Inefficient Best Invariant Tests

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Abstract
Test statistics which are invariant under various subgroups of the orthogonal group are shown to provide tests whose powers are asymptotically equal to their level against the usual type of contiguous alternative in models where the number of parameters is allowed to grow as the sample size increases. The result is applied to the usual analysis of variance test in the Neyman-Scott many means problem and to an analogous problem in exponential families. Proofs are based on a method used by Čibisov(1961) to study spacings statistics in a goodness-of-fit problem. We review the scope of the technique in this context.

Keywords. Asymptotic relative efficiency, Neyman-Scott many means problem, goodness-of-fit, spacings statistics, many parameter problems, permutation central limit theorem, bootstrap.

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1 Introduction

Consider the problem of data $X$ coming from a model indexed by a parameter space $\mathcal{M}$. Suppose there is a group $G$ which acts both on the data and on the parameter space so that $gX$ has the same distribution under $m \in \mathcal{M}$ as $X$ has under $g^{-1}m$. The problem of testing $H_0: m \in \mathcal{M}_0 \subset \mathcal{M}$ is invariant under $G$ if $m \in \mathcal{M}_0$ iff $gm \in \mathcal{M}_0$ for all $g \in G$. In what follows we shall impose the stronger condition that $gm = m$ for all $m \in \mathcal{M}_0$ and all $g \in G$.

In this note we use the observation that for an invariant test the power at alternative $m$ minus the level of the test is the covariance between the test function and the likelihood ratio averaged over the orbit of $m$ under $G$ to study the power of invariant tests under contiguous alternatives. The technique was used in Čibisov(1961) to study the asymptotic behaviour of tests of uniformity based on sample spacings.

Our results may be summarized as follows. When for a given point $m$ in the null hypothesis, the number of possible directions of departure from $m$ into the alternative hypothesis grows with the sample size the power of invariant tests may be expected to be low. We will exhibit a variety of examples in which the power minus the level of invariant tests converges to 0 uniformly in the class of invariant tests.

We begin in section 2 with a simple normal example to illustrate the technique and set down the basic identities and inequalities. In section 3 we extend the normal example to general exponential families using a version of the permutation central limit theorem. In section 4 we examine the Neyman Scott many means problem and extend our results to more general models. In section 5 we revisit Čibisov’s example to show that a variety of goodness-of-fit tests have ARE 0 over a large class of alternatives. The results of the first 5 sections suggest that either invariance is not always desirable or that contiguity calculations are not always the right way to compare powers of tests in such models. Section 6 is a discussion of the relevance of contiguity calculations in this context, together with some open problems and suggestions for further work. An appendix contains some technical steps in the proofs.
2 Basic Results: A Normal Example

Suppose $X \in \mathcal{R}^n$ has a multivariate normal distribution with mean vector $m \in \mathcal{R}^n$ and identity covariance matrix. The problem of testing the null hypothesis $H_0: m = 0$ is invariant under the group of orthogonal transformations. Under $H_0$, $X$ and $PX$ have the same distribution for any orthogonal matrix $P$. Consider a simple alternative $m \neq 0$. The likelihood ratio of $m$ to 0 is $L(X) = \exp\{m^T X - \|m\|^2/2\}$. Thus the Neyman Pearson test rejects when $m^T X/\|m\|$ is too large. Under $H_0$ this statistic has a standard normal distribution while under the alternative $m$ the mean is shifted to $\|m\|$. Thus as $n \to \infty$ non-trivial limiting power (i.e. a limiting power larger than the level and less than 1) results when $\|m\| \to \delta \neq 0$. (Throughout this paper objects named by Roman letters depend on $n$; wherever possible the dependence is suppressed in the notation. Objects named by Greek letters do not depend on $n$.)

We may analyse the efficiency of an invariant test relative to the Neyman Pearson test for a given sequence of alternatives as follows. Let $\mathcal{T}$ be the class of all test functions $T(X)$ for testing $H_0$ which are invariant under the group of orthogonal transformations, that is, for which $T(PX) = T(X)$ for each orthogonal transformation $P$. We have the following theorem.

**Theorem 1** As $n \to \infty$,  
$$\sup \{|E_m(T) - E_0(T)| : T \in \mathcal{T}, \|m\| \leq \delta\} \to 0.$$ 

Since the parameter space depends on the sample size the usual definition of relative efficiency does not make sense. Instead, given an alternative $m$ we will define the efficiency of a (level $\alpha$ where $\alpha$ is fixed) test relative to the Neyman Pearson test to be $1/c^2$ where $c$ is chosen so that the test under consideration has power against $cm$ equal to the Neyman Pearson power against the alternative $m$. In traditional finite dimensional parametric models this notion agrees with the usual notion of Pitman relative efficiency (asymptotically). Thus ARE will be the limit of $1/c^2$; the alternative sequence $m$ must be contiguous so that the asymptotic power of the Neyman Pearson test is not 1. For a discussion of the relevance of the Neyman-Pearson test as a standard for the relative efficiency see section 6.
Corollary 1  The best invariant test of \( H_o \) has ARE 0.

Proof of Theorem 1

Let \( L(X) \) be the likelihood ratio for \( m \) to 0. Then

\[
E_m (T(X)) = E_0 (T(X)L(X)).
\]

Since \( PX \) and \( X \) have the same distribution under \( H_o \) we have

\[
E_m (T(X)) = E_0 (T(PX)L(PX)) = E_0 (T(X)L(PX)) \tag{1}
\]

for all orthogonal \( P \). Since \( P \) appears only on the right hand side of (1) we may average over orthogonal \( P \) to obtain

\[
E_m (T(X)) = \int E_0 (T(X)L(PX)) F(dP) = E_0 \left( T(X) \int L(PX)F(dP) \right) \tag{2}
\]

where \( F \) is any probability measure on the compact group of orthogonal matrices. Let

\[
\overline{L}(X) = \int L(PX)F(dP).
\]

Then

\[
|E_m(T) - E_0(T)| = |E_0(T(\overline{L} - 1))| \leq E_0(|\overline{L} - 1|). \tag{3}
\]

Since the last quantity is free of \( T \) we need only show that

\[
\sup \{ E_0(|\overline{L} - 1|) : \|m\| \leq \delta \} \to 0.
\]

Since \( E(\overline{L}) = 1 \) the dominated convergence theorem shows that it suffices to prove for an arbitrary sequence of alternatives \( m \) with \( \|m\| \leq \delta \) and for a suitably chosen sequence of measures \( F \) that \( \overline{L} \to 1 \) in probability. We take \( F \) to be Haar measure on the compact group of orthogonal transformations; that is, we give \( P \) a uniform distribution.
For each fixed $X$, when $P$ has the distribution for which $F$ is Haar measure on the orthogonal group, the vector $PX$ has the uniform distribution on the sphere of radius $\|X\|$. Using the fact that a standard multivariate normal vector divided by its length is also uniform on a sphere we find that

$$
\overline{L} = H(\|m\| \cdot \|X\|)/H(0)
$$

where

$$
H(t) = \int_0^\pi \exp(t \cos \theta) \sin^{n-2} \theta d\theta.
$$

Standard asymptotic expansions of Bessel functions (see, e.g. Abramowitz and Stegun, 1965, p 376ff) then make it easy to show that $\overline{L} \to 1$ in probability, finishing the proof.

A test invariant under the group of orthogonal transformations is a function of $\|X\|^2$ and the analysis above can be made directly and easily using the fact that this statistic has a chi-squared distribution with non-centrality parameter $\|m\|^2$. Our interest centres on the technique of proof. Equations (1-3) and the argument following (3) use only the group structure of the problem and the absolute continuity of the alternative with respect the null. The remainder of the argument depends on an asymptotic approximation to the likelihood ratio averaged over alternatives. Whenever such an approximation is available we can expect to obtain efficiency results for the family of invariant tests. In the next section we apply the technique to a more general model by replacing the explicit Bessel function calculation with a version of the permutation central limit theorem.

