THE UNIVERSAL MINIMAL FLOW OF THE HOMEOMORPHISM GROUP OF THE LELEK FAN

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Abstract. We compute the universal minimal flow of the homeomorphism group of the Lelek fan – a one-dimensional tree-like continuum with many symmetries.

1. Introduction

Let $G$ be a topological group and let $X$ be a compact space. A continuous action $G \act X$ is called a $G$-flow (or just a flow, if the group $G$ is understood from the context). A $G$-map between two flows $G \act X$ and $G \act Y$ is a map $f : X \to Y$ such that for every $g \in G$ and $x \in G$ we have $f(gx) = gf(x)$. A flow is minimal if all of its orbits are dense. It is a general result in topological dynamics, due to Ellis, that for any topological group $G$ there is a universal minimal flow $M(G)$, that is, a minimal flow $G \act M(G)$ such that for any minimal flow $G \act X$ there is a continuous $G$-map from $M(G)$ onto $X$. For a compact group $G$, the flow $G \act M(G)$ can be identified with the action of $G$ on itself by left translations. If $G$ is locally compact, but not compact (such as $G$ discrete) $M(G)$ is a very large space, in particular, it is always non-metrizable. For example, when $G$ is the set of integers $\mathbb{Z}$, $M(G)$ is the Gleason space of $2^c$, that is the Stone space of the Boolean algebra of regular open sets of $2^c$, where $c$ is the cardinality of real numbers.

Many groups that have been studied by descriptive set-theorists and model-theorists, as it turned out in the last 10-20 years, have metrizable universal minimal flows that can be computed explicitly and a surprising number of them have a trivial universal minimal flow, we call such groups extremely amenable. Pestov [P] applied the finite Ramsey theorem to show that $\text{Aut}(\mathbb{Q}, <)$, the group of order-preserving bijections of rationals with the pointwise convergence topology, is extremely amenable. Glasner and Weiss [GW] used the finite Ramsey theorem to identify the universal minimal flow of $S_\infty$, the group of all permutations of natural numbers $\mathbb{N}$, with its canonical action on the space of all linear orderings on $\mathbb{N}$. In 2005, Kechris, Pestov, and Todorcevic [KPT] developed a general powerful tool to compute universal minimal flows of automorphism groups of countable model-theoretic structures via establishing a strong connection between the dynamics of such groups and the structural Ramsey theory.

The focus of our paper is on homeomorphism groups of compact spaces. We compute the universal minimal flow of the homeomorphism group $H(L)$ of the Lelek fan $L$. We will show that it is equal to the action of $H(L)$ on the space of all maximal chains on $L$ consisting of continua containing the top point of $L$, which is induced from the evaluation action of $H(L)$ on $L$.

An important motivation for our work was the following (still open) question due to Uspenskij from 2000.

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Question 1.1 (Uspenskij, [U]). Let \( P \) be the pseudo-arc and let \( H(P) \) be its homeomorphism group. What is the universal minimal flow of \( H(P) \)? In particular, is the action \( H(P) \looparrowright P \) given by \( (h, x) \to h(x) \) the universal minimal flow of \( H(P) \)?

Both the pseudo-arc and the Lelek fan are well known very homogeneous continua that can be constructed as natural quotients of projective Fraïssé limits of finite structures.

Pestov [P] showed as a consequence of the Ramsey theorem that the group of increasing homeomorphisms of the unit interval is extremely amenable and identified the universal minimal flow of the orientation preserving homeomorphisms of the circle \( S^1 \) with its natural action on \( S^1 \). Glasner and Weiss [GW2] proved that the universal minimal flow of the homeomorphism group of the Cantor set is the action on the space of maximal chains of closed subsets of the Cantor set, which is induced from the evaluation action. These seem to be the only examples of homeomorphism groups for which the universal minimal flow was computed. In each of these examples, the description of the universal minimal flow follows directly from a description of the universal minimal flow of a certain automorphism group of a countable structure (rational numbers in the case of \( S^1 \) and \([0, 1]\), and the countable atomless Boolean algebra in the case of the Cantor set). Universal minimal flows of homeomorphism groups of many very simple compact spaces, such as \([0, 1]^2\) or the sphere \( S^2\), are unknown.

Unlike for the automorphism groups of countable structures, there are no general techniques to compute universal minimal flows of homeomorphism groups of compact spaces. To obtain the universal minimal flow of \( H(L) \), we will use our earlier construction, presented in [BK], of the Lelek fan \( L \) as a quotient of a projective Fraïssé limit \( L \). First, we compute the universal minimal flow of the automorphism group \( \text{Aut}(L) \); we will use tools provided by Kechris, Pestov, and Todorcevic [KPT] and a new Ramsey theorem, which we prove using the Dual Ramsey theorem [GR]. Second, by relating \( L \) and \( L \), we compute the universal minimal flow of \( H(L) \). This second step is novel and nontrivial, and we hope it will find applications to homeomorphism groups of other compact spaces.

2. Discussion of results

A continuum is a compact connected metric space. Denoting by \( C \) the Cantor set and by \([0, 1]\) the unit interval, the Cantor fan is the quotient of \( C \times [0, 1] \) by the equivalence relation \( \sim \) given by \((a, b) \sim (c, d)\) if and only if either \((a, b) = (c, d)\) or \( b = d = 0\).

For a continuum \( X \), a point \( x \in X \) is an endpoint in \( X \) if for every homeomorphic embedding \( h : [0, 1] \to X \) with \( x \) in the image of \( h \) either \( x = h(0) \) or \( x = h(1) \). The Lelek fan \( L \), constructed by Lelek in [L], can be characterized as the unique non-degenerate subcontinuum of the Cantor fan whose endpoints are dense in \( L \) (see [BO] and [C]). Denote by \( v \) the top (which we will also sometime call the root) \((0, 0)/\sim \) of the Lelek fan.

If \( K \) is a compact topological space, a chain \( C \) on \( K \) is a family of closed subsets of \( K \) such that for every \( C_1, C_2 \in C \), either \( C_1 \subset C_2 \) or \( C_2 \subset C_1 \). We say that a chain \( C \) is maximal if for every closed set \( C \subset K \), if \( \{C\} \cup C \) is a chain then \( C \in C \). The set \( \text{Exp}(K) \) of all closed subsets of a compact topological space \( K \) equipped with the Vietoris topology is a compact space, which we introduce in Section 3.4.

Let \( Y^* \subset \text{Exp}(\text{Exp}(K)) \) be the space of all maximal chains \( C \) on \( L \) such that each \( C \subseteq C \) is connected and it contains the root of \( L \). This space is compact, which we prove in Proposition 3.13 and the natural action of \( H(L) \) – the homeomorphism group of the Lelek fan \( L \) on \( L \) given by \((g, x) \to g(x)\) induces an action on \( \text{Exp}(L) \) which further
induces an action on $\text{Exp}(\text{Exp}(L))$ which is invariant on $Y^*$. The main result of this article is the following:

**Theorem 2.1.** The universal minimal flow of $H(L)$ – the homeomorphism group of the Lelek fan $L$ – is

$$H(L) \acts Y^*.$$

To prove Theorem 2.1, we will first find a “quotient” description of the universal minimal flow of $H(L)$. Let $H$ be the closed subgroup of $H(L)$ consisting of homeomorphisms that preserve the “generic” maximal chain in $Y^*$. Such a chain is constructed explicitly, we expand the projective Fraïssé family $\mathcal{F}$ of finite fans, whose limit gives the Lelek fan, to the projective Fraïssé family $\mathcal{F}_c$ of finite fans expanded by a maximal chain of connected sets containing the root. The limit of this new family gives the Lelek fan equipped with the required “generic” chain. The details and necessary definitions are contained in the next sections. The quotient space $H(L)/H$ is precompact in the quotient of the right uniformity on $H(L)$ and consequently its completion $\hat{H(L)}/H$ is compact. The group $H(L)$ acts on itself by left translations. This actions induces an action on the quotient, which extends to the completion. Theorem 2.1 will follow from Theorem 2.2

**Theorem 2.2.** The universal minimal flow of $H(L)$ – the homeomorphism group of the Lelek fan $L$ is

$$H(L) \acts \hat{H(L)}/H.$$

Let $\mathcal{L}$ and $\mathcal{L}_c$ be the projective Fraïssé limits of $\mathcal{F}$ and $\mathcal{F}_c$ respectively and let $	ext{Aut}(\mathcal{L})$ and $\text{Aut}(\mathcal{L}_c)$ be their automorphism groups. In Section 4.2 we show that $\text{Aut}(\mathcal{L}_c)$ is extremely amenable and in Section 4.3 we provide two equivalent descriptions of the universal minimal flow of $\text{Aut}(\mathcal{L})$. We prove our main result in Section 5.

### 3. Preliminaries

We first review the Fraïssé and the projective Fraïssé constructions, as well as the construction of the Lelek fan in the projective Fraïssé framework the authors introduced in [BK] (Sections 3.1, 3.2, and 3.3). We then discuss topics specifically relevant to studying the universal minimal flow of the homeomorphism group of the Lelek fan: maximal chains on compact spaces (Section 3.4), uniform spaces (Section 3.5), and the Kechris-Pestov-Todorcevic correspondence for Fraïssé-HP families (Section 3.6).

#### 3.1. Fraïssé families

Given a first-order language $\mathcal{L}$ that consists of relation symbols $r_i$, with arity $m_i$, $i \in I$, and function symbols $f_j$, with arity $n_j$, $j \in J$, and two structures $A$ and $B$ in $\mathcal{L}$, say that $i : A \to B$ is an embedding if it is an injection such that for a function symbol $f$ in $\mathcal{L}$ of arity $n$ and $x_1, \ldots, x_n \in A$ we have $i(f^A(x_1, \ldots, x_n)) = f^B(i(x_1), \ldots, i(x_n))$; and for a relation symbol $r$ in $\mathcal{L}$ of arity $m$ and $x_1, \ldots, x_m \in A$ we require $r^A(x_1, \ldots, x_m)$ iff $r^B(i(x_1), \ldots, i(x_m))$. For a relation symbol $r \in \mathcal{L}$ with arity $k$ and a function $f : A \to B$, say that $f$ is $R$-preserving if for every $x_1, \ldots, x_k \in A$ we have $r^A(x_1, \ldots, x_k)$ iff $r^B(f(x_1), \ldots, f(x_k))$.

A countable first order structure $M$ in $\mathcal{L}$ is locally finite if every finite subset of $M$ generates a finite substructure. It is ultrahomogeneous if every isomorphism between finite substructures of $M$ can be extended to an automorphism of $M$. In that case, $\mathcal{F} = \text{Age}(M)$, the family of all finite substructures of $M$, has the following three
properties: the hereditary property (HP), that is, if $A \in \mathcal{F}$ and $B$ is a substructure of $A$, then $B \in \mathcal{F}$; the joint embedding property (JEP), that is, for any $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ such that $A$ embeds both into $B$ and into $C$; and the amalgamation property (AP), that is, for any $A, B_1, B_2 \in \mathcal{F}$, any embeddings $\phi_1 : A \to B_1$ and $\phi_2 : A \to B_2$, there exist $C \in \mathcal{F}$ and embeddings $\psi_1 : B_1 \to C$ and $\psi_2 : B_2 \to C$ such that $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$.

Conversely, by a classical theorem due to Fraïssé, if a countable family of finite structures $\mathcal{F}$ in some language $\mathcal{L}$ has the HP, the JEP and the AP, then there is a unique countable locally finite ultrahomogeneous structure $M$ such that $\mathcal{F} = \text{Age}(M)$.

In this paper, we will call a countable family of finite structures that satisfies the JEP and the AP a Fraïssé-HP family (read as Fraïssé minus HP family), and a countable family of finite structures that satisfies the HP, the JEP, and the AP we will call a Fraïssé family.

A Fraïssé limit of a Fraïssé family $\mathcal{F}$ is a countable locally finite ultrahomogeneous structure $M$ such that $\mathcal{F} = \text{Age}(M)$, and a Fraïssé limit of a Fraïssé-HP family $\mathcal{F}$ is a countable structure $M$ such that every structure in $\mathcal{F}$ embeds into $M$, for every finite subset $X$ of $M$ there is $A \in \mathcal{F}$ and an embedding $i : A \to M$ such that $X \subset i(A)$, and $M$ is ultrahomogeneous with respect to $\mathcal{F}$, that is, every isomorphism between finite substructures of $M$ which are isomorphic to a structure in $\mathcal{F}$ can be extended to an automorphism of $M$. Clearly every Fraïssé family is also a Fraïssé-HP family. If $\mathcal{F}$ is a Fraïssé family or it is a Fraïssé-HP family, the Fraïssé limit of $\mathcal{F}$ always exists and it is unique up to an isomorphism.

For example, the rationals with the usual ordering is the Fraïssé limit of the family of finite linear orders, the Rado graph is the Fraïssé limit of the family of finite graphs, and the countable atomless Boolean algebra is the Fraïssé limit of the family of finite Boolean algebras.

We say that a family $\mathcal{G}_1$ is cofinal in a family $\mathcal{G}_2$ if for every $A \in \mathcal{G}_2$ there are $B \in \mathcal{G}_1$ and an embedding $\phi : A \to B$.

Remark 3.1. Suppose that a family $\mathcal{G}_1$ is contained in and cofinal in a Fraïssé-HP family $\mathcal{G}_2$. Then $\mathcal{G}_1$ is also a Fraïssé-HP family and moreover Fraïssé limits of $\mathcal{G}_1$ and of $\mathcal{G}_2$ are isomorphic.

3.2. Projective Fraïssé families. Given a first-order language $\mathcal{L}$ that consists of relation symbols $r_i$, with arity $m_i$, $i \in I$, and function symbols $f_j$, with arity $n_j$, $j \in J$, a topological $\mathcal{L}$-structure is a compact zero-dimensional second-countable space $A$ equipped with closed (in the product topology) relations $r_i^A \subset A^{m_i}$ and continuous functions $f_j^A : A^{n_j} \to A$, $i \in I$, $j \in J$. A continuous surjection $\phi : B \to A$ between two topological $\mathcal{L}$-structures is an epimorphism if it preserves the structure, that is, for a function symbol $f$ in $\mathcal{L}$ of arity $n$ and $x_1, \ldots, x_n \in B$ we require:

$$f^A(\phi(x_1), \ldots, \phi(x_n)) = \phi(f^B(x_1, \ldots, x_n));$$

and for a relation symbol $r$ in $\mathcal{L}$ of arity $m$ and $x_1, \ldots, x_m \in A$ we require:

$$r^A(x_1, \ldots, x_m) \iff \exists y_1, \ldots, y_m \in B \left( \phi(y_1) = x_1, \ldots, \phi(y_m) = x_m, \text{ and } r^B(y_1, \ldots, y_m) \right).$$

By an isomorphism we mean a bijective epimorphism.

Let $\mathcal{G}$ be a countable family of finite topological $\mathcal{L}$-structures. We say that $\mathcal{G}$ is a projective Fraïssé family if the following two conditions hold:
(JPP) (the joint projection property) for any $A, B \in \mathcal{G}$ there are $C \in \mathcal{G}$ and epimorphisms from $C$ onto $A$ and from $C$ onto $B$;

(AP) (the amalgamation property) for $A, B_1, B_2 \in \mathcal{G}$ and any epimorphisms $\phi_1 : B_1 \to A$ and $\phi_2 : B_2 \to A$, there exists $C \in \mathcal{G}$ with epimorphisms $\psi_1 : C \to B_1$ and $\psi_2 : C \to B_2$ such that $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$.

A topological $\mathcal{L}$-structure $G$ is a projective Fraïssé limit of a projective Fraïssé family $\mathcal{G}$ if the following three conditions hold:

(L1) (the projective universality) for any $A \in \mathcal{G}$ there is an epimorphism from $G$ onto $A$;

(L2) (the projective universality) for any finite discrete topological space $X$ and any continuous function $f : G \to X$ there are $A \in \mathcal{G}$, an epimorphism $\phi : G \to A$, and a function $f_0 : A \to X$ such that $f = f_0 \circ \phi$;

(L3) (the projective ultrahomogeneity) for any $A \in \mathcal{G}$ and any epimorphisms $\phi_1 : G \to A$ and $\phi_2 : G \to A$ there exists an isomorphism $\psi : G \to G$ such that $\phi_2 = \phi_1 \circ \psi$.

Remark 3.2. It follows from (L2) above that if $G$ is the projective Fraïssé limit of $\mathcal{G}$, then every finite open cover can be refined by an epimorphism, i.e. for every open cover $U$ of $G$ there is an epimorphism $\phi : G \to A$, for some $A \in \mathcal{G}$, such that for every $a \in A$, $\phi^{-1}(a)$ is contained in an open set in $U$.

Theorem 3.3 (Irwin-Solecki, [IS]). Let $\mathcal{G}$ be a projective Fraïssé family of finite topological $\mathcal{L}$-structures. Then:

(1) there exists a projective Fraïssé limit of $\mathcal{G}$;

(2) any two projective Fraïssé limits of $\mathcal{G}$ are isomorphic.

Theorem below is a folklore, nevertheless it has not been published. It says that the projective Fraïssé theory is a special case of an (injective) Fraïssé theory via a generalization of Stone duality.

Theorem 3.4. For a projective Fraïssé family $\mathcal{F}$ in a relational language with a projective Fraïssé limit $F$ there is an equivalent via a contravariant functor (defined on $\mathcal{F} \cup \{F\}$ and on all epimorphisms between structures in $\mathcal{F} \cup \{F\}$) Fraïssé-HP family $\mathcal{G}$ with a Fraïssé limit $G$.

