TOEPLITZ AND HANKEL OPERATORS FROM BERGMAN TO ANALYTIC BESOV SPACES OF TUBE DOMAINS OVER SYMMETRIC CONES.

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Abstract. We characterize bounded Toeplitz and Hankel operators from weighted Bergman spaces to weighted Besov spaces in tube domains over symmetric cones. We deduce weak factorization results for some Bergman spaces of this setting.

1. Introduction

Let \( n \geq 3 \) and \( D = \mathbb{R}^n + i\Omega \) be the tube domain over an irreducible symmetric cone \( \Omega \) in \( \mathbb{R}^n \). Following the notations of [14] we denote the rank of the cone \( \Omega \) by \( r \) and by \( \Delta \) the determinant function of \( \mathbb{R}^n \). As example of symmetric cone on \( \mathbb{R}^n \) we have the Lorentz cone \( \Lambda^n \) which is a rank 2 cone defined for \( n \geq 3 \) by

\[
\Lambda^n = \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : y_1^2 - \cdots - y_n^2 > 0, \ y_1 > 0 \right\},
\]

the determinant function in this case is given by the Lorentz form

\[
\Delta(y) = y_1^2 - \cdots - y_n^2.
\]

We shall denote by \( H(D) \) the space of holomorphic functions on \( D \).

For \( 1 \leq p < \infty \) and \( \nu \in \mathbb{R} \), let \( L^p_\nu(D) = L^p(D, \Delta^{\nu - \frac{n}{r}}(y)dx\,dy) \) denotes the space of functions \( f \) satisfying the condition

\[
\|f\|_{L^p_\nu} = \left( \int_D |f(x + iy)|^p \Delta^{\nu - \frac{n}{r}}(y)\,dxdy \right)^{1/p} < \infty.
\]

Its closed subspace consisting of holomorphic functions in \( D \) is the weighted Bergman space \( A^p_\nu(D) \). This space is not trivial i.e. \( A^p_\nu(D) \neq \{0\} \) only for \( \nu > \frac{n}{r} - 1 \) (see [13]).

The weighted Bergman projection \( P_\nu \) is the orthogonal projection of the Hilbert space \( L^2_\nu(D) \) onto its closed subspace \( A^2_\nu(D) \). It is well known that \( P_\nu \) is an integral operator given by

\[
P_\nu f(z) = \int_D K_\nu(z, w) f(w)\,dV_\nu(w),
\]

where \( K_\nu(z, w) = c_\nu \Delta^{-\nu + \frac{n}{r}}((z - \overline{w})/i) \) is the weighted Bergman kernel, i.e. the reproducing kernel of \( A^2_\nu(D) \) (see [14]). Here, we use the notation \( dV_\nu(w) := \Delta^{\nu - \frac{n}{r}}(v)dv \,dv \), where \( w = u + iv \) is an element of \( D \). The unweighted case corresponds to \( \nu = \frac{n}{r} \).

2000 MATH SUBJECT CLASSIFICATION: 42B35, 32M15.

KEYWORDS: BERGMAN PROJECTION, HANKEL OPERATOR, TOEPLITZ OPERATOR, BESOV SPACE, SYMMETRIC CONE.
It is known that the weighted Bergman projection $P_\nu$ cannot be bounded on $L^p_\nu(\mathcal{D})$ for $p$ large (see [5], [25]). Consequently, the natural mapping from $A^p_\nu(\mathcal{D}) \left( \frac{1}{p} + \frac{1}{\nu} = 1 \right)$ into the dual space $(A^p_\nu(\mathcal{D}))^*$ of $A^p_\nu(\mathcal{D})$ is not an isomorphism for such values of the exponent $p$ (see [6]). In [6], it is also shown that the boundedness of $P_\nu$ on $L^p_\nu(\mathcal{D})$ for $p \geq 2$ is equivalent to the validity of the following Hardy-type inequality for $f \in A^p_\nu(\mathcal{D})$:

\begin{equation}
\int_{\mathcal{D}} \left| f(x + iy) \right|^p \Delta^{\nu-\frac{p}{2}}(y) \, dx \, dy \leq C \int_{\mathcal{D}} \left| \Delta(y) \Box f(x + iy) \right|^p \Delta^{\nu-\frac{p}{2}}(y) \, dx \, dy,
\end{equation}

where $\Box = \Delta\left(\frac{1}{\sqrt{x^2 + y^2}}\right)$ denotes the differential operator of degree $r$ in $\mathbb{R}^n$ defined by the equality:

\begin{equation}
\Box [e^{i\langle x, \xi \rangle}] = \Delta(\xi) e^{i\langle x, \xi \rangle}, \quad x, \xi \in \mathbb{R}^n.
\end{equation}

When (1.1) holds for some $p$ and $\nu$, we speak of Hardy inequality for $(p, \nu)$. This leads to the following definition of analytic Besov space $B^p_\nu(\mathcal{D})$, $1 \leq p < \infty, \nu \in \mathbb{R}$.

Let $m \in \mathbb{N}$, we denote

\[ \mathcal{N}_m := \{ F \in \mathcal{H}(\mathcal{D}) : \Box^m F = 0 \} \]

and set

\[ \mathcal{H}_m(\mathcal{D}) = \mathcal{H}(\mathcal{D})/\mathcal{N}_m. \]

For simplicity, we use the following notation for the normalized operator Box operator: we write

\begin{equation}
\Delta^m \Box F(z) := \Delta^m(\Box m z) \Box F(z).
\end{equation}

We will use the same notations for holomorphic functions and for equivalence classes in $\mathcal{H}_m$. We observe that, for $F \in \mathcal{H}_m$, we can speak of the function $\Box^m F$. Given $\nu \in \mathbb{R}$, $1 \leq p < \infty$ the Besov space $B^p_\nu(\mathcal{D})$ is defined as follows

\[ B^p_\nu := \{ F \in \mathcal{H}_{k_0}(\mathcal{D}) : \Delta^0 \Box^{k_0} F \in L^p_\nu \} \]

endowed with the norm $\| F \|_{B^p_\nu} = \| \Delta^{k_0} \Box^{k_0} F \|_{L^p_\nu}$. Here $k_0 = k_0(p, \nu)$ is fixed by

\begin{equation}
k_0(p, \nu) := \min \{ k \geq 0 : \nu + k p > \frac{n}{r} - 1 \} \quad \text{and Hardy’s inequality holds for } (p, \nu + pk) \).
\end{equation}

Each element of $B^p_\nu$ is the equivalence class of all analytic solutions of the equation $\Box^{k_0} F = g$, for some $g \in A^p_{\nu+k_0p}$. Consequently, these spaces are null when $\nu + k_0 p \leq \frac{n}{r} - 1$. We also observe that for $1 \leq p \leq 2$ and $\nu > \frac{p}{r} - 1$, $B^p_\nu(\mathcal{D}) = A^p_\nu(\mathcal{D})$ since the Hardy inequality (1.1) always holds in this range (see [7]). When $p > 2$, we have the embedding of $A^p_\nu(\mathcal{D})$ into $B^p_\nu(\mathcal{D})$.

For $p = \infty$, we define the Bloch $\mathcal{B} = \mathcal{B}^\infty$ of $\mathcal{D}$ as follows: Let $m_\nu$ be the smallest integer $m$ such $m > \frac{n}{r} - 1$,

\[ \mathcal{B} := \{ F \in \mathcal{H}_{m_\nu}(\mathcal{D}) : \Delta^m \Box^m F \in L^\infty \}. \]

Our interest in this paper is the boundedness of two operators, namely the Toeplitz operator with symbol a measure $\mu$ and the (small) Hankel operator with symbol $b$ from the Bergman space $A^p_\nu(\mathcal{D})$ into the Besov space $B^p_\nu(\mathcal{D})$. Note that usually in bounded domains as the unit ball, for the study of the boundedness of Toeplitz and Hankel operators between Bergman spaces, one needs among others a good understanding of the topological dual space of the target space. The choice of $B^p_\nu(\mathcal{D})$ as the target space is suggested not only
by the lack of continuity of the projection $P_\nu$ on $L^q_0(D)$ for large values of $q$ but also by the fact that $\mathbb{B}^q_0(D)$ is the dual space of of the Bergman space $A^q_0(D) \left( \frac{1}{q} + \frac{1}{\sigma'} = 1 \right)$ for an adapted duality pairing.

2. Statement of results

In this section, we present the main results of the paper. Our first interest in this paper concerns Toeplitz operator from a Bergman space to a Besov space. Recall that for a positive Borel measure $\mu$ on $D$ and $\nu > \frac{n}{p} - 1$, the Toeplitz operator $T_\nu^\mu$ is the operator defined for any function $f$ with compact support by
\[
T_\nu^\mu f(z) := \int_D K_\nu(z, w) f(w) d\mu(w),
\]
where $K_\nu$ is the (weighted) Bergman kernel. Boundeness of Toeplitz operators between Bergman spaces of the unit ball was treated in [22], the method used Carleson embedding and thus for estimations with loss, techniques of Luecking [19]. We will essentially make use of the same idea. We note that Carleson embeddings for Bergman spaces of tube domains over symmetric were obtained by the authors in [21]. We shall then prove the following result.

**THEOREM 2.6.** Let $1 < p \leq q < \infty$, $q \geq 2$, $\alpha, \beta, \nu > \frac{n}{p} - 1$ with $\nu > \frac{n}{p} - 1 + \frac{\beta - \frac{n}{p} + 1}{q} - \frac{\alpha - \frac{n}{p} + 1}{p}$.
Define the numbers $\lambda = 1 + \frac{1}{p} - \frac{1}{q}$ and $\lambda\gamma = \nu + \frac{\alpha - \frac{n}{p}}{q}$. Assume that $\nu > \frac{n}{p} - 1$. Then the following assertions are equivalent.
(a) The operator $T_\nu^\mu$ extends as a bounded operator from $A^p_0(D)$ to $\mathbb{B}^q_0(D)$.
(b) There is a constant $C > 0$ such that for any $\delta \in (0, 1)$ and any $z \in D$,
\[
\mu(B_\delta(z)) \leq C \Delta^{\lambda(\gamma + \frac{\alpha}{p})} (\Im z).
\]

We have also this estimation with loss result.

**THEOREM 2.8.** Let $2 \leq q < p < \infty$, $\alpha, \beta, \nu > \frac{n}{p} - 1$ with $\nu > \frac{n}{p} - 1 + \frac{\beta - \frac{n}{p} + 1}{q} - \frac{\alpha - \frac{n}{p} + 1}{p}$.
Define the numbers $\lambda = 1 + \frac{1}{p} - \frac{1}{q}$ and $\lambda\gamma = \nu + \frac{\alpha - \frac{n}{p}}{q}$. Assume $P_{\nu + m}$ is bounded on $L^p_0(D)$ for some integer $m$. Then the following assertions are equivalent.
(a) The operator $T_\nu^\mu$ extends as a bounded operator from $A^p_0(D)$ to $\mathbb{B}^q_0(D)$.
(b) For any $\delta \in (0, 1)$, the function
\[
\mathcal{D} \ni z \mapsto \frac{\mu(B_\delta(z))}{\Delta^{\lambda(\gamma + \frac{\alpha}{p})} (\Im z)}
\]

belongs to $L^{1/(1-\lambda)}(\mathcal{D})$.

Our second interest concerns (small) Hankel operator. For $b \in A^2_0(D)$, the (small) Hankel operator with symbol $b$ is defined for $f \in \mathcal{H}^\infty(D)$ by
\[
h_b^{(0)}(f) = h_b(f) := P_\nu(bf).
\]

Boundeness of Hankel operators between Bergman spaces on the unit ball was considered in [11] (see also the references therein) where a full characterization has been obtained for estimates without loss, i.e $h_b : A^p_0 \rightarrow A^q_0$ with $1 \leq p \leq q < \infty$. The estimations with loss (i.e. the case $p > q$) were recently handled in [23] closing the question for the unit ball. Note
that to deal with this last case, the authors of [23] for the necessary part used an approach due to Luecking for Carleson embeddings [19]. We are interested here in the question of the boundedness of $h_b$ from the Bergman space $A^p_\nu(D)$ into the Besov space $B^q_\gamma(D)$. For the case of estimations with loss, we also use an adaptation of Luecking techniques for the necessary part as in [24]. Let us denote

$$q_\nu = 1 + \frac{\nu}{p} - 1, \quad \tilde{q}_\nu = \frac{\nu + 2\tilde{q}}{\nu - 1}.$$

We shall the prove the following.

**Theorem 2.9.** Let $q_\nu < p < \tilde{q}_\nu$ and $\nu > \frac{n}{r} - 1$. Then the Hankel operator $h_b$ is bounded from $A^p_\nu(D)$ into $B^q_\gamma(D)$ if and only if $b = P_\nu g$ for some $g \in L^\infty(D)$.

**Theorem 2.10.** Let $1 \leq p < q < \infty$ and $\alpha, \beta, \nu > \frac{n}{r} - 1$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{\nu} + \frac{1}{\nu'} = 1$. Define $\beta' = \beta + (\nu - \beta)\gamma'$; $\frac{1}{p} + \frac{1}{q'} = \frac{1}{\nu} < 1$; $\frac{1}{p} + \frac{\beta'}{\nu} = \frac{1}{\nu}$; $\frac{1}{s} + \frac{1}{s'} = 1$ and $\frac{1}{s} + \frac{\beta'}{\nu} = \nu$. Assume moreover that

$$\max\{1, \frac{\beta - \nu + 1}{\nu - \frac{n}{r} + 1}\} < q < \infty.$$

Then the following hold.

(i) $h_b^{(\nu)}$ extends into a bounded operator from $A^p_\nu(D)$ to $B^q_\gamma(D)$.

