SOME PROPERTIES ON THE UNBOUNDED ABSOLUTE WEAK CONVERGENCE IN BANACH LATTICES

AZIZ ELBOUR

Abstract. In this paper, we investigate more about relationship between \(uaw\)-convergence (resp. \(un\)-convergence) and the weak convergence. More precisely, we characterize Banach lattices on which every weak null sequence is \(uaw\)-null. Also, we characterize order continuous Banach lattices under which every norm bounded \(un\)-null net (resp. sequence) is weakly null. As a consequence, we study relationship between sequentially \(uaw\)-compact operators and weakly compact operators. Also, it is proved that every continuous operator, from a Banach lattice \(E\) into a non-zero Banach space \(X\), is unbounded continuous if and only if \(E'\) is order continuous. Finally, we give a new characterization of \(b\)-weakly compact operators using the \(uaw\)-convergence sequences.

1. Introduction

The unbounded convergences received much attention. Several papers about “unbounded order convergence, unbounded norm convergence” and “unbounded absolute weak convergence” have been published, see [8, 10, 11, 12, 15]. Throughout this paper, \(E\) will stand for a Banach lattice. A net \((x_\alpha)\) in \(E\) is said to be:

- \(\text{unbounded norm convergent (} un\text{-convergent, for short)}\) if for each \(u \in E^+, \|x_\alpha - x\vee u\| \to 0\), in notation \(x_\alpha \overset{\text{un}}{\to} x\).
- \(\text{unbounded absolute weak convergent (} uaw\text{-convergent, in brief)}\) to \(x \in E\) if for each \(u \in E^+, |x_\alpha - x|\vee u \to 0\) weakly, in notation \(x_\alpha \overset{\text{uaw}}{\to} x\).

Note that both convergences are topological [8, 15]. Clearly, \(un\)-convergence implies the \(uaw\)-convergence, but the converse is false in general. However, \(un\)-convergence agrees with the \(uaw\)-convergence iff \(E\) is order continuous [15, Theorem 4].

2010 Mathematics Subject Classification. Primary 46B07; Secondary 46B42, 47B50.

Key words and phrases. Banach lattice, \(Un\)-convergence, \(Uaw\)-convergence, \(Uaw\)-compact operator, \(b\)-weakly compact operator.
Moreover, several investigations on the relationship between these convergences and the weak convergence are given in [8, 12, 15]. It is proved in [15, Theorem 7] that $E'$ is order continuous iff every norm bounded uaw-null net (resp. sequence) is weak null. In particular, if $E'$ is order continuous then every norm bounded un-null net is weak null [8, Theorem 2.4]. Also, it is proved that $E$ is discrete and order continuous iff every weak null net (resp. sequence) is un-null [12, Proposition 4.16] (see also [8, Proposition 6.2]).

In this paper, we investigate more about the relationship between uaw-convergence (resp. un-convergence) and the weak convergence. More precisely, we characterize Banach lattices on which every weak null sequence is uaw-null (Theorem 2.1). Also, we characterize order continuous Banach lattices under which every norm bounded un-null net (resp. sequence) is weak null (Proposition 2.5). Furthermore, we show that for norm bounded sequence, the uaw-convergence, un-convergence and the weak convergence coincide on a Banach lattice $E$ iff $E$ is discrete and both $E$ and $E'$ are order continuous (corollaries 2.4, and 2.6).

Next, we discuss when weakly compact operators are sequentially uaw-compact (Proposition 2.10 and Theorem 2.12). Also, it is proved that every continuous operator, from a Banach lattice $E$ into a non-zero Banach space $X$, is (sequentially) unbounded continuous iff $E'$ is order continuous (Theorem 2.16). Finally, we give a new characterization of $b$-weakly compact operators using the uaw-convergence sequences (Theorem 2.22).

We refer to [1, 2, 13] for all unexplained terminology and standard facts on vector and Banach lattices.

2. Main Results

We start this section by the following result which tells us when weak convergent sequences are uaw-convergent.

