SOME RESULTS ON RICCI-BOURGUIGNON AND RICCI-BOURGUIGNON ALMOST SOLITONS

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Abstract. We prove some results for the solitons of the Ricci-Bourguignon flow, generalizing corresponding results for Ricci solitons. Taking motivation from Ricci almost solitons, we then introduce the notion of Ricci-Bourguignon almost solitons and prove some results about them which generalize previous results for Ricci almost solitons. We also derive integral formulas for compact gradient Ricci-Bourguignon solitons and compact gradient Ricci-Bourguignon almost solitons. Finally, using the integral formula we show that a compact gradient Ricci-Bourguignon almost soliton is isometric to an Euclidean sphere if it has constant scalar curvature or its associated vector field is conformal.

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1. Introduction

Ricci solitons play a major role in Ricci flow where they correspond to self-similar solutions of the flow. Thus, given a geometric flow it is natural to study the solitons associated to that flow.

A family of metrics $g(t)$ on an $n$-dimensional Riemannian manifold $(M^n, g)$ is said to evolve by the Ricci-Bourguignon flow (RB flow for short) if $g(t)$ satisfies the following evolution equation

$$\frac{\partial g}{\partial t} = -2(\text{Ric} - \rho Rg)$$  \hspace{1cm} (1.1)

where $\text{Ric}$ is the Ricci tensor of the metric, $R$ is the scalar curvature and $\rho \in \mathbb{R}$ is a constant. The flow in equation (1.1) was first introduced by Jean-Pierre Bourguignon [Bou81], building on some unpublished work of Lichnerowicz and a paper of Aubin [Aub70]. We note that (1.1) is precisely the Ricci flow for $\rho = 0$. In particular, the right hand side of the evolution equation (1.1) is of special interest for different values of $\rho$, for example

- $\rho = \frac{1}{2}$, the Einstein tensor $\text{Ric} - \frac{R}{n}g$
- $\rho = \frac{1}{n}$, the traceless Ricci tensor $\text{Ric} - \frac{R}{n}g$
- $\rho = \frac{1}{2(n-1)}$, the Schouten tensor $\text{Ric} - \frac{R}{2(n-1)}g$
- $\rho = 0$, the Ricci tensor $\text{Ric}$.
A systematic study of the parabolic theory of the RB flow was initiated in [CCD+17].
In that paper, the authors proved, along with many other results, the short time existence of the flow (1.1) on any closed \( n \)-dimensional manifold starting with an arbitrary initial metric \( g_0 \) for \( \rho < \frac{1}{2(n-1)} \). Just like the Ricci flow case, we make the following

**Definition 1.1.** A Ricci-Bourguignon soliton (RB soliton for short) is a Riemannian manifold \((M^n, g)\) endowed with a vector field \( X \) on \( M \) that satisfies

\[
R_{ij} + \frac{1}{2} (L_X g)_{ij} = \lambda g_{ij} + \rho R g_{ij}
\]

where \( L_X g \) denotes the Lie derivative of the metric \( g \) with respect to the vector field \( X \) and \( \lambda \in \mathbb{R} \) is a constant.

When \( X = \nabla f \) for some smooth \( f : M \to \mathbb{R} \) then \((M, g)\) is called a gradient RB soliton. The soliton is called

1. expanding when \( \lambda < 0 \)
2. steady when \( \lambda = 0 \)
3. shrinking when \( \lambda > 0 \)

RB solitons correspond to self-similar solutions of the RB flow. An RB soliton is called trivial if \( X \) is a Killing vector field, i.e., \( L_X g = 0 \). We remark that even though the short time existence result for the flow (1.1) is for \( \rho < \frac{1}{2(n-1)} \), any value of \( \rho \) is possible for the considerations of self-similar solutions of the flow.

Gradient RB solitons were studied in detail, for example in [CM16] and [CMM15] where the authors called them gradient \( p \)-Einstein solitons. Various classification and rigidity results about gradient RB solitons were proved in those papers and we refer the reader to those papers for precise statements and proofs of the results.

