Gevrey Smoothing Effect for Solutions of the Non-Cutoff Boltzmann Equation in Maxwellian Molecules Case

Teng-Fei Zhang¹  Zhaoyang Yin²

Department of Mathematics, Sun Yat-sen University,
510275, Guangzhou, P. R. China.
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Abstract

In this paper we study the Gevrey regularity for the weak solutions to the Cauchy problem of the non-cutoff spatially homogeneous Botlzmann equation for the Maxwellian molecules model with the singularity exponent $s \in (0, 1)$. We establish that any weak solution belongs to the Gevrey spaces for any positive time.

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¹Corresponding author. Email: fgeyirui@163.com
²Email: mcszy@mail.sysu.com.cn
1 Introduction

1.1. The Boltzmann equation

In this paper we are concerned with the Cauchy problem of the non-cutoff Boltzmann equation. First we introduce the Cauchy problem of the full (or, spatially inhomogeneous) Boltzmann equation without angular cutoff, with a $T > 0$,

$$
\begin{cases}
  f_t(t, x, v) + v \cdot \nabla_x f(t, x, v) = Q(f, f)(v), & t \in (0, T], \; x \in \mathbb{T}^3, \; v \in \mathbb{R}^3, \\
  f(0, x, v) = f_0(x, v).
\end{cases}
$$

Above, $f = f(t, x, v)$ describes the density distribution function of particles located around position $x \in \mathbb{T}^3$ with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The right-hand side of the first equation is the so-called Boltzmann bilinear collision operator acting only on the velocity variable $v$:

$$
Q(g, f) = \int_{\mathbb{R}^3} \int_{S^2} B(v - v_*, \sigma) \{g'_* f' - g_* f\} \, d\sigma dv_*.
$$

Note that we use the well-known shorthands $f = f(t, x, v), \; f_* = f(t, x, v_*), \; f' = f(t, x, v'), \; f'_* = f(t, x, v'_*)$ throughout this paper.

Then, we consider the Cauchy problem of the Boltzmann equation in the spatially homogeneous case, that is, for a $T > 0$,

$$
\begin{cases}
  f_t(t, v) = Q(f, f)(v), & t \in (0, T], \; v \in \mathbb{R}^3, \\
  f(0, v) = f_0(v),
\end{cases}
$$

where “spatially homogeneous” means that $f$ depends only on $t$ and $v$. 

2 Preliminaries

2.1 The mollifier operator

2.2 Coercivity estimates for collision operator

2.3 Estimates for Commutator with weights

3 Sobolev regularity for weak solutions

3.1 Commutator estimates with Sobolev mollifier

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By using the $\sigma$-representation, we can describe the relations between the post- and pre-collisional velocities as follows, for $\sigma \in S^2$,

$$v' = \frac{v + v_s}{2} + \frac{|v - v_s|}{2} \sigma, \quad v'_s = \frac{v + v_s}{2} - \frac{|v - v_s|}{2} \sigma.$$  

We point out that the collision process satisfies the conservation of momentum and kinetic energy, i.e.

$$v + v_s = v' + v'_s, \quad |v|^2 + |v_s|^2 = |v'|^2 + |v'_s|^2.$$  

The collision cross section $B(z, \sigma)$ is a given non-negative function depending only on the interaction law between particles. From a mathematical viewpoint, that means $B(z, \sigma)$ depends only on the relative velocity $|z| = |v - v_s|$ and the deviation angle $\theta$ through the scalar product $\cos \theta = \frac{\langle z, \sigma \rangle}{|z|}$.  

The cross section $B$ is assumed here to be of the type:

$$B(v - v_s, \cos \theta) = \Phi(|v - v_s|) b(\cos \theta), \quad \cos \theta = \frac{v - v_s}{|v - v_s|} \cdot \sigma, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

where, $\Phi$ stands for the kinetic factor which is of the form:

$$\Phi(|v - v_s|) = |v - v_s|^\gamma,$$

and $b$ denotes the angular part with singularity such that,

$$\sin \theta b(\cos \theta) \sim K\theta^{-1 - 2s}, \quad \text{as} \quad \theta \to 0^+,$$

for some positive constant $K$ and $0 < s < 1$.  

We remark that if the inter-molecule potential satisfies specifically the inverse-power law $U(\rho) = \rho^{-(p-1)}$ (where $p > 2$), it holds $\gamma = \frac{p-5}{p-1}$, $s = \frac{1}{p-1}$. Generally, the cases $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$ correspond to so-called hard, Maxwellian, and soft potential respectively. And the cases $0 < s < 1/2$, $1/2 \leq s < 1$ correspond to so-called mild singularity and strong singularity respectively.

1.2. Review of related references

Now we give a brief review about some related researches. Firstly we refer the reader to Villani’s review book [20] for the physical background and the mathematical theories of the Boltzmann equation. And for more information about the non-cutoff theories, one can consult Alexandre’s review paper [1].

Before continuing the statement, we provide the definition of Gevrey spaces $G^s(\Omega)$ where $\Omega$ is an open subset of $\mathbb{R}^3$. (It could be found in many references, e.g. [17, 21].)

**Definition 1.1.** For $0 < s < +\infty$, we say that $f \in G^s(\Omega)$, if $f \in C^\infty(\Omega)$, and there exist $C > 0$, $N_0 > 0$ such that

$$\|\partial^\alpha f\|_{L^2(\Omega)} \leq C^{(|\alpha|+1)} \{\alpha!\}^s, \quad \forall \alpha \in \mathbb{N}^3, \quad |\alpha| \geq N_0.$$  

If the boundary of $\Omega$ is smooth, by using the Sobolev embedding theorem, we have the same type estimate with $L^2$ norm replaced by any $L^p$ norm for $2 < p \leq +\infty$. More specifically, on the whole space $\Omega = \mathbb{R}^3$, it is also equivalent to

$$e^{c_0(-\Delta)^{1/(2s)}}(\partial^{\beta_0} f) \in L^2(\mathbb{R}^3),$$

for some $c_0 > 0$ and $\beta_0 \in \mathbb{N}^3$, where $e^{c_0(-\Delta)^{1/(2s)}}$ is the Fourier multiplier defined by

$$e^{c_0(-\Delta)^{1/(2s)}}u(x) = \mathcal{F}^{-1}\left(e^{c_0|\xi|^{1/2}}\hat{u}(\xi)\right).$$

When $s = 1$, it is usual analytic function. If $s > 1$, it is Gevrey class function. And for $0 < s < 1$, it is called ultra-analytic function.

In 1984 Ukai showed in [19] that there exists a unique local solution to the Cauchy problem for the Boltzmann equation in Gevrey classes for both spatially homogeneous and inhomogeneous cases, under the assumption on the cross section:

$$|B(|z|, \cos \theta)| \leq K(1 + |z|^{-\gamma'} + |z|^\gamma)\theta^{-n+1-2s}, \quad n \text{ is dimensionality,}$$

$$(0 \leq \gamma' < n, \ 0 \leq \gamma < 2, \ 0 \leq s < 1/2, \ \gamma + 6s < 2).$$

By introducing the norm of Gevrey space

$$\|f\|_{U_{\delta, \rho, \nu}} = \sum_{\alpha} \rho^{\alpha} \left|\alpha!\right|^{\nu} \|e^{\delta(v)}\partial^\alpha v f\|_{L^\infty(\mathbb{R}^2)},$$

Ukai proved that in the spatially homogeneous case, for instance, under some assumptions for $\nu$ and the initial datum $f_0(v)$, the Cauchy problem (1.2) has a unique solution $f(t, v)$ for $t \in (0, T]$.

In [8] Desvillettes and Wennberg studied firstly the $C^\infty$ smoothing effect for solutions of Cauchy problem in spatially homogeneous non-cutoff case, and conjectured Gevrey smoothing effect. And later, Desvillettes et al. proved in [7] the propagation of Gevrey regularity for solutions for Maxwellian molecules case.

In 2009 Morimoto et al. considered in [16] the Gevrey regularity for the solutions to the Cauchy problem of the linearized Boltzmann equation, for the Maxwellian molecules model and around the absolute Maxwellian distribution, by virtue of the following mollifier:

$$G_{\delta}(t, D_v) = \frac{e^{t(D_v)\gamma/\alpha}}{1 + \delta e^{t(D_v)\gamma/\alpha}}, \quad 0 < \delta < 1.$$  

Therein the authors proved that the solutions belong to the Gevrey spaces $G^{1/\alpha}$ for any $0 < \alpha < 1$, when the singularity exponent $s \in (0, 1)$.

In [13] Lekrine and Xu proved that, using the same method, the Gevrey regularity for solutions to the Kac’s equation (a simplification of the Boltzmann equation to one dimensional case), and Gevrey
regularity for the radially symmetric weak solutions to the Boltzmann equation. Under the mild singularity assumption \( s \in (0, 1/2) \), they proved the radially symmetric weak solutions are in the Gevrey spaces \( G^{1/(2s')} \) for any \( s' \in (0, s) \) and any time \( t > 0 \). Recently, Glangetas and Najeme complemented their results for the strong singularity case \( s \in [1/2, 1) \), and established the analytic smoothing effect in \([9]\). The similar mollifier was used by Morimoto and Xu, to prove the ultra-analytic smoothing effect for spatially homogeneous nonlinear Landau equation and the linear and non-linear Fokker-Planck equations (see \([17]\)). And Lerner et al. proved in \([14]\) that the Cauchy problem of the radially symmetric spatially homogeneous non-cutoff Boltzmann equation with Maxwellian molecules enjoys the same Gelfand-Shilov regularizing effect as the Cauchy problem of some kind of evolution equation associated to a fractional harmonic oscillator.

In the mild singularity case of \( 0 < s < 1/2 \), Huo et al. proved in \([12]\) that any weak solution \( f(t, v) \) to the Cauchy problem (1.2) satisfying the natural boundedness on mass, energy and entropy, belongs to \( H^{+\infty}(\mathbb{R}^n) \) for any \( 0 < t \leq T \).

In 2010 Morimoto and Ukai considered in \([15]\) the Gevrey regularity (precisely, \( G^{1/(2s')} \)), of \( C^\infty \) solutions with the Maxwellian decay to the Cauchy problem of spatially homogeneous Boltzmann equation, with a modified kinetic factor \( \Phi(v) = \langle v \rangle^\gamma \). Recently, Zhang and Yin extended this result for the general kinetic factor \( \Phi(v) = |v|^\gamma \) (see \([21]\)), and as a continuation of that, they studied the problem for the spatially inhomogeneous case, for extending to a larger range of the exponent \( \gamma \), and for a special critical singularity case, respectively. Through these works we attempt to give an almost whole description of the Gevrey regularity for the so-called smooth Maxwellian decay solution. For more details, one can consult \([22, 23]\).

In this present work, we consider the Gevrey smoothing effect for the weak solutions to the Cauchy problem of spatially homogeneous Boltzmann equation without cut-off. Because of the difficulty coming from the interaction between the generally kinetic factor \( \Phi(v) = |v|^\gamma \) and the mollifier operator defined below, we will restrict our attention to the Maxwellian molecules model \( \Phi \equiv 1 \). Further, we consider not only the mild singularity case \( 0 < s < 1/2 \) but also strong singularity case \( 1/2 \leq s < 1 \), the latter case of which, as is known to all, is difficulty to deal with and thus there are few of research about that. Additionally, we point out that it’s necessary to consider an improved commutator estimate with weight owning one more higher order than before. We establish in the present paper that any weak solution belongs to the Gevrey spaces for any positive time.

1.3. Statement of the main result

Now we give our main result of Gevrey regularity for the spatially homogeneous Boltzmann equation for the Maxwellian molecules model, as follows:
Theorem 1.2. Suppose that the initial datum $f_0 \in L^{1+2s} \cap L\log L(\mathbb{R}^3)$. If $f \in L^\infty((0, +\infty); L^{3+2s} \cap L\log L(\mathbb{R}^3))$ is a non-negative weak solution to the Cauchy problem (1.2), then

i) for the mild singularity case $0 < s < \frac{1}{2}$, we have $f(t, \cdot) \in G^{\frac{1}{12}}(\mathbb{R}^3)$ for any $0 < \alpha < s$ and $t > 0$;

ii) in the critical case of $s = \frac{1}{2}$, we have $f(t, \cdot) \in G^{s'}(\mathbb{R}^3)$ for any $s' > \frac{3}{2}$ and $t > 0$;

iii) for the strictly strong singularity case $\frac{1}{2} < s < 1$, we have $f(t, \cdot) \in G^{\frac{1}{2}}(\mathbb{R}^3)$ for any $t > 0$.

Remark 1.3. From the argument in Section 4, we can claim precisely that $f(t, \cdot) \in G^{\frac{1}{12} + \varepsilon}(\mathbb{R}^3)$ for any $t > 0$ and $\varepsilon > 0$ for the strong singularity case $s \in [\frac{1}{2}, 1)$.

1.4. The structure of the paper

The rest of the paper is organized as follows. In the next section we give some preliminaries including some properties of the Gevrey mollifier, and commutator estimates between the collision operator and the mollifier. In Section 3 we establish the Sobolev smoothing effect for the weak solutions in some weighted Sobolev spaces, by taking advantage of the Sobolev mollifier operator. The last section is devoted to state the Gevrey regularizing effect for the weak solutions.

2 Preliminaries

2.1. The mollifier operator

To study the Gevrey regularizing effect for weak solutions of the Boltzmann equation, we consider the following exponential type mollifier (compare [13]):

$$G_\delta(t, \xi) = \frac{e^{c_0 t (\xi)^2}}{1 + \delta e^{c_0 t (\xi)^2}},$$

with $(\xi) = (1 + |\xi|^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}^3$ and $c_0 > 0$, $0 < \delta < 1$. It is an easy matter to check, for any $0 < \delta < 1$, that

$$G_\delta(t, \xi) \in L^\infty \left((0, T) \times \mathbb{R}^3\right),$$

and

$$\lim_{\delta \to 0} G_\delta(t, \xi) = e^{c_0 t (\xi)^2}.$$
where $\hat{h}$ represents the Fourier transform of $h$.

We aim to state the uniform bound of the term $\|G_\delta(t, D_\nu)f(t, v)\|_{L^2}$ for the weak solutions to the Cauchy problem \([1,2]\), with respect to $\delta$. In all that follows, the same notation $G_\delta$ will stand for the pseudo-differential operators $G_\delta(t, D_\nu)$ or alternatively, its symbol $G_\delta(t, \xi)$, for their meanings may be inferred from the context.

We then give some properties about $G_\delta(t, \xi)$, as follows:

**Lemma 2.1.** Let $T > 0$, then for any $t \in [0, T]$ and $\xi \in \mathbb{R}^3$, we have

(2.4) \[ |\partial_t G_\delta(t, \xi)| \leq c_0(\xi)^{2\alpha} G_\delta(t, \xi), \]

(2.5) \[ |\partial_\xi G_\delta(t, \xi)| \leq 2\alpha c_0 t |\xi|^{2\alpha-1} G_\delta(t, \xi), \]

(2.6) \[ |\partial^2_{\xi^2} G_\delta(t, \xi)| \leq C(\xi)^{2(2\alpha-1)} G_\delta(t, \xi), \]

(2.7) \[ |\partial^3_{\xi^3} G_\delta(t, \xi)| \leq C(\xi)^{3(2\alpha-1)} G_\delta(t, \xi), \]

(2.8) \[ |\partial^4_{\xi^4} G_\delta(t, \xi)| \leq C(\xi)^{4(2\alpha-1)} G_\delta(t, \xi), \]

with $C > 0$ independent of $\delta$.

