Maps and fields with compressible density

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Abstract
Properties of steady compressible flow for which geometric constraints have been placed on the potential function are derived, under hypotheses on the flow density and the singular set. Some related unconstrained problems are also considered, including the estimation of a class of fields having nonzero vorticity. 2000 MSC: 58E20, 58E99, 75N10.

1 Introduction
The study of certain classical fields leads to a generalization of harmonic maps in which the Dirichlet energy is replaced by the functional

$$E = \int_M \int_0^{Q(du)} \rho(s) ds dM;$$

here $M$ is a Riemannian manifold; $du$ is the differential of a map $u : M \rightarrow N$, where $N$ is another Riemannian manifold;

$$Q(du) = \langle du, du \rangle |_{T^*M \otimes u^{-1}TN};$$

$\rho : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+$ is a $C^{1,\alpha}$ function of $Q$ satisfying the differential inequality

$$0 < \frac{4}{\rho(Q)} [Q\rho^2(Q)] < \infty$$

for $Q \in [0, Q_{crit})$.

In the typical case, the manifold $N$ represents a geometric constraint placed on the flow potential of a steady, irrotational, polytropic ideal fluid for which the closed 1-form $du \in T^*M$ is dual to the flow velocity. In this case we choose

$$\rho(Q) = \left(1 - \frac{\gamma - 1}{2}Q\right)^{1/(\gamma - 1)},$$

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where $\gamma > 1$ is the adiabatic constant of the medium and $Q_{\text{crit}}$ is the square of the sonic flow speed. These choices transform (1) into a condition for subsonic flow of mass density $\rho$.

The study of functionals of this kind in the unconstrained Euclidean case, in which $N = \mathbb{R}^k$ and $M$ is a domain of $\mathbb{R}^n$, goes back at least to work on planar flow by Bateman in the 1920s [Ba]. An extensive bibliography covering the first half of the last century is given in [Be]. A more recent bibliography of mathematical work in compressible fluid dynamics (not necessarily connected with variational theory) appears in [Ch]; see also [DO] and the bibliographic remarks in [CF].

A discussion of unconstrained compressible flow in a local chart on a manifold appears in Sec. III.3 of [Sed]. A global existence theorem for steady, unconstrained subsonic flow on a compact Riemannian manifold is given in [SS1]; for subsequent research employing this nonlinear Hodge approach see, e.g., [SS2], [SS3], [Si], [Sm], and [ISS]. In those works the curvature of the manifold introduces geometry into the domain of the velocity field. By considering potentials subject to a geometric constraint, as we do here and in [O1], we introduce geometry into the range of the velocity field. In [O1] we emphasized this connection to the preceding literature by calling such potentials nonlinear Hodge maps. But the potentials studied in [O1] are not associated with a cohomology class and neither the geometric construction nor the physical interpretation extend automatically to higher-degree forms. So it is perhaps more accurate to call mappings which are critical points of $E$ compressible-density maps.

There is already a considerable literature on maps for which a nonquadratic energy functional is given by the $L^p$ norm of the gradient of the map $u$; see, e.g., [FH], [HL], [F] and the references therein. Those works are motivated by the mathematical observation that the harmonic map energy is the squared $L^2$ norm of the gradient, raising the question of whether a corresponding theory can be derived for stationary points of the nonmetrizable $L^p$ norms, $p > 1$. Our starting point, on the other hand, is the physical observation that harmonic maps model a geometric constraint on a field of constant mass density. This prompts one to ask whether a corresponding theory can be derived for fields having mass density $\rho$ which depends in a nonlinear way on field strength. That question leads to the replacement of the harmonic map energy by the functional $E$. In applications to fluid dynamics, the variational equations of $E$ correspond to continuity equations for the flow. In the incompressible (hydrodynamic) limit $\rho \equiv 1$, the variational equations reduce to the harmonic map equations. This incompressible special case has also been interpreted in the context of nonlinear elasticity [T].

The analysis of critical points of $E$ is necessarily somewhat different from analysis of the harmonic map energy or of the other $L^p$ norms of the gradient. In those cases scaling arguments are natural, whereas they may be unnatural for $E$, as they involve a choice of conformal behavior for $\rho$. This is a point of commonality between our problem and certain other recent extensions of the harmonic map equations, e.g., [A] and [LM]; see also [EL]. The density of maps which are $L^p$-critical points of their gradient tends to zero (cavitates) as elliptic-
ity degenerates; this behavior is atypical of the mass density of fluids, for which the sonic value lies at the supremum of the range of subsonic speeds. [Compare conditions (2), above, and (23), below]. Finally, the references cited at the beginning of the preceding paragraph assume an energy minimizing property. We assume only that Euler-Lagrange equations are satisfied on a given subdomain.

In Section 2 an $L^\infty$ estimate is derived for nonuniformly elliptic, scalar velocity fields. Section 3 concerns technical aspects of the uniformly elliptic case which we studied earlier ([O1], Theorem 3). We also present in Section 3 a somewhat different proof of the result in [O1], one which is simpler in that it avoids certain smoothness assertions which were necessary in the original argument. In Section 4, Corollary 8 of [O1] is extended from the uniformly elliptic case to the nonuniformly elliptic case under somewhat different hypotheses.

We note that the analysis literature tends to treat the velocity as a section of the cotangent bundle, whereas the physics and geometry literature puts this object in the tangent bundle. The local arguments of Sec. 3 are the same in both representations. The usefulness of the cotangent representation for considering fields with vorticity is apparent in Sec. 4; for consistency we employ this notation in Sec. 3 as well.

2 Near-sonic maps into a line

It is known [D] that if $u \in H^{1,p}(\Omega)$ is a weak solution of the scalar equation

$$\text{div} \left( |\nabla u|^{p-2} \nabla u \right) = 0$$

(3)

for $p > 1$ in an open domain $\Omega$ of $\mathbb{R}^n$, then for every $n$-disc $B_R \subset \Omega$ of radius $R$ and every number $\delta \in (0, 1)$ there is a constant $k(\delta)$ independent of $R$ for which

$$\|\nabla u\|_{L^\infty(B_{R-\delta R})}^{p/2} \leq k(\delta) R^{-N} \int_{B_R} |\nabla u|^p \, dx.$$  

(4)

(See also [E], [Le], and references therein.)

This is a useful result to have, as the semi-norm on the right is the energy integral associated to weak solutions of (3). Thus inequality (4) derives an $L^\infty$ bound on the weak solution, which is unnatural to impose directly, from a condition of finite energy, which is the natural condition to impose on solutions of equations with variational structure. A uniform $L^\infty$ bound on weak solutions plays important roles in smoothness estimates and numerical analysis. Inequality (4) can be significantly generalized within the class of $L^p$-stationary gradients (see, e.g., [HL]).

In this section we derive an analogue of inequality (4) for solutions of the scalar equation

$$\text{div} \left[ \rho(Q) \nabla u \right] = 0$$

(5)

for which $\rho$ satisfies condition (1). Equation (5) is the Euler-Lagrange equation for the functional $E$ in the special case in which $u$ is a scalar function on $\mathbb{R}^n$. 

