Parity-violating electromagnetic interactions in QED$_3$ at finite temperature

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Abstract

We study the parity-breaking terms generated by the box diagram in 2+1 dimensional thermal QED. These lead, in the long wave limit, to a gauge invariant extensive action which behaves as $1/T$ at high temperature. In contrast, the effective action in the static limit involves leading non-extensive terms proportional to $1/T^3$ at high temperature, which violate large gauge invariance. We derive a non-linear large gauge Ward identity, which relates the leading static terms of different order in perturbation theory and whose solution coincides with the all order effective action proposed earlier.
1 Introduction:

It is well known by now that, in odd space-time dimensions, one can add a topological term to the Lagrangian density of a gauge field, in addition to the usual Maxwell term. Such a term is known as the Chern-Simons term [1] and a theory with such a term is conventionally called a Chern-Simons theory [2]. In 2 + 1 dimensions, for example, the Chern-Simons (CS) action has the form [3, 4]

\[ S_{\text{CS}} = M \int d^3x \text{tr} \epsilon^{\mu
u\lambda} A_\mu \left( \partial_\nu A_\lambda + \frac{2g}{3} A_\nu A_\lambda \right). \] (1)

Here “tr” denotes trace over the matrix indices of the gauge fields, \( g \) the coupling constant while \( M \) is an arbitrary parameter with the dimensions of mass.

The CS action has several interesting features. Unlike the standard Maxwell action for the gauge fields, it is a topological action. In a theory with a Maxwell term, the CS action generates a mass for the gauge fields [3]. While it is invariant under small gauge transformations, the CS action, for a non-Abelian theory, is not invariant under topologically nontrivial large gauge transformations. Rather, its change is proportional to the winding number associated with the gauge transformation. Explicitly, under

\[ A_\mu \rightarrow U^{-1} A_\mu U + \frac{1}{g} U^{-1} \partial_\mu U, \] (2)

the CS action transforms as

\[ S_{\text{CS}} \rightarrow S_{\text{CS}} + \frac{8\pi^2 M}{g^2} W, \] (3)

where

\[ W = \frac{1}{24\pi^2} \int d^3x \text{Tr} \epsilon^{\mu\nu\lambda} \partial_\mu U^{-1} \partial_\nu U^{-1} \partial_\lambda U U^{-1} \] (4)

is a topological integer known as the winding number of the gauge transformation. For vanishing winding number, the gauge transformations are called small gauge transformations, while for any nontrivial value of the winding number, they are known as large gauge transformations. The CS action clearly is not invariant under large gauge transformations. However, the path integral and, therefore, the theory is, provided the coefficient of the CS term is quantized as [3]

\[ \frac{4\pi M}{g^2} = n, \] (5)
with \( n \) an integer.

The CS action, in 2+1 dimensions, is known to violate discrete symmetries like \( P \) and \( T \). Furthermore, the mass term for a fermion (in the irreducible two component representation) is also known to violate these symmetries. Therefore, if we have massive fermions interacting with a background non-Abelian gauge field, one expects the radiative corrections due to fermions to generate a CS term in the effective action. In fact, it is known that radiative corrections, at zero temperature, shift the value of the tree level CS coefficient \( [5] \) such that (assume, for simplicity, \( m > 0 \))

\[
M \rightarrow M - \frac{g^2 N_f}{8\pi}, \tag{6}
\]

where \( N_f \) represents the number of fermion flavors. It is clear now that, even if we start with a consistent theory with tree level quantization given by Eq. \( [5] \), the radiative corrections change this coefficient and the effective theory will continue to be invariant under large gauge transformations only for an even number of fermion flavors. An even number of fermion flavors is also required to cancel a global anomaly \( [6] \) in such theories and, therefore, we see that, in such a case, once the tree level CS coefficient is quantized, the quantum theory continues to have large gauge invariance at the one loop level.

In such a theory, it is also known that there is no higher loop corrections to the CS coefficient at zero temperature \( [7] \) so that the full quantum theory continues to be invariant under large gauge transformations.

In contrast, it was observed that, at finite temperature, the one loop radiative corrections due to fermions shift the tree level CS coefficient as \( [8, 9, 10] \) (We would see later that this corresponds to a particular limit.)

\[
M \rightarrow M - \frac{g^2 N_f}{8\pi} \tanh \frac{\beta m}{2}, \tag{7}
\]

where \( \beta = \frac{1}{T} \) in units of the Boltzmann constant. This, of course, reduces to Eq. \( [5] \) when \( T \rightarrow 0 \). However, for any nonzero temperature, this is a continuous function and, therefore, even when the tree level CS coefficient is quantized and the number of fermion flavors is even, it cannot take a discrete value as would be required for large gauge invariance to hold. It would appear, therefore, that large gauge invariance would be violated at finite temperature. On the other hand, this is rather strange since temperature is not expected to affect gauge invariance, small or large.
The possible understanding of this puzzle has led to a lot of interest in this topic and only recently, a mechanism for its resolution has been found within the context of the 0+1 dimensional Abelian CS theory [11]. Basically, the resolution of the puzzle in the 0+1 dimensional model goes as follows. For $N_f$ flavors of fermions interacting with an Abelian gauge background, at zero temperature, the radiative corrections due to fermions generate only the CS term (namely, only the one point function). On the other hand, at finite temperature, the effective action due to fermions can be exactly evaluated and has the form

$$\Gamma_f = -i N_f \log \left( \cos \frac{a}{2} + i \tanh \frac{\beta m}{2} \sin \frac{a}{2} \right), \quad (8)$$

where

$$a = \int dt A(t), \quad (9)$$

with $A(t)$ representing the gauge field. This shows that, unlike at zero temperature, all possible amplitudes are generated in the effective action at finite temperature. Second, all the terms in the effective action are non-extensive and, while every individual term in the effective action violates large gauge invariance, for an even number of fermion flavors, the full effective action is invariant under

$$a \to a + 2\pi N, \quad (10)$$

which represents the large gauge transformation in this case.