Theorem 3.4 will follow from Proposition 3.6, a generalization of the classical Stone duality between Boolean algebras with embeddings and compact totally disconnected spaces with continuous surjections, which we recall here. In this paper, we will not consider $\mathcal{F}$ in a language that contains function symbols.

Proposition 3.5. The family of compact totally disconnected spaces $\mathcal{F}_0$ with continuous surjections is equivalent via a contravariant functor to the family $\mathcal{G}_0$ of Boolean algebras with embeddings.

In the Stone duality, to $K \in \mathcal{F}_0$ we associate the Boolean algebra Clop($K$) of clopen sets of $K$ with the usual operations of the union $\cup^{\text{Clop}(K)}$, 0 is the empty set and 1 is identified with $K$, the intersection $\cap^{\text{Clop}(K)}$ and the complement $-\text{Clop}(K)$, and to a continuous surjection $f : L \to K$ we associate an embedding $F : \text{Clop}(K) \to \text{Clop}(L)$ given by $F(X) = f^{-1}(X)$.

Proposition 3.6. Let $\mathcal{L}$ be a relational language and let $\mathcal{F}_1$ be a family of topological $\mathcal{L}$-structures, maps between structures are epimorphisms. Then there is a family $\mathcal{G}_1$ of countable structures in the language equal to the union of the language of Boolean algebras and of $\mathcal{L}$, maps between structures are embeddings, such that $\mathcal{F}_1$ is equivalent to $\mathcal{G}_1$ via a contravariant functor.
Proof. Let \( L = \{ R_1, \ldots, R_n \} \), where \( R_i \) is a relation symbol of the arity \( m_i \), be the language of \( \mathcal{F}_1 \). Let \( L' = \{ S_1, \ldots, S_n, \cup, \cap, 0, 1 \} \) be the language where \( S_i \) is a relation symbol of the arity \( m_i \) and \( \{ \cup, \cap, 0, 1 \} \) is the language of Boolean algebras. For \( K = (K, R^K_1, \ldots, R^K_n) \in \mathcal{F}_1 \), let \( M = (M, S^M_1, \ldots, S^M_n, \cup^M, \cap^M, \cap^M - M, 0^M, 1^M) \) be the structure such that \( M = \text{Clop}(K) \) is the family of all clopen sets of \( K \), \( \cap^M \) is the intersection, \( \cap^M \) is the complement, \( 0^M \) is the empty set and \( 1^M = M \). Moreover, we require that for every \( i \), \( S^M_i(X_1, \ldots, X_m_i) \) iff for some \( c_1 \in X_1, \ldots, c_{m_i} \in X_{m_i} \), we have \( R^K_i(c_1, \ldots, c_{m_i}) \).

Let \( G_i \) be the family of all \( M \)'s obtained in this way from a \( K \in \mathcal{F}_1 \) with the embeddings.

Let \( f: L \to K \), where \( K, L \in \mathcal{F}_1 \), be a continuous surjection and let \( F: \text{Clop}(K) \to \text{Clop}(L) \) be the map given by \( F(X) = f^{-1}(X) \). In view of Proposition 3.5 all we have to check is that \( f \) is \( R^L \)-preserving if and only if \( F \) is \( S^L \)-preserving, and that will follow from the two claims below.

Claim. If \( f \) is \( R^L \)-preserving then \( F \) is \( S^L \)-preserving.

Proof. If \( S^L_i(\text{Clop}(K)) (X_1, \ldots, X_{m_i}) \) then \( R^L_i(a_1, \ldots, a_{m_i}) \) for some \( a_i \in X_i \), which implies that for some \( c_i \in f^{-1}(a_i) \subset f^{-1}(X_i), R^K_i(c_1, \ldots, c_{m_i}) \), hence \( S^L_i(\text{Clop}(L)) (f^{-1}(X_1), \ldots, f^{-1}(X_{m_i})) \), i.e. \( S^L_i(\text{Clop}(L)) (f(X_1), \ldots, f(X_{m_i})) \).

Conversely, if \( S^L_i(\text{Clop}(L)) (f(X_1), \ldots, f(X_{m_i})) \), then for some \( c_i \in f^{-1}(X_i), R^K_i(c_1, \ldots, c_{m_i}) \), therefore \( R^K_i(f(c_1), \ldots, f(c_{m_i})) \), which gives \( S^L_i(\text{Clop}(K)) (X_1, \ldots, X_{m_i}) \).

Claim. If \( F \) is \( S^L \)-preserving then \( f \) is \( R^L \)-preserving.

Proof. We have \( R^K_i(a_1, \ldots, a_{m_i}) \) iff for every \( X_i \in \text{Clop}(K) \) such that \( a_i \in X_i \) we have \( S^L_i(\text{Clop}(K)) (X_1, \ldots, X_{m_i}) \) iff for every \( X_i \in \text{Clop}(K) \) such that \( a_i \in X_i \) we have \( S^L_i(\text{Clop}(L)) (f^{-1}(X_1), \ldots, f^{-1}(X_{m_i})) \) iff for every \( X_i \in \text{Clop}(K) \) such that \( a_i \in X_i \) there exist \( c_i \in f^{-1}(X_i) \) for which we have \( R^K_i(c_1, \ldots, c_{m_i}) \) iff there exist \( c_i \in f^{-1}(a_i) \) for which we have \( R^K_i(c_1, \ldots, c_{m_i}) \). In the first and last equivalences we used that the relations \( R^K_i \) and \( R^L_i \) are closed in \( K^m_i \) and \( L^m_i \), respectively.

Now one may ask why we study projective Fraïssé families at all. The reason is that it is more natural to use projective Fraïssé families to construct and study compact spaces, like the Lelek fan or the pseudo-arc, rather than to study them via families of finite Boolean algebras equipped with relations.

In further sections, we will introduce families \( \mathcal{F}_c \) and \( \mathcal{F}_{nc} \) of finite fans expanded by an additional structure, which will be neither a function nor a relation, for which we will have to prove an analog of Theorem 3.3. These families will not exactly fall into the framework of the projective Fraïssé theory discussed in this section. Nevertheless, we will still call them projective Fraïssé families, and their limits we will call projective Fraïssé limits.

3.3. Construction of the Lelek fan. For completeness, we repeat here more or less Section 3.1 from [BK2], where we review the construction of the Lelek fan from [BK].
Unlike in [BK] and [BK2], we will not assume that all branches in a finite fan are of the same length.

By a fan we mean an undirected connected simple graph with all loops, with no cycles of the length greater than one, and with a distinguished point $r$, called the root, such that all elements other than $r$ have degree at most 2. On a fan $T$, there is a natural partial tree order $\preceq_T$: for $t, s \in T$ we let $s \preceq_T t$ if and only if $s$ belongs to the path connecting $t$ and the root. We say that $t$ is a successor of $s$ if $s \preceq_T t$ and $s \neq t$. It is an immediate successor if additionally there is no $p \in T$, $p \neq s, t$, with $s \preceq_T p \preceq_T t$. For a fan $T$ and $x, y \in T$ which are on the same branch and $x \preceq_T y$, by $[x, y]_{\preceq_T}$ we denote the interval $\{z \in T : x \preceq_T z \preceq_T y\}$.

A chain in a fan $T$ is a subset of $T$ on which the order $\preceq_T$ is linear. A branch of a fan $T$ is a maximal chain in $(T, \preceq_T)$. If $b$ is a branch in $T$ with $n + 1$ elements, we will sometimes enumerate $b$ as $(b^0, \ldots, b^n)$, where $b^0$ is the root of $T$, and $b^i$ is an immediate successor of $b^{i-1}$, for every $i = 1, 2, \ldots, n$. In that case, $n$ will be called the height of the branch $b$. Define the height of the fan to be the maximum of the heights of all of its branches and define the width of the fan to be the number of its branches.

Let $L = \{R\}$ be the language with $R$ a binary relation symbol. For a fan $T$ and $s, t \in T$, we let $R^T(s, t)$ if and only if $s = t$ or $t$ is an immediate successor of $s$. Let $F$ be the family of all finite fans, viewed as topological $L$-structures, equipped with the discrete topology.

**Remark 3.7.** For two fans $(S, R^S)$ and $(T, R^T)$ in $F$, a function $\phi : (S, R^S) \to (T, R^T)$ is an epimorphisms if and only if it is a surjective homomorphism, i.e., for every $s_1, s_2 \in S$, $R^S(s_1, s_2)$ implies $R^T(\phi(s_1), \phi(s_2))$.

We say that a projective Fraïssé family $G_1$ is coinitial in a projective Fraïssé family $G_2$ if for every $A \in G_2$ there are $B \in G_1$ and an epimorphism $\phi : B \to A$.

**Proposition 3.8.** The family $F$ is a projective Fraïssé family.

In [BK] Proposition 2.3], we proved that the family, which we call now $F_1$, of finite fans with all branches of the same length, is a projective Fraïssé family. The proof of Proposition 3.8 is essentially the same as the proof that $F_1$ is a projective Fraïssé family. By Theorem 3.3 there exists a unique projective Fraïssé limit of $F$, which we denote by $L = (L, R^L)$. The underlying set $L$ is homeomorphic to the Cantor set. The family $F_1$ is coinitial in $F$, and this implies (by Remark 3.1 and Theorem 3.4) that the projective Fraïssé limits of $F$ and $F_1$ are isomorphic. Let $R^L_S$ be the symmetrization of $R^L$, that is, $R^L_S(s, t)$ if and only if $R^L(s, t)$ or $R^L(t, s)$, for $s, t \in L$.

**Theorem 3.9 (Theorem 2.5, [BK]).** The relation $R^L_S$ is an equivalence relation which has only one and two element equivalence classes.

**Theorem 3.10 (Theorem 2.6, [BK]).** The quotient space $L/R^L_S$ is homeomorphic to the Lelek fan $L$.

Let $\pi : L \to L$ denote the quotient map given by $R^L_S$. We denote by $\text{Aut}(L)$ the group of all automorphisms of $L$, that is, the group of all homeomorphisms of $L$ that preserve the relation $R$. This is a topological group when equipped with the compact-open topology inherited from $H(L)$, the group of all homeomorphisms of the Cantor set underlying the structure $L$. Since $R^L$ is closed in $L \times L$, the group $\text{Aut}(L)$ is closed in $H(L)$.

Let $\pi^*$ be the map that takes $h \in \text{Aut}(L)$ to $h^* \in H(L)$ and $h^* \pi(x) = \pi h(x)$ for every $h \in \text{Aut}(L)$ and $x \in L$. We will frequently identify $\text{Aut}(L)$ with the corresponding...
3.4. Spaces of maximal chains. We will assume throughout the paper that every compact space is Hausdorff. Let $K$ be a compact topological space. A chain $C$ on $K$ is a family of closed subsets of $K$ such that for every $C_1, C_2 \in C$, either $C_1 \subset C_2$ or $C_2 \subset C_1$. Sometimes we will call the sets in a chain links. We say that a chain $C$ is maximal if for every closed set $C \subset K$, if $\{C\} \cup C$ is a chain then $C \in C$. Note that if $C$ is a maximal chain and $A \subset C$, then $\bigcap A \in C$ and $\bigcup A \in C$.

The set $\text{Exp}(K)$ of all closed subsets of $K$ is equipped with the Vietoris topology generated by the sets
\[ \{U_1, \ldots, U_n\} = \{F \in \text{Exp}(K) : F \subset U_1 \cup \ldots \cup U_n \text{ and for every } i = 1, \ldots, n, F \cap U_i \neq \emptyset\} , \]
where $n \in \mathbb{N}$ and $U_1, \ldots, U_n$ are open in $K$. Without loss of generality, $U_1, \ldots, U_n$ are only taken from some fixed basis of $K$. If $K$ is metrizable by a metric $d_0$ then the space $\text{Exp}(K)$ is metrizable by the Hausdorff metric given by
\[ d(X,Y) = \max \{\sup_{x \in X} \inf_{y \in Y} d_0(x,y), \sup_{y \in Y} \inf_{x \in X} d_0(x,y)\} . \]

It is not hard to show (see [P2] Lemma 6.4.7) that every maximal chain is closed in the Vietoris topology. It is well known that $\text{Exp}(K)$ is compact. Uspenskij [U] showed that the set of maximal chains on $K$ is closed in $\text{Exp}(\text{Exp}(K))$, and therefore it is compact.

Let $\leq$ be a partial order on $K$. We say that a set $C \subset K$ is downwards closed if it is closed and for every $x, y \in K$, if $y \in C$ and $x \leq y$ then $x \in C$, and a chain $C$ on $K$ be downwards closed if every $C \in C$ is downwards closed. A downwards closed chain $C$ is downwards closed maximal if for every downwards closed set $C \subset K$, if $\{C\} \cup C$ is a chain then $C \in C$.

For a map $f : Y \to X$ and a chain $C$ on $Y$, by $f(C)$ we will denote the chain $\{f(C) : C \in C\}$. We start with the following observations.

**Lemma 3.11.** Let $K, M$ be compact sets and let $f : M \to K$ be a continuous surjection. If $C$ is a maximal chain in $M$, then the chain $f(C)$ is also maximal.

**Proof.** Suppose a closed set $D \subset K$ is such that $\{D\} \cup f(C)$ is a chain. We will show that there is $J \in C$ satisfying $f(J) = D$. Let $K_1 = \{C \in f(C) : C \supset D\}$ and let $M_1 = \{E \in C : f(E) \in K_1\}$. As $C$ is maximal, $M = \bigcap M_1 \in C$. Since $D \subset f(M)$, we have that $J = f^{-1}(D) \cap M$ satisfies $f(J) = D$ and has the property that $\{J\} \cup C$ is a chain, and hence by the maximality of $C$, $J \in C$ as required.

Using Zorn’s Lemma, we get the following.

**Lemma 3.12.** Let $K$ be a compact set and let $D$ be a chain on $K$. Then there is a maximal chain on $K$ that extends $D$.

Let $F^*$ be the family of all topological $\mathcal{L}$-structures that are countable inverse limits of finite fans in $F$. If $P$ is the inverse limit of $(A_n, f_n^A)$, the relation $R^P$ on $P$ is defined as follows
\[ R^P(x,y) \text{ iff for every } n, R^{A_n}(f_n^\infty(x), f_n^\infty(y)) . \]
Clearly we can identify $\mathcal{F}$ with a subset of $\mathcal{F}^\ast$ by assigning to $A$ the inverse limit of $(A, \text{Id}^m_n)$. Recall that $A \in \mathcal{F}$ is equipped with the tree partial order $\preceq_A$; we let $x \preceq_A y$ iff $x$ belongs to the segment joining the root of $A$, $v_A$, with $y$. We let 
\[ x \preceq_P y \text{ iff for every } n, f_n^\infty(x) \preceq_A f_n^\infty(y). \]

In particular, we have just defined a partial order on $\mathbb{L}$, the projective Fraïssé limit of the family $\mathcal{F}$ of finite fans. This in turn defines a partial order on $L$ by $x \preceq_L y$ if and only if for some (equivalently, for any) $v,w \in \mathbb{L}$ such that $\pi(v) = x$ and $\pi(w) = y$, we have $v \preceq_L w$, where $\pi : \mathbb{L} \to L$ is the quotient map. Whenever we talk about downwards closed sets on $P \in \mathcal{F}^\ast$ or on $L$, we will understand that they are downwards closed with respect to $\preceq_P$ or $\preceq_L$, respectively.

**Remark 3.13.** A closed set in $L$ is downwards closed if and only if it is connected and it contains the root of $L$.

**Lemma 3.14.** Every downwards closed maximal chain on $P \in \mathcal{F}^\ast$ is maximal.

**Proof.** It is not hard to see that the conclusion is true for a structure in $\mathcal{F}$. Let $C$ be a downwards closed maximal chain on $P = \lim(A_n, f_n^m) \in \mathcal{F}^\ast$. Then for each $n$, $C^A_n = \{f_n^\infty(C) : C \in C\}$ is a downwards closed chain which is maximal, by the same argument as in Lemma 3.11. If a closed set $D \subset P$ is such that $C \cup \{D\}$ is a chain, then for each $n$, $f_n^\infty(D) \subset C^A_n$ by the maximality of $C^A_n$ and therefore $f_n^\infty(D)$ is downwards closed, and consequently so is $D$, which implies $D \in C$. \hfill $\square$

**Proposition 3.15.** For every $P \in \mathcal{F}^\ast$ the set of all downwards closed maximal chains on $P$ is compact. In particular, the set of all downwards closed maximal chains on $\mathbb{L}$ is compact.

**Proof.** We first show that the set $C\text{Exp}(P)$ of all downwards closed closed subsets of $P = \lim(A_n, f_n^m)$ is closed in $\text{Exp}(P)$. Let $K \subset P$ be closed but not downwards closed, witnessed by $x \notin K$ and $y \in K$ be such that $x \preceq_P y$. Pick $n$ and $a \in A_n$ such that $a = f_n^\infty(x) \neq f_n^\infty(y)$ and $A = (f_n^\infty)^{-1}(a) \cap K = \emptyset$. Let $B = (f_n^\infty)^{-1}\{b \in A_n : a \prec_A b\}$. Clearly $B$ is open and $y \in B$. Then
\[ V := [B, P] \cap [P \setminus A] = \{L \in \text{Exp}(P) : L \cap B \neq \emptyset \text{ and } L \subset P \setminus A\} \]
is such that $K \in V$ and all sets in $V$ are not downwards closed, which finishes the proof that $C\text{Exp}(P)$ is closed.

