Geometric Structure of Exact Triangles
Consisting of Projectively Flat Bundles on Higher Dimensional Complex Tori

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Abstract
The mirror dual objects corresponding to affine Lagrangian (multi) sections of a trivial special Lagrangian torus fibration $T^{2n} \to T^n$ are holomorphic vector bundles on a mirror dual complex torus of dimension $n$ via the homological mirror symmetry. In this paper, we construct a one-to-one correspondence between those holomorphic vector bundles and a certain kind of projectively flat bundles explicitly, by using the result of the classification of factors of automorphy of projectively flat bundles on complex tori. Furthermore, we give a geometric interpretation of the exact triangles consisting of projectively flat bundles on complex tori by focusing on the dimension of intersections of the corresponding affine Lagrangian submanifolds.

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1 Introduction
In this paper, we construct a mirror pair of tori as an analogue of the SYZ construction [12], and consider geometric structures of the exact triangles which appear in the discussions of the homological mirror symmetry [7]. The SYZ...
construction is conjectured by Strominger, Yau and Zaslow in 1996, and it proposes a way of constructing mirror pairs geometrically. Roughly speaking, this construction is the following. A mirror pair of Calabi-Yau manifolds \((M, \tilde{M})\) is realized as special Lagrangian torus fibrations \(\pi : M \to B\) and \(\tilde{\pi} : \tilde{M} \to B\) on the same base space \(B\). In particular, for each \(b \in B\), the special Lagrangian torus fibers \(\pi^{-1}(b)\) and \(\tilde{\pi}^{-1}(b)\) are related by the T-duality. On the other hand, the homological mirror symmetry is conjectured by Kontsevich in 1994, and it states the following. For each Calabi-Yau manifold \(M\), there exists a Calabi-Yau manifold \(\tilde{M}\) such that the derived category of the Fukaya category \([2]\) on \(M\) is equivalent to the derived category of coherent sheaves on \(\tilde{M}\) as triangulated categories. One of the most fundamental examples of mirror pairs is a pair \((T^{2n}, \tilde{T}^{2n})\) of tori, where \(T^{2n}\) is a symplectic torus and \(\tilde{T}^{2n}\) is a complex torus, so there are many studies of the homological mirror symmetry for tori. For example, Polishchuk and Zaslow discuss the homological mirror symmetry in the case of \(n = 1\), i.e., \((T^2, \tilde{T}^2)\) in \([11]\) (the details of the higher \(A_\infty\) product structures are studied in \([10]\)), and Fukaya studied the homological mirror symmetry for abelian varieties via the SYZ construction in \([3]\). In particular, in \([3]\), he discussed the homological mirror symmetry by focusing on the cases that the objects of the Fukaya category are restricted to affine Lagrangian submanifolds endowed with unitary local systems in the symplectic geometry side, and then, the holomorphic vector bundles corresponding to them are projectively flat. Thus, projectively flat bundles play a fundamental role in the homological mirror symmetry for tori. Let \((L, \mathcal{L})\) be an object of the Fukaya category \(\text{Fuk}(T^{2n})\), where \(L \cong T^n\) is an affine Lagrangian (multi) section of the trivial special Lagrangian torus fibration \(T^{2n} \to T^n\) and \(\mathcal{L} \to L\) is a unitary local system along \(L\). The objects \((L, \mathcal{L})\) correspond to holomorphic vector bundles on \(\tilde{T}^{2n}\) via the homological mirror symmetry, so we denote by \(E(L, \mathcal{L})\) a holomorphic vector bundle corresponding to \((L, \mathcal{L})\). Here, the special Lagrangian torus fibers with unitary local systems along them correspond to skyscraper sheaves on \(\tilde{T}^{2n}\). Hereafter, we call the affine Lagrangian (multi) section simply the affine Lagrangian submanifold. By definition, a holomorphic vector bundle \(E\) is projectively flat if and only if the curvature form of \(E\) is expressed locally as \(\alpha \cdot I_E\), where \(\alpha\) is a complex 2-form and \(I_E\) is the identity endomorphism of \(E\). Furthermore, the result of the classification of factors of automorphy of projectively flat bundles on complex tori is given in \([9]\), \([6]\), \([13]\). The purposes of this paper are to characterize holomorphic vector bundles \(E(L, \mathcal{L})\) by using the factors of automorphy of projectively flat bundles on \(\tilde{T}^{2n}\), and to give a geometric interpretation of the exact triangles consisting of projectively flat bundles \(E(L, \mathcal{L})\) by focusing on the dimension of intersections of the corresponding affine Lagrangian submanifolds \(L\).

Now, we explain the statements discussed in the body of this paper briefly. For each holomorphic vector bundle \(E(L, \mathcal{L})\), we can check easily that the curvature form of \(E(L, \mathcal{L})\) is expressed locally as \(\alpha \cdot I_{E(L, \mathcal{L})}\), where \(\alpha\) is a complex 2-form and \(I_{E(L, \mathcal{L})}\) is the identity endomorphism of \(E(L, \mathcal{L})\), so \(E(L, \mathcal{L})\) is projectively flat. But, the expression of transition functions of \(E(L, \mathcal{L})\) differs from the expression of factor of automorphy of the projectively flat bundle \(E(L, \mathcal{L})\)
which should be isomorphic to $E(L, L)$, so interpreting holomorphic vector bundles $E(L, L)$ in the language of factors of automorphy is a non-trivial problem. Thus, we interpret $E(L, L)$ in the language of factor of automorphy by constructing an isomorphism $E(L, L) \xrightarrow{\sim} \mathcal{E}(L, L)$ explicitly. Next, we discuss the exact triangles consisting of these holomorphic vector bundles and their shifts. Here, the triangulated category we treat is the one obtained from a DG-category $DG_{\mathcal{T}^2}n$ consisting of holomorphic vector bundles $E(L, L)$ on $\mathcal{T}^2n$ by the Bondal-Kapranov construction [1], instead of the derived category of coherent sheaves on $\mathcal{T}^2n$. In this triangulated category, we consider the following exact triangle.

$$\cdots E(L_a, L_a) \xrightarrow{} C(\psi) \xrightarrow{} E(L_b, L_b) \xrightarrow{\psi} TE(L_a, L_a) \cdots \quad (1)$$

Here, $T$ is the shift functor and $C(\psi)$ denotes the mapping cone of $\psi : E(L_b, L_b) \to TE(L_a, L_a)$. By the definition of the DG-category $DG_{\mathcal{T}^2}n$, the degrees of morphisms between holomorphic vector bundles $E(L, L)$ are equal to or larger than 0 in $DG_{\mathcal{T}^2}n$. This fact implies that the exact triangle consisting of projectively flat bundles and their shifts is always expressed as the exact triangle (1). In the above setting, we prove $\text{codim}(L_a \cap L_b) \leq 1$ if $C(\psi)$ is projectively flat, namely, if the exact triangle (1) becomes the exact triangle consisting of three projectively flat bundles and their shifts.

This paper is organized as follows. In section 2, we explain relations between the objects $(L, L)$ of the Fukaya category $\text{Fuk}(T^{2n})$ and holomorphic vector bundles $E(L, L)$. Furthermore, we construct the DG-category $DG_{\mathcal{T}^2}n$ consisting of those holomorphic vector bundles $E(L, L)$. In section 3, for each holomorphic vector bundle $E(L, L)$, we find the projectively flat bundle $\mathcal{E}(L, L)$ which should be isomorphic to $E(L, L)$, and construct an isomorphism $E(L, L) \xrightarrow{\sim} \mathcal{E}(L, L)$ explicitly. This result is given in Theorem 3.6. In section 4, for the exact triangle (1), we prove $\text{codim}(L_a \cap L_b) \leq 1$ if the exact triangle (1) becomes the exact triangle consisting of three projectively flat bundles and their shifts. This result is given in Theorem 4.1.

## 2 Holomorphic vector bundles and Lagrangian submanifolds

In this section, we consider a mirror pair $(T^{2n}, \mathcal{T}^{2n})$, where $T^{2n}$ is a symplectic torus and $\mathcal{T}^{2n}$ is a complex torus of $T^{2n}$, and discuss relations between affine Lagrangian submanifolds in $T^{2n}$ and holomorphic vector bundles on $\mathcal{T}^{2n}$. These are based on the SYZ construction [12] (see also [8]). Furthermore, we define the DG-category consisting of these holomorphic vector bundles.