By now, the 0+1 dimensional models have been studied from various points of view [12, 13, 14, 15]. First of all, since we do not expect to be able to evaluate the effective action in closed form in the 2+1 dimensional case, the 0+1 dimensional theory has been studied exhaustively in the perturbative approach [12]. This gives rise to many interesting features. Similarly, if we were to study the 2+1 dimensional theory perturbatively, a signature of large gauge invariance may lie in the large gauge Ward identity. With this in mind, large gauge Ward identities have been derived for the 0+1 dimensional theories [15] which have quite distinctive features. To better understand whether the non-extensive structure is special to 0+1 dimension, the effective action for a fermion, in 1+1 dimensions, interacting with an Abelian gauge background has also been evaluated at finite temperature [16] and it turns out that the effective action, in this case, is extensive although non-local and non-analytic as would be expected in a thermal background.
The analysis of the 0+1 dimensional model has also been generalized to 2+1 dimensional models for a restrictive gauge background \[17, 18\]. Namely, it has been shown that for a single fermion interacting with an Abelian gauge background of the form \( A_0 = A_0(t) \) and \( \vec{A} = \vec{A}(\vec{x}) \), the effective action has the form \[17, 18\]

\[
\Gamma' = \frac{e}{2\pi} \int d^2 x \arctan \left( \tanh \frac{\beta m}{2} \tan \frac{ea}{2} \right) B, \tag{11}
\]

where the magnetic field is defined to be \( B = \epsilon^{ij} \partial_i A_j \).

It is natural to believe that the effective action in Eq. (11) does not represent the complete effective action of the fermion theory in 2+1 dimensions. In fact, it is quite clear that the gauge background is quite restrictive. And, more importantly, the effective action in Eq. (11) does not exhibit non-locality or non-analyticity as would be expected from a thermal effective action. On the other hand, it does represent an all order calculation, be it for a very specific gauge background. It is, of course, quite clear that an exact evaluation of the effective action in a general gauge background is impossible. The only way to go beyond the CS action is through perturbation theory and possibly through the use of the large gauge Ward identity. With this in mind, we have decided to evaluate the parity violating part of the box diagram for a fermion interacting with an arbitrary Abelian gauge background which may serve as a first step towards understanding the question of the effective action and, therefore, large gauge invariance in the 2+1 dimensional theory. Even the calculation of the simple box diagram turns out to be extremely difficult and we had to make use of symbolic computer programs in the intermediate steps. However, the calculation does bring out some interesting features of the theory. The main results of our analysis were already reported in \[19\]. In this paper, we describe the details of our calculation.

The paper is organized as follows. In section 2, we compile our notation as well as various identities in 2+1 dimensions, which lead to the fact that all the odd point functions vanish in this theory. (This is really a consequence of \( C \) invariance.) Consequently, one need to look at only even point functions. In section 3, we exhibit the small gauge invariance of the fermion loop at finite temperature. In section 4, we discuss the choice of a small gauge invariant tensor basis which simplifies the calculations. We obtain the parity violating part of the box diagram at zero temperature as well as the quartic effective action associated with this. We evaluate the finite temperature amplitude
in two distinct limits, namely, the long wave and the static limits, which shows that the thermal amplitude is indeed non-analytic. We discuss various features of the result and construct the corresponding effective actions. We show that it is really in the static limit that the question of large gauge invariance comes up. In section 5, we derive a large gauge Ward identity and solve for the leading terms in the static limit, which coincides with the effective action, Eq. (11), obtained in the restrictive gauge background. In section 6, we present a brief conclusion along with future directions.

2 Notations and Conventions:

Let us consider a single flavor of fermion interacting with a background Abelian gauge field described by the Lagrangian density

\[ \mathcal{L} = \overline{\psi} \left( \gamma^\mu (i \partial_\mu - e A_\mu) - m \right) \psi. \]  

(12)

Here, \( e \) represents the electromagnetic coupling strength and we use a diagonal metric with signatures \((+,-,-)\) as well as assume that \( m > 0 \). The spinors are two component complex spinors and the Dirac matrices can be represented in terms of the Pauli matrices \( \vec{\sigma} \) as follows

\[ \gamma^0 = \sigma_2, \quad \gamma^1 = i \sigma_1, \quad \gamma^2 = i \sigma_3, \]

(13)

so that

\[ (\gamma^0)^\dagger = \gamma^0, \quad (\gamma^1)^\dagger = -\gamma^1, \quad (\gamma^2)^\dagger = -\gamma^2, \]

(14)

and

\[ (\gamma^0)^2 = 1 = -(\gamma^1)^2 = -(\gamma^2)^2. \]

(15)

The 2 \times 2 gamma matrices satisfy some interesting relations such as

\[ \gamma^\mu \gamma^\nu = \eta^{\mu\nu} + i e^{\mu\nu\lambda} \gamma^\lambda, \]

(16)

where \( e^{\mu\nu\lambda} \) represents the totally anti-symmetric Levi-Civita tensor with \( e^{012} = 1 \). Relation (16) shows that, unlike in four dimensions, in 2 + 1 dimensions, we have

\[ \text{Tr} \, \gamma^\mu \gamma^\nu \gamma^\lambda = 2 i e^{\mu\nu\lambda}. \]

(17)

It is worth noting here that the gamma matrices satisfy the relation

\[ \text{Tr} \, \gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_{2n+1}} = -\text{Tr} \, \gamma^{\mu_{2n+1}} \gamma^{\mu_{2n}} \cdots \gamma^{\mu_1}, \]

(18)
which is quite useful in showing that all the odd point functions, in this theory, vanish which, in turn, is a reflection of charge conjugation invariance of the theory. (A word of caution here, namely, that this holds only in the Abelian theory. The presence of internal symmetry generators invalidates this for non-Abelian theories.) Similarly, for an even number of gamma matrices, we have

\[ \text{Tr} \gamma^{\mu_1} \gamma^{\mu_2} \cdots \gamma^{\mu_{2n}} = \text{Tr} \gamma^{\mu_{2n}} \gamma^{\mu_{2n-1}} \cdots \gamma^{\mu_1}, \]  

which helps simplify the calculation of even point functions. There is one other 2 + 1 dimensional identity which is quite useful in simplifying the calculations, namely, for any arbitrary vector, \( A^\mu \), we have

\[ A^\mu \epsilon^{\nu\lambda\sigma} + A^\nu \epsilon^{\lambda\mu\sigma} + A^\lambda \epsilon^{\mu\nu\sigma} - A^\sigma \epsilon^{\mu\nu\lambda} = 0, \]  

which is really a statement of the fact that in 2 + 1 dimensions, we cannot have a fourth rank anti-symmetric tensor.

### 3 Gauge Invariance of the Fermion Loop:

In trying to evaluate the effective action due to the fermions, let us next show that, at finite temperature, the \( n \)-point amplitude generated by the fermion loop is gauge invariant, at least under small gauge transformations.

It is simpler to see the small gauge invariance in the real time formalism where there is a doubling of fields \[20]. (We will use the closed time path formalism \[20, 21\].) In this case, the propagator acquires a 2×2 matrix structure and an \( n \)-point amplitude can have both + and − type of external vertices. For simplicity, we will show gauge invariance only for amplitudes containing vertices of + type (namely, the original vertices) although everything can be carried over to vertices of other kind.