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If the middle term of (1) is bounded below away from zero on the entire range of values for \( Q \), then this analogue has already been derived in considerable generality (see, e.g., Theorem 4.3 of [SS], Proposition 3.1 of [Si], Lemma 3 of [DO], or Theorem 9 of [O1]). In each of these cases, however, the constant analogous to \( k(\delta) \) of inequality (4) tends to infinity as \( Q \) tends to \( Q_{crit} \), so these inequalities are not uniform unless eq. (5) is itself uniformly elliptic. Rather, the cited inequalities contribute indirectly to uniform Hölder estimates, by way of a delicate limiting argument introduced by Shiffman [Sh] in the planar case and extended to higher dimensions in [SS]. Direct arguments should suffice to estimate weak solutions of eq. (5) which, unlike the equations studied in the works cited, has scalar solutions. Our goal is to derive estimates for (5) which are manifestly uniform over the entire subcritical range of values for \( Q \).

By a weak solution in this scalar case we mean a function \( u \) having finite energy \( E \) and satisfying \( \forall \psi \in H^1_0(\Omega) \) the integral identity

\[
\int_{\Omega} \langle \rho(Q)du, d\psi \rangle dx = 0,
\]

where \( x \) is a vector in a bounded type-A domain \( \Omega \subset \mathbb{R}^n \), \( n > 2 \), and the angle brackets denote the euclidean inner product on 1-forms. For a definition of type-A domain see, e.g., p. 68 of [G]; our intention is to insure that a ball in the interior of \( \Omega \) does not become trapped in an outward cusp. As an example, any Lipschitz domain is type-A. In order for the following theorem to make sense in terms of fluid dynamics, we must additionally impose the condition that \( \Omega \) be topologically trivial so that the flow potential remains single-valued.

**Theorem 1**

Let the scalar function \( u(x), x \in \Omega, \) be a weak solution in the sense of (6) for \( \rho \) satisfying condition (1). In addition, assume that \( \rho'(Q) \leq 0 \ \forall Q \in [0, Q_{crit}] \) and that on this range, \( 0 < \kappa_0 \leq \rho(Q) \leq \rho(0) < \infty \). Then for every \( n \)-disc \( B_R \) strictly contained in \( \Omega \) and every \( \delta \in (0, 1) \) there exists a constant \( \kappa_1(n, \delta, \kappa_0, \rho(0)) \) for which

\[
\sup_{x \in B_R(1-\delta)} Q(x) \leq \kappa_1 R^{-n} E_{|B_R|}.
\]

The constant \( \kappa_1 \) in Theorem 1 depends neither on the radius \( R \) nor on any ellipticity parameter [such as the parameter \( \kappa_3 \) of condition (15), below]. Thus in particular, inequality (7) does not necessarily follow from the uniform bound of \( \sqrt{2/(\gamma + 1)} \) on the subsonic flow speed in (2). At the same time, it is satisfying to have a bound on weak solutions that results only from mathematical hypotheses on the equation itself rather than relying on a bound, such as the sonic speed, which is imposed on the mathematics by a physical model. The noncavitation hypothesis and the other hypotheses of the theorem are satisfied on the subsonic range by mass densities of the form (2).

The proof of Theorem 1 is elementary. The idea is to choose the test function in (6) to be a local restriction of the antiderivative \( F(Q) \) for the function

\[
f(Q) \equiv \rho^2(Q) + 2Q \rho(Q) \rho'(Q).
\]
The ellipticity condition is then interpreted, where it appears, as a piece of the chain rule applied to the gradient of $F$. This relieves us of the necessity to bound $f(Q)$ below away from zero, but obliges us to translate statements about $F(Q)$ and its $L^2$-norm into statements about $Q$ and its energy functional. Such an approach combines ideas from Sec. 1 of [U] and Sec. 3 of [D]. Those papers, as well as [Ev] and [Le], concern weak solutions satisfying hypotheses similar to inequality (26) of Sec. 4.

Proof. We initially assume that $u$ is twice continuously differentiable. It is then easy to verify that the results are unaffected if the derivatives are replaced by limits of finite differences. Taking a weak derivative of (6) yields

$$\sum_{i=1}^{n} \int_{\Omega} (\rho(Q)u_{x^i})_{x^i} \psi_{x^i} \, dx = 0$$

(8)

where, here and below, repeated indices are summed from 1 to $n$. For a function $\zeta \in C_0^\infty(B_R)$ and positive parameters $\alpha$ and $\beta$, choose

$$\psi(x) = u_{x^i} \left[ Q\rho^2(Q) + \beta \right]^{\alpha/2} \zeta^2(x).$$

Expanding the integrand of (8) yields a sum of six terms:

$$\sum_{i=1}^{n} (\rho(Q)u_{x^i})_{x^i} \psi_{x^i} = \frac{\alpha}{2} \rho'(Q)Q_{x^i}u_{x^i}u_{x^i} \left[ Q\rho^2(Q) + \beta \right]^{(\alpha-2)/2} \left[ \rho^2(Q) + 2Q\rho(Q)\rho'(Q) \right] Q_{x^i}\zeta^2 +$$

$$\frac{\alpha}{2} \rho'(Q)Q_{x^i}u_{x^i}u_{x^i} \left[ Q\rho^2(Q) + \beta \right]^{\alpha/2} 2\zeta_{x^i} +$$

$$\frac{\alpha}{2} \rho(Q)u_{x^i}u_{x^i} \left[ Q\rho^2(Q) + \beta \right]^{(\alpha-2)/2} \left[ \rho^2(Q) + 2Q\rho(Q)\rho'(Q) \right] Q_{x^i}\zeta^2 +$$

$$\rho(Q)u_{x^i}u_{x^i} \left[ Q\rho^2(Q) + \beta \right]^{\alpha/2} 2\zeta_{x^i} \equiv \sum_{r=1}^{6} i_r.$$

We estimate the terms of this sum individually. The following estimates should be interpreted as occurring “under the integral sign.” The hypothesis on the sign of $\rho'(Q)$ implies, using Kato’s inequality, that

$$i_1 + i_4 \geq \frac{\kappa_0}{4\rho^2(0)} \left( \frac{4}{\alpha + 2} \right)^2 \left| \nabla \left( [Q\rho^2(Q) + \beta]^{(\alpha+2)/4} \right) \right|^2 \zeta^2.$$

Because the range of $\rho(Q)/\rho(0)$ is contained in the interval $(0, 1]$, we also have

$$i_2 + i_5 \geq$$
Moreover, there exists a positive constant \( \varepsilon \) for which
\[
i_3 + i_6 \geq \frac{\varepsilon}{2} \left( \frac{4}{\alpha + 2} \right)^2 \left| \nabla \left( [Q\rho^2(Q) + \beta]^{(\alpha+2)/4} \right) \right|^2 \cdot 1.
\]
Claim 9

Choose \( \varepsilon \) to equal \([\kappa_0 + \alpha\rho(0)]/4\rho^2(0)\). Then we obtain the integral inequality
\[
\int_{\Omega} \left| \nabla \left( [Q\rho^2(Q) + \beta]^{(\alpha+2)/4} \right) \right|^2 \cdot 1 \leq m \int_{\Omega} \left| \nabla \left( [Q\rho^2(Q) + \beta]^{(\alpha+2)/4} \right) \right|^2 \cdot 1
\]
for
\[
m = \left[ \frac{2\rho^2(0)}{\kappa_0} \left( \frac{\alpha + 2}{\kappa_0 + \alpha\rho(0)} \right) \right]^{\frac{1}{4}}.
\]
As \( \alpha \) tends to either zero or infinity, \( m \) tends to a finite constant \( \kappa_2 \) which depends only on the upper and lower bounds on \( \rho(Q) \).

