QUANTUM DIMENSIONS AND FUSION RULES
FOR PARAFERMION VERTEX OPERATOR ALGEBRAS

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Abstract. The quantum dimensions and the fusion rules for the parafermion vertex operator algebra associated to the irreducible highest weight module for the affine Kac-Moody algebra $A_1^{(1)}$ of level $k$ are determined.

1. Introduction

Parafermion vertex operator algebra is the commutant of the Heisenberg vertex operator algebra in the affine vertex operator algebra. It comes from a special kind of coset construction [23]. Precisely speaking, let $L_{\hat{g}}(k,0)$ be the level $k$ integrable highest weight module with weight zero for affine Kac-Moody algebra $\hat{g}$ associated to a finite-dimensional simple Lie algebra $g$. Then $L_{\hat{g}}(k,0)$ contains the Heisenberg vertex operator subalgebra generated by the Cartan subalgebra $h$ of $g$. The commutant $K(g,k)$ of the Heisenberg vertex operator subalgebra in $L_{\hat{g}}(k,0)$ is the parafermion vertex operator algebra. The structure and representation theory of parafermion vertex operator algebras has been widely studied since 2009 (see [14], [15], [17], [18], [4]). In particular, [17] and [18] show that the role of parafermion vertex operator algebra $K(sl_2,k)$ in the parafermion vertex algebra $K(g,k)$ is similar to the role of 3-dimensional simple Lie algebra $sl_2$ played in Kac-Moody Lie algebras. We denote $K(sl_2,k)$ by $K_0$ in this paper. Moreover, it was proved that $K_0$ coincides with a certain $W$-algebra in [14] and [15]. Later in [4], the $C_2$-cofiniteness of parafermion vertex operator algebra $K(g,k)$ has been established by proving the $C_2$-cofiniteness of parafermion vertex operator algebra $K_0$, and irreducible modules for parafermion vertex operator algebra $K_0$ were also classified therein. Recently, the rationality of $K_0$ was established [5] by identifying the parafermion vertex operator algebra $K_0$ with certain $W$-algebra [3]. Also see [10] for the rationality of parafermion vertex operator algebra $K(g,k)$ for any $g$ and classification of irreducible modules.

The notion of quantum dimensions of modules for vertex operator algebras was introduced in [8]. It was proved therein for rational and $C_2$-cofinite vertex operator algebras that quantum dimensions do exist. In this paper, we first determine the...
quantum dimensions for the parafermion vertex operator algebra $K_0$. Then by using the important formula obtained in [8] which shows that quantum dimensions are multiplicative under tensor product, we give the fusion rules for the parafermion vertex operator algebra $K_0$. The quantum dimensions of irreducible modules and the fusion rules for any $K(g, k)$ were determined recently in [2].

The paper is organized as follows. In Section 2, we recall some results about parafermion vertex operator algebra $K_0$. In Section 3, after reviewing the notion and properties of quantum dimensions of modules for vertex operator algebras, we give the quantum dimensions of parafermion vertex operator algebra $K_0$. In the final section, we obtained the fusion rules of parafermion vertex operator algebra $K_0$ by using the results of quantum dimensions of parafermion vertex operator algebra $K_0$.

2. Preliminary

In this section, we recall from [14], [15] and [4] some basic results on the parafermion vertex operator algebra associated to the irreducible highest weight module of the affine Kac-Moody algebra $A_1^{(1)}$ of level $k$ with $k \geq 2$ being an integer. First we recall the notion of the parafermion vertex operator algebra.

We are working in the setting of [14]. Let $\{h, e, f\}$ be a standard Chevalley basis of $sl_2$ with brackets $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Let $\hat{sl}_2 = sl_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be the affine Lie algebra associated to $sl_2$. Let $k \geq 2$ be an integer and let

$$V(k, 0) = V_{\hat{sl}_2}(k, 0) = \text{Ind}_{sl_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}K}^{\hat{sl}_2} \mathbb{C}$$

be an induced $\hat{sl}_2$-module such that on $1 = 1$, $sl_2 \otimes \mathbb{C}[t]$ acts as 0 and $K$ acts as $k$. We denote by $a(n)$ the operator on $V(k, 0)$ corresponding to the action of $a \otimes t^n$. Then

