ON THE FINITE \((Q - 1)\)-HAUSDORFF MEASURE OF THE FREE
BOUNDARY IN THE SUBELLIPTIC OBSTACLE PROBLEM

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Abstract. In this note, we prove the finite \((Q - 1)\)-Hausdorff measure of the free boundary in the obstacle problem in a Carnot group \(G\). Here, \(Q\) represents the homogeneous dimension of \(G\). Our main result, Theorem 1.1, constitutes the subelliptic counterpart of the Euclidean result due to Caffarelli, but the analysis is complicated by the lack of commutation of the left-invariant vector fields. This obstruction is compensated by the use of right-invariant derivatives, and by a delicate compactness argument inspired to Caffarelli’s fundamental works.

1. Introduction

Variational inequalities occupy a central place in calculus of variations and in the applied sciences. Perhaps the most important example is the so-called obstacle problem in which, given a bounded open set \(\Omega \subset \mathbb{R}^n\), a source term \(f\), boundary values \(g\) and an obstacle \(\psi\) (with \(g \geq \psi\) on \(\partial \Omega\)), one looks for a solution to the minimisation problem

\[
\min_v \int_\Omega |\nabla v|^2 + f v,
\]

on the class of competing functions \(v\) which are above the obstacle \(\psi\), and with prescribed boundary values \(g\). It is well-known that such a problem has a unique variational solution \(u\) that possesses the optimal interior \(C^{1,1}\) regularity, see [20] and [16]. Furthermore, by the fundamental results of Caffarelli in his groundbreaking paper [4], near every point of positive density of the coincidence set the free boundary is a \(C^{1,\alpha}\) hypersurface provided the obstacle is strictly superharmonic. Combined with the higher-regularity of the free boundary proved in the same year by Kinderlehrer and Nirenberg in [19], this gives a complete picture of the so-called regular part of the free boundary. We also mention that in his note [5] Caffarelli showed that the free boundary has a locally finite \((n - 1)\)-Hausdorff measure.

Prompted by some questions in mechanical engineering and robotics asked by Chirikjian (see also [10]), in the work [12] Danielli, Salsa and the second named author began the study of the obstacle problem in a stratified, nilpotent Lie group \(G\), also known as Carnot group. These geometric ambients are tailor made to model media with non-holonomic constraints: motion at any point is only allowed along a limited set of directions which are prescribed by the physical problem at hand (when the set of directions coincides with the whole tangent space one has Riemannian geometry). In [12] the authors were able to prove that solutions to the subelliptic obstacle problem possess interior \(C^{1,1}\) regularity, where now smoothness is
measured along a basis of left-invariant vector fields which generate the Lie algebra of $G$, see [14] and [15]. Similarly to the above mentioned $C^{1,1}$ result, this $\Gamma^{1,1}$ regularity is best possible in this non-elliptic framework.

In the subsequent work [13] Danielli, Petrosyan and the second named author were able to extend the above mentioned positive density to $C^{1,\alpha}$ result of Caffarelli to the subelliptic obstacle problem in a Carnot group of step two. They proved that, under a suitable thickness condition, the free boundary is locally a non-characteristic $C^{1,\alpha}$ hypersurface (for the relevant notions see [13]).

The purpose of this note is to establish the analogue of Caffarelli’s above mentioned finite $(n - 1)$-Hausdorff measure result for the subelliptic obstacle problem in a Carnot group $G$ of arbitrary step. More precisely, we show that the free boundary has finite $(Q - 1)$-Hausdorff measure $\mathcal{H}^{Q - 1}$ with respect to the Carnot-Carathéodory metric associated with the horizontal bundle of $G$. Here, the number $Q$ indicates the so-called homogeneous dimension of the group $G$ defined by (2.4) below. After suitable normalisation (see [12]), we consider a local normalised solution to the following obstacle problem with zero obstacle

\[
\begin{align*}
\Delta_H u &= 1_{\{u > 0\}} & \text{in } B(2), \\
e &\in \Gamma(u),
\end{align*}
\]

where $\Delta_H$ indicates a horizontal Laplacian in $G$, and we have denoted by $1_E$ the indicator function of $E \subset G$, and with $e$ the group identity. The notation $\Gamma(u) = \partial\{u = 0\}$ indicates the boundary of the coincidence set in the intrinsic ball $B(2)$, i.e., the free boundary of $u$. Here is our main result.

**Theorem 1.1.** Let $u$ be a solution to the problem (1.1). Then, the free boundary $\Gamma(u)$ has finite $\mathcal{H}^{Q - 1}$ measure in $B(1)$.

Theorem 1.1 is best possible. For instance, if we use the logarithmic coordinates (2.11) below, then one can prove that the function $u(p) = \frac{1}{2}(x_1^+)^2$ is a global normalised solution to the obstacle problem (1.1) in $G$, with free boundary given by the non-characteristic vertical hyperplane $\Gamma(u) = \{p \in G \mid x_1 = 0\}$. If one takes into account the non-isotropic group dilations in (2.2) below, then it is not difficult to prove that, in this example, the metric Hausdorff dimension of $\Gamma(u)$ is exactly $Q - 1$.