Apply inequality (9.5.8) of [LU] to expression (9), taking the quantities \( u \) and \( \varepsilon \) of that reference to equal, respectively, the quantities \( \sqrt{Q\rho^2(Q) + \beta} \) and \( \alpha/2 \) of expression (9). Construct a Moser iteration along the lines of expressions (9.5.8)-(9.5.12) in [LU]. We obtain in the limit the inequality
\[
\sup_{x \in B_{R(1-\delta)}} \left[ Q(x)\rho^2(Q(x)) + \beta \right] \leq \kappa_2 R^{-n} \int_{B_R} \left[ Q\rho^2(Q) + \beta \right] \cdot 1. \quad (10)
\]
We have
\[
\int_{B_R} Q\rho^2(Q) \cdot 1 = \int_{B_R} \int_0^Q \frac{d}{ds} \left( s\rho^2(s) \right) \, ds \cdot 1 \leq \int_{B_R} \rho(0) \int_0^Q \rho(s) \, ds \cdot 1 = 2\rho(0)E_{1B_R}. \quad (11)
\]
Regarding the left-hand side of inequality (10), condition (1) implies that \( Q\rho^2(Q) \) is an increasing function of \( Q \). Thus the suprema in \( B_{R(1-\delta)} \) of \( Q(x) \) and of \( Q\rho^2(Q(x)) \) occur at the same value of \( x \). Because the mass density is noncavitating,
\[
\kappa_0 \sup_{x \in B_{R(1-\delta)}} Q(x) \leq \sup_{x \in B_{R(1-\delta)}} \left[ Q(x)\rho^2(Q(x)) \right]. \quad (12)
\]
Comparing inequalities (10)-(12), we conclude that there is a constant \( \kappa_1 \) such that
\[
\sup_{x \in B_{R(1-\delta)}} Q(x) + \beta \leq \kappa_1 R^{-n} [E_{|B_R} + \beta \text{vol}(B_R)],
\]
where \( \kappa_1 \) depends on \( n, \delta, \kappa_0, \) and \( \rho(0) \). Because \( \beta \) is an arbitrary positive number, we can let it tend to zero without affecting the other constants.

We now remove the smoothness assumption on \( u \). Replace the admissible test function \( \psi(x) \) in eq. (6) by the admissible test function \( \psi(x - he_j) \), where \( e_j \) is the \( j \)th basis vector for \( \mathbb{R}^n \), \( j = 1, \ldots, n \), and \( h \) is a positive constant. Then (6) assumes the form
\[
\int_B (\rho(Q(x)) du(x), d\psi(x - he_j)) dx = 0. \tag{13}
\]
Apply the coordinate transformation \( y = x - he_j \) to eq. (13) and subtract (6) from (13) to obtain
\[
h^{-1} \int_B (\rho(Q(x+he_j)) du(x+he_j) - \rho(Q(x)) du(x), d\psi(x)) dx = 0.
\]
The limiting case is an expression of the form (8). The expressions leading to inequality (9) remain true in the finite difference approximation. Because the right-hand side of (9) does not depend on \( h \), we can allow the parameter \( h \) to tend to zero in this approximation, completing the proof of Theorem 1.

Remarks. 1. In Sec. 9.5 of [LU] the Moser iteration is illustrated for linear equations of the form
\[
(\rho_{ij}(x) u_{x_j})_{x_i} = 0,
\]
where
\[
\nu \sum_{i=1}^n \xi_i^2 \leq \rho_{ij}(x) \xi_i \xi_j \leq \mu \sum_{i=1}^n \xi_i^2.
\]
In this case noncavitation is equivalent to ellipticity, whereas the two conditions are distinct for the quasilinear density \( \rho(Q) \). Thus the ratio \( \mu/\nu \) in expression (9.5.8) of [LU], which is analogous to the factor \( m \) in our expression (9), introduces a dependence on ellipticity in the linear case but not in the quasilinear case.

2. Theorem 9 of [O1] is a subparabolic Moser estimate for multivalued flow potentials possessing geometric constraints. The preceding proof is too simple to work in that case, and the constants obtained in the proof of Theorem 9 depend on ellipticity. However, one can replace, in the line preceding inequality (69) of that proof, the function \( Q^{r-1} \) for \( r > 2 \) by the function \( (Q + \beta)^{r-1} \) for \( r > 1 \), allowing \( \beta \) to tend to zero at the end as in the preceding proof. This avoids eventual difficulties in the Moser iteration.
3 Uniformly elliptic maps

We now consider the more difficult cases in which the target has nontrivial geometry. In what follows the symbol $C$ will denote generic positive constants unless otherwise indicated.

3.1 Effects of geometric constraints

In studying critical points of $E$, it is natural to obtain the admissible class of maps from the condition of finite energy. We seek a class of bounded maps having integrable density $e(u)$. In order to integrate this object, it is necessary to choose local coordinates for $e(u)$ on $N$ and it is not a priori true that $u$ takes a coordinate chart on $M$ into a coordinate chart on $N$. If however we restrict our attention to maps from $M$ into $N$ which are Hölder continuous, then the local oscillations of the map are controlled on the target, and the image of a sufficiently small region of $M$ will lie in a coordinate chart of $N$. In this case we can write

$$Q = \frac{1}{2} \gamma^{\alpha\beta}(x) g_{ij} (u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta},$$

where for $n = \text{dim}(M)$, $x = (x^1, ..., x^n)$ is a coordinate chart on the manifold $M$ having metric tensor $\gamma_{\alpha\beta}(x)$; $u = (u^1, ..., u^n)$ is a coordinate chart on the manifold $N$ having metric tensor $g_{ij}(u)$, where $m = \text{dim}(N)$; repeated Greek indices are summed from 1 to $n$; repeated Latin indices are summed from 1 to $m$.

This continuity assumption severely restricts the kinds of questions that we can ask about the map. Moreover, the geometric constraint re-emerges as a problem, even if the map is continuous, when we attempt to extremize the energy functional by taking variations. This is because the test functions $\psi$ might take the image of $u + t\psi$ off of $N$, even for small values of $t$. This can be immediately seen if, for example, we take $N$ to be the unit sphere $|u| = 1$.

The conventional solution to both problems, that of defining an admissible class of finite-energy maps and of varying the energy on the target manifold, is to embed $N$ isometrically into a higher-dimensional Euclidean space $\mathbb{R}^k$ by the Nash Embedding Theorem. The manifold $N$ emerges as a system of $k - m$ independent constraints,

$$\Phi(u) = (\Phi_1(u), ..., \Phi_{k-m}(u)) = 0.$$

In this case

$$Q = \frac{1}{2} \gamma^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta},$$

and the incompressible energy integral reduces to the classical Dirichlet integral. In taking variations, a suitable Euclidean neighborhood $O(N)$ of $N$ is projected onto $N$ by nearest point projection $\Pi$. If $t$ is small enough and $N$ is a $C^1$ submanifold of $\mathbb{R}^k$, then the variations $\Pi \circ (u + t\psi)$ will be constrained to lie on
for almost every \(x\) in \(M\), where \(\psi \in C^\infty_0 (M, \mathbb{R}^k)\), and for every \(x\) in \(M\) if \(u\) is continuous. Now the variational equations of \(E\) are given by

\[
\frac{d}{dt}|_{t=0} \int_M \int_Q t \rho(s) \, ds \, dM = 0,
\]

where

\[
Q_t = |d [\Pi \circ (u + t\psi)]|^2.
\]

The variational equations in the ambient space assume the explicit form

\[
\delta [\rho(Q) \, du] = \rho(Q) \, A(du, du),
\]

where \(\delta\) is the formal adjoint of the exterior derivative \(d\) and \(A\) is the second fundamental form of \(N\).

See \[Sch\] and \[ScU\] for detailed discussions of these issues in the harmonic map case.