(ii) For some integer $m$ large enough,

$$\Delta^m + R^{(\mu + \frac{n}{r})}b \in L^\infty(D).$$

**Theorem 2.11.** Let $2 \leq q < \infty$ and $\alpha, \beta, \nu > \frac{n}{r} - 1$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{\nu} + \frac{1}{\nu'} = 1$. Define $\beta' = \beta + (\nu - \beta)\gamma'$; $\frac{1}{p} + \frac{1}{q'} = \frac{1}{\nu} < 1$; $\frac{1}{p} + \frac{\beta'}{\nu} = \frac{1}{\nu}$; $\frac{1}{s} + \frac{1}{s'} = 1$ and $\frac{1}{s} + \frac{\beta'}{\nu} = \nu$. Assume that

$$\frac{\beta - \nu + 1}{\nu - \frac{n}{r} + 1} < q \quad (2.12)$$

and

$$\frac{1}{p} (\alpha - \frac{n}{r} + 1) + \frac{1}{q'} (\beta' - \frac{n}{r} + 1) < \nu - \frac{n}{r} + 1 \quad (2.13).$$

Then the following assertions hold.

(i) If $b$ is the representative of a class in $B^\mu_{\beta'}$, then the Hankel operator $h_b^{(\nu)}$ extends into a bounded operator from $A^p_\nu(D)$ to $B^q_\gamma(D)$.

(ii) If there exists $\sigma > \frac{n}{r} - 1$ such that $P_{\sigma}$ is bounded on $L^\infty(D)$ and if $h_b^{(\nu)}$ extends into a bounded operator from $A^p_\nu(D)$ to $B^q_{\gamma}(D)$, then $b$ is the representative of a class in $B^\mu_{\beta'}$.

Given a function $f$ in a Bergman space of some domain, if we can write $f = gh$ where $g$ and $h$ are in some different Bergman spaces, then we say $f$ admits a strong factorization. C. Horowitz has proved that this is the case for Bergman spaces of the unit disc [18]. This does not longer happen in higher dimension as proved in [17]. However, it is still possible to obtain the so-called weak factorization.
For two Banach spaces of functions $A$ and $B$ defined on the same domain, the weak factored space $A \otimes B$ is defined as the completion of finite sums

$$f = \sum_j g_j h_j, \{g_j\} \subset A, \{h_j\} \subset B$$

equipped with the following norm

$$\|f\|_{A \otimes B} = \inf\{\sum_j \|g_j\|_A \|h_j\|_B : f = \sum_j g_j h_j\}.$$ 

Given $0 < p, q, s < \infty$ and $\alpha, \beta, \gamma$, whenever the equality $A_s^\gamma(D) = A_p^\alpha(D) \otimes A_q^\beta(D)$ holds, we say $A_s^\gamma(D)$ admits a weak factorization. In the case of the unit ball $B_n$ of $\mathbb{C}^n$, that weak factorization holds for Bergman spaces with small exponent (i.e. $0 < s \leq 1$) is a consequence of the atomic decomposition of these spaces. This result was quite recently extended to large exponents ($s > 1$) by J. Pau and R. Zhao in [23] as a consequence of estimations with loss for the Hankel operators. As a consequence of our characterization of bounded Hankel operators, we have the following result for weighted Bergman spaces of our setting.

**Theorem 2.14.** Let $1 < p, q < \infty$ and $\alpha, \beta, \nu > \frac{n}{r} - 1$ so that

$$\frac{1}{p}(\alpha - \frac{n}{r} + 1) + \frac{1}{q}(\beta - \frac{n}{r} + 1) < \nu - \frac{n}{r} + 1$$

holds. Define $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} < 1$ and $\frac{\gamma}{s} = \frac{\alpha}{p} + \frac{\beta}{q}$. Assume that $P_\sigma$ is bounded on $A_p^\alpha(D)$ for some $\sigma > \frac{n}{r} - 1$, and that

$$\max\{q', \frac{\gamma' + \frac{n}{r} - 1}{\nu}, \frac{\gamma' - \frac{n}{r} + 1}{\nu - \frac{n}{r} + 1}\} < s < q_\gamma$$

and

$$\max\{q'_\beta, \frac{\beta + \frac{n}{r} - 1}{\nu}, \frac{\beta - \frac{n}{r} + 1}{\nu - \frac{n}{r} + 1}\} < q < q_\beta;$$

or

$$1 < s < 2 \quad \text{and} \quad 1 < q \leq 2.$$ 

Then

$$A_s^\gamma(D) = A_p^\alpha(D) \otimes A_q^\beta(D).$$

We also have the following when the weights are the same.

**Theorem 2.17.** Let $1 < p, q < \infty$ and $\nu > \frac{n}{r} - 1$. Assume that $P_\nu$ is bounded on both $L_p^\nu(D)$ and $L_q^\nu(D)$, and put $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} < 1$. Then

$$A_s^\nu(D) = A_p^\nu(D) \otimes A_q^\nu(D).$$

As usual, given two positive quantities $A$ and $B$, the notation $A \lesssim B$ (resp. $A \gtrsim B$) means that there is an absolute positive constant $C$ such that $A \leq CB$ (resp. $A \geq CB$). When $A \lesssim B$ and $B \lesssim A$, we write $A \simeq B$ and say $A$ and $B$ are equivalent. Finally, all over the text, $C, C_k, C_{k,j}$ will denote positive constants depending only on the displayed parameters but not necessarily the same at distinct occurrences.
3. Some useful notions and results

3.1. Symmetric cones and Bergman metric. Let Ω be an irreducible symmetric cone in the vector space $V \equiv \mathbb{R}^n$. Denote by $G(\Omega)$ be the group of transformations of $\mathbb{R}^n$ leaving invariant the cone $\Omega$, and by $e$ the identity element in $V$. We recall that $\Omega$ induces in $V$ a structure of Euclidean Jordan algebra, in which $\overline{\Omega} = \{ x^2 : x \in V \}$. It is well known that there is a subgroup $H$ of $G(\Omega)$ that acts simply transitively on $\Omega$, that is for $x, y \in \Omega$ there is a unique $h \in H$ such that $y = hx$. In particular, $\Omega = H \cdot e$.

If we denote by $\mathbb{R}^n$ the group of translation by vectors in $\mathbb{R}^n$, then the group $G(D) = \mathbb{R}^n \times H$ acts simply transitively on $D$.

Let us consider the matrix function $\{g_{jk}\}_{1 \leq j, k \leq n}$ on $D$ given by

$$g_{jk}(z) = \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log K(z, z)$$

where $K(w, z) = c(n/r) \Delta^{-2n/r} \left( \frac{w z - \overline{z}_i}{n/r} \right)$ is the (unweighted) Bergman kernel of $D$. The map $z \in D \mapsto G_z$ with

$$G_z(u, v) = \sum_{1 \leq j, k \leq n} g_{jk}(z) u_j \overline{v}_k, \quad u = (u_1, \ldots, u_n), \quad v = (v_1, \ldots, v_n) \in \mathbb{C}^n$$

defines a Hermitian metric on $\mathbb{C}^n$, called the Bergman metric. The Bergman length of a smooth path $\gamma : [0, 1] \to D$ is given by

$$l(\gamma) = \int_0^1 \left( G_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) \right)^{1/2} dt$$

and the Bergman distance $d(z_1, z_2)$ between two points $z_1, z_2 \in D$ is defined by

$$d(z_1, z_2) = \inf_{\gamma} l(\gamma)$$

where the infimum is taken over all smooth paths $\gamma : [0, 1] \to D$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$.

For $\delta > 0$, we denote by

$$B_\delta(z) = \{ w \in D : d(z, w) < \delta \}$$

the Bergman ball centered at $z$ with radius $\delta$.

We refer to [1, Theorem 5.4] for the following.

**Lemma 3.1.** Given $\delta \in (0, 1)$, there exists a sequence $\{z_j\}$ of points of $D$ called $\delta$-lattice such that, if $B_j = B_\delta(z_j)$ and $B'_j = B_{\delta/2}(z_j)$,

(i) the balls $B'_j$ are pairwise disjoint;

(ii) the balls $B_j$ cover $D$ with finite overlapping, i.e. there is an integer $N$ (depending only on $D$) such that each point of $D$ belongs to at most $N$ of these balls.

The above balls have the following properties:

$$\int_{B_j} dV_{\nu}(z) \approx \int_{B'_j} dV_{\nu}(z) \approx C_\delta \Delta^{n/r}(\Im z_j).$$

We recall that the measure $d\lambda(z) = \Delta^{-2n/r}(\Im z) dV(z)$ is an invariant measure on $D$ under the actions of $G(D) = \mathbb{R}^n \times H$. 
Let us denote by $l_0^p$, the space of complex sequences $\beta = \{\beta_j\}_{j \in \mathbb{N}}$ such that

$$||\beta||_{l_0^p}^p = \sum_j |\beta_j|^p \Delta^\nu \mathbb{D}(\Im \beta z_j) < \infty,$$

where $\{z_j\}_{j \in \mathbb{N}}$ is a $\delta$-lattice.

The following is quite easy to check.

**Lemma 3.2.** [10] Lemma 2.11.1 | Suppose $1 \leq p < \infty$, $\nu$ and $\mu$ are reals. Then, the dual space $(l^p_0)^*$ of the space $l_0^p$, identifies with $l^p_{\nu+(\mu-\nu)p'}$ under the sum pairing

$$\langle \eta, \beta \rangle_\mu = \sum_j \eta_j \beta_j \Delta^\nu \mathbb{D}(\Im \beta z_j),$$

where $\eta = \{\eta_j\}$ belongs to $l^p_0$ and $\beta = \{\beta_j\}$ belongs to $l^{p'}_{\nu+(\mu-\nu)p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

Note that in the text, the space $l^p_0$, is the usual sequence space.

The following result known as the sampling theorem.

**Lemma 3.3.** [4] Theorem 5.6 | Let $\{z_j\}_{j \in \mathbb{N}}$ be a $\delta$-lattice in $\mathbb{D}$, $\delta \in (0, 1)$ with $z_j = x_j + iy_j$.

The following assertions hold.

1. There is a positive constant $C_\delta$ such that every $f \in A_0^\nu(\mathbb{D})$ satisfies

$$||\{f(z_j)\}||_{l_0^p} \leq C_\delta ||f||_{p,\nu}.$$

2. Conversely, if $\delta$ is small enough, there is a positive constant $C_\delta$ such that every $f \in A_0^\nu(\mathbb{D})$ satisfies

$$||f||_{p,\nu} \leq C_\delta ||\{f(z_j)\}||_{l_0^p}.$$

We will need the following consequence of the mean value theorem (see [4])

**Lemma 3.4.** There exists a constant $C > 0$ such that for any $f \in \mathcal{H}(\mathbb{D})$ and $\delta \in (0, 1)$, the following holds

$$|f(z)|^p \leq C \delta^{-n} \int_{B_\delta(z)} |f(\zeta)|^p \frac{d\mathcal{V}(\zeta)}{\Delta^{2n/r}(\Im \beta \zeta)}.$$

We finish this subsection with the following pointwise estimate of functions in Bergman spaces.

**Lemma 3.6.** [4] Proposition 3.5 | Let $1 \leq p < \infty$, and $\nu > \frac{n}{p} - 1$. Then there is a constant $C > 0$ such that for any $f \in A_0^\nu(\mathbb{D})$ the following pointwise estimate holds

$$|f(z)| \leq C \delta^{-\frac{2}{r}} \Delta^{\nu} \mathbb{D}(\Im \beta \zeta) ||f||_{A_0^\nu}, \text{ for all } z \in \mathbb{D}.$$

3.2. **Isomorphism of Besov spaces.** Let us denote by $\mathbb{B}_{\nu}^{p,(k)}(\mathbb{D})$ (respectively $\mathbb{B}_{\nu}^{\infty,(k)}$) the space obtained by replacing $k_0$ (respectively $m_*$) by $k \geq k_0$ (respectively $k \geq m_*$) in the definition of $\mathbb{B}_{\nu}^{p}(\mathbb{D})$ (respectively $\mathbb{B}_{\nu}^{\infty}(\mathbb{D})$). Then it is not hard to prove the following result (see [4] for details).

**Proposition 3.8.** The natural projection

$$\mathbb{B}_{\nu}^{p,(k)}(\mathbb{D}) \ (\text{respectively } \mathbb{B}_{\nu}^{\infty,(k)}) \longrightarrow \mathbb{B}_{\nu}^{p,(m)}(\mathbb{D}) \ (\text{respectively } \mathbb{B}_{\nu}^{\infty,(m)})$$

$$F + N_k \mapsto F + N_m$$

is an isomorphism of Banach spaces.
We will heavily make use of the above proposition.

3.3. Bergman kernel and reproducing formulas. The (weighted) Bergman projection $P_\nu$ is defined by

$$P_\nu f(z) = \int_D K_\nu(z, w)f(w)dV_\nu(w),$$

where $K_\nu(z, w) = c_\nu \Delta^{-(\nu+\frac{1}{2})}((z - \overline{w})/i)$ is the Bergman kernel, i.e., the reproducing kernel of $A^2_\nu$ (see [14]). Here, we use the notation $dV_\nu(w) := \Delta^{\nu-\frac{1}{2}}(v)du dv$, where $w = u + iv$ is an element of $D$.

We recall that the Box operator $\square$ acts on the Bergman kernel in the following way:

$$\square^m K_\nu(z, .) = C_{\nu, m} K_{\nu + m}(z, .)$$

(see [7]).

