A COMPLEMENTARITY MODEL AND ALGORITHM FOR DIRECT MULTI-COMMODITY FLOW SUPPLY CHAIN NETWORK EQUILIBRIUM PROBLEM

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Abstract. In this paper, a three-level supply chain network equilibrium problem with direct selling and multi-commodity flow is considered. To this end, we first present equilibrium conditions which satisfy decision-making behaviors for manufacturers, retailers and consumer markets, respectively. Based on this, a nonlinear complementarity model of supply chain network equilibrium problem is established. In addition, we propose a new projection-type algorithm to solve this model without the backtracking line search, and global convergence result and $R$–linearly convergence rate for the new algorithm are established under weaker conditions, respectively. We also illustrate the efficiency of given algorithm through some numerical examples.

1. Introduction. With the development of science and technology, the issue of the supply chain network equilibrium problem has received more and more attention. The function of the supply chain network equilibrium is to coordinate the relationship among the members of the system, so that the profit maximization for its each member can be guaranteed, and thus achieve a win-win situation. The supply chain network is a system composed of different interests member with multiple products, multiple levels, competition and cooperation. Each member first considers own the maximum benefit, and then considers benefits of the whole system, so there is a game among the supply chain members. It is only necessary to find the optimal solution for the supply chain network equilibrium problem. Up to now, the issues of numerical methods and existence of the solution for the problem were discussed in the literature([8, 9, 3, 4, 17, 15]). Nagurney([8, 9]) first established a multi-layer supply chain network model via the variational inequality with a single commodity, and considered the individual independent decision-making behaviors of each layer and the interaction with network members. Dong, Zhang

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and Nagurney ([3]), for a single commodity, developed a three-tiered supply chain model consisting of manufacturers, retailers, and consumers by variational inequality formulation, and the product outputs, shipments, and consumption quantities, along with the prices of the product associated with the different tiers could be determined. Zhang ([17]) studied a nonlinear complementarity formulation of the supply chain network equilibrium problem, and a smoothing Newton method is proposed for solving this model. Nagurney, Daniele and Shukla ([10]) developed a supply chain network game theory model consisting of retailers and demand markets, and provided a novel alternative variational inequality formulation and algorithm, which yields closed form expressions in product transactions, security levels, and Lagrange multipliers associated with the budget constraints. Javad et al. ([6]) presented a constrained vendor-managed inventory model with fuzzy demand for a single-vendor multi retailer supply chain. Zhang et al. ([18]) studied a closed-loop supply chain network equilibrium problem in multi period planning horizons with consideration of product lifetime and carbon emission constraints. Fu et al. ([5]) studied the centralized decision-making mode of supply chain network, and gave the decision-making process and characteristics of the centralized decision-making. Liu and He ([7]) developed a logistics service supply chain network equilibrium problem, the variational inequality model and a new smoothing self-adaptive L-M algorithm are established for this problems. The motivations of this paper due to the following reasons: First, with the development of electronic commerce, more and more people choose to shop online, the most of the goods are shipped directly from the manufacturer to the consumer, the intermediate circulation is reduced. Thus, it is very necessary to establish a direct sales multi-commodity supply chain network equilibrium model. Second, this paper is a follow-up to [8, 9, 3], as in this paper will establish the new algorithm under weaker conditions than that needed in [8, 9, 3]. These constitute the main topics of this paper.

The main contribution of this paper is to establish a nonlinear complementarity model and method for direct multi-commodity flow supply chain network equilibrium problem. To this end, we first establish a complementarity model of supply chain network equilibrium problem with direct selling and multi-commodity flow. To this end, we first establish a complementarity model of supply chain network equilibrium problem with direct sales multi-commodity flow via equilibrium conditions of the manufacturer, the retailer, and the demand market. Second, we propose a new projection-type algorithm for solving this model without the backtracking line search to find a suitable step size. Global convergence of the new algorithm is proved under pseudo-monotonic mapping in detail, and we also establish \( R \)-linearly convergence rate for this new method if in addition a projection type error bound holds locally. In addition, the equilibrium price and product shipment pattern in the supply chain network equilibrium problem be determined. Finally, some experimental examples are given to verify the validity of the model and the feasibility of the algorithm.

2. Complementarity model of supply chain network equilibrium problem. In this section, we will establish a nonlinear complementarity model for a three-level supply chain network problem with direct selling and multi-commodity flow. To this end, we first give equilibrium conditions that satisfy decision-making behaviors for manufacturers, retailers, and consumer markets, respectively. Based on these, we establish a nonlinear complementarity model for this supply chain network equilibrium problem.
In the following, we give the supply chain network equilibrium problem with \( m \) manufacturers, \( n \) vendors, \( o \) consumer markets and \( s \) products (see figure 1). The \( i, j, k, l \) denote \( i \)-th manufacturer, \( j \)-th vendor, \( k \)-th consumer market and \( l \)-th product, respectively.

2.1. **Competition behavior of the manufacturers and their equilibrium conditions.** In this subsection, we discuss the competition behavior of the manufacturers, let \( q_l^i \) denote the amount of product \( l \) produced by manufacturer \( i \) (\( i = 1, 2, \cdots, m \)), and \( q_l := (q_l^1, q_l^2, \cdots, q_l^m)^T \in \mathbb{R}_+^m \).

The amount of product \( l \) of the manufacturer \( i \) supply to the vendor \( j \) denoted by \( q_{lj}^i \), and it offers directly to the consumer market \( k \) denoted by \( \tilde{q}_{lk}^i \). For convenience of discussion, we denote the column vector \( Q_l := \left( q_{lj}^i, q_{lk}^i \right) \ in \mathbb{R}_+^{mn} \) (\( l = 1, 2, \cdots, s \)).

We use \( f_l^i \) to denote that each manufacturer \( i \) is faced with a production cost function of product \( l \), which can depend, in general, on the entire vector of production outputs, that is,

\[
 f_l^i = f_l^i(Q_l), \quad \forall i.
\]

The transaction costs on the product \( l \) between manufacturer \( i \) and vendor \( j \), denoted by \( c_{lj}^i \), is a function of \( q_{lj}^i \), and assume that

\[
 c_{lj}^i = c_{lj}^i(q_{lj}^i),
\]

where \( c_{lj}^i \) including transportation costs. In a similar way, the transaction costs on the product \( l \) between manufacturer \( i \) and consumer market \( k \), denoted by \( \tilde{c}_{lk}^i \), is a function of \( \tilde{q}_{lk}^i \), and assume that

\[
 \tilde{c}_{lk}^i = \tilde{c}_{lk}^i(\tilde{q}_{lk}^i),
\]

where \( \tilde{c}_{lk}^i \) including transportation costs. Obviously,

\[
 q_l^i = \sum_{j=1}^n q_{lj}^i + \sum_{k=1}^o \tilde{q}_{lk}^i, \quad i = 1, 2, \cdots, m; l = 1, 2, \cdots, s.
\]

Thus, the total cost incurred by the manufacturer \( i \) is equal to the cost of production plus the total transaction cost. The manufacturer’s income is equal to the number of goods that the manufacturer sells to retailers and consumer markets multiplied by their respective selling prices. Let \( \rho_{lj}^i \) denote the price charged for unit \( l \)-th product by manufacturer \( i \) to retailer \( j \), and \( \rho_{lk}^i \) denote the price charged for unit...
$l$–th product by manufacturer $i$ to consumer marker $k$. The supply chain network structure of $i$–th manufacturer is as depicted in Figure 2. Then, the revenue maximization problem of manufacturer $i$ can be expressed as

$$\max \left\{ \sum_{j=1}^{n} \sum_{l=1}^{s} \rho_{ij} q_{lj} + \sum_{k=1}^{o} \sum_{l=1}^{s} \rho_{ik} q_{lk} - \sum_{l=1}^{s} f_{l}(Q_{l}) - \sum_{j=1}^{n} \sum_{l=1}^{s} c_{lj}(q_{lj}) - \sum_{k=1}^{o} \sum_{l=1}^{s} c_{lk}(q_{lk}) \right\}$$

s.t. $q_{ij} \geq 0, q_{lk} \geq 0, j = 1, 2, \ldots, n; k = 1, 2, \ldots, o; l = 1, 2, \ldots, s.$

(1)

Suppose that the competitions of manufacturers are non-cooperative, the production cost function $f_{l}(Q_{l})$ and the transaction cost function $c_{lj}$ are continuously differentiable convex functions, then (1) is a convex programming. Applying KKT conditions, (1) can be written as

$$\left\{ \begin{array}{l}
\sum_{l=1}^{s} \nabla f_{l}(Q_{l}) + \sum_{j=1}^{n} \sum_{l=1}^{s} \nabla c_{lj}(q_{lj}) + \sum_{k=1}^{o} \sum_{l=1}^{s} \nabla c_{lk}(q_{lk}) - \sum_{j=1}^{n} \sum_{l=1}^{s} \rho_{lj} q_{lj} - \sum_{k=1}^{o} \sum_{l=1}^{s} \rho_{ik} q_{lk} \\
= (\lambda_{11}, \ldots, \lambda_{1n}, \lambda_{21}, \ldots, \lambda_{2n}, \tilde{\lambda}_{11}, \ldots, \tilde{\lambda}_{1n}, \tilde{\lambda}_{21}, \ldots, \tilde{\lambda}_{2n})^T \geq 0,
q_{ij} \geq 0, q_{lk} \geq 0, \lambda_{ij} q_{ij} = 0, \tilde{\lambda}_{ik} q_{lk} = 0,
\end{array} \right. \quad (2)$$

where $\lambda_{ij}$ ($j = 1, 2, \ldots, n; l = 1, 2, \ldots, s$) and $\tilde{\lambda}_{ik}$ ($k = 1, 2, \ldots, o; l = 1, 2, \ldots, s$) are lagrange multiplier,

$$e_{j} = (0, \ldots, 0, 1, \ldots, 0)^T, \quad e_{k} = (0, \ldots, 0, 1, \ldots, 0)^T.$$  