We stress that the subelliptic obstacle problem poses remarkable challenges with respect to the classical case, and it is far from being fully understood. A crucial aspect is that the left-invariant horizontal vector fields do not commute, and thus in particular (left-) derivatives of a harmonic function are not harmonic, a fact that plays a critical role in the theory and, in particular, in the approach in [5]. Interestingly, we are able to resolve this obstruction by considering the derivatives along right-invariant vector fields, following an idea first introduced in [13]. We then combine this idea with a subtle compactness argument inspired by the fundamental works [3] and [7].

This note is organised as follows. In Section 2, we introduce the relevant notions and notations and gather some basic properties of Carnot groups. In Section 3, we collect some important auxiliary results. In Section 4, we finally prove our main result.

2. **Preliminaries**

In this section we collect some basic properties of Carnot groups which will be used in the rest of the paper. For a brief and self-contained introduction to these geometric ambients, the reader is referred to Chap. 1 & 2 in [17]. We begin with the relevant definition.
**Definition 2.1.** Given \( r \in \mathbb{N} \), a Carnot group of step \( r \) is a simply-connected real Lie group \((\mathbb{G}, \circ)\) whose Lie algebra \( \mathfrak{g} \) is stratified and \( r \)-nilpotent. This means that there exist vector spaces \( \mathfrak{g}_1, \ldots, \mathfrak{g}_r \) such that

\[
\begin{align*}
(i) & \mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r; \\
(ii) & [\mathfrak{g}_j, \mathfrak{g}_j] = \mathfrak{g}_{j+1}, \quad j = 1, \ldots, r-1, \quad [\mathfrak{g}_1, \mathfrak{g}_r] = \{0\}.
\end{align*}
\]

We note that when \( r = 1 \), then the group is Abelian and we are back in the familiar Euclidean situation of [5]. For this reason, we assume throughout that \( r > 1 \). By the assumption that \( \mathbb{G} \) be simply-connected we know that the exponential mapping \( \exp : \mathfrak{g} \to \mathbb{G} \) is a global analytic diffeomorphism onto, see [23] and [11]. We will use this global chart to identify the point \( p = \exp \xi \in \mathbb{G} \) with its logarithmic preimage \( \xi \in \mathfrak{g} \). Once the bracket relations at the level of the Lie algebra are assigned, the group law is too. This follows from the Baker-Campbell-Hausdorff formula, see, e.g., [23, Sec. 2.15],

\[
(2.1) \quad \exp(\xi) \circ \exp(\eta) = \exp \left( \xi + \eta + \frac{1}{2}[[\xi, \eta]] + \frac{1}{12} \left\{ [[\xi, [\xi, \eta]] - [\eta, [\xi, \eta]]] \right\} + \ldots \right),
\]

where the dots indicate commutators of order three and higher. Furthermore, since by (ii) in Definition 2.1 above all commutators of order \( r \) and higher are trivial, in every Carnot group the Baker-Campbell-Hausdorff series in the right-hand side of (2.1) is finite. According to (ii) the first layer \( \mathfrak{g}_1 \) plays a special role since it bracket-generates the whole Lie algebra \( \mathfrak{g} \). It is traditionally referred to as the horizontal layer of \( \mathfrak{g} \). We assume that a scalar product \( \langle \cdot, \cdot \rangle \) is given on \( \mathfrak{g} \) for which the \( \mathfrak{g}_j \)'s are mutually orthogonal. We let \( m_j = \dim \mathfrak{g}_j, \ j = 1, \ldots, r \), and denote by \( N = m_1 + \ldots + m_r \) the topological dimension of \( \mathbb{G} \). For notational simplicity we will hereafter denote with \( m = m_1 \) the dimension of the horizontal layer \( \mathfrak{g}_1 \).

Every Carnot group is naturally equipped with translations and dilations. Using the group law \( \circ \) in (2.1) we can respectively define the left- and right-translations by an element \( p_0 \in \mathbb{G} \) by \( L_{p_0}(p) = p_0 \circ p \) and \( R_{p_0}(p) = p \circ p_0 \). Given a function \( f : \mathbb{G} \to \mathbb{R} \), the action of \( L_{p_0} \) and \( R_{p_0} \) on \( f \) are defined by

\[
L_{p_0}f(p) = f(L_{p_0}(p)), \quad R_{p_0}f(p) = f(R_{p_0}(p)), \quad p \in \mathbb{G}.
\]

A vector field \( Y \) on \( \mathbb{G} \) is called left-invariant (or right-invariant) if for any \( f \in C^\infty(\mathbb{G}) \) and any \( p_0 \in \mathbb{G} \) one has

\[
Y(L_{p_0}f) = L_{p_0}(Yf), \quad Y(R_{p_0}f) = R_{p_0}(Yf).
\]

To define the non-isotropic dilations in \( \mathbb{G} \) one assigns the formal degree \( j \) to the \( j \)-th layer \( \mathfrak{g}_j \) of the Lie algebra, and defines \( \Delta_\lambda : \mathfrak{g} \to \mathfrak{g} \) by setting for every \( \xi = \xi_1 + \ldots + \xi_r \in \mathfrak{g} \), with \( \xi_j \in \mathfrak{g}_j, \ j = 1, \ldots, r \),

\[
\Delta_\lambda \xi = \lambda \xi_1 + \cdots + \lambda^r \xi_r.
\]

One then uses the exponential mapping to lift \( \Delta_\lambda \) to a one-parameter family \( \{\delta_\lambda\}_{\lambda > 0} \) in the group \( \mathbb{G} \) by letting for \( \lambda > 0 \)

\[
(2.2) \quad \delta_\lambda(p) = \exp \circ \Delta_\lambda \circ \exp^{-1}(p), \quad p \in \mathbb{G}.
\]