We call \(u \in L^\infty (M, \mathbb{R}^k) \cap H^{1,2} (M, \mathbb{R}^k)\) a weak solution of eqs. (14) in a coordinate chart \(\Omega\) of \(M\) if \(u\) has finite energy \(E\) and satisfies, \(\forall \, \zeta \in H_0^{1,2} (\Omega, \mathbb{R}^k) \cap L^\infty (\Omega, \mathbb{R}^k)\), the identity

\[
\int_\Omega \langle d\zeta, \rho(Q) \, du \rangle \ast 1 = \int_\Omega \langle \zeta, \rho(Q) \, A(du, du) \rangle \ast 1.
\]

The existence of weak solutions to the unconstrained problem in the elliptic range follows, by lower semicontinuity, from the convexity of the energy functional under condition (1). Weak solutions of the constrained problem may not exist for certain choices of \(\rho\) and \(N\). To see this, let \(\rho(Q) = Q^{(p-2)/2}, p > 1\), and consider the counterexample of \[HL\], Sec. 6.3.

### 3.2 Maps with apparent singularities

The literature on removable singularities is too large for even a superficial review. We mention that the removability of singularities in harmonic maps is considered in, e.g., \[SaU\], \[EP\], \[Me\], \[Li\], and \[CL\]. Obstacles to the extension of methods used in those references to our case include, in addition to the dependence of the scaling behavior of \(E\) on the choice of \(\rho\), the absence of an obvious analogue to the a priori Hölder estimate of \[HJW\], which forms the basis for many smoothness results in the harmonic map literature. Removable singularities theorems and related a priori estimates for mappings which are critical points for the \(L^p\)-norm of their gradient are reviewed in \[F\]. Those arguments also strongly depend on the scaling behavior of the energy. Removability of singularities in systems which resemble the unconstrained case of eqs. (14) can be found in, for example, \[ISS\]. The application of such results to the constrained case is limited by the presence of quadratic nonlinearities arising from the target curvature.

The removability of an apparent singularity can be proven either by showing the existence of a continuous transformation to a nonsingular domain, or by
ruling out the existence of the singular set on a priori grounds. We adopt the latter approach in the following theorem.

**Theorem 2** Let \( u : \Omega \to N \) be a \( C^2 \) stationary point of the energy \( E \) on \( \Omega/\Sigma \), where \( \Omega \) is an open bounded, type-A domain of \( \mathbb{R}^n \), \( n > 2 \); \( N \) is a smooth, compact \( m \)-dimensional Riemannian manifold, \( m \leq n \), \( \partial N = 0 \); \( \Sigma \subset B \subset \subset \Omega \) is a compact singular set, completely contained in a sufficiently small \( n \)-disc \( B \), which is itself completely contained in \( \Omega \). Suppose that \( \rho \) satisfies

\[
\kappa_3 < \rho(Q) + 2Q|\rho'(Q) < \kappa_4
\]

for constants \( 0 < \kappa_3 < \kappa_4 < \infty \). If \( n > 4 \), let \( 2n/(n-2) < \mu \leq n \), where \( \mu \) is the codimension of \( \Sigma \), and let \( du \in L^n(B) \); if \( n = 3, 4 \), let \( du \in L^{2q_0\beta}(B) \cap L^{4q}(B) \), where \( \beta = (\mu - \varepsilon)/(\mu - 2 - \varepsilon) \) for \( 2 < \mu \leq n \), \( \varepsilon > 0 \), and \( \frac{1}{4} < q_0 < q \). Then \( du \) is Hölder continuous on \( B \).

Because the singular set is assumed small, the choice of a Euclidean domain \( \Omega \) entails little reduction in generality. In distinction to the harmonic map case, Theorem 2 does not immediately imply any higher degree of smoothness. The theorem immediately extends to the case of a finite number of singular sets having the same properties as \( \Sigma \).

Theorem 2 is stated and proven in [O1] (Theorem 3). We begin by briefly reviewing that proof, adding details on the underlying elliptic theory in Lemmas 4 and 5. An alternate method of proof, which avoids Lemmas 4 and 5 altogether, is given in Sec. 3.3.1. We show in the proof that the modulus of continuity for \( du \) depends on \( \rho, u, N, n \), and on the \( L^n \)-norm of \( du \). A metric can be chosen on \( \Omega \) in which the \( L^n \)-norm of \( du \) over \( \Omega \) is smaller than any given fixed number.

There are choices of \( \rho \), however, under which the variational equations fail to be invariant under this transformation; c.f. [KFL].

**Lemma 3** Under the hypotheses of Theorem 2, \( u \) is Hölder continuous on \( B \).

**Proof.** Away from the singular set, \( Q \) is sufficiently smooth that local coordinates can be chosen on \( N \) and one can show ([O1], Theorem 2) that

\[
L(Q) + C_N Q(Q + 1) \geq 0,
\]

where \( L \) is an elliptic operator under hypothesis (15). Integrate inequality (16) by parts over \( B \) against a test function \( (\eta/\psi)^2 \Xi(Q) \); here \( \eta, \psi \geq 0 \); \( \psi(x) = 0 \) \( \forall x \) in a neighborhood of \( \Sigma; \eta \in C^\infty_0(B') \) where \( \Sigma \subset B' \subset B; \Xi(Q) = h(Q)h'(Q) \), where for \( k = 0, 1, \ldots \),

\[
h(Q) = \begin{cases}
Q^{n/(n-2)+k/n/4} & \text{for } 0 \leq Q \leq \ell, \\
\frac{\mu-\varepsilon}{\mu-2-\varepsilon} \left[ (\ell \cdot Q^{-2-\varepsilon/2})^{n/(n-2)+k\mu/2} - \frac{2}{\mu-\varepsilon} \ell^{n/(n-2)+k/n/4} \right] & \text{for } Q \geq \ell
\end{cases}
\]

if \( n > 4 \); \( h \) is an analogous test function ([Se], p. 280) when \( n \) is 3 or 4. Let \( \psi \) be the limit of a sequence \( 1 - \xi^{(v)} \), where \( \xi^{(v)} \) is the sequence \( \eta^{(v)} \) of [Se], Lemma
8. This sequence has the property that \( \xi^{(\nu)} = 0 \) a.e. in a neighborhood of \( \Sigma \), but \( \xi^{(\nu)} \) tends to 1 a.e. and \( \nabla \xi^{(\nu)} \) tends to zero in \( L^{\mu-\varepsilon} \) as \( \nu \) tends to infinity. It can be shown(\[O1\], (28)-(35)) that these choices imply the inequality

\[
\int_{B'} \eta^2 \left| \nabla \left( \frac{Q^{(k)/4}}{\tau(k)} \right) \right|^2 \leq \int_{B'} |\nabla \eta|^2 Q^{(k)/2} \ast 1,
\]

where \( \tau(k) = n|n/(n-2)|^k \). Taking \( k \) to equal zero, the right-hand side of this expression is bounded by the \( L^n \)-norm of \( du \) over \( \Omega \). Applying the Sobolev inequality to the left-hand side allows us to repeat the preceding integration by parts for \( k = 1 \). Applying the Sobolev inequality to the resulting inequality allows iterations for progressively higher values of \( k \). In this way any finite \( L^p \)-norm for \( du \) can be bounded by the \( L^n \)-norm of \( du \) over \( \Omega \). We conclude that \( du \) lies in the space \( L^p(B) \) for any finite value of \( p \) and is an \( H^{1,2} \) weak elliptic subsolution on \( B' \). Then \( u \) is Hölder continuous and the proof is complete.

