Odd Poisson Bracket in Hamilton’s Dynamics

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ABSTRACT

Some applications of the odd Poisson bracket to the description of the classical and quantum dynamics are represented.

1. Introduction

Mathematicians (first of them was Buttin) proved that in the phase superspace apart from the usual even Poisson bracket there also exists another bracket of the Poisson type namely the odd Poisson bracket (OB) having the nontrivial Grassmann grading. In physics the OB has firstly appeared as an adequate language for the quantization of the gauge theories in the well-known Batalin-Vilkovisky scheme. However, the apparent dynamical role of the OB was not understood quite well till papers in which a possibility of the reformulation of Hamilton dynamics on the basis of the OB was proved for the classical systems having an equal number of pairs of even and odd (relative to the Grassmann grading) phase coordinates. Earlier, the prescription for the canonical quantization of the OB was suggested, and several odd-bracket quantum representations for the canonical variables were also obtained. In contrast with the even Poisson bracket case, some of the odd-bracket quantum representations turned out to be no equivalent. Recently the direct connection of the odd-bracket quantum representations for the canonical variables with the quantization of the classical Hamilton dynamics based on the OB has been established.

I concentrate my attention in the report on the dynamical aspect of the OB, that is on the description with the help of the OB of both the classical and quantum dynamics for the systems in superspace.

The report is organized as follows. The main properties of the odd bracket are presented in Section 2. In Section 3 it is shown that Hamilton’s equations of motion obtained by means of the even Poisson bracket with the help of the even Hamiltonian can be reproduced by the odd bracket using the equivalent odd Hamiltonian. The odd-bracket quantum representations for the canonical variables are described in Section 4.

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In Section 5 the problem of quantization of the systems with odd bracket is considered on the simplest example of the supersymmetric one-dimensional oscillator.

2. Properties of the odd Poisson bracket

First, we recall the necessary properties of various graded Poisson brackets. The even and odd brackets in terms of the real even \( y_i = (q^a, p_a) \) and odd \( \eta^i = \theta^\alpha \) canonical variables have, respectively, the form

\[
\{A, B\}_o = A \sum_{a=1}^{n} \left( \partial_{q^a} \partial_{p_a} - \partial_{p_a} \partial_{q^a} \right) - i \sum_{a=1}^{2n} \partial_{\theta^\alpha} \partial_{\theta^\alpha} B ;
\]

\[
\{A, B\}_1 = A \sum_{i=1}^{N} \left( \partial_{y_i} \partial_{\eta^i} - \partial_{\eta^i} \partial_{y_i} \right) B ,
\]

where \( \partial \) and \( \bar{\partial} \) are the right and left derivatives, and the notation \( \partial_x = \frac{\partial}{\partial x} \) is introduced. By introducing apart from the Grassmann grading \( g(A) \) of any quantity \( A \) its corresponding bracket grading \( g_\epsilon(A) = g(A) + \epsilon \pmod{2} \) (\( \epsilon = 0, 1 \)), the grading and symmetry properties, the Jacobi identities and the Leibnitz rule are uniformly expressed for the both brackets (1,2) as

\[
g_\epsilon(\{A, B\}_\epsilon) = g_\epsilon(A) + g_\epsilon(B) \pmod{2} ,
\]

\[
\{A, B\}_\epsilon = -(-1)^{g_\epsilon(A)g_\epsilon(B)} \{B, A\}_\epsilon ,
\]

\[
\sum_{(ABC)} (-1)^{g_\epsilon(A)g_\epsilon(C)} \{A, \{B, C\}_\epsilon \} = 0 ,
\]

\[
\{A, BC\}_\epsilon = \{A, B\}_\epsilon C + (-1)^{g_\epsilon(A)g(B)} B\{A, C\}_\epsilon ,
\]

where (3a)–(3c) have the shape of the Lie superalgebra relations in their canonical form\(^8\) with \( g_\epsilon(A) \) being the canonical grading for the corresponding bracket.

