Average shadowing revisited

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Abstract
We propose a novel unifying approach to study the shadowing property for a broad class of dynamical systems (in particular, discontinuous and non-invertible) under a variety of perturbations. In distinction to known constructions, our approach is local: it is based on the gluing property which takes into account the shadowing under a single (not necessarily small) perturbation.

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1. Introduction

When modelling a time-evolving process, we obtain only its approximate realisations. This proximity is due to several reasons. First, we never know exactly the description of the process itself, and second, the presence of various kinds of errors from purely random to rounding errors when implemented on a computer are inevitable. The question of the adequacy of the simulation results is primarily associated with the presence of a real trajectory of this process in the vicinity of the obtained realisation over the longest possible time interval. This question is especially nontrivial in the case of a chaotic system, since for such systems close trajectories diverge very quickly (often exponentially fast).

At the level of correspondence between the individual trajectories of a hyperbolic system and the pseudo-trajectories\textsuperscript{4}, this problem was first posed by Anosov [2, 3, (1967–1970)] as a key step in analysing the structural stability of diffeomorphisms. A similar but much

\textsuperscript{4} Approximate trajectories of a system under small perturbations, already considered by Birkhoff [4, (1927)] for a completely different purpose.
less intuitive approach, called ‘specification’, was proposed in the same setting by Bowen [9, (1975)]. Informally, both approaches guarantee that errors do not accumulate during the modelling process: in systems with the shadowing property, each approximate trajectory can be uniformly traced by the true trajectory over an arbitrary long period of time. Naturally, this is of great importance in chaotic systems, where even an arbitrary small error in the initial position leads to (exponentially in time) a large divergence of trajectories.

Further development (see main results, generalisations and numerous references in monographs dedicated to this subject [15, 16] and the textbook [11]) demonstrated deep connections between the shadowing property and various ergodic characteristics of dynamical systems, in particular, to properties related to uniform hyperbolicity. To some extent, the latter restricts the theory of uniform shadowing to an important but very special class of hyperbolic dynamical systems.

The concept of average shadowing introduced in [5] about 30 years ago gave a possibility to extend significantly the range of perturbations under consideration in the theory of shadowing, in particular to be able to deal with perturbations which are small only on average but not uniformly. However, the original idea under this concept was twofold: (i) to extend the range of perturbations and (ii) to be able to deal with non-hyperbolic systems. While the first objective was largely achieved (see discussion of unresolved issues below), the second objective was not resolved: the proof was given only for smooth hyperbolic systems. Nevertheless this concept generated a number of subsequent works where various versions of shadowing similar in idea to the average shadowing were introduced (see, for example, [12–14, 18–20]) and the connections between them were studied in detail.

The most notorious in the variety of obstacles in the analysis of the shadowing property is that one needs to take into account an infinite number of independent perturbations of the original system. This makes the problem highly nonlocal. It is therefore very desirable to reduce the shadowing problem to the situation with a single perturbation, albeit with tighter control of the approximation accuracy.

To realise this idea in our recent paper [7] we developed a fundamentally new ‘gluing’ construction⁵, consisting in the effective approximation of a pair of consecutive segments of true trajectories. See exact definitions and details of the construction in section 2.

In [7] using this construction we were able to study systems under perturbations being small in various senses (see section 2 for details). Moreover, using it we studied non-hyperbolic and even discontinuous systems under some combinations of types of perturbations and types of shadowing⁶. Still the most interesting Gaussian perturbations were out of reach (it was expected that more sophisticated estimates will allow to achieve this objective). In this paper we indeed overcome this difficulty.

The paper is organised as follows. In section 2 we give general definitions related to the shadowing property and introduce the key tool of our analysis—the gluing property. In section 3 we formulate and prove the main result—theorem 2.1, which deduces various versions of shadowing from the gluing property. The remaining part of the paper is devoted to the analysis of the applicability of the gluing property for various classes of discrete time dynamical systems, starting with hyperbolic diffeomorphisms (section 4), for non-uniformly hyperbolic endomorphisms with singularities and discontinuities (section 5), and finally for a special class of multivalued maps induced by symbolic dynamics (section 6).

⁵ When this article was already written, I became aware that the concept of ‘gluing’ was used earlier in [8] in the context of specifications, that does not allow to study the situations considered here.

⁶ Previously, a separate method was developed to analyse each specific system and combinations of a perturbation and a type of shadowing.
Among other things, the examples of the systems under study demonstrate the difference between strong (1) and weak (2) versions of the gluing property, which shows that even with uniformly small perturbations it is possible that only the average shadowing takes place (but not the uniform one). Similarly, it is shown that there are systems, belonging to the class \( S(R,A) \), but not to \( S(A,A) \) (see definition of \( S(\cdot,\cdot) \) in section 2).

### 2. Setting and main result

We restrict ourselves to discrete time dynamical systems, leaving the extension of our approach to continuous time systems (flows) for future research. A discrete time dynamical system is completely defined by a non-necessarily invertible map \( T: X \to X \) from a complete metric space \((X, \rho)\) into itself.

**Definition 2.1.** A trajectory of the map \( T \) starting at a point \( x \in X \) is a sequence of points \( \bar{x} := \{x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots \} \subset X \), for which \( x_0 = x \) and \( T x_i = x_{i+1} \) for all available indices \( i \). The part of \( \bar{x} \) corresponding to non-negative indices is called the forward (semi-)trajectory, while the part corresponding to non-positive indices is called the backward (semi-)trajectory.

Observe that although the forward trajectory is always uniquely determined by \( x = x_0 \) and infinite, the backward trajectory might be finite (if its ‘last’ point has no preimages)\(^7\) and for a given \( x = x_0 \) there might be arbitrary many admissible backward trajectories.

**Remark 2.1.** Despite the introduction of the backward trajectory of a non-invertible dynamical system looks somewhat unusual, we inevitably have to go back and through in time when constructing the true trajectory approximating the trajectory of the perturbed system. Therefore it is more convenient to define bi-infinite trajectories from the very beginning.

**Definition 2.2.** A pseudo-trajectory of the map \( T \) is a sequence of points \( \bar{y} := \{\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots \} \subset X \), for which the sequence of distances \( \{\rho(T y_i, y_{i+1})\} \) for all available indices \( i \) satisfies a certain ‘smallness’ condition. The parts corresponding to non-negative or non-positive indices are referred as forward or backward pseudo-trajectories.

Introduce the set of ‘moments of perturbations’:

\[
\mathcal{N}(\bar{y}) := \{ t_i : \gamma_i := \rho(T y_i, y_{i+1}) > 0, \ i \in \mathbb{Z} \}
\]

ordered with respect to their values, i.e. \( t_i < t_{i+1} \forall i \). We refer to the amplitudes of perturbations \( \gamma_i \) as gaps between consecutive segments of true trajectories.

**Definition 2.3.** For a given \( \varepsilon > 0 \) we say that a pseudo-trajectory \( \bar{y} \) is of

(U) uniform type, if \( \rho(T y_i, y_{i+1}) \leq \varepsilon \) for all available indices \( i \).

(A) small on average type, if \( \limsup_{n \to \infty} \frac{1}{n+1} \sum_{i=-n}^{0} \rho(T y_i, y_{i+1}) \leq \varepsilon \).

(R) rare perturbations type, if the upper density of the set \( \mathcal{N}(\bar{y}) \) does not exceed \( \varepsilon \). Namely, \( \limsup_{n \to \infty} \frac{1}{n+1} \#(\mathcal{N}(\bar{y}) \cap [-n,n]) \leq \varepsilon \).