The dilations are group automorphisms, and thus we have for any \( p, p' \in \mathbb{G} \) and \( \lambda > 0 \)

\[
(2.3) \quad (\delta_\lambda(p))^{-1} = \delta_{\lambda^{-1}}(p), \quad \delta_\lambda(p) \circ \delta_\lambda(p') = \delta_\lambda(p \circ p').
\]

Given \( f : \mathbb{G} \to \mathbb{R} \), the action of \( \{\delta_\lambda\}_{\lambda > 0} \) on \( f \) is defined by

\[
\delta_\lambda f(p) = f(\delta_\lambda(p)), \quad p \in \mathbb{G}.
\]
A function \( f : G \to \mathbb{R} \) is called homogeneous of degree \( \kappa \in \mathbb{R} \) if for every \( \lambda > 0 \) one has \( \delta_\lambda f = \lambda^\kappa f \). A vector field \( Y \) on \( G \) is called homogeneous of degree \( \kappa \) if for every \( f \in C^\infty(G) \) one has \( Y(\delta_\lambda f) = \lambda^\kappa \delta_\lambda(Y f) \).

The \emph{homogeneous dimension} of \( G \) with respect to the dilations (2.2) is defined as follows

\begin{equation}
Q = \sum_{j=1}^r j \dim g_j.
\end{equation}

The name comes from the fact that the bi-invariant Haar measure \( dp \) on \( G \) (which is obtained by pushing forward via the exponential mapping the Lebesgue measure on \( g \)) interacts with \( \{\delta_\lambda\}_{\lambda > 0} \) according to the formula

\begin{equation}
d \circ \delta_\lambda(p) = \lambda^Q dp.
\end{equation}

We note that in the non-Abelian setting of this note (we are assuming \( r > 1 \)) the number \( Q \) in (2.4) is strictly bigger than the topological dimension \( N \) of \( G \). Such number \( Q \) plays a pervasive role in the analysis and geometry of \( G \) and, as the statement of Theorem 1.1 shows, it also determines the dimension of the free-boundary in the obstacle problem (1.1).

With any given orthonormal basis \( \{e_1, \ldots, e_m\} \) of \( g_1 \) we associate a family \( \{X_1, \ldots, X_m\} \) of left-invariant vector fields on \( G \) by letting for \( j = 1, \ldots, m \) and \( p \in G \)

\begin{equation}
X_j(p) = dL_p(e_j),
\end{equation}

where we have denoted by \( dL_p \) the differential of \( L_p \). In view of (ii) in Definition 2.1 the vector fields \( \{X_1, \ldots, X_m\} \) bracket generate the whole Lie algebra of left-invariant vector fields on \( G \), and they constitute a basis for the so-called horizontal subbundle \( H \) of the tangent bundle \( TG \). We note that the action of \( X_j \) on a function \( f \in C^\infty(G) \) is specified by the Lie derivative

\begin{equation}
X_j f(p) = \lim_{t \to 0} \frac{f(p \exp(te_j)) - f(p)}{t} = \frac{d}{dt} f(p \exp(te_j))\bigg|_{t=0}.
\end{equation}

In a similar way, the right-invariant vector field \( \tilde{X}_j \) is defined by

\begin{equation}
\tilde{X}_j(p) = dR_p(e_j),
\end{equation}

and we have

\begin{equation}
\tilde{X}_j f(p) = \lim_{t \to 0} \frac{f(\exp(te_j)p) - f(p)}{t} = \frac{d}{dt} f(\exp(te_j)p)\bigg|_{t=0},
\end{equation}

see e.g. p. 20 in [15]. The operation \( X_j \to \tilde{X}_j \) is an anti-isomorphism of Lie algebras, in the sense that \( [X_i, X_j] = [\tilde{X}_j, \tilde{X}_i] \). The vector fields \( X_j, \tilde{X}_j \) are homogeneous of degree \( \kappa = 1 \), see p. 21 in [15]. One easily checks, see e.g. [13], that

\begin{equation}
[X_i, \tilde{X}_j] = 0, \quad i, j = 1, \ldots, m.
\end{equation}

Denote by \( \{e_{j_1}, \ldots, e_{j_m}\} \) an orthonormal basis of the layer \( g_j \), \( j = 2, \ldots, r \). If \( p = \exp(\xi_1 + \ldots + \xi_r) \), where \( \xi_1 = x_1 e_1 + \ldots + x_m e_m, \xi_2 = x_{2,1} e_{2,1} + \ldots + x_{2,m} e_{2,m}, \ldots, \xi_r = x_{r,1} e_{r,1} + \ldots + x_{r,m} e_{r,m} \), we will routinely identify \( p \) with its logarithmic coordinates, i.e.,

\begin{equation}
p \cong (\xi_1, \ldots, \xi_r) = (x_1, \ldots, x_m, x_{2,1}, \ldots, x_{r,1}, \ldots, x_{r,m}).
\end{equation}