**Proof.** By direct calculus, we can infer that

(2.9) \[ \partial_t G_\delta(t, \xi) = c_0(\xi)^{2\alpha} G_\delta(t, \xi) \frac{1}{1 + \delta e^{\alpha t}(\xi)^{2\alpha}}, \]

(2.10) \[ \partial_\xi G_\delta(t, \xi) = 2\alpha c_0 t (1 + |\xi|^2)^{\alpha-1} \xi G_\delta(t, \xi) \frac{1}{1 + \delta e^{\alpha t}(\xi)^{2\alpha}}, \]

(2.11) \[ \partial^2_{\xi, \xi_j} G_\delta(t, \xi) = [2\alpha c_0 t (1 + |\xi|^2)^{\alpha-1}]^2 \xi \xi_j G_\delta(t, \xi) \frac{1 - \delta e^{\alpha t}(\xi)^{2\alpha}}{(1 + \delta e^{\alpha t}(\xi)^{2\alpha})^2} \]
\[ + 2\alpha c_0 t [(1 + |\xi|^2)^{\alpha-2} \xi_j \xi_j (1 + |\xi|^2)^{\alpha-2}] G_\delta(t, \xi) \frac{1}{1 + \delta e^{\alpha t}(\xi)^{2\alpha}}, \]

and

(2.12) \[ \partial^3_{\xi, \xi_j, \xi_k} G_\delta(t, \xi) \]
\[ = [2\alpha c_0 t (1 + |\xi|^2)^{\alpha-1}]^3 \xi \xi_j \xi_k G_\delta(t, \xi) \frac{(1 - \delta e^{\alpha t}(\xi)^{2\alpha})^2}{(1 + \delta e^{\alpha t}(\xi)^{2\alpha})^3} \]
\[ + [2\alpha c_0 t (1 + |\xi|^2)^{\alpha-1}]^3 (\xi_j \delta_j + \xi_j \delta_k + \xi_k \delta_j) G_\delta(t, \xi) \frac{1 - \delta e^{\alpha t}(\xi)^{2\alpha}}{(1 + \delta e^{\alpha t}(\xi)^{2\alpha})^2} \]
\[ + (2\alpha c_0 t)^2 6(\alpha - 1)(1 + |\xi|^2)^{2\alpha-3} \xi \xi_j \xi_k G_\delta(t, \xi) \frac{1 - \delta e^{\alpha t}(\xi)^{2\alpha}}{(1 + \delta e^{\alpha t}(\xi)^{2\alpha})^2} \]
\[ + 2\alpha c_0 t [(2(\alpha - 1)(1 + |\xi|^2)^{\alpha-2} (\xi_j \delta_j + \xi_j \delta_k + \xi_k \delta_j) + 4(\alpha - 1)(\alpha - 2)\xi_j \xi_j (1 + |\xi|^2)^{\alpha-3}] G_\delta(t, \xi) \frac{1}{1 + \delta e^{\alpha t}(\xi)^{2\alpha}}. \]

\[ \Delta A + B + C + D. \]
Then the former four results of Lemma 2.2 follow easily. As for the 4-th derivations in the last equality, for the sake of simplicity, we only take the term $\partial_{\xi} A$ for example, as follows:

$$\partial_{\xi}A = \left[2\alpha c_0 t (1 + |\xi|^2)^{\alpha - 1}\right]^4 \xi_i \xi_j \xi_k \xi_l G_{\delta}(t, \xi) \frac{(1 - \delta e^{\alpha t (\xi)^2})^2}{(1 + \delta e^{\alpha t (\xi)^2})^4}$$

$$+ \left[2\alpha c_0 t (1 + |\xi|^2)^{\alpha - 1}\right]^3 (\delta_{ij} \xi_k + \delta_{jk} \xi_i + \delta_{ki} \xi_j) G_{\delta}(t, \xi) \frac{(1 - \delta e^{\alpha t (\xi)^2})^2}{(1 + \delta e^{\alpha t (\xi)^2})^3}$$

$$+ (2\alpha c_0 t)^3 \delta(\alpha - 1) (1 + |\xi|^2)^{2\alpha - 4} \xi_j \xi_k \xi_l G_{\delta}(t, \xi) \frac{(1 - \delta e^{\alpha t (\xi)^2})^3}{(1 + \delta e^{\alpha t (\xi)^2})^3},$$

combining with computations for the other three terms $\partial_{\xi} B$, $\partial_{\xi} C$, and $\partial_{\xi} D$, this yields the desired result.

**Lemma 2.2.** For all $0 < \delta < 1$ and $\xi \in \mathbb{R}^3$, we have

$$|G_{\delta}(\xi) - G_{\delta}(\xi^+)| \lesssim \sin^2\left(\frac{\theta}{2}\right)|\xi|^{2\alpha} G_{\delta}(\xi^+) G_{\delta}(\xi^-).$$

**Proof.** Noticing the fact $G_{\delta}(t, \xi) = G_{\delta}(t, |\xi|)$, and denoting $s = |\xi|^2$, $s^+ = |\xi^+|^2$, we have

$$G_{\delta}(\xi) = \tilde{G}_{\delta}(s) = \frac{e^{c_0 t + s} \alpha c_0 t \tilde{G}_{\delta}(s)(1 + s)^{\alpha - 1}}{1 + \delta e^{c_0 t + s} \alpha c_0 t}.$$

Moreover, we compute

$$\frac{d}{ds} \tilde{G}_{\delta}(s) = \alpha c_0 t \tilde{G}_{\delta}(s)(1 + s)^{\alpha - 1} \frac{1}{1 + \delta e^{c_0 t + s} \alpha c_0 t} > 0.$$

By virtue of the Taylor formula, it holds,

$$|G_{\delta}(\xi) - G_{\delta}(\xi^+)| = |\tilde{G}_{\delta}(s) - \tilde{G}_{\delta}(s^+)| = |s - s^+| \int_0^1 \frac{d}{ds} \tilde{G}_{\delta}(s_r) ds_r d\tau$$

$$\lesssim |s - s^+| \int_0^1 \tilde{G}_{\delta}(s_r)(1 + s_r)^{\alpha - 1} d\tau,$$

where $s_r = (1 - \tau)s^+ + \tau s$ with $\tau \in [0, 1]$.

On the other hand, the fact $|\xi^+|^2 = |\xi|^2 \cos^2\left(\frac{\theta}{2}\right)$ with $\theta \in [0, \pi/2]$ implies that

$$s_r = (1 - \tau)s^+ + \tau s = (1 - \tau)|\xi^+|^2 + \tau |\xi|^2 \in \left[\frac{1}{2} |\xi|^2, |\xi|^2 \right],$$

thereby we have $(1 + s_r)^{\alpha - 1} \lesssim (1 + s)^{\alpha - 1}$ for $\alpha \in (0, 1/2)$, and

$$\tilde{G}_{\delta}(s_r) \leq \tilde{G}_{\delta}(s) = G_{\delta}(\xi).$$

Recalling the formula $|\xi|^2 = |\xi^+|^2 + |\xi^-|^2$, and the following facts,

$$(1 + a + b)^\alpha \leq (1 + a)^\alpha + (1 + b)^\alpha, \quad (1 + \delta e^\alpha)(1 + \delta e^\beta) \leq 3(1 + \delta e^{\alpha\beta}),$$

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we get
\begin{equation}
G_\delta(\xi) \leq 3G_\delta(\xi^+)G_\delta(\xi^-).
\end{equation}

Thus we can obtain, from (2.15),
\begin{equation}
|G_\delta(\xi) - G_\delta(\xi^+)| \lesssim |\xi|^2 - |\xi^+|^2 \left| G_\delta(\xi)(1 + |\xi|^2)^{\alpha-1} \right| \lesssim \sin^2\left(\frac{\theta}{2}\right)(\xi)^{2\alpha}G_\delta(\xi^+)G_\delta(\xi^-).
\end{equation}

\section{Coercivity estimates for collision operator}

We introduce the following coercivity estimates for the collision operator of the Boltzmann equation (see [8, 16]).

\textbf{Lemma 2.3.} Assume that $g \geq 0, g \not\equiv 0$, and further $g \in L^1_2 \cap LlogL(R^3)$, then there exists a constant $C_g > 0$ depending only on $B, \|g\|_{L^1_2}$ and $\|g\|_{L^1_2 \cap LlogL}$ such that
\begin{equation}
\|f\|^2_{H^s} \leq C_g \langle -Q(g,f), f \rangle + C\|g\|_{L^1_2}\|f\|^2_{L^2}
\end{equation}
for any smooth function $f \in H^2(R^3)$.

\section{Estimates for Commutator with weights}

In this subsection, we will give some estimates for commutators between the Boltzmann collision operator and the mollifier operator.

\textbf{Proposition 2.4.} Assume that $0 < s < 1$ and $0 < \alpha \leq 1/2$. For a suitable function $f$, we have,
\begin{equation}
|\langle G_\delta Q(f,f) - Q(f,G_\delta f), G_\delta f \rangle| \lesssim \|G_\delta f\|_{L^2_2}\|G_\delta f\|^2_{H^s}.
\end{equation}

\textbf{Proof.} Thanks to the Bobylev identity and Plancherel formula, we write that
\begin{equation}
\langle G_\delta Q(f,f) - Q(f,G_\delta f), G_\delta f \rangle
\end{equation}
where we have used the following notations:
\begin{equation}
\xi^- = \frac{\xi - |\xi|\sigma}{2}, \quad \xi^+ = \frac{\xi + |\xi|\sigma}{2}, \quad \cos \theta = \frac{\xi}{|\xi|} \cdot \sigma \triangleq \omega \cdot \sigma.
\end{equation}
By using Lemma 2.2, we get

\[(2.24) \quad |\langle G_\delta Q(f, f) - Q(f, G_\delta f), G_\delta f \rangle| \leq \int \int b(\cos \theta) \sin^2 \frac{\theta}{2} |G_\delta(f^-)\hat{f}(\xi^-)| \, d\xi d\sigma \]

\[\leq \frac{1}{2} \|G_\delta f\|_{L^2} \|G_\delta f\|_{H^s} \]

where in the last inequality we have used the embedding $L^2_{\delta/2+\varepsilon}(\mathbb{R}^3) \subset L^1(\mathbb{R}^3)$ for any $\varepsilon > 0$.

**Proposition 2.5.** Assume that $1/2 \leq s < 1$ and $0 < \alpha \leq 1/2$. For a suitable function $f$, we have,

\[(2.25) \quad |\langle vG_\delta Q(f, f) - Q(f, vG_\delta f), vG_\delta f \rangle| \leq \frac{1}{2} \|G_\delta f\|_{L^2} \|G_\delta f\|_{H^s} \|G_\delta f\|_{H^s} \|G_\delta f\|_{H^s}.\]

**Proof.** Applying again the Bobylev identity and Plancherel formula, we get

\[(2.26) \quad - \langle vG_\delta Q(f, f) - Q(f, vG_\delta f), vG_\delta f \rangle \]

\[= C \int \int b(\cos \theta) \left\{ \partial_\xi \left( G_\delta(f^-)\hat{f}(\xi^-) - \hat{f}(\xi^-) \partial_\xi (G_\delta\hat{f})(\xi^-) \right) \right\} d\xi d\sigma \]

\[= C \int \int b(\cos \theta) \frac{\partial \xi^-}{\partial \xi} \left( \partial_\xi \hat{f}(\xi^-) G_\delta(f)(\xi^-) \partial_\xi (G_\delta\hat{f})(\xi^-) \right) d\xi d\sigma \]

\[+ C \int \int b(\cos \theta) \left\{ \partial_\xi \left( G_\delta(f)\hat{f}(\xi^+) - (\partial_\xi (G_\delta\hat{f})(\xi^+) \right) \right\} d\xi d\sigma \]

\[= I_1 + I_2.\]

We now treat the term $I_1$. Firstly, the fact $\xi^- = \frac{\xi - |\xi| \omega}{2}$ implies

\[(2.27) \quad \frac{\partial \xi^-}{\partial \xi} = \frac{I - \sigma \otimes \omega}{2},\]

then we have

\[(2.28) \quad \left| \frac{\partial \xi^-}{\partial \xi} \right| = \frac{1 - \sigma \cdot \omega}{2} = \sin^2 \frac{\theta}{2}.\]

Furthermore, we can split $I_1$ into two terms, as follows:

\[(2.29) \quad I_1 = \int \int b(\cos \theta) \frac{\partial \xi^-}{\partial \xi} \left( \partial_\xi \hat{f}(\xi^-) G_\delta(f)(\xi^-) \partial_\xi (G_\delta\hat{f})(\xi^-) \right) d\xi d\sigma \]

\[+ \int \int b(\cos \theta) \frac{\partial \xi^-}{\partial \xi} \left( \partial_\xi \hat{f}(\xi^-) \left\{ G_\delta(f) - G_\delta(f)(\xi^+) \right\} \partial_\xi (G_\delta\hat{f})(\xi^-) \right) d\xi d\sigma \]

Here we use the notation $\xi \otimes \eta = (\xi, \eta)$ for two vectors $\xi = (\xi_1, \xi_2, \xi_3)$ and $\eta = (\eta_1, \eta_2, \eta_3)$.\]
By virtue of Lemma 2.2, we have

\[ I_{11} + I_{12}, \]

thereby we compute

\[ |I_{11}| \lesssim \int \int b(\cos \theta) \sin^2 \frac{\theta}{2} \left| \left( \partial_\xi \hat{f} \right)(\xi^-) \right| \left| G_\delta (\xi^+, \hat{f}(\xi^-)) \right| \left| \partial_\xi \left( G_\delta (\xi) \hat{f}(\xi) \right) \right| \ d\xi \ d\sigma \]

\[ \lesssim \| \partial_\xi \hat{f} \|_{L^\infty} \int \frac{b(\cos \theta) \sin^2 \frac{\theta}{2}}{\cos \frac{\theta}{2}} \ d\sigma \left( \int |G_\delta (\xi^+, \hat{f}(\xi^-))|^2 d\xi^+ \right)^{\frac{1}{2}} \| v G_\delta f \|_{L^2} \]

\[ \lesssim \| f \|_{L^1} \| G_\delta f \|^2_{H^\alpha}, \]

and recalling Lemma 2.2 we have

\[ |I_{12}| = \int \int b(\cos \theta) \sin^4 \frac{\theta}{2} \left| G_\delta (\xi^-) \left( \partial_\xi \hat{f} \right)(\xi^-) \right| \left| \left| G_\delta (\xi^+, \hat{f}(\xi^-)) \right| \right| \left| \left| \xi \right| \right|^{2\alpha} \left| \partial_\xi \left( G_\delta (\xi) \hat{f}(\xi) \right) \right| \ d\xi \ d\sigma \]

\[ \lesssim \| G_\delta f \|_{L^\infty} \int \frac{b(\cos \theta) \sin^4 \frac{\theta}{2}}{\sin^{\alpha+1} \frac{\theta}{2}} \ d\sigma \left( \int |\xi|^{2\alpha} \left| G_\delta (\xi^-) \left( \partial_\xi \hat{f} \right)(\xi^-) \right| \right. \left| \left| \xi \right| \right|^{2\alpha} \left| \partial_\xi \left( G_\delta (\xi) \hat{f}(\xi) \right) \right| \ d\xi \ d\sigma \]

\[ \lesssim \| G_\delta f \|_{L^1} \| \| \xi \|^\alpha G_\delta (\partial_\xi \hat{f}) \|_{L_2}^2 \| G_\delta f \|_{H^\alpha} \]

\[ \lesssim \| G_\delta f \|_{L^1} \| G_\delta f \|^2_{H^\alpha}. \]

Note that in the last inequality we have used the assumption \( 0 < \alpha \leq 1/2 \), which implies

\[ \| \xi \|^\alpha G_\delta (\partial_\xi \hat{f}) \|_{L_2}^2 = \| G_\delta (\partial_\xi \hat{f}) \|_{L_2}^2 \lesssim \| \partial_\xi (G_\delta \hat{f}) \|_{L_2}^4 + \| (\partial_\xi G_\delta) \hat{f} \|_{L_2}^4 \]

\[ \lesssim \| \partial_\xi (G_\delta \hat{f}) \|_{L_2}^4 + \| \xi \|^{2\alpha-1} G_\delta \hat{f} \|_{L_2}^2 \lesssim \| G_\delta f \|_{H^\alpha} + \| G_\delta f \|_{H^\alpha} \]

\[ \lesssim \| G_\delta f \|_{H^\alpha}. \]

Thus we get the estimate for \( I_1 \),

\[ |I_1| \lesssim \| I_{11} \| + \| I_{12} \| \lesssim \| f \|_{L^1} \| G_\delta f \|^2_{L_2^2} + \| G_\delta f \|_{L^2} \| G_\delta f \|_{H^\alpha}^2. \]

On the other hand, since

\[ \partial_\xi \left( G_\delta (\xi) \hat{f}(\xi^+) \right) - \left( \partial_\xi (G_\delta \hat{f}) \right)(\xi^+) \]

\[ = \left\{ G_\delta (\xi) - G_\delta (\xi^+) \right\} \left( \partial_\xi \hat{f} \right)(\xi^+) + G_\delta (\xi) \left( \frac{\partial \xi^+}{\partial \xi} - 1 \right) \left( \partial_\xi \hat{f} \right)(\xi^+) + \left\{ (\partial_\xi G_\delta) (\xi) - (\partial_\xi G_\delta)(\xi^+) \right\} \hat{f}(\xi^+), \]

we can split correspondingly \( I_2 \) into three terms \( I_2 = I_{21} + I_{22} + I_{23} \).

