STRANGE EXPECTATIONS

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Abstract. Let \( \gcd(a, b) = 1 \). J. Olsson and D. Stanton proved that the maximum number of boxes in a simultaneous \((a, b)\)-core is

\[
\max_{\lambda \in \text{core}(a,b)} (\text{size}(\lambda)) = \frac{(a^2 - 1)(b^2 - 1)}{24},
\]

and that this maximum was achieved by a unique core. P. Johnson combined Ehrhart theory with the polynomial method to prove D. Armstrong’s conjecture that the expected number of boxes in a simultaneous \((a, b)\)-core is

\[
\mathbb{E}_{\lambda \in \text{core}(a,b)} (\text{size}(\lambda)) = \frac{(a - 1)(b - 1)(a + b + 1)}{24}.
\]

We extend P. Johnson’s method to compute the variance to be

\[
\mathbb{V}_{\lambda \in \text{core}(a,b)} (\text{size}(\lambda)) = \frac{ab(a - 1)(b - 1)(a + b + 1)(a + b + 2)}{1440}.
\]

By extending the definitions of “simultaneous cores” and “number of boxes” to affine Weyl groups, we give uniform generalizations of all three formulae above to simply-laced affine types. We further explain the appearance of the number 24 using the “strange formula” of H. Freudenthal and H. de Vries.

1. Introduction

1.1. Motivation: Simultaneous Cores. An \( a \)-core is an integer partition with no hook-length divisible by \( a \). As a first example, observe that the \( 2 \)-cores are exactly those partitions of staircase shape. According to the notes in G. James and A. Kerber [JK81], cores were originally developed by T. Nakayama in his study of the modular representation theory of the symmetric group [Nak40]. For \( a \) a partition of \( k \), we write \( \text{size}(\lambda) := k \). Let \( \text{core}(a) \) be the set of all \( a \)-cores, and define

\[
\text{core}_k(a) := \{ \lambda \in \text{core}(a) : \text{size}(\lambda) = k \}.
\]

The following identity relating integer partitions and \( a \)-cores is a fun exercise using the abacus:

**Theorem 1.1** (Generating function for size on \( \text{core}(a) \): [JKS90]).

\[
\sum_{k=0}^{\infty} \frac{\text{core}_k(a)}{q^k} = \prod_{i=1}^{\infty} \frac{(1 - q^{ai})^a}{1 - q^i}.
\]

An \((a, b)\)-core is a partition that is both an \( a \)-core and a \( b \)-core. We denote the set of \((a, b)\)-cores by \( \text{core}(a, b) \). When \( a \) and \( b \) are coprime, J. Anderson proved the surprising fact that there are only finitely many \((a, b)\)-cores.

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\[1\text{The relationship arises as follows. The irreducible representations of } S_n \text{ over a field of characteristic zero are parametrized by integer partitions of } n. \text{ T. Nakayama conjectured that the } p\text{-blocks for } S_n \text{ are in bijection with } p\text{-cores—more specifically, that the } p\text{-block corresponding to a } p\text{-core } \lambda \text{ contains exactly those representations whose indexing partitions have core } \lambda \text{ [Nak40]. This conjecture was proven by R. Brauer and G. Robinson [BR47].}\]
Theorem 1.2 (Number of simultaneous \((a,b)\)-cores; J. Anderson [And02]). For \(\gcd(a,b) = 1\),

\[
|\text{core}(a,b)| = \frac{1}{a+b} \binom{a+b}{b}.
\]

In part due to their connection with rational Dyck paths (and hence diagonal harmonics and the zeta map) [ST14, CDH15, BGLX14] and rational Catalan combinatorics [ARW13, GMV14, ALW14], simultaneous cores have recently attracted attention. Furthermore, the study of simultaneous cores has now transcended these original motivations, and they have become combinatorial objects worthy of study in their own right [Nat08, AKS09, Fay11, AL14, YZZ14, Nat14, CHW14, Agg14b, Fay14, Xio14, Agg15, Fay15]. In this direction, there are two main results on the statistic size.

Theorem 1.3 (Maximum size of an \((a,b)\)-core; J. Olsson and D. Stanton [OS07]). For \(\gcd(a,b) = 1\),

\[
\max_{\lambda \in \text{core}(a,b)} \left( \text{size}(\lambda) \right) = \frac{(a^2-1)(b^2-1)}{24}.
\]

This maximum is attained by a unique \((a,b)\)-core.

A stronger statement is actually true: J. Vandehey proved that the diagram of this unique \((a,b)\)-core maximizing size contains the diagrams of all other \((a,b)\)-cores [Van08, Fay11, OS07, Remark 4.11].

D. Armstrong conjectured the following attractive formula [Arm15a, AHJ14], which was proven for \(b = a+1\) by R. Stanley and F. Zanello [SZ13]; for \(b = ma+1\) by A. Aggarwal [Agg14a]; and in full generality by P. Johnson [Joh15].

Theorem 1.4 (Expected size of an \((a,b)\)-core; P. Johnson [Joh15]). For \(\gcd(a,b) = 1\),

\[
\mathbb{E}_{\lambda \in \text{core}(a,b)} \left( \text{size}(\lambda) \right) = \frac{(a-1)(b-1)(a+b+1)}{24}.
\]

Our first new result is to extend P. Johnson’s technique\(^2\) to compute the variance of \(\text{core}(a,b)\).

Theorem 1.5 (Variance of size on \((a,b)\)-cores). For \(\gcd(a,b) = 1\),

\[
\mathbb{V}_{\lambda \in \text{core}(a,b)} \left( \text{size}(\lambda) \right) = \frac{ab(a-1)(b-1)(a+b)(a+b+1)}{1440}.
\]

With more effort, we also compute the third moment, which was conjectured by D. Armstrong in 2013 [Arm15b].

Theorem 1.6 (Third moment of size on \((a,b)\)-cores). For \(\gcd(a,b) = 1\), let \(\mu := \mathbb{E}_{\lambda \in \text{core}(a,b)} \left( \text{size}(\lambda) \right)\). Then

\[
\sum_{\lambda \in \text{core}(a,b)} \left( \text{size}(\lambda) - \mu \right)^3 = \frac{ab(a-1)(b-1)(a+b)(a+b+1) \left( 2a^2b - 3a^2 + 2ab^2 - 3ab - 3b^2 - 3 \right)}{60480}.
\]

D. Armstrong also conjectured a formula for the fourth cumulant on the basis of extensive computations. We will not state or prove his conjecture, but the interested reader might enjoy Section 8.1.

\(^2\)This is of course the well-known Paulynomial method.
1.2. Simply-Laced Generalizations. The main purpose of this paper is to give generalizations of Theorems 1.1 to 1.5 for all simply-laced types. This simply-laced requirement arises from a simplification that only happens in those types, and is explained in Section 6.4.

To this end, we fix the following notation, which is fully reviewed in Section 2. Let \( \Phi \) be an irreducible crystallographic root system of rank \( n \) with ambient space \( V \) and Weyl group \( W \). Let \( \tilde{\Phi} \) be the set of affine roots and denote the affine Weyl group by \( \tilde{W} \). Choose a set of simple roots \( \Delta \) for \( \Phi \) and let \( \Phi^+ \) be the corresponding set of positive roots. Say \( \Phi \) has exponents \( e_1 \leq e_2 \leq \cdots \leq e_n \), Coxeter number \( h := e_n + 1 \), and dual Coxeter number \( g \). Theorems 1.1 to 1.5 will be recovered in this notation by specializing to type \( A_{a-1} \), in which case \( n = a - 1 \), \( h = a \) and \( \tilde{W} = \tilde{S}_a \).

A useful analogue of a core for \( \tilde{W} \) turns out to be a point of the coroot lattice \( \tilde{Q} \), which we emphasize with the notation \( \text{core}(\tilde{W}) := \tilde{Q} \).

In Section 3, we recall how this definition recovers \( a \)-cores when \( \tilde{W} = \tilde{S}_a \).

In order to generalize Theorems 1.1 and 1.3 to 1.5, we require a notion of the statistic \( \text{size} \), defined combinatorially for \( \tilde{S}_a \) using the Ferrers diagram of a core. For the purposes of the introduction, we pull this out of a hat (but see Definition 6.1 and Example 6.2): for any point \( x \in V \)—and in particular for any point in the coroot lattice \( \tilde{Q} \)—let

\[
\text{size}(x) := \frac{g}{2}||x||^2 - \langle x, \rho \rangle,
\]

where \( \rho := \frac{1}{2} \sum_{\alpha \in \Phi} \alpha \) is half the sum of all positive roots.

The statistic \( \text{size} \) on \( \text{core}(\tilde{S}_a) \) recovers the number of boxes in the corresponding core in \( \text{core}(a) \) (Definition 6.6 and Proposition 6.4).

Define \( \text{core}_k(\tilde{W}) := \{ \lambda \in \text{core}(\tilde{W}) : \text{size}(\lambda) = k \} \).

By specializing his character formula at a primitive \( h \)th root of unity, I. G. MacDonald has uniformly generalized Theorem 1.1 to all simply-laced types.

**Theorem 1.7** ([Mac71, Theorem 8.16]). For \( \tilde{W} \) the affine Weyl group of a simply-laced irreducible crystallographic root system \( \Phi \) with Weyl group \( W \), let \( f(q) \) be the characteristic polynomial of a Coxeter element in \( W \) (in the reflection representation). Then

\[
\sum_{k=0}^{\infty} |\text{core}_k(\tilde{W})| q^k = \prod_{i=1}^{\infty} \left( f(q^i)(1 - q^{hi})^n \right).
\]

Having generalized the notion of core and the statistic \( \text{size} \), we still require a definition of simultaneous cores. In Section 4 for any positive integer \( b \) that is relatively prime to \( h \), we define the Sommers region

\[
\mathcal{S}_\Phi(b) := \{ x \in V : \langle x, \alpha \rangle \geq -t \text{ for all } \alpha \in \Phi_r \text{ and } \langle x, \alpha \rangle \leq t+1 \text{ for all } \alpha \in \Phi_{h-r} \}.
\]

This is the region in \( V \) bounded by all the affine hyperplanes corresponding to affine roots of height \( b \).

M. Haiman has uniformly proven (for all affine Weyl groups) the following generalization of Theorem 1.2 [Hai94], which we state in terms of \( \mathcal{S}_\Phi(b) \) using a result of

\[\text{3The last equality in [Mac71] Theorem 8.16 appears to have a small typo.}\]
E. Sommers [Som05, Theorem 5.7]. We remark that R. Suter has also (presumably independently) observed essentially the same formula type-by-type [Sut98].

**Theorem 1.8** (Number of \(\tilde{W}, b\)-cores; M. Haiman [Hai94]).

For \(\gcd(h, b) = 1\),

\[
\left| \text{core}(\tilde{W}, b) \right| = \frac{1}{|\tilde{W}|} \prod_{i=1}^{n} (b + e_i).
\]

We now state generalizations of Theorems 1.3 to 1.5.

**Theorem 1.9** (Maximum size of a \(\tilde{W}, b\)-core).

For \(\tilde{W}\) a simply-laced affine Weyl group with \(\gcd(h, b) = 1\),

\[
\max_{\lambda \in \text{core}(\tilde{W}, b)} \text{(size(\(\lambda\))} = \frac{n(b^2 - 1)(h + 1)}{24}.
\]

This maximum is attained by a unique \(\lambda \in \text{core}(\tilde{W}, b)\).

In Conjecture 6.14, we conjecture an analogue of J. Vandehey’s result for \((a, b)\)-cores, using the inversion sets of the dominant affine elements corresponding to \((\tilde{W}, b)\)-cores.

**Theorem 1.10** (Expected size of a \(\tilde{W}, b\)-core).

For \(\tilde{W}\) a simply-laced affine Weyl group with \(\gcd(h, b) = 1\),

\[
\mathbb{E}_{\lambda \in \text{core}(\tilde{W}, b)} \text{(size(\(\lambda\))} = \frac{n(b - 1)(h + b + 1)}{24}.
\]

The appearence of the number 24 in Theorems 1.9 and 1.10 is explained by Equation (2) and Definition 6.6, where we relate the statistic size to an easily-computed quadratic form \(Q\) whose value at 0 is \(-\frac{\langle \rho, \rho \rangle}{2g}\). By the “strange formula” of H. Freudenthal and H. de Vries (Theorem 2.6),

\[
-\frac{\langle \rho, \rho \rangle}{2g} = \frac{n(h + 1)}{24},
\]

which accounts for the constant term in both theorems.

We also compute a uniform formula for the variance \(\mathbb{V}\) of the statistic size over \(\text{core}(\tilde{W}, b)\).

**Theorem 1.11** (Variance of size on \((\tilde{W}, b)\)-cores).

For \(\tilde{W}\) a simply-laced affine Weyl group with \(\gcd(h, b) = 1\),

\[
\mathbb{V}_{\lambda \in \text{core}(\tilde{W}, b)} \text{(size(\(\lambda\))} = \frac{nhb(b - 1)(h + b)(h + b + 1)}{1440}.
\]

**Remark 1.12.** For affine types outside of \(\tilde{A}_n\), we did not compute any moments beyond the second (but see Section 8.1). Our justification is that we verified that there is no possible assignment of \(a \mapsto \{n+1, h\}\) in Theorem 1.6—where each factor of \(a\) is assigned independently—that results in a uniform product formula simultaneously valid for all simply-laced affine Weyl groups. This leaves open the possibility that there are “hidden” factors of powers of \(\frac{n+1}{h}\), though we suspect that this is not the case.