Let \( D \) be an \( n \)-disc of radius \( R \), completely contained in the \( n \)-disc \( B' \), completely containing the singular set \( \Sigma \) and centered at a point \( x_0 \in \Sigma \). We require a classical result on linear boundary-value problems:

**Lemma 4** If \( \rho \in C^{1,\alpha}(D) \) and \( w \in C^0(\partial D) \), then there exists \( \varphi : D \to \mathbb{R}^k \) such that \( \varphi \in C^{2,\alpha}(D) \cap C^0(\overline{D}) \) and \( \varphi \) satisfies the linear boundary-value problem

\[
\delta \left[ \rho(|x|^2) \varphi \right] = 0 \quad \text{in } D_R(x_0), \quad \varphi_\vartheta = w_\vartheta \quad \text{on } \partial D,
\]

where the subscripted \( \vartheta \) denotes the tangential component of the map in coordinates \( (r, \vartheta_1, \ldots, \vartheta_{n-1}) \).

**Proof.** Condition (15) implies [U] that

\[
\rho \left( |x|^2 \right) \geq K
\]

for some positive constant \( K \). This inequality implies strict ellipticity of the linearized equations (17). The result now follows from Theorem 6.13 of [GT], although that result is stated for scalar equations, because the differential operator in (17) is diagonal. This completes the proof.

Define a map \( \varphi : D \to \mathbb{R}^k \) and consider the nonlinear boundary-value problem

\[
\delta \left( \rho(|d\varphi|^2) d\varphi \right) = 0 \quad \text{in } D_R(x_0), \quad \varphi_\vartheta = u_\vartheta \quad \text{on } \partial D.
\]

**Lemma 5** If \( u \) satisfies the hypotheses of Theorem 2, then the boundary-value problem (18) has a solution \( \varphi \) in the space \( C^{2,\beta}(D) \cap C^{0,\alpha}(\overline{D}) \).
Remarks. If \( u \) lies in the space \( C^{2,\beta}(D/\Sigma) \) and if \( \Sigma \) is an isolated point (or by extension, a finite point set), then the smoothness of \( \varphi \) follows from Schauder estimates, and as the radius \( R \) of \( D \) shrinks to a point, the boundary conditions of problem (18) remain smooth. In this case one can compare \( du \) to \( d\varphi \) with the goal of applying Theorem III.1.3 of [G] exactly as in [O1], and no further remarks are necessary. If, however, \( \Sigma \) is not a point, then for sufficiently small \( R, D_r \) will intersect \( \Sigma \) and we have only the result of Lemma 3, that \( u \) is Hölder continuous on a domain that includes the singular set. It is not explicitly shown in [O1] that this is sufficient boundary regularity for completing the proof of Theorem 2; but that is in fact the case, as we will show here.

Proof of Lemma 5. Consider the boundary-value problem (17) for \( w = u \) on \( D_r(x_0) \), where \( r \in (0, R] \). The boundary data are Hölder continuous by Lemma 3, so the solution \( v \) lies in the space \( C^{2,\alpha}(D_r) \cap C^0(\overline{D}_r) \) by Lemma 4. Now we extend to systems the proof of [LU], Theorem 4.8.7. That is, we solve a sequence of boundary-value problems having the form

\[
\delta \left( \rho(|d\varphi|^2)d\varphi \right) = 0 \quad \text{in } D_{r_i}(x_0), \quad D_{r_i} \subset D_R, \tag{19}
\]

\[
\varphi_{\theta} = v_{\theta} \quad \text{on } \partial D_{r_i},
\]

where \( \{r_i\} \to R \). A \( C^{2,\alpha} \) solution \( \varphi_i \) to this problem exists for every \( i \) by Theorem 1 of [SS2]. (The differentiability requirements on the boundary are encapsulated in the definition of the space \( D_2 \) of that paper.) Also, by hypothesis \( u \) is bounded by a constant depending only on \( N \). This gives a uniform bound on the boundary data \( v_{\theta} \) on each \( \partial D_{r_i} \). Solutions \( \varphi_i \) of (19) satisfy a maximum principle, for each \( i \), by Sec. 2 of [SS3]. Thus the sequence \( \{\varphi_i\} \) possesses a subsequence which converges, as \( r_i \) tends to \( R \), to a solution \( \varphi \in C^{2,\beta}(D_R) \cap C^{0,\alpha}(\overline{D}_R) \), as required. This completes the proof of Lemma 5.

We now complete the proof of Theorem 2 by showing that the differential \( du \) is Hölder continuous in a domain that includes the singular set.

For sufficiently small \( B \), we can construct a suitable \( n \)-disc, on the boundary of which the tangential component of a comparison vector can be forced to agree with the tangential component of \( u \) (c.f. [Li], Sec. 3).

Consider a solution \( \varphi \) to the boundary-value problem (19). Combining Lemma 5 with Theorem III.1.2 of [G], we find that if \( (d\varphi)_{R,x_0} \) denotes the mean value of the 1-form \( d\varphi \) on \( D_R(x_0) \), then for any sufficiently small \( R \), \( d\varphi \) satisfies

\[
\int_{D_R(x_0)} |d\varphi - (d\varphi)_{R,x_0}|^2 \leq CR^{n+2\lambda}
\]

for some number \( \lambda \in (0, 1] \). Then

\[
\int_{D_R(x_0)} \left( d(u - \varphi), \rho(|du|^2)du - \rho(|d\varphi|^2)d\varphi \right) * 1
\]

\[
= \int_{D_R(x_0)} \left( (u - \varphi), \rho(|du|^2)A(du, du) \right) * 1.
\]
Apply a generalized mean-value formula to the 1-form $\rho(|ds|^2)ds$ as in [Si], Lemma 1.1. We obtain

$$\int_{D_R(x_0)} |d(u - \varphi)|^2 \star 1 \leq C \left( \int_{D_R(x_0)} (|du| + |d\varphi|) |x| \star 1 + \int_{D_R(x_0)} |u - \varphi| \rho(|du|^2) |u| |du|^2 \star 1 \right)$$

$$\equiv i_1 + i_2. \quad (20)$$

We have

$$i_1 \leq \int_{D_R(x_0)} (|d(u - \varphi)| + 2 |d\varphi|) |x| \star 1 \leq \varepsilon \int_{D_R(x_0)} |d(\varphi - u)|^2 \star 1 + C \left( S^n \varepsilon, \|d\varphi\|_{\infty} \right) \int_0^R |x|^{n+1} d|x|,$$

where the sup norm of $d\varphi$ depends on the modulus of continuity for $u$ through eq. (18).

$$i_2 \leq C(\rho) \int_{D_R(x_0)} |u - \varphi| |u| |du|^2 \star 1 \leq R^{-\nu} \int_{D_R(x_0)} |u - \varphi|^2 |u|^2 \star 1 + R^{\nu} \int_{D_R(x_0)} |du|^4 \star 1 \leq R^{-\nu} \int_{D_R(x_0)} |u - \varphi|^2 |u|^2 \star 1 + C \left( \|du\|_{4p}^4, n \right) R^{n(p-1)/p+\nu}$$

for a constant $\nu$ to be chosen and $p$ so large that $\nu p > n$. We have by the Sobolev Theorem

$$R^{-\nu} \int_{D_R(x_0)} |u - \varphi|^2 |u|^2 \star 1 \leq R^{-\nu} \left( \int_{D_R(x_0)} |u - \varphi|^{2n/(n-2)} \star 1 \right)^{(n-2)/n} \left( \int_{D_R(x_0)} |u|^n \star 1 \right)^{2/n} \leq R^{-\nu} C_{Sobolev} \int_{D_R(x_0)} |d(u - \varphi)|^2 \star 1 \left( \int_{D_R(x_0)} |u|^n \star 1 \right)^{2/n} \leq R^{-\nu} C \left\| u^2 \right\|_{C^{0, \gamma}(D)} \int_{D_R(x_0)} |d(u - \varphi)|^2 \star 1 \left( \int_0^R |x|^{n-1} d|x| \right)^{2/n} \leq R^2 R^{-\nu} C \int_{D_R(x_0)} |d(u - \varphi)|^2 \star 1.$$
Choose \( \nu \in (0, 2) \). Substitute the estimates for \( i_1 \) and \( i_2 \) into the right-hand side of (20) and absorb small terms on the left in (20) to obtain

\[
\int_{D_R(x_0)} |d(u - \varphi)|^2 \ast 1 \leq CR^{n+\lambda'}
\]

for some positive \( \lambda' \).