By virtue of Lemma 2.2 we have

\[ |I_{21}| \lesssim \int \int b(\cos \theta) \sin^2 \frac{\theta}{2} \left| G_\delta (\xi^-) \left( \partial_\xi \hat{f} \right)(\xi^-) \right| \left| G_\delta (\xi^+) \left( \partial_\xi \hat{f} \right)(\xi^+) \right| \left| \xi \right|^{2\alpha} \left| \partial_\xi \left( G_\delta (\xi) \hat{f}(\xi) \right) \right| \ d\xi \ d\sigma \]

\[ \lesssim \| G_\delta \hat{f} \|_{L^\infty} \int \frac{\int \int b(\cos \theta) \sin^2 \frac{\theta}{2} \ d\sigma}{\cos^{\alpha+1} \frac{\theta}{2}} \ d\sigma \left( \int \xi^{2\alpha} \left| G_\delta (\xi^+) \left( \partial_\xi \hat{f} \right)(\xi^+) \right| \right. \left| \xi \right|^{2\alpha} \left| \partial_\xi \left( G_\delta (\xi) \hat{f}(\xi) \right) \right| \ d\xi \ d\sigma \]

\[ \lesssim \| G_\delta \hat{f} \|_{L^\infty} \int \frac{\int \int b(\cos \theta) \sin^2 \frac{\theta}{2} \ d\sigma}{\cos^{\alpha+1} \frac{\theta}{2}} \ d\sigma \left( \int \xi^{2\alpha} \left| G_\delta (\xi^+) \left( \partial_\xi \hat{f} \right)(\xi^+) \right| \right. \left| \xi \right|^{2\alpha} \left| \partial_\xi \left( G_\delta (\xi) \hat{f}(\xi) \right) \right| \ d\xi \ d\sigma \]

\[ \lesssim \| G_\delta \hat{f} \|_{L^\infty} \int \frac{\int \int b(\cos \theta) \sin^2 \frac{\theta}{2} \ d\sigma}{\cos^{\alpha+1} \frac{\theta}{2}} \ d\sigma \left( \int \xi^{2\alpha} \left| G_\delta (\xi^+) \left( \partial_\xi \hat{f} \right)(\xi^+) \right| \right. \left| \xi \right|^{2\alpha} \left| \partial_\xi \left( G_\delta (\xi) \hat{f}(\xi) \right) \right| \ d\xi \ d\sigma \]

\[ \lesssim \| G_\delta \hat{f} \|_{L^\infty} \int \frac{\int \int b(\cos \theta) \sin^2 \frac{\theta}{2} \ d\sigma}{\cos^{\alpha+1} \frac{\theta}{2}} \ d\sigma \left( \int \xi^{2\alpha} \left| G_\delta (\xi^+) \left( \partial_\xi \hat{f} \right)(\xi^+) \right| \right. \left| \xi \right|^{2\alpha} \left| \partial_\xi \left( G_\delta (\xi) \hat{f}(\xi) \right) \right| \ d\xi \ d\sigma \]

\[ \lesssim \| G_\delta \hat{f} \|_{L^\infty} \int \frac{\int \int b(\cos \theta) \sin^2 \frac{\theta}{2} \ d\sigma}{\cos^{\alpha+1} \frac{\theta}{2}} \ d\sigma \left( \int \xi^{2\alpha} \left| G_\delta (\xi^+) \left( \partial_\xi \hat{f} \right)(\xi^+) \right| \right. \left| \xi \right|^{2\alpha} \left| \partial_\xi \left( G_\delta (\xi) \hat{f}(\xi) \right) \right| \ d\xi \ d\sigma \]

\[ \lesssim \| G_\delta \hat{f} \|_{L^\infty} \int \frac{\int \int b(\cos \theta) \sin^2 \frac{\theta}{2} \ d\sigma}{\cos^{\alpha+1} \frac{\theta}{2}} \ d\sigma \left( \int \xi^{2\alpha} \left| G_\delta (\xi^+) \left( \partial_\xi \hat{f} \right)(\xi^+) \right| \right. \left| \xi \right|^{2\alpha} \left| \partial_\xi \left( G_\delta (\xi) \hat{f}(\xi) \right) \right| \ d\xi \ d\sigma \]
Due to the fact 
\[
\left| \frac{\partial \xi^+}{\partial \xi} - I \right| = \left| -\frac{I - \sigma \otimes \omega}{2} \right| = \sin^2 \frac{\theta}{2},
\]
we get
\[
|I_{22}| \lesssim \int \int b(\cos \theta) \sin^2 \frac{\theta}{2} |G_\delta(\xi^-)\hat{f}(\xi^-) - G_\delta(\xi^+)\hat{f}(\xi^+)| \left| \partial_\xi \left( G_\delta(\xi)\hat{f}(\xi) \right) \right| d\xi d\sigma
\]
\[
\lesssim \|G_\delta f\|_{L^1} \|G_\delta f\|_{L^2} \|G_\delta f\|_{L^2^2}
\]
\[
\lesssim \|G_\delta f\|_{L^2} \|G_\delta f\|_{L^2}^2.
\]

Thanks to the Taylor expansion up to order 2, we have
\[
(\partial_\xi G_\delta)(\xi) = (\xi - \xi^+) \cdot \left( \partial^2_{\xi\xi} G_\delta \right) (\xi^+) + \int_0^1 (1 - \tau)(\xi - \xi^+) \otimes (\xi - \xi^+) : \left( \partial^3_{\xi\xi\xi} G_\delta \right) (\xi_\tau) d\tau
\]
with \( \tau \in [0, 1] \) and \( \xi_\tau = (1 - \tau)\xi^+ + \tau \xi \). Correspondingly, we can rewrite \( I_{23} \) as follows:
\[
I_{23} = \int \int b(\cos \theta)\hat{f}(\xi^-) (\xi - \xi^+) \cdot \left( \partial^2_{\xi\xi} G_\delta \right) (\xi^+) \hat{f}(\xi^+) \left| \partial_\xi \left( G_\delta(\xi)\hat{f}(\xi) \right) \right| d\xi d\sigma
\]
\[
+ \int \int \int_0^1 (1 - \tau)b(\cos \theta)\hat{f}(\xi^-) (\xi - \xi^+) \otimes (\xi - \xi^+) : \left( \partial^3_{\xi\xi\xi} G_\delta \right) (\xi_\tau)\hat{f}(\xi^+) \left| \partial_\xi \left( G_\delta(\xi)\hat{f}(\xi) \right) \right| d\tau d\xi d\sigma
\]
\[
= I_{231} + I_{232}.
\]

For the estimate of \( I_{231} \), we use the symmetry of cross-section \( b \) with respect to \( \sigma \) around the direction \( \xi/|\xi| \) (see [3, 10]), which forces all components of \( \xi - \xi^+ \) to vanish except the component in the symmetry direction. Noticing \( \xi^- \perp \xi^+ \), we can take the place of \( \xi - \xi^+ \) in \( I_{231} \) by
\[
\left\langle \xi - \xi^+, \frac{\xi}{|\xi|} \right\rangle \cdot \frac{\xi}{|\xi|} = \left\langle \xi^-, \frac{\xi^- + \xi^+}{|\xi|} \right\rangle \cdot \frac{\xi}{|\xi|} = \xi \frac{|\xi^-|^2}{|\xi|^2} = \xi \sin^2 \frac{\theta}{2}.
\]

Combining Lemma 2.34 with the fact \( 4\alpha - 1 \leq 2\alpha \) for \( \alpha \leq 1/2 \), this yields
\[
|I_{231}| \lesssim \int \int b(\cos \theta) \sin^2 \frac{\theta}{2} |\hat{f}(\xi^-)| |G_\delta(\xi^+)\hat{f}(\xi^+)| |\xi||\xi^+|^{2(2\alpha - 1)} \left| \partial_\xi \left( G_\delta(\xi)\hat{f}(\xi) \right) \right| d\xi d\sigma
\]
\[
\lesssim \|\hat{f}\|_{L^\infty} \int \frac{b(\cos \theta)}{\cos \frac{\theta}{2}} \sin^2 \frac{\theta}{2} d\sigma \|\xi^+\|^\alpha G_\delta(\xi^+)\hat{f}(\xi^+)\|_{L^2} \|v G_\delta f\|_{H^\alpha}
\]
\[
\lesssim \|\hat{f}\|_{L^1} \|G_\delta f\|_{H^\alpha}^2.
\]

\[\text{For matrices } A = (a_{ij}), B = (b_{ij}), i, j \in \{1, 2, 3\}, \text{ we agree that } A : B = (a_{ij}b_{ij}).\]
Concerning the term $I_{232}$, we have

\begin{equation}
|I_{232}| \lesssim \int b(\cos \theta) \sin^{2} \frac{\theta}{2} \left| G_{\delta}(\xi)^{-1} \hat{f}(\xi) \right| \left| G_{\delta}(\xi)^{+} \hat{f}(\xi)^{+} \right| \left( \xi \right)^{2(\alpha - \frac{1}{2})} \left| \partial_{\xi} \left( G_{\delta}(\xi) \hat{f}(\xi) \right) \right| d\xi d\sigma
\end{equation}

\begin{align*}
&\lesssim \|G_{\delta} \hat{f}\|_{L^{\infty}} \int b(\cos \theta) \sin^{2} \frac{\theta}{2} \cos^{\alpha + \frac{1}{2}} \left( \xi \right)^{3(\alpha - \frac{1}{2})} \left| G_{\delta}(\xi)^{+} \hat{f}(\xi)^{+} \right|_{L^{2}} \|vG_{\delta} \hat{f}\|_{H^{(3\alpha - \frac{1}{2})^{+}}},
\end{align*}

Thus we obtain the estimate

\begin{equation}
|I_{2}| \lesssim |I_{21}| + |I_{22}| + |I_{231}| + |I_{232}|
\end{equation}

\begin{align*}
&\lesssim \left( \|f\|_{L^{1}} + \|G_{\delta} f\|_{L^{2}} \right) \|G_{\delta} \hat{f}\|_{H^{\alpha}} + \|G_{\delta} f\|_{L^{2}} \|G_{\delta} \hat{f}\|_{H^{\alpha}}^{2} + \|G_{\delta} f\|_{L^{2}} \|G_{\delta} \hat{f}\|_{H^{\alpha}}^{2}.
\end{align*}

Together with the estimates (2.32) and (2.41), we obtain the desired result. \hfill \Box

**Remark 2.6.** For $0 < \alpha < s < 1/2$, we have

\begin{equation}
|\langle vG_{\delta} Q(f, f) - Q(f, vG_{\delta} f), vG_{\delta} f \rangle| \lesssim \left( \|f\|_{L^{1}} + \|G_{\delta} f\|_{L^{2}} \right) \|G_{\delta} \hat{f}\|_{H^{\alpha}}^{2}.
\end{equation}

**Proof.** We need only to revise the estimate for $I_{23}$ in the above process. The Taylor formula gives,

\begin{align*}
&\left| (\partial_{\xi} G_{\delta})(\xi) - (\partial_{\xi} G_{\delta})(\xi^{+}) \right| = \left| \int_{0}^{1} (\xi - \xi^{+}) (\partial_{\xi}^{2} G_{\delta})(\xi^{+}) d\tau \right|
\end{align*}

\begin{align*}
&\lesssim |\xi| \left\| (\xi^{+})^{2(\alpha - 1)} G_{\delta}(\xi^{+}) d\tau \right|
\end{align*}

\begin{align*}
&\lesssim \sin \frac{\theta}{2} (\xi^{+})^{(3\alpha - 1)} G_{\delta}(\xi^{+}) G_{\delta}(\xi^{+}).
\end{align*}

Noticing the fact $(4\alpha - 1)^{+} \leq 2\alpha$ for $\alpha \leq 1/2$, hence we have

\begin{equation}
|I_{23}| \lesssim \int \int b(\cos \theta) \sin^{2} \frac{\theta}{2} \left| G_{\delta}(\xi)^{-1} \hat{f}(\xi) \right| \left| G_{\delta}(\xi)^{+} \hat{f}(\xi)^{+} \right| \left( \xi \right)^{2(\alpha - \frac{1}{2})} \left| \partial_{\xi} \left( G_{\delta}(\xi) \hat{f}(\xi) \right) \right| d\xi d\sigma
\end{equation}

\begin{align*}
&\lesssim \|G_{\delta} \hat{f}\|_{L^{\infty}} \int b(\cos \theta) \sin^{2} \frac{\theta}{2} \cos^{\alpha + \frac{1}{2}} \left( \xi \right)^{3(\alpha - \frac{1}{2})} \left| G_{\delta}(\xi)^{+} \hat{f}(\xi)^{+} \right|_{L^{2}} \|vG_{\delta} \hat{f}\|_{H^{\alpha}}
\end{align*}

\begin{align*}
&\lesssim \|G_{\delta} f\|_{L^{2}} \|G_{\delta} \hat{f}\|_{H^{\alpha}} + \|G_{\delta} f\|_{L^{2}} \|G_{\delta} \hat{f}\|_{H^{\alpha}}^{2}
\end{align*}

Therefore we get

\begin{equation}
|I_{2}| \lesssim |I_{21}| + |I_{22}| + |I_{231}| + |I_{232}| \lesssim \|G_{\delta} f\|_{L^{2}} \|G_{\delta} \hat{f}\|_{H^{\alpha}}^{2}.
\end{equation}

Together with the estimates for $I_{1}$ (see (2.32)), this completes the proof of Remark 2.6. \hfill \Box

**Proposition 2.7.** Assume that $1/2 \leq s < 1$ and $0 < \alpha \leq 1/2$. For a suitable function $f$, we have,

\begin{equation}
|\langle v \otimes vG_{\delta} Q(f, f) - Q(f, v \otimes vG_{\delta} f), v \otimes vG_{\delta} f \rangle| \lesssim \|G_{\delta} f\|_{L^{2}}^{2} + \left( \|f\|_{L^{1}} + \|f\|_{L^{1}} + \|G_{\delta} f\|_{L^{2}} \right) \|G_{\delta} f\|_{L^{2}}^{2} \|G_{\delta} f\|_{H^{(3\alpha - \frac{1}{2})^{+}}}.
\end{equation}
Proof. From the Bobylev identity and Plancherel formula, we deduce that

\[(2.47) \quad \langle v \otimes v G_\delta Q(f, f) - Q(f, v \otimes v G_\delta f), v \otimes v G_\delta f \rangle \]
\[= C \int b(\cos \theta) \left\{ \partial_{\xi \xi}^2 \left( G_\delta(\xi) \hat{f}(\xi^-) \hat{f}(\xi^+) \right) - \hat{f}(\xi^-) \left( \partial_{\xi \xi}^2 (G_\delta \hat{f}) \right)(\xi^+) \right\} \left( \partial_{\xi \xi}^2 (G_\delta \hat{f}) \right)\left( \xi \right) d\xi d\sigma \]
\[= C \int b(\cos \theta) \left\{ \left( \partial_{\xi \xi}^2 \hat{f} \right)(\xi^-) \left( \partial_{\xi \xi}^2 \hat{f} \right)(\xi^+) \right\} \left( \partial_{\xi \xi}^2 (G_\delta \hat{f}) \right)\left( \xi \right) d\xi d\sigma \]
\[+ 2C \int b(\cos \theta) \left( \partial_{\xi \xi}^2 \hat{f} \right)(\xi^-) \left( \partial_{\xi \xi}^2 \hat{f} \right)(\xi^+) \left( \partial_{\xi \xi}^2 (G_\delta \hat{f}) \right)\left( \xi \right) d\xi d\sigma \]
\[+ C \int b(\cos \theta) \hat{f}(\xi^-) \left( \partial_{\xi \xi}^2 (G_\delta(\xi) \hat{f}(\xi^+)) - \left( \partial_{\xi \xi}^2 (G_\delta \hat{f}) \right)(\xi^+) \right) \left( \partial_{\xi \xi}^2 (G_\delta \hat{f}) \right)\left( \xi \right) d\xi d\sigma \]
\[\triangleq II_1 + II_2 + II_3. \]