We stress that although the statements of Theorems 1.9 to 1.11 are uniform for simply-laced types, many of our proofs (especially the computations in Section 7) are very much type-dependent. It would be desirable to have uniform proofs.
1.3. Proof Strategy and Summary. We outline here the two technical difficulties (both already present in the type $A_n$ case studied in [Joh15]), the explanation and resolution of which will occupy much of Sections 2.2, 4, and 6. Given a vector space $V$ and an $n$-dimensional polytope $P$ in $V$ whose vertices are elements of a lattice $L$—that is, $P$ is an integer polytope with respect to $L$—Ehrhart theory tells us that the number of lattice points of $L$ inside the $b$-th dilation of $P$ is given by a polynomial $P^L_p(b)$ of degree $n$ in $b$. Ehrhart theory extends to Euler-Maclaurin theory (see Section 7.1), which says that given a polynomial $p$ on $V$ of degree $m$, the sum

$$
\mathcal{P}^L_p(b) := \sum_{x \in bP \cap L} p(x).
$$

over these lattice points gives a polynomial $\mathcal{P}^L_p(b)$ of degree $n + m$ in $b$.

To prove Theorems 1.6, 1.10 and 1.11 we wish to use Ehrhart theory combined with the polynomial method to determine

$$
\sum_{\lambda \in S_b(b) \cap Q} \text{size}(\lambda) \text{ for } i = 1, 2, 3.
$$

The trouble is that Ehrhart theory manifestly does not apply: $S_b(b)$ is neither the dilation of a polytope, nor are its vertices in the coroot lattice $Q$ for general values of $b$.

The first obstacle is that $S_b(b)$ is not the dilation of a polytope—as the residue class of $b$ mod $h$ changes, so does the orientation of $S_b(b)$. We therefore first translate the study of $S_b(b)$ to the study of $bA$—the $b$-fold dilation of the fundamental alcove—which remains in a fixed orientation as $b$ varies:

- Theorem 4.2 uniformly proves that $S_b(b)$ may be mapped bijectively to $bA$ via an explicit rigid motion $\tilde{w}_b$ (filling a gap in the literature); and
- Using the rigid motion $\tilde{w}_b$, we translate the statistic size on core($\tilde{W}, b$) onto a statistic size on $bA \cap Q$ in Corollary 6.8.

The second obstacle is that $S_b(b)$—and therefore also $bA$—is not an integer polytope with respect to the coroot lattice $Q$. Following P. Johnson, Ehrhart theory extends to rational polytopes, at the cost of trading polynomiality for quasipolynomiality (with an explicit period). It is somewhat easier to translate the study of the coweights $bA \cap Q$ to the study of the coweights $bA \cap \Lambda$:

- We recall in Proposition 7.2 that the polytope $bA_0$ is a rational polytope in the coweight lattice $\Lambda$;
- The coroot lattice $Q$ is a lattice of index $f \in \mathbb{N}$ (the index of connection) inside $\Lambda$. We define the group $\Omega = \Lambda/\tilde{Q}$ in Section 2.4, and we prove in Theorem 2.5 that each $b\Omega$-orbit of $bA \cap \Lambda$ contains exactly one point of $bA \cap Q$; and
- We show in Lemma 6.11 that the action of $b\Omega$ preserves size.

The remainder of this paper is structured as follows. In Section 2 we review the basic notions of finite and affine Weyl groups. In Section 3 we review how $a$-cores fit into the framework of affine Weyl groups as the special case $\tilde{W} = \tilde{S}_\Lambda$. In Section 4 we generalize $a$-cores to $\tilde{W}$ using the Sommers region $S_b(b)$, and we relate $S_b(b)$ and $bA$. We also recall M. Haiman’s Theorem 1.8 and prove Theorem 1.9. In Section 6 we generalize the statistic size to $S_b(b)$ for all affine Weyl groups, and we study how it transforms to a statistic on $bA$.

In Section 7 for $b \coprime h$, we compute the relevant residue classes of the Ehrhart quasipolynomial $A^\text{size}_b(b)$ to conclude Theorems 1.3, 1.6, 1.10 and 1.11. In Section 8 we state some open problems and conjectures regarding higher moments, non-simply-laced types, and combinatorial models.
2. Affine Weyl Groups

In this section, we introduce finite and affine root systems (Sections 2.1 and 2.3) and associated data. We also define their associated hyperplane arrangements and Weyl groups (Section 2.2). Finally, we define the abelian group \( \Omega \), which allows us to relate the coroot and coweight lattices in Theorem 2.5.

2.1. Root Systems. Let \( \Phi \) be an irreducible crystallographic root system of rank \( n \) with ambient space \( V \). Define the root lattice \( Q \) of \( \Phi \) as the lattice in \( V \) generated by \( \Phi \). Let \( \Phi^+ \) be a system of positive roots for it and let \( \Delta = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \) be the corresponding system of simple roots. Then \( \Phi \) is the disjoint union of \( \Phi^+ \) and \( -\Phi^+ \), and \( \Delta \) is a basis for \( V \).

For \( \alpha \in \Phi \), we may write \( \alpha \) in the basis of simple roots as \( \alpha = \sum_{i=1}^{n} a_i \alpha_i \), where the coefficients \( a_i \) are either all nonnegative or all nonpositive. We define the height of \( \alpha \) as the sum of the coefficients: \( \text{ht}(\alpha) := \sum_{i=1}^{n} a_i \). Notice that \( \text{ht}(\alpha) > 0 \) if and only if \( \alpha \in \Phi^+ \) and \( \text{ht}(\alpha) = 0 \) if and only if \( \alpha \in \Delta \). There is a unique root

\[
\tilde{\alpha} = \sum_{i=1}^{n} c_i \alpha_i \in \Phi
\]

of maximal height, which we call the highest root of \( \Phi \). We choose to normalize the inner product \( \langle \cdot, \cdot \rangle \) on \( V \) in such a way that \( ||\tilde{\alpha}||^2 = 2 \). We define the Coxeter number of \( \Phi \) as \( h := 1 + \text{ht}(\tilde{\alpha}) = 1 + \sum_{i=1}^{n} c_i \). Let \( \rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \).

For a root \( \alpha \in \Phi \), define its coroot as \( \check{\alpha} := \frac{2\alpha}{\langle \alpha, \alpha \rangle} \). Define the dual root system of \( \Phi \) as \( \Phi^\vee := \{ \check{\alpha} : \alpha \in \Phi \} \). It is itself an irreducible crystallographic root system. We say that \( \Phi \) is simply-laced if all roots \( \alpha \in \Phi \) satisfy \( ||\alpha||^2 = 2 \). So in this case \( \check{\alpha} = \alpha \) for all \( \alpha \in \Phi \) and thus \( \Phi = \Phi^\vee \).

Define the coroot lattice \( \hat{Q} \) of \( \Phi \) as the lattice in \( V \) generated by \( \Phi^\vee \). Let \( \hat{\rho} := \frac{1}{2} \sum_{\alpha \in \Phi^+} \check{\alpha} \).

Finally, let \( (\check{\omega}_1, \check{\omega}_2, \ldots, \check{\omega}_n) \) be the basis that is dual to the basis \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \) of \( V \) consisting of the simple roots, so that \( \langle \check{\omega}_i, \alpha_j \rangle = \delta_{i,j} \). Then \( \check{\omega}_1, \check{\omega}_2, \ldots, \check{\omega}_n \) are the fundamental coweights. They are a basis of the coweight lattice

\[
\check{\Lambda} := \{ x \in V : \langle x, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi \}
\]

of \( \Phi \). We also have

\[
\hat{\rho} = \sum_{i=1}^{n} \check{\omega}_i,
\]

so that \( \langle \hat{\rho}, \alpha \rangle = 1 \) for all \( \alpha \in \Delta \) and thus \( \langle \hat{\rho}, \alpha \rangle = \text{ht}(\alpha) \) for all \( \alpha \in \Phi \).

We can write the highest root \( \hat{\alpha} \) (which is its own coroot) in terms of the coroots corresponding to the simple roots:

\[
\hat{\alpha} = \sum_{i=1}^{n} d_i \check{\omega}_i.
\]

Then we define the dual Coxeter number of \( \Phi \) as \( g := 1 + \sum_{i=1}^{n} d_i \).

2.2. Weyl Groups. For \( \alpha \in \Phi \) and \( k \in \mathbb{Z} \), define the affine hyperplane

\[
H^k_\alpha := \{ x \in V : \langle x, \alpha \rangle = k \}
\]

and let

\[
s^k_\alpha : x \mapsto x - \frac{2\langle x, \alpha \rangle - k}{\langle \alpha, \alpha \rangle} \langle \alpha, x \rangle
\]

be the reflection through \( H^k_\alpha \). We write \( H_\alpha \) for the hyperplane \( H^0_\alpha \) and \( s_\alpha \) for the reflection \( s^0_\alpha \).
Let $W$ be the group generated by $\{s_\alpha : \alpha \in \Phi\}$, called the Weyl group of $\Phi$. It acts on $\Phi$ and is minimally generated by the set $S := \{s_\alpha, s_{\alpha_2}, \ldots, s_{\alpha_n}\}$ of simple reflections of $\Phi$. A Coxeter element is a product of the simple reflections in any order, each appearing exactly once.

The Coxeter arrangement of $\Phi$ is the central hyperplane arrangement in $V$ given by all the hyperplanes $H_\alpha$ for $\alpha \in \Phi$. The complement $V \setminus \{H_\alpha\}_{\alpha \in \Phi}$ falls apart into connected components, which we call chambers. The Weyl group $W$ acts simply transitively on the set of chambers, so we define the dominant chamber

$$C := \{x \in V : \langle x, \alpha \rangle > 0 \text{ for all } \alpha \in \Delta\}$$

and write any chamber as $wC$ for a unique $w \in W$.

Let $\tilde{W}$ be the group generated by $\{s_\alpha^k : \alpha \in \Phi, k \in \mathbb{Z}\}$, called the affine Weyl group of $\Phi$. It is minimally generated by the set $\tilde{S} := S \cup \{s_\alpha^1\}$ of affine simple reflections of $\Phi$. So we may write any $\tilde{w} \in \tilde{W}$ as a word in the generators on $\tilde{S}$. The minimal length of such a word is called the length $l(\tilde{w})$ of $\tilde{w}$. It is not hard to see that $\tilde{W}$ acts $\tilde{Q}$. To any $y \in V$, there is an associated translation

$$t_y : V \to V$$

$$x \mapsto x + y.$$ 

If we identify $\tilde{Q}$ with the corresponding group of translations acting on the affine space $V$, then we may write $\tilde{W} = W \rtimes \tilde{Q}$ as a semidirect product.

The affine Coxeter arrangement of $\Phi$ is the affine hyperplane arrangement in $V$ given by all the affine hyperplanes $H_\alpha^k$ for $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Its complement falls apart into connected components, which we call alcoves. The affine Weyl group $\tilde{W}$ acts simply transitively on the set of alcoves, so we define the (closed) fundamental alcove as

$$A := \{x \in V : \langle x, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta \text{ and } \langle x, \tilde{\alpha} \rangle \leq 1\}$$

and write any alcove as $\tilde{w}A^\circ$ for a unique $\tilde{w} \in \tilde{W}$, where $A^\circ$ is the interior of $A$. We call $\tilde{w}$ dominant if $\tilde{w}A^\circ \subseteq C$.

2.3. Affine Root Systems. We may also understand $\tilde{W}$ in terms of its action on the set of affine roots $\tilde{\Phi}$ of $\Phi$. To do this, let $\delta$ be a formal variable and define $\tilde{V} := V \oplus \mathbb{R}\delta$. Define the set of affine roots as

$$\tilde{\Phi} := \{\alpha + k\delta : \alpha \in \Phi \text{ and } k \in \mathbb{Z}\}.$$ 

If $\tilde{w} \in \tilde{W}$, write it as $\tilde{w} = wt_\mu$ for unique $w \in W$ and $\mu \in \tilde{Q}$ and define

$$\tilde{w}(\alpha + k\delta) = w(\alpha) + (k - \langle \mu, \alpha \rangle)\delta.$$
This defines an action of $\tilde{W}$ on $\tilde{\Phi}$. It imitates the action of $\tilde{W}$ on the half-spaces of $V$ defined by the hyperplanes of the affine Coxeter arrangement. To see this, define the half-space

$$\mathcal{H}_\alpha^k := \{ x \in V : (x, \alpha) > -k \}.$$ 

Then for $\tilde{w} \in \tilde{W}$ we have $\tilde{w}(\mathcal{H}_\alpha^k) = \mathcal{H}_{\tilde{\alpha}}^k$ if and only if $\tilde{w}(\alpha + k\delta) = \beta + l\delta$. Define the set of positive affine roots as

$$\tilde{\Phi}^+ := \{ \alpha + k\delta : \alpha \in \Phi^+ \text{ and } k \geq 0 \} \cup \{ \alpha + k\delta : \alpha \in -\Phi^+ \text{ and } k > 0 \},$$

the set of affine roots corresponding to half-spaces that contain $A^\circ$. So $\tilde{\Phi}$ is the disjoint union of $\tilde{\Phi}^+$ and $-\tilde{\Phi}^+$.

