The radiative transfer equations for Compton scattering of polarized low frequency radiation on a hot electron gas

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21 March 2022

ABSTRACT
We deduce the equations that describe how polarized radiation is Comptonized by a hot electron gas. Low frequencies are considered, and the equations are expanded to second order in electron velocities. Induced scattering terms are included and a Maxwellian velocity distribution for the electrons is assumed. The special case of an axisymmetric radiation field is also considered, and the corresponding radiative transfer equations are found. Our results correct errors and misprints in previously published transfer equations. The extension to a moving electron gas is made, and the radiative transfer equations are deduced to second order in gas velocity. We use the equations to study polarization in the Sunyaev-Zeldovich effect.

Key words: polarization – radiative transfer – scattering – galaxies: clusters: general – cosmic microwave background

1 INTRODUCTION
A study of the spectral evolution of a radiation field caused by Compton scattering on an electron gas was first published by Kompaneets (1956). He obtained the kinetic equation describing the evolution of an isotropic and homogenous radiation field in an infinite, homogenous, thermal and nearly nonrelativistic electron gas. The Kompaneets equation was later extended to include a non-homogenous radiation field in a plane parallel atmosphere (Babuel-Peyrissac & Rouvillois 1969). More recently the extension to arbitrarily high orders in frequency and temperature has been discussed (Stebbins 1997; Challinor & Lasenby 1998).

Compton scattering of polarized radiation was studied by Chandrasekhar (1960) in the zero temperature limit. Stark (1981) attempted to extend this work to nonzero temperatures. He obtained radiative transfer equations for the $I$ and $Q$ Stokes parameters for low frequency radiation scattered on a hot, nearly nonrelativistic thermal electron gas, including induced scattering and assuming an axisymmetric radiation field. The equations were derived by expanding to first order in $kT/mc^2$ and $h\nu/mc^2$, where $h\nu$ is photon energy, $kT$ is thermal energy and $mc^2$ is the electron rest energy.

The fully relativistic equations for all four Stokes parameters were obtained by Nagirner and Poutanen (1993) and further extended to include induced scattering by Nagirner (1994a). These equations are very complicated and hard to use in practice. Therefore, when working with low frequencies and electron velocities, a simplified form is preferred. Nagirner did also reduce the fully relativistic equations in this limit, to the same order as Stark (1981). However, when the simplified equations of Nagirner (1994a) are integrated assuming an axisymmetric field, the result is very different from the equations obtained by Stark (1981). The two sets of equations are supposed to describe the same phenomenon, but several terms disagree. To make the situation even more confusing, the equations in the Russian original paper by Nagirner (1994b) are different from those in the English translation (Nagirner 1994a). As we show in this paper, the reason for the disagreements is that some relativistic corrections to the rotation angles between different polarization bases were neglected by Stark. Some small integration errors in both papers also contribute to the differences.

In this paper we present the correct radiative transfer equations for polarized radiation in an electron gas to first order in $h\nu/mc^2$ and $kT/mc^2$ [equation (13)]. The equations are derived from the Boltzmann collisional equation. The special case of an axisymmetric field is considered [equations (14) and (15)]. We then extend the equations to include a moving electron gas, to second order in gas velocity [equation (17)].

The transfer equations for Compton scattering of low frequency radiation on a relatively hot electron gas is of high importance in cosmology. Clusters of galaxies contain...
a hot electron gas, and the cosmic background radiation is changed because of Compton scattering on this gas. This is the so called Sunyaev-Zeldovich effect (SZ effect) (Zeldovich & Sunyaev 1969; Rephaeli 1995). This effect together with X-ray data of clusters of galaxies is used e.g. to measure the Hubble constant. The SZ-effect for a moving electron gas (Sunyaev & Zeldovich 1980; Rephaeli 1995; Sazonov & Sunyaev 1998) can be used to measure peculiar velocities of clusters. Such measurements have so far only used the intensity of the cosmic background radiation (CBR). Upcoming satellite experiments like MAP and Planck will try to measure polarization in the CBR. Therefore the equations describing the change in the Stokes parameters of the CBR caused by Compton scattering in clusters [equations (13) and (17) in this paper] will be important. We investigate if there is any temperature or velocity dependent polarization produced. We also briefly discuss scattering of radiation from extended radio sources in the cluster on the intracluster gas and possible polarization produced from this effect.

Here we derive the radiative transfer equations, starting in Section 2 with the Boltzmann collisional equations in the rest frame of an electron. Using Lorentz transformations, we transform the equations to the rest frame of the electron gas, and then expand to second order in electron velocity and first order in \(\frac{h\nu}{mc^2}\), before averaging over electron velocities (Section 3). The final result is presented in equation (13). We use different bases for the Stokes vector in the electron rest frame and in the rest frame of the gas. In Section 3 we discuss the transformation between these two bases. In Section 4, the equations are integrated in the case of an axisymmetric field to compare with the equations of Stark [equations (14) and (15)]. Then, in Section 5, equation (13) is extended to a moving electron gas. We derive the radiative transfer equations to second order in gas velocity [equation (17)]. Finally, applications of the formalism deduced in the previous sections to the SZ effect is discussed in Section 7. In the conclusions, we discuss the flaws in the previously published results. Throughout this paper, the relativistic quantum system of units, in which the electron mass, speed of light and Planck’s constant are given by \(m = c = \hbar = 1\), is used.

