The Dynamical Yang-Baxter Relation and the Minimal Representation of the Elliptic Quantum Group

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Abstract

In this paper, we give the general forms of the minimal $L$ matrix (the elements of the $L$-matrix are $c$ numbers) associated with the Boltzmann weights of the $A_{n-1}^1$ interaction-round-a-face (IRF) model and the minimal representation of the $A_{n-1}$ series elliptic quantum group given by Felder and Varchenko. The explicit dependence of elements of $L$-matrices on spectral parameter $z$ are given. They are of five different forms (A(1-4) and B). The algebra for the coefficients (which do not depend on $z$) are given. The algebra of form A is proved to be trivial, while that of form B obey Yang-Baxter equation (YBE). We also give the PBW base and the centers for the algebra of form B.

1 Introduction

Recently, many papers have focused on the many-body long-distance integrable dynamical system, such as the Ruijsenaar Schneider model and the Calogero Moser (CM) model\cite{1-3}. They are closely connected with the quantum Hall effect in the condense matter physics and the Seiberg Witten (SW) theory in the field theory, especially for the equations of the spectral curve in the SW theory, namely, the modified eigenvalue equations of the Lax matrices in the above integrable models\cite{4-6}. These Lax matrices are the classical limit of the $L$ matrices which is associated with the interaction-round-a-face (IRF) model of Lie group\cite{7-9} and the modified Yang-Baxter relation (NSF equation)\cite{10-13}. All these $L$-matrices are corresponding to the representation of the elliptic quantum group which was proposed by Felder and Varchenko\cite{11, 12}. So it is very interesting to study the general solution of the $L$-matrices.

In this paper, we study the simplest case of $L$-matrices which satisfy the Dynamical Yang-Baxter Relation (DYBR) for the $A_{n-1}$ group. The deep study of the $A_{n-1}$ group
case can help us to understand the other Lie group cases because a subset of the other Lie groups can be constructed by the $A_{n-1}$ group. We only study the simplest case of $L$ matrices, that is to say, the Hilbert space of the $L$-operator is a scale function space. We find that the $L$ matrices can only have five possible forms, form A(1), A(2), A(3), A(4) and form B. The form A(1) and B can be constructed by the factorized $L$ matrices[14-17]. And the coefficient part of form B obeys a set of quadratic equations which can be related to the Shibukawa-Ueno operator[18]. The algebra of these quadratic relations have explicit PBW base and satisfy the YBE without spectral parameter $z$.

We find that all known $L$-matrices[9, 12, 16] of related problem are equivalent to one representation of this algebra. But it is still an open question that whether it is the unique one.

The present paper is organized as follows. In section 2, we study the dependence of the elements of $L$-matrix with spectral parameter $z$. In section 3, we study the dependence of the essential part of these elements, which are functions of both $z$ and $(a, b)$, with respect to the indices of the elements. We prove that there are five possible classes of the minimal $L$-matrices. We then relate the elements of adjacent lattice points $(a, b)$ and $(a+\hat{i}, b+\hat{j})$ in the end of this section. Section 4 is written for the equations of the coefficient part of the elements of $L$-matrices, as a necessary and sufficient condition of $L$-matrices to satisfy the DYBR. This leads to two kinds of algebras. The A algebra, which is corresponding to the form A(i) (i=1,2,3,4), is trivially commutative and thus the coefficients of form A(i) can be determined completely. The algebra for form B coefficients is studied further in section 5, which satisfies YBE, and we establish the PBW base for it. We also give center elements for this algebra. In the last section, we give a known solution to the equations of form B. Throughout this paper, we always assume all elements of $L$-matrix are c number functions which are not identically zero and $n \geq 4$.

2 DYBR and the relation between factorized $L$-matrix and minimal $L$-matrix

It is well known that the Boltzmann weight of the $A^{(1)}_{n-1}$ IRF[7, 8, 9] model can be written as

$$R(a|z)^{ij}_{ij} = \frac{\sigma(z+w)}{\sigma(a)}$$,

$$R(a|z)^{ij}_{ij} = \frac{\sigma(z)\sigma(a_{ij} - w)}{\sigma(w)\sigma(a_{ij})} \quad \text{for } i \neq j,$$

$$R(a|z)^{ij}_{ij} = \frac{\sigma(z + a_{ij})}{\sigma(a_{ij})} \quad \text{for } i \neq j, \quad R(a|z)^{ij}_{ij} = 0 \quad \text{for other cases},$$

where $a \equiv (m_0, m_1, \cdots, m_{n-1})$ is an n-vector, and $a_{ij} = a_i - a_j$, $a_i = w(m_i - \frac{1}{n}\sum m_i + w_i)$, $m_i$ ($i = 0, 1, \cdots, n-1$) are integers which describe the state of model, while $\{w, w_i\}$ are generic c-numbers which are the parameters of the model, and $\sigma(z) \equiv \theta^*_{a,b}(z, \tau)$, with

$$\theta^*_{a,b}(z, \tau) \equiv \sum_{m \in \mathbb{Z}} e^{i\pi(m+a)^2\tau + 2i\pi(m+a)(z+b)}.$$
We define an n-dimension vector \( \hat{j} = (0, 0, \cdots, 0, 1, 0, \cdots) \), in which \( j \)th component is 1.

We consider a matrix whose elements are linear operators. We denote the elements of the matrix as \( L(\hat{b} \mid \hat{z})^j_i \). The \( R \)-matrix and the \( L \)-matrix can also be depicted by the following figures,

**Figure 1**: The elements of \( R \)-matrix
\[
R(a \mid z_1 - z_2)^{ij'}_{ij}.
\]

**Figure 2**: The element of \( L \)-matrix,
\[
L(a, h \mid z)^j_i \equiv L(\hat{a} \mid \hat{z})^j_i.
\]

**Figure 3**: The dynamical Yang-Baxter relation.

The dynamical Yang-Baxter relation (DYBR) is written as (also see figure 3)
\[
\sum_{i',j'} R(b \mid z_1 - z_2)^{i'j'}_{ij} L(\hat{b} \mid \hat{z}_1)^{i''}_{i'} L(\hat{a} + \hat{j'})_{j'} = \sum_{i',j'} L(\hat{a} + \hat{b})_{j'}^{i''} L(\hat{b}_+ \mid \hat{z}_2)^{i'}_{i''} R(a \mid z_1 - z_2)^{i'}_{i''},
\]
where \( b \equiv (m_0^b, m_1^b, \cdots, m_{n-1}^b) \), \( a \equiv (m_0^a, m_1^a, \cdots, m_{n-1}^a) \). We note that Eq.(2) gives the quadratic relation of the elements of \( L \). If we let \( b = a + h \), the form of the equation will be the same as that given in the Ref.[11, 12], and the relation which \( L \) satisfies is the definition relation of the elliptic quantum group proposed by Felder and Varchenko. Here, the elements of the \( L \)-matrix are operators, and Eq.(2) is the algebra of these operators. In this paper, we only discuss the minimal form of the operators, namely, we only consider the simplest case that all elements are \( c \) numbers. In this situation, the \( L(\hat{b} \mid \hat{z})^j_i \) is scalar functions of \((a, b, z, i, j)\). We will try to find the general form of such \( L \)-matrix. From Eq.(2), we have

\[
\frac{L(\hat{a} + \hat{b})_{j'}^{i''} L(\hat{b}_+ \mid \hat{z}_2)^{i'}_{i''}}{L(\hat{b} \mid \hat{z}_1)^{i'}_{i''}} = \frac{L(a)_{j'}^{i''} L(b^+_i \mid \hat{z}_2)^{i'}_{i''}}{L(\hat{b} \mid \hat{z}_1)^{i'}_{i''}},
\]

\[
R(b \mid z_1 - z_2)^{i''}_{i'} = R(a \mid z_1 - z_2)^{i''}_{i'} \frac{L(\hat{a} + \hat{b})_{j'}^{i''} L(\hat{b}_+ \mid \hat{z}_2)^{i'}_{i''}}{L(\hat{b} \mid \hat{z}_1)^{i'}_{i''}}.
\]

\( \sum_{i',j'} \)
Hence the above equation gives

$$R(a|z_1 - z_2)_{i'j'}^{i''j''} = R(b|z_1 - z_2)_{ij}^{ij} \frac{L(a^{+}i'|z_2)^{i'}_{i}}{L(b^{+}i'|z_2)^{i'}_{i}} \frac{L(a^{+}j'|z_1)^{j'}_{j}}{L(b^{+}j'|z_1)^{j'}_{j}} (i' \neq j'),$$

(4)

$$R(a|z_1 - z_2)_{i'j'}^{i''j''} = R(b|z_1 - z_2)_{ij}^{ij} \frac{L(a^{+}i'|z_1)^{i'}_{i}}{L(b^{+}i'|z_1)^{i'}_{i}} \frac{L(a^{+}j'|z_2)^{j'}_{j}}{L(b^{+}j'|z_2)^{j'}_{j}} + R(b|z_1 - z_2)_{ij}^{ij} \frac{L(b^{+}i'|z_1)^{i'}_{i}}{L(b^{+}j'|z_1)^{j'}_{j}} \frac{L(a^{+}j'|z_2)^{j'}_{j}}{L(b^{+}j'|z_2)^{j'}_{j}} (i \neq j).$$

(5)