Moreover, for any $i$ ($i = 1, 2, \ldots, m$), (2) can be further written as

$$\left\{ \begin{array}{l}
\frac{\partial f_{i}(Q_{i})}{\partial q_{ij}} + \frac{\partial c_{lj}(q_{lj})}{\partial q_{ij}} - \rho_{lj} \geq 0, q_{ij} \geq 0, \quad \frac{\partial f_{i}(Q_{i})}{\partial q_{ik}} + \frac{\partial c_{lk}(q_{lk})}{\partial q_{ik}} - \rho_{lk} \geq 0, q_{lk} \geq 0, \quad (3)
\end{array} \right. \quad q_{ij} = 0, \quad \left( \frac{\partial f_{i}(Q_{i})}{\partial q_{ik}} + \frac{\partial c_{lk}(q_{lk})}{\partial q_{ik}} - \rho_{lk} \right) q_{lk} = 0,
\end{array} \right.$$

$$\begin{array}{l}
\quad \begin{cases}
\left( \frac{\partial f_{l}(Q_{l})}{\partial q_{lj}} + \frac{\partial c_{lj}(q_{lj})}{\partial q_{lj}} - \rho_{lj} \right) q_{lj} = 0, \quad \left( \frac{\partial f_{l}(Q_{l})}{\partial q_{lk}} + \frac{\partial c_{lk}(q_{lk})}{\partial q_{lk}} - \rho_{lk} \right) q_{lk} = 0,
\end{cases}
\end{array}$$

$$\begin{array}{l}
\quad j = 1, 2, \ldots, n; k = 1, 2, \ldots, o; l = 1, 2, \ldots, s.
\end{array}$$

(3)

For the convenience of the following description, the (3) is abbreviated as

$$f(w_{1}) \geq 0, \quad w_{1} \geq 0, \quad w_{1}^{T} f(w_{1}) = 0,$$

(4)
where

\[
f(w_1) = \begin{pmatrix}
\frac{\partial f^j_l(Q^l_j)}{\partial q^l_{ij}} + \frac{\partial c^j_{l}(Q^l_j)}{\partial q^l_{ij}} - \rho^{ij} \\
\frac{\partial f^j_l(Q^l_j)}{\partial q^l_{ik}} + \frac{\partial c^j_{l}(Q^l_j)}{\partial q^l_{ik}} - \rho^{ik}
\end{pmatrix},
\]

\[
w_1 = \begin{pmatrix}
q^l_{ij} \\
q^l_{ik}
\end{pmatrix}, \quad i = 1, 2, \ldots, m,
\quad j = 1, 2, \ldots, n,
\quad k = 1, 2, \ldots, o,
\quad l = 1, 2, \ldots, s.
\]

2.2. Competition behavior of the retailers and their equilibrium conditions. The retailers, in turn, must decide how much to order from the manufacturers in order to seek to maximize their profits. At the same time, it is necessary to trade with customers in the consumer market because their final products are sold to consumers. The supply chain network structure of \(j\)-th retailer is as depicted in Figure 3. In this subsection, we discuss the competition behavior of the vendors.

![Figure 3. The network structure of \(j\)’s retailer](image)

The retailer \(j\) purchased \(l\)-th product from the manufacturer \(i\), with the amount of the product \(l\) is \(q^l_{ij}\). A vendor may ship the product to the customer markets, with the amount of the product \(l\) shipped (or transacted) between vendor \(j\) and consumer market \(k\) denoted by \(q^l_{jk}\). There are transaction costs between retailers, manufacturers and consumer markets. For manufacturers, retailers face management costs, such as the cost of displaying, selling and storing goods. For the consumer market, retailers face transportation costs and so on. Thus, we assume that \(c^j_l\) denote the transaction costs that retailer \(j\) pays for commodity \(l\), and the simplest form is to think of \(c^j_l\) as a function of \(q^l_{1j}, \ldots, q^l_{mj}, q^l_{j1}, \ldots, q^l_{jo}\). For the convenience of narration, let

\[
Q_j := \begin{pmatrix}
q^l_{ij} \\
q^l_{jk}
\end{pmatrix}, \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, n,
\quad j = 1, 2, \ldots, n, k = 1, 2, \ldots, o,
\quad l = 1, 2, \ldots, s.
\]

Therefore, \(c^j_l\) can be expressed as

\[
c^j_l = c^j_l(Q_j).
\]
Let \( \hat{p}_{lj}^j \) denote the price at which retailer \( j \) sells unit commodity \( l \) to consumer market \( k \). Then the profit maximization model of retailer \( j \) is

\[
\max \sum_{k=1}^{o} \sum_{l=1}^{s} q_{lj}^j \hat{p}_{lj}^j - \sum_{l=1}^{s} c_l^j (Q_j) - \sum_{i=1}^{m} \sum_{l=1}^{s} q_{ij}^l \rho_{lj}^i \\
\text{s.t.} \sum_{i=1}^{m} q_{ij}^l \geq \sum_{k=1}^{o} q_{lj}^j, q_{ij}^l \geq 0, q_{lj}^j \geq 0, i = 1, 2, \ldots, m, k = 1, 2, \ldots, o, l = 1, 2, \ldots, s.
\]

Let the transaction cost function \( c_l^j \) of each retailer be a continuous differentiable convex function, so the above optimization problem is a convex programming problem. Applying KKT conditions, (5) can be written as

\[
\sum_{i=1}^{o} \nabla c_l^j (Q_j) + \sum_{l=1}^{s} \rho_{lj}^i \partial c_l^j \partial q_{ij}^l - \sum_{l=1}^{s} \sum_{k=1}^{o} \rho_{lj}^k \partial c_l^j = \begin{pmatrix} \mu_{ij}^l, \cdots, \mu_{ij}^s, \cdots, \mu_{ij}^{m_j}, \cdots, \mu_{ij}^{o_j} \end{pmatrix}^\top + \sum_{i=1}^{o} \gamma_l^j \partial c_l^j
\]

\[
\mu_{ij}^l \geq 0, q_{ij}^l \geq 0, \mu_{ij}^l q_{ij}^l = 0, \quad i = 1, 2, \cdots, m, \quad l = 1, 2, \cdots, s
\]

\[
\mu_{ij}^k \geq 0, q_{ij}^k \geq 0, \mu_{ij}^k q_{ij}^k = 0, \quad k = 1, 2, \cdots, o, \quad l = 1, 2, \cdots, s
\]

\[
\sum_{l=1}^{s} q_{ij}^l - \sum_{k=1}^{o} q_{lj}^k \geq 0, \quad \gamma_l^j \geq 0, \quad \gamma_l^j (\sum_{l=1}^{s} q_{ij}^l - \sum_{k=1}^{o} q_{lj}^k) = 0, \quad l = 1, 2, \cdots, s
\]

where \( \mu_{ij}^l (i = 1, 2, \cdots, m; \ l = 1, 2, \cdots, s) \), \( \mu_{ij}^k (k = 1, 2, \cdots, o; \ l = 1, 2, \cdots, s) \) and \( \gamma_l^j (l = 1, 2, \cdots, s) \) are lagrange multiplier, \( \partial c_l^j = (0, 0, \cdots, 0, 1, 0, \cdots, 0) \), \( \partial c_l^k = (0, 0, \cdots, 0, 1, 0, \cdots, 0) \), for any \( j (j = 1, 2, \cdots, n) \), (6) is equivalent to

\[
\frac{\partial c_l^j (Q_j)}{\partial q_{ij}^l} + \rho_{lj}^i - \gamma_l^j = \mu_{ij}^l \geq 0, \quad q_{ij}^l \geq 0, \quad q_{ij}^l (\frac{\partial c_l^j (Q_j)}{\partial q_{ij}^l} + \rho_{lj}^i - \gamma_l^j) = 0
\]

\[
\frac{\partial c_l^j (Q_j)}{\partial q_{lj}^k} - \rho_{lj}^k + \gamma_l^j = \mu_{lj}^k \geq 0, \quad q_{lj}^k \geq 0, \quad q_{lj}^k (\frac{\partial c_l^j (Q_j)}{\partial q_{lj}^k} - \rho_{lj}^k + \gamma_l^j) = 0
\]

\[
\sum_{l=1}^{s} q_{ij}^l - \sum_{k=1}^{o} q_{lj}^k \geq 0, \quad \gamma_l^j \geq 0, \quad \gamma_l^j (\sum_{l=1}^{s} q_{ij}^l - \sum_{k=1}^{o} q_{lj}^k) = 0
\]

For the convenience of the following description, the (7) is abbreviated as

\[
g(w_2) \geq 0, \quad w_2 \geq 0, \quad w_2^T g(w_2) = 0.
\]
where

\[
g(w^2) = \begin{pmatrix}
\frac{\partial c_i^l(Q_j)}{\partial q_{ij}} + \rho^l_{ij} - \gamma^l_j \\
\frac{\partial c_i^l(Q_j)}{\partial q_{jk}} - \rho^l_{jk} + \gamma^l_j \\
\sum_{i=1}^{m} q^l_{ij} - \sum_{k=1}^{o} q^l_{jk} \\
\sum_{i=1}^{m} q^l_{ij} - \sum_{k=1}^{o} q^l_{jk} \\
\end{pmatrix},
\]

\[
w^2 = \begin{pmatrix}
q^l_{ij} \\
q^l_{jk} \\
\gamma^l_j \\
\end{pmatrix},
\]

\[
i = 1, 2, \ldots, m, \\
j = 1, 2, \ldots, n, \\
k = 1, 2, \ldots, o, \\
l = 1, 2, \ldots, s.
\]

### 2.3. Competition behavior of the consumer market and their equilibrium conditions.

In this subsection, we turn to a discussion of the market equilibrium conditions. For the consumer market, consumers should not only make consumption decisions to buy the products from retailers, but also make consumption decisions to buy the products directly from manufacturers, and also consider the transaction costs they have to pay when they get the products they need. The consumer market \(k\) and the transaction network structure of all manufacturers and retailers, as depicted in Figure 4.

