MINIMAL SPACE WITH NON-MINIMAL SQUARE

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Abstract. We completely solve the problem whether the product of two compact metric spaces admitting minimal maps also admits a minimal map. Recently Boronski, Clark and Oprocha gave a negative answer in the particular case when homeomorphisms rather than continuous maps are considered. In the present paper we show that there is a metric continuum $X$ admitting a minimal map, in fact a minimal homeomorphism, such that $X \times X$ does not admit any minimal map.

1. Introduction

By a dynamical system we mean a pair $(X,T)$ where $X$ is a metric space and $T: X \to X$ is a continuous map (not necessarily a homeomorphism). It is called minimal if there is no proper subset $M \subseteq X$ which is nonempty, closed and $T$-invariant (i.e., $T(M) \subseteq M$). In such a case we also say that the map $T$ itself is minimal. Clearly, a system $(X,T)$ is minimal if and only if the (forward) orbit $\text{Orb}_T(x) = \{x, T(x), T^2(x), \ldots\}$ of every point $x \in X$ is dense in $X$. Recall that if $T$ is a homeomorphism then in the compact case it is sufficient to check the density of all full orbits — in compact spaces, a homeomorphism has all forward orbits dense if and only if it has all full orbits dense.

Throughout the present paper, a metric space admitting a minimal map is sometimes said to be a minimal space. The classification of (compact) minimal spaces is a well-known open problem in topological dynamics, solved only in some particular cases; for some references see e.g. [KS]. Even such a basic question as whether the product of two compact minimal spaces is a minimal space, has not been answered so far in its full generality, though recently a negative answer was provided in the particular case when homeomorphisms rather than continuous maps are considered. In fact, Boronski, Clark and Oprocha [BCO] have found a compact metric space $Y$ admitting a minimal homeomorphism, such that $Y \times Y$ does not admit any minimal homeomorphism. The aim of the present paper is to solve the problem completely. We prove the following theorem (recall that a continuum is a compact connected space).

Theorem 1. There is a metric continuum $X$ admitting a minimal homeomorphism, such that $X \times X$ does not admit any minimal continuous map.

We show that this property has each of those Slovak spaces, which have been constructed in [DST, Section 4]. Recall that a compact metric space $X$ is called a Slovak space if it has at least three elements, admits a minimal homeomorphism $T$ and the group of homeomorphisms $X \to X$ is $H(X,X) = \{T^n : n \in \mathbb{Z}\}$. In [DST] it is also proved that if $X$ is a Slovak space then it is a continuum and the cyclic group $H(X,X)$ is infinite (i.e., isomorphic to $\mathbb{Z}$) and all its elements, except identity, are minimal homeomorphisms.

We do not know whether in Theorem 1 the space $X$ can be any Slovak space, but those constructed in [DST, Section 4] do work. The authors of [BCO], when constructing a minimal space $Y$ with non-minimal square $Y^2$, were apparently also inspired by the construction of those Slovak spaces, therefore $Y$ shares some features with them. In particular, similarly as in the case of

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the Slovak spaces constructed in [DST, Section 4], with one exception all componsants of $Y$ are continuous injective images of the real line. However, the exceptional compont of $Y$ consists of countably many pseudo-arcs connected by arcs, while the exceptional compont of the Slovak spaces constructed in [DST, Section 4] is the union of countably many topologist's sine curves, see Figure 1. Therefore each nondegenerate proper subcontinuum of our space $X$ from Theorem 1 is decomposable, while the space $Y$ from [BCO] does not have this nice property (both $X$ and $Y$, having uncountably many componsants, are indecomposable continua). Maybe even the space $Y$ has the property that its square does not admit any minimal map. However, in [BCO] it is only proved that $Y^2$ does not admit minimal homeomorphisms. In the present paper we show that our minimal space $X$ is such that $X^2$ does not admit minimal maps at all (to prove that $X^2$ does not admit minimal homeomorphisms is much easier, see Remark 22).

Let us also mention that for continuous flows such a counterexample is not possible. In fact, the class of compact metrizable spaces admitting minimal continuous flows is closed with respect to at most countable products, see [Di, Theorem 25] for even a stronger result.

2. Square of an appropriate Slovak space $X$

From now on we fix one of those Slovak spaces which have been constructed in [DST, Section 4] and we denote it by $X$; we will call it the Slovak space. To show that $X$ can serve as a counterexample required by Theorem 1, we are going first to describe its topological structure. We also introduce notation which will be used throughout the paper.

2.1. Description of the Slovak space $X$ used in the proof of Theorem 1. We are going to describe some properties of the Slovak space (for more details the reader is referred to [DST]).

First basic fact is that the Slovak space $X$ is a subset of $X_0 \times [0, 1]$, where $X_0$ is a generalized solenoid

$$X_0 = (C \times [0, 1]) / (y, 1) \sim (h(y), 0), \quad (2.1)$$

with $C$ being a Cantor set and $h : C \to C$ being a minimal homeomorphism. The continuum $X_0$ has uncountably many componsants, each of them is dense in $X_0$ and is a continuous injective image of the real line. The only nondegenerate proper subcontinua of $X_0$ are arcs.

