1 The majorant property: Some generalities

This paper is concerned with the majorant property of various randomly generated subsets of \([1, N]\). More precisely, suppose \(A_N \subset [1, N]\) is a sequence of sets so that \(|A_N| \asymp N^p\) for some fixed \(0 < p < 1\) as \(N \to \infty\). For example, one can take \(A_N\) to be the squares, cubes, etc., or (multi-dimensional) arithmetic progressions. As in [M], given \(p \geq 2\), one asks for the smallest power \(\gamma = \gamma(p) > 0\) (which might be also specific to the sequence \(A_N\)) such that

\[
\sup_{|a_n| \leq 1} \left\| \sum_{n \in A_N} a_n e(n \cdot) \right\|_p \leq C \frac{N^{\gamma}}{N^p} \left\| \sum_{n \in A_N} e(n \cdot) \right\|_p.
\]

This is only one out of several ways of stating the majorant problem. [M] asks for a power \(\gamma\) that applies to all \(A_N \subset [1, N]\) simultaneously. If \(p\) is an even integer, then one can take \(C = 1\) and \(\gamma = 0\) as realized by Hardy and Littlewood. On the other hand, it has also been known for some time that one cannot take \(\gamma = 0\) if \(p\) is not an even integer. Moreover, a quantitative lower bound of \(\exp(c \log N / \log \log N)\) is obtained in [M] for (1.1) with a particular choice of \(A_N\). If (1.1) holds for all \(\gamma > 0\) and appropriate \(p\), then it would imply the restriction and therefore also the Kakeya conjecture, see [M] for those matters. One always has the bound

\[
\sup_{|a_n| \leq 1} \left\| \sum_{n \in A_N} a_n e(n \cdot) \right\|_p \leq C \left( \frac{N}{|A_N|} \right)^{\frac{1}{2-p}} \left\| \sum_{n \in A_N} e(n \cdot) \right\|_p
\]

by Hausdorff-Young and the obvious lower bound \(\left\| \sum_{n \in A_N} e(n \cdot) \right\|_p^p \geq |A_N|^{p-1} N^{-1}\). This settles the case of any sequence of large sets, i.e., \(\rho = 1\), as well as all arithmetic progressions. Another easy estimate can be obtained by interpolation. Indeed, if \(2 < p < 4\), say, then interpolating between 2 and 4 yields \(\gamma \leq (1 - \frac{2}{p})(1 - \frac{2}{3})\). It turns out that this interpolation can be done more carefully, which gives optimal results for sets \(A_N\) whose Dirichlet kernel satisfies a certain “reverse interpolation inequality.” To this end, let \(P_A := \{ \sum_{n \in A} a_n e(n \theta) \mid |a_n| \leq 1 \}\). Then, with \(A = A_N\) for simplicity, for any odd integer \(p > 2\),

\[
\sup_{|a_n| \leq 1} \int_0^1 \left\| \sum_{n \in A} a_n e(n \theta) \right\|^p d\theta = \sup_{|a_n| \leq 1} \int_0^1 e(n \theta) \sum_{k \in A} a_k e(-k \theta) \left( \sum_{\ell \in A} a_\ell e(\ell \theta) \right)^{p-2} d\theta
\]

\[
\leq \sup_{g \in P_A} \sqrt{|A|} \left( \sum_{n \in A} |g|^{p-2} |(n \cdot)|^2 \right)^{\frac{1}{p}} \leq \sup_{g \in P_A} \sqrt{|A|} \|g\|_{2(p-1)}^{p-1}
\]

\[
\leq \left\| \sum_{n \in A} e(n \cdot) \right\|_2 \left\| \sum_{n \in A} e(n \cdot) \right\|_{2(p-1)}^{p-1}
\]

On the Hardy-Littlewood majorant problem for random sets

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Here the first inequality sign in (1.2) follows by putting absolute values inside and Cauchy-Schwarz, the second is Plancherel, and (1.3) uses the majorant property on \(2(p-1)\). Now assume the following condition

\[
\left\| \sum_{n \in A} e(n \cdot) \right\|_2 \leq C_\varepsilon N^\varepsilon \varepsilon \left\| \sum_{n \in A} e(n \cdot) \right\|_p^{p-1} \leq C_\varepsilon N^\varepsilon \varepsilon \left\| \sum_{n \in A} e(n \cdot) \right\|_p^p.
\]

In view of the preceding, one then has (1.1) for any \(\gamma > 0\). This condition, which is of basic importance for most of our work, is basically the reverse of the usual interpolation inequality. One checks immediately that arithmetic progressions satisfy (1.4). Also, observe that any sequence \(A_N\) for which (1.4) holds for all \(p\) satisfies (1.1) for all \(p\) with \(\gamma > 0\). Indeed, this follows inductively from the argument leading up to (1.3) using the majorant property from the previous stage \(2(p-1)\) to pass to the next stage \(p\). Finally, interpolation is required to obtain the desired bound for all \(p\) (at the cost of \(N^\varepsilon\)). Another case which is covered by this argument, but not the previous one based on Hausdorff-Young, are multi-dimensional arithmetic progressions. For example, one easily checks that

\[
A = \{b+j_1a_1 + j_2a_2 \mid 0 \leq j_1 < L_1, 0 \leq j_2 < L_2\}
\]

with \(a_1L_1 < a_2\), satisfies

\[
\left\| \sum_{n \in A} e(n \cdot) \right\|_p \asymp (L_1L_2)^{p-1}
\]

for \(p > 1\). Another interesting case are the squares \(A_N = \{n^2 \mid 1 \leq n \leq \sqrt{N}\}\). In this case it is well-known that the there is a “kink” at \(p = 4\),

\[
\left\| \sum_{n \in A} e(n \cdot) \right\|_p \leq C_\varepsilon N^\varepsilon + \frac{1}{2} \text{ if } 2 \leq p \leq 4,
\]

\[
\left\| \sum_{n \in A} e(n \cdot) \right\|_p \leq C_\varepsilon N^{1-\frac{2}{p}+\varepsilon} \text{ if } p \geq 4,
\]

so that (1.4) holds only for \(2 \leq p \leq 3\). In particular, the argument leading up to (1.3) gives the (trivial) statement that the majorant property holds at \(p = 3\) for the squares. A nontrivial statement can be obtained by improving on the use of Plancherel in (1.2). Indeed, it is a well-known fact that

\[
\left\| \sum_{n=1}^N a_n e(n^2 \theta) \right\|_4 \leq C_\varepsilon N^\varepsilon \left( \sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}} \iff \left( \sum_{n=1}^N |\hat{f}(n^2)|^2 \right)^{\frac{3}{2}} \leq C_\varepsilon N^\varepsilon \|f\|_{L^4(T)},
\]

the second statement being the dual of the first. This can be checked by reducing the \(L^4\)-norm to an \(L^2\)-norm by squaring, and then using Cauchy-Schwarz and the \(N^\varepsilon\)-bound on the divisor function, see [B3]. We now repeat the argument leading up to (1.3) to conclude the following. Let

\[
\mathcal{P} := \{ \sum_{n=1}^N a_n e(n^2 \theta) \mid |a_n| \leq 1 \}.
\]
If $p = 3k + 1$, then one can apply the majorant property at $\frac{4}{3}(p - 1)$ so that

\[
\sup_{|a_n| \leq 1} \int_0^1 \left| \sum_{n=1}^N a_n e(n^2 \theta) \right|^p d\theta = \sup_{|a_n| \leq 1} \sum_{n=1}^N a_n \int_0^1 e(n^2 \theta) e(-k^2 \theta) \left| \sum_{\ell=1}^N a_\ell e(\ell^2 \theta) \right|^{p-2} d\theta
\]

(1.7) \leq \sup_{g \in \mathcal{P}} \sqrt{|A|} \left( \sum_{n=1}^N |g[n]^{p-2}(n^2)|^2 \right)^{\frac{1}{2}} \leq \sup_{g \in \mathcal{P}} \sqrt{|A|} \|g\|^{p-1}_{\frac{4}{3}(p-1)}

\leq \left\| \sum_{n=1}^N e(n^2 \cdot) \right\|_2 \left\| \sum_{n=1}^N e(n^2 \cdot) \right\|_{p-1}^{p-1} \leq C_{\varepsilon} N^\varepsilon N^\frac{1}{2} N^{p-\frac{5}{2}} \leq C_{\varepsilon} N^\varepsilon N^{p-2}

\leq C_{\varepsilon} N^\varepsilon \left\| \sum_{n=1}^N e(n^2 \cdot) \right\|_p.

Here we used (L.6) in (1.7). This implies that for the sequence of squares (1.1) holds with any $\gamma > 0$ at $p = 7, 13, 19$ etc.

Another case of sets $A_N$ that do not satisfy (1.4) are random subsets $A_N \subset [1, N]$. Indeed, we show below that random sets $A_N$ which are obtained by selecting each integer $1 \leq n \leq N$ with probability $\tau$ have the property that for $p > 1$

\[
E \left\| \sum_{n \in A_N} e(n \theta) \right\|_p^p \asymp \tau^p N^{p-1} + (\tau N)^{\frac{p}{2}},
\]

see Theorem 2.1. The two terms on the right balance at $\tau_{\text{crit}} = N^{-1+\gamma}$ so that it is clear that (1.4) cannot hold in general. The main objective of the following section is to show that nevertheless, such random subsets do satisfy (1.3) with large probability. The method to some extent resembles the calculation from (1.4), but is of course more involved. We rely on a probabilistic lemma from Bourgain’s work [B1].

It is possible to abstract the arguments below, and then verify that various examples satisfy the conditions of such an abstract theorem, the most important one being condition (1.4). More precisely, starting with a deterministic sequence $A_N$, define $S_N(\omega) = \{n \in A_N \mid \xi_n = 1\}$ where $\xi_n$ are i.i.d. selector variables satisfying $\mathbb{P}[\xi_n = 1] = \tau = 1 - \mathbb{P}[\xi_n = 0]$. If, amongst other things, (1.4) holds for $A_N$, then much of what is done in the following section goes through. On the other hand, some improvements which we obtain below for the case of arithmetic progressions are not easily axiomatized. Moreover, since we do not have any examples apart from (multi-dimensional) arithmetic progressions, we have decided against casting this into a more general framework. Thus, we write out the main argument only for arithmetic progressions. If (1.4) is violated, then our method applies only to certain $p$ or after suitable modifications. For example, one can check that the machinery which we develop below shows that with high probability random subset of the squares satisfy (1.1) at $p = 7$ for any $\gamma > 0$. This requires invoking the (almost) $\Lambda(4)$ property of the squares as in (1.7). It seems difficult to obtain the desired bound for all $p$ in case of the squares.

In addition to random subsets we also consider perturbations of arithmetic progressions. This means that each element of a given arithmetic progression is shifted independently and randomly by some small amount. We again show that most sets obtained in this fashion satisfy (1.1) for any $\gamma > 0$,
see Theorem 3.6. As before, the method can be presented abstractly for perturbations of arbitrary sets $A_N$ that satisfy condition (1.4).

2 Random subsets have the majorant property

**Theorem 2.1.** Let $0 < \delta < 1$ be fixed. For every positive integer $N$ we let $\xi_j = \xi_j(\omega)$ be i.i.d. variables with $\mathbb{P}[\xi_j = 1] = \tau$, $\mathbb{P}[\xi_j = 0] = 1 - \tau$ where $\tau = N^{-\delta}$. Define a random subset

$$S(\omega) = \{ j \in [1, N] \mid \xi_j(\omega) = 1 \}.$$ 

Then for every $\varepsilon > 0$ and $7 \geq p \geq 2$ one has

$$\mathbb{P}\left[ \sup_{|a_n| \leq 1} \left\| \sum_{n \in S(\omega)} a_n e(n\theta) \right\|_{L^p(T)} \geq N^{\varepsilon} \left\| \sum_{n \in S(\omega)} e(n\theta) \right\|_{L^p(T)} \right] \to 0$$

as $N \to \infty$. Moreover, under the additional restriction $\delta \leq \frac{1}{2}$, (2.1) holds for all $p \geq 7$.

We show below that the $N^{\varepsilon}$-factor can be removed in certain cases, for example when $p = 3$. The restriction $\delta \leq \frac{1}{2}$ for $p \geq 7$ appears to be of a purely technical nature, and we believe that the theorem should hold for all $\delta \in (0, 1)$.

The proof of Theorem 2.1 relies on a method that Bourgain developed for the $\Lambda(p)$ problem, see [B1] and [B2]. In fact, in this situation we can avoid several complications that arose in Bourgain’s work. Notice that our Theorem 2.1 is implied by Bourgain’s existence theorem of $\Lambda(p)$ sets provided $\delta \geq 1 - \frac{2}{p}$, but not for $\delta < 1 - \frac{2}{p}$. Indeed, in the former case the random set $S$ will typically have cardinality $N^{\frac{2}{p}}$ or smaller, and such sets were shown by Bourgain [B1] to be $\Lambda(p)$-sets with large probability.

2.1 Random sums over asymmetric Bernoulli variables

We first dispense with some simple technical statements about the behavior of random sums with asymmetric Bernoulli variables as summands. They are definitely standard, but lacking a precise reference we prefer to present them.

**Lemma 2.2.** Let $\eta_j$ be i.i.d. variables so that $\mathbb{P}[\eta_j = 1 - \tau] = \tau$, $\mathbb{P}[\eta_j = -\tau] = 1 - \tau$. Here $0 < \tau < 1$ is arbitrary. Let $N \geq 1$ and $\{a_j\}_{j=1}^N \in \mathbb{C}$ be given. Define $\sigma^2 = \tau(1 - \tau) \sum_{j=1}^N |a_j|^2$. Then for $\lambda > 0$,

$$\mathbb{P}\left[ \left| \sum_{j=1}^N a_j \eta_j \right| > \lambda \sigma \right] \leq 4 e^{-\lambda^2 \sigma^2}$$

provided

$$\max_{1 \leq j \leq N} \lambda |a_j| \leq 4 \sigma.$$
Proof. Assume first that all \( a_j \in \mathbb{R} \). Then for any \( t > 0 \)

\[
\Pr\left[ \sum_{j=1}^{N} a_j \eta_j > \lambda \sigma \right] \leq e^{-t \lambda \sigma} \mathbb{E} \exp\left(t \sum_{j=1}^{N} a_j \eta_j \right)
\]

(2.3)

\[
eq e^{-t \lambda \sigma} \prod_{j=1}^{N} \left[ \tau e^{(1-\tau) a_j} + (1-\tau) e^{-\tau a_j} \right]
\]

(2.4)

Next, we claim that

\[
\tau e^{(1-\tau)x} + (1-\tau) e^{-\tau x} \leq \exp(2(1-\tau)x^2) \text{ for all } |x| \leq 1.
\]

(2.5)

Observe that this property fails for \( x = \tau - \frac{1}{2} \). To prove this, set

\[
\phi_\tau(x) = \exp(2(1-\tau)x^2) - \tau e^{(1-\tau)x} - (1-\tau) e^{-\tau x}.
\]

By symmetry it suffices to consider the case \( 0 \leq x \leq 1 \) and to show that \( \phi_\tau \geq 0 \) there. Clearly,

\[
\phi'_\tau(x) = \tau(1-\tau)[4x \exp(2(1-\tau)x^2) - e^{(1-\tau)x} + e^{-\tau x}]
\]

(2.6)

\[
\geq \tau(1-\tau)[4x - e^{(1-\tau)x} + e^{-\tau x}]
\]

Differentiating the expression in brackets yields

\[
4 - (1-\tau)e^{(1-\tau)x} - \tau e^{-\tau x} \geq 4 - (1-\tau)e^{(1-\tau)x} - \tau e^{(1-\tau)x} \geq 4 - e > 0
\]

for all \( 0 \leq x \leq 1 \). It follows that \( \phi'_\tau(x) \geq \phi'_\tau(0) = 0 \) for \( 0 \leq x \leq 1 \). Hence also \( \phi_\tau(x) \geq \phi_\tau(0) = 0 \) for \( 0 \leq x \leq 1 \), as desired. Inserting (2.3) into (2.4) gives

\[
\Pr\left[ \sum_{j=1}^{N} a_j \eta_j > \lambda \sigma \right] \leq \min_{t>0} e^{-t \lambda \sigma} \exp(2t^2 \sigma^2) = e^{-\frac{\lambda^2}{8}}
\]

provided for the minimizing choice of \( t = t_0 \) one has \( \max_j |t_0 a_j| \leq 1 \). But \( t_0 = \frac{\lambda}{4\sigma} \) and this condition therefore reads

\[
\max_{1 \leq j \leq N} \frac{|\lambda||a_j|}{4\sigma} \leq 1,
\]

which is precisely (2.2). Evidently, the same bound also holds for deviations less than \(-\lambda \sigma\), which gives \( 2e^{-\lambda^2/8} \) as an upper bound on the large deviation probability in the real case. Finally, if \( a_n \in \mathbb{C} \), then one splits into real and complex parts.

Lemma 2.2 immediately leads to the following version of the Salem–Zygmund inequality for asymmetric variables.

[5]
Corollary 2.3. With $\eta_n$ and $\sigma$ as in the previous lemma

$$\mathbb{P}\left[ \sup_{\theta \in \mathbb{T}} \left| \sum_{n=1}^{N} a_n \eta_n e(n\theta) \right| > 20 \sigma \sqrt{\log N} \right] \leq 4N^{-8}$$

for any $a_n \in \mathbb{C}$ provided the following conditions hold:

\begin{align*}
\sup_{1 \leq n \leq N} 10 |a_n|^2 \log N &\leq \sigma^2 = \tau(1 - \tau) \sum_{k=1}^{N} |a_k|^2 \\
10 &\leq \tau(1 - \tau) N \log N. 
\end{align*}

(2.7)

Proof. Let $\{\theta_j\}_{j=1}^{N^2} \subset \mathbb{T}$ be a $N^{-2}$-net. Denote

$$T_{N,\omega}(\theta) := \sum_{n=1}^{N} a_n \eta_n(\omega) e(n\theta).$$

Then, by Bernstein’s inequality, and with the usual de la Vallee-Poussin kernel $V_N$,

\begin{align*}
\min_j |T_{N,\omega}(\theta) - T_{N,\omega}(\theta_j)| &\leq N^{-2} \|T^\prime_{N,\omega}\|_\infty \\
&\leq N^{-2} \|T_{N,\omega}\|_2 \|V_N^\prime\|_2 \\
&\leq N^{-2} \left( \sum_{n=1}^{N} |a_n|^2 \right)^{1/2} 8\pi N^{3/2} \\
&= \frac{8\pi \sigma \sqrt{N}}{\sqrt{\tau(1 - \tau)}} \leq 10 \sigma \sqrt{\log N}.
\end{align*}

The final inequality here follows from our assumption (2.7). Therefore, by Lemma 2.2

\begin{align*}
\mathbb{P}\left[ \sup_{\theta \in \mathbb{T}} \left| \sum_{n=1}^{N} a_n \eta_n e(n\theta) \right| > 20 \sigma \sqrt{\log N} \right] &\leq \sum_{j=1}^{N^2} \mathbb{P}\left[ \left| \sum_{n=1}^{N} a_n \eta_n e(n\theta_j) \right| > 10 \sigma \sqrt{\log N} \right] \\
&\leq 4N^2 \exp(-100 \log N/8) \leq 4N^{-8},
\end{align*}

which is precisely the bound claimed in the lemma. The first condition in (2.7) ensures that (2.2) holds.