**Theorem 2.1.** Let $E$ be a Banach lattice. The following are equivalent:

1. The lattice operations of $E$ are sequentially weakly continuous;
2. For any sequence $(x_n) \subset E$, $x_n \overset{w}{\to} 0$ implies $x_n \overset{uaw}{\to} 0$.

**Proof.** (1) $\Rightarrow$ (2) Obvious because $|x_n| \overset{w}{\to} 0$ implies $x_n \overset{uaw}{\to} 0$.

(2) $\Rightarrow$ (1) Assumes (2) holds. Let $(x_n) \subset E$ be a weakly null sequence. So by hypothesis, $x_n \overset{uaw}{\to} 0$. We need to show that $|x_n| \overset{w}{\to} 0$. To this end, let $x' \in (E')^+$ and $\varepsilon > 0$. By Theorem 4.37 of [1], there exists $u \in E^+$ such that, for all $n$,

$$x' (|x_n| - |x_n| \wedge u) = x' (|x_n| - u)^+ < \varepsilon.$$
Since \( x' (\max |x_n|, u) \to 0 \), we conclude that \( x' (\max |x_n|) \to 0 \).

Therefore, \( |x_n| \xrightarrow{w} 0 \), as desired. \( \square \)

We know that the lattice operations in every AM-space are sequentially weakly continuous. Also, if \( E \) is a Banach lattice with an order continuous norm then the lattice operations of \( E \) are sequentially weakly continuous if and only if \( E \) is discrete (see Proposition 2.5.23 of [13] and Corollary 2.3 of [7]).

**Corollary 2.2.** For a Banach lattice \( E \), the following are equivalent:

1. The lattice operations of \( E \) are sequentially weakly continuous and \( E' \) is order continuous;
2. \( x_n \xrightarrow{w} 0 \iff x_n \xrightarrow{uaw} 0 \) for every norm bounded sequence \( (x_n) \subset E \).

**Proof.** Follows immediately from [15, Theorem 7] and Theorem 2.1. \( \square \)

**Corollary 2.3.** For a norm bounded sequence \( (x_n) \) in every AM-space, we have \( x_n \xrightarrow{w} 0 \iff x_n \xrightarrow{uaw} 0 \).

It is easy to see from [15, Theorem 4] and its proof that a Banach lattice \( E \) is order continuous iff \( x_n \xrightarrow{uaw} 0 \iff x_n \xrightarrow{un} 0 \) for every norm bounded (resp. order bounded) sequence \( (x_n) \subset E \).

**Corollary 2.4.** For a Banach lattice \( E \), the following are equivalent:

1. \( E \) is discrete and both \( E \) and \( E' \) are order continuous;
2. \( x_n \xrightarrow{w} 0 \iff x_n \xrightarrow{uaw} 0 \iff x_n \xrightarrow{un} 0 \) for every norm bounded sequence \( (x_n) \subset E \).

**Proof.** Follows immediately from [15, Theorem 4], [7, Corollary 2.3], and Corollary 2.2. \( \square \)

**Proposition 2.5.** For an order continuous Banach lattice \( E \), the following are equivalent:

1. \( E' \) is order continuous;
2. \( x_\alpha \xrightarrow{un} 0 \Rightarrow x_\alpha \xrightarrow{w} 0 \) for every norm bounded net \( (x_\alpha) \subset E \).
3. \( x_n \xrightarrow{um} 0 \Rightarrow x_n \xrightarrow{w} 0 \) for every norm bounded sequence \( (x_n) \subset E \).

**Proof.** (1) \( \Rightarrow \) (2) Follows from [8, Theorem 6.4].

(2) \( \Rightarrow \) (3) Obvious.

