Bouncing Universe in the Contexts of Generalized Cosmic Chaplygin Gas and Variable Modified Chaplygin Gas

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In this work, we have considered the Friedmann-Robertson-Walker (FRW) model of the universe where bounce occurs and the universe is filled with Generalized Cosmic Chaplygin Gas (GCCG) or Variable Modified Chaplygin Gas (VMCG). We have studied the stability analysis through dynamical system for both models and found the critical points in flat, open and closed universe. In presence of scalar field, the dynamical behavior of scale factor and Hubble parameter is described in both models. Finally, we have analyzed the energy conditions for both the models in bouncing universe.

Keywords: Bouncing Universe, Dynamical Model, Scalar Field, Energy Conditions.

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I. INTRODUCTION

The initial singularity in the cosmological models is the big problem in the background of General Relativity (GR). The inflation theories suffer from the initial singularity problem at the origin $t = 0$. The inflationary cosmology for early-time is consistent but many theoretical inconsistencies of the big bang cosmology had occurred. Inflationary paradigm may resolve several kinds of problems in standard big bang cosmology in the early universe but the initial singularity cannot be avoided. Since the inflation is described by the dynamics of the scalar fields coupled with the Einstein’s gravity, so a new gravitational theory may be required to describe the beginning of the universe to avoid the initial singularity. Many researchers have attempted to resolve this singularity problem through the generalized/modified general relativity theory say for example, Superstring theory, loop quantum gravity etc. The initial singularity can be avoided in frames of non-singular bouncing cosmological models. The key feature of such models is modification of standard Einstein-Hilbert action. The oscillating universe is an alternative to standard big bang cosmology to avoid the big bang singularity by replacing it with a cyclical evolution. A bouncing universe has an initial narrow state by a minimal radius and then develops to an expanding phase, which means that the universe was arriving to the Big Bang era after the bouncing and so the equation of state parameter should cross from $\omega < -1$ to $\omega > -1$. Several studies in bouncing cosmology could be found in literatures. On the other hand, the bouncing cosmology has also been studied in the framework of some modified theories of gravity eg. $f(R)$ gravity, $f(T)$ gravity, Gauss-Bonnet gravity, Brans-Dicke theory, braneworld gravity and loop quantum cosmology.

Several works on extended Chaplygin gas have been done in references. The variable generalized and modified Chaplygin gas models have been discussed in. Moreover there are some other models like viscous Chaplygin gas, cosmic Chaplygin gas discussed in. Recently Salehi has studied the bouncing universe in presence of extended Chaplygin gas in the framework of Friedmann-Robertson-Walker (FRW) model. Motivated by the work, we have investigated the bouncing universe model in generalized cosmic Chaplygin gas (GCCG) and variable modified Chaplygin gas (VMCG). We have organized the paper in the following way. In section II, we have considered the bouncing framework of FRW model with GCCG or VMCG. In section III, we have studied the dynamical system analysis in both cases and shown the phase space analysis. In Sec. IV, we have discussed the possibility of an oscillating universe in the presence of a scalar field. In section V, we have examined the null energy conditions (NEC). Finally, we have drawn the conclusion in section VI.

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II. MAIN EQUATIONS IN FRW UNIVERSE AND BOUNCE CONDITIONS

We assume the non-flat FRW model of the universe which is given by

\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right) \]  

(1)

where \( a(t) \) is the scale factor and \( k (= 0, \pm 1) \) is the curvature index of the universe described as flat, closed and open respectively.

If the FRW universe is undergoing a ‘bounce’ then it attains a minimum and at this minimum, the strong energy condition (SEC) of classical gravity must be violated (which is necessary but not the sufficient condition) [34]. In a bouncing universe, the universe first undergoes a collapse to attain a minimum and then subsequently expands. For the bounce in FRW model, during contraction phase of the universe, \( a(t) \) decreases i.e., \( \dot{a}(t) < 0 \) and then in the expanding phase of the universe, \( a(t) \) increases i.e., \( \dot{a}(t) > 0 \). At the bounce point, \( t = t_b \), the minimal necessary condition is \( \dot{a}(t) = 0 \) and \( \ddot{a}(t) > 0 \) for \( t \in (t_b - \epsilon, t_b) \cup (t_b, t_b + \epsilon) \) for small \( \epsilon > 0 \). For a non-singular bounce \( a(t_b) \neq 0 \). The conditions may not be sufficient for a non-singular bounce.

