Dimensional Regularization for the Standard Model by Rightmost Positioning of $\gamma_5$

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Abstract

We present here a treatment of the $\gamma_5$ matrix in dimensional regularization that is able to preserve gauge symmetry for diagrams up to two-loop order. The first part of our scheme is moving all $\gamma_5$ matrices to the rightmost position before we analytically continue the dimension. This renders all Feynman amplitudes corresponding to diagrams without fermion loops regulated and consistent with gauge invariance. Next we extend our scheme to amplitudes corresponding to diagrams with chiral fermion loops, on which the rightmost positions can only be found by cutting open the fermion loops.

In contrast to the $\gamma_5$ scheme by Breitenlohner and Maison, we show that, by choosing the cut point properly located outside the divergent self-energy or vertex correction sub-diagram, all one and two loop diagrams in the standard model can be regularized gauge invariantly and their renormalized amplitudes obtained via minimal subtractions do not require further finite counter-term renormalization.

1 Introduction

It is well known that $\gamma_5$ matrix is an intrinsically four dimensional object and no definition of $\gamma_5$ is available under dimensional regularization [1] such that the anti-commutation relationship

$$\gamma_5 \gamma_\mu + \gamma_\mu \gamma_5 = 0$$

(1)

is preserved for space-time dimension $n \neq 4$. This difficulty of defining $\gamma_5$ for $n \neq 4$ poses a difficulty of applying dimensional regularization and minimal subtraction to gauge field theories with chiral fermions [2, 3]. This is because

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we are no longer assured that a regularized gauge theory involving $\gamma_5$ still preserves the gauge symmetry of the original formal theory. In fact, it is known that the triangular Ward-Takahashi identity \[4\] is prone to be broken for gauge theories involving $\gamma_5$. This anomaly \[5\] is also known to be confined to the one-loop order \[6, 7, 8, 9\].

In the dimensional regularization scheme of Breitenlohner and Maison \[10, 11\], the $\gamma_5$ matrix is defined in a way such that it anti-commutates with $\gamma^\mu$ for $\mu \in \{0, 1, 2, 3\}$ but commutes with $\gamma^\mu$ when $\mu$ is continued beyond the first four dimensions. With this scheme for $\gamma_5$, Breitenlohner and Maison were able to show that dimensional regularization and minimal-subtraction renormalization can be implemented consistently \[12, 13\] for theories involving $\gamma_5$. But there is a major deficiency of this BM scheme: it is not a gauge invariant scheme \[14\]. Consequently, amplitudes obtained therewith do not satisfy Ward-Takahashi or BRST \[15\] identities, and finite counter-term renormalizations are required to restore the validities of these identities \[16, 17, 18, 19, 20, 21\]. This in fact renders the application of dimensional regularization for chiral gauge theories rather complicated in practical calculation.

In this paper, we will present a scheme \[22\] that maximizes the usefulness of the anti-commutation relationship \[1\]. In an open fermion line, we move all the $\gamma_5$ matrices to the rightmost position among the Feynman factors of this line before continuing to $n \neq 4$. We will show that the amplitudes corresponding to diagrams without fermion loops obtained with this prescription are consistent with gauge symmetry. For diagrams with one or more fermion loops, we choose a proper position on each fermion loop, to be defined below, to fill the role of the rightmost position. We shall demonstrate that, with this $\gamma_5$ scheme, it is possible to regularize a chiral gauge theory without breaking Ward-Takahashi or BRST identities up to two-loop order provided the theory is free of the one-loop anomaly. The standard model is such a theory. Therefore, it can be dimensionally regulated and renormalized according to our $\gamma_5$ scheme up to two-loop order without resorting to additional finite counter term renormalization.

In dimensional regularization, it is known that the condition
\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \tag{2}
\]
can be analytically continued and consistently utilized in $n$ dimensional space.
In a four-dimensional space, $\gamma_5$ is defined as

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

(3)

This definition of $\gamma_5$ satisfies $\gamma_5^2 = 1$ and the anti-commutation relationship $\gamma_5\gamma_\mu + \gamma_\mu\gamma_5 = 0$ for $\mu \in \{0, 1, 2, 3\}$. Let us introduce the notation $p^\mu$ for the component of $p^\mu$ vector in the first 4 dimensions and the notation $p^\mu_\Delta$ for the component in the remaining dimensions. i.e.,

$$p^\mu = p^\mu + p^\mu_\Delta,$$

with

$$p^\mu_\Delta = 0 \text{ if } \mu \in \{0, 1, 2, 3\}, \quad p^\mu = 0 \text{ if } \mu \notin \{0, 1, 2, 3\}.$$

Likewise, the Dirac matrix $\gamma^\mu$ is decomposed as

$$\gamma^\mu = \gamma^\mu + \gamma^\mu_\Delta$$

with $\gamma^\mu_\Delta = 0$ when $\mu \in \{0, 1, 2, 3\}$ and $\gamma^\mu = 0$ when $\mu \notin \{0, 1, 2, 3\}$.

In our scheme, we will insist on maintaining the definition (3) for $\gamma_5$ even when $n$ departs from 4. We therefore have

$$\gamma_5\gamma^\mu + \gamma^\mu\gamma_5 = 2\gamma^\mu_\Delta\gamma_5,$$

(4)

which means that $\gamma_5$ does not anti-commute with $\gamma^\mu$ when $\mu$ is not in $\{0, 1, 2, 3\}$. In a four-dimensional space, any matrix product

$$\hat{M} = \gamma_{\omega_1}\gamma_{\omega_2}\ldots\gamma_{\omega_n} \text{ with } \omega_i \in \{0, 1, 2, 3, 5\}$$

may be reduced, by anti-commuting $\gamma_5$ to the rightmost position, to either the form of $\pm\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_m}$ with $\mu_i \in \{0, 1, 2, 3\}$ if $\hat{M}$ contains even $\gamma_5$ factors, or the form $\pm\gamma_{\nu_1}\gamma_{\nu_2}\ldots\gamma_{\nu_p}\gamma_5$ with $\nu_i \in \{0, 1, 2, 3\}$ if the $\gamma_5$ count is odd. As the $\gamma_\mu$ matrix is analytically continued and consistently defined when the component $\mu$ runs out of the range $\{0, 1, 2, 3\}$ under the dimensional regularization scheme, the matrix product $\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_m}$ is also unambiguously defined under dimensional regularization. We may also analytically continue the product $\gamma_{\nu_1}\gamma_{\nu_2}\ldots\gamma_{\nu_p}\gamma_5$ with one $\gamma_5$ on the right by defining it to be the product of the analytically continued $\gamma_{\nu_1}\gamma_{\nu_2}\ldots\gamma_{\nu_p}$ and the $\gamma_5$ defined in (3). Similarly, we may analytically continue the product $\gamma_{\nu_1}\gamma_{\nu_2}\ldots\gamma_{\nu_i}\gamma_5\gamma_{\nu_{i+1}}\ldots\gamma_{\nu_p}$ by defining it to be the analytically continued $\gamma_{\nu_1}\gamma_{\nu_2}\ldots\gamma_{\nu_i}$ times $\gamma_5$ then times the analytically continued $\gamma_{\nu_{i+1}}\ldots\gamma_{\nu_p}$. 

3
In a \( n = 4 \) dimensional space, a matrix product involving one \( \gamma_5 \) has more than one equivalent expressions corresponding to different positionings of the \( \gamma_5 \) matrix such as

\[
\gamma_{\nu_1} \gamma_{\nu_2} \cdots \gamma_{\nu_p} \gamma_5 = (-1)^{p-i} \gamma_{\nu_1} \gamma_{\nu_2} \cdots \gamma_{\nu_{i+1}} \gamma_{\nu_i} \gamma_{\nu_{i+1}} \cdots \gamma_{\nu_p}
\]

for \( i = 0, 1, \ldots, p - 1 \). When \( n \neq 4 \), the above equation does not always hold because the anti-commutator (1) becomes commutator (4) and does not vanish. In particular, we have

\[
\gamma_{\nu_1} \gamma_{\nu_2} \cdots \gamma_{\nu_p} \gamma_5 = -\gamma_{\nu_1} \gamma_{\nu_2} \cdots \gamma_5 \gamma_{\nu_p} + 2\gamma_{\nu_1} \gamma_{\nu_2} \cdots \gamma_{\nu_{p-1}} \gamma_5 \Delta_{\nu_p}
\]

Thus a matrix product that contains an odd number of \( \gamma_5 \) is not unambiguously continued from its value at \( n = 4 \).

Before analytic continuation is made, a \( \gamma_5 \)-odd matrix product may always be reduced to a matrix product with only one \( \gamma_5 \) factor. To analytically continue such a matrix product, we need an extra information which is specifying the location of the \( \gamma_5 \) factor within the matrix product. We adopt the default continuation by anti-commuting the \( \gamma_5 \) matrix to the rightmost position before making the analytical continuation.

## 2 Proper \( \gamma_5 \) Position and Cut Point

While the scheme of rightmost ordering defines uniquely the amplitude corresponding to a diagram without chiral fermion loops, it is not the case with a diagram with one or more chiral fermion loops. This because there is no starting point in a chiral fermion loop, and the rightmost position in the product of the corresponding gamma matrices is not defined. We may choose a point on a fermion loop as the starting point and move all the \( \gamma_5 \) matrices to this point. In general, continuations from different choices of the starting point give different values for the Feynman trace associated with the fermion loop when \( n \neq 4 \).

We note that a fermion loop opens up and becomes a fermion line if we make a cut at some point on the loop. We shall always choose as the cut point either the beginning point or the endpoint of an internal fermion line on the loop. An internal fermion line begins from a vertex and ends at another vertex. When the cut point is chosen to be the endpoint of an internal fermion line, the vertex factor will be assigned to appear as the beginning factor and
stands at the right end of the matrix product for the entire open fermion line. And when the cut point is chosen to be the beginning point of an internal fermion line that emits from a vertex, the matrix factor corresponding to that vertex will be assigned to be the terminating factor and stands at the left end of the matrix product for the entire open fermion line. With the cut point on a fermion loop chosen and with the fermion loop turned into a fermion line, we may apply the rule of rightmost ordering for $\gamma_5$.

Although we have multiple continuations for a matrix product or the trace of a matrix product, they differ with one another either by terms that are $O(n - 4)$ or by terms containing at least a factor of $\gamma_\Delta$. In the tree order and in the limit $n \rightarrow 4$, they are all restored to the same result because $\gamma_\Delta$ will disappear when $n \rightarrow 4$. For higher loop orders, $\gamma_\Delta$ contribution may not be ignored in the limit $n \rightarrow 4$. This is because the factor $\gamma_\mu g_{\mu\nu} = (n - 4)$ multiplied by a simple pole factor $\frac{1}{(n-4)}$ or a higher-order pole term becomes finite or even infinite in the limit $n \rightarrow 4$. Thus $\gamma_5$ located within a divergent diagram or sub-diagram in general yields different regulated amplitude from that given by rightmost $\gamma_5$. To avoid such differences as much as possible in our $\gamma_5$ scheme, we will refrain from positioning $\gamma_5$ inside a divergent sub-diagram, of which the pole terms will be removed after renormalization.

A position for $\gamma_5$ will be called proper if it is not located within a divergent 1PI sub-diagram such as a self-energy insertion or a vertex correction. Likewise, for a fermion loop, a cut and the corresponding cut point will be called proper if the cut is not made within a divergent self-energy insertion or vertex correction sub-diagram.

To form a Levi-Civita tensor, we need at least 4 different indices in the first 4 dimensions. A diagram with 2 external non-fermion lines has at most two indices provided by the polarizations of the external lines. The other two indices of the Levi-Civita tensor have to be contracted with two different external momenta to give a non-zero value. Since there is only one external momentum available for a 2-point function, the Levi-Civita tensor term is absent in such a function. This is to say that a term in the 2-point function involving the trace of a $\gamma_5$-odd matrix product for a fermion loop vanishes. As to the trace of a $\gamma_5$-even matrix product, it is void of $\gamma_5$ and therefore uniquely defined under dimensional regularization. Thus the amplitude for a 2-point function with non-fermion external fields and with only one fermion loop is independent of the cut point chosen on the fermion loop. Such a cut point, which may reside in a divergent sub-diagram, will be called proper in any case.
The minimal-subtraction prescription, which subtracts out the pole terms for all possible forests of non-overlapping sub-diagrams \[23, 24, 11\], is a convenient renormalization procedure. For a superficially convergent diagram with an open fermion line or with a closed fermion loop, positioning $\gamma_5$ at all proper locations gives the same amplitude as the default continuation with rightmost $\gamma_5$ in the limit $n \to 4$ provided that all the divergent sub-diagrams have been renormalized. We therefore have

**Theorem 1** The renormalized amplitudes for a superficially convergent 1PI diagram obtained with different proper $\gamma_5$ locations approach the same $n \to 4$ limit.

More care is needed to treat a superficially divergent diagram. This is because, even with all proper sub-diagrams renormalized, pole terms may still arise from overall integrations. If we expand the overall amplitude as a Taylor series with respect to the external momenta, the pole terms occur only in the first few terms in the series because the degree of divergence from power counting for each term in the Taylor series is progressively decreased by the power of the external momenta. For example, while the vertex correction function is logarithmically divergent, only the first term in this Taylor series, henceforth called the $T_0$ term, may have pole terms on condition that all proper sub-diagrams have been renormalized. The overall subtraction of pole terms does not remove the finite difference stemming from multiplying these overall pole terms to the $\gamma_\Delta$ or $O(n-4)$ difference even if we position $\gamma_5$ at two different proper locations.

If we rely on the Ward identities to determine these ambiguous finite terms, as is done in the BM scheme, we can choose whichever position or cut point for $\gamma_5$ as long as it is a proper one. The renormalized amplitudes so calculated are consistent with those obtained from the BM scheme. As we have mentioned, this method of finite counter-term renormalization is rather complicated and difficult to implement in practical calculation. Fortunately, we will show there is a cut-point prescription which is capable of regularizing amplitudes gauge invariantly under the rightmost $\gamma_5$ scheme for diagrams up to 2-loop order, provided that the 1-loop triangular anomaly is absent. This result is significant in that it greatly reduces the complexity of calculations for amplitudes in the standard model. Furthermore, by moving all the $\gamma_5$ to the rightmost position, the matrix product in front of $\gamma_5$ is a fully $n$-dimensional covariant expression and, in contrast to the non-covariant treatment of the
\( \gamma \) matrix indices in the BM scheme, we are spared the chore of splitting the \( n \) dimensional space into 4 and \((n - 4)\) spaces in practical calculations.

## 3 Ward Identity and rightmost \( \gamma_5 \) ordering

A Ward-Takahashi identity involving divergent amplitudes is not meaningful unless these amplitudes have been regularized. Take, for example, the formal identity \( k_\mu \Pi^{(1)}_{\mu\nu} = 0 \) for the one-loop photon self-energy \( \Pi^{(1)}_{\mu\nu} \) in QED. The function \( \Pi^{(1)}_{\mu\nu} \) may be formally written as

\[
\Pi^{(1)}_{\mu\nu} = e^2 \int \frac{d^4 \ell}{(2\pi)^4} Tr \left( \frac{1}{\ell + k - m} \gamma^\mu \frac{1}{\ell + m} \gamma^\nu \right)
\]

where \( k \) is the external momentum. The identity

\[
\frac{1}{\ell + k - m} \frac{k}{\ell - m} = \frac{1}{\ell - m} - \frac{1}{\ell + k - m}
\]

allows us to express \( k_\mu \Pi^{(1)}_{\mu\nu} \) as

\[
k_\mu \Pi^{(1)}_{\mu\nu} = e^2 \int \frac{d^4 \ell}{(2\pi)^4} Tr \left( \frac{1}{\ell - m} \gamma^\nu - \frac{1}{\ell + k - m} \gamma^\nu \right).
\]

The above integral is divergent and hence meaningless. Therefore, shifting loop variables at \( n = 4 \) is not always a legitimate operation, and the identity \( k_\mu \Pi^{(1)}_{\mu\nu} = 0 \) is merely formal.