In the proof of Theorem 2.1 we shall need to know the typical size of the easier norm in (2.1). We determine this norm in the following lemma.

Lemma 2.4. Let $\xi_j$ be selector variables as above with $\tau = N^{-\delta}$, $0 < \delta < 1$ fixed. Let $p \geq 2$ and define

$$I_{p,N}(\omega) = \int_{0}^{1} \left| \sum_{n=1}^{N} \xi_n(\omega)e(n\theta) \right|^p \, d\theta.$$
Then for some constants $C_p$,

$$C_p^{-1} \left( \tau_p^p N^{p-1} + (\tau N)^{\frac{p}{2}} \right) \leq \mathbb{E} I_{p,N} \leq C_p \left( \tau_p^p N^{p-1} + (\tau N)^{\frac{p}{2}} \right).$$

Moreover, there is some small constant $c_p$ such that

$$\mathbb{P} \left[ I_{p,N} \leq c_p (\tau_p^p N^{p-1} + (\tau N)^{\frac{p}{2}}) \right] \to 0$$

as $N \to \infty$.

Proof. Let $\eta_n(\omega) = \xi_n(\omega) - \tau$, so that $\mathbb{E} \eta_n = 0$ and $\mathbb{E} \eta_n^2 = \tau(1 - \tau)$. Then

$$I_{p,N}(\omega) \lesssim \int_0^1 \left| \sum_{n=1}^N \eta_n e(n \theta) \right|^p \, d\theta + \int_0^1 \left| \sum_{n=1}^N \eta_n e(n \theta) \right|^p \, d\theta$$

(2.8)

$$\lesssim \tau^p N^{p-1} + \int_0^1 \left| \sum_{n=1}^N \eta_n e(n \theta) \right|^p \, d\theta.$$ 

One now checks that

$$\mathbb{E} \int_0^1 \left| \sum_{n=1}^N \eta_n e(n \theta) \right|^p \, d\theta \leq C_p (N \tau(1 - \tau))^{\frac{p}{2}}.$$ 

This can be verified by expanding the norm for even $p$ and then interpolating. Indeed,

$$\mathbb{E} \int_0^1 \left| \sum_{n=1}^N \eta_n e(n \theta) \right|^{2k} \, d\theta = \mathbb{E} \int_0^1 \left| \sum_{n_1, \ldots, n_k=1}^N \eta_{n_1} \cdots \eta_{n_k} e((n_1 + \cdots + n_k) \theta) \right|^2 \, d\theta$$

$$= \sum_n \mathbb{E} \left| \sum_{n_1 + \cdots + n_k = n} \eta_{n_1} \cdots \eta_{n_k} \right|^2 = \sum_{n_1 + \cdots + n_k = m_1 + \cdots + m_k} \mathbb{E} [\eta_{n_1} \cdots \eta_{n_k} \eta_{m_1} \cdots \eta_{m_k}]$$

(2.9)

$$\leq C_k \sum_{r=1}^k \sum_{n_1, \ldots, n_r=1}^N \mathbb{E} |\eta_{n_1}|^{s_1} \cdots \mathbb{E} |\eta_{n_r}|^{s_r}$$

(2.10)

$$\leq C_k \sum_{r=1}^k N^{r}(\tau(1 - \tau))^r \leq C_k (N \tau(1 - \tau))^k.$$ 

The constants in (2.9) and (2.10) are of a combinatorial nature and not necessarily the same. The relevant point in (2.9) is that $s_i \geq 2$ which is due to independence and $\mathbb{E} \eta_j = 0$. In particular, $s_i \geq 2$ implies the important fact $r \leq k$. Moreover, to pass to the last line we used that for every positive integer $s \geq 2$

$$\tau(1 - \tau) \geq \mathbb{E} \eta_j^s = \tau(1 - \tau)(\tau^{s-1} + (1 - \tau)^{s-1}) \geq 2^{s-1}(1 - \tau).$$
To obtain the lower bound on the expectation, one splits the integral in $\theta$ into the region where the Dirichlet kernel dominates the mean zero random sum and vice versa. More precisely, with $h = \sqrt{\tau N^{-1}} = N^{-\frac{1+\delta}{2}}$,

$$I_{p,N} \geq \int_{|\theta| < \frac{1}{h}} \left| \sum_{n=1}^{N} \tau e(n\theta) \right|^p d\theta - \int_{|\theta| < \frac{1}{h}} \left| \sum_{n=1}^{N} \eta_n e(n\theta) \right|^p d\theta + \int_{|\theta| > h} \left| \sum_{n=1}^{N} \tau e(n\theta) \right|^p d\theta
\quad + \int_{h}^{1-h} \left| \sum_{n=1}^{N} \eta_n e(n\theta) \right|^p d\theta - \int_{h}^{1-h} \left| \sum_{n=1}^{N} \tau e(n\theta) \right|^p d\theta$$

(2.11) $\geq \tau^p N^{p-1} - C \int_{|\theta| < \frac{1}{h}} \left| \sum_{n=1}^{N} \eta_n e(n\theta) \right|^p d\theta + \int_{|\theta| > h} \left| \sum_{n=1}^{N} \eta_n e(n\theta) \right|^p d\theta - C \tau^p h^{1-p}$.

According to Corollary 2.3, the first integral in (2.11) is

(2.12) $\leq N^{-1} (\log N)^{\frac{p}{2}} (\tau(1-\tau)N)^{\frac{p}{2}}$

up to a negligible probability. For the second, one has because of $p \geq 2$

$$\int_{h}^{1-h} \left| \sum_{n=1}^{N} \eta_n e(n\theta) \right|^p d\theta \geq \left( \int_{h}^{1-h} \left| \sum_{n=1}^{N} \eta_n e(n\theta) \right|^2 d\theta \right)^{\frac{p}{2}} - \int_{|\theta| \leq h} \left| \sum_{n=1}^{N} \eta_n e(n\theta) \right|^p d\theta$$

(2.13) $\geq \left( \sum_{n=1}^{N} \eta_n^2 \right)^{\frac{p}{2}} - C h (N \tau \log N)^{\frac{p}{2}}$

where the last term in (2.13) is obtained from Corollary 2.3. Using $p \geq 2$ again,

$$\mathbb{E} \left( \sum_{n=1}^{N} \eta_n^2 \right)^{\frac{p}{2}} \geq \left( \mathbb{E} \sum_{n=1}^{N} \eta_n^2 \right)^{\frac{p}{2}} \geq \left( N \tau(1-\tau) \right)^{\frac{p}{2}}.$$

In fact, Lemma 2.2 gives the following more precise estimate:

(2.14) $\Pr \left[ \left| \sum_{n=1}^{N} (\eta_n^2 - \mathbb{E} \eta_n^2) \right| \geq \lambda \sqrt{N \mathbb{E} (|\eta_1^2 - \mathbb{E} \eta_1^2|^2)} \right] \leq 4e^{-\lambda^2/8}$

provided the conditions (2.2) hold. One checks that $\mathbb{E} (|\eta_1^2 - \mathbb{E} \eta_1^2|^2) \propto \tau(1-\tau)$. Hence it follows from (2.14) that for large $N$

$$\Pr \left[ \sum_{n=1}^{N} \eta_n^2 \leq \frac{1}{2} \mathbb{E} \sum_{n=1}^{N} \eta_n^2 = \frac{1}{2} N \tau(1-\tau) \right] \leq \Pr \left[ \left| \sum_{n=1}^{N} \eta_n^2 - \mathbb{E} \sum_{n=1}^{N} \eta_n^2 \right| \geq \frac{1}{2} N \tau(1-\tau) \right]
\leq \Pr \left[ \left| \sum_{n=1}^{N} \eta_n^2 - \mathbb{E} \sum_{n=1}^{N} \eta_n^2 \right| \geq \log N \sqrt{N \tau(1-\tau)} \right] \leq 4e^{-\left(\log N\right)^2/8},$$
since with our choice of parameters (2.2) hold for large $N$. Inserting this bound into (2.13) now yields (recall that $h = \sqrt{\tau N^{-1}} = N^{-\frac{1+\delta}{2}}$)

$$
\int_{-h}^{1-h} \left| \sum_{n=1}^{N} \eta_n e(n\theta) \right|^p d\theta \geq \left( \frac{1}{2} N\tau (1 - \tau) \right)^{\frac{p}{2}} - C N^{-\frac{1+\delta}{2}} (N\tau \log N)^{\frac{p}{2}} \gtrsim (N\tau)^{\frac{p}{2}}
$$

up to negligible probability. In view of this bound and (2.12), one obtains from (2.11) that

$$
I_{p,N} \gtrsim \tau^p N^{p-1} - C \int_{|\theta| < \frac{1}{N}} \left| \sum_{n=1}^{N} \eta_n e(n\theta) \right|^p d\theta + \int_{|\theta| > h} \left| \sum_{n=1}^{N} \eta_n e(n\theta) \right|^p d\theta - C\tau^p h^{1-p}
$$

up to negligible probability. To remove the final term in the first line we used that $(N\tau)^{\frac{p}{2}} \gtrsim \tau^p h^{1-p}$ which follows from our choice of $h$ provided $N$ is big. \hfill \square

### 2.2 Suprema of random processes

We now collect the statements from Bourgain’s paper that we will need. The first is Lemma 1 from [13] with $q_0 = 1$. In fact, Bourgain’s lemma is slightly stronger because of certain $\log \frac{1}{\tau}$-factors. While these factors are important for his purposes, they play no role in our argument. We present the proof for the reader’s convenience, following Bourgain’s original argument. Another proof was found by Ledoux and Talagrand [11] which is close to the ideology surrounding Dudley’s theorem on suprema of Gaussian processes. While their point of view is perhaps more conceptual, we have found it advantageous to follow [11]. Throughout, if $x \in \mathbb{R}^N$, then $|x| = |x|_{\ell_2} = \left( \sum_{j=1}^{N} x_j^2 \right)^{\frac{1}{2}}$ is the Euclidean norm. Secondly, $N_2(\mathcal{E}, t)$ refers to the $L^2$-entropy of the set $\mathcal{E}$ at scale $t$. Recall that this is defined to be the minimal number of $L^2$-balls of radius $t$ needed to cover $\mathcal{E}$.

**Lemma 2.5.** Let $\mathcal{E} \subset \mathbb{R}^N$, $B = \sup_{x \in \mathcal{E}} |x|$, and $\xi_j$ be selector variables as above with $\mathbb{P}[\xi_j = 1] = \tau$, $\mathbb{P}[\xi_j = 0] = 1 - \tau$, and $0 < \tau < 1$ arbitrary. Let $1 \leq m \leq N$. Then

$$
\mathbb{E} \sup_{x \in \mathcal{E}, |x| = m} \left[ \sum_{j \in A} \xi_j x_j \right] \lesssim (\tau m + 1)^{\frac{p}{2}} B + \int_0^B \sqrt{\log N_2(\mathcal{E}, t)} \, dt
$$

where $N_2$ refers to the $L^2$ entropy.

**Proof.** Let $\mathcal{E}_k$ be minimal $2^{-k}$-nets for $\mathcal{E}$ with $2^{-k} \leq B$. Let $B = 2^{-k_0}$. Then every $x \in \mathcal{E}$ can be written as

$$
x = x_{k_0} + \sum_{k=k_0}^{\infty} (x_{k+1} - x_k) = x_{k_0} + \sum_{k=k_0}^{\infty} 2^{-k+1} y_k
$$

where $x_k \in \mathcal{E}_k$ for every $k \geq k_0$. We can and do set $x_{k_0} = 0$. Now, $y_k \in \mathcal{F}_k$ where $\text{diam}(\mathcal{F}_k) \leq 1$ and $\#(\mathcal{F}_k) \leq \#(\mathcal{E}_k) \cdot \#(\mathcal{E}_{k+1})$. Hence

$$
(2.15) \quad \log \#\mathcal{F}_k \leq C \log \#\mathcal{E}_{k+1},
$$
and thus
\begin{equation}
(2.16) \quad \mathbb{E} \sup_{x \in \mathcal{E}, |A| = m} \left[ \sum_{j \in A} \xi_j x_j \right] \leq \sum_{k \geq k_0} 2^{-k+1} \mathbb{E} \sup_{y \in \mathcal{F}_k, |A| \leq m} \sum_{i \in A} \xi_i |y_i|.
\end{equation}

Now fix some $k \geq k_0$ and write $\mathcal{F}$ instead of $\mathcal{F}_k$. Moreover, replacing every vector $y = \{y_j\}_{j=1}^N \in \mathcal{F}$ with the vector $\{|y_i|\}_{i=1}^N$, we may assume that $\mathcal{F} \subset \mathbb{R}_+$. Note that this changes neither the diameter nor the cardinality of $\mathcal{F}$. With $0 < \rho_1 < \rho_2$ to be determined, one has
\begin{equation}
\sum_{i \in A} \xi_i y_i \leq \sum_{y_i \geq \rho_2} y_i + \sum_{i \in A, y_i \leq \rho_1, \rho_1 < y_i < \rho_2} y_i + \sum_{\rho_1 < y_i < \rho_2} \xi_i y_i \leq \rho_2^{-1} \sum_{y_i \geq \rho_2} y_i^2 + m \rho_1 + \sum_{\rho_1 < y_i < \rho_2} \xi_i y_i.
\end{equation}

Let $q = 1 + \lfloor \log \mathcal{F} \rfloor$. Since $|y| \leq 1$, one concludes that
\begin{equation}
\mathbb{E} \sup_{y \in \mathcal{F}, |A| \leq m} \sum_{i \in A} \xi_i y_i \leq \rho_2^{-1} + m \rho_1 + \sup_{y \in \mathcal{F}, \rho_1 < y_i < \rho_2} \sum_{i \in A} \xi_i y_i
\end{equation}
\begin{equation}
\lesssim \rho_2^{-1} + m \rho_1 + \mathbb{E} \left[ \sum_{y \in \mathcal{F}} \left( \sum_{\rho_1 < y_i < \rho_2} \xi_i y_i \right)^q \right]^{\frac{1}{q}}
\end{equation}
\begin{equation}
\lesssim \rho_2^{-1} + m \rho_1 + \mathbb{E} \left[ \sum_{\rho_1 < y_i < \rho_2} \left( \sum_{\rho_1 < y_i < \rho_2} \xi_i y_i \right)^q \right]^{\frac{1}{q}}
\end{equation}
\begin{equation}
\lesssim \rho_2^{-1} + m \rho_1 + \mathbb{E} \left[ \sum_{\rho_1 < y_i < \rho_2} \left( \sum_{\rho_1 < y_i < \rho_2} \xi_i y_i \right)^q \right]^{\frac{1}{q}}
\end{equation}
\begin{equation}
\lesssim \rho_2^{-1} + m \rho_1 + \mathbb{E} \left[ \sum_{\rho_1 < y_i < \rho_2} \left( \sum_{\rho_1 < y_i < \rho_2} \xi_i y_i \right)^q \right]^{\frac{1}{q}}
\end{equation}
\begin{equation}
\lesssim \rho_2^{-1} + m \rho_1 + \sum_{|y| \leq 1} \mathbb{E} \left[ \sum_{\rho_1 < y_i < \rho_2} \xi_i(\omega) y_i \right]_q \sup_{|y| \leq 1} \mathbb{E} \left[ \sum_{\rho_1 < y_i < \rho_2} \xi_i(\omega) y_i \right]_q.
\end{equation}

Here (2.17) follows from the embedding $\ell^q(\mathcal{F}) \hookrightarrow \ell^\infty(\mathcal{F})$, (2.18) follows from Hölder’s inequality, and to pass from (2.19) to (2.20) one uses that
\begin{equation}
(\# \mathcal{F})^{\frac{1}{q}} = \exp(\log \# \mathcal{F})/q \leq e
\end{equation}
by our choice of $q = 1 + \lfloor \log \mathcal{F} \rfloor$. To control the last term in (2.20), we need the following simple estimate, see Lemma 2 in [31]. By the multinomial theorem (for any positive integer $q$),
\begin{equation}
\mathbb{E} \left[ \sum_{j=1}^n \xi_j \right]^q = \sum_{q_1 + \ldots + q_n = q} \binom{q}{q_1, \ldots, q_n} \mathbb{E} \xi_1^{q_1} \cdot \ldots \cdot \mathbb{E} \xi_n^{q_n}
\end{equation}
\begin{equation}
= \sum_{\ell=1}^q \sum_{1 \leq i_1 < i_2 < \ldots < i_\ell \leq n} \sum_{q_{i_1} + \ldots + q_{i_\ell} = q} \binom{q}{q_{i_1}, \ldots, q_{i_\ell}} \left( \prod_{i=1}^\ell \xi_i^{q_{i_\ell}} \right) \leq \sum_{\ell=1}^q \sum_{\ell_{i_\ell} \geq 1} \binom{q}{q_{i_1}, \ldots, q_{i_\ell}} \left( \prod_{i=1}^\ell \xi_i^{q_{i_\ell}} \right)
\end{equation}
\begin{equation}
\leq (q + e \tau n)^q.
\end{equation}
It is perhaps more natural (and also more precise) to estimate $q$th moments by means of the Bernoulli law
\[ \mathbb{E} \left[ \sum_{j=1}^{n} \xi_j \right]^q = \sum_{\ell=0}^{n} \binom{n}{\ell} \ell^q \tau^{\ell} (1 - \tau)^{n-\ell}. \]

But we have found the approach leading to (2.21) more flexible since it also applies to non Bernoulli cases. Continuing with the final term in (2.20) one concludes from (2.21) that
\[
(2.22) \quad \sup_{|y| \leq 1} \left\| \sum_{\rho_1 < y_i < \rho_2} \xi_i(\omega) y_i \right\|_{L^q(\omega)} \leq 2 \sum_{\rho_1 < \rho_2} 2^{-j} \left\| \sum_{i=1}^{2j} \xi_i(\omega) \right\|_{L^q(\omega)} \leq 2 \sum_{\rho_1 < \rho_2} 2^{-j} (q + e \tau 2^j) \leq q \rho_2 + \tau \rho_1^{-1}.
\]

