Normal forms, inner products, and Maslov indices of general multimode squeezings

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Abstract. In this paper we present a pure algebraic construction of the normal factorization of multimode squeezed states and calculate their inner products. This procedure allows one to orthonormalize bases generated by squeezed states. We calculate several correct representations of the normalizing constant for the normal factorization, discuss an analogue of the Maslov index for squeezed states, and show that the Jordan decomposition is a useful mathematical tool for problems with degenerate Hamiltonians. As an application of this theory we consider a non-trivial class of squeezing problems which are solvable in any dimension.

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Contents

1 Introduction 2
2 Canonical transformations and normal representation of squeezings 3
3 Integral representations of $s_t$ and the index problem 6
4 Algebraic forms of $e^{st}$ and normal symbols of squeezings 9
5 Inner product of squeezed states and composition of squeezings 13
6 The Jordan decomposition of squeezings 14
7 An example of normal decomposition 16
8 Numerical tests for integral and algebraic representations of $s_t$ 17
1. Introduction

In this paper we derive a correct expression for the normal ordering of the unitary group $U_t = e^{i\hat{H}t}$ generated by the Hamiltonian

$$\hat{H} = \frac{i}{2}\left((a^\dagger, AA^\dagger) - (a, A^\dagger A)\right) + (a^\dagger, Ba) + i(a^\dagger, h) - i(a, \overline{h}) = \hat{H}_2 + \hat{H}_1,$$

where $a^\dagger = \{a^\dagger_i\}_i^n$, $a = \{a_i\}_i^n$ are the multimodeCreation and annihilation operators with canonical commutation relation (CCR) $[a_i, a_j^\dagger] = \delta_{ij}$, $A = A^T = \{A_{ij}\}$ is a complex symmetric $n \times n$ matrix, $B = B^* = \{B_{ij}\}$ is a Hermitian matrix of the same size, and $h \in \mathbb{C}^n$. We use the standard notation: the star $B^*$ denotes the Hermite conjugation of $B$, the bar $\overline{A}$ stands for the complex conjugation, and $A^T$ means the transposed matrix $A$. By $(\cdot, \cdot)$ we denote the bilinear inner product in $\mathbb{R}^n$ and the corresponding bilinear form in $\mathbb{C}^n$; the sesquilinear inner product in the Hilbert state space $\mathcal{H} = \otimes_1^n \ell_2$ will be denoted by $\langle \cdot, \cdot \rangle$.

In section 2, the normal decomposition of generalized squeezings $U_t = e^{i\hat{H}t}$ is constructed for Hamiltonians (1) with $A \neq 0$, $B \neq 0$. To this end, a system of algebraic equations is derived for $R_t$, $\rho_t$, $C_t \in \mathbb{C}^{n \times n}$, $g_t$, $f_t \in \mathbb{C}^n$, and $s_t \in \mathbb{C}$ such that

$$U_t = e^{i\hat{H}t} = e^{\eta t} e^{-\frac{1}{2}(a^\dagger R a^\dagger - (g^t, a^\dagger))} e^{(a^\dagger, C_t a)} e^{\frac{1}{2}((a^\dagger, f_t) + (g, a))}, \quad U_0 = I.$$  

(2)

The solutions are represented in terms of $(n \times n)$-matrices $\Phi_t$ and $\Psi_t$ of canonical transformations preserving canonical commutation relations (1). Decomposition (2) allows one to calculate the normal symbol of squeezing and the inner products of squeezed states. The last procedure is necessary for constructing a basis generated by squeezed states.

For single mode quantum systems, the normal ordered factorization of the unitary exponent $U_t = e^{i\hat{H}t}$ follows form a formula proved by D.A.Kirznic in [2]. Applications of this formula to quantum statistics are considered in monograph of N.Bogoliubov and D.Shirkov [3]. The multimode versions of (2) for $B = 0$ was derived by H.-Y.Fan [5]. For the theory and recent investigations related to multimode squeezed states see the monograph of C.Gardiner and P.Zoller [6] and the papers of V.Dodonov [8], G.Agarwal [9], N.Schuch et al. [10]. In [11] we describe the normal factorization (2) of squeezed states in terms of canonical variables $\Phi_t$ and $\Psi_t$ introduced by F.Berezin in [1]. We reconsider his proof and suggest new expressions for $s_t$ which preserve the norm of the corresponding squeezed states.

Note that the assumption $B = 0$ is typical for the standard definition of a squeezed state. The factorization of squeezings (2) with general matrix $B \neq 0$ was described in [12]. If $[C_t, \dot{C}_t] \neq 0$, difficulties arise when one tries to derive an evolution equation for $C_t$ in decomposition (2) (see [14] and [15], pp. 274–275, Eq. (1.10)). The advantage of canonical variables $\Phi_t$, $\Psi_t$ is that they allow one to derive and to solve just algebraic equations for matrices $R_t$, $C_t$, $\rho_t$ in (2), but not a nonlinear ODE, which can not be written for $C_t$ as a local ODE, when $[C_t, \dot{C}_t] \neq 0$ (see [12]).
A short proof of the normal factorization (2) and explicit representations of the matrix valued coefficients for this decomposition in terms of canonical transformations are considered in section 2.

In section 3, we derive integral representations for the scalar function $s_t$ which defines the norm and the phase of the normal decomposition and discuss the index problem, which is essential for systems with $B \neq 0$ and implies continuity of $s_t$. The algebraic representations of $s_t$ can be calculated faster than the corresponding integral expressions.

Algebraic expressions for $s_t$ and the formula for the normal symbol of squeezings are discussed in section 4.

In section 5, we recall some useful facts on $L_2(\mathbb{R}^n)$-representations of multimode squeezings and establish equations representing the inner product of squeezings and compositions of squeezed states. In this way, the orthonormalization procedure for squeezed states can be reduced to standard problems of linear algebra.

The algebraic expressions for components of the Jordan decomposition of matrices generating the canonical transformations are derived in section 6. This procedure is helpful for solving the problems with degenerate Hamiltonians.

In section 7, we note that in the class of problems with $A$ and $B$ such that $[B, AA] = 0$, the factorization problem reduces to the eigenvalue problem for the Hermitian matrix $AA - B^2$.

Numerical tests are considered in section 8. The basic equalities have been checked either analytically or numerically by using Wolfram Mathematica, and these interactive tests are available at [19].

2. Canonical transformations and normal representation of squeezings

Hamiltonian (1) defines the $(2n \times 2n)$-block matrix $G = \begin{pmatrix} -i B & A \\ \overline{A} & iB \end{pmatrix}$ and the group of symplectic matrices $e^{Gt}$ (see [1], [13]) such that

\[
i [\hat{H}, \begin{pmatrix} a & a^\dagger \end{pmatrix}] = G \begin{pmatrix} a & a^\dagger \end{pmatrix} + \begin{pmatrix} \hbar \\ \overline{\hbar} \end{pmatrix}, \quad S_t \overset{\text{def}}{=} e^{Gt} = \begin{pmatrix} \Phi_t & \Psi_t \\ \overline{\Psi_t} & \overline{\Phi_t} \end{pmatrix}, \quad t \in \mathbb{R}. \tag{3}\]

The matrices $S_t$ preserve $(2n \times 2n)$-block structure (3) and possess the following properties: $\det S_t = 1$,

\[
S_{-t} = S_t^{-1} = \begin{pmatrix} \Phi_{-t} & \Psi_{-t} \\ \overline{\Psi_{-t}} & \overline{\Phi_{-t}} \end{pmatrix} = \begin{pmatrix} \Phi_t^* & -\Psi_t^T \\ -\Psi_t & \Phi_t^T \end{pmatrix}, \quad S_t^T J S_t = J, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{4}\]

Equations (3) define the evolution $\dot{S}_t = S_t G - G S_t$, initial values $\Phi_0 = I$, $\Psi_0 = 0$, and algebraic representations for $a_t = U_t a U_t^*$, $a_t^\dagger = U_t a^\dagger U_t^*$, $h_t$, and $\overline{h_t}$:

\[
\begin{pmatrix} a_t \\ a_t^\dagger \end{pmatrix} = S_t \begin{pmatrix} a \\ a^\dagger \end{pmatrix} + \begin{pmatrix} h_t \\ \overline{h_t} \end{pmatrix}, \quad \begin{pmatrix} h_t \\ \overline{h_t} \end{pmatrix} \overset{\text{def}}{=} \int_0^t S_{\tau} \begin{pmatrix} h \\ \overline{h} \end{pmatrix} d\tau = \frac{S_t - I}{G} \begin{pmatrix} h \\ \overline{h} \end{pmatrix}. \tag{5}\]
The matrices
\[ G^{-1}(\exp Gt - I) = I + \frac{1}{2!}G + \frac{1}{3!}G^2 + \ldots, \quad G^{-2}(\exp Gt - I - Gt) = \frac{1}{2!}I + \frac{1}{3!}G + \ldots \]
remain well defined for degenerate \( G \).

