A CONICAL APPROACH TO LAURENT EXPANSIONS FOR MULTIVARIATE MEROMORPHIC GERMS WITH LINEAR POLES

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ABSTRACT. We use convex polyhedral cones to study a large class of multivariate meromorphic germs, namely those with linear poles, which naturally arise in various contexts in mathematics and physics. We express such a germ as a sum of a holomorphic germ and a linear combination of special non-holomorphic germs called polar germs. In analyzing the supporting cones – cones that reflect the pole structure of the polar germs – we obtain a geometric criterion for the non-holomorphicity of linear combinations of polar germs. This yields the uniqueness of the above sum when required to be supported on a suitable family of cones and assigns a Laurent expansion to the germ. Laurent expansions provide various decompositions of such germs and thereby a uniformized proof of known results on decompositions of rational fractions. These Laurent expansions also yield new concepts on the space of such germs, all of which are independent of the choice of the specific Laurent expansion. These include a generalization of Jeffrey-Kirwan’s residue, a filtered residue and a coproduct in the space of such germs. When applied to exponential sums on rational convex polyhedral cones, the filtered residue yields back exponential integrals.

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1. Introduction

Our study aims at an extension of the classical Laurent theory in one variable to multivariate meromorphic germs.

The classical Laurent theory assigns to a meromorphic germ at zero in one variable a unique Laurent expansion. So the space $\mathcal{M}_C(\mathbb{C})$ of meromorphic germs at zero consists of the polar part of meromorphic germs:

$$\mathcal{M}_C(\mathbb{C}) = \mathbb{C}[[\varepsilon^{-1}]] \oplus \mathcal{M}_{C,+,\mathbb{C}}(\mathbb{C}),$$

the direct sum on the right hand side hosting Laurent expansions. We note that the direct sum decomposition comes as a consequence of the Laurent expansions.

For multivariate meromorphic germs, one naturally asks the same questions, namely how to

(a) define a “polar part” as a linear complement of the space of holomorphic germs.
(b) obtain a canonical basis or building blocks of this linear complement from which to assign Laurent expansions to meromorphic germs.

While these problems remain open in general, in this paper we provide an answer for another generalization of the Laurent series to multivariate meromorphic germs, called multivariate meromorphic germs at zero with linear poles, i.e., whose poles lie on unions of hyperplanes. Such meromorphic germs arise in various areas of mathematics and in physics as “regularized” integrals or sums with linear or conical constraints in the variables, and more specifically in

- perturbative quantum field theory when computing Feynman integrals by means of analytic regularization à la Speer [24, 25] (see also the recent work by Dang [8]), where the linear constraints in the integration variables correspond to conservation of momentum;
- number theory with multiple zeta functions [16, 28] (see also [15, 22, 23, 29]) and their generalizations such as cyclotomic [27] and Witten multiple zeta functions [20] - the conical constraint $n_1 > \cdots > n_k$ on the variables in the summand used to define multiple zeta functions, is responsible for the linearity of the poles;
- the combinatorics on cones when evaluating exponential integrals or sums on cones following Berline and Vergne [2] (see also [10]) in the context of Euler-Maclaurin formula;
- algebraic geometry, in particular with the celebrated Jeffrey-Kirwan residue [17, 18], see also [6] for a review.

In our approach of multivariate meromorphic germs with linear poles, the classical Laurent theory generalizes to a local Laurent theory in the sense that one can cover the space of meromorphic germs at zero with linear poles by what we call Laurent subspaces. To define these Laurent subspaces, we use the geometry of cones. Laurent subspaces are indexed by properly positioned families of simplicial cones, a family of cones being properly positioned when the cones meet along faces and their union does not contain any nontrivial subspace. So a meromorphic germ in a Laurent subspace has a Laurent expansion supported by the corresponding properly positioned family of simplicial cones, called supporting cones. In the framework of certain generalized subdivisions we call pan-subdivisions, one can build a direct system of properly positioned families of simplicial cones. Thus the corresponding set of Laurent subspaces inherits a direct system structure whose direct limit is the whole space of meromorphic germs at zero with linear poles.

We explicit below our geometric approach and at the same time give an outline of the paper.

To distinguish the polar part from the holomorphic part, we fix an inner product and use the orthogonal complement to define the key concept of polar germs, which serve as building blocks for...
the polar part. A polar germ is a non-holomorphic germ represented by a fraction \( \frac{h(L_1, \ldots, L_m)}{L_1^{s_1} \cdots L_m^{s_n}} \), with \( h \) is a holomorphic germ at zero in variables \( \ell_1, \cdots, \ell_m \) orthogonal to the linear forms \( L_1, \cdots, L_m \) in the pole with multiplicities \( s_1, \cdots, s_n \) (Definition 2.3). We then decompose a meromorphic germ as a sum of polar germs and a holomorphic germ (Theorem 2.10) by operations on fractions.

However such a decomposition is not unique, as the equation \( \frac{1}{L_1 L_2} = \frac{1}{L_1 (L_1 + L_2)} + \frac{1}{L_2 (L_1 + L_2)} \) indicates. We use the geometry underlying meromorphic germs to address the uniqueness of the decompositions. To a polar germ \( \frac{h(L_1, \ldots, L_m)}{L_1^{s_1} \cdots L_m^{s_n}} \), we assign a supporting cone (Definition 3.1) \( \langle L_1, \cdots, L_m \rangle \) generated by the vectors \( L_1, \cdots, L_m \).

The notion of supporting cones provides a geometric criterion for the non-holomorphicity, in particular the linear independence, of a sum of polar germs in Theorem 3.6. More precisely, if the family of supporting cones of a linear combination of polar germs is “properly positioned” (Definition 3.2), then this linear combination cannot be holomorphic. Properly positioned supporting cones are essential to assign Laurent expansions to meromorphic germs (Definition 4.5) with the help of a surjective “forgetful map” (16) from the space of formal Laurent expansions to the space of meromorphic germs. We first identify the Laurent subspaces in Proposition 4.3 and build from there the Laurent expansion supported on an appropriate properly positioned family of cones. With the notations of the previous example, \( \frac{1}{L_1 L_2} \) is the Laurent expansion supported on the properly positioned family \( \{ \langle L_1, L_2 \rangle \} \) while \( \frac{1}{L_1 (L_1 + L_2)} + \frac{1}{L_2 (L_1 + L_2)} \) is the Laurent expansion supported on the properly positioned family \( \{ \langle L_1, L_1 + L_2 \rangle, \langle L_2, L_1 + L_2 \rangle \} \).

As with the one variable case, an immediate consequence of the Laurent expansions is a splitting of the space of meromorphic germs with linear poles into a direct sum of the space of holomorphic ones and the space spanned by the polar germs (Corollary 4.18). This direct sum defines in turn a projection onto the holomorphic part along the polar part, a “multivariate subtraction”, which is multiplicative on orthogonally variate germs (Corollary 4.20), generalizes the minimal subtraction projection operator for meromorphic germs in one variable. This projection is a key ingredient for the algebraic Birkhoff factorization [13].

On the grounds of the Laurent expansions and homogeneity properties of the kernel of the forgetful map (Theorem 4.22), we equip the space of meromorphic germs with multiple gradings, given by total orders of the poles – called the p-order – of the polar germs arising in the expansion, the spaces spanned by the supporting cones, as well as the dimensions of the supporting cones (Theorem 5.3). These gradings yield several further applications by means of a uniformized approach. More precisely,

(a) we generalize the Brion-Vergne decomposition [4] \( R_\Delta = G_\Delta \oplus NG_\Delta \) of rational germs at zero with poles lying in unions of hyperplanes in a given hyperplane arrangement \( \Delta \). See Corollary 5.5.

(b) we obtain the decomposition [3, Theorem 7.3] of Berline and Vergne as a consequence of a more refined decomposition in Theorem 5.3. In contrast to their approach which applies to a meromorphic germ with a prescribed set of poles determined by a given hyperplane arrangement \( \Delta \), here we consider the whole class of meromorphic germs at zero with linear poles.

(c) through a projection to a suitable components from one of the gradings, we define a generalized Jeffrey-Kirwan residue [4, 17] valid for all meromorphic germs at zero with linear poles in stead of for \( R_\Delta \). See Corollary 5.7 and Definition 5.8.

The Laurent expansions have further interesting consequences leading to new results. The “p-order” generalized to meromorphic germs at zero gives a filtration of the meromorphic germs
which generalizes the filtration by the order of the poles on meromorphic germs at zero in one variable and defines a valuation on the division ring of Laurent series [9, Example 4.2.2]. We further introduce a filtered residue, the “p-residue” (Definition 6.1) which, for Laurent expansions \( \sum_{n=-N}^{\infty} a_n x^n \) in one variable filtered by the valuation given by the order \( N \geq 0 \) of the poles, corresponds to \( \frac{d}{dx} \). Composed with the exponential sum \( S \) [1] on a lattice cone \( p\text{-res} \circ S \), the p-residue turns out to be compatible with subdivisions (Proposition 6.8), as a result of which (see Corollary 6.9), the p-residue yields back the corresponding exponential integral on the lattice cone, related to the former by the Euler-Maclaurin formula studied in [13].

Finally, using Laurent expansions, we define in Section 6.3 a coproduct on the space of meromorphic germs at zero with linear poles, which is closely related to the coproduct on cones derived in [14]. This relation is most relevant in the context of renormalization à la Connes and Kreimer [5] who regarded a renormalized map as a map defined on a coalgebra and taking values in meromorphic functions.

Throughout this paper, \( \mathbb{F} \) denotes a fixed subfield of \( \mathbb{R} \).

2. A decomposition of meromorphic germs

In order to show the existence of a decomposition of the space of meromorphic germs, we first introduce the concept of a polar germ which will later serve as the building blocks of the linear complement of holomorphic germs.

2.1. Polar germs. We begin with some necessary preliminary concepts.

**Definition 2.1.**  
(a) A **lattice (vector) space** is a pair \((V, \Lambda_V)\) where \(V\) is a finite dimensional real vector space and \(\Lambda_V\) is a lattice in \(V\), that is, a finitely generated abelian subgroup of \(V\) whose \(\mathbb{R}\)-linear span is \(V\);
(b) An **\(\mathbb{F}\)-inner product** on a lattice space \((V, \Lambda_V)\) is an inner product \(Q\) on \(V\) such that the restriction of \(Q\) to \(\Lambda_V \otimes \mathbb{F} \subseteq \Lambda_V \otimes \mathbb{R} = V\) and hence to \(\Lambda_V\) takes values in \(\mathbb{F}\);
(c) A lattice space with an \(\mathbb{F}\)-inner product is called an **\(\mathbb{F}\)-Euclidean lattice space**;
(d) A **filtered space** is a real vector space \(V\) with a filtration \(V_1 \subseteq V_2 \subseteq \cdots\) of finitely dimensional real vector subspaces such that \(V = \bigcup_{k \geq 1} V_k\). Let \(j_k : V_k \to V_{k+1}\) denote the inclusion;
(e) A **filtered lattice space** is a filtered space \(V = \bigcup_{k \geq 1} V_k\) with lattices \(\Lambda_k := \Lambda_{V_k}\) of \(V_k\) such that \(\Lambda_{k+1}v_k = \Lambda_k, k \geq 1\). Then we denote the filtered lattice space by \((V, \Lambda_V) = \bigcup_{k \geq 1}(V_k, \Lambda_{V_k})\) where \(\Lambda_V = \bigcup_{k \geq 1} \Lambda_{V_k}\);
(f) An **inner product** \(Q\) on a filtered space \(V = \bigcup_{k \geq 1} V_k\) is a sequence of inner products \(Q_k(\cdot, \cdot) = (\cdot, \cdot)_k : V_k \otimes V_k \to \mathbb{R}, k \geq 1\), that is compatible with the inclusions \(j_k, k \geq 1\);
(g) An **\(\mathbb{F}\)-inner product** on a filtered lattice space \((V, \Lambda_V)\) is an inner product \(Q := \{Q_k\}_{k \geq 1}\) on the filtered space \(V = \bigcup_{k \geq 1} V_k\) such that \(Q_k\) is an \(\mathbb{F}\)-inner product for each \(k \geq 1\). A filtered lattice space together with an \(\mathbb{F}\)-inner product is called a **filtered \(\mathbb{F}\)-Euclidean lattice space**.

We now assume that \((V, \Lambda_V) = \bigcup_{k \geq 1}(V_k, \Lambda_k)\) is a filtered \(\mathbb{F}\)-Euclidean lattice space. Let \(V^*_k := \text{Hom}(V_k, \mathbb{R})\) be the dual space of \(V_k\). The \(\mathbb{F}\)-inner product \(Q_k : V_k \otimes V_k \to \mathbb{R}\) induces an isomorphism \(Q^*_k : V_k \to V_k^*\). This yields an embedding \(V_k^* \hookrightarrow V_{k+1}^*\) induced from \(j_k : V_k \to V_{k+1}\). The
direct limit

\[ V^\oplus := \lim_{\to} V^*_k = \bigcup_{k=0}^{\infty} V^*_k \]

is called the **filtered dual space** of \( V = \bigcup_{k=0}^{\infty} V_k \). Notice that \( V^\oplus \) is a proper subspace of the usual dual space \( V^\ast \) unless \( V \) is finite dimensional.

**Definition 2.2.** Let \( \bigcup_{k \geq 1} (V_k, \Lambda_k) \) be a filtered lattice space.

(a) A meromorphic germ \( f(\xi) \) on \( V_k^\ast \otimes \mathbb{C} \) is said to have **linear poles at zero with coefficients in** \( \mathbb{F} \) if there exist vectors \( L_1, \cdots, L_n \in \Lambda_V \otimes \mathbb{F} \) (possibly with repetitions) such that \( f \Pi_{i=1}^n L_i \) is a holomorphic germ at zero whose Taylor expansion for coordinates in the dual basis \( \{ e_1^\ast, \cdots, e_k^\ast \} \) of a given (and hence every) basis \( \{ e_1, \cdots, e_k \} \) of \( \Lambda_k \) has coefficients in \( \mathbb{F} \).

(b) Let \( M_{\mathbb{F}}(V_k^\ast \otimes \mathbb{C}) \) denote the set of germs of meromorphic functions on \( V_k^\ast \otimes \mathbb{C} \) with linear poles at zero and with coefficients in \( \mathbb{F} \). It is a linear space over \( \mathbb{F} \).

(c) A germ of meromorphic functions of the form \( \frac{1}{L_1^{s_1} \cdots L_n^{s_n}} \) with linearly independent vectors \( L_1, \cdots, L_n \) in \( \Lambda_k \otimes \mathbb{F} \) and \( s_1, \cdots, s_n \geq 1 \) is called a **simplicial fraction** with coefficients in \( \mathbb{F} \) or simplicial \( \mathbb{F} \)-fraction. Such a fraction is called **simple** if \( s_1 = \cdots = s_n = 1 \).

Since a set of vectors in \( \Lambda_k \otimes \mathbb{F} \) is \( \mathbb{F} \)-linearly independent if and only if it is \( \mathbb{R} \)-linearly independent in \( V_k \), from now on we just call it linearly independent without specifying the type of coefficients.

