Instability of shallow open channel flow with lateral velocity gradients

A C Lima and N Izumi
River and Watershed Engineering Laboratory, Hokkaido University, Sapporo, 060-8628, Japan
E-mail: adriano@eng.hokudai.ac.jp; nizumi@eng.hokudai.ac.jp

Abstract. The turbulent flow in a wide rectangular open channel partially covered with vegetation is studied using linear stability analysis. In the base state normal flow condition, the water depth is constant and the transverse velocity vanishes, while there is a lateral gradient in the streamwise velocity with an inflexion point at the boundary between the vegetated zone and the main channel. The Reynolds stress is expressed by introducing the eddy viscosity, which is obtained from assuming a logarithmic distribution of the velocity near the bed. Perturbation expansions are introduced to the streamwise and transverse velocities, as well as to the water depth. The system of governing equations was solved in order to determine the maximum growth rate of the perturbations as a function of parameters which describe physical characteristics of the channel and the flow.

1. Introduction

Active transverse mixing is likely to occur in flows with an inflexion point in the depth-averaged streamwise velocity, such as in a watercourse with bank vegetation or in a compound channel. This situation generates instabilities which may grow into discrete horizontal vortices centred at the edge of the vegetated region or the edge of the flood plain, as observed in the experiments of Ikeda et al. (1991), Tsujimoto (1991), Tsujimoto & Kitamura (1992), Kitamura et al. (1998), White & Nepf (2007), which simulated partially vegetated channels, and Tamai et al. (1986), which featured compound channels.

Ikeda et al. (1994) performed linear stability analysis of a wide open channel partially covered with vegetation using the shallow-water approach. The eddy viscosity was assumed to be empirically related to the ratio of the velocities of the two parallel streams, which are far from the inflexion point. When these velocities are measured, they are already affected by the vortices generated by the instabilities and, as a consequence, the value of the eddy viscosity and the base state include the effects of the instabilities.

Linear stability analysis has also been employed by White & Nepf (2007), who concluded that the instability is fundamentally two-dimensional, and requires only a penetrable roughness layer to be initiated. Both Ikeda et al. (1994) and White & Nepf (2007) have introduced perturbations to the streamwise and transverse velocities, while herein the perturbations are also introduced to the water depth.
2. Formulation

It is assumed that water is flowing through a wide rectangular open channel which features two zones: a main channel and a zone covered by emerging vegetation (trees), according to Figure 1. The vegetation acts as a source of drag forces which provide resistance to the flow. The vegetation is approximated into regularly spaced cylinders. The channel has a small constant slope along its streamwise direction.

\[ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \beta - F^{-2} \frac{\partial D}{\partial x} - \beta (U^2 + V^2)^{1/2} U D^{-1} + \epsilon \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \]  

(1)

\[ \frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = -F^{-2} \frac{\partial D}{\partial y} - \beta (U^2 + V^2)^{1/2} U D^{-1} + \epsilon \left( \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \]  

(2)

\[ \frac{\partial D}{\partial t} + \frac{\partial UD}{\partial x} + \frac{\partial V D}{\partial y} = 0 \]  

(3)

In the vegetated zone \((-B_v \leq y \leq 0\)), the governing equations are

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \beta - F^{-2} \frac{\partial d}{\partial x} - \beta (d^{-1} + \alpha) (u^2 + v^2)^{1/2} u + \epsilon \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]  

(4)

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -F^{-2} \frac{\partial d}{\partial y} - \beta (d^{-1} + \alpha) (u^2 + v^2)^{1/2} u + \epsilon \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \]  

(5)

\[ \frac{\partial d}{\partial t} + \frac{\partial ud}{\partial x} + \frac{\partial vd}{\partial y} = 0 \]  

(6)

where \( t \) is time, \( x \) is the streamwise direction and \( y \) is the transverse direction. In the non-vegetated zone, \( U \) is the streamwise velocity, \( V \) is the transverse velocity and \( D \) is the water depth, while in the vegetated zone these variables are denoted by small letters \( (u, v \) and \( d)\). \( F \) is the Froude number sufficiently far from the vegetated zone. The governing equations are normalized such that the width of the non-vegetated zone assumes the value of unity, and, in the main channel far from the interface, the streamwise velocity and the water depth also assume the value of unity. The dimensionless parameters \( \beta, \epsilon \) and \( \alpha \) are given by

\[ \beta = \frac{C_f \tilde{B} \tilde{D}_n}{\tilde{D}_n}, \quad \epsilon = \frac{C_f^{1/2}}{15} \left( \frac{\tilde{D}_n}{\tilde{B}} \right), \quad \alpha = \frac{C_D \delta \tilde{D}_n}{2 C_f} \]  

(7)
where $C_f$ is a weak function of the flow depth relative to the roughness height, but assumed as a constant for simplicity hereafter, $\bar{B}$ is the dimensional width of the vegetated zone, $\bar{D}_n$ is the dimensional flow depth at the non-vegetated zone far from the interface, $C_D$ is the drag coefficient of the vegetation and $\bar{a}$ is the dimensional total projected width of vegetation per unit bed area.

