SUBADDITIVITY, STRAND CONNECTIVITY AND MULTIGRADED BETTI NUMBERS OF MONOMIAL IDEALS

A V JAYANTHAN AND ARVIND KUMAR

Abstract. Let \( R = \mathbb{K}[x_1, \ldots, x_n] \) and \( I \subset R \) be a homogeneous ideal. In this article, we first obtain certain sufficient conditions for the subadditivity of \( R/I \). As a consequence, we prove that if \( I \) is generated by homogeneous complete intersection, then subadditivity holds for \( R/I \). We then study a conjecture of Avramov, Conca and Iyengar on subadditivity, when \( I \) is a monomial ideal with \( R/I \) Koszul. We identify several classes of edge ideals of graphs \( G \) such that the subadditivity holds for \( R/I(G) \). We then study the strand connectivity of edge ideals and obtain several classes of graphs whose edge ideals are strand connected. Finally, we compute upper bounds for multigraded Betti numbers of several classes of edge ideals.

1. Introduction

Let \( R = \mathbb{K}[x_1, \ldots, x_n] \) be a standard graded polynomial ring over a field \( \mathbb{K} \). Let \( M \) be a finitely generated graded \( R \)-module. Let

\[
(F_\bullet, \partial_\bullet) : 0 \rightarrow F_p \xrightarrow{\partial_p} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \rightarrow M \rightarrow 0
\]

be a graded free resolution of \( M \) (not necessarily minimal) with \( F_i = \oplus_j R(-j)^{b_{ij}} \) for some \( b_{ij} \in \mathbb{Z}_{\geq 0} \). For \( i \geq 0 \), set

\[
t^R_i(F_\bullet) = \sup \{ j : b_{ij} \neq 0 \}.
\]

We say subadditivity holds for \( (F_\bullet, \partial_\bullet) \) if for all \( a, b \geq 0 \) with \( a + b \leq p \),

\[
t^R_{a+b}(F_\bullet) \leq t^R_a(F_\bullet) + t^R_b(F_\bullet).
\]

If \( (F_\bullet, \partial_\bullet) \) is the graded minimal free resolution of \( M \), then write \( F_i = \oplus_j R(-j)^{\beta_{ij}(M)} \), where the number \( \beta_{ij}(M) \) is called the \((i,j)\)-th graded Betti number of \( M \), and in this case we write \( t^R_i(M) \) for \( t^R_i(F_\bullet) \).

It is known that the subadditivity does not always hold for homogeneous ideals in polynomial rings. Eisenbud, Huneke and Ulrich gave an example of an ideal \( I \) in a polynomial ring \( R \) for which \( t^R_2(R/I) > 2t^R_1(R/I) \), [11, Example 4.4]. In the same paper they proved that if \( \dim(R/I) \leq 1 \), then \( t^R_a(R/I) \leq t^R_a(R/I) + t^R_{n-a}(R/I) \), for all \( a \geq 1 \). McCullough proved without any restriction on dimension, [21], that \( t^R_p(R/I) \leq \max\{t^R_i(R/I) + t^R_{p-i}(R/I) : 1 \leq i \leq p-1\} \), where \( p = \text{pd}(R/I) \). Herzog and Srinivasan, [13], improved this result further and showed that \( t^R_p(R/I) \leq t^R_{p-1}(R/I) + t^R_1(R/I) \).

They, and independently Yazdan Pour [25], proved that if \( I \) is a monomial ideal, then \( t^R_a(R/I) \leq t^R_{a-1}(R/I) + t^R_1(R/I) \) for all \( a \geq 1 \). Recently, McCullough and Seceleanu proved that the subadditivity holds for quotients of complete intersection ideals, [22, Proposition 4.1]. They also gave a family of Gorenstein cyclic \( R \)-modules for which subadditivity does not hold. For some recent developments in this direction, see [11, 12, 13]. In [3], Avramov,
Conca and Iyengar also gave an example to show that the subadditivity does not hold in general. They conjectured:

**Conjecture 1.1.** [3 Conjecture 6.4] If $a, b \geq 1$ such that $a+b \leq \text{pd}(R/I)$ and $\text{reg}_{a+b+1}^{R/I}(K) = 0$, then $t_{a+b}^R(R/I) \leq t_a^R(R/I) + t_b^R(R/I)$.

In the above conjecture, $\text{reg}_{a+b+1}^{R/I}(K) = \sup_{i \leq a} \{ j-i : \text{Tor}_i^{R/I}(K, K)_j \neq 0 \}$. They proved that if $\text{reg}_{a+b+1}^{R/I}(K) = 0$ and $(a+b)$ is invertible in $K$, then for $a, b \leq \text{ht}(I)$ with $a+b \leq \text{pd}(R/I)$, $t_{a+b}^R(R/I) \leq t_a^R(R/I) + t_b^R(R/I) + 1$. An important instance for $\text{reg}_{a+b+1}^{R/I}(K) = 0$ to happen is when $R/I$ is a Koszul algebra. In this article, we study the above conjecture for monomial ideals $I$ such that $R/I$ is Koszul. It is well known that if $I$ is a monomial ideal, then $R/I$ is Koszul if and only if $I$ is a quadratic monomial ideal. Any quadratic monomial ideal can be polarized to get a squarefree quadratic monomial ideal having same Betti numbers.

Thus, to study the subadditivity problem of Koszul algebras which are quotients of monomial ideals, it is enough to study the problem for squarefree monomial ideals. Note that quadratic squarefree monomial ideals are in one to one correspondence with finite simple graphs. This gives us the extra leverage of using combinatorial tools to study algebraic invariants of these ideals. Let $G$ be a finite simple graph on $V(G) = \{x_1, \ldots, x_n\}$. Then the edge ideal of $G$, denoted by $I(G)$, is the ideal generated by the set $\{x_ix_j : \{x_i, x_j\} \in E(G)\}$. Abedelfatah and Nevo, [2], proved that for any graph $G$ on $V(G)$ over any field $K$, $t_a^R(R/I(G)) \leq t_a^R(R/I(G)) + t_b^R(R/I(G))$ for all $a \geq 1$ and $i = 1, 2, 3$. Bigdeli and Herzog, [3], showed the subadditivity holds for edge ideals of chordal graphs and whisker graphs.

In this article, we prove that the subadditivity holds for several classes of edge ideals. First we study the problem for homogeneous ideals with some extra hypotheses, Theorems 2.1, 2.3, 2.5. As a consequence of our results, we reprove a result of McCullough and Seceleanu that subadditivity holds for homogeneous complete intersections, Corollary 2.4. We then move on to study the subadditivity problem for edge ideals of finite simple graphs. We first show that if $G$ is a graph and $I(G)$ its edge ideal, then $t_{a+b}^R(R/I(G)) \leq t_a^R(R/I(G)) + t_b^R(R/I(G))$ if $a \leq \nu(G) + 1$, Propositions 2.7, 2.9, where $\nu(G)$ denotes the induced matching number (see Section 2 for the definition). We then study the (multi)graded Betti numbers of a hereditary class of graphs under some hypotheses and show that the subadditivity holds for the edge ideals of this class. As a consequence, we prove that the subadditivity holds for clique sum of a cycle and chordal graphs (in particular, unicyclic graphs), Wheel graphs, Jahangir graphs, complete multipartite graphs and fan graphs, Theorem 2.18, Corollary 2.20, Corollary 2.22. Our methods give new ways of constructing more classes of graphs having subadditivity. We also consider $t$-path ideals, which is a generalization of edge ideals (or can be thought of as certain $t$-uniform hypergraphs) of rooted trees and prove that the subadditivity holds for these ideals, Theorem 2.24.

A closely related problem on vanishing of Betti numbers is the strand connectivity. For a finitely generated graded $R$-module $M$, the set $\{i : \beta_{i,i,j}^R(M) \neq 0\}$ is called the $j$-strand of $M$. If $j$-strand of $M$ is non-empty, then set

$$p_j(M) := \max\{i : \beta_{i,i,j}^R(M) \neq 0\} \quad \text{and} \quad q_j(M) := \min\{i : \beta_{i,i,j}^R(M) \neq 0\}.$$
A non-empty \( j \)-strand of \( M \) is said to be connected if \( j \)-strand = \([q_j(M), p_j(M)]\). The module \( M \) is said to be strand connected if every non-empty strand of \( M \) is connected. For a homogeneous ideal \( I \subset R \), we set these terminologies for \( I \) by taking \( M = R/I \).

What are strand connected homogeneous ideals? Well, there are some obvious classes in this category. For example, if \( I \) has a pure resolution, then the non-empty strands are always connected. It is not very difficult to see that the strands of not all monomial ideals are connected, see Example 3.1. So, one is interested in identifying classes of monomial ideals which are strand connected. Even in the case of quadratic monomial ideals, not many classes of ideals have been identified which are strand connected. In this context, Conca asked:

**Question 1.2.** [2, Question 1.1] *If \( I \) is a quadratic monomial ideal, then is \( I \) strand connected?*

In [2], Abedelfatah and Nevo gave a class of quadratic monomial ideals that are not strand connected. They also proved that the 2-strand of \( I \) is connected for any quadratic monomial ideal. In [5], Bigdeli and Herzog proved that edge ideals of chordal graphs and cycles are strand connected. Our goal is to identify more classes of edge ideals which are strand connected.

We begin with an example of a monomial ideal which is not strand connected. We then identify a hereditary class of graphs whose edge ideals are strand connected, Theorem 2.18. Then we prove that from a given edge ideal which is strand connected, one can obtain stand connected edge ideals by doing certain combinatorial operations on it, Theorems 3.5, 3.6. As a consequence, we prove strand connectivity for several important classes of graphs, Corollary 3.7.

Most of the important homological invariants associated with finitely generated modules are read off from the Betti numbers. Graded Betti numbers of edge ideals of some classes of graphs are known, see for example [15], [18], [19]. The structure of the minimal free resolution can be further refined by considering multigraded resolution and multigraded Betti numbers. If \( G \) is a forest on \( n \) vertices, then Bouchat, in [7], proved that the \( \mathbb{N}^n \)-graded Betti numbers of \( I(G) \) are either 0 or 1. Boocher et al. showed that the \( \mathbb{N}^n \)-graded Betti numbers of cycles are bounded above by 2. In this article, we generalize this result to the case of unicyclic graphs and show that the \( \mathbb{N}^n \)-graded Betti numbers are bounded above by 2, Theorem 4.1. We also obtain upper bounds for the \( \mathbb{N}^n \)-graded Betti numbers of Fan graphs, Jahangir graphs and complete multipartite graphs, Corollaries 4.4, 4.3, 4.5.

The paper is organized as follows: In Section 2, we prove the results concerning the subadditivity of monomial ideals. We collect all the results on strand connectivity in the next section and final section contains the results on multigraded Betti numbers.

### 2. Subadditivity of syzygies of homogeneous ideals

In this section, we study the subadditivity of maximal shifts in the finite graded free resolution of homogeneous ideals. We first discuss certain sufficient conditions for the subadditivity of monomial ideals.

**Theorem 2.1.** *Let \( I \subset R \) be a homogeneous ideal, and \( f \in R \) be a homogeneous polynomial of degree \( d > 0 \) such that \( f \notin I \). Assume that \( t_1^R(R/(I : f)) \leq d \). If free resolutions*
of $R/I$ and $R/(I : f)$ satisfy the subadditivity condition, then the resolution obtained by the mapping cone construction applied to the map $[R/(I : f)](-d)\to R/I$ satisfies the subadditivity condition.

