LEONHARD EULER
AND A $q$-ANALOGUE OF THE LOGARITHM

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On the 300th anniversary of Euler’s birth

Abstract. We study a $q$-logarithm which was introduced by Euler and give some of its properties. This $q$-logarithm has not received much attention in the recent literature. We derive basic properties, some of which were already given by Euler in a 1751 paper and in a 1734 letter to Daniel Bernoulli. The corresponding $q$-analogue of the dilogarithm is introduced. The relation to the values at 1 and 2 of a $q$-analogue of the zeta function is given. We briefly describe some other $q$-logarithms that have appeared in the recent literature.

1. Introduction

In a paper from 1751, Leonhard Euler (1707–1783) introduced the series \[ s = \sum_{k=1}^{\infty} \frac{(1-x)(1-x/a) \cdots (1-x/a^{k-1})}{1-a^k}. \] (1.1)

We will take $q = 1/a$. Then this series is convergent for $|q| < 1$ and $x \in \mathbb{C}$. In this paper we will assume $0 < q < 1$. Then this becomes

\[ S_q(x) = -\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} (x;q)_k, \]

where $(x;q)_0 = 1$, $(x;q)_k = (1-x)(1-xq) \cdots (1-xq^{k-1})$. This can be written as a basic hypergeometric series

\[ S_q(x) = -\frac{q(1-x)}{1-q} \, \, _3\phi_2 \left( \begin{array}{c} q, qx \end{array} q^2 q \, \right| q, q \right). \]

Euler had come across this series much earlier in an attempt to interpolate the logarithm at powers $a^k$ (or $q^{-k}$); see, e.g., Gautschi’s comment [11] discussing Euler’s letter to Daniel Bernoulli where Euler introduced the function for $a = 10$. Euler was aware that this interpolation did not work very well; see [11] §§3-4. The function in [12] does not seem to appear in the recent literature, even though it has some nice properties. We will prove some of its properties, some already obtained...
We also use the q-exponential functions \[ e_q(z) = \frac{1}{(z; q)_\infty} = \sum_{n=0}^\infty (q; q)_n z^n, \quad |z| < 1, \]
and
\[ E_q(z) = (-z; q)_\infty = \sum_{n=0}^\infty q^{n(n-1)/2} (q; q)_n z^n. \]

2. The q-logarithm as an entire function

First of all we will show that the function \( S_q \) in (1.2) is an entire function, and as such it is a nicer function than the logarithm, which has a cut along the negative real axis.

**Property 2.1.** The function \( S_q \) defined in (1.2) is an entire function of order zero.

**Proof.** For \( k \in \mathbb{N} \) the q-Pochhammer \((z; q)_k\) is a polynomial of degree \( k \) with zeros at \( 1, 1/q, \ldots, 1/q^{k-1} \). For \(|z| \leq r \) we have the simple bound
\[
|z|_k \leq (1+r)(1+r|q|) \cdots (1+r|q|^{k-1}) = (-r; |q|)_k < (-r; |q|)_\infty,
\]
and hence the partial sums are uniformly bounded on the ball \(|z| \leq r\):
\[
\left| -\sum_{k=1}^n \frac{q^k}{1-q^k} (z; q)_k \right| \leq (-r; |q|)_\infty \sum_{k=1}^\infty \frac{|q|^k}{1-|q|^k}.
\]
The partial sums therefore are a normal family and are uniformly convergent on every compact subset of the complex plane. The limit of these partial sums is \( S_q(z) \) and is therefore an entire function of the complex variable \( z \).

Let \( M(r) = \max_{|z| \leq r} |S_q(z)| \). Then
\[
M(r) \leq (-r; |q|)_\infty \sum_{k=1}^\infty \frac{|q|^k}{1-|q|^k}.
\]
and \((-r; q)_\infty = E_{-q}(r)\) is the maximum of \(E_{-q}(z)\) on the ball \(|z| \leq r\). The function \(E_q\) is an entire function of order zero, which can be seen from the coefficients \(a_n\) of its Taylor series and the formula \[\text{[2], Theorem 2.2.2}\]

\[(2.1) \lim \sup_{n \to \infty} \frac{n \log n}{\log(1/|a_n|)}\]

for the order of \(\sum_{n=0}^\infty a_n z^n\). Hence also \(S_q\) has order zero.