For the type of amplitude that we are interested in (namely, ones with + vertices), we need only one component of the fermion propagator, namely,

\[ S_{++}(k) = (\not{k} + m) \left[ \frac{1}{k^2 - m^2 + i\epsilon} + 2i\pi n_F(|k^0|)\delta(k^2 - m^2) \right]. \]  

Here, \( n_F \) represents the fermion distribution function. Defining \( q = k' - k \), we have
Figure 1: Fermion loop with $N$ photon lines plus an extra attached line with momentum $q$ and index $\mu$. Dotted lines represent photons, and solid lines stand for electrons.

\[
S_{++}(k') \& S_{++}(k) = \left[ \frac{1}{k'^2 - m^2 + i\epsilon} + 2i\pi n_F(|k'^0|)\delta(k'^2 - m^2) \right] \times (k' + m)((k' - m) - (k - m))(k + m) \\
\times \left[ \frac{1}{k^2 - m^2 + i\epsilon} + 2i\pi n_F(|k^0|)\delta(k^2 - m^2) \right] \\
= S_{++}(k) - S_{++}(k').
\] (22)

This relation is identical to the one at zero temperature.

Let us next consider a fermion loop with $N$ photons, carrying momenta $p_1, p_2 \cdots p_N$ and indices $\mu_1, \mu_2, \cdots, \mu_N$ in an ordered way and define

\[
k_r = k + p_1 + p_2 + \cdots + p_r,
\]

where $k$ represents the momentum in the loop. Let us next attach an extra photon line with momentum $q$ and index $\mu$ (see Fig.1) between the photon lines carrying momenta $p_r$ and $p_{r+1}$ (all the lines are of $+$ type). Contracting this diagram with $q_\mu$, we obtain (we are going to neglect the coupling constants as well as an overall sign coming from the fermion loop)

\[
g_{\mu_1 \cdots \mu_N}^r = \int \frac{d^3k}{(2\pi)^3} \text{Tr} \ S_{++}(k) \gamma^{\mu_1} S_{++}(k_1) \gamma^{\mu_2} \cdots \gamma^{\mu_r} \\
\times S_{++}(k_r) \& S_{++}(k + q) \gamma^{\mu_{r+1}} \cdots S_{++}(k_{N-1} + q) \gamma^{\mu_N} \\
= \int \frac{d^3k}{(2\pi)^3} \text{Tr} \ S_{++}(k) \gamma^{\mu_1} S_{++}(k_1) \cdots \gamma^{\mu_r}
\]

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\[ \times (S_{++}(k_r) - S_{++}(k_r + q)) \gamma^{\mu_{r+1}} \cdots S_{++}(k_{N-1} + q) \gamma^{\mu_N} \]
\[ = \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ S_{++}(k) \gamma^{\mu_1} \cdots \gamma^{\mu_r} S_{++}(k_r) \gamma^{\mu_{r+1}} \cdots S_{++}(k_{N-1} + q) \gamma^{\mu_N} \right. \]
\[ - S_{++}(k) \gamma^{\mu_1} \cdots \gamma^{\mu_r} S_{++}(k_r + q) \gamma^{\mu_{r+1}} \cdots S_{++}(k_{N-1} + q) \gamma^{\mu_N} \]. \quad (23) \]

Here, we have used the relation Eq. (22) in the intermediate steps.

If we now sum over all the possible insertions of the photon line with momentum \( q \), terms cancel pairwise to give
\[ \sum_{r=1}^{N} g^{\mu_1 \cdots \mu_N}_r = \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ S_{++}(k) \gamma^{\mu_1} S_{++}(k_1) \cdots S_{++}(k_{N-1}) \gamma^{\mu_N} \right. \]
\[ - S_{++}(k + q) \gamma^{\mu_1} S_{++}(k_1 + q) \cdots S_{++}(k_{N-1} + q) \gamma^{\mu_N} \]
\[ = 0. \quad (24) \]

Here, we have shifted the variable of integration in the second term by \( k \rightarrow k - q \) to obtain the final result. Such a shift is, of course, meaningful if the integrand is well behaved. We note here that, at finite temperature, the temperature dependent terms are ultraviolet finite and, therefore, such a shift is allowed if the zero temperature part is well behaved, which we know to be true. This argument parallels the zero temperature argument \[22\] and shows that the \( n \)-point amplitudes generated by the fermion loop are gauge invariant even at finite temperature. Note that this argument can be easily extended to any number of dimensions.

As we have said, the gauge invariance of amplitudes with mixed vertices can be shown in an analogous manner. We simply note here that with a matrix propagator of the form
\[ S(k) = \left( \begin{array}{cc} S_{++}(k) & S_{+-}(k) \\ S_{-+}(k) & S_{--}(k) \end{array} \right), \quad (25) \]
where,
\[ S_{++}(k) = 2i\pi(k + m)(n_F(|k^0|) - \theta(-k^0))\delta(k^2 - m^2) \]
\[ S_{-+}(k) = 2i\pi(k + m)(n_F(|k^0|) - \theta(k^0))\delta(k^2 - m^2) \]
\[ S_{--}(k) = (k + m) \left[ \frac{1}{k^2 - m^2 - i\epsilon} + 2i\pi n_F(|k^0|)\delta(k^2 - m^2) \right], \quad (26) \]
and a matrix vertex of the form
\[ \Gamma^\mu = \left( \begin{array}{cc} \gamma^\mu & 0 \\ 0 & -\gamma^\mu \end{array} \right), \quad (27) \]
it is easy to see that the generalization of Eq. (22) takes the form
\[ S(k')(q, \Gamma^\mu) S(k) = S(k) - S(k'). \] (28)

Furthermore, along with the identities \( q = k' - k \),
\[ S_{\pm\mp}(k') q S_{\pm\mp}(k) = 0, \] (29)
the gauge invariance of any mixed amplitude follows in a completely analogous manner.

