ON TOTALLY GEODESIC SUBMANIFOLDS IN THE JACOBIAN LOCUS

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Abstract. We study submanifolds of $A_g$ that are totally geodesic for the locally symmetric metric and which are contained in the closure of the Jacobian locus but not in its boundary. In the first section we recall a formula for the second fundamental form of the period map $M_g \hookrightarrow A_g$ due to Pirola, Tortora and the first author. We show that this result can be stated quite neatly using a line bundle over the product of the curve with itself. We give an upper bound for the dimension of a germ of a totally geodesic submanifold passing through $[C] \in M_g$ in terms of the gonality of $C$. This yields an upper bound for the dimension of a germ of a totally geodesic submanifold contained in the Jacobian locus, which only depends on the genus. We also study the submanifolds of $A_g$ obtained from cyclic covers of $\mathbb{P}^1$. These have been studied by various authors. Moonen determined which of them are Shimura varieties using deep results in positive characteristic. Using our methods we show that many of the submanifolds which are not Shimura varieties are not even totally geodesic.

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1. Introduction

1.1. Denote by $A_g$ the moduli space of principally polarized abelian varieties of dimension $g$, by $M_g$ the moduli space of smooth curves of genus $g$ and by $j : M_g \to A_g$ the period mapping or Torelli mapping. Both $M_g$ and $A_g$ are complex orbifolds (or smooth stacks) and $A_g$ is endowed with a locally symmetric metric, the so-called Siegel metric. One expects the Jacobian locus, that is the image $j(M_g) \subset A_g$, to be rather curved with respect to the Siegel metric. In particular, it should contain very few totally geodesic submanifolds of $A_g$. Another reason for this expectation comes from arithmetic geometry. Indeed for a special class of totally geodesic submanifolds (Shimura varieties) it has been conjectured by Coleman and Oort that for large genus no positive dimensional Shimura variety is contained in the closure of the Jacobian locus (in $A_g$) and meets the Jacobian locus itself. Moonen [20] has proven that an algebraic totally geodesic submanifold is a Shimura subvariety if and only if it contains a complex multiplication point. For results on Shimura subvarieties contained in $j(M_g)$ we refer to [13, 20, 22, 25, 21, 1, 19, 15, 18].

Outside the hyperelliptic locus the period map is an orbifold immersion. For $g \geq 4$ the Jacobian locus $j(M_g)$ has dimension strictly smaller than $A_g$. Therefore it makes sense to compute the second fundamental form of $j(M_g) \subset A_g$ and to study its metric properties by infinitesimal methods. The second fundamental form has been studied by Pirola, Tortora and the first author [6], where an expression for it is given and it is proven that the second fundamental form lifts the second Gaussian map, as stated in an unpublished paper by Green and Griffiths [12]. In particular the computation of the second fundamental form on $\xi_p \odot \xi_p$ (where $\xi_p$ is a Schiffer variation at the point $p$ on the curve) reduces to the evaluation of the second gaussian map at the point $p$. These results have been used in [5] to compute the curvature of the restriction to $M_g$ of the Siegel metric. In [5] there is an explicit formula for the holomorphic sectional curvature of $M_g$ in the direction $\xi_p$ in terms of the holomorphic sectional curvature of $A_g$ and the second Gaussian map.

It is much harder to use the formula in [6] to compute the second fundamental form on $\xi_p \odot \xi_q$ when $p \neq q$. In fact the formula contains the evaluation at $q$ of a meromorphic 1-form on the curve, called $\eta_p$, which has a double pole at $p$ and is defined by Hodge theory. In general it seems rather hard to control the behaviour of $\eta_p$, in a way to get constraints on the second fundamental form.

1.2. In this paper we give a global and more intrinsic description of this form. We show that as $p$ varies on the curve the forms $\eta_p$ glue to give a holomorphic section $\hat{\eta}$ of the line bundle $K_S(2\Delta)$, where $S = C \times C$ and $\Delta \subset S$ is the diagonal. With this interpretation we are able to prove that the second fundamental form coincides with the multiplication by $\hat{\eta}$.

More precisely, fix a genus $g$, which will always be assumed greater than 3, and fix $[C] \in M_g$ outside the hyperelliptic locus. The conormal bundle
of \( j : M_g \subset A_g \) at \([C]\) can be identified with \( I_2(K_C)\), which is the kernel of the multiplication map \( S^2H^0(C,K_C) \rightarrow H^0(C,2K_C)\). Hence the second fundamental form can be seen as a map \( \rho : I_2(K_C) \rightarrow S^2H^0(C,2K_C)\). Let \( S = C \times C \) and let \( \Delta \) be the diagonal. By Künneth formula \( H^0(S,K_S) = H^0(C,K_C) \otimes H^0(C,K_C) \) and \( H^0(S,2K_S) = H^0(C,2K_C) \otimes H^0(C,2K_C)\). In particular \( I_2(K_C) \subset H^0(S,K_S(-2\Delta))\).

**Theorem A.** (See Theorem 3.13). The second fundamental form \( \rho \) is the restriction to \( I_2 \) of the multiplication map

\[
H^0(S,K_S(-2\Delta)) \rightarrow H^0(S,2K_S) \quad Q \mapsto Q \cdot \hat{\eta}.
\]

1.3. Based on these results on the second fundamental form we get some constraints on the existence of totally geodesic submanifolds of \( A_g \) contained in \( M_g \). Since our methods are local in nature, the results apply to germs of such submanifolds. We get upper bounds for the dimension of totally geodesic germs passing through \([C] \in M_g\) in terms of the gonality of the curve \( C\).

**Theorem B.** (See Theorem 4.3). Assume that \( C \) is a \( k\)-gonal curve of genus \( g \) with \( g \geq 4 \) and \( k \geq 3 \). Let \( Y \) be a germ of a totally geodesic submanifold of \( A_g \) which is contained in the jacobian locus and passes through \( j([C]) = [J(C)] \). Then \( \dim(Y) \leq 2g + k - 4 \).

This immediately yields a bound which only depends on \( g \).

**Theorem C.** (See Theorem 4.5). If \( g \geq 4 \) and \( Y \) is a germ of a totally geodesic submanifold of \( A_g \) contained in the jacobian locus, then \( \dim Y \leq \frac{5}{2}(g-1) \).

1.4. For low genus one can construct examples of totally geodesic submanifolds contained in \( M_g \) using cyclic covers of \( \mathbb{P}^1 \), see e.g. [7, 21, 25]. These are in fact Shimura varieties. A complete list of the Shimura varieties that can be obtained in this way has been given in [21] using deep results in positive characteristic. With our methods we check directly that these examples are indeed totally geodesic and we show that a large class of cyclic covers, which are not in the list of Shimura varieties, are not even totally geodesic (see Proposition 5.7 and Corollary 5.8).

1.5. Other works studying totally geodesic submanifolds contained in the Jacobian locus include [26, 13, 8]. In particular Hain [13] proves the following. Let \( X \) be an irreducible symmetric domain and consider the locally symmetric variety \( \Gamma \backslash X \) (where \( \Gamma \) is a lattice). If there is a totally geodesic immersion \( \Gamma \backslash X \rightarrow j(M_g)\), and if some additional conditions are satisfied, then \( X \) must be the complex ball. De Jong and Zhang [8] prove a similar result under milder conditions, but still retaining the irreducibility assumption on \( X \). The techniques used in these works are global and are based on group cohomology and on a rigidity theorem for the mapping class group due to Farb and Masur [9]. Our result instead applies to germs of totally
geodesic submanifolds, since it is local in nature and does not require irreducibility assumptions. The same local point of view is present in [17], where the object of study are totally geodesic submanifolds contained in an algebraic subvariety of a complex hyperbolic space form.

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2. Notation and preliminary results

2.1. Second fundamental form. Denote by $A_g$ the moduli space of principally polarized abelian varieties of dimension $g$, by $M_g$ the moduli space of smooth curves of genus $g$ and by $j : M_g \to A_g$ the period mapping or Torelli mapping. By the Torelli theorem $j$ is injective. To study $j$ one can fix a level structure with $n \geq 3$ and consider $M_g^{(n)} \to A_g^{(n)}$ which is a smooth map between manifolds. Since level structures play no role in what we are doing, it is more appropriate to think of $M_g$ and $A_g$ as complex orbifolds or smooth stacks, see e.g. [2, XII, 4]. The period map is smooth in the orbifold sense. Moreover its restriction to the set of non-hyperelliptic curves is an orbifold immersion [23]. By abuse of terminology we will henceforth omit the word ”orbifold”. The moduli space $A_g$ is endowed with the Siegel metric, which is the metric induced on $A_g$ from the symmetric metric on the Siegel upper halfspace $H_0^g$. Outside the hyperelliptic locus we have the sequence of tangent bundles:

$$0 \to TM_g \to j^*(TA_g) \xrightarrow{\pi} N \to 0,$$

whose dual, at $[C] \in M_g$ is

$$0 \to I_2 \to S^2 H^0(C, K_C) \xrightarrow{m} H^0(C, 2K_C) \to 0,$$

where $I_2 := I_2(K_C)$ is the set of quadrics containing the canonical curve and $m$ is the multiplication map (see [5] for more details). Denote by

$$\Pi : S^2 T_{[C]} M_g = S^2 H^1(C, T_C) \to N_{[C]}$$

the second fundamental form of the period map with respect to the Siegel metric on $A_g$. Denote by

$$\rho : I_2 \to S^2 H^0(C, 2K_C)$$

the dual of $\Pi$. We will refer both to $\Pi$ and to $\rho$ as second fundamental forms.

