CONSERVED QUANTITIES IN GENERAL RELATIVITY: THE CASE OF INITIAL DATA SETS WITH A NONCOMPACT BOUNDARY

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ABSTRACT. It is well-known that considerations of symmetry lead to the definition of a host of conserved quantities (energy, linear momentum, center of mass, etc.) for an asymptotically flat initial data set, and a great deal of progress in Mathematical Relativity in recent decades essentially amounts to establishing fundamental properties for such quantities (positive mass theorems, Penrose inequalities, geometric representation of the center of mass by means of isoperimetric foliations at infinity, etc.) under suitable energy conditions. In this article I first review certain aspects of this classical theory and then describe how they can be (partially) extended to the setting in which the initial data set carries a non-compact boundary. In this case, lower bounds for the scalar curvature in the interior and for the mean curvature along the boundary both play a key role. Our presentation aims to highlight various rigidity/flexibility phenomena coming from the validity, or lack thereof, of the corresponding positive mass theorems and/or Penrose inequalities.

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1. INTRODUCTION

The ADM formalism in General Relativity has provided a systematic way to attach to certain solutions of Einstein field equations an array of conserved quantities, thus resolving, at least in the special cases where the strategy succeeds, an old controversy related to the notorious difficulties in making sense of such invariants. The underlying idea consists in requiring that in the asymptotic region the corresponding initial data set decays in a suitable sense towards some reference solution that, besides enjoying other nice properties which are irrelevant for the present discussion, is assumed to carry a non-trivial isometry group. After transplanting the corresponding Killing vector fields to the given initial data set by means of the chosen identification at infinity, one is able to apply our favorite rendition of Noether’s principle relating symmetries to conservation laws in order to exhibit the desired quantities, which in general are expressed in terms of certain flux integrals over the boundary at infinity. It then

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follows that these quantities are conserved under time evolution of the system whenever the matter
fields decay fast enough (in particular, for vacuum solutions). In addition to playing a paramount role
in the understanding of the dynamics of solutions, the study of these quantities (notably the energy
and the center of mass) reveals deep connections with Geometric Analysis (positive mass theorems
and their applications to the Yamabe problem, Penrose inequalities and the inverse mean curvature
flow, geometric representation of the center of mass by means of isoperimetric foliations at infinity,
etc.). In this setting, the scalar curvature emerges as the fundamental concept linking together these
quite disparate realms, as it not only may be viewed as the energy density along initial data sets but
also appears prominently in the zero order term of the relevant differential operators (the conformal
Laplacian, the Jacobi operator on minimal surfaces, the Dirac Laplacian acting on spinors, etc.). The
interactions between General Relativity and Geometric Analysis arising from these ideas are countless
and the aim of this survey is to convey certain aspects of this narrative from a rather personal perspec-
tive, with an emphasis towards positive mass theorems and the various rigidity/flexibility phenomena
stemming from them.

We start our journey in Section 2 by recalling how the classical attempts to make sense of the total
energy of a gravitational system in General Relativity were incorporated into the ADM approach to
Canonical Gravity \[10\] so as to allow a precise formulation of the so-called Positive Mass Theorem
(PMT) for initial data sets meeting suitable energy conditions. A proof of this central result was even-
tually obtained by Schoen and Yau using minimal surfaces \[50\] and soon afterwards Witten presented
an alternate proof based on spinors \[56\]. Although the spin assumption is quite restrictive in higher
dimensions, Witten’s method revealed itself quite flexible, being particularly useful when trying to
extend this circle of ideas to initial data sets carrying a noncompact boundary \[3\], \[5\], \[6\]. A distinctive
feature of this latter set of results, which is reviewed in Section 3, is that besides a lower bound on the
scalar curvature in the interior we also require, quite naturally, a lower bound on the mean curvature
along the boundary \[1\]. Similarly to what happens in the boundaryless case, the (time-symmetric) PMT
presented in Theorem 3.4 also finds notable applications in Geometric Analysis, especially in regard to
the Yamabe problem on manifolds with boundary \[1\], \[2\], \[7\], besides inspiring further developments \[38\].
In Section 4 certain rigidity/flexibility phenomena are summarized in case the initial data set yields a
solution with nonzero cosmological constant. In the asymptotically hyperbolic case we present a rigid-
ity result for conformally compact Einstein manifolds carrying a minimal inner boundary obtained in
\[5\], which may be of some interest in connection with the so-called AdS/BCFT correspondence. Also,
we discuss a flexibility result for the de Sitter-Schwarzschild metric \[20\] in the line of the famous (neg-
ative) solution of Min-Oo’s conjecture by Brendle-Marques-Neves \[15\]. In Section 5 we complete our
survey by explaining how the well-known connections between the center of mass and isoperimetry in
the boundaryless setting ensuing from the seminal article by Huisken and Yau \[37\] also admit suitable
extensions in the presence of a boundary \[4\]. Finally, with a view towards the future we intersperse
along the text a few interesting problems in this area of research.

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2. THE CLASSICAL LEGACY

The scalar curvature has played a significant role in General Relativity (GR) since its inception by
A. Einstein in 1915. Its use in the Lagrangian formulation of the theory was justified by H. Vermeil,
a student of F. Klein at Göttingen, who checked the allegedly folklore result that the scalar curvature

\[1\] In General Relativity, this intimate relationship between scalar and mean curvatures makes its debut in the Gibbons-
Hawking-York action, which plays a central role in the so-called path integral approach to Quantum Gravity \[58\], \[27\]. See \[31\]
below. For an in-depth account of the geometric side of this story, see \[15\].
$R_{\mathcal{g}}$ is the unique Riemannian/Lorentzian scalar invariant which depends on derivatives of the metric up to second order, being linear in the second derivatives. This naturally led to the selection of the so-called Hilbert-Einstein action

$$\mathcal{S}_{\mathcal{g}, \psi} \mapsto \int_{\mathcal{M}} \left( R_{\mathcal{g}} + \eta T_{\mathcal{g}, \psi} \right) d\mathcal{M},$$

where $\mathcal{g}$ is a Lorentzian metric on a given 4-manifold $\mathcal{M}$ and $\mathcal{T}$ is the matter-energy Lagrangian density, which is assumed to depend on the metric and on some external matter field $\psi$. By extremizing this action with respect to the metric and discarding boundary terms we obtain Einstein field equations

$$-G_{\mathcal{g}} + 8\pi T_{\mathcal{g}, \psi} = 0,$$

whose solutions $(\mathcal{g}, \psi)$ encode the dynamical features of the theory. Here,

$$G_{\mathcal{g}} = \text{Ric}_{\mathcal{g}} - \frac{R_{\mathcal{g}}}{2}\mathcal{g}$$

is the Einstein tensor of $\mathcal{g}$, $T_{\mathcal{g}, \psi}$ is the stress-energy-momentum tensor and we have adjusted the coupling constant $\eta$ conveniently. We should also add to (2.2) the system of equations $U_{\mathcal{g}, \psi} = 0$ obtained by extremizing (2.1) with respect to $\psi$, but this should not concern us here.

