A convergent numerical scheme to a parabolic equation with a nonlocal boundary condition

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Abstract

In this paper, a numerical scheme for a nonlinear McKendrick–von Foerster equation with diffusion in age (MV-D) with the Dirichlet boundary condition is proposed. The main idea to derive the scheme is to use the discretization based on the method of characteristics to the convection part, and the finite difference method to the rest of the terms. The nonlocal terms are dealt with the quadrature methods. As a result, an implicit scheme is obtained for the boundary value problem under consideration. The consistency, and the convergence of the proposed numerical scheme is established. Moreover, numerical simulations are presented to validate the theoretical results.

Keywords Age-structured population model, Convergence, Interpolation.

AMS Subject Classification 35A01, 35A02, 35K20, 65N12, 92D25.

1 Introduction

The structured population models distinguish individuals from one another based on characteristics such as size, age, and so on. Moreover the birth, and the death rates also depend on these characteristics. Structured models are broadly classified as age structured models [12, 13, 17, 20, 35, 36, 40, 42], gender structured models [11, 23, 44], size structured models [10, 18, 19, 34], and maturity structured models [14, 15, 37, 43].

Many size structured models describing the behavior of the cell populations have been introduced and studied [11] and the references therein). There are four important phases in the mammalian cell cycle, they are $G_1$ - phase, $S$ - phase, $G_2$ - phase and $M$ - phase. In $S$-phase and $M$-phase there will

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be DNA replication and in $M$-phase cell division occurs. The authors in [8] considered a size structured model and used DNA content as a measure of the genetic term ‘cell size’, because the vital transitions (i.e., from $G_1$-phase to $S$-phase, from $M$-phase to cell division, etc.) are accompanied in DNA content. These changes in the DNA content can be measured by flow cytometry, but sometimes the cells are not evenly illuminated when they pass through the flow cytometer. Hence, in order to account for these changes in the cell cycle model, the authors introduced the diffusive term.

In this paper, we consider the McKendrick–von Foerster equation with diffusion in the age variable with the Dirichlet boundary condition. Let $u(t,x)$ denote the population density with age $x$ at time $t$. Let $d$ and $B$ denote the mortality rate and the fertility rate, respectively. Let $\psi$ and $S$ denote the competition weight and weighted population, respectively. Let $g$ denote a non-negative $C^1$ function defined on $[0, \infty)$ that satisfies some assumptions which will be described later.

With these notations, we consider the following nonlinear MV-D in $Q := (0, T) \times (0, a_1),$

\[
\begin{cases}
    u_t(t,x) + u_x(t,x) + d(x,S(t))u(t,x) = \epsilon u_{xx}(t,x), \quad (t,x) \in Q, \\
    u(t,0) = g\left(\int_0^{a_1} B(x)u(t,x)dx\right), \quad t \in (0,T), \\
    u(t,a_1) = 0, \quad t \in (0,T), \\
    u(0,x) = u_0(x), \quad x \in (0,a_1),
\end{cases}
\]  

(1)

where

\[
S(t) = \int_0^{a_1} \psi(x)u(t,x)dx.
\]

(2)

We assume that $u \in W^{3,\infty}(Q)$ and the compatibility criterion, i.e.,

\[
u_0(0) = g\left(\int_0^{a_1} B(x)u_0(x)dx\right).
\]

Different variants of MV-D equations attracted the attention of many mathematicians. In [1], the authors considered the linear MV-D equation with Robin boundary data and studied the existence, uniqueness, and positivity of the solution. In addition, they proved the exponential decay of the solution to MV-D equation for large time to the steady state. The existence, uniqueness, and the long time behavior of the solution for a particular type of non-linear MV-D were proved in [38]. The authors in [25, 39] considered the nonlinear MV-D with Robin boundary data and proved the existence and uniqueness of its solution using the fixed point arguments. Moreover, the authors studied the asymptotic behavior of the solution towards its steady
state using the notion of the General Relative Entropy. There is a lot of literature available in the numerical study of McKendrick–von Foerster equation (MV) with Dirichlet boundary data and there are very few papers available related to MV-D with Robin boundary data ([26]). In [31, 32], the authors developed the concept of stability to certain class of nonlinear problems. They proved that for a smooth discretization, the stability was equivalent to the stability of its linearization around the theoretical solution. This concept of stability works for MV type equation with Dirichlet boundary data. In [2, 6, 7, 33], the authors considered age structured model with Dirichlet boundary data and developed a new numerical method to approximate the solution to the equation that they considered. The analysis involved theory of discretizations based on the notion of stability thresholds. In [21, 22], Iannelli et al. presented finite difference methods to find the numerical solution to the linear Lotka-McKendrick equation. Furthermore, they showed that the scheme was convergent in the maximum norm and also discussed the stability of the scheme. In [27], Kim et al. used the collocation method along the characteristics to approximate the solution to the nonlinear MV equation with Dirichlet boundary data. Their scheme indeed converges and it is of the fourth order accuracy in the supremum norm. In [30], the authors presented a finite difference numerical scheme to a class of nonlinear nonlocal equations with Robin type boundary conditions. By the method of reduction of order, the authors proved that the numerical scheme is second order accurate.