First, we explain the complex geometry side. We define a complex torus $\mathcal{T}^{2n}$ as follows. Let us denote the coordinates of the covering space $\mathbb{R}^{2n}$ of $\mathcal{T}^{2n}$ by $(x_1, \cdots, x_n, y_1, \cdots, y_n)^t$, and we define

$$x := (x_1, \cdots, x_n)^t, \quad y := (y_1, \cdots, y_n)^t.$$
We also regard \((x_1, \ldots, x_n, y_1, \ldots, y_n)^t\) as a point of \(\tilde{T}^{2n}\) by identifying \(x_i \sim x_i + 2\pi\) and \(y_i \sim y_i + 2\pi\) for each \(i = 1, \ldots, n\). We fix an \(\varepsilon > 0\) small enough and let

\[
O^i_{m_1 \cdots m_n} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \tilde{T}^{2n} \mid \begin{array}{c}
\frac{2}{3}\pi(l_j - 1) - \varepsilon < x_j < \frac{2}{3}\pi l_j + \varepsilon, \\
\frac{2}{3}\pi(m_k - 1) - \varepsilon < y_k < \frac{2}{3}\pi m_k + \varepsilon, \quad j, k = 1, \ldots, n
\end{array} \right\}
\]

be a subset of \(\tilde{T}^{2n}\), where \(l_j, m_k = 1, 2, 3\). Then, \(\{O^i_{m_1 \cdots m_n} \}_{l_j, m_k = 1, 2, 3}\) is an open cover of \(\tilde{T}^{2n}\). Sometimes we denote \(O^i_{m_1 \cdots (l_j = l) \cdots m_n}\) instead of \(O^i_{m_1 \cdots m_n}\) in order to specify the values \(l_j = l, m_k = m\). We can regard each \(O^i_{m_1 \cdots m_n}\) as an open set of \(\mathbb{R}^{2n}\), and we define the local coordinates of \(O^i_{m_1 \cdots m_n}\) by

\[
(x_1, \ldots, x_n, y_1, \ldots, y_n)^t \in \mathbb{R}^{2n}.
\]

We locally express the complex coordinates \(z := (z_1, \ldots, z_n)^t\) of \(\tilde{T}^{2n}\) by \(z = x + Ty\), where \(T\) is a complex matrix of order \(n\) such that \(\text{Im}T\) is positive definite. We denote by \(t_{ij}\) the \((i, j)\) component of \(T\). Then, for the lattice \(L\) in \(\mathbb{C}^n\) generated by

\[
\gamma_1 := (2\pi, 0, \cdots, 0)^t, \quad \gamma_n := (0, \cdots, 0, 2\pi)^t,
\]

\[
\gamma'_1 := (2\pi t_{11}, \cdots, 2\pi t_{1n})^t, \quad \gamma'_n := (2\pi t_{n1}, \cdots, 2\pi t_{nn})^t,
\]

\(\tilde{T}^{2n}\) is isomorphic to \(\mathbb{C}^n/L = \mathbb{C}^n/2\pi(\mathbb{Z}^n \oplus T\mathbb{Z}^n)\). In this paper, we further assume that \(T\) is a non-singular matrix. Actually, in our setting described below, the mirror partner of \(\tilde{T}^{2n}\) turns not to exist if \(\text{det}T = 0\). However, we can avoid this problem and discuss the homological mirror symmetry even if \(\text{det}T = 0\) by modifying the definition of the mirror partner of \(\tilde{T}^{2n}\) and a class of holomorphic vector bundles which we treat. This fact will be discussed in [5].

Now, we define a class of holomorphic vector bundles \(E_{(r, A, \mu, \mathcal{L})}\) on \(\tilde{T}^{2n}\). This \(E_{(r, A, \mu, \mathcal{L})}\) corresponds to \(E(L, \mathcal{L})\) in the introduction. We first construct it as a complex vector bundle, and then discuss when it becomes a holomorphic vector bundle later in Proposition 2.2. However, since the notations of transition functions of \(E_{(r, A, \mu, \mathcal{L})}\) are complicated, before giving the strict definition of \(E_{(r, A, \mu, \mathcal{L})}\), we explain the idea of the construction of \(E_{(r, A, \mu, \mathcal{L})}\). We assume \(r \in \mathbb{N}, A \in M(n; \mathbb{Z})\) and \(\mu := (\mu_1, \cdots, \mu_n)^t \in \mathbb{C}^n\). This \(r \in \mathbb{N}\) denotes the rank of \(E_{(r, A, \mu, \mathcal{L})}\). Hereafter, sometimes we denote \(\mu = p + Tq\) with \(p := (p_1, \cdots, p_n)^t \in \mathbb{R}^n\), \(q := (q_1, \cdots, q_n)^t \in \mathbb{R}^n\). In general, the affine Lagrangian submanifold corresponding to a holomorphic vector bundle \(E_{(r, A, \mu, \mathcal{L})}\) is the following (we will explain the details of the symplectic geometry side again later).

\[
\left\{ \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \in (T^{2n}, \tilde{\omega}) \mid \hat{y} = \frac{1}{r} A\hat{x} + \frac{1}{r} p \right\}.
\]

Here, \(\hat{x} := (x^1, \cdots, x^n)^t\), \(\hat{y} := (y^1, \cdots, y^n)^t\) are the coordinates of the (complexified) symplectic torus \((T^{2n}, \tilde{\omega})\). In this situation, if \(x^j \mapsto x^j + 2\pi\) \((j = 1, \cdots, n)\),
then
\[ \hat{y} \mapsto \hat{y} + \frac{2\pi}{r}(a_{1j}, \ldots, a_{nj})^t. \]

We decide the transition functions of \( E_{(r, A, \mu, \mathcal{U})} \) by using this \( \frac{1}{r}(a_{1j}, \ldots, a_{nj})^t \in \mathbb{Q}^n \). This construction is a generalization of the case of \( (T^2, T^2) \) to the higher dimensional case in the paper [4] (see section 2). The strict definition of \( E_{(r, A, \mu, \mathcal{U})} \) is given as follows. Let

\[
\psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} : O_{m_1 \cdots m_n}^{l_1 \cdots l_n} \rightarrow \mathbb{C}^r, \quad l_j, m_k = 1, 2, 3
\]

be a smooth section of \( E_{(r, A, \mu, \mathcal{U})} \). The transition functions of \( E_{(r, A, \mu, \mathcal{U})} \) are non-trivial on

\[
O_{m_1 \cdots m_n}^{(l_1=3) \cdots l_n} \cap O_{m_1 \cdots m_n}^{(l_1=1) \cdots l_n}, \quad O_{m_1 \cdots m_n}^{(l_1=2) \cdots l_n} \cap O_{m_1 \cdots m_n}^{(l_2=1) \cdots l_n}, \ldots, \quad O_{m_1 \cdots m_n}^{(l_1=3) \cdots l_n} \cap O_{m_1 \cdots m_n}^{(l_n=1) \cdots l_n},
\]

and otherwise are trivial. We define the transition function on \( O_{m_1 \cdots m_n}^{l_1 \cdots l_n} \cap O_{m_1 \cdots m_n}^{l_1 \cdots l_n} \) by

\[
\psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} \big|_{O_{m_1 \cdots m_n}^{l_1 \cdots l_n} \cap O_{m_1 \cdots m_n}^{l_1 \cdots l_n}} = e^{\frac{r}{2}a_j y} V_j \psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} \big|_{O_{m_1 \cdots m_n}^{l_1 \cdots l_n} \cap O_{m_1 \cdots m_n}^{l_1 \cdots l_n}},
\]

where \( i = \sqrt{-1}, \ a_j := (a_{1j}, \ldots, a_{nj}) \) and \( V_j \in GL(r; \mathbb{C}) \). Similarly, we define the transition function on \( O_{m_1 \cdots m_n}^{l_1 \cdots l_n} \cap O_{m_1 \cdots m_n}^{l_1 \cdots l_n} \) by

\[
\psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} \big|_{O_{m_1 \cdots m_n}^{l_1 \cdots l_n} \cap O_{m_1 \cdots m_n}^{l_1 \cdots l_n}} = U_k \psi_{m_1 \cdots m_n}^{l_1 \cdots l_n} \big|_{O_{m_1 \cdots m_n}^{l_1 \cdots l_n} \cap O_{m_1 \cdots m_n}^{l_1 \cdots l_n}},
\]

where \( U_k \in GL(r; \mathbb{C}) \). Moreover, when we define

\[
\psi_{m_1 \cdots m_n}^{l_1 \cdots (l_j=3) \cdots l_n} \big|_{O_{m_1 \cdots m_n}^{l_1 \cdots (l_j=3) \cdots l_n} \cap O_{m_1 \cdots m_n}^{l_1 \cdots (l_j=1) \cdots l_n}} = U_k \psi_{m_1 \cdots m_n}^{l_1 \cdots (l_j=3) \cdots l_n} \big|_{O_{m_1 \cdots m_n}^{l_1 \cdots (l_j=3) \cdots l_n} \cap O_{m_1 \cdots m_n}^{l_1 \cdots (l_j=1) \cdots l_n}},
\]

the cocycle condition is expressed as

\[
V_j V_k = V_k V_j, \quad U_j U_k = U_k U_j, \quad \omega^{-a_k} U_k V_j = V_j U_k,
\]

5
where $\omega$ is the $r$-th root of 1 and $j,k = 1, \ldots, n$. We define the set $\mathcal{U}$ of non-singular matrices by

$$\mathcal{U} := \{ V_j, U_k \in GL(r; \mathbb{C}) \mid V_j V_k = V_k V_j, \ U_j U_k = U_k U_j, \ \omega^{-\alpha_{ij}} U_k V_j = V_j U_k, \ j,k = 1, \ldots, n \}.$$ 

**Remark 2.1.** The set $\mathcal{U}$ can be empty. For example, when we consider the case of $T^4$ and assume $(r, A) = (2, I_2)$, no four matrices $V_1, V_2, U_1, U_2 \in GL(2; \mathbb{C})$ satisfy the relations $V_1 V_2 = V_2 V_1$, $U_1 U_2 = U_2 U_1$, $-U_1 V_1 = V_1 U_1$, $U_1 V_2 = V_2 U_1$, $U_2 V_1 = V_1 U_2$, $-U_2 V_2 = V_2 U_2$, where $I_r$ denotes the identity matrix of order $r$. Thus, the holomorphic vector bundles such that $(r, A) = (2, I_2)$ do not exist on $T^4$.

