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Effect of the Planetesimal Belt on the Dynamics of the Restricted Problem of 2 + 2 Bodies

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Abstract: In this paper, we study the existence and stability of collinear and noncollinear equilibrium points within the frame of the perturbed restricted problem of 2 + 2 bodies by a planetesimal belt. We compare and investigate the corresponding results of the perturbed and unperturbed models. The impact of the planetesimal belt is observed on collinear and noncollinear equilibrium points. We demonstrate that all equilibrium points are unstable, and we numerically investigate the noncollinear equilibrium points. Finally, we emphasize that the proposed problem is a credible model for describing the capture of small bodies by a planet.

Keywords: R2+2BP; planetesimal belt; equilibrium points; stability analysis

1. Introduction

In the field of celestial mechanics, the circular restricted three-body problem (CRTBP), where an infinitesimal body moves under the influence of two primaries without affecting the motion of the primaries, has been extensive used. For details and investigations, see [1–4]. The dynamics of the combined system of two primaries and two minor bodies, called the restricted 2 + 2 body problem (R2+2BP), was analyzed in [5,6]. In this model, two equations of the motion of the CRTBP are coupled with each other by mutual gravitational interactions. This model, or the dual-satellite-like model, helps with studying the dynamic behavior of binary asteroids in the presence of two primaries. Whipple studied R2+2BP and found 14 pairs of equilibrium points [5]. The study of R2+2BP with different disturbing forces produced more precise and accurate data about the system’s dynamic behavior.

Many researchers incorporated different effects of R2+2BP to analyze the perturbed dynamics of the system. For example, Kumar et al. studied the generalized R2+2BP by considering the second primary as a straight segment, and showed that length parameter has a subsequent effect on the location of all equilibrium points [7]. Kalvouridis and Mavraganis [8] found R2+2BP dynamics in the presence of the photo gravitational effect, and Kalvouridis [9] studied the impact of oblate in R2+2BP. The families of a periodic orbit in R2+2BP were discussed by Spurgin [10], who found that the orbits are stable. The stability of R2+2BP was evaluated by Milani and Nobili [11], who showed that the integral of the system, which is similar to RTBP, does not result in hill stability. The restricted 2 + 2 problem with a homogeneous axis-symmetric ellipsoid was described by El-Shaboury [12], who also found 16 solutions in the neighborhood of triangular equilibrium points. A ring-shaped disk-like region formed from dust, comets, asteroids, etc., in the space having considerable mass exerts some gravitational force and affects the dynamic behavior of infinitesimal bodies.
In our solar system, the asteroid belt and kuiper belt are dust-belt-like structures. Dust-belt-like structures are also present in the Proxima Centauri system. Many researchers have studied the effect of the asteroid belt in CRTBP [13–16], and found that these perturbations exhibit significant changes in the equilibrium position. In this paper, we investigate some new aspects of R2+2BP, along with the disk-like belt effect on the potential function; as such, we found the change in the equilibrium positions. Moreover, we analyze the variation in the distance of the equilibrium location of a minor body $P_1$ and $P_2$ for $\mu_1 = \mu_2$ from its neighboring equilibrium point $L_i$, $i = 1, 2, 3, 4, 5$ versus mass ratio $\mu$. The effects of perturbed equilibrium points are compared with those of the unperturbed 2 + 2 body problem. We discuss the stability of the perturbed system with the help of eigenvalues for the particle $P_1$.

In general, the restricted 2 + 2 bodies problem is formulated as a credible model to show the capture of small bodies by a planet. In particular, two primaries are considered to revolve in a circular mutual orbit and two infinitesimal bodies, where neither of them affects the primaries’ motion. If the small bodies are temporarily captured in the Hill sphere of the smaller primary, they may become close enough to each other to exchange energy so that one of them becomes regularly and permanently captured. The aforementioned descriptions are considered the major applications of the restricted 2 + 2 bodies problem, which motivated us to study a more generalized model for this problem.

This paper is organized as follows: In Section 2, we formulate the restricted 2 + 2 body problem in the presence of the planetesimal belt effect. In Section 3, the variation in equilibrium points against mass ratio $\mu$ is described and we compare the perturbed equilibrium points with unperturbed equilibrium points. Furthermore, a stability analysis is performed in Section 4, and in Section 5, we discuss the results and provide our conclusions.

2. Formulation of the Model

The restricted 2 + 2 body problem consists of two primaries, $M_1$ and $M_2$, with unit-less masses $1 - \mu$ and $\mu$, respectively. They are assumed to move on circular Keplerian orbits around their common center of mass under their mutual gravitational force. Two infinitesimal bodies, $P_1$ and $P_2$, of dimensionless masses $\mu_1$ and $\mu_2$, respectively, move in the gravitational field while mutually attracting each other without perturbing the primaries. The perturbed mean motion $n$ can be considered as in [15–18],

$$n^2 = 1 + \frac{2M_b r_c}{(r_c^2 + T^2)^2},$$

where $M_b$ is the total mass of the planetesimal belt, and $r_c$ is the dimensionless reference radius of the planetesimal belt. The gravitational potential of the planetesimal-belt-like system is expressed as in [19,20],

$$\varphi_b(R, z) = \frac{M_b}{(R^2 + [a + \sqrt{z^2 + b^2}]^2)^{1/2}},$$

where $R$ is the minor body’s radial distance, $a$ is the flatness parameter, $b$ is the core parameter, and $z$ is the coordinate of the planetesimal belt in a direction of $Z$-axis. The potential reduces to the system of a point mass with the condition $a = b = 0$. Restricting the condition to the $XY$ plane (i.e., $Z = 0$) and defining $T = a + b$, we have the unit less potential written as,

$$\varphi_b(R, 0) = \frac{M_b}{(R^2 + T^2)^{1/2}}.$$

According to the formulation of the restricted 2 + 2 body problem given by [5,21,22], R2 + 2BP is characterized by three parameters, $\mu$, $\mu_1$, and $\mu_2$, which are the mass parameters of $M_2$, $P_1$, and $P_2$, respectively. We consider a perturbed R2+2BP in the rotating coordinate
system. Let \((\mu, 0, 0), (\mu - 1, 0, 0), (x_1, y_1, z_1), \) and \((x_2, y_2, z_2)\) be the coordinates of \(M_1, M_2, P_1, \) and \(P_2, \) respectively, in a rotating frame, as shown in Figure 1.

