Failure of Equilibrium Selection Methods for Multiple-Principal, Multiple-Agent Problems with Non-Rivalrous Goods:
An Analysis of Data Markets

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Abstract—The advent of machine learning tools has led to the rise of data markets. These data markets are characterized by multiple data purchasers interacting with a set of data sources. Data sources have more information about the quality of data than the data purchasers; additionally, data itself is a non-rivalrous good that can be shared with multiple parties at negligible marginal cost. In this paper, we study the multiple-principal, multiple-agent problem with non-rivalrous goods. Under the assumption that the principal’s payoff is quasilinear in the payments given to agents, we show that there is a fundamental degeneracy in the market of non-rivalrous goods. Specifically, for a general class of payment contracts, there will be an infinite set of generalized Nash equilibria. This multiplicity of equilibria also affects common refinements of equilibrium definitions intended to uniquely select an equilibrium: both variational equilibria and normalized equilibria will be non-unique in general. This implies that most existing equilibrium concepts cannot provide predictions on the outcomes of data markets emerging today. The results support the idea that modifications to payment contracts themselves are unlikely to yield a unique equilibrium, and either changes to the models of study or new equilibrium concepts will be required to determine unique equilibria in settings with multiple principals and a non-rivalrous good.

I. INTRODUCTION

The widespread adoption of machine learning tools has led to data becoming a marketable good in its own right. This has led to the rise of data markets where companies and people can put forth payment contracts to incentivize workers to harvest data. Several platforms act as intermediaries in this exchange, such as the Amazon Mechanical Turk, Witkey, and Terbine.

The exchange of data often gives rise to problems of asymmetric information. In these exchanges, the data seller has more information about the quality of data than the data purchaser. This asymmetric information often gets modeled in a principal-agent setting: the principal (data purchaser) wishes to incentivize the agent (data source) to harvest and share high quality data, but cannot directly observe the quality of data or the effort exerted when harvesting data.

Additionally, data as a good is special: it is a non-rivalrous good. (This is also sometimes referred to as a public good in the economics literature.) Once data is produced, it can be shared with multiple data purchasers at negligible marginal cost. Non-rivalrous goods often lead to externalities between agents, and thus free-riding and social inefficiencies are a common consequence [1]. Data markets are no exception [2].

Finally, as data increases in importance, multiple firms will attempt to enter the data market. This leads to our problem formulation, where multiple principals (data purchasers) interact with the same set of agents (data sources) to incentivize the collection and sharing of a non-rivalrous good.

The contribution of this paper is as follows. To the best of our knowledge, this is the first work to generally study the structure of multiple-principal, multiple-agent problems with non-rivalrous goods. We look at a general class of payment functions, and show that whenever the principal’s payoffs are quasilinear in the payments given, there will be an infinite number of generalized Nash equilibria. Additionally, neither the variational equilibria concept nor the normalized equilibria concept can act as an equilibrium selection method: we show that both these equilibrium concepts will yield a multiplicity of equilibria. These results point at a fundamental ambiguity that exists under very mild conditions for the multiple-principal, multiple-agent problem when the underlying good produced is non-rivalrous.

The rest of this paper is organized as follows. In Section II, we contextualize our contributions among the existing literature. In Section III, we outline our model and formally state our problem of study. Additionally, we show how our model includes some recently studied models for data markets as a special case. In Section IV, we introduce the equilibrium concepts that will be applied throughout the paper. In Section V, we prove that multiple-principal, multiple-agent problems with non-rivalrous goods will have an infinite number of generalized Nash equilibria, variational equilibria, and normalized equilibria. This section contains the main theoretical contributions of this paper. We provide closing remarks in Section VI.

II. BACKGROUND

The most relevant work in this area can be divided into the categories: 1) the study of principal-agent problems with a single principal, 2) the study of multiple-principal, multiple-agent problems, and 3) the literature on equilibrium selection methods.

In principal-agent problems with a single principal, the strategic response of agents to a payment contract is studied...
to design contract mechanisms. These works seek to ensure truthful reporting from agents while maximizing the principal’s utility. In [3], a peer-prediction based contract mechanism is developed for a single data purchaser buying data from multiple data sellers. In [4], an electricity market where a single purchaser acquires electricity from multiple sources to satisfy its demand is studied. In [5], the agent’s strategy to contract only a subset of sellers considering data as a non-rivalrous good are studied in [2] and the existence of a multiplicity of equilibria is shown. In [7], failure of the revelation principle in the multiple-principal multiple-agent scenario when principals can resort to a complex communication scheme is shown. In [8], the agent’s strategy to contract only a subset of principals in order to maximize its utility is studied.