In non-flat FRW model, the Einstein’s field equations are given by (assuming \( 8\pi G = 1 \))

\[ 3 \left( H^2 + \frac{k}{a^2} \right) = \rho, \]  

(2)

\[ \left( 2\dot{H} + 3H^2 + \frac{k}{a^2} \right) = -p \]  

(3)

where \( H = \frac{\dot{a}}{a} \) is the Hubble parameter. At the bounce time \( t_b \), \( H = 0 \) and \( \dot{H} > 0 \) in a small time interval near the bounce time. For \( \dot{H} > 0 \), we have \( \rho + 3p < 0 \), so SEC must be violated. The energy conservation equation is given by

\[ \dot{\rho} + 3H(\rho + p) = 0 \]  

(4)

Now let us assume that the universe is filled with generalized cosmic Chaplygin gas (GCCG) or variable modified Chaplygin gas (VMCG). So here \( \rho \) and \( p \) represent respectively the energy density and pressure of GCCG or VMCG. The equations of state for the two models GCCG [70, 71] and VMCG [72] are respectively

\[ p_{GCCG} = -\rho_{GCCG}^{-\alpha} \left[ C + (\rho_{GCCG}^{1+\alpha} - C)^{-\omega} \right] \]  

(5)

and

\[ p_{VMCG} = A\rho_{VMCG} - \frac{B(a)}{\rho_{VMCG}^\alpha} \]  

(6)

where \( C = \frac{\dot{A}}{1+\omega} - 1 \) with \( A \) a constant which can take on both positive and negative values and \( -l < \omega < 0 \), \( l \) being a positive definite constant which can take on values larger than unity, \( A > 0 \) and \( 0 < \alpha < 1 \). Also \( B(a) = B_0 a^{-n} \) [72], with \( B_0 > 0 \) and \( n \) is a positive constant. It should be noted that VMCG starts from radiation era but GCCG starts from dust era. Thus the solutions for the two different models GCCG and VMCG become

\[ \rho_{GCCG} = \left[ C + \left( 1 + \frac{D}{a^{3(1+\alpha)(1+\omega)}} \right)^{\frac{1}{1+\alpha}} \right]^{\frac{1}{1+\alpha}} \]  

(7)

and

\[ \rho_{VMCG} = \left[ \frac{3(1+\alpha)B_0}{a^n[3(1+\alpha)(1+A) - n]} + \frac{E}{a^{3(1+A)(1+\alpha)}} \right]^{\frac{1}{1+\alpha}} \]  

(8)

where \( D > 0 \) and \( E > 0 \) are integration constants with \( 3(1+\alpha)(1+A) > n \).
III. DYNAMICAL SYSTEM AND STABILITY ANALYSIS

A theoretically viable model under investigation can be conceived easily by the study of its dynamical system. It means that the analytical solutions for the model needs to be stable (or asymptotically stable under small perturbations). This can be visualized from the phase space analysis and the behaviour of the system can be well-understood. Many works exist in the literature where extensive studies were performed to different dark energy models by forming their dynamical systems and further analysing their stability criteria. In \[73\], a phantom scalar field was studied with a non-coupled perfect fluid having a constant equation of state. Inhomogeneous dust with a positive cosmological constant given by the Lemaitre-Tolman-Bondi (LTB) model was investigated in \[74\] and the critical points were examined in a unique way. Further recent studies in the inhomogeneous sector was conducted with interactive mixture of dark fluids \[75\]. Interactive holographic dark energy models were explored in the light of stability analysis in Einstein’s gravity \[76\] and other theories of gravity \[77\]. A case study of tachyonic scalar field dark energy, non-minimally coupled to the Barotropic fluid as the matter of the universe, was carried out in a novel way by best fitting the stability parameters to the observational data \[78\]. Very recently, a bouncing Bianchi-IX model was proposed in Hořava-Lifshitz gravity \[79\] with a positive cosmological constant along with non-interacting dust and radiation as the dark energy and matter component.