We begin with the estimate for \(II_1\), noticing the facts

\[(2.48) \quad \frac{\partial \xi^-}{\partial \xi} = \frac{I - \sigma \otimes \omega}{2} = \frac{\delta_{ij} - \sigma_{i\omega_j}}{2}, \quad i, j \in \{1, 2, 3\}, \]

and

\[(2.49) \quad \frac{\partial^2 \xi^-}{\partial \xi \partial \xi} = \left( \frac{\partial^2 \xi^-}{\partial \xi_j \partial \xi_k} \right)_{3 \times 3} = \frac{\sigma_{ij}(\delta_{jk} - \omega_j \omega_k)}{2|\xi|}, \quad i, j, k \in \{1, 2, 3\}, \]

by definition of the determinant, we can deduce that

\[(2.50) \quad \left| \frac{\partial^2 \xi^-}{\partial \xi \partial \xi} \right| = 0. \]

Then we have

\[(2.51) \quad |II_1| \lesssim \int b(\cos \theta) \sin^4 \frac{\theta}{2} \left| G_\delta(\xi^-) \left( \partial_{\xi \xi}^2 \hat{f} \right)(\xi^-) \right| \left| G_\delta(\xi^+) \hat{f}(\xi^+) \right| \left| \left( \partial_{\xi \xi}^2 (G_\delta \hat{f}) \right)(\xi) \right| d\xi d\sigma \]
\[\lesssim \|G_\delta \hat{f}\|_{L^\infty} \int b(\cos \theta) \sin^4 \frac{\theta}{2} \|G_\delta(\partial_{\xi \xi}^2 \hat{f})\|_{L^2} \|\partial_{\xi \xi}^2 (G_\delta \hat{f})\|_{L^2} \]
\[\lesssim \|G_\delta \hat{f}\|_{L^1} \|G_\delta \hat{f}\|_{L^2}^2 \]
\[\lesssim \|G_\delta \hat{f}\|^3_{L^2}. \]

Herein, we have used the following fact, in the last second inequality,

\[(2.52) \quad \|G_\delta(\partial_{\xi \xi}^2 \hat{f})\|_{L^2} \leq \|\partial_{\xi \xi}^2 (G_\delta \hat{f})\|_{L^2} + \|\partial_{\xi \xi}^2 G_\delta \hat{f}\|_{L^2} + 2\|\partial_{\xi} G_\delta(\partial_{\xi} \hat{f})\|_{L^2} \]
\[\lesssim \|\partial_{\xi \xi}^2 (G_\delta \hat{f})\|_{L^2} + \|\xi\|^{2(\alpha - 1)} G_\delta \hat{f}\|_{L^2} + \|\xi\|^{2\alpha - 1} G_\delta(\partial_{\xi} \hat{f})\|_{L^2} \]
\[\lesssim \|\partial_{\xi \xi}^2 (G_\delta \hat{f})\|_{L^2} + \|G_\delta \hat{f}\|_{L^2} + \|G_\delta(\partial_{\xi} \hat{f})\|_{L^2} \]

\footnote{For a \(3 \times 3\) matrix \(A = (a_{ijk})\), the determinant is given by the formula
\[
\det(A) = \sum (-1)^{\tau(i_1i_2i_3) + \tau(j_1j_2j_3)} a_{i_1j_1k_1} a_{i_2j_2k_2} a_{i_3j_3k_3}
\]
with \(\tau\) being the inversion function.}
\[
\lesssim \||\partial^2_{\xi}(G_\delta \hat{f})\|_{L^2} + \|G_\delta \hat{f}\|_{L^2}
\lesssim (\nu)^2 \|G_\delta \hat{f}\|_{L^2}.
\]

As for the term \(I_{H2}\), we rewrite it as
\[
I_{H2} = C \int \int b(\cos \theta) \frac{\partial}{\partial \xi} \left( \partial \hat{f}_\delta \right) (\xi^-) \{G_\delta(\xi^-) - G_\delta(\xi^+)\} \left( \partial \hat{f}_\delta \right)(\xi^+) \left( \frac{\partial^2_{\xi}(G_\delta \hat{f})}{\partial \xi} \right)(\xi^-) d\xi d\sigma
\]
\[
+ C \int \int b(\cos \theta) \frac{\partial}{\partial \xi} \left( \partial \hat{f}_\delta \right)(\xi^-) G_\delta(\xi^-) \left( \partial \hat{f}_\delta \right)(\xi^+) \left( \frac{\partial^2_{\xi}(G_\delta \hat{f})}{\partial \xi} - \frac{1}{2} \right) d\xi d\sigma
\]
\[
+ C \int \int b(\cos \theta) \frac{\partial}{\partial \xi} \left( \partial \hat{f}_\delta \right)(\xi^-) \{G_\delta(\xi^-) - G_\delta(\xi^+)\} \left( \partial \hat{f}_\delta \right)(\xi^+) \left( \frac{\partial^2_{\xi}(G_\delta \hat{f})}{\partial \xi} \right)(\xi^-) d\xi d\sigma
\]
\[
\approx I_{H21} + I_{H22} + I_{H23} + I_{H24}.
\]

Combining the fact \(\left| \frac{\partial}{\partial \xi} \right| = \sin^2 \frac{\theta}{2}\) and Lemma 2.2, it follows that
\[
|I_{H21}| \lesssim \int \int b(\cos \theta) \sin^4 \frac{\theta}{2} \left( \xi^{-2} \left| G_\delta(\xi^-) \left( \partial \hat{f}_\delta \right)(\xi^-) \right| \right) \left| G_\delta(\xi^+) \left( \partial \hat{f}_\delta \right)(\xi^+) \right| \left| \left( \frac{\partial^2_{\xi}(G_\delta \hat{f})}{\partial \xi} \right)(\xi^-) \right| d\xi d\sigma
\]
\[
\lesssim \int \int b(\cos \theta) \sin^4 \frac{\theta}{2} \frac{2}{\cos^4 + 1} d\xi d\sigma \|G_\delta(\partial \hat{f}_\delta)\|_{L^4} \|\xi^2 \|_{G_\delta(\partial \hat{f}_\delta)}\|_{L^4} \|\xi^2 \|_{\partial^2_{\xi}(G_\delta \hat{f})}\|_{L^2}
\]
\[
\lesssim \|G_\delta(\partial \hat{f}_\delta)\|_{L^4} \|G_\delta(\partial \hat{f}_\delta)\|_{L^4} \|G_\delta \hat{f} \|_{H^2}.
\]

Thanks to the Gagliardo-Nirenberg inequality, (see, for instance, \[11\] \[18\]),
\[
\|\Lambda^2_{\xi}(G_\delta \hat{f})\|_{L^4} \lesssim \|\Lambda^2_{\xi}(G_\delta \hat{f})\|_{L^4} \|G_\delta \hat{f} \|_{L^2},
\]
we obtain that
\[
\|G_\delta(\partial \hat{f}_\delta)\|_{L^4} \lesssim \|\partial \hat{f}_\delta\|_{L^4} + \|\partial \hat{f}_\delta\|_{L^4} + \|\partial \hat{f}_\delta\|_{L^4} \|\xi^2 \|_{L^4} \approx \|G_\delta \hat{f} \|_{L^4}
\]
\[
\lesssim \|\Lambda^2_{\xi}(G_\delta \hat{f})\|_{L^4} \|G_\delta \hat{f} \|_{L^4} \|G_\delta \hat{f} \|_{L^2} \lesssim \|G_\delta \hat{f} \|_{H^2},
\]
and similarly,
\[
\|G_\delta(\partial \hat{f}_\delta)\|_{L^4} \lesssim \|\Lambda^2_{\xi}(G_\delta \hat{f})\|_{L^4} \|G_\delta \hat{f} \|_{L^2} \lesssim \|G_\delta \hat{f} \|_{H^2}.
\]

Thus, we get the estimate for \(I_{H21}\):
\[
|I_{H21}| \lesssim \|G_\delta \hat{f} \|_{L^2} \|G_\delta \hat{f} \|_{H^2}^2.
\]

\[\text{We agree that } \Lambda f = F^{-1} \left( (1 + | \cdot |^2)^{1/2} f \right).\]
By virtue of the fact \( \left| \frac{\partial \xi^+}{\partial \xi} - f \right| = \sin^2 \frac{\theta}{2} \), we have

\[
|I_{22}| \lesssim \int b(\cos \theta) \sin^2 \frac{\theta}{2} \left| G_\delta(\xi^-) \left( \partial_\xi f \right)(\xi^-) \right| \left| G_\delta(\xi^+) \left( \partial_\xi f \right)(\xi^+) \right| \left| \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^-) \right| \left| \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^+) \right| \, d\xi d\sigma
\]

\[
\lesssim \int \frac{b(\cos \theta) \sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}} \left| G_\delta(\partial_\xi f) \right|_{L^2} \left| \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^-) \right| \left| \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^+) \right| \, d\xi d\sigma
\]

\[
\lesssim \left| G_\delta(\partial_\xi f) \right|_{L^2} \left| G_\delta f \right|_{L^2}^2
\]

\[
\lesssim \left| G_\delta f \right|_{L^2}^3.
\]

Considering the estimate of \( I_{23} \), by the Taylor formula, we have

\[
|I_{23}| \lesssim \left| \int_0^1 (\xi - \xi^+) \cdot \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^-) \, d\tau \right|
\]

\[
\lesssim |\xi^-| \left| \int_0^1 (\xi^-)^2(2\alpha - 1)G_\delta(\xi^-) \, d\tau \right|
\]

\[
\lesssim \sin \frac{\theta}{2} |(\xi^{4\alpha - 1}^+)G_\delta(\xi^-)G_\delta(\xi^+)|,
\]

where \( \xi^- = (1 - \tau)\xi^+ + \tau \xi \) with \( \tau \in [0, 1] \).

Observing the fact \( (4\alpha - 1)^+ \leq 2\alpha \) for \( 0 < \alpha \leq 1/2 \), it follows that

\[
|I_{23}| \lesssim \left| \int_0^1 (\xi^-)^2 \cdot \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^-) \, d\tau \right|
\]

\[
\lesssim \left| G_\delta(\partial_\xi f) \right|_{L^2} \left| \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^-) \right| \left| \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^+) \right| \, d\xi d\sigma
\]

\[
\lesssim \left| G_\delta(\partial_\xi f) \right|_{L^2} \left| \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^-) \right| \left| \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^+) \right| \, d\xi d\sigma
\]

\[
\lesssim \left| G_\delta f \right|_{L^2} \left| G_\delta f \right|_{L^2}^2
\]

\[
\lesssim \left| G_\delta f \right|_{L^2}^3.
\]

For the term \( I_{24} \), we infer that

\[
|I_{24}| \lesssim \int b(\cos \theta) \sin^2 \frac{\theta}{2} \left| \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^-) \right| \left| \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^+) \right| \, d\xi d\sigma
\]

\[
\lesssim \left| G_\delta f \right|_{L^2} \left| \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^-) \right| \left| \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f) \right)(\xi^+) \right| \, d\xi d\sigma
\]

\[
\lesssim \left| G_\delta f \right|_{L^2} \left| G_\delta f \right|_{L^2}^2
\]

\[
\lesssim \left| G_\delta f \right|_{L^2} \left| G_\delta f \right|_{L^2}^2.
\]

Thus, the inequalities \( (5.13), (5.15), (5.16) \), and \( (2.61) \) enable us to obtain the estimate for \( I_2 \):

\[
|I_2| \leq |I_{21}| + |I_{22}| + |I_{23}| + |I_{24}| \lesssim \left( \left| f \right|_{L^1} + \left| G_\delta f \right|_{L^2} \right) \left| G_\delta f \right|_{L^2}^2.
\]

Now we deal with the term \( I_3 \), firstly we write that

\[
I_3 = C \int b(\cos \theta) \hat{f}(\xi^-) \left\{ \left( \frac{\partial^2}{\partial \xi^2}(G_\delta(\xi^-)\hat{f}(\xi^+)) \right)(\xi^-) - \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f)(\xi^-) \right)(\xi^-) \right\} \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f)(\xi^-) \right)(\xi^-) \, d\xi d\sigma
\]

\[
= C \int b(\cos \theta) \hat{f}(\xi^-) \left\{ \left( \frac{\partial^2}{\partial \xi^2}(G_\delta(\xi^-)\hat{f}(\xi^+)) \right)(\xi^-) - \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f)(\xi^-) \right)(\xi^-) \right\} \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f)(\xi^-) \right)(\xi^-) \, d\xi d\sigma
\]

\[
= C \int b(\cos \theta) \hat{f}(\xi^-) \left\{ \left( \frac{\partial^2}{\partial \xi^2}(G_\delta(\xi^-)\hat{f}(\xi^+)) \right)(\xi^-) - \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f)(\xi^-) \right)(\xi^-) \right\} \left( \frac{\partial^2}{\partial \xi^2}(G_\delta f)(\xi^-) \right)(\xi^-) \, d\xi d\sigma.
\]

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\[= C \int b(\cos \theta) \hat{f}(\xi^-) \{ (\partial^{2}_{\xi} G_{\delta})(\xi) - (\partial^{2}_{\xi} G_{\delta})(\xi^+) \} \hat{f}(\xi^+) \left( \partial^{2}_{\xi}(G_{\delta} \hat{f}) \right)(\xi) \, d\xi d\sigma \]
\[+ C \int b(\cos \theta) \hat{f}(\xi^-) G_{\delta}(\xi) \left( \partial_{\xi} \hat{f} \right)(\xi^+) \left( \partial^{2}_{\xi}(G_{\delta} \hat{f}) \right)(\xi) \, d\xi d\sigma \]
\[+ C \int b(\cos \theta) \hat{f}(\xi^-) \left\{ (\partial_{\xi} G_{\delta})(\xi) \left( \frac{\partial^{+}_{\xi}}{\partial \xi} \right) - (\partial_{\xi} G_{\delta})(\xi^+) \right\} \left( \partial_{\xi} \hat{f} \right)(\xi^+) \left( \partial^{2}_{\xi}(G_{\delta} \hat{f}) \right)(\xi) \, d\xi d\sigma \]
\[+ C \int b(\cos \theta) \hat{f}(\xi^-) \left\{ (\partial^{+}_{\xi} G_{\delta})(\xi) \left( \frac{\partial^{+}_{\xi}}{\partial \xi} \right)^2 - G_{\delta}(\xi^+) \right\} \left( \partial^{2}_{\xi}(G_{\delta} \hat{f}) \right)(\xi) \, d\xi d\sigma \]
\[\triangleq I_{31} + b + I_{32} + I_{33}.\]

We then turn to the term \(I_{31}\). The Taylor formula up to order 2 gives that

\[(2.63)\]
\[(\partial^{2}_{\xi} G_{\delta})(\xi) - (\partial^{2}_{\xi} G_{\delta})(\xi^+) = (\xi - \xi^+) \cdot (\partial^{3}_{\xi\xi\xi} G_{\delta})(\xi^+) + \int_{0}^{1} (1 - \tau)(\xi - \xi^+) \otimes (\xi - \xi^+) : (\partial^{4}_{\xi\xi\xi\xi} G_{\delta})(\xi^+) \, d\tau\]

with \(\tau \in [0,1]\) and \(\xi_{\tau} = (1 - \tau)\xi^+ + \tau\xi\). Then we can decompose \(I_{31}\) into two corresponding terms \(I_{31} = I_{311} + I_{312}\).

By the symmetry of \(b\) with respect to \(\sigma\) mentioned before, we can take the place of \(\xi - \xi^+\) in \(I_{311}\) by

\[(2.64)\]
\[\left\langle \xi - \xi^+, \frac{\xi}{|\xi|} \right\rangle \cdot \frac{\xi}{|\xi|} = \xi \sin^2 \frac{\theta}{2},\]

then it follows that

\[(2.65)\]
\[|I_{311}| \lesssim \int b(\cos \theta) \sin^2 \frac{\theta}{2} \left| \hat{f}(\xi^-) \right| \left| \xi \right| |\xi|^6 \left| G_{\delta}(\xi^+) \hat{f}(\xi^+) \right| \left| \partial^{2}_{\xi}(G_{\delta} \hat{f}) \right|(\xi) \, d\xi d\sigma \]
\[\lesssim \left\| \hat{f} \right\|_{L^\infty} \int b(\cos \theta) \sin^2 \frac{\theta}{2} \left( \frac{1}{\cos \frac{\theta}{2}} \right) \, d\sigma \left( \left\langle \xi \right\rangle^3 G_{\delta} \hat{f} \right)_{L^2} \left( \left\langle \xi \right\rangle^3 G_{\delta} \hat{f} \right)_{L^2} \]
\[\lesssim \left\| \hat{f} \right\|_{L^1} \left\| G_{\delta} \hat{f} \right\|_{L^2} \left\| G_{\delta} \hat{f} \right\|_{L^2}^2,\]

due to the assumption \(\alpha \leq 1/2\).