Composing with the map \( j'_k : V_{k+1}^\ast \to V_k^\ast \) dual to \( j_k : V_k \to V_{k+1} \), yields the embedding

\[ M_{\mathbb{F}}(V_k^\ast \otimes \mathbb{C}) \hookrightarrow M_{\mathbb{F}}(V_{k+1}^\ast \otimes \mathbb{C}), \]

giving rise to the direct limit

\[ M_{\mathbb{F}}(V^\ast \otimes \mathbb{C}) := \lim_{\to} M_{\mathbb{F}}(V^*_k \otimes \mathbb{C}) = \bigcup_{k=1}^{\infty} M_{\mathbb{F}}(V^*_k \otimes \mathbb{C}). \]

Let \( M_{\mathbb{F},+}(V_k^\ast \otimes \mathbb{C}) \) denote the space of germs of holomorphic functions at zero in \( V_k^\ast \otimes \mathbb{C} \) whose Taylor expansions at zero under the dual basis of a basis of \( \Lambda_k \) have coefficients in \( \mathbb{F} \). We set

\[ M_{\mathbb{F},+}(V^\ast \otimes \mathbb{C}) := \bigcup_{k=1}^{\infty} M_{\mathbb{F},+}(V^*_k \otimes \mathbb{C}). \]

When \( \mathbb{F} = \mathbb{R} \), we usually drop the subscript \( \mathbb{F} \) from the notation.

When \( V_k \) is taken to be \( \mathbb{R}^k \) and is equipped with its standard lattice \( \mathbb{Z}^k \), the dual space \( V_k^\ast \) is identified with \( \mathbb{R}^k \) equipped with the standard lattice. Then the space \( M_{\mathbb{R},+}(\mathbb{C}^k) = M_{\mathbb{R},+}(V_k^\ast \otimes \mathbb{C}) \) corresponds to the space of germs of holomorphic functions at zero in \( \mathbb{C}^k \) whose Taylor expansions at zero have coefficients in \( \mathbb{R} \) with respect to the canonical basis of \( \mathbb{R}^k \).

We next identify a linear complement of \( M_{\mathbb{R},+}(V_k^\ast \otimes \mathbb{C}) \) which is canonical upon fixing an inner product on \( V_k \). It is spanned by a class of germs which then can be regarded as purely non-holomorphic germs. More importantly, they will also serve as the building blocks for our Laurent expansions of meromorphic germs in multiple variables with linear poles. See Section 4. For notational simplicity, we will call them polar germs.

**Definition 2.3.** Let \( (V, \Lambda_V) = \bigcup_{k} (V_k, \Lambda_k) \) be a filtered \( \mathbb{F} \)-Euclidean lattice space with its \( \mathbb{F} \)-inner product \( Q \). A **polar germ with \( \mathbb{F} \)-coefficients** or simply a **\( \mathbb{F} \)-polar germ** in \( V_k^\ast \otimes \mathbb{C} \) is a germ of...
meromorphic functions at zero of the form
\[
\frac{h(\ell_1, \ldots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}},
\]
where
(a) \( h \) lies in \( M_{\mathbb{F}^+}(\mathbb{C}^m) \),
(b) \( \ell_1, \ldots, \ell_m, L_1, \ldots, L_n \) lie in \( \Lambda \otimes \mathbb{F} \), with \( \ell_1, \ldots, \ell_m \) and \( L_1, \ldots, L_n \) linearly independent, such that
\[
Q(\ell_i, L_j) = 0 \quad \text{for all } (i, j) \in [m] \times [n],
\]
where for a positive integer \( k \), we have set \( [k] := \{1, \ldots, k\} \).
(c) \( s_1, \ldots, s_n \) are positive integers.

For notational simplicity, we shall also set \( \tilde{L}^s := L_1^{s_1} \cdots L_n^{s_n} \) and write
\[
\frac{h(\tilde{\ell})}{\tilde{L}^s} := \frac{h(\ell_1, \ldots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}}.
\]

**Definition 2.4.** We let \( M_{\mathbb{F}^-}^Q(V_k \otimes \mathbb{C}) \) denote the linear subspace of \( M_{\mathbb{F}}(V_k \otimes \mathbb{C}) \) spanned by \( \mathbb{F} \)-polar germs and set
\[
M_{\mathbb{F}^-}^Q(V \otimes \mathbb{C}) := \bigcup_{k=1}^{\infty} M_{\mathbb{F}^-}^Q(V_k \otimes \mathbb{C}) = \lim_{\longrightarrow} M_{\mathbb{F}^-}^Q(V_k \otimes \mathbb{C}),
\]
regarding \( \{M_{\mathbb{F}^-}^Q(V_k \otimes \mathbb{C})\}_k \) as a sub-direct system of \( \{M_{\mathbb{F}}(V_k \otimes \mathbb{C})\}_k \).

**Remark 2.5.** The space \( M_{\mathbb{F}^-}(V \otimes \mathbb{C}) \) is not closed under the function product. As a simple example, for the canonical inner product on \( \mathbb{R}^2 \), both \( f(\varepsilon_1, \varepsilon_2) := \varepsilon_1/\varepsilon_2 \) and \( g(\varepsilon_1, \varepsilon_2) := \varepsilon_2/\varepsilon_1 \) are polar germs. But their product 1 is not.

**Example 2.6.**
(a) For linearly independent vectors \( L_1, \ldots, L_k \in \Lambda \otimes \mathbb{F} \) and \( s_1, \ldots, s_k > 0 \),
\[
\frac{1}{L_1^{s_1} \cdots L_k^{s_k}}
\]
lies in \( M_{\mathbb{F}^-}^Q(V_k \otimes \mathbb{C}) \) for any inner product \( Q \).
(b) For the canonical Euclidean inner product on \( \mathbb{R}^2 \), the functions \( f(\varepsilon_1 e_1^s + \varepsilon_2 e_2^s) = \frac{(\varepsilon_1 e_1^s + \varepsilon_2 e_2^s)}{(\varepsilon_1 + \varepsilon_2)^s} \), \( s > 0, t \geq 0 \), lie in \( M_{\mathbb{F}^-}^Q(\mathbb{R}^2) \).

**Remark 2.7.** We will mostly be working with a filtered \( \mathbb{F} \)-Euclidean lattice space given by a fixed \( \mathbb{F} \)-inner product \( Q \). Thus we often drop the superscript \( Q \) to simplify notations.

The following lemma shows the uniqueness of the expression of a polar germ.

**Lemma 2.8.** If a polar germ can be written as \( \frac{h(\ell_1, \ldots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}} \) and \( \frac{g(\ell'_1, \ldots, \ell'_n)}{(L'_1)^{t_1} \cdots (L'_n)^{t_n}} \), both in a form satisfying the conditions in Definition 2.3, then \( n = \ell \) and \( L'_1, \cdots, L'_n \) can be rearranged in such a way that \( L_i \) is a multiple of \( L'_i \) and \( s_i = t_i \) for \( 1 \leq i \leq n \).

**Proof.** We implement an induction on \( M := \max(s_1 + \cdots + s_n, t_1 + \cdots + t_i) \).

To deal with the case when \( M = 1 \), assume \( \frac{h(\ell_1, \ldots, \ell_m)}{L} = \frac{g(\ell'_1, \ldots, \ell'_n)}{L} \). Since \( h \) and \( g \) holomorphic. Extend \( \{L, \ell_1, \cdots, \ell_m\} \) to a basis \( \{z_1, \cdots, z_{m+1}, \cdots, z_k\} \), with \( z_1 = L, z_2 = \ell_1, \cdots, z_{m+1} = \ell_m \). If \( \frac{h(\ell_1, \ldots, \ell_m)}{L} = \frac{g(\ell'_1, \ldots, \ell'_n)}{L} \), \( L' \) is not a multiple of \( L \), then
\[
L'(z_1, z_2, \cdots, z_k) = L''(z_2, \cdots, z_k) + cz_1,
\]
with \( L''(z_2, \cdots, z_k) \) not identically zero, so we can pick \( z^0_2, \cdots, z^0_k \), such that \( h(z^0_2, \cdots, z^0_{m+1}) \neq 0 \) and \( L''(z^0_2, \cdots, z^0_k) \neq 0 \). Consider the restriction to \((z_1, z^0_2, \cdots, z^0_k)\) of the equality \( \frac{h(\ell_1, \cdots, \ell_m)}{L_1} = \frac{g(\ell'_1, \cdots, \ell'_{m-1})}{L'_{m-1}} \). The left hand side of the equality is singular in \( z_1 \) while the right hand side is holomorphic in \( z_1 \). This is a contradiction, showing that \( L \) must be a multiple of \( L' \). The same argument shows that \( \frac{h(\ell_1, \cdots, \ell_m)}{L_1} = \frac{g(\ell'_1, \cdots, \ell'_{m-1})}{L'_{m-1}} \) is impossible.

For the inductive step, suppose that none of the linear forms \( L'_1, \cdots, L'_{r-1} \) is a multiple of \( L_1 \). As in the case for \( M = 1 \), when

\[
\frac{h(\ell_1, \cdots, \ell_m)}{L_1^{s_1} \cdots L_m^{s_n}} = \frac{g(\ell'_1, \cdots, \ell'_{m-1})L_2^{s_2} \cdots L_n^{s_n}}{(L'_1)^{r_1} \cdots (L'_{m-1})^{r_{m-1}}}
\]

is restricted to a proper choice of \( z^0_2, \cdots, z^0_k \), the left hand side of the equality has a non-trivial singular part in \( z_1 \), while the right hand side is holomorphic in \( z_1 \), which leads to a contradiction. Therefore we can rearrange \( L'_1, \cdots, L'_{r-1} \) so that \( L_1 = cL'_1 \) for some constant \( c \neq 0 \). Thus from

\[
\frac{h(\ell_1, \cdots, \ell_m)}{L_1^{s_1} \cdots L_m^{s_n}} = \frac{g(\ell'_1, \cdots, \ell'_{m-1})}{(L'_1)^{r_1} \cdots (L'_{m-1})^{r_{m-1}}},
\]

we obtain

\[
\frac{h(\ell_1, \cdots, \ell_m)}{L_1^{s_1-1} \cdots L_m^{s_n}} = \frac{cg(\ell'_1, \cdots, \ell'_{m-1})}{(L'_1)^{r_1-1} \cdots (L'_{m-1})^{r_{m-1}}},
\]

By the inductive hypothesis, the conclusion holds for the two sides of the equation. This completes the induction. \( \square \)

### 2.2. Decomposition of a meromorphic germ into polar germs.

In this subsection we consider any lattice space \((V, \Lambda)\) which can be taken to be \((V_k, \Lambda_k)\) from a filtered lattice space. The notions for \( V_k \) such as \( M\mathbb{F}(V_k^* \otimes \mathbb{C}) \) and \( M_{\leq 0}(V_k^* \otimes \mathbb{C}) \) can be defined in the same way for \( V \).

Before giving the decomposition, we first provide some preliminary results.

**Lemma 2.9.** Let \((V, \Lambda)\) be a lattice space. Let \( L_1, \cdots, L_n, n \geq 2 \), be vectors in \( \Lambda \otimes \mathbb{F} \) and let \( s_1, \cdots, s_n \) be positive integers.

(a) If \( L_1, \cdots, L_n \) are \( \mathbb{F} \)-linearly independent and \( L_{n+1} = \sum_{i=1}^n c_i L_i \) with nonzero \( c_i \in \mathbb{F}, 1 \leq i \leq n \). Then

\[
\frac{1}{L_1^{s_1} \cdots L_{n+1}^{s_{n+1}}} = \sum_{j} \frac{b_j}{N_j^{s_1} \cdots N_j^{s_{n+1}}},
\]

where, for each \( j \), \( b_j \) is in \( \mathbb{F} \) and \( \{N_{j1}, \cdots, N_{jn}\} \) is one of the sets \( \{L_1, \cdots, L_i, \cdots, L_{n+1}\}, 1 \leq i \leq n \) (where \( \tilde{L}_i \) means that the factor \( L_i \) is omitted) and hence is a basis of the linear span \( \text{lin}[L_1, \cdots, L_{n+1}] \) of \( L_1, \cdots, L_{n+1} \).

(b) In general, the fraction \( \frac{1}{L_1^{s_1} \cdots L_n^{s_n}} \) can be rewritten as a linear combination

\[
\sum_{i} \frac{a_i}{M_i^{s_1} \cdots M_i^{s_{n}}},
\]

with \( a_i \in \mathbb{F} \) and linear independent subsets \( \{M_{i1}, \cdots, M_{in}\} \) of \( \{L_1, \cdots, L_n\} \).

**Proof.** (a) The statement easily follows from the straightforward identity

\[
\frac{1}{L_1 \cdots L_r} = \sum_{i=1}^r \frac{c_i}{L_1 \cdots \tilde{L}_i \cdots L_r L_{r+1}},
\]
by induction on the sum \( m := \sum_{j=1}^{s} s_j \).

(b) Combining factors of linear forms that are multiples of each other if necessary, we can assume that the \( L_i \)'s are not multiples of each other. The statement then follows from an induction on the difference \( d := n - \dim(\text{lin}(L_1, \cdots, L_n)) \) using Eq. (5) applied to a subset \( L_{i_1}, \cdots, L_{i_r} \) of linearly independent forms such that \( L_{i_{r+1}} = \sum_{j=1}^{r} c_j L_{i_j} \) for some \( 2 \leq r \leq n \).

We are now ready to prove the existence of a decomposition of meromorphic germs at zero into a sum of holomorphic germs and polar germs.

**Theorem 2.10.** Let \((V, \Lambda)\) be an \(\mathbb{F}\)-Euclidean lattice space with an \(\mathbb{F}\)-inner product \(Q\). For any \(f \in M_\mathbb{F}(V^* \otimes \mathbb{C})\), there exists a finite set of \(\mathbb{F}\)-polar germs \(\{S_j\}_{j \in J}\) and a holomorphic germ \(h\) in \(M_\mathbb{F}(V^* \otimes \mathbb{C})\) such that

\[
f = \left( \sum_{j \in J} S_j \right) + h.
\]

Furthermore, the \(\mathbb{F}\)-polar germs \(S_j\) can be chosen to satisfy the following properties.

- their linear poles are taken from the linear poles of \(f\).
- if the germ \(f\) can be written in the form \(\tilde{f}(\ell_1, \cdots, \ell_n)\) for some function \(\tilde{f}\) on \(\mathbb{C}^n\) and linearly independent linear forms \(\ell_1, \cdots, \ell_n\) on \((\Lambda \otimes \mathbb{F})^\ast\), then the polar germs \(S_j\) and the holomorphic germ \(h\) can be written as compositions of functions on \(\mathbb{C}^n\) and linearly independent linear forms in \(\text{span}(\ell_1, \cdots, \ell_n)\).

**Remark 2.11.** Whereas the holomorphic part will turn out to be uniquely defined, the individual polar germs arising in this decomposition are not. In Corollary 3.7 we provide geometric conditions under which the polar germs can be uniquely determined, leading to Laurent expansions.

