Optimal dividends under Erlang(2) inter-dividend decision times with nonlinear surplus

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ABSTRACT
In the classic dividend surplus model, initial surplus are generally fixed and premiums or incomes increased linearly. In our research, we consider that the initial surplus and income in the Erlang(2) dividend model will also change over time. This model contains both the continuous-time and the discrete-time risk model as a limit and represents a certain type of bridge between them which still enables the explicit calculation of moments of total discounted dividend payments until ruin. In this paper, we adopt Statistical methods to convert the differential equations with variable coefficients to ordinary differential equation and further study the optimal periodic barrier strategies when the initial surplus is dynamic and the income is nonlinear under the condition of the inter-dividend decision times follow Erlang(2) distribution.

1. Introduction
In actuarial risk theory, the fact that stochastic processes are used to model surplus of a company and evaluate it's stability can be traced back to the early twentieth century. Because of the inverse relationship between probability of ruin and surplus, in order to minimize the ruin probability, someone accumulated the surplus to infinity. In response to this unrealistic feature, De Finetti (1957) introduced dividends to make the dynamic behaviour of the surplus process more realistic in 1957. Subsequently, many papers have adopted this criterion and developed it. Such as Avanzi and Thonhauser (2009) and Avanzi (2009) etc. Generally speaking, optimal dividend strategy is determined by maximizing the expected present value of dividends. But Gerber (1974) thought that may not logical. In many previous studies, optimal dividend strategies are generally of barrier or threshold types. But in fact, corporate dividends are usually paid at pre-determined time, which is introduced as periodic dividend strategies by Albrecher, Gerber, and Shiu (2011). This has attracted the attention of many scholars.

With the research going on, Asmussen, Avram, and Usabel (2002) proposed an ‘Erlangisation’ technique to solve finite-time ruin problems. While dividends can be paid and ruin can be observed, Albrecher, Cheung, and Thonhauser (2011) studied periodic dividend barrier strategies by that technique. Wei et al. (2012) studied the optimality of periodic barrier strategy under switched Brownian risk model. In addition, the expected present value of dividends at ruin time under a periodic barrier strategy in the dual model with diffusion and Erlang(n) inter-decision times is studied by Avanzi, Cheung, Wong, and Woo (2013). Under the conditions of the ruin is allowed to happen at anytime, Avanzi et al. (2013) introduced a dividend barrier strategy in which dividend decisions are only made periodically. Avanzi et al. (2014) studied the optimal strategy under a periodic dividend strategy in a dual model with diffusion, and pointed out that the first liquidation strategy under this strategy is the optimal strategy. Choi and Chueng (2014) obtained the present value of expected dividend before ruin under the strategy of dividend barrier in the classical Cramér-Lundberd risk model, and gave a numerical example of maximizing dividend. Schmidli (2015) extended the classical risk model to the condition that both earnings and costs are non-linear, and studied the Gerber-Shiu discount function. Avanzi, Tu, and Wong (2018) studied optimal dividend strategies and probability of ruin by adding a Brownian motion on the basis of Avanzi et al. (2013) under Erlang(2) inter-dividend decision times.

We mainly get the optimal dividend strategies through two steps. First of all, we can gain the relevant expected present value of dividends until ruin by proposing a candidate barrier or threshold strategies type. The next step inspects whether properties of the proposed solutions satisfy conditions of the verification lemma after
we derived it. When a Brownian motion is involved in the risk surplus model, the fact that the surplus less than zero will lead to ruin of the company. Avanzi et al. (2013) completed the first step described above by solving a system of integro-differential equations to discover implicit solutions of the dividend expected present value under a periodic barrier strategy with Erlang( n) decision times. It is regrettable that so far the explicit solutions of expected present value of dividends are only applied to periodic strategies with Erlang(1) inter-decisions times from Albrecher, Gerber, et al. (2011). Pérez and Yamazaki (2017) made a contribution to develop optimal periodic barrier strategies in a spectrally positive Lévy process. Matric, Badescu, and Stanford (2012) considered the absolute ruin problem in a risk model with debit and credit interest, to renewal and non-renewal structures. They solved the equation by setting up a random variable subject to a distribution with rational Laplace transform in Section 2 in Matric et al. (2012). Recently, we found that periodic dividend barrier strategies are globally optimal as soon as inter-dividend-decision times follow Erlang(2) distribution with rational Laplace transform in Section 2.1 in Avanzi et al. (2018). We obtain the jumps between states arrive according to a Poisson process \( \Lambda, \mathcal{F}_t, \{ \mathcal{F}_t \}, P \) is assumed at the same time. Dividends can be distributed periodically, at time points \( \{ T_k \}_{k=1}^{\infty} \). We also assume that the times between successive dividend decision times \( (T_k - T_{k-1}) \) is a sequence of independent and identically distributed Erlang( n) random variables with positive scale parameter \( \gamma \) possessing common density

