Quasi-exactly solvable periodic potentials with three known eigenstates

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Abstract

Supersymmetric method of the constructing well-like quasi exactly solvable (QES) potentials with three known eigenstates has been extended to the case of periodic potentials. The explicit examples are presented. New QES potential with two known eigenstates has been obtained.

1 Introduction

Description of the electron’s motion on a lattice has been investigated for a long time as a central problem of the condensed matter physics. Such quantum problem is reduced to the solving of the Schrödinger equation with some model potential which is periodic often. Therefore, the periodic quantum mechanics problems remain at the investigation’s focus up to now.

The general properties of the solutions of Schrödinger equation with periodic potential energy are described by the oscillation theorem \( \boxed{1} \). Energy spectrum of the periodic potential has band structure, i.e. eigenvalues belong to the allowed bands (energy bands) \([E_0, E_1], [E'_1, E_2], \ldots\) . The wave functions are the Bloch functions, which are bounded and extended on the full real axe

\[
\psi(x + L) = \exp^{ikL} \psi(x),
\]  

(1)
where $L$ is potential period and $k$ is a so-called quasi-momentum. The limits of the energy bands are given by the equation $kL = \{0, \pi\}$, and the wave functions, which belong to the limiting energy values, satisfy the condition $\psi(x + L) = \pm \psi(x)$. These energy values and wave functions are often called the eigenvalues and the eigenfunctions of the described above problem.

Oscillation theorem claims that in the case of periodic potentials the eigenfunctions, which belong to the limits of the energy bands and are arranged in the energy of the increasing order $E_0 \leq E_1 \leq E_1' \leq E_2 \leq E_2' \leq E_3...$, are the periodic functions with the period $L, 2L, 2L, L, 2L, 2L, ...$ and have $0, 1, 1, 2, 2, 3, 3, ...$ nodes in the interval $L$ respectively.

Despite long term investigations, there is rather a limited number of exactly solvable periodic potentials even in one dimension. The classical examples are the Kronig-Penney model potential [2] or Lamé’s potentials [3].

Because of limited number of the exactly solvable potentials, recently much attention has been given to the quasi exactly solvable (QES) potentials for which a finite number of the energy levels and the corresponding wave functions are known explicitly. A general treatment of the quasi exact solvability has been introduced by Turbiner and Ushveridze [4]. The class of QES trigonometric potentials was presented in [5]. In [6] it was shown that the Lamé equation is a peculiar example of QES systems. The authors of the paper [7] considered a family of spectral equation which extends those of [6]. Authors of [8] has applied quantum Hamilton-Jacobi formalism to the QES periodic potentials. In the latest paper [9] an unified treatment of quasi exactly solvable potentials was proposed.

The powerful tool for studying the problem of exact solvability of the Schrödinger equation is the supersymmetric (SUSY) quantum mechanics introduced by Witten [10] (for a review of SUSY quantum mechanics see [11]). The SUSY method for constructing QES potentials was used for the first time in [12] - [14]. The idea of this method starts from some initial QES potential with $n + 1$ known eigenstates and using the properties of the unbroken supersymmetry to obtain the SUSY partner potential, which is a new QES one with $n$ known eigenstates.

In [15]- [17] using the formalism of SUSY quantum mechanics a large number of new solvable and QES periodic potential was proposed. It is worth mentioning recent paper [18], where the highest order SUSY transformations was applied for studying periodic potential.

In recent Tkachuk’s papers [20]- [22] a new SUSY method for constructing of the QES potentials with two and three known eigenstates has been pro-

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posed. This method does not require knowledge of the initial QES potential in order to generate a new QES one. Within the frame of this method QES potentials has been obtained for which the explicit form of the energy levels and the wave functions of the ground and the excited states can be found. After the paper [23] by Dolya and Zaslavskii, where they showed how to generate QES potentials with arbitrary two known eigenstates without resorting to the SUSY quantum mechanics, SUSY method has been extended [24] for constructing QES potentials with arbitrary two known eigenstates. In our recent works using the SUSY method periodic [25] and disordered [26] QES potentials were obtained.

