Subvarieties of the Hilbert scheme
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Trianalytic subvarieties of the Hilbert scheme of points on a K3 surface

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Let $X$ be a hyperkähler manifold. Trianalytic subvarieties of $X$ are subvarieties which are complex analytic with respect to all complex structures induced by the hyperkähler structure. Given a K3 surface $M$, the Hilbert scheme classifying zero-dimensional subschemes of $M$ admits a hyperkähler structure. We show that for $M$ generic, there are no trianalytic subvarieties of the Hilbert scheme. This implies that a generic deformation of the Hilbert scheme of K3 has no proper complex subvarieties.

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1 Introduction

For basic results and definitions of hyperkähler geometry, see [Bes].

This Introduction is independent from the rest of this paper.
1.1 An overview

An almost hypercomplex manifold \( M \) is a manifold equipped with an action of the quaternion algebra \( \mathbb{H} \) on its tangent bundle. The manifold \( M \) is called hypercomplex if for all algebra embedding \( \mathbb{C} \hookrightarrow \mathbb{H} \), the corresponding almost complex structure \( I_\iota \) is integrable. A manifold \( M \) is called hyperkähler if, on top of that, \( M \) is equipped with a Riemannian metric which is Kähler with respect to the complex structures \( I_\iota \), for all embeddings \( \mathbb{C} \hookrightarrow \mathbb{H} \). The complex structures \( I_\iota \) are called induced complex structures; the corresponding Kähler manifold is denoted by \((M, I_\iota)\).

For a more formal definition of a hyperkähler manifold, see Definition 2.1. The notion of a hyperkähler manifold was introduced by E. Calabi, \[\text{[C]}\].

Clearly, the real dimension of \( M \) is divisible by 4. For \( \dim \mathbb{R} M = 4 \), there are only two classes of compact hyperkähler manifolds: compact tori and K3 surfaces.

Let \( M \) be a complex surface and \( M^{(n)} \) be its \( n \)-th symmetric power, \( M^{(n)} = M^n / S_n \). The variety \( M^{(n)} \) admits a natural desingularization \( M^{[n]} \), called Hilbert scheme of points, or Hilbert scheme for short. For its construction and additional results, see the lectures of H. Nakajima, \[\text{[N]}\].

Most importantly, \( M^{[n]} \) admits a hyperkähler metrics whenever the surface \( M \) is compact and hyperkähler (\[\text{[Bea]}\]). This way, Beauville constructed two series of examples of hyperkähler manifolds, associated with a torus \[\text{[Bea]}\] and a K3 surface. It was conjectured that all hyperkähler manifolds \( X \) with \( H^1(X) = 0, H^{2,0}(X) = \mathbb{C} \) are deformationally equivalent to one of these examples. We study the complex and hyperkähler geometry of \( M^{[n]} \) for \( M \) a “sufficiently generic” K3 surface, in order to construct counterexamples to this conjecture.

Let \( M \) be a hyperkähler manifold. A trianalytic subvariety of \( M \) is a closed subset which is complex analytic with respect to any of the induced complex structures. It was proven in \[\text{[V2]}\] that for all induced complex structures \( I \), except maybe a countable number, all complex subvarieties of \((M, I)\) are trianalytic (see also Proposition 2.11). This reduces the study of complex subvarieties of “sufficiently generic” deformations\[\text{[Bea]}\] of \( M \) to the study of trianalytic subvarieties.

Trianalytic subvarieties of hyperkähler manifolds were studied at length\[\text{[Bea]}\]. There is a natural torus action on its Hilbert scheme; to obtain the Beauville’s hyperkähler manifold, we must take the quotient by this action.

\[\text{[Bea]}\] By deformations of \( M \) we understand complex manifolds which are deformationally equivalent to \( M \).
in \([V2]\) and \([V-d2]\). Of the results obtained in this study, the most important ones are Desingularization Theorem (Theorem 2.13) and the cohomological criterion of trianalyticity (Theorem 2.8). The aim of the present paper is to obtain the following theorem.

**Theorem 1.1:** Let \(M\) be a complex K3 surface without automorphisms, \(H\) a hyperkähler structure on \(M\), and \(I\) an induced complex structure on \(M\) which is Mumford-Tate generic in the class of induced complex structures. Let \(M^{[n]}\) be a Hilbert scheme of points on the complex surface \((M, I)\). Pick any hyperkähler structure on \(M^{[n]}\), compatible with the complex structure. Then, \(M^{[n]}\) has no proper trianalytic subvarieties.

**Proof:** This is Theorem 9.12.

In the forthcoming paper, we construct a 21-dimensional family of compact hyperkähler manifolds \(M_x\), with

\[
H^1(M) = 0, \quad H^{2,0}(M) = \mathbb{C}
\]

which have proper trianalytic subvarieties. This leads to assertion that these manifolds are not deformations of \(M^{[n]}\).

As another application of Theorem 1.1 we obtain that a generic complex deformation of \(M^{[n]}\) has no proper closed complex subvarieties (Corollary 9.14).

A version of this result is true for a compact torus. For a generic complex structure \(I\) on a complex torus \(T\), the complex manifold \((T, I)\) has no proper subvarieties. This is easy to see from the fact that the group \(H^{2p, 2p}(T, I) \cap H^{2p}(T, \mathbb{Z})\) is empty. For a Hilbert scheme of K3, this Hodge-theoretic argument does not work. In fact, there are integer cycles in \(H^{2p, 2p}(M^{[n]}, I)\) for all complex structures on \(M^{[n]}\). As Corollary 9.14 implies, these cycles cannot be represented by subvarieties. This gives a counterexample to the Hodge conjecture.

Of course, for a generic complex structure \(I\), the manifold \((M^{[n]}, I)\) is not algebraic. There are many other counterexamples to the Hodge conjecture for non-algebraic manifolds.

In our approach to the study of trianalytic subvarieties of the Hilbert scheme, we introduce the concept of the **universal subvariety** of the Hilbert scheme (Definition 5.1). For a complex surface \(M\), the local automorphisms \(\gamma : U \rightarrow U\) of \(M \supset U\) act on the corresponding open subsets.
\[ U^{[n]} \subset M^{[n]} \] of the Hilbert scheme. A universal subvariety of \( M^{[n]} \) is a subvariety which is preserved by all automorphisms obtained this way (see \textbf{Definition 5.1} for a more precise statement).

We show that a trianalytic subvariety of a Hilbert scheme of a K3 surface \( M \) is universal, assuming that \( M \) is Mumford-Tate generic with respect to some hyperkähler structure and has no complex automorphisms (\textbf{Theorem 8.5}).

1.2 Contents

The paper is organized as follows.

- **Section 2** is taken with preliminary conventions and basic theorems. We define hyperkähler manifolds and formulate Yau’s theorem on the existence of hyperkähler structures on compact holomorphically symplectic manifolds of Kähler type. Furthermore, we define trianalytic subvarieties of hyperkähler manifolds and recall the basic properties of trianalytic subvarieties, following \[V2\] and \[V-d2\]. There are no new results in Section 2, and nothing unknown to the reader acquainted with the literature.

- In **Section 3**, we apply the desingularization theorem to the subvarieties of \( M^n \), where \( M \) is a generic K3 surface. We classify trianalytic subvarieties of \( M^n \) and describe them explicitly. This section is independent from the rest of this paper.

- We study the Hilbert scheme \( M^{[n]} \) of a smooth holomorphic symplectic complex surface \( M \) in **Section 4**. We give its definition and explain the construction of the holomorphic symplectic structure on \( M^{[n]} \). By Yau’s proof of Calabi conjecture (\textbf{Theorem 2.4}), this implies that \( M^{[n]} \) admits a hyperkähler structure.

Using perverse sheaves, we write down the cohomology of \( M^{[n]} \) in terms of diagonals in the symmetric power \( M^{(n)} \). This is done using the fact that the standard projection \( \pi : M^{[n]} \to M^{(n)} \) is a semi-small resolution of the symmetric power \( M^{(n)} \). These results are well known (\[N\]).

Further on, we apply the same type arguments to the trianalytic subvarieties \( X \subset M^{[n]} \). Using the holomorphic symplectic geometry, we show that the map \( \tilde{X} \xrightarrow{\tilde{\pi}} \pi(X) \) is a semi-small resolution, where \( \tilde{X} \xrightarrow{n} X \) is the hyperkähler desingularization of \( X \) (\textbf{Theorem 2.13}).
This gives an expression for the cohomology of $\tilde{X}$. We don’t use this result anywhere in this paper.

- Section 4 is heavily based on perverse sheaves ([BBD]), and does not use results of hyperkähler geometry, except Desingularization Theorem (Theorem 2.13).

- The following four sections (Sections 5–8) are dedicated to the study of universal properties of the Hilbert scheme.
  - In Section 5, we give a definition of a universal subvariety of a Hilbert scheme. A relative dimension of a universal subvariety is a dimension of the generic fiber of the projection $\pi : X \to \pi(X)$, where $\pi : M^{[n]} \to M^{(n)}$ is the standard projection of the Hilbert scheme to the symmetric power of $M$. We classify and describe explicitly the universal subvarieties of relative dimension 0. Results of Section 5 are in no way related to the hyperkähler geometry.
  - In Section 6, we study the Hilbert scheme of a K3 surface $M$, assuming that $M$ is Mumford-Tate generic with respect to some hyperkähler structure. We consider subvarieties $X \subset M^{[n]}$, such that $X$ is projected generically finite to $\pi(X) \subset M$, and $\pi(X)$ is a diagonal in $M^{(n)}$. We use the theory of Yang-Mills connections and Uhlenbeck–Yau theorem, in order to show that such subvarieties are universal, in the sense of Section 5.
  - Section 7 is completely parallel to Section 5. We define special subvarieties of the Hilbert scheme, which are similar to the universal subvarieties, with some conditions relaxed. Whereas universal subvarieties are subvarieties which are fixed by all local automorphisms of $M^{[n]}$ coming from $M$, special subvarieties are the subvarieties fixed by all the local automorphisms coming from $M$ which preserve a finite subset $S \subset M$. As in Section 5, we classify and describe explicitly the special subvarieties of relative dimension zero. Using results of Section 5, we study the subvarieties $X \subset M^{[n]}$, such that $X$ is projected generically finite to $\pi(X)$, where $M$ is a generic K3 surface. We show that all such subvarieties are special of relative dimension zero.
  - In Section 8, we study the deformations of subvarieties of the Hilbert scheme of a K3 surface. The deformations of special subvarieties of relative dimension zero are easy to study using the
explicit description given in Section 7. The deformations of trianalytic subvarieties were studied at length in [V3]. In conjunction, these results lead to the assertion that all trianalytic subvarieties of $M^{[n]}$ are universal, in the sense of Section 5.

- In Section 9, we study the second cohomology of universal subvarieties $X_\alpha \hookrightarrow M^{[n]}$ of the Hilbert scheme $M^{[n]}$, in assumption that $X_\alpha$ is trianalytic. First of all, we show that $X_\alpha$ is birationally equivalent to a hyperkähler manifold which is a product of Hilbert schemes of $M$. Using Mukai’s theorem, which states that second cohomology of hyperkähler manifolds is a birational invariant, we obtain a structure theorem for $H^2(X_\alpha)$. Assuming that $X_\alpha$ is not a product of hyperkähler manifolds, we show that the pullback map $\varphi^*: H^2(M^{[n]}) \rightarrow H^2(X_\alpha)$ is an isomorphism, and compute this map explicitly. The second cohomology of a hyperkähler manifold $X$ is equipped with a canonical non-degenerate quadratic form $\langle \cdot, \cdot \rangle_B$, defined up to a constant multiplier. This form is invariant under the natural $SU(2)$-action on $H^2(X)$. We compute the pullback of the form $\langle \cdot, \cdot \rangle_B$ under the map $\varphi^*: H^2(M^{[n]}) \rightarrow H^2(X_\alpha)$, and show that it cannot be $SU(2)$-invariant. Thus, $\varphi^*$ is not compatible with the $SU(2)$-action on the second cohomology. This implies that $\varphi$ cannot be compatible with the hyperkähler structures on $X_\alpha$, $M^{[n]}$. Therefore, $M^{[n]}$ contains no trianalytic subvarieties.

2 Hyperkähler manifolds

2.1 Hyperkähler manifolds

This subsection contains a compression of the basic and best known results and definitions from hyperkähler geometry, found, for instance, in [Bes] or in [Bea].

Definition 2.1: (Bes) A hyperkähler manifold is a Riemannian manifold $M$ endowed with three complex structures $I$, $J$ and $K$, such that the following holds.

(i) the metric on $M$ is Kähler with respect to these complex structures and

(ii) $I$, $J$ and $K$, considered as endomorphisms of a real tangent bundle, satisfy the relation $I \circ J = -J \circ I = K$. 

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The notion of a hyperkähler manifold was introduced by E. Calabi ([C]).

Clearly, a hyperkähler manifold has a natural action of the quaternion algebra \( \mathbb{H} \) in its real tangent bundle \( TM \). Therefore its complex dimension is even. For each quaternion \( L \in \mathbb{H} \), \( L^2 = -1 \), the corresponding automorphism of \( TM \) is an almost complex structure. It is easy to check that this almost complex structure is integrable ([Bes]).

**Definition 2.2:** Let \( M \) be a hyperkähler manifold, \( L \) a quaternion satisfying \( L^2 = -1 \). The corresponding complex structure on \( M \) is called an **induced complex structure**. The \( M \) considered as a Kähler manifold is denoted by \( (M, L) \). In this case, the hyperkähler structure is called **compatible with the complex structure** \( L \).

Let \( M \) be a compact complex variety. We say that \( M \) is of hyperkähler **type** if \( M \) admits a hyperkähler structure compatible with the complex structure.

**Definition 2.3:** Let \( M \) be a complex manifold and \( \Theta \) a closed holomorphic 2-form over \( M \) such that \( \Theta^n = \Theta \wedge \Theta \wedge \ldots \), is a nowhere degenerate section of a canonical class of \( M \) \((2n = \dim_{\mathbb{C}}(M))\). Then \( M \) is called **holomorphically symplectic**.

Let \( M \) be a hyperkähler manifold; denote the Riemannian form on \( M \) by \(< \cdot, \cdot >\). Let the form \( \omega_I := < I(\cdot), \cdot > \) be the usual Kähler form which is closed and parallel (with respect to the Levi-Civitta connection). Analogously defined forms \( \omega_J \) and \( \omega_K \) are also closed and parallel.

A simple linear algebraic consideration ([Bes]) shows that the form \( \Theta := \omega_J + \sqrt{-1} \omega_K \) is of type \((2,0)\) and, being closed, this form is also holomorphic. Also, the form \( \Theta \) is nowhere degenerate, as another linear algebraic argument shows. It is called the **canonical holomorphic symplectic form of a manifold** \( M \). Thus, for each hyperkähler manifold \( M \), and an induced complex structure \( L \), the underlying complex manifold \( (M, L) \) is holomorphically symplectic. The converse assertion is also true:

**Theorem 2.4:** ([Bea], [Bes]) Let \( M \) be a compact holomorphically symplectic Kähler manifold with the holomorphic symplectic form \( \Theta \), a Kähler class \([\omega] \in H^{1,1}(M)\) and a complex structure \( I \). Let \( n = \dim_{\mathbb{C}} M \). Assume that \( \int_M \omega^n = \int_M (Re \Theta)^n \). Then there is a unique hyperkähler structure
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(I, J, K, (·, ·)) over M such that the cohomology class of the symplectic form \( \omega_I = (\cdot, I\cdot) \) is equal to \([\omega]\) and the canonical symplectic form \( \omega_J + \sqrt{-1}\omega_K \) is equal to \( \Theta \).

Theorem 2.4 immediately follows from the conjecture of Calabi, proven by Yau ([Y]).

Let \( M \) be a hyperkähler manifold. We identify the group \( SU(2) \) with the group of unitary quaternions. This gives a canonical action of \( SU(2) \) on the tangent bundle, and all its tensor powers. In particular, we obtain a natural action of \( SU(2) \) on the bundle of differential forms.

**Lemma 2.5:** The action of \( SU(2) \) on differential forms commutes with the Laplacian.

**Proof:** This is Proposition 1.1 of [V1].

Thus, for compact \( M \), we may speak of the natural action of \( SU(2) \) in cohomology.

Further in this article, we use the following statement.

**Lemma 2.6:** Let \( \omega \) be a differential form over a hyperkähler manifold \( M \). The form \( \omega \) is \( SU(2) \)-invariant if and only if it is of Hodge type \((p, p)\) with respect to all induced complex structures on \( M \).

**Proof:** This is [V-bun], Proposition 1.2.

### 2.2 Trianalytic subvarieties in compact hyperkähler manifolds.

In this subsection, we give a definition and a few basic properties of trianalytic subvarieties of hyperkähler manifolds. We follow [V2].

Let \( M \) be a compact hyperkähler manifold, \( \dim_{\mathbb{R}} M = 2m \).

**Definition 2.7:** Let \( N \subset M \) be a closed subset of \( M \). Then \( N \) is called **trianalytic** if \( N \) is a complex analytic subset of \((M, I)\) for any induced complex structure \( I \).

Let \( I \) be an induced complex structure on \( M \), and \( N \subset (M, I) \) be a closed analytic subvariety of \((M, I)\), \( \dim_{\mathbb{C}} N = n \). Consider the homology
class represented by \( N \). Let \( [N] \in H^{2m-2n}(M) \) denote the Poincare dual cohomology class. Recall that the hyperkähler structure induces the action of the group \( SU(2) \) on the space \( H^{2m-2n}(M) \).

**Theorem 2.8:** Assume that \( [N] \in H^{2m-2n}(M) \) is invariant with respect to the action of \( SU(2) \) on \( H^{2m-2n}(M) \). Then \( N \) is trianalytic.

**Proof:** This is Theorem 4.1 of [V2].

**Remark 2.9:** Trianalytic subvarieties have an action of quaternion algebra in the tangent bundle. In particular, the real dimension of such subvarieties is divisible by 4.

**Definition 2.10:** Let \( M \) be a complex manifold admitting a hyperkähler structure \( \mathcal{H} \). We say that \( M \) is of general type or generic with respect to \( \mathcal{H} \) if all elements of the group

\[
\bigoplus_p H^{p,p}(M) \cap H^{2p}(M, \mathbb{Z}) \subset H^*(M)
\]

are \( SU(2) \)-invariant. We say that \( M \) is Mumford–Tate generic if for all \( n \in \mathbb{Z}^{>0} \), all the cohomology classes

\[
\alpha \in \bigoplus_p H^{p,p}(M^n) \cap H^{2p}(M^n, \mathbb{Z}) \subset H^*(M^n)
\]

are \( SU(2) \)-invariant. In other words, \( M \) is Mumford–Tate generic if for all \( n \in \mathbb{Z}^{>0} \), the \( n \)-th power \( M^n \) is generic. Clearly, Mumford–Tate generic implies generic.

**Proposition 2.11:** Let \( M \) be a compact manifold, \( \mathcal{H} \) a hyperkähler structure on \( M \) and \( S \) be the set of induced complex structures over \( M \). Denote by \( S_0 \subset S \) the set of \( L \in S \) such that \( (M, L) \) is Mumford-Tate generic with respect to \( \mathcal{H} \). Then \( S_0 \) is dense in \( S \). Moreover, the complement \( S \setminus S_0 \) is countable.

