Before discussing the Comment by Thuneberg [1] (referred to later as the Comment) I describe the problem addressed in the paper [2] (further called the Paper). The goal was to calculate the current-phase curve of the long ballistic SNS sandwich by the method of the self-consistent field [3]. In this method an effective pairing potential is introduced, which transforms the second-quantization Hamiltonian with an electron interaction term into an effective Hamiltonian, which is quadratic in creation and annihilation electron operators. The effective Hamiltonian can be diagonalized by the Bogolyubov–Valatin transformation. The parameters of the transformation are the two-component wave functions, which are solutions of the Bogolyubov–de Gennes equations. The pairing potential in the effective Hamiltonian must satisfy the integral self-consistency equation. In the previous investigations [4–7] and in the Paper this step was skipped. Instead a simple profile of the pairing potential was postulated: the pairing potential (gap) of the constant modulus $\Delta_0$ in superconducting layers, and zero pairing potential in the normal layer. Then the problem reduces to the analytical solution of the Bogolyubov–de Gennes equations and subsequent summations over all quasiparticle states. Eventually one obtains the analytical solution of the problem, which is exact in the limit of small ratios of the gap to the Fermi energy and of the coherence length to the thickness $L$ of the normal layer.

The Comment argues that my analysis contradicts “a vast literature on the topic” [4–7]. In fact, the Paper confirmed the current-phase curve obtained in previous calculations for multidimensional (2D and 3D) systems for any temperature and for the 1D system for zero temperature. Different results were obtained only for the 1D case at high temperatures (high compared with the Andreev level energy spacing but still low compared to the temperature. Different results were obtained only for the 1D case at high temperatures (high compared with the Andreev level energy spacing but still low compared to the superconducting gap). But the Paper presented a new approach to the problem and another physical picture of superconducting gap). But the Paper presented a new approach to the problem and another physical picture of the phenomenon. Now I explain why I was not satisfied by the approach of the paper and does not provide any reasonable argument that it is incorrect.

2. The effect of parity of Andreev levels (odd versus even number of states) was not considered or even mentioned. The effect can be essential for the 1D case (in analogy with effect of parity of electron numbers in normal 1D rings).

3. There was no clarity about the effect of continuum states on the current in the normal layer, despite statements to the contrary by the Comment. Ishii [5] argued that this effect was important. This view was supported in a number of later publications and repeated in the Comment. But none of them compared contributions to the current from bound Andreev and continuum states. Moreover, Bardeen and Johnson [7] reproduced the result of Ishii [9] without taking into account continuum states.

Dealing with Problem #1 the Paper suggested a remedy for the absence of the conservation law. Instead of the strict law, the Paper imposed a softer condition that at least the total currents deep in all layers are the same. The condition can be satisfied taking into account three contributions to the total current $J = J_s + J_v + J_q$: (i) The current $J_s$ induced by the phase gradient in the superconducting layers. In the Paper it was called the Cooper-pair condensate, or simply the condensate current. (ii) The current $J_v$, which can flow in the normal layer even if the Cooper-pair condensate is at rest and all Andreev states are empty. In the Paper it was called the vacuum current. (iii) The current $J_q$ induced by nonzero occupation of Andreev states, i.e., by creation of quasiparticles. It was called the excitation current. The condensate motion produces the same current $J_s$ in superconducting and normal layers of the SNS sandwich, while the vacuum and excitation currents exist only in the normal layer. Thus, the charge conservation law requires that the sum of the vacuum and the excitation currents $J_v + J_q$ always vanishes.

The phase variation in states with the condensate and vacuum currents is shown in Fig. 1. The condensate current $J_s = evn_s$ [Fig. 1(a)] appears if there is the phase gradient $\nabla \theta$ in superconducting leads, which determines the superfluid velocity $v_s = \frac{\hbar}{2m} \nabla \theta$ ($n$ is the electron density). The same condensate current in the normal layer requires the phase difference $\theta_s = \frac{L}{v} \nabla \theta$ across the normal layer called the superfluid phase. This is directly confirmed by the solution of the Bogolyubov–de Gennes equations. The vacuum current is determined by the vacuum phase $\theta_0$ [Fig. 1(b)]. Figure 1(c) shows the phase...
variation at the coexistence of the condensate and the vacuum current. The total phase difference across the normal layer is the Josephson phase \( \theta = \theta_s + \theta_0 \). The phase profiles in the normal layer are shown in Fig. 1 by dashed lines since this phase is not and cannot be determined because it is a phase of the order parameter \( \Delta \), which vanishes in the normal layer. Only the total phase difference across the normal layer appears in the Bogolyubov–de Gennes equations. Dashed lines simply show what the phase gradient would be if a normal metal were replaced by a superconductor.