In the present paper using the results of previous study [20]-[26] we extend Tkachuk’s SUSY method for constructing QES periodic potentials with three known eigenstates.

2 The Witten model of SUSY quantum mechanics

Witten model of supersymmetric quantum mechanics is a quantum mechanics of the matrix Hamiltonian

\[ H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \]

where Hamiltonians

\[ H_\pm = -\frac{1}{2} \frac{d^2}{dx^2} + V_\pm(x) = B^\pm B^\pm \]

are supersymmetric partners and

\[ B^\pm = \frac{1}{\sqrt{2}} \left( \mp \frac{d}{dx} + W(x) \right). \]

Here \( \hbar = m = 1 \) units are used. Function \( W(x) \) is referred to as superpotential, \( V_\pm(x) \) are so-called supersymmetric partner potentials

\[ 2V_\pm(x) = W^2(x) \pm W'(x). \]

Energy spectrum of the supersymmetric partners \( H_+ \) and \( H_- \) is identical except for zero-energy ground state which exists in the case of the unbroken
supersymmetry. This leads to twofold degeneracy of the energy spectrum of $H$, except for the unique zero-energy ground state. Only one of the Hamiltonians $H_\pm$ has zero-energy eigenvalue. We shall use the convention that the zero-energy eigenstate belongs to $H_-$

$$
\begin{align*}
E_{n+1}^- &= E_n^+ \\
E_0^- &= 0
\end{align*}
$$

(6)

where $n = 0, 1, 2, \ldots$. The wave functions of the supersymmetric partners $H_\pm$ are related by the supersymmetric transformations

$$
\begin{align*}
\psi_{n+1}^- (x) &= \frac{1}{\sqrt{E_n^+}} B^+ \psi_n^+ (x) \\
\psi_n^+ (x) &= \frac{1}{\sqrt{E_{n+1}^-}} B^- \psi_{n+1}^- (x)
\end{align*}
$$

(7)

Due to the factorization $H_- = B^+ B^-$, we can find solution of the Schrödinger problem for the eigenstate with zero energy

$$
H_- \psi_0^- (x) = E_0 \psi_0^- (x) = 0.
$$

(8)

It is easy to see that

$$
\psi_0^- (x) = C_0^- \exp \left(- \int W(x) dx\right),
$$

(9)

where $C_0^-$ is an arbitrary constant.

In the present paper we shall consider the systems on the full real axe $-\infty < x < \infty$ with periodic superpotential. The periodic superpotential $W(x + L) = W(x)$ leads to the periodic potential energy $V_\pm (x + L) = V_\pm$, which results in the bounded and extended eigenfunction. A satisfactory condition for the existence of periodic eigenfunctions, written in the terms of the SUSY quantum mechanics, is

$$
\int_0^L W(x) = 0.
$$

(10)

In [27, 28] a detailed analysis of the SUSY quantum mechanics was made for this case.
3 SUSY constructing QES potentials

We shall study the Hamiltonian $H_-$ with the potential energy

$$V_-(x) = W_0^2(x) - W_0'(x),$$

the ground state of which is given by (9).

Let us consider Hamiltonian $H_+$ which is the SUSY partner of Hamiltonian $H_-$. If we calculate the ground state of $H_+$ we immediately find the first excited state of $H_-$ using the degeneracy of the spectrum of SUSY Hamiltonian and SUSY transformations (7). In order to calculate the ground state of $H_+$ let us rewrite Hamiltonian in the following form

$$H_+ = H_-^{(1)} + \epsilon, \quad \epsilon > 0$$

where

$$H_-^{(1)} = B_1^+ B_1^-,$$

and $W_1(x)$ is a some new function. Note that $\epsilon$ is the energy of the ground state of $H_+$ since $H_-^{(1)}$ has zero-energy ground state.