Since, Uspenskij proved in [U] that the set of maximal chains is closed in $\text{Exp}(\text{Exp}(P))$, by Lemma 3.14, it is enough to show that the set of points in $\text{Exp}(\text{Exp}(P))$ consisting of sets contained $C\text{Exp}(P)$ is again a closed sets. This last thing follows from the following simple general observation: If $K$ is a compact space and $D \subset \text{Exp}(K)$, then $\{E \in \text{Exp}(K) : E \subset D\}$ is closed in $\text{Exp}(K)$. Finally, take $K = \text{Exp}(P)$ and $D = C\text{Exp}(P)$. \hfill $\square$

### 3.5. Precompact uniform spaces.

A good introduction to uniform spaces can be found in Engelking [E], Chapter 8 (precompact spaces are called totally bounded there). Below we briefly review the very minimum that is needed for the paper, all undefined concepts are in Engelking [E].

A **uniformity** is a set $X$ together with a family $\mathcal{U}$ of subsets of $X \times X$ having the following properties:

1. each $U \in \mathcal{U}$ contains the diagonal $\{(x,x) : x \in X\}$;
called uniformly continuous open if and only if for every \( x \in X \) there exists \( U \in \mathcal{U} \) such that \( x \in U \cap V \in \mathcal{U} \).

Every uniform space \((X, \mathcal{U})\) becomes a topological space if we declare \( U \subseteq X \) to be open if and only if for every \( x \in U \) there exists an \( V \in \mathcal{U} \) such that \( V[x] = \{ y \in X : (x, y) \in V \} \subseteq U \). A function \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) between uniform spaces is called uniformly continuous if for every \( V \in \mathcal{V} \) there exists \( U \in \mathcal{U} \) such that \( f(U) \subseteq V \).

We say that a uniform space \((X, \mathcal{U})\) is precompact if and only if for every \( U \in \mathcal{U} \) there are finitely many \( x_1, \ldots, x_n \in X \) such that \( X = \{ x \in X : \exists i(x, x_i) \in U \} \). Equivalently, a uniform space \((X, \mathcal{U})\) is precompact if its completion is compact. If \((X, \mathcal{U})\) is metrizable by a metric \( d \) (i.e. the topology induced by \((X, \mathcal{U})\) is equal to the topology induced by \( d \)) then the completion of \((X, \mathcal{U})\) is equal to the completion of \((X, d)\) (see Lemma 8.3.7 and Proposition 8.3.5 in Engelking [E]). This implies, \((X, \mathcal{U})\) is precompact if and only if the metric space \((X, d)\) is precompact.

Recall that any compact space \( X \) has the unique uniformity compatible with the topology. This uniformity consists of all symmetric neighbourhoods of the diagonal in \( X \times X \).

A topological group \( G \) admits a few natural uniform structures compatible with its topology. We will be working with the right uniformity, which is generated by the sets

\[
O_V = \{ (x, y) : xy^{-1} \in V \},
\]

where \( V \) is an open symmetric neighbourhood of the identity in \( G \). For a closed subgroup \( H \) of \( G \) we consider the quotient space \( G/H \) with the quotient uniformity generated by the sets

\[
U_V = \{ (xH, yH) : xy^{-1} \in V \} = \{ (gH, vgH) : g \in H, v \in V \},
\]

where \( V \) is an open symmetric neighbourhood of the identity in \( G \). This uniformity is compatible with the quotient topology of \( G/H \) and \( G/H \) is precompact if and only if for every open symmetric neighbourhood \( V \) of the identity in \( G \), there exist finitely many \( x_1, \ldots, x_n \in G \) such that \( G = \bigcup_{i=1}^n Vx_iH \).

If \( G \) is a Polish group and \( d_R \) is a right-invariant metric on \( G \), then the uniformity on \( G/H \) is metrizable by the metric

\[
d(g_1H, g_2H) = \inf_{h \in H} d_R(g_1h, g_2).
\]

The following is a folklore, but we could not find a proof, therefore we include it here.

**Proposition 3.16.** Suppose that \( G/H \) is precompact, where \( G, H \) are Polish groups, \( H \) is a closed subgroup of \( G \). Then the continuous action of \( G \) on \( G/H \) by left translations, \( g_1 \cdot (g_2H) = (g_1g_2)H \), extends to a continuous action of \( G \) on the completion \( \overline{G/H} \).

Suppose that \( G \) acts on a uniform space \( X = (X, \mathcal{U}_X) \) by uniform space isomorphisms. The action is called bounded or motion equicontinuous (see Pestov [P2], page 70, and references therein) if for every \( U \in \mathcal{U}_X \), the set \( \{ g \in G : \forall x \in X \ (x, g \cdot x) \in U \} \) is a neighbourhood of \( 1 \) in \( G \).

Assuming additionally that \( X \) is a Polish space and \( G \) is a Polish group, we immediately see the following.
(1) If $X$ is a completion of an invariant space $X_0$ then if the action of $G$ on $X_0$ is motion equicontinuous then so is the action of $G$ on $X$.

(2) If the action of $G$ on $X$ is motion equicontinuous then for a fixed $x \in X$ the function $g \mapsto g \cdot x$ is continuous with respect to the compatible topologies. (The definition immediately implies that $g \mapsto g \cdot x$ is continuous at the identity, which implies that this function is in fact continuous.)

**Proof of Proposition 3.16.** First observe that for a fixed $g \in G$, the bijection $f_g : G/H \to G/H$ given by $f_g(hH) = ghH$ is uniformly continuous. Indeed, for any open symmetric neighbourhood $1 \in V$ in $G$, we have $(f_g \times f_g)(U_{g^{-1}Vg}) \subset U_V$. Similarly, $f_g^{-1}$ is uniformly continuous. Therefore $f_g$ extends to a uniform isomorphism of $G/H$. This gives an action of $G$ on $G/H$, which is continuous if we fix $g \in G$. Since separately continuous functions are continuous (see [BKe], Proposition 2.2.1), it suffices to show that it is continuous if we fix $x \in G/H$.

By the remarks before the proof, it suffices to check that for a fixed $hH$ the function from $G$ to $G/H$, $g \mapsto ghH$ is motion equicontinuous. However, this is clear, as for any open symmetric neighbourhood $1 \in V$ in $G$, $h \in G$, and $g \in V$ we have $(hH, ghH) \in U_V$. □

**3.6. Kechris-Pestov-Todorcevic correspondence.** In this section, we review the Kechris-Pestov-Todorcevic correspondence between the structural Ramsey theory of a Fraïssé-HP family and the dynamics (extreme amenability, the universal minimal flow) of the automorphism group of its Fraïssé limit.

A topological group $G$ is **extremely amenable** if every $G$-flow has a fixed point. A **colouring** of a set $X$ is any function $c : X \to \{1, 2, \ldots, r\}$, for some $r \geq 2$; we say that $Y \subset X$ is $c$-monochromatic (or just monochromatic) if $r \nmid |Y|$ is constant.

Let $\mathcal{G}$ be a family of finite structures in a language $\mathcal{L}$. For $A, B$ in $\mathcal{G}$, let $(B)^A$ denotes the set of all embeddings of $A$ into $B$. We say that $\mathcal{G}$ is a **Ramsey class** if for every integer $r \geq 2$ and for $A, B \in \mathcal{G}$ there exists $C \in \mathcal{G}$ such that for every colouring $c : (C^A)^C \to \{1, 2, \ldots, r\}$ there exists $h \in (C)^B$ such that $\{h \circ f : f \in (B)^A\}$ is monochromatic. We say that $A \in \mathcal{G}$ is **rigid** if it has a trivial automorphism groups.

Kechris-Pestov-Todorcevic [KPT] worked with Fraïssé families and their ordered Fraïssé expansions, which was generalized by then Nguyen Van Thé [NVT] to Fraïssé families and to arbitrary relational Fraïssé expansions. The Kechris-Pestov-Todorcevic correspondence remains true for Fraïssé-HP families, which was checked by several people, and it appears in [Z].

**Theorem 3.17.** (Kechris-Pestov-Todorcevic [KPT], see Theorem 5.1 in [Z]). Let $\mathcal{G}$ be a Fraïssé-HP, let $\mathbb{G}$ be its Fraïssé limit, and let $G = \text{Aut}(\mathbb{G})$. Then the following are equivalent:

1. The group $G$ is extremely amenable.
2. The family $\mathcal{G}$ is a Ramsey class and it consists of rigid structures.

Let $\mathcal{G}$ be a Fraïssé-HP family in a language $\mathcal{L}$, let $\mathbb{G}$ be its Fraïssé limit, and let $G = \text{Aut}(\mathbb{G})$. Let $G^*$ be a Fraïssé-HP family, in a language $\mathcal{L}^* \supset \mathcal{L}$, $\mathcal{L}^* \setminus \mathcal{L}$ relational, such that for every $A^* \in G^*$, $A^* \upharpoonright \mathcal{L} \in \mathcal{G}$, that is, every $A^* \in G^*$ is an expansion of some $A \in \mathcal{G}$, or $G^*$ is an expansion of $\mathcal{G}$. Let $\mathbb{G}^*$ be the Fraïssé limit of $\mathcal{G}$, and let $G^* = \text{Aut}(\mathbb{G}^*)$.

We say that the expansion $G^*$ of $\mathcal{G}$ is **reasonable**, that is, for any $A, B \in \mathcal{G}$, an embedding $\alpha : A \to B$ and an expansion $A^* \in G^*$ of $A$, there is an expansion $B^* \in G^*$ of $B$ such that
α : A* → B* is an embedding. It is precompact if for every A ∈ 𝒢 there are only finitely many A* ∈ 𝒢* such that A* ↾ ℒ = A. We say that 𝒢* has the expansion property relative to 𝒢 if for any A* ∈ 𝒢* there is B ∈ 𝒢 such that for any expansion B* ∈ 𝒢*, there is an embedding α : A* → B*.

**Proposition 3.18** ([KPT], [NVT], see Proposition 5.3 in [Z]). The expansion 𝒢* of 𝒢 is reasonable if and only if 𝒢* ↾ ℒ = 𝒢.

From now on till the end of this section, we will assume that the expansion 𝒢* of 𝒢 is reasonable, precompact, and satisfies the property (*) below.

(*) For any A ∈ 𝒢 and an embedding i : A → 𝒢 there is an expansion A* ∈ 𝒢* of A such that i : A* → (𝒢, ⃗R) is an embedding.

Below (𝐺, ⃗R), (𝑮, ⃗S), etc. denote an expansion of 𝒢 to a structure in 𝒢*. Instead of (𝐺, ⃗R) we will often just write ⃗R.

Define

\[ X_{G^*} = \{ \tilde{R} : \text{for every } A \in 𝒢, \text{ and an embedding } i : A \to 𝒢 \text{ there exists } A^* \in 𝒢^*, \text{ such that } i : A^* \to (𝑮, \tilde{R}) \text{ is an embedding} \}. \]

We make 𝑥_{G^*} a topological space by declaring sets

\[ V_{i,A^*} = \{ \tilde{R} \in X_{G^*} : i : A^* \to (郜, \tilde{R}) \text{ is an embedding} \}, \]

where \( i : A \to 𝒢 \) is an embedding, \( A^* \in 𝒢^* \), and \( A^* \upharpoonright ℒ = A \), to be open. The group \( \text{Aut}(郜^*) \) acts continuously on 𝑥_{G^*} via

\[ g \cdot \tilde{R}(\bar{a}) = \tilde{R}(g^{-1}(\bar{a})). \]

Reasonability and precompactness of the expansion 𝒢* of 𝒢 imply that the space 𝑥_{G^*} is compact, zero-dimensional, and it is nonempty as 𝒢* ∈ 𝑥_{G^*}.

**Theorem 3.19** ([KPT], [NVT], see Proposition 5.5 in [Z]). The following are equivalent:

1. The flow 𝑔 ↾ 𝑥_{G^*} is minimal.
2. The family 𝒢* has the expansion property relative to 𝒢.

**Theorem 3.20** (Kečrhis-Pestov-Todorcevic [KPT], Nguyen Van Thé [NVT], see Theorem 5.7 in [Z]). The following are equivalent:

1. The flow 𝑔 ↾ 𝑥_{G^*} is the universal minimal flow of 𝑔.
2. The family 𝒢* is a rigid Ramsey class and has the expansion property relative to 𝒢.

We finish this section with several general observations, which are adaptations of those in [NVT] (pages 6-8) to the framework of the Fraïssé-HP theory.

For an embedding α : A → 𝒢, A ∈ 𝒢, let \( V_α \) denote the pointwise stabilizer of α, that is,

\[ V_α = \{ g \in \text{Aut}(𝑮) : \text{ for every } a \in A, g(α(a)) = α(a) \}. \]

Then \( V_α \) is a symmetric clopen neighbourhood of the identity in \( \text{Aut}(𝑮) \), in fact it is also a subgroup of \( \text{Aut}(𝑮) \), and sets of this form constitute a neighbourhood basis of the identity in \( \text{Aut}(𝑮) \).

**Lemma 3.21.** The right uniform space \( \text{Aut}(𝑮)/\text{Aut}(郜^*) \) is precompact.
Proof. Let $V = V_{\alpha}$ for some embedding $\alpha : A \to G$. Enumerate all expansions of $A$ in $G^*$ as $A_1^*, A_2^*, \ldots, A_N^*$. For each $i = 1, 2, \ldots, N$, using the universality of $G^*$, pick an embedding $y_i : A_i^* \to G^*$. The ultrahomogeneity of $G$ with respect to $G$ implies that there are $x_i \in \text{Aut}(G)$ such that $y_i = x_i \circ \alpha : A_i^* \to G^*$. Pick any $g \in \text{Aut}(G)$ and we will show that $g^{-1} \in V x_i^{-1} \text{Aut}(G^*)$, for some $i = 1, 2, \ldots, N$, which will finish the proof of the lemma. Using the property ($\ast$), take $i$ such that $g \circ \alpha : A_i^* \to G^*$ is an embedding. From the ultrahomogeneity of $G^*$, we get $h \in \text{Aut}(G^*)$ such that $h \circ g \circ \alpha = x_i \circ \alpha$. That implies $x_i^{-1} \circ h \circ g \in V$, and hence we get $g^{-1} \in V x_i^{-1} \text{Aut}(G^*)$.

As we have just seen, the space $X_{G^*}$ is compact and that the right uniform space $\text{Aut}(G)/\text{Aut}(G^*)$ is precompact, we show in Theorem 3.22 that $X_{G^*}$ and $\text{Aut}(G)/\text{Aut}(G^*)$ are isomorphic. Call a map $t : X \to Y$, such that $X, Y$ are uniform spaces and a topological group $G$ acts continuously on both $X$ and $Y$ a uniform $G$-isomorphism if it is a $G$-map which is an isomorphism between the uniform spaces $X$ and $Y$.

**Theorem 3.22.** The map $g\text{Aut}(G^*) \to g \cdot \tilde{R}^G$ from $\text{Aut}(G)/\text{Aut}(G^*)$ to $X_{G^*}$, is a uniform $G$-isomorphism from $G \curvearrowright \text{Aut}(G)/\text{Aut}(G^*)$ to $G \curvearrowright X_{G^*}$.

We will say that flows $G \curvearrowright X$ and $G \curvearrowright Y$ are isomorphic if there is a homeomorphism from $X$ onto $Y$ which is a $G$-map.

**Corollary 3.23.** The flow $G \curvearrowright \text{Aut}(G)/\text{Aut}(G^*)$ is isomorphic to the flow $G \curvearrowright X_{G^*}$.

To prove Theorem 3.22, first we show the following lemma.

**Lemma 3.24.** (1) The uniformity on $X_{G^*}$ is generated by sets

$$U^\alpha = \{(\tilde{R}, \tilde{S}) : \text{for some } A^* \in G^* \text{ with } A^* \upharpoonright \mathcal{L} = A$$

$$\alpha : A^* \to (G, \tilde{R}) \text{ and } \alpha : A^* \to (G, \tilde{S}) \text{ are embeddings}\},$$

where $\alpha : A \to G$ is an embedding.

(2) The quotient uniformity on $\text{Aut}(G)/\text{Aut}(G^*)$ given by the right uniformity on $\text{Aut}(G)$ is generated by sets

$$U_\alpha = \{(x\text{Aut}(G^*), y\text{Aut}(G^*)) : x^{-1} \circ \alpha = y^{-1} \circ \alpha\},$$

where $\alpha : A \to G$ is an embedding.

**Proof.** To see (1), using the compactness of $X_{G^*}$, note that for each open neighbourhood $U$ of the diagonal of $X_{G^*}$ we can find $A \in \mathcal{G}$ and an embedding $\alpha : A \to G$ such that the partition of $X_{G^*}$ into clopen sets

$$V_{\alpha, A^*} = \{\tilde{R} \in X_{G^*} : \alpha : A^* \to (G, \tilde{R}) \text{ is an embedding}\},$$

where $A^* \in \mathcal{L}^*$ and $A^* \upharpoonright \mathcal{L} = A$, has the property that

$$\bigcup_{A^* \in G^*, \ A^* \upharpoonright \mathcal{L} = A} V_{\alpha, A^*} \times V_{\alpha, A^*} \subseteq U.$$

Part (2) follows immediately from the definition of the uniformity on $\text{Aut}(G)/\text{Aut}(G^*)$.