**Proof:** This is Proposition 2.2 from [V2].

Theorem 2.8 has the following immediate corollary:

**Corollary 2.12:** Let \( M \) be a compact holomorphically symplectic manifold. Assume that \( M \) is of general type with respect to a hyperkähler structure \( \mathcal{H} \). Let \( S \subset M \) be closed complex analytic subvariety. Then \( S \) is trianalytic with respect to \( \mathcal{H} \).
In [V-d3], [V-d], [V-d2], we gave a number of equivalent definitions of a singular hyperkähler and hypercomplex variety. We refer the reader to [V-d2] for the precise definition; for our present purposes it suffices to say that all trianalytic subvarieties are hyperkähler varieties. The following Desingularization Theorem is very useful in the study of trianalytic subvarieties.

**Theorem 2.13:** ([V-d2]) Let $M$ be a hyperkähler or a hypercomplex variety, $I$ an induced complex structure. Let

$$\tilde{(M,I)} \xrightarrow{n} (M,I)$$

be the normalization of $(M,I)$. Then $(\tilde{M},I)$ is smooth and has a natural hyperkähler structure $\mathcal{H}$, such that the associated map $n : (\tilde{M},I) \to (M,I)$ agrees with $\mathcal{H}$. Moreover, the hyperkähler manifold $\tilde{M} := (M,I)$ is independent from the choice of induced complex structure $I$.

2.3 Simple hyperkähler manifolds

**Definition 2.14:** ([Bea]) A connected simply connected compact hyperkähler manifold $M$ is called **simple** if $M$ cannot be represented as a product of two hyperkähler manifolds:

$$M \neq M_1 \times M_2, \text{ where } \dim M_1 > 0 \text{ and } \dim M_2 > 0$$

Bogomolov proved that every compact hyperkähler manifold has a finite covering which is a product of a compact torus and several simple hyperkähler manifolds. Bogomolov’s theorem implies the following result ([Bea]):

**Theorem 2.15:** Let $M$ be a compact hyperkähler manifold. Then the following conditions are equivalent.

(i) $M$ is simple

(ii) $M$ satisfies $H^1(M, \mathbb{R}) = 0$, $H^{2,0}(M) = \mathbb{C}$, where $H^{2,0}(M)$ is the space of $(2,0)$-classes taken with respect to any of induced complex structures.

\[\square\]
2.4 Hyperholomorphic bundles

This subsection contains several versions of a definition of hyperholomorphic connection in a complex vector bundle over a hyperkähler manifold. We follow \[\text{	extit{V-bun}}\].

Let $B$ be a holomorphic vector bundle over a complex manifold $M$, $\nabla$ a connection in $B$ and $\Theta \in \Lambda^2 \otimes \text{End}(B)$ be its curvature. This connection is called \textit{compatible with a holomorphic structure} if $\nabla_X(\zeta) = 0$ for any holomorphic section $\zeta$ and any antiholomorphic tangent vector $X$. If there exist a holomorphic structure compatible with the given Hermitian connection then this connection is called \textit{integrable}.

One can define a \textit{Hodge decomposition} in the space of differential forms with coefficients in any complex bundle, in particular, $\text{End}(B)$.

\textbf{Theorem 2.16}: Let $\nabla$ be a Hermitian connection in a complex vector bundle $B$ over a complex manifold. Then $\nabla$ is integrable if and only if $\Theta \in \Lambda^{1,1}(M, \text{End}(B))$, where $\Lambda^{1,1}(M, \text{End}(B))$ denotes the forms of Hodge type $(1,1)$. Also, the holomorphic structure compatible with $\nabla$ is unique.

\textbf{Proof}: This is Proposition 4.17 of \[\text{\textit{Ko}}\], Chapter I. \hfill $\blacksquare$

\textbf{Definition 2.17}: Let $B$ be a Hermitian vector bundle with a connection $\nabla$ over a hyperkähler manifold $M$. Then $\nabla$ is called \textit{hyperholomorphic} if $\nabla$ is integrable with respect to each of the complex structures induced by the hyperkähler structure.

As follows from \[\text{Theorem 2.16}\], $\nabla$ is hyperholomorphic if and only if its curvature $\Theta$ is of Hodge type $(1,1)$ with respect to any of complex structures induced by a hyperkähler structure.

As follows from \[\text{Lemma 2.6}\], $\nabla$ is hyperholomorphic if and only if $\Theta$ is a $SU(2)$-invariant differential form.

\textbf{Example 2.18}: (Examples of hyperholomorphic bundles)

(i) Let $M$ be a hyperkähler manifold, $TM \otimes \mathbb{R} \mathbb{C}$ a complexification of its tangent bundle equipped with Levi–Civita connection $\nabla$. Then $\nabla$ is integrable with respect to each induced complex structure, and hence, Yang–Mills.

(ii) For $B$ a hyperholomorphic bundle, all its tensor powers are also hyperholomorphic.
Thus, the bundles of differential forms on a hyperkähler manifold are also hyperholomorphic.

### 2.5 Stable bundles and Yang–Mills connections.

This subsection is a compendium of the most basic results and definitions from the Yang–Mills theory over Kähler manifolds, concluding in the fundamental theorem of Uhlenbeck–Yau [UY].

**Definition 2.19:** Let $F$ be a coherent sheaf over an $n$-dimensional compact Kähler manifold $M$. We define $\text{deg}(F)$ as

$$
\text{deg}(F) = \int_M c_1(F) \wedge \omega^{n-1} \frac{\omega}{\text{vol}(M)}
$$

and $\text{slope}(F)$ as

$$
\text{slope}(F) = \frac{1}{\text{rank}(F)} \cdot \text{deg}(F).
$$

The number $\text{slope}(F)$ depends only on a cohomology class of $c_1(F)$.

Let $F$ be a coherent sheaf on $M$ and $F' \subset F$ its proper subsheaf. Then $F'$ is called **destabilizing subsheaf** if $\text{slope}(F') \geq \text{slope}(F)$.

A holomorphic vector bundle $B$ is called **stable** if it has no destabilizing subsheaves.

Later on, we usually consider the bundles $B$ with $\text{deg}(B) = 0$.

Let $M$ be a Kähler manifold with a Kähler form $\omega$. For differential forms with coefficients in any vector bundle there is a Hodge operator $L: \eta \rightarrow \omega \wedge \eta$. There is also a fiberwise-adjoint Hodge operator $\Lambda$ (see [GH]).

**Definition 2.20:** Let $B$ be a holomorphic bundle over a Kähler manifold $M$ with a holomorphic Hermitian connection $\nabla$ and a curvature $\Theta \in \Lambda^{1,1} \otimes \text{End}(B)$. The Hermitian metric on $B$ and the connection $\nabla$ defined by this metric are called **Yang–Mills** if

$$
\Lambda(\Theta) = \text{constant} \cdot \text{Id} \big|_B,
$$

where $\Lambda$ is a Hodge operator and $\text{Id} \big|_B$ is the identity endomorphism which is a section of $\text{End}(B)$.

---

1 In the sense of Mumford-Takemoto
Further on, we consider only these Yang–Mills connections for which this constant is zero.

A holomorphic bundle is called **indecomposable** if it cannot be decomposed onto a direct sum of two or more holomorphic bundles.

The following fundamental theorem provides examples of Yang–Mills bundles.

**Theorem 2.21:** (Uhlenbeck-Yau) Let B be an indecomposable holomorphic bundle over a compact Kähler manifold. Then B admits a Hermitian Yang-Mills connection if and only if it is stable, and this connection is unique.

**Proof:** [UY]. ■

**Proposition 2.22:** Let $M$ be a hyperkähler manifold, $L$ an induced complex structure and $B$ be a complex vector bundle over $(M, L)$. Then every hyperholomorphic connection $\nabla$ in $B$ is Yang-Mills and satisfies $\Lambda(\Theta) = 0$ where $\Theta$ is a curvature of $\nabla$.

**Proof:** We use the definition of a hyperholomorphic connection as one with $SU(2)$-invariant curvature. Then Proposition 2.22 follows from the

**Lemma 2.23:** Let $\Theta \in \Lambda^2(M)$ be a $SU(2)$-invariant differential 2-form on $M$. Then $\Lambda_L(\Theta) = 0$ for each induced complex structure $L$.

**Proof:** This is Lemma 2.1 of [V-bun]. ■

Let $M$ be a compact hyperkähler manifold, $I$ an induced complex structure. For any stable holomorphic bundle on $(M, I)$ there exists a unique Hermitian Yang-Mills connection which, for some bundles, turns out to be hyperholomorphic. It is possible to tell when this happens (though in the present paper we never use this knowledge).

**Theorem 2.24:** Let $B$ be a stable holomorphic bundle over $(M, I)$, where $M$ is a hyperkähler manifold and $I$ is an induced complex structure over $M$. Then $B$ admits a compatible hyperholomorphic connection if and only if the first two Chern classes $c_1(B)$ and $c_2(B)$ are $SU(2)$-invariant.

**Proof:** This is Theorem 2.5 of [V-bun]. ■

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2 By $\Lambda_L$ we understand the Hodge operator $\Lambda$ associated with the Kähler complex structure $L$.

3 We use Lemma 2.5 to speak of action of $SU(2)$ in cohomology of $M$. 
3 Trianalytic subvarieties of powers of K3 surfaces

3.1 Trianalytic subvarieties of a product of a K3 surface with itself

Let $M$ be any manifold, $M^n = M \times \ldots \times M$ its $n$-th product with itself. We define the “natural” subvarieties of $M$, recursively, as follows.

(i) Natural subvarieties of $M$ are $M$ and points.

(ii) Let $Z \subset M^n$ by a natural subvariety. The following subvarieties of $M^{n+1}$ are natural.

a. $Z_M := Z \times M \subset M^n \times M = M^{n+1}$

b. $Z_t := Z \times \{t\} \subset M^n \times M = M^{n+1}$, depending on a point $t \in M$.

c. $Z_i := \{(m_1, \ldots, m_{n+1}) \in Z \times M \mid m_i = m_{n+1}\}$, depending on a number $i \in \{1, \ldots, n\}$

The main result of this section is the following theorem.

**Theorem 3.1:** Let $M$ be a hyperkähler K3-surface which has no hyperkähler automorphisms, and $X \subset M^n$ an irreducible trianalytic subvariety. Then $X$ is “natural”, in the sense of (3.1).

**Proof:** Let

$$\tilde{\Pi}_{n+1} : M^{n+1} \rightarrow M^n$$

be the natural projection $m_1, \ldots, m_{n+1} \rightarrow m_1, \ldots, m_n$. Clearly, $\tilde{\Pi}_{n+1}(X)$ is irreducible and trianalytic. Using induction, we may assume that

a trianalytic subvariety $X \subset M^k$ is natural, in the sense of (3.1), for $k \leq n$.

Clearly, by (3.2) $\tilde{\Pi}_{n+1}(X)$ is of the type (3.1). All varieties $Z$ of type (3.1) are isomorphic to $M^k$, for $k = \dim_{\mathbb{H}} Z$. Thus, $X$ is realized as a subvariety in

$$\tilde{\Pi}_{n+1}(X) \times M = M^{\dim_{\mathbb{H}} \tilde{\Pi}_{n+1}(X)+1}.$$ 

Unless $\dim_{\mathbb{H}} \tilde{\Pi}_{n+1}(X) = n$, (3.2) implies that $X$ is a “natural” subvariety. Thus, to prove Theorem 3.1, we may assume that $\tilde{\Pi}_{n+1}(X) = M^n$. 

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For a point \( t \in M \), let \( X_t := \left\{ (m_1, \ldots, m_{n+1}) \in X \mid m_{n+1} = t \right\} \). The subvariety \( X_t \subset M^n \) is not necessarily irreducible, because the components of \( X_t \) may “flow together” as \( t \) changes, so that \( X = \bigcup_{t \in M} X_t \) is still irreducible, while the \( X_t \)’s are not. However, all components of \( X_t \) must be deformationally equivalent in the class of “natural” subvarieties of \( M^n \), in order for \( X \) to be irreducible.

Since \( X \) is irreducible and \( X = \bigcup_t X_t \), then either \( X_t = X \) for one value of \( t \) (and in this case \( X = M^n \times \{t\} \)), or \( X_t \neq \emptyset \) for \( t \) in a positive-dimensional trianalytic subset of \( M \). Since \( \dim_H M = 1 \), this subset coincides with \( M \).

Using (3.2), we obtain that all irreducible components of \( X_t \) are of the type (3.1). All “natural” subvarieties of \( M^n \) of complex codimension \( 2 = \dim_{\mathbb{C}} M \) are given by either \( m_i = m_j \) for some distinct fixed indices \( i, j \), or by \( m_i = t \) for a fixed index \( i \) and a fixed point \( t \in M \). We proceed on case-by-case basis.

(i) For some \( t \), \( X_t \) contains a component \( X_t^{i,j} \) given by \( m_i = m_j \) for distinct fixed indices \( i, j \). Since \( X_t^{i,j} \) is rigid in the class of natural subvarieties, and \( X \) is irreducible, this implies that \( X_t \) contains \( M_{i,j}^n \times \{t\} \) for all \( t \), where \( M_{i,j}^n \subset M^n \) is a subvariety given by \( m_i = m_j \). Since \( \dim X = \dim M_{i,j}^n \times M \), \( X \) irreducible and \( M_{i,j}^n \times M \subset X \), this implies that \( X = M_{i,j}^n \times M \). This proves Theorem 3.1 (case (i)).

(ii) For some \( t \), \( X_t \) contains a component \( X_t^i(m) \), given by \( m_i = m \), for a fixed index \( i \) and a fixed point \( m \in M \). Deforming \( X_t^i(m) \) in the class of natural subvarieties, we obtain again \( X_t^i(m') \), with different \( m' \). Taking a union of all \( X_t^i(m) \subset X_t \), for some fixed \( i \) and varying \( t \) and \( m \), we obtain a closed subvariety of \( X \) of the same dimension as \( X \). Since \( X \) is irreducible, \( all \) components of \( X_t \) are given by \( m_i = m \), for a fixed index \( i \) and a fixed point \( m \in M \). Consider a trianalytic subvariety \( Z \subset M^2 \),

\[
Z := \{ (m, t) \in M^2 \mid X_t^i(m) \subset X_t \}
\]

To prove Theorem 3.1 (case (ii)), it suffices to show that \( Z \) is natural, in the sense of (3.1). We reduced Theorem 3.1 to the case of trianalytic subvarieties in \( M^2 \).

The following lemma finishes the proof of Theorem 3.1.
**Lemma 3.2:** Let $M$ be a hyperkähler K3-surface which has no hyperkähler automorphisms, and $X \subset M^2$ a closed irreducible trianalytic subvariety of $M^2$. Then $X$ is a “natural” subvariety of $M^2$, in the sense of (3.1).

**Proof:** Let $\pi_1, \pi_2 : M^2 \rightarrow M$ be the natural projections. Assume that neither $\pi_1(X)$ nor $\pi_2(X)$ is a point, and $X \not\subset M^2$ (otherwise $X$ obviously satisfies (3.1)). Let $\tilde{X} \rightarrow X$ be the desingularization of $X$, given by Theorem 2.13.

Consider the maps $p_i : \tilde{X} \rightarrow M$, $i = 1, 2$, given by $p_i := n \circ \pi_i$. Since $\dim X = \dim M$, and $p_i$ is non-trivial, these maps have non-degenerate Jacobians in general point. Fix an induced complex structure $I$ on $M$, and consider $X, \tilde{X}, M^2$ as complex varieties and $p_i$ as holomorphic maps. Let $\Theta_M$ be the holomorphic symplectic form of $M$. Then $p_i^* \Theta_M$ gives a section of the canonical class of $\tilde{X}$. Since $\tilde{X}$ is compact and hyperkähler, any non-zero section of the canonical class is nowhere degenerate. Thus, $p_i^* \Theta_M$ is nowhere degenerate, and the Jacobian of $p_i$ nowhere vanishes. Therefore, $p_i$ is a covering. Since $X$ is irreducible, $\tilde{X}$ is connected, and since $M$ is simply connected, $p_i$ is isomorphism. Since $M$ has no hyperkähler automorphisms, except identity, $X$ is a graph of an identity map. This proves Lemma 3.2 and Theorem 3.1. ■

**Corollary 3.3:** Let $M$ be a complex K3 surface with no complex automorphisms. Assume that $M$ admits a hyperkähler structure $\mathcal{H}$ such that $M$ is Mumford-Tate generic with respect to $\mathcal{H}$. Let $X$ be an irreducible complex subvariety of $M^n$. Then $X$ is a “natural” subvariety of $M^n$, in the sense of (3.1).

**Proof:** By Corollary 2.12, $X$ is trianalytic. Now Corollary 3.3 is implied by Theorem 3.1. ■

### 3.2 Subvarieties of symmetric powers of varieties

In this section, we fix the notation regarding the “natural” subvarieties of the symmetric powers of complex varieties.

Let $M$ be a complex variety and $M^{(n)}$ its symmetric power, $M^{(n)} = M^n/S_n$. The space $M^{(n)}$ has a natural stratification by **diagonals** $\Delta_{(\alpha)}$, which are numbered by Young diagrams

$$\alpha = (n_1 \geq n_2 \geq \ldots \geq n_k), \quad \sum n_i = n.$$
This stratification is constructed as follows. Let \( \sigma : M^n \to M^{(n)} \) be the natural finite map (a quotient by the symmetric group). Then

\[
\Delta_\alpha := \sigma \left( \left\{ (x_1, x_2, ..., x_n) \in M^n \mid \begin{array}{l}
    x_1 = x_2 = ... = x_{n_1}, \\
    x_{n_1+1} = x_{n_1+2} = ... = x_{n_1+n_2}, \\
    ..., \\
    x_{\sum_{i=1}^{k-1} n_i+1} = x_{\sum_{i=1}^{k-1} n_i+2} = ... = x_n, \\
\end{array} \right\} \right)
\]  

(3.3)

where \( \sigma : M^n \to M^{(n)} \) is the natural quotient map.

Consider a Young diagram \( \alpha \),

\[ \alpha = (n_1 \geq n_2 \geq ... \geq n_k), \quad \sum n_i = n. \]

As in (3.3), \( \alpha \) corresponds to a diagonal \( \Delta_\alpha \), which is a closed subvariety of \( M^{(n)} \). Fix a subset \( A \subset \{1, ..., k\} \), and let \( \varphi : A \to M \) be an arbitrary map. Then \( \Delta_\alpha(A, \varphi) \subset \Delta_\alpha \) is given by

\[
\Delta_\alpha(A, \varphi) := \sigma \left( \left\{ (x_1, x_2, ..., x_n) \in M^n \mid \begin{array}{l}
    x_1 = x_2 = ... = x_{n_1}, \\
    x_{n_1+1} = x_{n_1+2} = ... = x_{n_1+n_2}, \\
    ..., \\
    x_{\sum_{i=1}^{k-1} n_i+1} = x_{\sum_{i=1}^{k-1} n_i+2} = ... = x_n, \\
    \text{and } \forall i \in A, \ x_i = \varphi(i) \end{array} \right\} \right),
\]  

(3.4)

where \( \sigma : M^n \to M^{(n)} \) is the standard projection. For “sufficiently generic” K3 surfaces, all complex subvarieties in \( M^n \) are given by Corollary 3.3. From Corollary 3.3, it is easy to deduce the following result.