If for every \( j = 1, \ldots, r \), we assign the formal degree \( j \) to the layer \( g_j \) in the stratification of \( g \), then a homogeneous monomial \( \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_r^{\alpha_r} \), with multi-indices \( \alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,m_j}) \), \( j =
therefore by Hörmander's theorem \[ \text{(2.1)} \], states that in every Carnot group \( G \), the distribution \( u \) is said to have weighted degree \( \kappa \) if

\[
\sum_{j=1}^{r} j|\alpha_j| = \sum_{j=1}^{r} j\left(\sum_{s=1}^{m_j} \alpha_{j,s}\right) = \kappa.
\]

Using the Baker-Campbell-Hausdorff formula \( \text{(2.1)} \) one obtains from \( \text{(2.7)} \), \( \text{(2.9)} \) the following expressions in the logarithmic coordinates

\[
X_i = \frac{\partial}{\partial x_i} + \sum_{j=2}^{r} \sum_{s=1}^{m_j} b_{j,i}^s(\xi_1, \ldots, \xi_{j-1}) \frac{\partial}{\partial x_{j,s}}, \quad \tilde{X}_i = \frac{\partial}{\partial x_i} + \sum_{j=2}^{r} \sum_{s=1}^{m_j} \tilde{b}_{j,i}^s(\xi_1, \ldots, \xi_{j-1}) \frac{\partial}{\partial x_{j,s}}
\]

where each \( b_{j,i}^s, \tilde{b}_{j,i}^s \) is a homogeneous polynomial of weighted degree \( j-1 \). From \( \text{(2.12)} \) we obtain

\[
\tilde{X}_i = X_i + \sum_{j=2}^{r} \sum_{s=1}^{m_j} c_{j,i}^s(\xi_1, \ldots, \xi_{j-1}) \frac{\partial}{\partial x_{j,s}},
\]

where \( c_{j,i}^s \) is a homogeneous polynomial of weighted degree \( j-1 \).

We will henceforth routinely assume that \( G \) has been endowed with a left-invariant Riemannian metric with respect to which the system \( \{X_1, \ldots, X_m\} \) is orthonormal. For a function \( f \in C^\infty(G) \), we denote by \( \nabla_H f = \sum_{j=1}^{m} X_j f x_j \) its horizontal gradient, i.e., the projection of the Riemannian connection \( \nabla \) of \( G \) onto the horizontal bundle \( H \). We clearly have \( |\nabla_H f|^2 = \sum_{j=1}^{m} (X_j f)^2 \). We also note that, if \( X_j^\ast \) indicates the formal adjoint of \( X_j \) in \( L^2(G) \), then \( X_j^\ast = -X_j \), see \[ 14 \].

**Definition 2.2.** The horizontal Laplacian associated with the orthonormal basis \( \{e_1, \ldots, e_m\} \) of the horizontal layer \( g_1 \) is the left-invariant second-order partial differential operator in \( G \) defined by

\[
\Delta_H = -\sum_{j=1}^{m} X_j^\ast X_j = \sum_{j=1}^{m} X_j^2.
\]

A distribution \( u \) is called harmonic in an open set \( \Omega \subset G \) if \( \Delta_H u = 0 \) in \( D'(\Omega) \).

Every horizontal Laplacian is homogeneous of degree \( \kappa = 2 \), i.e., one has \( \Delta_H(\delta_f) = \lambda^2 \delta(f(\Delta_H f)) \), for every \( f \in C^\infty(G) \). Furthermore, by the assumptions (i) and (ii) in Definition \( \text{2.1} \) one immediately sees that the system \( \{X_1, \ldots, X_m\} \) satisfies the finite rank condition

\[
\text{rank Lie}[X_1, \ldots, X_m] = N,
\]

therefore by Hörmander’s theorem \[ 18 \] the operator \( \Delta_H \) is hypoelliptic. However, since we are assuming that \( r > 1 \) this operator fails to be elliptic at every point \( p \in G \).

Denote by \( \delta \in D'(G) \) the distribution such that \( <\delta, \varphi> = -\varphi(e) \), where we have denoted by \( e \in G \) the group identity. A distribution \( E \in D'(G) \) such that \( \Delta_H E = -\delta \) is called a fundamental solution for \( \Delta_H \) with singularity at \( e \). A basic result due to Folland, see \[ 14 \], Théor. 2.1], states that in every Carnot group \( G \) there exists a unique fundamental solution \( E \) of \( \Delta_H \) vanishing at infinity which is a homogeneous distribution of degree \( 2 - Q \), where \( Q \) is the homogeneous dimension of \( G \). By Hörmander’s hypoellipticity theorem one has \( E \in C^\infty(G \setminus \{e\}) \).

Finally, we recall that in a Carnot group there exists an intrinsic left-invariant control distance \( d(p, p') \) associated with the horizontal subbundle \( H \) (also known as the Carnot-Carathéodory distance). For its precise definition and its basic properties we refer the reader to \[ 21 \], \[ 1 \] and \[ 17 \], Chap. 4]. We denote with \( B(p, R) = \{ p' \in G \mid d(p', p) < R \} \) the ball
centred at $p$ with radius $R$ in such metric. When the center is the group identity $e$, we will simply write $B(R)$ instead of $B(e, R)$. We note that by left-translation and rescaling, for every $p \in \mathbb{G}$ and $R > 0$ we have $|B(p, R)| = \omega R^Q$, where $\omega = |B(e, 1)| > 0$ is a universal constant. Given $s > 0$, we will denote by $\mathcal{H}^s$ the $s$-Hausdorff measure with respect to the metric $d(p, p')$, see [22].