By solving Eq.(4) and Eq.(5), we can determine \( L(b^{+}|z)^{i}_{i} \) as the function of \( z \). Let

$$\frac{L(a^{+}i'|z_2)^{i'}_{i}}{L(b^{+}i'|z_2)^{i'}_{i}} \equiv g(z_2), \quad \frac{L(a^{+}j'|z_1)^{j'}_{j}}{L(b^{+}j'|z_1)^{j'}_{j}} \equiv h(z_1), \quad \frac{L(a^{+}j'|z_2)^{j'}_{j}}{L(b^{+}j'|z_2)^{j'}_{j}} \equiv f(z),$$

(6)

$$\frac{R(b|z_1 - z_2)_{ij}^{ij}}{R(a|z_1 - z_2)_{i'j'}^{i''j''}} \equiv A(z_1 - z_2), \quad \frac{R(b|z_1 - z_2)_{ij}^{ij}}{R(a|z_1 - z_2)_{i'j'}^{i''j''}} \equiv B(z_1 - z_2).$$

(7)

We then rewrite Eq.(4) as

$$g(z_2)h(z_1) = A(z_1 - z_2) + B(z_1 - z_2)f(z_1)/f(z_2) \equiv F(z_1, z_2).$$

(8)

We find that the left hand side of the above equation is factorized by the functions of \( z_1 \) and \( z_2 \). So taking logarithm to the both sides of the above equation and taking the derivative with respect to \( z_1 \) and \( z_2 \), we have

$$\frac{\partial^2}{\partial z_1 \partial z_2} \ln F(z_1, z_2) = 0.$$

(9)

Hence the above equation gives

$$F(z_1, z_2) \frac{\partial^2}{\partial z_1 \partial z_2} F(z_1, z_2) - \frac{\partial}{\partial z_1} F(z_1, z_2) \frac{\partial}{\partial z_2} F(z_1, z_2) = 0.$$

(10)

By using Eq.(8) and Eq.(10), we can get an algebraic equation of 2nd order about \( f(z_1) \)

$$f(z_1)^2 \left[ d_1 f'(z_2) + d_2 f(z_2) \right] + f(z_1) \left[ d_3 f'(z_2)^2 + d_4 f'(z_2) f(z_2) + d_5 f(z_2)^2 \right] + d_6 f'(z_2) f(z_2)^2 + d_7 f(z_2)^3 = 0,$$

(11)

where \( d_i \) ( \( i = 1, 2, \cdots, 7 \) ) are known functions of \( z_1 - z_2 \). Define

$$y = \frac{f(z_1)}{f(z_2)},$$

$$\theta = \frac{f'(z_2)}{f(z_2)} = \frac{\partial}{\partial z_2} \ln \left\{ \frac{L(a^{+}i'|z_2)^{i'}_{i}}{L(b^{+}i'|z_2)^{i'}_{i}} \right\}.$$

Then, Eq.(11) can be rewritten as

$$y^2 (d_1 \theta + d_2) + y (d_3 \theta^2 + d_4 \theta + d_5) + (d_6 \theta + d_7) = 0.$$

(12)
When $z_2$ is fixed, the coefficients of Eq.(12) are the functions of $z_1$. So $y$ is also a function of $z_1$. Since Eq.(12) is of 2nd order, the $y$ can only have two solutions $y_1(z_1, z_2)$ and $y_2(z_1, z_2)$ at most. If we can find two different $L$-matrices $L_1(\theta|z)$ and $L_2(\theta|z)$ which satisfy the DYBR with same $\theta$, we must have $f(z_1)/f(z_2) = f_1(z_1)/f_1(z_2)$ or $f(z_1)/f(z_2) = f_2(z_1)/f_2(z_2)$, where $f_1$ and $f_2$ are obtained by the two different $L$’s. Then, we can obtain $f(z_1) \sim f_1(z_1)$ or $f(z_1) \sim f_2(z_1)$, where “$\sim$” implies that as the function of $z_1$, two sides of it can only be different with a constant respect to $z_1$. Thus, we can conclude that if there are two $L_1(\theta|z_i) (i = 1, 2)$ which satisfy the DYBR and are not proportion to each other, and when $z = z_2$, they have same $\theta = \theta_1(z_2) = \theta_2(z_2)$, then every $f(z)$ related with $L(\theta|z)$ satisfying $f'(z)/f(z) = \theta$ when $z = z_2$, must satisfy

\[
\frac{f(z)}{f(z_2)} = f_1(z_1)\text{const.} \quad \text{or} \quad \frac{f(z)}{f(z_2)} = f_2(z_2)\text{const.},
\]

(13)

where the constants do not depend on $z$.

Now we consider the factorized $L$-matrix[14-17] which has an adjustable parameter $\delta$. We will show that for the given $z_2$ and $\theta$, there are generally two different $\delta$’s which can give $f_1(z_2)/f_2(z_2) = f_1(z_2)/f_2(z_2) = \theta$.

Considering the intertwiner of the $Z_n$ Belavin model and the $A_{n-1}$ IRF model[19, 20], we have

\[
\varphi^{(j)}_{a+i,a}(z) = \theta \left[ \frac{1}{2} - \frac{i}{n} \right] (z + n(a + i)_i, n\tau) \equiv \theta^{(j)}(nz_i),
\]

\[
(a + i)_i = w(m_i + 1 - \frac{1}{n} \sum_l (m_l + \delta_{il}) + w_i) = a_i + w(1 - \frac{1}{n}).
\]

Define $\tilde{\varphi}^{(j)}_{a+i,a}(z)$ which satisfies

\[
\sum_{j=0}^{n-1} \tilde{\varphi}^{(j)}_{a+i,a}(z) \varphi^{(j)}_{a+i,a}(z) = \delta_{\mu\nu}.
\]

Let

\[
\tilde{L}_s(\theta|z)\nu = \sum_{j=0}^{n-1} \tilde{\varphi}^{(j)}_{a+i,a}(z) \varphi^{(j)}_{a+i,a}(z + s),
\]

(14)

where $s$ is an arbitrary parameter. Then by using the correspondence relation between face and vertex[20], we can prove that the $L$-matrix above satisfies the DYBR Eq.(2). After some derivation, we have[17]

\[
\tilde{L}_s(\theta|z)\nu = \frac{\sigma(z + \Delta + (n - 1)w - n\frac{1}{2} + \frac{a - 1}{n} + b - a)}{\sigma(z + \Delta + (n - 1)w - n\frac{1}{2})} \prod_{j(\neq \nu)} \frac{\sigma(\delta + b - a)}{\sigma(a - a_j)}
\]

with $\Delta = w \sum_j w_j$. Let $\delta = \Delta + (n - 1)w - (n - 1)/2 + s/n = \delta(s)$, $\delta' = s/n$. Since $\sigma(z + \Delta + (n - 1)w - (n - 1)/2)$ is irrelevant with $a, b, \mu, \nu$, from the above formula, we can prove that

\[
L_\delta(\theta|z)\nu = \tilde{L}_s(\theta|z)\nu \sigma(z + \Delta + (n - 1)w - n\frac{1}{2})
\]

\[
= \sigma(z + \delta + b - a) \prod_{j(\neq \nu)} \frac{\sigma(\delta' + b - a)}{\sigma(a - a_j)}
\]

(15)
also satisfy the DYBR (Eq.(2)).

Considering the definition of $\theta$, we have

$$\theta(z) = \frac{f'_d(z)}{f_d(z)} = \frac{\sigma'(z + \delta + b_i - a_j' + w)}{\sigma(z + \delta + b_i - a_j' + w)} - \frac{\sigma'(z + \delta + b_i - a_j)}{\sigma(z + \delta + b_i - a_j')} \quad (16)$$

By using the properties of the $\theta$-function, one can show that for a given $\theta$, there generally exist two different $\delta$’s satisfying Eq.(16).

From Eq.(13), we know that for the $L$-matrix which satisfies the DYBR,

$$f(z) \sim f_d(z) \quad (17)$$

must be held for certain $\delta$. And from Eq.(8), we know that $g(z)$ and $h(z)$ can be determined completely by $f(z)$ up to a scale. So we have

$$g(z) \sim g_d(z), \quad h(z) \sim h_d(z) \quad (18)$$

Here the parameter $\delta$ is the same as that in Eq.(17). Then, from Eq.(17) and Eq.(18), we have

$$\frac{L_{(b+1)}^j}{L_{b}^j} \sim \frac{\sigma(z + \delta + b_i - a_j' + w)}{\sigma(z + \delta + b_i - a_j')} \quad (19)$$

$$\frac{L_{(b)}^j}{L_{b}^j} \sim \frac{\sigma(z + \delta + b_i - a_j')}{\sigma(z + \delta + b_i - a_j')} \quad (20)$$

So from Eq.(19) and Eq.(21), we can obtain

$$\frac{L_{(b+1)}^j}{L_{b}^j} \sim \frac{\sigma(z + \delta + b_i - a_j')}{\sigma(z + \delta + b_i - a_j')} \quad (22)$$

In Eqs.(19)-(22), all $\delta$’s are the same. We note here that the $\delta$ may depend on $i, i', j', a, b$, but it does not depend on $z$, i.e. $\delta = \delta_i(abij)$. One sees from Eqs.(19), (20) and (22) $\delta_i(i' j') \equiv \delta_i(j' i') \pmod{\Lambda_r}$.