\(\square\)

Observe that for \(0 < q < 1\) we have

\[
M(r) = \max_{|z| \leq r} |S_q(z)| = \sum_{k=1}^\infty \frac{q^k}{1-q^k} (-r; q)_k
\]

and some simple bounds give

\[
(q; q)_\infty \sum_{k=1}^\infty \frac{q^k}{(q; q)_k} (-r; q)_k \leq M(r) \leq (-r; q)_\infty \sum_{k=1}^\infty \frac{q^k}{1-q^k}.
\]

For the lower bound we can use the \(q\)-binomial theorem \[\text{[1.3]}\] to find

\[
(-rq; q)_\infty - (q; q)_\infty \leq M(r) \leq (-r; q)_\infty \sum_{k=1}^\infty \frac{q^k}{1-q^k},
\]

which shows that \(M(r)\) behaves like \(E_q(qr) - C_1 \leq M(r) \leq C_2 E_q(r)\), where \(C_1\) and \(C_2\) are constants (which depend on \(q\)).

Euler \[\text{[8, \S\S 14-15]}\] essentially also stated the following Taylor expansion.

**Property 2.2.** The \(q\)-logarithm \[\text{[1.2]}\] has the following Taylor series around \(x = 0\):

\[
S_q(x) = - \sum_{k=1}^\infty \frac{q^k}{1-q^k} \left( 1 + q^{k(k-1)/2} (-x)^{k} \frac{(q; q)_k}{(q; q)_j (q; q)_{k-j}} \right).
\]

**Proof.** Use the \(q\)-binomial theorem \[\text{[1.3]}\] with \(x = q^k \) and \(a = q^{-k}\) to find

\[(2.2) (z; q)_k = \sum_{j=0}^k \frac{k!}{j!} q^{(j-1)/2} (-z)^j, \quad \sum_{j=0}^k \frac{k!}{j!} = \frac{(q; q)_k}{(q; q)_j (q; q)_{k-j}}.
\]

Use this in \[\text{[1.2]}\], and change the order of summation to find

\[
S_q(x) = - \sum_{k=1}^\infty \frac{q^k}{1-q^k} - \sum_{j=1}^\infty \frac{q^j}{1-q^j} q^{(j-1)/2} (-x)^j \sum_{k=j}^\infty \frac{q^k}{1-q^k} (q; q)_k (q; q)_{k-j}.
\]

With a new summation index \(k = j + \ell\) this becomes

\[
S_q(x) = - \sum_{k=1}^\infty \frac{q^k}{1-q^k} - \sum_{j=1}^\infty \frac{q^j}{1-q^j} q^{(j-1)/2} (-x)^j \sum_{\ell=0}^\infty \frac{q^{\ell} (q; q)_\ell}{(q; q)_j}.
\]

Now use the \(q\)-binomial theorem \[\text{[1.3]}\] to sum over \(\ell\) to find

\[
S_q(x) = - \sum_{k=1}^\infty \frac{q^k}{1-q^k} - \sum_{j=1}^\infty \frac{q^j}{1-q^j} q^{(j-1)/2} (-x)^j \frac{(q; q)_\ell}{(q; q)_j}.
\]

If we combine both series, the required expansion follows.  \(\square\)
This result can be written in terms of basic hypergeometric series as
\[
S_q(x) = -\frac{q}{1 - q} \, {}_2\phi_1 \left( \frac{q, q}{q^2} ; q, q \right) - \frac{q x}{(1 - q)^2} \, {}_2\phi_2 \left( \frac{q, q}{q^2}, q^2 ; q, q^2 x \right).
\]

The growth of the coefficients in this Taylor series again shows that \( S_q \) is an entire function of order zero if we use the formula \( (2.1) \) for the order of \( \sum_{n=0}^{\infty} a_n z^n \); see also \( [11] \, \S 4 \).

Next we mention the following \( q \)-integral representation, where we use Jackson’s \( q \)-integral (see \( [10] \, \S 1.11 \))
\[
(2.3) \quad \int_0^a f(t) \, dq \, t = (1 - q) a \sum_{k=0}^{\infty} f(a q^k) \, q^k,
\]
defined for functions \( f \) whenever the right-hand side converges.

**Property 2.3.** For every \( x \in \mathbb{C} \) we have
\[
S_q(x) = -\frac{q(1 - x)}{1 - q} \int_0^1 G_q(q x, qt) \, dq \, t,
\]
with
\[
G_q(x, t) = \sum_{k=0}^{\infty} t^k(x; q)_k = {}_2\phi_1 \left( \frac{x, q}{1 - q} ; q, qt \right) = \frac{1}{1 - t} \, {}_1\phi_1 \left( \frac{q}{qt} ; q, xt \right).
\]

Since \( \int_0^a f(t) \, dq \, t \to \int_0^a f(t) \, dt \) when \( q \to 1 \) and \( G_q(x, t) \to 1/(1 - t(1 - x)) \) when \( q \to 1 \) for \( x > 0 \), we see (at least formally) that Property 2.3 is a \( q \)-analogue of the integral representation
\[
\log(x) = -\int_0^1 \frac{1 - x}{1 - t(1 - x)} \, dt, \quad x \notin (-\infty, 0]
\]
for the logarithm.