In terms of arbitrary real dynamical variables \( x^M = (x^m, x^\alpha) = x^M(y, \eta) \) with the same number of Grassmann even \( x^m \) and odd \( x^\alpha \) coordinates the odd bracket (2) takes the form

\[
\{A, B\}_1 = A \bar{\partial}_M \bar{\omega}^{MN}(x) \partial_N B .
\]

The matrix \( \bar{\omega}_{MN} \), inverse to \( \bar{\omega}^{MN} \)

\[
\bar{\omega}_{MN}\bar{\omega}^{NL} = \delta^L_M ,
\]

and consisting of the coefficients of the odd closed 2-form, in view of the odd bracket properties (3a)-(3c) can be represented in the form of the grading strength

\[
\bar{\omega}_{MN} = \partial_M \bar{A}_N - (-1)^{g(M)g(N)} \partial_N \bar{A}_M ,
\]
where \( g(M) = g(x^M) \) and \( \partial_M = \partial/\partial x^M \). The coefficients of the 1-form \( \tilde{A}(d) = dx^M \tilde{A}_M \) satisfy the conditions

\[
g(\tilde{A}_M) = g(M) + 1, \quad (\tilde{A}_M)^+ = \tilde{A}_M. \tag{7}\]

As can be seen from (6) \( \tilde{\omega}_{MN} \) is invariant under gauge transformations

\[
\tilde{A}'_M = \tilde{A}_M + \partial_M \tilde{\chi} \tag{8}
\]

with functions \( \tilde{\chi} \) as parameters.

3. Classical dynamics in terms of the odd Poisson bracket

Let us consider the Hamilton system containing an equal number \( n \) of pairs of even and odd with respect to the Grassmann grading real canonical variables. We require that the equations of motion of the system be reproduced both in the even Poisson-Martin bracket (1) with the help of the even Hamiltonian \( H \) and in the odd bracket (2) with the Grassmann-odd Hamiltonian \( \tilde{H} \), that is\(^{3,4}\),

\[
\frac{dx^M}{dt} = \{ x^M, H \}_0 = \{ x^M, \tilde{H} \}_1, \tag{9}\]

where \( t \) is the proper time. Using definitions (4) and (1) together with (5),(6) the relations (9) can be represented as the equations

\[
(\partial_M \tilde{A}_N - (-1)^{g(M)g(N)} \partial_N \tilde{A}_M) \omega^{NL} \partial_L H = \partial_M \tilde{H} \tag{10}
\]

to derive the unknown \( \tilde{H} \) and \( \tilde{A}_M \) under the given \( H \) and the even matrix \( \omega^{MN} \) corresponding to the even bracket (1).

In order to solve Eqs. (10) it is convenient to use such real canonical in the even bracket (1) coordinates \( x^M \) which contain among canonically conjugate pairs the pair consisting of the proper time \( t \) and the Hamiltonian \( H \). It follows from Eqs.(9) that the rest of the canonical quantities \( z^M \) would be the integrals of motion for the system considered: even \( I_1, ..., I_{2(n-1)} \) and odd \( \Theta^1, ..., \Theta^{2n} \). In terms of these coordinates \( x^M \) Eqs. (10) take the form

\[
(\partial_M \tilde{A}_t - \partial_t \tilde{A}_M) = \partial_M \tilde{H}. \tag{11}
\]

The quantities \( \tilde{A}_M, \tilde{\chi} \) and \( \tilde{H} \) can be expanded in powers of the Grassmann variables \( \Theta^a \) as

\[
\tilde{A}_m = \sum_{k=1}^{n} \frac{i^{(k-1)(2k-1)}}{(2k-1)!} A_{m\alpha_1...\alpha_{2k-1}} \Theta^{\alpha_1} ... \Theta^{\alpha_{2k-1}}, \tag{12a}
\]
\[ \tilde{A}_\alpha = \sum_{k=0}^{n} \frac{i^{k(2k+1)}}{(2k)!} B_{\alpha a_1 \ldots a_{2k}} \Theta^{a_1} \ldots \Theta^{a_{2k}}, \quad (12b) \]

\[ \tilde{\chi} = \sum_{k=1}^{n} \frac{i^{(k-1)(2k-1)}}{(2k-1)!} \chi_{\alpha a_1 \ldots a_{2k-1}} \Theta^{a_1} \ldots \Theta^{a_{2k-1}}, \quad (12c) \]

\[ \tilde{H} = \sum_{k=1}^{n} \frac{i^{(k-1)(2k-1)}}{(2k-1)!} h_{\alpha a_1 \ldots a_{2k-1}} \Theta^{a_1} \ldots \Theta^{a_{2k-1}}. \quad (12d) \]

