Abstract

In this paper, a multilevel Monte Carlo theta EM scheme is provided for stochastic differential delay equations with small noise. Under a global Lipschitz condition, the variance of two coupled paths is derived. Then, the global Lipschitz condition is replaced by one-sided Lipschitz condition, in order to guarantee the moment finiteness of numerical scheme, a modified multilevel Monte Carlo theta EM scheme is put forward and the second moment of two coupled paths is estimated.

Key words: multilevel Monte Carlo theta EM scheme; stochastic differential delay equations; small noise; global Lipschitz condition; one-sided Lipschitz condition

1 Introduction

Small noise stochastic differential equations (SDEs) are widely used in economics, finance, computational fluid dynamics, ecology, population dynamics and etc, and many customized numerical methods have been developed for small noise SDEs with the aim of improving efficiency [1 2 3]. The mostly used numerical methods are the Euler-Maruyama (EM) scheme, the Monte Carlo method, the Milstein method and the Runge-Kutta method. There are a lot of results for numerical schemes of SDEs under the global Lipschitz condition, see [4 5], etc. Since the global Lipschitz condition is too strong for most equations, more and more works on SDEs with the non-global Lipschitz conditions are established in recent years. For SDEs under the non-global Lipschitz condition, the numerical schemes may not reproduce the behaviour of exact solutions [6], or the moments of numerical solutions may
even explode in a finite time\cite{7}. Thus, the classical numerical schemes are modified or improved to guarantee the finiteness of numerical solutions or to improve efficiency under non-global Lipschitz conditions, for example the tamed EM scheme \cite{7}, the truncated EM scheme \cite{8}, the theta EM scheme \cite{9,10}, the tamed Milstein method \cite{11}, the multilevel Monte Carlo method \cite{12,13,14,15}.

In \cite{1}, the authors proposed a multilevel Monte Carlo EM method for stochastic differential equations with small noise, analyzed the variance between two coupled paths, and discovered that the computational complexity of multilevel Monte Carlo method combined with standard EM scheme was lower than the standard Monte Carlo. However, the results of \cite{1} are obtained under the global Lipschitz condition. If the global Lipschitz condition weakens to one-sided Lipschitz condition, will it remain the same property? Motivated by \cite{1}, we combine multilevel Monte Carlo method with the theta EM scheme and consider the variance between two coupled paths for stochastic differential delay equations (SDDEs) with small noise under global Lipschitz condition. Then, we replace the global Lipschitz condition by the one-sided Lipschitz condition, give a modified multilevel Monte Carlo theta EM scheme in order to guarantee the moment finiteness of the scheme. The second moment of two coupled paths is estimated under one-sided Lipschitz condition.

Throughout this paper, unless otherwise specified, we let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t\geq 0}\) satisfying the usual conditions. Let \(W(t) = (W_1(t), \ldots, W_d(t))^T\) be an \(d\)-dimensional Brownian motion defined on the probability space. Let \(\tau > 0\) be a delay. Consider the stochastic differential delay equation with small noise of the form

\[
dX^\varepsilon(t) = f(X^\varepsilon(t), X^\varepsilon(t-\tau))dt + \varepsilon g(X^\varepsilon(t), X^\varepsilon(t-\tau))dW(t), t \geq 0 \tag{1.1}
\]

with initial data \(X(\theta) = \xi(\theta), \theta \in [-\tau, 0]\), where \(\varepsilon \in (0, 1)\) and

\[
f : \mathbb{R}^a \times \mathbb{R}^a \to \mathbb{R}^a \quad \text{and} \quad g : \mathbb{R}^a \times \mathbb{R}^a \to \mathbb{R}^{a \times d}.
\]

In the following, we will analyze multilevel Monte Carlo EM solution of \cite{1.1} under the global Lipschitz condition and one-sided Lipschitz condition respectively.

\section{SDDEs with Global Lipschitz Condition}

We shall impose the following hypothesis:

\textbf{(H)} Both \(f\) and \(g\) satisfy the global Lipschitz condition. That is, there exists an \(\alpha > 1\) such that

\[
|f(x, y) - f(\bar{x}, \bar{y})| + |g(x, y) - g(\bar{x}, \bar{y})| \leq \alpha(|x - \bar{x}| + |y - \bar{y}|)
\]

for all \(x, y, \bar{x}, \bar{y} \in \mathbb{R}^a\). Moreover, for all \(x, y \in \mathbb{R}^a\)

\[
|\nabla f(x, y)|^2 \vee |\nabla^2 f(x, y)|^2 \leq \alpha.
\]
Lemma 2.1  Let assumption (H) hold. Then, for any $T > 0$ and $p \geq 2$, we have

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^\varepsilon(t)|^p \right] \leq C.
$$

Remark 2.1  Assumption (H) implies the existence and uniqueness of equation (1.1). Moreover, if (H) holds, then for any $x, y \in \mathbb{R}^a$

$$
|f(x, y)| + |g(x, y)| \leq \beta(1 + |x| + |y|)
$$

where $\beta = \max\{\alpha, |f(0, 0)|, |g(0, 0)|\}$, and for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^a$

$$
\langle x - \bar{x}, f(x, y) - f(\bar{x}, \bar{y}) \rangle \leq \bar{\alpha}(|x - \bar{x}|^2 + |y - \bar{y}|^2)
$$

where $\bar{\alpha} = \frac{1}{2} + \alpha^2$.

2.1 The theta EM Scheme

We now introduce theta EM scheme for (1.1). Given any time $T > 0$, assume that there exist two positive integers such that $h = \frac{T}{M} = \frac{\tau}{Nh}$, where $h \in (0, 1)$ is the step size. For $n = -m, \cdots, 0$, set $X^\varepsilon_h(t_n) = \xi(nh)$; For $n = 0, 1, \cdots, M - 1$, we form

$$
X^\varepsilon_h(t_{n+1}) = X^\varepsilon_h(t_n) + \theta f(X^\varepsilon_h(t_{n+1}), X^\varepsilon_h(t_{n+1-m}))(h) + (1 - \theta) f(X^\varepsilon_h(t_n), X^\varepsilon_h(t_{n-m}))(h) + \varepsilon g(X^\varepsilon_h(t_n), X^\varepsilon_h(t_{n-m}))(\Delta W(t_n)),
$$

(2.1)

where $t_n = nh$, $\Delta W(t_n) = W(t_{n+1}) - W(t_n)$. Here $\theta \in [0, 1]$ is an additional parameter that allows us to control the implicitness of the numerical scheme. For $\theta = 0$, the theta EM scheme reduces to the EM scheme, and for $\theta = 1$, it is exactly the backward EM scheme. For a given $X^\varepsilon_h(t_n)$, in order to guarantee a unique solution $X^\varepsilon_h(t_{n+1})$ to (2.1), the step size is required to satisfy $\theta h < \frac{1}{\alpha}$ according to the monotone operator [16], where $\bar{\alpha}$ is defined as in Remark 2.1. In addition, to guarantee the moment finiteness of numerical solutions, we also require $h\theta < \frac{1}{\alpha^2}$ in this section. Thus, in Section 2, we set $h^* \in \left(0, \frac{1}{\bar{\alpha}(\alpha + 2\beta)}\right)$, and let $h, h_t \in (0, h^*)$ for $\theta \in (0, 1]$, while for $\theta = 0$, we only need $h, h_t \in (0, 1)$, where $h_t$ is a step size defined in Section 2.2.

We find it is convenient to work with a continuous form of a numerical method. Rewrite (2.1) with a continuous form as follows:

$$
X^\varepsilon_h(t) - \theta f(X^\varepsilon_h(t), X^\varepsilon_h(t - \tau))(h) = \xi(0) - \theta f(\xi(0), \xi(-\tau))(h)

+ \int_0^t f(X^\varepsilon_h(\eta_h(s), X^\varepsilon_h(s - \tau)))(s) + \varepsilon \int_0^t g(X^\varepsilon_h(\eta_h(s), X^\varepsilon_h(s - \tau)))(s) dW(s),
$$

(2.2)

where $\eta_h(s) = \lfloor s/h \rfloor h$.

Lemma 2.2  Let assumption (H) hold. Then, for any $T > 0$ and $p \geq 2$, we have

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^\varepsilon_h(t)|^p \right] \leq C.
$$
Proof. Denote \(Y_h^\varepsilon(t) = X_h^\varepsilon(t) - \theta f(X_h^\varepsilon(t), X_h^\varepsilon(t - \tau))h\). For any \(t \in [0, T]\), by (2.2) and the Burkholder-Davis-Gundy (BDG) inequality, we get

\[
\mathbb{E}\left[\sup_{0 \leq s \leq t} |Y_h^\varepsilon(s)|^p\right] \leq C|Y_h^\varepsilon(0)|^p + C t^{p-1} \mathbb{E}\left[\int_0^t |f(X_h^\varepsilon(\eta_h(s)), X_h^\varepsilon(\eta_h(s - \tau)))|^p ds\right]
\]

\[
+ C t^{p-1} \mathbb{E}\left[\int_0^t |g(X_h^\varepsilon(\eta_h(u)), X_h^\varepsilon(\eta_h(u - \tau)))| dW(u)\right]^p
\]

(2.3)

By the discrete Gronwall inequality, we have

\[
\mathbb{E}\left[\sup_{\nu \leq n} |Y_h^\varepsilon(\nu h)|^p\right] \leq C + C \sum_{i=0}^{n-1} \mathbb{E}\left[\sup_{\nu \leq i} |X_h^\varepsilon(\nu h)|^p\right] h.
\]

Since \(\theta h < \frac{1}{63}\), by \(|x - y|^p \geq 2^{1-p}|x|^p - |y|^p\), we have

\[
|Y_h^\varepsilon(\nu h)|^p \geq 2^{1-p}|X_h^\varepsilon(\nu h)|^p - 3^{p-1}\beta\theta h^p(1 + |X_h^\varepsilon(\nu h)|^p + |X_h^\varepsilon(\nu h - mh)|^p),
\]

which implies that

\[
\mathbb{E}\left[\sup_{\nu \leq n} |X_h^\varepsilon(\nu h)|^p\right] \leq C + C \mathbb{E}\left[\sup_{\nu \leq n} |Y_h^\varepsilon(\nu h)|^p\right]
\]

(2.4)

\[
\leq C + C \sum_{i=0}^{n-1} \mathbb{E}\left[\sup_{\nu \leq i} |X_h^\varepsilon(\nu h)|^p\right] h.
\]

By the discrete Gronwall inequality,

\[
\mathbb{E}\left[\sup_{\nu \leq n} |X_h^\varepsilon(\nu h)|^p\right] \leq C.
\]

(2.5)

Furthermore, by (2.3) and (2.5),

\[
\mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_h^\varepsilon(t)|^p\right] \leq C + C T \mathbb{E}\left[|X_h^\varepsilon(\eta_h(s))|^p + |X_h^\varepsilon(\eta_h(s - \tau))|^p\right] ds \leq C
\]

In the same way as (2.4), we derive

\[
\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_h^\varepsilon(t)|^p\right] \leq C + C \mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_h^\varepsilon(t)|^p\right] \leq C.
\]

This completes the proof. \(\square\)
Lemma 2.3 Let assumption (H) hold. Then, for any \( p \geq 2 \), we have
\[
\sup_{0 \leq n \leq M-1} \mathbb{E}[|X^\varepsilon_h(t_{n+1}) - X^\varepsilon_h(t_n)|^p] \leq Ch^p + C\varepsilon^p h^{p/2}.
\]

Proof. Use the same notation \( Y^\varepsilon_h(t) \) as in Lemma 2.2. We derive from (2.2), assumption (H), Lemma 2.2 and the BDG inequality that for \( p \geq 2 \)
\[
\begin{align*}
\mathbb{E}|Y^\varepsilon_h(t_{n+1}) - Y^\varepsilon_h(t_n)|^p &\leq 2^{p-1}h^{p-1} \int_{t_n}^{t_{n+1}} \left| f(X^\varepsilon_h(\eta_h(s)), X^\varepsilon_h(\eta_h(s - \tau))) \right|^p ds \\
&\quad + \varepsilon^p h^{p-1} \int_{t_n}^{t_{n+1}} \left| g(X^\varepsilon_h(\eta_h(s)), X^\varepsilon_h(\eta_h(s - \tau))) \right|^p ds \\
&\leq Ch^p + C\varepsilon^p h^{p/2}.
\end{align*}
\]

With the relationship between \( X^\varepsilon_h(t) \) and \( Y^\varepsilon_h(t) \), we obtain
\[
X^\varepsilon_h(t_{n+1}) - X^\varepsilon_h(t_n) = Y^\varepsilon_h(t_{n+1}) - Y^\varepsilon_h(t_n) + \theta f(X^\varepsilon_h(t_{n+1}), X^\varepsilon_h(t_{n+1} - \tau)) h \\
- \theta f(X^\varepsilon_h(t_n), X^\varepsilon_h(t_n - \tau)) h.
\]

By Lemma 2.2 it is easy to show that
\[
\mathbb{E}|X^\varepsilon_h(t_{n+1}) - X^\varepsilon_h(t_n)|^p \leq C\mathbb{E}|Y^\varepsilon_h(t_{n+1}) - Y^\varepsilon_h(t_n)|^p + Ch^p \leq Ch^p + C\varepsilon^p h^{p/2}.
\]

We now reveal the error between the numerical solution (2.2) and the exact solution (1.1).

Theorem 2.1 Let assumption (H) hold, assume that \( \Psi : \mathbb{R}^a \to \mathbb{R} \) has continuous second order derivative and there exists a constant \( C \) such that
\[
\left| \frac{\partial \Psi}{\partial x_i} \right| \leq C \quad \text{and} \quad \left| \frac{\partial^2 \Psi}{\partial x_i \partial x_j} \right| \leq C
\]
for any \( i, j = 1, 2, \ldots, a \). Then, we have
\[
\sup_{0 \leq t \leq T} \mathbb{E} |\Psi(\varepsilon(t)) - \Psi(X^\varepsilon_h(t))|^2 = O(h^2 + \varepsilon^2 h).
\]

Proof. Set \( I(t) = X^\varepsilon_h(t) - X^\varepsilon(t) - \theta f(X^\varepsilon_h(t), X^\varepsilon_h(t - \tau)) h \), then
\[
I(t) = I(0) + \int_0^t [f(X^\varepsilon_h(\eta_h(s)), X^\varepsilon_h(\eta_h(s - \tau))) - f(\varepsilon(s), \varepsilon(s - \tau))] ds \\
+ \varepsilon \int_0^t [g(X^\varepsilon_h(\eta_h(s)), X^\varepsilon_h(\eta_h(s - \tau))) - g(\varepsilon(s), \varepsilon(s - \tau))] dW(s),
\]
where \( I(0) = -\theta f(\xi(0),\xi(-\tau))h \). By the assumption (H) and Lemma 2.3, we see
\[
\sup_{0 \leq t \leq T} \mathbb{E}|I(t)|^2 \leq 3|I(0)|^2 + 3T\mathbb{E} \int_0^T |f(X_h^\varepsilon(\eta_h(s)), X_h^\varepsilon(\eta_h(s - \tau))) - f(X^\varepsilon(s), X^\varepsilon(s - \tau))|^2 ds \\
+ C\varepsilon^2 \mathbb{E} \int_0^T |g(X_h^\varepsilon(\eta_h(s)), X_h^\varepsilon(\eta_h(s - \tau))) - g(X^\varepsilon(s), X^\varepsilon(s - \tau))|^2 ds \\
\leq Ch^2 + C\varepsilon^2 h + C\varepsilon^2 \int_0^T \sup_{0 \leq t \leq s} \mathbb{E}|X^\varepsilon(t) - X_h^\varepsilon(t)|^2 ds + C\varepsilon^2 h^2 + C\varepsilon^4 h.
\]
\tag{2.6}

Since \(|x - y|^p \geq \frac{1}{2}|x|^p - |y|^p\), we get
\[
|I(t)|^2 \geq \frac{1}{2} |X^\varepsilon(t) - X_h^\varepsilon(t)|^2 - |\theta f(X_h^\varepsilon(t), X_h^\varepsilon(t - \tau)) h|^2
\]
We then derive from Lemma 2.2 (2.6) and the Gronwall inequality that
\[
\sup_{0 \leq t \leq T} \mathbb{E}|X^\varepsilon(t) - X_h^\varepsilon(t)|^2 \leq Ch^2 + \varepsilon^2 h.
\]
Since \( \Psi \) has continuous bounded first order derivative, we immediately get
\[
\sup_{0 \leq t \leq T} \mathbb{E}|\Psi(X^\varepsilon(t)) - \Psi(X_h^\varepsilon(t))|^2 \leq C \sup_{0 \leq t \leq T} \mathbb{E}|X^\varepsilon(t) - X_h^\varepsilon(t)|^2.
\]
The desired result then follows. \(\square\)

**Corollary 2.1** Assume that the conditions of Theorem 2.1 hold. Let \( M \geq 2, l \geq 1, h_l = T \cdot M^{-l}, h_{l-1} = T \cdot M^{-(l-1)} \). Then
\[
\sup_{0 \leq n < M^{l-1}} \text{Var}(\Psi(X_{h_l}^\varepsilon(t_n)) - \Psi(X_{h_{l-1}}^\varepsilon(t_n))) \leq Ch_{l-1}^2 + C\varepsilon^2 h_{l-1}.
\]
**Proof.** For \( 0 \leq n \leq M^{l-1} - 1 \), by Theorem 2.1
\[
\text{Var}(\Psi(X_{h_l}^\varepsilon(t_n)) - \Psi(X_{h_{l-1}}^\varepsilon(t_n))) \leq \mathbb{E}|\Psi(X_{h_l}^\varepsilon(t_n)) - \Psi(X_{h_{l-1}}^\varepsilon(t_n))|^2 \\
\leq 2\mathbb{E}|\Psi(X_{h_l}^\varepsilon(t_n)) - \Psi(X_{h_{l-1}}^\varepsilon(t_n))|^2 + 2\mathbb{E}|\Psi(X_{h_l}^\varepsilon(t_n)) - \Psi(X_{h_{l-1}}^\varepsilon(t_n))|^2 \\
\leq Ch_{l-1}^2 + C\varepsilon^2 h_{l-1}.
\]
\( \square \)