3 Exponential Families

Suppose now that $X = (X_1, \ldots, X_n)^T$ with the $X_i$ independent and $X_i$ having the exponential family density $\exp(m_i x_i - \beta(m_i))$ relative to some fixed measure. We assume the $m_i$’s take values in $\Theta$ an open subset of $\mathcal{R}$. The parameter space is then $\mathcal{M} = \Theta^n$. Let $\Theta_0$ be a fixed compact subset of $\Theta$.

Consider the null hypothesis $H_0$: $m_1 = \cdots = m_n$. This problem is invariant under the subgroup of the orthogonal group consisting of all permutation
matrices $P$. Let $\bar{m} = \sum m_i/n$; we also use $\bar{m}$ to denote the vector of length $n$ all of whose entries are $\bar{m}$. The calculations leading to (1-3) establish that for any test $T$ in $\mathcal{T}$, the family of all tests invariant under permutations of the entries of $X$, we have

$$|E_m(T) - E_{\bar{m}}(T)| \leq E_{\bar{m}}(|\mathcal{L} - 1|).$$

where now

$$\mathcal{L} = \exp \left\{ - \sum (\beta(m_i) - \beta(\bar{m})) \right\} \sum_P \exp \left( (m - \bar{m})^T P X \right) / n!$$

is the likelihood ratio averaged over all permutations of the alternative vector $m$.

### 3.1 Heuristic Computations

Think now of $X$ as fixed and $P$ as a randomly chosen permutation matrix. Then $(m - \bar{m})^T P X = (m - \bar{m})^T P (X - \bar{X})$ has moment generating function

$$g(s) = \sum_P \exp \{ s(m - \bar{m})^T P (X - \bar{X}) \} / n!.$$

If $\max\{|m_i - \bar{m}| : 1 \leq i \leq n\} \to 0$ the permutation central limit theorem suggests that $(m - \bar{m})^T P X$ has approximately a normal distribution with mean 0 and variance $\sum (m_i - \bar{m})^2 \sum (X_i - \bar{X})^2 / n$. Thus heuristically

$$g(s) \approx \exp \{ s^2 \sum (m_i - \bar{m})^2 \sum (X_i - \bar{X})^2 / (2n) \}.$$

Under the same condition, $\max\{|m_i - \bar{m}| : 1 \leq i \leq n\} \to 0$, we may expand

$$\sum (\beta(m_i) - \beta(\bar{m})) \approx \sum (m_i - \bar{m})^2 \beta''(\bar{m}) / 2.$$

We are led to the heuristic calculation

$$\mathcal{L}(X) \approx \exp \{ \sum (m_i - \bar{m})^2 (S^2 - \beta''(\bar{m})) / 2 \}$$

where $S^2 = \sum (X_i - \bar{X})^2 / n$ is the sample variance. Since $\text{Var}_{\bar{m}}(X) = \beta''(\bar{m})$ we see that $\mathcal{L}$ should converge to 1 provided $\sum (m_i - \bar{m})^2 = o(n^{1/2})$ or
$\|m - \overline{m}\| = o(n^{1/4})$. In the simple normal example of the previous section we are actually able to prove (again using asymptotic expansions of Bessel functions) this strengthened version of Theorem 1, namely, if $a = o(n^{1/4})$ then

$$\sup\{|E_m(T) - E_0(T)| : T \in T, \|m\| \leq a\} \to 0.$$ 

In the more general exponential family problem we do not know a permutation central limit theorem which extends in a useful way to give convergence of moment generating functions. For contiguous alternative sequences we are able to replace the moment generating function by a characteristic function calculation. We can then prove that invariant tests have power converging to their level uniformly on compact subsets of $\Theta$.

**Theorem 2**  
As $n \to \infty$,

$$\sup\{|E_m(T) - E_{\overline{m}}(T)| : T \in T, \|m - \overline{m}\| \leq \delta, m_i \in \Theta_0\} \to 0.$$ 

### 3.2 Proof of Theorem 2

Our proof uses contiguity techniques to replace the moment generating function used in the heuristics above by a corresponding characteristic function calculation. A standard compactness argument reduces the problem to showing that

$$E_m(T) - E_{\overline{m}}(T) \to 0$$

for an arbitrary sequence of alternatives $m$ satisfying $\sum(m_i - \overline{m})^2 \leq \delta^2$, $m_i \in \Theta_0$, and an arbitrary sequence $T \in T$. Our proof is now in three stages. First we prove that any such sequence is contiguous to the null sequence indexed by $\overline{m}$. The second step is to eliminate the particular statistics, $T$, as at (3), reducing the problem to showing that the permutation characteristic function of the log-likelihood ratio is nearly non-random. The final step is to establish the latter fact appealing to Theorem 3.

**Step 1: Contiguity of $m$ to $\overline{m}$**

The log-likelihood ratio of $m$ to $\overline{m}$ is

$$\ell(X) = \sum (m_i - \overline{m})(X_i - \beta(\overline{m})) - \sum (\beta(m_i) - \beta(\overline{m})).$$

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Under $\overline{m}$ the mean of $\ell(X)$ is

$$- \sum (\beta(m_i) - \beta(\overline{m})) = - \sum (m_i - \overline{m})^2 \beta''(t_i)$$

for some $t_i$ between $m_i$ and $\overline{m}$. In view of the conditions on $m$ and the compactness of $\Theta_0$ this sequence of means is bounded. Also under $\overline{m}$ the variance of $\ell(x)$ is

$$\sum (m_i - \overline{m})^2 \beta''(\overline{m}).$$

Again this is bounded. Thus the sequence of log-likelihood ratios is tight under the null sequence $\overline{m}$ and so the sequence of alternatives, $m$, is contiguous to the null sequence $\overline{m}$.

**Lemma 1** Suppose $Q$ is a sequence of measures contiguous to a sequence of measures $P$. If $T$ is a bounded sequence of statistics such that

$$E_P(T \exp(i\tau \log dQ/dP)) - E_P(T)E_P(\exp(i\tau \log dQ/dP)) \to 0$$

for each real $\tau$ then

$$E_Q(T) - E_P(T) \to 0.$$

**Remark:** In the Lemma the random variable $\log dQ/dP$ can be replaced by any random variable $S$ such that $\log dQ/dP - S$ tends to 0 in probability under $P$.

**Remark:** The Lemma is very closely connected with LeCam’s third lemma (see Hájek and Šidák, 1967, page 209, their formula 4) which could also be applied here to the $Q$ characteristic function of $T$.

Before proving the lemma we finish the theorem.

**Step 2:** Elimination of the sequence $T$.

Let $H(t, X) = \exp(it\ell(X))$. According to the lemma we must show

$$E_{\overline{m}}(T(X)H(\tau, X)) - E_{\overline{m}}(T)E_{\overline{m}}(H(\tau, X)) \to 0.$$  \hspace{1cm} (4)

Arguing as in (1-3) the quantity in (4) may be seen to be

$$E_{\overline{m}}(T(X)\overline{H}(\tau, X)) - E_{\overline{m}}(T(X))E_{\overline{m}}(\overline{H}(\tau, X))$$

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where \( H(\tau, X) = \sum P H(\tau, PX)/n! = E(H(\tau, PX)|X) \). Since both \( T \) and \( H(\tau, X) \) are bounded it suffices to prove that

\[
H(\tau, X) - E(\overline{H}(\tau, X)) \to 0
\]

in probability.

**Remark:** If \( \ell(X) = S(X) + o_P(1) \) under \( \overline{m} \) then in view of the remark following the lemma the random variable \( \ell(X) \) can be replaced in the definition of \( H \) by the random variable \( S(X) \). Moreover, if \( S^*(X) = S(X) + a \) then (5) will hold for \( S^* \) replacing \( \ell \) if and only if it holds for \( S \) replacing \( \ell \) because the sequence \( \exp(i\tau a) \) is bounded in modulus.

**Step 3:** Application of the Permutation Central Limit Theorem

Using the last remark take \( S^*(X) = (m - \overline{m})^T P x \). The variable \( \overline{H} \) then becomes \( \sum P \exp(i\tau (m - \overline{m})^T P x)/n! \) whereas \( E(\overline{H}(\tau, X)) \) becomes \( E(\overline{m})(\exp(i\tau (m - \overline{m})^T X)) \).