2.2. Schiffer variations. If $C$ is a curve and $x \in C$, the coboundary of the exact sequence $0 \to T_C \to T_C(x) \to T_C(x)|_x \to 0$ yields an injection $H^0(T_C(x)|_x) \cong \mathbb{C} \hookrightarrow H^1(C, T_C)$. Elements in the image are called Schiffer
Lemma 2.3. Let \( \beta \in H^0(C, 2K_C) \) and let \((U, z)\) be a chart centred at \( x \in C \). If \( \beta = f(z)(dz)^2 \) on \( U \), then \( \beta(\frac{\partial}{\partial z}) = 2\pi i f(0) \).

Proof.

\[
\beta(\frac{\partial}{\partial z}) = \int_C \beta \cup \theta = \int_U f(z)dz \wedge \frac{\partial b}{\partial z} = -\int_{U-\{x\}} \bar{\partial} \left( \frac{b(z)f(z)}{z}dz \right).
\]

If \( \varepsilon > 0 \) is small enough, \( b \equiv 1 \) on \( \{|z| \leq \varepsilon\} \). Using Stokes and Cauchy theorems we get

\[
\beta(\frac{\partial}{\partial z}) = -\lim_{\varepsilon \to 0} \int_{U\cap\{|z|>\varepsilon\}} \bar{\partial} \left( \frac{b(z)f(z)}{z}dz \right) = \lim_{\varepsilon \to 0} \int_{|z|=\varepsilon} \frac{f(z)}{z}dz = 2\pi i f(0).
\]

\( \Box \)

2.4. Gaussian maps. We briefly recall the definition of Gaussian maps for curves. Let \( N \) and \( M \) be line bundles on \( C \). Set \( S := C \times C \) and let \( \Delta \subset S \) be the diagonal. For a non-negative integer \( k \) the \( k \)-th Gaussian or Wahl map associated to these data is the map given by restriction to the diagonal

\[
H^0(S, N \boxtimes M(-k\Delta)) \xrightarrow{\mu_{k,M}} H^0(S, N \boxtimes M(-k\Delta)|_{\Delta}) \cong H^0(C, N \otimes M \otimes K_C^k).
\]

We are only interested in the case \( N = M \). In this case we set \( \mu_{k,M} := \mu_{k,M,M}^k \).

With the identification \( H^0(S, N \boxtimes M) \cong H^0(C, N) \otimes H^0(C, M) \) the map \( \mu_{0,M} \) is the multiplication map of global sections

\[
H^0(C, M) \otimes H^0(C, M) \to H^0(C, M^2),
\]

which obviously vanishes identically on \( \wedge^2 H^0(C, M) \). Consequently \( \ker \mu_{0,M} = H^0(S, M \boxtimes M(-\Delta)) \) decomposes as \( \wedge^2 H^0(C, M) \oplus I_2(M) \), where \( I_2(M) \) is the kernel of \( S^2 H^0(C, M) \to H^0(C, M^2) \). Since \( \mu_{1,M} \) vanishes on symmetric tensors, one usually writes

\[
\mu_{1,M} : \wedge^2 H^0(M) \to H^0(K_C \otimes M^2).
\]

If \( \sigma \) is a local frame for \( M \) and \( z \) is a local coordinate, given sections \( s_1, s_2 \in H^0(C, M) \) with \( s_i = f_i(z)\sigma \), we have

\[
(2.1) \quad \mu_{1,M}(s_1 \wedge s_2) = (f_1 f_2 - f_2 f_1)dz \otimes \sigma^2.
\]
Consequently, the zero divisor of $\mu_{1,M}(s_1 \wedge s_2)$ is twice the base locus of the pencil $\langle s_1, s_2 \rangle$ plus the ramification divisor of the associated morphism.

Again $H^0(S, M \otimes M(-2\Delta))$ decomposes as the sum of $I_2(M)$ and the kernel of $\mu_{1,M}$. Since $\mu_{2,M}$ vanishes identically on skew-symmetric tensors, one usually writes

$$\mu_{2,M} : I_2(M) \to H^0(C, M^2 \otimes K_C^2).$$

By $\mu_2$ we denote the second gaussian map of the canonical line bundle $K_C$ on $C$:

$$\mu_2 := \mu_{2,K_C} : I_2(K_C) \to H^0(K_C^1).$$

2.5. The form $\eta_x$ and the second fundamental form. We now recall the definition of $\eta_x$. Let $C$ be a smooth complex projective curve of genus $g \geq 4$. Fix a point $x \in C$. The space $H^0(C, K_C(2x))$ is contained in the space of closed 1-forms on $C - \{x\}$. The induced map $H^0(C, K_C(2x)) \to H^1(C - \{x\}, \mathbb{C})$ is injective as soon as $g > 0$. By the Mayer-Vietoris sequence, the inclusion $C - \{x\} \to C$ induces an isomorphism $H^1(C, \mathbb{C}) \cong H^1(C - \{x\}, \mathbb{C})$. Thus we get an injection

$$j_x : H^0(C, K_C(2x)) \hookrightarrow H^1(C, \mathbb{C}).$$

$H^{1,0}(C)$ is contained in the image of $j_x$ and $h^0(C, K_C(2x)) = g + 1$, so $j_x^{-1}(H^{1,0}(C))$ is a line. If $(U, z)$ is a chart centred at $x$, there is a unique element $\varphi$ in this line such that on $U - \{x\}$

$$\varphi = \left(\frac{1}{z^2} + h(z)\right)dz$$

with $h \in \mathcal{O}_C(U)$. (Applying Stokes theorem on $C$ minus a disc around $x$ shows that the residue at $x$ vanishes, so there is no term in $1/z$.) Define a linear map

$$\eta_x : T_x C \to H^0(C, K_C(2x))$$

by the rule

$$u = \lambda \frac{\partial}{\partial z}(x) \mapsto \eta_x(u) := \lambda \varphi.$$ 

We will often drop $x$ and simply write $\eta_u$ for $\eta_x(u)$. An easy computation shows that $\eta_x$ does not depend on the choice of the local coordinate.

**Theorem 2.6** ([5 Thm. 3.1], [5 Lemma 3.5]). Let $C$ be a non-hyperelliptic curve of genus $g \geq 4$. Given points $x \neq y$ in $C$ and tangent vectors $u \in T_x C$ and $v \in T_y C$ we have

$$\rho(Q)(\xi_u \otimes \xi_v) = -4\pi i \cdot \eta_x(u)(v)Q(u, v),$$

$$\rho(Q)(\xi_u \otimes \xi_u) = -2\pi i \cdot \mu_2(Q)(u \otimes 4).$$

This theorem is basic to the whole paper. For the reader’s convenience we recall its proof.
Proof. First of all we take \( w \in H^1(C, T_C) \) and we compute \( \rho(Q)(w) \in H^0(C, 2K_C) \) for \( Q \in I_2(K_C) \). We can assume that there is a one-dimensional deformation \( \mathcal{X} \xrightarrow{f} \Delta \) with \( C = f^{-1}(0) \) and \( w = \kappa(\frac{\partial}{\partial t}) \), where \( \Delta = \{ |t| < 1 \} \) and \( \kappa \) is the Kodaira-Spencer map. Take a \( C^\infty \) lifting \( \hat{Y} \) of the holomorphic vector field \( \frac{\partial}{\partial t} \). So we have a \( C^\infty \) trivialization \( \tau : \Delta \times C \to \mathcal{X} \), \( \tau(t, x) = \tau_t(x) := \Phi_t(x) \), where \( \{ \Phi_t \} \) is the flow of the vector field \( \hat{Y} \). Then \( \theta := \bar{\partial}Y|_\mathcal{C} \) is a \( \partial \)-closed form in \( A^{0,1}(C, T_C) \) such that \( [\theta] = w \in H^1(C, T_C) \).

