Symmetries of the Boundary of $AdS_5 \times S^5$

and Harmonic Superspace

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We study the boundary limit of the bulk isometries of $AdS \times S$. The superconformal symmetry is realized on the coordinates of the $AdS$ boundary, the fermionic superspace coordinates, and the harmonics on the sphere. We show how these may be related to the coordinates of an off-shell harmonic superspace of SCFT living on the boundary. In the special case of $d = 4, \mathcal{N} = 2$ super Yang-Mills theory, a truncation of the $\mathcal{N} = 4$ SYM dual to Type IIB string theory compactified on $AdS_5 \times S^5$, we identify the bosonic space $SU(2)/U(1)$ of the $\mathcal{N} = 2$ harmonic superspace (known before as auxiliary) with the $S^2$ submanifold of the $S^5$.

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There is a growing interest in the \( AdS/CFT \) correspondence conjecture \([1]\). One of the basic arguments in favor of this conjecture concerning \( \mathcal{N} = 4 \) Yang-Mills theory is the fact that the superisometry of the \( AdS \) superspace forms the same \( SU(2, 2|4) \) supergroup as the symmetries of the \( \mathcal{N} = 4 \) Yang-Mills field theory, which supposedly lives on the boundary of the \( AdS \) space. The superspaces with the bosonic \( AdS_{p+2} \times S^{d-p-2} \) geometry and the proper form field were recently constructed in \([2]\), and their superisometries were found in closed form in \([3]\). The purpose of this note is to motivate a connection between the limit to the boundary (\( \rho \to \infty \)) of the bulk \( AdS_5 \times S^5 \) superspace with the harmonic superspace \([4]\) of the off-shell 4-dimensional \( \mathcal{N} = 2 \) Yang-Mills theory.

The superisometries found in \([3]\) for the \( AdS_5 \times S^5 \) case are governed by the supercoset structure \( G/H = SU(2, 2|4)/(SO(1, 4) \times SO(5)) \) \([5, 2]\). They are of the form

\[
\begin{align*}
\delta \theta |_{\text{bulk}} &= \delta \theta(x, \rho, u, \theta | \epsilon, \eta, a, \lambda_M, \lambda_D, \Lambda_K, \Lambda_R), \\
\delta x |_{\text{bulk}} &= \delta x(x, \rho, u, \theta | \epsilon, \eta, a, \lambda_M, \lambda_D, \Lambda_K, \Lambda_R), \\
\delta \rho |_{\text{bulk}} &= \delta \rho(x, \rho, u, \theta | \epsilon, \eta, a, \lambda_M, \lambda_D, \Lambda_K, \Lambda_R), \\
\delta u |_{\text{bulk}} &= \delta u(u, \theta | \epsilon, \eta, a, \lambda_M, \lambda_D, \Lambda_K, \Lambda_R).
\end{align*}
\]

Here, \( x \) are the coordinates of the 4-dimensional space parallel to the boundary of the \( AdS_5 \) space, and \( \rho \) is the \( AdS_5 \) coordinate orthogonal to the boundary, which is located at \( \rho = \infty \). The 32 fermionic coordinates \( \theta \) are the fermionic coordinates of type IIB string theory/supergravity, and finally \( u \) are harmonics of the five-sphere, which may be given as functions of the angles parametrizing the sphere, i.e. \( u(\phi) \). The superisometries are rigid transformations, i.e. they are well defined functions of the superspace coordinates and \emph{global} parameters. These are naturally the parameters of the superconformal transformations, namely supersymmetry \( \epsilon \), special supersymmetry \( \eta \), translations \( a \), Lorentz transformations \( \lambda_M \), dilatations \( \lambda_D \), special conformal transformations \( \Lambda_K \) and \( R \)-symmetry \( \Lambda_R \).