The notion of Ricci almost solitons was introduced in [PRRS11] where the authors modified the definition of a Ricci soliton by considering the parameter \( \lambda \) in the definition of a Ricci soliton to be a function rather than a constant. Motivated by the Ricci flow case we make the following

**Definition 1.2.** A Riemannian manifold \((M^n, g)\) is a Ricci-Bourguignon almost soliton (RB almost soliton for short) if there is a vector field \( X \) and a soliton function \( \lambda : M \to \mathbb{R} \) satisfying

\[
\text{Ric} + \frac{1}{2} L_X g = \lambda g + \rho R g
\]

An RB almost soliton is called a gradient RB almost soliton if \( X = \nabla f \) for some smooth function \( f \) on \( M \) and is expanding, steady or shrinking if \( \lambda < 0 \), \( \lambda = 0 \) or \( \lambda > 0 \) respectively. We note that if \( X \) is a Killing vector field then a RB almost soliton is just a RB soliton as it forces \( \lambda \) to be a constant.

Recall that a vector field \( Y \) on a Riemannian manifold \((M, g)\) is called a conformal vector field if there exists a function \( \psi : M \to \mathbb{R} \) such that

\[
L_Y g = 2\psi g
\]

The conformal vector field is non-trivial if \( \psi \neq 0 \).

Some characterization results for compact Ricci and Ricci almost solitons were obtained in [ABR11] and [BR12] respectively. The goal of the present paper is to generalize the results obtained in those papers to RB and RB almost solitons. More precisely, in §3 we prove the following theorems.
Theorem 1.3. Let \((M^n, g, X, \lambda, \rho)\) be a RB soliton with \(n \geq 3\) and suppose that the vector field \(X\) is a conformal vector field.

(1) If \(M\) is compact then \(X\) is a Killing vector field and hence \((M^n, g, X, \lambda, \rho)\) is a trivial RB soliton.

(2) If \(M\) is non-compact, complete and gradient RB soliton then either \(X\) is a Killing vector field or \((M^n, g, X, \lambda, \rho)\) is isometric to the Euclidean space.

This generalizes Theorem 3 in [ABR11] and characterizes compact RB solitons when \(X\) is a conformal vector field. The following corollary gives a lower bound for the first eigenvalue of the Laplacian on a compact RB soliton when \(X\) is a conformal vector field and generalizes Theorem 4 in [ABR11].

Corollary 1.4. Let \((M^n, g, X, \lambda, \rho)\) be a compact RB soliton with \(X\) a conformal vector field. If \(n \geq 3\) and \(\lambda + \rho R > 0\) then the first eigenvalue \(\lambda_1\) of the Laplacian satisfies \(\lambda_1 \geq (\lambda + \rho R)^{\frac{n}{n-1}}\). Moreover, equality occurs if and only if \(M^n\) is isometric to a standard sphere.

The next theorem characterizes compact RB almost solitons with \(X\) a conformal vector field and generalizes Theorem 2 in [BR12].

Theorem 1.5. Let \((M^n, g, X, \lambda, \rho)\) be a compact RB almost soliton with \(n \geq 3\). If \(X\) is a nontrivial conformal vector field then \(M^n\) is isometric to an Euclidean sphere.

The next theorem generalizes Theorem 3 in [BR12] obtained for compact Ricci almost solitons, which is the case when \(\rho = 0\).

Theorem 1.6. Let \((M^n, g, X, \lambda, \rho)\) be a compact RB almost soliton with \(n \geq 3\). If \(\rho \neq \frac{1}{n}\) and

\[
\int_M [\text{Ric}(X, X) + \frac{n \rho}{n \rho - 1} \nabla_X \text{div} X - 2 \rho g(\nabla R, X) - \frac{(n(2\rho + 1) - 2)}{n \rho - 1} g(\nabla \lambda, X)] \, dv \leq 0
\]

then \(X\) is a Killing vector field and \(M^n\) is a trivial RB soliton.

Since every RB almost soliton is also a RB soliton for constant \(\lambda\), hence using \(\nabla \lambda = 0\) we get the following corollary for compact RB soliton

Corollary 1.7. Let \((M^n, g, X, \lambda, \rho)\) be a compact RB soliton with \(n \geq 3\). If \(\rho \neq \frac{1}{n}\) and

\[
\int_M [\text{Ric}(X, X) + \frac{n \rho}{n \rho - 1} \nabla_X \text{div} X - 2 \rho g(\nabla R, X)] \, dv \leq 0
\]

then \(X\) is a Killing vector field and \(M^n\) is a trivial RB soliton.

Remark 1.8. Corollary 1.7 is an analog of Theorem 1.1 in [PW09] which was for the case of compact Ricci solitons. We obtain Petersen-Wylie’s result from ours by setting \(\rho = 0\). In fact, the condition in (1.5) is analogous to the condition in [PW09, Theorem 1.1], which is obtained when \(\rho = 0\) in (1.5).