The \( \Theta^\alpha \) coefficients are the real Grassmann-even functions of the even variables \( x^m = (t, H, I_1, \ldots, I_{2(n-1)}) \) and are chosen to be antisymmetric in the indices contracted with \( \Theta^\alpha \). In terms of these functions the gauge transformations (8) have the form

\[ A'_{ma_1 \ldots a_{2k}-1} = A_{ma_1 \ldots a_{2k}-1} + \partial_m \chi_{a_1 \ldots a_{2k}-1}, \quad (k = 1, \ldots, n); \]

\[ B'_{[aa_1 \ldots a_{2k}]} = B_{[aa_1 \ldots a_{2k}]} + \chi_{aa_1 \ldots a_{2k}}, \quad (k = 0, 1, \ldots, n-1); \]

\[ B'_{(aa_j)\alpha_1 \ldots a_{j-1}\alpha_{j+1} \ldots a_{2k}} = B_{(aa_j)\alpha_1 \ldots a_{j-1}\alpha_{j+1} \ldots a_{2k}}, \quad \left( k = 0, 1, \ldots, n; \quad j = 1, \ldots, 2k \right). \quad (13) \]

where the expansion in the components with different symmetries of the indices has been used for the tensor antisymmetric in all indices but the first

\[ B_{\alpha a_1 \ldots a_{2k}} = B_{[aa_1 \ldots a_{2k}]} + \frac{2}{N+1} \sum_{j=1}^{N} B_{(aa_j)\alpha_1 \ldots a_{j-1}\alpha_{j+1} \ldots a_{2k}}. \]

The additive character of the transformations for the functions \( B_{[aa_1 \ldots a_{2k}]}(k = 0, 1, \ldots, n-1) \) allows us to put them equal to zero in the expression (12b) for \( \tilde{A}_\alpha \) by choosing \( \chi_{aa_1 \ldots a_{2k}} = -B_{[aa_1 \ldots a_{2k}]} \). This gauge choice amounts to the following gauge condition

\[ \Theta^\alpha \tilde{A}_\alpha = 0. \]

Using this condition and Eqs.(11), we obtain the equality

\[ \tilde{H} = \tilde{A}_t. \quad (14) \]

which, being substituted again into Eqs.(11), leads to the simple equations

\[ \partial_t \tilde{A}_M = 0. \quad (15) \]

Thus, in consequence of (15), the solution of Eqs.(11) for \( \tilde{A}_M \) and \( \tilde{H} \) in the chosen gauge resides in that the nonzero coefficients \( A_{ma_1 \ldots a_{2k-1}} \) and \( B_{(aa_j)\alpha_1 \ldots a_{j-1}\alpha_{j+1} \ldots a_{2k}} \) in expansions (12a,b) for \( \tilde{A}_M \) are the arbitrary functions (denoted as \( a_{ma_1 \ldots a_{2k-1}} \) and \( b_{(aa_j)\alpha_1 \ldots a_{j-1}\alpha_{j+1} \ldots a_{2k}} \), respectively) of all, except the proper time \( t \), even variables \( H \) and \( I_1, \ldots, I_{2(n-1)} \), and the odd Hamiltonian is expressed in terms of these functions with the help of Eq.(14).
Using the gauge transformations (13) with the arbitrary functions $\chi_{\alpha_1...\alpha_{2k-1}}(t, H, I)$, we obtain the general solution of Eqs.(11) in the arbitrary gauge:

$$A_{m\alpha_1...\alpha_{2k-1}} = a_{m\alpha_1...\alpha_{2k-1}}(H, I) + \partial_m \chi_{\alpha_1...\alpha_{2k-1}}(t, H, I);$$

$$B_{aa_1...\alpha_{2k}} = \frac{2}{2k + 1} \sum_{j=1}^{2k} b_{(aa_j)a_1...a_{j-1}a_{j+1}...\alpha_{2k}}(H, I) + \chi_{aa_1...\alpha_{2k}}(t, H, I);$$

$$h_{\alpha_1...\alpha_{2k-1}} = a_{t\alpha_1...\alpha_{2k-1}}(H, I).$$

Note that the solution of the analogous problem of finding the even brackets and the corresponding even Hamiltonians, which lead to the same equations of motion

$$\frac{dx^M}{dt} = \{x^M, H\}_0 = \{x^M, \tilde{H}\}_0,$$

has a similar structure but with the difference that the odd quantities $\tilde{A}_M, \tilde{\chi}$ and $\tilde{H}$ has to be replaced by the even ones.

