Magic labelings of distance at most 2

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Abstract

For an arbitrary set of distances \( D \subseteq \{0, 1, \ldots, d\} \), a graph \( G \) is said to be \( D \)-distance magic if there exists a bijection \( f : V \to \{1, 2, \ldots, v\} \) and a constant \( k \) such that for any vertex \( x \), \( \sum_{y \in N_D(x)} f(y) = k \), where \( N_D(x) = \{y \in V | d(x, y) \in D\} \).

In this paper we study some necessary or sufficient conditions for the existence of \( D \)-distance magic graphs, some of which are generalization of conditions for the existence of \( \{1\} \)-distance magic graphs. More specifically, we study \( D \)-distance magic labelings for cycles and \( D \)-distance magic graphs for \( D \subseteq \{0, 1, 2\} \).

1 Introduction

As standard notation, assume that \( G=G(V, E) \) is a finite, simple, and undirected graph with \( v \) vertices, \( e \) edges, and diameter \( d \). By a labeling we mean a one-to-one mapping that carries a set of graph elements onto a set of numbers, called labels.

The notion of distance magic labeling was introduced separately in the PhD thesis of Vilfred [28] in 1994 and an article by Miller et. al [18] in 2003. A distance magic labeling, or \( \Sigma \) labeling, is a bijection \( f : V \to \{1, 2, \ldots, v\} \) with the property that there is a constant \( k \) such that at any vertex \( x \), \( \sum_{y \in N(x)} f(y) = k \), where \( N(x) \) is the open neighborhood of \( x \), i.e., the set of vertices adjacent to \( x \). This labeling was introduced due to two different motivations; as a tool in utilizing magic squares into graphs and as a natural extension of previously known graph labelings: magic labeling [24, 15] and radio labeling (which is distance-based) [13].

In the last decade, many results on distance magic labeling have been published. Several families of graphs have been showed to admit the labeling, for instance circulant graphs [7], bipartite graphs [18, 16], tripartite graphs [18], regular multipartite graphs [28, 18]. Cartesian product graphs [14, 23], lexicographic product graphs [18, 26, 2, 3], and joint product graphs [25]. Constructions of distance magic graphs have also been studied: construction producing regular graphs was studied in [9, 10, 11, 16] and non-regular graphs in [27, 17]; the constructions utilize Kotzig array and magic rectangle.

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It was proved in [28] that every graph is a subgraph of a distance magic graph. A stronger result that every graph is an induced subgraph of a regular distance magic graph was then proved in [1]. A yet stronger result can also be found in [22] where it is stated that every graph is an induced subgraph of a regular distance magic graph. Additionally, an application of the labeling in designing incomplete tournament is introduced in [10]. For more results, please refer to a recent survey article in [4].

Jinah [14] introduced a variation of distance magic labeling. A Σ’ labeling, is a bijection $f : V \rightarrow \{1, 2, \ldots, v\}$ with the property that there is a constant $k$ such that at any vertex $x$, $\sum_{y \in N[x]} f(y) = k$, where $N[x]$ is the closed neighborhood of $x$, i.e., the set containing $x$ and all vertices adjacent to $x$. It was stated in [20] that there does not exist a graph of even order that admits both distance magic and Σ’ labelings. As for graphs of odd order, the path $P_3$ is one example of a graph admitting both labelings. In the same article, it was also showed that a graph is distance magic if and only if its complement is Σ’-labeled.

Recently O’Neal and Slater [20] generalized the notion of distance magic labeling to an arbitrary set of distances $D \subseteq \{0, 1, \ldots, d\}$. As in distance magic labeling, the domain of the new labeling is the set of all vertices and the codomain is $\{1, 2, \ldots, v\}$. We define the $D$-weight of each vertex $x$ in $G$, denoted by $w(x)$, to be the sum of labels of the vertices at distance $k$ to $x$, where $k \in D$. If all vertices in $G$ have the same weight, we call the labeling a $D$-distance magic labeling. More formally, we have the following definition.

**Definition 1.** [20] A bijection $f : V \rightarrow \{1, 2, \ldots, v\}$ is said to be a $D$-distance magic labeling if there exists a $D$-distance magic constant $k$ such that for any vertex $x$, $w(x) = \sum_{y \in N_D(x)} f(y) = k$, where $N_D(x) = \{y \in V | d(x, y) \in D\}$. A graph admitting a $D$-distance magic labeling is called $D$-distance magic.