4 Box Diagram:

Since all the odd point functions vanish in this theory and the two point function is already known (we will come back to the non-analyticity in the two point function later), the next meaningful amplitude to evaluate is the four point function. Furthermore, we are interested only in the parity violating part of this amplitude. This calculation, of course, is extremely cumbersome. However, as we have seen in the last section, the four point function has to be invariant under small gauge transformations and we expect this to be of help. While small gauge invariance alone predicts uniquely the form of the \( n \)-point amplitude in the 0 + 1 dimensional theory, it is not so in 2 + 1 dimensions. For example, we know that the \( n \)-point amplitude in 0 + 1 dimensions has to be of the form
\[ \Pi(n) = \alpha_n \delta(p_1) \delta(p_2) \cdots \delta(p_{n-1}), \]
if small gauge invariance has to hold. However, let us note that in the 2 + 1 dimensional theory, even at the level of the parity violating four point amplitude, there are several possible structures that are compatible with small gauge invariance. Each of the structures below (and possibly more), for example,
\[ \Pi_{PV}^{\mu\nu\lambda\rho} \sim u^\mu u^\nu u^\lambda \epsilon^{\rho\sigma\tau} p_{4,\tau} \delta(u \cdot p_1) \delta(u \cdot p_2) \delta(u \cdot p_3) + \text{perm.} \]
\[ \Pi_{PV}^{\mu\nu\lambda\rho} \sim u^\mu u^\nu \epsilon^{\lambda\rho\sigma} p_{4,\tau} \delta(u \cdot p_1) \delta(u \cdot p_2) \delta^3(p_3 + p_4) + \text{perm.} \]
\[ \Pi_{PV}^{\mu\nu\lambda\rho} \sim u^\mu u^\nu \epsilon^{\lambda\rho\sigma} u_{\sigma}(u \cdot (p_3 - p_4)) \delta(u \cdot p_1) \delta(u \cdot p_2) \delta(p_3^4) \delta(p_4^4) + \text{perm.} \]
where \( u^\mu \) denotes the velocity of the heat bath, is compatible with small gauge invariance. This is what makes the calculation hard. However, one can simplify the calculation somewhat by choosing a (small) gauge invariant tensor basis for this amplitude.
Figure 2: Box diagrams which contribute to the four photon function.

Figure 3: One of the four forward scattering amplitudes corresponding to the first diagram in Fig. 2.

4.1 The Calculation:

The graphs which contribute to the four photon amplitude are shown in Fig. 2. There are three other contributions obtained by charge conjugation. To evaluate these diagrams, we use the analytically continued imaginary-time thermal perturbation theory [20, 24, 25]. This approach can be formulated so as to express the thermal Greens function in terms of forward scattering amplitudes [20] of an on-shell fermion in an external electromagnetic field, as depicted in Fig. 3. Each of these forward scattering amplitude diagrams is obtained by cutting one of the internal lines of the box diagrams in Fig. 2. This, therefore, generates a total of $4 \times 6 = 24$ diagrams, which can be systematically obtained from the graph in Fig. 3, by permutations of the external momenta and polarizations. The contribution of the box diagrams,
at finite temperature, can then be written in the form

$$\Pi^{\mu\nu\lambda\rho}(p_1, p_2, p_3, p_4) = \frac{e^4}{(2\pi)^2} \int \frac{d^2k}{2\omega_k} \left( n_F(\omega_k) - \frac{1}{2} \right) \left[ \sum_{ijkl} B^{\mu\nu\lambda\rho}_{(ijkl)} + (k \leftrightarrow -k) \right].$$  \hspace{1cm} (30)

Here \( \omega_k = \sqrt{k^2 + m^2} \), \( n_F(\omega_k) = (e^{\omega_k/T} + 1)^{-1} \), and the sum is over the permutations \((ijkl)\) of \((1234)\). Each \( B \) has a numerator which involves a trace over the Dirac indices. For example, we have

$$B^{\mu\nu\lambda\rho}_{(1234)} = \left. \frac{\text{tr} \left[ (\not p + m) \gamma^\mu (\not p + m) \gamma^\nu (\not p + m) \gamma^\lambda (\not p + m) \gamma^\rho \right]}{\left(2k \cdot p_1 + p_1^2\right) \left(2k \cdot p_12 + p_12^2\right) \left(2k \cdot p_123 + p_123^2\right)} \right|_{k_0 = \omega_k},$$  \hspace{1cm} (31)

where \( p_{12} = p_1 + p_2 \), etc. Here, we are only interested in the contributions from the trace in Eq. \((31)\) which contain odd powers of the mass, since these will lead to parity-breaking terms (remember that the fermion mass breaks parity).

Let us first study the zero temperature contribution coming from the box diagram, which is associated with the factor \(1/2\) in the first bracket of Eq. \((30)\), as \( n_F(\omega_k) \) vanishes in this limit. The computation can be performed explicitly in the low momentum region, where \( |p_\mu| \ll m \). The result can then be expressed in terms of a series in powers of \( p/m \), which begins with the leading contribution

$$\Pi^{\mu\nu\lambda\rho}_{PV,T=0} = -\frac{i e^4}{16\pi m^6} \left[ \epsilon^{\mu\alpha\beta} p_1^\alpha (p_2)^2 + \epsilon^{\mu\alpha\beta} p_1^\alpha p_2^\beta p_2^\gamma \right] \times \left[ \eta^{\lambda\rho} p_3 \cdot p_4 - p_3^\rho p_4^\lambda \right] + \text{permutations.}$$  \hspace{1cm} (32)

It is interesting to note that this result is consistent with the Coleman-Hill theorem \([7]\) which implies that, in the four point Greens function at zero temperature, the terms of order \( p \) should be absent. In fact, the above structure shows that the parity-violating contributions, generated by the box diagram at \( T = 0 \), begin only with terms of order \((p/m)^5\). In the configuration space, the low-energy effective action associated with Eq. \((32)\) can be written in the form

$$\Gamma^4_{PV,T=0} = -\frac{e^4}{64\pi m^6} \int d^3x \epsilon^{\mu\nu\lambda} F_{\mu\nu} (\partial^\tau F_{\tau\lambda}) F^{\rho\sigma} F_{\rho\sigma},$$  \hspace{1cm} (33)
which is manifestly Lorentz and gauge invariant (small and large).

It is worth pointing out that this is the unique, lowest order (in derivatives) parity violating quartic action that one can construct at zero temperature and can be thought of as the generalization of the result of Karplus and Neuman [27] (to the parity violating amplitude in 2 + 1 dimensions). Of course, one can naively write down other possible structures, for example, of the form

$$S_{PV,T=0}^4 = \int d^3x \epsilon^{\mu\nu\lambda} \partial_\nu F_\lambda T F_{\mu\rho} F^{\rho\sigma} F_\sigma.$$

(34)

However, using identities such as in Eq. (20) as well as the Bianchi identity, it is straightforward to show that the two structures in Eqs. (33)-(34) are related by a simple multiplicative constant. One can, of course, also construct structures with three epsilon tensors, but they reduce to one of the two forms above. This shows that the lowest order, parity violating quartic action, at zero temperature, has a unique form given in Eq. (33). In fact, the identities, in 2+1 dimensions, are so restrictive that the general form of the lowest order (in derivatives) parity violating effective action can be determined to have the form

$$\Gamma_{PV} = \Gamma_{CS} + \sum_{n=1} a_n \epsilon^{\mu\nu\lambda} F_{\mu\nu} (\partial^\tau F_\tau T) F^{\rho\sigma} F_{\rho\sigma}.$$

(35)

with the coefficient $a_n$ to be determined perturbatively ($a_1$ is already determined in Eq. (33)).