Thus, we extended the notion of the bi-Hamiltonian systems onto the case when the pairs of the Hamiltonian-bracket, giving the same equations of motion, have an opposite Grassmann grading.

4. Quantum representations of the odd Poisson bracket

The procedure of the odd-bracket canonical quantization given in $^5,^6$ resides in splitting all the canonical variables into two sets, in the division of all the functions dependent on the canonical variables into classes, and in the introduction of the quantum multiplication $*$, which is either the common product or the bracket composition, in dependence on what the classes the co-factors belong to. Under this, one of the classes has to contain the normalized wave functions, and the result of the multiplication $*$ for any quantity on the wave function $\Psi$ must belong to the class containing $\Psi$. This procedure is the generalization on the odd bracket case of the canonical quantization rules for the usual Poisson bracket $\{,\}$ Pois., which, for example, in the coordinate representation for the canonical variables $q$ and $p$ is defined as

$$q * \Psi(q) = q\Psi(q), \quad p * \Psi(q) = i\hbar \{p, \Psi(q)\}_{\text{Pois.}} = -i\hbar \frac{\partial \Psi}{\partial q},$$

where $\Psi(q)$ is the normalized wave function depending on the coordinate $q$.

In $^5,^6$ two nonequivalent odd-bracket quantum representations for the canonical variables were obtained by using two different ways of the function division. But these ways do not exhaust all the possibilities. In $^7$ a more general way of the division is proposed, which contains as the limiting cases the ones given in $^5,^6$.

Let us build quantum representations for an arbitrary graded bracket under its canonical quantization. To this end, all canonical variables are split into two equal in
the number sets, so that none of them should contain the pairs of canonical conjugates. Note that to make such a splitting possible for the even bracket (1), the transition has to be done from the real canonical self-conjugate odd variables to some pairs of odd variables, which simultaneously are complex and canonical conjugate to each other. Composing from the integer degrees of the variables from the one set (we call it the first set) the monomials of the odd $2s + 1$ and even $2s$ uniformity degrees and multiplying them by the arbitrary functions dependent on the variables from the other (second) set, we thus divide all the functions of the canonical variables into the classes designated as $\hat{O}_s$ and $\hat{E}_s$, respectively. For instance, in the general case the odd-bracket canonical variables can be split, so that the first set would contain the even $y_i (i = 1, \ldots, n \leq N)$ and odd $\eta^{n+\alpha} (\alpha = 1, \ldots, N - n)$ variables, while the second set would involve the rest variables. Then the classes of the functions obtained under this splitting have the form

$$\hat{O}_s = \left(y_i, \eta^{n+\alpha}\right)^{2s+1} f \left(y^i, y_{n+\alpha}\right); \quad \hat{E}_s = \left(y_i, \eta^{n+\alpha}\right)^{2s} f \left(y^i, y_{n+\alpha}\right),$$

where the factors before the arbitrary function $f \left(y^i, y_{n+\alpha}\right)$ denote the monomials having the uniformity degrees indicated in the exponents. These classes satisfy the corresponding bracket relations

$$\{\hat{O}_s, \hat{O}_{s'}\}_\epsilon = \hat{O}_{s+s'}; \quad \{\hat{O}_s, \hat{E}_{s'}\}_\epsilon = \hat{E}_{s+s'}; \quad \{\hat{E}_s, \hat{E}_{s'}\}_\epsilon = \hat{E}_{s+s'-1}, \quad (16)$$

and the relations of the ordinary Grassmann multiplication

$$\hat{O}_s \cdot \hat{O}_{s'} = \hat{E}_{s+s'+1}; \quad \hat{O}_s \cdot \hat{E}_{s'} = \hat{O}_{s+s'}; \quad \hat{E}_s \cdot \hat{E}_{s'} = \hat{E}_{s+s'}. \quad (17)$$

