Replica Symmetric Bound for Restricted Isometry Constant

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Abstract—We develop a method for evaluating restricted isometry constants (RICs). This evaluation is reduced to the identification of the zero-points of entropy (complexity), which is defined for submatrices that are composed of columns selected from a given measurement matrix. Using the replica method developed in statistical mechanics, we assess RICs for Gaussian random matrices under the replica symmetric (RS) assumption. In order to numerically validate the adequacy of our analysis, we employ the exchange Monte Carlo (EMC) method, which has been empirically demonstrated to achieve much higher numerical accuracy than naive Monte Carlo methods. The EMC method enables effective sampling, avoiding entrapment at local minima, which limits the effectiveness of naive Monte Carlo sampling to capture the true behavior [10]. Numerical results suggest that our scheme currently provides the tightest RIC upper bound, which could be further tightened by taking into account the replica symmetry breaking (RSB).

I. INTRODUCTION

The signal processing paradigm of compressed sensing (CS) enables a substantially more effective sampling than that required by the conventional sampling theorem [1]. CS is applied to problems in various fields, in which the acquisition of data is quite costly, such as astronomical and medical imaging [2], [3]. The CS performance is mathematically analyzed using the problem settings of a randomized linear observation [4], [5]. Here, $A \in \mathbb{R}^{M \times N}$ is the given observation matrix, and CS endeavors to reconstruct the $S$-sparse signal $x \in \mathbb{R}^N$ that has $S(< N)$ nonzero components from observation $y = Ax$.

A widely used strategy for the reconstruction of this signal is the $\ell_0$ minimization, which corresponds to the relaxed problem of $\ell_0$ minimization. A key quantity used to mathematically analyze the $\ell_0$ and $\ell_1$ minimization strategies is the restricted isometry constant (RIC) [6]. Directly evaluating an RIC requires the computation of maximum and minimum eigenvalues of $N!/S!(N-S)!$ submatrices that are generated by extracting $S$-columns from $A$, which is computationally infeasible. In the case of Gaussian random matrices of $A$, the upper bound for the RIC is estimated using the large deviation property without direct computation of the eigenvalues [6], [7], [8].

This paper proposes a theoretical scheme for the direct estimation of the RICs. In order to do this, we evaluate the entropy of the submatrices that offers a given value of the maximum and minimum eigenvalues, which we call complexity. An RIC of matrix $A$ is offered by the condition that the corresponding complexity vanishes. Furthermore, in order to demonstrate our method’s utility, we apply our scheme to Gaussian random matrices, using the replica method, and compare the obtained result with that of earlier studies.

Our theoretical evaluation is also numerically assessed using the exchange Monte Carlo (EMC) method [9], which is expected to achieve much higher numerical accuracy than those of naive Monte Carlo schemes. The EMC method enables effective sampling, avoiding entrapment at local minima, which limits the effectiveness of naive Monte Carlo sampling to capture the true behavior [10]. Numerical results suggest that our scheme currently provides the tightest RIC upper bound, which could be further tightened by taking into account the replica symmetry breaking (RSB).

II. RESTRICTED ISOMETRY CONSTANT

Definition 1 (Restricted isometry constants). A matrix $A \in \mathbb{R}^{M \times N}$ satisfies the restricted isometry property (RIP) with $RIC 0 < \delta_S^\text{min} \leq \delta_S^\text{max}$ if

$$1 - \delta_S^\text{min} ||x||_2^2 \leq ||Ax||_2^2 \leq (1 + \delta_S^\text{max}) ||x||_2^2$$

holds for any $S$-sparse vector $x \in \mathbb{R}^N$, in which $S$ is the number of non-zero components.

The original work presented by Candès et al. [4] addresses symmetric RIC $\delta_S = \max[\delta_S^\text{min}, \delta_S^\text{max}]$. An RIC indicates how close the space, which is spanned by the $S$-columns of $A$, is to an orthonormal system. If an RIC is small, the linear transformation performed using $A$ is nearly an orthogonal transformation. The symmetric RIC provides sufficient conditions for the reconstruction of $S$-sparse vector $x$ in underdetermined linear system $y = Ax$ using $\ell_0$ and $\ell_1$ minimization [6].

