Sewing Constraints and Non-Orientable Open Strings

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ABSTRACT

We extend to non-orientable surfaces previous work on sewing constraints in Conformal Field Theory. A new constraint, related to the real projective plane, is described and is used to illustrate the correspondence with a previous construction of open-string spectra.
Introduction

The main ingredient of Conformal Field Theory, central to all its applications in String Theory and critical phenomena, is the extensive use of notions and techniques from the classical theory of analytic functions\textsuperscript{[1]}. Thus, amplitudes may be defined in terms of power series, their full structure being determined by analytic continuation. The global aspects of the geometry manifest themselves in a number of conditions that the expansion coefficients are to satisfy. These are the “sewing constraints”. In soluble models, they yield simple algebraic conditions on the data of the conformal theory.

For amplitudes defined on closed orientable surfaces, the case of interest for models of oriented closed strings, only two sewing constraints are needed\textsuperscript{[2]}. The first one is the non-planar duality of the four-point amplitude on the sphere, a condition on the OPE coefficients $C_{ijk}$\textsuperscript{[1]}, while the second one is (essentially) the modular invariance\textsuperscript{[3]} of the torus amplitude. One may then argue by induction in the number of moduli that all amplitudes are consistent with the geometry.

Similar inductive arguments\textsuperscript{[4],[5]} show that, in order to define a conformal theory on surfaces with an arbitrary number of holes, four additional constraints need be satisfied. Now there are both “bulk” and “boundary” operators (the conformal counterparts of closed-string and open-string emission vertices), and the latter may also mediate changes of boundary conditions. Since in general there is some freedom in the choice of boundary conditions\textsuperscript{[6]}, the corresponding OPE coefficients $C_{abc}^{ijk}$ contain additional labels that specify them. The normalization coefficients of the one-point functions in the presence of a boundary, $C_{a}^{b}$\textsuperscript{[4]}, and the one-point functions of the identity in the presence of a boundary, $\alpha^{a}$, are new data of the theory. The four additional constraints typify their possible choices.

This letter is aimed at extending the results to non-orientable surfaces with an arbitrary number of boundaries, thus relating refs. [4] and [5] to the constructions of open-string descendants\textsuperscript{[7]} of closed-string models of refs.\textsuperscript{[8],[9],[10]}. Standard

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results on the topology of Riemann surfaces\textsuperscript{[11]}, first used in String Theory in ref.\textsuperscript{[12]}, imply that their type is fully specified by the total number of handles, the total number of holes and the total number of crosscaps, three crosscaps being topologically equivalent to one handle and one crosscap. A crosscap may be introduced by cutting a hole on the surface and by gluing to it a Möbius strip along its unique boundary. The crosscap is our main interest here, because it provides a new building block of the construction. It may be modelled as the real projective plane, \textit{i.e.} the plane augmented by a line at infinity. Alternatively, it may be defined as an orbifold of the sphere under the involution that identifies all pairs of antipodal points. Our results may be summarized as follows. First of all, crosscaps allow for \textit{three} new types of cuts (fig. 1). Two of them join a crosscap to another crosscap or to a hole, and first present themselves in Klein bottle and Möbius amplitudes, where they relate vacuum channels to “twisted” three-point functions. The resulting constraints were both implicitly taken into account in refs. \textsuperscript{[8]}, \textsuperscript{[9]} and \textsuperscript{[10]}. On the other hand, the last cut first presents itself in the real projective plane. It begins on one crosscap and comes back to it after enclosing one puncture. The resulting constraint is new, and is related rather neatly to the fundamental group of the twice punctured real projective plane. In the next Section we sketch the extension of the inductive argument of ref. \textsuperscript{[5]} and derive the crosscap constraint. In the last Section we illustrate the new constraint using the Ising model as an example.
The Real Projective Plane and the Crosscap Constraint

Fig. 1 displays the three new types of cuts allowed by the presence of crosscaps. They all connect pairs of identified points, since any line surrounding the cap may be deformed into a pair of these. If the two resulting ends are joined after enclosing at least one puncture, an allowed cut results. Alternatively, the two ends may terminate at a hole or at another crosscap. The last two settings require, respectively, a Möbius strip and a Klein bottle, and were both implicitly taken into account in refs. [8], [9] and [10]. Let us justify this statement for the Möbius amplitude*, whose vacuum channel exhibits the propagation of a closed string between a hole (one of the building blocks of ref. [5]) and a crosscap. The crosscap amplitude for a closed-string puncture is a new building block of the construction. It should be appreciated that the prescription for the vacuum channels of refs. [8], [9] and [10] endows the Möbius amplitude with reflection coefficients that, for each sector of the spectrum, are geometric means of those for the other two vacuum channels. This is precisely as demanded by the compatibility of the cuts. The dual interpretation corresponds to the ultraviolet limit of the vacuum channel, and exhibits the open-string three-point function with one open-string “twist”. Though in principle an independent building block, this amplitude is in fact proportional to the usual three-point function for each set of three physical states. In a similar fashion, one may relate the Klein bottle amplitude of fig. 1 both to the propagation of a closed string between two crosscaps and to the three-point function on the sphere with one closed-string “twist”. These “twist” operations are discussed in detail, within the old operator formalism, in ref. [13]. A reformulation in the language of conformal field theory will be presented elsewhere. We now turn to the main topic of this letter, the first type of cut and the corresponding crosscap constraint.