We will need the following integration by parts formula which follows from the density of the intersection of two Bergman spaces in each of them (see [6], [7]). For $\nu > \frac{n}{2} - 1$, $1 \leq p \leq \infty$ and $f \in A^p_\nu$, $g \in A^p_{\nu}$, we have the formula

$$\int_D f(z)\overline{g}(z)dV_\nu(z) = c_{\nu, m} \int_D f(z)\square^m g(z)\Delta^m(\Im m z)dV_\nu(z).$$

Using an adapted version of the above formula for the Box of functions, we can prove the following (see [6] for details).

**Proposition 3.11.** Let $\nu > \frac{n}{2} - 1$ and $1 \leq p \leq \infty$. For all $f \in A^p_\nu$ we have the formula

$$\square^\ell f(z) = c \int_D K_{\nu + \ell}(z, w)\square^m f(w)\Delta^m(\Im m w)dV_\nu(w)$$

for $m \geq 0$ and $\ell$ large enough so that $K_{\nu + \ell}(z, .)$ is in $L^p_\nu$. In particular, when $1 \leq p < \tilde{p}_\nu$, the formula is valid with $\ell = 0$.

**Corollary 3.13.** Let $1 \leq p < \tilde{p}_\nu$ and $\nu > \frac{n}{2} - 1$. Then, any $f \in A^p_\nu$ satisfies the formula

$$f(z) = \int_D K_\nu(z, w)f(w)dV_\nu(w).$$

The following is Proposition 2.19 in [6].

**Proposition 3.15.** Let $\mu, \nu, \alpha \in \mathbb{R}$ and $1 \leq p < \infty$ satisfying

$$\nu + \alpha > \frac{n}{r} - 1, \quad \nu p - \mu > (p - 1)(\frac{n}{r} - 1) \quad \text{and} \quad \mu + \alpha p > (p - 1)(\frac{n}{r} - 1) - \frac{n}{r}.$$ 

Then, the function $z \mapsto \Delta^{\nu-\mu}(\Im m z)K_{\nu + \alpha}(z, .)$ is in $L^p_\mu(D)$, and for all holomorphic function $f$ such that the function $z \mapsto \Delta^{\alpha}(\Im m z)f(z)$ is in $L^p_\mu$, we have

$$f(z) = \int_D K_{\nu + \alpha}(z, w)f(w)\Delta^\alpha(\Im m w)dV_\nu(w).$$

The next lemma gives integrability properties of the determinants and Bergman kernels.

**Lemma 3.17.** Let $\alpha, \beta, \nu$ be real. Then
1) for $y \in \Omega$, the integral

$$J_\alpha(y) = \int_{\mathbb{R}^n} |\Delta^{-\alpha}((x + iy)/i)| \, dx$$

converges if and only if $\alpha > \frac{2n}{r} - 1$. In this case, $J_\alpha(y) = C_\alpha \Delta^{-\alpha + \frac{n}{r}}(y)$, where $C_\alpha$ is a constant depending only on $\alpha$.

2) The function $f(z) = \Delta^{-\alpha}(\frac{z + it}{i})$, with $t \in \Omega$, belongs to $A^p_\nu$ if and only if

$$\nu > \frac{n}{r} - 1 \quad \text{and} \quad \alpha > \frac{1}{p}(\nu + \frac{2n}{r} - 1).$$

In this case, $||f||_{A^p_\nu} = C_{\alpha,p} \Delta^{-\alpha + \frac{\nu}{p}}(t)$.

**Proof:** See [4].

3.4. Integral operators and duality. Let us now consider the following integral operators $T = T_{\mu,\alpha}$ and $T^+ = T^+_{\mu,\alpha}$, defined by

$$T f(z) = \Delta^\alpha(\Im z) \int_D K_{\mu+\alpha}(z, w) f(w) dV_\mu(w),$$

and

$$T^+ f(z) = \Delta^\alpha(\Im z) \int_D |K_{\mu+\alpha}(z, w)| f(w) dV_\mu(w),$$

provided these integrals make sense. Observe that $P_\mu = T_{\mu,0}$.

The following result is in [25].

**Lemma 3.20.** Let $\alpha, \mu, \nu \in \mathbb{R}$ and $1 \leq p < \infty$. Then the following conditions are equivalent:

(a) The operator $T^+_{\mu,\alpha}$ is well defined and bounded on $L^p_\nu(D)$.

(b) The parameters satisfy $\mu + \alpha > \frac{n}{r} - 1$ and the inequalities

$$\mu p - \nu > \left(\frac{n}{r} - 1\right) \max\{1, p - 1\}, \quad \alpha p + \nu > \left(\frac{n}{r} - 1\right) \max\{1, p - 1\}.$$

Also, from [25] we have the following.

**Lemma 3.21.** Let $\alpha, \mu \in \mathbb{R}$. Then both $T$ and $T^+$ are bounded on $L^\infty(D)$ if and only if $\alpha > \frac{n}{r} - 1$ and $\mu > \frac{n}{r} - 1$.

We will also need the following result.

**Lemma 3.22** (Békolé 1986 [1]). Let $\mu > \frac{n}{r} - 1$ and $m$ an integer such that $m > \frac{n}{r} - 1$. Then the dual space $(A^1_\mu)^*$ identifies with the Bloch space $\mathcal{B}^\infty$ under the integral pairing

$$\langle f, g \rangle_{\mu,m} = \int_D f(z) \Delta^m \sqrt{m} g(z) dV_\mu(z), \quad f \in A^1_\mu, \ g \in \mathcal{B}^\infty.$$

The following duality with change of weights can be proved in the same way as the previous one.
Lemma 3.24. Let \( \nu, \mu > \frac{m}{p} - 1 \) and \( 1 < p \leq 2 \). Then, \( (A_p^\mu)^* \) identifies with \( \mathbb{B}_{\nu+(\mu-\nu)p'}^{p'} \) under the integral pairing

\[
(f, g)_{\nu,m} = \int_D f(z) \Delta_m \overline{g(z)} \, dV_\mu(z), \quad f \in A_p^\mu, \quad g \in \mathbb{B}_{\nu+(\mu-\nu)p'}^{p'},
\]

for any integer \( m \geq k_0(p', \nu + (\mu - \nu)p') \).

We recall the following notations:

\[
\tilde{q}_{\nu,p} = \frac{\nu + \frac{m}{p} - 1}{\left( \frac{m}{p'} - 1 \right)_+}, \quad q_{\nu,p} = \min\{p, p'\} q_{\nu}, \quad q_{\nu} = 1 + \frac{\nu}{p - 1}
\]

with \( \tilde{q}_{\nu,p} = \infty \), if \( n/r \leq p' \). It is clear that \( 1 < q_{\nu} < q_{\nu,p} < \tilde{q}_{\nu,p} \). By density of the intersection \( A_p^\nu \cap A_\mu^m \) in \( A_p^\mu \), we have the following reproducing formula for all \( \alpha > \frac{m}{p} - 1 \) and \( f \in A_p^\nu \) with \( 1 \leq p < \tilde{q}_{\nu,p} \).

\[
f(z) = \int_D K_\alpha(z, w) f(w) \Delta_{\alpha - \frac{m}{p}}(3w) \, dV(w), \quad z \in D.
\]

The following theorem obtained in [25] characterizes the topological dual space of the Bergman space \( A_p^\nu \) for some values of \( p \) and \( \nu \) for which the Bergman projection is not necessarily bounded.

Theorem 3.27. Let \( \nu > \frac{m}{p} - 1 \) be real, \( 1 < p < q_{\nu} \). If \( \mu \) is a sufficiently large real number so that \( \mu > \frac{m}{p} - 1 \) and \( 1 < p' < q_{\nu} \), then the topological dual space \( (A_p^\nu)^* \) of the Bergman space \( A_p^\nu \) identifies with \( A_{p'}^\mu \) under the integral pairing

\[
\langle f, g \rangle_{\alpha} = \int_D f(w) \overline{g(w)} \Delta_{\alpha - \frac{m}{p}}(3w) \, dV(w)
\]

where \( \alpha = \frac{m}{p} + \frac{m}{p'}, \frac{1}{p} + \frac{1}{p'} = 1 \).

3.5. Atomic decomposition. The following general atomic decomposition of functions in weighted Bergman space is from [21].

Theorem 3.28. Let \( 1 < p < \infty \) and let \( \mu, \nu > \frac{m}{p} - 1 \) satisfying \( \nu + (\mu - \nu)p' > \frac{m}{p} - 1 \). Assume that the operator \( P_{\mu} \) is bounded on \( L_p^\nu(D) \) and let \( \{z_j\}_{j \in \mathbb{N}} \) be a \( \delta \)-lattice in \( D \). Then the following assertions hold.

(i) For every complex sequence \( \{\lambda_j\}_{j \in \mathbb{N}} \) in \( l_p^\nu \), the series \( \sum_j \lambda_j K_\mu(z, z_j) \Delta_{\mu + \frac{m}{p}}(y_j) \) is convergent in \( A_p^\nu(D) \). Moreover, its sum \( f \) satisfies the inequality

\[
\|f\|_{p,\nu} \leq C_{\delta} \|\{\lambda_j\}\|_{l_p^\nu},
\]

where \( C_{\delta} \) is a positive constant.

(ii) For \( \delta \) small enough, every function \( f \in A_p^\nu(D) \) may be written as

\[
f(z) = \sum_j \lambda_j K_\mu(z, z_j) \Delta_{\mu + \frac{m}{p}}(y_j)
\]

with

\[
\|\{\lambda_j\}\|_{l_p^\nu} \leq C_{\delta} \|f\|_{p,\nu}
\]

where \( C_{\delta} \) is a positive constant.
3.6. **Properties of** $B^p_\nu(D)$, **image of the Bergman operator.** The notion of the Bergman projection has been extended in [6] allowing to see $B^p_\nu(D)$ as a projection of $L^p_\nu(D)$ in some sense. More precisely, for $1 \leq p < \infty$ and $\nu \in \mathbb{R}$ and $m$ large enough, the Bergman projection $P^{(m)}_\nu(f)$ of $f \in L^p_\nu(D)$ is defined as the equivalent class (in $H_m$) of all holomorphic solutions of

$$\Box^m F = C_{\nu,m} \int_D K_{\nu+m}(\cdot, w)f(w)dV_\nu(w).$$

The constant $C_{\nu,m}$ is as in (3.9). One easily observes that for any $F$ in the equivalent class $P^{(m)}_\nu(f)$,

$$\Delta^m \Box^m F = C_{\nu,m}T_{\nu,m}f$$

and consequently, $P^{(m)}_\nu$ is well defined and bounded from $L^p_\nu(D)$ to $B^p_\nu(D)$ if and only if $T_{\nu,m}$ is bounded in $L^p_\nu(D)$. It follows in particular with the help of Proposition 3.11 that every $F \in B^p_\nu(D)$ satisfies

$$\Box^m F = C_{\nu,m} \int_D K_{\nu+m}(\cdot, w)\Delta^m(\Im w)\Box^m F(w)dV_\nu(w)$$

whenever $m$ is sufficiently large.

3.7. **Korányi’s Lemma and averaging of a measure.** Let $\mu$ be a positive measure on $\mathcal{D}$. For $z \in \mathcal{D}$ and $\delta \in (0, 1)$, we define the average of the positive measure $\mu$ at $z$ by

$$\hat{\mu}_\delta(z) = \frac{\mu(B_\delta(z))}{V_\nu(B_\delta(z))}.$$ 

The following is also needed here.

**Lemma 3.32.** Let $1 \leq p \leq \infty$, $\nu \in \mathbb{R}$, $\beta, \delta \in (0, 1)$. For $z \in \mathcal{D}$ we define the average of the positive measure $\mu$ at $z$ by

$$\hat{\mu}_\delta(z) = \frac{\mu(B_\delta(z))}{V_\nu(B_\delta(z))}.$$ 

Let $\{z_j\}_{j \in \mathbb{N}}$ be a $\delta$-lattice in $\mathcal{D}$. Then the following assertions are equivalent.

(i) $\hat{\mu}_\beta \in L^p_\nu(D)$.
(ii) $\{\hat{\mu}_\delta(z_j)\}_{j \in \mathbb{N}} \in l^p_\nu$.

For $\nu > \frac{n}{r} - 1$ and $w \in \mathcal{D}$, the normalized reproducing kernel is

$$k_\nu(\cdot, w) = \frac{K_\nu(\cdot, w)}{\|K_\nu(\cdot, w)\|_{2,\nu}} = \Delta^{-\frac{\nu}{r}} \left(\frac{\cdot - \bar{w}}{i}\right) \Delta^{\frac{1}{2}\left(\nu + \frac{n}{r}\right)}(\Im w).$$

We have the following result known as Korányi’s lemma.

**Lemma 3.34.** [8 Theorem 1.1] For every $\delta > 0$, there is a constant $C_\delta > 0$ such that

$$\left|\frac{K(\zeta, z)}{K(\zeta, w)} - 1\right| \leq C_\delta d(z, w)$$

for all $\zeta, z, w \in \mathcal{D}$, with $d(z, w) \leq \delta$.