The set of canonical commutation relations and the rules for inversion of time
\[ \Phi_t \Phi_t^* - \Psi_t \Psi_t^* = \Phi_t^2 \Phi_t - \Psi_t^2 \Psi_t = I, \quad \Phi_t \Psi_t^T - \Psi_t \Phi_t^T = \Phi_t^* \Psi_t - \Psi_t^* \Phi_t = 0, \quad (6) \]
\[ R_t = -\rho_{-t}, \quad \Phi_t = \Phi_{-t}^*, \quad \Psi_t = -\Psi_t^* \quad (7) \]
is a corollary of equations (4) and the identity \( S_t S_{-t} = S_{-t} S_t = I \). More generally, from \( e^{G(t\pm s)} = e^{Gt} e^{\pm Gs} = e^{\pm Gs} e^{Gt} \) and (4), the matrix analogue of addition-subtraction formulæ for sine and cosine follow:
\[ \Phi_{t+s} = \Phi_t \Phi_s + \Psi_t \Psi_s = \Phi_s \Phi_t + \Psi_s \Psi_t, \quad \Psi_{t+s} = \Psi_t \Phi_s + \Phi_t \Psi_s = \Phi_s \Psi_t + \Psi_s \Phi_t, \]
\[ \Phi_{t-s} = \Phi_t \Phi_s^* - \Psi_t \Psi_s^* = \Phi_s^* \Phi_t - \Psi_s^* \Psi_t, \quad \Psi_{t-s} = \Psi_t \Phi_s^* - \Phi_t \Psi_s^* = \Phi_s^* \Psi_t - \Psi_s^* \Phi_t. \quad (8) \]

Identities (6) imply the inequality \( \Phi_t \Phi_t^* \geq I \), so that the inverse matrix \(|\Phi_t| \geq I\), i.e. \( \Phi_t^{-1} \) exists. The second identity (6) proves that the matrices
\[ R_t = \Phi_t^{-1} \Psi_t = \Psi_t^T (\Phi_t^T)^{-1} = R_t^*, \quad \rho_t = \Psi_t \Phi_t^{-1} = (\Phi_t^*)^{-1} \Psi_t^T \]
are symmetric and well defined for all \( t \geq 0 \). Moreover, equation (6) implies
\[ R_t R_t = R_t R_t^* = I - (\Phi_t^* \Phi_t)^{-1} \leq I, \quad |R_t|^2 = R_t^* R_t \leq I. \]

Therefore, the operator \( e^{\pm \frac{1}{2}(\alpha^1, R_t \alpha^1)} \) in (2) is densely defined at any time \( t \in \mathbb{R} \).

If \( A = A^* \) is zero, then
\[ e^{Gt} = \begin{pmatrix} \Phi_t & 0 \\ 0 & \overline{\Phi_t} \end{pmatrix} = \begin{pmatrix} e^{-itB} & 0 \\ 0 & e^{itB} \end{pmatrix} \]
and the unitary group \( U_t = e^{itH} \) can be rewritten as a normally ordered composition
\[ e^{iH} = e^{\sigma_t} e^{-\gamma_t \alpha^1} e^{(\alpha^1, C_t \alpha)} e^{(\overline{T}_t, \alpha)} = e^{\sigma_t} e^{-\gamma_t \alpha^1} : e^{(\alpha^1, (e^{C_t - I} \alpha))} : e^{(\overline{T}_t, \alpha)}, \]
where the creation and annihilation operators inside the colon brackets act in the normal order. Explicit equations for \( s_t, g_t, f_t, \) and \( C_t \) readily follow from CCR and the pair of equivalent representations of \( a_t = U_t a U_t^* \) and \( a_t^\dagger = U_t a^\dagger U_t^* \):
\[ \Phi_t a + h_t = e^{\overline{H}} a e^{-\overline{H}} = e^{-(g_t, \alpha^1)} e^{(\alpha^1, C_t \alpha)} e^{\overline{T}_t \alpha} = e^{-C_t} (a + g_t), \]
\[ \Phi_t a^\dagger + \overline{h}_t = e^{itH} a^\dagger e^{-itH} = e^{a^\dagger, C_t \alpha} \alpha^1 e^{\overline{T}_t \alpha} = e^{C_t^T} a^\dagger + \overline{f}_t. \]

By equating the coefficients at \( a, a^\dagger \), and at the operator of multiplication by scalar on the left and right hand sides of these equalities, we obtain
\[ g_t = \Phi_t^{-1} h_t, \quad f_t = h_t, \quad C_t : \Phi_t = e^{-itB}, \quad C_t = itB. \quad \quad (9) \]

In order to calculate \( e^{s_t} = \langle 0 | e^{itH} | 0 \rangle \), one can use the following equation:
\[ \dot{s_t} e^{s_t} = \langle 0 | e^{itH} i \overline{H} | 0 \rangle = -\langle 0 | e^{itH} (h, \alpha^1) | 0 \rangle = -\langle 0 | (h, \overline{\Phi}_t a^\dagger + \overline{h}_t) e^{itH} | 0 \rangle = -e^{s_t} (h, \overline{h}_t). \quad \quad (10) \]
Therefore, \( s_t = -\int_0^t (h, \overline{h}_t) d\tau \), and finally we obtain the normal decomposition for squeezings with \( A = 0 \):
\[ e^{itH} = e^{-\int_0^t (h, \overline{h}_t) d\tau} e^{-(\Phi_t^{-1} h_t a^\dagger)} : e^{(a^\dagger, (e^{itB - I} a))} : e^{(\overline{T}_t, \alpha)}. \quad \quad (11) \]
For Hamiltonian \([1]\), the general form of the normal decomposition
\[
e^{it\hat{H}} = e^{st}e^{-\frac{1}{2}(a^\dagger,Ra^\dagger)-(g_r,a^\dagger) + (\Phi, a^\dagger)}e^{\frac{1}{2}(a^\dagger ,Ra^\dagger)+(\overline{t_1},a^\dagger)}.
\]
is more similar to an expression used for \(B=0\) (see [11]):
\[
e^{Gt} = e^{t \left( \begin{array}{cc} 0 & A \\ \overline{A} & 0 \end{array} \right) } = \left( \begin{array}{cc} \Phi_t & \Psi_t \\ \Psi_t^\dagger & \Phi_t^\dagger \end{array} \right), \quad \Phi_t = \Phi_t^\dagger = \cosh(A\overline{A})^{\frac{1}{2}}t, \quad \Psi_t = \Psi_t^T = \frac{\sinh(A\overline{A})^{\frac{1}{2}}t}{(A\overline{A})^{\frac{1}{2}}}A.
\]

The proof of \((12)\) in the general case uses the commutation rules
\[
e^{-\frac{1}{2}(a^\dagger ,Ra^\dagger)-(g_r,a^\dagger) + (\Phi, a^\dagger)}e^{-\frac{1}{2}(a^\dagger ,Ra^\dagger)+(g_r,a^\dagger)} = a + Ra^\dagger + g_t,
\]
and the equations for parameters of the normal decomposition which follow from the commutation relations
\[
\Phi_t a + \Psi_t + h_t = e^{it\hat{H}}a e^{-it\hat{H}} = e^{-\frac{1}{2}(a^\dagger,Ra^\dagger)-(g_r,a^\dagger) + (\Phi, a^\dagger)}e^{\frac{1}{2}(a^\dagger,Ra^\dagger)+(g_r,a^\dagger)} = e^{-Ct}(a + Ra^\dagger + g_t),
\]
\[
\Phi_t a^\dagger + \overline{\Psi}_t a + \overline{\Phi}_t = e^{it\hat{H}}a^\dagger e^{-it\hat{H}} = e^{-\frac{1}{2}(a^\dagger ,Ra^\dagger)-(g_r,a^\dagger) + (\Phi, a^\dagger)}e^{\frac{1}{2}(a^\dagger ,Ra^\dagger)+(g_r,a^\dagger)} = e^{Ct}a^\dagger + \overline{\Phi}_t e^{-Ct}(a + Ra^\dagger + g_t) + \overline{f}_t.
\]

These relations imply equations for parameters \(R_t, \rho_t, C_t, g_t, h_t\) of the normal decomposition \((2)\):
\[
\Phi_t = e^{-Ct}, \quad \Phi_t^\dagger = e^{Ct} + \overline{\Phi}_t e^{-Ct}R_t, \quad h_t = e^{-Ct}g_t,
\]
\[
\Psi_t = e^{-Ct}R_t, \quad \overline{\Psi}_t = \overline{\Phi}_t e^{-Ct}, \quad \overline{\Phi}_t = \overline{\Phi}_t e^{-Ct}g_t + \overline{f}_t.
\]

This system of equations possesses the following solution:
\[
C_t = -\ln \Phi_t, \quad R_t = \Phi_t^{-1} \Psi_t, \quad \overline{\rho}_t = \overline{\Psi}_t \Phi_t^{-1}, \quad g_t = \Phi_t^{-1}h_t, \quad f_t = h_t - \rho_t \overline{f}_t.
\]
The compatibility of equations \((14)\) for \(\Phi_t, \Psi_t, \overline{\Psi}_t\) and \(\overline{\Phi}_t\) is a remarkable fact:
\[
e^{Ct} + \overline{\Phi}_t e^{-Ct}R_t = (\Phi_t^T)^{-1} + \overline{\Psi}_t \Phi_t^{-1} \Psi_t = (\Phi_t^T)^{-1} + (\Phi_t^T)^{-1} \Psi_t \Phi_t = (\Phi_t^T)^{-1} + (\Phi_t^T)^{-1} \Psi_t \Phi_t = \overline{\Phi}_t e^{-Ct}.
\]

Thus, the following theorem is proved.