In [28], the authors considered an age structured alcoholism model. They investigated the global behavior of the solution of the model using the basic reproduction number, and analysed the nature of the nontrivial equilibrium when the basic reproduction number is greater than one. The authors in [9, 29] studied a nonlinear epidemic model and obtained the existence result for the solutions. Using the basic reproduction number they provided a necessary and sufficient condition for global asymptotic stability of the free-equilibrium. In [10], Bentout et al. presented a mathematical model that predicts the spread of the pandemic COVID-19. By considering the age as a factor of the progress and severity of the disease, the authors proposed an age structured model. Using this model they predicted the size of the epidemic in the USA, the UAE, and Algeria.

There are many more mathematical models which are age structured. For instance, the model of stress erythropoiesis (see [4] and references therein), the model which describes cell dwarfism [3]. Here, the authors proposed a numerical method to obtain an approximation to its solution and established the convergence of the numerical method. The authors in [5] considered a size structured cell population model. They proposed a second order numerical method to approximate its solution and showed the convergence of the proposed scheme.

In this paper, we present a numerical method to find an approximate so-
olution to equation (1)–(2). The present numerical method is based on the method of characteristics (MOC) (see [24]). It is easy to compute the solution to nonlinear boundary value problem (1)–(2) using the method that we propose because there is no need to solve any nonlinear equation at the discrete level. On the other hand, we take the central difference approximation for the diffusion term at the n-th stage so that we can have the convergence of the scheme.

This paper is organised as follows. In Section 2, we propose a numerical scheme to equations (1)–(2) and prove that the numerical solution obtained from the proposed scheme converges to the solution to the MV-D equation. Numerical results are presented in Section 3.

2 The finite difference scheme

Let $h$ and $\Delta t$ denote the uniform step size of age variable and time variable, respectively. Moreover, we denote

$$
x_j = jh, \ t^n = n\Delta t, \ u^n_j = u(t^n, x_j), \ j = 1, \ldots, M, \ n = 1, \ldots, N, 
$$

and

$$
\bar{x}_j = x_j - \Delta t, \ \bar{u}_n^j = u(t^n, \bar{x}_j), \ j = 0, 1, \ldots, M \text{ and } n = 0, 1, \ldots, N.
$$

Observe that (1)–(2) reduces to a hyperbolic equation when $\epsilon = 0$. Thus we use the method of characteristics to discretize the convection part (namely, $u_t + u_x$). The standard central difference approximation is used to deal with the diffusion part (namely $u_{xx}$). We discretize the hyperbolic part using the backward finite difference, i.e.,

$$
u(t, x) + u_x(t, x) \sim \frac{1}{\Delta t} (u(t, x) - u(t - \Delta t, x - \Delta t)).
$$

Let $U^n_i$ denote the approximation of $u(t^n, x_i)$ at every grid $(t^n, x_j)$. Assume that $q_j$’s are the Newton-Cote numbers such that the approximation of the integral is

$$
\int_{a}^{b} B(x) u(t, x) dx \sim \sum_{j=1}^{M} hq_j B(x_j) U^n_j,
$$

which is of the order $h^3$. With this notation, we now discretize (1)–(2) to
get

\[
\begin{split}
U^n_j &= U^{n-1}_j - \Delta t d(x_j, S^{n-1})U^{n-1}_j + \frac{h^2}{2} (U^n_{j+1} - 2U^n_j + U^n_{j-1}), \\
\text{for } i = 1, 2, \ldots, M - 1, j = 1, 2, \ldots, N, \\
U^n_0 &= g \left( \sum_{j=1}^M h q_j B(x_j) U^{n-1}_j \right), \quad n = 1, \ldots, N, \\
U^n_n &= 0, \quad n = 1, \ldots, N, \\
U^n_j &= u_0(x_j), \quad j = 1, \ldots, M, \\
S^n &= \sum_{i=1}^M h q_j \psi_j U^n_j,
\end{split}
\]

(3)

where \(\bar{U}^{n-1}_j\) is the evaluation of the linear interpolation value taken between \(U^{n-1}_j\) and \(U^{n-1}_{j-1}\) at \(\bar{x}\).

