Analog of the Gravitational Anomaly
in Topological Chiral Superconductors

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Received March 19, 2021; revised March 22, 2021; accepted March 23, 2021

It is known that the contribution of torsion to the equation for the chiral Weyl fermions can be equivalently considered in terms of the axial gauge field. In this scenario, the gravitational field transforms to the gauge field. Here, we show that in chiral superconductors the opposite scenario takes place: the electromagnetic field serves as the spin connection for the Bogoliubov fermionic quasiparticles. As a result, the electromagnetic field gives rise to the gravitational anomaly, which contains an extra factor of 1/3 in the corresponding Adler–Bell–Jackiw equation as compared to the conventional chiral anomaly. We also consider the gravitational anomaly produced in neutral Weyl superfluids by the analog of the gravitational instanton, the process of creation and annihilation of hopfions, which are three-dimensional topological objects. The gravitational instanton leads to creation of the chiral charge.

DOI: 10.1134/S0021364021080014

1. INTRODUCTION

Topological materials with Weyl fermions provide the possibility to study quantum anomalies, such as the Adler–Bell–Jackiw (ABJ) chiral anomaly [1–3]. A condensed matter analog of the ABJ anomaly was experimentally probed in the chiral A-phase of liquid \(^3\)He, which is a Weyl superfluid [4]. In this electrically neutral matter, the chiral anomaly is caused by the effective electromagnetic field produced by deformations, which lead to the space–time dependence of the superfluid order parameter. Recent discussion of anomalies in Weyl materials can be found in [5]. Reviews on chiral anomaly effects in the heavy-ion collisions see in [6, 7].

The gravitational anomalies in Weyl materials are also discussed. These anomalies are produced by the effective gravitational fields acting on Weyl fermions. These fields are represented by tetrads, spin connection, and torsion fields. In particular, the analog of the Nieh–Yan anomaly in terms of torsion [8–11] was developed [12–18].

Here, we consider the chiral superconductors and show that the \(U(1)\) electromagnetic field plays the role of the spin connection in the effective tetrad gravity. As a result, these superconductors experience the analog of gravitational anomaly. As distinct from the conventional chiral anomaly, the corresponding ABJ equation contains an extra factor of 1/3.

2. FROM GAUGE FIELD TO THE SPIN CONNECTION

The Hamiltonian for Bogoliubov quasiparticles in the \(p\)-wave homogeneous superfluid \(^3\)He-A in a simplest model, which neglects Fermi-liquid corrections, has the form [19]

\[
H(p) = \left\{ \begin{array}{c}
\frac{p^2}{2m} - \mu_0 - e_1 \cdot p + ie_2 \cdot p \\
e_1 \cdot p - ie_2 \cdot p - \frac{p^2}{2m} + \mu_0
\end{array} \right\}.
\]

(1)

Here, \(e_1\) and \(e_2\) are the real vectors satisfying the conditions \(e_1 \cdot e_1 = 0\) and \(e_2 \cdot e_2 = e_2^2\). For simplicity, we do not consider the spin structure; i.e., our quasiparticles are considered as spinless. The Hamiltonian (1) describes the homogeneous system, and \(\mu_0 = p_0^2/2m\) is the equilibrium chemical potential.

This Hamiltonian contains two Weyl points, i.e., topologically stable point nodes in the energy spectrum, which represent monopoles in the Berry phase in momentum space [20] (classification in terms of two different topological invariants in the interacting systems can be found in [21]). The nodes are at \(p_{\pm} = \pm p_0 \hat{l}\), where \(\hat{l} \parallel e_1 \times e_2\) is the unit vector. The space and time dependence of the Weyl points in the inhomogeneous superfluid produces the effective electromagnetic field \(A^{\text{eff}}(r,t) = p_0 \hat{l}(r,t)\), which acts...
on the Weyl fermions (the pseudo-electromagnetic field in terminology of [22]):

$$H(p) = e_i^a \tilde{e}^a \left( p_i - A_i^{\text{eff}} \right). \quad (2)$$

Here, $\tilde{e}^a$ are Pauli matrices in the particle–hole space; and $e_i^a = (e_i^1, e_i^2, e_i^3 = v_i \hat{l})$ are the space components of the gravitational tetrads, where $v_i = p_0/m$ is the Fermi velocity in the normal Fermi liquid.