The Slovak space $X$ is the closure of the graph of a function from $X_0$ to $[0, 1]$. Denote by $\pi : X \to X_0$ the natural projection. It is almost 1-1 and the only nondegenerate point inverses are arcs $W_n (n \in \mathbb{Z})$, with

$$\lim_{n \to \pm\infty} \text{diam}(W_n) = 0. \quad (2.2)$$

The space $X$ has uncountably many componsants, each of them being dense in $X$. One of them, denote it by $\gamma$, is not path connected. Its path components are $C_n (n \in \mathbb{Z})$, where each $C_n$ is homeomorphic to the graph of $\sin(1/x), x \in (0, 1]$; see Figure 1. Moreover, for every $n$, we have

$$C_n = C_n \cup W_{n+1} \subseteq C_n \cup C_{n+1}, \quad C_n \cap C_{n+1} = W_{n+1}, \quad (2.3)$$

where $\cup$ denotes disjoint union and $C_n$ stands for the closure of the set $C_n$. The family of all the other componsants of $X$ will be denoted by $\mathcal{A}$; every compont $\alpha \in \mathcal{A}$ is a continuous injective image of the real line.

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1We warn the reader that our notation differs from that used in [DST]. Instead of the notation $X, F, T, \tilde{T}$ in [DST] we are going to write $X_d, X, T_d, T$, respectively. Here lower index $d$ can be read as “down” since the system $(X_d, T_d)$ will be a factor of $(X, T)$.

2The solenoids, as well as the circle, are compact connected metrizable abelian, hence monothetic, groups. On the other hand, by [HA], a nondegenerate continuum is a solenoid if and only if it is indecomposable, homogeneous and all of its proper subcontinua are arcs. Therefore, if a generalized solenoid is not a solenoid, then it is not homogeneous and so it is not a topological group.
There are special minimal homeomorphisms $T_d: X_d \to X_d$ and $T: X \to X$ such that $\pi: (X,T) \to (X_d,T_d)$ is a factor map (i.e., $\pi \circ T = T_d \circ \pi$) and for every $n \in \mathbb{Z}$ it holds that

$$T(C_n) = C_{n+1}, \quad T(W_n) = W_{n+1}.$$  \hspace{1cm} (2.4)

Since both $T$ and $T^{-1}$ are minimal, in view of (2.4) we have that for every $n_0 \in \mathbb{Z}$ the disjoint unions

$$\bigcup_{n \geq n_0} W_n \text{ and } \bigcup_{n \leq n_0} W_n \text{ are dense in } X.$$ \hspace{1cm} (2.5)

Now, we describe how $T_d$ has been constructed in [DST]. Start with the suspension flow $\phi = (\phi_t)_{t \in \mathbb{R}}$ on $X_d$; i.e. $\phi_t: X_d \to X_d$ is defined by

$$\phi_t(y,s) = (h^{t+s}(y), \{t+s\}),$$ \hspace{1cm} (2.6)

where $[\cdot]$ and $\{\cdot\}$ denote the integer and fractional part of a real number, respectively. Then $T_d = \phi_{t_0}$, where $t_0 \neq 0$ is such that $\phi_{t_0}$ is a minimal homeomorphism on $X_d$; since the suspension flow $\phi$ is minimal, such $t_0$ exists, see e.g. [Ex], [Fa].

2.2. Some facts related to $X \times X$. For pairs of integers $m,n$ put

$$C_{m,n} = C_m \times C_n \quad \text{and} \quad W_{m,n} = W_m \times W_n.$$ \hspace{1cm} (2.7)

Notice that $C_{m,n}$ is homeomorphic to a quadrant of the plane and $W_{m,n}$ is homeomorphic to the square.

For maps $f,g: Y \to Z$, we denote by $\text{Coin}(f,g)$ the set of points of coincidence of $f$ and $g$, i.e., $\text{Coin}(f,g) = \{y \in Y: f(y) = g(y)\}$.

**Lemma 2.** Let $Y$ and $Z$ be topological spaces, $Y$ having the fixed point property. Let $f,g: Y \to Z$ be continuous maps, $g$ being a homeomorphism. Then $f$ and $g$ have a point of coincidence in $Y$.

**Proof.** The map $F = g^{-1} \circ f$ is a continuous map $Y \to Y$ and so it has a fixed point $y_0$. Thus $g^{-1} \circ f)(y_0) = y_0$ whence $f(y_0) = g(y_0)$.

For integers $a,b$ put $T_a \times b = T_d^a \times T_d^b$ and $T_a \times T_b = T_d^a \times T_d^b$.

**Lemma 3.** Let $F: X \times X \to X \times X$ be a continuous map. If $m,n$ are integers such that $F(W_{m,n}) \subseteq W_{m+a,n+b}$ then $\text{Coin}(F,T_a \times b) \cap W_{m,n} \neq \emptyset$.

