Renewal Aging as Emerging Property of Phase Synchronization

Simone Bianco$^1$, Elvis Geneston$^1$, Paolo Grigolini$^{1,2,3}$, and Massimiliano Ignaccolo$^1$

$^1$Center for Nonlinear Science, University of North Texas, P.O. Box 311427, Denton, Texas 76203-1427, USA
$^2$Istituto dei Processi Chimico Fisici del CNR, Area della Ricerca di Pisa, Via G. Moruzzi, 56124, Pisa, Italy and
$^3$Dipartimento di Fisica "E.Fermi" - Università di Pisa, Largo Pontecorvo, 3 56127 PISA

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In this letter we examine a model recently proposed to produce phase synchronization [K. Wood et al, Phys. Rev. Lett. 96, 145701 (2006)] and we show that the onset to synchronization corresponds to the emergence of an intermittent process that is non-Poisson and renewal at the same time. We argue that this makes the model appropriate for the physics of blinking quantum dots, and the dynamics of human brain as well.

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Phase synchronization of coupled clocks (oscillators) is a growing field of research, which is fast developing from the seminal work of Winfree [1] and Kuramoto [2]. Using the model of coupled clocks of these authors it is possible to derive [4] the Turing structure [3], which, in turn, triggered the research work in the field of diffusion reaction [5]. Physiologists are using a clock as a representation of a single neuron. In the work of Ref. [6] a neuron is a clock whose behavior is described by a chaotic attractor: the Rössler oscillator [5]. The authors of Ref. [6] and of Ref. [7] have shown that coupled stochastic clocks can show cooperative (synchronized) behavior. Brain functions, such as a cognitive act, rest on the cooperative behavior of a collection of many neurons [8]. The authors of Ref. [9] study the dynamic of the cooperation of a collection of many neurons by mapping the brain activity into a network. They find that the changes in the topology of the network describing the brain activity are driven by a non-Poisson renewal process operating in the non-ergodic regime [10]. A collection of Blinking Quantum Dots (BQDs) [11] has the same dynamical property as the brain activity. The non-Poisson non-ergodic character of the distribution of the sojourn times of the BQDs in the "light" and in the "dark" state is a well established property [12]. The renewal character of the BQDs dynamic has been established only recently [13].

The similarity between the BQDs and the brain activity dynamics makes it plausible to search for a dynamic model accounting for both complex systems. In this letter we show that a simplified version of the model of Ref. [1] affords significant suggestions on how to realize this important purpose. The authors of Ref. [6] use a system of coupled 3-state stochastic clocks, while we use a system coupled 2-state stochastic clocks. We shall prove that at the onset of phase synchronization the dynamics of the system has the same properties as the BQDs [11, 14] and brain dynamics [11]. They are non-Poisson renewal processes operating in the non-ergodic regime.

We consider a Gibbs ensemble of systems with $N$ 2-state stochastic clocks, each of them coupled to $N_c$ clocks. We denote by $|1\rangle$ and $|2\rangle$ the two states of the clock, corresponding to the phases $\Phi=0$ and $\Phi=\pi$, respectively. The master equation for a single clock of a system of the Gibbs ensemble is

$$
\begin{aligned}
\frac{d}{dt} P_1 &= -g_{12} P_1 + g_{21} P_2 \\
\frac{d}{dt} P_2 &= -g_{21} P_2 + g_{12} P_1 .
\end{aligned}
$$

(1)

$P_1$ ($P_2$) is the probability of finding the clock in the state $|1\rangle$ ($|2\rangle$), and $g_{12}$ ($g_{21}$) is the rate of transitions from the state $|1\rangle$ ($|2\rangle$) to the state $|2\rangle$ ($|1\rangle$). The transition rates $g_{12}$ and $g_{21}$ are defined by means of the prescription of Ref. [3]:

$$
g_{12(21)} = g \exp \left[ K(\pi_{2(1)} - \pi_{1(2)}) \right].
$$

(2)