When we give $r, A, \mu$ and $\mathcal{U}$, a complex vector bundle $E_{(r, A, \mu, \mathcal{U})}$ is defined. Furthermore, we define a connection $\nabla_{(r, A, \mu, \mathcal{U})}$ on $E_{(r, A, \mu, \mathcal{U})}$ locally as

$$\nabla_{(r, A, \mu, \mathcal{U})} = d + \omega_{(r, A, \mu, \mathcal{U})} := d - \frac{i}{2\pi} \left( \frac{1}{r} x^t A^t + \frac{1}{r} \mu^t \right) dy \cdot I_r,$$

where $dy := (dy_1, \ldots, dy_n)^t$ and $d$ denotes the exterior derivative. In fact, $\nabla_{(r, A, \mu, \mathcal{U})}$ is compatible with the transition functions and so defines a global connection. Then, its curvature form $\Omega_{(r, A, \mu, \mathcal{U})}$ is

$$\Omega_{(r, A, \mu, \mathcal{U})} = -\frac{i}{2\pi r} dx^i A^i dy \cdot I_r,$$

where $dx := (dx_1, \ldots, dx_n)^t$. Here, we consider the condition such that $E_{(r, A, \mu, \mathcal{U})}$ is holomorphic. We see that the following proposition holds.

**Proposition 2.2.** A complex vector bundle $E_{(r, A, \mu, \mathcal{U})}$ is holomorphic if and only if $AT$ is a symmetric matrix.

**Proof.** In general, a complex vector bundle is holomorphic if and only if the $(0,2)$-part of its curvature form is vanishes, so we calculate the $(0,2)$-part of $\Omega_{(r, A, \mu, \mathcal{U})}$. It turns out to be

$$\Omega^{(0,2)}_{(r, A, \mu, \mathcal{U})} = \frac{i}{2\pi r} d\tilde{z}^i \{ T(T - \bar{T})^{-1} \}^t A^t (T - \bar{T})^{-1} d\tilde{z} \cdot I_r,$$

where $d\tilde{z} := (d\tilde{z}_1, \ldots, d\tilde{z}_n)^t$. Thus, $\Omega^{(0,2)}_{(r, A, \mu, \mathcal{U})} = 0$ is equivalent to that $\{ T(T - \bar{T})^{-1} \}^t A^t (T - \bar{T})^{-1}$ is a symmetric matrix, i.e., $AT = (AT)^t$. \hfill $\Box$

Next, we explain the symplectic geometry side. We denote by $(x^1, \ldots, x^n, y^1, \ldots, y^n)^t$ the coordinates of the covering space $\mathbb{R}^{2n}$ of $T^{2n}$, and define

$$\tilde{x} := (x^1, \ldots, x^n)^t, \ \tilde{y} := (y^1, \ldots, y^n)^t.$$
Similarly as in the case $\tilde{T}^{2n}$, we regard $(x^1, \cdots, x^n, y^1, \cdots, y^n)^t$ as a point of $T^{2n}$ by identifying $x^i \sim x^i + 2\pi$ and $y^i \sim y^i + 2\pi$ for each $i = 1, \cdots, n$. We define a complexified symplectic form $\tilde{\omega}$ on $T^{2n}$ by

$$\tilde{\omega} := -d\tilde{x}^t(T^{-1})^t d\tilde{y},$$

where $d\tilde{x} := (dx^1, \cdots, dx^n)^t$ and $d\tilde{y} := (dy^1, \cdots, dy^n)^t$. We decompose $\tilde{\omega}$ into

$$\tilde{\omega} = -d\tilde{x}^t \Re(T^{-1})^t d\tilde{y} - i d\tilde{x}^t \Im(T^{-1})^t d\tilde{y},$$

and define

$$\omega := \Im(T^{-1})^t, \quad B := \Re(T^{-1})^t.$$

Sometimes we identify $\omega$ and $B$ with the 2-forms $d\tilde{x}^t \omega d\tilde{y}$ and $d\tilde{x}^t B d\tilde{y}$, respectively. Then, $-\omega$ defines a symplectic form on $T^{2n}$. The closed 2-form $-B$ is often called the $B$-field. We explain the objects of the Fukaya category $\text{Fuk}(T^{2n}, \tilde{\omega})$ on $(T^{2n}, \tilde{\omega})$ corresponding to holomorphic vector bundles $(\tilde{E}_{(r,A,p,t)}, \nabla_{(r,A,p,t)})$ on $T^{2n}$, namely, the pairs of Lagrangian submanifolds and unitary local systems on them. We define a map $s_{(r,A,p)} : \mathbb{R}^n \to \mathbb{C}^n$ by

$$s_{(r,A,p)}(\tilde{x}) := \frac{1}{r} A\tilde{x} + \frac{1}{r} \mu = \frac{1}{r} A\tilde{x} + \frac{1}{r} (p + T^t q).$$

We remove the term $\frac{1}{r} T^t q$ from the above $s_{(r,A,p)}(\tilde{x})$, and consider the graph of

$$\tilde{y} = \frac{1}{r} A\tilde{x} + \frac{1}{r} p.$$

By a direct calculation, we see that the graph of $\tilde{y} = \frac{1}{r} A\tilde{x} + \frac{1}{r} p$ becomes a Lagrangian submanifold $\tilde{L}_{(r,A,p)}$ in $\mathbb{R}^{2n}$ if and only if $\omega A = (\omega A)^t$ holds. Then, for the covering map $\pi : \mathbb{R}^{2n} \to T^{2n}$, $L_{(r,A,p)} := \pi(\tilde{L}_{(r,A,p)})$ defines a Lagrangian submanifold in $(T^{2n}, \omega)$. Furthermore, we consider a flat complex vector bundle $\mathcal{L}_{(r,A,p,t)} : L_{(r,A,p)} \to L_{(r,A,p)}$, with a unitary holonomy. Note that $q \in \mathbb{R}^n$ is the unitary holonomy of $\mathcal{L}_{(r,A,p,t)}$ along $L_{(r,A,p)} \cong T^n$. These $L_{(r,A,p)}$, $\mathcal{L}_{(r,A,p,t)}$ correspond to $L$, $\mathcal{L}$ in the introduction, respectively. By the definition of the Fukaya category, we have

$$\Omega_{\mathcal{L}_{(r,A,p,t)}} = -d\tilde{x}^t B d\tilde{y} \big|_{L_{(r,A,p)}},$$

where $\Omega_{\mathcal{L}_{(r,A,p,t)}}$ is the curvature form of the flat connection of $\mathcal{L}_{(r,A,p,t)}$, i.e., $\Omega_{\mathcal{L}_{(r,A,p,t)}} = 0$. Hence, we see

$$-d\tilde{x}^t B d\tilde{y} \big|_{L_{(r,A,p)}} = -\frac{1}{r} d\tilde{x}^t B A d\tilde{x} = 0,$$

so one has $BA = (BA)^t$. Note that $\omega A = (\omega A)^t$ and $BA = (BA)^t$ hold if and only if $AT = (AT)^t$ holds, i.e., $E_{(r,A,p,t)}$ becomes a holomorphic vector bundle on $\tilde{T}^{2n}$ (Proposition 2.2). Hereafter, for $(L_{(r,A,p)}, \mathcal{L}_{(r,A,p,t)}) \in \text{Ob}(\text{Fuk}(T^{2n}, \tilde{\omega}))$, we often forget the additional structure $\mathcal{L}_{(r,A,p,t)}$ and treat $L_{(r,A,p)}$ only, because
we mainly focus on the Lagrangian submanifolds when we discuss symplectic geometry side in this paper.

We define the DG-category $\text{DG}_{\mathcal{T}^2}$ consisting of holomorphic vector bundles $(E(\mathcal{r},A,\mu,\mathcal{U}), \nabla(\mathcal{r},A,\mu,\mathcal{U}))$. This definition is an extension of the case of $(\mathcal{T}^2, \mathcal{T}^2)$ to the higher dimensional case in the paper [4] (see section 3). The objects of $\text{DG}_{\mathcal{T}^2}$ are holomorphic vector bundles $E(\mathcal{r},A,\mu,\mathcal{U})$ with $U(\mathcal{r})$-connections $\nabla(\mathcal{r},A,\mu,\mathcal{U})$. Of course, we assume $A T = (A T)^t$. Sometimes we simply denote by $E(\mathcal{r},A,\mu,\mathcal{U})$ a holomorphic vector bundle with a $U(\mathcal{r})$-connection $(E(\mathcal{r},A,\mu,\mathcal{U}), \nabla(\mathcal{r},A,\mu,\mathcal{U}))$.