$$(2.1) \quad [a(m), b(n)] = [a, b](m + n) + m(a, b) \delta_{m+n,0} k$$

for $a, b \in sl_2$ and $m, n \in \mathbb{Z}$. It is well known [22] that there is a vertex operator algebra structure on $V(k, 0)$ and it has a unique maximal ideal $\mathcal{J}$, which is generated by a weight $k+1$ vector $e(-1)^{k+1}1$ [25]. The quotient algebra $L(k, 0) = V(k, 0)/\mathcal{J}$ is the simple vertex operator algebra associated to an affine Lie algebra $\hat{sl}_2$ of type $A_1^{(1)}$ with level $k$. The subspace $V_0^s(k, 0)$ of $V(k, 0)$ spanned by $h(-i_1) \cdots h(-i_p)1$ for $i_1 \geq \cdots \geq i_p \geq 1$ and $p \geq 0$ is a vertex operator subalgebra of $V(k, 0)$ associated to the Heisenberg algebra. The parafermion vertex operator algebra $K_0$ is defined as the commutant of $V_0^s(k, 0)$ in $L(k, 0)$, that is,

$$K_0 = \{v \in L(k, 0) \mid h(m)v = 0 \text{ for } m \geq 0\}.$$ 

It was proved that $K_0$ is a simple vertex operator algebra and the irreducible $K_0$-modules $M^{i,j}$ for $0 \leq i \leq k, 0 \leq j \leq k - 1$ were constructed in [14]. Note that $K_0 = M^{0,0}$. It was also proved that $M^{i,j} \cong M^{k-i, k-j}$ as $K_0$-module in [14] Theorem 4.4] and moreover, Theorem 8.2 in [4] showed that the irreducible $K_0$-modules $M^{i,j}$ for $0 \leq i \leq k, 0 \leq j \leq k - 1$ constructed in [14] form a complete set of isomorphism classes of irreducible $K_0$-module. Moreover, $K_0$ is $C_2$-cofinite [4] and rational [16].

Recall from [14] that $L = \mathbb{Z} \alpha_1 + \cdots + \mathbb{Z} \alpha_k$ with $\langle \alpha_p, \alpha_q \rangle = 2\delta_{pq}$. $V_L = M(1) \otimes \mathbb{C}[L]$ is the lattice vertex operator algebra associated with the lattice $L$. Let $\gamma = \alpha_1 + \cdots + \alpha_k$. Thus $\langle \gamma, \gamma \rangle = 2k$. It is well known that the vertex operator algebra
associated with a positive definite even lattice is rational \[^7\]. And any irreducible
module for the lattice vertex operator algebra \(V_{\mathbb{Z}}\) is isomorphic to one of \(V_{\mathbb{Z}+n\gamma/2k}\), \(0 \leq n \leq 2k - 1\ \[^7\]. Let \(L(k,i)\) for \(0 \leq i \leq k\) be the irreducible modules for the
rational vertex operator algebra \(L(k,0)\). The following result was due to \[^14\].

**Lemma 2.1.** \(L(k,i) = \bigoplus_{j=0}^{k-1} V_{\mathbb{Z}+(i-2j)\gamma/2k} \otimes M^{i,j}\) as \(V_{\mathbb{Z}+\gamma}\)-modules.

3. QUANTUM DIMENSIONS FOR IRREDUCIBLE \(K_0\)-MODULES

In this section, we first recall the notion and some basic facts about quantum
dimension from \[^8\]. Then we determine the quantum dimensions of the irreducible
\(K_0\)-modules.

Let \((V,Y,1,\omega)\) be a vertex operator algebra (see \[^21\], \[^27\]). We define weak module, module and admissible module following \[^10\], \[^11\]. Let \(W\{z\}\) denote the
space of \(W\)-valued formal series in arbitrary complex powers of \(z\) for a vector space \(W\).