**Proof of Theorem 3.22.** Let $\alpha : A \to G$ be an embedding. It is enough to show that

$$U_\alpha = \{(x\text{Aut}(G^*), y\text{Aut}(G^*)) : x^{-1} \circ \alpha = y^{-1} \circ \alpha\}$$

and we will finish the proof of the lemma.
on $\text{Aut}(\mathcal{G})/\text{Aut}(\mathcal{G}^*)$ is mapped to
\[ U^\alpha = \{(x \cdot \vec{R}^G, y \cdot \vec{R}^G) : \text{for some } A^* \in \mathcal{G}^* \text{ with } A^* \upharpoonright \mathcal{L} = A \}
\[ \alpha : A^* \to (\mathcal{G}, x \cdot \vec{R}^G) \text{ and } \alpha : A^* \to (\mathcal{G}, y \cdot \vec{R}^G) \text{ are embeddings} \]
on $\text{Aut}(\mathcal{G}) \cdot \vec{R}^G$. Clearly, the set $U_\alpha$ is mapped to
\[ \overline{U}_\alpha = \{(x \cdot \vec{R}^G, y \cdot \vec{R}^G) : x^{-1} \circ \alpha = y^{-1} \circ \alpha \} \]

Let $(x \cdot \vec{R}^G, y \cdot \vec{R}^G) \in U^\alpha$. Let $A^*$ be such that $\alpha : A^* \to (\mathcal{G}, x \cdot \vec{R}^G)$ and $\alpha : A^* \to (\mathcal{G}, y \cdot \vec{R}^G)$ are embeddings. Then $x^{-1} \circ \alpha : A^* \to (\mathcal{G}, \vec{R}^G)$ and $y^{-1} \circ \alpha : A^* \to (\mathcal{G}, \vec{R}^G)$ are embeddings. By the projective ultrahomogeneity property for $\mathcal{G}^*$, there is $h \in \text{Aut}(\mathcal{G}^*)$ such that $h \circ x^{-1} \circ \alpha = y^{-1} \circ \alpha$. This implies $((x \circ h^{-1}) \cdot \vec{R}^G, y \cdot \vec{R}^G) \in U^\alpha$, and since $h^{-1} \in \text{Aut}(\mathcal{G}^*)$ and so $h^{-1} \cdot \vec{R}^G = \vec{R}^G$, we get $(x \cdot \vec{R}^G, y \cdot \vec{R}^G) \in \overline{U}_\alpha$.

Now suppose that $(x \cdot \vec{R}^G, y \cdot \vec{R}^G) \in \overline{U}_\alpha$. From the property (\star), it follows that there is $A^* \in \mathcal{G}^*$ with $A^* \upharpoonright \mathcal{L} = A$ such that $x^{-1} \circ \alpha : A^* \to (\mathcal{G}, \vec{R}^G)$ is an embedding. Since $x^{-1} \circ \alpha = y^{-1} \circ \alpha$, clearly $y^{-1} \circ \alpha : A^* \to (\mathcal{G}, \vec{R}^G)$ is an embedding. This implies that $\alpha : A^* \to (\mathcal{G}, x \cdot \vec{R}^G)$ and $\alpha : A^* \to (\mathcal{G}, y \cdot \vec{R}^G)$ are embeddings, which gives $(x \cdot \vec{R}^G, y \cdot \vec{R}^G) \in U^\alpha$.

\[ \square \]

4. The universal minimal flow of $\text{Aut}(\mathcal{L})$

We will expand each finite fan in $\mathcal{F}$ by a maximal chain of downwards closed subsets and obtain a class $\mathcal{F}_c$, which does not directly fall into the framework of the projective Fraïssé theory. However, we will show that $\mathcal{F}_c$ is equivalent to a Fraïssé-HP class of first-order structures, which is reasonable and precompact with respect to $\mathcal{F}$.

In Section 4.2, we prove the main combinatorial result that $\mathcal{F}_c$ is a Ramsey class. We do so indirectly by showing that a certain class $\mathcal{F}_{cc}$ coinitial in $\mathcal{F}_c$ is Ramsey. In our proof we will use the dual Ramsey theorem of Graham and Rothschild.

Finally, in Section 4.3, we apply methods from Section 3.6 to compute the universal minimal flow of $\text{Aut}(\mathcal{L})$ in two ways - as a completion of a precompact space equal to the quotient of $\text{Aut}(\mathcal{L})$ by an extremely amenable subgroup, and as the space of maximal downwards closed chains on $\mathcal{L}$ - and we exhibit an explicit isomorphism between them.

4.1. Finite fans equipped with chains – the family $\mathcal{F}_c$. Define
\[ \mathcal{F}_c = \{(A, \mathcal{C}^A) : \text{A } \in \mathcal{F} \text{ and } \mathcal{C}^A \text{ is a downwards closed maximal chain}\} \]
and for $(A, \mathcal{C}^A), (B, \mathcal{C}^B) \in \mathcal{F}_c$ we say that $f : (B, \mathcal{C}^B) \to (A, \mathcal{C}^A)$ is an epimorphism iff $f : B \to A$ is an epimorphism and for every $C \in \mathcal{C}^B$, we have $f(C) \in \mathcal{C}^A$ (short: $f(C^B) = \mathcal{C}^A$). If $A \in \mathcal{F}$ and $\mathcal{C}^A$ are such that $A_c = (A, \mathcal{C}^A) \in \mathcal{F}_c$, then $\mathcal{C}^A$ induces a linear order $\leq_{A_c}$ on $A$ given by $x \leq_{A_c} y$ iff for some $C \in \mathcal{C}^A$, $x \in C$ and $y \notin C$. For $(A, \mathcal{C}^A) \in \mathcal{F}_c$ we say that $\mathcal{C}^A$ is canonical on $A$ if for some ordering of branches $b_1 < \ldots < b_n$ of $A$, it holds that whenever $C \in \mathcal{C}^A$ and $C \cap b_j \neq \emptyset$, then $b_i \subset C$ for every $1 \leq i < j \leq n$.

Analogously as for topological $\mathcal{L}$-structures, we define the JPP and the AP for the family $\mathcal{F}_c$ with the epimorphisms as above.

**Lemma 4.1.** The family $\mathcal{F}_c$ has the JPP and the AP.
Remark 4.2. In particular, the following natural attempt fails. For the AP, let \( f : B \to A \) and \( g : D \to A \) be epimorphisms. We find \( E, k : E \to B \) and \( l : E \to D \) such that \( f \circ k = g \circ l \).

It is straightforward to see that the conclusion holds when each of \( A, B, D \) has exactly one branch.

In the general situation, we take \( E \) that has the width \( w = b + d - a \), where \( b, d, a \) are numbers of elements in \( B, D, A \), respectively, and it has all branches of the height \( h \), where \( h \) is the height of any \( E_0 \in \mathcal{F}_c \) that witnesses the AP for \( A_0, B_0, D_0 \in \mathcal{F}_c \) all with one branch only whose heights are the same as of \( A, B, D \), respectively, and any epimorphisms \( f_0 : B_0 \to A_0 \) and \( g_0 : D_0 \to A_0 \). Let \( \mathcal{C}^E \) be the canonical chain on \( E \) with respect to some enumeration \( e_1, \ldots, e_w \) of branches of \( E \).

Let \((a')_{1 \leq i \leq a}\) be the increasing enumeration of \( A \) according to the linear order \( \leq^A \) induced by \( \mathcal{C}^A \). For every \( i \) let \( b^i \in B \) and \( d^i \in D \) be the the smallest (with respect to the linear orders \( \leq^B \) induced by \( \mathcal{C}^B \) and \( \leq^D \) induced by \( \mathcal{C}^D \), respectively), such that \( f(b^i) = a^i \) and \( g(d^i) = a^i \). Let \( v_A, v_B, v_D \) be the roots of \( A, B, D \), respectively.

We now proceed to define epimorphisms \( k : E \to B \) and \( l : E \to D \) such that \( f \circ k = g \circ l \).

For every \( i = 1, 2, \ldots, a \), let \( J_i = \{ b \in B : b^i <^B b^i+1 \} \} + \{ d \in D : d^i <^D d^i+1 \} - (i - 1).

For \( j = J_i + 1 \), let \( k \upharpoonright e_j \) and \( l \upharpoonright e_j \) be chosen so that \((k \upharpoonright e_j)(e_j) = [v_B, b^i]_{\leq^B}, (l \upharpoonright e_j)(e_j) = [v_D, d^i]_{\leq_D}, \) and \((f \circ k) \upharpoonright e_j = (g \circ l) \upharpoonright e_j, \) which we can do by the choice of \( h \). Let \( b^{i,1}, \ldots, b^{i,m_i} \) be the increasing (with respect to \( \leq^B \)) enumeration of \( \{ b \in B : b^i <^B b^i+1 \} \) and \( d^{i,1}, \ldots, d^{i,m_i} \) be the increasing (with respect to \( \leq^D \)) enumeration of \( \{ d \in D : d^i <^D d^i+1 \} \). For \( j = J_i + m_i + 1 \), let \( k \upharpoonright e_j \) and \( l \upharpoonright e_j \) be such that \((k \upharpoonright e_j)(e_j) = [v_B, b^{i,m_i}]_{\leq^B}, (l \upharpoonright e_j)(e_j) = [v_D, d^{i,m_i}]_{\leq_D}, \) and \((f \circ k) \upharpoonright e_j = (g \circ l) \upharpoonright e_j. This defines the required epimorphisms \( k \) and \( l \).

\( \square \)

At this point we would like to say that there is a limit of the family \( \mathcal{F}_c \), i.e. there is some \( \mathcal{F}_c \) which satisfies properties (L1), (L2), and (L3). However, structures in \( \mathcal{F}_c \) are not first order structures, chains are neither functions nor relations, therefore we cannot directly apply the Irwin-Solecki theorem about the existence and uniqueness of projective Fraïssé limits.

It does not seem that we can realize the family \( \mathcal{F}_c \) as a projective Fraïssé family. In particular, the following natural attempt fails.

**Remark 4.2.** To a structure \((A, \mathcal{C}^A) \in \mathcal{F}_c \) we associate a structure \((A, <^A)\), where \(<^A \) is the linear order induced by \( \mathcal{C}^A \), and we consider a family \( \mathcal{F}_c \) of all \((A, <^A)\) obtained in this way. However, epimorphisms between structures in \( \mathcal{F}_c \) and epimorphisms between structures in \( \mathcal{F}_c \) are not the same. For example, let \( A = \{a_1, a_2\} \) consist of a single branch and let \( \mathcal{C}^A = \{\{a_1\}, \{a_1, a_2\}\} \), let \( B = \{b_1, b_2, b_3\} \) consist of a single branch and let \( \mathcal{C}^B = \{\{b_1\}, \{b_1, b_2\}, \{b_1, b_2, b_3\}\} \). Let \( \phi \) satisfy \( \phi(b_1) = \phi(b_3) = a_1 \) and \( \phi(b_2) = a_2 \). Then \( \phi : (B, \mathcal{C}^B) \to (A, \mathcal{C}^A) \) is an epimorphism, whereas \( \phi : (B, <^B) \to (A, <^A) \) is not an epimorphism.

Nevertheless, we will show that the family \( \mathcal{F}_c \) can be identified with a Fraïssé-HP family similarly as in Theorem 3.3. For this we will have to consider inverse limits of structures in \( \mathcal{F}_c \).
Recall from Section 3.3 that by $\mathcal{F}^*$ we denoted the family of all topological $\mathcal{L}$-structures that are countable inverse limits of finite fans in $\mathcal{F}$. If $P \in \mathcal{F}^*$ is the inverse limit of an inverse sequence $(A_n, f_n^*)$, denoted by $P = \varprojlim (A_n, f_n^*)$, we consider the partial order on $P$ given by

$$x \preceq_P y \text{ iff for every } n, \ f_n^*(x) \preceq_{A_n} f_n^*(y),$$

where $\preceq_{A_n}$ is the tree partial order on $A_n$. Downwards closed sets on $P \in \mathcal{F}^*$ will be taken with respect to $\preceq_P$. If $((A_n, C_{A_n}), f_n^*)$ is an inverse sequence in $\mathcal{F}_c$, we say that $(P, C^P)$ is its inverse limit if $P = \varprojlim (A_n, f_n^*)$ and $C^P$ is the collection of all closed subsets $C$ of $P$ such that $f_n^*(C) \in C_{A_n}$ for every $n$. We will show that $C^P$ is a downwards closed maximal chain. To see that $C^P$ is a chain, note that if $C_1, C_2 \in C^P$, then either for all $n$, $f_n^*(C_1) \subseteq f_n^*(C_2)$ or for all $n$, $f_n^*(C_2) \subseteq f_n^*(C_1)$. In the first case, $C_1 \subseteq C_2$, and in the second, $C_2 \subseteq C_1$. The chain $C^P$ is downwards closed. Finally, the chain $C^P$ is maximal. Indeed, if a downwards closed set $C$ is such that $\{C\} \cup C^P$ is a chain, then for each $n$, by the maximality of $C_{A_n}$, $f_n^*(C) \in C_{A_n}$, which by the definition of $C^P$ gives $C \subseteq C^P$.

Let $\mathcal{F}^*_c$ be the family of all inverse limits of structures from $\mathcal{F}_c$. Clearly, we can identify $\mathcal{F}_c$ with a subfamily of $\mathcal{F}^*_c$ by assigning to $(A, C^A)$ the inverse limit of $((A, C^A), \mathbb{I}^m)$.

**Theorem 4.3.** The family $\mathcal{F}^*_c$ with epimorphisms is equivalent via a contravariant functor to a family of first order structures with embeddings.

**Proof.** Let $R$ be a binary relation symbol and take the language $\{S, \leq_{BA}, \cup, \cap, 0, 1\}$, where $S$ and $\leq_{BA}$ are binary relation symbols and $\{\cup, \cap, 0, 1\}$ is the language of Boolean algebras. For $K = (K, R^K, C^K) \in \mathcal{F}^*_c$, let $M = (M, S^M, \leq_M^K, \cup^M, \cap^M, -^M, 0^M, 1^M)$ be the structure such that $M = \text{Clop}(K)$ is the family of all clopen sets of $K$, $\cup^M$ is the union, $\cap^M$ is the intersection, $-^M$ is the complement, $0^M$ is the empty set and $1^M = M$. As in Proposition 3.6, we set for every $X, Y \in M$, $S^M(X, Y)$ if and only if for some $a \in X, b \in Y$, we have $R^K(a, b)$.

We first define $\leq_{BA}$ for $K \in \mathcal{F}_c$, and then we provide a definition for $K \in \mathcal{F}^*_c$. Let then $K \in \mathcal{F}_c$. As before, denote by $\leq^K$ the linear order on $K$ induced by $C^K$ by letting $x <^K y$ iff there exists $C \in C^K$ such that $x \in C$ and $y \notin C$. From the maximality of $C^K$, the order $\leq^K$ is total. For $K \in \mathcal{F}_c$, let $\leq_{op}$ denote the opposite to the order $\leq^K$; that is, we let $x \leq_{op} y$ iff $y \leq^K x$.

Take $\leq_{BA}^M$ to be the antilexicographical order with respect to $\leq_{op}^M$, that is for $X, Y \in M = \text{Clop}(K) = P(K)$, where $P(K)$ denotes the power set of $K$, let $X <^M_{BA} Y$ iff for $a \in K$ which is the largest with respect to $\leq_{op}^M$ such that $a \in X \Delta Y$, we have $a \in Y$.

Let $f : L \to K$, where $K, L \in \mathcal{F}$ be a continuous surjection and let $F : P(K) \to P(L)$ be the map given by $F(X) = f^{-1}(X)$.

**Claim.** $F$ is $\leq_{BA}$-preserving iff $f(C^L) = C^K$.

**Proof.** The function $f$ is chains preserving iff $f$ maps cofinal segments in $\leq_{op}^L$ to cofinal segments in $\leq_{op}^K$. Hence $f$ maps sets $\{z \in L : a \leq_{op}^L z\}$, some $a \in L$ to $\{z \in K : b \leq_{op}^K z\}$, some $b \in K$ iff $F$ is $\leq_{BA}$-preserving. $\square$
Proposition 3.6 together with the claim above, already imply the conclusion of the theorem for the family $\mathcal{F}_c$.

Let $\mathcal{G}$ be the family of all $M$’s obtained in this way from some $K \in \mathcal{F}_c$. Maps we consider between structures in $\mathcal{G}$ are embeddings.

Now let us come to the general situation where the structures come from $\mathcal{F}_c^*$. The claim above implies that an inverse sequence $((K_n, C^{K_n}), f_m^n)$ in $\mathcal{F}_c$ corresponds to a direct sequence $((M_n, \preceq_{BA}), g_m^n)$ in $\mathcal{G}$. Let $(K, C^K)$ together with epimorphisms $f_m^n : (K, C^K) \to (K_n, C^{K_n})$ be the inverse limit of $((K_n, C^{K_n}), f_m^n)$. Let $M$ together with embeddings $g_m^n : M_n \to M$ be the direct limit of $((M_n, \preceq_{BA}), g_m^n)$.

By the definition of the direct limit, $X \leq_{BA}^Y$ iff for some (equivalently every) $n$ such that there are $X_n, Y_n \in M_n$ with $g_m^n(X_n) = X$ and $g_m^n(Y_n) = Y$, we have $X_n \preceq_{BA} Y_n$.

Then $(M, \leq_{BA}^M)$ together with embeddings $g_m^n : M_n \to M$ is the direct limit of $((M_n, \preceq_{BA}), g_m^n)$.

The following claim will finish the proof.

**Claim.** Let $f : L \to K$, where $K, L \in \mathcal{F}_c^*$, be a continuous surjection and let $F : \text{Clop}(K) \to \text{Clop}(L)$ be the map given by $F(X) = f^{-1}(X)$. Then $f$ preserves chains iff $F$ preserves $\leq_{BA}$.