**Proof.** Observe that
\[
\frac{1 - q}{1 - q^{k+1}} = (1 - q) \sum_{p=0}^{\infty} q^{(k+1)p} = \int_0^1 t^k \, dq \, t.
\]
Inserting this in the definition \( \{1.2\} \) of \( S_q \) and interchanging summations, which is justified by the absolute convergence of the double sum, give the result. The identity between the basic hypergeometric series representing \( G_q(x, t) \) is the case \( c = 0 \) of \( [10] \, (III.4) \).

Note that, as in the proof of Property 2.2, one can show that
\[
(2.4) \quad G_q(x, t) = \sum_{j=0}^{\infty} \frac{(-xt)^j q^j/2}{(t; q)_{j+1}}.
\]

3. The \( q \)-difference equation

The function \( S_q \) satisfies a simple \( q \)-difference equation:

**Property 3.1.** The \( q \)-logarithm \( \{1.2\} \) satisfies
\[
(3.1) \quad S_q(x/q) - S_q(x) = 1 - (x; q)_\infty.
\]
Proof. Recall the $q$-difference operator

$$D_q f(x) = \frac{f(qx) - f(x)}{x(q - 1)}.$$  

Then a simple exercise is

$$D_{1/q}(x; q)_k = -\frac{1 - q^k}{1 - q} (x; q)_{k-1}.$$  

Use this in (1.2) to find

$$D_{1/q} S_q(x) = \sum_{k=1}^{\infty} q^k \frac{1 - q^k}{1 - q} (x; q)_{k-1} = \frac{q}{1 - q} \sum_{k=0}^{\infty} q^k (x; q)_k.$$  

Observe that $(x; q)_{k+1} - (x; q)_k = (x; q)_k[1 - xq^k - 1] = -xq^k(x; q)_k$. Summing we find $-x \sum_{k=0}^{n} q^k(x; q)_k = (x; q)_{n+1} - (x; q)_0$, and when $n \to \infty$,

$$\sum_{k=0}^{\infty} q^k(x; q)_k = \frac{1 - (x; q)_{\infty}}{x}.$$  

If we use this result, then

$$D_{1/q} S_q(x) = \frac{q}{1 - q} \frac{1 - (x; q)_{\infty}}{x},$$  

which is (3.1).

In order to see how this is related to the classical derivative of $\log x$, one may rewrite this as

$$D_q((1 - q)S_q(x)) = \frac{1}{x} - \frac{(qx; q)_{\infty}}{x}.$$  

This $q$-difference equation can already be found in [8 §6], where Euler writes $s = S_q(x)$ and $t = S_q(x/q)$ and gives the relation

$$1 + s - t = (1 - x) \left(1 - \frac{x}{a}\right) \left(1 - \frac{x}{a^2}\right) \left(1 - \frac{x}{a^3}\right) \left(1 - \frac{x}{a^4}\right) \cdots,$$

where $q = 1/a$.

As a corollary one has [8 §7].

**Property 3.2.** For every positive integer $n$ one has $S_q(q^{-n}) = n$.

*Proof.* Use (3.1) with $x = q^{-n+1}$ to find $S_q(q^{-n}) - S_q(q^{-n+1}) = 1$, since $(x; q)_{\infty}$ vanishes whenever $x = q^{-n}$ for $n \geq 0$. The result then follows by induction and $S_q(1) = 0$.

It is this property, which is quite similar to $\log_a a^n = n$, where $\log_a$ is the logarithm with base $a$, which gives $S_q$ the flavor of a $q$-logarithm and which made Euler consider this function as an interpolation of the logarithm; see [11 §1]. Observe that this interpolation property can be stated as follows: $- \log q S_q(x)$ approximates $\log x$ as $q \uparrow 1$ and for fixed $q$ this approximation is perfect if $x = q^{-n} (n = 1, 2, \ldots)$.

Another interesting value is

$$S_q(0) = -\sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} = -\zeta_q(1),$$
which is a $q$-analogue of the harmonic series, where the $q$-analogue of the $\zeta$-function is defined by

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{n^{s-1}q^n}{1-q^n}.$$ 

It has been proved (see Erdős [7], Borwein [3, 4], Van Assche [27]) that this quantity is irrational whenever $q = 1/p$ with $p$ an integer $\geq 2$. For the specific argument $1$ this coincides, up to a factor, with the value at $1$ of the $q$-$\zeta$-function considered by Ueno and Nishizawa [24].

The values of $S_q(q^n)$ for $n \in \mathbb{N}$ are distinctly different, and for these values we do not get the same flavor as the logarithm.