Now let \( \hat{F} \) be the empirical distribution of the \( X_1, \ldots, X_n \). Let \( X_1^*, \ldots, X_n^* \) be independent and identically distributed according to \( \hat{F} \). We will show in the next section that

\[
\sum P \exp(i\tau m^T P x)/n! - E_{\hat{F}} \left( \exp(i\tau \sum m_j X_j^*) \right) \to 0
\]

and

\[
E_{\hat{F}} \left( \exp(i\tau \sum m_j X_j^*) \right) - E(\overline{m})(\exp(i\tau \sum m_j X_j)) \to 0
\]

in probability for each fixed \( \tau \). This will finish the proof of Theorem 2 except for establishing the lemma.

To prove the (undoubtedly well-known) lemma argue as follows. Letting \( \ell = \log dQ/dP, \) the condition shows that \( E_P(T\phi(\ell)) - E_P(T)E_P(\phi(\ell)) \to 0 \) for each bounded continuous function \( \phi \). There is then a sequence \( a \) tending to infinity so slowly that \( E_P(Tf(\ell)) - E_P(T)E_P(f(\ell)) \to 0 \) where \( f(x) = \min(e^x, a) \). In view of contiguity \( E_Q(T) - E_P(Te^\ell) \to 0 \) and for any sequence \( a \) tending to infinity \( |E_P(T(e^\ell - f(\ell)))| \leq E_P(|e^\ell - f(\ell)|) \to 0 \). The lemma follows.
3.3 The Permutation Limit Theorem

Suppose \( m \in \mathbb{R}^n \) and \( x \in \mathbb{R}^n \) are two (non-random) vectors with \( \mathbf{m} = 0 \) for convenience. Suppose \( P \) is a random \( n \times n \) permutation matrix. The random variable \( m^T P x \) has mean \( n \mathbf{m}^T \mathbf{x} \) and variance \( s^2 = \sum (m_i - \mathbf{m})^2 \sum (x_i - \mathbf{x})^2 / (n - 1) \). An alternative description of the random variable \( m^T P x \) is as follows. Let \( J_1, \ldots, J_n \) be a simple random sample drawn without replacement from the set \( \{1, \ldots, n\} \). Then \( J_1, \ldots, J_n \) is a random permutation of \( \{1, \ldots, n\} \) and \( m^T P x \) has the same distribution as \( \sum m_i x(J_i) \); we now use functional rather than subscript notation for legibility. Hájek (1961, see his formula 3.11) shows that it is possible to construct, on a single probability space, \( J_1, \ldots, J_n \) together with \( J_1^*, \ldots, J_n^* \), a random sample drawn with replacement from \( \{1, \ldots, n\} \) in such a way that

\[
E[(\sum m_i x(J_i) - \sum m_i x(J_i^*))^2] \leq 3 s \max |x_i - \mathbf{x}|/(n - 1)^{1/2}. \tag{8}
\]

Since the \( X_i \) have an exponential family distribution and \( \Theta \) is compact it is straightforward to check that

\[
S \max |X_i - \mathbf{X}|/(n - 1)^{1/2} = O_P(\log n/n^{1/2}) = o_P(1)
\]

under \( \mathbf{m} \) where \( S \) denotes the sample standard deviation. In view of the elementary inequality

\[
|E(exp(itW)) - E(exp(itW'))| \leq t^2 E((W - W')^2) + |t|E^{1/2}((W - W')^2) \tag{9}
\]

this establishes (6).

It seems easiest to deal with (7) after recalling some facts about weak convergence in the presence of moment conditions. Consider the set, \( \Delta \), of distribution functions \( F \) on \( \mathbb{R}^n \) which have finite variance. Throughout this paper we write \( F \overset{\Delta}{\rightarrow} \Phi \) if \( E_F(\gamma(X)) \rightarrow E_{\Phi}(\gamma(X)) \) for each fixed continuous function \( \gamma \) such that \( \gamma(x)/(1 + \|x\|^2) \) is bounded. (The notation is \( E_F(\gamma(X)) = \int \gamma(x)F(dx) \).) This notion of convergence defines a topology on \( \Delta \) which can be metrized by a metric \( \rho_2 \) in such a way that \( \Delta \) becomes a complete separable metric space. In fact the metric \( \rho_2 \) may be taken to be the Wasserstein metric; see Shorack and Wellner (1986, pp 62-65).

Suppose that \( \Delta_0 \) is a subset of \( \Delta \). The following are equivalent:
i. $\Delta_0$ has compact closure.

ii. for each $\epsilon > 0$ there is a fixed compact subset $\Psi$ of $\mathbb{R}^\pi$ such that $E_F(\|X\|^2 1(X \not\in \Psi)) \leq \epsilon$ for all $F \in \Delta_0$.

iii. there is a fixed function $\Psi$ such that $E_F(\|X\|^2 1(|X| \geq t)) \leq \Psi(t)$ for all $F \in \Delta_0$ and all $t$ where $\Psi$ has the property that $\lim_{t \to \infty} \Psi(t) = 0$.

iv. the family $\Delta_0$ makes $\|X\|^2$ uniformly integrable. 

Notice that for each fixed $\Psi$ such that $\lim_{t \to \infty} \Psi(t) = 0$ the family of distributions $F$ with $E_F(\|X\|^2 1(|X| \geq t)) \leq \Psi(t)$ for all $t$ is a compact metric space.

Finally note that $F \overset{D}{\Rightarrow} \Phi$ if and only if $F$ converges in distribution to $\Phi$ and $E_F(\|X\|^2) \to E_\Phi(\|X\|^2)$. All the foregoing results are trivial modifications of the usual results for convergence in distribution; see Billingsley (1968) pp 31-41. It will be useful to let $\rho_0$ denote a metric analogous to $\rho_2$ for which the space of all distributions on $\mathbb{R}^\pi$ becomes a complete separable metric space with the topology of convergence in distribution. If $\Delta_0$ is a compact subset of $\Delta$ for the metric $\rho_2$ then it is a compact subset of the space of all distributions for the metric $\rho_0$.

To state the result let $\hat{F}$ be the empirical distribution of the numbers $X_1, \ldots, X_n$. Let $X_1^*, \ldots, X_n^*$ be independent and identically distributed according to $\hat{F}$. Let $L_{\hat{F}}$ be the (conditional given $X_1, \ldots, X_n$) law of $\sum m_i X_i^*$ and $L_F$ be the law of $\sum m_i X_i$.

**Theorem 3** Let $\Delta_0$ be a fixed compact subset of $\Delta$. Suppose $F$ is any sequence in $\Delta_0$ and that for each $n$ the vector $X = (X_1, \ldots, X_n)^T$ has independent entries distributed according to $F$. Assume $m$ is an arbitrary sequence satisfying $\sum m_i = 0$ and $\sum m_i^2 \leq \delta^2$. If $\hat{F}$ is the empirical distribution function of $X_1, \ldots, X_n$ then $\rho_2(L_F, L_{\hat{F}}) \to 0$ in probability.

**Corollary 2** Under the conditions of Theorem 3

$$E_F\left(\exp(i\tau \sum m_j X_j)\right) - E_F\left(\exp(i\tau \sum m_j X_j)\right) \to 0$$

in probability for each fixed $\tau$.  

It is straightforward to rephrase a (weakened) version of Hájek’s permutation central limit theorem in the notation of Theorem 3. Let $L_{P}$ be the law of $m^{T}Px$.

**Theorem 4** Suppose that $m \in \mathcal{R}^{n}$ and $x \in \mathcal{R}^{n}$ are two sequences of vectors such that

\[ n\bar{m}\bar{x} \to 0, \tag{10} \]
\[ \sum (m_i - \bar{m})^2 \leq \delta^2 \tag{11} \]

for a fixed $\delta$ and such that $x$ satisfies

\[ \sum (x_i - \bar{x})^2 1(|x_i - \bar{x}| \geq t)/n \leq \Psi(t) \tag{12} \]

where $\Psi$ is a fixed function such that $\lim_{t \to \infty} \Psi(t) = 0$. If $\hat{F}$ is the empirical distribution of the numbers $\{1, \ldots , x_n\}$ then $\rho_2(L_P, L_{\hat{F}}) \to 0$.

