Stability of Fractional Chern Insulators in the Effective Continuum Limit of $|C| > 1$
Harper-Hofstadter Bands

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Outline

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- Composite Fermion Theory
- Scaling of Energies

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- Thermodynamic Effective Continuum

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- Many-body Gaps
- Correlation Functions
- Particle Entanglement Spectra

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Introduction

- **Fractional Chern Insulators (FCIs)** generalize the FQHE to systems with non-trivial Chern number, \( C \).
- The Harper-Hofstadter model has provided some of the first examples of FCIs (Sørensen et al., 2005), and hosts a fractal energy spectrum with any desired Chern number.
- Examine states of the composite fermion (CF) series predicted by Möller & Cooper, 2015.
- Generalize the \( n_\phi \to 0 \) continuum limit to the **effective continuum limit** at \( n_\phi \to 1/|C| \) (Möller & Cooper, 2015).
- Investigate the **stability** (i.e. robustness in the effective continuum limit) of the many-body gap, \( \Delta \).
Harper-Hofstadter Model

We consider $N$ spinless particles hopping on an $N_x \times N_y$ square lattice with a constant effective magnetic flux.

$$H = \sum_{i,j} \left[ t_{ij} e^{\phi_{ij}} c_j^\dagger c_i + \text{h.c.} \right] + \mathcal{P}_{LB} \left[ \sum_{i<j} V_{ij} :\rho(r_i)\rho(r_j) : \right] \mathcal{P}_{LB}$$

- hopping parameter
- lowest-band projection operator
- bosons $\Rightarrow$ on-site interactions
- fermions $\Rightarrow$ nearest-neighbour interactions
Composite Fermion Theory

Predicted **filling fraction** from CF theory on the lattice for a well-isolated lowest band (Möller & Cooper, 2015):

\[ \nu = \frac{r}{|kC|r + 1} \equiv \frac{r}{s}, \text{ where } r \text{ and } s \text{ are co-prime} \]

- \( C \) = Chern number of the band
- \( k \) = number of flux quanta attached to the particles
- \( |r| \) = number of bands filled in the CF spectrum
- \( \text{sgn}(r) = \text{sgn}(C^*) \) for the CF band relative to \( C \)
- \( |s| \) = ground state degeneracy
Scaling & Stability

Aim to consider 2D isotropic limit \( \Rightarrow \) demand \( N_x = N_y \).

- Note: \( \frac{\text{Nbr. Sites}}{\text{Nbr. MUCs}} = q \) is a measure of MUC size.

Scaling relations (Bauer et al., 2016):

\[
\Delta \propto q^{-1} \quad \text{for bosons (contact interactions)},
\]
\[
\Delta \propto q^{-2} \quad \text{for fermions (NN interactions)}.
\]

Investigate stability (robustness of many-body gap)...

1. ...in the effective continuum limit: \( q \rightarrow \infty \).
2. ...in the thermodynamic limit: \( N \rightarrow \infty \).
Harper-Hofstadter Model
Approaching the Effective Continuum

\[ n_{\phi} = \frac{p}{|C|p - \text{sgn}(C)} \equiv \frac{p}{q}, \quad p \in \mathbb{N}. \]

\[ \lim_{q \to \infty} \frac{1}{q} \]

\[ \lim_{p \to \infty} \frac{p}{2p \pm 1} \]

\[ (\text{e.g. Hormozi et al., 2012}) \]
Basic Method

1. Plot the many-body energy spectrum for a particular \( \{C, r, N\} \) configuration and for a variety of MUC sizes, \( q \). Identify the ground states, predicted by CF theory.

2. Read off the many-body gap, \( \Delta \), for each energy spectrum.

3. Plot \( \Delta \) against \( q \). Read off \( \lim_{q \to \infty} (q^2 \Delta) \).

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**Illustration:**

- Plot of \( (E-E_0)/10^{-7} \) vs. \( k_x L_y + k_y \).
- Graph of \( \Delta = 0.82/q^2 \) vs. \( q^{-2}/10^{-6} \).
- Graph of \( q^2 \Delta \approx 0.82 \) vs. \( q^{-1}/10^{-3} \).
Basic Method

e.g. $N = 8$ fermions in the $|C| = 2$ band at $\nu = 1/3$ filling

1. Plot the many-body energy spectrum for a particular $\{C, r, N\}$ configuration and for a variety of MUC sizes, $q$. Identify the ground states, predicted by CF theory.