The boundary conditions of the governing equations are such that the lateral velocity vanishes at the walls,

$V = 0$ at $y = 1$

$v = 0$ at $y = -B_v$ (8)

and right at the boundary between the two zones, the velocity, flow depth and Reynolds stress are continuous,

$U = u$, $V = v$, $D = d$, $\epsilon \frac{\partial U}{\partial x} = \epsilon \frac{\partial u}{\partial x}$, $\epsilon \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) = \epsilon \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$, $\epsilon \frac{\partial V}{\partial y} = \epsilon \frac{\partial v}{\partial y}$ at $y = 0$ (9)

3. Asymptotic expansions

The velocity components ($U$, $u$, $V$ and $v$), and the flow depth ($D$, $d$) are decomposed into the base state and the perturbations as

\[
\begin{pmatrix}
U(x, y, t) \\
V(x, y, t) \\
D(x, y, t)
\end{pmatrix} = 
\begin{pmatrix}
U_0(y) \\
0 \\
1
\end{pmatrix} + A
\begin{pmatrix}
U_1(y) \\
V_1(y) \\
D_1(y)
\end{pmatrix} e^{i(kx - \omega_c t)}
\]

(11)

\[
\begin{pmatrix}
u(x, y, t) \\
v(x, y, t) \\
d(x, y, t)
\end{pmatrix} = 
\begin{pmatrix}
u_0(y) \\
0 \\
1
\end{pmatrix} + A
\begin{pmatrix}
u_1(y) \\
v_1(y) \\
d_1(y)
\end{pmatrix} e^{i(kx - \omega_c t)}
\]

(12)

where index 0 indicates components of the base state and index 1 denotes components of the perturbations. In the base state normal flow condition, the transverse velocity vanishes, the streamwise velocity is assumed as a function of the transverse direction, only, and the water depth assumes the value of 1. Concerning the perturbations, $A$ is the amplitude, assumed to be small, $k$ is the wavenumber and $\omega_c$ is the angular frequency, which is complex as $\omega_c = \omega + i\Omega$. By introducing this relation of $\omega_c$ into (11) and (12), the obtained term $e^{i(kx - \omega_c t)}$ denotes a wave propagating in the $x$ direction at wave speed $\omega/k$ and wavelength $2\pi/k$. The parameter $\Omega$ denotes the growth rate of perturbation. When $\Omega > 0$, the perturbation grows as time progresses while when $\Omega < 0$, the perturbation decays to vanish. When $\Omega = 0$, the perturbation does not grow nor decay.

4. Solutions in the base state

Explicit analytical solutions in the base state are obtained by introducing (11) and (12) without the perturbation terms into the governing equations (1) to (6). In the non-vegetated zone, the solution is

$U_0 = 3 \tanh^2 \left[ \left( \frac{\beta}{2\epsilon} \right)^{1/2} y + \tanh^{-1} \left( \frac{\psi + 2}{3} \right)^{1/2} \right] - 2$ (13)

In the vegetated zone

$u_0 = 3\phi \coth^2 \left[ - \left( \frac{\beta}{2\epsilon\phi} \right)^{1/2} y + \coth^{-1} \left( \frac{\psi + 2\phi}{3\phi} \right)^{1/2} \right] - 2\phi$ (14)
where $\phi$ is the streamwise velocity at the vegetated zone far from the interface with the non-vegetated zone, given by

$$\phi = \frac{1}{(1 + \alpha)^{1/2}}$$

and $\psi$ is the streamwise velocity at interface,

$$\psi = \left(\frac{2\phi^2}{1 + \phi}\right)^{1/3}$$

5. Solutions of the perturbations

A spectral method with the Chebyshev polynomials is employed in order to solve the governing equations (1) to (6), after the perturbation expressions (11) and (12) have been introduced into them (as well as into the boundary and matching conditions (8) to (10)). The variables are then expanded by the Chebyshev polynomials as

$$U_1 = \sum_{n=0}^{N} A_n T_n(w), \quad V_1 = \sum_{n=0}^{N} B_n T_n(w) \quad \text{and} \quad D_1 = \sum_{n=0}^{N} C_n T_n(w)$$

in the non-vegetated zone, and as

$$u_1 = \sum_{n=0}^{N} a_n T_n(w), \quad v_1 = \sum_{n=0}^{N} b_n T_n(w) \quad \text{and} \quad d_1 = \sum_{n=0}^{N} c_n T_n(w)$$

in the vegetated zone. $A_n, B_n, C_n, a_n, b_n$ and $c_n$ are constants to be determined and $T_n(w)$ are the Chebyshev polynomials, with $w$ ranging from -1 to 1. The variable $w$ is related to the variable $y$ as $w = 2y - 1$ in the non-vegetated zone and as $w = 2y/B_v - 1$ in the vegetated zone.