Inserting this bound into (2.21) and setting $\rho_1 = \sqrt{\tau/m}$ and $\rho_2 = q^{-1/2}$ yields
\[
\mathbb{E} \sup_{y \in F, |A| \leq m} \sum_{i \in A} \xi_i y_i \lesssim \sqrt{m \tau} + \sqrt{q} \lesssim \sqrt{m \tau} + 1 + \sqrt{\log \#F}.
\]

The lemma now follows in view of (2.15) and (2.16).

\[\square\]

### 2.3 Entropy bounds

As in [B1], we will need bounds on certain covering numbers, also called entropies. We recall those bounds starting with the so-called “dual Sudakov inequality” for the reader’s convenience. More on this can be found in Pisier [P] and Bourgain, Lindenstrauss, Milman [BLM], Section 4. Consider $\mathbb{R}^n$ with two norms, the Euclidean norm $| \cdot |$ and some other (semi)norm $\| \cdot \|$. We set $X = (\mathbb{R}^n, \| \cdot \|)$ and denote the unit ball in this space by $B_X$, whereas the Euclidean unit ball will be $B^n$. As usual, for any set $U \subset \mathbb{R}^n$ and $t > 0$ one sets
\[
(2.24) \quad E(U, B_X, t) := \inf \left\{ N \geq 1 \mid \exists x_j \in \mathbb{R}^n, 1 \leq j \leq N, U \subset \bigcup_{j=1}^{N} (x_j + tB_X) \right\}.
\]

There are two closely related quantities, namely
\[
(2.25) \quad \tilde{E}(U, B_X, t) := \inf \left\{ N \geq 1 \mid \exists x_j \in U, 1 \leq j \leq N, U \subset \bigcup_{j=1}^{N} (x_j + tB_X) \right\} \quad \quad D(U, B_X, t) := \sup \left\{ M \geq 1 \mid \exists y_j \in U, 1 \leq j \leq M, \| y_j - y_k \| \geq t, j \neq k \right\}.
\]

There are the following comparisons between these quantities:
\[
(2.26) \quad D(U, B_X, t) \geq \tilde{E}(U, B_X, t) \geq E(U, B_X, t) \geq D(U, B_X, 2t) \quad \text{and} \quad E(U, B_X, t) \geq \tilde{E}(U, B_X, 2t).
\]
The final inequality holds because every covering of \( U \) by arbitrary \( t \)-balls gives rise to a covering by \( 2t \)-balls with centers in \( U \). To see that \( E(U, B_X, t) \geq D(U, B_X, 2t) \), let \( \{y_j\}_{j=1}^M \subset U \) be \( 2t \)-separated and \( U \subset \bigcup_{i=1}^N (x_i + tB_X) \). Then every \( y_j \in x_i + tB_X \) for some \( i = i(j) \). Moreover, \( j \neq k \implies i(j) \neq i(k) \). Hence \( N \geq M \).

The “dual Sudakov inequality” Lemma 2.6 bounds \( E(B^n, B_X, t) \) in terms of the Levy mean

\[
M_X := \int_{S^{n-1}} \|x\| \, d\sigma(x),
\]

where \( \sigma \) is the normalized measure on \( S^{n-1} \). Alternatively, one has

\[
M_X = \alpha_n (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\|x\|^2} \, dx
\]

\[
M_X = \alpha_n \int \left\| \sum_{i=1}^n g_i(\omega) \mathbf{e}_i \right\| \, d\mathbb{P}(\omega),
\]

where

\[
\alpha_n = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \sqrt{\frac{2}{n}} \sim n^{-\frac{1}{2}}
\]

and \( g_i \) are i.i.d. standard normal variables, and \( \mathbf{e}_i \) is an ONS. The probabilistic form (2.23) is of course just a restatement of (2.28), whereas the latter can be obtained from the definition (2.27) by means of polar coordinates. The following lemma is due to Pajor and Tomczak-Jaegerman [PT-J] but the proof given below is due to Pajor and Talagrand, see [BLM].

**Lemma 2.6.** For any \( t > 0 \)

\[
\log E(B^n, B_X, t) \leq C n \left( \frac{M_X}{t} \right)^2,
\]

where \( C \) is an absolute constant.

**Proof.** Let \( \{x_i\}_{i=1}^N \subset B^n \), \( \|x_i - x_j\| \geq t \) for \( i \neq j \) and \( N \) maximal. Then \( E(B^n, B_X, t) \leq N \). Let \( \mu(dx) = (2\pi)^{-\frac{n}{2}} e^{-\|x\|^2/2} \, dx \). Then by definition (2.27),

\[
\mu(\|x\| > 2M_X \alpha^{-1}) < \frac{1}{2} \implies \mu(\|x\| \leq 2M_X \alpha^{-1}) > \frac{1}{2}.
\]

Moreover, \( \{x_i + \frac{t}{2}B_X\}_{i=1}^N \) and therefore also \( \{y_i + 2M_X \alpha^{-1} B_X\}_{i=1}^N \) have mutually disjoint interiors, where we have set \( y_i = 4M_X (t \alpha^{-1}) x_i \). Now, by symmetry of \( B_X \) and convexity of \( e^{-u} \),

\[
\mu(y_i + 2M_X \alpha^{-1} B_X) = (2\pi)^{-\frac{n}{2}} \int_{2M_X \alpha^{-1} B_X} e^{-|y-y_i|^2/2} \, dy
\]

\[
= (2\pi)^{-\frac{n}{2}} \int_{2M_X \alpha^{-1} B_X} \frac{1}{2} \left[ e^{-|y-y_i|^2/2} + e^{-|y+y_i|^2/2} \right] \, dy
\]

\[
\geq (2\pi)^{-\frac{n}{2}} \int_{2M_X \alpha^{-1} B_X} e^{-\left(|y-y_i|^2 + |y+y_i|^2\right)/4} \, dy
\]

\[
= (2\pi)^{-\frac{n}{2}} \int_{2M_X \alpha^{-1} B_X} e^{-\left(|y|^2 + |y_i|^2\right)/2} \, dy \geq \frac{1}{2} e^{-|y_i|^2/2},
\]

12
where the last step follows from (2.31). Since $|y_i| \leq 4M_X(t\alpha_n)^{-1}$,

$$
\mu(y_i + 2M_X\alpha_n^{-1}B_X) \geq \frac{1}{2} \exp \left( -\frac{1}{2}(4M_X)^2(t\alpha_n)^{-2} \right).
$$

Hence

$$
1 \geq \sum_{i=1}^{N} \mu(y_i + 2M_X\alpha_n^{-1}B_X) \geq \frac{1}{2} N \exp \left( -\frac{1}{2}(4M_X)^2(t\alpha_n)^{-2} \right),
$$

and the lemma follows since $\alpha_n \asymp n^{-\frac{1}{2}}$.

Observe that (2.30) is a poor bound as $t \to 0$. Indeed, rather than the $\exp(t^{-2})$ behavior exhibited by (2.30) the true asymptotics is $t^{-n}$ as $t \to 0$. The point of Lemma 2.6 is to relate the size of $t$ to both $M_X$ and $n$. This is best illustrated by some standard examples.

- Firstly, take $X = \ell_1^n$. In that case,

$$
\alpha_n^{-1} M_X = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \sum_{i=1}^{n} |x_i| e^{-\frac{|x|^2}{2}} dx = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x_1| e^{-\frac{x^2}{2}} dx_1 = \frac{2n}{\sqrt{2\pi}}.
$$

Therefore, $M_X \asymp \sqrt{n}$. By (2.30),

$$
\sup_{n} E(B^n, B_{\ell_1^n}, n) \leq C.
$$

This bound is somewhat wasteful. Indeed, since $\sqrt{n}B_{\ell_1^n} \supset B^n$, one actually has

$$
\sup_{n} E(B^n, B_{\ell_1^n}, \sqrt{n}) \leq C.
$$

The reason for this “overshoot” is that the major contribution to $M_X$ comes from the corners of $B_{\ell_1^n}$. On the other hand, these corners do not determine the smallest $r$ for which $rB_X \supset B^n$.

- Secondly, consider $X = \ell_\infty^n$. Using (2.29),

$$
\alpha_n^{-1} M_X = \mathbb{E} \sup_{1 \leq i \leq n} |g_i| \asymp \sqrt{\log n},
$$

where the latter bound is a rather obvious and well-known fact. Hence

$$
M_X \asymp \sqrt{\frac{\log n}{n}}
$$

which implies via (2.30) that

$$
\sup_{n} E(B^n, B_{\ell_\infty^n}, \sqrt{\log n}) \leq C.
$$

This is the correct behavior up to the $\log n$-factor since $B^n \subset B_{\ell_\infty^n}$. In contrast to the previous case, the bulk of the contribution to $M_X$ comes from that part of $B_{\ell_\infty^n}$ that is also the most relevant for the covering of the Euclidean ball.
Finally, and most relevantly for our purposes, identify $\mathbb{R}^n$ with the space of trigonometric polynomials with real coefficients of degree $n$, i.e.,

$$\mathbb{R}^n \simeq \left\{ \sum_{j=1}^{n} a_j e(j\theta) \mid a_j \in \mathbb{R} \right\}.$$  

(2.32)

Furthermore, define $\| \cdot \| = \| \cdot \|_{L^q(\mathbb{T})}$ where $q \geq 2$ is fixed. Then

$$M_X = \alpha_n \int_{\Omega} \left\| \sum_{j=1}^{n} g_j(\omega)e(j\theta) \right\|_{L^q(\mathbb{T})} dP(\omega)$$

(2.33)

$$= \alpha_n \mathbb{E} \int_{\Omega} \left\| \sum_{j=1}^{n} \pm g_j(\omega)e(j\theta) \right\|_{L^q(\mathbb{T})} dP(\omega)$$

(2.34)

$$\leq C \alpha_n \sqrt{q} \int_{\Omega} \left( \sum_{j=1}^{n} g_j^2(\omega) \right)^{\frac{1}{2}} dP(\omega)$$

$$\leq C \alpha_n \sqrt{q} \left( \int_{\Omega} \sum_{j=1}^{n} g_j^2(\omega) dP(\omega) \right)^{\frac{1}{2}} = C \alpha_n \sqrt{q} \sqrt{n} \leq C \sqrt{q}.$$

In (2.33) the expectation $\mathbb{E}$ refers to the random and symmetric choice of signs $\pm$, whereas the $\sqrt{q}$-factor in (2.34) is due to the fact that the constant in Khinchin’s inequality grows like $\sqrt{q}$.

Hence

$$\log E(B^n, B_X, t) \leq C q n t^{-2}$$

(2.35)

in this case.

The proof of Theorem 2.1 requires estimating $N_q(P_A, t) := E(P_A, B_{L^q(\mathbb{T})}, t)$. Here

$$P_A := \left\{ \sum_{n \in A} a_n e(n\theta) \mid |a| = |a|_{\ell^2_N} \leq 1 \right\}$$

where $A \subset [1, N]$. Invoking (2.35) leads to

$$\log N_q(P_A, t) \leq C q |A| t^{-2}.$$  

(2.36)

This bound is basically optimal when $t \sim 1$, but it can be improved for very small and very large $t$.

**Corollary 2.7.** For $q \geq 2$ and any $A \subset [1, N]$

$$\log N_q(P_A, t) \leq C q |A| \left[ 1 + \log \frac{1}{t} \right] \text{ if } 0 < t \leq \frac{1}{2}.$$  

(2.37)

**Proof.** Let $m = |A|$. Thus $1 \leq m \leq N$. Notice firstly that

$$\log N_q\left( \left\{ \sum_{n \in A} a_n e(n\theta) \mid |a| \leq 1 \right\}, t \right) \leq C m \log \frac{1}{t} + \log N_q\left( \left\{ \sum_{n \in A} a_n e(n\theta) \mid |a| \leq 1 \right\}, 1 \right).$$  

(2.38)
This follows from the fact that for any norm $\| \cdot \|$ in $\mathbb{R}^m$ with unit-balls $B_X$ one has

\begin{equation}
D(B_X, B_X, t) \leq (4/t)^m \quad \text{for all } 0 < t < 1
\end{equation}

by scaling and volume counting, see (2.23) for the definition of $D(B_X, B_X, t)$. Indeed, suppose $M = D(B_X, B_X, t)$. Then there are $M$ disjoint balls $\{x_j + \frac{1}{2}tB_X\}_{j=1}^M$ with centers $x_j \in B_X$. Since $x_j + \frac{1}{2}tB_X \subset 2B_X$ if $t < 1$, it follows that

$$
\sum_{j=1}^{M} |\frac{1}{2}tB_X| \leq |2B_X| \implies M(t/2)^m \leq 2^m,
$$

as claimed. Here $| \cdot |$ stands for Lebesgue measure. Thus (2.39) holds, and therefore also (2.38) in view of (2.26). Hence

$$
\log N_q(P_A, t) \leq Cm \log \frac{1}{t} + \log N_q(\left\{ \sum_{n \in A} a_n e(n\theta) \right\} \left| a \right| \leq 1, 1) \\
\leq Cm \log \frac{1}{t} + Cqm,
$$

where the final term follows from (2.35).

We now turn to large $t$. The following corollary slightly improves on the rate of decay.

**Corollary 2.8.** Let $q \geq 2$ and $A \subset [1, N]$. With $P_A$ as above one has

\begin{equation}
\log N_q(P_A, t) \leq Cq |A| t^{-\nu} \quad \text{if } t > \frac{1}{2}
\end{equation}

where $\nu = \nu(q) > 2$.

**Proof.** Recall that $N_q(P_A, t) = E(P_A, B_{L^q}, t)$. Using (2.26), one obtains from (2.36) that also

\begin{equation}
\log \tilde{E}(P_A, B_{L^q}, t) \leq Cq |A| t^{-2}.
\end{equation}

Let $q < r$, $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{r}$. Since for any $f, g \in P_A$

$$
\|f - g\|_q \leq \|f - g\|_{1-\theta}^{1-\theta} \|f - g\|_r^\theta \leq 2 \|f - g\|_r^\theta,
$$

one concludes from (2.41) that

$$
\log \tilde{E}(P_A, B_{L^q}, t) \leq \log \tilde{E}(P_A, B_{L^r}, (t/2)^{1/\theta}) \leq Cq |A| t^{-2/\theta}.
$$

Applying (2.26) again yields (2.44).
2.4 Decoupling lemma

Lastly, we require a version of Bourgain’s decoupling technique, cf. Lemma 4 in [B1]. In contrast to his case we only need to decouple into two sets rather than three.

**Lemma 2.9.** Let real-valued functions $h_\alpha(u)$ on $\mathbb{R}$ be given for $\alpha = 1, 2, 3$ that satisfy

$$
|h_\alpha(u)| \leq C(1 + |u|)^{p_\alpha}, \quad |h_\alpha(u) - h_\alpha(v)| \leq C(1 + |u| + |v|)^{p_\alpha - \delta}|u - v|
$$

for all $u, v \in \mathbb{R}$ and some fixed choice of $p_\alpha > 0, \delta > 0$. Let $x, y, z \in \ell^2_\mathbb{N}$ be sequences so that $|x|, |y|, |z| \leq 1$ and suppose $\zeta_j = \zeta_j(t)$ are i.i.d. random variables with $\mathbb{P}(\zeta_j = 1) = \mathbb{P}(\zeta_j = 0) = \frac{1}{2}$. We assume that $\mathbb{P}(dt) = dt$ on $[0, 1]$, say. Set $R^j_1 = \{1 \leq j \leq N\zeta_j(t) = 1\}$, $R^j_2 = \{1 \leq j \leq N\zeta_j(t) = 0\}$. Then

$$
\left| \int h_1(\sum_{i \in R^1_1} x_i)h_2(\sum_{i \in R^1_2} y_i)h_3(\sum_{i \in R^1_2} z_i) dt - h_1(\frac{1}{2}\sum_{i} x_i)h_2(\frac{1}{2}\sum_{i} y_i)h_3(\frac{1}{2}\sum_{i} z_i) \right|
$$

\begin{equation}
\leq C \left(1 + \left|\sum_{i} x_i\right| + \left|\sum_{i} y_i\right| + \left|\sum_{i} z_i\right|\right)^{p - \delta}
\end{equation}

(2.42)

where $p = p_1 + p_2 + p_3$ and $C$ is some absolute constant.

**Proof.** By assumption,

$$
\left| h_\alpha(\sum_{i \in R^1_1} x_i) - h_\alpha(\frac{1}{2}\sum_{i = 1}^{N} x_i) \right| \leq C \left(1 + \left|\sum_{i = 1}^{N} x_i\right| + \sum_{i = 1}^{N}(\zeta_i - \frac{1}{2})x_i\right)^{p_\alpha - \delta}\left|\sum_{i = 1}^{N}(\zeta_i - \frac{1}{2})x_i\right|^{\delta}
$$

\begin{equation}
\leq C \left(1 + \left|\sum_{i = 1}^{N} x_i\right|\right)^{p_\alpha - \delta}\left(1 + \sum_{i = 1}^{N}(\zeta_i - \frac{1}{2})x_i\right)^{p}
\end{equation}

$$
\left| h_\alpha(\sum_{i \in R^1_1} x_i) + h_\alpha(\frac{1}{2}\sum_{i = 1}^{N} x_i) \right| \leq C \left(1 + \left|\sum_{i = 1}^{N} x_i\right|\right)^{p_\alpha}\left(1 + \sum_{i = 1}^{N}(\zeta_i - \frac{1}{2})x_i\right)^{p}
$$

for $\alpha = 1, 2, 3$. Hence

$$
\left| \int h_1(\sum_{i \in R^1_1} x_i)h_2(\sum_{i \in R^1_2} y_i)h_3(\sum_{i \in R^1_2} z_i) dt - h_1(\frac{1}{2}\sum_{i} x_i)h_2(\frac{1}{2}\sum_{i} y_i)h_3(\frac{1}{2}\sum_{i} z_i) \right|
$$

\begin{equation}
\leq C \left(1 + \left|\sum_{i} x_i\right| + \left|\sum_{i} y_i\right| + \left|\sum_{i} z_i\right|\right)^{p - \delta}
\end{equation}

$$
\cdot \int \left(1 + \sum_{i = 1}^{N}(\zeta_i - \frac{1}{2})x_i\right)^{p_\alpha}\left(1 + \sum_{i = 1}^{N}(\zeta_i - \frac{1}{2})y_i\right)\left|\sum_{i = 1}^{N}(\zeta_i - \frac{1}{2})z_i\right|^p dt.
$$

(2.43)

The lemma now follows from Khinchin’s inequality. Indeed,

$$
\int \left|\sum_{i = 1}^{N}(\zeta_i - \frac{1}{2})x_i\right|^p dt \leq C_p |x|^p \leq C_p,
$$

by assumption. 

\[\square\]
2.5 The proof of Theorem 2.1 for \( p = 3 \)

We now start the proof of Theorem 2.1 for \( p = 3 \). In fact, we state a somewhat more precise form of this theorem for \( p = 3 \).