It follows from (16),(17), that $\hat{O} = \{\hat{O}_s\}$ and $\hat{E} = \{\hat{E}_s\}$ form a superalgebra with respect to the addition and the quantum multiplication $\hat{*} (\epsilon = 0, 1)$ defined for the corresponding bracket as

$$\hat{O}_s \hat{*} \hat{O}_{s'} = \{\hat{O}, \hat{O}\}_\epsilon \in \hat{O}; \quad \hat{O}_s \hat{*} \hat{E} = \{\hat{O}, \hat{E}\}_\epsilon \in \hat{E}; \quad \hat{E}_s \hat{*} \hat{E}_{s'} = \hat{E}_s \hat{*} \hat{E}_{s'} \in \hat{E}, \quad (18)$$

where $\hat{O}, \hat{O} \in \hat{O}$ and $\hat{E}, \hat{E} \in \hat{E}$. Note, that the classes $\hat{O}_0$ and $\hat{E}_0$ form the sub-superalgebra. In terms of the quantum grading $q_\epsilon (A)$ of any quantity $A$

$$q_\epsilon (A) = \begin{cases} g_\epsilon (A), & \text{for } A \in \hat{O}; \\ g(A), & \text{for } A \in \hat{E}, \end{cases}$$

introduced for the appropriate bracket, the grading and symmetry properties of the quantum multiplication $\hat{*}$, arising from the corresponding properties for the bracket (3a,b) and Grassmann composition of any two quantities $A$ and $B$, are uniformly written as
With the use of the quantum multiplication \( \ast \) and the quantum grading \( q_\epsilon \), let us define for any two quantities \( A, B \) the quantum bracket ((anti)commutator) \( [A, B]_\epsilon \) (under its action on the wave function \( \Psi \) that is considered to belong to the class \( E \)) in the form\(^{5-7}\)

\[
[A, B]_\epsilon \ast \Psi = A \ast (B \ast \Psi) - (-1)^{q_\epsilon(A)q_\epsilon(B)} B \ast (A \ast \Psi) .
\]

If \( A, B \in E \), then, due to (19c), the quantum bracket between them equals zero. In particular, the wave functions are (anti)commutative. If \( A \) or both of the quantities \( A \) and \( B \) belong to the class \( O \), then in the first case, due to the Leibnitz rule (3d), and in the second one, because of the Jacobi identities (3c), the relation follows from the definitions (18) and (20)

\[
[A, B]_\epsilon \ast \Psi = \{A, B\}_\epsilon \ast \Psi = (A \ast B)_\epsilon \ast \Psi ,
\]

that establishes the connection between the classical and quantum brackets of the corresponding Grassmann parity. Note, that the quantization procedure also admits the reduction to \( O_o \cup E_o \).

The grading \( q_\epsilon \) determines the symmetry properties of the quantum bracket (20). Under above-mentioned splitting of the odd-bracket canonical variables into two sets, the grading \( q_1 \) equals unity for the variables \( y_i \in O, \eta^i \in E \) \((i = 1, \ldots, n \leq N)\) and equal to zero for the rest canonical variables \( y_{n+\alpha} \in E, \eta^{n+\alpha} \in O \) \((\alpha = 1, \ldots, N - n)\). Therefore, in this case the quantum odd bracket is represented with the anticommutators between the quantities \( y_i, \eta^i \) and with the commutators for the remaining relations of the canonical variables. If the roles of the first and the second sets of the canonical variables change, then the quantum bracket is represented with the anticommutators between \( y_{n+\alpha}, \eta^{n+\alpha} \) and with the commutators in the other relations. In\(^{5,6}\) the odd-bracket quantum representations were obtained for the cases \( n = 0, N \), containing, respectively, only commutators or anticommutators.