The evaluation of the temperature dependent part of the box diagram, on the other hand, is extremely cumbersome and, as we have mentioned earlier, we would like to systematize the calculation by first selecting a gauge invariant basis which we do next.

4.2 Gauge Invariant Tensor Basis:

Let us next construct a set of gauge invariant tensor basis for the parity violating part of the four point amplitude. We note that the tensors in this basis must be linear in the Levi-Civita tensor (odd number of epsilons are, of course, allowed, but reduce to a single epsilon upon using various identities). Furthermore, the tensor basis should also reflect symmetry under exchange of external photon lines. At finite temperature, in addition to the usual tensor structures, we also have the velocity $u^\mu$ of the heat bath and, thus, there are, in general, many such structures that one can construct. However, it is practically impossible to carry out the calculation for a general
configuration of momenta. For this reason, we have chosen to work with a special configuration of momenta, namely,

\[ p_1 = p_2 = p_3 = p = -\frac{1}{3}p_4. \]  

(36)

In this special configuration, the number of linearly independent, gauge invariant tensor structures is rather easy to determine. For example, for tensor structures where the Levi-Civita tensor has two free indices, there are only two linearly independent structures possible, namely,

\[
T_{1}^{\mu\nu\lambda\rho} = \epsilon^{\sigma\lambda\rho} p_{\sigma} \left( \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) + \text{perm.}
\]

\[
T_{2}^{\mu\nu\lambda\rho} = \epsilon^{\sigma\lambda\rho} p_{\sigma} \left( u^\mu - \frac{p \cdot u}{p^2} p^\mu \right) \left( u^\nu - \frac{p \cdot u}{p^2} p^\nu \right) + \text{perm.}
\]

(37)

These two independent structures are, in fact, quite easy to understand intuitively. Let us recall that, for the parity conserving part of the two point function, there are two independent tensor structures at finite temperature (there are really three structures with a constraint) and the parity violating part of the self-energy has a unique structure with the epsilon tensor. The two structures above simply arise as products of the parity violating structure with the two independent parity conserving structures.

One can similarly look for tensor structures where two of the indices of the Levi-Civita tensor are contracted. There are again only two linearly independent, gauge invariant tensor structures of this kind that one can construct and they have the forms

\[
T_{3}^{\mu\nu\lambda\rho} = \epsilon^{\sigma\tau\rho} u_{\sigma} p_{\tau} \left( u^\lambda - \frac{p \cdot u}{p^2} p^\lambda \right) \left( \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) + \text{perm.}
\]

\[
T_{4}^{\mu\nu\lambda\rho} = \epsilon^{\sigma\tau\rho} u_{\sigma} p_{\tau} \left( u^\lambda - \frac{p \cdot u}{p^2} p^\lambda \right) \left( u^\mu - \frac{p \cdot u}{p^2} p^\mu \right) + \text{perm.}
\]

(38)

The four structures in Eqs. (37)-(38) represent a complete set of linearly independent, gauge invariant basis for the parity violating part of the box diagram in this special momentum configuration. (There are other structures possible, but they are not linearly independent.) Therefore, the parity violating part of the four point amplitude can be written as

\[
\Pi_{(4),PV}^{\mu\nu\lambda\rho} = \sum_{i=1}^{4} C_{i} T_{i}^{\mu\nu\lambda\rho}.
\]

(39)
where the coefficients $C_i$ are to be determined from the actual evaluation of the Feynman diagrams. Explicit calculation shows that $C_3 = C_4 = 0$, so that the parity violating part of the four point amplitude can be expressed in terms of only the first two structures in Eq. (37).

### 4.3 Non-analyticity:

In evaluating the box amplitude at finite temperature, one faces yet another difficulty. Namely, thermal amplitudes are known to be non-analytic at the origin in the energy-momentum plane [29], which is understood as resulting from new branch cuts that develop at finite temperature due to the possibility of additional channels of reaction. This also translates to the fact that the temperature dependent effective action has a non-analytic structure [30]. This is not of importance in 0+1 dimension where there is no non-analyticity. But, this becomes quite crucial in higher dimensions. Thus, for example, the parity violating part of the self-energy, in 2+1 dimensions, has the form

$$\Pi_{\mu\nu}^{PV}(p) = -\frac{i e^2}{8\pi} \epsilon_{\lambda\mu\nu} p_\lambda \int \frac{d^2 k}{\omega_k} \tanh \left( \frac{\omega_k}{2T} \right) \left( \frac{1}{p^2 + 2 k \cdot p} + k \leftrightarrow -k \right). \quad (40)$$

Although it is not widely appreciated, the integrand in Eq. (40) is non-analytic at $p^\mu = 0$ and depending on how one evaluates the integral, the result would be different. For example, in the long wave limit (LW), the leading term, at high temperature, of the parity violating part of the self-energy takes the form

$$\Pi_{\mu\nu}^{PV(LW)}(p^0, \vec{p} = 0) = -i e^2 \frac{m}{8\pi T} \ln \left( \frac{m}{T} \right) \epsilon^{0\mu\nu} p^0 + \cdots, \quad (41)$$

giving rise to a leading quadratic effective action of the form

$$\Gamma^{(LW)}_{CS} = \frac{e^2 m}{16\pi T} \ln \frac{m}{T} \int d^3 x \epsilon^{0ij} A_i E_j. \quad (42)$$

In contrast, in the static limit (S), the leading behavior of the parity violating term in the self-energy has the form

$$\Pi_{\mu\nu}^{PV(S)}(p^0 = 0, \vec{p}) = \frac{i e^2}{4\pi} \tanh \left( \frac{m}{2T} \right) \epsilon^{\mu\nu\lambda} p_\lambda + \cdots \quad (43)$$
giving rise to a leading quadratic effective action of the form

$$\Gamma_{CS}^{(s)} = \frac{e^2}{4\pi} \tanh \left( \frac{m}{2T} \right) \int d^3 x A_0 B.$$ (44)

There are several things to note from this analysis. First, the form of the effective actions, in the two different limits, are quite different. Their leading temperature dependence is also quite distinct. And, finally, if we think of finite temperature as compactifying the time direction and, thereby, inducing a large gauge transformation, then, the effective action in the long wave limit is invariant under such large gauge transformations while the problem of large gauge invariance really manifests in the static limit. Without going into detail, we would like to point out here that the leading contribution to the CS term vanishes if we approach the origin $p^\mu = 0$ along a light-like direction. It is also worth pointing out here that the original calculation of the CS term \[8\] corresponds to evaluating it in the static limit (the question of non-analyticity was not very much appreciated then).