Similarly, from the Eq.(5), we have

$$\frac{L_{(b+1)}^j}{L_{b}^j} \sim \frac{\sigma(z + \delta + b_i - a_j' - w)}{\sigma(z + \delta + b_j - a_i')} \quad (23)$$

$$\frac{L_{(b)}^j}{L_{b}^j} \sim \frac{\sigma(z + \delta + b_i - a_j')}{\sigma(z + \delta + b_j - a_i')} \quad (24)$$

$$\frac{L_{(b+1)}^j}{L_{b}^j} \sim \frac{\sigma(z + \delta + b_i - a_i')}{\sigma(z + \delta + b_i - a_i')} \quad (25)$$

$$\frac{L_{(b)}^j}{L_{b}^j} \sim \frac{\sigma(z + \delta + b_j - a_i')}{\sigma(z + \delta + b_j - a_i')} \quad (26)$$

Here the dependence of the $\delta$’s are similar with the former. We also have $\delta = \delta^ij(abij)$ and $\delta^ij(i) \equiv \delta^ij(ji) \pmod{\Lambda_r}$.
3 Dependence of elements of $L$-matrix with spectral parameter $z$

In this section, we study the dependence of $L(b | z)_{ij}$ with respect to $z$. It is found that there are only five possible forms of $L$-matrices in the whole lattice. We prove this in the following steps.

Step 1. Assume $i \neq i'$, $j \neq j'$. From Eq.(22) and Eq.(26), we have

$$\frac{L(b | z)_{ij}^2}{L(b | z)_{ij}^2} = \frac{L(b | z)_{i'j}^2}{L(b | z)_{i'j}^2} \sim \frac{\delta(z + \delta_i + b_i - a_j) \sigma(z + \delta' + b_i - a_j)}{\sigma(z + \delta + b_i - a_j) \sigma(z + \delta' + b_i - a_j)}$$

(27)

$$\frac{L(b | z)_{ij}^3}{L(b | z)_{ij}^3} = \frac{L(b | z)_{i'j}^3}{L(b | z)_{i'j}^3} \sim \frac{\delta(z + \delta_i + b_i - a_j) \sigma(z + \delta' + b_i - a_j)}{\sigma(z + \delta_i + b_i - a_j) \sigma(z + \delta' + b_i - a_j)}$$

(28)

giving

$$\frac{\delta(z + \delta_i + b_i - a_j) \sigma(z + \delta_i + b_i - a_j)}{\sigma(z + \delta_i + b_i - a_j) \sigma(z + \delta_i + b_i - a_j)} = 1,$$

(29)

where

$$\delta_i = \delta_i(a, b, j, j'), \quad \delta_j = \delta_j(a, b, i, i'),$$

$$\delta_i = \delta_i(a, b, j, j'), \quad \delta_j = \delta_j(a, b, i, i').$$

Obviously in Eq.(29), the zeroes of numerator must coincide with those of denominator. From this fact and noticing that $a_j$ and $a_{j'}$, $b_i$ and $b_i'$ are generic complex numbers, we analyze all cases and obtain

$$\delta_i - \delta_i' \cong K(b_i - b_i') \quad \text{and} \quad \delta_j - \delta_j' \cong K(a_j - a_{j'}).$$

(30)

where $\delta_i = \delta_i(j, j')$, $\delta_i' = \delta_i(j, j')$, $\delta_j = \delta_j(i, i')$, $\delta_j' = \delta_j(i, i')$ and $K = K(i, i', j, j')$.

From Eq.(29), we also have

$$\delta_i \cong \delta_i' \cong \delta_i \cong \delta_i' \quad \text{when} \quad K = 0,$$

(31)

$$\delta_i(j, j') - \delta_i'(i, i') \cong b_i - b_i' + a_j - a_{j'}, \quad \text{when} \quad K = 2.$$  

(32)

Step 2. Since the dimension $n \geq 4$, we may choose three different $i_1, i_2, i_3$ and substitute $\{i_1, i_2\}, \{i_2, i_3\}, \{i_1, i_3\}$ as $\{i, i'\}$ into Eq.(30). This leads to the conclusion that $K$ is independent of the indices $i, i', j$ and $j'$.

These are the rules for the differences between $\delta_i(j, j')$ and $\delta_i'(j, j')$ and for the differences between $\delta_j(i, i')$ and $\delta_j'(i, i')$.

Step 3. Now let us study the differences between $\delta_i(j_1, j_2)$ and $\delta_i(j_3, j_4)$. Consider different indices $j_1, j_2, j_3, j_4$. We have from Eq.(22)

$$\frac{L(b | z)_{i_1}^{j_1} L(b | z)_{i_2}^{j_2}}{L(b | z)_{i_1}^{j_2} L(b | z)_{i_2}^{j_1}} \sim \frac{\sigma(z + \delta_i(j_1, j_2) + b_i - a_{j_1}) \sigma(z + \delta_i(j_2, j_3) + b_i - a_{j_2})}{\sigma(z + \delta_i(j_1, j_2) + b_i - a_{j_2}) \sigma(z + \delta_i(j_2, j_3) + b_i - a_{j_3})},$$

$$\frac{L(b | z)_{i_1}^{j_1} L(b | z)_{i_2}^{j_2}}{L(b | z)_{i_1}^{j_2} L(b | z)_{i_2}^{j_1}} \sim \frac{\sigma(z + \delta_i(j_1, j_3) + b_i - a_{j_1})}{\sigma(z + \delta_i(j_1, j_3) + b_i - a_{j_3}).}$$

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This implies
\[
\frac{\sigma(z + \delta_i (j_1 j_2) + b_i - a_{j_1})}{\sigma(z + \delta_i (j_1 j_2) + b_i - a_{j_2})} \frac{\sigma(z + \delta_i (j_2 j_3) + b_i - a_{j_2})}{\sigma(z + \delta_i (j_2 j_3) + b_i - a_{j_3})} \frac{\sigma(z + \delta_i (j_1 j_3) + b_i - a_{j_3})}{\sigma(z + \delta_i (j_1 j_3) + b_i - a_{j_1})}
\equiv \frac{(1) (2) (3)}{(1') (2') (3')} \sim 1.
\]

From this equation, we obtain,
\[
d_i (j_1 j_2) - k(a_{j_1} + a_{j_2}) \equiv d_i (j_2 j_3) - k(a_{j_2} + a_{j_3}) \equiv d_i (j_1 j_3) - k(a_{j_1} + a_{j_3}) \quad k = 0, 1, (33)
\]
where \( k = k(i j_1 j_2 j_3) \).

Step 4. Consider unequal \( j_a, j_b, j_c, j_d \), and substitute \( \{j_a, j_b, j_c\}; \{j_b, j_c, j_d\}; \{j_a, j_c, j_d\} \) as \( \{j_1, j_2, j_3\} \) into Eq.(33), we may show that \( k \) is also independent of the indices.

Therefore, from Eq.(30) and Eq.(33), we conclude that one can always find a number \( C \) independent of indices \( i j j' \) such that
\[
C \equiv d_i (j j') - k_0 (a_j + a_{j'}) + K b_i.
\]

Similarly, we also find a number \( D \) satisfying
\[
D \equiv d^j (i i') + k^0 (b_i + b_{i'}) - K a_j,
\]
where \( D, K, k_0, k^0 \) are independent of indices, and are fixed for a given lattice point \( (a, b) \).

Step 5. In the following, we discuss the cases \( K = 0 \) and 2. For \( K = 0 \), one has from Eq.(30), Eq.(31), Eq.(34) and Eq.(35)
\[
\delta_i (j j') \equiv C + k_0 (a_j + a_{j'}) \equiv d^j (i i') \equiv D - k^0 (b_i + b_{i'})
\Rightarrow D - C = k_0 (a_j + a_{j'}) + k^0 (b_i + b_{i'}).
\]
Thus, the \( k_0 \) and \( k^0 \) must be zero since \( C \) and \( D \) are independent of the indices. We have
\[
\delta \equiv C \equiv D \equiv \delta_i \equiv \delta^j.
\]
When \( K = 2 \), From Eq.(32), we can find a number \( E \) satisfying
\[
E \equiv C \equiv D \quad \text{and} \quad k_0 = k^0 = 1.
\]

Step 6. We next study the relations for \( C, D, K, k_0, k^0 \) between adjacent lattice point \( (a, b) \) and \( (a + \hat{i}', b + \hat{i}) \). Eq.(19) and Eq.(23) intertwine two lattice points. Notice that in Eqs.(19)-(22) (or in Eqs.(23)-(26)) the \( \delta \)’s are the same. By using these equations, we can prove that \( k, k_0, k^0 \) are unchanged for adjacent lattice points while
\[
C(a + \hat{i}', b + \hat{i}) - C(a, b) = C' - C \equiv -k_0 w(1 - \frac{2}{n}) + Kw(1 - \frac{1}{n}),
\]
\[
D(a + \hat{i}', b + \hat{i}) - D(a, b) = D' - D \equiv k^0 w(1 - \frac{2}{n}) - Kw(1 - \frac{1}{n}).
\]
These equations imply that Eq.(37) can not be realized in two adjacent lattice points. Thus \( K = 2 \) must be discarded.

According to \( K, k_0, k^0 \), when \((a,b)\) is given, the elements of the \( L \)-matrix can take five forms.