2. Read off the many-body gap, $\Delta$, for each energy spectrum.

3. Plot $\Delta$ against $q$. Read off $\lim_{q \to \infty} (q(2)\Delta)$.

4. Plot $\lim_{q \to \infty} (q(2)\Delta)$ against $N$. Read off $\lim_{N,q \to \infty} (q(2)\Delta)$. 
Constraints

1. We are only interested in filled CF levels:

   \[ N \text{ must be a multiple of } r. \]

2. \( \nu = N/N_c \Rightarrow N = \nu N_c: \)

   \[ N_c \text{ must be a multiple of } s. \]

3. Isolated lowest Chern number \( C \) band at

   \[ n_\phi = \frac{p}{|C|p - \text{sgn}(C)} \equiv \frac{p}{q}, \quad p \in \mathbb{N}. \]

4. Consider **2D systems** \( \Rightarrow \) approximately unit aspect ratios:

   \[ \left| 1 - \frac{N_x}{N_y} \right| \leq \epsilon, \quad \text{for small } \epsilon. \]

5. Limited computation time:

   \[ \dim\{\mathcal{H}\} < 10^7. \]
**Approaching the Thermodynamic Effective Continuum**

Q: In which order should we take the $N \to \infty$ and $q \to \infty$ limits?

(a) $\nu = 1/2$ bosons

(b) $\nu = 1/3$ fermions

**Figure:** Finite-size scaling of the gap for Laughlin states
Approaching the Thermodynamic Effective Continuum

Q: In which order should we take the $N \to \infty$ and $q \to \infty$ limits?
A: Doesn't matter. We take the effective continuum limit first.

$N, q \to \infty$ limits commute! (if both limits can be taken)

(a) $\nu = 1/2$ bosons
(b) $\nu = 1/3$ fermions

Figure: Finite-size scaling of the gap for Laughlin states
Warm-up: $|C| = 1$ Band
Bosons - Stability in the Continuum

Figure: Finite-size scaling of the gap to the thermodynamic continuum limit at fixed aspect ratio
Warm-up: $|C| = 1$ Band
Bosons - Stability in the Continuum

Figure: Finite-size scaling of the gap to the thermodynamic continuum limit at fixed aspect ratio

agrees with Bauer et al. √

$\nu = 2$ DMRG BIQHE:
more than just LLL involved in stabilizing the state
(He et al., 2017)

$\nu = 2$ competition expected from continuum results
(Cooper & Rezayi, 2007)
Warm-up: $|C| = 1$ Band

Bosons - Stability in the Continuum

$$\nu = 1/2 \quad \nu = 2/3 \quad \nu = 2 \quad \nu = 3/4 \quad \nu = 3/2$$

**Figure:** Finite-size scaling of the gap to the thermodynamic continuum limit at fixed aspect ratio

$$\lim_{q \to \infty} (q\Delta)/10^{-1}$$

$$L_x = 1, L_y = 7, \text{MUC: } p = 848, q = 77 \times 11$$

$$\nu = 2: \neq \text{BIQHE!}$$

$$\nu = 3/2: \text{indications for a stable RR state (as in LLL)}$$
Warm-up: \(|C| = 1\) Band

Bosons - Pair Correlation Functions

Figure: Pair correlation functions for the lowest-lying ground state in the \((k_x, k_y) = (0, 0)\) momentum sector

- Laughlin state
- finite-size effects
- charge density wave

| Results for \(|C| = 1, 2, 3\) |
|--------------------------------|

- \(\nu = 1/2\)  ■
- \(\nu = 2/3\) ◦ \(\nu = 2\) ○
- \(\nu = 3/4\) ▲ \(\nu = 3/2\) △

- no perfect correlation hole at \(\nu > 1/2\)
- no correlation hole
Warm-up: $|C| = 1$ Band
Fermions - Stability in the Continuum