**Property 3.3.** For every positive integer $n$ one has

$$S_q(q^n) = -n + (q; q)_\infty \sum_{k=0}^{n-1} \frac{1}{(q; q)_k}. \tag{3.2}$$

**Proof.** Choose $x = q^{k+1}$ in (3.1). Then $S_q(q^k) - S_q(q^{k+1}) = 1 - (q^{k+1}; q)_\infty$. Summing and using the telescoping property give

$$S_q(q^0) - S_q(q^n) = \sum_{k=0}^{n-1} (S_q(q^k) - S_q(q^{k+1})) = n - \sum_{k=0}^{n-1} (q^{k+1}; q)_\infty.$$ 

By Property 3.2 we have $S_q(1) = 0$. Now $(q^{k+1}; q)_\infty = (q; q)_\infty/(q; q)_k$ gives the required expression (3.2). \qed

In order to see how this approximates $\log x$, one may reformulate this as

$$-\log q \; S_q(q^n) = \log q^n - \log q \sum_{k=0}^{n-1} (q^{k+1}; q)_\infty.$$ 

In [8, §10] Euler writes $s = S_q(q^n)$, $t = S_q(q^{n-1})$, $u = S_q(q^{n-2})$, and he writes the recursion

$$s = \frac{2t - u + aq^n(1 - t)}{1 - aq^n},$$

where $q = 1/a$. In contemporary notation we write $y_n = S_q(q^n)$ and obtain the recurrence relation

$$y_n(1 - q^{-1}) - (2 - q^{-1})y_{n-1} + y_{n-2} = q^{n-1}.$$ 

One can verify that this recurrence relation indeed holds for $y_n = S_q(q^n)$ given in (3.2). More generally one in fact has

$$(1 - qx)S_q(q^2x) - (2 - qx)S_q(qx) + S_q(x) = qx,$$

which is a second order non-homogeneous $q$-difference equation for $S_q$.

Note that the explicit evaluation $S(q^{-n}) = n$, $n \in \mathbb{N}$, gives the following summation formulas:

$$\sum_{k=1}^{n} \frac{(q^{-n}; q)_k}{1-q^k} q^k = -n, \quad \sum_{k=1}^{\infty} \frac{q^{k(k+1)/2}(-1)^{k-1}q^{-nk}}{(1-q^k)(q; q)_k} = n + \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}, \tag{3.3}$$
using the definition of $S_q(x)$ and the Taylor expansion in Property 2.2. Similarly, the evaluation at $q^n$, \( n \in \mathbb{N} \), given in (5.2) gives the summation formulas

\[
\sum_{k=1}^{\infty} \frac{(q^n; q)_k}{1 - q^k} q^k = n - \sum_{k=0}^{n-1} (q^{k+1}; q)_\infty, \tag{3.4}
\]

\[
\sum_{k=1}^{\infty} \frac{q^{k(k+1)/2}(-1)^{k-1} q^{nk}}{(1 - q^k) (q; q)_k} = -n + \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} + \sum_{k=0}^{n-1} (q^{k+1}; q)_\infty.
\]

Note that all infinite series are absolutely convergent and that for \( n = 0 \) the results in (3.3) and (3.4) coincide. The first sums become trivial, and the second sum gives the following expansion for $\zeta_q(1)$:

\[
\zeta_q(1) = \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} = \sum_{k=1}^{\infty} \frac{q^{k(k+1)/2}(-1)^{k-1}}{(1 - q^k) (q; q)_k}. \tag{3.5}
\]

Using (3.5) in Property 2.2 gives the expansion

\[
S_q(x) = -\sum_{k=1}^{\infty} \frac{q^{k(k+1)/2}(-1)^{k-1}(1 - x^k)}{(1 - q^k) (q; q)_k},
\]

so that in particular

\[
\frac{dS_q}{dx}(1) = \lim_{x \to 1} \frac{S_q(x)}{1 - x} = -\sum_{k=1}^{\infty} \frac{k q^{k(k+1)/2}(-1)^{k-1}}{(1 - q^k) (q; q)_k}.
\]

4. An extension of the $q$-logarithm and Lambert series

If we have the definition of $S_q(x)$ resembling Lambert series, it is natural to look for the extension

\[
F_q(x, t) = -\sum_{k=1}^{\infty} (x; q)_k \frac{t^k}{1 - t^k},
\]

which is a Lambert series; see [15, §58.C]. Since \(|(x; q)_k| \leq (\mathfrak{r} - \mathfrak{s}); |q|)_\infty \leq (-r; q)_\infty\) for \( x \in \mathbb{C} \) \(|x| \leq r\), the convergence in (4.1) is uniform on compact sets in \( x \) and on compact subsets of the open unit disk in \( t \). Also since the series

\[-\sum_{k=1}^{\infty} (x; q)_k t^k\]

is absolutely convergent for \(|t| < 1\) uniformly in \( x \) in compact sets, it follows by [13 Satz 259] that \( F_q \) is analytic for \((x, t) \in \mathbb{C} \times \{ t \in \mathbb{C} \mid |t| < 1 \}\). Observe that \( S_q(x) = F_q(x, q) \).