Under these transformations (which have to be supplemented by the compensating transformations of the frame from the stability group \( (SO(4, 1) \times SO(5)) \)) the geometry
of the $AdS_5 \times S^5$ superspace is invariant. The expressions for the superisometries $\delta Z^M$ (where $Z$ denotes all superspace coordinates) are rather complicated, and they are explicitly constructed in [3]. In short, they are given in terms of components of a covariantly constant Super-Killing field $\Sigma(Z)$ and the inverse of the supervielbein $E^M_M$ via

$$\delta Z^M = \Sigma^M E^M_M. \quad (5)$$

At $\theta = 0$, the supervielbein reduces to the standard vielbein, the gravitino, and a fermion-fermion sector. The Killing field $\Sigma$ depends linearly on the Killing vectors and Killing spinors $\Sigma_0(x, \rho, u(\phi))$ of the background geometry (and the corresponding compensating transformations). Naturally, both $E$ and $\Sigma$ are also functions of the $SU(2, 2|4)$ structure constants and $\theta$.

At first glance the expressions for the isometries do not have a well defined limit at $\rho \to \infty$. For example, the metric, and hence the vielbein, has terms with positive and negative powers of $\rho$:

$$g_{mn} = \rho^2 \delta_{mn} \quad g_{\rho\rho} = \left(\frac{R}{\rho}\right)^2, \quad (6)$$

where $R$ is the radius of curvature of the $AdS_5$ space as well as of the sphere $S^5$. Also, the translations $\Sigma^m_0$ parallel to the boundary and the compensating Lorentz transformations $\Sigma^{m\rho}_0$ which rotate in the bulk-boundary plane $(x^m, \rho)$ have a non-trivial $\rho$ dependence. They are of the form

$$\Sigma^m_0 = \rho \xi^m(x) + \left(\frac{R^2}{\rho}\right) \Lambda^m_K \quad (7)$$

$$\Sigma^{m\rho}_0 = \left(\frac{\rho}{R}\right) \xi^m(x) - \left(\frac{R}{\rho}\right) \Lambda^m_K. \quad (8)$$

In addition, we see the dependence on $\rho$ in the Killing spinors of the $AdS_5 \times S^5$ space,

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1For details and notations see [3]. Note that we changed a sign relative to (3.1) of [3] for readability.
which depend on the harmonics of the five sphere \( u_i^I(\phi) \):

\[
\left( \begin{array}{c}
\Sigma_0^I \\
\tilde{\Sigma}^I_0 \\
\end{array} \right) = \left( \begin{array}{c}
\left( \frac{\rho}{R} \right)^{1/2} \epsilon(x)_\alpha^I \\
\left( \frac{\rho}{R} \right)^{1/2} \bar{\eta}^{\dot{\alpha} i} \\
\end{array} \right) u_i^I(\phi),
\]

where

\[
\epsilon(x)_\alpha^i = \epsilon^{\alpha i} + \left( x^m \sigma_m \right)_{\alpha \dot{\alpha}} \bar{\eta}^{\dot{\alpha} i},
\]

and where \( \epsilon \) and \( \bar{\eta} \) are the parameters of the Q-supersymmetry and S-supersymmetry resp. We have found that this irregular dependence of the bulk isometries on \( \rho \) can be perfectly organized if we use two important technical tools.

- The superisometries in the bulk were found in the AdS basis of the superalgebra which treats the bulk direction on equal footing with the boundary directions since there is a manifest \( SO(4,1) \times SO(5) \) symmetry, mixing bulk and boundary directions. The limit to the boundary, however, should be performed in the superconformal basis of the algebra which has only manifest \( SO(1,3) \) symmetry in the \( AdS_5 \) sector, reflecting the split of the \( AdS \) coordinates into \( x \) and \( \rho \).

- We use the Killing gauge in the superspace \(^2\), since in this gauge the gravitino vanishes not only at \( \theta = 0 \) for the bosonic background but also to all orders in \( \theta \) which makes various calculations doable and easy. In this gauge we find the limit to the boundary \( \rho \to \infty \) in which the radial direction \( \rho \) decouples.

