Properties and Applications of a Transmuted Power Gompertz Distribution

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Authors’ contributions

This work was carried out in collaboration among all authors. Authors IBE and AFC designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors BSY and AHB managed the analyses of the study. Author AIA managed the literature searches. All authors read and approved the final manuscript.

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Abstract

This article introduces and studies a new probability distribution called “Transmuted Power Gompertz distribution”. It looks at the properties of the transmuted power Gompertz distribution. The article also estimates the four parameters of the new model using the method of maximum likelihood estimation. The article further evaluates the goodness-of-fit of the proposed distribution compared to other distributions by means of applications of the model to two real life datasets and the result show that the proposed distribution is more flexible than the fitted existing distributions.

Keywords: Transmuted power Gompertz distribution; statistical properties; estimation of parameters; application and performance.

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1 Introduction

The Gompertz distribution is both skewed to the right and to the left. It is a generalization of the exponential distribution and is commonly used in many applied problems, particularly in lifetime data analysis [1]. The Gompertz distribution has been applied in the analysis of survival, in some sciences such as gerontology [2], computer [3], biology [4] and marketing science [5]. The hazard rate function of the Gompertz distribution is an increasing function and often applied to describe the distribution of adult life spans by actuaries and demographers [6].

New families of distributions are produced day by day and are useful for adding parameters to all forms of probability distributions which makes the resulting distribution more flexible for modeling heavily skewed dataset. Some of these families of distributions include the beta generated family (Beta-G) [7], Transmuted family of distributions [8], Gamma-G (type 1) [9], the Kumaraswamy-G family [10], McDonald-G family [11], Gamma-G (type 2) family [12], Gamma-G (type 3) family [13], Log-gamma-G family [14], Exponentiated T-X family [15], Exponentiated-G (EG) family [16], Weibull-X family [17], Weibull-G family [18], Logistic-G family [19], Gamma-X family [20], a Lomax-G family [21], a new generalized Weibull-G family [22], a Beta Marshall-Olkin family of distributions [23], Logistic-X family [24], a new Weibull-G family [25], a Lindley-G family [26], a Gompertz-G family [27] and Odd Lindley-G family [28] and so on.

As a result of the introduction of these families of probability distribution and the desire to improve the flexibility of classical distributions especially the Gompertz distribution, a number of authors have introduced different extensions of the Gompertz distribution and some of the recent and known studies include the generalized Gompertz distribution [29] which was based on an idea of [30], the Beta Gompertz distribution [31], the odd generalized Exponential-Gompertz distribution [32], the Transmuted Gompertz distribution [33], the Lomax-Gompertz distribution [34], the Lomax-Gompertz distribution [35] and Power Gompertz distribution [35].

It has been discovered that using the quadratic rank transmutation map by [8] brings about more skewed and flexible probability distributions with an additional shape parameter usually known as the transmuted parameter and this process has led to many flexible compound distributions such as the Transmuted Gompertz distribution [33], the transmuted exponential Lomax distribution by [36], the transmuted odd generalized exponential exponential distribution [37], the transmuted Weibull-Exponential distribution [38], the transmuted Lindley-Exponential distribution [39] and the transmuted Weibull-Rayleigh distribution [40].

Therefore this article aims at modifying the power Gompertz distribution [35] by introducing a new extension of the Gompertz distribution (transmuted power Gompertz distribution) using the method of quadratic rank transmutation map [8] considered previously by the above listed authors and hope that it will result to more better model compared to the power Gompertz distribution and will be useful for analyzing real life situations.

According to [35], the cumulative distribution function (cdf) of the Power Gompertz distribution (PGD) with parameters $\alpha$, $\beta$ and $\theta$ and its probability density function (pdf) are given as:

$$F(x) = 1 - e^{-\theta x^{\alpha - 1}}$$

(1)

and

$$f(x) = \alpha \theta x^{\alpha - 1} e^{\beta x} e^{-\theta x^{\alpha - 1}}$$

(2)
respectively. For $x > 0, \alpha > 0, \beta > 0, \theta > 0$ where $\alpha$ and $\beta$ are scale and shape parameters of the model respectively and $\theta$ is the power parameter.