**Definition 3.1.** A weak \(V\)-module \(M\) is a vector space with a linear map
\[
Y_M : V \to (\text{End}M)\{z\}
\]
\[
v \mapsto Y_M(v,z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End}M),
\]
which satisfies the following conditions for \(u, v \in V, w \in M\):
\[
u_n w = 0 \quad \text{for} \quad n \gg 0,
\]
\[
Y_M(1,z) = \text{Id}_M,
\]
\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u,z_1) Y_M(v,z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_M(v,z_2) Y_M(u,z_1)
\]
\[
= \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u,z_0)v,z_2),
\]
where \(\delta(z) = \sum_{n \in \mathbb{Z}} z^n\).

Recall that the canonical central element \(C\) of the Virasoro algebra acts on \(V\) as a scalar \(c \in \mathbb{C}\), called the central charge of \(V\). We remark that the component operators of
\(Y_M(\omega,z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}\) still satisfy the Virasoro algebra relation
on \(M\ \[^{10}\] with the same central charge \(c\).

**Definition 3.2.** A modular \(V\)-module is a weak \(V\)-module \(M\) which carries a \(\mathbb{Z}\)-grading
\(M = \bigoplus_{\lambda \in \mathbb{Z}} M_{\lambda}\), where \(M_{\lambda} = \{ w \in M | L(0)w = \lambda w \}\). Moreover, we require that \(\dim M_{\lambda}\) is finite and for fixed \(\lambda\), \(M_{\lambda+n} = 0\) for all small enough integers \(n\).

**Definition 3.3.** An admissible \(V\)-module \(M = \bigoplus_{n \in \mathbb{Z}+} M(n)\) is a \(\mathbb{Z}+\)-graded weak
module such that \(u_m M(n) \subset M(wt u - m - 1 + n)\) for homogeneous \(u \in V\) and
\(m,n \in \mathbb{Z}\).

**Definition 3.4.** (1) A vertex operator algebra \(V\) is called rational if the admissible
module category is semisimple.

(2) \(V\) is called \(C_2\)-cofinite if \(\dim V/C_2(V) < \infty\) where \(C_2(V)\) is a subspace of \(V\)
spanned by \(u_{-2} v\) for \(u,v \in V\).

(3) \(V\) is of CFT type if \(V = \bigoplus_{n \geq 0} V_n\) and \(\dim V_0 = 1\).

The following lemma about rational vertex operator algebras is well known \[^{12}\].
Lemma 3.5. If $V$ is rational and $M$ is an irreducible admissible $V$-module, then
(1) $M$ is a $V$-module and there exists a $\lambda \in \mathbb{C}$ such that $M = \bigoplus_{n \in \mathbb{Z}_+} M_{\lambda+n}$
where $M_\lambda \neq 0$. And $\lambda$ is called the conformal weight of $M$.
(2) There are only finitely many irreducible admissible $V$-modules up to isomorphism.

Let $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ be a $V$-module. Set $M' = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda^*$, the restricted dual of $M$, where $M_\lambda^* = \text{Hom}_\mathbb{C}(M_\lambda, \mathbb{C})$. It was proved in [20] that $M'$ is naturally a $V$-module where the vertex operator $Y_{M'}(v, z)$ is defined for $v \in V$ via

$$\langle Y_{M'}(v, z)f, u \rangle = \langle f, Y_M(e^{zL(1)}(-z^{-2})L(0)v, z^{-1})u \rangle,$$

where $\langle f, w \rangle = f(w)$ is the natural pairing $M' \times M \to \mathbb{C}$. The $V$-module $M'$ is called the contragredient module of $M$. A $V$-module $M$ is called self-dual if $M$ and $M'$ are isomorphic $V$-modules. The following result was proved in [26].

Lemma 3.6. Let $V$ be a simple vertex operator algebra such that $L(1)V_1 \neq V_0$. Then $V$ is self-dual.

Remark 3.7. Note that the weight one subspace of $K_0$ is zero. By using lemma 3.6 parafermion vertex operator algebra $K_0$ is obviously self-dual.

Now we recall from [20] the notions of intertwining operators and fusion rules.