The motion of the Bogoliubov quasiparticles in this effective field gives rise to the anomalous nonconservation of the chiral current, which is an analog of the chiral anomaly [1–3]. For $^3$He-A, the ABJ equation has the form

$$\partial_\mu J_5^\mu = \frac{1}{32\pi^2} e^{\mu\nu\rho\sigma} F_\mu^{\text{eff}} F_\nu^{\text{eff}}. \quad (3)$$

The prefactor in the ABJ equation (3) has been confirmed in the experiments on vortex dynamics [4], where the space and time dependence of the effective gauge field $A_i^{\text{eff}}(r, t) = p_0 \hat{l}(r, t)$ has been created by moving continuous vortices, i.e., skyrmions in the $\hat{l}$ field.

Let us now consider the electrically charged $p$-wave superfluid, i.e., the superconductor with the same order parameter, and consider the role of the real electromagnetic field in the chiral anomaly. Here, we are interested in the influence of the electromagnetic field only, and neglect the effective electromagnetic field, $F_\mu^{\text{eff}} = 0$; i.e., the order parameter is assumed to be fixed.

The inverse Green’s function of Bogoliubov quasiparticles in the external electromagnetic field is:

$$G^{-1} = -i\partial_\tau + \tau_3 g A_0(r, t) + H(p + \tau_3 g A(r, t)), \quad (4)$$

where $A_0$ is the vector potential of the electromagnetic field. This field acts in the opposite ways on particle and hole components of Bogoliubov spinor; for this reason, the vector potential is accompanied by the matrix $\tau_3$. Here, $q = -1$ is electric charge.

Since $\tau_3 = \frac{1}{2i} (\tau_1 \tau_2 - \tau_2 \tau_1)$, the Green’s function can be rewritten in terms of the “covariant derivative”:

$$D_i = \partial_\tau + \frac{1}{8} C_i^{ab} (\tau_a \tau_b - \tau_b \tau_a),$$

$$D_i = \partial_\tau + \frac{1}{8} C_i^{ab} (\tau_a \tau_b - \tau_b \tau_a), \quad (5)$$

where $C_i^{ab}$ are the elements of the spin connection, see, e.g., [15]. Let us show that the nonzero components of the spin connection in our case are expressed in terms of the vector potential of the electromagnetic field:

$$C_i^{12} = -C_i^{21} = 2A_i(r, t), \quad C_0^{12} = -C_0^{21} = 2A_0(r, t). \quad (6)$$

Expanding the Green’s function (4) near the Weyl node at $p_0 \hat{l}$, one obtains

$$G^{-1} = -i\partial_\tau + \tau_3 \left( A_0 + \frac{A^2}{2m} \right)$$

$$+ \frac{1}{2} \left( \tau e_1 \cdot (p - A \tau_3 - p_0 \hat{l}) + e_1 \cdot (p - A \tau_3 - p_0 \hat{l}) \tau_1 \right)$$

$$+ \frac{1}{2} \left( \tau e_2 \cdot (p - A \tau_3 - p_0 \hat{l}) + e_2 \cdot (p - A \tau_3 - p_0 \hat{l}) \tau_2 \right)$$

$$+ \frac{1}{m} A \cdot (p - A \tau_3 - p_0 \hat{l})$$

$$+ \frac{(p - p_0 \hat{l})^3}{2m}.$$}

This expansion is not gauge invariant, because we omitted the gradients of the order parameter. When they are included, the gauge invariance is recovered.

The quadratic term $(p - p_0 \hat{l})^3/2m$ can be neglected. Then the resulting Lagrangian can be expressed in terms of effective tetrads $e_\mu^a$, which in our case are constant fields except for the vector $e_0^a = A'/m$. This element may produce only the higher order terms in anomaly equation and thus can be neglected. The relevant nonzero elements of the effective spin connection are $C_0^{12} = -C_0^{21} = (A, A_0)/2$ in Eq. (6). As we already mentioned, the effective vector potential $A^{\text{eff}} = p_0 \hat{l}$ is kept constant and thus does not contribute to the anomaly.

