The double-Kerr equilibrium configurations involving
one extreme object

J A Rueda¹, V S Manko², E Ruiz³ and J D Sanabria-Gómez¹

¹ Escuela de Física, Universidad Industrial de Santander, AA 678, Bucaramanga, Colombia
² Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, AP 14-740, 07000 México DF, Mexico
³ Departamento de Física Fundamental, Universidad de Salamanca, 37008 Salamanca, Spain

E-mail: jrueda@tux.uis.edu.com, vsmanko@fis.cinvestav.mx, eruiz@usal.es
and jsanabria@uis.edu.co

Received 22 August 2005, in final form 30 September 2005
Published 31 October 2005
Online at stacks.iop.org/CQG/22/4887

Abstract
We demonstrate the existence of equilibrium states in the limiting cases of the
double-Kerr solution when one of the constituents is an extreme object. In
the ‘extreme–subextreme’ case the negative mass of one of the constituents is
required for the balance, whereas in the ‘extreme–superextreme’ equilibrium
configurations both Kerr particles may have positive masses. We also show that
the well-known relation $|J| = M^2$ between the mass and angular momentum
in the extreme single Kerr solution ceases to be a characteristic property of the
extreme Kerr particle in a binary system.

PACS number: 04.20.Jb

1. Introduction

In the paper [1], a unified approach to the solution of the double-Kerr equilibrium problem
[2] in its extended form involving arbitrary combinations of the sub- and superextreme Kerr
particles was developed. This permitted later on to rigorously prove the non-existence of
equilibrium states of two Kerr black holes with positive masses [3] and, furthermore, to
establish the general law determining equilibrium of two arbitrary Kerr particles [4]. In the
paper [1], the explicit formulae defining the double-Kerr solution in the limiting case when
one of the constituents is an extreme object and the other a sub- or superextreme Kerr particle
were also worked out. At the same time, attempts to solve numerically the corresponding
balance equations failed to result in any equilibrium configuration, which forced the authors
of [1] to put forward a conjecture on the impossibility of balance between an extreme and a
non-extreme Kerr particle via the gravitational spin–spin interaction.

However, a recent approximation analysis of the double-Kerr equilibrium problem carried
out in [5], especially the last approximation scheme in which, on the one hand, the upper
A constituent looks very much like an extreme object and, on the other hand, the corresponding binary equilibrium configurations do exist, strongly motivated us to undertake a revision of the limiting case from [1] involving one extreme Kerr particle and thus continue the search for the equilibrium configurations which were not found before. As a result, we have been able to arrive, after scrupulously modifying our schemes for finding numerical roots of the analytical system of balance equations, at the desired equilibrium configurations which can be considered as the first examples of the balance of that kind in the literature. As expected, the equilibrium states between one extreme and one subextreme Kerr particle (the ‘extreme–subextreme’ configurations) necessarily imply the negative mass of one of the constituents; in the ‘extreme–superextreme’ case equilibrium configurations are possible when both Kerr particles possess positive Komar masses.

The rest of the paper is organized as follows. In section 2, we write the metric describing the stationary axisymmetric double-Kerr systems with one extreme constituent, as well as the corresponding balance conditions. Particular equilibrium states are considered in section 3 and some of their physical characteristics such as Komar masses and angular momenta [6] are calculated. Section 4 is devoted to the discussion of the results obtained.

2. The ‘extreme–non-extreme’ systems and the balance conditions

For tackling the two-Kerr configurations with one extreme particle we shall make use of the solution obtained in [1] with the aid of Sibgatullin’s method [7, 8] which is defined by the Ernst complex potential $\mathcal{E}$ [9] of the form

$$\mathcal{E} = \frac{\Lambda + \Gamma}{\Lambda - \Gamma},$$

$$\Lambda = \frac{r_1}{\alpha_1 - \beta_1} \begin{vmatrix} r_1 & r_2 & r_3 & -r_1^2 \frac{1}{(\alpha_1 - \beta_1)^2} \\ \alpha_1 - \beta_1 & \alpha_2 - \beta_1 & \alpha_3 - \beta_1 \end{vmatrix},$$