The proof of Theorem 3 is in the Appendix.

Remark: Theorem 3 asserts the validity of the bootstrap approximation to $L_F$. Theorem 4 says that the bootstrapping carried out in Theorem 3 by sampling from the list $X_1, \ldots , X_n$ with replacement can also be carried out without replacement. Notice that the result applies only to *contrasts* in the $X_i$; the condition $\bar{m} = 0$ is crucial to this bootstrap interpretation of Theorem 4.

### 4 Extensions

#### 4.1 Non-exponential families

The exponential family model gives the log-likelihood ratio a rather special form. For general models, however, a Taylor expansion can be used to show that the log-likelihood ratio has almost that form. Rather than try to discover the weakest possible conditions on a general model permitting the conclusion we try to illustrate the idea with assumptions which are far from best possible.
Suppose $X = (X_1, \ldots, X_n)^T$ with the individual $X_i$ independent and $X_i$ having density $\exp(\phi(\cdot; m_i))$ where $m_i \in \Theta$ an open subset of $\mathcal{R}$. Again consider the null hypothesis $H_0: m_1 = \cdots = m_m$. Let $\Theta_0$ denote some fixed compact subset of $\Theta$. The log-likelihood ratio for $m$ to $\bar{m}$ is $\ell(X) = \sum \{\phi(X_i; m_i) - \phi(X_i; \bar{m})\}$. Assume that $\phi$ is twice differentiable with respect to the parameter. Let $\phi_i$ denote the $i$-th derivative of $\phi$ with respect to the parameter. Assume that the usual identities of large sample theory of likelihood hold, namely, that $\phi_1(X; \bar{m})$ has mean 0 and finite variance $\iota(\bar{m})$ under $\bar{m}$ and that $\phi_2(X; \bar{m})$ has mean $-\iota(\bar{m})$. Then we may write

$$
\ell(X) = \sum (m_i - \bar{m})\phi_1(X_i; \bar{m}) - \sum (m_i - \bar{m})^2\iota(\bar{m})/2 \\
+ \sum (m_i - \bar{m})^2\{\phi_2(X_i; \bar{m}) + \iota(\bar{m})\}/2 \\
+ \sum (m_i - \bar{m})^2\{\phi_2(X_i, t_i) - \phi_2(X_i, \bar{m})\}
$$

where $t_i$ is between $m_i$ and $\bar{m}$.

We see that, under the usual sort of regularity conditions, there will exist a constant $\alpha$ such that

$$
|E_m(\phi(X_i; m^*) - \phi(X_i; m))| \leq \alpha(m^* - m)^2
$$

(14)

and such that

$$
\text{Var}_m(\phi(X_i; m^*) - \phi(X_i; m)) \leq \alpha(m^* - m)^2
$$

(15)

for all $m$ and $m^*$ in $\Theta_0$. Under these two conditions we see that $|E(\ell(X))| \leq \alpha \sum (m_i - \bar{m})^2$ and $\text{Var}(\ell(X)) \leq \alpha \sum (m_i - \bar{m})^2$. Thus any sequence $m$ with $\sum (m_i - \bar{m})^2 \leq \delta^2$ and all $m_i \in \Theta_0$ is contiguous to the null sequence $\bar{m}$. The first two steps in the proof of Theorem 3 may therefore be seen to apply to general one-parameter models under regularity conditions, specifically whenever (14) and (15) hold.

Further regularity assumptions are necessary in order for Step 3 of Theorem 2 to go through in the present context. To get some insight consider the situation where $\max\{|m_i - \bar{m}|; 1 \leq i \leq n\} \to 0$. Under further regularity conditions we will have

$$
\sum (m_i - \bar{m})^2\{\phi_2(X_i, t_i) - \phi_2(X_i, \bar{m})\} \to 0
$$

(16)
and
\[ \sum (m_i - \overline{m})^2 \{ \phi_2(X_i; \overline{m}) + \iota(\overline{m}) \} \to 0 \] (17)
in probability. Assuming that (16) and (17) hold we see
\[ \ell(X) = \sum (m_i - \overline{m}) \phi_1(X_i; \overline{m}) - \sum (m_i - \overline{m})^2 \iota(\overline{m}) / 2 + o_P(1). \] (18)

Define \( S^*(X) = \sum (m_i - \overline{m}) \phi_1(X_i; \overline{m}) \). Step 3 of the proof of Theorem 2 may now be carried through with \( X_i \) replaced by \( U_i = \phi_1(X_i; \overline{m}) \) provided that the map from \( \Theta_0 \) to \( \Delta \) which associates \( m \) with the \( m \) distribution of \( \phi_1(X_i; \overline{m}) \) is continuous.

The assumption that \( \max \{|m_i - \overline{m}|; 1 \leq i \leq n\} \to 0 \) can be avoided; we now make this assertion precise. We will need three more assumptions:
\[ \sup \{ E_m(|\phi_2(X, m) + \iota(m)|1(|\phi_2(X, m) + \iota(m)| \geq t)); m \in \Theta_0\} \to 0 \] (19)
as \( t \to \infty \). Define
\[ W(X, \epsilon, m) = \sup \{|\phi_2(X, m') - \phi_2(X, m)|; |m' - m| \leq \epsilon, m' \in \Theta_0\}. \]
The second assumption will then be
\[ \limsup_{\epsilon \to 0} \{ E_m(W(X, \epsilon, m)); m \in \Theta_0\} = 0. \] (20)

Finally we will assume that the map
\[ (m, \overline{m}) \mapsto \begin{cases} \mathcal{L}((\phi(X, m) - \phi(X; \overline{m}))/|m - \overline{m}|); & m \neq \overline{m} \\ \mathcal{L}(\phi_1(X; \overline{m})); & m = \overline{m} \end{cases} \] (21)
is continuous from \( \Theta_0 \times \Theta_0 \) to \( \Delta \) where \( \mathcal{L}(X|m) \) denotes the law of \( X \) when \( m \) is true.

**Theorem 5** Assume conditions (14, 15, 19, 20, 21). Then as \( n \to \infty \),
\[ \sup \{|E_m(T) - E_{\overline{m}}(T)| : T \in \mathcal{T}, \|m - \overline{m}\| \leq \delta, m_i \in \Theta_0\} \to 0. \]

**Proof**

Assume, without loss of generality that the entries in \( m \) have been sorted so that \( |m_1 - \overline{m}| \geq \cdots \geq |m_n - \overline{m}| \). Let \( k = k(n) \) be any sequence tending to
infinity. Then \( \sum (m_i - \overline{m})^2 \leq \delta^2 \) implies that \( \max \{ |m_i - \overline{m}|; k \leq i \leq n \} \to 0 \). The assumptions now imply that

\[
\ell(X) = \sum_{i \leq k} \{ \phi(X_i, m_i) - \phi(X_i, \overline{m}) \} + \sum_{i > k} (m_i - \overline{m})\phi_1(X_i, m_i)
- \sum_{i > k} (m_i - \overline{m})^2 \nu(\overline{m})/2 + o_P(1).
\]

Define

\[
S^*(X) = \sum_{i \leq k} \{ \phi(X_i, m_i) - \phi(X_i, \overline{m}) - \mathbb{E}_{\overline{m}}(\phi(X_i, m_i) - \phi(X_i, \overline{m})) \}
+ \sum_{i > k} (m_i - \overline{m})\phi_1(X_i, m_i).
\]

Let \( \mathbb{H}(\tau, X) = \sum_{\mathbf{P}} \exp(i\tau S^*(\mathbf{P} X))/n! \). As before it suffices to prove that

\[
\mathbb{H}(\tau, X) - \mathbb{E}_{\overline{m}}(\mathbb{H}(\tau, X)) \to 0
\]

in probability. The proof may be found in the Appendix; its length is due to our failure to impose the condition that \( \max \{ |m_i - \overline{m}|; k \leq i \leq n \} \to 0 \).