Denote by \( C_t \) the fibre of \( f \) over \( t \) and let \( \omega(t) \) be a section of the Hodge bundle, i.e. \( \forall t \in \Delta, \omega(t) \in H^0(K_{C_t}) \). Since \( \omega(t) \) is closed, also \( \tau^*_t(\omega(t)) \) is closed, and so \( \tau^*_t(\omega(t)) = \omega + (\alpha + dh)t + o(t) \), where \( \omega := \omega(0) \), \( \alpha \) is harmonic and \( h \) is a \( C^\infty \) function. Denote by \( \nabla^{GM} \) the flat Gauss-Manin connection on \( \Omega^1 f_*C \). So we have \( \nabla^{GM}_{\partial/\partial t}|_t = [\alpha] \). By Griffiths’ results (see e.g. [29, pp. 234ff]) \( \theta \cdot \omega = \omega^{0,1} + \bar{\partial}h \) and \( \kappa(\frac{\partial}{\partial t}) \cdot [\omega] = \alpha^{0,1} \), where \( \alpha^{0,1} \) is the \( (0, 1) \) component of \( \alpha \). Now assume that \( \{ \omega_i \}_{i=1,\ldots,g} \) is a basis of \( H^0(K_C) \). Take a quadric \( Q = \sum_{i,j} a_{ij} \omega_i \otimes \omega_j \in I_2(K_C) \). Denote by \( \nabla^{1,0} \) the Gauss-Manin connection on the Hodge bundle \( f_*\omega_{X/\Delta} \), i.e. \( \nabla^{1,0} = \pi \nabla^{GM} \), where \( \pi \) is the projection of \( H^1(C, T_C) \) onto \( H^0(C_t, K_{C_t}) \). Then for all \( i \) we have \( \nabla^{1,0}_{\partial/\partial t}|_t = [\alpha_i^{1,0}] \). Denote by \( \nabla \) the induced connection on \( S^2 f_*\omega_{X/\Delta} \). If \( \bar{Q}(t) = \sum_{i,j} a_{ij}(t) \omega_i(t) \otimes \omega_j(t) \in I_2(K_C) \) is a section of the conormal bundle such that \( Q(0) = Q \), then

\[
\rho(Q)(w) = m(\nabla_{\partial/\partial t} \bar{Q})|_{t=0} = \sum_{i,j} a'_{ij}(0)\omega_i \omega_j + 2 \sum_{i,j} a_{ij} \alpha_i^{1,0} \omega_j.
\]

Since \( \sum_{i,j} a_{ij}(t)\omega_i(t)\omega_j(t) \equiv 0 \), also its derivative at \( t = 0 \) vanishes, i.e. \( 2 \sum_{i,j} a_{ij}(t)\omega_i(t)\omega_j(t) + \sum_{i,j} a'_{ij}(0)\omega_i \omega_j \equiv 0 \), and if we take the \( (1, 0) \) part we have \( 2 \sum_{i,j} a_{ij}(\alpha_i^{1,0} + \partial h_i) \omega_j + \sum_{i,j} a'_{ij}\omega_i \omega_j \equiv 0 \), so

\[
\rho(Q)(w) = -2 \sum_{i,j} a_{ij}\omega_j \partial h_i.
\]

This is an instance of a Hodge-Gaussian map, see [6] and also [24, §4].

Now fix a point \( x \in C \) and a chart \((U, z)\) centred at \( x \). Let \( w \) be the Schiffer variation \( \xi_{\frac{\partial}{\partial z}}(x) \) at \( x \) with Dolbeault representative \( \theta := \frac{\partial}{\partial z} \), where \( b \in C^\infty_0(U) \) be a bump function equal to \( 1 \) on a neighbourhood of \( x \). Let \( \omega_i = f_i(z)dz \) be the local expression of \( \omega_i \) in \( U \). On \( C - \{ x \} \) we have \( \theta \cdot \omega_i = \frac{f_i}{z} \partial b \), so \( \alpha_i^{0,1} + \partial h_i = \bar{\partial} \left( \frac{bf_i}{z} \right) \). Set \( g_i := \frac{bf_i}{z} - h_i \). Then \( \alpha_i^{0,1} = \bar{\partial} g_i \).

Define \( \eta_i := \partial g_i \).

Now we will show that \( \sum a_{ij}\omega_i \partial h_j = -\sum a_{ij}\omega_i \eta_j \). In the first place, note that \( \eta_j \) is holomorphic in \( C - \{ x \} \). Indeed, \( \alpha_i \) is harmonic, thus \( \partial \eta_i = \partial \partial g_i = -\partial \bar{\partial} g_i = -\partial \alpha_i^{0,1} = 0 \). Hence, \( \sum a_{ij}\omega_i \partial (\frac{bf_i}{z}) = \sum a_{ij}\omega_i \partial h_j + \sum a_{ij}\omega_i \eta_j \) is holomorphic on \( C - \{ x \} \), because the first term is a holomorphic section of \( 2K_C \) by (23) and the second is holomorphic on \( C - \{ x \} \) since \( \eta_j \) is holomorphic. In a neighborhood of \( x \) where \( b \equiv 1 \), this expression has the
form
\[ \sum a_{ij} f_i \frac{\partial}{\partial z} \left( \frac{f_j}{z} \right) \, dz^2 = \sum a_{ij} f_i \left( -\frac{f_j}{z^2} \frac{f_j'}{z} \right) \, dz^2 = 0, \]
because \[ \sum a_{ij} f_i f_j = \sum a_{ij} f_i f_j' = 0. \] So \[ \sum a_{ij} \omega_i \partial_b (\frac{h f_j}{z^2}) \] is identically zero.

Thus we have
\[ \rho(Q)(\xi) (\frac{\partial}{\partial z}(x)) = -\frac{2}{z} \sum a_{ij} \omega_i \partial h_j = 2 \sum a_{ij} \omega_i \eta_j. \]

Now we claim that \[ \eta_i = -f_i(x) \eta_x (\frac{\partial}{\partial z}). \] In fact \[ \eta_i \in H^0(K_C(2x)), \] since on \( U, \) where \( b \equiv 1, \) \( \eta_i \) has the form
\[ \eta_i = -f_i(x) - f_i'(z) \psi_i(z) \, dz \]
with \( \psi_i(z) \) a holomorphic function, hence \( \eta_i \) is a meromorphic form, with a double pole at \( x. \) By definition, \( \eta_i + \alpha_{i}^{0,1} = \partial g_i + \partial g_i = dg_i, \) so
\[ j_x(\eta_i) = -[\alpha_{i}^{0,1}] \in j_x(H^0(K_C(2x))) \cap H^{0,1}(C). \]
This proves the claim.

We can assume that the chart \((U, z)\) contains also \( y. \) Applying Lemma 2.3 the first statement immediately follows for \( u = \partial / \partial z(x) \) and \( v = \partial / \partial z(y). \) It is clear that this is enough. For the second statement it suffices to use the local expression of \( \mu_2. \) \( \square \)

**Remark 2.7.** We remark that Schiffer variations, the forms \( \eta_x, Q \) and \( \mu_2(Q) \) are sections of vector bundles, but they become functions as soon as a coordinate chart is fixed. Because of this many statements, like the one above, are usually stated for simplicity, as if these sections were evaluated at points instead of tangent vectors. We will follow this notation when it is convenient. In the next section instead it is better to stick a more formal notation.

3. The second fundamental form as a multiplication map

**3.1.** In this section we show that as \( x \) varies on the curve the form \( \eta_x \) varies holomorphically in an appropriate sense and gives rise to a section of a vector bundle on \( C \) and to a corresponding section \( \hat{\eta} \) of the line bundle \( K_S(2\Delta) \) on \( S = C \times C. \) The two main points are Theorem 3.13 and the invariance of \( \hat{\eta} \) with respect to the action of \( \text{Aut}(C). \)

**3.2.** Let \((U, z)\) be a chart centred at \( x \in C. \) Set \( u = \frac{\partial}{\partial z}(x). \) It is a classical result that there is a harmonic function \( f_u \in C^\infty(C - \{x\}) \) such that \( f_u = -1/z + g(z) \) on \( U - \{x\} \) for some \( g \in C^\infty(U). \) This function is unique up to an additive constant and is called elementary potential. Its existence can be proven for example using the (real) Hodge decomposition theorem and the Weyl lemma (see e.g. [10] p. 46-48) or using the Perron method (see e.g. [16] p. 213ff)].
Lemma 3.3. If \( f_u \) is an elementary potential, then \( \partial f_u = \eta_u \), \( \bar{\partial} f_u \) is smooth on \( C \) and \( j_x(\eta_u) = [-\bar{\partial} f_u] \).