Figure: Finite-size scaling of the gap to the thermodynamic continuum limit at fixed aspect ratio
Warm-up: $|C| = 1$ Band

Fermions - Pair Correlation Functions

**Laughlin state**

- $N = 9$
- $\nu = 1/3$
- $\nu = 2/5$ (red dots)
- $\nu = 2/3$ (orange dots)
- $\nu = 3/7$ (green triangles)
- $\nu = 3/5$ (pink triangles)

**charge density wave**

- $N = 8$
- $N = 18$

**finite-size effects**

- $N = 9$

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**Figure:** Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0,0)$ momentum sector.
$|C| = 2$ Band
Bosons - Stability in the Effective Continuum

$\nu = 1/3$: Laughlin-like state

\[
\lim_{q \to \infty} \left( \frac{q \Delta}{10} \right) = \frac{1}{3}
\]

\[
\nu = 1: \quad \frac{1}{\nu} = \frac{2}{3}, \quad \frac{1}{\nu} = \frac{3}{5}, \quad \frac{1}{\nu} = \frac{1}{3}:	ext{ Laughlin-like state}
\]

in-line with size dependence of BIQHE in optical flux lattices (Sterdyniak et al., 2015)

**Figure:** Finite-size scaling of the gap to the thermodynamic effective continuum limit at fixed aspect ratio
$|C| = 2$ Band

Bosons - Stability in the Effective Continuum

$\nu = 1/3$: Laughlin-like state

\[ \lim_{q \to \infty} \left( \frac{q \Delta}{N} \right) = \begin{cases} \nu & \text{for } |C| = 1, 2, 3 \end{cases} \]

\[ \nu = 1/3, \quad \nu = 2/5, \quad \nu = 3/7 \]

\[ \nu = 1/3, \quad \nu = 2/5, \quad \nu = 3/7 \]

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**Figure:** Finite-size scaling of the gap to the thermodynamic effective continuum limit at fixed aspect ratio
$|C| = 2$ Band
Bosons - Correlation Function

\( (x \mod |C|, y \mod |C|) \)

\begin{align*}
(0,0) & \quad (1,0) \\
(0,1) & \quad (1,1)
\end{align*}

\( \nu = 1/3 \)

\( N = 7 \)

Figure: Pair correlation functions for the lowest-lying ground state in the \((k_x, k_y) = (0,0)\) momentum sector
$|C| = 2$ Band

Bosons - Correlation Function

$$g(r) = \frac{1}{3}$$

$$N = 7$$

Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0, 0)$ momentum sector
$|C| = 2$ Band

Bosons - Correlation Function

Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0, 0)$ momentum sector
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Bosons - Correlation Function

Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0, 0)$ momentum sector
$|C| = 2$ Band
Bosons - Correlation Functions & Entanglement Spectra

$|\psi\rangle = \sum_{k,n} \lambda_{k,n} |\psi^A_{k,n}\rangle \otimes |\psi^B_{k,n}\rangle$

$\xi \equiv -\ln \lambda^2$

Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0, 0)$ momentum sector, and corresponding entanglement spectra
$|C| = 2$ Band
Fermions - Stability in the Effective Continuum

Figure: Finite-size scaling of the gap to the thermodynamic effective continuum limit at fixed aspect ratio
|C| = 2 Band
Fermions - Correlation Functions & Entanglement Spectra

Figure: Pair correlation functions for the lowest-lying ground state in the $(k_x, k_y) = (0, 0)$ momentum sector, and corresponding entanglement spectra.
$|C| = 3$ Band
Bosons - Stability in the Effective Continuum

Figure: Finite-size scaling of the gap to the thermodynamic effective continuum limit at fixed aspect ratio
|C| = 3 Band
Bosons - Correlation Functions

Figure: Pair correlation functions for the lowest-lying ground state in the
\((k_x, k_y) = (0, 0)\) momentum sector
Stability in the Effective Continuum

Summary

![Graphs showing finite-size scaling of the gap at fixed aspect ratio for bosonic Laughlin states](image)