Define the set of simple affine roots as

$$\tilde{\Delta} := \Delta \cup \{ -\tilde{\alpha} + \delta \},$$

the set of affine roots corresponding to half-spaces that contain $A^\circ$ and share one of its defining inequalities. We will also write $a_0 := -\tilde{\alpha} + \delta$.

For $\tilde{w} \in \tilde{W}$, we say that $\alpha + k\delta \in \tilde{\Phi}^+$ is an inversion of $\tilde{w}$ if $\tilde{w}^{-1}(\alpha + k\delta) \in -\tilde{\Phi}^+$, and we write

$$\text{inv}(\tilde{w}) := \{ \alpha + k\delta \in \tilde{\Phi}^+ : \tilde{w}^{-1}(\alpha + k\delta) \in -\tilde{\Phi}^+ \}$$

as the set of inversions of $\tilde{w}$.

**Theorem 2.1.** The positive affine root $\alpha + k\delta \in \tilde{\Phi}^+$ is an inversion of $\tilde{w}$ if and only if the hyperplane $H_{\alpha}^{-k}$ separates $\tilde{w}A^\circ$ from $A^\circ$.

**Proof.** If $\alpha + k\delta \in \tilde{\Phi}^+$ is an inversion of $\tilde{w}$, then $A^\circ \subseteq H_{\alpha}^k$ and $A^\circ \not\subseteq \tilde{w}^{-1}(H_{\alpha}^k)$. Thus $\tilde{w}A^\circ \not\subseteq H_{\alpha}^k$ and therefore $H_{\alpha}^{-k}$ separates $\tilde{w}A^\circ$ from $A^\circ$.

Conversely, if $\alpha + k\delta \in \tilde{\Phi}^+$ and $H_{\alpha}^{-k}$ separates $\tilde{w}A^\circ$ from $A^\circ$, then $A^\circ \subseteq H_{\alpha}^k$ and $\tilde{w}A^\circ \not\subseteq H_{\alpha}^k$. Therefore $A^\circ \not\subseteq \tilde{w}^{-1}(H_{\alpha}^k)$ and thus $\tilde{w}^{-1}(\alpha + k\delta) \in -\tilde{\Phi}^+$. So $\alpha + k\delta$ is an inversion of $\tilde{w}$.

Define the height of an affine root $\alpha + k\delta$ as $\text{ht}(\alpha + k\delta) = \text{ht}(\alpha) + kh$. So $\text{ht}(\alpha + k\delta) > 0$ if and only if $\alpha + k\delta \in \tilde{\Phi}^+$ and $\text{ht}(\alpha + k\delta) = 1$ if and only if $\alpha + k\delta \in \tilde{\Delta}$.

For an integer $l$ with $-h < l < h$, let $\Phi_l$ be the set of roots in $\Phi$ of height $l$. Similarly, for any positive integer $b$, let $\tilde{\Phi}_b$ be the set of affine roots in $\tilde{\Phi}$ of height $b$. If we write $b = th + r$ with $t, r \in \mathbb{Z}$ and $0 \leq r < h$, then

$$\tilde{\Phi}_b = \{ \alpha + t\delta : \alpha \in \Phi_r \} \cup \{ \alpha + (t + 1)\delta : \alpha \in \Phi_{r-h} \}.$$ 

### 2.4. Symmetry of the Affine Diagram

Define $\tilde{W}_{\text{ex}} := W \ltimes \tilde{\Lambda}$ to be the extended affine Weyl group of $\Phi$. Let

$$\Omega := \{ \tilde{w} \in \tilde{W}_{\text{ex}} : \tilde{w}A = A \}.$$ 

Then $\Omega \cong \tilde{W}_{\text{ex}}/\tilde{W} \cong \tilde{\Lambda}/Q$ is an abelian group of order $f$, the index of connection of $\Phi$. It can be thought of as a group of symmetries of the fundamental alcove $A$, or—dually—as a group of symmetries of the affine Dynkin diagram. The structure of $\Omega$ in simply-laced types is given in Figure 3.

**Proposition 2.2** (B. Kostant [Kos76, Lemma 3.4.1]). If $M$ is the Cartan matrix and $c$ is a Coxeter element of $W$, then

$$|\det(M)| = \det(1 - c) = |\Omega| = f.$$
The extended affine Weyl group acts on $\mathbf{\check{\Lambda}}$ in affine simply-laced types $[IM65]$, where $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. Compare with the symmetries of Figure 1.

We next relate $\mathbf{\check{\Lambda}}$ to a subset of the finite Weyl group $W$. To do this, we first need a lemma due to Kostant.

**Lemma 2.3** ([LP12] Lemma 3.6). Every alcove $\mathbf{\check{w}} A$ contains exactly one point in $\frac{1}{h} \mathbf{\check{\Lambda}}$. For the fundamental alcove $A$, this point is $\frac{\mathbf{\check{e}}}{h}$.

**Proof.** We have $\check{\rho} = \sum_{i=1}^{n} \check{\omega}_i \in \mathbf{\check{\Lambda}}$, so $\frac{\check{\rho}}{h} \in \frac{1}{h} \mathbf{\check{\Lambda}}$. We also have that $\langle \frac{\check{\rho}}{h}, \check{\alpha} \rangle = \check{h}t(\check{\alpha})/h \in (0, 1)$ for all $\check{\alpha} \in \Phi^+$. Thus $\frac{\check{\rho}}{h}$ lies in $A^+$—in fact, it is the only element in $A^+ \cap \frac{1}{h} \mathbf{\check{\Lambda}}$.

Indeed, suppose that $\nu \in A^+ \cap \frac{1}{h} \mathbf{\check{\Lambda}}$. Then for all $\check{\alpha}_i \in \Delta$ we have $\langle \nu, \check{\alpha}_i \rangle = a_i/h$ for some $a_i \in \mathbb{Z}_+$. But we also have $\langle \nu, \check{\alpha} \rangle = (\sum_{i=1}^{n} a_i c_i)/h < 1$, so $a_i = 1$ for all $i \in [n]$ and thus $\nu = \frac{\check{\rho}}{h}$.

Since $W$ acts on $\frac{1}{h} \mathbf{\check{\Lambda}}$, there is exactly one element of $\frac{1}{h} \mathbf{\check{\Lambda}}$ in any alcove $\mathbf{\check{w}} A^\circ$. □

**Theorem 2.4.** The point $\frac{\mathbf{\check{e}}}{h}$ is a fixed point of the action of $\Omega$. Any element $\check{w}$ of $\Omega$ can be written as $\check{w} = t_{\frac{\mathbf{\check{e}}}{h}} w_{\mathbf{\check{e}}}$ for a unique $w \in W$.

**Proof.** The extended affine Weyl group acts on $\frac{1}{h} \mathbf{\check{\Lambda}}$. Thus if $\check{w} \in \mathbf{\check{\Lambda}}$, then $\check{w}(\frac{\mathbf{\check{e}}}{h}) \in (\frac{1}{h} \mathbf{\check{\Lambda}}) \cap A$, so by Lemma 2.3 we have $\check{w}(\frac{\mathbf{\check{e}}}{h}) = \frac{\mathbf{\check{e}}}{h}$. Writing $\check{w} = t_{\mu} w$ for $\mu \in \mathbf{\check{\Lambda}}$ and $w \in W$, we have

$$w \left( \frac{\check{\rho}}{h} \right) + \mu = \frac{\check{\rho}}{h}.$$

Therefore,

$$\check{w} = t_{\mu} w = t_{\frac{\mathbf{\check{e}}}{h} - w(\frac{\mathbf{\check{e}}}{h})} w = t_{\frac{\mathbf{\check{e}}}{h} - \check{w}(\frac{\mathbf{\check{e}}}{h})} w = t_{\frac{\mathbf{\check{e}}}{h}} w_{\mathbf{\check{e}}},$$

as required. □

We now show that in each $\Omega$-orbit of $\mathbf{\check{\Lambda}}$, there is exactly one point of $\check{Q}$. Starting from $\Omega$, let

$$b\Omega = \{ t_{b\mu} w : t_{\mu} w \in \Omega \}$$

be the analogous group of automorphisms of $bA$.

**Theorem 2.5.** Let $b$ be a positive integer relatively prime to the index of connection $f$. Then the group $b\Omega$ acts freely on $\mathbf{\check{\Lambda}}$ and every $b\Omega$-orbit contains a unique point in $\check{Q}$.

**Proof.** The set of vertices of the fundamental alcove $\Gamma \subset \{0\} \cup \{ \check{\omega}_i : \langle \check{\omega}_i, \check{\alpha} \rangle = 1 \}$ is a set of representatives of $\mathbf{\check{\Lambda}}/\check{Q}$. Furthermore, we have that $\Omega = \{ t_{\mu} w_{\mu} : \mu \in \Gamma \}$, where $w_{\mu} \in W$ for all $\mu \in \Gamma$, and $\Omega$ acts simply transitively on $\Gamma$ $[IM65]$ Proposition 1.18).

Since $b$ is coprime to $f = [\mathbf{\check{\Lambda}} : \check{Q}]$, the map

$$\mathbf{\check{\Lambda}}/\check{Q} \to \mathbf{\check{\Lambda}}/\check{Q}$$

$$x + \check{Q} \mapsto bx + \check{Q}$$

is invertible. The set $b\Gamma$ is therefore also a set of representatives of $\mathbf{\check{\Lambda}}/\check{Q}$. The group $b\Omega$ acts simply transitively on it. We conclude that for any $\mu \in \mathbf{\check{\Lambda}}$ the orbit $(b\Omega)\mu$ is a set of representatives of $\mathbf{\check{\Lambda}}/\check{Q}$. In particular, it is free and contains exactly one point in $\check{Q}$. □
One can check case-by-case that any prime that divides $f$ also divides the Coxeter number $h$. Thus, the conclusion of the theorem also holds when $b$ is relatively prime to $h$.

2.5. Strange Identity. To conclude this section, we recall the “strange formula” of H. Freudenthal and H. de Vries.

**Theorem 2.6.** Let $\Phi$ be an irreducible crystallographic root system of rank $n$. Let $h$ be the Coxeter number of $\Phi$, $g$ the dual Coxeter number and recall that $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. Then
\[
\|\rho\|^2 = \frac{n(h + 1)}{24}.
\]

The appearance of the number 24 in the denominator of the “strange formula” will explain its appearance in Theorems 1.9 and 1.10.

3. Cores

In this section, we recall the bijection between $a$-cores and the minimal-length coset representatives for the parabolic quotient $\tilde{S}_a / S_a$ (Theorem 3.1). Using the isomorphism between $\tilde{W}/W$ and $\hat{Q}$ for $W = S_a$, we interpret and generalize cores as points in the coroot lattice $\hat{Q}$.

3.1. The Affine Symmetric Group and Cores. The **affine symmetric group** has presentation
\[
\tilde{S}_a := \langle s_0, s_1, \ldots, s_{a-1} : (s_is_{i+1})^3 = e, (s_is_j)^2 = e \text{ if } |i - j| > 1 \rangle,
\]
where indices will always be taken modulo $a$. The elements $\tilde{w}\tilde{S}_a$ such that $\tilde{w}^{-1} A^0 \subseteq C$ are the minimal-length right coset representatives for the parabolic quotient $\tilde{S}_a / S_a$. By abuse of notation, we will associate elements of $\tilde{S}_a / S_a$ with their minimal-length right coset representatives.

There is a bijection between $\tilde{S}_a / S_a$ and $a$-cores, given as follows. Label the $(i,j)$th box of the Ferrers diagram of an $a$-core $\lambda$ by its content $(j - i) \mod a$. We define an action $\tilde{S}_a$ on the set of $a$-cores by defining how the simple reflections $s_i$ act. Given an $a$-core $\lambda$, we define $s_i\lambda$ (for $0 \leq i \leq a - 1$) to be the unique $a$-core that differs from $\lambda$ only by boxes with content $i$. The partial order on core$(a)$ is given by letting $\lambda$ cover $\mu$ if and only if size$(\lambda) > size(\mu)$ and $\lambda = s_i\mu$ for some $i$.

**Theorem 3.1 ([Las01]).** There is a poset isomorphism core$_a$ between the weak order on the parabolic quotient $\tilde{S}_a / S_a$ and the poset on core$(a)$ defined above.

Thus, $a$-cores are identified with elements of $\tilde{S}_a / S_a$. **Theorem 3.1** is illustrated in Figure 3 in the case $a = 3$.

3.2. Affine Weyl Groups and Cores. There are two fundamentally different ways to think of $\tilde{W}$. The first, mentioned in Section 2.2, is as
\[
\tilde{W} := W \rtimes \hat{Q},
\]
which we think of as tiling $V$ using bounded copies of $W$ centered at each point of the coroot lattice. The second way is as
\[
\tilde{W} := (\tilde{W}/W) \rtimes W,
\]
which may be visualized as replicating a copy of the parabolic quotient $(\tilde{W}/W)$ in each region of the Coxeter arrangement of $W$. These two constructions are related as follows.
Figure 3. The weak order on the minimal-length representatives $\tilde{w} \in \tilde{S}_3/\tilde{S}_3$ displayed as the dominant alcoves $\tilde{w}^{-1}A^o$ and the poset of 3-cores. The five simultaneous $(3, 4)$-cores are the empty core and the four cores shaded gray.