We now present a consistency result which holds whenever the mortality rate satisfies the Lipschitz condition given in the following lemma.

**Lemma 2.1.** Assume that there exists \(L > 0\) such that

\[
|d(x, S_1) - d(x, S_2)| \leq L|S_1 - S_2|, \quad S_1, S_2 \geq 0, \quad x \in (0, a^\dagger).
\]

(4)

Then following local truncation results hold:

1. \(u_t(t^n, x_j) + u_x(t^n, x_j) = \frac{u^n_j - \bar{u}^{n-1}_j}{\Delta t} + O(\Delta t),\)

2. \(d(x, S(t^n)) = d(x, \sum_{i=1}^M h q_j \psi_j u^n_j) + O(h^3),\)

3. \(u_{xx}(t^n, x_j) = \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{h^2} + O(h^2).\)

**Proof.** Let \(u\) be the solution to (1). Let \(\tau := \tau(x)\) be the characteristic direction associated with the operator \(\partial_t + \partial_x\). Then we have

\[
\frac{\partial u}{\partial \tau} = \frac{1}{\sqrt{2}} \frac{\partial u}{\partial t} + \frac{1}{\sqrt{2}} \frac{\partial u}{\partial x}.
\]

We approximate the characteristic derivative as

\[
\sqrt{2} \frac{\partial u(t^n, x_j)}{\partial \tau} = \sqrt{2} \frac{u(t^n, x_j) - u(t^{n-1}, \bar{x}_j)}{\sqrt{(x_j - \bar{x}_j)^2 + (t^n - t^{n-1})^2}} + O(\Delta t) \approx \frac{u^n_j - \bar{u}^{n-1}_j}{\Delta t} + O(\Delta t).
\]
In order to prove (ii), we use that \( d \) satisfies Lipschitz condition (4). Since \( q_j \)'s are chosen such that
\[
|S(t^n) - h \sum_{j=1}^{M} q_j \psi_j u_{j}^n| = O(h^3),
\]
it follows that
\[
|d(x_j, S(t^n)) - d(x_j, \sum_{i=1}^{M} h q_j \psi_j u_{j}^n)| = O(h^3).
\]
Note that (iii) is straightforward from the formal Taylor series expansion of \( u \) about \((t^n, x_j)\).

We now state and prove the main result of this paper, i.e., the convergence of the numerical solution \( U^n_j \) to the solution \( u \) to (1)–(2) at the grid points.

**Theorem 2.1.** Assume the hypothesis of Lemma 2.1. Moreover assume that
\[
\|g'\|\|B\|_\infty a_{\alpha} < 1, \quad \frac{\Delta t}{h^2} \leq \frac{1}{2}.
\]
Fix \( T > 0 \) and let \( N \in \mathbb{N}, \Delta t > 0 \) be such that \( N \Delta t = T \). Then the solution to numerical scheme (3) converges to the solution to (1)–(2) as \( h \to 0, \Delta t \to 0 \).

Moreover, there exists a positive constant \( \tilde{C} \), independent of \( h, \Delta t \), such that
\[
\max_{0 \leq j \leq M, 0 \leq n \leq N} |u^n_j - U_j| \leq \tilde{C}(h + \Delta t).
\]

**Proof.** In view of Lemma 2.1, we get the consistency of the scheme as follows:
\[
\begin{cases}
\frac{u^n_j - u^{n-1}_j}{\Delta t} + d(x_j, S(t^n))u^{n-1}_j = \frac{e_j^n}{h^2}(u^n_{j+1} - 2u^n_j + u^n_{j-1}) + e^n_j, \\
\quad j = 1, 2, \ldots, M - 1, \ n = 1, 2, \ldots, N, \\
u^0_0 = g\left(\sum_{j=1}^{M} h q_j B(x_j) u^{n-1}_j\right) + \bar{e}^n_j, \ n = 1, \ldots, N, \\
u^0_n = 0, \ n = 1, \ldots, N, \\
u^n_0 = u_0(x_j), \ j = 1, \ldots, M, \\
S(t^n) = \sum_{i=1}^{M} h q_j \psi(x_j) u^n_j + \bar{e}_j, \ n = 1, \ldots, N,
\end{cases}
\]
where \( e_j^n = O(h + \Delta t), \ \bar{e}_j^n = O(h^3) \) and \( \bar{e}_j = O(h^3) \).
Let \( \rho^n_j = u^n_j - U^n_j \), \( \bar{\rho}^n_j = \bar{u}^n_j - \bar{U}^n_j \) and \( S^n = \sum_{i=1}^{M} h q_j \psi(x_j) U^n_j \). From (3)–(6), it follows that