Theorem 1. Let $A \in \mathbb{R}^{M \times N}$ and $x \in \mathbb{R}^N$ with $M < N$, and consider the linear equation $y = Ax$. If $\delta_{2S} < 1$, a unique $S$-sparse solution exists and is the sparsest solution to $\ell_0$ problem

$$\min_x ||x||_0, \text{ subject to } y = Ax. \quad (2)$$

Also, if $\delta_{2S} < \sqrt{2} - 1$, the $S$-sparse solution to $\ell_1$ problem

$$\min_x ||x||_1, \text{ subject to } y = Ax$$

(3)
is uniquely identified as the sparsest solution and equals the \( \ell_0 \) problem's solution.

It should be noted that \( \delta_{S}^{\text{min}} \) and \( \delta_{S}^{\text{max}} \) do not increase or decrease at the same rate, and asymmetric RICs improve the condition of \( \ell_1 \) reconstruction [11].

**Theorem 2.** Let us consider the same problem settings as in Theorem [7] If \( (4\sqrt{2} - 3)\delta_{S}^{\text{min}} + \delta_{S}^{\text{max}} < 4(\sqrt{2} - 1) \), then the unique \( S \)-sparse solution is the sparsest solution to the \( \ell_1 \) problem andequals the solution to the \( \ell_0 \) problem.

RIC evaluation is also a fundamental linear algebra problem [7], [8] because RICs clearly relate to the eigenvalues of Gram matrices. Let \( T \subseteq V = \{1, \cdots , N\}, |T| = S \) be the position of the nonzero elements of \( S \)-sparse vector \( x \). The product \( AX \) equals \( A_T x_T \), where \( A_T \) is the submatrix that consists of \( i \in T \) columns of \( A \) and where \( x_T = \{x_i | i \in T \} \). For any realization of \( T \), the following holds.

\[
\lambda_{\min}(A_T^T A_T) ||x_T||_F^2 \leq ||A_T x_T||_F^2 \leq \lambda_{\max}(A_T^T A_T) ||x_T||_F^2
\]

Here, \( \lambda_{\min}(B) \) and \( \lambda_{\max}(B) \) denote the minimum and maximum eigenvalues of \( B \), respectively, and superscript \( T \) denotes the matrix transpose. Therefore, the following expression of the RIC is equivalent to \[11\]:

\[
\delta_{S}^{\text{min}} = 1 - \frac{\lambda_{\min}(A; S)}{|T| \subseteq V, |T| = S} \min \lambda_{\min}(A_T^T A_T), \delta_{S}^{\text{max}} = \frac{\lambda_{\max}(A; S)}{|T| \subseteq V, |T| = S} \max \lambda_{\max}(A_T^T A_T)
\]

in which

\[
\lambda_{\min}(A; S) = \min_{T: T \subseteq V, |T| = S} \lambda_{\min}(A_T^T A_T), \lambda_{\max}(A; S) = \max_{T: T \subseteq V, |T| = S} \lambda_{\max}(A_T^T A_T).
\]

Direct evaluation of eq. \[12\] requires the calculations of the maximum and minimum eigenvalues of the \( N! / (S!(N - S)!) \) Gram matrices \( \{A_T^T A_T\} \), which is computationally difficult when \( N \) and \( S \) are large. For typical Gaussian random matrices \( A \), the RIC's upper bound is estimated using large deviation properties of the maximum and minimum eigenvalues of the Wishart matrix [6], [7], [8].

**III. PROBLEM SETUP AND FORMALISM**

We estimate RICs in a different manner, and the following theorem is fundamental to our approach.