Proposition 3.2. There is a canonical bijection
\[ \text{crt} : (\tilde{W}/W) \to \tilde{Q}, \]
\[ w \mapsto w(0). \]

Therefore, the coroot points in $\tilde{S}_a$ are in bijection with the set of $a$-cores. This suggests that the set of coroot points is the correct generalization of cores to any affine Weyl group $\tilde{W}$. It is natural to write $\text{core}(\tilde{W}) := \tilde{Q}$.

4. Two Simplices

In this section, we recall the definitions of two simplices associated to $\tilde{W}$ (Section 4), and show that they are equivalent up to an explicit rigid transformation (Theorem 4.2).

4.1. Dilations of the Fundamental Alcove. We write
\[ bA := \{ x \in V : \langle x, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta \text{ and } \langle x, \tilde{\alpha} \rangle \leq b \} \]
for the $b$-fold dilation of the fundamental alcove, defined for any $b \in \mathbb{R}_{\geq 0}$. This region is bounded by the hyperplanes
\[ \{ H_\alpha : \alpha \in \Delta \} \cup \{ H_{\alpha_0, b} \}. \]
Its volume is $b^n$ times that of the fundamental alcove $A$, so it contains $b^n$ alcoves.

4.2. Sommers Regions. The second simplex, $\mathcal{S}_\Phi(b)$, is defined only for $b$ relatively prime to $h$. In this case write $b = th + r$ with $t, r \in \mathbb{Z}_{\geq 0}$ and $0 < r < h$. Define the Sommers region $\mathcal{S}_\Phi(b)$ as
\[ \mathcal{S}_\Phi(b) := \{ x \in V : \langle x, \alpha \rangle \geq -t \text{ for all } \alpha \in \Phi_r \text{ and } \langle x, \alpha \rangle \leq t+1 \text{ for all } \alpha \in \Phi_{h-r} \}. \]
The significance of the Sommers region is as follows. Define $\tilde{W}^b$ as the set of $\tilde{w} \in \tilde{W}$ that have no inversions of height $b$. So $\tilde{W}^b = \{ \tilde{w} \in \tilde{W} : \tilde{w}^{-1}(\tilde{\Phi}_b) \subseteq \tilde{\Phi}^+ \}$. By Theorem 4.2, we have that $\tilde{w} \in \tilde{W}^b$ if and only if none of the affine hyperplanes in

$$\left\{ H_\alpha^{-t} : \alpha \in \Phi_\tau \right\} \cup \left\{ H_\alpha^{t-1} : \alpha \in \Phi_{\tau-h} \right\} = \{ H_\alpha^{-t} : \alpha \in \Phi_\tau \} \cup \{ H_\alpha^{t+1} : \alpha \in \Phi_{h+\tau} \}$$

separate $\tilde{w}A^\circ$ from $A^\circ$. So $\tilde{w} \in \tilde{W}^b$ if and only if $\tilde{w}A^\circ \subseteq S_b(b)$.

4.3. From $S_b(b)$ to $bA$. It turns out that $bA$ and $S_b(b)$ are equivalent up to a rigid transformation, which may be realized as an element $\tilde{w}_b \in \tilde{W}$.

**Theorem 4.1.** For $b$ relatively prime to $h$, there exists a unique element $\tilde{w}_b = t_\mu w \in \tilde{W}$ with

$$b_\mu = \tilde{w}_b \left( \frac{\rho}{h} \right).$$

**Proof.** For all $\alpha \in \Phi^+$, we have that

$$\left\langle b_\mu, \alpha \right\rangle = b \frac{\text{ht}(\alpha)}{h} \not\in \mathbb{Z},$$

since $b$ is relatively prime to $h$ and $h$ does not divide $\text{ht}(\alpha)$. Thus $b_\mu$ lies on no hyperplane of the affine Coxeter arrangement, so it is contained in some alcove $\tilde{w}_bA^\circ$. Since $b_\mu \in \frac{1}{h}A$ we have that $b_\mu = \tilde{w}_b(A_\mu)$ by Lemma 2.3.

We were unable to find the following result—which explicitly identifies the rigid transformation that sends $S_b(b)$ to $bA$ as an element of $\tilde{W}$—in the literature, although it is probably well-known to the experts.

**Theorem 4.2.** The affine Weyl group element $\tilde{w}_b = t_\mu w$ maps $S_b(b)$ bijectively to $bA$.

**Proof.** We calculate that

$$\frac{\text{ht}(\alpha)}{h} = \left\langle b_\mu, \alpha \right\rangle = \left\langle w \left( \frac{\rho}{h} \right), w(\alpha) \right\rangle = \left\langle b_\mu - \mu, w(\alpha) \right\rangle = b_\mu \frac{\text{ht}(w(\alpha))}{h} - \left\langle \mu, w(\alpha) \right\rangle.$$

Thus $\text{ht}(\alpha) = b_\mu \text{ht}(w(\alpha)) - h(\mu, w(\alpha))$. Again write $b = th + r$ with $t, r \in \mathbb{Z}$ and $0 < r < h$. So reducing modulo $h$ we get $\text{ht}(\alpha) \equiv r \text{ht}(w(\alpha)) \mod h$. Thus $\text{ht}(\alpha) \equiv r \mod h$ if and only if $\text{ht}(w(\alpha)) \equiv 1 \mod h$. So

$$w(\Phi_\tau \cup \Phi_{\tau-h}) = \Phi_1 \cup \Phi_{-h} = \Delta \cup \{ -\alpha \}.$$  

For $\alpha \in \Delta$, we have

$$\frac{\text{ht}(w^{-1}(\alpha))}{h} = b \frac{\text{ht}(\alpha)}{h} - \left\langle \mu, \alpha \right\rangle = \frac{b}{h} - \left\langle \mu, \alpha \right\rangle.$$

Now $\text{ht}(w^{-1}(\alpha))$ equals either $r$ or $r - h$, so $\left\langle \mu, \alpha \right\rangle = t$ if $w^{-1}(\alpha) \in \Phi^+$ and $\left\langle \mu, \alpha \right\rangle = t + 1$ if $w^{-1}(\alpha) \in -\Phi^+$. Comparing with [Pan96, Section 2.3] $(w = w', \mu = \nu)$ gives the result.

We stress that this bijection sends the set of coweight lattice points in $S_b(b)$ to the set of coweight lattice points in $bA$, and similarly for coroot lattice points.

Now [Thi15, Theorem 8.2] implies that this element is unique.

**Theorem 4.3.** $\tilde{w}_b$ is the unique $\tilde{w} \in \tilde{W}$ with $\tilde{w}(S_b(b)) = bA$.

4The closest result to Theorem 4.2 we were able to find was an existence statement in [Som05, Proof of Theorem 5.7], which relies upon Lemma 2.2 and the end of Section 2.3 in [Pan96]. See also [Ath04, Theorem 4.2].
5. Simultaneous Cores

We generalize simultaneous cores to the points $S_\Phi(b) \cap \check{Q}$ (Definition 5.2). In Section 5.2, we recall M. Haiman’s result on the number of such simultaneous cores (Theorem 1.8).

5.1. Definition. We now recall how to identify simultaneous $(a, b)$-cores using the bijection of Proposition 3.2. Let $\text{core}_\check{Q} := \text{core}_\Phi \circ \text{crt}^{-1} : \check{Q} \to \text{core}(a)$.

**Proposition 5.1** ([GKS90, Joh15]). For $\gcd(a, b) = 1$,

$$\text{core}(a, b) = \{\text{core}_\check{Q}(\lambda) : \lambda \in S_\Phi(b) \cap \check{Q}\},$$

where $\Phi$ is a root system for $S_a$.

We conclude that the coroot points in $S_\Phi(b)$ generalize the set of simultaneous $(a, b)$-cores to all $\check{W}$. We emphasize this with the following definition.

**Definition 5.2.** For $b$ relatively prime to $h$, we write $\text{core}(\check{W}, b) := S_\Phi(b) \cap \check{Q}$.

5.2. Enumeration: Theorem 1.8. We could count $|\text{core}(\check{W}, b)|$ using the relationship between the lattices $\check{\Lambda}$ and $\check{Q}$ given in Theorem 2.5 and between the simplices $S_\Phi(b)$ and $bA$ in Theorem 4.2. If we write a coweight lattice point in the coweight basis as $(x_0, x_1, x_2, \ldots, x_n)$, then the coweight points $bA \cap \check{\Lambda}$ are exactly the positive integral solutions to the linear equation

$$\sum_{i=0}^{n} c_i x_i = b, \quad 0 \leq x_i \in \mathbb{Z},$$

where $\alpha = \sum_{i=0}^{n} c_i \alpha_i$ and we take $c_0 := 1$. This last reformulation is easily counted [Sut98]: simply expand the generating function

$$\prod_{i=0}^{n} \frac{1}{1 - q^{c_i}} = \sum_{b=0}^{\infty} |bA \cap \check{\Lambda}| q^b.$$

**Example 5.3.** For an affine root system of type $S_a$, we have that $c_i = 1$. It is clear that there are $\binom{a+b-1}{b}$ coweight lattice points that are solutions to Equation (1).

Dividing by the index of connection, we obtain the corresponding number of coroot lattice points $|\text{core}(a, b)| = \frac{1}{\alpha} \binom{a+b-1}{b}$.

To do this sort of type-by-type analysis, however, is to overlook M. Haiman’s beautiful nearly uniform proof which combines Pólya theory, the Shephard-Todd formula, and—perhaps most surprisingly—Dirichlet’s theorem that any infinite arithmetic sequence of positive integers contains an infinite number of primes [Hai94]. See also [Som05].

**Theorem 1.8** (Number of $(\check{W}, b)$-cores; M. Haiman [Hai94]). For $\gcd(h, b) = 1$,

$$|\text{core}(\check{W}, b)| = \frac{1}{|W|} \prod_{i=1}^{n} (b + e_i).$$

This number has since become known as the rational Catalan number associated to $W$ and $b$ (see, for example, [ARW13]).

---

5M. Haiman’s proof uses the fact that any prime that divides a coefficient of the highest root also divides the Coxeter number $h$, for which we don’t know a uniform proof.
6. The Statistic size

In this section, we interpret the statistic size, which counts the number of boxes in the Ferrers diagram of an \( a \)-core, in the language of root systems. We then extend this statistic to any affine Weyl group.

6.1. size on Elements of \( \tilde{W} \).

**Definition 6.1.** For \( \tilde{w} \in \tilde{W} \), define
\[
\text{size}(\tilde{w}) := \sum_{\alpha + k\delta \in \text{inv}(\tilde{w}^{-1})} k.
\]

**Example 6.2.** The core \( \lambda = \begin{array}{c}
\boxed{1} \\
\boxed{1} \\
\boxed{1}
\end{array} \) in \( \tilde{A}_2 \) has 5 boxes. The corresponding affine element \( \tilde{w} = s_1 s_2 s_1 s_0 \) —illustrated in Figure 3—has size(\( \tilde{w} \)) = 5 because the inversions of \( \tilde{w}^{-1} \) are the affine roots \( -\tilde{\alpha} + 1 \cdot \delta, -\alpha_1 + 1 \cdot \delta, -\alpha_2 + 1 \cdot \delta, -\tilde{\alpha} + 2 \cdot \delta \), and \( 5 = 1 + 1 + 1 + 2 \).

**Remark 6.3.** We specify here that the statistic size on elements of \( \tilde{Q} \) in type \( \tilde{C}_n \) is not equal to the number of boxes of the corresponding self-conjugate core (a model studied, for example, in [BDF 06, FMIS09, HN13, CHW14, Alp14]). For example, one can compute that the element \( \tilde{w} = s_0 s_1 s_0 s_1 s_2 s_1 s_0 \in \tilde{C}_2 \) has size(\( \tilde{w} \)) = 11, but that \( \tilde{w} \) corresponds to the self-conjugate core
\[
\begin{array}{c}
\boxed{1} \\
\boxed{1} \\
\boxed{1}
\end{array}
\]
, which has 15 boxes.

There is a simple way to read off size in \( \tilde{C}_n \) on a self-conjugate core, which we state here without proof: weight by 2 those boxes \( (i, j) \) such that \( i < j \) and \( j - i = 0 \mod n \), by 1 the remaining boxes \( (i, j) \) such that \( i \leq j \), and by 0 all other boxes. Then size is given by the sum of the weights of the boxes. For \( \tilde{w} \) as above, we have the weighting
\[
\begin{array}{cccc}
1 & 1 & 2 & 1 \\
0 & 1 & 1 & \\
0 & 0 & 1 & \\
0 & \\
0 & \\
0 & \\
0
\end{array}
\]
. The sum of these weights is the desired
\[ 11 = \text{size}(\tilde{w}). \]

The statistic size is preserved under the bijection between minimal coset representatives of \( \tilde{S}_a/\tilde{S}_a \) and \( a \)-cores.