In Eq. (2), $g$ is the unperturbed transition rate of a single clock, $K>0$ is the coupling constant and $\pi_1$ ($\pi_2$) is the fraction of the $N_c$ coupled clocks that are in the state $|1\rangle$ ($|2\rangle$). The authors of Ref. [3] place their clocks in a $d$-dimensional lattice where only the nearest neighbors are coupled. Thus, every clock is coupled to $N_c=2d$ clocks. We adopt, instead, an all to all coupling: $N_c=N-1$. With this choice, when $N \to \infty$, the mean field approximation [15]

$$
\pi_{1(2)} = P_{1(2)}
$$

(3)

is an exact property. Thanks to the mean field approximation of Eq. (3) and to the normalization condition $P_1+P_2=1$, the master equation Eq. (1) reduces to

$$
\frac{d\Pi(t)}{dt} = -2g \cosh(K\Pi)\Pi + 2g \sinh(K\Pi) = -\frac{\partial V(\Pi)}{\partial \Pi},
$$

(4)

where $\Pi=P_1-P_2$ and $\Pi\in[-1,1]$. Eq. (4) describes the overdamped motion of a particle, whose position is $\Pi$, within the potential $V(\Pi)$ [16]. Using Eq. (4), we find that the potential $V(\Pi)$ is symmetric and the values of its minima depend only on the coupling constant $K$. Moreover, we find that there is a critical value $K_c$ of the coupling parameter $K$ such that: 1) If $K \leq K_c$ the potential $V(\Pi)$ has only one minimum for $\Pi_{\min}=0$; 2) if $K > K_c$ the potential $V(\Pi)$

$$
K = K_c = 1
$$

(5)
is symmetric and has two minima ±\(\Pi_{\text{min}}\) separated by a barrier with the maximum centered in \(\Pi=0\). As shown in Fig. 1 the value \(\Pi_{\text{min}}\) and the height of the barrier \(V(0)\) are increasing function of the coupling constant \(K\). In particular \(\Pi_{\text{min}}\rightarrow 1\) and \(V(0)\rightarrow +\infty\) when \(K\rightarrow +\infty\).

The time evolution of the variable \(\Pi\) is determined by the minima and maxima of the potential \(V(\Pi)\). Thus, two kinds of dynamical evolution are possible: 1) If \(K<K_c\), \(\Pi(t)\) will reach, after a transient, an asymptotic value \(\Pi(\infty)=0\) not depending on the initial conditions \(\Pi(0)\); 2) if \(K>K_c\), \(\Pi(t)\) will reach, after a transient, an asymptotic value \(\Pi(\infty)=+\Pi_{\text{min}}\neq 0\) (\(\Pi(\infty)=-\Pi_{\text{min}}\neq 0\)) for an initial condition \(\Pi(0)>0\) (\(\Pi(0)<0\)), while the initial condition \(\Pi(0)=0\) will result in \(\Pi(t)=0\ \forall t\). In Fig. 2 we compare the minima ±\(\Pi_{\text{min}}\) of the potential \(V(\Pi)\) and the numerical evaluation of \(\Pi(\infty)\) for different values of the coupling constant \(K\). Fig. 2 shows that a phase transition occurs at \(K=K_c=1\). For a single clock the condition \(\Pi(\infty)=+\Pi_{\text{min}}=P_1-P_2\neq 0\) corresponds to the statistical "preference" to be either in the state |1\rangle or |2\rangle. This is a consequence of the fact that the transition rates \(g_{12}\) and \(g_{21}\) of Eq. (2) are different if \(K>K_c\).

Plugging Eq. (3) into Eq. (2) and allowing \(\Pi\) to reach its asymptotic value, we get

\[
g_{12} = g \exp(-K\Pi(\infty)) \neq g_{21} = g \exp(K\Pi(\infty)).
\]  

(6)

Fig. 3 confirms the prediction of Eq. (6) showing that if \(\Pi(\infty)=+\Pi_{\text{min}}\) (\(\Pi(\infty)=-\Pi_{\text{min}}\)) the single clock spends on average more time in the state |1\rangle (|2\rangle). The probability density function for the sojourn times in both the preferred and not-preferred state are exponential functions with different mean sojourn times.