The previous literature [4–7] and the Comment considered the states without phase gradients in superconducting layers. Thus, they considered the case shown in Fig. 1(b) when the current is determined by the vacuum phase \( \theta_0 \).

Ignoring the effect of the phase gradient in superconducting leads on the Josephson current was common in the past. By default it was supposed that this is not an issue because the gradients in leads are very small. This is true for a weak link inside which the phase varies much faster than in leads as shown in Fig. 1(d). As argued in the Paper, the long SNS junction is not a weak link, and the phase gradient in leads does affect the current in the normal layer. Thus, one should determine currents in normal and superconducting layers self-consistently.

The Comment denies the very existence of the problem with the charge conservation law (Problem #1) for vacuum current. The Comment discusses the vacuum current [see Fig. 1(c) in the Comment] not for 1D leads as in the Paper but for multidimensional leads. The goal of this “trick” (by the definition of the Comment itself) is to show that the vacuum current can flow without violation of the conservation law. This sweeps the problem under the carpet. A problem revealed in one case is denied because it does not exist in another case. The logics of such an argument looks strange. The reason why the conservation law problem is absent (or, more carefully, is not so important) for multidimensional leads is clear: with multidimensional leads the junction becomes a weak link (see the previous paragraph).

The Comment does not see any difference between the condensate and the vacuum currents. The Paper discusses this difference in details. Tuning of the phase \( \theta_s \) producing the condensate current shifts Andreev levels together with the gap edges, so that their relative positions do not vary [see Fig. 4(b) in the Paper]. But at tuning the phase \( \theta_0 \) connected with the vacuum current Andreev levels move with the respect to gap edges and can cross them [see Fig. 4(a) in the Paper]. It is interesting that this phenomenon is mentioned in the Comment. I quote: "... the levels are shifted relative to the gap edge, and some new discrete levels may appear and some others disappear." But the Comment failed to notice that these processes are not possible at tuning of the phase \( \theta_s \). This justifies an introduction of two phases in addition to their sum (the Josephson phase) \( \theta = \theta_0 + \theta_s \), which is eventually present in the final current-phase relation.

FIG. 1. The phase variation across the SNS sandwich. (a) The condensate current produced by the phase gradient \( \nabla \theta \) in the superconducting layers. In all layers the electric current is equal to \( nev_s \), where \( n \) is the electron density and \( v_s = \frac{k}{2m} \nabla \theta \) is the superfluid velocity. The phase \( \theta_s = L \nabla \theta \), which is called the superfluid phase. (b) The vacuum current produced by the phase \( \theta_0 \) called vacuum phase. The current is confined to the normal layer, there is no current in superconducting layers. (c) The superposition of the condensate and the vacuum current. (d) The phase variation across a weak link.