The problem remains of extracting physical information out of the highly nonlinear system (2.2). This already afflicted the founding fathers of GR, especially in regard to the status of energy conservation in the theory. In a by now well documented story, both Hilbert and Klein commissioned the algebraist E. Noether to clarify the role played by certain differential identities satisfied by the variational derivative of the action [39]. In a stroke of genius, Noether derived these identities in the general framework of her celebrated Second Theorem, which applies, for instance, to any Lagrangian theory whose symmetry group depends locally on finitely many functions of the spacetime variables. In the specific setting of GR as formulated above, these identities read as

$$\text{div}_{\mathcal{g}} \left( -G_{\mathcal{g}} + 8\pi T_{\mathcal{g}, \psi} \right) - \frac{1}{2} \mathcal{L}\psi^* U_{\mathcal{g}, \psi} = 0,$$

where $\mathcal{L}\psi^*$ is the formal adjoint of the Lie derivative $\mathcal{L}\psi$ acting on vector fields [11]. We stress that this holds true for any pair $(\mathcal{g}, \psi)$, irrespective of it being a solution. Notice also that in vacuum (absence of matter fields) this reduces to the contracted Bianchi identities for an arbitrary metric $\mathcal{g}$. As explained elsewhere, the aftereffect of this discussion is that even though GR admits a huge symmetry group encompassing the diffeomorphisms of $\mathcal{M}$, none of these yields a nontrivial conservation law on a given solution by the standard procedure [44]. In this way, full covariance, which is a treasured feature of the theory and happens to be the mechanism behind the validity of (2.3), actually entails the rather disappointing conclusion that conservation laws in GR are distinct in nature from those appearing for instance in Classical Mechanics or Special Relativity, which lie both in the confines of Noether’s First Theorem and hence are amenable to the more conventional treatment.

Despite this rather discouraging outcome, the formalism above already contains the clue towards an alternate construction of nontrivial conserved quantities in GR, at least for a restricted class of solutions. The key observation is that the differential identities in (2.3) suggest that only six out of the ten

\[\text{If most of the results in Sections 2-5 hold true in any spacetime dimension \( n + 1 \geq 4 \), we momentarily work in the physical dimension \( n = 3 \) in order to simplify the exposition. Also, we assume that all Lorentzian manifolds are time-oriented. Moreover, when representing tensor quantities in local coordinates, we will make use of the index ranges \( 0 \leq \alpha, \beta, \ldots \leq 3 \) and \( 1 \leq i, j, \ldots \leq 3 \), with the convention that the label 0 is reserved for a chosen time coordinate.}\]

\[\text{For our purposes, it suffices to assume that \( \psi \) varies among the sections of a natural vector bundle over \( \mathcal{M} \), so the action (2.1) is generally covariant, i.e. preserved by the action of the diffeomorphism group of \( \mathcal{M} \). This choice already includes many interesting examples.}\]

\[\text{The notable exception here takes place in the rather special cases where the underlying space-time carries Killing vector fields; this covers the stationary case, for instance [12].}\]
equations in (2.2) actually carry dynamical content, whereas the remaining four equations correspond to constraints relating the initial values of the fields instead of determining how these fields evolve. The standard way to confirm this intuition is to consider a spacelike slice \((M, g, h) \to (\mathcal{M}, \mathcal{g})\), which means that \(M\) is an embedded spacelike hypersurface, \(g = \mathcal{g}|_M\) is the induced Riemannian metric and \(h\) is the associated second fundamental form (with respect to the future directed, timelike unit normal vector field to \(M\)). If we set
\[
\mu(g, h) = \frac{1}{16\pi} \left( R_g - |h|^2 + (\text{tr} g h)^2 \right), \quad J(g, h) = \frac{1}{8\pi} \left( \text{div}_g h - d\text{tr}_g h \right),
\]
then the Gauss and Codazzi equations of hypersurface theory applied to (2.2) yield the constraint equations
\[
(2.4) \quad \mu = T_{00}, \quad J_i = T_{0i}.
\]
Interestingly enough, we see here the scalar curvature \(R_g\) of \(g\) resurfacing as a multiple of the energy density as measured by an observer comoving with the slice in the time-symmetric case \((h = 0)\).

A major breakthrough in Mathematical Relativity took place when Mme. Y. Choquet-Bruhat reversed this line of thought and proved that, at least in vacuum, (2.4) is a sufficient condition for the existence of solutions. More precisely, she showed that if one is given a triple \((M, g, h)\), where \((M, g)\) is a Riemannian 3-manifold and \(h\) is a twice covariant symmetric tensor on \(M\) satisfying (2.4) with \(T = 0\) then there exists a Lorentzian 4-manifold \((\mathcal{M}, \mathcal{g})\) satisfying \(G_\mathcal{g} = 0\) and a spacelike isometric embedding \((M, g) \to (\mathcal{M}, \mathcal{g})\) such that \(h\) is the induced second fundamental form. Roughly, this is proved by propagating in time the given initial data \((M, g, h)\) by means of the evolution system corresponding to the \((i \leq j)\)-components of the field equations and then checking that the constraints (in this case, \(\mu = 0\) and \(J = 0\)) are preserved; see [46, 18] for recent accounts of her work, including extensions by herself and others in the presence of matter fields.

Choquet-Bruhat’s theorem and its variants provided an initial value formulation for GR which shifted the focus from a solution \((\mathcal{M}, \mathcal{g})\) to an initial data set \((M, g, h, \mu, J)\), thus launching a whole new perspective on the subject. The following definition, which is an outgrowth of the so-called ADM approach to Canonical Gravity [10], illustrates this viewpoint in the context of the search for conserved quantities for a class of initial data sets modeling isolated gravitational systems.

**Definition 2.1.** An initial data set (IDS) \((M, g, h, \mu, J)\) is asymptotically flat with decay rate \(\tau \in (1/2, 1]\) if there exists an exterior region \(M_{\text{ext}}\), with \(M \setminus M_{\text{ext}}\) compact, and a diffeomorphism \(\mathbb{R}^3 \setminus B_1(0) \cong M_{\text{ext}}\) such that, in the corresponding asymptotic coordinate \(x\), there hold
\[
g_{ij}(x) = \delta_{ij} + O_3(r^{-\tau}), \quad h_{ij}(x) = O_1(r^{-\tau - 1}),
\]
and
\[
\mu(x) = O(r^{-2\tau - 2}), \quad J(x) = O(r^{-2\tau - 2}),
\]
as \(r = |x|_\delta \to +\infty\).

Intuitively, this means that as one approaches spatial infinity, \((M, g, h, \mu, J)\) converges in a suitable sense to \((\mathbb{R}^3, \delta, 0, 0, 0)\), an IDS of Minkowski vacuum space \((\mathbb{I}^1, \delta, 0, 0)\) of Special Relativity. Now, since this maximally symmetric vacuum solution carries a 10-dimensional space of isometries (the Poincaré group), we are tempted to use the restrictions of the corresponding Killing fields to the slice \(t = x_0 = 0\) in order to construct conserved quantities in the asymptotic limit via the appropriate version of Noether’s First Theorem [19, 31]. This works fine indeed and Table I provides the final outcome.

