NON-EXISTENCE OF TRANSLATION-INVARIANT DERIVATIONS ON ALGEBRAS OF MEASURABLE FUNCTIONS

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Abstract. Let $S(0,1)$ be the $*$-algebra of all classes of Lebesgue measurable functions on the unit interval $(0,1)$ and let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a complete symmetric $\Delta$-normed $*$-subalgebra of $S(0,1)$, in which simple functions are dense, e.g., $L_\infty(0,1)$, $L_{\text{log}}(0,1)$, $S(0,1)$ and the Arens algebra $L^\infty(0,1)$ equipped with their natural $\Delta$-norms. We show that there exists no non-trivial derivation $\delta : \mathcal{A} \rightarrow S(0,1)$ commuting with all dyadic translations of the unit interval. Let $\mathcal{M}$ be a type II (or I$_\infty$) von Neumann algebra, $\mathcal{A}$ be an arbitrary abelian von Neumann subalgebra of $\mathcal{M}$, let $S(\mathcal{M})$ be the algebra of all measurable operators affiliated with $\mathcal{M}$. We show that there exists no non-trivial derivation $\delta : \mathcal{A} \rightarrow S(\mathcal{A})$ which admits an extension to a derivation on $S(\mathcal{M})$. In particular, we answer an untreated question in [8].

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1. Introduction. Let $A$ be an algebra over the field of complex numbers. A linear operator $\delta : A \to A$ is called a derivation if $\delta$ satisfies the Leibniz rule, i.e., $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$. The theory of derivations is an important and well studied part of the general theory of operator algebras, with significant applications in mathematical physics (see, e.g., [13], [40]). The development of non-commutative integration theory initiated in [41] has brought about new classes of (not necessarily Banach) algebras of unbounded operators, which by their algebraic and order-topological structure are still somewhat similar to $C^*$, $W^*$ and $AW^*$-algebras. Special importance here is attached to the algebras $S(M)$ ($S(M, \tau)$) of all measurable ($\tau$-measurable, respectively) operators affiliated with a von Neumann algebra $M$ with a faithful normal semifinite trace $\tau$. The two classes certainly agree in the case of a finite von Neumann algebra equipped with a faithful normal tracial state, but not in general (see for example [20, Theorem 2.46(5)]. In the classical case, when $M = L_\infty(0, 1)$, the algebra $S(M)$ coincides with the familiar space $S(0, 1)$ of all (classes of) measurable functions on $(0, 1)$. This development has naturally led to the question concerning the description of derivations on algebras $S(M)$ and their properties. One of the most important examples of derivations is the usual differential operator $\frac{d}{dt}$ on the algebra $D(0, 1)$ consisting of all classes in $S(0, 1)$ which contain functions having finite derivative almost everywhere in $(0, 1)$. In [6] (see also [5]), the problem of existence of non-trivial derivations in the setting of von Neumann regular commutative algebras was considered. As an application, it was established in [6] that the algebra $S(0, 1)$ of all Lebesgue measurable complex functions (with identification $m$-a.e.) on the interval $(0, 1)$ admits non-trivial derivations [6, Theorem 3.1] (see also [32, Remark 6.3] for an alternative proof), where $m$ is the Lebesgue measure. In particular, it is established in [6, Theorem 3.1] that there exist derivations on the algebra $S(0, 1)$ of all classes of measurable functions on $(0, 1)$ which extend the unbounded derivation $\frac{d}{dt}$ on the algebra $D(0, 1)$. A natural question is

what properties of $\frac{d}{dt}$ on $D(0, 1)$ are shared by its extension?

A very important property of $\frac{d}{dt}$ is the translation-invariance property, which has been widely studied since the 1970s. In particular, S. Sakai [39, Proposition 1.17] proved that a closed derivation on $C(\mathbb{T})$ commuting with translations by elements of $\mathbb{T}$ is a constant multiple of $\frac{d}{dt}$, where $\mathbb{T}$ is the one-dimensional torus.

Let $x \in S(0, 1)$. Set

\[
(\alpha_n(x))(t) = x \left( \left\{ t - \frac{1}{2^n} \right\} \right), \quad n \in \mathbb{N},
\]

where \{t\} stands for the fractional part of the number $t$. Note \{\alpha_n\} generates the group $G$ of dyadic-rational translations of $S(0, 1)$ (see Section 3.2 for the definition), which is a subgroup in the group $Aut(S(0, 1))$ of all automorphisms of $S(0, 1)$.

It is well-known (see Section 2.3) that the (approximately-)differential operator $\frac{d}{dt}$ (respectively, $\partial_{AD}$) is translation-invariant on $D(0, 1)$ (respectively, the algebra $AD(0, 1)$ of all classes of approximately differentiable functions on $(0, 1)$). We are interested in studying the translation-invariance property of derivations on
the larger algebra $S(0,1)$:

$$
\begin{align*}
\alpha_n & \quad \Omega \\
D(0,1) \subset AD(0,1) \subset S(0,1) \\
\frac{d}{dt} & \subset \partial_{AD} \subset \delta \\
S(0,1) = S(0,1) \subset S(0,1)
\end{align*}
$$

where $\delta|_{AD(0,1)} = \partial_{AD}$ and $\partial_{AD}|_{D(0,1)} = \frac{d}{dt}$.

The main result of the present paper is an interesting property of derivations on $S(0,1)$, which shows that non-trivial derivations on $S(0,1)$ do not commute with all $\alpha_n$. This is in strong contrast with the result by Sakai[39].

**Theorem 1.1.** Let $(\mathcal{A}, \|\cdot\|_\mathcal{A})$ be a complete symmetric $\Delta$-normed $*$-subalgebra of $S(0,1)$ in which simple functions are dense in the $\|\cdot\|_\mathcal{A}$-norm topology. Let $\delta$ be a derivation from $\mathcal{A}$ into $S(0,1)$ commuting with all $\alpha_n$, $n \in \mathbb{N}$. Then $\delta$ is trivial. In particular, the approximately differential operator $\partial_{AD}$ has no translation-invariant extension as a derivation on the algebra $S(0,1)$.