**Theorem 1.** The vector-valued and matrix-valued coefficients of the squeezing with Hamiltonian \((7)\)
\[
R_t = \Phi_t^{-1} \Psi_t, \quad \overline{\rho}_t = \overline{\Psi}_t \Phi_t^{-1}, \quad C_t = -\ln \Phi_t, \quad g_t = \Phi_t^{-1}h_t, \quad f_t = h_t - \rho_t \overline{f}_t
\]
are well defined in terms of \(\Phi_t, \Psi_t\) by \([3]\). The matrices \(\Phi_t^{-1}\) and \(G^{-1}(e^{G} - 1)\) are well defined for any given \(A = A^T, B = B^*\) at any time \(t \in \mathbb{R}\).
3. Integral representations of $s_t$ and the index problem

Let us calculate $e^{st}$ by using the vacuum expectation $e^{st} = \langle 0 | e^{it\hat{H}} | 0 \rangle$ (see (2)) and one of the two obvious equations: $\dot{s}_t = ie^{-st}\langle 0 | e^{it\hat{H}} \hat{H} | 0 \rangle$ or $\dot{s}_t = ie^{-st}\langle 0 | He^{it\hat{H}} | 0 \rangle$.

By definition of the vacuum state, we have
\[ \langle 0 | e^{i\hat{H}t}(a, \bar{h}) | 0 \rangle = 0, \quad \langle 0 | e^{i\hat{H}t}(a^\dagger, Ba) | 0 \rangle = 0, \quad \langle 0 | e^{i\hat{H}t}(a, \bar{A}a) | 0 \rangle = 0. \]

Definition (15) of $\bar{p}_t$ and canonical transformations (13) justify the relationship
\[ e^{i\hat{H}t}(a^\dagger - \bar{p}_t a) e^{-i\hat{H}t} = (\Phi^T_t)^{-1}a^\dagger + \bar{f}_t. \]

As a corollary of (17) we find the two basic vacuum expectations:
\[ \langle 0 | e^{i\hat{H}t}(a^\dagger, h) | 0 \rangle = \langle 0 | e^{i\hat{H}t}((a^\dagger - \bar{p}_t a), h) | 0 \rangle = \langle 0 | (\Phi^T_t)^{-1}a^\dagger + \bar{f}_t, h \rangle e^{i\hat{H}t} | 0 \rangle = e^{st}(\bar{f}_t, h), \]
\[ \langle 0 | e^{i\hat{H}t}(a^\dagger, Aa^\dagger) | 0 \rangle = \langle 0 | e^{i\hat{H}t}((a^\dagger - \bar{p}_t a), A(a^\dagger - \bar{p}_t a)) | 0 \rangle + \langle 0 | e^{i\hat{H}t} | 0 \rangle \text{tr } \bar{p}_t A = \langle 0 | (\bar{f}_t, A\bar{f}_t) e^{i\hat{H}t} | 0 \rangle + e^{st}\text{tr } \bar{p}_t A = e^{st}(\bar{f}_t, A\bar{f}_t + \text{tr } \bar{p}_t A). \]

Therefore, $\dot{s}_t = ie^{-st}\langle 0 | e^{i\hat{H}t} \hat{H} | 0 \rangle = -(\bar{f}_t, h) - \frac{1}{2}(\bar{f}_t, A\bar{f}_t + \text{tr } \bar{p}_t A)$ and this equality proves Lemma 2.

**Lemma 2.** For $f_t$ and $\rho_t$ defined by theorem 1, we have
\[ s_t = -\int_0^t \left( (\bar{f}_\tau, h) + \frac{1}{2} (\bar{f}_\tau, A\bar{f}_\tau) + \frac{1}{2} \text{tr } \bar{p}_\tau A \right) d\tau, \quad \bar{f}_t = \bar{h}_t - \bar{p}_t h_t. \] (18)

If $A = 0$, then $\rho_t = 0$, $f_t = h_t$, and (10) coincides with function (18).

An equivalent representation of $s_t$ follows from (17) and the equality $e^{-it\hat{H}}(a + R_0 a^\dagger)e^{it\hat{H}} = -(\Phi_{-t})^{-1}a + h_{-t} - \rho_{-t}h_{-t} = \Phi_t^{-1}a + f_{-t}$:
\[ s_t = \int_0^t \left( (\bar{f}_\tau, \bar{h}) + \frac{1}{2} (\bar{f}_\tau, A\bar{f}_\tau) - \frac{1}{2} \text{tr } R_\tau \bar{A} \right) d\tau, \quad \bar{f}_t = h_{-t} + R_t \bar{h}_{-t}. \] (19)

Equivalence of (18) and (19) was also tested numerically for randomly simulated $A$, $B$, and $h$ (see [19]).

The expression for $s_t$ in Berezin’s book (see [1], (6.24) in p. 143) differs from (18) and (19). Taking into account the correspondence of notations, his expression of the

| Table 1. |
|-----------|
| Ber | $\rho$ | $A$ | $iA$ | $C$ | $f$ | $i\bar{f}$ | $g_t$ |
|---------|--------|------|-------|-----|------|----------|-------|
| Che-Tl  | $\bar{A}$ | $\bar{A}$ | $B$ | $ih$ | $\bar{h}$ | $h_t$ |

normalizing factor is equal to
\[ e^{it_{\text{Ber}}} = \frac{e^{-\frac{1}{2}t \text{tr } B}}{\sqrt{\det(\Phi_t)}} \exp\left\{ \int_0^t \left( (\Phi_t^{-1}h_\tau, A\Phi_t^{-1}h_\tau) - (\Phi_t^{-1}h_\tau, \bar{h}) \right) d\tau \right\}. \] (20)

At least, the factor $1/2$ at quadratic form in exponential (20) is missed and numerical values of (19) and (20) are different. As a consequence, the normalization condition $||e^{it\hat{H}}|0||^2 = 1$ for evolution of the vacuum state is violated if $h \neq 0$ (see section 8), and
perhaps this was the reason for physicists to ignore [1] and look for alternative theories (see [6], [7]). Numerical tests of normalization conditions (18), (19), (20) are given in [19].

Recall that $\text{tr}(X + Y) = \text{tr} X + \text{tr} Y$ and $\text{tr} XY = \text{tr} YX$. In order to represent the integral $\int_0^t \text{tr} \overline{\rho} \alpha \alpha d\tau$ as an algebraic expression, we apply the R. Feynman formula [14] for the left and right derivatives, whose traces coincide:

$$C_t^L = \left( \frac{d}{dt} e^{C_t} \right) e^{-C_t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_0^1 ds \frac{d}{ds} e^{s(C_t + \Delta t C_t)} e^{-sC_t} = \int_0^1 ds e^{sC_t} \dot{C}_t e^{-sC_t};$$

$$C_t^R = \int_0^1 ds e^{-sC_t} \dot{C}_t e^{sC_t}, \quad \text{tr} \dot{C}_t^R = \int_0^1 ds \text{tr}(e^{-sC_t} \dot{C}_t e^{sC_t}) = \int_0^1 ds \text{tr} \dot{C}_t = \text{tr} \dot{C}_t.$$

These equalities prove that $\text{tr} \dot{C}_t^L = \text{tr} \dot{C}_t^R = \text{tr} \dot{C}_t$ because $\text{tr} e^{-sC_t} \dot{C}_t e^{sC_t} = \text{tr} \dot{C}_t$.

Consider the set of relationships which follow from commutativity of the group $e^{Gt}$ and its generator $G$. Equations (16) imply explicit algebraic representations of the left and the right derivatives $C_t^L$ and $C_t^R$:

$$\dot{C}_t^L = \left( \frac{d}{dt} \Phi_t^{-1} \right) \Phi_t = -\Phi_t^{-1} \dot{\Phi}_t = -\Phi_t^{-1}(\Psi_t A - i \Phi_t B) = iB - R_t A,$$

$$\dot{C}_t^R = \dot{\Phi}_t \left( \frac{d}{dt} \Phi_t^{-1} \right) = -\Phi_t^{-1}(\Phi_t A - iB) - (A \Phi_t - iB \Phi_t) \Phi_t^{-1} = iB - A \overline{\rho}_t.$$

Since $\text{tr} C_t^L = \text{tr} C_t^R = \text{tr} C_t$, $e^{C_t^L} = (\text{det} C_t)^{-1}$, we obtain an algebraic value for the integral of the last summands in (18)-(19):

$$\int_0^t \text{tr} \overline{\rho}_\tau A d\tau = \int_0^t \text{tr} R_t \overline{A} d\tau = \text{tr}(iBt - C_t),$$

$$e^{i \frac{\text{tr} B}{2} t} \int_0^t \text{tr} \overline{\rho}_\tau A d\tau = e^{-\frac{\text{tr} B}{2} \text{tr}(iBt - C_t)} = \frac{e^{-\frac{\text{tr} B}{2} t}}{\sqrt{\text{det} \Phi_t}},$$

$$e^{i \frac{\text{tr} B}{2} t} \mid_{t=0} = e^{i \frac{\text{tr} B}{2}} = \frac{e^{-\frac{\text{tr} B}{2} t}}{\sqrt{\text{det} \Phi_t}} e^{-\frac{1}{2}(a^t, R_0 a^t)} ; e^{a^t, (\Phi_t^{-1} - I)a} ; e^{\frac{1}{2} (a, \overline{\rho}_0 a)}$$

with correctly chosen sign $(\pm)$ which implies the continuity of expressions (23)-(24) in $t$. If the values of $e^{i \frac{\text{tr} B}{2} t} \int_0^t \text{tr} \overline{\rho}_\tau A d\tau - \frac{\text{tr} B}{2} \text{tr} B$ are not calculated at a given instant of time $t$, the local choice of the corresponding branch of the root $(\pm)\sqrt{\text{det} \Phi_t}$ is impossible [3].