\[
f(t) = \frac{\gamma^n t^{n-1} e^{-\gamma t}}{(n-1)!}, \quad t > 0,
\]

We can see random variable \( T_i \) as the sum of \( n \) i.i.d exponential variables with mean \( 1/\gamma \), which help us to get benefits from their memoryless properties. Then we can use Erlang( n) random variables to approximate deterministic the time horizons by ‘Erlangisation’ technique. We only focus on the case of \( n=2 \) in this article.

We obtain the jumps between states arrive according to a Poisson process \( \{ N_t \} \) with constant intensity \( \gamma \) and the set of times when the Markov chain enters state \( n \) represents the set of dividend decision times (see Section 2.1 in Avanzi et al. (2018) for a derivation).

The surplus after dividends are distributed from the surplus according to a periodic strategy at each dividend decision time is defined as

\[
U(t) = x \left( e^{\alpha t} + \frac{\mu}{\alpha} (e^{\alpha t} - 1) + \sigma W(t) - D(t) \right),
\]

where \( D(0) = 0 \), \( D(t), t \geq 0 \) is the total dividends process. Dividends paid at time \( t \) are denoted as \( D_t, t \geq 0 \). We can write the aggregate dividends process associated with an Erlang( n) periodic strategy as

\[
D(t) = \int_0^t \varphi \eta_n(s) dN_t(s),
\]

where the process \( \eta_n(t) \) is a auxiliary process defined in Avanzi et al. (2018). Assuming that the time until the next
dividend follows Erlang($i$) distribution, we can write the expected present value of dividends paid until ruin under a periodic strategy $\Theta$ with Erlang($n$) inter-decision times as the following conditional expectation

$$ J(x, i; \Theta) = \mathbb{E}_{x,i} \left[ \int_0^\infty e^{-\delta s} \varphi (s) \, dN_p(s) \right], \quad x \geq 0, i = 1, 2, \ldots, n - 1, n, $$

where $\delta$ is the discount factor which represents the time preference of investors. In addition, $\mathbb{E}_{x,i}$ is the conditional expectation with the initial surplus $X(0) = x$ and the initial state space $\epsilon(0) = i$.

The jump times associated with $\mathcal{F}, D_1, D_2, \ldots, D_n$ are stopping times with respect to $\{\mathcal{F}_t\}$. We define $D_n$ as $\{T_1, T_2, \ldots\}$ and satisfy

$$ 0 < T_1 < T_2 < \ldots \quad \text{a.s.} \quad (6) $$

Let $V(x_0, i)$ be the expected present value of dividends of the optimal periodic dividend strategy with Erlang($n$) inter-decision times.

$$ V(x, i) = \sup_{\Theta \in \mathcal{D}} J(x, i; \Theta), \quad i = 1, 2, \ldots, n, $$

where $\mathcal{D}$ represents all of the admissible periodic strategy. The optimal dividend strategy is defined as $\Theta^* = \varphi_{T_1}^*, \varphi_{T_2}^*, \varphi_{T_3}^*, \ldots$ for all $i = 1, 2, \ldots, n$.

$$ V(x, i) = J(x, i; \Theta^*) = \mathbb{E}_{x,i} \left[ \int_0^\infty e^{-\delta s} \varphi (s) \, dN_p(s) \right], \quad x \geq 0. \quad (8) $$

We define the ruin time $\tau$ as

$$ \tau = \inf \{t \geq 0 : X(t) = 0\}, \quad (9) $$

This means ruin is assumed to occur as soon the surplus is exhausted. Suppose that we follow a periodic barrier strategy (barrier level $b < 0$) with Erlang($n$) inter-decision times. Then we can expressed the probability of ultimate ruin as

$$ \psi_b(x, i) = \mathbb{P}_{x,i} [\tau < \infty], \quad i = 1, 2, 3, \ldots, n, \quad (10) $$

where $\mathbb{P}_{x,i}$ is the probability under the condition of the initial surplus is $X(0) = x$ and initial Erlang clock is $\epsilon(0) = i$.