Regarding this table, a few comments are in order:
Killing vector field | Conserved quantity | Surface integral
---|---|---
∂₀ (time translation) | energy E | $16\pi E = \int_{S_\infty^2} U_1(\nu) dS_{\infty}^2$
∂ᵢ (spatial translation) | linear momentum P | $8\pi P_i = \int_{S_\infty^2} (i_0, \Pi)(\nu) dS_{\infty}^2$
$x_j\partial_0 + x_0\partial_j$ (boost) | center of mass C | $16\pi EC_i = \int_{S_\infty^2} U_i(\nu) dS_{\infty}^2$
$X_{ij} = x_i\partial_j - x_j\partial_i$ (spatial rotation) | angular momentum Q | $8\pi Q_{ij} = \int_{S_\infty^2} (iX_{ij}, \Pi)(\nu) dS_{\infty}^2$

Table 1. Conserved quantities for asymptotically flat IDS’s

- We set $U_w = w(\text{div}s e - dt\text{tr}s e) - i\nu_x w e + (\text{tr}s e)dw$, and $\Pi = h - (\text{tr}s h)g$,

where $e = g - \delta$, $w : \mathbb{R}^3 \to \mathbb{R}$ is a function, $1$ represents the function identically equal to $1$ and the infinity symbol means that we first compute the flux integrals over a large coordinate sphere $S^2_r$ and then pass the limit as $r \to +\infty$. In particular, $\nu$ is the outward pointing unit normal vector field to this sphere.

- $E$ and $C$ only depend on the intrinsic geometry of the initial data set. Notice also that $C$ only makes sense if $E \neq 0$.

- $(E, P)$ is the energy-momentum vector of the corresponding gravitational system. It is well defined under the ADM decay conditions in Definition 2.1.

- $C$ and $Q$ are well defined only if we further assume the so-called Regge-Teitelboim conditions:

$$g^{\text{odd}}(x) = O(r^{-\tau-1}), \quad h^{\text{even}}(x) = O(r^{-\tau-2}),$$

and

$$\mu^{\text{odd}}(x) = O(r^{-2\tau-3}), \quad J^{\text{odd}}(x) = O(r^{-2\tau-3}).$$

Here, $f^{\text{odd}}(x) = (f(x) - f(-x))/2$ and $f^{\text{even}} = f - f^{\text{odd}}$ for any function in the asymptotic region.

- These quantities are conserved under time evolution if the matter fields decay fast enough at infinity, except for the center of mass that satisfies $dC/dt = P$, as expected [19]; this also follows from the quasi-local approach in [54].

A great deal of progress in Mathematical Relativity in recent decades amounts to establishing fundamental properties of these quantities. The prominent result in this area is the Positive Mass Theorem (PMT) due to R. Schoen and S.-T. Yau. To state it, first we need single out those initial data sets which are viable from a physical viewpoint.

**Definition 2.2.** An initial data set $(M, g, h, \mu, J)$ satisfies the dominant energy condition (DEC) if there holds

$$\mu \geq |J|_g$$

everywhere along $M$.

Note that, by (2.4), this means that the 4-vector $T^0\alpha$ is causal and future directed. In words: a comoving observer never sees the energy-momentum density $(\mu, J)$ flowing faster than light!
Theorem 2.3. [50] If \((M, g, h, \mu, J)\) is an asymptotically flat IDS satisfying the DEC (2.5) then \(E \geq |P|_\delta\), with the equality holding if and only if \(E = |P|_\delta = 0\) and \((M, g, h)\) can be isometrically embedded in Minkowski space.

Thus, under (2.5) we see that \((E, P)\) is causal and future directed as a 4-vector in Minkowski space. In words: viewed from infinity the gravitational system mode led by the IDS moves as a massive particle in Special Relativity. This remarkable result was first proved using minimal surfaces techniques [50] and soon after by using spinors [56]. We note that the rigidity statement in arbitrary dimensions has been settled in full generality only recently [35]; we refer to this latter article for details on the history of this subject.

From our perspective the next corollary is worth mentioning.

Corollary 2.4. Let \((M, g)\) be a time-symmetric, asymptotically flat IDS (i.e. an asymptotically flat Riemannian manifold) satisfying \(R_g \geq 0\) everywhere. Then the ADM mass \(m_{\text{ADM}} := E\) is nonnegative and vanishes only if \((M, g) = (\mathbb{R}^3, \delta)\) isometrically.

This result, which actually holds true in any dimension \(n \geq 3\) [51, 41], has notable applications to many topics in Geometric Analysis, including the Yamabe problem and its developments [48, 40, 14]. But perhaps as important as the direct applications themselves is the injection of new methods in a honorable area of research. For instance, the tools employed to establish a rather special case of Corollary 2.4, namely, the so-called Geroch conjecture [5], when combined with the celebrated “controlled surgery” technique, eventually led to a huge collection of remarkable accomplishments, including the classification of closed, simply connected manifolds of dimension \(n \geq 5\) carrying a metric with positive scalar curvature [49, 28, 29, 52].

3. CONSERVED QUANTITIES IN THE PRESENCE OF A NONCOMPACT BOUNDARY

A version of the Yamabe problem on compact manifolds with boundary asks for a conformal metric which is scalar flat in the interior and has constant mean curvature along the boundary [25]. The strategy of attack here mimics the boundaryless situation: in the simpler case, the test function to be inserted in the relevant Yamabe quotient is devised by local methods, whereas in the harder case the construction of the test function is global in nature and in principle requires a version of the PMT in the presence of a noncompact boundary [1]. Even though this latter case was eventually settled by alternate methods [42], the use of a PMT is definitely more conceptual and seems to be indispensable when dealing with a parabolic version of the problem [2]. The appropriate version of the PMT applies to the class of Riemannian manifolds described below.

Definition 3.1. [3] A 3-manifold \((M, g)\) is asymptotically flat with a non-compact boundary \(\Sigma\) if there exists a diffeomorphism \(\mathbb{R}^3_+ \setminus B_1(0) \cong M_{\text{ext}}\) such that, in the corresponding asymptotic coordinates \(x\), there hold
\[
e^+ := g - \delta^+ = O_\delta(r^{-\tau}), \quad \tau > \frac{1}{2},
\]
and
\[
R_g = O(r^{-3 - \sigma}), \quad H_g = O(r^{-2 - \sigma}), \quad \sigma > 0,
\]
as \(r = |x|_\delta \to +\infty\). Here, \(H_g\) is the mean curvature of \(\Sigma, \mathbb{R}^3_+ = \{x \in \mathbb{R}^3; x_3 \geq 0\}\) and \(\delta^+ = \delta|_{\mathbb{R}^3_+}\).

For this kind of manifold we may define a notion of mass (or energy) \(\delta^+\) as indicated in the first line of Table 2 below, where \(\partial S^2_{\infty, +}\) is the large scale limit of coordinate hemispheres \(S^2_{\tau, +}\) of radius \(\tau\) centered at the origin of the model space \(\mathbb{R}^3_+\) and \(S^1_{\infty} = \partial S^2_{\infty, +}\) is the boundary at infinity of \(\Sigma\). However,

5This says that a metric with nonnegative scalar curvature on a torus is necessarily flat.