The following corollary is a straightforward consequence of [4 Corollary 3.4] and Lemma 3.34.
Corollary 3.35. Let \( \nu > \frac{n}{r} - 1 \), \( \delta > 0 \) and \( z, w \in D \). There is a positive constant \( C_\delta \) such that for all \( z \in B_\delta(w) \),
\[
V_\nu(B_\delta(w))|k_\nu(z, w)|^2 \leq C_\delta.
\]
If \( \delta \) is sufficiently small, then there is \( C > 0 \) such that for all \( z \in B_\delta(w) \),
\[
V_\nu(B_\delta(w))|k_\nu(z, w)|^2 \geq (1 - C\delta).
\]

3.8. Carleson embeddings. We start by recalling the following from [21, Theorem 3.5].

Theorem 3.36. Suppose that \( \mu \) is a positive measure on \( D \), \( 1 \leq p < \infty \), \( \nu > \frac{n}{r} - 1 \) and \( \lambda \geq 1 \). Then there exists a constant \( C > 0 \) such that
\[
\int_D |f(z)|^{p\lambda}d\mu(z) \leq C\|f\|_{p, \lambda}^{p\lambda}, \quad f \in A_p^\nu(D)
\]
if and only if
\[
\mu(B_\delta(z)) \leq C'\Delta^{\lambda(\nu + \frac{n}{r})}(\Im z), \quad z \in D,
\]
for some \( C' > 0 \) independent of \( z \), and for some \( \delta \in (0, 1) \).

Note that condition (3.38) is still sufficient for (3.37) to hold in the case \( 0 < p \leq 1 \). The following embedding with loss was also obtained in [21, Theorem 3.8].

Theorem 3.39. Suppose that \( \mu \) is a positive measure on \( D \), \( \nu > \frac{n}{r} - 1 \) and \( 0 < \lambda < 1 \). Then the following assertions hold.

(i) Let \( 0 < p < \infty \). If the function
\[
D \ni z \mapsto \frac{\mu(B_\delta(z))}{\Delta^{\nu + \frac{n}{r}}(\Im z)}
\]
belongs to \( L^s_\nu(D) \), \( s = \frac{1}{\lambda} \) for some \( \delta \in (0, 1) \), then there exists a constant \( C > 0 \) such that
\[
\int_D |f(z)|^{p\lambda}d\mu(z) \leq C\|f\|_{p, \lambda}^{p\lambda}, \quad f \in A_p^\nu(D).
\]

(ii) Let \( 1 \leq p < \infty \). If the Bergman projection \( P_\alpha \) is bounded on \( L^p_\nu(D) \) and (3.40) holds, then the function
\[
D \ni z \mapsto \frac{\mu(B_\delta(z))}{\Delta^{\nu + \frac{n}{r}}(\Im z)}
\]
belongs to \( L^s_\nu(D) \), \( s = \frac{1}{1-\lambda} \) for some \( \delta \in (0, 1) \).

Any measure satisfying (3.37) or (3.40) is called \((\lambda, \nu)\)-Carleson measure.

4. Toeplitz operators from \( A_p^\nu(D) \) to \( B_\beta^\nu(D) \).

In this section we provide criteria for the boundedness of the Toeplitz operators from a weighted Bergman space to an analytic Besov space. In particular, we prove here Theorem 2.6 and Theorem 2.8.
Proof of Theorem 2.6. Let us start with the sufficiency part. Put \( \beta' = \beta + (\nu - \beta)q' > \frac{a}{p} - 1 \). We observe that by Lemma 3.21, \( A_{\beta'}^q(D) \) is the dual space of \( \mathbb{B}_\beta^q \) under the duality pairing

\[
(f, g)_{\nu,m} = \int_D f(z) \Delta^m(mz) \text{B}_{\mu} g(z) dV(z)
\]

for \( m \) large enough.

We also observe the condition \( \nu > \frac{a}{p} - 1 + \frac{\beta - \frac{a}{p} + 1}{q} - \frac{\alpha - \frac{a}{p} + 1}{p} \) provides that \( \gamma > \frac{a}{p} - 1 \), and that as \( \lambda \geq 1 \), so by Theorem 3.36, for every \( t \geq 1 \), there exists a constant \( C > 0 \) such that

\[
\int_D |h(z)|^{t\lambda} d\mu(z) \leq C\|h\|_{\frac{\lambda}{t\gamma}}^{t\lambda}
\]

for all \( h \in A_\gamma^t(D) \). Taking \( t\lambda = 1 \), this is, in particular, equivalent to saying that there is a constant \( C > 0 \) such that for any \( h \in L_\gamma^+ (D) \),

\[
(4.41) \quad \int_D |h(z)| d\mu(z) \leq C\|h\|_{\frac{1}{\gamma}}.
\]

Let \( f \in A_\gamma^p(D) \) and \( g \in A_{\beta'}^q(D) \). Using Fubini’s Theorem and reproducing formula for weighted Bergman space, we obtain that

\[
(4.42) \quad \langle g, T_{\mu}^\nu f \rangle_{\nu,m} = \int_D \overline{f(z)} g(z) d\mu(z).
\]

Also \( fg \in L_\gamma^+ (D) \). As a matter of fact, by Hölder’s inequality, we have

\[
\|fg\|_{\frac{1}{\gamma}} = \left( \int_D |f(z)|^{1/\lambda} |g(z)|^{1/\lambda} \Delta^{\frac{\lambda}{\lambda}} \frac{dV(z)}{\Delta^{\frac{\lambda}{\lambda}}(mz)} \right)^{\lambda} \leq \left( \int_D |f(z)|^p dV_\alpha(z) \right)^{1/p} \left( \int_D |g(z)|^{q'} \Delta^{\frac{q'}{\lambda}} \frac{dV(z)}{\Delta^{\frac{q'}{\lambda}}(mz)} \right)^{1/q'} \leq \|f\|_{\rho,\alpha} \|g\|_{q',\beta'}.
\]

Therefore, from (4.42) and (4.41), we get

\[
|\langle g, T_{\mu}^\nu f \rangle_{\nu,m}| \leq \int_D |f(z)g(z)| d\mu(z) \leq C\|fg\|_{\frac{1}{\gamma}} \leq \|f\|_{\rho,\alpha} \|g\|_{q',\beta'}.
\]

Thus, \( T_{\mu}^\nu \) is bounded from \( A_\alpha^p(D) \) to \( \mathbb{B}_\beta^q(D) \) whenever (b) holds.

Next, suppose that \( T_{\mu}^\nu \) extends as a bounded operator from \( A_\alpha^p(D) \) to \( \mathbb{B}_\beta^q(D) \). For \( a \in D \) given, we define

\[
f_a(z) = K_{\nu+m}(z,a)
\]

where \( m \) is an integer large enough. From Lemma 3.17, we obtain that \( f_a \) belongs to \( A_\alpha^p(D) \) with

\[
(4.43) \quad \|f_a\|_{\rho,\alpha} = C_{\rho,m,\nu,a} \Delta^{-(m+\nu+\frac{a}{p})+(\alpha+\frac{a}{p})\frac{1}{\lambda}}(m\alpha).
\]
We have
\[
T^\nu_\mu f_a(z) = \int_D f_a(w) K_\nu(z, w) d\mu(w)
\]
\[
= \int_D K_{\nu+m}(w, a) K_\nu(z, w) d\mu(w)
\]
hence
\[
\Box^m T^\nu_\mu f_a(z) = C_m \int_D |K_{\nu+m}(z, w)|^2 d\mu(z).
\]
It follows in particular, using Corollary 3.35 that for any \(\delta \in (0, 1)\) and any \(z \in D\),
\[
(4.44) \quad \Box^m T^\nu_\mu f_a(z) \geq C_m \frac{\mu(B_\delta(z))}{\Delta^{2(m+\nu+\frac{1}{p})(3mz)}}.
\]
On the other hand, we have from our hypothesis that \(\Box^m T^\nu_\mu f_a \in A^\eta_{\beta+mq}(D)\), thus from the pointwise estimate (3.7) and (4.43) that
\[
|\Box^m T^\nu_\mu f_a(z)| \leq C \Delta^{-\beta(m+mq)} \frac{1}{(3mz)} \Box^m T^\nu_\mu f_a \| q, \beta+mq
\]
\[
\leq C \Delta^{-\beta(m+mq)} \frac{1}{(3mz)} \Box^\nu_\mu \| f_a \| p, \alpha
\]
\[
\leq C \| T^\nu_\mu \| \Delta^{-\nu+2m+\frac{1}{p}} \frac{\alpha+\beta}{\alpha+\beta+\frac{2}{p}+\frac{1}{q}} (3mz).
\]
Combining the latter with (4.44) we obtain that
\[
\frac{\mu(B_\delta(z))}{\Delta^{2(m+\nu+\frac{1}{p})(3mz)}} \leq C \| T^\nu_\mu \| \Delta^{-\nu+2m+\frac{1}{p}} \frac{\alpha+\beta}{\alpha+\beta+\frac{2}{p}+\frac{1}{q}} (3mz)
\]
that is
\[
\mu(B_\delta(z)) \leq C \| T^\nu_\mu \| \Delta^{\nu+2m+\frac{1}{p}} \frac{\alpha+\beta}{\alpha+\beta+\frac{2}{p}+\frac{1}{q}} (3mz)
\]
or equivalently
\[
\mu(B_\delta(z)) \leq C \| T^\nu_\mu \| \Delta^{\nu+2m+\frac{1}{p}} \frac{\alpha+\beta}{\alpha+\beta+\frac{2}{p}+\frac{1}{q}} (3mz)
\]
The proof is complete. \(\Box\)

The following can be proved the same using the duality result of Theorem 3.27.

**Theorem 4.45.** Let \(1 < p \leq q < q_0\), \(\alpha, \beta, \nu > \frac{n}{p} + 1\) with \(\nu > \frac{n}{p} + 1 + \frac{\beta - \frac{n+1}{q}}{p}\). Define the numbers \(\lambda = 1 + \frac{1}{p} - \frac{1}{q}\) and \(\lambda_{\gamma} = \nu + \frac{\alpha}{p} - \frac{\beta}{q}\). Assume that \(1 < q' < q_0\) where \(\beta' = \beta + (\nu - \beta)q' > \frac{n}{p} - 1\), then the following assertions are equivalent.

(a) The operator \(T^\nu_\mu\) extends as a bounded operator from \(A^\alpha_{\alpha}(D)\) to \(\mathcal{B}^\gamma_{\beta}(D)\).

(b) There is a constant \(C > 0\) such that for any \(\delta \in (0, 1)\) and any \(z \in D\).

\[
(4.46) \quad \mu(B_\delta(z)) \leq C \Delta^{\lambda+\gamma+\frac{1}{p}} (3mz)
\]

We now prove the case of estimations with loss.

**Proof of Theorem 4.45.** That \((b) \Rightarrow (a)\) follows as in the proof of Theorem 3.39 using Theorem 3.39. Let us prove that \((a) \Rightarrow (b)\). We start by recalling that the Rademacher functions \(r_j\) are given by \(r_j(t) = r_0(2^j t)\), for \(j \geq 1\) and \(r_0\) is defined as follows
\[
r_0(t) = \begin{cases} 
1 & \text{if } 0 \leq t - |t| < 1/2 \\
-1 & \text{if } 1/2 \leq t - |t| < 1 
\end{cases}
\]
$|t|$ is the smallest integer $k$ such $k \leq t < k + 1$. We have the following.

**Lemma 4.47** (Kinchine’s inequality). For $0 < p < \infty$ there exist constants $0 < L_p \leq M_p < \infty$ such that, for all natural numbers $m$ and all complex numbers $c_1, c_2, \ldots, c_m$, we have

$$L_p \left( \sum_{j=1}^{m} |c_j|^2 \right)^{p/2} \leq \int_{0}^{1} \left| \sum_{j=1}^{m} c_j \varphi_j(t) \right|^p \, dt \leq M_p \left( \sum_{j=1}^{m} |c_j|^2 \right)^{p/2}.$$

Thus if $r_j(t)$ is a sequence of Rademacher functions and $\{\lambda_j\} \in l^p_\alpha$, the series

$$\sum_j \lambda_j K_{\nu+m}(z, z_j) \Delta^{\nu+m+n/r}(\Im z_j)$$

is convergent in $A^p_\alpha(D)$ and its sum $f$ satisfies

$$\|f\|_{p,\alpha} \leq C \left( \sum_j |\lambda_j|^p \Delta^{\alpha+n/r}(\Im z_j) \right)^{1/p}.$$

Thus if $r_j(t)$ is a sequence of Rademacher functions and $\{\lambda_j\} \in l^p_\alpha$,

$$f_t(z) = \sum_j \lambda_j r_j(t) K_{\nu+m}(z, z_j) \Delta^{\nu+m+n/r}(\Im z_j)$$

also converges in $A^p_\alpha(D)$ with $\|f_t\|_{p,\alpha} \leq C \|\{\lambda_j\}\|_{l^p_\alpha}$.

