Lp-improving properties of certain singular measures on the Heisenberg group

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Abstract. Let μA be the singular measure on the Heisenberg group Hn supported on the graph of the quadratic function ϕ(y) = y t Ay, where A is a 2n × 2n real symmetric matrix. If det(2A ± J) ≠ 0, we prove that the operator of convolution by μA on the right is bounded from L(2n+2)/(2n+1)(Hn) to L2n+2(Hn). We also study the type set of the measures dνγ(y, s) = η(y)|y|−γdμA(y, s), for 0 ≤ γ < 2n, where η is a cut-off function around the origin on R2n. Moreover, for γ = 0 we characterize the type set of ν0.

Keywords: Heisenberg group; singular Borel measure; Lp-improving property

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1. Introduction

Let In be the n × n identity matrix and J be the 2n × 2n skew-symmetric matrix given by

(1) J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.

The Heisenberg group is Hn = R2n × R endowed with the group law (non-commutative)

(x, t) · (y, s) = (x + y, t + s + ⟨x, y⟩),

where ⟨x, y⟩ is the standard symplectic form on R2n, i.e. ⟨x, y⟩ = xtJy with neutral element (0, 0) and with inverse (x, t)−1 = (−x, −t). The topology in Hn is induced by R2n+1, so the borelian sets of Hn are identified with those of R2n+1. The Haar measure in Hn is the Lebesgue measure of R2n+1, thus Lp(Hn) ≡ Lp(R2n+1). Given

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a borelian function $f : \mathbb{H}^n \to \mathbb{C}$ and a Borel measure $\mu$ on $\mathbb{H}^n$, define the convolution by $\mu$ on the right by

$$(f * \mu)(x, t) = \int_{\mathbb{H}^n} f((x, t) \cdot (y, s))^{-1}) \, d\mu(y, s),$$

provided the integral exists.

A Borel measure $\mu$ on the Heisenberg group $\mathbb{H}^n$ is said to be $L^p$-improving if the operator $T_\mu : f \mapsto f * \mu$ is bounded from $L^p(\mathbb{H}^n)$ into $L^q(\mathbb{H}^n)$ for some $1 \leq p < q < \infty$. A remarkable fact is that singular measures can be $L^p$-improving. If in (2) we replace the Heisenberg group $\mathbb{H}^n$ by $\mathbb{R}^n$ with the ordinary convolution in $\mathbb{R}^n$ and considering there $\mu = \eta \sigma_M$, where $\sigma_M$ is the surface measure on a given manifold $M$ (in $\mathbb{R}^n$) and $\eta$ is a smooth cut-off function, then the $L^p$-improving properties of a measure of this type are closely related to the existence of a certain amount of curvature of the manifold $M$ (see [5], [6], [7]). A similar result holds on general Lie groups (see Theorem 1.1, page 362 in [9]).

A more delicate problem consists in determining the exact range of pairs $(p, q)$ for which $L^p * \mu \subseteq L^q$ embeds continuously. Given a manifold $M$ (in $\mathbb{H}^n$), define the type set $E_{\eta \sigma_M}$ by

$$E_{\eta \sigma_M} = \left\{ \left( \frac{1}{p}, \frac{1}{q} \right) \in [0, 1] \times [0, 1] : \|T_{\eta \sigma_M}\|_{p,q} < \infty \right\}.$$  

A very interesting survey of results concerning the type sets for convolution operators with singular measures in $\mathbb{R}^n$ can be found in [8].

In the $\mathbb{H}^n$ setting, Secco in [10] and [11] obtained $L^p$-improving properties of measures supported on curves in $\mathbb{H}^1$, under certain assumptions. In [9], Ricci and Stein showed that the type set of the measure given by (3) for the case $\varphi \equiv 0$, $\gamma = 0$ and $n = 1$ is the triangle with vertices $(0, 0), (1, 1)$ and $(\frac{3}{4}, \frac{1}{4})$. In [3] and [4], the author jointly with Godoy generalized the work of Ricci and Stein for the case $\varphi(w) = w^t A w = \sum_{j=1}^{n} \alpha_j |w_j|^2$, where $A$ is a $2n \times 2n$ real diagonal matrix such that $a_{ii} = a_{(i+1)(i+1)}$ for $i = 2j - 1$ with $j = 1, 2, \ldots, n$, $a_j = a_{(2j-1)(2j-1)}$, $w_j \in \mathbb{R}^2$, $0 \leq \gamma < 2n$ and $n \in \mathbb{N}$. There we also gave some examples of surfaces with degenerate curvature at the origin.

Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}$ be the function defined by $\varphi(y) = y^t A y$, where $A$ is a $2n \times 2n$ real symmetric matrix. It is well known that if $A$ is an arbitrary matrix, then there exists a symmetric matrix $\tilde{A}$ such that $y^t A y = y^t \tilde{A} y$ for all $y$. We consider two borelian measures on $\mathbb{H}^n$ supported on the graph of $\varphi$, $\mu_A$ and $\nu_{\gamma}$, $0 \leq \gamma < 2n$, given by

$$\mu_A(E) = \int_{\mathbb{R}^{2n}} \chi_E(y, \varphi(y)) \, dy.$$
\[\nu_\gamma(E) = \int_{\mathbb{R}^{2n}} \chi_{E}(y, \varphi(y))\eta(y)|y|^{-\gamma} \, dy,\]

where \(\eta : \mathbb{R}^{2n} \to [0,1]\) is a smooth cut-off function such that \(\eta(y) = 1\) if \(|y| \leq 1\), \(\eta(y) = 0\) if \(|y| \geq 2\), and \(E\) is a borelian set of \(\mathbb{H}^{n}\). Let \(T_{\mu_A}f = f \ast \mu_A\) and \(T_{\nu_\gamma}f = f \ast \nu_\gamma\) be the operators of convolution by \(\mu_A\) and \(\nu_\gamma\) on the right, respectively.

We are interested in studying the \(L^p\)-improving properties of the operator \(T_{\mu_A}\) and in the characterization of the type set \(E_{\nu_\gamma}\). We point out that our measure \(\mu_A\) is not the surface measure on the graph \(\text{gr}(\varphi)\) of \(\varphi\), however the measures \(\eta \mu_A\) and \(\eta \sigma_{\text{gr}(\varphi)}\) are equivalent, see Proposition 2 below, so \(E_{\eta \mu_A} = E_{\eta \sigma_{\text{gr}(\varphi)}}\).

The following restrictions for the type sets \(E_{\nu_\gamma}, 0 \leq \gamma < 2n\), were proved in [3] and [4] for the case \(\varphi(w_1, \ldots, w_n) = \sum_{j=1}^{n} \alpha_j |w_j|^2\) with \(w_j \in \mathbb{R}^2\). It is easy to see that such an argument works as well for our function \(\varphi(y) = y^T Ay\). Thus, if \((1/p, 1/q) \in E_{\nu_\gamma}, 0 \leq \gamma < 2n\), then

\[p \leq q, \quad \frac{1}{q} \geq \frac{2n+1}{p} - 2n, \quad \frac{1}{q} \geq \frac{1}{(2n+1)p}.\]  

Another necessary condition for the pair \((1/p, 1/q)\) to be in \(E_{\nu_\gamma}\) is the following:

\[\frac{1}{q} \geq 1 - \frac{2n - \gamma}{2n + 2}.\]

This last condition is relevant only for the case \(0 < \gamma < 2n\). Let \(D\) be the point of intersection, in the \((1/p, 1/q)\) plane, of the lines \(1/q = (2n+1)/p - 2n\), \(1/q = (1/p - (2n - \gamma)/(2n+2)\), and let \(D'\) be its symmetric image with respect to the symmetry axis \(1/q = 1 - 1/p\). So

\[D = \left(\frac{4n^2 + 2n + \gamma}{2n(2n+2)}, \frac{2n + (2n+1)\gamma}{2n(2n+2)}\right) = \left(\frac{1}{pD}, \frac{1}{qD}\right)\quad \text{and} \quad D' = \left(1 - \frac{1}{qD}, 1 - \frac{1}{pD}\right).\]

Since \(0 \leq \gamma < 2n\), it is clear that \(\|T_{\nu_\gamma}f\|_p \leq c\|f\|_p\) for all Borel functions \(f \in L^p(\mathbb{H}^n)\) and all \(1 \leq p \leq \infty\), so \((1/p, 1/p) \in E_{\mu_A}\). Thus, for \(0 < \gamma < 2n\) the set \(E_{\nu_\gamma}\) is contained in the closed trapezoid with vertices \((0,0), (1,1), D\) and \(D'\), and the set \(E_{\nu_0}\) is contained in the closed triangle with vertices \((0,0), (1,1)\) and \((2n+1)/(2n+2), 1/(2n+2)\)).