The minimizing property of the mean value with respect to location parameters implies

\[
\int_{D_R(x_0)} |du - (du)_{R,x_0}|^2 \ast 1 \leq \int_{D_R(x_0)} |du - (d\varphi)_{R,x_0}|^2 \ast 1
\]

\[
\leq \int_{D_R(x_0)} |du - d\varphi|^2 \ast 1 + \int_{D_R(x_0)} |d\varphi - (d\varphi)_{R,x_0}|^2 \ast 1
\]

\[
\leq CR^{n+\ell}
\]

(21)

for some \( \ell > 0 \). Because these estimates can be repeated for any sufficiently small value of \( R \), the proof of Theorem 2 is completed by the local form of Campanato’s Theorem (Theorem III.1.3 of [G]).

### 3.3 Weak solutions of eqs. (11) and (15)

If we make no assumptions about the singular set but assume that \( u \) satisfies (14) weakly in \( B \), it is possible to show by estimating difference quotients that \( du \) is an element of the space \( H^{1,2}(B) \). The next logical step would be to show \( du \in L^\infty(B) \). This step cannot be taken in the constrained case by following an analogy to the unconstrained case. The latter arguments proceed from a scalar inequality, as in Sec. 1 of [U]; but in order to adapt that argument to the constrained case it is necessary to choose local coordinates on the tangent space of \( N \), as in Theorem 2 of [O1]. This requires some \( a \ priori \) information about the singular set of \( u \).

Theorem 2 of the preceding section implies that if an \( E \)-critical map \( u \) is bounded and Hölder continuous on an open Euclidean domain, then \( du \) is Hölder continuous on small compact subdomains. The initial continuity assumption means that the target geometry will play little role in the analysis beyond its contribution to the nonlinearity of the variational equations.

#### 3.3.1 An alternate proof of Theorem 2

The arguments of [U] imply that weak solutions of (18) are Hölder continuous, but the Hölder estimate implied by that work cannot be continued up to the boundary. Nevertheless, it is possible to show that interior smoothness of weak solutions to (18) is sufficient to complete the proof of Theorem 2 using a modification of the preceding arguments.

The weak form of problem (18) can be written
\[
\int_{D_R(x_0)} \left\langle d\zeta, \rho \left( |d\varphi|^2 \right) d\varphi \right\rangle \ast 1 = 0, \tag{22}
\]
where \( d\zeta \) is a closed 1-form in \( L^2(D_R) \) having vanishing tangential component on \( \partial D_R \). Applying standard function-theoretic arguments on \( \mathbb{R}^n \), we consider \( \zeta \) to be an admissible test function; c.f. eq. (1.2) of [Si]. Writing (22) as the weak variational equations of the energy functional \( E \) with \( N \) replaced by \( \mathbb{R}^k \), we have

\[
\int_{D_R(x_0)} \int_0^{|d\varphi|^2} \rho(s) ds \ast 1 \geq K \int_{D_R(x_0)} |d\varphi|^2 \ast 1,
\]
so \( d\varphi \) lies in the space \( L^2(B_R) \) by ellipticity and finite energy. (See also Sec. 1 of [U].) The proof of Lemma 3 implies that \( du \) lies in \( L^2(D_R) \). Because \( d(u - \varphi) \) is in \( L^2 \), we can choose \( \zeta = u - \varphi \) in (22). The resulting weak Dirichlet problem is solvable by Proposition 4.3 of [Si]; see also [ISS]. The 1-form \( d\varphi \) is Hölder continuous in the interior of \( D \) by Proposition 4.4 of [Si], which is derived from [U]. The Campanato Theorem implies that

\[
\int_{D_R/2(x_0)} |d\varphi - (d\varphi)_{R/2,x_0}|^2 \ast 1 \leq CR^{n+\alpha}
\]
for some \( \alpha \in (0,2] \). Estimating (20) as in the preceding section, we find that

\[
\int_{D_R(x_0)} |d(u - \varphi)|^2 \ast 1 \leq CR^{n+\mu}
\]
for some positive \( \mu \). Then of course

\[
\int_{D_R/2(x_0)} |d(u - \varphi)|^2 \ast 1 \leq CR^{n+\mu}.
\]
Rewrite inequality (21) over \( D_{R/2}(x_0) \) to obtain

\[
\int_{D_{R/2}(x_0)} |du - (du)_{R/2,x_0}|^2 \ast 1 \leq \int_{D_{R/2}(x_0)} |du - (d\varphi)_{R/2,x_0}|^2 \ast 1
\]
\[
\leq \int_{D_{R/2}(x_0)} |du - d\varphi|^2 \ast 1 + \int_{D_{R/2}(x_0)} |d\varphi - (d\varphi)_{R/2,x_0}|^2 \ast 1
\]
\[
\leq CR^{n+\ell}
\]
for some \( \ell > 0 \). This completes the alternate proof of Theorem 2.

An application of this argument to an unconstrained problem for bundle curvature is given in [O2].
4 Nonuniformly elliptic solutions having nonzero vorticity

Note that the map $u$ enters into the problem of the preceding section only through its geometry. The variational equations in the unconstrained case are written in terms of $du$, and the map $u$ does not directly appear in them. This raises the question of how much of the theory can be deduced in the unconstrained case without assuming the existence of a potential.

Thus we consider systems having the form $[O1]$

$$\delta (\rho (Q) \omega) = 0, \quad (23)$$

$$d\omega = \nu \wedge \omega, \quad (24)$$

for $\nu \in \Lambda^1(V)$, where $V$ is a smooth section of a vector bundle over an open, bounded domain $\Omega$ of $\mathbb{R}^n$; $Q = \omega \wedge \ast \omega$, where $\ast : \Lambda^p \to \Lambda^{n-p}$ is the Hodge involution; $\rho$ is defined as in the preceding sections, but will be assumed to satisfy an inequality somewhat different from (1).

The condition

$$d\omega = 0 \quad (25)$$

implies, by the converse of the Poincaré Lemma, the local existence of a potential $u \in \Lambda^{p-1}(V)$ such that $du = \omega$. Thus solutions of (25) lie in a cohomology class, whereas solutions of (24) do not in general. However, the integrability condition (24) generates a closed ideal when $p = 1$. Obvious modifications of condition (24) generate a closed ideal for solutions of higher degree (see, e.g., [Ed], Theorem 4-2.1).

In the interpretation of eqs. (23), (25) in which $\omega$ is dual to the flow velocity, the vanishing of the vorticity $d\omega$ expresses the property that $\omega$ must integrate to zero along any curve homologous to zero. The 0-form $u$ is the flow potential. One-form solutions to (24) have only the weaker property that $\omega = \ell du$ for some 0-form $\ell$.