**4.2 The Neyman-Scott Problem**

Consider now the Neyman-Scott many means problem in the following form. Let \( \{ X_{ij}; 1 \leq j \leq \nu, 1 \leq i \leq n \} \) be independent normals with mean \( m_i \) and standard deviation \( \sigma \). The usual Analysis of Variance \( F \)-test of the hypothesis that \( m_1 = \cdots = m_n \) is invariant under permutations of the indices \( i \). Any level \( \alpha \) test of this null hypothesis for this model with \( \sigma \) unknown is a level \( \alpha \) test in any submodel with a known value of \( \sigma \). When \( \sigma \) is known the argument of section 3 can be applied to the vector \( X = (X_1, \ldots, X_n) \) of cell means to conclude that the ARE of ANOVA is 0 along any contiguous sequence of alternatives.

This Analysis of Variance problem may be extended to the following multiparameter exponential family setting. Suppose that for \( i = 1, \ldots, n \) the \( \mathbb{R}^\pi \)-valued random variable \( X_i \) has density \( \exp\{m_i^T x_i - \beta(m_i)\} \) relative to some fixed measure on \( \mathbb{R}^\pi \). The natural parameter space for a single
observation $X_i$ is some $\Theta \subset \mathcal{R}^\pi$. Let $a(m)$ be some parameter of interest and consider the problem of testing $H_0: a(m_1) = \cdots = a(m_n)$. The problem is again permutation invariant.

Let $M$ be the $n \times \pi$ matrix with $i$th row $m_i$ and $\overline{M}$ be the $n \times \pi$ matrix with $i$th row $\overline{m}$. Let $X$ be the $n \times \pi$ matrix with $i$th row $X_i^T$. The log likelihood ratio of $M$ to $\overline{M}$ is

$$\ell(X) = \text{tr}((M - \overline{M})^T X) - \sum_i (\beta(m_i) - \beta(\overline{m}))$$

Denote $\|M\|^2 = \text{tr}(M^T M)$. Let $\Theta_0$ be some fixed compact subset of $\Theta$. Let $\mathcal{T}$ be the family of all permutation invariant test functions, $T(X)$.

**Theorem 6** As $n \to \infty$,

$$\sup\{|E_M(T) - E_{\overline{M}}(T)| : T \in \mathcal{T}, \|M - \overline{M}\| \leq \delta, m_i \in \Theta_0\} \to 0.$$

The proof of this theorem is entirely analogous to that of Theorem 2 needing only a multivariate extension of Theorems 3 and 4. Suppose $M$ and $x$ are sequences of $n \times \pi$ matrices. Let $M_1, \ldots, M_\pi$ and $x_1, \ldots, x_\pi$ be the columns of $M$ and $x$ respectively. Let $P$ be a random $n \times n$ permutation matrix and let $\hat{F}$ be the empirical distribution (measure on $\mathcal{R}^\pi$) of the $n$ rows of $x$. Let $X$ be an $n \times \pi$ matrix whose rows are iid according to $\hat{F}$. Let $L_P$ denote the joint law of $(M_1^T P x_1, \ldots, M_\pi^T P x_\pi)$. Let $L_{\hat{F}}$ denote the law of $M_i^T X_i$ where $X_i$ is the $i$th column of $X$.

**Theorem 7** Suppose that each $M_i$ satisfies (11) and has $\overline{M_i} = 0$. Suppose that each $x_i$ satisfies (12). Then $\rho_2(L_P, L_{\hat{F}}) \to 0$.

The obvious analogue of Theorem 3 also holds.

**Theorem 8** Let $\Delta_0$ be a fixed compact subset of $\Delta$. Suppose $F$ is any sequence in $\Delta_0$ and that for each $n$ the $n \times \pi$ matrix $X$ has independent rows distributed according to $F$. Assume $M$ is an arbitrary sequence of $n \times \pi$ matrices whose columns $M_i$ each satisfy (11) and have $\overline{M_i} = 0$. If $\tilde{F}$ is the empirical distribution function of the rows of $X$ then $\rho_2(L_F, L_{\tilde{F}}) \to 0$ in probability.
It should be noted that the actual null hypothesis plays no role in these theorems. If the theorems are to be used to deduce that any particular sequence of permutation invariant tests has poor power properties it is necessary that \( m_1 = \ldots = m_n \) imply the assertion that the null hypothesis is true and that there be some alternative sequence satisfying the conditions of the preceding theorems.

5 Spacings Statistics

Suppose \( U_1 \leq \cdots \leq U_n \) are the order statistics for a sample of size \( n \) from a distribution on the unit interval. To test the null hypothesis that this distribution is uniform many authors have suggested tests based on the sample spacings \( D_i = U_i - U_{i-1} \) where we take \( U_0 = 0 \) and \( U_{n+1} = 1 \). Examples of statistics include Moran’s statistic \( \sum \log(D_i) \) and Greenwood’s statistic \( \sum D_i^2 \). See Guttorp and Lockhart (1989) and the references therein for a detailed discussion. Notice that these statistics are invariant under permutations of the \( D_i \). Also note that the joint distribution of the \( D_i \) is permutation invariant.

Consider a sequence of alternative densities \( 1 + h(x)/n^{1/2} \). Čibisov(1961) showed (though his proof seems to rely on convergence of the permutation moment generating function which does not seem to me to follow from the form of the permutation central limit theorem which he cites) under differentiability conditions on \( h \) that the power of any spacings statistic invariant under permutations of the \( D_i \) is asymptotically equal to its level using essentially the method of proof used above. We can relax the conditions on \( h \) somewhat to achieve the following.

**Theorem 9** Let \( \Delta_0 \) be a compact subset of \( L_2 \), the Hilbert space of square integrable functions on the unit interval. Let \( \mathcal{T} \) be the family of permutation invariant test functions \( T(D) \). As \( n \to \infty \),

\[
\sup\{|E_h(T) - E_0(T)| : T \in \mathcal{T}, h \in \Delta_0\} \to 0
\]
In Guttorp and Lockhart (1988) it is established that, under the conditions of the theorem, $\ell(D)$, the log-likelihood ratio, is equal to $\sum h_i(D_i - 1/(n+1)) - \int h^2(x) dx/2 + O_P(1)$ for any sequence of alternatives $h$ converging in $L_2$ where the $h_i$ are suitable constants derived from $h$. Using the remark following Lemma 1, the fact that the joint distribution of the $D_i$ is permutation invariant under the null hypothesis, and the characterization of the spacings as $n+1$ independent exponentials divided by their total, the theorem may be proved by following the argument leading to Theorem 2.

6 Discussion

6.1 Relevance of Contiguity

All the theorems establish that permutation invariant tests are much less powerful than the Neyman-Pearson likelihood ratio test for alternatives which are sufficiently different from the null that the Neyman-Pearson test has nontrivial power. Thus if, in practice, it is suspected which parameters $m_i$ are the ones most likely to be different from all the others there will be scope for much more sensitive tests than the invariant tests.

On the other hand, some readers will argue that the analysis of variance is often used in situations where no such prior information is available. Such readers, I suspect, will be inclined to argue that this sort of contiguity calculation is irrelevant to practical people. Some readers may feel that a user would need unreasonable amounts of prior knowledge to derive a better test than the $F$-test. Consider the situation of the normal example in the first section. Suppose that the $m_i$ can be sorted so that adjacent entries are rather similar. Define $h(s) = m_{[ns]}$. If the sequence of functions are reasonably close to some square integrable limit $\eta$ then, without knowing $\eta$, we can construct a test whose power stays larger than its level if the Neyman-Pearson test has the same property. Specifically consider the exponential family example. Let $\gamma_i; i = 1, 2, \ldots$ be an orthogonal basis of $L_2[0, 1]$ with each $\gamma_i$ continuous and let $\lambda_i$ be any sequence of summable positive constants. Define a test statistic of the form $T(X) = \sum \lambda_i(\sum \gamma_i(j/n)X_j)/(\sum \gamma_i^2(j/n))^{1/2}$. If the sequence $h$
converges to some $\eta \neq 0$ in $L^2[0,1]$ then the asymptotic power of $T$ will be larger than its level. The test is the analogue of the usual sort of quadratic goodness-of-fit test of the Cramer-von Mises type.