Definition 3.8. Let $(V, Y)$ be a vertex operator algebra and let $(W^1, Y^1)$, $(W^2, Y^2)$ and $(W^3, Y^3)$ be $V$-modules. An intertwining operator of type

$$\begin{pmatrix} W^3 \\
W^1 & W^2 \end{pmatrix}$$

is a linear map

$I(\cdot, z) : W^1 \to \text{Hom}(W^2, W^3)\{z\}$

$$w^1 \to I(w^1, z) = \sum_{n \in \mathbb{C}} w_1^n z^{-n-1},$$

satisfying for $v \in V, w^i \in W^i$ with $i = 1, 2$ and $\lambda \in \mathbb{C}$:
(1) $w^1_{\lambda+n}w^2 = 0$ for $n \in \mathbb{Z}$ sufficiently large.
(2) $I(L(-1)w^1, z) = \frac{d}{dz} I(w^1, z)$.
(3) The Jacobi identity holds:

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y^3(v, z_1)I(w^1, z_2) - z_0^{-1} \delta \left( \frac{-z_2 + z_1}{z_0} \right) I(w^1, z_2)Y^2(v, z_1)$$

$$= z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right) I(Y^1(w^1, z_0)v, z_2).$$

The space of all intertwining operators of type $\begin{pmatrix} W^3 \\
W^1 & W^2 \end{pmatrix}$ is denoted by

$I_V \begin{pmatrix} W^3 \\
W^1 & W^2 \end{pmatrix}$.

Let $N^{W_3}_{W^1, W^2} = \dim I_V \begin{pmatrix} W^3 \\
W^1 & W^2 \end{pmatrix}$. These integers $N^{W_3}_{W^1, W^2}$ are usually called the fusion rules.
Definition 3.9. Let $V$ be a vertex operator algebra, and $W^1, W^2$ be two $V$-modules. A module $(W, I)$, where $I \in I_V \left( \begin{array}{cc} W & M \\ W^1 & W^2 \end{array} \right)$, is called a tensor product (or fusion product) of $W^1$ and $W^2$ if for any $V$-module $M$ and $\mathcal{Y} \in I_V \left( \begin{array}{cc} M & \mathcal{Y} \\ W^1 & W^2 \end{array} \right)$, there is a unique $V$-module homomorphism $f : W \to M$, such that $\mathcal{Y} = f \circ I$. As usual, we denote $(W, I)$ by $W^1 \boxtimes_V W^2$.

Remark 3.10. It is well known that if $V$ is rational, then for any two irreducible $V$-modules $W^1$ and $W^2$, the fusion product $W^1 \boxtimes_V W^2$ exists and $W^1 \boxtimes_V W^2 = \sum W \ N_{W^1, W^2}^W$, where $W$ runs over the set of equivalence classes of irreducible $V$-modules.

Now we recall some notions about quantum dimensions.

Definition 3.11. Let $M = \bigoplus_{n \in \mathbb{Z}^+} M_{\lambda+n}$ be a $V$-module, the formal character of $M$ is defined as $\text{ch}_q M = \text{tr}_M q^{L(0)-c/24} = q^{\lambda-c/24} \sum_{n \in \mathbb{Z}^+} (\dim M_{\lambda+n}) q^n$, where $c$ is the central charge of the vertex operator algebra $V$ and $\lambda$ is the conformal weight of $M$.

We now assume that the vertex operator algebra $V$ is rational and $C_2$-cofinite. It is proved [29] that $\text{ch}_q M$ converges to a holomorphic function in the domain $|q| < 1$. We denote the holomorphic function $\text{ch}_q M$ by $Z_M(\tau)$. Here and below, $\tau$ is in the upper half plane $\mathbb{H}$ and $q = e^{2\pi i \tau}$.

Let $M^0, \cdots, M^d$ be the inequivalent irreducible $V$-modules with corresponding conformal weights $\lambda_i$ and $M^0 \cong V$. It is proved in [29] that for any $i$

$$Z_{M^i} \left( \frac{-1}{\tau} \right) = \sum_{j=0}^d S_{i,j} Z_{M^j}(\tau).$$

The complex matrix $S = (S_{i,j})_{i,j=0}^d$ is called the $S$-matrix.

The following definition of quantum dimension was introduced in [8].