**Remark on terminology.** The distinction between curl-free and rotation-free fields is sometimes used to characterize velocity fields corresponding to (25) and (24), respectively (c.f. pp. 123, 124 of [MTW]; p. 28 of [So]). In [O1] the term irrotational field is used to denote a curl-free field, and any field which is not curl-free is called rotational. While that terminology may be misleading physically, the term curl-free is not mathematically correct in higher dimensions, so either choice of terms is open to criticism.

In Theorem 7 and Corollary 8 of [O1] a Hölder estimate is derived for the variant (23), (24) of the nonlinear Hodge equations on a possibly singular domain. As the solution approaches the critical value at which the ellipticity of the differential operator breaks down, the elliptic estimate of [O1] also breaks down. In this section we derive an estimate which is uniform over the entire subcritical range.
We assume that \( \omega \) is a classical solution of eqs. (23), (24) outside a singular set of prescribed dimension and that the density \( \rho \) satisfies
\[
\kappa_5^{-1}(Q + k_0)^q \leq \rho(Q) + 2Q \rho'(Q) \leq \kappa_5(Q + k_0)^q,
\]
for constants \( \kappa_5, q > 0 \) and \( k_0 \geq 0 \). Condition (26) was introduced in [U] in connection with a generalized version of eqs. (23), (25). That condition implies that there is a possibly larger value of \( \kappa_5 \) for which
\[
\kappa_5^{-1}(Q + k_0)^q \leq \rho(Q) \leq \kappa_5(Q + k_0)^q
\]
and
\[
|Q \rho'(Q)| \leq \kappa_5(Q + k_0)^q.
\]
In the sequel we denote by \( \kappa \) a number so large that it satisfies (26), (27), and (28). Condition (26) is an ellipticity condition for eqs. (23). If \( k = 0 \), then ellipticity degenerates as \( Q \) tends to zero; condition (27) implies that the density \( \rho \) also tends to zero (cavitates) in this limit. Thus ellipticity and noncavitation are equivalent under condition (26). In applications to compressible flow, the degeneration of ellipticity need not imply cavitation, and in cases in which these two phenomena are equivalent, as in the Chaplygin approximation, the degeneracy occurs at infinity rather than at zero. Moreover, condition (26) is not associated with a sonic transition. For these reasons, condition (26) does not appear to be appropriate for applications to fluid dynamics. However, it arises in certain natural generalizations of the Dirichlet energy; see [HL] and the references cited therein for details.

The methods used to study eq. (23) also apply to systems in which (23) is replaced by an equation of the form
\[
\delta (\rho(Q) \omega) = \xi(\omega),
\]
where \( p = 1 \) and \( \xi \) is a scalar function of \( \omega \) satisfying
\[
|\xi'(\omega)| \leq \kappa(Q + k)^\alpha
\]
for \( \alpha \in \mathbb{R}^+ \). For simplicity we take \( \alpha = q \). Obvious algebraic modifications lead to results analogous to inequality (34), Theorem 6, and Corollary 7 for general \( \alpha > 0 \). In that case, inequality (34) may no longer be linear in its terms of zero order.

Certain properties of eqs. (29), (24) can be obtained by deriving a differential inequality for an appropriate scalar function of the solution. The case \( \nu = \xi = 0 \) is due to Uhlenbeck [U], who framed the argument in the context of a broadly defined elliptic complex. We initially present a version of Uhlenbeck’s argument in simpler notation for solutions of (29) and (25), and then indicate how to modify the proof for the case of solutions of the system (29), (24).

Denote by \( H(Q) \) a \( C^1 \) function of \( Q \) such that
\[
H'(Q) = \frac{1}{2} \rho(Q) + Q \rho'(Q).
\]

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Then \([U]\)

\[
\langle \omega, \Delta (\rho(Q)\omega) \rangle = \partial_i \langle \omega, \partial_i (\rho(Q)\omega) \rangle - \langle \partial_i \omega, \partial_i (\rho(Q)\omega) \rangle \\
= \Delta H(Q) - [\rho(Q) \langle \partial_i \omega, \partial_i \omega \rangle + \rho'(Q) \langle \partial_i \omega, \omega \rangle \partial_i Q],
\]

where \(\partial_i = \partial/\partial x^i, x = x^1, \ldots, x^n \in \Omega,\) and

\[
\rho'(Q) \langle \partial_i \omega, \omega \rangle \partial_i Q = \sum_i 2\rho'(Q) \langle \partial_i \omega, \omega \rangle^2.
\]

We have [for either sign of \(\rho'(Q)\)]

\[
\rho(Q) \langle \partial_i \omega, \omega \rangle + \rho'(Q) \langle \partial_i \omega, \omega \rangle \partial_i Q \geq \kappa^{-1}(Q + k)^q |\nabla \omega|^2
\]

and

\[
\langle \omega, \Delta (\rho(Q)\omega) \rangle \leq \Delta H(Q) - \kappa^{-1}(Q + k)^q |\nabla \omega|^2.
\]

In addition,

\[
\langle \omega, \Delta (\rho(Q)\omega) \rangle = \langle \omega, \delta d (\rho(Q)\omega) \rangle + \langle \omega, d \delta (\rho(Q)\omega) \rangle = \langle \omega, \delta d (\rho(Q)\omega) \rangle + \langle \omega, d \xi \rangle
\]

for solutions of (29) and (25), yielding

\[
\kappa^{-1}(Q + k)^q |\nabla \omega|^2 \leq \Delta H(Q) - \langle \omega, \delta d (\rho(Q)\omega) \rangle - \langle \omega, d \xi \rangle.
\] (31)

Define a map \(\beta_\omega : \Lambda^0 \to \Lambda^{p+1}\) by the formula \(\beta_\omega : \zeta \to d\zeta \wedge \omega,\) for \(\zeta \in \Lambda^0\) and \(\omega \in \Lambda^p.\) If \(\nu = 0,\)

\[
\beta_\omega (\zeta) = d(\zeta \omega),
\]

but we do not use this property. Define the map \(\beta^*_\omega : \Lambda^{p+1} \to \Lambda^0\) by the formula

\[
\beta^*_\omega (\zeta) = \delta * (\omega \wedge * \zeta)
\]

for \(\zeta \in \Lambda^{p+1}.\) Writing

\[
\langle \omega, \delta d (\rho(Q)\omega) \rangle \equiv * [\omega \wedge * \delta d (\rho(Q)\omega)]
\]

\[
= * d [\omega \wedge * (\rho'(Q) dQ \wedge \omega)] = \delta * [\omega \wedge * (\rho'(Q) dQ \wedge \omega)]
\]

(c.f. Lemma 2.1.4 of [J]), we can write (31) in the form

\[
\kappa^{-1}(Q + k)^q |\nabla \omega|^2 \leq \Delta H(Q) - \beta^*_\omega \beta_\omega [\rho(Q)] - \langle \omega, d \xi \rangle.
\]

Because

\[
dQ = \frac{dH}{H'(Q)},
\]

we can rewrite this inequality, in terms of \(H,\) as

\[
\kappa^{-1}(Q + k)^q |\nabla \omega|^2 \leq \Delta H - \text{div} \left\{ \star \left[ \omega \wedge * \left( \frac{\rho'(Q)}{H'(Q)} dH \wedge \omega \right) \right] \right\}
\]

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\[
- \langle \omega, d\xi \rangle = \Delta H - \beta^* \varsigma^\omega(H) - \langle \omega, d\xi \rangle.
\]

for
\[
\varsigma^\omega(H) = \frac{\rho'(Q)}{H'(Q)} dH \wedge \omega.
\]