Proof. Let $(F_\bullet, \delta F)$ and $(G_\bullet, \delta G)$ be free resolutions of $R/I$ and $R/(I : f)$, respectively. Note that $(G_\bullet(-d), \delta G)$ is a free resolution of $[R/(I : f)](-d)$. Let $(\mathfrak{S}_\bullet, \xi)$ be the mapping cone construction applied to the map $[R/(I : f)](-d)\to R/I$. Then $\mathfrak{S}_i = F_i \oplus G_{i-1}(-d)$, for all $i \geq 1$. Therefore, for all $a, b \geq 1$.

$$t_a^R(\mathfrak{S}_\bullet) + t_b^R(\mathfrak{S}_\bullet) = \max\{t_a^R(F_\bullet), t_{a-1}^R(G_\bullet(-d))\} + \max\{t_b^R(F_\bullet), t_{b-1}^R(G_\bullet(-d))\}$$

$$= \max\{t_a^R(F_\bullet), t_{a-1}^R(G_\bullet) + d\} + \max\{t_b^R(F_\bullet), t_{b-1}^R(G_\bullet) + d\}$$

$$= \max\{t_a^R(F_\bullet) + t_b^R(F_\bullet), t_{a-1}^R(G_\bullet) + t_{b-1}^R(G_\bullet) + d\}$$

$$\geq \max\{t_a^R(F_\bullet), t_{a-1}^R(G_\bullet) + d\} + d + \max\{t_b^R(F_\bullet), t_{b-1}^R(G_\bullet) + d\}$$

$$\geq \max\{t_a^R(F_\bullet), t_{a-1}^R(G_\bullet) + d\} + d$$

$$= \max\{t_{a+b}^R(F_\bullet), t_{a+b-1}^R(G_\bullet(-d))\}$$

$$= t_{a+b}^R(\mathfrak{S}_\bullet).$$

Hence, the assertion follows. \qed

In Theorem 2.1 if we drop the condition $t_i^R(R/(I : f)) \leq d$, then the result need no be true. We illustrate this by the following example.

Example 2.2. Let $R = \mathbb{K}[x_1, x_2, x_3, x_4, x_5, x_6, x_7, y_1, y_2, y_3, y_4, y_5, y_6]$. Let $f = x_1y_6x_7 - x_6y_1x_7$ and $I = (x_1y_2 - x_2y_3, x_2y_3, x_3y_1, x_3y_2 - x_3y_4, x_4y_3 - x_4y_5, x_4y_5 - x_5y_4, x_5y_6 - x_6y_5)$. Since $x_7$ is regular modulo $I$, $I : f = I : (x_1y_6 - x_6y_1)$. Therefore, it follows from [24, Theorem 2.4] that the mapping cone applied to $0 \to [R/(I : f)](-3)\to R/I$ gives the minimal free resolution of $R/(I, f)$. Using Macaulay2 [14], we get

| Table 1. Betti diagram of $R/I$ | Table 2. Betti diagram of $R/(I : f)$ |
|--------------------------------|----------------------------------|
| 0: 1 | 0: 0 |
| 1: 5 | 1: 5 |
| 2: 10 | 2: 10 |
| 3: 10 | 3: 10 |
| 4: 5 | 4: 5 |
| 5: 1 | 5: 1 |
Table 3. Betti diagram of \( R/(I, f) \)

|       | 0  | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|----|---|---|---|---|---|---|
| 0:    | 1  | . | . | . | . | . | . |
| 1:    | .  | 5 | . | . | . | . | . |
| 2:    | .  | 1 | 10| . | . | . | . |
| 3:    | .  | . | 5 | 10| . | . | . |
| 4:    | .  | . | . | 10| 5 | . | . |
| 5:    | .  | . | 5 | 24| 55| 41| 10|

It follows from Tables 1 and 2 that subadditivity holds for \( R/I \) and \( R/(I : f) \). Also, \( t_1^R(R/(I : f)) = 4 > 3 \). Observe from Table 3 that \( t_1^R(R/(I, f)) = 3 \) and \( t_2^R(R/(I, f)) = 7 > 2t_1^R(R/(I, f)) \). Thus, subadditivity does not hold for \( R/(I, f) \). □

In the following result, we prove that we can drop the condition \( t_1^R(R/(I : f)) \leq d \), if \( I : f = I \).

**Theorem 2.3.** Let \( I \subset R \) be a homogeneous ideal, and \( f \in R \) be a homogeneous polynomial of degree \( d > 0 \) such that \( I : f = I \). If subadditivity holds for \( R/I \), then the subadditivity holds for \( R/(I, f) \).

**Proof.** Let \((F_\bullet, \delta_\bullet)\) be the minimal free resolution of \( R/I \). Note that \((F_\bullet(-d), \delta_\bullet)\) is the minimal free resolution of \([R/I](-d)\). Since \( I : f = I \), the mapping cone construction applied to the map \([R/(I : f)](-d) \to R/I\) gives the minimal free resolution of \( R/(I, f) \). Therefore, for each \( 1 \leq i \leq \pd(R/(I, f)) \),

\[
t_i^R \left( \frac{R}{(I, f)} \right) = \max \left\{ t_i^R \left( \frac{R}{I} \right), t_{i-1}^R \left( \frac{R}{I}(-d) \right) \right\} = \max \left\{ t_i^R \left( \frac{R}{I} \right), t_i^R \left( \frac{R}{I} \right) + d \right\}.
\]

Thus, for all \( a, b \geq 1 \) with \( a + b \leq \pd(R/(I, f)) \),

\[
t_{a+b}^R \left( \frac{R}{(I, f)} \right) = \max \left\{ t_{a+b}^R \left( \frac{R}{I} \right), t_{a+b-1}^R \left( \frac{R}{I} \right) + d \right\}
\]

\[
\leq \max \left\{ t_a^R \left( \frac{R}{I} \right) + t_b^R \left( \frac{R}{I} \right), t_a^R \left( \frac{R}{I} \right) + t_{b-1}^R \left( \frac{R}{I} \right) + d \right\}
\]

\[
\leq \max \left\{ t_a^R \left( \frac{R}{I} \right), t_{a-1}^R \left( \frac{R}{I} \right) + d \right\} + \max \left\{ t_b^R \left( \frac{R}{I} \right), t_{b-1}^R \left( \frac{R}{I} \right) + d \right\}
\]

\[
= t_a^R \left( \frac{R}{(I, f)} \right) + t_b^R \left( \frac{R}{(I, f)} \right)
\]

Hence, the assertion follows. □

As an immediate consequence, we derive a result of McCullough and Secleanu:

**Corollary 2.4.** [22, Proposition 4.1] If \( I \) is a homogeneous complete intersection, then subadditivity holds for \( R/I \).

The following result says that if \( I \) and \( J \) are homogeneous ideals in distinct polynomial rings over same field and subadditivity holds for these two ideals, then subadditivity holds for the ideal generated by their sum in the tensor product of these two polynomial rings.
Theorem 2.5. Let $I, J$ be homogeneous ideals of $R$ such that there exist minimal generating sets for $I$ and $J$ in disjoint sets of variables. If subadditivity holds for $R/I$ and $R/J$, then subadditivity holds for $R/(I + J)$.

Proof. Since the minimal generating sets of $I$ and $J$ are in disjoint variables, the tensor product of the minimal free resolution of $R/I$ and $R/J$ provides the minimal free resolution of $R/(I + J)$. In particular,

$$
\beta_{i,j}^R \left( \frac{R}{I + J} \right) = \sum_{0 \leq r \leq i, s \leq j} \beta_{r,s}^R \left( \frac{R}{I} \right) \beta_{i-r,j-s}^R \left( \frac{R}{J} \right).
$$

Thus, it is straightforward to verify that for each $i \geq 1$,

$$
t_i^R \left( \frac{R}{I + J} \right) = \max_{0 \leq r \leq i} \left\{ t_r^R \left( \frac{R}{I} \right) + t_{i-r}^R \left( \frac{R}{J} \right) \right\}.
$$

Let $a, b \geq 1$ such that $a + b \leq \text{pd}(R/(I + J))$. Then, we have

$$
t_{a+b}^R \left( \frac{R}{I + J} \right) = \max_{0 \leq r \leq a+b} \left\{ t_r^R \left( \frac{R}{I} \right) + t_{a+b-r}^R \left( \frac{R}{J} \right) \right\}
= \max_{0 \leq i \leq a, 0 \leq k \leq b} \left\{ t_i^R \left( \frac{R}{I} \right) + t_k^R \left( \frac{R}{I} \right) + t_{a+b-i-k}^R \left( \frac{R}{J} \right) \right\}
\leq \max_{0 \leq i \leq a, 0 \leq k \leq b} \left\{ t_i^R \left( \frac{R}{I} \right) + t_k^R \left( \frac{R}{I} \right) + t_{a-i}^R \left( \frac{R}{J} \right) + t_{b-k}^R \left( \frac{R}{J} \right) \right\}
\leq \max_{0 \leq i \leq a} \left\{ t_i^R \left( \frac{R}{I} \right) + t_{a-i}^R \left( \frac{R}{J} \right) \right\} + \max_{0 \leq k \leq b} \left\{ t_k^R \left( \frac{R}{I} \right) + t_{b-k}^R \left( \frac{R}{J} \right) \right\}
= t_a^R \left( \frac{R}{I + J} \right) + t_b^R \left( \frac{R}{I + J} \right).
$$

This completes the proof. \hfill \Box

2.1. Edge ideals of graphs. In this subsection, we study the subadditivity problem for quadratic squarefree monomial ideals. First, we recall some notion from graph theory.

Let $G$ be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $E(G) = \emptyset$, then we say that $G$ is a trivial (or empty) graph. For $A \subseteq V(G)$, $G[A]$ denotes the induced subgraph of $G$ on the vertex set $A$, i.e., for $i, j \in A$, $\{i, j\} \in E(G[A])$ if and only if $\{i, j\} \in E(G)$. For $A \subset V(G)$, $G \setminus A$ denotes the induced subgraph of $G$ on the vertex set $V(G) \setminus A$. The neighborhood of a vertex $v$, denoted by $N_G(v)$, is defined as $\{u \in V(G) : \{u, v\} \in E(G)\}$. We set $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v$ is $|N_G(v)|$, and it is denoted by $\deg_G(v)$. If $\deg_G(v) = 1$, then we say that $v$ is a pendant vertex. For $e \in E(G)$, $G \setminus e$ is the graph on the vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$.

A connected graph $G$ is said to be a cycle if $\deg_G(v) = 2$, for all $v \in V(G)$. A cycle $G$ is a $k$-cycle if $|V(G)| = k$, and it is denoted by $C_k$. A tree is a connected graph $G$ such that $k$-cycle is not an induced subgraph of $G$, for all $k \geq 3$. A path is a tree which has exactly two pendant vertices. We say that $G$ is a chordal graph if $k$-cycle is not an induced subgraph of $G$, for all $k \geq 4$.

A set of pairwise disjoint edges in a graph $G$ is called a matching. If a matching is an induced subgraph, then such matching is called an induced matching. The largest size of
an induced matching in $G$ is called induced matching number of $G$, and it is denoted by $\nu(G)$. A subset $C \subseteq V(G)$ is said to be a vertex cover of $G$ if for each $e \in E(G)$, $e \cap C \neq \emptyset$. If $C$ is minimal with respect to inclusion, then $C$ is called a minimal vertex cover of $G$.

Below, we fix some notation for the rest of the paper.

**Notation 2.6.** If $G$ is a graph on $n$ vertices, then we set $V(G) = \{x_1, \ldots, x_n\}$ and the edge ideal $I(G) = (x_ix_j : \{x_i, x_j\} \in E(G))$ to be an ideal of $R = \mathbb{K}[x_1, \ldots, x_n]$. Also, we set $t_a(G) := t_a^R(R/I(G))$. For a graph $G$, by subadditivity of $G$ we mean subadditivity of $R/I(G)$.

We now begin the study of the subadditivity problem for edge ideals of graphs.

**Proposition 2.7.** For $b \leq \nu(G)$, and $a \geq 1$ with $a + b \leq \text{pd}(R/I(G))$,

$$t_{a+b}(G) \leq t_a(G) + t_b(G).$$

**Proof.** Let $b \leq \nu(G)$, and $a \geq 1$ be such that $a + b \leq \text{pd}(R/I(G))$. It follows from [19, Lemma 2.2], $t_b(G) = 2b = b \cdot t_1(G)$. Now, apply [16, Corollary 4] $b$ times, we get $t_{a+b}(G) \leq t_a(G) + b \cdot t_1(G)$, which completes the proof. 

It follows from [19, Lemma 2.2] that $t_a(G) < 2a$ for all $a > \nu(G)$. In the following auxiliary lemma, we compute $t_a(G)$ for $a = \nu(G) + 1$.