In the present work, we propose to perform a systematic stability analysis for both GCCG and VMCG models and conclude about their theoretical viability. In order to do so, let us introduce the following variables:

\[ \chi = H, \quad \zeta = a, \quad \eta^2 = \rho \]  

Then from the field equations with proper substitution of the expressions of energy densities for both cases, we get the final expressions of the autonomous system as

\[ \dot{\chi}_{\text{GCCG}}| = \frac{k}{\zeta^2} + \frac{C + \left\{ (3\chi^2 + \frac{3k}{\zeta})^{(1+\alpha)} - C \right\}^{-\omega}}{2 \left(3\chi^2 + \frac{3k}{\zeta}\right)^{(1+\alpha)}} \]  

\[ \dot{x}_{\text{VMCG}}| = -\frac{k(1 + 3A)}{2\zeta^2} - \frac{3(1 + A)}{2} \chi^2 + \frac{B_0}{2 \frac{3^a \zeta^n}{\left(\chi^2 + \frac{k}{\zeta}\right)^{\alpha}}} \]  

and

\[ \dot{\zeta} = \zeta \chi \]  

Equation (12) is applicable for both the models. Solving them, the critical point \((\chi_c, \zeta_c)\) can be obtained as

\[ \chi_c = 0 \]  

and \(\zeta_c\) is the root of the equations

\[ 3\zeta_c^{2(2+\alpha)} \left[ C + \left\{ (3k)^{1+\alpha} \zeta_c^{-2(1+\alpha)} - C \right\}^{-\omega} \right] = (3k)^{2+\alpha} \]  

and

\[ B_0 \zeta_c^{(2\alpha - n) + 2} = (3k)^{\alpha+1} k(1 + 3A) \]  

for the GCCG and the VMCG models respectively.
It can be seen that the critical points depend on a number of parameters for both the models. In case of GCCG, both $\alpha$ and $\omega$ are crucial for the determination of critical points whereas in the case of VMCG, both $\alpha$ and $n$ play major roles. Here we shall study the dynamical system of equations (10), (11) and (12) to obtain the critical points of the system. In this context, the Jacobian Matrices for the two models are (for $\chi_c = 0$)

\[
J_1 = \begin{pmatrix}
0 & \frac{U}{3k\zeta_c} \\
\zeta_c & 0
\end{pmatrix}
\]

and

\[
J_2 = \begin{pmatrix}
0 & \frac{(1+3\lambda)k}{2(3k)^n\zeta_c^{(n-2\alpha)+1}} - \frac{B_0(n-2\alpha)}{(3k)^n\zeta_c^{(n-2\alpha)+1}} \\
\zeta_c & 0
\end{pmatrix}
\]

where $U$ is given by

\[
U = 3k^2 + \zeta_c^2(1 + \alpha)P^{-(\alpha+1)} \left[ (3k)^{-\alpha}\zeta_c^{2(\alpha+1)}P(1 + CP^\omega) + 3k\omega \right]
\]

with

\[
P = -C + (3k)^{1+\alpha}\zeta_c^{-2(1+\alpha)}
\]

Hence the eigenvalues for the two models are given by

\[
\lambda|_{GCCG} = \pm \frac{1}{\zeta_c} \sqrt{\frac{U}{3k}}
\]

and

\[
\lambda|_{VMCG} = \pm \sqrt{k} \zeta_c \sqrt{(1 + 3A) + \frac{B_0(2\alpha - n)\zeta_c^{(2\alpha-n)+2}}{2 3^\alpha k^{\alpha+1}}}
\]

For the GCCG model, we can not state that $U$ is positive definite as it’s value depends on many parameters viz. $\alpha$, $\omega$, $k$ and $A$. Therefore the only pair of complex roots with the zero real part do not come by assuming $k = -1$ only. In the VMCG model, for the case of $\alpha = 1$, if $n \leq 2$, i.e., either $n = 1$ or $n = 2$, then for $k = -1$, we shall obtain two stable critical points in the phase space of the system.

In the following the critical points are analyzed for different values of the parameters involved. The critical points can be obtained analytically from equations (14) and (15) respectively for the two models. For GCCG model, analytical expressions are too complicated to express here, so we provide the analysis by studying the phase space diagrams for both the models. Note that, in this model, the case of $\omega = -1$ should be avoided as the term $C$ diverges in this case.