The general theory of Lambert series then gives the power series of \( F \) in powers of \( t \):

\[
F_q(x, t) = \sum_{\ell=1}^{\infty} \left( \sum_{k|\ell} (x; q)_k \right) t^\ell \implies S_q(x) = \sum_{\ell=1}^{\infty} \left( \sum_{k|\ell} (x; q)_k \right) q^\ell.
\]

We are mainly interested in the power series development with respect to \( x \).

**Property 4.1.** For \(|t| < 1\) one has

\[
F_q(x, t) = -\sum_{k=1}^{\infty} \frac{t^k}{1 - t^k} - \sum_{\ell=1}^{\infty} x^\ell (-1)^{\ell/2} \left( \sum_{n=1}^{\infty} n^\ell (t^n q^{\ell+1}; q)_\infty \right). \]
The proof is along the same lines as the proof of Property 2.2. We find

\[ \text{Proof.} \quad (1.5.1) \].

\[ n \]

which justifies the interchange of summations. Using this and replacing

\[ \ell \]

for \( q \)-binomial theorem again and the absolute convergence of the double sum,

\[ \sum_{n=1}^{\infty} \frac{(q^\ell q^{-n+1}; q)_n}{(q^\ell; q)_n} q^{n\ell} = \frac{q^\ell}{1 - q^\ell} \]

for \( \ell \in \mathbb{N}, \ell \geq 1 \). This can be obtained as a special case of the \( q \)-Gauss sum \[10 \] (1.5.1)].

\[ \text{Proof.} \] The proof is along the same lines as the proof of Property 2.2. We find similarly

\[ F_q(x, t) = -\sum_{k=1}^{\infty} \frac{q^{xk}}{1 - q^{2k}} - \sum_{j=1}^{\infty} q^{j(j-1)/2}(-xt)^j \sum_{\ell=0}^{\infty} \frac{(q^{j+1}; q)_\ell}{(q; q)_\ell} \frac{t^\ell}{1 - t^{j+\ell}} \]

and we write

\[ \sum_{\ell=0}^{\infty} \frac{(q^{j+1}; q)_\ell}{(q; q)_\ell} \frac{t^\ell}{1 - t^{j+\ell}} = \sum_{p=0}^{\infty} \frac{t^{p(j+\ell)}}{(1+q)^{p(j+\ell)}} = \sum_{p=0}^{\infty} t^{j+\ell(p+1)} \]

using the \( q \)-binomial theorem again and the absolute convergence of the double sum, which justifies the interchange of summations. Using this and replacing \( n = p + 1 \) we get the result.

Consider the case \( t = q^2 \). Following the line of proof of Property 2.2 we write

\[ \sum_{k=1}^{\infty} \frac{q^{2k}(x; q)_k}{1 - q^{2k}} = -\sum_{k=1}^{\infty} \frac{q^{2k}}{1 - q^{2k}} - \sum_{j=1}^{\infty} \frac{(-1)^j q^{j(j-1)/2} x^j}{(q; q)_j} \sum_{\ell=0}^{\infty} \frac{(q; q)_{\ell+j} q^{2j+2\ell}}{(q; q)_\ell (1 - q^{2\ell+2j})} \]

and we can write the inner sum over \( \ell \) as

\[ \sum_{\ell=0}^{\infty} \frac{(q; q)_{\ell+j}}{(q; q)_\ell (1 + q^{2\ell+j})} = \frac{(q; q)_j}{1 + q^{j+1}} \sum_{\ell=0}^{\infty} \frac{(q; q)_\ell (-q^{-j}; q)_\ell q^{2\ell}}{(-q^{1+1}; q)_\ell} q^{2\ell}. \]

Using Property 4.11 for \( t = q^2 \) then gives

\[ \sum_{n=1}^{\infty} \frac{q^{2n+1}}{(q^2; q)_n} = \frac{q^{2j}}{(1 - q^{2j})} \sum_{\ell=0}^{\infty} \frac{(q; q)_\ell (-q^{-j}; q)_\ell q^{2\ell}}{(-q^{1+1}; q)_\ell} q^{2\ell}. \]

This can also be proved directly using the \( q \)-binomial theorem and geometric series. We can rewrite (4.3) in standard basic hypergeometric series form (see [10]) as the quadratic transformation

\[ \frac{(1 - q^{2j})}{(q^2; q)_j+1} \] \( 3\phi_2 \left( \begin{array}{c} q^2, q^2, q^3 \\ q^{j+3}, q^{j+4} \end{array} ; q^2 \right) = 2\phi_1 \left( \begin{array}{c} q^j, -q^j \\ -q^{j+1} ; q, q^2 \right) \right). \]

Analogous to Property 2.3 and using the notation of Property 2.3 we have the following.