Furthermore, by Lemma 2.1 we can derive that

\[(2.66)\]
\[|I_{312}| \lesssim \int b(\cos \theta) \sin^2 \frac{\theta}{2} \left( G_{\delta}(\xi^-) \hat{f}(\xi^-) \right) \left( \left\langle \xi \right\rangle^8 \hat{f}(\xi^+) \right) \left( \partial^{2}_{\xi}(G_{\delta} \hat{f}) \right)(\xi) \, d\xi d\sigma \]
\[\lesssim \left\| G_{\delta} \hat{f} \right\|_{L^\infty} \int b(\cos \theta) \sin^2 \frac{\theta}{2} \left( \frac{1}{\cos^4 \frac{\theta}{2}} \right) \, d\sigma \left( \left\langle \xi \right\rangle^4 G_{\delta} \hat{f} \right)_{L^2} \left( \left\langle \xi \right\rangle^4 G_{\delta} \hat{f} \right)_{L^2} \]
\[\lesssim \left\| G_{\delta} \hat{f} \right\|_{L^1} \left\| G_{\delta} \hat{f} \right\|_{H^{(4\alpha-1)+}} \left\| G_{\delta} \hat{f} \right\|_{H^{(4\alpha-1)+}} \]
\[\lesssim \left\| G_{\delta} \hat{f} \right\|_{L^2}^2 \left\| G_{\delta} \hat{f} \right\|_{H^{(4\alpha-1)+}}^2,\]

Combining the above two inequalities gives that

\[(2.67)\]
\[|I_{31}| \lesssim \left\| f \right\|_{L^1} \left\| G_{\delta} f \right\|_{L^2}^2 + \left\| G_{\delta} f \right\|_{L^2} \left\| G_{\delta} f \right\|_{H^{(4\alpha-1)+}}^2.\]
As for the term $II_{32}$, we rewrite it as

\[
(2.68) \quad II_{32} = 2C \int b(\cos \theta) \hat{f}(\xi^-) \left\{ (\partial_\xi G_\delta)(\xi) - (\partial_\xi G_\delta)(\xi^+) \right\} \left( \partial_{\xi} \hat{f} \right)(\xi^+) \left( \frac{\partial \xi^+}{\partial \xi} - I \right) \left( \partial_{\xi}^2 (G_\delta \hat{f}) \right)(\xi) \, d\xi d\sigma
\]

\[
+ 2C \int b(\cos \theta) \hat{f}(\xi^-) \left( \partial_{\xi} G_\delta \right)(\xi^+) \left( \partial_{\xi} \hat{f} \right)(\xi^+) \left( \partial_{\xi}^2 (G_\delta \hat{f}) \right)(\xi) \, d\xi d\sigma
\]

\[
\triangleq II_{321} + II_{322}.
\]

Thanks to the Taylor formula \((2.36)\), we can split $II_{321}$ into two terms $II_{321} = II_{3211} + II_{3212}$, correspondingly. Following along the same lines of that of treating $II_{31}$, we have firstly

\[
(2.69) \quad |II_{3211}| \lesssim \int b(\cos \theta) \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} \left| \hat{f}(\xi^-) \right| |\xi| |\xi^+|^{4\alpha - 2} \left| G_\delta(\xi^+) \right| \left| \left( \partial_{\xi}^2 (G_\delta \hat{f}) \right)(\xi) \right| \, d\xi d\sigma
\]

\[
\lesssim \| \hat{f} \|_{L^\infty} \int b(\cos \theta) \sin^2 \frac{\theta}{2} \frac{\theta}{\cos \frac{\theta}{2}} \, d\sigma \left\| G_\delta(\partial_{\xi} \hat{f}) \right\|_{L^\infty} \left\| \left( \partial_{\xi}^2 (G_\delta \hat{f}) \right) \right\|_{L^2}
\]

\[
\lesssim \| \hat{f} \|_{L^1} \left\| G_\delta \hat{f} \right\|_{H^\alpha} \left\| G_\delta^2 f \right\|_{H^2},
\]

where we have used the facts $4\alpha - 1 \leq 2\alpha$ and $\left\| G_\delta(\partial_{\xi} \hat{f}) \right\|_{L^2} \lesssim \left\| G_\delta^2 f \right\|_{L^2}$ for $\alpha \leq 1/2$.

Secondly, we have

\[
(2.70) \quad |II_{3212}| \lesssim \int b(\cos \theta) \sin^2 \frac{\theta}{2} \frac{\theta}{\cos \frac{\theta}{2}} \left| \hat{f}(\xi^-) \right| |\xi| |\xi^+|^{6\alpha - 3} \left| G_\delta(\xi^+) \right| \left( \partial_{\xi} \hat{f} \right)(\xi^+) \left( \partial_{\xi}^2 (G_\delta \hat{f}) \right)(\xi) \, d\xi d\sigma
\]

\[
\lesssim \| \hat{f} \|_{L^\infty} \int b(\cos \theta) \sin^2 \frac{\theta}{2} \frac{\theta}{\cos \frac{\theta}{2}} \, d\sigma \left\| G_\delta(\partial_{\xi} \hat{f}) \right\|_{L^2} \left\| \partial_{\xi}^2 (G_\delta \hat{f}) \right\|_{H^{(3\alpha - 1)/2} +}
\]

\[
\lesssim \| \hat{f} \|_{L^1} \left\| G_\delta \hat{f} \right\|_{L^2} \left\| G_\delta^2 f \right\|_{H^{(3\alpha - 1)/2} +}.
\]

On the other hand, by the fact $\left| \frac{\partial \xi^+}{\partial \xi} - I \right| = \sin^2 \frac{\theta}{2}$, we get

\[
(2.71) \quad |II_{322}| \lesssim \int b(\cos \theta) \sin^2 \frac{\theta}{2} \left| \hat{f}(\xi^-) \right| |\xi^+|^{2\alpha - 1} \left| G_\delta \xi^+ \right| \left( \partial_{\xi} \hat{f} \right)(\xi^+) \left( \partial_{\xi}^2 (G_\delta \hat{f}) \right)(\xi) \, d\xi d\sigma
\]

\[
\lesssim \| \hat{f} \|_{L^\infty} \int b(\cos \theta) \sin^2 \frac{\theta}{2} \frac{\theta}{\cos \frac{\theta}{2}} \, d\sigma \left\| G_\delta(\partial_{\xi} \hat{f}) \right\|_{L^2} \left\| \partial_{\xi}^2 (G_\delta \hat{f}) \right\|_{L^2}
\]

\[
\lesssim \| \hat{f} \|_{L^1} \left\| G_\delta \hat{f} \right\|_{L^2}^2.
\]

Together with the three above inequalities, we obtain

\[
(2.72) \quad |II_{32}| \leq |II_{3211}| + |II_{3212}| + |II_{322} | \lesssim \| \hat{f} \|_{L^1} \left\| G_\delta \hat{f} \right\|_{H^\alpha} + \left\| G_\delta \hat{f} \right\|_{L^2} \left\| G_\delta^2 f \right\|_{L^{2(3\alpha - 1)/2} +}.
\]

Considering the last term $II_{33}$, we have

\[
II_{33} = C \int b(\cos \theta) \hat{f}(\xi^-) \left\{ G_\delta(\xi) - G_\delta(\xi^+) \right\} \left( \partial_{\xi} \hat{f} \right)(\xi^+) \left( \partial_{\xi}^2 (G_\delta \hat{f}) \right)(\xi) \, d\xi d\sigma
\]
\[ + C \int b(\cos \theta) \tilde{f}(\xi^-) G_\delta(\xi^+) \left( \partial_{\xi^+}^2 \tilde{f}(\xi^+) \right) \left( \left( \frac{\partial \xi^+}{\partial \xi^-} \right)^2 - I \right) \{ \partial_{\xi^+}^2 (G_\delta \tilde{f}) \}(\xi) \, d\xi d\sigma \]

\[ \equiv II_{331} + II_{332}. \]

Then, Lemma 2.2 gives that

\[ (2.73) \]

\[ |II_{331}| \lesssim \int b(\cos \theta) \sin^{\alpha} \theta \left| G_\delta(\xi^-) \tilde{f}(\xi^-) \right| |\xi|^2 \left( \frac{\partial_{\xi^+}^2 \tilde{f}(\xi^+)}{\partial_{\xi^-}^2 \tilde{f}(\xi^+)} \right) (\xi) \, d\xi d\sigma \]

\[ \lesssim \|G_\delta \tilde{f}\|_{L^\infty} \int b(\cos \theta) \sin^{\alpha} \theta \left( \frac{\partial_{\xi^+}^2 \tilde{f}(\xi^+)}{\partial_{\xi^-}^2 \tilde{f}(\xi^+)} \right) (\xi) \, d\xi d\sigma \]

\[ \lesssim \|G_\delta \tilde{f}\|_{L^2} \|G_\delta f\|_{H^2}, \]

where we have used the estimate \( \|G_\delta (\partial_{\xi^+}^2 \tilde{f})\|_{L^2} \lesssim \|G_\delta f\|_{H^2} \) due to (2.52).

As for the term \( II_{332} \), since

\[ (2.74) \]

\[ \left| \left( \frac{\partial \xi^+}{\partial \xi^-} \right)^2 - I \right| = \left| \left( \frac{\partial \xi^+}{\partial \xi^-} - I \right) \left( \frac{\partial \xi^+}{\partial \xi^-} - I \right) \right| \leq C \sin^{\alpha} \theta, \]

then we obtain

\[ (2.75) \]

\[ |II_{332}| \lesssim \int b(\cos \theta) \sin^{\alpha} \theta \left| \tilde{f}(\xi^-) \right| \left| G_\delta(\xi^+) \right| \left( \frac{\partial_{\xi^+}^2 \tilde{f}(\xi^+)}{\partial_{\xi^-}^2 \tilde{f}(\xi^+)} \right) \left( \frac{\partial_{\xi^-}^2 G_\delta \tilde{f}}{\partial_{\xi^+}^2 G_\delta \tilde{f}} \right) (\xi) \, d\xi d\sigma \]

\[ \lesssim \|\tilde{f}\|_{L^\infty} \int b(\cos \theta) \sin^{\alpha} \theta \left( \frac{\partial_{\xi^+}^2 \tilde{f}(\xi^+)}{\partial_{\xi^-}^2 \tilde{f}(\xi^+)} \right) (\xi) \, d\xi d\sigma \]

\[ \lesssim \|f\|_{L^1} \|G_\delta f\|_{L^2}^2. \]

Thereby, we get the estimate for \( II_{33} \),

\[ (2.76) \]

\[ |II_{33}| \leq |II_{331}| + |II_{332}| \lesssim (\|f\|_{L^1} + \|G_\delta f\|_{L^2}) \|G_\delta f\|_{H^2}. \]

Together with the estimates (2.67), (2.72), and (2.76), and observing the fact \( \frac{\partial_{\xi^+}^2 \tilde{f}}{\partial_{\xi^-}^2 \tilde{f}} = 0 \) implies that

\[ (2.77) \]

\[ |\Psi| = 0, \]

we can conclude the estimate of \( II_3 \):

\[ (2.78) \]

\[ |II_3| \leq |II_{31}| + |\Psi| + |II_{32}| + |II_{33}| \]

\[ \lesssim \left( \|f\|_{L^1} + \|G_\delta f\|_{L^2} \right) \|G_\delta f\|_{H^2} + \|G_\delta f\|_{L^2} \|G_\delta f\|_{H^2}^{(3s - \frac{1}{2})}. \]

Combining this inequality with (2.51), (2.72) completes the whole proof.

**Remark 2.8.** For \( 0 < \alpha < s < 1/2 \), we have,

\[ (2.79) \]

\[ |(v \otimes v G_\delta Q(f, f) - Q(f, v \otimes v G_\delta f), v \otimes v G_\delta f)| \lesssim \left( \|f\|_{L^1} + \|f\|_{L^1} + \|G_\delta f\|_{L^2} \right) \|G_\delta f\|_{H^2}. \]
Proof. It suffices to revise the estimates for $II_{31}$ and $II_{321}$, relying on Taylor expansion of order 1.

Because

$$
|\{\partial_{\xi}^2 G_\delta(\xi) - (\partial_{\xi}^2 G_\delta)(\xi^+)\}| = |\int_{0}^{1} (\xi - \xi^+) \cdot (\partial_{\xi}^3 G_\delta)(\xi) d\tau|
$$

$$
\lesssim |\xi|\int_{0}^{1} (\xi^+)^{3(2\alpha - 1)} G_\delta(\xi) d\tau \lesssim \sin^{\frac{\theta}{2}}(\xi) (6\alpha - 2)^+ \|G_\delta(\xi^{-})\| G_\delta(\xi^+),
$$

using the fact $(6\alpha - 2)^+ \leq 2\alpha$, we can derive that

$$
II_{31} \lesssim \int b(\cos \theta) \sin^{\frac{\theta}{2}} \|G_\delta f(\xi^{-})\| \|G_\delta(\xi^+)\| \|\partial_{\xi}^2 (G_\delta f)\| d\xi d\sigma
$$

$$
\lesssim \|G_\delta f\|_{L_0^2} \|G_\delta f\|_{L^2}^{2}.
$$

For $II_{321}$, applying the Taylor expansion and the inequality ensures that

$$
II_{321} \lesssim \int b(\cos \theta) \sin^{\frac{\theta}{2}} \|G_\delta f(\xi^{-})\| \|G_\delta(\xi^+)\| \|\partial_{\xi}^2 (G_\delta f)\| d\xi d\sigma
$$

$$
\lesssim \|G_\delta f\|_{L_0^1} \|G_\delta f\|_{H_0^1} \|G_\delta f\|_{H_0^2}.
$$

Finally, we can obtain the desired result:

$$
|\{v \otimes v G_\delta Q(f, f) - Q(f, v \otimes v G_\delta f), v \otimes v G_\delta f\}| \lesssim \left(\|f\|_{L_0^1} + \|f\|_{L_0^1} + \|G_\delta f\|_{L_0^2}\right) \|G_\delta f\|_{H_0^2}.
$$

\[\square\]

3 Sobolev regularity for weak solutions

In this section, we will study the regularizing effect of the weak solutions for the Cauchy problem of the Boltzmann equation in some weighted Sobolev spaces.

**Theorem 3.1.** Assume that the initial datum $f_0 \in L^{1}_{2+2s} \cap L\log L(\mathbb{R}^3)$. Let $f \in L^\infty((0, +\infty); L^1_{2} \cap L\log L(\mathbb{R}^3))$ be a non-negative weak solution of the Cauchy problem of the Boltzmann equation \[L2,\] then

$$f(t, \cdot) \in H^s_{2}(\mathbb{R}^3)$$

for any $t > 0$. 

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We introduce the following mollifier to help us prove the regularity of weak solutions in Sobolev spaces,
\begin{equation}
M_\delta(t, \xi) = \langle \xi \rangle^{N_t - 2} \frac{\langle \xi \rangle^{NT_0 + 3}}{(1 + \delta|\xi|^2)^{N_0}} \tag{3.1}
\end{equation}
for $0 < \delta < 1$ and $2N_0 = NT_0 + 3$, $t \in [0, T_0]$. Then we agree that, in what follows, the notation $M_\delta(t, D_v)$ stands for the Fourier multiplier of symbol $M_\delta(t, \xi)$, that is to say,
\begin{equation}
M_\delta h(t, v) = M_\delta(t, D_v)h(t, v) = F_{\xi \rightarrow \nu}^{-1} \left( M_\delta(t, \xi)\hat{h}(t, \xi) \right). \tag{3.2}
\end{equation}

We give some properties about $M_\delta(t, \xi)$ at first:

**Lemma 3.2.** Let $T > 0$, then for any $t \in [0, T]$ and $\xi \in \mathbb{R}^3$, we have
\begin{align}
|\partial_t M_\delta(t, \xi)| &\leq N \log(\langle \xi \rangle)M_\delta(t, \xi), \tag{3.2} \\
|\partial_\xi^k M_\delta(t, \xi)| &\leq C_k \langle \xi \rangle^{-k}M_\delta(t, \xi), \tag{3.3} \\
|M_\delta(t, \xi)| &\leq CM_\delta(t, \xi^+), \tag{3.4}
\end{align}
with $C, C_k > 0$ independent of $\delta$.