(3) \( \Rightarrow \) (1) Let \( (x_n) \subset E^+ \) be any norm bounded disjoint sequence and let \( u \in E^+ \). Since \( E \) is order continuous, it follows from [12, Proposition 3.5] that \( x_n \xrightarrow{um} 0 \). So, by hypothesis \( x_n \xrightarrow{w} 0 \). Now Theorem 116.1 of [14] (see also [13, Theorem 2.4.14]) finish the proof. \( \square \)
Corollary 2.6. For a Banach lattice $E$, the following assertions are equivalent:

1. $E$ is discrete and both $E$ and $E'$ are order continuous;
2. $x_\alpha \xrightarrow{un} 0 \iff x_\alpha \xrightarrow{uaw} 0 \iff x_\alpha \xrightarrow{w} 0$ for every norm bounded net $(x_\alpha) \subset E$.
3. $x_n \xrightarrow{w} 0 \iff x_n \xrightarrow{uaw} 0 \iff x_n \xrightarrow{un} 0$ for every norm bounded sequence $(x_n) \subset E$.
4. $x_\alpha \xrightarrow{un} 0 \iff x_\alpha \xrightarrow{w} 0$ for every norm bounded net $(x_\alpha) \subset E$.
5. $x_n \xrightarrow{un} 0 \iff x_n \xrightarrow{w} 0$ for every norm bounded sequence $(x_n) \subset E$.

Proof. (1) $\Rightarrow$ (2) Follows from Theorems 4 and 7 of [15] and [12, Proposition 4.16].

(2) $\Rightarrow$ (4) $\Rightarrow$ (5) and (2) $\Rightarrow$ (3) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (1) Follows from [12, Proposition 4.16] and Proposition 2.5 by noting that every weak null sequence is norm bounded. □

Remark 1. It follows from [15, Theorem 4] (resp. Corollary 2.2) that the condition (3) of Corollary 2.6 cannot be replaced by the following condition

$x_n \xrightarrow{uaw} 0 \iff x_n \xrightarrow{un} 0$, (resp. $x_n \xrightarrow{w} 0 \iff x_n \xrightarrow{uaw} 0$).

Proposition 2.7. Let $(x_\alpha)$ be a net in a Banach lattice $E$. If $(x_\alpha)$ is almost order bounded or relatively weakly compact, then $x_\alpha \xrightarrow{uaw} 0 \iff |x_\alpha| \xrightarrow{w} 0$.

Proof. Clearly $|x_\alpha| \xrightarrow{w} 0$ implies $x_\alpha \xrightarrow{uaw} 0$. Now assumes that $x_\alpha \xrightarrow{uaw} 0$ and pick any $x' \in (E')^+$ and $\varepsilon > 0$.

- If $(x_\alpha)$ is almost order bounded then there exists some $u \in E^+$ such that, for all $\alpha$,

  \[ \|(x_\alpha| - u)^+\| < \varepsilon. \]

So,

\[ x'(|x_\alpha|) = x'(|x_\alpha| - u)^+ + x'(|x_\alpha| \wedge u) \leq \varepsilon \|x'\| + x'(|x_\alpha| \wedge u). \]

Since $x'(|x_\alpha| \wedge u) \xrightarrow{0}$, we conclude that $x'(|x_\alpha|) \xrightarrow{0}$.

- If $(x_\alpha)$ is relatively weakly compact then, by Theorem 4.37 of [1], there exists $u \in E^+$ such that, for all $\alpha$,

  \[ x'(|x_\alpha| - |x_\alpha| \wedge u) = x'(|x_\alpha| - u)^+ < \varepsilon. \]
Since $x'(\|x_\alpha\| \land u) \longrightarrow 0$, we conclude that $x'(\|x_\alpha\|) \longrightarrow 0$.
Therefore, $\|x_\alpha\| \overset{w}{\longrightarrow} 0$, as desired. 

**Corollary 2.8.** Let $E$ be a Banach lattice. If $(x_\alpha) \subset E$ is a disjoint net and almost order bounded in $E''$, then $\|x_\alpha\| \overset{w}{\longrightarrow} 0$.

**Proof.** Since $(x_\alpha)$ is a disjoint net in $E''$, it follows from Lemma 2 of [15] that $x_\alpha \overset{uw}{\longrightarrow} 0$ (as a net in $E''$). So, by proposition 2.7, $\|x_\alpha\| \longrightarrow 0$ for $\sigma(E'', E'''')$, and hence $\|x_\alpha\| \longrightarrow 0$ for $\sigma(E, E')$. 