The system composed of the governing equations and the boundary and matching conditions is rewritten as

$$Mc = 0$$

where

$$c = \begin{bmatrix} A_0 & \ldots & A_n & B_0 & \ldots & B_n & C_0 & \ldots & C_n & a_0 & \ldots & a_n & b_0 & \ldots & b_n & c_0 & \ldots & c_n \end{bmatrix}^T,$$

and $M$ is the square matrix of the corresponding coefficients. Equation (19) forms an eigenvalue problem with the eigenvalue $\omega_c$. The condition for a non-trivial solution is that $M$ should be singular. Thus its determinant should vanish. In order to determine $\omega_c$, the parameters $\beta, \epsilon, \alpha, F, B_v$ and $k$ must be specified. Thus $\omega_c$ takes the functional form $\omega_c = \omega_c(\beta, \epsilon, \alpha, F, B_v, k)$.

6. Application and results

6.1. Characteristic wavenumber for maximum instability

The dependence of $\Omega$ on parameters $\beta, \epsilon, \alpha, F, B_v$ and $k$ is studied in Figure 2.

All curves present a similar behaviour, where a strong dependence of $\Omega$ on $k$ can be visualized. As $k$ becomes small, $\Omega$ tends to a negative value. As $k$ increases, $\Omega$ increases, reaches a peak value, and decreases below zero. Vortices would be generated in most cases, since almost all of them have a maximum positive $\Omega$.

Once all curves were obtained after the same $C_f$, smaller values of $\beta$ (and thus higher values of $\epsilon$) represent channels with smaller aspect ratios ($\bar{B}/\bar{D}$). Thus, from (a), it is found that the
growth rate of perturbations increases with the aspect ratio. When $\epsilon$ decreases, the effect of the Reynolds stress also decreases, and the variation of the streamwise velocity at the boundary between the vegetated and non-vegetated zones becomes more abrupt, generating waves with shorter wavelength (and thus, higher values of $k$ corresponding to the maximum $\Omega$).

The parameter $\alpha$ strongly depends on the density of vegetation. Since the vegetation is the source of the perturbations, a small $\alpha$ is less likely to lead to the growth of perturbations. On the other side, if $\alpha$ is too large, the maximum $\Omega$ decreases, due to the decrease in the Reynolds stress (Figure 2(b)).

When $B_v$ increases, the growth rate of perturbations increases. However, when $B_v$ is sufficiently large, its further increase will lead to a small or null increment of the maximum $\Omega$ (Figure 2(c)).

According to (d), the maximum $\Omega$ decreases with the increase of the Froude number. When $F$ is larger than 2.0, the maximum $\Omega$ is likely to be small or negative.

6.2. Period of vortex generation

The dimensional period of vortex generation ($\tilde{T}$) is determined by

$$\tilde{T} = \frac{2\pi \tilde{B}}{\omega_{\text{max}} U}$$

where $\omega_{\text{max}}$ is the $\omega$ which corresponds to the maximum $\Omega$. 
By employing (21), the period was determined herein for experimental cases described in Ikeda et al. (1991), Tsujimoto (1991) and Tsujimoto & Kitamura (1992). These predicted periods were then compared to the periods measured in the experiments.

Vortices were generated in the experiments, and in accordance to this, positive maximum growth rates of perturbation were determined. The predicted periods have the same order of magnitude of the measured periods, although all predicted values were smaller than the experimental values (Figure 3).

Figure 3. Measured $\tilde{T}$ versus predicted $\tilde{T}$.

7. Conclusion

Linear stability analysis of the flow in a partially vegetated channel was performed to study the maximum growth of the perturbations originated from the active transverse mixing at the boundary between the non-vegetated and vegetated zones.

The perturbations, assumed to be small, were introduced to the streamwise and transverse velocities and to the water depth. The generation of vortices could be predicted using data from experiments, and agreement was found between theoretical and measured values. The values of the maximum growth rate of perturbation were coherent towards parameters such as water depth, width of non-vegetated and vegetated zones, and vegetation density.

References

Ikeda, S., Izumi, N. & Ito, R. 1991 Effects of pile dikes on flow retardation and sediment transport. *Journal of Hydraulic Engineering* 117, 1459–78.

Ikeda, S., Ohta, K. & Hasegawa, H. 1994 Instability-induced horizontal vortices in shallow open-channel flows with an inflexion point in skewed velocity profile. *Journal of Hydroscience and Hydraulic Engineering* 12, 69–84.

Kitamura, T., Jia, Y., Tsujimoto, T. & Wang, S. 1998 Sediment transport capacity in channels with vegetation zone. *Water Resource Engineering* 1.

Tamai, N., Asaeda, T. & Ikeda, H. 1986 Study on generation of periodic large surface eddies in a composite channel flow. *Water Resources Research* 22, 1129–1138.

Tsujimoto, T. 1991 Open channel flow with bank vegetation. *KHL Communication* 2, 41–54.

Tsujimoto, T. & Kitamura, T. 1992 Transverse mixing associated with surface wave in open-channel flow with longitudinal zone of vegetation. *Proc. of Hydraulic Engineering* 36, 273–280.

White, B. L. & Nepf, H. M. 2007 Shear instability and coherent structures in shallow flow adjacent to a porous layer. *Journal of Fluid Mechanics* 593, 1–32.