(1). Form A(1). \( K = 1, k_0 = k^0 = 0 \), from Eq.(22), Eq.(34) and Eq.(35), we have

\[
\frac{L(\delta_i|z)_i^j}{L(\delta_i|z)_0^0} \sim \frac{\sigma(z + \delta_i(0j) + b_i - a_j)}{\sigma(z + C - b_i - a_j)} \sim \frac{\sigma(z + C - b_i + b_i - a_j)}{\sigma(z + C - b_i + b_i - a_j)} = \frac{\sigma(z + C - a_j)}{\sigma(z + C - a_j)},
\]

and from Eq.(26), we have

\[
\frac{L(\delta_i|z)_0^0}{L(\delta_i|z)_0^0} \sim \frac{\sigma(z + \delta^0(0j) + b_i - a_0)}{\sigma(z + D + a_0 + b_i - a_0)} \sim \frac{\sigma(z + D + a_0 + b_i - a_0)}{\sigma(z + D + b_i)} = \frac{\sigma(z + D + b_i)}{\sigma(z + D + b_i)}.
\]

Therefore, we obtain

\[
L(\delta_i|z)_i^j \sim \frac{\sigma(z + C - a_j) \sigma(z + D + b_i)}{\sigma(z + C - a_0) \sigma(z + D + b_0)} L(\delta_i|z)_0^0 \sim \frac{\sigma(z + C - a_j) \sigma(z + D + b_i) F(\delta_i|z)}{\sigma(z + C - a_0) \sigma(z + D + b_0)}.
\]

By using Eq.(27), Eq.(34) and Eq.(35), we can similarly derive other forms as follows,

(2). Form A(2). \( K = 1, k_0 = 0, k^0 = 1 \), we have

\[
L(\delta_i|z)_i^j \sim \frac{\sigma(z + C - a_j) F(\delta_i|z)}{\sigma(z + D + b_i)}.
\]

(3). Form A(3). \( K = 1, k_0 = 1, k^0 = 1 \), we have

\[
L(\delta_i|z)_i^j \sim \frac{1}{\sigma(z + C + a_j) \sigma(z + D - b_i)} F(\delta_i|z).
\]

(4). Form A(4). \( K = 1, k_0 = 1, k^0 = 0 \), we have

\[
L(\delta_i|z)_i^j \sim \frac{\sigma(z + D + b_i)}{\sigma(z + C + a_j)} F(\delta_i|z).
\]

(5). Form B. \( K = 0, k_0 = 0, k^0 = 0 \), we have from Eq.(27) and Eq.(36), one obtains

\[
L(\delta_i|z)_i^j \sim \sigma(z + \delta + b_i - a_j) F(\delta_i|z).
\]

The relation of \( F(z) \) between adjacent lattice points \((a,b)\) and \((a',b')\) are discussed in the appendix A.

In conclusion, there can at most five classes of \( L \)-matrices in the whole lattice. Each of them is of the same form at all lattice points.

We must check that if these inductive relations are integrable in the whole lattice. That is, if one goes from \((a,b)\) to \((a'' = a + i' + j', b'' = b + i + j')\) via different paths, the resulting \( C'' D'' F''(z) \) should be the same. The conclusion is affirmative.
For \( a \equiv (m_0, m_1, \ldots, m_{n-1}) \), define \( m \equiv \sum_i m_i \). Thus \( m(a' = a + \hat{i}', b' = b + \hat{i}) = m(a, b) + 1 \). We can express five forms as follows, which satisfy all relations of adjacent lattice points,

1. Form A(1). Let \( C = C_0 + mw(1 - 1/n), \quad D = D_0 - mw(1 - 1/n) \). Then
\[
L^{(a)}_{(b)}[z]_k \sim \sigma(z + C_0 + mw(1 - \frac{1}{n}) - a_i)\sigma(z + D_0 - mw(1 - \frac{1}{n}) + b_k)F_0(z) \quad (45)
\]
and \( C_0, \ D_0, \ F_0(z) \) are unchanged in the whole lattice.

2. Form A(2). Let
\[
C = C_0 + mw(1 - \frac{1}{n}), \quad D = D_0 - mw, \quad F(z) = F_0(z) \prod_{j=0}^{n-1} \sigma(z + D_0 - mw - b_j).
\]
We then have
\[
\frac{F'(z)}{F(z)} = \frac{\sigma(z + D_0 - (m + 1)\frac{w}{n} - b_i - w(1 - \frac{1}{n}))}{\sigma(z + D_0 - m\frac{w}{n} - b_i)} = \frac{\sigma(z + D - b_i - w)}{\sigma(z + D - b_i)}.
\]
Thus,
\[
L^{(a)}_{(b)}[z]_k \sim \sigma(z + C_0 + mw(1 - \frac{1}{n}) - a_i) \prod_{j(\neq k)} \sigma(z + D_0 - mw - b_j)F_0(z) \quad (46)
\]
and \( C_0, \ D_0, \ F_0(z) \) are unchanged in the whole lattice.

3. Form A(3). Let
\[
C = C_0 + \frac{w}{n}, \quad D = D_0 - \frac{w}{n}, \quad F(z) = F_0(z) \prod_{j=0}^{n-1} \sigma(z + C_0 + \frac{w}{n} + a_j)\sigma(z + D_0 - \frac{w}{n} - b_j).
\]
We then have
\[
\frac{F'(z)}{F(z)} = \frac{\sigma(z + C + a_i + w)\sigma(z + D - b_i - w)}{\sigma(z + C + a_i)\sigma(z + D - b_i)}.
\]
Thus,
\[
L^{(a)}_{(b)}[z]_k \sim \prod_{j(\neq i)} \sigma(z + C_0 + \frac{w}{n} + a_j) \prod_{j(\neq k)} \sigma(z + D_0 - \frac{w}{n} - b_j)F_0(z) \quad (47)
\]
and \( C_0, \ D_0, \ F_0(z) \) are unchanged in the whole lattice.

4. Form A(4). Let
\[
C = C_0 + \frac{w}{n}, \quad D = D_0 - mw(1 - \frac{1}{n}), \quad F(z) = F_0(z) \prod_{j=0}^{n-1} \sigma(z + C_0 + m\frac{w}{n} + a_j).
\]
We then have
\[ L(z|z)_k^l \sim \sigma(z + D_0 - mw(1 - \frac{1}{n}) + b_k) \prod_{j \neq i} \sigma(z + C_0 + m\frac{w}{n} + a_j)F_0(z) \]  
(48)
and \( C_0, D_0, F_0(z) \) are unchanged in the whole lattice.

(5). Form B.
\[ L(z|z)_k^l \sim \sigma(z + \delta + b_k - a_l)F_0(z) \]  
(49)
and \( \delta_0, F_0(z) \) are unchanged in the whole lattice.

Thus we can establish the \( L \)-matrix in the whole lattice, if we can properly choose the coefficients of the elements of \( L \)-matrix. We will discuss this problem in the next section.

4 The coefficients irrelevant with \( z \) of the elements of \( L \)-matrix

In this section, we study the sufficient condition of \( L \)-matrices for DYBR and derive the equations satisfied by the coefficients irrelevant with \( z \) of the elements of \( L \)-matrix.

As an example, we study the form B which is useful in the later. From the Eq.(44) for the form B, The \( L \)-matrix takes the form
\[ L(z|z)_k^l = (z)_{b+i}^l \sigma(z + \delta + b_l - a_j)F(z), \]  
(50)
\[ L(z|z)_k^l = (z)_{b+i}^l \sigma(z + \delta + b_j - a_j)F(z). \]  
(51)

Then, substituting the above equation and the Eq.(1) for the \( R \)-matrix into the DYBR Eq.(2) and noticing the fact
\[ a'_{j'} = a_j + w(\delta_{i'j'} - \frac{1}{n}), \quad b'_{j'} = b_j + w(\delta_{ij} - \frac{1}{n}), \quad (\text{for } a' = a + \iota', b' = b + \hat{i}) \]
we obtain the equations for the coefficients:
\[ \left( \begin{array}{c} a'_{j'} \\ b'_{j'} \end{array} \right)_i \left( \begin{array}{c} a + \iota' \\ b + \hat{i} \end{array} \right)_i = \left( \begin{array}{c} a \\ b \end{array} \right)_i \left( \begin{array}{c} a + \iota \\ b + \hat{i} \end{array} \right)_i, \]  
(52)
which is trivially satisfied, and
\[ (a)_{b}^l (a')_{b+i}^l - \frac{\sigma(a_{r-j} - w)}{\sigma(a_{r-j} + w)} (a)_{b}^l (a')_{b+i}^l = 0 \quad (i' \neq j'), \]  
(53)
\[ (a)'_{b}^l (a')_{b+i}^l - (a)'_{b}^l (a')_{b+i}^l = 0 \quad (i \neq j), \]  
(54)
\[ (a)'_{b}^l (a')_{b+i}^l \sigma(a_{r-j} + b_{ij})\sigma(w) + (a)_{b}^l (a')_{b+i}^l \sigma(b_j - w)\sigma(a_{r-j}) \]  
\[ - (a)'_{b}^l (a')_{b+i}^l \sigma(a_{r-j} - w)\sigma(b_{ij}) = 0 \quad (i \neq j, \ i' \neq j'), \]  
(55)
respectively. In the derivation, we have used the addition formula
\[ \sigma(u + x)\sigma(u - x)\sigma(v + y)\sigma(v - y) - \sigma(u + y)\sigma(u - y)\sigma(v + x)\sigma(v - x) = \sigma(u + v)\sigma(u - v)\sigma(x + y)\sigma(x - y), \] (56)