The next result is a $uw$ variant of [12, Lemma 9.10]; its proof follows from Proposition 2.7.

**Lemma 2.9.** Let $(x_n)$ be a sequence in a Banach lattice $E$. If $x_n \overset{w}{\longrightarrow} x$ and $x_n \overset{uw}{\longrightarrow} y$ then $x = y$.

**Proof.** Put $z_n = x_n - y$ and $z = x - y$. Then $z_n \overset{w}{\longrightarrow} z$ and $z_n \overset{uw}{\longrightarrow} 0$. Since $(z_n)$ is relatively weakly compact, then by Proposition 2.7, we have $\|z_n\| \overset{w}{\longrightarrow} 0$ and hence $z_n \overset{w}{\longrightarrow} 0$. So $z = 0$, a desired. 

In [12] (resp. [9]), the concept of (sequentially) un-compact (resp. $uw$-compact) operator is defined and studied. An operator $T: X \rightarrow E$, where $X$ is a Banach space and $E$ is a Banach lattice, is called un-compact (resp. $uw$-compact) if $T(B_X)$ is relatively un-compact (resp. $uw$-compact), where $B_X$ denotes the closed unit ball of $X$. Equivalently, for every norm bounded net $(x_\alpha)$ its image has a subnet, which is un-convergent (resp. $uw$-convergent).

And $T$ is called sequentially un-compact (resp. sequentially $uw$-compact) if $T(B_X)$ is relatively sequentially un-compact (resp. sequentially $uw$-compact). Equivalently, for every norm bounded sequence $(x_n)$ its image has a subsequence, which is un-convergent (resp. $uw$-convergent).

In [12], the authors discussed when weakly compact operators are sequentially un-compact. In the next, we discuss when weakly compact operators are sequentially $uw$-compact.

Theorem 7 of [15], Theorem 2.1 and Corollary 2.6 yield the following.

**Proposition 2.10.** Let $T: X \rightarrow E$ be an operator from a Banach space $X$ into a Banach lattice $E$.

1. If $E'$ is order continuous and $T$ is (sequentially) $uw$-compact, then $T$ is weakly compact;
2. If the lattice operations of $E$ are sequentially weakly continuous, and $T$ is weakly compact, then $T$ is sequentially $uw$-compact;
(3) If the lattice operations of $E$ are sequentially weakly continuous and $E'$ is order continuous, then $T$ is weakly compact iff it is sequentially uaw-compact;

(4) If $E$ is discrete and both $E$ and $E'$ are order continuous, then $T: X \to E$ is weakly compact iff it is (sequentially) uaw-compact iff it (sequentially) un-compact.

**Remark 2.** Note that a weakly compact operator is not necessary sequentially uaw-compact. In fact, the identity operator $I: L^2[0,1] \to L^2[0,1]$ is weakly compact but it is not sequentially uaw-compact. To see this, let $(r_n)$ be the sequence of Rademacher function in $L^2[0,1]$. This sequence is order bounded satisfying $r_n \xrightarrow{w} 0$, $|r_n| = 1$, and $\|I(r_n)\| = \|r_n\| = 1$ (see [13, p. 196]). So $(r_n)$ has no subsequence uaw-convergent.

Also, a (sequentially) uaw-compact operator is not necessary weakly compact. In fact, by Proposition 9.1 of [12], the identity operator $I: \ell^1 \to \ell^1$ is (sequentially) un-compact (and hence, it is (sequentially) uaw-compact) but is not weakly compact.

**Corollary 2.11.** An operator $T: X \to E$ from a Banach space $X$ into an AM-space $E$ is weakly compact iff it is sequentially uaw-compact.

**Theorem 2.12.** For a Banach lattice $E$, the following assertions are equivalent:

1. The lattice operations of $E$ are sequentially weakly continuous;
2. For every Banach space $X$, every weakly compact operator $T: X \to E$ is sequentially uaw-compact;
3. Every weakly compact operator $T: \ell^1 \to E$ is sequentially uaw-compact.