The \( SU(2,2|4) \) superalgebra with generators \( T_\Lambda \) and structure constants \( f_{\Lambda \Sigma}^{\Delta} \)

\[
[T_\Lambda, T_\Sigma] = f_{\Lambda \Sigma}^{\Delta} T_\Delta
\]

is defined up to linear redefinitions of the generators. The most useful basis for our purposes is the superconformal basis, whose most important property is that

\(^2\)We use a two-component notation, which means that we work with a representation of \( \gamma \)-matrices in 5 dimensions (AdS) for which \( \gamma_5 \) is diagonal. The \( AdS_5 \times S^5 \) Killing spinor consists of four 4-dimensional Dirac spinors transforming under the \( SU(4) \) with index \( i = 1,2,3,4 \) and the index \( I = 1,2,3,4 \) labels the spinorial representation of the stability group on the sphere \( SO(5) \). More details on Killing spinors in terms of the harmonics will be given in a future publication \(^3\).
there exists a grading operator $D$, under which every generator $T_\Lambda$ has a particular conformal weight:

$$[D, T_\Lambda] = W(T_\Lambda) T_\Lambda.$$  \hfill (12)

In fact, the translation in the $\rho$-direction becomes the dilatation operator $D$, translations along the boundary coordinates $x$ combined with the Lorentz transformations mixing the $\rho$ and $x$ directions give 4 translation and 4 special conformal transformations along the boundary directions. Translations $P_m$ have weight +1, special conformal transformations $K_m$ have -1, dilatations $D$, Lorentz transformations $M_{mn}$ and R-symmetry transformations $U_{ij}$ have weight 0. The supersymmetries $Q$ have weight +1/2 and special supersymmetries $S$ have weight -1/2. It turns out that the superconformal basis of the superalgebra displays the $\rho$ dependence of isometries in a very clear way. We have found that all isometries $\Sigma$ at $\theta = 0$ in the superconformal basis are homogeneous in $\rho$, which is expected because the $\rho$ direction is associated with dilatations and every conformal operator has a specific Weyl weight. They scale according to the conformal weight of the corresponding operator:

$$\Sigma^\Lambda_m(x^m, \rho, u(\phi)) \equiv \rho^{W(\Sigma)} \Sigma^\Lambda_m(x^m, u(\phi)) = \Sigma^\Lambda_m(x^m, u(\phi)) R^\Lambda_m(\rho).$$ \hfill (13)

Now, the matrix $R^\Lambda_m(\rho)$ carries all dependence on the radial variable and the fields $\Sigma^\Lambda_m(x^m, u(\phi))$ depend only on the bosonic coordinates of the boundary, and the harmonics of the sphere.

The dependence of $\Sigma$ and $E$ on $\theta$ is such that use of the Killing spinor gauge \cite{2} together with (13) allows all non-trivial $\rho$ dependence in $\delta Z$ to be absorbed into new $\rho$-dependent structure “constants” via the matrix $R(\rho)$ defined in (13). The crucial observation at this point is that the structure constants of the algebra have the property that for any component $f^\Lambda_{\Lambda\Sigma}$

$$W(T_\Lambda) + W(T_\Sigma) = W(T_\Delta)$$ \hfill (14)

holds. This identity shows then that in the ‘underlined’ basis the structure “constants” $f^\Lambda_{\Lambda\Sigma}(\rho)$ are numerically nothing but the old constants $f^\Lambda_{\Lambda\Sigma}$, since

$$f^\Lambda_{\Lambda\Sigma}(\rho) \equiv (R^{-1})^\Lambda_\Delta(\rho) R^\Lambda_\Delta(\rho) f^\Lambda_{\Lambda\Sigma} R^\Sigma_\Sigma(\rho)$$

$$= \rho^{W(T_\Lambda)+W(T_\Sigma)-W(T_\Delta)} \delta^\Lambda_\Delta \delta^\Sigma_\Sigma f^\Lambda_{\Lambda\Sigma} = \delta^\Lambda_\Delta \delta^\Sigma_\Sigma f^\Lambda_{\Lambda\Sigma},$$ \hfill (15)
i.e. the $\rho$-dependence cancels out. This gives us control over the limit $\rho \to \infty$ in all expressions for superisometries. We find that the bulk-boundary isometry correspondence has the following form:

$$
\delta \theta|_{\text{bulk}} = \delta \theta|_{\text{boundary}}(x, u, \theta),
$$

