I. INTRODUCTION

Generalized Parton Distributions (GPDs) are very interesting physical objects, as they encode in a natural way the rich information on the internal structure of hadrons (for extensive reviews see e.g. [1, 2, 3, 4, 5, 6, 7, 8] and references therein). Moments of the GPDs in the X variable form polynomials in the ξ variable (see the following), with the t-dependent coefficients known as the generalized form factors (GFFs). Experimentally, the GPDs, especially for the pion, are rather elusive quantities, as they show up in difficult to measure hard exclusive processes such as Deeply Virtual Compton Scattering (DVCS) or hard electroproduction of mesons (HMP). On the other hand, the GFFs are accessible to lattice QCD measurements. In particular, the lowest-order vector and tensor GFFs of the pion have recently been determined from full-QCD lattice calculations and more similar results are announced to appear in the future.

In this paper we evaluate the vector GFFs of the pion in a chiral quark model, namely the Nambu–Jona-Lasinio (NJL) model with the Pauli-Villars (PV) regularization, using the techniques described in Ref. [11]. Importantly, the results for the GPDs obtained in this model obey all formal requirements, such as normalization, proper support, polynomiality and positivity constraints. At the same time, the obtained expressions have a non-trivial form which is not of a factorizable in the t-variable. We compare the lowest form factors, namely the electromagnetic and the gravitational form factor, to the recent lattice data [9] and find good qualitative agreement. Predictions for higher-order moments are also made. Relevant features of the generalized form factors in the chiral quark models are highlighted and the role of the QCD evolution for the higher-order GFFs is stressed.

II. GENERALIZED PARTON DISTRIBUTIONS AND GENERALIZED FORM FACTORS

We begin with the description of kinematics of GPDs. The assignment of the momenta in the considered process is depicted in Fig. 1. The standard notation is

\[ p^2 = m^2, \quad q^2 = -2p \cdot q = t, \]
\[ n^2 = 0, \quad p \cdot n = 1, \quad q \cdot n = -\zeta. \]  

(1)

The null vector \( n \) defines the light cone. Throughout this paper we work for simplicity in the strict chiral limit with

\[ m_\pi = 0, \]  

(2)

although an extension to the physical pion mass is straightforward in the NJL model. The two isospin projections of the GPDs of the pion are defined through the matrix elements of bilinears of quark fields displaced along the light cone, namely

\[ \delta_{ab} \mathcal{H}^{l=0}(x, \zeta, t) = \int \frac{dz^+dz^-}{4\pi} e^{ixp^+z^-} \times \]  

(3)

\[ \langle \pi(p + q) | \bar{\psi}(0) \gamma \cdot n \psi(z) | \pi(p) \rangle |_{z^+=0, z^-=0}. \]

\[ i\epsilon_{3ab} \mathcal{H}^{l=1}(x, \zeta, t) = \int \frac{dz^+dz^-}{4\pi} e^{ixp^+z^-} \times \]  

(4)

\[ \langle \pi(p + q) | \bar{\psi}(0) \gamma \cdot n \psi(z) \tau_3 | \pi(p) \rangle |_{z^+=0, z^-=0}. \]

where \( z \) is on the light cone, and \( a \) and \( b \) are the pion isospin indices. At the scale pertaining to the chiral quark models the gluons are integrated out, hence \( \mathcal{H}^{l=0}(x, \zeta) = 0 \).

In chiral quark models at the leading-\( N_c \) level the calculation of the GPDs proceeds according to the oneloop diagrams of Fig. 2. Extensive details of the quark-model evaluation are given in [11]. The pion GPDs were...
Λ = 871 MeV, which yields the results presented here. They are qualitatively similar to particular in the Spectral Quark Model [19, 20], will also be presented elsewhere. They are qualitatively similar to calculations in other variants of the chiral quark models, in particular in the Spectral Quark Model [19, 20], will be presented elsewhere. They are qualitatively similar to the results presented here.