For any two objects $E(\mathcal{r},A,\mu,\mathcal{U}) = (E(\mathcal{r},A,\mu,\mathcal{U}), \nabla(\mathcal{r},A,\mu,\mathcal{U}))$, $E(\mathcal{s},\mathcal{B},\mathcal{\nu},\mathcal{V}) = (E(\mathcal{s},\mathcal{B},\mathcal{\nu},\mathcal{V}), \nabla(\mathcal{s},\mathcal{B},\mathcal{\nu},\mathcal{V}))$, the space of morphisms is defined by

$$\text{Hom}_{\text{DG}_{\mathcal{T}^2}}(E(\mathcal{r},A,\mu,\mathcal{U}), E(\mathcal{s},\mathcal{B},\mathcal{\nu},\mathcal{V})) := \Gamma(E(\mathcal{r},A,\mu,\mathcal{U}), E(\mathcal{s},\mathcal{B},\mathcal{\nu},\mathcal{V})) \otimes_{C^\infty(\mathcal{T}^2)} \Omega^{0,*}(\mathcal{T}^2),$$

where $\Omega^{0,*}(\mathcal{T}^2)$ is the space of anti-holomorphic differential forms, and $\Gamma(E(\mathcal{r},A,\mu,\mathcal{U}), E(\mathcal{s},\mathcal{B},\mathcal{\nu},\mathcal{V}))$ is the space of homomorphisms from $E(\mathcal{r},A,\mu,\mathcal{U})$ to $E(\mathcal{s},\mathcal{B},\mathcal{\nu},\mathcal{V})$. The space of morphisms $\text{Hom}_{\text{DG}_{\mathcal{T}^2}}(E(\mathcal{r},A,\mu,\mathcal{U}), E(\mathcal{s},\mathcal{B},\mathcal{\nu},\mathcal{V}))$ is a $\mathbb{Z}$-graded vector space, where the grading is defined as the degree of the anti-holomorphic differential forms. The degree $r$ part is denoted $\text{Hom}_{\text{DG}_{\mathcal{T}^2}}^r(E(\mathcal{r},A,\mu,\mathcal{U}), E(\mathcal{s},\mathcal{B},\mathcal{\nu},\mathcal{V}))$. We decompose $\nabla(\mathcal{r},A,\mu,\mathcal{U})$ into its holomorphic part and anti-holomorphic part $\nabla(\mathcal{r},A,\mu,\mathcal{U}) = \nabla^{(1,0)}(\mathcal{r},A,\mu,\mathcal{U}) + \nabla^{(0,1)}(\mathcal{r},A,\mu,\mathcal{U})$, and define a linear map

$$\text{Hom}_{\text{DG}_{\mathcal{T}^2}}^r(E(\mathcal{r},A,\mu,\mathcal{U}), E(\mathcal{s},\mathcal{B},\mathcal{\nu},\mathcal{V})) \to \text{Hom}_{\text{DG}_{\mathcal{T}^2}}^{r+1}(E(\mathcal{r},A,\mu,\mathcal{U}), E(\mathcal{s},\mathcal{B},\mathcal{\nu},\mathcal{V}))$$

by

$$\psi \mapsto (2 \nabla^{(0,1)}(\mathcal{s},\mathcal{B},\mathcal{\nu},\mathcal{V}))(\psi) - (-1)^r \psi(2 \nabla^{(0,1)}(\mathcal{r},A,\mu,\mathcal{U})).$$

We can check that this linear map is a differential. Furthermore, the product structure is defined by the composition of homomorphisms of vector bundles together with the wedge product for the anti-holomorphic differential forms. Then, these differential and product structure satisfy the Leibniz rule. Thus, $\text{DG}_{\mathcal{T}^2}$ forms a DG-category.

3 The construction of an isomorphism $E(\mathcal{r},A,\mu,\mathcal{U}) \cong E(\mathcal{r},A,\mu,\mathcal{U})$

In this section, we first recall the definition of projectively flat bundles and some properties of them. Next, we construct a one-to-one correspondence between holomorphic vector bundles $(E(\mathcal{r},A,\mu,\mathcal{U}), \nabla(\mathcal{r},A,\mu,\mathcal{U}))$ and a certain kind of projectively flat bundles. In general, the factors of automorphy of projectively
flat bundles on complex tori are classified concretely, so we interpret holomorphic vector bundles \( (E_{(r,A,\mu,U)}, \nabla_{(r,A,\mu,U)}) \) in the language of those factors of automorphy. This result is given in Theorem 3.6.

We recall the definition of factors of automorphy for holomorphic vector bundles following [6]. Let \( M \) be a complex manifold such that its universal covering space \( \tilde{M} \) is a topologically trivial (contractible) Stein manifold (\( \mathbb{C}^n \) is an example of a Stein manifold). Let \( p: \tilde{M} \to M \) be the covering projection and \( \Gamma \) the covering transformation group acting on \( \tilde{M} \) so that \( M = \tilde{M}/\Gamma \). Let \( E \) be a holomorphic vector bundle of rank \( r \) over \( M \). Then its pull-back \( \tilde{E} = p^*E \) is a holomorphic vector bundle of the same rank over \( \tilde{M} \). Since \( \tilde{M} \) is topologically trivial, \( \tilde{E} \) is topologically a product bundle. Since \( \tilde{M} \) is Stein, by Oka’s principle, \( \tilde{E} \) is holomorphically a product bundle, i.e., \( \tilde{E} = \tilde{M} \times \mathbb{C}^r \). Having fixed this isomorphism, we define a holomorphic map \( j: \Gamma \times \tilde{M} \to GL(r; \mathbb{C}) \) by the following commutative diagram

\[
\begin{array}{ccc}
\tilde{E}_{\gamma(x)} & \cong & \mathbb{C}^r \\
\downarrow & & \downarrow \\
\tilde{E}_{p(x)} & \cong & \mathbb{C}^r \\
\end{array}
\]

where \( x \in \tilde{M}, \gamma \in \Gamma \). Then, for \( x \in \tilde{M}, \gamma, \gamma' \in \Gamma \), the following relation holds.

\[
j(\gamma + \gamma', x) = j(\gamma', x + \gamma) \circ j(\gamma, x).
\]

The map \( j: \Gamma \times \tilde{M} \to GL(r; \mathbb{C}) \) is called the factor of automorphy for the holomorphic vector bundle \( E \).

For a complex vector bundle \( E \) of rank \( r \) with a connection \( D \) over a compact Kähler manifold \( M \), it is known that the following proposition holds.

**Proposition 3.2** ([9], [6], [13]). Let \( R \) be a curvature of \( (E,D) \). Then, \( E \) is projectively flat if and only if \( R \) takes values in scalar multiples of the identity endomorphism \( I_E \) of \( E \), i.e., if and only if there exists a complex 2-form \( \alpha \) on \( M \) such that \( R = \alpha \cdot I_E \).

There are many studies of projectively flat bundles on complex tori, i.e., the cases \( M = \mathbb{C}^n/\Gamma \), where \( \Gamma \) is a nondegenerate lattice of rank \( 2n \) of \( \mathbb{C}^n \) ([9], [6], [13] etc.). Let us denote the coordinates of \( \mathbb{C}^n \) by \( (z_1, \cdots, z_n)^t \). If we consider the case of \( M = \mathbb{C}^n/\Gamma \), the curvature form \( \Omega \) of \( E \) in Proposition 3.2 is expressed
locally as
\[ \Omega = \alpha \cdot I_r, \quad \alpha = \frac{1}{r} \sum_{i,j=1}^{n} R_{ij} dz_i \wedge d\bar{z}_j, \]
where \( R_{ij} = \bar{R}_{ji} \). By using this Hermitian matrix \((R_{ij})\), we define an Hermitian bilinear form on \( \mathbb{C}^n \) by
\[ R(z, w) = \sum_{i,j=1}^{n} R_{ij} z_i \bar{w}_j, \]
where \( z = (z_1, \ldots, z_n)^t \) and \( w = (w_1, \ldots, w_n)^t \). Then, a connection 1-form \( \omega \) of \( E \) is expressed locally as
\[ \omega = -\frac{1}{r} R(dz, z) \cdot I_r + dz^t b \cdot I_r, \]
where \( dz := (dz_1, \ldots, dz_n)^t \) and \( b := (b_1, \ldots, b_n)^t \in \mathbb{C}^n \). Moreover, it is known that a factor of automorphy \( j : \Gamma \times \mathbb{C}^n \to GL(r; \mathbb{C}) \) for a projectively flat bundle \( E \) over \( M = \mathbb{C}^n / \Gamma \) is expressed as follows (see [6], [13]).
\[ j(\gamma, z) = U(\gamma) \exp \left\{ \frac{1}{r} R(z, \gamma) + \frac{1}{2r} R(\gamma, \gamma) \right\}, \]
where \( (\gamma, z) \in \Gamma \times \mathbb{C}^n \), \( \text{Im} R(\gamma, \gamma') \in \pi \mathbb{Z} \), \( \gamma, \gamma' \in \Gamma \) and \( U : \Gamma \to GL(r; \mathbb{C}) \) is a semi-representation in the sense that it satisfies
\[ U(\gamma + \gamma') = U(\gamma) U(\gamma') e^{\frac{1}{2} \text{Im} R(\gamma', \gamma)}. \]

We consider the case of \( \tilde{T}^{2n} \cong \mathbb{C}^n / L = \mathbb{C}^n / 2\pi (\mathbb{Z}^n \oplus T\mathbb{Z}^n) \), and discuss the relations between holomorphic vector bundles \((E_{(r, A, \mu, U)}, \nabla_{(r, A, \mu, U)})\) and projectively flat bundles. Note that the curvature form \( \Omega_{(r, A, \mu, U)} \) of a holomorphic vector bundle \( E_{(r, A, \mu, U)} \) is expressed locally as
\[ \Omega_{(r, A, \mu, U)} = \frac{i}{2\pi r} dz^t (T - \bar{T})^{-1} A d\bar{z} \cdot I_r. \]
Now, we define
\[ R := \frac{i}{2\pi} (T - \bar{T})^{-1} A, \]
namely,
\[ \Omega_{(r, A, \mu, U)} = \frac{1}{r} dz^t R d\bar{z} \cdot I_r. \]
Then, the following lemma holds.