The remaining parts of this article are presented in sections as follows: the new distribution is defined with its graphical analysis in section 2. Section 3 derived some properties of the new distribution. The estimation of parameters using maximum likelihood estimation (MLE) is provided in section 4. An application of the new model with other existing distributions to two datasets is done in section 5 and a useful summary and conclusion is given in section 6.

2 The Transmuted Power Gompertz Distribution (TPGD)

According to [8], the cdf and pdf of the quadratic rank transmutation map are respectively given by:

\[
F(x) = (1 + \lambda)G(x) - \lambda[G(x)]^2
\]  \hspace{1cm} (3)

and

\[
f(x) = g(x)[1 + \lambda - 2\lambda G(x)]
\]  \hspace{1cm} (4)

where; $x > 0$, and $-1 \leq \lambda \leq 1$ is the transmuted parameter, $G(x)$ and $g(x)$ are the cdf and pdf of the continuous distribution to be extended respectively.

Using the above method by [8], the cdf and pdf of the transmuted power Gompertz distribution are defined for a random variable $X$ by substituting equation (1) and (2) in (3) and (4) and simplifying as:

\[
F(x) = (1 + \lambda)[1 - e^{-\frac{\alpha}{\theta}}(e^{\beta} - 1)] - \lambda[1 - e^{-\frac{\alpha}{\theta}}(e^{\beta} - 1)]^2
\]  \hspace{1cm} (5)

\[
f(x) = \alpha \theta x^{\beta - 1} e^{-\theta x} e^{-\frac{\alpha}{\theta}}[1 + \lambda - 2\lambda \left(1 - e^{-\frac{\alpha}{\theta}}(e^{\beta} - 1)\right)]
\]  \hspace{1cm} (6)

respectively. Where, $x > 0, \alpha > 0, \theta > 0, -1 \leq \lambda \leq 1$ in which $\alpha$ is the shape parameter respectively and $\theta$ is the exponential parameter while $\lambda$ is called the transmuted parameter.

The pdf and cdf of the TPGD using some parameter values are displayed in Fig. 1 as follows:
Fig. 1. PDF and CDF of the TransPGomD for different parameter values

Fig. 1 indicates that the TransPGomD distribution is positively skewed and takes various shapes depending on the parameter values. Also, from the above cdf plot, it is clear that the cdf approaches one (1) when X tends to infinity and equals zero when X tends to zero as normally expected.

3 Statistical Properties of TPGD

In this section, we derived, study and discuss some properties of the TransPGomD distribution. They are as follows:

3.1 Quantile function

Hyndman and Fan [41] defined the quantile function for any distribution in the form 
\[ Q(u) = X_q = F^{-1}(u) \] where \( Q(u) \) is the quantile function of the random variable \( X \) for the cdf \( F(x) \) and \( u \) is uniform with the interval \( 0 < u < 1 \).

Taking \( F(x) \) to be the cdf of the TransPGomD and inverting it as defined above will give us the quantile function as follows:

\[
F(x) = 1 - \left(1 - \lambda\right)e^{\frac{\alpha x^{\alpha - 1}}{\lambda}} - \lambda e^{\frac{2\alpha x^{\alpha - 1}}{\beta}} = u \tag{7}
\]

Simplifying (7) above and solving for \( X \) presents the quantile function of the TransPGomD as:

\[
Q(u) = \left(\frac{1}{\beta} \log\left(1 - \frac{\beta(1-u)}{3\alpha}\right)\right)^{\frac{1}{\alpha}} \tag{8}
\]
where \( u \) is a uniform variate on the unit interval \((0,1)\). This function is used for obtaining some moments like skewness and kurtosis as well as the median and for generation of random variables from the distribution in question as shown below.

Using (8) above, the median of \( X \) from the TransPGomD is simply obtained by setting \( u = 0.5 \) and this substitution of \( u = 0.5 \) in (8) gives:

\[
\text{Median} = \left( \frac{1}{\beta} \log \left( 1 - \frac{\beta(0.5)}{3\alpha} \right) \right)^{\frac{1}{\beta}}
\]

Similarly, random numbers can be simulated from the TransPGomD by setting \( Q(u) = X \) and this process is called simulation using inverse transformation method. This means for any \( \alpha, \beta, \theta > 0 \) and \( u \in (0,1) \):

\[
X = \left( \frac{1}{\beta} \log \left( 1 - \frac{\beta(1-u)}{3\alpha} \right) \right)^{\frac{1}{\beta}}
\]

“where \( u \) is a uniform variate on the unit interval \((0,1)\) and \( W_1(\cdot) \) represents the negative branch of the Lambert function”.