**Figure:** Finite-size scaling of the gap at fixed aspect ratio for bosonic Laughlin states

| $|C|$ | $r$ | $\nu$ | $\lim_{N,q \to \infty} (q\Delta)$ | $|C|$ | $r$ | $\nu$ | $\lim_{N,q \to \infty} (q^2\Delta)$ |
|-----|-----|-----|---------------------|-----|-----|-----|---------------------|
| 1   | 1   | 1/2 | 0.64 ± 0.01         | 1   | 1   | 1/3 | 2.56 ± 0.02         |
| 2   | 1   | 1/3 | 0.27 ± 0.005        | 1   | −2  | 2/3 | 2.56 ± 0.02         |
| 3   | 1   | 1/4 | 0.13 ± 0.01         | 2   | 1   | 1/5 | 0.46 ± 0.02         |
| −1  | 1/2 |     | 0.18 ± 0.07         | −1  | 1/3 |     | 0.65 ± 0.16         |

**Table:** States with (effective) continuum limits that could be extrapolated to the thermodynamic limit
Conclusion

- Scaling to the effective continuum limit at fixed aspect ratio converges faster than scaling at fixed flux density.
- Vast majority of finite-size spectra produce the ground state degeneracy predicted by CF theory.
- Laughlin-like states with $\nu = 1/(|kC| + 1)$ are the most robust, and yield a clear gap in the effective continuum limit.
- Instability may be caused by competing topological phases, charge density waves, or finite-size effects.
- Stable FCIs found with clear entanglement gaps in $|C| > 1$ bands - largest gaps seen for $|C| = 2$ fermions.
- Pair-correlations are smooth functions modulated by $|C|$ sites along both axes, giving rise to the appearance of $|C|^2$ sheets.
Conclusion

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- Stable FCIs found with clear entanglement gaps in $|C| > 1$ bands - largest gaps seen for $|C| = 2$ fermions.
- Pair-correlations are smooth functions modulated by $|C|$ sites along both axes, giving rise to the appearance of $|C|^2$ sheets.
Supplementary Slides

1. Approaching the Effective Continuum
2. Warm-up: $|C| = 1$ Band: Bosons & Fermions - Scaling of the Gap with MUC Size
3. $|C| = 2$ Band: Bosons - Rectangular Geometries
4. $|C| = 3$ Band: Bosons - Rectangular Geometries
Approaching the Effective Continuum

Q: Should we fix $n_\phi$ or fix aspect ratio?

hollow symbols $\Rightarrow$ fixed $n_\phi$
filled symbols $\Rightarrow$ fixed aspect ratio

Figure: Finite-size scaling of the gap in the $|C| = 2$ band
Approaching the Effective Continuum

Q: Should we fix $n_\phi$ or fix aspect ratio?
A: Fix aspect ratio

scaling at fixed aspect ratio is more robust!

Figure: Finite-size scaling of the gap in the $|C| = 2$ band
Warm-up: $|C| = 1$ Band
Bosons & Fermions - Scaling of the Gap with MUC Size

Figure: Finite-size scaling of $q^{(2)}\Delta$ to a constant value in the continuum limit for 8-particle Laughlin states
Warm-up: $|C| = 1$ Band

Bosons & Fermions - Scaling of the Gap with MUC Size

agrees with Bauer et al. ✓

\[ \lim_{q \to \infty} (q\Delta) = 0.62 \]

\[ \lim_{q \to \infty} (q^2\Delta) = 2.60 \]

**Figure:** Finite-size scaling of $q^{(2)}\Delta$ to a constant value in the continuum limit for 8-particle Laughlin states
$|C| = 2$ Band
Bosons - Rectangular Geometries

Figure: Magnitude of the gap for the 12-particle state at $\nu = 2/3$
$|C| = 2$ Band
Bosons - Rectangular Geometries

Figure: Magnitude of the gap for the 12-particle state at $\nu = 2/3$
$|C| = 3$ Band
Bosons - Rectangular Geometries

Figure: Magnitude of the gap for the 6-particle state at $\nu = 3/8$
$|C| = 3$ Band

Bosons - Rectangular Geometries

$q\Delta \approx 0.03$  

Figure: Magnitude of the gap for the 6-particle state at $\nu = 3/8$
$|C| = 3$ Band
Bosons - Rectangular Geometries

Figure: Magnitude of the gap for the 6-particle state at $\nu = 3/8$