### 2.2 The Multilevel Monte Carlo theta EM Scheme

We now define the multilevel Monte Carlo theta EM scheme. Given any \( T > 0 \), let \( M \geq 2, l \geq 1, h_l = T \cdot M^{-l}, h_{l-1} = T \cdot M^{-(l-1)} \), assume there exists an \( m_l \) such that \( \tau = m_l h_l \). Let
\[
X_{h_l}^\varepsilon(t) - \theta f(X_{h_l}^\varepsilon(t), X_{h_l}^\varepsilon(t - \tau)) h_l \\
= \xi(0) - \theta f(\xi(0),\xi(-\tau)) h_l + \int_0^t f(X_{h_l}^\varepsilon(\eta_{h_l}(s)), X_{h_l}^\varepsilon(\eta_{h_l}(s - \tau))) ds \\
+ \varepsilon \int_0^t g(X_{h_l}^\varepsilon(\eta_{h_l}(s)), X_{h_l}^\varepsilon(\eta_{h_l}(s - \tau)))dW(s),
\]
\tag{2.7}
\[
X_{h_{t-1}}^\varepsilon(t) - \theta f(X_{h_{t-1}}^\varepsilon(t), X_{h_{t-1}}^\varepsilon(t - \tau))h_{t-1}
\]
\[
= \xi(0) - \theta f(\xi(0), \xi(-\tau))h_{t-1} + \int_0^t f(X_{h_{s-1}}^\varepsilon(\eta_{h_{s-1}}(s)), X_{h_{s-1}}^\varepsilon(\eta_{h_{s-1}}(s - \tau)))ds
\]
\[
+ \varepsilon \int_0^t g(X_{h_{s-1}}^\varepsilon(\eta_{h_{s-1}}(s)), X_{h_{s-1}}^\varepsilon(\eta_{h_{s-1}}(s - \tau)))dW(s),
\]
where \(\eta_{h_{s}}(s) = \lfloor s/h_{s} \rfloor h_{s}\). Here \(\theta \in [0, 1]\) is a parameter to control the implicitness. For \(n \in \{0, 1, \ldots, M^{l-1} - 1\}\) and \(k \in \{0, \ldots, M\}\), let
\[
t_n = nh_{t-1} \quad \text{and} \quad t_n^k = nh_{t-1} + kh_{t}.
\]
This means we divide the interval \([t_n, t_{n+1}]\) into \(M\) equal parts, we have \(t_n^0 = t_n, t_n^M = t_{n+1}\). We can rewrite (2.7) and (2.8) as the following discretization schemes. For \(n \in \{0, 1, \ldots, M^{l-1} - 1\}\) and \(k \in \{0, \ldots, M - 1\}\), let
\[
X_{h_{n}}^\varepsilon(t_{n+1}) - \theta f(X_{h_{n}}^\varepsilon(t_{n+1}), X_{h_{n}}^\varepsilon(t_{n+1} - m_{h_{n}})))h_{l}
\]
\[
=X_{h_{n}}^\varepsilon(t_{n}) - \theta f(X_{h_{n}}^\varepsilon(t_{n}), X_{h_{n}}^\varepsilon(t_{n} - m_{h_{n}})))h_{l} + \sum_{k=0}^{M-1} f(X_{h_{n}}^\varepsilon(t_{n}^k), X_{h_{n}}^\varepsilon(t_{n}^k - m_{h_{n}})))h_{l}
\]
\[
+ \varepsilon \sqrt{h_{l}}g(X_{h_{n}}^\varepsilon(t_{n}^k), X_{h_{n}}^\varepsilon(t_{n}^k - m_{h_{n}}))\xi_{n}^k,
\]
where the random vector \(\xi_{n}^k \in \mathbb{R}^d\) has independent components, and each component is distributed as \(N(0, 1)\). This implies
\[
X_{h_{n}}^\varepsilon(t_{n+1}) - \theta f(X_{h_{n}}^\varepsilon(t_{n+1}), X_{h_{n}}^\varepsilon(t_{n+1} - m_{h_{n}})))h_{l}
\]
\[
=X_{h_{n}}^\varepsilon(t_{n}) - \theta f(X_{h_{n}}^\varepsilon(t_{n}), X_{h_{n}}^\varepsilon(t_{n} - m_{h_{n}})))h_{l} + \sum_{k=0}^{M-1} f(X_{h_{n}}^\varepsilon(t_{n}^k), X_{h_{n}}^\varepsilon(t_{n}^k - m_{h_{n}})))h_{l}
\]
\[
+ \varepsilon \sqrt{h_{l}}\sum_{k=0}^{M-1} g(X_{h_{n}}^\varepsilon(t_{n}^k), X_{h_{n}}^\varepsilon(t_{n}^k - m_{h_{n}}))\xi_{n}^k.
\]
To simulate \(X_{h_{t-1}}^\varepsilon\), we use
\[
X_{h_{t-1}}^\varepsilon(t_{n+1}) - \theta f(X_{h_{t-1}}^\varepsilon(t_{n+1}), X_{h_{t-1}}^\varepsilon(t_{n+1} - m_{h_{t-1}})))h_{t-1}
\]
\[
=X_{h_{t-1}}^\varepsilon(t_{n}) - \theta f(X_{h_{t-1}}^\varepsilon(t_{n}), X_{h_{t-1}}^\varepsilon(t_{n} - m_{h_{t-1}})))h_{t-1} + \sum_{k=0}^{M-1} f(X_{h_{t-1}}^\varepsilon(t_{n}^k), X_{h_{t-1}}^\varepsilon(t_{n}^k - m_{h_{t-1}})))h_{t-1}
\]
\[
+ \varepsilon \sqrt{h_{t-1}}g(X_{h_{t-1}}^\varepsilon(t_{n}^k), X_{h_{t-1}}^\varepsilon(t_{n}^k - m_{h_{t-1}}))\sum_{k=0}^{M-1} \xi_{n}^k.
\]
For convenience, let
\[
Y_{h_{t}}^\varepsilon(t) := X_{h_{t}}^\varepsilon(t) - \theta f(X_{h_{t}}^\varepsilon(t), X_{h_{t}}^\varepsilon(t - m_{h_{t}})))h_{t},
\]
and
\[
Y_{h_{t-1}}^\varepsilon(t) := X_{h_{t-1}}^\varepsilon(t) - \theta f(X_{h_{t-1}}^\varepsilon(t), X_{h_{t-1}}^\varepsilon(t - m_{h_{t}})))h_{t-1}.
\]
We now have the following estimates.
Lemma 2.4 Let assumption (H) hold. Then, for any $T > 0$ and $p \geq 2$, we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_h^\varepsilon(t)|^p \right] \leq C,
\]
and
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_{h_{i-1}}^\varepsilon(t)|^p \right] \leq C.
\]

Proof. We omit the proof here since it is similar to that of Lemma 2.2. □

Let $Z_h$ be the deterministic solution to
\[
Z_h(t) - \theta f(Z_h(t), Z_h(t-\tau))h = \xi(0) - \theta f(\xi(0), \xi(-\tau))h + \int_0^t f(Z_h(\eta_h(s)), Z_h(\eta_h(s-\tau)))ds,
\]
which is the theta EM approximation to the ordinary differential delay equation obtained from (1.1) by taking $\varepsilon = 0$.

Lemma 2.5 Let assumption (H) hold. Then, for any $T > 0$ and $p \geq 2$, we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Z_h(t)|^p \right] \leq C,
\]
and
\[
\sup_{0 \leq n < M^{t-1}, 1 \leq k \leq M} \mathbb{E} |Z_{hi}(t_n^k) - Z_{hi}(t_n)|^p \leq CM_p h_t^p.
\]

Proof. Following the proof of Lemma 2.2 under global Lipschitz condition (H), the first part is obvious. Denote by $\tilde{Z}_{hi}(t) = Z_{hi}(t) - \theta f(Z_{hi}(t), Z_{hi}(t-\tau))h$. By the result of the first part,
\[
\mathbb{E} |\tilde{Z}_{hi}(t_n^k) - Z_{hi}(t_n)|^p \leq |kh|^p \mathbb{E} \int_{t_n}^{t_n^k} |f(Z_{hi}(\eta_{hi}(s)), Z_{hi}(\eta_{hi}(s-\tau)))|^p ds \leq CM_p h_t^p.
\]
On the other side, we see
\[
\mathbb{E} |Z_{hi}(t_n^k) - Z_{hi}(t_n)|^p \leq C \mathbb{E} |\tilde{Z}_{hi}(t_n^k) - \tilde{Z}_{hi}(t_n)|^p + CM_p h_t^p.
\]
Thus, the desired assertion follows. □

Lemma 2.6 Let assumption (H) hold. Then, for any $T > 0$ and $p \geq 2$, we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_{hi}^\varepsilon(t) - Z_{hi}(t)|^p \right] \leq C\varepsilon^p,
\]
and
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_{hi_{i-1}}^\varepsilon(t) - Z_{hi_{i-1}}(t)|^p \right] \leq C\varepsilon^p.
\]
Proof. Use the notation $\bar{Z}_{h_t}(t)$ defined in Lemma 2.3. By the definition of $Y_{h_t}(t)$ and $\bar{Z}_{h_t}(t)$,

$$Y_{h_t}(t) - Z_{h_t}(t) = \int_0^t [f(X_{h_t}^\varepsilon(\eta_{h_t}(s)), X_{h_t}^\varepsilon(\eta_{h_t}(s - \tau))) - f(Z_{h_t}(\eta_{h_t}(s)), Z_{h_t}(\eta_{h_t}(s - \tau)))]ds
$$

$$+ \varepsilon \int_0^t g(X_{h_t}^\varepsilon(\eta_{h_t}(s)), X_{h_t}^\varepsilon(\eta_{h_t}(s - \tau)))dW(s),$$

thus, by the BDG inequality, we get

$$E \left[ \sup_{0 \leq s \leq t} |Y_{h_t}^\varepsilon(s) - \bar{Z}_{h_t}(s)|^p \right]$$

$$\leq C \int_0^t |f(X_{h_t}^\varepsilon(\eta_{h_t}(s)), X_{h_t}^\varepsilon(\eta_{h_t}(s - \tau))) - f(Z_{h_t}(\eta_{h_t}(s)), Z_{h_t}(\eta_{h_t}(s - \tau)))]^pds$$

(2.12)

$$+ C\varepsilon^p \int_0^t |g(X_{h_t}^\varepsilon(\eta_{h_t}(s)), X_{h_t}^\varepsilon(\eta_{h_t}(s - \tau)))|^pds.$$ 

Let $t = nh_t$ and $s = \bar{n}h_t$, where $n$ and $\bar{n}$ are nonnegative integers such that $\bar{n}h_t \leq nh_t \leq T$, then by assumption (H),

$$E \left[ \sup_{\bar{n} \leq n} |Y_{h_t}^\varepsilon(\bar{n}h_t) - \bar{Z}_{h_t}(\bar{n}h_t)|^p \right] \leq C \sum_{i=0}^{n-1} E \left[ \sup_{\bar{n} \leq i} |X_{h_t}^\varepsilon(\bar{n}h_t) - Z_{h_t}(\bar{n}h_t)|^p \right] h_t$$

(2.13)

$$+ C\varepsilon^p + C\varepsilon^p E \left[ \sup_{0 \leq s \leq t} |X_{h_t}^\varepsilon(s)|^p \right].$$

By using $|x - y|^p \geq 2^{1-p}|x|^p - |y|^p$ and assumption (H) again, we see

$$E \left[ \sup_{\bar{n} \leq n} |X_{h_t}^\varepsilon(\bar{n}h_t) - Z_{h_t}(\bar{n}h_t)|^p \right] \leq C E \left[ \sup_{\bar{n} \leq n} |Y_{h_t}^\varepsilon(\bar{n}h_t) - \bar{Z}_{h_t}(\bar{n}h_t)|^p \right],$$

then, Lemma 2.4 and (2.13) give that

$$E \left[ \sup_{\bar{n} \leq n} |X_{h_t}^\varepsilon(\bar{n}h_t) - Z_{h_t}(\bar{n}h_t)|^p \right]$$

$$\leq C \sum_{i=0}^{n-1} E \left[ \sup_{\bar{n} \leq i} |X_{h_t}^\varepsilon(\bar{n}h_t) - Z_{h_t}(\bar{n}h_t)|^p \right] h_t + C\varepsilon^p.$$

The discrete Gronwall inequality leads to

$$E \left[ \sup_{\bar{n} \leq n} |X_{h_t}^\varepsilon(\bar{n}h_t) - Z_{h_t}(\bar{n}h_t)|^p \right] \leq C\varepsilon^p.$$

Furthermore, with assumption (H), we derive from (2.12) and Lemma 2.4 that

$$E \left[ \sup_{0 \leq s \leq T} |Y_{h_t}^\varepsilon(s) - \bar{Z}_{h_t}(s)|^p \right] \leq C\varepsilon^p.$$

Then, the first part follows by using the relationship between $Y_{h_t}^\varepsilon(t)$ and $X_{h_t}^\varepsilon(t)$ together with Lemma 2.4. By the same technique, the second part can be verified. □
Lemma 2.7 Let assumption (H) hold. Then, we have
\[
\sup_{0 \leq n < M^{l-1}, 1 \leq k \leq M} |\mathbb{E}[X^\varepsilon_{h_i}(t^k_n) - X^\varepsilon_{h_i}(t_n)]| \leq CMh_l.
\]

Proof. By (2.9), for \( k \in \{1, 2, \ldots, M\} \),
\[
Y^\varepsilon_{h_i}(t^k_n) = Y^\varepsilon_{h_i}(t^0_n) + \sum_{q=0}^{k-1} f(X^\varepsilon_{h_i}(t^q_n), X^\varepsilon_{h_i}(t^q_n - m_lh_l))h_l
\]
\[+ \varepsilon \sqrt{h_l} \sum_{q=0}^{k-1} g(X^\varepsilon_{h_i}(t^q_n), X^\varepsilon_{h_i}(t^q_n - m_lh_l))\xi^q_n. \tag{2.14}
\]

Taking expectation on both sides, together with Lemma 2.4, yields
\[
|\mathbb{E}[Y^\varepsilon_{h_i}(t^k_n) - Y^\varepsilon_{h_i}(t^0_n)]| \leq \left| \mathbb{E} \left[ \sum_{q=0}^{k-1} f(X^\varepsilon_{h_i}(t^q_n), X^\varepsilon_{h_i}(t^q_n - m_lh_l))h_l \right] \right|
\]
\[+ \left| \mathbb{E} \left[ \varepsilon \sqrt{h_l} \sum_{q=0}^{k-1} g(X^\varepsilon_{h_i}(t^q_n), X^\varepsilon_{h_i}(t^q_n - m_lh_l))\xi^q_n \right] \right| \tag{2.15}
\]
\[\leq Ch_l \sum_{q=0}^{k-1} \left( 1 + \mathbb{E}|X^\varepsilon_{h_i}(t^q_n)| + \mathbb{E}|X^\varepsilon_{h_i}(t^q_n - m_lh_l)| \right)
\]
\[\leq CMh_l.
\]

Since we have
\[
X^\varepsilon_{h_i}(t^k_n) - X^\varepsilon_{h_i}(t_n) = Y^\varepsilon_{h_i}(t^k_n) - Y^\varepsilon_{h_i}(t^0_n) + \theta f(X^\varepsilon_{h_i}(t^k_n), X^\varepsilon_{h_i}(t^k_n - m_lh_l))h_l
\]
\[- \theta f(X^\varepsilon_{h_i}(t^0_n), X^\varepsilon_{h_i}(t^0_n - m_lh_l))h_l. \tag{2.16}
\]

Combining (2.15) and (2.16), it is easy to show the desired result by Lemma 2.4. \( \square \)

Lemma 2.8 Let assumption (H) hold. Then, for any \( p > 0 \), we have
\[
\sup_{0 \leq n < M^{l-1}, 1 \leq k \leq M} \mathbb{E}[|X^\varepsilon_{h_i}(t^k_n) - X^\varepsilon_{h_i}(t_n)|^p] \leq CM^p h^p_l + C\varepsilon^p M^{p/2} h^{p/2}_l.
\]

Proof. We derive from (2.14), assumption (H), Lemma 2.4 and the discrete BDG inequality.
Lemma 2.9 Let the Young inequality to get the results for $A$ where

$\text{Proof.}$ Let Lemma 2.4 again, we conclude that the desired result follows for $p$.

Taylor expansion of the drift coefficient.

that for $p \geq 2$

$$
\mathbb{E}|Y_{h_i}(t_n^k) - Y_{h_i}(t_0^k)|^p \leq 2^{p-1} M^{p-1} h_t^p \sum_{q=0}^{k-1} \mathbb{E}|f(X_{h_i}(t_n^q), X_{h_i}(t_n^q - m_i h_t))|^p 
$$

$$
+ 2^{p-1} \varepsilon h_t^{p/2} \mathbb{E} \left| \sum_{q=0}^{k-1} f(X_{h_i}(t_n^q), X_{h_i}(t_n^q - m_i h_t)) \xi_{n}^{q} \right|^{p} 
$$

$$
\leq CM^{p} h_{t}^{p} + C \varepsilon h_{t}^{p/2} \mathbb{E} \left\| \sum_{q=0}^{k-1} \sum_{i=1}^{d} |g^{i}(X_{h_i}(t_n^q), X_{h_i}(t_n^q - m_i h_t))|^{2} \right\|^{p/2} 
$$

$$
\leq CM^{p} h_{t}^{p} + C \varepsilon M^{p/2} h_{t}^{p/2} \left( 1 + \sup_{0 \leq q \leq M} \mathbb{E}|X_{h_i}(t_n^q)|^{p} \right) 
$$

$$
\leq CM^{p} h_{t}^{p} + C \varepsilon M^{p/2} h_{t}^{p/2},
$$

where $g^{i}$ is the $i$-th column of $g$ and in the last step we have used Lemma 2.4. By (2.16) and Lemma 2.4 again, we conclude that the desired result follows for $p \geq 2$. Finally, one can use the Young inequality to get the results for $p \in (0, 2)$.

Taylor expansion of the drift coefficient.

Lemma 2.9 Let $\nabla$ and $\nabla^2$ be the first and second order derivatives respectively. Then

$$
f(X_{h_i}(t_n^k), X_{h_i}(t_n^k - m_i h_t)) - f(X_{h_i}(t_n), X_{h_i}(t_n - m_i h_t)) = A_n^k + B_n^k + C_n^k,
$$

where $A_n^k, B_n^k, C_n^k$ are defined as in the proof.