Then we can derive that the ruin probability $\psi_b(x, 1) = \psi_b(x, 2) = \cdots = \psi_b(x, n) = 1$.

We can use the production to express the probability of never ruin as

$$ 1 - \psi_b(x, i) = \mathbb{P}_{x,i} \left[ \inf_{0 \leq s \leq T_1} X(s) > 0 \right | X > 0], \quad (11) $$

where $T_1, T_2, \ldots$ are the dividend decision times. We can build the following inequality as the surplus process modified starts at a level less than or equal to $b$ at each dividend decision.

$$ \mathbb{P}_{x,n} \left[ \inf_{T_k \leq s \leq T_{k+1}} X(s) > 0 \right | X(T_k > b)] := p, \quad k = 1, 2, 3, \ldots, n. \quad (12) $$

Obviously $p$ is strictly less than 1. Substituting (11) into (10), We have the following inequality

$$ 1 - \psi_b(x, i) \leq \mathbb{P}_{x,i} \left[ \inf_{0 \leq s \leq T_1} X(s) > 0 \right | X > 0] \lim_{k \to \infty} p^k = 0. \quad (13) $$

Therefore the equation $\psi_b(x, 1) = \psi_b(x, 2) = \cdots = \psi_b(x, n) = 1$ is proved.

3. Main results

Next we can construct a verification lemma that include the necessary conditions for an Erlang($n$) periodic optimal dividend strategy. There two features in the method. One is that dynamics of the surplus are not influenced by the changing state space and secondly if and only if the state space changes from 1 to $n$ dividends are paid.

Note that when $n = 2$ or $n > 2$, it is essentially same between the lemma and its proof. Hence we formulate it for any $n$.

See Section 3.1 in Avanzi et al. (2018), we get the following lemma

**Lemma 3.1.** Suppose that we follow a periodic dividend strategy $\{\xi_t\}$ where dividend decision times occur in the set $\{D_n\}$. There exist functions $\{H(x, j)\}_{j=1}^n \geq 0$ that are linear bounded and twice continuously differentiable a.e., and satisfy following conditions

$$ (1) \quad (B - \delta - \gamma)H(x, j) + \gamma H(x, j - 1) = 0, \quad j = 2, 3, \ldots, n, \quad (14) $$

where $B$ is the ceiling function and $\gamma$ is the dividend tax rate.
2. \((B - \delta - \gamma)H(x, 1) + \gamma \max_{0 \leq l < x}(l + H(x - l, n)) \leq 0\),
3. \(H(0, j) = 0\),
4. \(0 \leq H'(x, j) < \infty\),
5. \(H''(x, j) \leq 0\)

for \(j = 1, 2, \ldots, n\) and some constant \(l\). \(B\) is a new differential operator we constructed for the twice continuously differentiable function

\[Bf(x) = (\omega x + \mu)f'(x) + \frac{\sigma^2}{2}f''(x).\]  

we obtain

\[H(x, j) \geq V(x, j).\]  

Then we use above verification lemma to determine sufficient and necessary conditions for the global optimality of liquidation-at-first-opportunity strategies with \(Erlang(2)\) inter-decision times. The expected present value of dividends until ruin is denoted as \(F(x, i)\), where \(i = 1, 2\). We satisfy the following system of differential equations about \(F(x, 1)\) and \(F(x, 2)\) (see Avanzi et al., 2013, 2018, for a derivation by Taylor expansion)

\[\frac{\sigma^2}{2}F''(x, 2) + (\omega x + \mu)F'(x, 2) - (\gamma + \delta)F(x, 2)\]
\[+ \gamma F(x, 1) = 0.\]  