**Figure 4.** The network structure of \(k\)-th consumer

We assume that the transaction cost between the consumer market \(k\) and the retailer \(j\), denoted by \(\hat{c}^l_{jk}\), which the transaction cost be paid by the retailer \(j\) in the consumer market, that is:

\[
\hat{c}^l_{jk} = \hat{c}^l_{jk}(\hat{q}^l_{jk}),
\]

where \(\hat{q}^l_{jk}\) defined in Subsection 2.2.

Let \(\rho^j_k\) denote the demand price of the product \(l\) associated with consumer market \(k\). We assume that \(d^l_k\) denote the amount of the demand for the product \(l\) from consumer market \(k\), and allow the function to, in general, depend also on the demand price of the product \(l\) held by consumer market, therefore, we may write:

\[
d^l_k = d^l_k(\rho^1_k, \rho^2_k, \ldots, \rho^o_k).
\]

Consumers pay price for purchasing products that they need from retailers and manufacturers. We borrow the user’s optimum choice behavior by traffic network equilibrium, i.e. Wardop user criterion, and the user’s optimum equilibrium condition is as follows:
For all retailers \( j (j = 1, 2, \cdots, n) \),
\[
\bar{p}_{jk}^l + d_{jk}^l (q_{jk}^l) \begin{cases} 
\rho_k^l, & \text{a.e., if } \bar{q}_{jk}^l \geq 0, \\
\geq \rho_k^l, & \text{a.e., if } \bar{q}_{jk}^l = 0,
\end{cases}
\]
(9)
where a.e. means that the corresponding equality or inequality holds almost everywhere, \( \bar{p}_{jk}^l \) and \( \bar{q}_{jk}^l \) defined in Subsection 2.2.

For all manufacturers \( i (i = 1, 2, \cdots, m) \),
\[
\bar{p}_{ik}^l + c_{ik}^l (q_{ik}^l) \begin{cases} 
\rho_k^l, & \text{a.e., if } \bar{q}_{ik}^l \geq 0, \\
\geq \rho_k^l, & \text{a.e., if } \bar{q}_{ik}^l = 0.
\end{cases}
\]
(10)
where \( \bar{p}_{ik}^l \), \( \bar{q}_{ik}^l \) and \( c_{ik}^l (q_{ik}^l) \) defined in Subsection 2.1.

For all consumers \( k (k = 1, 2, \cdots, o) \),
\[
d_k^l \begin{cases} 
\leq \sum_{i=1}^{m} q_{ik}^l + \sum_{j=1}^{n} q_{jk}^l, & \text{a.e., if } \rho_k^l = 0, \\
= \sum_{i=1}^{m} q_{ik}^l + \sum_{j=1}^{n} q_{jk}^l, & \text{a.e., if } \rho_k^l \geq 0.
\end{cases}
\]
(11)

Firstly, conditions (9) indicate, in equilibrium, if the consumer in consumer market \( k \) buys the product from retailer \( j \), the retailer’s selling price plus transaction cost will not exceed the price that the consumer is willing to pay for the product, with exceptions of zero probability. Secondly, conditions (10) indicate, in equilibrium, if the consumer in consumer market \( k \) buys the product directly from the manufacturer \( i \), the price of the manufacturer’s direct sale plus the transaction cost will not exceed the price that the consumer is willing to pay for the product, with exceptions of zero probability. Finally, for (11), if the demand price is positive, then the total amount of products purchased by the consumers directly from the manufacturers and from the retailers is equal to the demand in the consumer market, with exceptions of zero probability.

Equilibrium conditions (9), (10) and (11) are equivalent to the following complementarity problem:
\[
\bar{p}_{jk}^l + d_{jk}^l (q_{jk}^l) - \rho_k^l \geq 0, \bar{q}_{jk}^l \geq 0, \left[ \bar{p}_{jk}^l + d_{jk}^l (q_{jk}^l) - \rho_k^l \right] \bar{q}_{jk}^l = 0,
\]
(12)
\[
\bar{p}_{ik}^l + c_{ik}^l (q_{ik}^l) - \rho_k^l \geq 0, \bar{q}_{ik}^l \geq 0, \left[ \bar{p}_{ik}^l + c_{ik}^l (q_{ik}^l) - \rho_k^l \right] \bar{q}_{ik}^l = 0,
\]
(13)
\[
\sum_{i=1}^{m} q_{ik}^l + \sum_{j=1}^{n} q_{jk}^l - d_k^l \geq 0, \rho_k^l \geq 0, \left[ \sum_{i=1}^{m} q_{ik}^l + \sum_{j=1}^{n} q_{jk}^l - d_k^l \right] \rho_k^l = 0,
\]
(14)
\[
i = 1, 2, \cdots, m; j = 1, 2, \cdots, n; k = 1, 2, \cdots, o; l = 1, 2, \cdots, s.
\]

For the convenience of the following description, the (12), (13) and (14) is abbreviated as
\[
h(w_3) \geq 0, w_3 \geq 0, w_3^T h(w_3) = 0.
\]
(15)
where

\[
\begin{pmatrix}
\hat{\rho}_{lk} + \hat{c}_{lk}(\hat{q}_{lk}) - \rho_{lk} \\
\hat{\rho}_{lk} + \hat{c}_{lk}(\hat{q}_{lk}) - \rho_{lk} \\
\sum_{i=1}^{m} \hat{q}_{ik} + \sum_{j=1}^{n} \hat{q}_{jk} - d_{lk} \\
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\rho_{lk} \\
\rho_{lk} \\
\rho_{lk} \\
\end{pmatrix}
\begin{pmatrix}
\hat{q}_{lk} \\
\hat{q}_{lk} \\
\hat{q}_{lk} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\gamma_{l} \\
\gamma_{l} \\
\gamma_{l} \\
\end{pmatrix}
\]

2.4. Complementary model for supply chain network. Based on the above (4), (8) and (15), it can be concluded that the network equilibrium model with direct selling single commodity flow is:

\[
X := \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \geq 0, \quad F(X) := \begin{pmatrix} f(w_1) \\ g(w_2) \\ h(w_3) \end{pmatrix} \geq 0, \quad X^\top F(X) = 0. \quad (16)
\]

We use \( \Omega^* \) to denotes the solution set of (16), and assume that \( \Omega^* \) is not empty.

Note that the variables in the model are: the equilibrium product shipments between manufacturers and retailers given by \( \hat{q}_{ij} \), and the equilibrium product shipments between manufacturers and consumer markets given by \( \hat{q}_{ik} \), the equilibrium product shipments transacted between retailers and consumer markets denoted by \( \hat{q}_{jk} \), as well as the equilibrium demand prices \( \hat{\rho}_{lk} \) and the lagrange multiplier \( \gamma_{lj} \).

Now, we consider how to recover the price \( \hat{\rho}_{ij} \) and \( \hat{\rho}_{ik} \) associated with the top tier of nodes of the supply chain network and the prices \( \hat{\rho}_{jk} \) associated with the middle tier.

Firstly, we discuss the transaction price \( \hat{\rho}_{ij} \), \( \hat{\rho}_{ik} \) between the retailer, the consumer and the manufacturer for the commodity \( l \), respectively. From (3), we obtain that if \( \hat{q}_{ij} > 0 \), then the price

\[
\hat{\rho}_{ij} = \frac{\partial f_i(Q_i)}{\partial \hat{q}_{ij}} + \frac{\partial c_{ij}(\hat{q}_{ij})}{\partial \hat{q}_{ij}}.
\]

Also, from (3) it follows that if \( \hat{q}_{ik} > 0 \), then the price

\[
\hat{\rho}_{ik} = \frac{\partial f_i(Q_i)}{\partial \hat{q}_{ik}} + \frac{\partial c_{ik}(\hat{q}_{ik})}{\partial \hat{q}_{ik}}.
\]

Hence, the product is priced at the manufacturer’s level according to whether it has been transacted with the retailer or the consumer.

Secondly, from (7), we obtained the transaction price \( \hat{\rho}_{jk} \) between the consumer and the retailer for the commodity \( l \), it follows that if \( \hat{q}_{jk} > 0 \), then

\[
\hat{\rho}_{jk} = \frac{\partial c_{jk}(Q_j)}{\partial \hat{q}_{jk}} + \gamma_j.
\]
Based on analysis above, using (3), (7), (12) and (13), we have

\[
\begin{align*}
\rho_{ij}^l & \left\{ \begin{array}{l}
\frac{\partial f_i^l(Q_i)}{\partial q_{ij}} + \frac{\partial c_i^l(q_{ij})}{\partial q_{ij}}, \\
\gamma_j - \frac{\partial f_i^l(Q_i)}{\partial q_{ij}} + \frac{\partial c_i^l(q_{ij})}{\partial q_{ij}},
\end{array} \right. \\
q_{ij}^l > 0 & , \\
q_{ij}^l = 0. \\
\rho_{ik}^l & \left\{ \begin{array}{l}
\frac{\partial f_i^l(Q_i)}{\partial q_{ik}} + \frac{\partial c_i^l(q_{ik})}{\partial q_{ik}}, \\
\rho_k - \frac{\partial f_i^l(Q_i)}{\partial q_{ik}}(q_{ik}) + \frac{\partial c_i^l(q_{ik})}{\partial q_{ik}},
\end{array} \right. \\
\dot{q}_{ik}^l > 0, & , \\
\dot{q}_{ik}^l = 0. \\
\rho_{jk}^l & \left\{ \begin{array}{l}
\frac{\partial c_i^l(q_{ik})}{\partial q_{jk}} + \gamma_j, \\
\rho_k - \frac{\partial c_i^l(q_{ik})}{\partial q_{jk}}(q_{ik}) + \gamma_j,
\end{array} \right. \\
\dot{q}_{jk}^l > 0, & , \\
\dot{q}_{jk}^l = 0.
\end{align*}
\]

(17)

Thus, in this model, the equilibrium prices associated with the manufacturer, the retailer and the consumer market are endogenous to the model with the manufacturers', the retailers' and the consumer markets' product shipments at equilibrium being determined at the equilibrium price vectors.

