Propagation of time-truncated Airy-type pulses in media with quadratic and cubic dispersion

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Abstract

In this paper, we describe analytically the propagation of Airy-type pulses truncated by a finite-time aperture when second and third order dispersion effects are considered. The mathematical method presented here, based on the superposition of exponentially truncated Airy pulses, is very effective, allowing us to avoid the use of time-consuming numerical simulations. We analyze the behavior of the time truncated Ideal-Airy pulse and also the interesting case of a time truncated Airy pulse with a “defect” in its initial profile, which reveals the self-healing property of this kind of pulse solution.

1 Introduction

For a few years a new kind of localized wave solution has attracted the attention of researchers. It is the non-dispersive Airy pulse \([1]\). Contrary to other types of non-dispersive pulses, the Airy one does not require a spectral space-time coupling to become resistant to the dispersive effects. Instead, it just uses a cubic phase frequency spectrum, which greatly facilitates its experimental generation.

An ideal Airy pulse is immune to dispersion effects of second order, but it possesses infinite energy making its experimental generation impossible. However, it is possible to obtain a finite-energy version of the ideal Airy pulse by modulating it with a time exponential function on the initial plane \(z = 0\) \([2,3]\). The analysis of a finite-energy Airy pulse considering the effects of second and third order dispersion was first made in \([3]\) and subsequently in \([4,5]\). Additional work on finite-energy Airy beams and pulses is provided in Refs. \([7,11]\).

All the previous analyses about finite-energy Airy pulses have considered exponential or Gaussian (initial) time apodization, but until now no one has studied the case of Airy-type pulses with truncation in time, that is, a temporal apodization made with a rectangular function. Such truncation, besides providing finite energy solutions, enables the resulting pulse to present, up to a certain distance, the very same characteristics of the non-truncated version. Other types of apodizations, such as the exponential and Gaussian, are not so efficient in this sense.

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In this paper, we present an analytical description of Airy-type pulses truncated in time when second and third order dispersion effects are considered. The mathematical method used here is an extension of one used for describing spatially truncated Airy beams \[12\].

We analyze the behavior of the time truncated Ideal-Airy pulse and also the interesting case of a time truncated Airy pulse with a “defect” in its initial profile, which reveals the self-healing property of this kind of pulse solution.

2 Theoretical description of Airy-type pulses truncated in time

When quadratic and cubic dispersion effects are predominant in the evolution of a pulse in a linear and homogeneous material medium, the differential equation describing the pulse envelope propagation can be written as \[13\]

\[
\frac{i}{\partial z} U = \frac{\beta_2}{2} \frac{\partial^2 U}{\partial T^2} + \frac{i \beta_3}{6} \frac{\partial^3 U}{\partial T^3}
\]

where \(T\) is the retarded time in a frame reference moving with the group velocity \(T = t - z/v_g\), and \(\beta_2\) and \(\beta_3\) represent the second and third-order dispersion parameters, respectively, both depending on the central frequency of the pulse.

Let us define \(b = -\beta_3/(2\beta_2 T_0)\) and the new normalized variables \(s = T/T_0\) and \(Z = z\beta_2/T_0^2\), where \(T_0\) is a constant that will be related with the time width of the initial pulse.

Now, by considering an initial temporal pulse profile given by an Airy function apodized by an exponential, i.e,

\[
U_n(s, Z = 0) = \text{Ai} (s) \exp (a_n s)
\]

with \(\text{Re}(a_n) > 0\), we can obtain, by solving eq. (1), the finite energy Airy pulse given by

\[
U_n(s, Z) = \frac{1}{(1 + bZ)^{1/3}} \times \exp \left[ \frac{6a_n}{12 (1 + bZ)^2} \left( 2s - Z^2 \right) + i Z \left( -6a_n^2 - 6s + Z^2 \right) \right] \\
\times \exp \left[ \frac{b}{12 (1 + bZ)^2} \left( 4Za_n \left( -a_n^2 + 3s + a_n^2 bZ \right) + i 6Z^2 \left( a_n^2 - s \right) \right) \right] \\
\times \text{Ai} \left[ \frac{s - Z^2 + i a_n Z}{(1 + bZ)^{4/3}} + bZ \left( s - a_n^2 \right) \right]
\]

which was first obtained and analyzed in detail by Besieris et al. in \[3\]. Briefly, when \(a_n = 0\) (infinite energy), \(\beta_3 = 0\) and \(\beta_2 \neq 0\), we have the Ideal-Airy pulse, which is immune to the dispersion effects. In the case \(a_n = 0\), \(\beta_3 \neq 0\) and \(\beta_2 \neq 0\), the Ideal-Airy pulse can present resistance to the dispersion effects for long (finite) distances.