**Property 4.2.** For \( |p| < 1 \) one has

\[ F_q(x, p) = -\frac{p(1-x)}{1-p} \int_0^1 G(qx, pt) \, dt. \]
5. A q-analogue of the dilogarithm

Euler’s dilogarithm is defined by the first equality in

\[ \text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-t)}{t} \, dt = -\int_{1-x}^1 \frac{\log(t)}{1-t} \, dt = \frac{\pi^2}{6} - \text{Li}_2(1-x) \]

for \( 0 \leq x \leq 1 \); see [18], [14] for more information and references. Here we use \( \text{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6} \). In particular, \( x \frac{d}{dx} \text{Li}_2(x) = -\log(1-x) \), and the definition by the series can be extended to complex \( x \) being absolutely convergent for \( |x| \leq 1 \).

We define the \( q \)-dilogarithm by

\[ \text{Li}_2(x; q) = \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2} (x; q)_k. \]

We have \( \lim_{q \downarrow 1} (1-q)^2 \text{Li}_2(x; q) = \sum_{k=1}^{\infty} (1-x)^k/k^2 = \text{Li}_2(1-x) \). In this case we can justify the interchange of the limit and summation using dominated convergence. We assume \( 0 < q < 1 \), and we first observe that \( |(x; q)_k| \leq 1 \) for \( |1-x| \leq 1 \). Next we use

\[ \frac{1-q^k}{1-q} = \sum_{j=0}^{k-1} q^j = q^{(k-1)/2} \left( \sum_{j=0}^{k-1} q^{j+\frac{1}{2}} + q^{-j-\frac{1}{2}} \right), \quad k \text{ even}, \]
\[ = 1 + \sum_{j=0}^{k-1} (q^{j+1} + q^{-j-1}), \quad k \text{ odd}, \]

and \( x + 1/x \geq 2 \) for \( x \in [0, 1] \) then gives

\[ \frac{1-q^k}{1-q} \geq k q^{(k-1)/2}, \]

so that

\[ q^k \frac{(1-q)^2}{(1-q^k)^2} \leq \frac{1}{k^2}. \]

Combining both estimates gives

\[ \left| \frac{q^k}{(1-q^k)^2} (x; q)_k \right| \leq \frac{1}{k^2} \]

for \( |1-x| \leq 1 \), and dominated convergence is established.

We list some properties of the \( q \)-dilogarithm. In the following we use \( \zeta_q(2) = \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2} \) as an analogue of \( \frac{1}{6} \pi^2 \). This is equal to the \( q \)-\( \zeta \)-function

\[ \zeta_q(s) = \sum_{n=1}^{\infty} \frac{n^{s-1} q^n}{1 - q^n} \]

for \( s = 2 \) since

\[ \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} n q^{nk} = \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2}, \]

(see, e.g., [21] Part VIII, Chapter 1, problem 75). This quantity was considered by Zudilin [28, 29], Krattenthaler et al. [17], Postelmans and Van Assche [22], who studied its irrationality when \( 1/q \) is an integer \( \geq 2 \). Note that this no longer corresponds to Ueno and Nishizawa [26], who essentially have \( \sum_{k=1}^{\infty} \frac{q^{2k}}{(1-q^{2k})} \) as the value at 2 for their \( q \)-\( \zeta \)-function.
**Property 5.1.** \( \text{Li}_2(\cdot; q) \) is an entire function of order zero. Moreover, we have the special values

\[
\text{Li}_2(1; q) = 0, \quad \text{Li}_2(0; q) = \zeta_q(2), \quad \text{Li}_2(q^{-n}; q) = -\sum_{k=1}^{n} \frac{k}{1 - q^k},
\]

and \((1 - q)(1 - x) (D_q\text{Li}_2(\cdot; q))(x) = S_q(x)\) and

\[
\text{Li}_2(x; q) = \zeta_q(2) + \frac{1}{1 - q} \int_0^x S_q(t) \, dt.
\]

Moreover, the \( q \)-dilogarithm has the Taylor expansion

\[
\text{Li}_2(x; q) = \zeta_q(2) + \sum_{j=1}^\infty (-1)^j q^{j(j+1)/2} \frac{x^j}{(1 - q^j)^2} 2\phi_1 \left( \frac{q^j, q^j}{q^{j+1}; q, q} \right).
\]

Here the \( 2\phi_1 \)-series is defined by

\[
2\phi_1 \left( \frac{q^j, q^j}{q^{j+1}; q, q} \right) = \sum_{\ell=0}^\infty \frac{(q^j; q)_{\ell}(q^j; q)_{\ell}}{(q; q)_{\ell}(q^{j+1}; q)_{\ell}} q^\ell = \sum_{\ell=0}^\infty \frac{(q^j; q)_{\ell}(1 - q^\ell)}{(q; q)_{\ell}(1 - q^{\ell+\ell})} q^\ell.
\]

Unfortunately, this series cannot be summed using the (non-terminating) \( q \)-Chu-Vandermonde sum.