Definition 3.12. Let $V$ be a vertex operator algebra and $M$ a $V$-module such that $Z_V(\tau)$ and $Z_M(\tau)$ exist. The quantum dimension of $M$ over $V$ is defined as

$$\text{qdim}_V M = \lim_{y \to 0} \frac{Z_M(iy)}{Z_V(iy)},$$

where $y$ is real and positive.

The following result was obtained in [8 Lemma 4.2].

Lemma 3.13. Let $V$ be a simple, rational and $C_2$-cofinite vertex operator algebra of CFT type with $V \cong V'$. Let $M^i$ for $0 \leq i \leq d$ be the inequivalent irreducible $V$-modules with corresponding conformal weights $\lambda_i$ and $M^0 \cong V$. Assume $\lambda_0 = 0$ and $\lambda_i > 0 \forall i \neq 0$. Then $\text{qdim}_V M^i = \frac{S_{i,0}}{S_{0,0}}$. 
We now also assume that $V$ is of CFT type with $V \cong V'$, the conformal weights
\( \lambda_i \) of $M^i$ are positive for all $i > 0$. From Remark 3.7 and statements in Section 2, the parafermion vertex operator algebra $K_0$ satisfies all the assumptions.

The following result shows that the quantum dimensions are multiplicative under tensor product \[8\].

**Proposition 3.14.** Let $V$ and $M_i$ for $0 \leq i \leq d$ be as in Lemma 3.13 Then
\[
\text{qdim}_V (M^i \boxtimes M^j) = \text{qdim}_V M^i \cdot \text{qdim}_V M^j
\]
for $i, j = 0, \cdots, d$.

Before giving the main result of this section, we recall the following character formula of irreducible $K_0$-modules $M^{i,j}$ which is given in [6][24]:
\[
\text{ch} M^{i,j} = \eta(\tau) c^i_{-2j}(\tau),
\]
where $\eta(\tau) = q^{-1/24} \prod_{n \geq 1} (1 - q^n)$ is the Dedekind $\eta$-function and $c^i_{-2j}(\tau)$ are the string functions \[25\]. Note that $k, l$ and $m$ in \[24\] are $k$, $i$ and $i - 2j$, respectively in our notation.

**Theorem 3.15.** The quantum dimensions for all irreducible $K_0$-modules $M^{m,n}$ are
\[
\text{qdim}_{K_0} M^{m,n} = \frac{\sin \frac{\pi (m+1)}{k+2}}{\sin \frac{\pi}{k+2}}
\]
for $0 \leq m \leq k$, $0 \leq n \leq k - 1$, where $M^{m,n}$ are the irreducible modules of $K_0$ constructed in [14].

**Proof.** Let $M^{m,n}(\tau)$ denote the character of $M^{m,n}$ for $0 \leq m \leq k$, $0 \leq n \leq k - 1$. The $S$-modular transformation of characters has the following form \[21\], \[25\]:
\[
M^{m,n}(-\frac{1}{\tau}) = \sum_{m',n'} S_{m,n}^{m',n'} M^{m',n'}(\tau),
\]
where $S_{m,n}^{m',n'} = (k(k+2))^{\frac{1}{2}} \exp \frac{i\pi(m-2n)(m'-2n')}{k} \sin \frac{\pi(m+1)(m'+1)}{k+2}$. From [14] and [4], we see that $K_0$ has $\frac{k(k+2)}{2}$ irreducible modules $M^{m,n}$ with the conformal weights
\[
\lambda_{m,n} = \frac{1}{2k(k+2)}(k(m-2n) - (m-2n)^2 + 2kn(m-n+1))
\]
for $0 \leq m \leq k$, $0 \leq n \leq m - 1$. It is easy to check that $\lambda_{k,0} = 0$ and $\lambda_{m,n} > 0$ for $(m,n) \neq (k,0)$. Thus by using Lemma 3.13, we have
\[
\text{qdim}_{K_0} M^{m,n} = \frac{S_{m,n}^{0,0}}{S_{0,0}^{0,0}} = \frac{\sin \frac{\pi (m+1)}{k+2}}{\sin \frac{\pi}{k+2}}.
\]

\[
\square
\]