Clearly, a distance-magic labeling is a $\{1\}$-distance magic labeling and a Σ’ labeling is a $\{0, 1\}$-distance magic labeling. Rewriting the results in [20], we have the following relations between $\{1\}$-distance magic and $\{0, 1\}$-distance magic labelings.

**Lemma 1.** [20] There does not exist a graph of even order that admits both $\{1\}$-distance magic and $\{0, 1\}$-distance magic labelings.

**Lemma 2.** [20] A graph is $\{1\}$-distance magic if and only if its complement is $\{0, 1\}$-distance magic.

In this paper we study properties of $D$-distance magic labelings for a distance set $D$, where $D \subseteq \{0, 1, \ldots, d\}$. Obviously, the only $\{0\}$-distance magic graph is the trivial graph, and so we exclude $D = \{0\}$ from our consideration. Additionally, we also study $D$-distance magic labelings for $D \subseteq \{0, 1, 2\}$.

## 2 Some general results

In this section, we study some necessary and sufficient conditions for the existence of $D$-distance magic graphs for particular distance sets $D, D \subseteq \{0, 1, \ldots, d\}$ and $D \neq \{0\}$. Unless stated, we shall exclude the trivial graph from consideration. We start by generalizing some properties of $\{1\}$-distance magic graphs presented in [18].
In [18] it was proved that there does not exist a \( \{1\} \)-distance magic labeling for \( r \)-regular graph with odd \( r \). The next result generalize this idea to arbitrary neighborhood sets. Graph \( G \) is defined to be \((D, r)\)-regular if for all \( v \in V(G) \), \(|N_D(v)| = r\), that is, all \( D \)-neighborhoods have the same cardinality.

**Lemma 3.** [20] Let \( G \) be a graph of even order. If \( G \) is \((D, r)\)-regular with odd \( r \), then \( G \) is not \( D \)-distance magic.

Another result can be found in [18] is that if a graph \( G \) contains two vertices \( x \) and \( y \) such that \(|N(x) \cap N(y)| = d(x) - 1 = d(y) - 1\) then \( G \) is not \( \{1\} \)-distance magic. We shall generalize the idea to \( D \)-distance magic graphs.

**Lemma 4.** If a graph \( G \) contains two distinct vertices \( x \) and \( y \) such that \(|N_D(x) \cap N_D(y)| = |N_D(x)| - 1 = |N_D(y)| - 1\) then \( G \) is not \( D \)-distance magic.

**Proof.** Suppose \( G \) is \( D \)-distance magic and let \( x' \) (\( y' \), respectively) be the one vertex in \( N_D(x) - N_D(y) \) (\( N_D(y) - N_D(x) \), respectively). Then \( \sum_{u \in N_D(x)} f(u) = w(x) = w(y) = \sum_{u \in N_D(y)} f(u) \), and so \( f(x') = f(y') \), a contradiction. \( \Box \)

The following two lemmas also give necessary conditions connected to the \( D \)-neighborhood of vertices in the graph.

**Lemma 5.** If \( G \) contains a vertex \( x \) with \( N_D(x) = \emptyset \) then \( G \) is not \( D \)-distance magic.

**Proof.** Suppose \( G \) is \( D \)-distance magic. Since \( D \subseteq \{0, 1, \ldots, d\} \) then there is a vertex \( y \) where \( N_D(y) \neq \emptyset \) and so \( w(y) \neq 0 \). However \( w(x) = 0 \), a contradiction. \( \Box \)

**Lemma 6.** If \( G \) contains two distinct vertices \( x \) and \( y \) such that \( N_D(x) \subseteq N_D(y) \) then \( G \) is not \( D \)-distance magic.

**Proof.** Suppose \( G \) is \( D \)-distance magic. Since \( w(x) = w(y) \), then \( \sum_{u \in N_D(y) - N_D(x)} f(u) = 0 \), a contradiction. \( \Box \)

Properties of \( D \)-distance magic graphs can also be found in [21], the most important is the uniqueness of the \( D \)-distance magic constant.

**Definition 2.** A function \( g : V(G) \to R^+ = [0, \infty) \) is said to be a \( D \)-neighborhood fractional dominating function if for every \( v \in V(G) \), \( \sum_{u \in N_D(v)} g(u) \geq 1 \). The \( D \)-neighborhood fractional domination number of \( G \), denoted by \( \gamma_f(G; D) \), is defined as \( \gamma_f(G; D) = \min \{ \sum_{v \in V(G)} g(v) | g \text{ is a } D\text{-neighborhood fractional dominating function} \} \).