Again using the function above, the quantile based measures of skewness and kurtosis are obtained as follows:

Kennedy and Keeping [42] defined the Bowley’s measure of skewness based on quartiles as:

\[
SK = \frac{Q(\frac{3}{4}) - 2Q(\frac{1}{2}) + Q(\frac{1}{4})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}
\]

And Moors [43] presented the Moors’ kurtosis based on octiles by:

\[
KT = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) - Q(\frac{3}{8}) + Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{7}{8})}
\]

“where \( Q(\cdot) \) is calculated by using the quantile function from equation (8).

### 3.2 Moments

Let \( X \) denote a continuous random variable, the \( n^{th} \) moment of \( X \) is given by:

\[
\mu_n = E(X^n) = \int_0^\infty x^n f(x) \, dx
\]
Before substitution in (8), we perform the expansion and simplification of the pdf as follows:

\[ f(x) = \alpha (1 - \lambda) \theta x^{\theta - 1} e^{\beta x} e^{-e^{\beta x}} + 2 \alpha \lambda \theta x^{\theta - 1} e^{\beta x} e^{\frac{\alpha}{\beta} (e^{\beta x} - 1)} \]

\[ f(x) = \alpha (1 - \lambda) \theta x^{\theta - 1} e^{\beta x} e^{\frac{\alpha}{\beta} (e^{\beta x} - 1)} + 2 \alpha \lambda \theta x^{\theta - 1} e^{\beta x} e^{\frac{2 \alpha}{\beta} (e^{\beta x} - 1)} \]

(14)

First, by expanding the exponential term in (14) using power series, we obtain:

\[ e^{\frac{\alpha}{\beta} (e^{\beta x} - 1)} = \sum_{k=0}^{\infty} \frac{\alpha^k}{\beta^k k!} (1 - e^{\beta x})^k \]

(15)

and

\[ e^{\frac{2 \alpha}{\beta} (e^{\beta x} - 1)} = \sum_{k=0}^{\infty} \frac{2^k \alpha^k}{\beta^k k!} (1 - e^{\beta x})^k \]

(16)

Making use of the result in (15) and (16) above, (14) becomes

\[ f(x) = \alpha (1 - \lambda) \theta \sum_{k=0}^{\infty} \frac{\alpha^k}{\beta^k k!} x^{\theta - 1} e^{\beta x} e^{\frac{\alpha}{\beta} (1 - e^{\beta x})^k} + 2 \alpha \lambda \theta \sum_{k=0}^{\infty} \frac{2^k \alpha^k}{\beta^k k!} x^{\theta - 1} e^{\beta x} e^{\frac{2 \alpha}{\beta} (1 - e^{\beta x})^k} \]

\[ f(x) = \alpha (1 - \lambda) \theta \sum_{k=0}^{\infty} \frac{\alpha^k}{\beta^k k!} x^{\theta - 1} e^{\beta x} e^{\frac{\alpha}{\beta} (1 - e^{\beta x})^k} + 2 \alpha \lambda \theta \sum_{k=0}^{\infty} \frac{2^k \alpha^k}{\beta^k k!} x^{\theta - 1} e^{\beta x} e^{\frac{2 \alpha}{\beta} (1 - e^{\beta x})^k} \]

(17)

Also, using the generalized binomial theorem, we can write the last term from the above result as:

\[ (1 - e^{\beta x})^k = \sum_{l=0}^{\infty} \binom{k}{l} e^{l \beta x} \]

(18)