Proof. The \((1,0)\)-form \( \partial f_u \) is holomorphic on \( C - \{x\} \) since \( f_u \) is harmonic. Moreover \( \partial f_u = z^{-2} dz + dg \) on \( U - \{x\} \). The form \( \partial g \) is holomorphic on \( U - \{x\} \), but also smooth on \( U \). Hence it is holomorphic on \( U \). This shows that \( \partial f_u = \eta_u + \omega \) for some \( \omega \in H^0(K_C) \). Set \( \alpha := -\bar{\partial} f_u \). Then \( \alpha \) is smooth on \( C \) by the definition of \( f_u \). It is closed, since \( f_u \) is harmonic and of type \((0,1)\). On \( C - \{x\} \) we have \( \partial f_u - \alpha = df_u \), so \( [\partial f_u] = [\alpha] \) in \( H^1(C - \{x\}) \). Therefore \( j_x(\partial f_u) = [\alpha] \). Since \([\alpha] \in H^{0,1}(C)\), this shows that \( j_x(\partial f_u) \in H^{0,1}(C) \). Therefore \( \omega = 0 \) and \( \eta_u = \partial f_u \).

\[ \square \]

Remark 3.4. Notice that one could prove the existence of \( f_u \) using \( \eta_u \) and the fact that \( C - \{x\} \) is Stein.

Lemma 3.5. If \( H^{0,1}(C) \) is identified with \( H^0(C, K_C)^\ast \) using Serre duality, then \( x \mapsto \text{Im} j_x \cap H^{0,1}(C) \in \mathbb{P}(H^{0,1}(C)) \) coincides with the canonical map.

Proof. Let \((U, z)\) be a coordinate centred at \( x \). Set \( u = \frac{\partial}{\partial z}(x) \). If \( \omega \in H^0(C, K_C) \), let \( \omega = \varphi(z) dz \) be its local expression on \( U \). By Lemma 3.3 \( j_x(\eta_u) = -[\partial f_u] \). Therefore

\[
\int_C \omega \wedge j_x(\eta_u) = - \int_C \omega \wedge \partial f_u = \int_{C - \{x\}} d(f_u \omega) = \\
(3.1) = \lim_{\varepsilon \to 0} \int_{|z| = \varepsilon} \varphi(z) \left( -\frac{1}{z} + g(z) \right) dz = -\varphi(0) = -\omega(u).
\]

\[ \square \]

3.6. Let \( S := C \times C \), let \( \Delta \) denote the diagonal and let \( p, q : S \to C \) be the projections \( p(x, y) = x \), \( q(x, y) = y \). Then \( K_S = p^*K_C \otimes q^*K_C \). Consider the line bundle \( L := K_S(2\Delta) \) on \( S \) and set

\[ V := p_*(q^*K_C(2\Delta)) \quad \quad E := p_*L. \]

By the projection formula \( E = K_C \otimes V \). Since \( q^*K_C(2\Delta)|_{\{x\} \times C} = q^*K_C(2x) \), we have \( H^0(p^{-1}(x), q^*K_C(2\Delta)) \cong H^0(C, K_C(2x)) \). By Grauert semicontinuity theorem \( V \) is a holomorphic vector bundle on \( C \) with fibre \( V_x \cong H^0(C, K_C(2x)) \) and the map \( x \mapsto \eta_x \) is a section of \( E \). We call this section \( \eta \).

Proposition 3.7. \( \eta \) is a holomorphic section of \( E \).

Proof. Let \( W \to C \) denote the trivial vector bundle with fibre \( H^1(C, \mathbb{C}) \). We claim that the injection \( j : V \hookrightarrow W \) defined in (2.2) is holomorphic. Fix \( \alpha \in H^0(C, K_C(2x)) \) and a smooth singular 1-cycle \( c \) on \( C \). If \( x \) does not lie in the support of \( c \), the integral \( \int_c \alpha \) is well-defined. It does not change if \( \alpha \) is replaced by \( \alpha + df \) with \( f \in C^\infty(C - \{x\}) \). Therefore \( \int_c \alpha = \langle [c], j(\alpha) \rangle \).

Fix \( x_0 \in C \) and choose smooth 1-cycles \( c_1, \ldots, c_{2g} \), that do not touch \( x_0 \) and whose classes form a basis of \( H_1(C, \mathbb{C}) \). Let \( A \) be an open subset of \( C \), such
that $V$ is trivial over $A$ and $\overline{A}$ does not intersect the supports of the cycles $c_i$. Fix $s \in H^0(A, V)$. To show that $j(s)$ is a holomorphic section of $W$ over $A$, it is enough to prove that the functions

$$x \mapsto \langle [c_i], j_x(s(x)) \rangle = \int_{c_i} s(x)$$

are holomorphic on $A$. Since $V = p_*q^*K_C(2\Delta)$, $s$ corresponds to a section $s \in H^0(A \times C, q^*K_C(2\Delta))$. So $s(x, \cdot)|_{C - A}$ is a holomorphic 1-form on $C - A$ depending holomorphically on the parameter $x \in A$ and its integral over the 1-cycle $c_i$ (which is contained in $C - A$) is a holomorphic function of $x$. Therefore $j(s)$ is holomorphic. Since $s$ is arbitrary, this proves that $j$ is holomorphic, as claimed. Next fix a chart $(U, z)$ on $C$. To show that $\eta$ is holomorphic on $U$ it is enough to prove that $\eta(\frac{\partial}{\partial z})$ is holomorphic or - by the above - that $j(\eta(\frac{\partial}{\partial z}))$ is a holomorphic function $U \to H^1(C, \mathbb{C})$. Fix a basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(C, K_C)$. The functionals $\int_C \cdot \wedge \omega_j$ and $\int_C \cdot \wedge \omega_j$ form a basis of $H^1(C, \mathbb{C})^*$. Since $j(\eta(\frac{\partial}{\partial z})) \in H^{0,1}(C)$ the latter $g$ functionals vanish on $j(\eta(\frac{\partial}{\partial z}))$. Assume that $\omega_j(z) = f_j(z)dz$ on $U$. By (3.1)

$$\int_C j(\eta(\frac{\partial}{\partial z})) \wedge \omega_j = f_j(z).$$

This proves that $j(\eta(\frac{\partial}{\partial z}))$ is holomorphic. \hfill \Box

3.8. Since $E = p_* L$ there is an isomorphism $H^0(C, E) \cong H^0(S, L)$ that associates to $\alpha \in H^0(C, E)$ the section $\hat{\alpha} \in H^0(S, L)$ such that

$$\alpha_x = \hat{\alpha}|_{\{x\} \times C} \in T_x^* C \otimes H^0(C, K_C(2x)) = E_x.$$ 

It follows that $\alpha_x(u)(v) = \hat{\alpha}(u, 0, (0, v))$ for $x \neq y$, $u \in T_x C$ and $v \in T_y C$.

Thus there is a well-defined section $\hat{\eta} \in H^0(S, L)$ corresponding to $\eta$ and for $u \in T_x C$ and $v \in T_y C$ with $x \neq y$, we have $\eta_x(u)(v) = \hat{\eta}(u, v)$.

Lemma 3.9. The form $\hat{\eta}$ is symmetric, i.e. $\hat{\eta}((u, 0), (0, v)) = \hat{\eta}((v, 0), (0, u))$.

Proof. Fix points $x \neq y$ and tangent vectors $u \in T_x C, y \in T_y C$. Using the identity $d\left(f_u(\bar{\partial}f_v - \partial f_v) - f_v(\bar{\partial}f_u - \partial f_u)\right) = 2(f_u \bar{\partial}f_v - f_v \bar{\partial}f_u) = 0$ and applying Stokes theorem on $C - \{|z| < \varepsilon\} \cup \{|w| < \varepsilon\}$ we get

$$0 = \int_{|z| = \varepsilon} \left(f_u(\bar{\partial}f_v - \partial f_v) - f_v(\bar{\partial}f_u - \partial f_u)\right) + \int_{|w| = \varepsilon} \left(f_u(\bar{\partial}f_v - \partial f_v) - f_v(\bar{\partial}f_u - \partial f_u)\right)$$