**Theorem 3.** Let \( A \in \mathbb{R}^{M \times N} \). Then the maximum and minimum eigenvalues of \( A^T A \) are given by

\[
\lambda_{\min}(A^T A) = -\beta \rightarrow +\infty \frac{2}{N\beta} \log Z(A; \beta), \lambda_{\min}(A^T A) = -\beta \rightarrow +\infty \frac{2}{N\beta} \log Z(A; \beta),
\]

respectively, where \( Z(A; \beta) \) is defined using \( u \in \mathbb{R}^N \):

\[
Z(A; \beta) = \int du e^{u^T A^T A u / 2} \delta(||u||_F^2 - N).
\]

**Proof:** Applying identity \( \delta(|u||_F^2 - N) = \beta/(4\pi) \times \int dy e^{-\beta y^2/2} \delta(||y||_F^2 - N) \) gives us

\[
Z(A; \beta) = \frac{(2\pi)^{\frac{N}{2} - 1}}{2\beta^{\frac{N}{2}}} \int dy e^{\beta \left( \frac{N\eta}{2} - \frac{1}{2\beta} \sum_{i} \ln(\eta + \lambda_i) \right)}
\]

in which \( \{\lambda_i\} \) is the \( i \)-th eigenvector of \( A^T A \). As \( \beta \rightarrow +\infty \), the integral can be evaluated using the saddle point method, which is dominated by \( \eta = -\lambda_{\min}(A^T A) + (N\beta)^{-1} + o(\beta^{-1}) \), where \( o(\beta^{-1}) \) represents the contribution from negligible terms compared with \( \beta^{-1} \). This yields eq. \[7\], and eq. \[8\] is similarly obtained by applying the saddle point method for \( \beta \rightarrow -\infty \).

**Theorem 3** holds for all submatrices \( A_T \). For mathematical convenience, we introduce variables \( \epsilon \in \{0, 1\}^N \) and define

\[
Z_0\epsilon(c, A; \beta) = \int |u|^2 dP(u|c) \exp \left( -\frac{\beta}{2} ||A(c \oplus u)||_F^2 \right) \times \delta(||c \oplus u||_F^2 - N),
\]

where \( \oplus \) denotes the component-wise product, and \( P(u|c) \propto e^{-\sum_{i=1}^N (1 - c_i)u_i^2/2} \) is introduced in order to avoid the divergence caused by integrating \( u_i \) when \( c_i = 0 \). Let us define \( c(T) \in \{0, 1\}^{|T|} \) to be \( c(T) \) for \( i \in T \) and to be \( c(T)_i = 0 \) otherwise. The two functions \( Z(A_T; \beta) \) and \( Z_0(c(T)_c, A; \beta) \) have a one-to-one correspondence: \( Z(A_T; \beta) = Z_0(c(T)_c, A; \beta) \). We write \( \lambda_{\min}(c, A) \) and \( \lambda_{\min}(c, A) \), which are obtained by substituting \( Z(c, A; \beta) \) into eq. \[7\] and eq. \[8\], respectively. Because \( \lambda_{\min}(A_T^T A_T) = \lambda_{\max}(c(T), A) \) and \( \lambda_{\min}(c(T), A; \beta) = \lambda_{\min}(c(T), A; \beta) \) naturally hold, eqs. \[7\] and \[8\] can be rewritten as

\[
\lambda_{\min}(A; S) = \min_{c \in \mathbb{R}^N} \lambda_{\min}(c, A), \lambda_{\max}(A; S) = \max_{c \in \mathbb{R}^N} \lambda_{\max}(c, A),
\]

where \( c \epsilon \) denotes the set of configurations of \( c \) that satisfy \( \sum_i c_i = S \).

Let us define the energies of \( c \) to be \( \Lambda_0(c, A) = \lambda_{\min}(c, A) / 2 \) and \( \Lambda_0(c, A) = \lambda_{\max}(c, A) / 2 \). Based on this, we introduce a free entropy density as \( \phi(\mu; A, S) = N^{-1} \sum_{c} e^{-N \mu \lambda_{\min}(c, A)} \delta \left( \sum_{i=1}^N c_i - S \right) \). Here, \( \mu \) denotes the sign of \( \mu \). Eqs. \[7\] and \[8\] offer its alternative expression

\[
\phi(\mu; A, S) = \lim_{\beta \rightarrow +\infty} N^{-1} \sum_{c} Z_0^\delta(c, A; \beta) \delta \left( \sum_{i=1}^N c_i - S \right).
\]