6See also [2] and the references therein for applications of this PMT to the compactness of the space of solutions.
where the triple \((M, g, \Sigma)\) is isometrically embedded in some Lorentzian manifold \((\overline{M}, \overline{g})\) carrying a 3-dimensional time-like boundary \(\Sigma\) such that \(\Sigma = M \cap \overline{\Sigma}\). In order to recover this as the IDS associated to a solution of appropriate field equations we must replace (2.1) by the Gibbons-Hawking-York action

\[
\langle \overline{g}, \psi, \xi \rangle \mapsto \int_M (R_{\overline{g}} + \eta T_{\overline{g}}) \, d\overline{M} + \int_{\overline{\Sigma}} (2H_{\overline{g}} + \eta \mathcal{J}_{\overline{g}}) \, d\overline{\Sigma},
\]

where \(H_{\overline{g}}\) is the mean curvature of the embedding \(\Sigma \hookrightarrow \overline{M}\) and \(\mathcal{J}\) is the matter-energy Lagrangian density due to a matter field \(\xi\) distributed along \(\Sigma\) \([8, 27]\). After extremizing this with respect to \(\overline{g}\), but this time carefully taking into account the boundary terms, we obtain the corresponding field equations:

\[
\begin{aligned}
\overline{G}_{\overline{g}} &= 8\pi T \text{ in } \overline{M}, \\
\overline{\Pi}_{\overline{g}} &= 8\pi S \text{ on } \overline{\Sigma}.
\end{aligned}
\]

Here, \(\overline{\Pi}_{\overline{g}} = \overline{B} - H_{\overline{g}} \overline{\eta}_{\overline{g}}\overline{\eta}_{\overline{g}}\), where \(\overline{B}\) is the second fundamental form of \(\overline{\Sigma}\) with respect to the inward pointing unit normal vector \(\overline{\eta}\). \(S\) is the boundary stress-energy-momentum tensor on \(\overline{\Sigma}\) induced by the matter distribution associated to \(\xi\), and we have set \(\overline{\eta} = \eta\). As we have seen, restriction of the first system of equations in (3.2) to \(M\) yields the interior constraint equations in (2.4). On the other hand, if we further assume that \(M\) meets \(\overline{\Sigma}\) orthogonally along \(\Sigma\), then a computation shows that the restriction of the second system of equations in (3.2) to \(\Sigma\) gives the boundary constraint equation:

\[
\begin{aligned}
H_g &= 8\pi S_{00}, \\
(i_\nu \Pi)_a &= 8\pi S_{0a},
\end{aligned}
\]

where the indexes \(1 \leq a, b, \ldots \leq 2\) refer to directions tangential to \(\Sigma\). Needless to say, the reader’s attention should be drawn to the distinctive role played by the mean curvature in the discussion above.

Now let \(\mathbb{L}^{1,3}_+ = \{ x \in \mathbb{L}^{1,3}_+; x_3 \geq 0 \}\) be the Minkowski half-space, whose boundary \(\partial \mathbb{L}^{1,3}_+\) is a time-like hypersurface. Notice that \(\mathbb{L}^{1,3}_+\) carries the totally geodesic spacelike hypersurface \(\mathbb{R}^3_+ = \{ x \in \mathbb{L}^{1,3}_+; x_0 = 0 \}\), the Euclidean half-space, which is endowed with the standard flat metric \(\delta = \delta_{ij}\mathbb{R}^3_+\). Notice also that \(\mathbb{R}^3_+\) carries a totally geodesic boundary \(\partial \mathbb{R}^3_+\). We now make precise the requirement that the spatial infinity of \(\overline{M}\), as observed along the initial data set \((M, g, h, \Sigma)\), is modeled on the inclusion \(\mathbb{R}^3_+ \hookrightarrow \mathbb{L}^{1,3}_+\).

**Definition 3.2.** \([6]\) We say that the IDS \((M, g, h, \Sigma)\) is asymptotically flat (with a non-compact boundary \(\Sigma\)) if there exists a diffeomorphism \(\mathbb{R}^3_+ \setminus B_1(0) \cong \mathbb{M}_{\text{ext}}\) such that, as \(r \to +\infty,
\[
g_{ij}(x) = \delta_{ij} + O_3(r^{-\gamma}), \quad h_{ij}(x) = O_1(r^{-\gamma-1}), \\
\mu(x) = O(r^{-2\gamma-2}), \quad J(x) = O(r^{-2\gamma-2}),
\]
and
\[
H_g \sim O(r^{-2-\sigma}), \quad (i_\nu \Pi)^\top \sim O(r^{-2-\sigma}),
\]
where \(^\top\) means orthogonal projection onto \(\Sigma\).

We may now proceed as in the boundaryless case and use the Killing vector fields on \(\mathbb{L}^{1,3}_+\) associated to the isometries leaving the boundary invariant\(^4\) in order to define an array of conserved quantities

\(^4\)We stress that \(\mathcal{J}\) keeps no relationship with \(\mathcal{J}\).  
\(^5\)The reason for the terminology is that, similarly to what happens in \([24, 33]\), seems to relate the initial values of fields along \(\Sigma\) instead of determining how these fields evolve. It remains to investigate whether these boundary constraints play a role in the corresponding initial-boundary value Cauchy problem. 
\(^6\)Note that this “half-Poincaré algebra” is isomorphic to the full Poincaré algebra of \(\mathbb{L}^{1,2}_+\). Thus, from a physical viewpoint, our model at infinity corresponds to Special Relativity with one less spatial dimension.
as portrayed in Table 2. Here, \( e^+ = g - \delta^+ \), \( \vartheta \) is the outward co-normal unit vector field to \( S^1_r \to \Sigma \) (computed with respect to \( \delta \)) and \( a, b = 1, 2 \). We also note that the extra flux integrals over \( S^1_{\infty, \varphi} \) which reflect the presence of the boundary \( \Sigma \), only appear in the expressions for the energy and the center of mass, which correspond to those Killing vector fields which are normal to the IDS.

We may now state the main result in [6], which provides a PMT for the energy-momentum 3-vector \( (\mathcal{E}, \mathcal{P}) \) under suitable DECs on the interior and along the boundary.

**Theorem 3.3.** [6] Let \((M, g, h, \Sigma)\) be an asymptotically flat IDS with a noncompact boundary as above and assume that it satisfies the interior DEC

\[
(3.4) \quad \mu \geq |J|_g \quad \text{in} \quad M,
\]

and the boundary DEC

\[
(3.5) \quad H_g \geq |(i_g \Pi)\top|_g \quad \text{on} \quad \Sigma.
\]

Then \( \mathcal{E} \geq |\mathcal{P}|_\delta \), with the equality holding only if \( \mathcal{E} = |\mathcal{P}|_\delta = 0 \) and \((M, g)\) can be isometrically embedded in \( \mathbb{L}^{1,3}_+ \) in such a way that \( h \) is the induced second fundamental form, \( \Sigma \) is totally geodesic (as a hypersurface in \( M \)), lies on \( \partial \mathbb{L}^{1,3}_+ \) and \( M \) is orthogonal to \( \partial \mathbb{L}^{1,3}_+ \) along \( \Sigma \).