The pion electromagnetic form factor in the NJL model is equal to

\[ F_{\nu}^{\text{NJL}}(t) = 1 + \frac{N_c M^2}{8\pi^2 f^2} \left( 2 \sqrt{4(M^2 + \Lambda^2) - t - \sqrt{t}} \log \left( \frac{\sqrt{4(M^2 + \Lambda^2) - t - \sqrt{t}}}{\sqrt{4(M^2 + \Lambda^2) - t + \sqrt{t}}} \right) \right)_{\text{reg}}. \]

The condition \( \lim_{t \to -\infty} F_{\nu}^{\text{NJL}}(t) = 0 \) is satisfied due to Eq. (6).

In the so-called symmetric notation for the GPDs, more convenient for our study, one introduces

\[ \xi = \frac{\zeta}{2 - \zeta}, \quad X = \frac{x - \zeta/2}{1 - \zeta/2}, \]

where \( 0 \leq \xi \leq 1 \) and \( -1 \leq X \leq 1 \). Then one defines

\[ H^{I=0,1}(X, \xi, t) = H^{I=0,1} \left( \frac{\xi + X}{\xi + 1}, \frac{2\xi}{\xi + 1}, t \right), \]

which exhibit the reflection properties about the \( X = 0 \) point,

\[ H^{I=0}(X, \xi, t) = -H^{I=0}(-X, \xi, t), \]
\[ H^{I=1}(X, \xi, t) = H^{I=1}(-X, \xi, t). \]

In addition, for \( X \geq 0 \) one has

\[ H^{I=0,1}(X, 0, 0) = q(X), \]

relating the distributions to the the pion’s forward diagonal parton distribution function (PDF), \( q(X) \).

The polynomiality conditions [1, 2] state that the moments of the GPDs can be written as

\[ \int_{-1}^{1}dX X^{2j} H^{I=1}(X, \xi, t) = 2 \sum_{i=0}^{j} A_{2j+1,i}(t) \xi^{2i}, \]
\[ \int_{-1}^{1}dX X^{2j+1} H^{I=0}(X, \xi, t) = 2 \sum_{i=0}^{j+1} A_{2j+2,i}(t) \xi^{2i}, \]

where \( A_{2j+1,i}(t) \) are the generalized form factors, depending on \( j = 0, 1, \ldots \) and \( i \). The polynomiality property follows from very basic field-theoretic assumptions such as the Lorentz invariance, time reversal, and hermiticity, hence is automatically satisfied in approaches that obey these requirements. In our approach polynomiality is manifest from the use of the double distributions [11].

The notation and normalization factors in Eq. (11) are adjusted in order to agree with the conventions of Ref. [9], except for the subscript \( i \) which in our case labels the powers of \( \xi^2 \) and not \( \xi \). We note that for the isovector (non-singlet) GPD only even, and for the isoscalar (singlet) GPD only odd moments are non-zero. For the few

FIG. 1: The direct (a), crossed (b), and contact (c) Feynman diagrams for the quark-model evaluation of the GPD of the pion. The contact contribution is responsible for the D-term.
owing to the momentum sum rule in the deep
regime, while (13)

\begin{align}
\int_0^1 dX X H^{I=0}(X, \xi, t) &= A_{20}(t) + A_{21}(t)\xi^2 \\
\int_0^1 dX X^2 H^{I=0}(X, \xi, t) &= A_{30}(t) + A_{31}(t)\xi^2 \\
\int_0^1 dX X^3 H^{I=0}(X, \xi, t) &= A_{40}(t) + A_{41}(t)\xi^2 + A_{42}(t)\xi^4.
\end{align}

In Eq. (12) we have introduced the electromagnetic form factor $F_V(t) = A_{10}(t)$, while $\theta_1(t) = 2A_{20}$ and $\theta_2(t) = -2A_{21}$ in Eq. (13) are the gravitational form factors of the pion, discussed in more detail in [11]. In the chiral limit these form factors satisfy the low energy theorem $\theta_1(0) = \theta_2(0)$ [21]. In our quark model calculation in the chiral limit

\[ \theta_1(t) = \theta_2(t) \equiv \theta(t), \]

hence, consequently, $A_{20}(t) = -A_{21}(t)$. The sum rule (12) expresses the electric charge conservation, while (13) is responsible for the momentum sum rule in the deep inelastic scattering.