Finally, we obtain an integral formula for compact gradient RB almost solitons generalizing corresponding result for compact gradient Ricci almost solitons from [BR12].
Theorem 1.9. Let \((M^n, g, \nabla f, \lambda, \rho)\) be a compact gradient RB almost soliton. Then

\[
\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 dv = \frac{(n-2)}{2n} \int_M g(\nabla R, \nabla f) dv \tag{1.6}
\]

\[
\int_M |\text{Ric} - \frac{R}{n} g|^2 dv = \frac{(n-2)}{2n} \int_M g(\nabla R, \nabla f) dv \tag{1.7}
\]

As an application of the previous theorem we state some conditions for a compact gradient RB almost soliton to be isometric to an Euclidean sphere.

Corollary 1.10. A nontrivial compact gradient RB almost soliton \((M^n, g, \nabla f, \lambda, \rho)\), \(n \geq 3\) is isometric to an Euclidean sphere if any one of the following holds

1. \(M^n\) has constant scalar curvature.
2. \(\int_M g(\nabla R, \nabla f) dv \leq 0\).
3. \(M^n\) is a homogenous manifold.

The paper is organized as follows. In §2 we state and prove some identities for RB solitons and RB almost solitons which will be used to prove the main results. The corresponding analogs for Ricci and Ricci almost solitons can be found, for example, in [ABR11] and [BR12] respectively. In §3 we will prove the main theorems and their corollaries.

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2. Preliminaries

In this section we prove some general results about RB and RB almost solitons. The proofs of some of these results in the compact gradient case can also be found in [CM16] or [CMM15]. Let us first recall the Ricci identities for a \((0,2)\)-tensor \(\alpha\):

\[
\nabla_i \nabla_j \alpha_{kl} - \nabla_k \nabla_i \alpha_{jl} = -R_{ijkm} \alpha_{ml} - R_{ijlm} \alpha_{km}
\]

where \(R_{ijkl}\) is the Riemann curvature tensor. The Ricci curvature is obtained from the Riemann curvature tensor by contracting on the first and last index

\[
R_{ij} = g^{kl} R_{kijl}
\]

and the contracted second Bianchi identity is

\[
\nabla_i R_{ij} = \frac{1}{2} \nabla_i R .
\]

We start with the following

Proposition 2.1. Let \((M^n, g, \nabla f, \lambda, \rho)\) be a gradient RB almost soliton. Then the following identities hold

\[
(1 - n\rho) R + \Delta f = n\lambda \tag{2.1}
\]

\[
(1 - 2\rho(n - 1)) \nabla_i R = 2 R_{id} \nabla_d f + 2(n - 1) \nabla_i \lambda \tag{2.2}
\]

\[
\nabla_j R_{ik} - \nabla_k R_{ij} = R_{ijkl} \nabla_l f + \rho (\nabla_j R_{gk} - \nabla_k R_{gj}) + (\nabla_j \lambda g_{ik} - \nabla_k \lambda g_{ij}) \tag{2.3}
\]

\[
\nabla_i \left[ (1 - 2\rho(n - 1)) R + |\nabla f|^2 - 2(n - 1) \lambda \right] = (2\rho R + 2\lambda) \nabla_i f \tag{2.4}
\]
Proof. For a gradient RB almost soliton we have
\[ R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij} + \rho R g_{ij} \] (2.5)
Taking trace of the above equation gives (2.1).

Taking the covariant derivative of (2.1) in an orthonormal frame gives
\[ (1 - n\rho)\nabla_i R + \nabla_i \nabla_j \nabla_j f = n \nabla_i \lambda \]
Commuting covariant derivatives and using the contracted second Bianchi identity give
\[ (1 - n\rho)\nabla_i R = -\nabla_j \nabla_j f + \nabla_i f + n \nabla_i \lambda \]
which proves (2.1).