Introduction of two phases \( \theta_0 \) and \( \theta_s \) essentially revised the physical picture of charge transport through the ballistic SNS junction. Let us consider the current-phase curve at zero temperature (Fig. 2), which was the same in the previous literature [4–7] and in the Paper. But, as mentioned above, the previous literature and the Comment considered the states without phase gradients in superconducting layers, i.e., they calculated the vacuum current as a function of the vacuum phase \( \theta_0 \) at \( \theta_s = 0 \) [the case shown in Fig. 2(b)]. Meanwhile, the charge conservation law does not allow the vacuum current without the excitation current compensating it. At sloped segments of the \( T = 0 \) current-phase curve there
are no quasiparticles in Andreev states because all their energies are positive. Thus, complicated calculations of the vacuum current in the previous literature are not relevant for sloped segments of the current-phase curve. According to the Paper, at sloped segments only the condensate current determined by the phase $\theta_s$ flows without violation of charge conservation law. The condensate current is simply determined from the principle of Galilean invariance valid at Andreev reflection at interfaces between layers despite the absence of translational invariance. This principle was formulated by Bardeen and Johnson [7], but they applied it only to the normal layer ignoring that this principle requires that the same faces between layers despite the absence of translational invariance. This principle was formulated by Bardeen and Johnson [7], but they applied it only to the normal layer ignoring that this principle requires that the same current flows also in superconducting layers. The vacuum current at zero temperature appears only at vertical segments of the $T = 0$ current-phase curve at $\theta = \pi(2s + 1)$ (s is an integer) when the energy of the lowest Andreev state reaches zero and its occupation becomes possible. This allows one to satisfy the condition that the sum of the vacuum and the excitation current must vanish. At $\theta \neq \pi(2s + 1)$ (slope segments) the charge transport through the junction does not differ from that in a uniform superconductor.

According to the Comment, the Paper is incorrect because it ignored the contribution of continuum states to the current through the SNS junction [8]. Formulating Problem #3 I mentioned that the statement of Ishii [9] about importance of this contribution was nothing more than a declaration, which was not supported by any calculation or estimation. Moreover, the end of the second paragraph of the Comment warned against calculating contributions to the current from bound Andreev states and continuum states separately. Thus, it is not only unknown from the Comment, which contribution to the current is more important, but also the Comment does not recommend checking it. Without knowing how large the current in continuum states is there are no grounds for judging whether I was correct or not ignoring the current in continuum states.

Meanwhile, there is no problem to receive an exact expression for the contribution of continuum states to the vacuum current in the normal layer since the wave functions of continuum states satisfying the Bogolyubov–de Gennes equations are known:

$$J_{\nu C} = \frac{e}{\pi \hbar} \int_{\Delta}^\infty \left[ T(-\theta_0) - T(\theta_0) \right] d\xi,$$

where

$$T(\theta_0) = \frac{2\xi^2}{2\xi^2 + \Delta_0^2 \left[ 1 - \cos \left( \frac{2\pi m L}{h^2 k_f} - \theta_0 \right) \right]}$$

$$= \frac{2\xi^2}{2\xi^2 + \Delta_0^2 \left[ 1 - \cos \left( \frac{2(\varepsilon - \Delta_0)mL}{h^2 k_f} + 2\pi \alpha - \theta_0 \right) \right]}$$

is the transmission probability [9] for a quasiparticle with the energy $\varepsilon$, and $\xi = \sqrt{\varepsilon^2 - \Delta_0^2}$. The incommensurability parameter $\alpha$ is a fractional part of the ratio of the gap to the Andreev interlevel spacing:

$$\frac{\Delta_0 L}{\pi v f} = s_m + \alpha.$$ (3)

An integer $s_m$ is chosen so that $0 < \alpha < 1$. In multidimensional cases integration over wave vectors transverse to the current requires averaging of $J_{\nu C}$ over $\alpha$. After this two terms in Eq. (1) become independent from $\theta$, and the current $J_{\nu C}$ totally vanishes. This refutes the claim of the Comment that the Paper erroneously ignored the contribution of continuum states to the current at least for multidimensional systems.

In the 1D case one should not average over $\alpha$, and the current $J_{\nu C}$ does not vanish. After some crude estimations I concluded in the Paper that nevertheless in the limit $L \to \infty$ this current can be ignored also in the 1D case. A more quantitative calculation would be useful, and the work in this direction is in progress.

Apparently the most nontrivial prediction of the Paper was the conclusion about possibility of the anomalous current–phase relation with the ground state at nonzero phase $\theta$. In the Paper the junction with such an anomaly was called the $\theta$ junction. But later I became aware that earlier these junctions were known as $\varphi_0$ junctions [10]. The anomaly can be explained by simple arguments. In the ground state there is no condensate current ($\theta_s = 0$), and it is sufficient to investigate the dependence vacuum current versus phase $\theta_0$ [the case shown in Fig. 1(b)]. Definitely there is an energy extremum at $\theta_0 = 0$, but one should check whether it is a minimum or maximum. The anomaly appears if the extremum is a maximum and the second derivative of the energy, i.e., the first derivative of the current $dL/d\theta_0 + dJ/d\theta_0$, with respect to $\theta_0$ is negative.