In [8], a noncommutative analogue $AD(\mathcal{R})$ of the algebra $AD(0,1)$ for the hyperfinite type $II_1$ factor $\mathcal{R}$ was introduced (all necessary definitions can be found in Section 2.3 below or in [8]). It was also established that the classical approximately differential operator on the algebra $AD(0,1)$ admits an extension to a derivation $\delta$ from $AD(\mathcal{R})$ into $S(\mathcal{R})$ with $\delta|_{AD(0,1)} = \partial_{AD}$, where $AD(0,1)$ can be viewed as a subalgebra of $S(\mathcal{R})$ (see Section 2.3 below or [8]).

In [8], the question whether the approximately differential operator $\partial_{AD}$ on $AD(\mathcal{R})$ has an extension to the algebra $S(\mathcal{R})$ was left unanswered. Now using the main result of the paper [9], we are able to answer this question.

**Proposition 1.2.** Let $\mathcal{M}$ be a type II (or $I_\infty$) von Neumann algebra, $\mathcal{A}$ be an arbitrary abelian von Neumann subalgebra of $\mathcal{M}$. Suppose that $\delta$ is a derivation on $S(\mathcal{M})$ such that the range $\delta|_{\mathcal{P}(\mathcal{A})}$ of the projection lattice $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ is contained in $S(\mathcal{A})$. Then $\delta$ vanishes on $S(\mathcal{A})$. In particular,

(i). if a derivation $\delta : \mathcal{A} \to S(\mathcal{A})$ can be extended to a derivation on $S(\mathcal{M})$, then $\delta$ is trivial;

(ii). there exists no derivation $\delta : S(\mathcal{R}) \to S(\mathcal{R})$ such that $\delta|_{AD(0,1)} = \partial_{AD}$, where $\mathcal{R}$ is the type $II_1$ hyperfinite factor.

Observe that item (i) of the above proposition is in strong contrast with the main result in [6] for commutative algebras, showing that $\frac{d}{dt} : D(0,1) \to S(0,1)$ admits an extension to the whole algebra $S(0,1)$.
2. Preliminaries. In this section, we briefly list some necessary facts concerning algebras of measurable operators.

Let $H$ be a Hilbert space and let $B(H)$ be the $*$-algebra of all bounded linear operators on $H$. A von Neumann algebra $\mathcal{M}$ is a weakly closed unital $*$-subalgebra in $B(H)$. For details on von Neumann algebra theory, the reader is referred to [15, 27, 30, 42, 43]. General facts concerning measurable operators may be found in [35, 41] (see also [44, Chapter IX] and the forthcoming book [17]). For convenience of the reader, some of the basic definitions are recalled below.

2.1. Murray-von Neumann algebras. Let $\mathcal{M}$ be a semifinite von Neumann algebra. A densely defined closed linear operator $x: \text{dom}(x) \to H$ (here the domain $\text{dom}(x)$ of $x$ is a linear subspace in $H$) is said to be affiliated with $\mathcal{M}$ if $yx \subset xy$ for all $y$ from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$.

Denote by $P(\mathcal{M})$ the set of all projections in $\mathcal{M}$. Recall that two projections $e, f \in P(\mathcal{M})$ are called equivalent (denoted by $p \sim q$) if there exists an element $u \in \mathcal{M}$ such that $u^*u = e$ and $uu^* = f$. For projections $e, f \in \mathcal{M}$, the notation $e \leq f$ means that there exists a projection $q \in \mathcal{M}$ such that $e \sim q \leq f$. A projection $p \in \mathcal{M}$ is called finite, if the conditions $q \leq p$ and $q$ is equivalent to $p$ imply that $q = p$. If the unit $1$ of the von Neumann algebra $\mathcal{M}$ is a finite projection in $\mathcal{M}$, then $\mathcal{M}$ is called finite.

A linear operator $x$ affiliated with $\mathcal{M}$ is called measurable with respect to $\mathcal{M}$ if $e^{[x]}(\lambda, \infty)$ is a finite projection\(^1\) for some $\lambda > 0$. Here $e^{[x]}(\lambda, \infty)$ is the spectral projection of $|x|$ corresponding to the interval $(\lambda, +\infty)$.

The development of non-commutative integration theory was initiated by Murray and von Neumann [34] and by Segal [41], who introduced new classes of (not necessarily Banach) algebras of unbounded operators, in particular the algebra $S(\mathcal{M})$ of all measurable operators affiliated with a von Neumann algebra $\mathcal{M}$. The specific interest of the study of $S(\mathcal{M})$ when $\mathcal{M}$ is a $II_1$ von Neumann algebra, is also recorded in von Neumann’s talk at the International Congress of Mathematicians, Amsterdam, 1954 [37, pp. 231–246]. In the special case when $\mathcal{M}$ is a finite von Neumann algebra, the algebra $S(\mathcal{M})$ of all densely defined closed operators affiliated with $\mathcal{M}$ is frequently referred as the Murray-von Neumann algebra associated with $\mathcal{M}$, which is the algebra of all densely defined closed operators affiliated with $\mathcal{M}$ (see e.g. [28, 29]).

Let $x, y \in S(\mathcal{M})$. It is well known that $x + y$ and $xy$ are densely-defined and preclosed operators. Moreover, the (closures of) operators $x + y, xy$ and $x^*$ are also in $S(\mathcal{M})$. When equipped with these operations, $S(\mathcal{M})$ becomes a unital $*$-algebra over $\mathbb{C}$ (see [14]). It is clear that $\mathcal{M}$ is a $*$-subalgebra of $S(\mathcal{M})$.

From now on, we shall always assume that $\mathcal{M}$ is a finite von Neumann algebra equipped with a faithful normal finite trace $\tau$. Consider the measure topology $t_\tau$ on $S(\mathcal{M})$, which is defined by the following neighborhoods of zero:

$$N(\varepsilon, \delta) = \{x \in S(\mathcal{M}) : \exists \varepsilon \in P(\mathcal{M}), \tau(1 - e) \leq \delta, xe \in \mathcal{M}, \|xe\|_\infty \leq \varepsilon\},$$

where $\varepsilon, \delta$ are positive numbers, $1$ is the unit in $\mathcal{M}$ and $\|\cdot\|_\infty$ denotes the operator

\(^1\)Note that $e^{[x]}(\lambda, \infty)$ is not necessarily a $\tau$-finite projection.
norm on $\mathcal{M}$. The algebra $S(\mathcal{M})$ equipped with the measure topology is a topological algebra.

Let $x \in S(\mathcal{M})$ and let $x = v|x|$ be the polar decomposition of $x$ [16, 33]. Then $l(x) = vv^*$ and $r(x) = v^*v$ are left and right supports of the element $x$, respectively.