(16)

$$
\delta x|_{\text{bulk}} = \delta x|_{\text{boundary}}(x, u, \theta) + \frac{1}{\rho^2} f(x, u, \theta),
$$

(17)

$$
\delta u|_{\text{bulk}} = \delta u|_{\text{boundary}}(u, \theta).
$$

(18)

The isometry in the radial direction blows up in the limit $\rho \to \infty$:

$$
\delta \rho|_{\text{bulk}} = \rho g(x, u, \theta),
$$

(19)

but decouples from the boundary isometries.

We do not specify here the functions of $(x, u, \theta)$ introduced above as they are still rather complicated, but they can be read off from the generic expressions for the isometries in the bulk, given explicitly in [3].

Thus we have extracted the $\rho$-dependence of the isometries in the superconformal basis and realized the algebra on coordinates of the boundary space $(x, u, \theta)$. In the bulk we have a spontaneously broken form of the superconformal symmetry since $\rho$ occurs explicitly in the transformations. However at the boundary $\rho \to \infty$ where the $\rho$ dependence is absent, we have a realization of the superconformal symmetry without spontaneous breaking. Note that the boundary of the superspace naturally includes the 4 bosonic coordinates of the boundary of the $AdS_5$ space, the fermionic coordinates $\theta$, and the harmonics of the $S^5$ sphere, which are not decoupled at the boundary!

One can now try to understand what the role of these harmonics is from the point of view of the SCFT in 4 dimensions. Their structure and transformation laws suggest a connection to the auxiliary coordinates of the off-shell harmonic superspace of the 4-dimensional Super Yang-Mills theory. We will study the simplest example and derive the $\mathcal{N}=2$ $d=4$ off-shell harmonic superspace [4] of the super Yang-Mills theory from the string theory/supergravity $AdS_5 \times S^5$ superspace.

To find the correspondence with the SYM theory we will require only $Q$ supersymmetry to be realized manifestly. The Lagrangians of the 4-dimensional $\mathcal{N} = 2$
SYM theory have also a superconformal symmetry but this does not have to be a manifest property of the off-shell superfields. To do the truncation consistently we do not require that the special supersymmetry $S$, dilatation $D$, and the special conformal symmetry $K$ are symmetries anymore. It means that we are fixing special supersymmetry to keep the $\bar{\lambda}^{\hat{\alpha}}$, which is the component of $\theta$ along the special supersymmetry direction, at $\bar{\lambda}^{\hat{\alpha}} = 0$. Thus we choose $\eta = 0, \lambda_D = \Lambda_K = 0$ simultaneously with $\bar{\lambda}^{\hat{\alpha}} = 0$. The expressions for the isometries are greatly simplified, in particular, the matrix $M^2$, appearing in the isometries as given in [3], drops out in the truncated theory. We are left with $1/2$ of the fermionic coordinates, 4 bosonic coordinates $x^m$ and harmonics $u_I^i$ on $S^5$. The fermionic coordinates related to $Q$-supersymmetry can be taken as four complex 2-component left-handed $d = 4$ spinors which we denote by $\theta_i^a, i = 1, 2, 3, 4$ and $\alpha = 1, 2$. We derive from the superisometries in the bulk [3] that after truncation the remaining symmetries at the boundary are:

$$\delta \theta^i_\alpha = \epsilon^i_\alpha + \frac{1}{4}(\lambda_{mn}\sigma^{mn}\theta^i_\alpha) + \theta^j_\alpha \Lambda^j_i,$$  \hspace{1cm} (20)

$$\delta x^m = a^m + \lambda^m x_n + i(\epsilon^i\sigma^m\tilde{\theta}_i - \theta^i\sigma^m\epsilon_i).$$  \hspace{1cm} (21)