2 THE BOLTZMANN COLLISIONAL EQUATION FOR THE STOKES PARAMETERS

In this section, we study the evolution of a beam of polarized radiation in the ‘system frame’, in which the mean momentum of the electrons in the gas is zero. In this frame we introduce a fixed coordinate system \((x, y, z)\) with arbitrary directions of the axes. A photon beam in the direction \((\theta, \phi)\) (see Fig. 1) is described by the ‘number’ Stokes vector

\[
\tilde{n}(\nu, \theta, \phi, t, \mathbf{x}) = \begin{pmatrix} n \\ n_Q \\ n_U \\ n_V \end{pmatrix},
\]

where \(\nu\) is the frequency, \(\mathbf{x}\) is position in space and \(t\) is time. The number Stokes parameters \(n, n_Q, n_U\) and \(n_V\) are related to the usual Stokes parameters by

\[
\tilde{\mathbf{I}} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = 4\pi\nu^3\tilde{n}.
\]

First we study the photon beam in the rest frame of an electron moving with velocity \(v\). All quantities taken in this frame are denoted by subscript ‘e’. The number Stokes parameters \(n, n_Q, n_U, n_V\) are relativistically invariant, but
where (\(\alpha, \Phi\)) are the polar and azimuthal angles between \(\mathbf{v}\) and the photon [see Fig. 3] and \(d\Omega\) is an infinitesimal solid angle. The direction unit vector \(\mathbf{\omega}\) gives the direction of the photon beam and \(\Phi\) is the angle between two photons which have angles \(\alpha\) and \(\alpha'\) with the electron velocity \(\mathbf{v}\).

The Boltzmann collisional equation describes the change of \(\bar{n}_e(\nu_e, \theta_e, \phi_e)\) caused by scattering on particles with velocity \(\mathbf{v}\) [Acquista & Anderson 1974]. In the rest frame the equation takes the form

\[
\frac{\partial \bar{n}_e}{\partial t} + \mathbf{\omega} \cdot \nabla \bar{n}_e = \int (d\bar{n}_e - d\bar{n}_e) F_e(v^2, \rho_e),
\]

where \(\bar{n}_e^+\) and \(\bar{n}_e^-\) are the number of photons per phase space volume scattered into the beam (\(\nu_e, \theta_e, \phi_e\)) and out of it, respectively, from each scattering. Assuming an isotropic electron gas, the number of electrons with velocity \(\mathbf{v}\) is given by the electron distribution function \(F_e(v^2, \rho_e)\), where \(\rho_e\) is the electron density in the rest frame. Next, we assume that the distribution function can be written as \(F_e(v^2, \rho_e) = \rho_e f_e(v^2)\), which in the system frame is \(F(v^2, \rho) = \rho(1 - \cos \Theta) f(v^2)\).

Here \(f(v^2)\) is normalised as \(4\pi \int_0^\infty f(v^2)v^2dv = 1\). Going to the system frame, using the Lorentz transformations [equation (4)] above, we find that

\[
\frac{\partial \bar{n}_e}{\partial t} + \mathbf{\omega} \cdot \nabla \bar{n}_e = \int (d\bar{n}_e^+ - d\bar{n}_e^-)(1 - \cos \Theta) \rho f(v^2).
\]

The rate of scattering from one photon beam (\(\nu_e, \theta_e, \phi_e\)) into (\(\nu_e, \theta_e, \phi_e\)) is given by

\[
d\bar{n}_e^+ \nu_e d\nu_e = (1 + \mathbf{N}) \mathbf{L}_1 \mathbf{L}_2 \nu_e^2 \nu_e d\nu_e d\mathbf{\omega}_e,
\]

where \(\bar{n}' = \bar{n}(\nu', \theta', \phi')\), \(d\mathbf{\omega}_e = d\cos \theta_e d\phi_e\), and \(\mathbf{L}_1\) is the scattering matrix in the rest frame, given by [Berestetskii & Lifshitz & Pitaevskii 1971]

\[
R_e = \frac{3}{16\pi} \sigma_T \left( \frac{\nu_e}{\nu_e'} \right)^2 \begin{pmatrix} R_{11} & R_{12} & 0 & 0 \\ R_{12} & R_{22} & 0 & 0 \\ 0 & 0 & R_{33} & 0 \\ 0 & 0 & 0 & R_{44} \end{pmatrix},
\]

where

\[
R_{11} = \left( \frac{\nu_e'}{\nu_e} \right)^2 \frac{16\pi}{3\sigma_T} R_e, \quad R_{12} = \cos^2 \Theta_e - 1,
\]

\[
R_{22} = \cos^2 \Theta_e + 1, \quad R_{33} = 2 \cos \Theta_e
\]

and

\[
R_{44} = \cos \Theta_e \left( \frac{\nu_e'}{\nu_e} + \frac{\nu_e}{\nu_e'} \right).
\]

The Klein-Nishina cross section \(R_e\) is given by

\[
R_e = \frac{3}{16\pi} \sigma_T \left( \frac{\nu_e}{\nu_e'} \right)^2 \frac{\nu_e + \nu_e'}{\nu_e' + \cos^2 \Theta_e - 1},
\]

where \(\sigma_T\) is the Thompson cross section. The \(\mathbf{L}\) matrices transform between different polarization bases as discussed in the next section. The more photons there are in the state (\(\nu_e, \theta_e, \phi_e\)), the more likely it is that photons are scattered into this state. This fact (induced scattering) is included by the factor \((1 + \mathbf{N})\), where \(\mathbf{N}\) is given by [Nagirner 1994a]

\[
\mathbf{N} = \begin{pmatrix} n & n_Q & n_V & n_V \\ n_Q & n & 0 & 0 \\ n_U & 0 & n & 0 \\ n_V & 0 & 0 & n \end{pmatrix}.
\]

The incoming and outgoing frequencies are related by the usual formula for the Compton frequency shift,

\[
\nu_e' = \frac{\nu_e}{1 - \nu_e(1 - \cos \Theta_e)}.
\]

The rate of scattering out of the beam (\(\nu_e, \theta_e, \phi_e\)) is [Nagirner 1994a]

\[
d\bar{n}_e^- = (R_e \bar{n}_e + \mathbf{N} \mathbf{L}_1 \mathbf{L}_2 \nu_e') d\mathbf{\omega}_e,
\]

where \(\nu_e' = \bar{n}(\nu', \theta', \phi')\). Note that the spontaneous scattering term (the first term on the right hand side) in this equation contains the Klein-Nishina cross section and not the scattering matrix. This is because of the isotropy of the medium [Acquista & Anderson 1974]. The cross section \(R_e'\) and the scattering matrix \(\mathbf{R}_e\) are given by equation (5) and (6), except that \(\nu_e'\) and \(\nu_e\) are replaced by \(\nu_e'\) and \(\nu_e\) respectively. The frequencies \(\nu_e'\) and \(\nu_e\) are related by the Compton formula for the scattering \(\nu_e \rightarrow \nu_e'\),

\[
\nu_e' = \frac{\nu_e}{1 + \nu_e(1 - \cos \Theta_e)}.
\]