**Theorem 2.10.** Let \( 0 < \delta < 1 \) be fixed. For every positive integer \( N \) we let \( \xi_j = \xi_j(\omega) \) be i.i.d. variables with \( \mathbb{P}[\xi_j = 1] = \tau, \mathbb{P}[\xi_j = 0] = 1 - \tau \) where \( \tau = N^{-\delta} \). Define a random subset

\[
S(\omega) = \{ j \in [1, N] | \xi_j(\omega) = 1 \}.
\]

Then for every \( \gamma > 0 \) there is a constant \( C_\gamma \) so that

\[
\sup_{N \geq 1} \mathbb{P} \left( \sup_{|a_n| \leq 1} \left\| \sum_{n \in S(\omega)} a_n e(n\theta) \right\|_{L^3(\mathbb{T})} \geq C_\gamma \right) \leq \gamma.
\]

**Proof.** Firstly, note that for fixed \( 0 < \delta < 1 \) and large \( N \) Lemma 2.2 implies that

\[
\mathbb{P} \left[ \sum_{n=1}^{N} \xi_n \geq 2\tau N \right] \lesssim \exp(-c\tau N).
\]

Let \( \mathbb{E}' \) denote the restricted expectation

\[
\mathbb{E}' \sup_{|a_n| \leq 1} \left\| \sum_{n=1}^{N} \xi_n a_n e(n\theta) \right\|_{L^3(\mathbb{T})} := \mathbb{E} \chi[\sum_n \xi_n \leq 2\tau N] \sup_{|a_n| \leq 1} \left\| \sum_{n=1}^{N} \xi_n a_n e(n\theta) \right\|_{L^3(\mathbb{T})}.
\]

Then

\[
\mathbb{E} \sup_{|a_n| \leq 1} \left\| \sum_{n=1}^{N} \xi_n a_n e(n\theta) \right\|_{L^3(\mathbb{T})} \leq N \exp(-c\tau N) + \mathbb{E}' \sup_{|a_n| \leq 1} \left\| \sum_{n=1}^{N} \xi_n a_n e(n\theta) \right\|_{L^3(\mathbb{T})}
\]

\[
\leq O(1) + \mathbb{E}' \sup_{|a_n| \leq 1} \left\| \sum_{n=1}^{N} \xi_n a_n e(n\theta) \right\|_{L^3(\mathbb{T})}.
\]

From now on, we set \( m = 2\tau N \), and we will mostly work with \( \mathbb{E}' \) instead of \( \mathbb{E} \). Next, fix some \( \{ a_n \}_{n=1}^{N} \) with \( |a_n| \leq 1 \). Then, rescaling Lemma 2.9 (with \( h_1(x) = h_2(x) = x \) and \( h_3(x) = |x| \)) one obtains that

\[
\frac{1}{8} \int_{0}^{1} \left\| \sum_{n=1}^{N} a_n \xi_n e(n\theta) \right\|_{\mathbb{T}}^{3} \, d\theta = \int_{0}^{1} \left( \sum_{n \in R_{t}^{1}} a_n \xi_n e(n\theta) \right) \sum_{k \in R_{t}^{2}} \bar{a}_k \xi_k e(-k\theta) \sum_{\ell \in R_{t}^{3}} a_\ell \xi_\ell e(\ell\theta) \, d\theta \, dt
\]

\[
+ O \left( \frac{3}{m} \int_{0}^{1} \left( \frac{N}{m} \sum_{n=1}^{N} \frac{a_n \xi_n e(n\theta)}{\sqrt{m}} \right)^{2} \, d\theta \right).
\]

The \( O \)-term in \((2.45)\) is \( O(m^{\frac{3}{2}}) \) by construction. Let \( \{ \xi_n(\omega_1) \}_{n=1}^{N} \) and \( \{ \xi_n(\omega_2) \}_{n=1}^{N} \) denote two independent copies of \( \{ \xi_n(\omega) \}_{n=1}^{N} \). Recall that \( R_{t}^{1} \) and \( R_{t}^{2} \) are disjoint for every \( t \). Therefore, for
\[ \mathbb{E}_{\omega} \sup_{|a_n| \leq 1} \left| \int_0^1 \sum_{n \in R_1^2} a_n \xi_n(\omega) e(n\theta) \sum_{k \in R_2^2} \hat{a}_k \xi_k(\omega) e(-k\theta) \sum_{\ell \in R_2^2} a_{\ell} \xi_\ell(\omega) e(\ell\theta) \right| d\theta \]

This leads to

\[ \mathbb{E}_{\omega} \sup_{|a_n| \leq 1} \left( \int_0^1 \left| \sum_{n=1}^N a_n \xi_n(\omega) e(n\theta) \right|^3 d\theta \right) \]

\[ \lesssim m^{\frac{3}{2}} + \int \mathbb{E}_{\omega_1, \omega_2} \sup_{|a_n| \leq 1} \left( \int_0^1 \left| \sum_{n \in R_1^2} a_n \xi_n(\omega_1) e(n\theta) \sum_{k \in R_2^2} \hat{b}_k \xi_k(\omega_2) e(-k\theta) \sum_{\ell \in R_2^2} b_{\ell} \xi_\ell(\omega_2) e(\ell\theta) \right| d\theta \right) dt \]

\[ \lesssim m^{\frac{3}{2}} + \int \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \sup_{|a_n| \leq 1} \left( \int_0^1 \left| \sum_{n=1}^N a_n \xi_n(\omega_1) e(n\theta) \sum_{k=1}^N \hat{b}_k \xi_k(\omega_2) e(-k\theta) \sum_{\ell=1}^N b_{\ell} \xi_\ell(\omega_2) e(\ell\theta) \right| d\theta \right) dt \]

\[ \lesssim m^{\frac{3}{2}} + \mathbb{E}_{\omega_2} \mathbb{E}_{\omega_1} \sup_{x \in \mathcal{E}(\omega_2)} \sup_{|a_n| \leq 1} \sum_{n=1}^N \xi_n(\omega_1) x_n. \]

Here

\[ \mathcal{E}(\omega_2) := \left\{ \left( \left| e(n) \cdot \sum_{k=1}^N \hat{b}_k \xi_k(\omega_2) e(-k\cdot) \right| \left| \sum_{\ell=1}^N b_{\ell} \xi_\ell(\omega_2) e(\ell\cdot) \right| \right)^n_{n=1} \sup_{1 \leq n \leq N} |b_n| \leq 1 \right\} \subset \mathbb{R}^N_+. \]

In the calculation leading up to (2.47) we firstly used (2.46), secondly the obvious fact that the supremum only increases if we introduce \( \{b_n\}_{n=1}^N \) in addition to \( \{a_n\}_{n=1}^N \), thirdly that one can remove the restrictions to the sets \( R_1^2 \) and \( R_2^2 \) because they can be absorbed into the choice of the sequences \( a_n, b_n \), and lastly that \( \sum_n \xi_n \leq m \) which allows us to introduce \( A \subset [1, N], |A| = m \). If \( x \in \mathcal{E}(\omega_2) \), then

\[ |x|_{AN}^2 \leq \sup_{|a_n| \leq 1} \left\| \sum_k a_k \xi_k(\omega_2) e(k\cdot) \right\|_4^4 \leq \left\| \sum_k \xi_k(\omega_2) e(k\cdot) \right\|_4^4 =: B_2^2(\omega_2) \]

by the \( L^4 \) majorant property. By Lemma 2.4,

\[ \mathbb{E} B_4 \leq \left( \mathbb{E} I_{4,N} \right)^{\frac{1}{2}} \lesssim \tau^2 N^{\frac{3}{2}} + \tau N. \]
We now apply Lemma 2.5 to (2.47). This yields
\[
\mathbb{E}_\omega \sup_{|a_n| \leq 1} \int_0^1 \left| \sum_{n=1}^N a_n \xi_n(\omega)e(n\theta) \right|^3 \, d\theta \lesssim m^3 + \mathbb{E}_{\omega_2} \left[ (\sqrt{\tau m} + 1) B_4(\omega_2) + \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2), t)} \, dt \right]
\]
(2.50) \quad \lesssim (\tau N)^{\frac{3}{2}} + (1 + \tau N^\frac{3}{2})(\tau^2 N^\frac{3}{2} + \tau N) + \mathbb{E}_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2), t)} \, dt.

It remains to deal with the entropy integral in (2.50). To this end, observe that the distance between any two elements in \(\mathcal{E}(\omega_2)\) is of the form
\[
\|g - h\|_2 \lesssim \|g - h\|_\infty(\|g\|_2 + \|h\|_2)
\]
\[
\lesssim N^\varepsilon \|g - h\|_q(\|g\|_2 + \|h\|_2) \lesssim N^\varepsilon \sqrt{m}\|g - h\|_q,
\]
where we chose \(q\) very large depending on \(\varepsilon\) (the factor \(N^\varepsilon\) comes from Bernstein’s inequality). Here \(g, h \in \sqrt{m} \mathcal{P}_A\) where \(A = A(\omega_2) = \{n \in [1, N] \mid \xi_n(\omega_2) = 1\}\) and
\[
\mathcal{P}_A = \left\{ \sum_{n \in A} a_n e(n\cdot) \mid |a|_{\ell_N^2} \leq 1 \right\}.
\]

Actually, our coefficients are in the unit-ball of \(\ell_n^\infty\), but we have embedded this into \(\ell_m^2\) in the obvious way, which leads to the \(\sqrt{m}\)-factor in front of \(\mathcal{P}_A\) (at this point recall that we are working with \(\mathbb{E}_{\omega_2}'\)). One concludes that, for \(\varepsilon > 0\) small and \(q < \infty\) large depending on \(\varepsilon\),
\[
\log N_2(\mathcal{E}(\omega_2), t) \leq \log N_q(\mathcal{P}_A, N^{-\varepsilon}m^{-1}t)
\]
(2.52) \quad \leq C q m \begin{cases} 1 + \log \frac{t}{m} & 0 < t < mN^\varepsilon \\ (m^{-1}N^{-\varepsilon}t)^{-\nu} & t > N^\varepsilon m \end{cases}
\]
where \(\nu > 2\), see Corollary 2.7 and Corollary 2.8. It follows that the last term in (2.50) is at most
\[
\mathbb{E}_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2), t)} \, dt \lesssim N^\varepsilon m^\frac{3}{2}.
\]

Plugging this into (2.50) yields
\[
\mathbb{E}_\omega \sup_{|a_n| \leq 1} \int_0^1 \left| \sum_{n=1}^N a_n \xi_n(\omega)e(n\theta) \right|^3 \, d\theta \lesssim (\tau N)^{\frac{3}{2}} + (1 + \tau N^\frac{3}{2})(\tau^2 N^\frac{3}{2} + \tau N) + N^\varepsilon (\tau N)^{\frac{3}{2}}
\]
(2.53) \quad \lesssim \tau^3 N^2 + N^\varepsilon (\tau N)^{\frac{3}{2}}.

Now suppose \(\delta < \frac{1}{3}\). Then \(\tau^3 N^2 > N^\varepsilon (\tau N)^{\frac{3}{2}}\) provided \(\varepsilon > 0\) is small and fixed, and provided \(N\) is large. Hence, combining (2.53) with Lemma 2.4 leads to Theorem 2.10 at least if \(\delta < \frac{1}{3}\). If one is willing to loose a \(N^\varepsilon\)-factor, then (2.53) in combination with Lemma 2.4 leads to the desired bounds in all cases. On the other hand, if \(\delta \geq \frac{1}{3}\) so that typically \(#(S(\omega)) \lesssim N^\frac{2}{3}\), then Bourgain showed that \(S(\omega)\) is a \(\Lambda_3\) set with large probability. More precisely, he showed that the constant
\[
K_3(\omega) := \sup_{|a|_{\ell_N^2} \leq 1} \left\| \sum_{n \in S(\omega)} a_n e(n\cdot) \right\|_3
\]
satisfies $E K_3^3 \leq C$, see also Theorem 2.13 below. Hence, in our case,

$$E \sup_{|a_n| \leq 1} \left\| \sum_{n=1}^{N} a_n \xi_n(\omega)e(n\cdot) \right\|_3^3 \lesssim (\tau N)^{3/2}.$$ 

Clearly,

$$\left\| \sum_{n=1}^{N} \xi_n(\omega)e(n\cdot) \right\|_3 \geq \left\| \sum_{n=1}^{N} \xi_n(\omega)e(n\cdot) \right\|_2 = \#(S(\omega))^{1/2},$$

and we have thus proved (2.44) for $\delta \geq \frac{1}{3}$ as well.

It is perhaps worth pointing out that interpolation of the $L^4$ bound with the $L^2$ bound gives

$$\tau^{3/2}N^2 + (\tau N)^{3/2},$$

so that the estimate we just obtained is better by the initial $\tau^3$-factor (note that this is due to the $\sqrt{\tau m}$-factor in Lemma 2.5 as compared to a $\sqrt{\tau N}$-factor).

### 2.6 The case of general $p$

The strategy is to first generalize the previous argument to all odd integers using the fact that the majorant property holds for all even integers (for $p = 3$ we used this fact with $p = 4$). Then one runs the same argument again, using now that the (random) majorant property holds for all integers $p$ and so on. For a given $\varepsilon > 0$ this yields that there is a set of $p$ that is $\varepsilon$-dense in $[2, \infty)$ and for which the majorant property holds. This is enough by interpolation, since we are allowing a loss of $N^\varepsilon$ in (2.1). Unfortunately, there are certain technical complications in carrying out this program having to do with the size of $\delta$. In this section we deal with $\delta \leq \frac{1}{2}$, and in the following section we discuss a refinement of the method that allows one to relax this condition in some cases.

Lemma 2.11 formalizes the main probabilistic argument from the previous section. Let $p \geq 2$. In this section, we say that the random majorant property (or RMP in short) holds at $p$ if and only if for every $\varepsilon > 0$ there exists a constant $C_\varepsilon$ so that

$$(2.54) \quad E \sup_{|a_n| \leq 1} \left\| \sum_{n=1}^{N} a_n \xi_n e(n\theta) \right\|_p^p \leq C_\varepsilon N^\varepsilon \left\| \sum_{n=1}^{N} \xi_n e(n\theta) \right\|_p^p$$

for all $N \geq 1$. Note that (the proof of) Theorem 2.10 establishes that the random majorant property holds at $p = 3$. Moreover, if (2.54) holds for some $p$, then (2.1) also holds for that value of $p$, see Lemma 2.4.

**Lemma 2.11.** Let $2 \leq p \leq 3$. Suppose the random majorant property (2.54) holds at $2(p-1)$. Then it also holds at $p$. Furthermore, suppose the RMP holds at $p-1$, $2(p-1)$ and $2(p-2)$. If $4 \geq p \geq 3$, then it also holds at $p$. If $p > 4$ and $\delta \leq \frac{1}{2}$ (i.e., $\tau = N^{-\delta} \geq N^{-\frac{1}{2}}$), then it also holds at $p$. 

20
\[ 2^{-p} \int_0^1 \left| \sum_{n=1}^N a_n \xi_n e(n\theta) \right|^p \ d\theta = \int_0^1 \left| \sum_{n \in R_1^2} a_n \xi_n e(n\theta) \right|^2 \left| \sum_{\ell \in R_1^2} a_{\ell} \xi_{\ell} e(\ell\theta) \right|^2 \ d\theta \ dt \]

(2.55)

\[ + O \left( m^2 \int_0^1 \left( 1 + \sum_{n=1}^N \frac{a_n}{\sqrt{m}} |\xi_n e(n\theta)|^{p-1} \right) d\theta \right). \]

To bound the \(O\)-term in (2.55) note that by the RMP for \(p - 1 \geq 2\),

(2.56) \[ \mathbb{E} \sup_{|a_n| \leq 1} \int_0^1 \left| \sum_{n=1}^N a_n \xi_n e(n\theta) \right|^{p-1} d\theta \leq C_\varepsilon N^\varepsilon \mathbb{E} \int_0^1 \left| \sum_{n=1}^N \xi_n e(n\theta) \right|^{p-1} d\theta = C_\varepsilon N^\varepsilon \mathbb{E} I_{p-1,N}. \]

A calculation analogous to that leading up to (2.47) therefore yields

(2.57) \[ \mathbb{E}_\omega \sup_{|a_n| \leq 1} \int_0^1 \left| \sum_{n=1}^N a_n \xi_n(\omega) e(n\theta) \right|^p d\theta \leq m^\frac{p}{2} + C_\varepsilon N^\varepsilon m^\frac{1}{2} \mathbb{E} I_{p-1,N} + \mathbb{E}_{\omega_2} \mathbb{E}_{\omega_1} \sup_{x \in \mathcal{E}(\omega_2)} \sup_{|A| = m} \sum_{n \in A} \xi_n(\omega)x_n, \]

where now \( \mathcal{E}(\omega_2) = \left\{ \left( \left( \langle e(n), \sum_{k=1}^N \tilde{b}_k \xi_k(\omega) e(-k) \rangle \left( \sum_{\ell=1}^N b_{\ell} \xi_{\ell}(\omega) e(\ell) \right)^{p-2} \right) \right)_{n=1}^N \right\} \sup_{1 \leq n \leq N} |b_n| \leq 1 \subset \mathbb{R}_+^N. \)

If \( x \in \mathcal{E}(\omega_2) \), then by Plancherel and the RMP at 2\((p-1)\),

(2.58) \[ \mathbb{E} \sup_{x \in \mathcal{E}(\omega_2)} |x|^2_{L^2(N)} \leq \mathbb{E}_{\omega_2} \sup_{|a_n| \leq 1} \int_0^1 \left| \sum_{k} a_k \xi_k(\omega_2) e(k\theta) \right|^{2(p-1)} d\theta \]

\[ \leq C_\varepsilon N^\varepsilon \mathbb{E}_{\omega_2} \int_0^1 \left| \sum_k \xi_k(\omega_2) e(k\theta) \right|^{2(p-1)} d\theta \leq C_\varepsilon N^\varepsilon \mathbb{E} I_{2(p-1),N}. \]