The symmetries of the sphere are given in terms of the coset representative (i.e. the harmonics $u(\phi)$) as follows

$$gu(\phi) = u(\phi')h(\phi, g),$$  \hspace{1cm} (22)

where, $g$ is an element of $SO(6)$, $h(\phi)$ is an element of $SO(5)$, and $\phi'$ are the transformed angles of the sphere. Explicitly, the infinitesimal transformations take the form

$$- \Lambda^i_J u^J_i(\phi) = u^K_i(\phi + \delta \phi) - u^K_i(\phi) - u^L_i(\phi)H^K_L(\phi, \Lambda),$$  \hspace{1cm} (23)

where $\Lambda^i_J$ are the parameters of the transformation from the $SO(6)$ group in spinorial

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3Here we use the Killing Spinor gauge [2], which means that we decompose the 32 component fermionic coordinates $\theta$ in the same way as the Killing spinor [1], i.e.

$$\theta^{\alpha}_I = \begin{pmatrix} (\frac{\epsilon}{\pi})^{1/2}(\theta^i_\alpha + (x^m \sigma_m)_{\alpha \hat{\alpha}} \bar{\lambda}^{\hat{\alpha} i}) \\ (\frac{\pi}{\epsilon})^{1/2}(\bar{\lambda}^{\hat{\alpha} i}) \end{pmatrix} u_I^i(\phi).$$
representation and $H^K_L(\phi)$ are the parameters of the transformation from the stability group $SO(5)$.

The truncated system of coordinates at the boundary has a symmetry realizing the truncated $SU(2, 2|4)$ algebra without $S, D, K$, which is the simply the extended $\mathcal{N} = 4$ super Poincaré algebra in $d = 4$. It is realized on 4-dimensional coordinates $x^m$, on complex spinors $\theta_i^\alpha$ and on harmonics $u_i^I(\phi)$ of the $S^5$ depending on 5 angles of the sphere.

Now, we would like to perform a further truncation of our boundary superspace to reproduce the $d = 4\, \mathcal{N} = 2$ harmonic superspace of GIKOS. To do so we have to restrict ourselves to an $S^2$ subsphere of the full $S^5$ and reduce the supersymmetries. The symmetries of the reduced system then are:

\begin{align*}
\delta \theta_i^\alpha &= \epsilon^i_\alpha + \frac{1}{2} (\lambda_{mn} \sigma^{mn} \theta^i)_{\alpha} + \theta^j_{\alpha} \Lambda_{j}^i, \quad (24) \\
\delta x^m &= a^m + \lambda^{mn} x_n + i (e^i \sigma^m \bar{\theta}_i - \theta^i \sigma^m \epsilon_i), \quad (25) \\
\delta u_i^K &= -\Lambda_{i}^j u_j^K + u_i^L H^K_L. \quad (26)
\end{align*}

where $\Lambda^i_j$ with $i, j = 1, 2$ are now the parameters of the $SO(3) \sim SU(2)$ subgroup of $SO(6)$ and $H_i^J \equiv \Lambda(\sigma_3)_i^J$ ($I, J = 1, 2$) is the parameter of the transformation of the stability subgroup $SO(2) \sim U(1)$. These are precisely the super Poincaré transformations of the harmonic superspace in the central basis, as given in [4]. This observation constitutes our main result.