(This is just Green formula.) Let us denote by $A_\varepsilon$ and $B_\varepsilon$ these two integrals. Observe that $f_v \bar{\partial}f_u = \partial(f_u f_v) - f_u \bar{\partial}f_v$ and $f_u \bar{\partial}f_v = \bar{\partial}(f_u f_v) - f_v \partial f_u$. Therefore

$$A_\varepsilon = \int_{|z| = \varepsilon} \left(d(f_u f_v) - 2f_v \bar{\partial}f_u - 2f_u \bar{\partial}f_v\right) = -2 \int_{|z| = \varepsilon} \left(f_v \bar{\partial}f_u + f_u \partial f_v\right).$$
Since $f, \bar{\partial}f_u$ is a smooth form near $x$, we have $\lim_{\varepsilon \to 0} \int_{|z| = \varepsilon} f_u \bar{\partial}f_u = 0$. Choose a coordinate $(U, z)$ centred at $x$ such that $u = \partial/\partial z(x)$ and let $g(z)$ be as in 3.2. Near $x$ we have $\eta_y(v) = \partial f_v = h(z)dz$ for some holomorphic function $h$. So

$$
\lim_{\varepsilon \to 0} A_\varepsilon = -2 \lim_{\varepsilon \to 0} \int_{|z| = \varepsilon} f_u \partial f_v = -2 \lim_{\varepsilon \to 0} \int_{|z| = \varepsilon} \left( -\frac{1}{z} + g(z) \right) h(z)dz = 4\pi ih(0) = 4\pi i\eta_y(v)(u).
$$

By the corresponding computation, $\lim_{\varepsilon \to 0} B_\varepsilon = -4\pi i\eta_x(v)(u)$, so $\eta_y(v)(u) = \eta_x(u)(v)$ as desired. \hfill \Box

3.10. $\text{Aut}(C)$ acts diagonally on $C \times C$ and preserves $\Delta$. Therefore the action lifts to $K_S$ and also to $L = K_S(2\Delta)$. This yields a representation of $\text{Aut}(C)$ on $H^0(S, L)$. On the other hand, if $\sigma \in \text{Aut}(C)$, then $(\sigma^{-1})^*$ is a map from $H^0(C, K_C(2\sigma(x))) = V_{\sigma(x)}$ to $H^0(C, K_C(2\sigma(x)))) = V_{\sigma(x)}$. Tensoring it with $(d\sigma^{-1})^*: T_x^* C \to T_{\sigma(x)}^* C$ we get a map $T_x^* C \otimes V_x \to T_{\sigma(x)}^* C \otimes V_{\sigma(x)}$. This yields an action of $\text{Aut}(C)$ on the total space of $E = K_C \otimes V$ which covers the action of $\text{Aut}(C)$ on $C$. This means that $E$ is an equivariant bundle. In this way we get a representation of $\text{Aut}(C)$ on $H^0(C, E)$. The map $\alpha \mapsto \hat{\alpha}$ considered in 3.8 is an isomorphism of $\text{Aut}(C)$–representations.

Lemma 3.11. $\eta$ and $\hat{\eta}$ are invariant with respect to the action of $\text{Aut}(C)$.

Proof. By the above it is enough to show that $\eta$ is invariant. For $\tau \in \text{Aut}(C)$ we wish to prove that $(d\tau^{-1})^* \otimes (\tau^{-1})^* (\eta_y) = \eta_{\tau(y)}$ for any $y \in C$. For simplicity, set $\sigma := \tau^{-1}$, $x := \tau(y)$. So we wish to prove that $\sigma^*(\eta_y(\partial f(u))) = \eta_x(u)$ for any $u \in C$. By continuity it is enough to prove this for $x$ such that $\sigma(x) \neq x$. Choose a coordinate patch $(U, z)$ centred at $x$, such that $U \cap \sigma(U) \neq \emptyset$ and $u = \frac{\partial}{\partial z}|_x$. Then $(\sigma(U), w := z \circ \sigma^{-1})$ is a coordinate system centred at $\sigma(x)$ and $\frac{\partial}{\partial w}(\sigma(x)) = d\sigma(u)$. Assume that

$$
\eta_x(u) = \left( \frac{1}{z^2} + h(z) \right) dz
$$
on $U$. Then

$$
\tau^* \eta_x(u) = \left( \frac{1}{w^2} + h(w) \right) dw
$$on $\sigma(U)$. Moreover $j_y \tau^* = \tau^* j_x$. So $j_y \tau^* (\eta_x(u)) = \tau^* j_x (\eta_x(u))$. Since $j_x(\eta_x(u)) \in H^{0,1}(C)$, also $j_y \tau^* (\eta_x(u)) \in H^{0,1}(C)$. Hence $\tau^* \eta_x(u) = \eta_y(d\sigma(u))$ as desired. \hfill \Box

3.12. If we identify $H^0(C, K_C) \otimes H^0(C, K_C)$ with $H^0(S, K_S)$, then $I_2$ becomes a subspace of $H^0(S, K_S(-\Delta))$. Since elements of $I_2$ are symmetric, they are in fact in $H^0(S, K_S(-2\Delta))$. So if $Q \in I_2$ the product section $Q \cdot \hat{\eta}$ lies in $H^0(S, 2K_S) \cong H^0(C, 2K_C) \otimes H^0(C, 2K_C)$. 


Theorem 3.13. With the above identifications, if $C$ is non-hyperelliptic and of genus $g \geq 4$, then $\rho : I_2 \to S^2H^0(C, 2K_C)$ is the restriction to $I_2$ of the multiplication map

$$H^0(S, K_S(-2\Delta)) \longrightarrow H^0(S, 2K_S) \quad Q \mapsto Q \cdot \hat{\eta}.$$ 

Proof. The identification $H^0(C, 2K_C) \otimes H^0(C, 2K_C) \cong H^0(S, 2K_S)$ is compatible with Lemma 2.3, i.e. for $\alpha \in H^0(S, 2K_S)$, $u \in T_2C$ and $v \in T_yC$ we have $\langle \alpha, \xi_u \otimes \xi_v \rangle = \alpha((u, 0), (0, v))$. If $x \neq y$, by Theorem 2.6

$$\rho(Q)(\xi_u \otimes \xi_v) = Q(u, v)\eta_x(u)(v) = (Q \cdot \hat{\eta})(u, 0, (0, v)) = (Q \cdot \hat{\eta})(\xi_u \otimes \xi_v).$$

(In the last identity we use the fact that both $Q$ and $\hat{\eta}$ are symmetric). So $\rho(Q) - Q \cdot \hat{\eta}$ vanishes on tensors of the form $\xi_u \otimes \xi_v$ with $x \neq y$. Clearly we can choose a basis of $S^2H^1(C, T_C)$ formed by such elements. \qed

3.14. Since the second fundamental form is symmetric, using this theorem we get another proof of Lemma 3.9

4. Totally geodesic submanifolds and gonality

4.1. In this section we will give an upper bound for the dimension of a germ of a totally geodesic submanifold contained in the Jacobian locus.

Theorem 4.2. Assume that $C$ is a $k$-gonal curve of genus $g$, with $g \geq 4$ and $k \geq 3$. Then there exists a quadric $Q \in I_2(K_C)$ such that rank $\rho(Q) > 2g - 2k$.

Proof. Here we follow the simplified notation mentioned in 2.7. So we understand that a local coordinate has been fixed at the relevant points and we write $\xi_P$ for a Schiffer variation at $P$. Let $F$ be a line bundle on $C$ such that $|F|$ is a $g^1_1$ and choose a basis $\{x, y\}$ of $H^0(F)$. Set $M = K_C - F$ and denote by $B$ the base locus of $|M|$. By Clifford theorem $\deg(B) < k - 2$.

Assume that $(t_1, t_2)$ is a pencil in $H^0(M)$, with base locus $B$. We can write $t_i = s_i$ for a section $s \in H^0(C, \mathcal{O}_C(B))$ with div$(s) = B$. Then $(t_1, t_2)$ is a base point free pencil in $|M - B|$. Let $\psi : C \to \mathbb{P}^1$ be the morphism induced by this pencil.