**Proof.** The set $W_{m,n}$, being a square, has the fixed point property. Since $T_a \times b(W_{m,n}) = W_{m+a,n+b}$, it is sufficient to use Lemma 2. \hfill $\square$

**Remark 4.** It is of some interest to mention that, analogously, $\text{Coin}(F,T_a \times b) \cap \overline{C}_{m,n} \neq \emptyset$ provided $F(\overline{C}_{m,n}) \subseteq \overline{C}_{m+a,n+b}$. In fact, the closure $\overline{C}_{m,n}$, being the Cartesian product of arc-like continua, has the fixed point property [Dy].

**Figure 1.** The composant $\gamma$
Lemma 5. Let \( Y, Z \) be compact metric spaces and \( \varphi, \tau : Y \to Z \) be continuous maps. Assume that there are nowhere dense sets \( A_n \subseteq Y \) (\( n \in \mathbb{N} \)) with diameters converging to zero such that their union is dense in \( Y \) and, for every \( n \), \( A_n \cap \text{Coin}(\varphi, \tau) \neq \emptyset \). Then \( \varphi = \tau \).

Proof. The assumptions imply that every nonempty open set in \( Y \) contains some \( A_n \). Hence \( \text{Coin}(\varphi, \tau) \) is dense in \( Y \). \( \square \)

Since \( \pi : (X, T) \to (X_d, T_d) \) is an (almost 1-1) factor map, we immediately get the following.

Lemma 6. For every \( a, b \in \mathbb{Z}, \pi \times \pi : (X \times X, T^{a \times b}) \to (X_d \times X_d, T_d^{a \times b}) \) is an (almost 1-1) factor map.

2.3. Direct products \( T^{a \times b} : X \times X \to X \times X \) are not minimal. Recall that our aim is to show that \( X \times X \) does not admit a minimal map. In this subsection we show that the particular homeomorphisms \( T^{a \times b} \) are not minimal.

Lemma 7. The map \( T_d^{a \times b} \) is not minimal on \( X_d \times X_d \) for any \( a, b \in \mathbb{Z} \).

Proof. It follows from (4.7) that the map \( T_d = \phi_{t_0} : X_d \to X_d \) is an extension of the rotation \( s \mapsto t_0 + s \) of the circle \( \mathbb{R}/\mathbb{Z} \), the corresponding factor map being the projection onto the second coordinate. The map \( T_d^{a \times b} \) is not minimal since its factor \( (s, s') \mapsto (at_0 + s, bt_0 + s') \) on the torus is not minimal. \( \square \)

Proposition 8. The map \( T_d^{a \times b} \) is not minimal on \( X \times X \) for any \( a, b \in \mathbb{Z} \).

Proof. Since minimality is preserved by factors, just use Lemmas 6 and 7. \( \square \)

3. Continuous surjections on \( X \times X \)

Throughout this section assume that \( F : X \times X \to X \times X \) is a continuous surjective map.

We use the notation from Subsection 2.1. Notice that \( C_n \cap C_m \) (\( n < m \)) is nonempty if and only if \( m = n + 1 \); if this is the case, the intersection is the arc \( W_{n+1} \). The path components of \( X \times X \), being products of path components of \( X \), are of four types:

- \( \alpha \times \beta \), where \( \alpha, \beta \in \mathcal{A} \);
- \( C_m \times \alpha \), where \( m \in \mathbb{Z} \) and \( \alpha \in \mathcal{A} \);
- \( \alpha \times C_m \), where \( m \in \mathbb{Z} \) and \( \alpha \in \mathcal{A} \);
- \( C_m \times C_n \), where \( m, n \in \mathbb{Z} \).

Note that every path component of the first type is dense in \( X \times X \), while any other path component is nowhere dense in \( X \times X \).

Lemma 9. For every \( \alpha, \beta \in \mathcal{A} \) there are \( \alpha', \beta' \in \mathcal{A} \) such that

\[
F(\alpha \times \beta) \subseteq \alpha' \times \beta'.
\]

Proof. A continuous image of a path connected set is path connected, and every continuous surjection maps dense sets onto dense sets. Since only the path components of the first type are dense, we get that such \( \alpha' \) and \( \beta' \) necessarily exist. \( \square \)