In order to ensure the continuity of functions in (23)-(24), we consider the definition of the index. A similar problem in quasi-classical quantum theory was solved by V. P. Maslov [4], who introduced the index for classical trajectories as the difference between the number of positive and negative eigenvalues of the Hessian matrix of the action along the classical trajectory in configuration space associated to the Hamiltonian, the set of initial data and independent variables associated to quantum Hamiltonian.

Recall that numerical values of $\sqrt{z}$ calculated by Wolfram Mathematica (or MatLab, or other modern computational tool) are nonassociative v. r. t. multiplication and discontinuous along the negative cut:

$$\sqrt{z} = \{ \sqrt{|z|} e^{i \phi}, \text{ if } \phi \in [0, \pi]; \sqrt{|z|} e^{i \phi - \pi}, \text{ if } \phi \in (\pi, 2\pi) \},$$

$\dagger$ Concerning the proper choice of the sign in (23)-(24), F. Berezin wrote: "It is impossible to remove the remaining non uniqueness in sign." (see [1], p. 136, line 5 from the bottom.)
Figure 1. The first two lines of figures illustrate the complex square root calculated by MatLab or Wolfram Mathematica. The complex square root is implemented as a function which is continuous at the positive half line $z > 0$ and discontinuous at $z < 0$. The last two lines of panels show successful reconstructions of the continuous phase functions based on the (mod $2\pi$)-continuity condition (see equation (26) below and the algorithm in [19]).
that is $\phi = \pi$ is the point, where the phase functions are discontinuous [20]. At the same time $\sqrt{\Phi_t}$ in (23) is continuous at the origin and $\Phi_0 = I$. In the general case, $\sqrt{\text{Arg} \det \Phi_t}$ is a discontinuous function (see the second line of panels in fig. 1 and the first line of panels in fig. 2), because its values must belong to half a circle (for example, either to $[0, \pi]$, or to $[-\pi/2, \pi/2]$), and at each given instant of time $t$ we have no physical or mathematical reasons to choose either positive or negative branch of the root.

On the other hand, the matrix elements of $e^{G_t} = \left( \frac{\Phi_t}{\Psi_t} \Psi_t \Phi_t \right)$ and the left hand sides in (23)-(24) are continuous in $t$ (see the second and the third lines of panels in fig. 2). The only disadvantage of representations (23)-(24) is that the numerical integration of $\text{tr} \rho_{\tau} A$ converges very slow for multimode systems ($n > 3$). Therefore, the continuity and smoothness of (24) can be ensured either by using non-local integral representation (23) of $s_t$, or by a global continuity construction based on the integer valued index.

An analogue of the construction of index introduced by V. P. Maslov for non-degenerate (recall that $\det |\Phi_t| \geq 1$) and non-Hermitian matrices $\Phi_t$ can be formulated in terms of the polar decomposition $\Phi_t = U_t |\Phi_t|$. The index can be defined correctly, if the arguments of unitary eigenvalues of $U_s$ have a finite number of jumps in a finite time interval $(0, t)$.

Set $\varphi(t) = \frac{1}{2} \sum_k \lambda_k(t) \in (-\pi, \pi]$, where $\lambda_k(t)$ are the arguments of eigenvalues $e^{i\lambda_k(t)}$ of $U_t$. Let $\{T_k\} : 0 < T_1 < T_2 < \ldots < T_n(t) < t$ be the set of instants of $2\pi$-jumps of $\varphi(t)$ from one side of the interval $(-\pi, \pi]$ to another during time $t$ (see fig. 2, panel 1). If $\varphi(t)$ decreases, its jumps from $-\pi$ to $\pi$ are positive, and if $\varphi(t)$ increases, the jumps of the argument are negative (see fig. 1, panels 2 and 5, fig. 2, panels 2 and 5). Then

$$\text{Ind}(s,t) \overset{\text{def}}{=} - \sum_{T_n \in (s,t)} \text{sign}(\varphi(T_n + 0) - \varphi(T_n - 0)),$$

$$e^{i\hat{H}_2 t} = \frac{e^{-\frac{1}{2} \text{tr} B + i\pi \text{Ind}(0,t)}}{\sqrt{\det \Phi_t}} e^{-\frac{1}{2} (a^\dagger R_t a) \cdot e^{(a^\dagger (\Phi_t^{-1} - I) a)} \cdot e^{\frac{1}{2} (a R_t a)}}$$

is a continuous function, where the square root and the index are calculated according to (25) and (26) respectively. Examples of continuous reconstructions of the phase functions are shown in the last two lines of panels in fig. 1.

In next section, we derive a pure algebraic representation of $s_t$ for fast numerical implementation.

4. Algebraic forms of $e^{s_t}$ and normal symbols of squeezings

For $z \in \mathbb{C}^n$, define the normalized coherent vector $\psi(z) = e^{(z,a^\dagger) - (\pi,a)} |0\rangle = |z\rangle$. By (24) and by definition of the normal ordering, the normal symbol of $e^{i\hat{H}_2 t}$ is equal to

$$\langle z | e^{i\hat{H}_2 t} | z \rangle = \frac{e^{\text{Ind}(0,t)} - \frac{1}{2} \text{tr} B}{\sqrt{\det \Phi_t}} e^{-\frac{1}{2} \langle z, R_t z \rangle} e^{\langle z, (\Phi_t^{-1} - I) z \rangle} e^{\frac{1}{2} \langle z R_t z \rangle},$$
Figure 2. The first two panels show the discontinuous functions \( \text{ArgDet } \Phi(t) \in [-\pi, \pi] \) and \( \text{Arg } \frac{e^{-\frac{1}{2}i \text{tr } B}}{\sqrt{\det \Phi_t}} \in [-\pi, \pi] \) in (23). The second line shows continuous functions \( \text{Im } s(t) \in \mathbb{R} \) (see Eqs. (18), (19)) and \( \text{Im } S_t, \) where \( S_t = \text{Arg } \frac{e^{-\frac{1}{2}i \text{tr } B + i \pi \text{ Ind}(0, t)}}{\sqrt{\det \Phi_t}} \) differs from the corresponding picture in the first line by \( i\pi \text{ Ind}(0, t) \) in the exponential. The third line illustrates coincidence and continuity of the real part of \( \int_0^t \frac{1}{2} \text{tr } \rho \tau A d\tau \) in Eqs. (18) and \( \text{Log } \left| \frac{e^{-\frac{1}{2}i \text{tr } B}}{\sqrt{\det \Phi_t}} \right| \) in the right hand side of (23). The last two panels show \( \text{Ind}(0, t) \) defined by (26) and reconstructed continuity of the \( \text{ArgDet } \Phi(t) \) (c. f. panel 1 in this figure.)
and the commutation rule $e^{-(a^\dagger,x)+(a,x)}F(a^\dagger,a)e^{(a^\dagger,x)-(a,x)} = F(a^\dagger+z,a+z)$ implies that

$$\frac{e^{\text{Ind}(0,t)-\frac{\pi}{4} tr B}}{\sqrt{\det \Phi_t}} e^{-\frac{1}{2}(\tau,R \tau)} e^{\tau(\Phi_t^{-1}-I)} e^{\frac{1}{2}(z,\tau)} = \langle z | e^{i\hat{H}_2 t} | z \rangle = \langle 0 | e^{i\hat{H}_2 (a^\dagger+\tau,a+z)t} | 0 \rangle$$

$$= \langle 0 | e^{i(z,\tau)}e^{i(t)(a^\dagger,x)+(a,x)} | 0 \rangle e^{i\tau(\Phi_t^{-1}-\frac{1}{2}(\tau,\tau)+\frac{1}{2}(z,\tau))} = e^{st} e^{i\text{Im}(z,\tau)},$$

where we set $x = A\tau - iBz$ and $\tau = \bar{A}z + i\bar{B}$. If det $G \neq 0$, the equations $A\tau - iBz = h$, $\bar{A}z + i\bar{B} = \bar{h}$ are solvable with respect to $\{z, \tau\}$ so that

$$\begin{pmatrix} z(h,\bar{h}) \\ \tau(h,\bar{h}) \end{pmatrix} = G^{-1} \begin{pmatrix} h \\ \bar{h} \end{pmatrix}, \quad \hat{H}_2 + i(a^\dagger, (A\tau - iBz)) - i(a, (\bar{A}z + i\bar{B}z)) |_{z(h,\bar{h})} = \hat{H}. \quad (28)$$