It is worth noting that the calculations compute the power function by averaging over alternative vectors which are a permutation of a basic vector $m$. Another approach to problems with large numbers of different populations (labelled here by the index $i$) is to model the $m_i$ themselves as an iid sequence chosen from some measure $G$. In this case the null hypothesis is that $G$ is point mass at some unknown value $\overline{m}$. I note that alternative measures $G$ which make the resulting model contiguous to the null make $\text{Var}_G(m_i) = O(n^{-1/2})$ which means that a typical $m_i$ deviates from $\overline{m}$ by the $n^{-1/4}$ discrepancy which arises in our first example and in analysis of spacings tests. In other words when any ordering of the $m_i$ is equally likely vectors $m$ differing from $\overline{m}$ by the amount we have used here are indistinguishable from the null hypothesis according to this empirical Bayes model. It is important to note, however, that for this empirical Bayes model the hypothesis of permutation invariance of the statistic $T$ is unimportant: if $\text{Var}_G(m_i) = o(n^{-1/2})$ then every test statistic, permutation invariant or not, has power approaching its level.

The proofs hinge rather critically on the exact invariance properties of the statistics considered. In the Neyman-Scott problem for instance if a single sample size were to differ from all the others the whole argument would come apart. As long as the sample sizes are bounded the ARE of ANOVA is 0 nevertheless, as may be seen by direct calculation with the alternative non-central $F$-distribution. In the spacings problem of section 5 the sample 2-spacings defined by $D_i = U_{i+2} - U_i$ still provide tests with non-trivial power only at alternatives at the $n^{-1/4}$ distance; the joint distribution of these 2-spacings is not permutation invariant and our ideas do not help. Our ideas do apply, however, to the non-overlapping statistics of Del Pino(1971).

The definition of ARE offered here may well be challenged since the comparison is relative to the Neyman-Pearson test which would not be used for a composite null versus composite alternative situation. Nevertheless there seems to us to be a sharp distinction between procedures for which our definition yields an ARE of 0 and the quadratic tests mentioned above whose
6.2 Open Problems and Conjectures

The results presented here lead to some open problems and obvious areas for further work. Hájek’s proof of the permutation central limit theorem guarantees convergence of the characteristic function and moments up to order 2 of the variables $m^T P x$. Our heuristic calculations suggest that a good deal more information could be extracted if the characteristic function could be replaced by the moment generating function and if convergence of the moment generating function could be established not only for fixed arguments but for arguments growing at rates slower than $n^{1/2}$. Such an extension would eliminate, as in the normal example, the need for considering only contiguous alternatives. Large sample theory for spacings statistics suggests that the results presented here hold out to alternatives at such distances. If, in addition, approximations were available to the moment generating function for arguments actually growing at the rate $n^{1/2}$ the technique might extend to providing power calculations for alternatives so distant that permutation invariant tests have non-trivial limiting power. Another possible extension would use Edgeworth type expansions in the permutation central limit theorem to get approximations for the difference between the power and the level in the situation, covered by our theorems, where this difference tends to 0.

Consider the exponential family model of section 3 for the special case of the normal distribution. The problem of testing $m_1 = \cdots = m_n$ is invariant under the permutation group and under the sub-group of the orthogonal group composed of all orthogonal transformations fixing the vector 1 all of whose entries are 1. It is instructive to compare our results for the two different groups. The example illustrates the trade-off. For statistics invariant under larger groups it may be easier to prove the required convergence of the average likelihood ratio; the easier proof is balanced against applying the conclusion to a smaller family of statistics.

For statistics invariant under the larger group of orthogonal transformations fixing 1 we can modify the argument of section 2 and extend the conclusion described in the heuristic problem of section three to relatively
large values for $\|m - \overline{m}\|$. Rather than describe the details we follow a suggestion made to us by Peter Hooper. Suppose $Y = Xb + se$ where $e$ has a multivariate normal distribution with mean vector $m$ and variance covariance matrix the identity and where $s$ is an unknown constant, $X$ is an $n \times p$ matrix of regression covariate values of rank $p$ and $b$ is an unknown $p$ dimensional vector. Let $H = X(X^TX)^{-1}X^T$ be the hat matrix. Suppose we wish to test the hypothesis $(I - H)m = 0$. (If $X = 1$ this is equivalent to the problem mentioned above of testing $m_1 = \cdots = m_n$. The problem is identifiable only if $Hm = 0$ or equivalently if $X^Tm = 0$.) The problem is invariant under the group $O_X$ of orthogonal matrices $P$ for which $PX = X$.

Suppose $T(Y)$ is a family of statistics such that $T(YP) = T(Y)$ for any $P$ in $O_X$. Consider the likelihood ratio of $m$ to $Hm$ (the latter is a point in the null). Following equations (1-3) we are lead to study

$$\overline{L}(Y) = \int L(YP)F(dP),$$

where now $F$ is Haar measure on $O_X$ and $L(Y) = \exp(m^T(I - H)(Y - Xb) - \|(I - H)m\|^2/2)$. Since $PX = X$ we see that $L(YP) = \exp(m^T(I - H)P(Y - Xb) - \|(I - H)m\|^2/2)$. If $P$ is distributed according to $F$ and $Z$ is standard multivariate normal then $P^T(I - H)m/\|(I - H)m\|$ and $(I - H)Z/\|(I - H)Z\|$ have the same distribution. This fact and expansions of Bessel functions show that $\overline{L} \to 1$ in probability provided $\|(I - H)m\| = o((n - p)^{1/4})$.

The family of statistics invariant under the group of permutations of the entries of $Y$ will be different than the family invariant under $O_X$. When $X$ is simply a vector which is a non-zero multiple of $1$ the family of statistics invariant under the permutation group is much larger than the family invariant under $O_X$. For this case we are led to study the variable

$$\overline{T} = \sum_P \exp((m - \overline{m})^TPe - \|(m - \overline{m})\|^2/2)/n!.$$  

We find that $E_{\overline{m}}(\overline{T}) = 1$ and $\text{Var}_{\overline{m}}(\overline{T}) = \sum_P \exp((m - \overline{m})^TP(m - \overline{m}))/n! - 1$. Just when this variance goes to 0 depends on extending the permutation central limit theorem to give convergence of moment generating functions. Since the random variable $(m - \overline{m})^TP(m - \overline{m})$ has mean 0 and variance $\|(m - \overline{m})\|^4/n$ we are again led to the heuristic rate $\|(m - \overline{m})\| = o(n^{1/4})$. However, by taking $m$ to have exactly one non-zero entry it is not too hard to check that this heuristic calculation cannot be made rigorous without further conditions on $m$ to control
the largest entries.

Finally, if $X$ is not a scale multiple of 1 the problem is not invariant under the permutation group. Is there some natural extension of our techniques to this context for a group smaller than $O_X$?

Appendix

Proof of Theorem 3

We prove below (cf Shorack and Wellner, p 63 their formula 5, except that there the distribution $F$ does not depend on $n$) that

$$\rho_2(F, \hat{F}) \to 0$$

(23)

in probability. If Theorem 3 were false then from any counterexample sequence we could extract a subsequence which is a counterexample and along which the convergence in (23) is almost sure. The theorem then follows from the assertion that

$$\rho_2(F, G) \to 0 \quad \text{implies} \quad \rho_2(L_F, L_G) \to 0$$

(24)

whenever $F$ is any sequence of distributions with compact closure in $\Delta$.

Assertion (24) is a consequence of Lemma 1 of Guttman and Lockhart (1988).