Now consider the rank 4 quadric $Q := xt_1 \otimes yt_2 - xt_2 \otimes yt_1$. Clearly $Q \in I_2(K_C)$. Set $d := \deg(M - B) = 2g - 2 - k - \deg(B)$. We need the following fact: if $\{P_1, \ldots, P_d\}$ is a fibre of the morphism $\psi$ over a regular value, then the Schiffer variations $\xi_{P_1}, \ldots, \xi_{P_d}$ are linearly independent in $H^1(C, T_C)$. In fact, denote by $W := \langle \xi_{P_1}, \ldots, \xi_{P_d} \rangle$. We want to show that the annihilator Ann$(W)$ of $W$ has dimension $3g - 3 - d$. By lemma 2.3 Ann$(W) = \{\alpha \in H^0(C, 2K_C) \mid \alpha(P_i) = 0, \ i = 1, \ldots, d\}$, hence by Riemann-Roch dim Ann$(W) = h^0(2K_C - P_1 - \cdots - P_d) = h^0(2K_C - M + B) = g - 1 + k + \deg(B) = 3g - 3 - d$. The claim is proven. Next denote by $\varphi$ the morphism induced by the pencil $|F|$ and consider the set $E := \psi(Crit(\varphi) \cup Crit(\psi) \cup B)$ where Crit$(\varphi)$ (resp. Crit$(\psi)$) denote the set of critical points of $\varphi$ (resp. $\psi$).
Let \( z \in \mathbb{P}^1 - E \) and let \( \{P_1, \ldots, P_d\} \) be the fibre of \( \psi \) over \( z \). By changing coordinates on \( \mathbb{P}^1 \) we can assume \( z = [0, 1] \), i.e. \( t'_1(P_i) = 0 \) for \( i = 1, \ldots, d \). Then clearly \( t_1(P_i) = 0 \), so \( Q(P_i, P_j) = 0 \) for all \( i, j \). Applying Theorem 4.3 one immediately obtains that the restriction of \( \rho(Q) \) to the subspace \( W \) is represented in the basis \( \{\xi_{P_1}, \ldots, \xi_{P_d}\} \) by a diagonal matrix with entries \( \pi_i \cdot \mu_2(Q)(P_i) \) on the diagonal. For a rank 4 quadric the second Gaussian map can be computed as follows: \( \mu_2(Q) = \mu_1,F(x \wedge y)\mu_1,M(t_1 \wedge t_2) \) see [4, Lemma 2.2]. Now \( \mu_1,F(x \wedge y)(P_i) \neq 0 \), because \( P_i \notin \text{Crit}(\varphi) \) by the choice of \( z \), see (2.1). Moreover \( P_i \notin B \). On \( C - B \) the morphism \( \psi \) coincides with the map associated to \( \langle t_1, t_2 \rangle \). Since \( P_i \notin \text{Crit}(\psi) \), it is not a critical point for the latter map. Therefore also \( \mu_1,M(t_1 \wedge t_2)(P_i) \neq 0 \) see (2.1). Thus \( \mu_2(Q)(P_i) = \mu_1,F(x \wedge y)(P_i)\mu_1,M(t_1 \wedge t_2)(P_i) \neq 0 \) for every \( i = 1, \ldots, d \). This shows that in the basis \( \{\xi_{P_1}, \ldots, \xi_{P_d}\} \) the quadric \( \rho(Q)|_W \) is represented by a diagonal matrix with non-zero diagonal entries. So \( \rho(Q) \) has rank at least \( d = 2g - 2 - k - \deg B > 2g - 2k \).

**Theorem 4.3.** Assume that \( C \) is a \( k \)-gonal curve of genus \( g \) with \( g \geq 4 \) and \( k \geq 3 \). Let \( Y \) be a germ of a totally geodesic submanifold of \( A_g \) which is contained in the jacobian locus and passes through \([C]\). Then \( \dim Y \leq 2g + k - 4 \).

**Proof.** By Theorem 4.2 we know that there exists a quadric \( Q \in I_2 \) such that the rank of \( \rho(Q) \) is at least \( 2g - 2k + 1 \). By assumption for any \( v \in T_{[C]}Y \) we must have that \( \rho(Q)(v \circ v) = 0 \), so \( v \) is isotropic for \( \rho(Q) \), hence

\[
\dim T_{[C]}Y \leq 3g - 3 - \frac{(2g - 2k + 1)}{2} = 2g + k - \frac{7}{2}.
\]

**Remark 4.4.** In Theorem 4.2 if \( |M| \) is base point free – this happens in particular if \( C \) is generic in the locus of \( k \)-gonal curves – the above proof shows that rank \( \rho(Q) \geq 2g - 2 - k \). So if \( Y \) is a germ of a totally geodesic submanifold of \( A_g \) contained in the jacobian locus and passing through a generic \( k \)-gonal curve, then \( \dim Y \leq 2g - 2 + k/2 \).

**Theorem 4.5.** If \( g \geq 4 \) and \( Y \) is a germ of a totally geodesic submanifold of \( A_g \) contained in the jacobian locus, then \( \dim Y \leq \frac{5}{2}(g - 1) \).

**Proof.** This immediately follows from Theorem 4.3 since the gonality of a genus \( g \) curve is at most \([g + 3]/2\). \( \square \)

**Corollary 4.6.** For any \( g \geq 4 \) and \( k \geq 2 \) the \( k \)-gonal locus is not totally geodesic in \( A_g \).

**Proof.** For \( k = 2 \) this is Proposition 5.1 in [5]. If \( k \geq 3 \) the dimension of the \( k \)-gonal locus is \( 2g + 2k - 5 > 2g + k - 4 \). Hence the statement follows immediately from Theorem 4.3. \( \square \)

**Remark 4.7.** As it is evident from the proof, gonality is used to construct a quadric \( Q \in I_2(KC) \) of rank 4 such that \( \rho(Q) \) has large rank. It seems...
unlikely that gonality plays any role in this problem. In fact we expect the existence of $Q \in I_2(K_C)$ with image $\rho(Q)$ a nondegenerate quadric on $H^1(C,T_C)$. This would give the upper bound $\frac{3}{2}(g - 1)$ for the dimension of a germ of a totally geodesic submanifold. On the other hand, the map $\rho$ is injective. This can be deduced from [5, Cor. 3.4] or from Theorem 3.13 since the form $\hat{\eta}$ is non-zero. Therefore $\rho(I_2(K_C))$ is a linear system of quadrics of dimension $(g - 1)(g - 4)$ on $\mathbb{P}(H^1(C,T_C)) = \mathbb{P}^{3g-4}$. This already gives an upper bound for the dimension of a submanifold $Y$ as in Theorem 4.5. Indeed for any point $[C] \in Y$, the tangent space $T_{[C]}Y \subset H^1(C,T_C)$ is contained in the base locus of $\rho(I_2(K_C))$. This means that $\rho(I_2(K_C))$ is contained in the space of quadrics $q \in S^2 H_0^0(C,2K_C)$ that vanish on $T_{[C]}Y$. This yields the bound
\[ \dim Y \leq -1 + \frac{\sqrt{32g^2 - 40g + 1}}{2}. \]

Nevertheless for any $g \geq 2$ this bound is weaker than the one provided in Theorem 4.5. At any case the study of the base locus of the linear system $\rho(I_2(K_C))$ should clearly improve the understanding of totally geodesic submanifolds. If one could prove that the base locus is empty, one would rule out the existence of totally geodesic submanifolds passing through $[C]$. However to proceed in this direction it is probably necessary to better understand the form $\hat{\eta}$.

**Remark 4.8.** Observe that for $g \geq 5$, the non existence of germs of totally geodesic hypersurfaces follows directly from Theorem 2.6 and [4, Thm. 6.1]. In fact if $Y$ is a hypersurface in $M_g$, it passes through a non-trigonal curve $[C]$. Since $\mathbb{P}T_{[C]}Y$ intersects the bicanonical curve in $\mathbb{P}H^1(T_C)$, $T_{[C]}Y$ contains a Schiffer variation $\xi_x$, for some $x \in C$. By [4, Thm. 6.1], there exists a quadric $Q \in I_2(K_C)$ such that $\mu_2(Q)(x) \neq 0$, so $\rho(Q)(\xi_x \circ \xi_x) \neq 0$ by Theorem 2.6.

5. Families of cyclic covers of the projective line

5.1. Let $C$ be a curve of genus $g \geq 4$ and let $G$ be a subgroup of the group of automorphisms of $C$. This yields an inclusion of $G$ in the mapping class group $\Gamma_g$ [11]. So $G$ acts on the Teichmüller space $T_g$ and we denote by $T_g^G$ the set of fixed points of $G$ on $T_g$, which is a nonempty by the solution of Nielsen realization problem [14, 27, 3], and is a complex submanifold of $T_g$. The tangent space to $T_g^G$ at a point $[C]$ is given by $H^1(C,T_C)^G$. Moreover $T_g^G$ parametrizes marked curves $C$ endowed with a holomorphic action of $G$ of the given topological type and there is a universal family $C \to T_g^G$ containing all such curves. If $t \in T_g^G$ we denote by $C_t$ the corresponding curve.

5.2. Let us now consider the period map at the level of Teichmüller spaces $T_g \to H_g$. This is an immersion outside the hyperelliptic locus, and we will
Consider its restriction to $T^G_g$. Denote by $Z(G)$ its image in $A_g$. Given a point $t \in T^G_g$ such that $C_t$ is non-hyperelliptic, the cotangent spaces fit in the exact sequence

$$0 \to N^* \to S^2 H^0(K_{C_t}) \xrightarrow{\pi} H^0(2K_{C_t})^G \to 0,$$

where $N^*$ is the conormal space to $Z(G) \subset A_g$ at the point $J(C_t)$. Since $\pi$ is $G$-invariant, $N^*$ is a $G$-submodule. Let

$$\tilde{\rho} : N^* \to H^0(2K_{C_t})^G \otimes H^0(2K_{C_t})^G$$

denote the second fundamental form of $Z(G)$ at point $J(C_t)$.