Taking into account (2), (23), and (28), under assumption det $G \neq 0$, we obtain an algebraic expression for the scalar multiplier $e^{st}$:

$$e^{st} = \langle 0 | e^{i\hat{H}_t} | 0 \rangle = \frac{e^{-\frac{\pi}{4} tr B + Q_t}}{\sqrt{\det \Phi_t}}, \quad (29)$$

$$Q_t = \text{Ind}(0, t) + \frac{1}{2}(z, (\bar{\rho}_t - A\bar{t})z) - \frac{1}{2}(\tau, (R_t - A\bar{t})z) + (\tau, (\Phi_t^{-1} - I - iBt)z) |_{z(h,\bar{h})}.$$ 

Note that the second exponential can be represented as a symmetric quadratic form in terms of algebraic operations:

$$Q_t = \frac{1}{2} \begin{pmatrix} G^{-1} \begin{pmatrix} h \\ \bar{h} \end{pmatrix} \end{pmatrix}, \quad \begin{pmatrix} \bar{\rho}_t - A\bar{t} \\ (\Phi_t)^{-1} - I - iBt \end{pmatrix} \begin{pmatrix} \tau \end{pmatrix}, \quad G^{-1} \begin{pmatrix} h \\ \bar{h} \end{pmatrix} \end{pmatrix}.$$

The final result of this section is an algebraic representation of $s_t$ in terms of the matrices $e^{Gt}$, $G^{-1}(e^{Gt} - I)$, and $G^{-2}(e^{Gt} - I - Gt)$ which are well defined for degenerate and nondegenerate matrices $G$.

For $\hat{H} = \hat{H}_2 - (a^\dagger, h) + (a, \bar{h})$, we have $e^{st} = \langle 0 | e^{i\hat{H}_t} | 0 \rangle = \langle \phi_t | e^{-i\hat{H}_t} e^{i\hat{H}_t} | 0 \rangle$ with $\phi_t \equiv e^{-i\hat{H}_t} | 0 \rangle \in \otimes_1^2 \ell_2$. By unitary isomorphism between $\otimes_1^2 \ell_2$ and $L_2(\mathbb{R}^n)$

$$\otimes_1^2 \ell_2 \ni | 0 \rangle \leftrightarrow \frac{e^{-\frac{1}{2}|x|^2}}{\pi^{\frac{n}{2}}} \in L_2(\mathbb{R}^n), \quad a \leftrightarrow x + \frac{\partial_x}{\sqrt{2}}, \quad a^\dagger \leftrightarrow x - \frac{\partial_x}{\sqrt{2}},$$

decomposition (12) implies

$$e^{-i\hat{H}_t} | 0 \rangle = \frac{e^{\text{Ind}(0,t)+\frac{\pi}{4} tr B}}{\sqrt{\det \Phi_t}} e^{-\frac{1}{2}(a^\dagger R - a^\dagger)^2} | 0 \rangle \leftrightarrow e^{\text{Ind}(0,t)+\frac{|x|^2}{2}-(x, (I-R_t)^{-1}x)} \frac{\pi^{\frac{n}{4}}}{\sqrt{\det \Phi_t \det(I-R_t)}} \equiv \phi_t(x).$$

$$= e^{\text{Ind}(0,t)+\frac{|x|^2}{2}-(x, (I+\rho_t)^{-1}x)} \frac{\pi^{\frac{n}{4}}}{\sqrt{\det \Phi_t \det(I+\rho_t)}} = \phi_t(x), \quad (30)$$

where $\phi_t(x) \in L_2(\mathbb{R}^n)$, and $R_t = -\rho_t$, $\Phi_t = \Phi_t^*$. Moreover, the CCR (6) implies two useful identities for determinants: det $\Phi_t$ det $\Phi_t^*$ det $(I - R_t R_t^*) = 1$ and

$$\det \Phi_t \det \Phi_t^* \det(I-R_t) \det(I-R_t^*) \det((I-R_t)^{-1} + (I-R_t^*)^{-1} - I) = 1. \quad (31)$$

Let us prove the unitary equivalence of exponential vectors from $\ell_2$ and $L_2$:

$$\psi_t = e^{-i\hat{H}_2 t} e^{i(\hat{H}_2 - (a^\dagger, h) + (a, \bar{h})) t} | 0 \rangle = e^{-i(a^\dagger h_t) + i\tau_t - \frac{|h_t|^2}{2}} | 0 \rangle \leftrightarrow$$

$$\leftrightarrow e^{i\tau_t - \frac{(h_t, \bar{h}_t - h_t)}{2}} \frac{e^{-\frac{1}{2}(x + \sqrt{2}h_t, x + \sqrt{2}h_t)}}{\pi^{\frac{n}{2}}} \equiv \psi_t(x), \quad (32)$$
where \( h_t, \bar{h}_t \) are the same as in (5), and

\[
\gamma_t = \text{Im} \int_0^t \langle \dot{h}_s, \bar{h}_s \rangle ds, \quad e^{*t} = \langle \phi_t, \psi_t \rangle_{\ell_2} = \int_{\mathbb{R}^n} \bar{\phi}_t(x) \psi_t(x) d^n x = \int_{\mathbb{R}^n} \phi_{-t}(x) \psi_t(x) d^n x. \quad (33)
\]

By taking the time derivative of the left hand side of (32) in \( \ell_2 \) representation, we obtain

\[
(\Phi_{-t} a + \Psi_{-t} a^\dagger, \bar{h}_t) - (\bar{\Phi}_{-t} a^\dagger + \bar{\Psi}_{-t} a, h_t) - (a + h_t, \bar{h}_t) + (a^\dagger, \dot{h}_t) + i\dot{\gamma}_t - \frac{(h_t, \bar{h}_t) - (\dot{h}_t, \bar{h}_t)}{2} = 0.
\]

Note that zero values of coefficients at \( a, a^\dagger, \) and \( I \) are the necessary conditions for this equality. Taking into account the identities \( \Phi_{-t} = \Phi_t^*, \Psi_{-t} = -\Psi_t^T, \) we obtain the following equations

\[
\begin{pmatrix} \dot{h}_t \\ \bar{h}_t \end{pmatrix} = \begin{pmatrix} \Phi_t & \Psi_t \\ \bar{\Psi}_t & \bar{\Phi}_t \end{pmatrix} \begin{pmatrix} h_t \\ \bar{h}_t \end{pmatrix}, \quad i\dot{\gamma}_t = \frac{(\dot{h}_t, \bar{h}_t) - (h_t, \bar{h}_t)}{2} = \text{Im} (h_t, \bar{h}_t).
\]

Consider the integral representation of (33)

\[
\begin{pmatrix} h_t \\ \bar{h}_t \end{pmatrix} = \frac{e^{Gt} - I}{G^2} \begin{pmatrix} h \\ \bar{h} \end{pmatrix}, \quad i\gamma_t = \text{Im} \int_0^t \langle \dot{h}_s, \bar{h}_s \rangle ds = \frac{1}{2} \int_0^t \text{det} \begin{pmatrix} \dot{h}_t & \bar{h}_t \\ h_t & \bar{h}_t \end{pmatrix} ds
\]

and let us transform the above integral to algebraic form:

\[
i\gamma_t = \frac{1}{2} \left( \frac{I - Gt - e^{-Gt}}{G^2} \begin{pmatrix} h \\ \bar{h} \end{pmatrix}, \begin{pmatrix} \bar{h} \\ -h \end{pmatrix} \right), \quad e^{Gt} \equiv \begin{pmatrix} \Phi_t & \Psi_t \\ \bar{\Psi}_t & \bar{\Phi}_t \end{pmatrix}.
\]

The symplectic property of canonical transformations (4) implies

\[
\begin{pmatrix} \Phi_t & \Psi_t \\ \bar{\Psi}_t & \bar{\Phi}_t \end{pmatrix}^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \Phi_t & \Psi_t \\ \bar{\Psi}_t & \bar{\Phi}_t \end{pmatrix} = \begin{pmatrix} \Phi_t^T & \Psi_t^T \\ -\Psi_t & -\Phi_t \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} e^{-G^T t} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\]

Hence from equation (5) we have

\[
2 \text{Im} (h_s, \bar{h}_s) = \text{det} \begin{pmatrix} h_s & h_s \\ \bar{h}_s & \bar{h}_s \end{pmatrix} = \frac{e^{Gt} - I}{G} \begin{pmatrix} h \\ \bar{h} \end{pmatrix}, \quad \frac{e^{G^T t}}{G} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} e^{-Gt} \begin{pmatrix} h \\ \bar{h} \end{pmatrix}
\]

Integration of this equality in \( s \) over \([0,t]\) readily implies (34). Finally, by combining (30) and (32), we obtain an algebraic expression for \( e^{*t} = \langle 0 | e^{i\hat{H}t} | 0 \rangle \) and also expressions for \( \psi_z = e^{S_t - \frac{1}{2} (a^\dagger, R a^\dagger - \langle G_t, a^\dagger \rangle | z \rangle \rangle \) and \( N_{A,B,H}(z, z) = \langle z | \psi_z \rangle \) as corollaries.