Put

\[
\begin{align*}
D_{11} &= \{ m \in \mathbb{Z} : F(C_m \times \alpha) \subseteq C_k \times X \text{ for some } k \in \mathbb{Z} \text{ and } \alpha \in \mathcal{A} \}, \\
D_{12} &= \{ m \in \mathbb{Z} : F(C_m \times \alpha) \subseteq X \times C_k \text{ for some } k \in \mathbb{Z} \text{ and } \alpha \in \mathcal{A} \}, \\
D_{21} &= \{ m \in \mathbb{Z} : F(\alpha \times C_m) \subseteq C_k \times X \text{ for some } k \in \mathbb{Z} \text{ and } \alpha \in \mathcal{A} \}, \\
D_{22} &= \{ m \in \mathbb{Z} : F(\alpha \times C_m) \subseteq X \times C_k \text{ for some } k \in \mathbb{Z} \text{ and } \alpha \in \mathcal{A} \}.
\end{align*}
\]

\[\text{Note that the space } X_d \times X_d, \text{ though in general it is not a topological group, does admit a minimal homeomorphism. In fact, } X_d \text{ admits a minimal continuous flow, see (4.7), and so } X_d \times X_d \text{ also admits a minimal continuous flow by [Di Theorem 25]. By passing to an appropriate time } t\text{-map we get a minimal homeomorphism [Ev] [Es].}\]
We are going to show that if \( m \in D_{ij} \) then the corresponding \( k \in \mathbb{Z} \) is unique and does not depend on \( \alpha \).

**Lemma 10.** Let \( m, k \in \mathbb{Z} \) and \( \alpha \in \mathcal{A} \).

1. If \( F(C_m \times \alpha) \subseteq C_k \times X \), then \( F(C_m \times \beta) \subseteq C_k \times X \) for every \( \beta \in \mathcal{A} \).
2. If \( F(C_m \times \alpha) \subseteq X \times C_k \), then \( F(C_m \times \beta) \subseteq X \times C_k \) for every \( \beta \in \mathcal{A} \).
3. If \( F(\alpha \times C_m) \subseteq C_k \times X \), then \( F(\beta \times C_m) \subseteq C_k \times X \) for every \( \beta \in \mathcal{A} \).
4. If \( F(\alpha \times C_m) \subseteq X \times C_k \), then \( F(\beta \times C_m) \subseteq X \times C_k \) for every \( \beta \in \mathcal{A} \).

**Proof.** We prove only (1). Let \( F(C_m \times \alpha) \subseteq C_k \times X \). The fact that \( \alpha \) is dense gives \( F(C_m \times X) \subseteq \overline{C_k \times X} \). By (2,3), \( F(C_m \times X) \subseteq (C_k \times X) \cup (C_{k+1} \times X) \). Now let \( \beta \in \mathcal{A} \). Then the first projection of \( F(C_m \times \beta) \) is a path connected subset of \( C_k \cup C_{k+1} \). So we have proved that if \( F(C_m \times \alpha) \subseteq C_k \times X \) then, for every \( \beta \in \mathcal{A} \), \( F(C_m \times \beta) \) is a subset of either \( C_k \times X \) or \( C_{k+1} \times X \).

We claim that \( F(C_m \times \beta) \) is in fact a subset of \( C_k \times X \). Suppose, on the contrary, that \( F(C_m \times \beta) \subseteq C_{k+1} \times X \) for some \( \beta \). Applying what we have just proved we get that \( F(C_m \times \alpha) \) is a subset of either \( C_{k+1} \times X \) or \( C_{k+2} \times X \). This contradicts our assumption on \( \alpha \) in (1).

For \( i, j \in \{1, 2\} \) and \( m \in D_{ij} \) denote the corresponding \( k \) from the definition of \( D_{ij} \) by \( \psi_{ij}(m) \); it is unique by Lemma 10. So \( \psi_{ij} \) is a function defined on \( D_{ij} \); we denote its range by \( R_{ij} \).

**Lemma 11.** Let \( i, j \in \{1, 2\} \).

1. Either \( D_{ij} = \mathbb{Z} \) or \( D_{ij} = \emptyset \) or \( D_{ij} = [m_0, \infty) \cap \mathbb{Z} \) for some \( m_0 \).
2. If \( m \in D_{ij} \) then also \( m + 1 \in D_{ij} \) and \( \psi_{ij}(m+1) - \psi_{ij}(m) \in \{0, 1\} \); hence the function \( \psi_{ij} \) is nondecreasing.

**Proof.** Let \( i = j = 1 \); the other cases are similar. Let \( m \in D_{11} \) and put \( k = \psi_{11}(m) \); that is, \( F(C_m \times \alpha) \subseteq C_k \times X \) for some \( \alpha \in \mathcal{A} \). Since \( \alpha \) is dense and \( W_{m+1} \subseteq \overline{C_m} \) we get \( F(W_{m+1} \times \alpha) \subseteq \overline{C_k \times X} \). Due to path connectedness, \( F(C_m \times \alpha) \) is a subset of either \( C_k \times X \) or \( C_{k+1} \times X \). Thus \( m + 1 \in D_{11} \) and \( \psi_{11}(m+1) \in \{k, k+1\} \). So (1) and (2) for \( i = j = 1 \) are proved.

**Lemma 12.** Let \( j \in \{1, 2\} \). Then \( R_{1j} \cup R_{2j} = \mathbb{Z} \), and at least one of \( D_{1j} \), \( D_{2j} \) is equal to \( \mathbb{Z} \).