In problems with multiple principals, the added effect of coupling between the decisions of the principals and their effect on the agent’s response is studied. In [6], spectrum markets with multiple buyers and sellers is modeled as a finite horizon dynamic game and optimal behaviour is determined using a combination of short-term and long-term contracts. Data markets with multiple buyers and multiple sellers considering data as a non-rivalrous good are studied in [2] and the existence of a multiplicity of equilibria is shown. In [7], failure of the revelation principle in the multiple-principal multiple-agent scenario when principals can resort to a complex communication scheme is shown. In [8], the agent’s strategy to contract only a subset of principals in order to maximize its utility is studied.

The literature on equilibrium selection focuses on refinement of equilibrium concepts due to the issue of multiplicity of equilibria that often plagues generalized Nash equilibrium (GNE) problems. Variational equilibrium (VE) and normalized equilibrium (NoE) are the most popular equilibrium selection methods. [9] provides sufficient conditions for ensuring that VE is a refinement to GNE. In [10], sufficient conditions for equivalence of VE and NoE are presented, however these conditions do not necessarily hold in our model and we treat the two as independent selection methods. Quasi-Nash equilibrium and constrained Nash equilibrium as selection methods are introduced in [11] and [12]. [12] shows the relationship of a constrained Nash equilibrium with VE and GNE. [13] shows that under the linear independent constraint qualification, the quasi-Nash equilibrium coincides with NoE for their model.

Our work is aimed at studying the consequences of non-rivalrous nature of data in realistic data markets with multiple buyers and multiple sellers. We generalize this study to a market with any non-rivalrous good and show that the nature of the good leads to an infinite number of generalized Nash equilibria. Furthermore, this set cannot be refined by the concepts of variational equilibrium or normalized equilibrium.

We model this scenario as a two-stage game. In the first stage principals decide on contract parameters and in the second stage, agents decide their efforts based on the contracts available to them. We show that for all quasilinear payment contracts that induce a unique dominant strategy equilibrium amongst the agents, this game does not admit a unique solution and the principal bearing the cost of the public good is uncertain. Payment contracts of this form are commonly used in crowdsourcing [2], [14], [15], [16].

Our results substantiate the idea that modification to payment contracts are unlikely to admit a unique equilibrium and changes in modeling techniques or new equilibrium concepts will be required for analysis of settings with multiple principals and a non-rivalrous good.

III. Model and Problem Statement

In this section, we introduce our model for the multiple-principal, multiple-agent problem with a non-rivalrous good, and state the problem of equilibrium selection. Before we present the details in full, we provide a high-level overview of our model.

The structure of the interaction is as follows. First, let $B = \{1, \ldots, M\}$ denote the set of principals and let $\mathcal{N} = \{1, \ldots, N\}$ denote the set of agents. Each principal $j \in B$ announces payment contracts $(p_j^i)_{i \in \mathcal{N}}$, where $p_j^i$ is the payment contract offered to agent $i \in \mathcal{N}$. The principals announce their payment contracts simultaneously. Based on the payment contracts, the agents choose to exert some effort to produce a good. The more effort an agent exerts, the higher the quality of good they produce. This good is non-rivalrous in nature, and all principals derive some value from the production of the good.

One example of a non-rivalrous good is data. Once an agent produces data of a certain quality, they can infinitely share this data with multiple principals; sharing their data with one more principal has negligible marginal cost. Thus, our framework captures many interactions that take place in crowdsourcing applications, and the model encompasses the payment contracts used by [2], [14], [15] and [16].

An issue in this interaction is the asymmetric information between the principals and the agents. That is, the principals cannot directly observe the effort exerted by agents, and the payment contracts must be functions of some observable quantity, rather than the exerted effort. In our motivating data markets example, a principal may not know the variance of the reported data’s distribution, but they know the realized value of the reported data. The asymmetric information between principals and agents motivates the need for incentive-compatible payment contracts, to motivate the agents to exert effort in producing the good.

A. Model

In this subsection, we will introduce the payoff functions for the principals and the agents, as well as the payment contracts under consideration.