**Lemma 2.8.** If $G$ is not a disjoint union of edges, then $\nu(G)+1(G) = 2\nu(G) + 1$.

**Proof.** Set $b = \nu(G) + 1$. It is easy to see that if $C$ is a minimal vertex cover of $G$, then $\nu(G) \leq |C|$. Therefore, $\nu(G) \leq \text{ht}(I(G))$. By [19, Lemma 2.2], $t_{b-1}(G) = 2b - 2$ and $t_b(G) < 2b$. If $\nu(G) < \text{ht}(I(G))$, then it follows from [19, Lemma 6.1] that $t_{b-1}(G) < t_b(G)$. Thus, $t_b(G) = 2b - 1 = 2\nu(G) + 1$. Suppose $\nu(G) = \text{ht}(I(G))$. Let $\{e_1, \ldots, e_{\nu(G)}\}$ be an induced matching in $G$. Since $G$ is not a disjoint union of edges, there is an edge $f$ such that $f \cap e_i \neq \emptyset$ for some $1 \leq i \leq \nu(G)$. Set $e_i = \{x_i, y_i\}$. Without loss of generality, we may assume that $\{x_1, \ldots, x_{\nu(G)}\}$ is a vertex cover and $f = \{x_1, z\}$. Since $\text{ht}(I(G)) = \nu(G)$, $\{y_i, z\} \notin E(G)$ for any $1 \leq i \leq \nu(G)$. Without loss of generality, we may assume that $N_G(z) = \{x_1, \ldots, x_k\}$. Let $H$ denote the induced subgraph of $G$ on the vertex set $\{x_1, \ldots, x_{\nu(G)}, y_1, \ldots, y_{\nu(G)}, z\}$. Then $H$ is a graph as shown in Figure 1. Let $H_1$ be the induced subgraph on $\{z, x_1, \ldots, x_k, y_1, \ldots, y_k\}$ and $H_2$ be the induced subgraph on $\{x_{k+1}, \ldots, x_{\nu(G)}, y_{k+1}, \ldots, y_{\nu(G)}\}$. Since the minimal free resolution of $R/I(H)$ is given by the tensor product of minimal free resolutions of $R/I(H_1)$ and $R/I(H_2)$, it can be seen that $\beta^{R}_{i,j}(R/I(H)) = \sum_{0 \leq r \leq s} \beta^{R}_{i-r,j-s}(R/I(H_1)) \beta^{R}_{r,s}(R/I(H_2))$. To prove the main assertion, it is enough to prove that $\beta^{R}_{b,2b-1}(R/I(H)) \neq 0$ and to prove this statement, we prove that $\beta^{R}_{b+1,2b+1}(R/I(H_1)) \neq 0$ and $\beta^{R}_{b-1,k,2b-1-k}(R/I(H_2)) \neq 0$. 

Let $H'$ be the induced subgraph of $H_1$ on $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ and $H''$ be the induced subgraph on $\{z, x_1, \ldots, x_k\}$. Then, it follows from \cite{15} Theorem 4.6 that

$$\beta_{i,j}^R(R/I(H_1)) = \beta_{i,j}^R(R/I(H')) + \beta_{i,j}^R(R/I(H'')) + \beta_{i-1,j-1}(R/I(H')).$$  

Since $H'$ is a disjoint union of $k$ edges, $I(H')$ is a complete intersection ideal, and thus, $\beta_{k,2k}^R(R/I(H')) \neq 0$. Consequently, $\beta_{k+1,2k+1}^R(R/I(H_1)) \neq 0$. Since $H_2$ is a disjoint union of $b - 1 - k$ edges, $I(H_2)$ is complete intersection, $\beta_{b-1-k,2(b-1-k)}^R(R/I(H_2)) \neq 0$. Hence, $\beta_{b,2b-1}^R(R/I(H)) \neq 0$, which concludes the proof. \hfill \Box

Let $\Delta$ be a simplicial complex on $\{x_1, \ldots, x_n\}$. For $V \subseteq \{x_1, \ldots, x_n\}$, the subcomplex of $\Delta$ on $V$ is $\Delta[V] = \{F \in \Delta : F \subseteq V\}$. The \textit{Stanly-Reisner ideal} $I_\Delta$ of the simplicial complex $\Delta$ is the ideal generated by squarefree monomials $x_F = \prod_{x \in F} x_i$ with $F \notin \Delta$, $F \subseteq \{x_1, \ldots, x_n\}$. Let $G$ be a graph on the vertex set $V(G)$. A subset $U \subseteq V(G)$ is said to be an \textit{independent set} if $G[U]$ is a trivial graph. Let $\Delta_G = \{U \subseteq V(G) : U$ is an independent set in $G\}$. Then $\Delta_G$ is a simplicial complex, called the \textit{independence complex} of $G$. It is easy to observe that $I(G) = I_{\Delta_G}$.

**Proposition 2.9.** Let $b = \nu(G) + 1$. Then, for all $a \geq 1$ with $a + b \leq \text{pd}(R/I(G))$, $t_{a+b}(G) \leq t_a(G) + t_b(G)$.

**Proof.** Let $\Delta$ denote the independent complex of $G$. Then $I_\Delta = I(G)$. By Hochster’s formula \cite{17}, we get

$$\beta_{l,j}^R(R/I_\Delta) = \sum_{W \subseteq V(G), |W| = j} \dim_k \widetilde{H}_{j-i-1}(\Delta[W]; \mathbb{K}).$$  

Taking $i = a + b$ and $j = t_{a+b}(G)$, we get $\widetilde{H}_l(\Delta[W]; \mathbb{K}) \neq 0$ for some $W \subseteq V(G)$, where $l = t_{a+b}(G) - (a + b) - 1$. Set $\Delta' = \Delta[W]$ and $H = G[W]$. We claim that $H$ has no isolated vertices. If $H$ has an isolated vertex, say $x$, then $x \in F$ for every facet $F$ of $\Delta'$. Hence $\Delta'$ is a cone of a simplicial complex and hence an acyclic complex. Consequently, $\widetilde{H}_l(\Delta'; \mathbb{K}) = 0$, which is a contradiction. Thus, $H$ has no isolated vertices. Now, we claim that there is a vertex of degree at least two in $H$. If not, then every vertex of $H$ has degree one, i.e., $H$ is a disjoint union of edges. Therefore, $I(H)$ is a complete intersection. Since $\beta_{a+b,t_{a+b}(G)}^R(R/I(H)) \neq 0$, $t_{a+b}(G) = 2(a + b)$, which is a contradiction to the fact

![Figure 1. H = H1 ∪ H2](image-url)
that \( t_i(G) < 2i \) for \( i > \nu(G) \), see [19] Lemma 2.2]. Thus, \( H \) has a vertex of degree at least two.

Let \( v \) be a vertex of degree at least two in \( H \). Let \( x, y \in N_H(v) \) such that \( x \neq y \). Observe that \( \Delta' = (\Delta' \setminus \{v\}) \cup (\Delta' \setminus \{x, y\}) \). Set \( \Delta_1 = \Delta' \setminus \{v\} \) and \( \Delta_2 = \Delta' \setminus \{x, y\} \). Consider the long exact sequence of reduced homologies,

\[
\cdots \to \tilde{H}_i(\Delta_1 \cap \Delta_2; \mathbb{K}) \to \tilde{H}_i(\Delta_1; \mathbb{K}) \oplus \tilde{H}_i(\Delta_2; \mathbb{K}) \to \tilde{H}_i(\Delta'; \mathbb{K}) \to \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; \mathbb{K}) \to \cdots
\]

If \( \tilde{H}_i(\Delta_1; \mathbb{K}) \neq 0 \), then by (1), \( \beta_{a+b-1,t_{a+b}}(G) - 1(R/I(G)) \neq 0 \). Consequently, \( t_{a+b}(G) \leq t_{a+b-1}(G) + 1 \). Now, by Proposition 2.7, we have

\[
t_{a+b}(G) \leq t_a(G) + t_{b-1}(G) + 1 = t_a(G) + 2(b - 1) + 1 = t_a(G) + t_b(G).
\]

If \( \tilde{H}_i(\Delta_1; \mathbb{K}) = 0 \), then \( \tilde{H}_i(\Delta_2; \mathbb{K}) \neq 0 \) or \( \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; \mathbb{K}) \neq 0 \). Assume that \( \tilde{H}_i(\Delta_2; \mathbb{K}) \neq 0 \). Then, by (1), \( \beta_{a+b-2,t_{a+b}}(G) - 2(R/I(G)) \neq 0 \) which further implies that \( t_{a+b}(G) \leq t_{a+b-2}(G) + 2 \). Again, by Proposition 2.7 we get

\[
t_{a+b}(G) \leq t_a(G) + t_{b-2}(G) + 2 = t_a(G) + 2(b - 2) + 2 < t_a(G) + t_b(G).
\]

Finally, if \( \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; \mathbb{K}) \neq 0 \), then by (1), \( \beta_{a+b-3,t_{a+b}}(G) - 3(R/I(G)) \neq 0 \). Thus, \( t_{a+b}(G) \leq t_{a+b-2}(G) + 3 \). Thus, by Proposition 2.7

\[
t_{a+b}(G) \leq t_a(G) + t_{b-2}(G) + 3 = t_a(G) + 2(b - 2) + 3 < t_a(G) + t_b(G).
\]

Hence, the assertion follows. \( \square \)

\textbf{Remark 2.10.} The above two propositions together prove that if one of the indices is bounded above by \( \nu(G) + 1 \), then \( t_{a+b}(G) \leq t_a(G) + t_b(G) \). Comparing with [3, Theorem 6.2] in the case of Koszul monomial ideals, we can see that while they have a possibly bigger upper bound on \( a \) and \( b \), our hypothesis puts restriction only on one of \( a \) and \( b \). While we may not be able to directly achieve subadditivity from their result, in our case, we obtain subadditivity.

We have immediate consequence of Propositions 2.7 and 2.9

\textbf{Corollary 2.11.} If \( G \) is a graph such that \( \text{pd}(R/I(G)) \leq 2\nu(G) + 2 \), then subadditivity holds for \( G \).

It follows from the above corollary and [18, Corollary 7.6.30] that the subadditivity holds for cycles.

We now introduce a hereditary class of graphs along with some extra hypotheses.

\textbf{Definition 2.12.} Let \( \mathcal{G} \) be a hereditary class of finite simple graphs with the property that \( G \in \mathcal{G} \) if and only if

1. \( G = C_n \) for some \( n \geq 3 \) or
2. \( G \neq C_n \) for any \( n \geq 3 \), there exists\( \ e = \{x, y\} \in E(G) \) such that \( N_G(x) \subset N_G[y] \) and \( G \setminus e \in \mathcal{G} \).

\textbf{Remark 2.13.} Let \( G \) be a disconnected graph. It is easy to observe that \( G \in \mathcal{G} \) if and only if each component of \( G \) is in \( \mathcal{G} \).
It is immediate from the definition that \( \mathcal{G} \) contains cycles and forests. We now show that \( \mathcal{G} \) contains some important classes of graphs. A subset \( U \) of \( V(G) \) is said to be a clique if \( G[U] \) is a complete graph. A vertex \( v \) is said to be a simplicial vertex if it belongs to exactly one maximal clique of \( G \).

**Lemma 2.14.** Let \( G \) be a chordal graph. Then there exist an edge \( e = \{x, y\} \in E(G) \) such that \( N_G(x) \subset N_G[y] \) and \( G \setminus e \) is a chordal graph.

*Proof.* It follows from [10] that \( G \) has a simplicial vertex, say \( x \). Let \( y \in N_G(x) \). Then \( e = \{x, y\} \in E(G) \) is an edge such that \( N_G(x) \subset N_G[y] \). We claim that \( G \setminus e \) is a chordal graph. Suppose \( G \setminus e \) is not a chordal graph. Then \( G \setminus e \) contains an induced cycle \( C \) of length at least 4. Since \( G \) is chordal, this implies that \( e \) is a chord in \( G \) connecting two vertices of \( C \). This contradicts the fact that \( x \) is a simplicial vertex. Therefore, \( G \setminus e \) is a chordal graph. \( \square \)

Let \( G \) and \( H \) be two graphs. The clique-sum of \( G \) and \( H \) along a complete graph \( K_m \) is a graph with the vertex set \( V(G) \cup V(H) \) and edge set \( E(G) \cup E(H) \) such that the induced subgraph on the vertex set \( V(G) \cap V(H) \) is the complete graph \( K_m \).

**Lemma 2.15.** Let \( G \) be a graph obtained by clique sum of a cycle \( C_m \), \( m \geq 4 \), and some chordal graphs. If \( G \neq C_m \), then there exists an edge \( e = \{x, y\} \in E(G) \) such that \( N_G(x) \subset N_G[y] \) and \( G \setminus e \) is a graph obtained by clique sum of \( C_m \) and some chordal graphs.