Let \(s_x = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x - \mu)^2}{2\sigma^2}} dx\)

we obtain a general solution for \(F(x, 1)\), but it include indefinite integrals of non-elementary functions

\[F(x, 1) = \left[ c_1 e^{s_x} \left[ \frac{(\omega x + \mu)^2}{4\sigma^2} - \frac{2(\gamma + \delta) - \alpha}{\sigma^2} \right] + c_2 e^{-s_x} \left[ \frac{(\omega x + \mu)^2}{4\sigma^2} - \frac{2(\gamma + \delta) - \alpha}{\sigma^2} \right] \right.\]
\[+ c_3(x) e^{s_x} \left[ \frac{(\omega x + \mu)^2}{4\sigma^2} - \frac{2(\gamma + \delta) - \alpha}{\sigma^2} \right] \left. + c_4(x) e^{-s_x} \left[ \frac{(\omega x + \mu)^2}{4\sigma^2} - \frac{2(\gamma + \delta) - \alpha}{\sigma^2} \right] \right]^{-1} \int_{-\infty}^{x} \frac{1}{\sigma^2} e^{-\frac{(x - \mu)^2}{2\sigma^2}} dx,\]  

where \(c_1, c_2\) is constant, \(c_3(x)\) and \(c_4(x)\) are functions which include indefinite integrals of non-elementary functions as mentioned above

\[c_3(x) =\]
\[\int_{-\infty}^{x} \frac{1}{\gamma + \delta} \left[ \frac{(\omega x + \mu)^2}{4\sigma^2} - \frac{2(\gamma + \delta) - \alpha}{\sigma^2} \right] \]  

\[c_4(x) = -\int_{-\infty}^{x} \frac{1}{\gamma + \delta} \left[ \frac{(\omega x + \mu)^2}{4\sigma^2} - \frac{2(\gamma + \delta) - \alpha}{\sigma^2} \right] \]  

We can’t get the explicit form of \(c_3(x)\) and \(c_4(x)\), this will bring enormous difficulties to the next work. Hence we adopt Statistical methods to convert the differential equations with variable coefficients to ordinary differential equation by drawing on the ideas of in literature (Matric et al., 2012).

We suppose that the initial surplus \(x\) is a random variables following a normal distribution with mean \(\mu\) and finite variance \(\sigma^2\). Let the coefficient with independent variables \((\omega x + \mu)\) → \((\omega x + \mu)\), we obtain the following system of ordinary differential equations by (16) and (17)

\[\frac{\sigma^2}{2}F''(x, 2) + (\omega + \mu)F'(x, 2) - (\gamma + \delta)F(x, 2)\]
\[+ \gamma F(x, 1) = 0.\]  

Assuming \(F(0, 1) = F(0, 2) = 0\) and the two functions approach some linear functions as \(x \to \infty\), then we gain an explicit solution for \(F(x, 1)\)

\[F(x, 1) = \left[ \frac{\gamma}{\gamma + \delta} \left( \frac{\omega + \mu}{\gamma + \delta} \right)^x + x \right] + \frac{(\omega + \mu)^2}{\gamma + \delta} e^{\gamma \xi x},\]  

where \(r_\gamma > 0\) and \(s_\gamma < 0\) are roots of the following equation and the remainder term include \(r_\gamma\) is cut off because it does not converge

\[\frac{\sigma^2}{2} \xi^2 + (\omega + \mu) \xi - (\gamma + \delta) = 0,\]  

then we can obtain a general solution for \(F(x, 2)\) by substituting (19) to (17)

\[F(x, 2) = \left( \frac{\gamma}{\gamma + \delta} \right)^2 \left[ \frac{(\omega + \mu)^2}{\gamma + \delta} + x \right] - A e^{s_\gamma x} + B x e^{s_\gamma x},\]
using the initial condition \( F(0, 2) = 0 \) to figure out \( A \)

\[
A = -\frac{2(\alpha \rho + \mu)}{\gamma + \delta} \left( \frac{\gamma}{\gamma + \delta} \right)^2,
\]

(26)

we obtain the following equation by substituting (23) to (21)

\[
F(x, 1) = \frac{\gamma}{\gamma + \delta} \left( \frac{(\alpha \rho + \mu)}{\gamma + \delta} + x \right)
- \frac{(\alpha \rho + \mu)}{\gamma} \frac{\sigma^2 s_y}{s_y - r_y} B e^{r_x}. \tag{27}
\]

Comparing (27) with (23), we can determine explicit coefficient \( B \)

\[
B = -\frac{s_y r_y}{s_y - r_y} \frac{(\alpha \rho + \mu)}{\gamma + \delta} \left( \frac{\gamma}{\gamma + \delta} \right)^2.
\tag{28}
\]

Obviously both function \( F(x, 1) \) and \( F(x, 2) \) are increasing and concave.