**Proposition 6.4.** The bijection
\[
\text{core}_{\tilde{S}} : \tilde{S}_a/\tilde{S}_a \to \text{core}(a)
\]
\[ \tilde{w} \mapsto \tilde{w} \cdot \emptyset \]
preserves size.
Proof. By [FY10, Proposition 8.2], if \( \kappa = \text{core}_S(\tilde{w}) \) is an \( a \)-core, \( k > 0 \) and \( s_i \in \Delta \) is such that \( l(s_i \tilde{w}) > l(\tilde{w}) \), then \( s_i \kappa \) has \( k \) more boxes than \( \kappa \) if and only if the unique hyperplane that separates \( \tilde{w}^{-1} s_i A^o \) from \( \tilde{w}^{-1} A^o \) is of the form \( H^k_\alpha \) for some \( \alpha \in \Phi^+ \). Thus by induction on \( l(\tilde{w}) \) the size of the core \( \tilde{w} \cdot \emptyset \) is

\[
\sum_{\alpha \in \Phi^+} \sum_{k} k, \quad \text{for } H^k_\alpha \text{ separates } \tilde{w}^{-1} A^o \text{ from } A^o
\]

which equals

\[
\sum_{\alpha + k\delta \in \text{Inv}(\tilde{w}^{-1})} k
\]

by Theorem [2.1]. Therefore \( \text{size}(\tilde{w} \cdot \emptyset) = \text{size}(\tilde{w}) \). \( \square \)

The statistic \( \text{size} \) of Definition 6.1 therefore generalizes the statistic \( \text{size} \) on \( a \)-cores to all elements of an affine Weyl group.

6.2. \text{size} as a Quadratic Form. We can also view \( \text{size} \) as a statistic on the coroot lattice \( \tilde{Q} \). To do this, we follow [Mac71], although it is possible to argue directly using the “strange identities” of Section 2.5.

I. G. Macdonald views the affine root \( \alpha + k\delta \in \tilde{\Phi} \) as the affine linear functional

\[
\alpha + k\delta : V \to \mathbb{R}, \quad x \mapsto \langle x, \alpha \rangle + k.
\]

For \( \tilde{w} \in \tilde{W} \) he defines \( s(\tilde{w}) := \sum_{\alpha + k\delta \in \text{Inv}(\tilde{w})} \alpha + k\delta \). Furthermore, I. G. Macdonald introduces a quadratic form \( \Psi \) as

\[
\Psi(x) := \frac{g}{2} \left\| x - \frac{\rho}{g} \right\|^2.
\]

Lemma 6.5 (Proposition 7.5 in Macdonald). We have

\[
s(\tilde{w}) = \Psi \circ \tilde{w}^{-1} - \Psi
\]

for all \( \tilde{w} \in \tilde{W} \).

Using this, we can calculate \( \text{size}(\tilde{w}) \) as follows:

\[
\text{size}(\tilde{w}) = \sum_{\alpha + k\delta \in \text{Inv}(\tilde{w}^{-1})} k
\]

\[
= \left( \sum_{\alpha + k\delta \in \text{Inv}(\tilde{w}^{-1})} \alpha + k\delta \right)(0)
\]

\[
= s(\tilde{w}^{-1})(0)
\]

\[
= (\Psi \circ \tilde{w} - \Psi)(0)
\]

\[
= \frac{g}{2} \left\| \tilde{w}(0) - \frac{\rho}{g} \right\|^2 - \frac{g}{2} \left\| \frac{\rho}{g} \right\|^2
\]

\[
= \frac{g}{2} \left\| \tilde{w}(0) \right\|^2 - (\tilde{w}(0), \rho).
\]

This suggests the following definition of \( \text{size} \) as a quadratic form on \( V \).

Definition 6.6. For any \( x \in V \), define

\[
\text{size}(x) := \frac{g}{2} \left\| x - \frac{\rho}{g} \right\|^2 - \frac{g}{2} \left\| \frac{\rho}{g} \right\|^2
\]

\[
= \frac{g}{2} \| x \|^2 - (x, \rho).
\]
Note that by the “strange” formula (Theorem 2.6), we have
\[ g \frac{\|\rho\|^2}{2} = \frac{n(h+1)}{24}. \]

The computation directly after Lemma 6.5 shows that we have the following analogue of Proposition 6.4.

**Corollary 6.7.** The bijection
\[ \text{crt}: \tilde{W}/W \to \tilde{Q}, \quad \tilde{w} \mapsto \tilde{w}(0) \]
preserves size.

6.3. **Translating size from** $S_{\Phi}(b)$ **to** $bA$. To resolve the first obstacle raised in Section 1.3—that the orientation of the Sommers region changes with the residue class of $b \mod h$—we wish to transfer the size statistic from the Sommers region $S_{\Phi}(b)$ to the dilated fundamental alcove $bA$. Using the results of Section 4.3, we define
\[ \text{size}(x) := \text{size}(w_b^{-1}(x)) \]
for all $x \in V$, so that the bijection $w_b$ sends $\text{size}$ on $S_{\Phi}(b)$ to $\text{size}$ on $bA$.

**Corollary 6.8.** For $b$ coprime to $h$ we have
\[ \left\{ \text{size}(\lambda) : \lambda \in \text{core}(\tilde{W}, b) \right\} = \left\{ \text{size}(\lambda) : \lambda \in bA \cap \tilde{Q} \right\}. \]

6.4. **The simply-laced condition.** It will be useful to define the $W$-invariant quadratic form $Q$ on $V$ by
\[
Q(x) := g \frac{\|x\|^2}{2} - \frac{g}{2} \frac{\|\rho\|^2}{g} \\
= \frac{g}{2} \|x\|^2 - \frac{n(h+1)}{24}
\]
so that $\text{size}(x) = Q(x - \frac{\rho}{g})$ for all $x \in V$.

Using the quadratic form $Q$, we find a considerable simplification of $\text{size}$ in the case where $\Phi$ is simply laced.

**Theorem 6.9.** For $\tilde{W}$ simply laced,
\[ \text{size}(x) = \frac{h}{2} \left\| x - b \frac{\rho}{h} \right\|^2 - \frac{n(h+1)}{24} \]

**Proof.** From Theorems 4.1 and 4.2 we have that $\tilde{w}_b = t_{\mu}w$ for $\mu \in \tilde{Q}$ and $w \in W$ such that $b \frac{\rho}{h} = w(\frac{\rho}{h}) + \mu$. We calculate
\[ \tilde{w}_b = t_{\mu}w = t_b t_{w(\frac{\rho}{h})}w = t_b t_{w(\frac{\rho}{h})} - w(\frac{\rho}{h})w = t_{w(\frac{\rho}{h})} - w(\frac{\rho}{h})w. \]

We conclude that
\[
\text{size}(x) = \text{size}\left( \frac{\tilde{w}_b^{-1}(x)}{x - b \frac{\rho}{h}} \right) \\
= \text{size}\left( t_b t_{w(\frac{\rho}{h})}^{-1}(x) \right) \\
= \text{size}\left( w^{-1} \left( x - b \frac{\rho}{h} + \frac{\rho}{h} \right) \right) \\
= Q \left( w^{-1} \left( x - b \frac{\rho}{h} + \frac{\rho}{h} - \frac{\rho}{g} \right) \right).
\]
By assumption $\bar{W}$ is simply-laced, so that $\rho = \bar{\rho}$ and $g = h$. Then
\[
\text{size}(x) = Q \left( w^{-1} \left( x - b\frac{\rho}{h} \right) \right) = Q \left( x - b\frac{\rho}{h} \right),
\]
since $Q$ is $W$-invariant.
\[\square\]

Remark 6.10. The simplification of size in Theorem 6.9 is the origin of the simply-laced condition in Theorems 1.7, 1.9, 1.10 and 1.11.

6.5. size and Symmetry. We show that the action of $b\Omega$ on $b\mathcal{A}$ preserves size. In particular, this dispenses with the second obstacle from Section 1.3, allowing us to study the coweights $b\mathcal{A} \cap \bar{A}$ instead of the coroots $b\mathcal{A} \cap \bar{Q}$.

Lemma 6.11. The group $b\Omega$ preserves the statistic size on $b\mathcal{A}$.

Proof. For $\bar{w} = t_{y_b} wt_{-y_b} \in b\Omega$ and $x \in b\mathcal{A}$ we have
\[
\text{size}(\bar{w}(x)) = Q \left( \bar{w}(x) - b\frac{\rho}{h} \right) = Q \left( w \left( x - b\frac{\rho}{h} \right) \right) = Q \left( x - b\frac{\rho}{h} \right) = \text{size}(x).
\]
\[\square\]

6.6. Theorem 1.9 (Maximum size of a $(\bar{W}, b)$-core).

For $\bar{W}$ a simply-laced affine Weyl group with $\operatorname{gcd}(h, b) = 1$,
\[
\max_{\lambda \in \mathcal{S}(b) \cap \bar{Q}} \text{size}(\lambda) = \frac{n(b^2 - 1)(h + 1)}{24}.
\]
This maximum is attained by a unique point $\lambda \in \mathcal{S}(b)$.

Proof. We claim that the maximum is obtained at $\lambda = \bar{w}_b^{-1}(0)$. First note that since $\bar{w}_b$ maps $\mathcal{S}(b)$ bijectively to $b\mathcal{A}$, $\bar{w}_b^{-1}(0)$ is indeed in $\text{core}(\bar{W}, b) \cap \mathcal{S}(b) \cap \bar{Q}$.

Since $\bar{w}_b$ maps size to size (Corollary 6.8), we will show the equivalent statement that $0$ is the unique element of $b\mathcal{A} \cap \bar{Q}$ of maximum size. We have that
\[
\text{size}(x) = Q \left( x - b\frac{\rho}{h} \right) = \frac{h}{2} \left\| x - b\frac{\rho}{h} \right\|^2 - \frac{n(h + 1)}{24}
\]
is a strictly convex function in $x$, and so it can only be maximized at a vertex of the convex polytope $b\mathcal{A}$. We will show that among all the vertices of $b\mathcal{A}$, the vertex $0$ has maximal size. Together with the fact that $0$ is the only vertex of $b\mathcal{A}$ that is in the coroot lattice $\bar{Q}$ this implies the result.

Let $x_0, x_1, \ldots, x_n$ be the vertices of $\mathcal{A}$, where $x_0 = 0$ and $x_i$ is the vertex with $(x_i, \alpha_i) > 0$ for $i \in [n]$. So $bx_0, bx_1, \ldots, bx_n$ are the vertices of $b\mathcal{A}$. We wish to show that $\|bx_i - b\frac{\rho}{h}\|^2$ is maximal for $i = 0$. For this it is sufficient to show that $\|x_i - \frac{\rho}{h}\|^2$ is maximal for $i = 0$.

Define $\alpha_0 = -\bar{\alpha}$ and for any $i = 0, 1, \ldots, n$ let $\Phi_i$ be the root system whose set of simple roots is $\{\alpha_0, \alpha_1, \ldots, \alpha_n\} \setminus \{\alpha_i\}$. Define $\rho_i = \frac{1}{2} \sum_{\alpha \in \Phi_i^+} \alpha$. Then by [Mac71] Proposition 7.3 (using that $\rho = \bar{\rho}$ and $g = h$) we have $x_i - \frac{\rho}{h} = -\frac{\rho}{h}$ for all $i \in \{0, 1, \ldots, n\}$. So we just need to check case-by-case that $\|\rho_i\|^2$ is maximized
when $i = 0$. This is easily accomplished.

We explicitly compute this maximum:

$$\text{size}(\tilde{w}_b^{-1}(0)) = \text{size}(0) = Q \left( -\frac{b\hat{\rho}}{h} \right) = \frac{q}{2} \left\| \frac{\hat{\rho}}{h} \right\|^2 - \frac{q}{2} \left\| \frac{\hat{\rho}}{h} \right\|^2$$

$$= (b^2 - 1) \frac{q}{2} \left\| \frac{\hat{\rho}}{h} \right\|^2 = (b^2 - 1) \frac{q^2 n(h + 1)}{24h^2}$$

$$= \frac{n(b^2 - 1)(h + 1)}{24}.$$

\[\square\]

We now characterize the inversion set of the affine element $\tilde{w}_b$.

**Theorem 6.12.** For $\tilde{W}$ a simply-laced affine Weyl group $b$ relatively prime to $h$ and $\tilde{w}_b$ as in Theorem 4.2,

$$\text{inv}(\tilde{w}_b) = \left\{ -\alpha + k\delta : \alpha \in \Phi^+, 0 < k < \frac{b}{h} \text{ht}(\alpha) \right\}.$$

**Proof.** From Theorem 4.1, we know that $\tilde{w}_b \left( \frac{\hat{\rho}}{h} \right) = b \frac{\hat{\rho}}{h}$. Then the result follows from calculating $\langle b \frac{\hat{\rho}}{h}, \alpha \rangle = b h \text{ht}(\alpha)$ for $\alpha \in \Phi^+$. \[\square\]

Let $\gcd(h, b) = 1$. If we draw a line of rational slope in $\mathbb{R}^2$ from the point $(0,0)$ to the point $(h,b)$, then for $1 \leq i \leq b - 1$, the number of boxes with $y$-coordinate equal to $i$ is given by the sequence

$$\left\lfloor \frac{ih}{b} \right\rfloor_{i=1}^{b-1}.$$

By Theorem 6.12, this sequence characterizes the inversion set of $\tilde{w}_b$.