**Proof.** We may assume that \( j = 1 \); the other case is analogous. To prove that \( R_{11} \cup R_{21} = \mathbb{Z} \) fix \( k \in \mathbb{Z} \). We need to show that there exists \( m \in \mathbb{Z} \) and \( \alpha \in \mathcal{A} \) such that \( F(C_m \times \alpha) \subseteq C_k \times X \) or \( F(\alpha \times C_m) \subseteq C_k \times X \). This follows by a cardinality argument. The set \( C_k \times X \) contains uncountably many path-components \( C_k \times \beta \) (\( \beta \in \mathcal{A} \)). By surjectivity of \( F \), for every \( C_k \times \beta \) there is a path component of \( X \times X \) which is mapped by \( F \) into \( C_k \times \beta \). Thus there are uncountably many path components of \( X \times X \) which are mapped by \( F \) into \( C_k \times X \). In view of Lemma 9 none of them is of the form \( \alpha \times \beta \) (\( \alpha, \beta \in \mathcal{A} \)). Further, there are only countably many path components of the form \( C_m \times C_n \) (\( m, n \in \mathbb{Z} \)). Hence necessarily a path component of the form \( C_m \times \alpha \) or \( \alpha \times C_m \) is mapped to \( C_k \times X \).

If both \( D_{11} \) and \( D_{21} \) are different from \( \mathbb{Z} \) then, by Lemma 11, their union is bounded from below. Hence, by Lemma 11, the union of \( R_{11} \) and \( R_{21} \) is also bounded from below, a contradiction.

**Lemma 13.** Let \( j \in \{1, 2\} \). Then one of \( D_{1j} \), \( D_{2j} \) is \( \mathbb{Z} \) and the other one is empty.

**Proof.** Again assume that \( j = 1 \). By Lemma 12, at least one of \( D_{11} \) and \( D_{21} \) equals \( \mathbb{Z} \). Suppose that both are nonempty. Fix \( m \in D_{11} \) and \( n \in D_{21} \). Put \( k = \psi_{11}(m) \) and \( l = \psi_{21}(n) \). Then \( F(C_m \times X) \subseteq \overline{C_k} \times X \) and \( F(X \times C_n) \subseteq \overline{C_l} \times X \). Consequently, \( F(C_m \times C_n) \subseteq (\overline{C_k} \cap \overline{C_l}) \times X \) and so \( |k - l| \leq 1 \). Thus we have proved that

\[
|\psi_{11}(m) - \psi_{21}(n)| \leq 1
\]

for every \( m \in D_{11} \) and \( n \in D_{21} \).

It follows that both \( \psi_{11} \) and \( \psi_{21} \) are bounded, which contradicts Lemma 12. \( \square \)
Lemma 14. Let \( i \in \{1, 2\} \). Then one of \( D_{11}, D_{i2} \) is \( \mathbb{Z} \) and the other one is empty.

Proof. Assume that \( i = 1 \); the other case is analogous. By Lemma 13, each of \( D_{11} \) and \( D_{12} \) is \( \mathbb{Z} \) or empty. To prove that one of them is \( \mathbb{Z} \) and the other one is empty, suppose on the contrary that both are nonempty. Then \( D_{11} = D_{12} = \mathbb{Z} \) and so, by Lemma 13, \( D_{21} = D_{22} = \emptyset \).

In view of Lemma 10 and definitions of \( D_{11} \) and \( D_{12} \), for every \( m \in \mathbb{Z} \) and \( \alpha \in A \) we have \( F(C_m \times \alpha) \subseteq C_{\psi_{11}(m)} \times C_{\psi_{12}(m)} \), and so \( F(C_m \times X) \subseteq \overline{C}_{\psi_{11}(m)} \times \overline{C}_{\psi_{12}(m)} \subseteq \gamma \times \gamma \). Since \( m \) was arbitrary,

\[
F(\gamma \times X) \subseteq \gamma \times \gamma. \tag{3.1}
\]

Fix a path component \( C_k \times \beta \), where \( k \in \mathbb{Z} \) and \( \beta \in A \). By surjectivity of \( F \) there is a path component mapped into \( C_k \times \beta \). In view of Lemma 9 and (3.1), there are integers \( m, \alpha \in A \) with \( F(\alpha \times C_m) \subseteq C_k \times \beta \). Hence \( m \in D_{21} = \emptyset \), a contradiction. \( \square \)

Proposition 15. Exactly one of the following two possibilities is true:

(11-22) \( D_{11} = D_{22} = \mathbb{Z} \) and \( D_{12} = D_{21} = \emptyset \), hence \( R_{11} = R_{22} = \mathbb{Z} \) and \( R_{12} = R_{21} = \emptyset \);

(12-21) \( D_{11} = D_{22} = \emptyset \) and \( D_{12} = D_{21} = \mathbb{Z} \), hence \( R_{11} = R_{22} = \emptyset \) and \( R_{12} = R_{21} = \mathbb{Z} \).

