Constants of Motion for Constrained Hamiltonian Systems

— A Particle around a Charged Rotating Black Hole —

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Abstract

We discuss constants of motion of a particle under an external field in a curved spacetime, taking into account the Hamiltonian constraint which arises from reparametrization invariance of the particle orbit. As the necessary and sufficient condition for the existence of a constant of motion, we obtain a set of equations with a hierarchical structure, which is understood as a generalization of the Killing tensor equation. It is also a generalization of the conventional argument in that it includes the case when the conservation condition holds only on the constraint surface in the phase space. In that case, it is shown that the constant of motion is associated with a conformal Killing tensor. We apply the hierarchical equations and find constants of motion in the case of a charged particle in an electro-magnetic field in black hole spacetimes. We also demonstrate that gravitational and electro-magnetic fields exist in which a charged particle has a constant of motion associated with a conformal Killing tensor.

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I. INTRODUCTION

Black holes have great importance in modern physics. In astrophysics, black holes are thought to be central engines of active galactic nuclei. The gravitational and electromagnetic fields of black holes play crucial roles for production of the energy actually observed. On the other hand, higher-dimensional black holes, in recent years, gather much attention in the context of unified theories of interactions. An important task at present is to reveal their properties because existence of the extra dimensions could be verified by observations of the phenomena concerned with black holes.

A test particle is an important probe of black hole spacetimes because the study of motions of a test particle gives an important insight into physical properties of black holes. In the case of a charged test particle, the motion gives information on both the gravitational and electro-magnetic fields. The constants of motion, i.e., the conserved quantities along a particle trajectory, are useful for analysis of the motion of the particle. One can conclude that the equations of motion are integrable if one knows sufficient numbers of constants of motion satisfying a certain property.

In the case of a particle without charge when the trajectory is a geodesic on a curved spacetime, existence of a one-parameter group of isometries generated by a Killing vector implies the existence of a constant of motion which is linear in the momentum. The constants of motion which is non-linear in the momentum arise from Killing tensors. For example, in the Kerr spacetime, there exists a constant of motion quadratic in the momentum \[1\] arises from a Killing tensor of rank 2 \[2\]. Furthermore, the existence of a rank-2 Killing tensor was recently shown in the Kerr-NUT-de Sitter black holes in any dimensionality \[3–8\].

A constant of motion which is quadratic in the momentum for a charged particle in the Kerr-Newman spacetime was also found \[4\] by the method of Hamilton-Jacobi equation. It was shown that the constant of motion is related to a rank-2 Killing tensor also in this case \[4, 10\], and a set of coupled equations is obtained which should be satisfied by the constants of motion for the charged particle \[10\] (see also \[11\] for recent works).

The motion of a test particle in a curved spacetime is described by a world line with arbitrary parametrization. This reparametrization invariance gives rise to the Hamiltonian constraint. In this paper, we discuss a generalization of the conservation condition for the system of a particle which is subject to external fields in the Hamiltonian formalism. We
consider the conservation condition with the constraint taken into account. Namely, we require that the conservation equation hold under the constraint condition. As a result, we obtain a generalized set of equations which is the necessary and sufficient condition for the existence of a constant of motion for a particle in an external field. The equations have hierarchical structure, and the topmost equation in the hierarchy is the conformal Killing tensor equation.

As applications, we first consider systems of a charged particle in electro-magnetic fields around black holes, namely, a test Maxwell field on the Kerr spacetime, the Kerr-Newman spacetime, and a five-dimensional charged rotating black hole. In these cases, the Hamiltonian constraint does not play any role in finding constants of motion since the metrics admit rank-2 Killing tensors. In the final example, we demonstrate that a constant of motion can exist which is related to a conservation condition holding only on the constraint surface. The example is constructed by conformal transformation of a spacetime with a test Maxwell field.

The organization of the paper is as follows. In the following section, we review the relation between constants of motion of a free particle and geometrical quantities by using the Hamiltonian formalism. In section III, we formulate the condition for existence of a constant of motion for a particle in an external field when there is a constraint condition. We obtain a set of equations with a hierarchical structure as a result. The equations are applied to systems of a charged particle in section IV. Finally, section V is devoted to a summary.

II. CONSERVED QUANTITIES OF A FREE PARTICLE IN THE HAMILTONIAN FORMALISM

In this section, we review the relation between geometrical quantities of a curved spacetime \((\mathcal{M}, g_{\mu\nu})\) and conserved quantities of a free particle in \((\mathcal{M}, g_{\mu\nu})\). It is well known that the solutions of Killing equation, Killing fields, give conserved quantities along the trajectory of the particle, which is a geodesic. We shall derive the relation by using the Hamiltonian formalism. The relation will be generalized in the following section.
Let $H$ be the Hamiltonian of a free particle given by

$$H = \frac{1}{2m} \left( g^{\mu \nu} p_\mu p_\nu + m^2 \right), \tag{1}$$

where $m$ is the mass and $p_\mu$ is the canonical momentum of the particle. The Hamilton equation for (1) leads to the geodesic equation. Let $F$ be a dynamical quantity of the free particle represented by a function on the phase space with coordinates $(x^\mu, p_\nu)$. In the Hamiltonian formalism, if $F$ is a constant of motion, i.e., a conserved quantity along the orbit of the particle, it commutes with $H$ under the Poisson bracket,

$$\frac{dF}{d\tau} = \{ F, H \}_{\text{PB}} := \frac{\partial F}{\partial x^\mu} \frac{\partial H}{\partial p_\mu} - \frac{\partial H}{\partial x^\mu} \frac{\partial F}{\partial p_\mu} = 0, \tag{2}$$

where the bracket with PB denotes the Poisson bracket, and $\tau$ is the proper time of the particle.