\[
\begin{align*}
\dot{x}_i - 2ny_i &= \frac{1}{\mu_i} \frac{\partial U}{\partial x_i}, \\
\dot{y}_i + 2nx_i &= \frac{1}{\mu_i} \frac{\partial U}{\partial y_i}, \\
\dot{z}_i &= \frac{1}{\mu_i} \frac{\partial U}{\partial z_i}, \\
U &= \sum_{i=1}^{2} \mu_i \left[ \frac{1}{2} n^2 \left( x_i^2 + y_i^2 \right) + \frac{1 - \mu}{r_{1i}} + \frac{\mu}{r_{2i}} + \frac{1}{2} \frac{\mu_3 - 1}{r} \right] + \frac{M_b}{(R_i^2 + T_i^2)^{3/2}}.
\end{align*}
\]

where

\[
\begin{align*}
n^2 &= 1 + \frac{2M_br_c}{(r^2 + T^2)^{3/2}}, \\
\mu &= \frac{M_2}{M_1 + M_2}, \\
\mu_i &= \frac{m_i}{M_1 + M_2} \\
r_{1i}^2 &= 1 - \mu + \mu^2, \\
r_{2i}^2 &= x_i^2 + y_i^2, \\
r_1^2 &= (x_1 - \mu)^2 + y_1^2 + z_1^2, \\
r_2^2 &= (x_1 - \mu + 1)^2 + y_1^2 + z_1^2, \\
r^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.
\end{align*}
\]

3. Equilibrium Points

The equilibrium points are the positions of an infinitesimal body where the motion of the minor vanishes. Thus, the velocity and acceleration of \(P_i, i = 1, 2\) are zero, i.e., \(\dot{x}_i = \dot{y}_i = \dot{z}_i = x_i = y_i = z_i = 0.\) Applying these conditions in Equation (2), we obtain,

\[
U_{x_1} = \mu_1 \left( n^2x_1 - \frac{(1-\mu)(x_1-\mu)}{r_{11}^3} - \frac{\mu(x_1-\mu+1)}{r_{21}^3} - \frac{\mu_3(x_1-x_2)}{r^3} \right) - \frac{M_b x_1}{(r_1^2 + T_1^2)^{3/2}} = 0,
\]

Figure 1. Restricted 2 + 2 body problem model.

Using dimensionless variables and considering the effect of the planetesimal belt, the restricted 2 + 2 problem is described by differential equations [5]:
with small parameters $[5,8]$. Using the value of $\mu$ by [5]. The solutions of Equations (10) and (11) are using Equations (3) and (6) with conditions.

3.1. Collinear Equilibrium Points

The solution of the present model obtained by the perturbation method was proposed from (9) in (8), we obtain either $z_2 = 0$ or $c = 0$, where

$$\begin{align*}
    c &= \left[\frac{(1-\mu)}{r_{21}} + \frac{\mu}{r_{22}} + \frac{\mu_1 \mu_2}{r_6}\left(\frac{(1-\mu)}{r_{11}} + \frac{\mu}{r_{21}} + \frac{\mu_2}{r_{22}}\right)\right].
\end{align*}$$

However, we observe that $c$ is nonzero for all values of $\mu$, $\mu_1$, and $\mu_2$. Therefore, $z_2$ must be zero and hence $z_1 = 0$. Consequently, all equilibrium points of the restricted 2 + 2 body problem lie on the XY plane. Hence, the solutions of the equilibrium points can be obtained from Equations (3), (4), (6) and (7).

$$\begin{align*}
    U_{y_1} &= \mu_1 \left(n^2 y_1 - \frac{(1-\mu)y_1}{r_{11}^3} - \frac{\mu y_1}{r_{21}^3} - \frac{\mu_2 (y_1 - y_2)}{r^3} \right) - \frac{M_0 y_1}{(R_1^2 + T^2)^2} = 0, \\
    U_{z_1} &= -\frac{(1-\mu)z_1}{r_{11}^3} - \frac{\mu z_1}{r_{21}^3} - \frac{\mu_2 (z_1 - z_2)}{r^3} = 0, \\
    U_{x_2} &= \mu_2 \left(n^2 x_2 - \frac{(1-\mu)(x_2 - \mu)}{r_{12}^3} - \frac{\mu (x_2 - \mu + 1)}{r_{22}^3} - \frac{\mu_1 (x_2 - x_1)}{r^3} \right) \\
    & \quad - \frac{M_0 x_2}{(R_2^2 + T^2)^2} = 0, \\
    U_{y_2} &= \mu_2 \left(n^2 y_2 - \frac{(1-\mu)y_2}{r_{12}^3} - \frac{\mu y_2}{r_{22}^3} - \frac{\mu_2 (y_2 - y_1)}{r^3} \right) - \frac{M_0 y_2}{(R_2^2 + T^2)^2} = 0, \\
    U_{z_2} &= -\frac{(1-\mu)z_2}{r_{21}^3} - \frac{\mu z_2}{r_{22}^3} - \frac{\mu_1 (z_2 - z_1)}{r^3} = 0.
\end{align*}$$

Simplifying Equation (5), we obtain,

$$\begin{align*}
    z_1 &= z_2 \frac{\mu_2}{r^3 \left[\frac{(1-\mu)}{r_{11}^3} + \frac{\mu_1 \mu_2}{r_{21}^3 + r_{22}^3} \right]}.
\end{align*}$$

Using the value of $z_1$ from (9) in (8), we obtain either $z_2 = 0$ or $c = 0$, where

$$\begin{align*}
    c &= \left[\frac{(1-\mu)}{r_{21}} + \frac{\mu}{r_{22}} + \frac{\mu_1 \mu_2}{r_6}\left(\frac{(1-\mu)}{r_{11}} + \frac{\mu}{r_{21}} + \frac{\mu_2}{r_{22}}\right)\right].
\end{align*}$$

The collinear equilibrium points appear on the X-axis. These points can be calculated using Equations (3) and (6) with conditions $y_1 = 0$ and $y_2 = 0$. Then, we have,

$$\begin{align*}
    \mu_1 \left(nx_1 - \frac{(1-\mu)(x_1 - \mu)}{|x_1 - \mu|^3} - \frac{\mu(x_1 - \mu + 1)}{|x_1 - \mu + 1|^3} - \frac{\mu_2 (x_1 - x_2)}{|x_1 - x_2|^3} \right) \\
    & \quad - \frac{M_0 x_1}{x_1^2} \left(1 - \frac{3 r_2^2}{2 x_2^2} + \frac{15 r_4^4}{8 x_2^4}\right) = 0,
\end{align*}$$

$$\begin{align*}
    \mu_1 \left(nx_1 - \frac{(1-\mu)(x_1 - \mu)}{|x_1 - \mu|^3} - \frac{\mu(x_1 - \mu + 1)}{|x_1 - \mu + 1|^3} - \frac{\mu_2 (x_1 - x_2)}{|x_1 - x_2|^3} \right) \\
    & \quad - \frac{M_0 x_1}{x_1^2} \left(1 - \frac{3 r_2^2}{2 x_2^2} + \frac{15 r_4^4}{8 x_2^4}\right) = 0.
\end{align*}$$

The solution of the present model obtained by the perturbation method was proposed by [5]. The solutions of Equations (10) and (11) are $x_1$ and $x_2$, expressed in a power series with small parameters [5,8].