If the backward pseudo-trajectory is finite, only positive indices \( i \) are taken into account, which leads to one-sided sums \( \frac{1}{n+1} \sum_{i=0}^{n} \rho(T y_i, y_{i+1}) \) instead of two-sided ones.

\(^{7}\) In this case we are speaking only about available indices \( i \) in the definition.
The U-type pseudo-trajectory is the classical one, introduced by Birkhoff [4] and Anosov [2]. A stronger (compared to (A)) version

\[(A') \text{ small on average (strong) type, if } \exists N \text{ such that } \frac{1}{2n+1} \sum_{i=-n}^{n} \rho(Ty_i, y_{i+1}) \leq \varepsilon \ \forall n \geq N.\]

was proposed by Blank [5] in order to take care about perturbations small only on average. It is unnecessarily strong in the sense that the value of $N$ is universal for all pseudo-trajectories satisfying (A'). For example, consider a sequence of pseudo-trajectories $\{\tilde{y}^n\}_n$ such that $\rho(T\tilde{y}_i^n, \tilde{y}_{i+1}^n) = n$ if $i = n$ and zero otherwise. Then every $\tilde{y}^n$ satisfy the assumption (A) with $\varepsilon = 0$, but there is no finite $N$, for which all of them satisfy the assumption (A').

In particular, for true Gaussian perturbations, the probability that the trajectories of the perturbed system satisfy (A') is zero. The reason for the stronger assumption is purely technical: this was necessary for the techniques applied in [5] for the analysis of shadowing. Thus, the weaker version (A), despite being more natural, is completely new. The R-type was introduced as an intermediate version in our recent publication [7], and it allows to consider large but rare perturbations.

Clearly $U \subset A' \subset A$ and $R \subset A$, but $R \setminus A \neq \emptyset$. Despite $R \subset A$ we will demonstrate that their separate analysis is worth doing since in some situations the R-type perturbations, but not of general A-type, are shadowed.

To simplify notation we will speak about $\varepsilon$-pseudo-trajectories, when the corresponding property is satisfied with the accuracy $\varepsilon$.

The idea of shadowing in the theory of dynamical systems boils down to the question, is it possible to approximate the pseudo-trajectories of a given dynamical system with true trajectories? Naturally, the answer depends on the type of approximation.

**Definition 2.4.** We say that a true trajectory $\tilde{x}$ shadows a pseudo-trajectory $\tilde{y}$ with accuracy $\delta$ (notation $\delta$-shadows):

1. \textbf{(U) uniformly, if } $\rho(x_i, y_i) \leq \delta$ for all available indices $i$.
2. \textbf{(A) on average, if } $\limsup_{n \to \infty} \frac{1}{2n+1} \sum_{i=-n}^{n} \rho(x_i, y_i) \leq \delta$.

If the backward pseudo-trajectory is finite, only positive indices $i$ are taken into account, which leads to the one-sided sum $\frac{1}{n+1} \sum_{i=0}^{n} \rho(x_i, y_i)$.

The U-type shadowing was originally proposed by Anosov [2], while the A-type was introduced\(^8\) by Blank [5]. Naturally, the types of pseudo-trajectories and the types of shadowing may be paired in an arbitrary way.

**Definition 2.5.** We say that a dynamical system $(T, X, \rho)$ satisfies the $(\alpha + \beta)$-shadowing property (notation $T \in S(\alpha, \beta)$) with $\alpha \in \{U, A, R\}$, $\beta \in \{U, A\}$ if $\forall \delta > 0 \ \exists \varepsilon > 0$ such that each $\varepsilon$-pseudo-trajectory of $\alpha$-type can be shadowed in the $\beta$ sense with the corresponding accuracy $\delta$.

For example, $S(U, U)$ stands for the classical situation of the uniform shadowing of uniformly perturbed systems, while $S(A, A)$ corresponds to the average shadowing in the case of small on average perturbations.

One of the most interesting open questions related to the shadowing problem is to find out under what conditions on the map does the presence of a certain type of shadowing for each pseudo-trajectory with a single perturbation implies one or another type of shadowing property for the system. The reason for this question is that the case of a single perturbation is much

\(^8\) The reason was that pseudo-trajectories with large perturbations cannot be uniformly shadowed.
simpler, and therefore the idea of getting information about other types of perturbations from this fact is quite attractive. The answer is known (although very partially, see [1]) only in the case of U-shadowing of the so-called positively expansive\(^9\) dynamical systems, if additionally one assumes that the single perturbation does not exceed \(0 < \varepsilon \ll 1\).

In order to give the answer to this question we introduce the following property.

**Definition 2.6.** We say that a trajectory \(\vec{x}\) *glues* together semi-trajectories \(\vec{x}, \vec{y}\) with accuracy rate \(\varphi : \mathbb{Z} \to \mathbb{R}_+\) strongly if
\[
\rho(x_k, z_k) \leq \varphi(k) \rho(x_0, y_0) \quad \forall k < 0, \quad \rho(y_k, z_k) \leq \varphi(k) \rho(x_0, y_0) \quad \forall k \geq 0
\]
and weakly if
\[
\rho(x_k, z_k) \leq \varphi(k) \quad \forall k < 0, \quad \rho(y_k, z_k) \leq \varphi(k) \quad \forall k \geq 0.
\]

In other words \(\vec{x}\) approximates both the backward part of \(\vec{x}\) and the forward part of \(\vec{y}\) with accuracy controlled by the rate function \(\varphi\), and in the strong version the accuracy additionally depends multiplicatively on the distance between the ‘end-points’ of the glued segments of trajectories.

Without loss of generality, we assume that the functions \(\varphi(|k|)\) and \(\varphi(-|k|)\) are monotonically decreasing\(^{10}\). Indeed, replacing a general \(\varphi\) by its monotone envelope
\[
\tilde{\varphi}(k) := \begin{cases} \sup_{i \leq k} \varphi(i) & \text{if } k < 0 \\ \sup_{i \geq k} \varphi(i) & \text{if } k \geq 0 \end{cases},
\]
we get the result.

**Definition 2.7.** We say that the dynamical system \((T, X, \rho)\) satisfies the (strong/weak) gluing property with the rate-function \(\varphi : \mathbb{Z} \to \mathbb{R}\) (notation \(T \in G_{\tilde{\varphi}}(\varphi)\)) if for any pair of trajectories \(\vec{x}, \vec{y}\) there is a trajectory \(\vec{z}\) which glues them at time \(t = 0\) with accuracy \(\varphi\) in the strong/weak sense.

**Remark 2.2.** If \(T \in G_{\tilde{\varphi}}(\varphi)\), then \(\forall \tau \in \mathbb{Z}\) and for any pair of trajectories \(\vec{x}, \vec{y}\) there exists a trajectory \(\vec{z}\) which glues them (strongly/weakly) at time \(t = \tau\) with accuracy \(\varphi\). Therefore it is enough to check the gluing property for \(\tau = 0\).

Indeed, for a given \(\tau\) consider a pair of trajectories \(\vec{x}^\tau, \vec{y}^\tau\) obtained from \(\vec{x}, \vec{y}\) by the time shift by \(\tau\), namely \(x^\tau_i := x_{i+\tau}, y^\tau_i := y_{i+\tau}, \forall i\). Then since \(\vec{x}^\tau, \vec{y}^\tau\) may be glued together at time \(t = 0\) with accuracy \(\varphi\), we deduce the same property for \(\vec{x}, \vec{y}\) by the time \(t = \tau\). \(\square\)

**Remark 2.3 (necessity).** Observe that any type of shadowing includes the situation with a single perturbation, and, therefore, the gluing property is necessarily satisfied, although with a not necessarily small (and not even necessarily vanishing at \(n \to \pm \infty\)) accuracy rate-function \(\varphi(n)\).