Dimensional regularization has the advantage of giving the amplitude \( \Pi^{(1)}_{\mu\nu} \) a well-defined expression in which loop momentum shifting is allowed. As a result, the difference of two terms related by a shift of loop momentum variable is equal to zero. Thus the dimensionally regularized amplitude \( \Pi^{(1)}_{\mu\nu} \) satisfies \( k_\mu \Pi^{(1)}_{\mu\nu} = 0 \). But it works only when \( \Pi^{(1)}_{\mu\nu} \) involves no \( \gamma_5 \). For diagrams with chiral fermion lines, the \( \gamma_5 \) difficulty is not resolved by conventional dimensional regularization and as a consequence, Ward identities are not always obeyed. We shall show why this is so and that this \( \gamma_5 \) difficulty is solved by the additional adoption of rightmost \( \gamma_5 \) ordering before analytical continuation of dimension. This is because, as we will explain below, keeping the matrices in the rightmost \( \gamma_5 \) order helps the preservation of gauge invariance.
For a gauge theory involving $\gamma_5$, there is a basic identity similar to (5) for verifying Ward identities:

$$\frac{1}{\ell + k - m} (k - 2m) \gamma_5 \frac{1}{\ell - m} = \gamma_5 \frac{1}{\ell - m} + \frac{1}{\ell + k - m} \gamma_5 \quad (6)$$

The above identity valid at $n = 4$ is derived by decomposing the vertex factor $(k - 2m) \gamma_5$ into $(\ell + k - m) \gamma_5$ and $\gamma_5 (\ell - m)$ that annihilate respectively the propagators of the outgoing fermion with momentum $\ell + k$ and the incoming fermion with momentum $\ell$. Positioning $\gamma_5$ at the rightmost site, the above identity at $n = 4$ becomes

$$\frac{1}{\ell + k - m} (k - 2m) \frac{1}{\ell - m} \gamma_5 = \left( \frac{1}{\ell - m} + \frac{1}{\ell + k - m} \right) \gamma_5. \quad (7)$$

If we disregard the rightmost $\gamma_5$ on both sides of the above identity, we obtain another identity that is valid at $n = 4$. This new identity, which is void of $\gamma_5$, may be analytically continued to hold when $n \neq 4$. We then multiply $\gamma_5$ on the right to every analytically continued term of this $\gamma_5$-free identity to yield the analytic continuation of the identity (6).

As a side remark, we note that when we go to the dimension of $n \neq 4$, (6) in the form presented above is not valid. This is because $\gamma_5$ does not always anti-commute with $\gamma^\mu$ if $n \neq 4$. Instead, the identity needs to be modified by including an additional vertex factor $2 \ell_\Delta \gamma_5$, as shown below, if it is to hold for $n \neq 4$.

$$\frac{1}{\ell + k - m} (k + 2 \ell_\Delta - 2m) \gamma_5 \frac{1}{\ell - m} = \gamma_5 \frac{1}{\ell - m} + \frac{1}{\ell + k - m} \gamma_5$$

Adopting the rightmost $\gamma_5$ ordering avoids this difficulty, as the validity of the identity in the form of rightmost $\gamma_5$ ordering no longer depends on $\gamma_5$ anti-commuting with the $\gamma$ matrices.

For an amplitude corresponding to a diagram involving no fermion loops, we shall move all $\gamma_5$ matrices to the rightmost position before we continue analytically the dimension $n$. Subsequent application of dimensional regularization gives us regulated amplitudes satisfying the Ward identities.

The same conclusion cannot be drawn if $\gamma_5$ appears in a fermion loop. For example, the following identity involving traces of matrix products is valid at $n = 4$.

$$Tr \left( \gamma^\mu \gamma^\nu \hat{M} \gamma_5 \right) - Tr \left( \gamma^\mu \hat{M} \gamma^\nu \gamma_5 \right) = 2g^{\mu\nu} Tr \left( \hat{M} \gamma_5 \right) \quad (8)$$
where \( \hat{M} \) is a matrix product of \( \gamma \) matrices and is free of \( \gamma_5 \). Although all \( \gamma_5 \) in the above identity appear rightmost positioned, this identity does not always hold when \( n \neq 4 \). Specifically, condition (2) ensures that the following matrix equation continues to hold when \( n \neq 4 \).

\[
\gamma^\mu \gamma^\nu \hat{M} \gamma_5 + \gamma^\nu \gamma^\mu \hat{M} \gamma_5 = 2g^{\mu \nu} \hat{M} \gamma_5
\]  

(9)

The identity derived by taking the trace of the above equation thus also holds for \( n \neq 4 \). But the second term in (8) no longer remains equal to the trace of the second term in (9) when the index \( \nu \) is continued beyond the first 4 dimensions. This is because

\[
-\text{Tr} \left( \gamma^\mu \hat{M} \gamma^\nu \gamma_5 \right) = \text{Tr} \left( \gamma^\nu \gamma^\mu \hat{M} \gamma_5 \right) - 2\text{Tr} \left( \gamma^\mu \hat{M} \gamma^\nu \Delta \gamma_5 \right)
\]

Thus (8) does not hold when \( n \neq 4 \), and similarly for other identities.

On the other hand, if we are able to find a matrix equation, such as (9), with every \( \gamma_5 \) rightmost positioned, then the equation corresponding to the analytic continuation of the trace of this matrix equation also holds. A Ward identity involving \( \gamma_5 \) in fermion loops may be regularized and continued for \( n \neq 4 \) if the un-traced matrix equation for the identity can be established before taking the trace. This is how we will proceed to construct dimensionally regularized amplitudes to satisfy the Ward identities when chiral fermion loops are present.

An identity relating the traces of matrix products without \( \gamma_5 \) at \( n = 4 \) can always be analytically continued to hold when \( n \neq 4 \). Therefore, the portion of an amplitude in which the count of \( \gamma_5 \) on every loop is even has no \( \gamma_5 \) difficulty [25]. But to calculate amplitudes with an odd count of \( \gamma_5 \), we need an additional prescription. This is because, as we have mentioned, the rightmost position on a fermion loop is not defined a-priori. Divergent diagrams with fermion loops are the only type of diagrams that may be ambiguous with respect to the \( \gamma_5 \) positioning. Not incidentally, they are also the diagrams that may be plagued by anomaly problem. We will show that such diagrams up to 2-loop order can also be handled by our \( \gamma_5 \) scheme. Before proceeding to the prescription for the target theory of the standard model, we will first treat the simpler theory of chiral Abelian-Higgs gauge theory defined in the following section.
4 Abelian-Higgs Gauge Theory with Chiral Fermion

The Lagrangian for the Abelian-Higgs gauge theory \[26, 15\] with chiral fermion is

\[
L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^\mu \phi)^\dagger (D_\mu \phi) - \frac{1}{2} \lambda g^2 \left( \phi^\dagger \phi - \frac{1}{2} v^2 \right)^2 + \bar{\psi}_L (i \not\!D) \psi_L + \bar{\psi}_R (i \not\!D) \psi_R - \sqrt{2} f \left( \bar{\psi}_L \phi \psi_R + \bar{\psi}_R \phi^\dagger \psi_L \right),
\]

where

\[
F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu,
\]

\[
D_\mu \phi \equiv (\partial_\mu + igA_\mu) \phi,
\]

\[
\psi_L = L\psi, \quad \psi_R = R\psi
\]

with the chiral projection operators \(L\) and \(R\) defined as

\[
L = \frac{1}{2} (1 - \gamma_5), \quad R = \frac{1}{2} (1 + \gamma_5).
\]

We define two Hermitian fields \(H\) and \(\phi_2\) for the real and imaginary parts of the complex scalar field by

\[
\phi = \frac{H + i\phi_2 + v}{\sqrt{2}}.
\]

We also introduce two mass parameters \(M\) and \(m\) defined by

\[
M = g v, \quad m = f v
\]

Both \(M\) and \(m\) will be regarded as zero order quantities in perturbation. To quantize this theory, we add to the Lagrangian \(L\) gauge fixing terms in the \(\alpha\) gauge as well as the associated ghost terms \[27\]. The sum will be called the effective Lagrangian \(L_{\text{eff}}\), and is invariant under the following BRS variations:

\[
\delta A_\mu = \partial_\mu \xi,
\]

\[
\delta \phi_2 = -M \xi - g\xi H,
\]

\[
\delta H = g\xi \phi_2,
\]

\[
\delta \bar{\psi}_L = -ig\xi \bar{\psi}_L, \delta \bar{\psi}_R = 0,
\]

\[
\delta \eta = -\frac{i}{\alpha} (\partial^\mu A_\mu), \delta \xi = 0.
\]
where $\xi$ is the ghost field and $\eta$ is the anti-ghost field. The gauge fixing term is

$$L_{gf} = -\frac{1}{2\alpha} (\partial_{\mu} A^\mu)^2$$

and the ghost term is

$$L_{\text{ghost}} = i\eta \delta (\partial_{\mu} A^\mu) = i\eta (\partial_{\mu} \partial^{\mu}) \xi.$$  

The BRS invariant effective Lagrangian is

$$L_{\text{eff}} = L + L_{gf} + L_{\text{ghost}}$$

### 4.1 Charge Conjugation Transformation

If we disregard terms involving fermion fields, the effective Lagrangian is invariant under the following charge conjugation transformation:

$$H \rightarrow H$$

$$\phi_2 \rightarrow -\phi_2, A^\mu \rightarrow -A^\mu, \eta \rightarrow -\eta, \xi \rightarrow -\xi.$$  

Fields that have odd (even) charge parity shorthanded as $C$-parity under this transformation are classified as $C$-odd ($C$-even) fields. For non-fermion fields, $H$ is $C$-even and non-$H$ fields are $C$-odd. We define the $C$-parity of a Feynman diagram to be the product of the $C$-parities of its non-fermion external lines. A vertex without fermion lines attached is always $C$-even, so is a fermionless Feynman diagram. It is therefore impossible to construct a $C$-odd diagram without including fermion lines or loops. For a theory that does not involve $\gamma_5$, such as QED, the charge conjugation transformation is a symmetry of its Lagrangian. A consequence of this symmetry is the Furry theorem which states that any amplitude for an odd number of external vector fields such as the $\text{AAA}$ amplitude vanishes in QED.

In four dimensional space, the charge conjugation transformation $\psi \rightarrow C\bar{\psi}^T$ for fermion fields is effected by the matrix

$$C = i\gamma^2\gamma^0$$

that satisfies

$$C\gamma^\mu C^{-1} = - (\gamma^\mu)^T.$$  

The above identity is based on the property that $\gamma^0$ and $\gamma^2$ are symmetric matrices while $\gamma^1$ and $\gamma^3$ are antisymmetric in four dimensional space. It is
not guaranteed that this property specific to \( n = 4 \) may be dimensionally continued such that (19) holds when that \( n \neq 4 \).

We shall not assume the validity of (19) when \( n \neq 4 \) and define instead the charge conjugation for a matrix product of \( N \gamma \) matrices \( \hat{M} = \gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_N} \) as \( \hat{M}^C = (-\gamma^N) \cdots (-\gamma^{\mu_2}) (-\gamma^{\mu_1}) \) which is the product of the negative of these \( N \gamma \) matrices in reversed order. When \( n = 4 \), we may make use of (19) to verify straightforwardly that the trace of \( \hat{M} \) is the same as that of \( \hat{M}^C \) or

\[
\text{Tr} (\gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_N}) = \text{Tr} ((-\gamma^N) \cdots (-\gamma^{\mu_2}) (-\gamma^{\mu_1})) \tag{20}
\]

Since both sides in (20) consist of terms that are product of \( g^{\mu_1,\mu_2} \) metric tensors, the polarizations \( \mu_1, \mu_2, \ldots, \mu_N \) may be dimensionally continued beyond the first 4 dimensions so that (20) is also valid when \( n \neq 4 \). The validity of

\[
\text{Tr} \left( \gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_N} \gamma^0 \gamma^1 \gamma^2 \gamma^3 \right) = \text{Tr} \left( \gamma^3 \gamma^2 \gamma^1 \gamma^0 (-\gamma^N) \cdots (-\gamma^{\mu_2}) (-\gamma^{\mu_1}) \right)
\]

also yields

\[
\text{Tr} (\gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_N} \gamma_5) = \text{Tr} (\gamma_5 (-\gamma^N) \cdots (-\gamma^{\mu_2}) (-\gamma^{\mu_1})) \tag{21}
\]

where \( \mu_1, \mu_2, \ldots, \mu_N \) are allowed to be polarizations in arbitrary \( n \) dimensional space.

We will use the notation \( \hat{M}_{DR} \) with the sub-index \( DR \) to indicate that \( \hat{M}_{DR} \) is the matrix product obtained from \( \hat{M} \) by anti-commuting all the \( \gamma_5 \) matrices with \( \gamma \) matrices to the right and then continuing to \( n \neq 4 \). Conditions (20) and (21) in the above may be summarized as

\[
\text{Tr} \left( \hat{M}_{DR} \right) = \text{Tr} \left( \left( \hat{M}_{DR} \right)^C \right) \tag{22}
\]

The accumulated sign change resulting from moving a \( \gamma_5 \) in \( \hat{M} \) to the right-most position is equal to that from moving the corresponding \( \gamma_5 \) in \( \hat{M}^C \) to the leftmost position. We thus have

\[
\left( \hat{M}_{DR} \right)^C = \left( \hat{M}^C \right)_{DL} \tag{23}
\]

where the subscript \( DL \) means that the analytical continuation to \( n \neq 4 \) starts from the expression obtained after anti-commuting all the \( \gamma_5 \) factors to the leftmost position. If the count of \( \gamma^\mu \) matrices with \( \mu \in \{0, 1, 2, 3\} \) in a
matrix product is odd, the trace of the matrix product is zero and so is its continuation from any form. Hence, the $\gamma_5$ factor at the leftmost position of $(\hat{M}^C)_{DL}$ in a trace may be moved to the rightmost position to yield

$$Tr\left((\hat{M}^C)_{DL}\right) = Tr\left((\hat{M}^C)_{DR}\right) \quad (24)$$

Combining (22), (23) and (24) in the above, we get

$$Tr\left(\hat{M}_{DR}\right) = Tr\left((\hat{M}^C)_{DR}\right) \quad (25)$$

Let $G$ be a Feynman diagram with a fermion loop that has been cut open at the point $P$. The conjugate diagram $G^C$ is defined to be the diagram obtained by reversing the direction of the fermion loop in $G$. The point $P$ remains to be the cut point of $G^C$. If the cut point $P$ on $G$ is the endpoint of a certain fermion line on the loop, it becomes the beginning point of the reversed fermion line in $G^C$, and vice versa.

The identity (25) may be utilized to show that dimensionally regularized amplitudes for $G$ and $G^C$ are related. To be more specific, let $F$ be a fermion loop attached by fields in the sequence $\varphi_1, \varphi_2, ... \varphi_n$ with inward momenta $k_1, k_2, ... k_n$ and the cut point is chosen to be the endpoint of the internal fermion line flowing into the vertex of $\varphi_1$. The Feynman integrand $I(F)$ for $F$ may be written as

$$I(F) = Tr\left(\frac{i}{\varphi-m} \varpi(\varphi_n) \frac{i}{\varphi+\varphi_1+b_{2n-1-m}} \varpi(\varphi_{n-1}) \times \frac{i}{\varphi+b_{2n-2-m}} \varpi(\varphi_2) \frac{i}{\varphi+b_{n-1-m}} \varpi(\varphi_1)\right)_{DR} \quad (26)$$

where $\ell$ is the loop momentum variable and the vertex factors are

$$\varpi(A^\mu) = -igR\gamma^\mu L, \varpi(\phi_2) = f\gamma_5 \text{ and } \varpi(H) = -if.$$ 

According to the identity (25), performing the charge conjugation operation on the matrix product inside the trace of (26) leaves the value of $I(F)$ unchanged. Thus,

$$I(F) = Tr\left(\frac{i}{\varphi-m} \tilde{\varpi}(\varphi_1) \frac{i}{-(\varphi+\varphi_1)+b_{1-m}} \tilde{\varpi}(\varphi_2) \frac{i}{-(\varphi+\varphi_1+b_2)-m} \times \frac{i}{-(\varphi+b_2+b_{2n-1-m})-m} \tilde{\varpi}(\varphi_{n-1}) \frac{i}{-(\varphi+b_{n-1}-m)} \tilde{\varpi}(\varphi_n) \frac{i}{\varphi-m}\right)_{DR}$$

where

$$\tilde{\varpi}(A^\mu) = igL\gamma^\mu R, \tilde{\varpi}(\phi_2) = f\gamma_5 \text{ and } \tilde{\varpi}(H) = -if.$$
We are allowed to make the transformation $\ell \to -\ell$ in carrying out the $\int d^n \ell$ loop integration and arrive at

$$\int d^n \ell I (F) = \int d^n \ell Tr \left( \frac{\tilde{\varphi} (\varphi_1) \varphi_{-k_1-m} \tilde{\varphi} (\varphi_2) \varphi_{-k_1-k_2-m} \cdots \times \tilde{\varphi} (\varphi_{n-1}) \varphi_{-k_1-k_2-k_3-m} \tilde{\varphi} (\varphi_n) \varphi_{-k_1-k_2-k_3-m} \cdots}{\varphi_{-k_1-k_2-k_3-m} \tilde{\varphi} (\varphi_{n-1}) \varphi_{-k_1-k_2-k_3-m} \tilde{\varphi} (\varphi_n) \varphi_{-k_1-k_2-k_3-m} \cdots} \right)_{DR}$$