Here, we assume that $F$ is written in the form

$$F(x^\mu, p_\mu) = \xi^\mu p_\mu, \tag{3}$$

where $\xi^\mu$ is a vector field on $\mathcal{M}$. For $F$ in the form of (3), the equation (2) takes the form

$$\{ F, H \}_{\text{PB}} = \frac{1}{m} \xi^{\mu,\nu} p_\mu p_\nu = 0, \tag{4}$$

where the semicolon denotes the covariant derivative. Hence we have the Killing equation

$$\xi^{\mu,\nu} = 0. \tag{5}$$

A solution $\xi^\mu$ of the equation, a Killing vector, gives a constant of motion $F$.

We can generalize the Killing equation to a higher-rank tensor equation. Let us assume that $F$ has the form of

$$F = \frac{(k)}{K} \mu_1 \cdots \mu_k p_{\mu_1} \cdots p_{\mu_k}, \tag{6}$$

where $\frac{(k)}{K}$ denotes a completely symmetric tensor field of rank $k$. Then the equation (2) yields the Killing tensor equation for rank-$k$ tensor

$$\frac{(k)}{K} \left( \mu_1 \cdots \mu_k ; \mu_{k+1} \right) = 0. \tag{7}$$

That is, Killing tensors, which are geometrical quantities, are related with constants of motion.
III. FORMULATION OF GENERALIZED KILLING EQUATIONS

In this section, we generalize the Killing tensor equation derived in the previous section to include the case when the conservation condition holds only on the constraint surface in the phase space.

Let us first derive the Hamiltonian for a free particle. The geodesic is obtained from the variational principle with the action being the arc length of a curve connecting two points,

\[ S = -m \int_C \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \, d\lambda. \] (8)

The action \( S \) has the reparametrization invariance \( \lambda \to \lambda' \). Introducing a Lagrange multiplier \( N(\lambda) \), we can construct an equivalent action in the quadratic form,

\[ S = \int_C \left( \frac{m}{2} g_{\mu\nu} \frac{dx^\mu}{N d\lambda} \frac{dx^\nu}{N d\lambda} - \frac{m}{2} \right) N d\lambda. \] (9)

By the standard procedure of moving from the Lagrange formalism to the Hamilton formalism, we can derive the Hamiltonian,

\[ H = \frac{N}{2m} \left( g^{\mu\nu} p_\mu p_\nu + m^2 \right). \] (10)

Variation by \( N \) yields the mass shell condition or the Hamiltonian constraint,

\[ \mathcal{H}(x^\mu, p_\mu) = g^{\mu\nu} p_\mu p_\nu + m^2 \approx 0. \] (11)

The last equality, weak equality, defines a constraint surface in the phase space where the actual particle motion is confined on the hypersurface. The action (11) is restricted to be used with an affine parameter, while an arbitrary parameter is allowed in (9).

Let us generalize the Killing tensor equation, taking into account the case when the conservation condition holds only on the constraint surface. We begin with a Hamiltonian slightly more general than (10),

\[ H = \frac{N}{2m} \left( g^{\mu\nu} p_\mu p_\nu + B^\rho p_\rho + V \right), \] (12)

to describe a system of a particle in an external field, where \( B^\rho \) and \( V \) are a vector field and a scalar field, respectively, on \( \mathcal{M} \). By setting the differentiation of \( H \) by \( N \) equal to zero, we get the Hamiltonian constraint equation

\[ \mathcal{H} = g^{\mu\nu} p_\mu p_\nu + B^\rho p_\rho + V \approx 0. \] (13)
We shall examine the conservation condition for a function \( F(x^\mu, p_\mu) \) on the constrained system. Since the particle motion is realized only on the constraint surface, then it suffices that \( F \) commutes with \( H \) only on the constraint surface, i.e.,

\[
\{ F, H \}_\text{PB} = N \{ F, \mathcal{H} \}_\text{PB} \approx 0.
\]  

(14)

This is equivalent to

\[
\{ F, \mathcal{H} \}_\text{PB} + \phi \mathcal{H} = 0,
\]  

(15)

where \( \phi \) is an arbitrary function on the phase space.

We assume that \( F \) and \( \phi \) are expanded in the form,

\[
F = \sum_k (k) K^{\mu_1 \cdots \mu_k} p_{\mu_1} \cdots p_{\mu_k} =: \sum_k (k) K \cdot p^k,
\]  

(16)

\[
\phi = \sum_l (l) \lambda^{\mu_1 \cdots \mu_l} p_{\mu_1} \cdots p_{\mu_l} =: \sum_l (l) \lambda \cdot p^l,
\]  

(17)

where \( K^{\mu_1 \cdots \mu_k} \) and \( \lambda^{\mu_1 \cdots \mu_l} \) are symmetric tensor fields of rank \( k \) and rank \( l \), respectively, on \( \mathcal{M} \). The right-hand sides of (16) and (17) are abbreviations of contraction with \( p \)'s. Substituting these expressions into (15), we have

\[
\sum_k \left( -\left[ (k-1) K, (2) g \right]_\text{S} - \left[ (k) K, (1) B \right]_\text{S} - \left[ (k+1) K, (0) V \right]_\text{S} \\
+ (k-2) \lambda \otimes g + (k-1) \lambda \otimes B + (k) \lambda V \right) \cdot p^k = 0
\]  

(18)

where we understand that \( (k) K = 0 \) and \( (k) \lambda = 0 \) for \( k < 0 \), and \( \otimes \) denotes symmetric tensor product. The bracket with the subscript \( S \) is the Schouten bracket [12], which is the map from symmetric tensor fields \( X \) and \( Y \) of rank \( k \) and rank \( l \), respectively, to a rank-(\( k+l-1 \)) symmetric tensor field \( [ (k), (l) ]_\text{S} \) defined by

\[
[k, l]_\text{S} \cdot p^{k+l-1} = -\{ (k) X, (l) Y \}_\text{PB}.
\]  

(19)

Since (18) should be satisfied for any \( p^k = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_k} \), then the each coefficient of \( p^k \) vanishes, i.e.,

\[
-\left[ (k-1) K, (2) g \right]_\text{S} - \left[ (k) K, (1) B \right]_\text{S} - \left[ (k+1) K, (0) V \right]_\text{S} \\
+ (k-2) \lambda \otimes g + (k-1) \lambda \otimes B + (k) \lambda V = 0.
\]  

(20)
Let us call the set of equations (20) the *Killing hierarchy* because it is a generalization of the Killing equation.