Proof. The claims on the domains follow from Lemmas 13 and 14. For the claims on the ranges then use Lemma 12. \( \square \)

Compare the following lemma with [DST, Lemma 4].

Lemma 16. \( F(\gamma \times \gamma) = F^{-1}(\gamma \times \gamma) = \gamma \times \gamma \).

Proof. Fix \( \alpha \in A \). If \( F(\gamma \times \alpha) \) intersects \( \gamma \times \gamma \), there are integers \( m, k, l \) with \( F(C_m \times \alpha) \subseteq C_k \times C_l \). Hence \( m \in D_{11} \cap D_{12} \), which contradicts Proposition 15. Thus \( F(\gamma \times \alpha) \) is disjoint from \( \gamma \times \gamma \).

Analogously \( F(\alpha \times \gamma) \cap (\gamma \times \gamma) = \emptyset \).

We have proved that both \( F(\gamma \times (X \setminus \gamma)) \) and \( F((X \setminus \gamma) \times \gamma) \) are disjoint from \( \gamma \times \gamma \). By Lemma 9, \( F((X \setminus \gamma) \times (X \setminus \gamma)) \) is a subset of \( (X \setminus \gamma) \times (X \setminus \gamma) \). Hence \( F^{-1}(\gamma \times \gamma) \subseteq \gamma \times \gamma \).

We are going to prove that \( F(\gamma \times \gamma) \subseteq \gamma \times \gamma \). Fix \( m, n \in \mathbb{Z} \). Assume that we are in Case (11-22) from Proposition 15. Since \( m \in D_{11} \) and \( n \in D_{22} \), \( F(C_m \times C_n) \subseteq \overline{C}_{\psi_{11}(m)} \times \overline{C}_{\psi_{22}(n)} \subseteq \gamma \times \gamma \). In Case (12-21) we similarly get \( F(C_m \times C_n) \subseteq \overline{C}_{\psi_{21}(m)} \times \overline{C}_{\psi_{12}(m)} \subseteq \gamma \times \gamma \). This finishes the proof of \( F(\gamma \times \gamma) \subseteq \gamma \times \gamma \).

So, for \( A = \gamma \times \gamma \) we have proved \( F^{-1}(A) \subseteq A \) and \( F(A) \subseteq A \). Moreover, trivially \( F^{-1}(F(A)) \supseteq A \) and, since \( F \) is surjective, \( F(F^{-1}(A)) = A \). It follows that \( F(A) = F^{-1}(A) = A \). \( \square \)

4. Non-existence of minimal maps on \( X \times X \)

In this section suppose that \( F : X \times X \rightarrow X \times X \) is a minimal map. To get a contradiction, we show that then \( F^2 \) is necessarily a direct product, in fact \( F^2 = T^{c \times d} \) for some integers \( c \neq 0 \neq d \). Then Proposition 8 will be used.

We distinguish two cases.

4.1. Case (11-22) from Proposition 15. Assume that (11-22) is true. Then for every \( m, n \in \mathbb{Z} \) and \( \alpha \in A \),

\[
F(C_m \times \alpha) \subseteq C_{\psi_{11}(m)} \times X, \quad F(\alpha \times C_n) \subseteq X \times C_{\psi_{22}(n)}. \tag{4.1}
\]

In the considered case, \( \varphi = \psi_{11} \times \psi_{22} \) is a surjective selfmap of \( \mathbb{Z} \times \mathbb{Z} \). It follows from (4.1), by passing to closures, that the map \( \varphi \) has the property

\[
F(C_{mn}) \subseteq \overline{C}_{\varphi(m,n)} \tag{4.2}
\]

for every \( m, n \in \mathbb{Z} \) (we use the notation from (2.7)).
For integers $a \neq 0$ and $m$ consider the following ray of integers:

$$I_{m}^{\text{sgn}(a)} = \begin{cases} 
\mathbb{Z} \cap [m, \infty) & \text{if } a > 0; \\
\mathbb{Z} \cap (-\infty, m] & \text{if } a < 0
\end{cases}$$

(the notation is due to the fact that it depends only on $m$ and on the sign of $a$). Further, for non-zero integers $a, b$ denote

$$E_{a,b} = \{(m, n) \in \mathbb{Z}^2 : \varphi(m, n) = (m, n) + (a, b)\}.$$  

**Lemma 17.** Assume that $F$ satisfies (11-22) from Proposition 15. Then there are integers $a, m_a$ and $b, n_b$ such that $a \neq 0 \neq b$ and

$$\psi_1(m) = m + a, \quad \psi_2(n) = n + b$$

for all integers $m \in I_{m_a}^{\text{sgn}(a)}$ and $n \in I_{n_b}^{\text{sgn}(b)}$. Hence