*Proof.* Suppose \( G \neq C_m \). Then \( G \) is a clique sum of \( C_m \) and some chordal graphs, say \( G_1, \ldots, G_t \). By a theorem of Dirac, [10], each \( G_i \) is either a clique or contains two simplicial vertices that are non-adjacent. In either case, \( G \) will contain a simplicial vertex, say \( x \). Without loss of generality, assume that \( x \in V(G_1) \setminus V(C_m) \). Let \( y \in N_{G_1}(x) \) and set \( e = \{x, y\} \). Then \( N_G(x) = N_{G_1}(x) \subset N_{G_1}[y] \subset N_G[y] \). As in the proof of Lemma 2.14, one can see that \( G_1 \setminus e \) is a chordal graph. Therefore, \( G \setminus e \) is a clique sum of \( C_m \) and the chordal graphs \( G_1 \setminus e, G_2, \ldots, G_t \). \( \square \)

We now show that \( \mathcal{G} \) contains chordal graphs and their clique sum with a cycle.

**Theorem 2.16.** We have the followings:

1. If \( G \) is a chordal graph, then \( G \in \mathcal{G} \).
2. If \( G \) be a graph obtained by taking clique sum of a cycle, \( C_m \), and some chordal graphs, then \( G \in \mathcal{G} \).

*Proof.* (1) Let \( G \) be a chordal graph. We prove this by induction on \( |E(G)| \). If \( |E(G)| = 1 \), then the assertion is true. Assume that \( |E(G)| > 1 \) and for every chordal graph \( H \) with \( |E(H)| < |E(G)| \), \( H \in \mathcal{G} \). Every induced subgraph of a chordal graph is chordal. If \( H \) is a proper induced subgraph of \( G \), then \( |E(H)| < |E(G)| \) and \( H \) is a chordal graph. Thus, by induction, \( H \in \mathcal{G} \). By Lemma 2.14, there exist an edge \( e = \{x, y\} \) such that \( N_G(x) \subset N_G[y] \) and \( G \setminus e \) is a chordal graph. Now, by induction, \( G \setminus e \in \mathcal{G} \). Hence, \( G \in \mathcal{G} \).

(2) We prove this by induction on \( |E(G)| \geq m \). If \( |E(G)| = m \), then \( G \) is a cycle and hence, \( G \in \mathcal{G} \). Assume that \( |E(G)| > m \). Let \( H \) be a proper induced subgraph of \( G \). If \( H \) is a chordal graph, then by (1), \( H \in \mathcal{G} \). Suppose \( H \) is not a chordal graph. Then \( H \) is a disjoint union of a chordal graph and a graph obtained from clique sum of \( C_m \) and some
chordal graphs. By Remark 2.13 and the first part, it is enough to prove that if $H$ is a clique sum of a cycle and some chordal graphs, then $H \in \mathcal{G}$, and this follows by induction, as $H$ is a proper induced subgraph of $G$. As $G$ is not $C_m$, by Lemma 2.15, there exist an edge $e = \{x, y\} \in E(G)$ such that $N_G(x) \subset N_G[y]$ and $G \setminus e$ is a graph obtained by clique sum of $C_m$ and some chordal graphs. Since $|E(G \setminus e)| < |E(G)|$, by induction $G \setminus e \in \mathcal{G}$. Hence, $G \in \mathcal{G}$.

For the following result, we consider the ring $R$ to be $\mathbb{N}^n$-graded. We require only the graded version of this result in this and the next section. We require the multigraded version in the last section while studying multigraded Betti numbers. Set $\deg x_i = e_i$, the standard basis vector with 1 at the $i$-th place and zero everywhere else. For $u \in V(G)$, set $e_u = e_i$ if $u = x_i$.

**Proposition 2.17.** Let $G$ be a graph on $V(G)$ and $e = \{x, y\} \in E(G)$. If $N_G(x) \subset N_G[y]$, then the mapping cone applied to

$$0 \to \frac{R}{I(G \setminus e) : xy}(-e_x - e_y) \xrightarrow{x_{xy}} \frac{R}{I(G \setminus e)}$$

gives the minimal free resolution of $R/I(G)$. In particular, for all $i \geq 0$ and $a \in \mathbb{N}^n$,

$$\beta^{R}_{i, a} \left( \frac{R}{I(G)} \right) = \beta^{R}_{i, a} \left( \frac{R}{I(G \setminus e)} \right) + \beta^{R}_{i-1, a} \left( \frac{R}{I(G \setminus e) : xy}(-e_x - e_y) \right)$$

$$= \beta^{R}_{i, a} \left( \frac{R}{I(G \setminus e)} \right) + \beta^{R}_{i-1, a-e_x-e_y} \left( \frac{R}{I(G \setminus e) : xy} \right).$$

**Proof.** We claim that $I(G \setminus e) : xy = I(G \setminus e) : y$. Note that $I(G \setminus e) : xy = (I(G \setminus e) : y) : x$. Since $N_G(x) \subset N_G[y]$, it follows from [9] Lemma 3.1, that $x$ does not divide any of the minimal monomial generators of $I(G \setminus e) : y$. Hence $I(G \setminus e) : xy = I(G \setminus e) : y$. Consider, the short exact sequence

$$0 \to \frac{R}{I(G \setminus e) : xy}(-e_x - e_y) \xrightarrow{x_{xy}} \frac{R}{I(G \setminus e)} \to \frac{R}{I(G)} \to 0. \quad (2)$$

It follows from [24] Theorem 2.4] that the minimal free resolution of $R/I(G)$ is obtained by mapping cone construction on $0 \to [R/(I(G \setminus e) : xy)](-e_x - e_y) \xrightarrow{x_{xy}} R/I(G \setminus e)$. Hence, the assertion follows.

**Theorem 2.18.** Subadditivity holds for every $G \in \mathcal{G}$.

**Proof.** We prove this by induction on the number of edges. Let $G \in \mathcal{G}$. If $|E(G)| = 1$, then the assertion clearly holds. Assume that $|E(G)| > 1$ and the result is true for all graphs $H \in \mathcal{G}$ with $|E(H)| < |E(G)|$. If $G$ is a cycle, then the subadditivity holds for $G$, [5] Example 1(b)]. Assume that $G$ is not a cycle graph. Then, by Definition 2.12(3), $G$ has an edge $e = \{x, y\}$ such that $N_G(x) \subset N_G[y]$. Consider the following short exact sequence:

$$0 \to \frac{R}{I(G \setminus e) : xy}(-2) \xrightarrow{x_{xy}} \frac{R}{I(G \setminus e)} \to \frac{R}{I(G)} \to 0. \quad (3)$$

Note that $G \setminus e$ is graph with $|E(G \setminus e)| < |E(G)|$ and $G \setminus e \in \mathcal{G}$. Therefore by induction, the subadditivity holds for $G \setminus e$. From the proof of Proposition 2.17 we have $I(G \setminus e) : xy = I(G \setminus e) : y = I(G \setminus N_G[y]) + (N_G \setminus e)(y))$, where the last equality follows from [9]
Theorem 2.21. Let $G = G \setminus N_G[y]$ be an induced subgraph of $G$, therefore, $H \in \mathcal{G}$. Since $|E(H)| < |E(G)|$, by induction, the subadditivity holds for $H$. As, $(N_{G \setminus e}(y))$ is generated by a regular sequence, the subadditivity holds for $(N_{G \setminus e}(y))$, by Corollary 2.4. Since tensor product of the minimal free resolutions of $R/I(H)$ and $R/(N_{G \setminus e}(y))$ resolves $I(G \setminus e) : xy$, the subadditivity holds for $R/(I(G \setminus e) : xy)$, by Theorem 2.5. Note that $i_1^R(R/(I(G \setminus e) : xy)) \leq 2$. Therefore, by Theorem 2.11 the resolution obtained by the mapping cone construction applied to the map $[R/(I(G \setminus e) : xy)](-2) \rightarrow R/I(G \setminus e)$ satisfies the subadditivity condition. Now, by Proposition 2.17 the subadditivity holds for $G$.

Remark 2.19. In the proof of the above theorem, the properties that we used to derive the subadditivity property of $R/I(G)$ are that the mapping cone applied to $[R/(I(G \setminus e) : e)](-2) \rightarrow R/I(G \setminus e)$ gives the minimal free resolution of $R/I(G)$ and that subadditivity holds for $R/(I(G \setminus e) : e)$ and $R/I(G \setminus e)$. Hence, if $G$ has an edge $e$ satisfying these properties, then $R/I(G)$ has subadditivity property. This way, one can possibly get more classes of graphs whose edge ideals have subadditivity property. For example, if we take $G$ to be the graph $C_6$ along with an edge $\{x_1, x_4\}$ and $e = \{x_2, x_3\}$, then $G \setminus e$ is a unicyclic graph and $I(G \setminus e) : e$ is a tree. Hence both these ideals satisfy subadditivity. In this case, it is easy to verify that the mapping cone gives a minimal free resolution of $R/I(G)$. Hence $R/I(G)$ satisfies subadditivity. Note that $G \notin \mathcal{G}$.

We say that a vertex $v \in V(G)$ is a cut vertex if $G \setminus v$ has more components than $G$. A block of a graph is a maximal nontrivial connected induced subgraph which has no cut vertex. If exactly one block of a graph $G$ is a cycle and each other block is an edge, then we say that $G$ is a unicyclic graph.

There is a containment of graph classes:

$$\{\text{unicyclic graphs}\} \subset \{\text{semi-block graphs}\} \subset \{\text{clique sum of a cycle and chordal graphs}\}.$$ 

A unicyclic graph is a clique sum of a cycle and some trees. The notion of semi-block graph was introduced in [20]. We refer the readers to [20] Section 3 for the definition of semi-block graphs. Now, as a consequence of Theorems 2.16 and 2.18 we show that for some important classes of graphs, the subadditivity holds. This includes chordal graphs for which the subadditivity was proved by Bigdeli and Herzog in [5].

Corollary 2.20. If $G$ is a chordal graph, semi-block graph or a unicyclic graph, then the subadditivity holds for $G$.

Let $H$ be a graph and $U \subset V(H)$. The cone of $H$ along $U$, denoted by $x \ast_U H$, is the graph on the vertex set $V(H) \cup \{x\}$ and edge set $E(H) \cup \{\{x, u\} : u \in U\}$. If $U = V(H)$, then we simply write $x \ast H$.

Theorem 2.21. Let $H$ be a non-trivial graph, and $U \subset V(H)$ be a vertex cover of $H$. Let $G = x \ast_U H$. If subadditivity holds for $H$, then it holds for $G$.

Proof. Note that $I(G) = I(H) + (xu : u \in U)$. It follows from [15] Corollary 4.3 that

$$I(H) \cap (xu : u \in U) = xI(H).$$
Let \( H' \) be the subgraph of \( G \) on the vertex set \( \{x\} \cup U \) and edge set \( \{x, u\} : u \in U \}. \) It follows from Theorem [15, Theorem 4.6] that
\[
\beta_{i,j}^R \left( \frac{R}{I(G)} \right) = \beta_{i,j}^R \left( \frac{R}{I(H)} \right) + \beta_{i-1,j}^R \left( \frac{R}{xI(H)} \right) + \beta_{i,j}^R \left( \frac{R}{I(H')} \right).
\]
Therefore, for each \( i \geq 1 \), \( t_i(G) = \max\{t_i(H), t_{i-1}(H) + 1, t_i(H')\} \). Note that \( H' \) is a star graph. Thus, by [5, Corollary 3], for all \( a, b \geq 1 \) with \( a + b \leq \text{pd}(R/I(H')) \), \( t_{a+b}(H') \leq t_a(H') + t_b(H') \). For all \( a, b \geq 1 \) with \( a + b \leq \text{pd}(R/I(G)) \),
\[
t_a(G) + t_b(G) = \max\{t_a(H), t_{a-1}(H) + 1, t_a(H')\} + \max\{t_b(H), t_{b-1}(H) + 1, t_b(H')\}
\geq \max\{t_a(H) + t_b(H), t_{a-1}(H) + t_b(H) + 1, t_a(H') + t_b(H')\}
\geq \max\{t_{a+b}(H), t_{a+b-1}(H) + 1, t_{a+b}(H')\} = t_{a+b}(G).
\]
Hence, the subadditivity holds for \( G \).