(27)

On the other hand, the conjugate diagram $F^C$ is the fermion loop with the external fields attached on the loop in the order of $\varphi_n, \varphi_{n-1}, \ldots, \varphi_1$, and with the cut point being the beginning point of the fermion line that leaves the vertex of $\varphi_1$. The Feynman integrand for $F^C$ may be written as

$$I (F^C) = Tr \left( \frac{\varphi (\varphi_1) \varphi_{-k_1-m} \varphi (\varphi_2) \varphi_{-k_1-k_2-m} \cdots \times \varphi (\varphi_{n-1}) \varphi_{-k_1-k_2-k_3-m} \varphi (\varphi_n) \varphi_{-k_1-k_2-k_3-m} \cdots}{\varphi_{-k_1-k_2-k_3-m} \varphi (\varphi_{n-1}) \varphi_{-k_1-k_2-k_3-m} \varphi (\varphi_n) \varphi_{-k_1-k_2-k_3-m} \cdots} \right)_{DR}$$

(28)

Let us observe that

$$\tilde{\varphi} (A^\mu) = -\varphi (A^\mu) \mid_{\gamma_5 \to -\gamma_5},$$

$$\tilde{\varphi} (\phi_2) = -\varphi (\phi_2) \mid_{\gamma_5 \to -\gamma_5},$$

$$\tilde{\varphi} (H) = \varphi (H) \mid_{\gamma_5 \to -\gamma_5}.$$

These relationships demonstrate that if we insert an additional negative sign in front of every $\gamma_5$, the vertex factors $\tilde{\varphi} (A^\mu)$, $\tilde{\varphi} (\phi_2)$ and $\tilde{\varphi} (H)$ become $-\varphi (A^\mu)$, $-\varphi (\phi_2)$ and $\varphi (H)$ respectively. Note also that the integrand in (27) becomes the integrand $I (F^C)$ in (28) if all the vertex factors $\tilde{\varphi} (\varphi)$ in (27) are replaced by $\varphi (\varphi)$. Thus we have

$$\int d^n \ell I (F^C) = (-1)^{N_C (F)} \int d^n \ell I (F) \mid_{\gamma_5 \to -\gamma_5}$$

(29)

where $N_C (F)$ is the number of $C$-odd fields in $\{ \varphi_1, \varphi_2, \ldots, \varphi_n \}$ and $(-1)^{N_C (F)}$ is the $C$-parity of the diagram $F$ or $F^C$. Decomposing the identity (29) into the $\gamma_5$-even part and the $\gamma_5$-odd part, we get

$$\gamma_5$$-even part of $\int d^n \ell I (F^C) = \gamma_5$$-even part of $(-1)^{N_C (F)} \int d^n \ell I (F)$

(30)

and

$$\gamma_5$$-odd part of $\int d^n \ell I (F^C) = \gamma_5$$-odd part of $(-1)^{N_C (F)+1} \int d^n \ell I (F)$.

(31)
If the fermion loop $F$ is a sub-diagram of a larger diagram $G$ that contains no other fermion lines than those in $F$, then the $C$-parity of $G$ is equal to the $C$-parity of $F$. Since the Feynman integrand for the complement of $F$ in $G$ is the same as that for the complement of $F^C$ in $G^C$, (29)-(31) are also valid if we replace $F$ with $G$. We have thus proved the following theorem:

**Theorem 2** For a diagram $G$ without any external fermion line and with a fermion loop that has been cut open, the conjugate diagram $G^C$ is defined to be the diagram obtained by reversing the direction of the fermion loop in $G$ without altering the location of the cut point. If $G$ is $C$-even, the $\gamma_5$-even part of the dimensionally regularized amplitude of $G$ is equal to the $\gamma_5$-even part of $G^C$ but the $\gamma_5$-odd part of $G$ is the negative of the $\gamma_5$-odd part of $G^C$. If $G$ is $C$-odd, the $\gamma_5$-odd part of $G$ is equal to the $\gamma_5$-odd part of $G^C$ and the $\gamma_5$-even part of $G$ is the negative of the $\gamma_5$-even part of $G^C$.

In our dimensional regularization scheme, we will make it a rule that if a diagram $G$ is included as one of the component diagrams, the conjugate diagram $G^C$ must also be included (with, of course, suitable adjustment of weighting factors). Since the $\gamma_5$-odd parts are cancelled between $G$ and $G^C$ when $G$ is $C$-even, the following corollary is obvious.

**Corollary 3** No Levi-Civita tensor term is possible for $C$-even functions.

For $C$-odd functions, the $\gamma_5$-even parts are cancelled between $G$ and $G^C$. If we discard the $\gamma_5$-even part, either $G$ or $G^C$ suffices for the evaluation of the $C$-odd function and we choose the one whose cut point is located at the endpoint of an internal fermion line.

**Corollary 4** Only Levi-Civita tensor terms survive in the regularized $C$-odd functions. These Levi-Civita tensor terms may be evaluated by diagrams whose cut points are restricted to the subset of endpoints of internal fermion lines on the fermion loops.

From now on, unless specified otherwise, we limit the cut points to be those residing at the end of fermion lines.
4.2 Graphical Identities

The prescription of the rightmost $\gamma_5$ ordering under dimensional regularization offers a scheme to construct amplitudes when $n \neq 4$. We now introduce some graphical notations for verifying diagrammatically if the regularized amplitudes so obtained satisfy Ward-Takahashi identities.

According to the Feynman rules, one assigns the factor $-ig R\gamma^\mu L$ to the vertex $\bar{\psi} - A^\mu - \psi$ and the factor $-f (L - R)$ to the vertex $\bar{\psi} - \phi_2 - \psi$, as these factors correspond to the terms $-g\bar{\psi}_L A\psi_L$ and $-i f (\bar{\psi}_L \phi_2 \psi_R - \bar{\psi}_R \phi_2 \psi_L)$ in the interaction Lagrangian of (10). Let us define the following two graphical notations for these two vertices:

$$\bigotimes_\mu = -ig R\gamma^\mu L, \quad \bigotimes = -f (L - R).$$

(32)

We also introduce the notation

$$\bigotimes_k = -g R k L + mg (L - R)$$

(33)

which represents the sum of $-ik_\mu$ times the $\bar{\psi} - A^\mu - \psi$ vertex factor, with $k$ the momentum of the vector particle flowing into the vertex and $-M$ times the $\bar{\psi} - \phi_2 - \psi$ vertex factor. Note that $Mf$ may be equated to $mg$ according to (12). The identity

$$R k L - m (L - R) = (kL + k - m) L - R (k - m),$$

(34)

valid in a four-dimensional space, will be our building block for verifying various Ward identities involving fermion lines. Indeed, if we set $L = R = 1$, the identity above becomes the familiar identity used in verifying Ward identities in QED.

Sandwiching equation (34) between two fermion propagators, we get

$$\frac{1}{(L + k - m)} (R k L - m (L - R)) \frac{1}{(k - m)} = L \frac{1}{(L - m)} - \frac{1}{(L + k - m)} R$$

(35)

Note the similarity of this identity with its familiar counterpart (5) in QED. As noted before, when we go to the dimension of $n \neq 4$, (35) in the form presented above is not valid. This is because $\gamma_5$ does not always anti-commute with $\gamma^\mu$ if $n \neq 4$. Adopting the rightmost $\gamma_5$ ordering avoids this difficulty.
The above equation multiplied by the coupling constant $g$ may be expressed graphically as

\[ \ell + k = \ell k + \ell k \quad (36) \]

where the double line emitting from the composite vertex $\otimes$ indicates that the fermion propagator is annihilated. In addition, the double line together with the composite vertex is to be replaced by $-igL$ if the arrow points to the left, and to be replaced by $igR$ if the arrow points to the right. Thus the following two diagrams cancel each other if the corresponding external momenta are the same:

\[ \otimes + \otimes = 0 \quad (37) \]

In our convention, the direction of any horizontal fermion line is assumed to be pointing to the left side unless indicated otherwise.

### 4.3 One-Loop Order

At 1-loop level, the problem with multiple values of analytical continuation only occurs when the amplitude associated with the fermion loop is divergent. This happens when there are 2, 3 or 4 external lines attached to the fermion loop.

A diagram with 2 external lines has at most two indices provided by the polarizations of the external lines and one index provided by the external momentum. Thus there are insufficient indices to form a Levi-Civita tensor term for a 2-point function and the $\gamma_5$-odd amplitude for such a function vanishes. As to the remaining $\gamma_5$-even amplitude, it is uniquely defined and can be calculated in consistency with gauge invariance. Hence the Ward identities for 2-point functions with one fermion loop are always satisfied and are free from the anomaly problem.

For a 3-point 1PI function, the Levi-Civita tensor needs to be contracted to at least two components of the external momenta if there are less than 3 external polarizations. The 3-point function is linearly divergent and the 2nd order term of its Taylor series expanded with respect to its external momenta, henceforth called the $T_2$ term, is convergent. At 1-loop level, the Levi-Civita tensor term of such a 3-point function is well defined in the $n \to 4$ limit although its continuation when $n \neq 4$ may be ambiguous. Such convergent
terms also satisfy the Ward identities they are supposed to obey. Thus, the possibility for 1-loop anomaly occurs when the Ward identity involves an $AAA$ amplitude for three external vector fields. Furthermore, for this 1-loop $AAA$ function, the $T_0$ term with all the external momenta equal to zero has insufficient indices to form a Levi-Civita tensor and only the $T_1$ term (1st order term in the Taylor series expansion) may contribute to the ambiguous Levi-Civita tensor terms.

The 4 point 1PI function is logarithmically divergent. Only the $T_0$ term of an $AAAA$ 1PI amplitude for 4 vector fields with 4 different polarizations has enough indices to form a Levi-Civita tensor term that may be divergent. Such 4-point functions are $C$-even and free of any Levi-Civita tensor term according to Corollary 3 of Theorem 2. All the Ward identities for 4-point functions in the Abelian-Higgs theory are thus free of 1-loop anomaly.

In the following section, we will attempt to construct, with our prescription of rightmost $\gamma_5$ dimensional regularization, the 1-loop amplitudes for the triangular Ward identity. The Ward identities for two-loop diagrams will be discussed later.

### 4.4 One-Loop Triangular Diagrams

Let $\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho; k_1, k_2, k_3)$ denote the 1PI amplitude with one fermion loop and three external fields $A^\mu, A^\nu, A^\rho$, with $k_1, k_2, k_3 = -k_1 - k_2$ the momenta of $A^\mu, A^\nu, A^\rho$, respectively. We may omit the momentum variables $k_1, k_2, k_3$ if there is no confusion. The superscript $^{(1)}$ signifies that the amplitude is of one loop, while the subscript $F$ signifies the presence of a fermion loop. The directions of the external momenta are inward. As noted before, the $T_1$ term of $\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)$, denoted by $T_1 \left[ \Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho) \right]$, is ambiguous.

One possible way to resolve the ambiguity is by relating $\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)$ to $\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)$ through the Ward identity

$$-i k_3^\rho \Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho; k_1, k_2, k_3) - M \Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2; k_1, k_2, k_3) = 0. \quad (38)$$

We will show that there are 1-loop diagrams that may be constructed to yield amplitudes which satisfy the above identity. Formally, amplitude for the left side of the above identity (38) is represented by the sum of the following two
Feynman diagrams:

\[
\begin{array}{c}
\times \\
\nu & \mu \\
\end{array}
\begin{array}{c}
\times \\
\mu & \nu
\end{array}
\]

(39)

If we replace the circled cross \(\times\) in the two diagrams above by the uncircled cross \(\not{\times}\) defined in (32), then these two diagrams become the 1-loop diagrams for the AAA amplitude in (38). Similarly, if we replace the circled cross \(\times\) by the black dot \(\bullet\) defined in (32), then the two diagrams become the 1-loop diagrams for the AA\(\phi_2\) amplitude in (38). Thus the two diagrams in (39) represent the left side of (38). Furthermore, since both AAA and AA\(\phi_2\) functions are \(C\)-odd, only Levi-Civita tensor terms survive in their regularized amplitudes by Corollary 4 of Theorem 2. By power counting, \(T_2\left[\Gamma_F^{(1)}(A^\mu, A'^\nu, \phi_2)\right]\) is convergent and may be easily evaluated with any cut point by rightmost \(\gamma_5\) dimensional regularization. The result

\[
\lim_{n \to 4} T_2 \left[\Gamma_F^{(1)}(A^\mu, A'^\nu, \phi_2)\right] = -\frac{1}{12\pi^2 M} g^3 \epsilon^{\mu\nu\rho\sigma} k_1 k_2
\]

is unambiguously defined.

If we detach the composite vertex \(\otimes\) from each of the two diagrams in (39), both diagrams then become the same as

\[
\begin{array}{c}
\otimes \\
\nu & \mu
\end{array}
\]

(41)

Since the component diagrams in (39) may be generated by all possible insertions of the composite vertex \(\otimes\) into the internal lines of the above diagram in (41), this diagram in (41) will be called the generator for the Ward identity (38).

By making a cut at the \(\bar{\psi} - A^\nu - \psi\) vertex and then by repeated use of (36) and (37), the sum of the two diagrams in (39) becomes

\[
\begin{array}{c}
\otimes \\
\mu & \nu
\end{array} + \begin{array}{c}
\otimes \\
\mu & \nu
\end{array} = \begin{array}{c}
\otimes \\
\mu & \nu
\end{array} + \begin{array}{c}
\otimes \\
\mu & \nu
\end{array}
\]

(42)

in which the horizontal line is supposed to be an open fermion line flowing to the left. We emphasize that the identity (42) remains satisfied when \(n \neq 4\) if
we adopt the rightmost $\gamma_5$ dimensional regularization for every term in the identity. Calling the momentum for the fermion line entering the cut point as $\ell$, we find that these two amplitudes are, respectively,

$$(-ig^3 L) \frac{1}{\ell + k_1 + k_2 - m} \gamma^\mu L \frac{1}{\ell + k_2 - m} \gamma^\nu L = L\hat{M}(\ell) L$$

and

$$\frac{1}{\ell - m} \gamma^\mu L \frac{1}{\ell + k_2 + k_3 - m} (ig^3 R) \gamma^\nu L = -\hat{M}(\ell + k_3) L,$$

where $\hat{M}(\ell)$ stands for $-ig^3 \frac{1}{\bar{\psi} + k_1 + k_2 - m} \gamma^\mu L \frac{1}{\psi + k_2 - m} \gamma^\nu$. Since the fermion lines form a closed loop, the trace of the expressions above will be taken. Because $Tr\left(L\hat{M}(\ell) L\right)_{DR}$ may be reduced to $Tr\left(\hat{M}(\ell) L\right)_{DR}$, the amplitudes corresponding to the last two diagrams in (42) are related by a shift of the momentum variable. Since it is legitimate to shift the loop momentum by a finite amount after regularization, the regularized amplitude of (42) vanishes after integration.

Note that the first two cut diagrams in (42) may be generated by attaching the composite vertex $\otimes$ in all possible manners consistent with Feynman rules to the cut diagram

$\begin{array}{c}
\otimes \\
\mu \nu
\end{array}$

obtained by cutting the generator diagram (41) at the $\bar{\psi} - A^\nu - \psi$ vertex. It is convenient to view the identity that the regularized amplitude of (42) vanishes as being generated by the cut generator in (43). To summarize, if we choose the $\bar{\psi} - A^\nu - \psi$ vertex as the cut point for the generator (41), construct the component diagrams by attaching $\otimes$, anti-commute $\gamma_5$ to the rightmost position, and then dimensionally regularize the coefficients in front of $\gamma_5$, the regularized amplitudes so obtained satisfy the Ward identity (38).