**Proof.** (1) $\Rightarrow$ (2) Follows from Proposition 2.10 (2).
(2) $\Rightarrow$ (3) Obvious.
(3) $\Rightarrow$ (1) The proof is very similar to that [12, Theorem 9.11] (by using Theorem 2.1 and Lemma 2.9). □

Recall that an operator $T: E \to X$, from a Banach lattice into a Banach space, is called AM-compact (resp. order weakly compact) if it maps order intervals to relatively compact (resp. weakly compact) sets. It is proved in [12, Proposition 9.9] that every order bounded un-compact operator is AM-compact. In the next, we prove that this result is also true for order bounded sequentially compact operators. Also, we prove that order bounded (sequentially) uaw-compact operators are order weakly compacts.
Proposition 2.13. Let $T : E \to F$ be an order bounded operator between Banach lattices.

1. If $T$ is sequentially un-compact operator then $T$ is AM-compact.
2. If $T$ is (sequentially) uaw-compact operator then $T$ is order weakly compact.

Proof. 1) Assume that $T$ is sequentially un-compact. It suffices to show that every order bounded sequence $(x_n)_{n \in \mathbb{N}}$ in $E$ its image has a norm convergent subsequence. Since $T$ is sequentially un-compact, there is a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $T(x_{n_k}) \xrightarrow{\text{un}} x$ for some $x \in F$. As $T$ is order bounded, $(T(x_{n_k}))_{k \in \mathbb{N}}$ is order bounded, and, therefore, $T(x_{n_k}) \xrightarrow{\|\cdot\|} x$ as desired.

2) We establish the result when $T$ is uaw-compact; the other case is similar. It suffices to show that every order bounded net $(x_\alpha)_{\alpha \in A}$ in $E$ its image has a weak convergent subnet. Since $T$ is uaw-compact, there is a subnet $(y_\beta)_{\beta \in B}$ of $(x_\alpha)$ such that $T(y_\beta) \xrightarrow{\text{uaw}} x$ for some $x \in F$. As $T$ is order bounded, $(T(y_\beta))_{\beta \in B}$ is order bounded, and, therefore, $T(y_\beta) \xrightarrow{w} x$ for the absolute weak topology $|\sigma|(E, E')$. Thus, $T(y_\beta) \xrightarrow{w} x$, as desired. \hfill \Box

Note that the converse of (2) in Proposition 2.13 is false: the identity operator on $c_0$ is order weakly compact but is neither uaw-compact nor sequentially uaw-compact (by Proposition 13 of [15]).

In [12, Example 9.7], the authors presented an example to show that, in general, un-compactness is not inherited under domination. In fact, there exist two operators $S, T : \ell^1 \to L^1[0,1]$ satisfying $0 \leq T \leq S$ with $S$ is (compact) un-compact and $T$ is neither un-compact nor sequentially un-compact. Then $S$ is uaw-compact, and since $L^1[0,1]$ is order continuous, it follows from [15, Theorem 4] that $T$ is neither uaw-compact nor sequentially uaw-compact.

Proposition 2.14. Let $S, T : E \to F$ be two positive operators between Banach lattices satisfying $0 \leq S \leq T$.

1. Suppose that the lattice operations of $F$ are sequentially weakly continuous and and both $E'$ and $F'$ are order continuous. If $T$ is sequentially uaw-compact then so is $S$.
2. Suppose that $F$ is discrete and both $F$ and $F'$ are order continuous. If $T$ is (sequentially) uaw-compact (resp. un-compact) then so is $S$.

Proof. Follows immediately from Proposition 2.10 and [1, Theorem 5.31]. \hfill \Box
Remark 3. The assumption “$E'$ is order continuous” of (1) (resp. “$F$ is order continuous” of (2)) in the previous result cannot be removed. Indeed, it follows from [1, Example 5.30] that there exist two operators $S, T : L^1[0, 1] \to \ell^\infty$ satisfying $0 \leq S \leq T$ with $T$ is (compact) weakly compact and $S$ is not weakly compact. Then $T$ is (sequentially) uaw-compact, and since $(\ell^\infty)'$ is order continuous, it follows from Proposition 2.10 (1) that $S$ is neither uaw-compact nor sequentially uaw-compact.