The derivative of the excitation current $dJ/d\theta_0$ is always negative, and according to the Paper at high temperatures its absolute value exceeds the positive derivative $dL/d\theta_0$. Thus, the energy extremum at $\theta_0 = 0$ is a maximum. This means that the total current decreases with growing small phase, as we show now.
The condition that \( J_v + J_q \) vanishes must be checked taking into account that the excitation current is determined by the total phase \( \theta = \theta_0 + \theta_s \):

\[
J_v + J_q = \frac{dJ_v}{d\theta_0} \theta_0 + \frac{dJ_q}{d\theta_0}(\theta_0 + \theta_s) = 0. \tag{4}
\]

Then the total current at small \( \theta = \theta_0 + \theta_s \) is

\[
J = J_s = ev_s = \frac{ev_f}{\pi L} \theta_s = \frac{ev_f}{\pi L} \left( \frac{dJ_v}{d\theta_0} + \frac{dJ_q}{d\theta_0} \right) \theta. \tag{5}
\]

The sign of the current at growing positive \( \theta \) is determined by the sign of \( \frac{dJ_v}{d\theta_0} + \frac{dJ_q}{d\theta_0} \) indeed.

In the past \( \varphi_0 \) junctions (\( \pi \) junction with \( \varphi_0 = \pi \) is the most known example) usually were explained by magnetism of the normal layer \([10]\). In our case there is no magnetism. This is an example of spontaneously broken time-reversal symmetry: if a \( \varphi_0 \) junction is put into a superconducting ring, a persistent current and a related magnetic moment appear in the ground state of the ring.

According to the Comment the most reliable method to solve the problem is the quasiclassical Green’s function formalism but not the approach chosen in the Paper. I fully respect the Green’s function formalism, which is useful for problems with interaction or disorder solved by the perturbation theory. However, we deal with the case without disorder or interaction. The problem is reduced to quadratures. There is an analytical solution expressed in \( \textit{ab initio} \) sums and integrals, which, however, are not so simple for calculation. The Green’s function formalism suggests a complicated chain of transformations dealing not with one but with a family of Green’s functions. In the end after a number of assumptions one ends with other sums and integrals, which also require some efforts for calculations. I do not understand why this approach is more reliable than direct calculations of \( \textit{ab initio} \) sums and integrals.

The Paper presented a \( \textit{quantitative} \) calculation of \( \textit{ab initio} \) sums and integrals. One might expect that for a claim that a calculation is wrong some \( \textit{quantitative} \) arguments were necessary—namely, a calculation or at least a crude estimation disproving some step or assumption of the original calculation. One cannot find a single quantitative argument in the Comment. The only explanation why the calculation is incorrect is that the Paper ignored the contribution of continuum states. This is together with the statement that this contribution is unknown and does not deserve calculation separately from the contribution of bound states.

In summary, the Comment does not contain anything that puts in doubt the method and the conclusions of the Paper. No evidence that they are incorrect was presented. That said, I thank Erkki Thuneberg for his the Comment and discussions, which urged me to treat the continuum states more carefully.

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[7] J. Bardeen and J. L. Johnson, Josephson current flow in pure Superconducting-Normal-Superconducting junctions, Phys. Rev B 5, 72 (1972).
[8] Here we discuss only the contribution of continuum states to the vacuum current, which flows in the normal layer while in superconducting leads the electron fluid is at rest. Continuum states give the main contribution to the condensate current, which flows in all layers without breaking the charge conservation law.
[9] The parameters of continuum scattering states were calculated by Bardeen and Johnson [7] and reproduced in Paper. They are determined by Andreev reflection at interfaces between layers. But Bardeen and Johnson [7] did not estimate the current in continuum states. This was done in Paper.
[10] A. Buzdin, Direct coupling between magnetism and superconducting current in the Josephson \( \varphi_0 \) junction, Phys. Rev. Lett. 101, 107005 (2008)