Making use of the result in (18) above in (17) and simplifying, we obtain:

\[ f(x) = \alpha (1 - \lambda) \theta \sum_{k=0}^{\infty} \frac{\alpha^k}{\beta^k k!} x^{\theta - 1} e^{\beta x} e^{\frac{\alpha}{\beta} \sum_{l=0}^{\infty} \binom{k}{l} e^{l \beta x}} + 2 \alpha \lambda \theta \sum_{k=0}^{\infty} \frac{2^k \alpha^k}{\beta^k k!} x^{\theta - 1} e^{\beta x} e^{\frac{2 \alpha}{\beta} \sum_{l=0}^{\infty} \binom{k}{l} e^{l \beta x}} \]

\[ f(x) = \alpha (1 - \lambda) \theta \sum_{k=0}^{\infty} \frac{\alpha^k (1)^l}{\beta^k k!} x^{\theta - 1} e^{\beta x} e^{\frac{\alpha}{\beta} \sum_{l=0}^{\infty} \binom{k}{l} e^{l \beta x}} + 2 \alpha \lambda \theta \sum_{k=0}^{\infty} \frac{2^k \alpha^k (1)^l}{\beta^k k!} x^{\theta - 1} e^{\beta x} e^{\frac{2 \alpha}{\beta} \sum_{l=0}^{\infty} \binom{k}{l} e^{l \beta x}} \]

\[ f(x) = \left[ \frac{(1 - \lambda)}{(\alpha \theta)} \sum_{k=0}^{\infty} \frac{\alpha^k (1)^l}{\beta^k k!} \left( \frac{k}{l} \right) + \frac{2 \lambda}{(\alpha \theta)} \sum_{k=0}^{\infty} \frac{2^k \alpha^k (1)^l}{\beta^k k!} \left( \frac{k}{l} \right) \right] x^{\theta - 1} e^{\beta x} e^{\frac{\alpha}{\beta} \sum_{l=0}^{\infty} \binom{k}{l} e^{l \beta x}} \]

(19)
Now, using the simplified pdf of the TransPGomD in equation (19), the $n^{th}$ ordinary moment of the TransPGomD is derived as follows:

$$
\mu_n = E\left(X^n\right) = \int_0^{\infty} x^n f(x) dx = \left[\frac{(1-\lambda)}{(\alpha \theta)} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{\alpha^k (-1)^l}{\beta^k k! l} \left(\frac{k}{l}\right) + \frac{2\lambda}{(\alpha \theta)} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{\alpha^k 2^l (-1)^l}{\beta^k k! l} \left(\frac{k}{l}\right)\right] \int_0^{\infty} x^{n+1} e^{\theta(1+l)x} dx
$$

(20)

Making use of integration by substitution method in equation (20), we perform the following operations:

Let $-u = \beta (1+l)x^\theta \Rightarrow x = u^{\frac{1}{\theta}} \left(\frac{-1}{\beta (1+l)}\right)^{\frac{1}{\theta}}$ which implies that $\frac{du}{dx} = \theta \beta (1+l)x^{\theta-1} \Rightarrow dx = -\frac{x^{(\theta-1)} du}{\theta \beta (1+l)}$.

Substituting for $X^\theta$ and $dx$ in equation (20) and simplifying; we have:

$$
\mu_n = \left[\frac{(1-\lambda)}{(\alpha \theta)} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{\alpha^k (-1)^l}{\beta^k k! l} \left(\frac{k}{l}\right) + \frac{2\lambda}{(\alpha \theta)} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{\alpha^k 2^l (-1)^l}{\beta^k k! l} \left(\frac{k}{l}\right)\right] \int_0^{\infty} u^{\frac{n+1}{\theta}} e^{-u} du
$$

(21)

Using the definition of complete Gamma function, we obtain the $n^{th}$ ordinary moment of $X$ for the TransPGomD as:

$$
\mu_n = \left[\frac{(1-\lambda)}{(\alpha \theta)} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{\alpha^k (-1)^l}{\beta^k k! l} \left(\frac{k}{l}\right) + \frac{2\lambda}{(\alpha \theta)} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{\alpha^k 2^l (-1)^l}{\beta^k k! l} \left(\frac{k}{l}\right)\right] \Gamma\left(\frac{n+1}{\theta}\right) \left(\frac{-1}{\theta \beta (1+l)}\right)^{\frac{n+1}{\theta}}
$$

(22)

The mean ($\mu_1$), variance ($\sigma^2$), coefficient of variation (CV), coefficient of skewness (CS), coefficient of kurtosis (CK), moment generating function (mgf) and characteristics function (cf) can all be calculated based on the ordinary moments in equation (22) using some simple and well-defined relationships (see details in [44]).