$$\begin{align*}
    x_1 &= L_j + a_{11} \varepsilon_2 + a_{12} \varepsilon_2^2 + \ldots, \\
    x_2 &= L_j + a_{21} \varepsilon_1 + a_{22} \varepsilon_1^2 + \ldots,
\end{align*}$$
where \( \varepsilon_i = \frac{\mu_i}{(\mu_1 + \mu_2)^{\frac{3}{2}}} \) and \( L_j, j = 1, 2, 3 \) are the collinear Lagrangian points of CRTBP. Hence, Equations (10) and (11) can be written as:

\[
\begin{align*}
a_{11} W_{xx}^{0} \varepsilon_2 - \frac{\mu_2 (x_1 - x_2)}{|x_1 - x_2|^3} & = 0, \\
a_{21} W_{xx}^{0} \varepsilon_1 - \frac{\mu_1 (x_2 - x_1)}{|x_1 - x_2|^3} & = 0,
\end{align*}
\]

where

\[
W = \frac{1}{2} \left( \frac{x^2 + y^2}{1 + \frac{\mu}{r_1} + \frac{\mu}{r_2} + \frac{M_b}{(R^2 + T^2)^2}} \right).
\]

Using Equations (12) and (13), we have \( a_{11} = -a_{21} \) and \( a_{11} = \pm \frac{1}{(W_{xx})^{\frac{1}{3}}} \). Thus, the equilibrium points of R2+2BP are:

\[
\begin{align*}
L_{j\pm}^0 & = L_j \pm \frac{\mu_2}{[(\mu_1 + \mu_2)^2 W_{xx}]^{\frac{1}{3}}}, \\
L_{j\pm}^1 & = L_j \pm \frac{-\mu_1}{[(\mu_1 + \mu_2)^2 W_{xx}]^{\frac{1}{3}}}.
\end{align*}
\]

Equations (15) and (16) yield paired equilibrium positions near the collinear equilibrium points \( L_j, j = 1, 2, 3 \) of R2+2BP. We denote the collinear equilibrium points as \( L_{j\pm}^1 \), \( i = 1, 2, j = 1, 2, 3 \). Superscript \( P_i \) denotes the equilibrium points for an infinitesimal body; \( P_j, j = 1, 2 \) denotes the equilibrium points near \( L_1, L_2 \), and \( L_3 \). Subscript \( j \) denotes the relative position of equilibrium points, where + indicates the right and - indicates the left position with respect to \( L_j, j = 1, 2, 3 \). Figure 2 shows the positions of the equilibrium points when \( \mu = 0.1, \mu_1 = 0.01 \), and \( \mu_2 = 0.001 \) with the planetesimal belt effect \( M_b = 3 \times 10^{-7} \) and parameter \( T = 0.11 \). The positions of the equilibrium points are shown as \( P_1 \) (green) and \( P_2 \) (red). In the presence of the planetesimal belt effect, we can observe that the first equilibrium position of \( P_1 \) \( \left( L_{1\pm}^1 \right) \) is to the left of \( L_1 \). The equilibrium position of \( P_2 \) \( \left( L_{1\pm}^2 \right) \) is to the right of \( L_1 \). Similarly, the second equilibrium position of \( P_1 \), i.e., \( \left( L_{1\pm}^3 \right) \), is to the right of \( L_1 \). The equilibrium position of \( P_2 \), i.e., \( \left( L_{1\pm}^4 \right) \), is to the left of \( L_1 \). As such, we also found four collinear equilibrium points near \( L_2 \) and \( L_3 \).

The positions of the collinear equilibrium points were also calculated numerically, as shown in Tables 1–3 for \( \mu_1 = 10^{-10}, \mu_2 = 10^{-12}, M_b = 3.7 \times 10^{-7} \), and \( T = 0.11 \) with the variation in \( \mu \). Table 1 shows that when \( \mu \) increases \( L_{1\pm}^1, L_{1\pm}^2 \) decreases. In Tables 2 and 3, \( L_{2\pm}^1, L_{2\pm}^2, L_{2\pm}^3, L_{3\pm}^1, \) and \( L_{3\pm}^2 \) increase with increasing \( \mu \). Moreover, we analyzed the effect of the planetesimal belt on R2+2BP. The equilibrium positions of two minor bodies are the same if the masses are equal, i.e., \( \mu_1 = \mu_2 \). Consequently, the distance from \( L_j \) to \( L_{j\pm}^1 \) for \( i = 1, 2 \) is equal, i.e., if \( d(x, y) \) is the distance between points \( x \) and \( y \), then \( d(L_j, L_{j\pm}^1) = d(L_j, L_{j\pm}^2) \). Therefore, we plot the distance versus \( \mu \) in Figure 2 by considering \( M_b = 3.7 \times 10^{-7}, T = 0.11 \) and \( \mu_1 = \mu_2 = 10^{-2} \); in Figure 3, \( d(L_1, L_{1\pm}^1) \) increases as \( \mu \) increases in \( 0 < \mu < 0.5 \). Concurrently, \( d(L_2, L_{2\pm}^1) \), and \( d(L_3, L_{3\pm}^1) \) decrease as \( \mu \) increases from 0 to 0.5.
Figure 2. The position of equilibrium points when $\mu = 0.1$, $\mu_1 = 0.01$, $\mu_2 = 0.001$, $M_x = 3 \times 10^{-7}$, and parameter $T = 0.11$. The positions of the primaries are $M_1$ and $M_2$, represented by an asterisk and blue dot, respectively. $L_1$, $L_2$, $L_3$, $L_4$, and $L_5$ with black dots are the Lagrangian points of CRTBP. The equilibrium points of R2+2BP with the planetesimal belt effect are shown in green and red dots for $P_1$ and $P_2$, respectively.

Figure 3. Distance versus mass ratio $\mu$ plot in the presence of the planetesimal belt effect for collinear equilibrium points.
Table 1. Equilibrium solution of R2+2BP near $L_1$.

| $\mu$ | $L_{1+}^{P_1}$ | $L_{1-}^{P_1}$ | $L_{1+}^{P_2}$ | $L_{1-}^{P_2}$ |
|-------|----------------|----------------|----------------|----------------|
| 0.0001 | $-1.03220039257577$ | $-1.0324743926651$ | $-1.03242294329243$ | $-1.03242294329243$ |
| 0.0010 | $-1.06968776321577$ | $-1.07014443189694$ | $-1.069911838099976$ | $-1.06991381421295$ |
| 0.0010 | $-1.14652917405563$ | $-1.14700990922027$ | $-1.146764704031377$ | $-1.146762826926213$ |
| 0.0100 | $-1.25944627441467$ | $-1.25995339005807$ | $-1.25970236781459$ | $-1.25969729665815$ |

Table 2. Equilibrium solution of R2+2BP near $L_2$.