First, let us introduce the notation to be used throughout this section. For each agent $i \in \mathcal{N}$, let $e_i \in \mathbb{R}$ denote the effort exerted by agent $i$, and let $e = (e_i)_{i \in \mathcal{N}}$ denote the vector of efforts. Furthermore, each principal $j \in B$ does

1 We use $B$ to denote the set of principals as they are the ‘data buyers’ in our motivating application, and to avoid confusion with the payment contracts $p$.
not directly observe the efforts \(e\), but rather observes some (potentially random) function of the efforts, \(y^j(e)\).

**Assumption 1** (Payment structure). We consider payment contracts of the following form, where the payment from principal \(j\) to agent \(i\) is parameterized by \((c^j_i, a^j_i)\):

\[
p^j_i(y^j(e); c^j_i, a^j_i) = c^j_i + h^j(a^j_i, y^j(e))
\]

For notational simplicity, we will combine \(h^j\) and \(y^j\) into a single (potentially random) function \(f^j\):

\[
p^j_i(y^j(e); c^j_i, a^j_i) = c^j_i + f^j(a^j_i, e)
\]

We assume \(f^j : (a^j_i, e) \mapsto \mathbb{E}[f^j(a^j_i, e)]\) is a continuously differentiable function.

Note that the payment to agent \(i\) depends on the entire effort vector \(e\). In other words, the payment received by agent \(i\) is affected by the efforts exerted by the other agents. Put another way, the payment contracts induce a game between the agents. Payment contracts of this form are studied in [2], [14], [15] and [16].

Now, let us consider the agent’s payoffs.

**Assumption 2** (Agent model). We assume that the agent’s utility function is quasilinear in efforts and payments received; that is, for each agent \(i\), their payoff is given by:

\[
u_i(e) = \sum_{j \in \mathcal{B}} p^j_i(y^j(e); c^j_i, a^j_i) - e_i
\]

When selecting \(e_i\), the agent knows the functions \(p^j_i\) and \(y^j\).

As mentioned previously, our model captures a non-rivalrous good, so once agent \(i\) exerts effort \(e_i\), every principal will derive some value from the produced good. Consequently, once agent \(i\) exerts effort \(e_i\), they will receive payments from all the principals.

Next, let us consider the principal’s payoffs. Let \(c^j = (c^j_i)\) and \(a^j = (a^j_i)\) denote the vector of payment contract parameters for all agents \(i\). Similarly, let \(c = (c^j)\) and \(a = (a^j)\) denote the matrices of the contract parameter vectors across all principals \(j\).

**Assumption 3** (Principal’s loss function). The cost of each principal \(j\) is given by:

\[
L_j(c^j, a^j; e) = \sum_{i \in \mathcal{N}} p^j_i(y^j(e); c^j_i, a^j_i) - v^j(e)
\]

Here, \(v^j\) is a (potentially random) function, and we assume \(v^j : e \mapsto \mathbb{E}[v^j(e)]\) is a continuously differentiable function.

The cost for each principal is the sum of all payments made by this principal to the agents, minus the value received when the agents exert effort \(e\). In a data markets setting, \(v^j(e)\) can represent the quality of statistical inference when the reported dataset is \(y^j(e)\).

**Assumption 4** (Risk neutrality). We will assume that the principals and agents make their decisions ex-ante. In other words, the principals must decide \((c^j, a^j)\) and the agents must decide \(e\) prior to the realization of the \((y^j(e))\) and thus prior to the realization of the payment values) as well as \(v^j(e)\), the value derived from the goods.

Furthermore, all parties are risk-neutral, so their ex-ante decisions are made to optimize the expected value of their cost:

\[
\mathbb{E}[u_i(e)] = \sum_{j \in \mathcal{B}} \left( c^j_i + f^j(a^j_i, e) \right) - e_i
\]

\[
\mathbb{E}[L_j(c^j, a^j; e)] = \sum_{i \in \mathcal{N}} \left( c^j_i + f^j(a^j_i, e) \right) - v^j(e)
\]

Thus, this setting can be thought of as a two-stage game. In the first stage, the principals decide the payment contract parameters \((c, a) = (c^j, a^j)_{j \in \mathcal{B}}\). In the second stage, the agents decide the effort levels \(e = (e_i)_{i \in \mathcal{N}}\). Put another way, in the first stage, the principals choose payment contracts that jointly determine which game is induced between the agents in the second stage.