It is proved in [12, p. 278] that if an operator $T : X \to E$ is sequentially un-compact and semi-compact then $T$ is compact.

Proposition 2.15. If an operator $T : X \to E$, from a Banach space $X$ into a Banach lattice $E$, is (sequentially) uaw-compact and semi-compact then $T$ is weakly compact.

Proof. We will prove the statement for the sequential case; the other case is analogous. Let $(x_n)$ be a norm bounded sequence in $X$. There is a subsequence $(x_{n_k})$ such that $T(x_{n_k}) \xrightarrow{\text{uaw}} x$ for some $x \in E$. Since $T$ is semi-compact, the sequence $(T(x_{n_k}))$ is almost order bounded (and so is $(T(x_{n_k}) - x)$). Therefore, $|T(x_{n_k}) - x| \xrightarrow{\text{w}} 0$ (by Proposition 2.7). Thus, $T(x_{n_k}) \xrightarrow{\text{w}} x$, as desired.

Following O. Zabeti [16], we shall say that a continuous operator $T : E \to X$, where $X$ is a Banach space, is called unbounded continuous if for each bounded net $(x_\alpha) \subset E$, $x_\alpha \xrightarrow{\text{uaw}} 0$ implies $T(x_\alpha) \xrightarrow{\text{w}} 0$. It is sequentially unbounded continuous provided that the property happens for sequences. Moreover, a continuous operator $T : E \to F$ between Banach lattices, is said to be uaw-continuous (sequentially uaw-continuous) if $T$ maps every norm bounded uaw-null net (sequence) into a uaw-null net (sequence).

The following result tells us when every continuous operator is unbounded continuous.

Theorem 2.16. Let $E$ be a Banach lattice and $X$ be a non-zero Banach space. The following are equivalent:

1. $E'$ is order continuous;
2. Every continuous operator $T : E \to X$ is unbounded continuous;
3. Every continuous operator $T : E \to X$ is sequentially unbounded continuous.

Proof. (1) $\Rightarrow$ (2) Follows immediately from Theorem 7 of [15].
(2) $\Rightarrow$ (3) Obvious.
(3) ⇒ (1) Assume that $E'$ is not order continuous. To finish the proof, we have to construct a continuous operator $T : E \to X$ which is not sequentially unbounded continuous.

Since the norm of $E'$ is not order continuous, it follows from Theorem 116.1 of [14] that there is a norm bounded disjoint sequence $(u_n)$ of positive elements in $E$ which does not converge weakly to zero. Hence, we may assume that $\|u_n\| \leq 1$ for all $n$ and also that for some $0 \leq \varphi \in E'$ and some $\varepsilon > 0$ we have $\varphi (u_n) > \varepsilon$ for all $n$. Moreover, by Theorem 116.3 (i) of [14], the components $\varphi_n$ of $\varphi$ in the carrier $C_{u_n}$ form an order bounded disjoint sequence in $(E')^+$ such that

$$\varphi_n (u_n) = \varphi (u_n) \text{ for all } n \text{ and } \varphi_n (u_m) = 0 \text{ if } n \neq m.$$  

(*)

Note that $0 \leq \varphi_n \leq \varphi$ holds for all $n$.

On the other hand, let $y$ be a non-zero vector in $X$ and consider the operator $T : E \to X$ defined by

$$T(x) = \left( \sum_{n=1}^{\infty} \frac{\varphi (u_n)}{\varphi_n (x)} \right) y$$

for all $x \in E$. Since

$$\sum_{n=1}^{\infty} \frac{\varphi_n (x)}{\varphi (u_n)} \leq \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \varphi_n (|x|) \leq \frac{1}{\varepsilon} \varphi (|x|)$$

holds for each $x \in E$, the operator $T$ is well defined.