Summing the inversions of Theorem 6.12 rank-by-rank gives the following corollary.

**Corollary 6.13.** For $\tilde{W}$ a simply-laced affine Weyl group with $\gcd(h, b) = 1$,

$$\sum_{i=1}^{b-1} (b - i) \sum_{j=1}^{\left\lfloor \frac{i}{b} \right\rfloor} |\Phi_{h-j}| = \frac{n(b^2 - 1)(h + 1)}{24},$$

where $\Phi_{\geq i}$ is the set of positive roots of height greater than or equal to $h - i$.

Since we can easily write down explicit formulas for $|\Phi_{h-j}|$, we obtain apparently nontrivial identities involving the floor function. For example, in type $\tilde{A}_n$ with $\gcd(n + 1, b) = 1$, we obtain the equality

$$\sum_{i=1}^{b-1} \frac{b - i}{2} \left( \frac{i(n + 1)}{b} \right) \left( 1 + \left\lfloor \frac{i(n + 1)}{b} \right\rfloor \right) = \frac{n(b^2 - 1)(n + 2)}{24}.$$

In type $\tilde{D}_n$ with $\gcd(2n - 2, b) = 1$, we have

$$\sum_{i=1}^{b-1} (b - i) \left( \min\left\{ \left\lfloor \frac{j+1}{2} \right\rfloor_{j=1}^{\frac{j+1}{2}}, \left\lfloor \frac{j+3}{2} \right\rfloor_{j=n-2}^{\frac{j+3}{2}} \right\} \right) = \frac{n(b^2 - 1)(2n - 1)}{24}.$$

We challenge the reader to prove these equalities directly!
Although Theorem 1.9 proves that $\text{size}(\tilde{w}^{-1}_b(0))$ is the maximum that the statistic $\text{size}$ can take on $\text{core}(\tilde{W}, b)$, we believe that the inversion set of $w_b$—specified in Theorem 6.12—contains the inversion sets of all other affine elements corresponding to elements of $\text{core}(W, b)$. This conjecture generalizes J. Vandehey’s result that the largest $(a, b)$-core contains all other $(a, b)$-cores as subdiagrams (see [Van08, Fay11]).

**Conjecture 6.14.** The element $\tilde{w}_b$ is maximal in the weak order on $\tilde{W}/W$ among all dominant elements $\{\tilde{w} \in \tilde{W}/W : \tilde{w}^{-1}(0) \in S_b(\beta)\}$.

### 7. Calculations

We begin with a review of weighted Ehrhart theory, which extends the quasipolynomiality and reciprocity theorems of usual Ehrhart theory to weighted sums over lattice points in a rational polytope (Section 7.1). In Section 7.2, we outline the calculations we will perform, pulling together the theory from the previous parts of the paper. We work out these calculations by hand in Sections 7.3 and 7.4 to find the variance in type $\tilde{A}_n$ and the expected value in type $\tilde{D}_n$. In Section 7.5, we detail our methodology for automating these computations, which allows us to compute the third moment in type $\tilde{A}_n$ and the variance in type $\tilde{D}_n$. In Section 7.6, we explain how to verify Theorem 1.10 in types $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ using the freely available program Normaliz [BK01, BIS10].

#### 7.1. Weighted Ehrhart Theory

Fix $P$ a $n$-dimensional rational convex polytope in a lattice $L$ (with generators a basis of $\mathbb{R}^n$), and let $h : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $r$. The weighted lattice-point enumerator for the $b$th dilate of $P$ is

$$P_h^L(b) := \sum_{x \in bP \cap L} h(x).$$

It turns out that $P_h^L(b)$ is not only a quasipolynomial, but also satisfies a reciprocity relation.

**Theorem 7.1** ([Bar06, BV07, BBDL+12, ABL+14, Theorem 4.6]). For $P$, $L$, and $h$ as above,

1. $P_h^L(b)$ is a quasipolynomial in $b$ of degree $n + r$. Its period divides the least common multiple of the denominators of the coordinates (in the generators of $L$) of the vertices of $P$.
2. If $P^o$ is the interior of $P$, then

$$P_h^L(-b) = (-1)^n(P^o_h)^L(b).$$

When $h(x) = 1$, we have $P_h^L(b) = |bP \cap L|$ and we therefore recover the well-known theorems of E. Ehrhart and I. G. MacDonald. We refer the reader to [BJ07] for further information and definitions. We will use the notation $P_h^L(b)_i$ to refer to the $i$th component of the quasipolynomial $P_h^L(b)$.

#### 7.2. Outline of Calculations

Drawing heavily from P. Johnson’s proof in [Joh15] of the expected size of a simultaneous core in type $\tilde{A}_n$, we outline our methodology.

In Equation (2), we proved that the statistic $\text{size}$ is a quadratic form. Corollary 6.8 then transfers $\text{size}$ to a statistic $\text{size}$ on $bA \cap \Lambda$, resolving the first obstacle outlined in Section 1.3 (that our desired region was changing orientation as we changed the dilation factor).

---

Even though these results are well-known to the experts, it is difficult to find explicit statements in the literature that apply at once to rational polytopes and weighted lattice-point enumerators; see the remark in [Joh15, Section 1.2.2].
The following proposition identifies $A$ as a rational polytope, so that the theory in Section 7.1 resolves the second obstacle in Section 1.3 (that $S_6(b)$ is not an integer polytope).

**Proposition 7.2.** The polytope $A$ is rational in the coweight lattice $\tilde{\Lambda}$.

**Proof.** The vertices of $A$ are given by the set

$$\Gamma := \{0\} \cup \{\tilde{\omega}_i : 1 \leq i \leq n\},$$

where we remind the reader that the $\tilde{\omega}_i$ are the fundamental coweights and $\tilde{\alpha}$ is the highest root. These vertices are rational in the lattice $\tilde{\Lambda}$. $\Box$

Part (1) of Theorem 7.1 now allows us to conclude that the weighted lattice-point enumerator

$$A^\Lambda_{\text{size}}(b) = \sum_{\mu \in S_6(b) \cap \Lambda} \text{size}^k(x) = \sum_{\mu \in S_6(b) \cap \Lambda} \text{size}^k(x)$$

is a quasipolynomial of of degree $n + 2k$ of period \(m(\tilde{W}) := \text{lcm}(c_1, \ldots, c_n)\), where the $c_i$ are the coefficients of the simple roots in the highest root—by Proposition 7.2 the $c_i$ are the denominators of the coordinates of the vertices of $A$. One can check that $m(\tilde{A}_n) = 1$, $m(\tilde{D}_n) = 2$, $m(\tilde{E}_6) = 6$, $m(\tilde{E}_7) = 12$, and $m(\tilde{E}_8) = 60$. Outside of type $\tilde{A}_n$, this imposes an additional constraint: we want to pick $b$ that is in the correct residue class modulo $m(\tilde{W})$. On the other hand, what was only a priori a quasipolynomial actually collapses to a polynomial in type $\tilde{A}_n$.

To deduce the desired formula for

$$A^Q_{\text{size}}(b) = \sum_{\lambda \in bA \cap \tilde{\Lambda}} \text{size}^k(x) = \sum_{\lambda \in S_6(b) \cap \tilde{Q}} \text{size}^k(\lambda)$$

from the quasipolynomial $A^\Lambda_{\text{size}}(b)$, we use the results of Section 2.4; the coroot lattice $\tilde{Q}$ is a lattice of index $f$ inside $\tilde{\Lambda}$, with a group $\Omega$ of order $f$ acting freely (Theorem 2.5). By Lemma 6.11, size is invariant under the action of $b\Omega$. Specifically, we have the simple relationship

$$\frac{1}{f} \sum_{\lambda \in bA \cap \tilde{\Lambda}} \text{size}^k(x) = \sum_{\lambda \in bA \cap \tilde{Q}} \text{size}^k(x).$$

We are therefore now in the desirable position of needing to collect enough points to fully determine the polynomial $A^\Lambda_{\text{size}}(b)_j$, for all $j$ coprime to $h$.

**Remark 7.3.** We note here that since $|bA \cap \tilde{\Lambda}| = |(h + b)A' \cap \tilde{\Lambda}|$, by Part (2) of Theorem 7.1 we have that $A^\Lambda_{\text{size}}(b) = A^\Lambda_{\text{size}}(-h - b)$. Thus, for each point $b$ for which we can evaluate $A^\Lambda_{\text{size}}(b)$, we get the second point $-h - b$ “for free.”

By Equation (1), the points $bA \cap \tilde{\Lambda}$ are the positive integral solutions to the linear equation

$$\sum_{i=0}^n c_i x_i = b,$$

Restricting to types $\tilde{A}_n$ and $\tilde{D}_n$, by fixing a small $b$ but letting $n$ be arbitrary, we can explicitly describe these points and sum $\text{size}^k$ over this description for all $n$ simultaneously.

Over the next two sections, we compute by hand the variance in type $\tilde{A}_n$ (Theorem 1.11) and expected value in type $\tilde{D}_n$ (Theorem 1.10), after which we discuss automation and the computation for type $\tilde{E}_n$. 
7.3. **Theorem 1.5 or Theorem 1.11** in type $\tilde{A}_n$. In type $\tilde{A}_n$, $m(\tilde{A}_n) = 1$, and so $A^{\Lambda}_{\text{zise}}(b) = A^{\Lambda}_{\text{zise}}(b)_0$ is a polynomial. We have the relation of polynomials

$$A^{\Lambda}_{\text{zise}}(b) = \frac{1}{n+1} A^\Lambda_{\text{zise}}(b),$$

since this equality holds for all $b$ coprime to $h$, and therefore for all $b$. We may therefore choose $b$ without concern as to its residue class modulo $h$.

All exponents $e_1, e_2, \ldots, e_n$ are coprime to $h$ and—as they are less than $h$—have the property that integral dilations $e_i \Lambda$ do not contain any interior lattice points. P. Johnson used these properties along with Ehrhart reciprocity to identify $n$ zeroes of the polynomial $A^\Lambda_{\text{zise}}(b)$ of degree $n + 2$ [Joh15, Corollary 3.8].

Furthermore, it is easy to see that $A^\Lambda_{\text{zise}}(1) = 0$, so that by Remark 7.3 we also have $A^\Lambda_{\text{zise}}(-h + 1) = 0$. This gives $(n + 2)$ zeroes of $g(b)$, and it remains only to check that the constant term $A^\Lambda_{\text{zise}}(0) = \text{zise}(0) = -\frac{n(h+1)}{24}$ by Equation (2).

To compute the variance, we will evaluate

$$A^\Lambda_{\text{zise}}(b) := \sum_{\lambda \in b \Lambda} \text{zise}(\lambda)^2,$$

which is a polynomial of degree $n + 4$, by Theorem 7.1. The same reasoning as above gives us $(n + 2)$ zeroes of $v(b)$, as well as the constant term

$$A^\Lambda_{\text{zise}}(0) = \left(\frac{n(h+1)}{24}\right)^2.$$

We now have

$$A^\Lambda_{\text{zise}}(b) = \left(\prod_{i=1}^{n}(b + i)\right) (b - 1)(b + h + 1)(b - b_1)(b - b_2)c,$$

and we have identified the value of $A^\Lambda_{\text{zise}}(b)$ at $b = 0$.

We require two additional points, which we obtain in the next two subsections by explicitly evaluating $A^\Lambda_{\text{zise}}(2)$ separately for $\Lambda_n$ with $n$ even and $n$ odd.

There are $\binom{n+1}{2}$ coweight points in $2\Lambda$. These are given explicitly as follows, where each line corresponds to a $\Omega$-orbit of coweights.

- $w_i + w_j, w_{i+1} + w_{j+1}, \ldots, w_{i+n-1} + w_{j-1}$, for $0 \leq i \leq j \leq n$.

7.3.1. $n = 0 \mod 2$. In this case $h = n + 1$ is coprime to 2, and for each $1 \leq i \leq \frac{n+2}{2}$, there are $(n + 1)$ points $\mu \in 2\Lambda \cap \Lambda$ with $\text{zise}(\mu) = \binom{i}{2}$.

**Remark 7.4.** We can see this combinatorially, by noting (as in the introduction) that the set of 2-cores consists of exactly those partitions of staircase shape $(k, k - 1, \ldots, 1)$ for $k \in \mathbb{N}$, along with the empty partition. When $n + 1$ is odd, the simultaneous $(2, n + 1)$ cores will then be those 2-cores with fewer than $\frac{n+2}{2}$ rows.

We compute that

$$A^\Lambda_{\text{zise}}(2) = (n + 1) \sum_{i=1}^{\frac{n+2}{2}} \binom{i}{2}^2 = \frac{(3n^2 + 12n + 4)(n + 4)(n + 2)(n + 1)(n)}{1920}.$$
7.3.2. \( n = 1 \mod 2 \). In this case there are the same \( \frac{(n+1)(n+2)}{2} \) coweight points in \( 2A \) as before, but the evaluation of size on these points changes (since 2 is not relatively prime to \( h \), we no longer have the combinatorial interpretation as a sum over 2-cores). In particular, one can check that there are

- \( \frac{n+1}{2} \) coweight points \( \mu \in 2A \cap \Lambda \) with \( \text{size}(\mu) = \frac{1}{8} \), and
- for \( 1 \leq i \leq \frac{n+1}{2} \), there are \( n+1 \) coweight points \( \mu \in 2A \cap \Lambda \) with \( \text{size}(\mu) = \frac{4i^2-1}{8} \).