Finally, we outline desirable properties for our payment contracts.

We wish to have voluntary participation of the agents. Formally, this means that their ex-ante utilities should be positive. We will refer to ‘ex-ante individual rationality’ as simply ‘individual rationality’ throughout this paper.

**Definition 1** (Individual rationality). The payment contracts are individually rational with respect to an effort level \(e^*\) if, for every \(i \in \mathcal{N}\):

\[
\mathbb{E}[u_i(e^*)] \geq 0
\]

Whereas individual rationality incentivizes participation, each principal must almost incentivize the agents to share the non-rivalrous good with them. Thus, we have another desirable condition.

**Definition 2** (Ex-ante positive payments). The payment contracts are positive with respect to an effort level \(e^*\) if, for all \(i \in \mathcal{N}\) and \(j \in \mathcal{B}\):

\[
\mathbb{E}[p^j_i(y^j(e); c^j, a^j)] \geq 0
\]

The last desirable property is stability of chosen effort levels \(e^*\). In our case, we enforce that \(e^*\) must be a dominant strategy equilibrium between the agents. Intuitively, this means that each agent’s payoff is maximized by choosing \(e^*_i\), regardless of the effort levels \(e_{-i}\) chosen by other agents.

**Definition 3** (Dominant strategy equilibrium). For the second stage, we say \(e^*\) is a dominant strategy equilibrium for payment contract parameters \((c, a)\) if for any \(i\) and any other potential effort vector \(e\):

\[
\mathbb{E}[u_i(e^*_i, e_{-i})] \geq \mathbb{E}[u_i(e)]
\]

We say \(e^*\) is unique if no other \(e'\) satisfies this property.

This provides our desired behavior in the second stage of the game: the payment contracts should induce a unique dominant strategy equilibrium \(e^*\), and \(e^*\) should be individually rational and have positive payments in expectation.

Our problem statement concerns the first stage of the game. Given this enforced structure on the second stage
(i.e. the interaction between agents), how can we specify the behavior of the principals in the first stage? In addition to the constraints on the payment contracts outlined, the contract parameters \((c, a)\) must be chosen to minimize the principal’s cost in some sense.

Importantly, having uniqueness of some equilibrium concept for this would provide insight into what outcomes to expect in real world applications such as data markets.

**Problem Statement.** What equilibrium selection methods can uniquely identify payment contract parameters \((c, a)\) and an induced effort level \(e^*\)?

To be a reasonable model for interactions between strategic parties, the parameters \((c, a)\) should form some equilibrium concept in the first stage between principals. Additionally, the resulting payment contracts must satisfy these properties in the second stage: the payment contracts induce a unique dominant strategy equilibrium \(e^*\) between agents, and be individually rational with respect to \(e^*\), with positive ex-ante payments.

Our paper shows that the most common equilibrium concepts and refinements fail to uniquely select an equilibrium in this problem setting. There is a fundamental ambiguity in multiple-principal, multiple-agent problems with non-rivalrous goods, due to the ambiguity in which principals can free-ride off others, and neither variational equilibrium or normalized equilibria can resolve this ambiguity.

**B. Example: effort-averse data sources in data markets**

In this subsection, we show how our framework encapsulates the model of a data market presented in [2]. In this setting, the principals are data aggregators and the agents are effort-averse data sources. A data source produces data by drawing one data sample from a distribution; the higher the level of effort, the lower the variance of the underlying distribution. The data aggregators have access to the reported data, but not the variance of the distributions that generated the data.

More formally, in this example, the data aggregators are trying to estimate some function \(\phi\). Each data source \(i \in \mathcal{N}\) generates a data point at \(x_i\):

\[ y_i(e_i) = \phi(x_i) + \epsilon_i \]

Here, \(\epsilon_i\) is zero mean noise with variance \(\sigma_i^2(\epsilon_i)\). The variance \(\sigma_i^2(\epsilon_i)\) decreases as the effort \(\epsilon_i\) increases.