Similarly, we may open up the fermion loop by choosing the $\bar{\psi} - A^\mu - \psi$ vertex as the cut point, follow through the same arguments, and arrive at another set of amplitudes for $\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)$ and $\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)$. Such amplitudes may also be obtained from the interchange of $(\mu, k_1) \leftrightarrow (\nu, k_2)$ on the previously defined $\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)$ and $\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)$. Since $\epsilon^{\mu\nu\rho\sigma} k_{1\rho} k_{2\sigma}$ is invariant under such exchange, the amplitude $T_2 \left[\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)\right]$ remains the same. Therefore, it may appear in order that the result of (10) is consistent with the Ward identity (38), $T_1 \left[\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)\right]$ should be defined.
such that
\[ -i k_3^\rho \lim_{n \to 4} T_1 \left[ \Gamma_F^{(1)} (A^\mu, A^\nu, A^\rho) \right] = -\frac{1}{12\pi^2} g^3 \epsilon^{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma. \] (44)

However, we will show, in the immediate following, that the above condition (44) for AAA amplitude is inconsistent with the Bose permutation symmetry. By definition, \( T_1 \left[ \Gamma_F^{(1)} (A^\mu, A^\nu, A^\rho) \right] \) is a product of a Levi-Civita tensor and a linear combination of the independent external momenta \( k_1 \) and \( k_2 \). From relativistic covariance, we must have
\[ T_1 \left[ \Gamma_F^{(1)} (A^\mu, A^\nu, A^\rho; k_1, k_2, k_3) \right] = \epsilon^{\mu\nu\rho\sigma} (C_1 k_1^\sigma + C_2 k_2^\sigma) \] (45)
where \( C_1 \) and \( C_2 \) are dimensionless constants. The Bose symmetry under the exchange of \((A^\mu, k_1) \leftrightarrow (A^\nu, k_2)\) gives
\[ T_1 \left[ \Gamma_F^{(1)} (A^\mu, A^\nu, A^\rho; k_1, k_2, k_3) \right] = T_1 \left[ \Gamma_F^{(1)} (A^\nu, A^\mu, A^\rho; k_2, k_1, k_3) \right], \]
which is equivalent to
\[ \epsilon^{\mu\nu\rho\sigma} (C_1 k_1^\sigma + C_2 k_2^\sigma) = \epsilon^{\mu\nu\rho\sigma} (C_1 k_2^\sigma + C_2 k_1^\sigma) = \epsilon^{\mu\nu\rho\sigma} (-C_2 k_1^\sigma - C_1 k_2^\sigma) \] (46)
Similarly, the Bose symmetry for the exchange of \((A^\nu, k_2) \leftrightarrow (A^\rho, k_3)\) yields
\[ T_1 \left[ \Gamma_F^{(1)} (A^\mu, A^\nu, A^\rho; k_1, k_2, k_3) \right] = T_1 \left[ \Gamma_F^{(1)} (A^\mu, A^\rho, A^\nu; k_1, k_3, k_2) \right] \]
and
\[ \epsilon^{\mu\nu\rho\sigma} (C_1 k_1^\sigma + C_2 k_2^\sigma) = \epsilon^{\mu\nu\rho\sigma} (C_1 k_1^\sigma + C_2 k_3^\sigma) = \epsilon^{\mu\nu\rho\sigma} ((C_2 - C_1) k_1^\sigma + C_2 k_2^\sigma). \] (47)
To meet (46) and (47), we must have \( C_1 = C_2 = 0 \). Consequently,
\[ T_1 \left[ \Gamma_F^{(1)} (A^\mu, A^\nu, A^\rho) \right] = 0. \] (48)
This result contradicts the result (44) derived on the basis of the validity of the Ward identity (38), showing that this Ward identity for the triangular diagrams is not consistent with the Bose permutation symmetry.

Note that the diagrams used in the graphical identity (42) with the \( \bar{\psi} - A^\nu - \psi \) vertex as the cut point are not Bose symmetric. Nor are those with
the $\bar\psi - A^\mu - \psi$ vertex as the cut point. Nor are the sum of these two sets of diagrams. This is because we have left out the third vertex as a cut point. In fact, if we choose the vertex $\bar\psi - A^\rho - \psi$ to be the cut point for $\Gamma_F^{(1)}(A^\mu, A^\nu, A^\rho)$ and the vertex $\bar\psi - \phi_2 - \psi$ to be the cut point for $\Gamma_F^{(1)}(A^\mu, A^\nu, \phi_2)$, the left side of (38) is now diagrammatically expressed as

\[
\begin{array}{c}
\includegraphics{diagram1} \\
\mu & \nu \\
\end{array} +
\begin{array}{c}
\includegraphics{diagram2} \\
\nu & \mu \\
\end{array}
\] (49)

which, by making use of (36), is expanded into

\[
\begin{array}{c}
\includegraphics{diagram3} \\
\mu & \nu \\
\end{array} +
\begin{array}{c}
\includegraphics{diagram4} \\
\mu & \nu \\
\end{array} +
\begin{array}{c}
\includegraphics{diagram5} \\
\nu & \mu \\
\end{array} +
\begin{array}{c}
\includegraphics{diagram6} \\
\nu & \mu \\
\end{array}
\] (50)

Let $\ell$ be the momentum of the fermion line entering the cut point. The symbol $\otimes^*$ on the right of either the second diagram or the fourth diagram in the above figure is to be replaced by $gR(\ell - m)$ at the right-end before moving $\gamma_5$ to the rightmost position. If the factor of $(\ell - m)$ at the right-end of the second (fourth) diagram annihilates the fermion propagator $i(\ell - m)$ at the left-end, the amplitude so obtained will cancel the amplitude of the third (first) diagram. But in our scheme of dimensional regularization, we continue to $n \neq 4$ after positioning $\gamma_5$ at the rightmost site. This rightmost $\gamma_5$ may stand between the $\frac{1}{\ell - m}$ at the left-end and the $(\ell - m)$ at the right-end to prevent their annihilation in the trace. In fact, when $n \neq 4$, the symbol $\otimes^*$ should be replaced by the expression

\[
(gR(\ell - m))_{DR} = g(\ell L - mR) = gR(\ell - m) - g \ell \Delta \gamma_5.
\] (51)

The last term $-g \ell \Delta \gamma_5$ in the above is the leftover after the cancellation. To evaluate the total amplitude of (50) by dimensional regularization, we only need to take into account the contribution from the leftover terms. Furthermore, due to the presence of $\ell \Delta \gamma_5$, only the divergent orders in the Taylor series expansion with respect to the external momenta may contribute to the $n \to 4$ limit. In particular, the leftover amplitude from the second diagram of (50) is

\[
-ig^3 \int \frac{d^n \ell}{(2\pi)^n} Tr \left( \frac{1}{\ell - m} \gamma^\mu \frac{\ell - k_1}{(\ell - k_1)^2 - m^2} \gamma^\nu \frac{\ell + k_3}{(\ell + k_3)^2 - m^2} \ell \Delta \gamma_5 \right)
\]
The evaluation of the above amplitude in the limit \( n \to 4 \) is greatly simplified by keeping only the \( T_2 \) order term to yield the result

\[
\frac{1}{8\pi^2} g^3 \epsilon^{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma \tag{52}
\]

In figure (50), the fourth diagram may be obtained from the second diagram by the exchange \((\mu, k_1) \leftrightarrow (\nu, k_2)\). Therefore, the leftover amplitude due to the former diagram is the same as that due to the latter diagram and the total amplitude of (49) in the limit \( n \to 4 \) is equal to twice the amount of (52),

\[
\frac{1}{4\pi^2} g^3 \epsilon^{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma, \tag{53}
\]

which is also equal to \(-M\) times thrice the amplitude of \( T_2 \left[ \Gamma_F^{(1)} (A^\mu, A^\nu, \phi_2) \right] \) in (40). We have thus shown that the Ward identity (38) is not satisfied if the fermion loops are cut open as in (49).

The amplitude \( \Gamma_F^{(1)} (A^\mu, A^\nu, A^\rho) \) obtained by averaging over the three amplitudes corresponding to the three different cut points chosen at the vertices of \( \bar{\psi} - A^\mu - \psi, \bar{\psi} - A^\nu - \psi \), and \( \bar{\psi} - A^\rho - \psi \) satisfies the permutation symmetry. The amplitude \( T_1 \left[ \Gamma_F^{(1)} (A^\mu, A^\nu, A^\rho) \right] \) so obtained therefore vanishes.

The amplitude \( T_2 \left[ \Gamma_F^{(1)} (A^\mu, A^\nu, \phi_2) \right] \) is convergent and its value, which is independent of the cut point chosen, remains equal to (40). If we take the average of the left-hand side of the Ward identity (38) over the three cuts, this average value does not vanish but is equal to \(-MT_2 \left[ \Gamma_F^{(1)} (A^\mu, A^\nu, \phi_2) \right] \).

This observation is consistent with the result of (53) for (49) and the vanishing sum of (42). We have learned that \( T_1 \left[ \Gamma_F^{(1)} (A^\mu, A^\nu, A^\rho) \right] \) vanishes only if total permutation symmetry is built into its component diagrams. Since this symmetry of Bose statistics must be obeyed, we have to pay the price of losing the validity of a Ward identity, and conclude that there exists an anomaly.

### 4.5 Anomaly Compensating Fermion Field

We have just observed that the ambiguity in choosing the cut point for the 1-loop \( AAA \) amplitude owes its origin to the singular behavior of its integrand. As a result, the Ward identity for this amplitude is not obeyed and there
is an anomaly. Let us add to the theory another fermion field $\psi'$ with a coupling constant $-g$ for $\psi'_L$ equal to the negative of the coupling constant $g$ for $\psi_L$. The covariant derivative for $\psi'_L$ is

$$D_\mu \psi'_L = (\partial_\mu - igA_\mu) \psi'_L$$

in contrast to the covariant derivative for $\psi_L$:

$$D_\mu \psi_L = (\partial_\mu + igA_\mu) \psi_L.$$ The Lagrangian for such a theory is given by

$$L'_{\text{eff}} = L_{\text{eff}} + \bar{\psi}'_L (i \not{D}) \psi'_L + \bar{\psi}'_R (i \not{D}) \psi'_R - \sqrt{2} f' (\bar{\psi}'_L \phi^\dagger \psi'_R + \bar{\psi}'_R \phi \psi'_L)$$

where $L_{\text{eff}}$ is defined in (16). Note the coupling $f'$ does not need to be the same as the $f$ in (10) and the masses for the two fermion fields may not be equal. The amplitude for a 1-loop $AAA$ diagram with a fermion loop due to the $\psi'$ field is proportional to $(-g)^3$ and cancels the logarithmically divergent term of the amplitude due to the $\psi$ field. Therefore, the 1-loop $AAA$ amplitude in the theory with the additional $\psi'$ field is convergent and cut point independent. The theory of (54) is free of the 1-loop anomaly.

### 4.6 Two-Loop Diagrams with External Fermion Fields

For the theory of (54) with the anomaly compensating fermion field $\psi'$, we have shown that all one–fermion-loop diagrams are convergent and are well defined in the limit $n \to 4$. But there are $O(n-4)$ terms stemming from different positioning of $\gamma_5$ on the fermion loop. These $O(n-4)$ terms may not be ignored if the fermion loop is embedded as a sub-diagram in a divergent diagram that may give rise to a $\frac{1}{n-4}$ pole term factor.

At 2-loop order, divergent 1PI diagrams with a fermion loop that may be plagued by $\gamma_5$ ambiguity are the fermion self-energy, vertex correction, three point $AAA$ and four point $AAAA$ diagrams. The fermion self-energy, as shown below, is a diagram with two internal lines attached to a fermion loop and to an open fermion line.
The amplitude for the fermion-loop sub-diagram in the above, having at most two indices and one momentum available, does not have sufficient indices to form a Levi-Civita tensor term. Consequently, rightmost positioning of $\gamma_5$ on the open fermion line ensures that the regularized amplitude for the above diagram obeys all the relevant Ward identities.

The vertex function is logarithmically divergent by power counting. Only its $T_0$ term may be ambiguous. A triangular fermion loop, with zero momentum for one of the three fields attached, will not have enough indices to form a Levi-Civita tensor term unless all the fields attached to the fermion loop are vector fields. At 2-loop order, only the following diagram with a sub-diagram of a triangular fermion loop attached by three vector fields may be ambiguous.

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram (56)}
\end{array}
\end{array}
\]

If the cuts for both the triangular $\psi$ and $\psi'$ fermion loops are made at the same point, the divergence due to the $\psi$ fermion loop cancels the divergence due to the $\psi'$ fermion loop and only convergent terms proportional to the mass difference between $\psi$ and $\psi'$ may survive in the amplitude for the fermion loop. With this synchronized choice of cut points for both $\psi$ and $\psi'$ loops, the 2-loop vertex-correction diagram of (56) is convergent and satisfies all the relevant Ward identities.

We are left with the possibility that only the type of 2-loop diagrams with a fermion-loop sub-diagram but without external fermion lines may be ambiguous. For this type of diagrams, a $\gamma_5$ positioning prescription that is able to preserve the validities of Ward identities will be given. Let us first deal with the 2-loop triangular diagrams in the following section.

### 4.7 Two-Loop Triangular Diagrams

We now proceed to construct the triangular Ward identity at the 2-loop order. To simplify the presentation in this section, we will only consider the subset of 2-loop triangular diagrams with one fermion loop and one internal vector meson line. Other types of 2-loop triangular diagrams can be handled similarly without additional difficulty and will be addressed in the Appendix. For this restricted type of diagrams, the triangular Ward identity is the identity that equates the sum of amplitudes for the following 12 diagrams to
In the above figure, the fermion loop is the arrowed loop and the wavy lines are vector meson lines. These 12 diagrams are also the ones generated by attaching the composite vertex in all possible manners consistent with Feynman rules to the following three generator diagrams:

To preserve the permutation symmetry, we will not use cut points at the vertices connecting to external fields in order not to give an asymmetric treatment to any of the external fields. For convenience, cut point positioned at a vertex connecting to an external field line is called illegitimate, otherwise it is called legitimate. For the diagrams in (57), cut points at or vertices are illegitimate. Because we do not position \(\gamma_5\) inside a self-energy or vertex-correction sub-diagram on an open fermion line in our prescription, it is also appropriate to avoid cutting the fermion loops at improper positions. Furthermore, according to Corollary 4 of Theorem 2, cut points chosen at the endpoints of fermion lines are sufficient for the purpose of evaluating regularized Levi-Civita tensor terms. For each of the 12 diagrams in (57), there is one and only one cut point that is proper, legitimate and located at the end of a fermion line.

Any of the three generator diagrams in (58) may be cut open at the endpoint of a fermion line and then used as a cut generator to construct a Ward identity with four component diagrams. For example, the cut point indicated by the arrow on a dotted line for the first diagram in (58) gives the
cut generator

\begin{equation}
\begin{aligned}
\mu & \nu \\
\end{aligned}
\end{equation}

By attaching $\otimes$ in all possible manners that are consistent with Feynman rules to the above generator, we obtain the following four proper diagrams:

\begin{equation}
\begin{aligned}
\otimes & \otimes \mu \nu \\
\otimes & \otimes \mu \nu \\
\otimes & \otimes \mu \nu \\
\otimes & \otimes \mu \nu \\
\end{aligned}
\end{equation}

Using (36) and (37) repeatedly, the above expression can be reduced to

\begin{equation}
\begin{aligned}
\mu & \nu \\
\end{aligned}
\end{equation}

If we identify $f(\ell_1, \ell_2)$ as the Feynman integrand for the last diagram, where $\ell_2$ is the momentum of the vector meson line and $\ell_1$ is the momentum of the leftmost fermion line, the Feynman integrand corresponding to (61) is the difference of two terms related by a shift of the loop momentum $\ell_1$. Specifically, this sum is

\begin{equation}
\begin{aligned}
\mu & \nu \\
\end{aligned}
\end{equation}

which vanishes upon carrying out the integration $\int d^n \ell_1 d^n \ell_2$ under our scheme of rightmost $\gamma_5$ dimensional regularization. The sum of amplitudes for the four diagrams in (60) therefore vanishes. In (58), the exchange $(\mu, k_1) \leftrightarrow (\nu, k_2)$ transforms the first diagram into the second diagram and thus the corresponding identity generated by the second diagram has the same component diagrams as those obtained from (60) by making the exchange $(\mu, k_1) \leftrightarrow (\nu, k_2)$.

Unlike the first two diagrams in (58), no proper cut point is available for the third diagram. Making the cut at the position pointed by the dotted arrowed line for the third diagram in (58), we get the cut generator

\begin{equation}
\begin{aligned}
\mu & \nu \\
\end{aligned}
\end{equation}
that yields the identity

\[ \sum_{\mu, \nu} X_{\mu, \nu} + \sum_{\mu, \nu} X_{\mu, \nu} + \sum_{\mu, \nu} X_{\mu, \nu} + \sum_{\mu, \nu} X_{\mu, \nu} = 0 \]  

(64)

For the first two diagrams in the above, the cut points are proper. But for each of the last two diagrams, if we reconnect the beginning point and the endpoint of the open fermion line to restore the original fermion loop, we see that there is a sub-diagram of radiative correction for the vertex \( \bar{\psi} - A^\mu - \psi \).