**Proof.** Let $f_i(x)$ be the $i$th component of $f(x)$. By the Taylor expansion, for $i = 1, 2, \cdots, a$,

$$
f_i(X_{h_i}(t_n^k), X_{h_i}(t_n^k - m_i h_t)) - f_i(X_{h_i}(t_n), X_{h_i}(t_n - m_i h_t)) 
$$

$$
= \int_{0}^{1} \nabla f_i((X_{h_i}(t_n^k), X_{h_i}(t_n - m_i h_t))) + s((X_{h_i}(t_n^k), X_{h_i}(t_n - m_i h_t)) 
$$

$$
- (X_{h_i}(t_n), X_{h_i}(t_n - m_i h_t))]ds \times \left( X_{h_i}(t_n^k) - X_{h_i}(t_n) \right) \left( X_{h_i}(t_n^k - m_i h_t) - X_{h_i}(t_n - m_i h_t) \right) 
$$

$$
= \int_{0}^{1} \nabla f_i((X_{h_i}(t_n), X_{h_i}(t_n - m_i h_t)) + s((X_{h_i}(t_n^k), X_{h_i}(t_n^k - m_i h_t)) 
$$

$$
- (X_{h_i}(t_n), X_{h_i}(t_n - m_i h_t))]ds \times \left( \frac{\sigma_{11}^{13} + \sigma_{12}^{13} + \sigma_{13}^{13}}{\sigma_{21}^{13} + \sigma_{22}^{13} + \sigma_{23}^{13}} \right),
$$

11
where
\[
\sigma_n^{11} = \sum_{q=0}^{k-1} f(X_t^n(t_n^q), X_t^n(t_n^q - m_1 h_1)) h_1,
\]
\[
\sigma_n^{12} = \varepsilon \sqrt{h_i} \sum_{q=0}^{k-1} g(X_t^n(t_n^q), X_t^n(t_n^q - m_1 h_1)) \xi_q,
\]
\[
\sigma_n^{13} = \theta f(X_t^n(t_n^k), X_t^n(t_n^k - m_1 h_1)) h_1 - \theta f(X_t^n(t_n), X_t^n(t_n - m_1 h_1)) h_1,
\]
\[
\sigma_n^{21} = \sum_{q=0}^{k-1} f(X_t^n(t_n^q - m_1 h_1), X_t^n(t_n^q - 2m_1 h_1)) h_1,
\]
\[
\sigma_n^{22} = \varepsilon \sqrt{h_i} \sum_{q=0}^{k-1} g(X_t^n(t_n^q - m_1 h_1), X_t^n(t_n^q - 2m_1 h_1)) \xi_q,
\]
\[
\sigma_n^{23} = \theta f(X_t^n(t_n^k - m_1 h_1), X_t^n(t_n^k - 2m_1 h_1)) h_1 - \theta f(X_t^n(t_n - m_1 h_1), X_t^n(t_n - 2m_1 h_1)) h_1.
\]

Again by the Taylor expansion we derive
\[
\int_0^1 \nabla f_i((X_t^n(t_n), X_t^n(t_n - m_1 h_1)) + s[(X_t^n(t_n^k), X_t^n(t_n^k - m_1 h_1))
- (X_t^n(t_n), X_t^n(t_n - m_1 h_1))] ds \times \left( \frac{\sigma_n^{12}}{\sigma_n^{21}} \right)
= \nabla f_i(X_t^n(t_n), X_t^n(t_n - m_1 h_1)) \times \left( \frac{\sigma_n^{12}}{\sigma_n^{22}} \right) + [X_t^n(t_n^k) - X_t^n(t_n), X_t^n(t_n^k - m_1 h_1) - X_t^n(t_n - m_1 h_1)]
\cdot \int_0^1 \int_0^s \nabla^2 f_i((X_t^n(t_n), X_t^n(t_n - m_1 h_1)) + u[(X_t^n(t_n^k), X_t^n(t_n^k - m_1 h_1))
- (X_t^n(t_n), X_t^n(t_n - m_1 h_1))] du ds \times \left( \frac{\sigma_n^{12}}{\sigma_n^{22}} \right).
\]

These implies
\[
f_i(X_t^n(t_n^k), X_t^n(t_n^k - m_1 h_1)) - f_i(X_t^n(t_n), X_t^n(t_n - m_1 h_1))
= \int_0^1 \nabla f_i((X_t^n(t_n), X_t^n(t_n - m_1 h_1)) + s[(X_t^n(t_n^k), X_t^n(t_n^k - m_1 h_1))
- (X_t^n(t_n), X_t^n(t_n - m_1 h_1))] ds \times \left( \frac{\sigma_n^{11} + \sigma_n^{13}}{\sigma_n^{21} + \sigma_n^{23}} + \nabla f_i(X_t^n(t_n), X_t^n(t_n - m_1 h_1)) \times \left( \frac{\sigma_n^{12}}{\sigma_n^{22}} \right)
+ [X_t^n(t_n^k) - X_t^n(t_n), X_t^n(t_n^k - m_1 h_1) - X_t^n(t_n - m_1 h_1)]
\cdot \int_0^1 \int_0^s \nabla^2 f_i((X_t^n(t_n), X_t^n(t_n - m_1 h_1)) + u[(X_t^n(t_n^k), X_t^n(t_n^k - m_1 h_1))
- (X_t^n(t_n), X_t^n(t_n - m_1 h_1))] du ds \times \left( \frac{\sigma_n^{12}}{\sigma_n^{22}} \right)
= A_n^{ik} + B_n^{ik} + C_n^{ik}.
\]
Denote by $A_n^k = (A_{n1}^k, A_{n2}^k, \ldots, A_{nk}^k)^T$, $B_n^k = (B_{n1}^k, B_{n2}^k, \ldots, B_{nk}^k)^T$ and $C_n^k = (C_{n1}^k, C_{n2}^k, \ldots, C_{nk}^k)^T$, then we can rewrite $f(X_{n1}^{\varepsilon}(t_n^k), X_{n2}^{\varepsilon}(t_n^k)) = f(X_{n1}^{\varepsilon}(t_n), X_{n2}^{\varepsilon}(t_n - m_1h_1))$ as $A_n^k + B_n^k + C_n^k$.

This completes the proof.

\textbf{Theorem 2.2} Let assumption (H) hold. Then we have

$$\sup_{0 \leq n < M^{l-1}} \mathbb{E}[|X_{\varepsilon}^{\varepsilon}(t_n) - X_{\varepsilon}^{\varepsilon}(t_n)|^2] \leq CM^2h^2 + C\varepsilon^4Mh_1.$$ 

\textbf{Proof.} For any $n \leq M^{l-1} - 1$, by (2.10) and (2.11), we get

$$Y_{h_1}^{\varepsilon}(t_{n+1}) - Y_{h_1-1}^{\varepsilon}(t_{n+1}) = Y_{h_1}^{\varepsilon}(t_n) - Y_{h_1-1}^{\varepsilon}(t_n)$$

$$+ h_1 \sum_{k=0}^{M-1} [f(X_{h_1}^{\varepsilon}(t_n^k), X_{h_1}^{\varepsilon}(t_n^k - m_1h_1)) - f(X_{h_1-1}^{\varepsilon}(t_n), X_{h_1-1}^{\varepsilon}(t_n - m_1h_1))]$$

$$+ \varepsilon \sqrt{h_1} \sum_{k=0}^{M-1} [g(X_{h_1}^{\varepsilon}(t_n^k), X_{h_1}^{\varepsilon}(t_n^k - m_1h_1)) - g(X_{h_1-1}^{\varepsilon}(t_n), X_{h_1-1}^{\varepsilon}(t_n - m_1h_1))]\xi_n$$

$$= Y_{h_1}^{\varepsilon}(t_{n+1}) - Y_{h_1-1}^{\varepsilon}(t_{n+1}) + h_1 \sum_{k=0}^{M-1} [f(X_{h_1}^{\varepsilon}(t_n^k), X_{h_1}^{\varepsilon}(t_n^k - m_1h_1)) - f(X_{h_1}^{\varepsilon}(t_n), X_{h_1}^{\varepsilon}(t_n - m_1h_1))]$$

$$+ h_1 \sum_{k=0}^{M-1} [f(X_{h_1}^{\varepsilon}(t_n), X_{h_1}^{\varepsilon}(t_n - m_1h_1)) - f(X_{h_1-1}^{\varepsilon}(t_n), X_{h_1-1}^{\varepsilon}(t_n - m_1h_1))]$$

$$+ \varepsilon \sqrt{h_1} \sum_{k=0}^{M-1} [g(X_{h_1}^{\varepsilon}(t_n^k), X_{h_1}^{\varepsilon}(t_n^k - m_1h_1)) - g(X_{h_1}^{\varepsilon}(t_n), X_{h_1}^{\varepsilon}(t_n - m_1h_1))]\xi_n$$

$$+ \varepsilon \sqrt{h_1} \sum_{k=0}^{M-1} [g(X_{h_1}^{\varepsilon}(t_n), X_{h_1}^{\varepsilon}(t_n - m_1h_1)) - g(X_{h_1-1}^{\varepsilon}(t_n), X_{h_1-1}^{\varepsilon}(t_n - m_1h_1))]\xi_n.$$

13
By the elementary inequality \( \left( \sum_{i=1}^{n} x_i \right)^2 \leq x_1^2 + (n-1) \sum_{i=2}^{n} |x_i|^2 + 2 \sum_{i=2}^{n} \langle x_1, x_i \rangle \), we compute

\[
|Y_{h_1}^\varepsilon (t_{n+1}) - Y_{h_{l-1}}^\varepsilon (t_{n+1})|^2 \leq |Y_{h_1}^\varepsilon (t_n) - Y_{h_{l-1}}^\varepsilon (t_n)|^2 \\
+ 4Mh_t^2 \sum_{k=0}^{M-1} \left| f(X_{h_1}^\varepsilon (t_n^k), X_{h_1}^\varepsilon (t_n^k - m_l h_l)) - f(X_{h_1}^\varepsilon (t_n), X_{h_1}^\varepsilon (t_n - m_l h_l)) \right|^2 \\
+ 4Mh_t^2 \sum_{k=0}^{M-1} \left| f(X_{h_1}^\varepsilon (t_n), X_{h_1}^\varepsilon (t_n - m_l h_l)) - f(X_{h_{l-1}}^\varepsilon (t_n), X_{h_{l-1}}^\varepsilon (t_n - m_l h_l)) \right|^2 \\
+ 4\varepsilon^2 h_t \sum_{k=0}^{M-1} \left| g(X_{h_1}^\varepsilon (t_n^k), X_{h_1}^\varepsilon (t_n^k - m_l h_l)) - g(X_{h_1}^\varepsilon (t_n), X_{h_1}^\varepsilon (t_n - m_l h_l)) \right| \xi_n^k \right|^2 \\
+ 4\varepsilon^2 h_t \sum_{k=0}^{M-1} \left| g(X_{h_1}^\varepsilon (t_n), X_{h_1}^\varepsilon (t_n - m_l h_l)) - g(X_{h_{l-1}}^\varepsilon (t_n), X_{h_{l-1}}^\varepsilon (t_n - m_l h_l)) \right| \xi_n^k \right|^2 \\
+ 2h_t \sum_{k=0}^{M-1} \langle Y_{h_1}^\varepsilon (t_n) - Y_{h_{l-1}}^\varepsilon (t_n), f(X_{h_1}^\varepsilon (t_n^k), X_{h_1}^\varepsilon (t_n^k - m_l h_l)) - f(X_{h_1}^\varepsilon (t_n), X_{h_1}^\varepsilon (t_n - m_l h_l)) \rangle \\
+ 2h_t \sum_{k=0}^{M-1} \langle Y_{h_{l-1}}^\varepsilon (t_n) - Y_{h_{l-1}}^\varepsilon (t_n), f(X_{h_{l-1}}^\varepsilon (t_n^k), X_{h_{l-1}}^\varepsilon (t_n^k - m_l h_l)) - f(X_{h_{l-1}}^\varepsilon (t_n), X_{h_{l-1}}^\varepsilon (t_n - m_l h_l)) \rangle \\
+ 2\varepsilon \sqrt{h_t} \sum_{k=0}^{M-1} \langle Y_{h_1}^\varepsilon (t_n) - Y_{h_{l-1}}^\varepsilon (t_n), [g(X_{h_1}^\varepsilon (t_n^k), X_{h_1}^\varepsilon (t_n^k - m_l h_l)) - g(X_{h_1}^\varepsilon (t_n), X_{h_1}^\varepsilon (t_n - m_l h_l))] \xi_n^k \rangle \\
+ 2\varepsilon \sqrt{h_t} \sum_{k=0}^{M-1} \langle Y_{h_{l-1}}^\varepsilon (t_n) - Y_{h_{l-1}}^\varepsilon (t_n), [g(X_{h_{l-1}}^\varepsilon (t_n^k), X_{h_{l-1}}^\varepsilon (t_n^k - m_l h_l)) - g(X_{h_{l-1}}^\varepsilon (t_n), X_{h_{l-1}}^\varepsilon (t_n - m_l h_l))] \xi_n^k \rangle.
\]

Taking expectation, then summing both sides, using assumption (H) and the Young ineq-
ity, we obtain that for \( \Lambda \leq M^{t-1} - 1 \)

\[
\sup_{0 \leq n \leq \Lambda + 1} \mathbb{E}[Y^\varepsilon_{h_1}(t_n) - Y^\varepsilon_{h_{t-1}}(t_n)]^2 
\leq 8\alpha^2 M h_t^2 \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} \mathbb{E} \left( |X^\varepsilon_{h_i}(t_j^k) - X^\varepsilon_{h_{t-1}}(t_j)|^2 + |X^\varepsilon_{h_i}(t_j^k - m_i h_t) - X^\varepsilon_{h_{t-1}}(t_j - m_i h_t)|^2 \right)
\]

\[
+ 8\alpha^2 M h_t^2 \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} \mathbb{E} \left( |X^\varepsilon_{h_i}(t_j) - X^\varepsilon_{h_{t-1}}(t_j)|^2 + |X^\varepsilon_{h_i}(t_j - m_i h_t) - X^\varepsilon_{h_{t-1}}(t_j - m_i h_t)|^2 \right)
\]

\[
+ 8\alpha^2 \varepsilon^2 h_t \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} \mathbb{E} \left( |X^\varepsilon_{h_i}(t_j) - X^\varepsilon_{h_{t-1}}(t_j)|^2 + |X^\varepsilon_{h_i}(t_j - m_i h_t) - X^\varepsilon_{h_{t-1}}(t_j - m_i h_t)|^2 \right)
\]

By Lemma 2.28, we immediately get

\[
\sup_{0 \leq n \leq \Lambda + 1} \mathbb{E}[Y^\varepsilon_{h_i}(t_n) - Y^\varepsilon_{h_{t-1}}(t_n)]^2 
\leq CM h_t \Lambda (M^2 h_t^2 + \varepsilon^2 M h_t) + C \varepsilon^2 (M^2 h_t^2 + \varepsilon^2 M h_t)
\]

\[
+ C(M^2 h_t^2 + \varepsilon^2 M h_t) \sum_{j=0}^{\Lambda} \mathbb{E}[X^\varepsilon_{h_i}(t_j) - X^\varepsilon_{h_{t-1}}(t_j)]^2 + \frac{1}{4} \sup_{0 \leq n \leq \Lambda + 1} \mathbb{E}[Y^\varepsilon_{h_i}(t_n) - Y^\varepsilon_{h_{t-1}}(t_n)]^2
\]

\[
+ C(M^2 h_t^2 + \varepsilon^2 M h_t) \sum_{j=0}^{\Lambda} \mathbb{E}[X^\varepsilon_{h_i}(t_j - m_i h_t) - X^\varepsilon_{h_{t-1}}(t_j - m_i h_t)]^2
\]

\[
+ 2h_t \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} \mathbb{E}[Y^\varepsilon_{h_i}(t_j) - Y^\varepsilon_{h_{t-1}}(t_j), f(X^\varepsilon_{h_i}(t_j^k), X^\varepsilon_{h_i}(t_j^k - m_i h_t)) - f(X^\varepsilon_{h_i}(t_j), X^\varepsilon_{h_i}(t_j - m_i h_t))] + CM h_t \sum_{j=0}^{\Lambda} \mathbb{E}[X^\varepsilon_{h_i}(t_j - m_i h_t) - X^\varepsilon_{h_{t-1}}(t_j - m_i h_t)]^2.
\]

(2.17)
Applying the Young inequality and Lemma 2.9, we see since by Lemma 2.4 and Lemma 2.8, it is easy to see

\[ \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} E \langle Y_{h_i}^\varepsilon(t_j) - Y_{h_{i-1}}^\varepsilon(t_j), A_j^k \rangle + 2h_t \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} E \langle Y_{h_i}^\varepsilon(t_j) - Y_{h_{i-1}}^\varepsilon(t_j), B_j^k \rangle \]

\[ + 2h_t \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} E \langle Y_{h_i}^\varepsilon(t_j) - Y_{h_{i-1}}^\varepsilon(t_j), C_j^k \rangle \]

\[ \leq \frac{1}{4} \sup_{0 \leq n \leq \Lambda+1} E |Y_{h_i}^\varepsilon(t_n) - Y_{h_{i-1}}^\varepsilon(t_n)|^2 + Ch_t \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} E |A_j^k|^2 + Ch_t \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} E |C_j^k|^2 \]

\[ \leq \frac{1}{4} \sup_{0 \leq n \leq \Lambda+1} E |Y_{h_i}^\varepsilon(t_n) - Y_{h_{i-1}}^\varepsilon(t_n)|^2 + C M^2 h_t^2 + C \varepsilon^2 M^3 h_t^3 + C \varepsilon^4 M^2 h_t^2, \]

(2.18)

since by Lemma 2.4 and Lemma 2.8 it is easy to see \( E |A_j^k|^2 \leq C M^2 h_t^2 + C \varepsilon^2 M^3 h_t^3 \), moreover, by the Hölder inequality and Lemmas 2.4, 2.8 we have

\[ E |C_j^k|^2 \leq C \varepsilon^2 h_t E \left[ |X_{h_i}^\varepsilon(t_j^k) - X_{h_{i-1}}^\varepsilon(t_j)|^2 + |X_{h_i}^\varepsilon(t_j^k - m_i h_t) - X_{h_{i-1}}^\varepsilon(t_j - m_i h_t)|^2 \right] \]

\[ \cdot \left[ \sum_{q=0}^{k-1} |g(X_{h_i}^\varepsilon(t_j^q), X_{h_{i-1}}^\varepsilon(t_j^q - m_i h_t))|\xi_j^q|^2 + \sum_{q=0}^{k-1} |g(X_{h_i}^\varepsilon(t_j^q - m_i h_t), X_{h_{i-1}}^\varepsilon(t_j^q - 2m_i h_t))|\xi_j^q|^2 \right] \]

\[ \leq C \varepsilon^2 h_t \left( E |X_{h_i}^\varepsilon(t_j^k) - X_{h_{i-1}}^\varepsilon(t_j)|^4 + E |X_{h_i}^\varepsilon(t_j^k - m_i h_t) - X_{h_{i-1}}^\varepsilon(t_j - m_i h_t)|^4 \right)^{1/2} \]

\[ \cdot \left( \sum_{q=0}^{k-1} E |g(X_{h_i}^\varepsilon(t_j^q), X_{h_{i-1}}^\varepsilon(t_j^q - m_i h_t))|\xi_j^q|^4 + \sum_{q=0}^{k-1} E |g(X_{h_i}^\varepsilon(t_j^q - m_i h_t), X_{h_{i-1}}^\varepsilon(t_j^q - 2m_i h_t))|\xi_j^q|^4 \right)^{1/2} \]

\[ \leq C \varepsilon^2 M^3 h_t^3 + C \varepsilon^4 M^2 h_t^2. \]

By the definition of \( Y_{h_i}^\varepsilon(t_n) \) and \( Y_{h_{i-1}}^\varepsilon(t_n) \), we have

\[ Y_{h_i}^\varepsilon(t_n) - Y_{h_{i-1}}^\varepsilon(t_n) = X_{h_i}^\varepsilon(t_n) - X_{h_{i-1}}^\varepsilon(t_n) \]

\[ - \theta f(X_{h_i}^\varepsilon(t_n), X_{h_{i-1}}^\varepsilon(t_n) - m_i h_t))h_t + \theta f(X_{h_{i-1}}^\varepsilon(t_n), X_{h_{i-1}}^\varepsilon(t_n) - m_i h_t))h_{i-1}. \]

Taking advantage of the elementary equality \( 2(|a|^2 + |b|^2) \geq |a - b|^2 \geq |a|^2 - |b|^2 \), we get

\[ |Y_{h_i}^\varepsilon(t_n) - Y_{h_{i-1}}^\varepsilon(t_n)|^2 \geq |X_{h_i}^\varepsilon(t_n) - X_{h_{i-1}}^\varepsilon(t_n)|^2 \]

\[ - \theta f(X_{h_i}^\varepsilon(t_n), X_{h_{i-1}}^\varepsilon(t_n) - m_i h_t))h_t - \theta f(X_{h_{i-1}}^\varepsilon(t_n), X_{h_{i-1}}^\varepsilon(t_n) - m_i h_t))h_{i-1}|^2 \]

\[ \geq |X_{h_i}^\varepsilon(t_n) - X_{h_{i-1}}^\varepsilon(t_n)|^2 - 2\theta^2 h_t^2 |f(X_{h_i}^\varepsilon(t_n), X_{h_{i-1}}^\varepsilon(t_n) - m_i h_t))|^2 \]

\[ - 2\theta^2 h_{i-1}^2 |f(X_{h_{i-1}}^\varepsilon(t_n), X_{h_{i-1}}^\varepsilon(t_n) - m_i h_t))|^2. \]
This, together with Lemma 2.4 imply
\[
\sup_{0 \leq n \leq \Lambda + 1} \mathbb{E} \left| X_{h_i}^\varepsilon(t_n) - X_{h_{i-1}}^\varepsilon(t_n) \right|^2 \leq \sup_{0 \leq n \leq \Lambda + 1} \mathbb{E} \left| Y_{h_i}^\varepsilon(t_n) - Y_{h_{i-1}}^\varepsilon(t_n) \right|^2 + Ch_{l-1}^2.
\] (2.19)

Combining (2.17)-(2.19) yields
\[
\sup_{0 \leq n \leq \Lambda + 1} \mathbb{E} \left| X_{h_i}^\varepsilon(t_n) - X_{h_{i-1}}^\varepsilon(t_n) \right|^2 \leq C(M^3 h_l^3 + \varepsilon^2 M^2 h_l^2 + \varepsilon^4 M h_l + M^2 h_l^2 + \varepsilon^2 M^3 h_l^3 + \varepsilon^4 M^2 h_l^2)
\]
\[
\quad + C(M^2 h_l^2 + \varepsilon^2 M h_l + M h_l) \sum_{j=0}^{\Lambda} \sup_{0 \leq n \leq j} \mathbb{E} \left| X_{h_i}^\varepsilon(t_n) - X_{h_{i-1}}^\varepsilon(t_n) \right|^2.
\]

By the discrete Gronwall inequality, the desired result can be obtained since the dominant term above is of order $M^2 h_l^2$ and $\varepsilon^4 M h_l$.