| $\mu$ | $L_{2+}^{P_1}$ | $L_{2-}^{P_1}$ | $L_{2+}^{P_2}$ | $L_{2-}^{P_2}$ |
|-------|----------------|----------------|----------------|----------------|
| 0.0001 | $-0.96784674731841$ | $-0.96806302118760$ | $-0.96806739035667$ | $-0.96806739035667$ |
| 0.0010 | $-0.93107229302071$ | $-0.93150165725965$ | $-0.93128482832217$ | $-0.93128912196449$ |
| 0.0100 | $-0.84787219955956$ | $-0.848076647451765$ | $-0.84808077777943$ | $-0.84808077777943$ |
| 0.0100 | $-0.60884456030787$ | $-0.60922565908104$ | $-0.60903320420095$ | $-0.60903701518832$ |

Table 3. Equilibrium solution of R2+2BP near $L_3$.

| $\mu$ | $L_{3+}^{P_1}$ | $L_{3-}^{P_1}$ | $L_{3+}^{P_2}$ | $L_{3-}^{P_2}$ |
|-------|----------------|----------------|----------------|----------------|
| 0.0001 | 1.00036136204178 | 0.99972197050552 | 1.0000486323133 | 1.00003846931597 |
| 0.0010 | 1.00073606083426 | 1.0009702640224 | 1.0001986261641 | 1.0001346828009 |
| 0.0100 | 1.00485869150204 | 1.00384735168538 | 1.00416980239190 | 1.00416342073972 |
| 0.0100 | 1.04192234926276 | 1.04129546687074 | 1.04161204247871 | 1.04160577365479 |

In Figure 4, we compare the effect of planetesimal belt perturbation on the collinear equilibrium points with unperturbed collinear equilibrium points in consideration of a distance function. In Figure 4a–d, the red line shows the distance $d_1(L_j, L_{j+}^{P_1}), (j = 1, 2, 3)$ and $(i = 1, 2)$ with the planetesimal belt effect, and the green line shows the distance $d_2(L_j, L_{j+}^{P_1})$ without the planetesimal belt effect. Here, $d_1$ is used to show the effect on distance with planetesimal belt perturbation, whereas $d_2$ is used for unperturbed collinear points. Figure 4a,d shows that $d_1(L_1, L_{1+}^{P_1}) > d_2(L_1, L_{1+}^{P_1})$ and $d_1(L_3, L_{3+}^{P_1}) > d_2(L_3, L_{3+}^{P_1})$ when $0 < \mu < 0.5$. We used a step length $h = 0.001$ for the variation in $\mu$. We found that $d_1(L_2, L_{2+}^{P_1}) < d_2(L_2, L_{2+}^{P_1})$ for $\mu < 0.153$ and $d_1(L_2, L_{2+}^{P_1}) > d_2(L_2, L_{2+}^{P_1})$ for $\mu > 0.153$, as shown in Figure 4a–c.

Tables 4–6 show the collinear equilibrium points of the system: Sun–Saturn with the Kuiper belt ($\mu = 0.000286, M_B = 3.00 \times 10^{-7}$, $T = 0.11$), Sun–Mars with an asteroid belt ($\mu = 0.000003, M_B = 1.6 \times 10^{-9}$, $T = 0.11$), and the Proxima Centauri system with a dust disc ($\mu = 0.000031, M_B = 2.50 \times 10^{-7}$, $T = 0.11$), as described by the authors in [16,23,24] for the restricted 2 + 2 body problem having $\mu_1 = 10^{-10}$ and $\mu_2 = 10^{-12}$. Table 4 shows the equilibrium points near $L_1$, Table 5 represents the equilibrium points near $L_2$, and the equilibrium points near $L_3$ are depicted in Table 6.

Table 4. Equilibrium solution near $L_1$ of different planetary systems.

| $\mu$ | $L_{1+}^{P_1}$ | $L_{1-}^{P_1}$ | $L_{1+}^{P_2}$ | $L_{1-}^{P_2}$ |
|-------|----------------|----------------|----------------|----------------|
| 0.0002860 | $-1.04608390234869$ | $-1.04613298051897$ | $-1.04608402825123$ | $-1.04603497798094$ |
| 0.0000310 | $-1.02197011476239$ | $-1.02195566697454$ | $-1.02197021176980$ | $-1.02185865955764$ |
| 0.0000003 | $-1.00464841079892$ | $-1.00469660080715$ | $-1.00464850708266$ | $-1.00460031707443$ |

Table 5. Equilibrium solution near $L_2$ of different planetary systems.

| $\mu$ | $L_{2+}^{P_1}$ | $L_{2-}^{P_1}$ | $L_{2+}^{P_2}$ | $L_{2-}^{P_2}$ |
|-------|----------------|----------------|----------------|----------------|
| 0.0002860 | $-0.95473464120842$ | $-0.95478173944278$ | $-0.95473475331078$ | $-0.95468763707642$ |
| 0.0000310 | $-0.97834688257383$ | $-0.97839450389731$ | $-0.97834697743511$ | $-0.97829935599925$ |
| 0.0000003 | $-0.99536525492820$ | $-0.99541324652273$ | $-0.99536535081551$ | $-0.9953173921598$ |
Table 6. Equilibrium solution near $L_3$ of different planetary systems.

| $\mu$          | $L_{3+}^P$          | $L_{3-}^P$          | $L_{3+}^P$          | $L_{3-}^P$          |
|----------------|---------------------|---------------------|---------------------|---------------------|
| 0.0002860      | 1.00011913765467    | 1.00004978229365    | 1.00011899908252    | 1.00018835444353    |
| 0.0000310      | 1.00001290410545    | 0.99994354530304    | 1.00001276552642    | 1.00008212432883    |
| 0.0000003      | 1.00000019376614    | 0.99993083453848    | 1.00000005518627    | 1.00006941441393    |

3.2. Noncollinear Equilibrium Points

The noncollinear equilibrium points of R2+2BP can be found by solving Equations (3), (4), (6) and (7) with $y_1 \neq 0$ and $y_2 \neq 0$. The solution can be obtained by the power series perturbation method. Let $x_i$ and $y_i$, $i = 1, 2$ be the solutions; then,