5. Quantization of the systems with the odd Poisson bracket

As the simplest example of using of the odd-bracket quantum representations under the quantization of the classical systems based on the odd bracket\(^7\), let us consider the one-dimensional supersymmetric oscillator, whose phase superspace \( x^A \)
contains a pair of even $q, p$ and a pair of odd $\eta^1, \eta^2$ real canonical coordinates. In terms of more suitable complex coordinates $z = (p - iq)/\sqrt{2}, \eta = (\eta^1 - i\eta^2)/\sqrt{2}$ and their complex conjugates $\bar{z}, \bar{\eta}$, the even bracket is written as

$$\{A, B\}_0 = iA \left[ \partial_{\bar{z}} \partial_{\bar{z}} - \partial_{\bar{z}} \partial_{\bar{z}} - \left( \partial_{\bar{\eta}} \partial_{\eta} + \partial_{\bar{\eta}} \partial_{\eta} \right) \right] B$$

and the even Hamiltonian $H$, the supercharges $Q_1, Q_2$ and the fermionic charge $F$ have the forms

$$H = z\bar{z} + \eta \bar{\eta} ; \quad Q_1 = \bar{z} \eta + z \bar{\eta} ; \quad Q_2 = i(\bar{z} \eta - z \bar{\eta}) ; \quad F = \eta \bar{\eta}.$$  

The odd Hamiltonian $\bar{H}$ and the appropriate odd bracket, which reproduce the same Hamilton equations of motion, as those resulting from (21) with the even Hamiltonian $H$ (22), i.e., which satisfy the condition (9), can be taken as $\bar{H} = Q_1$ and

$$\{A, B\}_1 = iA \left( \partial_{\bar{z}} \partial_{\eta} - \partial_{\bar{z}} \partial_{\eta} + \partial_{\bar{\eta}} \partial_{\bar{z}} - \partial_{\bar{\eta}} \partial_{\bar{z}} \right) B. \tag{23}$$

The complex variables have the advantage over the real ones, because with their use the splitting of the canonical variables into two sets $\bar{z}, \bar{\eta}$ and $z, \eta$ satisfies simultaneously the requirements necessary for the quantization both of the brackets (21), (23). Besides, any of the vector fields $\dot{X}_A = -i\{A_i, \ldots \}_\epsilon$ for the quantities $\{A_i\} = (H, Q_1, Q_2, F)$, describing the dynamics and the symmetry of the system under consideration, is split into the sum of two differential operators dependent on either $\bar{z}, \bar{\eta}$ or $z, \eta$. For instance, from (21)-(23) we have

$$0^0 X_H = X_{\bar{H}} = z \partial_z + \eta \partial_\eta - \bar{z} \partial_{\bar{z}} - \bar{\eta} \partial_{\bar{\eta}}. \tag{24}$$

The diagonalization does not take place in terms of the variables $x^A = (q, p; \eta^1, \eta^2)$.

In accordance with the above-mentioned splitting of the complex variables, we can perform one of the two possible divisions all of the functions into the classes, which are common for both of the brackets (21),(23), playing a crucial role under their canonical quantization and leading to the same quantum dynamics for the system under consideration. If $\bar{z}, \bar{\eta}$ are attributed to the first set, then the corresponding function division is

$$\hat{O}_s = (\bar{z} \bar{\eta})^{2s+1} f(z, \eta) ; \quad \hat{E}_s = (\bar{z} \bar{\eta})^{2s} f(z, \eta).$$

If we restrict ourselves to the classes $O_o$ and $E_o$, then $\Psi \in E_o$ and depends only on $z, \eta$ and $A_i \in O_o$. According to the definition (18), the results of the quantum multiplications $^*_i$ and $^*_o$ of $z, \eta \in E_o$ and $\bar{z}, \bar{\eta} \in O_o$ on the wave function $\Psi$ are

$$z ^*_i \Psi = z ^*_o \Psi = z \cdot \Psi ; \quad \bar{\eta} ^*_i \Psi = \bar{z} ^*_o \Psi = \partial_{\bar{z}} \Psi ;$$

$$\eta ^*_i \Psi = \eta ^*_o \Psi = \eta \cdot \Psi ; \quad \bar{z} ^*_i \Psi = -\bar{\eta} ^*_o \Psi = \partial_\eta \Psi.$$  

$$8$$
The positive definite scalar product of the wave functions $\Psi_1(z, \eta)$ and $\Psi_2(z, \eta)$ can be determined in the form

$$
(\Psi_1, \Psi_2) = \frac{1}{\pi} \int \exp\left[-(|z|^2 + \bar{\theta}\eta)\right] \Psi_1(z, \eta) ^\dagger \Psi_2(z, \theta) d\bar{\theta} d\eta d(Rez) d(Imz),
$$

(26)

where $\theta$ is the auxiliary complex Grassmann quantity anticommuting with $\eta$, and the integration over the real and imaginary components of $z$ is performed in the limits $(-\infty, \infty)$. It is easy to see that with respect to the scalar product (26) the pairs of the canonical variables, being Hermitian conjugated to each other under the multiplication $\ast$, are $z, \bar{\eta}$ and $\bar{z}, \eta$, but under $\ast_0$ are $z, \bar{z}$ and $\eta, -\bar{\eta}$.