In this case, the data aggregators issue payments of the form:

\[ p_i^j(y, c_i^j, a_i^j) = c_i^j - a_i^j(y_i - \hat{\phi}_{-i}(x_i; y_{-i}))^2 \]

Here, \(\hat{\phi}_{-i}\) is the leave-one-out estimator [3], where \(\phi(x_i)\) is estimated using every other data source’s data \(y_{-i}\). Intuitively, this setting uses the other data sources \(-i\) to generate an estimate of what data source \(i\) should be reporting. The term on the right decomposes into two independent terms: one which increases the payment when the variance decreases, and another term which agent \(i\) has no control over.

In this example:

\[ f^j = -a_i^j E[(y_i - \hat{\phi}_{-i}(x_i))^2] \]

From the perspective of the data aggregators, their loss is a combination of three terms: a penalty for poor statistical estimation, the cost of payments issued, and an optional penalty for the estimation quality of competing firms. For simplicity, we consider the estimation loss at a single point \(x\):

\[ E[L_j((c^j, a^j))] = E[(\phi(x) - \hat{\phi}^j(x))^2] - \sum_{k \neq j} \zeta_k^j E[(\phi(x) - \hat{\phi}^k(x))^2] + \sum_{i \in \mathcal{N}} E[p_i^j] \]

Here, \((\zeta_k^j)_{k \geq 0}\) are parameters that act as weights in this loss function. Thus, in this example:

\[ v^j = -E[(\phi(x) - \hat{\phi}^j(x))^2] + \sum_{k \neq j} \zeta_k^j E[(\phi(x) - \hat{\phi}^k(x))^2] \]

Note that \(v^j\) depends on \(e\) through the realized data \((y^k(e))^k_{k \in \mathcal{B}}\) and the resulting estimators \((\hat{\phi}^k)_{k \in \mathcal{B}}\).

We note that there are several other approaches to incentive design for strategic data sources which similarly fall into our framework. For example, in [14], rather than using a leave-one-out estimator, the authors assume access to queries of known quality, referred to as ‘gold standard’ queries.

**IV. Equilibrium Concepts**

With our model in place, in this section, we introduce the equilibrium concepts we consider. In Section [V] we will show that none of these concepts will successfully determine a unique set of parameters \((c, a)\).

First, let us introduce some notations. Let \(\mathcal{P}_c(e^*)\) denote the set of payment contract parameters such that \(e^*\) is a unique dominant strategy equilibrium, and the payment contracts are individually rational and positive with respect to \(e^*\). Similarly, let \(\mathcal{P} = \cup_c \mathcal{P}_c(e^*)\) denote the set of payment contracts such that there exists an \(e^*\) that satisfies these properties.

We introduce the function \(\mu\), which maps payment contract parameters to the dominant strategy effort levels. Note that if a payment contract induces a unique dominant strategy equilibrium between agents, this equilibrium depends only on the \(a\) parameters. That is, the parameters \(c\) have no effect on the value of the dominant strategy equilibrium.

More formally, let \(\mu: \mathcal{P} \to \mathbb{R}^\mathcal{N}\) denote the mapping from payment contract parameters \((c, a)\) to the dominant strategy equilibrium \(e^*\).

**Proposition 1.** The mapping from contract parameters \((c, a)\) to the dominant strategy equilibrium \(e^*\) depends only on \(a\). That is, the function \(\mu(\cdot, a)\) is constant on \(\mathcal{P}\).

**Proof.** By combining Equations (3) and (1), we can see that \(e^*\) is a dominant strategy equilibrium if and only if:

\[ \sum_{j \in \mathcal{B}} \mathbf{P}^j(a^j, (e^*_i, e_{-i})) \geq \sum_{j \in \mathcal{B}} \mathbf{P}^j(a^j, e) \]

The desired result follows. \(\square\)
In light of Proposition 1, we will use the notation $\mu(a)$ without any loss of generality, dropping the $c$ argument entirely.

Intuitively, Proposition 1 holds because the parameters $c$ act as a constant shift and do not affect the strategic nature of the game at all. Indeed, one can thinking of this as follows: the inclusion of the $c$ parameters ensures individual rationality while the inclusion of the $a$ parameters ensures the dominant strategy equilibrium is as desired.

Now, we can introduce the concept of generalized Nash equilibrium. The introduction of $\mu$ allows us to note how the dominant strategy effort levels $e^*$ vary with the $a$ parameters in the following definition.