**Proof.** A direct calculation gives that
\begin{align*}
\log M_\delta(t, \xi) &= \frac{Nt}{2} \log(1 + |\xi|^2) - N_0 \log(1 + \delta|\xi|^2), \\
\partial_t M_\delta &= \frac{N}{2} \log(1 + |\xi|^2)M_\delta(t, \xi),
\end{align*}
which yields the first result. The left two results are easy to check, and thus omitted here. \qed

### 3.1. Commutator estimates with Sobolev mollifier

We will estimate the commutator between the collision operator and the Sobolev mollifier operator, as follows:

**Proposition 3.3.** Suppose that $0 < s < 1$. For a suitable function $f$, we have
\begin{equation}
|\langle M_\delta Q(f, f) - Q(f, M_\delta f), M_\delta f \rangle| \lesssim \|f\|_{L^1} \|M_\delta f\|_{L^2}^2. \tag{3.5}
\end{equation}

**Proof.** We can write that,
\begin{align}
\langle M_\delta Q(f, f) - Q(f, M_\delta f), M_\delta f \rangle &= C \left\{ \int \int M_\delta(\xi)b(\cos \theta) \left[ \hat{f}(\xi^-)\hat{f}(\xi^+) - \hat{f}(0)\hat{f}(\xi) \right] (M_\delta f)^\dagger(\xi) d\xi d\sigma \\
&\quad - \int \int b(\cos \theta) \left[ \hat{f}(\xi^-)M_\delta(\xi^+)\hat{f}(\xi^+) - \hat{f}(0)M_\delta(\xi)\hat{f}(\xi) \right] (M_\delta f)^\dagger(\xi) d\xi d\sigma \right\} \tag{3.6}
\end{align}
Suppose that Proposition 3.4.

Proof. With respect to \( \tau \), we write that

\[
\langle M_\delta Q(f, f) - Q(f, M_\delta f), M_\delta f \rangle = C \int \int b(\cos \theta) \hat{f}(\xi^-) \{M_\delta(\xi) - M_\delta(\xi^+)\} \hat{f}(\xi^+) M_\delta(\xi) \hat{f}(\xi) d\xi d\sigma.
\]

From the following Taylor expansion up to order 2,

\[
M_\delta(\xi) - M_\delta(\xi^+) = (\xi - \xi^+) (\partial_\xi M_\delta)(\xi^+) + \int_0^1 (1 - \tau) (\xi - \xi^+) \otimes (\xi - \xi^+): (\partial_\xi^2 M_\delta)(\xi_\tau) d\tau,
\]

with \( \xi_\tau = (1 - \tau) \xi^+ + \tau \xi \) for \( \tau \in [0, 1] \), we get

\[
\langle M_\delta Q(f, f) - Q(f, M_\delta f), M_\delta f \rangle = C \int \int b(\cos \theta) \hat{f}(\xi^-) (\xi - \xi^+) (\partial_\xi M_\delta)(\xi^+) \hat{f}(\xi^+) M_\delta(\xi) \hat{f}(\xi) d\xi d\sigma
\]

\[
+ C \int \int \int_0^1 (1 - \tau) b(\cos \theta) \hat{f}(\xi^-) (\xi - \xi^+) \otimes (\xi - \xi^+) : (\partial_\xi^2 M_\delta)(\xi_\tau) \hat{f}(\xi^+) M_\delta(\xi) \hat{f}(\xi) d\tau d\xi d\sigma
\]

\[\triangleq A_1 + A_2.\]

By the symmetry property of \( b \) with respect to \( \sigma \), we can substitute \( \sin^2 \frac{\theta}{2} \xi \) for \( \xi - \xi^+ \) in \( A_1 \), and get

\[
|A_1| \leq \left| \int \int b(\cos \theta) \sin^2 \frac{\theta}{2} |\hat{f}(\xi^-)| |\xi| |\partial_\xi M_\delta(\xi^+)| |\hat{f}(\xi^+)| |M_\delta(\xi)\hat{f}(\xi)| d\xi d\sigma \right|
\]

\[
\lesssim \int \int b(\cos \theta) \sin^2 \frac{\theta}{2} |\hat{f}(\xi^-)| |\xi| |\xi(\xi^+) |^{-1} |M_\delta(\xi^+)\hat{f}(\xi^+)| |M_\delta(\xi)\hat{f}(\xi)| d\xi d\sigma
\]

\[
\lesssim \|\hat{f}\|_{L^\infty} \int b(\cos \theta) \sin^2 \frac{\theta}{2} \frac{|\xi|}{\cos \frac{\theta}{2}} d\sigma \|M_\delta\hat{f}\|_{L^2}^2
\]

\[
\lesssim \|f\|_{L^1} \|M_\delta\hat{f}\|_{L^2}^2.
\]

As for the term \( A_2 \), we have

\[
|A_2| \leq \left| \int \int \int_0^1 (1 - \tau) b(\cos \theta) |\xi^-|^2 |\hat{f}(\xi^-)| |\partial_\xi^2 M_\delta(\xi^+)| |\hat{f}(\xi^+)| |M_\delta(\xi)\hat{f}(\xi)| |\tau d\xi d\sigma \right|
\]

\[
\lesssim \int \int b(\cos \theta) \sin^2 \frac{\theta}{2} |\hat{f}(\xi^-)| |M_\delta(\xi^+)| |\hat{f}(\xi^+)| |M_\delta(\xi)\hat{f}(\xi)| d\xi d\sigma
\]

\[
\lesssim |\hat{f}(\xi^-)|_{L^\infty} \int b(\cos \theta) \sin^2 \frac{\theta}{2} \frac{|\xi|}{\cos \frac{\theta}{2}} d\sigma \|M_\delta\hat{f}\|_{L^2}^2
\]

\[
\lesssim \|f\|_{L^1} \|M_\delta\hat{f}\|_{L^2}^2.
\]

Therefore, we can obtain the needed result from the above two inequalities. \( \square \)

**Proposition 3.4.** Suppose that \( 0 < s < 1 \). For a suitable function \( f \), we have,

\[
|\langle v M_\delta Q(f, f) - Q(f, v M_\delta f), v M_\delta f \rangle| \lesssim \left( \|f\|_{L^1} + \|f\|_{L^1}^2 \right) \|M_\delta f\|_{L^2}^2.
\]

**Proof.** Arguing as Proposition 2.5, we write that

\[
- \langle v M_\delta Q(f, f) - Q(f, v M_\delta f), v M_\delta f \rangle
\]
where we have used the estimate (3.14)
\[ \{ \partial \xi \big( M_\delta(\xi)\hat{f}(\xi^+) \big) \big( \xi^+ \big) \} \partial \xi \big( M_\delta(\xi)\hat{f}(\xi) \big) d\xi d\sigma \]

Firstly, the fact \( M_\delta(\xi) \lesssim M_\delta(\xi^+) \) gives that

(3.13) \[ |I_1| \lesssim \iint b(\cos \theta) \sin^2 \frac{\theta}{2} \left| \big( \partial \xi \big( M_\delta(\xi)\hat{f}(\xi^+) \big) \big( \xi^+ \big) \big) \right| d\xi d\sigma \]
\[ \lesssim \| \partial \xi \|_{L^\infty} \iint \frac{b(\cos \theta) \sin^2 \frac{\theta}{2}}{\cos \frac{\theta}{2}} d\sigma \left( \iint |M_\delta(\xi)\hat{f}(\xi^+)|^2 d\xi^+ \right)^{\frac{1}{2}} \| v M_\delta f \|_{L^2} \]
\[ \lesssim \| f \|_{L^1} \| M_\delta f \|_{L^2}^2. \]

Furthermore, observe the fact

(3.14) \[ \partial \xi \big( M_\delta(\xi)\hat{f}(\xi^+) \big) - \big( \partial \xi \big( M_\delta(\xi)\hat{f}(\xi^+) \big) \big) \big( \xi^+ \big) \]
\[ = \big\{ M_\delta(\xi) - M_\delta(\xi^+) \big\} \big( \partial \xi \big( M_\delta(\xi)\hat{f}(\xi^+) \big) \big( \xi^+ \big) \big) + M_\delta(\xi) \left( \frac{\partial \xi^+}{\partial \xi} - I \right) \big( \partial \xi \big( M_\delta(\xi)\hat{f}(\xi^+) \big) \big) + \{( \partial \xi M_\delta(\xi) \} \big( \xi^+ \big) \big( M_\delta(\xi)\hat{f}(\xi^+) \big), \]

then correspondingly, the term \( I_2 \) can be reformulated as \( I_2 = I_{21} + I_{22} + I_{23} \).

The Taylor expansion (3.7) yields that

(3.15) \[ I_{21} = \iint b(\cos \theta) \hat{f}(\xi^-) \big( \xi - \xi^- \big) \partial \xi M_\delta(\xi) \big( \xi^+ \big) \partial \xi \big( M_\delta(\xi)\hat{f}(\xi) \big) d\xi d\sigma \]
\[ + \iint_0^1 (1 - \tau) b(\cos \theta) \hat{f}(\xi^-) \big( \xi - \xi^- \big) \partial \xi^2 \infty \xi M_\delta(\xi) \big( \xi^+ \big) \partial \xi \big( M_\delta(\xi)\hat{f}(\xi) \big) d\xi d\sigma \]
\[ \equiv I_{211} + I_{212}. \]

The symmetry property enables us to take the place of \( \xi - \xi^+ \) by \( \sin^2 \frac{\theta}{2} \cdot \xi \) in \( I_{211} \), thereby we get

(3.16) \[ |I_{211}| \lesssim \iint b(\cos \theta) \sin^2 \frac{\theta}{2} \left| \hat{f}(\xi^-) \right| \left| \xi \right| \left( \xi^+ \right)^{-1} \left| \partial \xi \big( M_\delta(\xi)\hat{f}(\xi^+) \big) \right| d\xi d\sigma \]
\[ \lesssim \| \hat{f} \|_{L^\infty} \iint \frac{b(\cos \theta) \sin^2 \frac{\theta}{2}}{\cos \frac{\theta}{2}} d\sigma \| M_\delta(\partial \xi \hat{f}) \|_{L^2} \| v M_\delta f \|_{L^2} \]
\[ \lesssim \| f \|_{L^1} \| M_\delta f \|_{L^2}^2, \]

where we have used the estimate

(3.17) \[ \| M_\delta(\partial \xi \hat{f}) \|_{L^2} \leq \| \partial \xi (M_\delta f) \|_{L^2} + \| (\partial \xi M_\delta \hat{f}) \|_{L^2} \]

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As for \( I_{212} \), we have

\[
(I.18) \quad |I_{212}| \leq \oint \int_0^1 (1 - \tau) b(\cos \theta) \sin^2 \frac{\theta}{2} |\hat{f}(\xi^-)| |\xi|^2 \langle \xi^+ \rangle^{-2} |M_\delta(\xi\tau)| \left| (\partial_\xi \hat{f})(\xi^+) \right| \left| \partial_\xi \left( M_\delta(\xi) \hat{f}(\xi) \right) \right| d\xi d\sigma
\]

\[
\lesssim \|\hat{f}\|_{L^\infty} \oint \int \frac{b(\cos \theta)}{\cos \frac{\theta}{2}} \sin^2 \frac{\theta}{2} d\sigma \|M_\delta(\partial_\xi \hat{f})\|_{L^2} \|vM_\delta f\|_{L^2}
\]

\[
\lesssim \|f\|_{L^1} \|M_\delta f\|_{L^2}^2.
\]

Then it follows

\[
(I.19) \quad |I_{211}| + |I_{212}| \lesssim \|f\|_{L^1} \|M_\delta f\|_{L^2}^2.
\]

Concerning the term \( I_{22} \), we deduce that

\[
(I.20) \quad |I_{22}| \lesssim \oint \int b(\cos \theta) \sin^2 \frac{\theta}{2} |\hat{f}(\xi^-)| \left| M_\delta(\xi\tau) \right| \left| \partial_\xi \left( M_\delta(\xi) \hat{f}(\xi) \right) \right| d\xi d\sigma
\]

\[
\lesssim \|f\|_{L^1} \|M_\delta f\|_{L^2}^2.
\]

As for \( I_{23} \), we use the Taylor expansion up to order 2 to get,

\[
(I.21) \quad \left( \partial_\xi M_\delta \right) (\xi) - \left( \partial_\xi M_\delta \right) (\xi^+) = (\xi - \xi^+) \cdot \left( \partial_\xi^2 M_\delta \right) (\xi^+) + \oint_0^1 (1 - \tau)(\xi - \xi^+) \otimes (\xi - \xi^+) : \left( \partial_\xi^3 M_\delta \right) (\xi\tau)d\tau
\]

with \( \tau \in [0, 1] \) and \( \xi_\tau = (1 - \tau)\xi^+ + \tau \xi \). Then we can rewrite \( I_{23} = I_{231} + I_{232} \), correspondingly.

A similar process as above ensures that,

\[
(I.22) \quad |I_{231}| \lesssim \oint \int b(\cos \theta) \sin^2 \frac{\theta}{2} |\hat{f}(\xi^-)| |\xi| \langle \xi^+ \rangle^{-2} \left| M_\delta(\xi\tau) \hat{f}(\xi^+) \right| \left| \partial_\xi \left( M_\delta(\xi) \hat{f}(\xi) \right) \right| d\xi d\sigma
\]

\[
\lesssim \|\hat{f}\|_{L^\infty} \|M_\delta \hat{f}\|_{L^2} \|vM_\delta f\|_{L^2}
\]

\[
\lesssim \|f\|_{L^1} \|M_\delta f\|_{L^2}^2,
\]

and

\[
(I.23) \quad |I_{232}| \lesssim \oint \int_0^1 b(\cos \theta) \sin^2 \frac{\theta}{2} |\hat{f}(\xi^-)| \left| \langle \xi^+ \rangle^{-3} \right| \left| M_\delta(\xi\tau) \hat{f}(\xi^+) \right| \left| \partial_\xi \left( M_\delta(\xi) \hat{f}(\xi) \right) \right| d\xi d\sigma
\]

\[
\lesssim \|\hat{f}\|_{L^\infty} \|M_\delta \hat{f}\|_{L^2} \|vM_\delta f\|_{L^2}
\]

\[
\lesssim \|f\|_{L^1} \|M_\delta f\|_{L^2}^2.
\]

The two estimates imply that,

\[
(I.24) \quad |I_{23}| \leq |I_{231}| + |I_{232}| \lesssim \|f\|_{L^1} \|M_\delta f\|_{L^2}^2.
\]
Thus, from \((3.19), (3.20),\) and \((3.21),\) we have,

\[
|I_2| \lesssim ||f||_{L^1} ||M_{\delta}f||^2_{L^2}.
\]

Combining with \((3.13),\) this completes the proof of Proposition \((3.4).\)

**Proposition 3.5.** Suppose that \(0 < s < 1.\) For a suitable function \(f,\) we have,

\[
|\langle v \otimes v M_{\delta}Q(f, f) - Q(f, v \otimes v M_{\delta}f), v \otimes v M_{\delta}f \rangle| \lesssim \left(||f||_{L^1} + ||f||_{L^1} + ||f||_{L^2}\right)||M_{\delta}f||^2_{L^2}.
\]