As $T_\mu^\nu$ extends as a bounded operator from $A^p_\alpha(D)$ to $B^q_\beta(D)$, we obtain for $m$ integer large enough (actually, $P_{\nu+m}$ is bounded on $L^p_{\alpha}(D)$),

$$\|\Box^m T_\mu^\nu f_t\|_{q,\beta+mq} = \left( \int_D \left| \sum_j \lambda_j r_j(t) \Box^m T_\mu^\nu K_{\nu+m}(z, z_j) \Delta^{\nu+m+n/r}(\Im z_j) \right|^q \, dV_{\beta+mq}(z) \right)^{1/q} \leq \|T_\mu^\nu\|_q \|f_t\|_{p,\alpha} \leq \|T_\mu^\nu\|_q \|\{\lambda_j\}\|_{l^p_\alpha}.$$  

That is

$$(4.48) \int_D \left| \sum_j \lambda_j r_j(t) \Box^m T_\mu^\nu K_{\nu+m}(z, z_j) \Delta^{\nu+m+n/r}(\Im z_j) \right|^q \, dV_{\beta+mq}(z) \leq \|T_\mu^\nu\|_q \|\{\lambda_j\}\|_{l^p_\alpha}^q.$$  

By Kinchine’s inequality, we have

$$\int_0^1 \left| \sum_j \lambda_j r_j(t) \Box^m T_\mu^\nu K_{\nu+m}(z, z_j) \Delta^{\nu+m+n/r}(\Im z_j) \right|^q \, dt \geq L_q \left( \sum_j |\lambda_j|^2 \|\Box^m T_\mu^\nu K_{\nu+m}(z, z_j)\|^2 \Delta^{2(\nu+m+n/r)}(\Im z_j) \right)^{q/2}.$$
Integrating both sides of (4.48) from 0 to 1 with respect to \( dt \), we obtain using Fubini’s Theorem and the last inequality that
\[
\int_D \left( \sum_j |\lambda_j|^2 |\Box^m T^\nu_{\mu} K_{\nu+m}(z, z_j)|^2 \Delta^{2(\nu+m+n/r)}(\Im m z_j) \right)^{q/2} dV_{\beta+mq}(z) \lesssim \|T^\nu_{\mu}\|^q \|\{\lambda_j\}\|_{l^q_m}^q.
\]
Observe that for a sequence \( \{a_j\} \) of complex numbers and for \( \frac{2}{q} < 1 \),
\[
\sum_j |a_j|^q \chi_{B_j}(z) \leq \left( \sum_j |a_j|^2 \chi_{B_j}(z) \right)^{\frac{q}{2}}.
\]
Thus using the latter observation, we obtain from (4.49),
\[
\sum_j |\lambda_j|^q \Delta^{q(\nu+m+n/r)}(\Im m z_j) \int_{B_j} |\Box^m T^\nu_{\mu} K_{\nu+m}(z, z_j)|^q dV_{\beta+mq}
\[
= \int_D \sum_j |\lambda_j|^q \Delta^{q(\nu+m+n/r)}(\Im m z_j) |\Box^m T^\nu_{\mu} K_{\nu+m}(z, z_j)|^q \chi_{B_j}(z) dV_{\beta+mq}
\[
\lesssim \int_D \left( \sum_j |\lambda_j|^2 \Delta^{2(\nu+m+n/r)}(\Im m z_j) |\Box^m T^\nu_{\mu} K_{\nu+m}(z, z_j)|^2 \right)^{q/2} dV_{\beta+mq}
\[
\lesssim \|T^\nu_{\mu}\|^q \|\{\lambda_j\}\|_{l^q_m}^q.
\]
But from Lemma 3.4 we have
\[
|\Box^m T^\nu_{\mu} K_{\nu+m}(z, z_j)|^q \lesssim \frac{1}{\Delta^{mq+\beta+n/r}(\Im m z_j)} \int_{B_j} |\Box^m T^\nu_{\mu} K_{\nu+m}(z, z_j)|^q dV_{\beta+mq}(z).
\]
Thus
\[
\sum_j |\lambda_j|^q \Delta^{q(\nu+m+n/r)+mq+\beta+n/r}(\Im m z_j) |\Box^m T^\nu_{\mu} K_{\nu+m}(z, z_j)|^q \lesssim \|T^\nu_{\mu}\|^q \|\{\lambda_j\}\|_{l^q_m}^q.
\]
Now note that
\[
\Box^m T^\nu_{\mu} K_{\nu+m}(z, z_j) = C_m \int_D K_{\nu+m}(w, z_j) K_{\nu+m}(z, w) d\mu(w).
\]
Hence
\[
\frac{\mu(B_j)}{\Delta^{2(\nu+m+n/r)}(\Im m z_j)} \lesssim \Box^m T^\nu_{\mu} K_{\nu+m}(z, z_j).
\]
Combining (4.50) and (4.51) we obtain
\[
\sum_j |\lambda_j|^q \left( \frac{\mu(B_j)}{\Delta^{\lambda(\gamma+n/r)}(\Im m z_j)} \right)^q \Delta^{\frac{\alpha}{\gamma}(\gamma+n/r)}(\Im m z_j) \lesssim \|T^\nu_{\mu}\|^q \|\{\lambda_j\}\|_{l^q_m}^q.
\]
Thus as the sequence \( \{ |\lambda_j| \Delta^{\frac{\alpha}{r}}(\Delta w m z_j) \} \) belongs to \( L^{p/q} \), it follows by duality that the sequence \( \left\{ \left( \frac{\mu(B_j)}{\Delta^{\frac{\alpha}{r}}(\Delta w m z_j)} \right)^q \right\} \) belongs to \( L^{p/p-q} \) which is the dual of \( L^{p/q} \) by the sum pairing

\[
\{a_j\}, \{b_j\}_\nu := \sum_j a_j b_j.
\]

Hence

\[
\sum_j \left( \frac{\mu(B_j)}{\Delta^{\frac{\alpha}{r}}(\Delta w m z_j)} \right)^{\frac{\nu}{q}} \lesssim \|T^\nu\|_L \quad \text{i.e.} \quad \sum_j \left( \frac{\mu(B_j)}{\Delta^{\frac{\alpha}{r}}(\Delta w m z_j)} \right)^{\frac{1}{\nu}} \lesssim \|T^\nu\|_L
\]

Thus, \( \{\hat{\mu}_\delta(z_j)\} \in L^{\frac{1}{\nu}} \) and from Lemma 3.32 this means that (b) holds. The proof is complete. \( \square \)

We obtain in the same way the following.

**THEOREM 4.52.** Let \( 1 < q < q_\alpha \), \( q < p < \infty \), \( \alpha, \beta, \nu > \frac{\nu}{r} - 1 \) with \( \nu > \frac{\nu}{r} - 1 + \frac{\beta - \alpha + 1}{q} \). Define the numbers \( \lambda = 1 + \frac{\nu}{q} - \frac{1}{q} \) and \( \lambda' = \nu + \frac{\nu}{q} - \frac{\beta}{\nu} \). Assume \( \nu \) is large enough so that \( P_\nu \) is bounded on \( L^p(D) \) and \( 1 < q' < q_\beta \) where \( \beta' = \beta + (\nu - \beta)q' > \frac{\nu}{r} - 1 \). Then the following assertions are equivalent.

(a) The operator \( T^\nu_{\beta}(D) \) extends as a bounded operator from \( A^p_\alpha(D) \) to \( B^q_\beta(D) \).

(b) For any \( \delta \in (0, 1) \), the function

\[
D \ni z \mapsto \frac{\mu(B_\delta(z))}{\Delta^{\frac{\alpha}{r}}(\Delta w m z)}
\]

belongs to \( L^{\frac{1}{\nu}}(D) \).

5. **Hankel operators from \( A^p_\alpha(D) \) to \( B^q_\beta(D) \)**

In this section, we provide criteria of boundedness of (small) Hankel operators from weighted Bergman spaces to weighted Besov spaces.

5.1. **Boundedness of \( h^\nu_\delta : A^p_\alpha(D) \to B^q_\beta(D) \), \( p > 2 \).** We would like to start this section with a general result involving a change of weights. To avoid any ambiguity, we recall that we write \( h^{(\nu)}_\delta f = P_\nu(bT) \). The superscript may be removed when the result involves the same weight. Let us also recall the notation

\[
q_\nu = \frac{\nu + \frac{\nu}{r} - 1}{\frac{\nu}{r} - 1}.
\]

We have the following result.

**THEOREM 5.53.** Let \( \nu, \mu > \frac{\nu}{r} - 1 \) and \( p > \max \left\{ \frac{\nu + \frac{\nu}{r} - 1}{\mu}, \frac{\nu + \frac{\nu}{r} + 1}{\mu - \frac{\nu}{r} + 1} \right\} \). Then the Hankel operator \( h^{(\nu)}_\delta \) is bounded from \( A^p_\alpha(D) \) into \( B^q_\beta(D) \) if \( b = P_\nu g \) for some \( g \in L^\infty(D) \) for which \( P_\nu g \) makes sense.
**Proof:** We first observe that if \( b = P_\nu g \) with \( g \in L^\infty(\mathcal D) \), then for \( m > \frac{\nu}{\nu + 1} - 1 \), as
\[
\Delta^m \Box^m b = T_{\nu, m} g,
\]
we have from Lemma 3.21 that
\[
\sup_{z \in \hat{\mathcal D}} \Delta^m(\Im z) \Box^m b(z) < \infty.
\]
Now, we prove the sufficient condition. We suppose that \( b = P_\nu g \) with \( g \in L^\infty(\mathcal D) \); recall that \( h^\mu_b(f) \) is a representative of a class in \( \mathbb B^p_\nu(\mathcal D) \) is equivalent to saying that \( \Delta^m \Box^m h^\mu_b(f) \in L^p_\nu(\mathcal D) \) with \( m \) large enough. Let us put \( k = f \Delta^m \Box^m b \). Observe that for \( f \in A^p_\nu(\mathcal D) \), \( k \in L^p_\nu(\mathcal D) \) as \( \Delta^m \Box^m b \) is bounded. Next, using the formula (3.16), one easily sees that
\[
\Delta^m \Box^m h^\mu_b(f) = T_{\mu, m} k.
\]
Since \( m \) is taken large enough and \( p > \max\{\frac{\nu + \frac{\mu}{\nu + 1} - 1}{\mu - \frac{\nu + 1}{\nu + 1}}\} \), Lemma 3.20 gives that
\[
||\Delta^m \Box^m h^\mu_b(f)||_{L^p_\nu} = ||T_{\mu, m} k||_{L^p_\nu} \lesssim ||k||_{L^p_\nu} \lesssim ||g||_{L^\infty} ||f||_{A^p_\nu}.
\]
The proof is complete. \( \square \)

We now prove Theorem 2.9.

**Proof of Theorem 2.9.** The sufficiency is a special case of the previous theorem corresponding to \( \mu = \nu \). Conversely, if \( h_b \) is bounded, then for any \( f \in A^p_\nu(\mathcal D) \) and any \( g \in A^p_\nu(\mathcal D) \), using the reproduction formula (3.16), we have
\[
||f||_{A^p_\nu} \lesssim ||f||_{A^p_\nu} ||g||_{A^p_\nu}.
\]
We take \( f(z) = f_w(z) = \Delta^{-\frac{\nu}{\nu + 1}}(\frac{z}{\nu + 1}) \) and \( g(z) = g_w(z) = \Delta^{-\frac{\nu}{\nu + 1}}(\frac{z}{\nu + 1}) \) with \( m \) large enough. Following Lemma 3.17, we have
\[
||f||_{A^p_\nu} = C_{\nu, m, p} \Delta^{-\frac{\nu}{\nu + 1} + \frac{\nu}{\nu + 1} + \frac{2}{\nu + 1}}(\Im m w),
\]
and
\[
||g||_{A^p_\nu} = C_{\nu, m, p} \Delta^{-\frac{\nu}{\nu + 1} + \frac{\nu}{\nu + 1} + \frac{2}{
u + 1}}(\Im m w).
\]
Now, since by the reproduction formula (3.10), we have \( P_{\nu + m} g = g \), replacing \( f \) and \( g \) in (5.51), we obtain
\[
\Delta^m(\Im m w) \left| \int_\mathcal D K_{\nu + m}(w, z) \Box^m b(z) dV_{\nu + m}(z) \right| \leq C.
\]
Taking \( \ell = m \) in (3.12), this is equivalent to
\[
\Delta^m(\Im m w) \Box^m b(w) = \Delta^m(\Im m w) |P_{\nu + m}(\Box^m b)(w)| \leq C.
\]
Let \( h = \Delta^m \Box^m b \), we have just obtained that \( h \in L^\infty(\mathcal D) \). Moreover, as \( b \in A^p_\nu(\mathcal D) \), \( P_\nu h \) makes sense and we have from the reproducing formula, taking \( \ell = 0 \) in (3.12) that
\[
P_\nu h = \int_\mathcal D K_\nu(w, z) \Delta^m(\Im m w) \Box^m b(w) dV_\nu(w) = b.
\]
The proof is complete. \( \square \)

As a consequence of the above result, we have the following corollary.
Corollary 5.55. Let $1 < p < \infty$ and $\nu > \frac{n}{r} - 1$. If $P_{\nu}$ is bounded on $L^p(V)$ then the Hankel operator $h_b$ extends as a bounded operator on $A^p_\alpha(V)$ if and only if $b = P_{\nu}g$ for some $g \in L^\infty(V)$ for which $P_{\nu}g$ makes sense.

When $p = \infty$, we can prove in the same way the following result.

Theorem 5.56. The Hankel operator $h_b$ is bounded from $H^\infty(V)$ into $B$ if and only if $b = P_{\nu}g$ for some $g \in L^\infty(V)$ for which $P_{\nu}g$ makes sense.

5.2. Boundedness of $h_b : A^p_\alpha(V) \rightarrow B^p_\beta(V)$, $1 \leq p < \infty$. Let us first recall the following pointwise estimate from [13].

Lemma 5.57. Let $1 \leq p < \infty$ and $\alpha, \beta > \frac{n}{q} - 1$. Assume that $\frac{1}{p}(\alpha + \frac{n}{q}) = \frac{1}{q}(\beta + \frac{n}{p})$. Then there exists a positive constant $C$ such that for any $f \in A^p_\alpha(V)$,

$$\int_V |f(z)|^q D^{\beta - \frac{n}{q}}(\Delta m z) dV(z) \leq C \|f\|_{p, \alpha}^q.$$  

Proof: Let us recall that there is a constant $C > 0$ such that for any $f \in A^p_\alpha(V)$ the following pointwise estimate holds:

$$(5.58) \quad |f(z)| \leq C D^{\beta - \frac{n}{q}}(\Delta m z) \|f\|_{p, \alpha}, \quad \text{for all } z \in V.$$  

It follows easily that

$$I := \int_V |f(z)|^q D^{\beta - \frac{n}{q}}(\Delta m z) dV(z)$$  

$$= \int_V |f(z)|^q |f(z)|^q D^{\beta - \frac{n}{q}}(\Delta m z) dV(z)$$  

$$\leq C \|f\|_{p, \alpha}^q \int_V |f(z)|^q dV(z) = C \|f\|_{p, \alpha}^q.$$

We can now prove Theorem 5.56.