It is clear, therefore, that in evaluating the box diagram, at finite temperature, we expect the amplitude to be non-analytic as well. In fact, as in the case of the self-energy, we are going to evaluate this amplitude only in the long wave and the static limits. Let us first concentrate on the long wave limit. In this limit ($\vec{p} = 0$) we have the relation $p^\mu = (p \cdot u) u^\mu$ and, therefore, it is clear that three of the four basis tensor structures in Eqs. (37)-(38) identically vanish (as we have mentioned earlier, the last two structures do not contribute to the parity violating part of the four point amplitude at all), namely,

$$T_{2}^{\mu\nu\lambda\rho} = 0 = T_{3}^{\mu\nu\lambda\rho} = T_{4}^{\mu\nu\lambda\rho},$$

and the only non-vanishing basis tensor takes the simple form (with a multiplicative factor taken out)

$$T_{1}^{\mu\nu\lambda\rho} = \epsilon^{\sigma\lambda\rho} u_\sigma (\eta^{\mu\nu} - u^\mu u^\nu) + \text{perm}.$$ (45)

In this case, the amplitude can be written as

$$\Pi_{PV}^{\mu\nu\lambda\rho (LW)} = C_1 \epsilon^{\sigma\lambda\rho} u_\sigma (\eta^{\mu\nu} - u^\mu u^\nu) + \text{perm}.$$ (46)

We note here from the explicit structure in Eq. (45) that in the long wave limit, the amplitude is nontrivial only when all the external indices take spatial values. Furthermore, the coefficients $C_1$ which depend on the momenta,
temperature etc, are to be evaluated from the Feynman diagram and have the form

\[ C_1 = -\frac{6ie^4 m p_0}{\pi} \int_0^\infty d|\vec{k}| \frac{|\vec{k}|}{\omega_k} \tanh \frac{\omega_k}{2T} \frac{(3\omega_k^2 - 5m^2 + 2p_0^2)}{(p_0^2 - \omega_k^2)(p_0^2 - 4\omega_k^2)(9p_0^2 - 4\omega_k^2)}. \] (47)

For \(|p_0| \ll m\), we can expand this in a series of the form

\[ C_1 = \frac{i e^4 m T p_0}{16} \sum_{l=-\infty}^{\infty} \left\{ \left( \frac{5m^2}{\Delta_l^6} + \frac{3}{\Delta_l^4} \right) \ln \left( 1 + \frac{\Delta_l^2}{m^2} \right) - \frac{5}{\Delta_l^4} - \frac{1}{2 m^2 \Delta_l^2} \right\}, \] (48)

where \(\Delta_l \equiv (2l + 1) \pi T\). In the high temperature limit, the leading contribution comes from the last term in Eq. (48). Performing the summation over \(l\), we then obtain that

\[ C_1(T \gg m) = -\frac{i e^4 p_0}{128} \frac{1}{mT}. \] (49)

Therefore, we see that the leading contribution, in the long wave limit, comes from an extensive effective action of the form

\[ \tilde{\Gamma}_{PV}^{(LW)} = \frac{e^4}{512 m T} \int d^3x \epsilon^{0ij} E_i \left( \partial_t^{-1} E_j \right) \left( \partial_t^{-1} E_k \right), \] (50)

where \(\vec{E}\) denotes the electric field. This action is non-local and manifestly gauge invariant (both under small and large gauge transformations) much like the quadratic effective action in the long wave limit. We would like to note here that, in the long wave limit, we have evaluated the amplitude for arbitrary values of the energies, but have chosen to present the results only for the special configuration of Eq. (36) for simplicity.

Next, let us turn to the discussion of the thermal behavior of the box diagram in the static limit, where \(p_i^0 = 0\). In this case, due to the very complicated angular integrations, the calculations are extremely difficult, even when using computer algebra. As a result, we have restricted ourselves, in this calculation, to the special configuration of the external spatial momenta (of Eq. (36)), where \(\vec{p}_1 = \vec{p}_2 = \vec{p}_3 = \vec{p} = -\frac{1}{3} \vec{p}_4\). In this case, we note that the tensor structure \(T_{\mu\nu}^{\mu\nu}\) would give contributions only to \(\Pi_{PV}^{00i}\) whereas \(T_{\mu\nu}^{\mu\nu}\) would give contributions to both \(\Pi_{PV}^{00i}\) as well as \(\Pi_{PV}^{ijk}\) (in the rest frame of the heat bath). Thus, we see that, in the static limit, the amplitude can have
only an odd number of temporal indices (unlike the long wave limit). Let us parameterize the two nontrivial amplitudes as

$$\Pi^{0ij(S)}_{PV} = 2(C_1 + C_2)\epsilon^{0ij} \frac{p_j}{|\vec{p}|} = -\frac{1}{4|\vec{p}|^2}\epsilon^{0ij}p_j\Pi_1(\vec{p}, T)$$ (51)

$$\Pi^{0ijk(S)}_{PV} = 2C_1\epsilon^{0kl} \frac{p_l}{|\vec{p}|} \left( \frac{\vec{p}^i \vec{p}^j}{\vec{p}^2} + \eta^{ij} \right) = \frac{1}{12|\vec{p}|^4}\epsilon^{0kl}p_l \left( |\vec{p}|^2 \eta^{ij} + p^i p^j \right) \Pi_2(\vec{p}, T)$$, (52)

where $$\Pi_{1,2}(\vec{p}, T)$$ are rather complicated functions of the momenta and the temperature. However, for small momenta, namely, $|\vec{p}| \ll m, T$, we can expand these in a powers series in the momenta and each term in the series can be evaluated in a straightforward manner. Thus, for example, the leading term in $$\Pi_1$$, in this domain, can be evaluated to have the form

$$\Pi_1(\vec{p}, T) = \frac{6i e^4}{4\pi} \left[ \tanh \left( \frac{m}{2T} \right) - \tanh^3 \left( \frac{m}{2T} \right) \right] |\vec{p}|^2 \frac{1}{T^2} + O \left( \frac{|\vec{p}|^4}{m^2 T^2} \right).$$ (53)

In the high temperature limit, this behaves as $$\frac{1}{T^3}$$, which is quite different from the leading $$\frac{1}{T}$$ behavior of the result (49) in the long wave limit. Let us note here that $$\Pi_2$$ can also be evaluated in a similar fashion and has the leading high temperature behavior

$$\Pi_2(\vec{p}, T) = -\frac{17i e^4}{16800\pi} \frac{m^3}{T^7} p^4.$$ (54)

It is interesting that terms with lower powers of momentum in $$\Pi_2$$ identically vanish. As a result, we see that the leading term in $$\Pi^{0ijk(S)}_{PV}$$ is strongly suppressed at high temperature compared with $$\Pi^{000i(S)}_{PV}$$ which, in turn, is suppressed relative to $$\Pi^{ijkl(LW)}_{PV}$$.