Thus, by (2.57) and Lemma 2.5,

(2.59) \[ \mathbb{E}_\omega \sup_{|a_n| \leq 1} \int_0^1 \left| \sum_{n=1}^N a_n \xi_n(\omega) e(n\theta) \right|^p d\theta \]

\[ \leq C_\varepsilon N^\varepsilon \left[ m^\frac{p}{2} + m^\frac{1}{2} \mathbb{E} I_{p-1,N} + (1 + \sqrt{m\tau}) \sqrt{\mathbb{E} I_{2(p-1),N}} + \mathbb{E}_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2), t)} \ dt \right]. \]

To estimate the entropy term, let \( q \) be very large depending on \( \varepsilon \). Then the distance between any two elements in \( \mathcal{E}(\omega_2) \) is of the form

\[ \|g\|_{2(p-2)} - h\|_{2(p-2)} \|_2 \leq \|g - h\|_{2(p-2)} (\|g\|_{2(p-2)} + \|h\|_{2(p-2)}) \]

\[ \leq C_\varepsilon N^\varepsilon \|g - h\|_q (\|g\|_{p-2} + \|h\|_{p-2}) \]

\[ \leq C_\varepsilon N^\varepsilon \sup_{|a_n| \leq 1} \left| \sum_{n=1}^N a_n \xi_n(\omega_2) e(n\cdot) \right|^{p-2} \|g - h\|_q. \]

(2.60) \[ =: C_\varepsilon N^\varepsilon \mathbb{J}_{2(p-2),N}(\omega_2) \|g - h\|_q, \]

21
where the $N^\varepsilon$-term follows from Bernstein’s inequality and we have set

$$
\sup_{|a_n|\leq 1} \left\| \sum_{n=1}^N a_n \xi_n(\omega) e(n\theta) \right\|^{2(p-2)}_{2(p-2)} =: J_{2(p-2),N}(\omega).
$$

As before, $g, h \in \sqrt{m}P_A$, $A = A(\omega) = \{ n \in [1,N] \mid \xi_n(\omega) = 1 \}$, see (2.51). One concludes that, for $\varepsilon > 0$ small and $q < \infty$ large depending on $\varepsilon$,

$$
\log N_2(\mathcal{E}(\omega), t) \leq \log N_q(\mathcal{P}_A(\omega), N^{-\varepsilon}m^{-\frac{1}{2}}J_{2(p-2),N}(\omega) t)
$$

$$
\leq C_q m \begin{cases} 
1 + \log \frac{1}{t} & \text{if } 0 < t < N^\varepsilon \sqrt{m J_{2(p-2),N}(\omega)} \\
(m^{-\frac{1}{2}}J_{2(p-2),N}(\omega) N^{-\varepsilon}t)^{-\nu} & \text{if } t > N^\varepsilon \sqrt{m J_{2(p-2),N}(\omega)}
\end{cases}
$$

where $\nu > 2$, see Corollary 2.7 and Corollary 2.8. Inserting this estimate into the last term of (2.59) yields by the random majorant property on $2(p-2) \geq 2$,

$$
(2.61) \quad \mathbb{E}_{\omega} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega), t)} \, dt \leq C_\varepsilon N^\varepsilon m \sqrt{\mathbb{E} I_{2(p-2),N}}
$$

and therefore finally, by Lemma 2.4,

$$
\mathbb{E}_\omega \sup_{|a_n|\leq 1} \int_0^1 \left\| \sum_{n=1}^N a_n \xi_n(\omega) e(n\theta) \right\|^p d\theta
$$

$$
\leq C_\varepsilon N^\varepsilon \left[ \frac{m^{\frac{3}{2}}}{2} + m \frac{1}{2} \mathbb{E} I_{p-1,N} + (1 + \sqrt{m\tau}) \sqrt{\mathbb{E} I_{2(p-1),N}} + m \sqrt{\mathbb{E} I_{2(p-2),N}} \right]
$$

$$
\leq C_\varepsilon N^\varepsilon \left[ (\tau N)^{\frac{p}{2}} + (\tau N)^{\frac{p-1}{2}} \left( \tau^{p-1} N^{p-2} + (\tau N)^{\frac{p-1}{2}} \right) \right]

+ (1 + \tau \sqrt{N}) \left( \tau^{2(p-1)} N^{2p-3} + (\tau N)^{p-1} \right)^{\frac{1}{2}} + \tau N \left( \tau^{2(p-2)} N^{2p-5} + (\tau N)^{p-2} \right)^{\frac{1}{2}}
$$

$$
(2.62) \quad \leq C_\varepsilon N^\varepsilon \left[ \tau^{p-1} N^{p-2} + \tau^{p-1} N^{p-2} + (\tau N)^{\frac{p}{2}} \right].
$$

If $\tau \geq N^{-\frac{1}{2}}$, then $\tau^{p-1} N^{p-2} \geq \tau^{p-1} N^{p-2}$. Moreover, if $\tau \leq N^{\frac{3-p}{2}}$, then $\tau^{p-1} N^{p-2} \leq (\tau N)^{\frac{p}{2}}$. In particular, if $3 \leq p \leq 4$, then $\tau^{p-1} N^{p-2} \leq \mathbb{E} I_{p,N}$, and the result follows. On the other hand, if $p \geq 4$, then $\tau \geq N^{-\frac{1}{2}}$ insures that $\tau^{p-1} N^{p-2} \leq \tau^{p-1} N^{p-1} \leq \mathbb{E} I_{p,N}$, as claimed.

It remains to discuss $2 \leq p \leq 3$. In that case, Lemma 2.3 implies that

$$
2^{-p} \int_0^1 \left\| \sum_{n=1}^N a_n \xi_n e(n\theta) \right\|^p d\theta = \int_0^1 \int_0^1 \sum_{n \in R_1^2} a_n \xi_n e(n\theta) \sum_{k \in R_1^2} \bar{a}_k \xi_k e(-k\theta) \sum_{\ell \in R_1^2} a_{\ell} \xi_{\ell} e(\ell\theta) d\theta d\theta
$$

$$
(2.63) \quad + O \left( m^\frac{p}{2} \int_0^1 \left( 1 + \left| \sum_{n=1}^N \frac{a_n}{\sqrt{m}} \xi_n e(n\theta) \right|^2 \right) d\theta \right).
$$

22
The integral in (2.63) is $O(1)$. Hence (2.57) changes to

\[(2.64) \quad \mathbb{E}_\omega \sup_{|a_n| \leq 1} \int_0^1 \left| \sum_{n=1}^N a_n \xi_n(\omega)e(n\theta) \right|^p d\theta \lesssim m_\omega^2 + \mathbb{E}'_{\omega_2} \mathbb{E}'_{\omega_1} \sup_{x \in \mathcal{E}(\omega_2)} \sup_{|A|=m} \sum_{n \in A} \xi_n(\omega_1)x_n,\]

with the same $\mathcal{E}(\omega_2)$, and (2.58) becomes

\[(2.65) \quad \mathbb{E}_\omega \sup_{|a_n| \leq 1} \int_0^1 \left| \sum_{n=1}^N a_n \xi_n(\omega)e(n\theta) \right|^p d\theta \leq C_\varepsilon N^\varepsilon \left[ m_\omega^2 + (1 + \sqrt{m\tau}) \sqrt{\mathbb{E} I_{2(p-1),N} + \mathbb{E}'_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2),t)} \, dt} \right].\]

Finally, the entropy estimate simplifies as $2(p-2) \leq 2$ in this case: If $g|g|^{p-2}, h|h|^{p-2} \in \mathcal{E}(\omega_2)$, then $g, h \in \mathcal{P}_{A(\omega_2)}$ and thus

\[
\|g|g|^{p-2} - h|h|^{p-2}\|_2 \lesssim \|g - h\|_\infty (\|g\|_{2(p-2)}^{p-2} + \|h\|_{2(p-2)}^{p-2}) \lesssim C_\varepsilon N^\varepsilon \|g - h\|_q (\|g\|_2^{p-2} + \|h\|_2^{p-2}) \lesssim C_\varepsilon N^\varepsilon m_\omega^\frac{p-2}{2} \|g - h\|_q,
\]

so that now

\[
\mathbb{E}'_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2),t)} \, dt \leq C_\varepsilon N^\varepsilon m_\omega^\frac{p}{2}.
\]

We leave it to the reader to check that this again leads to (2.62). As already mentioned above, the term $\tau^{p-1}N^{p-\frac{3}{2}}$ can be absorbed into $(\tau N)^\frac{p}{2}$, since $p \leq 3$.

This lemma quickly leads to a proof of Theorem 2.1 in case $\delta \leq \frac{1}{2}$ for $p > 4$, and for all $0 < \delta < 1$ if $2 < p < 4$.

**Corollary 2.12.** Suppose $0 < \delta \leq \frac{1}{2}$ and assume otherwise that the hypotheses of Theorem 2.1 are satisfied. Then (2.54) holds for all $p \geq 4$. If $2 < p < 4$, then (2.54) holds for all $0 < \delta < 1$. In particular, Theorem 2.1 is valid in these cases.

**Proof.** As a first step, note that Lemma 2.11 immediately implies that all odd integers satisfy (2.54). Next, one checks that (2.54) holds at $p = \frac{3}{2}$ since $2(p-1) = 3$ in that case. Now Lemma 2.11 implies that (2.54) holds at all other values $p = \frac{2\ell + 1}{2}$, for all integers $\ell \geq 3$. Generally speaking, one checks by means of induction that (2.54) holds at all

\[p \in \left\{ 2 + \frac{\ell}{2^j} \mid \ell \in \mathbb{Z}^+ \right\} =: \mathcal{P}_j.\]

Indeed, we just verified that this holds for $j = 0, 1$. Now assume that it holds up to some integer $j$ and we will prove it for $j + 1$. Thus take $p = 2 + \frac{\ell}{2^j} \in \mathcal{P}_{j+1}$ such that $2 < p < 3$. Then $2(p-1) = 2 + \frac{\ell}{2^j}$ for which (2.54) holds by assumption. Hence Lemma 2.11 applies. Now suppose $p \in \mathcal{P}_{j+1}$ is such that
3 < p < 4. Then (2.54) holds at p − 1 by what we just did, and at 2(p − 1), 2(p − 2) by assumption. Hence Lemma 2.11 applies again. One now continues with 4 < p < 5 etc., and we are done. Given any ε > 0 and p > 2 one can find p_1 < p < p_2 with p_1, p_2 ∈ P_j where p_2 − p_1 < ε. Hence (2.54) holds for all p by interpolation, as desired. It remains to deal with δ > 1/2 if 2 < p < 4. Fix such a p. Then by Bourgain’s theorem on random Λ(p) sets, δ > 1/2 implies that the random set S(ω) is a Λ(p) set. More precisely,

\[ E \sup_{|a_n| \leq 1} \left\| \sum_{n=1}^{N} a_n \xi_n(\omega) e(n \cdot) \right\|_p^p \lesssim (\tau N)^{p/2}, \]

see Theorem 2.13 below. Clearly,

\[ \left\| \sum_{n=1}^{N} \xi_n(\omega) e(n \cdot) \right\|_p \geq \left\| \sum_{n=1}^{N} \xi_n(\omega) e(n \cdot) \right\|_2 = \#(S(\omega))^{1/2}, \]

and we are done.

2.7 Some improvements and δ > 1/2

It is clear that the proof of Lemma 2.11 in its present form does not allow us to deal with the case δ > 1/2. The difficulty arises from the use of Plancherel in (2.58) and (2.60). Indeed, once the \( L^2 \) bound is used, the estimates in the proof of Lemma 2.11 are optimal and they produce the unwanted \( \tau^{p-1} N^{p-1/2} \) term in (2.62). In order to improve this step, one can invoke Bourgain’s theorem on random Λ(p) sets. Recall the main theorem from [B1]:

**Theorem 2.13.** Fix some \( p > 2 \). Let \( \{\xi_j\}_{j=1}^{N} \) be selector variables as in Theorem 2.1 with \( \delta = 1 - \frac{2}{p} \).

Define

\[ K_p(\omega) = \sup_{|a| \leq 1} \left\| \sum_{n=1}^{N} a_n \xi_n(\omega) e(n \cdot) \right\|_p \]

Then \( E K_p^p \leq C_p < \infty \).

Although the main theorem in [B1] is formulated for generic sets rather than in terms of expected values, this statement appears implicitly in [B1], see page 241 (especially the last line on that page), as well as Section 5 of that paper. We will need the following dual version of (2.66). With \( S(\omega) = \{n \in [1, N] \mid \xi_n(\omega) = 1\} \),

\[ \left( \sum_{n \in S(\omega)} |\hat{f}(n)|^2 \right)^{1/2} \leq K_p(\omega) \|f\|_{L^p(\mathbb{T})}, \]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \).

**Proposition 2.14.** If 4 < p ≤ 7, then the random majorant property (2.54) and therefore (2.1) hold for all 0 < δ < 1.
Proof. It suffices to consider \( \delta > \frac{1}{2} \). This will be done in several steps. For the sake of clarity, we first present the case \( p = 5 \), and then indicate how to pass to the range \( 4 < p \leq \frac{15}{8} \). We will then refine the argument even further to obtain the specified range. The idea is to factor through a \( \Lambda(3) \) set, i.e., in order to generate a random subset \( S(\omega) \subset [1, N] \) of cardinality roughly \( N^{1-\delta} \) one first chooses a random subset \( S_0(\omega) \subset [1, N] \) of cardinality about \( N^{\frac{3}{4}} \), and then generates \( S(\omega) \subset S_0(\omega) \).

Hence, we let \( \xi_j = \xi_j^{(0)} \xi_j^{(1)} \) where \( \mathbb{E} \xi_j^{(0)} = N^{-\frac{1}{2}} \), and \( \mathbb{E} \xi_j^{(1)} = \tau' \) so that \( \tau = N^{-\frac{1}{2}} \tau' \). Moreover, we of course choose all these random variables to be independent. The set \( S_0(\omega) := \{ n \in [1, N] | \xi_j^{(0)}(\omega) = 1 \} \) satisfies (2.66) and also its dual version (2.66) at \( p = 3 \). The argument is similar to those in Theorems 2.10 and Lemma 2.11, so we will only indicate those places that are different. Starting the argument as before, one arrives at

\[
(2.68) \quad \mathbb{E}_\omega \sup_{|a_n| \leq 1} \left| \sum_{n=1}^{N} a_n \xi_n(\omega) e(n\theta) \right|^5 d\theta \lesssim m^{\frac{5}{2}} + m^{\frac{3}{2}} \mathbb{E} I_{4,N} + \mathbb{E} \omega_2 \mathbb{E} \omega_0 \mathbb{E} \omega_1 \sup \sup \sum \xi_n^{(1)}(\omega_1) x_n,
\]

in place of (2.57), where now

\[
(2.69) \quad \mathbb{E} (\omega_0, \omega_2) = \left\{ \left( \left| e(n \cdot), \sum_{k=1}^{N} b_k \xi_k(\omega_2) e(-k \cdot) \right| \sum_{\ell=1}^{N} b_\ell \xi_\ell(\omega_2) e(\ell \cdot) \right|^3 \right\} \chi_{S_0(\omega_0)}(n) \right\} \sup_{1 \leq n \leq N} |b_n| \leq 1 \subset \mathbb{R}^N.
\]

Using (2.67) with \( p' = \frac{3}{2} \) instead of Plancherel and the majorant property at \( p = 6 \) leads to

\[
\sup_{x \in \mathcal{E}(\omega_0, \omega_2)} |x|_{\ell_6^N} \leq K_3(\omega_0) \sup_{|a_k| \leq 1} \left\| \sum_{k} a_k \xi_k(\omega_2) e(k\theta) \right\|_6^4 \leq K_3(\omega_0) \left\| \sum_{k} \xi_k(\omega_2) e(k\theta) \right\|_6^4 = K_3(\omega_0) \left( I_{6,N}(\omega_2) \right)^{\frac{2}{3}},
\]

and thus

\[
(2.70) \quad \mathbb{E} \sup_{x \in \mathcal{E}(\omega_0, \omega_2)} |x|_{\ell_6^N} \leq C \left( \mathbb{E} I_{6,N} \right)^{\frac{1}{3}}.
\]

In the next step we use Lemma 2.3 to bound the last term in (2.68) for fixed \( \omega_0, \omega_2 \). Since \( \sqrt{m\tau'} = N \frac{\sqrt{2}}{2} \sqrt{m\tau} \), one obtains from that lemma that

\[
(2.70) \quad \mathbb{E} I_{6,N}(\omega_0, \omega_2) = \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_0, \omega_2), t)} \, dt.
\]

To control the entropy term, observe that the distance between any two elements in \( \mathcal{E}(\omega_0, \omega_2) \) is bounded by

\[
K_3(\omega_0) \|g\|_3^3 - h\|h\|_3^3 \leq K_3(\omega_0) \|g - h\|_\infty (\|g\|_5^3 + \|h\|_5^3)
\]

\[
\leq K_3(\omega_0) C_\varepsilon N^\varepsilon \|g - h\|_q (\|g\|_5^3 + \|h\|_5^3)
\]

\[
\leq K_3(\omega_0) C_\varepsilon N^\varepsilon \sup_{|a_n| \leq 1} \left\| \sum_{n=1}^{N} a_n \xi_n(\omega_2) e(n\cdot) \right\|_5^3 \|g - h\|_q.
\]

25
As before, \( q \) is large depending on \( \varepsilon > 0 \), \( g, h \in \sqrt{m}P_A \), \( A = A(\omega_2) = \{ n \in [1, N] \mid \xi_n(\omega_2) = 1 \} \). Using Corollaries 2.7 and 2.8, one now arrives at

\[
\mathbb{E}_{\omega_0} \mathbb{E}_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_0, \omega_2), t)} \, dt
\]

(2.71)

\[
\leq \left( \mathbb{E}_{\omega_0} K_3(\omega_0)^3 \right)^\frac{1}{3} C_\varepsilon N^\varepsilon m \left( \mathbb{E}_{\omega_2} \sup_\{|a_n| \leq 1| \sum_{n=1}^N a_n \xi_n(\omega_2)e(n\cdot) \|^5 \right)^\frac{3}{5}.
\]