As a consequence of Theorem 2.21, we obtain the following result:

**Corollary 2.22.** Let \( G \) be a graph on \( V(G) \). Then, the subadditivity holds for \( G \) if

1. \( G = W_n = x \ast C_n \), the wheel graph on \( n + 1 \) vertices;
2. \( G = J_{2,n} = x \ast_U C_{2n} \), Jahangir graph on \( 2n + 1 \) vertices, where \( U \) is a vertex cover of \( C_{2n} \) of size \( n \);
3. \( G \) is a complete multipartite graph;
4. \( G = F_{m,n} = (x_1 \ast_U \cdots \ast_U (x_m \ast_U P_n)) \), where \( U = V(P_n) \), is a fan graph.

Note that the classes of the graph listed above are not in \( G \). In similar manner, one can keep constructing several graphs \( G \notin G \) satisfying subadditivity.

Taking a join of two graphs is an important operation in graph theory. For two graphs \( G \) and \( H \), \( G \ast H \) is the graph on the vertex set \( V(G \ast H) = V(G) \cup V(H) \) and with the edge set \( E(G \ast H) = E(G) \cup E(H) \cup \{\{x, y\} : x \in V(G) \text{ and } y \in V(H)\} \). It is natural to ask how the subadditivity of \( G \) and \( H \) gets translated to \( G \ast H \). We answer this question below.

**Theorem 2.23.** Let \( G \) and \( H \) be graphs on \( m \) and \( n \) vertices, respectively. If subadditivity holds for \( G \) and \( H \), then so for \( G \ast H \).

**Proof.** It follows from [23, Corollary 3.4] that for all \( i, j \),
\[
\beta_{i,j}^R \left( \frac{R}{I(G \ast H)} \right) = \sum_{k=0}^{j-2} \left[ \binom{n}{k} \beta_{i-k,j-k}^R \left( \frac{R}{I(G)} \right) + \binom{m}{k} \beta_{i-k,j-k}^R \left( \frac{R}{I(H)} \right) \right].
\]
First, we claim that for each \( 1 \leq i \leq n + m - 1 \),
\[
t_i(G \ast H) = \max\{t_{i-k}(G) + k : 0 \leq k \leq \min\{i, n\}\} \cup \{t_{i-l}(H) + l : 0 \leq l \leq \min\{i, m\}\}.
\]
Set \( p = \max\{t_{i-k}(G) + k : 0 \leq k \leq \min\{i, n\}\} \cup \{t_{i-l}(H) + l : 0 \leq l \leq \min\{i, m\}\} \). If \( q > p \), then for each \( 0 \leq k \leq \min\{i, n\}, t_{i-k}(G) < q - k \), and for each \( 0 \leq l \leq \min\{i, m\}, \)
It follows from (5) that $\beta^{R}_{i,q}(R/I(G \ast H)) = 0$. Thus, $t_{i}(G \ast H) \leq p$. Now, to prove our claim, it is enough to prove that $\beta^{R}_{i,p}(R/I(G \ast H)) \neq 0$. Note that for some $0 \leq k \leq \min\{i, n\}$, $t_{i-k}(G) = p - k$ or for some $0 \leq l \leq \min\{i, m\}$, $t_{i-l}(H) = p - l$. Hence, using (5), we get $\beta^{R}_{i,p}(R/I(G \ast H)) \neq 0$. This proves the claim.

Now, consider for $a, b \geq 1$,

\[
t_{a+b}(G \ast H)
= \max \left\{ \left\{ t_{a+b-k}(G) + k : 0 \leq k \leq \min\{a, b, n\} \right\} \right.
\cup \left\{ t_{a+b-l}(H) + l : 0 \leq l \leq \min\{a, b, m\} \right\} \right.
\cup \left. \left\{ t_{a+b-k'}(G) + k + k' : 0 \leq k + k' \leq \min\{a, b, n\} \right\} \right.
\cup \left. \left\{ t_{a+b-l'}(H) + l + l' : 0 \leq l + l' \leq \min\{a, b, m\} \right\} \right\}
\leq \max \left\{ \left\{ t_{a+b-k'}(G) + k + k' : 0 \leq k \leq \min\{a, n\}, 0 \leq k' \leq \min\{b, n\} \right\} \right.
\cup \left. \left\{ t_{a+b-l'}(H) + l + l' : 0 \leq l \leq \min\{a, m\}, 0 \leq l' \leq \min\{b, m\} \right\} \right\}
\leq \max \left\{ \left\{ t_{a-k}(G) + k + t_{b-k'}(G) + k' : 0 \leq k \leq \min\{a, n\}, 0 \leq k' \leq \min\{b, n\} \right\} \right.
\cup \left. \left\{ t_{a-l}(H) + l + t_{b-l'}(H) + l' : 0 \leq l \leq \min\{a, m\}, 0 \leq l' \leq \min\{b, m\} \right\} \right\}
\leq \max \left\{ \left\{ t_{a-k}(G) + k : 0 \leq k \leq \min\{a, n\} \right\} \cup \left( t_{a-l}(H) + l : 0 \leq l \leq \min\{a, m\} \right) \right\}
+ \max \left\{ \left\{ t_{b-k'}(G) + k' : 0 \leq k' \leq \min\{b, n\} \right\} \cup \left( t_{b-l'}(H) + l' : 0 \leq l' \leq \min\{b, m\} \right) \right\}
= t_{a}(G \ast H) + t_{b}(G \ast H).
\]

This completes the proof. \qed

### 2.2. Path ideals of rooted tree

Path ideals are generalization of edge ideals. In this subsection, we prove that the subadditivity for path ideals of rooted trees.

A tree together with a fixed vertex is called a **rooted tree**, and the fixed vertex of that tree is called a **root**. In a tree, there exists a unique path between any two given vertices. Thus, we can see that there is a unique path between the root and any other vertex in a rooted tree. We can also view a rooted tree as a directed graph by assigning to each edge the direction that goes “away” from the root. From now onward, $\Gamma$ denotes a rooted tree with $x$ as root and $\Gamma$ is viewed as a directed rooted tree in the above sense. An edge $\{u, v\}$ in a rooted tree whose direction is from $u$ to $v$ is denoted by $(u, v)$. Let $\Gamma$ be a rooted tree on $\{x_1, \ldots, x_n\}$ with root $x = x_i$ for some $i$. Let $t \geq 1$. A **directed path** of length $(t - 1)$ in $\Gamma$ is a sequence of distinct vertices $x_{i_1}, \ldots, x_{i_t}$ such that $(x_{i_j}, x_{i_{j+1}}) \in E(\Gamma)$ for each $j \in \{1, \ldots, t - 1\}$. The **$t$-path ideal** of $\Gamma$ is denoted by $I_t(\Gamma)$ and is defined as

\[ I_t(\Gamma) := (x_{i_1} \cdots x_{i_t} : x_{i_1}, \ldots, x_{i_t} \text{ is a path of length } (t - 1) \text{ in } \Gamma). \]

For a vertex $u$ of $\Gamma$, the **level of $u$** is denoted by $\text{level}(u)$ and is length of the unique path from $x$ to $u$. The **height of $\Gamma$**, denoted by $\text{height}(\Gamma)$, is $\max_{v \in V(\Gamma)} \text{level}(v)$. A vertex $u$ is said to be **parent** of a vertex $v$ if $(u, v) \in E(\Gamma)$, and a vertex $w$ is said to be **child** of a vertex $v$ if $(v, w) \in E(\Gamma)$. A vertex $v$ is said to be **descendant** of a vertex $u$ if there is
a path from \(u\) to \(v\). An induced subtree of \(\Gamma\) with root \(u \in V(\Gamma)\) is an induced subtree of \(\Gamma\) on the vertex set \(\{u\} \cup \{v : v \text{ is descendant of } u\}\). We conclude this section by discussing subadditivity of \(t\)-path ideals of rooted trees:

**Theorem 2.24.** Let \(\Gamma\) be a rooted tree on \(\{x_1, \ldots, x_n\}\). Then, for all \(1 \leq t \leq \text{height}(\Gamma) + 1\), subadditivity holds for \(R/I_\Gamma(t)\).

**Proof.** We prove this by induction on the number of vertices of \(\Gamma\). Let \(x_{i_1}\) be a vertex such that \(\text{level}(x_{i_1}) = \text{height}(\Gamma)\). Let \(x_{i_1}, \ldots, x_{i_t}\) be a path of length \((t - 1)\) terminating at \(x_{i_t}\). Consider the following short exact sequence

\[
0 \rightarrow \frac{R}{I_t(\Gamma \setminus \{x_{i_t}\}) : x_{i_1} \cdots x_{i_t}} \rightarrow \frac{R}{I_t(\Gamma \setminus \{x_{i_t}\})} \rightarrow \frac{R}{I_t(\Gamma)} \rightarrow 0. \tag{6}
\]

By induction, the subadditivity holds for \(R/I_t(\Gamma \setminus \{x_{i_t}\})\). Let \(x_{i_0}\) be the only parent of \(x_{i_t}\), if it exists. For \(j \in \{0, \ldots, t\}\), let \(\Gamma_j\) be the induced subtree of \(\Gamma\) rooted at \(x_{i_j}\), and for \(j \in \{0, \ldots, t - 1\}\), let \(\Delta_j = \Gamma[V(\Gamma_j) \setminus V(\Gamma_{j+1})]\). By [8, Lemma 2.8],

\[
I_t(\Gamma \setminus \{x_{i_t}\}) : x_{i_1} \cdots x_{i_t} = I_t(\Gamma \setminus \{x_{i_0}, \ldots, x_{i_t}\}) + (x_{i_0}) + \sum_{j=0}^{t-1} I_{t-j}(\Delta_j \setminus \{x_{i_0}, \ldots, x_{i_t}\}).
\]

Observe that \(t_1(R/(I_t(\Gamma \setminus \{x_{i_t}\}) : x_{i_1} \cdots x_{i_t})) \leq t\). It follows from [8, Remark 2.9] that the minimal free resolution of \(R/(I_t(\Gamma \setminus \{x_{i_t}\}) : x_{i_1} \cdots x_{i_t})\) is obtained by the tensor product of the minimal free resolution of \(R/I_t(\Gamma \setminus \{x_{i_0}, \ldots, x_{i_t}\})\), \(R/(x_{i_0})\) and \(R/I_{t-j}(\Delta_j \setminus \{x_{i_0}, \ldots, x_{i_t}\})\) for \(j \in \{0, \ldots, t - 1\}\). Now, by induction, the subadditivity holds for \(R/I_t(\Gamma \setminus \{x_{i_0}, \ldots, x_{i_t}\})\) and for \(j \in \{0, \ldots, t - 1\}\), \(R/I_{t-j}(\Delta_j \setminus \{x_{i_0}, \ldots, x_{i_t}\})\). Thus, subadditivity holds for \(R/(I_t(\Gamma \setminus \{x_{i_t}\}) : x_{i_1} \cdots x_{i_t})\), by Theorem 2.3. It follows from [8, Theorem 2.7] that the mapping cone construction applied to (6) gives a minimal free resolution of \(R/I_t(\Gamma)\). Hence, by Theorem 2.4, the subadditivity holds for \(R/I_t(\Gamma)\). \qed

### 3. Strand connectivity of edge ideals

In this section, we discuss the strand connectivity of edge ideals. It is known that the tensor product of minimal free resolutions of two homogeneous ideals on disjoint set of variables gives a minimal free resolution of their sum. It is interesting to ask if this property translates to strand connectivity. The following example shows that this is not the case.

**Example 3.1.** Let \(I = I(C_5) = (x_1x_2, \ldots, x_5x_1) \subset R_1 = \mathbb{K}[x_1, \ldots, x_5]\) and \(J = (y_1y_2y_3) \subset R_2 \subset \mathbb{K}[y_1, y_2, y_3]\). Then \(I\) is strand connected, [5, Example 1(b)] and \(J\) being a complete intersection, it is strand connected. It can be seen that for \(I + J \subset R = \mathbb{K}[x_1, \ldots, x_5, y_1, y_2, y_3]\), \(\beta^R_1(I + J) = 1\), \(\beta^R_2(I + J) = 0\) and \(\beta^R_3(I + J) = 1\). This shows that the 2-strand of \(I + J\) is not connected.

However, we show that we can get strand connectivity if one of the ideals is generated by linear forms.