The cut point, being the endpoint of the fermion line in this vertex correction sub-diagram, is improper. For convenience, from here on in this section, we will identify \( S \) as the sum of the last two diagrams in (64). The sub-diagram of radiative correction for the vertex \( \bar{\psi} - A^\mu - \psi \) in each diagram of \( S \) will be denoted by \( H \).

For both diagrams in \( S \), if the improper cut point inside \( H \) is moved out of \( H \) to the endpoint of the fermion line connecting to \( H \), the relocated cut becomes proper and \( S \) becomes

\[ \sum_{\mu, \nu} X_{\mu, \nu} + \sum_{\mu, \nu} X_{\mu, \nu} \]  

(65)

Since all the fermion lines and vertex factors sandwiched between the original cut points in \( S \) and the relocated ones in (65) lie within the sub-diagram \( H \), the difference between \( S \) and (65) may be expressed as a combination of terms with \( \gamma_\Delta \) factors stemming from the matrix product in \( H \). These \( \gamma_\Delta \) factors cannot be ignored if they are multiplied by pole terms arising from divergent loop integrations. Although both (65) and \( S \) appear divergent, we will nevertheless be able to demonstrate shortly that their regularized amplitudes differ only by terms of \( O(n - 4) \) and are equal in the limit \( n \to 4 \).

Making repetitive use of (36) and (37), \( S \) can be transformed into

\[ \sum_{\mu, \nu} X_{\mu, \nu} + \sum_{\mu, \nu} X_{\mu, \nu} \]  

(66)

There is a similar transformation for (65). The pole terms generated from the loop integration of the sub-diagram \( H \) only occur in the \( T_0 \) term of \( H \), denoted by \( T_0[H] \), which is the amplitude of \( H \) with all the external momenta relative to \( H \) set to zero. If we substitute \( T_0[H] \) for \( H \) in each
diagram of (66), the resulting amplitude for the entire diagram, depending only one external momentum $k_2$ and two external polarizations $\mu$ and $\nu$, does not have sufficient indices to form a Levi-Civita tensor term. Thus the pole terms of $H$ do not contribute to (66) or $S$. Similarly, the pole terms of the vertex-correction sub-diagram do not contribute to (65). The remaining possibility for the survival of the $\gamma_\Delta$ factors in the limit of $n \to 4$ is that they are multiplied by some pole terms due to the overall divergence of the entire diagram.

In the Taylor series expansion with respect to the external momenta for $S$, only the second order term denoted by $T_2[S]$ may have divergent Levi-Civita tensor terms. If the composite vertex $\otimes$ in $S$ is replaced by the vertex $\oslash$, the two diagrams of $S$ become two of the component diagrams associated with the $A^\mu - A^\nu - A^\rho$ amplitude. This amplitude denoted by $S^\rho$, as can be examined diagrammatically, is symmetric with respect to the exchange $(\nu, k_2) \leftrightarrow (\rho, k_3)$. Only $T_1[S^\rho]$, the $T_1$ term of this $AAA$ amplitude, may have divergent term proportional to $\epsilon^{\mu\nu\rho\sigma}(k_{2\sigma} - k_{3\sigma})$ to contribute to the pole terms of $S$. Let $\Gamma^\mu(\ell, k_1)$ be the amplitude for the sub-diagram of $H$ in $S$ where $\ell$ is the fermion momentum entering $H$. Because $T_1[S^\rho]$ is invariant under $(\nu, k_2) \leftrightarrow (\rho, k_3)$, if $H$ is expanded in a Taylor series with respect to the variable $k_1$, only the term $\Gamma^\mu(\ell, 0)$ without any power of $k_1$ may have a non-vanishing contribution to $T_1[S^\rho]$. Thus the pole terms of $S$ are not altered if $\Gamma^\mu(\ell, 0)$ instead of $\Gamma^\mu(\ell, k_1)$ is substituted for the amplitude of $H$. Furthermore, the transformation of $S$ into (66) by the use of (36) and (37) is independent of the expression for the sub-diagram $H$. With $\Gamma^\mu(\ell, 0)$ substituted for $\Gamma^\mu(\ell, k_1)$, the amplitude for the $T_2$ term of (66), with $\ell$ identified as the momentum of the fermion line leaving the $\overline{\psi} - A^\nu - \psi$ vertex, is proportional to

$$T_2 \left[ \text{Tr} \int d^n \ell \left[ \Gamma^\mu(\ell + k_3, 0) - \Gamma^\mu(\ell, 0) \right] L \frac{1}{\ell - m} \gamma^\nu L \frac{1}{\ell - k_2 - m} R \right]$$

in which all the $\gamma_5$ factors are supposed to be consolidated and positioned at the end of the fermion line inside $H$. In the power series expansion of $\Gamma^\mu(\ell + k_3, 0) - \Gamma^\mu(\ell, 0)$ with respect to the variable $k_3$, the symmetric tensor $k_3\rho k_3\sigma$ in the second order term cannot have non-zero contraction with the antisymmetric Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$. Only the term linear in $k_3$ may have non-zero contribution. Thus the pole terms of $S$ are contained in

$$k_3\rho T_1 \left[ \text{Tr} \int d^n \ell \frac{\partial \Gamma^\mu(\ell, 0)}{\partial \ell_\rho} L \frac{1}{\ell - m} \gamma^\nu L \frac{1}{\ell - k_2 - m} R \right] \quad (67)$$
Next, consider the following 1-loop Ward identity that relates the fermion self-energy insertion $\Sigma(\ell)$ to the radiative vertex corrections of $\psi - \phi_2 - \bar{\psi}$ and $\psi - A^\mu - \bar{\psi}$:

$$ig\Sigma(\ell + k) - ik_\mu \Gamma(\ell, k) - ig\Sigma(\ell) = \text{terms proportional to } m$$

In each of the above three diagrams, the $\gamma_5$ position, as marked by an arrow on a dotted line, is chosen to be the endpoint of the fermion line connecting to the internal vector meson line. Since the 1-loop vertex correction for $\psi - \phi_2 - \bar{\psi}$ is proportional to the fermion mass $m$ and is convergent, the identity (68) may be written as

$$g \frac{\partial}{\partial \ell_\mu} \Sigma(\ell) = \text{terms proportional to } m$$

where $\Gamma^\mu(\ell, k)$ is the amplitude for the 1-loop vertex correction of $\psi - A^\mu - \bar{\psi}$. In the limit $k \to 0$, the above identity yields

$$\Gamma^\mu(\ell, 0) - g \frac{\partial}{\partial \ell_\mu} \Sigma(\ell) = \text{terms proportional to } m$$

The terms on the right-hand side in the above will not contribute to any pole terms because the mass factor $m$ lowers the degree of divergence not only for the vertex correction sub-diagram but also for the entire diagram. Thus the term $g \frac{\partial \Gamma^\mu(\ell, 0)}{\partial \ell_\rho}$ in (67) may be replaced by $g \frac{\partial \Sigma(\ell)}{\partial \ell_\mu}$, which is symmetric under the interchange $(\mu \leftrightarrow \rho)$. Since the $\epsilon^{\mu\nu\rho\sigma}$ tensor is antisymmetric under $(\mu \leftrightarrow \rho)$, (67) must vanish and the occurrence of pole terms in $S$ is prohibited. In a similar manner, (65) can be shown to be convergent. We have thus shown that there is no divergent pole term to prevent the difference between (65) and $S$ from vanishing in the limit $n \to 4$.

Note that the first two diagrams in (64) after the $(\mu, k_1) \leftrightarrow (\nu, k_2)$ interchange become the two diagrams in (65) which are equal to the last two diagrams in (64) when $n \to 4$. The identity (64) thus leads to the result that the amplitude of (65) after the symmetrization of $(\mu, k_1) \leftrightarrow (\nu, k_2)$ will vanish in the limit $n \to 4$. Knowing that (61) vanishes, the sum of amplitudes symmetrized with respect to the interchange of $(\mu, k_1) \leftrightarrow (\nu, k_2)$ for the 6 proper diagrams consisting of the four diagrams in (60) and the two diagrams in (65) also vanishes. Since none of the external fields is given a preferential
role, the Ward identity constructed will not contradict the Bose permutation symmetry. \textit{i.e.,} the cut diagrams for the $AAA$ amplitude obtained by replacing the composite vertex $\otimes$ with the vertex $\check{\rho}$ in all the 12 properly cut diagrams for (57) are symmetric with respect to the permutation of the three external vector fields $A^\mu$, $A^\nu$ and $A^\rho$.

In the 1-loop case, one of the three external vertices must be used as the cut point and we have shown it is impossible to construct a set of diagrams to satisfy both the triangular Ward identity and Bose permutation symmetry. For the two-loop diagrams we have discussed here, there is the additional freedom of choosing proper cut points at a vertex connecting to the internal vector meson line. As a result, we are able to construct diagrams that satisfy both the triangular Ward identity and Bose permutation symmetry.

The $AA\phi_2$ function is superficially convergent and its renormalized amplitude can be calculated with any convenient choice of proper cut point according to Theorem 1. Since the 2-loop triangular Ward identity can be regularized and renormalized by minimal subtractions without violating Bose permutation symmetry, the $T_2$ term of the renormalized $AA\phi_2$ amplitude can be expressed as a linear combination of the $T_1$ term of the renormalized $AAA$ amplitude. Knowing that $T_1 [AAA]$ vanishes on the sole account of permutation symmetry, $T_2 [AA\phi_2]$ must vanish as well. This condition has been verified by direct calculation [28] without using dimensional regularization.

4.8 Two-Loop $N$-Point Diagrams with $N > 3$

A $N$-point Ward identity can be generated by attaching the composite vertex $\otimes$ to a $(N - 1)$-point generator diagram. As before, if only proper and legitimate cut points are used, the corresponding $N$-point Ward identity can be regularized without anomaly. Unlike the absence of a proper and legitimate cut for the third generator diagram in (58) with two overlapping vertex corrections, it is always possible to locate a proper and legitimate cut point on the fermion loop for a 2-loop $(N - 1)$-point generator when $N > 3$. For this reason, the corresponding regularized $N$-point Ward identity can always be satisfied by component diagrams obtained with proper and legitimate cut points. Furthermore, the amplitude for the Levi-Civita tensor terms of a $N$-point 1PI function with $N > 3$ is superficially convergent and according to Theorem 1 its renormalized amplitude can be calculated with any convenient choice of proper cut point.
5 Standard Model

The gauge group for the standard model \([29]\) is \(SU(3) \times SU(2) \times U(1)\) with three kinds of vector gauge bosons: \(G_\mu^a, a = 1, 2, ..8\) for \(SU(3)\); \(W_\mu^a, a = 1, 2, 3\) for \(SU(2)\); and \(B_\mu\) for \(U(1)\). Let \(S^a, a = 1, 2, ..8\) and \(T^a, a = 1, 2, 3\) be the traceless and Hermitian generators for \(SU(3)\) and \(SU(2)\) in the adjoint representation. They are normalized as

\[
\text{Tr}(S^a S^b) = \frac{1}{2} \delta^{ab}, \quad \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}\]

and the commutators are

\[
[S^a, S^b] = i f^{abc} T^c, \quad [T^a, T^b] = i \epsilon^{abc} T^c
\]

We choose \(T^a = \frac{\sigma_a}{2}\) as the \(SU(2)\) generator with \(\sigma_a\) being the Pauli matrix. Define the matrix fields

\[
G^\mu = \sum_{a=1}^{8} G_\mu^a S^a, \quad W^\mu = \sum_{a=1}^{3} W_\mu^a T^a
\]

and the covariant derivatives

\[
D_S^\mu = \partial^\mu + ig_S G^\mu, \quad D_W^\mu = \partial^\mu + ig_W W^\mu, \quad D_B^\mu = \partial^\mu + ig_B B^\mu,
\]

for \(SU(3), SU(2)\) and \(U(1)\) with coupling constants \(g_S, g_W\) and \(g_B\) respectively. Let

\[
G^{\mu\nu} = \frac{1}{i g_S} [D_S^\mu, D_S^\nu] = \partial^\mu G^\nu - \partial^\nu G^\mu + i g_S [G^\mu, G^\nu],
\]

\[
W^{\mu\nu} = \frac{1}{i g_W} [D_W^\mu, D_W^\nu] = \partial^\mu W^\nu - \partial^\nu W^\mu + i g_W [W^\mu, W^\nu],
\]

and

\[
B^{\mu\nu} = \frac{1}{i g_B} [D_B^\mu, D_B^\nu] = \partial^\mu B^\nu - \partial^\nu B^\mu.
\]

The Lagrangian for the standard model without including matter fields is

\[
L_1 = -\frac{1}{2} Tr(G_\mu G^{\mu\nu}) - \frac{1}{2} Tr(W_\mu W^{\mu\nu}) - \frac{1}{4} B_\mu B^{\mu\nu} - \frac{1}{8} g^2 \left( \phi^\dagger \phi - \frac{v^2}{2} \right)^2
\]

\[
+ (D_{H\mu}^\dagger D_{H\mu} \phi) - \frac{\lambda}{8} g^2 \left( \phi^\dagger \phi - \frac{v^2}{2} \right)^2
\]

\[
+ \text{(terms involving fermions)}
\]

where \(v\) is the vacuum expectation value of the Higgs field. The above Lagrangian includes the kinetic terms for the gauge fields and the kinetic term for the Higgs field, as well as the Higgs potential term.
where the Higgs $\phi$ is a two component complex scalar field coupled to $W$ and $B$ gauge bosons with

$$D_H^\mu \phi = (\partial^\mu + ig_W W^\mu - i g_B B^\mu) \phi$$

$\phi$ is assumed to have the vacuum expectation value:

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ v \end{bmatrix}$$

Express $\phi$ in terms of four real components $H$ and $\phi_a$, $a = 1, 2, 3$:

$$\phi = \frac{1}{\sqrt{2}} \begin{bmatrix} i\phi_1 + \phi_2 \\ H + v - i\phi_3 \end{bmatrix} = \frac{1}{\sqrt{2}} (H + v + i\phi_a \sigma^a) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hat{\phi} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(71)

where $\hat{\phi}$ is defined as

$$\hat{\phi} = \frac{1}{\sqrt{2}} (H + v + i\phi_a \sigma^a).$$

Note that

$$\hat{\phi} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} H + v + i\phi_3 \\ i\phi_1 - \phi_2 \end{bmatrix} = i\sigma_2 \left( \frac{1}{\sqrt{2}} \begin{bmatrix} i\phi_1 + \phi_2 \\ H + v - i\phi_3 \end{bmatrix} \right)^* = i\sigma_2 \hat{\phi}^* \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Under a $SU(2) \times U(1)$ transformation

$$\hat{\phi} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow e^{-ig_W \theta_a T^a} e^{ig_B \chi} \hat{\phi} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(72)

and, since $(i\sigma_2) \bar{\sigma}^* = -\bar{\sigma} (i\sigma_2)$,

$$\hat{\phi} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow i\sigma_2 e^{ig_W \theta_a T^a} e^{-ig_B \chi} \hat{\phi}^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{-ig_W \theta_a T^a} e^{-ig_B \chi} \hat{\phi} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(73)

The Lagrangian (70) is invariant under the following BRS variations with Grassmann ghost fields $\xi_S = \sum_{a=1}^8 \xi^a_S S^a$, $\xi_W = \sum_{a=1}^3 \xi^a W^a$, $\xi_B$ as the parameters for the $SU(3)$, $SU(2)$, $U(1)$ groups:

$$\delta G^\mu = [D^\mu_S, \xi_S], \delta W^\mu = [D^\mu_W, \xi_W], \delta B^\mu = \partial_\mu \xi_B, \delta \phi = -i (g_W \xi_W - g_B \xi_B) \phi$$

(74)
The gauge fixing and corresponding ghost terms \[27\] in the pure alpha gauge are

\[
L_{gf} = -\frac{1}{\alpha_S} Tr \left( \partial_\mu G^\mu \right)^2 - \frac{1}{\alpha_W} Tr \left( \partial_\mu W^\mu \right)^2 - \frac{1}{2\alpha_B} \left( \partial_\mu B^\mu \right)^2 
+ 2 Tr \left( i\eta_S \delta (\partial_\mu G^\mu) \right) + 2 Tr \left( i\eta_W \delta (\partial_\mu W^\mu) \right) + i\eta_B \delta (\partial_\mu B^\mu)
\]

(75)

where \(\eta_S, \eta_W, \eta_B\) are the anti-ghosts corresponding to \(\xi_S, \xi_W, \xi_B\) and the BRS variations for ghost and anti-ghost fields are

\[
\delta \xi^a_S = \frac{g_S}{2} f^{abc} \xi^b_S \xi^c_S, \delta \xi^a_W = \frac{g_W}{2} \epsilon^{abc} \xi^b_W \xi^c_W, \delta \xi_B = 0,
\]

\[
\delta \eta_S = -\frac{i}{\alpha_S} \partial_\mu G^\mu, \delta \eta_W = -\frac{i}{\alpha_W} \partial_\mu W^\mu, \delta \eta_B = -\frac{i}{\alpha_B} \partial_\mu B^\mu.
\]

There are three generations of fermion matter fields consisting of quarks

\[
\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}
\]

and leptons

\[
\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}
\]

Note for simplicity, we have suppressed the color indices of quarks. Note also we have assumed the existence of right-handed neutrinos. In this formalism, right-handed neutrinos are free fields if they are massless. We will use the notation \(\psi^i = \begin{pmatrix} \psi^i_u \\ \psi^i_d \end{pmatrix}\) indexed by \(i\) to denote one of the above fermion fields.