Inserting the expressions for \(d\bar{n}_e^-\) and \(d\bar{n}_e^+\) into equation (5), we find that

\[
\frac{\partial \bar{n}_e}{\partial t} + \mathbf{\omega} \cdot \nabla \bar{n}_e = \int \left\{ \rho(1 - \cos \Theta) f(v^2) \left[ (1 + \mathbf{N}) \mathbf{L}_1 \mathbf{L}_2 \right] \left( \frac{\nu_e'}{\nu_e} \right)^2 \frac{d\nu_e'}{d\nu_e} \right\} d\mathbf{Y}_e.
\]

3 POLARIZATION BASES

So far, the polarization vector basis has not been considered. The scattering matrix \(\mathbf{R}_e\) describes the scattering of photons \(\bar{n}_e\) in a certain polarization basis which we call the 'scattering basis'. This basis is dependent on the photons and electrons involved in each scattering event [Berestetskii & Lifshitz & Pitaevskii 1971], and is therefore inconvenient when considering several scatterings. We wish to describe the polarization relative to the fixed set of axes \((x, y, z)\) in the system frame. When we use the vector \(\bar{n}\) in this frame, we use a polarization basis which we call the 'system basis'. In this basis, the Stokes parameters are measured relative to the meridian planes. The matrix \(\mathbf{L}_1\) rotates the vector \(\bar{n}\) from the system basis to the scattering basis, so that the matrix \(\mathbf{R}_e\) can be applied. Then \(\mathbf{L}_1\) rotates back to the system basis [Chandrasekhar 1960]. A Stokes vector can be rotated from one basis to another with the rotation matrix [Chandrasekhar 1960].

\[
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\]
where $\chi$ is the rotation angle of the polarization basis vectors. Using the notation of Chandrasekhar, we write

$$L_{1}R_{2}L_{2} = L(\pi - i\varepsilon)R_{2}L(-i\varepsilon),$$

where $\pi - i\varepsilon$ and $-i\varepsilon$ are the rotation angles in the rest frame of the electron. These angles must be transformed to the system frame and expanded to second order in electron velocity. We wish to express $i_{1}$ and $i_{2}$ in terms of the zeroth order rotation angles $i_{1}$ and $i_{2}$ used by Chandrasekhar (1960). We separate the factors which are dependent on electron directions (which we call $x$ and $y$) from $i_{1}$ and $i_{2}$ which are not. This enables us, as will be shown in the next section, to collect all dependency on electron directions (which we average over) in one matrix, making the integration in Section 1 easier. The Lorentz transformed rotation angles are given by (Nagirner & Poutanen 1993):

$$\begin{align*}
\cos i_{1} & = x \cos i_{1} + y \sin i_{1}, \\
\sin i_{1} & = x \sin i_{1} - y \cos i_{1}, \\
\cos i_{2} & = x \cos i_{2} + y \sin i_{2}, \\
\sin i_{2} & = x \sin i_{2} - y \cos i_{2}, \\
x & = \frac{1}{\Delta} \left( \frac{\gamma(1-v\zeta)}{\sqrt{1-\zeta^{2} \cos \Phi}} - \frac{1+\mu}{1-\mu} \right), \\
y & = \frac{\gamma v \sqrt{1-\zeta^{2} \sin \Phi}}{\Delta}, \\
x^{2} + y^{2} & = 1.
\end{align*}$$

Here the following definitions were used:

$$\Delta = \sqrt{\frac{2\gamma^{2}(1-v\zeta)(1-v\zeta')}{(1-\mu)}} - 1,$$

$$\zeta = \cos \alpha, \quad \zeta' = \cos \alpha',$$

$$\eta = \cos \theta, \quad \eta' = \cos \theta',$$

$$\mu = \eta \eta' + \sqrt{1-\eta^{2}} \sqrt{1-\eta'^{2}} \cos (\phi - \phi'),$$

where $\alpha'$ is the angle between the photon $(\theta', \phi')$ and the electron velocity $\mathbf{v}$. The angle between the two photons is $\Theta$ and $\mu = \cos \Theta$. The angles $i_{1}$ and $i_{2}$ are the zeroth order rotation angles given by Chandrasekhar (1960),

$$\begin{align*}
\cos i_{1} & = \frac{\eta - \eta' \mu}{\sqrt{1-\eta^{2}} \sqrt{1-\mu^{2}}}, \\
\cos i_{2} & = \frac{\eta' - \eta \mu}{\sqrt{1-\eta'^{2}} \sqrt{1-\mu^{2}}}, \\
\sin i_{1} & = \frac{1-\eta^{2}}{1-\mu^{2}} \sin (\phi - \phi'), \\
\sin i_{2} & = \frac{1-\eta'^{2}}{1-\mu^{2}} \sin (\phi - \phi').
\end{align*}$$

### 4 INTEGRATION OVER ELECTRON VELOCITIES

The next step is to average over all possible electron directions by integrating the right side of equation (8) over $\zeta$ from 0 to 1 and $\Phi$ from 0 to $2\pi$ and divide by $2\pi$. Before integrating, all terms are expanded to second order in $v$ and first order in $\nu$. Expanding to second order in $v$ is equivalent to first order in $kT/mc^{2}$ when using a Maxwellian distribution function for the electrons. We expand $\vec{n}'$ and $\vec{n}''$ near $\nu \approx \nu'$ and $\nu \approx \nu''$:

$$\begin{align*}
\vec{n}(\nu', \theta', \phi') & \approx \vec{n}' + \frac{\partial \vec{n}'}{\partial \nu} (\nu' - \nu) + \frac{1}{2} \frac{\partial^{2} \vec{n}'}{\partial \nu^{2}} (\nu' - \nu)^{2}, \\
\vec{n}(\nu'', \theta', \phi') & \approx \vec{n}'' + \frac{\partial \vec{n}''}{\partial \nu} (\nu'' - \nu) + \frac{1}{2} \frac{\partial^{2} \vec{n}''}{\partial \nu^{2}} (\nu'' - \nu)^{2}.
\end{align*}$$