The leading contribution given in Eqs. (51) and (53) can be associated with the effective non-extensive action (remember that $$p_4 = -3p$$)

$$\tilde{\Gamma}^{4(S)}_{PV} = \frac{e^4 T}{48\pi} \left[ \tanh \left( \frac{m}{2T} \right) - \tanh^3 \left( \frac{m}{2T} \right) \right] \int d^3 x a_0^3 B, $$ (55)

where we have defined

$$a_0 = \int_0^\beta dt A_0(t, \vec{x})$$ (56)
and $B$ is the magnetic field. This form, which may also hold in the quasi-static limit, is consistent with the result derived from the all-orders effective action noted earlier in the special gauge background (see Eq. (11)). We note here that the effective action that would give rise to the amplitude $\Pi_{PV}^{0ijk(S)}$ in Eqs. (52) and (54) can also be determined in a similar manner, but is highly suppressed at high temperature and, unlike the non-extensive action in Eq. (55), would have an extensive, be it non-local structure characteristic of thermal actions. Let us note here that our calculations have been done in the small momentum approximation (which would correspond to a derivative expansion of the effective action). It is well known that [14], in such an expansion, it is impossible to pick out delta function structures characteristic of non-extensive actions unless one sums the series, which in the present case is simply impossible. In fact, even the evaluation of the leading term in the small momentum expansion already pushes us to the limit of our computational abilities (we really mean even with the use of computers). Therefore, in isolating delta function structures, we have been guided by our earlier experience from the studies in $0 + 1$ dimension [12], namely, that if an amplitude has a delta function structure, then, in the small momentum expansion, the amplitude vanishes if the variable has a nonzero value and is nonzero only when the variable assumes a vanishing value. This we have checked explicitly. It remains an open question as to whether one can find a better way of isolating delta function structures from a calculation of the leading term in the small momentum expansion.

To conclude this section, therefore, we have found that the temperature dependent part of the parity violating four point amplitude is non-analytic, much like the self-energy. The effective actions, in general, contain both extensive as well as non-extensive terms. In the long wave limit, the leading term at high temperature goes as $\frac{1}{T}$ and the effective action associated with this is extensive. Furthermore, this action is invariant under both small and large gauge transformations, much like the CS action in the long wave limit. In the static limit, the leading term in the effective action is non-extensive and behaves as $\frac{1}{T^3}$ at high temperature. Furthermore, while this action is invariant under small gauge transformations, it violates large gauge invariance. Thus, large gauge invariance seems to hold order by order in the long wave limit, while it is the static limit where large gauge invariance appears to be an issue at every order.
5 Large Gauge Ward Identity:

To a given order, the quasi-static perturbative contributions are not invariant under large gauge transformations generated by $ea_0 \rightarrow ea_0 + 2\pi N$, where $N$ is a topological integer. But one can derive, in this case, a Ward identity for large gauge invariance, which relates the amplitudes obtained in perturbation theory. To this end, motivated by the structure of Eq. (53), let us write the corresponding all order effective action in the form

$$\tilde{\Gamma}^{(S)} = \frac{e}{2\pi} \int d^3x \tilde{\Gamma}(\tilde{a}) B,$$

(57)

where $\tilde{a} = ea_0$. It has been noted in [12] that in the special background $A_0 = A_0(t)$ and $\tilde{B} = \tilde{A}(x)$, $\tilde{\Gamma}(\tilde{a})$ corresponds to the real part of the effective action $\Gamma_f$ in Eq. (8), with the identification $a \rightarrow \tilde{a}$. This action obeys, for a single fermion flavor, the large gauge Ward identity [13]

$$\partial^2 \Gamma^{(1)} / \partial \tilde{a}^2 = \frac{i}{4} \left[ \frac{1}{4} - \left( \frac{\partial \Gamma^{(1)}}{\partial \tilde{a}} \right)^2 \right],$$

(58)

where the one point function has the value

$$\frac{\partial \Gamma^{(1)}}{\partial \tilde{a}} \bigg|_{\tilde{a}=0} = \frac{1}{2} \tanh \frac{\beta m}{2}.$$

(59)

In order to derive the large gauge Ward identity satisfied by $\tilde{\Gamma}(\tilde{a}) = \Re \left[ \Gamma^{(1)}(\tilde{a}) \right]$, we write

$$\Gamma^{(1)}(\tilde{a}) = \tilde{\Gamma}(\tilde{a}) + i I(\tilde{a}),$$

(60)

where $I$ denotes the imaginary part of the action $\Gamma^{(1)}$, and substitute this relation into the nonlinear equation (58). Equating to zero the resulting real and imaginary parts, we obtain the following system of coupled equations

$$\frac{\partial^2 \tilde{\Gamma}}{\partial \tilde{a}^2} = 2 \frac{\partial \tilde{\Gamma}}{\partial \tilde{a}} \frac{\partial I}{\partial \tilde{a}}; \quad \frac{\partial^2 I}{\partial \tilde{a}^2} = \frac{1}{4} + \left( \frac{\partial I}{\partial \tilde{a}} \right)^2 - \left( \frac{\partial \tilde{\Gamma}}{\partial \tilde{a}} \right)^2.$$

(61)

We must now eliminate from the first equation $\partial I / \partial \tilde{a}$, so as to express $\partial^2 \tilde{\Gamma} / \partial \tilde{a}^2$ solely in terms of functionals of $\tilde{\Gamma}$. After some analysis, it turns out that a consistent solution of the above set of equations requires $\partial I / \partial \tilde{a}$ to have the form

$$\frac{\partial I}{\partial \tilde{a}} = A \sin(\omega \tilde{\Gamma}) + B \cos(\omega \tilde{\Gamma}),$$