We define the so-called rank metric $\rho$ on $S(\mathcal{M})$ by setting

$$\rho(x, y) = \tau(r((x - y))) = \tau(l(x - y)), \quad x, y \in \mathcal{A}.$$  

In fact, the rank-metric $\rho$ was firstly introduced in a general case of regular rings by von Neumann in [36], where it was shown that $\rho$ is a metric on $S(\mathcal{M})$. By [14, Proposition 2.1], the algebra $S(\mathcal{M})$ equipped with the metric $\rho$ is a complete topological ring. We note that if $\{x_n\}_{n=1}^{\infty}$ is a sequence of self-adjoint operators in $S(\mathcal{M})$ having pairwise orthogonal supports, then $\sum_{n=1}^{\infty} x_n$ exists in the topology induced by $\rho$ and also in measure. In the special case when $\mathcal{M} = L^2(0, 1)$, the metric $\rho$ on the regular algebra $S(\mathcal{M}) = S(0, 1)$ is the same as in [6, 9].

2.2. Symmetrically $\Delta$-normed spaces of measurable functions. For convenience of the reader, we recall the definition of $\Delta$-norms. Let $E$ be a linear space over the field $\mathbb{C}$. A function $\|\cdot\|$ from $E$ to $\mathbb{R}$ is a $\Delta$-norm, if for all $x, y \in E$ the following properties hold:

$$\begin{align*}
\|x\| &\geq 0, \quad \|x\| = 0 \iff x = 0; \\
\|\alpha x\| &\leq \|x\|, \quad \forall \alpha \in \mathbb{C}, |\alpha| \leq 1; \\
\lim_{\alpha \to 0} \|\alpha x\| & = 0; \\
\|x + y\| &\leq C_E \cdot (\|x\| + \|y\|)
\end{align*}$$

for a constant $C_E \geq 1$ independent of $x, y$. The couple $(E, \|\cdot\|)$ is called a $\Delta$-normed space. We note that the definition of a $\Delta$-norm given above is the same as in [31]. It is well-known that every $\Delta$-normed space $(E, \|\cdot\|)$ is metrizable and conversely every metrizable topological linear space can be equipped with a $\Delta$-norm [31, p. 5]. In particular, when $C_E = 1$, $E$ is called an $F$-normed space [31, p. 3]. We note that every $\Delta$-norm has an equivalent $F$-norm [31, Chapter 1.2]. We say that a $\Delta$-norm $\|\cdot\|_\mathcal{A}$ on a subspace $\mathcal{A}$ of $S(0, 1)$ is invariant with respect to translations, if for any translation $\alpha$ on $[0, 1)$, we have $\alpha(\mathcal{A}) \subseteq \mathcal{A}$ (i.e., $\mathcal{A}$ is translation-invariant) and $\|\alpha(x)\|_\mathcal{A} = \|x\|_\mathcal{A}$.

We now come to the definition of the main object of this paper.

**Definition 2.1.** Let $\mathcal{E}$ be a linear subspace in $S(0, 1)$ equipped with a $\Delta$-norm $\|\cdot\|_\mathcal{E}$. We say that $\mathcal{E}$ is a symmetrically $\Delta$-normed space if for $x \in \mathcal{E}$, $y \in S(0, 1)$ and $\mu(y) \leq \mu(x)$ imply that $y \in \mathcal{E}$ and $\|y\|_\mathcal{E} \leq \|x\|_\mathcal{E}$ [3, 22, 21]. Here, $\mu(f)$ stands for the decreasing rearrangement of $f \in S(0, 1)$ [33, 16].

Clearly, symmetric $\Delta$-norms are invariant with respect to translation. We note that convergence in the topology induced by any symmetric $\Delta$-norm implies convergence in the measure topology on $S(0, 1)$ [3, 22, 21]. It is also known [3, 22] that any symmetric $\Delta$-normed space contains all simple functions in $S(0, 1)$. In particular, if the symmetric $\Delta$-norm is order continuous (see [23] for the definition), then
all simple functions are dense in this symmetrically Δ-normed space [23, Remark 2.9].

It is well-known [22, 24, 3] that the *-algebra $S(0, 1)$ can be equipped with a complete symmetric Δ-norm. Indeed, by defining that

$$\|X\|_S = \inf_{t > 0} [t + \mu(t; x)], \ X \in S(0, 1),$$

we obtain a symmetric $F$-norm $\|\cdot\|_S$ on $S(0, 1)$ [22, Remark 3.4] (indeed, the constant for the quasi-triangle inequality is 1). Moreover, the topology induced by $\|\cdot\|_S$ is equivalent to the measure topology [22, Proposition 4.1]. Important examples of subalgebras of $S(0, 1)$, such as $L_\infty(0, 1)$, $L_{\log}(0, 1)$ [18] and the Arens algebra $L^\omega(0, 1)$ [2], can be equipped with complete symmetric $F$-norms.

### 2.3. Approximately differentiable functions.

Let us recall the concept of approximately differentiable functions. Consider a Lebesgue measurable set $E \subset \mathbb{R}$, a measurable function $f : E \to \mathbb{R}$ and a point $t_0 \in E$, where $E$ has Lebesgue density equal to 1. If the approximate limit

$$f'_{ap}(t_0) := \text{ap} - \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

exists and it is finite, then it is called approximate derivative of the function $f$ at $t_0$ and the function is called approximately differentiable at $t_0$ (see [19] for the details). We note that all simple functions on $(0, 1)$ are approximately differentiable. However, it is clear that simple functions are not dense in $(AD(0, 1), \rho)$.

**Remark 2.1.** Let $AD(0, 1)$ be the set of all classes $[f] \in S(0, 1)$, for which $f$ have finite approximate derivatives almost everywhere in $(0, 1)$ (for simplicity, we denote $[f]$ by $f$). We note that the algebra $AD(0, 1)$ is the $\rho$-completion of the subalgebra in $S(0, 1)$, generated by the algebra $C^{(1)}(0, 1)$ of all (classes of) continuously differentiable functions on $(0, 1)$ and by the algebra of all simple functions on $(0, 1)$. Moreover, for any $x \in AD(0, 1)$, there exist a partition of the unit $\{\chi_{A_n}\}_{n \geq 1}$ and a sequence $\{x_n\}_{n \geq 1}$ in $C^{(1)}(0, 1)$ such that $\chi_{A_n} x = \chi_{A_n} x_n$ for all $n \geq 1$ [8, Proposition 4.7].