**Theorem 1.** [21] If graph \( G \) is \( D \)-distance magic, then its \( D \)-distance magic constant \( k = \frac{n(D+1)}{2\gamma_f(G; D)} \).

The following two lemmas deal with existence of \( D \)-distance magic graphs for particular \( D \).

**Lemma 7.** If each vertex in \( G \) has a unique vertex at distance \( d \) then \( G \) is \( \{1, 2, \ldots, d-1\} \)-distance magic.
Proof. We define a labeling \( f \) such that if a vertex \( x \) is labeled with \( i \) then the unique vertex at distance \( d \) from \( x \) is labeled with \( v + 1 - i \). Thus, for every vertex \( x \) in \( G \), the weight of \( x \), \( w(x) = \sum_{x \in V(G)} f(x) - (i + (v + 1 - i)) = \sum_{x \in V(G)} f(x) - (v + 1) \), which is independent of the choice of \( x \). Therefore, \( G \) is \( \{1, 2, \ldots, d - 1\} \)-distance magic. \( \blacksquare \)

**Lemma 8.** Every connected graph is \( \{0, 1, \ldots, d\} \)-distance magic.

Proof. The proof is straightforward since under the \( \{0, 1, \ldots, d\} \)-distance magic labeling, we sum all labels in the graph in counting the weight of a vertex. \( \blacksquare \)

For obvious reason, we shall call the \( \{0, 1, \ldots, d\} \)-distance magic of \( G \) the *trivial D*-distance magic labeling of \( G \). The following lemma deals with similar result for non-connected graphs.

**Lemma 9.** Let \( G \) be a non-connected graph having connected components \( G_1, G_2, \ldots, G_p \), each of diameter \( d_1, d_2, \ldots, d_p \), respectively. Let \( d_{\text{max}} = \max d_i \) and \( |V(G_i)| = n \) for each \( i \). \( G \) is \( \{0, 1, \ldots, d_{\text{max}}\} \)-distance magic if and only if \( n \) is even or both \( n \) and \( p \) are odd.

Proof. Suppose \( G \) is \( \{0, 1, \ldots, d_{\text{max}}\} \)-distance magic. Since the weight of a vertex \( x \) is the sum of all labels in the component containing \( x \), then such a sum must equal to the magic constant \( k \). Now we count the sum of all labels by two different ways of counting:

\[
kp = 1 + \ldots + np \\
kp = \frac{(np + 1)(np)}{2} \\
k = \frac{(np + 1)n}{2}.
\]

To guarantee that both sides are integers then \( n \) has to be even or both \( n \) and \( p \) must be odd.

To prove the sufficiency, let \( x_{ij}, 1 \leq j \leq n \), be the vertices in the component \( G_i \). If \( n \) is even, label the vertices in the following way

\[
f(x_{ij}) = \begin{cases} 
  i + (j - 1)p, & i \text{ odd}, \\
  p - i + 1 + (j - 1)p, & i \text{ even}.
\end{cases}
\]

With this labeling, the sum of all labels in the component \( G_i \) is \( \frac{n}{2}(np + 1) \), which is equal to \( w(x) \), for \( x \) a vertex in \( G_i \).

If \( n \) is odd, consider \( n = 2k + 1, p = 2m + 1 \), and the labeling \( f \) as defined below.

\[
f(x_{ij}) = \begin{cases} 
  2i - 1, & 1 \leq i \leq m + 1 \text{ and } j = 1, \\
  2(i - m - 1), & m + 2 \leq i \leq 2m + 1 \text{ and } j = 1, \\
  4m + 3 - i, & 1 \leq i \leq 2m + 1 \text{ and } j = 2, \\
  5m + 4 - i, & 1 \leq i \leq m + 1 \text{ and } j = 3, \\
  7m + 5 - i, & m + 2 \leq i \leq 2m + 1 \text{ and } j = 3, \\
  i + (j - 1)(2m + 1), & 1 \leq i \leq 2m + 1 \text{ and } j > 3, \text{ } j \text{ even,} \\
  2m + 2 - i + (j - 1)(2m + 1), & 1 \leq i \leq 2m + 1 \text{ and } j > 3, \text{ } j \text{ odd.}
\end{cases}
\]

Thus, the sum of all labels in the component \( G_i \) is \( (9m + 6) + (k - 1)(2m + 2) + (k - 1)(2k + 3)(2m + 1) \). \( \blacksquare \)
In the previous lemma, we only consider graphs having all connected components of the same order. As to graphs having connected components with different order, we have $K_2 \cup K_1$ as an example of $\{0,1\}$-distance magic graph. Whether there are other graphs remains a question.