Combining (2.68), (2.70), (2.69), (2.71) one obtains

\[
\mathbb{E}_\omega \sup_\{|a_n| \leq 1| \sum_{n=1}^N a_n \xi_n(\omega)e(n\theta) \|^5 \leq (\tau N)^\frac{\delta}{2} + (\tau N)^{\frac{1}{2}}(\tau^4 N^3 + \tau^2 N^2)
\]

\[
+ (1 + N^\frac{1}{6} \sqrt{N \tau^2})(\tau^6 N^5 + \tau^3 N^3)^\frac{3}{8}
\]

\[
+ C_\varepsilon N^\varepsilon N \tau \left( \mathbb{E}_{\omega_2} \sum_{n=1}^N a_n \xi_n(\omega_2)e(n\cdot) \right)^\frac{\delta}{5}
\]

\[
\leq C_\varepsilon N^\varepsilon (\tau N)^\frac{\delta}{2} + \tau^5 N^4 + \tau^3 N^2 + \tau^4 N^3 + N^\frac{5}{2} \tau^3
\]

\[
+ \frac{1}{2} \mathbb{E}_\omega \sup_\{|a_n| \leq 1| \sum_{n=1}^N a_n \xi_n(\omega)e(n\theta) \|^5.
\]

We leave it to the reader to check that the expressions with fractional exponents are dominated by \( \tau^5 N^4 \) provided \( \tau > N^{-\frac{5}{2}} \). Hence, for those \( \tau \), we have proved

\[
\mathbb{E}_\omega \sup_\{|a_n| \leq 1| \sum_{n=1}^N a_n \xi_n(\omega)e(n\theta) \|^5 \leq C_\varepsilon N^\varepsilon \mathbb{E} I_{5,N},
\]

and thus the RMP at \( p = 5 \) and therefore also (2.1) with \( p = 5 \) holds for all \( 0 < \delta < \frac{3}{5} \). On the other hand, if \( \tau \leq N^{-\frac{2}{3}} \) and \( \tau \leq N^{-\frac{1}{2}} \), then \( S(\omega) \) is a \( \Lambda(5) \) set with large probability by Bourgain’s theorem. More precisely, (2.68) holds with \( p = 5 \) and thus (2.1) follows with \( p = 5 \) for the range \( \delta \geq \frac{3}{5} \) as well. We now indicate how to obtain the range \( 4 < p < \frac{11}{4} \). Instead of factoring through a \( \Lambda(3) \)-set, one factors through a \( \Lambda(q) \)-set where \( q'(p - 1) = 6 \). Since we need to cover the range \( \frac{1}{2} < \delta < 1 \), one needs to allow all \( N^{-1} < \tau < N^{-\frac{1}{2}} \). On the other hand, the factorization means that \( \tau = N^{-1+\frac{\delta}{4}} \tau' \) with some \( \tau' < 1 \). This implies that necessarily \( N^{-1+\frac{\delta}{4}} \geq N^{-\frac{1}{2}} \) or \( 2 \leq q \leq 4 \). Hence \( q' \geq \frac{4}{3} \), and thus \( \frac{4}{3}(p - 1) \leq q'(p - 1) = 6 \iff p \leq \frac{11}{2} \). Inspection of the previous argument reveals that we also need \( q'(p - 2) < p \), which by our choice of \( q' \) is the same as \( p^2 - 7p + 12 = (p - 4)(p - 3) > 0 \). But this holds for all \( p > 4 \). We are now ready to run the same argument as before. Observe that the first step already requires the (random) majorant property at \( p - 1 \). Therefore, we start with the range \( 4 < p < 5 \) so that this property is ensured by Corollary 2.13. Analogously to the case \( p = 5 \) one.
arrives at the bound
\[
M_p^p := \mathbb{E}_\omega \sup_{|a_n| \leq 1} \left\| \sum_{n=1}^N a_n \xi_n(\omega) e(n\theta) \right\|^p_p \lesssim \left( \tau N \right)^{\frac{p}{2}} + C_\varepsilon N^\varepsilon (\tau N)^{\frac{1}{2}} \left( \tau^{p-1} N^{p-2} + (\tau N)^{\frac{p-1}{2}} \right) + \left( 1 + N^{-\frac{1}{p}} \tau \right) \left( \tau^6 N^5 + \tau^3 N^3 \right)^{\frac{p-1}{2}} + C_\varepsilon N^\varepsilon N \tau M_p^{p-2}
\]
\[
\lesssim C_\varepsilon N^\varepsilon \left[ (\tau N)^{\frac{p}{2}} + \tau^p N^{p-1} \right] + \tau^{p-1} N^{\frac{p}{2}+\varepsilon} + \frac{1}{2} M_p^p.
\]
One now checks that the two unwanted terms in the final expression are dominated by \( \tau^p N^{p-1} \)

provided \( \tau > N^{-\frac{p-1}{6}} \) and \( \tau > N^{-\frac{p}{4}} \), respectively. By the usual reduction to the (random) \( \Lambda(p) \)-

property, it suffices to consider the range \( \tau > N^{-\frac{p-1}{7}} \). But since for \( p \geq 4 \)

\[
1 - \frac{2}{p} \leq \frac{p-1}{6} \iff p \geq 4, \quad 1 - \frac{2}{p} = \frac{2}{3} \iff p \leq 6,
\]

we are done with the case \( 4 < p \leq 5 \). Finally, if \( 5 < p < \frac{11}{2} \), then we just showed that the RMP holds at \( p - 1 \), and so we can repeat the exact same argument.

Next, we increase \( p \) even further. For example, take \( p = 7 \) and factor through a \( \Lambda(3) \) set. More precisely, set \( q' = \frac{1}{3} \). Then \( q'(p-1) = 8 \) and \( q'(p-2) < p \). Since the majorant property holds at \( p-1 = 6 \), one can repeat the argument for \( p = 5 \) mutatis mutandis (use \( L^8 \) instead of \( L^6 \)). We leave it to the reader to check that this leads to

\[
M_7^7 \lesssim C_\varepsilon N^\varepsilon \left[ \tau^7 N^6 + (\tau N)^{7/2} \right] + \tau^6 N^{21/4} + \tau^4 N^{15/4}.
\]

Moreover, the unwanted terms are dominated by \( \tau^7 N^6 \) provided \( \tau > N^{-\frac{4}{7}} \). But since \( 1 - \frac{2}{p} = \frac{5}{7} < \frac{4}{7} \), the remaining range of \( \tau \) is covered by the random \( \Lambda(7) \) property as before. Using \( L^8 \) instead of \( L^6 \)

of course gives a larger range of \( p \)'s. Indeed, let now \( \frac{1}{7} \leq q' \leq 2 \) be general such that \( q'(p-1) = 8 \).

This is possible for all \( 5 \leq p \leq 7 \). On the other hand, we also require \( q'(p-2) < p \). This reduces to

\[ 0 < p^2 - 9p + 16 \text{ which means that } p > \frac{1}{2} (9 + \sqrt{17}). \]

Finally, to run the argument we also need to know the RMP at \( p - 1 \). This was clear for \( p = 7 \), but it is not if \( p \) is below 7. However, we will show in the next step that the RMP at \( p = 7 \) allows us to increase the range of \( p \) from \( \frac{11}{2} \) (which would be insufficient for our purposes) to \( \frac{25}{4} > 6 \). This in turn settles the issue of \( p - 1 \) if \( p < 7 \). Thus one does indeed obtain the RMP for all \( \frac{1}{2} (9 + \sqrt{17}) \leq p \leq 7 \).

Next, we argue that the RMP at \( p = 7 \) makes it possible to increase \( p \) from \( \frac{11}{2} \) to \( \frac{25}{4} \). To be precise, set \( q'(p-1) = 7 \). The restriction \( \frac{4}{7} \leq q' \leq 2 \) yields \( \frac{9}{7} \leq p \leq \frac{25}{4} \). On the other hand, \( q'(p-2) < p \) is the same as \( 0 < p^2 - 8p + 14 \), which holds for all \( p > 4 + \sqrt{2} \) and thus, in particular, for \( p \geq \frac{11}{2} \).

Finally, if \( p \leq \frac{25}{4} \), then the random majorant property (2.54) holds at \( p - 1 \leq \frac{21}{4} < \frac{11}{2} \) by the first part of the proof. We can now run the same argument as before to conclude that

\[
M_p^p \lesssim (\tau N)^{\frac{p}{2}} + C_\varepsilon N^\varepsilon (\tau N)^{\frac{1}{2}} \left( \tau^{p-1} N^{p-2} + (\tau N)^{\frac{p-1}{2}} \right) + C_\varepsilon N^\varepsilon \left( 1 + N^{-\frac{1}{p}} \tau \right) \left( \tau^7 N^6 + (\tau N)^{7/2} \right)^{\frac{p-1}{2}} + C_\varepsilon N^\varepsilon N \tau M_p^{p-2}
\]
\[
\lesssim C_\varepsilon N^\varepsilon \left[ (\tau N)^{\frac{p}{2}} + \tau^p N^{p-1} + \tau^{p-1} N^{\frac{p}{2}+\varepsilon} + \tau^{\frac{p-1}{2}} N^{\frac{p-1}{2}+\varepsilon} + \frac{1}{2} M_p^p \right].
\]
One now checks that the two unwanted terms in the final expression are dominated by $\tau^p N^{p-1}$, provided $\tau > N^{-\frac{p-1}{r}}$ and $\tau > N^{-\frac{1}{4}}$, respectively. By the usual reduction to the (random) $\Lambda(p)$-property, it suffices to consider the range $\tau > N^{-1+\frac{1}{r}}$. But since for $p \geq 4$

$$1 - \frac{2}{p} \leq \frac{p-1}{7} \iff p \geq 4 + \sqrt{2}, \quad 1 - \frac{2}{p} \leq \frac{5}{7} \iff p \leq 7,$$

we are done with this case as well.

It remains to close the gap $\frac{25}{4} < p < \frac{1}{2}(9 + \sqrt{17})$. The idea is to forfeit the requirement $q'(p-2) < p$ and instead replace $q'(p-2)$ by the smallest number $r$ to the right of $q'(p-2)$ for which the random majorant property is known. One then uses Hölder’s inequality which brings in $\mathbb{E} I_{r,N}$ instead of $M_p$.

In our case the best choice of $r$ is $r = \frac{1}{2}(9 + \sqrt{17})$. To be precise, we set $q'(p-1) = 8$ which by $q' \leq \frac{4}{3}$ can be done for $p \leq 7$. But we are only interested in $p < r$, which is equivalent to $q'(p-2) = 8\frac{p-2}{p-1} < r$.

Since we already know that the RMP holds at $p-1$, we can proceed as before, but using Hölder’s inequality to pass from $q'(p-2)$ to $r$. One checks that this leads to

\[ M_p^p \lesssim (\tau N)^{\frac{p}{2}} + C_{\epsilon} N^\epsilon (\tau N)^{\frac{1}{2}} \left( \tau^p N^{p-2} + (\tau N)^{\frac{p-1}{r}} \right) + \left( 1 + N^{\frac{1}{7}} \tau \right) \left( \tau^8 N^7 + (\tau N)^4 \right)^{\frac{p-1}{2}} \]

\[ + N^{\frac{1}{7}} \tau (\tau^p N^{p-1} + (\tau N)^{\frac{p-1}{r}}) + N^{\frac{1}{7}} \tau (\tau N)^{\frac{p-1}{2}} + \tau^{p-1} N^{\frac{p-2}{r}} N^\frac{1}{7}. \]

This yields the desired bound under the conditions $\tau > N^{-\frac{1}{4}}$, $\tau > N^{-\frac{p-1}{8}}$, $\tau > N^{-\frac{p-2}{r}}$. Since $\frac{3}{4} \geq \frac{p-1}{8} \geq \frac{p-2}{r}$, this reduces to $\delta \leq \frac{p-2}{r}$. To recapitulate, for the range $\frac{25}{4} \leq p \leq r$ we have raised the admissible values of $\delta$ from $\frac{1}{2}$ to $\frac{p-2}{r}$, which is at least $\frac{17}{4}$. The point is now that this allows us to factor through $\Lambda(q)$-sets for values of $q$ larger than 4. Indeed, define $q_0$ by $1 - \frac{25}{4} = \frac{17}{16}$. Going back to the argument involving the random majorant property at $p = 7$, we see that we can apply it for all $p$ for which $q'(p-1) = 7$ with $2 \leq q \leq q_0$. Recall that $q'(p-2) < p$ holds if $p > 4 + \sqrt{2}$, and this for all $p$ in the range under consideration. In order to close the gap $[\frac{25}{4}, r]$ we therefore only need to check that $q_0'(r-1) \leq 7$. But since $\frac{1}{q_0} = \frac{1}{2} + \frac{17}{8r}$, this is the same as $8r^2 - 36r - 119 \leq 0$. One explicitly checks that with $r = \frac{1}{2}(9 + \sqrt{17})$ one has $8r^2 - 36r - 119 < -10$, and we are done. \(\square\)

### 2.8 Choosing subsets by means of correlated selectors

To conclude this section, we want to address the issue of obtaining a version of Theorem 2.1 for subsets which are obtained by means of selectors $\xi_j$ that are allowed to have some degree of dependence. More precisely, we will work with the selectors from the following definition.

**Definition 2.15.** Let $0 < \tau < 1$ be fixed. Define $\xi_j(\omega) = \chi_{[0,\tau]}(2^j \omega)$ for $j \geq 1$. Here $\omega \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ with probability measure $\mathbb{P}(d\omega) = d\omega$ equal to normalized Lebesgue measure.

Since the doubling map $\omega \mapsto 2\omega \mod 1$ is measure preserving, it follows that $\mathbb{E} \xi_j = \tau$ and $\mathbb{P}[\xi = 1] = \tau$, $\mathbb{P}[\xi_j = 0] = 1 - \tau$, as in the random case. However, these selector variables are no longer independent. Nevertheless, they are close enough to being independent to make the following theorem accessible to the methods of the previous section.
Theorem 2.16. Let $0 < \delta < 1$ be fixed. For every positive integer $N$ we let $\xi_j = \chi_{[0, \tau]}(2^j \omega)$ be as in Definition 2.13 with $\tau = N^{-\delta}$. Define a subset
\begin{equation}
S(\omega) = \{ j \in [1, N] \mid \xi_j(\omega) = 1 \}
\end{equation}
for every $\omega \in \mathbb{T}$. Then for every $\varepsilon > 0$ and $7 \geq p \geq 2$ one has
\begin{equation}
P\left[ \sup_{|a_n| \leq 1} \left\| \sum_{n \in S(\omega)} a_ne(n\theta) \right\|_{L^p(\mathbb{T})} \geq N^{\varepsilon} \left\| \sum_{n \in S(\omega)} e(n\theta) \right\|_{L^p(\mathbb{T})} \right] \to 0
\end{equation}
as $N \to \infty$. Moreover, under the additional restriction $\delta \leq \frac{1}{7}$, (2.73) holds for all $p \geq 7$.

To prove this theorem we may of course assume that $\tau = 2^{-k}$ for some positive integer $k$. Then $\xi_j$ is measurable with respect to the dyadic intervals of length $2^{-k-j}$ on the unit interval $\mathbb{T}$, denoted by $\mathcal{D}_{j+k}$. Moreover, it is easy to see that $\xi_j$ and $\xi_{j+ak}$ are independent variables.

Lemma 2.17. Fix $j \geq 0$ and $k \geq 1$. Let $\tau = 2^{-k}$ and $\xi_i$ be as in Definition 2.14. Then the sequence $\{ \xi_{j+ak} \}_{a=1}^{\infty}$ is a realization of a $0,1$-valued Bernoulli sequence with $\mathbb{E} \xi_i = \tau$.

Proof. Fix $a > 1$ and note that the variable $\xi_{j+ak}(\omega)$ is $2^{-(j+ak)}$-periodic. On the other hand, each of the variables $\xi_{j+bk}$ with $b < a$ is constant on intervals from $\mathcal{D}_{j+ak}$ (which is the same as saying that these variables are all $\mathcal{D}_{j+ak}$ measurable). It follows that
\[
P[\xi_{j+ak} = 1 \mid \xi_{j+bk} = \varepsilon_b, \ 0 \leq b \leq a-1] = \tau = P[\xi_{j+ak} = 1],
\]
for any choice of $\varepsilon_b = 0, 1, 0 \leq b \leq a-1$. This implies independence. \hfill \square

From now on, let $\tau = N^{-\delta}$ for some fixed $0 < \delta < 1$. In view of Lemma 2.17 we can decompose the sequence $\{\xi_j\}_{j=1}^{N}$ into about $\log N$ many subsequences, where the indices run along arithmetic progressions $\mathcal{P}_i$ of step-size equal to $\sim \log N$, and $1 \leq i \lesssim \log N$. Each of the subsequences consists of i.i.d. variables, but variables from different subsequences are not independent. This easily shows that Lemma 2.3 remains valid here, possibly with a logarithmic loss in the upper bound for $\mathbb{E} I_{p,N}$. Indeed, recall that the proof of that Lemma is based upon splitting a random trigonometric polynomial into its expectation and a mean-zero part. Since the $L^p$-norm of the Dirichlet kernel on an arithmetic progression of length $K$ is about $K^{1/p}$, and here $\# \mathcal{P}_i \sim \frac{N}{\log N}$, one sees immediately that the upper bound from (2.8) is the same up to logarithmic factors. As far as the lower bound of Lemma 2.4 is concerned, note that the proof relies on obtaining upper bounds on certain error terms, cf. (2.11)-(2.14). However, these upper bounds are again immediate corollaries of the random case by virtue of the splitting into the progressions $\mathcal{P}_i$.

The consequence of this is that basically all the main estimates from the previous section remain valid here, up to possibly an extra factor of $\log N$. Clearly, such factors are irrelevant in this context. More precisely, with $\xi_j$ as in Definition 2.13 and $S(\omega)$ as in (2.72), it is a corollary of the proof of Theorem 2.1 that
\begin{equation}
\mathbb{E} \sup_{|a_n| \leq 1} \left\| \sum_{n \in S(\omega)} a_n e(n\theta) \right\|_{L^p(\mathbb{T})}^p \leq C\varepsilon N^{\varepsilon} (\tau^p N^{p-1} + (\tau N)^{\frac{p}{2}}).
\end{equation}
The proof of Theorem 2.16 is therefore completed as before by appealing to (the adapted version) of Lemma 2.4.
Remark 2.18. Other examples of much more strongly correlated selectors are $\xi_j(\omega) = \chi_{[0, \tau]}(j^s \omega)$ where $s$ is a fixed positive integer and $\omega \in \mathbb{T}$. It appears to be rather difficult to prove a version of Theorem 2.1 for these types of selectors.

3 Perturbing arithmetic progressions

Let $P \subset [1, N]$ be an arithmetic progression of length $L$, i.e.,
$$P = \{b + a\ell \mid 0 < b < a, \ 0 \leq \ell < L := \lfloor N/a \rfloor \} \subset [1, N].$$

Fix some arbitrary $\varepsilon_0 > 0$. Suppose $N^{\varepsilon_0} < s < a$ and let $\{\xi_j\}_{j \in P}$ be i.i.d. variables, integer valued and uniformly distributed in $[-s, s]$. We define a random subset
\begin{equation}
S(\omega) := \{j + \xi_j(\omega) \mid j \in P\}.
\end{equation}

For future reference, we set $I_j := [j - s, j + s]$ for each $j \in P$. By construction, $S(\omega) \subset \bigcup_{j \in P} I_j$, and the intervals $I_j$ are congruent and pairwise disjoint.