Since the differential operator $\frac{d}{dt}$ commutes with all dyadic-rational translations of the unit interval $(0, 1)$, it follows that the approximately differential operator $\partial_{AD} : f \mapsto f'_{ap}$ also commutes with all $\alpha_n$, that is,

$$\partial_{AD} \circ \alpha_n = \alpha_n \circ \partial_{AD}.$$
view $S(D) = S(0, 1)$ as a $\ast$-subalgebra of $S(\mathcal{R})$ [8]. It is established in [9] that the classical approximately differential operator on $AD(0, 1)$ admits an extension to a derivation $\delta$ from $AD(\mathcal{R})$ into $S(\mathcal{R})$ with $\delta|_{AD(0, 1)} = \delta_{AD}$.

3. Translation-invariance of derivations. Let $\mathcal{A}$ be a $\ast$-algebra and let $\delta$ be a derivation on $\mathcal{A}$. Set

$$\delta_1(x) = \frac{\delta(x) + \delta(x^*)}{2}, \quad x \in \mathcal{A}$$

and

$$\delta_2(x) = \frac{\delta(x) - \delta(x^*)}{2i}, \quad x \in \mathcal{A}.$$ 

Then, $\delta_1, \delta_2$ are $\ast$-derivations (that is, $\delta_1(x^*) = \delta_1(x)^*$ and $\delta_2(x^*) = \delta_2(x)^*$) and $\delta = \delta_1 + i\delta_2$. Without loss of generality, from now on, we may assume that all derivations in this section are $\ast$-derivations.

3.1. The lack of extension of the differential operator $\frac{d}{dt}$ up to $S(\mathcal{R})$. Each element $a$ in an algebra $\mathcal{A}$ implements a derivation $ad(a)$ on $\mathcal{A}$ defined as

$$ad(a)(x) = [a, x] = ax - xa, \quad x \in \mathcal{A}.$$ 

Such derivations $ad(a)$ are called inner derivations. For a detailed exposition of the theory of derivations on operator algebras we refer to the monograph of Sakai [40].

It is known that every derivation on a von Neumann algebra $\mathcal{M}$ is necessarily inner [25, 38] (see [11, 12] for more general results for derivations with values into ideals of a von Neumann algebra). However, the properties of derivations of the algebra $S(\mathcal{M})$ are far from being similar to those exhibited by derivations on von Neumann algebras $\mathcal{M}$. In [4], Ayupov asked for a full description of derivations on $S(\mathcal{M})$ for any von Neumann algebra has been obtained recently in [9]. In particular, for any type $II$ or $I_\infty$ von Neumann algebra $\mathcal{M}$, derivations on $S(\mathcal{M})$ are automatically inner. However, for the commutative von Neumann algebra $\mathcal{M} = L_\infty(0, 1)$, the algebra $S(\mathcal{M})$ coincides with $S(0, 1)$, and the latter algebra admits non-trivial (and hence, non-inner) derivations [5, 6].

In general, problems in the non-commutative setting are more complicated than their commutative counterparts. However, due to the fact that there exist non-inner derivations on $S(0, 1)$ [6] and any derivations on $S(\mathcal{M})$ are inner when $\mathcal{M}$ is a type $II$ (or $I_\infty$) von Neumann algebra [9], the proof yielding the lack of extension of differential operator up to $S(\mathcal{R})$ is much simpler than that for $S(0, 1)$.

We now present a proof of Proposition 1.2.

Proof of Proposition 1.2. Let $\delta : S(\mathcal{M}) \to S(\mathcal{M})$ be a derivation such that $\delta(P(\mathcal{A})) \subset S(\mathcal{A})$. By [9], $\delta$ is inner, in particular, it is continuous in the measure
topology. Since \( \delta(\mathcal{P}(\mathcal{A})) \subset S(\mathcal{A}) \), for any projection \( e \in \mathcal{A} \) we have \( \delta(e) = 0 \). Indeed, we have
\[
\delta(e) = \delta(e^2) = \delta(e)e + e\delta(e) = 2e\delta(e).
\]
Multiplying the above equality by \( e \), we obtain \( e\delta(e) = 2e\delta(e) \). Hence, \( e\delta(e) = 0 \), and therefore \( \delta(e) = 0 \). Thus, \( \delta \) vanishes on the set of all linear combinations of mutually orthogonal projections from \( \mathcal{A} \). Since \( \delta \) is continuous in measure and the set of all linear combinations of mutually orthogonal projections from \( \mathcal{A} \) is dense in measure in the real part \( S(\mathcal{A})_h \), it follows that \( \delta \) also vanishes on \( S(\mathcal{A})_h \). By linearity of derivations, \( \delta \) vanishes on \( S(\mathcal{A}) \).

Let \( \mathcal{R} \) be the hyperfinite type II\(_1\) factor and \( \mathcal{D} \) be its diagonal masa. Recall that \( \partial_{\mathcal{AD}} \) maps \( \mathcal{AD}(\mathcal{D})(\cong \mathcal{AD}(0,1)) \) into \( \mathcal{S}(\mathcal{D})(\cong \mathcal{S}(0,1)) \) and \( \mathcal{P}(\mathcal{D}) \subset \mathcal{AD}(\mathcal{D}) \) (see Section 2.3). Let \( \delta \) be a derivation on \( \mathcal{S}(\mathcal{R}) \) as an extension of \( \partial_{\mathcal{AD}} \). Setting \( \mathcal{M} = \mathcal{R} \) and \( \mathcal{A} = \mathcal{D} \), the first assertion of the proposition yields that \( \partial_{\mathcal{AD}} = \delta|_{\mathcal{AD}(\mathcal{D})} = 0 \), which is a contradiction.

We note that for a derivation vanishing on the abelian subalgebra \( \mathcal{A} \), there exist non-trivial extensions of this derivation on \( \mathcal{S}(\mathcal{M}) \).