4. Fusion rule for irreducible $K_0$-modules

In this section, we give the fusion rules for irreducible $K_0$-modules. First we fix some notation. Let $W^1, W^2, W^3$ be irreducible $K_0$-modules. In the following, we use $I \begin{pmatrix} W^3 \\ W^1 W^2 \end{pmatrix}$ to denote the space $I_{K_0} \begin{pmatrix} W^3 \\ W^1 W^2 \end{pmatrix}$ of all intertwining
operators of type \( \begin{pmatrix} W^3 \\ W^1 W^2 \end{pmatrix} \), and use \( W^1 \otimes W^2 \) to denote the fusion product \( W^1 \boxtimes_{K_0} W^2 \) for simplicity.

We recall the fusion rules for the affine vertex operator algebra of type \( A_1^{(1)} \) [28] for later use.

**Lemma 4.1.**

\[
L(k, i) \boxtimes_{L(k,0)} L(k, j) = \sum_l L(k, l),
\]

where \( |i - j| \leq l \leq i + j, \ i + j + l \in 2\mathbb{Z}, \ i + j + l \leq 2k. \)

**Theorem 4.2.** The fusion rule for the irreducible modules of parafermion vertex operator algebra \( K_0 \) is as follows:

\[
M^{i,i'} \boxtimes M^{j,j'} = \sum_l M^{i',j'} \boxtimes \sum_{(i,j) \in \mathbb{Z}} D((i,i'), (j,j'), (l,l')), \tag{4.1}
\]

where \( \bar{a} \) means the residue of the integer \( a \) modulo \( k \), \( 0 \leq i, j, i' \leq k, 0 \leq i', j' \leq k - 1, \ |i - j| \leq l \leq i + j, \ i + j + l \in 2\mathbb{Z}, \ i + j + l \leq 2k. \) Moreover, with fixed \( i, i', j, j' \), \( M^{i,j} = M^{i',j'} = 0 \) for \( |i - j| \leq l \leq i + j, \ i + j + l \leq 2k \) are inequivalent irreducible modules.

**Proof.** We take \( V = L(k, 0), U = V_{Z\gamma} \otimes K_0 \) in Proposition 2.9 of [1], from Lemma 2.1 we see that

\[
\dim I_V \left( \frac{L(k, l)}{L(k, i) L(k, j)} \right) \leq \dim I_U \left( V_{Z\gamma + (i - 2j)/2k} \otimes M^{i,i'} V_{Z\gamma + (j - i')/2k} \otimes M^{j,j'} \right)
\]

for \( 0 \leq i, j, l \leq k, 0 \leq i', j' \leq k - 1. \) Note that \( L(k, l) = \bigoplus_{l'=0}^{k-1} V_{Z\gamma + (l-2l')/2k} \otimes M^{l,l'}, \)

thus we have

\[
\dim I_U \left( V_{Z\gamma + (i - 2j)/2k} \otimes M^{i,i'} V_{Z\gamma + (j - i')/2k} \otimes M^{j,j'} \right) = \sum_{l'=0}^{k-1} D((i, i'), (j, j'), (l,l')),
\]

where

\[
D((i, i'), (j, j'), (l,l')) = \dim I_U \left( V_{Z\gamma + (i - 2j)/2k} \otimes M^{i,i'} \right). V_{Z\gamma + (j - i')/2k} \otimes M^{j,j'} \right).
\]

Using Theorem 2.10 of [1], we have

\[
D((i, i'), (j, j'), (l,l')) = \dim I_{V_{Z\gamma}} \left( V_{Z\gamma + (i - 2j)/2k} \otimes V_{Z\gamma + (j - i')/2k} \right).
\]

Recall from [9] that the fusion rule for lattice vertex operator algebras is given by

\[
V_{Z\gamma + \lambda} \boxtimes V_{Z\gamma + \mu} = V_{Z\gamma + \lambda + \mu}.
\]
for $\lambda, \mu \in (\mathbb{Z}\gamma)^\circ$, where $(\mathbb{Z}\gamma)^\circ$ is the dual lattice of $\mathbb{Z}\gamma$. Using this together with the fusion rule for the affine vertex operator algebra given in Lemma 4.1, we see that if $l' \neq \frac{l-i+2i' - j+2j'}{2}$, then

$$D((i, i'), (j, j'), (l, l')) = 0.$$  

Thus for any $l$ satisfying $|i-j| \leq l \leq i+j$, $i+j+l \in 2\mathbb{Z}$, $i+j+l \leq 2k$, we have

$$\dim I_{K_0} \left( M_{l, \frac{1}{2}(2i' - i+2j'-j+l)} M_{i, i'} M_{j, j'} \right) \geq 1.$$