Let us now explore the collective behavior of a single system of \(N\) clocks under the all to all coupling condition. Following the authors of Ref. [2], we define the global clock variable \(\xi(t)\) as

\[
\xi(t) = \frac{1}{N} \sum_{j=1}^{N} \exp(i \Phi_j(t)) = \frac{N_1(t) - N_2(t)}{N}.
\]  

(7)

The symbol \(i\) indicates the imaginary unit, \(\Phi_j\) is the phase of the j-th clock, 0 (\(\pi\)) if the clock is in the state |1\rangle (|2\rangle), and \(N_1(t)\) (\(N_2(t)\)) is the number of clocks of the system in state |1\rangle (|2\rangle) at a time \(t\). When \(N\rightarrow \infty\), the single system becomes a Gibbs ensemble on its own as all the clocks are identical, and, at the same time, the mean field approximation (Eq. 8) becomes valid. In
The condition \( \xi(\infty) \neq 0 \) of Eq. (4) proves that a global phase synchronization occurs in the system at the onset of phase transition \((K > K_c)\). Moreover, the time evolution of a single clock of the system is the one depicted by Fig. 3 state \(|1\rangle \rangle (|2\rangle \rangle)\) is statistically preferred if \( \xi(\infty) = +\Pi_{\text{min}} \) \( \xi(\infty) = -\Pi_{\text{min}} \).

Is the phase synchronization of Eq. (4) present in a system with a finite number of clocks? From the definition of transition rate and of probability, it follows that

\[
\begin{align*}
g_{12}P_1 &= \lim_{N \to \infty} \frac{N_{1 \to 2}}{N}, \\
g_{21}P_2 &= \lim_{N \to \infty} \frac{N_{2 \to 1}}{N},
\end{align*}
\tag{10}
\]

where \(N_{1 \to 2}(N_{2 \to 1})\) is the number of clocks that undergo a transition from state \(|1\rangle \rangle (|2\rangle \rangle)\) to state \(|2\rangle \rangle (|1\rangle \rangle)\) per unit of time, and \(N\) is the number of clocks of the system. Using the law of large numbers \([18]\), we get that, for any \(N\) fitting the condition \(\infty > N \gg 1\),

\[
\left\{ \begin{array}{l}
\frac{N_{1 \to 2}}{N} = g_{12}P_1 + \varepsilon_{12}P_1 \\
\frac{N_{2 \to 1}}{N} = g_{21}P_2 + \varepsilon_{21}P_2
\end{array} \right.
\tag{11}
\]

where \(\varepsilon_{12}\) and \(\varepsilon_{21}\) are fluctuating variables whose intensities are \(\propto 1/\sqrt{N}\). From Eqs. (10) and (11), we conclude that the master equation of a system with a finite number of clocks is equivalent to that of a system with an infinite number of clocks whose transition rates fluctuate:

\[
\left\{ \begin{array}{l}
\frac{d}{dt}P_1 = -(g_{12} + \varepsilon_{12})P_1 + (g_{21} + \varepsilon_{21})P_2 \\
\frac{d}{dt}P_2 = -(g_{21} + \varepsilon_{21})P_2 + (g_{12} + \varepsilon_{12})P_1
\end{array} \right.
\tag{12}
\]

If \(\infty > N \gg 1\), we can still consider the mean field approximation of Eq. (5) to be valid. Using the master equation Eq. (12) and the normalization condition \(P_1 + P_2 = 1\), we get for the variable \(\Pi = P_1 - P_2\) the following equation of motion

\[
\frac{d\Pi(t)}{dt} = -\frac{\partial V(\Pi)}{\partial \Pi} - \eta(t)\Pi(t) + \theta(t).
\tag{13}
\]

Thus, for a system with a finite number of clocks the phase synchronization of Eq. (4) is not stable. The global clock variable \(\xi\) of Eq. (4) fluctuates for \((K > K_c)\) between the two minima, \(\pm \Pi_{\text{min}}\), of the potential \(V(\Pi)\), as confirmed by Fig. 4. The single clock follows the fluctuations of the global clock variable \(\xi\), switching back and forth from the condition where the state \(|1\rangle \rangle\) is statistically preferred (time evolution described by the full line of Fig. 4) to that where the state \(|2\rangle \rangle\) is (time evolution described by the dashed line of Fig. 4).