**Example 3.1.** Let \( \mathcal{R} \) be the hyperfinite II\(_1\) factor and let \( \mathcal{D} \) be its diagonal masa. Recall that every derivation \( \delta: \mathcal{S}(\mathcal{R}) \to \mathcal{S}(\mathcal{R}) \) is implemented by an element in \( \mathcal{S}(\mathcal{R}) \) [9]. Let \( a \in \mathcal{D}\setminus \mathbb{C}1 \). We define a derivation \( \text{ad}(a)(x), x \in \mathcal{S}(\mathcal{R}) \). In particular, \( \text{ad}(a) = 0 \) on \( \mathcal{D} \) and therefore on \( \mathcal{S}(\mathcal{D}) \). However, it is non-trivial on \( \mathcal{S}(\mathcal{R}) \). Indeed, if \( \text{ad}(a) \) is trivial on \( \mathcal{S}(\mathcal{R}) \), then \( a \) is in \( \mathcal{R}' \). However, \( \mathcal{R} \) is a factor. Hence, \( a \in \mathbb{C}1 \), which is a contradiction with the assumption of \( a \).

**3.2. The proof of Theorem 1.1.** From now on, we concentrate on the algebra \( \mathcal{S}(0,1) \) of all classes of Lebesgue measurable functions on \( (0,1) \).

Recall that \( \mathcal{G} \) is the group of all automorphisms of \( \mathcal{S}(0,1) \) generated by the dyadic translations of the unit interval \( (0,1) \), that is, any element \( \alpha \) of \( \mathcal{G} \) is defined as follows
\[
\alpha(x)(t) = x(t-r), \quad x \in \mathcal{S}(0,1), \quad t \in (0,1),
\]
where \( r \) is the dyadic number from \( [0,1) \) and \( \{a\} \) is the fractional part of the real number \( a \). We say that a derivation \( \delta \) of \( \mathcal{S}(0,1) \) commutes with \( \mathcal{G} \) or is \( \mathcal{G} \)-invariant, if
\[
\alpha \circ \delta = \delta \circ \alpha
\]
for all \( \alpha \in \mathcal{G} \).

The following theorem is the main result of the present section.

**Theorem 3.2.** Let \( (\mathcal{A}, \| \cdot \|_\mathcal{A}) \) be a complete symmetric \( \Delta \)-normed *-subalgebra of \( \mathcal{S}(0,1) \) in which simple functions are dense in the \( \| \cdot \|_\mathcal{A} \)-norm topology. Let \( \delta \) be a derivation from \( \mathcal{A} \) into \( \mathcal{S}(0,1) \) commuting with \( \mathcal{G} \). Then \( \delta \) is trivial.

From now on, we always assume that \( \delta \) is a non-trivial *-derivation from \( \mathcal{A} \) into \( \mathcal{S}(0,1) \) commuting with \( \mathcal{G} \). We construct below an element \( h \) from \( \mathcal{A} \) such that
\[ \delta(h) \notin S(0,1). \] After that, we will present some properties of \( h \), which allow us to show that the \( G \)-invariance of \( \delta \) fails at this element.

For any real-valued \( f \in S(0,1) \), we have \( \delta(f) = \delta(f)^* \), that is, \( \delta(f) \) is also real. Since \( \delta \) is a non-trivial \( * \)-derivation, there exists an element \( f \in \mathcal{A} \) such that \( \delta(f) \neq 0 \). If necessary, replacing \( f \) with \(-f\), we can assume that the positive part of \( \delta(f) \) is non zero. Then we can find positive numbers \( \lambda < \mu \) and a measurable subset \( A \subset (0,1) \) with a positive measure such that

\[ \lambda \chi_A \leq \chi_A \delta(f) \leq \mu \chi_A. \]

Note that \( \delta(e) = 0 \) for any projection \( e \in \mathcal{A} \) (see e.g. [6, Proposition 2.3. (iii)]). Therefore, \( \delta(\chi_A f) = \chi_A \delta(f) \) and we obtain that

\[ \lambda \chi_A \leq \delta(\chi_A f) \leq \mu \chi_A. \]

By replacing \( f \) with \( \frac{1}{\chi_A f} \), we may assume that \( f \) is a function such that

\[ \chi_A \leq \delta(f) \leq \gamma \chi_A, \quad (3.2) \]

where \( \gamma := \frac{\mu}{\lambda} > 1. \)

Since there are countably many dyadic numbers, it follows that we can numerate all dyadic translations of \((0,1)\) as \( \beta_n, n = 0, 1, \cdots \), i.e., \( G = \{ \beta_n : n \geq 0 \} \) (\( \beta_0 \) is the identity map on \((0,1)\)). Set

\[ B = \bigcup_{n \geq 0} \beta_n(A). \]

Observe that for each \( k \geq 0 \), we have \( \beta_k(B) = B \). Since the group \( G \) acts on \((0,1)\) ergodically (see [30, p. 927]), it follows that either \( m(B) = 0 \) or \( m((0,1) \setminus B) = 0 \). Since \( m(B) > 0 \), it follows that

\[ m((0,1) \setminus B) = 0. \quad (3.3) \]

Set

\[ B_0 = \beta_0(A) = A, \]
\[ B_n = \beta_n(A) \setminus \bigcup_{k=0}^{n-1} \beta_k(A), \quad n \geq 1. \]

We note that all \( B_n, n \geq 0 \), are pairwise disjoint (note that \( B_n \) may be the empty set for some \( n \)) and

\[ B = \bigcup_{n \geq 0} B_n. \]

Taking into account into (3.3) and the last equality, we obtain that

\[ \sum_{n \geq 0} \chi_{B_n} = \chi_{[0,1)} \text{ a.e.} \]
Denote by \( \tilde{\beta}_k \) an automorphism of \( S(0,1) \) generated by the translation \( \beta_k, \ k \geq 0 \), that is, \( \tilde{\beta}_k(x) = x \circ \beta_k^{-1}, \ x \in S(0,1) \). Set

\[
(3.4) \quad g = \sum_{k=0}^{\infty} \chi_{B_k} \tilde{\beta}_k(f),
\]

where the series is considered in \( \rho \)-topology (see Section 2.1). Using (3.2) and the translation-invariance of \( \delta \), we have

\[
(3.5) \quad \chi_{B_n} \leq \chi_{B_n} \tilde{\beta}_n(\delta(f)) = \chi_{B_n} \delta(\tilde{\beta}_n(f)) \leq \gamma \chi_{B_n}
\]

for all \( n \geq 0 \). Taking into account that \( \delta \) is \( \rho \)-continuous [8, Proposition 2.4], we obtain that

\[
\delta(g) = \sum_{k=0}^{\infty} \chi_{B_k} \tilde{\beta}_k(\delta(f)) \leq \sum_{k=0}^{\infty} \chi_{B_k} \tilde{\beta}_k(f).
\]

Then, summing the above inequalities over all \( n \), we obtain that

\[
1 \leq \delta(g) \leq \gamma.
\]

Due to the assumption that simple functions are dense in \( A \), for every \( k \geq 1 \), there exists a simple function \( s_k \in S(0,1) \) such that \( \|g - s_k\|_A \leq \frac{1}{(2\gamma)^{k^2} C_A^{k^2+1}} \).