**Proposition 3.4:** Let \( M \) be a complex K3 surface with no complex automorphisms. Assume that \( M \) admits a hyperkähler structure \( \mathcal{H} \) such that \( M \) is Mumford-Tate generic with respect to \( \mathcal{H} \). Let \( X \) be an irreducible complex subvariety of \( M^n \). Then \( X = \Delta_\alpha(A, \varphi) \) for appropriate \( \alpha, A, \varphi \).

\[ \blacksquare \]

### 4 Hilbert scheme of points

For the definitions and results related to the Hilbert scheme of points on a surface, see the excellent lecture notes of H. Nakajima [N].
4.1 Symplectic structure of the Hilbert scheme

Definition 4.1: Let $M$ be a complex surface. The $n$-th Hilbert scheme of points, also called Hilbert scheme of $M$ (denoted by $M^{[n]}$) is the scheme classifying the zero-dimensional subschemes of $M$ of length $n$.

There is a natural projection $\pi: M^{[n]} \to M^{(n)}$ associating to a subscheme $S \subset M$ the set of points $x_i \in \text{Sup}(S)$ of support of $S$, taken with the multiplicities equal to the length of $S$ in $x_i$.

It is well-known that a Hilbert scheme of a smooth surface is a smooth manifold, and the fibers of $\pi$ are irreducible and reduced (see, e. g. \[N\]).

For our purposes, the most important property of the Hilbert scheme is the existence of the non-degenerate holomorphic symplectic structure, for $M$ holomorphically symplectic.

Let $X$ be an irreducible complex analytic space, which is reduced in generic point, and $\Omega^1X$ the sheaf of Kahler differentials on $X$. We denote the exterior square $\Lambda^2_{\mathcal{O}_X} \Omega^1X$ by $\Omega^2X$. The sections of $\Lambda^2_{\mathcal{O}_X} \Omega^1X$ are called 2-forms on $X$. We say that two-forms $\omega_1, \omega_2$ are equal up to a torsion if $\omega_1 = \omega_2$ in the generic point of $X$.

Proposition 4.2: (Beauville) Let $M$ be a smooth complex surface equipped with a nowhere degenerate holomorphic 2-form. Then

(i) the Hilbert scheme $M^{[n]}$ is equipped with a natural, nowhere degenerate holomorphic 2-form $\Theta_{M^{[n]}}$.

(ii) Consider the Cartesian square

\[
\begin{array}{ccc}
\tilde{M}^{[n]} & \xrightarrow{\tilde{\pi}} & M^{[n]} \\
\downarrow{\tilde{\sigma}} & & \downarrow{\sigma} \\
M^{[n]} & \xrightarrow{\pi} & M^{(n)} \\
\end{array}
\]

Let $\Theta_{M^n}$ be the natural symplectic form on $M^n$. Then the pullback $\tilde{\sigma}^*\Theta_{M^{[n]}}$ is equal to the pullback $\tilde{\pi}^*\Theta_{M^n}$, outside of the subvariety $D \subset \tilde{M}^{[n]}$ of codimension 2.

(iii) The complex analytic space $\tilde{M}^{[n]}$ is irreducible, and $\tilde{\sigma}^*\Theta_{M^{[n]}}$ is equal up to a torsion to $\tilde{\pi}^*\Theta_{M^n}$.
Proof: In [Bea], A. Beauville proved the conditions (i) and (ii). Clearly, (ii) implies that $\tilde{\sigma}^* \Theta_{M^{[n]}}$ is equal up to a torsion to $\tilde{\pi}^* \Theta_{M^n}$, assuming that $\tilde{M}^{[n]}$ is irreducible. It remains to show that $\tilde{M}^{[n]}$ is irreducible.

The following argument can be easily generalized to a more general type of Cartesian squares. We only use that the arrow $\pi$ is rational, $\sigma$ is finite and generically etale, and the varieties $M^n, M^{[n]}$ are irreducible.

Let $U$ be the general open stratum of $\tilde{M}^{[n]}$,

$$U := \tilde{M}^{[n]} \setminus \bigcup_{\alpha} \tilde{\sigma}^{-1} \Delta_{[\alpha]}$$

The map $\tilde{\pi} : U \to M^n$ is an open embedding. Therefore, the variety $U$ is irreducible. To prove that $\tilde{M}^{[n]}$ is irreducible, we need to show that for all points $x \in \tilde{M}^{[n]}$, there exists a sequence $\{x_i\} \subset U$ which converges to $x$. Since $M^{[n]}$ is irreducible, there exists a sequence $\{x_i\} \in \tilde{\sigma}(U)$ converging to $\tilde{\sigma}(x)$. Consider the sequence $\{\pi(x_i)\} \subset M^{(n)}$. The general stratum $\pi(U)$ of $M^n$ is identified with $U$, since $\pi|_U$ is an isomorphism. Lifting $\{\pi(x_i)\}$ to $M^n$, we obtain a sequence $\{x_i\} \subset \pi(U) = U$. Taking a subsequence of $\{x_i\}$, we can assure that it converges to a point in a finite set $\tilde{\sigma}^{-1}(\tilde{\sigma}(x))$. By an appropriate choice of the lifting, we obtain a sequence converging to any point in $\tilde{\sigma}^{-1}(\tilde{\sigma}(x))$, in particular, $x$. This proves that $\tilde{M}^{[n]}$ is irreducible. ■

Remark 4.3: From Proposition 4.2 and Calabi-Yau theorem (Theorem 2.4), it follows immediately that $M^{[n]}$ admits a hyperkähler structure, if $M$ is a K3 surface or a compact torus. However, this hyperkähler structure is not in any way related to the hyperkähler structures on $M$.

Remark 4.4: In the preliminary version of [N], it was stated without proof that $\tilde{\sigma}^* \Theta_{M^{[n]}} = \tilde{\pi}^* \Theta_{M^n}$. This statement seems to be subtle, and I was unable to find the proof. However, a weaker version of this equality can be proven.

Proposition 4.5: Let $A_{M^{[n]}} := \tilde{\sigma}^* \Theta_{M^{[n]}}$, $A_{M^n} := \tilde{\pi}^* \Theta_{M^n}$, be the forms defined in Proposition 4.2. Then for all closed complex subvarieties $X \hookrightarrow \tilde{M}^{[n]}$, the 2-forms $i^* A_{M^{[n]}}$, $i^* A_{M^n} \in \Omega^2 X$ are equal outside of singularities of $X$. 

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The forms $A_{M^n}, A_{M^n}$ are equal up to torsion. Therefore, their difference lies in the torsion subsheaf of $\Omega^2 \widetilde{M}^{[n]}$. To prove that the 2-forms $i^* A_{M^n}, i^* A_{M^n} \in \Omega^2 X$ are equal outside of singularities of $X$, it suffices to show the following: for all torsion sections $\omega \in \Omega^2 \widetilde{M}^{[n]}$, the pullback $i^* \omega$ lies in the torsion of $\Omega^2 X$.

Let $\Delta_{[\alpha]}$ be a stratum of $M^{[n]}$, defined by a Young diagram $\alpha$ as in Subsection 5.1, and $X \hookrightarrow \widetilde{M}^{[n]}$ an irreducible component of $\tilde{\sigma}^{-1}(X)$, considered as a complex subvariety of $\widetilde{M}^{[n]}$. As a first step in proving Proposition 4.3, we show that for all torsion sections $\omega \in \Omega^2 \widetilde{M}^{[n]}$, the pullback $i^* \omega$ lies in the torsion of $\Omega^2 X$, for this particular choice of $X$. Consider a generic point $x \in \Delta_{[\alpha]}$, and let $V \subset M^{[n]}$ be a neighbourhood of $x$ in $M^{[n]}$. Let $U \subset \Delta_{[\alpha]}$ be a neighbourhood of $x \in \Delta_{[\alpha]}$. For an appropriate choice of $V, U$, these varieties are equipped with a locally trivial fibration $V \rightarrow U$, inverse to a natural embedding $U \hookrightarrow V$. Assume also that $U$ consists entirely of generic points of $\Delta_{[\alpha]}$. Let $\bar{x}$ be a point of $\overline{\sigma}^{-1}(x) \cap X$, and $\bar{V}$ be a component of $\overline{\sigma}^{-1}(V)$ which contains $\bar{x}$, and $\bar{U} := \overline{\sigma}^{-1}(U) \cap \bar{V}$. Since $U$ consists of generic points of $\Delta_{[\alpha]}$, and $\overline{\sigma}$ is finite, the map $\overline{\sigma} : \bar{U} \rightarrow U$ is etale. Therefore, $\bar{V}$ is equipped with a locally trivial fibration $\bar{V} \rightarrow \bar{U}$, inverse to a natural embedding $\bar{U} \hookrightarrow \bar{V}$. Using this fibration, we decompose the sheaf of differentials on $\bar{V}$ as follows:

$$\Omega^1 \bar{V} = \bar{p}^* \Omega^1 \bar{U} \oplus \Omega^1_{\bar{p}} \bar{V}$$

where $\Omega^1_{\bar{p}} \bar{V}$ is the sheaf of relative differentials of $\bar{V}$ along $\bar{p}$. Clearly, for all sections $\omega \in \Omega^1_{\bar{p}} \bar{V}$, the pullback of $\omega$ under $\bar{U} \hookrightarrow \bar{V}$ is zero. On the other hand, $\bar{U}$ is smooth, and therefore $\bar{p}^* \Omega^1 \bar{U}$ has no torsion. Thus, the torsion-component of $\Omega^1 \bar{V}$ is contained in $\Omega^1_{\bar{p}} \bar{V}$ and vanishes on $\bar{U}$. A similar argument implies that the torsion-component of $\Omega^2 \bar{V}$ is contained in

$$\Omega^2_{\bar{p}} \bar{V} \oplus \bar{p}^* \Omega^1 \bar{U} \otimes \Omega^1_{\bar{p}} \bar{V} \subset \Omega^2 \bar{V}$$

and also vanishes on $\bar{U}$. Therefore, all torsion components on $\Omega^2 \widetilde{M}^{[n]}$ vanish on $X$, where $X$ is an irreducible component of the preimage of the stratum of $M^{[n]}$. Consider a stratification of $\widetilde{M}^{[n]}$ by such $X$’s. For any subvariety $Y \subset \widetilde{M}^{[n]}$, let $\Delta_{[\alpha]}$ be the smallest stratum of $\widetilde{M}^{[n]}$ which contains $Y$. Then, the set $Y_\alpha$ of generic points of $Y$ is contained in the set $\bar{U}_{[\alpha]}$ of the generic points of $\Delta_{[\alpha]}$. But, as we have seen, the restrictions of the forms $A_{M^n}$,
A_{M^n} to \tilde{U}_{[a]} coincide. Therefore, restrictions of $A_{M^n}, A_{M^n}$ to $Y_g \subset \tilde{U}_{[a]}$ are equal. This proves Proposition 4.3.

In the situation similar to the above, we will say that the forms $A_{M^n}, A_{M^n}$ are equal on subvarieties.

From the fact that $A_{M^n}, A_{M^n}$ are equal on subvarieties we immediately obtain the following.

Claim 4.6: Let $X \subset M^{[n]}$ be a complex subvariety of the Hilbert scheme. Assume that the holomorphic symplectic form is non-degenerate in the generic point of $X$ (this happens, for instance, when $X$ is trianalytic). Then the restriction $\pi|_X$ of $\pi : M^{[n]} \to M^{(n)}$ to $X$ is finite in generic point of $X$.

Proof: This result is easily implied by Proposition 4.3. For details of the proof, the reader is referred to Proposition 4.16, Corollary 4.18.

The rest of this section is not used directly anywhere in this paper. A reader who does not like perverse sheaves is invited to skip the rest and proceed to Section 5.

4.2 Cohomology of the Hilbert scheme

For perverse sheaves, we freely use terminology and results of [BBD]. For the computation of cohomology of the Hilbert scheme via perverse sheaves, see [N].

Definition 4.7: Let $X$ be an irreducible complex variety and $F$ a perverse sheaf on $M$. The $F$ is called a Goresky-MacPherson sheaf, or a sheaf of GM-type if it has no proper subquotient perverse sheaves with support in $Z \subset X$. For an arbitrary perverse sheaf $F$, consider the Goresky-MacPherson subquotient $F_{GM}$ of $F$, such that for a nonempty Zariski open set $U \subset X$, $F|_U = F_{GM}|_U$. Such subquotient is obviously unique; we call it the Goresky-MacPherson extension of $F$. The Intersection Cohomology sheaf $IC(X)$ is the Goresky-MacPherson extension of the constant sheaf $\mathbb{C}_X$.

Definition 4.8: Let $X$ be a complex variety. The $X$ is called homology rational if the constant sheaf $\mathbb{C}_X$ on $X$, considered as a complex of sheaves,
is isomorphic to the Intersection Cohomology perverse sheaf. The variety $X$ is called **weakly homology rational** if the Intersection Cohomology sheaf (considered as a complex of sheaves) is constructible (i.e., the cohomology of this complex of sheaves are zero in all but one degree).

**Remark 4.9:** Clearly, for a homology rational variety, the Intersection Cohomology coincides with the standard cohomology.

**Claim 4.10:** Let $f : X \rightarrow Y$ be a finite dominant morphism of complex varieties. Assume that $X$ is smooth. Then $Y$ is weakly homology rational. Moreover, if $Y$ is also normal, then $Y$ is homology rational.

**Proof:** Well known. ■

**Definition 4.11:** Let $\pi : X \rightarrow Y$ be a morphism of complex varieties. Consider a stratification $\{C_i\}$ of $Y$, defined in such a way that the restriction of $f$ to the open strata $\pi^{-1}(C_i)$ is a locally trivial fibration. The map $f$ is called a **semismall resolution of $Y$** if $X$ is smooth, and for all $i$, the dimension of the fibers of $\pi : \pi^{-1}(C_i) \rightarrow C_i$ is at most half the codimension of $C_i \subset Y$.

**Proposition 4.12:** Let $\pi : X \rightarrow Y$ be a semismall resolution, associated with the stratification $Y = \bigsqcup C_i$. Let $Y_i$ be the closed strata of the corresponding stratification of $Y$, $Y_i = \overline{C_i}$, and $X_i$ the corresponding closed strata of $X$, $X_i = \pi^{-1}(Y_i)$. Consider the Goresky-MacPherson sheaf $V_i$ associated with the sheaf $R^l \pi_* \mathbb{C}X_i$ where $l_i = \frac{\text{codim}_{Y_i} \overline{C_i}}{2}$ (for $\text{codim}_{Y_i} \overline{C_i}$ odd, we put $V_i = 0$) and $\mathbb{C}X_i$ a constant sheaf on $X_i$. Assume that $X$ is a Kähler manifold. Then $R^* \pi_* \mathbb{C}X$ is a direct sum of the perverse sheaves $V_i$ shifted by $l_i$.

**Proof:** In algebraic situation, this is proven using the weight arguments and $l$-adic cohomology ([BBD]). To adapt this argument in Kähler situation, one uses the mixed Hodge modules of M. Saito, [S]. ■

**Theorem 4.13:** [GS] Let $M$ be a complex surface, $M^{[n]}$ its Hilbert scheme, $M^{(n)}$ the symmetric power and

$$\pi : M^{[n]} \rightarrow M^{(n)}$$

the standard projection map. Consider the stratification of $M^{(n)}$ by the diagonals $\Delta_{(\alpha)}$, parametrized by the Young diagrams $\alpha$ (see [33]). Then $\pi :$
$M^{[n]} \to M^{(n)}$ is a semismall resolution associated with this stratification. Moreover, the sheaves $V_i$ of Proposition 4.12 are constant sheaves $C_{\Delta(\alpha)}$.

**Proof:** The map $\pi : M^{[n]} \to M^{(n)}$ is a semismall resolution, which is easy to check by counting dimensions (see Proposition 4.16 for a conceptual proof). Now, the first assertion of Theorem 4.13 is a straightforward application of Proposition 4.12. The second assertion is much more subtle; see [N] for details and further reference.

**Corollary 4.14:** The $i$-th cohomology of $M^{[n]}$ are isomorphic to

$$\bigoplus_{\alpha} H^i_{\text{codim} \Delta(\alpha)} \left( \Delta(\alpha) \right)$$

**Proof:** By Theorem 4.13,

$$R^\bullet \pi_* C_{M^{[n]}} = \bigoplus C_{\Delta(\alpha)} \left[ \frac{\text{codim} \Delta(\alpha)}{2} \right],$$

where $[\cdots]$ denotes the shift by that number.

### 4.3 Holomorphically symplectic manifolds and semismall resolutions

**Definition 4.15:** Let $\pi : X \to Y$ be a morphism of complex varieties, and $\mathcal{Y}, \mathcal{X}$ be stratification of $Y, X$. Denote by $Y_i$ the strata of $\mathcal{Y}$. We say that $\mathcal{Y}$ and $\mathcal{X}$ are compatible, if the preimages $\pi^{-1}(Y_i)$ coincide with the strata $X_i$ of $\mathcal{X}$, all the strata of $\mathcal{Y}$ are non-singular, and the maps $\pi|_{X_i} : X_i \to Y_i$ are locally trivial fibrations.

**Proposition 4.16:** Let $\pi : X \to Y$ be a generically finite, dominant morphism of complex varieties. Assume that $X$ is smooth and equipped with a holomorphically symplectic form $\Theta_X$. Moreover, assume that there exists a Cartesian square

\[
\begin{array}{ccc}
\tilde{X} & \overset{\tilde{\pi}}{\longrightarrow} & \tilde{Y} \\
\downarrow \tilde{\sigma} & & \downarrow \sigma \\
X & \overset{\pi}{\longrightarrow} & Y
\end{array}
\]

$^2$The varieties $\Delta(\alpha)$ are normal and obtained as quotients of smooth manifolds by group action. Thus, all $\Delta(\alpha)$ are homology rational by Claim 4.10. Thus, the sheaves $C_{\Delta(\alpha)}$ are GM-type.
with finite dominant morphisms as vertical arrows and birational morphisms as horizontal arrows. Assume that $\tilde{Y}$ is a holomorphically symplectic manifold, and the pullbacks of the holomorphic symplectic forms $\Theta_X$, $\Theta_{\tilde{Y}}$ via $\tilde{\pi}$ and $\tilde{\sigma}$ are equal on subvarieties, in the sense of Subsection 4.1. Assume, finally, that there exist compatible stratifications $\{X_i\}$, $\{\tilde{X}_i\}$, $\{\tilde{Y}_i\}$ such that $\Theta_{\tilde{Y}}\bigl|_{\tilde{Y}_i}$ is non-degenerate in the generic points of $\tilde{Y}_i$. Then $\pi : X \to Y$ is a semismall resolution.

**Proof:** Let $r(X_i)$ be the rank of the radical of $\Theta_X\bigl|_{\tilde{X}_i}$ in the generic point of the stratum $X_i$. Similarly, let $r(\tilde{X}_i)$ the rank of the radical of $\tilde{\sigma}^*\Theta_X\bigl|_{\tilde{X}_i}$ in the generic point of the stratum $\tilde{X}_i$. Since $\tilde{\sigma}$ is finite dominant, we have $r(\tilde{X}_i) = r(X_i)$. By definition, $\tilde{Y}_i = \tilde{\pi}(\tilde{X}_i)$ is (generically) a non-degenerate symplectic subvariety of $\tilde{Y}$. Since the forms $\Theta_X$ and $\tilde{\pi}^*\Theta_{\tilde{Y}}$ are equal on subvarieties, and $\Theta_{\tilde{Y}}\bigl|_{\tilde{Y}_i}$ is non-degenerate in the generic points of $\tilde{Y}_i$, we have

$$r(\tilde{X}_i) = \dim_C (\tilde{\pi}^{-1}(y)), \quad (4.3)$$

for $y \in \tilde{Y}_i$ a generic point. Let $w(X_i)$ be the number $\dim(X_i) - r(X_i)$. The following linear-algebraic claim immediately implies that

$$\dim_C X - w(X_i) \geq 2r(X_i) \quad (4.4)$$

**Claim 4.17:** Let $W$ be a symplectic vector space, $\Theta$ the symplectic form, $V \subset W$ a subspace, $r(V)$ the rank of the radical $\Theta\bigl|_V$ and $w(V) := \dim V - r(V)$. Then $\dim W - w(V) \geq 2r(V)$.