In Section 3, our main result appears. There we prove that the operator \(T_{\mu_A}\) is bounded from \(L^{(2n+2)/(2n+1)}(\mathbb{H}^n)\) to \(L^{2n+2}(\mathbb{H}^n)\), see Theorem 3 below. This result allows us to characterize the type set \(E_{\nu_0}\) as well as the interior of \(E_{\nu_\gamma}\) for \(0 < \gamma < 2n\).
More precisely, we show that $E_{\nu_0}$ is the closed triangle with vertices $(0, 0)$, $(1, 1)$ and \((2n + 1)/(2n + 2), 1/(2n + 2)\) and the interior of $E_{\nu_0}$ coincides with the interior of the closed trapezoid with vertices $(0, 0)$, $(1, 1)$, $D$ and $D'$, see Theorem 4 and Theorem 6 below.

Throughout this paper, $c$ will denote a positive real constant not necessarily the same at each occurrence. The symbol $A \lesssim B$ stands for the inequality $A \leq cB$ for a constant $c$. We use the following convention for the Fourier transform in $\mathbb{R}^n$: $\hat{f}(\xi) = \int f(x)e^{-i\xi \cdot x} \, dx$. The Fourier transform $\hat{u}$ of a distribution $u$ on $\mathbb{R}^n$ is the distribution defined by $(\hat{u}, \phi) = (u, \hat{\phi})$ for all rapidly decreasing functions $\phi$ on $\mathbb{R}^n$.

2. Preliminaries

In the sequel $J$ will denote the $2n \times 2n$ skew-symmetric matrix defined in (1). It is easy to check that

\begin{enumerate}[(a)]
  \item $J^2 = -I$,
  \item $J^t = -J$,
  \item $x^t J x = 0$ for all $x \in \mathbb{R}^{2n}$,
  \item $x^t J y = -y^t J x$ for all $x, y \in \mathbb{R}^{2n}$.
\end{enumerate}

\textbf{Lemma 1.} Let $A$ be a $2n \times 2n$ real diagonal matrix. Then

$$\det(A \pm J) = (a_{11}a_{(n+1)(n+1)} + 1) \cdot (a_{22}a_{(n+2)(n+2)} + 1) \cdot \ldots \cdot (a_{nn}a_{(2n)(2n)} + 1),$$

where the $a_{ii}$'s are the diagonal entries of $A$.

\textbf{Proof.} Since $\det(A + J) = \det((A + J)^t) = \det(A - J)$, it is sufficient to prove the statement of the lemma for $\det(A + J)$. Applying induction on $n$, the lemma follows. \hfill \Box

\textbf{Proposition 2.} Let $A$ be a $2n \times 2n$ real symmetric matrix. Then the graph of the function $\varphi(y) = y^t Ay$ generates all the group $\mathbb{H}^n$. Moreover, the measure $\nu_0 = \eta \mu_A$ is equivalent to the measure $\eta \sigma$, where $\eta$ is a cut-off function and $\sigma$ is the surface measure on the graph of $\varphi$.

\textbf{Proof.} The first statement will follow if we prove that $(x, 0)$ and $(0, t)$ belong to the set $G_{\text{gr}(\varphi)}$ generated by the graph $\text{gr}(\varphi)$ of $\varphi$, since $(x, t) = (x, 0) \cdot (0, t)$. It is clear that $(x, \varphi(x)) \in G_{\text{gr}(\varphi)}$, so $(-t^{1/2}x, \varphi(t^{1/2}x)) = (-t^{1/2}x, \varphi(-t^{1/2}x)) \in G_{\text{gr}(\varphi)}$ for all $x \in \mathbb{R}^{2n}$ and all $t > 0$. From that it follows that $(0, t\varphi(x)) \in G_{\text{gr}(\varphi)}$ for all $t > 0$ and all $x$. If $A$ is a non-null matrix, then $(0, -t) = (0, t)^{-1} \in G_{\text{gr}(\varphi)}$ and $(x, 0) = (x, \varphi(x)) \cdot (0, -\varphi(x)) \in G_{\text{gr}(\varphi)}$. If $A$ is the null matrix, it is sufficient to
prove that \((0,t) \in G_{gr(\varphi)}\) for all \(t\). Indeed, for \(x\) and \(y\) such that \(\langle x, y \rangle \neq 0\) we have 
\[(0,t) = (x,0) \cdot (ty/\langle x, y \rangle, 0) \cdot (-x - ty/\langle x, y \rangle, 0) \in G_{gr(\varphi)}.
\] So \(G_{gr(\varphi)} = \mathbb{H}^n\).