**Remark 2.4.** Additional assumptions about the rate-function \(\varphi\) are necessary in order to obtain meaningful applications of the gluing property. In what follows, we will assume summability of this function:
\[
\Phi := \sum_k \varphi(k) < \infty. \tag{3}
\]

Our main result is the following statement.

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\(^9\) Roughly speaking, this means that if two forward trajectories are uniformly close enough to each other, then they coincide. In particular, this property is satisfied for expanding maps.

\(^{10}\) As we will see, this makes no restriction to the types of perturbations under consideration.
Theorem 2.1. Let $T: X \rightarrow X$ be a map from a metric space $(X, \rho)$ into itself, and let $T \in G_s(\varphi)$ with $\Phi := \sum_k \varphi(k) < \infty$. Then

(a) $T \in S(U, U)$,
(b) $T \in S(A, A)$,

with $\delta = \varepsilon \Phi e^0$, where $\varepsilon$ is the perturbations amplitude, while $\delta$ stands for the approximation accuracy.

Remark 2.5. 1. The part (a) has been proven under the same assumption in [7], as well as that $T \in S(U, U)$ if $T \in G_w(\varphi)$ and the perturbations are uniformly bounded. Therefore we will prove only the (b) part here.

2. The proof of the situation $T \in S(R, A)$ in [7] was based on a very crude technical assumption that all the perturbations are equal to some constant $D$. This is not sufficient to get the claim (b), which we do in the present paper.

Proposition 2.6. 1. The part (a) of theorem 2.1 follows from the proof of (b) if one assumes that additionally the perturbations are bounded by $0 < \varepsilon \ll 1$.

2. For $S(U, U)$ it is enough to check the gluing property for $\tilde{x}, \tilde{y}$ with $\rho(x_0, y_0) \leq \varepsilon_0 \ll 1$.

3. $S(U, A) \setminus S(U, U) \neq \emptyset$.

3. Proof of theorem 2.1

Proof. We will prove a stronger ‘linear’ version of the shadowing, namely that there is a constant $K = K(\varphi) < \infty$, such that for each $\varepsilon > 0$ small enough there is a true trajectory average approximating with accuracy $\delta \leq K\varepsilon$ each $\varepsilon$-pseudo-trajectory of U or A type.

The scheme of the proof goes as follows. For a pseudo-trajectory $\tilde{y} := \{y_i\}_{i \in \mathbb{Z}}$ we consider in detail the set of moments of perturbations

$$N(\tilde{y}) := \{t_i : \tau_{t_i} := \rho(Ty_{t_i}, y_{t_i+1}) > 0, \ i \in \mathbb{Z}\}.$$

Between the moments of time $t_k$ there are no perturbations and hence $\tilde{y}$ can be divided into segments of true trajectories. Thanks to the $G_s(\varphi)$ property each pair of consecutive segments of true trajectories can be ‘glued’ together by a true trajectory with the controlled accuracy.

Without loss of generality, we will assume that perturbations occur at every moment of time and therefore $t_i = i \ \forall i \in \mathbb{Z}$.

In our construction (see figure 1) we first glue together pairs of segments around the moments of perturbations $t_i$ with even indices, $i \equiv 2k$, obtaining longer segments of the true ‘gluing’ trajectories. At each next step, we simultaneously ‘glue’ together consecutive pairs of already obtained segments.

Namely, on the $n$th step of the construction we choose the ‘central’ end-point $t_{j(n)}$ of the segments obtained on the previous step, and simultaneously ‘glue’ together pairs of already obtained segments around the points $t_{j(n)+k2^n}, \ k \in \mathbb{Z}$. To make this construction as symmetric (around the origin) as possible, we set $j(0)$ to be the smallest on modulus index $j$ among the end-points. If there are two of them, we choose the one with the positive index. The ‘central’ end-points $t_{j(n)}$ satisfy the following recurrent relation $j(n+1) = -2j(n) + 1$ with $j(0) := 0$. To prove this, observe that from this relation it follows that $j(n) = (1 - (-2)^n) / 3$. Therefore $|j(n) - j(n+1)| \leq 2^n$, which confirms that $t_{j(n+1)}$ is one of the boundary points of the segment obtained after the gluing of the segments around the point $t_{j(n)}$. 


Consequently, at each step of the construction, we get a new pseudo-trajectory, consisting of half the number of segments of true trajectories (i.e. only half of the perturbation moments remain) with exponentially increasing lengths, but with larger gaps between them (compared to the original gaps). Moreover, we see that on the \( n \)th step of the construction the segment \([-2^{n-1}, 2^{n-1}]\) belongs to one of the already glued segments. In the limit as \( n \to \infty \), we obtain an approximation of the entire pseudo-trajectory.

**Remark 3.1.** • The above scheme corresponds to the parallel gluing construction introduced in our previous paper [7]. Here we apply the same scheme with an additional advantage that on each step of the construction all the lengths of gluing segments coincide and those lengths grow exponentially with the step number.

• Another possibility is to use sequential gluing, which can be described as follows. Starting from a certain segment of the trajectory (say, between the time moments from \( t_0 \) to \( t_1 \)), we glue it first with the right neighbour, then with the left one (or vice versa). Therefore at each step of the construction a new segment of the trajectory is glued to the already approximated ones. In fact, the construction used in [5] to prove the average shadowing property for Anosov systems in the terminology given above, is precisely sequential gluing. The advantage of the sequential gluing construction is that the corresponding calculations are much simpler, but on closer examination it turns out that for their application it is necessary to make much stronger assumptions on the rate function \( \phi \), in particular, that \( \phi(\pm1) < 1 \). Even for uniformly hyperbolic systems, this can only be done for so-called Lyapunov metric \( \rho \), and not for the general one. In most of the examples discussed in sections 4–6 the value \( \phi(\pm1) \) turns out to be quite large.

To estimate approximation errors, we find the accuracy of gluing a pair of segments of true trajectories: \( v_{-N-}, v_{-N-+1}, \ldots, v_{-1} \) and \( v_0, v_1, \ldots, v_{N+} \). By the property \( G_\epsilon(\phi) \) there exists a trajectory \( \vec{z} \subset X \) such that
\[
\rho(v_k, z_k) \leq \varphi(k) \rho(Tv_{-1}, v_0) \quad \forall k \in \{-N^-, \ldots, N^+\}.
\]
Therefore
\[
\sum_{k=-N^-}^{N^+} \rho(z_k, v_k) \leq \rho(Tv_{-1}, v_0) \sum_k \varphi(k) = \Phi \cdot \rho(Tv_{-1}, v_0).
\]

There are several important points here:

(1) The approximation accuracy depends only on the gap \( \rho(Tv_{-1}, v_0) \) between the ‘end-points’ of the segments of trajectories glued together.
(2) After gluing a pair of trajectory segments, the gaps between the end-points in the next step of the procedure may become larger than the original gaps in $\vec{y}$.

(3) Each moment of perturbation $t_i$ is taken into account only once during the gluing process.

(4) There is an advantage of the ($A+A$) case compared with the ($R+A$) case: all segments of the true trajectories in the process of gluing has the same length (in distinction to the ($R+A$) case). As we will see, this helps a lot in the analysis.

In [7], in proving a similar result for the case ($R+A$), we have used the simplifying assumption that the initial gaps are uniformly bounded by some constant $D < \infty$. Under this assumption, it has been proven that, although the gaps may increase during the gluing process they remain uniformly bounded. On the final stage, this fact, together with the assumption that perturbations occur very rarely, allow to estimate the average approximation error.