**Definition 4** (Generalized Nash equilibrium (GNE) [10]). A set of contract parameters $(c, a)$ form a generalized Nash equilibrium for the first stage of the game if for every $j$, $(e^j, a^j)$ solves the minimization problem:

$$
\min_{(e^j, a^j)} \mathbb{E}[L(e^j, a^j; \mu(a))]
$$

s.t. $(c, a) \in \mathcal{P}$

In other words, holding the other agent’s actions constant, $(e^j, a^j)$ minimizes the principal $j$’s cost subject to the constraint that $(c, a)$ induce a unique dominant strategy equilibrium that is individually rational and ex-ante positive.

It is not uncommon that games admit an infinite set of GNEs [12]. This has motivated the refinements of equilibrium concepts. We present two here: the variational equilibrium and normalized equilibrium.

**Definition 5** (Variational equilibria (VE) [9]). We say $(c, a)$ is a variational equilibrium if $(c, a) \in \mathcal{P}$ and, for any $(\tilde{c}, \tilde{a}) \in \mathcal{P}$:

$$
\nabla_{(e^j, a^j)} \mathbb{E}[L_1(e^j, a^j; \mu(a))] \geq 0
$$

Furthermore, note that $\mu$ allows us to write the set $\mathcal{P}$ in terms of a finite number of constraints:

$$
\mathcal{P} = \{(c, a) : \mathbb{E}[u_i(\mu(a))] \geq 0 \text{ for all } i \in \mathcal{N}, \mathbb{E}[p_i'(\mu(a))] \geq 0 \text{ for all } i \in \mathcal{N} \text{ and } j \in \mathcal{B}\}
$$

Thus, the constraint set in Equation (4) can be thought of as a finite number of inequality constraints, and admits a finite number of Lagrange multipliers. This allows us to introduce another equilibrium refinement method.

**Definition 6** (Normalized equilibria (NoE) [17]). Let $\gamma \in \mathbb{R}^{|\mathcal{N}|}$. Let $\lambda^j$ denote the KKT multipliers from the Equation (4) for principal $j$. We say a GNE is a normalized equilibrium with weights $\gamma$ if the KKT multipliers $\lambda^j$ satisfy the normalization condition:

$$
\gamma_1 \lambda^1 = \gamma_2 \lambda^2 = \cdots = \gamma_M \lambda^M
$$

V. Failure of Equilibrium Concepts

In this section, we outline why the equilibrium concepts in Section IV fail to uniquely select an equilibrium for multiple-principal, multiple-agent problems with a non-rivalrous good. The inability of variational equilibria to select a unique GNE in Section V-B and the non-uniqueness of normalized equilibria in Section V-C are the main results of this paper. In doing so, we highlight fundamental degeneracies when multiple principals attempt to incentivize multiple agents to produce a non-rivalrous good.

A. Non-uniqueness of GNE

First, we note one property which is the underlying reason why the aforementioned equilibrium selection methods fail. Whenever the principal’s loss function is quasilinear in the payments given, the individual rationality constraints are always binding at generalized Nash equilibria. This is given in the next proposition.

**Proposition 2.** Let $(c, a)$ be any generalized Nash equilibrium with associated effort level $e^*$. Then the individual rationality constraints are binding. That is:

$$
\mathbb{E}[u_i(e^*)] = \sum_{j \in \mathcal{B}} (c^j_i + f^j(a^j_i, e^*)) - e^*_i = 0
$$

**Proof.** Suppose there exists an $i \in \mathcal{N}$ such that the individual rationality constraint is not binding. Define $s_i$ as:

$$
s_i = \sum_{j \in \mathcal{B}} (c^j_i + f^j(a^j_i, e^*)) - e^*_i > 0
$$

Then, any principal $j$ can decrease their loss by changing from $c^j_i$ to $c^j_i - s_i$. Thus, $(c, a)$ is not an optimizer of Equation (4), and cannot be a GNE.

Now, we will explicitly calculate the simplex of generalized Nash equilibria for the game between principals. It is common for GNE to not be unique, and, in fact, for there to be a manifold of GNEs. This lack of uniqueness is one of the commonly stated issues in the study of GNEs [12], and has motivated the study of equilibrium refinements such as VE and NoE.