3.1 Suprema of random processes

The following lemma is related to Lemma 2.5.

Lemma 3.1. Let $E \subset \mathbb{R}_+^N$, $B = \sup_{x \in E} |x|$, and $S(\omega)$ be as in (3.1). Then
\begin{equation}
\mathbb{E}_{\omega} \sup_{x \in E} \sum_{j \in S(\omega)} x_j \lesssim B(1 + \sqrt{L/s}) + \int_0^B \sqrt{\log N_2(E, t)} \, dt
\end{equation}
where $N_2$ refers to the $L^2$ entropy.

Proof. As in the proof of Lemma 2.5, we introduce $2^{-k}$-nets $E_k$ and $F_k \subset \mathbb{R}^N$ so that $\text{diam}(F_k) \leq 1$,
\begin{equation}
\log \#F_k \leq C \log \#E_{k+1},
\end{equation}
and
\begin{equation}
\mathbb{E} \sup_{x \in E} \sum_{n \in S(\omega)} x_n \leq \sum_{k \geq k_0} 2^{-k+1} \mathbb{E} \sup_{y \in F_k} \sum_{n \in S(\omega)} |y_n|.
\end{equation}

Now fix some $k \geq k_0$ and write $F$ instead of $F_k$. With $0 < \rho_2$ to be determined, one has for any $|y| \leq 1$
\begin{equation}
\sum_{i \in S(\omega)} y_i \leq \sum_{y_i \geq \rho_2} y_i + \sum_{y_i < \rho_2} \chi_{S(\omega)}(i) y_i \leq \rho_2^{-1} \sum_{y_i < \rho_2} \chi_{S(\omega)}(i) y_i.
\end{equation}

Let $q := 1 + \lfloor \log F \rfloor$. Then, as in (2.20),
\begin{equation}
\mathbb{E} \sup_{y \in F} \sum_{i \in S(\omega)} y_i \lesssim \rho_2^{-1} + \sup_{|y| \leq 1} \left\| \sum_{y_i < \rho_2} \chi_{S(\omega)}(i) y_i \right\|_{L^q(\omega)}.
\end{equation}
To control the last term in (3.4), we need the following analogue of (2.21). By the multinomial theorem (for any positive integer \( q \)),

\[
\mathbb{E} \left[ \sum_{n \in S(\omega)} \chi_A(n) \right]^q = \mathbb{E} \left[ \sum_{j \in P} \chi_A(j + \xi_j(\omega)) \right]^q = \sum_{q_1 + \ldots + q_L = q} \left( q \right)_{q_1, \ldots, q_L} \mathbb{E} \prod_{j \in P} \chi_A(j + \xi_j(\omega))
\]

\[
= \sum_{\nu=1}^q \nu! \left( \sum_{j \in P} \frac{|A \cap I_j|}{|I_j|} \right)^{\nu} \leq \sum_{\nu=1}^q \nu! \left( \frac{|A \cap \bigcup_{j \in P} I_j|}{2s+1} \right)^{\nu}
\]

\[
\leq \sum_{\nu=1}^q \left( q \right)^{q-\nu} \left( \frac{e|A \cap \bigcup_{j \in P} I_j|}{2s+1} \right)^{\nu} \leq \left( q + \frac{|A \cap \bigcup_{j \in P} I_j|}{2s+1} \right)^q.
\]

Continuing with the final term in (3.4) one concludes that

\[
\sup_{|y| \leq 1} \left\| \sum_{y \in \rho_2} \chi_S(i) y_i \right\|_{L^q(\omega)} \lesssim \sum_{\rho_2 < 2^{\frac{1}{2}}} \sup_{|A| = 2^{\frac{1}{2}}} \sum_{n \in S(\omega)} \chi_A(n) \left\|_{L^q(\omega)} \lesssim 2^{-\frac{1}{2}} \left( q + \frac{\min(L_2, 2^{\frac{1}{2}})}{s} \right) \lesssim q \rho_2 + \sqrt{L/s}.
\]

Let \( \rho_2 = q^{-\frac{1}{2}} = (1 + \log \# F)^{-\frac{1}{2}} \). Inserting this bound into (3.4) therefore yields

\[
\mathbb{E} \sup_{y \in F} \sum_{i \in S(\omega)} y_i \lesssim \sqrt{q} + \sqrt{L/s} \lesssim \sqrt{L/s} + 1 + \sqrt{\log \# F}.
\]

The lemma now follows in view of (3.2) and (3.3).

3.2 The \( L^p \) norm of the Dirichlet kernel over \( S(\omega) \)

The following lemma determines an upper bound on the typical size of the Dirichlet kernel over \( S(\omega) \) in the \( L^p \)-norm, with \( 2 \leq p \leq 4 \). The lower bound, as well as the case \( p > 4 \) will be dealt with below.

**Lemma 3.2.** With \( S(\omega) \) as in (3.4), there exists a constant \( C_p \) so that

\[
\mathbb{E} \left\| \sum_{n \in S(\omega)} e(n \cdot) \right\|_p \leq C_p \left( L_2^\frac{p}{2} + \frac{L_2^{p-1}}{s} \right)
\]

for all \( 2 \leq p \leq 4 \).
Proof. For every \( \ell \in \mathbb{Z} \) define

\[
A_\ell(\omega) := \# \{ n, m \in \mathcal{S}(\omega) \mid n - m = \ell \} = \sum_{j, k \in \mathcal{P}} \chi_{[j - k + \xi_j - \xi_k = \ell]}.
\]

Clearly, \( \mathcal{P} - \mathcal{P} \subset \bigcup_i J_i \) where \( i \in a\mathbb{Z} \) and \( J_i := [i - 2s, i + 2s] \). These intervals are mutually disjoint since \( s \ll a \). This means that

\[
\ell \in J_i \implies A_\ell(\omega) = \sum_{j \in \mathcal{P}} \chi_{[j - i \in \mathcal{P}]} \chi_{[\xi_j - \xi_{j-i} = \ell-i]}.
\]

Let us denote the unique \( i \) for which \( \ell \in J_i \) by \( i(\ell) \). For simplicity, we shall mostly write \( i \). If \( i = 0 \), then \( A_\ell(\omega) = L\delta_0(\ell) \) (recall that \#\( \mathcal{P} = L \)). Otherwise, if \( i \neq 0 \), then one finds that

\[
\begin{aligned}
\mathbb{E} A_\ell &= \sum_{j \in \mathcal{P}} \frac{2}{2s+1} \left( 1 - \frac{|\ell - i|}{s} \right)_+ \chi_\mathcal{P}(j - i) = (L - |i|/a)_+ \frac{2}{2s+1} \left( 1 - \frac{|\ell - i|}{s} \right)_+ \\
&= \frac{2L}{2s+1} \hat{K}_s(\ell - i(\ell)) \hat{K}_L(|i|/a)
\end{aligned}
\]

where \( \hat{K}_s(k) = (1 - |k|/n)_+ \) denotes the Fejér kernel. Moreover, if \( i \neq 0 \), then

\[
\begin{aligned}
\mathbb{E} A^2_\ell &= \mathbb{E} \sum_{j, k \in \mathcal{P}} \chi_{[j - \xi_{j-i} = \ell-i]} \chi_{[\xi_k - \xi_{k-i} = \ell-i]} \\
&= \sum_{j, k \in \mathcal{P}} \chi_{[j \neq k, j \neq k \pm i]} \mathbb{E} \chi_{[\xi_j - \xi_{j-i} = \ell-i]} \mathbb{E} \chi_{[\xi_k - \xi_{k-i} = \ell-i]} \\
&\quad + \mathbb{E} \left( \chi_{[j = k, j \neq k \pm i]} + \chi_{[j \neq k, k+i, j = k-i]} + \chi_{[j \neq k, k-i, j = k+i]} \right) \mathbb{E} \chi_{[\xi_j - \xi_{j-i} = \ell-i]} \mathbb{E} \chi_{[\xi_k - \xi_{k-i} = \ell-i]}.
\end{aligned}
\]

Hence

\[
\begin{aligned}
\mathbb{E} A^2_\ell &= \sum_{j, k \in \mathcal{P}} \mathbb{E} \chi_{[\xi_j - \xi_{j-i} = \ell-i]} \mathbb{E} \chi_{[\xi_k - \xi_{k-i} = \ell-i]} \\
&\quad + \sum_{j, k \in \mathcal{P}} \chi_{[j = k, j \neq k \pm i]} \mathbb{E} \chi_{[\xi_j - \xi_{j-i} = \ell-i]} \mathbb{E} \chi_{[\xi_k - \xi_{k-i} = \ell-i]} \\
&\quad + \mathbb{E} \left( \chi_{[j = k, j \neq k \pm i]} + \chi_{[j \neq k, k+i, j = k-i]} + \chi_{[j \neq k, k-i, j = k+i]} \right) \mathbb{E} \chi_{[\xi_j - \xi_{j-i} = \ell-i]} \mathbb{E} \chi_{[\xi_k - \xi_{k-i} = \ell-i]} \\
&\quad + \mathbb{E} \mathbb{E} \left( \frac{L}{s} \left( 1 - \frac{|\ell - i|}{s} \right)_+ \right).
\end{aligned}
\]

The \( O \)-term in (3.9) arises because the error terms in (3.7) and (3.8) basically reduce to the computation of a single expectation as in (3.3). Now consider

\[
V_{p,N} := \int_0^1 \left| \sum_{\ell \in \mathbb{Z}} (A_\ell(\omega) - \mathbb{E} A_\ell) e(\ell \theta) \right|^2 d\theta.
\]
Since $p \leq 4$ by assumption, $EV_{p,N} \leq (EV_{4,N})^{\frac{p}{4}}$. Moreover, by (3.9),

$$EV_{4,N} = E \sum_{\ell \in \mathbb{Z}} |A_\ell(\omega) - E A_\ell|^2 = E A_0^2 - (E A_\ell)^2 + \sum_{\ell \neq 0} (E A_\ell)^2 + O(L_s(1 - \frac{|\ell - i|}{2s + 1})) - \sum_{\ell \neq 0} (E A_\ell)^2$$

$$\lesssim L^2$$

and therefore

$$EV_{p,N} \lesssim L^\frac{p}{2}.$$  

(3.10)

In view of (3.6),

$$\sum_{\ell \in \mathbb{Z}} E A_\ell e(\ell \theta) = \sum_{\ell \in \mathbb{Z}} 2L_{2s+1} K_s(\ell - i(\ell)) K_L(|i(\ell)|/a)e((\ell - i(\ell))\theta)e(i(\ell)\theta)$$

$$= \frac{2L}{2s+1} \sum_{k \in \mathbb{Z}} K_s(k)e(k\theta) \sum_{j \in \mathbb{Z}} K_L(j)e(ja\theta) = \frac{2L}{2s+1} K_s(\theta) K_L(a\theta).$$

It follows that

$$\int_0^1 \left| \sum_{\ell \in \mathbb{Z}} E A_\ell e(\ell \theta) \right|^\frac{p}{2} d\theta \lesssim \left( \frac{L}{s} \right)^\frac{p}{2} \int_0^1 \left( \frac{1}{s} \min(s^2, \theta^{-2}) \right)^{\frac{p}{2}} |K_L(a\theta)|^{\frac{p}{2}} d\theta$$

$$\lesssim \left( \frac{L}{s} \right)^\frac{p}{2} \left\{ s^{\frac{p}{2}} a^{-1} L^{\frac{p}{2}-1} + \sum_{j=1}^a \left( \frac{1}{s} \min(s^2, (j/a)^{-2}) \right)^{\frac{p}{2}} a^{-1} L^{\frac{p}{2}-1} \right\}$$

(3.11)

Combining (3.10) with (3.11) one obtains for $2 \leq p \leq 4$

$$\mathbb{E} \int_0^1 \left| \sum_{\ell \in \mathbb{Z}} e(n\ell) \right|^p d\theta = \mathbb{E} \int_0^1 \left| \sum_{\ell \in \mathbb{Z}} A_\ell(\omega) e(\ell \theta) \right|^\frac{p}{2} d\theta$$

$$\lesssim \int_0^1 \left| \sum_{\ell \in \mathbb{Z}} E A_\ell e(\ell \theta) \right|^\frac{p}{2} d\theta + \mathbb{E} \int_0^1 \left| \sum_{\ell \in \mathbb{Z}} [A_\ell(\omega) - E A_\ell] e(\ell \theta) \right|^\frac{p}{2} d\theta$$

(3.12)

$$\lesssim \frac{L^{p-1}}{s} + L^\frac{p}{2},$$

as claimed.

The following lemma is a special case of a well-known large deviation estimate for martingales with bounded increments. The norm $\| \cdot \|_\infty$ refers to the supremum norm with respect to the probability space.
Lemma 3.3. Suppose \( \{X_j\}_{j=1}^M \) are complex-valued independent variables with \( \mathbb{E} X_j = 0 \). Then for all \( \lambda > 0 \)
\[
\mathbb{P}\left[ \left| \sum_{j=1}^M X_j \right| > \lambda \left( \sum_{j=1}^M \|X_j\|_\infty^2 \right)^{\frac{1}{2}} \right] < C e^{-c\lambda^2}
\]
with some absolute constants \( c, C \).

Lemma 3.3 implies the following simple generalization of the Salem-Zygmund bound.

Corollary 3.4. Let \( s, L \) be positive integers. Suppose \( T_L \) is a trigonometric polynomial with random coefficients that can be written in the form
\[
T_L(\theta) = \sum_{j=-L}^L a_j(\theta) e(j\theta)
\]
where \( a_j(\theta) \) are trigonometric polynomials of degree at most \( s \), and such that for fixed \( \theta \) they are independent random variables with \( \mathbb{E} a_j(\theta) = 0 \). Moreover, we assume that \( \sup_{\theta \in \mathbb{T}} |a_j(\theta)| \leq 1 \) for each \( j \). Then for every \( A > 1 \)
\[
\mathbb{P}[\|T_L\|_\infty > C \sqrt{\log(s + L)} \sqrt{L}] \leq (s + L)^{-A},
\]
with some constant \( C = C(A) \).

Proof. Fix \( \theta \in \mathbb{T} \) and apply Lemma 3.3 with \( X_j = a_j(\theta) e(j\theta) \). By assumption, these are complex valued independent mean-zero variables with \( \|X_j\|_\infty \leq 1 \). Therefore,
\[
\sup_{\theta \in \mathbb{T}} \mathbb{P}\left[ \left| \sum_{j=-L}^L a_j(\theta) e(j\theta) \right| > \lambda \sqrt{L} \right] < C e^{-c\lambda^2}. \tag{3.13}
\]
If \( |\theta - \theta'| < (s + L)^{-2} \), then by Bernstein’s inequality
\[
|T_L(\theta) - T_L(\theta')| \leq (s + L)\|T_L\|_\infty |\theta - \theta'| \lesssim (s + L)L(s + L)^{-2} \lesssim 1.
\]
Now pick a \((s + L)^{-2}\)-net on the circle. The corollary follows by setting \( \lambda = C \log(s + L) \) with \( C \) large, and summing (3.13) over the elements of the net. \( \square \)

We can now state the general version of Lemma 3.2. It is possible to remove the log-term from the upper bound, but the bound given below suffices for our purposes.

Lemma 3.5. For all \( p \geq 2 \) there exists \( C_p \) so that
\[
\mathbb{E} \left\| \sum_{n \in S(\omega)} e(n \cdot) \right\|_p^p \leq C_p \left( \frac{L^{p-1}}{s} + (L \log N)^{\frac{p}{2}} \right). \tag{3.14}
\]

Moreover, there is \( c_p > 0 \) small so that
\[
\mathbb{P}\left[ \left\| \sum_{n \in S(\omega)} e(n \cdot) \right\|_p^p < c_p \left( L^{\frac{p}{2}} + \frac{L^{p-1}}{s} \right) \right] \to 0
\]
as \( N \to \infty \).
Proof. We work with the following splitting:

\[
\sum_{n \in S(\omega)} e(n\theta) = \sum_{n \in \mathbb{Z}} \mathbb{E} \chi_{S(\omega)}(n)e(n\theta) + \sum_{n \in \mathbb{Z}} \left[ \chi_{S(\omega)}(n) - \mathbb{E} \chi_{S(\omega)}(n) \right] e(n\theta).
\]

Clearly,

\[
\sum_{n \in \mathbb{Z}} \mathbb{E} \chi_{S(\omega)}(n)e(n\theta) = \frac{1}{2s+1} D_s(\theta) \sum_{j \in \mathcal{P}} e(j\theta),
\]

and thus

\[
\left\| \sum_{n \in \mathbb{Z}} \mathbb{E} \chi_{S(\omega)}(n)e(n\theta) \right\|_p^p \lesssim s^{-p} \int_0^1 \left| \min(s, \theta^{-1}) \sum_{j=1}^L e(ja\theta) \right|^p d\theta
\]

\[
\lesssim s^{-p} \left[ \sum_{k=1}^L \min(s, a/k)^p + s^p \right] \frac{L^{p-1}}{a} \lesssim \frac{L^{p-1}}{s}.
\]

Conversely,

\[
\left\| \sum_{n \in \mathbb{Z}} \mathbb{E} \chi_{S(\omega)}(n)e(n\theta) \right\|_p^p \gtrsim s^{-p} \int_0^{1/s} \left| D_s(\theta) \sum_{j=1}^L e(ja\theta) \right|^p d\theta
\]

\[
\gtrsim \frac{a}{s} \frac{L^{p-1}}{a} = \frac{L^{p-1}}{s}.
\]