In the end of this section, based on (17), we will discuss a way to find equilibrium solutions.

**Proposition 2.1.** The vector \((Q^1, Q^2, Q^3, \gamma^*, \xi^*)\) is an equilibrium solution of supply chain network, if and only if \((Q^1, Q^2, Q^3, \gamma^*, \xi^*)\) is the solution of the following complementary problems:

\[
\begin{align*}
\begin{cases}
(Q_1, Q_2, Q_3, \gamma, \xi) \geq 0, \\
F(Q_1, Q_2, Q_3, \gamma, \xi) = \\
\begin{pmatrix}
F_1(Q_1, \gamma) \\
F_2(Q_2, \xi) \\
F_3(Q_3, \gamma, \xi) \\
F_4(Q_1, Q_3) \\
F_5(Q_2, \xi)
\end{pmatrix} \geq 0, \\
(Q_1, Q_2, Q_3, \gamma, \xi)^\top F(Q_1, Q_2, Q_3, \gamma, \xi) = 0.
\end{cases}
\end{align*}
\]

where the vectors

\[
\begin{align*}
Q^1 & = (q_{ij}^l, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; l = 1, 2, \ldots, s) \in R^{mns}, \\
Q^2 & = (q_{ik}^l, i = 1, 2, \ldots, m; k = 1, 2, \ldots, o; l = 1, 2, \ldots, s) \in R^{mos}, \\
Q^3 & = (q_{jk}^l, j = 1, 2, \ldots, n; k = 1, 2, \ldots, o; l = 1, 2, \ldots, s) \in R^{nos}, \\
\gamma & = (\gamma_j^l, j = 1, 2, \ldots, n; l = 1, 2, \ldots, s) \in R^{ns}, \\
\xi & = (\rho_k^l, k = 1, 2, \ldots, o; l = 1, 2, \ldots, s) \in R^{os},
\end{align*}
\]
Algorithm and global convergence. In this section, based on the network equilibrium model with direct selling single described in Section 2, we shall propose a new projection-type algorithm to solve (16) without the backtracking line search to find a suitable step size. Under weaker conditions, global convergence of the new projection-type algorithm is proved in detail. In addition, we also establish R-linear convergence rate of the method. We first give the following needed assumption.

**Assumption 3.1.** For (16), the mapping $F$ is pseudo-monotone on $R^N$, i.e.,

$$(X - X^*)^\top F(X^*) \geq 0 \Rightarrow (X - X^*)^\top F(X) \geq 0, \quad \forall X \in R^N,$$

and the mapping $F$ is Lipschitz-continuous with constant $L > 0$, i.e., there exists $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in R^N,$$

where $N = 2mns + 2mos + 2nos + os + ns$.

Now, we present a fundamental existence result for (16) defined over a compact convex region. Since the production capacity of manufacturers is limited, we guarantee the existence of solutions by imposing a relatively weak restriction on the flow of goods. Let

$$\Omega_b = \{X \mid 0 \leq X \leq b_1\}$$

where $b_1 \geq 0$ is constant. Then $\Omega_b$ is a bounded closed convex subset of $R^N$. Thus, we immediately obtain the following conclusion.

**Proposition 3.1.** ([4, 2]) Suppose that Assumption 3.1 holds. Then, the solution set of (16) is nonempty.

In the following, we recall the definition of projection operator and some related properties ([19, 1]).

For a nonempty closed convex set $K \subset R^N$ and vector $x \in R^N$, the orthogonal projection of $x$ onto $K$, i.e., $\arg \min \{\|y - x\| \mid y \in K\}$, is denoted by $P_K(x)$. We use $x_+$ to denote the projections of vector $x$ onto $R^+_1$.
Proposition 3.2. Let $K$ be a closed convex subset of $\mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$ and $z \in K$, the following statements hold.

(i) \[ \langle P_K(x) - x, z - P_K(x) \rangle \geq 0. \]

(ii) \[ \|P_K(x) - P_K(y)\|^2 \leq \|x - y\|^2 - \|(P_K(x) - x) - (P_K(y) - y)\|^2. \]

(iii) \[ \|P_K(x) - x\|^2 \leq \|x - z\|^2 - \|P_K(x) - z\|^2. \]

For (16) and $X \in \mathbb{R}^N$, define the projection residue
\[ r(X, \beta) := X - P_{R^N}(X - \beta F(X)) = \min \{ X, \beta F(X) \}, \quad (18) \]
where $\beta > 0$ is a constant.

The projection residue is intimately related to the solution of (16) as shown by the following well-known result, which is due to Noor ([11]).

Proposition 3.3. $\Omega^*$ is a solution of (16) if and only if $r(X^*, \beta) = 0$ with some $\beta > 0$.

Now, we can also give the needed result for our analysis.

Proposition 3.4. Suppose that $H = \{ X \in \mathbb{R}^N \mid \alpha^\top X - b \leq 0 \}$, for any $z \notin H$, one has
\[ P_H(z) = z - \frac{\alpha^\top z - b}{\|\alpha\|^2} \alpha, \quad (19) \]
where $z, \alpha \in \mathbb{R}^N, \alpha \neq 0, b \in \mathbb{R}$.

Now, we formally state our algorithm.

Algorithm 3.1.

Step0. Select any $\tau, \rho \geq 0, 0 < \beta < \frac{1-|\rho+\tau-1|}{(\rho+\tau)^2}, X^0 \in \mathbb{R}^N_+$. Let $k := 0$.

Step1. Compute
\[ Z^k = \{ X^k - \beta F(X^k) \}_+. \quad (20) \]
If $\|r(X^k, \beta)\| = 0$, stop. Otherwise, go to Step 2.

Step2. Compute
\[ X^{k+1} = P_{H_k}(X^k - \beta d(X^k)), \quad (21) \]
where
\[ H_k := \{ X \in \mathbb{R}^N \mid \|r(X^k, \beta) - \beta F(X^k)\| X - Z^k \leq 0 \}, \quad (22) \]
\[ d(X^k) = \frac{\tau}{\beta} [(X^k - \beta F(X^k)) - (Z^k - \beta F(Z^k))] + \rho F(Z^k). \quad (23) \]

Step3. Go to Step 1 with $k \triangleq k + 1$.

Remark 3.1. Firstly, we discuss the solution method of $X^{k+1}$.

Case 1. If $[X^k - \beta d(X^k) - Z^k]^\top [r(X^k, \beta) - \beta F(X^k)] \leq 0$, one has $X^k - \beta d(X^k) \in H_k$. Thus, we take
\[ X^{k+1} = X^k - \beta d(X^k). \quad (24) \]

Case 2. If $[X^k - \beta d(X^k) - Z^k]^\top [r(X^k, \beta) - \beta F(X^k)] > 0$, one has
\[ r(X^k, \beta) - \beta F(X^k) \neq 0. \]
Combining this with Proposition 3.4, we obtain

\[ X^{k+1} = [X^k - \beta d(X^k)] \]

\[ -\frac{[X^k - \beta d(X^k) - Z^k]^	op [r(X^k, \beta) - \beta F(X^k)]}{\|r(X^k, \beta) - \beta F(X^k)\|^2} \]

\[ = [X^k - \beta d(X^k)] \]

\[ -\frac{[r(X^k, \beta) - \beta d(X^k)]^\top [r(X^k, \beta) - \beta F(X^k)]}{\|r(X^k, \beta) - \beta F(X^k)\|^2} \]

\[ \|r(X^k, \beta) - \beta F(X^k)\|^2 \]

\[ \geq 0. \] (25)

Secondly, we can obtain that \( R_N^+ \subseteq H_k. \) In fact, for any \( X \in R_N^+. \) By Proposition 3.2 (i) with \( x := X^k - \beta F(X^k), z = X, \) one has

\[ [r(X^k, \beta) - \beta F(X^k)]^\top [X - Z^k] = [X^k - \beta F(X^k) - Z^k]^	op [X - Z^k] \leq 0. \]

Thus, \( X \in H_k. \)

Next, we discuss the convergence and convergence rate of Algorithm 3.1, we need the following some lemmas.