We compute that

\[
A_{\text{size}}^\Lambda(2) = (n+1) \left( \frac{1}{2} \right)^2 \left( -\frac{1}{8} \right)^2 + \sum_{i=1}^{\frac{n+1}{2}} \left( \frac{4i^2-1}{8} \right)^2 = \frac{(3n^2 + 12n + 4)(n+4)(n+2)(n+1)n}{1920}.
\]

Comparing Equation (7) with Equation (6), we see that the formula for \( A_{\text{size}}^\Lambda(2) \) does not depend on the parity of \( n \). By Ehrhart reciprocity, we have also found the value of \( A_{\text{size}}^\Lambda(-(n+3)) \). A straightforward computation with Equation (4) now yields that

\[
(b - b_1)(b - b_2)c = \frac{n(n+1)(2b^2 + 2b^2 + 10n + 9bn + 7b^2n - 5n^2 + 7bn^2)}{2880|A_n|}.
\]

Using the relation \( V(X) = E(X^2) - E(X)^2 \), Equation (8), and Theorem 1.4, we conclude the following theorem after substituting \( a = n + 1 \).

**Theorem 1.5** (Theorem 1.11 in type \( \widetilde{A}_n \)): Variance of size on \( (a, b) \)-cores. For \( \gcd(a, b) = 1 \),

\[
\mathbb{V}_{\lambda \in \text{core}(a, b)} \text{(size}(\lambda)) = \frac{ab(a - 1)(b - 1)(a + b)(a + b + 1)}{1440}.
\]

7.4. **Theorem 1.10** in type \( \widetilde{D}_n \). In type \( \widetilde{D}_n \), the period \( m(\widetilde{D}_n) = 2 \), and so \( A_{\text{size}}^\Lambda(b) \) has two polynomial components, \( A_{\text{size}}^\Lambda(b)_0 \) and \( A_{\text{size}}^\Lambda(b)_1 \). As the Coxeter number \( h = 2n - 2 \) is even, we are interested only in \( A_{\text{size}}^\Lambda(b)_1 \). We may therefore only choose odd \( b \) when trying to determine the desired component of the quasipolynomial.

There are \( n - 1 \) distinct odd exponents, which lie in the desired residue class modulo 2 (since they are coprime to \( h = 2n - 2 \)). As in Section 7.3 by considering the dilations \( e_i A \) for these \( n - 1 \) exponents, we identify \( n - 1 \) zeroes of the 1 mod 2 component of the polynomial \( A_{\text{size}}^\Lambda(b)_1 \). It is similarly easy to evaluate \( A_{\text{size}}^\Lambda(1)_1 = 0 \), so that also \( A_{\text{size}}^\Lambda(-2n+1)_1 = 0 \). We therefore have found \( n+1 \) zeroes of \( A_{\text{size}}^\Lambda(b)_1 \), and can write

\[
A_{\text{size}}^\Lambda(b)_1 = \left( \prod_{i=1}^{n-1} (b + 2i - 1) \right) (b - 1)(b + 2n - 1)(b + b_1)c,
\]

so that we require two additional points to find the unknowns \( b_1 \) and \( c \) and fully determine \( A_{\text{size}}^\Lambda(b)_1 \).

Sadly, the remaining exponent is equal to \( n - 1 \), which is either repeated (when \( n \) is even) or even (when \( n \) is odd). We are therefore unable to use this exponent to find a zero of \( A_{\text{size}}^\Lambda(b)_1 \). Furthermore, we cannot use the evaluation \( A_{\text{size}}^\Lambda(0)_1 \), since the dilation of the factor \( b = 0 \) is not in the desired 1 mod 2 residue class.
We are left with no choice but to compute \( A_{\text{size}}(b) \) for some additional odd value of \( b \). The smallest unidentified such \( b \) is 3, and we now determine \( A_{\text{size}}(3) \).

There are \( 4(n + 2) \) coweight points inside \( \Delta \), and \( (n + 2) \) corresponding coroot points. These coweights are given explicitly as follows, where each line corresponds to a \( \Omega \)-orbit of coweights.

There are 20 coweight points in \( \Delta \) arranged in \( \Omega \) orbits of size 4:

- \( 3w_0, 3w_1, 3w_n, 3w_{n-1} \),
- \( 2w_0 + w_1, w_0 + 2w_1, 2w_n + w_n, w_{n-1} + 2w_n \),
- \( w_0 + w_{n-1} + w_n, w_0 + w_1 + w_n, w_0 + w_1 + w_{n-1}, w_1 + w_{n-1} + w_n \),
- \( w_0 + 2w_n, 2w_0 + w_{n-1}, 2w_1 + w_n, w_1 + 2w_{n-1} \).

There are an additional \( 4(n - 3) \) coweight points of the form

- \( w_0 + w_i, w_{n-i} + w_n, w_1 + w_i, w_{n-i} + w_1 \) for \( 2 \leq i \leq n - 2 \).

7.4.1. \( n \neq 1 \mod 3 \). In this case, 3 is coprime to \( b = 2n - 2 \).

Let \( \text{pent}(i) := \frac{1}{2} \left( 3 \left\lfloor \frac{i+1}{2} \right\rfloor^2 + (-1)^i \left\lfloor \frac{i+1}{2} \right\rfloor \right) \) be the \( i \)th largest pentagonal number.

One can check that there are

- four coweight points with \( \text{size}(\mu) = \text{pent}(i) \) for \( 0 \leq i \leq n - 2 \) \( - \frac{n+1}{3} \);
- eight coweight points with \( \text{size}(\mu) = \text{pent}(n - 1 - \left\lfloor \frac{n+1}{3} \right\rfloor) \) and
- alternatingly four or zero coweight points (starting with four) \( \mu \) with \( \text{size}(\mu) = \text{pent}(i) \) for \( n - \frac{n+1}{3} \leq i \leq 2n - 2 \frac{n+1}{3} \).

Thus,

\[
A_{\text{size}}(3) = \sum_{i=0}^{n-2-\left\lfloor \frac{n+1}{3} \right\rfloor} 4 \cdot \text{pent}(i) + 8 \cdot \text{pent} \left( n - 1 - \left\lfloor \frac{n+1}{3} \right\rfloor \right) + \sum_{i=n-\frac{n+1}{3}}^{2n-2-\left\lfloor \frac{n+1}{3} \right\rfloor} 4 \cdot \left( 1 + (-1)^{i-n+\frac{n+1}{3}} \right) \cdot \text{pent}(i).
\]

This expression simplifies to

\[
A_{\text{size}}(3) = 4 \cdot \frac{n(n + 1)(n + 2)}{6}.
\]

7.4.2. \( n = 1 \mod 3 \). Let \( \text{binom}(i) := 3\left\lfloor \frac{i+1}{2} \right\rfloor + \frac{i}{3} \), so that \( \text{binom}(0) = \frac{1}{3} \). We check that there are

- eight coweight points with \( \text{size}(x) = \text{binom}(i) \) for \( 0 \leq i \leq \frac{n+1}{3} \);
- 12 coweight points with \( \text{size}(x) = \text{binom} \left( \frac{n-1}{3} \right) \) and
- four coweight points with \( \text{size}(x) = \text{binom}(i) \) for \( \frac{n-2}{3} \leq i \leq \frac{2n-2}{3} \).

Thus,

\[
A_{\text{size}}(3) = \sum_{i=0}^{n-4} 8 \cdot \text{binom}(i) + 12 \cdot \text{binom} \left( \frac{n-1}{3} \right) + \sum_{i=\frac{n+2}{3}}^{2n-2} 4 \cdot \text{binom}(i).
\]

As before, this expression simplifies to

\[
A_{\text{size}}(3) = 4 \cdot \frac{n(n + 1)(n + 2)}{6}.
\]

Comparing Equation (10) with Equation (10), we see that the formula for \( A_{\text{size}}(3) \) does not depend on the residue class of \( n \) mod 3. By Ehrhart reciprocity, this also determines the value of \( A_{\text{size}}(-2n-1) \). A straightforward computation with Equation (9) now yields Theorem 1.10.
Theorem 1.10 (Expected size of a \((\tilde{D}_n, b)\)-core).
For \(\gcd(h, b) = 1\),
\[
\mathbb{E}_{\lambda \in \text{core}(\tilde{D}_n, b)} (\text{size}(\lambda)) = \frac{n(b - 1)(b + h + 1)}{24}.
\]

7.5. Automation: Theorem 1.6 and Theorem 1.11 in type \(\tilde{D}_n\). We now describe how we automated the computations to compute the third moment in \(\tilde{D}_n\) (Theorem 1.6) and variance in type \(\tilde{D}_n\) (Theorem 1.11).

Let \(C_\Phi = (\langle \alpha_i, \alpha_j \rangle)_{1 \leq i, j \leq n}\) be the Cartan matrix for the root system \(\Phi\). Then, if \(x = \sum_{i=1}^{n} x_i w_i\) is expressed in terms of the coweight basis,
\[
\|x\| = (x_1, \ldots, x_n)^T C^{-1}_\Phi \cdot (x_1, \ldots, x_n).
\]

Proposition 7.5 ([Hum72 Table 1]). The \((i, j)\)th entry of \(C^{-1}_\Phi\) is given

- in type \(A_n\) by
\[
C^{-1}_{i,j} = \begin{cases} \frac{n+1-i}{n+1} & \text{if } i \leq j, \\
\frac{n+1-j}{n+1} & \text{otherwise; and}
\end{cases}
\]

- in type \(D_n\) by
\[
C^{-1}_{i,j} = \begin{cases} \min(i, j) & \text{if } i, j \leq n - 2, \\
\frac{\min(i, j)}{2} & \text{if } \max(i, j) > n - 2 \text{ and } \min(i, j) \leq n - 2, \\
\frac{n}{2} & \text{if } n - 1 \leq i = j, \\
\frac{n-2}{2} & \text{otherwise.}
\end{cases}
\]

Proposition 7.6. The difference \(\|x - \frac{b}{h} \rho\| - \|x\|\) is a linear function of the \(x_i\) given

- in type \(\tilde{A}_n\) by
\[
\|x - \frac{b}{h} \rho\| = \|x\| + \left(\frac{b}{n+1}\right)^2 \frac{1}{2} \left(\frac{n+2}{3}\right) - \sum_{i=1}^{n} \frac{b(n+1-i)}{n+1} x_i; \quad \text{and}
\]

- in type \(\tilde{D}_n\) by
\[
\|x - \frac{b}{h} \rho\| = \|x\| + \left(\frac{b}{2n-2}\right)^2 \frac{1}{2} \left(\frac{2n+1}{4}\right) - \left(\sum_{i=1}^{n-2} \frac{b(2n-1-i)}{2n-2} x_i\right) + \frac{bn}{4}(x_{n-1}+x_n).
\]

Proof. This follows from direct computation with Proposition 7.5. \(\square\)

We now describe our automation of the calculations in Sections 7.3 and 7.4, taking type \(\tilde{A}_n\) as our example. In type \(\tilde{A}_n\), the coweight points \((x_1, x_2, \ldots, x_n)\) (where \(x_i\) is the \(i\)th coordinate in the coweight basis) contained in \(bA\) are exactly the nonnegative solutions to the linear equation
\[
\sum_{i=0}^{n} x_i = b.
\]

Let \(\text{comp}(b) := \{c = (c_1, c_2, \ldots, c_{\ell(c)}) : \sum_{i=1}^{\ell(c)} c_i = b\}\) be all compositions of \(b\). For a fixed dilation factor \(b\), Equation (12) ensures that there will only be at most \(b\) nonzero coordinates \(x_i\) when computing \(\|x\|\). By Proposition 7.6, we can calculate \(\text{size}(x) = \frac{n}{2} \|x - \frac{b}{h} \rho\| - \frac{a(h+1)}{24}\) from \(\|x\|\) and a linear function in \(x\). Let
\[ \mu := \mathbb{E}_{\lambda \in \text{comp}(b)} \left( \text{size}(\lambda) \right) \]. Using the explicit formulas for \( C_{i,j}^{-1} \), for \( c \in \text{comp}(b) \) the summation

\[ \sum_{0 \leq i_1 < i_2 < \cdots < i_{\ell(c)} \leq n} \left( \text{size} \left( \sum_{j=1}^{\ell(c)} c_{i_j} w_{i_j} \right) - \mu \right)^k \]

for \( k \geq 1 \) may therefore be explicitly evaluated (either by hand, or by computer) as a polynomial of degree \( 2k + \ell(c) \). Summing now over all \( 2^{b-1} \) compositions for a fixed \( b \), we can determine the polynomial of degree \( 2k + b \)

\[ A^h_{(\text{size} - \mu)}(b) = \sum_{c \in \text{comp}(b)} \sum_{0 \leq i_1 < i_2 < \cdots < i_{\ell(c)} \leq n} \left( \text{size} \left( \sum_{j=1}^{\ell(c)} c_{i_j} w_{i_j} \right) - \mu \right)^k \]

One can write a similar sum in type \( \tilde{D}_n \), treating the four simple roots in the orbit of the affine node differently from the rest. We wrote Mathematica code to find \( A^h_{(\text{size} - \mu)}(b) \) for \( b = 2, 3, 4, 5 \) simultaneously for all ranks \( n \) and determine the third moment in type \( \tilde{A}_n \) ([Vol]).