Setting \( g_k := g - s_k \), we have that

\[
(3.7) \quad \|g_k\|_A \leq \frac{1}{(2\gamma)^{k^2} C_A^{k^2+1}}
\]

where \( C_A \) is the constant for the quasi-triangle inequality for \( \|\cdot\|_A \). By (3.6) and the fact that \( \delta(s_k) = 0 \) (see e.g. [6, Proposition 2.3, (ii)]), we obtain that

\[
(3.8) \quad 1 \leq \delta(g_k) \leq \gamma, \ \forall k \geq 1.
\]

Recall that for any \( k \geq 1 \), we have defined automorphism \( \alpha_k \) of \( S(0,1) \) by

\[
(\alpha_k(x))(t) = x \left( \left\{ t - \frac{1}{2^k} \right\} \right).
\]

By the definition of \( \alpha_k \) (see (1.1)), it is clear that \( \alpha_n = \alpha_k^{2^{k-n}}, \ k > n \). Here, we denote the composition of \( \alpha_m \) with itself \( i \)-times by \( \alpha_m^i \), that is, \( \alpha_m^i = \alpha_{m-1} \circ \alpha \) for \( i \geq 2 \) with \( \alpha_m^1 = \alpha_m \). Set

\[
(3.9) \quad h_k = \gamma^{k^2} \sum_{i=1}^{2^k} (-1)^i \alpha_k^i \left( \chi_{[0, \frac{1}{2^k})} g_k \right).
\]
Appealing to the definition of symmetric \( \Delta \)-norms, we obtain that
\[
\left\| \alpha_k^i \left( \chi_{[0, \frac{1}{2^k})} g_k \right) \right\|_A \leq \|g_k\|_A \leq \frac{1}{(2\gamma)^{k^2} C_A^{2k+1}},
\]
for all \( 1 \leq i \leq 2^k \). Hence, by the quasi-triangle inequality, we obtain that
\[
\|h_k\|_A \leq C_A^k \sum_{i=1}^{2^k} \frac{\gamma^k}{(2\gamma)^{k^2} C_A^{2k+1}} = \frac{1}{2^{k^2-k} C_A^{2k}}.
\]

Note that
\[
\left\| \sum_{k=l}^{n} h_k \right\|_A \leq \sum_{k=l}^{n} C_A^k \|h_k\|_A \leq \sum_{k=l}^{n} C_A^k \frac{1}{2^{k^2-k} C_A^{2k}} \leq \sum_{k=l}^{n} \frac{1}{2^{k^2-k}} \to 0 \text{ as } l \to \infty.
\]

Thus, \( \left\{ \sum_{k=1}^{n} h_k \right\}_{n \geq 1} \) is a Cauchy sequence in \( (A, \| \cdot \|_A) \). We define
\[
(3.10) \quad h := \lim_{n \to \infty} \sum_{k=1}^{n} h_k
\]
converges in \( (A, \| \cdot \|_A) \). In particular, \( h \in S(0, 1) \).

Having constructed the element \( h \in S(0, 1) \), we shall now show that \( h \) is the required element, at which the \( G \)-invariance of \( \delta \) fails. Before proceeding to the proof of Theorem 3.2, we collect some relations between \( h_k \) and \( \alpha_n, k > n \).

From now on, the notations \( h_k, k \in \mathbb{N} \), and \( h \) always stand for the functions defined in (3.9) and (3.10), respectively.

The following lemma shows that the elements \( h_k, k \geq 1 \), are well-behaved with respect to translations \( \alpha_n, n \geq 1 \).

**LEMMA 3.3.** Let \( n \geq 1 \). Let \( \alpha_n \) be defined as (1.1) and \( h_k, k \in \mathbb{N} \), be as defined in (3.9). Then
\[
(3.11) \quad \alpha_n(h_n) = -h_n
\]
and
\[
(3.12) \quad \alpha_n(h_k) = h_k
\]
for all \( k > n \).

**Proof.** When \( k = n \), we have
\[
\alpha_n(h_n) \overset{(3.9)}{=} \gamma n^2 \sum_{i=1}^{2^n} (-1)^i \alpha_n \left( \alpha_n^i \left( \chi_{[0, \frac{1}{2^n})} g_n \right) \right) = \gamma n^2 \sum_{i=1}^{2^n} (-1)^i \alpha_n^{i+1} \left( \chi_{[0, \frac{1}{2^n})} g_n \right) = -\gamma n^2 \sum_{i=1}^{2^n} (-1)^{i+1} \alpha_n^{i+1} \left( \chi_{[0, \frac{1}{2^n})} g_n \right) = -h_n.
\]
Recall that \( \alpha_n = \alpha_k^{2^n}, k > n, \) and \( \alpha_k^{2^k+i} = \alpha_k^{i}. \) Hence, we obtain that

\[
\alpha_n(h_k) = \sum_{i=1}^{2^k} (-1)^i \alpha_n \left( \alpha_k^i \left( \chi_{[0, \frac{1}{2^k})} g_k \right) \right) = \sum_{i=1}^{2^k} (-1)^i \alpha_k^{2^n+i} \left( \chi_{[0, \frac{1}{2^k})} g_k \right) = h_k.
\]

We provide below uniform estimates for the differences between \( \alpha_n(\delta(h_k)) \) and \( \delta(h_k), n, k \geq 1. \) Recall that \( \delta \) is a derivation on \( S(0, 1) \) commuting with \( G. \)

**Lemma 3.4.** Let \( n \geq 1 \) be fixed. For every \( k \geq 1, \) we have\(^2\)

\[
|\alpha_n(\delta(h_k)) - \delta(h_k)| \leq 2\gamma^{k^2+1},
\]

and

\[
|\alpha_n(\delta(h_n)) - \delta(h_n)| \geq 2\gamma^{n^2}.
\]