In order to have the action of the Hamiltonian operator, obtained from the system quantization, on the wave function, we need, as it is well known, to replace the canonical variables in the classical Hamiltonian by the respective operators or, which is the same, to define their action with the help of the corresponding quantum multiplication $\ast$. In this connection, in view of (24), (25), we see that the self-consistent quantum Hamilton operators in the even and odd cases, being in agreement with the classical expressions (22) for the equivalent Hamiltonians $H$ and $\bar{H}$ and giving the same result at the action on $\Psi(z, \eta)$, will be respectively

$$
H \ast_0 \Psi = z \ast_0 (\bar{z} \ast_0 \Psi) - \eta \ast_0 (\bar{\eta} \ast_0 \Psi),
$$

(27)

$$
\bar{H} \ast_1 \Psi = z \ast_1 (\bar{\eta} \ast_1 \Psi) + \eta \ast_1 (\bar{z} \ast_1 \Psi).
$$

(28)

The Hamiltonians (27), (28) are Hermitian relative to the scalar product (26) and both, due to (25), are reduced to the Hamilton operator for the one-dimensional supersymmetric oscillator $H = a^+ a + b^+ b$ expressed in terms of the creation and annihilation operators for the bosons $a^+ = z$, $a = \partial_z$, and fermions $b^+ = \eta$, $b = \partial_\eta$ respectively, in the Fock-Bargmann representation (see, for example\textsuperscript{10}). The normalized with respect to (26) eigenfunctions $\Psi_{k,n}(z, \eta)$ of the Hamiltonians (27), (28), corresponding to energy eigenvalues $E_{k,n} = k + n$ ($k = 0, 1; n = 0, 1, \ldots, \infty$) have the form

$$
\Psi_{k,n}(z, \eta) = \frac{1}{\sqrt{n!}} (\eta \ast)^k (z \ast)^n 1.
$$

Note, that another equivalent representation of the quantum supersymmetric oscillator can be obtained, if the canonical variables $z, \eta$ are chosen as the first set. Let us also note that the consideration described above can be extended to the quantization of a set non-interacting supersymmetric oscillators by supplement all the canonical variables $z, \bar{z}, \eta, \bar{\eta}$ with the index $i$ ($i = 1, \ldots, N$) over which a summation have to be performed in the bilinear combinations of the variables in all the formulas of this section.

Thus, we have demonstrated that the use of the quantum representations found for the odd bracket\textsuperscript{7} leads to the self-consistent quantization of the classical Hamilton
systems based on this bracket. We should apparently expect that these representa-
tions are also applicable for the quantization of more complicated classical systems
with the odd bracket.

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7. References

1. C. Buttin *C. Acad. Sci. Paris* **269** (1969) A87.
2. I.A.Batalin, G.A.Vilkovisky, *Phys. Lett.* **102B** (1981) 27.
3. D.V.Volkov, A.I.Pashnev, V.A.Soroka and V.I.Tkach, *JETP Lett.* **44** (1986)
   70; *Teor. Mat. Fiz.* **79** (1989) 117 (in Russian).
4. V.A.Soroka, *Lett. Math. Phys.* **17** (1989) 201.
5. D.V.Volkov, V.A.Soroka, V.I.Tkach, *Yad. Fiz.* **44** (1986) 810 (in Russian).
6. D.V.Volkov, V.A.Soroka, *Yad. Fiz.* **46** (1987) 110 (in Russian).
7. V.A.Soroka, *JETP Lett.* **59** (1994) 219.
8. F.A.Berezin, *Introduction in algebra and analysis with anticommuting variables*
   (Moscow State University, 1983) (in Russian).
9. F.A.Berezin, *The method of secondary quantization* (Moscow, Nauka, 1965) (in
   Russian).
10. A.M.Perelomov, *Generalized coherent states and their applications* (Moscow,
    Nauka, 1987) (in Russian).