**Proof.** Let $(L_n, l_m^n)$ be an inverse sequence in $\mathcal{F}_c$ with the limit $L$ and let $(K_n, k_m^n)$ be an inverse sequence in $\mathcal{F}_c$ with the limit $K$. Let $p_m^n : \text{Clop}(K_m) \to \text{Clop}(K_n)$ be the dual map to $k_m^n$ and let $q_m^n : \text{Clop}(L_m) \to \text{Clop}(L_n)$ be the dual map to $l_m^n$. Then $(\text{Clop}(K_m), p_m^n)$ is an inverse sequence with the limit $\text{Clop}(K)$ and $(\text{Clop}(L_m), q_m^n)$ is an inverse sequence with the limit $\text{Clop}(L)$.

Having $f : L \to K$ find a strictly increasing sequence $(t_n)$ and continuous surjections $h_n : L_{t_n} \to K_n$ such that $h_n \circ l_{t_n+1} = k_{t_n+1} \circ h_{t_n+1}$. Let $H_n : \text{Clop}(K_n) \to \text{Clop}(L_{t_n})$ be dual maps to $h_n$.

Then we have: $f : L \to K$ preserves chains iff for every $n$, $h_n$ preserve chains iff for every $n$, $H_n$ preserve linear orders iff $F : \text{Clop}(K) \to \text{Clop}(L)$ preserves linear orders.

\[ \square \]

Thanks to Theorem 1.3, we not only know that there is a unique up to an isomorphism structure in $\mathcal{F}_c^*$ that satisfies conditions (L1), (L2) and (L3) for the family $\mathcal{F}_c$, but also all theorems that were proved for Fraïssé-HP families are available to us.

Let $\mathbb{L}_c$ denote the limit of the family $\mathcal{F}_c$. The proposition below tells us that $\mathbb{L}_c$ is equal to $(\mathbb{L}, C^L)$ for some downwards closed maximal chain $C^L$.

**Lemma 4.4.** The expansion $\mathcal{F}_c$ of $\mathcal{F}$ is reasonable, that is, for every $A, B \in \mathcal{F}$, an epimorphism $\phi : B \to A$, and $A_c \in \mathcal{F}_c$ such that $A_c \upharpoonright L = A$, there is $B_c \in \mathcal{F}_c$ such that $B_c \upharpoonright \mathcal{L} = B$ and $\phi : B_c \to A_c$ is an epimorphism.

**Proof.** Let $A, B \in \mathcal{F}$, an epimorphism $\phi : B \to A$, and $A_c \in \mathcal{F}_c$ such that $A_c \upharpoonright \mathcal{L} = A$, be given. We get $C^B$ by extending in an arbitrary way the downwards closed chain $\{\phi^{-1}(C) : C \in C^A\}$ into a downwards closed maximal chain. \[ \square \]

Lemma 1.4 and Proposition 3.18 immediately imply the following corollary.

**Corollary 4.5.** We have $\mathbb{L}_c \upharpoonright \mathcal{L} = \mathbb{L}$.  

Corollary 4.5 implies that Aut(Lc), the automorphism group of Lc, is a subgroup of Aut(L), the automorphism group of L. Recall that the group Aut(L) is equipped with the compact open topology inherited from H(L), the homeomorphism group of L.

Lemma 4.6. The group Aut(Lc) is a closed subgroup of Aut(L).

Proof. For any D ∈ Exp(L), the map that takes f ∈ Aut(L) and assigns to it f(D) ∈ Exp(L) is continuous. Since Cb is maximal, it is closed in Exp(L), which implies that Aut(Lc) is closed in Aut(L). □

Lemma 4.6 also follows from Proposition 4.17, which we prove later.

4.2. Extreme amenability of Aut(Lc). In this section, we define a family \( F_{cc} \) coinitial in \( \mathcal{F}_c \) and show that it is a Ramsey class. This will imply that Aut(Lc) is extremely amenable and that \( \mathcal{F}_c \) is a Ramsey class.

Recall that for \( A_c \in \mathcal{F}_c \), \( C^A \) is canonical if there is an order on branches of \( A_c \), which we denote by \( \leq_{ac} \), given by \( b \leq_{ac} c \) iff for every \( x \in b \) and \( y \in c \), we have \( x \leq_{ac} y \). Let \( \mathcal{F}_{cc} = \{ (A, C^A) \in \mathcal{F}_c : \text{C}^A \text{ is canonical and all branches in A have the same height} \} \).

For \( A, B \in \mathcal{F}_{cc} \), let \( \{ B \} \) denote the set of all epimorphisms from \( B \) onto \( A \).

The main result of this section is the following theorem.

Theorem 4.7. The class \( \mathcal{F}_{cc} \) is a Ramsey class, that is, for every integer \( r \geq 2 \) and for \( S, T \in F_{cc} \) with \( \{ S \} \neq \emptyset \) there exists \( U \in F_{cc} \) such that for every colouring \( e : \{ U \} \to \{ 1, 2, \ldots, r \} \) there is \( g \in \{ T \} \) such that \( \{ h \circ g : h \in \{ T \} \} \) is monochromatic.

Proposition 4.8. The family \( \mathcal{F}_{cc} \) is coinitial in \( \mathcal{F}_c \), that is, for every \( A_c \in \mathcal{F}_c \) there exist \( B_c \in \mathcal{F}_{cc} \) and an epimorphism from \( B_c \) onto \( A_c \). Moreover, we can choose \( B_c \) in a way that its height and width depend only on the height and width of \( A_c \).

Proof. Let \( A_c \in \mathcal{F}_c \) be of height \( k \) and let \( v_{A_c} \) denote its root. Let \( B_c \in \mathcal{F}_{cc} \) be of height \( k \) and width \( l \) equal to the number of elements in \( A_c \), and let \( C^{B_c} \) be canonical. Enumerate \( A_c \) according to \( \leq_{ac} \) into \( a_1, \ldots, a_l \), and enumerate branches in \( B_c \) according to \( \leq_{bc} \) into \( b_1, \ldots, b_l \). Now let for each \( i = 1, \ldots, l \), the branch \( b_i \) be mapped onto the segment \([v_{A_c}, a_i]_{\leq_a} \in A_c \) in an \( R \)-preserving way. This defines a required epimorphism from \( B_c \) onto \( A_c \). □

Remark 4.9. Proposition 4.8 implies (by Lemma 4.1 and Remark 3.1) that \( \mathcal{F}_{cc} \) satisfies the JPP and the AP and that the limits of \( \mathcal{F}_c \) and \( \mathcal{F}_{cc} \) are isomorphic to \( L_c \).

From Theorems 4.17 and 4.7 using Remark 4.9, we will obtain the following corollary.

Corollary 4.10. The automorphism group Aut(Lc) is extremely amenable.

The family \( \mathcal{F}_{cc} \) is easier to work with than the family \( \mathcal{F}_c \). Nevertheless, \( \mathcal{F}_c \) is a Ramsey class as well, which follows from the following proposition.

Proposition 4.11. Let \( \mathcal{G}_1 \subset \mathcal{G}_2 \) be Fraïssé-HP families and suppose that \( \mathcal{G}_1 \) is coinitial in \( \mathcal{G}_2 \). If \( \mathcal{G}_1 \) is a Ramsey class, so is \( \mathcal{G}_2 \).

Proof. Let \( \mathcal{G} \) be the Fraïssé limit of both \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) and let \( G = \text{Aut}(\mathcal{G}) \). As \( \mathcal{G}_1 \) is a Ramsey class, Theorem 3.17 (applied to \( G \) and \( \mathcal{G}_1 \)) implies that \( G \) is extremely amenable. Then again applying Theorem 3.17 this time to \( G \) and \( \mathcal{G}_2 \), we get that \( \mathcal{G}_2 \) is a Ramsey class. □
Corollary 4.12. The family $\mathcal{F}_c$ is a Ramsey class.

The main two ingredients in the proof of Theorem 4.7 will be Theorem 4.13 and Corollary 4.16.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers and let $k \in \mathbb{N}$. For a function $p : \mathbb{N} \to \{0, 1, \ldots, k\}$, we define the support of $p$ to be the set $\text{supp}(p) = \{l \in \mathbb{N} : p(l) \neq 0\}$.

Let $\text{FIN}_k = \{p : \mathbb{N} \to \{0, 1, \ldots, k\} : \text{supp}(p) \text{ is finite and } \exists l \in \text{supp}(p) \ (p(l) = k)\}$, and for each $n \in \mathbb{N}$, let

$$\text{FIN}_k(n) = \{p \in \text{FIN}_k : \text{supp}(p) \subset \{1, 2, \ldots, n\}\}.$$ We equip $\text{FIN}_k$ and each $\text{FIN}_k(n)$ with a partial semigroup operation $+$ defined for $p$ and $q$ whenever $\text{supp}(p) \cap \text{supp}(q) = \emptyset$ by $(p + q)(x) = p(x) + q(x)$.

The Gowers’ tetris operation is the function $T : \text{FIN}_k \to \text{FIN}_{k-1}$ defined by

$$T(p)(l) = \max\{0, p(l) - 1\}.$$ We define for every $0 < i \leq k$ a function $T^{(k)}_i : \text{FIN}_k \to \text{FIN}_{k-1}$, which behaves like the identity up to the value $i - 1$ and like the tetris above it as follows.

$$T_i^{(k)}(p)(l) = \begin{cases} p(l) & \text{if } p(l) < i \\ p(l) - 1 & \text{if } p(l) \geq i. \end{cases}$$

We also define $T_0^{(k)} = \text{Id}|_{\text{FIN}_k}$. It may seem more natural to denote the identity by $T_{k+1}^{(k)}$ or $T_\infty^{(k)}$, only for notational convenience later on, we will be using $T_0^{(k)}$. Note that $T_1^{(k)}$ is the Gowers’ tetris operation. We will usually drop superscripts and write $T_i$ rather than $T_i^{(k)}$.

Define

$$\text{FIN}_k^{(d)}(n) = \{(p_1, \ldots, p_d) : p_i \in \text{FIN}_k(n) \text{ and } \forall i < j \ (\text{supp}(p_i) \cap \text{supp}(p_j) = \emptyset \text{ and } \min(\text{supp}(p_i)) < \min(\text{supp}(p_j)))\}$$

and

$$\text{FIN}_k^{*(d)}(n) = \{(p_1, \ldots, p_d) \in \text{FIN}_k^{(d)} : \min(\text{supp}_k(p_i)) < \min(\text{supp}_k(p_{i+1}))\},$$

where $\text{supp}_j(p) = \{l \in \{1, \ldots, n\} : p(l) = j\}$ for $p \in \text{FIN}_k(n)$ and $j = 1, \ldots, k$.

Let $P_k = \prod_{j=1}^k \{0, 1, \ldots, j\}$. For any $\vec{i} = (i(1), \ldots, i(k)) \in P_k$ denote

$$T_\vec{i} = T_{i(1)} \circ \ldots \circ T_{i(k)}.$$ For $l > k$, let $P_{k+1} = \prod_{j=k+1}^l \{1, 2, \ldots, j\}$, and let $P_{k+1}$ contain only the constant sequence $(0, \ldots, 0)$. Note that if $p \in \text{FIN}_l$ and $\vec{i} \in P_{k+1}$, then $T_{\vec{i}}(p) \in \text{FIN}_k$.

Let $l \geq k$ and let $B = (b_s)_{s=1}^m \in \text{FIN}_l^{*(m)}(n)$, we denote by $\left(\bigcup_{i \in P_{k+1, l}} T_i(B)\right)_{P_k}$ the partial subsemigroup of $\text{FIN}_k$ consisting of elements of the form

$$\sum_{s=1}^m T_{\vec{i}_s} \circ T_{\vec{i}_s}^*(b_s).$$
where \( \vec{r}_1, \ldots, \vec{r}_m \in P_{k+1}^d \), \( \vec{r}_1, \ldots, \vec{r}_m \in P_k \), and there is an \( s \) such that all entries of \( t_s \) are 0. By \( \left( \bigcup_{i \in P_{k+1}^d} T_i(B) \right)_{P_k}^{s(d)} \), we denote the set of all \( (p_1, \ldots, p_d) \in \text{FIN}_k^{s(d)} \) such that \( p_i \in \left( \bigcup_{i \in P_{k+1}^d} T_i(B) \right)_{P_k}^{s(d)} \).

The following theorem is the combinatorial core of Theorem 4.7, its proof was inspired by the proof a Ramsey Theorem in [BLLM].

**Theorem 4.13.** Let \( k \geq 1 \). Then for every \( m \geq d \), for every \( l \geq k \), and for every \( r \geq 2 \) there exists a natural number \( n \) such that for every colouring \( c : \text{FIN}_k^{s(d)}(n) \to \{1, 2, \ldots, r\} \), there is \( B \in \text{FIN}_k^{s(m)}(n) \) such that the partial semigroup \( \left( \bigcup_{i \in P_{k+1}^d} T_i(B) \right)_{P_k}^{s(d)} \) is \( c \)-monochromatic. Denote the smallest such \( n \) by \( L(d, m, k, l, r) \).

In the proof of Theorem 4.13, we will use the Graham-Rothschild theorem about colouring of partitions [GR]. For natural numbers \( d, n \) let \( P^d(n) \) denote the set of all partitions of the set \( \{1, 2, \ldots, n\} \) into exactly \( d \) non-empty sets. We say that a partition \( P \) is a coarsening of a partition \( Q \) if for every \( Q \in Q \) there is \( P \in P \) such that \( Q \subset P \). Every partition \( P \in P^d(n) \) carries a canonical enumeration, where for \( P, Q \in P \) we set ..

**Theorem 4.14** (Graham-Rothschild, [GR]). Let \( k < l \) and \( r \geq 2 \) be given natural numbers. Then there is a natural number \( n \) such that for any colouring of \( P^k(n) \) into \( r \) colours there is a partition \( P \in P^d(n) \) such that the set \( \{ Q \in P^k(n) : P \text{ is a coarsening of } Q \} \) is monochromatic. Let \( GR(k, l, r) \) denote the smallest such natural number \( n \).

**Proof of Theorem 4.13.** We set \( n = GR(dk + 1, ml + 1, r) \) and let \( c : \text{FIN}_k^{s(d)}(n) \to \{1, 2, \ldots, r\} \) be an arbitrary colouring. Define the map \( \Phi : P^{dk+1}(n) \to \text{FIN}_k^{s(d)}(n) \) that to a canonically enumerated partition \( P = (P_i)_{i=0}^{dk} \) assigns

\[
\Phi(P)_j = \sum_{s=1}^{k} s \cdot \mathbb{1}_{P_{(j-1)k+s}}
\]

for \( j = 1, \ldots, d \). Then \( c \circ \Phi \) is a colouring of \( P^{dk+1}(n) \).

Let a canonically enumerated partition \( Q = (Q_i)_{i=0}^{ml} \) be \( c \circ \Phi \)-monochromatic. We define \( B = (b_j)_{j=1}^{ml} \) by

\[
b_j = \sum_{s=1}^{l} s \cdot \mathbb{1}_{P_{(j-1)l+s}}.
\]

The following claim verifies that \( \left( \bigcup_{i \in P_{k+1}^d} T_i(B) \right)_{P_k}^{s(d)} \) is contained in the image of coarsenings of \( Q \) of size \( dk + 1 \) under \( \Phi \), which implies that \( \left( \bigcup_{i \in P_{k+1}^d} T_i(B) \right)_{P_k}^{s(d)} \) is \( c \)-monochromatic, which is what we wanted to show.

**Claim.** Let \( A = (A_1, \ldots, A_d) \in \left( \bigcup_{i \in P_{k+1}^d} T_i(B) \right)_{P_k}^{s(d)}. \) Then \( A = \Phi(P) \) for \( P = (P_j)_{j=0}^{dk} \) given by \( P_{(s-1)k+i} := \text{supp}_i(A_s) \) for \( s = 1, \ldots, d \) and \( i = 1, \ldots, k \), and \( P_0 = \{1, \ldots, n\} \setminus \bigcup_{j=1}^{dk} P_j \).
Proof. Clearly, $\mathcal{P}$ is a coarsening of $\mathcal{Q}$. Therefore all we have to show is that $\mathcal{P}$ is a partition into exactly $dk + 1$ nonempty sets (see (1) below) and that the enumeration of sets in $\mathcal{P}$ is the canonical enumeration (see (2), (3), and (4) below).

1. For every $i = 1, \ldots, k$ and $s = 1, \ldots, d$, we have $\text{supp}_i(A_s) \neq \emptyset$.
2. We have $Q_0 \subseteq P_0$.
3. For every $i, i' = 1, \ldots, k$, $i \neq i'$, and $s = 1, \ldots, d$,
\[
\min \text{supp}_i(A_s) < \min \text{supp}_{i'}(A_s).
\]
4. For every $s = 1, \ldots, d - 1$,
\[
\min \text{supp}_s(A_s) < \min \text{supp}(A_{s+1}).
\]

Property (2) is clear. Properties (1) and (3) follow from the definition of $B$ and that we can write
\[
A_s = \sum_{j \in J_s} T_{ij} b_j
\]
for some $\vec{t}_j \in P_k P_{k+1}$ (where $P_k P_{k+1}$ is the set of concatenations of sequences in $P_k$ and $P_{k+1}$) and $J_s \subset \{1, \ldots, m\}$. Property (4) follows from $A \in \text{FIN}_k^{\ast(d)}(n)$.