For the second part of the proposition, we have that the surface measure on the graph of \(\varphi\) is given by
\[
\sigma(E) = \int_{\varphi^{-1}(E)} \sqrt{\text{det}[(\partial_{x_i} \varphi, \partial_{x_j} \varphi)_x]} \, dx,
\]
where \(\varphi(x) = (x, \varphi(x))\) and \(E\) is a borelian set of \(\mathbb{R}^{2n+1}\) (see pages 43–45 in [1]). A computation gives
\[
\text{det}[(\partial_{x_i} \varphi, \partial_{x_j} \varphi)_x] = 1 + \sum_{j=1}^{2n} (\partial_{x_j} \varphi(x))^2 \quad \forall x.
\]
So
\[
\int_{\mathbb{R}^{2n}} \chi_E(\varphi(x)) \eta(x) \, dx \leq \int_{\varphi^{-1}(E)} \sqrt{\text{det}[(\partial_{x_i} \varphi, \partial_{x_j} \varphi)_x]} \eta(x) \, dx \leq \int_{\mathbb{R}^{2n}} \chi_E(\varphi(x)) \eta(x) \, dx.
\]
Then \(\nu_0\) is equivalent to \(\eta \sigma\). \(\square\)

The \(\lambda\)-twisted convolution is defined by
\[
(f \times_\lambda g)(x) = \int_{\mathbb{R}^{2n}} f(x - y)g(y)e^{-i\lambda x^tJy} \, dy.
\]
Given a \(2n \times 2n\) real symmetric matrix \(A\), we put
\[
e_A(x) = e^{ix^tAx}.
\]
It is easy to check, using the properties (b) and (c) of the matrix \(J\), that
\[
(f \times_{\lambda} e_A)(x) = e_{\lambda A}(x)(e_{\lambda A}(\cdot)f(\cdot)) \hat{\lambda}(2A + J)x),
\]
where \(\hat{f}(\xi) = \int_{\mathbb{R}^{2n}} f(x)e^{-ix^t\xi} \, dx\) is the Fourier transform of \(f\). Thus, for each \(f \in L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n})\) we have
\[
\|f \times_{\lambda} e_A\|_{L^2(\mathbb{R}^{2n})} = (2\pi)^n|\lambda|^{-n} \det(2A \pm J)^{-1/2}\|f\|_{L^2(\mathbb{R}^{2n})}
\]
if \(\det(2A \pm J) \neq 0\).
3. Main result

To prove the $L^{(2n+2)/(2n+1)}(\mathbb{H}^n) - L^{2n+2}(\mathbb{H}^n)$ boundedness of the operator $T_{\mu_A}$ we embed our operator in an analytic family $\{T_z\}$ of operators on the strip $-n \leq \Re(z) \leq 1$, and then we apply the complex interpolation theorem.

**Theorem 3.** If $\det(2A \pm J) \neq 0$, then the operator $T_{\mu_A}$ is bounded from $L^{(2n+2)/(2n+1)}(\mathbb{H}^n)$ to $L^{2n+2}(\mathbb{H}^n)$.

**Proof.** To prove the statement of the theorem we consider the family $\{|s|z-1\}$ of functions initially defined when $\Re(z) > 0$ and $s \in \mathbb{R}\{0\}$. This family of functions can be extended in the $z$ variable to an analytic family of distributions on $\mathbb{C}\{-2k: k \in \mathbb{N} \cup \{0\}\}$. By abuse of notation, we denote this extension by $|s|z-1$. The family $\{|s|z-1\}$ has simple poles in $z = -2k$ for $k \in \mathbb{N} \cup \{0\}$. Since the meromorphic continuation of the function $\Gamma(\frac{1}{2}z)$ (we keep the notation for his continuation) has simple poles at the same points (i.e. $z = -2k$), the family $\{I_z\}$ of distributions defined by

$$I_z(s) = \frac{2^{-z/2}}{\Gamma(\frac{1}{2}z)}|s|^{z-1}$$

results in an entire family of distributions (see pages 55–56 in [2]).