Formulas (11) are equivalent to the definition

\[ \langle \pi^+(p') | \pi(0) | \gamma_{\mu_1} iD_{\mu_2} \ldots iD_{\mu_N} | u(0) \rangle \pi^+(p) \rangle = 2P^\mu \prod_{\mu_i} P_{\mu_i} A_{j+1,0}(t) + \sum_{i=1}^{2j} q_{\mu_1} \ldots q_{\mu_i} P_{\mu_{i+1}} \ldots P_{\mu_N} A_{j+1,(i+1)/2}(t), \]

where $P = (p+p')/2$, $\vec{D} = (\vec{D} - \vec{D})$, and $\{ \ldots \}$ denotes symmetrization and subtraction of traces for each pair of indices. Equivalence of Eq. (15) and (11) is easily seen by contracting (15) with the null vectors $n^{\mu_1} \ldots n^{\mu_N}$ and applying the definitions (11).

III. RESULTS AT THE QUARK-MODEL SCALE

The calculation of the GPDs made in the NJL model according to the diagrams of Fig. 1 is straightforward [11] and is most efficiently done via the double distributions. Then the GFFs are extracted from the full GPDs by evaluating the moments (11). The obtained expressions are rather lengthy, hence we do not present them here. They have the form similar to Eq. (7), involving logs and rational functions in the $t$ variable.

Our determination of GPDs and GFFs corresponds to the quark model scale $Q_0$, where matching to QCD is made. The reader is referred to Ref. [11], where the issue is discussed in detail. The value of $Q_0$ may be estimated by performing the evolution of the parton distribution functions (PDF) or the parton distribution amplitude (PDA) to higher scales and comparing the results to the available data. It turns out to be low, $Q_0 \approx 320$ MeV.

The results for the electromagnetic and gravitational form factors are independent of the scale. They are shown in Fig. 2. The dots are the data points from lattice calculations of Ref. [9]. One has to bare in mind that the full-QCD lattice calculations need to be extrapolated to the chiral limit. Also, our model incorporates only the leading-$N_c$ contributions. Nevertheless, we note a rather remarkable qualitative agreement, in particular the feature of a much slower decay of the gravitational form factor compared to the electromagnetic form factor occurs both in the model and the data. Of course, for the electromagnetic form factor accurate experimental data could be used for the comparison. However, the focus of this work is on higher-order GFFs, where the information comes from the lattices.

In Fig. 3 we show the higher level GFFs, $A_{3,i}$ and $A_{4,i}$ obtained at the quark model scale. Since in chiral quark models in the chiral limit one has at $t = 0$

\[ H^{I=1}(X, \xi, 0) = \theta(1 - X^2) \]

\[ H^{I=0}(X, \xi, 0) = \theta((1 - X)(X - \xi)) - \theta((X + 1)(-\xi - X)), \]

it follows from the definition (11) that at the quark-model scale

\begin{align}
A_{2j+1,i}(0) &= \begin{cases} 
1 & \text{for } i = 0 \\
0 & \text{otherwise}
\end{cases} \\
A_{2j+2,i}(0) &= \begin{cases} 
1 & \text{for } i = 0 \\
-1 & \text{for } i = j + 1 \\
0 & \text{otherwise}
\end{cases}
\end{align}

(17)

This behavior is seen in Fig. 3. We note that the model GFFs go to zero very slowly at large $-t$. Another prop-
property follows from the fact that in the considered model
\[ H^{I=0}(X,1,t) = 0 \] for any value of \( t \). Then Eq. (11) yields
\[ \sum_{i=0}^{j+1} A_{2j+2,i}(t) = 0. \] (18)
This feature can be seen in the lower panel of Fig. 3.