**Open problem 1.** Let $G$ be a non-connected graph having connected components $G_1, G_2, \ldots, G_p$, each of diameter $d_1, d_2, \ldots, d_p$, respectively. Let $d_{\text{max}} = \max_i d_i$ and there exist $i, j$ such that $|V(G_i)| \neq |V(G_j)|$. Does there exist $G$ admitting $\{0,1,\ldots,d_{\text{max}}\}$-distance magic labeling other than $K_2 \cup K_1$?

The last result in this section provides connection between $D$-distance magic labelings with different $D$s.

**Lemma 10.** [21] Let $D \subseteq \{0,1,\ldots,d\}$ and $D^* = \{0,1,\ldots,d\} - D$. Then $G$ is $D$-distance magic if and only if $G$ is $D^*$-distance magic.

As a consequence of Lemma 10, we have the following.

**Lemma 11.** A graph of diameter $d$ is not $\{1,2,\ldots,d\}$-distance magic.

We shall call the $D^*$-distance magic labeling in Lemma 10 the complement labeling of $D$-distance magic labeling. In the following we extend the result to non-connected graphs.

**Lemma 12.** Let $G$ be a graph having connected components $G_1, G_2, \ldots, G_p$ of diameters $d_1, d_2, \ldots, d_p$, respectively. Let $D \subseteq \{0,1,\ldots,d_{\text{max}}\}$ and $D^* = \{0,1,\ldots,d_{\text{max}}\} - D$, where $d_{\text{max}} = \max_i d_i$. If $G$ admits a $D$-distance magic labeling $f$ such that $\sum_{x \in G_i} f(x)$ is constant for each $i$ then $G$ is $D^*$-distance magic. Conversely, if $G$ admits a $D^*$-distance magic labeling $f^*$ such that $\sum_{x \in G_i} f^*(x)$ is constant for each $i$ then $G$ is $D$-distance magic.

**Proof.** For each $x \in V(G)$, we define $w(x) = \sum_{u \in N_D(x)} f(u)$ and $w^*(x) = \sum_{u \in N_{D^*}(x)} f(u)$. Clearly $w^*(x) = \sum_{u \in G_x} f(u) - w(x)$, where $G_x$ is the component containing $x$. If $w(x)$ is constant for each vertex $x$, then so is $w^*(x)$. The converse can be proved similarly.

In the next section, we study the existence of $D$-distance magic labelings with various $D$ for cycles.

## 3 $D$-distance magic labelings for cycles

We shall start with cycles of even order.

**Theorem 2.** Every even cycle $C_{2k}$ is $\{1,2,\ldots,k-1\}$-distance magic.

**Proof.** Each vertex in $C_{2k}$ is at distance $k$ from exactly one other vertex and so $C_{2k}$ is $\{1,2,\ldots,k-1\}$-distance magic by Lemma 7.

As a direct consequence of Lemma 10, we obtain

**Corollary 1.** Every even cycle $C_{2k}$ is $\{0,k\}$-distance magic.
The next result is a characterization of cycles admitting \( D \)-distance magic labelings where \( D \) is a singleton.

**Theorem 3.** For \( k \) a positive integer, a cycle \( C_n \) is \( \{ k \} \)-distance magic if and only if \( n = 4k \).

**Proof.** Suppose that \( f \) is a \( \{ k \} \)-distance magic labeling of \( C_n \). Let \( x \) be an arbitrary vertex in \( C_n \), then there exist exactly two vertices of distance \( k \) from \( x \), say \( x_1 \) and \( x_2 \). There also exists another vertex of distance \( k \) from \( x_1 \) beside \( x \), say \( y \), and similarly there exists another vertex of distance \( k \) from \( y \) beside \( x_1 \), say \( y_2 \). If \( n \neq 4k \) then \( x, x_1, x_2, y, y_2 \) are all distinct. Thus we obtain a contradiction by Lemma 4.