For proving (2.3), we use (2.5) and commute covariant derivatives to get
\[ \nabla_j R_{ik} - \nabla_k R_{ij} = (\nabla_k \nabla_i \nabla_j f - \nabla_j \nabla_i \nabla_k f) + \rho(\nabla_j R g_{ik} - \nabla_k R g_{ij}) \]
\[ + (\nabla_j \lambda g_{ik} - \nabla_k \lambda g_{ij}) \]
\[ = (\nabla_k \nabla_j \nabla_i f - \nabla_j \nabla_k \nabla_i f) + \rho(\nabla_j R g_{ik} - \nabla_k R g_{ij}) \]
\[ + (\nabla_j \lambda g_{ik} - \nabla_k \lambda g_{ij}) \]
\[ = R_{jkl} \nabla_i f + \rho(\nabla_j R g_{ik} - \nabla_k R g_{ij}) + (\nabla_j \lambda g_{ik} - \nabla_k \lambda g_{ij}) \] (2.7)
Finally from (2.2) we get
\[ (1 - 2\rho(n - 1))\nabla_i R = 2\nabla_i f(-\nabla_i \nabla_i f + \lambda g_{ii} + \rho R g_{ii}) + 2(n - 1)\nabla_i \lambda \]
\[ = -2\nabla_i f\nabla_i f + 2\lambda \nabla_i f + 2\rho R \nabla_i f + 2(n - 1)\nabla_i \lambda \]
\[ = -\nabla_i |\nabla f|^2 + 2\lambda \nabla_i f + 2\rho R \nabla_i f + 2(n - 1)\nabla_i \lambda \]
so we get
\[ \nabla_i [(1 - 2\rho(n - 1)) R + |\nabla f|^2 - 2(n - 1)\lambda] = (2\rho R + 2\lambda) \nabla_i f \] (2.8)
which proves (2.4).

Remark 2.2. The analogous identities for gradient RB solitons \((M^n, g, \nabla f, \lambda, \rho)\) are
\[ (1 - n\rho) R + \Delta f = n \lambda \] (2.9)
\[ (1 - 2\rho(n - 1))\nabla_i R = 2 R_{ii} \nabla_i f \] (2.10)
\[ \nabla_j R_{ik} - \nabla_k R_{ij} = R_{jkl} \nabla_i f + \rho(\nabla_j R g_{ik} - \nabla_k R g_{ij}) \] (2.11)
\[ \nabla_i [(1 - 2\rho(n - 1) R + |\nabla f|^2 - 2\lambda f] = 2\rho R \nabla_i f \] (2.12)
The proofs of these identities are special cases of the previous result as \(\nabla \lambda = 0\).

We recall the following lemma from [PW09, Lemma 2.1]
Lemma 2.3. Let $X$ be a vector field on a Riemannian manifold $(M^n, g)$. Then
\[
\text{div}(\mathcal{L}_X g)(X) = \frac{1}{2} \Delta |X|^2 - |\nabla X|^2 + \text{Ric}(X, X) + \nabla X \text{ div } X \tag{2.13}
\]
When $X = \nabla f$ and $Z$ is any vector field then
\[
\text{div}(\mathcal{L}_{\nabla f} g)(Z) = 2 \text{Ric}(Z, \nabla f) + 2 \nabla Z \text{ div } \nabla f \tag{2.14}
\]
We use the preceding lemma to prove the following

Lemma 2.4. Let $(M^n, g, X, \lambda, \rho)$ be a RB almost soliton. Then
\[
\frac{(1 - n\rho)}{2} \Delta |X|^2 = (1 - n\rho) |\nabla X|^2 + (n\rho - 1) \text{Ric}(X, X) + n\rho \nabla X \text{ div } X + 2\rho(1 - n\rho) g(\nabla R, X) - (n(2\rho + 1) - 2) g(\nabla \lambda, X) \tag{2.15}
\]
and
\[
\frac{(1 - n\rho)}{2} (\Delta - \nabla X)|X|^2 = (1 - n\rho) |\nabla X|^2 + \lambda(n\rho - 1)|X|^2 + \rho(n\rho - 1) R |X|^2 + n\rho \nabla X \text{ div } X + 2\rho(1 - n\rho) g(\nabla R, X) - (n(2\rho + 1) - 2) g(\nabla \lambda, X) \tag{2.16}
\]