**Lemma 3.1.** For \( Z^k \) and \( d(X^k) \) defined in Algorithm 3.1, one has

\[ \langle Z^k - X^*, d(X^k) \rangle \geq 0, \] (26)

where \( X^* \in \Omega^*. \)

**Proof.** By Assumption 3.1, one has

\[ F(Z^k)^	op (Z^k - X^*) \geq 0. \]

Combining this with Proposition 3.2 (i), a direct computation yields that

\[ \langle Z^k - X^*, d(X^k) \rangle \]

\[ = \langle Z^k - X^*, \rho F(Z^k) \rangle + \langle Z^k - X^*, \frac{\beta}{\rho} (X^k - \beta F(X^k) - Z^k) \rangle \]

\[ = \frac{\beta}{\rho} \langle Z^k - X^*, X^k - \beta F(X^k) - Z^k \rangle + \rho + \tau \langle Z^k - X^*, F(Z^k) \rangle \]

\[ \geq 0. \]

\( \square \)

**Lemma 3.2.** Suppose that Assumption 3.1 holds, and an infinite sequence \( \{X^k\} \) generated by Algorithm 3.1. Then, for any \( X^* \in \Omega^*, \) one has

\[ \|X^{k+1} - X^*\|^2 \leq \|X^k - X^*\|^2 - \sigma \|r(X^k, \beta)\|^2, \] (27)

where \( \sigma = 1 - |\tau + \rho - 1| + \beta (\rho + \tau) L^2 > 0 \) is a constant.
Proof. By a direct computation yields that
\[ \|X^{k+1} - X^*\|^2 \]
\[ = \|P_{H_k} (X^k - \beta d(X^k)) - X^*\|^2 \]
\[ \leq \|X^k - \beta d(X^k) - X^*\|^2 \]
\[ - \|P_{H_k} (X^k - \beta d(X^k)) - [X^k - \beta d(X^k)]\|^2 \]
\[ = \|X^k - X^* - \beta d(X^k)\|^2 \]
\[ - \|X^{k+1} - X^k\|^2 + \beta d(X^k)\|^2 \]
\[ = \|X^k - X^* - 2\beta (X^k - X^*, d(X^k)) + \beta^2 \|d(X^k)\|^2 \]
\[ - \|X^k - X^{k+1}\|^2 - 2\beta (X^k - X^{k+1}, d(X^k)) + \beta^2 \|d(X^k)\|^2 \]
\[ = \|X^k - X^* - 2\beta X^{k+1} - X^* - d(X^k)\|^2 \]
\[ - 2\beta \|X^{k+1} - Z^k + Z^k - X^*, d(X^k)\|^2 \]
\[ = \|X^k - X^* - Z^k, d(X^k)\|^2 \]
\[ \leq \|X^k - Z^k + Z^k - X^{k+1}\|^2 \]
\[ - 2\beta (Z^{k+1} - Z^k, \tau [(X^k - \beta F(X^k)) - (Z^k - \beta F(Z^k))] + \rho F(Z^k)) \]
\[ = \|X^k - X^* - Z^k, \tau [(X^k - \beta F(X^k)) - (Z^k - \beta F(Z^k))] + \rho F(Z^k)) \]
\[ = \|X^k - X^* - Z^k, \tau [(X^k - \beta F(X^k)) - (Z^k - \beta F(Z^k))] + \rho F(Z^k) - (X^k - Z^k)\|^2 \]
\[ = \|X^k - X^* - Z^k, -r(X^k, \beta) + \beta (\rho + \tau) F(Z^k)\|^2 \]
\[ = \|X^k - X^* - Z^k, -r(X^k, \beta) + \beta (\rho + \tau) F(Z^k)\|^2 \]
\[ \leq \|X^k - X^* - Z^k, +2\rho (X^{k+1} - Z^k, r(X^k, \beta) - \beta F(X^k)) \]
\[ - 2\tau (X^{k+1} - Z^k, \tau [(X^k - \beta F(X^k)) - (Z^k - \beta F(Z^k))] + \rho F(Z^k) - (X^k - Z^k)) \]
\[ = \|X^k - X^* - Z^k, -r(X^k, \beta) + \beta (\rho + \tau) F(Z^k)\|^2 \]
\[ \leq \|X^k - X^* - Z^k, -r(X^k, \beta) + \beta (\rho + \tau) F(Z^k)\|^2 \]
\[ = \|X^k - X^*\|^2 - \|X^k - Z_k\|^2 - \|Z_k - X^{k+1}\|^2 \\
+ 2(Z_k - X^{k+1}, (\tau + \rho - 1)r(X^k, \beta) - \beta(\rho + \tau)(F(X^k) - F(Z_k))) \\
= \|X^k - X^*\|^2 - \|X^k - Z_k\|^2 \\
- \|Z_k - X^{k+1}\| - \{(\tau + \rho - 1)r(X^k, \beta) - \beta(\rho + \tau)(F(X^k) - F(Z_k))\|^2 \\
+ \|(\tau + \rho - 1)r(X^k, \beta) - \beta(\rho + \tau)(F(X^k) - F(Z_k))\|^2 \\
\leq \|X^k - X^*\|^2 - \|X^k - Z_k\|^2 \\
+ \|(\tau + \rho - 1)r(X^k, \beta) - \beta(\rho + \tau)(F(X^k) - F(Z_k))\|^2 \\
\leq \|X^k - X^*\|^2 - \|X^k - Z_k\|^2 \\
+ \{(\tau + \rho - 1)r(X^k, \beta) - \beta(\rho + \tau)(F(X^k) - F(Z_k))\|^2 \\
\leq \|X^k - X^*\|^2 - \{1 - |\tau + \rho - 1| + \beta(\rho + \tau)L\}|r(X^k, \beta)|^2 \\
= \|X^k - X^*\|^2 - \{1 - |\tau + \rho - 1| + \beta(\rho + \tau)L\}|r(X^k, \beta)|^2. \]

where the first equality is obtained by (21), the first inequality uses Proposition 3.2 (iii), the second inequality follows from (26), the third inequality follows from (22) with \(X^{k+1} \in H_k\), letting \(\sigma = 1 - |\tau + \rho - 1| + \beta(\rho + \tau)L\|^2\), and the desired result follows.

Now, we give the global convergence result of Algorithm 3.1.

**Theorem 3.1.** Suppose that Assumption 3.1 holds, and \(0 \leq |\tau + \rho - 1| + \beta(\rho + \tau)L \leq 1\). Then, the sequence \(\{X^k\}\) generated by Algorithm 3.1 globally converges to a solution of (16).

**Proof.** From (27) and \(0 \leq |\tau + \rho - 1| + \beta(\rho + \tau)L \leq 1\), one has

\[\|X^{k+1} - X^*\|^2 \leq \|X^k - X^*\|^2. \quad (28)\]

Therefore, the sequence \(\{\|X^k - X^*\|\}\) is non-increasing and bounded. Thus, it converges, and one has

\[\lim_{k \to \infty} \|r(X^k, \beta)|^2 = \lim_{k \to \infty} \frac{1}{\sigma} \left[\|X^k - X^*\|^2 - \|X^{k+1} - X^*\|^2\right] \quad (29)\]

Since the sequence \(\{X^k\}\) is bounded. Then there exists convergent subsequence \(\{X^{k_j}\}\) of \(\{X^k\}\), denote its limit by \(\bar{X}\), combining this with (29), one has

\[\|r(\bar{X}, \beta)|^2 = \lim_{j \to \infty} \|r(X^{k_j}, \beta)|^2 = 0, \quad (30)\]

and thus \(\bar{X}\) is a solution of (16).

We take \(X^* = \bar{X}\) in the preceding arguments, in particular, in (27). Thus the sequence \(\{\|X^k - X^*\|\}\) converges. Since \(\lim_{k \to \infty} \|X^k - \bar{X}\| = 0\) again, so we obtain that \(\|X^k - \bar{X}\|\) converges to zero, i.e., that \(\{X^k\}\) converges to \(\bar{X} \in \Omega^*\), and the desired result follows.

To establish \(R\) — linearly convergence rate of Algorithm 3.1, we need the following assumption.
Assumption 3.2. For (16), there exists constant $\varrho > 0, \epsilon > 0$ such that
\begin{equation}
\text{dist}(X, \Omega^*) \leq \varrho r(X, \beta),
\end{equation}
for any $X \in \Lambda$, where $\Lambda = \{X | r(X, \beta) \leq \epsilon\}$, dist$(X, \Omega^*)$ denotes the closest distance from point $X$ to the solution set $\Omega^*$ of (16).

In the following, we give some sufficient conditions to ensure Assumption 3.2 holds. The error bound $\|r(X, \beta)\|$ has been used in the convergence rate analysis of various methods ([4]) and is known to hold whenever $F$ is affine ([12, 13, 14]) and either $F$ has certain strong monotonicity structure or $F$ is a uniform $P$-function ([16]). Moreover, under additional assumption on $F$, this condition holds with $\epsilon = \infty$.

Theorem 3.2. Suppose that Assumption 3.1 and 3.2 hold, and $0 < \sigma < \varrho$. Then, the sequence \{X$^k$\} generated by Algorithm 3.1 converges to a solution of (16) R-linearly, where $\sigma, \varrho$ are constants defined by Lemma 3.2 and Assumption 3.2, respectively.

Proof. From Theorem 3.1, one has
\begin{equation}
\lim_{k \to \infty} X^k = \bar{X},
\end{equation}
and $\bar{X} \in \Omega^*$. we take $X^* = \bar{X}$ in (27), one has
\begin{equation}
\|X^{k+1} - \bar{X}\|^2 \leq \|X^k - \bar{X}\|^2 - \sigma \|r(X^k, \beta)\|^2.
\end{equation}
Combining (32) with (31), one has
\begin{equation}
\|X^k - \bar{X}\| \leq \varrho \|r(X^k, \beta)\|.
\end{equation}
From (33) and (34), we obtain
\begin{equation}
\frac{\|X^{k+1} - \bar{X}\|^2}{\|X^k - \bar{X}\|^2} \leq 1 - \left(\frac{\epsilon}{\varrho}\right)^2
\end{equation}
i.e.,
\begin{equation}
\frac{\|X^{k+1} - \bar{X}\|^2}{\|X^k - \bar{X}\|^2} \leq 1 - \left(\frac{\epsilon}{\varrho}\right)^2.
\end{equation}
Since $0 < \sigma < \varrho$, then one has $0 < 1 - \left(\frac{\epsilon}{\varrho}\right)^2 < 1$. Thus, the desired result follows.