\[
\frac{\rho^n_j - \bar{\rho}^{n-1}_j}{\Delta t} + d(x_j, S(t^n-1)) u^n_j - d(x_j, S^{n-1}) U^n_j \leq \frac{\epsilon}{h^2} (\rho^n_{j+1} - 2\rho^n_j + \rho^n_{j-1}) + \epsilon^n_j,
\]

\( j = 1, 2, \ldots, M - 1, \ n = 1, 2, \ldots, N, \)

\[
\rho^n_0 = g \left( \sum_{j=1}^{M} h q_j B(x_j) u^n_{j-1} \right) - g \left( \sum_{j=1}^{M} h q_j B(x_j) U^n_{j-1} \right) + \bar{\epsilon}^n_1, \quad n = 1, \ldots, N,
\]

\[
\rho^n_M = 0, \ n = 1, \ldots, N,
\]

\[
\rho^0_j = 0, \ j = 1, \ldots, M.
\]

We first estimate

\[
|S(t^n) - S^n| \leq \left| \int_0^{a^n_1} \psi(x) u(t^n, x) dx - \sum_{i=1}^{M} h q_j \psi(x_j) U^n_j \right|
\leq \left| \int_0^{a^n_1} \psi(x) u(t^n, x) dx - \sum_{i=1}^{M} h q_j \psi(x_j) u^n_j \right| + \left| \sum_{i=1}^{M} h q_j \psi(x_j) \rho^n_j \right|
\leq O(h^3) + \|\psi\|_{\infty} a_1 \max_{1 \leq j \leq M} |\rho^n_j|, \ n \in \mathbb{N}.
\]

Furthermore, using the above estimate, we obtain

\[
|d(x_j, S(t^n-1)) u^n_{j-1} - d(x_j, S^{n-1}) U^n_{j-1}| \leq |d(x_j, S^n_{j-1}) \rho^n_{j-1}| + |d(x_j, S(t^n-1)) - d(x_j, S^{n-1})||u^n_{j-1}|
\leq (|d|_{\infty} + L\|u\|_{\infty} \|\psi\|_{\infty} a_1) \max_{1 \leq j \leq M} |\rho^n_{j-1}| + O(h^3). \quad (8)
\]

We now estimate the boundary term in (7). In order to do that, we consider

\[
|\rho^n_j| = |g \left( \sum_{j=1}^{M} h q_j B(x_j) u^n_{j-1} \right) - g \left( \sum_{j=1}^{M} h q_j B(x_j) U^n_{j-1} \right) + \bar{\epsilon}^n_j|
\leq |g'(\xi_j)| \sum_{j=1}^{M} h q_j B(x_j) \rho^n_{j-1} + O(h^3)
\leq \|g'\|_{\infty} B \|\psi\|_{\infty} a_1 \max_{1 \leq j \leq M} |\rho^n_{j-1}| + O(h^3). \quad (9)
\]
Let \( I \) and \( I_1 \) denote the identity operator and the linear interpolation operator, respectively. We observe that
\[
\bar{\rho}^{n-1}_j = I_1(\rho^{n-1})(\bar{x}_j) + (I - I_1)(u^{n-1})(\bar{x}_j),
\]
or
\[
|\bar{\rho}^{n-1}_j| \leq \max_{0 \leq j \leq M} |\rho^{n-1}_j| + |(I - I_1)(u^{n-1})(\bar{x}_j)|.
\]
In view of the Peano kernel theorem, we have
\[
\max_{1 \leq j \leq M} |\rho^{n-1}_j| \leq \max_{0 \leq j \leq M} |\rho^{n-1}_j| + \|u^{n-1}\|_{2,\infty} h \Delta t. \tag{10}
\]
From the first equation of (7) and estimates (8)–(10), it follows that
\[
\max_{1 \leq j \leq M} |\rho^n_j| \leq \max_{0 \leq j \leq M} |\rho^{n-1}_j| + C \Delta t \max_{0 \leq j \leq M} |\rho^{n-1}_j| + O(h + \Delta t) \Delta t
\leq (1 + C \Delta t) \max_{0 \leq j \leq M} |\rho^{n-1}_j| + O(h + \Delta t) \Delta t. \tag{11}
\]
From (9) and (11), we get
\[
|\rho^n_0| \leq M \left\{ h^3 + (h + \Delta t) \Delta t \left[ 1 + (1 + C \Delta t) + \cdots + (1 + C \Delta t)^{N-2} \right] \right\}
\leq M \left\{ h^3 + (h + \Delta t) \Delta t Ne^{CT} \right\}
\leq M \left\{ h^3 + (h + \Delta t) Te^{CT} \right\},
\]
and
\[
\max_{1 \leq j \leq M} |\rho^n_j| \leq M(h + \Delta t) \Delta t \left[ 1 + (1 + C \Delta t) + \cdots + (1 + C \Delta t)^{N-1} \right]
\leq M(h + \Delta t) \Delta t Ne^{CT}
\leq M(h + \Delta t) Te^{CT}.
\]
Hence we find that
\[
\max_{0 \leq j \leq M} |\rho^n_j| = O(h + \Delta t) \text{ as } h \to 0, \Delta t \to 0. \tag{12}
\]
This completes the proof of the promised result. \(\square\)