**Proof:** Clear. $\blacksquare$

Comparing (4.3) and (4.4), we obtain that

$$\text{codim}_C Y_i \geq 2 \dim_C (\tilde{\pi}^{-1}(y)), \quad (4.3)$$

for $y \in \tilde{Y}_i$ a generic point. Finally, since $\sigma : \tilde{Y} \to Y$ is finite dominant, we have $\dim_C (\tilde{\pi}^{-1}(y)) = \dim_C (\pi^{-1}(\sigma(y)))$. Thus, $\text{codim}_C Y_i \geq$ 3By a **birational morphism** we understand a morphism $\varphi : X_1 \to X_2$ of complex varieties such that the inverse of $\varphi$ is rational.
2 \dim \mathbb{C} (\pi^{-1}(y)) \text{ for } y \in Y_i \text{ a generic point. This finishes the proof of Proposition 4.16.} 

**Corollary 4.18:** Let $M$ be a complex K3 surface or a compact complex 2-dimensional torus, $M^{[n]}$ its Hilbert scheme and $M^{(n)}$ the symmetric power of $M$, with $\pi : M^{[n]} \to M^{(n)}$ being the standard map. Consider an arbitrary hyperkähler structure $\mathcal{H}$ on $M^{[n]}$ compatible with the complex structure. Let $Z \subset M^{[n]}$ be a subvariety which is trianalytic with respect to $\mathcal{H}$, and $n : X \to Z$ be the desingularization of $Z$. Assume that $M$ is Mumford-Tate generic with respect to some hyperkähler structure. Then $\pi \circ n : X \to Y$ is a semismall resolution of $Y := \pi(Z)$.

**Proof:** Assume that $Z$ is irreducible. Since the desingularization $X$ is hyperkähler, this manifold is holomorphically symplectic, and the holomorphic symplectic form $\Theta_X$ on $X$ is obtained as a pullback of the holomorphic symplectic form $\Theta_{M^{[n]}}$ on $M^{[n]}$. To simplify notations, we denote $\pi \circ n$ by $\pi$. Let

$$
\begin{array}{ccc}
\tilde{X} & \xleftarrow{\tilde{\pi}} & \tilde{Y} \\
\downarrow{\tilde{\sigma}} & & \downarrow{\sigma} \\
X & \xrightarrow{\pi} & Y
\end{array}
$$

be the Cartesian square, with $\tilde{Y}$ obtained as an irreducible component of the preimage $\sigma^{-1}(Y) \subset M^n$. We intend to show that the square (4.5) satisfies the assumptions of Proposition 4.16. For each morphism of complex varieties, there exists a stratification, compatible with this morphism. Take a set of compatible stratifications $\{X_i\}$, $\{\tilde{Y}_i\}$, $\{\tilde{X}_i\}$. By Corollary 2.12, any stratification of $\tilde{Y}$ consists of trianalytic subvarieties because all closed complex subvarieties of $M^n$ are trianalytic. Applying Proposition 4.16 to the map $\pi : X \to Y$ and the Cartesian square (4.5), we immediately obtain Corollary 4.18. 

**5 Universal subvarieties of the Hilbert scheme**

The Sections 5–8 are independent from the rest of this paper. The only result of Sections 5–8 that we use is Corollary 7.7.

Let $M$ be a smooth complex surface, $M^{[n]}$ its Hilbert scheme. An automorphism $\gamma$ of $M$ gives an automorphism $\gamma^{[n]}$ of $M^{[n]}$; similarly, an in-
finitesimal automorphism of $M$ (that is, a holomorphic vector field) gives an infinitesimal automorphism of $M^{[n]}$.

**Definition 5.1:** Let $M$ be a surface, $M^{[n]}$ its Hilbert scheme and $X \subset M^{[n]}$ a closed complex subvariety. Then $X$ is called **universal** if for all open $U \subset M$, and all global or infinitesimal automorphisms $\gamma \in \Gamma(T(M))$, the subvariety $X_U := X \cap U^{[n]}$ is preserved by $\gamma^{[n]}$.

The universal subvarieties are described more explicitly in the following subsection

### 5.1 Young diagrams and universal subvarieties of the Hilbert scheme

Let $M$ be a smooth surface, $M^{(n)}$ its symmetric power, $M^{[n]}$ its Hilbert scheme and $\pi : M^{[n]} \rightarrow M^{(n)}$ the natural map. For a Young diagram

$$\alpha = (n_1 \geq n_2 \geq ... \geq n_k), \quad \sum n_i = n,$$

we defined a subvariety $\Delta(\alpha) \subset M^{(n)}$ (3.3). Let $\Delta_{[\alpha]} := \pi^{-1}(\Delta(\alpha))$ the the corresponding subvariety in $M^{[n]}$.

Let $a$ be the general point of $\Delta(\alpha)$, i.e. the one satisfying

$$a = \sigma(a_1, ..., a_n), \quad \text{where} \quad a_1 = a_2 = ... = a_{n_1}$$
$$a_{n_1+1} = a_{n_1+2} = ... = a_{n_1+n_2} = ...$$
$$a_{\sum_{i=1}^{k-1} n_i + 1} = a_{\sum_{i=1}^{k-1} n_i + 2} = ... = a_n$$

and the points $a_1, a_{n_1 + 1}, ..., a_{(\sum_{i=1}^{k-1} n_i) + 1}$ are pairwise unequal (5.1)

Let $F_{\alpha}(a) := \pi^{-1}(a) \subset \Delta_{[\alpha]}$ be the general fiber of the projection $\pi : \Delta_{[\alpha]} \rightarrow \Delta(\alpha)$. By definition, $F_{\alpha}(a)$ parametrizes 0-dimensional subschemes $S \subset M$ with $Sup(S) = \{a_i\}$ and prescribed multiplicities

$$\text{length}_{a_i} S = n_i.$$

Let $\mathcal{O}_{a_i}$ be the adic completion of $\mathcal{O}_M$ at $a_i$, and $G_{a_i} := \text{Aut}(\mathcal{O}_{a_i})$. Clearly, the group $G_{\alpha} := \prod_i G_{a_i}$ acts naturally on $F_{\alpha}(a)$. We are interested in $G_{\alpha}$-invariant subvarieties of $F_{\alpha}(a)$.

**Lemma 5.2:** Let $\alpha$ be a Young diagram, $\Delta(\alpha)$ the corresponding subvariety of $M^{(n)}$ and $a, b$ the points of $\Delta(\alpha)$ satisfying (5.1). Let $F_{\alpha}(a)$, $F_{\alpha}(b)$
be the corresponding fibers of $\pi: \Delta_{[a]} \to \Delta_{(a)}$. Consider the groups $G_a$, $G_b$ acting on $F_\alpha(a)$, $F_\alpha(b)$ as above. Then

(i) there exist a canonical bijective correspondence $\theta$ between $G_a$-invariant subvarieties in $F_\alpha(a)$ and $G_b$-invariant subvarieties in $F_\alpha(b)$.

(ii) For any complex automorphism $\gamma: M \to M$ such that $\gamma(a) = b$, the corresponding map $\gamma: F_\alpha(a) \to F_\alpha(b)$ maps $G_a$-invariant subvarieties of $F_\alpha(a)$ to $G_b$-invariant subvarieties of $F_\alpha(b)$ and induces the correspondence $\theta$.

**Proof:** Let $(a_1,\ldots,a_n)$, $(b_1,\ldots,b_n)$ be the points of $M^n$ satisfying (5.1), such that $a = \sigma(b_1,\ldots,b_n)$, $b = \sigma(b_1,\ldots,b_n)$. Let $U$ be an open subset of $M$ containing $a_i, b_i, i = 1,\ldots,n$. Let $\gamma: U \to U$ be a complex automorphism of $U$ such that $\gamma(a_i) = b_i$. Since $a, b$ satisfy (5.1), for an appropriate choice of $U$, such $\gamma$ always exists. Clearly, $\gamma$ identifies $F_\alpha(a)$ and $F_\alpha(b)$. This identification is not unique, since it depends on the choice of $\gamma$, but every two such identifications differ by a twist by $G_a, G_b$. This proves Lemma 5.2.

By Lemma 5.2, the set of $G_a$-invariant subvarieties of $F_\alpha(a)$ is independent from $a$. Denote this set by $Z_\alpha$. For each $Y \in Z_\alpha$, and a generic point $a \in \Delta_{(a)}$, denote the corresponding subvariety of $F_\alpha(a)$ by $Y(a)$. Let $Z_\alpha(Y)$ be the union of $Y(a)$ for all $a \in \Delta_{(a)}$ satisfying (5.1).

**Theorem 5.3:** Let $\alpha$ be a Young diagram, $\Delta_{(a)} \subset M^{(n)}$ the corresponding diagonal in $M^{(n)}$ and $a \in \Delta_{(a)}$ a general point (that is, one satisfying (5.1)). Let $Y \in Z_\alpha$ be a $G_a$-invariant subvariety of $F_\alpha(a) = \pi^{-1}(a) \subset M^{[a]}$, and $Z_\alpha(Y)$ the corresponding subvariety of $M^{[a]}$. Then $Z_\alpha(Y)$ is a universal subvariety of $M^{[a]}$, in the sense of Definition 5.1. Moreover, all irreducible universal subvarieties of $M^{[a]}$ can be obtained this way.

**Proof:** The statement of Theorem 5.3 is local by $M$. Thus, to prove that $Z_\alpha(Y)$ is preserved by infinitesimal automorphisms, it suffices to show that $Z_\alpha(Y)$ is preserved by all global automorphisms of $M$. Let $\gamma: M \to M$ be an automorphism. Denote by $\Delta^\circ_{(a)} \subset \Delta_{(a)}$ the set of all $a$ satisfying (5.1). Clearly, $\gamma$ preserves

$$\Delta^\circ_{[a]} := \pi^{-1}\left(\Delta^\circ_{(a)}\right).$$

To show that $\gamma$ preserves $Z_\alpha(Y)$, it suffices to prove that, for all $a, b \in \Delta^\circ_{(a)}$, $\gamma(a) = b$, the automorphism $\gamma$ maps $F_\alpha(a)$ to $F_\alpha(b)$. This is Lemma 5.2 (ii).
We obtained that $Z_\alpha(Y)$ is universal.

Let $X$ be an irreducible universal subvariety in $M^{[n]}$. Then $\pi(X) \subset M^{(n)}$ is preserved by the automorphisms of $M^{(n)}$ coming from $M$. For $M$ Stein, the only subvarieties of $M^{(n)}$ preserved by infinitesimal automorphisms are unions of diagonals. Since $X$ is irreducible, so is $\pi(X)$. We obtain that $\pi(X)$ is a diagonal $\Delta_\alpha$ corresponding to some Young diagram $\alpha$. It remains to prove that $X \cap F_\alpha(a)$ is $G_a$-invariant, for all $a \in \Delta_\alpha^\circ$. This is clear, because infinitesimal automorphisms of $M$ fixing $\{a_i\}$ generate the group $G_a = \prod_i \text{Aut}(\hat{\mathcal{O}}_{a_i})$, and since $X$ is invariant under such automorphisms, $X \cap F_\alpha(a)$ is $G_a$-invariant. [Theorem 5.3] is proven.

5.2 Universal subvarieties of relative dimension 0

Definition 5.4: Let $M$ be a smooth complex surface, $M^{[n]}$ its Hilbert scheme, $\alpha$ a Young diagram corresponding to a diagonal $\Delta_\alpha \subset M^{(n)}$. Let $a \in \Delta_\alpha^\circ$ be a general point, and $F_\alpha(a) := \pi^{-1}(a)$ the corresponding fiber of $\pi : M^{[n]} \to M^{(n)}$. Consider a $G_\alpha(a)$-invariant subvariety $Y$ of $F_\alpha(a)$. Let $Z \subset M^{[n]}$ be a corresponding universal subvariety, $Z = Z_\alpha(Y)$ ([Theorem 5.3]). Then the relative dimension of $Z$ is the dimension of $Y$.

In this subsection, we classify the universal subvarieties of relative dimension 0.

Let $\alpha = (n_1 \geq n_2 \geq \ldots \geq n_k)$ be a Young diagram, $\sum n_i = n$. Clearly,

$$F_\alpha(a) \cong F_0(n_1) \times F_0(n_2) \times \ldots,$$

(5.2)

where $F_0(i)$ is the classifying space of 0-dimensional subschemes of length $i$ in $\mathbb{C}^2$ with support in $0 \in \mathbb{C}^2$. Let $G_0 = \text{Aut}(\mathbb{C}[[x, y]])$ be the group of automorphisms of the ring of formal series, acting on $F_0(i)$. By (5.2), the $k$-th power of $G_0$ acts on $F_\alpha(a)$. This gives an isomorphism $G_0^k \cong G_\alpha(a)$.

Assume that $n_i = \frac{m_i(m_i+1)}{2}$, for some positive integer $m_i$. Consider a $G_0$-invariant point $s_i \in F_0(n_i)$, given by

$$s_{m_i} = \mathbb{C}[[x, y]]/\mathfrak{m}^{m_i},$$

(5.3)

where $\mathfrak{m} \subset \mathbb{C}[[x, y]]$ is the maximal ideal generated by $x$ and $y$. Let $\{s_1\} \times \{s_2\} \times \ldots \times \{s_k\}$ be the $G_0^k$-invariant point of $\prod F_0(n_i)$. Using the isomorphism (5.2), we obtain a $G_\alpha(a)$-invariant point $a$ of $F_\alpha(a)$. Denote by $X_\alpha$ the corresponding universal subvariety of $M^{[n]}$. It has relative dimension 29.
0. The aim of this subsection is to show that all universal subvarieties of relative dimension 0 are obtained this way.

**Proposition 5.5:** Let $X \subset M^{[n]}$ be a universal subvariety of relative dimension 0. Then where exists a Young diagram

$$\alpha = (n_1 \geq n_2 \geq ... \geq n_k), \quad \sum n_i = n,$$

and positive integers $m_1, ..., m_k$, such that $n_i = \frac{m_i(m_i+1)}{2}$, and $X = \mathcal{X}_\alpha$.

**Proof:** Let $a$ be a general point of $\Delta_\alpha$, and $s \in F_\alpha(a)$ a point of the zero-dimensional variety $F_\alpha(a) \cap X$. Consider the varieties $F_0(i)$ defined above, and the action of $G_0 = \text{Aut}(\mathbb{C}[[x,y]])$ on $F_0(i)$. Let $x_i \in F_0(n_i)$ be the points of $F_0(k)$, such that under an isomorphism (5.2), $s$ corresponds to $\{x_1\} \times \{x_2\} \times ... \times \{x_k\}$. Then the points $x_i$ are $G_0$-invariant. To finish the proof of Proposition 5.5, it remains to prove the following lemma.

**Lemma 5.6:** Let $s \in F_0(i)$ be a $G_0$-invariant point. Then $i = \frac{j(j+1)}{2}$ and $s$ is given by (5.3).

**Proof:** The group $GL(2, \mathbb{C})$ acts on $\mathbb{C}[[x,y]]$ by automorphisms. Clearly, this $GL(2, \mathbb{C})$-action is factorized through the natural action of

$$G_0 = \text{Aut}(\mathbb{C}[[x,y]]).$$

We show (Sublemma 5.7 below) that all $GL(2, \mathbb{C})$-invariant ideals in $\mathbb{C}[[x,y]]$ are powers of the maximal ideal. Since $x = \mathbb{C}[[x,y]]/I$ for some $G_0$-, and hence, $GL(2, \mathbb{C})$-invariant ideal of $\mathbb{C}[[x,y]]$, this will finish the proof of Lemma 5.6. We reduced Proposition 5.5 to the following result.

**Sublemma 5.7:** Consider the natural action of $GL(2, \mathbb{C})$ on

$$A = \mathbb{C}[[x,y]].$$

Let $I$ be a proper $GL(2, \mathbb{C})$-invariant ideal in $A$. Then $I$ is a power of the maximal ideal.

**Proof:** Consider the $GL(2)$-invariant filtration

$$A_0 \subset A_0 \oplus A_1 \subset A_0 \oplus A_1 \oplus A_2 \subset ... \subset A$$

where $A_i \subset A$ consists of homogeneous polynomials of degree $i$. Let $l$ be the minimal number for which $I \cap A_l \neq 0$. Since $I$ and $A_l$ are $GL(2)$-invariant,
the intersection $I \cap A_l$ is also $GL(2)$-invariant. The space $A_l$ is an irreducible representation of $GL(2)$, and thus, $I \supset A_l$. Therefore, $I = A_l \cdot A_l$, and $I$ is $l$-th power of the maximal ideal. This finishes the proof of Sublemma 5.7, Lemma 5.6, and Proposition 5.5.

6 Subvarieties of $M^{[n]}$ which are generically finite over $M^{(n)}$, for $M$ a generic K3 surface

Throughout this section, $M$ is a smooth complex surface, $M^{[n]}$ the Hilbert scheme of $M$, $M^{(n)}$ the $n$-th symmetric power of $M$ and $\pi : M^{[n]} \rightarrow M^{(n)}$ the natural map.

Let $f : X \rightarrow Y$ be a morphism of complex varieties. We say that $f$ is generically finite if there exist an open dense subset $X_0 \subset X$ such that the map $f|_{X_0} : X_0 \rightarrow f(X_0)$ is finite. The morphism $f$ is called generically one-to-one if there exist an open dense subset $X_0 \subset X$ such that the map $f|_{X_0} : X_0 \rightarrow f(X_0)$ is an isomorphism.

The main result of this section is the following theorem.

Theorem 6.1: Let $M$ be a complex K3 surface. Assume that $M$ admits a hyperkähler structure $H$ such that $M$ is Mumford-Tate generic with respect to $H$ (Definition 2.10). Let $X \subset M^{[n]}$ be an irreducible complex analytic subvariety such that the restriction of $\pi$ to $X$ is generically finite. Assume that there exists a Young diagram $\alpha$ such that the subvariety $\pi(X) \subset M^{(n)}$ coincides with $\Delta(\alpha)$. Then $X$ is a universal subvariety (Definition 5.1) of $M^{[n]}$.

Remark 6.2: The relative dimension of the universal subvariety $X \subset M^{[n]}$ is zero, because $\pi|_X$ is generically finite. Thus, Proposition 5.5 can be applied to this situation. We obtain that, under assumptions of Theorem 5.1, $\pi|_X : X \rightarrow \Delta(\alpha)$ is generically one-to-one.

The proof of Theorem 6.1 takes the rest of this section.