**FIG. 4:** The global variable \(\xi(t)\) as a function of time, for \(K = 1.05, g = 0.01\) and a system of 1000 clocks.

The probability density functions of the sojourn times in the state \(|\xi > 0\rangle \rangle \xi < 0\rangle \rangle (Fig. 3) are identical since the potential \(V(\Pi)\) of Fig. 4 is symmetric. Thus, we denote them with the same symbol \(\psi(\tau)\). Let us consider a condition where the coupling constant \(K\) is close to the critical value \(K - K_c < 1\). In this case, the height of the barrier \(V(0)\) dividing the two wells of the potential \(V(\Pi)\) of Fig. 4 is smaller than or comparable to the intensity of the fluctuations \(\eta\) and \(\theta\) of Eq. (13). Under this condition, we expect \(\psi(\tau) \propto 1/\tau^{1.5}\) for an extended interval of sojourn times. This is exactly what we observe in Fig. 4, where the full line denotes the survival probability \(\Psi(\tau)\), namely, the probability of observing a sojourn time larger than \(\tau\) [24]. For any fixed value of the unperturbed rate \(g\) (Eq. (2)), the height \(V(0)\) of the barrier dividing the two wells of the potential \(V(\Pi)\) (Fig. 4) increases as the value of the coupling parameter increases. Eventually, the height \(V(0)\) will become much larger than the intensity of the fluctuations \(\eta\) and \(\theta\) (Eq. (13)). As a consequence, the power law behavior \((\psi(\tau) \propto 1/\tau^{1.5})\) of Fig. 4 disappears, the theoretical arguments of [14] loses validity, and an exponential behavior emerges, as predicted by the Kramers theory [17].

Finally, we show that the transition between the states \(\xi > 0\) and \(\xi < 0\) shown in Fig. 4 is a renewal process. For this purpose we use the aging experiment of Ref. [14]. We evaluate the survival probability \(\Psi(t_a, \tau)\) of age \(t_a\). This is the probability of observing a sojourn time larger than \(\tau\) if the observation starts at a time \(t_a\) after a cross-
FIG. 5: The survival probability $\Psi(\tau)$ (full line), the survival probability $\Psi(t_a, \tau)$ (dashed line) and the survival probability in the renewal case $\Psi_r(t_a, \tau)$ (full thick line) as a function of sojourn time $\tau$. Here, we plot for clarity only the survival probability of age $t_a=500$. 

If $\Psi(\tau)$ is not an exponential function, $\Psi(t_a, \tau)$ yields a slower decay than $\Psi(\tau)$, a condition that is denoted as “aging” [14]. Fig. 5 shows that the transition between the states $\xi>0$ and $\xi<0$ (Fig. 4) is a renewal process and that there is aging. With increasing values of coupling parameter $K$ the renewal property is not lost, but the aging property is because the survival probability $\Psi(\tau)$ becomes an exponential function [14].