When working in the plane, the stereographic projection turns the two-fold

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* Following ref. [5], we have put one open-string puncture on the hole.
identification that defines the crosscap into the anti-conformal involution (fig. 2)

\[ I(z) = -\frac{1}{\bar{z}} \quad , \]  

(1)

that has no fixed points and is compatible with the \( SU(2) \) group that transforms \( z \) according to

\[ z \rightarrow \frac{az + b}{-bz + \bar{a}} \quad . \]  

(2)

This is the global symmetry of our problem.

In the more familiar case of a unit disk centered at the origin, the involution lacks the “minus” sign in eq. (1), and the resulting line of fixed points is the unit circle \(|z| = 1\). The global symmetry, an \( SU(1, 1) \) subgroup of \( SL(2, C) \), is related by conjugation to the \( SL(2, R) \) subgroup that obtains if the disk is identified with the upper-half plane. In the latter construction the involution is \( I(z) = \bar{z} \), and the line of fixed points is the real axis. The inversion present in eq. (1) plays an important role in the discussion that follows.

Despite its apparent simplicity, the crosscap is not topologically trivial. This non-orientable surface may be viewed as a disk where two halves of the boundary are identified according to their (opposite) orientations (fig. 3), and has a rather curious feature: its fundamental group has a single non-trivial generator, \( \alpha \), that becomes contractible if ran along twice. Indeed, if \( \alpha \) were slipped across the right edge, it would emerge from the left edge with a reverted orientation. The fundamental group of the crosscap is therefore \( Z_2 \), the additive group of integers modulo two. Amplitudes consistent with it require pairs of image punctures lying on pairs of opposite rays through the origin. For convenience, we confine one puncture to the upper-half plane, though the unit circle would be an equally good fundamental domain.

If \( n \) punctures are present, any one of them may probe the fundamental group of the surface with the other \( n - 1 \) punctures. The simplest non-trivial surface involves two punctures. Referring to fig. 4, the puncture \( A \) “sees” a fundamental
group with two generators $\alpha$ and $\beta$, where $\alpha^2 = \beta$. The crosscap constraint is the condition that all two-point amplitudes be single valued if one puncture, $A$ if you will, is moved along $\alpha$. A recursive argument along the lines of ref. [5] then shows that amplitudes with arbitrary numbers of punctures are consistent as well.

In order to derive the crosscap constraint, let us begin by defining the one-point function in the presence of a crosscap. In the notation and conventions of ref. [5], one would be tempted to write it

$$< \phi_k >_a = \frac{\Gamma^a_k}{(z + \frac{1}{z})^{2h}},$$

where the condition $h = \bar{h}$ is implicit and where $\Gamma^a_k$ are new data of the conformal theory associated to the crosscap. Though natural in view of eq. (1), this choice is not a convenient one, since in the basis $(dz, dI(z))$ an $(h, \bar{h})$ differential has spurious monodromies. On the other hand, referring all differentials to the standard basis $(dz, d\bar{z})$, the resulting one-point function,

$$< \phi_k >_a = \frac{\Gamma^a_k}{(1 + z\bar{z})^{2h}},$$

is manifestly single-valued.