### 3 Numerical simulations

In this section, we present some examples in which we have performed numerical simulations. In the first two examples, we have taken the linear version of (1)–(2), i.e., \( d(x, S) = d(x) \), \( g(x) = x \), and \( B \) is chosen such that the hypotheses of Theorem 2.1 hold. On the other hand, in the last two examples we have considered nonlinear equations.
Example 3.1. We first assume that $a_\dagger = 1$. Moreover assume that the vital rates $d, B$, and the initial data $u_0$ are given by
\[ d(x, S) = \frac{3e^{-x} - e^{-1}}{e^{-x} - e^{-1}}, \quad B(x) = 1 + \frac{e^{-1}}{1 - 2e^{-1}}, \quad \epsilon = 1.0, \quad u_0(x) = e^{-x} - e^{-1}. \]
In this case, it is easy to verify that the solution to (1)–(2) is given by
\[ u(t, x) = e^{-t}(e^{-x} - e^{-1}), \quad t > 0, \quad x \in (0, 1). \]

For the numerical solution, we have taken $h = 0.0025, \Delta t = 3.125 \times 10^{-6}$. In Figure 1(a), we have shown the analytical solution and the numerical solution when $N = 16000$, and $64000$. The absolute error is shown in Figure 1(b). From these figures, it is clear that the numerical solution is very close to the analytical solution. In Figure 2 we have presented the numerical plots for different values of $h$ and $\Delta t$ at $t = 0.1$ and also compared the same with the corresponding analytical solution. It is evident from Figure 2 that as $h$ and $\Delta t$ tend to zero, the numerical solution converges to the analytical solution.

Example 3.2. In this example also, we take $a_\dagger = 1$. The vital rates $d, B$, and the initial data $u_0$ are given by
\[ d(x, S) = 3 \left( 1 + \frac{1}{1 - x} \right), \quad B(x) = \frac{4}{1 + e^{-2}e^{-x}}, \quad \epsilon = 1.0, \quad u_0(x) = e^{-x}(1 - x). \]
It is straightforward to verify that the solution to (1)–(2) in this example is given by
\[ u(t, x) = e^{-t}e^{-x}(1 - x), \quad t > 0, \quad x \in (0, 1). \]

For the numerical computations, we have taken $h = 0.002, \Delta t = 2.0 \times 10^{-6}$. In Figure 3(a), we plot the analytical solution and the numerical solution when $N = 9375$, and $75000$. The absolute error is presented in Figure 3(b). This example also validates that the numerical method proposed in this article gives an approximate solution which is in a good agreement with the exact solution. In Figure 4 the numerical solutions for different values of $h$ and $\Delta t$ at $t = 0.15$ are shown, Moreover, the corresponding analytical solutions are also presented in the same figure for the comparison. We conclude from Figure 4 that as $h$ and $\Delta t$ go to zero, the numerical solution converges to the analytical solution.