$$x_i = xL_j + a_i \epsilon_{3-i} + a_i^2 \epsilon_{3-i}^2 + \ldots,$$

$$y_i = yL_j + a_i \epsilon_{3-i} + a_i^2 \epsilon_{3-i}^2 + \ldots.$$
we concluded that the distance increases due to the perturbation. $M$ with the planetesimal belt effect
where, $j$ perpendicular position, $d$ perpendicular equilibrium points, i.e., $x$ prefixes distances, respectively. Furthermore, when the noncollinear equilibrium points are in the $d$ noncollinear equilibrium points. Here, we use $\mu$ increases as $d$ shown in Figure 5a,b. In the case of the perpendicular equilibrium position, the distance $\mu$ increases, as shown in Figure 5a. In Figure 5b, in the case of the inline equilibrium position, the prefixes $\mu$ increases as
respectively. The noncollinear equilibrium points can be distinguished as the perpendicular section, we use different notations to represent equilibrium points near $L$ and $P$. In this
Equations (17)–(20) represent the noncollinear equilibrium points in $R+2BP$. In this
solving the above, we obtain
\[ x_i^p = \mu - \frac{1}{2} \pm \frac{a_i(-1)^j\mu_{3-i}}{[(\mu_1 + \mu_2)^2(W_{xy}a_j + W_{yy})]^\frac{1}{2}(1 + \alpha_j^2)\frac{1}{2}}, \]  
(17)
\[ y_i^p = \frac{(-1)^{k+1}\sqrt{3}}{2} \pm \frac{\mu_{3-i}}{[(\mu_1 + \mu_2)^2(W_{xx} + W_{yy})]^\frac{1}{2}(1 + \alpha_j^2)\frac{1}{2}}, \]  
(18)
\[ x_i^l = \mu - \frac{1}{2} \pm \frac{a_i\mu_{3-i}}{[(\mu_1 + \mu_2)^2(W_{xx} + W_{yy})]^\frac{1}{2}(1 + \alpha_j^2)\frac{1}{2}}, \]  
(19)
\[ y_i^l = \frac{(-1)^{k+1}\sqrt{3}}{2} \pm \frac{(-1)^{j+1}\mu_{3-i}}{[(\mu_1 + \mu_2)^2(W_{xx} + W_{yy})]^\frac{1}{2}(1 + \alpha_j^2)\frac{1}{2}}, \]  
(20)
where, $j = 1, k = 1$ at $L_4$ and $j = 2, k = 2$ at $L_5$
\[ a_{1,2} = \frac{(-1)^{k+1} \pm (-1)^{k} \sqrt{1 - 12(\mu - 0.5)}}{2\sqrt{3}(\mu - 0.5)}. \]

Equations (17)–(20) represent the noncollinear equilibrium points in $R+2BP$. In this
section, we use different notations to represent equilibrium points near $L_4$ and $L_5$. The
prefixes $x$ and $y$ are used in $L_{i+}$ to denote the X and Y coordinates of equilibrium points,
respectively. The noncollinear equilibrium points can be distinguished as the perpendicular
and inline equilibrium solutions in [9]. Again, the prefixes $I$ and $P$ are used for inline and
perpendicular equilibrium points, i.e., $1xL_{i+}$ is the X-coordinate of the equilibrium point
of the first infinitesimal body toward the origin.

The distance versus $\mu$ of the perpendicular and inline equilibrium positions to $L_{4,5}$
with the planetesimal belt effect $M_b = 3.7 \times 10^{-7}$, $T = 0.11$ and $\mu_1 = \mu_2 = 10^{-2}$
are shown in Figure 5a,b. In the case of the noncollinear equilibrium position $d(L_{4,5}, PL_{4,5\pm})$
increases as $\mu$ increases, as shown in Figure 5a. In Figure 5b, in the case of the inline
equilibrium position, the distance $d(L_{4,5}, IL_{4,5\pm})$ increases rapidly when $\mu$
approaches $\frac{1}{2}$. Figure 6a shows that $d_1(L_{4,5}, PL_{4,5\pm}) > d_2(L_{4,5}, PL_{4,5\pm})$, i.e., due to perturbation, the
distance increases with respect to the unperturbed distance in the case of perpendicular
noncollinear equilibrium points. Here, we use $d_1$ and $d_2$ for the perturbed and unperturbed
distances, respectively. Furthermore, when the noncollinear equilibrium points are in the
perpendicular position, $d_1(L_{4,5}, IL_{4,5\pm}) > d_2(L_{4,5}, IL_{4,5\pm})$, as shown in Figure 6b. Hence,
we concluded that the distance increases due to the perturbation.
The positions of the noncollinear equilibrium points were calculated numerically and are shown in Tables 7–10 for $\mu_1 = 10^{-10}$, $\mu_2 = 10^{-12}$, $M_B = 3.7 \times 10^{-7}$, and $T = 0.11$. Table 7 shows the perpendicular equilibrium points, where the coordinates were taken as $(P_{xL_{4,5}^{i+1}}, P_{yL_{4,5}^{i+1}})$ and $(P_{xL_{4,5}^{i-1}}, P_{yL_{4,5}^{i-1}})$, $i = 1, 2$. We observed that the coordinates $P_{xL_{4,5}^{4+}}$, $P_{yL_{4,5}^{4+}}$, and $P_{xL_{4,5}^{4-}}$ increased, and $P_{yL_{4,5}^{4-}}$ decreased as we increased the value of $\mu$. The inline equilibrium points $(Ix_{L_{4,5}^{i-1}}, Iy_{L_{4,5}^{i-1}})$ are shown in Table 8.

Furthermore, we observed that $Ix_{L_{4,5}^{4+}}$, $Iy_{L_{4,5}^{4+}}$, and $Ix_{L_{4,5}^{4-}}$ increased as $\mu$ increased, whereas $Iy_{L_{4,5}^{4+}}$ decreased as $\mu$ increased. Table 9 represents the perpendicular equilibrium points near $L_5$, and $(P_{xL_{5}^{i+1}}, P_{yL_{5}^{i+1}})$ and $(P_{xL_{5}^{i-1}}, P_{yL_{5}^{i-1}})$ are coordinates in which $P_{xL_{5}^{i+1}}$, $P_{yL_{5}^{i+1}}$, and $P_{xL_{5}^{i-1}}$ increased and $P_{yL_{5}^{i-1}}$ decreased as $\mu$ increased. Finally, the coordinates of the inline equilibrium points are shown in Table 10. $(Ix_{L_{5}^{i+1}}, Iy_{L_{5}^{i+1}})$ and $(Ix_{L_{5}^{i-1}}, Iy_{L_{5}^{i-1}})$

Figure 5. Distance versus $\mu$ in the presence of the planetesimal belt effect for noncollinear equilibrium points. (a) Perpendicular equilibrium points. (b) Inline equilibrium points.

Figure 6. Comparison of distance versus $\mu$ between perturbed (red line) and unperturbed (green line) noncollinear equilibrium points in R2+2BP. (a) Perpendicular equilibrium points. (b) Inline equilibrium points.
are the coordinates, and we observed that $I_x L_{5+}^P$, $I_y L_{5+}^P$ and $(I_x L_{5-}^P)$ increased and $I_y L_{5-}^P$ decreased as the value of $\mu$ increased.