To prove the sufficiency, suppose that \( n = 4k \). Notice that each vertex \( x \) in \( C_{4k} \) has a distinct pair of vertices of distance \( k \), say \( x_1 \) and \( x_2 \). We label such a pair with a labeling \( f \) such that \( f(x_1) = 4k + 1 - f(x_2) \). Thus the weight of \( x \), \( w(x) = f(x_1) + f(x_2) = 4k \) (independent of the choice of \( x \)) and so \( C_{4k} \) is \( \{ k \} \)-distance magic.

As a direct consequence of Lemma 10, we obtain

**Corollary 2.** For \( k \) a positive integer, a cycle \( C_n \) is \( \{ 0, 1, \ldots, k-1, k+1, \ldots, \lfloor \frac{n}{2} \rfloor \} \)-distance magic if and only if \( n = 4k \).

We could generalize the result in Theorem 3 to 2-regular graphs which is a generalization of a result in [19].

**Theorem 4.** [19] A 2-regular graph is \( \{ 1 \} \)-distance magic if and only if it is the union of 4-cycles.

**Theorem 5.** For \( k \) a positive integer, a 2-regular graph is \( \{ k \} \)-distance magic if and only if it is a disjoint union of \( C_{4k} \)s.

**Proof.** The proof is similar to that of Theorem 3 except for proving the sufficiency, where we use the labeling \( f \) such that \( f(x_1) = m4k + 1 - f(x_2) \), where \( m \) is the number of copies of \( C_{4k} \).

Some additional negative results for cycles are presented in the following theorem and corollary. The next result is proved by using Lemma 4.

**Theorem 6.** For \( n \geq 2k + 2 \), a cycle \( C_n \) is not \( \{ 0, 1, \ldots, k \} \)-distance magic.

By Lemma 10 we obtain

**Corollary 3.** For \( n \geq 2k + 2 \), a cycle \( C_n \) is not \( \{ k + 1, k + 2, \ldots, \lfloor \frac{n}{2} \rfloor \} \)-distance magic.

We then have the problem of characterizing \( D \)-distance magic cycles, or more generally, \( D \)-distance magic 2-regular graphs.

**Open problem 2.** Given a particular distance set \( D \), what are the necessary and sufficient conditions for 2-regular graphs to have \( D \)-distance magic labeling?
4  D-distance magic labelings with \( D \subseteq \{0, 1, 2\} \)

A well-known result of Blass and Harary [6] stated that almost all graphs have diameter 2. Therefore in this section we dedicate our study to \( D \)-distance magic labelings where \( D \subseteq \{0, 1, 2\} \). Since \{1\}-distance magic and \{0,1\}-distance magic labelings have been studied extensively, we only consider \( D \in \{\{2\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\} \).

In the next lemma, we shall present necessary conditions of the existence of \( D \)-distance magic graphs with \( D \) containing 2 but not 0.

**Lemma 13.** Let \( D \) be a distance set containing 2 but not 0. If \( G \) is a graph of diameter at least 2 containing either

1. two adjacent pendants, or
2. two vertices of distance 2 having the same neighborhood,

then \( G \) is not \( D \)-distance magic.

**Proof.** Suppose \( G \) is \( D \)-distance magic and let \( x, y \) be the two adjacent pendants (in case 1) or the two vertices of distance 2 having the same neighborhood (in case 2). In both cases, since \( D \) containing 2 but not 0, \( N_D(x) \) and \( N_D(y) \) containing exactly the same vertices except \( x \) for \( N_D(x) \) and \( y \) for \( N_D(y) \). Thus since \( w(x) = w(y) \), we have \( f(x) = f(y) \), a contradiction.

By the aforementioned lemma, many trees do not have \( D \)-labelings, where \( D \) containing 2 but not 0. However, to characterize trees admitting such a labeling needs further study. More specifically, it is interesting to determine which trees have \( D \)-distance magic labelings where \( D \subseteq \{0, 1, 2\} \).

**Open problem 3.** What are the necessary and sufficient conditions for trees to have \( D \)-distance magic labelings where \( D \subseteq \{0, 1, 2\} \)?

4.1 \{2\}-distance magic labelings

**Theorem 7.** A complete multipartite graph is not \{2\}-distance magic.

**Proof.** Let \( x \) and \( y \) be two vertices in the same partite set of a multipartite graph \( G \). If we name the partite set \( V_0 \) then \( N_{\{2\}}(x) = V_0 - \{x\} \) and \( N_{\{2\}}(y) = V_0 - \{y\} \). By Lemma 4, \( G \) is not \{2\}-distance magic.