**Proof.** (sketch) Since \( M \) is spin, we may use the DECs to ensure that there exists a spinor \( \psi \) on \( M \) which is harmonic with respect to the corresponding Dirac-Witten operator and which satisfies suitable boundary conditions both at infinity (it asymptotes a previously chosen “constant” spinor \( \phi \) in the model \( \mathbb{L}^{1,3}_+ \)) and along the boundary (it satisfies the so-called MIT bag boundary condition). With these spinors at hand, a somewhat involved computation establishes a nice extension of Witten’s celebrated formula [56] for the energy-momentum vector in the presence of a boundary:

\[
\frac{1}{4} \left( \mathcal{E}|\phi|^2 - \langle \phi, \mathcal{P} \partial_{x_0} \cdot \partial_{x_2} \cdot \phi \rangle \right) = \int_M (|\nabla \psi|^2 + \langle \mathcal{R} \psi, \psi \rangle) \, dM + \int_\Sigma \langle \mathcal{H} \psi, \psi \rangle \, d\Sigma,
\]

where \( \mathcal{R} \) and \( \mathcal{H} \) are certain self-adjoint endomorphims acting on spinors. Since the DECs (3.4) and (3.5) imply that \( \mathcal{R} \geq 0 \) and \( \mathcal{H} \geq 0 \), respectively, the mass inequality \( \mathcal{E} \geq |\mathcal{P}|_\delta \) follows after choosing \( \phi \) properly. The rigidity statement is a bit more involved as we are supposed to start with the equality \( \mathcal{E} = |\mathcal{P}|_\delta \) and then to prove that \( \mathcal{E} = |\mathcal{P}|_\delta = 0 \) indeed. Once this is accomplished, we are able conclude the argument by means of a reflection method that allows us to apply the rigidity part of the PMT in

\[\text{[56]} \text{We first learned about the energy } \mathcal{E} \text{ in conversations with F. Marques.}\]
for manifolds without boundary. This last step employs in a crucial way an Ashtekar-Hansen-type formula for $\mathcal{E}$.

As it is apparent from this proof, Theorem 3.3 holds true in any dimension $n \geq 3$ if we assume that $M$ is spin. Note also that in the time-symmetric case ($\Pi = 0$), we check that $\mathcal{R} = R_g/4, \mathcal{H} = H_g/2$ and $\mathcal{E} = 0$, so Theorem 3.3 reduces to a purely Riemannian assertion.

**Theorem 3.4.** If $(M, g, \Sigma)$ is asymptotically flat as in Definition 3.1 with $R_g \geq 0$ and $H_g \geq 0$ then its mass $m := \mathcal{E}$ is nonnegative and vanishes only if $(M, g, \Sigma) = (\mathbb{R}^3_+, \delta^+, \partial \mathbb{R}^3_+)$ isometrically.

The following rigidity result, which is the analogue of Geroch’s conjecture in our setting, is worth mentioning.

**Corollary 3.5.** Under the conditions of Theorem 3.4, if $g = \delta^+$ outside a compact set then $(M, g, \Sigma) = (\mathbb{R}^3_+, \delta^+, \partial \mathbb{R}^3_+)$ isometrically.

We remark that the proof of Theorem 3.4 in [3] involves a doubling argument to reduce it to the standard (time-symmetric) PMT. Thus, in view of recent breakthroughs [51, 41] it actually holds true in any dimension (with no need for the spin assumption). We remark that an alternative approach in low dimensions, based on the theory of free boundary minimal hypersurfaces, is presented in [16]. Also, it should be emphasized that the mean convexity condition $H_g \geq 0$ is derived as an immediate consequence of the boundary DEC, thus acquiring, likewise the interior DEC $R_g \geq 0$, a justification on purely physical grounds: (3.5) means that the 3-vector $S_0$ is causal and future directed.

### 4. The Case of a Nonzero Cosmological Constant

If we now work in an arbitrary spatial dimension $n \geq 3$ and in the actions (2.1) and (3.1) we replace $R_g$ by $R_g - 2\Lambda n_\epsilon$, where $\Lambda_{n, \epsilon} = cn(n-1)/2, \epsilon = \pm 1$, is a cosmological constant, then $G_g$ gets replaced by $G_g + \Lambda_{n, \epsilon}g$ in the field equations (2.2) and (3.2). The model spacelike metric is now given by

$$g_{m, \epsilon} = \frac{ds^2}{1 - \epsilon s^2 - \frac{2m}{s^{n-2}}} + s^2 h_0,$$

where $m \geq 0$ is a real parameter, $h_0$ is the standard metric on $S^{n-1}$ or $S^{n-1}_+$ and $s$ varies in a certain interval $I_{m, \epsilon} \subset [0, +\infty]$. One easily verifies that (4.1) is the vacuum time-symmetric IDS associated to the Lorentzian metric

$$\mathcal{G}_m, \epsilon = -\left(1 - \epsilon s^2 - \frac{2m}{s^{n-2}}\right) dx_0^2 + g_{m, \epsilon}, \quad x_0 \in \mathbb{R},$$

which satisfies $G_{\mathcal{G}_{m, \epsilon}} + \Lambda_{n, \epsilon} \mathcal{G}_{m, \epsilon} = 0$. We next consider rigidity/flexibility phenomena, in the spirit of Corollary 3.5 where these models appear prominently.

**Remark 4.1.** The case $\epsilon = 0$ in (4.1) corresponds to the famous Schwarzschild metric, which models an uncharged, non-spinning black hole in GR with a (minimal) horizon located at $s = (2m)^{-\frac{1}{n-2}}$. Another way of expressing this metric involves introducing a new radial parameter $r$ by

$$s = r \left(1 + \frac{m}{2r^{n-2}}\right)^{\frac{n-2}{n}},$$

so that a direct computation gives

$$g_{m, 0} = \left(1 + \frac{m}{2r^{n-2}}\right)^{-\frac{n-2}{n}} \delta.$$

In these isotropic coordinates, the horizon is located at $r = (m/2)^{-\frac{1}{n-2}}$. 

4.1. The asymptotically hyperbolic case ($\epsilon = -1$). Due partially to potential applications to the so-called ADS/CFT correspondence, in recent years there has been a lot of interest in establishing positive mass theorems in case the asymptotic geometry at spatial infinity of the given spacetime is anti-deSitter space (or a quotient thereof) [13]; this corresponds to choosing $\epsilon = -1$ in (4.1). A notable result in this direction was put forward by Wang [55], who worked in the spin category and treated the conformally compact case. In particular, the rigidity statement in his main theorem recovers a previous result characterizing the standard hyperbolic space as the unique conformally compact, asymptotically hyperbolic spin manifold which is Einstein and whose conformal boundary at infinity is the canonical conformal structure on the sphere [9]. Since our chief interest in this section is to explain how to extend this result in the presence of a noncompact boundary, we refrain from discussing the general theory of large scale invariants for asymptotically hyperbolic manifolds. In any case, a good summary of recent results in this area may be found in [34].