Proof. We first notice that (1.3) gives
\[
2 \text{div Ric} + \text{div}(\mathcal{L}_X g) = 2 \nabla \lambda + 2\rho \nabla R \tag{2.17}
\]
Taking the trace of (1.3) gives $(1 - n\rho) R + \text{div } X = n\lambda$ and thus
\[
(1 - n\rho) \nabla X R + \nabla X (\text{div } X) = n \nabla X \lambda \tag{2.18}
\]
So using (2.13), (2.17), (2.18) and the contracted second Bianchi identity, we get
\[
\nabla X (\text{div } X) = (n\rho - 1) \nabla X R + ng(\nabla \lambda, X) = 2(n\rho - 1) \text{div Ric}(X) + ng(\nabla \lambda, X) = -(n\rho - 1) \text{div}(\mathcal{L}_X g)(X) + 2\rho(n\rho - 1) g(\nabla R, X) + 2(n\rho - 1) g(\nabla \lambda, X) + ng(\nabla \lambda, X) = (1 - n\rho) \left( \frac{1}{2} \Delta |X|^2 - |\nabla X|^2 + \text{Ric}(X, X) + \nabla X \text{ div } X \right) + 2\rho(n\rho - 1) g(\nabla R, X) + (n(2\rho + 1) - 2) g(\nabla \lambda, X) = \frac{(1 - n\rho)}{2} \Delta |X|^2 - (1 - n\rho) |\nabla X|^2 + (1 - n\rho) \text{Ric}(X, X) + (1 - n\rho) \nabla X \text{ div } X + 2\rho(n\rho - 1) g(\nabla R, X) + (n(2\rho + 1) - 2) g(\nabla \lambda, X)
\]
which gives
\[
\frac{(1 - n\rho)}{2} \Delta |X|^2 = (1 - n\rho) |\nabla X|^2 + (n\rho - 1) \text{Ric}(X, X) + n\rho \nabla X \text{ div } X + 2\rho(1 - n\rho) g(\nabla R, X) - (n(2\rho + 1) - 2) g(\nabla \lambda, X) \tag{2.19}
\]
thus proving (2.15).

Using (1.3) to write $\text{Ric}(X, X) = -\frac{1}{2} (\mathcal{L}_X g)(X, X) + \lambda |X|^2 + \rho R |X|^2$ in (2.15) we get
\[ \frac{(1 - n\rho)}{2} \Delta |X|^2 = (1 - n\rho) |\nabla X|^2 + (n\rho - 1) \left( -\frac{1}{2} (\mathcal{L}_X g)(X, X) + \lambda |X|^2 + \rho R |X|^2 \right) \\
+ n\rho \nabla_X \operatorname{div} X + 2\rho(1 - n\rho) g(\nabla R, X) - (n(2\rho + 1) - 2) g(\nabla \lambda, X) \]

which gives
\[ \frac{(1 - n\rho)}{2} (\Delta - \nabla_X) |X|^2 = (1 - n\rho) |\nabla X|^2 + (n\rho - 1)|X|^2 + \rho(n\rho - 1)R|X|^2 \\
+ n\rho \nabla_X \operatorname{div} X + 2\rho(1 - n\rho) g(\nabla R, X) \\
- (n(2\rho + 1) - 2) g(\nabla \lambda, X) \] (2.20)

proving (2.16).

If we consider the diffusion operator \( \Delta_X = \Delta - \nabla_X \) then the previous lemma with \( X = \nabla f \) and \( \Delta_f = \Delta - \nabla \nabla f \) gives the following corollary

**Corollary 2.5.** For a gradient RB almost soliton \( (M, g, \nabla f, \lambda, \rho) \) we have
\[ \frac{(1 - n\rho)}{2} \Delta_f |\nabla f|^2 = (1 - n\rho) |\nabla^2 f|^2 + (n\rho - 1) |\nabla f|^2 + \rho(n\rho - 1)R |\nabla f|^2 \\
+ n\rho \nabla_{\nabla f} (\Delta f) + 2\rho(1 - n\rho) g(\nabla R, \nabla f) \\
- (n(2\rho + 1) - 2) g(\nabla \lambda, \nabla f) \] (2.21)

**Remark 2.6.** The analogs of (2.15) and (2.16) for a RB soliton \( (M^n, g, X, \lambda, \rho) \) are
\[ \frac{(1 - n\rho)}{2} \Delta |X|^2 = (1 - n\rho) |\nabla X|^2 + (n\rho - 1) \operatorname{Ric}(X, X) + n\rho \nabla_X \operatorname{div} X \\
+ 2\rho(1 - n\rho) g(\nabla R, X) \] (2.22)

and
\[ \frac{(1 - n\rho)}{2} (\Delta - \nabla_X) |X|^2 = (1 - n\rho) |\nabla X|^2 + (n\rho - 1)|X|^2 + \rho(n\rho - 1)R|X|^2 \\
+ n\rho \nabla_X \operatorname{div} X + 2\rho(1 - n\rho) g(\nabla R, X) \] (2.23)

The proofs are special cases of the proof of Lemma 2.4 with \( \nabla \lambda = 0 \).

### 3. Proofs of the Results

We start this section by proving the following lemma which will be used in the proof of Theorem 1.3 and Theorem 1.5.