In conclusion, we have shown that, for a Gibbs ensemble of systems with $N=\infty$ coupled 2-state clocks, a phase transition occurs at the critical value of coupling parameter $K_c=1$ (Fig. 2). The phase transition mirrors a statistical “preference” for a clock of one of the Gibbs ensemble systems to be in one of the two possible states (Eq. (6)). For a single system with $N=\infty$ coupled 2-state clocks, the phase transition of Fig. 2 signals the onset of a global phase synchronization (Eqs. (12), (13) and (14)). If the number of clocks of the system is finite the phase synchronization is not stable. In this case, the variable $\xi$ (Eq. (7)) describing the collective motion of the system (Fig. 4) is characterized by non-Poisson intermittent behavior. At the onset of the phase transition ($K-K_c<1$), the zero crossings of the global variable $\xi$ (Fig. 4) define a series of events with the following properties: 1) The probability density function of inter events intervals (full line of Fig. 4) has a non-Poisson non-ergodic character ($\psi(\tau)\propto 1/\tau^{1.5} \Rightarrow$ infinite mean inter event interval). 2) The sequence of inter events intervals satisfy the renewal aging condition (Fig. 4). The properties 1) and 2) are observed properties of the events in both BQDs [12, 13, 14] and brain activity [11], suggesting that a system consisting of a finite number of coupled 2-state clocks may be a good model for the dynamics of both processes.

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[1] A. T. Winfree, The Geometry of Biological Time, Springer-Verlag, Berlin (1990).
[2] Y. Kuramoto, Rhythms and Turbulence in Populations of Chemical Clocks Physica A 126, 128 (1981).
[3] G. Biosa, S. Bastianoni, and M. Rustici, Chem. Eur. J. 12, 3430 (2006).
[4] T. Prager, B. Naundorf, and L. Schimansky-Geier, Physica A 325, 176 (2003).
[5] I. Prigogine and R. Lefever, J. Chem. Phys. 48, 1695 (1967).
[6] C. J. Stam, Clinical Neurophysiology 116, 2266 (2005).
[7] O. E. Rössler, Phys. Lett. 57 A, 397 (1976).
[8] T. Prager, B. Naundorf, and L. Schimansky-Geier, Physica A 325, 176 (2003).
[9] K. Wood, C. Van den Broeck, R. Kawai, and K. Lindenberg, Phys. Rev. Lett. 96, 145701 (2006).
[10] F. Varela, J.-P. Lachaux, E. Rodriguez and J. Martinerie, Nature Review Neuroscience, 2, 229 (2001).
[11] S. Bianco, P. Grigolini, M. Ignaccolo, M. Rider, M. Ross, P. Winsor, http://arxiv.org/abs/q-bio.NC/0610037 (submitted to PRE).
[12] G. Margolin and E. Barkai, Phys. Rev. Lett. 94 080601 (2005); G. Bel and E. Barkai, Phys. Rev. Lett. 94, 240602 (2005); G. Margolin and E. Barkai, J. Stat. Phys. 122, 137 (2006); G. Bel and E. Barkai, Phys. Rev. E 73, 016125 (2006).
[13] M. Nirmal, B.O. Dabbousi, M. G. Bawendi, J.J. Macklin, J. K. Trautman, T. D. Harris, L. E. Brus, Nature 383, 802 (1996); M. Kuno, D.P. Fromm, S.T. Johnson, A. Gallagher and D.J. Nesbitt, Phys. Rev. B 67, 125304 (2003); K.T. Shimizu, R.G. Neuhauer, C. A. Leatherdale, S. A. Empedocles, W.K. Woo and M. G. Bawendi, Phys. Rev. B 63, 205316 (2001).
[14] S. Bianco, P. Grigolini, P. Paradisi, J. Chem. Phys. 123, 174704 (2005).
[15] H. E. Stanley, Introduction to Phase Transitions and Critical Phenomena, Oxford University Press, Oxford (1971).
[16] H. A. Kramers, Physica 7, 284 (1940).
[17] The arbitrary constant in the definition of the potential $V(\Pi)$ is chosen to satisfy the condition $V(\pm \Pi_{\min})=0 \forall K<K_c$.
[18] L. E. Reichl, A Modern Course in Statistical Physics, John Wiley, New York (1998).
[19] G. Margolin and E. Barkai, Phys. Rev. E 72, 025101 (R) (2005).
[20] Note that $\psi(\tau)\propto 1/\tau^{1.5} \Rightarrow \Psi(\tau)\propto 1/\tau^{0.5}$.