In the present setting, the perturbations are neither uniformly bounded nor sparse. Therefore, a more sophisticated approach is needed. Namely, we first show that the average values of gaps during the gluing procedure cannot exceed $K \varepsilon$ for some finite constant $K \neq K(\varepsilon)$.

Using this estimate, we prove that there exists a finite constant $C$ such that for any $0 < \delta \ll 1$ small enough and $\varepsilon := C\delta$ for each $\varepsilon$-pseudo-trajectory $\vec{y}$ of $A$ type there is a true trajectory $\vec{z}$, which on average approximates $\vec{y}$ with accuracy $\delta$.

Now we are ready to proceed. On the $n$th step of the process of gluing we have a bi-infinite collection of gaps $\{\gamma_i^{(n)}\}_{i \in \mathbb{Z}}$. By the assumption that the perturbations are small on average, we get

$$\limsup_{k \to \infty} \frac{1}{2k+1} \sum_{i=-k}^{k} \gamma_i^{(0)} \leq \varepsilon.$$  

Our aim is to show that $\exists C \neq C(\varepsilon)$ such that

$$\limsup_{k \to \infty} \frac{1}{2k+1} \sum_{i=-k}^{k} \gamma_i^{(n)} \leq C\varepsilon \quad \forall n.$$  

Using this inequality we can apply the machinery similar to the one developed for the analysis of the ($R+A$) case in [7].

We start with a recursive upper estimate for the gaps:

$$\gamma_i^{(n+1)} \leq \gamma_i^{(n)} + \phi^+(\tau)^{\gamma_i^{(n)}} + \phi^-(\tau)^{\gamma_i^{(n)}},$$  

where $\phi^{\pm}(\tau) = \phi(\pm 2\tau)$. Indeed, the lengths of the glued segments of trajectories on the $n$th step of the procedure is equal to $2^n$ and $\phi^-(\tau)^{\gamma_i^{(n)}}$ is the upper estimate for the approximation error coming from the left, while $\phi^+(\tau)^{\gamma_i^{(n+1)}}$ is the upper estimate for the approximation error coming from the right.

In the case of uniformly small perturbations we proceed as follows. Setting $\gamma^{(n)} := \sup \gamma_i^{(n)}$ and $\tau^{(n)} := 2^n$, using the inequality (4) and the monotonicity of the functions $\phi(\pm k)$ we get

$$\gamma^{(n+1)} \leq \gamma^{(n)} + \phi(\tau^{(n)}) \gamma^{(n)} + \phi^+(\tau^{(n)}) \gamma^{(n)} = \gamma^{(n)} \cdot \left(1 + \phi^-(\tau^{(n)}) + \phi^+(\tau^{(n)})\right).$$  

Continuing this and passing from $n$ to $n-1$, etc gives a recursive estimate

$$\gamma^{(n+1)} \leq \gamma^{(0)} \cdot \prod_{k=0}^{n-1} \left(1 + \phi^-(\tau^{(k)}) + \phi^+(\tau^{(k)})\right).$$
To estimate the right hand side we need a simple real analysis inequality.

**Lemma 3.2.** For any sequence of nonnegative real numbers \( \{b_k\}_{k \geq 1} \) we have

\[
\limsup_{n \to \infty} \prod_{k \geq 1} (1 + b_k) \leq \limsup_{n \to \infty} \sum_{k=1}^{n} b_k.
\]

**Proof.** Denote \( B_n := \prod_{k \geq 1} (1 + b_k) \), \( S_n := \sum_{k=1}^{n} b_k \). We proceed by induction on \( n \). For \( n = 1 \), the question boils down to

\[
1 + v \leq e^v \quad \forall v \in \mathbb{R}.
\]

\[
\frac{d}{dv}(1 + v) = \begin{cases} 
1 > e^v = \frac{d}{dv} e^v & \text{if } v < 0 \\
1 = e^v = \frac{d}{dv} e^v & \text{if } v = 0 \\
1 < e^v = \frac{d}{dv} e^v & \text{if } v > 0.
\end{cases}
\]

Therefore the graph of \( e^v \) lies above the straight line \( 1 + v \) with the only tangent point at the origin.

This implies that \( B_1 \leq e^{S_1} \). Assume that \( B_n \leq e^{S_n} \) for some \( n \in \mathbb{Z}^+ \) and prove the same inequality for \( n + 1 \). We get

\[
B_{n+1} = (1 + b_{n+1}) B_n \leq (1 + b_{n+1}) e^{S_n} \leq e^{b_{n+1}} e^{S_n} = e^{S_{n+1}}.
\]

Lemma is proven. \( \square \)

Setting \( b_k := \phi(-\tau(k)) + \phi(\tau(k)) \), we apply lemma 3.2 to get the uniform on \( n \) bound for the gaps, using additionally that \( \gamma(t_i) \leq \varepsilon \) \( \forall i \in \mathbb{Z} \):

\[
\gamma^{(n+1)} \leq \varepsilon \exp \left( \sum_k \phi(k) \right) = \varepsilon e^\Phi.
\]

It is important here that each moment of perturbation \( t_i \) is taken into account only once during the entire approximation process.

The distance between \( z_i^{(n)} \) and \( y_i = z_i^{(0)} \) can be estimating from above by adding together the distances between consecutive pairs of approximating pseudo-trajectories \( z_i^{(k)} \), \( z_i^{(k+1)} \), which are completely controlled by the corresponding gaps and the rate function \( \phi \) (see figure 2). Using this, we get

\[
\rho(z_i^{(n)}, y_i) \leq \sum_i \gamma_i^{(n)} \phi(t - t_i) \leq \varepsilon e^\Phi \sum_i \phi(t - t_i) \leq \varepsilon \Phi e^\Phi \quad \forall t \in \mathbb{Z}. \tag{6}
\]

Moreover, by a similar argument for each \( k > 0 \)

\[
\rho(z_i^{(n)}, z_i^{(n+k)}) \leq \varepsilon e^\Phi \sum_{|j| \geq 2^{k-1}} \phi(j) \xrightarrow{n \to \infty} 0,
\]
due to the summability of the function $\varphi$ and since by the construction at the step $n$ the segment $[-2^{n-1}, 2^{n-1}]$ belongs to one of the already glued segments.

Therefore, for any given $t$ the sequence $\{z^{(n)}_t\}_n$ is fundamental and converges as $n \to \infty$ to the limit $z_t$, where $\{z_t\}$ is the true trajectory of our system.

Since the estimate (6) is uniform on $n$ we may use it for $z_t$ as well, which gives

$$\rho(z_t, y_t) \leq e^{\Phi} e^{\rho} \quad \forall t \in \mathbb{Z}.$$  

This finishes the proof of the part (a) of the theorem that $T \in S(U, U)$.

The proof of the part (b) is more involved, since we do not have a uniform upper bound for perturbations, and hence for the gaps as well.

Rewriting (4) as follows:

$$\gamma^{(n+1)}_0 \leq \left( \varphi_{\gamma}^{(n)} + \varphi_{\gamma}^{(n)} \right) \gamma^{(n)}_0 + \left( \left( 1 - \varphi_{\gamma}^{(n)} - \varphi_{\gamma}^{(n)} \right) \gamma^{(n)}_0 + \varphi_{\gamma}^{(n)} \gamma^{(n)}_0 + \varphi_{\gamma}^{(n)} \gamma^{(n)}_0 \right),$$  

we will show that in the 1st term the factor $\varphi_{\gamma}^{(n)} + \varphi_{\gamma}^{(n)}$ vanishes with $n$, while the 2nd term corresponds to the averaging operator of type $v_t \to (1 - a - b)v_t + av_{t-1} + bv_{t+1}$, the recursive application of which will flatten a sequence $\{v_t\}$ to a constant.