The following two lemmas are from [1].

Lemma 2.10 Suppose $X_1(t)$ and $X_2(t)$ are stochastic processes on $\mathbb{R}^a$ and that $x_1(t)$ and $x_2(t)$ are deterministic processes on $\mathbb{R}^a$. Further, suppose that
\[
\sup_{t \leq T} \mathbb{E} \left| X_1(t) - x_1(t) \right|^2 \leq C_1 \varepsilon^2, \quad \sup_{s \leq T} \mathbb{E} \left| X_2(t) - x_2(t) \right|^2 \leq C_2 \varepsilon^2,
\]
for some $C_1, C_2$ and any $\varepsilon \in (0,1)$. Assume that $\Phi : \mathbb{R}^a \rightarrow \mathbb{R}$ is Lipschitz with Lipschitz constant $C_L$. Then
\[
\sup_{t \leq T} \text{Var} \left( \int_0^t \Phi(X_2(s) + s(X_1(t) - X_2(t))) ds \right) \leq C_L^2 C_1 C_2 \varepsilon^2.
\]

Lemma 2.11 Suppose that $A^{\varepsilon h}$ and $B^{\varepsilon h}$ are families of random variables determined by scaling parameters $\varepsilon$ and $h$. Further, suppose that there are positive constants $C_1, C_2, C_3$ such that for any $\varepsilon \in (0,1)$, the following three conditions hold:
(i) $\text{Var}(A^{\varepsilon h}) \leq C_1 \varepsilon^2$ uniformly in $h$.
(ii) $|A^{\varepsilon h}| \leq C_2$ uniformly in $h$.
(iii) $|E B^{\varepsilon h}| \leq C_3 h$.

Then
\[
\text{Var}(A^{\varepsilon h} B^{\varepsilon h}) \leq 3C_3^2 C_1 \varepsilon^2 h^2 + 15C_2^2 \text{Var}(B^{\varepsilon h}).
\]

Theorem 2.3 Let assumption (H) hold, assume that $\Psi : \mathbb{R}^a \rightarrow \mathbb{R}$ has continuous second order derivative and there exists a constant $C$ such that
\[
\left| \frac{\partial \Psi}{\partial x_i} \right| \leq C \quad \text{and} \quad \left| \frac{\partial^2 \Psi}{\partial x_i \partial x_j} \right| \leq C
\]
for any $i, j = 1, 2, \cdots, a$. Then, we have
\[
\sup_{0 \leq n < M^{l-1}} \text{Var}(\Psi(X_{h_i}^\varepsilon(t_n)) - \Psi(X_{h_{i-1}}^\varepsilon(t_n))) \leq Ch_{l-1}^4 + C\varepsilon^2 h_{l-1}^2 + C\varepsilon^4 h_{l-1}.
\]
Proof. By the Taylor expansion, we see
\[
\Psi(X^\varepsilon_{h_l}(t_n)) - \Psi(X^\varepsilon_{h_{l-1}}(t_n)) = \int_0^1 [\nabla \Psi(X^\varepsilon_{h_{l-1}}(t_n) + s(X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n))]ds \cdot (X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n)).
\]

Moreover, we have
\[
\text{Var} \left( \int_0^1 [\nabla \Psi(X^\varepsilon_{h_{l-1}}(t_n) + s(X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n))]ds \cdot (X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n)) \right) \\
\leq a \sum_{i=0}^a \text{Var} \left( \int_0^1 [\nabla_i \Psi(X^\varepsilon_{h_{l-1}}(t_n) + s(X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n))]ds \cdot [X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n)]_i \right),
\]

(2.20)

where \( \nabla_i \) is the \( i \)-th component of first derivatives vector and \( [X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n)]_i \) is the \( i \)-th component of \( X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n) \). By Lemma 2.6 it is obvious to get
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^\varepsilon_{h_l}(t) - Z_{h_l}(t)|^2 \right] \leq C\varepsilon^2,
\]

and
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^\varepsilon_{h_{l-1}}(t) - Z_{h_{l-1}}(t)|^2 \right] \leq C\varepsilon^2.
\]

Thus, application of Lemma 2.10 leads to
\[
\text{Var} \left( \int_0^1 [\nabla_i \Psi(X^\varepsilon_{h_{l-1}}(t_n) + s(X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n))]ds \cdot [X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n)]_i \right) \leq C\varepsilon^2.
\]

Then by Theorem 2.2 and Lemma 2.11 we get
\[
\text{Var} \left( \int_0^1 [\nabla_i \Psi(X^\varepsilon_{h_{l-1}}(t_n) + s(X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n))]ds \cdot [X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n)]_i \right) \\
\leq C\varepsilon^2 (M^2 h_l^2 + \varepsilon^4 M h_l) + C\text{Var} \left( [X^\varepsilon_{h_l}(t_n) - X^\varepsilon_{h_{l-1}}(t_n)]_i \right).
\]

(2.21)
Now we concentrate on \( \text{Var} \left( [X_{h_t}^\varepsilon(t_n) - X_{h_{t-1}}^\varepsilon(t_n)]_i \right) \). For \( n \leq M^{l-1} - 1, \ i = 1, 2, \cdots, a \)

\[
[Y_{h_t}^\varepsilon(t_{n+1}) - Y_{h_{t-1}}^\varepsilon(t_{n+1})]_i = [Y_{h_t}^\varepsilon(t_n) - Y_{h_{t-1}}^\varepsilon(t_n)]_i \\
+ h_t \sum_{k=0}^{M-1} [f_i(X_{h_t}^\varepsilon(t_n^k), X_{h_t}^\varepsilon(t_n^k - m_l h_t)) - f_i(X_{h_t}^\varepsilon(t_n), X_{h_t}^\varepsilon(t_n - m_l h_t))]
\\
+ h_t \sum_{k=0}^{M-1} [f_i(X_{h_{t-1}}^\varepsilon(t_n), X_{h_{t-1}}^\varepsilon(t_n - m_l h_t)) - f_i(X_{h_{t-1}}^\varepsilon(t_n), X_{h_{t-1}}^\varepsilon(t_n - m_l h_t))]
\\
+ \varepsilon \sqrt{h_t} \sum_{k=0}^{M-1} [g_i(X_{h_t}^\varepsilon(t_n^k), X_{h_t}^\varepsilon(t_n^k - m_l h_t)) - g_i(X_{h_t}^\varepsilon(t_n), X_{h_t}^\varepsilon(t_n - m_l h_t))] \xi^k_n
\\
+ \varepsilon \sqrt{h_t} \sum_{k=0}^{M-1} [g_i(X_{h_{t-1}}^\varepsilon(t_n), X_{h_{t-1}}^\varepsilon(t_n - m_l h_t)) - g_i(X_{h_{t-1}}^\varepsilon(t_n), X_{h_{t-1}}^\varepsilon(t_n - m_l h_t))] \xi^k_n,
\]

where \( f_i \) is the \( i \)-th component of \( f \) and \( g_i \) is the \( i \)-th row of \( g \). By computation,

\[
\text{Var}[Y_{h_t}^\varepsilon(t_{n+1}) - Y_{h_{t-1}}^\varepsilon(t_{n+1})]_i \leq \text{Var}[Y_{h_t}^\varepsilon(t_n) - Y_{h_{t-1}}^\varepsilon(t_n)]_i \\
+ 4Mh_t^2 \sum_{k=0}^{M-1} \text{Var}[f_i(X_{h_t}^\varepsilon(t_n^k), X_{h_t}^\varepsilon(t_n^k - m_l h_t)) - f_i(X_{h_t}^\varepsilon(t_n), X_{h_t}^\varepsilon(t_n - m_l h_t))]
\\
+ 4M^2h_t^2 \text{Var}[f_i(X_{h_t}^\varepsilon(t_n), X_{h_{t-1}}^\varepsilon(t_n - m_l h_t)) - f_i(X_{h_{t-1}}^\varepsilon(t_n), X_{h_{t-1}}^\varepsilon(t_n - m_l h_t))]
\\
+ 4\varepsilon^2 h_t \text{Var} \left\{ \sum_{k=0}^{M-1} [g_i(X_{h_t}^\varepsilon(t_n^k), X_{h_t}^\varepsilon(t_n^k - m_l h_t)) - g_i(X_{h_t}^\varepsilon(t_n), X_{h_t}^\varepsilon(t_n - m_l h_t))] \xi^k_n \right\}
\\
+ 4\varepsilon^2 h_t \text{Var} \left\{ \sum_{k=0}^{M-1} [g_i(X_{h_{t-1}}^\varepsilon(t_n), X_{h_{t-1}}^\varepsilon(t_n - m_l h_t)) - g_i(X_{h_{t-1}}^\varepsilon(t_n), X_{h_{t-1}}^\varepsilon(t_n - m_l h_t))] \xi^k_n \right\}
\\
+ 2\text{Cov} \left( [Y_{h_t}^\varepsilon(t_n) - Y_{h_{t-1}}^\varepsilon(t_n)]_i, h_t \sum_{k=0}^{M-1} [f_i(X_{h_t}^\varepsilon(t_n^k), X_{h_t}^\varepsilon(t_n^k - m_l h_t)) - f_i(X_{h_t}^\varepsilon(t_n), X_{h_t}^\varepsilon(t_n - m_l h_t))] \right)
\\
+ 2\text{Cov} \left( [Y_{h_t}^\varepsilon(t_n) - Y_{h_{t-1}}^\varepsilon(t_n)]_i, h_t \sum_{k=0}^{M-1} [f_i(X_{h_{t-1}}^\varepsilon(t_n), X_{h_{t-1}}^\varepsilon(t_n - m_l h_t)) - f_i(X_{h_{t-1}}^\varepsilon(t_n), X_{h_{t-1}}^\varepsilon(t_n - m_l h_t))] \right).
\]
Summing both sides, for $0 \leq \Lambda \leq M^{l-1} - 1$,

$$\sup_{0 \leq t \leq \Lambda + 1} \text{Var}[Y^{\varepsilon}_{h_i}(t_n) - Y^{\varepsilon}_{h_{i-1}}(t_n)]_i \leq 4Mh^2 \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} \text{Var}[f_i(X^{\varepsilon}_{h_i}(t_j^k), X^{\varepsilon}_{h_i}(t_j^k - m_i h_i)) - f_i(X^{\varepsilon}_{h_i}(t_j), X^{\varepsilon}_{h_i}(t_j - m_i h_i))]
+ 4M^2 h_i \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} \text{Var}[f_i(X^{\varepsilon}_{h_i}(t_j), X^{\varepsilon}_{h_i}(t_j - m_i h_i)) - f_i(X^{\varepsilon}_{h_{i-1}}(t_j), X^{\varepsilon}_{h_{i-1}}(t_j - m_i h_i))]
+ 4\varepsilon^2 h_i \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} \text{Var}[g_i(X^{\varepsilon}_{h_i}(t_j^k), X^{\varepsilon}_{h_i}(t_j^k - m_i h_i)) - g_i(X^{\varepsilon}_{h_i}(t_j), X^{\varepsilon}_{h_i}(t_j - m_i h_i))] \xi_j^k
+ 4\varepsilon^2 h_i \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} \text{Var}[g_i(X^{\varepsilon}_{h_i}(t_j), X^{\varepsilon}_{h_i}(t_j - m_i h_i)) - g_i(X^{\varepsilon}_{h_{i-1}}(t_j), X^{\varepsilon}_{h_{i-1}}(t_j - m_i h_i))] \xi_j^k
+ 2 \sum_{j=0}^{\Lambda} \text{Cov}(Y^{\varepsilon}_{h_i}(t_j) - Y^{\varepsilon}_{h_{i-1}}(t_j), X^{\varepsilon}_{h_i}(t_j^k - m_i h_i)) - f_i(X^{\varepsilon}_{h_i}(t_j), X^{\varepsilon}_{h_i}(t_j - m_i h_i))]
+ 2 \sum_{j=0}^{\Lambda} \text{Cov}(Y^{\varepsilon}_{h_i}(t_j) - Y^{\varepsilon}_{h_{i-1}}(t_j), X^{\varepsilon}_{h_{i-1}}(t_j - m_i h_i)) - f_i(X^{\varepsilon}_{h_{i-1}}(t_j), X^{\varepsilon}_{h_{i-1}}(t_j - m_i h_i))]
:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.

**Lemma 2.12** There exists a positive constant $C$ such that

$$I_1 \leq C\varepsilon^2 M^2 h_i^2 + CM^5 h_i^5.$$

**Proof.** By the Taylor expansion,

$$f_i(X^{\varepsilon}_{h_i}(t_j^k), X^{\varepsilon}_{h_i}(t_j^k - m_i h_i)) - f_i(X^{\varepsilon}_{h_i}(t_j), X^{\varepsilon}_{h_i}(t_j - m_i h_i))$$

$$= \int_0^1 \{ \nabla f_i((X^{\varepsilon}_{h_i}(t_j), X^{\varepsilon}_{h_i}(t_j - m_i h_i)) + s((X^{\varepsilon}_{h_i}(t_j^k), X^{\varepsilon}_{h_i}(t_j^k - m_i h_i))
- (X^{\varepsilon}_{h_i}(t_j), X^{\varepsilon}_{h_i}(t_j - m_i h_i)))\} ds \times \left( X^{\varepsilon}_{h_i}(t_j^k) - X^{\varepsilon}_{h_i}(t_j) \right) \left( X^{\varepsilon}_{h_i}(t_j^k - m_i h_i) - X^{\varepsilon}_{h_i}(t_j - m_i h_i) \right).$$
By virtue of properties of expectation, for \( r = 1, 2, \cdots, 2a, \)
\[
\text{Var} \left[ f_i(X_{h_i}^\varepsilon(t_j^k), X_{h_i}^\varepsilon(t_j^k - m_i h_i)) - f_i(X_{h_i}^\varepsilon(t_j), X_{h_i}^\varepsilon(t_j - m_i h_i)) \right] \\
\leq 2a \sum_{r=1}^{2a} \text{Var} \left[ \int_0^1 \left\{ \nabla_r f_i((X_{h_i}^\varepsilon(t_j), X_{h_i}^\varepsilon(t_j - m_i h_i)) + s[(X_{h_i}^\varepsilon(t_j^k), X_{h_i}^\varepsilon(t_j^k - m_i h_i)) \\
- (X_{h_i}^\varepsilon(t_j), X_{h_i}^\varepsilon(t_j - m_i h_i)))] \right\} ds \times \left( X_{h_i}^\varepsilon(t_j^k - m_i h_i) - X_{h_i}^\varepsilon(t_j - m_i h_i) \right)_r \right].
\]
\[(2.22)\]
where \( \nabla_r f_i \) is the \( r \)-th component of first derivatives to \( f_i \), and \((\cdot)_r \) is the \( r \)-th component of a vector. We apply Lemma 2.6 and Lemma 2.10 to get

\[
\text{Var} \left[ \int_0^1 \left\{ \nabla_r f_i((X_{h_i}^\varepsilon(t_j), X_{h_i}^\varepsilon(t_j - m_i h_i)) + s[(X_{h_i}^\varepsilon(t_j^k), X_{h_i}^\varepsilon(t_j^k - m_i h_i)) \\
- (X_{h_i}^\varepsilon(t_j), X_{h_i}^\varepsilon(t_j - m_i h_i)))] \right\} ds \right] \leq C\varepsilon^2.
\]
Moreover, for \( r = 1, \cdots, a, \) taking advantage of assumption (H) and Lemmas 2.4, 2.6

\[
\text{Var} \left[ \left( X_{h_i}^\varepsilon(t_j^k) - X_{h_i}^\varepsilon(t_j) \right)_r \right] \\
\leq 3\text{Var} \left( \sum_{q=0}^{k-1} f_r(X_{h_i}^\varepsilon(t_j^q), X_{h_i}^\varepsilon(t_j^q - m_i h_i))h_i \right)_r + 3\text{Var} \left( \varepsilon \sqrt{h_i} \sum_{q=0}^{k-1} g_r(X_{h_i}^\varepsilon(t_j^q), X_{h_i}^\varepsilon(t_j^q - m_i h_i))\xi_j^q \right) \\
+ 3\text{Var} \left[ \theta f_r(X_{h_i}^\varepsilon(t_j^k), X_{h_i}^\varepsilon(t_j^k - m_i h_i))h_i - \theta f_r(X_{h_i}^\varepsilon(t_j), X_{h_i}^\varepsilon(t_j - m_i h_i))h_i \right] \\
\leq 3h_i^2\text{Var} \left( \sum_{q=0}^{k-1} [f_r(X_{h_i}^\varepsilon(t_j^q), X_{h_i}^\varepsilon(t_j^q - m_i h_i)) - f_r(Z_{h_i}^\varepsilon(t_j^q), Z_{h_i}^\varepsilon(t_j^q - m_i h_i))] \right)_r \\
+ 3\varepsilon^2 h_i E \left( \sum_{q=0}^{k-1} g_r(X_{h_i}^\varepsilon(t_j^q), X_{h_i}^\varepsilon(t_j^q - m_i h_i))\xi_j^q \right)^2 \\
+ 9\text{Var} \left[ \theta f_r(X_{h_i}^\varepsilon(t_j^k), X_{h_i}^\varepsilon(t_j^k - m_i h_i))h_i - \theta f_r(Z_{h_i}^\varepsilon(t_j^k), Z_{h_i}^\varepsilon(t_j^k - m_i h_i))h_i \right] \\
+ 9\text{Var} \left[ \theta f_r(Z_{h_i}^\varepsilon(t_j^k), Z_{h_i}^\varepsilon(t_j^k - m_i h_i))h_i - \theta f_r(Z_{h_i}^\varepsilon(t_j), Z_{h_i}^\varepsilon(t_j - m_i h_i))h_i \right] \\
+ 9\text{Var} \left[ \theta f_r(Z_{h_i}^\varepsilon(t_j), Z_{h_i}^\varepsilon(t_j - m_i h_i))h_i - \theta f_r(X_{h_i}^\varepsilon(t_j), X_{h_i}^\varepsilon(t_j - m_i h_i))h_i \right] \\
\leq C\varepsilon^2 M^2 h_i^2 + C\varepsilon^2 M h_i + C\varepsilon^2 h_i^2 + Ch_i^4.
\]
Similarly, for \( r = a + 1, \ldots, 2a \),
\[
\Var \left[ \left( \frac{X_{h_i}^ε(t_j^k) - X_{h_i}^ε(t_j)}{m_i h_i} - X_{h_i}^ε(t_j - m_i h_i) \right) \right]_r \leq 3 \Var \left( \sum_{q=0}^{k-1} f_{r-a}(X_{h_i}^ε(t_j^q - m_i h_i), X_{h_i}^ε(t_j^q - 2m_i h_i)) h_i \right)
\]
\[
+ 3 \Var \left( \varepsilon \sqrt{h_i} \sum_{q=0}^{k-1} g_{r-a}(X_{h_i}^ε(t_j^q - m_i h_i), X_{h_i}^ε(t_j^q - 2m_i h_i)) \xi_q^q \right)
\]
\[
+ 3 \Var \left[ \theta f_{r-a}(X_{h_i}^ε(t_j^k - m_i h_i), X_{h_i}^ε(t_j^k - 2m_i h_i)) h_i - \theta f_{r-a}(X_{h_i}^ε(t_j - m_i h_i), X_{h_i}^ε(t_j - 2m_i h_i)) h_i \right]
\]}
\[
\leq C \varepsilon^2 M^2 h_i^2 + C \varepsilon^2 M h_i + Ch_i^4.
\]