By Proposition 2 we know that the individual rationality constraints always bind. This can be rewritten for any $i \in \mathcal{N}$:

$$
\sum_{j \in \mathcal{B}} c^j_i = e^*_i - \sum_{j \in \mathcal{B}} f^j(a^j_i, e^*) = \mu(a)_i - \sum_{j \in \mathcal{B}} f^j(a^j_i, \mu(a))
$$

Let’s define this quantity as $g_i(a)$:

$$
g_i(a) = \mu(a)_i - \sum_{j \in \mathcal{B}} f^j(a^j_i, \mu(a))
$$

Thus, by re-arranging terms, we can write:

$$
c^j_i = g_i(a) - \sum_{k \neq j} c^k_i
$$

This becomes an equality constraint in the optimization problem of principal $j$. In particular, each principal’s ex-ante cost can be re-written to be independent of $e^j$. 
\[ E[L_j(a^j; \mu(a))] = \sum_{i \in \mathcal{N}} \left( g_i(a) - \sum_{k \neq j} c_{ij}^k + \mathbf{f}^j(a_i^j, \mu(a)) \right) - \mathbf{v}^j(\mu(a)) \] \tag{7}

In light of this observation, we can think of game between principals as having two parts. On one hand, in the \( a \) parameters, they decide which game to induce among agents, and what the equilibrium effort levels \( e^* \) should be. On the other hand, in the \( c \) parameters, they divide the expected gains from the agents accordingly.

**Proposition 3** (Simplex of GNE). Let \((c, a)\) be any generalized Nash equilibrium. Then, take any \( \tilde{c} \) such that:

\[
\tilde{c}_i^j = g_i(a) - \sum_{k \neq j} c_{ik}^j \geq -f^j(a_i^j, \mu(a)) \tag{8}
\]

Then \((\tilde{c}, a)\) is also a generalized Nash equilibrium. Furthermore, there are no other generalized Nash equilibria with the same \( a \) parameters.

**Proof.** The change from \( c \) to \( \tilde{c} \) does not change the \( a \) parameters that optimize Equation (7), and we can see that Equation (8) ensures that the new \((\tilde{c}, a)\) remains feasible for each principal’s optimization.

**B. Failure of variational equilibria as a selection method**

Now that we have noted that there is generally not a unique generalized Nash equilibrium, we will show that variational refinements will not allow us to select among the GNE.

**Theorem 1.** If one generalized Nash equilibrium is a variational equilibrium, then all generalized Nash equilibria with the same \( a \) parameters are variational equilibria.

**Proof.** First, let’s define \( \tilde{f}^j \) as:

\[
\tilde{f}^j(a) = \sum_{j \in \mathcal{B}} f^j(a_i^j, \mu(a)) - \mathbf{v}^j(\mu(a))
\]

Principal \( j \)’s ex-ante cost in Equation (2) can be written:

\[
E[L_j(c^j, a^j; \mu(a))] = \sum_{i \in \mathcal{N}} c_{ij}^j + \tilde{f}_j^j(a)
\]

Now, let \((c, a)\) be a generalized Nash equilibrium and a variational equilibrium. We can see that, for any variation \((\tilde{c}, \tilde{a})\):

\[
\begin{bmatrix}
\nabla_{(c^1, a^1)} E[L_1(c^1, a^1; \mu(a))] \\
\vdots \\
\nabla_{(c^M, a^M)} E[L_M(c^M, a^M; \mu(a))]
\end{bmatrix}^T [(\tilde{c}, \tilde{a}) - (c, a)] = 
\sum_{j \in \mathcal{B}} \sum_{i \in \mathcal{N}} \left( c_{ij}^j - c_{ij}^\tilde{j} \right) + \sum_{j \in \mathcal{B}} \sum_{i \in \mathcal{N}} \frac{\partial \tilde{f}^j}{\partial a_i^j} (\tilde{a}_i^j - a_i^j)
\]

Let \((\tilde{c}, \tilde{a})\) be any other GNE with the same \( a \) parameters. By Proposition 3, Equation 8 must be satisfied. Thus:

\[
\sum_{j \in \mathcal{B}} \sum_{i \in \mathcal{N}} (\tilde{c}_{ij}^j - c_{ij}^j) = 0
\]

Additionally, since the \( a \) parameters are unchanged, the second term is 0. Thus:

\[
\begin{bmatrix}
\nabla_{(c^1, a^1)} E[L_1(c^1, a^1; \mu(a))] \\
\vdots \\
\nabla_{(c^M, a^M)} E[L_M(c^M, a^M; \mu(a))]
\end{bmatrix}^T [(\tilde{c}, \tilde{a}) - (c, a)] = 0
\]

Thus, the simplex of GNEs are all VEs as well.