The moment generating function of a random variable $X$ can be obtained as

$$
M_x(t) = E\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx} f(x) dx
$$

(23)

Applying power series expansion and simplifying (23) gives the following:

$$
M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} E\left(X^r\right) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\mu_r\right]
$$

(24)

Using the result in (24) and simplifying the integral in (23) therefore we have:

$$
M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\frac{(1-\lambda)}{(\alpha \theta)} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{\alpha^k (-1)^l}{\beta^k k! l} \left(\frac{k}{l}\right) + \frac{2\lambda}{(\alpha \theta)} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{\alpha^k 2^l (-1)^l}{\beta^k k! l} \left(\frac{k}{l}\right)\right] \Gamma\left(\frac{r+1}{\theta}\right) \left(\frac{-1}{\theta \beta (1+l)}\right)^{\frac{r+1}{\theta}}
$$

(25)

The characteristics function of a random variable $X$ is defined by:
Again, applying power series expansion and simplifying equation (26), we obtained the characteristics function of $X$ as:

$$\varphi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left\{ \left(1-\lambda\right) \sum_{k=0}^{\infty} \frac{\alpha^k (-1)^l}{\beta^k} \left(\frac{k}{l}\right) + \frac{2\lambda}{(\alpha \theta)^r} \sum_{k=0}^{\infty} \frac{\alpha^k 2^k (-1)^l}{\beta^k} \left(\frac{k}{l}\right) \right\} \frac{\Gamma\left(\frac{z+1}{\theta}\right)}{\beta(1+l)}$$  

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left\{ \left(1-\lambda\right) \sum_{k=0}^{\infty} \frac{\alpha^k (-1)^l}{\beta^k} \left(\frac{k}{l}\right) + \frac{2\lambda}{(\alpha \theta)^r} \sum_{k=0}^{\infty} \frac{\alpha^k 2^k (-1)^l}{\beta^k} \left(\frac{k}{l}\right) \right\} \frac{\Gamma\left(\frac{z+1}{\theta}\right)}{\beta(1+l)}$$  

The Cumulant generating function (CGF) is obtained as:

$$K(t) = \log \left\{ \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left\{ \left(1-\lambda\right) \sum_{k=0}^{\infty} \frac{\alpha^k (-1)^l}{\beta^k} \left(\frac{k}{l}\right) + \frac{2\lambda}{(\alpha \theta)^r} \sum_{k=0}^{\infty} \frac{\alpha^k 2^k (-1)^l}{\beta^k} \left(\frac{k}{l}\right) \right\} \frac{\Gamma\left(\frac{z+1}{\theta}\right)}{\beta(1+l)} \right\}$$  

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left\{ \left(1-\lambda\right) \sum_{k=0}^{\infty} \frac{\alpha^k (-1)^l}{\beta^k} \left(\frac{k}{l}\right) + \frac{2\lambda}{(\alpha \theta)^r} \sum_{k=0}^{\infty} \frac{\alpha^k 2^k (-1)^l}{\beta^k} \left(\frac{k}{l}\right) \right\} \frac{\Gamma\left(\frac{z+1}{\theta}\right)}{\beta(1+l)}$$  

### 3.3 Reliability analysis of the TransPGomD

The Survival function (SF) describes the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x)$$  

Applying the cdf of the TPGD in (29) and simplifying, the survival function for the TransPGomD is obtained as:

$$S(x) = e^{-\frac{\alpha (\rho^\theta - 1)}{\beta}} \left[ 1 - \lambda + \lambda e^{-\frac{\alpha (\rho^\theta - 1)}{\beta}} \right]$$  

The hazard function (HF) is the probability that a component will fail or die for an interval of time. The hazard function is defined as:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}$$  

Meanwhile, the expression for the hazard rate of the TransPGomD is given by

$$h(x) = \frac{\alpha \theta x^{\theta-1} e^{\beta x} \left[ 1 - \lambda + 2\lambda e^{-\frac{\alpha (\rho^\theta - 1)}{\beta}} \right]}{1 - \lambda + \lambda e^{-\frac{\alpha (\rho^\theta - 1)}{\beta}}}$$  

Hence, equation (30) and (32) are the survival and the hazard functions of the transmuted power Gompertz distribution (TransPGomD) respectively and their plots are given in the figure below.
The plots in Fig. 2(a) show that the probability of survival equals one (1) at initial time or early age and it decreases as time increases and equals zero (0) as time approaches infinity.