Define
\[ (a) \quad Y_{i j}^{i'j'} = \prod_{l(i \neq i')} \sigma(a_l - a_{i'}), \]
\[ [a]^{i'j'} \quad \left[ b_{\hat{b} + i} \right]_{j} = Y_{i j}^{i'j'}. \] (57)

Then for form B, we rewrite the Eqs.(53)-(55) as
\[ Y_{i i}^{i'j'} - Y_{i i}^{i'j'} = 0 \quad (i' \neq j'), \] (58)
\[ Y_{i j}^{i'j'} - Y_{i j}^{i'j'} = 0 \quad (i \neq j), \] (59)
\[ \sigma(w)\sigma(a_{i'j'} + b_{ij})Y_{i i}^{i'j'} + \sigma(a_{i'j'})\sigma(b_{ij} - w)Y_{i i}^{i'j'} \]
\[ - \sigma(a_{i'j'} + w)\sigma(b_{ij})Y_{i i}^{i'j'} = 0 \quad (i \neq j, i' \neq j'). \] (60)

With same procedure, one can also show that all A forms (form A(1)-A(4)) share a common coefficient relations
\[ Y_{i i}^{i'j'} - \frac{\sigma(a_{i'j'} - w)}{\sigma(a_{i'j'} + w)} Y_{i i}^{i'j'} = 0 \quad (i' \neq j'), \] (61)
\[ Y_{i j}^{i'j'} - Y_{i j}^{i'j'} = 0 \quad (i \neq j), \] (62)
\[ Y_{i j}^{i'j'} = Y_{i j}^{i'j'} = \frac{\sigma(a_{i'j'} - w)}{\sigma(a_{i'j'} + w)} Y_{i j}^{i'j'} \quad (i \neq j, i' \neq j'). \] (63)

For the coefficients of form A(i) (i=1,2,3,4), we can easily find the rule. Consider a function \( G(a, b) \) on a lattice points \( a = \sum_j m^j_i, b = \sum_i m^i_j \). From the lattice \( (a, b) \), by using the relation \( G(a+\hat{i}', b+\hat{j}) = G(a, b)\left[ b \right]^{i'}_{j} \), we can construct the function on the other lattice point. Because of the Eqs.(61)-(63), we can obtain same \( G(a+\hat{i}' + \hat{j}', b + \hat{i} + \hat{j}) \) through different paths from \( (a, b) \) to \( (a + \hat{i}' + \hat{j}', b + \hat{i} + \hat{j}) \). So this procedure is integrable. This implies that there must exist a function \( G(a, b) \) which can determine \( [a]^{i'}_{j} \) with
\[ [a]^{i'}_{j} = G(a + \hat{i}', b + \hat{j})/G(a, b). \] (64)

Hence, we can solve the problem of form A completely. However, to the coefficients of the form B, its rule is more complicated and we will discuss it in the next section.

Obviously, if we take a gauge transformation
\[ [a]^{i'}_{j} \rightarrow \tilde{[a]}^{i'}_{j} = \frac{\tilde{g}(a + \hat{j}, b + \hat{i})}{g(a, b)}, \]
and if \( \tilde{[a]}^{i'}_{j} \) satisfies Eqs.(61)-(63), \( [a]^{i'}_{j} \) also satisfies these equations. In this sense, all form A coefficients are gauge equivalent to a constant.
5 The algebra for form (B) coefficients

5.1 The PBW base of the algebra

In this section, we give the PBW base of the algebra for form (B) coefficients. The main result is Theorem 1. We also give the center of this algebra. Eqs.(58)-(60) can be regarded as the algebraic relations which are satisfied by the operators in the lattice \( a = \sum_{j=0}^{n-1} m_j \hat{j}, \) \( b = \sum_{i=0}^{n-1} m_i \hat{i} \). We define a new operator

\[
A_i^{i'} \equiv \left[ \frac{n}{[\hat{b}]}_{i} \right] \Gamma_i^{i'},
\]

where

\[
\Gamma_i^{i'} f(a, b) = f(a + \hat{i}', b + \hat{i}) \Gamma_i^{i'}.
\]

Namely, we regard the \( a, b \) as operators, the function \( \Gamma_i^{i'} \) is not commutative with the function of \( a, b \). In this way, we have the following exchange relations of the operators \( \{ A_i^{i'} \} \)

\[
\begin{align*}
(a) \quad & A_i^{i'} A_j^{j'} = A_j^{j'} A_i^{i'} \quad (i' \neq j'), \\
(b) \quad & \sigma(\alpha_{i'j'} + w) \sigma(b_{ij}) A_j^{j'} A_i^{i'} \\
& = \sigma(\alpha_{i'j'}) \sigma(b_{ij} - w) A_i^{i'} A_j^{j'} + \sigma(w) \sigma(\alpha_{i'j'} + b_{ij}) A_j^{j'} A_i^{i'} \quad (i \neq i', j \neq j'), \\
(c) \quad & A_i^{i'} A_j^{i'} = A_j^{i'} A_i^{i'} \quad (i \neq j).
\end{align*}
\]

These equations are equivalent relations to the Felder and Varchenko’s elliptic quantum algebra under special condition. It is worth noting that in the Eq.(67b), the coefficients should be regarded as the functions of operators and they do not commute with \( A_i^{i'} \). These equations are irrelevant with the parameter \( z \). This situation is similar to the relation between the Sklyanin algebra[21-25] and the YBR of the Belavin model[26-28].

In formulation, Eq.(67b) is also similar to the function \( R \)-matrices given by Shibukawa and Ueno[18].

Using the (a) and (b) of Eq.(67), we can exchange the order of the up-indices of a pair of operators \( A_i^{i'} A_j^{j'} \). So \( A_i^{i'} A_j^{j'} \) can be written as linear combination of \( A_i^{i'} A_j^{j'} \) and \( A_j^{j'} A_i^{i'} \). Therefore, we can write the product of three operators \( A_i^{i'} A_j^{j'} A_k^{k'} \) as the linear combination of \( A_i^{i'} A_j^{j'} A_k^{k'} \). This procedure can be done in two different ways. For the two ways, by using Eqs.(58)-(60), we can show that according to the rules Eq.(67a) and Eq.(67b) (we will simplify it as \( ab \)), if the product of three operators \( A_i^{i'} A_j^{j'} A_k^{k'} \) is changed to the linear combination of \( A_i^{i'} A_j^{j'} A_k^{k'} \) by two different paths, their results are equal. The paths are as follows:

\[
\begin{align*}
(A) & \quad i'j'k' \rightarrow i'k'j' \rightarrow k'i'j' \rightarrow k'j'i', \\
(B) & \quad i'j'k' \rightarrow j'i'k' \rightarrow j'k'i' \rightarrow k'j'i'.
\end{align*}
\]

In the above transformation, we think that the result of the \( ab \) transformation on two adjacent operators with same up-indices does not change the order of them, namely, \( A_i^{i'} A_j^{j'} \Rightarrow A_j^{j'} A_i^{i'} \). And we think the associative and the distributive law are satisfied in the transformation.

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Further more, if we consider the rule Eq.(67c), the linear expansions of operator products \( A_i' A_j' A_k' \) and \( A_i' A_j' A_k' \) by \( A_i' A_j' A_k' \) via the (ab) transformation are equal. Therefore, we also call this fact Yang-Baxter equation (YBE).

Similarly, after \( A_i' A_j' A_k' \) and the \( A_j' A_i' A_k' \) change to the linear combination of the \( A_i' A_j' A_k' \) by (ab), these two expansion are equal via the rule Eq.(67c).

For the coefficient algebra (or Yang-Baxter algebra) which we discussed above, we will give it a PBW base in the following. We first give some definitions for establishing the base.

**Definition 1: Bunch.** A bunch is a polynomial (or monomial) of operator \( A \)'s, in which all terms has the same number of \( A \)'s and the upper indices of \( A \)'s in all terms are arranged in the same way.

**Example:**

\[
B = \sum_{i_1, i_2, i_3, i_4} c_{i_1, i_2, i_3, i_4} A_{i_1}^{i_1} A_{i_2}^{i_2} A_{i_3}^{i_3} A_{i_4}^{i_4}
\]

is a bunch. A polynomial is always a bunch.

**Definition 2: Inverse order number.** To any two integers \( i', j' \) with a given order, we call the inverse order number is 1 if \( i' > j' \), is 0 if \( i' \leq j' \). And the inverse order number of a successive product \( A_i' A_j' A_k' \cdots \) is the sum of the inverse order numbers of all up-index pairs.

**Definition 3: Normal order product.** The (ab) normal order product is a successive product of operators in which the up-indices are arranged from smaller to bigger when inspecting from the left to the right, while the arrangement of the down-indices can be arbitrary. The (abc) normal order product is that the up-indices are arranged from the smaller to the bigger and the down-indices of the operators with the same up-indices are also arranged from smaller to bigger. Their inverse order numbers are zero.

**Example:** \( A_2^1 A_1^2 A_3^2 A_3^1 A_4^1 A_5^3 A_5^1 \) is an (ab) normal order product but is not an (abc) normal order product. By using the rule Eq.(67c), we can change it to the (abc) normal order product \( A_1^1 A_2^1 A_3^2 A_3^3 A_4^1 A_5^1 A_5^2 \).

**Definition 4: Normal order expansion.** The (ab) normal order expansion of a polynomial of \( A \)'s is a procedure in which we change each term of the polynomial into a bunch of (ab) normal order products by only using rules Eq.(67a) and Eq.(67b). We also call the final resulting polynomial as the (ab) normal expansion of the original polynomial.