$$E_{a,b} \supseteq I_{m_a}^{\text{sgn}(a)} \times I_{n_b}^{\text{sgn}(b)}.$$  

**Proof.** We prove only the claim for $\psi_1$. First realize that $\psi_1$ has no fixed point due to minimality of $F$ (in fact, $\psi_1(k) = k$ implies that $C_k \times X$ is $F$-invariant by (4.1)). Further, $\psi_1$ is nondecreasing and $(\psi_1 - \text{Id}_Z)$ is nonincreasing by Lemma 11. Further, assume first that there is $m_1$ such that $\psi_1(m_1) > m_1$, i.e. $(\psi_1 - \text{Id}_Z)(m_1) > 0$. Since $\psi_1 - \text{Id}_Z$ is nonincreasing and does not vanish (otherwise $\psi_1$ would have a fixed point), we clearly have that there are integers $a > 0$ and $m_a$ such that $(\psi_1 - \text{Id}_Z)(m) = a$, i.e. $\psi_1(m) = m + a$, for all $m \geq m_a$.

If there is no such $m_1$ then $\psi_1 < \text{Id}_Z$. Then $\psi_1 - \text{Id}_Z$ is negative and nonincreasing. Hence there are integers $a < 0$ and $m_a$ such that $(\psi_1 - \text{Id}_Z)(m) = a$, i.e. $\psi_1(m) = m + a$, for all $m \leq m_a$.  \hfill $\square$

**Lemma 18.** Let $m, n \in \mathbb{Z}$ be such that

$$\varphi(m + 1, n + 1) = \varphi(m, n) + (1, 1).$$

Then $F(W_{m, n+1}) \subseteq W_{\varphi(m, n+1)}$.

**Proof.** By (2.3) and (2.7), $W_{m, n+1} = C_{m, n} \cap C_{m, n+1}$. Then the assumption and (4.2) yield $F(W_{m, n+1}) \subseteq C_{\varphi(m, n)} \cap C_{\varphi(m, n+1)} = W_{\varphi(m, n+1)}$.  \hfill $\square$

The reason why it is useful to consider the sets $E_{a,b}$ lies in the following simple lemma.

**Lemma 19.** Let $m, n \in \mathbb{Z}$ be such that both $(m, n)$ and $(m + 1, n + 1)$ belong to $E_{a,b}$. Then we have

$$\text{Coin}(F, T_{a,b}) \cap W_{m, n+1} \neq \emptyset.$$  

**Proof.** The first claim follows from the definition of $E_{a,b}$. So, by Lemma 18 $F(W_{m+1, n+1}) \subseteq W_{\varphi(m+1, n+1)} = W_{(m+1, n+1) + (a, b)}$. Now use Lemma 3.  \hfill $\square$

**Lemma 20.** Let $F$ satisfy (11-22) from Proposition 15. Then $F = T_{a,b}$ for some integers $a \neq 0 \neq b$.

**Proof.** Take $a \neq 0 \neq b$ from Lemma 17. Let $J$ be the set of all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that both $(m, n)$ and $(m + 1, n + 1)$ belong to $E_{a,b}$. By Lemma 17 the set $J$ contains a quadrant in $\mathbb{Z} \times \mathbb{Z}$. Then using (2.5) we get that $\bigcup_{(m, n) \in J} W_{m, n+1}$ is dense in $X \times X$. Further, the sets $W_{m, n+1}$ as $(m, n) \in J$ are nowhere dense in $X \times X$ (because they are closed and are subsets of the set $\gamma \times \gamma$ with empty interior) and their diameters converge to zero by (2.2). Now Lemmas 19 and 5 yield that $F = T_{a,b}$.  \hfill $\square$
4.2. Case (12-21) from Proposition 15. Assume that (12-21) is true. Then, for every \( m, n \in \mathbb{Z} \) and \( \alpha \in \mathcal{A} \),
\[
F(C_m \times \alpha) \subseteq X \times C_{\psi_{12}(m)}, \quad F(\alpha \times C_n) \subseteq C_{\psi_{21}(n)} \times X.
\]
Due to Lemma 16, we in fact have
\[
F(C_m \times \alpha) \subseteq (X \setminus \gamma) \times C_{\psi_{12}(m)}, \quad F(\alpha \times C_n) \subseteq C_{\psi_{21}(n)} \times (X \setminus \gamma).
\]
These two inclusions imply that, for every \( m, n \in \mathbb{Z} \) and \( \alpha \in \mathcal{A} \),
\[
F^2(C_m \times \alpha) \subseteq C_{\psi_{21}(\psi_{12}(m))} \times X \quad \text{and} \quad F^2(\alpha \times C_n) \subseteq X \times C_{\psi_{12}(\psi_{21}(n))}.
\]
So \( F^2 \) satisfies (11-22) from Proposition 15. Moreover, \( F^2 \) is also minimal since \( F \) is minimal and \( X \times X \) is a continuum. Thus, applying Lemma 20 to \( F^2 \), we immediately obtain the following.

**Lemma 21.** Let \( F \) satisfy (12-21) from Proposition 15. Then \( F^2 = T^{a \times b} \) for some integers \( a \neq 0 \neq b \).