**Lemma 3.2.** Let \(I\) and \(J\) be homogeneous ideals of \(R\) such that there exist minimal generating sets for \(I\) and \(J\) in disjoint sets of variables. If \(I\) is generated by \(k\) linear forms and \(J\) is strand connected, then \(I + J\) is strand connected.
Proof. Since the minimal generating sets of \( I \) and \( J \) are in disjoint variables, the tensor product of the minimal free resolution of \( R/I \) and \( R/J \) gives the minimal free resolution of \( R/(I+J) \). Since \( I \) is generated by \( k \) linear forms, for all \( i, j \), we have

\[
\beta^R_{i,j} \left( \frac{R}{I+J} \right) = \sum_{0 \leq r \leq \min\{i,k\}} \beta^R_{r,r} \left( \frac{R}{I} \right) \beta^R_{i-r,j-r+j} \left( \frac{R}{J} \right). \tag{7}
\]

Thus, if \( j \)-strand of \( I+J \) is nonempty, then we claim that the \( j \)-strand of \( J \) is nonempty and \( j \)-strand of \( I + J = [q_j(J), p_j(J) + \min\{k, q_j(J)\}] \). If \( j \)-strand of \( J \) is empty, then it follows from (7) that \( j \)-strand of \( I+J \) is empty. Thus, \( j \)-strand of \( J \) is nonempty. Let \( i \in [q_j(J), p_j(J) + \min\{k, q_j(J)\}] \). If \( i \leq p_j(J) \), then \( \beta^R_{i,j} R/I + \beta^R_{i,j} R/J \neq 0 \), and hence it follows from (7) that \( \beta^R_{i,j} R/(I+J) \neq 0 \). If \( i > p_j(J) \), then for some \( l \leq \min\{k, q_j(J)\} \), \( i-l = p_j(J) \). Therefore, \( \beta^R_{i,j} R/I + \beta^R_{i,j} R/J \neq 0 \), and hence, \( \beta^R_{i,j} R/(I+J) \neq 0 \). Thus, for \( i \in [q_j(J), p_j(J) + \min\{k, q_j(J)\}] \), \( \beta^R_{i,j} R/(I+J) \neq 0 \). Now, let \( i \notin [q_j(J), p_j(J) + \min\{k, q_j(J)\}] \). If \( i < q_j(J) \), then for each \( 0 \leq r \leq \min\{k, i\} \), \( i-r < q_j(J) \), and hence, \( \beta^R_{i,j} R/(I+J) = 0 \). If \( i > p_j(J) + \min\{k, q_j(J)\} \), then for any \( 0 \leq r \leq \min\{i,k\} \), \( i-r > p_j(J) \). Hence so that \( \beta^R_{i-r,i-r+j} R/J = 0 \). Therefore \( \beta^R_{i,j} R/(I+J) = 0 \). This completes the proof. \( \square \)

In the example above, one of the ideals is generated in degree 3. If both ideals are generated in degree 2 and are strand connected, then can one say that their sum is strand connected? This question has a graph theoretic analogue too. If \( G \) and \( H \) are two disjoint graphs whose edge ideals are strand connected, then is \( I(G \cup H) \) strand connected?

Now we begin the study of strand connectivity of edge ideals of graphs. If \( j > \ \text{reg}(R/I(G)) \), then the \( j \)-strand is empty. Therefore, to study the strand connectivity, we can restrict ourselves to \( j \leq \text{reg}(R/I(G)) \). It is known that \( \nu(G) \leq \text{reg}(R/I(G)) \), [19]. We have earlier shown that the graphs in \( G \) satisfy subadditivity. We now consider a subcollection of \( G \). Let

\[ G' := \{ G \in G : C_n \text{ is not an induced subgraph of } G \text{ for } n \equiv 2 \pmod{3} \}. \]

We first show that graphs in this collection have minimal regularity.

**Proposition 3.3.** Let \( G \in G' \). Then, \( \text{reg}(R/I(G)) = \nu(G) \).

**Proof.** The inequality \( \nu(G) \leq \text{reg}(R/I(G)) \) follows from [19] Lemma 2.2. We now prove that \( \text{reg}(R/I(G)) \leq \nu(G) \). We proceed by induction on the number of vertices. Let \( G \in G' \). If \( |V(G)| = 2 \), then the assertion is true. Assume that \( |V(G)| > 2 \) and the result is true for all graphs \( H \in G' \) with \( |V(H)| < |V(G)| \). If \( G \) is a cycle, then \( |V(G)| \not\equiv 2 \pmod{3} \), and thus, by [18] Theorem 7.6.28, \( \text{reg}(R/I(G)) = \nu(G) \). Suppose that \( G \) is not a cycle. Then, by Definition 2.12(2b), \( G \) has an edge \( e = \{x,y\} \) such that \( N_G(x) \subset N_G[y] \) and \( G \setminus e \in G' \). Since \( G \setminus \{y\} \) and \( G \setminus N_G[y] \) are induced subgraphs of \( G \), by Definition 2.12(2a), \( G \setminus \{y\}, G \setminus N_G[y] \in G' \). Thus, by induction,

\[ \text{reg} \left( \frac{R}{I(G \setminus \{y\})} \right) \leq \nu(G \setminus \{y\}) \]

and

\[ \text{reg} \left( \frac{R}{I(G \setminus N_G[y])} \right) \leq \nu(G \setminus N_G[y]). \]
Since $G \setminus \{y\}$ is an induced subgraph of $G$, $\nu(G \setminus \{y\}) \leq \nu(G)$. If $\{e_1, \ldots, e_s\}$ is an induced matching in $G \setminus N_G[y]$, then $\{e_1, \ldots, e_s, \{x, y\}\}$ is an induced matching in $G$. Therefore, $\nu(G \setminus N_G[y]) + 1 \leq \nu(G)$. It follows from [4, Theorem 2.7] that

$$\operatorname{reg} \left( \frac{R}{I(G)} \right) \leq \max \left\{ \operatorname{reg} \left( \frac{R}{I(G \setminus \{y\})} \right), \operatorname{reg} \left( \frac{R}{I(G \setminus N_G[y])} \right) + 1 \right\} \leq \nu(G).$$

Hence, the assertion follows. \hspace{1cm} \Box

**Theorem 3.4.** If $G \in \mathcal{G}'$, then $I(G)$ is strand connected.

**Proof.** We prove this by induction on the number of edges. Let $G \in \mathcal{G}'$. If $|E(G)| = 1$, then the assertion is true. Assume that $|E(G)| > 1$ and the result is true for all graphs $H \in \mathcal{G}'$ with $|E(H)| < |E(G)|$. If $G$ is a cycle, then by [4, Example 1(b)], $j$-strand of $I(G)$ is connected. Assume that $G$ is not a cycle. Then, by Definition 2.12(2b), $G$ has an edge $e = \{x, y\}$ such that $N_G(x) \subset N_G[y]$. Since $N_G(x) \subset N_G[y]$, the edge $e$ cannot be a chord of an induced cycle of length at least 5. Therefore, $G \setminus e$ does not contain $C_5$ as an induced cycle for any $n \equiv 2 \pmod{3}$ so that $G \setminus e \in \mathcal{G}'$. Since by Proposition 3.3, $\operatorname{reg}(R/I(G)) = \nu(G)$ for all $G \in \mathcal{G}'$, $j$-strand is empty for all $j > \nu(G)$.

Let $1 \leq j \leq \nu(G)$. It follows from [19, Lemma 2.2] that $g_j(I(G)) = j$. Suppose that for some $i > j$, $\beta^R_{i,j}(R/I(G)) = 0$. It is enough to show that $\beta^R_{i+1,i+1+j}(R/I(G)) = 0$. By Proposition 2.17 we have

$$\beta^R_{i,i+j} \left( \frac{R}{I(G)} \right) = \beta^R_{i,i+j} \left( \frac{R}{I(G \setminus e)} \right) + \beta^R_{i-1,i+j-2} \left( \frac{R}{I(G \setminus e) : e} \right).$$

(8)

Since $\beta^R_{i,j}(R/I(G)) = 0$, we have $\beta^R_{i,i+j}(R/I(G \setminus e)) = 0$. By induction, $I(G \setminus e)$ is strand connected. Moreover, it follows from (8) that $\nu(G \setminus e) \leq \nu(G)$. Hence, $\beta^R_{i+1,i+1+j}(R/I(G \setminus e)) = 0$ as $i > j$. This implies that $\beta^R_{i+1,i+1+j}(R/I(G)) = \beta^R_{i+1,i+1+j}(R/I(G \setminus e))$. Since $G \setminus N_G[y]$ is an induced subgraph of $G$, $G \setminus N_G[y] \in \mathcal{G}'$. Thus, by induction, $I(G \setminus N_G[y])$ is strand connected. Note that $I(G \setminus e) : e = I(G \setminus N_G[y]) + (N_G[e])[y]$. Therefore, by Lemma 3.2, $I(G \setminus e) : e$ is strand connected. Since $i - 1 > j$ and $\beta^R_{i-1,i+j-1}(R/I(G \setminus e)) = 0$, we have $\beta^R_{i,i+j-1}(R/I(G \setminus e)) = 0$. Hence from (8), we get $\beta^R_{i+1,i+1+j}(R/I(G)) = 0$. This implies that the $j$-strand of $I(G)$ is connected. Hence, $I(G)$ is strand connected. \hspace{1cm} \Box

The above result gives several important classes of graphs whose edge ideals are strand connected. Before we go on listing them, we prove two more results in this direction which enable us to construct more classes of graphs whose edge ideals are strand connected.

**Theorem 3.5.** Let $H$ be a non-trivial graph on $n$ vertices, $U$ be a vertex cover of $H$ and $x$ be a new vertex. Let $G = x \ast_U H$. If $j$-strand of $I(H)$ is connected, then $j$-strand of $I(G)$ is connected. In particular, if $I(H)$ is strand connected, then $I(G)$ is strand connected.

**Proof.** Let $j > 1$ be such that $j$-strand of $I(H)$ is connected. Let $H'$ be the subgraph of $G$ on the vertex set $\{x\} \cup U$ and edge set $\{\{x, u\} : u \in U\}$. Since $H' = K_{1,|U|}$, by [18, Theorem 5.2.4], $\beta_{i,j}(R/I(H')) = 0$ if $j > 1$. Therefore, we get from (4)

$$\beta^R_{i,i+j} \left( \frac{R}{I(G)} \right) = \beta^R_{i,i+j} \left( \frac{R}{I(H)} \right) + \beta^R_{i-1,i-1+j} \left( \frac{R}{I(H)} \right).$$

(9)
This implies that for each $1 \leq j \leq \text{reg}(R/I(G))$, $q_j(I(G)) = q_j(I(H))$. Assume that for some $i > q_j(I(G)) = q_j(I(H))$, $\beta^R_{i,i+1}(R/I(G)) = 0$. It is enough to prove that $\beta^R_{i+1,i+i+1+j}(R/I(G)) = 0$. It follows from Equation (9) that $\beta^R_{i,i+1+j}(R/I(H)) = 0$. Since $j$-strand of $I(H)$ is connected, this implies that $\beta^R_{i+1,i+1+i+j}(R/I(H)) = 0$. Hence, from Equation (9), we get $\beta^R_{i+1,i+1+j}(R/I(G)) = 0$. This shows that $j$-strand of $I(G)$ is connected. □

**Theorem 3.6.** Let $G$ and $H$ be graphs on $m$ and $n$ vertices such that $I(G)$ and $I(H)$ are strand connected. Then, $I(G \ast H)$ is strand connected.