The (abc) normal order expansion is a procedure, in which we first perform the (ab) normal order expansion and then we rearrange each term of the resulting polynomial into (abc) normal order product by using rule Eq.(67c). We also call the final result as an (abc) normal expansion of the original polynomial.

Then, we have a theorem.

**Theorem 1:** Transforming on a polynomial of operators \( A_i^j \) by using the rules (abc) of Eq.(67) does not change its (abc) normal order expansion.

It is worth noting that the coefficients of the expansions are functions of the parameters \( \{a, b\} \), they do not commute with operator \( A_i^j \).

The detailed proof of the theorem will be given in the appendix B.
**Corollary:** The (abc) normal order products are linearly independent.

**Proof:** If there were a linear relation \( \sum C_i g_i = 0 \), where \( g_i \) are (abc) normal order products. The LHS must be able to be changed to zero via Eq.(67). However, each operation does not change the (abc) expansion. Thus it is impossible since \( C_i \) are not all zero. \( \Delta \)

Thus the set of all (abc) normal order products is the PBW base of the algebra defined by Eq.(67).

### 5.2 The center of the algebra

By standard procedure, we may obtain the center of elliptic quantum group (the detail will be given elsewhere).

\[
I = \frac{\Delta(a)}{\Delta(b)} \text{Det} L(\alpha|z),
\]

where \( \Delta(a) = \prod_{i<j} \sigma(a_{ij}) \), \( \Delta(b) = \prod_{i<j} \sigma(b_{ij}) \),

\[
\text{Det} L(\alpha|z) = \sum_P (-1)^{\text{Sign}(\mu_0 \mu_1 \cdots \mu_{n-1})} \\
\times L_{(\mu_0 \mu_1 \cdots \mu_{n-1})}^{(a \cdots b \cdots)}(z + w)^{\mu_1} \cdots L_{(\mu_0 \mu_1 \cdots \mu_{n-2} + \mu_{n-1} - a_{n-1})}^{(a \cdots b \cdots)}(z + (n - 1)w)^{\mu_{n-1}},
\]

and \( P \)'s are permutations of integers 0, 1, \cdots, \( n - 1 \). This agrees with that of Ref.[12] for \( n = 2 \).

In the case of

\[
L(\alpha|z)_i' = \sigma(z + \delta + b_i - a_i') A_i',
\]

the quantum determinant can be written as

\[
I(\alpha|z) = \sum_P (-1)^{\text{Sign}(\mu_0 \mu_1 \cdots \mu_{n-1})} \\
\times \sigma(z + \delta + b_{\mu_0} - a_0) \sigma(z + w + \delta + b_{\mu_1} - a_1) \cdots \\
\times \sigma(z + (n - 1)w + \delta + b_{\mu_{n-1} - a_{n-1}}) A_{\mu_0}^0 A_{\mu_1}^1 \cdots A_{\mu_{n-1}}^{n-1}.
\]

It is easy to check

\[
\Phi(z)_{\mu_0 \cdots \mu_{n-1}} = \sigma(z + \delta + b_{\mu_0} - a_0) \cdots \sigma(z + (n - 1)w + \delta + b_{\mu_{n-1} - a_{n-1}})
\]

is quasi doubly periodic in \( \tau \) and 1:

\[
\Phi(z + 1) = (-1)^n \Phi(z), \\
\Phi(z + \tau) = \exp \left[ -2\pi i \left( \frac{n\tau}{2} + n\delta + nz + \frac{n(n-1)}{2}w + \frac{n}{2} + \sum_i b_{\mu_i} - \sum_i a_{\nu_i} \right) \right] \Phi(z)
\]

\[
= \exp \left[ -2\pi i \left( \frac{n\tau}{2} + n\delta + nz + \frac{n(n-1)}{2}w + \frac{n}{2} \right) \right] \Phi(z)
\]
for all $\mu_0, \cdots, \mu_{n-1}$ being a permutation of $(0, 1, \cdots, n-1)$. Therefore, from a theorem of such function (see D. Mumford, Tata Lectures on Theta, Birkhauser 1983), we have

$$\Phi(z)_{\mu_0, \cdots, \mu_{n-1}} = \sum_{i=0}^{n-1} C^i_{\mu_0, \cdots, \mu_{n-1}} f_i(z),$$

(68)

where $\{f_i(z)\}$ are base functions of the space of such quasi double periodic function. For example, we may choose

$$f_i(z) = \theta \left[ \frac{z-n\delta}{2} \right] \left( nz + n(n-1)w + \frac{n-1}{2}, n\tau \right).$$

One can obtain $C^i_{\mu_0, \cdots, \mu_{n-1}}$ by choosing $n$ points $z_1, \cdots, z_n$ in the above equation and solve a set of $n$ linear equations. We then derive the $n$ center elements for the algebra.

$$I_{(a|b)}(z) = \sum_i f_i(z) \left\{ \sum_P (-1)^P \text{Sign}(P(1, \cdots, n-1)) C^i_{\mu_0, \cdots, \mu_{n-1}} A^0_{\mu_0} A^1_{\mu_1} \cdots A^{n-1}_{\mu_{n-1}} \right\}$$

$$\equiv \sum_i f_i(z) J_i,$$

we see that $[\Delta(a)/\Delta(b)] J_i$ are the center elements of the algebra.

6 A known solution for the form B coefficients

The equations (Eqs.(58)-(60)) of form B coefficients seem simple but they interrelate the values of the coefficients $[a_{ij}']$ between different lattice points. To our best knowledge, we only know the analytic solution

$$[a_{ij}'] = \prod_{j(\neq i')} \sigma(\delta' + b_i - a_{i'}),$$

(69)

which can be derived by the factorized operator of Eq.(15)

$$L_\delta(a|b)_i^{(a')_i} = \sigma(z + \delta + b_i - a_{i'}) \prod_{j(\neq i')} \frac{\sigma(\delta' + b_i - a_j)}{\sigma(a_{i'} - a_j)}$$

$$\equiv (-1)^{n-1} \sigma(z + \delta + b_i - a_{i'}) [a_{i'}^{(a')_i}]$$

and

$$[a_{ij}'] = [a_{ij}]_i^{(a')_i} \prod_{j(\neq i')} \sigma(a_j - a_{i'}).$$

The corresponding $Y_{i,j}^{i',j'}$ is,

$$Y_{i,j}^{i',j'} = [a_{ij}]^{(a')_i}_{[a_{i'}^{(a')_i}]_j}$$

$$= \prod_{l(\neq i')} \sigma(\delta' + b_i - a_l) \prod_{m(\neq j')} \sigma(\delta' + b_j' - a_m').$$

(70)
By using the addition formula Eq.(56), we can check that the solution satisfies Eqs.(58)-(60) directly.

This solution can be proved to be equivalent with the results obtained by using the symmetry fusion method for the $A_{n-1}^{(1)}$ model in the Ref.[9]. And it is also equivalent with the evaluation modules ($n = 2$) obtained by Felder and Varchenko in the Ref.[12].

Eq.(69) is the only known solution for the form B coefficients. We do not know if there are other analytic solutions. This is still a worthy studying open question.

**Appendix A  The relation $F(z)$ between adjacent lattice points**

Suppose we go from $(a, b)$ to $(a + i', b + i)$, then we have $a' = a + w(\delta_{ij} - \frac{1}{n})$, $b' = b + w(\delta_{ij} - \frac{1}{n})$. From Eq.(38) and (39), we may choose

$$C' - C = -k_0 w(1 - \frac{2}{n}) + K w(1 - \frac{1}{n}), \quad (A.1)$$

$$D' - D = k_0 w(1 - \frac{2}{n}) - K w(1 - \frac{1}{n}) \quad (A.2)$$

without loss of generality. This is the explicit relations of $C, D, \delta$ between adjacent lattice points for each form of $L$-matrices. From Eq.(19)

$$\frac{L(a+i'|z)_{j}'}{L(b+i'|z)_{i}'} \sim \frac{\sigma(z + C + (1 - K)b_i + ka_i + (k_0 - 1)a_{j'} + w)}{\sigma(z + C + (1 - K)b_i + (k_0 - 1)a_{j'} + k_0 a_{j'})}.$$

(A.3)

The relations of $F(z)$ and $F'(z)$ (the new function at lattice point $(a', b')$) can be obtained by putting the explicit forms of five forms of $L$-matrices (Eqs.(40)-(44)) into Eq.(A.3). For example, we study the A(1) form.

(1) $A(1) \rightarrow A(1) \quad K = 1, \quad k_0 = k^0 = 0$

From Eq.(A.1) and Eq.(A.2), one has

$$C' = C + w(1 - \frac{1}{n}), \quad D' = D - w(1 - \frac{1}{n}). \quad (A.4)$$

Then Eq.(40) and Eq.(A.3) yield

$$\frac{L(a+i'|z)_{j}'}{L(a+i'|z)_{i}'} \sim \frac{\sigma(z + C' - a_{j'})\sigma(z + D' + b_i')F'(z)}{\sigma(z + C - a_{j'})\sigma(z + D + b_i)F(z)}$$

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\[
\sigma(z + C - a_j' + w)\sigma(z + D + b_i)F'(z) \\
\sigma(z + C - a_i' )\sigma(z + D + b_i)F(z) \\
\sim \frac{\sigma(z + C - a_j' + w)}{\sigma(z + C - a_i' )} \\
\Rightarrow \frac{F'(z)}{F(z)} \sim 1. \quad (A.5)
\]

Other A(i)'s are similar. We list them in the following.