Table 7. Perpendicular equilibrium solution of the restricted 2 + 2 body problem near $L_4$.

| $\mu$   | $P_x L_{4+}^P$ | $P_y L_{4+}^P$ | $P_x L_{4-}^P$ | $P_y L_{4-}^P$ |
|---------|----------------|----------------|----------------|----------------|
| 0.0001  | -0.49990159842961 | 0.86602249507265 | -0.49989401570399 | 0.8660283274910 |
| -0.0005 | -0.49990594296138 | 0.86574838008869 | -0.49970415703862 | 0.86632014773306 |

Table 8. In-line equilibrium solution of the restricted 2 + 2 body problem near $L_4$.

| $\mu$   | $I_x L_{4+}^P$ | $I_y L_{4+}^P$ | $I_x L_{4-}^P$ | $I_y L_{4-}^P$ |
|---------|----------------|----------------|----------------|----------------|
| 0.0001  | -0.49978629067686 | 0.86595962036838 | -0.50001370932314 | 0.86609090745337 |
| -0.0010 | 0.49937148790105 | 0.86297693011322 | -0.50482512098495 | 0.86907357468918 |

Table 9. Perpendicular equilibrium solution of the restricted 2 + 2 body problem near $L_5$.

| $\mu$   | $P_x L_{5+}^P$ | $P_y L_{5+}^P$ | $P_x L_{5-}^P$ | $P_y L_{5-}^P$ |
|---------|----------------|----------------|----------------|----------------|
| 0.0001  | -0.49998401570399 | -0.86602803274910 | -0.49990159842961 | 0.86602249507265 |
| -0.0010 | -0.49880422918461 | -0.86602803935550 | -0.49900159770815 | 0.86602249376500 |

Table 10. The inline equilibrium solution of the restricted 2 + 2 body problem near $L_5$.

| $\mu$   | $I_x L_{5+}^P$ | $I_y L_{5+}^P$ | $I_x L_{5-}^P$ | $I_y L_{5-}^P$ |
|---------|----------------|----------------|----------------|----------------|
4. Stability of Motion Near Equilibrium Points

We found four paired equilibrium positions around each of \( L_4 \) and \( L_5 \). In this section, we analyze the stability of the infinitesimal body \( P_1 \). Let \( (x_0, y_0) \) be any equilibrium point of particle \( P_1 \), and \( \delta \) and \( \zeta \) be small perturbations in the \( x \) and \( y \) directions, respectively, i.e., \( x_1 = x_0 + \delta \) and \( y_1 = y_0 + \zeta \).

\[
\delta - 2n\zeta = \frac{1}{\mu_1} U_{x_1}(x_0 + \delta, y_0 + \zeta) = \frac{1}{\mu_1} (\delta U_{x_1}^0 + \zeta U_{y_1}^0), \quad (21)
\]

\[
\zeta + 2n\delta = \frac{1}{\mu_1} U_{y_1}(x_0 + \delta, y_0 + \zeta) = \frac{1}{\mu_1} (\delta U_{y_1}^0 + \zeta U_{x_1}^0). \quad (22)
\]

The equations can be written in matrix form as \( \dot{X} = AX \), i.e.,

\[
\begin{bmatrix}
\delta \\
\zeta \\
\dot{\delta} \\
\dot{\zeta}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{1}{\mu_1} U_{x_1}^0 & \frac{1}{\mu_1} U_{y_1}^0 & 0 & 2n \\
\frac{1}{\mu_1} U_{y_1}^0 & \frac{1}{\mu_1} U_{x_1}^0 & -2n & 0
\end{bmatrix} \begin{bmatrix}
\delta \\
\zeta \\
\dot{\delta} \\
\dot{\zeta}
\end{bmatrix}.
\]

The characteristic equation of the matrix may be reduced to

\[
\lambda_1^2 + (4n^2 - \frac{1}{\mu_1} U_{x_1}^0 - \frac{1}{\mu_1} U_{y_1}^0) \lambda_2^2 + \frac{1}{\mu_1} (U_{x_1}^0 U_{y_1}^0 - U_{y_1}^0)^2 = 0. \quad (23)
\]

Solving Equation (23), we have,

\[
\lambda_{1,2} = \pm \left( \frac{b - \sqrt{b^2 - 4c}}{2} \right)^{\frac{1}{2}}, \quad (24)
\]

\[
\lambda_{3,4} = \pm \left( \frac{b + \sqrt{b^2 - 4c}}{2} \right)^{\frac{1}{2}}, \quad (25)
\]

where, \( b = 4n^2 - \frac{1}{\mu_1} U_{x_1}^0 - \frac{1}{\mu_1} U_{y_1}^0 \) and \( c = \frac{1}{\mu_1} (U_{x_1}^0 U_{y_1}^0 - U_{y_1}^0)^2 \).

We note that Equations (24) and (25) are the eigenvalues of the characteristic Equation (23). With the help of these eigenvalues, we can find the stability of particle \( P_1 \).

4.1. Stability at Collinear Points

The stability of the collinear equilibrium point of R2+2BP can be approximated by Equation (23), with condition \( y_1 = y_2 = 0 \), i.e.,

\[
U_{x_1 x_1}^0 = \mu_1 \left( n + \frac{2(1 - \mu)}{|x_1 - \mu|^3} + \frac{2\mu}{|x_1 - \mu + 1|^3} + \frac{\mu_2}{|x_1 - x_2|^3} \right) + \frac{3M_b x_1^2}{(T^2 + x_1^2)^{\frac{3}{2}}},
\]

\[
U_{y_1 y_1}^0 = \mu_1 \left( n - \frac{(1 - \mu)}{|x_1 - \mu|^3} - \frac{\mu}{|x_1 - \mu + 1|^3} - \frac{\mu_2}{|x_1 - x_2|^3} \right) - \frac{M_b}{(T^2 + x_1^2)^{\frac{3}{2}}},
\]

and

\[
U_{x_1 y_1}^0 = 0.
\]

for \( i = 1 \). Consequently, \( b^2 - 4c > 0 \), and hence, the characteristic equation yields at least one positive real root. This results in the instability at the collinear points.
Tables 4–6 provide the collinear equilibrium points for the following systems: Sun–Saturn with the Kuiper belt, Sun–Mars with an asteroid belt, and the Proxima Centauri system with a dust disc. With the help of these collinear equilibrium points, we calculated the stability of these systems using Equations (23)–(28). We found that real nonzero eigenvalues occur with opposite signs and purely imaginary eigenvalues with opposite signs. Thus, the collinear equilibrium points have a saddle-center behavior [25], i.e., all collinear equilibrium points are unstable. Again, as shown in Figure 7, we selected the Sun–Saturn system with the Kuiper belt with \( \mu_1 = 10^{-7} \), and plotted the eigenvalues against \( \mu_2 \in (10^{-10}, 10^{-7}) \). Continuous lines indicate the nonzero real parts of eigenvalues, denoted as \( \alpha \), and the dashed lines represent the nonzero imaginary parts of eigenvalues, denoted as \( \beta \). Figure 7a,b presents the stability near \( L_1 \), Figure 7b,e displays the stability near \( L_2 \), and the stability near \( L_3 \) is shown in Figure 7c,f. We can observe that all figures have a nonzero positive \( \alpha \), and thus all collinear points are unstable in the considered system.
4.2. Stability at Noncollinear Points