It is also shown in Fig. 2(b) that the TransPGomD has increasing failure rate which implies that the probability of failure for any random variable following a TransPGomD increases as time increases, that is, probability of failure or death increases as life ages.

### 3.4 Order statistics

Suppose \( X_1, X_2, \ldots, X_n \) is a random sample from the TransPGomD (TPGomD) and let \( X_{1:n}, X_{2:n}, \ldots, X_{n:n} \) denote the corresponding order statistic obtained from this same sample. The pdf, \( f_{i:n}(x) \) of the \( i^{th} \) order statistic can be obtained by
\[ f_{x_n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} f(x) F(x)^{k+1} \]

Using (5) and (6), the pdf of the \( i^{th} \) order statistics \( X_{(i)} \), can be expressed from (33) as;

\[ f_{x_n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \frac{\alpha \theta x^{\theta-1} e^{\beta x}}{e^{\beta x^{\theta-1}}} \left[ 1 - \lambda e^{-\frac{\beta}{\beta^\theta x^{\theta-1}}} \right] \left[ 1 - \frac{(1-\lambda)}{e^{\beta x^{\theta-1}}} - \frac{\lambda}{e^{\beta x^{\theta-1}}} \right]^{k+1} \]

Hence, the pdf of the minimum order statistic \( X_{(1)} \) and maximum order statistic \( X_{(n)} \) of the TransPGomD are respectively given by;

\[ f_{x_1}(x) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{\alpha \theta x^{\theta-1} e^{\beta x}}{e^{\beta x^{\theta-1}}} \left[ 1 - \lambda e^{-\frac{\beta}{\beta^\theta x^{\theta-1}}} \right] \left[ 1 - \frac{(1-\lambda)}{e^{\beta x^{\theta-1}}} - \frac{\lambda}{e^{\beta x^{\theta-1}}} \right]^{k} \]

and

\[ f_{x_n}(x) = n \left[ \frac{\alpha \theta x^{\theta-1} e^{\beta x}}{e^{\beta x^{\theta-1}}} \left[ 1 - \lambda e^{-\frac{\beta}{\beta^\theta x^{\theta-1}}} \right] \left[ 1 - \frac{(1-\lambda)}{e^{\beta x^{\theta-1}}} - \frac{\lambda}{e^{\beta x^{\theta-1}}} \right]^{n-1} \right] \]

### 4 Estimation of Unknown Parameters of the TransPGomD

In this section, the estimation of the parameters of the TransPGomD is done by using the method of maximum likelihood estimation (MLE). Let \( X_1, X_2, \ldots, X_n \) be a sample of size \( n \) independently and identically distributed random variables from the TransPGomD with unknown parameters \( \alpha, \beta, \theta \) and \( \lambda \) all defined previously in section one.

The likelihood function of the TransPGomD using the pdf in equation (6) is given by;

\[ L(X | \alpha, \beta, \theta, \lambda) = (\alpha \theta)^n \prod_{i=1}^{n} \left[ e^{\beta x^{\theta-1}} \left[ 1 - \lambda e^{-\frac{\beta}{\beta^\theta x^{\theta-1}}} \right] \right] \]

Let the natural logarithm of the likelihood function be, \( l(\eta) = \log L(X | \alpha, \beta, \theta, \lambda) \), therefore, taking the natural logarithm of the function above gives:

\[ l(\eta) = n \log \alpha + n \log \beta + \theta - 1 + \sum_{i=1}^{n} \log x_i + \beta \sum_{i=1}^{n} x_i^{\theta-1} - \frac{\beta}{\beta^\theta} \sum_{i=1}^{n} x_i^{\theta-1} + \sum_{i=1}^{n} \left[ 1 - \lambda e^{-\frac{\beta}{\beta^\theta x_i^{\theta-1}}} \right] \]

Differentiating \( l(\eta) \) partially with respect to parameters \( \alpha, \beta, \theta \) and \( \lambda \) respectively gives the following results;
\[
\frac{\partial l(\eta)}{\partial \alpha} = n - \frac{1}{\beta} \sum_{i=1}^{n} \left( e^{\beta\alpha} - 1 \right) - 2\lambda \beta \sum_{i=1}^{n} \left( e^{\beta\alpha} - 1 \right) \left( e^{\beta\alpha} - 1 \right) \left( 1 - \lambda + 2\lambda e^{\beta\alpha} \right) \]