The ground state wave function of $H_+$ with the energy $E = \epsilon$ is also zero energy wavefunction of $H_-^{(1)}$ and it satisfies the equation

$$B_1^- \psi^+_0(x) = 0.$$  (15)

The solution of this equation is

$$\psi^+_0(x) = C_0^+ \exp \left( - \int W_1(x) dx \right),$$

where $C_0^+$ is an arbitrary constant.

Using the SUSY transformation (7) we can calculate the wavefunction of the first excited state of $H_-$. Repeating the described procedure for $H_-^{(1)}$ we can obtain the second excited state for $H_-$ and so on. This procedure is well known in the SUSY quantum mechanics (see review [11] for example). The wavefunctions and corresponding energy levels read

$$\left\{ \begin{array}{l}
\psi^-_n(x) = C_n^- B_0^+ \cdots B_{n-2}^+ B_{n-1}^+ \exp \left( - \int W_n(x) dx \right) \\
E_n^- = \sum_{i=0}^{n-1} \epsilon_i
\end{array} \right.$$  (17)
where \( n = 1, 2, ..., N; \, \epsilon_0 = \epsilon, \, B_0^\pm = B^\pm, \, W_0(x) = W(x), \, C_n \) are an arbitrary constants. Operators \( B_n^\pm \) are given by (11) with the superpotentials \( W_n(x) \).

Equation (12) rewritten for \( N \) steps

\[
H^{(n)} = H^{(n+1)} + \epsilon_n, \tag{18}
\]

where \( n = 0, 1, ..., N - 1 \), leads to the set of equations for superpotentials

\[
W_n^2(x) + W_n'(x) = W_{n+1}^2(x) - W_{n+1}'(x) + 2\epsilon_n, \tag{19}
\]

where \( n = 0, 1, ..., N - 1 \).

Unfortunately, each of the equations in (19) are the Rikatti equation, which can not be solved in the general case. Previously this set of equations was solved in special cases of shape-invariant potentials [29] and self-similar potentials for arbitrary \( N \) (see review [30]). For \( N = 1 \) in the context of parasupersymmetric quantum mechanics one can obtain a general solution of (19) without restricting ourselves to shape-invariant and self-similar potentials [31]. In recent papers [20]-[26] a solution of (19) for \( N = 1 \) and \( N = 2 \) in order to obtain non-singular QES potentials with two and three known eigenstates respectively has been constructed.

Let us write set of equations (19) for the case \( N = 2 \) in the explicit form

\[
\begin{align*}
W_0^2(x) + W_0'(x) &= W_1^2(x) - W_1'(x) + 2\epsilon_0 \\
W_1^2(x) + W_1'(x) &= W_2^2(x) - W_2'(x) + 2\epsilon_1.
\end{align*}
\tag{20}
\]

It is convenient to introduce new functions

\[
\begin{align*}
W_+(x) &= W_1(x) + W_0(x) \\
W_-(x) &= W_1(x) - W_0(x) \\
\tilde{W}_+(x) &= W_2(x) + W_1(x) \\
\tilde{W}_-(x) &= W_2(x) - W_1(x),
\end{align*}
\tag{21}
\]

then superpotentials can be rewritten in the following form

\[
\begin{align*}
2W_0(x) &= W_+(x) - W_-(x) \quad 2W_1(x) = \tilde{W}_+(x) - \tilde{W}_-(x) \\
2W_1(x) &= W_+(x) + W_-(x) \quad 2W_2(x) = \tilde{W}_+(x) + \tilde{W}_-(x).
\end{align*}
\tag{22}
\]