To make this reasoning accurate, we need some calculations. Without loss of generality, we assume that the function $\varphi$ is even (i.e. $\varphi(-k) = \varphi(k) \forall k$). Indeed, replacing a general $\varphi$ by $\tilde{\varphi}(k) := \max(\varphi(-k), \varphi(k)) \forall k$, we get the result.

Denote $R^{(n)}_k := \sum_{i=-k}^{k} \gamma^{(n)}_i$. Then using (7) we get

$$R^{(n+1)}_k = \sum_{i=-k}^{k} \gamma^{(n+1)}_i \leq \sum_{i=-k}^{k} \gamma^{(n)}_i + \sum_{i=-k}^{k} \varphi_{\gamma}^{(n)} \gamma^{(n)}_{i-1} + \sum_{i=-k}^{k} \varphi_{\gamma}^{(n)} \gamma^{(n)}_{i+1}$$

$$= R^{(n)}_k + \varphi_{\gamma}^{(n)} (\gamma^{(n)}_{i-1} + \gamma^{(n)}_{i+1}) + \varphi_{\gamma}^{(n)} (\gamma^{(n)}_{i-1} + \gamma^{(n)}_{i+1})$$

$$= (1 + \varphi_{\gamma}^{(n)} + \varphi_{\gamma}^{(n)}) R^{(n)}_k + \varphi_{\gamma}^{(n)} (\gamma^{(n)}_{i-1} - \gamma^{(n)}_{i+1})$$

$$= (1 + \varphi_{\gamma}^{(n)} + \varphi_{\gamma}^{(n)}) R^{(n)}_k \quad \text{(since $\varphi_{\gamma}^{(n)} = \varphi_{\gamma}^{(n)}$)}$$

$$\leq \cdots \leq R^{(0)}_k \prod_{i=0}^{n} (1 + \varphi_{\gamma}^{(i)} + \varphi_{\gamma}^{(i)}).$$

Contributions to the upper bound of the gluing error are coming from two different sources: the estimates of the gaps (changing during the steps of the parallel gluing construction) and the summation over error contributions from the gluing of pairs of consecutive segments of true trajectories (see figure 2).

Now we are ready to obtain an upper bound for the sum of gaps. Setting $b_n := \varphi_{\gamma}^{(n)} + \varphi_{\gamma}^{(n)}$, by lemma 3.2 we get

$$R^{(n+1)}_k \leq \prod_{i=1}^{n} (1 + b_i) R^{(0)}_k \leq e^{\sum_{i=0}^{n} b_i} R^{(0)}_0 \leq e^{\Phi} R^{(0)}_k.$$  

(8)

Under the additional assumption that the perturbations are uniformly bounded

$$\Gamma := \sup_i \gamma^{(0)}_i < \infty$$  

(9)

one gets an even stronger estimate

$$\Gamma^{(n)} := \sup_i \gamma^{(n)}_i \leq e^{\Phi} \Gamma^{(0)} = e^{\Phi} \Gamma.$$  

(10)
Indeed, (4) implies
\[ \Gamma^{(n+1)} \leq \left( 1 + \varphi^{(n)}_+ + \varphi^{(n)}_- \right) \Gamma^{(n)}. \]

Applying the lemma 3.2 again, we get the result.
For the \( n \)th approximating pseudo-trajectory \( \vec{z}^{(n)} \) denote
\[ Q_k^{(n)} := \frac{1}{2k+1} \sum_{t=-k}^k \rho \left( z^{(n)}_t, y_t \right). \]

Then
\[ Q_k^{(n)} \leq \frac{1}{2k+1} \sum_{t=-k}^k \sum_{i=-t}^i \varphi(i) \gamma^{(n)}_{t+i}, \]
\[ = \sum_i \varphi(i) \cdot \left( \frac{1}{2k+1} \sum_{t=-k}^k \gamma^{(n)}_{t+i} \right), \]
\[ = \sum_i \varphi(i) \cdot K_k^{(n)}(t), \]
where \( K_k^{(n)}(t) := \sum_{i=-t}^{t} \gamma^{(n)}_{i}. \)

Therefore, using (8) we obtain an upper bound
\[ \limsup_{k \to \infty} Q_k^{(n)} \leq \varepsilon \Phi e^\Phi, \quad (11) \]
which does not depend on the step number \( n \).

Now for each \( k > 0 \) one estimates the distance between pseudo-trajectories \( \vec{z}^{(n)} \) and \( \vec{z}^{(n+k)} \) as follows. Recall that the pseudo-trajectory \( \vec{z}^{(n)} \) consists of pieces of true trajectories of length \( 2^n \). Therefore
\[ \rho \left( z^{(n)}_t, z^{(n+k)}_t \right) \leq \sum_{|j| \geq 2^n} \varphi(j) \gamma^{(n)}_j \]
similarly to the analysis of uniformly small perturbations.

Since the function \( \varphi \) is summable, this implies that
\[ \sum_{|j| \geq 2^n} \varphi(j) \xrightarrow{n \to \infty} 0. \]

Therefore, under the additional assumption that the phase space \( X \) is compact (and hence \( \sup_{n,j} \gamma^{(n)}_j < \infty \)), for any given \( t \) the sequence \( \{z^{(n)}_t\}_n \) is fundamental and converges as \( n \to \infty \) to the limit \( z_t \), where \( \{z_t\} \) is the true trajectory of our system.

In the non-compact case there are two options. If the perturbations are uniformly bounded (see (9)), then (10) implies the same result as the the compactness. Otherwise, only weaker average convergence to the limit trajectory \( \vec{z} \) is available:
\[ \limsup_{m \to \infty} \frac{1}{2m+1} \sum_{t=-m}^m \rho \left( z^{(n)}_t, z^{(n+k)}_t \right) \leq \varepsilon e^\Phi \sum_{|j| \geq 2^{n+1}} \varphi(j) \xrightarrow{n \to \infty} 0. \]
Since the estimate (11) is uniform on $n$ we may use it for $\vec{z}$ instead of $\vec{z}(n)$, getting

$$\limsup_{k \to \infty} \frac{1}{2k+1} \sum_{t=-k}^{k} \rho(z_t, y_t) \leq \epsilon \Phi e^\Phi.$$

Theorem is proven. \hfill $\square$

Proof of proposition 2.6. (1) Observe that if the perturbations are uniformly bounded by $\epsilon \ll 1$, then from the proof of theorem 2.1 on each step of the gluing procedure we get that all gaps are bounded from above by $\epsilon \Phi e^\Phi$, while the uniform approximation error is estimated from above by the same constant.

(2) Follows from the observation above that all gaps are bounded from above by $\epsilon \Phi e^\Phi$.

(3) See proposition 5.1 item (3).

4. The gluing property for diffeomorphisms

The gluing property may be explained in terms similar to those which are actively used in the theory of smooth hyperbolic dynamical systems. Denote by $\vec{x}^- := \{ x_{-2}, x_{-1}, x_0 = x \}$ and $\vec{x}^+ := \{ x, x_1, x_2, \ldots \}$ be backward and forward semi-trajectories of the point $x \in X$, and consider the sets:

$$W^-(\vec{x}^-) := \left\{ \vec{z} \subseteq X : \rho(x_k, z_k) \xrightarrow{k \to \infty} 0 \right\}.$$  

$$W^+(\vec{x}^+) := \left\{ \vec{z} \subseteq X : \rho(x_k, z_k) \xrightarrow{k \to \infty} 0 \right\}.$$  

Then the gluing property means that for each pair of semi-trajectories $\vec{x}^-$ and $\vec{x}^+$ the sets $W^+(\vec{x}^+)$ and $W^-(\vec{x}^-)$ have a non-empty intersection. Additionally the rate of convergence in the definition of the sets $W^\pm$ is controlled by the rate function $\varphi$.