where we redefine $\vec{n}' = \vec{n}(\nu, \nu', \nu'')$. Using the Lorentz transformation equations (1), $\nu'/\nu$ and $d\nu'/d\nu$ are transformed to give

$$\begin{align*}
\frac{\nu''}{\nu'} & = \frac{(1-v\zeta')}{(1-v\zeta)} \frac{(1-\zeta'' - \nu(1-\mu))}{(1-\zeta' - \nu(1-\mu))}, \\
\frac{d\nu'}{d\nu} & = \left( \frac{\nu'}{\nu} \right)^{2} = \frac{\gamma^{2}(1-v\zeta')^{2}}{(\gamma(1-v\zeta') - \nu(1-\mu))^{2}}, \\
\frac{d\Phi'}{d\Phi} & = \frac{1}{\gamma^{2}(1-v\zeta')^{2}}.
\end{align*}$$

In Appendix A we have expanded all quantities we need in equations (8) to first order in $\nu$ (low frequencies) and second order in $\nu$ (equivalent to first order in temperature). The quantities $u = 2xy$ and $w = (2x^{2} - 1)$ are expanded instead of $x$ and $y$ for reasons which will become obvious.

The rotation matrices $L_{1}$ and $L_{2}$ contain cosine and sine of the angles $2(i_{2} - \pi)$ and $2(-i_{1})$. Again we wish to separate the factors that are dependent on electron directions from those that are not. We use equations (1) to write

$$\begin{align*}
\cos 2(\pi - i\varepsilon) & = (2x^{2} - 1) \cos (2\pi - i\varepsilon) - 2xy \sin (2\pi - i\varepsilon), \\
\sin 2(\pi - i\varepsilon) & = (2x^{2} - 1) \sin (2\pi - i\varepsilon) + 2xy \cos (2\pi - i\varepsilon), \\
\cos 2(-i\varepsilon) & = (2x^{2} - 1) \cos (2\pi - i) - 2xy \sin (2\pi - i), \\
\sin 2(-i\varepsilon) & = (2x^{2} - 1) \sin (2\pi - i) + 2xy \cos (2\pi - i).
\end{align*}$$

Using these equations we can write $L_{1}R_{2}L_{2}$ as

$$L_{1} \mathbf{R} L_{2} = L(\pi - i\varepsilon) \mathbf{A} L(-i\varepsilon),$$

where $\mathbf{A}$ is given by,

$$\mathbf{A} = \frac{3}{16\pi} \sigma_{T} \left( \frac{\nu'}{\nu} \right)^{2} \times \left( \begin{array}{cccc}
R_{11} & R_{12}w & R_{13}u & 0 \\
R_{12}w & R_{22}w^{2} - R_{33}u^{2} & (R_{22} + R_{33})uw & 0 \\
R_{13}u & -(R_{22} + R_{33})uw & R_{33}u^{2} - R_{22}w^{2} & 0 \\
0 & 0 & 0 & R_{33}
\end{array} \right).$$

First of all, one can see that all terms proportional to $\sin \Phi$ can be omitted since they will disappear in the averaging over $\Phi$. The $\mathbf{R}$ matrices are now independent of electron directions, so they can be treated as constants in the integration. Next, we insert the expanded expressions into equation (8) and use the relation (which can be found from geometric considerations)

$$\begin{align*}
\zeta' & = \mu \zeta + \sqrt{1-\mu^{2}} \sqrt{1-\zeta^{2} \cos \Phi}.
\end{align*}$$

Finally, Mathematica(R) is used to integrate over $\Phi$ and $\zeta$. Then we average over velocities, using a non relativistic Maxwellian distribution,

$$f(v^{2}) = (2\pi kT)^{-3/2} e^{-v^{2}/2kT}.$$
where \( k \) is the Boltzmann constant, and \( T \) is the temperature of the gas. Remembering that \( \langle \nu \rangle = 0 \) and \( \langle \nu^2 \rangle = 3kT \equiv 3\tau \),

\[
(X) = \int_{-\infty}^{\infty} X f(\nu^2) e^{\frac{\nu^2}{2\tau}} d\nu,
\]

we find that

\[
\frac{\partial \hat{n}}{\partial t} + \omega \cdot \nabla \hat{n} = -\sigma_T \rho (1 - 2\nu) \hat{n}
\]

\[
+ \sigma_T \rho \frac{3}{16\pi} \int \left\{ \mathbf{L}(\pi - i\omega) \mathbf{B}(\nu) \left[ \hat{n} + (1 - \mu)(2\nu \hat{n}') \right] + (\nu + 4\tau) \nu \frac{\partial \hat{n}' \nu}{\partial \nu} + \tau \nu^2 \left( \frac{\partial^2 \hat{n}'}{\partial \nu^2} \right) \right\}
\]

\[
- 2\tau \mathbf{L}(\pi - i\omega) \mathbf{C}(1 \hat{n})
\]

\[
+ 2(1 - \mu) \nu \mathbf{N}(\pi - i\omega) \mathbf{B}(\nu) \left[ \hat{n} + \nu \left( \frac{\partial \hat{n}' \nu}{\partial \nu} \right) \right] d\nu,
\]

\[
(13)
\]

where the \( \mathbf{B} \) and \( \mathbf{C} \) matrices are defined in appendix \[3\].

These are the radiative transfer equations for all four Stokes parameters in a rather hot thermal isotropic electron gas (first order in \( kT/mc^2 \)) for low frequency (first order in \( h\nu/mc^2 \)) radiation in the rest frame of the electron gas. If the equation for the I-Stokes parameter is integrated assuming no dependency on angles in the radiation field \( \hat{n} \) (isotropic radiation field) we recover the Kompaneets equation (Kompaneet, 1956).