4. Numerical examples. In order to obtain the equilibrium price and product shipment pattern in the supply chain network equilibrium problem, and to verify the effectiveness and the reliability of the model, some examples are given in this section. All the program codes were written in Matlab and run in Matlab a environment. All numerical experiments were done at a PC with Intel(R) Core (TM) i5-5257U CPU @ 2.70GHZ and RAM of 4G. Throughout the computational experiments, the parameters used in the algorithm are set as $\tau=0.5, \rho=0.5, \beta = 0.5$.

Example 4.1. The supply chain network equilibrium problem consisted of three manufacturers, three vendors, three consumer markets and two products, as depicted in Figure 5.
The production cost functions for the manufacturers were given by:

\[ f_1^1(Q^1) = 2.5q_1^1q_2^1 + q_1^1q_2^1 + 2q_1^1, \]
\[ f_2^1(Q^1) = 2.5q_2^1q_3^1 + q_1^1q_2^1 + 2q_2^1, \]
\[ f_3^1(Q^1) = 2.5q_3^1q_4^1 + q_1^1q_3^1 + 2q_3^1, \]
\[ f_1^2(Q^2) = 3q_1^2q_2^2 + q_1^2q_2^2 + 2q_1^2, \]
\[ f_2^2(Q^2) = 3(q_1^2q_2^2) + q_1^2q_2^2 + 2q_2^2, \]
\[ f_3^2(Q^2) = 3(q_1^2q_2^2) + q_1^2q_2^2 + 2q_3^2. \]

where \( q_i^j = \sum_{j=1}^3 q_{ij}^j + \sum_{k=1}^3 \tilde{q}_{ik}^j. \)

The transaction cost functions faced by the manufacturers and associated with transacting with the consumer markets were given by:

\[ c_{11}^1(q_{11}^1) = (q_{11}^1)^2 + 3.5q_{11}^1, \]
\[ c_{12}^1(q_{12}^1) = (q_{12}^1)^2 + 3q_{12}^1, \]
\[ c_{13}^1(q_{13}^1) = (q_{13}^1)^2 + 2q_{13}^1, \]
\[ c_{21}^1(q_{21}^1) = (q_{21}^1)^2 + 2q_{21}^1, \]
\[ c_{22}^1(q_{22}^1) = (q_{22}^1)^2 + 4q_{22}^1, \]
\[ c_{23}^1(q_{23}^1) = (q_{23}^1)^2 + q_{23}^1, \]
\[ c_{31}^1(q_{31}^1) = 2(q_{31}^1)^2 + q_{31}^1, \]
\[ c_{32}^1(q_{32}^1) = (q_{32}^1)^2 + 2q_{32}^1, \]
\[ c_{33}^1(q_{33}^1) = (q_{33}^1)^2 + q_{33}^1, \]
\[ c_{11}^2(q_{11}^2) = (q_{11}^2)^2 + 3.5q_{11}^2, \]
\[ c_{12}^2(q_{12}^2) = 2(q_{12}^2)^2 + 3q_{12}^2, \]
\[ c_{13}^2(q_{13}^2) = 3(q_{13}^2)^2 + 2q_{13}^2, \]
\[ c_{21}^2(q_{21}^2) = (q_{21}^2)^2 + 2q_{21}^2, \]
\[ c_{22}^2(q_{22}^2) = 2(q_{22}^2)^2 + 4q_{22}^2, \]
\[ c_{23}^2(q_{23}^2) = 3(q_{23}^2)^2 + q_{23}^2, \]
\[ c_{31}^2(q_{31}^2) = (q_{31}^2)^2 + q_{31}^2, \]
\[ c_{32}^2(q_{32}^2) = (q_{32}^2)^2 + 2q_{32}^2, \]
\[ c_{33}^2(q_{33}^2) = 3(q_{33}^2)^2 + q_{33}^2. \]

The transaction cost functions faced by the retailers and associated with transacting with the consumer markets were given by:

\[ c_{11}^1(Q_1) = 5q_{11}^1 + 4q_{12}^1 + q_{13}^1 + (q_{11}^1)^2 + q_{11}^1 + (q_{12}^1)^2 + 2q_{12}^1 + (q_{13}^1)^2 + 3q_{13}^1 + 6, \]
\[ c_{12}^1(Q_2) = 2q_{11}^1 + 3q_{12}^1 + 2q_{13}^1 + 2(q_{11}^1)^2 + q_{11}^1 + 2(q_{12}^1)^2 + 4q_{12}^1 + 2(q_{13}^1)^2 + 3q_{13}^1 + 7, \]
\[ c_{13}^1(Q_3) = 3q_{11}^1 + 3q_{12}^1 + 3q_{13}^1 + 3(q_{11}^1)^2 + q_{11}^1 + 3(q_{12}^1)^2 + 2q_{12}^1 + 3(q_{13}^1)^2 + 3q_{13}^1 + 6. \]
\[ \hat{c}_2(Q_1) = q_{11}^2 + 2q_{12}^2 + 3q_{13}^2 + (\hat{q}_{11}^2)^2 + \hat{q}_{11}^2 + 2(\hat{q}_{12}^2)^2 + 3(\hat{q}_{13}^2)^2 + 3d_{13}^2 + 6, \]
\[ \hat{c}_2(Q_2) = q_{21}^2 + 2q_{22}^2 + 3q_{23}^2 + (\hat{q}_{21}^2)^2 + \hat{q}_{21}^2 + 2(\hat{q}_{22}^2)^2 + 4(\hat{q}_{23}^2)^2 + 3d_{23}^2 + 7, \]
\[ \hat{c}_2(Q_3) = q_{31}^2 + 2q_{32}^2 + 3q_{33}^2 + (\hat{q}_{31}^2)^2 + \hat{q}_{31}^2 + 2(\hat{q}_{32}^2)^2 + 2(\hat{q}_{33}^2)^2 + 3d_{33}^2 + 7. \]

The demand for the product \( l \) from consumer market \( k \) at the demand price were given by:
\[ d_{11} = \frac{1}{4} \rho_{11} + \frac{3}{8} \rho_{12} - 125, \]
\[ d_{12} = \frac{1}{2} \rho_{11} + \frac{1}{2} \rho_{12} - 200, \]
\[ d_{13} = -\frac{3}{2} \rho_{11} - \frac{9}{8} \rho_{12} + 625, \]
\[ d_{21} = \frac{1}{4} \rho_{21} + \frac{1}{2} \rho_{22} - 200, \]
\[ d_{22} = \frac{5}{8} \rho_{21} + \frac{3}{8} \rho_{22} - 200, \]
\[ d_{23} = \frac{3}{8} \rho_{21} + \frac{5}{8} \rho_{22} - 200. \]

For this problem, apply Algorithm 3.1 proposed in this example, the calculation results are shown in Table 1-7. The product shipments between three manufacturers, three retailers and three consumer markets are shown in Table 1,2,3, respectively. The equilibrium price pattern between three manufacturers, three retailers and three consumer markets were given by Table 4,5,6, respectively. From Table 2-5, we can conclude that the demand price of consumer market will decrease as the demand of consumer market increases. From Table 4,5, it can be concluded that the price of direct selling is lower than that of the retailer in the consumer market, because direct selling reduces the cost in the transaction process, and the quantity purchased from the manufacturer in the consumer market is more than that from the retailer. At the same time, the results of the table show that the equilibrium conditions of supply chain can be better satisfied between manufacturer and retailer, retailer and consumer market, and between manufacturer and consumer market. The total computational time is 3.69 second.

**Table 1.** Productions from manufacturers to retailers

| \((q_{ij}^1/q_{ij}^2)\) | Retailer 1 | Retailer 2 | Retailer 3 |
|---------------------------|-----------|-----------|-----------|
| Manufacturer 1            | 7.9043/7.8023 | 7.9295/7.8056 | 7.9800/7.1543 |
| Manufacturer 2            | 7.9800/7.8100 | 7.8790/7.8432 | 8.0305/8.0025 |
| Manufacturer 3            | 7.9295/7.6265 | 7.9800/7.6770 | 8.0305/7.7275 |

**Table 2.** Productions from manufacturers to consumer markets

| \((\tilde{q}_{1k}^1/\tilde{q}_{1k}^2)\) | Consumer market 1 | Consumer market 2 | Consumer market 3 |
|-------------------------------|-----------------|-----------------|-----------------|
| Manufacturer 1                | 9.4445/9.1415   | 9.4950/9.1920   | 9.5455/9.2425   |
| Manufacturer 2                | 9.4445/9.1415   | 9.5455/9.2425   | 9.4950/9.1920   |
| Manufacturer 3                | 9.5455/9.5455   | 9.3435/9.3435   | 9.4950/9.4950   |

**Table 3.** Productions from retailers to consumer markets

| \((\tilde{q}_{jk}^1/\tilde{q}_{jk}^2)\) | Consumer market 1 | Consumer market 2 | Consumer market 3 |
|-------------------------------|-----------------|-----------------|-----------------|
| Retailer 1                    | 1.2475/1.2345   | 1.2424/1.2323   | 1.2374/1.2172   |
| Retailer 2                    | 1.2374/1.2475   | 1.2323/1.2233   | 1.2273/1.2172   |
| Retailer 3                    | 1.2273/1.2475   | 1.2222/1.2323   | 1.2172/1.2172   |
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Table 4. Price from manufacturers to consumer markets

|                  | Consumer market 1 | Consumer market 2 | Consumer market 3 |
|------------------|-------------------|-------------------|-------------------|
| Manufacturer 1   | 198.2578/228.0022| 200.0245/230.5247| 201.1178/228.7279|
| Manufacturer 2   | 198.2445/228.0022| 200.8785/227.2225| 200.0110/225.7920|
| Manufacturer 3   | 196.3135/220.1133| 211.8830/234.5435| 200.0110/220.4950|