(2) \( A(2) \xrightarrow{i,i'} A(2) \quad K = 1, \; k_0 = 0, \; k^0 = 1 \)

Eq.(A.1) and Eq.(A.2) give
\[
C' = C + w(1 - \frac{1}{n}), \quad D' = D - \frac{w}{n}, \quad (A.6)
\]

From Eq.(A.1) and Eq.(A.3), we have
\[
\frac{F'(z)}{F(z)} \sim \frac{\sigma(z + D - b_i - w)}{\sigma(z + D - b_i)}. 
\]

(3) \( A(3) \xrightarrow{i,i'} A(3) \quad K = 1, \; k_0 = k^0 = 1 \)

\[
\frac{F'(z)}{F(z)} \sim \frac{\sigma(z + C + a_i' + w)\sigma(z + D - b_i - w)}{\sigma(z + C + a_i')\sigma(z + D - b_i)}. \quad (A.7)
\]

(4) \( A(4) \xrightarrow{i,i'} A(4) \quad K = 1, \; k_0 = 1, \; k^0 = 0 \)

\[
\frac{F'(z)}{F(z)} \sim \frac{\sigma(z + C + a_i' + w)}{\sigma(z + C + a_i')}. \quad (A.8)
\]

(5) \( B \xrightarrow{i,i'} B \quad K = 0, \; k_0 = k^0 = 0 \)

From Eq.(A.1) and noting \( \delta \equiv C \equiv D \) for this class, we have \( C' \equiv D' \equiv \delta' \equiv C \equiv D \equiv \delta \). We may choose \( \delta' = \delta \) without loss of generality. Eq.(44) and Eq.(A.3) imply
\[
\frac{L(y^* | z)_i^{i'} \sigma(z + \delta' + b_i' - a_j')}{L(y^* | z)_i^{i} \sigma(z + \delta + b_i - a_j')} \\
= \frac{\sigma(z + \delta + b_i - a_j' + w)F'(z)}{\sigma(z + \delta + b_i - a_j')F(z)} \\
\sim \frac{\sigma(z + \delta + b_i - a_j' + w)}{\sigma(z + \delta + b_i - a_j')} \\
\Rightarrow \frac{F'(z)}{F(z)} \sim 1. \quad (A.9)
\]

### Appendix B The proof of the theorem 1

To prove the theorem, firstly, we prove the following lemma.
**Lemma 1:** To any successive product of operators, if we transform it by using Eq.(67a) and Eq.(67b) such that at each step its inverse order number is reduced (the adjacent up-indices is exchanged when the left one is bigger than that of the right one) the final result of the (ab) normal order expansion is unique.

Here we assume that in this transformation, two adjacent operators with same up-indices do not change the order. And we think that in every step of the transformation, the location of two exchanged operators in all terms of the linear combination after previous step are same.

**Proof:** We can do the procedure by different paths. For example, if we want to obtain (ab) normal order expansion of $A^4A^4A^5A^2A^2$, we may do this in following different paths:

1. $A^4A^4A^6A^5A^2A^2 \equiv (446522) \overset{Q_{1,5}}{\rightarrow} (446252) \overset{Q_{3,4}}{\rightarrow} (442652) \overset{Q_{5,6}}{\rightarrow} (442625) \overset{Q_{4,5}}{\rightarrow} (442265) \overset{Q_{2,3}}{\rightarrow} (242465) \overset{Q_{3,4}}{\rightarrow} (242456) \overset{Q_{5,6}}{\rightarrow} (244566) \overset{Q_{4,5}}{\rightarrow} (442562) \overset{Q_{2,3}}{\rightarrow} (244562) \overset{Q_{5,6}}{\rightarrow} (244256) \overset{Q_{4,5}}{\rightarrow} (244256) \overset{Q_{3,4}}{\rightarrow} (244265) \overset{Q_{2,3}}{\rightarrow} (244566)$, where $Q_{i,i+1}$ denotes the exchange of the $i$th operator $A$ and $i + 1$th operator $A$ by using rules Eq.(67a) and Eq.(67b). We may denote such procedure by product of a set of exchange operators $\{Q_{i,i+1}\}$ acting on the bunch. For the path (1) in the example, we have

$$Q_{5,6}Q_{2,3}Q_{3,4}Q_{2,3}Q_{5,6}Q_{3,4}Q_{4,5}A^4A^4A^6A^5A^2A^2 = \sum \cdots A^2A^2A^4A^4A^5A^6.$$ 

For the path (2), we have

$$Q_{2,3}Q_{3,4}Q_{4,5}Q_{5,6}Q_{1,2}Q_{2,3}Q_{4,5}Q_{3,4}A^4A^4A^6A^5A^2A^2 = \sum \cdots A^2A^2A^4A^4A^5A^6.$$ 

In general cases, a path of such procedure is denoted by

$$Q_{i_1,i_1+1}Q_{i_2,i_2+1} \cdots Q_{i_s,i_s+1} \left(A_{k_1}^{j_1}A_{k_2}^{j_2} \cdots A_{k_l}^{j_l}\right) = \sum_{j'_{k_1}}c_{j_1 \cdots j_l}A_{k_1}^{j_1'}A_{k_2}^{j_2'} \cdots A_{k_l}^{j_l'} \quad (B.1)$$

with $j_{i_1} \leq j_{i_2} \leq \cdots \leq j_{i_s}$. Note that the original arrangement $\{j_1j_2 \cdots j_l\}$ and the final arrangement $\{j_{i_1}j_{i_2} \cdots j_{i_s}\}$ are same for whatever path of the (ab) normal product expansion we choose.

Assume there is another path for (ab) normal product expansion

$$Q_{i',i'+1}Q_{i',i'+1} \cdots Q_{i',i'+1} \left(A_{k_1}^{j_1}A_{k_2}^{j_2} \cdots A_{k_l}^{j_l}\right) = \sum_{j'_{k_1}}d_{j_1 \cdots j_l}A_{k_1}^{j_1'}A_{k_2}^{j_2'} \cdots A_{k_l}^{j_l'} \quad (B.2)$$

Consider the corresponding two products of exchange operators in the permutation group

$$P^{(1)} = P_{i_1,i_1+1}P_{i_2,i_2+1} \cdots P_{i_s,i_s+1}$$

and

$$P^{(2)} = P_{i',i'+1}P_{i',i'+1} \cdots P_{i',i'+1}.$$ 

They must all be able to permute the arrangement $\{j_1 \cdots j_l\}$ into $\{j_{i_1}j_{i_2} \cdots j_{i_s}\}$. Although some of the $j$'s may be the same, the permutation $\{1_{i_1}2_{i_2} \cdots s_{i_s}\}$ is unique however.
This is due to the rule we do not exchange adjacent operators with same upper indices. In permutation group, we can express an arbitrary element by product of exchange operators in different ways. However, we can always make them equal step by step using the following equations.

\[ P_{i,i+1}P_{i,i+1} = id, \]  
\[ P_{i,i+1}P_{j,j+1} = P_{j,j+1}P_{i,i+1} \quad (i + 1 < j), \]  
\[ P_{i,i+1}P_{i+1,i+2}P_{i,i+1} = P_{i+1,i+2}P_{i,i+1}P_{i+1,i+2}. \]  

Thus \( P^{(1)} \) can be changed to \( P^{(2)} \) by using these equations step by step.

On the other hand, the \( \{Q_{i,i+1}\} \) operators have the same properties. We have checked

\[ Q_{i,i+1}Q_{i,i+1} = id \]  
for two adjacent operators \( A_{j_1}^{(a)}A_{k_2}^{(b)} \) (\( j_1 \neq j_2 \)), and thus it is also valid for all bunches due to distribution law. We also have

\[ Q_{i,i+1}Q_{j,j+1} = Q_{j,j+1}Q_{i,i+1} \quad (i + 1 < j) \]  

because of the distribution law. Finally we have

\[ Q_{i,i+1}Q_{i,i+1}Q_{i,i+1} = Q_{i,i+1}Q_{i,i+1}Q_{i,i+1} \]  
due to YBE for any polynomial \( A_{j_1}^{(a)}A_{k_2}^{(b)} \) with different indices. Due to distribution law, this equation is also true for any bunch. Therefore, we can also change \( Q^{(1)} = Q_{i,i+1}Q_{i,i+1}Q_{i,i+1} \) into \( Q^{(2)} = Q_{i,i+1}Q_{i,i+1}Q_{i,i+1} \) in Eq.(B.1) and Eq.(B.2), respectively, by using Eqs.(B.6)-(B.8) step by step since \( P^{(1)} \) and \( P^{(2)} \)
can be equaled in such way by using Eqs.(B.3)-(B.5), respectively. Thus we have

\[ C_{j_1\cdots k_{i-1}'} \cdots = d_{j_1\cdots k_{i-1}'} \cdots. \]

We then conclude that the resulting (ab) normal order expansion of the two paths give the same result. Therefore, all paths give the same result. \( \Delta \)

Corollaries then follows:

**Corollary 1:** If in a product of successive product of operators \( CA_{i}^{(a)}A_{j}^{(b)}D \) where \( C, D \) are all products of operators, we obtain the combination of \( CA_{i}^{(a)}A_{j}^{(b)}D \) (it is, \( C(\alpha A_{i}^{(a)}A_{j}^{(b)} + \beta A_{j}^{(a)}A_{i}^{(b)})D) \) by changing (with rule (ab) in Eq.(67)) two adjacent operators whose up-indices are unequal, the results of their (ab) normal order expansions are same, if the procedure is done according to the rules described in lemma 1.