$$\Gamma = \frac{1}{\alpha_1 - \beta_1} \begin{vmatrix} 1 & 1 & 1 & (z - \alpha_1)/r_1 \\ \alpha_1 - \beta_1 & \alpha_2 - \beta_1 & \alpha_3 - \beta_1 & -r_1^2 \frac{1}{(\alpha_1 - \beta_1)^2} \\ \alpha_1 - \beta_1 & \alpha_2 - \beta_1 & \alpha_3 - \beta_1 & -r_1^2 \frac{1}{(\alpha_1 - \beta_2)^2} \\ 0 & 1 & 1 & 1 & r_1^2 \frac{\partial}{\partial \alpha_1} \left[ \frac{1}{(\alpha_1 - \beta_1)/r_1} \right] \\ 0 & 1 & 1 & 1 & r_1^2 \frac{\partial}{\partial \alpha_1} \left[ \frac{1}{(\alpha_1 - \beta_2)/r_1} \right] \end{vmatrix}.$$ 

$$r_n = \sqrt{\rho^2 + (z - \alpha_n)^2}, \quad (1)$$

The wrong sign of $B$ in [1] is rectified.
where $\alpha_1$ is an arbitrary real constant, the constants $\alpha_2$ and $\alpha_3$ can assume arbitrary real values or be an arbitrary complex conjugate pair $\alpha_2 = \bar{\alpha}_2$, a bar over a symbol denoting complex conjugation; $\beta_1$ and $\beta_2$ are arbitrary complex parameters; $\rho$ and $z$ are the usual Weyl–Papapetrou cylindrical coordinates in the Papapetrou stationary axisymmetric line element

$$ds^2 = e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2 - f(dt - \omega d\varphi)^2.$$  

The double-Kerr equilibrium configurations consist of two major groups: (a) binary systems composed of one extreme and one superextreme Kerr particle (figure 1(a)), and (b) configurations composed of one extreme and one superextreme Kerr particle (figure 1(b)). The position of the extreme constituent on the $z$-axis is defined by $\alpha_1$; the part of the symmetry axis between the real-valued $\alpha_2$ and $\alpha_3$ represents the location of the subextreme constituent, whereas the cut joining the complex conjugates $\alpha_2$ and $\alpha_3$ defines a superextreme Kerr particle. To tackle the case of two separated objects, in what follows we assign, without loss of generality, the following order to $\alpha_n$:

$$\alpha_1 > \text{Re}(\alpha_2) \geq \text{Re}(\alpha_3).$$  

The equilibrium of two constituents due to balance of the gravitational attraction and spin–spin repulsion forces implies the regularity of the symmetry axis outside the location of these constituents, i.e. regularity of the regions I, II and III shown in figure 1 which correspond to the parts $z > \alpha_1$, $\text{Re}(\alpha_2) < z < \alpha_1$ and $z < \text{Re}(\alpha_3)$ of the symmetry axis, respectively.
As is well-known [10, 11], the regularity of regions I, II, III will be achieved if the metric functions $\gamma$ and $\omega$ are equal to zero there:

$$\gamma(I,II,III) = 0, \quad \omega(I,II,III) = 0.$$  

(5)

Conditions $\gamma^{(I)} = 0, \gamma^{(III)} = 0$ are satisfied automatically by construction of formulae (3). The fulfilment of the condition $\omega^{(III)} = 0$ is equivalent to the requirement of asymptotic flatness of the metric under consideration. Taking into account that on the upper part of the symmetry axis ($z > \alpha_1$) the potential $E$ assumes the form

$$E(0, z) = 1 + \frac{e_1}{z - \bar{\beta}_1} + \frac{e_2}{z - \bar{\beta}_2},$$

$$e_1 = \frac{2(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)(\beta_1 - \alpha_3)}{(\beta_1 - \bar{\beta}_1)(\beta_1 - \bar{\beta}_2)(\beta_1 - \bar{\beta}_3)},$$

$$e_2 = \frac{2(\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)(\beta_2 - \alpha_3)}{(\beta_2 - \bar{\beta}_1)(\beta_2 - \bar{\beta}_2)(\beta_2 - \bar{\beta}_3)},$$

the asymptotic flatness condition (vanishing of the angular momentum monopole moment) reduces to

$$\text{Im}(e_1 + e_2) = 0.$$  

(7)

Therefore, the latter equation together with the remaining two conditions

$$\gamma^{(II)} = 0 \quad \text{and} \quad \omega^{(II)} = 0$$

(8)

constitutes the entire system of the balance equations whose solutions define equilibrium configurations of the extreme and non-extreme Kerr particles.