**Proof.** Thanks to Lemma 2.9.(b), without loss of generality we can reduce the proof to meromorphic germs at zero of the form

\[
f = \frac{h}{L_{s_1} \cdots L_{s_m}}
\]

with \(h \in M_{\mathbb{F}, +}(V^* \otimes \mathbb{C}), L_{s_1}, \cdots, L_{s_m} \in \Lambda \otimes \mathbb{F}\) linearly independent and \(s_1, \cdots, s_m\) positive integers. Then we extend \(\{L_1, \cdots, L_{m}\}\) to a basis \(\{L_1, \cdots, L_m, \ell_1, \cdots, \ell_{k-m}\}\) of \(\Lambda \otimes \mathbb{F}\) satisfying

\[Q(L_i, \ell_j) = 0, \quad 1 \leq i \leq m, 1 \leq j \leq k - m.\]

We proceed by induction on the sum \(s := s_1 + \cdots + s_m\). If \(s = 1\), then \(m = 1\) and \(s_1 = 1\). Under these conditions we have

\[
f = \frac{h(L_1, \ell_1, \cdots, \ell_{k-1})}{L_1} = h(0, \ell_1, \cdots, \ell_{k-1}) + \frac{h(L_1, \ell_1, \cdots, \ell_{k-1}) - h(0, \ell_1, \cdots, \ell_{k-1})}{L_1}.
\]

The first term lies in \(M_{\mathbb{F}, +}(V^* \otimes \mathbb{C})\) as a consequence of the orthogonality of \(L_1\) with the \(\ell_j\)'s. The second term is holomorphic at 0. This yields the required decomposition.

For \(t \geq 1\), assume that the decomposition exists when \(s \leq t\) and consider \(f = \frac{h(L_1, \cdots, L_m, \ell_1, \cdots, \ell_{k-m})}{L_1^{s_1} \cdots L_m^{s_m}}\) with \(s = s_1 + \cdots + s_m = t + 1\). We note that [11] the Taylor expansion of \(h\) gives

\[
h(L_1, \cdots, L_m, \ell_1, \cdots, \ell_{k-m}) := h(0, \cdots, 0, \ell_1, \cdots, \ell_{k-m}) + \sum_{j=1}^{m} L_j g_j,
\]

where the \(g_j\)'s are holomorphic. Thus \(S_0 := h(0, \cdots, 0, \ell_1, \cdots, \ell_{k-m})/(L_1^{s_1} \cdots L_m^{s_m})\) is in \(M_{\mathbb{F}, -}(V^* \otimes \mathbb{C})\) while by the induction hypothesis, \((L_j g_j)/(L_1^{s_1} \cdots L_m^{s_m}) = h_i + \sum_{j \in J_i} S_{j_i}\) with \(h_i\) a holomorphic
germ at zero and $S_j$, polar germs in $M_\mathbb{C}(V^* \otimes \mathbb{C})$. Hence $f = S_0 + \sum_{i=1}^{m} h_i + \sum_{j \in J_{m}} S_j$ is the sum of a holomorphic germ $\sum_{i=1}^{m} h_i$ and finitely many polar germs $S_j$.

Now for a germ $f$ expressed in the form $f(\ell_1, \cdots, \ell_n)$ as given in the theorem, replace the lattice space $(V, \Lambda)$ by its lattice subspace $(W, \Lambda \cap W)$ where $W := \text{span}(\ell_1, \cdots, \ell_n)$. Then $f$ is in $M_\mathbb{C}(W^* \otimes \mathbb{C})$ and applying the first part of the theorem yields the second part of the theorem. \hfill \Box

3. A geometric criterion for non-holomorphicity

In this section, we pursue our geometric approach initiated in [12] to study meromorphic germs at zero through the cones associated to the germs. By means of the supporting cone of a polar germ, we first give a geometric criterion for the linear independence of simplicial fractions in Section 3.1. We then obtain the main Non-holomorphicity Theorem in Section 3.2.

3.1. A geometric criterion for the linear independence of simplicial fractions. We briefly recall the notations and terminology of [12] and use the results obtained there on the geometry of cones underlying the decomposition of fractions, further refined to require that the coefficients lie in the subfield $\mathbb{F}$.

As in [12] we consider closed convex polyhedral cones henceforth simply called cones in a filtered lattice space $(V, \Lambda_V) = \bigcup_{k \geq 1} (V_k, \Lambda_k)$. We call $\mathbb{F}$-cones the ones whose generators lie in $\Lambda_k \otimes \mathbb{F}$. A $\mathbb{Q}$-cone is called rational.

We recall that a subdivision of a cone $C$ is a set $\{C_1, \cdots, C_r\}$ of cones which have the same dimension as $C$, whose union is $C$, and that intersect along their faces, i.e., $C_i \cap C_j$ is a face of both $C_i$ and $C_j$. Such a subdivision is simplicial (resp. smooth, in the case when $C$ is rational) if all $C_i$’s are simplicial (resp. smooth). An $\mathbb{F}$-subdivision of an $\mathbb{F}$-cone is a subdivision such that every $C_i$ is an $\mathbb{F}$-cone.

On the grounds of Lemma 2.8, we can assign a simplicial cone to a polar germ.

**Definition 3.1.** Let

$$f := \frac{h(\ell_1, \cdots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}},$$

be a polar germ, as defined by the conditions in Definition 2.3. We shall say that the cone $\langle L_1, \cdots, L_m \rangle$ supports the germ; it is a supporting cone of the germ.

By Lemma 2.8, the supporting cones of a polar germ is defined up to the choice of a sign of each of the vectors $L_1, \cdots, L_m$. Indeed, any cone $\langle \pm L_1, \cdots, \pm L_m \rangle$ delimited by the hyperplanes in the hyperplane arrangement $\{H_1, \cdots, H_n\}$ with $H_i = \{L_i = 0\}$ is also a supporting cone. For example, in the standard Euclidean space $\mathbb{R}^2$, the polar germ $\frac{1}{e_1 e_2}$ has four supporting cones given by the four quadrants of the Euclidean plane, cut out by the two lines spanned by the basis vectors $e_1$ and $e_2$ respectively.

We now introduce the key concepts concerning families of cones.

**Definition 3.2.**

(a) A family of cones is said to be properly positioned if the cones meet along faces and the union does not contain any nonzero linear subspace.

(b) A family of polar germs is called properly positioned if there is a choice of a supporting cone for each of the polar germs such that the resulting family of cones is properly positioned.

(c) A family of polar germs is called projectively properly positioned if it is properly positioned and none of the denominators of the polar germs is proportional to another.
So a family of polar germs is projectively properly positioned if it is properly positioned when viewed in the projective space of polar germs (that is, modulo scalar multiples), hence the terminology. It would be interesting to find a simple criterion for a projectively properly positioned family of polar germs.

We next give a reinterpretation of a related result in [12] for which we recall some preliminary notations and results. See also [2, 10, 21].

Let \( C \) be a simplicial cone in \( V_k \) with \( \mathbb{R} \)-linearly independent generators \( v_1, \ldots, v_n \) expressed in a fixed basis \( \{ e_1, \ldots, e_k \} \) as \( v_i = \sum_{j=1}^{n} a_{ij} e_j \), for \( 1 \leq i \leq n \). Define linear functions \( L_i, 1 \leq i \leq n \), on \( V_k^* \otimes \mathbb{C} \) by \( L_i(\vec{e}) := L_{v_i}(\vec{e}) := \sum_{j=1}^{n} a_{ij} e_j \), where \( \vec{e} := \sum_{i=1}^{n} e_i e_i^* \in V_k^* \otimes \mathbb{C} \) and \( \{ e_1^*, \ldots, e_n^* \} \) is the dual basis in \( V_k^* \). Let \( A_C = \{ a_{ij} \} \) denote the associated matrix in \( M_{k \times n}(\mathbb{R}) \). Let \( w(v_1, \ldots, v_n) \) or \( w(C) \) denote the sum of absolute values of the determinants of all minors of \( A_C \) of rank \( n \). As in [12] except a different notation \( \Phi \) instead of \( I \) and a sign convention, define

\[
I(C) := (-1)^{w(v_1, \ldots, v_n)} \frac{w(v_1, \ldots, v_n)}{L_1 \cdots L_n}.
\] (7)

Let \( C \) be a cone in \( V_k \) and let \( \{ C_i \} \) be a simplicial subdivision of \( C \). By [12, Lemma 3.3], the sum

\[
I(C) := \sum_i I(C_i)
\] (8)

is well-defined, independent of the choice of simplicial subdivisions, hence yielding a linear map

\[
I : \mathcal{RC}(\mathbb{R}) \to \mathcal{M}_{\mathbb{R}^+}(V^* \otimes \mathbb{C}),
\] (9)

where \( \mathcal{RC}(\mathbb{R}) \) is the \( \mathbb{R} \)-linear space spanned by the set \( \mathcal{C}(\mathbb{R}) \) of cones in \( V \).

Now we are ready for our first geometric criterion for the linear independence of fractions.

**Lemma 3.3.** A projectively properly positioned family of simple \( \mathbb{F} \)-fractions whose supporting cones span the same linear subspace is \( \mathbb{F} \)-linearly independent.

**Proof.** We choose the supporting cones \( \{ C_i \} \) in such a way that the family is properly positioned. Since the simple \( \mathbb{F} \)-fractions are not pairwise proportional, these supporting cones are distinct. Since the cones are properly positioned, their union does not contain any nonzero linear subspace. Thus the union of the cones has a topological boundary. Thus by [12, Lemma 3.5], the set \( \{ I(C_i) \} \) is linearly independent. But each \( I(C_i) \) is a nonzero multiple of the original fraction. Thus the original family of simple fractions is linearly independent. \( \square \)

Before the treatment of more general fractions, we give the following “locality” lemma.

**Lemma 3.4.** Let \( \frac{h_i}{L_i}, i = 1, \ldots, r \) be \( \mathbb{F} \)-polar germs and \( h_0 \) a holomorphic germ at zero satisfying

\[
\sum_{i=1}^{r} a_i \frac{h_i}{L_i} = h_0
\] (10)

with \( a_1, \ldots, a_r \in \mathbb{F} \). For any linear \( \mathbb{F} \)-subspace \( W \) of \( V \) and \( N \in \mathbb{Z}_{>0} \), denote

\[
I(W, N) := \{ i \in [r] | \text{span}(L_{i1}, \ldots, L_{in}) = W, [s_i] := s_{i1} + \cdots + s_{in} = N \}.
\]

Then

\[
\sum_{i \in I(W, N)} a_i \frac{h_i}{L_i} = 0,
\]

with the convention that the sum over an empty set is zero.
Proof. For distinct pairs \((W, N)\) and \((W', N')\) arising in the expression Eq. (10) we have \(I(W, N) \cap I(W', N') = \emptyset\). Thus \([\mathcal{I}]\) is partitioned into finitely many non-empty and disjoint subsets \(I(W_1, N_1), \ldots, I(W_p, N_p)\). Then

\[
\sum_{j=1}^{p} \sum_{i \in I(W_j, N_j)} a_i \frac{h_i}{L_i^j} = \sum_{i=1}^{r} a_i \frac{h_i}{L_i} = h_0.
\]

Suppose that an expression in Eq. (10) is a counter example to the lemma. Then

\[
\sum_{i \in I(W_j, N_j)} a_i \frac{h_i}{L_i^j} \neq 0
\]

for some \(j \in [p]\). By dropping those \(j \in [p]\) with \(\sum_{i \in I(W_j, N_j)} a_i \frac{h_i}{L_i^j} = 0\) if necessary, we can assume that Eq. (11) holds for all \(j \in [p]\).

Let \(N := \max ||s_j|| i \in [r]\) and let \(W\) be one of those \(\text{span}(L_{i1}, \ldots, L_{in})\) with \(|s_j| = N\) whose dimension is minimal. Reordering the terms of the sum in Eq. (10) if necessary, we can assume that \(I(W, N) = [t]\) for some \(t \geq 1\). Thus \(|s_j| = N\) and \(\text{span}(L_{i1}, \ldots, L_{in}) = W\) precisely for \(i \in [t]\).

We extend the linearly independent linear forms \(L_{i1}, \ldots, L_{in}\) to a basis \(e_1, \ldots, e_k\) of \(\Lambda_k \otimes \mathbb{C}\) with \(e_i = L_{ij}\) for \(i \in [n]\) such that \(Q(e_i, e_j) = 0\) for \(1 \leq i, j \leq k\). Write the polar germs \(h_i \frac{h_i}{L_i} = \frac{h_i(l_{i1}, \ldots, l_{in})}{L_i l_{i1} \cdots l_{in}}\) as in Definition 2.3. Since \(e_1, \ldots, e_k\) is also a basis of \(\text{span}(L_{i1}, \ldots, L_{in})\) for \(i \in [t]\), we have

\[
Q(e_j, \ell_{ij}) = 0, 1 \leq j \leq n_i, 1 \leq \ell \leq m_i, 1 \leq i \leq t.
\]

So the linear forms \(\ell_{i1}, \ldots, \ell_{in}\) lie in \(\text{span}(e_{n+1}, \ldots, e_k)\). Thus with respect to the dual basis \(\{e^*_1, \ldots, e^*_k\}\) of the basis \(\{e_1, \ldots, e_k\}\) of \(V_i \otimes \mathbb{C}\), the functions \(h_1(\ell_{i1}, \ldots, \ell_{im}), \ldots, h_k(\ell_{i1}, \ldots, \ell_{im})\) as functions in the variables \(\vec{c} = \sum \epsilon_i e^*_i\) are in fact germs at zero of holomorphic functions depending only on the variables \(e_{n+1}, \ldots, e_k\), which we write as \(h_i(\epsilon_{n+1}, \ldots, \epsilon_k), \ldots, h_k(\epsilon_{n+1}, \ldots, \epsilon_k)\).

Fix \(i > t\). For any \(j \in [n]\), write \(L_{ij} = L_{ij}' + L_{ij}''\), where \(L_{ij}'\) is a linear combination of \(e_1, \ldots, e_{n_i}\) and \(L_{ij}''\) is a linear combination of \(e_{n+1}, \ldots, e_k\). Thus \(L_{ij}''(\vec{c})\) is a linear function in \(\epsilon_{n+1}, \ldots, \epsilon_k\).

We note that \(i > t\) if and only if either \(\sum_{j=1}^{m_i} s_{ij} < N\), or there is an index \(j\) such that \(L_{ij}'' \neq 0\) as a result of the fact that \(\{L_{i1}, \ldots, L_{in}\}\) and \(\{L_{11}, \ldots, L_{1n}\}\) do not span the same linear space.

Since \(h_i(\epsilon_{n+1}, \ldots, \epsilon_k), 1 \leq i \leq t\), are not identically zero, there are fixed values \(\epsilon^0_{n+1}, \ldots, \epsilon^0_k\) of \(\epsilon_{n+1}, \ldots, \epsilon_k\) for which \(h_1(\epsilon^0_{n+1}, \ldots, \epsilon^0_k) \neq 0, 1 \leq i \leq t, L_{ij}''(\epsilon^0_{n+1}, \ldots, \epsilon^0_k) \neq 0, i > t\) for those \(L_{ij}'' \neq 0\). These values form a non-empty open subset.