(39)

\[
\frac{\partial l(\eta)}{\partial \beta} = \sum_{i=1}^{n} x_i^\alpha \beta^{-\alpha} \sum_{i=1}^{n} x_i^\alpha e^{\beta\alpha} - \alpha \sum_{i=1}^{n} x_i^\alpha \ln x_i e^{\beta\alpha} - 2\alpha \sum_{i=1}^{n} \left( x_i^\alpha \ln x_i e^{\beta\alpha} - 1 \right) \left( 1 - \lambda + 2\lambda e^{\beta\alpha} \right) \]

(40)

\[
\frac{\partial l(\eta)}{\partial \theta} = n - \sum_{i=1}^{n} \ln x_i - \alpha \sum_{i=1}^{n} x_i^\alpha \ln x_i e^{\beta\alpha} - 2\alpha \sum_{i=1}^{n} \left( x_i^\alpha \ln x_i e^{\beta\alpha} - 1 \right) \left( 1 - \lambda + 2\lambda e^{\beta\alpha} \right) \]

(41)

\[
\frac{\partial l(\eta)}{\partial \lambda} = \sum_{i=1}^{n} \left( 2e^{\beta\alpha} - 1 \right) \left( 1 - \lambda + 2\lambda e^{\beta\alpha} \right) \]

(42)

Making equation (39), (40), (41) and (42) equal to zero (0) and solving for the solution of the non-linear system of equations produce the maximum likelihood estimates of parameters \( \hat{\alpha}, \hat{\beta}, \hat{\theta} \) and \( \hat{\lambda} \). However, these solutions cannot be obtained manually except numerically with the aid of suitable statistical software like R, SAS, MATHEMATICA etc. Hence, some datasets are being considered in the next section to fit the new distribution with other related distributions using “maxLik” package in R software.

### 5 Applications to Three Real Life Datasets

This section presents two real life datasets, their descriptive statistics, graphical summary and some distributions fitted to the datasets. The section presents and compares the fits of the transmuted Power Gompertz Distribution (TPGD), transmuted Gompertz distribution (TGD), Power Gompertz Distribution (PGD) and the Gompertz Distribution (GD) based on the two datasets mentioned above.

To check the fitness and flexibility of the models listed above, the Akaike Information Criterion (AIC) is chosen and used for selecting the best model. The formula for this criterion is given as:

\[
AIC = -2\ell + 2k
\]

Where \( \ell \) denotes the value of log-likelihood function evaluated at the MLEs, \( k \) is the number of model parameters and \( n \) is the sample size. Meanwhile, when taking our decisions the model with the lowest values of AIC would be chosen as the best model to fit the data.

Tables 2 and 4 list the Maximum Likelihood Estimates of the model parameters and the AIC values for the fitted models based on the first and second datasets respectively.

**Data set I**: This data represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929-1938 from [45,46] and [47]. Its summary is given as follows:
Table 1. Summary statistics for the dataset I

| n   | Minimum | Q₁ | Median | Q₃ | Mean | Maximum | Variance | Skewness | Kurtosis |
|-----|---------|----|--------|----|------|---------|----------|----------|----------|
| 121 | 0.30    | 17.5 | 40.00  | 60.0 | 46.33 | 154.00  | 1244.464 | 1.04318  | 0.40214  |

From the descriptive statistics in table 1, it is seen that dataset I is positively skewed and not normally distributed.