In addition, we let \( \exp(N \Sigma_\pm(\lambda; A; S)) \) represent the number of \( c \) that correspond to \( \Lambda_{\pm}(c, A) = \lambda / 2 \) and satisfy \( \sum_i c_i = S \). Here, \( \Sigma_\pm(\lambda; A; S) \) is called the complexity, which is naturally assumed to be a convex function of \( \lambda \). Summation over the microscopic states of \( c \) is replaced with the integral of \( \lambda \) over the possible value of \( \Lambda_\pm(c, A) \):

\[
\phi(\mu; A, S) = \frac{1}{N} \log \left[ \int d\lambda e^{-N \mu \lambda} + N \Sigma_\text{sgn}(\mu)(\lambda; A; S) \right] \rightarrow \max \{-\mu \lambda + \Sigma_\text{sgn}(\mu)(\lambda; A; S)\},
\]
in which the saddle point method is employed. The maximizer of \( \lambda \), which corresponds to the typical energy value of \( e \) that is sampled following the weight \( e^{-N\mu\lambda + \Sigma_{\text{ges}}(\mu)} \delta(\sum c_i - S) \), must satisfy

\[
-\mu + \frac{\partial \Sigma_{\text{ges}}(\mu)}{\partial \lambda} = 0. \tag{15}
\]

Eq. (14) implies that \( \phi(\mu; A; S) \) is obtained using the Legendre transformation of \( \Sigma_\pm(\lambda; A; S) \), and the inverse Legendre transformation converts \( \phi(\mu; A; S) \) to \( \Sigma_\pm(\lambda; A; S) \):

\[
\Sigma_{\text{ges}}(\mu)(\lambda; A; S) = \phi(\mu; A; S) - \mu \frac{\partial \phi(\mu; A; S)}{\partial \mu}, \tag{16}
\]

from the convexity assumption of \( \Sigma_\pm(\lambda; A; S) \). A similar formalism has been introduced for investigating the geometrical structure of weight space in learning of multilayer neural networks [12].

The relationships among \( \mu \), \( \lambda \), and \( \Sigma_\pm \) are illustrated in Fig. 1. Complexities \( \Sigma_+ \) and \( \Sigma_- \) are convex increasing and decreasing functions of \( \lambda \), respectively. According to eq. (15), the value of \( \lambda \) at \( \mu \) represents the point in which the gradient of \( \Sigma_\pm \) equals \( \mu \). By definition, negative complexity values are not allowed, and \( \lambda_+ \) is the maximum complexity.

\[
\lambda_{\text{max}}(A; S) = \lambda_+, \quad \lambda_{\text{min}}(A; S) = \lambda_-, \tag{17}
\]

which are the typical values for \( \mu = \mu_{\text{max}} \) and \( \mu = \mu_{\text{min}} \), respectively (Fig. 1).

**IV. RS ANALYSIS FOR GAUSSIAN RANDOM MATRIX**

This section applies the methodology introduced in the previous section to the case in which \( A \) is composed of entries that are independently generated using a Gaussian distribution with mean 0 and variance \( N^{-1} \). In this case, \( \phi(\mu; A; S) \) and \( \Sigma_\pm(\lambda; A; S) \) randomly fluctuate depending on \( A \). However, for all \( \epsilon > 0 \), the probability that deviation from the typical values, \( \phi(\mu; S) \equiv [\phi(\mu; A; S)]_A \) and \( \Sigma_\pm(\lambda; S) \equiv [\Sigma_\pm(\lambda; A; S)]_A \), is larger than \( \epsilon \) tends to vanish as \( N \to \infty \). Here, \( [\cdot]_A \) denotes the average of \( A \). Therefore, typical properties can be characterized by evaluating the typical values, \( \phi(\mu; \rho) \) and \( \Sigma_\pm(\lambda) \), using the replica method with the identity [13], [14]:

\[
\log f(A)_A = \lim_{n \to 0} \frac{\partial}{\partial \mu} \log f^n(A)_A \tag{18}
\]