In the terms of new functions (21) the set of equations (20) read as follows

\[
\begin{align*}
W_1'(x) &= W_-(x) W_+(x) + 2\epsilon_0 \\
\tilde{W}_1'(x) &= \tilde{W}_-(x) \tilde{W}_+(x) + 2\epsilon_1.
\end{align*}
\tag{23}
\]
Note, that there are two terms for the $W_1(x)$ in the equations (22) with respect to $W_-(x)$ and with respect to $\tilde{W}_+(x)$. This gives us a possibility to obtain relation between $W_+(x)$ and $\tilde{W}_+(x)$

$$W_+(x) + \frac{W'_+(x)}{W_+(x)} - 2\epsilon_0 = \tilde{W}_+(x) - \frac{\tilde{W}'_+(x)}{\tilde{W}_+(x)} - 2\epsilon_1,$$

(24)

here (23) are used. It is easy to rewrite this equation as follows

$$W_+(x)\tilde{W}_+(x)[\tilde{W}_+(x) - W_+(x)] - [W_+(x)\tilde{W}_+(x)]' + 2[\epsilon_1 W_+(x) + \epsilon_0 \tilde{W}_+(x)] = 0,$$

or

$$U(x)\left(\frac{U(x)}{W_+(x)} - W_+(x)\right) - U'(x) + 2\left(\epsilon_1 W_+(x) + \epsilon_0 \frac{U(x)}{W_+(x)}\right) = 0,$$

(25)

where we have introduced a new function

$$U(x) = W_+(x)\tilde{W}_+(x).$$

(26)

We arrive again to the Riccati equation with respect to $U(x)$. On the other hand, this is an algebraic equation with respect to $W_+(x)$, which can be solved explicitly

$$\begin{cases}
W_+(x) = \frac{2U(x)(U(x) + 2\epsilon_0)}{U'(x)(1 + R(x))}, \\
\tilde{W}_+(x) = \frac{2U(x)(U(x) + 2\epsilon_1)}{U'(x)(1 + R(x))},
\end{cases}$$

(27)

where

$$R(x) = 1 + 4\frac{U(x)(\epsilon_1 W_+(x) + 2\epsilon_0)(U(x) - 2\epsilon_1)}{U'(x)^2}, \quad R(x) = \pm \sqrt{\mathfrak{R}(x)}. \quad (28)$$

The square root $\mathfrak{R}(x)$ is a positively defined value, while the function $R(x)$ can be chosen in the form of $\mathfrak{R}(x)$ or $-\mathfrak{R}(x)$ within different intervals separated by zeros of the function $\mathfrak{R}(x)$.

Thus, we can start from an arbitrary function $U(x)$ to construct the functions $W_+(x)$ and $\tilde{W}_+(x)$ given by (27). Using (22) we obtain three consequent superpotentials

$$\begin{cases}
W_0(x) = \frac{1}{2} \left(W_+(x) - \frac{W'_+(x) - 2\epsilon_0}{W_+(x)}\right), \\
W_1(x) = \frac{1}{2} \left(W_+(x) + \frac{W'_+(x) - 2\epsilon_0}{W_+(x)}\right), \\
W_2(x) = \frac{1}{2} \left(\tilde{W}_+(x) + \frac{\tilde{W}'_+(x) - 2\epsilon_1}{\tilde{W}_+(x)}\right),
\end{cases}$$

(29)
Then because of (17), we can find the wavefunctions of three explicitly known eigenstates of the Hamiltonian $H_-$

$$
\begin{align*}
\psi_0^-(x) &= C_0^- e^{-\int W(x) dx} \\
\psi_1^-(x) &= C_1^- W_+(x) e^{-\int W_1(x) dx} \\
\psi_2^-(x) &= C_2^- \left( (W_0(x) + W_2(x))\tilde{W}_+(x) - \tilde{W}_+''(x) \right) e^{-\int W_2(x) dx}
\end{align*}
$$

(30)

where energy values are $E_0^- = 0$, $E_1^- = \epsilon_0$, $E_2^- = \epsilon_0 + \epsilon_1$ and potential energy

$$
V_-(x) = \frac{1}{2} (W_0(x)^2 - W_0''(x)).
$$

(31)