**Lemma 5.3.** The second fundamental form $\tilde{\rho}$ is $G$-equivariant.

**Proof.** Recall that by definition, given $a \in N^*$, we have $\tilde{\rho}(a) = \pi(\nabla(a))$, where $\nabla$ is the Gauss-Manin connection on $S^2 f_\ast \omega_C/T^G_g$ of the family $\mathcal{C} \xrightarrow{f} T^G_g$. Since $\pi$ is $G$-invariant, it suffices to prove that $\nabla$ is $G$-invariant, i.e. $g^{-1} \nabla g = \nabla$ for every $g \in G$. Observe that the flat connection $\nabla^{\text{GM}}$ on $R^1 f_\ast \mathcal{C}$ is $G$-invariant, since the $G$-action maps flat sections to flat sections. Let $\pi^{1,0} : H^1(C_t, \mathcal{C}) \to H^{1,0}(C_t)$ be the projection. Then $\pi^{1,0} \circ \nabla^{\text{GM}}$ is the connection on the Hodge bundle $f_\ast \omega_C/T^G_g$. Since the action of $G$ on $\mathcal{C}$ is holomorphic, the projection $\pi^{1,0}$ is $G$-equivariant, hence $\pi^{1,0} \circ \nabla^{\text{GM}}$ is also a $G$-invariant connection. Finally $\nabla$ is the connection induced by $\pi^{1,0} \circ \nabla^{\text{GM}}$ on $S^2 f_\ast \omega_C/T^G_g$. The result follows.

**Proposition 5.4.** If there are no nonzero quadrics in $I_2(K_{C_t})$, which are invariant under the action of the group $G$, then $Z(G)$ is totally geodesic.

**Proof.** Consider the cotangent sequence of the period map $j : T_g \to H_g$ at the point $[C_t]$ and its restriction to $T^G_g$:

$$0 \to I_2(K_{C_t}) \to S^2 H^0(K_{C_t}) \xrightarrow{m} H^0(2K_{C_t}) \to 0$$

(5.1)

(The notation is as in [5.2]) Since the map $m$ is $G$-equivariant, any $G$-invariant element in $N^*$ lies in $I_2(K_{C_t})$. Hence by the assumption there are no nontrivial invariant elements in $N^*$, i.e. the trivial representation does not appear in the decomposition of $N^*$. On the other hand the representation $H^0(2K_{C_t})^G$ is trivial. By Lemma 5.3, $\tilde{\rho}$ is a morphism of $G$-representations. Hence Schur lemma implies that $\tilde{\rho}$ is the trivial map.

**5.5.** Now we will consider the case in which $G = \mathbb{Z}/m\mathbb{Z}$, $m \geq 3$ and $C/G \cong \mathbb{P}^1$. These families have been studied by various authors, e.g. [7 [21 [25], since they provide the only known examples of totally geodesic submanifolds contained in the Jacobian locus. These examples are in fact
Shimura varieties. A complete list of the Shimura varieties that can be obtained in this way has been given in \[21\], which we follow for the notation in the rest of the paper.

We identify \(G\) also with the group of \(m\)-th roots of unity. Fix an integer \(N \geq 4\), together with an \(N\)-tuple \(a = (a_1,...,a_N)\) of positive integers, such that \(\gcd(m,a_1,...,a_N) = 1\), \(a_i \neq 0\) mod \(m\) and \(\sum_{i=1}^{N} a_i \equiv 0\) mod \(m\). Given distinct points \(t_1,...,t_N \in \mathbb{P}^1\) there is a well-defined curve \(C_t\) which is a cyclic cover of \(\mathbb{P}^1\) with covering group \(\mathbb{Z}/m\mathbb{Z}\), branch points \(t_i\) and local monodromy \(a_i\) at \(t_i\). It is the normalization of the affine curve

\[
y^m = \prod_{i=1}^{N} (x - t_i)^{a_i}.
\]

Varying the branch points \(t = (t_1,\ldots,t_N)\) one obtains a \((N-3)\)-dimensional family of curves \(C \to B\). There is an action of \(G\) on \(C\) given by the rule \(\zeta \cdot (x,y,t) := (x,\zeta \cdot y,t)\), \(\zeta \in G\). Thus triples \((m,N,a)\) parametrize these families and two triples \((m,N,a)\) and \((m',N',a')\) yield equivalent families if and only if \(m = m', N = N'\) and the classes of \(a\) and \(a'\) in \((\mathbb{Z}/m\mathbb{Z})^N\) are in the same orbit under the action of \((\mathbb{Z}/m\mathbb{Z})^* \times \Sigma_N\), where \((\mathbb{Z}/m\mathbb{Z})^*\) acts diagonally by multiplication and the symmetric group \(\Sigma_N\) acts by permutation of the indices. Notice that to fix the class of the triple \((m,N,a)\) is equivalent to fixing the topological type of the \(G\)-action. Thus \(B\) and \(T'_g\) have the same image in \(M_g\) and we will consider \(Z(m,N,a) := Z(G) \subset A_g\) the \((N-3)\)-dimensional subvariety given by the Jacobians of the curves \(C_t\), \(t \in B\). By the Hurwitz formula

\[
g = 1 + \frac{(N-2)m - \sum_{i=1}^{N} r_i}{2}
\]

with \(r_i := \gcd(m,a_i)\). A basis of \(H^0(C_t,K_{C_t})\) is given as follows. For \(i \in \{1,\ldots,N\}\) and \(n \in \mathbb{Z}\) set

\[
l(i,n) := \left\lfloor \frac{-na_i}{m} \right\rfloor \quad d_n = -1 + \sum_{i=1}^{N} \left\lfloor \frac{-na_i}{m} \right\rfloor
\]

(Here \([a]\) and \((a)\) denote the integral and fractional parts of \(a \in \mathbb{R}\).) Since \(G\) acts on \(C_t\), there is a decomposition \(H^0(C_t,K_{C_t}) = \bigoplus_{n=0}^{m-1} V_n\), where \(V_n\) is the subspace of 1-forms \(\omega\) such that \(\zeta \cdot \omega = \zeta^n \omega\). Then \(V_0 = \{0\}\), while for \(n = 1,\ldots,m-1\) a basis of \(V_n\) is provided by the forms that have the following expression in the model \((5.2)\):

\[
\omega_{n,\nu} := y^n \cdot (x - t_1)^\nu \cdot \prod_{i=1}^{N} (x - t_i)^{(l(i,n)} \cdot dx \quad 0 \leq \nu \leq d_n - 1.
\]

Remark 5.6. In \[21\], Moonen proved that there is a finite list of triples \((m,N,a)\) such that the corresponding subvariety \(Z = Z(m,N,a) \subset A_g\) is a Shimura variety. By \[20\] a Shimura variety is a totally geodesic algebraic submanifold of \(A_g\) that contains a complex multiplication point. Therefore
the families in Moonen’s list are totally geodesic. Using Proposition 5.4 we can immediately verify that all these families are totally geodesic. In fact in all those cases there are no nontrivial quadrics in $I_2(K_{C_t})$ which are invariant under the action of the cyclic group.

**Proposition 5.7.** Consider the family $C \to B$ associated to the triple $(m, N, a)$ as above and the corresponding subvariety $Z = Z(m, N, a) \subset A_g$ given by the Jacobians of the curves $C_t$. Assume that not all the curves $C_t$ are hyperelliptic. If there exists an integer $n \in \{1, \ldots, m-1\}$ such that $d_n \geq 2$, $d_{m-n} \geq 2$ and $n \neq m-n$, then $Z$ is not totally geodesic.