Note that after multiplying the integral representation for \( \text{Li}_2(x; q) \) by \((1 - q)^2\), we can take a formal limit \( q \uparrow 1 \) to get

\[
\text{Li}_2(1 - x) = \frac{\pi^2}{6} + \int_0^x \log(t) \, dt - \int_0^{1-x} \frac{\log(1-t)}{t} \, dt,
\]

so that we recover the integral representation for the dilogarithm.

**Proof.** The proof of \( \text{Li}_2(\cdot; q) \) being an entire function of order zero is derived as in Property 2.1. Since \( (q;x;q)_k - (x; q)_k = x(1 - q^2)(qx; q)_{k-1} \) we obtain

\[
(5.2) \quad \text{Li}_2(qx; q) - \text{Li}_2(x; q) = \frac{x}{1 - x} \sum_{k=1}^\infty \frac{q^k(x; q)_k}{1 - q^k} = \frac{-x}{1 - x} S_q(x).
\]

This implies \((1 - q)(1 - x) (D_q\text{Li}_2(\cdot; q))(x) = S_q(x)\).

Using (5.2) for \( x = q^{-n}, \, n \in \mathbb{N}, \) and \( \text{Li}_2(1; q) = 0, \, S(q^{-n}) = n, \) we find the value for \( \text{Li}_2(q^{-n}; q) \). Iterating (5.2) we get

\[
\text{Li}_2(x; q) = \sum_{k=0}^N \frac{xq^k}{1 - xq^k} S_q(xq^k) + \text{Li}_2(xq^{N+1}; q),
\]

and by letting \( N \to \infty \) we get the convergent series expansion

\[
\text{Li}_2(x; q) = \text{Li}_2(0; q) + \sum_{k=0}^\infty \frac{xq^k}{1 - xq^k} S_q(xq^k) = \zeta_q(2) + \frac{1}{1 - q} \int_0^x S_q(t) \, dt.
\]

Finally, the Taylor expansion proceeds as in the proof of Property 2.2 and we find

\[
\text{Li}_2(x; q) = \sum_{k=1}^\infty \frac{q^k}{(1 - q^k)^2} + \sum_{j=1}^\infty \frac{(-x)^j q^{j(j-1)/2}}{(q; q)_j} \sum_{\ell=0}^\infty \frac{(q; q)_{j+\ell} q^{j+\ell}}{(1 - q^{j+\ell})^2}.
\]
The inner sum over \( \ell \) can be rewritten as
\[
\frac{q^{-1} (q; q)_{j-1}}{1 - q^{j}} \sum_{\ell=0}^{\infty} \frac{(q^j q^{j+1}; q)_{\ell}}{(q; q)_{\ell}} q^{\ell},
\]
and this gives the result. \( \square \)

The evaluation of the \( q \)-dilogarithm gives the following summation (cf. \((3.3)\)):
\[
\sum_{k=1}^{n} \frac{(q^{-n}; q)_{k} q^{k}}{(1 - q^{k})^{2}} = - \sum_{k=1}^{n} \frac{k}{1 - q^{k}} = \sum_{j=1}^{\infty} \frac{q^{j}}{(1 - q^{j})^{2}} + \sum_{j=1}^{\infty} \frac{(-1)^{j} q^{j(j+1)/2} q^{-nj}}{(1 - q^{j})^{2}} 2 \phi_{1} \left( \frac{q^{j}, q^{j}}{q^{j+1} ; q, q} \right).
\]

In particular, for \( n = 0 \) we obtain an alternating series representation for \( \zeta_{q}(2) \):
\[
\zeta_{q}(2) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} q^{j(j+1)/2}}{(1 - q^{j})^{2}} 2 \phi_{1} \left( \frac{q^{j}, q^{j}}{q^{j+1} ; q, q} \right).
\]

If we write \( \text{Li}_{2}(x; q) = \sum_{n=0}^{\infty} a_{n} x^{n} \); \( S_{q}(x) = \sum_{n=0}^{\infty} b_{n} x^{n} \) temporarily, then \((5.2)\) implies that \( q^{n} a_{n} - a_{n} \) equals the coefficient, say \( c_{n} \), of \( x^{n} \) in \(-S_{q}(x) x/(1 - x)\). If we use \(-x/(1 - x) = \sum_{k=1}^{\infty} -x^{k} \), it follows that \( c_{n} = - \sum_{p=0}^{n-1} b_{p} \). Note that the relation is trivial in case \( n = 0 \), and for integer \( n \geq 1 \) we find from the explicit Taylor expansions for \( S_{q}(\cdot ; q) \) and \( \text{Li}_{2}(\cdot ; q) \) the relation
\[
\frac{(-1)^{n-1} q^{n(n+1)/2}}{(1 - q^{n})} 2 \phi_{1} \left( \frac{q^{n}, q^{n}}{q^{n+1} ; q, q} \right) = \sum_{k=1}^{\infty} \frac{q^{k}}{1 - q^{k}} + \sum_{j=1}^{n-1} \frac{(-1)^{j} q^{j(j+1)/2}}{(1 - q^{j}) (q; q)_{j}}.
\]