We claim that $T$ is not sequentially unbounded continuous. To see this, note that $(u_n)$ is a norm bounded disjoint sequence and $u_n \xrightarrow{\text{uaw}} 0$ (by Lemma 2 of [15]). But, from (*) we have $T(u_n) = y \xrightarrow{w} 0$. Thus $T$ is not sequentially unbounded continuous, and this ends the proof of the Theorem.

\[ \square \]

**Remark 4.** The previous result improve Theorem 1 of [16]. In fact, if every continuous operator $T : \ell^1 \to X$ is (sequentially) unbounded continuous then $X = \{0\}$.

**Corollary 2.17.** For a Banach lattice $E$, the following are equivalent:

1. Every continuous operator $T : E \to E$ is unbounded continuous;
2. $E'$ is order continuous.

The following result follows immediately from Theorems 2.1 and 2.16.

**Corollary 2.18.** Let $E$ and $F$ be two Banach lattices. If $E'$ is order continuous and the lattice operations of $F$ are weakly sequentially continuous, then every continuous operator $T : E \to F$ is sequentially uaw-continuous.
The following result follows immediately from Corollary 2.2 and it generalize Corollary 4 of [16].

**Proposition 2.19.** Let $E$ and $F$ be two Banach lattices. If $F'$ is order continuous and the lattice operations of $F$ are weakly sequentially continuous, then a continuous operator $T : E \to F$ is sequentially unbounded continuous if and only if it is sequentially $uaw$-continuous.

**Theorem 2.20.** Let $E$ be a Banach lattice and $F$ be a non-zero Banach lattice. If every continuous operator $T : E \to F$ is (sequentially) $uaw$-continuous then $E'$ is order continuous.

**Proof.** The proof is similar to that (3) $\Rightarrow$ (1) of Theorem 2.16. □

**Remark 5.** The previous result improve Theorem 7 of [16]. In fact, if every continuous operator $T : L_1 [0, 1] \to F$ is (sequentially) $uaw$-continuous then $F = \{0\}$.

An operator $T$ from a Banach lattice $E$ into a Banach space $X$ is said to be $b$-weakly compact, if it maps each subset of $E$ which is $b$-order bounded (i.e. order bounded in the topological bidual $E''$) into a relatively weakly compact subset in $X$ [3]. Several interesting characterizations of this class of operators are given in [3, 4, 5, 6].

To prove next result, we need the following lemma.

**Lemma 2.21.** Let $T$ be a $b$-weakly compact operator from a Banach lattice $E$ into a Banach space $X$. If $(x_n)$ is a $b$-order bounded sequence of $E$, then for each $\varepsilon > 0$ there exists some $u \in E^+$ such that

$$q_T((|x_n| - u)^+) \leq \varepsilon$$

holds for all $n$, where $q_T$ is the lattice seminorm on $E$ defined by

$$q_T(x) = \sup\{\|T(y)\| : |y| \leq |x|\}$$

for each $x \in E$.

In the following, we give our characterizations of $b$-weakly compact operators using the $uaw$-convergence sequences.

**Theorem 2.22.** Let $T$ be an operator from a Banach lattice $E$ into a Banach space $X$. Then the following assertions are equivalent:

1. $T$ is $b$-weakly compact;
2. For every $b$-order bounded sequence $(x_n) \subset E^+$, $x_n \xrightarrow{uaw} 0$ implies $q_T(x_n) \to 0$;
3. For every $b$-order bounded sequence $(x_n) \subset E^+$, $x_n \xrightarrow{uaw} 0$ implies $\|T(x_n)\| \to 0$;
(4) For every $b$-order bounded disjoint sequence $(x_n) \subset E^+$, $\|T(x_n)\| \to 0$.

Proof. $(1) \iff (4)$ Follows from Proposition 2.8 of [3].

$(1) \implies (2)$ Assume that $T$ is $b$-weakly compact and let $(x_n) \subset E^+$ be a $b$-order bounded sequence such that $x_n \overset{\text{uaw}}{\to} 0$. Pick $\varepsilon > 0$. So, by Lemma 2.21, there exists some $u \in E^+$ such that
\[ q_T(x_n - x_n \wedge u) = q_T((x_n - u)^+) \leq \varepsilon \]
holds for all $n$. On the other hand, since $(x_n \wedge u) \subset E^+$ is order bounded and weakly null, it follows from Theorem 2.2 of [6] that $q_T(x_n \wedge u) \to 0$. Therefore,
\[ q_T(x_n) \leq \varepsilon + q_T(x_n \wedge u) \leq 2\varepsilon \]
for all sufficiently large $n$. So $q_T(x_n) \to 0$.