**Lemma 3.3.** The matrix \( R \) is a real symmetric matrix of order \( n \).
Thus, are symmetric because $AT$ definite, one has 

$$R = \frac{i}{2\pi} \{(T - \bar{T})^{-1}\}^t A(T - \bar{T})(T - \bar{T})^{-1}$$

and it is clear that the two matrices

$$\frac{i}{2\pi} \{(T - \bar{T})^{-1}\}^t A(T - \bar{T})^{-1}, \frac{i}{2\pi} \{(T - \bar{T})^{-1}\}^t \bar{A}(T - \bar{T})^{-1},$$

are symmetric because $AT = (AT)^t$. Hence, $R$ is a symmetric matrix. Furthermore, for a matrix $T = X + iY$, where $X, Y \in M(n; \mathbb{R})$ and $Y$ is positive definite, one has

$$R = \frac{1}{4\pi} (Y^{-1})^t A.$$ 

This relation implies $R \in M(n; \mathbb{R})$.

By a direct calculation,

$$R = \frac{i}{2\pi} \{(T - \bar{T})^{-1}\}^t A(T - \bar{T})(T - \bar{T})^{-1}$$

and it is clear that the two matrices

$$\frac{i}{2\pi} \{(T - \bar{T})^{-1}\}^t A(T - \bar{T})^{-1}, \frac{i}{2\pi} \{(T - \bar{T})^{-1}\}^t \bar{A}(T - \bar{T})^{-1},$$

are symmetric because $AT = (AT)^t$. Hence, $R$ is a symmetric matrix. Furthermore, for a matrix $T = X + iY$, where $X, Y \in M(n; \mathbb{R})$ and $Y$ is positive definite, one has

$$R = \frac{1}{4\pi} (Y^{-1})^t A.$$ 

By using this real symmetric matrix $R$ of order $n$, we define an Hermitian bilinear form $\mathcal{R} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ by

$$\mathcal{R}(z, w) := \sum_{i,j=1}^{n} R_{ij} z_i \bar{w}_j,$$

where $z = (z_1, \cdots, z_n)^t, w = (w_1, \cdots, w_n)^t$. Then, the following propositions hold.

**Proposition 3.4.** For $\gamma_1, \cdots, \gamma_n$ and $\gamma_1', \cdots, \gamma_n'$, $\text{Im} \mathcal{R}(\gamma_j, \gamma_k) = 0$, $\text{Im} \mathcal{R}(\gamma_j', \gamma_k') = 0$, where $j, k = 1, \cdots, n$.

**Proof.** By definition, $\mathcal{R}(\gamma_j, \gamma_k) = 4\pi^2 R_{jk}$, where $R_{jk} \in \mathbb{R}$, so $\text{Im} \mathcal{R}(\gamma_j, \gamma_k) = 0$. On the other hand, we see $\mathcal{R}(\gamma_j', \gamma_k') = 4\pi^2 (T^t \bar{R} T)_{jk}$, so for $T = X + iY$, it turns out to be

$$4\pi^2 T^t \bar{R} T = 4\pi^2 (X^t + iY^t) \cdot \frac{1}{4\pi} (Y^{-1})^t A \cdot (X - iY)$$

$$= \pi \{X^t (Y^{-1})^t AX + AY + i(AX - X^t (Y^{-1})^t AY)\}.$$ 

Thus,

$$\text{Im} \mathcal{R}(\gamma_j', \gamma_k') = \{\pi (AX - X^t (Y^{-1})^t AY)\}_{jk}$$

$$= \{\pi (AX - AX)\}_{jk}$$

$$= O_{jk}.$$ 

Here, the second equality follows from $AT = (AT)^t$. 

**Proposition 3.5.** For $\gamma_1, \cdots, \gamma_n$ and $\gamma_1', \cdots, \gamma_n'$, $\text{Im} \mathcal{R}(\gamma_j, \gamma_k) = -\pi a_{kj}$, $\text{Im} \mathcal{R}(\gamma_k', \gamma_j') = \pi a_{kj}$, where $j, k = 1, \cdots, n$. 

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Proof. First, we prove $\text{Im} R\tilde{T} = -\frac{1}{4\pi} A'$. For $T = X + iY,$
\[
R\tilde{T} = \frac{1}{4\pi} (Y^{-1})^t A X - \frac{1}{4\pi} (Y^{-1})^t A Y,
\]
so we see
\[
\text{Im} R\tilde{T} = -\frac{1}{4\pi} (Y^{-1})^t A Y = -\frac{1}{4\pi} A'.
\]
Here, we used $AT = (AT)^t$. Similarly as above, we can prove
\[
\text{Im} R\tilde{T} = \frac{1}{4\pi} A'.
\]
On the other hand, the following relations hold.
\[
\mathcal{R}(\gamma_j, \gamma'_k) = (4\pi^2 R\tilde{T})_{jk}, \quad \mathcal{R}(\gamma'_k, \gamma_j) = (4\pi^2 \tilde{R}T)_{jk}.
\]
Thus, by using $\text{Im} R\tilde{T} = -\frac{1}{4\pi} A'$ and $\text{Im} \tilde{R}T = \frac{1}{4\pi} A'$, we obtain
\[
\text{Im} \mathcal{R}(\gamma_j, \gamma'_k) = -\pi a_{kj}, \quad \text{Im} \mathcal{R}(\gamma'_k, \gamma_j) = \pi a_{kj}.
\]
\[\square\]

Now, we consider a projectively flat bundle $\mathcal{E}_{(r, A, \mu, \xi)}$ of rank $r$ whose factor of automorphy $j : L \times \mathbb{C}^n \to GL(r; \mathbb{C})$ and connection $\nabla_{(r, A, \mu, \xi)} = d + \omega_{(r, A, \mu, \xi)}$ are expressed locally as follows.
\[
j(\gamma, z) = U(\gamma) \exp \left\{ \frac{1}{r} \mathcal{R}(z, \gamma) + \frac{1}{2r} \mathcal{R}(\gamma, \gamma) \right\},
\]
\[
\omega_{(r, A, \mu, \xi)} = -\frac{1}{r} dz^\gamma R\mathbf{z} \cdot I_r - \frac{i}{2\pi r} \mu^t (T - \bar{T})^{-1} dz \cdot I_r.
\]
In particular, $U(\gamma_j), U(\gamma'_k) \in GL(r; \mathbb{C})$ $(j, k = 1, \cdots, n)$ satisfy the following relations (see Proposition 3.4, Proposition 3.5).
\[
U(\gamma_j) U(\gamma_k) = U(\gamma_k) U(\gamma_j), \quad U(\gamma'_j) U(\gamma'_k) = U(\gamma'_k) U(\gamma'_j), \quad \omega^{-a_{kj}} U(\gamma'_k) U(\gamma_j) = U(\gamma_j) U(\gamma'_k).
\]
Note that these relations are equivalent to the cocycle condition of $E_{(r, A, \mu, \xi)}$. Therefore, we can denote
\[
\mathcal{U} = \left\{ U(\gamma_j), U(\gamma'_k) \in GL(r; \mathbb{C}) \mid U(\gamma_j) U(\gamma_k) = U(\gamma_k) U(\gamma_j), \quad U(\gamma'_j) U(\gamma'_k) = U(\gamma'_k) U(\gamma'_j), \quad \omega^{-a_{kj}} U(\gamma'_k) U(\gamma_j) = U(\gamma_j) U(\gamma'_k), \quad j, k = 1, \cdots, n \right\}.
\]
This $\mathcal{E}_{(r, A, \mu, \xi)}$ correspond to $\mathcal{E}(L, \xi)$ in the introduction. The purpose of this section is to prove $E_{(r, A, \mu, \xi)} \cong \mathcal{E}_{(r, A, \mu, \xi)}$ (Theorem 3.6). Similarly as in the case of $E_{(r, A, \mu, \xi)}$, when we give $r, A, \mu$ and $\mathcal{U}$, a projectively flat bundle $\mathcal{E}_{(r, A, \mu, \xi)}$ is
defined. It is clear that the curvature form $\tilde{\Omega}_{(r,A,\mu,U)}$ of $E_{(r,A,\mu,U)}$ is expressed locally as
\[ \Omega_{(r,A,\mu,U)} = \frac{1}{r} dz^i R dz^i \cdot I_r. \]
Hence, we fix $r, A, \mu$ and by comparing the definition of $E_{(r,A,\mu,U)}$ with the definition of $E_{(r,A,\mu,U)}$, we see that the cardinality of the set $\{ (E_{(r,A,\mu,U)}, \nabla_{(r,A,\mu,U)}) \}$ is equal to the cardinality of the set $\{ (E_{(r,A,\mu,U)}, \nabla_{(r,A,\mu,U)}) \}$. Thus, we expect that there exists an isomorphism $\Psi : E_{(r,A,\mu,U)} \cong E_{(r,A,\mu,U)}$ which gives a correspondence between $\{ (E_{(r,A,\mu,U)}, \nabla_{(r,A,\mu,U)}) \}$ and $\{ (E_{(r,A,\mu,U)}, \nabla_{(r,A,\mu,U)}) \}$ such that
\[ U(\gamma_j) = c_j(r, A, \mu, T) \cdot V_j, \quad U(\gamma'_k) = c'_k(r, A, \mu, T) \cdot U_k, \]
\[ c_j(r, A, \mu, T), c'_k(r, A, \mu, T) \in \mathbb{C}^n, \quad j, k = 1, \cdots, n. \]
Actually, the following theorem holds, and this is the main theorem in this section.