Table 5. Price from retailers to consumer markets

|                  | Consumer market 1 | Consumer market 2 | Consumer market 3 |
|------------------|-------------------|-------------------|-------------------|
| Retailer 1       | 220.2273/250.2245| 218.2323/248.2475| 221.7715/252.2172|
| Retailer 2       | 218.2374/248.5455| 219.9423/249.1415| 220.1920/250.2172|
| Retailer 3       | 222.7415/247.1420| 222.9435/249.1835| 223.1920/251.3570|

Table 6. Price from manufacturers to retailers

|                  | Retailer 1 | Retailer 2 | Retailer 3 |
|------------------|------------|------------|------------|
| Manufacturer 1   | 157.5213/186.7230| 160.1246/187.4950| 158.2275/186.2711|
| Manufacturer 2   | 155.2117/184.1917| 159.3287/190.0058| 158.2365/185.5003|
| Manufacturer 3   | 156.2226/185.2459| 159.4962/189.2234| 159.1435/189.3872|

Table 7. Consumer market demand price

| Product | Consumer market 1 | Consumer market 2 | Consumer market 3 |
|---------|-------------------|-------------------|-------------------|
| Product 1 | 236.2227 | 225.5642 | 215.2359 |
| Product 2 | 209.2117 | 201.4962 | 189.1165 |

Example 4.2. The supply chain network equilibrium problem consisted of two manufacturers, two retailers, two consumer markets and a single commodity, as depicted in Figure 6.

The production cost functions for the manufacturers were given by:

\[ f_1(Q^1) = 2.5(q_1)^2 + q_1q_2 + 2q_1, \]
\[ f_2(Q^2) = 2.5(q_2)^2 + q_1q_2 + 2q_2, \]

where \( q_i \) denote the output of manufacturer \( i \) and \( q_i = \sum_{j=1}^{2} q_{ij} + \sum_{k=1}^{2} \hat{q}_{ik} \).

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers were given by:

\[ c_{ij}(q_{ij}) = 5(q_{ij})^2 + 3.5q_{ij} \quad (i = 1, 2; j = 1, 2), \]

where \( q_{ij} \) denote the amount of manufacturer \( i \) supply to vendor \( j \).

The transaction cost functions faced by the manufacturer and associated with transacting with the consumer markets were given by:

\[ \hat{c}_{ik}(\hat{q}_{ik}) = 5(\hat{q}_{ik})^2 + 5\hat{q}_{ik} \quad (i = 1, 2; k = 1, 2), \]

where \( \hat{q}_{ik} \) denote the amount of manufacturer \( i \) supply to consumer market \( k \).
The transaction cost functions faced by the retailers and associated with transacting with manufacturers and consumer markets were given by:

\[ \hat{c}_j(Q_j) = 5 \left( \sum_{i=1}^{2} q_{ij}^2 + \sum_{k=1}^{2} \hat{q}_{jk}^2 \right) \quad (j = 1, 2). \]

The demand for the product from consumer market at the demand price were given by:

\[ d_1 = -5\rho_1 + 4\rho_2 + \frac{200}{3}, \quad d_2 = -\frac{9}{4}\rho_1 + 5\rho_2 + 100. \]

Algorithm 3.1 in this paper is used to solve this problem, and yields the following equilibrium pattern: manufacturer 1,2’s production capacity are \( q_1 = q_2 = 1135.2 \) (\( q_i \) denote the total output of manufacturer \( i, \ i = 1, 2 \)), the product shipments between the two manufacturers and the two retailers are: \( q_{11} = q_{12} = q_{21} = q_{22} = 245.925 \), the product shipments between the two manufacturers and the two consumer markets are: \( \hat{q}_{11} = \hat{q}_{12} = \hat{q}_{21} = \hat{q}_{22} = 321.675 \), the price shipments between the two manufacturers and the two retailers are: \( \rho_{11} = \rho_{12} = \rho_{21} = \rho_{22} = 0.6172 \), the price shipments between the two manufacturers and the two consumer markets are: \( \hat{\rho}_{11} = \hat{\rho}_{12} = \hat{\rho}_{21} = \hat{\rho}_{22} = 0.7512 \), the price shipments between the two retailers and the two consumer markets are: \( \rho_{11} = \rho_{12} = \rho_{21} = \rho_{22} = 0.8246 \), and the demand price in the consumer market are \( \rho_1 = \rho_2 = 45.85 \) (\( \rho_i \) denote the demand price of consumer market \( i, \ i = 1, 2 \)). The total number of iterations is:101. The total computational time is 0.15 second. From experimental result, we can see that the number of goods that manufacturers sell directly to the consumer market is more than that to retailers. This is because direct sales reduces transaction costs, making the price paid by the consumer market smaller.

Now, we will implement Algorithm 3.1 in the following some examples, which first present by Dong, Zhang and Nagurney in [3], and also used by Zhang in [17]. From the numerical results, we conclude that our algorithm performs well for some problems.

**Example 4.3.** This is the second example in [3, 17], respectively, which includes two manufacturers, two retailers and a single commodity, as depicted in Figure 7.

The production cost functions for the manufacturers were given by:

\[ f_1(Q) = 2.5(q_1)^2 + q_1q_2 + 2q_1, \quad f_2(Q) = 2.5(q_2)^2 + q_1q_2 + 2q_2, \]
where \( q_1 = q_{11} + q_{12}, q_2 = q_{21} + q_{22} \). The transaction cost functions faced by the manufacturers and associated with transacting with the retailers were given by:

\[
c_{ij}(q_j) = 0.5(q_j)^2 + 3.5q_j, \quad \text{for } i = 1, 2; j = 1, 2.
\]

The handling costs of the retailers, in turn, were given by:

\[
c_1(Q) = 0.5(\sum_{i=1}^{2} q_{ij})^2, \quad c_2(Q) = 0.5(\sum_{i=1}^{2} q_{2i})^2.
\]

Applying (19) in [3] with \( \lambda^+_j = \lambda^-_j = 1(j = 1, 2) \), we obtain

\[
X \geq 0, \quad F(X) \geq 0, \quad X^TF(X) = 0.
\]

where \( X = (q_{11}, q_{12}, q_{21}, q_{22}, \rho_1, \rho_2)^T \),

\[
F(X) = \begin{pmatrix}
\frac{\partial f_1}{\partial q_{11}} + \frac{\partial c_{11}}{\partial q_{11}} + \frac{\partial c_1}{\partial q_{11}} + P_1 - (1 + \rho_1)(1 - P_1) \\
\frac{\partial f_1}{\partial q_{12}} + \frac{\partial c_{12}}{\partial q_{12}} + \frac{\partial c_1}{\partial q_{12}} + P_2 - (1 + \rho_2)(1 - P_2) \\
\frac{\partial f_2}{\partial q_{21}} + \frac{\partial c_{21}}{\partial q_{21}} + \frac{\partial c_1}{\partial q_{21}} + P_1 - (1 + \rho_1)(1 - P_1) \\
\frac{\partial f_2}{\partial q_{22}} + \frac{\partial c_{22}}{\partial q_{22}} + \frac{\partial c_1}{\partial q_{22}} + P_2 - (1 + \rho_2)(1 - P_2) \\
q_{11} + q_{21} - d_1(\rho_1) \\
q_{12} + q_{22} - d_2(\rho_2) \\
(7 + \frac{\rho_1}{50})q_{11} + 5q_{12} + (2 + \frac{\rho_2}{50})q_{21} + 2q_{22} - \rho_1 + 4.5 \\
5q_{11} + (7 + \frac{\rho_1}{50})q_{12} + q_{21} + (2 + \frac{\rho_2}{50})q_{22} - \rho_2 + 4.5 \\
(2 + \frac{\rho_1}{50})q_{11} + q_{12} + (7 + \frac{\rho_1}{50})q_{21} + 5q_{22} - \rho_1 + 4.5 \\
q_{11} + (2 + \frac{\rho_1}{50})q_{12} + 5q_{21} + (7 + \frac{\rho_1}{50})q_{22} - \rho_2 + 4.5 \\
q_{11} + q_{21} - \frac{50}{\rho_1} \\
q_{12} + q_{22} - \frac{50}{\rho_2}
\end{pmatrix}
\]

where the second equality follows from

\[
P_j = \frac{(q_{1j} + q_{2j})\rho_j}{b_j}, \quad d_j(\rho_j) = \frac{b_j}{2\rho_j}, \quad j = 1, 2, \tag{36}
\]

with \( b_j = 100, j = 1, 2 \).

Use Algorithm 3.1 in this example, and yields the following equilibrium pattern: the product shipments between the two manufacturers and the two retailers are: \( q_{11} = q_{12} = q_{21} = q_{22} = 0.75(q_j \) denote the amount of the manufacturer \( i \) supply to the vendor \( j \), and the demand prices at the retailers are; \( \rho_1 = \rho_2 = 33.02 \); the total number of iterations is: 64; the total computational time is 0.43 second. Compared with the calculated results in Example 2 ([17]), and are shown in Table 8.

| Table 8. Compared with the results in Example 2([17]) |
|-----------------------------------------------|
| Iteration steps | 12 | 64 |
| Running time    | 0.47 | 0.43 |

Example 4.4. This is the third example in [3, 17], respectively. We kept the data as in Example 4.3 in this paper, but now we set \( b_1 = b_2 = 1000 \), the structure of the supply chain network remained as in Figure 7.
Algorithm 3.1 in this paper for this example is convergent and yields the following equilibrium pattern: the product shipments between the two manufacturers and the two retailers are \( q_{11} = q_{12} = q_{21} = q_{22} = 2.7417 \), and the equilibrium demand price pattern given by: \( \rho_1 = \rho_2 = 91.8165 \); the total number of iterations is: 67; the total computational time is 0.07 second.