In order to define the two-point function, one needs the OPE of two bulk operators,

$$\phi_i(z) \phi_j(w) \sim \sum_k C_{ijk} (z-w)^{h_k-h_i-h_j} (\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j} \phi_k(w),$$

In the conventions defined above, this may be used to deduce the OPE between one field and the image of the other. The result,

$$\phi_i(z) I(\phi_j(w)) \sim \sum_k C_{ijk} \Gamma^a_k (1 + z\bar{w})^{h_k-h_i-\bar{h}_j} (1 + \bar{z}w)^{\bar{h}_k-\bar{h}_i-\bar{h}_j} \phi_k(w),$$

contains $\Gamma^a_k$ since, as in eq. (4), the image of a field need not coincide precisely with the field itself.
The two-point function contains two factors. The first factor, $P$, depends on the conformal weights and determines the behavior of the amplitude under the residual projective group. In the conventions of ref. [5], and in the $(dz, d\bar{z})$ basis for the differentials,

$$P = (z_1 - z_2)^{r-h_1-h_2}(\bar{z}_1 - \bar{z}_2)^{\bar{r}-\bar{h}_1-\bar{h}_2}(1 + |z_1|^2)^{r-h_1-\bar{h}_1}(1 + |z_2|^2)^{r-h_2-\bar{h}_2}(1 + |z_1\bar{z}_2|^2)^{r-h_1-\bar{h}_1}(1 + |z_2\bar{z}_1|^2)^{r-h_2-\bar{h}_2},$$

where

$$r = \frac{1}{3} (h_1 + h_2 + \bar{h}_1 + \bar{h}_2).$$

The second factor, $Y$, is a function of the (real) cross-ratio

$$\eta = \frac{|z_1 - z_2|^2}{(1 + |z_1|^2)(1 + |z_2|^2)},$$

and is thus manifestly invariant under projective transformations.

The crosscap constraint results from the comparison of two distinct limits of the two-point function. In the first case of fig. 4 ($\eta \to 0$) two bulk operators approach one another, eq. (5) applies, and the limiting amplitude is the product of a three-point function on the sphere and of the one-point function of eq. (4). As in ref. [5], $Y$ may then be related to the conformal blocks $F^k$, and the result is

$$Y(\eta) = \sum_k C_{ijk} \Gamma_{ik}^a F^k(\eta).$$

In the second case of fig. 4 ($\eta \to 1$) one bulk operator approaches the image of the other after moving along $\alpha$. Since the quotient topology allows one to effectively “rotate” the crosscap, the limiting amplitude, determined by the OPE of eq. (6), is similar to the previous one. Namely, it is again the product of a three-point function on the sphere and of the one-point function of eq. (4), where one of
the punctures is now replaced by its image. The limiting behavior provides an independent determination of \(Y\),

\[
Y(\eta) = \sum_{k} (-1)^{h_\eta - \bar{h}_i + h_j - \bar{h}_j} C_{ijk} \Gamma^a_k F^k(1 - \eta)
\]

still linear in \(\Gamma\), since in this case the limiting one-point function involves an image originally in the upper-half plane, and therefore has the conventional normalization. The crosscap constraint is the condition that the two definitions of eqs. (10) and (11) coincide. In a rational model duality matrices relate the different forms of the conformal blocks, and the crosscap constraint becomes a linear equation for the \(\Gamma^a_k\),

\[
\sum_{k} C_{ijk} \Gamma^a_k M_{ij}^{\bar{j}\bar{i}} = (-1)^{h_\eta - \bar{h}_i + h_j - \bar{h}_j} C_{ijp} \Gamma^a_p
\]

This is the main result of this work. An inductive argument along the lines of ref. [5] suggests that eq. (12) completes the sewing constraints for (rational) conformal models on arbitrary Riemann surfaces.

**An Example**

In order to illustrate the content of eq. (12), let us consider the “open-string descendants” of the Ising model [9]. In this case, the spectrum of bulk operators is obtained combining the familiar torus partition function

\[
T = |\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2
\]

properly halved to account for the projection, with the Klein-bottle contribution

\[
K = \frac{1}{2} (\chi_0 + \chi_{1/2} + \chi_{1/16})
\]

As in the usual case, the bulk spectrum contains the primary fields 1, \(\epsilon\) and \(\sigma\), of dimensions \((0,0)\), \((1/2,1/2)\) and \((1/16,1/16)\), but their Verma modules are now
truncated according to the “parameter-space” projection. In addition, the spectrum of boundary operators includes three types of primary fields, of dimensions 0, 1/2 and 1/16, with an associated pattern of Chan-Paton charges\[14\] determined by combining the annulus

\[
A = \frac{1}{2} \left( n_0^2 + n_{1/2}^2 + n_{1/16}^2 \right) \chi_0 \\
+ \left( n_0 n_{1/2} + \frac{1}{2} n_{1/16}^2 \right) \chi_{1/2} \\
+ \left( n_0 n_{1/16} + n_{1/2} n_{1/16} \right) \chi_{1/16}
\] (3)

and Möbius partition functions

\[
M = \pm \frac{1}{2} \left[ \left( n_0 + n_{1/2} \right) \chi_0 + n_{1/16} \chi_{1/2} \right],
\] (4)

where the overall factors enforce the projection in the Chan-Paton charge space. The three types of charges, of multiplicities \(n_0, n_{1/2}\) and \(n_{1/16}\), correspond to as many types of boundaries [6], and the charge assignments are manifestly compatible with the factorization of amplitudes.