6.1 Fibrations arising from the Hilbert scheme

We work in assumptions of Theorem 6.1. Let $\Delta^0_{(\alpha)} \subset \Delta(\alpha)$ be the set of general points of $\Delta(\alpha)$, defined by (5.1). Consider the fibration $\pi :
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Let $M_0(l)$ be the $M(l)$ with all diagonals deleted:

$$M_0(l) = \{ (x_1, \ldots, x_l) \in M(l) \mid x_i \neq x_j \text{ for all } i \neq j \}$$

We write $\alpha = (n_1 \geq n_2 \geq \ldots \geq n_k)$ as

$$\alpha = (n_1 = n_2 = \ldots = n_{n_1}' > n_{n_1}'+1 = \ldots = n_{n_2}'+n_1 > \ldots > n_{\sum_{i=1}^{k'} n_i'-1} = n_{\sum_{i=1}^{k'} n_i'})$$

where $\sum_{i=1}^{k'} n_i' = k$.

**Claim 6.3:** The manifold $\Delta_{(\alpha)}^o$ is naturally isomorphic to $\prod_i M_0(n_i')$.  

**Proof:** Clear. ■

Let $D_{(\alpha)}^o$ be the universal covering of $\Delta_{(\alpha)}^o$. From Claim 6.3 it is clear that

$$D_{(\alpha)}^o = \prod_i M_0(n_i') \subset M^{'k'},$$

where $M_0^{n_i'}$ is $M^{n_i'}$ without diagonals. We define $D_{[\alpha]}^o$ as a fibered product, in such a way that the square

$$\begin{array}{ccc}
D_{[\alpha]}^o & \longrightarrow & \Delta_{[\alpha]}^o \\
\downarrow p & & \downarrow \pi \\
D_{(\alpha)}^o & \longrightarrow & \Delta_{(\alpha)}^o
\end{array}$$

is Cartesian. The map $D_{[\alpha]}^o \rightarrow D_{(\alpha)}^o$ is a locally trivial fibration. We determine the fibers of $p$ in terms of the isomorphism (6.2) as follows.

Consider the vector bundle $J^i(M)$ over $M$, with the fibers $J^i(M)\big|_x = O_M/m_x^i$, where $m_x$ is the maximal ideal on $O_M$ corresponding to $x$. Clearly, the bundle $J^i(M)$ has a natural ring structure. Let $G^i(M)$ be the fibration over $M$ with the fibers $G^i(M)\big|_x$ classifying the ideals $I \subset J^i(M)$ of codimension $i$. Consider again the equation (6.1). Let $N : \{1, \ldots, k'\} \rightarrow \mathbb{Z}^+$ be the map

$$l \rightarrow n_{\sum_{i=1}^{l-1} i},$$

$$\Delta_{[\alpha]}^o \rightarrow \Delta_{(\alpha)}^o,$$ where $\Delta_{[\alpha]}^o = \pi^{-1}(\Delta_{(\alpha)}^o)$. 

Let $M_0^{(l)}$ be the $M^{(l)}$ with all diagonals deleted:

$$M_0^{(l)} = \{ (x_1, \ldots, x_l) \in M^{(l)} \mid x_i \neq x_j \text{ for all } i \neq j \}$$

We write $\alpha = (n_1 \geq n_2 \geq \ldots \geq n_k)$ as

$$\alpha = (n_1 = n_2 = \ldots = n_{n_1}' > n_{n_1}'+1 = \ldots = n_{n_2}'+n_1 > \ldots > n_{n_{k-1}}'+1 = n_{n_{k-1}}'+n_{k-1} = n_{n_{k-1}}')$$

where $\sum_{i=1}^{k'} n_i' = k$.
i.e., 1 is mapped to $n_1$, 2 to the biggest value of $n_i$ not equal to $n_1$, 3 to the third biggest, etc.

For a locally trivial fibrations $Y_1, Y_2$ over $X_1, X_2$, we denote the external product by $Y_1 \boxtimes Y_2$. This is a fibration over $X_1 \times X_2$, with the fibers which are products of fibers of $Y_1, Y_2$. The iterations of $\boxtimes$ (for three or more fibrations) are defined in the same spirit.

**Claim 6.4:** Under the isomorphism (6.2), the locally trivial fibration $p : D(\alpha) \rightarrow M^k$ is isomorphic to the fibration

$$\boxtimes_{i=1}^{k'} G^{N(i)}(M) \big|_{D(\alpha)}$$

over $D(\alpha) \subset M^{k'}$.

**Proof:** Clear.

Let $D(\alpha) \rightarrow D(\alpha)$ be the fibration $\boxtimes_{i=1}^{k'} G^{N(i)}(M) \rightarrow M^{k'}$. We consider $D(\alpha)$, $D(\alpha)$ as open subsets in $D(\alpha)$, $D(\alpha)$.

Let $X \subset M^{[n]}$ be a closed subvariety, $\pi(X) = \Delta(\alpha)$, and $\pi : X \rightarrow \Delta(\alpha)$ generically finite. Consider $X \cap \Delta(\alpha)$ as a closed subvariety of $\Delta(\alpha)$. Let $\tilde{X}$ be an irreducible component of $n^{-1}(X) \subset \Delta(\alpha)$, where $n : D(\alpha) \rightarrow \Delta(\alpha)$ is the horisontal arrow of (6.3). Clearly, the closure of $\tilde{X}$ in $D(\alpha)$ is a closed complex subvariety of $D(\alpha)$. We denote this subvariety by $Z$. By construction, $Z$ is irreducible (it is an image of an irreducible variety) and generically finite over $D(\alpha) = M^{k'}$.

Consider the fibration $G^m(M) \rightarrow M$ constructed above. Assume that $m = \frac{l(l+1)}{2}$ for a positive integer $l$. Then the fibration $G^m(M) \rightarrow M$ has a canonical section $s : M \rightarrow G^m(M)$, defined by $x \mapsto \mathcal{O}_M/m_x^l$, where $m_x \subset \mathcal{O}_M$ is the maximal ideal corresponding to $x$.

**Proposition 6.5:** Let $M$ be a complex K3 surface. Assume that $M$ admits a hyperkähler structure $\mathcal{H}$ such that $M$ is generic with respect to $\mathcal{H}$. Let $Y \subset G^m(M)$ be a closed irreducible subvariety of the total space of the fibration $G^m(M) \rightarrow M$. Assume that $Y$ is generically finite over $M$. Then $m = \frac{l(l+1)}{2}$ for some positive integer $l$, and $Y$ is the image of the natural map $s : M \rightarrow G^m(M)$ constructed above.
Proposition 6.5 is proven in Subsection 6.3. Presently, we are going to explain how Proposition 6.5 implies Theorem 6.1.

Consider the map $p : D_{[\alpha]} \to M^{k'}$, and the closed subvariety $Z \subset D_{[\alpha]}$ constructed from $X$ as above. Let $(m_1, \ldots, m_{k'-1}) \in M^{k'-1}$ be a point such that the map

$$p : Z \cap p^{-1}(\{(m_1, \ldots, m_{k'-1})\} \times M) \to \{(m_1, \ldots, m_{k'-1})\} \times M$$

is generically finite. The set $S$ of such $(m_1, \ldots, m_{k'-1})$ is open and dense in $M^{k'-1}$. Let $\Psi_i : D_{[\alpha]} \to G^{N(i)}(M)$ be the natural projection to the $i$-th component of the product $D_{[\alpha]} = \bigotimes_{i=1}^{k'} G^{N(i)}(M)$. By Proposition 6.5, $N(k') = \frac{l(l-1)}{2}$ and the subvariety

$$\Psi(Z \cap p^{-1}(\{(m_1, \ldots, m_{k'-1})\} \times M))$$

coinsides with image of the map

$$s_{k'} : M \to G^{N(k')}(M).$$

Since $Z$ is irreducible,

$$\Psi_{k'}(Z \cap p^{-1}(\{(m_1, \ldots, m_{k'-1})\} \times M)) \subset G^{N(k')}(M)$$

is independent from the choice of $(m_1, \ldots, m_{k'-1}) \in S$. Therefore, $\Psi_{k'}(Z) = \text{im}(s_{k'})$. A similar argument shows that $\Psi_i(Z) = \text{im}(s_i)$, for all $i = 1, \ldots, k'$. Thus, $Z$ is an image of the section of the map $p : D_{[\alpha]} \to M^{k'}$ given by $\bigotimes_{i=1}^{k'} s_i$. This implies Theorem 6.1. We reduced Theorem 6.1 to Proposition 6.5.

### 6.2 Projectivization of stable bundles

Let $M$ be a compact Kähler manifold. We understand stability of holomorphic vector bundles over $M$ in the sense of Mumford–Takemoto (Definition 2.19). A polystable bundle is a direct sum of stable bundles of the same slope. Let $V$ be a polystable bundle, and $\mathbb{P}V$ its projectivization. Consider the unique Yang-Mills connection on $V$ (Definition 2.20). This gives a natural connection $\nabla_V$ on the fibration $\mathbb{P}V \to M$. 

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**Proposition 6.6:** Let \( M \) be a compact complex simply connected manifold of hyperkähler type. Assume that \( M \) admits a hyperkähler structure \( \mathcal{H} \) such that \( M \) is generic with respect to \( \mathcal{H} \). Consider \( M \) as a Kähler manifold, with the Kähler metric induced from \( \mathcal{H} \). Let \( V \) be a polystable bundle over \( M \), and \( \mathbb{P}V \to M \) its projectivization. Consider a closed irreducible subvariety \( X \subset \mathbb{P}V \) such that \( \pi(X) = M \). Then \( X \) is preserved by the connection \( \nabla_V \) in \( \mathbb{P}V \).

**Proof:** Let \( x \in M \) be a point of \( M \) such that in a neighbourhood \( U \subset M \) of \( x \), the projection \( \pi : X \to M \) is a locally trivial fibration. Assume that \( U \) is open and dense in \( M \). Let \( X_x \) be the fiber of \( \pi : X \to M \) in \( x \), and \( V_x := V|_x \) the fiber of \( V \). Consider the Hilbert scheme \( H \) classifying the subvarieties \( Y \subset \mathbb{P}V_x \) with the same Hilbert polynomial as \( Y \). Then \( H \) can be naturally embedded to the projectivization of a linear space \( W_x \), where \( W_x \) is a certain tensor power of \( V_x \), depending on the Hilbert polynomial of \( X_x \). Consider the corresponding bundle \( W \), which is related to \( V \) in the same way as \( W_x \) to \( V_x \). Then, \( W \) is a tensor power of \( V \), and hence, \( W \) is equipped with a unique Yang-Mills connection. Consider the corresponding connection \( \nabla_W \) on the projectivization \( \mathbb{P}W \). Let \( X_0 \) denote \( \pi^{-1}(U) \cap X \). The locally trivial fibration \( \pi|_{X_0} : X_0 \to U \) gives a section \( s \) of \( \mathbb{P}W|_{X_0} \). To prove that \( X \) is preserved by the connection \( \nabla_V \) in \( \mathbb{P}V \), it suffices to show that \( X_0 \) is preserved by \( \nabla_V \), or that \( \text{im} \ s \) is preserved by \( \nabla_W \). This is implied by the following lemma, which finishes the proof of Proposition 6.6.

**Lemma 6.7:** In assumptions of Proposition 6.6, let \( U \subset M \) be a dense open set, such that \( \pi|_{X_0} : X_0 \to U \) is an isomorphism, where
\[
X_0 = \pi^{-1}(U) \cap X \subset \mathbb{P}V.
\]
Then \( \nabla_V \) preserves \( X \).

**Proof:** Since \( M \) is generic with respect to \( \mathcal{H} \), all its complex subvarieties have complex codimension at least 2. Thus, we may assume that the complement \( M \setminus U \) is a complex subvariety of codimension at least 2.

Consider the restriction \( V|_U \). Then \( X_0 \) gives a one-dimensional subbundle \( L \) of \( V|_U \). Let \( V' = i_*L \subset i_*V|_U \) be the direct image of \( L \) under the embedding \( U \to M \). Since \( M \setminus U \) is a complex subvariety of codimension at least 2, the natural map \( V \to i_*V|_U \) is an isomorphism. Therefore, \( V' \) is
a coherent subsheaf in $V$. To prove [Lemma 6.7] it suffices to show that $V'$ is preserved by the connection. Since $M$ is generic with respect to $\mathcal{H}$, all integer $(1, 1)$-classes of cohomology have degree 0 ([Lemma 2.23]). Therefore, slope($V'$) = slope($V$) = 0 and $V'$ is a destabilising subsheaf of $V$. Since $V$ is polystable, this implies that $V'$ is a direct summand of $V$, and the Yang-Mills connection in $V$ preserves the decomposition $V = V' \oplus V'^\perp$, where $V'^\perp$ is the orthogonal complement of $V'$ with respect to any Yang-Mills metric on $V$. [Lemma 6.7] is proven. This finishes the proof of Proposition 6.6.

**Corollary 6.8:** In assumptions of [Proposition 6.6], let $X$ be generically finite over $M$. Assume that $M$ is simply connected. Then $\pi : X \to M$ is an isomorphism.

**Proof:** Since $X$ is preserved by the connection, the map $\pi : X \to M$ is a finite covering. Since $M$ is simply connected, and $X$ is irreducible, $\pi : X \to M$ is one-to-one. ■

### 6.3 Fibrations over K3 surfaces

The purpose of this subsection is to prove [Proposition 6.5]. Consider the fibration $G^m(M) \to M$ over the K3 surface $M$. Recall that $G^m(M)$ was defined as a fibration with fibers classifying the codimension-$m$ ideals in $J^m(M)$, where $J^m(M)$ is the bundle of rings $J^m(M)|_x = \mathcal{O}_M/m_x^m$. There is a decreasing filtration

$$J^i(M) \supset \mathfrak{M}(M) \supset \mathfrak{M}^2(M) \supset ...,$$

with

$$\mathfrak{M}^i(M)|_x = m_x^i \cdot \mathcal{O}_M/m_x^m$$

Consider the bundle $V = \mathfrak{M}^{i-1}(M)/\mathfrak{M}^i(M)$.

**Lemma 6.9:** Let $M$ be a complex K3 surface which is generic with respect to some hyperkähler structure. Let $V$ the holomorphic vector bundle defined above, $V = \mathfrak{M}^{i-1}(M)/\mathfrak{M}^i(M)$. Then $V$ is isomorphic to a symmetric power of the cotangent bundle of $M$. Moreover, $V$ is (Mumford-Takemoto) stable for all Kähler structures on $M$, and has no proper subbundles.

**Proof:** The first assertion is clear. Let us prove stability of $V$. From Yau’s proof of Calabi conjecture, it follows that for all Kähler classes on $M$, $M$ is equipped with the hyperkähler metric in the same Kähler class. The Levi-Civita connection on the cotangent bundle $\Lambda^1(M)$ of a hyperkähler manifold $M$ is hyperholomorphic ([Definition 2.17]), and hence Yang-Mills
Therefore, \( \Lambda^1(M) \) is stable, and \( V \) polystable (a tensor power of a Yang-Mills bundle is again Yang-Mills). The holonomy group of \( \Lambda^1(M) \) is obviously isomorphic to \( SU(2) \). Therefore, the holonomy group of \( V = S^l(\Lambda^1(M)) \) is also \( SU(2) \). The symmetric power of the tautological representation of \( SU(2) \) is obviously irreducible. Therefore, the holonomy representation of \( V \) is irreducible, and \( V \) cannot be represented as a direct sum of vector bundles. To show that \( V \) has no proper subbundles, we notice that \( H^{1,1}(M) \cap H^2(M, \mathbb{Z}) = 0 \) because \( M \) is generic with respect to \( \mathcal{H} \). Therefore, all coherent sheaves on \( M \) have first Chern class zero. We obtain that a proper subbundle of \( V \) is destabilizing, which contradicts stability of \( V \).

Let \( J^m_{gr}(M) \) be the graded sheaf of rings associated with the filtration (6.4). Consider the fibration \( G^m_{gr}(M) \) with the points classifying codimension-\( m \) ideals in the fibers of \( J^m_{gr}(M) \). There is a natural map \( G^m(M) \xrightarrow{\varphi} G^m_{gr}(M) \) associating to an ideal its associated graded quotient. Composing \( \varphi \) with the map \( s : M \to G^{(i-1)/2}(M) \), we obtain the section \( s_{gr} : M \to G^{(i-1)/2}_{gr}(M) \) of the natural projection \( p_{gr} : G^{(i-1)/2}_{gr}(M) \to M \). The following Proposition 6.10 obviously implies Proposition 6.5.

**Proposition 6.10:** Let \( M \) be a complex K3 surface. Assume that \( M \) admits a hyperkähler structure \( \mathcal{H} \) such that \( M \) is generic with respect to \( \mathcal{H} \). Let \( Y \subset G^m_{gr}(M) \) be a closed irreducible subvariety of the total space of the fibration \( G^m_{gr}(M) \xrightarrow{\pi} M \). Assume that \( Y \) is generically finite over \( M \). Then \( m = \frac{l(l+1)}{2} \) for some positive integer \( l \), and \( Y \) is the image of the natural map \( s_{gr} : M \to G^m_{gr}(M) \) constructed above.

**Proof:** By Lemma 6.4, the bundle \( J^m_{gr}(M) \) is polystable. As usually, applying the Uhlenbeck–Yau theorem (Theorem 2.21), we endow the fibration \( G^m_{gr}(M) \xrightarrow{p} M \) with a natural connection \( \nabla \). From Proposition 6.6 it is easy to deduce that the image of \( \pi : X \to G^m_{gr}(M) \) is preserved by the connection \( \nabla \). Since \( Y \) is generically finite over \( M \), the natural projection \( Y \xrightarrow{p} M \) is a finite covering. Since \( M \) is simply connected, this map is an isomorphism.

For \( x \in M \), let \( t_x \in X \) be the ideal of \( J^m_{gr}(M) \) such that \( p(t_x) = x \). Denote by \( l \) the maximal number such that \( t_x \not\supseteq \mathfrak{m}^{l-1}(M) \) for some \( x \).
Consider the space

\[ A_x := \mathbb{W}^{l-1}(M) \cap t_x / \mathbb{W}^{l}(M) \subset \mathbb{W}^{l-1}(M) / \mathbb{W}^{l}(M). \]

Let \( w = \dim A_x \). Since \( t_x \) is preserved by the connection \( \nabla \), the number \( w \) does not depend on \( x \). This gives a \( w \)-dimensional subbundle \( A \) in \( V = \mathbb{W}^{l}(M) \subset \mathbb{W}^{l-1}(M) \). By Lemma 6.9, \( A \) is either \( V \) or empty. Since \( t_x \not\supset \mathbb{W}^{l-1}(M) \), \( A = 0 \). Since \( t_x \) is an ideal, this implies that \( t_x \subset \mathbb{W}^{l}(M) \). By definition of \( l \), it is the maximal number for which \( t_x \not\supset \mathbb{W}^{l-1}(M) \), and thus, \( t_x = l \). Therefore, \( t_x = l \). This proves Proposition 6.10. We finished the proof of Proposition 6.5 and Theorem 6.1.

## 7 Special subvarieties of the Hilbert scheme

### 7.1 Special subvarieties

**Definition 7.1:** (See also Definition 5.1). Let \( M \) be a complex surface, \( \mathcal{A} \) a finite set, \( \varphi : \mathcal{A} \to M \) an arbitrary map. For \( i \in \mathcal{A} \), consider the local ring \( O_{\varphi(i)} \) of germs of holomorphic functions in \( \varphi(i) \). For \( U \subset M \), consider the set \( A_U \) of all automorphisms (global or infinitesimal) of \( U \) which fix the image \( \text{im} \varphi \subset M \) and act trivially on \( O_{\varphi(i)} \). For \( \gamma \in A_U \), we denote by \( \gamma^{[n]} \) the corresponding automorphism of the Hilbert scheme \( U^{[n]} \). A closed subvariety \( X \subset M^{[n]} \) is called **special** if for all \( U \subset M \), all \( \gamma \in A_U \), \( X \cap U^{[n]} \) is fixed by \( \gamma^{[n]} \).