### 5 THE RADIATIVE TRANSFER EQUATION FOR AN AXISYMMETRIC FIELD

Now the field \( \hat{n} \) is assumed homogenous and symmetric about the \( z \)-axis, so there is no \( \phi \) dependence. With this symmetry, there is no \( U \)-polarization, so only the \( I \) and \( Q \) components of the Stokes vector are considered. We also change to the intensity parameters \( (I, Q) \) by means of the relations

\[
\hat{n} = \frac{i}{4\pi \nu}, \quad \frac{\partial \hat{n}}{\partial \nu} = \frac{1}{4\pi \nu} \left( \frac{\partial \hat{I}}{\partial \nu} - \frac{3}{\nu} \hat{I} \right),
\]

\[
\frac{\partial^2 \hat{n}'}{\partial \nu^2} = \frac{12}{4\pi \nu^5} \hat{I} - \frac{6}{4\pi \nu^4} \frac{\partial \hat{I}}{\partial \nu} + \frac{1}{4\pi \nu^3} \frac{\partial^2 \hat{I}}{\partial \nu^2}.
\]

Making the substitution \( z = \cos(\phi - \phi') \), we define

\[
\mathbf{D} = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{1 - z^2}} \left\{ \mathbf{L}(\pi - i\omega) \mathbf{B}(\nu) \right\} \frac{\partial \hat{I}}{\partial \nu} \right|_{\nu = \mu, \, \omega = 0},
\]

\[
\mathbf{E} = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{1 - z^2}} \left\{ \mu \mathbf{L}(\pi - i\omega) \mathbf{B}(\nu) \right\} \frac{\partial \hat{I}}{\partial \nu} \right|_{\nu = \mu, \, \omega = 0},
\]

\[
\mathbf{F} = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{1 - z^2}} \left\{ \mathbf{L}(\pi - i\omega) \mathbf{C}(\nu) \right\} \frac{\partial \hat{I}}{\partial \nu} \right|_{\nu = \mu, \, \omega = 0},
\]

where \( \mathbf{M} \), denotes the upper left minor of the matrix \( \mathbf{M} \). Now, using \( \mu = \eta \nu^2 + \sqrt{1 - \eta^2} \sqrt{1 - \eta^2} z \) to integrate over \( z \), we find

\[
\mathbf{D} = \left( \begin{array}{ccc} 3 - \eta^2 - \eta'^2(3\nu^2) & 3\nu^2 + \nu' - 1 & 3(1 - \eta^2)(1 - \eta'^2) \\
(\eta^2 - 1)(3\nu^2 - 1) & (3\nu^2 - 1)(\eta^2 - 1) & 3(1 - \eta^2)(1 - \eta'^2) 
\end{array} \right),
\]

\[
\mathbf{E} = \eta \left( \begin{array}{ccc} 5 - 3\eta^2 - \eta'^2(3 - 5\eta^2) & (5\eta^2 - 3)(\eta'^2 - 1) \\
(\eta^2 - 1)(5\eta^2 - 3) & 5(1 - \eta^2)(1 - \eta'^2) 
\end{array} \right)
\]

and

\[
\mathbf{F}_{11} = 1 - 3\eta^2 + 3\eta^2(3\nu^2 - 1) + 2\eta^3 \nu(3 - 5\eta^2),
\]

\[
\mathbf{F}_{21} = -2(\eta^2 - 1)(1 - 3\eta^2 - 3\eta^2 + 5\eta^2 \nu'),
\]

\[
\mathbf{F}_{22} = (\eta^2 - 1)(\eta'^2 - 1)(3 - 10\eta^2).
\]

Finally we define (using the notation of Stark 1981)

\[
X_n \equiv \int_{-\infty}^{\infty} X n dx',
\]

and

\[
F \equiv \nu \left( \begin{array}{ccc} 1 + i \frac{\partial}{\partial \nu} & \tau \left( \nu \frac{\partial^2}{\partial \nu^2} - 2\nu \frac{\partial}{\partial \nu} \right) 
\end{array} \right),
\]

where \( X = (I, Q) \) and \( n = (0, 1, 2, 3) \). With this notation, the radiative transfer equations for \( I \) and \( Q \) take the form

\[
\frac{\partial I}{\partial s} = -I(1 - 2\nu) + \frac{3}{16} \left\{ (3 - \eta^2) I_0 + (1 - \eta^2) (Q_0 - I_2) - Q_2 \right\} + F[G_I] + 2\tau [3\nu^2 - 1] (I_0 - 2\eta I_1 - 3I_2)
\]

\[
+ 2(3\eta^2 - 1) (I_3 + Q_3 - Q_1) + 2(3\eta^2 - 1)(Q_0 - Q_2) + \nu \left[ I(1 - 3\nu^2 - 3\eta^2 + 5\eta^2 \nu') \right],
\]

\[
(14)
\]

\[
\frac{\partial Q}{\partial s} = -Q(1 - 2\nu) + \frac{3}{16} \left\{ (1 - \eta^2) I_0 + (1 - 3\eta^2)(Q_0 + Q_2 - 3I_2) + F[G_Q] + 2\tau [3\nu^2 - 1] (I_0 - 2\eta I_1 - 3I_2)
\]

\[
+ 3(Q_0 - Q_2) + 10\eta (Q_1 - I_2 - Q_3)]
\]

\[
+ \nu \left[ Q(1 + \nu \frac{\partial}{\partial \nu}) G_I + I(1 - \nu \frac{\partial}{\partial \nu}) G_Q \right],
\]

\[
(15)
\]

where \( s = \sigma_T \rho t \), and

\[
G_I \equiv (3 - \eta^2) I_0 + (1 - 3\eta^2)(Q_0 - I_2 - Q_2) - \eta((5 - 3\eta^2) I_1 + (3 - 5\eta^2)(Q_1 - I_3 - Q_3)),
\]

\[
G_Q \equiv (1 - \eta^2)(I_0 + 3Q_0 - 3Q_2 - 3I_2) - \eta(3I_1 + 5Q_4 - 5Q_2 - 5I_4).
\]

Equations (14) and (15) are in Section 6 compared with the results of Stark (1981).