From this construction and by taking the ratios of the corresponding residues at $z = 0$, we have $I_0 = \delta$, where $\delta$ is the Dirac distribution at the origin on $\mathbb{R}$ (see equation (3), page 57 in [2]), also $\hat{I}_z = cI_{1-z}$ for a real constant $c$ independent of $z$ (see equation (12′), page 173 in [2]).

For $z \in \mathbb{C}$, we also define $U_z$ as the distribution on $\mathbb{H}^n$ given by the tensor product

$$U_z = \delta_{\mathbb{R}2n} \otimes I_z,$$

where $\delta_{\mathbb{R}2n}$ is the Dirac distribution at the origin on $\mathbb{R}2n$ and $I_z$ is given by (7). Let $\{T_z\}$ be the analytic family of operators on the strip $-n \leq \Re(z) \leq 1$, given by

$$T_z f = f \ast \mu_A \ast U_z.$$

It is clear that $T_0 = T_{\mu_A}$. For $\Re(z) = 1$ we have

$$\|T_z f\|_{\infty} = \|f \ast \mu_A \ast U_z\|_{\infty} \leq \|f\|_1 \|\mu_A \ast U_z\|_{\infty}.$$

Since $\mu_A \ast U_{1+ib}(x, t) = I_{1+ib}(t - \varphi(x)) = (2^{-(1+ib)/2}/\Gamma(\frac{1}{2}(1+ib))|t - \varphi(x)|^{ib}$, it follows that

$$\|T_{1+ib}\|_{1,\infty} \leq \left|\frac{2^{-(1+ib)/2}}{\Gamma(\frac{1}{2}(1+ib))}\right| \quad \forall b \in \mathbb{R}.$$
For \(\Re(z) = -n\) we will prove that the operator \(T_z\) is bounded on \(L^2(\mathbb{H}^n)\). This is equivalent to showing that

\[
\int_{\mathbb{R}^{2n}} |(T_zf)^\lambda(x)|^2 \, dx \leq c \int_{\mathbb{R}^{2n}} |f^\lambda(x)|^2 \, dx,
\]

where \(h^\lambda(x) := \int_{\mathbb{R}} h(x, t)e^{-\lambda t} \, dt\). A computation gives

\[
(T_{-n+ib}f)^\lambda(x) = \hat{T}_{-n+ib}(\lambda) \int_{\mathbb{R}^{2n}} f^\lambda(x - y)e^{\lambda x^t y} \, dy
\]

\[
= \hat{T}_{-n+ib}(\lambda)(f^\lambda \times_{\lambda, c} e_{\lambda A})(x).
\]

From the identity in (6) and since \(\hat{T}_z = cI_{1-z}\), we get

\[
\|(T_{-n+ib}f)^\lambda\|_{L^2(\mathbb{R}^{2n})} = \left| c^{2^{-\left(1+n-ib\right)/2}} (2\pi)^n \left| \det(2A \pm J) \right|^{-1/2} \|f^\lambda\|_{L^2(\mathbb{R}^{2n})} \right|
\]

for each \(b \in \mathbb{R}\). So \(T_{-n+ib}\) is bounded on \(L^2(\mathbb{H}^n)\) if \(\det(2A \pm J) \neq 0\). Finally, it is easy to see, with the aid of the Stirling formula (see e.g. [12]), that the family \(\{T_z\}\) satisfies, on the strip \(-n \leq \Re(z) \leq 1\), the hypothesis of the complex interpolation theorem (see [13], page 205) and so \(T_0 = T_{\mu_A}\) is bounded from \(L^{(2n+2)/(2n+1)}(\mathbb{H}^n)\) into \(L^{2n+2}(\mathbb{H}^n)\).

\[\square\]

**Theorem 4.** Let \(\nu_0\) be the measure defined by (3) with \(\gamma = 0\). If \(\det(2A \pm J) \neq 0\), then the type set \(E_{\nu_0}\) is the closed triangle with vertices \((0, 0), (1, 1)\) and \(((2n + 1)/(2n + 2), 0)\).

**Proof.** Since the inequality \(T_{\nu_0}f \leq T_{\mu_A}f\) holds for each borelian function \(f \geq 0\), the theorem follows from the restrictions that appear in (4), Theorem 3 and the Riesz convexity theorem.

\[\square\]

**Corollary 5.** If \(\det(2A \pm J) \neq 0\), then the operator \(T_{\mu_A}\) is bounded from \(L^p(\mathbb{H}^n)\) into \(L^q(\mathbb{H}^n)\) if and only if \(p = (2n + 2)/(2n + 1)\) and \(q = 2n + 2\).