An optimal liquidation-at-first-opportunity strategy should satisfy the following inequality according to the condition (2) of the above verification lemma

\[
(B - \delta - \gamma)F(x, 1) + \gamma(l + F(x - l, 2)) \leq 0,
\]

(29)

we substitute (23) into the above inequality yields

\[
I + F(x - l, 2) \leq \frac{F(x - l, 2) - F(0, 2)}{x - l} \leq 1,
\]

(30)

for \( l \in [0, x] \). We can obtain that \( F'(0, 2) \leq 1 \) and \( F'(x, 2) \leq 1 \), then the optimal periodic strategy with Erlang(2) inter-decision time is the liquidation-at-first-opportunity strategy.

Next we focus on periodic barrier strategies with Erlang(2) inter-decision times, where \( b^* > 0 \) is the optimal barrier. The expected present value of dividends until ruin are denote as \( G_i(x, b^*) \) for \( i = 1, 2 \), after that we have

\[
G(x, 2; b^*) = \begin{cases} G_L(x, 2; b^*) , & x \in [0, b^*) \\ G_U(x, 2; b^*) , & x \in [b^*, \infty) \end{cases}
\]

(31)

and

\[
G(x, 1; b^*) = \begin{cases} G_L(x, 1; b^*) , & x \in [0, b^*) \\ G_U(x, 1; b^*) , & x \in [b^*, \infty) \end{cases}
\]

(32)

Similarly \( G_L(x, 2; b^*) \) and \( G_U(x, 1; b^*) \) satisfy the following system of differential equations for \( x \in [0, b^*) \)

\[
\frac{\sigma^2}{2} G''_L(x, 2; b^*) + (\alpha \rho + \mu) G_L(x, 2; b^*)
- (\gamma + \delta) G_L(x, 2; b^*) + \gamma G_L(x, 1; b^*) = 0,
\]

(33)

\[
\frac{\sigma^2}{2} G''_U(x, 1; b^*) + (\alpha \rho + \mu) G_U(x, 1; b^*)
- (\gamma + \delta) G_U(x, 1; b^*) + \gamma G_L(x, 2; b^*) = 0,
\]

(34)

where \( G_L(0, 2; b^*) = G_U(0, 1; b^*) = 0 \). Afterwards we substitute (34) into (33) and get the following fourth-order homogeneous differential equation for \( G_L(x, 2; b^*) \)

\[
(B - \delta - \gamma)^2 G_L(x, 2; b^*) - \gamma^2 G_U(x, 2; b^*) = 0,
\]

(35)

the characteristic equation is

\[
\left( \frac{\sigma^2}{2} \xi^2 + (\alpha \rho + \mu) \xi - \delta \right) \left( \frac{\sigma^2}{2} \xi^2 + (\alpha \rho + \mu) \xi - (2\gamma + \delta) \right) = 0.
\]

(36)

Hence the general solution for \( G_L(x, 2; b^*) \) is

\[
G_L(x, 2; b^*) = A_r e^{r_0 x} + A_s e^{s_0 x} + B_r e^{r_2 x} + B_s e^{s_2 x}, \quad x \in [0, b^*),
\]

(37)

where \( r_0 > 0, s_0 > 0, r_2, > 0, s_2, > 0 \) are the roots of the characteristic equation, \( A_r, A_s, B_r, B_s \) are constants.

Then we substitute (37) into (34) and obtain

\[
G_L(x, 2; b^*) = -\frac{1}{\gamma} (B - \delta - \gamma)(A_r e^{r_0 x} + A_s e^{s_0 x} + B_r e^{r_2 x} + B_s e^{s_2 x})
+ B_r e^{r_2 x} + B_s e^{s_2 x} = A_r e^{r_0 x} + A_s e^{s_0 x} - B_r e^{r_2 x} - B_s e^{s_2 x}.
\]

(38)

Next we can write the general solution for \( G_L(x, 2; b^*) \) and \( G_U(x, 1; b^*) \) as

\[
G_L(x, 2; b^*) = A \cdot h_0(x) + B \cdot h_2 y(x),
\]

(39)

\[
G_U(x, 1; b^*) = A \cdot h_0(x) - B \cdot h_2 y(x),
\]

(40)

where \( x \in [0, b^*), \) and \( A, B \) are constants, \( h_0(x) = e^{r_0 x} - e^{s_0 x}, h_2 y(x) = e^{r_2 x} - e^{s_2 x} \).