Based on this result and the results of O’Neal and Slater [20] on extremal graphs of diameter 2 and 3, we suspect that graphs with diameter 2 are not \{2\}-distance magic and more generally, graphs with diameter \( d \) are not \{\( d \)\}-distance magic.

**Conjecture 1.** Graphs with diameter 2 are not \{2\}-distance magic. More generally, graphs with diameter \( d \) are not \{\( d \)\}-distance magic.
4.2 \{0, 2\}-distance magic labelings

By Lemma\[10\] we have the following results as consequences of the existence of \{1\}-distance magic labelings for particular graphs of diameter 2.

**Theorem 8.** [10] Let $H_{n,p}, n > 1$ and $p > 1$, denote the complete symmetric multipartite graph with $p$ parts, each of which contains $n$ vertices. $H_{n,p}$ is \{0, 2\}-distance magic if and only if either $n$ is even or both $n$ and $p$ are odd.

**Theorem 9.** [10] Let $1 \leq a_1 \leq a_2 \leq a_3$. Let $s_i = \sum_{j=1}^{i} a_j$, $p = 2$ for complete bipartite graph $K_{a_1,a_2}$, and $p = 3$ for complete tripartite graph $K_{a_1,a_2,a_3}$. There exist \{0, 2\}-distance magic labelings for $K_{a_1,a_2}$ and $K_{a_1,a_2,a_3}$ if and only if the following conditions hold.

(a) $a_2 \geq 2$,
(b) $v(v + 1) \equiv 0 \mod 2p$, and
(c) $\sum_{j=1}^{i}(n + 1 - j) \geq \frac{v(v+1)}{2p}$ for $1 \leq i \leq p$.

**Theorem 10.** [10] An odd order $r$-regular graph of diameter 2 is \{0, 2\}-distance magic if and only if $r$ is even and $2 \leq r \leq n - 2$.

**Theorem 11.** [10] Let $G$ be an odd order regular graph of diameter 2 and $n$ be an odd positive integer. Then the graph $G[K_n]$ is \{0, 2\}-distance magic.

The aforementioned theorem deal with odd order $G$; however for even order $G$, we have an example in which the composition of $G$ with $K_n$ does have a \{0, 2\}-distance magic labeling.

**Theorem 12.** [26] For $n \geq 1, C_4[K_n]$ is \{0, 2\}-distance magic.

For graphs of diameter other than 2, we have the path of order 4, which is of diameter 3, admitting a \{0, 2\}-distance magic labeling. This leads to the following question.

**Open problem 4.** Does there exist a graph of diameter larger than 2, other than $P_4$, admitting \{0, 2\}-distance magic labeling?

4.3 \{1, 2\}-distance magic labelings

By Lemma\[11\] \{1, 2\}-distance magic labelings do not exist for graphs of diameter 2, and so in the following theorem we construct \{1, 2\}-distance magic labelings for infinite families of graphs with diameter larger than 2.

**Theorem 13.** There exists an infinite family of regular graphs with diameter 3 admitting \{1, 2\}-distance magic labeling.

**Proof.** We construct a graph $G$ with $V(G) = \{x, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and $E(G) = \{xx_i, yy_j | 1 \leq i \leq n\} \cup \{x,y_j | 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$. We can see that $G$ is an $n$-regular graph of order $2n + 2$ and diameter 3. Moreover, each vertex has a unique vertex of distance 3: $y$ for $x$ and $y_i$ for $x_i$, $1 \leq i \leq n$. By Lemma\[7\] $G$ is \{1, 2\}-distance magic.

The existence of non-regular \{1, 2\}-distance magic graphs or \{1, 2\}-distance magic graphs of larger diameter remain open as stated in the following.
Open problem 5. Does there exists an infinite family of non-regular graphs admitting \(\{1,2\}\)-distance magic labeling?

Open problem 6. Does there exists an infinite family of graphs with diameter at least 4 admitting \(\{1,2\}\)-distance magic labeling?

4.4 \(\{0,1,2\}\)-distance magic labelings

By Theorem 8, every graph of diameter 2 admits the trivial \(D\)-distance magic labeling, i.e., an \(\{0,1,2\}\)-distance magic labeling. We could not find \(\{0,1,2\}\)-distance magic graphs of larger diameter, and so we ask the following question.

Open problem 7. Does there exist a graph of diameter at least 3 admitting an \(\{0,1,2\}\)-distance magic labeling?

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