3. Auxiliary results

In this section we collect some auxiliary results that will be needed in the proof of Theorem 1.1. We begin with some basic interior Schauder estimates for solutions of the horizontal Laplacian, see [25] and also [8] for a generalisation. The reader should keep in mind that, as we have already previously stressed, in the context of this note it is not true that derivatives of a harmonic function are harmonic.

**Theorem 3.1.** Suppose that $\Delta_H u = 0$ in $B(p, 2r) \subset \mathbb{G}$. For any $s \in \mathbb{N}$ there exists a universal constant $C = C(\mathbb{G}, s) > 0$, such that one has

$$|X_{j_1}X_{j_2}...X_{j_s} u(p)| \leq \frac{C}{r^s} \max_{B(p, r)} |u|.$$

In the above estimate, for every $i = 1, ..., s$, the index $j_i$ runs in the set $\{1, ..., m\}$.

Next, we collect some facts from [12]. We fix a bounded domain $D \subset \mathbb{G}$ and a continuous function $\varphi : \overline{D} \to \mathbb{R}$, an obstacle, satisfying $\max_D \varphi > 0$, $\varphi \leq 0$ on $\partial D$. We introduce the Hilbert space $W^{1,2}_H(D) = \{f \in L^2(D) \mid X_j f \in L^2(D), j = 1, ..., m\}$, endowed with the obvious inner product norm $||f||_{W^{1,2}_H(D)} = ||f||_{L^2(D)}^2 + \sum_{j=1}^m ||X_j f||_{L^2(D)}^2$, and let $W^{1,2}_{H,0}(D) = C^\infty_0(D)^{|||}W^{1,2}_H(D)$. Consider the closed convex set $\mathcal{K}_\varphi = \{v \in W^{1,2}_{H,0}(D) \mid v \geq \varphi \text{ a.e. in } D\}$. By standard variational methods there exists a unique solution $u \in \mathcal{K}_\varphi$ to the variational inequality

$$\int_D <\nabla_H u, \nabla_H (v - u)> \geq 0$$

for $v \in \mathcal{K}_\varphi$. If $\zeta \in C^\infty_0(D)$, $\zeta \geq 0$, then $u = u + \zeta \in \mathcal{K}_\varphi$, and therefore $u$ is $\Delta_H$-superharmonic in $D$. One also knows that $u \in C(D)$, and therefore the set $\Omega(u, \varphi) = \{p \in D \mid u(p) > \varphi(p)\}$ is open in $D$. Since $\Delta_H u = 0$ in $\Omega(u, \varphi)$, by hypoellipticity we have $u \in C^\infty(\Omega(u, \varphi))$. The coincidence set $A(u, \varphi) = \{p \in D \mid u(p) = \varphi(p)\}$ is closed relative to $D$, and so is the free boundary $\Gamma(u, \varphi) = \partial A(u, \varphi)$. When the obstacle $\varphi = 0$, we simply write $\Omega(u)$, $\Lambda(u)$ and $\Gamma(u)$.

Before stating the next result we recall the definition of the Folland-Stein class $\Gamma^{1,1}_D = \{f \in C(D) \mid X_j f \in C(D), j = 1, ..., m\}$, see [14], [15].

**Theorem 3.2** (see Theor. 4.3 in [12]). Suppose that $\varphi \in \Gamma^{1,1}(\overline{D})$, then the solution $u$ to the obstacle problem can be modified on a set of measure zero in $D$ so that the resulting function is in $\Gamma^{1,1}_{loc}(D)$. From this property, and from the horizontal Rademacher-Stepanov theorem, we infer that for every $\omega \subset D$ one has $u \in W^{2,\infty}_H(\omega) = \{f \in L^\infty(\omega) \mid X_i f, X_i X_j f \in L^\infty(\omega), i, j = 1, ..., m\}$.

Let $u$ be the solution to the obstacle problem, and normalise it by letting $w = u - \varphi$. Since $\Delta_H u \leq 0$ in $D$, by Bony’s strong maximum principle (see [2]) $u$ cannot touch $\varphi$ at a point where $\Delta_H \varphi > 0$, so near the free boundary one has $\Delta_H \varphi \leq 0$. This shows that in the set
where \( u \) is above the obstacle, one has \( \Delta_H u = \Delta_H (u - \varphi) = -\Delta_H \varphi \geq 0 \). Now it turns out that for the development of the theory of free boundaries (see for instance [4]), one requires that the obstacle \( \varphi \) be strictly superharmonic at the points where the solution \( u \) touches \( \varphi \). In the context of a Carnot group \( \mathbb{G} \), the analogous requirement would be \( \Delta_H \varphi < 0 \) on the coincidence set \( \Lambda(u, \varphi) \). Therefore, following [6] and [12], we normalise the problem by assuming \( \Delta_H \varphi \equiv -1 \). Thus, we henceforth consider a normalised solution \( u \) to the obstacle problem in the metric ball \( B(2) \). We mean by this a function \( u \) which in the ball \( B(2) \subset \mathbb{G} \) satisfies the following properties:

(i) \( u \geq 0 \) in \( B(2) \);
(ii) \( \Delta_H u = 1 \) on the set \( \Omega(u) = \{ p \in B(2) \mid u(p) > 0 \} \);
(iii) we also assume without loss of generality that \( ||u||_{L^\infty(B(2))} = 1 \) and that \( e \in \Gamma(u) \).