We now turn to concrete examples.

3. Particular equilibrium states

The approach which eventually proved to be successful for solving numerically the algebraic system of equations (7)–(8) is essentially based on the following two ingredients: (i) the fact that the metric coefficients $\gamma$ and $\omega$ of the double-Kerr solution are step functions on the symmetry axis [10], so that in particular they take constant values on the part of the $z$-axis between the particles, and (ii) a fortunate concrete choice of the unknowns in the balance system for fixing the remaining parameters.
With regard to (i) one can show, after expanding determinants with the aid of Laplace's rule, that equation \( \psi''(\Omega) = 0 \) reduces to solving the equation

\[
\text{Re}(D) = 0, \quad D = \begin{vmatrix}
\frac{1}{\alpha_2 - \beta_1} & \frac{1}{\alpha_3 - \beta_1} & \frac{1}{\alpha_1 - \beta_1} \\
\frac{1}{\alpha_2 - \beta_2} & \frac{1}{\alpha_3 - \beta_2} & \frac{1}{\alpha_1 - \beta_2} \\
\frac{1}{\alpha_2 - \beta_1} & \frac{1}{\alpha_3 - \beta_1} & \frac{1}{\alpha_1 - \beta_1} \\
\end{vmatrix},
\]

(9)

if \( \alpha_2 \) and \( \alpha_3 \) are real, and equation

\[
\text{Im}(D) = 0,
\]

(10)

if \( \alpha_3 = \alpha_2 \). The easiest practical way of obtaining the explicit form of \( \omega''(\Omega) \) is to consider the coefficient at the leading power of \( z \) after substituting appropriately \( r_n \) by \( |z - a_n| \) into the expression of \( \omega \), but we do not write here the resulting expression because of its cumbersome form. Where (ii) is concerned, one has to remember that the system (7)–(8) is underdetermined (three equations for seven unknowns \( \alpha_1, \alpha_2, \alpha_3, \text{Re}(\beta_1), \text{Re}(\beta_2), \text{Im}(\beta_1), \text{Im}(\beta_2) \)). To get the numerical roots, we fixed the values of \( \alpha_1, \alpha_2, \alpha_3 \) and \( \text{Re}(\beta_1) \), then searched for \( \text{Im}(\beta_1), \text{Re}(\beta_2) \) and \( \text{Im}(\beta_2) \).

The next step after finding numerical values of \( \alpha \) and \( \beta \) which define an equilibrium configuration is the estimation of individual masses and angular momenta of the constituents with the aid of Komar integrals [6]. In the case of two subextreme Kerr constituents this can easily be done via very concise Tomimatsu’s formulae [12]; however, in the presence of an extreme object which can be accompanied by a superextreme Kerr particle the integration should be carried out over the surface of a cylinder surrounding one or another constituent, and the explicit formulae for the individual masses \( M_i \) and angular momenta \( J_i \) of the particles are [1]

\[
M_i = \frac{1}{4} \left( \int_{z_u}^{z_d} \left[ \rho (\ln f)_{,\rho} - \omega \Omega_{,z} \right]_{\rho = \rho_0} dz + \int_{0}^{\rho_0} \left[ \rho (\ln f)_{,z} + \omega \Omega_{,\rho} \right]_{z = z_0} d\rho \\
- \int_{0}^{\rho_0} \left[ \rho (\ln f)_{,z} + \omega \Omega_{,\rho} \right]_{z = z_0} d\rho \right),
\]