**Lemma 3.1.** Let \( (M^n, g, X, \lambda, \rho) \) be a RB almost soliton with \( n \geq 3 \). If \( X \) is a nontrivial conformal vector field with \( \mathcal{L}_X g = 2\psi g \) then \( R \) and \( \lambda - \psi \) are constant.

**Proof.** The soliton equation is
\[ R_{ij} + \frac{1}{2} (\mathcal{L}_X g)_{ij} = \lambda g_{ij} + \rho g_{ij} \] (3.1)
where \( \lambda : M \to \mathbb{R} \) is a function. If \( X \) is a nontrivial conformal vector field then we have
\[ \mathcal{L}_X g = 2\psi g \] (3.2)
for some function $\psi : M \to \mathbb{R}$, $\psi \neq 0$. So (3.1) becomes

$$R_{ij} = (\lambda - \psi + \rho R)g_{ij}$$ (3.3)

Taking the divergence of (3.3) we get

$$\nabla_i R_{ij} = \nabla_i (\lambda - \psi + \rho R)g_{ij}$$

$$\implies \left( \frac{1}{2} - \rho \right) \nabla_j R = \nabla_j (\lambda - \psi)$$ (3.4)

On the other hand, tracing (3.3) and taking the covariant derivative we get

$$(1 - n\rho) \nabla_j R = n \nabla_j (\lambda - \rho)$$ (3.5)

So from (3.4) and (3.5) we get

$$(1 - n\rho) \nabla_j R = n \left( \frac{1}{2} - \rho \right) \nabla_j R$$ (3.6)

So if $M$ is connected then $R$ is a constant and hence $\lambda - \psi$ is a constant.

**Remark 3.2.** If $(M^n, g, X, \lambda, \rho)$ is a RB soliton with $n \geq 3$ and $X$ is a conformal vector field with $L_X g = 2\psi g$ for some function $\psi : M \to \mathbb{R}$ then the proof of Lemma 3.1 shows that $R$ and $\psi$ are constant as in this case $\nabla \lambda = 0$.

We prove Theorem 1.3 which we restate here

**Theorem 3.3.** Let $(M^n, g, X, \lambda, \rho)$ be a RB soliton with $n \geq 3$ and suppose that the vector field $X$ is a conformal vector field.

1. If $M$ is compact then $X$ is a Killing vector field and hence $(M^n, g, X, \lambda, \rho)$ is a trivial RB soliton.
2. If $M$ is non-compact, complete and a gradient RB soliton then either $X$ is a Killing vector field or $(M^n, g, X, \lambda, \rho)$ is isometric to the Euclidean space.

Proof. Suppose $X$ is a conformal vector field with potential $\psi : M \to \mathbb{R}$, i.e.,

$$L_X g = 2\psi g$$ (3.7)

then from Remark 3.2 we know that $R$ and $\psi$ are constant.

Taking trace of (3.7) we get

$$2 \text{div} X = 2n\psi$$

which upon integration over compact $M$ gives

$$0 = \int_M 2 \text{div} X d\nu = 2n \text{Vol}(M)\psi$$ (3.8)

i.e., $\psi = 0$. So $X$ is a Killing vector field and hence $(M^n, X, g, \lambda, \rho)$ is a trivial RB soliton.

If $M$ is noncompact and a gradient RB soliton with $X = \nabla f$, then $X$ being conformal implies

$$\nabla_i \nabla_j f = \psi g_{ij}$$

and by Remark 3.2, $\psi$ is constant. If $\psi = 0$ then $X$ is a Killing vector field and $M$ is a trivial RB soliton. If $\psi \neq 0$, then from [Tas65, Theorem 2], we conclude that $M^n$ is isometric to the Euclidean space.

Next we prove Corollary 1.4.
Proof. Since $M^n$ is compact, from Theorem 1.3 we know that $X$ is a Killing vector field and hence $\text{Ric} = (\lambda + \rho R)g$. So we can apply a classical theorem due to Lichnerowicz \[Lic58\] which states that if $\text{Ric} \geq k$ where $k > 0$ is a constant then the first eigenvalue of the Laplacian $\lambda_1$ satisfies $\lambda_1 \geq \frac{n}{n-1} k$. So we get

$$\lambda_1 \geq (\lambda + \rho R)\frac{n}{n-1}$$

Moreover, for the equality case we can apply Obata’s theorem \[Oba62\] to conclude that equality occurs in the above inequality if and only if $M^n$ is isometric to a sphere of constant curvature $(\lambda + \rho R)\frac{n}{n-1}$.