**Proof.** Recall that \( \delta(e) = 0 \) for any projection \( e \in \mathcal{A} \) (see e.g. \cite[Proposition 2.3. (iii)]{6}, see also the proof of Proposition 1.2). Using (3.8) we have

\[
\chi_{\left[\frac{1}{2^n}, \frac{k+i}{2^n}\right]} \leq \alpha_k^i \left( \delta \left( \chi_{\left[0, \frac{1}{2^n}\right]} g_k \right) \right) = \alpha_k^i \left( \chi_{\left[0, \frac{1}{2^n}\right]} \delta(g_k) \right) \leq \gamma \chi_{\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right]}
\]

for all \( i = 1, \ldots, 2^k - 1 \) and

\[
\chi_{\left[0, \frac{1}{2^n}\right]} \leq \alpha_k^{2^k} \left( \delta \left( \chi_{\left[0, \frac{1}{2^n}\right]} g_k \right) \right) \leq \gamma \chi_{\left[0, \frac{1}{2^n}\right]}.
\]

Since the derivation \( \delta \) is \( G \)-invariant, it follows that

\[
\delta(h_k) = \sum_{i=1}^{2^k} (-1)^i \alpha_k^i \left( \chi_{\left[0, \frac{1}{2^n}\right]} g_k \right) \left( \chi_{\left[0, \frac{1}{2^n}\right]} g_k \right) = \sum_{i=1}^{2^k} (-1)^i \alpha_k^i \left( \chi_{\left[0, \frac{1}{2^n}\right]} g_k \right)
\]

(3.16)

and, therefore, by (3.14) and (3.15), we have

\[
|\delta(h_k)| \leq \gamma^{k^2+1}.
\]

Hence, \( |\alpha_n(\delta(h_k))| \leq \gamma^{k^2+1}. \) By the triangle inequality, we obtain that

\[
|\alpha_n(\delta(h_k)) - \delta(h_k)| \leq 2\gamma^{k^2+1}.
\]

\(^2\)We note that when \( k > n, \) we have \( \alpha_n(\delta(h_k)) - \delta(h_k) = \delta(\alpha_n(h_k)) - \delta(h_k) \stackrel{(3.12)}{=} 0. \)
On the other hand, we have
\[
|\delta(h_n)\chi_{[0, \frac{1}{2\pi})}^{(3.16)}| = |\gamma n^2 \sum_{i=1}^{2^n} (-1)^i \alpha_n^i \left(\delta \left(\chi_{[0, \frac{1}{2\pi})} g_n\right)\right)\chi_{[0, \frac{1}{2\pi})}^{(3.15)} \geq \gamma n^2 \chi_{[0, \frac{1}{2\pi})},
\]
and for any \(i = 1, 2, \ldots, 2^{n-1}\), we have
\[
|\delta(h_n)\chi_{[\frac{i}{2^n}, \frac{i+1}{2^n})}^{(3.16)}| = |\gamma n^2 \sum_{i=1}^{2^n} (-1)^i \alpha_n^i \left(\delta \left(\chi_{[\frac{i}{2^n}, \frac{i+1}{2^n})} g_n\right)\right)\chi_{[\frac{i}{2^n}, \frac{i+1}{2^n})}^{(3.14)} \geq \gamma n^2 \chi_{[\frac{i}{2^n}, \frac{i+1}{2^n})}.
\]
That is,
\[
|\delta(h_n)| \geq \gamma n^2.
\]
Taking (3.11) into account, i.e., \(\alpha_n(h_n) = -h_n\), we obtain that
\[
|\alpha_n(\delta(h_n)) - \delta(h_n)| = |\delta(\alpha_n(h_n)) - \delta(h_n)| = |-2\delta(h_n)| \geq 2\gamma n^2,
\]
which completes the proof. \(\square\)

In the next lemma, we provide a uniform estimate for the sum of the difference between \(\delta(h_k)\) and \(\alpha_n(\delta(h_k))\) over all \(k = 1, \ldots, n\).

**Lemma 3.5.** There exists a number \(N\) such that for any \(n \geq N\), we have
\[
(3.17) \quad \left| \sum_{k=1}^{n} \alpha_n(\delta(h_k)) - \delta(h_k) \right| \geq \gamma n^2.
\]
**Proof.** By Lemma 3.4, we have
\[
|\alpha_n(\delta(h_k)) - \delta(h_k)| \leq 2\gamma k^2 + 1.
\]
Recall that \(\gamma > 1\) and note that for sufficiently large \(n\), we have
\[
\sum_{k=1}^{n-1} \gamma k^2 + 1 \leq (n - 1)\gamma (n-1)^2 + 1 \leq \gamma n \cdot \gamma (n-1)^2 + 1 = \gamma n^2 - n + 2.
\]
We infer that
\[
(3.18) \quad \left| \sum_{k=1}^{n-1} \alpha_n(\delta(h_k)) - \delta(h_k) \right| \leq \sum_{k=1}^{n-1} |\alpha_n(\delta(h_k)) - \delta(h_k)| \leq 2 \sum_{k=1}^{n-1} \gamma k^2 + 1 \leq 2\gamma n^2 - n + 2.
\]
When \(k = n\), by (3.13), we have that
\[
(3.19) \quad |\alpha_n(\delta(h_n)) - \delta(h_n)| \geq 2\gamma n^2.
\]
Combining inequalities (3.18) and (3.19), we conclude that

\[
\left| \sum_{k=1}^{n} \alpha_n(\delta(h_k)) - \delta(h_k) \right| \geq |\alpha_n(\delta(h_n)) - \delta(h_n)| - \left| \sum_{k=1}^{n-1} \alpha_n(\delta(h_k)) - \delta(h_k) \right|
\]

\[
\geq 2\gamma^{n^2} - 2\gamma^{n^2-n+2} = \gamma^{n^2} + (\gamma^{n^2} - 2\gamma^{n^2-n+2})
\]

\[
= \gamma^{n^2} + \gamma^{n^2-n+2}(\gamma^{n-2} - 2) \geq \gamma^{n^2}
\]

for all sufficiently large \( n \).

The following lemma is the key ingredient in our proof. It shows that an estimate similar to that of Lemma 3.5 holds for the infinite sum \( h = \sum_{k=1}^{\infty} h_k \in S(0,1) \).