We next introduce a new set of variables \(r_m, 1 \leq m \leq n_1\), and \(\epsilon\), and apply the substitution \((\epsilon_1, \ldots, \epsilon_k) = (r_1 \epsilon, \ldots, r_{n_1} \epsilon, \epsilon^0_{n+1}, \ldots, \epsilon^0_k)\) in Eq. (10). This gives rise to a Laurent series in \(\epsilon\) that is holomorphic at zero by the choice of the germ. Thus the coefficient of every given negative power of \(\epsilon\) is 0. In particular the coefficient of the least possible power \(\epsilon^{-N}\) is zero. In order for a term \(h_i L_{i1}^{s_1} \cdots L_{in}^{s_n}\) in the sum to contribute to this coefficient, we must have \(\sum_j s_{ij} = N\) and \(L_{ij}'' = 0\), that is, \(1 \leq i \leq t\) as a result of the definition of \(t\). On the other hand, for \(1 \leq i \leq t, L_{i1}, \ldots, L_{in}\) are linear homogeneous in \(L_{11}(\vec{c}) = \epsilon_1, \ldots, L_{1n}(\vec{c}) = \epsilon_n\). Hence under the above substitution, they give \(\epsilon L_{i1}, \ldots, \epsilon L_{in}\) in the variables \(r_1, \ldots, r_{n_1}\) and the coefficient of \(\epsilon^{-N}\) in the Laurent series reads

\[
\sum_{i=1}^{t} a_i \frac{h_i(\epsilon^0_{n+1}, \ldots, \epsilon^0_k)}{L_{i1}^{s_1} \cdots L_{in}^{s_n}}.
\]
Hence it is zero as a sum of fractions in variables \( r_1, \ldots, r_n \). Thus

\[
\sum_{i=1}^{t} a_i \frac{h_i(\varepsilon_{r_{1}+1}^0, \ldots, \varepsilon_{r_{m}}^0)}{L_{r_{1},i}^{s_{r_{1},i}} \cdots L_{r_{n},i}^{s_{r_{n},i}}} = 0
\]

for any generic point \((\varepsilon_{r_{1}+1}^0, \ldots, \varepsilon_{r_{m}}^0)\). Comparing with Eq. (11) gives the desired contradiction. \(\Box\)

Based on this lemma, Lemma 3.3 can be generalized to the following statement.

**Proposition 3.5.** A projectively properly positioned family of simplicial \(\mathbb{F}\)-fractions is \(\mathbb{F}\)-linearly independent.

**Proof.** We only need to prove that a contradiction follows from any linear relation

\[
\sum_{i=1}^{r} a_i \frac{1}{L_{s_{1},i}^{r_{1}} \cdots L_{s_{n},i}^{r_{m}} L_{i_{1},i}^{s_{i_{1},i}} \cdots L_{i_{n},i}^{s_{i_{n},i}}} = 0, \quad 0 \neq a_i \in \mathbb{F},
\]

of a projectively properly positioned family of \(\mathbb{F}\)-fractions \(G_i := \frac{1}{L_{s_{1},i}^{r_{1}} \cdots L_{s_{n},i}^{r_{m}} L_{i_{1},i}^{s_{i_{1},i}} \cdots L_{i_{n},i}^{s_{i_{n},i}}} \).

By Lemma 3.4, we can assume that, for each \(1 \leq i \leq r\), the weight \(|s_i| = s_{i_{1},i} + \cdots + s_{i_{n},i}\) is the same and the linear forms in the denominators span the same space. In particular, \(n_1 = \cdots = n_r = n\).

We next proceed by induction on \(s := |s|\). So \(s \geq n\). If \(s = n\), then the powers of all the linear forms are equal to 1. It then follows from Lemma 3.3 that \(a_i = 0\) for all indices \(i\), leading to the expected contradiction. Assume that a contradiction arises for any relation in Eq. (13) with \(s = N \geq k\) and consider such a relation with \(s = N + 1\). In this case, at least one linear form, say \(L_{1}\), has exponent greater than one.

Let \(r_1\) be the maximal power of \(L_{1}\) in all the simplicial fractions \(G_i, 1 \leq i \leq r\). We split these fractions into three disjoint sets. Let \(G_{1}, \ldots, G_{m}\) be all the simplicial fractions with \(L_{1}\) raised to the power of \(r_1\). Let \(G_{m+1}, \ldots, G_{m+\ell}\) be all the simplicial fractions, if any, with \(L_{1}\) raised a positive power less than \(r_1\). Let \(G_{m+\ell+1}, \ldots, G_{r}\) be all the simplicial fractions, if any, that do not contain \(L_{1}\) in their denominator. Thus

\[
0 = L_{1} \sum_{i=1}^{r} a_i G_i = \sum_{i=1}^{m} a_i L_{1} G_i + \sum_{i=m+1}^{m+\ell} a_i L_{1} G_i + \sum_{i=m+\ell+1}^{r} a_i L_{1} G_i.
\]

For any \(m + 1 \leq i \leq m + \ell\), the power of \(1/L_{1}\) in \(L_{1} G_i\) is less than \(r_1 - 1\). Since the linear forms in the denominators are assumed to span the same spaces, we write \(L_{1}\) as a linear combination of the linear forms \(L_{i_{1}}, \ldots, L_{i_{n}}\) of \(G_i\) for \(m + \ell + 1 \leq i \leq r\):

\[
L_{1} = a_{i_{1}} L_{i_{1}} + \cdots + a_{i_{n}} L_{i_{n}}.
\]

Thus each \(L_{1} G_i = \sum_{j_{1}=1}^{k} \frac{a_{j_{1}}}{L_{i_{j_{1}}}^{s_{i_{j_{1}},j_{1}}} \cdots L_{i_{n},j_{1}}^{s_{i_{n},j_{1}}}}\) for \(m + \ell + 1 \leq i \leq r\) is a linear combination of fractions that do not contain \(L_{1}\) as a linear form in the denominator. In summary, each fraction in \(\sum_{i=m+1}^{r} a_i L_{1} G_i\) has its power of \(1/L_{1}\) less than \(r_1 - 1\), so that no such monomial can cancel with any fraction in \(\sum_{i=1}^{m} a_i L_{1} G_i\).

With the notation of Lemma 3.4, in \(\sum_{i=1}^{r} a_i L_{1} G_i = 0\) we can use the space spanned by \(L_{i_{1}}, \ldots, L_{i_{n}}\) to single out an equation

\[
\sum_{i=1}^{m} a_i L_{1} G_i + \sum_{i=m+1}^{m+\ell} a_i' L_{1} G_i + \cdots = 0.
\]
In this equality, simplicial fractions $L_1 G_1, \cdots, L_1 G_m$ have non-zero coefficients $a_1, \cdots, a_m$, and the weight of each terms in the sum is $N$. Thus by the induction hypothesis we must have $a_i = 0, i = 1, \cdots, m$, which yields the expected contradiction. □

3.2. The non-holomorphicity of polar germs. Based on Proposition 3.5, we prove the following non-holomorphicity of polar germs at zero, a central result of this paper.

**Theorem 3.6. (Non-holomorphicity Theorem)** A projectively properly positioned family

$$\left\{ \frac{h_i}{L_{i1}^{s_{i1}} \cdots L_{in}^{s_{in}}} \mid 1 \leq i \leq p \right\}$$

of $\mathbb{F}$-polar germs at zero is non-holomorphic in the sense that, if a linear combination

$$\sum_{i=1}^{p} a_i \frac{h_i}{L_{i1}^{s_{i1}} \cdots L_{in}^{s_{in}}}, \quad a_i \in \mathbb{F}, 1 \leq i \leq p,$$

is holomorphic, then $a_i = 0$ for $1 \leq i \leq p$. In particular, this family of polar germs is linearly independent and the holomorphic function for the linear combination is identically zero.

**Proof.** It suffices to show the theorem for $\mathbb{F} = \mathbb{R}$ since the result then follows for any subfield $\mathbb{F}$ of $\mathbb{R}$.

By Lemma 3.4, we can assume that, for each $1 \leq i \leq r$, the weight $|s_i| = s_{i1} + \cdots + s_{in}$ is the same and the linear forms in the denominators span the same space. As in the proof of Lemma 3.4, we can pick values $\ell_{n1}^0, \cdots, \ell_k^0$ of $\ell_{n1}^0, \cdots, \ell_k^0$ such that $h_i(\ell_{n1}^0, \cdots, \ell_k^0) \neq 0, 1 \leq i \leq t$, and

$$\sum_{i=1}^{t} a_i \frac{h_i(\ell_{n1}^0, \cdots, \ell_k^0)}{L_{s1}^{s_1} \cdots L_{sn}^{s_n}} = 0.$$

But the set of fractions $\frac{1}{L_{s1}^{s_1} \cdots L_{sn}^{s_n}}, 1 \leq i \leq t$, is projectively properly positioned. Thus by Proposition 3.5, the coefficients $a_i h_i(\ell_{n1}^0, \cdots, \ell_k^0)$ and hence the coefficients $a_i, 1 \leq i \leq t$, are zero which leads to a contradiction. □

A direct consequence of Theorem 3.6 is the following uniqueness result.

**Corollary 3.7.** Let $\{S_i\}_{1 \leq i \leq t}$ and $\{T_j\}_{1 \leq j \leq m}$ be projectively properly positioned families of $\mathbb{F}$-polar germs at zero sharing the same properly positioned family of supporting cones (upon a suitable choice of signs of the linear forms). If

$$\sum_{i=1}^{t} S_i + g_0 = \sum_{j=1}^{m} T_j + h_0$$

for holomorphic germs $g_0$ and $h_0$, then $\{S_i\}_{1 \leq i \leq t} = \{T_j\}_{1 \leq j \leq m}$ and $g_0 = h_0$.

**Proof.** Let $\{C_1, \cdots, C_r\}$ be a properly positioned family of supporting cones of $\{S_i\}_{1 \leq i \leq t}$ and of $\{T_j\}_{1 \leq j \leq m}$. For $1 \leq i \leq r$, let $L_{i1}, \cdots, L_{im}$ be fixed generators of the $\mathbb{F}$-cones $C_i$. Let $N$ be the largest sum of powers in the denominators of $\{S_i\}_{1 \leq i \leq t}$ and $\{T_j\}_{1 \leq j \leq m}$ and denote

$$\{M_1, \cdots, M_t\} = \left\{ L_{i1}^{s_{i1}} \cdots L_{in}^{s_{in}} \mid i = 1, \cdots, r, \ |\vec{s}| := \sum_j s_j \leq N \right\}.$$
Then we have
\[ \sum_{i=1}^{t} S_i = \sum_{k=1}^{n} \frac{g_k}{M_k} \quad \text{and} \quad \sum_{j=1}^{m} T_j = \sum_{k=1}^{n} \frac{h_k}{M_k}, \]
where for \( 1 \leq k \leq t \), \( g_k \) and \( h_k \), some of which can be zero, are holomorphic in some linear forms orthogonal to the linear forms in \( M_k \) with respect to the given inner product. Thus Eq. (15) gives
\[ \sum_{k=1}^{t} \frac{g_k - h_k}{M_k} = h_0 - g_0. \]
But the terms in the sum satisfy the conditions of Theorem 3.6. Thus we have \( g_k = h_k \) for \( 0 \leq k \leq t \) which implies that the \( S_i \)'s match with the \( T_j \)'s, giving the identification we want. \( \square \)

4. Laurent expansions of meromorphic germs at zero with linear poles

In this section, we apply the Non-holomorphicity Theorem 3.6 and Corollary 3.7 to develop a notion of Laurent expansions for multivariate meromorphic germs at zero with linear poles.

Central to the notion of Laurent expansions is the forgetful map in Definition 4.2 which gives formal expansions of meromorphic germs at zero. Taking local cross sections of this map, we first identify the Laurent subspaces in Proposition 4.3, formally defined in Definition 4.5. We then show in Theorem 4.15 that these Laurent subspaces cover the whole space \( \mathcal{M}_{\mathbb{F}}(V^\otimes \mathbb{C}) \) with the help of the surjectivity of \( \varphi \) proved in Theorem 2.10. We then establish a consistency property of these Laurent subspaces in Proposition 4.17. Finally, we determine the kernel of the forgetful map in Section 4.3.

4.1. The space of formal expansions. We first generalize the concept of decorated smooth cones in [12].

**Definition 4.1.** A decorated simplicial \( \mathbb{F} \)-cone is a formal monomial \( \overline{C} := \langle v_1 \rangle^{s_1} \cdots \langle v_n \rangle^{s_n} \) where \( v_1, \ldots, v_n \) are linearly independent \( \mathbb{F} \)-vectors and \( s_1, \ldots, s_n \) are in \( \mathbb{Z}_{\geq 1} \). The simplicial \( \mathbb{F} \)-cone \( \langle v_1, \ldots, v_n \rangle \) generated by \( v_1, \ldots, v_n \) is called the geometric cone of the decorated cone \( \overline{C} \) and is denoted by \( G(\overline{C}) \).

As before, these generators define linear functions \( L_1, \ldots, L_n \) on \( V_k^* \otimes \mathbb{C} \). For a different choice of the spanning vectors \( v_1, \ldots, v_n \), the function \( \overline{L}_{\overline{C}} := L_1^{s_1} \cdots L_n^{s_n} \) alters by a constant. Thus for any subspace \( U \) of \( V \otimes \mathbb{C} \), the subspace \( \frac{1}{L_{\overline{C}}} \mathcal{M}\mathbb{F}(U^\ast) \) does not depend on the choice of the spanning vectors \( v_1, \ldots, v_n \). In particular this holds for \( U = \text{lin}^+(G(\overline{C})) \), the orthogonal complement of the linear span of \( G(\overline{C}) \) in \( V \otimes \mathbb{C} \) with respect to the given inner product \( Q \). The space
\[ \mathcal{M}_{\overline{C}} := \frac{1}{L_{\overline{C}}} \mathcal{M}\mathbb{F}^+((\text{lin}^+(G(\overline{C})))^\ast) \subseteq \mathcal{M}\mathbb{F}(V^\otimes \mathbb{C}) \]
is precisely the space spanned by polar germs whose support is \( G(\overline{C}) \) and with the fixed denominator \( L_{\overline{C}} \).

**Definition 4.2.** (a) Define the space of formal expansions in polar germs to be
\[ \mathcal{M}\mathbb{F}(V^\otimes \mathbb{C}) := \left( \bigoplus_{\overline{C}} \mathcal{M}_{\overline{C}} \right) \bigoplus \mathcal{M}\mathbb{F}^+((V^\otimes \mathbb{C})^\ast) = \left( \bigoplus_{\overline{C}} \frac{1}{L_{\overline{C}}} \mathcal{M}\mathbb{F}^+((\text{lin}^+(G(\overline{C})))^\ast) \right) \bigoplus \mathcal{M}\mathbb{F}^+((V^\otimes \mathbb{C})^\ast), \]
where the sum is taken over decorated simplicial \( \mathbb{F} \)-cones \( \overline{C} \).
(b) Define the forgetful map

\[ \varphi : \mathcal{M}(V^\otimes \mathbb{C}) \rightarrow \mathcal{M}(V^\otimes \mathbb{C}), \quad \bigoplus S_C \otimes h \mapsto \sum C \in \bigoplus S_C + h, S_C \in \mathcal{M}, h \in \mathcal{M}(V^\otimes \mathbb{C}), \]

sending direct sums in \( \mathcal{M}(V^\otimes \mathbb{C}) \) to sums of functions in \( \mathcal{M}(V^\otimes \mathbb{C}) \).