**Theorem 3.6.** One has $E_{(r,A,\mu,U)} \cong E_{(r,A,\mu,U)}$, where an isomorphism $\Psi : E_{(r,A,\mu,U)} \cong E_{(r,A,\mu,U)}$ is expressed locally as
\[ \Psi(z, \bar{z}) = \exp \left\{ \frac{i}{4\pi r} z^i A z + \frac{i}{4\pi r} \bar{z}^i A \bar{z} - \frac{i}{2\pi r} z^i \bar{A} \bar{z} + \frac{i}{2\pi r} \bar{z}^i (T - \bar{T})^{-1} \mu \right\} \cdot I_r, \]
\[ A := \{ (T - \bar{T})^{-1} \}^t T A^t (T - \bar{T})^{-1}. \]

**Proof.** Note that $A$ is a symmetric matrix because $AT = (AT)^t$. By a direct calculation, we see
\[ \omega_{(r,A,\mu,U)} = \frac{i}{2\pi} z^i A dz \cdot I_r - \frac{i}{2\pi} z^i A dz \cdot I_r - \frac{i}{2\pi} \bar{z}^i \bar{A} d\bar{z} \cdot I_r + \frac{i}{2\pi} \bar{z}^i \bar{A} d\bar{z} \cdot I_r \]
\[ - \frac{i}{2\pi} \mu^t (T - \bar{T})^{-1} dz \cdot I_r + \frac{i}{2\pi} \mu^t (T - \bar{T})^{-1} d\bar{z} \cdot I_r. \]
We construct an isomorphism $\Psi : E_{(r,A,\mu,U)} \rightarrow E_{(r,A,\mu,U)}$ explicitly such that its local expression is
\[ \Psi(z, \bar{z}) = \psi(z, \bar{z}) \cdot I_r, \]
where $\psi(z, \bar{z})$ is a function defined locally. We consider the following differential equation.
\[ \nabla_{(r,A,\mu,U)} \Psi(z, \bar{z}) - \Psi(z, \bar{z}) \nabla_{(r,A,\mu,U)} = O. \tag{2} \]
For $A$ and $R$, we see
\[ \frac{i}{2\pi} \bar{A} - R \]
\[ = \frac{i}{2\pi} \{ (T - \bar{T})^{-1} \}^t T A^t (T - \bar{T})^{-1} - \frac{i}{2\pi} \{ (T - \bar{T})^{-1} \}^t (T - \bar{T}) A^t (T - \bar{T})^{-1} \]
\[ = \frac{i}{2\pi} \{ (T - \bar{T})^{-1} \}^t T A^t (T - \bar{T})^{-1} \]
\[ = \frac{i}{2\pi} A, \]
so one has
\[ \frac{i}{2\pi} (\hat{A} - A) = R. \]  

(3)

By using the identity (3), the differential equation (2) turns out to be
\[(\partial + \bar{\partial})(\psi(z, \bar{z})) \cdot I_r = \frac{i}{2\pi\sigma} \left\{ z^tA\sigma - z^t\bar{A}\sigma - \bar{z}^t\bar{A}\sigma + \bar{z}^t\bar{A}\sigma + \mu^t(T - \bar{T})^{-1}d\bar{z}\right\} \wedge \psi(z, \bar{z}) \cdot I_r = O.\]

Thus, \[\psi(z, \bar{z}) = c \cdot \exp \left\{ \frac{i}{4\pi\sigma} z^tA\sigma + \frac{i}{4\pi\sigma} \bar{z}^t\bar{A}\sigma - \frac{i}{2\pi\sigma} z^t\bar{A}\sigma + \frac{i}{2\pi\sigma} \bar{z}^t\bar{A}\sigma + \bar{z}^t\{(T - \bar{T})^{-1}\}^t\mu \right\},\]
where \(c\) is an arbitrary constant, so by setting \(c = 1\), we obtain
\[\Psi(z, \bar{z}) = \exp \left\{ \frac{i}{4\pi\sigma} z^tA\sigma + \frac{i}{4\pi\sigma} \bar{z}^t\bar{A}\sigma - \frac{i}{2\pi\sigma} z^t\bar{A}\sigma + \frac{i}{2\pi\sigma} \bar{z}^t\{(T - \bar{T})^{-1}\}^t\mu \right\} \cdot I_r.\]

By using this \(\Psi : E_{(r, A, \mu, \mathcal{U})} \rightarrow E_{(r, A, \mu, \mathcal{U})}\), we transform the transition functions of \(E_{(r, A, \mu, \mathcal{U})}\). We see
\[e^{\frac{1}{4\pi\sigma} a_j(T - \bar{T})^{-1}\sigma - \frac{1}{2\pi\sigma} a_j(T - \bar{T})^{-1}\sigma} = e^{\frac{1}{4\pi\sigma} (\gamma_j) - \frac{1}{2\pi\sigma} (\gamma_j)} \cdot e^{\frac{1}{4\pi\sigma} \bar{a}_j(T - \bar{T})^{-1}\bar{\sigma} - \frac{1}{2\pi\sigma} \bar{a}_j(T - \bar{T})^{-1}\bar{\sigma}}\]
so, we calculate the following formula.
\[(\Psi(z + \gamma_j, \bar{z} + \bar{\gamma}_j)) \left( e^{\frac{1}{4\pi\sigma} a_j(T - \bar{T})^{-1}\sigma - \frac{1}{2\pi\sigma} a_j(T - \bar{T})^{-1}\sigma} V_j \right) \left( \Psi^{-1}(z, \bar{z}) \right),\]  

(4)

where \(j = 1, \ldots, n\). We set
\[(\hat{A} - A)_j := (\hat{A}_{1j} - A_{1j}, \ldots, \hat{A}_{nj} - A_{nj}), R_j := (R_{1j}, \ldots, R_{nj}),\]
and by using the identity (3), the formula (4) turns out to be
\[
\begin{align*}
\exp \left\{ \frac{i}{r} (\hat{A} - A)_j \bar{z} + \frac{\pi i}{r} (\hat{A} - A)_{jj} + \frac{i}{r} \{(T - \bar{T})^{-1}\}^t \mu_j \right\} \\
+ \frac{i}{r} a_j(T - \bar{T})^{-1}\bar{z} - \frac{i}{r} a_j(T - \bar{T})^{-1}\bar{z} V_j \\
= \exp \left\{ \frac{2\pi}{r} R_j z + \frac{2\pi^2}{r} R_{jj} + \frac{i}{r} \{(T - \bar{T})^{-1}\}^t \mu_j \right\} V_j.
\end{align*}
\]

On the other hand,
\[j(\gamma_j, z) = U(\gamma_j) \exp \left\{ \frac{1}{r} \mathcal{R}(z, \gamma_j) + \frac{1}{2r} \mathcal{R}(\gamma_j, \gamma_j) \right\}
= U(\gamma_j) \exp \left\{ \frac{2\pi}{r} R_j z + \frac{2\pi^2}{r} R_{jj} \right\},\]
Next, we calculate the following formula.

\[
\Psi(z + \gamma_k, \bar{z} + \gamma_k')(U_k)(\Psi^{-1}(z, \bar{z})),
\]

where \( k = 1, \cdots, n \). In order to calculate the formula (5), we prove the following relations.

\[
AT = \overline{AT}, \tag{6}
\]

\[
A(T - \bar{T}) = -2\pi iRT. \tag{7}
\]

We can show the identity (6) as follows. For \( T = X + iY \),

\[
AT = \{(T - \bar{T})^{-1}\}^{t}T^{t}A^{t}(T - \bar{T})^{-1}T
\]

\[
= -\frac{1}{4}(Y^{-1})^{t}X^{t}A^{t}Y^{-1}X - \frac{1}{4}A^{t} - \frac{i}{4}((Y^{-1})^{t}X^{t}A^{t} - A^{t}Y^{-1}X),
\]

and since \( AT = (AT)^{t} \),

\[
\text{Im}AT = -\frac{1}{4}((Y^{-1})^{t}X^{t}A^{t} - A^{t}Y^{-1}X) = O.
\]

This implies \( AT = \overline{AT} \). Furthermore, by a direct calculation,

\[
A(T - \bar{T}) = \{(T - \bar{T})^{-1}\}^{t}T^{t}A^{t}(T - \bar{T})^{-1}(T - \bar{T})
\]

\[
= \{(T - \bar{T})^{-1}\}^{t}A\bar{T}
\]

\[
= -2\pi iRT,
\]

so we obtain the identity (7). Now, we calculate the formula (5). We set

\[
(A(T - \bar{T}))_{k} := (\{A(T - \bar{T})\}_{1k}, \cdots , (A(T - \bar{T}))_{nk}),
\]

\[
(R\bar{T})_{k} := ((R\bar{T})_{1k}, \cdots , (R\bar{T})_{nk}).
\]