Thus, combining (2.22) and Lemmas 2.8, 2.11 we see
\[
\Var \left[ f_i(X_{h_i}^ε(t_j^k), X_{h_i}^ε(t_j^k - m_i h_i)) - f_i(X_{h_i}^ε(t_j), X_{h_i}^ε(t_j - m_i h_i)) \right]
\]
\[
\leq 2a \sum_{r=1}^{2a} \left\{ C \varepsilon^2 (M^2 h_i^2 + \varepsilon^2 M h_i) + C \Var \left[ \left( \frac{X_{h_i}^ε(t_j^k) - X_{h_i}^ε(t_j)}{m_i h_i} - X_{h_i}^ε(t_j - m_i h_i) \right) \right]_r \right\}
\]
\[
\leq C \varepsilon^2 M h_i + Ch_i^4,
\]

which leads to
\[
I_1 \leq C \varepsilon^2 M^2 h_i^2 + C M h_i^5.
\]

\[ \square \]

**Lemma 2.13** There exists a positive constant \( C \) such that
\[
I_2 \leq C \varepsilon^2 M^3 h_i^3 + C \varepsilon^4 M^2 h_i^2 + C M^2 h_i^2 \sum_{j=0}^{\Lambda} \sum_{i=0}^a \sup_{0 \leq n \leq j} \Var \left( [X_{h_i}^ε(t_n) - X_{h_i-1}^ε(t_n)]_i \right).
\]

**Proof.** Application of the Taylor expansion gives that
\[
f_i(X_{h_i}^ε(t_j), X_{h_i}^ε(t_j - m_i h_i)) - f_i(X_{h_i-1}^ε(t_j), X_{h_i-1}^ε(t_j - m_i h_i))
\]
\[
= \int_0^1 \left\{ \nabla f_i((X_{h_i-1}^ε(t_j), X_{h_i-1}^ε(t_j - m_i h_i)) + s((X_{h_i}^ε(t_j), X_{h_i}^ε(t_j - m_i h_i))
\]
\[
- (X_{h_i-1}^ε(t_j), X_{h_i-1}^ε(t_j - m_i h_i))) \right\} ds \times \left( \frac{X_{h_i}^ε(t_j) - X_{h_i-1}^ε(t_j)}{X_{h_i}^ε(t_j - m_i h_i) - X_{h_i-1}^ε(t_j - m_i h_i)} \right).
\]

Taking similar steps as in Lemma 2.12, Lemma 2.16 together with Lemma 2.10 yield
\[
\Var \left[ \int_0^1 \left\{ \nabla f_i((X_{h_i-1}^ε(t_j), X_{h_i-1}^ε(t_j - m_i h_i)) + s((X_{h_i}^ε(t_j), X_{h_i}^ε(t_j - m_i h_i))
\]
\[
- (X_{h_i-1}^ε(t_j), X_{h_i-1}^ε(t_j - m_i h_i))) \right\} ds \right] \leq C \varepsilon^2
\]

22
for \( r = 1, 2, \ldots, 2a \). Further, Theorem 2.2 and Lemma 2.11 yield

\[
I_2 \leq CM^2 \varepsilon^2 (h_{l_1}^2 + \varepsilon^4 M h_l) + CM^2 h_l^2 \sum_{j=0}^{\Lambda} \sum_{r=1}^{2a} \text{Var} \left[ \left( X_{h_1}(t_j) - X_{h_{l-1}}(t_j) \right) \right] \\
\leq C\varepsilon^2 M^3 h_l^3 + C\varepsilon^6 M^2 h_l^2 + CM^2 h_l^2 \sum_{j=0}^{\Lambda} \sum_{r=1}^{2a} \text{Var} \left( X_{h_1}(t_n) - X_{h_{l-1}}(t_n) \right) 
\]

This completes the proof.

**Lemma 2.14** There exists a positive constant \( C \) such that

\[
I_3 \leq C\varepsilon^2 M^2 h_l^2 + C\varepsilon^4 M h_l.
\]

**Proof.** By assumption (H) and Lemma 2.8

\[
I_3 \leq 4\varepsilon^2 h_l \sum_{j=0}^{\Lambda} \mathbb{E} \left[ \left| \sum_{k=0}^{M-1} [g_i(X_{h_1}(t_j), X_{h_{l-1}}(t_j) - m_l h_l)) - g_i(X_{h_1}(t_j), X_{h_{l-1}}(t_j) - m_l h_l)] \xi_{j}^k \right|^2 \right] \\
\leq C\varepsilon^2 (M^2 h_l^2 + \varepsilon^4 M h_l).
\]

This completes the proof.

**Lemma 2.15** There exists a positive constant \( C \) such that

\[
I_4 \leq C\varepsilon^2 M^2 h_l^2 + C\varepsilon^6 M h_l.
\]

**Proof.** By assumption (H) and Theorem 2.2

\[
I_4 \leq 4\varepsilon^2 h_l \sum_{j=0}^{\Lambda} \mathbb{E} \left[ \left| \sum_{k=0}^{M-1} [g_i(X_{h_1}(t_j), X_{h_{l-1}}(t_j) - m_l h_l)) - g_i(X_{h_1}(t_j), X_{h_{l-1}}(t_j) - m_l h_l)] \xi_{j}^k \right|^2 \right] \\
\leq C\varepsilon^2 (M^2 h_l^2 + \varepsilon^4 M h_l).
\]

**Lemma 2.16** There exists a positive constant \( C \) such that

\[
I_5 \leq C\varepsilon^2 M^2 h_l^2 + CM^4 h_l^4 + \frac{1}{4} \sup_{0 \leq n \leq \Lambda+1} \text{Var} \left( [Y_{h_1}(t_n) - Y_{h_{l-1}}(t_n)] \right).
\]

**Proof.** Recall from Lemma 2.9 that

\[
I_5 = 2\sum_{j=0}^{\Lambda} \text{Cov} \left( [Y_{h_1}(t_j) - Y_{h_{l-1}}(t_j)], h_l \sum_{k=0}^{M-1} (A_j^k + B_j^k + C_j^k) \right) \\
= 2\sum_{j=0}^{\Lambda} \text{Cov} \left( [Y_{h_1}(t_j) - Y_{h_{l-1}}(t_j)], h_l \sum_{k=0}^{M-1} (A_j^k + C_j^k) \right) \\
\leq \frac{1}{4} \sup_{0 \leq n \leq \Lambda+1} \text{Var} \left( [Y_{h_1}(t_n) - Y_{h_{l-1}}(t_n)] \right) + Ch_l \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} \text{Var}(A_j^k) + Ch_l \sum_{j=0}^{\Lambda} \sum_{k=0}^{M-1} \text{Var}(C_j^k).
\]

(2.23)
We are now going to estimate $\text{Var}(A_j^{ik})$ and $\text{Var}(C_j^{ik})$. By the Taylor expansion,

$$\text{Var}(A_j^{ik}) = \text{Var} \left[ \int_0^1 \{ \nabla f_i((X_h^\varepsilon(t_j), X_h^\varepsilon(t_j - m_1 h_1)) + s[(X_h^\varepsilon(t_j^k), X_h^\varepsilon(t_j^k - m_1 h_1)) - (X_h^\varepsilon(t_j), X_h^\varepsilon(t_j - m_1 h_1)) \} \right] \times \left( \frac{\sigma_{j1}^1 + \sigma_{j3}^1}{\sigma_{j2}^1 + \sigma_{j3}^1} \right)

\leq 2a \sum_{r=1}^{2a} \text{Var} \left[ \int_0^1 \{ \nabla f_i((X_h^\varepsilon(t_j), X_h^\varepsilon(t_j - m_1 h_1)) + s[(X_h^\varepsilon(t_j^k), X_h^\varepsilon(t_j^k - m_1 h_1)) - (X_h^\varepsilon(t_j), X_h^\varepsilon(t_j - m_1 h_1)) \} \right] \times \left( \frac{\sigma_{j1}^1 + \sigma_{j3}^1}{\sigma_{j2}^1 + \sigma_{j3}^1} \right)

\leq C\varepsilon^2(M^2 h_1^2 + \varepsilon^2 M^2 h_1^3) + C \sum_{r=1}^{2a} \text{Var} \left[ \left( \frac{\sigma_{j1}^1 + \sigma_{j3}^1}{\sigma_{j2}^1 + \sigma_{j3}^1} \right) \right]

\leq C\varepsilon^2 M^2 h_1^2 + C\varepsilon^2 M^3 h_1^3 + Ch_4^4.

Since for $r = 1, \cdots, a$, similar to the procedure of Lemma 2.12 by assumption (H) and Lemmas 2.4, 2.6 we get

$$\text{Var} \left[ \left( \frac{\sigma_{j1}^{11} + \sigma_{j3}^{11}}{\sigma_{j2}^{21} + \sigma_{j3}^{21}} \right) \right] \leq 2\text{Var} \left( \sum_{q=0}^{k-1} f_r(X_h^\varepsilon(t_j^q), X_h^\varepsilon(t_j^q - m_1 h_1))h_1 \right)

+ 2\text{Var} \left[ \theta f_r(X_h^\varepsilon(t_j^q), X_h^\varepsilon(t_j^q - m_1 h_1))h_1 - \theta f_r(X_h^\varepsilon(t_j), X_h^\varepsilon(t_j - m_1 h_1))h_1 \right]

\leq C\varepsilon^2 M^2 h_1^2 + C\varepsilon^2 h_1^2 + Ch_4^4.

Similarly, for $r = a + 1, \cdots, 2a$,

$$\text{Var} \left[ \left( \frac{\sigma_{j1}^{13} + \sigma_{j3}^{13}}{\sigma_{j2}^{23} + \sigma_{j3}^{23}} \right) \right] \leq C\varepsilon^2 M^2 h_1^2 + C\varepsilon^2 h_1^2 + Ch_4^4.

Similar to the estimation of $\mathbb{E}|C_j^{ik}|^2$ in Theorem 2.2, we easily get

$$\text{Var}(C_j^{ik}) \leq \mathbb{E}|C_j^{ik}|^2 \leq C\varepsilon^2 M^2 h_1^3 + C\varepsilon^4 M^2 h_1^2.

Then, we derive from (2.23) that

$$I_5 \leq \frac{1}{4} \sup_{0 \leq n \leq A+1} \text{Var} \left( [Y_{h_1}^\varepsilon(t_n) - Y_{h_1-1}^\varepsilon(t_n)]_i \right)\right]

+ C(\varepsilon^2 M^2 h_1^2 + \varepsilon^4 M^3 h_1^3 + h_1^4) + C(\varepsilon^2 M^3 h_1^3 + \varepsilon^4 M^2 h_1^2).

\square

**Lemma 2.17** There exists a positive constant $C$ such that

$$I_6 \leq C\varepsilon^2 M^2 h_1^2 + C\varepsilon^0 M h_1 + \frac{1}{4} \sup_{0 \leq n \leq A+1} \text{Var} \left( [Y_{h_1}^\varepsilon(t_n) - Y_{h_1-1}^\varepsilon(t_n)]_i \right)

+ C M h_1 \sum_{j=0}^{A} \sum_{i=1}^{a} \sup_{0 \leq n \leq j} \text{Var} \left( [X_{h_1}^\varepsilon(t_n) - X_{h_1-1}^\varepsilon(t_n)]_i \right).
Proof. Obviously, by the result of Lemma 2.13

\[
I_6 \leq \frac{1}{4} \sup_{0 \leq n \leq \Lambda + 1} \text{Var} \left( [Y_{h_l}(t_n) - Y_{h_{l-1}}(t_n)]_i \right)
+ CMh_l \sum_{j=0}^{\Lambda} \text{Var} \left( f_i(X_{h_l}(t_j), x_{h_l}(t_j - m_it_l)) - f_i(X_{h_{l-1}}(t_j), x_{h_{l-1}}(t_j - m_it_l)) \right)
\leq \frac{1}{4} \sup_{0 \leq n \leq \Lambda + 1} \text{Var} \left( [Y_{h_l}(t_n) - Y_{h_{l-1}}(t_n)]_i \right) + C(\varepsilon^2 M^2 h_l^2 + C\varepsilon Mh_l)
+ CMh_l \sum_{j=0}^{\Lambda} \sum_{a=0}^{a} \sup_{0 \leq n \leq j} \text{Var} \left( [X_{h_l}(t_n) - X_{h_{l-1}}(t_n)]_i \right).
\]

Continue of Theorem 2.3. By Lemmas 2.12-2.17, we see

\[
\sup_{0 \leq n \leq \Lambda + 1} \text{Var} [Y_{h_l}(t_n) - Y_{h_{l-1}}(t_n)]_i \leq C\varepsilon^2 M^2 h_l^2 + C\varepsilon^4 Mh_l + CM^4 h_l^4
+ CMh_l \sum_{j=0}^{\Lambda} \sum_{a=0}^{a} \sup_{0 \leq n \leq j} \text{Var} \left( [X_{h_l}(t_n) - X_{h_{l-1}}(t_n)]_i \right).
\]

Since by Lemma 2.6 we have

\[
\text{Var} [X_{h_l}(t_n) - X_{h_{l-1}}(t_n)]_i \\
\leq 3\text{Var} [Y_{h_l}(t_n) - Y_{h_{l-1}}(t_n)]_i + 3\theta^2 h_l^2 \text{Var} [f_i(X_{h_l}(t_n), X_{h_l}(t_n - m_it_l))]
+ 3\theta^2 h_{l-1}^2 \text{Var} [f_i(X_{h_{l-1}}(t_n), X_{h_{l-1}}(t_n - m_it_l))]
\leq 3\text{Var} [Y_{h_l}(t_n) - Y_{h_{l-1}}(t_n)]_i
+ 3\theta^2 h_l^2 \text{Var} [f_i(X_{h_l}(t_n), X_{h_l}(t_n - m_it_l)) - f_i(Z_{h_l}(t_n), Z_{h_l}(t_n - m_it_l))]
+ 3\theta^2 h_{l-1}^2 \text{Var} [f_i(X_{h_{l-1}}(t_n), X_{h_{l-1}}(t_n - m_it_l)) - f_i(Z_{h_{l-1}}(t_n), Z_{h_{l-1}}(t_n - m_it_l))]
\leq 3\text{Var} [Y_{h_l}(t_n) - Y_{h_{l-1}}(t_n)]_i + C\varepsilon^2 M^2 h_l^2.
\]

Then, by (2.24) and (2.25), for \( \Lambda \leq M^{l-1} - 1 \)

\[
\sup_{0 \leq n \leq \Lambda + 1} \text{Var} [X_{h_l}(t_n) - X_{h_{l-1}}(t_n)]_i \\
\leq C\varepsilon^2 M^2 h_l^2 + C\varepsilon^4 Mh_l + CM^4 h_l^4
+ CMh_l \sum_{j=0}^{\Lambda} \sum_{a=0}^{a} \sup_{0 \leq n \leq j} \text{Var} \left( [X_{h_l}(t_n) - X_{h_{l-1}}(t_n)]_i \right).
\]

The Gronwall inequality leads to

\[
\sup_{0 \leq n \leq M^{l-1}} \sup_{0 \leq n \leq a} \text{Var} [X_{h_l}(t_n) - X_{h_{l-1}}(t_n)]_i \leq C\varepsilon^2 M^2 h_l^2 + C\varepsilon^4 Mh_l + CM^4 h_l^4.
\]

The desired result then follows from (2.20)-(2.21). □
Remark 2.2 From Section 2.1 we see that the theta EM scheme has the following property
\[ \sup_{0 \leq n < M^{l-1}} \text{Var}(\Psi(X^{\varepsilon}_{h_l}(t_n)) - \Psi(X^{\varepsilon}_{h_{l-1}}(t_n))) \leq Ch_l^2 + C\varepsilon^2 h_{l-1}, \]
while for the multilevel Monte Carlo theta EM scheme, the variance is bounded by \( O(h_l^4 + \varepsilon^2 h_{l-1}^2 + \varepsilon^4 h_{l-1}) \). That is, the multilevel Monte Carlo theta EM scheme is more efficient than the theta EM scheme.

3 SDDEs under One-side Lipschitz Condition

In this section, instead of the global Lipschitz condition \((H)\), we impose weaker assumptions to \((1.1)\). We assume that:

(H1) There exist \(\alpha_1, \alpha_2 > 1\) such that for some \(p \geq 2, r \geq 1\)
\[ 2\langle x - \bar{x}, f(x, y) - f(\bar{x}, \bar{y}) \rangle + (p - 1)\varepsilon^2|g(x, y) - g(\bar{x}, \bar{y})|^2 \leq \alpha_1(|x - \bar{x}|^2 + |y - \bar{y}|^2) \]
and
\[ |f(x, y) - f(\bar{x}, \bar{y})| \leq \alpha_2(1 + |x|^r + |\bar{x}|^r + |y|^r + |\bar{y}|^r)(|x - \bar{x}| + |y - \bar{y}|) \]
for all \(x, y, \bar{x}, \bar{y} \in \mathbb{R}^a\).