**Proof.** Fix an arbitrary non-hyperelliptic curve $C = C_t$ belonging to the family. We use the representation (5.2). Since $d_n \geq 2$ and $d_{m-n} \geq 2$ and $n \neq m-n$, there are four distinct forms $\omega_{n,0}$, $\omega_{n,1}$, $\omega_{m-n,0}$, $\omega_{m-n,1}$ given in (5.3). We form the quadric

$$Q := \omega_{n,0} \circ \omega_{m-n,1} - \omega_{n,1} \circ \omega_{m-n,0}.$$ 

One immediately sees from the definition that $Q \in I_2(K_C)$ and that $Q$ is $G$-invariant. Let $D$ be the divisor of poles of the meromorphic function $x \in \mathcal{M}(C)$. Let $\sigma_0, \sigma_1 \in H^0(C, \mathcal{O}_C(D))$ be the sections corresponding to the meromorphic functions $1$ and $x$. Assume for simplicity that $t_1 = 0$, so we have $\omega_{n,1} = x \cdot \omega_{n,0}$ and $\omega_{m-n,1} = x \cdot \omega_{m-n,0}$. Hence the forms $\omega_{n,0}, \omega_{m-n,0}$ can be seen as elements in $H^0(C, K_C(-D))$. Set $\tau_0 := \omega_{n,0}$, $\tau_1 := \omega_{m-n,0}$. The quadric $Q$ can be written as follows:

$$(5.4) \quad Q = \sigma_0 \tau_0 \circ \sigma_1 \tau_1 - \sigma_0 \tau_1 \circ \sigma_1 \tau_0.$$ 

Denote by $\varphi : C \to \mathbb{P}^1$ our covering; it corresponds to the pencil $\langle 1, x \rangle = \langle \sigma_0, \sigma_1 \rangle$. Denote by $\psi$ the map to $\mathbb{P}^1$ induced by the other pencil $\langle \tau_0, \tau_1 \rangle$. Take a point $p \in C$ that does not belong to the set $\text{Crit}(\varphi) \cup G \cdot \text{Crit}(\psi) \cup B$, where $B$ is the base locus of $\psi$. Fix a nonzero vector $u \in T_p C$. The vector $v := \sum_{g \in G} \xi_{dg(u)} \in H^1(C, \mathcal{T}_C)$ is clearly $G$-invariant. So it is a tangent vector to $T_g^2$ at the point corresponding to $C$. Hence by diagram (5.1) we have $\rho(Q)(v \circ v) = \bar{\rho}(Q)(v \circ v)$. By Lemma (3.11) the map $\rho$ is also $G$-equivariant. So, using Theorem (2.6) we get

$$\bar{\rho}(Q)(v \circ v) = \rho(Q)(v \circ v) = \sum_{g_1, g_2 \in G} \rho(Q)(\xi_{dg_1(u)} \circ \xi_{dg_2(u)}) =$$

$$= |G| \cdot \left( \sum_{g \neq 1} \rho(Q)(\xi_u \circ \xi_{dg(u)}) - 2\pi i \mu_2(Q)(p) \right) =$$

$$= -2\pi i m \cdot \left( 2 \sum_{g \neq 1} Q(u, dg(u)) \cdot \eta_u(dg(u)) + \mu_2(Q)(p) \right).$$

By (5.4) $Q$ is the quadric associated to the pencils $\langle \sigma_0, \sigma_1 \rangle$ and $\langle \tau_0, \tau_1 \rangle$, corresponding to the maps $\varphi$ and $\psi$ respectively. Since the fibre of $\varphi$ containing...
$p$ is the orbit $G \cdot p$, $Q(u, g, u) = 0$, forall $g \in G - \{1\}$. So finally we get
\[
\hat{p}(Q)(v \circ v) = -2\pi i m \cdot \mu_2(Q)(p),
\]
and $\mu_2(Q)(p) \neq 0$ by our choice of $p$. Hence $Z$ is not totally geodesic. $\square$

**Corollary 5.8.** For any $m$, there is only a finite number of families which can be totally geodesic.

**Proof.** By Proposition [5.7] if there exists an integer $n \in \{1, \ldots, m - 1\}$ such that $d_n, d_{m-n} \geq 2$, $n \neq n - m$, then the family is not totally geodesic. So we can assume that for all $n \in \{1, \ldots, m - 1\}$ either $d_n \leq 1$, or $d_{m-n} \leq 1$. In particular either $d_1 \leq 1$, or $d_{m-1} \leq 1$. Denote by $N_i$ the cardinality of the set $\{ j \mid a_j = i \}$. Then $\sum_{i=1}^{m-1} N_i = N$ and the relation $\sum_{i=1}^{N} a_i \equiv 0 \mod m$ becomes $\sum_{i=1}^{m-1} iN_i \equiv 0 \mod m$. But
\[
d_l = -1 + \sum_{i=1}^{N} \left\langle \frac{-da_i}{m} \right\rangle = -1 + \sum_{i=1}^{m-1} N_i \left\langle \frac{-li}{m} \right\rangle, \quad \text{for } l = 1, \ldots, m - 1,
\]

so
\[
d_1 = -1 + \sum_{i=1}^{m-1} \frac{m-i}{m} N_i, \quad d_{m-1} = -1 + \sum_{i=1}^{m-1} \frac{i}{m} N_i.
\]

Hence for $l \in \{1, m - 1\}$ we have $d_l \geq -1 + \sum_{i=1}^{m-1} \frac{N_i}{m} = -1 + \frac{N}{m}$. So $d_l \leq 1$ implies $N \leq 2m$. Hence only a finite number of families can be totally geodesic. $\square$

Now we give some examples of computations for low degree ($m = 3$ and $m = 5$), showing that by the previous results most of the families are not totally geodesic.

**Corollary 5.9.** If $m = 3$ and $g \geq 5$, then the varieties $Z(3, N, a)$ are not totally geodesic. If $g = 4$ there is only one totally geodesic family given by the triple $(3, 6, a)$ where $a = (1, 1, 1, 1, 1, 1)$.

**Proof.** If $d_1, d_2 \geq 2$, we know by Proposition [5.7] that the family is not totally geodesic. So we can assume that there exists $n \in \{1, 2\}$ such that $d_n = 1$. In this case the space $S^2 H^0(K_{C_t})^{(0)}$ given by the invariant elements equals $\langle \omega_{n, 0} \circ V_{3-n} \rangle$. So there are no nonzero invariant elements in $I_2(K_{C_t})$ and by Proposition [5.4] we conclude that the family is totally geodesic. So we have to show that $g = 4$, $N = 6$ and $a = (1, 1, 1, 1, 1, 1)$. Denote as above by $N_i$ the cardinality of the set $\{ j \mid a_j = i \}$. Then $N_1 + N_2 = N$ and $N_1 + 2N_2 \equiv 0 \mod 3$. So
\[
d_1 = -1 + \sum_{i=1}^{N} \left\langle \frac{-a_i}{3} \right\rangle = -1 + N_1 \left\langle \frac{-1}{3} \right\rangle + N_2 \left\langle \frac{-2}{3} \right\rangle = -1 + \frac{2}{3} N_1 + \frac{1}{3} N_2,
\]
\[
d_2 = -1 + \sum_{i=1}^{N} \left\langle \frac{-2a_i}{3} \right\rangle = -1 + N_1 \left\langle \frac{-2}{3} \right\rangle + N_2 \left\langle \frac{-4}{3} \right\rangle = -1 + \frac{1}{3} N_1 + \frac{2}{3} N_2.
\]
By Hurwitz formula \( g = N - 2 \). Since we are assume \( g \geq 4 \) we get \( N_1 + N_2 = N \geq 6 \). If \( d_1 = 1 \),

\[
6 \leq N_1 + N_2 \leq 2N_1 + N_2 = 6,
\]

so \( N = 6 \), \( g = 4 \), \( N_1 = 0 \), \( N_2 = 6 \), hence \( a = (2, 2, 2, 2, 2) \) which is equivalent to \((1,1,1,1,1,1)\). If \( d_2 = 1 \) we have

\[
6 \leq N_1 + N_2 \leq N_1 + 2N_2 = 6,
\]

so \( N = 6 \), \( g = 4 \), \( N_2 = 0 \), \( N_1 = 6 \), \( a = (1,1,1,1,1) \).

\( \square \)

**Remark 5.10.** If \( m = 5 \), the following families are totally geodesic (see the list in [21]):

1. \( g = 4 \), \( N = 4 \), \( a = (1,3,3,3) \),
2. \( g = 6 \), \( N = 5 \), \( a = (2,2,2,2,2) \).

(1) is the family constructed by de Jong-Noot [7]. Applying the criterion given in Proposition 5.7 we are able to prove that all other families are not totally geodesic except possibly for the following 4 cases

3. \( g = 4 \), \( N = 4 \), \( a = (1,1,4,4) \),
4. \( g = 4 \), \( N = 4 \), \( a = (1,2,3,4) \),
5. \( g = 6 \), \( N = 5 \), \( a = (1,1,1,3,4) \),
6. \( g = 6 \), \( N = 5 \), \( a = (1,1,2,2,4) \),
7. \( g = 8 \), \( N = 6 \), \( a = (1,1,2,2,2,2) \).

(3) is contained in the hyperelliptic locus, see [21, (5.7)]. (4) has been studied in detail in [28]. Since one can check that it contains a CM point, it is not totally geodesic by Moonen’s classification.

It would be interesting to get a complete list of the families of cyclic coverings which are totally geodesic and to compare it with the list in [21].

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