Proof of Theorem 5.56. We first prove the sufficiency. Let $f \in A^p_\alpha(V)$. From the integration by parts formula, we obtain

$$\Box^m h^\nu_b f(z) = C_{\nu, m} \int_V K_{\nu + m}(z, w) b(w) \overline{f(w)} dV(w)$$  

$$= C_{\nu, m} \int_V K_{\nu + m}(z, w) \Box^m b(w) \overline{f(w)} dV(w)$$  

$$= P_{\nu + m} T_b f(z),$$

where the operator $T_b$ is defined by

$$T_b f(z) = C_{\nu, m} \overline{f(z)} \Box^m b(z), \quad z \in V.$$  

Let us prove that $T_b$ is bounded from $A^p_\alpha(V)$ to $L^p_{\beta + \nu q}(V)$. Observe that if $\vartheta = -\frac{1}{q}(\mu + \frac{n}{p}) + \beta$, then $\frac{1}{p}(\vartheta + \frac{n}{p}) = \frac{1}{p}(\alpha + \frac{n}{p})$. Using the embedding of $A^p_\alpha(V)$ into $L^p_{\beta + \nu q}(V)$, we easily...
obtain
\[
\int_{\mathcal{D}} |T_b f(z)|^q dV_{\beta+mq}(z) = C \int_{\mathcal{D}} |f(z)|^q |\bigtriangleup^m b(z)|^q dV_{\beta+mq}(z) \\
\leq C \int_{\mathcal{D}} |f(z)|^q \Delta^{-\frac{1}{q}(\mu+\frac{\beta}{q})}(\Im z) dV_{\beta}(z) \\
\leq C ||f||_{p,\alpha}.
\]

Now, since by Lemma 3.20, for $m$ large enough and for $q$ as in our assumptions, $P_{\nu+m}$ is bounded on $L^q_{\beta+mq}(\mathcal{D})$, we conclude that for any $f \in A^p_{\alpha}(\mathcal{D})$, $\bigtriangleup^m b f \in A^q_{\beta+mq}(\mathcal{D})$.

For the converse, we consider the two different situations: $1 \leq q \leq 2$ and $2 < q < \infty$.

**Case** $2 < q < \infty$. If $h^{(\nu)}_b$ is bounded then as we have seen in the previous section, there is a positive constant $C$ such that for any $f \in A^p_{\alpha}(\mathcal{D})$ and any $g \in A^q_{\beta'}(\mathcal{D})$ inequality (5.54) holds. We take again $f(z) = f_w(z) = \Delta^{-(\nu+\frac{q}{q})}(\frac{z-w}{i})$ and $g(z) = g_w(z) = \Delta^{-(\nu+\frac{q}{q})}(\frac{z-w}{i})$ with $m$ large enough. From Lemma 3.17 we have

\[
||f||_{p,\alpha} = C_{\nu,m,q} \Delta^{-(\nu+\frac{q}{q})} + (\alpha + \frac{\beta}{q})\frac{1}{q} (\Im m w),
\]

and

\[
||g||_{p',\beta'} = C_{\nu,m,q} \Delta^{-(\nu+\frac{q}{q})} + (\beta' + \frac{\beta}{q})\frac{1}{q} (\Im m w).
\]

Replacing $f$ and $g$ in (5.54), we obtain

\[
\Delta^{m-\frac{1}{q}(\alpha+\frac{\beta}{q})-\frac{1}{q}(\beta'+\frac{\beta}{q})+\nu+\frac{\beta}{q}}(\Im m w) \left| \int_{\mathcal{D}} K_{\nu+m}(w, z) \bigtriangleup^m b(z) dV_{\nu+m}(z) \right| \leq C.
\]

By (3.12), this is equivalent to

\[
\Delta^{m+\frac{1}{q}(\mu+\frac{\beta}{q})}(\Im m w) |\bigtriangleup^m b(w)| \leq C,
\]

for any $w \in \mathcal{D}$.

**Case** $1 \leq q \leq 2$. Note that in this case $\mathbb{B}^q_{\beta}(\mathcal{D}) = A^q_{\beta}(\mathcal{D})$. That $h^{(\nu)}_b$ is bounded, is equivalent to saying that there exists a constant $C > 0$ so that for any $f \in A^p_{\alpha}(\mathcal{D})$ and any representative $g$ of any class in $g \in A^q_{\beta'}(\mathcal{D})$,

\[(5.59) \quad |\langle h^{(\nu)}_b f, g \rangle| \leq C ||f||_{p,\alpha} ||g||_{q',\beta'}
\]
Now, taking $g(z) = g_\zeta(z) = K_\nu(z, \zeta)$ and $f(z) = f_\zeta(z) = \Delta^{-m}(\frac{z}{\zeta})$, we obtain
\[
\langle h_b^{(\nu)} f, g \rangle_\nu = \int_D \left( h_b^{(\nu)} f(z) \right) \overline{g(z)} dV_\nu(z)
\]
\[
= C_\nu \int_D K_\nu(\zeta, z) \left( h_b^{(\nu)} f(z) \right) dV_\nu(z)
\]
\[
= C_\nu P_\nu(h_b^{(\nu)} f)(\zeta)
\]
\[
= C_\nu h_b^{(\nu)} f(\zeta) = C_\nu \int_D K_\nu(\zeta, z) \square^m b(z) dV_{\nu+m}(z)
\]
\[
= C_\nu \int_D K_{\nu+m}(\zeta, z) \square^m b(z) dV_{\nu+m}(z)
\]
\[
= C_\nu \nu, m \square^m b(\zeta).
\]

Now, choosing $m$ large enough, we obtain from Lemma 5.17 that
\[
\|f\|_{p, \alpha} = C_{\nu, m, p} \Delta^{-m+(\alpha+\frac{\nu}{r}) \frac{1}{r}}(\Im \zeta)
\]
and since the conditions on $q$ insure that $g \in A_q^{\beta}$,
\[
\|g\|_{q, \beta} = C_{\nu, m, q} \Delta^{-(\nu+\frac{\nu}{r}) + \frac{1}{r} \left( \beta' + \frac{\nu}{r} \right)}(\Im \zeta).
\]

Taking all the above observations in (5.69), we obtain that
\[
\Delta^{m+(\nu+\frac{\nu}{r}) - \frac{1}{r} \left( \alpha + \frac{\nu}{r} \right) - \frac{1}{r} \left( \beta' + \frac{\nu}{r} \right)}(\Im \zeta) |\square^m b(\zeta)| \leq C < \infty, \quad \zeta \in \mathcal{D}.
\]

The proof is complete. \(\square\)

**Corollary 5.60.** Let $1 \leq p < q < \infty$ and $\nu > \frac{n}{r} - 1$. If $P_\nu$ is bounded on $L^p_0(D)$, then the Hankel operator $h_b$ extends into a bounded operator from $A^p_\alpha(D)$ to $A^q_\beta(D)$ if and only for some $m$ large,
\[
\Delta^{m+(\nu+\frac{\nu}{r}) - \frac{1}{r} \left( \alpha + \frac{\nu}{r} \right) - \frac{1}{r} \left( \beta' + \frac{\nu}{r} \right)}(\Im \zeta) |\square^m b(\zeta)| \leq C < \infty, \quad \zeta \in \mathcal{D}.
\]

### 5.3. Boundedness of $h_b^{(\nu)} : A^p_\alpha(D) \to B^q_\beta(D)$, $1 < q < p < \infty$. We begin with the case $p = \infty$.

**Theorem 5.61.** Let $2 \leq p < \infty$, $\nu > \frac{n}{r} - 1$, and $\alpha \in \mathbb{R}$. Assume that
\[
\max \left\{ \frac{\alpha + \frac{\nu}{r} - 1}{\nu}, \frac{\alpha - \frac{n}{r} + 1}{\nu - \frac{\nu}{r} + 1} \right\} < p.
\]

Then the Hankel operator $h_b^{(\nu)}$ extends into a bounded operator from $H^\infty(D)$ to $B^p_\alpha(D)$ if and only if $b = P_\alpha^{(m)}(f)$ for some $f \in L^p_\alpha(D)$ and $m$ a large enough integer.

**Proof.** First suppose that for $m$ large, $\Delta^m \square^m b \in L^p_\alpha(D)$. Then for any $f \in H^\infty(D)$ and any $g \in A^p_\alpha(D)$, $\alpha' = \alpha + (\nu - \alpha)p > \frac{n}{r} - 1$, we easily have
\[
|\langle h_b^{(\nu)} f, g \rangle_{\nu, m}| = \left| \int_D \overline{f(z)} g(z) \square^m b(z) dV_{\nu+m}(z) \right|
\]
\[
\leq \||\Delta^m \square^m b||_{p, \alpha}\|f\|_{\infty}\|g\|_{q', \alpha'} < \infty.
\]
Conversely, if $h_b^{(\nu)}$ is bounded then as before there exists $C > 0$ such that for any $f \in H^\infty(D)$ and any $g \in A^p_\alpha(D)$,

$$
|\langle h_b^{(\nu)}f, g \rangle_{\nu,m}| = \left| \int_D f(z)g(z)\Box^m b(z)dV_{\nu+m}(z) \right| \leq C||f||_\infty||g||_{p',\alpha'}.
$$

Taking $f(z) = 1$ for any $z \in D$, we obtain that there exists a constant $C > 0$ such that for any $g \in A^p_\alpha(D)$,

$$
\left| \int_D g(z)\Box^m b(z)dV_{\nu+m}(z) \right| \leq C||g||_{p',\alpha'};
$$

i.e.

$$
|\langle b, g \rangle_{\nu,m}| \leq C||g||_{p',\alpha'}.
$$

It follows from Lemma 3.24 that $b$ is a representative of a class in $B^p_\alpha(D) = (A^p_\alpha(D))^*$ with respect to the pairing $\langle \cdot, \cdot \rangle_{\nu,m}$. As the condition $p > \max\left\{ \frac{\alpha+\nu-1}{\nu}, \frac{\alpha-\nu+1}{\nu+1} \right\}$ implies that $P_{\nu+m}$ is bounded on $L^p_{\alpha+mp}(D)$, for $m$ large enough, the equality 3.31 allows us to conclude that $b = cP^{(m)}_\alpha(f)$ with $f = \Delta^m\Box^m b$. The proof is complete. \qed

We have in particular the following.

**Corollary 5.62.** Let $1 < p < \infty$ and $\nu > \frac{\mu}{r} - 1$. If $P_\nu$ is bounded on $L^p_{\alpha}(D)$, then the Hankel operator $h_b$ extends into a bounded operator from $H^\infty(D)$ to $A^p_\alpha(D)$ if and only if $b$ is in $A^p_\alpha(D)$.

In the same way, with the help of Theorem 3.27 we obtain the following.

**Theorem 5.63.** Let $\max\left\{ q'_\alpha, \frac{\alpha+\nu-1}{\nu}, \frac{\alpha-\nu+1}{\nu+1} \right\} < p < q_\alpha$, $\alpha, \nu > \frac{\mu}{r} - 1$. Then the Hankel operator $h_b^{(\nu)}$ extends into a bounded operator from $H^\infty(D)$ to $A^p_\alpha(D)$ if and only if $b \in A^p_\alpha(D)$.

The following endpoint case follows also the same using the duality result in Lemma 3.22.

**Theorem 5.64.** Let $\nu > \frac{\mu}{r} - 1$. Then the Hankel operator $h_b^{(\nu)}$ extends into a bounded operator from $H^\infty(D)$ to $A^p_\alpha(D)$ if and only if $b \in A^p_\alpha(D)$.

We next prove the following lemma.

**Lemma 5.65.** Let $2 \leq q < \infty$, $\alpha, \nu > \frac{\mu}{r} - 1$, $\mu \in \mathbb{R}$. Assume that $\frac{\mu-\nu+1}{\nu+1} < q < \infty$. For $m$ an integer such that $P_\gamma$ is bounded on $L^p_{\beta}$, $\gamma = \alpha+\nu+m+\frac{\mu}{r}$, $\beta = \mu+(\nu-\nu)q'$, $\frac{1}{q} + \frac{1}{q'} = 1$, consider the operator

$$
T^{m}_{\alpha,\beta}f(z) = \int_D K_{\alpha+\nu+m+\frac{\mu}{r}}(z,w)\Box^m f(w)dV_{\nu+m}(w)
$$

define for functions with compact support. Then if there exists a constant $C > 0$ such that for any $\delta \in (0,1)$ and any $\delta$-lattice $\{z_j\}_{j \in \mathbb{N}_0}$ of points in $D$,

$$
\sum_j \Delta^{q(\alpha+\nu+\frac{\mu}{r})+(\nu+m)}(\Im z_j)\Delta^{m}(\Im m z_j)T^{m}_{\alpha,\beta}f(z_j)|^q \leq C^q,
$$

then $f$ is a representative of a class $F \in B^q_\mu$ with $||F||_{B^q_\mu} \leq C$. 