For a natural number $N$ we let $N^{[j]}$ denote the collection of all $j$-element subsets of $\{1, \ldots, N\}$ and let $N^{[\leq j]}$ denote the collection of all at most $j$-element subsets of $\{1, \ldots, N\}$. Note that $N^{[\leq j]} = \bigcup_{i=0}^{j} N^{[i]}$. We will often write $N$ instead of $N^{[1]}$.

Let $m, k_1, \ldots, k_m, l_1, \ldots, l_m, r \geq 2$, and $N$ be natural numbers and let
\[
c: \prod_{i=1}^{m} N^{[\leq k_i]} \to \{1, 2, \ldots, r\}
\]
be a colouring. Given $B_i \subseteq N$ for $i = 1, 2, \ldots, m$, we say that $c$ is size-determined on $(B_i)_{i=1}^{m}$ if whenever $A_i, A'_i \subseteq B_i$ are such that $0 \leq |A_i| = |A'_i| \leq k_i$ for $i = 1, 2, \ldots, m$ then
\[
c(A_1, \ldots, A_m) = c(A'_1, \ldots, A'_m).
\]

**Theorem 4.15** (Theorem 4.10, [BK2]). Let $m, k_1, \ldots, k_m, l_1, \ldots, l_m$ and $r \geq 2$ be natural numbers such that $k_i \leq l_i$ for every $i = 1, 2, \ldots, m$. Then there exists $N$ such that for every colouring
\[
c: \prod_{i=1}^{m} N^{[\leq k_i]} \to \{1, 2, \ldots, r\}
\]
there exist $B_1, \ldots, B_m \subseteq N$ with $|B_i| = l_i$ such that $c$ is size-determined on $(B_i)_{i=1}^{m}$. Denote by $S(m, k_1, \ldots, k_m, l_1, \ldots, l_m, r)$ the minimal such $N$.

For $f \in \prod_{i=1}^{m} N^{[\leq k_i]}$, we define $\text{supp}(f) = \{i : f(i) \neq \emptyset\}$. For a natural number $d$, let $(\prod_{i=1}^{m} N^{[\leq k_i]})^{\ast(d)}$ be the set of all sequences $(f_s)_{s=1}^{d}$ with $f_s \in \prod_{i=1}^{m} N^{[\leq k_i]}$ and $\text{supp}(f_s) \cap \text{supp}(f_{s+1}) = \emptyset$, for each $s$. Note that supports of some of the $f_s$ may be empty.

Then, more generally, if
\[
\chi: \left(\prod_{i=1}^{m} N^{[\leq k_i]}\right)^{\ast(d)} \to \{1, 2, \ldots, r\}
\]
is a colouring and \( B_i \subseteq N \) for \( i = 1, 2, \ldots, m \), we say that \( \chi \) is size-determined on \((B_i)_{i=1}^m\) if whenever \((f_s)_{s=1}^d\) and \((g_s)_{s=1}^d\) are such that for each \( i \) and \( s \), \( \text{supp}(f_s) = \text{supp}(g_s) \), \( f_s(i), g_s(i) \subseteq B_i \), and \( |f_s(i)| = |g_s(i)| \), then
\[
\chi \left( (f_s)_{s=1}^d \right) = \chi \left( (g_s)_{s=1}^d \right).
\]

Corollary 4.16 is a multidimensional version of Theorem 4.15.

**Corollary 4.16.** Let \( d \leq m \), and \( k_1 \leq l_1, \ldots, k_m \leq l_m \), and \( r \geq 2 \) be natural numbers. Then there exists \( N \) such that for every colouring
\[
\chi : \left( \prod_{i=1}^m N_{\leq k_i} \right)^{\ast(d)} \to \{1, 2, \ldots, r\}
\]
there exist \( B_1, \ldots, B_m \subseteq N \) with \( |B_i| = l_i \) such that \( \chi \) is size-determined on \((B_i)_{i=1}^m\). Denote by \( S(d, m, k_1, \ldots, k_m, l_1, \ldots, l_m, r) \) the minimal such \( N \).

**Proof.** Denote by \( \Gamma \) the set of all ordered partitions \( \gamma : \{1, \ldots, m\} \to \{1, \ldots, d\} \) into \( d \) non-empty pieces. Let \( c : \prod_{i=1}^m N_{\leq k_i} \to r^\Gamma \) be the colouring given by
\[
c(A_1, \ldots, A_m)(\gamma) = \chi \left( (f_s^\gamma)_{s=1}^d \right),
\]
where
\[
f_s^\gamma(n) = \begin{cases} A_n & \text{if } n \in \gamma(s), \\ \emptyset & \text{otherwise.} \end{cases}
\]

Applying Theorem 4.15 we get \( B_1, \ldots, B_m \subseteq N \) with \( |B_i| = l_i \) such that \( c \) is size-determined on \((B_i)_{i=1}^m\). It follows that \( \chi \) is size-determined on \((B_i)_{i=1}^m\). \( \square \)

The proof below is similar to the proof of Theorem 4.12 in Bartošová-Kwiatkowska [BK2].

**Proof of Theorem 4.17.** Let \( S \in \mathcal{F}_{cc} \) be of height \( k \) and width \( d \), and let \( T \in \mathcal{F}_{cc} \) be of height \( l \geq k \) and width \( m \geq d \) (so that \( \left( \frac{t}{k} \right) \neq \emptyset \)). Let \( r \geq 2 \) be the number of colours. Let \( n \) be as in Theorem 4.13 for \( d, m, k, l, r \), that is \( n = L(d, m, k, l, r) \), and let \( N \) be as in Corollary 4.16 for \( d, n, k, \ldots, l \), \( d \) is size-determined on \((B_i)_{i=1}^m\). It follows that \( \chi \) is size-determined on \((B_i)_{i=1}^m\). \( \square \)

Let \( a_1, \ldots, a_d \) and \( c_1, \ldots, c_n \) be the increasing (according to \( \prec_{cc}^S \) and \( \prec_{cc}^U \)) enumerations of branches in \( S \) and \( U \), respectively. Let \( (a_j^i)_{i=0}^k \) be the increasing enumeration of the branch \( a_j \), \( j = 1, \ldots, d \), and let \( (c_j^i)_{i=0}^N \) be the increasing enumeration of the branch \( c_j \) for \( j = 1, \ldots, n \).

To each \( f \in \{U \}_{S} \), we associate \( f^* = (p_i^f)_{i=0}^d \in \text{FIN}^{\ast(d)}(n) \) such that
\[
\text{supp}(p_i^f) = \{ j : a_i^j \in f(c_j) \}
\]
and for \( j \in \text{supp}(p_i^f) \)
\[
p_i^f(j) = z \iff f(c_j^N) = a_i^j.
\]

We moreover associate to \( f \) a sequence \( (F_i^f)_{i=1}^d \in \left( \prod_{j=1}^n (c_j \setminus \{c_j^0\}) \right)^{\ast(d)} \) such that for each \( i \) there is \( j \) with \( F_i^f(j) \in (c_j \setminus \{c_j^0\})^{\leq k} \) as follows. For \( j \in \text{supp}(p_i^f) \), we let
\[
F_i^f(j) = \{ \min \{ c_j^0 \in c_j : f(c_j^0) = a_i^j \} : 0 < x \leq p_i^f(j) \},
\]
where the min above is taken with respect to the partial order \( \preceq_U \) on the fan \( U \).


Let us remark that \( f \mapsto (F_i^j)^d_{i=1} \) is an injection from \((U_\mathcal{S})^n\) to \((\Pi_{j=1}^n(c_j \setminus \{c_j^0\})^{[\leq k]} \ast^{(d)}\) and \( f \mapsto f^* \) is a surjection from \((U_\mathcal{S})^n\) to \(\text{FIN}_k^{*(m)}(n)\). Note that if \( f_1^* = f_2^* \) then \(|F_i^j_1(j)| = |F_i^j_2(j)| \) for all \( i, j \).

Analogously, to any \( g \in (U_\mathcal{S})^n \), we associate \( g^* \in \text{FIN}_k^{*(m)}(n) \) and \((F_i^g)^m_{i=1} \in ([\Pi_{j=1}^n(c_j \setminus \{c_j^0\})^{[\leq k]} \ast^{*(m)})\).

Let \( e : (U_\mathcal{T})^n \to \{1, \ldots, r\} \) be a colouring. Let \( e_0 \) be a colouring of \((\Pi_{j=0}^n(c_j \setminus \{c_j^0\})^{[\leq k]} \ast^{(d)}\) induced by the colouring \( e \) via the injection \( f \mapsto (F_i^f)^d_{i=1} \). We colour elements in \((\Pi_{j=0}^n(c_j \setminus \{c_j^0\})^{[\leq k]} \ast^{(d)}\) not of the form \((F_i^g)^m_{i=1}\) in an arbitrary way by one of the colours in \(\{1, \ldots, r\}\). Applying Corollary \ref{cor:exponential}, we can find \( C_j \subset c_j \setminus \{c_j^0\} \) of size \( l \) for \( j = 1, \ldots, n \) such that \( e_0 \) is size-determined on \((C_j)_{j=1}^n\). It follows that the colouring \( e^* : \text{FIN}_k^{*(d)}(n) \to \{1, 2, \ldots, r\} \) given by \( e^*(f^*) = e(f) \) for \( f \in (U_\mathcal{S})^n \) with \((F_i^f)^d_{i=1} \in ([\Pi_{j=1}^n(C_j^{[\leq k]} \ast^{(d)})\) is well-defined.

Now we can apply Theorem \ref{thm:projective} to get \( D = (d_j)^m_{j=1} \in \text{FIN}_k^{*(m)}(n) \) such that \( \langle \bigcup_{\mathcal{F}_i \mathcal{P}_k} T_i(D) \rangle_{\mathcal{P}_k}^{*(d)} \) is \( e^* \)-monochromatic. Let \( g \in (U_\mathcal{T})^n \) be any epimorphism such that \((F_i^g)^m_{i=1} \in ([\Pi_{j=1}^n(C_j^{[\leq k]} \ast^{(m)})\) and \( g^* = D \). Then for every \( h \in (T_\mathcal{S})^n \), we have \((h \circ g)^* \in \langle \bigcup_{\mathcal{F}_i \mathcal{P}_k} T_i(D) \rangle_{\mathcal{P}_k}^{*(d)} \). Since \( \langle \bigcup_{\mathcal{F}_i \mathcal{P}_k} T_i(D) \rangle_{\mathcal{P}_k}^{*(d)} \) is \( e^* \)-monochromatic, we conclude that \((d_j)_{j=1}^n \circ g \) is \( e \)-monochromatic.

\( \square \)

4.3. The universal minimal flow of \( \text{Aut}(\mathcal{L}) \). In this section, we present two descriptions of the universal minimal flow of \( \text{Aut}(\mathcal{L}) \) and we exhibit an explicit isomorphism between them.

Let \( X^* \) be the set of downwards closed maximal chains on \( \mathcal{L} \) equipped with the topology inherited from \( \text{Exp}(\text{Exp}(\mathcal{L})) \). This is a compact space, which we proved in Proposition \ref{prop:compact}. The group \( \text{Aut}(\mathcal{L}) \) acts on \( X^* \) by left translations

\[ g \cdot C_1 = C_2 \iff C_2 = \{ g(C) : C \in C_1 \}. \]

**Proposition 4.17.** The action \( \text{Aut}(\mathcal{L}) \actson X^* \) is continuous.

**Proof.** We have the following general fact: Whenever a Polish group \( H \) acts continuously on a compact space \( K \), then the corresponding action of \( H \) on \( \text{Exp}(K) \) by left translations is also continuous (see page 20 in [BKc], Example (ii)).

It follows, using the fact above twice, that the action of \( \text{Aut}(\mathcal{L}) \) on \( \text{Exp}(\text{Exp}(\mathcal{L})) \) by left translations is continuous. Therefore the restriction of this action to the closed invariant set \( X^* \) is also continuous, which is what we wanted to show. \( \square \)

In this section, we prove the following:

**Theorem 4.18.** The universal minimal flow of \( \text{Aut}(\mathcal{L}) \) – the automorphism group of the projective Fraïssé limit of finite fans – is equal to

\[ \text{Aut}(\mathcal{L}) \actson X^* \]

and it is isomorphic to

\[ \text{Aut}(\mathcal{L}) \actson \text{Aut}(\mathcal{L})/\text{Aut}(\mathcal{L}_c). \]
Let
\[ X_{L_e} = \{ C \in X^* : \text{ for every } A \in \mathcal{F} \text{ and an epimorphism } \phi : \mathbb{L} \to A \text{ there exists } A_e \in \mathcal{F}_{c}, \text{ such that } \phi : (\mathbb{L}, C) \to A_e \text{ is an epimorphism} \}. \]

We make \( X_{L_e} \) a topological space by declaring sets
\[ V_{\phi,A_e} = \{ C \in X_{L_e} : \phi : (\mathbb{L}, C) \to A_e \text{ is an epimorphism} \}, \]
where \( \phi : \mathbb{L} \to A \) is an epimorphism, \( A_e \in \mathcal{F}_{c} \), and \( A_e \upharpoonright \mathcal{L} = A \), to be open.

**Proposition 4.19.** We have \( X_{L_e} = X^* \).

**Proof.** We have to show two things: \( X^* \subset X_{L_e} \) and the topologies on \( X^* \) and \( X_{L_e} \) agree.

Lemma 3.11 implies that if \( C \in X^* \) and \( \phi : \mathbb{L} \to A \) is an epimorphism, then there is a (necessarily unique) \( A_e \in \mathcal{F}_{c} \) with \( A_e \upharpoonright \mathcal{L} = A \) such that \( \phi : (\mathbb{L}, C) \to A_e \) is an epimorphism, from which it follows that \( X^* \subset X_{L_e} \).

Since \( X^* \) and \( X_{L_e} \) are both compact, it suffices to show that the identity map from \( X^* \) to \( X_{L_e} \) is continuous. For this we show that sets of the form \( V_{\phi,A_e} \) are open in \( X^* \). For any partition \( P \) of \( \mathbb{L} \) into clopen sets and any \( P_1, \ldots, P_n \subset P \), by the definition of the Vietoris topology, each of the sets
\[
\overline{P}_i = \{ D \in \text{Exp}(\mathbb{L}) : D \subset \bigcup P_i \text{ and for every } p \in P_i, D \cap p \neq \emptyset \}
\]
\[
= \{ D \in \text{Exp}(\mathbb{L}) : P_i = \{ p \in D : D \cap p \neq \emptyset \} \}
\]
is open in \( \text{Exp}(\mathbb{L}) \) and therefore
\[
V_{P_1,\ldots,P_n} = \{ C \in \text{Exp}(\text{Exp}(\mathbb{L})) : C \subset \overline{P}_1 \cup \ldots \cup \overline{P}_n \text{ and for every } i, C \cap \overline{P}_i \neq \emptyset \}
\]
\[
= \{ C \in \text{Exp}(\text{Exp}(\mathbb{L})) : \{ P_1, \ldots, P_n \} = \{ Q \subset P : \text{ for some } D \in C, Q = \{ p \in D : D \cap p \neq \emptyset \} \} \}
\]
is open in \( \text{Exp}(\text{Exp}(\mathbb{L})) \). Now if \( P = \{ \phi^{-1}(a) : a \in A \} \), where \( \phi : \mathbb{L} \to A \) is an epimorphism, and if \( P_i = \{ \phi^{-1}(a_j) : a_j \leq_{A_e} a_i \} \), where \( A = \{ a_1, \ldots, a_n \} \) is linearly ordered by \( \leq_{A_e} \), the order on \( A \) induced by the chain on \( A_e \), then \( V_{\phi,A_e} = V_{P_1,\ldots,P_n} \cap X^* \), which implies that \( V_{\phi,A_e} \) is open.

\[ \square \]

**Lemma 4.20.** The family \( \mathcal{F}_{c} \) has the expansion property with respect to \( \mathcal{F} \), that is, for any \( A_e \in \mathcal{F}_{c} \), there is \( D \in \mathcal{F} \) such that for any expansion \( D_e \in \mathcal{F}_{c} \) of \( D \), there is an epimorphism \( \phi : D_c \to A_e \).

**Proof.** Using Proposition 4.18 find \( B_c \in \mathcal{F}_{cc} \) such that there is an epimorphism from \( B_c \) onto \( A_e \). Let \( k \) and \( l \) be the height and width of \( B_c \), respectively. We show that \( D \in \mathcal{F} \) of height \( m = kl \) and width \( l \) is as required, that is, for any \( D_e \in \mathcal{F}_{c} \) with \( D_e \upharpoonright \mathcal{L} = D \), there is an epimorphism from \( D_e \) onto \( B_c \).

Let \( b_1 \prec_{cc} b_2 \prec_{cc} \ldots \prec_{cc} b_l \) be the increasing enumeration of branches of \( B_c \) and let \( d_1, d_2, \ldots, d_l \) be a list of all branches of \( D \). Let for each \( i \), \( (d_i^j)_{j=0}^m \) be the \( \preceq_D \)-increasing enumeration of the branch \( d_i \) and let for each \( i \), \( (b_i^j)_{j=0}^k \) be the \( \preceq_B \)-increasing enumeration of the branch \( b_i \). Let \( v_B \) and \( v_D \) denote the roots of \( B \) and \( D \), respectively.