### 6 THE RADIATIVE TRANSFER EQUATIONS FOR COMPTON SCATTERING OF POLARIZED LOW FREQUENCY RADIATION ON A MOVING ELECTRON GAS

In Sections 5 and 6, we transformed the Boltzmann collisional equations from the rest frame of the electron [equation (3)] to the rest frame of the electron gas. Now we obtain the
where the electron has a velocity which is a sum of the thermal velocity (the velocity in the rest frame of the electron gas) and the velocity of the gas.

We expand to second order in thermal velocity ($v$) and electron gas velocity ($v_g$). To this order, relativistic addition of velocities reduces to $v_t = v + v_g$ where $v_t$ is the total electron velocity observed from the moving frame, $v$ is the velocity in the rest frame of the gas (thermal velocity) and $v_g$ is the velocity of the gas observed in the moving frame. Using spherical coordinates, we add each of the components to get

\[
\begin{align*}
v_t \sin \alpha \cos \Phi &= v \sin \alpha \cos \Phi + v_g \sin \alpha_g \cos \Phi, \\
v_t \sin \alpha \sin \Phi &= v \sin \alpha \sin \Phi + v_g \sin \alpha_g \sin \Phi, \\
v_t \cos \alpha &= v \cos \alpha + v_g \cos \alpha_g,
\end{align*}
\]

where $v_t = |v_t|$, $v_g = |v_g|$. Here $(\alpha, \Phi)$ are the angles between the total velocity vector and the photon, $(\alpha_g, \Phi)$ are the angles between the electron velocity and the photon in the rest frame of the gas and $(\alpha_g, \Phi_g)$ are the angles between the gas velocity and the photon.

We follow the same steps that we used to obtain the equations in the rest frame of the electron gas. The difference is that we now transform to a frame where the electron velocity is $v_t = v + v_g$ instead of $v$. So, in equation (16) and in the expanded quantities [equation (A1)], we have to make the following replacements [using equations (10)]:

\[
\begin{align*}
v \sqrt{1 - \zeta^2} \cos \Phi &= \sqrt{1 - \zeta_g^2} \cos \Phi + v_g \sqrt{1 - \zeta_g^2} \cos \Phi, \\
v \sqrt{1 - \zeta^2} \sin \Phi &= \sqrt{1 - \zeta_g^2} \sin \Phi + v_g \sqrt{1 - \zeta_g^2} \sin \Phi, \\
v \zeta &= v \zeta + v_g \zeta_g,
\end{align*}
\]

where $\zeta_g = \cos \alpha_g$.

Making these replacements, we write the right hand side of equation (13) as

\[
F_0 + F_1(\zeta_g, \Phi_g) v_g + F_1(\zeta_g, \Phi_g) v + F_2(\zeta_g, \Phi_g) v^2_g + F_2(\zeta, \Phi) v^2 + F(\zeta, \Phi, \zeta_g, \Phi_g) v v_g,
\]

where the $F$-functions are just collections of terms proportional to different orders of $v$. When averaging over electron velocities, the term $F(\zeta, \Phi, \zeta_g, \Phi_g)$ disappears since $\langle v \rangle = 0$ by definition. This is important, because the terms that are left are just a sum of terms dependent on gas velocity $(v_g, \zeta_g, \Phi_g)$ and terms dependent on thermal velocity $(v, \zeta, \Phi)$. There are no cross terms dependent on both! So in the radiative transfer equations for a moving electron gas, the part dependent on temperature (thermal velocity) and the part dependent on gas velocity are independent.

This simplifies the calculations since one can calculate the terms dependent on thermal velocity and the terms dependent on gas velocity separately. The terms containing the gas temperature can be found assuming that the gas velocity is zero, and similarly the terms dependent on gas velocity can be found assuming that the temperature is zero. Then the different terms can be added to get the complete radiative transfer equations for a moving electron gas with nonzero temperature.

We now find the terms dependent on gas velocity assuming the temperature to be zero. Going back to the Boltzman collisional equation in the rest frame of an electron [equations (1)], one has to transform all quantities to the frame of reference where the electron has a velocity $v_g$. This is exactly what was done previously, except that one has to replace $(v, \zeta, \Phi)$ with $(v_g, \zeta_g, \Phi_g)$ in all the expressions and expansions used, since the only velocity now is the gas velocity.

Since there is no averaging over electron velocities and electron directions this time (the electron gas has a constant velocity which all the electrons are following), one can change to a coordinate system in which the expressions and calculations are more easy. Previously, a coordinate system $(x, y, z)$ with arbitrary axes was introduced. The incoming and outgoing photon directions were described with polar and azimuthal angles $(\eta = \cos \theta, \phi)$ and $(\eta' = \cos \theta', \phi')$ with respect to this coordinate system. But since the directions of the axes were arbitrary, one can choose the $z$-axis to be in the direction of the gas velocity. With this choice of $z$-axis, the angle $\zeta_g = \cos \alpha_g$ between the photon and electron. The azimuthal angles are still called $\phi$ and $\phi'$ for the photons with frequency $\nu$, and $\nu'$. With this choice of axes, the change $\eta \rightarrow \zeta$ and $\eta' \rightarrow \zeta'$ is made everywhere in the equations.

The correction factors $w$ and $u$ from the Lorentz transformations of the angles $\zeta_1$ and $\zeta_2$ need special attention. These are expressed (see appendix A) through the angle $\Phi$ that was a convenient angle when there was an integration over electron directions. Now however, there is only an integration over incoming photon directions (note that $d\Omega' = d\zeta' d\phi'$ with the current choice of axes), so the angle $\phi - \phi'$ is a better choice than $\Phi$. The following relations between $\Phi$ and $(\phi - \phi')$ can be found from geometrical considerations:

\[
\begin{align*}
\cos \Phi &= \frac{\zeta - \zeta \eta \mu}{\sqrt{1 - \zeta_g^2} \sqrt{1 - \mu^2}}, \\
\sin \Phi &= -\frac{\sqrt{1 - \zeta_g^2} \sin (\phi - \phi')}{\sqrt{1 - \mu^2}}.
\end{align*}
\]