By using the identities (6), (7), the formula (5) turns out to be

\[
\exp\left\{ \frac{i}{r}(A(T - \bar{T}))_{k}z + \frac{\pi i}{r}((AT)^{t}T)_{kk} + \frac{\pi i}{r}((AT)^{t}\bar{T})_{kk} - \frac{2\pi i}{r}((AT)^{t}\bar{T})_{kk} \right\}U_k
\]

\[
= \exp\left\{ \frac{i}{r}(A(T - \bar{T}))_{k}z + \frac{\pi i}{r}(T^{t}A(T - \bar{T}))_{kk} + \frac{\pi i}{r}(\mu^{t}(T - \bar{T})^{-1}T)_{k} \right\}U_k
\]

\[
= \exp\left\{ \frac{2\pi i}{r}(R\bar{T})_{k}z + \frac{2\pi^{2} i}{r}(T^{t}R\bar{T})_{kk} + \frac{i}{r}(\mu^{t}(T - \bar{T})^{-1}T)_{k} \right\}U_k
\]

\[
= \exp\left\{ \frac{2\pi i}{r}(R\bar{T})_{k}z + \frac{2\pi^{2} i}{r}(\bar{T}^{t}RT)_{kk} + \frac{i}{r}(\mu^{t}(T - \bar{T})^{-1}T)_{k} \right\}U_k.
\]
On the other hand,
\[ j(\gamma_k', z) = U(\gamma_k') \exp \left\{ \frac{1}{r} R(z, \gamma_k') + \frac{1}{2r} R(\gamma_k', \gamma_k') \right\} = U(\gamma_k') \exp \left\{ \frac{2\pi}{r} (R\bar{T})_k z + \frac{2\pi^2}{r} (\bar{T}^t R T)_k \right\}, \]
so one has
\[ U(\gamma_k') = e^{\frac{1}{r}(\mu'(T - \bar{T})^{-1}\bar{T})_k} U_k. \]

4 Exact triangles consisting of projectively flat bundles on \( \hat{T}^{2n} \)

In this section, we give a geometric interpretation of exact triangles consisting of projectively flat bundles \( E_{(r,A,\mu,\mathcal{U})} \) on \( \hat{T}^{2n} \). In general, we can construct a triangulated category from a given DG-category [1], so we can obtain a triangulated category \( Tr(DG_{\hat{T}^{2n}}) \) from \( DG_{\hat{T}^{2n}} \). Note that exact triangles in \( Tr(DG_{\hat{T}^{2n}}) \) are defined as exact triangles associated to mapping cones. Here, we consider the mapping cones of morphisms between projectively flat bundles.

Theorem 4.1. We take two projectively flat bundles \( E_{(r,A,\mu,\mathcal{U})} \), \( E_{(s,B,\nu,\mathcal{V})} \) \((\mu = p + T^t q, \nu = u + T^t v)\) on \( \hat{T}^{2n} \), and consider the following exact triangle associated to the mapping cone of \( \psi \):
\[ \cdots \rightarrow E_{(r,A,\mu,\mathcal{U})} \rightarrow C(\psi) \rightarrow E_{(s,B,\nu,\mathcal{V})} \rightarrow T E_{(r,A,\mu,\mathcal{U})} \rightarrow \cdots, \] \hspace{1cm} (8)
where \( T \) is the shift functor and \( C(\psi) \) denotes the mapping cone of \( \psi \). Furthermore, we assume that there exists a projectively flat bundle \( E_{(t,C,\eta,\mathcal{W})} \) such that \( C(\psi) \cong E_{(t,C,\eta,\mathcal{W})} \), namely, the exact triangle (8) becomes the exact triangle consisting of three projectively flat bundles and their shifts. Then, \( \text{codim}(L_{(r,A,p)} \cap L_{(s,B,u)}) \leq 1 \) holds.

Proof. For convenience, we set
\[ \alpha := \frac{1}{r} A - \frac{1}{s} B, \quad \beta := \frac{1}{s} u - \frac{1}{r} p, \]
and consider the following two cases.

Case 1. \( \alpha = 0. \)

In this case, if \( \beta \in 2\pi \mathbb{Z}^n \), we see \( \text{codim}(L_{(r,A,p)} \cap L_{(s,B,u)}) = 0 \) because \( L_{(r,A,p)} = L_{(s,B,u)} \), and otherwise \( L_{(r,A,p)} \cap L_{(s,B,u)} = \emptyset. \)
\textbf{case 2.} \( \alpha \neq O \).

We can check \( L_{(r,A,p)} \cap L_{(s,B,u)} \neq \emptyset \) as follows. Note that \( L_{(r,A,p)} \cap L_{(s,B,u)} \neq \emptyset \) holds if \( \text{Ext}^1(E_{(s,B,v,V)}, E_{(r,A,\mu,\mathcal{U})}) \neq 0 \) via the homological mirror symmetry [3], so we prove \( \text{Ext}^1(E_{(s,B,v,V)}, E_{(r,A,\mu,\mathcal{U})}) \neq 0 \) when \( \alpha \neq O \). We assume \( \text{Ext}^1(E_{(s,B,v,V)}, E_{(r,A,\mu,\mathcal{U})}) = 0 \). Then \( C(0) = E_{(r,A,\mu,\mathcal{U})} \oplus E_{(s,B,v,V)} \) and its curvature form is expressed locally as

\[ \begin{pmatrix} \Omega_{(r,A,\mu,\mathcal{U})} & 0 \\ 0 & \Omega_{(s,B,v,V)} \end{pmatrix}. \]

Now we assume that there exists a projectively flat bundle \( E_{(t,C,\eta,\mathcal{W})} \) such that \( C(0) \cong E_{(t,C,\eta,\mathcal{W})} \), so \( \Omega_{(r,A,\mu,\mathcal{U})} = \Omega_{(s,B,v,V)} \), i.e., \( \alpha = O \), but this fact contradicts \( \alpha \neq O \). Thus, we see \( L_{(r,A,p)} \cap L_{(s,B,u)} \neq \emptyset \).

We can prove \( \text{codim}(L_{(r,A,p)} \cap L_{(s,B,u)}) = 1 \) in this case. We define the 2-forms \( \Omega'_{(r,A,\mu,\mathcal{U})} \), \( \Omega'_{(s,B,v,V)} \) and \( \Omega'_{(t,C,\eta,\mathcal{W})} \) by

\[ \Omega'_{(r,A,\mu,\mathcal{U})} := \frac{1}{4\pi^2 r} d^2 A' dx dy, \]

\[ \Omega'_{(s,B,v,V)} := \frac{1}{4\pi^2 s} d^2 B' dx dy, \]

\[ \Omega'_{(t,C,\eta,\mathcal{W})} := \frac{1}{4\pi^2 t} d^2 C' dx dy, \]

namely,

\[ -\frac{1}{2\pi i} \Omega_{(r,A,\mu,\mathcal{U})} = \Omega'_{(r,A,\mu,\mathcal{U})} \cdot I_r, \]

\[ -\frac{1}{2\pi i} \Omega_{(s,B,v,V)} = \Omega'_{(s,B,v,V)} \cdot I_s, \]

\[ -\frac{1}{2\pi i} \Omega_{(t,C,\eta,\mathcal{W})} = \Omega'_{(t,C,\eta,\mathcal{W})} \cdot I_t. \]

Since we assume \( C(\psi) \cong E_{(t,C,\eta,\mathcal{W})} \), one has \( ch_i(C(\psi)) = ch_i(E_{(t,C,\eta,\mathcal{W})}) \), where \( i = 1, \ldots, n \) and \( ch_i(E) \) denotes the \( i \)-th chern character of a vector bundle \( E \). In particular,

\[ ch_i(C(\psi)) = ch_i(E_{(r,A,\mu,\mathcal{U})}) + ch_i(E_{(s,B,v,V)}), \]

so \( ch_i(C(\psi)) = ch_i(E_{(t,C,\eta,\mathcal{W})}) \) is equivalent to

\[ ch_i(E_{(r,A,\mu,\mathcal{U})}) + ch_i(E_{(s,B,v,V)}) = ch_i(E_{(t,C,\eta,\mathcal{W})}). \]  \hspace{1cm} (9)

Now we calculate \( ch_i(C(\psi)) \), \( ch_i(E_{(t,C,\eta,\mathcal{W})}) \) and consider the equality (9). It is clear that the equality (9) in the cases \( i = 0, 1 \) are equivalent to

\[ r + s = t, \]

\[ r\Omega'_{(r,A,\mu,\mathcal{U})} + s\Omega'_{(s,B,v,V)} = t\Omega'_{(t,C,\eta,\mathcal{W})}. \]  \hspace{1cm} (10)

(11)
respectively. We consider the equality (9) in the case \( i = 2 \). By a direct calculation, the equality (9) turns out to be

\[
\frac{r}{2}(\Omega_{(r,A,\mu,U)}')^2 + \frac{s}{2}(\Omega_{(s,B,\nu,V)}')^2 = \frac{t}{2}(\Omega_{(t,C,\eta,W)}')^2,
\]

(12)

and we obtain the following relation by substituting the equality (11) into the equality (12).

\[
(rt - r^2)(\Omega_{(r,A,\mu,U)}')^2 + (st - s^2)(\Omega_{(s,B,\nu,V)}')^2 = 2rs\Omega_{(r,A,\mu,U)}' \land \Omega_{(s,B,\nu,V)}'.
\]

(13)