The origin of the gluing property is the so-called local product structure, introduced by D V Anosov for uniformly hyperbolic dynamical system. The local product structure means that for a pair of close enough points their stable and unstable manifolds intersect, and the orbit of the point of intersection approximates the corresponding semi-trajectories with an error decreasing exponentially in time. In our notation this means $\varphi(k) = C e^{-|k|}$. Later in [5] this property has been extended to the global one, but only for the uniformly hyperbolic dynamical system.

Now we are going to check the gluing property for some important classes of smooth invertible dynamical systems.

Example 4.1 (affine mapping). Let $X := \mathbb{R}^d$ with $d \geq 1$ with the Euclidean metric $\rho$, $A$ be an invertible $d \times d$ matrix, and $a \in \mathbb{R}^d$. Then $T_x := Ax + a$ is an affine map from $X$ into itself.

An invertible $d \times d$ real-valued matrix $A$ decomposes the Euclidean space $\mathbb{R}^d$ into a direct sum of three linear subspaces $E^s, E^u, E^n$ (stable, unstable and neutral):

$$E^s := \left\{ v \in \mathbb{R}^d : ||A^n v|| \leq C \lambda_s^n ||v|| \forall n \in \mathbb{Z}_+ \right\},$$  

$$E^u := \left\{ v \in \mathbb{R}^d : ||A^{-n} v|| \leq C \lambda_u^{-n} ||v|| \forall n \in \mathbb{Z}_+ \right\},$$  

$$E^n := \left\{ v \in \mathbb{R}^d : ||v|| \leq C \lambda_s^n ||v|| \forall n \in \mathbb{Z}_+ \right\},$$  

where $\lambda_s$ and $\lambda_u$ denote the spectral radii of $A$.
Proposition 4.1. 1. If \( E^a \neq \emptyset \) then \( \forall x \exists y \) with an arbitrary small \( \rho(x_0, y_0) < 1 \) which cannot be glued together with a summable accuracy rate \( \varphi \).

2. If \( E^a = E^g = \emptyset \) then \( T \in G_s(\varphi) \) with \( \varphi(k) := C\lambda^{|k|} \).

3. If \( E^a = E^s = \emptyset \) then \( T \in G_s(\varphi) \) with \( \varphi(k) := C\lambda^{|k|} \).

4. If \( E^a = \emptyset \) and \( E^s, E^g \neq \emptyset \) then \( T \in G_s(\varphi) \) with \( \varphi(k) := C(\lambda^{|k|} + \lambda^{-|k|}) \).

Proof. Let \( \bar{x} \) be an arbitrary trajectory of the map \( T \) and let \( 0 \neq v \in E^a \). Consider a trajectory \( \bar{y} \), defined by the relations \( y_0 := x_0 + v \), \( y_k := T^k y_0 \) \( \forall k \in \mathbb{Z} \) (see figure 3). The existence of a trajectory \( \bar{z} \) which glues \( \bar{x}, \bar{y} \) with a summable accuracy rate implies that

\[
\liminf_{n \to -\infty} \rho(z_n, x_n) + \liminf_{n \to -\infty} \rho(z_n, y_n) > 0.
\]

Therefore \( z_0 - x_0 \in E^a \), \( z_0 - y_0 = z_0 - (x_0 + v) \in E^s \) with \( v \in E^a \). Therefore \( (E^a - v) \cap E^g \neq \emptyset \). We came to a contradiction, which proves the case (1).

In the case (2) the map \( T \) is expanding and the verification of the gluing property for a pair of a backward semi-trajectory \( \bar{x} := \{ \ldots, x_{-1}, x_0 \} \) and a forward semi-trajectory \( \bar{y} := \{ y_0, y_1, \ldots \} \) of the map \( T \) is straightforward. Setting \( z_0 := y_0 \), we get \( z_k = z_k \quad \forall k \geq 0 \) and \( \rho(z_k, x_k) \leq \lambda^{|k|} \quad \forall k < 0 \). Observe, that this is the only possibility to construct the approximating trajectory.

In the case (3) the map \( T \) is contracting and hence \( T^{-1} \) is expanding. Therefore the proof is exactly the same as in the previous case with the only change to the inverse time.

It remains to consider the generic case (4). In this case there is a fixed point \( O := (I - A)^{-1} a \). Consider an arbitrary pair of trajectories \( \bar{x} \) and \( \bar{y} \) (see figure 3). By the assumptions the intersection \( I \) of the sets \( x_0 + E^a \) and \( y_0 + E^s \) is nonempty. Choosing any point \( z_0 \in I \), for its trajectory we get
\[
\rho(z_k, x_k) \leq C \left( \lambda_k^{|k|} + \lambda_{-|k|}^{|k|} \right) \quad \forall k < 0,
\]
\[
\rho(z_k, y_k) \leq C \left( \lambda_k^{|k|} + \lambda_{-|k|}^{|k|} \right) \quad \forall k \geq 0.
\]

An important point is that the constant \( C = C(A, \rho) \) can be arbitrary large here independently on the optimisation by the proper choice of the point \( z_0 \in I \).

**Remark 4.2.** If \( d = 2 \), then for the case of a hyperbolic matrix, namely, if \( \vec{x}, \vec{y} \) belong to the same \( T \)-invariant half-hyperplane, then \( \vec{z} := \vec{y} \) glues \( \vec{x}, \vec{y} \) together (see figure 3). Unfortunately, when the dimension of \( E^o \) or \( E^s \) is greater than 1, this simple recipe does not work.

**Example 4.2 (Anosov diffeomorphism).** Let \( X := \mathbb{T}^2 \) be a unit two-dimensional torus and let \( T : X \to X \) be a uniformly hyperbolic diffeomorphism.

The simplest map that satisfies the above properties is \( T \) \( := Ax \pmod{1} \), where \( A \) is an integer matrix with the determinant equal 1 on modulus. For exact definition of the uniformly hyperbolic system we refer the reader to numerous publications to the subject (see, for example, \([3, 6, 9, 10]\)).

**Proposition 4.3.** For the map \( T : \mathbb{T}^2 \to \mathbb{T}^2 \) of the example 4.2 there exists a special (Lyapunov) metric \( \rho \) and \( \lambda > 1 \), for which this system satisfies the \( G_{s}(\varphi) \) property with \( \varphi(k) := e^{-\lambda |k|} \) \( \forall n \in \mathbb{Z} \).

This result follows from the global product structure for a hyperbolic system proven in \([5]\). The local version of this property, which asserts the intersection of stable and unstable local manifolds of sufficiently close points, is well known (see, for example, \([3, 9, 10]\)). In the global version, the locality assumption is dropped.

It is worth noting that none of these results follow from proposition 4.1. Indeed, the diffeomorphism under study is nonlinear and the proof of proposition 4.3 is based on the construction of arbitrary thin Markov partitions and their mixing properties. On the other hand, these constructions fail in the example 4.1. Moreover, proposition 4.1 holds for an arbitrary metric \( \rho \) (induced by a norm), while proposition 4.3 holds only for a special (Lyapunov) metric.

5. The gluing property for non-uniformly hyperbolic endomorphisms with singularities

Now we are ready to turn to discontinuous and non-invertible mappings. So far the maps in all the examples under consideration were linear, and when the gluing property was satisfied for them, the rate function was exponential. The following example demonstrates that this is not necessary.