Take any \( D_e \in \mathcal{F}_{c} \) with \( D_e \upharpoonright \mathcal{L} = D \). We recursively define a sequence \( (x^i)_{i=1}^l \) as follows. Let \( x^i \) be the least (with respect to the linear order \( \preceq_{D_e} \) induced by \( D_e \)) element from the set \( \{ d^k_j : j = 1, 2, \ldots, l \} \) which is not on the same branch as \( x^1, \ldots, x^{i-1} \) are.
Let \(\tilde{d}_i\) denote the branch of \(D\) on which \(x^i\) is, so \(x^i = \tilde{d}_i^k\). Let \(\psi : D_c \to B_c\) be the \(R\)-preserving map satisfying for each \(i\),
\[
\begin{align*}
\psi(\tilde{d}_i^{(i-1)k+1}) &= b_i^k, \quad t = 1, 2, \ldots, k, \\
\psi([d_i^k, \hat{d}_i^m]_{\leq D}) &= b_i^k, \quad \text{and} \\
\psi([v_D, \hat{d}_i^{(i-1)k}]_{\leq D}) &= v_B.
\end{align*}
\]

Then \(\psi\) preserves chains and therefore it is a required epimorphism. \(\square\)

**Proof of Theorem 4.18.** We will apply Theorem 3.20 via Theorem 4.3 to families \(\mathcal{F}\) and \(\mathcal{F}_c\) to show that the flow \(\text{Aut}(\mathbb{L}) \curvearrowright \hat{X}_{L_c}\) is the universal minimal flow of \(\text{Aut}(\mathbb{L})\). Clearly, \(\mathcal{F}_c\) is a precompact expansion of \(\mathcal{F}\) and from Lemma 3.11 it follows that the property (*) holds for \(\mathcal{F}\) and \(\mathcal{F}_c\). Further, the expansion \(\mathcal{F}_c\) of \(\mathcal{F}\) is reasonable (Lemma 4.4), \(\mathcal{F}_c\) has the expansion property with respect to \(\mathcal{F}\) (Lemma 4.20), and \(\mathcal{F}_c\) is a Ramsey class (Theorem 4.7 and Corollary 4.12).

Finally, Corollary 3.23 implies that \(\text{Aut}(\mathbb{L}) \curvearrowright \text{Aut}(\mathbb{L})/\text{Aut}(\mathbb{L}_c)\) and Proposition 4.19 implies that \(\text{Aut}(\mathbb{L}) \curvearrowright X^*\) describe the universal minimal flow of \(\text{Aut}(\mathbb{L})\). \(\square\)

## 5. The Universal Minimal Flow of \(H(L)\)

We will compute the universal minimal flow of \(H(L)\) – the homeomorphism group of the Lelek fan \(L\) – in two ways, as we did for \(\text{Aut}(\mathbb{L})\); one description will correspond to the quotient description, and the other to the chain description.

### 5.1. The 1st description

Let \(\pi : \mathbb{L} \to L\) denote the continuous surjection and let \(\pi^* : \text{Aut}(\mathbb{L}) \to H(L)\) denote the continuous homomorphism with a dense image, both defined in Section 3.3. The chain \(C^L = \pi(C^c)\) is downwards closed and it is maximal by Lemma 3.11. Let \(L_c = (L, C^L)\) and set
\[
H(L_c) = \{h \in H(L) : h(C^L) = C^L\}.
\]

Note that \(H(L_c)\) is closed in \(H(L)\). Indeed, since for any \(D \in \text{Exp}(L)\) the map that takes \(h \in H(L)\) and assigns to it \(h(D) \in \text{Exp}(L)\) is continuous, and since \(C^L\) is maximal and hence closed in \(\text{Exp}(L)\), we get that \(H(L_c)\) is closed.

In this section, we will prove the following theorem.

**Theorem 5.1.** The universal minimal flow of \(H(L)\) is equal to \(H(L) \curvearrowright \hat{H(L_c)}\).

Theorem 5.3 below will immediately imply that the universal minimal flow of \(H(L)\) is equal to \(H(L)/H_1\), where \(H_1 = \pi^*(\text{Aut}(\mathbb{L}_c))\). We will identify \(H_1\) with \(H(L_c)\) in Proposition 5.4.

First, we need Theorem 5.2; its proof is in [NVT] (the proof of \(ii\) \to \(i\)) of Theorem 5).

We will say that a flow \(G \curvearrowright X\) is universal in a family of \(G\)-flows \(\mathcal{F}\) if \(G \curvearrowright X \in \mathcal{F}\) and for every \(G\)-flow \(G \curvearrowright Y \in \mathcal{F}\) there is a continuous \(G\)-map from \(X\) onto \(Y\).

**Theorem 5.2.** Let \(G\) be a Polish group and let \(H\) be its closed subgroup. Suppose that \(H\) is extremely amenable and that \(G/H\) is precompact. Then the \(G\)-flow \(G \curvearrowright \hat{G/H}\) is universal in the family of \(G\)-flows in which there is an \(H\)-fixed point with a dense orbit.
**Theorem 5.3.** Let $G$ be a Polish group and let $H$ be a closed subgroup of $G$. Suppose that $H$ is extremely amenable, $G/H$ is precompact, and $M(G) = G/H$. Let $G_1$ be a Polish group and let $\phi : G \to G_1$ be a continuous homomorphism with a dense image. Then $M(G_1) = \hat{G}_1/\hat{H}_1$, where $\hat{H}_1 = \phi(G)$.

**Proof.** First note that $H_1$ is extremely amenable (see Lemma 6.18 in [KPT]) and that $\phi$ is uniformly continuous. We show that $G_1/H_1$ is precompact. Consider $\hat{\phi} : G/H \to G_1/H_1$ given by $\hat{\phi}(gH) = \phi(g)H_1$. This map is well defined, uniformly continuous (with respect to quotients of the right uniformities), and has a dense image. As the image of a precompact space by a uniformly continuous map is precompact (a straightforward calculation, this property is also stated in Engelking [E], page 445, the 2nd paragraph), $\hat{\phi}(G/H)$ is precompact in the uniformity inherited from $G_1/H_1$. Moreover, as $\hat{\phi}(G/H)$ is dense in $G_1/H_1$, completions of both spaces are equal (see [E] 8.3.12). This gives that $G_1/H_1$ is precompact.

Since $\hat{\phi}$ is uniformly continuous, $\hat{\phi}$ extends to a $G$-map $\hat{\Phi} : G/H \to \hat{G}_1/\hat{H}_1$ (see 8.3.10 in [E]). As $\hat{\phi}(G/H)$ is dense in $G_1/H_1$ and the image $\hat{\Phi}(G/H)$ is closed in $\hat{G}_1/\hat{H}_1$, $\hat{\Phi}$ is onto. The continuous action $G_1 \cap \hat{G}_1/\hat{H}_1$ and $\phi : G \to G_1$ induce a continuous action $G \cap \hat{G}_1/\hat{H}_1$. As $G \cap \hat{G}_1/\hat{H}_1$ is minimal, so is $G \cap \hat{G}_1/\hat{H}_1$, therefore the flow $G_1 \cap G_1/\hat{H}_1$ is minimal as well. By Theorem 5.2, the flow $G_1 \cap G_1/\hat{H}_1$ is universal in the family of minimal $G_1$-flows.

To finish the proof of Theorem 5.3, we show the proposition below.

**Proposition 5.4.** We have $H_1 = H(L_c)$.

To show Proposition 5.4, we need Lemma 5.6 which generalizes the following lemma.

**Lemma 5.5** (Bartošová-Kwiatkowska [BK], Lemma 2.14). Let $d < 1$ be any metric on $L$. Let $\epsilon > 0$ and let $v$ be the root of $L$. Then there is $A \in \mathcal{F}$ and an open cover $(U_a)_{a \in A}$ of $L$ such that

(C1) for each $a \in A$, $diam(U_a) < \epsilon$,

(C2) for each $a, a' \in A$, if $U_a \cap U_{a'} \neq \emptyset$ then $R^A(a, a')$ or $R^A(a', a)$,

(C3) for each $x, y \in L$ with $x \in [v, y]$, if $x \in U_a$ and $y \in U_b$, $a \neq b$, and $\{x, y\} \not\subset U_a \cap U_b$, then $a \preceq_A b$, where $\preceq_A$ is the partial order on $A$,

(C4) for every $a \in A$ there is $x \in L$ such that $x \in U_a \setminus (\bigcup \{U_{a'} : a' \in A, a' \neq a\})$.

**Lemma 5.6.** Let $d < 1$ be any metric on $L$. Let $\epsilon > 0$ and let $v$ be the root of $L$. Then there is $A_c = (A, C^A) \in \mathcal{F}_c$ and an open cover $(U_a)_{a \in A_c}$ of $L_c$ such that (C1)-(C4) from Lemma 5.5 hold and additionally we have the following.

(C5) For any $c \in C^L$, the set $\{a \in A : c \cap U_a \neq \emptyset\}$ is in $C^A$.

**Proof.** Take the cover $(U_a)_{a \in A}$ of $L$ as in Lemma 5.5. Then the set $\{a \in A : c \cap U_a \neq \emptyset\} : c \in C^L$ is a downwards closed chain in $A$. To get $A_c$ extend this chain to a downwards closed maximal one.

**Proof of Proposition 5.4**. Since $H(L_c)$ is closed, it follows that $H_1 \subset H(L_c)$.

To show the converse, take $h \in H(L_c)$ and $\epsilon > 0$. Let $d < 1$ be any metric on $L$ and let $d_{sup}$ be the corresponding supremum metric on $H(L)$. We will find $\gamma \in Aut(L_c)$ such that $d_{sup}(h, \gamma^*) < \epsilon$, which will finish the proof as $\gamma^* \in H_1$ and since $H_1$ is closed.
Let $A_c \in \mathcal{F}_c$ and $(U_a)_{a \in A}$, an open cover of $L$, be as in Lemma 5.6 taken for $d$ and $\epsilon$. Since $h$ is uniformly continuous, we can assume additionally that for each $a \in A$, $\text{diam}(h(U_a)) < \epsilon$. Let $(V^1_a)_{a \in A}$ and $(V^2_a)_{a \in A}$ be the open covers of $L$ given by $V^1_a = \pi^{-1}(U_a)$ and $V^2_a = \pi^{-1}(h(U_a))$.

For $B^i \in \mathcal{F}$ and an epimorphism $\phi^i : L \to B^i$ such that $\{((\phi^i)^{-1}(b) : b \in B^i\}$ refines $(V^i_a)_{a \in A}$, $i = 1, 2$, denote by $(W^i_a(\phi^i))_{a \in A}$ the clopen partition of $L$ such that for every $a$, $W^i_a(\phi^i)$ is the union of all $((\phi^i)^{-1}(b))$ which lie in $V^i_a$ and do not lie in a $V^i_{a'}$ for some $a'$ with $R^A(a, a')$. This partition defines an epimorphism $\psi_{\phi^i}$ from $L$ to $A$. Indeed, by (C4) $\psi_{\phi^i}$ is onto. The properties (C2) and (C3) imply that if $x, y \in L$ satisfy $R^L(x, y)$ then $R^A(\psi_{\phi^i}(x), \psi_{\phi^i}(y))$, which already gives that $\psi_{\phi^i} : L \to A$ is an epimorphism. Observe that if we take arbitrary $B^i \in \mathcal{F}$ and epimorphism $\phi^i : L \to B^i$, $i = 1, 2$, such that $\{((\phi^i)^{-1}(b) : b \in B^i\}$ refines $(V^i_a)_{a \in A}$, then for any $c \in \mathcal{C}^L$ and $a \in A$, we have $a \in \psi_{\phi^i}(a)$ iff $c \cap W^i_a(\phi^i) \neq \emptyset$.

Define $\mathcal{D} = \{a : c \cap V^i_a \neq \emptyset : c \in \mathcal{C}^L\}$ and observe that $\mathcal{D} = \{a : c \cap U_a \neq \emptyset : c \in \mathcal{C}^L\}$, and hence $\mathcal{D} \subseteq \mathcal{C}^A$, by the definition of $\mathcal{C}^A$. This follows for $i = 1$ from that for any $c \in \mathcal{C}^L$ and $a \in A$, $c \cap V^1_a \neq \emptyset$ iff $c \cap \pi^{-1}(U_a) \neq \emptyset$ iff $\pi(c) \cap U_a \neq \emptyset$, and for $i = 2$ it follows from that for any $c \in \mathcal{C}^L$ and $a \in A$, $c \cap V^2_a \neq \emptyset$ iff $c \cap \pi^{-1}(h(U_a)) \neq \emptyset$ iff $\pi(c) \cap h(U_a) \neq \emptyset$, and use that $h^{-1}$ preserves $\mathcal{C}^L$.

**Claim.** There are $B^i \in \mathcal{F}$ and epimorphisms $\phi^i : L \to B^i$, $i = 1, 2$, such that $\{((\phi^i)^{-1}(b) : b \in B^i\}$ refines $(V^i_a)_{a \in A}$, satisfying $\mathcal{C}^A = \psi_{\phi^1}(\mathcal{C}^L)$.

**Proof of the Claim.** We fix $i = 1, 2$. Let $k$ be the number of elements in $A$ and let $\mathcal{D}$ be as before. Note that $\mathcal{D}$ may not be maximal. Inductively on $1 \leq m \leq k$ we construct $B_m \in \mathcal{F}$ together with an epimorphism $\phi_m : L \to B_m$ such that the partition $P_m = \{\phi_m^{-1}(b) : b \in B_m\}$ refines $P_{m-1}$ and the the first $m$ links of the maximal chain $\psi_{\phi_m}(\mathcal{C}^L)$ are equal to the first $m$ links of $\mathcal{C}^A$. Then $\phi^i = \phi_k$ will be as required.

We let $\phi_1 : L \to B_1$ be an arbitrary epimorphism such that $\{\phi_1^{-1}(b) : b \in B_1\}$ refines $(V^1_a)_{a \in A}$. Suppose that we already constructed $\phi_1 : L \to B_1, \ldots, \phi_m : L \to B_m, m < k$, and that there is an $m$-element link $M$ in $\mathcal{D}$. We construct $\phi_{m+1} : L \to B_{m+1}, \ldots, \phi_{m+l} : L \to B_{m+l}$, where $l > 0$ is the smallest such that there is an $(m + l)$-element link $N$ in $\mathcal{D}$. If $l = 1$ and the $(m + 1)$-st link of the maximal chain $\psi_{\phi_m}(\mathcal{C}^L)$ is equal to the $(m + 1)$st link of $\mathcal{C}^A$ (which may not be the case even if $l = 1$), we let $\phi_{m+1} = \phi_m$.

Otherwise, let us see that

$$S = \{d \in \mathcal{C}^L : N = \{a : d \cap V^i_a \neq \emptyset\} \text{ and } \psi_{\phi_m}(d) = M\}$$

has at least $l$ elements.

If $|M| + 1 = |N|$ and $S = \emptyset$, then using that for every $d \in \mathcal{C}^L$, $\psi_{\phi_m}(d) \subseteq \{a : d \cap V^i_a \neq \emptyset\}$, we obtain that the $(m + 1)$-st link of $\psi_{\phi_m}(\mathcal{C}^L)$ is equal to the $(m + 1)$-st link of $\mathcal{C}^A$ (which in this case is equal to $N$). But we explicitly ruled out this happening.

If $|M| + 2 \leq |N|$, we will show that $S$ is infinite. For this it will suffice to show that there is no smallest $d \in \mathcal{C}^L$ (with respect to the inclusion) such that $N = \{a : d \cap V^i_a \neq \emptyset\}$ and that simultaneously there is the smallest $d \in \mathcal{C}^L$ such that $\psi_{\phi_m}(d) = M$. The second claim is clear, since $B_m$ is a clopen partition. For the first claim, suppose towards a contradiction that there is the smallest $d_l \in \mathcal{C}^L$ such that $N = \{a : d_l \cap V^i_a \neq \emptyset\}$ and that simultaneously there is the smallest $d_l \in \mathcal{C}^L$ such that $\psi_{\phi_m}(d_l) = M$. The second claim is clear, since $B_m$ is a clopen partition. For the first claim, suppose towards a contradiction that there is the smallest $d_l \in \mathcal{C}^L$ such that $M = \{a : d_l \cap V^i_a \neq \emptyset\}$ and $d_l \subseteq d_s$. However, this contradicts the maximality of $\mathcal{C}^L$. If $a_1 \neq a_2 \in N \setminus M$, then $d_s \setminus V^i_{a_2}$ is downwards closed, properly contained in $d_s$, and it properly contains $d_l$. 


Therefore we can find \( c_1 \subset \ldots \subset c_l \subset C^L \) with \( c_j \in S \). Let \( M = M_0 \subset \ldots \subset M_l = N \) be links in \( C^A \) such that \( |M_{j+1} \setminus M_j| = 1 \). We can now successively define \( \phi_{m+1}, \ldots, \phi_{m+l} \) in a way that \( \phi_{m+j}(c_j) = M_j \) and the first \( m + j \) links of the maximal chain \( \phi_{m+j}(C^L) \) are equal to the first \( m + j \) links of \( C^A \), \( j = 1, \ldots, l \). To get \( \phi_{m+j} \), we take \( a \in M_j \setminus M_{j-1} \) and we pick \( p \in P_{m+j-1} \) such that \( p \cap c_j \neq \emptyset \), \( p \cap V^a_1 \neq \emptyset \) and \( p \cap V^{a-}_1 \neq \emptyset \), where \( a^- \in A \) is such that \( R^3(a^-,a) \). We then partition \( p \) into clopen sets \( r \) and \( s \) such that \( s \subset V^a_1 \) and \( \phi_{m+j} \) with \( P_{m+j} = (P_{m+j-1} \setminus \{p\}) \cup \{r,s\} \) is an epimorphism. Then \( \phi_{m+j} \) is as required. We are done with steps \( m + 1, \ldots, m + l \). 