The \(G^\mu\) gluons couple only to the quark fields with equal strength for left-handed and right-handed quarks. \(W\) and \(B\) gauge bosons couple to both left-handed quarks and left-handed leptons. The right-handed fermion \(\bar{\psi}^i\) is a \(SU(2)\) singlet and thus is not coupled to \(W\). The covariant derivative for a left-handed quark is

\[
D^\mu_{g,L} \psi^i = (\partial^\mu + ig_S G^\mu + ig_W W^\mu - iY_i g_B B^\mu) \psi^i
\]

(76)

and that for a left-handed lepton is

\[
D^\mu_{l,L} \psi_i = (\partial^\mu + ig_W W^\mu - iY_i g_B B^\mu) \psi_i
\]

(77)

where \(Y_i\) is the weak hypercharge. The weak hypercharge for the right-handed fermions must also be \((Y_i + \sigma_3) g_B\) so that the electric charges for
the left-handed and right-handed fermions are the same [30, 31]. Thus, the covariant derivative for a right-handed quark is

$$D_{q,R}^\mu R\psi_i = (\partial^\mu + ig_S G^\mu - i (Y_i + \sigma_3) g_B B^\mu) R\psi_i$$  \hspace{1cm} (78)$$

and that for a right-handed lepton is

$$D_{l,R}^\mu R\psi_i = (\partial^\mu - i (Y_i + \sigma_3) g_B B^\mu) R\psi_i.$$  \hspace{1cm} (79)$$

It is known that $Y_i = -1$ for all leptons and $Y_i = \frac{1}{3}$ for all quarks. If $Y_i$ is summed over the fields of lepton and quarks with 3 different colors in each fermion generation, we get

$$\sum_i Y_i = -1 + 3 \times \left(\frac{1}{3}\right) = 0 \hspace{1cm} (80)$$

This vanishing result plays a key role, as will be illustrated below, in the cancellation of anomaly for divergent one-fermion-loop diagrams.

The transformations (72) and (73) for $\hat{\phi}$ can be utilized to show that the following four types of Yukawa terms

$$\bar{\psi}_i^d \left( \hat{\phi} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)^\dagger L\psi_j, \bar{\psi}_i^u \hat{\phi} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^\dagger R\psi_j, \bar{\psi}_i^d \hat{\phi} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^\dagger L\psi_j, \bar{\psi}_i^u \hat{\phi} \begin{bmatrix} 1 \\ 0 \end{bmatrix} R\psi_j$$

are gauge invariant provided the $\psi_i$ and $\psi_j$ fields in the above have the same weak hypercharge. The Yukawa interaction for quarks can be written as

$$L_{YQ} = -\sum_{\text{quarks } (i,j)} \sqrt{2} \left( \bar{\psi}_i \hat{f}_{ij} \hat{\phi}^\dagger L\psi_j + \bar{\psi}_i \hat{\phi} \hat{f}_{ji}^* R\psi_j \right) \hspace{1cm} (81)$$

where the summation is over the three different flavors of quark fields for both $\psi_i$ and $\psi_j$ and

$$\hat{f}_{ij} = \begin{bmatrix} f^u_{ij} & 0 \\ 0 & f^d_{ij} \end{bmatrix}.$$  \hspace{1cm} (82)$$

is a $2 \times 2$ diagonal matrix. If we combine the three flavors of quark fields $\psi_i$ with $i = 1, 2, 3$ into a large field $\Psi_q$ such that the $ith$ component of $\Psi_q$ is $\psi_i$, then $L_{YQ}$ in (81) may be written as

$$L_{YQ} = -\sqrt{2} \left( \bar{\Psi}_q \hat{f}_q \hat{\phi}^\dagger L\Psi_q + \bar{\Psi}_q \hat{\phi} \hat{f}_q^* R\Psi_q \right) \hspace{1cm} (83)$$
with
\[ \hat{f}_q = \begin{bmatrix} \hat{f}_q^u & 0 \\ 0 & \hat{f}_q^d \end{bmatrix} \]

where \( \hat{f}_q^u \) and \( \hat{f}_q^d \) are \( 3 \times 3 \) matrices whose \( i,j \) elements equal to \( f_{ij}^u \) and \( f_{ij}^d \) in (82), respectively.

The Yukawa interaction for leptons does not have terms with mixed generations and is equal to
\[ L_{Y_L} = -\sum_{\text{leptons } (i)} \sqrt{2} \left( \tilde{\psi}_i \hat{f}_i \phi^+ L \psi_i + \tilde{\psi}_i \phi \hat{f}_i R \psi_i \right) \] (84)

where the matrix
\[ \hat{f}_i = \begin{bmatrix} f_i^u & 0 \\ 0 & f_i^d \end{bmatrix} \] (85)
is real and diagonal. The gauge invariant Lagrangian for the fermion fields is
\[ L_F = \bar{\Psi}_q (i \not\partial - \hat{m}_q L - \hat{m}_q^+ R) \Psi_q + \sum_{\text{leptons } (i)} \bar{\psi}_i (i \not\partial - \hat{m}_i) \psi_i \] (86)

which yields the free Lagrangian:
\[ L_{F}^{(0)} = \bar{\Psi}_q (i \not\partial - \hat{m}_q L - \hat{m}_q^+ R) \Psi_q + \sum_{\text{leptons } (i)} \bar{\psi}_i (i \not\partial - \hat{m}_i) \psi_i \] (87)

In the above, the mass matrices for leptons and quarks are
\[ \hat{m}_i = \begin{bmatrix} m_i^u & 0 \\ 0 & m_i^d \end{bmatrix} = v \hat{f}_i, \] (88)
and
\[ \hat{m}_q = \begin{bmatrix} m_q^u & 0 \\ 0 & m_q^d \end{bmatrix} = v \hat{f}_q \] (89)
The lepton masses are \( m_i^u = v f_i^u \) and \( m_i^d = v f_i^d \) for \( \psi_i^u \) and \( \psi_i^d \) respectively. Note if \( m_i^u = 0 \) for neutrino field \( \psi_i^u \) with \( Y_i = -1 \), then the corresponding right-handed component \( R \psi_i^u \) only occurs in the free Lagrangian (87) and does not interact with any other field components.
5.1 Charge Conjugation in the Standard Model

The Lagrangian (70) is invariant under the transformation

\[ G^\mu \to - (G^\mu)^*, \xi_S \to - (\xi_S)^*, \eta_S \to - (\eta_S)^* \]
\[ W^\mu \to - (W^\mu)^*, \xi_W \to - (\xi_W)^*, \eta_W \to - (\eta_W)^* \]
\[ B^\mu \to - B^\mu, \xi_B \to - \xi_B, \eta_B \to - \eta_B \]
\[ \phi \to \phi^* \text{ or } \hat{\phi} \to \hat{\phi}^* \]

Since \( \sigma_2 \) is an imaginary matrix while both \( \sigma_1 \) and \( \sigma_3 \) are real. The fields

\[ W_2^\mu, \xi_W^2, \eta_W^2, H, \phi_2 \]

are invariant under the charge conjugation transformation and are \( C \)-even. The remaining fields

\[ B^\mu, \xi_B, \eta_B, W_1^\mu, \xi_W^1, \eta_W^1, \phi_1, W_3^\mu, \xi_W^3, \eta_W^3, \phi_3 \]

are \( C \)-odd. Theorem 2 and Corollaries 3\textendash}4 in Section 4.1 can be straightforwardly extended and proved for the standard model. As in the Abelian gauge theory, only \( C \)-odd amplitudes with fermion loops may contribute to the violation of Ward identities in the standard model.

5.2 One-Loop Order in the Standard Model

There are many kinds of fermion fields in the standard model. We shall require that all the cut points are synchronized at the same position for the fermion loops that differ only in the types of fermion fields but are otherwise equivalent. At 1-loop order, only the amplitude for a \( C \)-odd fermion loop attached by three or four external vector fields may have logarithmically divergent Levi-Civita tensor terms. For such a fermion loop in the standard model, the amplitude for the difference stemming from changing the mass matrix \( \hat{m} \), which may be either \( \hat{m}_i \) or \( \hat{m}_q \) in (87), of a fermion propagator in the loop to another value, say \( m_0 \), is convergent because terms in the difference are proportional to \( (\hat{m} - m_0) \) which lowers the degree of divergence and ensures that the accompanying integrals are convergent. Thus, to calculate the part of amplitude that is divergent for a fermion loop, we may assume that all the fermion propagators in the loop have a uniform mass \( m_0 \). The fermion propagators then become diagonal and the divergent part of the
amplitude for the fermion loop is proportional to the product of the vertex isospin factors on the loop.

Since the generators $S^a$, $a = 1, 2, \ldots, 8$ for $SU(3)$ and $T^a$, $a = 1, 2, 3$ for $SU(2)$ are all traceless, the amplitude for the fermion loop with only one $G$ or one $W$ among the attached vector fields is convergent. If there are two $G$ or two $W$ attached to the fermion loop, the isospin indices for both $G$ or both $W$, according to (69), must be the same to prevent the trace of the isospin factors from vanishing. Furthermore, since the gluon’s coupling to the left-handed quarks is the same as that to the right-handed quarks, there is no $\gamma_5$ in the vertex factor of $\bar{\psi} - G - \psi$ and it is not possible for the amplitude of the fermion loop to have Levi-Civita tensor terms when all the external lines are $G$ field lines. For the $C$-odd functions with at least one external $G$ fields, we are left with the possibility that only $GGGB$ and $GGB$ may be divergent. According to (76)-(79), the vertex factor for $\bar{\psi} - B^\mu - \psi$ may be expressed as $i\gamma^\mu (Y_i + \sigma_3 R) g_B$. Hence in $GGGB$ or $GGB$, there is only one $\gamma_5$ which appears together with the $\sigma_3$ matrix in the vertex factor of $\bar{\psi} - B - \psi$. Since $\sigma_3$ is traceless, both $GGGB$ and $GGB$ are in fact convergent.

We have shown that in order to have a divergent amplitude for a $C$-odd fermion loop, there must not be any external $G$ vector field. $WWWW$ and $WWW$ are two such functions that are also void of external $B$ field. To have a non-zero trace for the $SU(2)$ isospin factors, the four Pauli matrices associated with the four vertices for the $WWWW$ function must be paired into two pairs and the product of the three Pauli matrices for the $WWW$ function must be proportional to $\sigma_1 \sigma_2 \sigma_3$. Consequently, both $WWWW$ and $WWW$ are $C$-even and are free of Levi-Civita tensor terms. To have a divergent amplitude of Levi-Civita tensor terms, there must be at least one external $B$ field attached to the fermion loop. We are left with the following $C$-odd functions that may be divergent:

$$WWWB, WWB \text{ and } BBB.$$  

Note $WBB$ is excluded from the above because the Pauli matrix associated with the vertex isospin factor for $W$ is traceless. Since $W$ is not coupled to the right-handed fermions, only the $g_B Y_i$ coupling of left-handed fermion to $B$ in (76) or (77) may contribute to the divergence of $WWWB$ or $WWB$. But if we sum over the $Y_i$ for all lepton and quark loops, the result vanishes according to (80). For the $BBB$ function, as can be seen from (76)-(79), the left-handed fermion loop contributes a factor $Tr (g_B Y_i)^3$ while the right-handed fermion contributes $-Tr (g_B (Y_i + \sigma_3))^3$ (the negative sign is due to
the different signs of $\gamma_5$ in $L$ and $R$). The sum of these two factor is proportional to $Y_i$.

$$Tr (g_B Y_i)^3 - Tr (g_B (Y_i + \sigma_3))^3 = -Tr (3g_B Y_i)$$

Again the vanishing sum of (80) indicates that the divergent part of $BBB$ vanishes if contributions from all loops of leptons and quarks are added together.

We have thus shown that the amplitudes for all 1-loop diagrams are well defined regardless how we position the cut points provided that we have synchronized the cut-point positions for all the fermion loops with identical external fields. Furthermore, for a triangular fermion loop attached by three vector fields, the amplitude after the cancellation of divergent terms either vanishes or contains a $(\hat{m} - m_0)$ factor. If such a fermion loop is embedded as a sub-diagram in a larger diagram, this factor of $(\hat{m} - m_0)$ also effectively lowers the degree of overall divergence for the entire diagram to a value less than the one indicated by naive power counting.

5.3 Two-Loop Order in the Standard Model

As in the Abelian-Higgs theory, the amplitude for the 2-loop fermion self-energy of (55) is unambiguously defined because the fermion-loop sub-diagram in (55) does not have sufficient indices to give a Levi-Civita tensor term. For the 2-loop vertex correction (56), only the $T_0$ term may be ambiguous. Unless the three vertices on the fermion-loop sub-diagram in (56) are all connected to vector fields, no Levi-Civita tensor term is possible due to the insufficiency of indices when one of the three incoming momenta for the fermion loop is zero. We have shown in the preceding section that the contribution of the Levi-Civita tensor terms from a triangular fermion-loop sub-diagram attached by three vector fields to the entire diagram of (56) is actually convergent, as opposed to being logarithmically divergent by naive power counting, and is unambiguously defined. Thus there is no 2-loop anomaly for any one-fermion-loop diagram with external fermion lines. At 2-loop order, one-fermion-loop diagrams without external fermion lines are the ones that may invalidate the Ward identities.

There are three kinds, corresponding to the three gauge groups $SU (3)$, $SU (2)$ and $U (1)$, of composite vertices that may be attached to fermion lines in the standard model instead of only one kind defined in (33) for the
Abelian-Higgs theory. The vertex factors for these three composite vertices are defined in the following:

\[ \xi^a_G \otimes = -g_S S^a k, \ a = 1, 2, ..., 8 \]  
(90)

\[ \xi^a_W \otimes = -\frac{g_W}{2} (\sigma_a k L - \hat{m}\sigma_a L - \sigma_a \hat{m} R), \ a = 1, 2, 3 \]  
(91)

\[ \xi^a_B \otimes = g_B (\sigma \dot{Y}_i + \sigma_3 R) + \sigma_3 \hat{m} L - \hat{m} \sigma_3 R \]  
(92)

With the above definitions, the graphical identity (36) for the Abelian-Higgs theory can be generalized with any of the above vertices in (90)-(92) provided that the combination of \( \otimes \) with the arrowed double line is defined as in the following lookup table:

\[ \xi^a_G \rightarrow = i g_S S_a \]  
\[ \xi^a_G \otimes = i g_S S_a \]

\[ \xi^a_W \rightarrow = -i g_W \sigma a L \]  
\[ \xi^a_W \otimes = i g_W \sigma a R \]

\[ \xi^a_B \rightarrow = i g_B (\sigma \dot{Y}_i + \sigma_3 R) \]  
\[ \xi^a_B \otimes = -i g_B (\sigma \dot{Y}_i + \sigma_3 L) \]

As in the Abelian-Higgs theory, we construct the component diagrams for a Ward identity by attaching \( \otimes \) in all possible manners consistent with Feynman rules to a cut generator diagram. For a 2-loop generator with three or more external lines, a cut point that is proper and legitimate is always available to generate component diagrams whose cut points are all proper and legitimate.

Only a generator, such as the third diagram in (58), with two overlapping vertex corrections may generate a 2-loop triangular Ward identity containing both proper and improper component diagrams. For an improper component diagram, moving the cut point to a proper position will result in a difference that is composed of terms with \( \gamma_\Delta \) matrix factors. In this difference, the mass matrix for every composite vertex and for every fermion propagator may be changed to a uniform value \( m_0 \) so that all vertex isospin matrix factors may be moved and placed together. The verification that the difference between
and the sum of the last two diagrams in (64) is \( O(n-4) \) for the Abelian-Higgs theory in Section 4.7. can now be straightforwardly carried over to the standard model, without being hindered by the isospin factors, to show that neither the sub-divergence of the 1-loop vertex correction nor the overall divergence is capable of resurrecting the \( \gamma_\Delta \) terms in the limit \( n \to 4 \).