Since there is no integration over electron velocities, it is now easy to find the equations describing the radiative transfer of polarized radiation in a moving zero temperature electron gas. All that is necessary is to put the expanded Lorentz transformed quantities (appendix A) into the Boltzmann equation (1), and replace $(\eta, \eta', \Phi)$ in the way described above. The resulting equations are the radiative transfer equations for a moving zero temperature electron gas. Adding the temperature dependent terms from equation (13) (with a coordinate system where the $z$ axis is chosen to be along the gas velocity direction) we find that

\[
\frac{\partial n}{\partial t} + \omega \cdot \nabla n = -\sigma_T \rho (1 - v - 3\nu) n + \sigma_T \frac{3}{16\pi} \int_{-1}^{1} d\zeta' \int_{0}^{2\pi} d\phi' \left( L_1 \left( (1 + 2\nu(1 - \mu)) |\mathbf{B} - 2\pi \mathbf{C} + |\mathbf{G} + (2\zeta' - \zeta) |\mathbf{B} |\mathbf{v} + |\mathbf{H} + (2\zeta' - \zeta) |\mathbf{G} - (1 - 3\zeta'^2) + 2\zeta' \mathbf{B} |\mathbf{v}^2 + L_2 |\mathbf{L}_2 | \mathbf{v} \frac{\partial n}{\partial t} + \frac{1}{2} \left( v^2 (\zeta' - \zeta)^2 \right)
\right)
\]

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\[ +2\tau(1 - \mu)\left\{ \nu^2 L_1 B L_2 \frac{\partial^2 \tilde{n}'}{\partial v^2} + 2 N L_1 B L_2 (1 - \mu) \nu |\tilde{n}'| \right\} \]
\[ + \nu \frac{\partial \tilde{n}'}{\partial v}, \]

where we have written \( \zeta \) instead of \( \zeta' \) for simplicity. Here the matrices \( G \) and \( H \) that expand from the array of \( A, A = B + Gv + Hv^2 \). These matrices are given in appendix 2. The \( L \) matrices are again \( L_1 = L_2 = (1 - 1) \), where the angles \( i_1 \) and \( i_2 \) are given by equations (3) (remember to let \( \eta \rightarrow \zeta \) and \( \eta' \rightarrow \zeta' \)). This is the equations describing the change of a radiation field upon interaction with a hot (first order in \( kT/mc^2 \)), moving (second order in \( v_0/c \)) electron gas to first order in \( h\nu/mc^2 \). To be able to perform the integral over incoming electron directions, one can use the relation \( \nu = \zeta \zeta' + \sqrt{1 - \zeta'^2} \sqrt{1 - \zeta^2} \cos (\phi - \phi') \).

7 APPLICATION TO THE SZ EFFECT

One important application of the equations deduced here, is the interaction of the cosmic background radiation with the hot electron gas in clusters of galaxies. In this case, the frequency is so low that only terms to zeroth order in frequency need be taken into account. For small changes in the radiation field, the initial field can be inserted for \( \tilde{n} \) on the right hand side in equation (17) and integrated along a path through the cluster (Zeldovich & Sunyaev 1969). The initial radiation field is the cosmic background radiation which can be assumed to be isotropic and homogenous. In this case, the integration in equation (17) over incoming photons can easily be made using the techniques of Section 3.

First we assume that the gas velocity is zero, studying only temperature dependent effects. In this case the solution of equation (13) for small changes in the radiation field gives the following change \( \Delta \tilde{n} \) to first order in optical depth:

\[ \Delta \tilde{n} = -s \tilde{n}_0 + \frac{3}{16\pi} \int ds \left( \int \int \int d\zeta' \int \int d\phi' \left\{ L_1 \mathbf{B} \right. \right. \]
\[ -2\tau \mathbf{C} L_2 \tilde{n}_0 + 4\tau (1 - \mu) L_1 B L_2 \nu \frac{\partial \tilde{n}_0}{\partial \nu} \]
\[ \left. + (1 - \mu) L_1 B L_2 \nu \left( \frac{\partial^2 \tilde{n}_0}{\partial \nu^2} \right) \right\}, \]

where \( s = \sigma_T \rho \) is the optical depth and \( \int ds \) is an integral over the optical depth of the cluster. The initial radiation field \( \tilde{n}_0 \) is given by

\[ \tilde{n}_0 = \frac{1}{\nu^{2/\tau} - 1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \]

This is a matrix equation equivalent to four scalar equations each describing the change in the Stokes parameters \( I, Q, U \) and \( V \). Making the integrations, the first equation gives the thermal SZ effect (Zeldovich & Sunyaev 1969). The other equations all give zero polarization to first order in optical depth as was expected because of the isotropy of the cluster gas and the radiation field.

The equation can also be used to study the effect of the intracluster medium on radiation from extended radio sources within the cluster. The state of polarization of the radiation coming from radio sources will be changed upon scattering with electrons in the intracluster gas. The change in the degree of polarization will of course be proportional to the optical depth. It will also be dependent on the temperature of the cluster gas, as can be seen by considering the change of \( \tilde{n} \) from a single scattering [from equation (17) considering only the inscattering term]:

\[ \Delta \tilde{n} = \frac{3}{16\pi} \int_{-1}^{1} d\zeta' \int_{0}^{2\pi} d\phi' \left\{ L_1 \mathbf{B} - 2\tau \mathbf{C} L_2 \tilde{n}_0 \right. \]
\[ + 4\tau (1 - \mu) L_1 B L_2 \nu \frac{\partial \tilde{n}_0}{\partial \nu} + \tau (1 - \mu) L_1 B L_2 \nu \left( \frac{\partial^2 \tilde{n}_0}{\partial \nu^2} \right) \right\}, \]

where \( \tilde{n}_0 \) now is the initial spectrum of radiation coming from the radio source. This time \( \tilde{n}_0 \) is not isotropic since the radiation is only coming from some angles. A more detailed study is necessary to determine if the change of polarization of radiation from a radio source can be used to determine density and temperature of the cluster gas.