Table 2. Performance of the distributions using the value of AIC based on dataset I

| Distributions | Parameter estimates | log-likelihood value | AIC | Rank of models |
|---------------|--------------------|----------------------|-----|----------------|
| TPGomD        | \( \hat{\alpha} = 0.09434 \)  
                | \( \hat{\beta} = 0.36095 \)  
                | \( \hat{\theta} = 0.42241 \)  
                | \( \hat{\lambda} = -0.96519 \)  | -579.3027 | 1166.605 | 1st |
| PGomD         | \( \hat{\alpha} = 0.020775 \)  
                | \( \hat{\beta} = 0.021019 \)  
                | \( \hat{\theta} = 0.870416 \)  |  | -581.2547 | 1168.509 | 3rd |
| TGomD         | \( \hat{\alpha} = 0.011294 \)  
                | \( \hat{\beta} = 0.008866 \)  
                | \( \hat{\lambda} = 0.297067 \)  |  | -580.4765 | 1166.953 | 2nd |
| GomD          | \( \hat{\alpha} = 0.018400 \)  
                | \( \hat{\beta} = 0.007703 \)  |  |  | -583.1125 | 1170.225 | 4th |

Fig. 3. Histogram and estimated densities (Pdfs) and cdfs of the models fitted to dataset I
Fig. 4. Probability plots for the fit of the TPGomD, TGomD, PGomD and GomD based on dataset I

**Data set II:** This dataset represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli reported by [48] and also used by [39]. The summary is presented in Table 3 below.

| n   | Minimum | Q1 | Median | Q3 | Mean | Maximum | Variance | Skewness | Kurtosis |
|-----|---------|----|--------|----|------|---------|----------|----------|----------|
| 72  | 10.0    | 108.0 | 149.5  | 224.0 | 176.8 | 555.0   | 10705.1 | 1.34128  | 1.98852  |

From the descriptive statistics in Table 3, it can be observed that dataset II is also positively skewed.

**Table 4. Performance of the distributions using the value of AIC based on dataset II**

| Distributions | Parameter estimates | log-likelihood value | AIC   | Rank of models |
|---------------|---------------------|----------------------|-------|----------------|
| TPGomD        | $\hat{\alpha}=0.18958$ | -390.8596            | 789.7193 | 1st            |
|               | $\hat{\beta}=0.37333$ |                      |       |                |
|               | $\hat{\theta}=0.28835$ |                      |       |                |
|               | $\hat{\lambda}=-5.80601$ |                    |       |                |
| PGomD         | $\hat{\alpha}=0.006765$ | -433.2963            | 872.5925 | 2nd            |
|               | $\hat{\beta}=0.902166$ |                      |       |                |
|               | $\hat{\theta}=0.319790$ |                      |       |                |
| TGomD         | $\hat{\alpha}=0.0032698$ | -433.3389            | 872.6778 | 3rd            |
|               | $\hat{\beta}=-0.0038057$ |                      |       |                |
|               | $\hat{\lambda}=3.1017073$ |                    |       |                |
| GomD          | $\hat{\alpha}=0.0034090$ | -438.6373            | 881.2746 | 4th            |
|               | $\hat{\beta}=0.0039771$ |                      |       |                |
Fig. 5. Histogram and estimated densities (pdfs) and cdfs of the models fitted to dataset II

Fig. 6. Probability plots for the fit of the TPGomD, TGomD, PGomD and GomD based on dataset II

Tables 2 and 4 present the parameter estimates and the values of AIC for the four fitted distributions using dataset I and dataset II respectively. Based on the values of AIC in Table 2 and those in Table 4, it can be seen that AIC values of TPGD are smaller compared to those of TGomD, PGomD and the GomD and these results indicate that the transmuted Power Gompertz distribution (TPGomD) fits the two datasets better than the other three distributions (TGomD, PGomD and the GomD). This result and performance is confirmed from the estimated pdfs and cdfs in Fig. 3 and 5 as well as the probability plots presented in Figs. 4 and 6. The result is an affirmation to the fact that the method quadratic rank transmutation map for adding a single or transmuted parameter has increased the flexibility of the power Gompertz distribution. Hence, this study is in line with the conclusions in [33,36,37,38, 39,40,49,50,51].
6 Summary and Conclusion

In conclusion, the present article has used the method of quadratic rank transmutation map to defined a transmuted power Gompertz distribution and study its properties such as quantile function, moments, moment generating function, characteristics function, survival function, hazard function, distribution of order statistics among others. The research has discussed analytically and in practice the usefulness and applications of these properties. The study has also demonstrated earlier in the preceding section with applications of the model to real life datasets that the new distribution (TPGomD) has a better fit to the two datasets compared to the transmuted Gompertz, Power Gompertz and the conventional Gompertz distributions.

Competing Interests

Authors have declared that no competing interests exist.

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