Simultaneously we can find the wave function of two explicitly known eigenstates of the Hamiltonian $H_+$

$$
\begin{align*}
\psi_1^+(x) &= B_0^- \psi_1^-(x) \\
\psi_2^+(x) &= B_0^- \psi_2^-(x)
\end{align*}
$$

(32)

with energy values $E_1^+ = \epsilon_0$, $E_2^+ = \epsilon_0 + \epsilon_1$ and potential energy

$$
V_+(x) = \frac{1}{2} (W_0(x)^2 + W_0''(x)).
$$

(33)

Note that obtained terms for the superpotentials, potentials and wave functions allow existence of two different solutions depending on the selected sign before the square root $\pm \sqrt{R(x)}$ in the $W_+(x)$ and $\tilde{W}_+(x)$ definitions. Here and later we shall distinguish solutions which were obtained for different signs, by superscript in the parenthesis after the function designation, for example $Y(x)^{(+)}$. We will denote as $Y(x)$ solutions which are identical for different signs $Y(x)^{(+)} = Y(x)^{(-)}$.

Choosing different generating functions $U(x)$ we will obtain different QES potentials (31) with three explicitly known eigenstates (30) and QES potentials (33) with two explicitly known eigenstates (32). Of course, function $U(x)$ must satisfy some conditions to provide physical solutions of the Schrödinger equation.

The main obvious condition imposed on the function $U(x)$ is a positivity of the expression under the square root (28)

$$
1 + \frac{4U(x)(U(x) + 2\epsilon_0)(U(x) - 2\epsilon_1)}{U'(x)^2} \geq 0
$$

(34)
on the all periodicity interval.

Another set of restrictions imposed on function \( U(x) \) appears due to the requirement of the non-singularity of resulting potential \( V_-(x) \). The full analysis of the properties of superpotential \( W_0(x) \) which provides non-singular potential \( V_-(x) \) was done in [24]-[26] for the case of the quasi exactly solvable potentials with two exactly known eigenstates. Below we extend this analysis to the case of the quasi exactly solvable potential with three exactly known eigenstates.

As we can see from the superpotentials \( W_0(x) \), \( W_1(x) \) and \( W_2(x) \) definitions, potential \( V_-(x) \) can have poles at the points \( x_0 \) where \( W_+(x_0) = 0 \) or \( W_+(x_0) = 0 \). Fortunately, such poles can be removed when

\[
\begin{align*}
W'_+(x_0) &= \pm 2\epsilon_0, \\
\tilde{W}'_+(x_0) &= \pm 2\epsilon_1.
\end{align*}
\] (35)

Besides, potential energy \( V_-(x) \) can have poles at the points of singularity \( x_\infty \) of the function \( W_+(x) \). As it was shown in [26], if function \( W_+(x) \) at the singularity points \( x_\infty \) has the behavior

\[ W_+(x) = \text{const} + \frac{-1}{x - x_\infty} + o(x - x_\infty), \] (36)

or

\[ W_+(x) = \frac{-3}{x - x_\infty} + o(x - x_\infty), \] (37)

obtained potential energy and wave functions will be continuous functions at the points \( x_\infty \).

To provide bounded and extended wave functions \( \psi_0^-(x) \), \( \psi_1^-(x) \), \( \psi_2^-(x) \) the conditions [110] should be satisfied

\[
\begin{align*}
\int_0^L W_0(x)dx &= 0 \\
\int_0^L W_1(x)dx &= 0 \\
\int_0^L W_2(x)dx &= 0.
\end{align*}
\] (38)