**Theorem 2.**

1. For arbitrary symmetric matrix \( A \), Hermitian matrix \( B \), and complex vector \( h \), the vacuum expectation of the unitary group \( e^{i H t} \) (4) is equal to

\[
e^{*t} = \langle 0 | e^{i H t} | 0 \rangle = e^{i \text{Ind}(0,t) + i \gamma_t - \frac{1}{2} (h_t, h_t) - \frac{1}{2} (\bar{h}_t, \bar{h}_t)} \int d^n x \frac{e^{-\sqrt{2}(h_t, x) - (x, (I + \rho_t^\dagger)^{-1} x)}}{\pi^{n/4} \sqrt{\text{det}(I + \rho_t^\dagger) \text{det} \Phi_t}}
\]

\[
= \frac{e^{i \text{Ind}(0,t) + i \gamma_t - \frac{1}{2} (h_t, (\bar{h}_t, \bar{h}_t)) + it \text{tr} B}}{\sqrt{\text{det} \Phi_t}} = \frac{e^{i \text{Ind}(0,t) + i \gamma_t - \frac{1}{2} (\bar{h}_t, \bar{h}_t)) + it \text{tr} B}}{\sqrt{\text{det} \Phi_t}},
\]

where \( \bar{\rho_t} = \rho_t^\dagger, \) \( h_t \) and \( \gamma_t \) are given by (5) and (34).
2. The state $\psi_z = e^{S_t - \frac{1}{2}(a^\dagger R_t a)}(G_t, a^\dagger)|z\rangle$ is a unit vector in $\otimes^n \ell_2, e^{S_t}$ and its image in $L_2(\mathbb{R}^n)$ is equal to the Gaussian function

$$\psi_t(x) = \frac{e^{S_t}}{\pi^n \sqrt{\det(I - R_t)}} e^{\frac{1}{2}|x|^2 -(x+\frac{a_t}{\sqrt{2}})(I-R_t)^{-1}(x+\frac{a_t}{\sqrt{2}}))} \in L_2(\mathbb{R}^n),$$

where $G_t = \Phi_t^{-1}(h_t - z), S_t = s_t + (z, \bar{f}_t - \frac{1}{2}(z - \bar{p}_t z))$, and $f_t = h_t - \rho_t \bar{f}_t$ (see (15)).

3. The normal symbol of squeezing $[\hat{z}]$ is equal to

$$N_{A,B,h}(z, z) \overset{\text{def}}{=} \langle z| e^{i\hat{H}t}|z\rangle = e^{\text{Ind}_t + s_t - |z|^2 - \frac{1}{2}(\bar{z}, R_t \bar{z}) - \frac{1}{2}(z, (\Phi_t^{-1} - I)z) + \frac{1}{2}(z, \bar{p}_t z) + (f_t, z)}.$$ (37)

The coincidence of expressions (18), (19), (29), (35) was tested numerically. The testing modules are available for users of Wolfram Mathematica at [19].

5. Inner product of squeezed states and composition of squeezings

The inner products of squeezed states are necessary for constructing orthonormal bases, and the symbols of compositions of squeezings allow one to represent in algebraic terms the quantum evolution of multimode systems in some important cases.

In this section we use the well known canonical isometric isomorphism between $\otimes^n \ell_2$ and $L_2(\mathbb{R}^n)$, so that $|0\rangle \leftrightarrow \frac{e^{-|z|^2}}{\pi^{n/2}}$ and $a \leftrightarrow \frac{z}{\sqrt{2}}, a^\dagger \leftrightarrow \frac{\bar{z}}{\sqrt{2}}$. According to equation (4.1) from [11], the multimode squeezed state

$$e^{i\hat{H}t}|z\rangle = e^{S_t - \frac{1}{2}(a^\dagger R_t a)}(g_t, a^\dagger)|0\rangle \in \otimes^n \ell_2, \quad z \in \mathbb{C}^n$$

is unitary equivalent to the Gaussian $\psi$-function

$$\psi_t(x) = \frac{e^{S_t}}{\pi^n \sqrt{\det(I - R_t)}} e^{\frac{1}{2}|x|^2 -(x+\frac{g_t}{\sqrt{2}})(I-R_t)^{-1}(x+\frac{g_t}{\sqrt{2}}))} \in L_2(\mathbb{R}^n),$$

where $g_t = \Phi_t^{-1} h_t$.

The calculation of the norm $||\psi_t||^2_{L_2}$ reduces to integration of the Gaussian function $\overline{\psi_t}(x)\psi_t(x)$. Note that $R_t = R_t^T, \rho_t = \rho_t^T$, and [0] imply a set of useful identities:

$$I - R_t R_t^* = I - R_t \overline{R_t} = \frac{1}{\det(I - R_t \overline{R_t})} \det \Phi_t \det \overline{\Phi_t} = I,$n$$

$$\Omega_t = (I - \overline{R_t})^{-1} + (I - R_t)^{-1} - I = (I - \overline{R_t} R_t)(I - R_t)^{-1} = \overline{\Omega}_t = \Omega_t^T,$$

and $\Omega_t^{-1} = (\Phi_t^T - \Psi_t^T)(\Phi_t - \overline{\Psi_t}) = (\Phi_t^* - \Psi_t^*)(\Phi_t - \overline{\Psi_t})$. Therefore, $\overline{\psi_t}(x)\psi_t(x)$ is a well defined Gaussian density with correlation matrix $\Omega_t > 0$. After integration of a product of Gaussian functions [36] we obtain

$$||\psi_t||^2_{L_2} = \frac{1}{\sqrt{\det(I - R_t \overline{R_t})}} e^{2\text{Re} s_t + 2\text{Re} (I - R_t)^{-1} g_t, \Omega_t^{-1} \text{Re} (I - R_t)^{-1} g_t)} = 1$$ (38)

because from $e^{2\text{Re} s_t} = \sqrt{\det(I - \overline{R_t} R_t)}$ and det $M = \text{det} M^T$ we have

$$\frac{e^{2\text{Re} s_t}}{\sqrt{\det(I - \overline{R_t} R_t)}} = \sqrt{\text{det}(\Phi_t \Phi_t^* - \Psi_t \Psi_t^*)^{-1}} = 1.$$
On the other hand,
\[ \text{Re} \left( (I - R_t)^{-1} g_t, \Omega^{-1} \text{Re} \left( (I - R_t)^{-1} g_t \right) - \text{Re} \left( g_t, (I - R_t)^{-1} g_t \right) \right) = 0. \]

Similarly, for \( G_k = \Phi_1^{-1}(h - z_k), S_k = s_k + (\bar{f}_k, z_k) + \frac{(z_k, \bar{z}_k, z_k)}{2} - \frac{1}{2} |z_k|^2, R_k = \Phi_k^{-1} \Psi_k \) \((k = 1, 2), \) and \( Y = (I - R_1)^{-1} G_1 + (I - R_2)^{-1} G_2, \) we calculate the inner product of squeezed states in \( L_2(\mathbb{R}^n) \) or \( \otimes_1 R_2 \) representation:
\[
\langle \psi_1, \psi_2 \rangle_{L_2} = e^{S_1 + S_2} \int e^{\frac{1}{2} - \frac{1}{2}(x + \frac{1}{2}(\varphi_1^2 Y)), \Omega_1(x + \frac{1}{2}(\varphi_1^2 Y))} \frac{d^n x}{\sqrt{\det(I - R_1) \det(I - R_2)}} \end{equation}
\]
\[ \Omega_{12} = \Omega_{12}^T = (I - R_1)^{-1} + (I - R_2)^{-1} - I = (I - R_1)^{-1}(I - R_1 R_2)(I - R_2)^{-1}, \]
\[ \sigma_{12} = S_1 + S_2 - \frac{1}{2} ((G_1, (I - R_1)^{-1} G_1) - \frac{1}{2} ((G_2, (I - R_2)^{-1} G_2) + \frac{1}{2} (Y, \Omega_{12} Y). \]