Now, when \(\text{Re}(a_n) > 0\) the corresponding pulse given in eq. (3) possesses finite energy content and it is dispersion resistant for long distances when \(\text{Re}(a_n) >> 1/T_0\). However, even in these cases, the entire pulse starts to be destorted from the beginning and this occurs because with the exponential apodization at \(z = 0\), all time sidelobes are affected (damped) and, as it is well known, the latter are responsible for the pulse reconstruction.
Based on these observations, we can envisage that a time apodization at $z = 0$ made by a rectangular function (i.e., a double step function) will preserve, within it, exactly the same initial field structure of the non-apodized pulse. As a result, we can expect that the resulting pulse will maintain the very same characteristics of its non-truncated version if the time width of the truncation is much greater than $T_0$. Actually, in the self-healing process, the inner lobes (i.e., those closer to the main peak) are fed by the outer ones. The latter are consumed until only the main peak remains, which then suffers distortions due to the dispersion effects. At this point, the pulse will have reached its maximum distance of resistance from the dispersion effects (depth of field).

Next, we are going to present a mathematical method capable of describing Airy-type pulses apodized by a time rectangular function i.e., truncated in time, propagating in a linear medium characterized by both quadratic and cubic dispersion. This approach is a variant/extension of one developed to deal with spatially truncated Airy beams \[12\], where the equivalent to the third order term in eq.(4) was not considered.

Mathematically, we wish to solve (approximately) eq.(1), when the following initial pulse profile is considered at $z = 0$

$$ F(s) = \text{Ai}(s) m(s) [u(s + S) - u(s - S)] \quad (4) $$

Here, $m(s)$ is an arbitrary function modulating the Airy function and the time truncation is represented by the difference between the Heaviside unit step functions $u(s + S)$ and $u(s - S)$.

We start by considering as a solution to eq.(1) a superposition of pulses given by

$$ U(s, Z) = \sum_{n=\pm\infty} B_n U_n(s, Z) \quad , \quad (5) $$

where the pulses $U_n$ are given by \[3\], with $B_n$ and $a_n$ complex constants yet unknown.

The pulse solution (5) at $z = 0$ is written as

$$ U(s, 0) = \sum_{n=\pm\infty} B_n U_n(s, 0) = \sum_{n=\pm\infty} B_n \text{Ai}(s) \exp(a_n s) \quad (6) $$

Our problem is to determine the values of $B_n$ and $a_n$ so that our proposed solution at $z = 0$ is (approximately) equal to the desired truncated Airy-type pattern given by eq.(4). Once this is done, the resulting pulse propagating in the second and third-order dispersion medium is given by eq.(5).

Let us make the following choice:

$$ a_n = a_R + \frac{2\pi}{L} n \quad (7) $$

where $a_R$ and $L$ are positive constants. By using (7) in (6), we get

$$ U(s, 0) = \text{Ai}(s) \exp(a_R s) \sum_{n=\pm\infty} B_n \exp\left(\frac{2\pi}{L} ns\right) \quad (8) $$

The Fourier series appearing in eq.(8) is suggestive. In the case we define

$$ \Lambda(s) = \sum_{n=\pm\infty} B_n \exp\left(\frac{2\pi}{L} ns\right) \quad , \quad (9) $$
with $S < L/2$, and the coefficients $B_n$ as

$$B_n = \frac{1}{L} \int_{-S}^{S} m(s) \exp(-a_R s) \exp \left( -i \frac{2 \pi}{L} ns \right) ds,$$  

(10)

the series (9) will represent, within the domain $-L/2 \leq s \leq L/2$, the following function

$$\Lambda(s) = \begin{cases} 
  m(s) \exp(-a_R s) & \text{for } -S \leq s \leq S \\
  0 & \text{for } S < |s| \leq L/2 
\end{cases}$$  

(11)

In this way, by using (8,9,10,11) and with appropriate values for $a_R$ and $L$, we can get the following result:

$$U(s, Z = 0) = \begin{cases} 
  \text{Ai}(s) m(s) & \text{for } |s| \leq S \\
  \text{Ai}(s) \exp(a_R s) \Lambda(s) \approx 0 & \text{for } |s| > L/2 
\end{cases} \approx F(s)$$  

(12)

The suitable values for $L$ and $a_R$ serve to guarantee the result given above when $|s| > L/2$. Actually, as $\Lambda(s)$ is the Fourier series with period $L$ representing the function given in eq.(11) and since $L/2 > S$, for appropriate choices of $L$ and $a_R$ we have that $\text{Ai}(s) \exp(a_R s) \Lambda(s) \approx 0$ for $|s| > L/2$ due to the behavior of the functions $\text{Ai}(s)$ and $\exp(a_R s)$ for positive and negative values of $s$, respectively. A good criterion is to choose values of $a_R$ and $L$ such that $\exp(a_R L/2) >> \text{Max}[\Lambda(s)]$, with $\Lambda(s)$ given in Eq. (11).