**A. Case 1: $k = -1$, $\alpha = 1/2$**

In GCCG model, we do not get any stable critical points assuming $\omega = -0.5$ and $C = 1$. Similar solution appears in VMCG model as well when we choose $A = 0$, $n = 2$ and $B_0 = 0.5$. For GCCG and VMCG model, the graphs of $a$ vs $H$ are shown in Figs. 1 and 2 and $a$, $H$, $\rho$ vs $t$ are shown in Figs. 3-8 respectively.
Figs. 1 and 2 show the dynamical behavior of the system for GCCG (left image/blue line) and VMCG (right image/red line) models around the critical point where the parameters are chosen as $C = 1$ and $\omega = -0.5$ (Case 1).
Figs. 9 and 10 show the stable centers which have been found for both GCCG (left image) and VMCG (right image) models around the critical point where the parameters are chosen as $C = 1$ and $\omega = -0.5$ (Case 2).

B. Case 2: $k = 0, \alpha = 1/2$

In GCCG model, we get stable centers assuming $\omega = -0.5$ and $C = 1$. Similar solution appears in VMCG model as well when we choose $A = 0, n = 2$ and $B_0 = 0.5$. For GCCG and VMCG model, the graphs of $a$ vs $H$ are shown in Figs. 9 and 10.

C. Case 3: $k = 1$

For GCCG and VMCG model, the graphs of $a$ vs $H$ are shown in Figs. 11 and 12. Stable centers could not be obtained in this case. If for some values of parameters, we get real eigenvalues in both the models, then the universe does not oscillate. But a single bounce can occur at time $t_b$ under the condition $\dot{a}_b = 0, \ddot{a}_b \geq 0$ or $\chi_b = 0$ and $\frac{d\chi_b}{dt} > 0$ in terms of new variables. This gives us the conditions for the two models from equations (10) and (11) as

$$C + \left[ \left( \frac{3k}{a_b^2} \right)^{1+\alpha} - C \right]^{-\omega} > \frac{1}{3} \left( \frac{3k}{a_b^2} \right)^{2+\alpha}$$

and

$$B_0 > \frac{3\alpha k^{\alpha+1}(1+3A)}{a_b^{2\alpha-n}+2}$$

with the energy density at the bounce for both cases as

$$\rho_b = \frac{3k}{a_b^2}$$

To avoid the negative energy density, $\rho_b$ must be positive, which is evident from equation \ref{24}.

IV. BOUNCING UNIVERSE IN PRESENCE OF SCALAR FIELD

With the effective Lagrangian $L_\phi = \frac{1}{2} \dot{\phi}^2 - V$, the field equations for both the cases are (assuming $8\pi G = 1$)
Figs. 11 and 12 show the GCCG (left image) and VMCG (right image) models around the critical point where the parameters are chosen as $C = 1$ and $\omega = -0.5$.

\[ 3 \left( H^2 + \frac{k}{a^2} \right) = \rho + \frac{1}{2} \dot{\phi}^2 + V \]  
(25)

\[ \left( 2\dot{H} + 3H^2 + \frac{k}{a^2} \right) = -p - \frac{1}{2} \dot{\phi}^2 + V \]  
(26)

The energy conservation equation is given by

\[ \dot{\rho} + 3H(\rho + p) = 0 \]  
(27)

The field equation for the scalar field is

\[ \ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \]  
(28)

Let us define the new variables

\[ x = H, \quad y = \rho, \quad \dot{\phi} = g(t), \quad z = V \]  
(29)

Then we get the new set of equations of motion as

\[ \frac{dx}{dt}\big|_{GCCG} = -\frac{f_1(y)}{2} - \frac{1}{2}g(t)^2 + \frac{k}{a^2}, \quad \frac{dx}{dt}\big|_{VMCG} = -\frac{f_2(y)}{2a^n} - \frac{1}{2}g(t)^2 + \frac{k}{a^2} \]  
(30)

\[ \frac{dy}{dt}\big|_{GCCG} = -3xf_1(y), \quad \frac{dy}{dt}\big|_{VMCG} = -3xf_2(y) \]  
(31)

\[ \frac{da}{dt} = ax \]  
(32)

\[ \frac{dz}{dt} = -g(t)[g(t) + 3xg(t)] \]  
(33)
Figs. 13-15 show the dynamical behavior of the scale factor and the Hubble parameter in GCCG (blue line/Fig.13) and VMCG (red line/Figs.14 and 15) models for $k = 0$ and $g(t) = 0$. Figs. 13 and 14 signify that in the absence of the scalar field in a flat universe, the models undergo a single bounce only and do not show any oscillations.

where $f_1(y) = (\rho_{GCCG} + p_{GCCG}) = \left(\frac{y^{1+\alpha} - C}{y^{\alpha}} - \frac{y^{1+\alpha} - C}{y^{\alpha}}\right)^{-\omega}$ and $f_2(y) = (\rho_{VMCG} + p_{VMCG}) = \frac{(1+A)y^{1+\alpha} - B_0}{y^\alpha}$.