These conditions are satisfied in the simplest way if the corresponding superpotentials \( W_0(x) \), \( W_1(x) \), \( W_2(x) \) are odd function with regard to the middle of the periodicity interval \( x_m \). To obtain odd superpotentials it is enough to expect the odd behavior of the function \( W_+(x) \).
Let us choose the $U(x)$ as even function with regard to the middle of the periodicity interval $x_m$. Then, if we apply solutions with the different signs before square root to the parts of the periodicity interval from the left and from the right of $x_m$, $W_+(x)$ will be odd function. It is easy to see if we rewrite term (27) for the $W_+(x)$ as follows

$$W_+(x) = \frac{2U(x)(U + 2\epsilon_0)}{U'(x) \pm \sqrt{U'(x)^2 + 4U(x)(U(x) + 2\epsilon_0)(U(x) - 2\epsilon_1)}}.$$ \hspace{1cm} (39)

Application of the solutions with the different signs leads to the finite breaks of the function $W_+(x)$. These breaks can be removed if the value of the function $W_+(x)$ will tend to zero both from the left and right direction.

Thus, to provide existence of the bounded extended wave functions, $U(x)$ should be even function with regard to the middle of the periodicity interval $x_m$, and obtained function $W_+(x)$ should have zero at the point $x_m$. Function $W_+(x)$ can have zeros at the points, where $U(x) = 0$, and, since $U(x)$ should be even function with regard to $x_m$, function $U(x)$ can have at the point $x_m$ zero of the even-order only.

Of course, function $U(x)$ can have zeros at the other points of the periodicity interval too. Let us analyze in details the behavior of the superpotentials, potential energy and the wave functions in the vicinity of the $U(x)$ zeros.

Let the function $U(x)$ have the first-order zeros at the points $x_0^a$

$$U(x) = U'(x_0^a)(x - x_0^a) + \frac{1}{2}U''(x_0^a)(x - x_0^a)^2 + o(x - x_0^a)^3.$$ \hspace{1cm} (40)

Then behavior of the functions $W_+(x)$, $\tilde{W}_+(x)$ in the vicinity of the points $x_0^a$ will be as follows

$$\begin{align*}
W_+(x)^+ &= 2\epsilon_0(x - x_0^a) + o(x - x_0^a)^2 \\
W_+(x)^- &= \frac{U'(x_0^a)}{2\epsilon_0} + o(x - x_0^a) \\
\tilde{W}_+(x)^+ &= \frac{U'(x_0^a)}{2\epsilon_1} + o(x - x_0^a) \\
\tilde{W}_+(x)^- &= 2\epsilon_1(x - x_0^a) + o(x - x_0^a)^2.
\end{align*}$$ \hspace{1cm} (41)

It is easy to see that functions $W_+(x)$ and $\tilde{W}_+(x)$ at the points $x_0^a$ will have non-zero values or will have zeros which satisfy (35).

Superpotentials $W_0(x)$, $W_1(x)$, $W_2(x)$ will be the following

$$\begin{align*}
W_0(x)^{\pm} &= A_0^{(\pm)} + o(x - x_0^a) \\
W_1(x)^{\pm} &= A_1^{(\pm)} + o(x - x_0^a) \\
W_2(x)^{\pm} &= A_2^{(\pm)} + o(x - x_0^a).
\end{align*}$$ \hspace{1cm} (42)
where
\[
\begin{align*}
A_0^{(+)} &= -A_1^{(+)} = -\frac{8\epsilon_0^2+U'(x_0^a)^2-\epsilon_0 U''(x_0^a)}{2\epsilon_0 U'(x_0^a)} \\
A_1^{(-)} &= -A_2^{(-)} = \frac{U'(x_0^a)^2+\epsilon_1(U''(x_0^a)-8\epsilon_0 \epsilon_1)}{2\epsilon_1 U'(x_0^a)} \\
A_0^{(-)} &= -A_2^{(+)} = \frac{U'(x_0^a)^2-8\epsilon_0 \epsilon_1}{2U'(x_0^a)}
\end{align*}
\]  
(43)