**Example 5.1 (nonuniform hyperbolicity).** \( X := [0, 1], \alpha, \beta > 0, 0 < c < 1 \)

\[
T_x := \begin{cases} 
    x(1 + ax^\alpha) & \text{if } x \leq c \\
    1 - (1 - x) \left( 1 + b(1 - x)^\beta \right) & \text{if } x > c
\end{cases}
\]

This example is of particular interest because it exhibits very different behaviours for different regions in the parameter space \((a, b, c, \alpha, \beta)\). Observe, that if \( \alpha, \beta = 0 \), \( a, b > 0 \) we are getting a piecewise expanding map, while for \( \alpha, \beta = 0 \), \( a, b < 0 \) this is a contracting map. The
most interesting situation (nonuniform hyperbolicity)) corresponds to the case \( \alpha, \beta > 0 \), when there are two neutrally expanding fixed points at the end-points 0 and 1. From the point of view of ergodic theory, there are additional peculiarities in the properties of invariant measures. It is known (see, for example, [17]) that for a slightly more simple system having a single neutral fixed point with the exponent \( 0 < \alpha < 1 \) there is an absolutely continuous probabilistic invariant measure, while for \( \alpha > 1 \) there are exactly two ergodic probabilistic invariant measures—Dirac-measures at points 0 and 1 (apart from \( \sigma \)-finite absolutely continuous invariant measures). We expect that the same property remains valid in the setting of the example 5.1. As we will see, this important distinction is reflected by the presence or absence of the shadowing property.

An important feature for our analysis of shadowing, common for all admissible choices of the parameters \((a, b, c, \alpha, \beta)\), is the presence of at least two periodic trajectories—fixed points at 0 and 1.

**Proposition 5.1.** The map \( T \) in the example 5.1 (see figure 4) satisfies

1. if \( \alpha, \beta \geq 0 \) and \( T \in G_{s/w}(\varphi) \) then
   
   \[ c(1 + ac^\alpha) = (1 - c)(1 + b)\beta = 1; \tag{12} \]

2. if \( \alpha = \beta = 0 \) and (12) holds true, then \( T \in G_s(\varphi) \) with an exponentially decreasing rate function \( \varphi \);

3. if \( 0 < \alpha, \beta < 1 \) and (12) holds true, then \( T \in G_w(\varphi) \) with the summable rate function
   
   \[ \varphi(k) := \begin{cases} C|k|^{-\gamma} & \text{if } k \leq 0, \\ 0 & \text{if } k \geq 0, \end{cases} \]
   
   where \( C = C(a, b, c, \alpha, \beta) < \infty, \gamma > 1/\min(\alpha, \beta) \);

4. if \( \alpha, \beta > 1 \) and \( ab \neq 0 \), then the strong gluing property (1) with a summable rate function \( \varphi \) cannot hold.

**Remark 5.2.** (a) \( \alpha = \beta = 0 \) means that the function \( T \) is piecewise linear.

(b) Equality (12) is equivalent to \( T[0, c] = [0, 1], T[c, 1] = (0, 1] \), which implies that \( a, b > 0 \).
Proof. If any version of the gluing property holds, then any pair of trajectories of the map $T$ may be glued together. Consider a pair of trajectories $\vec{x} := \{0\}, \vec{y} := \{1\}$. Here by $\{0\}$ and $\{1\}$ we mean trajectories staying at fixed points 0 and 1 correspondingly $\forall t \in \mathbb{Z}$.

There are several possibilities:

1. $a, b < 0$ or $a, b > 0$. Then for $c < 1$ then forward trajectory, starting from a point $u$ belonging to a neighbourhood of 0, cannot assume a value greater $Tc < 1$ (here we are using monotonicity of the branches of $T$). Similarly, if $K := \lim_{x \to c, x < c} T(x) > 0$ then any backward trajectory, which starts from a point $v$ belonging to a neighborhood of 0, cannot get out of this neighborhood. Therefore we came to the same conclusion.

2. $a < 0 < b$. Then a forward trajectory, starting from a point $u$ belonging to a neighborhood of 0, cannot get out of this neighborhood. Therefore we came to the same conclusion.

3. $a > 0 > b$. Similarly to the previous item, but one needs to consider the neighborhood of 1.

This analysis proves item (1).

To prove item (2) consider two arbitrary trajectories $\vec{x}, \vec{y}$ of this map. We define the gluing trajectory $\vec{z}$ as follows: $z_0 := y_0, z_k := T^k y_0$ for $k > 0$, and $z_{k-1} := T_{a, b}^{-1} z_k$ for $k \leq 0$. In fact, since the map $T$ is piecewise expanding, there are no other options for $z_0$, similarly to the example 4.1.

By this construction $z_k = y_k \ \forall k \geq 0$, while for negative $k$ the distances between $z_k$ and $x_k$ decrease at exponential rate, since each time we are applying for their calculations the same inverse branch of the expanding map $T$.

Nevertheless, observe that the distance between $Tx_0$ and $Ty_0$ (i.e. the gap between the backward trajectory $\vec{x}$ and the forward trajectory $\vec{y}$) might be arbitrary close to 1 when the points $x_0$ and $y_0$ are close to the point $c$.

Thus if $\min(a, b) > 0$ we get the summable rate function

$$\varphi(k) := \begin{cases} (1 + \min(a, b))^k & \text{if } k \leq 0 \\ 0 & \text{if } k \geq 0 \end{cases}.$$  

Item (2) is proven.

In all situations considered so far the rate function $\varphi$ has exponential tails. In general this is absolutely not the case, which will be demonstrated in the case $\alpha, \beta > 0$. Moreover, we will show that the neutrally expanding map, considered in this example, for $0 < \alpha, \beta < 1$ satisfies only the weak gluing property (2), but the strong one (1) breaks down, which excludes the uniform (U+U) shadowing property.

If $\alpha, \beta > 0$ the map $T$ is uniformly piecewise expanding everywhere except the neighbourhoods of the neutral fixed points 0 and 1. Therefore the analysis of the forward part of the gluing property does not differ much from the situation in item (2), namely, we choose $z_0 := y_0$. Still we need to show that proper chosen pre-images of the point $z_0$ will approximate the backward semi-trajectory well enough.

To this end we need some estimates of the rate of convergence in backward time of a map with a neutral fixed point obtained in [7].

**Lemma 5.3 ([7])**. Let $\tau(v) := v + R^{1+\alpha}, R > 0, \alpha \geq 0, v \geq 0$. Then
(a) \( \tau^{-n}(v) \leq Kn^{-\gamma} \forall v \in [0,1], n \in \mathbb{Z}_+ \) and some \( K < \infty, \gamma > 1/\alpha. \)
(b) \( \tau^{-n}(v) \geq Kvn^{-\gamma} \forall v \in [0,1], n \in \mathbb{Z}_+ \) and some \( K < \infty, \gamma < 1/\alpha. \)
(c) if \( \alpha = 0 \) then \( \tau^{-n}(v) \leq (1 + R)^{-n}v \forall v \in [0,1], n \in \mathbb{Z}_+. \)

Applying the assertion (a) of lemma 5.3 to the inverse branches of the map \( T \), and using that \( 0 \leq u \leq 1 \) we get

\[
\rho(T^{-n}u, \{0,1\}) < Cn^{-\gamma} \forall u \in X, n \in \mathbb{Z}_+,
\]

where \( C = C(a,b,c,\alpha,\beta) < \infty, \gamma > 1/\min(\alpha,\beta) \) and

\[
\rho(u,A) := \inf_{a \in A} \rho(u,a), \quad \rho(u,v) := |u - v|.
\]

Now we are ready to estimate \( \rho(x_{-n},z_{-n}) \) for \( n \in \mathbb{Z}_+ \). By the triangle inequality, using (13), we get

\[
\rho(x_{-n},z_{-n}) \leq \rho(x_{-n},0) + \rho(0,z_{-n}) \leq 2Cn^{-\gamma}.
\]

Therefore, since \( z_n \equiv y_n \forall n \in \mathbb{Z}_+ \), \( \varphi(k) := \begin{cases} 2C|k|^{-\gamma} & \text{if } k \leq 0 \\ 0 & \text{if } k \geq 0 \end{cases} \) defines the rate function for the (weak) gluing property (2). Moreover, if \( \alpha < 1 \) then \( \gamma > 1 \) and hence \( \varphi \) is summable. This proves item (3).