In particular, this gives for every \( 0 < r < 1/2 \)

\[
||u||_{L^\infty(B(p,r))} \leq c_1 r^2,
\]

see [12].

**Proposition 3.3.** There exists a universal constant \( C > 0 \), depending only on \( \mathbb{G} \), such that the function \( u \) and its \( \Gamma^{1,1} \) norm in \( B_{1/2} \) are bounded by \( C \).

We finally recall the following basic result, which is [12, Theor. 5.4].

**Theorem 3.4 (Maximum growth).** There exists a universal constant \( c_0 > 0 \), depending only on \( \mathbb{G} \), such that for for every \( 0 < r < 1 \) and \( p \in B(1) \cap \Gamma(u) \) one has

\[
||u||_{L^\infty(B(p,r))} \geq c_0 r^2.
\]

4. PROOF OF THEOREM 1.1

Let \( u \) be a normalised solution to the subelliptic obstacle problem. For a given horizontal direction \( e_i \in g_1 \), let \( \tilde{X}_i \) denote the corresponding right invariant derivative as defined in (2.8), (2.9) above. Our first step is the following main non-degeneracy lemma.

**Lemma 4.1.** There exists a universal number \( \eta \in (0,1) \), depending exclusively on \( c_0, c_1 \), such that

\[
\sum_{i=1}^{\infty} |\nabla_H \tilde{X}_i u|^2 \geq \eta
\]
on a subset \( E \subset B(1) \) such that \( |E| \geq \eta |B(1)| \).

**Proof.** We argue by contradiction and assume that (4.1) does not hold. Then, for every \( k \in \mathbb{N} \) there exists a normalised solution \( u_k \), and a subset \( E_k \subset B(1) \), such that \( |E_k| \geq (1 - \frac{1}{k})|B(1)| \) and

\[
\sum_{i=1}^{\infty} |\nabla_H \tilde{X}_i u_k|^2 \leq \frac{1}{k} \quad \text{on } E_k.
\]

We first note that since \( e \in \Gamma(u_k) \), by Theorem 3.4 we infer that for every \( 0 < r < \frac{3}{4} \) one has

\[
||u_k||_{L^\infty(B(r))} \geq c_0 r^2,
\]

with \( c_0 \) independent of \( k \). Since \( \Delta_H u_k = 1 \) in the open set \( \Omega(u_k) = B(2) \setminus \Lambda(u_k) \), if we let \( x = (x_1, ..., x_m) \) denote the logarithmic coordinates of a point \( p \in \mathbb{G} \) with respect to the orthonormal basis \( \{e_1, ..., e_m\} \) of the horizontal layer \( g_1 \), and we let \( \psi(p) = -\frac{|x|^2}{2m} \), then from
(2.12) we obviously have \( \Delta_H \psi \equiv -1 \). Therefore, the function \( v_k = u_k + \psi \) is harmonic in the open set \( \Omega(u_k) \). Let \( p \in \Omega(u_k) \), and denote by \( r = \frac{1}{2}d(p) \), where \( d(p) = \text{dist}(p, \Gamma(u_k)) \), so that \( B(p, 2r) \subset \Omega(u_k) \). If \( Y \) is a left-invariant vector field of homogeneous order \( \ell \), from Theorem 3.1 and from (3.1) we have \( |Y v_k(p)| \leq \rho d(p)^{2-\ell} \), where \( \rho = c(G, \ell, c_1) > 0 \). Since \( \psi \) is homogeneous of degree \( \kappa = 2 \), we clearly have \( |Y \psi(p)| \leq \rho' d(p)^{2-\ell} \), where \( \rho' = \rho'(G, \ell) > 0 \). From these considerations we conclude that

\[
|Y u_k(p)| \leq C d(p)^{2-\ell}.
\]

where \( C > 0 \) depends only on \( G, \ell \) and \( c_1 \). Consequently, from (4.4) we have that for every \( p \in \Omega(u_k) \),

\[
|X_i u_k(p)| \leq C d(p), \quad i = 1, \ldots, m,
\]

where \( C > 0 \) depends only on \( G \) and \( c_1 \). From (4.4), (4.5) and (2.13) it follows that for every \( p \in \Omega(u_k) \), we also have

\[
|X_i u_k(p)| \leq \tilde{C} d(p),
\]

where \( \tilde{C} > 0 \) depends only on \( G \) and \( c_1 \). We note that (4.6) implies, in particular, that \( X_i u_k \) vanishes on the free boundary \( \Gamma(u_k) \). Since by (2.10) \( X_i u_k \) is harmonic in \( \Omega(u_k) \), from Theorem 3.1 and (4.6) we obtain for \( j = 1, \ldots, m \) and \( p \in \Omega(u_k) \),

\[
|X_j X_i u_k(p)| \leq C.
\]

Thus, \( ||\nabla_H X_i u_k||_{L^\infty(\Omega(u_k))} \leq C \), where \( C > 0 \) is independent of \( k \). Arguing as in the proof of [9, Theor. 6.6], we obtain that \( X_i u_k \) is horizontally Lipschitz in \( \Omega(u_k) \) (i.e., up to \( \Gamma(u_k) \)), with Lipschitz bounds independent of \( k \). Since \( X_i u_k \) vanishes on \( B(1) \setminus \Omega(u_k) \), by a standard argument it follows that \( X_i u_k \) is uniformly horizontally Lipschitz in \( B(1) \). Consequently, the following estimate holds for all \( p \in B(1) \),

\[
|X_j X_i u_k(p)| \leq C.
\]