(H2) There exists a positive constant \(\alpha_3\) such that
\[ |g(x, y)|^2 \leq \alpha_3(1 + |x|^2 + |y|^2) \]
for all \(x, y \in \mathbb{R}^a\).

Lemma 3.1 Let assumptions (H1) and (H2) hold. Then, for any \(T > 0\) and \(p \geq 2\), we have
\[ \sup_{0 \leq t \leq T} \mathbb{E}|X^\varepsilon(t)|^p \leq C. \]

Remark 3.1 Assumption (H1) implies that for any \(x, y \in \mathbb{R}^a\)
\[ \langle x, f(x, y) \rangle \leq \tilde{\alpha}_1(1 + |x|^2 + |y|^2) \]
where \(\tilde{\alpha}_1 = \alpha_1 \vee \frac{1}{2}|f(0, 0)|^2\). Moreover, assumption (H1) also implies that
\[ (p - 1)\varepsilon^2|g(x, y) - g(\bar{x}, \bar{y})|^2 \leq \alpha_1(|x - \bar{x}|^2 + |y - \bar{y}|^2) + 2|x - \bar{x}||f(x, y) - f(\bar{x}, \bar{y})| \leq \tilde{\alpha}(1 + |x|^r + |\bar{x}|^r + |y|^r + |\bar{y}|^r)(|x - \bar{x}|^2 + |y - \bar{y}|^2) \]
where \(\tilde{\alpha}\) is a constant depends on \(\alpha_1\) and \(\alpha_2\).

Remark 3.2 Assumptions (H1)-(H2) guarantee the existence and uniqueness of the solution to \((1.1)\).
In order to guarantee the finiteness of $p$-th moment of the numerical solutions to (3.1), we make a tiny modification to the drift coefficient. Given any $T > 0$, let $M \geq 2, l > 1$, $h_l = T \cdot M^{-l}$, $h_{l-1} = T \cdot M^{-(l-1)}$, define

$$f_{h_l}(x, y) := \frac{f(x, y)}{1 + h_{l-1}^{-\delta} |f(x, y)|}$$

(3.3)

for any $x, y \in \mathbb{R}^a$ and some $\delta \in (0, \frac{1}{2}]$.

**Remark 3.3** With the definition of $f_{h_l}$, it is easy to show that under assumption (H1) the following condition hold:

$$|f_{h_l}(x, y)| \leq \min \left( |f(x, y)|, h_{l-1}^{-\delta} \right).$$

(3.4)

Moreover, one can verify that $f_{h_l}$ satisfies the following properties:

$$\langle x - \bar{x}, f_{h_l}(x, y) - f_{h_l}(\bar{x}, \bar{y}) \rangle \leq \frac{\alpha_1}{2}(|x - \bar{x}|^2 + |y - \bar{y}|^2),$$

(3.5)

and

$$\langle x, f_{h_l}(x, y) \rangle \leq \bar{\alpha}_1 (1 + |x|^2 + |y|^2).$$

(3.6)

Furthermore,

$$|f(x, y) - f_{h_l}(x, y)|^p \leq \bar{\alpha}_2 h_{l-1}^{\delta p} \left[ 1 + |x|^{2(r+1)p} + |y|^{2(r+1)p} \right]$$

(3.7)

where $\bar{\alpha}_2 = [\alpha_2 + |f(0, 0)|]^2$.

Similar to the global Lipschitz case, assume there exists an $m_l$ such that $\tau = m_l h_l$ and define

$$X^\varepsilon_{h_l}(t) - \theta f_{h_l}(X^\varepsilon_{h_l}(t), X^\varepsilon_{h_l}(t - \tau))h_l$$

(3.8)

$$= \xi(0) - \theta f_{h_l}(\xi(0), \xi(-\tau))h_l + \int_0^t f_{h_l}(X^\varepsilon_{h_l}(t_1), X^\varepsilon_{h_l}(t_1 - \tau))ds$$

$$+ \varepsilon \int_0^t g(X^\varepsilon_{h_l}(t_1), X^\varepsilon_{h_l}(t_1 - \tau))dW(s),$$

and

$$X^\varepsilon_{h_{l-1}}(t) - \theta f_{h_{l-1}}(X^\varepsilon_{h_{l-1}}(t), X^\varepsilon_{h_{l-1}}(t - \tau))h_{l-1}$$

(3.9)

$$= \xi(0) - \theta f_{h_{l-1}}(\xi(0), \xi(-\tau))h_{l-1} + \int_0^t f_{h_{l-1}}(X^\varepsilon_{h_{l-1}}(t_1), X^\varepsilon_{h_{l-1}}(t_1 - \tau))ds$$

$$+ \varepsilon \int_0^t g(X^\varepsilon_{h_{l-1}}(t_1), X^\varepsilon_{h_{l-1}}(t_1 - \tau))dW(s),$$

where $\eta_{h_l}(s) = \lfloor s/h_l \rfloor h_l$ and $\eta_{h_{l-1}}(s) = \lfloor s/h_{l-1} \rfloor h_{l-1}$. Here $\theta \in [0, 1]$ is a parameter to control the implicitness. For $n \in \{0, 1, \ldots, M^{l-1} - 1\}$ and $k \in \{0, \ldots, M\}$, let

$$t_n = nh_{l-1}$$

and

$$t_n^k = nh_{l-1} + kh_l.$$
This means we divide the interval \([t_n, t_{n+1}]\) into \(M\) equal parts, we have \(t^0_n = t_n, t^M_n = t_{n+1}\). We can rewrite (3.8) and (3.9) as the following discretization schemes. For \(n \in \{0, 1, \ldots, M^l - 1\}\) and \(k \in \{0, \ldots, M - 1\}\), let

\[
X_h^c(t^{k+1}_n) = \theta f_h(X_h^c(t^k_n), X_h^c(t^k_n - m_i h_i))h_i
\]

(3.10)

\[
= X_h^c(t^k_n) - \theta f_h(X_h^c(t^k_n), X_h^c(t^k_n - m_i h_i))h_i + \sum_{k=0}^{M-1} f_h(X_h^c(t^k_n), X_h^c(t^k_n - m_i h_i))h_i
\]

\[
+ \varepsilon \sqrt{h_l g(X_h^c(t^k_n), X_h^c(t^k_n - m_i h_i))}\xi^k
\]

where the random vector \(\xi^k \in \mathbb{R}^d\) has independent components, and each component is distributed as \(N(0, 1)\). This implies

\[
X_h^c(t^{k+1}_n) = X_h^c(t^k_n) - \theta f_h(X_h^c(t^k_n), X_h^c(t^k_n - m_i h_i))h_i + \sum_{k=0}^{M-1} f_h(X_h^c(t^k_n), X_h^c(t^k_n - m_i h_i))h_i
\]

(3.11)

\[
+ \varepsilon \sqrt{h_l g(X_h^c(t^k_n), X_h^c(t^k_n - m_i h_i))}\xi^k
\]

To simulate \(X_h^c(t^{k+1}_n)\), we use

\[
X_h^c(t^{k+1}_n) = X_h^c(t^k_n) - \theta f_h(X_h^c(t^k_n), X_h^c(t^k_n - m_i h_i))h_i - \sum_{k=0}^{M-1} f_h(X_h^c(t^k_n), X_h^c(t^k_n - m_i h_i))h_i
\]

(3.12)

For convenience, let

\[
Y_h^c(t) := X_h^c(t) - \theta f_h(X_h^c(t), X_h^c(t - \tau))h_i,
\]

and

\[
Y_h^c(t) := X_h^c(t) - \theta f_h(X_h^c(t), X_h^c(t - \tau))h_i.
\]

Furthermore, in order to ensure the existence and uniqueness of solutions to implicit equations (3.8) and (3.9), we assume that \(h_{l-1} \theta < \frac{2}{a_1}\) according to the monotone operator [16].

Thus, in this section, we set \(h^* \in \left(0, \frac{2}{a_1}\right]\), and let \(h_{l-1} \in (0, h^*]\) for \(\theta \in (0, 1]\), while for \(\theta = 0\), let \(h_{l-1} \in (0, 1]\).

We now have the following estimates.

**Lemma 3.2** Let assumptions (H1) and (H2) hold. Then, for any \(T > 0\) and \(p \geq 2\), we have

\[
\sup_{0 \leq t \leq T} \mathbb{E}|X_h^c(t)|^p \leq C,
\]

and

\[
\sup_{0 \leq t \leq T} \mathbb{E}|X_h^c(t)|^p \leq C.
\]
Proof. Here we concentrate on the first part, since the second part can be proved similarly. For $x > 0$, let $[x]$ be the integer part of $x$. For any $t \in [0, T]$, applying the Itô formula to $[1 + |Y_{h_1}^\varepsilon(t)|^2]^{\frac{p}{2}}$, we obtain

\[
\begin{align*}
E[1 + |Y_{h_1}^\varepsilon(t)|^2]^{\frac{p}{2}} &\leq E[1 + |Y_{h_1}^\varepsilon(0)|^2]^{\frac{p}{2}} \\
+ pE \int_0^t [1 + |Y_{h_1}^\varepsilon(s)|^2]^{\frac{p-2}{2}} \langle Y_{h_1}^\varepsilon(s), f_{h_1}(X_{h_1}^\varepsilon(\eta_{h_1}(s)), X_{h_1}^\varepsilon(\eta_{h_1}(s - \tau))) \rangle \, ds \\
+ \frac{1}{2}p(p-1)E \int_0^t [1 + |Y_{h_1}^\varepsilon(s)|^2]^{\frac{p-2}{2}} |\varepsilon g(X_{h_1}^\varepsilon(\eta_{h_1}(s)), X_{h_1}^\varepsilon(\eta_{h_1}(s - \tau)))|^2 \, ds \\
\leq &\, E[1 + |Y_{h_1}^\varepsilon(0)|^2]^{\frac{p}{2}} + \frac{1}{2}p(p-1)E \int_0^t [1 + |Y_{h_1}^\varepsilon(s)|^2]^{\frac{p-2}{2}} |\varepsilon g(X_{h_1}^\varepsilon(\eta_{h_1}(s)), X_{h_1}^\varepsilon(\eta_{h_1}(s - \tau)))|^2 \, ds \\
+ pE \int_0^t [1 + |Y_{h_1}^\varepsilon(s)|^2]^{\frac{p-2}{2}} \langle X_{h_1}^\varepsilon(\eta_{h_1}(s)), f_{h_1}(X_{h_1}^\varepsilon(\eta_{h_1}(s)), X_{h_1}^\varepsilon(\eta_{h_1}(s - \tau))) \rangle \, ds \\
+ pE \int_0^t [1 + |Y_{h_1}^\varepsilon(s)|^2]^{\frac{p-2}{2}} \langle Y_{h_1}^\varepsilon(s) - X_{h_1}^\varepsilon(\eta_{h_1}(s)), f_{h_1}(X_{h_1}^\varepsilon(\eta_{h_1}(s)), X_{h_1}^\varepsilon(\eta_{h_1}(s - \tau))) \rangle \, ds \\
=: &\, E[1 + |Y_{h_1}^\varepsilon(0)|^2]^{\frac{p}{2}} + E_1(t) + E_2(t) + E_3(t),
\end{align*}
\]

where $Y_{h_1}^\varepsilon(0) = \xi(0) - \theta f_{h_1}(\xi(0), \xi(-\tau))h_1$. With (H2), (3.4), (3.6) and the Young inequality, we have

\[
E_1(t) + E_2(t) \leq C \int_0^t \left[1 + |Y_{h_1}^\varepsilon(s)|^2 \right]^{\frac{p-2}{2}} \left(1 + |X_{h_1}^\varepsilon(\eta_{h_1}(s))|^2 + |X_{h_1}^\varepsilon(\eta_{h_1}(s - \tau))|^2 \right) \, ds
\]

\[
\leq C + C \int_0^t \left[1 + |Y_{h_1}^\varepsilon(s)|^2 \right]^{\frac{p-2}{2}} \left(|X_{h_1}^\varepsilon(\eta_{h_1}(s))|^p + |X_{h_1}^\varepsilon(\eta_{h_1}(s - \tau))|^p \right) \, ds
\]

\[
\leq C + C \int_0^t \left[1 + |X_{h_1}^\varepsilon(s)|^p + |\theta f_{h_1}(X_{h_1}^\varepsilon(s), X_{h_1}^\varepsilon(\eta_{h_1}(s - \tau)))h_1|^p
\right. \\
\left. + |X_{h_1}^\varepsilon(\eta_{h_1}(s))|^p + |X_{h_1}^\varepsilon(\eta_{h_1}(s - \tau))|^p \right] \, ds
\]

\[
\leq C + C \int_0^t \left(|X_{h_1}^\varepsilon(s)|^p + |X_{h_1}^\varepsilon(\eta_{h_1}(s))|^p + |X_{h_1}^\varepsilon(\eta_{h_1}(s - \tau))|^p \right) \, ds
\]

\[
\leq C + C \int_0^t \sup_{0 \leq u \leq s} E|X_{h_1}^\varepsilon(u)|^p \, ds.
\]
Due to (3.4) and the Young inequality,

Furthermore, it is easy to observe that,

\[ E_3(t) \leq p\mathbb{E} \int_0^t \left[ 1 + |Y_{h_i}^\varepsilon(s)|^2 \right]^{\frac{p-2}{2}} \langle Y_{h_i}^\varepsilon(s) - Y_{h_i}^\varepsilon(\eta_{h_i}(s)), f_{h_i}(X_{h_i}^\varepsilon(\eta_{h_i}(s)), X_{h_i}^\varepsilon(\eta_{h_i}(s - \tau))) \rangle ds \]

\[ = p\mathbb{E} \int_0^t \left[ 1 + |Y_{h_i}^\varepsilon(\eta_{h_i}(s))|^2 \right]^{\frac{p-2}{2}} \langle Y_{h_i}^\varepsilon(\eta_{h_i}(s)), f_{h_i}(X_{h_i}^\varepsilon(\eta_{h_i}(s)), X_{h_i}^\varepsilon(\eta_{h_i}(s - \tau))) \rangle ds \]

\[ + p\mathbb{E} \int_0^t \left\{ \left[ 1 + |Y_{h_i}^\varepsilon(\eta_{h_i}(s))|^2 \right]^{\frac{p-2}{2}} - \left[ 1 + |Y_{h_i}^\varepsilon(\eta_{h_i}(s))|^2 \right]^{\frac{p-2}{2}} \right\} \langle Y_{h_i}^\varepsilon(s) - Y_{h_i}^\varepsilon(\eta_{h_i}(s)), f_{h_i}(X_{h_i}^\varepsilon(\eta_{h_i}(s)), X_{h_i}^\varepsilon(\eta_{h_i}(s - \tau))) \rangle ds \]

\[ =: pE_{31}(t) + pE_{32}(t), \]

where

\[ Y_{h_i}^\varepsilon(s) - Y_{h_i}^\varepsilon(\eta_{h_i}(s)) = \int_{\eta_{h_i}(s)}^s f_{h_i}(X_{h_i}^\varepsilon(\eta_{h_i}(u)), X_{h_i}^\varepsilon(\eta_{h_i}(u - \tau))) du \]

\[ + \int_{\eta_{h_i}(s)}^s \varepsilon g(X_{h_i}^\varepsilon(\eta_{h_i}(u)), X_{h_i}^\varepsilon(\eta_{h_i}(u - \tau))) dW(u). \]

Due to (3.4) and the Young inequality,

\[ E_{31}(t) = \mathbb{E} \int_0^t \left[ 1 + |Y_{h_i}^\varepsilon(s)|^2 \right]^{\frac{p-2}{2}} \left\langle f_{h_i}(X_{h_i}^\varepsilon(\eta_{h_i}(s)), X_{h_i}^\varepsilon(\eta_{h_i}(s - \tau))) \right\rangle ds \]

\[ + \mathbb{E} \left\langle \int_{\eta_{h_i}(s)}^s \varepsilon g(X_{h_i}^\varepsilon(\eta_{h_i}(u)), X_{h_i}^\varepsilon(\eta_{h_i}(u - \tau))) dW(u) \right\rangle \]

\[ \leq \mathbb{E} \int_0^t \left[ 1 + |Y_{h_i}^\varepsilon(s)|^2 \right]^{\frac{p-2}{2}} \int_{\eta_{h_i}(s)}^s |f_{h_i}(X_{h_i}^\varepsilon(\eta_{h_i}(u)), X_{h_i}^\varepsilon(\eta_{h_i}(u - \tau)))| du \]

\[ \leq \mathbb{E} \int_0^t \left[ 1 + |Y_{h_i}^\varepsilon(s)|^2 \right]^{\frac{p-2}{2}} \int_{\eta_{h_i}(s)}^s |f_{h_i}(X_{h_i}^\varepsilon(\eta_{h_i}(s)), X_{h_i}^\varepsilon(\eta_{h_i}(s - \tau)))| ds \]

\[ \leq \mathbb{E} \int_0^t \left[ 1 + |Y_{h_i}^\varepsilon(s)|^2 \right]^{\frac{p-2}{2}} \left| f_{h_i}(X_{h_i}^\varepsilon(\eta_{h_i}(s)), X_{h_i}^\varepsilon(\eta_{h_i}(s - \tau))) \right|^2 ds \]

\[ \leq Ch_{t-1}^{1-2\delta} \mathbb{E} \int_0^t (1 + |X_{h_i}^\varepsilon(\eta_{h_i}(s))|)^p ds + Ch_{t-1}^{1-2\delta} h_{t-1}^{(1-\delta)p} \]

\[ \leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |X_{h_i}^\varepsilon(u)|^p ds. \]
Applying the Itô formula again, we obtain

\[
\begin{align*}
&[1 + |Y_{h_{1}}^{\varepsilon}(s)|^2]^\frac{p-2}{2} - [1 + |Y_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(s))|^{2}]^\frac{p-2}{2} \\
&+ (p-2) \int_{\eta_{h_{1}}(s)}^{s} [1 + |Y_{h_{1}}^{\varepsilon}(u)|^2]^\frac{p-2}{2} \langle Y_{h_{1}}^{\varepsilon}(u), f_{h_{1}}(X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(u)), X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(u - \tau))) \rangle \, du \\
&+ \frac{1}{2}(p-2)(p-3) \int_{\eta_{h_{1}}(s)}^{s} [1 + |Y_{h_{1}}^{\varepsilon}(u)|^2]^\frac{p-4}{2} |\varepsilon| g(X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(u)), X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(u - \tau)))) \rangle^2 \, du \\
&+ (p-2) \int_{\eta_{h_{1}}(s)}^{s} [1 + |Y_{h_{1}}^{\varepsilon}(u)|^2]^\frac{p-4}{2} \langle Y_{h_{1}}^{\varepsilon}(u), \varepsilon g(X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(u)), X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(u - \tau))) \rangle dW(u).
\end{align*}
\]