4.3. Proof of Theorem 1. We are finally ready to prove Theorem 1.

**Theorem 1.** There is a metric continuum \( X \) admitting a minimal homeomorphism, such that \( X \times X \) does not admit any minimal continuous map.

**Proof.** Let \( X \) and \( T \) be the Slovak space and the minimal homeomorphism \( X \to X \) from Subsection 2.1. Assume that \( X \times X \) admits a minimal map \( F \). By Lemmas 20 and 21, the map \( F^2 \) is of the form \( T^{c \times d} \) for some integers \( c, d \). This contradicts Proposition 8. \( \square \)

**Remark 22** (Proof of Theorem 1 for homeomorphisms). The weaker result that, for the Slovak space \( X \), \( X \times X \) does not admit a minimal homeomorphism can be proved in a much easier way. In fact, one can show that any (not necessarily minimal) homeomorphism \( F \) of \( X \times X \) is factor preserving, even that there are integers \( a \) and \( b \) such that
\[
either \ F = T^{a \times b} \quad \text{or} \quad F = R \circ T^{b \times a}, \tag{4.5}
\]
where \( R \) is the homeomorphism sending \((x_1, x_2)\) to \((x_2, x_1)\). Then \( F^2 = T^{c \times d} \) for some \( c \) and \( d \) and, by Proposition 8, \( F^2 \) is not minimal. Hence neither \( F \) is not minimal because \( X \times X \) is not connected.

It remains to prove that any homeomorphism \( F \) of \( X \times X \) satisfies (4.5). First realize that \( F \) maps (in a bijective way) the path components of the type \( C_{m,n} \) onto the path components of the same type; to see this, it suffices to use that every \( C_{m,n} \) has the fixed point property (see Remark 4), while none of the sets \( C_m \times X, X \times C_m, X \times X \) (the closures of path components of the other three types) has this property. Thus \( F \) induces a bijection \( \varphi \) of \( Z \times Z \) such that
\[
F(C_{m,n}) = C_{\varphi(m,n)} \quad \text{for every} \quad m, n \in \mathbb{Z}. \tag{4.6}
\]
Then, similarly as in the proof of Lemma 11(2), one can show that, for every \( m, n \in \mathbb{Z} \) and \( i, j \in \{0, 1\} \),
\[
\varphi(m + i, n + j) - \varphi(m, n) \quad \text{belongs to the set} \quad \{0, 1\}^2. \tag{4.7}
\]
We can view \( Z \times Z \) as a poset with the partial ordering
\[
(m, n) \leq (m', n') \quad \text{if} \quad (m \leq m') \text{ and } (n \leq n'). \tag{4.3}
\]
It follows from (4.7) that the bijection \( \varphi \) on the poset \((Z \times Z, \leq)\) is order-preserving.
Fix \( m, n \in \mathbb{Z} \). Since \( \varphi \) is bijective, the formula (4.7) gives that the set of four points \((m, n) + \{0, 1\}^2\) is mapped onto the set of four points \(\varphi(m, n) + \{0, 1\}^2\). However, \( \varphi \) is order preserving and so we necessarily have that (4.3) holds. Moreover, the set of two points \((m, n) + \{(0, 1), (1, 0)\}\) is mapped
onto the set of two points \( \varphi(m, n) + \{(0, 1), (1, 0)\} \). Further, by easy argument as in the proof of Lemma \[ \text{4.6} \] and the just obtained property \[ \text{4.3} \] give

\[
F(W_{m,n}) = W_{\varphi(m,n)} \quad \text{for every } m, n \in \mathbb{Z}. \tag{4.8}
\]

Call \((m, n)\) nice if \((m, n) + (0, 1)\) is mapped to \(\varphi(m, n) + (0, 1)\), otherwise call it ugly. It is straightforward to check that if \((m, n)\) is nice or ugly then \((m, n) + (0, 1)\) and \((m, n) + (1, 0)\) are both nice or both ugly, respectively. So there are two cases:

1. all elements of \(\mathbb{Z} \times \mathbb{Z}\) are nice;
2. all elements of \(\mathbb{Z} \times \mathbb{Z}\) are ugly.

Denote \((a, b) = \varphi(0, 0)\).

In case 1 clearly \(\varphi(m, n) = (m, n) + (a, b)\) for every \(m, n \in \mathbb{Z}\). Then, by \[ \text{4.8}, \]

\[
F(W_{m,n}) = W_{m+a,n+b} = T^{a \times b}(W_{m,n}).
\]

Hence \(F = T^{a \times b}\) by Lemmas \[ \text{3} \] and \[ \text{5} \].

In case 2 one can check that \(\varphi(m, n) = (n, m) + (a, b)\) for every \(m, n \in \mathbb{Z}\). Analogously as above, \(F = R \circ T^{b \times a}\). This finishes the proof of \[ \text{4.5} \] and so, as explained above, this implies that \(X \times X\) does not admit any minimal homeomorphism.

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