Notice that different decorated simplicial \( \mathbb{F} \)-cones might give the same space \( \mathcal{M} \), for example when the generators of a cone change signs, giving multiple copies of identical summand in \( \mathcal{M}(V^\otimes \mathbb{C}) \). For instance, \( \mathcal{M}(C(1)) = \mathcal{M}(C(-1)) \) but they give distinct summands in \( \mathcal{M}(V^\otimes \mathbb{C}) \).

By definition, the restriction of \( \varphi \) to \( \frac{1}{2} \mathcal{M}(V^\otimes \mathbb{C}) \) for each decorated simplicial cones \( C \), as well as to \( \mathcal{M}^+(V^\otimes \mathbb{C}) \), is injective. The Non-holomorphicity Theorem 3.6 shows that this injectivity of \( \varphi \) holds for much larger subspaces of \( \mathcal{M}(V^\otimes \mathbb{C}) \).

**Proposition 4.3.** Let \( \mathcal{C} \) be a properly positioned family of cones in \( V \). Denote

\[ \mathcal{M}(\mathcal{C}) := \bigoplus_{G(C) \in \mathcal{C}} \mathcal{M}(C) = \bigoplus_{G(C) \in \mathcal{C}} \frac{1}{2} \mathcal{M}(V^\otimes \mathbb{C}) \)

The restriction of \( \varphi \) to

\[ \mathcal{M}(\mathcal{C}) := \mathcal{M}(\mathcal{C}) \otimes \mathcal{M}(V^\otimes \mathbb{C}) \subseteq \mathcal{M}(V^\otimes \mathbb{C}) \]

is injective.

**Proof.** This follows directly from Corollary 3.7.

**Remark 4.4.**

(a) As a consequence of the proposition, we have

\[ \mathcal{M}(\mathcal{C}) := \varphi(\mathcal{M}(\mathcal{C})) \cong \mathcal{M}(\mathcal{C}). \]

(b) Note that \( \mathcal{M}(\mathcal{C}) \) is the space spanned by polar germs whose supporting cone is contained in \( \mathcal{C} \).

**Definition 4.5.** Let \( \mathcal{C} \) be a properly positioned family of simplicial cones. A meromorphic germ \( f \in \mathcal{M}(V^\otimes \mathbb{C}) \) is said to admit a Laurent expansion supported on \( \mathcal{C} \) if it is contained in \( \varphi(\mathcal{M}(\mathcal{C})) \) or, more precisely, if there exists a projectively properly positioned family \( \{S_j\}_{j \in J} \) of polar germs whose supporting cones are contained in \( \mathcal{C} \), together with a holomorphic germ \( h \), all with coefficients in \( \mathbb{F} \), such that

\[ f = \varphi\left( \bigoplus_{j \in J} S_j \otimes h \right). \]

The element \( \bigoplus_{j \in J} S_j \otimes h \in \mathcal{M}(V^\otimes \mathbb{C}) \) with this property, unique by the injectivity in Proposition 4.3, is called the \( \mathcal{C} \)-supported Laurent expansion of \( f \), denoted by \( \mathcal{L}_\mathcal{C}(f) \), that is,

\[ \mathcal{L}_\mathcal{C}(f) := \bigoplus_{j \in J} S_j \otimes h. \]

The subspace \( \varphi(\mathcal{M}(\mathcal{C})) \) of \( \mathcal{M}(V^\otimes \mathbb{C}) \) is called the Laurent subspace supported by \( \mathcal{C} \).

**Remark 4.6.**

(a) Clearly, for a polar germ \( f = \frac{h(c_1, \ldots, c_m)}{t_1^\alpha \cdots t_m^\alpha} \) with supporting cone \( C \) and a properly positioned family \( \mathcal{C} \) of simplicial cones containing \( C \), we have \( \mathcal{L}_\mathcal{C}(f) = f \).

(b) For any \( f \in \mathcal{M}(V^\otimes \mathbb{C}) \) which admits a \( \mathcal{C} \)-supported Laurent expansion, we have

\[ \varphi \circ \mathcal{L}_\mathcal{C}(f) = f. \]
Example 4.7. Take $\mathbb{C} = \langle (e_1) \rangle$ in the standard Euclidean space. Then polar germs at the variables given by $z = \sum \epsilon_i e_i'$ and supported on $\mathbb{C}$ are $h_i/e_i', i \geq 0$, for holomorphic functions $h_i$ in variables other than $e_1$. Thus the Laurent subspace supported by $\mathbb{C}$, restricted to $V_1 := \mathbb{R}e_1$, is precisely

$$
\mathcal{M}_e(V_1 \otimes \mathbb{C}) := \mathcal{M}_e(V^0 \otimes \mathbb{C}) \cap \mathcal{M}(V_1 \otimes \mathbb{C}) = \bigoplus_{i \geq 1} \mathbb{C}e_i' \oplus \mathbb{C}\langle \epsilon \rangle,
$$

recovering the classical Laurent series expansions.

4.2. The subdivision operators. In order to show that every element in $\mathcal{M}_e(V^0 \otimes \mathbb{C})$ admits a Laurent expansion, we want to cover $\mathcal{M}_e(V^0 \otimes \mathbb{C})$ with Laurent subspaces. This is achieved by the subdivision operators which will also take care of the consistency on overlaps of Laurent subspaces.

The following definition generalizes the concept of a subdivision of a cone.

Definition 4.8. A subdivision of a family of cones $\{C_i\}$ is a set $\{D_1, \cdots, D_r\}$ of cones such that

(a) $D_1, \cdots, D_r$ intersect along their faces,
(b) for any $i$, there is $I_i \subset [r]$ such that $\{D_\ell\}_{\ell \in I_i}$ is a subdivision of $C_i$, and
(c) $\bigcup_i I_i = [r]$.

We introduce a notion which will be convenient for later discussions.

Definition 4.9. Fix an ordered basis $\{e_i\}$ of a filtered space $V = \bigcup_{k \geq 1} V_k$ such that $\{e_i\} \cap V_k$ is a basis of $V_k$. A nonzero vector $v = \sum_i c_i e_i$ is called pseudo-positive if the leading coefficient of $v$, namely the nonzero coefficient of $v$ with the largest subscript $i$, is positive. By convention, 0 is taken to be a pseudo-positive vector. Let $P$ denote the set of pseudo-positive vectors.

As can be easily checked, the set $P$ is the union of the increasing filtration consisting of the strictly convex sets $P_n \subseteq \mathbb{R}\{e_1, \cdots, e_n\}, n \geq 0$, where, by convention, $P_0 := \{0\}$ and recursively,

$$
P_{n+1} := P_n \cup (\mathbb{R}\{e_1, \cdots, e_n\} \times \mathbb{R}_{>0}e_{n+1}), \quad n \geq 0.
$$

Consequently, $P$ is a strictly convex set.

Lemma 4.10. Any finite family of cones whose union does not contain a nonzero linear subspace has a properly positioned family of cones as a subdivision. The union of the family of the cones does not change with the subdivision. In particular, if a finite family of cones is in $P$, then so is any of its properly positioned families of subdivisions.

Proof. The existence of a subdivision follows the proof of Lemma 2.3(a) in [12], noting that the assumption made there, namely that the cones span the same linear subspace, is redundant. Then the assumption that the union of the family does not contain a nonzero linear subspace guarantees that the resulting family is properly positioned. The second other statement also follows from the proof of [12, Lemma 2.3(a)]. \qed

Lemma 4.11. Given any finite family of polar germs, there is a choice of the family of supporting cones whose union does not contain a nonzero linear subspace.

Proof. Fix an ordered basis of $V$. By rescaling if necessary, we can assume that all linear forms in the denominators of the polar germs are pseudo-positive. The supporting cones of the polar germs spanned by vectors corresponding to these linear forms are therefore contained in the strictly convex set $P$ and hence does not contain any non-zero linear subspace. \qed
Definition 4.12. A pan-subdivision of a family of cones \( \mathcal{C} = \{ C_i \} \) is a set \( \mathcal{D} = \{ D_1, \ldots, D_r \} \) of cones that satisfies conditions (a) and (b) in Definition 4.8, namely

(a) \( D_1, \ldots, D_r \) intersect along their faces,
(b) for any \( i \), there is \( I_i \subset [r] \) such that \( \{ D_{\ell} \}_{\ell \in I_i} \) is a subdivision of \( C_i \).

If all the cones are \( \mathbb{F} \)-cones, then the pan-subdivision is called a \( \mathbb{F} \)-pan-subdivision.

Example 4.13. A subdivision for a family \( \mathcal{C} \) of cones is a pan-subdivision for a sub-family of \( \mathcal{C} \).

Let \( \mathcal{C} \) be a properly positioned family of cones and \( \mathcal{D} \) a simplicial pan-subdivision of \( \mathcal{C} \). We next define a subdivision operator

\[
\Xi(\varepsilon, \bar{x}) : M(\mathbb{F} \otimes \mathbb{C}) \to M_2(\mathbb{F} \otimes \mathbb{C}).
\]

Since \( M(\mathbb{F} \otimes \mathbb{C}) := \bigoplus_{G(C) \in \mathcal{C}} M_C \), we only need to define its action on \( M_C \) for a decorated cone \( C := \langle v_1 \rangle \cdots \langle v_n \rangle \) as in Definition 4.1, with \( G(C) \) in \( \mathcal{C} \).

We first consider the action when \( s_1 = \cdots = s_n = 1 \). Then a polar germ in \( M_C \) is of the form \( g \frac{L_1 \cdots L_n}{L_{\ell_1} \cdots L_{\ell_n}} \) for a simple fraction \( \frac{1}{L_{\ell_1} \cdots L_{\ell_n}} \) and a holomorphic germ \( g \) in a set of variables orthogonal to the linear span of \( L_1, \ldots, L_n \). Let \( G(C) = C \). There is a unique subset \( \{ D_{\mu} \}_{\mu \in J} \) of \( \mathcal{D} \) that gives a subdivision of \( C \). As in Eq. (7), we have

\[
I(C) = (-1)^n \frac{a}{L_1 \cdots L_n}, \quad I(D_\mu) = (-1)^n \frac{b_\mu}{M_{\mu_1} \cdots M_{\mu_n}},
\]

where \( a, b_\mu \) are constants in \( \mathbb{F} \). By Eq. (8),

\[
(21) \quad \frac{1}{L_1 \cdots L_n} = \frac{(-1)^n}{a} I(C) = \frac{(-1)^n}{a} \sum_{\mu \in J} I(D_\mu) = \frac{\sum_{\mu \in J} b_\mu}{a} \frac{1}{M_{\mu_1} \cdots M_{\mu_n}}.
\]

Note that \( I(D_\mu) \) is supported on \( D_\mu \), and by Lemma 3.3, such a decomposition with support on \( \mathcal{D} \) is unique. Thus we can define

\[
(22) \quad \Xi(\varepsilon, \bar{x}) \left( \frac{g}{L_1 \cdots L_n} \right) := \bigoplus_{\mu \in J} \frac{b_\mu}{a} \frac{g}{M_{\mu_1} \cdots M_{\mu_n}} \in \bigoplus_{\mu \in J} M_{D_\mu} \subseteq M_2(\mathbb{F} \otimes \mathbb{C}).
\]

We next introduce a class of differential operators in order to treat general decorated cones. Let \( \{ e_1, e_2, \cdots \} \) be a basis of the filtered space \( V \) and let \( \{ e_1^*, e_2^*, \cdots \} \) be the dual basis. Let \( \bar{e} = \sum \varepsilon_i e_i^* \) be a generic vector in \( \mathbb{F} \otimes \mathbb{C} \). Then respect to the variables \( \varepsilon_i \), we have differential operators

\[
\partial_i := \frac{\partial}{\partial \varepsilon_i} : M(\mathbb{F} \otimes \mathbb{C}) \to M(\mathbb{F} \otimes \mathbb{C}).
\]

For a fixed vector \( v^* = \sum c_i e_i^* \in V^* \), denote \( \partial_{v^*} := \sum c_i \partial_i \), the negative of the directional derivation. Then for any function \( f \) in linear independent linear forms \( K_1, \cdots, K_m \), the chain rule gives

\[
(23) \quad \partial_{v^*} f(K_1, \cdots, K_m) = - \sum_{i=1}^m \langle v^*, K_m \rangle \frac{\partial f}{\partial K_m}.
\]

Now for any given decorated cone \( C \) with \( G(C) \in \mathcal{C} \) and polar germ \( \frac{g}{L_{\ell_1} \cdots L_{\ell_n}} \) in \( M_C \), let \( \{ L_{\ell} \} = \sum_j c_{ij} e_j^* \) be dual to the linear forms \( \{ L_{\ell} \} \) in the sense that \( \langle L_{\ell}, L_{\ell'} \rangle = \delta_{ij} \), \( 1 \leq i, j \leq n \). By Eq. (23) we obtain

\[
\partial_{v^*} \frac{1}{M_{\mu_1} \cdots M_{\mu_n}} = \sum_{j=1}^n \frac{c_{ij}}{M_{\mu_1} \cdots M_{\mu_j}^{r_j} \cdots M_{\mu_n}^{r_n}}.
\]
for some constants \( c_{i1}, \ldots, c_{in} \) depending on the poles \( M_1, \ldots, M_n \) and on \( L_i \). Since \( g \) is in a set of variables orthogonal to \( M_1, \ldots, M_n \), we further obtain
\[
\partial_{L_i} \frac{g}{M_1^{s_1} \cdots M_n^{s_n}} = \sum_{j=1}^{n} \frac{c_{ij} g}{M_1^{s_1} \cdots M_{j-1}^{s_{j-1}} M_j^{s_j} \cdots M_n^{s_n}} = \frac{g}{M_1^{s_1} \cdots M_n^{s_n}}.
\]

We then define
\[
\delta_{L_i} : M_D \to \bigoplus_{G \subseteq D} M_E \subseteq M_D(V^\otimes \mathbb{C}), \quad \frac{g}{M_1^{s_1} \cdots M_n^{s_n}} \mapsto \bigoplus_{j=1}^{n} \frac{c_{ij} g}{M_1^{s_1} \cdots M_{j-1}^{s_{j-1}} M_j^{s_j} \cdots M_n^{s_n}},
\]
which, by acting componentwise in \( M_D(V^\otimes \mathbb{C}) := \bigoplus_{D \subseteq G} M_D \), gives rise to an operator
\[
\delta_{L_i} : M_D(V^\otimes \mathbb{C}) \to M_D(V^\otimes \mathbb{C}).
\]

By [12, Proposition 4.8 (b)], we have
\[
\frac{1}{L_1^{s_1} \cdots L_n^{s_n}} = \frac{1}{(s_1 - 1)! \cdots (s_n - 1)!} \partial_{L_1}^{s_1-1} \cdots \partial_{L_n}^{s_n-1} \frac{1}{L_1 \cdots L_n}.
\]

We accordingly apply Eqs. (22) and (25) to define
\[
\mathcal{E}_{(\xi, \mathcal{D})}\left(\frac{g}{L_1^{s_1} \cdots L_n^{s_n}}\right) := \frac{1}{(s_1 - 1)! \cdots (s_n - 1)!} \delta_{L_1}^{s_1-1} \cdots \delta_{L_n}^{s_n-1} \left(\mathcal{E}_{(\xi, \mathcal{D})}\left(\frac{g}{L_1 \cdots L_n}\right)\right),
\]
completing the definition of the subdivision operator
\[
\mathcal{E}_{(\xi, \mathcal{D})} : M_D(V^\otimes \mathbb{C}) \to M_D(V^\otimes \mathbb{C}).
\]

Notice that by Eq. (24),
\[
\mathcal{E}_{(\xi, \mathcal{D})}\left(\frac{g}{L_1^{s_1} \cdots L_n^{s_n}}\right) = g \mathcal{E}_{(\xi, \mathcal{D})}\left(\frac{1}{L_1^{s_1} \cdots L_n^{s_n}}\right).
\]

In the definition of the subdivision operator, we choose a basis of \( V \) and a dual of the linear forms in the polar germs. The following proposition shows that this operator does not actually depend on such choices.