**IV. QCD EVOLUTION**

As already mentioned, the crucial role of the QCD evolution in chiral quark model calculations has been discussed in Ref. [11]. We carry the leading-order DGLAP-ERBL evolution from the quark-model scale
\[ Q_0 = 313 \text{ MeV} \] (19)
to the scale of the lattice calculation of Ref. [9]. This scale can be inferred from the value \( 2A_{20}(t) = 0.63 \) in [9], which is reproduced in our calculation when we evolve the isoscalar GPD to the scale \( Q^2 = 0.71 \text{ GeV}^2 \). Thus we estimate the lattice scale as
\[ Q = 843 \text{ MeV}. \] (20)

We use the method and code described in [23] to evolve the GPDs taken at several selected values of \( \xi \). From this one may disentangle the coefficients of the powers of \( \xi \), namely the GFFs, at the lattice scale \( Q \). The results of this procedure are presented in Fig. 4. We note a sizable change, both in the value at \( t = 0 \) and in shape, compared to the behavior of Fig. 3 which shows that in general the GFFs do evolve with the scale, except for the protected form factors as those in Fig. 2 which are invariants of the evolution.

For the evolution scale \( Q^2 \rightarrow \infty \), the GPDs tend to their asymptotic forms located entirely in the ERBL region \( |X| < \xi \). Explicitly, we have in this limit [11]
\[ H^{I=1} = \frac{3}{2\xi} \left( 1 - \frac{X^2}{\xi^2} \right) F_V(t) \] (21)
\[ H^{I=0} = (1 - \xi^2) \frac{15}{4\xi^2} \frac{N_f}{4C_F + N_f} \frac{X}{\xi} \left( 1 - \frac{X^2}{\xi^2} \right) \theta(t) \]
\[ XH^{G} = (1 - \xi^2) \frac{15}{4\xi^2} \frac{C_F}{4C_F + N_f} \left( 1 - \frac{X^2}{\xi^2} \right)^2 \theta(t), \]
where \( C_F = (N_c^2 - 1)/(2N_c) \) and \( N_f = 3 \) is the number of active flavors. The proportionality factors reflect the normalization at the initial quark-model scale \( Q_0 \), as the charge- and momentum-conservation sum rules are
We have studied the form factors of the pion in the Nambu–Jona-Lasinio model with the Pauli-Villars regularization. We have shown that the condition which carries no gluons, which are then built from the fact that we start from the quark-model initial condition, which carries no gluons, which are then built in the process of evolution. Asymptotically, for $N_c = 3$ and $N_f = 2$ the quark to gluon ratio in the momentum sum rule equals $9/16$.

In summary, we have computed the generalized vector form factors of the pion in the Nambu–Jona-Lasinio model with the Pauli-Villars regularization. We have proceeded through the $X$-moments of the generalized parton distribution functions, evolved from the quark-model scale to the scale of the lattice calculations. The model GPDs exhibit no factorization in the $t$-variable. Comparison to the lattice results for the electromagnetic and gravitational form factors of the the pion have been made, with proper agreement. The QCD evolution has been carried out and its role for the higher-order generalized form factors has been discussed. Our predictions for higher-order GFFs may be compared to lattice results when these become available.

VI. MANOJ

Since this volume is devoted to the memory of Manoj K. Banerjee, our unforgettable teacher and friend, let me finish with an anecdote. When I started my graduate work, Manoj gave me a code written together with Mike Birse who was then a postdoc at Maryland. The code was used to solve the chiral soliton model with valence quarks (the Birse-Banerjee model [24, 25]), which was a major achievement. My assignment was to introduce vector mesons to the model [26, 27]. When reading down the FORTRAN lines I noticed that the encoded radial differential equation for the pion had a seriously-looking mistake: in one of the terms instead of $p/r*2$ there was $p/r**2$. Omission of one asterisk changed the square into the multiplication by 2, a potentially devastating mistake. When I showed this to my advisor, he rushed to the computer and ran the corrected code. The first output quantity was the soliton mass, which changed by a tiny amount at the relative level of $10^{-4}$ or so. Other observables were also practically unaffected . . . – “You see, young man, good physics is immune to such silly mistakes as omission of an asterisk!”

Let Manoj’s kindness, brilliance, enthusiasm, and confidence be with us!

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