We now prove Theorem 1.5 which we restate here

**Theorem 3.4.** Let $(M^n, g, X, \lambda, \rho)$ be a compact RB almost soliton with $n \geq 3$. If $X$ is a nontrivial conformal vector field then $M^n$ is isometric to an Euclidean sphere.

Proof. Suppose $X$ is a nontrivial conformal vector field with potential function $\psi : M \rightarrow \mathbb{R}$, i.e.,

$$\mathcal{L}_X g = 2\psi g$$

with $\psi \neq 0$. Since $(M^n, g, X, \lambda, \rho)$ is a compact RB almost soliton with $n \geq 3$, Lemma 3.1 tells us that $R$ and $\lambda - \psi$ are constant. So from Lemma 2.3 in \[Yan70, pg.52\] we conclude that $R \neq 0$ or else $\psi$ would be 0. Taking the Lie derivative of (3.3) we get

$$\mathcal{L}_X \text{Ric} = \mathcal{L}_X (\lambda - \psi + \rho R)g$$

and since $(\lambda - \psi)$, $\rho$ and $R$ are all constant so we get

$$\mathcal{L}_X \text{Ric} = 2(\lambda - \psi + \rho R)\psi g$$

(3.9)

Now we can apply Theorem 4.2 of \[Yan70, pg. 54\] to conclude that $M$ is isometric to an Euclidean sphere. □

We proceed to the proof of Theorem 1.6.

Proof. We see from (2.15) of Lemma 2.4 that

$$\frac{1}{2}(1 - n\rho)\Delta |X|^2 = (1 - n\rho)|\nabla X|^2 + (n\rho - 1) \text{Ric}(X, X) + n\rho \nabla_X \text{div} X$$

$$+ 2\rho(1 - n\rho)g(\nabla R, X) - (n(2\rho + 1) - 2)g(\nabla \lambda, X)$$

Integrating above over compact $M$ we get

$$0 = \int_M [(1 - n\rho)|\nabla X|^2 + (n\rho - 1) \text{Ric}(X, X) + n\rho \nabla_X \text{div} X$$

$$+ 2\rho(1 - n\rho)g(\nabla R, X) - (n(2\rho + 1) - 2)g(\nabla \lambda, X)]dv$$

(3.10)

Since $\rho \neq \frac{1}{n}$, we get

$$\int_M |\nabla X|^2 dv = \int_M [\text{Ric}(X, X) + \frac{n\rho}{n\rho - 1} \nabla_X \text{div} X - 2\rho g(\nabla R, X)$$

$$- \frac{(n(2\rho + 1) - 2)}{n\rho - 1} g(\nabla \lambda, X)]dv$$

(3.11)

so if (1.4) holds then $|\nabla X|^2 = 0$ and hence $X$ is a Killing vector field. Thus $(M^n, g, X, \lambda, \rho)$ is trivial. □
The proof of Corollary 1.7 is a special case of the proof of Theorem 1.6 where we use (2.22) of Remark 2.6.

Next, we prove Theorem 1.9 which we restate here

**Theorem 3.5.** Let $(M^n, g, \nabla f, \lambda, \rho)$ be a compact gradient RB almost soliton. Then

\[
\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 dv = \frac{(n-2)}{2n} \int_M g(\nabla R, \nabla f) dv \tag{3.12}
\]

\[
\int_M |\text{Ric} - \frac{R}{n} g|^2 dv = \frac{(n-2)}{2n} \int_M g(\nabla R, \nabla f) dv \tag{3.13}
\]

**Proof.** For proving (3.12) we take the divergence of (2.4) of Proposition 2.1 to get

\[
(1 - 2\rho(n-1)) \Delta R + \Delta |\nabla f|^2 - 2(n-1)\Delta \lambda = 2 \rho g(\nabla R, \nabla f) + 2g(\nabla \lambda, \nabla f) + (2\rho R + 2\lambda) \Delta f \tag{3.14}
\]

By commuting covariant derivatives we have

\[
\nabla_i \nabla_i (g(\nabla_j f, \nabla_j f)) = 2 \nabla_i (g(\nabla_i \nabla_j f, \nabla_j f)) = 2g(\nabla_i \nabla_j \nabla_i f, \nabla_j f) + 2|\nabla^2 f|^2
\]

so (3.14) becomes

\[
(1 - 2\rho(n-1)) \Delta R + 2g(\nabla (\Delta f), \nabla f) + 2 \text{Ric}(\nabla f, \nabla f) + 2|\nabla^2 f|^2 - 2(n-1)\Delta \lambda = 2 \rho g(\nabla R, \nabla f) + 2g(\nabla \lambda, \nabla f) + (2\rho R + 2\lambda) \Delta f \tag{3.15}
\]