\[
J_i = -\frac{1}{8} \left( \int_{z_u}^{z_d} \left[ 2\omega - 2\rho \omega (\ln f)_{,\rho} + (\rho^2 f^{-2} + \omega^2) \Omega_{,z} \right]_{\rho = \rho_0} dz \\
- \int_{0}^{\rho_0} \left[ 2\rho \omega (\ln f)_{,z} + (\rho^2 f^{-2} + \omega^2) \Omega_{,\rho} \right]_{z = z_0} d\rho \\
+ \int_{0}^{\rho_0} \left[ 2\rho \omega (\ln f)_{,z} + (\rho^2 f^{-2} + \omega^2) \Omega_{,\rho} \right]_{z = z_0} d\rho \right),
\]

(11)

\( z_u \) and \( z_d \) denoting locations on the \( z \)-axis of the centres of the cylinder’s upper and lower bases, \( \rho_0 \) being the radius of the bases. Mention that the sum of individual masses and the sum of individual angular momenta should coincide, respectively, with the total mass \( M \) and total angular momentum \( J \) calculated from the asymptotic form of the potential (1):

\[
M = -\frac{1}{2} \text{Re}(e_1 + e_2), \quad J = \frac{1}{2} \text{Im}(e_1 \beta_1 + e_2 \beta_2).
\]

(12)

It is highly important for the total and both individual masses to assume positive values because only in this case can the corresponding equilibrium configuration be considered physically acceptable. The particular equilibrium states obtained by us can be described as follows.

*Extreme–subextreme equilibrium configurations.* The general characteristic feature of all such configurations is that they necessarily involve at least one constituent with negative mass, which is in line with the Manko–Ruiz theorem [3] in the absence of two-Kerr black
hole equilibrium states with positive Komar masses. Equilibrium is possible between two constituents with negative masses, and also when the mass of either the extreme or the subextreme particle is negative. In table 1 one finds particular $\alpha$ and $\beta$ determining three extreme–subextreme equilibrium configurations (the approximate numerical values are given up to three decimal places), and in table 2 the corresponding physical characteristics of the constituents are shown (index 1 refers to the upper, extreme constituent, and index 2 to the lower, subextreme one).

In figure 2 we have plotted the stationary limit surfaces ($f = 0$) for the above equilibrium states, and one can see that the constituents with negative masses develop the massless ring singularities located on the $f = 0$ surface.

3.1. Extreme–superextreme equilibrium configurations

In this kind of binary system equilibrium can take place when (a) both constituents have negative masses, (b) one constituent has negative mass and the other has positive mass or (c) both Kerr particles possess positive Komar masses. Of course, only the latter case represents physical interest, so we shall restrict further consideration exclusively to it.

In table 3 three different numerical solutions of the system (7)–(8) are given, and in table 4 the corresponding masses and angular momenta calculated with the aid of the formulae (11), (12) are shown. In figure 3 we have plotted the stationary limit surfaces for the equilibrium states from table 3, and one can see that because of the positiveness of the masses of the extreme and superextreme constituents no massless ring singularities appear on these surfaces. A sort
The double-Kerr equilibrium configurations

(i) (ii) (iii)

Figure 3. Stationary limit surfaces in the ‘extreme–superextreme’ equilibrium configurations from table 3.

Table 3. Particular numerical solutions of the balance equations in the ‘extreme–superextreme’ case.

| $\alpha_1$ | $\alpha_2 = \bar{\alpha}_3$ | $\beta_1$ | $\beta_2$ |
|------------|-----------------------------|-----------|-----------|
| 3.9        | $-1.9 - 6.45i$              | $-3.7 + 7.6i$ | $0.312 + 3.588i$ |
| 4.1        | $-0.8 - 6.7i$               | $-3.8 + 7.9i$ | $2.68 + 1.42i$   |
| 3.6        | $-6.3i$                     | $-3.9 + 7.5i$ | $3.43 + 0.173i$  |

Table 4. Masses and angular momenta in the ‘extreme–superextreme’ equilibrium configurations from table 3.

| $M_1$ | $J_1$ | $M_2$ | $J_2$ | $M$ | $J$ |
|-------|-------|-------|-------|-----|-----|
| 4.204 | 19.767| 1.184 | 13.247| 5.388| 33.014|
| 1.6   | 3.653 | 2.82  | 26.323| 4.42 | 29.976|
| 0.178 | 0.064 | 3.895 | 29.891| 4.073| 29.955|

of kink on the stationary limit surfaces of superextreme objects present in these plots could be attributed to the insufficient differentiability of the surface $f = 0$ due to the discontinuity of the derivative $\partial f / \partial z$ on the circle $z = \text{Re}(\alpha_3)$, $0 < \rho < \text{Im}(\alpha_3)$.