The stability of the triangular restricted problem of 2 + 2 bodies may be approximated by Equation (23), which is calculated for $i = 1$ with the help of the following equations:

\[
\begin{align*}
U^0_{x_1x_1} &= \mu_1 \left( n + \frac{3(1-\mu)(x_1-\mu)^2}{((x_1-\mu)^2 + y_1^2)^2} \right) - \frac{1-\mu}{((x_1-\mu)^2 + y_1^2)^2} + \frac{3\mu(x_1-\mu+1)^2}{((x_1-\mu+1)^2 + y_1^2)^2} \\
&\quad - \frac{\mu}{((x_1-\mu+1)^2 + y_1^2)^2} + \frac{3\mu_1\mu_2(x_1-x_2)^2}{((x_1-x_2)^2 + (y_1-y_2)^2)^2} - \frac{3M_0\lambda_1^2}{(T^2 + x_1^2 + y_1^2)^2} \tag{29}
\end{align*}
\]

\[
\begin{align*}
U^0_{x_1y_1} &= \mu_1 \left( n + \frac{3(1-\mu)(x_1-\mu)y_1}{((x_1-\mu)^2 + y_1^2)^2} \right) - \frac{1-\mu}{((x_1-\mu)^2 + y_1^2)^2} + \frac{3\mu(x_1-\mu+1)y_1}{((x_1-\mu+1)^2 + y_1^2)^2} \\
&\quad + \frac{3\mu_1\mu_2(x_1-x_2)(y_1-y_2)}{((x_1-x_2)^2 + (y_1-y_2)^2)^2} - \frac{3M_0x_1y_1}{(T^2 + x_1^2 + y_1^2)^2} \tag{30}
\end{align*}
\]

\[
\begin{align*}
U^0_{y_1y_1} &= \mu_1 \left( n + \frac{3(1-\mu)y_1^2}{((x_1-\mu)^2 + y_1^2)^2} \right) - \frac{1-\mu}{((x_1-\mu)^2 + y_1^2)^2} + \frac{3\mu y_1^2}{((x_1-\mu+1)^2 + y_1^2)^2} \\
&\quad - \frac{\mu}{((x_1-\mu+1)^2 + y_1^2)^2} + \frac{3\mu_1\mu_2(y_1-y_2)^2}{((x_1-x_2)^2 + (y_1-y_2)^2)^2} - \frac{3M_0y_1^2}{(T^2 + x_1^2 + y_1^2)^2} \tag{31}
\end{align*}
\]
The characteristic Equation (23) provides the eigenvalues to study the stability of the noncollinear equilibrium points in the presence of the planetesimal belt effect. The kinds of roots of Equation (23) depend on the parameters $\mu$, $\mu_1$, $\mu_2$, $M_b$, and $T$. The noncollinear equilibrium points are shown in Tables 11–14 for different planetary systems. We analyzed the stability of the noncollinear equilibrium points using Equations (23)–(28), and we found that the real nonzero eigenvalues occur with opposite signs and purely imaginary eigenvalues occur with opposite signs. Thus, the noncollinear equilibrium points have a saddle-center behavior.

In Figures 8 and 9, for the Sun–Saturn system with the Kuiper belt with $\mu_1 = 10^{-7}$, we plot the eigenvalues against $\mu_2 \in (10^{-10}, 10^{-7})$. Figure 8 describes the stability near $L_4$. Figure 8a,b presents the eigenvalues of the perpendicular equilibrium points, and Figure 8c,d depicts the eigenvalues of the inline equilibrium points near $L_4$. It can be observed in Figure 8a–d that real nonzero positive eigenvalues exist; thus, all noncollinear equilibrium points in the presence of the planetesimal belt effect. The real nonzero eigenvalues occur with opposite signs and purely imaginary eigenvalues occur with opposite signs. Thus, the noncollinear equilibrium points have a saddle-center behavior.

In Figures 8 and 9, for the Sun–Saturn system with the Kuiper belt with $\mu_1 = 10^{-7}$, we plot the eigenvalues against $\mu_2 \in (10^{-10}, 10^{-7})$. Figure 8 describes the stability near $L_4$. Figure 8a,b presents the eigenvalues of the perpendicular equilibrium points, and Figure 8c,d depicts the eigenvalues of the inline equilibrium points near $L_4$. It can be observed in Figure 8a–d that real nonzero positive eigenvalues exist; thus, all noncollinear equilibrium points in the presence of the planetesimal belt effect. The real nonzero eigenvalues occur with opposite signs and purely imaginary eigenvalues occur with opposite signs. Thus, the noncollinear equilibrium points have a saddle-center behavior.

Table 11. Perpendicular equilibrium solution near $L_4$ of different planetary systems.

| $\mu$     | $P_xL_{4-}$ | $P_yL_{4-}$ | $P_xL_{4+}$ | $P_yL_{4+}$ |
|-----------|-------------|-------------|-------------|-------------|
| 0.000286  | -0.4997140346001 | 0.86602523034786 | -0.4997139653999 | 0.86602535037870 |
| -0.49974864000649 | 0.86596527494152 | -0.4996793999351 | 0.86608530578504 |
| 0.000031 | -0.49996903464443 | 0.86602524927065 | -0.4996896533557 | 0.86602536928619 |
| -0.50000364442855 | 0.86596530150765 | -0.49993453557145 | 0.86608531704919 |
| -0.50003434496569 | 0.86596539632081 | -0.49996505303431 | 0.86608541003841 |

Table 12. Inline equilibrium solution near $L_4$ of different planetary systems.

| $\mu$     | $I_xL_{4-}$ | $I_yL_{4-}$ | $I_xL_{4+}$ | $I_yL_{4+}$ |
|-----------|-------------|-------------|-------------|-------------|
| 0.000286  | -0.49971226314910 | 0.86602428787884 | -0.49971573685090 | 0.86602629284772 |
| -0.499771491001 | 0.86502280592385 | -0.50145085089989 | 0.86702777480271 |
| 0.000031 | -0.49996359716080 | 0.86602320762467 | -0.49997264028392 | 0.86602741093216 |
| -0.49632871608103 | 0.86392365553369 | -0.50360928391897 | 0.86812696302315 |
| -0.4999261604195 | 0.86601539754749 | -0.50001678395805 | 0.86603526640443 |
| -0.48921574195066 | 0.85616197836003 | -0.51708365804938 | 0.8758888799919 |

Table 13. Perpendicular equilibrium solution near $L_5$ of different planetary systems.