We are going to characterize special subvarieties more explicitly, in the spirit of Theorem 5.3.

Let \( M \) be a complex surface, \( \alpha \) a Young diagram.

\[ \alpha = (n_1 \geq n_2 \ldots \geq n_k), \quad \sum n_i = n, \]

\( \mathcal{A} \subset \{1, \ldots, k\} \), and \( \varphi : \mathcal{A} \to M \) an arbitrary map. Consider a subvariety \( \Delta_{(\alpha)}(\mathcal{A}, \varphi) \) of \( M^{(n)} \) defined as in (5.4). A generic point \( a \in \Delta_{(\alpha)}(\mathcal{A}, \varphi) \) is
the one satisfying

\[ a = \sigma(x_1, ..., x_n), \quad \text{where} \quad x_1 = x_2 = ... = x_n \]

\[ x_{n_1 + 1} = x_{n_1 + 2} = ... = x_{n_1 + n_2} = ..., \]

\[ x_{\sum_{i=1}^{k-1} n_i + 1} = x_{\sum_{i=1}^{k-1} n_i + 2} = ... = x_n \]

and

\[ (i) \quad x_i = \varphi(i) \text{ for all } i \in A \]

\[ (ii) \quad \text{the points } x_1, x_{n_1 + 1}, ..., x_{(\sum_{i=1}^{j-1} n_i) + 1}, ..., \text{ are pairwise unequal for all } j \notin A \]

\[ (iii) \quad \text{the points } x_1, x_{n_1 + 1}, ..., x_{(\sum_{i=1}^{j-1} n_i) + 1}, ..., \text{ don't belong to the set } \varphi(A), \text{ for all } j \notin A \]

We split the Young diagram \( \alpha = (n_1 \geq n_2 \geq ... \geq n_k) \), \( \sum n_i = n \), onto two diagrams, \( \alpha_A = (n_{a_1} \geq n_{a_2} \geq ... \geq n_{a_l}) \), with \( a_i \) running through \( A \), and \( \tilde{\alpha}_A = (n_{b_1} \geq n_{b_2} \geq ... \geq n_{b_{k-l}}) \), where \( b_i \) runs through \( \{1, ..., k\} \setminus A \). Consider the Hilbert scheme \( M^{[n_a]} \), where \( n_a = \sum n_{a_i} \). The map \( \varphi : A \to M \) gives a point \( \Phi \in \Delta(\alpha_A) \subset M^{[n_a]} \) (see Subsection 7.2 for details). Let \( F_\varphi := \pi^{-1}(\Phi) \) be the fiber of the standard projection \( \pi : M^{[n_a]} \to M^{(n_a)} \). For a generic point \( y \in \Delta(\tilde{\alpha}_A) \) a generic point of \( \Delta(\tilde{\alpha}_A) \), let \( F_{\tilde{\alpha}_A}(y) \) be the fiber of \( \pi : M^{[n_b]} \to M^{(n_b)} \) over \( y \). Clearly, for \( z \in \Delta(\alpha_A, \varphi) \) a generic point, the fiber of \( \pi : M^{[n]} \to M^{(n)} \) over \( z \) is isomorphic to \( F_\varphi \times F_{\tilde{\alpha}_A}(y) \). This isomorphism is not canonical, but is defined up to a twist by the action of the group \( G_y \) (see Lemma 5.2 for details).

Fix a \( G_y \)-invariant subvariety \( E \subset F_\varphi \times F_{\tilde{\alpha}_A}(y) \). For a generic point \( z \in \Delta(\alpha_A, \varphi) \), consider a subvariety \( E_z \subset \pi^{-1}(z) \subset M^{[n]} \) corresponding to \( E \) under the isomorphism

\[ \pi^{-1}(z) \cong F_\varphi \times F_{\tilde{\alpha}_A}(y). \quad (7.2) \]

Let \( \Delta(\alpha_A, \varphi, E) \) be the closure of the union of \( E_z \) for all \( z \in \Delta(\alpha_A, \varphi) \) satisfying (7.1). Clearly, \( \Delta(\alpha_A, \varphi, E) \) is a closed subvariety in \( M^{[n]} \).
Theorem 7.2: Let $M$ be a complex surface, $A$ a finite set, $\varphi : A \to M$ an arbitrary map, and $X \subset M^{[n]}$ a special subvariety, associated with $\varphi$. Then

(i) there exist a Young diagram $\alpha$

$$\alpha = (n_1 \geq n_2 \geq \ldots \geq n_k), \quad \sum n_i = n,$$

an injection $A \hookrightarrow \{1, \ldots, k\}$, and a $G_y$-invariant subvariety

$$E \subset F_\varphi \times F_{\bar{\alpha},A}(y),$$

such that $X = \Delta_{[\alpha]}(A, \varphi, E)$, where $F_\varphi \times F_{\bar{\alpha},A}(y)$ and $\Delta_{[\alpha]}(A, \varphi, E)$ are varieties constructed above.

(ii) Conversely, $\Delta_{[\alpha]}(A, \varphi, E)$ is a special subvariety of $M^{[n]}$ for all $A, \pi, E$.

Proof: We use the notation of Definition 7.1. For sufficiently small $U$, the automorphisms from $A_U \setminus \text{im } \varphi$ act $n$-transitively on $U \setminus \text{im } \varphi$. This implies that $\pi(X) = \Delta_{[\alpha]}(A, \varphi)$, for an appropriate Young diagram

$$\alpha = (n_1 \geq n_2 \geq \ldots \geq n_k), \quad \sum n_i = n,$$

and an embedding $A \hookrightarrow \{1, 2, \ldots, k\}$. Let $x$ be a generic point of $\Delta_{[\alpha]}(A, \varphi)$. Consider the isomorphism $\pi^{-1}(x) \cong F_\varphi \times F_{\bar{\alpha},A}(y)$ of (7.2), and the action of $G_y$ on $F_\varphi \times F_{\bar{\alpha},A}(y)$. Clearly, $A_U$ acts on $\pi^{-1}(x)$ as $G_y$. Therefore, the intersection $E_x := X \cap \pi^{-1}(x)$ is $G_y$-invariant. We intend to show that $X = \Delta_{[\alpha]}(A, \varphi, E_x)$

For $x, y \in M^{(n)}$ generic points of $\Delta_{[\alpha]}(A, \varphi)$, there exists $U \supset \{x, y\}$ and an automorphism $\gamma : U \to U$ such that $\gamma^{(n)}(x) = y$, for $\gamma^{(n)} : U^{(n)} \to U^{(n)}$ the induced by $\gamma$ automorphism of $U^{(n)}$. Since $X$ is a special subvariety, $\gamma^{[n]}$ maps $E_x$ to $E_y := X \cap \pi^{-1}(y)$. By definition, $\Delta_{[\alpha]}(A, \varphi, E_x)$ is a closure of the union of all $\gamma^{[n]}(E_x)$, for all $U \subset M$ and $\gamma \in A_U$. On the other hand, $X$ is a closure of the union of all $E_y$, where $y$ runs through all generic points of $\Delta_{[\alpha]}(A, \varphi)$. Thus, $X$ and $\Delta_{[\alpha]}(A, \varphi, E_x)$ coincide. This proves Theorem 7.2 (i). Theorem 7.2 (ii) is clear.
7.2 Special subvarieties of the Hilbert scheme of K3

**Theorem 7.3:** Let $M$ be a complex K3 surface admitting a hyperkähler structure $H$ such that $M$ is generic with respect to $H$, $M^{[n]}$ its Hilbert scheme and $M^{(n)}$ its symmetric power. Let $X \subset M^{[n]}$ be a closed irreducible subvariety such that $X$ is generically finite over $\pi(X) \subset M^{(n)}$. Assume that $M$ has no holomorphic automorphisms. Then $X$ is a special subvariety of $M^{[n]}$, in the sense of Definition 7.1.

**Proof:** From Proposition 3.4 it follows that $\pi(X) = \Delta_{(\alpha)}(A, \varphi)$ for appropriate $A$, $\alpha$ and $\varphi$. As previously, we split the Young diagram $\alpha$ onto $\alpha_A = (n_{a_1} \geq n_{a_2} \geq \ldots \geq n_{a_l})$, with $a_i$ running through $A$, and $\alpha_A = (n_{b_1} \geq n_{b_2} \geq \ldots \geq n_{b_k-l})$, where $b_i$ runs through $\{1, \ldots, k\} \setminus A$. Let $n_a := \sum n_{a_i}$, $n_b := \sum n_{b_i}$. Consider the natural map

$$M^{(n_a)} \times M^{(n_b)} \xrightarrow{s} M^{(n)},$$

defined in such a way as that to map $\Delta_{(\alpha)} \times \Delta_{(\tilde{\alpha}_A)}$ to $\Delta_{(\alpha)}$. This map is obviously finite. Let $x = (x_1, \ldots, x_{n_a}) \in \Delta_{(\alpha_A)}$ and $y = (y_1, \ldots, y_{n_a}) \in \Delta_{(\tilde{\alpha}_A)}$ be the points satisfying $x_i \neq y_j \forall i, j$. Then the fiber of $\pi : M^{[n]} \to M^{(n)}$ in $s(x, y)$ is naturally isomorphic to the product $\pi^{-1}(x) \times \pi^{-1}(y)$, where the first $\pi$ is the standard projection $\pi : M^{[n_a]} \to M^{(n_a)}$ and the second one is the standard projection $\pi : M^{[n_b]} \to M^{(n_b)}$. Denote thus obtained map

$$\pi^{-1}(x) \times \pi^{-1}(y) \xrightarrow{\theta} \pi^{-1}(s(x, y))$$

by $\theta$. Together, the maps (7.3), (7.4) give a correspondence

$$D \subset \left( \Delta_{[\alpha_A]} \times \Delta_{[\tilde{\alpha}_A]} \right) \times \Delta_{[\alpha]}$$

which is generically one-to-one over the first component and generically finite over the second one. Denote the corresponding projections from $D$ by $\pi_1$, $\pi_2$. Consider $X$ (the subvariety of $M^{[n]}$ given as data of Theorem 7.3) as a subvariety of $\Delta_{[\alpha]}$. Let $D_X := \pi_1(\pi_2^{-1}(X))$. And $\Phi \in \Delta_{(\alpha,A)}$ be the point given by $\varphi$,

$$\Phi = \begin{pmatrix} \varphi(a_1), \ldots, \varphi(a_1), \varphi(a_2), \ldots, \varphi(a_2), \ldots \\ n_{a_1} \text{ times} \hspace{1cm} n_{a_2} \text{ times} \end{pmatrix}. \quad 41$$
Let \( p_1, p_2 \) be the projections of \( \Delta_{[\alpha,A]} \times \Delta_{[\hat{\alpha},\hat{A}]} \) to its components. Since \( \pi(X) = \Delta_{(\alpha)}(A, \varphi) \), and \( X \) is generically finite over \( \pi(X) \), the subvariety

\[
D_X \subset \Delta_{[\alpha,A]} \times \Delta_{[\hat{\alpha},\hat{A}]} \]

is generically finite over \( \{\Phi\} \times \Delta_{(\hat{\alpha},\hat{A})} \). Therefore, \( p_2(D_2) \subset \Delta_{[\hat{\alpha},\hat{A}]} \) is generically finite over \( \{\Phi\} \times \Delta_{(\hat{\alpha},\hat{A})} \). Applying Theorem 6.1, we obtain that \( p_2(D_X) \) is a universal subvariety of \( \Delta \) The varieties \( \Delta_{[\alpha,A]}, \Delta_{[\hat{\alpha},\hat{A}]} \) are equipped with the local action of the automorphisms \( A_U \) (see Definition 7.1). Since \( p_2(D_X) \) is universal,

\[
D_X \subset \{\Phi\} \times \Delta_{[\hat{\alpha},\hat{A}]} \subset \Delta_{[\alpha,A]} \times \Delta_{[\hat{\alpha},\hat{A}]}
\]

is fixed by the \( A_U \) -action. Therefore, \( p_2(D_2) \subset \Delta_{[\hat{\alpha},\hat{A}]} \) is also fixed by \( A_U \). By construction, \( \pi_2(D_X) = X \), and thus, \( X \) is fixed by \( A_U \), i.e., special. 

### 7.3 Special subvarieties of relative dimension 0

**Definition 7.4:** Let \( M \) be a complex surface, \( M^{[n]} \) its Hilbert scheme and \( X \subset M^{[n]} \) an irreducible special subvariety. The **relative dimension of** \( X \) is the dimension of the generic fiber of the projection \( \pi|_X : X \rightarrow \pi(X) \), where \( \pi : M^{[n]} \rightarrow M^{(n)} \) is the standard morphism.

Let \( \Delta_{(\alpha)}(A, \varphi) \subset M^{(n)} \) be the subvariety defined as in Subsection 6.2. Split \( \alpha \) onto \( \alpha_A \) and \( \hat{\alpha}, \hat{A} \), as in Subsection 7.2: \( \alpha_A = (n_{a_1} \geq n_{a_2} \geq \ldots \geq n_{a_t}) \), with \( a_i \) running through \( A \) and \( \hat{\alpha}, \hat{A} = (n_{b_1} \geq n_{b_2} \geq \ldots \geq n_{b_{k-l}}) \), where \( b_i \) runs through \( \{1, \ldots, k\} \setminus A \). Let \( \Phi \in \Delta_{(\alpha,A)} \) be the point defined in Subsection 7.2. Consider the variety \( \pi^{-1}(\Phi) \subset M^{[n_a]} \), where \( n_a = \sum n_{a_i} \).

**Proposition 7.5:**

(i) There exists a special subvariety \( X \subset M^{[n]} \) such that \( \pi(X) = \Delta_{(\alpha)}(A, \varphi) \) if and only if all the numbers \( n_{b_i} \) are of form \( \frac{i(l+1)}{2} \), for integer \( l \)’s.

(ii) Let \( \mathcal{S}(\alpha, A, \varphi) \) be the set of all such subvarieties. Assume that all the numbers \( n_{b_i} \) are of form \( \frac{i(l+1)}{2} \), for integer \( l \)’s. Then \( \mathcal{S}(\alpha, A, \varphi) \) is in bijective correspondence with the set of points of \( \pi^{-1}(\Phi) \subset M^{[n_a]} \).

**Proof:** Using notation of the proof of Theorem 7.3, we consider the subvariety \( p_2(D_X) \subset \Delta_{(\hat{\alpha},\hat{A})} \). We have shown that this subvariety is universal of relative dimension 0. Therefore, \( \mathcal{S}(\alpha, A, \varphi) \) is nonempty if and only if
all $n_b$ are of the form $\frac{l(l+1)}{2}$, for integer $l$’s. Assume that $\mathcal{G}(\alpha, \mathcal{A}, \varphi)$ is nonempty. Consider the unique universal subvariety $S \subset \Delta[\alpha, \mathcal{A}]$ of relative dimension 0, constructed in Proposition 5.5. As in Theorem 5.3, a universal subvariety of $\Delta[\alpha, \mathcal{A}]$ corresponds to a $G_y$-invariant subvariety of the general fiber of the projection $\Delta[\alpha, \mathcal{A}] \rightarrow \Delta(\alpha, \mathcal{A})$. Since $S$ is of relative dimension 0, the corresponding $G_y$-invariant subvariety is a point. Denote this point by $s$. Choose a point $f \in \pi^{-1}(\Phi)$. Using the notation of Theorem 7.2, and an isomorphism (7.4), we construct a special subvariety $X \subset M[n]$, $X = \Delta[\alpha, \mathcal{A}, \{s\} \times \{f\}]$. From Theorem 7.2 it follows that all special subvarieties of relative dimension 0 are obtained this way. Since $s$ is defined canonically, the only freedom of choice we have after $\alpha, \mathcal{A}, \varphi$ are fixed is the choice of $f \in \pi^{-1}(\Phi)$. This finishes the proof of Proposition 7.5.

**Proposition 7.6:** Let $M$ be a complex K3 surface with no complex automorphisms, $M[n]$ its Hilbert scheme and $\Omega$ be the canonical holomorphic symplectic form on $M[n]$. Assume that $M$ admits a hyperkähler structure $\mathcal{H}$ such that $M$ is Mumford-Tate generic with respect to $\mathcal{H}$. Let $X$ be an irreducible complex subvariety of $M^n$, such that the restriction $\Omega|_X$ is non-degenerate somewhere in $X$. Then $X \subset M[n]$ is a special subvariety of relative dimension 0.

**Proof:** By Theorem 7.3, to prove that $X$ is special it suffices to show that $X$ is generically finite over $\pi(X)$. This follows Claim 4.6.

**Corollary 7.7:** Let $M$ be a complex K3 surface which is Mumford-Tate generic with respect to some hyperkahler structure, $M[n]$ its Hilbert scheme and $M(n)$ its symmetric power. Assume that $M$ has no holomorphic automorphisms. Consider an arbitrary hyperkähler structure on $M^n$ which is compatible with the complex structure. Let $X \subset M[n]$ be a trianalytic subvariety of $M[n]$. Then $X$ is generically one-to-one over $\pi(X) \subset M(n)$.

**Proof:** By Proposition 7.6, $X$ is a special subvariety of relative dimension 0. Now Corollary 7.7 follows from an explicit description of special subvarieties of relative dimension 0, given in the proof of Proposition 7.5.

1. Clearly, for $X$ trianalytic, $X_{ns}$ the non-singular part of $X$, $\Omega|_{X_{ns}}$ is nowhere degenerate.
8 Trianalytic and universal subvarieties of the Hilbert scheme of a general K3 surface

The aim of this section is to show that all trianalytic subvarieties of the Hilbert scheme of a generic K3 surface are universal (Theorem 8.3).

8.1 Deformations of trianalytic subvarieties

We need the following general results on the structure of deformations of trianalytic subvarieties, proven in [V3].

Let $M$ be a compact hyperkähler manifold, $I$ an induced complex structure and $X \subset M$ a trianalytic subvariety. Consider $(X, I)$ as a closed complex subvariety of $(M, I)$. Let $\text{Def}_I(X)$ be a Douady space of $(X, I) \subset (M, I)$, that is, a space of deformations of $(X, I)$ inside of $(M, I)$. By Theorem 2.8 for $X' \subset (M, I)$ a complex deformation of $X$, the subvariety $X' \subset M$ is trianalytic. In particular, $X'$ is equipped with a natural singular hyperkähler structure ([V3]), i.e., with a metric and a compatible quaternionic structure.

Theorem 8.1:

(i) The $\text{Def}_I(X)$ is a singular hyperkähler variety, which is independent from the choice of an induced complex structure $I$.

(ii) Consider the universal family $\pi: \mathcal{X} \longrightarrow \text{Def}_I(X)$ of subvarieties of $(M, I)$, parametrized by $\text{Def}_I(X)$. Then the fibers of $\pi$ are isomorphic as hyperkähler varieties.