### 3. GRAVITATIONAL ANOMALY FROM THE ELECTROMAGNETIC FIELD

Since we neglect the effective gauge field, $F_\mu^{\text{eff}} = 0$, the chiral anomaly comes only from the curvature of the effective gravitational field. For a single Weyl node, one has the following equation for the gravitational anomaly (see, e.g., [23]):

$$\partial_\mu J_5^\mu = \frac{1}{768\pi^2} e^{\mu\nu\rho\sigma} R_\mu^{ab} R_\nu^{cd} \eta_{ad} \eta_{bc}, \quad (8)$$

where the curvature is

$$R_\mu^{ab} = \nabla_\mu C_\nu^{ab} - \nabla_\nu C_\mu^{ab} + C_\mu^{ac} C_\nu^{bd} - C_\nu^{ac} C_\mu^{bd} \eta_{cd}. \quad (9)$$

According to Eq. (6), the nonzero elements of the curvature tensor are

$$R_{2\mu\nu} = -R_{2\nu\mu} = \nabla_\mu C_\nu^{12} - \nabla_\nu C_\mu^{12}. \quad (10)$$

Since the elements of spin connections are expressed in terms of the vector potential of the electromagnetic field, the components of curvature tensor are
expressed in terms of electric and magnetic fields:

\[ R_{\gamma\nu} = 2(\partial_{\mu}A_\gamma - \partial_{\gamma}A_\nu) = 2F_{\gamma\nu}. \]  

(11)

Then, from the anomaly Eq. (8), one obtains:

\[ \partial_{\mu}J_{\phi} = -\frac{1}{384\pi^2}e^{i\phi}\rho_{\text{eff}} = \frac{1}{24\pi^2}\frac{e^{i\phi}}{e^{i\phi}} F_{\mu\nu}F^{\mu\nu}. \]  

(12)

\[ \partial_{\mu}J_{\phi} = \frac{1}{32\pi^2}e^{i\phi} F_{\mu\nu}F^{\mu\nu}, \]  

(13)

or

\[ \partial_{\mu}J_{\phi} = \frac{1}{24\pi^2}\mathbf{E} \cdot \mathbf{B} = \frac{1}{32\pi^2}\mathbf{E} \cdot \mathbf{B}. \]  

(14)

This means that in the superconducting states with Weyl points, the gravitational anomaly becomes the gauge anomaly, but with an extra factor of 1/3 when compared to ABJ equation (3) coming from the effective gauge field \( A_{\text{eff}} = p_0\hat{l}. \)

The factor 1/3 may have relation to the consistent anomaly [22] and to a factor of 1/3 obtained in [24] for the electromagnetic response and \( \theta \)-term in the gapped topological superconductors [25–27].

4. GRAVITATIONAL ANOMALY IN NEUTRAL CHIRAL SUPERFLUID AND CREATION OF HOPFIONS

In the electrically neutral superfluids, such as superfluid \(^3\)He-A, the vector potential of external field is replaced by superfluid velocity: \( A \rightarrow mv_s, \) and the gravitational anomaly becomes

\[ \partial_{\mu}J_{\phi} = \frac{m^2}{24\pi^2}\partial_{\mu}v_s \cdot (\nabla \times v_s). \]  

(15)

Such anomaly can be produced by the instanton, which represents the process of creation or annihilation of three-dimensional skyrmions, i.e., hopfions in the vector field \( \hat{l} \) [28]. Hopfions are described by the \( \pi_3(S^3) = Z \) topological charge, which is a Hopf invariant [29]. Recent papers on hopfions in condensed matter can be found in [30].

In superfluid \(^3\)He-A, the density of the hopfion topological charge is expressed in terms of the helicity of the superfluid velocity [28]:

\[ n_H = \frac{m^2}{4\pi^2}(v_s \cdot (\nabla \times v_s)), \quad N_H = \int d^3 r n_H. \]  

(16)

Introducing the current density of the topological charge:

\[ n_H = \frac{m^2}{4\pi^2}(v_s \times \partial_{\mu}v_s). \]  

(17)

one obtains the (non)conservation law for the topological charge:

\[ \partial_{\mu}n_H^\mu = \frac{m^2}{2\pi^2} (\partial_yv_s \cdot (\nabla \times v_s)). \]  

(18)

The process of the change in the topological charge represents the \( \pi_3 \) instanton: \( \partial_{\mu}n_H^\mu = \delta(t)\delta(r). \) In high energy physics, this is the gravitational instanton, or the so-called torsion vortex [31, 32].