Example 3.3. Here we take the mortality rate, the fertility rate and the initial data to be
\[ d(x, S) = \frac{1 - e^{-a_\dagger} - a_\dagger e^{-a_\dagger}}{S} + \frac{e^{-x} - 2e^{-a_\dagger}}{2(e^{-x} - e^{-a_\dagger})}, \quad B(x) = 1 + \frac{a_\dagger e^{-a_\dagger}}{1 - e^{-a_\dagger} - a_\dagger e^{-a_\dagger}}, \]
\[ u_0(x) = \frac{e^{-x} - e^{-a_\dagger}}{2}, \quad \epsilon = 0.5, \quad g(x) = x. \]
In this example the solution to (1)–(2) is given by

\[ u(t, x) = \frac{1}{1 + e^{-t}(e^{-x} - e^{-a_t})}, \quad t > 0, \quad x \in (0, a_t). \]

In Figure 5(a), we present the analytical solution and the numerical solution when \( N = 20000 \). In Figure 5(b), we have shown the absolute error. For these computations, we have taken \( h = 0.001, \Delta t = 5 \times 10^{-7}, a_t = 2 \). In this example, we observe that \( \|g^r\|_\infty \|B\|_\infty a_t > 1 \) and this type of non-linearity does not satisfy the hypotheses of Theorem 2.1. However, we still notice that the numerical solution obtained using our scheme is indeed a good approximation to the analytical solution. This example suggests that assumption (5) is sufficient but not necessary for the convergence of the numerical scheme presented in this article.

**Example 3.4.** In this example, we assume that the vital rates \( d, B, \) and the initial data \( u_0 \) are given by

\[ d(x, S) = 1 + S, \quad B(x) = \frac{e^{-5x}}{10}, \quad \epsilon = 0.5, \quad g(x) = \sqrt{1 + x}, \quad a_t = 7, \]

\[ u_0(x) = f(x) * \eta_{0.1}(x), \]

where \( \eta_t(x) \) is the standard mollifier with support in \([-0.1, 0.1]\), \( f * \eta_{0.1} \) denote the convolution of \( f \) with \( \eta_{0.1} \), and

\[ f(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 6, \\ (x - 7)^2, & \text{if } 6 \leq x \leq 7. \end{cases} \]

We have chosen \( u_0(.) \) such that it is smooth. The step sizes for the simulations are \( \Delta t = 0.0002, \) and \( h = 0.02 \). The numerical solutions for different values of \( t \) are shown in Figure 6. From Figure 6 we observe that the numerical solution for \( t \geq 3 \) are very close to each other giving us an impression that the solution to (1)–(2) is converging to the corresponding steady state solution.

**4 Conclusions**

In this paper, we have presented an implicit numerical scheme given in equation (3), for a nonlinear McKendrick–von Foerster equation with diffusion in age (MV-D). We have used the discretization based on the method of characteristics to \( u_t + u_x \) and the finite difference method to the rest of the terms. We have also proved that the scheme we have considered is indeed convergent. As the approximation for the transport term is of the first order, the order of the proposed scheme turns out to be \( O(h + \Delta t) \) (see equation (12)). Some numerical experiments are presented to revalidate the main theorem. From these simulations, numerical solutions are in a good agreement with
the corresponding analytical solutions. We have also presented an example where the hypothesis of Theorem [2.1] is not satisfied. Here also the numerical solution obtained using our scheme provides a good approximation to the analytical solution. This shows that assumption (5) in Theorem [2.1] can be weakened.

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References

[1] Abdellaoui, B., Touaoula, T.M.: Decay solution for the renewal equation with diffusion. Nonlinear Differential Equations and Applications 17 (2010), pp. 271 - 288.

[2] Abia, L. M., Angulo, O., Lopez-Marcos, J. C.: Age-structured population dynamics models and their numerical solutions. Ecological Modelling 188 (2005), pp. 112 - 136.

[3] Abia, L. M., Angulo, O., López-Marcos, J. C., López-Marcos, M. A.: Numerical analysis of a cell dwarfism model. Journal of Computational and Applied Mathematics 349 (2019), pp. 82 - 92.

[4] Angulo, O., Crauste, F., López-Marcos, J. C.: Numerical integration of an erythropoiesis model with explicit growth factor dynamics. Journal of Computational and Applied Mathematics 330 (2018), pp. 770 - 782.

[5] Angulo, O., López-Marcos, J. C., López-Marcos, M. A.: A second-order method for the numerical integration of a size-structured cell population model. Abstract and Applied Analysis, Article ID 549168, 8 pages (2015).

[6] Angulo, O., Lopez Marcos, J. C., Lopez Marcos, M. A., Milner, F. A.: A numerical method for nonlinear age-structured population models with finite maximum age. Journal of Mathematical Analysis and Applications 36 (2010), pp. 150 - 160.