In this example, we increased the \( b_j \) even more than that in Example 4.3, by the second equality in (36), we have that the expected demand associated with the retailers was even higher than in Example 4.3. It is easy to see from numerical results that the production outputs of the manufacturers increased, as did the demand prices for the product at the retailers.

**Example 4.5.** This is the fourth example in [3, 17], respectively, which includes three manufacturers, two retailers and a single commodity, as depicted in Figure 8.

The data for this example were constructed from the data for Example 4.4 in this paper, but we added the necessary functions for the third manufacturer resulting in the following functions: The production cost functions for the manufacturers were given by:

\[
\begin{align*}
  f_1(Q) &= 2.5(q_1)^2 + q_1q_2 + 2q_1, \\
  f_2(Q) &= 2.5(q_2)^2 + q_1q_2 + 2q_2, \\
  f_3(Q) &= 0.5(q_3)^2 + 0.5q_1q_3 + 2q_3,
\end{align*}
\]

where \( q_1 = q_{11} + q_{12}, q_2 = q_{21} + q_{22}, q_3 = q_{31} + q_{32} \).

Note that the production cost function associated with the third manufacturer was distinct from those of the other two manufacturers.

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers were given by:

\[
  c_{ij}(q_{ij}) = 0.5(q_{ij})^2 + 3.5q_{ij}, \text{ for } i = 1, 2, 3; j = 1, 2.
\]

For this example, we retained the transaction cost functions utilized in Example 4.4 except that we now added new ones associated with the transactions between the new manufacturer and the two retailers. The handling costs of the retailers remained as in Example 4.4 as did the expected demand functions. This example will illustrates what may happen when a new manufacturer enters the market with lower production costs than the other manufacturers.

Applying (19) in [3] with \( \lambda_1^T = 1, \lambda_2^T = 1(j = 1, 2) \) again, we obtain

\[
X \geq 0, \quad F(X) \geq 0, \quad X^TF(X) = 0.
\]
where $X = \left( \begin{array}{c} q_{11}, q_{12}, q_{21}, q_{22}, q_{31}, q_{32}, \rho_1, \rho_2 \end{array} \right)^T$,

$$F(X) = \left( \begin{array}{c} \frac{\partial f_1(q_{11})}{\partial q_{11}} + \frac{\partial c_{11}(q_{11})}{\partial q_{11}} + \frac{\partial c_{11}(q_{11})}{\partial q_{11}} + P_1 - (1 + \rho_1)(1 - P_1) \\ \frac{\partial f_2(q_{21})}{\partial q_{21}} + \frac{\partial c_{21}(q_{21})}{\partial q_{21}} + \frac{\partial c_{21}(q_{21})}{\partial q_{21}} + P_2 - (1 + \rho_2)(1 - P_2) \\ \frac{\partial f_1(q_{21})}{\partial q_{21}} + \frac{\partial c_{21}(q_{21})}{\partial q_{21}} + \frac{\partial c_{21}(q_{21})}{\partial q_{21}} + P_1 - (1 + \rho_1)(1 - P_1) \\ \frac{\partial f_2(q_{31})}{\partial q_{31}} + \frac{\partial c_{31}(q_{31})}{\partial q_{31}} + \frac{\partial c_{31}(q_{31})}{\partial q_{31}} + P_2 - (1 + \rho_2)(1 - P_2) \\ \frac{\partial f_1(q_{31})}{\partial q_{31}} + \frac{\partial c_{31}(q_{31})}{\partial q_{31}} + \frac{\partial c_{31}(q_{31})}{\partial q_{31}} + P_1 - (1 + \rho_1)(1 - P_1) \\ \frac{\partial f_2(q_{32})}{\partial q_{32}} + \frac{\partial c_{32}(q_{32})}{\partial q_{32}} + \frac{\partial c_{32}(q_{32})}{\partial q_{32}} + P_2 - (1 + \rho_2)(1 - P_2) \\ \frac{\partial f_1(q_{11})}{\partial q_{11}} + \frac{\partial c_{11}(q_{11})}{\partial q_{11}} + \frac{\partial c_{11}(q_{11})}{\partial q_{11}} + \rho_1 - 4.5 \\ 6q_{31} + (6 + \omega)q_{12} + q_{21} + (2 + \omega)q_{12} + (1 + \omega)q_{32} - \rho_1 + 4.5 \\ (2 + \omega)q_{11} + q_{12} + (7 + \omega)q_{21} + 5q_{22} + (1 + \omega)q_{31} - \rho_1 - 4.5 \\ q_{11} + (2 + \omega)q_{21} + 5q_{21} + (7 + \omega)q_{22} + (1 + \omega)q_{32} - \rho_2 + 4.5 \\ \frac{1}{2}q_{11} + (1 + \omega)q_{12} + (1 + \omega)q_{21} + (3 + \omega)q_{31} + 5q_{31} - \rho_1 + 4.5 \\ \frac{1}{2}q_{11} + (1 + \omega)q_{21} + (3 + \omega)q_{22} - \rho_2 + 4.5 \\ q_{11} + q_{21} + q_{31} - \frac{500}{P_j} \\ q_{12} + q_{22} + q_{32} - \frac{500}{P_j} \end{array} \right)$$

where the second equality follows from $P_j = \frac{(q_{12} + q_{22} + q_{32})\rho_j}{1000}$, $d_j(\rho_j) = \frac{500}{\rho_j}$, $j = 1, 2$, and $\omega = \rho_1 + \frac{(\rho_1)^2}{1000}$.

Applying Algorithm 3.1 in this paper, the equilibrium product shipment pattern given by: $q_{11} = q_{12} = q_{21} = q_{22} = 1.3648$, $q_{31} = q_{32} = 5.3326$, and the equilibrium demand price pattern given by: $\rho_1 = \rho_2 = 61.2642$; the total number of iterations is: 58; the total computational time is 0.07 second. Compared with the results in Example 4 in [17], and are shown in Table 9.

The data for this example are constructed from the data for Example 4.4 in this paper, but we added the third manufacturer. Compared with numerical results in Example 4.4, with the addition of a new manufacturer, the price charged at the retail outlets is now lower, due to the competition and the increased supply of the product.

![Figure 8. The network structure of the supply chain for Example 4.5](image)

**Example 4.6.** This is the fifth example in [3, 17], respectively, which includes three manufacturers, two retailers and a single commodity, the structure of the supply chain network remained as depicted in Figure 8.

This numerical example was constructed from the Example 4.5 in this paper with the data retained but with the following change: we increase the weight associated
with oversupply at all retail outlets from 1 to 10, and decrease the weights associated with undersupply at all retail outlets from 1 to 10, i.e., $\lambda^+_j = 10, \lambda^-_j = 0 (j = 1, 2)$.

For this example, using (19) in [3], we obtain

$$X \geq 0, \quad F(X) \geq 0, \quad X^T F(X) = 0.$$ 

where $X = (q_{11}, q_{12}, q_{21}, q_{22}, q_{31}, q_{32}, \rho_1, \rho_2)^T$,

$$F(X) = \begin{pmatrix} \frac{\partial f_1(Q)}{\partial q_{11}} + \frac{\partial c_1(q_{11})}{\partial q_{11}} + \frac{\partial c_1(q_{11})}{\partial q_{12}} + 10P_1 - \rho_1(1 - P_1) \\ \frac{\partial f_1(Q)}{\partial q_{12}} + \frac{\partial c_1(q_{12})}{\partial q_{12}} + \frac{\partial c_1(q_{12})}{\partial q_{21}} + 10P_2 - \rho_2(1 - P_1) \\ \frac{\partial f_1(Q)}{\partial q_{21}} + \frac{\partial c_1(q_{21})}{\partial q_{21}} + \frac{\partial c_1(q_{21})}{\partial q_{22}} + 10P_2 - \rho_2(1 - P_2) \\ \frac{\partial f_1(Q)}{\partial q_{22}} + \frac{\partial c_1(q_{22})}{\partial q_{22}} + \frac{\partial c_1(q_{22})}{\partial q_{31}} + 10P_1 - \rho_1(1 - P_2) \\ \frac{\partial f_1(Q)}{\partial q_{31}} + \frac{\partial c_1(q_{31})}{\partial q_{31}} + \frac{\partial c_1(q_{31})}{\partial q_{32}} + 10P_2 - \rho_2(1 - P_2) \\ q_{11} + q_{21} + q_{31} - d_1(\rho_1) \\ q_{12} + q_{22} + q_{32} - d_2(\rho_2) \end{pmatrix}.$$ 

where the second equality follows from $P_j = \frac{(q_{11} + q_{21} + q_{31})\rho_j}{1000}, d_j(\rho_j) = \frac{500}{\rho_j}, j = 1, 2,$

and $\omega = \frac{\rho_1}{100} + \frac{(\rho_2)^2}{1000}$.

Applying Algorithm 3.1 in this paper, the equilibrium product shipment pattern given by: $q_{11} = q_{12} = q_{21} = q_{22} = 1.250$, $q_{31} = q_{32} = 5.42226$, and the equilibrium demand price pattern given by: $\rho_1 = \rho_2 = 63.5469$; the total number of iterations is 109; the total computational time is 0.086 second. Compared with the calculated results in Example 5 ([17]), and are shown in Table 10.

Since the penalty associated with excess supply increased and there was no penalty imposed on shortage by each retailer. Thus, each retailer reduced his order quantity. Compared with the results in Example 4.5, the price at each retailer increased due to the higher probability of having a shortage.

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Table 10. Compared with the results in Example 5 ([17])

|                      | Literature results | Results of this paper |
|----------------------|--------------------|-----------------------|
| Iteration steps      | 12                 | 109                   |
| Running time         | 0.11               | 0.086                 |

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