Denote \( \psi_1 = \psi_{\phi_1} \) and \( \psi_2 = \psi_{\phi_2} \). The (L3) provides us with \( \gamma \in \text{Aut}(\mathbb{L}_c) \) such that \( \psi_1 = \psi_2 \circ \gamma \). We will show that \( d_{\sup}(h,\gamma^*) < \epsilon \). Pick any \( x \in \mathbb{L}_c \), let \( a = \psi_1(x) \), and note that

\[
\gamma^*(\pi(x)) \in \gamma^*(\pi \circ \psi_1^{-1}(a)) = \pi \circ \gamma \circ \psi_1^{-1}(a) = \pi \circ \psi_2^{-1}(a) \subset h(U_a) .
\]

Therefore \( \gamma^*(\pi(x)) \in h(U_a) \), \( h(\pi(x)) \in h(U_a) \), and \( \text{diam}(h(U_a)) < \epsilon \), and we get the required conclusion.

5.2. The 2nd description. Let \( C \) be the Cantor set viewed as the middle third Cantor set in \([0,1]\). Each point of \( C \) can be expanded in a ternary sequence \( 0.a_1a_2a_3 \ldots \), where \( a_i \in \{0,2\} \) for every \( i \), and each point of the interval \([0,1]\) can be expanded in a binary sequence \( 0.b_1b_2b_3 \ldots \) where \( b_i \in \{0,1\} \) for every \( i \). Let \( \pi_0: C \to [0,1] \) be given by \( \pi_0(0.a_1a_2a_3 \ldots) = 0.b_1b_2b_3 \ldots \), where \( b_i = 0 \) if \( a_i = 0 \) and \( b_i = 1 \) if \( a_i = 2 \). We can view the Cantor fan \( F \) as the union of segments joining the point \( v = (\frac{1}{3},0) \in \mathbb{R}^2 \) and a point \((c,1) \in \mathbb{R}^2 \), where \( c \in C \). We first describe a topological \( \mathcal{L} \)-structure \( \mathbb{F} \) such that \( \pi(\mathbb{F}) = F \), where \( \pi \) is the continuous surjection such that \( \pi(x) = \pi(y) \) if and only if \( R^\mathbb{F}(x,y) \), and for every \((a,b) \in \mathbb{F} \), the coordinate of \( \pi(\mathbb{F}) \) is equal to \( \pi_0(b) \). For this, we let \( \mathbb{F} = F \cap (C \times C) \) and \( R^\mathbb{F}((a,b),(c,d)) \) if and only if \( a = c \) and \( b = d \), or if \( v, (a,b) \), and \((c,d) \) are collinear and \((b,d) \) is an interval removed from \([0,1]\) in the construction of \( C \).

We view the Lelek fan \( L \) as a subset of \( F \) with its root equal to \( v \) and we notice that \( L \) is isomorphic to \( \pi^{-1}(L) \).

Let \( m_0 \) denote the metric on \( L \), equal to the restriction of the Euclidean distance on \( \mathbb{R}^2 \). Let \( d_0 \) be the corresponding supremum metric on \( H(L) \). It is a right-invariant metric and it induces a right-invariant metric \( d \) on \( H(L)/H(L_c) \) via

\[
d(gH(L_c),hH(L_c)) = \inf\{d_0(gh,k) : k \in H(L_c)\} .
\]

Let \( m \) be the metric on \( \text{Exp}(\text{Exp}(L)) \) obtained from \( m_0 \). To be more specific, to get \( m \), we first take the Hausdorff metric on \( \text{Exp}(L) \) with respect to \( m_0 \) and then we take the Hausdorff metric on \( \text{Exp}(\text{Exp}(L)) \) with respect to that metric on \( \text{Exp}(L) \).

Let \( Y^* \) be the set of downwards closed maximal chains on \( L \). Note that \( \pi : \mathbb{L} \to L \) induces a continuous surjection from \( \text{Exp}(\mathbb{L}) \) to \( \text{Exp}(L) \), which further induces a continuous surjection from \( \text{Exp}(\text{Exp}(L)) \) to \( \text{Exp}(\text{Exp}(L)) \). Let \( \pi_c : Y^* \to Y^* \) be the restriction of this last map to \( X^* \). Note that \( \pi_c \) is onto. Indeed, take \( D \in Y^* \) and observe that \( \pi_c^{-1}(D) \) is a downwards closed chain in \( \mathbb{L} \). Using Zorn’s Lemma, extend it to a downwards closed maximal chain \( C \), and note that \( \pi_c(C) = D \). Consider the \( H(L) \)-flow \( H(L) \curvearrowright Y^* \) induced from the natural action of \( H(L) \) on \( L \) given by \((g,x) \to g(x) \). In Theorem 5.3.7 we will show that this flow is isomorphic to the flow \( H(L) \curvearrowright H(L)/H(L_c) \). The main ingredient of the proof will be Theorem 5.8.
Theorem 5.7. The universal minimal flow of $H(L)$ is isomorphic to $H(L) \bowtie Y^*$. 

Let $G = H(L)$ and let $H = H(L_e)$. 

Theorem 5.8. The bijection 
\[ gH \rightarrow g \cdot C^L \]

is a uniform $G$-isomorphism from $G/H$ to $G \cdot C^L$. 

Let us first see that Theorem 5.8 implies Theorem 5.7. 

Proof of Theorem 5.7. The uniform $G$-isomorphism $gH \rightarrow g \cdot C^L$ from $G/H$ to $G \cdot C^L$ extends to a uniform $G$-isomorphism $h$ between the completion $\hat{G}/\hat{H}$ of $G/H$ and the completion of $G \cdot C^L$, which is the closure of $G \cdot C^L$ in $Y^*$, which we have to show is equal to $Y^*$. 

Let $f : \text{Aut}(\mathbb{L})/\tilde{\text{Aut}}(\mathbb{L}_e) \rightarrow X^*$ be the uniform $G$-isomorphism that extends the map $g\text{Aut}(\mathbb{L}_e) \rightarrow g \cdot C^L$, let $\rho$ be the extension of the map $g\text{Aut}(\mathbb{L}_e) \rightarrow gH$ to the respective completions $\text{Aut}(\mathbb{L})/\tilde{\text{Aut}}(\mathbb{L}_e)$ and $\hat{G}/\hat{H}$. Let $\pi_e : X^* \rightarrow Y^*$, as before, be the map induced from $\pi : \mathbb{L} \rightarrow \mathbb{L}_e$. As $\pi_e f = h\rho$ and $\pi_e$ is onto, we obtain that $h$ is onto. 

We define the mesh of a finite open cover $\mathcal{U}$ of $L$ to be the minimum of diameters of the sets in $\mathcal{U}$ and denote it by $\text{mesh}(\mathcal{U})$, and we define the spread of $\mathcal{U}$ to be the minimum of distances between non-intersecting sets in $\mathcal{U}$ and denote it by $\text{spr}(\mathcal{U})$. 

Let $A \in \mathcal{F}$ and let $\mu_A$ be the path metric on $A$ in which $\mu_A(x,y)$ is equal to the length of the shortest path joining $x$ and $y$ in the undirected graph obtained from $A$ via symmetrization of the relation $R^A$. The length of a path is understood to be the number of edges in the path. The metric $\mu_A$ on $A$ induces the Hausdorff metric $\mu^A_1$ on $\text{Exp}(A)$, which in turn induces the Hausdorff metric $\mu^A = \mu^{1\star}_A$ on $\text{Exp}(\text{Exp}(A))$.

**Special covers:** We will need the following covers $\mathcal{U}_l$ of $L$, $l = 1, 2, \ldots$. Let $J_i = [x_i, y_i]$, $i = 0, 1, \ldots, 2^l - 1$, be the intervals we obtain in the $l$-th step of the construction of the middle third Cantor set. In particular, we have $y_i - x_i = \frac{1}{3^l}$. We assume that $y_i < x_{i+1}$, $i = 0, 1, \ldots, 2^l - 2$. Let $b_j = \frac{1}{2^l}$, $j = 0, 1, \ldots, 2^l$. We let

\[ \mathcal{U}_l = \left\{ \{ P(J_i, b_j, b_{j+1}) : i = 0, 1, \ldots, 2^l - 1, j = 1, \ldots, 2^l - 1 \} \cup \{ \emptyset \} \right\} \cup \{ T(b_1) \}, \]

where $P(J_i, b_j, b_{j+1})$ is the intersection of $L$ with the 4-gon determined by lines $y = b_j$, $y = b_{j+1}$, the line through $v$ and $(x_i, 1)$ and the line through $v$ and $(y_i, 1)$, and $T(b_1)$ is the the intersection of $L$ with the triangle determined by the lines $y = b_1$, the line through $v$ and $(0, 1)$, and the line through $v$ and $(1, 1)$. 

Any cover $\mathcal{U}_l$ obtained as above will be called a special cover. Note that, by the definition of $\pi$, for any $l$, the open cover $\mathcal{V}_l = \{ \pi^{-1}(U) : U \in \mathcal{U}_l \}$ consists of disjoint sets, and when $l \rightarrow \infty$, then mesh($\mathcal{U}_l$) $\rightarrow 0$ and spr($\mathcal{U}_l$) $\rightarrow 0$. 

For a special cover $\mathcal{U}_l$, let $A_l = (a_V)_{V \in \mathcal{V}_l} \in \mathcal{F}$ be such that the map $\theta_l : \mathbb{L} \rightarrow A_l$ given by $\theta_l(x) = a_V$ iff $x \in V$ is an epimorphism. We will call $\theta_l$ a special epimorphism. 

In the proof of Theorem 5.8, we will use the following lemma. 

**Lemma 5.9.** Let $A, B \in \mathcal{F}$, let $K$ be the number of elements in $A$, and let $\phi : B \rightarrow A$ be an epimorphism such that for every branch $b$ in $B$, and $a \in A$, if $\phi^{-1}(a) \cap b$ does not contain the endpoint of $b$ then it has at least $2K + 1$ elements. Let $\mathcal{C}$ and $\mathcal{D}$ be maximal chains on $B$ such that $\mu^B(\mathcal{C}, \mathcal{D}) \leq 1$. 

Then there is an epimorphism $\psi : B \to A$ such that the maximal chains $\psi(C)$ and $\psi(D)$ are equal and for every $a \in A$ 
$$\psi^{-1}(a) \subset \phi^{-1}(a) \cup \bigcup_{a' \in \text{is}(a)} \phi^{-1}(a'),$$
where is($a$) denotes the set of immediate successors of $a$ (and it is a singleton unless $a$ is the root).

**Proof.** We will construct $\psi$ by induction on $n \leq K$. In step $n$ we will construct $\psi_n : B \to A$ and a downwards closed set $D_n$ of $A$ such that the first $n$ links in both $\psi_n(C)$ and $\psi_n(D)$ are equal to $D_1 \subset \ldots \subset D_n$. Moreover, for any $a \in A$ and $n \leq K$ it will hold 
$$\psi_n^{-1}(a) \subset \phi^{-1}(a) \cup \{x : \exists y_0, \ldots, y_m \text{ such that } m \leq 2(n-1), y_0 \in \phi^{-1}(a), x = y_m \text{ and } \forall i<m R^\phi(y_i, y_{i+1})\}.$$ Note that this last containment implies that: 
$(\triangle_n)$ for every branch $b$ in $B$, and $a \in A$, if $\psi_n^{-1}(a) \cap b$ does not contain the endpoint of $b$ then it has at least $2(K - n + 1) + 1$ elements.

Then we set $\psi = \psi_K$ and we will have $\psi(C) = \psi(D) = \{D_n : n \leq K\}$. This $\psi$ will be as required.

Observe that by the definition of the metric $\mu^B$, $\mu^B(C, D) \leq 1$ means: for every $C \in \mathcal{C}$ there is $D \in \mathcal{D}$ such that $\mu^B(C, D) \leq 1$ and for every $D \in \mathcal{D}$ there is $C \in \mathcal{C}$ such that $\mu^B(C, D) \leq 1$.

**Step 1.** Take $\psi_1 = \phi$ and let $D_1$ be the set whose only element is the root of $A$.

**Step $n+1.** Let $E = \psi_n^{-1}(D_n)$, $E$ is clearly downwards closed. Let $C \in \mathcal{C}$ be the least (with respect to containment) such that there is a branch $b = (b^i)$ in $B$ and $i_0$ such that $p = b_{i_0} \notin E$ and $p, q = b_{i_0+1} \in C$, and let $D \in \mathcal{D}$ be such that $\mu^B(C, D) \leq 1$. Take $\psi_{n+1}$ such that $\psi_{n+1}(x) = \psi_n(x)$ for $x \in E \cup (B \setminus (C \cup D)) \cup b$. For $x \in (C \cup D) \setminus (E \cup b)$ let $c = (c^j)$ be the branch in $B$ such that for some $j_0$, we have $x = c^{j_0}$, and let $\psi_{n+1}(x) = \psi_n(z)$, where $z \in E \cap c$ is the largest with respect to the tree order on $B$. Take $D_{n+1} = D_n \cup \{\psi_n(p)\}$.

Note that $b \cap ((C \cup D) \setminus E)$ consists of 2 or 3 elements, in fact it is equal to $\{p, q, r\}$ or to $\{p, q, r\}$, where $r$ is the immediate successor of $q$. Further by $(\triangle_n)$, $\psi_{n+1}$ is constant on $b \cap ((C \cup D) \setminus E)$. □

**Proof of Theorem 5.9**. It is easy to see that $gH \to g \cdot C^L$ is uniformly continuous. It is straightforward, by the definitions of metrics $d$ and $m$, that if $d(gH, hH) < \epsilon$ then $m(g \cdot C^L, h \cdot C^L) < \epsilon$.

For the opposite direction, we show that for every $\epsilon$ there is $\delta$ such that for every $g, h \in G$ whenever $m(g \cdot C^L, h \cdot C^L) < \delta$, then $d(gH, hH) < 3\epsilon$, that is, for some $k \in H$ we have $d_0(gk, h) < 3\epsilon$. This direction requires more work, we will use Lemma 5.9 and the ultrahomogeneity of $\mathbb{L}_{\mu}$. Pick $\epsilon > 0$ and let $l$ be such that $\text{mesh}(U) < \epsilon$, where $U$ is the cover $\{U_a \cup \bigcup_{b \in \text{is}(a)} U_b : a \in A_l\}$ obtained from the cover $U_l = (U_a)_{a \in A_l}$.

Let $K = |A_l|$ and let $k \geq (2K + 1)l$. Let $\theta_l : \mathbb{L} \to A_l$ and $\theta_k : \mathbb{L} \to A_k$ be special epimorphisms, and $\phi : A_k \to A_l$ be the epimorphism such that $\phi \theta_k = \theta_l$. Denote $A = A_l$ and $B = A_k$. Take $\delta = \text{spr}(U_k)$ and suppose that $m(g \cdot C^L, h \cdot C^L) < \delta$.

Suppose first that there are $g_0, h_0 \in \text{Aut}(\mathbb{L})$ such that $g = \pi^*(g_0) \in H(L)$ and $h = \pi^*(h_0) \in H(L)$. Since $m(g \cdot C^L, h \cdot C^L) < \text{spr}(U_k)$, we obtain $\mu^B(\theta_k(g_0 \cdot C^L), \theta_k(h_0 \cdot C^L)) \leq 1$. Applying Lemma 5.9 to $\phi : B \to A$, $C = \theta_k(h_0 \cdot C^L)$ and $D = \theta_k(g_0 \cdot C^L)$, we get an
epimorphism \( \psi : B \to A \) such that \( \psi(C) = \psi(D) =: C^A \). The conclusion of Lemma 5.9 implies that \( \psi_\theta : (L, h_0 \cdot C^L) \to (A, C^A) \) and \( \psi_\theta : (L, g_0 \cdot C^L) \to (A, C^A) \) are epimorphisms, as well as that \( \psi^{-1}(a) \subset \phi^{-1}(a) \cup \bigcup_{b \in \pi(a)} \phi^{-1}(b) \), for every \( a \in A \), \( \operatorname{mesh}(\mathcal{U}_0) \leq \operatorname{mesh}(\mathcal{U}) < \epsilon \), where \( \mathcal{U}_0 = \{ \pi((\psi_\theta^{-1}(a)) : a \in A \}. \) From the ultrahomogeneity of \( L \), applied to epimorphisms \( \psi_\theta g_0 : (L, C^L) \to (A, C^A) \) and \( \psi_\theta h_0 : (L, C^L) \to (A, C^A) \), we get \( k_0 \in \operatorname{Aut}(L) \) such that \( \psi_\theta g_0 k_0 = \psi_\theta h_0 \), that is for every \( x \in L \), \( g_0 k_0(x) \) and \( h_0(x) \) are in the same set of the cover \( \{ (\psi_\theta^{-1}(a)) : a \in A \} \) of \( L \). This implies that, denoting \( k = \pi^*(k_0) \), for every \( y \in L \), \( gk(y) \) and \( h(y) \), are in the same set of the cover \( \mathcal{U}_0 \), and therefore, since \( \operatorname{mesh}(\mathcal{U}_0) < \epsilon \), we get \( d_0(gk, h) < \epsilon \), as needed.

In general, if \( g, h \in H(L) \), take \( g', h' \in \pi^*(\operatorname{Aut}(L)) \subset H(L) \) such that \( d_0(g, g') < \epsilon \) and \( d_0(h, h') < \epsilon \). Then, if \( k \) is such that \( d_0(gk, h') < \epsilon \), using the right-invariance of \( d_0 \) and the triangle inequality, we obtain \( d_0(gk, h) < 3\epsilon \). \( \square \)

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