Analogous to above equations, we have the following system of differential equations about \( G_L(x, 2; b^*) \) and \( G_U(x, 1; b^*) \) for \( x \in [0, b^*), \)

\[
\frac{\sigma^2}{2} G''_L(x, 2; b^*) + (\alpha \rho + \mu) G_L(x, 2; b^*)
- (\gamma + \delta) G_L(x, 2; b^*) + \gamma G_U(x, 1; b^*) = 0,
\]

(41)

\[
\frac{\sigma^2}{2} G''_U(x, 1; b^*) + (\alpha \rho + \mu) G_U(x, 1; b^*)
- (\gamma + \delta) G_U(x, 1; b^*) + \gamma (x - b^* + G_L(b^*, 2; b^*)) = 0.
\]

(42)
Using the results and algorithm in the literature (Avanzi et al., 2013, 2018), we have

\[
G_U(x,2;b^*) = \left( \frac{\gamma}{\gamma + \delta} \right)^2 \left[ 2(\mu + \alpha \rho) + x - b^* \right]
+ G_L(b^*,2;b^*) + (C + D(x - b^*)) \, e^{\gamma(x-b^*)},
\]  
(43)

\[
G_U(x,1;b^*) = \left( \frac{\gamma}{\gamma + \delta} \right)^2 \left[ \frac{\alpha + \mu}{\gamma + \delta} + x - b^* \right]
+ G_L(b^*,2;b^*) + D \, \frac{S_y - \gamma + \delta}{S_y \gamma} \, e^{\gamma(x-b^*)},
\]  
(44)

when the optimal barrier \( b^* \) is applied, \( G'_u(b^*,2;b^*) = G'_U(b^*,2;b^*) = 1, \); \( G'_u(b^*,2;b^*) = G'_U(b^*,2;b^*) = 1, \); \( i = 1,2 \), we have the following system of differential equations

\[
A \cdot h'_0(b^*) + B \cdot h'_2(b^*) = 2D_S \gamma + C S^2 \gamma
\]

\[
A \cdot h'_0(b^*) \gamma - B \cdot h'_2(b^*) = D_S \gamma - r \gamma \gamma + \delta S \gamma
\]

\[
A \cdot h'_0(b^*) + B \cdot h'_2(b^*) = 1
\]

\[
\left( \frac{\gamma}{\gamma + \delta} \right)^2 + D + C \gamma = 1.
\]  
(45)

Using the similar method, we figure out the explicit solution for \( A, B, C \) and \( D \), which expressed as the following forms by \( b^* \)

\[
A = \frac{\mathcal{H}_{2y}(b^*)}{\mathcal{H}_{0y}(b^*)},
\]

\[
B = \frac{\mathcal{H}_{2y}(b^*) + \mathcal{H}_{2y}(b^*) \mathcal{H}_{0y}(b^*)}{\mathcal{H}_{0y}(b^*)},
\]

\[
C = \frac{1}{2} \frac{h'_0(b^*) h'_2(b^*) + h'_0(b^*) h'_2(b^*) + h'_0(b^*) h'_2(b^*)}{\mathcal{H}_{0y}(b^*)} - \frac{r \gamma}{2} \, \frac{h'_0(b^*) h'_2(b^*) + h'_0(b^*) h'_2(b^*)}{\mathcal{H}_{0y}(b^*)}
\]

\[
D = \frac{r \gamma}{s \gamma - \gamma + \delta} \, \frac{h'_0(b^*) h'_2(b^*) + h'_0(b^*) h'_2(b^*)}{\mathcal{H}_{0y}(b^*)},
\]  
(46)

\[
\mathcal{H}_0(\xi) = h'_0(\xi) - s \gamma \left( 1 - \frac{\gamma}{\gamma + \delta} \right) h_0(\xi),
\]  
(47)