**Theorem 1.16** (Third moment of size on \((a, b)\)-cores). For \( \gcd(a, b) = 1 \), let \( \mu := \mathbb{E}_{\lambda \in \text{core}(a, b)} \left( \text{size}(\lambda) \right) \). Then

\[ \sum_{\lambda \in \text{core}(a, b)} (\text{size}(\lambda) - \mu)^3 = \frac{ab(a - 1)(b - 1)(a + b + 1)(2a^2b - 3a^2 + 2ab^2 - 3ab - 3b^2 - 3)}{60480} \]

We used similar code in type \( D_n \) to compute \( A^h_{(\text{size} - \mu)}(b) \) for \( b = 3, 5 \) for all ranks \( n \) to determine variance.

**Theorem 1.11** (Variance of size on \((\tilde{D}_n, b)\)-cores). For \( \gcd(h, b) = 1 \),

\[ \mathbb{V}_{\lambda \in \text{core}(\tilde{D}_n, b)} \left( \text{size}(\lambda) \right) = \frac{nhb(h - 1)(h + b)(h + b + 1)}{1440} \]

7.6. **Automation: Theorems 1.10 and 1.11 in types \( \tilde{E}_n \).** In the exceptional types \( \tilde{E}_6, \tilde{E}_7, \) and \( \tilde{E}_8 \), Theorems 1.10 and 1.11 are a finite check, which we accomplish using a similar method as in Section 7.5 with the freely available program Normaliz [BK01, BIS10].

Suppose \( \Phi \) is a root system of type \( E_6, E_7, \) or \( E_8 \). Using the fact that \( \Phi \) is simply laced (and therefore \( g = h \) and \( \rho = \tilde{\rho} \)) we calculate that

\[ \sum_{\lambda \in \text{core}(\tilde{W}, b)} \text{size}(\lambda) = \sum_{\lambda \in \text{core}(\tilde{W}, b)} \text{size}(\lambda) = \sum_{\lambda \in \text{core}(\tilde{W}, b)} \left( \frac{h}{2} \| \lambda \|^2 - b(\lambda, \rho) + (b^2 - 1) \frac{n(h + 1)}{24} \right) \]

Thus our task is to calculate the Euler-Maclaurin quasipolynomials \( \sum_{\lambda \in \text{core}(\tilde{W}, b)} \frac{h}{2} \| \lambda \|^2 \) and \( \sum_{\lambda \in \text{core}(\tilde{W}, b)} \lambda, \rho \). To be able to use Normaliz for this task, we need to interpret these sums as sums over \( \mathbb{Z}^n \) as follows. Let \( A = (\langle \alpha_i, \alpha_j \rangle)_{i,j} \) be the Cartan matrix of \( \Phi \). Write \( \tilde{\alpha} = \sum_{i=1}^n c_i \alpha_i \) and let \( c = (c_1, c_2, \ldots, c_n) \) be the row vector of coefficients.

\[ \sum_{\lambda \in \text{core}(\tilde{W}, b)} \frac{h}{2} \| \lambda \|^2 = \sum_{x=(x_1,x_2,\ldots,x_n) \in \mathbb{Z}^n} \sum_{\sum_{i=1}^n x_i \alpha_i \in bA} \frac{h}{2} \| x \tilde{\alpha} \| = \sum_{\sum_{i=1}^n x_i \alpha_i \in bA, x \geq 0} \frac{h}{2} x^T Ax, \]

\[ \sum_{\lambda \in \text{core}(\tilde{W}, b)} \lambda, \rho = \sum_{x=(x_1,x_2,\ldots,x_n) \in \mathbb{Z}^n} \sum_{\sum_{i=1}^n x_i \alpha_i \in bA, x \geq 0} \rho(x \tilde{\alpha}) = \sum_{\sum_{i=1}^n x_i \alpha_i \in bA, x \geq 0} \rho(x \tilde{\alpha}). \]
Similarly,
\[ \sum_{\lambda \in b \cap Q} \langle \lambda, \rho \rangle = \sum_{x=(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n} \sum_{\substack{A \mathbf{x} \geq 0 \quad \mathbf{b} - c^T \mathbf{A} \mathbf{x} \geq 0}} x_i. \]

For the purposes of using Normaliz, it is helpful to replace the set
\[ \{ (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n : A \mathbf{x} \geq 0 \text{ and } b - c^T A \mathbf{x} \geq 0 \} \]
with
\[ \{ (x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{Z}^{n+1} : A \mathbf{x} \geq 0, x_{n+1} - c^T A \mathbf{x} \geq 0 \text{ and } \lambda(x) = b \}, \]
where \( \lambda \) is the linear functional on \( \mathbb{R}^{n+1} \) defined by \( \lambda(x) = x_{n+1} \) for all \( x \in \mathbb{R}^{n+1} \). The linear functional \( \lambda \) is thus used as a grading.

As an example calculation in type \( \tilde{E}_6 \), in the visual interface sf jNormaliz, we input the matrix
\[
M = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
and the grading \((0, 0, 0, 0, 0, 0, 1)\). Then we use its generalized Ehrhart series functionality with the quadratic polynomial \( \frac{h}{2} x^T A \mathbf{x} \) to find
\[ \sum_{\lambda \in b \cap Q} \frac{h}{2} \| \lambda \|^2 = \sum_{x=(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n} \sum_{\substack{A \mathbf{x} \geq 0 \quad \mathbf{b} - c^T A \mathbf{x} \geq 0}} \frac{h}{2} x^T A \mathbf{x}, \]
as a quasipolynomial in \( b \) with period 6. Similarly we calculate
\[ \sum_{\lambda \in b \cap Q} \langle \lambda, \rho \rangle = \sum_{x=(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n} \sum_{\substack{A \mathbf{x} \geq 0 \quad \mathbf{b} - c^T A \mathbf{x} \geq 0}} x_i. \]
as a quasipolynomial in \( b \) with period 6. Somewhat miraculously, we find that
\[ \sum_{\lambda \in b \cap Q} \left( \frac{h}{2} \| \lambda \|^2 - b \langle \lambda, \rho \rangle + (b^2 - 1) \frac{n(b+1)}{24} \right) \]
is a \textit{polynomial} in \( b \) equal to
\[
\frac{1}{207360} (b-1)(b+1)(b+4)(b+5)(b+7)(b+8)(b+11)(b+13) = \frac{1}{24} n(b-1)(b+h+1) \times \frac{1}{|W|} \prod_{i=1}^{n} (b+e_i),
\]
proving Theorem 1.10 in type \( \tilde{E}_6 \).
8. Open Problems

In this section, we present some open problems and conjectures. Section 8.1 proposes a first step towards finding formulas beyond the third moment. Section 8.2 then asks if it is possible to extend our theorems to non-simply-laced types. Finally, Section 8.3 suggests that existing combinatorics associated to the representation theory of affine Lie algebras might be harnessed to develop combinatorial models for cores in other types.

8.1. Higher Moments and Integrals. It is natural to ask about formulas for higher moments, even though we do not believe there are uniform formulas (see Remark 1.12).

Open Problem 8.1. Extend Theorems 1.9 to 1.11 to higher moments.

Rather than compute the entire Ehrhart quasipolynomial \(\mathcal{A}^{\mathcal{Q}}_{\text{size} - \mu}(b)\), we could instead ask for its leading coefficient. In general, this leading coefficient turns out to be an integral over the polytope.

Theorem 8.2 ([BBDL+11, BBDL+12]). Fix \(P, L,\) and \(h\) as in Section 7.4. The leading coefficient of the Ehrhart quasipolynomial \(P^h_L(b)\) does not depend on \(b\) and is given by

\[
\int_{x \in \mathcal{P}} h(x).
\]

For example, the leading coefficients of Theorems 1.10 and 1.11 give the following formulae for the integral of the quadratic form \(\text{size}\). Here we have normalized so that \(\text{Vol}(b, \mathcal{A}) = 1\).

Corollary 8.3. For \(\gcd(h, b) = 1\),

\[
\int_{x \in \mathcal{S}_h(b)} \text{size}(x) = \frac{nb^2}{24}, \quad \text{and}
\]

\[
\int_{x \in \mathcal{S}_h(b)} \left(\text{size}(x) - \frac{nb^2}{24}\right)^2 = \frac{nhb^4}{1440}.
\]

Although Remark 1.12 suggests that there is no uniform formula for the third moment, we conjecture that the leading coefficient of the third moment does have a uniform formula.

Conjecture 8.4. For \(\gcd(h, b) = 1\), we have the following uniform integral.

\[
\int_{x \in \mathcal{S}_h(b)} \left(\text{size}(x) - \frac{nb^2}{24}\right)^3 = \frac{nhb^6}{60480}(2(h - 1) - 1).
\]

Computational evidence suggests that even this leading coefficient lacks a uniform formula beyond the third moment. We record a few of these leading coefficients here.

Conjecture 8.5. Let the leading coefficient of \(\mathcal{A}^{\mathcal{Q}}_{\text{size} - \mu}(b)\) be denoted by

\[
\text{top}_{h}^i(i) := \int_{x \in \mathcal{S}_h(b)} \left(\text{size}(x) - \frac{nb^2}{24}\right)^i.
\]
In type $\tilde{A}_n$,
\[
\text{top}_b \Phi(4) = \frac{nhb^8}{4838400} (19n^2 - 13n + 4),
\]
\[
\text{top}_b \Phi(5) = \frac{nhb^{10}}{95800320} (23n^2 - 25n + 12)(2n - 1),
\]
\[
\text{top}_b \Phi(6) = \frac{nhb^{12}}{4184557977600} (307561n^4 - 826062n^3 + 1048509n^2 - 647948n + 155040),
\]
and
\[
\text{top}_b \Phi(7) = \frac{nhb^{14}}{8230000640000} (15562n^5 - 64721n^4 + 129288n^3 - 142241n^2 + 82300n - 19488).
\]

In type $\tilde{D}_n$,
\[
\text{top}_b \Phi(4) = \frac{nhb^8}{2419200} (31n^2 - 99n + 86),
\]
\[
\text{top}_b \Phi(5) = \frac{nhb^{10}}{23950080} (70n^3 - 365n^2 + 667n - 426),
\]
and
\[
\text{top}_b \Phi(6) = \frac{nhb^{12}}{523069747200} (859445n^4 - 6449250n^3 + 19050243n^2 - 26075294n + 13852536).
\]

We do not even have a conjecture for the denominators of these expressions, although D. Armstrong has suggested a connection to the Dirichlet $\eta$ function [Arm15b].

8.2. Non-Simply-Laced Types. It is reasonable to ask for analogues of our results in non-simply-laced types.

Open Problem 8.6. Extend Theorems 1.9 to 1.11 to non-simply-laced types.

To whet the reader’s appetite, we offer a conjecture for the Fuß-Catalan case $b = mh + 1$ in type $\tilde{C}_n$, for which our open problem seems to be low-hanging fruit (see Remark 6.3).

Conjecture 8.7.
\[
E_{\lambda \in \text{core}(\tilde{C}_n, mh+1)} (\text{size}(\lambda)) = \frac{mn(2(m+1)n^2 + (m+3)n - (m+1))}{12}.
\]

8.3. Basic Representations and Combinatorial Models. In this section, we suggest that researchers interested in extending combinatorial models of cores to other types might benefit from existing combinatorial models arising in the representation theory of affine Lie algebras.

Fix an affine Lie algebra $\mathfrak{g}$. The highest weight module $L(\Lambda_0)$ is the basic representation of $\mathfrak{g}$. We refer the reader to [Kac94] for further details. The module $L(\Lambda_0)$ has an associated crystal $B(\Lambda_0)$, which is an infinite directed graph with a unique source (the highest weight), and with edges labeled by simple affine roots $\tilde{\Delta}$. For $\alpha \in \tilde{\Delta}$, an $\alpha$-string is a maximal connected chain of $B(\Lambda_0)$ whose edges are all labeled by $\alpha$. There is a $\tilde{W}$-action on the vertices of $B(\Lambda_0)$, where $s_\alpha$ acts by reversing all $\alpha$-strings.

Theorem 8.8 ([Kac94]). The $\tilde{W}$-orbit of the highest weight in $B(\Lambda_0)$ is in $\tilde{W}$-equivariant bijection with the coroot lattice $\tilde{Q}$.

Rather than redevelop the combinatorics of cores (or abaci) for other affine types—as in [HJ12, BNP+15]—we propose that it might be worthwhile to study the restriction of existing combinatorial models for the crystal $B(\Lambda_0)$ to the $\tilde{W}$-orbit of Theorem 8.8. For example, in type $\tilde{A}_n$ we observe that the “Young wall” model illustrated for $n = 2$ in [Kan03, Figure 22] recovers cores.
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