So, we have achieved our goal: The solution describing the propagation of an Airy-type pulse truncated in time according to eq.(12) in the presence of quadratic and cubic dispersion is given by eq.(5), with $a_n$ and $B_n$ given by eqs.(7,10), $a_R$ and $L$ being appropriately chosen according the above criterion.

We can also calculate the initial temporal spectrum of the time-truncated Airy-type pulse. Fourier transformation of (6) yields

$$\tilde{U}(0, \Omega) = \sum_{n=-\infty}^{\infty} B_n \exp(-a_n \Omega^2) \exp \left( \frac{i}{3} \left( \Omega^3 - 3 a_n^2 \Omega - ia_n^3 \right) \right).$$  

(13)

### Examples

Here, we are going to apply the method considering two situations where the material medium is fused silica, the central wavelength $\lambda_0 = 1550$nm (so $\beta_2 = -27.909$ps$^2$/km and $\beta_3 = 0.151$ps$^3$/km), $T_0 = 10$ps, $a_R = 0.1, S = 40$ and $L = 3S$. Of course a finite number of $2N + 1$ terms must be used in the summation of our solution given by eq.(5), corresponding to $-N \leq n \leq N$. Here we use $N = 80$.

1. Ideal Airy pulse truncated in time

Let us consider the case of an Ideal-Airy pulse with initial main peak of $T_0 = 10$ps, truncated by a finite time aperture of width forty times greater than $T_0$, i.e., $S = 40$. In this case we have $m(s) = 1$ and the pulse at $z = 0$ is given by eq.(12), where the coefficients $a_n$ and $B_n$ are defined by the equations (7) and (10), respectively. As already has been mentioned, many sets of values for $a_R$ and $L$ can yield good results and here we have chosen $L = 3S$ and $a_R = 0.1$. The resulting pulse is given by eq.(5).

Figure 1a shows the pulse intensity at $z = 0$ obtained from equation (6), and Figure 1b shows its frequency spectrum given by eq.(13).

*Of course there are many sets of values of $a_R$ and $L$ that satisfy this criterion, yielding a good result.*
Figure 1: (a) Field intensity, at $z = 0$, of the Ideal-Airy pulse truncated by a finite-time aperture according to example 1; (b) Frequency spectrum of the truncated Ideal Airy pulse.

Figure 2: (a) Temporal evolution of the time-truncated Ideal Airy pulse intensity of example 1 at different propagation distances; (b) The same pulse evolution through an orthogonal projection on the plane $(T/T_0, z)$.
Figure 3: Peak intensity evolution in range when $S = T/T_0 = 40$ and 150 respectively.

The temporal evolution of the resulting pulse at different distances is shown in Figure 2a. We can see that with this kind of time apodization the inner lobes, in particular the main one, are fed by the outer ones and maintain their amplitude and shape similarly to those of the Ideal-Airy pulse. This process continues until the distance (field depth) where only the main peak remains, which then is distorted by dispersion effects. We should note that at that distance the pulse has a shape that is very different from the initial one, given that there are no longer any sidelobes left, but just the main peak with the same time width $T_0$.

For comparison, a Gaussian pulse with the same initial time and central wavelength, would have a dispersion length $L_D = T_0^2/|\beta|^2 \approx 3.6\text{km}$, while the field depth of the present truncated Ideal Airy pulse is approximately 35km.

Figure 2b shows the same pulse evolution through an orthogonal projection on the plane $(T/T_0, z)$, where the accelerated character of the pulse is evident.

It should be clear that the greater the time truncation width, the greater the depth of field of the resulting pulse. Figures 3a and 3b show the peak intensity evolution along the Z-direction when $S = T/T_0 = 40$ and 150, respectively. Of course, a wider time width requires a larger amount of energy.

Many other Airy-type pulses truncated in time can be described through the present method just by choosing the appropriate modulation function $m(s)$ in eq. (12).