(62)
where the coefficients $A$ and $B$, as well as the frequency $\omega$, must be determined from the boundary conditions. One of these conditions can be read directly from (59) and the fact that $\hat{\Gamma}(\hat{\alpha}) = \Re\left[\Gamma^{(1)}(\hat{\alpha})\right]$. The other condition follows from the form (57) of the effective action $\hat{\Gamma}(\hat{S})$ which, as a consequence of invariance under charge conjugation, is a functional involving only even powers of $A_{\mu}$. Consequently, $\hat{\Gamma}(\hat{\alpha})$ must contain only odd powers of $\hat{\alpha}$ and therefore, in particular, $\partial^2 \hat{\Gamma}/\partial \hat{\alpha}^2$ should vanish at $\hat{\alpha} = 0$. These conditions, together with the set of Eqs. (61), determine uniquely $A$, $B$ and $\omega$ in Eq. (62), so that

$$\frac{\partial I}{\partial \hat{\alpha}} = \frac{1}{2} \sinh \beta m \sin(2 \hat{\Gamma}).$$

(63)

Using this form, we find from the first relation in Eq. (61) that $\hat{\Gamma}(\hat{\alpha})$ satisfies the large gauge Ward identity

$$\frac{\partial^2 \hat{\Gamma}}{\partial \hat{\alpha}^2} = \frac{1}{\sinh \beta m} \frac{\partial \hat{\Gamma}}{\partial \hat{\alpha}} \sin \left(2 \hat{\Gamma}\right).$$

(64)

This identity, which reflects the underlying large gauge invariance of the quasi-static QED$_3$ theory, relates higher point Greens functions to lower ones. However, unlike the Ward identity for small gauge transformations, the relation (64) is nonlinear. In some sense, this is expected for large gauge transformations which are topologically nontrivial. The relation (64), in fact, allows us to check for large gauge invariance perturbatively. Note from Eqs. (44), (55) and (57) that

$$\left. \frac{\partial \hat{\Gamma}}{\partial \hat{\alpha}} \right|_{\hat{\alpha} = 0} = \frac{1}{2} \tanh \frac{\beta m}{2}, \quad \left. \frac{\partial^3 \hat{\Gamma}}{\partial \hat{\alpha}^3} \right|_{\hat{\alpha} = 0} = \frac{1}{4} \left( \tanh \frac{\beta m}{2} - \tanh^3 \frac{\beta m}{2} \right).$$

(65)

The identity in Eq. (64) leads to (remember that $\hat{\Gamma}$ is odd in $\hat{\alpha}$ and hence vanishes for $\hat{\alpha} = 0$),

$$\left. \frac{\partial^3 \hat{\Gamma}}{\partial \hat{\alpha}^3} \right|_{\hat{\alpha} = 0} = \frac{2}{\sinh \beta m} \left( \left. \frac{\partial \hat{\Gamma}}{\partial \hat{\alpha}} \right|_{\hat{\alpha} = 0} \right)^2.$$

(66)

This can be easily seen to hold from the relations in Eq. (63). In fact, the solution of the Ward identity (64), subject to the above boundary conditions, is given by

$$\hat{\Gamma}(\hat{\alpha}) = \arctan \left[ \frac{\beta m}{2} \tan \left( \frac{\hat{\alpha}}{2} \right) \right].$$

(67)
Note that in this solution, which sums up the leading perturbative effects in this region, the tangent is invariant under the large gauge transformations \( \tilde{a} \rightarrow \tilde{a} + 2\pi N \). (Incidentally, large gauge invariance would also require quantization of the magnetic flux, which we do not get into here.) Substituting the form (67) in the expression (57), we obtain for \( \tilde{\Gamma}(s) \) a result which agrees, in the static limit of QED\(_3\), with the parity-breaking effective action previously discussed in the literature [17, 18].

6 Conclusion:

In this paper, we have studied the radiatively generated parity violating part of the four point amplitude in a theory of a single fermion interacting with an arbitrary Abelian gauge background in 2 + 1 dimensions at finite temperature. We have shown that the zero temperature part of the parity violating quartic action is unique and, in fact, so is the structure of the complete parity violating part of the effective action. In evaluating the temperature dependent contribution, we have pointed out various obstacles that one has to face and have systematically shown how one can handle these in a given calculation. Of importance is the non-analyticity of thermal amplitudes as well as of the thermal effective actions. We have discussed this in detail for the CS term (self-energy) as well as for the parity violating part of the four photon amplitude. In particular, we have shown that the behavior of the leading amplitudes and, therefore, the leading effective actions in the long wave and static limits are quite distinct at high temperature. Furthermore, while the leading term in the quartic effective action is extensive (but non-local) in the long wave limit, it is non-extensive in the static limit. We have found that, in the long wave limit, large gauge invariance is manifest order by order. In contrast, it appears to be violated order by order in the static limit.

These results can be understood intuitively from the following heuristic arguments. Note that, in 2 + 1 dimensions, we can always write

\[
A_0(t, \vec{x}) = \frac{1}{\beta} \int_0^\beta dt' A_0(t', \vec{x}) + \partial_t \Omega(t, \vec{x}),
\]

\[
A_i(t, \vec{x}) = \frac{1}{\sqrt{2}} \partial^j F_{ij}(0, \vec{x}) + \left( \int_0^t - \frac{t}{\beta} \int_0^\beta \right) dt' E_i(t', \vec{x}) + \partial_t \Omega(t, \vec{x}),
\]
where,
\[
\Omega(t, \vec{x}) = \left( \int_0^t - \frac{t}{\beta} \int_0^\beta \right) dt' A_0(t', \vec{x}) - \frac{1}{\nabla^2} \nabla \cdot \vec{A}(0, \vec{x})
\]  
(70)

Thus, in a particular gauge, we can think of \(a_0(\vec{x}), B(0, \vec{x})\) and \(\vec{E}(t, \vec{x})\) as representing the physically meaningful variables. From this, it is clear that, in the long wave limit, the only meaningful variable is the electric field which is both small and large gauge invariant. Consequently, the effective action, in this limit, would be large gauge invariant order by order. In contrast, in the static limit, all of the three variables are meaningful implying that the leading term (in derivatives) would involve an odd number of \(a_0(\vec{x})\) and a single \(B(0, \vec{x})\). Of course, there can be other terms, but they will be higher order in the number of derivatives. Furthermore, order by order, the leading term would violate large gauge invariance.

We have written down a large gauge Ward identity that the leading order terms of the parity violating effective action in the static limit must satisfy for large gauge invariance to hold. This identity can be solved to obtain the leading, all order parity violating effective action which coincides with the action proposed earlier in a restrictive gauge background. However, it is worth remembering that this does not represent the full effective action – rather, it only represents the leading term of the full parity violating effective action.

This study has been carried out within the context of an Abelian gauge theory as a first step towards understanding the question of large gauge invariance at finite temperature. The main interest is, of course, the study of this issue within the context of a non-Abelian gauge theory, which is work in progress.

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