From (2.1) of Proposition 2.1 we know that \(\Delta f = n\lambda + (n\rho - 1)R\) which on differentiation and using (2.5) becomes

\[
0 = \nabla_i \Delta f + (1 - n\rho) \nabla_i R - n \nabla_i \lambda
\]

\[
= (1 - n\rho) \nabla_i R + \nabla_j \nabla_j \nabla_i f - R_{ij} \nabla_i f - n \nabla_i \lambda
\]

\[
= (1 - n\rho) \nabla_i R + \nabla_j (- R_{ij} + \lambda g_{ij} + \rho R_{ij}) - R_{ij} \nabla_i f - n \nabla_i \lambda
\]

\[
= \left( \frac{1}{2} - \rho(n-1) \right) \nabla_i R - R_{ij} \nabla_i f + (1 - n) \nabla_i \lambda
\]

and hence

\[
2 \text{Ric}(\nabla f, \nabla f) = (1 - 2\rho(n-1)) g(\nabla R, \nabla f) + 2(1 - n) g(\nabla \lambda, \nabla f) \tag{3.16}
\]

So using (3.16) and \(\Delta f = n\lambda + (n\rho - 1)R\), the left hand side of (3.15) becomes

\[
(1 - 2\rho(n-1)) \Delta R + 2|\nabla^2 f|^2 - 2(n-1)\Delta \lambda + 2g(\nabla \lambda, \nabla f) + (2\rho - 1) g(\nabla R, \nabla f)
\]

and hence (3.15) becomes

\[
(1 - 2\rho(n-1)) \Delta R + 2|\nabla^2 f|^2 - 2(n-1)\Delta \lambda = g(\nabla R, \nabla f) + (2\rho R + 2\lambda) \Delta f \tag{3.17}
\]

Since \(|\nabla^2 f - \frac{\Delta f}{n} g|^2 = |\nabla^2 f|^2 - \frac{(\Delta f)^2}{n}\), (3.17) becomes
\[(1 - 2p(n - 1))\Delta R + 2|\nabla^2 f - \frac{\Delta f}{n} g|^2 = g(\nabla R, \nabla f) + (2pR + 2\lambda)\Delta f - 2\frac{(\Delta f)^2}{n} + 2(n - 1)\Delta \lambda
\]
\[= g(\nabla R, \nabla f) + (2pR + 2\lambda)\Delta f - 2\frac{(\Delta f)^2}{n} + 2R\Delta f + 2(n - 1)\Delta \lambda
\]
\[(3.18)\]

Integrating (3.18) over compact \(M\) gives
\[\int_M 2|\nabla^2 f - \frac{\Delta f}{n} g|^2 dv = \int_M \left[g(\nabla R, \nabla f) + \frac{2R\Delta f}{n}\right] dv
\]
\[= \frac{(n - 2)}{n} \int_M g(\nabla R, \nabla f) dv \quad (3.19)\]

where we have used integration by parts in the second equality. This proves (3.12).

For proving (3.13) note that
\[\text{Ric} - \frac{R}{n} g = -\nabla^2 f + \lambda g + \rho R g - \frac{R}{n} g
\]
\[= -\nabla^2 f + (\lambda + \rho R - \frac{R}{n}) g
\]
\[= -\nabla^2 f + \frac{\Delta f}{n} g \quad (3.20)
\]

and then (3.13) follows from (3.12).

\[\square\]

**Remark 3.6.** Since a gradient RB soliton is a special case of a gradient RB almost soliton, the proof of Theorem 1.9, with \(\nabla \lambda = 0\), shows that the same integral formulas (3.12) and (3.13) hold for a compact gradient RB soliton too.

Finally, using Theorem 1.9 we prove Corollary 1.10.

**Proof.** Observe that any of the assumptions of Corollary 1.10 enable us to conclude that the right hand side of (3.13) is less than or equal to zero and hence \(\text{Ric} = \frac{\Delta f}{n} g\). So from (2.5) we see that
\[\nabla_i \nabla_j f = (\lambda + R(\rho - \frac{1}{n})) g
\]
and hence \(\nabla f\) is a nontrivial conformal vector field so from Theorem 1.5 we get that \(M^n\) is isometric to an Euclidean sphere. \(\square\)

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