For $g(t) = 0$ in both the cases, the scale factor is seen to have a single bounce without any oscillations. Figs. 13-15 show that dynamical behavior of the scale factor and the Hubble parameter in GCCG and VMCG models.

For the second choice of $g(t) = e^{-\lambda t}$, ($\lambda$ constant, damping coefficient), we have the minimal conditions of bounce as $\chi_b = 0$, $\frac{dx}{dt} > 0$, which in terms of the new parameters for the two models become

$$C + (y_b^{1+\alpha} - C)^{-\omega} > \left[y_b + g_b^2 - \frac{2k}{a_b^2}\right] y_b^\alpha \quad (34)$$

and

$$B_0 > y_b^\alpha \left[(1+A)y_b + \left(g_b^2 - \frac{2k}{a_b^2}\right) a_b^n \right] \quad (35)$$

where $g_b = e^{-\lambda t}$. The dynamical behavior of this case has been shown in Figs. 16-23 with different $k$ values. In none of these cases, the scale factor has shown either a single bounce or an oscillating behaviour for different parameters.
Figs. 16-19 show the dynamical behavior of the scale factor and the Hubble parameter in GCCG (blue line/left images) and VMCG (red line/right images) models for $k = 0$ and $g(t) = e^{-\lambda t}$.

V. CONNECTION BETWEEN BOUNCE AND ENERGY CONDITIONS

In this section, we demonstrate the connection between bounce and energy conditions. Reframing the field equations, we have

$$\rho + p = 2 \left[ \frac{k}{a^2} + \frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a} \right]$$

$$\rho - p = 2 \left[ 2\frac{k}{a^2} + 2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} \right]$$

$$\rho + 3p = -6\frac{\ddot{a}}{a}$$

so that for bounce, the restrictions become

$$\rho_b + p_b < \frac{2k}{a_b^2}$$

$$\rho_b - p_b > \frac{4k}{a_b^2}$$
Figs. 20 and 21 show the dynamical behavior of the scale factor in GCCG (blue line/left image) and VMCG (red line/right image) models for $k = 1$ and $g(t) = e^{-\lambda t}$.

Figs. 22 and 23 show the dynamical behavior of the scale factor in GCCG (blue line/left image) and VMCG (red line/right image) models for $k = -1$ and $g(t) = e^{-\lambda t}$.

$$\rho_b + 3p_b < 0 \quad (41)$$

In the discussion of energy conditions, we mainly focus on NEC $\Leftrightarrow (\rho + p \geq 0)$. If this is violated, then all the other point-wise energy conditions are violated as well.

A. NEC in spatially flat ($k = 0$) and hyperbolic/open ($k = -1$) universe

From equation (39), NEC is definitely violated. Let us see the energy conservation equation for the bounce

$$\dot{\rho}_b = -3H_b(\rho_b + p_b) \quad (42)$$

Since $H_b = 0$, hence $\dot{\rho}_b = 0$. Thus the energy densities of both the models reach a point of extremum at the time of bounce. Now

$$\ddot{\rho}_b = -3\dot{H}_b(\rho_b + p_b) - 3H_b(\dot{\rho}_b + \dot{p}_b) = -3\dot{H}_b(\rho_b + p_b) \quad (43)$$
Now, for bounce $\ddot{a}_b > 0$, which implies $\dot{H}_b > 0$. Also restriction (39) gives violation of NEC. Hence from equation (43), we must have, $\dot{\rho}_b > 0 \Rightarrow \rho_b = \rho_{\text{min}} = \frac{3k}{a_b^2}$.