\[
\mathcal{H}_{2y}(\xi) = h'_2(\xi) - s \gamma \left( 1 + \frac{\gamma}{\gamma + \delta} \right) h_2(\xi),
\]  
(48)

then we obtain an implicit equation to solve for the optimal periodic barrier \( b^* \)

\[
\frac{h''_0(b^*)}{\mathcal{H}_{0y}(b^*)} - \frac{h''_{2y}(b^*)}{\mathcal{H}_{2y}(b^*)} = \frac{2}{\gamma \gamma} \, \frac{s \gamma - \gamma + \delta}{\gamma}.
\]  
(49)

According to the conclusion of the (Avanzi et al., 2018), we further the expected present value of optimal dividends with barrier level \( b^* \) until ruin as

\[
G(x,1;b^*) = \begin{cases} 
G_U(x,1;b^*) = \frac{h'_0(b^*) h'_2(b^*) \gamma + h'_0(b^*) h'_2(b^*) \gamma}{\mathcal{H}_{0y}(b^*) + \mathcal{H}_{2y}(b^*) \gamma}, & x \in [0,b^*] \\
+ D \mathcal{S} \gamma, & x \in [b^*,\infty] 
\end{cases}
\]  
(50)

\[
G(x,2;b^*) = \begin{cases} 
G_U(x,2;b^*) = \frac{h'_0(b^*) h'_2(b^*) \gamma + h'_0(b^*) h'_2(b^*) \gamma}{\mathcal{H}_{0y}(b^*) + \mathcal{H}_{2y}(b^*) \gamma}, & x \in [0,b^*] \\
+ D \mathcal{S} \gamma, & x \in [b^*,\infty] 
\end{cases}
\]  
(51)

Since

\[
E_{x_j}[G(x(t \wedge \tau), e(t \wedge \tau), b^*)] = E_{x_j} \left[ \int_0^t \sigma e^{-\delta s} G(X(S,1), b^*) \eta_1(s) \, dW(s) \right]
+ E_{x_j} \left[ \int_0^t \sigma e^{-\delta s} G(X(S,2), b^*) \eta_2(s) \, dW(s) \right]
+ E_{x_j} \left[ \int_0^t e^{-\delta s} G(X(S,1), b^*) \right]
-G(X(S,1), b^*) \eta_1(s) \, d\tilde{N}_y(s),
\]

\[
+ E_{x_j} \left[ \int_0^t e^{-\delta s} G(X(S,1), b^*) \eta_2(s) \, d\tilde{N}_y(s) \right],
\]

\[
E_{x_j} \left[ \int_0^t e^{-\delta s} \xi_s \eta_2(s) \, d\tilde{N}_y(s) \right],
\]  
(52)

then we let

\[
M_1(t) = \int_0^t \sigma e^{-\delta s} G(X(S,1), 1; b^*) \eta_1(s) \, dW(s),
\]  
(53)

\[
M_2(t) = \int_0^t \sigma e^{-\delta s} G(X(S,2), 2; b^*) \eta_2(s) \, dW(s),
\]  
(54)

\[
M_3(t) = \int_0^t e^{-\delta s} G(X(S,1), 1; b^*)
-G(X(S,1), 2; b^*) \eta_2(s) \, d\tilde{N}_y(s),
\]

\[
M_4(t) = \int_0^t e^{-\delta s} \xi_s \eta_2(s) \, d\tilde{N}_y(s),
\]  
(56)
we can learn that the 4 stochastic integrals $M_1(t), M_2(t), M_3(t), M_4(t)$ are uniformly integrable martingales and obtain

$$E_{x,i}[M_i(t)] = 0, \quad i = 1, 2, 3, 4,$$

so that

$$E_{x,i} [e^{-\delta(X(t) \land \tau)} G(X(t \land \tau), \epsilon(t \land \tau); b^*)] = G(x, i; b^*)$$

$$- E_{x,i} \left[ \int_0^{t \land \tau} e^{-\delta s \xi_2(s)} d\tau \right].$$

According to that the probability of ultimate ruin is 1, and $e^{-\delta(X(t \land \tau))} G(X(t \land \tau), \epsilon(t \land \tau); b^*) < R, E[R] < \infty$, where $R$ is an integrable random variable. Then we use the dominated convergence theorem to get the limit of (61) is

$$\lim_{t \to \infty} E_{x,i} = [e^{-\delta(X(t \land \tau))} G(X(t \land \tau), \epsilon(t \land \tau); b^*)] = 0. \quad (59)$$