2. Self reconstruction of a defective Airy pulse truncated in time

Here, we are going to present the analytical description of the evolution of a time truncated Airy pulse with a “defect” in its initial profile. This situation can be easily described by our method, since the defect in question can be inserted by the function $m(s)$ in the initial pulse profile, eq. (12). For instance, let us consider the Truncated Ideal Airy pulse of the previous example with a missing piece of its initial field. This defect can be represented by choosing

$$m(s) = \begin{cases} 0 & \text{for } -S_2 \leq s \leq -S_1 \\ 1 & \text{elsewhere} \end{cases} \quad (14)$$

1In the case of $S = 150$ we use $a_R = 0.01$. 
Figure 4: (a) Field intensity, at $z = 0$, of a defective Airy pulse truncated by a finite-time aperture; (b) Frequency spectrum of the truncated defective Airy pulse.

Figure 5: (a) Temporal evolution revealing the self-healing property of a defective Airy pulse truncated by a finite-time aperture; (b) The same pulse evolution through an orthogonal projection on the plane $T/T_0, z$. 
If this “gap” in the pulse profile is considerably shorter than the truncation time width, the Airy-type pulse is able to self-regenerate. To confirm this important property we set $S_1 = 5$ and $S_2 = 9$, such that the defect occupies 10% of the initial pulse.

This defect can be clearly seen in Fig.4a, which shows the initial ($z = 0$) pulse intensity, obtained from eq.(6). Figure 4b shows the correspondent frequency spectrum, given by eq.(13).

The temporal evolution of the resulting pulse at different distances is shown in Fig.4a, where the self-healing property is revealed. The main part of the pulse (close to the main peak) is severely affected in the range $10\,\text{km} \lesssim z \lesssim 25\,\text{km}$, after which the main peak is regenerated, lasting until $z \approx 35\,\text{km}$, when it gets to be distorted by dispersion effects.

Figure 5b shows the same pulse evolution through an orthogonal projection on the plane ($T/T_0, z$). Again, the accelerated character of the pulse is evident.

3 Conclusion

Contrary to previous studies of finite-energy Airy beams and pulses, as well as of other types of localized waves [15, 16], based on exponential or Gaussian spatial or temporal apodization techniques, a novel efficient time-truncation method has been developed in this article and has been used to examine the evolution of a time-truncated ideal Airy pulse in a homogeneous linear medium characterized by both second and third-order temporal dispersion. A detailed theoretical study has been undertaken of the depth of field, and illustrations have been provided clearly showing the acceleration property of such a pulse. It has been established that a properly truncated Airy pulse can maintain its salient characteristic features up to ranges that are much larger than those of a Gaussian pulse with a comparable initial spectral structure. The same method has been used to study the evolution of a "deficient" ideal Airy pulse in the same medium, illustrating the accompanying regeneration or self-healing effects.

The work of Chong et al. [14] on versatile light bullets limited to second-order temporal dispersion can be extended to accommodate both second and third-order dispersive effects. Eq.(1) is extended as follows:

$$i \frac{\partial \psi(r,t)}{\partial z} = \frac{\beta_2}{2} \frac{\partial^2 \psi(r,t)}{\partial T^2} + \frac{i \beta_3}{6} \frac{\partial^3 \psi(r,t)}{\partial T^3} - \frac{1}{2 \beta_0} \nabla^2_t \psi(r,t)$$

(15)

Here, $\nabla^2_t$ denotes the transverse (with respect to $z$) Laplacian operator and $\beta_0$ is the wavenumber computed at the central angular frequency $\omega_0$. Equation (15) allows a solution of the form

$$\psi(r,t) = U(z,T)Q(r)$$

(16)

with $U(z,T)$ satisfying Eq.(1) and $Q(r)$ governed by the 3D parabolic equation

$$i \frac{\partial Q(r)}{\partial z} + \frac{1}{2 \beta_0} \nabla^2_t Q(r) = 0$$

(17)

A time-truncated ideal Airy beam solution $U(z,T)$ together with a finite-energy solution of the paraxial equation (17), e.g., a Bessel-Gauss beam, will give rise to a light bullet $\psi(r,t)$ according to Eq.(16). In this case, the finite energy light bullet can be resistant to the diffraction and dispersion effects for a distance much greater than the field depth of the ordinary pulses in dispersive media. If an Airy beam solution to Eq.(17) is chosen, a nonlinear transverse bending due to diffraction will appear in addition to the longitudinal acceleration.

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