When the highest rank of the hierarchy is $N$, namely, when $^{(l+1)}K = 0$ and $^{(l)}\lambda = 0$ for $l \geq N$, the Killing hierarchy reads

\begin{align}
- \left[ \begin{array}{c}
^{(N)}K \\
^{(2)}g
\end{array} \right]_S + \ (N-1)S \otimes \ (2)g = 0, \\
- \left[ \begin{array}{c}
^{(N-1)}K \\
^{(2)}g
\end{array} \right]_S - \left[ \begin{array}{c}
^{(N)}K \\
^{(1)}B
\end{array} \right]_S + \ (N-2)S \otimes \ (2)g + \ (N-1)S \otimes \ (1)g = 0, \\
- \left[ \begin{array}{c}
^{(k-1)}K \\
^{(2)}g
\end{array} \right]_S - \left[ \begin{array}{c}
^{(k)}K \\
^{(1)}B
\end{array} \right]_S - \left[ \begin{array}{c}
^{(k+1)}K \\
^{(0)}V
\end{array} \right]_S \\
\quad + \ (k-2)S \otimes \ (2)g + \ (k-1)S \otimes \ (1)g + \ (k)S \otimes \ (0)g = 0, \quad 2 \leq k \leq N - 1, \\
- \left[ \begin{array}{c}
^{(0)}K \\
^{(1)}B
\end{array} \right]_S - \left[ \begin{array}{c}
^{(1)}K \\
^{(0)}V
\end{array} \right]_S + \ (0)S \otimes \ (1)g + \ (1)S \otimes \ (0)g = 0,
\end{align}

where $g \cdot p^2$ is assumed to be nonvanishing. The structure of the Killing hierarchy tells us that we should solve them from the highest-rank equation (21). Since the highest-rank equation is the conformal Killing equation, existence of a conformal Killing tensor is necessary for a constant of motion to exist. If the Killing hierarchy admits a non-trivial solution, then there exist a constant of motion associated with a conformal Killing tensor.

In the case of a free particle, i.e., $^{(1)}B = 0$ and $^{(0)}V = \text{const.}$, the Killing hierarchy reduces to a set of decoupled conformal Killing equations if the particle is massless, $^{(0)}V = 0$, and to a set of decoupled Killing equations if the particle is massive, $^{(0)}V = m^2$. This fact is shown in appendix A.

In the case of $^{(k)}\lambda = 0$ for all $k$ the Killing hierarchy reduces to

\begin{align}
- \left[ \begin{array}{c}
^{(N)}K \\
^{(2)}g
\end{array} \right]_S = 0, \\
- \left[ \begin{array}{c}
^{(N-1)}K \\
^{(2)}g
\end{array} \right]_S - \left[ \begin{array}{c}
^{(N)}K \\
^{(1)}B
\end{array} \right]_S = 0, \\
- \left[ \begin{array}{c}
^{(k-1)}K \\
^{(2)}g
\end{array} \right]_S - \left[ \begin{array}{c}
^{(k)}K \\
^{(1)}B
\end{array} \right]_S - \left[ \begin{array}{c}
^{(k+1)}K \\
^{(0)}V
\end{array} \right]_S = 0, \quad 2 \leq k \leq N - 1, \\
- \left[ \begin{array}{c}
^{(0)}K \\
^{(2)}g
\end{array} \right]_S - \left[ \begin{array}{c}
^{(1)}K \\
^{(1)}B
\end{array} \right]_S - \left[ \begin{array}{c}
^{(2)}K \\
^{(0)}V
\end{array} \right]_S = 0, \\
- \left[ \begin{array}{c}
^{(0)}K \\
^{(1)}B
\end{array} \right]_S - \left[ \begin{array}{c}
^{(1)}K \\
^{(0)}V
\end{array} \right]_S = 0.
\end{align}

These equations were obtained by Sommers [10] and van Holten [11]. Several applications are found in [13].
IV. KILLING HIERARCHY FOR A CHARGED PARTICLE

In this section, we apply the Killing hierarchy to the systems of an electrically charged particle subject to an external electro-magnetic field as an important application of our formalism. We consider the Hamiltonian of a charged particle in the form

\[ H = \frac{N}{2m} \left[ g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) + m^2 \right], \]  

where \( m \) and \( q \) are the mass and the electric charge of the particle, respectively, and \( A_\mu \) denotes the gauge potential. Substituting

\[ B_\mu = -2qA_\mu, \quad V = q^2 A_\mu A^\mu + m^2 \]  

in (21)-(25), we obtain the Killing hierarchy for a charged particle:

\[ \frac{N}{2m} \left[ g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) + m^2 \right] = 0, \]

where \( A^2 \) denotes the squared norm of \( A^\mu \). Summing up all equations contracted by \( A^\alpha \)'s, we have

\[ m^2 \left( q^{N-1} \frac{N-1}{N} \lambda A^{N-1} + q^{N-2} \frac{N-2}{N} \lambda A^{N-2} + \cdots + q \lambda A^1 + \frac{1}{N} \lambda \right) = 0. \]  

One can also derive \((35)\) by setting \( p_\mu = qA_\mu \) in \((15)\) after calculating the Poisson bracket.