Hence,

\[
E_{32}(t) \leq (p-2) \mathbb{E} \int_{0}^{t} \int_{\eta_{h_{1}}(s)}^{s} [1 + |Y_{h_{1}}^{\varepsilon}(u)|^2]^\frac{p-2}{2} \langle Y_{h_{1}}^{\varepsilon}(u), f_{h_{1}}(X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(u)), X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(u - \tau))) \rangle \, du \\
\times \langle Y_{h_{1}}^{\varepsilon}(s) - Y_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(s)), f_{h_{1}}(X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(s)), X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(s - \tau))) \rangle ds \\
+ \frac{1}{2}(p-2)(p-3) \mathbb{E} \int_{0}^{t} \int_{\eta_{h_{1}}(s)}^{s} [1 + |Y_{h_{1}}^{\varepsilon}(u)|^2]^\frac{p-4}{2} |\varepsilon| g(X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(u)), X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(u - \tau))) \rangle^2 \, du \\
\times \langle Y_{h_{1}}^{\varepsilon}(s) - Y_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(s)), f_{h_{1}}(X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(s)), X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(s - \tau))) \rangle ds \\
+ (p-2) \mathbb{E} \int_{0}^{t} \int_{\eta_{h_{1}}(s)}^{s} [1 + |Y_{h_{1}}^{\varepsilon}(u)|^2]^\frac{p-4}{2} \langle Y_{h_{1}}^{\varepsilon}(u), \varepsilon g(X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(u)), X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(u - \tau))) \rangle dW(u) \\
\times \langle Y_{h_{1}}^{\varepsilon}(s) - Y_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(s)), f_{h_{1}}(X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(s)), X_{h_{1}}^{\varepsilon}(\eta_{h_{1}}(s - \tau))) \rangle ds \\
=: (p-2) E_{321} + \frac{1}{2}(p-2)(p-3) E_{322} + (p-2) E_{323}.
\]

Using (H2), [3.3], the Young inequality, the Hölder inequality and the Burkholder-Davis-
Gundy (BDG) inequality, since \( \delta \in (0, 1/2] \), we compute

\[
E_{321}(t) \leq \mathbb{E} \int_{0}^{t} \int_{\eta_{h_{i}}(s)}^{s} \left[ 1 + |Y_{h_{i}}^{\varepsilon}(u)|^{2} \right] ^{\frac{p-3}{2}} \langle Y_{h_{i}}^{\varepsilon}(u), f_{h_{i}}(X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(u)), X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(u - \tau))) \rangle du ds
\times \left( \int_{\eta_{h_{i}}(s)}^{s} f_{h_{i}}(X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(u)), X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(u - \tau))) du, f_{h_{i}}(X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(s)), X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(s - \tau))) \right) ds
\]
\[
+ \mathbb{E} \int_{0}^{t} \int_{\eta_{h_{i}}(s)}^{s} \left[ 1 + |Y_{h_{i}}^{\varepsilon}(u)|^{2} \right] ^{\frac{p-3}{2}} \langle Y_{h_{i}}^{\varepsilon}(u), f_{h_{i}}(X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(u)), X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(u - \tau))) \rangle du ds
\times \left( \int_{\eta_{h_{i}}(s)}^{s} \varepsilon g(X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(u)), X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(u - \tau))) du, f_{h_{i}}(X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(s)), X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(s - \tau))) \right) ds
\]
\[
\leq h_{l-1} \mathbb{E} \int_{0}^{t} \int_{\eta_{h_{i}}(s)}^{s} \left[ 1 + |Y_{h_{i}}^{\varepsilon}(u)|^{2} \right] ^{\frac{p-3}{2}} |f_{h_{i}}(X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(s)), X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(s - \tau)))|^{3} du ds
\]
\[
+ C \mathbb{E} \int_{0}^{t} \left( \int_{\eta_{h_{i}}(s)}^{s} \left[ 1 + |Y_{h_{i}}^{\varepsilon}(u)|^{2} \right] ^{\frac{p-3}{2}} |f_{h_{i}}(X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(s)), X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(s - \tau)))|^{2} du \right)^{\frac{p}{p-1}} ds
\]
\[
+ C \mathbb{E} \int_{0}^{t} \left( \int_{\eta_{h_{i}}(s)}^{s} |\varepsilon g(X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(u)), X_{h_{i}}^{\varepsilon}(\eta_{h_{i}}(u - \tau)))|^{2} du \right)^{\frac{p}{2}} ds
\]
\[
\leq C + C \int_{0}^{t} \sup_{0 \leq u \leq s} \mathbb{E} |X_{h_{i}}^{\varepsilon}(u)|^{p} ds + C h_{l-1}^{1-2\delta} \mathbb{E} \int_{0}^{t} \sup_{0 \leq u \leq s} \mathbb{E} |X_{h_{i}}^{\varepsilon}(u)|^{p} ds
\]
\[
\leq C + C \int_{0}^{t} \sup_{0 \leq u \leq s} \mathbb{E} |X_{h_{i}}^{\varepsilon}(u)|^{p} ds.
\]

Using the same techniques in the way to the estimation of \( E_{321}(t) \), we get

\[
E_{322}(t) \leq C + C \int_{0}^{t} \sup_{0 \leq u \leq s} \mathbb{E} |X_{h_{i}}^{\varepsilon}(u)|^{p} ds.
\]
Furthermore, by (H2) and (3.4) again, we have

\[
E_{323}(t) = \mathbb{E} \int_0^t \int_{\eta_1(t)}^s \left[ 1 + |Y^\varepsilon_{h_1}(u)|^2 \right]^{1/2} \langle \varepsilon g(X^\varepsilon_{h_1}(\eta_1(u)), X^\varepsilon_{h_1}(\eta_1(u - \tau)))dW(u) \right. \\
\times \left. \left\langle f_{h_1}(X^\varepsilon_{h_1}(\eta_1(u)), X^\varepsilon_{h_1}(\eta_1(u - \tau)))du, f_{h_1}(X^\varepsilon_{h_1}(\eta_1(s)), X^\varepsilon_{h_1}(\eta_1(s - \tau))) \right\rangle \right\rangle ds \\
+ \mathbb{E} \int_0^t \int_{\eta_1(s)}^s \left[ 1 + |Y^\varepsilon_{h_1}(u)|^2 \right]^{-1/2} \langle \varepsilon g(X^\varepsilon_{h_1}(\eta_1(u)), X^\varepsilon_{h_1}(\eta_1(u - \tau)))dW(u) \\
\times \varepsilon g \left( X^\varepsilon_{h_1}(\eta_1(u)), X^\varepsilon_{h_1}(\eta_1(u - \tau)) \right) \rangle \rangle ds \\
= \mathbb{E} \int_0^t \int_{\eta_1(s)}^s \left[ 1 + |Y^\varepsilon_{h_1}(u)|^2 \right]^{-1/2} \langle \varepsilon g(X^\varepsilon_{h_1}(\eta_1(u)), X^\varepsilon_{h_1}(\eta_1(u - \tau)))dW(u) \\
\times \varepsilon g \left( X^\varepsilon_{h_1}(\eta_1(u)), X^\varepsilon_{h_1}(\eta_1(u - \tau)) \right) \rangle \rangle ds \\
\leq \mathbb{E} \int_0^t \int_{\eta_1(s)}^s \left[ 1 + |Y^\varepsilon_{h_1}(u)|^2 \right]^{-1/2} \langle \varepsilon g(X^\varepsilon_{h_1}(\eta_1(u)), X^\varepsilon_{h_1}(\eta_1(u - \tau))) \rangle \right\rangle ds \\
\leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |X^\varepsilon_{h_1}(u)|^p ds.
\]

By sorting these equations, we conclude that

\[
E_3(t) \leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |X^\varepsilon_{h_1}(u)|^p ds.
\]

Thus, the estimation of \( E_1(t) - E_3(t) \) results in

\[
\sup_{0 \leq u \leq t} \mathbb{E} |Y^\varepsilon_{h_1}(u)|^p \leq \sup_{0 \leq u \leq t} \mathbb{E} \left[ 1 + |Y^\varepsilon_{h_1}(u)|^2 \right]^{1/2} \leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |X^\varepsilon_{h_1}(u)|^p ds. \tag{3.13}
\]

By the relationship between \( X^\varepsilon_{h_1}(t) \) and \( Y^\varepsilon_{h_1}(t) \), it is easy to derive from (3.4) that

\[
\sup_{0 \leq u \leq t} \mathbb{E} |X^\varepsilon_{h_1}(u)|^p \leq C h_{l-1}^{1-\delta p} + \sup_{0 \leq u \leq t} \mathbb{E} |Y^\varepsilon_{h_1}(u)|^p \leq C + C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |X^\varepsilon_{h_1}(u)|^p ds.
\]

Finally, the desired result follows by the Gronwall inequality. \( \square \)

Let \( Z_{h_1} \) be the deterministic solution to

\[
Z_{h_1}(t) - \theta f_{h_1}(Z_{h_1}(t), Z_{h_1}(t-\tau))h_1 = \xi(0) - \theta f_{h_1}(\xi(0), \xi(-\tau))h_1 + \int_0^t f_{h_1}(Z_{h_1}(\eta_1(s)), Z_{h_1}(\eta_1(s-\tau)))ds,
\]

33
which is the corresponding theta EM approximation to the ordinary differential delay equation obtained from (1.1) when $\varepsilon = 0$.

**Lemma 3.3** Let assumptions (H1) and (H2) hold. Then, for any $T > 0$ and $p \geq 2$, we have

$$\sup_{0 \leq t \leq T} \mathbb{E}|Z_{h_i}(t)|^p \leq C,$$

and

$$\sup_{0 \leq n < M^{i-1}, 1 \leq k \leq M} \mathbb{E}|Z_{h_i}(t_n^k) - Z_{h_i}(t_n)|^p \leq CM^{(1-\delta)p}h_i^{(1-\delta)p}.$$

**Proof.** Following the proof of Lemma 3.2, the first part is obvious. Denote by $\bar{Z}_{h_i}(t) = Z_{h_i}(t) - \theta f_{h_i}(Z_{h_i}(t), Z_{h_i}(t - \tau))h_i$. For any $n \in \{0, 1, \ldots, M^{i-1} - 1\}$ and $k \in \{1, \ldots, M\}$, by (3.1), we have

$$\mathbb{E}|\bar{Z}_{h_i}(t_n^k) - \bar{Z}_{h_i}(t_n)|^p \leq |kh_i|^{p-1}\mathbb{E} \int_{t_n}^{t_n^k} |f_{h_i}(Z_{h_i}(\eta_{h_i}(s)), Z_{h_i}(\eta_{h_i}(s - \tau)))|^p ds \leq CM^{(1-\delta)p}h_i^{(1-\delta)p}.$$

On the other side, we see

$$\mathbb{E}|Z_{h_i}(t_n^k) - Z_{h_i}(t_n)|^p \leq C\mathbb{E}|\bar{Z}_{h_i}(t_n^k) - \bar{Z}_{h_i}(t_n)|^p + CM^{(1-\delta)p}h_i^{(1-\delta)p}.$$

Thus, the desired assertion follows. \qed

**Lemma 3.4** Let assumptions (H1) and (H2) hold. Then, for any $p > 0$, we have

$$\sup_{0 \leq n < M^{i-1}, 1 \leq k \leq M} \mathbb{E}[|X_{h_i}^\varepsilon(t_n^k) - X_{h_i}^\varepsilon(t_n)|^p] \leq CM^{(1-\delta)p}h_i^{(1-\delta)p} + C\varepsilon^p M^{p/2}h_i^{p/2}.$$

**Proof.** For $n \in \{0, 1, \ldots, M^{i-1} - 1\}$ and $k \in \{1, \ldots, M\}$, we see

$$Y_{h_i}^\varepsilon(t_n^k) = Y_{h_i}^\varepsilon(t_n) + \int_{t_n}^{t_n^k} f_{h_i}(X_{h_i}^\varepsilon(\eta_{h_i}(s)), X_{h_i}^\varepsilon(\eta_{h_i}(s - \tau)))ds$$

$$+ \varepsilon \int_{t_n}^{t_n^k} g(X_{h_i}^\varepsilon(\eta_{h_i}(s)), X_{h_i}^\varepsilon(\eta_{h_i}(s - \tau)))dW(s).$$

By the elementary inequality $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, $p \geq 1$, we compute for $p \geq 1$

$$\mathbb{E}|Y_{h_i}^\varepsilon(t_n^k) - Y_{h_i}^\varepsilon(t_n)|^p \leq 2^{p-1} \mathbb{E} \left[ \left| \int_{t_n}^{t_n^k} f_{h_i}(X_{h_i}^\varepsilon(\eta_{h_i}(s)), X_{h_i}^\varepsilon(\eta_{h_i}(s - \tau)))ds \right|^p \right]$$

$$+ 2^{p-1}\varepsilon^p \mathbb{E} \left[ \left| \int_{t_n}^{t_n^k} g(X_{h_i}^\varepsilon(\eta_{h_i}(s)), X_{h_i}^\varepsilon(\eta_{h_i}(s - \tau)))dW(s) \right|^p \right].$$

34
With (H2), (3.4), Lemma 3.2, the Hölder inequality and the BDG inequality, we derive

\[
\begin{align*}
\mathbb{E}[Y_h^\varepsilon(t_n^k) - Y_h^\varepsilon(t_n^k)]^p &\leq 2^{p-1}M^{p-1}h_t^{p-1}\mathbb{E}\int_{t_n}^{t_n^k} |f_h(X_h^\varepsilon(\eta_h(s)), X_h^\varepsilon(\eta_h(s-\tau)))|^p \,ds \\
&+ C\varepsilon^pM^{\frac{p}{2}-1}h_t^{\frac{p}{2}}\mathbb{E}\int_{t_n}^{t_n^k} |g(X_h^\varepsilon(\eta_h(s)), X_h^\varepsilon(\eta_h(s-\tau)))|^p \,ds \\
&\leq CM^{(1-\delta)p}h_t^{(1-\delta)p} + C\varepsilon^p M^{p/2}h_t^{p/2}.
\end{align*}
\]  

(3.14)

Since we have

\[X_h^\varepsilon(t_n^k) - X_h^\varepsilon(t_n^k) = Y_h^\varepsilon(t_n^k) - Y_h^\varepsilon(t_n^k) + \theta f_h(X_h^\varepsilon(t_n^k), X_h^\varepsilon(t_n^k - m\eta_h))h_t - \theta f_h(X_h^\varepsilon(t_n^k), X_h^\varepsilon(t_n^k - m\eta_h))h_t.\]

This combines with (3.14) lead to

\[
\mathbb{E}[X_h^\varepsilon(t_n^k)|X_h^\varepsilon(t_n)| |X_h^\varepsilon(n\eta_h)|]^p \leq CM^{(1-\delta)p}h_t^{(1-\delta)p} + C\varepsilon^p M^{p/2}h_t^{p/2}.
\]

The desired result then follows for \( p \geq 1 \). Finally, one can use the Young inequality to get the results for \( p \in (0, 1) \).

**Lemma 3.5** Let assumptions (H1) and (H2) hold. Then, for any \( T > 0 \) and \( p > 0 \), we have

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \sup_{0 \leq n < M^t \eta_h \leq t < (n+1)\eta_h} |X_h^\varepsilon(t) - X_h^\varepsilon(n\eta_h)|^p \right] \leq Ch_t^{(1-\delta)p} + C\varepsilon^p h_t^{p/2},
\]

and

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \sup_{0 \leq n < M^t \eta_h \leq t < (n+1)\eta_h} |Z_h(t) - Z_h(n\eta_h)|^p \right] \leq Ch_t^{(1-\delta)p}.
\]

**Proof.** Similar to the proof of Lemma 3.1, the result is obvious. \( \square \)

**Lemma 3.6** Let assumptions (H1) and (H2) hold. Then, for any \( T > 0 \) and \( p \geq 2 \), we have

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |X_h^\varepsilon(t) - Z_h(t)|^p \right] \leq CM^{\delta p/2}h_t^{\delta p/2} + C\varepsilon^p M^{\frac{1}{4} - \frac{1}{4} \delta} h_t^{\frac{1}{4} - \frac{2\delta}{4}} + C\varepsilon^p,
\]

and

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |X_h^{-1}(t) - Z_h^{-1}(t)|^p \right] \leq CM^{\delta p/2}h_t^{\delta p/2} + C\varepsilon^p M^{\frac{1}{4} - \frac{1}{4} \delta} h_t^{-\frac{1}{4} - \frac{2\delta}{4}} + C\varepsilon^p.
\]

**Proof.** By the definition of \( Y_h^\varepsilon(t) \) and \( Z_h(t) \),

\[Y_h^\varepsilon(t) - Z_h(t) = \int_0^t [f_h(X_h^\varepsilon(\eta_h(s)), X_h^\varepsilon(\eta_h(s-\tau))) - f_h(Z_h(\eta_h(s)), Z_h(\eta_h(s-\tau))))] ds + \varepsilon \int_0^t g(X_h^\varepsilon(\eta_h(s)), X_h^\varepsilon(\eta_h(s-\tau))) dW(s),\]