To prove (23) we may assume without loss, in view of the compactness of $\Delta_0$ that $F \xrightarrow{2} \Phi$ for some $\Phi$. Elementary moment calculations assure that $\hat{F}(\tau)$ converges in probability to $\Phi(\tau)$ for each $\tau$ which is a continuity point of $\Phi$. This guarantees that $\rho_0(\hat{F}, \Phi) \to 0$ in probability. We need only show that $E_F(X^2) \to E_\Phi(X^2)$. But $E_F(X^2) = \sum X_i^2 / n$. The triangular array version of the law of large numbers given in Lemma 2 of Guttman and Lockhart (1988) shows that $\sum X_i^2 / n - E_F(X^2) \to 0$ in probability. Since $F \xrightarrow{2} \Phi$ implies that $E_F(X^2) \to E_\Phi(X^2)$ we are done.
Proof of Theorem 5

It remains to choose a sequence \( k = k(n) \) in such a way that we can check (22). In view of permutation invariance we may assume without loss that \( |m_1 - \overline{m}| \geq \cdots \geq |m_n - \overline{m}| \). Define matrices \( C \) and \( D \) by setting \( C(i, j) = \phi(X_j, m_i) - \overline{m} \phi(X_j, m_i) - \phi(X_j, \overline{m}) \) and \( D(i, j) = (m_i - \overline{m}) \phi_1(X_j; \overline{m}) \). We will eventually choose a sequence \( k \) and put \( B(i, j) = C(i, j) \) for \( i \leq k \) and \( B(i, j) = D(i, j) \) for \( i > k \). Note that \( \sum_i B(i, i) \) is simply \( S^*(X) \).

If \( P \) is a random permutation matrix then in row \( i \) there is precisely 1 non-zero entry; let \( J_i \) be the column where this entry occurs. Then \( S^*(PX) = \sum_i B(i, J_i) \). The variables \( J_1, \ldots, J_n \) are a random permutation of the set \( \{1, \ldots, n\} \). As in the proof of Theorem 2 let \( J_1^*, \ldots, J_n^* \) be a set of independent random variables uniformly distributed on \( \{1, \ldots, n\} \). We will show that for each fixed \( \kappa \)

\[
\rho_2 \left( \mathcal{L}(\sum_{i \leq \kappa} C(i, J_i^*)|X), \mathcal{L}(\sum_{i \leq \kappa} C(i, i)) \right) \to 0 \quad (25)
\]

and

\[
\rho_2 \left( \mathcal{L}(\sum_{i \leq \kappa} C(i, J_i^*)|X), \mathcal{L}(\sum_{i \leq \kappa} C(i, J_i)|X) \right) \to 0 \quad (26)
\]

in probability. We will also show that for any sequence \( k \) tending to \( \infty \) with \( k^2 = o(n) \) we have

\[
\rho_2 \left( \mathcal{L}(\sum_{i > k} (m_i - \overline{m}) \phi_1(X(J_i^*), \overline{m})|X), \mathcal{L}(\sum_{i > k} D(i, i)) \right) \to 0 \quad (27)
\]

in probability. There is then a single sequence \( k \) tending to infinity so slowly that (25) and (26) hold with \( \kappa \) replaced by \( k \) and so that (27) holds. We use this sequence \( k \) to define \( B \).

We will then show that

\[
\rho_2 \left( \mathcal{L}(\sum_i B(i, J_i^*)|X), \mathcal{L}(\sum_i B(i, i)) \right) \to 0 \quad (28)
\]
in probability and that
\[ \rho_2 \left( \mathcal{L} \left( \sum_i B(i, J_i) | X \right), \mathcal{L} \left( \sum_i B(i, J_i^*) | X \right) \right) \rightarrow 0 \tag{29} \]
in probability. These two are enough to imply (22) as in Corollary 2 and the obvious (but unstated) corresponding corollary to Theorem 4.

**Proof of (25)**

For each fixed \( i \) we may apply (23) with the vector \( (X_1, \ldots, X_n) \) replaced by \( (C(i, 1), \ldots, C(i, n)) \) to conclude that
\[ \rho_2(\mathcal{L}(C(i, J_i^*)|X), \mathcal{L}(C(i, i))) \rightarrow 0 \]
in probability; the condition imposed on \( F \) leading to (23) is implied by (21). Use the independence properties to conclude
\[ \rho_2(\mathcal{L}(C(1, J_1^*), \ldots, C(\kappa, J_\kappa^*)|X), \mathcal{L}(C(1, 1), \ldots, C(\kappa, \kappa))) \rightarrow 0 \]
for each fixed \( \kappa \). Assertion (25) follows.

**Proof of (26)**

For each fixed \( i \) we have \( \mathcal{L}(C(i, J_i^*)|X) = \mathcal{L}(C(i, J_i)|X) \). Furthermore it is possible to construct \( J \) and \( J^* \) in such a way that for each fixed \( \kappa \) we have \( P(J_i = J_i^*; 1 \leq i \leq \kappa) \rightarrow 1 \). This establishes (26).

**Proof of (27)**

Let \( \overline{m}_{-k} = \sum_{i > k} m_i / (n - k) \) and let \( U \) be the vector with \( i \)-th entry \( \phi_1(X_i; \overline{m}) \). Arguing as in the proof of Theorem 3 (see 24 above) we see that
\[ \rho_2 \left( \mathcal{L} \left( \sum_{i > k} (m_i - \overline{m}_{-k}) U(J_i^*) | X \right), \mathcal{L} \left( \sum_{i > k} (m_i - \overline{m}_{-k}) U_i \right) \right) \rightarrow 0 \tag{30} \]
in probability. We need to replace \( \overline{m}_{-k} \) by \( \overline{m} \) in order to verify (27). Elementary algebra shows \( \overline{m}_{-k} - \overline{m} = O(k/(n - k)) \). Temporarily let
\[ T_1 = \sum_{i > k} (m_i - \overline{m}_{-k}) U(J_i^*) - \sum_{i > k} (m_i - \overline{m}) U(J_i^*), \]
and

\[ T_2 = \sum_{i>k} (m_i - \bar{m}_k)U_i - \sum_{i>k} (m_i - \bar{m})U_i \]

In view of (21) we see that \( \text{Var}(U_1) = O(1) \). Hence

\[ \text{Var}(T_2) = (\bar{m}_k - \bar{m})^2(n - k)\text{Var}(U_1) \to 0. \]

(31)

Since the central limit theorem shows that the sequence \( \mathcal{L}(\sum_{i>k} D(i, i)) \) has compact closure in \( \Delta \) we may use (31) to show that

\[ \rho_2 \left( \mathcal{L}(\sum_{i>k} (m_i - \bar{m}_k)U_i), \mathcal{L}(\sum_{i>k} D(i, i)) \right) \to 0. \]

(32)

Next

\[ \text{Var}(T_1|X) = (\bar{m}_k - \bar{m})^2(n - k)\text{Var}(U(J^*_1)|X). \]

(33)

Since

\[ \text{Var}(U(J^*_1)|X) = \sum U_i^2/n - (\sum U_i/n)^2 \]

we may apply the triangular array law of large numbers given in Guttrop and Lockhart(1988, Lemma 2) to conclude that \( \text{Var}(U(J^*_1)|X) = O_P(1) \). Since \( k^2 = o(n) \) we see that the right hand side of (33) tends to 0 in probability. Hence

\[ \rho_2 \left( \mathcal{L}(\sum_{i>k} (m_i - \bar{m}_k)U(J^*_1)|X), \mathcal{L}(\sum_{i>k} (m_i - \bar{m})U(J^*_1)|X) \right) \to 0 \]

(34)

in probability. Assembling (30), (32) and (34) we have established (27).

**Proof of (28)**

Given \( X \), the variables \( \sum_{i\leq k} C(i, J^*_i) \) and \( \sum_{i>k} (m_i - \bar{m})\phi_1(X(J^*_i), \bar{m}) \) are independent. Similarly \( \sum_{i\leq k} C(i, i) \) and \( \sum_{i>k} D(i, i) \) are independent. Statement (28) then follows from (25) and (27).