Note that this relation gives an explicit expression for the remainder if approximating \( \zeta_{q}(1) \) with the alternating series as in \((3.3)\). Of course, we get the same result if we use the Taylor expansion of \( S_{q} \) as in Property \((2.2)\) in the integral representation for the \( q \)-dilogarithm in Property \((6.1)\).

The classical dilogarithm satisfies many interesting properties, such as a simple functional equation, a five-term recursion, a characterisation by these first two properties, explicit evaluation at certain special points, etc.; see \([18, 14]\) for more information and references. It would be interesting to see if these interesting properties have appropriate analogues for the \( q \)-analogue of the dilogarithm discussed here.

6. Other \( q \)-LOGARITHMS

In the physics literature (see e.g. \([25]\)), one defines \( \ln_{q}(x) = \frac{x^{1-q} - 1}{1-q} \). There are no \( q \)-series, \( q \)-Pochhammer symbols, \( q \)-difference relations, etc. The choice of the letter \( q \) and the fact that \( \lim_{q \to 1} \ln_{q}(x) = \log x \) are not sufficient motivation to call this a \( q \)-analogue. It just shows that the logarithmic function is somewhere between the constant function and powers \( x^{\alpha} - 1 \) for \( \alpha > 0 \). This \( q \)-analogue of the logarithm plays a role in statistical mechanics and, as pointed out in \([13]\), was also introduced by Euler in 1779; see \([13]\) for more information and references.

Borwein \([4]\), Zudilin et al. \([19]\), and Van Assche \([27]\) consider
\[
\ln_{q}(1 + z) = \sum_{k=1}^{\infty} \frac{(-1)^{k} z^{k}}{1 - q^{k}}, \quad |z| < |q|,
\]
with \(|q| > 1\). They prove that \(\ln_q(1 + z)\) is irrational for \(z = \pm 1\) and \(q\) an integer greater than 2. For \(z = -1\) one has a \(q\)-analogue of the harmonic series, and this is essentially the generating function of \(d_n = \sum_{k|n} 1\), i.e. the number of divisors of \(n\). A similar formula, but now for \(0 < q < 1\),

\[
\log_q(z) = \sum_{k=1}^{\infty} \frac{z^n}{1 - q^n} = \frac{ze_q'(z)}{c_q(z)}, \quad |z| < 1,
\]

has been considered as a \(q\)-analogue of the logarithm by Kirillov [14] and Koornwinder [16]. This \(q\)-analogue is well adapted to non-commutative algebras; see [14, §2.5, Ex. 11], [16, Prop. 6.1], since

\[
\log_q(x + y - xy) = \log_q(x) + \log_q(y) \quad \text{for} \quad xy = qyx.
\]

The corresponding \(q\)-analogue of the dilogarithm, provisionally denoted by \(\widetilde{\text{Li}}_2(x; q)\), is defined by

\[
\widetilde{\text{Li}}_2(x; q) = \sum_{k=1}^{\infty} \frac{z^k}{k(1 - q^k)} = \log(c_q(z)) \implies \log_q(z) = z \widetilde{\text{Li}}_2'(z; q).
\]

Zudilin [29] considers a similar \(q\)-logarithm but a different \(q\)-dilogarithm,

\[
L_1(x; q) = \sum_{n=1}^{\infty} \frac{(xq)^n}{1 - q^n}, \quad L_2(x; q) = \sum_{n=1}^{\infty} \frac{n(xq)^n}{1 - q^n},
\]

and mainly studies simultaneous rational approximation to \(L_1\) and \(L_2\) in order to obtain quantitative linear independence over \(\mathbb{Q}\) for certain values of these functions.

Other \(q\)-logarithms are defined as inverses of \(q\)-exponential functions; see Nelson and Gartley [20] for two different cases viewed from complex function theory, and Chung et al. [5], where implicitly \(q\)-commuting variables are used. Fock and Goncharov [9, 12] introduce a \(q\)-logarithm of \(\ln(e^z + 1)\) by an integral. The corresponding \(q\)-dilogarithm is essentially Ruijsenaars’ hyperbolic \(\Gamma\)-function; see [23, II.A]. For other \(q\)-logarithms based on Jacobi theta functions, see Sauloy [24] and Duval [6], where the \(q\)-logarithms play a role in difference Galois theory in constructing the analogue of a unipotent monodromy representation.

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