Then from the anomaly Eq. (15) one obtains the connection between the creation of the chiral charge and the creation of the hopfion:

\[ \partial_{\mu}j_{\phi}^\mu = \frac{1}{6} \partial_{\mu}n_H^\mu. \]  

(19)

Here, we added a factor of 2 to take into account the contribution of two Weyl point into the gravitational anomaly. Equation (19) means that due to the gravitational anomaly, the gravitational instanton process of creation of single hopfion is accompanied by creation of 6 chiral fermions. This is the gravitational analog of the Kuzmin—Rubakov—Shaposhnikov scenario of the anomalous electroweak baryogenesis [33].

5. HIGHER VALUES OF TOPOLOGICAL INVARIANTS

Let us consider the gravitational anomaly in case of the higher order Weyl points. The Hamiltonian in [34] for Weyl points with topological charges \( N \) and \( -N \) in chiral superfluids/superconductors has the form

\[ H(p) = \begin{pmatrix} \frac{p^2}{2m} - \mu_0 & (e_1 \cdot p + ie_2 \cdot p)^N \\ (e_1 \cdot p - ie_2 \cdot p)^N & \frac{p^2}{2m} + \mu_0 \end{pmatrix}. \]  

(20)

The corresponding spectrum of Bogoliubov quasiparticles near the Weyl points

\[ E^2(\hat{p}) = (g^{ilk}\hat{p}_l\hat{p}_k)^N + g^{ilk}\hat{p}_l\hat{p}_k, \]  

(21)

where \( \hat{p} = p \mp p_0\hat{l} \) is the momentum measured from the Weyl points, and the elements of the effective metric tensor are:

\[ g^{ilk} = e^i e^k + e^i e^k = g^{ilk} - g^{ilk}, \quad g^{ilk} = e^i e^k. \]  

(22)

For corresponding superconductors the Hamiltonian is

\[ H(p) = \tau^3 e_1 \cdot (\hat{p} - \tau^3 A) \]  

\[ + (\tau^1 - i\tau^2)((e_1 + ie_2) \cdot (\hat{p} - \tau^3 A))^N \]  

\[ + (\tau^1 + i\tau^2)((e_1 - ie_2) \cdot (\hat{p} - \tau^3 A))^N. \]  

(23)

Equation (23) represents the analog of the Hořava gravity [35]. Without the first term this is the 2 + 1 Hořava gravity, which takes place in graphene, see [36]. With the first term, it becomes the anisotropic
extension of the 3 + 1 Hořava gravity. While the relativistic invariance is missing, the tetrad contribution to topology and anomaly is still valid. As before, the gravitational anomaly gives rise to the U(1) anomaly in the external electromagnetic field, again with the extra factor 1/3:

$$\partial_{\mu} J_5^\mu = \frac{N}{96\pi} e^{\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (24)$$

For the electric current, one has the same expression

$$\partial_{\mu} J_5^\mu = \frac{N}{96\pi} e^{\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \quad (25)$$

but the creation of the electric charge is cancelled by the anomaly at the opposite Weyl point with topological charge $-N$. The electric charge created at one node is annihilated at the opposite node.

The chiral currents for two nodes are added to give the total chiral current

$$\partial_{\mu} J_{5tot}^\mu = \frac{N}{48\pi^2} e^{\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \frac{1}{3} \frac{N}{16\pi^2} e^{\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (26)$$

6. CONCLUSIONS

It has been demonstrated that in the chiral superconductors with Weyl fermions the external electromagnetic field serves as the spin connection for the Bogoliubov fermionic quasiparticles. As a result, the electromagnetic field gives rise to the gravitational anomaly, which is described by the ABJ equation with an extra factor of 1/3 compared to the ABJ equation for the conventional chiral anomaly.

In neutral chiral superfluids, this gravitational anomaly takes place during creation and annihilation of hopfions, which are three-dimensional topological objects. A process analogous to the gravitational instanton leads to creation of the chiral charge. This is the gravitational analog of the Kuzmin–Rubakov–Shaposhnikov electroweak baryogenesis.

FUNDING

This work was supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement no. 694248).

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