**Proof.** The “if” part of the corollary is Theorem 3. To see the reciprocal we introduce the action of the dilation group \(\mathbb{R}_{>0}\) on \(\mathbb{H}^n\), i.e. \(\delta \cdot (x, t) = (\delta x, \delta^2 t)\), \(\delta > 0\). For a function \(f\) defined on \(\mathbb{H}^n\) we put \(f_\delta(x, t) = f(\delta \cdot (x, t))\). It is easy to check that

\[
(T_{\mu_A}f)_\delta = \delta^{2n} T_{\mu_A}(f_\delta).
\]

If \(\|T_{\mu_A}f\|_q \leq c_{p,q}\|f\|_p\), then

\[
\delta^{-2(2n+2)/q} \|T_{\mu_A}f\|_q = \|(T_{\mu_A}f)_\delta\|_q = \delta^{2n} \|T_{\mu_A}(f_\delta)\|_q \leq \delta^{2n} c\|f_\delta\|_p = \delta^{2n-2(2n+2)/p} c\|f\|_p
\]

for all \(\delta > 0\). So \(1/q = 1/p - 2n/(2n + 2)\). Since \(T_{\nu_0}f \leq T_{\mu_A}f\) for \(f \geq 0\), from Theorem 4 it follows that \(p = (2n + 2)/(2n + 1)\) and \(q = 2n + 2\).

\[\square\]
Theorem 6. Let $\nu_\gamma$ be the measure defined by equation (3) with $0 < \gamma < 2n$. If $\det(2A \pm J) \neq 0$, then the type set $E_{\nu_\gamma}$ is contained in the closed trapezoid with vertices $(0, 0)$, $(1, 1)$, $D$ and $D'$, where

$$D = \left(\frac{4n^2 + 2n + \gamma}{2n(2n + 2)}, \frac{2n + (2n + 1)\gamma}{2n(2n + 2)}\right) = \left(\frac{1}{p_D}, \frac{1}{q_D}\right) \text{ and } D' = \left(1 - \frac{1}{q_D}, 1 - \frac{1}{p_D}\right)$$

and with the only possible exception of the closed segment joining the two points $D$ and $D'$.

Proof. For each $k \in \mathbb{N} \cup \{0\}$ we define the sets $A_k \subset \mathbb{R}^{2n}$ by

$$A_k = \{y \in \mathbb{R}^{2n}: 2^{-k} < |y| \leq 2^{-k+1}\}.$$ 

Let $\nu_{\gamma,k}$ be the fractional Borel measure given by

$$\nu_{\gamma,k}(E) = \int_{A_k} \chi_E(y, \varphi(y)) \eta(y) |y|^{-\gamma} \, dy$$

and let $T_{\nu_{\gamma,k}}$ be its corresponding convolution operator, i.e. $T_{\nu_{\gamma,k}} f = f * \nu_{\gamma,k}$. Now, it is clear that $\nu_\gamma = \sum_k \nu_{\gamma,k}$ and $\|T_{\nu_\gamma}\|_{p,q} \leq \sum_k \|T_{\nu_{\gamma,k}}\|_{p,q}$. For $f \geq 0$ we have that

$$\int_{\mathbb{R}^{2n}} f(y, s) \, d\nu_{\gamma,k}(y, s) \leq 2^{k\gamma} \int_{\mathbb{R}^{2n}} f(y, \varphi(y)) \eta(y) \, dy.$$ 

Thus $\|T_{\nu_{\gamma,k}}\|_{p,q} \leq c 2^{k\gamma}\|T_{\nu_\gamma}\|_{p,q}$, from Theorem 4 it follows that

$$\|T_{\nu_{\gamma,k}}\|_{(2n+2)/(2n+1), 2n+2} \leq c 2^{k\gamma}.$$ 

It is easy to check that $\|T_{\nu_{\gamma,k}}\|_{1,1} \leq |\nu_{\gamma,k}(\mathbb{R}^{2n+1})| \sim \int_{A_k} |y|^{-\gamma} \, dy = c 2^{-k(2n-\gamma)}$. For $0 < \theta < 1$ we define

$$\left(\frac{1}{p_\theta}, \frac{1}{q_\theta}\right) = \left(\frac{2n + 1}{2n + 2}, \frac{1}{2n + 2}\right)(1 - \theta) + (1, 1)\theta.$$ 