We note that existence of constants of motion linear in momenta for a massive particle requires existence of a Killing vector. This is so because, when \( N = 1 \) and \( m \neq 0 \), the equation \((35)\) reduce to \((0)\lambda = 0\), so that the highest-rank equation \((31)\) becomes the Killing vector equation.

In what follows, we first consider three spacetimes with an electro-magnetic field: test electro-magnetic fields called the Wald solutions on the Kerr background, the four-dimensional Kerr-Newman black hole, and the five-dimensional charged rotating black hole.
Since these systems admit a rank-2 Killing tensor $^{(2)}K$, we consider rank-2 solutions of the Killing hierarchy with $\lambda = 0$,

\begin{align}
- \left[^{(2)}K, g\right]_{S} &= 0, \\
- \left[^{(1)}K, g\right]_{S} + 2q\left[^{(2)}K, A\right]_{S} &= 0, \\
- \left[^{(0)}K, g\right]_{S} + 2q\left[^{(1)}K, A\right]_{S} - q^2\left[^{(2)}K, A^2\right]_{S} &= 0.
\end{align}

Next, in the final subsection, we demonstrate existence of a constant of motion of a charged particle associated with a conformal Killing tensor. We construct a four-dimensional space-time which admits a non-trivial conformal Killing tensor by a conformal transformation of the Minkowski spacetime. Making use of the conformal invariance of the Maxwell theory in four dimensions, we construct a solution of the Maxwell field on the spacetime. We show that there exists a suitable conformal transformation such that the charged particle system has a constant of motion associated with the conformal Killing tensor, so that the conservation equation holds only on the constraint surface in the phase space.

A. Wald solutions on Kerr geometry

Let us consider an electro-magnetic field on the Kerr geometry. If a Ricci flat metric admits a Killing vector, the Killing vector solves the vacuum Maxwell equation as a test gauge 4-potential in the Lorentz gauge. This is called the Wald solution [14].

The Kerr metric is given by

\begin{equation}
\begin{aligned}
ds^2 &= - \left( \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \right) dt^2 - \frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma} dt d\phi \\
&\quad + \left[ \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \right] \sin^2 \theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2,
\end{aligned}
\end{equation}

\begin{align}
\Sigma &= r^2 + a^2 \cos^2 \theta, \\
\Delta &= r^2 + a^2 - 2Mr.
\end{align}

The spacetime admits two commuting Killing vectors $\xi = \partial_t$ and $\psi = \partial_\phi$, where $a$ and $M$ are the rotation and mass parameters, respectively. On the spacetime,

\begin{equation}
A^\mu = c_1 \xi^\mu + c_2 \psi^\mu
\end{equation}
is a solution of the vacuum Maxwell equations, where \( c_1 \) and \( c_2 \) are arbitrary constants. The solution (42) has no magnetic charge. In the case \( c_1 = 2ac_2 \), it has no electric charge either. Furthermore, the Kerr metric also admits the Killing tensor
\[
K'_{\mu\nu} = 2\Sigma l^{(\mu} n^{\nu)} + r^2 g_{\mu\nu},
\]
which commutes with both of \( \xi \) and \( \psi \), where
\[
l^{\mu} = \frac{r^2 + a^2}{\Delta} \xi^{\mu} + \frac{a}{\Delta} \psi^{\mu} + (\partial_{r})^{\mu}, \quad n^{\mu} = \frac{r^2 + a^2}{2\Sigma} \xi^{\mu} + \frac{a}{2\Sigma} \psi^{\mu} - \frac{\Delta}{2\Sigma} (\partial_{r})^{\mu}.
\]

Let us solve the Killing hierarchy (36). The Killing tensor (43), solves the highest-order equation of the hierarchy (36). We thus set
\[
(2)K'_{\mu\nu} = K_{\mu\nu}.
\]
Since \( (2)K \) commutes with \( A \), which is a linear combination of \( \xi \) and \( \psi \), the second equation (37) reduces to the Killing vector equation. Thus we have
\[
(1)K'_{\mu} = \alpha \xi^{\mu} + \beta \psi^{\mu},
\]
where \( \alpha \) and \( \beta \) are arbitrary constants. Then the last equation (38) can be written as
\[
(0)K'_{\nu|\mu} = q^2 K_{\mu\nu} A_{\nu,\mu}.
\]
By inspecting the integrability condition of the partial differential equation (47), we find that this equation is integrable only when \( c_2 = 0 \), i.e., \( A = \xi \). In the case, the solution is given by
\[
(0)K = q^2 K_{\mu\nu} \xi^{\mu} \xi^{\nu}.
\]
Thus we have found a constant of motion of a charged particle associated with a rank-2 Killing tensor, \( F = (\alpha \xi^{\mu} + \beta \psi^{\mu}) p_{\mu} + K_{\mu\nu}(p_{\mu} p_{\nu} + q^2 \xi^{\mu} \xi^{\nu}) \).