where \( f(A) \) is an arbitrary function. When both \( n \) and \( m = \mu/\beta \) are positive integers, regarding \( \sum_{i=1}^{N} c_i - S \) in eq. (13) as \( f(A) \) leads us to express \( [f^n(A)]_A \) as a summation/integration with respect to \( n \) and \( m n \) replica variables \( \{c^a \} \) and \( \{\epsilon^a \} \) (\( a \in \{1, 2, \ldots, n\}, \sigma \in \{1, 2, \ldots, m\} \)), which can be evaluated by the saddle point method for \( N, M, S \to \infty \), keeping \( a = M/N \) and \( \rho = S/N \) finite. Under the replica symmetric (RS) assumption, in which the dominant saddle point is assumed to be invariant against any permutation of the replica indices \( a \) and \( \sigma \) within each of their sets \( \{1, 2, \ldots, n\} \) and \( \{1, 2, \ldots, m\} \), respectively, the resulting functional form of \( N^{-1} \log f^n(A)_A \) becomes extendable for non-integer \( n \) and \( m \). Therefore, we insert the expression into eq. (13) employing the formula of eq. (18), which finally yields

\[
\phi(\mu; \rho) = -\frac{\alpha}{2} \log(\alpha + \mu(1 - \mu)) - \frac{\alpha}{2} \log(\alpha + \mu) + K \rho
\]

\[
- \frac{2(\alpha + \mu(1 - \mu))}{Q - 2Q - \hat{q}_0 + \hat{q}_0 + K \rho}
\]

\[
+ \int Dz \log \left\{ 1 + e^{-K} \int Dy \exp \left\{ \frac{(\sqrt{q_1 - q_0} + \sqrt{q_0})^2}{2Q} \right\} \right\},
\]

where \( \{q_1, q, \check{q}, \hat{q}_0, \hat{q}_1, K\} \) are determined to extremize the right hand side, and \( \int Dz = \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}} \exp(-z^2/2) \). Complexity \( \Sigma_\pm(\lambda; \rho) \) is derived by applying the inverse Legendre transformation to \( \phi(\mu; \rho) \).

**V. RESULTS**

In Fig. 2 complexity \( \Sigma_\pm \) with \( \alpha = 0.5 \) and \( \rho = 0.1 \) is shown for (a) \( \mu < 0 \) and (b) \( \mu > 0 \) results of the exchange Monte Carlo (EMC) sampling are represented by circles, and the EMC procedure is summarized in Appendix A.

The values of \( \lambda \) when \( \mu \to +0 \) and \( \mu \to -0 \), which are denoted using dashed lines, coincide with the respective minimum and maximum of the Marchenko-Pastur (MP) distribution’s support for the \( M \times S \) Gaussian random matrix [15]. As the limit of \( |\mu| \to 0 \) corresponds to unbiased generation of \( M \times S \) Gaussian random matrices, the coincidence theoretically supports the adequacy of our analysis. The slight discrepancy between the theoretical and EMC results in the complexity’s tails could be due to the insufficiency of the RS assumption. The convexity of our complexity suggests that the RS assumption exactly creates the complexity curve or extends it outward [16]. This is consistent with the EMC method’s result, which indicates that the exact complexity curve is
Fig. 2. Complexity for $\alpha = 0.5$ and $\rho = 0.1$ with (a) $\mu > 0$ and (b) $\mu < 0$. Circles denote EMC method results. Vertical lines represent (a) minimum and (b) maximum eigenvalues of MP distribution.

Fig. 3. Comparison of symmetric RIC for $\alpha = 0.5$. Numerical lower bound is estimated for $N = 1000$ and $M = 500$.

Fig. 4. $\ell_0$ and $\ell_1$ limits given by RS RIC. Black lines represent Bah and Tanner’s results, denoted BT.

VI. SUMMARY AND CONCLUSION

We proposed a theoretical scheme for the evaluation of restricted isometry constants. The problem was converted to the assessment of complexity, and the possible maximum and minimum eigenvalues, which produce the RIC, are the complexity’s zero-points. Given a Gaussian random matrix, we computed complexity using the replica method under the replica symmetric ansatz and estimated the value of the RIC. Physically, it has meaning as a bound and is tighter than existing bounds. Numerical experiments using the EMC sampling support our analysis.

A more accurate evaluation of the RIC is possible if the RSB is taken into account. Our scheme is applicable to more general matrices than Gaussian random matrices.