Let $\mathcal{N}$ be a compact $n$-manifold, $n \geq 3$, whose boundary $\partial \mathcal{N}$ decomposes as the union of two smooth hypersurfaces $S$ and $\Sigma$, with $S$ being connected and such that $S \cap \Sigma$ is a $(n - 2)$-dimensional “corner”. If $g$ is a Riemannian metric on $\mathcal{N} := \text{int} \mathcal{N} \cup \Sigma$ then we say that $(\mathcal{N}, g)$ is conformally compact if there exists a collar neighborhood $\mathcal{U} \subset \mathcal{N}$ of $S$ such that on int $\mathcal{U}$ we may write $g = \rho^{-2} \hat{g}$ with $\hat{g}$ extending to a sufficiently regular metric on $\mathcal{U}$ so that $S$ and $\Sigma$ meet orthogonally (with respect to $\hat{g}$) along their common boundary $\mathcal{U} \cap \Sigma$, where $\rho : \mathcal{U} \to \mathbb{R}$ is a defining function for $S$ in the sense that $\rho \geq 0, \rho^{-1}(0) = S$, $d\rho|_{S} \neq 0$ and $\nabla_{\hat{g}} \rho$ is tangent to $\Sigma$ along $\mathcal{U} \cap \Sigma$. Clearly, the restriction $\hat{g}|_{\Sigma}$ determines a metric which changes by a conformal factor in case the defining function is changed. Thus, the conformal class $[\hat{g}|_{\Sigma}]$ of $\hat{g}|_{\Sigma}$ is well defined. We then say that the pair $(S, [\hat{g}|_{\Sigma}])$ is the conformal infinity of $(\mathcal{N}, g)$.

One then computes, as $\rho \to 0$, that the curvature tensor of $g$ is given by

$$ R_{ijkl} = -|d\rho|_{\hat{g}}^2 (\hat{g}_{ik} \hat{g}_{jl} - \hat{g}_{il} \hat{g}_{jk}) + O(\rho). $$

Thus, if $|d\rho|_{\hat{g}} = 1$ along $S$ then $(\mathcal{N}, g)$ is weakly asymptotically hyperbolic in the sense that its sectional curvature converges to $-1$ as one approaches $S$. In this case, it is shown that if $h_0$ is a metric on $S$ representing the given conformal infinity then there exists a unique defining function $\theta$ in $\mathcal{U}$ so that

$$ g = \sinh^{-2} \theta \left( d\theta^2 + h_0 \right), $$

where $h_0$ is a $\theta$-dependent family of metrics on $S$ with $h_0|_{\theta=0} = h_0$.

**Definition 4.2.** Let $(\mathcal{N}, g)$ be a weakly asymptotically hyperbolic manifold satisfying (4.3). We say that $(\mathcal{N}, g)$ is asymptotically hyperbolic (in the conformally compact sense and with a non-compact boundary $\Sigma$) if its conformal infinity is $(S^{n-1}_+, [h_0])$, where $h_0$ is a round metric on $S^{n-1}_+$, the unit upper $(n - 1)$-hemisphere, and the following asymptotic expansion holds as $\theta \to 0$:

$$ h_\theta = h_0 + \frac{\theta^n}{n!} h + \mathcal{E}, $$

where $h$ and $\mathcal{E}$ are symmetric 2-tensors on $S^{n-1}_+$ and the remainder term $\mathcal{E}$ satisfies

$$ |\mathcal{E}| + |\nabla_{h_0} \mathcal{E}| + |\nabla^2_{h_0} \mathcal{E}| = o(\theta^{n+1}). $$

**Remark 4.3.** It is instructive to briefly discuss the model space underlying Definition 4.2. For this we fix a constant $m \geq 0$ and consider the metric in (4.1) with $\epsilon = -1$, which is defined on $\mathbb{H}^n_{+,m} := (s_{m,-1}, +\infty) \times S^{n-1}_+$, where $s_{0,-1} = 0$ and for $m > 0$, $s_{m,-1} > 0$ is the unique positive zero of the denominator in (4.1). Clearly, $(\mathbb{H}^n_{+,0}, g_{0,-1})$ is the hyperbolic half-space, which is obtained from the standard hyperbolic space by cutting along the totally geodesic hypersurface $(0, +\infty) \times S^{n-2}$. For $m > 0$, $(\mathbb{H}^n_{+,m}, g_{m,-1})$ is the anti-de Sitter-Schwarzschild (adSS) half-space. These latter metrics satisfy $R_{g_{m,-1}} = -n(n - 1)$ and are asymptotically hyperbolic in the sense that $g_{m,-1} = g_{0,-1} + O(s^{-n})$ as $s \to +\infty$. The adSS manifolds represent the gravitational field of a dynamical, noncompact, smooth membrane of internal dimension $n - 1$ moving with prescribed speed at spatial infinity, together with the corresponding Gauss curvature $-n(n-1)$.
$s \to +\infty$. In order to check that each $g_{m,-1}$ is conformally compact we rewrite it as

$$g_{m,-1} = \sinh^{-2} \theta (d\theta^2 + u(\theta)^2 \delta_0),$$

so that $s$ and $u$ satisfy the ODEs

$$\dot{s}(\theta) = - \sinh^{-1} \theta \sqrt{1 + s^2 - \frac{2m}{s^{n-2}}}$$

and

$$\cosh \theta u - \sinh \theta \dot{u} = \sqrt{\sinh^2 \theta + u^2 - \frac{2m}{u^{n-2}}} \sinh \theta,$$

with $s(0) = +\infty$ and $u(0) = 1$. A short computation then shows that, as $\theta \to 0$, $u$ expands as

$$u(\theta) = 1 + c_n m \theta^n + o(\theta^{n+1}), \quad c_n > 0,$$

as desired.

The parameter $m$ may be viewed as the total mass of the gravitational system modeled by the IDS $(\mathbb{H}^n_{+m}, g_{m,-1})$. More generally, a notion of mass may be assigned to any manifold as in Definition 4.2, which, as in the asymptotically flat case, captures the rate of convergence of $g$ towards the reference metric $g_{0,-1}$, with the corresponding PMT being established in the spin category $[5,14]$. As a consequence of its rigidity statement we obtain the next theorem, which extends to our setting a previous result due to Andersson-Dahl [9].

**Theorem 4.4.** [5] Let $(M, g, \Sigma)$ be a conformally compact, asymptotically hyperbolic spin $n$-manifold as above. Assume further that $g$ is Einstein and that the mean curvature of $\Sigma$ is everywhere nonnegative. Then $(M, g, \Sigma) = (\mathbb{H}^n_{+m}, g_{0,-1}, \partial \mathbb{H}^n_{+m})$ isometrically.