We conclude that, in the case \( A = \xi \), the system has independent Poisson-commuting constants of motion,
\[
\xi^{\mu} p_{\mu}, \quad \psi^{\mu} p_{\mu}, \quad \text{and} \quad K_{\mu\nu} u^{\mu} u^{\nu},
\]
where \( u^{\mu} := \frac{1}{m}(p^{\mu} - qA^{\mu}) \) is the four velocity of the particle. In (49), we used the fact that \( K_{\mu\nu} u^{\mu} u^{\nu} \) is a linear combination of \( \xi^{\mu} p_{\mu} \) and \( \psi^{\mu} p_{\mu} \), and \( K_{\mu\nu}(p_{\mu} p_{\nu} + q^2 \xi^{\mu} \xi^{\nu}) \). We remark that no constant of motion associated with the rank-2 Killing tensor exists in the electrically neutral case \( c_1 = 2ac_2 \).
B. Kerr-Newman black holes

The Kerr-Newman spacetime is the exact solution of electrically charged rotating black hole in the Einstein-Maxwell system. The spacetime metric is given by (39) and (40) with
\[ \Delta = r^2 + a^2 + e^2 - 2Mr, \] (50)
instead of (41), and electro-magnetic 4-potential is given by
\[ A = -e r \Sigma (dt - a \sin^2 \theta d\phi), \] (51)
where \( e \) is electric charge of the black hole. As the Kerr metric, the Kerr-Newman metric admits two Killing vectors \( \xi = \partial_t \) and \( \psi = \partial_r \) and the Killing tensor \( K^{\mu \nu} \) in (43), where \( \Delta \) in \( l \) and \( n \) is now given by (50).

In a manner similar to that in the previous section, we can find the solution to the set of equations (36), (37), and (38),
\[ K^{(2)}_{\mu \nu} = K_{\mu \nu}, \] (52)
\[ K^{(1)}_{\mu} = -2q K^{\nu}_{\mu} A_{\nu} + \alpha \xi_{\mu} + \beta \psi_{\mu}, \] (53)
\[ K^{(0)} = q^2 K^{\mu \nu} A_{\mu} A_{\nu}, \] (54)
where \( \alpha \) and \( \beta \) are arbitrary constants. Then the constant of motion associated with the Killing tensor is given by
\[ F = K^{\mu \nu} p_{\mu} p_{\nu} + (-2q K^{\nu}_{\mu} A_{\nu} + \alpha \xi_{\mu} + \beta \psi_{\mu}) p_{\mu} + q^2 K^{\mu \nu} A_{\mu} A_{\nu} \]
\[ = K_{\mu \nu} u^{\mu} u^{\nu} + \alpha \xi_{\mu} p_{\mu} + \beta \psi_{\mu} p_{\mu}. \] (55)
As the Wald solution \( A = \xi \) on the Kerr metric discussed in the previous section, the Kerr-Newman metric admits the independent constants of motion, \( \xi_{\mu} p_{\mu}, \psi_{\mu} p_{\mu}, \) and \( K_{\mu \nu} u^{\mu} u^{\nu}, \) for a charged particle. These two examples have common properties, i.e., the both are rotating black holes having an electric monopole and a magnetic field falling off toward infinity.

The constant of motion for a charged particle associated with the Killing tensor is referred to as Carter’s constant, which was first obtained by the Hamilton-Jacobi method in Ref. [1].

C. Five-dimensional Charged Black Holes

We demonstrate that our formalism is applicable to a charged particle moving around a five-dimensional charged black hole. Here, we consider the five-dimensional charged rotating
black hole with the following metric and the electromagnetic 5-potential \[ 15 \],
\[
 ds^2 = - \left( \frac{\rho^2 (dt + 2e\nu)}{\rho^2} \right)^2 + \frac{2 e \nu \omega}{\rho^2} (dt - \omega)^2 + \frac{\rho^2 d\Omega^2}{\Delta_r} + \rho^2 d\theta^2 \\
+ (r^2 + a^2) \sin^2 \theta d\phi^2 + (r^2 + b^2) \cos^2 \theta d\psi^2, 
\]
(56)
\[
 A = \frac{\sqrt{3} e}{2\rho^2} \left( dt - a \sin^2 \theta d\phi - b \cos^2 \theta d\psi \right), 
\]
(57)
where
\[
 S = a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \rho^2 = r^2 + S, \quad f = 2M \rho^2 - e^2; 
\]
(58)
\[
 \nu = b \sin^2 \theta d\phi + a \cos^2 \theta d\psi, \quad \omega = a \sin^2 \theta d\phi + b \cos^2 \theta d\psi; 
\]
(59)
\[
 \Delta_r = (r^2 + a^2)(r^2 + b^2) + e^2 + 2abe - 2M. 
\]
(60)
The black hole is characterized by mass parameter \( M \), charge parameter \( e \), and two spin parameters \( a \) and \( b \). The metric is an exact solution in the five-dimensional Einstein-Maxwell-Chern-Simons theory. The metric admits three Killing vectors \( \partial_t, \partial_\phi, \) and \( \partial_\psi \), and an irreducible Killing tensor \[ 16 \]
\[
 K^{\mu \nu} = - S g^{\mu \nu} - S (\partial_t)^\mu (\partial_t)^\nu + \frac{1}{\sin^2 \theta} (\partial_\phi)^\mu (\partial_\phi)^\nu + \frac{1}{\cos^2 \theta} (\partial_\psi)^\mu (\partial_\psi)^\nu + (\partial_\theta)^\mu (\partial_\theta)^\nu. 
\]
(61)
These Killing vectors and tensor commute with each other. Hence, we can discuss the existence of constants of motion of a charged particle associated with the three Killing vectors and the Killing tensor.

Let us consider the Killing hierarchy \[ 36 \], \[ 37 \], and \[ 38 \], with the rank-2 Killing tensor \( K^{\mu \nu} = K^{\nu \mu} \) which solves \[ 39 \]. We try to find the solution of the second equations \[ 37 \] of the form
\[
 (1) K^{\mu \nu} + (1) K^{\nu \mu} = B^{\mu \nu}, 
\]
(62)
where \( B^{\mu \nu} = 2q \left( A^\lambda K^{\mu \nu, \lambda} - 2K^{\lambda \mu, A^\nu, \lambda} \right) \). Since \( t, \phi, \) and \( \psi \) are the Killing coordinates, the equation \[ 62 \] has a simple form, which is shown in appendix \[ 13 \].