[7] Angulo, O., Lopez Marcos, J. C., Milner, F. A.: The application of an age-structured model with unbounded mortality to demography. Mathematical Biosciences 208 (2007), pp. 495 - 520.
[8] Basse, B., Baguley, B.C., Marshall, E.S., Joseph, W.R., VanBrunt, B., Wake, G., Wall, D.J.N.: A mathematical model for analysis of the cell cycle in cell lines derived from human tumors. Journal of Mathematical Biology 47(4) (2003), pp. 295 - 312.

[9] Bentout Soufiane, Tarik Mohammed Touaoula: Global analysis of an infection age model with a class of nonlinear incidence rates. Journal of Mathematical Analysis and Applications 434 (2016), pp. 1211 - 1239.

[10] Soufiane Bentout, Abdessamad Tridane, Salih Djilali, Tarik Mohammed Touaoula: Age-Structured Modeling of COVID-19 Epidemic in the USA, UAE and Algeria. Alexandria Engineering Journal 60 (2021), pp. 401 - 411.

[11] Calsina, A, Ripoll, J.: A general structured model for a sequential hermaphrodite population. Mathematical Biosciences 208(2) (2008), pp. 393 - 418.

[12] Chipot, M.: On the equations of age-dependent population dynamics. Archive for Rational Mechanics and Analysis 82(1) (1983), pp. 13 - 25.

[13] Cushing, J.M.: An introduction to structured population dynamics. SIAM, Philadelphia (1998).

[14] Dyson, J., Bressan, R.V., Webb, G.: A nonlinear age and maturity structured model of population dynamics i. basic theory. Journal of Mathematical Analysis and Applications 242 (2000), pp. 93 - 104.

[15] Dyson, J., Bressan, R.V., Webb, G.: A nonlinear age and maturity structured model of population dynamics ii. chaos. Journal of Mathematical Analysis and Applications 242 (2000), pp. 255 - 270.

[16] Escobedo, M., Laurencot, P., Mischler, S., Perthame, B.: Gelation and mass conservation in coagulation–fragmentation models. Journal of Differential Equations 195(1) (2003), pp. 143 - 174.

[17] Farkas, F.Z.: Stability conditions for the nonlinear McKendrick equations. Applied Mathematics and Computation 156 (2004), pp. 771 - 777.

[18] Farkas, F.Z.: Stability conditions for a nonlinear size structured model. Nonlinear Analysis: Real World Applications 6 (2005), pp. 962 - 969.

[19] Farkas, F.Z., Hagen, T.: Stability and regularity results for a size structured population model. Journal of Mathematical Analysis and Applications 328 (2007), pp. 119 - 13.
[20] Iannelli, M.: Mathematical theory of age-structured population dynamics, Applied Mathematics Monograph C.N.R. 7. Pisa: Giardini editore stampatori (1995).

[21] Iannelli, M., Kim, M.Y., Park, E.J.: Splitting methods for the numerical approximation of some models of age structured population dynamics and epidemiology. Applied Mathematics and Computation 87 (1997), pp. 69 - 93.

[22] Iannelli, M., Milner, F. A.: On the approximation of the Lotka-McKendrick equation with finite life-span. Journal of Computational and Applied Mathematics 136 (2001), pp. 245 - 254.

[23] Iannelli, M., Martcheva, M., Milner, F.A.: Gender structured population modeling: Mathematical Methods, Numerics, and Simulations 31. Frontiers in Applied Mathematics, SIAM, Philadelphia (2005).

[24] Jim Douglas Jr., Thomas F.R.: Numerical methods for Convection-Dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures. SIAM Journal on Numerical Analysis 19 (5) (1982), pp. 871 - 885.

[25] Kakumani B.K., Tumuluri, S. K.: On a nonlinear renewal equation with diffusion. Mathematical Methods in the Applied Sciences 39 (2016), pp. 697 - 708.

[26] Kakumani, B.K., Tumuluri, S.K.: A numerical scheme to the McKendrick-von Foerster equation with diffusion in age. Numerical methods for Partial Differential Equation 34(6) (2018), pp. 2113 - 2128.

[27] Kim, M. Y. , Park, E. J.: An upwind scheme for a nonlinear model in age-structured population dynamics. Computers and Mathematics with Applications 30 (1995), pp. 5 - 17.

[28] Soufiane Bentout, Salih Djilali, Abdenasser Chekroun: Global threshold dynamics of an age structured alcoholism model. International Journal of Biomathematics, DOI: https://doi.org/10.1142/S1793524521500133 (2021).

[29] Soufiane Bentout, Yuming Chen, Salih Djilali: Global Dynamics of an SEIR Model with Two Age Structures and a Nonlinear Incidence. Acta Applicandae Mathematicae, DOI: https://doi.org/10.1007/s10440-020-00369-z (2021).