**Proof of (29)**

To deal with (29) we must cope with the lack of independence among the \( J_i \). A random permutation of \( \{1, \ldots, n\} \) can be generated as follows.
Pick \( J_1, \ldots, J_k \) a simple random sample of size \( k \) from \( \{1, \ldots, n\} \). Let \( P_{-k} \), independent of \( J_1, \ldots, J_k \), be a random permutation of \( \{1, \ldots, n-k\} \). Then \( \sum B(i, J_i) \) has the same distribution (given \( X \)) as
\[
\sum_{i=1}^k C(i, J_i) + (m_{-k} - \overline{m}_{-k})^T P_{-k} U_{-k}
\]
where the subscript \(-k\) on \( m \) denotes deletion of \( m_1, \ldots, m_k \) while that on \( U \) denotes deletion of the entries \( U(J_1), \ldots, U(J_k) \).

Let \( Z_0 \) denote a random variable, independent of \( J \) and \( J^* \) whose conditional distribution given \( X \) is normal with mean 0 and variance \( \sum_{i>k} (m_i - \overline{m})^2 S \) where \( S \) is the sample variance of the \( U_j \)'s. Statement (29) is a consequence of the following 3 assertions:

\[
\rho_2 \left( \mathcal{L} \left( \sum_{i \leq k} B(i, J_i^*) \mid X \right), \mathcal{L} \left( \sum_{i \leq k} C(i, J_i^*) + Z_0 \mid X \right) \right) \to 0 \quad (35)
\]
in probability,\n
\[
\rho_2 \left( \mathcal{L} \left( \sum_{i \leq k} C(i, J_i) + Z_0 \mid X \right), \mathcal{L} \left( \sum_{i \leq k} C(i, J_i^*) + Z_0 \mid X \right) \right) \to 0 \quad (36)
\]
in probability, and

\[
\rho_2 \left( \mathcal{L} \left( \sum_{i \leq k} C(i, J_i) + Z_0 \mid X \right), \mathcal{L} \left( \sum_i B(i, J_i) \mid X \right) \right) \to 0 \quad (37)
\]
in probability.

Condition (36) follows from (26), the conditional independence of \( Z_0 \) and \( J, J^* \) and the fact that the conditional variance of \( Z_0 \) is bounded. Condition (35) is implicit in the proof of (28) after noting that the variance \( S \) of the entries in \( U \) is negligibly different from \( \nu(\overline{m}) \).

It remains to establish (37). We will condition on \((J_1, \ldots, J_k)\) as well as \( X \) and apply the Permutation Central Limit Theorem. The application of the conditions of that theorem is a bit delicate since the conditions will only be shown to hold in probability. We present the argument in the form of a technical lemma.
Lemma 2 Suppose \((W_1, W_2, W_3)\) is a sequence of random variables. Suppose \(g\) and \(h\) are sequences of measurable functions defined on the range spaces of \((W_1, W_2)\) and \((W_1, W_2, W_3)\). Let \(\zeta_1, \zeta_2\) be two independent real valued random variables. Suppose that there are functions \(f_i\) for \(i = 0, 1, \ldots\) (also indexed as usual by the hidden index \(n\)) such that

\[
f_0(w_1) \to 0 \quad \text{implies} \quad g(w_1, W_2) \overset{2}{\Rightarrow} \zeta_1
\]

and

\[
f_i(w_1, w_2) \to 0 \quad i = 1, 2, \ldots \quad \text{implies} \quad h(w_1, w_2, W_3) \overset{2}{\Rightarrow} \zeta_2.
\]

If \(f_0(W_1) \to 0\) in probability and \(f_i(W_1, W_2) \to 0\) in probability then

\[
\mathcal{L}(g(W_1, W_2) + h(W_1, W_2, W_3)|W_1) \Rightarrow \zeta_1 + \zeta_2
\]

in probability. If in addition

\[
\mathbb{E}(g(W_1, W_2) + h(W_1, W_2, W_3)|W_1) \to \mathbb{E}(\zeta_1 + \zeta_2)
\]

in probability, \(\text{Var}(g(W_1, W_2)|W_1) \to \text{Var}(\zeta_1)\) (41) in probability and

\[
\text{Var}(h(W_1, W_2, W_3)|W_1) \to \text{Var}(\zeta_2)
\]

in probability then

\[
\mathcal{L}(g(W_1, W_2) + h(W_1, W_2, W_3)|W_1) \overset{2}{\Rightarrow} \zeta_1 + \zeta_2
\]

in probability.

The lemma is to be applied with \(W_1 = X\), with \(W_2 = (J_1, \ldots, J_k)\) and with \(W_3 = P - k\). Conclusion (37) can be reduced to the form given by a compactness argument. The sequence of laws of \(\sum_{i \leq k} C(i, i)\) has compact closure in \(\Delta\) in view of (21). Then apply (25) and (26) to conclude that the sequence of laws of \(\sum_{i \leq k} C(i, J_i)\) also has compact closure. Let the distribution of \(\zeta_1\) be any limit point of this sequence of laws. The random variable \(\zeta_2\) will be the normal limit in distribution of \((m_{-k} - \bar{m})^T P^{-k} U_{-k}\).
In order to apply the lemma we must give the conditions of the Permutation Central Limit Theorem in a form in which we have only a countable family of convergences, as required in (39), to check. Note that (12) in Theorem 4 can be replaced by the assertion that there is a sequence \( \tau_1, \ldots \) of real numbers increasing to \( \infty \) such that

\[
\max(n^{-1} \sum (x_i - \bar{x})^2 1(|x_i - \bar{x}| > \tau_j), \Psi(\tau_j)) - \Psi(\tau_j) \to 0 \quad (43)
\]

for each \( j \geq 1 \).

We now apply Theorem 4 with \( x \) replaced by \( U_{-k} \), with \( m \) replaced by \( m_{-k} - \bar{m} \) and with \( P \) replaced by \( P_{-k} \). It is easy to check by conditioning on \( J \) that \( \mathbb{E}(U_{-k}) = 0 \) and \( \text{Var}(U_{-k}) = \iota(m) / (n - k) \). Hence \( (n - k)(\bar{m}_{-k} - \bar{m})U_{-k} \to 0 \) in probability. Set

\[
T_3 = \sum (U_{-k,i} - \bar{U}_{-k})^2 1(|U_{-k,i} - \bar{U}_{-k}| > t) / (n - k)
\]

Then

\[
T_3 \leq 2 \sum U_{-k,i}^2 1(|U_{-k,i} - \bar{U}_{-k}| > t) / (n - k) + 2\bar{U}_{-k}^2 \sum 1(|U_{-k,i} - \bar{U}_{-k}| > t) / (n - k).
\]

The second term on the right is \( O_p(1/(n - k)) = o_p(1) \). Since

\[
1(|U_{-k,i} - \bar{U}_{-k}| > t) \leq 1(|U_{-k,i}| > t/2) + 1(|\bar{U}_{-k}| > t/2)
\]

we see that

\[
T_3 \leq 2(n - k)^{-1} \sum U_i^2 1(|U_i| > t/2) + o_p(1).
\]

Take \( \Psi(t) = 2 \sup \{ \mathbb{E}_m(\phi_1(X_1, m)| > t/2); m \in \Theta_0 \} \) and apply Lemma 2 of Guttorp and Lockhart together with (21) to check that (43) holds. To finish the proof of (37) we need to check convergence of second moments as in (40), (41) and (42). This can be done using (25), (26) and direct calculation of the conditional mean and variance given \( X \) of \( \sum_{i>k} D(i, J_i) \). Theorem (5) follows.

The technical lemma itself may be proved as follows. From any counterexample sequence we may extract by a diagonalization argument a subsequence which is still a counterexample and for which \( f_0(W_1) \to 0 \) almost surely and
$f_j(W_1, W_2) \to 0$ almost surely for each $j \geq 1$. For any sample sequence for which all these convergences occur we have $\mathcal{L}(h(W_1, W_2, W_3)|W_1, W_2) \Rightarrow \zeta_2$ and $\mathcal{L}(g(W_1, W_2)|W_1) \Rightarrow \zeta_1$. Evaluation of the conditional characteristic function of $g(W_1, W_2) + h(W_1, W_2, W_3)$ given $W_1$ by further conditioning on $W_2$ yields the convergence in distribution asserted in the lemma. The remaining conclusions concerning moments are more elementary analogues of the same idea.

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