To find solutions of \[ 37 \], we assume that \( (1) K \) depends only on \( r \) and \( \theta \) as \( (2) K \) does. It turns out that
\[
 (1) K^r = 0, \quad (1) K^\theta = 0 
\]
(63)
from the explicit form of \[ 62 \], and the other components of \( (1) K \) satisfy the following equations,
\[
 g^{rr} (1) K^t_r = B^t_r, \quad g^{\theta \theta} (1) K^t_\theta = B^t_\theta, \quad g^{rr} (1) K^\phi_r = B^r_\phi, 
 g^{\theta \theta} (1) K^\phi_\theta = B^\phi_\theta, \quad g^{rr} (1) K^\psi_r = B^r_\psi, \quad g^{\theta \theta} (1) K^\psi_\theta = B^\psi_\theta. 
\]
(64)
After some calculations, we can find an explicit solution
\[
^{(1)} K^\mu = 2qSA^\mu + \alpha (\partial_t)^\mu + \beta (\partial_\phi)^\mu + \gamma (\partial_\psi)^\mu,
\]
where \(\alpha, \beta\) and \(\gamma\) are arbitrary constants.

Let us solve (38). It can be rewritten as
\[
^{(0)} K, t = ^{(0)} K, \phi = ^{(0)} K, \psi = 0,
\]
\[
g^{tt} ^{(0)} K, r = -\frac{3q^2 e^2 S}{2r^3 \Delta_r \rho^6} \left[ (r^4 - (ab + c)^2)S + r^4 (2r^2 - M) \right],
\]
\[
g^{\theta\theta} ^{(0)} K, \theta = -\frac{3q^2 e^2 r^2}{4\Delta_r \rho^6} (a + b)(a - b) \sin 2\theta.
\]
We can easily integrate these equations to get
\[
^{(0)} K = -q^2 SA^\mu A_\mu.
\]
As a result, we obtain a constant of motion associated with the Killing tensor,
\[
F = K^{\mu\nu} p_\mu p_\nu + 2qSA^\mu p_\mu - q^2 SA^\mu A_\mu
\]
for a charged particle moving in the five-dimensional charged rotating black holes in addition to the momentum components \(p_t, p_\phi\) and \(p_\psi\). These four constants of motions Poisson-commute mutually.

D. Constant of Motion associated with Conformal Killing Tensor

In this subsection, we demonstrate that the conservation equation can hold only on the constraint surface. In such a case, the constant of motion is associated with a conformal Killing tensor. Unfortunately, we have not found an example for this case as a solution of the Einstein equation, but we present some example of curved spacetime.

Let \((\mathcal{M}^4, \bar{g})\) be the Minkowski spacetime. Consider the metric \(\bar{g}\) spanned by the polar coordinates as
\[
d\bar{s}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]
The Minkowski spacetime admits a Killing tensor
\[
K^{\mu\nu} = (\partial_\theta)^\mu (\partial_\theta)^\nu + \frac{1}{\sin^2 \theta} (\partial_\phi)^\mu (\partial_\phi)^\nu.
\]
As a solution of the test electro-magnetic field on the flat background, we take $\bar{A} = \partial_\phi$. This is the special case of the Wald solution discussed in sec. [IV A] with $M = 0$ and $a = 0$. As was mentioned there, there is no constant of motion associated with the Killing tensor $K^{\mu\nu}$ for a charged particle.

Let $(\mathcal{M}^4, g)$ be a spacetime with $g_{\mu\nu} = e^\Phi \bar{g}_{\mu\nu}$ where $\Phi$ is a function on $\mathcal{M}^4$. The spacetime $(\mathcal{M}^4, g)$ is conformally flat. Since the Maxwell theory in a four-dimensional spacetime has conformal invariance, the gauge 4-potential $A_\mu := \bar{A}_\mu$ solves the Maxwell equations on $(\mathcal{M}^4, g)$. The tensor $K^{\mu\nu}$ given by (72), satisfying

$$- [ K , g ]_S = 2(K \partial \Phi) \otimes g, \quad (73)$$

is a conformal Killing tensor on $(\mathcal{M}^4, g)$, where $K \partial$ denotes the derivative operator $K^{\mu\nu} \partial_\nu$.

We show that there can exist a constant of motion associated with the conformal Killing tensor for a class of the conformal factor.

We shall try to find a solution of the Killing hierarchy which starts with a rank-2 conformal Killing tensor which is not a Killing tensor. Namely, we shall solve (29)-(34) with (35), i.e.,

$$- \left[ (2) K , g \right]_S + \lambda \otimes g = 0, \quad (74)$$

$$- \left[ (1) K , g \right]_S + 2q \left[ (2) K , A \right]_S + \lambda g - 2q \lambda \otimes A = 0, \quad (75)$$

$$- \left[ (0) K , g \right]_S + 2q \left[ (1) K , A \right]_S - q^2 \left[ (2) K , A^2 \right]_S - 2q \lambda A + \lambda (q^2 A^2 + m^2) = 0, \quad (76)$$

with

$$m^2 \lambda + q \lambda \cdot A = 0, \quad (77)$$

where $\lambda$ is not identically zero and the gauge potential $A^\mu$ is given by $A^\mu = e^{-\Phi} (\partial_\phi)^\mu$.