Making use of the matrix

\[
S = \frac{1}{2} \begin{pmatrix}
1 & 1 & \sqrt{2} \\
1 & 1 & -\sqrt{2} \\
\sqrt{2} & -\sqrt{2} & 0
\end{pmatrix}
\] (5)

in eqs. (2) and (3), and of the matrix \(P = T^{1/2} S T^2 S T^{1/2}\) in eqs. (4), one may turn them into the corresponding vacuum-channel contributions,

\[
\tilde{K} = \frac{1}{4} \left[ \left( 2 + \sqrt{2} \right) \chi_0 + \left( 2 - \sqrt{2} \right) \chi_{1/2} \right],
\] (6)

\[
\tilde{A} = \frac{1}{4} \left[ \left( n_0 + n_{1/2} + \sqrt{2} n_{1/16} \right)^2 \chi_0 \\
+ \left( n_0 + n_{1/2} - \sqrt{2} n_{1/16} \right)^2 \chi_{1/2} + \sqrt{2} \left( n_0 - n_{1/2} \right)^2 \chi_{1/16} \right],
\] (7)
\[ \tilde{M} = \pm \left[ \cos \left( \frac{\pi}{8} \right) (n_0 + n_{1/2} + \sqrt{2} n_{1/16}) \chi_0 + \sin \left( \frac{\pi}{8} \right) (n_0 + n_{1/2} - \sqrt{2} n_{1/16}) \chi_{1/2} \right] . \]

The consistency of this construction rests on the relation between eq. (8) and the other vacuum amplitudes of eqs. (6) and (7). Since all may be obtained by sewing bulk one-point amplitudes for boundaries and crosscaps, the coefficients in eq. (8) should be (and indeed are) geometric means of those in the other two amplitudes*. Actually, all the basic bulk one-point functions may be extracted from the vacuum amplitudes of eqs. (6), (7) and (8). Thus, apart from a combinatoric factor 1/2 related to the parameter-space orbifold construction, the square roots of the coefficients in eq. (7) are the normalizations of the boundary amplitudes. With a slight abuse of notation†, they are

\[
\begin{align*}
<1>_{b_1} &= \frac{1}{\sqrt{2}} & <1>_{\epsilon c} &= \frac{1}{\sqrt{2}} & <1>_{\sigma c} &= 1, \\
<\epsilon>_{b_1} &= \frac{1}{\sqrt{2}} & <\epsilon>_{\epsilon c} &= \frac{1}{\sqrt{2}} & <\epsilon>_{\sigma c} &= -1, \\
<\sigma>_{b_1} &= \frac{1}{\sqrt{2}} & <\sigma>_{\epsilon c} &= -\frac{1}{\sqrt{2}} & <\sigma>_{\sigma c} &= 0.
\end{align*}
\]

The ratios of these normalizations may be recovered from the sewing constraints for the disk geometry [5].

A similar argument relates the vacuum Klein-bottle amplitude to a single type of crosscap one-point functions,

\[
\begin{align*}
<1>_{c} &= \sqrt{\frac{2 + \sqrt{2}}{2}} & <\epsilon>_{c} &= \sqrt{\frac{2 - \sqrt{2}}{2}} & <\sigma>_{c} &= 0.
\end{align*}
\]

Clearly, the crosscap constraint does not fix the absolute normalizations in eq. (10). Moreover, the Klein-bottle vacuum channel defines a single one-point amplitude

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* As in ref. [9], we have inserted in eq. (8) the combinatoric factor that a proper measure over the moduli would generate.
† We are omitting the additional factors \((2\Im z)^{-2h}\).
for each primary field, and this amplitude may be regarded as a definition of the corresponding crosscap state. This is to be contrasted to the case of boundary states, where a preferred basis is selected by the fusion rules involving boundary fields (or, alternatively, by the Chan-Paton charge assignments). Still, eq. (12) allows only the parameter-space projection of eq. (2), since it forces \( \langle \sigma \rangle_c \) to vanish, while fixing the ratio of the other amplitudes. Most of the relevant data, OPE coefficients \( C_{ijk} \) and duality matrices, may be found in ref. [5].

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Figure Captions

Figure 1. New types of cuts and the simplest settings that allow them.

Figure 2. A puncture and its image under the involution of eq. (2.1).

Figure 3. The generators of the fundamental group of the crosscap.

Figure 4. Factorizations of the two-point amplitude.
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