[30] Liu, J., Sun, Z. Z.: Finite difference method for reaction diffusion equation with nonlocal boundary conditions. Numerical Mathematics: A journal of Chinese Universities 16 (2007), pp. 97 - 111.
[31] Lopez Marcos, J. C., Sanz-Serna, J. M.: A definition of stability for non-linear problems, In: K. Strehmel (ed.), Numerical treatment of differential equations 104, 216 - 226, Teubner-Texte zur Mathematik, Leipzig (1988).

[32] Lopez Marcos, J. C., Sanz-Serna, J. M.: Stability and convergence in numerical analysis, III: Linear investigation of nonlinear stability. IMA Journal of Numerical Analysis 8 (1988), pp. 71 - 84.

[33] Lopez Marcos, J. C.: An upwind scheme for a nonlinear hyperbolic integro-differential equation with integral boundary condition. Computers and Mathematics with Applications 22 (1991), pp. 15 - 28.

[34] Metz, J.A.Z., Diekmann, O.: The dynamics of physiologically structured populations. Springer, Verlag Berlin Heidelberg (1968).

[35] Michel, P.: General relative entropy in a nonlinear mckendrick model. Stochastic Analysis and Partial Differential Equations, American Mathematical Society, Providence, RI, 429 (2007), pp. 205 - 232.

[36] Michel, P., Mischler, S., Perthame, B.: General relative entropy inequality: an illustrations on growth models. Journal de Mathématiques Pures et Appliquées 84(9) (2005), pp. 1235 - 1260.

[37] Mischler, S., Perthame, B., Ryzhik, L.: Stability in a nonlinear population maturation model. Mathematical Models and Methods in Applied Sciences 12(12) (2002), pp. 1751 - 1772.

[38] Michel, P., Touaoula T.M.: Asymptotic behavior for a class of the renewal nonlinear equation with diffusion. Mathematical Methods in the Applied Sciences 36(3) (2012), pp. 323 - 335.

[39] Michel P., Kakumani, B.K.: GRE methods for nonlinear model of evolution equation and limited resource environment. Discrete and Continuous Dynamical Systems - Series B 24 (12) (2019), pp. 6653 - 6673.

[40] Perthame, B.: Transport equations in biology. LN Series Frontiers in Mathematics. Birkhauser (2007).

[41] Perthame, B., Ryzhik, L.: Exponential decay for the fragmentation or cell division equation. Journal of Differential Equations (210) (2005), pp. 155 - 177.

[42] Perthame, B., Tumuluri, S.K.: Nonlinear renewal equations. In: Selected Topics in Cancer Modeling, Modeling and Simulation in Science, Engineering and Technology 65 - 96, Birkhauser Boston, Boston, MA, (2008).
[43] Rubinow, S.I.: A maturity time representation for cell populations. Biophysical Journal 8(10) (1968), pp. 1055 - 1073.

[44] Thieme, H.R.: Mathematics in population biology. Princeton University Press, Princeton, NJ (2003).
Figure 1: (a) The numerical solution and the exact solution to (1)–(2) with $N = 16000$, 64000 with vital rates given in Example 3.1 (b) The absolute difference between $u(N \Delta t, x_i)$ and $U^N_{x_i}$ when $N = 16000$ and 64000.

Figure 2: Numerical solution with different values of $h$, $\Delta t$ and the corresponding analytical solution to (1)–(2) at $t = 0.1$ with vital rates given in Example 3.1.
Figure 3: (a) The numerical solution and the analytical solution to (1)–(2) with $N = 9375, 75000$ with $d, B, g, u_0$ are given in Example 3.2; (b) The absolute difference between $u(N\Delta t, x_i)$ and $U_{x_i}^{N\Delta t}$ with $N = 9375, 75000$.

Figure 4: Numerical solution with different values of $h$, $\Delta t$ and the corresponding analytical solution to (1)–(2) at $t = 0.15$ with $d, B, g, u_0$ given in Example 3.2.
Figure 5: (a) The numerical solution and the analytical solution to (1)–(2) with \( N = 20000 \) with vital rates given in Example 3.3. (b) The absolute difference between \( u(N \Delta t, x_i) \) and \( U_{x_i}^{N \Delta t} \) with \( N = 20000 \).

Figure 6: Numerical solution to (1)–(2) with \( N = 5000 \) with \( d, B, g \) and \( u_0 \) given in Example 3.4.