Next we apply the monotone convergence theorem and assume that $\tau$ is finite, then

$$\lim_{t \to \infty} E_{x,i} = [e^{-\delta(X(t \land \tau))} G(X(t \land \tau), \epsilon(t \land \tau); b^*)] = G(x, i; b^*)$$

$$- \lim_{t \to \infty} E_{x,i} \left[ \int_0^{t \land \tau} e^{-\delta s \xi_2(s)} d\tau \right].$$

(60)

It is obviously that

$$G(x, i; b^*) = E_{x,i} \left[ \int_0^{t \land \tau} e^{-\delta s \xi_2(s)} d\tau \right]$$

$$= V(x, i), \quad i = 1, 2.$$

Therefore the expected present value of dividends of the optimal periodic barrier strategies with Erlang(2) inter-decision times of the associated functions $G(x, 1; b^*)$ and $G(x, 1; b^*)$.

4. An illustrative example

In this section, a numerical simulation example is given to show the effectiveness of the derived results. We illustrate the existence of the optimal barriers $b^*$ using (49) while changing the following six parameters: risk-free interest rate $\alpha$, initial surplus' mean $\mu$, drift $\mu$, volatility $\sigma$, financial impatience $\delta$ and dividend frequency $\gamma$. In the six figures, we change one parameter while keeping others constant. For example in Figure 1, the rate parameter changes from 0 to 0.5 while keeping other parameters constant ($\rho = 10, \mu = 1, \sigma = 1, \delta = 0.1, \gamma = 0.1$). With the increase of interest rate $\alpha$, the optimal barrier $b^*$ decreases gradually but slowly. Figure 2, we find the influence of $\rho$ and $\alpha$ on optimal barrier $b^*$ shows the same trend. Figure 3, when the drift $\mu$ is small, we can observe that the optimal barrier start with a higher level, the difference between this and literature (Avanzi et al., 2018) is that $\mu$ just play a minor role relative to it in the literature (Avanzi et al., 2018). As the drift increases, the optimal barrier shows
Figure 5. $\alpha = 0.05, \rho = 10, \mu = 1, \sigma = 1, \delta = (0, 0.4), \gamma = 0.1$.

Figure 6. $\alpha = 0.05, \rho = 10, \mu = 1, \sigma = 1, \delta = 0.1, \gamma = (0, 0.5)$.

a trend of increasing first and then decreasing, and the fluctuation range is small. This means we can allow for more dividends to be paid at each decision time while the initial surplus $x$ generate revenue even if the drift is small. Figure 4, when volatility is very low, the optimal barrier is very small as there is not much risk for the surplus process to ruin. As volatility increases, the optimal barrier increases first and then decreases. Figure 5, with the financial impatience $\delta$ increasing, the optimal barrier decrease and more dividends are paid at each decision time. In Figure 6, as the dividend frequency $\gamma$ increases gradually, the inter-decision times gets smaller, the optimal barrier increases but means that dividend payments will be more frequent.

5. Conclusions

In summary, we use the ‘Erlangisation’ technique to study a periodic barrier strategy which is still optimal when inter-dividend decision times are Erlang(2) distributed in a Brownian risk model with nonlinear surplus in this paper. It is an extension of the idea in Avanzi et al. (2018), which is a special case of our extension when $\alpha = 0$. Finally, a numerical example has been provided to show the optimality of periodic barrier strategies with nonlinear surplus.

Compared with Avanzi et al. (2018), $\alpha$, $\rho$ and $\mu$ are equivalent to the separation of the ‘$\mu$’ in Avanzi et al. (2018). And several other factors $\delta, \gamma$ and $\sigma$ have the similar effect on $b^*$ although the data we selected are different from Avanzi et al. (2018).

The non-linear surplus considered in this paper is approximated by transforming the variable coefficient equation into the constant coefficient equation through the idea of statistical distribution. If it can be solved under the condition of variable coefficient, the result will be more accurate.

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