(iii) Applying the desingularization functor to $\pi: \mathcal{X} \longrightarrow \text{Def}_I(X)$, we obtain a projection $\pi: \widetilde{X} \times Y \longrightarrow Y$, where $Y$ is a desingularization of $\text{Def}_I(X)$ and $\widetilde{X}$ is a desingularization of $X$.

(iv) The variety $\widetilde{X} \times Y$ is equipped with a natural hyperkähler immersion to $M$.

Proof: Theorem 8.1 (i) and (ii) is proven in [V3], and Theorem 8.1 (iii) is a trivial consequence of Theorem 8.1 (ii) and the functorial property of the hyperkähler desingularization. To prove Theorem 8.1 (iv), we notice that $\mathcal{X}$

\footnote{The Desingularization Theorem (Theorem 2.13) significantly simplifies some of the proofs of [V3]. This simplification is straightforward.}
is equipped with a natural morphism $f : \mathcal{X} \to M$, which is compatible with the hyperkähler structure. Let $n : \tilde{X} \times Y \to \mathcal{X}$ be the desingularization map. Clearly, the composition $\tilde{X} \times Y \xrightarrow{n} \mathcal{X} \xrightarrow{f} M$ is compatible with the hyperkähler structure. A morphism compatible with a hyperkähler structure is necessarily an isometry, and an isometry is always an immersion.

**8.2 Deformations of trianalytic special subvarieties**

Let $M$ be a K3 surface, without automorphisms, which is Mumford-Tate generic with respect to some hyperkähler structure. In this Subsection, we study the deformations of the special subvarieties of $M^{[n]}$.

By Proposition 5.5, universal subvarieties of $M^{[n]}$ are rigid. For the special subvarieties, a description of its deformations is obtained as an easy consequence of Proposition 7.5.

**Claim 8.2**: Let $X \subset M^{[n]}$ be a special subvariety of relative dimension 0,  

$$X = \Delta_{[a]}(\mathcal{A}, \phi, \{s\} \times \{\psi\})$$

associated with $\Delta_{[a]}(\mathcal{A}, \phi)$ and $\psi \in F_{\phi}$ as in Proposition 7.5. Then the deformations of $X$ are locally parametrized by varying $\phi : \mathcal{A} \to M$ and $\psi \in F_{\phi}$.

Let $a_1, \ldots, a_r$ enumerate $\mathcal{A} \subset \{1, \ldots, k\}$. Unless all $n_{a_i} = 1$, the dimension of $F_{\phi}$ is non-zero. Thus, the union $\mathcal{X}$ of all deformation of $X = \Delta_{[a]}(\mathcal{A}, \phi, \{s\} \times \{\psi\})$ is not generically finite over $\pi(\mathcal{X}) = \Delta_{[a]}(\mathcal{A}, \phi)$. Together with Theorem 8.1 and Proposition 7.6, this suggests the following proposition.

**Proposition 8.3**: Let $M$ be a complex K3 surface with no automorphisms which is Mumford-Tate generic with respect to some hyperkähler structure. Consider the Hilbert scheme $M^{[n]}$ as a complex manifold. Let $\mathcal{H}$ be an arbitrary hyperkähler structure on $M^{[n]}$ agreeing with this complex structure. Consider an irreducible trianalytic subvariety $X \subset M^{[n]}$. By Proposition 7.6, $X$ is a special subvariety of $M^{[n]}$, $X = \Delta_{[a]}(\mathcal{A}, \phi, \{\psi\} \times \{s\})$. Then $n_i = 1$ for all $i \in \mathcal{A}$.
**Proof:** Consider $X$ as a complex subvariety in the complex variety $M^{[n]}$. The corresponding Douady space is described by Theorem 8.1. Consider the diagonal $\Delta_{(\alpha)} \subset M^{(n)}$. For a general point $a \in \Delta_{(\alpha)} \subset M^{(n)}$, the fiber $\pi^{-1}(a)$ is naturally decomposed as in (7.4):

$$\pi^{-1}(x) \times \pi^{-1}(y) \xrightarrow{\sim} \pi^{-1}(r(x, y)),$$

for $a = r(x, y)$, where $r : M^{(n_a)} \times M^{(n_b)} \to M^{(n)}$ is a morphism of (7.3). Consider the subvariety $\pi^{-1}(y) \times \{s\} \subset \pi^{-1}(a)$. This subvariety is clearly $G_a$-invariant, and applying Theorem 5.3, we obtain a universal subvariety of $M^{[n]}$. Denote this universal subvariety by $\Delta_{[\alpha]}(A)$. Let $X$ be the union of all complex deformations of $X \subset M^{[n]}$. From Theorem 8.1 it is clear that $X$ is trianalytic; from Proposition 7.5 it follows that $X = \Delta_{[\alpha]}(A)$. By Theorem 8.1, $X$ is trianalytic in $M^{[n]}$. From Proposition 7.6 it follows that all trianalytic subvarieties $X \subset M^{[n]}$ are generically finite over $\pi(X) \subset M^{(n)}$.

Consider the Young diagram $\alpha_A = (n_{a_1} \geq n_{a_2} \geq \ldots)$. The generic fiber of thus obtained generically finite map

$$\pi : \Delta_{[\alpha]}(A) \to \Delta_{(\alpha)}$$

is isomorphic to $\pi^{-1}(y)$, where $y$ is a generic point of $\Delta_{(\alpha_A)}$, and $\pi$ a projection $\pi : \Delta_{[\alpha_A]} \to \Delta_{(\alpha_A)}$. The dimension of this fiber is equal to $\sum(n_{a_i} - 1)$. By construction, $a_i$ enumerates $A \subset \{1, \ldots, k\}$. Thus, $n_i = 0$ for all $i \in A$.

This proves Proposition 8.3. \[\Box\]

### 8.3 Special subvarieties and holomorphic symplectic form

The aim of the Subsection is the following statement.

**Proposition 8.4:** Let $M$ be a complex surface equipped with a holomorphically symplectic form, $M^{[n]}$ its $n$-th Hilbert scheme, and

$$X = \Delta_{[\alpha]}(A, \varphi, \{\psi\} \times \{s\})$$

the special subvariety of relative dimension 0. Consider the holomorphic symplectic form $\Omega$ on $M^{[n]}$. Assume that for all $i \in A$, $n_i = 1$. Assume, furthermore, that the normalizarion $\tilde{X}$ of $X$ is smooth, and the pullback of $\Omega$ to $\tilde{X}$ is a nowhere degenerate holomorphic symplectic form on $\tilde{X}$. Then $A$ is empty. \[\Box\]

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\[2\] To say that $A$ is empty is the same as to say that $X$ is a universal subvariety of $M$. 46
Proof: Assume that \( \mathcal{A} \) is nonempty. By [Theorem 2.8], all complex deformations of \( X \) are trianalytic. Clearly, \( \Delta_{(n)}(\mathcal{A}, \varphi', \{ \psi \} \times \{ s \}) \) is a deformation of \( X \) for every \( \varphi' : \mathcal{A} \to M \). Therefore, we may assume that \( \varphi : \mathcal{A} \to M \) is an embedding.

Let \( x \in \Delta_{(n)} \) be an arbitrary point. We represent \( x \) as in (3.3). The \( i \)-th component of \( x \) is \( x(\sum_{i=1}^{n_1} + 1) \), in notation of (3.3). The components are defined up to a permutation \( i \to j \), for \( i, j \) satisfying \( n_i = n_j \). Fix \( p, q \in \{1, ..., k\}, p \in \mathcal{A}, q \notin \mathcal{A} \). Let \( \Pi_{pq} : X \to \{ \varphi(q) \} \times M \) be the map associating to \( x \in X \) the \( n_p \)-th and \( n_q \)-th components of \( \pi(x) \in \Delta_{(n)} \subset M^{(n)} \). Clearly, the map \( \Pi_{pq} \) is correctly defined. Let \( \Pi_{pq} : X \to M^{(\alpha-n_p-n_q)} \) be the map associating to \( x \in X \) the rest of components \( x(\sum_{i=1}^{n_1} + 1), i \neq p, q \) of \( x \). Let \( 2^M \) be the set of subsets of \( M \). Consider the map \( c : M^{(i)} \to 2^M \) associating to \( x \in M^{(i)} \) the corresponding subset of \( M \). For \( t = (\varphi(q), t_0) \in \{ \varphi(q) \} \times M \), denote by \( c(t) \) the subset \( \{ \varphi(q), t_0 \} \subset M \). Let \( X_0 \subset X \) be the set of all \( x \in X \) such that \( c(\Pi_{pq}(x)) \) does not intersect \( c(\Pi_{pq^{-1}}(x)) \). A subvariety \( C \subset X \) is called non-degenerately symplectic if the holomorphic symplectic form on \( X \) is nowhere degenerate on \( C \). For any \( t \in \Pi_{pq}(X) \subset M^{(\alpha-n_p-n_q)} \), the intersection \( X_t := \Pi_{pq}^{-1}(t) \cap X_0 \) is smooth. The holomorphically symplectic form in a tangent space to a zero-dimensional sheaf \( S \in M^{[n]} \), \( S_{\text{Sup}}(S) = A \coprod B \) can be computed separately for the part with support in \( A \) and the part with support in \( B \). Thus, \( X_t \) must be non-degenerately symplectic. On the other hand, \( X_t = \Pi_{pq}^{-1}(t) \cap X_0 \) is easy to describe explicitly. Let \( M_0 = M \setminus c(t) \), where \( c(t) \) is again \( t \) considered as a subset on \( M \). Then \( X_t \) is canonically isomorphic to a blow-up of \( M_0 \) in \( \{ \varphi(q) \} \). This blow-up is obviously not non-degenerately symplectic. We obtained a contradiction. This concludes the proof of [Proposition 8.4].

8.4 Applications for trianalytic subvarieties

Proposition 8.4 implies the following theorem, which is the main result of this section.

Theorem 8.5: Let \( M \) be a complex K3 surface with no automorphisms which is Mumford-Tate generic with respect to some hyperkähler structure. Consider the Hilbert scheme \( M^{[n]} \) as a complex manifold. Let \( \mathcal{H} \) be an arbitrary hyperkähler structure on \( M^{[n]} \) agreeing with this complex structure. Consider a trianalytic subvariety \( X \subset M^{[n]} \). Then \( X \) is a universal subvariety of \( M^{[n]} \) of relative dimension 0.

Proof: By [Proposition 8.3], \( X \) is a special subvariety of \( M^{[n]} \), \( X = \)
\[ \Delta_{\{a\}}(A, \varphi, \{\psi\} \times \{s\}). \] The \( X \) is non-degenerately symplectic because it is trianalytic. Applying Proposition 8.4, we obtain that \( A \) is empty, and \( X \) is universal in \( M^{[n]} \). □

Corollary 8.6: In assumptions of Theorem 8.5, \( \text{codim}_C X \geq 4 \), unless \( X = M^{[n]} \).

Proof: Theorem 7.2 classifies universal subvarieties of relative dimension 0. All such subvarieties correspond to diagonals \( \Delta_{\{a\}} \subset M^{(n)} \), with \( \alpha = (n_1 \geq n_2 \geq \ldots \geq n_k) \), \( \sum n_i = n \), with all \( n_i \) of form \( \frac{l(l+1)}{2} \), with integer \( l \)'s. Thus, for \( X \neq M^{[n]} \) we have \( n_1 \geq 3 \). On the other hand, \( \text{codim}_C \Delta_{\{a\}} = 2 \sum (n_i - 1) \), so \( \text{codim}_C \Delta_{\{a\}} \geq 4 \). Finally, since \( X \) is of relative dimension 0, \( \dim X = \dim \Delta_{\{a\}} \), so \( \text{codim}_C X = \text{codim}_C \Delta_{\{a\}} \geq 4 \). □

9 Universal subvarieties of the Hilbert scheme and algebraic properties of its cohomology

9.1 Birational types of algebraic subvarieties of relative dimension 0

Let \( M^{[n]} \) be a Hilbert scheme, \( \alpha \) a Young diagram satisfying assumptions of Proposition 5.5 and \( X_\alpha \subset M^{[n]} \) the corresponding universal subvariety of relative dimension 0. Consider the natural map \( \pi : M^{[n]} \rightarrow M^{(n)} \) mapping the Hilbert scheme to the symmetric product of \( n \) copies of \( M \). Clearly, \( \pi(X_\alpha) = \Delta_{\{a\}} \), where \( \Delta_{\{a\}} \) is the stratum of \( M^{(n)} \) corresponding to the Young diagram \( \alpha \) as in Subsection 3.2. Moreover, from the definition of \( X_\alpha \) it is evident that \( \pi : X_\alpha \rightarrow \Delta_{\{a\}} \) is a birational isomorphism.

Let

\[ \alpha = (n_1 = \ldots = n_{i_1} > n_{i_1+1} = \ldots = n_{i_1+i_2} > \ldots > n_{1+\sum_{j=1}^{l-1} i_j} = \ldots = n_{1+\sum_{j=1}^{l-1} i_j}) \]

be a diagram satisfying assumptions of Proposition 5.5. Then \( \Delta_{\{a\}} \) is birationally isomorphic to the product \( \prod_{j=1}^{l} M^{(i_j)} \). Thus, \( X_\alpha \) is birational to a hyperkähler manifold \( \prod_{j=1}^{l} M^{[i_j]} \).

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Theorem 9.1: (Mukai) Let \( f : X_1 \to X_2 \) be a birational isomorphism of compact complex manifolds of hyperkähler type. Then the second cohomology of \( X_1 \) is naturally isomorphic to the second cohomology of \( X_2 \), and this isomorphism is compatible with the Hodge structure and the Bogomolov-Beauville form on \( H^2(X_1) \), \( H^2(X_2) \).

Proof: Well known; see, e. g. [M], [H].

We obtain that \( H^2(X_\alpha) \) is isomorphic to \( \bigoplus_{j=1}^l H^2(M[i_j]) \). Therefore, \( \dim H^{2,0}(X_\alpha) > 1 \) unless \( l = 1 \). On the other hand, by Bogomolov’s decomposition theorem (Theorem 2.15), a hyperkähler manifold with \( H^1(X, \mathbb{R}) = 0 \), \( \dim H^{2,0}(X) > 1 \) is canonically isomorphic to a product of two hyperkähler manifolds of positive dimension. We obtain the following result.

Proposition 9.2: Let \( M \) be a complex K3 surface with no complex automorphisms, admitting a hyperkähler structure \( \mathcal{H} \) such that \( M \) is Mumford-Tate generic with respect to \( \mathcal{H} \). Let \( X_\alpha \) be a trianalytic subvariety of \( M^{[n]} \), which is by Theorem 8.5 universal and corresponds to a Young diagram \( \alpha = (n_1 \geq n_2 \geq ... \geq n_l) \). Assume that \( X_\alpha \) is not isomorphic to a product of two hyperkähler manifolds of positive dimension. Then \( n_1 = n_2 = ... = n_l \), and \( X_\alpha \) is birationally equivalent to \( M^{[l]} \).

9.2 The Bogomolov-Beauville form on the Hilbert scheme

Let \( X \) be a simple hyperkähler manifold. It is well known that \( H^2(X) \) is equipped with a natural non-degenerate symmetric pairing

\[
\langle \cdot, \cdot \rangle_B : H^2(X) \times H^2(X) \to \mathbb{C}
\]

which is compatible with the Hodge structure and with the \( SU(2) \)-action. This pairing is defined up to a constant multiplier, and it is a topological invariant of \( X \). For a formal definition and basic properties of this form, see [Bea] (Remarques, p. 775), and also [V], [V-a].

For a Hilbert scheme \( M^{[n]} \) of points on a K3 surface, the form \( \langle \cdot, \cdot \rangle_B \) can be computed explicitly as follows.

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1 For the definition and properties of Bogomolov-Beauville form, see Subsection 9.2.
2 See Theorem 2.15 for the definition of “simple”.

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Consider the map $\pi : M[n] \to M^{(n)}$ from the Hilbert scheme to the symmetric power of $M$. Clearly, the space $H^2(M^{(n)})$ is naturally isomorphic to $H^2(M)$. Let $\Delta_n \subset M[n]$ be the singular locus of the map $\pi$, and $[\Delta_n] \in H^2(M^{[n]})$ its fundamental class. The following proposition gives a full description of the Bogomolov-Beauville form on $H^2(M^{[n]})$ in terms of the Poincare form on $H^2(M)$.

**Proposition 9.3:** Let $M$ be a K3 surface, and $M^{[n]}$ its Hilbert scheme of points. Consider the pullback map $\pi^* : H^2(M) = H^2(M^{(n)}) \to H^2(M^{[n]})$.

Then

(i) The map $\pi^* : H^2(M) \to H^2(M^{[n]})$ is an embedding. We have a direct sum decomposition

$$H^2(M^{[n]}) = \pi^*(H^2(M)) \oplus \mathbb{C} \cdot [\Delta_n],$$

where $[\Delta_n] \in H^2(M^{[n]})$ is the cohomology class defined above.

(ii) The decomposition (9.1) is orthogonal with respect to the Bogomolov-Beauville form $(\cdot, \cdot)_B$. The restriction of $(\cdot, \cdot)_B$ to $H^2(M) = \pi^*(H^2(M)) \subset H^2(M^{[n]})$ is equal to the Poincare form times constant.

Since the form $(\cdot, \cdot)_B$ is defined up to a constant multiplier, we may assume that, after a rescaling, $(\cdot, \cdot)_B\big|_{H^2(M)}$ is equal to the Poincare form.

(iii) After a rescaling required by (ii), we have

$$([\Delta_n], [\Delta_n])_B = -2(n - 1).$$

**Proof:** Well known; see, for instance, [H2], 2.2. ■

Further on, we always normalize the Bogomolov-Beauville form on $H^2(M^{[n]})$ as in Proposition 9.3 (ii).

### 9.3 Frobenius algebras associated with vector spaces

Let $X$ be a compact hyperkähler manifold. The algebraic structure of $H^*(X)$ is studied using the general theory of Lefschetz-Frobenius algebras, introduced in [LL]. For details of definitions and computations, the reader is referred to [V], [V-a].
Definition 9.4: Let \( A = \bigoplus_{i=0}^{2d} A_i \) be a graded commutative associative algebra over a field of characteristic zero. Assume that \( A_{2d} \) is 1-dimensional, and the natural linear form \( \varepsilon : A \to A_{2d} \) projecting \( A \) to a \( A_{2d} \) gives a non-degenerate scalar product \( a, b \mapsto \varepsilon(ab) \). Then \( A \) is called a graded commutative Frobenius algebra, or Frobenius algebra for short.

Proposition 9.5: Let \( V \) be a vector space equipped with a non-degenerate scalar product, and \( n \) a positive integer number. Then there exist a unique up to an isomorphism Frobenius algebra \( A(V, n) = A_0 \oplus A_2 \oplus \ldots \oplus A_{4n} \) such that

(i) \[ A_{2i} = S^i(V), \quad \text{for } i \leq n \]
\[ A_{2i} = S^{2n-i}(V), \quad \text{for } i \geq n \]

and

(ii) For an operator \( g \in SO(V) \), consider the corresponding endomorphism of \( S^*(V) \). This way, \( g \) might be considered as a linear operator on \( A \).
Then \( g \) is an algebra automorphism.

Proof: Proposition 9.5 is elementary. For a complete proof of existence and uniqueness of \( A(V, n) \), see [V]. □

The importance of the algebra \( A(V, n) \) is explained by the following theorem.