Furthermore, by substituting the equality (10) into the equality (13), the equality (13) turns out to be

\[
rs(\Omega_{(r,A,\mu,U)}')^2 + rs(\Omega_{(s,B,\nu,V)}')^2 = 2rs\Omega_{(r,A,\mu,U)}' \land \Omega_{(s,B,\nu,V)}',
\]

and this relation is equivalent to

\[
(\Omega_{(r,A,\mu,U)}' - \Omega_{(s,B,\nu,V)}')^2 = 0.
\]

(14)

In general, for \( i \geq 3 \), we obtain the following equality by expanding the equality (9).

\[
\left( r \sum_{k=1}^{i-1} \binom{i-1}{k} r^{i-1-k} s^k \right) \left( \Omega_{(r,A,\mu,U)}' \right)^i + \left( s \sum_{k=0}^{i-2} \binom{i-1}{k} r^{i-1-k} s^k \right) \left( \Omega_{(s,B,\nu,V)}' \right)^i - \sum_{k=1}^{i-1} \binom{i}{k} \left( r\Omega_{(r,A,\mu,U)}' \right)^{i-k} \left( s\Omega_{(s,B,\nu,V)}' \right)^k = 0.
\]

(15)

Note that the left hand side of the equality (15) can be factored as

\[
(\Omega_{(r,A,\mu,U)}' - \Omega_{(s,B,\nu,V)}')^2 \times \sum_{l=0}^{i-2} \left\{ \left( \sum_{k=1}^{l} \binom{l}{k} (i - l - 1) \binom{i-1}{k} r^{i-k} s^k + (l + 1) \sum_{k=l+1}^{i-1} \binom{i-1}{k} r^{i-k} s^k \right) \times (\Omega_{(r,A,\mu,U)}')^{i-l-2} (\Omega_{(s,B,\nu,V)}')^l \right\}.
\]

Hence, when the equality (14) holds, the equality (15) holds automatically. Moreover, by definition,

\[
\Omega_{(r,A,\mu,U)}' - \Omega_{(s,B,\nu,V)}' = \frac{1}{4\pi^2} dx^t \left( \frac{1}{r} A^t - \frac{1}{s} B^t \right) dy = \frac{1}{4\pi^2} dx^t \alpha^t dy,
\]

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so by a direct calculation, one has

\[(\Omega_{(r,A,\mu,\lambda)} - \Omega_{(s,B,\nu,\gamma)})^2 = \frac{1}{8\pi^4} \sum_{1 \leq i < j \leq n, 1 \leq k < l \leq n} (\alpha_{ik} \alpha_{jl} - \alpha_{il} \alpha_{jk}) dx_k \wedge dy_i \wedge dx_l \wedge dy_j.\]

Thus, the equality (14) is equivalent to

\[\det \begin{pmatrix} \alpha_{ik} & \alpha_{jl} \\ \alpha_{jk} & \alpha_{ij} \end{pmatrix} = \alpha_{il} \alpha_{jk} - \alpha_{ik} \alpha_{jl} = 0,
\]

where \(1 \leq i < j \leq n, 1 \leq k < l \leq n\).

We consider the following system of linear equations.

\[\alpha \dot{x} = \beta. \tag{17}\]

Now we assume \(\alpha \neq 0\), so there exists an \(\alpha_{ij} \neq 0\). To an augmented matrix \((\alpha, \beta)\), we apply elementary row operations in order to solve the equation (17).

First, we multiply the first row of \((\alpha, \beta)\) by \(\alpha_{ij}\).

\[(\alpha, \beta) \rightarrow (\alpha, \beta)' := \begin{pmatrix} \alpha_{11} \alpha_{ij} & \ldots & \alpha_{1j} \alpha_{ij} & \ldots & \alpha_{1n} \alpha_{ij} & \beta_1 \alpha_{ij} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \alpha_{i1} & \ldots & \alpha_{ij} & \ldots & \alpha_{in} & \beta_i \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \alpha_{n1} & \ldots & \alpha_{nj} & \ldots & \alpha_{nn} & \beta_n \end{pmatrix}.
\]

Next, we add the \(i\)-th row of \((\alpha, \beta)'\) multiplied by \(-\alpha_{1j}\) to the first row of \((\alpha, \beta)'\).

\[(\alpha, \beta)' \rightarrow (\alpha, \beta)'' := \begin{pmatrix} \alpha_{11} \alpha_{ij} - \alpha_{i1} \alpha_{1j} & \ldots & 0 & \ldots & \alpha_{1n} \alpha_{ij} - \alpha_{in} \alpha_{1j} & \beta_1 \alpha_{ij} - \beta_i \alpha_{1j} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \alpha_{i1} & \ldots & \alpha_{ij} & \ldots & \alpha_{in} & \beta_i \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \alpha_{n1} & \ldots & \alpha_{nj} & \ldots & \alpha_{nn} & \beta_n \end{pmatrix}.
\]

Then, by using the equality (16), we see that all components of the first row of \((\alpha, \beta)''\) except \(\beta_1 \alpha_{ij} - \beta_i \alpha_{1j}\) are zero, namely,

\[(\alpha, \beta)'' = \begin{pmatrix} 0 & \ldots & 0 & 0 & \beta_1 \alpha_{ij} - \beta_i \alpha_{1j} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_{i1} & \ldots & \alpha_{ij} & \ldots & \alpha_{in} & \beta_i \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_{n1} & \ldots & \alpha_{nj} & \ldots & \alpha_{nn} & \beta_n \end{pmatrix}.
\]

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By applying elementary row operations to \((\alpha, \beta)^\prime\) similarly as above, \((\alpha, \beta)^\prime\) is transformed as follows finally.

\[
(\alpha, \beta)^\prime := \begin{pmatrix}
0 & \ldots & 0 & \ldots & 0 & \beta_1 \alpha_{ij} - \beta_i \alpha_{1j} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\alpha_{i1} & \ldots & \alpha_{ij} & \ldots & \alpha_{in} & \beta_i \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & \ldots & 0 & \beta_n \alpha_{ij} - \beta_i \alpha_{nj}
\end{pmatrix}.
\]

Since \(L_{(r,A,p)} \cap L_{(s,B,u)} \neq \emptyset\), i.e., \(\text{rank}(\alpha, \beta) = \text{rank} \alpha\) in the case \(\alpha \neq O\), we can regard all the components of the \((n+1)\)-th column of \((\alpha, \beta)^\prime\) except \(\beta_i\) as zero, namely, one obtains

\[
\alpha_i x^1 + \cdots + \alpha_{ij} x^j + \cdots + \alpha_n x^n = \beta_i.
\]

Thus, when we put \(x^i = k^i \in \mathbb{R} \ (i \neq j)\), \(x^j\) is expressed as

\[
x^j = -\frac{\alpha_{1j}}{\alpha_{ij}} k^1 - \cdots - \frac{\alpha_{nj}}{\alpha_{ij}} k^n + \frac{\beta_i}{\alpha_{ij}},
\]

where \(\alpha_{ij} \neq 0\), and this fact implies \(\text{codim}(L_{(r,A,p)} \cap L_{(s,B,u)}) = 1\), i.e., \(\text{codim}(L_{(r,A,p)} \cap L_{(s,B,u)}) = 1\).

Finally, we give an example in the case of \(n = 1\), i.e., \((T^2, \omega = -\frac{1}{\tau} dx^1 \wedge dy^1)\) and \(\tilde{T}^2 \cong \mathbb{C}/2\pi(\mathbb{Z} \oplus \tau \mathbb{Z})\), where \(\tau \in \mathbb{H}\). We take two maps

\[
s_{(1,0,\mu)}(x^1) = \mu = p + q \tau, \quad s_{(1,1,\nu)}(x^1) = x^1 + \nu = x^1 + u + v \tau,
\]

and consider the mapping cone of \(\psi : E_{(1,1,\nu \mathcal{V})} \to TE_{(1,0,\mu \mathcal{U})}\), namely, the following exact triangle.

\[
\cdots \longrightarrow E_{(1,0,\mu \mathcal{U})} \longrightarrow C(\psi) \longrightarrow E_{(1,1,\nu \mathcal{V})} \longrightarrow \psi \longrightarrow TE_{(1,0,\mu \mathcal{U})} \cdots.
\]

Here, note

\[
\mathcal{U} = \mathcal{V} = \left\{ V_1 = 1, U_1 = 1 \in \text{GL}(1; \mathbb{C}) = \mathbb{C}^* \right\}.
\]

Then, for a map

\[
s_{(2,1,\eta)}(x^1) = \frac{1}{2} x^1 + \frac{1}{2} \eta,
\]

\(C(\psi) \cong E_{(2,1,\eta \mathcal{W})}\) if and only if \(\eta \equiv \mu + \nu + \pi + \pi \tau \pmod{2\pi(\mathbb{Z} \oplus \tau \mathbb{Z})}\), where

\[
\mathcal{W} = \left\{ V_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, U_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{GL}(2; \mathbb{C}) \right\},
\]

and in fact, \(\text{codim}(L_{(1,0,p)} \cap L_{(1,1,u)}) = 1\). In particular, the fact that \(C(\psi) \cong E_{(2,1,\eta \mathcal{W})}\) if and only if \(\eta \equiv \mu + \nu + \pi + \pi \tau \pmod{2\pi(\mathbb{Z} \oplus \tau \mathbb{Z})}\) is the statement of Theorem 4.10 in [4].
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