**Proof.** The proof is similar as that of Proposition \(2.7,\) if we replace Lemma \(2.1,\) by Lemma \(3.2,\) and substitute \((2.18)\) for \((3.4).\) Firstly we write that,

\[
\begin{align*}
(v \otimes v M_{\delta}Q(f, f) - Q(f, v \otimes v M_{\delta}f), v \otimes v M_{\delta}f) & = C \int b(\cos \theta) \left\{ \partial_{\xi}^2 \left( M_{\delta}(\xi) \hat{f}(\xi^+) \right) - \hat{f}(\xi^+) \left( \partial_{\xi}^2 \left(M_{\delta} \hat{f}\right) (\xi^+) \right) \right\} \partial_{\xi}^2 \left(M_{\delta} \hat{f}\right) (\xi) \, d\xi d\sigma \\
& = C \int b(\cos \theta) \left\{ \left( \partial_{\xi}^2 \hat{f}(\xi^+) \right) \left( \partial_{\xi}^2 \left(M_{\delta} \hat{f}\right) (\xi^+) \right) \right\} \partial_{\xi}^2 \left(M_{\delta} \hat{f}\right) (\xi) \, d\xi d\sigma \\
& + 2C \int b(\cos \theta) \left( \partial_{\xi}^2 \hat{f}(\xi^-) \right) \left( \partial_{\xi}^2 \left(M_{\delta} \hat{f}\right) (\xi^-) \right) \partial_{\xi} \left(M_{\delta}(\xi) \hat{f}(\xi^+) \right) \, d\xi d\sigma \\
& + C \int b(\cos \theta) \hat{f}(\xi^-) \left( \partial_{\xi}^2 \left(M_{\delta}(\xi) \hat{f}(\xi^+) \right) - \partial_{\xi}^2 \left( M_{\delta} \hat{f}\right) (\xi^+) \right) \left( \partial_{\xi}^2 \left(M_{\delta} \hat{f}\right) (\xi^+) \right) \, d\xi d\sigma \\
& \triangleq II_1 + II_2 + II_3.
\end{align*}
\]

Noticing the fact

\[
\left| \frac{\partial^2 \xi^-}{\partial \xi \partial \xi} \right| = 0,
\]

we have

\[
|II_1| \lesssim \int b(\cos \theta) \sin \frac{\theta}{2} \left| \partial_{\xi}^2 \hat{f}(\xi^-) \right| \left| M_{\delta}(\xi) \hat{f}(\xi^+) \right| \left| \partial_{\xi}^2 \left(M_{\delta} \hat{f}\right) (\xi) \right| \, d\xi d\sigma \\
\lesssim \|\partial_{\xi}^2 \hat{f}\|_{L^\infty} \int b(\cos \theta) \sin ^4 \frac{\theta}{2} d\sigma \|M_{\delta} \hat{f}\|_{L^2} \|\partial_{\xi}^2 \left(M_{\delta} \hat{f}\right)\|_{L^2} \\
\lesssim \|\partial_{\xi}^2 \hat{f}\|_{L^1} \|M_{\delta}f\|^2_{L^2}.
\]

Above, we have used the following fact, in the last inequality,

\[
\|M_{\delta}(\partial_{\xi}^2 \hat{f})\|_{L^2} \leq \|\partial_{\xi}^2 \left(M_{\delta} \hat{f}\right)\|_{L^2} + \|\left(\partial_{\xi}^2 M_{\delta}\right) \hat{f}\|_{L^2} + 2\|\left(\partial_{\xi} M_{\delta}\right) (\partial_{\xi} \hat{f})\|_{L^2} \\
\lesssim \|\partial_{\xi}^2 \left(M_{\delta} \hat{f}\right)\|_{L^2} + \|\xi^{-2} M_{\delta} \hat{f}\|_{L^2} + \|\xi^{-1} M_{\delta} (\partial_{\xi} \hat{f})\|_{L^2} \\
\lesssim \|\partial_{\xi}^2 \left(M_{\delta} \hat{f}\right)\|_{L^2} + \|M_{\delta} \hat{f}\|_{L^2} + ||M_{\delta} \hat{f}\|_{L^2} \\
\lesssim \|(v)^2 (M_{\delta}f)\|_{L^2}.
\]

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Considering the term $II_2$, we reformulate it as

$$
(3.31) \quad II_2 = C \int \int b(\cos \theta) \left( \frac{\partial^2 \xi}{\partial \xi^2} \right) \left( \xi \right) \left( \partial_{\xi} M^2 \right) \left( \xi \right) \left( \partial_{\xi}^2 (M \hat{f}) \right) (\xi) \ d\xi d\sigma \\
+ C \int \int b(\cos \theta) \left( \frac{\partial^2 \xi}{\partial \xi^2} \right) \left( \xi \right) \left( \partial_{\xi} M^2 \right) \left( \xi \right) \left( \partial_{\xi}^2 (M \hat{f}) \right) (\xi) \ d\xi d\sigma \\
\triangleq II_{21} + II_{22}.
$$

Then we can deduce that

$$
(3.32) \quad |II_{21}| \lesssim \int \int b(\cos \theta) \sin^2 \frac{\theta}{2} \left( \left| \partial_{\xi} \hat{f} \right| (\xi) \right) \left( \xi \right) \left( \partial_{\xi} M^2 \hat{f} \right) (\xi) \ d\xi d\sigma \\
\lesssim \left\| \partial_{\xi} \hat{f} \right\|_{L^\infty} \left\| M \hat{f} \right\|_{L^2} \left\| \partial_{\xi}^2 (M \hat{f}) \right\|_{L^2} \\
\lesssim \left\| f \right\|_{L^1} \left\| M \hat{f} \right\|_{L^2}^2,
$$

and

$$
(3.33) \quad |II_{22}| \lesssim \int \int b(\cos \theta) \sin^2 \frac{\theta}{2} \left( \left| \partial_{\xi} \hat{f} \right| (\xi) \right) \left( \xi \right) \left( \partial_{\xi} M^2 \hat{f} \right) (\xi) \ d\xi d\sigma \\
\lesssim \left\| \partial_{\xi} \hat{f} \right\|_{L^\infty} \left\| M \hat{f} \right\|_{L^2} \left\| \partial_{\xi}^2 (M \hat{f}) \right\|_{L^2} \\
\lesssim \left\| f \right\|_{L^1} \left\| M \hat{f} \right\|_{L^2}^2.
$$

From the above two estimates it follows

$$
(3.34) \quad |II_2| \leq |II_{21}| + |II_{22}| \lesssim \left\| f \right\|_{L^1} \left\| M \hat{f} \right\|_{L^2}^2.
$$

We then turn to the estimate for $II_3$, we write

$$
(3.35) \quad II_3 = C \int \int b(\cos \theta) \hat{f}(\xi) \left( \partial_{\xi}^2 (M \hat{f}) \right) (\xi) \ d\xi d\sigma \\
+ C \int \int b(\cos \theta) \hat{f}(\xi) \left( \partial_{\xi} M^2 \hat{f} \right) (\xi) \left( \partial_{\xi}^2 (M \hat{f}) \right) (\xi) \ d\xi d\sigma \\
+ C \int \int b(\cos \theta) \hat{f}(\xi) \left( \partial_{\xi} M^2 \hat{f} \right) (\xi) \left( \partial_{\xi}^2 (M \hat{f}) \right) (\xi) \ d\xi d\sigma \\
\triangleq II_{31} + \Psi + II_{32} + II_{33}.
$$

The process of dealing with these above terms is similar as that of Proposition 2.7 and much simpler. Thus we omit it and give the following estimate,

$$
(3.35) \quad |II_3| \leq |II_{31}| + |\Psi| + |II_{32}| + |II_{33}| \lesssim \left\| f \right\|_{L^1} \left\| M \hat{f} \right\|_{L^2}^2.
$$

Combining the estimates (3.34), (3.35), and (3.35) yields the desired result.
3.2. Justification for Sobolev regularizing effect

Recalling the following upper bound for the collision operator (compare [1]):

\( \|Q(g, f)\|_{H^m_1(\mathbb{R}^3)} \lesssim \|g\|_{L_{1+2s}^1(\mathbb{R}^3)} \|f\|_{H^{m+2s}_{1+2s} (\mathbb{R}^3)} \)

with \( m = -4, \ l = 2 \) and \( 0 < s < 1 \), we get

\( \|Q(f, f)\|_{H^{-s}_{2+2s} (\mathbb{R}^3)} \lesssim \|f\|_{L_{1+2s}^1(\mathbb{R}^3)} \|f\|_{H^{s+2s}_{1+2s} (\mathbb{R}^3)} \lesssim \|f\|_{L_{1+2s}^1(\mathbb{R}^3)} \|f\|_{L_{1+2s}^1(\mathbb{R}^3)} \).

Let \( f \in L^\infty((0, T_0); L_{2+2s}^1 \cap LlogL(\mathbb{R}^3)) \) be a weak solution of the Cauchy problem ([12]), then we take

\( f_1(t, \cdot) = \left( M_\delta (v)^4 M_\delta f \right)(t, \cdot) \in L^\infty((0, T_0); H^{5}_{2+2s}(\mathbb{R}^3)) \)

as the test function. And moreover, a similar argument as that of [10] enables us to assume \( f_1 \in C^4([0, T_0]; H^{3}_{2+2s}(\mathbb{R}^3)) \).

Then we obtain the weak formulation:

\( \left\langle \partial_t f(t, \cdot), f_1(t, \cdot) \right\rangle = \left\langle Q(f, f), f_1 \right\rangle \).

We compute

\( L.H.S. = \frac{1}{2} \frac{d}{dt} \|M_\delta f\|_{L_{2}^2}^2 - \langle \langle \partial_t M_\delta \rangle f, M_\delta f \rangle - 2\langle v \langle \partial_t M_\delta \rangle f, v M_\delta f \rangle - \langle v^2 \langle \partial_t M_\delta \rangle f, v^2 M_\delta f \rangle, \)

\( R.H.S. = \langle M_\delta Q(f, f), (1 + 2|v|^2 + |v|^4)M_\delta f \rangle = \langle Q(f, M_\delta f), M_\delta f \rangle + 2\langle Q(f, v M_\delta f), v M_\delta f \rangle + \langle Q(f, v \otimes v M_\delta f), v \otimes v M_\delta f \rangle \)

\( + \langle M_\delta Q(f, f) - Q(f, M_\delta f), M_\delta f \rangle + 2\langle v M_\delta Q(f, f) - Q(f, v M_\delta f), v M_\delta f \rangle \)

\( + \langle v \otimes v M_\delta Q(f, f) - Q(f, v \otimes v M_\delta f), v \otimes v M_\delta f \rangle, \)

then we get the reformulation:

\( \frac{1}{2} \frac{d}{dt} \|M_\delta f\|_{L_{2}^2}^2 - \langle Q(f, M_\delta f), M_\delta f \rangle - 2\langle Q(f, v M_\delta f), v M_\delta f \rangle - \langle Q(f, v \otimes v M_\delta f), v \otimes v M_\delta f \rangle \)

\( = \langle \langle \partial_t M_\delta \rangle f, M_\delta f \rangle + 2\langle v \langle \partial_t M_\delta \rangle f, v M_\delta f \rangle + \langle v^2 \langle \partial_t M_\delta \rangle f, v^2 M_\delta f \rangle \)

\( + \langle M_\delta Q(f, f) - Q(f, M_\delta f), M_\delta f \rangle + 2\langle v M_\delta Q(f, f) - Q(f, v M_\delta f), v M_\delta f \rangle \)

\( + \langle v \otimes v M_\delta Q(f, f) - Q(f, v \otimes v M_\delta f), v \otimes v M_\delta f \rangle. \)

In the next, we need to handle with the three terms on the right-hand side, as follows:

**Lemma 3.6.** For the terms involving the derivative of the mollifier with respect to time, we have

\( \|\langle \partial_t M_\delta \rangle f, M_\delta f \| \leq \varepsilon \|M_\delta f\|_{H^s}^2 + C_\varepsilon \|M_\delta f\|_{L_2^2}^2, \)

\( \|v \langle \partial_t M_\delta \rangle f, v M_\delta f \| \leq \varepsilon \|M_\delta f\|_{H^s}^2 + C_\varepsilon \|M_\delta f\|_{L_2^1}^2, \)

\( \|v^2 \langle \partial_t M_\delta \rangle f, v^2 M_\delta f \| \leq \varepsilon \|M_\delta f\|_{H^s}^2 + C_\varepsilon \|M_\delta f\|_{L_2^1}^2. \)
Proof. By virtue of Lemma 3.2, we have

\begin{equation}
\langle (\partial_t M_\delta f), M_\delta f \rangle = \left\langle \frac{N}{2} \log(1 + |\xi|^2) M_\delta \hat{f}, M_\delta \hat{f} \right\rangle.
\end{equation}

Noticing the fact, with an \( \varepsilon > 0 \),

\begin{equation}
\frac{N}{2} \log(1 + |\xi|^2) \leq \varepsilon (1 + |\xi|^2)^s + C_\varepsilon,
\end{equation}

thus, we get

\begin{equation}
|\langle (\partial_t M_\delta f), M_\delta f \rangle| \leq \varepsilon \langle (1 + |\xi|^2)^s M_\delta \hat{f}, M_\delta \hat{f} \rangle + C_\varepsilon \langle M_\delta \hat{f}, M_\delta \hat{f} \rangle \leq \varepsilon \|M_\delta f\|_{L^s}^2 + C_\varepsilon \|M_\delta f\|_{L^2}^2.
\end{equation}

After a few calculations, we have,

\begin{equation}
\partial_\xi (\partial_t M_\delta \hat{f}) \leq \partial_\xi \left( \frac{N}{2} \log(1 + |\xi|^2) \right) M_\delta \hat{f} + \frac{N}{2} \log(1 + |\xi|^2) \partial_\xi (M_\delta \hat{f}),
\end{equation}

\begin{equation}
\partial_\xi^2 (\partial_t M_\delta \hat{f}) \leq \partial_\xi^2 \left( \frac{N}{2} \log(1 + |\xi|^2) \right) M_\delta \hat{f} + 2 \partial_\xi \left( \frac{N}{2} \log(1 + |\xi|^2) \right) \partial_\xi (M_\delta \hat{f}) + \frac{N}{2} \log(1 + |\xi|^2) \partial_\xi^2 (M_\delta \hat{f}).
\end{equation}

Observing that,

\begin{equation}
\left| \partial_\xi \left( \frac{N}{2} \log(1 + |\xi|^2) \right) \right| \leq C, \quad \left| \partial_\xi^2 \left( \frac{N}{2} \log(1 + |\xi|^2) \right) \right| \leq C,
\end{equation}

we can derive the latter two results by using of (3.47). \( \square \)

Now we resume to the proof of Theorem 3.1. From Lemma 2.3, it’s easy to check that

\begin{equation}
- \langle Q(f, M_\delta f), M_\delta f \rangle \geq c_f \|M_\delta f\|_{L^s}^2 - C\|f\|_{L^1} \|M_\delta f\|_{L^2}^2,
\end{equation}

\begin{equation}
- 2 \langle Q(f, v M_\delta f), v M_\delta f \rangle \geq c_f \|M_\delta f\|_{H^s}^2 - C\|f\|_{L^1} \|M_\delta f\|_{L^2}^2,
\end{equation}

\begin{equation}
- \langle Q(f, v \otimes v M_\delta f), v \otimes v M_\delta f \rangle \geq c_f \|M_\delta f\|_{H^s}^2 - C\|f\|_{L^1} \|M_\delta f\|_{L^2}^2.
\end{equation}

Combining these estimates, Lemma 3.6 and Propositions 3.3, 3.5 we obtain that

\begin{equation}
\frac{d}{dt} \|M_\delta f\|_{L^2}^2 + c_f \|M_\delta f\|_{H^s}^2 \leq \varepsilon \|M_\delta f\|_{L^2}^2 + C_\varepsilon \|M_\delta f\|_{L^2}^2 + C\left( \|f\|_{L^1} + \|f\|_{L^1} + \|M_\delta f\|_{L^2} \right) \|M_\delta f\|_{L^2}^2.
\end{equation}

Recalling the conservational properties for the Boltzmann equation implies that

\begin{equation}
\|f\|_{L^1} + \|f\|_{L^1} + \|M_\delta f\|_{L^2} \lesssim \|f_0\|_{L^1}, \quad c_f \geq c_f_0 > 0,
\end{equation}

and by choosing \( \varepsilon < c_f \), we get

\begin{equation}
\frac{d}{dt} \|M_\delta f\|_{L^2}^2 \leq C \|M_\delta f\|_{L^2}^2.
\end{equation}
from which it follows

\[(3.58) \quad \| (M_\delta f)(t) \|_{L^2}^2 \leq e^{Ct} \| M_\delta(0)f_0 \|_{L^2}^2. \]

Since

\[(3.59) \quad \| (M_\delta f)(t) \|_{L^2}^2 = \| (1 - \delta \Delta)^{-N_0} f(t) \|_{H^{N_0-2}}^2, \]
\[(3.60) \quad \| M_\delta(0)f_0 \|_{L^2}^2 = \| (1 - \delta \Delta)^{-N_0} f(0) \|_{H^{N_0-2}}^2 \leq C \| f_0 \|_{L^2}^2, \]

due to the embedding \( L^1_2(\mathbb{R}^3) \subset H^{-2}_2(\mathbb{R}^3) \), then we obtain

\[(3.61) \quad \| (1 - \delta \Delta)^{-N_0} f(t) \|_{H^{N_0-2}}^2 \leq Ce^{Ct} \| f_0 \|_{L^2}^2, \]

where \( C > 0 \) is independent of \( \delta \). Taking limit \( \delta \to 0 \), we get, finally, for \( t \in [0, \bar{T}] \),

\[(3.62) \quad \| f(t) \|_{H^{N_0-2}}^2 \leq Ce^{Ct} \| f_0 \|_{L^2}^2. \]

As \( N \) can be chosen arbitrarily large for any given \( t > 0 \), we conclude that

\[(3.63) \quad f(t) \in H^{+\infty}_2(\mathbb{R}^3). \]

This completes the whole proof of Theorem 3.1.