4. Discussion

Probably the physically most interesting result following from our investigation is that the well-known relation $|J| = M^2$ valid for a single extreme Kerr black hole [13] does not necessarily hold in a binary system of Kerr particles involving an extreme constituent. The data from tables 2 and 4 provide evidence that the extreme Kerr constituent in balance with the other non-extreme one is characterized by the inequality

$$|J_1| < M_1^2$$

in the ‘extreme–subextreme’ configurations, and by the inequality

$$|J_1| > M_1^2$$

in the ‘extreme–superextreme’ configurations. A somewhat analogous effect has been recently reported by Ansorg and Petroff [14] who showed that for a subextreme black hole the absolute value $|J|$ of the angular momentum can be greater than $M^2$ if the black hole is surrounded by fluid rings.
It should be pointed out that all the ‘extreme–non-extreme’ equilibrium configurations considered verify the general equilibrium law of two Kerr particles established by Manko and Ruiz [4]:

\[ J \pm (M + s)^2 + s \left( \frac{J_1}{M_1} + \frac{J_2}{M_2} \right) = 0, \]

where \( s \) is the coordinate distance between the particles, i.e., in our case \( s = (2\alpha_1 - \alpha_2 - \alpha_3)/2 \). This may be viewed as an independent confirmation of the correctness of our results.

The present paper, therefore, complements the study of equilibrium configurations in the extended double-Kerr solution and provides counter-examples to the conjecture about the non-existence of ‘extreme–non-extreme’ equilibrium states put forward in [1]. In view of the results obtained we may speculate that most probably the equilibrium states involving one extreme object can also arise in the double-Kerr–Newman systems; however, finding concrete equilibrium configurations of that kind remains a task for the future.

Acknowledgments

We thank the referees for several valuable suggestions. VSM is grateful to the Department of Fundamental Physics of Salamanca University for its kind hospitality and financial support of his visit. JDSG acknowledges financial support from COLCIENCIAS of Colombia. This work was also supported by Project 45946-F (CONACyT, Mexico), Project BFM2003-02121 (Ministerio de Ciencia y Tecnología, Spain) and by Project 5116 (DIF de Ciencias of Santander Industrial University, Colombia).

References

[1] Manko V S, Ruiz E and Sanabria-Gómez J D 2000 Class. Quantum Grav. 17 3881
[2] Kramer D and Neugebauer G 1980 Phys. Lett. A 75 259
[3] Manko V S and Ruiz E 2001 Class. Quantum Grav. 18 L11
[4] Manko V S and Ruiz E 2002 Class. Quantum Grav. 19 3077
[5] Manko V S and Ruiz E 2004 Class. Quantum Grav. 21 5849
[6] Komar A 1959 Phys. Rev. 113 934
[7] Sibgatullin N R 1991 Oscillations and Waves in Strong Gravitational and Electromagnetic Fields (Berlin: Springer)
[8] Manko V S and Sibgatullin N R 1993 Class. Quantum Grav. 10 1383
[9] Ernst F J 1968 Phys. Rev. 167 1175
[10] Tomimatsu A and Kihara M 1982 Prog. Theor. Phys. 67 1406
[11] Dietz W and Hoenselaers C 1985 Ann. Phys., NY 165 319
[12] Tomimatsu A 1983 Prog. Theor. Phys. 70 385
[13] Kerr R P 1963 Phys. Rev. Lett. 11 237
[14] Ansorg M and Petroff D 2005 Phys. Rev. D 72 024019