Finally we consider the kinematic SZ effect which is the change of the background radiation because of scattering on a moving cluster gas. This time we assume that the temperature of the gas is zero. Again, using equation (17), we find to first order in optical depth using an initial Planck function:

\[ \frac{\Delta n}{n} = -\frac{\nu}{c} \frac{\tau}{\zeta} \frac{x e^x}{e^{x} - 1} s \]
\[ \frac{\Delta n_Q}{n} = -\frac{1}{10} \frac{\nu^2}{c^2} (1 - e^2) s, \]

where \( x = \nu/\tau \). The first equation is just the formula for the kinematic SZ effect which was given by Sunyaev & Zeldovich (1982) without deductions. The second equation shows that some polarization is produced from the kinematic effect. The degree of polarization is proportional to the tangential velocity component of the cluster. This result was also given by Sunyaev & Zeldovich (1982), again without deductions. The second result can be used to determine the tangential velocity of clusters of galaxies, but at the moment the polarization degree produced (about \( 10^{-3} \)) is too small to be observed.

8 CONCLUSION

In equations (13) we have obtained the radiative transfer equations for the Stokes parameters to first order in \( h\nu/mc^2 \) and \( kT/mc^2 \). Comparing these to those obtained by Nagirner (1994a; 1994b), some small differences are seen. Comparing the radiative transfer equations for an axisymmetric field [equations (14) and (15) with those obtained by Stark (1981), we find several disagreements. However, setting \( Q = 0 \), studying only the intensity of radiation, the equations actually agree. This indicates that the difference has to do with polarization, and indeed, in the paper of Stark, we find that the zeroth order rotation angles \( i_1 \) and \( i_2 \) were used instead of \( i_{1e} \) and \( i_{2e} \). The relativistic corrections to these angles were neglected, which, as is seen from the Taylor expansion, is altering the integration substantially. Including these corrections, the equations in the paper of Stark become identical to those obtained here [equations (14) and (15)], except for the induced scattering terms. By carefully performing the last integration in that paper, we
find that the induced scattering terms also agree with what we have derived here. Manipulating the equations in the paper of Nagirner (1994b) before integration, it is possible to show that they agree with equation [1] of this paper. So there has to be an error somewhere in his integration. And indeed, by correcting the misprints in the integration, we find full agreement with the results presented here. So our equations (13) correct the errors of Stark (1981) and Nagirner (1994a; 1994b). We also transformed the equations to a frame where the electron gas is moving [equations (13)]. In this frame the equations were derived to second order in gas velocity. The transfer equations were tested by showing that they give zero polarization produced from the thermal SZ effect to first order in optical depth. We also showed that polarization from the kinematic SZ effect is very small. The formalism developed here can probably be used to study polarization of radiation from radio sources in the cluster upon scattering with the intracluster medium.

ACKNOWLEDGEMENTS

We thank Mark Birkinshaw for suggesting the study of polarization in the SZ-effect and Juri Poutanen for help and comments. We also thank the Research Council of Norway for a travel grant.

APPENDIX A: TAYLOR EXPANSIONS

The following quantities expanded to first order in \( \nu \) and second order in \( \nu \) are needed in Section 4:

\[
\begin{align*}
\left( \frac{v'_0}{v_0} \right)^2 & \approx 1 - 2\nu(1 - \mu), \\
\left( \frac{dv'}{dv_0} \right) & \approx 1 + 2\nu(1 - \mu), \\
\frac{(\nu' - \nu)}{\nu} & \approx \nu(1 - \mu) + v(\zeta' - \zeta) + v_0^2\zeta'(\zeta' - \zeta), \\
\frac{(\nu'' - \nu')}{\nu^2} & \approx \frac{1}{\nu^2} v^2(\zeta' - \zeta)^2, \\
\frac{d\gamma}{d\nu} & \approx 1 + 2\nu \zeta' - v^2(1 - 3\zeta^2), \\
\cos \Theta_e & \approx \mu(1 - \mu)(\nu(\zeta + \zeta')) + v^2(\zeta^2 + \zeta'^2 + \zeta' - 1)), \\
\cos^2 \Theta_e & \approx \mu^2 - \nu^2(1 - \mu)(\zeta + \zeta') - v^2(1 - \mu)(-2\nu - 3\zeta^2 + 3\mu\zeta^2), \\
\tan \phi' & \approx 1 + 2v^2(1 - \mu)(\zeta^2 - 1)(\cos^2 \phi - 1) \\
\tan \phi & \approx 2\nu(1 - \mu)\sqrt{1 + \frac{\zeta^2 - 1}{1 + \mu^2}} \sin \phi + 2\nu(1 - \mu)\sqrt{1 + \frac{\zeta^2 - 1}{1 + \mu^2}}(\zeta + \zeta') \sin \phi,
\end{align*}
\]

where \( u \equiv 2\tau y \) and \( w \equiv (2x^2 - 1) \).

APPENDIX B: MATRICES FOR THE RADIATIVE TRANSFER EQUATIONS

The following matrices were used in equation (13) and (14):

\[
\begin{align*}
B & = \begin{pmatrix}
\mu^2 + 1 & \mu^2 - 1 & 0 & 0 \\
\mu^2 - 1 & \mu^2 + 1 & 0 & 0 \\
0 & 0 & 2\mu & 0 \\
0 & 0 & 0 & 2\mu
\end{pmatrix}, \\
C & = \begin{pmatrix}
C_{11} & C_{12} & 0 & 0 \\
C_{12} & C_{22} & 0 & 0 \\
0 & 0 & C_{33} & 0 \\
0 & 0 & 0 & C_{44}
\end{pmatrix}, \\
G & = \begin{pmatrix}
G_{11} & G_{11} & G_{13} & 0 \\
G_{11} & G_{11} & G_{23} & 0 \\
-G_{13} & -G_{23} & G_{33} & 0 \\
0 & 0 & 0 & G_{33}
\end{pmatrix}, \\
H & = \begin{pmatrix}
H_{11} & H_{12} & H_{13} & 0 \\
H_{12} & H_{22} & H_{23} & 0 \\
-H_{13} & -H_{23} & H_{33} & 0 \\
0 & 0 & 0 & H_{44}
\end{pmatrix},
\end{align*}
\]

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