In words: the only way to fill in the conformal class $(\mathbb{S}^{n-1}_+,[\gamma_0])$ by an Einstein metric in the spin category and assuming further that the inner boundary is mean convex is the obvious one given by the hyperbolic half-space. This rigidity result may be of some interest in connection with recent developments involving the construction of a holographic dual to a conformal field theory defined on a manifold with boundary, the so-called ADS/BCFT correspondence $[3,5]$. In this context, the problem of determining the structure of the moduli space of conformally compact, Einstein manifolds with a given conformal infinity and having a minimal inner boundary plays a key role. A further development in this circle of ideas led to the formulation of the so-called Ryu-Takayanagi conjecture in Quantum Gravity $[17]$, which turns out to be a huge generalization of the celebrated Beckenstein-Hawking formula for the black hole entropy and proposes to compute the *entanglement entropy* determined by a proper region $S \subset W$ in a given conformal infinity $(W,[\gamma])$, where $W$ is closed and $[\gamma]$ is the conformal class of a metric $\gamma$ on $W$. In the static case the conjecture roughly says that this kind of entropy is proportional to the area of the minimal inner boundary $\Sigma_S$ which, together with $S$, bounds a conformally compact Einstein region $(M,g)$ which has $(S,[\gamma]|S)$ as its conformal infinity. Theorem 4.4 suggests that, at least for conformal infinities close to the conformal class $(\mathbb{S}^{n-1}_+, [\gamma_0])$ and for bulk regions parametrized by the half $n$-disk, the corresponding moduli space should be constituted by a unique configuration, thus eliminating any ambiguity in the choice of $\Sigma_S$. This would extend to this setting a recent breakthrough in the boundaryless case $[17]$.

**Remark 4.5.** The metric $g_{m,-1}, m > 0$, is *not* a model for Theorem 4.4 for at least two reasons: it is not Einstein and, more importantly, it carries an inner minimal horizon (black hole) given by $s = s_{m,-1}$. In fact, this family of metrics models the equality case of a (conjectured) Penrose inequality in the

\[\text{See also}[6]\text{ for a non-time-symmetric version of this result.}\]

\[\text{The “B” in BCFT means that the conformal boundary itself carries a nontrivial boundary.}\]
asymptotically hyperbolic setting which is still wide open to the best of our knowledge. For instance, in the boundaryless case this inequality has only been proved in two situations: graphs in any dimension \[21\] and small perturbations of the adSS metric for \(n = 3\) \[8\]. It would be interesting to investigate if the techniques in these works could be adapted in the presence of a boundary.

**Remark 4.6.** It also follows from the main result in \[3\] that the conclusion of Theorem \[4.4\] still holds if we replace the Einstein assumption by two other natural requirements: \(R_g \geq -n(n - 1)\) everywhere and \(g = \delta_{0,-1}\) outside a compact set.

**4.2. The case of positive cosmological constant** (\(\epsilon = 1\)). Here we consider the model metric in \(4.1\) with \(\epsilon = 1\). This metric, named after de Sitter-Schwarzschild (dSS), satisfies \(R_{\delta_{m,1}} = n(n - 1)\) and for each \(m\) such that

\[
0 < m < m_\ast := \frac{(n - 2)(n - 2)/2}{n^{n/2}},
\]

it is defined for \(s\) varying in a certain bounded interval \((s_{m,1}, s^\ast_{m,1}) \subset (0, +\infty)\). As \(m \to 0\), \(s_{m,1} \to 0\) and \(s^\ast_{m,1} \to 1\), so we recover the round hemisphere \((S^n_+, h_0)\) in the limit. Even though the lack of an asymptotic region prevents us from defining a notion of mass in this setting, we may still ask whether the dSS space is rigid with respect to a metric perturbation \(g\) which coincides with \(\delta_{m,1}\) up to first order along the totally geodesic boundary and satisfies the energy condition \(R_g \geq n(n - 1)\). Surprisingly, the next result confirms that this is never the case.

**Theorem 4.7.** \[20\] For each \(n \geq 3\) and for each value of the mass parameter as in \(4.4\), the dSS space \(M_{m,1} := (s_{m,1}, s^\ast_{m,1}) \times S^{n-1}\), carries metrics \(g_1\) and \(g_2\) with the following properties:

- \(R_{g_1} > n(n - 1)\) everywhere;
- \(g_1 = \delta_{m,1}\) along \(\partial M_{m,1}\);
- \(\partial M_{m,1}\) is totally geodesic with respect to \(\delta_{m,1}\),

and

- \(R_{g_2} \geq n(n - 1)\), with the strict inequality holding somewhere;
- \(g_2 = \delta_{m,1}\) in a whole neighborhood of \(\partial M_{m,1}\).

As \(m \to 0\) we recover the celebrated (negative) solution of Min-Oo’s conjecture \[15\]. Hence, flexibility seems to be a characteristic feature in the positive cosmological constant regime. Taken together, these results imply that there are no analogues of the rigidity statements of the positive mass and Penrose inequality in this case; this should be contrasted to Corollary \[5.5\] and Remark \[4.6\] where rigidity prevails thanks to the appropriate PMT.

**Remark 4.8.** As \(m \to m_\ast\), the dSS space converges to the cylinder \([0, s_\ast\pi] \times S^{n-1}(s^\ast)\) endowed with the metric \(g_{m,1} = dr^2 + s^2_\ast h_0\), where \(s_\ast^2 = (n - 2)/n\). Unfortunately, the argument leading to Theorem \[4.7\] above completely breaks down in this limit. It is an interesting question to examine the rigidity/flexibility of this IDS. Another nice question appears after consideration of the half dSS space \(M_{m,1,+} := (s_{m,1}, s^\ast_{m,1}) \times S^{n-2}_+\), which carries the totally geodesic inner boundary \(M_{m,1,+} := (s_{m,1}, s^\ast_{m,1}) \times S^{n-2}_+\). Again, it would be interesting to investigate the flexibility of this space with an eye towards extending Theorem \[4.7\] to this setting.

5. **Mass, center of mass and isoperimetry in the presence of a boundary**

Here we bring the center of mass \(c\) defined in Section \[3\] to the forefront of our discussion on rigidity phenomena and explain how it plays a central role in determining the large scale isoperimetric profile of an asymptotically flat 3-manifold \((M, g, \Sigma)\) of positive mass. The corresponding story in the boundaryless case is summarized in \[33\] \[24\].
For the sake of motivation, we start by fixing an interior point \( q \in M \). For each \( r > 0 \) small enough we may consider the isoperimetric quotient

\[
I^M_r (q) = \frac{A_r(q)^{3/2}}{V_r(q)},
\]

where \( A_r(q) \), respectively \( V_r(q) \), is the area of the geodesic sphere, respectively the volume of the geodesic ball, of radius \( r \) centered at \( q \). If \( I^{\mathbb{R}^3} = 6\pi^{1/2} \) is the corresponding quotient in \( \mathbb{R}^3 \), which obviously does not depend on \( (q, r) \), then a classical computation gives, as \( r \to 0 \),

\[
1 - \frac{I^M_r (q)}{I^{\mathbb{R}^3}} = \frac{R_g(q)}{20} r^2 + O(r^4).
\]