PROOF: First note that the condition on the exponent \( q \) implies that \( \beta > \frac{n}{r} - 1 \), and that \( A^{q'}_\beta \) is the dual space of \( \mathbb{B}^q_\mu \) under the pairing

\[
\langle h, F \rangle_{\nu,m} = \int_D h(z) \overline{F(z)} dV_{\nu+m}, \quad h \in A^{q'}_\beta \quad \text{and} \quad F \in \mathbb{B}^q_\mu.
\]

As the projector \( P_\gamma \) (\( \gamma = \alpha + \nu + m + \frac{n}{r} \)) is bounded on \( L^q_\beta(D) \), by Theorem 3.28 any \( h \in A^{q'}_\beta(D) \) can be represented as

\[
h(z) = \sum_j \lambda_j \Delta^{\gamma + \frac{n}{r}}(\Im m z_j) K_\gamma(z, z_j)
\]

with \( \|\{\lambda_j\}\|_{q'} \lesssim \|h\|_{q', \beta} \). It follows that if \( F \) is a class represented by \( f \),

\[
\langle h, F \rangle_{\nu,m} = \int_D \sum_j \lambda_j \Delta^{\gamma + \frac{n}{r}}(\Im m z_j) K_\gamma(z, z_j) \overline{F(z)} dV_{\nu+m}(z) = \sum_j \lambda_j \Delta^{\alpha + \nu + m + 2\frac{n}{r}}(\Im m z_j) T^{m}_{\alpha, \nu} f(z_j).
\]

It follows from Hölder’s inequality that

\[
\|F\|_{\mathbb{B}^q_\mu} := \sup_{h \in A^{q'}_\beta, \|h\|_{q', \beta} \leq 1} |\langle h, F \rangle_{\nu,m}| \leq \sup_{h \in A^{q'}_\beta, \|h\|_{q', \beta} \leq 1} \|\{\lambda_j\}\|_{q'} \left( \sum_j \Delta^{q(\alpha + \frac{\mu}{\nu})}(\Im m z_j) |\Delta^{m}(\Im m z_j) T_{\alpha, \nu}^{m} f(z_j)|^q \Delta^{\mu + \frac{n}{r}}(\Im m z_j) \right)^{1/q} \lesssim C.
\]

The following can be proved the same way with the help of the duality in Theorem 3.27.

**Lemma 5.66.** Let \( \max \left\{ q'_{\beta}, \frac{\mu + \frac{n}{r} - 1}{\nu}, \frac{\mu - \frac{n}{r} + 1}{\nu} \right\} < q < q_\mu, \alpha, \mu, \nu > \frac{n}{r} - 1 \). Define \( \beta = \mu + (\nu - \mu)q' \), \( \frac{1}{q} + \frac{1}{q'} = 1 \), and consider the operator

\[
T_{\alpha, \nu} f(z) = \int_D K_{\alpha + \nu + \frac{n}{r}}(z, w) f(w) dV_\nu(w)
\]

define for functions with compact support. Then if there exists a constant \( C > 0 \) such that for any \( \delta \in (0, 1) \) and any \( \delta \)-lattice \( \{z_j\}_{j \in \mathbb{N}_0} \) of points in \( D \),

\[
\sum_j \Delta^{q(\alpha + \frac{\mu}{\nu}) + \mu + \frac{n}{r}}(\Im m z_j) |T_{\alpha, \nu} f(z_j)|^q \leq C^q,
\]

then \( f \in A^q_\mu \) with \( \|f\|_{A^q_\mu} \leq C \).

Note that the condition \( \max \left\{ 1, \frac{\mu + \frac{n}{r} - 1}{\nu}, \frac{\mu - \frac{n}{r} + 1}{\nu} \right\} < q < q_\mu \) provides that \( \beta > \frac{n}{r} - 1 \) and that \( P_\nu \) is bounded on \( A^q_\mu(D) \). Also as \( 1 < q' < q_\beta \), by Theorem 3.27 \( A^{q'}_\beta(D) \) is the dual of \( A^q_\mu(D) \) under the pairing \( \langle \cdot, \cdot \rangle_\nu \).

Let us know prove Theorem 2.11.
Proof of Theorem 2.11. We start with the sufficiency which is the harmless part. Let \( m \) be a positive integer such that Hardy inequality holds for \( q, \beta + mq \). Recall that \( d\lambda(z) = \Delta^{-2n/r}(3m z) dV(z) \). Then for any \( f \in A^p_\alpha(D) \) and any \( g \in A^q_\beta(D) \),

\[
|\langle h_b^{(\nu)} f, g \rangle_{\nu,m}| = \left| \int_D f(z) g(z) \Box^m b(z) dV_{\nu+m}(z) \right| \\
= \left| \int_D \left[ \Delta^\frac{1}{2} (\nu + \frac{\beta}{2}) (3m z) f(z) g(z) \right] \|\Delta^{m+\frac{1}{r} (\mu+\frac{\beta}{r})} (3m z) \Box^m b(z) \| d\lambda(z) \right| \\
\leq \|f\|_{s,\gamma} \|\Delta^m \Box^m b\|_{s',\mu} \|g\|_{p,\alpha} \|g\|_{q',\beta'} < \infty.
\]

Thus

\[
\|h_b^{(\nu)}\| := \sup_{\|f\|_{A^p_\alpha} \leq 1, \|g\|_{A^q_\beta} \leq 1} \|\langle h_b^{(\nu)} f, g \rangle_{\nu,m}\| \leq \|\Delta^m \Box^m b\|_{s',\mu}.
\]

Let us now move to the necessity. We observe that as \( P_\sigma \) is bounded on \( L^p_\alpha(D) \), we have from Theorem 3.28 that given a \( \delta \)-lattice \( \{z_j\}_{j \in \mathbb{N}_0}, \delta \in (0,1) \), for every complex sequence \( \{\lambda_j\} \in l^p_\alpha \), the series

\[
\sum_j \lambda_j K_\sigma(z, z_j) \Delta^{\sigma+n/r}(3m z_j)
\]

is convergent in \( A^p_\alpha(D) \) and it sum \( f \) satisfies

\[
\|f\|_{p,\alpha} \leq C \left( \sum_j |\lambda_j|^p \Delta^{\alpha+n/r}(3m z_j) \right)^{1/p}.
\]

Thus if \( r_j(t) \) is a sequence of Rademacher functions and \( \{\lambda_j\} \in l^p_\alpha \),

\[
f_t(z) = \sum_j \lambda_j r_j(t) K_\sigma(z, z_j) \Delta^{\sigma+n/r}(3m z_j)
\]

also converges in \( A^p_\alpha(D) \) with \( \|f_t\|_{p,\alpha} \leq C \|\{\lambda_j\}\|_{l^p_\alpha} \).

As \( h_b^{(\nu)} \) extends as a bounded operator from \( A^p_\alpha(D) \) to \( \mathbb{B}_\beta^q(D) \), we obtain for \( m \) integer large enough,

\[
\|\Box^m h_b f_t\|_{q,\beta+mq}^q = \int_D \left| \sum_j \lambda_j r_j(t) \Box^m h_b K_\sigma(z, z_j) \Delta^{\sigma+n/r}(3m z_j) \right|^q dV_{\beta+mq}(z) \\
\leq \|h_b^{(\nu)}\| \|f_t\|_{p,\alpha}^q \leq \|h_b^{(\nu)}\| \|q\| \|\{\lambda_j\}\|_{l^p_\alpha}^q.
\]

That is

(5.67) \[
\int_D \left| \sum_j \lambda_j r_j(t) \Box^m h_b K_\sigma(z, z_j) \Delta^{\sigma+n/r}(3m z_j) \right|^q dV_{\beta+mq}(z) \lesssim \|h_b^{(\nu)}\| \|q\| \|\{\lambda_j\}\|_{l^p_\alpha}^q.
\]
Following the same reasoning as in the proof of the necessity part in Theorem 2.8 up to the inequality (4.50), we obtain that

\[
\sum_j |\lambda_j|^q \| q \Delta^{q(n/r)+m+\beta+n/r}(\Im m z_j) \| T^m b(\lambda_j, z_j) \|^q \lesssim \| h_b^{(v)} \| q \| \{ \lambda_j \} \|^q_{\mathcal{L}^q}.
\]

We next observe that

\[
\Box^m h_b^{(v)} K_\sigma(z, z_j) = C_m \int_D b(w)K_\sigma(z_j, w)K_{\nu+m}(z, w) dv_\nu(w) = C_m \int_D \Box^m b(w)K_\sigma(z_j, w)K_{\nu+m}(z, w) dv_\nu+m(w).
\]

Thus

\[
\Box^m h_b^{(v)} K_\sigma(z_j, z_j) = C_m \int_D \Box^m b(w)K_{\sigma+\nu+m+\frac{p}{q}}(z, w) dv_\nu+m(w) = C_m T^m_{\sigma, b}(z_j).
\]

Taking this in (5.62) we obtain

\[
\sum_j |\lambda_j|^q \| \Delta^{q(n/r)+mq+\beta+n/r}(\Im m z_j) T^m_{\sigma, b}(z_j) \|^q \lesssim \| h_b^{(v)} \| q \| \{ \lambda_j \} \|^q_{\mathcal{L}^q}.
\]

As \{\lambda_j\} is chosen arbitrary in \(\ell^q\), it follows by duality since \{|\lambda_j|^q\} belongs to \(l^{p/q}_\alpha\) that the sequence \(\{\Delta^{q(n/r)+mq+\beta-n/r}(\Im m z_j) T^m_{\sigma, b}(z_j)\}^{\infty}_{\sigma=1}\) belongs to \(l^{p/(p-q)}_{\alpha'}\), \(\alpha' = \alpha + \frac{p}{p-q}(\nu - \alpha)\), which is the dual of \(l^{p/q}_\alpha\) under the sum pairing

\[
\{\{a_j\}, \{b_j\}\} := \sum_j a_j b_j \Delta^{\nu+n/r}(\Im m z_j)
\]

with

\[
\| \{\Delta^{q(n/r)+mq+\beta-n/r}(\Im m z_j) T^m_{\sigma, b}(z_j)\}^{\infty}_{\sigma=1}\|_{p/(p-q)} \lesssim \| h_b^{(v)} \| q.
\]

That is

\[
\sum_j \| \Delta^{p/q}_{\nu+m+n/r}(\Im m z_j) \| T^m_{\sigma, b}(z_j) \| p/(p-q) \lesssim \| h_b^{(v)} \| q.
\]

Thus as \(\frac{p}{p-q} > q > 2\), using Lemma 5.65 we conclude that \(b\) is a representative of a class \(B \in \mathbb{B}_{\mu'}^{s'}, s' = \frac{p}{p-q}\), and

\[
\| B \|_{\mathbb{B}_{\mu'}^{s'}} \lesssim \| h_b^{(v)} \|.
\]

The proof is complete. \(\square\)

We obtain in the same way using the duality result in Theorem 3.27 and Lemma 5.65 the following.

**THEOREM 5.70.** Let \(\max \left\{ q' \beta, \frac{\beta+n-1}{\nu}, \frac{\beta-n+1}{\nu+1} \right\} < q < q_\beta\) and \(\alpha, \beta, \nu > \frac{n}{r} - 1, \frac{1}{p} = 1, \frac{1}{q} + \frac{1}{q'} = 1.\) Define \(\beta' = \beta + (\nu - \beta)q'; \frac{1}{2} < \frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1; \frac{\alpha}{p} + \frac{\beta'}{q'} = \frac{2}{s}; \frac{1}{2} + \frac{1}{s} = 1\) and \(\frac{2}{s} + \frac{1}{s'} = \nu.\) Assume that

\[
\frac{1}{p}(\alpha - n + 1) + \frac{1}{q'}(\beta' - n + 1) < \nu - n + 1.
\]

Then the following assertions hold.
(i) If \( b \) is the representative of a class in \( \mathbb{B}_\mu^{s'} \), then the Hankel operator \( h_b^{(\nu)} \) extends into a bounded operator from \( A^p_\alpha(D) \) to \( A^q_\beta(D) \).

(ii) If there exists \( \sigma > \frac{n}{p} - 1 \) such that \( P_\sigma \) is bounded on \( L^p_\alpha(D) \) and if \( h_b^{(\nu)} \) extends into a bounded operator from \( A^p_\alpha(D) \) to \( A^q_\beta(D) \), then \( b \) is the representative of a class in \( \mathbb{B}_\mu^{s'} \).

Using Theorem 3.27 and Lemma 5.66 we obtain in the same way the following.

**Theorem 5.72.** Let \( \max\left\{ q'_\beta, \frac{\beta+\frac{s-1}{\nu} - 1}{\nu - \frac{s+1}{r+1}} \right\} < q < q_\beta \) and \( \alpha, \beta, \nu > \frac{n}{r} - 1, \frac{1}{p} + \frac{1}{q'} = \frac{1}{q} + \frac{1}{q'} = 1 \). Define \( \beta' = \beta + (\nu - \beta)q' \); \( \frac{1}{p} + \frac{1}{q'} = \frac{1}{s}; \frac{\beta'}{q'} = \frac{s}{r}; \frac{1}{\nu} + \frac{1}{q'} = 1 \) and \( \frac{\beta'}{q'} = \nu \).

Assume that

\[
\frac{1}{p}(\alpha - \frac{n}{r} + 1) + \frac{1}{q'}(\beta' - \frac{n}{r} + 1) < \nu - \frac{n}{r} + 1,
\]

and

\[
\frac{1}{q'} < \frac{1}{s} < \max\left\{ \frac{1}{q'_\mu}, \frac{\nu}{\mu + \frac{s}{r} - 1}, \frac{\nu - \frac{n}{r} + 1}{\mu + \frac{n}{r} - 1} \right\}.
\]

Then the following hold.

(i) If \( b \in A^s_\mu \), then the Hankel operator \( h_b^{(\nu)} \) extends into a bounded operator from \( A^p_\alpha(D) \) to \( A^q_\beta(D) \).

(ii) If there exists \( \sigma > \frac{n}{p} - 1 \) such that \( P_\sigma \) is bounded on \( L^p_\alpha(D) \) and if \( h_b^{(\nu)} \) extends into a bounded operator from \( A^p_\alpha(D) \) to \( A^q_\beta(D) \), then \( b \in A^s_\mu \).