The tensor $(2) K^{\mu\nu} = K^{\mu\nu}$ given by (72) solves (74) with

$$\lambda = -2(K \partial \Phi). \quad (78)$$

From the algebraic condition (77), we have

$$\lambda = -qA \cdot (1) \lambda = 2qe^{-\Phi} r^2 \partial_\phi \Phi. \quad (79)$$

We assume that the function $\Phi$ is in the form $\Phi = \Phi(t, r, \theta)$, so that we have $\lambda = 0$ and $[ K , A ]_S = 0$. Then the second equation (75) reduces to the Killing vector equation

$$[ (1) K , g ]_S = 0. \quad (80)$$
For simplicity, we choose the trivial solution $K^{(1)} = 0$. The remaining equation (76) is given by

$$\left[[K, g]\right]_S = -2q^2 e^{-\Phi} K \partial g_{\phi \phi} - 2m^2 (K \partial \Phi), \quad (81)$$

or explicitly,

$$\partial_t K^{(0)} = \partial_r K^{(0)} = \partial_\phi K^{(0)} = 0, \quad (82)$$
$$\partial_\theta K^{(0)} = \bar{g}_{\theta \theta} (q^2 \partial_\theta \bar{g}_{\phi \phi} + m^2 e^{\Phi} \partial_\theta \Phi). \quad (83)$$

There exists a constant of motion if the function $\Phi$ is chosen such that the partial differential equations (82) and (83) for $K^{(0)}$ are integrable.

As the simplest case, we consider that the right-hand side of (83) vanishes, namely,

$$m^2 \partial_\theta e^{\Phi} + 2q^2 r^2 \sin \theta \cos \theta = 0. \quad (84)$$

In this case, equations (82) and (83) admit a trivial solution $K = 0$. We can integrate (84) easily to obtain

$$e^{\Phi} = \frac{q^2}{m^2} r^2 \cos^2 \theta + f(t, r), \quad (85)$$

where $f(t, r)$ is an arbitrary positive function.

Therefore, if we choose (85) as the conformal factor, then the quadratic quantity $F = K \cdot p^2$ is a constant of motion of a charged particle associated with the conformal Killing tensor. The quantity is conserved only on the constraint surface in the phase space.

**V. SUMMARY**

In this paper, we have discussed constants of motion for a test particle in a curved spacetime. For the particle which is subjected to an external field we have obtained the condition for existence of the constant of motion in the form of coupled equations with a hierarchical structure. There, we have taken the Hamiltonian constraint condition for the particle, which arises from the reparametrization invariance of particle’s world line, into consideration. The equation at the top of the hierarchy is the conformal Killing tensor equation. Then, the existence of constant of motion requires that the metric admits a
conformal Killing tensor. If the Killing hierarchy has a non-trivial solution, a constant of motion associated with the conformal Killing tensor exists.

As applications of the formalism, we have considered systems of a charged particle in Maxwell’s fields on black holes. In the case of a charged particle in the Kerr-Newman black holes, we have rediscovered a constant of motion quadratic in the canonical momenta, which has been found via the Hamilton-Jacobi method \([1]\). In the case of a charged particle in the electro-magnetic field without electric charge constructed by Wald’s method on the Kerr black holes, we have shown that the Killing hierarchy is not integrable. The non-existence of constant of motion does not depend on the choice of coordinate, in contrast to the fact that the discovery of constant of motion was due to suitable coordinates in the Hamilton-Jacobi method. We have found a new constant of motion as a solution of the hierarchical equations for a charged particle around a five-dimensional charged rotating black hole, which is a solution for the Einstein-Maxwell-Chern-Simons theory. Since these metrics admit Killing tensor of rank 2, the constants of motion in these examples are associated with the Killing tensors.

As the final example in this paper, we have considered Maxwell’s field on an artificial conformal flat spacetime. For a charged particle in this fields, we constructed a constant of motion which is associated with rank-2 conformal Killing tensor, \(i.e.,\) conservation equation holds only on the Hamiltonian constraint surface in the phase space. It would be interesting problem to find constants of motion for a particle moving in a solution of the Einstein-Maxwell system. The extension to a wide classes of interactions is an important future work.

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Appendix A: Killing hierarchy for a free particle

Let us consider the system of a free particle in the framework of the Killing hierarchy. The explicit form of the Hamiltonian is given by \([10]\) and the constraint equation is given
by (11). Then the Killing hierarchy (20) reads

\[- [(k-1) K, g]_S + (k-2) S \lambda \otimes g + m^2 (k) \lambda = 0, \quad k \geq 2,\]

\[- [(0) K, g]_S + m^2 (1) \lambda = 0,\]

\[- m^2 (0) \lambda = 0.\]  

(A1)

If the particle is massless, i.e. \( m = 0 \), then the constraint (11) becomes

\[ \mathcal{H} = g_{\mu \nu} p^\mu p^\nu \approx 0, \]  

(A2)

and the Killing hierarchy becomes

\[- [(k-1) K, g]_S + (k-2) S \lambda \otimes g = 0, \quad k \geq 2,\]

\[- [(0) K, g]_S = 0.\]  

(A3)

All the equations for \((k) \tilde{K} \quad (k \geq 1)\) become decoupled conformal Killing equations. Therefore a nontrivial conformal Killing tensor with non-vanishing \((l) \lambda\) gives a constant of motion conserved only on the constraint surface (A2).