**Proposition 4.14.**

(a) The subdivision operator \( \mathcal{E}_{(\xi, \mathcal{D})} \) is compatible with the forgetful map \( \varphi \), i.e., \( \varphi \circ \mathcal{E}_{(\xi, \mathcal{D})} = \varphi \).

(b) The subdivision operator \( \mathcal{E}_{(\xi, \mathcal{D})} \) does not depend on the choice of the basis of \( V \).

**Proof.**

(a). By Eqs. (21) and (22), the desired equation holds for polar germs with \( s_1 = \cdots = s_n = 1 \). Since \( \varphi \circ \delta_{L_j} = \partial_{L_j} \circ \varphi \) by construction, the desired equation follows from Eqs. (26) and (27).

(b). For a polar germ \( f \) supported on \( \mathfrak{C} \), and for any choice of the basis of \( V \), \( \mathcal{E}_{(\xi, \mathcal{D})}(f) \) is a sum of polar germs supported on \( \mathcal{D} \), which equals to \( f \) as a function by Item (a), so that by Corollary 3.7, \( \mathcal{E}_{(\xi, \mathcal{D})}(f) \) is unique.

Furthermore, for a simplicial \( \mathbb{P} \)-pan-subdivision \( \mathfrak{C} \) of \( \mathcal{D} \), by the transitivity of pan-subdivisions, we obtain
\[
\mathcal{E}_{(\mathfrak{D}, \mathfrak{E})} \circ \mathcal{E}_{(\xi, \mathfrak{D})} = \mathcal{E}_{(\xi, \mathfrak{E})}.
\]

Now we show that the Laurent subspaces cover the whole space \( M_D(V_k^\otimes \mathbb{C}) \), proving the existence of an Laurent expansion for any meromorphic germ.
**Theorem 4.15.** Let \( f \) be an element in \( \mathcal{M}_\rho(V_k \otimes \mathbb{C}) \). There exists a properly positioned family of simplicial cones \( \mathcal{C} \) such that \( f \) has a Laurent expansion supported on \( \mathcal{C} \). In other words, there is a projectively properly positioned family of polar germs \( \{ S_j \}_{j \in J} \) supported on \( \mathcal{C} \), together with a holomorphic germ \( h \), all with coefficients in \( \mathbb{F} \), such that

\[
(31) \quad f = \varphi \left( \bigoplus_{j \in J} S_j \oplus h \right),
\]

or as function decomposition,

\[
(32) \quad f = \sum_{j \in J} S_j + h.
\]

In fact, the family \( \mathcal{C} \) can be taken to be in \( \mathbf{P} \).

**Proof.** Take any decomposition of \( f \) as in Theorem 2.10, \( f = \sum_{i \in I} g_i + h \), where \( \{g_i\} \) is a finite set of polar germs and \( h \) is holomorphic at zero. By Lemma 4.11, there is a choice \( \mathcal{C} \) of the family of the supporting cones of the polar germs such that the union of the cones does not contain any nonzero linear subspace. In fact, the proof of Lemma 4.11 shows that \( \mathcal{C} \) can be chosen to be contained in \( \mathbf{P} \). By combining colinear terms, we can assume that the decorated cones of these polar germs are distinct. Hence we can write \( f = \varphi(\oplus_{i \in I} g_i \oplus h) \).

By Lemma 4.10, the family \( \mathcal{C} \) has a pan-subdivision \( \mathcal{D} \) that is properly positioned. Then through the subdivision operator \( \Xi(\mathcal{C}, \mathcal{D}) \), the sum \( \bigoplus_{i \in I} \Xi(\mathcal{C}, \mathcal{D})(g_i) \oplus h \) is a desired Laurent expansion of \( f \) supported on \( \mathcal{D} \).

**Example 4.16.** In the standard Euclidean space, we have

\[
\frac{z_1 + 2z_2}{z_1(z_1 + z_2)z_2} = \frac{1}{z_1z_2} + \frac{1}{z_1(z_1 + z_2)} = 2\frac{1}{z_1(z_1 + z_2)} + \frac{1}{(z_1 + z_2)z_2}.
\]

Here the first equation expressed the meromorphic germ as a sum of polar germs as in Theorem 2.10. The second equation rewrite the sum of polar germs as a sum of projectively properly positioned family of polar germs, as in Theorem 4.15.

We finally prove the coherence of Laurent expansions arising from different properly positioned family of cones, namely their compatibility with the subdivision operators.

**Proposition 4.17.**

(a) Assume that \( f \in \mathcal{M}_\rho(V^\circ \otimes \mathbb{C}) \) admits a \( \mathcal{C} \)-supported Laurent expansion and let \( \mathcal{D} \) be a simplicial \( \mathbb{F} \)-pan-subdivision of \( \mathcal{C} \). Then \( \Xi(\mathcal{C}, \mathcal{D})\mathcal{C}(f) \) is the \( \mathcal{D} \)-supported Laurent expansion of \( f \).

(b) With respect to the inclusion operators, the set of Laurent subspaces supported on cones in the set \( \mathbf{P} \) forms a direct system. Its direct limit is \( \mathcal{M}_\rho(V^\circ \otimes \mathbb{C}) \).

**Proof.** (a) follows in a straightforward manner from Proposition 4.14.(a).

(b) For two properly positioned families of simplicial cones in \( \mathbf{P} \), their union is contained in \( \mathbf{P} \). Thus by Lemma 4.10, their union has a properly positioned subdivision of simplicial cones, giving a common pan-subdivision of the two families. Thus the set of properly positioned families of simplicial cones in \( \mathbf{P} \) is direct with respect to pan-subdivisions. Through the subdivision operators, the set

\[
\{ \mathcal{M}_\rho(V^\circ \otimes \mathbb{C}) \mid \text{properly positioned families} \ \mathcal{C} \ \text{in} \ \mathbf{P} \}
\]

is a direct system. Then by Proposition 4.14.(a) the set of Laurent subspaces

\[
\{ \mathcal{M}_\rho(V^\circ \otimes \mathbb{C}) \mid \text{properly positioned families} \ \mathcal{C} \ \text{in} \ \mathbf{P} \}
\]
is a direct system with respect to the inclusion maps. Its direct limit is \( \mathcal{M}_F(V^* \otimes \mathbb{C}) \) since the union of Laurent subspaces supported in \( P \) is \( \mathcal{M}_F(V^* \otimes \mathbb{C}) \) by Theorem 4.15.

As an immediate consequence, we obtain the following Rota-Baxter type decomposition utilized in [13].

**Corollary 4.18.** Let \((V, \Lambda_V)\) be a filtered \( F \)-Euclidean lattice space. There is a direct sum decomposition

\[
\mathcal{M}_F(V^* \otimes \mathbb{C}) = \mathcal{M}_{F,-}(V^* \otimes \mathbb{C}) \oplus \mathcal{M}_{F,+}(V^* \otimes \mathbb{C}).
\]

In particular, the holomorphic part \( h \) and the polar part \( \sum_j S_j \) in Eq. (6) are uniquely determined by the germ \( f \).

**Proof.** For each properly positioned family \( \mathcal{C} \) of simplicial cones, Proposition 4.3 gives the direct sum decomposition

\[
\mathcal{M}_{F,\mathcal{C}}(V^* \otimes \mathbb{C}) = \mathcal{M}_{F,\mathcal{C},-}(V^* \otimes \mathbb{C}) \oplus \mathcal{M}_{F,\mathcal{C},+}(V^* \otimes \mathbb{C}).
\]

By Proposition 4.17, we obtain

\[
\mathcal{M}_F(V^* \otimes \mathbb{C}) = \lim \mathcal{M}_{F,\mathcal{C}}(V^* \otimes \mathbb{C}) = \lim \mathcal{M}_{F,\mathcal{C},-}(V^* \otimes \mathbb{C}) \oplus \mathcal{M}_{F,\mathcal{C},+}(V^* \otimes \mathbb{C}) = \mathcal{M}_{F,-}(V^* \otimes \mathbb{C}) \oplus \mathcal{M}_{F,+}(V^* \otimes \mathbb{C}),
\]

where the direct limits are taken over those \( \mathcal{C} \) in \( P \).

We introduce a notation before stating the next result.

**Definition 4.19.** Germs \( f, g \in \mathcal{M}_F(V^* \otimes \mathbb{C}) \) are said to be **orthogonally variate germs** if there are germs \( \tilde{f} \) on \( \mathbb{C}^n \) and \( \tilde{g} \) on \( \mathbb{C}^m \) such that \( f = \tilde{f}(L_1, \cdots, L_m) \) and \( g = \tilde{g}(M_1, \cdots, M_n) \) for linear independent linear forms \( \{L_1, \cdots, L_m\} \) and \( \{M_1, \cdots, M_n\} \) on \( V^* \otimes \mathbb{C} \) with \( Q(L_i, M_j) = 0 \) for \((i, j) \in [1, m] \times [1, n]\).

**Corollary 4.20.** (Multiplicativity of \( \pi_+ \) on orthogonally variate germs) Let

\[
\pi_+ : \mathcal{M}_F(V^* \otimes \mathbb{C}) \to \mathcal{M}_{F,+}(V^* \otimes \mathbb{C})
\]

denote the projection map onto \( \mathcal{M}_{F,+}(V^* \otimes \mathbb{C}) \) along \( \mathcal{M}_{F,-}(V^* \otimes \mathbb{C}) \). For orthogonally variate germs \( f \) and \( g \), we have

\[
\pi_+(fg) = \pi_+(f) \pi_+(g).
\]

**Proof.** Let \( f \) and \( g \) be in \( \mathcal{M}_{F,+}(V^* \otimes \mathbb{C}) \). Using Eq. (6), we decompose \( f = h + \sum_{j=1}^m S_j \) and \( g = k + \sum_{j=1}^n T_j \) with \( h, k \) holomorphic germs and \( S_j, T_j \) polar germs. Further by Theorem 2.10, with the notations in Definition 4.19, \( h \) and \( S_i \) (resp. \( g \) and \( T_j \)) can be written as functions in linear forms in \( \text{span}(L_1, \cdots, L_m) \) (resp. \( \text{span}(M_1, \cdots, M_n) \)). Now

\[
fg = hk + h\left(\sum_{j=1}^m T_j\right) + k\left(\sum_{i=1}^n S_i\right) + \sum_{i,j} S_i T_j.
\]

By the orthogonality of \( \text{span}(L_1, \cdots, L_m) \) and \( \text{span}(M_1, \cdots, M_n) \), the germs \( kT_j, kS_i \) and \( S_i T_j \) are all polar germs. Thus this is a decomposition of \( fg \) into the sum of a holomorphic germ \( hk \) and a linear combination of polar germs. Thus by Corollary 4.18, \( \pi_+(fg) = hk = \pi_+(f)\pi_+(g) \). □

**Remark 4.21.** The projection \( \pi_+ \) is a multivariate generalization of the minimal subtraction operator in one variable. The multiplicativity on orthogonally variate germs stated in Corollary 4.20 is closely related to locality in quantum field theory and central in renormalization issues.
4.3. The kernel of the forgetful map. We finally determine the kernel of the forgetful map \( \varphi : \mathcal{M}_E(V^\circ \otimes \mathbb{C}) \rightarrow \mathcal{M}_E(V^\circ \otimes \mathbb{C}) \) introduced in Eq. (16).

**Theorem 4.22.** The kernel of the map \( \varphi \) is the subspace of \( \mathcal{M}_E(V^\circ \otimes \mathbb{C}) \) spanned by elements of the following forms

I. \( \frac{h(t_1, \ldots, t_m)}{L_1^{1 \ell_1} \cdot \cdots \cdot L_n^{1 \ell_n}} \oplus (-1)^{s_1 + \cdots + s_m} \frac{t_1^{1+1} \cdots t_m^{1+1}}{(-L_1^{1+1} \cdots -L_n^{1+1})} \), for all polar germs of the form \( \frac{h(t_1, \ldots, t_m)}{L_1^{1 \ell_1} \cdot \cdots \cdot L_n^{1 \ell_n}} \).

II. \( \frac{h(t_1, \ldots, t_m)}{L_1^{1 \ell_1} \cdot \cdots \cdot L_n^{1 \ell_n}} \oplus \mathcal{Z}(C,E) \frac{h(t_1, \ldots, t_m)}{L_1^{1 \ell_1} \cdot \cdots \cdot L_n^{1 \ell_n}} \), for all polar germs of the form \( \frac{h(t_1, \ldots, t_m)}{L_1^{1 \ell_1} \cdot \cdots \cdot L_n^{1 \ell_n}} \), \( C := \langle L_1, \ldots, L_n \rangle \) and \( \mathcal{D} \) a simplicial subdivision of \( \langle L_1, \ldots, L_n \rangle \).

Thus modulo changing of signs, relations among polar germs amount to subdivision relations.

**Proof.** Clearly, the subspace \( W \) of \( \mathcal{M}_E(V^\circ \otimes \mathbb{C}) \) generated by elements of the forms I and II is a subspace of \( \text{ker} \varphi \). So we only need to prove that if \( G \oplus H \) is in \( \text{ker} \varphi \) with \( G = \oplus S_j \) a sum of polar germs \( S_j \) and \( H \) a holomorphic germ at zero as in Theorem 2.10, then \( G \) lies in \( W \) and \( H \) vanishes.

By Lemma 4.11, modulo elements of form I, we can assume that the union of the supporting cones of \( S_j \) does not contain any non-zero subspace. Let \( \mathcal{C} := \{C_j | j \in J\} \) be the family of supporting cones, and let \( \mathcal{D} \) be a simplicial subdivision of \( \mathcal{C} \). Then \( G + \mathcal{Z}(C,E) (-G) = \sum_j \left( S_j + \mathcal{Z}(C_j,E) (-S_j) \right) \) — where \( \mathcal{C}_j \) is the singleton \( \{C_j\} \) and \( \mathcal{D}_j \) is the subdivision of \( C_j \) induced by \( \mathcal{D} \) — is a sum of elements of type II and hence lies in \( W \). Since \( \mathcal{Z}(C,E) (-G) - H = -G - H \in \text{ker} \varphi \), we have \( \varphi(\mathcal{Z}(C,E) (-G)) - \varphi(H) = 0 \).