$(2) \implies (3)$ Follows immediately from the inequalities $\|T(x)\| \leq q_T(x)$ holds for all $x \in E$.

$(3) \implies (4)$ Follows immediately from the fact that for every $(b$-order bounded) disjoint sequence $(x_n) \subset E^+$, we have $x_n \overset{\text{uaw}}{\to} 0$ (Lemma 2 of [15]). \[ \square \]

It is proved in [3, Proposition 2.10] that $E$ is a KB-space if and only if the identity operator $\text{Id}_E : E \to E$ is a $b$-weakly compact operator.

Corollary 2.23. For a Banach lattice $E$, the following assertions are equivalent:

1. $E$ is a KB-space;
2. The identity operator $\text{Id}_E : E \to E$ is $b$-weakly compact;
3. For every $b$-order bounded sequence $(x_n) \subset E^+$, $x_n \overset{\text{uaw}}{\to} 0$ implies $\|x_n\| \to 0$.

References

[1] Aliprantis, C.D., Burkinshaw, O.: Positive Operators, 2nd edn. Springer, Berlin (2006)
[2] Aliprantis, C.D.,Burkinshaw, O.: Locally Solid Riesz Spaces with Applications to Economics, 2nd edn. AMS, Providence (2003)
[3] Alpay, S., Altin, B., Tonyali, C.: On property (b) of vector lattices, Positivity 7(1-2), 135–139 (2003)
[4] Alpay, S., Altin, B.: A note on $b$-weakly compact operators, Positivity 11(4), 575-582 (2007)
[5] Alpay, S., Ercan, Z.: Characterizations of Riesz spaces with $b$-property, Positivity 13(1), 21-30 (2009)
[6] Aqzzouz, B., Elbour, A.: Some Properties of the Class of $b$-Weakly Compact Operators, Complex Anal. Oper. Theory. 12, 1139-1145 (2010)
[7] Chen, Z. L., Wickstead, A. W.: Relative weak compactness of solid hulls in Banach lattices, Indag. Math. (N.S.) 9, no. 2, 187–196 (1998)
[8] Deng, Y., O’Brien M., Troitsky, V.G.: Unbounded norm convergence in Banach lattices. Positivity 21, 963–974 (2017)
[9] Erkurtun-Özcan, N., Anl Gezer, N., Zabeti, O.: Unbounded absolutely weak Dunford–Pettis operators, Turkish journal of mathematics, 43 (2019), pp. 2731–2740.
[10] Gao, N.: Unbounded order convergence in dual spaces, J. Math. Anal. Appl. 419(1), 347–354 (2014)
[11] Gao, N., Xanthos, F.: Unbounded order convergence and application to martingales without probability, J. Math. Anal. Appl. 415(2), 931–947 (2014)
[12] Kandić, M., Marabeh, M., Troitsky, V.G.: Unbounded norm topology in banach lattices, J. Math. Anal. Appl. 451(1), 259–279 (2017)
[13] Meyer-Nieberg, P.: Banach Lattices, Springer, Berlin (1991)
[14] Zaanen, A. C.: Riesz spaces II, vol. 30. North Holland Publishing Company (1983)
[15] Zabeti, O.: Unbounded absolute weak convergence in banach lattices, Positivity, 22(2):501-505 (2018)
[16] Zabeti, O.: Unbounded continuous operators in Banach lattices. Preprint https://arxiv.org/abs/1911.10015.

(A. Elbour) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, MOULAY ISMAIL UNIVERSITY OF MEKNES, P.O. BOX 509, BUTALAMINE 52000, ERRACHIDIA, MOROCCO
E-mail address: azizelbour@hotmail.com