The harmonics, introduced in [4] as auxiliary $S^2$ coordinates\footnote{The new bosonic coordinates relevant for extended supersymmetries were added to the superspace in the form of supertwistors and/or coset coordinates in [4]. But the origin of these additional coordinates which were used in [4] to construct an off-shell SYM theory was not clear so far to the best of our understanding. Now we seem to shed some light also on the physical origin of the supertwistors which may be related to the harmonic superspace.} are coset representatives of $SU(2)/U(1)$. A common parametrization is

\begin{align*}
\phi(\phi_1, \phi_2) = \begin{vmatrix}
\cos \phi_1 & i e^{-i\phi_2} \sin \phi_1 \\
i e^{i\phi_2} \sin \phi_1 & \cos \phi_1
\end{vmatrix} = \begin{vmatrix}
u_1^- & \nu_1^+ \\
u_2^- & \nu_2^+
\end{vmatrix}.
\end{align*}

(27)
They also appear explicitly in the Killing spinors on the $AdS \times S$ space \[3\]. Their main purpose is to allow for a change of fermionic variables from those transforming under $SU(2)$ to those transforming under the $U(1)$

$$\theta^\pm_\alpha \equiv \theta^i_\alpha u^\pm_i, \quad (28)$$

where we took $I = (+, -)$. It follows that

$$\delta \theta^\pm_\alpha = \epsilon^i_\alpha u^\pm_i + \frac{i}{4}(\lambda_{mn}\sigma^{mn}\theta^\pm)_\alpha \pm \theta^\pm_\alpha \Lambda. \quad (29)$$

The last term shows that $\theta^\pm$ transforms under the $U(1)$ symmetry. This term came from the cancellation of the $SU(2)$ transformation from the $\theta^i$ by the shift of angular variables and the remaining compensating $U(1)$ transformation is the one with the parameter $\Lambda$.

The ABC of the analytic $\mathcal{N} = 2$ superspace (which is reminiscent of the chiral subspace of $\mathcal{N} = 1$ susy) follows as in \[4\]. One can shift the real 4-dimensional coordinates $x$ by the bilinear combination of spinors so that there is an analytic subspace $x', \theta^+, \bar{\theta}^+, u$. Then one can construct unconstrained superfields which depend in this basis only on $\theta^+$. The complete $\mathcal{N} = 2$ SYM theory with manifest off-shell supersymmetry is then easily constructed. The SYM prepotential $V^{++}$ is an unconstrained analytic superfield of $U(1)$ charge +2, and the hypermultiplets $q^+$ are given by analytic superfields of charge +1.

The basic difference between \[4\] and present derivation lies in the fact that we have the coordinates of $S^2$ as part of the real boundary of the curved $AdS_5 \times S^5$ space.

Also, the somewhat tedious integration rules for the integration over $u$, $\int du \cdot 1 = 1$ and $\int du \cdot u^+_i u^-_j = 0$, find a simple explanation in the $AdS_5 \times S^5$ picture: they all can be derived from

$$du = \frac{1}{4\pi} \sin \phi_1 d\phi_1 d\phi_2, \quad (30)$$

i.e. from the integration over the physical sphere part of our boundary space.

This shows that the $S^2$ part of $S^5$ on which we compactify the superstring/supergravity theory plays a fundamental role in the existence of the off-shell hypermultiplet.
action and supergraph Feynman rules of $\mathcal{N} = 2$ SYM theory developed in [4]. We find here the long-anticipated presence of higher dimensions, however, they are not coming from the $d = 10$ SYM theory, but from string theory/supergravity on $AdS_5 \times S^5$ through the $AdS/SCFT$ correspondence. The harmonics of [4] find their place as the harmonics of the sphere which is part of the boundary of the $AdS_5 \times S^5$ space.

It would be very interesting to see whether a new (off-shell?) $\mathcal{N} = 4$ harmonic superspace can be constructed from the untruncated boundary isometries when the harmonics on the the full $S^5$ sphere are used.

Acknowledgements

We are grateful to S. Shenker for the suggestion to look at the boundary limit of the bulk isometries, to S. Shenker and L. Susskind and Y. Zunger for insightful discussions and to J. Maldacena for asking us about the role of the angles of the $S^5$ for the Yang-Mills theory in $d = 4$ which lead us to identification of the boundary superspace with the harmonic superspace. The work of R. K. and J. R. is supported by the NSF grant PHY-9870115. The work of P. C. is supported by the European Commission TMR programme ERBFMRX-CT96-0045.

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