**Proof.** Since the linear strand of a homogeneous ideal is connected, we assume that $j > 1$. It follows from [23] Corollary 3.4] that for all $i, j$,

$$\beta^R_{i,i+j}( \frac{R}{I(G \ast H)} ) = \sum_{k=0}^{i+j-2} \binom{n}{k} \beta^R_{i-k,i-k+j}( \frac{R}{I(G)} ) + \binom{m}{k} \beta^R_{i-k,i-k+j}( \frac{R}{I(H)} ) \right] . \quad (10)$$

First, we claim that for each $1 < j \leq \text{reg}(R/I(G \ast H))$,

$$q_j(I(G \ast H)) = \min \{q_j(I(G)), q_j(I(H))\}.$$ 

If $i < \min \{q_j(I(G)), q_j(I(H))\}$, then by Equation (10), $\beta^R_{i,i+j}(R/I(G \ast H)) = 0$, as $i - k < \min \{q_j(I(G)), q_j(I(H))\}$, for each $0 \leq k \leq i + j - 2$. If $i = \min \{q_j(I(G)), q_j(I(H))\}$, then either $\beta^R_{i,i+j}(R/I(G)) \neq 0$ or $\beta^R_{i,i+j}(R/I(H)) \neq 0$. Hence it follows from Equation (10) that $\beta^R_{i,i+j}(R/I(G \ast H)) \neq 0$. Thus, for each $1 < j \leq \text{reg}(R/I(G \ast H))$, $q_j(I(G \ast H)) = \min \{q_j(I(G)), q_j(I(H))\}.$

Let $q_j(I(G \ast H)) < i$. To prove that the $j$-strand of $I(G \ast H)$ is connected, it is enough to prove that if $\beta^R_{i,i+j}(R/I(G \ast H)) = 0$, then $\beta^R_{i+1,i+1+j}(R/I(G \ast H)) = 0$. Suppose $\beta^R_{i,i+j}(R/I(G \ast H)) = 0$. We claim that $i > \max \{q_j(I(G)), q_j(I(H))\}$. We may assume that $\max \{q_j(I(G)), q_j(I(H))\} = q_j(I(H))$. First, note that $i$ can not be equal either to $q_j(I(G))$ or to $q_j(I(H))$. Assume, if possible, that $i < q_j(I(H))$. Let $k > 0$ be such that $i - k = q_j(I(G))$. Then we have $0 < k < i < n$ so that \((n \choose k) \beta^R_{i-k,i-k+j}(R/I(G)) \neq 0\). Therefore, it follows from Equation (10) that $\beta^R_{i,i+j}(R/I(G \ast H)) \neq 0$ which is a contradiction to our assumption. Hence $i > \max \{q_j(I(G)), q_j(I(H))\}.$

From Equation (10), for $0 \leq k \leq i + j - 2$, we get

$$\binom{n}{k} \beta^R_{i-k,i-k+j}( \frac{R}{I(G)} ) = 0 = \binom{m}{k} \beta^R_{i-k,i-k+j}( \frac{R}{I(H)} ). \quad (11)$$

We need to prove that \((n \choose k) \beta^R_{i+1-k,i+1-k+j}(R/I(G)) = 0 = (m \choose k) \beta^R_{i+1-k,i+1-k+j}(R/I(H))\), for $0 \leq k \leq i+j-1$, i.e., we need to prove that for $-1 \leq k' \leq i+j-2$, \((n \choose k') \beta^R_{i-k',i-k'+j}(R/I(G)) = 0 = (m \choose k') \beta^R_{i-k',i-k'+j}(R/I(H))\). For $0 \leq k' \leq i+j-2$, this follows from Equation (11). And the case $k' = -1$ follows from the strand connectivity of $I(G)$ and $I(H)$. This completes the proof. □

As a consequence of our results, we obtain several classes of graphs whose edge ideals are strand connected.

**Corollary 3.7.** Let $G$ be a graph and $I(G)$ be its edge ideals. Then $I(G)$ is strand connected if

...
(1) $G$ is chordal graph; ([5] Proposition 5)
(2) $G = W_n = x \ast C_n$
(3) $G = J_{2,n}$, Jahangir graph on $2n + 1$ vertices;
(4) $G = F_{m,n}$, the fan graph;
(5) $G$ is a unicyclic graph with the induced cycle of length $n \neq 3k + 2$ for some $k \geq 1$.

Other than the named classes of graphs listed above, one can construct more graphs using Theorems 3.3, 3.6. It is known that not all edge ideals are strand connected. We expect that characterizing strand connected edge ideals will be a tough problem. There are even possibly simpler questions in this direction for which the answers are unknown:

**Question 3.8.**

1. If $I(G)$ is strand connected, then is $I(H)$ strand connected for every non-trivial induced subgraph $H$ of $G$?
2. If $\text{reg}(R/I(G)) = \nu(G)$, then is $I(G)$ strand connected?
3. If $j \leq \nu(G)$, is the $j$-strand of $I(G)$ connected?

4. **Multigraded Betti numbers of edge ideals of graphs**

In this section, we study multigraded Betti numbers of some classes of edge ideals. First we generalize the result of Boocher et al. to the case of unicyclic graphs. Girth of a unicyclic graph $G$ is the length of the induced cycle of $G$.

**Theorem 4.1.** Let $G$ be a unicyclic graph on $n$-vertices with girth $m$.

1. If $m$ is not a multiple of $3$, then $\beta_{i,a}(R/I(G)) \in \{0, 1\}$ for all $i \geq 1$ and $a \in \mathbb{N}^n$.
2. If $m = 3k$, then $\beta_{i,a}(R/I(G)) \in \{0, 1, 2\}$ for all $i \geq 1$ and $a \in \mathbb{N}^n$. Furthermore assume that every vertex in $G$ is at a distance at most two from the unique cycle in $G$. Then $\beta_{i,a}(R/I(G)) = 2$ if and only if $i = 2k$, and $a = \sum_{x \in V(C_m)} e_x$.

**Proof.** We prove this by induction on $n - m$. If $n - m = 0$, then the assertion follows from [6] Proposition 3.5. Now, assume that $n - m > 0$. Then, there exists $x \in V(G)$ such that $\text{deg}_G(x) = 1$. Let $N_G(x) = \{y\}$. Then, it follows from Proposition 2.17 that for $i \geq 1$ and $a \in \mathbb{N}^n$,

$$\beta_{i,a} \left( \frac{R}{I(G)} \right) = \beta_{i,a} \left( \frac{R}{I(G \setminus \{x\})} \right) + \beta_{i-1,a-e_x-e_y} \left( \frac{R}{I(G \setminus \{x\})} : xy \right).$$

(12)

If $a_x \neq 0$, then for any $j \geq 1$, $\beta_{j,a}(R/I(G \setminus \{x\})) = 0$ as $x$ does not divide any of the minimal monomial generators of $I(G \setminus \{x\})$. If $a_x = 0$, then $[a - e_x - e_y]_x$ is negative. Consequently, for any $j \geq 1$, we have $\beta_{j,a-e_x-e_y}(R/(I(G \setminus \{x\}) : xy)) = 0$. Thus only one term on the right hand side of Equation (12) will contribute to $\beta_{i,a}(R/I(G))$. Observe that $G \setminus \{x\}$ is a unicyclic graph on $(n - 1)$-vertices having girth $m$ and that

$$I(G \setminus \{x\}) : xy = I(G \setminus N_G[y]) + (N_{G\setminus\{x\}}(y)).$$

Since Koszul complex is the minimal free resolution of $(N_{G\setminus\{x\}}(y))$, $\beta_{i,a}(R/I(G \setminus \{x\})(y)) \in \{0, 1\}$ for all $i \geq 1$ and $a \in \mathbb{N}^n$. Observe that $G \setminus N_G[y]$ is either a forest or a unicyclic graph. If $G \setminus N_G[y]$ is a forest, then by [7] Theorem 2.2.2, $\beta_{i,a}(R/I(G \setminus N_G[y])) \in \{0, 1\}$ for all $i \geq 1$ and $a \in \mathbb{N}^n$.

1. Assume that $m$ is not a multiple of 3. Since $G \setminus \{x\}$ is a unicyclic graph on $n - 1$ vertices and having girth $m$, by induction, we have $\beta_{i,a}(R/I(G \setminus \{x\})) \in \{0, 1\}$ for all $i \geq 1$
and \(a \in \mathbb{N}^n\). If \(G \setminus N_G[y]\) is a unicyclic graph, then also we may conclude by induction that 
\[
\beta_{i,a}(R/(I(G \setminus N_G[y]))) \in \{0,1\} \quad \text{for all } i \geq 0 \text{ and } a \in \mathbb{N}^n.
\]
Since the generators of \((N_{G \setminus \{x\}}(y))\) and \(I(G \setminus N_G[y])\) are in disjoint variables, tensor product of the minimal free resolutions of 
\(R/(N_{G \setminus \{x\}}(y))\) and \(R/I(G \setminus N_G[y])\) gives the minimal free resolution of 
\(R/(I(G \setminus \{x\}) : xy)\). Thus, \(\beta_{i,a}(R/(I(G \setminus \{x\}) : xy)) \in \{0,1\} \quad \text{for all } i \geq 1 \text{ and } a \in \mathbb{N}^n.\)

Hence, it follows from \([12]\) that for each \(i \geq 1\) and \(a \in \mathbb{N}^n, \beta_{i,a}(R/I(G)) \in \{0,1\}\).

(2) Assume that \(m\) is a multiple of 3. Observe that \(G \setminus \{x\}\) is a unicyclic graph on \(n-1\) vertices with girth \(m\). By induction, we have \(\beta_{i,a}(R/I(G \setminus \{x\})) \in \{0,1,2\} \quad \text{for all } i \geq 1 \text{ and } a \in \mathbb{N}^n.\)

If \(G \setminus N_G[y]\) is a unicyclic graph, then by induction, \(\beta_{i,a}(R/I(G \setminus N_G[y])) \in \{0,1,2\} \quad \text{for all } i \geq 1 \text{ and } a \in \mathbb{N}^n.\)

Thus, \(\beta_{i,a}(R/I(G \setminus \{x\}) : xy) \in \{0,1,2\} \quad \text{for all } i \geq 1 \text{ and } a \in \mathbb{N}^n.\)

Hence, it follows from \([12]\) that for each \(i \geq 1\) and \(a \in \mathbb{N}^n, \beta_{i,a}(R/I(G)) \in \{0,1,2\}\).

Now, we prove second part. Since \(x\) is at a distance of at most 2 from the unique cycle, \(y\) is at a distance at most one from the unique cycle, and thus, \(G \setminus N_G[y]\) is a forest. Thus by eq. \([12], \beta_{i,a}(R/I(G)) = 2\) if and only if \(\beta_{i,a}(R/I(G \setminus \{x\})) = 2.\)

Since \(G \setminus \{x\}\) is a unicyclic graph on \(n-1\) vertices and having girth \(m\). Therefore, by induction, we have
\(\beta_{i,a}(I(G \setminus \{x\}) : xy) \in \{0,1,2\} \quad \text{for all } i \geq 1 \text{ and } a \in \mathbb{N}^n.\)

Hence, it follows from \([12]\) that for each \(i \geq 1\) and \(a \in \mathbb{N}^n, \beta_{i,a}(R/I(G)) \in \{0,1,2\}\).

Thus, the assertion follows.

In the previous two sections, we saw that the knowledge of Betti numbers of a graph would give information about Betti numbers of the cone of that graph along a vertex cover. We prove a similar result for multigraded Betti numbers here.

**Theorem 4.2.** Let \(H\) be a non-trivial graph on \(n\)-vertices. Let \(U\) be a vertex cover of \(H\) and \(x\) be a new vertex. Let \(G = x * U, H\). If
\[
\beta_{i,a}(R/I(H)) \in \{c \mid d\} \text{ if } |a| = i + 1, \quad \text{then } \beta_{i,a}(R/I(G)) \in \{c + 1 \mid d\} \text{ if } |a| > i + 1.
\]

**Proof.** Let \(H'\) be the subgraph of \(G\) on the vertex set \(\{x\} \cup U\) and edge set \(\{x, u\} : u \in U\). \(G\) is a graph and \(x\) is a new vertex. Let \(i \geq 1\) and \(a \in \mathbb{N}^{i+1}\) such that \(|a| = i + 1\). Since \(H'\) is a tree, by \([7], \beta_{i,a}(R/I(H')) \in \{0,1\}\). If \(a_x \neq 0\), then for any \(i \geq 1\), \(\beta_{i,a}(R/I(H)) = 0\) as \(x\) does not divide any of the minimal monomial generators of \(I(H)\). If \(a_x = 0\), then \(|a - e_x| < -1\). Consequently, for any \(i \geq 1\), we have \(\beta_{i,a-e_x}(R/I(H)) = 0\) and \(\beta_{i,a}(R/I(H')) = 0\). Thus, \(\beta_{i,a}(R/I(G)) \leq c + 1\).

Now, assume that \(a \in \mathbb{N}^{i+1}\) such that \(|a| > i + 1\). Since \(H' = K_{1,|U|}\), by \([18], \beta_{i,a}(R/I(H')) = 0\) as \(|a| > i + 1\). Therefore,
\[
\beta_{i,a}(R/I(G)) = \beta_{i,a}(R/I(H)) + \beta_{i,a}(R/I(H')).
\]
If \( a_x \neq 0 \), then for any \( i \geq 1 \), \( \beta_{i,a}^R(R/I(H)) = 0 \) as \( x \) does not divide any of the minimal monomial generators of \( I(H) \). If \( a_x = 0 \), then \( [a - e_x]_x = -1 \). Consequently, for any \( i \geq 1 \), we have \( \beta_{i,a-e_x}^R(R/I(H)) = 0 \). Thus only one term in Equation (13) will contribute to \( \beta_{i,a}^R(R/I(G)) \), and hence, \( \beta_{i,a}^R(R/I(G)) \leq d \). \( \Box \)

As a consequence, we obtain upper bounds for the multigraded Betti numbers of several classes of graphs.