At this point we observe that Proposition 3.3 and (4.7), along with the stratified mean value theorem (see [15, Theor. 1.33 on p. 28]), imply that the sequences \( \{u_k\} \) and \( \{X_i u_k\} \) are equibounded and equicontinuous. By the theorem of Ascoli-Arzelà we obtain, possibly passing to a subsequence, that for each \( i \),

\[
u_k \to u_0, \quad X_i u_k \to X_i u_0 \text{ uniformly in } B(1).
\]

Moreover, (4.7) implies (possibly passing to a further subsequence of \( \{u_k\} \))

\[
\nabla_H X_i u_k \to \nabla_H X_i u_0 \text{ weakly in } L^p(B(1)) \text{ for all } p < \infty.
\]

Furthermore, by a standard argument (see for instance the proof of [24, Prop. 3.17] in the Euclidean case), one can show that \( u_0 \) also solves (1.1). Also, from (4.3) and the uniform convergence of \( u_k \) to \( u_0 \) in \( B(1) \), it follows that

\[
||u_0||_{L^\infty(B(r))} \geq c r^2 \text{ for all } r < \frac{3}{4}.
\]

By the case \( p = 2 \) in (4.9), and by Fatou’s theorem, we have

\[
\int_{B(1)} |\nabla_H X_i u_0|^2 \leq \liminf_{k \to \infty} \int_{B(1)} |\nabla_H X_i u_k|^2.
\]
Keeping in mind that, from the way $E_k$ has been initially selected, we have $|B(1) \setminus E_k| \leq \frac{1}{k}|B(1)|$, we now observe that the integral in the right-hand side of (4.11) can be estimated in the following way using (4.2) and (4.7),

\[
\int_{B(1)} |\nabla_H \tilde{X}_i u_k|^2 = \int_{E_k} |\nabla_H \tilde{X}_i u_k|^2 + \int_{B(1) \setminus E_k} |\nabla_H \tilde{X}_i u_k|^2 \\
\leq \frac{1}{k} |E_k| + C|B(1) \setminus E_k| \leq \frac{C^*}{k} \quad k \to \infty.
\]

From (4.11) and (4.12) we conclude that $\int_{B(1)} |\nabla_H \tilde{X}_i u_0|^2 = 0$, and therefore, $|\nabla_H \tilde{X}_i u_0| \equiv 0$ in $B(1)$ for each $i$. By the connectedness of $B(1)$, this implies that, for each $i$, $\tilde{X}_i u_0 \equiv$ const. in $B(1)$. Since $\tilde{X}_i u_0(e) = 0$, we must have $\tilde{X}_i u_0 \equiv 0$ in $B(1)$. Since the right-invariant vector fields $\{\tilde{X}_i\}$ also generate the Lie algebra of $G$, this in turn implies that $u_0 \equiv$ const. in $B(1)$, and therefore we must have $u_0 \equiv 0$ (since $u_0(e) = 0$). This contradicts the non-degeneracy estimate (4.10) and the conclusion of the lemma thus follows.

\[\Box\]

Lemma 4.1 has the following consequence.

**Corollary 4.2.** There exists $\eta \in (0, 1)$, depending exclusively on $G, c_0$ and $c_1$, such that for any $r \in (0, 1)$, and $u$ that solves

\[
\begin{align*}
\Delta_H u = 1_{\{u > 0\}} & \quad \text{in } B(2r), \\
e \in \Gamma(u), \\
||u||_{L^\infty(B(2r))} & \leq r^2,
\end{align*}
\]

one has

\[
\sum_{i=1}^m |\nabla_H \tilde{X}_i u|^2 \geq \eta
\]

on a subset $E_r \subset B(r)$ such that $|E_r| \geq \eta|B(r)|$.

**Proof.** By considering the rescaled functions $u_r(g) = \frac{u(\delta_r(g))}{r^2}$, we are back in the situation of Lemma 4.1, from which the conclusion follows.

\[\Box\]

We now proceed with the proof of our main result Theorem 1.1. We closely follow the ideas in [5] and [24], but with certain delicate adjustments which are ultimately based on bringing the new Lemma 4.1 in the picture.

**Proof of Theorem 1.1.** Let $\phi$ be a compactly supported smooth cutoff such that $\phi \equiv 1$ in $B(7/6)$ and is 0 outside $B(5/4)$. Also for given $0 < \gamma < \delta$, let $T_{\gamma, \delta} : \mathbb{R} \to \mathbb{R}$ be the odd function defined on $[0, \infty)$ by

\[
T_{\gamma, \delta}(t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq \gamma/2, \\
2(t - \gamma/2) & \text{if } \gamma/2 \leq t \leq \gamma, \\
t & \text{if } \gamma \leq t \leq \delta, \\
\delta & \text{if } t \geq \delta.
\end{cases}
\]