Then from the equations of state (5) and (6), the restrictions for the NEC violation with bounce for GCCG and VMCG models will be

$$C + \left(\frac{3k}{a_b^2} - C\right)^{-\omega} > \left(\frac{3k}{a_b^2}\right)^{\alpha+1}$$

(44)

and

$$\frac{3k}{a_b^{\frac{2a - n + 2}{\alpha + 1}}} - \left(\frac{B_0}{A + 1}\right)^{\frac{\alpha}{\alpha + 1}} < 0$$

(45)

B. NEC in hyperspherical/closed ($k = 1$) universe

Here we do not get violation of NEC automatically. For bounce, here we rewrite equation (36) as follows

$$\rho_b + p_b = -2 \left[\frac{\ddot{a}_b}{a_b} - \frac{1}{a_b^2}\right]$$

(46)

This follows that NEC will be violated if $\ddot{a}_b \geq \frac{1}{a_b}$. Thus there are two possibilities

(a) $\ddot{a}_b \geq \frac{1}{a_b}$. This implies $\dot{H}_b > 0$ and $(\rho_b + p_b) < 0$. Hence we must have $\dot{\rho}_b > 0 \Rightarrow \rho_b = \rho_{\text{min}} = \frac{3}{a_b^2}$.

(b) $\ddot{a}_b < \frac{1}{a_b}$. This again implies $\dot{H}_b > 0$ but this time NEC is satisfied, i.e., $(\rho_b + p_b) > 0$. Therefore, $\ddot{\rho}_b < 0 \Rightarrow \rho_b = \rho_{\text{max}} = \frac{3}{a_b^2}$.

Thus for a bounce, violation of NEC is not necessary. Nevertheless, if NEC is violated, then the energy density reaches its minimum value at the bounce point. Thus for the EoS (5) and (6), the restrictions of NEC violation under the bounce for the GCCG and VMCG models will be two bi-conditionals viz.

$$C + \left(\frac{3}{a_b^2} - C\right)^{-\omega} > \left(\frac{3}{a_b^2}\right)^{\alpha+1} \Leftrightarrow \ddot{a}_b \geq \frac{1}{a_b}$$

(47)

and

$$B_0 > \frac{3^\alpha(A + 1)}{a_b^{(2a - n + 2)}}$$

(48)

GCCG Model

In this case, $\rho_{\text{GCCG}} \rightarrow (1 + C)^\frac{1}{\alpha} - \omega$, a constant value in the future. The Hubble parameter $H \rightarrow \pm \frac{1}{\sqrt{3}} (1 + C)^\frac{1}{\alpha - \omega}$. The positive and negative signs signify the expanding and contracting universes respectively. Also from the EoS (5), we get $p_{\text{GCCG}} \rightarrow -(1 + C)^\frac{1}{\alpha}$, $\omega_{\text{GCCG}} \rightarrow -1$ and $(\rho_{\text{GCCG}} + p_{\text{GCCG}}) \rightarrow 0$ in future. Therefore the energy conservation equation (42) gives $\frac{d\rho}{dt} \rightarrow 0$. Hence the energy density attains a point of extremum in future as well. Therefore we have two points ($t = t_b$ and $t \rightarrow \infty$) where the energy density of GCCG will attain its extremum values depending on the violation or satisfaction of the NEC. This can be classified in the following way:

(a) NEC is violated $\Rightarrow \rho_b = \rho_{\text{min}} = \frac{3}{a_b^2}$, $\rho_\infty = \rho_{\text{max}} = (1 + C)^\frac{1}{\alpha - \omega}$. 
Figs. 24-26 show the dynamical behavior of the scale factor $a$, the sum of the energy density and pressure $(\rho_{\text{GCCG}} + p_{\text{GCCG}})$ and the energy density $\rho_{\text{GCCG}}$ during the bounce period for $k = 1$, $\alpha = 0.5$, $\omega = -0.5$ and $a_b = 1$ for different values of $C$ and $D$ from (7). The plots show that for different values of $C$ and $D$, a bounce always appear and the NEC is always satisfied.

(b) NEC is satisfied $\Rightarrow \rho_b = \rho_{\text{max}} = \frac{3}{a_b^2}$, $\rho_\infty = \rho_{\text{min}} = (1 + C)\frac{1}{1 + a}$.

Figs. 24-26 show the dynamical behavior of the scale factor $a$, the sum of the energy density and pressure $(\rho_{\text{VMCG}} + p_{\text{VMCG}})$ and the energy density $\rho_{\text{VMCG}}$ during the bounce period for $k = 1$, $\alpha = 0.5$, $\omega = -0.5$ and $a_b = 1$ for different values of $C$ and $D$.

**VMCG Model**

In this case, $\rho_{\text{VMCG}} \rightarrow 0$ in the future. But in that case every other quantities as $p_{\text{VMCG}}$, $H$, $(\rho_{\text{VMCG}} + p_{\text{VMCG}})$, $\dot{\rho}_{\text{VMCG}}$ will also tend to zero.