| $\mu$     | $P_xL_{5-}$ | $P_yL_{5-}$ | $P_xL_{5+}$ | $P_yL_{5+}$ |
|-----------|-------------|-------------|-------------|-------------|
| 0.000286  | -0.4997139653999 | -0.86602535037870 | -0.4997140346001 | -0.86602523034786 |
| -0.4996793999351 | -0.86608530578504 | -0.49974864000649 | -0.8695627494152 |
| 0.000031 | -0.49996903464443 | -0.86602524927065 | -0.4996896533557 | -0.86956294207656 |
| -0.49993453557145 | -0.86601539754749 | -0.50001678395805 | -0.86956301507656 |
| -0.49999965335503 | -0.8660255318647 | -0.49999973455449 | -0.86956331727275 |
| -0.4996505503431 | -0.86608541003841 | -0.50003434496569 | -0.8695639632081 |
Figure 8. Eigenvalues versus $\mu_2 \in (10^{-10}, 10^{-7})$ of the Sun–Saturn system in the presence of the Kuiper belt effect near $L_4$ equilibrium point with $\mu_1 = 10^{-7}$. Continuous lines denote the real part of the eigenvalues, and dashed lines denote the imaginary parts of the eigenvalues for collinear points. (a) Eigenvalues of perpendicular equilibrium point near $L_4$ toward smaller primary. (b) Eigenvalues of perpendicular equilibrium point near $L_4$ away from smaller primary. (c) Eigenvalues of inline equilibrium point near $L_4$ away from center. (d) Eigenvalues of inline equilibrium point near $L_4$ toward center.

Table 14. Inline equilibrium solution near $L_5$ of different planetary systems.

| $\mu$     | $I_1 L_{P_1}^{P_1}$ | $I_1 L_{P_1}^{P_1}$ | $I_1 L_{P_1}^{P_1}$ | $I_1 L_{P_1}^{P_1}$ | $I_1 L_{P_1}^{P_1}$ | $I_1 L_{P_1}^{P_1}$ |
|-----------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 0.000286  | -0.49971396535999    | -0.86602535037870    | -0.49971403464001    | -0.86602523034786    |
|           | -0.49967935999351    | -0.86608530578504    | -0.49974864000649    | -0.86596527494152    |
| 0.000031  | -0.49996896535557    | -0.8660236928619     | -0.49996903464443    | -0.86602524927065    |
|           | -0.4999343357145     | -0.86608531704919    | -0.50000364428255    | -0.86596530150765    |
| 0.000000  | -0.49999656535553    | -0.86602546318647    | -0.49999734464497    | -0.866025343172275   |
|           | -0.49996505503431    | -0.86608541003841    | -0.50003434496569    | -0.86596539632081    |
Figure 9. Eigenvalues versus $\mu_2 \in (10^{-10}, 10^{-7})$ of the Sun–Saturn system in the presence of the Kuiper belt effect near $L_5$ equilibrium point with $\mu_1 = 10^{-7}$. Continuous lines denote the real part of the eigenvalues, and dashed lines denote the imaginary parts of the eigenvalues for collinear points. (a) Eigenvalues of perpendicular equilibrium point near $L_5$ toward smaller primary. (b) Eigenvalues of perpendicular equilibrium point near $L_5$ away from smaller primary. (c) Eigenvalues of inline equilibrium point near $L_5$ away from center. (d) Eigenvalues of inline equilibrium point near $L_5$ toward center.

5. Conclusions

In this study, we considered the effect of the planetesimal belt on the restricted problem of $2 + 2$ bodies in a rotating coordinate system. In the absence of the planetesimal belt effect, i.e., $M_b = 0$, the proposed system coincides with the system obtained by Whipple [5]. Again, if we consider $P_1$ and $P_2$ as the single minor body, the system will convert into the CRTBP. The problem possesses 14 paired equilibrium positions around the five equilibrium points of the CRTBP. The main highlights of work can be summarized as:

- Dynamic analysis of the perturbed restricted problem of $2 + 2$ bodies;
- Existence and stability analysis of both collinear and noncollinear equilibrium points;
- Description of the effect of the planetesimal belt on the motion in the proximity of equilibrium points.

The effect of the planetesimal belt on the equilibrium points and the variation in their distance from $L_j, j = 1, 2, 3, 4, 5$ were studied for some fixed parameter $M_b = 3.7 \times 10^{-7}$ and $T = 0.11$, which we compared with the unperturbed R2+2BP. For mass parameter
$\mu_1 = \mu_2 = 10^{-2}$, the distance of collinear equilibrium points $d(L_1, L_{P1}^{\pm})$, $i = 1, 2$ increases, and $d(L_2, L_{P2}^{\pm})$ and $d(L_3, L_{P3}^{\pm})$ decrease with the variation in $\mu$. The distance $d(L_4, PL_{P4}^{\pm})$ at the inline equilibrium points increases monotonically, but the distance $d(L_4, PL_{P4}^{\pm})$ at the perpendicular equilibrium points decreases initially for some $\mu$. After that, it increases when $\mu$ approaches one-half. The distance of equilibrium points in the presence of the planetesimal belt effect differs from the unperturbed distance. At the collinear point, the perturbed distance $d_1$ is greater than the unperturbed distance $d_2$ at $L_1$ and $L_3$. Near $L_2$, $d_1 < d_2$ for $\mu < 0.153$, and $d_1 < d_2$ for $\mu > 0.153$. In the case of an inline and perpendicular equilibrium point, $d_1 > d_2$ with the variation of $\mu$. Tables 1–3 show that the collinear equilibrium points $L_{P1}^{\pm}$ decrease and $L_{P2}^{\pm}, L_{P3}^{\pm}$ increases with the variation in $\mu \in (0, 0.5)$.

From the stability analysis, we found six paired collinear equilibrium points and eight paired noncollinear equilibrium points. The stability of all collinear equilibrium points was found to be unstable. The generalization of the stability of noncollinear equilibrium points is difficult as five different types of parameters $\mu$, $\mu_1$, $\mu_2$, $M_b$, and $T$ are present in the perturbed R2+2BP. The stability of noncollinear equilibrium points can be analyzed numerically. The different considered planetary systems with suitable $\mu_1$ and $\mu_2$ were found to be unstable.

In the framework of the perturbed restricted problem of 2 + 2 bodies by a planetesimal belt, our obtained results can be outlined as:

- The existence of equilibrium points was examined for both collinear and noncollinear points;
- The stability of motion around these points was studied;
- We compared the corresponding results of the perturbed and unperturbed models;
- The impact of a planetesimal belt was observed on collinear and noncollinear equilibrium points;
- All equilibrium points were found to be unstable, whereas the noncollinear equilibrium points were investigated numerically.

Furthermore, the restricted 2 + 2 bodies problem can be used as a credible model to describe the capture of small bodies by a planet. If the small bodies are temporarily captured in the Hill sphere of a smaller primary, they may become near enough to each other to exchange energy so that one of them becomes regularly and permanently captured.

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