By the Riesz convexity theorem we have

$$\|T_{\nu_{\gamma,k}}\|_{p_\theta, q_\theta} \leq c 2^{k\gamma(1-\theta) - k(2n-\gamma)\theta}.$$ 

Choosing $\theta$ such that $k\gamma(1-\theta) - k(2n-\gamma)\theta = 0$ yields $\sup_{k \in \mathbb{N}} \|T_{\nu_{\gamma,k}}\|_{p_\theta, q_\theta} \leq c < \infty$. A simple computation gives $\theta = (2n - \gamma)/(2n)$, then $(1/p_\theta, 1/q_\theta) = (1/p_D, 1/q_D)$, so
\[ \|T_{\nu_{\gamma,k}}\|_{p_D,q_D} \leq c, \text{ where } c \text{ is independent of } k. \] Interpolating once again, but now between the points \((1/p_D, 1/q_D)\) and \((1,1)\) we obtain for each \(0 < \tau < 1\) fixed
\[ ||T_{\nu_{\gamma,k}}||_{p_{\tau},q_{\tau}} \leq c 2^{-k(2n-\gamma)\tau}. \]
Since \( ||T_{\nu_{\gamma}}||_{p,q} \leq \sum_k ||T_{\nu_{\gamma,k}}||_{p,q} \) and \( 0 < \gamma < 2n \), it follows that
\[ ||T_{\nu_{\gamma}}||_{p_{\tau},q_{\tau}} \leq c \sum_{k \in \mathbb{N}} 2^{-k(2n-\gamma)\tau} < \infty. \]
By duality we also have
\[ ||T_{\nu_{\gamma}}||_{q_{\tau}/(q_{\tau}-1),p_{\tau}/(p_{\tau}-1)} \leq c_{\tau} < \infty. \]

Finally, the theorem follows from the Riesz convexity theorem, and the restrictions that appear in (4) and (5). \(\square\)

We conclude this note with the following remarks.

**Remark 7.** Let \( \nu_0 \) be the measure of compact support defined by (3), but now with \( \det(2A \pm J) = 0 \). In this case, by Theorem 1.1 in [9] and Proposition 2, we can be sure that the type set \( E_{\nu_0} \) has a nonempty interior.

**Remark 8.** Lemma 1 provides us with examples of diagonal matrices \( A \) such that \( \det(2A \pm J) = 0 \). By the above remark we know that the interior of the type set of measure \( \nu_0 = \eta \mu_A \) is nonempty. If \( n \geq 2 \) and \( A \) also satisfies that \( \varphi(y) = y^t A y = \sum_{j=1}^{n} \alpha_j |y_j|^2 \) \((\alpha_j \in \mathbb{R} \text{ and } y_j \in \mathbb{R}^2)\), then the type set of \( \nu_0 \) is the closed triangle with vertices \((0,0), (1,1)\) and \((\frac{(2n+1)}{2n+2}, \frac{1}{2n+2})\). This result is independent of the value of \( \det(2A \pm J) \) (see Theorem 1, page 102 in [3]).

These final comments illustrate the limits of the techniques used in this note as well as of those developed in the works [3] and [4].

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We conclude this note with the following remarks.

**Remark 7.** Let \( \nu_0 \) be the measure of compact support defined by (3), but now with \( \det(2A \pm J) = 0 \). In this case, by Theorem 1.1 in [9] and Proposition 2, we can be sure that the type set \( E_{\nu_0} \) has a nonempty interior.

**Remark 8.** Lemma 1 provides us with examples of diagonal matrices \( A \) such that \( \det(2A \pm J) = 0 \). By the above remark we know that the interior of the type set of measure \( \nu_0 = \eta \mu_A \) is nonempty. If \( n \geq 2 \) and \( A \) also satisfies that \( \varphi(y) = y^t A y = \sum_{j=1}^{n} \alpha_j |y_j|^2 \) \((\alpha_j \in \mathbb{R} \text{ and } y_j \in \mathbb{R}^2)\), then the type set of \( \nu_0 \) is the closed triangle with vertices \((0,0), (1,1)\) and \((\frac{(2n+1)}{2n+2}, \frac{1}{2n+2})\). This result is independent of the value of \( \det(2A \pm J) \) (see Theorem 1, page 102 in [3]).

These final comments illustrate the limits of the techniques used in this note as well as of those developed in the works [3] and [4].

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