A simple approach to the composition of squeezings can be given in terms of canonical transformations. Consider \( U_1 = e^{-iH_1}, U_2 = e^{iH_2} \) with unit time \( t = 1. \) We skip here the time dependence because the semigroup property does not hold for the composition \( U_1 U_2. \) The action of \( U_k \) on functions of \( a \) \( \dagger, a \) can be expressed in terms of \( \Phi_k \) and \( \Psi_k \) by \([5]. \) Since the scalar operators \( U_k \) commute with numerical expressions or matrices with scalar valued coefficients and act just on the creation-annihilation operators, we have
\[
\left( \begin{array}{c}
 a_2 \\
 a_2^\dagger \\
 \end{array} \right) = U_2 U_1 \left( \begin{array}{c}
 a \\
 a^\dagger \\
 \end{array} \right) U_1^* U_2^* = \left( \begin{array}{cc}
 \Phi_{12} & \Psi_{12} \\
 \Psi_{12}^* & \Phi_{12}^* \\
 \end{array} \right) \left( \begin{array}{c}
 a \\
 a^\dagger \\
 \end{array} \right) + \left( \begin{array}{c}
 h_{12} \\
 \bar{h}_{12} \\
 \end{array} \right), \]
\[ \Phi_{12} = \Phi_1 \Phi_2 + \Psi_1 \bar{\Psi}_2, \ \Psi_{12} = \Phi_1 \Psi_2 + \Phi_1 \bar{\Psi}_2, \ \ h_{12} = \Phi_{12} h_2 + \Psi_{12} \bar{h}_2 + h_1. \]

It can be readily proved that \( \Phi_{12} \) and \( \Psi_{12} \) possess the CCR property \([6]. \) Then
\[
U_{12} = e^{s_{12}} e^{\frac{1}{2}(a^\dagger R_{12} a^\dagger) - (g_{12}, a^\dagger a)} e^{\frac{1}{2}(a, \bar{R}_{12} a)} e^{\frac{1}{4}((a, R_{12} a), a^\dagger), \bar{R}_{12} a), \}
\]
where \( R_{12} = \Phi_{12}^\dagger \Psi_{12} \bar{\Psi}_{12} - \bar{\Psi}_{12} \Psi_{12} - g_{12} = \Phi_{12}^\dagger h_{12}, \ f_{12} = h_{12} - \bar{g}_{12} \bar{h}_{12}, \) and (see \([40]\))
\[
e^{s_{12}} = \langle 0 | e^{-i\hat{H}_1 t_1} e^{i\hat{H}_2 t_2} | 0 \rangle = \frac{e^{\sigma_{12}}}{\sqrt{\det(I - R_1 R_2)}}. \]

These collection of parameters describe the normal ordering of the composition of squeeings:
\[
U_1 U_2 = e^{s_{12}} e^{\frac{1}{2}(a^\dagger R_{12} a^\dagger) - (g_{12}, a^\dagger a)} e^{a^\dagger C_{12} a} e^{\frac{1}{4}(a, R_{12} a), (\bar{R}_{12} a), }
\]

6. The Jordan decomposition of squeeings

In the general case, the Jordan decomposition \( G = DJ D^{-1} \) justifies a useful representation of \((2n \times 2n)\)-matrix \( S_t = e^{G t} = D e^{J t} D^{-1} \) as the exponent of the Jordan matrix \( J \) with \((n_k \times n_k)\)-blocks \( J_k:\)
\[
J_k \overset{\text{def}}{=} \left( \begin{array}{cccc}
 \lambda_k & 1 & 0 & \cdots & 0 \\
 0 & \lambda_k & 1 & \cdots & 0 \\
 0 & 0 & \ddots & \cdots & \vdots \\
 0 & 0 & 0 & \cdots & \lambda_k \\
 \end{array} \right) \rightarrow e^{-J_k t} = e^{\lambda_k t} \Delta_k,
\]
\[ \Delta_k \overset{\text{def}}{=} \left( \begin{array}{cccc}
 1 & 1 & \frac{1}{2} & \cdots & \frac{1}{(n_k - 1)!} \\
 0 & 1 & 1 & \cdots & \frac{1}{(n_k - 2)!} \\
 0 & 0 & \ddots & \cdots & \vdots \\
 0 & 0 & 0 & \cdots & 1 \\
 \end{array} \right). \]
The multiplicity $n_k$ of $\lambda_k$ coincides with the rank of $J_k$, and decomposition of
\[ F^{(1)}(t) = \frac{e^{Gt} - I}{G} = D \int_0^t e^{J\tau} d\tau D^{-1} = D \frac{e^{gt} - I}{J} D^{-1} \]
(44)
is well defined in regular and degenerate cases. The Jordan blocks $J_k$ generate triangle matrices $F^{(1)}(t) = \frac{e^{J_k t} - I}{J_k}$:
\[
J_k^{-1} = \lambda_k^{-1} \begin{pmatrix}
1 & -\lambda_k & 0 & \cdots & 0 \\
0 & 1 & -\lambda_k & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix},
\frac{e^{gt} - I}{G} = D \begin{pmatrix}
F_1^{(1)}(t) & 0 & \cdots & 0 \\
0 & F_2^{(1)}(t) & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & F_K^{(1)}(t)
\end{pmatrix} D^{-1},
\]
(45)
for $i \geq j$; otherwise, $F^{(1)}(i,j) = 0$. The matrices $F^{(1)}(t)$ are well defined in the degenerate case because $(F^{(1)}(t))_{ij} \to \frac{e^{(j-i)\lambda_k t}}{(j-i)!}$ as $\lambda_k \to 0$.

The Jordan decomposition can be also used for calculation of $\frac{e^{gt} - I - J_k t}{G^2}$ because the algebraic form of $\frac{e^{J_k t} - I}{J_k}$ is well defined in nondegenerate and degenerate cases:
\[ F^{(2)}(t) = D \int_0^t d\tau \int_0^\tau e^{J\tau} d\tau D^{-1} = \frac{e^{gt} - I - Gt}{G^2}, \quad F^{(2)}(t) = \frac{e^{J_k t} - I - J_k t}{J_k^2}. \]
(46)
Moreover, the following expressions for components related to Jordan decomposition are satisfied:
\[
J^2 = \begin{pmatrix}
\lambda_k^2 & 2\lambda_k & 1 & 0 & \cdots & 0 \\
0 & \lambda_k^2 & 2\lambda_k & 1 & \cdots & 0 \\
0 & 0 & \lambda_k^2 & 2\lambda_k & \cdots & 0 \\
0 & 0 & 0 & \lambda_k^2 & \cdots & 0 \\
0 & 0 & 0 & 0 & \lambda_k^2 & \cdots
\end{pmatrix}, \quad J^{-2} = \begin{pmatrix}
\lambda_k^{-2} & -2\lambda_k^{-3} & 0 & \cdots & 0 \\
0 & \lambda_k^{-2} & -2\lambda_k^{-3} & \cdots & 0 \\
0 & 0 & \lambda_k^{-2} & -2\lambda_k^{-3} & \cdots \\
0 & 0 & 0 & \lambda_k^{-2} & \cdots \\
0 & 0 & 0 & 0 & \lambda_k^{-2}
\end{pmatrix},
\]
(47)
\[
\frac{e^{gt} - I - Gt}{G^2} = D \begin{pmatrix}
F_1^{(2)}(t) & 0 & \cdots & 0 \\
0 & F_2^{(2)}(t) & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & F_K^{(2)}(t)
\end{pmatrix} D^{-1},
\]
(48)
\[
(F^{(2)}(t))_{ij} = \frac{(-1)^j}{\lambda_k^{j+2}} \left( j - i + 1 + \lambda_k t - e^{\lambda_k t} \sum_{m=0}^{j-1} \frac{(j - i + 1 - m)(-\lambda_k t)^m}{m!} \right),
\]
for $i \geq j$; otherwise, $(F^{(2)}(t))_{ij} = 0$. The triangle matrices $F^{(2)}(t)$ are well defined in the degenerate case because $(F^{(2)}(t))_{ij} \to \frac{(-1)^{j-i+1}}{(j-i+1)!}$ as $\lambda_k \to 0$. 
This observation establishes an algebraic representation for \( h(t), \overline{h}(t), \gamma_t, \) and \( s_t \) which follows from (5) and (15) with constant matrices \( D \) and well-defined triangle matrices \( F_k^{(1)}(t), F_k^{(2)}(t) \). Implementation time for calculation of \( F(t) \) according to (45), (47) is faster than by (44), (46) (see [19]).

7. An example of normal decomposition

In this section, we consider Hamiltonian (1) such that \( G \) is invertible and all matrices in (16) and (29) can be described explicitly in terms of \( G \) and the spectral expansions of the Hermitian matrix \( D = A \overline{A} - B^2 \) in any dimension.