Then the same argument as of GCCG should be followed here too and the classification gives:

(a) NEC is violated $\Rightarrow \rho_b = \rho_{\text{min}} = \frac{3}{a_b^2}$, $\rho_\infty = 0$.

(b) NEC is satisfied $\Rightarrow \rho_b = \rho_{\text{max}} = \frac{3}{a_b^2}$, $\rho_\infty = \rho_{\text{min}} = 0$.

where $\rho_\infty$ denotes the energy density evaluated in the future. Figs. 27-29 show the dynamical behavior of the scale factor $a$, the sum of the energy density and pressure $(\rho_{\text{VMCG}} + p_{\text{VMCG}})$ and the energy density $\rho_{\text{VMCG}}$ during the bounce period for $k = 1$, $\alpha = 0.5$, and $a_b = 1$ for different values of $n$. 
Figs. 27-29 show the dynamical behavior of the scale factor $a$, the sum of the energy density and pressure ($\rho_{VMCG} + p_{VMCG}$) and the energy density $\rho_{VMCG}$ during the bounce period for $k = 1$, $\alpha = 0.5$, and $a_b = 1$ for different values of $n$ from $[5]$. The plots show that for all the variations in $n$, a bounce always occur. However in this case, the NEC is violated in late time.

C. NEC in presence of a scalar field

To study the NEC in both the models, we write the sum of the total energy density and pressure as

$$\rho_T + p_T = (\rho + p) + (\rho_\phi + p_\phi)$$

where

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

From equations (25) and (26), we obtain at the time of bounce

$$(\rho_T + p_T)_{b} = \frac{2}{a_b} \left[ k - \frac{\ddot{a}_b}{a_b^2} \right]$$

From this equation, NEC is violated automatically for $k = 0$ and $k = -1$. For $k = 1$, NEC is violated if $\ddot{a}_b > \frac{1}{a_b}$. In this case, let us substitute $\rho_{GCCG} + p_{GCCG} = (\rho^{1+n} - C) - (\rho^{1+n} - C)^2$, $\rho_{VMCG} + p_{VMCG} = \frac{(1+\alpha)\rho^{1+n} - B_0}{\alpha + \gamma}$ and $\rho_\phi + p_\phi = \dot{\phi}^2$ into equation (49) to get the expressions of $(\rho_T + p_T)_{b}$ for the GCCG and VMCG models as
\[(\rho_T + p_T)_{t_0} = \frac{(y_b^{1+\alpha} - C) - (y_b^{1+\alpha} - C)^{-\omega}}{y_b^{\alpha}} + g_b^2\]  

and

\[(\rho_T + p_T)_{t_0} = \frac{(1 + A)y_b^{1+\alpha} - B_0}{a_b^2y_b^{\alpha}} + g_b^2\]  

Therefore, for the violation of total NEC for both the models at the time of bounce we need

\[C + (y_b^{1+\alpha} - C)^{-\omega} > y_b^{\alpha}[y_b + g_b^2]\]  

and

\[B_0 > y_b^{\alpha}[(1 + A)y_b + a_b^2g_b^2]\]  

VI. DISCUSSIONS

In this work, we have considered the non-flat FRW model of the universe where bounce occurs and the universe is filled with GCCG or VMCG. At first, we studied the stability analysis through dynamical system for both models and found the critical points in flat, open and closed universe. For \(k = -1, \alpha = 1/2\), in GCCG model, we have not got any stable critical points assuming \(\omega = -0.5\) and \(C = 1\). Similar solution appeared in VMCG model as well when we chose \(A = 0, n = 2\) and \(B_0 = 0.5\). For GCCG and VMCG model, the graphs of \(a\) vs \(H\) are shown in Figs. 1 and 2, which are clearly not stable centers. Also in this case, \(a, H, \rho\) vs \(t\) are shown in Figs. 3-8 respectively to show the behaviour of the system. For \(k = 0, \alpha = 1/2\), in GCCG model, we got stable centers assuming \(\omega = -0.5\) and \(C = 1\). Similar solution appeared in VMCG model as well when we chose \(A = 0, n = 2\) and \(B_0 = 0.5\). These two stable solutions for the two models are plotted in Figs. 9 and 10. For \(k = 1\), in GCCG and VMCG models, the graphs of \(a\) vs \(H\) are shown in Figs. 11 and 12, which again are unstable in nature.