Both (3.17) and (3.18) hold for all \( p > 1 \). The second sum in (3.15) can be written as

\[
\sum_{n \in \mathbb{Z}} \left[ \chi_{S(\omega)}(n) - \mathbb{E} \chi_{S(\omega)}(n) \right] e(n\theta) = \sum_{j \in \mathcal{P}} a_j(\omega, \theta) e(j\theta),
\]

where \( a_j(\omega, \theta) = \chi_{I_j}(\xi_j(\omega))e(\xi_j(\omega)) - \frac{1}{2s+1} D_s(\theta) \). Clearly, \( \mathbb{E} a_j(\omega, \theta) = 0 \), \( \sup_{\theta} |a_j(\omega, \theta)| \leq 2 \) and for fixed \( \theta \) the random variables \( a_j(\omega, \theta) \) are independent. Thus Corollary 3.3 yields that

\[
\left\| \sum_{n \in \mathbb{Z}} \left[ \chi_{S(\omega)}(n) - \mathbb{E} \chi_{S(\omega)}(n) \right] e(n\theta) \right\|_\infty \lesssim \sqrt{L \log N}
\]

up to probability at most \( (s + L)^{-p} \). In particular,

\[
\mathbb{E} \left\| \sum_{n \in \mathbb{Z}} \left[ \chi_{S(\omega)}(n) - \mathbb{E} \chi_{S(\omega)}(n) \right] e(n\theta) \right\|_p^p \lesssim (L \log N)^{\frac{p}{2}} + L^p(s + L)^{-p} \lesssim (L \log N)^{\frac{p}{2}}.
\]

In conjunction with (3.17) this yields (3.14). For the lower bound, take \( N^{-\varepsilon_0/2} > h \gg \frac{1}{s} \). Then

\[
\int_0^1 \left| \sum_{n \in S(\omega)} e(n\theta) \right|^p d\theta \gtrsim \int_0^{1/s} \left| \sum_{n \in \mathbb{Z}} \mathbb{E} \chi_{S(\omega)}(n)e(n\theta) \right|^p d\theta - \int_0^{1/s} \left| \sum_{n \in \mathbb{Z}} \left[ \chi_{S(\omega)}(n) - \mathbb{E} \chi_{S(\omega)}(n) \right] e(n\theta) \right|^p d\theta
\]

\[
+ \int_h^{1-h} \left| \sum_{n \in \mathbb{Z}} \left[ \chi_{S(\omega)}(n) - \mathbb{E} \chi_{S(\omega)}(n) \right] e(n\theta) \right|^p d\theta - \int_h^{1-h} \left| \sum_{n \in \mathbb{Z}} \mathbb{E} \chi_{S(\omega)}(n)e(n\theta) \right|^p d\theta
\]

\[
=: I + II + III + IV.
\]
By (3.18), \( I \gtrsim \frac{L^{p-1}}{s} \). Secondly,

\[
IV \lesssim \int_h^{1-h} \left| \frac{1}{s} D_s(\theta) \sum_{j=0}^{L-1} e(ja\theta) \right|^p d\theta \\
\lesssim s^{-p} \sum_{j>ah} (j/a)^{-p} \frac{L^{p-1}}{a} \lesssim s^{-p} h^{p+1} L^{p-1} \ll \frac{L^{p-1}}{s}
\]

where the final estimate follows from \( hs \gg 1 \). Thirdly, in view of \( p \geq 2 \) and (3.19),

\[
III \gtrsim \int_0^1 \left| \sum_{n \in \mathbb{Z}} \left[ \chi_{S(\omega)}(n) - \mathbb{E} \chi_{S(\omega)}(n) \right] e(n\theta) \right|^p d\theta \\
- \left( \int_0^h + \int_{1-h}^1 \right) \left| \sum_{n \in \mathbb{Z}} \left[ \chi_{S(\omega)}(n) - \mathbb{E} \chi_{S(\omega)}(n) \right] e(n\theta) \right|^p d\theta \\
\gtrsim \left( \int_0^1 \left| \sum_{n \in \mathbb{Z}} \left[ \chi_{S(\omega)}(n) - \mathbb{E} \chi_{S(\omega)}(n) \right] e(n\theta) \right|^2 d\theta \right)^{\frac{p}{2}} - C h (L \log N)^{\frac{p}{2}}
\]

(3.22)

\[
\gtrsim L^\frac{p}{2} - C h (L \log N)^{\frac{p}{2}},
\]

up to probability \( (s + L)^{-p} = o(1) \) as \( N \to \infty \). Similarly, (3.11) implies that

\[
II \lesssim s^{-1} (L \log N)^{\frac{p}{2}}
\]

up to probability \( (s + L)^{-p} \). Combining this bound with (3.22), (3.21), and (3.20) implies that

\[
\int_0^1 \left| \sum_{n \in S(\omega)} e(n\theta) \right|^p d\theta \gtrsim \frac{L^{p-1}}{s} + L^\frac{p}{2} - C (h + s^{-1})(L \log N)^{\frac{p}{2}}
\]

asymptotically with probability one. Since \( h < N^{-\varepsilon} \) and \( s > N^{\varepsilon} \), the lemma follows.

\[ \square \]

3.3 The majorant property for randomly perturbed arithmetic progressions

We are now ready to state our first result for perturbed arithmetic progressions as defined in (3.1). In this section, if \( S \) is the perturbation of an arithmetic progression of length \( L \), then we write

\[
A_{p,L}(\omega) := \left\| \sum_{n \in S(\omega)} e(n\cdot) \right\|_p^p.
\]

Also, we say that the random majorant property (RMP) holds at \( p \) if

(3.23)

\[
\mathbb{E}_\omega \sup_{|a_n| \leq 1} \left\| \sum_{n \in S(\omega)} a_n e(n\cdot) \right\|_p^p \leq C_\varepsilon N^{\varepsilon} \mathbb{E}_\omega \left\| \sum_{n \in S(\omega)} e(n\theta) \right\|_p^p.
\]

Of course, this depends on the length \( L \) of the underlying arithmetic progression. Although \( L \) is arbitrary, it will be kept fixed in the course of any argument that uses (3.23).
Theorem 3.6. Let \( S \) be as in (3.1). Then for every \( \varepsilon > 0 \) and \( 4 \geq p \geq 2 \) one has

\[
(3.24) \quad \mathbb{P}\left[ \sup_{|a_n| \leq 1} \left\| \sum_{n \in S(\omega)} a_n e(n\theta) \right\|_{L^p(T)} \geq N^{\varepsilon} \sum_{n \in S(\omega)} e(n\theta) \right\|_{L^p(T)} \right] \to 0
\]

as \( N \to \infty \). Moreover, under the additional restriction \( L \geq s \), (3.24) holds for all \( p \geq 4 \).

Proof. The proof is similar to the random case of the previous section, so we shall be somewhat brief. We will show that the RMP holds at \( p \) provided either \( 2 \leq p \leq 3 \), or if the RMP holds at \( p - 1 \), \( 2(p - 1) \), and \( 2(p - 2) \). It is important to notice that the RMP at \( p \) implies (3.24). Firstly, recall that we can write \( S(\omega) = \{ j + \xi_j \mid j \in \mathcal{P} \} \). We apply the decoupling lemma, Lemma 2.9, to the progression \( \mathcal{P} \). I.e., in the notation of Lemma 2.9, \( R_1^1 := \{ j \in \mathcal{P} \mid \xi_j = 1 \} \), and \( R_1^2 := \{ j \in \mathcal{P} \mid \xi_j = 0 \} \).

Set

\[
S_1^1(\omega) := \{ j + \xi_j(\omega) \mid j \in R_1^1 \}, \quad S_2^2(\omega) := \{ j + \xi_j(\omega) \mid j \in R_1^2 \}.
\]

Therefore, by Lemma 2.9,

\[
\frac{1}{8} \int_0^1 \left| \sum_{n \in S(\omega)} a_n e(n\theta) \right|^p d\theta = \int_0^1 \left| \sum_{n \in S_1^1(\omega)} a_n e(n\theta) \sum_{k \in S_2^2(\omega)} \bar{a}_k e(-k\theta) \sum_{\ell \in S_2^2(\omega)} a_{\ell} e(\ell\theta) \right|^{p-2} d\theta d\ell + O \left( \frac{L^p}{8} \right) \int_0^1 \left( 1 + \left| \sum_{n \in S} a_n \sqrt{L} e(n\theta) \max(p-1,2) \right| \right) d\theta.
\]

(3.25)

If either \( p \leq 3 \), or if the RMP holds at \( p - 1 \), then the \( O \)-term in (3.25) is at most

\[
(3.26) \quad L^\frac{p}{2} + C_\varepsilon N^{\varepsilon} L^\frac{p}{2} \mathbb{E} A_{p-1,2} \lesssim N^{\varepsilon} L^\frac{p}{2},
\]

see Lemma 3.3. We therefore obtain as in (2.47),

\[
\mathbb{E} \sup_{|a_n| \leq 1} \int_0^1 \left| \sum_{n \in S(\omega)} a_n e(n\theta) \right|^p d\theta \lesssim C_\varepsilon N^{\varepsilon} L^\frac{p}{2} + \int \mathbb{E}_{\omega_1,\omega_2} \sup_{|a_n| \leq 1} \left| \sum_{n \in S_1^1(\omega_1)} a_n e(n\theta) \sum_{k \in S_2^2(\omega_2)} \bar{a}_k e(-k\theta) \sum_{\ell \in S_2^2(\omega_2)} a_{\ell} e(\ell\theta) \right|^{p-2} d\theta dt \lesssim C_\varepsilon N^{\varepsilon} L^\frac{p}{2} + \int \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \sup_{|a_n| \leq 1} \left| \sum_{n \in S_1(\omega_1)} a_n e(n\theta) \sum_{k \in S_2(\omega_2)} \bar{a}_k e(-k\theta) \sum_{\ell \in S_2(\omega_2)} a_{\ell} e(\ell\theta) \right|^{p-2} d\theta dt \lesssim C_\varepsilon N^{\varepsilon} L^\frac{p}{2} + \int \mathbb{E}_{\omega_1} \mathbb{E}_{\omega_2} \sup_{|a_n| \leq 1} \left| \sum_{n \in S(\omega_1)} a_n e(n\theta) \sum_{k \in S(\omega_2)} \bar{a}_k e(-k\theta) \sum_{\ell \in S(\omega_2)} a_{\ell} e(\ell\theta) \right|^{p-2} d\theta dt \lesssim C_\varepsilon N^{\varepsilon} L^\frac{p}{2} + \mathbb{E}_{\omega_2} \mathbb{E}_{\omega_1} \sup_{x \in E(\omega_2)} \sum_{n \in S(\omega_1)} x_n.
\]

(3.27)
Here

\[ \mathcal{E}(\omega_2) := \left\{ \left( \left| e(n\cdot) \right|, \sum_{k \in S(\omega_2)} \tilde{b}_k e(-k) \right| \sum_{\ell \in S(\omega_2)} |b_\ell e(\ell)|^{p-2} \right) \right\}^{N}_{n=1} \sup_{1 \leq n \leq N} |b_n| \leq 1 \right\} \subset \mathbb{R}_+^N. \]

By Lemma 3.1, it follows from (3.27) that

\[ (3.28) \quad \mathbb{E}_\omega \sup_{|a_n| \leq 1} \int_{0}^{1} \left| \sum_{n \in S(\omega)} a_n e(n\theta) \right|^p d\theta \]

\[ \lesssim C_\varepsilon N^{\varepsilon} \mathbb{E}_{\omega} \sup_{x \in \mathcal{E}(\omega_2)} |x| + \int_{0}^{\infty} \sqrt{\log N_2(\mathcal{E}(\omega_2), t)} dt. \]

Now suppose the RMP holds at 2(p − 1) (so this holds for sure if p is an odd integer). Then by Plancherel,

\[ \mathbb{E}_\omega \sup_{x \in \mathcal{E}(\omega_2)} |x| \leq C_\varepsilon N^{\varepsilon} \mathbb{E}_\omega \left| \sum_{n \in S(\omega)} e(n\cdot) \right|^{p-1}_{2(p-1)} \leq C_\varepsilon N^{\varepsilon} \sqrt{\mathbb{E}_\omega A_{2(p-1), L}(\omega)}. \]

As far as the entropy term in (3.28) is concerned, the same analysis as in the random case shows that if \( \cdot p \leq 3 \), then

\[ \mathbb{E}_\omega \int_{0}^{\infty} \sqrt{\log N_2(\mathcal{E}(\omega_2), t)} dt \leq C_\varepsilon N^{\varepsilon} L^{\frac{3}{2}}, \]

or if \( \cdot p > 3 \) and the RMP holds at 2(p − 2), then

\[ \mathbb{E}_\omega \int_{0}^{\infty} \sqrt{\log N_2(\mathcal{E}(\omega_2), t)} dt \leq C_\varepsilon N^{\varepsilon} L \sqrt{\mathbb{E} A_{2(p-2), L}}, \]

see (2.60) and (2.61) for the details. Inserting all of this into (3.28) yields, under the assumption that \( \cdot p > 3 \) and the RMP holds at \( p-1, 2(p-1), \) and \( 2(p-2) \) (the case \( \cdot p = 3 \) is similar),

\[ \mathbb{E}_\omega \sup_{|a_n| \leq 1} \int_{0}^{1} \left| \sum_{n \in S(\omega)} a_n e(n\theta) \right|^p d\theta \]

\[ \lesssim C_\varepsilon N^{\varepsilon} \left\{ L^{\frac{p}{2}} + (1 + \sqrt{L/s}) \mathbb{E}_\omega A_{2(p-1), L}(\omega) + L \sqrt{\mathbb{E} A_{2(p-2), L}} \right\} \]

\[ \lesssim C_\varepsilon N^{\varepsilon} \left\{ L^{\frac{p}{2}} + (1 + \sqrt{L/s}) \left( \frac{L^{2p-3}}{s} + L^{p-1} \right)^{\frac{1}{2}} + L \left( \frac{L^{2p-5}}{s} + L^{p-2} \right)^{\frac{1}{2}} \right\} \]

\[ \lesssim C_\varepsilon N^{\varepsilon} \left[ \frac{L^{p-1}}{s} + L^{\frac{p}{2}} + \frac{L^{p-\frac{3}{2}}}{\sqrt{s}} \right]. \]

Recall from Lemma 3.3 that the desired bound is \( L^{p-1} \) + \( L^{\frac{p}{2}} \). If \( p = 3 \), then (3.21) does indeed agree with this bound. Since the hypotheses involving the RMP hold in case \( p = 3 \), we are done with that case, regardless of the relative size of \( L \) and \( s \). Let us assume now that \( L \geq s \). Then (3.29) agrees with the desired bound for all \( p \). This means that we can run the same type of inductive argument.
as in Corollary 2.12. We leave it to the reader to check that this proves (3.24) for all \( p \geq 2 \) provided \( L \geq s \). Finally, if \( L < s \), then \( L < s \leq a \leq \frac{N}{L} \) and thus \( L \leq \sqrt{N} \). In particular, \( \# S \leq \sqrt{N} \) in that case. In analogy with the random subset case, this suggests that \( S(\omega) \) are \( \Lambda(p) \)-sets for \( 2 \leq p \leq 4 \) with high probability. Although perturbed arithmetic progressions are not covered by [B1] and [B2], it turns out that the strategy from \( \text{[B1]} \) and \( \text{[B2]} \) is still relevant. More precisely, suppose first that \( 2 \leq p \leq 3 \). Then (3.28) holds, even without the \( N^\varepsilon \)-term. By Plancherel, but without appealing to any RMP,

\[
(3.30) \quad \mathbb{E}_{\omega_2} \sup_{x \in \mathcal{E}(\omega_2)} |x| \leq \mathbb{E}_{\omega_2} \sup_{|a_n| \leq 1} \left\| \sum_{n \in S(\omega)} a_n e(n\theta) \right\|_2^{p-1} \leq K_p^p L^{\frac{p-2}{2}}.
\]

Here

\[
K_p := \mathbb{E}_\omega \sup_{|a_n| \leq 1} \int_0^1 \left| \sum_{n \in S(\omega)} a_n e(n\theta) \right|^p d\theta.
\]

To pass to (3.30), one writes \( 2(p-1) = p + (p-2) \) and then estimates the \( (p-2) \)-power in \( L^\infty \).

Secondly, to bound the entropy term, set \( q = \frac{2}{\delta - p} \). Then by Plancherel the distance between any two elements in \( \mathcal{E}(\omega_2) \) is at most

\[
\|g|g|^{p-2} - h|h|^{p-2}\|_2 \leq \|g - h\|_q (\|g\|_2^{p-2} + \|h\|_2^{p-2})
\]

\[
\leq L^{\frac{p-2}{q}} \|g - h\|_q,
\]

where \( g, h \in \sqrt{L}\mathcal{P}_{S(\omega_2)} \), see (2.51). As before, the entropy estimate therefore reads

\[
\mathbb{E}_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2), t)} dt \leq \sqrt{L L^{\frac{p-2}{q}}} \sqrt{L} = L^\frac{p}{2}.
\]

Inserting these bounds into (3.28) yields

\[
K_p^p \lesssim L^\frac{p}{2} + (1 + \sqrt{L/s})K_p^p L^{\frac{p-2}{2}} \leq C L^\frac{p}{2} + \frac{1}{2} K_p^p + C(1 + L/s)L^{p-2}.
\]

Since \( L^{p-2} \leq L^\frac{p}{2} \) in view of \( p \leq 4 \), one obtains the desired bound

\[
K_p^p \lesssim L^\frac{p}{2} + \frac{L^{p-1}}{s}
\]

if \( 2 \leq p \leq 3 \) and regardless of the relative size of \( L \) and \( s \). If \( 3 \leq p \leq 4 \), then the previous argument needs to be modified in two places: Firstly, there is the issue of the \( O \)-term in (3.23). However, we just showed that the RMP holds at \( p-1 \leq 3 \), and therefore (3.26) applies here as well (even without the \( N^\varepsilon \)-term). Secondly, the entropy bounds need to be modified. In case \( 3 \leq p \leq 4 \), one has \( 2(p-2) \leq p \). Hence

\[
\|g|g|^{p-2} - h|h|^{p-2}\|_2 \leq \|g - h\|_\infty (\|g\|_2^{p-2} + \|h\|_2^{p-2})
\]

\[
\leq C \varepsilon N^\varepsilon \|g - h\|_q (\|g\|_p^{p-2} + \|h\|_p^{p-2})
\]

\[
\leq C \varepsilon N^\varepsilon \sup_{|a_n| \leq 1} \left\| \sum_{n=1}^N a_n \xi_n(\omega_2) e(n\cdot) \right\|_p^{p-2} \|g - h\|_q,
\]

39
with $g, h$ as above. By the usual arguments, cf. (2.60), it follows that

$$E_{\omega_2} \int_0^\infty \sqrt{\log N_2(\mathcal{E}(\omega_2), t)}\, dt \lesssim L K_p^{\frac{p+2}{2p}} \lesssim \frac{1}{2} K_p^p + L^p.$$ 

Inserting these bounds into (3.28) implies the desired bound.

Remark 3.7. It is possible that one can make improvements on Theorem 3.6 similar to those in Theorem 2.1, thus removing the condition $L \geq s$ in some range of $p \geq 4$. This would require working with $\Lambda(p)$ type arguments as we just did in the end of the previous proof. But we do not pursue that issue here.

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