In the following, we use the quantum dimension of the irreducible modules over $K_0$ to prove that

$$\dim I_{K_0} \left( M_{l, \frac{1}{2}(2i' - i+2j'-j+l)} M_{i, i'} M_{j, j'} \right) = 1.$$

From Proposition 3.14, we know

$$\text{qdim}_{K_0} \left( M_{i, i'} \otimes M_{j, j'} \right) = \text{qdim}_{K_0} M_{i, i'} \cdot \text{qdim}_{K_0} M_{j, j'}.$$

We also have the fusion product

$$M_{i, i'} \otimes M_{j, j'} = \sum_{0 \leq l \leq k, 0 \leq l' \leq t-1} N_{(i,i'),(j,j')}^{(l,l')} M_{l,l'}.$$

Theorem 3.15 shows that

$$\text{qdim}_{K_0} M_{l, l'} = \frac{\sin \frac{\pi(i+l+1)}{k+2}}{\sin \frac{\pi}{k+2}}.$$

Thus we only need to prove that the following identity holds:

$$\frac{\sin \frac{\pi(i+j+2)}{k+2} \cdot \sin \frac{\pi(i-j+1)}{k+2}}{\sin \frac{\pi}{k+2} \cdot \sin \frac{\pi}{k+2}} = \sum_l \frac{\sin \frac{\pi(l+2)}{k+2}}{\sin \frac{\pi}{k+2}} = \sum_l (\cos \frac{\pi(l+2)}{k+2} - \cos \frac{\pi l}{k+2}).$$

for $0 \leq i, j \leq k, 0 \leq i', j' \leq k-1, |i-j| \leq l \leq i+j, i+j+l \in 2\mathbb{Z}$, $i+j+l \leq 2k$.

This identity is equivalent to the following identity:

$$(4.2) \quad \cos \frac{\pi(i+j+2)}{k+2} - \cos \frac{\pi(i-j)}{k+2} = \sum_{l} (\cos \frac{\pi(l+2)}{k+2} - \cos \frac{\pi l}{k+2}).$$

We note that if $i+j < k$, then $l_{\text{min}} = i-j$, $l_{\text{max}} = i+j$, thus (4.2) holds. If $i+j > k$, $l_{\text{min}} = i-j$, $l_{\text{max}} = i+j - 2n$ for some $n$ satisfying that $i+j-2n+i+j = 2k$, that is, $l_{\text{max}} = 2k - i - j$, thus (4.2) also holds.

Now we prove that the modules $M_{l, \frac{1}{2}(2i' - i+2j'-j+l)}$ appearing in the sum (4.1) are not isomorphic to each other. If $M_{l, \frac{1}{2}(2i' - i+2j'-j+l)} \cong M_{l_1, \frac{1}{2}(2i' - i+2j'-j+l_1)}$ for some $l, l_1$ satisfying $|i-j| \leq l, l_1 \leq i+j, i+j+l \in 2\mathbb{Z}$, $i+j+l \leq 2k, i+j+l_1 \leq 2k$. Since we know that $M_{i,j} \cong M_{k-i,k-i+j}$ as $K_0$-module for $0 \leq i \leq k, 0 \leq j \leq k-1$ and the $\frac{k(k+1)}{2}$ irreducible $K_0$-modules $M_{i,j}$ for $0 \leq i \leq k, 0 \leq j \leq i-1$ exhaust all the isomorphism classes of irreducible $K_0$-modules, it follows that $l_1 = k-l$ and

$$k-l + \frac{1}{2}(2i' - i+2j'-j+k-l) = \frac{1}{2}(2i' - i+2j'-j+k-l),$$

which is impossible by a direct calculation. This proves the assertion. □
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