Finally, let us note that if the composite vertex \( \otimes \) is attached to a vector field line in a generator, a term involving ghost-vector vertex may emerge. For example, the \( W^\mu_a W^\nu_b W^\rho_c \) vertex factor times \( k_\mu \), the incoming momentum for \( W^\mu_a \), is proportional to

\[
\epsilon_{abc} \left[ (g^{\nu\rho} p^2 - p^{\nu} p^{\rho}) - (g^{\nu\rho} q^2 - q^{\nu} q^{\rho}) \right]
\]

where \( p, q \) are the incoming momenta for \( W^\nu_b, W^\rho_c \). The terms proportional to \( p^{\nu} p^{\rho} \) and \( q^{\nu} q^{\rho} \) are related to the ghost-vector vertex and may yield a \( \otimes \) vertex connecting to the fermion line whose endpoint was originally chosen as the cut point. The \( \otimes \) vertex for either diagram in (49) is adjacent to such a cut point that gives rise to a leftover term involving \( f_\Delta \), as indicated by the last term of (51), where \( \ell \) is the fermion momentum entering the \( \otimes \) vertex. We have shown that such leftover terms invalidate the 1-loop triangular Ward identity for the Abelian-Higgs theory of (16). But for the theory of (54) with an additional fermion \( \psi' \), we have also shown that the leftover terms due to the two fermion loops of \( \psi \) and \( \psi' \) cancel out. For the standard model, the sum of the 2-loop leftover terms from all lepton and quark loops, as will be demonstrated in Appendix B, also vanishes in the \( n \to 4 \) limit.

To summarize, we have shown that by adopting legitimate and proper cut points, all the 2-loop Ward identities in the standard model remain valid under the rightmost \( \gamma_5 \) dimensional regularization scheme.

6 Conclusion

In this paper, we have found a simple and natural way to treat \( \gamma_5 \) in dimensional regularization: moving all \( \gamma_5 \) matrices to the rightmost position before analytically continuing the dimension. For amplitudes corresponding diagrams without fermion loops, the amplitudes obtained with our prescription automatically satisfy the Ward identities without further ado.

The rightmost position on a fermion loop is not defined. For this reason, we introduce the concept of a cut point. We have found that the choice of a cut point often conflicts with gauge invariance. From this vantage point,
this lack of a rightmost position is what breaks the Ward identities, leading
to triangular anomalies.

Applying our prescriptions to 1-loop triangular amplitudes, we reproduce
correctly the value of the triangular anomaly, verifying that our prescription
is applicable to diagrams with anomalies. For the Abelian-Higgs theory \( (54) \)
with an anomaly compensating fermion field or for the standard model, sum
of 1-loop anomalies vanishes. For these two theories, the Levi-Civita tensor
terms for the amplitude of a one-fermion-loop diagram is convergent and cut
point independent as the limit of \( n \to 4 \) is taken, provided that all the cut
points are synchronized at the same position.

For a 1-loop fermion self-energy diagram or a 1-loop vertex correction
diagram, positioning \( \gamma_5 \) within the divergent 1PI diagram gives an amplitude
differing from the amplitude obtained with rightmost \( \gamma_5 \) by a finite amount,
even after subtraction of pole terms. Thus for a 2-loop diagram with a
fermion loop and with a 1-loop self-energy insertion or a 1-loop radiative
vertex insertion, we do not assign a point inside a divergent 1-loop sub-
diagram as a cut point. Furthermore, in order not to give a preferential
role to any of the external lines, we do not choose the point of the vertex
connecting to an external field line as a cut point. We have shown that
this prescription of utilizing proper and legitimate cut points enables us to
regulate all amplitudes in a gauge invariant manner, and the use of minimal
subtractions then gives renormalized amplitudes for all diagrams up to two
loops. These renormalized amplitudes in the Abelian theory of \( (54) \) and in
the standard model satisfy Ward identities or gauge symmetry.

In the BM scheme, simply removing the pole terms from the amplitudes
of 1-loop diagrams does not yield renormalized amplitudes that satisfy Ward
identities. Instead, some finite renormalization terms have to be added.
These finite counter terms are determined from restoring the validities of
1-loop Ward identities. For the standard model, implementing this finite
renormalization in practical calculation is already a daunting task at 1-loop
order. To make the matter worse, restoring the validities of 2-loop Ward
identities in the BM scheme require another round of finite renormalization,
which is much more complicated than the first round at 1-loop order.

In contrast, we are spared the tedious finite renormalization procedures
for 1-loop or 2-loop amplitudes obtained with our rightmost \( \gamma_5 \) scheme. Fur-
thermore, since all the \( \gamma_5 \) matrices are moved to and consolidated at a single
position before continuing the dimension in our scheme, the burden of evalu-
ating the matrix products or trace of matrix products is considerably less
than that in the BM scheme. In our opinion, this rightmost $\gamma_5$ prescription is the simplest scheme available for calculating amplitudes in gauge theories involving $\gamma_5$.

Appendices

A Green Functions and Ward Identities

In the main context, we only consider Feynman diagrams in which the non-fermion internal lines are the vector meson lines. To handle other types of diagrams, we will make use of Green functions.

The Green function $G$ is the vacuum expectation value of a time-ordered product. Specifically,

$$G(O_1(x_1), O_2(x_2), \ldots O_n(x_n)) = T \langle O_1(x_1) O_2(x_2) \ldots O_n(x_n) \rangle, \quad (93)$$

where the operator $O_i(x_i)$ is either a field operator or a product of field operators at the same space-time point $x_i$. The connected Green function, denoted by $G_c$, is

$$G_c(...) = \text{all connected diagrams of } G(...)$$

We need a notation to indicate that some external lines of a Green function are amputated. To denote a truncated external line, we underline the corresponding field variable in the Green function. i.e.,

$$G(..., \varphi_i, ...) = D(\varphi_i, \varphi_j) G\left(..., \underline{\varphi_j}, ...ight) \quad (94)$$

$$G_c(..., \varphi_i, ...) = D(\varphi_i, \varphi_j) G_c\left(..., \underline{\varphi_j}, ...ight),$$

where the propagator $D(\varphi_i, \varphi_j)$ is also the two-point Green function,

$$D(\varphi_i, \varphi_j) = G(\varphi_i, \varphi_j).$$

Note that in (94) the space-time dependence of the field variable $\varphi_i$ is lumped into the index $i$ and the Einstein summation convention for the repeated index $j$ is extended to include summation over all possible field types and
integration of space-time points. The fully truncated Green function $\Gamma$ is the connected Green function with all field variables underlined.

$$\Gamma (\varphi_1, \varphi_2, ..., \varphi_n) = G_c (\varphi_1, \varphi_2, ..., \varphi_n)$$

In particular, $\Gamma (\varphi_i, \varphi_j)$ is the inverse propagator.

$$\Gamma (\varphi_i, \varphi_j) = G (\varphi_i, \varphi_j) = D^{-1} (\varphi_j, \varphi_i)$$

For a composite operator $\hat{O}$, which is a product of field operators at the same space-time point, we define

$$\Gamma (\varphi_1, \varphi_2, ..., \varphi_n, \hat{O}) = G_c (\varphi_1, \varphi_2, ..., \varphi_n, \hat{O}).$$

Note that to avoid misinterpretations, $\hat{O}$ is forbidden to be a single field operator in the above identification. The tree order part of a Green function $\hat{\phi}$, which may be any of the above $G$, $G_c$ or $\Gamma$ function, will be denoted by the notation $\hat{\phi}^{(0)}$ with the superscript $(0)$. The Fourier transform of a Green function $\hat{\phi}$ is labeled by an additional group of momentum variables and is related to its counterpart in the coordinate space by

$$\hat{\phi} (\varphi_1 (x_1), \varphi_2 (x_2), ..., \varphi_n (x_n))$$

$$= \int \frac{dk_1}{(2\pi)^4} \frac{dk_2}{(2\pi)^4} \cdots \frac{dk_{n-1}}{(2\pi)^4} e^{-i(k_1 x_1 + k_2 x_2 + ... + k_n x_n)} \hat{\phi} (\varphi_1, \varphi_2, ..., \varphi_n; k_1, k_2, ..., k_n),$$

where $k_1 + k_2 + ... + k_n = 0$. We will omit the momentum variables $k_1, k_2, ..., k_n$ for the Fourier transform if there is little chance of confusion.

### A.1 Basic Graphical Identities

The BRS invariance leads to a number of Ward identities which form an important part of the foundation on which renormalizability is based. These identities can be formally derived in the following way. The vacuum state $|0 >$ in the theory satisfies

$$Q |0 >= 0$$

where $Q$ is the BRS charge. The commutator (anticommutator) of $iQ$ with a non-ghost (ghost) field is equal to the BRS variation of the field. Because of (95), we have

$$T \langle 0 | iQ \varphi_1 (x_1) \varphi_2 (x_2) ... | 0 > = 0$$
where $\varphi_i$ is a field operator. By moving $iQ$ to the right until it operates on $|0\rangle$ and vanishes, we get

$$T \langle 0 | \delta (\varphi_1 (x_1) \varphi_2 (x_2) ... ) | 0 \rangle = T \langle 0 | \delta \varphi_1 (x_1) \varphi_2 (x_2) ... | 0 \rangle \pm T \langle 0 | \varphi_1 (x_1) \delta \varphi_2 (x_2) ... | 0 \rangle \pm ... = 0 \quad (96)$$

The relative sign between terms is determined by the positions of the ghost fields. The above BRST identity is formal and its renormalized version may not be satisfied when anomaly exists. But the tree order terms are finite and always satisfy the BRST identity provided the Lagrangian is BRS invariant. To facilitate the discussions for higher loop order terms, we will introduce graphical notations for some basic tree order identities. For simplicity’s sake, let us start with the Abelian-Higgs theory with the Lagrangian (16).

The BRST variation for any field variable $\varphi (x)$ in (13) may be decomposed as

$$\delta \varphi (x) = \delta_1 \varphi (x) + \delta_2 \varphi (x), \quad (97)$$

in which $\delta_1 \varphi (x)$ is a linear superposition of field variables and $\delta_2 \varphi (x)$ is a product of the ghost field $\xi (x)$ and another field variable at the same space-time point $x$. For the Abelian-Higgs theory, non-vanishing $\delta_1 \varphi$ are $\delta_1 A^\mu = \partial^\mu \xi$ and $\delta_1 \phi_2 = -M \xi$, and non-vanishing $\delta_2 \varphi$ are $\delta_2 H = g \xi \phi_2$, $\delta_2 \phi_2 = -g \xi H$ and $\delta_2 \psi_L = -ig \xi \psi_L$.

By (96), we have

$$T \langle 0 | (\delta \eta (z)) \varphi_i | 0 \rangle_{(0)} = T \langle 0 | \eta (z) \delta \varphi_i | 0 \rangle_{(0)} = \frac{\partial \delta \varphi_i}{\partial \xi (z')} D^{(0)} (\eta (z), \xi (z')) \quad (98)$$

where the subscript and superscript (0) refer to tree order terms and $\frac{\partial \delta \varphi_i}{\partial \xi}$ is a constant or constant operator. Next, let us assume that $\varphi_j$ and $\varphi_k$ are non-ghost fields. Then (96) yields

$$T \langle 0 | (\delta \eta (z)) \varphi_j \varphi_k | 0 \rangle_{(0)} = T \langle 0 | \eta (z) (\delta \varphi_j) \varphi_k | 0 \rangle_{(0)} + T \langle 0 | \eta (z) \varphi_j (\delta \varphi_k) | 0 \rangle_{(0)} \quad (99)$$

According to the definition (94) for the Green function with underlined arguments, the left side of (99) may be expressed as

$$T \langle 0 | (\delta \eta (z)) \varphi_j \varphi_k | 0 \rangle_{(0)} = D^{(0)} (\delta \eta (z), \varphi_i) G^{(0)} (\varphi_i, \varphi_j, \varphi_k)$$

$$= D^{(0)} (\eta (z), \xi (z')) \frac{\partial \delta \varphi_i}{\partial \xi (z')} G^{(0)} (\varphi_i, \varphi_j, \varphi_k)$$
In the tree order, the \( \eta(z) \) field in \( T \langle 0 | \eta(z) (\delta \varphi_j) \varphi_k | 0 \rangle_{(0)} \), which is the first term on the right side of (99), must be paired under Wick contraction with the \( \xi \) field in \( \delta \varphi_j \), otherwise it has to be paired with the ghost field from the interaction Lagrangian. For the Abelian theory of (16), the ghost field is free and we have

\[
T \langle 0 | \eta(z) (\delta \varphi_j) \varphi_k | 0 \rangle_{(0)} = D^{(0)}(\eta(z), \xi(z')) D^{(0)} \left( \frac{\partial \delta_2 \varphi_j}{\partial \xi(z')}, \varphi_k \right) \tag{100}
\]

Note that we have discarded \( D^{(0)} \left( \frac{\partial \delta_1 \varphi_i}{\partial \xi}, \varphi_k \right) \) owing to the vanishing vacuum expectation \( \langle \varphi_k \rangle = 0 \). For theories, such as the standard model, in which ghost fields are not free, terms with \( \eta(z) \) paired to the \( \xi \) field in the interaction Lagrangian may not be neglected and will be discussed below in Section B.

For the 2nd term on the right side of (99), there is an expression similar to (100). The identity (99), after factoring out the common ghost propagator \( D^{(0)}(\eta(z), \xi(z')) \) and then replacing \( z' \) by \( z \), becomes

\[
\frac{\partial \delta_1 \varphi_i}{\partial \xi(z)} G^{(0)}(\varphi_i, \varphi_j, \varphi_k) = D^{(0)} \left( \frac{\partial \delta_2 \varphi_j}{\partial \xi(z)}, \varphi_k \right) + D^{(0)} \left( \varphi(x), \frac{\partial \delta_2 \varphi_k}{\partial \xi(z)} \right) \tag{101}
\]

The definition (33) for the composite vertex \( \otimes \) on a fermion line may be extended to include other types of vertices. The extended composite vertex is defined as

\[
\otimes = \frac{\partial \delta_1 \varphi_i}{\partial \xi} \Gamma^{(0)}(\varphi_i, \varphi, \varphi') \tag{102}
\]

where the tree order amplitude \( \Gamma^{(0)}(\varphi_i, \varphi, \varphi') \) stands for the vertex factor of \( \varphi_i - \varphi - \varphi' \) and \( k \) is the incoming momentum of the vector field \( A^\mu \) or scalar field \( \phi_2 \). Note that this definition is the same as the restricted one in (33) when \( \varphi \) and \( \varphi' \) are the fermion fields \( \psi \) and \( \bar{\psi} \). The amplitude \( \frac{\partial \delta_1 \varphi_i}{\partial \xi} G^{(0)}(\varphi_i, \varphi_j, \varphi_k) \) can then be diagrammatically expressed as a composite vertex \( \otimes \) connected with two propagator lines to fields \( \varphi_j \) and \( \varphi_k \):
factoring out the $\xi$ field to retain the non-ghost factor. $D^{(0)}\left(\frac{\partial \delta_2 \varphi_j}{\partial \xi}, \varphi_k\right)$ is thus proportional to the free propagator that propagates the field $\varphi_k$ to the non-ghost field in $\delta_2 \varphi_j$. In particular, if $\varphi_j = \phi_2$ and $\varphi_k = H$, then $\delta_2 \phi_2 = -g\xi H$ and

$$D^{(0)}\left(\frac{\partial \delta_2 \phi_2}{\partial \xi}, H\right) = -gD^{(0)}(H, H).$$

We now graphically represent $D^{(0)}\left(\frac{\partial \delta_2 \varphi_j}{\partial \xi}, \varphi_k\right)$ by

$$\delta_2 \varphi_j \quad \varphi_k$$

where the single line stands for the free propagator from $\varphi_k$ to the non-ghost field in $\delta_2 \varphi_j$ and the arrowed double line emitting from the composite vertex is interpreted as that the original propagator connecting to field $\varphi_j$ as in (103) is annihilated and the composite vertex with the arrowed double line is to be replaced by the constant coefficient of the non-ghost field in $\frac{\partial \delta_2 \varphi_k}{\partial \xi}$.