In particular, we have the following result.

**Proposition 5.75.** Let \( 1 < q < p < \infty \) and \( \nu > \frac{n}{p} - 1 \). Suppose that \( P_\nu \) is bounded on both \( L^p_\alpha(D) \) and \( L^q_\beta(D) \). Then the Hankel operator \( h_b \) is bounded from \( A^p_\alpha(D) \) to \( A^q_\beta(D) \) if and only if \( b \in \mathbb{B}_\nu^{s'}(D) \), where \( s = \frac{pq}{p-q} \).

**Proof.** We note that \( q < \frac{pq}{p-q} = s \). If \( s \geq 2 \), everything follows as in the proof of the previous theorem. If \( q < s < 2 \), then by interpolation we also have that \( P_\nu \) is bounded on \( L^p_\alpha(D) \) and again the proof follows as for the theorem above. Note that in this last case, \( k_0(s, \nu) = 0 \).

The proof is complete.

### 6. Weak Factorization of Functions in Bergman Spaces of Tube Domains over Symmetric Cones

In this section, we prove Theorem 2.14 and Theorem 2.17. In fact we only have to prove equivalence between boundedness of Hankel operators and and weak factorization results as stated in the two theorems. We have the following result for weighted Bergman spaces of our setting.

**Theorem 6.76.** Let \( \gamma, \nu > \frac{n}{p} - 1 \) and \( 1 < s < q_\gamma \). Let \( 1 < p, q < \infty \) and \( \alpha, \beta > \frac{n}{p} - 1 \) so that \( \frac{1}{s} = \frac{1}{p} + \frac{1}{q} < 1 \), \( \frac{s}{n} = \frac{\alpha}{p} + \frac{\beta}{q} \) and

\[
\frac{1}{p}(\alpha - \frac{n}{r} + 1) + \frac{1}{q}(\beta - \frac{n}{r} + 1) < \nu - \frac{n}{r} + 1
\]
Then the following assertions are equivalent.

\[(6.77)\]
\[
\max\{q', \gamma' + \frac{\alpha}{\nu} - 1, \gamma' - \frac{\alpha}{\nu} + 1\} < s < q, \quad \text{and} \quad \max\{q', \frac{\beta + \alpha}{\nu} - 1, \beta - \frac{\alpha}{\nu} + 1\} < q < q';
\]
or
\[(6.78)\]

\[1 < s < 2 \quad \text{and} \quad 1 < q \leq 2.\]

Then the following assertions are equivalent.

(i) \(A_\alpha^q(D) = A_\alpha^p(D) \otimes A_\beta^q(D).\)

(ii) For any analytic function \(b\) on \(D\), \(h_b^{(\nu)}\) extends as a bounded operator from \(A_\alpha^p(D)\) to \(B_{q,\beta}(D)\) if and only if \(b\) is a representative of class in \(B_{s,\alpha}(D)\).

Proof. Recall that \(\gamma' = \gamma + (\nu - \gamma)s', \beta' = \beta + (\nu - \beta)q'.\) When condition \((6.77)\) holds, we have \(\gamma', \beta' > \frac{\alpha}{\nu} - 1\) and \(B_{q,\beta}(D) = A_\beta^q(D)\) and \(B_{s,\alpha}(D) = A_\alpha^s(D).\)

That (i) \(\Rightarrow\) (ii) is harmless. To prove (ii) \(\Rightarrow\) (i), we start by establishing the following:

**Lemma 6.79.** Let \(\gamma, \nu > \frac{\alpha}{\nu} - 1\) and \(1 < s < q\). Let \(1 < p, q < \infty\) and \(\alpha, \beta > \frac{\alpha}{\nu} - 1\) so that \(\frac{1}{s} = \frac{1}{p} + \frac{1}{q} < 1, \frac{s}{\alpha} = \frac{p}{\alpha} + \frac{q}{\alpha}\) and the hypotheses in Theorem 6.76 are satisfied.

Assume that for any analytic function \(b\) on \(D\), \(h_b^{(\nu)}\) extends as a bounded operator from \(A_\alpha^p(D)\) to \(B_{q,\beta}(D)\) if and only if \(b\) is a representative of class in \(B_{s,\alpha}(D)\). Then

\[A_\alpha^p(D) \otimes A_\beta^q(D) \subset A_\alpha^s(D).\]

**Proof of Lemma 6.79:**

Following the reasoning of [23], let \(F \in (A_\alpha^s(D))^*\). Thanks to Lemma 3.24, there is \(b \in B_{s,\alpha}(D)\) and \(m\) a large enough integer such that \(F(\varphi) = (b, \varphi)_{\nu,m}\) for all \(\varphi \in A_\alpha^s(D)\).

Let \(f \in A_\alpha^p(D) \otimes A_\beta^q(D)\). For every \(\varepsilon > 0\), there exist finite sequences \(\{g_j\} \in A_\alpha^p(D)\) and \(\{l_j\} \in A_\beta^q(D)\) such that \(f = \sum_j g_j l_j\) and \(\sum_j \|g_j\|_{p,\alpha} \|l_j\|_{p,\beta} \leq \|f\| + \varepsilon\). Using the reproduction formula \((5.14)\) and the boundedness of \(h_b^{(\nu)}\), we write

\[
|F(f)| = |(b, f)_{\nu,m}| = |(b, \sum_j g_j l_j)_{\nu,m}| = |\sum_j (b, g_j l_j)_{\nu,m}| = \sum_j |(g_j \otimes \mu b, l_j)_{\nu,m}| = \sum_j |\langle h_b^{(\nu)}(g_j), l_j \rangle_{\nu,m}| \leq \sum_j \|h_b^{(\nu)}(g_j)\|_{B_{q,\beta}} \|l_j\|_{q,\beta} \leq \|h_b\| \sum_j \|g_j\|_{p,\alpha} \|l_j\|_{q,\beta} < \|h_b\|(\|f\| + \varepsilon).
\]

Letting \(\varepsilon\) tend to 0 yields \(|F(f)| \leq \|h_b\| \|f\|\), that is \(F \in (A_\alpha^p(D) \otimes A_\beta^q(D))^*\). \(\square\)

To prove the reverse inclusion we need the following:
Lemma 6.80. ([ Proposition 5.1]) Assume that $1 \leq s < \infty$ and $\gamma, \sigma > \frac{n}{r} - 1$ and $P_\sigma$ is bounded on $L^s(\mathcal{D})$. Then $A^s_\sigma(\mathcal{D})$ is the closed linear span of the set $\{B_\sigma(\cdot, z) : z \in \mathcal{D}\}$. In particular, $B_\sigma(\cdot, z) \in A^s_\sigma(\mathcal{D})$ for any $z \in \mathcal{D}$.

We then deduce the reverse inclusion based on the Hahn-Banach Theorem.

Lemma 6.81. Let $\gamma > \frac{n}{r} - 1$ and $1 < s < q_\gamma$. Let $1 < p, q < \infty$ and $\alpha, \beta > \frac{n}{r} - 1$ so that $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} < 1$, $\frac{s}{\gamma} = \frac{p}{\gamma} + \frac{q}{\gamma}$. The space $A^p_\alpha(\mathcal{D}) \otimes A^q_\beta(\mathcal{D})$ is dense in $A^s_\gamma(\mathcal{D})$ and consequently

$$(A^p_\alpha(\mathcal{D}) \otimes A^q_\beta(\mathcal{D}))^* \subset (A^s_\gamma(\mathcal{D}))^*.$$  

Proof of Lemma 6.81. As $1 < s < q_\gamma$, taking $\sigma = m_1 + m_2$ where $m_1$ and $m_2$ are two large positive numbers, we have that by Lemma 3.20 that $P_\sigma$ is bounded on $A^p_\gamma(\mathcal{D})$. It follows from Lemma 6.80 that the space $A^s_\gamma(\mathcal{D})$ is the closed linear span of the set $\{B_{m_1+m_2}(\cdot, z) : z \in \mathcal{D}\}$. Thus, for every $f \in A^s_\gamma(\mathcal{D})$, there is a sequence $\{f_k\}$ defined by $f_k = \sum_{j=1}^{k} a_j B_{m_1+m_2}(\cdot, z_j)$, where $a_j$'s are complex constants, such that $\lim_{k \to \infty} \|f - f_k\|_{s, \gamma} = 0$. Observe now that $B_{m_1+m_2}(\cdot, z_j) = C_j \Delta^{-m_1-m_2-\frac{\beta}{s}} \left(\frac{-z_j}{s}\right) = g_j l_j$ where $g_j = C_j \Delta^{-m_1} \left(\frac{-z_j}{s}\right)$ and $l_j = \Delta^{-m_2-\frac{\beta}{s}} \left(\frac{-z_j}{s}\right)$. For $m_1$ and $m_2$ sufficiently large, we also have that $g_j \in A^p_\alpha(\mathcal{D})$ and $l_j \in A^q_\beta(\mathcal{D})$ for any $j = 1, 2, \cdots, k$, and $\|B_{\gamma}(\cdot, z_j)\|_{s, \gamma} = \|g_j\|_{p, \alpha} \|l_j\|_{q, \beta}$.

Hence for any $k \in \mathbb{N}$, $f_k = \sum_{j=1}^{k} a_j g_j l_j$ with $g_j \in A^p_\alpha(\mathcal{D})$ and $l_j \in A^q_\beta(\mathcal{D})$. This shows that the sequence $\{f_k\}$ lies in $A^p_\alpha(\mathcal{D}) \otimes A^q_\beta(\mathcal{D})$ and therefore we are done with the density.

Furthermore, if $F \in (A^p_\alpha(\mathcal{D}) \otimes A^q_\beta(\mathcal{D}))^*$, by the Hahn-Banach Theorem, there $G \in (A^s_\gamma(\mathcal{D}))^*$ that extends $F$ with $\|G\| = \|F\|$. By the density of $A^p_\alpha(\mathcal{D}) \otimes A^q_\beta(\mathcal{D})$ in $A^s_\gamma(\mathcal{D})$ and the boundedness of both $F$ and $G$, we conclude that $F = G \in (A^s_\gamma(\mathcal{D}))^*$.

The proof is complete.

For $\nu > \frac{n}{r} - 1$, let us denote by $\mathcal{R}_\nu$ the set of exponent $p \in [1, \infty)$ such that $P_\nu$ is bounded on $L^p(\mathcal{D})$. We have the following result.

Theorem 6.82. Let $\nu > \frac{n}{r} - 1$. Assume that $p, q, s \in \mathcal{R}_\nu$ and $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} < 1$. Then the following assertions are equivalent.

(i) $A^p_\nu(\mathcal{D}) = A^q_\nu(\mathcal{D}) \otimes A^p_\beta(\mathcal{D})$.

(ii) For any analytic function $b$ on $\mathcal{D}$, $h^\nu b$ extends as a bounded operator from $A^p_\nu(\mathcal{D})$ to $A^q_\nu(\mathcal{D})$ if and only if $b$ is a representative of class in $\mathcal{B}^\nu(\mathcal{D})$.

Proof. We only have to take care of the implication (ii) $\Rightarrow$ (i). This follows as above with the following lemma in place of Lemma 6.81.

Lemma 6.83. Let $\nu > \frac{n}{r} - 1$. Assume that $p, q, s \in \mathcal{R}_\nu$ and $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} < 1$. Then the space $A^p_\nu(\mathcal{D}) \otimes A^q_\nu(\mathcal{D})$ is dense in $A^s_\gamma(\mathcal{D})$ and consequently,

$$(A^p_\nu(\mathcal{D}) \otimes A^q_\nu(\mathcal{D}))^* \subset (A^s_\gamma(\mathcal{D}))^*.$$
Proof of Lemma 6.83. Let $F \in A^s_\nu(D)$, $\alpha_1, \alpha_2 > 0$, and $\epsilon > 0$. Put $\alpha = \alpha_1 + \alpha_2$, and define

$$F_{\epsilon,\alpha}(z) = F(z + i\epsilon \Delta^{-\alpha}(\frac{\epsilon z + i\epsilon}{\epsilon})).$$

We have the following facts (see the proof of [1] Theorem 3.23):

(a) $F_{\epsilon,\alpha} \in A^s_\nu(D)$ and $\|F_{\epsilon,\alpha}\|_{s,\nu} \leq \|F\|_{s,\nu}$.
(b) $\lim_{\epsilon \to 0} \|F - F_{\epsilon,\alpha}\|_{s,\nu} = 0$.
(c) For $\alpha_1$ large enough, $F_{\epsilon,\alpha_1} \in A^p_\nu(D)$.

Observe also with Lemma 3.17 that for $\alpha_2$ large enough, $\Delta^{-\alpha_2}(\frac{\epsilon^2 + i\epsilon}{\epsilon})$ is in $A^s_\nu(D)$. Thus

$$F_{\epsilon,\alpha} = F_{\epsilon,\alpha_1}G_{\epsilon,\alpha_2},$$

where $G_{\epsilon,\alpha_2}(z) := \Delta^{-\alpha_2}(\frac{\epsilon^2 + i\epsilon}{\epsilon})$, with $\|F_{\epsilon,\alpha_1}\|_{s,\nu} \leq \|F_{\epsilon,\alpha_1}\|_{p,\nu}\|G_{\epsilon,\alpha_2}\|_{q,\nu}$. Hence $A^s_\nu(D) \otimes A^p_\nu(D)$ is dense in $A^p_\nu(D)$.

That $(A^s_\nu(D) \otimes A^p_\nu(D))^* \subset (A^p_\nu(D))^*$ follows as in the proof of Lemma 6.81. □

The proof is complete. □

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