We also use the notation $T_\delta$ to indicate the odd function on $\mathbb{R}$ obtained by letting $\gamma \to 0$ in the definition of $T_{\gamma, \delta}$. We now cover $\Gamma(u) \cap B(1)$ by a finite family of balls of radius $\varepsilon$
centered at free boundary points. Since we eventually would want to send \( \varepsilon \to 0^+ \), without loss of generality we may assume that \( \varepsilon < \frac{1}{5} \). Since the volume of the metric balls is trivially doubling, by the Vitali covering lemma there exists a sub-collection of mutually disjoint balls \( \{B_k\}_{k=1}^{\ell} \) such that \( \Gamma(u) \cap B(1) \subset \bigcup_{k=1}^{\ell} B_k \). Next, from (4.6) there exists a universal \( C > 0 \) such that for every \( k = 1, \ldots, \ell \) we have \( |\tilde{X}_k u| \leq C \varepsilon \) on \( B_k \). With this constant \( C > 0 \), we choose \( \delta = C \varepsilon \) in the definition of \( T_{\gamma, \delta} \). By the harmonicity of \( \tilde{X}_k u \) in \( B(2) \setminus \Lambda(u) \), and also by using the fact that \( \tilde{X}_k u \equiv 0 \) in \( \Lambda(u) \), we have that

\[
\tag{4.16}
\sum_{i=1}^{m} \int T_{\gamma, C \varepsilon}(\tilde{X}_i u) \, \Delta_H \tilde{X}_i u \, \phi = 0.
\]

We note that, for any given \( \gamma > 0 \), the integrand in (4.16) in compactly supported in \( \Omega(u) \cap B(2) \). Therefore, by integrating by parts in the integral in (4.16), and using the fact that

\[
< \nabla_H \tilde{X}_i u, \nabla_H T_{\gamma, C \varepsilon}(\tilde{X}_i u) > = |\nabla_H T_{\gamma, C \varepsilon}(\tilde{X}_i u)|^2,
\]

we obtain

\[
\tag{4.17}
\sum_{i=1}^{m} \int |\nabla_H T_{\gamma, C \varepsilon}(\tilde{X}_i u)|^2 \phi = - \sum_{i=1}^{m} \int T_{\gamma, C \varepsilon}(\tilde{X}_i u) < \nabla_H \tilde{X}_i u, \nabla_H \phi > .
\]

If we now let \( \gamma \to 0 \) in (4.17), and by using the bound (4.7) to justify the application of Lebesgue dominated convergence theorem, we obtain

\[
\tag{4.18}
\sum_{i=1}^{m} \int |\nabla_H T_{C \varepsilon}(\tilde{X}_i u)|^2 \phi = - \sum_{i=1}^{m} \int T_{C \varepsilon}(\tilde{X}_i u) < \nabla_H \tilde{X}_i u, \nabla_H \phi > .
\]

Using the estimate \( |\tilde{X}_i u| \leq C \varepsilon \) in \( B_k \), and the fact that \( \phi \equiv 1 \) in \( B(7/6) \), from the definition of \( T_{C \varepsilon} \) we deduce that the integral in the left-hand side of (4.18) can be bounded from below as follows

\[
\tag{4.19}
\sum_{i=1}^{m} \int |\nabla_H T_{C \varepsilon}(\tilde{X}_i u)|^2 \phi \geq \sum_{i=1}^{m} \int_{B_k} |\nabla_H \tilde{X}_i u|^2 \\
= \sum_{i=1}^{m} \sum_{k=1}^{\ell} \int_{B_k} |\nabla_H \tilde{X}_i u|^2 \geq \sum_{k=1}^{\ell} \int_{E_k} \sum_{i=1}^{m} |\nabla_H \tilde{X}_i u|^2 \\
\geq \sum_{k=1}^{\ell} \eta |E_k| \geq \sum_{k=1}^{\ell} \eta^2 |B_k| = c_2 \sum_{k=1}^{\ell} \varepsilon^Q,
\]

where \( c_2 > 0 \) is a universal constant that also depends on \( \eta \). We note that in the chain of inequalities in (4.19), besides the mutual disjointness of the balls \( \{B_k\} \), we have crucially used the non-degeneracy estimate in Corollary 4.2 (with \( r = \varepsilon \)). We also observe that, thanks to the Lie group structure of \( \mathbb{G} \), Corollary 4.2 holds with respect to balls centred at any point on \( \Gamma(u) \).

To complete the proof we note that the integral in the right-hand side of (4.18) can be estimated from above in the following way

\[
\tag{4.20}
\left| \sum_{i=1}^{m} \int T_{C \varepsilon}(\tilde{X}_i u) < \nabla_H \tilde{X}_i u, \nabla_H \phi > \right| \leq C_1 \varepsilon.
\]
for some universal $C_1$ depending also on $C$. In (4.20) we have used the boundedness of $\nabla H_1 \tilde{X}_i u$ in (4.7) and the fact that $|T_{C\varepsilon}(\tilde{X}_i u)| \leq C\varepsilon$. From (4.18), (4.19) and (4.20) it finally follows that

$$\sum_{i=1}^\ell \varepsilon^{Q-1} \leq C_2,$$

for some $C_2 > 0$ depending on $c_2$ and $C_1$. Since $\{5B_i\}$ covers $\Gamma(u) \cap B(1)$, by taking the infimum on all possible covers and then letting $\varepsilon \to 0$, we conclude from (4.21) that

$$\mathcal{H}^{Q-1}(\Gamma(u) \cap B(1)) \leq C_4,$$

for some universal constant $C_4 > 0$. This is the desired conclusion. 

\[ \square \]

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