With the graphical elements defined in (102)-(104), the identity (101) can be diagrammatically expressed as

$$\delta_2 \varphi_j \varphi_k + \varphi_j \delta_2 \varphi_k = \varphi_j \varphi_k$$

Likewise, by expanding

$$T \langle 0 | \delta (\eta(z) \varphi_i \varphi_j \varphi_k) | 0 \rangle_{(0)} = 0$$

and utilizing (105), we get the identity

$$\varphi_k \varphi_k + \varphi_k \varphi_i + \varphi_i \varphi_k + \varphi_i \varphi_i = 0$$

(106)
The above two graphic identities (105) and (106) together with the condition
\[ T \langle 0 | \delta (\eta (z) \varphi_i \varphi_j \varphi_k \varphi_l) | 0 \rangle (0) = 0 \]
can be combined to yield the identity
\[ \varphi_l \varphi_k \varphi_j \varphi_i + \varphi_l \varphi_k \varphi_i \varphi_j + \varphi_l \varphi_k \varphi_j \varphi_i + \varphi_l \varphi_k \varphi_j \varphi_i = 0 \]  
(107)

We will need graphical notations to express two amputated external fields in a four-point function. In the following figure

\[ \varphi_i \varphi_j A^\mu \]
(108)
the amputated \( A^\mu \) and \( A^\nu \) fields are represented by two crosses that are stacked together. Similarly,

\[ \varphi_i \varphi_j A^\mu \]
(109)
represents a four-point function with an amputated external \( A^\mu \) and a composite vertex \( \bigotimes \).

We are now equipped with the graphical notations and identities needed to construct component diagrams for Ward identities without the restriction on the type of internal field lines.

**A.2 General Two-Loop Ward Identities**

First, let us construct regularized 2-loop amplitudes for the triangular Ward identity. If all the vertices for external fields are detached, a 3-point 2-loop
diagram in the presence of one fermion-loop sub-diagram becomes a 2-loop super-generator diagram

\[ (110) \]

composed of a fermion loop and a non-fermion internal line. Seven topologically different generator diagrams will result from all possible attachments of the vertices for \( A^\mu \) and \( A^\nu \) consistent with Feynman rules to this super-generator and are shown in the following:

\[ (111) \]

For each diagram in the above, either of the two legitimate cut points at the two vertices connecting to the non-fermion internal line is available to yield a cut generator for a regularized Ward identity. Since the component diagrams constructed from a generator obtained with proper cutting are all proper, we will choose the cut point for each generator in \((111)\) to be legitimate and proper if such a cut point is available. For example, if the cutting is made at the endpoint of the fermion line connecting to the lowest vertex on the last diagram in \((111)\), we obtain the proper generator

\[ (112) \]

The vertex for \( A^\nu \) is attached to the fermion line and the vertex for \( A^\mu \) is attached to the arc above the fermion line. We may attach \( \times \) to the cut generator \((112)\) in all possible manners to obtain the following collection of
A component diagram is constructed when we insert $\otimes$ in consistency with Feynman rules into one of the internal lines or vertices in the generator. The momentum entering the open fermion line from the right side is assumed to be equal to the momentum leaving the fermion line at the left end. Since the original closed fermion loop is restored by fusing the open fermion line, the amplitude of the cut diagram is calculated by taking the trace and carrying out the fermion-loop momentum integration.

There are many cancellations for the sum of component diagrams constructed from a cut generator. Making use of (105) and (106), the sum of the six diagrams in (113) becomes

$$
\nu \rightarrow \mu \nu
\rightarrow \mu
\rightarrow \nu
\rightarrow \nu
\rightarrow \mu
\rightarrow \nu
$$

(113)

The integrals for the first two diagrams in the above cancel each other after loop momentum shifting which is allowed under our scheme of dimensional regularization. The amplitude for the last diagram in (114) vanishes because the fermion loop that may produce Levi-Civita tensor terms are essentially embedded in a two point function that lacks sufficient indices to form a Levi-Civita tensor. Hence the sum of amplitudes for the six diagrams in (113) vanishes.

For the seven generators in (111), only the 3rd diagram on the first row does not have a legitimate and proper cut point available when the non-fermion internal line corresponds to a wavy line representing an internal vector meson line. But this is exactly the case of the 3rd generator diagram in (58) which has been discussed and taken care of in Section 4.7.
Since it is always possible to locate a proper and legitimate cut point for a \( N - 1 \) point, \( N > 3 \), generator diagram formed by a fermion loop and non-fermion internal lines, the corresponding 2-loop regularized \( N \)-point Ward identity can always be satisfied by component diagrams obtained with proper and legitimate cut points. Thus, we have succeeded in constructing 2-loop regularized amplitudes for the theory of (16), without the restriction on the type of non-fermion internal lines, while preserving the validities of relevant Ward identities and Bose permutation symmetry.

\section*{B Vertices Involving Ghost Fields}

For the Abelian-Higgs theory with the Lagrangian given by (16), the ghost field is free and de-coupled from other fields. For the standard model, there are vertices involving ghost fields and these vertices give further complications to the diagrammatic verification of Ward identities because of leftover terms (to be discussed below). The ghost interaction is not unique to the non-Abelian theory. It is possible to have a non-free ghost field for the Abelian theory by using appropriate gauge fixing term. For simplicity’s sake, we first illustrate the effect of ghost coupling with the Abelian-Higgs theory. We then proceed to the theory of the standard model to show that 2-loop Ward identities are not violated by these leftover terms.

\subsection*{B.1 Leftover Terms in Abelian-Higgs Theory}

For the Abelian-Higgs theory, if the gauge fixing term (14) is changed to

\[
L_{gf} = -\frac{1}{2\alpha} (\partial_\mu A^\mu - \alpha \Lambda \phi_2)^2
\]

(115)

where \( \Lambda \) an additional massive gauge parameter, the ghost term becomes

\[
L_{\text{ghost}} = i\eta \delta (\partial_\mu A^\mu - \alpha \Lambda \phi_2) = i\eta (\partial_\mu \partial^\mu + \alpha \Lambda M) \xi + ig\alpha \Lambda \eta H
\]

(116)

which gives the factor \(-g\alpha\Lambda\) to the vertex \( \eta - H - \xi \) such that the ghost field is no longer free.

Let us use a solid black box \[ \blacksquare \] to graphically represent the vertex factor \( \Gamma(0) (\eta, \varphi, \xi) \) for the vertex \( \eta - \varphi - \xi \). Then the Green function

\[
G(0) (\xi, \varphi, \xi; k_1, k_2, k_3) = D(0) (\xi, \eta; k_1) \Gamma(0) (\eta, \varphi; \xi) D(0) (\varphi, \varphi; k_2)
\]

(51)
can be diagrammatically expressed as

\[
\xi \varphi_i = G^{(0)}(\xi, \varphi_i, \xi)
\]  

(117)

Note in the above figure, the dotted arrowed line corresponds to the ghost propagator \(D^{(0)}(\xi, \eta)\). Let us also define

\[
\delta_1 \varphi_j \varphi_i = G^{(0)}(\delta_1 \varphi_j, \varphi_i, \xi) = \frac{\partial \delta_1 \varphi_j}{\partial \xi} G^{(0)}(\xi, \varphi_i, \xi)
\]  

(118)

The identity (100), with the additional pairing of \(\eta(z)\) with the \(\xi\) field in the interaction Lagrangian, becomes

\[
T \langle 0 | \eta(z) (\delta \varphi_j) \varphi_k | 0 \rangle = D^{(0)}(\eta(z), \xi(z')) \times \left[ D^{(0)}(\frac{\partial \delta_2 \varphi_j}{\partial \xi(z')}, \varphi_k) + G^{(0)}(\delta_1 \varphi_j, \varphi_i, \xi(z')) D^{(0)}(\varphi_i, \varphi_k) \right]
\]

The identity (105) needs to be appended by terms involving \(\delta_1 \varphi_j\) and \(\delta_1 \varphi_k\)

\[
\varphi_j \varphi_k \delta_2 \varphi_j \varphi_k \varphi_j \delta_2 \varphi_k \varphi_j \delta_1 \varphi_k \delta_1 \varphi_j \varphi_k
\]  

(119)

With the additional ghost coupling, the sum of the six diagrams in (113) becomes

\[
\nu + \nu + \nu + \nu
\]

(120)
The last three diagrams on the right hand side of the above identity vanish because each of them is effectively a two point function that is unable to produce a Levi-Civita tensor term. The second diagram on the right side is the only one that may be problematic because its cut point is located next to the composite vertex as in which the non-vanishing sum invalidates the basic identity to result in the 1-loop anomaly. To be more specific, under this circumstance, the double line pointing to the right together with the composite vertex in can no longer be replaced by because

\[(igR(\ell - m))_{DR} \frac{1}{(\ell - m)} = igR - ig\gamma_5 \Delta \frac{1}{(\ell - m)} \quad (121)\]

where \(\ell\) is the fermion momentum entering the composite vertex. But here, the massive \(\Lambda\) factor from the solid black box, which represents the vertex factor of \(\eta - H - \xi\) in the gauge fixing of (115), reduces the power counting. In fact, thanks to the reduced degree of divergence, all 2-loop diagrams involving the solid black box are convergent and their amplitudes are cut point independent in the \(n \to 4\) limit. Thus the extra ghost coupling introduced in (116) for the Abelian-Higgs theory does not give rise to any 2-loop anomaly in our \(\gamma_5\) scheme.

### B.2 Leftover Terms in Standard Model

For convenience, let us use the vector notations \(\vec{\phi} = (\phi_1, \phi_2, \phi_3), \vec{\xi}_W = (\xi_W^1, \xi_W^2, \xi_W^3)\) and \(\hat{e}_3 = (0, 0, 1)\). From (74), the BRS variation (74) for \(\phi\) may be expressed in terms of its real component fields.

\[
\delta H = \left(\frac{g_W}{2} \vec{\xi}_W + g_B \xi_B \hat{e}_3\right) \cdot \vec{\phi} \\
\delta \vec{\phi} = - \left(M_W \vec{\xi}_W + M_B \xi_B \hat{e}_3\right) - \left(\frac{g_W}{2} \vec{\xi}_W + g_B \xi_B \hat{e}_3\right) H \\
+ \left(\frac{g_W}{2} \vec{\xi}_W - g_B \xi_B \hat{e}_3\right) \times \vec{\phi}
\]

where

\[M_W = \frac{g_W}{2} v, M_B = g_B v\]
We have defined $\delta_1$ in \((97)\) as the part of BRS variation that is linear in field variables. For the standard model, the non-vanishing $\delta_1$ variations are

$$\begin{align*}
\delta_1 B^\mu &= \partial^\mu \xi_B, \\
\delta_1 W^\mu_a &= \partial^\mu \xi^a_W, \\
\delta_1 G^\mu_a &= \partial^\mu \xi^a_S, \\
\delta_1 \phi_1 &= -M_W \xi^1_W, \\
\delta_1 \phi_2 &= -M_W \xi^2_W, \\
\delta_1 \phi_3 &= -M_W \xi^3_W - M_B \xi_B.
\end{align*}$$

The graphical notations that we have defined for the Abelian-Higgs theory can be easily generalized to the standard model. In particular, the composite vertex \((122)\) can be defined for each component of ghost field

$$\begin{align*}
\xi_B \otimes &= \frac{\partial \delta_1 \varphi_i}{\partial \xi_B} \Gamma^{(0)}(\varphi_i, \varphi_j, \varphi_k) \\
&= -i k_\mu \Gamma^{(0)}(B^\mu, \varphi_j, \varphi_k) - M_B \Gamma^{(0)}(\phi_3, \varphi_j, \varphi_k) \tag{122}
\end{align*}$$

$$\begin{align*}
\xi^a_W \otimes &= \frac{\partial \delta_1 \varphi_i}{\partial \xi^a_W} \Gamma^{(0)}(\varphi_i, \varphi_j, \varphi_k) \\
&= -i k_\mu \Gamma^{(0)}(W^\mu_a, \varphi_j, \varphi_k) - M_W \Gamma^{(0)}(\phi_a, \varphi_j, \varphi_k) \tag{123}
\end{align*}$$

$$\begin{align*}
\xi^a_S \otimes &= \frac{\partial \delta_1 \varphi_i}{\partial \xi^a_S} \Gamma^{(0)}(\varphi_i, \varphi_j, \varphi_k) \tag{124}
\end{align*}$$

For the standard model with gauge fixing and ghost terms given by \((75)\), there are two kinds of ghost vertices stemming from $i f_{abc} g_S \partial^\mu \eta^a_S G^\mu_b \xi^c_S$ and $i \epsilon_{abc} g_W \partial^\mu \eta^a_W W^\mu_b \xi^c_W$ in the interaction Lagrangian. Since there is no risk of confusion, we use the solid black box $\Box$ to represent either the ghost vertex factor $\Gamma^{(0)}(\eta^a_S, G^\mu, \xi^c_S)$ or $\Gamma^{(0)}(\eta^a_W, W^\mu, \xi^c_W)$. These additional ghost coupling vertices may cause anomaly at 2-loop order only when they are involved in superficially divergent diagrams. Additionally, to violate a Ward identity in our $\gamma_5$ scheme, a cut point must be positioned next to the $\otimes$ vertex such as the cut point for the second diagram at the right hand side of \((120)\). There are four kinds of diagrams, as in the following, that may be responsible for the violations of Ward identities:

$$\begin{align*}
\otimes \quad \Box \quad \otimes \quad \Box \quad \otimes \quad \Box \quad \otimes \quad \Box \tag{125}
\end{align*}$$

Note that the composite vertex $\otimes$ may be any one of the three defined in \((122)-(124)\). Note also diagrams with improper cut points at $\otimes$ are not adopted in our scheme and are not included in the above figure.
According to (76)-(79), $-ig_S \gamma^\mu$, $-ig_W \gamma^\mu L$ and $ig_B \gamma^\mu (Y_i + \sigma_3 R)$ are the vertex factors for $\bar{\psi}_i - G^\mu - \psi_i$, $\bar{\psi}_i - W^\mu - \psi_i$ and $\bar{\psi}_i - B^\mu - \psi_i$. To have non-vanishing trace of $SU(3) \times U(2) \times U(1)$ isospin factors, the two external vector fields on any of the four diagrams in (125) must be either $(B^\mu, G^\nu_b)$ or $(B^\mu, W^\nu_b)$ with the solid black box corresponding to $\Gamma(0) (\eta^a_S, G^\mu, \xi^b_S)$ or $\Gamma(0) (\eta^a_W, W^\mu, \xi^b_W)$. The violation of a Ward identity may be traced back to the violation of the basic identity (36) due to the leftover term as specified by the last term in (121). This leftover term is proportional to the $\ell_\Delta$ component of the fermion momentum entering the $\otimes$ vertex and thus does not need to be taken into account in the power counting of the 1-loop sub-diagram containing the fermion line leaving $\otimes$ in any diagram of (125).

For the second diagram in (125), there is a logarithmically divergent 1-loop sub-diagram containing the $\ell_\Delta$ factor and the vertex for the external vector field on the left. If we keep only the $T_0$ term of this sub-diagram, two of the three external momenta will not be involved in the amplitude for the entire diagram and there are insufficient indices to form a Levi-Civita tensor term. This $T_0$ term may be subtracted such that there is effectively no divergent 1-loop sub-diagram. By power counting, the last two diagrams in (125) do not have divergent 1-loop sub-diagrams either. Only the 1-loop sub-diagram containing the fermion line leaving $\otimes$ on the first diagram in (125) may be logarithmically divergent. But this sub-divergence, which gives rise to a pole term independent of the mass of the fermion propagator, is removed by the minimal subtraction renormalization procedure. So essentially, for each of the four diagrams in (125), only the overall divergence may contribute a $\frac{1}{n-4}$ factor to resurrect the leftover term when $n \to 4$.

The mass matrix $\hat{m}$ for any fermion propagator in (125) may be replaced by a uniform $m_0$ because the difference is proportional to $(\hat{m} - m_0)$ which lowers the power count and annihilates the divergence of the entire diagram such that the $\ell_\Delta$ factor in the leftover term cannot survive. Thus for the evaluation of the leftover terms, we may assume that all the fermion propagators have identical mass $m_0$. If the external vector fields are $(B^\mu, G^\nu_b)$, there is only one $\gamma_5$ associated with the $ig_B \gamma^\mu \sigma_3 R$ term in the vertex factor of $\bar{\psi}_i - B^\mu - \psi_i$ and it does not contribute to the leftover term because $\sigma_3$ is traceless. If the external vector fields are $(B^\mu, W^\nu_b)$, the identity (80) also ensures that the amplitude for the leftover term vanishes if we sum over all lepton and quark loops. In essence, no leftover term is able to survive the $n \to 4$ limit in any diagram of (125). As a consequence, the coupling of
ghost fields to $G$ or $W$ vector fields in the standard model does not break the validity of any 2-loop Ward identity regularized according to our $\gamma_5$ scheme.

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