Theorem 3.6 and Proposition 4.3 then yield \( \mathcal{Z}(C,E) (-G) = 0 \) and \( H = 0 \). Therefore,

\[
G + H = G + \mathcal{Z}(C,E) (-G) - \mathcal{Z}(C,E) (-G) + H
\]

is in \( W \). \( \square \)

5. Refined gradings and applications

Laurent expansions have many useful applications, such as providing much finer decompositions of \( \mathcal{M}_E(V^\circ \otimes \mathbb{C}) \) than the one in Corollary 4.18. As applications, we obtain the Brion-Vergne decomposition and the Jeffery-Kirwan residue of a class of meromorphic germs.

5.1. Decompositions of meromorphic germs at zero. Let \( (V, \Lambda_V) \) be a filtered lattice space.

**Definition 5.1.**

(a) For a polar germ \( \frac{h(t_1, \ldots, t_m)}{L_1^{1 \ell_1} \cdot \cdots \cdot L_n^{1 \ell_n}} \), we call \( s_1 + \cdots + s_n \) the **p-order** of the polar germ.

(b) We call the **supporting subspace** of a polar germ the subspace spanned by the supporting cone of the polar germ.

(c) For \( p \in \mathbb{Z}_{\geq 0} \), let \( \mathcal{M}_E^p(V^\circ \otimes \mathbb{C}) \) denote the linear span of \( \mathbb{F} \)-polar germs with p-order \( p \).

(d) For any \( \mathbb{F} \)-subspace \( U \subset V \), let \( \mathcal{M}_{E,U}(V^\circ \otimes \mathbb{C}) \) denote the linear span of \( \mathbb{F} \)-polar germs with supporting subspace \( U \).

(e) For \( d \in \mathbb{Z}_{\geq 0} \), let \( \mathcal{M}_{E,d}(V^\circ \otimes \mathbb{C}) \) denote the linear span of \( \mathbb{F} \)-polar germs whose supporting subspaces have dimension \( d \).

(f) For any \( \mathbb{F} \)-subspace \( U \subset V \) and \( p \in \mathbb{Z}_{\geq 0} \), let \( \mathcal{M}_{E,U,d}(V^\circ \otimes \mathbb{C}) \) denote the linear span of \( \mathbb{F} \)-polar germs with supporting subspace \( U \) and p-order \( p \).

**Remark 5.2.** With these notations, we have \( \mathcal{M}_{E,(0)}(V^\circ \otimes \mathbb{C}) = \mathcal{M}_{E,0}(V^\circ \otimes \mathbb{C}) = \mathcal{M}_{E,4}(V^\circ \otimes \mathbb{C}) \) for the trivial cone \( \{0\} \) and integer \( d = 0 \).
**Theorem 5.3.** We have the decompositions

\[
\mathcal{M}_F(V^\otimes \mathbb{C}) = \bigoplus_{p \geq 0} \mathcal{M}_F^p(V^\otimes \mathbb{C}),
\]

\[
\mathcal{M}_F(V^\otimes \mathbb{C}) = \bigoplus_{U \subset V} \mathcal{M}_{F,U}(V^\otimes \mathbb{C}),
\]

\[
\mathcal{M}_F(V^\otimes \mathbb{C}) = \bigoplus_{d \geq 0} \mathcal{M}_{F,d}(V^\otimes \mathbb{C}),
\]

\[
\mathcal{M}_F(V^\otimes \mathbb{C}) = \bigoplus_{U \subset V, p \in \mathbb{Z}_{\geq 0}} \mathcal{M}_{F,U}^p(V^\otimes \mathbb{C}).
\]

Eq. (36) yields the decomposition in [3, Theorem 7.3] corresponding to a sum running over the set of subspaces spanned by elements of the hyperplane of arrangements corresponding to the poles.

**Proof.** By Theorem 4.22, the kernel of the surjective linear map \( \varphi : \mathcal{M}_F(V^\otimes \mathbb{C}) \rightarrow \mathcal{M}_F(V^\otimes \mathbb{C}) \) is linearly spanned by elements each of which is a linear combination of polar germs with the same p-order, the same supporting subspace, the same dimension of the supporting subspace. Then the equations follow. \( \square \)

On the grounds of Theorem 5.3, we can give the following definitions.

**Definition 5.4.** Let \( U \) be an \( F \)-subspace of \((V, \Lambda_V)\) and \( p \in \mathbb{Z}_{\geq 0} \). Define

\[
P_U^p : \mathcal{M}_F(V^\otimes \mathbb{C}) \rightarrow \mathcal{M}_F^p(V^\otimes \mathbb{C}) \subset \mathcal{M}_F(V^\otimes \mathbb{C})
\]

and

\[
P_U : \mathcal{M}_F(V^\otimes \mathbb{C}) \rightarrow \mathcal{M}_{F,U}(V^\otimes \mathbb{C}) \subset \mathcal{M}_F(V^\otimes \mathbb{C})
\]

to be the projections, called the **projection of \( f \) onto the space \( U \) of p-order \( p \)** and **projection of \( f \) onto the space \( U \)** respectively.

For \( d \in \mathbb{Z}_{\geq 0} \), setting

\[
\mathcal{M}_{F,\leq d}(V^\otimes \mathbb{C}) := \bigoplus_{0 \leq k \leq d} \mathcal{M}_{F,k}(V^\otimes \mathbb{C}),
\]

\[
\mathcal{M}_{F,> d}(V^\otimes \mathbb{C}) := \bigoplus_{k > d} \mathcal{M}_{F,k}(V^\otimes \mathbb{C}),
\]

then we have

\[
\mathcal{M}_F(V^\otimes \mathbb{C}) = \mathcal{M}_{F,\leq d}(V^\otimes \mathbb{C}) \oplus \mathcal{M}_{F,> d}(V^\otimes \mathbb{C}).
\]

This yields back the decomposition of Corollary 4.18 if we take \( d = 0 \).

The decomposition in Eq. (39) also yields back Brion-Vergne’s decomposition [4, Theorem 1] as follows. Let \( \Delta \) be a finite subset of lattice vectors in some \( V \) with coefficients in \( F \). Let

\[
U := \text{span}(\Delta), \quad r := \text{dim}(U).
\]

The symmetric algebra \( S(U) \) (over \( \mathbb{C} \)) can be viewed as the algebra of polynomial functions on \( U^* \). Following the notation of [4], let us denote by

\[
R_{\Delta} := \Delta^{-1} S(U)
\]

the localization of \( S(U) \) with respect to \( \Delta \) which is naturally regarded as a subset of \( S(U) \). It corresponds to the algebra of rational functions with linear poles in \( \Delta \). A subset \( \kappa \subset \Delta \) is called
generating if the linear span of $\kappa$ is $U$, and it is called a basis if it is a basis of $U$. Consider the following subspace of $R_\Delta$:

$$S_\Delta := \text{span}\left\{ \frac{1}{\Pi_{\alpha \in \Delta}} | \kappa \subseteq \Delta \text{ bases of } U \right\},$$

$$G_\Delta := \text{span}\left\{ \frac{1}{\Pi_{\alpha \in \Delta}} | \kappa \subseteq \Delta \text{ generating subsets of } U, n_\alpha \in \mathbb{Z}_{>0} \right\},$$

$$NG_\Delta := \text{span}\left\{ \frac{h}{\Pi_{\alpha \in \Delta}} | \kappa \subseteq \Delta \text{ non-generating subsets of } U, n_\alpha \in \mathbb{Z}_{\geq 0}, h \in S(U) \right\}.$$  

Clearly,

$$\mathcal{M}_{\mathbb{F},>r^{-1}}(V^\otimes \mathbb{C}) \cap R_\Delta = G_\Delta; \quad \mathcal{M}_{\mathbb{F},\leq r^{-1}}(V^\otimes \mathbb{C}) \cap R_\Delta = NG_\Delta.$$  

Thus Eq. (39) recovers the following decomposition of $R_\Delta$ obtained by Brion-Vergne.

Corollary 5.5. [4, Theorem 1] There is a direct sum decomposition

$$R_\Delta = G_\Delta \oplus NG_\Delta.$$  

5.2. The generalized Jeffrey-Kirwan residue. The Jeffrey-Kirwan residue introduced in [17] (see also [18]) in the study of localization for nonabelian compact group actions, is a powerful tool to compute intersection numbers for symplectic quotients.

There are several ways to define the Jeffrey-Kirwan residue, namely using iterated residues, inverse Laplace transforms or nested sets [4, 7, 17, 18, 19, 26]. We will use Brion-Vergne’s presentation [4], which we briefly recall here.

Taking total degrees gives a grading on the space $R_\Delta = \oplus_{j \in \mathbb{Z}} R_\Delta[j]$ and $G_\Delta$ is contained in $R_\Delta[\leq -r] := \oplus_{j \leq -r} R_\Delta[j]$. Thus from Corollary 5.5 we obtain

$$R_\Delta[\leq -r] = G_\Delta \oplus (NG_\Delta \cap R_\Delta[\leq -r]).$$  

Furthermore $S_\Delta = G_\Delta[-r]$ is the highest degree part of $G_\Delta$, giving the decomposition

$$G_\Delta = G_\Delta[-r] \oplus S_\Delta.$$  

Consider the localization

$$\hat{R}_\Delta := \Lambda^{-1} \hat{S}(U)$$

of the ring $\hat{S}(U)$ of formal power series by inverting the linear functions $\alpha \in \Delta$ and the natural decomposition

$$\hat{R}_\Delta = \hat{R}_\Delta[> -r] \oplus R_\Delta[\leq -r].$$  

Putting Eqs. (41) – (43) together yields the decomposition

$$\hat{R}_\Delta = \hat{R}_\Delta[> -r] \oplus (NG_\Delta \cap R_\Delta[\leq -r]) \oplus G_\Delta[-r] \oplus S_\Delta.$$  

Definition 5.6. The Jeffrey-Kirwan residue map

$$\text{Res}_\Delta : \hat{R}_\Delta \to S_\Delta$$

is defined to be the projection to the direct summand $S_\Delta$ in Eq. (44).

Since the Jeffrey-Kirwan residue of a Laurent power series is defined by that of the corresponding truncated Laurent polynomial, for the sake of simplicity, we focus here on $R_\Delta$ which is a subspace of $\mathcal{M}_{\mathbb{F}}(U^* \otimes \mathbb{C})$, and the decomposition

$$R_\Delta = R_\Delta[> -r] \oplus (NG_\Delta \cap R_\Delta[\leq -r]) \oplus G_\Delta[-r] \oplus S_\Delta.$$  

analogous to Eq. (44).
**Corollary 5.7.** Let $U = \text{span}(\Delta)$ and $r = \dim U$. Then for any $f \in R_\Delta$, the projection $P_U^r(f)$ from Definition 5.4 is the Jeffrey-Kirwan residue of $f$.

**Proof.** Let $\prod_{\alpha \in \kappa} \alpha$ be a spanning fraction of $S_\Lambda$. Then $\kappa$ is a basis of $U$ and $\prod_{\alpha \in \kappa} \alpha$ has degree $r$. Thus the fraction is in $M_U^r(V^\otimes C)$ and hence is fixed by $P_U^r$. On the other hand, the supporting cone for a polar germ in $R_\Delta[>-r]$ or $(NG_\Delta \cap R_\Delta[\leq -r])$ does not span $U$, while the polar germs in $G_\Lambda[<-r]$ do not have p-order $r$. Hence the polar germs are annihilated by $P_U^r$. □

Motivated by this fact, we set the following definition.

**Definition 5.8.** For a meromorphic germ $f$, and an $F$-subspace $U$ of $V$, let $d = \dim U$, then $P_U^d(f)$ is called the generalized Jeffrey-Kirwan residue of $f$ supported on $U$.

6. A filtered residue and a coproduct

In this part, we give two further applications of our Laurent theory developed in Section 4. We study the p-order of a meromorphic germ at zero, and defined an invariant, called p-residue for the germ. We show that for exponential sums, taking p-residue amounts to the exponential integrals. We also define a coproduct on the space of meromorphic germs at zero with linear poles.

6.1. The p-order and p-residue. The grading in Eq. (35) by p-orders of polar germs gives a p-order for any elements in $M(V^\otimes C)$.

**Definition 6.1.** Let $f \in M(V^\otimes C)$. Let

$$L_c(f) = \oplus_{j \in J} S_j \oplus h$$

be a $C$- supported Laurent expansion of $f$ for some appropriate family of supporting cones $C$ as in Definition 4.5.

(a) Define the polar order, or p-order in short, of $f$ to be

$$\text{p-ord}(f) := \max_j (\text{p-ord}(S_j)),$$

where p-ord($S_j$) is from Definition 5.1.

(b) Let $S_j = \frac{h_j}{L_{i_1}^{a_1} \cdots L_{i_n}^{a_n}}$, $1 \leq i \leq t$, be the polar germs in Eq. (46) We define the highest polar order residue, or the p-residue in short, of $f$ to be

$$\text{p-res}(f) = \sum_{j=1}^t \frac{h_j(0)}{L_{i_1}^{a_1} \cdots L_{i_n}^{a_n}}.$$

These notions are well-defined thanks to the following property.

**Proposition 6.2.** The p-order and p-residue of a meromorphic germ with linear poles depend neither on the choice of a Laurent expansion nor on the choice of the inner product used in the decomposition of $M(V^\otimes C)$ in Theorem 4.15.

Furthermore, for orthogonally variate $f$ and $g$ in the sense of Definition 4.19, we have

$$\text{p-res}(fg) = \text{p-res}(f) \text{p-res}(g).$$

Before giving the proof, let us recall the following elementary yet useful result.
Lemma 6.3. Let $I$ be a direct system and let $\varphi$ be a function on $I$. If $\varphi(i) = \varphi(j)$ for all $i \leq j$ in $I$, then $\varphi$ is a constant.

Proof. (of Proposition 6.2) The independence of the p-order on the choice of a Laurent expansion follows from the grading in Eq. (35). From Eq. (29), the numerator of a polar germ and hence the p-residue of $f$ does not change under the subdivision map $\Xi(\xi, \zeta)$. Then the independence of the p-residue on the choice of a Laurent expansion follows from Lemma 6.3.

We next prove the independence of the p-order on the inner product. For an inner product $Q$ in $V$ and $f \in M(V_0 \otimes \mathbb{C})$ with $\text{p-ord}(f) = p$. Following Eq. (46), we write the Laurent expansion of $f$ supported by $C$ as

\begin{equation}
\mathcal{L}_C(f) = \sum_{i=1}^{r} S_i + \sum_{j=r+1}^{n} S_j + h,
\end{equation}

with the polar germs sharing the largest p-order $p$ grouped in the first sum and those with lesser p-order in the second sum.

Relative to a different inner product $R$ on $V$, an $S_i$ might not be a polar germ any longer. Set

\begin{equation}
S_i = \frac{h_i(\ell_{i1}, \ldots, \ell_{im})}{L_{i1}^{s_{i1}} \cdots L_{im}^{s_{im}}}
\end{equation}

with $Q(\ell_{ip}, L_{iq}) = 0$. For $j = 1, \ldots, m_i$, there are coefficients $a_{ij}$ such that

\begin{equation}
\ell_{ij} = \ell'_{ij} - \sum_{k=1}^{n_i} a_{ij} L_{ik},
\end{equation}

where $R(\ell'_{ij}, L_{ik}) = 0$ for $k = 1, \ldots, n_i$. Then

\begin{equation}
S_i = \frac{h_i(\ell'_{i1}, \ldots, \ell'_{im})}{L_{i1}^{s_{i1}} \cdots L_{im}^{s_{im}}} + \text{terms of lower denominator degrees}.
\end{equation}

Thus

\begin{equation}
f = \sum_{i=1}^{r} h_i(\ell'_{i1}, \ldots, \ell'_{im}) \frac{L_{i1}^{s_{i1}} \cdots L_{im}^{s_{im}}}{L_{i1}^{s_{i1}} \cdots L_{im}^{s_{im}}} + \text{terms of lower denominator degrees}.
\end{equation}

This gives a decomposition of $f$ as a linear combination of polar germs for the inner product $R$.