**Lemma 3.6.** There exists a number \( N \) such that for any \( n \geq N \), we have

\[
(3.20) \quad |\alpha_n(\delta(h)) - \delta(h)| \geq \gamma^{n^2}.
\]

**Proof.** Let \( N \) be large enough such that (3.17) holds and let \( n \geq N \). Recall that \( \| \cdot \|_A \)-convergence implies measure-convergence (see Section 2.2). Recall the symmetric \( \Delta \)-norm \( \| \cdot \|_S \) on \( S(0,1) \) induced by the measure topology (see Section 2.2). Hence, \( \| \sum_{k=l}^{\infty} h_k \|_S \rightarrow 0 \) as \( l \rightarrow \infty \). For any \( l > n \), we have

\[
\| \alpha_n \left( \sum_{k=n+1}^{\infty} h_k \right) - \sum_{k=n+1}^{\infty} h_k \|_S \leq \| \alpha_n \left( \sum_{k=n+1}^{l} h_k \right) - \sum_{k=n+1}^{l} h_k \|_S + \| \alpha_n \left( \sum_{k=l}^{\infty} h_k \right) \|_S + \| \sum_{k=l}^{\infty} h_k \|_S \]

\[
\overset{(3.12)}{=} 2 \left\| \sum_{k=l}^{\infty} h_k \right\|_S \rightarrow 0 \quad \text{as} \quad l \rightarrow \infty.
\]

Hence, \( \alpha_n \left( \sum_{k=n+1}^{\infty} h_k \right) = \sum_{k=n+1}^{\infty} h_k \). By the assumption, \( \delta \) is \( G \)-invariant and hence it commutes with \( \alpha_n, n \geq 1 \). This implies that for any \( n \geq N \), we have

\[
|\alpha_n(\delta(h)) - \delta(h)| = |\delta(\alpha_n(h)) - \delta(h)| = |\delta(\alpha_n(h) - h)|
\]

\[
\overset{(3.10)}{=} \left| \delta \left( \alpha_n \left( \sum_{k=n+1}^{\infty} h_k \right) - \sum_{k=n+1}^{\infty} h_k \right) + \delta \left( \sum_{k=1}^{n} (\alpha_n(h_k) - h_k) \right) \right|
\]

\[
= \left| \delta \left( \sum_{k=1}^{n} (\alpha_n(h_k) - h_k) \right) \right| = \left| \sum_{k=1}^{n} \alpha_n(\delta(h_k)) - \delta(h_k) \right| \overset{(3.17)}{\geq} \gamma^{n^2}.
\]

Here, the series \( \sum_{k=n+1}^{\infty} \alpha_n(h_k) \) and \( \sum_{k=n+1}^{\infty} h_k \) are considered in the topology with respect to \( \| \cdot \|_A \) and therefore, in the measure topology. \( \square \)
The following lemma is a simple observation. For the sake of completeness, we incorporate a detailed proof for it.

**Lemma 3.7.** Let \( F : \mathbb{N} \to \mathbb{R} \) be an increasing function with \( \lim_{n \to \infty} F(n) = \infty \). For any \( y \in S(0,1) \), the inequality

\[
|\alpha_n(y) - y| \geq F(n)
\]

fails for all sufficiently large numbers \( n \).

**Proof.** Take a closed subset \( A \) in \((0,1)\) with the Lebesque measure \( m(A) > \frac{3}{4} \) such that

\[
|y|_{\chi_A} \leq c
\]

for some \( c > 0 \). Further, for each \( n \geq 1 \), the closed subset \( A_n := \left\{ t + \frac{1}{2^n} : t \in A \right\} \) satisfies \( m(A_n) = m(A) > \frac{3}{4} \) and

\[
|\alpha_n(y)|_{\chi_{A_n}} = |\alpha_n(y)|_{\alpha_n(\chi_A)} = \alpha_n(|y|_{\chi_A}) \leq c.
\]

For each \( n \geq 1 \), we have that \( m(A \cap A_n) > \frac{1}{2} \) and

\[
|\alpha_n(y) - y|_{\chi_{A \cap A_n}} \leq 2c_{\chi_{A \cap A_n}}.
\]

Assume by way of contradiction that \( F(n) \leq |\alpha_n(y) - y| \) for all sufficiently large \( n \). Then,

\[
F(n)_{\chi_{A \cap A_n}} \leq |\alpha_n(y) - y|_{\chi_{A \cap A_n}} \leq 2c_{\chi_{A \cap A_n}},
\]

which implies that \( F(n) \leq 2c \) for all sufficiently large \( n \). This is a contradiction with the assumption on the function \( F \).

Now we are in a position to present the proof of our main result, Theorem 3.2.

**Proof of Theorem 3.2.** Assume by contradiction that there exists a non-trivial \( \delta : \mathcal{A} \to S(0,1) \) commuting with \( G \). Let \( h \) be defined as in (3.10). By Lemma 3.6, the function \( y := \delta(h) \in \mathcal{A} \) satisfies inequality (3.20). Setting \( F(n) = \gamma n^2 \), \( n \in \mathbb{N} \), we obtain a contradiction with Lemma 3.7. Hence, there exists no non-trivial derivation \( \delta : \mathcal{A} \to S(0,1) \) commuting with \( G \).

The following result is an immediate consequence of Theorem 3.2, which shows that the translation invariance property of \( \frac{d}{dt} \) is not shared by its extension.

**Corollary 3.8.** The differential operator \( \frac{d}{dt} : D(0,1) \to S(0,1) \) has no translation-invariant extension to the algebra \( S(0,1) \).
Theorem 3.2 holds for a subalgebra \((A, \|\cdot\|_A)\) of \(S(0,1)\) in which simple functions are dense. However, simple functions are not dense in \((AD(0,1), \rho)\). By Theorem 3.2, for any complete symmetric \(\Delta\)-normed subalgebra \((A, \|\cdot\|_A)\) of \(S(0,1)\) in which simple functions are dense, there exists no derivation from \(A\) into \(S(0,1)\) commuting with all dyadic-rational translations on \((0,1)\). Also, recall that it is shown in [8] that the algebra \(AD(0,1)\) is the maximal subalgebra of \(S(0,1)\) admitting unique extension of \(\frac{d}{dt} : D(0,1) \rightarrow S(0,1)\). It is interesting to drop the “symmetric \(\Delta\)-normed” assumption and consider the following problem.

**Problem 3.9.** Is the algebra \(AD(0,1)\) a maximal subalgebra in \(S(0,1)\) admitting a translation-invariant derivation as an extension of \(\frac{d}{dt} : D(0,1) \rightarrow S(0,1)\)?

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