4 Completion of the proof of Gevrey regularity

Since for any \( t_0 > 0 \), the weak solution satisfies \( f \in L^\infty([t_0, T_0]; H^s_2(\mathbb{R}^3)) \), then \( f \) solves the following Cauchy problem:

\[(4.1) \begin{cases}
    f_t(t, v) = Q(f, f)(v), & t \in (t_0, T], \quad v \in \mathbb{R}^3, \\
    f|_{t=t_0} = f(t_0, \cdot) \in H^s_2(\mathbb{R}^3).
\end{cases} \]

When considering the Gevrey regularizing effect, we may assume \( t_0 = 0 \) by translation. Thus we state the result:

**Theorem 4.1.** Suppose the initial datum \( f_0 \in L^1_{2+2s} \cap H^2_2(\mathbb{R}^3) \). Let \( f \in L^\infty([0, T_0]; L^1_2 \cap H^s_2(\mathbb{R}^3)) \) be a non-negative weak solution of the Cauchy problem of the Boltzmann equation \((4.1)\) for some \( T_0 > 0 \), then

i) for the mild singularity case \( 0 < s < \frac{1}{2} \), there exists \( 0 < T_* \leq T_0 \) such that \( f(t, \cdot) \in G^{\frac{s}{s-s}}(\mathbb{R}^3) \) for any \( 0 < \alpha < s \) and \( 0 < t \leq T_* \), more precisely, there exists \( c_0 > 0 \) such that

\[(4.2) \quad e^{c_0 t(D_x)^{2s}} f \in L^\infty([0, T_*]; L^2_2(\mathbb{R}^3)); \]
ii) in the critical case of \( s = \frac{1}{2} \), there exists \( 0 < T_* \leq T_0 \) such that \( f(t, \cdot) \in G^{2+\varepsilon}(\mathbb{R}^3) \) for any \( \varepsilon > 0 \) and \( 0 < t \leq T_* \), moreover, there exists \( c_0 > 0 \), \( c' > 0 \) such that

\[
(4.3) \quad e^{ct(D_+)^{1/2}} f \in L^\infty([0, T_*]); L^2_2(\mathbb{R}^3));
\]

iii) for the strictly strong singularity case \( s \geq \frac{1}{2} \), there exists \( 0 < T_* \leq T_0 \) such that \( f(t, \cdot) \in G^{2}(\mathbb{R}^3) \) for any \( 0 < t \leq T_* \), in precise, there exists \( c_0 > 0 \) such that

\[
(4.4) \quad e^{ct(D_+)^{3/2}} f \in L^\infty([0, T_*]); L^2_2(\mathbb{R}^3)).
\]

**Remark 4.2.** By virtue of Theorem 1.2 of [7] describing the propagation of Gevrey regularity, the above theorem can lead to the Gevrey smoothing effect in global time showed in Theorem 1.2.

Let \( f \in L^\infty([0, T_0]; L^1_2 \cap H^2_0(\mathbb{R}^3)) \) be a weak solution of the Cauchy problem (1.2), then recall the following upper bound for the collision operator (compare [1]):

\[
(4.5) \quad \|Q(g, f)\|_{L^\infty(\mathbb{R}^3)} \lesssim \|g\|_{L^1_1(\mathbb{R}^3)} \|f\|_{L^{m+2s}_2(\mathbb{R}^3)};
\]

with \( m = l = 0 \) and \( 0 < s < 1 \), hence we have

\[
(4.6) \quad \|Q(f, f)\|_{L^2(\mathbb{R}^3)} \lesssim \|f\|_{L^1_1(\mathbb{R}^3)} \|f\|_{H^2_0(\mathbb{R}^3)} \lesssim \|f\|_{L^2_2(\mathbb{R}^3)} \|f\|_{H^2_0(\mathbb{R}^3)};
\]

which implies that \( Q(f, f) \in L^\infty([0, T_0]; L^2(\mathbb{R}^3)) \). Therefore we need to choose a test function \( \phi \in C^1([0, T_0]; L^2(\mathbb{R}^3)) \) to make sense \( \langle Q(f, f), \phi \rangle \).

We choose the mollified weak solution

\[
(4.7) \quad \tilde{f}(t, \cdot) = \left(G_\delta(v)^4 G_\delta f \right)(t, \cdot) \in L^\infty([0, T_0]; H^2(\mathbb{R}^3)).
\]

Furthermore, we suppose that \( \tilde{f}(t, \cdot) \in C^1([0, T_0]; H^2(\mathbb{R}^3)) \). Then we get

\[
(4.8) \quad \langle \partial_t f(t, \cdot), \tilde{f}(t, \cdot) \rangle = \langle Q(f, f), \tilde{f} \rangle = \langle G_\delta Q(f, f), (1 + 2|v|^2 + |v|^4)G_\delta f \rangle,
\]

which yields the reformulation:

\[
(4.9) \quad \frac{1}{2} \frac{d}{dt}\|G_\delta f\|_{L^2}^2 \cdots \cdots = \langle \partial_t G_\delta f, G_\delta f \rangle + 2\langle v(\partial_t G_\delta) f, vG_\delta f \rangle + 2\langle v^2(\partial_t G_\delta) f, v^2 G_\delta f \rangle
\]

Now it remains to estimate the three terms on the right-hand side, as follows:
Lemma 4.3. For the terms involving the derivative of the mollifier with respect to time, we have

\begin{align}
|\langle \partial_t G_\delta f, G_\delta f \rangle| & \lesssim \|G_\delta f\|_{L^2}^2, \\
|\langle v(\partial_t G_\delta) f, vG_\delta f \rangle| & \lesssim \|G_\delta f\|_{L^2}^2, \\
|\langle v^2(\partial_t G_\delta) f, v^2G_\delta f \rangle| & \lesssim \|G_\delta f\|_{L^2}^2.
\end{align}

Proof. Due to the Plancherel formula, we can deduce directly the first result by using Lemma 2.1.

As for the second result, we can write that

\begin{align}
|\langle v(\partial_t G_\delta) f, vG_\delta f \rangle|
& = \left| \int \left\{ \partial_\xi \left( c_0 \langle \xi \rangle^{2\alpha} G_\delta(\xi) \cdot \frac{1}{1 + \delta e^{c_0 t} \langle \xi \rangle^{2\alpha}} \hat{f}(\xi) \right) \right\} |vG_\delta f|^\alpha(\xi) d\xi \right| \\
& \lesssim \left| \int \langle \xi \rangle^{2\alpha} \partial_\xi \left( G_\delta(\xi) \hat{f}(\xi) \right) |vG_\delta f|^\alpha(\xi) d\xi \right| \\
& \quad + \left| \int \partial_\xi \left( \langle \xi \rangle^{2\alpha} \cdot \frac{1}{1 + \delta e^{c_0 t} \langle \xi \rangle^{2\alpha}} \right) \left| G_\delta(\xi) \hat{f}(\xi) \right| |(vG_\delta f)^\alpha(\xi)| d\xi \right| \\
& \lesssim \|G_\delta f\|_{L^2}^2,
\end{align}

in view of the estimate

\begin{align}
\left| \partial_\xi \left( \langle \xi \rangle^{2\alpha} \cdot \frac{1}{1 + \delta e^{c_0 t} \langle \xi \rangle^{2\alpha}} \right) \right| & \lesssim \langle \xi \rangle^{2\alpha}, \text{ for } \alpha < \frac{1}{2}.
\end{align}

By a similar but more slightly complicate scheme, the fact

\begin{align}
\left| \partial_\xi^2 \left( \langle \xi \rangle^{2\alpha} \cdot \frac{1}{1 + \delta e^{c_0 t} \langle \xi \rangle^{2\alpha}} \right) \right| & \lesssim \langle \xi \rangle^{2\alpha}
\end{align}

leads to the last result.

Now we resume to the proof of Theorem 4.1. From Lemma 2.3 it’s easy to check that

\begin{align}
- \langle Q(f, G_\delta f), G_\delta f \rangle & \geq c_f \|G_\delta f\|_{L^2}^2 - C\|f\|_{L^1} \|G_\delta f\|_{L^2}^2, \\
- 2\langle Q(f, vG_\delta f), vG_\delta f \rangle & \geq c_f \|G_\delta f\|_{L^2}^2 - C\|f\|_{L^1} \|G_\delta f\|_{L^2}^2, \\
- \langle Q(f, v \otimes v G_\delta f), v \otimes v G_\delta f \rangle & \geq c_f \|G_\delta f\|_{L^2}^2 - C\|f\|_{L^1} \|G_\delta f\|_{L^2}^2.
\end{align}

Combining these estimates, Lemma 4.3 and Propositions 2.1, 2.2, 2.6, we can infer that

\begin{align}
\frac{d}{dt} \|G_\delta f\|_{L^2}^2 + c_f \|G_\delta f\|_{L^2}^2 & \leq C\|G_\delta f\|_{H^2}^2 + C\|G_\delta f\|_{L^2}^2 + C\left( \|f\|_{L^1} + \|f\|_{L^1} + \|G_\delta f\|_{L^2} \right) \|G_\delta f\|_{H^2}^2 \\
& \quad + C\left( \|f\|_{L^1} + \|G_\delta f\|_{L^2} \right) \|G_\delta f\|_{L^2}^{3\alpha - \frac{1}{2}}. \\
\end{align}

Recalling that the conservational properties for the Boltzmann equation implies,

\begin{align}
\|f\|_{L^1} + \|f\|_{L^1} + \|G_\delta f\|_{L^2} \lesssim \|f_0\|_{L^2}, \quad c_f \geq c_{f_0} > 0,
\end{align}
we then have
\begin{equation}
\frac{d}{dt}\|G_\delta f\|^2_{L^2} + c_{f_0}\|G_\delta f\|^2_{H^s} \leq C_{f_0}\|G_\delta f\|^2_{H^s} + C\|G_\delta f\|^3_{L^3} + C_{f_0}\|G_\delta f\|^2_{H^s}^{(3\alpha - \frac{1}{2})^+} + C\|G_\delta f\|_{L^2}\|G_\delta f\|^2_{H^s} + C\|G_\delta f\|_{L^2}\|G_\delta f\|^2_{H^s}^{(3\alpha - \frac{1}{2})^+}.
\end{equation}

**Case** $\frac{1}{2} < s < 1$:
Take $\alpha = \frac{1}{3}$, then $(3\alpha - \frac{1}{2})^+ = \frac{1}{2} < s$, and we have
\begin{equation}
\frac{d}{dt}\|G_\delta f\|^2_{L^2} + c_{f_0}\|G_\delta f\|^2_{H^s} \leq C_{f_0}\|G_\delta f\|^2_{H^s} + C\|G_\delta f\|^3_{L^3} + C\|G_\delta f\|_{L^2}\|G_\delta f\|^2_{H^s}.
\end{equation}

Thanks to the following interpolation inequalities:
\begin{align}
\|G_\delta f\|^2_{H^s} &\leq \rho\|G_\delta f\|^2_{H^1} + \rho^{\frac{1+3}{2s}}\|G_\delta f\|^2_{L^3}, \\
\|G_\delta f\|_{L^2}\|G_\delta f\|^2_{H^s} &\leq \rho\|G_\delta f\|^2_{H^1} + C\rho\|G_\delta f\|^{2+\frac{2}{2s}}_{L^4},
\end{align}
and noticing the simple fact $2 + \frac{2}{2s} > 3$, it follows that, by choosing $2\rho = c_{f_0}/2$,
\begin{equation}
\frac{d}{dt}\|G_\delta f\|^2_{L^2} + \frac{c_{f_0}}{2}\|G_\delta f\|^2_{H^s} \leq C\|G_\delta f\|^2_{L^2} + C\|G_\delta f\|^{2+\frac{2}{2s}}_{L^4}.
\end{equation}

This is an ordinary differential equation of Bernoulli type including an extra term on the left-hand. Putting $g = e^{-Ct}\|G_\delta f\|^2_{L^2}$, we get
\begin{equation}
\frac{d}{dt}g \leq Ce^{\bar{C}t}g^{1+\frac{2}{2s}},
\end{equation}
with $\bar{C} = \frac{s}{2s-1}C$. Therefore, we can deduce that,
\begin{equation}
g(t) \leq \frac{g(0)}{1 + C\left(1 - e^{\bar{C}t}\right)g(0)^{\frac{2s}{2s-1}}}^{\frac{2s-1}{s}},
\end{equation}
which yields, for $0 < \delta < 1$,
\begin{equation}
\|G_\delta f\|^2_{L^2} \leq \frac{e^{Ct}\|f_0\|^2_{L^2}}{1 + C\left(1 - e^{\bar{C}t}\right)\|f_0\|^{\frac{2s}{2s-1}}}^{\frac{2s-1}{s}}.
\end{equation}

We choose $T_* \in (0, T_0]$ sufficiently small such that
\begin{equation}
\left\{1 + C\left(1 - e^{\bar{C}t}\right)\|f_0\|^{\frac{2s}{2s-1}}\right\}^{\frac{2s-1}{s}} \geq C_0 > 0, \quad t \in [0, T_*].
\end{equation}

Taking limit $\delta \to 0$, we obtain for $t \in [0, T_*]$:
\begin{equation}
\|e^{-C(t\Delta)^{2/3}}f\|^2_{L^\infty([0, T_*]; L^2_2(\mathbb{R}^3))} \leq C_0^{-1}e^{CT_*}\|f_0\|^2_{L^2_2(\mathbb{R}^3)}.
\end{equation}
This completes the justification for the case $s \in (1/2, 1)$.

-Case $s = 1/2$:
Given $\eta \in (0, \frac{1}{2})$, and by taking $\alpha = \frac{1-8\eta}{3} \in \left(\frac{1}{6}, \frac{1}{3}\right)$, we have $(3\alpha - \frac{1}{2})^+ = \frac{1}{2} - \eta \in (0, \frac{1}{2})$. Reasoning along exactly the same lines as above, we can state the whole proof in this case $s = 1/2$ and obtain the Gevrey smoothing effect in the space $G_{2(1-\eta)}^3$.

-Case $0 < \alpha < s < 1/2$:
Combining the coercivity estimates (4.16)-(4.18), Lemma 4.3, Remarks 2.6 and 2.8, we can get, corresponding to (4.19),

$$
\frac{d}{dt} \|G_\delta f\|_{L^2}^2 + c_f \|G_\delta f\|_{H^s_2}^2 \leq C\|G_\delta f\|_{H^2_2}^2 + C\left(\|f\|_{L^1} + \|f\|_{L^1_2} + \|G_\delta f\|_{L^2_2}\right) \|G_\delta f\|_{H^2_2}^2.
$$

Applying the conservation laws (4.20) and interpolation inequality, we can obtain,

$$
\frac{d}{dt} \|G_\delta f\|_{L^2}^2 + \frac{c_f}{2} \|G_\delta f\|_{H^2_2}^2 \leq C\|G_\delta f\|_{L^2_2}^2 + C\|G_\delta f\|_{L^2_2}^{2+\alpha/2}.
$$

After a few calculations, by choosing $T_* \in (0, T_0]$ small enough and taking limit $\delta \to 0$, we get finally, for $t \in [0, T_*)$:

$$
\|e^{\alpha(t)}_L f\|_{L^\infty([0, T_*] ; L^2_2(\mathbb{R}^3))} \leq C_0^{-1} e^{CT_*} \|f_0\|_{L^2_2(\mathbb{R}^3)}^2,
$$

which leads to the conclusion in the case $0 < \alpha < s < 1/2$ and thus, completes the whole proof of Theorem 4.1.

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