**Proof:** If \( i' > j' \), we can regard this changing procedure as the first step of the (ab) normal order expansion. Thus, we can prove it. If \( i' < j' \), we can do the (ab) normal order expansion of \( C(\alpha A_{i}^{(a)}A_{j}^{(b)} + \beta A_{j}^{(a)}A_{i}^{(b)})D \), and let the first step as the changing of \( A_{j}^{(b)}A_{i}^{(a)} \) into \( A_{i}^{(a)}A_{j}^{(b)} \). Then, By using the rule (ab), we can prove that \( i' j' \rightarrow j' i' \rightarrow i' j' \) is the identical transformation. So with the distributive law, the (ab) normal order expansion of bunch \( C(\alpha A_{i}^{(a)}A_{j}^{(b)} + \beta A_{j}^{(a)}A_{i}^{(b)})D \) = the (ab) normal order expansion of \( CA_{i}^{(a)}A_{j}^{(b)}D \). Therefore, this corollary is proved. \( \Delta \)

**Corollary 2:** With the rules of the Eq.(67a) and Eq.(67b), if a polynomial (a linear combination of products) of operators \( C \) can be changed to \( D \) (\( C \rightarrow D \)), the
of the (abc) normal order expansion, so the above two corollaries are also true for the (abc) normal order expansion.

Proposition (i). This lemma is true when the inverse order number is zero.

Proposition (ii). If the lemma is true when the inverse order number is smaller than \( i \), the down-indices for up-indices can be rearranged according to the rules described in lemma 1.

Lemma 2: The (abc) normal order expansion of the bunch \( CA_j^i A_k^j D \) and the bunch \( CA_k^i A_j^i D \) are same.

Proof: We need only to prove this when they are monomials. We prove the following propositions by using the mathematical inductive method:

Proposition (i). This lemma is true when the inverse order number is zero.

Proposition (ii). If the lemma is true when the inverse order number is smaller than \( i \), it is also true when the inverse order number is equal to \( i \).

The first proposition is obvious, because in this case, \( CA_j^i A_k^j D \) and \( CA_k^i A_j^i D \) are all (ab) normal order products. To obtain the (abc) normalization, we only need to rearrange the down-indices of the part of the product where the up-indices are same from the smaller to the bigger by rule Eq.(67c). Both of the bunches have same sets of the down-indices for up-indices \( i' \). Therefore, the (abc) normal order products of them are same.

To the second proposition, we have the following cases:

(a). If in \( C \) or \( D \), we can rearrange the up-indices \( \{i'\} \) of them to reduce the inverse order number, for example, \( D \xrightarrow{(ab)} D' \). We can obtain \( CA_j^i A_k^j D' \) and \( CA_k^i A_j^i D' \). According to the corollary 2 of the lemma 1, the (ab) normal order expansions of both of them will keep unchanged. However, because the inverse order number must be smaller that \( m \) now, so according to assumption of the proposition (ii), their (abc) normal order expansions are same. Therefore, the (abc) normal order expansions of the \( CA_j^i A_k^j D \) and the \( CA_k^i A_j^i D \) are same.

(b). If \( C \) and \( D \) have already been normalized but the inverse order number of the bunch as a whole can be reduced, namely, the bunch is not an (ab) normal order product. We can let \( C = C_1 A_{i_c}^{i_c}, D = A_{i_d}^{i_d} D_1 \). Then we must have \( i_c > i' \) or \( (and) i' > i_d \). Let us assume \( i_c > i' \). These two bunches can be rewritten as \( T_1 = C_1 A_{i_c}^{i_c} A_j^i A_k^j D \) and \( T_2 = C_1 A_{i_c}^{i_c} A_k^i A_j^i D \) respectively. According to the rule (ab) in Eq.(67), we can change them as \( T_1 \Rightarrow T'_1 = C_1 \sum_{rst} a_{rst} A_r^{i_r} A_s^{i_s} A_t^{i_t} D \) and \( T_2 \Rightarrow T'_2 = C_1 \sum_{rst} b_{rst} A_r^{i_r} A_s^{i_s} A_t^{i_t} D \), where \( a_{rst} \) and \( b_{rst} \) are some coefficients. With the help of the YBE which we studied in section 5, one can see that these two combinations \( \sum_{rst} a_{rst} A_r^{i_r} A_s^{i_s} A_t^{i_t} \) and \( \sum_{rst} b_{rst} A_r^{i_r} A_s^{i_s} A_t^{i_t} \) are the same if we take the rule Eq.(67c) into account. Thus we must have

\[
\sum_{rst} a_{rst} A_r^{i_r} A_s^{i_s} A_t^{i_t} - \sum_{rst} b_{rst} A_r^{i_r} A_s^{i_s} A_t^{i_t} = \sum_l \left( \sum_{rs} (a_{rst} - b_{rst}) A_r^{i_r} A_s^{i_s} \right) A_t^{i_t}
\]
with $\sum_{rs} (a_{rst} \cdot b_{rst}) A^r_i A^s_i = 0$ if we take the rule Eq.(67c) into account. This is to say
\[ a_{rst} + a_{srt} = b_{rst} + b_{srt} = 2c_{rst} \text{ for each } t. \] (B.9)

Thus we have
\[
T'_1 = \sum_{rst} C_1 a_{rst} A^r_i A^s_i A^t_i D \equiv \sum_t \sum_{rs} \left( C_1 a_{rst} A^r_i A^s_i D_t \right)
\]
and
\[
T'_2 = \sum_{rst} C_1 b_{rst} A^r_i A^s_i A^t_i D \equiv \sum_t \sum_{rs} \left( C_1 b_{rst} A^r_i A^s_i D_t \right).
\]

From Eq.(B.9) and due to the assumption of the proposition (ii), the (abc) normal order expansions of $T'_1$ and $T'_2$ are the same. According to the procedure of the (abc) normal order expansion, we see that the (abc) normal order expansions of $T_1$ and $T_2$ are same.

If $i' > i'_d$, the proof is similar. So we see that the proposition (ii) is true.

Thus, with the mathematical inductive method, we prove the lemma 2.

\[ \Delta \]

From the corollary 2 of lemma 1 and lemma 2, we obtain theorem 1.

References

[1] S.N.M.Ruijsenaars. Commun. Math. Phys. 110, 191 (1987).
[2] F.Calogero. Lett. Nuovo Cimento 13, 411-415 (1975).
[3] J.Moser. Adv. Math 16, 441-416 (1975).
[4] E.D’Hoker and D.H.Phong. Nucl. Phys. B513 405-444 (1998) and references there in.
[5] A.Gorsky, I.Krichever, A.Marshakov, A.Mironov and A.Morozov. Phys. Lett. B355, 466 (1995).
[6] H.W.Braden, A.Marshakov A.Mironov and A.Morozov. hep-th/9902205 and references there in.
[7] M.Jimbo, T.Miwa and M.Okado. Lett. Math. Phys. 14, 123-131 (1987).
[8] M.Jimbo, T.Miwa and M.Okado. Commun.Math.Phys. 116, 507-525 (1988).
[9] M.Jimbo, T.Miwa and M.Okado. Commun. Math. Phys. 119, 543-565 (1988)
[10] J.L.Gervais and A.Neveu. Nucl. Phys.B238, 125 (1984).
[11] G.Felder. hep-th/9412207.
[12] G.Felder, and A.Varchenko. Commun.Math.Phys 181, 741 (1996). hep-th/9601003.
[13] J.Avan, O.Babelon and E.Billey. hep-th/9505091.
[14] V.V.Bazhanov, R.M.Kashaev, V.V.Mangazeev and Y.G.Strogonov. Commun. Math. Phys. **138**, 393 (1991).
[15] Y.H.Quano and F.A.ujii. Tokyo Univ. preprint UT-603 (1992).
[16] K.Hasegawa. q-alg/9512029.
[17] B.Y.Hou, K.J.Shi and Z.X.Yang. J. Phys **A26**, 4951 (1993).
[18] Y.Shibukawa and K.Ueno. Completely Z symmetric R-matrix. Waseda University preprint, (1992).
[19] R.J.Baxter. Exactly Solved Models in Statistical Mechanics. Academic Press, New York, 1982 PP.202-272.
[20] M.Jimbo, T.Miwa and M.Okado. Nucl. Phys. **B300**, 74-108 (1988).
[21] P.P.Kulish, N.Y.Reshetikhin and E.K.Sklyanin. Lett. Math. Phys. **5** 393 (1981).
[22] E.K.Sklyanin. Funct. Anal. Appl. **16**, 263-270 (1982).
[23] I.V.Cherednik. Funct. Anal. Appl. **19**, 77-79 (1985).
[24] B.Y.Hou and H.Wei. J. Math. Phys. **30**, 2750-2755 (1989).
[25] Y.H.Quano and A.Fujii. Mod. Phys. Lett. **A6**, 3635 (1991).
[26] A.A.Belavin. Nucl. Phys. **B180**, 189-200 (1981).
[27] M.P.Richey and C.A.Tracy. J. Stat. phys. **42**, 311-348 (1986).
[28] C.A.Tracy. Physica **D16** 203-220 (1985).