The supporting cones from the right hand side of the above equation are easily seen to be faces of the supporting cones in the decomposition of $f$ under the inner product $Q$ arising in Eq. (47). So they remain properly positioned. Since $h_i(\ell_{i1}, \ldots, \ell_{im}) \neq 0$, we also have $h_i(\ell'_{i1}, \ldots, \ell'_{im}) \neq 0$. Therefore under the inner product $R$, the p-order of $f$ is again $p$.

Furthermore, Eq. (48) shows how the polar germs of p-order $\text{p-ord}(f)$ change for a different inner product. In particular the constant terms of the numerators remain the same. Thus the p-residue does not depend on the choice of inner products.

The second statement follows from the fact that the product of a polar germ with either a polar germ or a holomorphic germ which are orthogonally variate is again a polar germ. Thus the highest polar order part of $fg$ is the the product of the highest polar order part of $f$ and that of $g$.

Remark 6.4. The proof of this proposition actually shows how the terms of p-order $p$ change as the inner products change.
To simplify the notation, for a polar germ \( S = \frac{h(L_{1}\cdots L_{k})}{L_{1}\cdots L_{k}} \), we set
\[
S(0) := \frac{h(0)}{L_{1}\cdots L_{k}}.
\]

**Proposition 6.5.** Let \( f = \sum_{i} S_{i} + \sum_{j} T_{j} + h \), with \( S_{i} \), \( T_{j} \) polar germs at zero, \( h \) a holomorphic germ at zero, \( p\text{-ord}(S_{i}) \)'s all equal to \( r \), \( \sum_{i} S_{i} \neq 0 \) and \( p\text{-ord}(T_{j}) < k \). Then \( p\text{-ord}(f) = r \) and \( p\text{-res}(f) = \sum_{i} S_{i}(0) \).

**Proof.** Taking a subdivision of the set of supporting cones of the germs \( S_{i} \)'s and \( T_{j} \)'s, we have \( S_{i} = \sum_{it} S_{it} \) and \( T_{j} = \sum_{jm} T_{jm} \). Then \( f = \sum_{it} S_{it} + \sum_{jm} T_{jm} + h \). Combining terms that are proportional to one another, we can assume that this decomposition satisfies the conditions in Theorem 4.15. In this decomposition there are no terms of \( p \)-order greater than \( r \) and the sum of all the terms of \( p \)-order \( r \) is \( \sum_{i} S_{it} = \sum_{i} S_{i} \neq 0 \). Thus \( p\text{-ord}(f) = k \) and \( p\text{-res}(f) = \sum_{it} S_{it}(0) = \sum_{i} S_{i}(0) \). \( \Box \)

### 6.2. The \( p \)-residue of the exponential sum on a lattice cone

As in \([13]\), we can reinterpret the constructions of \([2, 10, 21]\) in terms of lattice cones.

We recall from \([13]\) that a **lattice cone** in \( V_{k} \) is a pair \((C, \Lambda_{C})\) with \( C \) a cone in \( V_{k} \) and \( \Lambda_{C} \) a lattice in \( \text{lin}(C) \) generated by lattice vectors. A lattice cone \((C, \Lambda_{C})\) is called **strongly convex** (resp. **simplicial**) if \( C \) is. A lattice cone \((C, \Lambda_{C})\) is called **smooth** if the additive monoid \( \Lambda_{C} \cap C \) has a monoid basis. In other words, there are linearly independent lattice vectors \( v_{1}, \cdots, v_{\ell} \) such that \( \Lambda_{C} \cap C = \mathbb{Z}_{\geq 0}\{v_{1}, \cdots, v_{\ell}\} \).

To a lattice cone \((C, \Lambda_{C})\) we can assign two meromorphic functions. One is the exponential sum \( S(C, \Lambda_{C}) \) \([1]\) (corresponding to \( S'(C, \Lambda_{C}) \) in \([13]\)), given in the strongly convex case by
\[
S(C, \Lambda_{C})(\vec{\varepsilon}) := \sum_{\vec{n} \in (C\cap \Lambda_{C})} e^{i\langle \vec{n}, \vec{\varepsilon} \rangle}.
\]

The other function is the exponential integral \( I(C, \Lambda_{C}) \) \([13]\), which is a generalization of Eq. (7), where the matrix \( A_{C} \) is with respect to a basis of \( \Lambda_{C} \).

**Lemma 6.6.** For a smooth lattice cone \((C, \Lambda_{C})\), we have
\[
p\text{-ord}(S(C, \Lambda_{C})) = p\text{-ord}(I(C, \Lambda_{C})) = \dim(C), \quad p\text{-res}(S(C, \Lambda_{C})) = I(C, \Lambda_{C}).
\]

In fact, we have
\[
S(C, \Lambda_{C}) = I(C, \Lambda_{C}) + \text{(terms of } p\text{-order } < \dim(C)).
\]

**Proof.** Let \( v_{1}, \cdots, v_{d} \) (where \( d = \dim(C) \)) be a basis of \( \Lambda_{C} \) that generates \( C \) as a cone. Then
\[
S(C, \Lambda_{C})(\vec{\varepsilon}) = \prod_{i=1}^{d} \frac{1}{1 - e^{i\langle \vec{v}_{i}, \vec{\varepsilon} \rangle}} = \prod_{i=1}^{d} \left( 1 - \frac{1}{\langle \vec{v}_{i}, \vec{\varepsilon} \rangle} + h(i\langle \vec{v}_{i}, \vec{\varepsilon} \rangle) \right),
\]
where \( h \) is holomorphic. So the highest \( p \)-order term is \( \prod_{i=1}^{d} \left( -\frac{1}{\langle \vec{v}_{i}, \vec{\varepsilon} \rangle} \right) \) which is \( I(C, \Lambda_{C}) \) and has \( p \)-order \( d \). \( \Box \)

**Lemma 6.7.** For a lattice cone \((C, \Lambda_{C})\),
\[
I(C, \Lambda_{C}) \neq 0 \iff S(C, \Lambda_{C}) \neq 0 \iff C \text{ is strongly convex}.
\]
Proof. We already know that \( I(C, \Lambda_C) = 0 \) and \( S(C, \Lambda_C) = 0 \) if \( C \) is not strongly convex [13]. So
we only need to prove that if \( C \) is strongly convex, then \( I(C, \Lambda_C) \neq 0 \) and \( S(C, \Lambda_C) \neq 0 \).

Take a smooth subdivision \( \{C_i\} \) of \( C \). Since \( C \) is strongly convex, \( \{C_i\} \) is properly positioned. So the \( I(C_i, \Lambda_C) \)'s are linearly independent by the Non-holomorphicity Theorem 3.6. Then \( I(C, \Lambda_C) \), as their sum, can not be 0.

Further, note that

\[
S(C, \Lambda_C) = \sum_i S(C_i, \Lambda_C) + (\text{terms with p-order < dim}(C))
= \sum_i S(C_i, \Lambda_C) = I(C_i, \Lambda_C) + (\text{terms with p-order < dim}(C)).
\]

By Proposition 6.5, \( p\text{-ord}(S(C, \Lambda_C)) < \text{dim}(C) \) implies \( \sum_i I(C_i, \Lambda_C) = 0 \), which is a contradiction. Then \( p\text{-ord}(S(C, \Lambda_C)) = \text{dim}(C) \) and so \( S(C, \Lambda_C) \neq 0 \). \( \Box \)

**Proposition 6.8.** For any subdivision \( \{(C_i, \Lambda_C)\} \) of a lattice cone \( (C, \Lambda_C) \), we have

\[
p\text{-res}(S(C, \Lambda_C)) = \sum_i p\text{-res}(S(C_i, \Lambda_C)).
\]

So the map \( p\text{-res} \circ S \) is compatible with subdivisions.

Proof. It is sufficient to consider a subdivision \( \{(C_i, \Lambda_C)\} \) of \( (C, \Lambda_C) \) with smooth cones \( C_i \), since any other subdivision can be further subdivided into one containing only smooth cones. It follows from the definition of \( S(C, \Lambda_C) \) that

\[
S(C, \Lambda_C) = \sum_{I \subseteq [r]} (-1)^{|I|+1} S(C_I, \Lambda_I) = \sum_i S(C_i, \Lambda_C) + (\text{terms of p-order < dim}(C)),
\]

where \( C_I := \cap_{i \in I} C_i \). Also by Lemma 6.6

\[
S(C_i, \Lambda_C) = \sum_j T_{ij} + (\text{terms of p-order < dim}(C)),
\]

where \( T_{ij} \) are polar germs at zero with \( p\text{-ord}(T_{ij}) = \text{dim}(C) \). Thus

\[
S(C, \Lambda_C) = \sum_{i,j} T_{ij} + (\text{terms of p-order < dim}(C)).
\]

If \( C \) is strongly convex, then \( p\text{-ord}(S(C, \Lambda_C)) = \text{dim}(C) \), and by Proposition 6.5,

\[
p\text{-res}(S(C, \Lambda_C)) = \sum_{i,j} T_{ij}(0) = \sum_i p\text{-res}(S(C_i, \Lambda_C)).
\]

If \( C \) is not strongly convex, then \( S(C, \Lambda_C) = 0 \); while by Proposition 6.5, this means \( \sum_{i,j} T_{ij} = 0 \), that is \( \sum_i p\text{-res}(S(C_i, \Lambda_C)) = 0 \). So the equality in the theorem holds in either case. \( \Box \)

Note that the operator \( I \) is also compatible with subdivisions [13]. Thus as a consequence of Proposition 6.8, we obtain

**Corollary 6.9.** For a lattice cone \( (C, \Lambda_C) \), we have \( p\text{-res}(S(C, \Lambda_C)) = I(C, \Lambda_C) \).

**Example 6.10.** Take \( \Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2 \) and \( C = \langle e_1, e_1 + e_2 \rangle \) with \( (e_1, e_2) \) the canonical orthonormal basis in \( \mathbb{R}^2 \). Then \( S^C(C, \Lambda_C) = \frac{1}{(1-e^{1_1})(1-e^{1_1+e_2})} \) has p-order 2 and p-residue \( I(C, \Lambda_C) = \frac{1}{e_1(e_1+e_2)} \).
6.3. **Further perspectives: a coproduct on meromorphic germs.** As an outlook for future study, we end the paper with some observations on a coproduct on \( \mathcal{M}(V^\otimes \otimes \mathbb{C}) \) derived from our Laurent theory of meromorphic germs at zero.

Let \( f \) be in \( \mathcal{M}(V^\otimes \otimes \mathbb{C}) \) with a Laurent expansion \( f = \sum_{i=1}^{N} S_i + h \) supported by a properly positioned family \( \mathcal{C} \). Let \( S_i = \frac{h}{E_i^q} \) and \( h_0 := h \). Define

\[
\Delta_{L}(f) := \sum_{i=1}^{N} h_i \otimes \frac{1}{E_i^q}.
\]

Using the compatibility with subdivisions, we see that \( \Delta_{L}(f) \) does not depend on the choice of Laurent expansions of \( f \) and so it defines a map

\[
\Delta_{M(V^\otimes \otimes \mathbb{C})} : \mathcal{M}(V^\otimes \otimes \mathbb{C}) \rightarrow \mathcal{M}(V^\otimes \otimes \mathbb{C}) \otimes \mathcal{M}(V^\otimes \otimes \mathbb{C}).
\]

Furthermore, since

\[
(\text{id} \otimes \Delta_{M(V^\otimes \otimes \mathbb{C})}) \circ \Delta_{M(V^\otimes \otimes \mathbb{C})}(f) = \sum_{i} h_i \otimes 1 \otimes \frac{1}{E_i^q} = (\Delta_{M(V^\otimes \otimes \mathbb{C})} \otimes \text{id}) \circ \Delta_{M(V^\otimes \otimes \mathbb{C})}(f),
\]

\( \Delta_{M(V^\otimes \otimes \mathbb{C})} \) is coassociative. Thus \( \Delta_{M(V^\otimes \otimes \mathbb{C})} \) defines a coproduct on \( \mathcal{M}(V^\otimes \otimes V) \). This coproduct is not compatible with the multiplication \( m_{M(V^\otimes \otimes \mathbb{C})} \) on \( \mathcal{M}(V^\otimes \otimes V) \), so we do not have a bialgebra. However, we do have the compatibility property

\[
(49) \quad m_{M(V^\otimes \otimes \mathbb{C})} \circ \Delta_{M(V^\otimes \otimes \mathbb{C})} = \text{id}_{M(V^\otimes \otimes \mathbb{C})}.
\]

We compare this coproduct with the coproduct on cones [14], especially in the context of renormalization à la Connes and Kreimer [5] who regarded a renormalized map as a map defined on a coalgebra and taking values in meromorphic functions. So fix a linear map

\[
\phi : \mathbb{Q} \mathbb{C} \rightarrow \mathcal{M}(V^\otimes \otimes \mathbb{C})
\]

and assume that the inner product to construct the coproduct in \( \mathbb{Q} \mathbb{C} \) coincides with the inner product used to define polar germs. Taking \( P \) to be the projection to \( M_+(V^\otimes \otimes \mathbb{C}) \), then Algebraic Birkhoff Factorization [13, 14] yields \( \phi = \phi_+ \star \phi_- \) for certain linear maps \( \phi_{\pm} : \mathbb{Q} \mathbb{C} \rightarrow M_\pm(V^\otimes \otimes \mathbb{C}) \). These maps fit in the following diagram:

\[
\begin{array}{ccc}
\mathbb{Q} \mathbb{C} & \xrightarrow{\phi} & \mathcal{M}(V^\otimes \otimes \mathbb{C}) \\
\Delta_{L} & \downarrow & \Delta_{M(V^\otimes \otimes \mathbb{C})} \\
\mathbb{Q} \mathbb{C} \otimes \mathbb{Q} \mathbb{C} & \rightleftharpoons & \mathcal{M}(V^\otimes \otimes \mathbb{C}) \otimes \mathcal{M}(V^\otimes \otimes \mathbb{C}) \\
\phi_+ \otimes \phi_- & \downarrow & \Delta_{M(V^\otimes \otimes \mathbb{C})}
\end{array}
\]

The commutativity of this diagram for a given \( \phi \) provides an alternative to the Algebraic Birkhoff Decomposition, without going through the coproduct of cones. This should be the case under suitable conditions, for example when the inner product used to construct the coproduct in \( \mathbb{Q} \mathbb{C} \) coincides with the inner product used to define polar germs, and when \( \phi \) is the exponential sum \( S(C, \Lambda_C) \), in which case one would recover the Euler-Maclaurin formula in [3, Theorem 7.15] and [13, Theorem 4.10 and Corollary 4.11].

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