Theorem 9.6: Let \( X \) be a compact connected simple hyperkähler manifold. Consider the space \( V = H^2(X) \), equipped with the natural scalar product of Bogomolov-Beauville (Subsection 9.2). Let \( A \) be a subalgebra of \( H^*(X) \) generated by \( H^2(M) \). Then \( A \) is naturally isomorphic to \( A(V, n) \).
□

9.4 A universal embedding from a K3 to its Hilbert scheme
Consider a universal embedding \( M \to M[n], \ n = \frac{k(k-1)}{2}, \ n > 1 \), mapping a point \( x \in M \) to subscheme given by the ideal \((m_x)^k\), where \( m_x \) is the
maximal ideal of $x$. Pick a hyperkähler structure $\mathcal{H}$ on $M^{[n]}$. The aim of this subsection is to prove the following result.

**Proposition 9.7:** The image of $\varphi$ is not trianalytic in $M^{[n]}$.

The proof of Proposition 9.7 takes the rest of this section. Together with Proposition 5.5 and Theorem 6.1, this result immediately implies the following corollary.

**Corollary 9.8:** Let $M$ be a complex K3 surface without automorphisms. Assume that $M$ admits a hyperkähler structure $\mathcal{H}$ such that $M$ is Mumford-Tate generic with respect to $\mathcal{H}$ (Definition 2.10). Pick a hyperkähler structure $\mathcal{H}'$ on its Hilbert scheme $M^{[n]}$ ($n > 1$). Let $X \subset M^{[n]}$ be a trianalytic subvariety of $M^{[n]}$. Then $\dim_{\mathbb{C}} X > 1$.

**Proof of Proposition 9.7:** Consider the map
\[ \varphi^* : H^4(M^{[n]}) \longrightarrow H^4(M) = \mathbb{C}. \]
To prove that $\text{im} \varphi$ is not trianalytic in $M^{[n]}$, it suffices to show that $\varphi^*$ is not $SU(2)$-invariant. Let $H^4_+(M^{[n]})$ be the subspace of $H^4(M^{[n]})$ generated by $H^2(M^{[n]})$, and $f : H^4_+(M^{[n]}) \longrightarrow \mathbb{C}$ the restriction of $\varphi^*$ to $H^4_+(M^{[n]}) \subset H^4(M^{[n]})$. Since the subspace $H^4_+(M^{[n]}) \subset H^4(M^{[n]})$ is $SU(2)$-invariant, the map $f$ should be $SU(2)$-invariant if $\varphi^* : H^4(M^{[n]}) \longrightarrow H^4(M) = \mathbb{C}$ is $SU(2)$-invariant.

By Theorem 9.6, the space $H^4_+(M^{[n]})$ is naturally isomorphic to
\[ S^2 H^2(M^{[n]}). \]
Thus, $f$ can be considered as a map $f : S^2 H^2(M^{[n]}) \longrightarrow \mathbb{C}$. Consider the Bogomolov-Beauville form as another such map $B : S^2 H^2(M^{[n]}) \longrightarrow \mathbb{C}$. From Proposition 9.3, it is clear that $f = B + 2(n - 1) d^2$, where $d : H^2(M^{[n]}) \longrightarrow \mathbb{C}$ is the projection of $H^2(M^{[n]})$ to the component $\mathbb{C} = \mathbb{C} \cdot [\Delta_n]$ of the decomposition (9.1). Since $B$ is $SU(2)$-invariant, the map $f$ is $SU(2)$-invariant if and only if the map $d^2 : S^2 H^2(M^{[n]}) \longrightarrow \mathbb{C}$ is $SU(2)$-invariant. Therefore, the following claim is sufficient to prove Proposition 9.7.

**Claim 9.9:** In the above notations, the vector $d^2 \in S^2 H^2(M^{[n]})^*$ is not $SU(2)$-invariant.
**Proof:** Let \( V \) be the \( SU(2) \)-subspace of \( S^2 H^2(M[n])^* \) generated by \( d^2 \). Acting on \( d^2 \) by various \( g \in su(2) \), we can obtain any element of type \( d \cdot g(d) \) (this follows from Leibnitz rule). Therefore, \( V = SU(2) \cdot d^2 \) contains \( d \otimes V_0 \), where \( V_0 \subset H^2(M[n])^* \) is the \( SU(2) \)-subspace of \( H^2(M[n])^* \) generated by \( d \). Acting on \( d \cdot g(d) \) by various \( h \in SU(2) \), we obtain any element of type \( h(d) \cdot h(gd) \). This implies that \( V = SU(2) \cdot d \otimes V_0 \) contains \( S^2(V_0) \). We obtained that \( V = S^2 V_0 \).

Clearly, \( d^2 \) is \( SU(2) \)-invariant if and only if \( V \) is 1-dimensional. Thus, \( d^2 \) is \( SU(2) \)-invariant if and only if \( V_0 \) is 1-dimensional, that is, if \( d \) is \( SU(2) \)-invariant. On the other hand, the map \( d \) is an orthogonal projection to \( \mathbb{C} \cdot [\Delta_n] \subset H^2(M[n]) \). Thus, \( d \) is \( SU(2) \)-invariant if and only if \( [\Delta_n] \in H^2(M[n]) \) is \( SU(2) \)-invariant. The class \( [\Delta_n] \in H^2(M[n]) \) is a fundamental class of a subvariety \( \Delta_n \subset M[n] \). By Theorem 2.8, \( [\Delta_n] \) is \( SU(2) \)-invariant if and only if \( \Delta_n \subset M[n] \) is trianalytic. The trianalytic subvarieties are hyperkahler, outside of singularities. Since \( \Delta_n \) is a divisor, it has odd complex dimension and cannot be hyperkahler. Thus, the class \( [\Delta_n] \) is not \( SU(2) \)-invariant, and the map \( d^2 : S^2 H^2(M[n]) \rightarrow \mathbb{C} \) is not \( SU(2) \)-invariant. This proves Claim 9.9. Proposition 9.7 is proven.

**9.5 Universal subvarieties of the Hilbert scheme and Bogomolov-Beauville form**

In this subsection, we show that the subvarieties \( X_{\alpha} \subset M[n] \), obtained as in Proposition 9.2, are not trianalytic. We prove this using the explicit calculation of the Bogomolov-Beauville form on \( M[i] \) (Proposition 9.3) and the following result.

**Claim 9.10:** Let \( \varphi : X \hookrightarrow Y \) be a morphism of compact hyperkahler manifolds. Consider the corresponding pullback map
\[
\varphi^* : H^2(Y) \rightarrow H^2(X).
\]
Let
\[
\Psi : S^2 H^2(Y) \rightarrow S^2 H^2(X)
\]
be the symmetric square of \( \varphi^* \), and \( B_Y \in S^2 H^2(Y) \) the vector corresponding to the Bogomolov-Beauville pairing. Then \( \Psi(B_Y) \) is \( SU(2) \)-invariant, with respect to the natural action of \( SU(2) \) on \( S^2 H^2(X) \).
Proof: It is well known that, for every morphism of hyperkahler varieties, the pullback map is compatible with the $SU(2)$-action in the cohomology. To see this, one may notice that the $SU(2)$-action is obtained from the Hodge-type grading associated with induced complex structures, and the pullback is compatible with the Hodge structure. Now, $B_Y$ is $SU(2)$-invariant, and therefore, $\Psi(B_Y)$ is also $SU(2)$-invariant. 

Let $M$ be a K3 surface, $M^{[n]}$ its Hilbert scheme and $X_\alpha$ be a universal subvariety of $M^{[n]}$ of relative dimension 0, obtained from the Young diagram $(n_1, ..., n_l)$ as in [Proposition 9.2]. Assume that $X_\alpha$ is trianalytic with respect to some hyperkahler structure on $M^{[n]}$. The manifold $X_\alpha$ is birational to $M[l]$ [Proposition 9.2]. By [Theorem 9.1], there is a natural isomorphism $H^2(X_\alpha) \cong H^2(M[l])$, and this isomorphism is compatible with the Bogomolov-Beauville form. Consider the map $\varphi^*: H^2(M^{[n]}) \to H^2(X_\alpha) = H^2(M[l])$. Recall that $H^2(M)$ is considered as a subspace of $H^2(M[l])$, for all $i$ [Proposition 9.3 (i)] Clearly, $\varphi^*$ acts as identity on the subspaces $H^2(M) \subset H^2(M^{[n]})$. $H^2(M) \subset H^2(M[l])$. Consider the pullback $\varphi^*([\Delta_n]) \in H^2(X_\alpha) = H^2(M[l])$ of $[\Delta_n] \in H^2(M[l])$.

Lemma 9.11: In the above notations, $\varphi^*([\Delta_n]) = \frac{\eta}{l} [\Delta]$. 

Proof: Let $t \in H^2(M) \subset H^2(M[l])$ be the component of $\varphi^*([\Delta_n])$ corresponding to the decomposition (9.1). Since the decomposition (9.1) is integer, $t$ is an integer cohomology class. Let $M$ be a universal K3 surface, considered as a fibration over the moduli space $D$ of marked K3 surfaces. The construction of the Hilbert scheme can be applied to the fibers of $M$. We obtain a universal Hilbert scheme $M^{[n]}$, which is a fibration over $D$. Since $X_\alpha$ is a universal subvariety of $M^{[n]}$, there is a corresponding fibration over $D$ as well. Consider the class $t \in H^2(M)$ as a function $t(I)$ of the complex structure $I$ on $M$. Since the cohomology class $t(I)$ is integer, and $D$ is connected, $t(I)$ is independent from $I \in D$. On the other hand, $t(I)$ has type $(1, 1)$ with respect to $I$. There are no non-zero cohomology classes $\eta \in H^2(M)$ which have type $(1, 1)$ with respect to all complex structures on $M$. Thus, $t = 0$. We obtain that $\varphi^*([\Delta_n]) = r[\Delta]$, where $r$ is some integer number. It remains to check that $r = \frac{\eta}{l}$. Recall that $\frac{\eta}{l}$ is an integer number which is equal to $\frac{k(k-1)}{2}$, for some $k \in \mathbb{Z}$, $k > 1$.

The points of the Hilbert scheme $M[l]$ correspond to ideals $I \in \mathcal{O}_M$, $\dim \mathcal{O}_M/I = i$. Consider a rational map $\xi: M[l] \to M^{[n]}$ mapping an ideal $I \subset \mathcal{O}_M$ to $I^k$. Let $S \subset M[l]$ be the union of all strata $\Delta_{[\alpha]}$ of
codimension more than 1. It is easy to check that \( \xi \) is well defined outside of \( S \): for all ideals \( I \in M^{[l]} \setminus S \), the ideal \( I^k \) satisfies \( \dim \mathcal{O}_M / I^k = n \).

Consider the pullback map on the cohomology associated with the morphism \( \psi : M^{[l]} \setminus S \to M^{[n]} \). Clearly, \( H^2(M^{[l]} \setminus S) = H^2(M^{[l]}) \). The map \( \varphi^* : H^2(M^{[n]}) \to H^2(X_\alpha) = H^2(M^{[l]}) \) is equal to \( \xi^* : H^2(M^{[n]}) \to H^2(M^{[l]} \setminus S) = H^2(M^{[l]}) \)

Let \( p \in \pi_2(M^{[l]} \setminus S) \) be an element of the second homotopy group corresponding to \([\Delta_1]\) under Gurevich isomorphism. It remains to show that \( \xi(p) \) is equal to \( \frac{n}{l} \) times the element of the second homotopy group \( \pi_2(M^{[n]}) \) corresponding to \([\Delta_n]\). The following observation is needed to understand the geometry of \( \Delta_i \).

Closed Artinian subschemes \( \xi \subset M \) of length 2 with support in \( x \in M \) are in one to one correspondence with the vectors of projectivization of \( T_x M \).

(9.2)

Therefore, generic points of \( \Delta_i \) correspond to the triples \((\mathcal{X}, x, \lambda)\), where \( \mathcal{X} \) is a non-ordered set of \((i - 1)\) distinct points of \( M \), \( x \in M \setminus \mathcal{X} \) a point of \( M \) and \( \lambda \in \mathbb{P}T_x M \) a line in \( T_x M \). Fix \( \mathcal{X}, x \) and an isomorphism \( \mathbb{P}T_x M \cong \mathbb{C}P^1 \). We pick a map \( p_x : S^2 \to M^{[l]} \) in such a way that \( p_x(\theta) = (\mathcal{X}, x, \theta) \). Clearly, the corresponding element of \( \pi_2(M^{[l]}) \) is mapped to \([\Delta_i]\) by Gurevich’s isomorphism.

To simplify notations, we assume that \( l = 2 \). It is easy to do the case of general \( l \) in the same spirit as we do \( l = 2 \).

Let \( U \) be a neighbourhood of \( x \) in \( M \). Taking \( U \) sufficiently small, we may assume that \( U \) is equipped with coordinates. Let \( R_\lambda \) be a parallel translation of \( U \) along these coordinates in the direction of \( \lambda \), for \( \lambda \in \mathbb{C}^2 \). The map \( R_\lambda \) is defined in a smaller neighbourhood \( U' \) of \( x \): \( R_\lambda : U_1 \to U \).

The coordinates give a natural identification \( \mathbb{C}P^1 \cong \mathbb{P}T_y M \), for all \( y \in U \). Let \( \lambda \in T_x M \setminus 0 \) be a vector corresponding to \( \theta \in \mathbb{P}T_x M \). Clearly, then,

\[
p_x(\theta) = \lim_{t \to 0, t \in \mathbb{R} \setminus 0} \left\{ x, R_{t\lambda}(x) \right\},
\]

where the pair \( \{ x, R_{t\lambda}(x) \} \) is considered as a point in \( M^{[2]} \).

Let \( y_i \in U_{1/2}^{[2]} \setminus \Delta_{\overline{2}} \) be a sequence of points converging to \( \varphi'(x) \), where \( \varphi' : M \to M_{\overline{2}} \) maps a maximal ideal of \( x \in M \) to its \( k \)-th power. Denote the support of \( y_i \) by \( S_{y_i} \). Clearly,

\[
\varphi(p_x(\theta)) = \lim_{t \to 0, t \in \mathbb{R} \setminus 0} \left( \lim_{i \to \infty} \{ y_i, R_{t\lambda}(y_i) \} \right),
\]

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Taking limits in different order, we obtain

\[ \varphi(p_x(\theta)) = \lim_{i \to \infty} \left( \lim_{t \to 0, t \in \mathbb{R} \setminus \{0\}} \{ y_i, R_{\lambda}(y_i) \} = \lim_{i \to \infty} y_i(\theta) \right), \]

where \( y_i(\theta) \in \Delta_n \) is a point corresponding to a closed subscheme in \( M^{[n]} \) with support \( S_{y_i} \), of length 2 at every point of its support. For \( y \in S_{y_i} \), consider the restriction \( y_i(\theta)|_y \) of \( y_i(\theta) \) to \( y \), which is a closed Artinian subscheme of length 2 with support in \( y \). Clearly, \( y_i(\theta)|_y \) corresponds to \( \theta \in \mathbb{P}T_yM \) as in (9.2). Varying \( \theta \), we obtain a map \( p_i : \mathbb{C}P^1 \to \Delta_n \setminus S, \theta \mapsto y_i(\theta) \). By construction, this map is homotopic to \( \varphi(p_x) \). On the other hand, it is clear that \( p_y \) is homotopic to \( \text{Card}(S_{y_i}) = \frac{n}{\ell} \) times a homotopy class represented by a map \( p_x : \mathbb{C}P^1 \to \Delta_n \setminus S \). This proves [Lemma 9.1].

The Beauville-Bogomolov form identifies the second cohomology with its dual. Thus, this form can be considered as a tensor in the symmetric square of the second cohomology. To show that the embedding \( X_\alpha \hookrightarrow M^{[n]} \) is not trianalytic, we compute the pullback \( \varphi^*B_{M^{[n]}} \) of the Beauville-Bogomolov tensor \( B_{M^{[n]}} \in S^2H^2(M^{[n]}) \). Consider the decomposition (9.1)

\[ H^2(M^{[i]}) = H^2(M) \oplus \mathbb{C} \cdot \Delta_i, \quad (9.3) \]

Let \( P \in S^2H^2(M) \) be the tensor corresponding to the Poincare pairing. Proposition 9.3 computes the form \( (\cdot, \cdot)_B \in S^2H^2(M)^* \) in terms of Poincare form and the decomposition (9.3): \( (\cdot, \cdot)_B = P - 2(n-1)d^2 \). Therefore, the dual tensor can be written as \( B_{M^{[l]}} = P - \frac{1}{(n-1)\ell} [\Delta_i]^2 \). We have shown that \( \varphi^* \) acts as an identity on the first summand of (9.3), and maps \( \Delta_i \) to \( \frac{n}{\ell}[\Delta_i] \).

Therefore,

\[ \varphi^*B_{M^{[n]}} = P - \frac{1}{2(n-1)\ell}[\Delta_i]^2 = B_{M^{[l]}} + \left( \frac{1}{2(l-1)} - \frac{1}{2(n-1)\ell} \right) \frac{n}{\ell}[\Delta_i]^2 \quad (9.4) \]

By [Claim 9.5], \( [\Delta_i]^2 \) is not \( SU(2) \)-invariant. Since \( B_{M^{[l]}} \) is \( SU(2) \)-invariant, \( \varphi^*B_{M^{[n]}} \) is \( SU(2) \)-invariant if and only if the coefficient of \( [\Delta_i]^2 \) in (9.4) vanishes:

\[ (l-1)^{-1} - (n-1)^{-1} \frac{n}{\ell} = 0 \quad (9.5) \]

Clearly, this happens only if \((n-\frac{n}{\ell}) = n-1\), i.e. when \( \frac{n}{\ell} = 1 \). By definition, \( \frac{n}{\ell} = \frac{k(k+1)}{2} \geq 3 \). Therefore, \( \varphi^*B_{M^{[n]}} \) is not \( SU(2) \)-invariant, and \( X_\alpha \) is not
trianalytic in $M^{[n]}$.\]

Comparing this with Claim 9.10, and Proposition 9.2, we obtain the following result.

**Theorem 9.12:** Let $M$ be a complex K3 surface without automorphisms. Assume that $M$ is Mumford-Tate generic with respect to some hyperkähler structure. Consider the Hilbert scheme $M^{[n]}$ of points on $M$. Pick a hyperkähler structure on $M^{[n]}$ which is compatible with the complex structure. Then $M^{[n]}$ has no proper trianalytic subvarieties.

\[\square\]

**Remark 9.13:** It is easy to see that a generic K3 surface has no complex automorphisms.

**Corollary 9.14:** Let $M$ be a complex K3 surface. Consider its Hilbert scheme $M^{[n]}$. Let $M$ be the generic deformation of $M^{[n]}$. Then $M$ has no complex subvarieties.

**Proof:** Follows immediately from Theorem 9.12 and Corollary 2.12.\[\square\]

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3 We do not need the whole strength of Lemma 9.11 to show that the tensor $\varphi^*B_{M^{[n]}}$ is not $SU(2)$-invariant. Let $t$ be the coefficient of Lemma 9.11, $\varphi^*(|\Delta_n|) = t|\Delta_l|$. To show that $\varphi^*B_{M^{[n]}}$ is not $SU(2)$-invariant, we need only to check that $n - 1 \neq t(l - 1)$, where $n = l(k+1)$. Since $l(k+1) - 1$ is not divisible by $l - 1$ for most $l, k$, the inequality $n - 1 \neq t(l - 1)$ holds automatically for most $l, n.$
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