35
thus, by the Itô formula,

\[
\begin{align*}
|Y_{ht}^{\varepsilon}(t) - \bar{Z}_{ht}(t)|^p & \leq p \int_0^t |Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s)|^{p-2} \langle Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s) \rangle \, ds \\

f_{ht}(X_{ht}^{\varepsilon}(\eta_{ht}(s)), X_{ht}^{\varepsilon}(\eta_{ht}(s - \tau))) - f_{ht}(Z_{ht}(\eta_{ht}(s)), Z_{ht}(\eta_{ht}(s - \tau))) \rangle & \, ds \\
+ \frac{p(p - 1)}{2} \int_0^t |Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s)|^{p-2} |\varepsilon g(X_{ht}^{\varepsilon}(\eta_{ht}(s)), X_{ht}^{\varepsilon}(\eta_{ht}(s - \tau)))|^2 \, ds \\
+ p \int_0^t |Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s)|^{p-2} \langle Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s) \rangle, \varepsilon g(X_{ht}^{\varepsilon}(\eta_{ht}(s)), X_{ht}^{\varepsilon}(\eta_{ht}(s - \tau))) \rangle dW(s) \rangle \\
\leq H_1(t) + H_2(t) + H_3(t) + H_4(t) + H_5(t) + H_6(t) + H_7(t) + H_8(t),
\end{align*}
\]

where

\[
H_1(t) = p \int_0^t |Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s)|^{p-2} \langle X_{ht}^{\varepsilon}(s) - X_{ht}^{\varepsilon}(\eta_{ht}(s)),

f_{ht}(X_{ht}^{\varepsilon}(\eta_{ht}(s)), X_{ht}^{\varepsilon}(\eta_{ht}(s - \tau))) - f_{ht}(Z_{ht}(\eta_{ht}(s)), Z_{ht}(\eta_{ht}(s - \tau))) \rangle \, ds,
\]

\[
H_2(t) = p \int_0^t |Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s)|^{p-2} \langle X_{ht}^{\varepsilon}(\eta_{ht}(s)) - Z_{ht}(\eta_{ht}(s)),

f_{ht}(X_{ht}^{\varepsilon}(\eta_{ht}(s)), X_{ht}^{\varepsilon}(\eta_{ht}(s - \tau))) - f(X_{ht}^{\varepsilon}(\eta_{ht}(s)), X_{ht}^{\varepsilon}(\eta_{ht}(s - \tau))) \rangle \, ds,
\]

\[
H_3(t) = p \int_0^t |Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s)|^{p-2} \langle X_{ht}^{\varepsilon}(\eta_{ht}(s)) - Z_{ht}(\eta_{ht}(s)),

f(X_{ht}^{\varepsilon}(\eta_{ht}(s)), X_{ht}^{\varepsilon}(\eta_{ht}(s - \tau))) - f(Z_{ht}(\eta_{ht}(s)), Z_{ht}(\eta_{ht}(s - \tau))) \rangle \, ds,
\]

\[
H_4(t) = p \int_0^t |Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s)|^{p-2} \langle Z_{ht}(\eta_{ht}(s)) - Z_{ht}(\eta_{ht}(s)),

f(Z_{ht}(\eta_{ht}(s)), Z_{ht}(\eta_{ht}(s - \tau))) - f(h_{ht}(\eta_{ht}(s)), Z_{ht}(\eta_{ht}(s - \tau))) \rangle \, ds,
\]

\[
H_5(t) = p \int_0^t |Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s)|^{p-2} \langle Z_{ht}(\eta_{ht}(s)) - Z_{ht}(s),

f_{ht}(X_{ht}^{\varepsilon}(\eta_{ht}(s)), X_{ht}^{\varepsilon}(\eta_{ht}(s - \tau))) - f_{ht}(Z_{ht}(\eta_{ht}(s)), Z_{ht}(\eta_{ht}(s - \tau))) \rangle \, ds,
\]

\[
H_6(t) = - \theta h_{ht} \int_0^t |Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s)|^{p-2} \langle f_{ht}(X_{ht}^{\varepsilon}(s), X_{ht}^{\varepsilon}(s - \tau)) - f_{ht}(Z_{ht}(s), Z_{ht}(s - \tau)),

f_{ht}(X_{ht}^{\varepsilon}(\eta_{ht}(s)), X_{ht}^{\varepsilon}(\eta_{ht}(s - \tau))) - f_{ht}(Z_{ht}(\eta_{ht}(s)), Z_{ht}(\eta_{ht}(s - \tau))) \rangle \, ds,
\]

\[
H_7(t) = \frac{p(p - 1)}{2} \int_0^t |Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s)|^{p-2} |\varepsilon g(X_{ht}^{\varepsilon}(\eta_{ht}(s)), X_{ht}^{\varepsilon}(\eta_{ht}(s - \tau)))|^2 \, ds,
\]

\[
H_8(t) = p \int_0^t |Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s)|^{p-2} \langle Y_{ht}^{\varepsilon}(s) - \bar{Z}_{ht}(s), \varepsilon g(X_{ht}^{\varepsilon}(\eta_{ht}(s)), X_{ht}^{\varepsilon}(\eta_{ht}(s - \tau))) \rangle dW(s).
\]
By the Hölder inequality, (3.4) and Lemma 3.5, we get

\[
\mathbb{E} \left( \sup_{0 \leq u \leq t} |H_1(u)|^p \right) \leq C \mathbb{E} \int_0^t |Y_{h_1}(s) - \bar{Z}_{h_1}(s)|^p ds + C \mathbb{E} \int_0^t |X_{\bar{h}_1}(s) - X_{h_1}(\eta_{h_1}(s))|^p/2 ds \\
|f_{h_1}(X_{\bar{h}_1}(\eta_{h_1}(s)), X_{h_1}(\eta_{h_1}(s - \tau))) - f_{h_1}(Z_{h_1}(\eta_{h_1}(s)), Z_{h_1}(\eta_{h_1}(s - \tau)))|^p/2 ds \\
\leq C \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} |X_{\bar{h}_1}(u) - Z_{h_1}(u)|^p \right] ds + CM \frac{1-2\delta}{2} \epsilon^\frac{1-2\delta}{\bar{h}_1} + C\epsilon^\frac{p}{2} M \frac{1-2\delta}{h_1} \epsilon^\frac{1-2\delta}{\bar{h}_1}.
\]

By (3.4), (3.7), Lemma 3.2, Lemma 3.3 and the Hölder inequality again,

\[
\mathbb{E} \left( \sup_{0 \leq u \leq t} |H_2(u)|^p \right) \leq C \mathbb{E} \int_0^t |Y_{h_1}(s) - \bar{Z}_{h_1}(s)|^p ds + C \mathbb{E} \int_0^t |X_{\bar{h}_1}(\eta_{h_1}(s)) - Z_{h_1}(\eta_{h_1}(s))|^p/2 ds \\
|f_{h_1}(X_{\bar{h}_1}(\eta_{h_1}(s)), X_{h_1}(\eta_{h_1}(s - \tau))) - f_{h_1}(X_{h_1}(\eta_{h_1}(s)), X_{h_1}(\eta_{h_1}(s - \tau)))|^p/2 ds \\
\leq C \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} |X_{\bar{h}_1}(u) - Z_{h_1}(u)|^p \right] ds + CM^{(1-\delta)p} h_1^{(1-\delta)p} + CM^{\delta p/2} h_1^{\delta p/2}.
\]

By assumption (H1) and the Hölder inequality,

\[
\mathbb{E} \left( \sup_{0 \leq u \leq t} |H_3(u)|^p \right) \leq C \mathbb{E} \int_0^t |Y_{h_1}(s) - Z_{h_1}(s)|^p ds \\
+ C \mathbb{E} \int_0^t |X_{\bar{h}_1}(\eta_{h_1}(s)) - Z_{h_1}(\eta_{h_1}(s))|^p ds + C \mathbb{E} \int_0^t |X_{\bar{h}_1}(\eta_{h_1}(s - \tau)) - Z_{h_1}(\eta_{h_1}(s - \tau))|^p ds \\
\leq C \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} |X_{\bar{h}_1}(u) - Z_{h_1}(u)|^p \right] ds + CM^{(1-\delta)p} h_1^{(1-\delta)p}.
\]

Similar to the estimation of $H_2(t)$, we derive

\[
\mathbb{E} \left( \sup_{0 \leq u \leq t} |H_4(u)|^p \right) \leq C \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} |X_{\bar{h}_1}(u) - Z_{h_1}(u)|^p \right] ds + CM^{(1-\delta)p} h_1^{(1-\delta)p} + CM^{\delta p/2} h_1^{\delta p/2}.
\]

With (3.4) and Lemma 3.5, we arrive at

\[
\mathbb{E} \left( \sup_{0 \leq u \leq t} |H_5(u)|^p \right) \leq C \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} |X_{\bar{h}_1}(u) - Z_{h_1}(u)|^p \right] ds + CM^{1-2\delta} \epsilon^\frac{1-2\delta}{h_1}.
\]

By (3.4) again, it is easy to see

\[
\mathbb{E} \left( \sup_{0 \leq u \leq t} |H_6(u)|^p \right) \leq C \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} |X_{\bar{h}_1}(u) - Z_{h_1}(u)|^p \right] ds + CM^{(1-\delta)p} h_1^{(1-\delta)p}.
\]

Then, (H2), (3.4), Lemma 3.2 and the Hölder inequality lead to

\[
\mathbb{E} \left( \sup_{0 \leq u \leq t} |H_7(u)|^p \right) \leq C \mathbb{E} \int_0^t |Y_{h_1}(s) - \bar{Z}_{h_1}(s)|^p ds + C\epsilon^p \\
\leq C \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} |X_{\bar{h}_1}(u) - Z_{h_1}(u)|^p \right] ds + CM^{(1-\delta)p} h_1^{(1-\delta)p} + C\epsilon^p.
\]
Moreover, with (H2), (3.4), Lemma 3.2 and the BDG inequality,

\[
\mathbb{E} \left( \sup_{0 \leq u \leq t} |H_8(u)|^p \right) \leq \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |Y_{h_1}^\varepsilon(s) - \bar{Z}_{h_1}(s)|^p \right] + CM^{(1-\delta)p} h_t^{(1-\delta)p} + C\varepsilon^p.
\]

Sorting the above inequations together leads to

\[
\mathbb{E} \left[ \sup_{0 \leq u \leq t} |Y_{h_1}^\varepsilon(u) - \bar{Z}_{h_1}(u)|^p \right] \leq C \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} |X_{h_1}^\varepsilon(u) - Z_{h_1}(u)|^p \right] ds + C\varepsilon^p + C\varepsilon^2 M^{\frac{1-2\delta}{4}p} h_t^{\frac{1-2\delta}{4}p} + CM^{\delta p/2} h_t^{\delta p/2}.
\]

Since we have

\[
\mathbb{E} \left[ \sup_{0 \leq u \leq t} |X_{h_1}^\varepsilon(u) - Z_{h_1}(u)|^p \right] \leq \mathbb{E} \left[ \sup_{0 \leq u \leq t} |Y_{h_1}^\varepsilon(u) - \bar{Z}_{h_1}(u)|^p \right] + CM^{(1-\delta)p} h_t^{(1-\delta)p}.
\]

The Gronwall inequality then leads to the desired result. By the same technique, the second part can be verified. ∎

**Theorem 3.1** Let assumptions (H1) and (H2) hold. Then

\[
\sup_{0 \leq t \leq T} \mathbb{E} |X_{h_1}^\varepsilon(t) - X_{h_{l-1}}^\varepsilon(t)|^2 \leq Ch_t^{1-2\delta} + Ch_{l-2}^{2\delta} + C\varepsilon^2 h_{l-1}.
\]

**Proof.** For any \( t \in [0, T] \), by (3.8) and (3.9), we get

\[
Y_{h_1}^\varepsilon(t) - Y_{h_{l-1}}^\varepsilon(t) = Y_{h_1}^\varepsilon(0) - Y_{h_{l-1}}^\varepsilon(0) + \int_0^t [f_{h_1}(X_{h_1}^\varepsilon(\eta_{h_1}(s)), X_{h_1}^\varepsilon(\eta_{h_1}(s - \tau))) - f_{h_{l-1}}(X_{h_{l-1}}^\varepsilon(\eta_{h_{l-1}}(s)), X_{h_{l-1}}^\varepsilon(\eta_{h_{l-1}}(s - \tau)))]) ds + \varepsilon \int_0^t [g(X_{h_1}^\varepsilon(\eta_{h_1}(s)), X_{h_1}^\varepsilon(\eta_{h_1}(s - \tau))) - g(X_{h_{l-1}}^\varepsilon(\eta_{h_{l-1}}(s)), X_{h_{l-1}}^\varepsilon(\eta_{h_{l-1}}(s - \tau)))] dW(s).
\]
By the Itô formula, we get

\[
\mathbb{E}|Y^\varepsilon_{h_i}(t) - Y^\varepsilon_{h_{i-1}}(t)|^2 \leq 2\mathbb{E}|Y^\varepsilon_{h_i}(0) - Y^\varepsilon_{h_{i-1}}(0)|^2 + 2\mathbb{E} \int_0^t \langle Y^\varepsilon_{h_i}(s) - Y^\varepsilon_{h_{i-1}}(s), \\
\mathbb{E}|f_{h_i}(X^\varepsilon_{h_i}(\eta_i(s)), X^\varepsilon_{h_i}(\eta_i(s - \tau))) - f_{h_{i-1}}(X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s)), X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s - \tau)))\rangle ds \\
+ C\varepsilon^2\mathbb{E} \int_0^t |g(X^\varepsilon_{h_i}(\eta_i(s)), X^\varepsilon_{h_i}(\eta_i(s - \tau))) - g(X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s)), X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s - \tau)))|^2 ds \\
= 2\mathbb{E}|Y^\varepsilon_{h_i}(0) - Y^\varepsilon_{h_{i-1}}(0)|^2 + 2\mathbb{E} \int_0^t \langle Y^\varepsilon_{h_i}(s) - Y^\varepsilon_{h_{i-1}}(s), \\
\mathbb{E}|f_{h_i}(X^\varepsilon_{h_i}(\eta_i(s)), X^\varepsilon_{h_i}(\eta_i(s - \tau))) - f_{h_{i-1}}(X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s)), X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s - \tau)))\rangle ds \\
+ 2\mathbb{E} \int_0^t \langle X^\varepsilon_{h_i}(s) - X^\varepsilon_{h_{i-1}}(s), f(X^\varepsilon_{h_i}(\eta_i(s)), X^\varepsilon_{h_i}(\eta_i(s - \tau)))\rangle ds \\
- f_{h_{i-1}}(X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s)), X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s - \tau))) ds + 2\mathbb{E} \int_0^t \langle X^\varepsilon_{h_i}(\eta_i(s)) - X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s)), \\
f_{h_{i-1}}(X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s)), X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s - \tau)))\rangle ds \\
+ 2\mathbb{E} \int_0^t \langle X^\varepsilon_{h_i}(s) - X^\varepsilon_{h_{i-1}}(s), X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s))\rangle ds \\
- f_{h_{i-1}}(X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s)), X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s - \tau))) ds + 2\theta h_i \mathbb{E} \int_0^t \langle f_h(X^\varepsilon_{h_i}(s), X^\varepsilon_{h_i}(s - \tau)) - f_{h_{i-1}}(X^\varepsilon_{h_{i-1}}(s), X^\varepsilon_{h_{i-1}}(s - \tau)), \\
f_{h_i}(X^\varepsilon_{h_i}(\eta_i(s)), X^\varepsilon_{h_i}(\eta_i(s - \tau))) - f_{h_{i-1}}(X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s)), X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s - \tau)))\rangle ds \\
+ C\varepsilon^2\mathbb{E} \int_0^t |g(X^\varepsilon_{h_i}(\eta_i(s)), X^\varepsilon_{h_i}(\eta_i(s - \tau))) - g(X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s)), X^\varepsilon_{h_{i-1}}(\eta_{i-1}(s - \tau)))|^2 ds \\
\leq Ch_{i-1}^{2-2\delta} + J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t) + J_6(t).
\]

By (3.4), (3.7) and Lemma 3.2 we see

\[
J_1(t) \leq C\varepsilon \mathbb{E} \int_0^t |X^\varepsilon_{h_i}(s) - X^\varepsilon_{h_{i-1}}(s)|^2 ds + C\varepsilon^2 + Ch_{i-1}^{2-2\delta}
+ \mathbb{E} \int_0^t |f_{h_i}(X^\varepsilon_{h_i}(\eta_i(s)), X^\varepsilon_{h_i}(\eta_i(s - \tau))) - f(X^\varepsilon_{h_i}(\eta_i(s)), X^\varepsilon_{h_i}(\eta_i(s - \tau)))|^2 ds
\leq C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|X^\varepsilon_{h_i}(u) - X^\varepsilon_{h_{i-1}}(u)|^2 ds + Ch_{i-1}^{2\delta}.
\]

Since we have \(\delta < 1/2\), similar to the estimation of \(J_1(t)\), we obtain

\[
J_2(t) \leq C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|X^\varepsilon_{h_i}(u) - X^\varepsilon_{h_{i-1}}(u)|^2 ds + Ch_{i-2}^{2\delta}.
\]

By (3.5), we compute

\[
J_3(t) \leq C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|X^\varepsilon_{h_i}(u) - X^\varepsilon_{h_{i-1}}(u)|^2 ds.
\]
By (3.4) and Lemma 3.5, we derive
\[ J_4(t) \leq Ch_1^{1-2\delta} + C\varepsilon h_1^{1-2\delta}. \]

With (3.4) again,
\[ J_5(t) \leq Ch_1^{1-2\delta}. \]

By (3.2), Lemma 3.2 and Lemma 3.4, we derive
\[ J_6(t) \leq C \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|X_{h_l}^\varepsilon(u) - X_{h_l-1}^\varepsilon(u)|^2 ds + Ch_l^{2-2\delta} + \varepsilon^2 h_{l-1}. \]

By sorting those inequalities and using the Gr"{o}nwall inequality, the desired result can be obtained.

\[ \square \]

Remark 3.4 Under one-sided Lipschitz condition, we show that the second moment of two coupled paths is bounded by \( O(h_l^{1-2\delta} + h_l^{2\delta} + \varepsilon^2 h_{l-1}) \). By Theorem 5.1, as the procedure of Theorem 2.3, we can show that under assumptions (H1) and (H2), we have
\[ \sup_{0 \leq n < M^{l-1}} \text{Var}(\Psi(X_{h_l}^\varepsilon(t_n)) - \Psi(X_{h_l-1}^\varepsilon(t_n))) \leq Ch_l^{1-2\delta} + Ch_{l-2}^{2\delta} + C\varepsilon^2 h_{l-1}. \]

Different from the global Lipschitz case, the efficiency can not be improved compared with the second moment since the drift coefficient is one-sided Lipschitz not global Lipschitz.

References

[1] Anderson D.F., Higham D.J., Sun Y., Multilevel Monte Carlo for stochastic differential equations with small noise. SIAM J. Numer. Anal., 54, 505-529, 2016.

[2] Milstein G.N., Tret’yakov M.V., Mean-square numerical methods for stochastic differential equations with small noises. SIAM J. Sci. Comput., 18, 1067-1087, 1997.

[3] R"{o}misch W., Winkler R., Stepwise control for mean-square numerical methods for stochastic differential equations with small noise. SIAM J. Sci. Comput., 28, 604-625, 2006.

[4] Kloeden P. E., Platen E., Numerical Solution of Stochastic Differential Equation. Springer-Verlag, Berlin Heidelberg, 1992.

[5] Platen E., Bruti-Liberati N., Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, 2011.

[6] Higham D.J., Mao X.R., Yuan C.G., Almost Sure and Moment Exponential Stability in the Numerical Simulation of Stochastic Differential Equations. SIAM Numer. Anal., 45, 592-609, 2007.
[7] Hutzenthaler M., Jentzen A., Kloeden P.E., Strong and weak divergence in finite time of Euler’s method for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Proc R Soc Lond Ser A Math Phys Eng Sci*, 467, 1563-1576, 2011.

[8] Mao X., The truncated Euler-Maruyama method for stochastic differential equations. *J Comput Appl Math*, 290, 370-383, 2015.

[9] Kloeden P.E., Platen E., Schurz H., The numerical solution of non-linear stochastic dynamical systems: A brief introduction. *Int. J. Bif. Chaos.*, 1, 277-286, 1991.

[10] Tan L., Yuan C.G., Strong convergence of a tamed theta scheme for NSDDEs with one-sided Lipschitz drift. *Appl. Math. Comput.*, 338, 607-623, 2018.

[11] Wang X., Gan S.Q., The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients. *J. Difference Equ. Appl.*, 19(3), 466-490, 2013.

[12] Giles M.B., Multi-level Monte Carlo path simulation. *Oper. Res.*, 56(3), 607-617, 2008.

[13] Giles M.B., *Improved Multilevel Monte Carlo Convergence using the Milstein Scheme, Monte Carlo and Quasi-Monte Carlo Methods*. Springer, Berlin, 2008.

[14] Hutzenthaler M., Jentzen A., Kloeden P.E., Divergence of the multilevel monte carlo Euler method for nonlinear stochastic differential equations. *Annals Appl. Probab.*, 23, 1913-1966, 2013.

[15] Guo Q., Liu W., Mao X., et al. Multi-level Monte Carlo methods with the truncated Euler-Maruyama scheme for stochastic differential equation. *International Journal of Computer Mathematics*, doi: 10.1080/00207160.2017.1329533, 2017.

[16] Zeidler E., *Nonlinear Functional Analysis and its Applications II*, Springer-Verlag, New York, 1990.