Notice that the left-hand side may be computed if the metric \( g \) is only assumed to be \( C^0 \). Thus, the validity of the local subisoperimetry property

\[
(5.1)
\]

may be interpreted as saying that \( R_g(q) \geq 0 \) in a weak sense. Interestingly enough, a similar interpretation holds for the mean convexity condition \( H_g(q) \geq 0 \) at a boundary point. Indeed, after introducing Fermi coordinates around \( q \in \Sigma \) we may consider the isoperimetric quotient

\[
I^\Sigma_r (q) = \frac{A_r^+(q)^{3/2}}{V_r^+(q)},
\]

where \( A_r^+(q) \) is the area of the coordinate hemisphere of radius \( r \) centered at \( q \) and \( V_r^+(q) \) is the volume of the region enclosed by this hemisphere and \( \Sigma \). This time we have

\[
1 - \frac{I^\Sigma_r (q)}{I^{\mathbb{R}^3, \mathbb{R}^2}} = \frac{3}{16} H_g(q) r + O(r^2),
\]

where \( I^{\mathbb{R}^3, \mathbb{R}^2} = 3 \cdot 2^{1/2} \cdot \pi^{1/2} \) is the corresponding quotient evaluated at hemispheres centered at \( \mathbb{R}^2 = \partial \mathbb{R}^3_+ \). Thus, we may think of the local boundary subisoperimetry property

\[
(5.2)
\]

as expressing mean convexity of \( \Sigma \) at \( q \) in a weak sense. We may now envisage a far-reaching generalization of Theorem 3.4.

**Question 5.1.** If \( (M, g, \Sigma) \) is a \( C^0 \) asymptotically flat 3-manifold satisfying (5.1) in the interior and (5.2) along the boundary, is it true that its “isoperimetric mass” is nonnegative and vanishes only if \( (M, g, \Sigma) = (\mathbb{R}^3_+, \delta^+, \mathbb{R}^2) \) isometrically? 

As it is always the case with a (perhaps too) optimistic proposal, the proper definition of the concepts involved is part of the problem. However, a little examination of the smooth case reveals a natural candidate for the isoperimetric mass.

**Definition 5.2.** For a (smooth) asymptotically flat 3-manifold \( (M, g, \Sigma) \) as in Definition 3.1 let \( A^+(r) \) (respectively \( V^+(r) \)) be the area of \( S^2_{r,+} \) (respectively, the volume of the compact region enclosed by \( S^2_{r,+} \) and \( \Sigma \)) and set

\[
I^M_r = \frac{V^+(r)}{A^+(r)} \left( 1 - \frac{I^{M, \Sigma}_r}{I^{\mathbb{R}^3, \mathbb{R}^2}} \right),
\]

\[13\] Curiously enough, this isoperimetric interpretation of the scalar curvature is due to the same H. Vermeil mentioned in the beginning of Section 3 and already appears in Pauli’s famous encyclopedia article on GR [55].
where

\[ I_{r}^{M;\Sigma} = \frac{A^{+}(r)^{\frac{3}{2}}}{V^{+}(r)}. \]

and \( I_{R_{3};R^{2}} = 3 \cdot 2^{1/2} \pi^{1/2} \) is the corresponding isoperimetric quotient computed at a hemisphere centered at a point in \( R^{2} = \partial R_{3}^{+} \). With this notation, the isoperimetric mass of \((M, g, \Sigma)\) is defined by

\[ m_{\text{iso}} = \lim_{r \to +\infty} I_{r}^{M;\Sigma}. \]

That we follow a promising route is manifested by the next result, which extends to our setting a famous remark by Huisken [36].

**Theorem 5.3.** [4] One has \( m_{\text{iso}} = m \).

**Corollary 5.4.** If \((M, g, \Sigma)\) satisfies \( R_{g} \geq 0 \) in the interior and \( H_{g} \geq 0 \) along the boundary then \( m_{\text{iso}} \geq 0 \), with the strict inequality holding unless \((M, g, \Sigma) = (R_{3}^{+}, \delta^{+}, R^{2})\) isometrically.

**Proof.** Apply Theorem 3.4. \( \square \)

This corollary may be viewed as a manifestation of the validity of Question 5.1 in the smooth world, thus supplying some evidence for its validity in general.

After this somewhat lengthy preamble, which emphasized the relationship between the (sign of the) mass and the large scale isoperimetric properties of an IDS with a noncompact boundary, we now investigate what the center of mass has to say in this respect. For this it is convenient to restrict ourselves to a special class of IDSs, whose behavior at infinity is modeled on the Schwarzschild metric (4.2) with \( n = 3 \).

**Definition 5.5.** An asymptotically flat 3-manifold with a noncompact boundary \((M, g, \Sigma)\) is asymptotically half-Schwarzschild (ahS) if a neighborhood of infinity is diffeomorphic to the complement of a hemisphere in \( R^{3} \) so that

\[ g = \left( 1 + \frac{2m}{r} \right) \delta^{+} + p^{+}, \quad p^{+} = O(r^{-2}) \]

holds in this asymptotic region.

As usual, if we further take it for granted that the corresponding Regge-Teitelboim conditions are met, then the limit defining ‘\( \mathcal{C} \)’ in Table 2 converges [22]. Now, Theorem 5.3 suggests that for an ahS manifold with \( m > 0 \), large coordinate hemispheres may be perturbed to yield global solutions of the corresponding relative isoperimetric problem, where each competing surface \( S \) satisfies \( \partial S \subset \Sigma \) and \( \text{int} \ S \cap \Sigma = \emptyset \), with the constrained volume being the one enclosed by \( S \) and \( \Sigma \). The next result, which identifies ‘\( \mathcal{C} \)’ to the center of a geometric foliation at infinity, turns out to be a first step towards this goal.

**Theorem 5.6.** [4] Assume that \((M, g)\) is an ahS 3-manifold with a noncompact boundary \( \Sigma \). If \( m = m_{/2} > 0 \) then there exists a neighborhood of infinity which is foliated by strictly stable free boundary constant mean curvature hemispheres. Moreover, the geometric center of this foliation coincides with the center of mass ‘\( \mathcal{C} \)’ of \((M, g, \Sigma)\).

The proof of this result makes use of the so-called implicit function method pioneered by Ye [57] and refined by Huang [32]. Our next contribution completely solves the relative isoperimetric problem referred to above by extending a celebrated result due to Eichmair-Metzger [23] to our setting.

**Theorem 5.7.** [4] Let \((M, g, \Sigma)\) be as in Theorem 5.6. Then for all sufficiently large volume there exists an associated bounded, relative isoperimetric region whose connected and smooth boundary remains close to a centered coordinate hemisphere, with the region sweeping out the whole manifold as the volume diverges towards infinity.
In particular, the corresponding isoperimetric surfaces coincide with the leaves of the foliation in Theorem 5.6, thus being unique for each value of the enclosed volume.

This result goes a long way towards determining an asymptotic expansion for the relative isoperimetric profile of \( \text{aH}S_3 \)-manifolds with positive mass for all sufficiently large values of the enclosed volume.

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