**C. Non-uniqueness of normalized equilibria**

Next, we note that normalized equilibria also fail to select a unique generalized Nash equilibrium. Recall the set of constraints were given by Equation (6). Thus, the individual rationality constraint is for each \( i \in \mathcal{N} \):

\[
\sum_{j \in \mathcal{B}} \left( c_{ij}^j + f^j(a_i^j, \mu(a)) \right) - \mu(a)_i \geq 0
\]

The positivity constraint is for each \( i \in \mathcal{N} \) and \( j \in \mathcal{B} \):

\[
c_{ij}^j + f^j(a_i^j, \mu(a)) \geq 0
\]

Let:

\[
\tilde{g}_i(c, a) = - \left( \sum_{j \in \mathcal{B}} \left( c_{ij}^j + f^j(a_i^j, \mu(a)) \right) - \mu(a)_i \right) = - \sum_{j \in \mathcal{B}} c_{ij}^j + g_i(a)
\]

Here, \( g_i(a) \) is as defined in Equation (6).

\[
\tilde{h}_{i,j}(c, a) = - \left( c_{ij}^j + f^j(a_i^j, \mu(a)) \right)
\]

Thus, our inequality constraints are simply \( \tilde{g}_i(c, a) \leq 0 \) for \( i \in \mathcal{N} \) and \( \tilde{h}_{i,j}(c, a) \leq 0 \) for \( i \in \mathcal{N} \) and \( j \in \mathcal{B} \).

**Theorem 2.** Consider any generalized Nash equilibrium where all payments are positive in expectation. This GNE is a normalized equilibria for \( \gamma_1 = \gamma_2 = \ldots = \gamma_N \).

**Proof.** For notational simplicity, let \( x^j = (c^j, a^j) \). The first-order stationarity condition for optimality of principal \( j \) is:

\[
\nabla_{x^j} E[L_j(c^j, a^j; \mu(a))] + \sum_{i \in \mathcal{N}} \lambda_i \nabla_{x^j} \tilde{g}_i(c, a) = 0
\]

Let \( \tilde{f}^j \) be as defined in the proof of Theorem 1. Then, the gradients in Equation (9) can be calculated:

\[
\nabla_{x^j} E[L_j(c^j, a^j; \mu(a))] = [\lambda_1 \nabla_{x^j} \tilde{g}_1(c, a) \ldots \nabla_{x^j} \tilde{g}_N(c, a)]^T
\]

(Here, \( [m \times n] \) denotes the matrix of dimension \( m \times n \) whose entries are all 1.) Similarly:

\[
\nabla_{x^j} \tilde{g}_i(c, a) = [0 \ldots -1_{i^{\text{pos}}} \ldots 0 \frac{\partial \tilde{g}_i}{\partial a_{i^1}} \ldots \frac{\partial \tilde{g}_i}{\partial a_{N}}]^T
\]

Now, consider any generalized Nash equilibrium in the relative interior of the simplex defined by Equation (8). By complementary slackness, at these interior points, \( \nu_{i,j} = 0 \) for all \( i \) and \( j \). Thus, solving Equation (9) for interior points would yield \( \lambda_i = 1 \) as the only possible values for all \( i \).

\[ \square \]
VI. CONCLUSION

Motivated by applications to data markets, we considered the multiple-principal, multiple-agents problem with a non-rivalrous good in this paper. We show that the payment contracts that have been studied in the literature thus far will, in the presence of multiple principals, leads to a multiplicity of equilibria, arising from ambiguity in which principals can free-ride off the others. This is structurally very different from situations where there is either a single principal or a rivalrous good. We have shown that this multiplicity of equilibria exists even for various refinements of equilibrium concepts. The proofs provide intuition for why we have this fundamental degeneracy.

This implies that most existing equilibrium concepts cannot provide predictions on the outcomes of data markets emerging today. Prior to this work, we believed that this multiplicity of equilibria could be addressed by modifications to the payment contracts. However, in this paper, we showed that this degeneracy holds even for a general class of payment contracts, and the proofs of our theorems in this paper outline technical reasons why. This shows that, in order to understand the behavior of strategic parties in data markets, we may need to explore new equilibrium concepts that provide uniqueness in the settings considered in this paper.

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