $H P^2$ is again a Hopf map, which is stably the generator $\nu$ of $\pi^{-3}(pt) \cong \mathbb{Z}/24$. Furthermore, $\eta^3 = 12\nu$.
4. The stable homotopy groups $\pi^{-4}(pt)$ and $\pi^{-5}(pt)$ vanish.
5. For even $d$, there is a projection of the complex projective to the quaternionic projective space.
6. The differentials in the spectral sequence are differentials for the algebra structure on the respective $E_i$-terms.
7. The spectral sequence is natural in $Y$. In particular one may use the map between spectral sequences induced by inclusions of the projective spaces into higher dimensional projective spaces and induced by the projection of complex projective spaces to quaternionic projective spaces.
8. By a result of I. M. James, cf. [1], the projection map

$$C P^{d-1}/C P^{d-4} \to C P^{d-1}/C P^{d-2} = S^{2d-2}$$

is stably split if and only if $d$ is divisible by 24.

Let $m(d, k)$ denote the order of the cokernel of the Hurewicz map $h^{2d-2-k}$ for even integers $k$. Since the integer valued Seiberg-Witten invariants are in the image of the Hurewicz map, one gets on particular:
Corollary 3.6. For a $K$-oriented 4-manifold $X$ with vanishing first Betti number and $b^+ = 2p + 1$ odd, the integer valued Seiberg-Witten invariant is divisible by $m(d, k)$ with $k = 2d - 2p - 2$.

Indeed, one gets estimates for the divisibility $m(d, k)$ of Seiberg-Witten invariants by comparing the Hurewicz maps from stable cohomotopy to $K$-theory and to singular cohomology. The following result was obtained in [10].

Theorem 3.7. The integers $m(d, 2\kappa)$ are divisible by the denominators of the rational numbers $a_{p,0}, a_{p,1}, \ldots, a_{p,\kappa}$, which appear as coefficients in the Taylor expansion

$$\log(1 - \xi) = \sum_{l=0}^{\infty} a_{p,l} \xi^{p+l}.$$ 

Comparison with 3.5 shows that the above estimates are not sharp in general.

Proof. For a stable map $\mu : \mathbb{C}P^{d-1} \to S^{2p}$, consider the commuting diagram

$$
\begin{align*}
K(S^{2p}) & \xrightarrow{\mu^*} K(\mathbb{C}P^{d-1}) \\
ch \Rightarrow & \\
H^*(S^{2p}; \mathbb{Z}) & \xrightarrow{\mu^*} H^*(\mathbb{C}P^{d-1}; \mathbb{Q})
\end{align*}
$$

with vertical arrows given by the Chern character and horizontal maps induced by $\mu$. The left vertical map is an isomorphism, mapping the class $\tau$ to the orientation class $[S^{2p}] \in H^*(S^{2p}; \mathbb{Z})$. The Hurewicz image in $K(\mathbb{C}P^{d-1}) \cong \mathbb{Z}[\xi]/(\xi^d)$ of the cohomotopy element $\mu$ is of the form $\mu^*(\tau) = \sum b_l \xi^{l+p}$ with integers $b_l$, because the composite of $\mu$ with the inclusion map $\mathbb{C}P^{p-1} \to \mathbb{C}P^{d-1}$ is nullhomotopic by dimension reasons. The Hurewicz image of $\mu$ in $H^*(\mathbb{C}P^{d-1}; \mathbb{Q}) \cong \mathbb{Q}[x]/(x^d)$ is of the form $nx^p$ for an integer $n$. The Chern character $\xi \mapsto (1 - \exp(x))$ on the right hand side is injective and becomes an isomorphism after tensoring with the rationals. Commutativity of the diagram now implies

$$\sum_{l=0}^{\kappa} b_l \xi^{l+p} = \mu^*(\tau) = ch^{-1}\mu^*([S^{2p}]) = ch^{-1}(nx^p) = d \log(1 - \xi)^p = n \sum_{l=0}^{\kappa} a_{p,l} \xi^{p+l}. $$

To show that the stable cohomotopy invariants are indeed effective refinements of the integer valued Seiberg-Witten invariants, it remains to find manifolds whose stable cohomotopy invariants are elements in the kernel of the Hurewicz-map. This will be done in [2].

Let’s finish by showing how to recapture Donaldon’s first theorem on manifolds with definite intersection form in the stable cohomotopy setting. This proof relies on the following fact about equivariant maps, which is well known and can be proved basically the same way as [3.7] by the use of the equivariant $K$-theory mapping degree (cf. e.g. [3]):
Lemma 3.8. Let \( f : (\mathbb{R}^n \oplus \mathbb{C}^m)^+ \to (\mathbb{R}^n \oplus \mathbb{C}^{m+k})^+ \) be an \( \mathbb{T} \)-equivariant map such that the restricted map on the fixed points has degree 1. Then \( k \geq 0 \) and \( f \) is homotopic to the inclusion.

We apply this lemma to the case where \( b = b_+ = 0 \), i.e. to manifolds \( N \) with negative definite intersection form. Note that for these manifolds the monopole map on the fixed point set is just a linear isomorphism in each fiber over \( \text{Pic}^0(N) \). As a consequence one gets:

Corollary 3.9. (Donaldson\cite{Donaldson}) Let \( N \) be a closed oriented four-manifold with negative definite intersection form. Then every characteristic \( c \in H_2(N; \mathbb{Z}) \) satisfies \( -c^2 \geq b_2(N) \). As a consequence, by \( \text{Pic}^0(N) \) the intersection pairing is diagonal.

Proof. The stable cohomotopy invariant associated to a \( \text{spin}^c \)-structure with determinant \( c \) is in \( \pi_0^{\mathbb{T},H}(\text{Pic}^0(N); \text{ind}(D)) \). Restricting it to a point in \( \text{Pic}^0(N) \) results in the stabilisation of a map as in \( \cite{Donaldson} \) with

\[
k = -\text{ind}_c(D) = \frac{-c^2 - b_2(N)}{8}.
\]

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