APPENDIX A
MONTE CARLO SAMPLING FOR RIC ESTIMATION

This process attempts to numerically compute the free entropy density $\phi(\mu|A)$, and obtain the complexity by applying the inverse Legendre transformation to $\phi(\mu|A)$. We employ the exchange Monte Carlo (EMC) sampling in order to avoid the trap of metastable states, which prevent correct sampling. In the EMC approach, replicating the system associated with a different parameter that is conjugate to “energy” accelerates the relaxation to equilibrium. In our problem, energy corresponds to $\Lambda_{\pm}(c|A)$ and its conjugate parameter is
μ. We prepare k systems, which have the same configuration of A, and assign configuration $c_i \in C_S$ and parameter $\mu_i$ to each system $i = 1, \cdots, k$. The signs of $\{\mu_i\}$ are set to be the same. Each step of the EMC process updates $c_i$ within each system, and attempts exchanges between configurations $c_i$ and $c_{i+1}$. The probability of transition from $c_i$ to $c_i'$ is given by

$$w(c_i, c_i') = \min\{\exp(\mu_i N \Delta_i), 1\},$$  \hspace{1cm} (20)

where $\Delta_i = A_{\text{sgn}(\mu_i)}(c_i | A) - A_{\text{sgn}(\mu_i)}(c_i' | A)$ is the energy difference. Eq. (20) satisfies the detailed balance condition, which is sufficient to maintain the systems at equilibrium: $P(c_i | A) \propto \exp(-N\mu_i A_{\text{sgn}(\mu_i)}(c_i | A))$. The probability of an exchange between systems $c_i$ and $c_{i+1}$ is given by

$$w_{\text{exc}}(c_i, c_{i+1}) = \min\{\exp(N(\mu_i - \mu_{i+1}) \Delta_{i,i+1}), 1\},$$  \hspace{1cm} (21)

in which $\Delta_{i,i+1} = A_{\text{sgn}(\mu_i)}(c_i | A) - A_{\text{sgn}(\mu_{i+1})}(c_{i+1} | A)$. After sufficient updates, the entire k-system is expected to converge to the equilibrium distribution $P_{\text{eq}}(c_i | A) \propto \prod_{i=1}^{k} \exp\{-N\mu_i A_{\text{sgn}(\mu_i)}(c_i | A)\}$.

We prepare histograms of each system $h_i(A = \lambda/2 | A; \mu_i)$ using the EMC sampling $t_i$ steps after relaxation. The density of states $W_{\pm}(\lambda | A) = \sum_{c \in C_S} \delta(\lambda/2 - \lambda(\pm | c | A))$ is obtained by applying the multihistogram method as in [18]:

$$W_{\text{sgn}(\mu_i)}(\lambda | A) = \frac{1}{t_i} \sum_{j=1}^{k} h_i(\lambda | A; \mu_i) p_i(\lambda) e^{-\mu_i \lambda/2},$$  \hspace{1cm} (22)

where

$$p_i(\lambda) = \frac{t_i \exp(\mu_i \lambda/2 - f_i)}{\sum_{j=1}^{k} t_j \exp(\mu_j \lambda/2 - f_j)}$$  \hspace{1cm} (23)

and $\{f_i\}$ are self-consistently determined to satisfy

$$\exp(f_i) = \int d\lambda \sum_{j=1}^{k} h_j(\lambda | A; \mu_j) \exp(\mu_j \lambda/2),$$  \hspace{1cm} (24)

Finally, the free entropy density is calculated:

$$\phi_{\text{sgn}(\mu)}(\mu | A) = \int d\lambda W_{\text{sgn}(\mu)}(\lambda | A) \exp(-N\mu \lambda/2).$$  \hspace{1cm} (25)

Fig. 5 exhibits numerically obtained $\phi(\mu)$ for $\alpha = 0.5$ and $\rho = 0.1$. We set $t_i = 10^5$ for all $i$, and the average of $A$ is implemented over 1000 realizations of Gaussian random matrix with mean 0 and variance $N^{-1}$. The results of $N \rightarrow \infty$ are extrapolated using $N = 30, 40, 60, 80, 100$ for $\mu < 0$ and $N = 30, 40, 60, 80, 100, 200$ for $\mu > 0$. Solid lines depict the free entropy derived using the RS analysis. The differences between the two methods when $|\mu|$ is large could be caused by the RS assumption. The corresponding complexity is shown in Fig. 2.

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