If the particle is massive, i.e. \( m \neq 0 \), we consider symmetric tensors \( \tilde{K} \) which satisfy the linear differential equations

\[- [(k) \tilde{K}, g]_S + m^2 (k+1) \lambda = 0.\]  

(A4)

The solutions of the linear differential equations (A4) have the form

\[ (k) \tilde{K} = (k) \tilde{K}_H + (k) \tilde{K}_I, \]  

(A5)

where the homogeneous part \((k) \tilde{K}_H\) is a solution of the Killing equation

\[ [(k) \tilde{K}_H, g]_S = 0, \]  

(A6)

and \((k) \tilde{K}_I\) is an inhomogeneous solution satisfying the original equations (A4),

\[- [(k) \tilde{K}_I, g]_S + m^2 (k+1) \lambda = 0.\]  

(A7)
Using $\tilde{K}_H$ and $\tilde{K}_I$, we can construct the solution of (A1) as

$$
\begin{align*}
(k) K = & \tilde{K}_H + \tilde{K}_I + \frac{1}{m^2} \tilde{K}_I \otimes g.
\end{align*}
$$

(A8)

For this solution, the conserved quantity has the form

$$
\begin{align*}
F &= \sum_k (k) K \cdot p^k \\
&= \sum_k (k) \tilde{K}_H \cdot p^k + \sum_k (k) \tilde{K}_I \cdot p^k \\
&+ \frac{1}{m^2} \sum_k (k) \tilde{K}_I \cdot (g \cdot p^2 + m^2) \\
&\approx \sum_k (k) \tilde{K}_H \cdot p^k.
\end{align*}
$$

(A9)

Therefore the homogeneous solutions, that is, solutions for the Killing equations, contribute to the conserved quantity. The constant of motion for a massive free particle is requires the existence of the Killing tensor.

**Appendix B: Equations in the case of five-dimensional black holes**

As a supplement to section IV C, we give the explicit form of equations (62) in terms of components:

\((t, r)\) : 

$$

\begin{align*}
K^{(1)}_{tr} + g^{(1)}_{\theta t} K^{(1)}_{\theta r} + g^{(1)}_{\phi t} K^{(1)}_{\phi r} + g^{(1)}_{\psi t} K^{(1)}_{\psi r} &= B_{tr}, \quad (B1)
\end{align*}

\((t, \theta)\) : 

$$

\begin{align*}
K^{(1)}_{\theta t} + g^{(1)}_{\theta \theta} K^{(1)}_{\theta \theta} + g^{(1)}_{\phi \theta} K^{(1)}_{\phi \theta} + g^{(1)}_{\psi \theta} K^{(1)}_{\psi \theta} &= B_{\theta \theta}, \quad (B2)
\end{align*}

\((r, \phi)\) : 

$$

\begin{align*}
K^{(1)}_{r \phi} + g^{(1)}_{r \phi} K^{(1)}_{r \phi} + g^{(1)}_{\psi \phi} K^{(1)}_{\psi \phi} + g^{(1)}_{\phi \phi} K^{(1)}_{\phi \phi} &= B_{r \phi}, \quad (B3)
\end{align*}

\((r, \psi)\) : 

$$

\begin{align*}
K^{(1)}_{r \psi} + g^{(1)}_{r \psi} K^{(1)}_{r \psi} + g^{(1)}_{\psi \psi} K^{(1)}_{\psi \psi} + g^{(1)}_{r r} K^{(1)}_{\psi r} &= B_{r \psi}, \quad (B4)
\end{align*}

\((\theta, \phi)\) : 

$$

\begin{align*}
K^{(1)}_{\theta \phi} + g^{(1)}_{\theta \phi} K^{(1)}_{\theta \phi} + g^{(1)}_{\psi \phi} K^{(1)}_{\psi \phi} + g^{(1)}_{\phi \phi} K^{(1)}_{\phi \phi} &= B_{\theta \phi}, \quad (B5)
\end{align*}

\((\theta, \psi)\) : 

$$

\begin{align*}
K^{(1)}_{\theta \psi} + g^{(1)}_{\theta \psi} K^{(1)}_{\theta \psi} + g^{(1)}_{\psi \psi} K^{(1)}_{\psi \psi} + g^{(1)}_{\theta \theta} K^{(1)}_{\psi \theta} &= B_{\theta \psi}, \quad (B6)
\end{align*}

\(\sum_{k} (k)\)
where components of $B^{\mu \nu}$ are given explicitly by

$$B^{t \theta} = \frac{\sqrt{3} q e}{\Delta_r \rho^6} \left[ (r^2 + a^2)(r^2 + b^2) + abe \right] (a + b)(a - b) \sin 2\theta,$$

(B7)

$$B^{\theta \phi} = \frac{\sqrt{3} q e}{\Delta_r \rho^6} \left[ b(ab + e) + ar^2 \right] (a + b)(a - b) \sin 2\theta,$$

(B8)

$$B^{\theta \psi} = \frac{\sqrt{3} q e}{\Delta_r \rho^6} \left[ a(ab + e) + br^2 \right] (a + b)(a - b) \sin 2\theta,$$

(B9)

$$B^{tr} = \frac{2\sqrt{3} q e S}{r^3 \Delta_r \rho^6} \left[ \left[ (r^2 + a^2)(r^2 + b^2) + abe \right] \left[ r^2 \Delta_r + \rho^2(a^2 + b^2 + 2(r^2 - M)) \right] 
- r^2 \rho^2 \Delta_r (a^2 + b^2 + 2r^2) \right],$$

(B10)

$$B^{r \phi} = \frac{2\sqrt{3} q e S}{r^3 \Delta_r \rho^6} \left[ \left[ b(e + ab) + ar^2 \right] \left[ r^2 \Delta_r + \rho^2(a^2 + b^2 + 2(r^2 - M)) \right] 
- ar^2 \Delta_r \rho^2 \right],$$

(B11)

$$B^{r \psi} = \frac{2\sqrt{3} q e S}{r^3 \Delta_r \rho^6} \left[ \left[ a(e + ab) + br^2 \right] \left[ r^2 \Delta_r + \rho^2(a^2 + b^2 + 2(r^2 - M)) \right] 
- br^2 \Delta_r \rho^2 \right].$$

(B12)

The other components of the equation are trivial.

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