**Corollary 4.3.** Let \( U \) be a vertex cover of \( C_n \), \( x \) be a vertex and \( G = x \ast_U C_n \). Then \( \beta_{i,a}^R(R/I(G)) \leq 2 \) for all \( i \geq 1 \) and \( a \in \mathbb{N}^{n+1} \). In particular, if \( G = W_n \), the wheel graph on \( n + 1 \) vertices or \( G = J_{2,n} \), the Jahangir graph, then \( \beta_{i,a}^R(R/I(G)) \leq 2 \) for all \( i \geq 1 \) and \( a \in \mathbb{N}^{V(G)} \).

**Proof.** Let \( i \geq 1 \) and \( a \in \mathbb{N}^{n+1} \). It follows from [6] Proposition 3.5 that \( \beta_{i,a}^R(R/I(C_n)) \leq 2 \). If \( |a| > i + 1 \), then, by Theorem 4.2 \( \beta_{i,a}^R(R/I(G)) \leq 2 \). Now, assume that \( |a| = i + 1 \). Then, following the notation and proof of Theorem 4.2 we get

\[
\beta_{i,a}^R(R/I(G)) = \begin{cases} 
\beta_{i,a}^R(R/I(C_n)) & \text{if } a_x = 0, \\
\beta_{i-1,a-e_x}^R(R/I(C_n)) + \beta_{i,a}^R(R/I(H')) & \text{if } a_x \neq 0.
\end{cases}
\]

Since \( H' \) is a tree, by [7] Theorem 2.2.2, \( \beta_{i,a}^R(R/I(H')) \in \{0,1\} \). Now the first part of the Corollary follows from [6] Proposition 3.5. The second part follows by observing that \( W_n = x \ast C_n \) and \( J_{2,n} = x \ast U C_{2n} \), where \( U = \{x_{2k-1} : 1 \leq k \leq n\} \). \( \Box \)

**Corollary 4.4.** Let \( G = F_{m,n} \) be the fan graph on \( n + m \) vertices, \( n \geq 2, m \geq 1 \). Then \( \beta_{i,a}^R(R/I(G)) \leq 2 \) for all \( i \geq 1 \) and \( a \in \mathbb{N}^{n+m} \).

**Proof.** We do this by induction on \( m \). If \( m = 1 \), then the assertion is immediate from Theorem 4.2 since \( F_{1,n} = x_1 \ast P_n \). Assume that \( \beta_{i,a}^R(R/I(F_{m-1,n})) \leq 2 \) for all \( i \geq 1 \) and \( a \in \mathbb{N}^{n+m} \). If \( |a| > i + 1 \), then it follows from Theorem 4.2 that \( \beta_{i,a}^R(R/I(F_{m,n})) \leq 2 \). If \( |a| = i + 1 \), then following the proof of Theorem 4.2 we get

\[
\beta_{i,a}^R(R/I(F_{m,n})) = \begin{cases} 
\beta_{i,a}^R(R/I(F_{m-1,n})) & \text{if } a_{x_m} = 0, \\
\beta_{i-1,a-e_{x_m}}^R(R/I(F_{m-1,n})) + \beta_{i,a}^R(R/I(H')) & \text{if } a_{x_m} \neq 0.
\end{cases}
\]

If \( a_{x_i} \neq 0 \) for some \( 1 \leq i \leq m - 1 \), then \( \beta_{i,a}^R(R/I(H')) = 0 \) so that the assertion holds true. If \( a_{x_i} = 0 \) for all \( 1 \leq i \leq m - 1 \) and \( a_{x_m} \neq 0 \), then \( \beta_{i,a}^R(R/I(H')) \leq 1 \). In this case \( [a-e_{x_m}]_{x_j} = 0 \) for all \( 1 \leq j \leq m \) so that \( \beta_{i,a-e_{x_m}}^R(R/I(F_{m-1,n})) = \beta_{i,a-e_{x_m}}^R(R/I(P_n)) \leq 1 \). This completes the proof. \( \Box \)

**Corollary 4.5.** Let \( G \) be a complete \( k \)-partite graph on \( n \)-vertices. Then, \( \beta_{i,a}^R(R/I(G)) \leq k-1 \), for all \( i \geq 1 \) and \( a \in \mathbb{N}^n \).

**Proof.** Let \( i \geq 1 \) and \( a \in \mathbb{N}^n \). If \( |a| > i + 1 \), then \( \beta_{i,a}^R(R/I(G)) = 0 \). Assume that \( |a| = i + 1 \). We prove this by induction on \( k \). Assume that \( k = 2 \). Then, \( G \) is a complete bipartite graph. Therefore, \( V(G) = V_1 \sqcup V_2 \) such that \( G[V_1] \) is a trivial graph for \( i = 1, 2 \) and \( G = G[V_1] \ast G[V_2] \). We now proceed by induction on \( |V_2| \). If \( |V_2| = 1 \), then \( G \) is a tree and thus, by [7] Theorem 2.2.2, the result holds. Let \( V_2 = \{x_1, \ldots, x_r\} \) with \( r > 1 \). Set \( V_2' = \{x_1, \ldots, x_r-1\} \). Note that \( G' = G[V_1] \ast G[V_2'] \) is a complete bipartite graph. By
induction, $\beta^R_{i,a}(R/I(G')) \leq 1$, for all $i$ and $a$. Since $V_1$ is a vertex cover of $G'$, it follows from the proof of Theorem 4.2 that

$$\beta_{i,a}\left(\frac{R}{I(G')}\right) = \begin{cases} 
\beta^R_{i,a}\left(\frac{R}{I(G')}\right) & \text{if } a_{x_r} = 0, \\
\beta^R_{i-1,a-e_{x_r}}\left(\frac{R}{I(G')}\right) + \beta^R_{i,a}\left(\frac{R}{I(H)}\right) & \text{if } a_{x_r} \neq 0.
\end{cases}$$

If $a_{x_j} \neq 0$ for some $1 \leq j \leq r - 1$, then $\beta_{i,a}(R/I(H')) = 0$. If $a_{x_j} = 0$ for $1 \leq j \leq r - 1$ and $a_{x_r} \neq 0$, then $\beta^R_{i-1,a-e_{x_r}}(R/I(G')) = \beta^R_{i-1,a-e_{x_r}}(R/I(G[V_1])) = 0$. Hence the assertion follows for the case $k = 2$.

Assume that $k > 2$. Let $V(G) = V_1 \sqcup \cdots \sqcup V_k$ and that the result holds true for any complete $(k-1)$-partite graph. Now we prove the assertion for $k$ by induction on $|V_k|$. Let $G' = G[V_1 \sqcup \cdots \sqcup V_{k-1}]$. Suppose $V_k = \{x\}$. Then $G = x \ast G'$. Hence the result follows from Theorem 4.2. Let $V_k = \{x_1, \ldots, x_r\}$, $r > 1$. Take $U = V_1 \sqcup \cdots \sqcup V_{k-1}$ and observe that $G = x_r \ast U G[U \sqcup \{x_1, \ldots, x_{r-1}\}]$. Now arguments similar to the proof of the case $k = 2$, we get the required assertion. \hfill $\Box$

We conclude the article with a question on multigraded Betti numbers:

**Question 4.6.** Given upper bounds for multigraded Betti numbers of $I(G)$ and $I(H)$, can one obtain an upper bound for multigraded Betti numbers of $I(G \ast H)$?

**References**

[1] Abed Abedelfatah. Some results on the subadditivity condition of syzygies. arXiv e-prints, page arXiv:2001.01136, January 2020.
[2] Abed Abedelfatah and Eran Nevo. On vanishing patterns in $j$-strands of edge ideals. *J. Algebraic Combin.*, 46(2):287–295, 2017.
[3] Luchezar L. Avramov, Aldo Conca, and Srikanth B. Iyengar. Subadditivity of syzygies of Koszul algebras. *Math. Ann.*, 361(1-2):511–534, 2015.
[4] Selvi Beyarslan, Huy Tài Hà, and Trân Nam Trung. Regularity of powers of forests and cycles. *J. Algebraic Combin.*, 42(4):1077–1095, 2015.
[5] Mina Bigdeli and Jürgen Herzog. Betti diagrams with special shape. In *Homological and computational methods in commutative algebra*, volume 20 of *Springer INdAM Ser.*, pages 33–52. Springer, Cham, 2017.
[6] Adam Boocher, Alessio D’Alì, Eloísa Grifo, Jonathon Montaño, and Alessio Sammartano. Edge ideals and DG algebra resolutions. *Matematiche (Catania)*, 70(1):215–238, 2015.
[7] Rachelle R. Bouchat. Free resolutions of some edge ideals of simple graphs. *J. Commut. Algebra*, 2(1):1–35, 2010.
[8] Rachelle R. Bouchat, Huy Tài Hà, and Augustine O’Keefe. Path ideals of rooted trees and their graded Betti numbers. *J. Combin. Theory Ser. A*, 118(8):2411–2425, 2011.
[9] Hailong Dao, Craig Huneke, and Jay Schweig. Bounds on the regularity and projective dimension of ideals associated to graphs. *J. Algebraic Combin.*, 38(1):37–55, 2013.
[10] G. A. Dirac. On rigid circuit graphs. *Abh. Math. Sem. Univ. Hamburg*, 25:71–76, 1961.
[11] David Eisenbud, Craig Huneke, and Bernd Ulrich. The regularity of Tor and graded Betti numbers. *Amer. J. Math.*, 128(3):573–605, 2006.
[12] Sara Faridi. Lattice complements and the subadditivity of syzygies of simplicial forests. *J. Commut. Algebra*, 11(4):535–546, 2019.
[13] Sara Faridi and Mayada Shahada. Breaking up Simplicial Homology and Subadditivity of Syzygies. *arXiv e-prints*, page arXiv:2003.00270, February 2020.
[14] Daniel R. Grayson and Michael E.Stillman. Macaulay2, a software system for research in algebraic geometry. Available at [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/).
[15] Huy Tài Hà and Adam Van Tuyl. Splittable ideals and the resolutions of monomial ideals. *J. Algebra*, 309(1):405–425, 2007.

[16] Jürgen Herzog and Hema Srinivasan. On the subadditivity problem for maximal shifts in free resolutions. In *Commutative algebra and noncommutative algebraic geometry. Vol. II*, volume 68 of *Math. Sci. Res. Inst. Publ.*, pages 245–249. Cambridge Univ. Press, New York, 2015.

[17] Melvin Hochster. Cohen-Macaulay rings, combinatorics, and simplicial complexes. In *Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975)*, pages 171–223. Lecture Notes in Pure and Appl. Math., Vol. 26, 1977.

[18] Sean Jacques. *Betti Numbers of Graph Ideals*. PhD thesis, -, October 2004.

[19] Mordechai Katzman. Characteristic-independence of Betti numbers of graph ideals. *J. Combin. Theory Ser. A*, 113(3):435–454, 2006.

[20] Arvind Kumar. Binomial edge ideals and bounds for their regularity. *Journal of Algebraic Combinatorics (To Appear)*, 2020.

[21] Jason McCullough. A polynomial bound on the regularity of an ideal in terms of half of the syzygies. *Math. Res. Lett.*, 19(3):555–565, 2012.

[22] Jason McCullough and Alexandra Seceleanu. Quadratic Gorenstein algebras with many surprising properties. *arXiv e-prints*, page arXiv:2004.10237, April 2020.

[23] Amir Mousivand. Algebraic properties of product of graphs. *Comm. Algebra*, 40(11):4177–4194, 2012.

[24] Leila Sharifan. Minimal free resolution of monomial ideals by iterated mapping cone. *Bull. Iranian Math. Soc.*, 44(4):1007–1024, 2018.

[25] Ali Akbar Yazdan Pour. Candidates for nonzero Betti numbers of monomial ideals. *Comm. Algebra*, 45(4):1483–1492, 2017.

E-mail address: jayanav@iitm.ac.in

E-mail address: arvkumar11@gmail.com

Department of Mathematics, 5th floor, New academic block, Indian Institute of Technology Madras, Chennai, INDIA - 600036