Write
\[
L_\omega(H) \equiv \Delta H - \beta^* \varsigma^\omega(H)
= \sum_{k,j} \partial_k (a_{kj} \partial_j) H.
\]

If \( \rho'(Q) \) is nonpositive, then the matrix \( a_{kj} \) satisfies
\[
1 \leq a_{kj} = 1 + \frac{Q |\rho'(Q)|}{H'(Q)} \leq 1 + \frac{2\kappa(Q + k)^q}{\kappa^{-1}(Q + k)^q}.
\]

Letting \( \pi = (\pi_1, \ldots, \pi_n) \) denote an \( n \)-vector, we have
\[
|\pi|^2 \leq \sum_{k,j} \pi_k a_{kj} \pi_j \leq (1 + 2\kappa)^2 |\pi|^2.
\]

If \( \rho'(Q) > 0 \), write
\[
div \left[ \left( 1 - \frac{Q \rho'(Q)}{H'(Q)} \right) \grad(H) \right] =
\]
\[
div \left[ \left( \frac{\rho(Q)}{H'(Q)} \right) \grad(H) \right]
= \frac{\rho(Q)}{2H'(Q)} \grad(H).
\]

The matrix \( a_{kj} \) now satisfies
\[
\frac{\kappa^{-1}}{2\kappa} \leq a_{kj} = \frac{\rho(Q)}{2H'(Q)} \leq \frac{2H'(Q)}{2H'(Q)} = 1.
\]

Letting \( \pi = (\pi_1, \ldots, \pi_n) \) denote an \( n \)-vector, we have
\[
\frac{|\pi|^2}{2\kappa^2} \leq \sum_{k,j} \pi_k a_{kj} \pi_j \leq |\pi|^2.
\]

Thus \( L \) is a uniformly elliptic operator on \( H \) for either sign of \( \rho'(Q) \).

It remains to estimate the lower-order nonlinear term \( \langle \omega, d\xi \rangle \) and to adjust for \( \nu \neq 0 \). We have
\[
\langle \omega, d\xi \rangle = \langle \omega, \xi' (\omega) d\omega \rangle = \xi' (\omega) \langle \omega, \nu \wedge \omega \rangle \geq
\]

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using (30) with \( \alpha = q \). Integrating condition (26) over a dummy variable in \([0, Q]\) and using \( H(0) = 0 \), we find that

\[
\kappa^{-1}(Q + k)^{q+1} \leq H(Q) \leq \kappa(Q + k)^{q+1}
\]

and obtain

\[
\langle \omega, d\xi \rangle \geq -\kappa^2 |\nu| H(Q).
\]

In the proof of Theorem 7 of [O1] it was shown that if \( \omega, \nu \) smoothly satisfies (23), (24), then there is an independent positive constant \( C \) and a sufficiently small constant \( \varepsilon(\kappa) \) for which

\[
0 \leq (\kappa^{-1} - \varepsilon\kappa)(Q + k)^q |\nabla \omega|^2
\]

\[
\leq \Delta H(Q) - *d [\omega \wedge * (\rho'(Q)dQ \wedge \omega)]
\]

\[
+ C(Q + k)^q (|\nabla \nu| + |\nu|^2) Q.
\]

We can convert this estimate to an inequality in \( H \), noticing first that

\[
(Q + k)^q (|\nabla \nu| + |\nu|^2) Q \leq \kappa (|\nabla \nu| + |\nu|^2) H(Q)
\]

by (32). Now taking into account the term \( -\langle \omega, d\xi \rangle \) and reasoning as in the curl-free case, we rewrite (33) in the form

\[
0 \leq L_\omega (H) + C(\kappa, q) \left(|\nabla \nu| + |\nu|^2 + t|\nu|\right) H,
\]

where

\[
L_\omega (H) = \Delta H - div \left\{ * \left[ \omega \wedge * \left( \frac{\rho'(Q)}{H'(Q)}dH \wedge \omega \right) \right] \right\}
\]

and \( t = 0 \) unless \( \xi \) is nonzero, in which case \( t = 1 \). This operator is clearly elliptic on \( H \), as we did not use the closure of \( \omega \) under \( d \) in establishing uniform ellipticity for the corresponding operator in the case \( \nu = 0 \).

Notice that the operator \( L_\omega (H) \) can be written as an operator on \( Q \) having the form

\[
\tilde{L}_\omega (Q) = \partial_h \left[ \left( \frac{1}{2} \rho(Q) + Q \rho'(Q) \right) \partial_h Q \right]
\]

\[- * d [\omega \wedge * (\rho'(Q)dQ \wedge \omega)].
\]

This operator is only elliptic on \( Q \) only if \( k \) exceeds zero. Thus for example, inequality (34) allows us to extend Corollary 8 of [O1], which was based on an elliptic inequality for \( \tilde{L}_\omega (Q) \). The bound on \( \omega \) established in that result is not uniform as \( Q \) tends to zero unless the constant \( k \) in condition (26) exceeds zero. We can remove that restriction if we place different \( L^p \) hypotheses on the solution. In comparison with the hypotheses of [O1], Corollary 8, the new \( L^p \) hypotheses placed directly on \( \omega \) are somewhat stronger, whereas those placed indirectly on \( \omega \), through the hypothesis on \( \nu \) and its derivatives, are considerably weaker.
Theorem 6 Let the pair $\omega, \nu$ smoothly satisfy eqs. (23), (24), with $\rho$ satisfying condition (26), on a domain $\Omega/\Sigma$. Let $\Omega$ be a type-A domain of $\mathbb{R}^n$, $n > 2$. Let $\Sigma$ be a compact singular set, completely contained in a sufficiently small $n$-disc $B$, which is itself completely contained in $\Omega$. If $n$ exceeds 4, let $2n/(n-2) < \mu < n$, where $\mu$ is the codimension of $\Sigma$, and let $\omega$ lie in $L^{n(q+1)}(B)$. If $n = 3$ or 4, let $\omega$ lie in $L^4(\beta)$ on $B$ for some $\beta$ exceeding $n/2$, then $\omega$ is bounded on compact subdomains of $\Omega$.

Proof. Integrate inequality (34) against the Serrin test function as in Lemma 3 of the preceding section. Using (32), the $L^p$ hypothesis on $\omega$ translates into $L^{(q+1)/2}$ hypotheses on $H$ which are sufficient for applying the arguments of Lemma 3 to $H$. These yield an integral inequality which can be iterated. After a finite number of iterations, we find that $H$ is in $L^p$ for all finite values of $P$ and is a weak $H^{1,2}$ subsolution on $B \cap \Sigma$. Theorem 5.3.1 of [Mo] implies that $H$ is bounded on compact subdomains of $\Omega$. Condition (26) extends this result to $Q$, and thus to $\omega$. This completes the proof of Theorem 6.

Corollary 7 Assume the conditions of Theorem 6, except let $\omega$ be a 1-form, replace eq. (23) with eq. (29), and let $\xi$ satisfy (30). Then the conclusion of Theorem 6 remains valid.

Proof. Clearly, $|\nabla v| + |v|^2 + |v| \in L^s(B)$ for some $s$ exceeding $n/2$. This completes the proof of Corollary 7.

Remark. We take this opportunity to correct a pair of misprints in the statement of Corollary 8 of [O1]: replace $L^{4q}(B)$ by $L^{4q_1}(B)$ and $1/2 < q_0 < q$ by $1/2 < q_0 < q_1$.

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