Finally, the fact that \( T \notin G_\gamma(\varphi) \) for any summable \( \varphi \) follows from lemma 5.3(b), which proves item (4).

\[ \square \]

6. The gluing property for symbolic dynamics

Example 6.1 (symbolic dynamics). Let \( T : X \to X \) be a map and let \( \{X_i\}_{i=1}^r \) be a partition of \( X \).

Then each (pseudo-)trajectory \( \vec{s} \) of the map \( T \) may be coded by a bi-infinite sequence \( \vec{a} \) of elements from the alphabet \( A := \{a_1, a_2, \ldots, a_r\} \) according to the rule: \( s_i := a_{k} \) if \( x_i \in X_k \). This gives a symbolic description of the dynamics, governed by a binary transition matrix \( \pi \), where \( \pi_{ij} = 1 \) iff \( TX_i \cap X_j \neq \emptyset \). The set of admissible sequences (corresponding to the transition matrix \( \pi \)) we denote by \( \Sigma_\pi \), while the set of all sequences from the alphabet \( A \) by \( \Sigma := A^{\mathbb{Z}} \).

When coding a pseudo-trajectory \( \vec{y} \) of the map \( T \) we get a sequence \( \vec{s} \notin \Sigma_\pi \) (in general). The question is, is there is an admissible sequence that ‘approximates’ \( \vec{s} \)?

This example differs from the previous ones in two important points. First, the ‘amplitude’ of the perturbation takes only a discrete set of values and thus cannot be uniformly small\(^{12}\). Second, despite the transition between letters from the alphabet \( A \) can be described in terms of a map, but this map is multi-valued and so our results about shadowing cannot be applied in this setting directly.

To this end we consider a shift map \( \sigma : \Sigma_\pi \to \Sigma_\pi \) with somewhat unusual perturbations. For a pair of admissible sequences \( \vec{x}, \vec{u} \) by the perturbed sequence we mean \( \vec{w} \in \Sigma_\pi \) such that

\[
w_i := \begin{cases} s_i & \text{if } i < 0 \\ u_i & \text{otherwise}. \end{cases}
\]

In words, we preserve the elements with negative indices of \( \vec{x} \), but

\(^{11}\) The partition needs not to be finite, i.e. it is possible that \( r = \infty \).

\(^{12}\) To measure perturbations, we endow the alphabet \( A \) with the discrete metric.
change the ones with nonnegative indices to \( \bar{u} \). Obviously, this is what happens under the perturbations in the example 6.1. In the space of all sequences \( \Sigma \) we consider a metric \( \rho(\bar{x}, \bar{u}) := \sum_{k=-\infty}^{\infty} 2^{-|k|} 1_{\alpha_k(u_k)} \), where \( 1_{\alpha_k}(b) := \begin{cases} 0 & \text{if } a = b \\ 1 & \text{otherwise.} \end{cases} \)

Again, the perturbations in this setting are not uniformly small, and the average shadowing in symbolic dynamics is equivalent to the average shadowing for the shift map \( \sigma \) acting in the metric space \( (\Sigma, \rho) \).

**Proposition 6.1.** Symbolic dynamics belongs to the class \( G_r(\varphi) \) with a finitely supported (and hence summable) function \( \varphi \) if and only if \( \exists M \in \mathbb{Z}_+ \) such that the \( M \)-power of the transition matrix (i.e. \( \pi^M \)) is positive. Therefore under the latter assumption the symbolic dynamics belongs to the class \( S(R, A) \).

Before to give the proof, let us demonstrate examples of the transition matrices, leading to the average shadowing \( \pi := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \), or its absence \( \pi := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

**Proof.** We start with the case of the finite alphabet \( A \), i.e. \( r < \infty \). According to the assumption \( \pi^M > 0 \) during time \( M \) we can go from any element of the alphabet \( A \) to any other. So setting the function \( \varphi \) equal to the indicator function of the integer segment \( [−M, M] \), we obtain the desired result. Namely, this demonstrates that each pair of admissible trajectories may be glued together with accuracy \( \varphi \).

To verify the necessary part, assume that the condition of the theorem is not satisfied and there is no \( M \) such that \( \pi^M := (\pi^{(M)}) > 0 \). It follows that

\[
\forall n \in \mathbb{Z}_+ \exists_{n, j_n} : \pi_{n, j_n}^{(n)} = 0.
\]

Indeed, if this were not the case, then there would be \( k \in \mathbb{Z}_+ \) such that \( \pi^k > 0 \). But in this case \( \pi^n > 0 \forall n > k \), which contradicts the assumption.

As we have already noted, the transition matrix \( \pi \) induces a multivalued mapping of the alphabet into itself by the following formula: \( \pi a := \{ b \in A : \pi_{ab} > 0 \} \). In these terms, since \( \pi \) is not strictly positive and \( A \) is finite, it follows that for some \( N \in \mathbb{Z}_+ \) there is a partition \( A := \sqcup_i A_i \) into non-empty \( \pi^N \)-invariant subsets, i.e. \( (\pi^N)^{-1} A_i = A_i \forall i \). It follows from this that for \( x_0, y_0 \) belonging to different elements of this partitions corresponding to admissible sequences \( \bar{x}, \bar{y} \) do not intersect, i.e. \( x_i \neq y_i \forall i \in \mathbb{Z} \). Therefore, there is no ‘gluing’ of them with any summable function \( \varphi \).

It remains to prove that the claim of theorem remains valid for \( r = \infty \). The sufficient part follows from the same argument as in the case \( r < \infty \). To prove the necessary part, consider a ‘circumcised’ finite alphabet \( \bar{A} := \{ a_1, a_2, \ldots, a_{\ell-1}, \bar{a}_\ell \} \) with \( \ell < \infty \) letters, such that \( \{a_1, a_2, \ldots, a_{\ell-1} \} \subset A \), while the letter \( \bar{a}_\ell \) corresponds to the union of the remaining elements of the partition \( \sqcup_{i \geq \ell} X_i \). Then considering the corresponding ‘circumcised’ transition matrix \( \bar{\pi} \), we get the shift map \( \bar{\sigma} : \Sigma_{\bar{A}} \to \Sigma_{\bar{A}} \), for which all results obtained in the first part of the proof are applicable. Therefore the average shadowing for \( \bar{\sigma} \) holds iff \( \exists M = M(\ell) < \infty : \bar{\pi}^M > 0 \).

According to the validity of the average shadowing for \( \bar{\sigma} \), the inequality \( \bar{\pi}^M > 0 \) does not hold \( \forall M \in \mathbb{Z}_+ \) for the complete infinite alphabet \( A \) if and only if \( M(\ell) \to \infty \) \( \bar{\pi}^M \). If the latter happens, one constructs a pair of admissible trajectories, having elements from \( A \) with arbitrary large indices, which cannot be glued together with the summable accuracy rate \( \varphi \). We came to the contradiction. \( \square \)
Note that despite the proof was reduced to the hyperbolic dynamics (under a special choice of a metric), the perturbations are small only on average. Moreover, proposition 6.1 gives both necessary and sufficient conditions for average shadowing for the symbolic dynamics with an infinite alphabet.

Data availability statement

No new data were created or analysed in this study.

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