The dynamical behaviour of the scale factor and the Hubble parameter in both models have also been analysed after introducing a scalar field. Initially when \(g(t) = 0\), the scale factor is seen to have a single bounce without any oscillation for both GCCG and VMCG models. Figs. 13 and 14 show this behavior for the two models. Later in presence of the scalar field, i.e., when \(g(t) = e^{-\lambda t}\), the dynamical behavior of the scale factor has shown neither a single bounce nor any oscillating behaviour. This has been clearly shown in Figs. 16-23 with different values of \(k\).

Finally, the energy conditions for both the models in bouncing universe have been investigated. The criteria for bounce with or without the violation of NEC were mentioned analytically and shown graphically for the two models. For GCCG model, Figs. 24-26 show the dynamical behavior of the scale factor \(a\), the sum of the energy density and pressure \((\rho_{GCCG} + p_{GCCG})\) and the energy density \(\rho_{GCCG}\) during the bounce period for \(k = 1, \alpha = 0.5, \omega = -0.5\) and \(a_b = 1\) for different values of \(C\) and \(D\). It can be clearly seen that for different choices of the parameters \(C\) and \(D\), a single bounce always appear and that never requires the violation of NEC. For VMVG model, Figs. 27-29 show the equivalent plots showing the dynamical behavior of the scale factor \(a\), the sum of the energy density and pressure \((\rho_{VMCG} + p_{VMCG})\) and the energy density \(\rho_{VMCG}\) during the bounce period for \(k = 1, \alpha = 0.5, \omega = -0.5,\) and \(a_b = 1\) for different values of \(n\). In this case, however a single bounce happens to exist with late time violation of NEC.

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[56] Naji, Jalil; Saadat, Hassan, Int. J. Theor. Phys. 53, 1547 (2014).
[57] Saadat, H., Int. J. Theor. Phys. 52, 3902 (2013).
[58] Saadat, H., Int. J. Theor. Phys. 52, 1696 (2013).
[59] Sadeghi, J.; Khurshudyan, M.; Farahani, H., Int. J. Theor. Phys. 55, 81 (2016).
[60] Saadat, H.; Pourhassan, B., Int. J. Theor. Phys. 52, 3712 (2013).
[61] Saadat, H.; Pourhassan, B.; Khurshudyan, M.; Farahani, H., Int. J. Theor. Phys. 53, 911 (2014).
[62] Saadat, H.; Pourhassan, B., Int. J. Theor. Phys. 53, 1168 (2014).
[63] A. Khodam-Mohammadi, E. Karimkhani, A. Alaei, Eur. Phys. J. Plus 131, 398 (2016).
[64] Kahya, E. O.; Pourhassan, B.; Uraz, S., Physical Review D 92, 103511 (2015).
[65] Avelino, P. P.; Bojelko, K.; Lewis, G. F., Physical Review D 89, 103004 (2014).
[66] Saadat, H.; Pourhassan, B, Astrophysics and Space Science 343, 783 (2013).
[67] Saadat, H.; Pourhassan, B, Astrophysics and Space Science 344, 237 (2013).
[68] B. POURHASSAN, International Journal of Modern Physics D 22, 1350061 (2013).
[69] A. Salehi, Phys. Rev. D 94, 123519 (2016).
[70] P. F. González-Díaz, Phys. Rev. D 68, 021303 (R) (2003).
[71] W. Chakraborty and U. Debnath, Gravitation and Cosmology, 13, 293 (2007).
[72] U. Debnath, Astrophys. Space Sci., Vol. 312, 295 (2007).
[73] L. Ureña-López, J. Cosmol. Astropart. Phys. 0509, 013 (2005).
[74] R. Sussman and G. Izquierdo, Class.Quant.Grav. 28, 045006 (2011).
[75] G. Izquierdo, R. C. Blanquet-Jaramillo and R. Sussman, Eur.Phys.J.C. 78, 233 (2018).
[76] R. Biswas, N. Mazumder and S. Chakraborty. [arXiv:1106.4620]
[77] S. Biswas and S. Chakraborty, Int.J.Mod.Phys.D. 24, 1550046 (2015).
[78] H. Farajollahi and A. Salehi, Phys.Rev.D 83, 124042 (2011).
[79] R. Maier and I. Soares, Phys.Rev.D 97, 049902 (2018).