Suppose that the matrix \( D = A \overline{A} - B^2 \) is not degenerate and \( BA = A \overline{B} \). Then 
\[
A \overline{A} = \overline{AB}, \quad BA \overline{A} = \overline{AB} \overline{A} = A \overline{AB}, \quad \text{and} \quad \overline{A}B^2 = \overline{B} \overline{A}B = \overline{B}^2 \overline{A}.
\]
These relationships imply that
\[
G^{2n} = \begin{pmatrix} D^n & 0 \\ 0 & \overline{D}^n \end{pmatrix}, \quad G^{2n+1} = G \begin{pmatrix} D^n & 0 \\ 0 & \overline{D}^n \end{pmatrix} = \begin{pmatrix} D^n & 0 \\ 0 & \overline{D}^n \end{pmatrix} G.
\]
(49)

Hence the matrix \( G^2 = \left( \begin{array}{cc} A \overline{A} - B^2 & 0 \\ 0 & \overline{A}A - \overline{B}^2 \end{array} \right) \) does not degenerate. Therefore, \( G \text{ def} \)
\[
\begin{pmatrix} -iB & A \\ \overline{A} & iB \end{pmatrix}
\]
do so does. Moreover, the matrices \( G^{-2} = \left( \begin{array}{cc} (A \overline{A} - B^2)^{-1} & 0 \\ 0 & (\overline{A}A - B^2)^{-1} \end{array} \right) \) and \( G^{-\frac{1}{2}} \) are well defined in terms of the spectral expansion of \( D \), and
\[
e^{Gt} = \begin{pmatrix} \Phi_t \\ \Psi_t \end{pmatrix} G \begin{pmatrix} D^{-\frac{1}{2}} \sinh D^\frac{1}{2}t & 0 \\ 0 & \overline{D}^{-\frac{1}{2}} \sinh \overline{D}^\frac{1}{2}t \end{pmatrix} + \begin{pmatrix} \cosh D^\frac{1}{2}t & 0 \\ 0 & \cosh \overline{D}^\frac{1}{2}t \end{pmatrix},
\]
(50)
\[
G^{-1} = G^{-2} G = \begin{pmatrix} -iD^{-1}B & D^{-1}A \\ \overline{D}^{-1}A & iD^{-1}B \end{pmatrix} = \begin{pmatrix} -iBD^{-1} & A \overline{D}^{-1} \\ A \overline{D}^{-1} & iBD^{-1} \end{pmatrix},
\]
\[
\begin{pmatrix} A & iB \\ iB & -A \end{pmatrix} \begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \begin{pmatrix} A & iB \\ iB & -A \end{pmatrix} \begin{pmatrix} -iB & A \\ \overline{A} & iB \end{pmatrix} G^{-2} \begin{pmatrix} h \\ \overline{h} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} h \\ \overline{h} \end{pmatrix}.
\]

Note that condition
\[
[A \overline{A}, B] = 0,
\]
(51)
does not imply that \( A \) and \( B \) commute, but if \( A \overline{A} \) has a simple spectrum, then \( BA = A \overline{B} \). Indeed, since \( [A \overline{A}, B] = 0 \), the matrices \( A \overline{A} \) and \( B \) must have a joint system of spectral projectors \( \{ \pi_k \} \) such that
\[
A \overline{A} = \sum_k d_k^2 \pi_k, \quad B = \sum_k \lambda_k \pi_k, \quad \sum_k \pi_k = I, \quad \pi_k^* = \pi_k^*, \quad \pi_k \pi_j = I \delta_{kj},
\]
where \( d_k^2 \) and \( \lambda_k \) are the eigenvalues of \( A \overline{A} \) and \( B \) respectively, and \( \pi_k \) are their common spectral projectors. If all \( \{ d_k^2 \} \) differ each other, then there exists the polynomial
\[
f(d) = \sum_k \lambda_k \prod_{d_m \neq d_k} \frac{d - d_m^2}{d_k^2 - d_m^2} = \sum_k f_k d^k, \quad f_k = f_k(\lambda, d) \in \mathbb{C}, \quad d \in \mathbb{R}_+.
\]
such that \( f(d_k^2) = \lambda_k \), and \( f(AA) = B \) follows from \( f(AA)\hat{\pi}_k = f(d_k^2)\hat{\pi}_k = \lambda_k\hat{\pi}_k \). Therefore, the “commutation relation”
\[
\overline{AB} = A\sum_k f_k(\overline{AA})^k = \sum_k f_k(\overline{AA})^k A = \overline{B}A
\]
(52)
is a consequence of (51) for matrices \( A\overline{A} \) with simple spectrum.

If the spectrum of \( A\overline{A} \) is multiple (for example, \( A = A\overline{A} = I \)) and \( B \neq \overline{B} \), then (52) clearly fails. On the other hand, (52) holds for the operators \( A \) such that the multiplicity of the spectrum of \( A\overline{A} \) is greater than or equal to the spectral multiplicity of \( B \), because in such case the polynomial representation \( B = f(A\overline{A}) \) remains well-defined and implies the equality \( BA = AB \) (see \[16\], sect. 4.4 for applications of this equality in linear algebra).

The relationship between the singular value decomposition of the Hermitian matrix \( A\overline{A} = U^{|D|^2}U \) (with unitary \( U \) and arbitrary diagonal matrix \( D \)), and the general representation of the symmetric matrix \( A \) follows from a modified version of the Takagi representation formula (see \[17\]): \( A = U^*DU \). In order to satisfy (51), we suppose that \( B = U^*\Lambda U \) with the same unitary \( U \) and arbitrary real diagonal matrix \( \Lambda \). Then \( A \) is symmetric, \( B \) is a Hermitian matrix, \( A\overline{A} = U^*D^2U \) and \( B = U^*\Lambda U \) commute, and \( A\overline{B} = BA = U^*\Lambda D\overline{U} \).

8. Numerical tests for integral and algebraic representations of \( s_t \)

Studying algebraic properties of the main objects related to symplectic matrices \([3]\), we have tested numerically non-trivial relations for randomly generated matrices \( A, B, \) and vectors \( h \).

1. The following representations for \( \gamma_t \) hold true:
\[
\gamma_t = \int_0^t ds \left( \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \left( \begin{pmatrix} h_s & 0 \\ 0 & \lambda_s \end{pmatrix} \right) e^{Gs} \left( \begin{pmatrix} h \end{pmatrix} \right) \right) = \left( \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \left( \begin{pmatrix} h \end{pmatrix} \right) \right) e^{\left( Gt - t - I \right)} \left( \begin{pmatrix} h \end{pmatrix} \right)
\]
2. For \( F_t = h_{-t} + R_t\overline{h}_{-t} \), the following representations of the vacuum expectation \( \langle 0 | e^{itH} | 0 \rangle = e^{s_t} \) are equivalent:
\[
e^{s_t} = e^{-\int_0^t ds \left( \frac{1}{2} (T_s, A_T) + \frac{1}{2} \text{tr}(\overline{A}) + (T_s, h_s) \right)} = e^{\int_0^t ds \left( \frac{1}{2} (F_s, A_F) - \frac{1}{2} \text{tr}(R_s, \overline{A}) + (F_s, h_s) \right) = \frac{e^{\text{Ind} \Phi_t}}{\sqrt{\text{det} \Phi_t}} e^{\frac{1}{2}(\gamma_t - it \text{tr} B + (\overline{h}_{-t}, \Phi^{-1} h_t))} = \frac{e^{\text{Ind} \Phi_t}}{\sqrt{\text{det} \Phi_t}} e^{\frac{1}{2}(\gamma_t - it \text{tr} B - (h_t, \overline{h}_{-t} - \rho_i h_t))}.
\]
3. Let \( \alpha = \frac{2}{|| (\Phi_t - \Psi_t) \text{Re} (\Phi_t - \Psi_t)^{-1} h_t ||^2} - \text{Re} \left( g_t, (\Phi_t - \Psi_t)^{-1} h_t \right) \). Then the unit norm of squeezed state \( |A, B, h_t = e^{itH} | 0 \rangle \) can be equivalently represented in terms of various objects:
\[
1 = ||A, B, h||^2 = \frac{1}{\sqrt{\text{det} (I - R_t^t R_t)}} e^{\alpha_t - \text{Re} \int_0^t ds (\langle T_s, A_T \rangle + \text{tr}(\overline{A}) + (T_s, h_s) \rangle
\]
\[
\frac{1}{\sqrt{\det(I - R_t R_t^* )}} e^{2\Re(s_t + \alpha_t)} = e^{\Re(\alpha_t + \gamma_t + (\overline{R_t} \Phi_t^* h_t))} = e^{\Re(\alpha_t + \gamma_t - (\overline{R_t} \Phi_t h_t, h_t))}.
\]

Note that the normalization conditions (54) are independent on the index function.

The graphs in fig. 1 and fig. 2 were created for randomly chosen \( A, B, \) and \( h \):

\[
A = \begin{pmatrix}
1.694 + 0.3276i & 0.317 + 0.54i & 0.509 + 0.331i \\
0.317 + 0.54i & 0.0031 + 1.9513i & 0.6619 + 0.0605i \\
0.509 + 0.331i & 0.6619 + 0.0605i & 0.5526 + 0.5576i
\end{pmatrix},
\]

\[
h = \begin{pmatrix}
-0.6898 + 0.8259i \\
-0.3758 + 0.0629i \\
-0.4417 - 0.5016i
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
1.3802 & 1.8946 + 0.5657i & 1.1696 + 1.1702i \\
1.8946 - 0.5657i & 1.2728 & 1.7892 + 1.3761i \\
1.1696 - 1.1702i & 1.7892 - 1.3761i & 0.5547
\end{pmatrix}.
\]

The coincidence of \( s_t \) in expressions (18) and (29) was tested for random matrices \( A, B, \) and the vector \( h \) generated numerically by using Wolfram Mathematica. For 3, 4, and 5-modes systems, the representation (29) was implemented 650, 8000, and 141000 times faster; the values of \( s_t \) calculated by (18) and (29) coincide with accuracy \( 10^{-11} \).

The functions (18) and (29) are deposited at [19] as Mathematica 7 modules.

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