Obtained potential will be regular function too
\[
V_-(x)^{\pm} = \alpha_-^{(\pm)} + o(x-x_0^a),
\]  
(44)

where
\[
\begin{align*}
\alpha_-^{(-)} &= -\frac{64\epsilon_0^2+8\epsilon_1 U'(x_0^a)^2+U''(x_0^a)^2-16\epsilon_0(U''(x_0^a)^2+\epsilon_1 U''(x_0^a))-2U'(x_0^a) U''(x_0^a)}{8U'(x_0^a)^2} \\
\alpha_-^{(+)} &= -3\alpha_-^{(-)} + 4\epsilon_0 + \frac{U''(x_0^a)}{2U'(x_0^a)}.
\end{align*}
\]  
(45)

The wave functions \(\psi_0^-(x), \psi_1^-(x), \psi_2^-(x)\) will read as follows
\[
\begin{align*}
\psi_0^-(x) &= 1 + o(x-x_0^a) \\
\psi_1^-(x)^{(+)} &= 2\epsilon_0 (x-x_0^a) + o(x-x_0^a)^2 \\
\psi_1^-(x)^{(-)} &= \frac{U'(x_0^a)}{2\epsilon_1} + o(x-x_0^a) \\
\psi_2^-(x) &= -2\epsilon_1 + o(x-x_0^a)
\end{align*}
\]  
(46)

Thus, at the points \(x_0^a\), where function \(U(x)\) has first-order zeros, potential \(V_-(x)\) and wave functions \(\psi_0^-(x), \psi_1^-(x), \psi_2^-(x)\) will be continuous functions, and wave function \(\psi_1^-(x)\) can have simple zeros at the points \(x_0^a\) depending on the selected sign before the square root.

Let us consider potential \(V_+(x)\), which is the supersymmetric partner of the obtained potential \(V_-(x)\). Potential \(V_+(x)\) will be regular function in the vicinity of the points \(x_0^a\) too
\[
\begin{align*}
V_+(x)^{\pm} &= \alpha_+^{(\pm)} + o(x-x_0^a) \\
\alpha_+^{(+) &= \alpha_-^{(-)} + 2\epsilon_1 + \frac{U'(x_0^a)^2-2\epsilon_0 U''(x_0^a)}{4\epsilon_0^2} } \\
\alpha_+^{(-)} &= \alpha_-^{(+)} - 2\epsilon_1
\end{align*}
\]  
(47)

with continuous wave functions
\[
\begin{align*}
\psi_1^+(x) &= \sqrt{2}\epsilon_0 + o(x-x_0^a) \\
\psi_1^+(x)^{(+)} &= 2\sqrt{2}\epsilon_1 (\epsilon_0 + \epsilon_1) (x-x_0^a) + o(x-x_0^a)^2 \\
\psi_1^+(x)^{(-)} &= \frac{(\epsilon_0 + \epsilon_1) U'(x_0^a)}{\sqrt{2}\epsilon_0} + o(x-x_0^a)
\end{align*}
\]  
(48)
Depending on the selected sign before the square root wave function $\psi_2^+(x)$
can have nodes at the points $x_0^b$.

Now let the function $U(x)$ have second-order zeros at the points $x_0^b$

$$U(x) = \frac{1}{2} U''(x_0^b)(x - x_0^b)^2 + \frac{1}{6} U^{(3)}(x_0^b)(x - x_0^b)^3 + o(x - x_0^b)^4,$$

then behavior of the functions $W_+(x)$ and $\tilde{W}_+(x)$ will be as follows

$$\begin{cases}
W_+(x) = 2\epsilon_0(x - x_0^b) + o(x - x_0^b)^{3/2} \\
\tilde{W}_+(x) = 2\epsilon_1(x - x_0^b) + o(x - x_0^b)^{3/2}
\end{cases}$$

i.e. at the points $x_0^b$ functions $W_+(x)$ and $\tilde{W}_+(x)$ will have zeros. Keeping in
mind (35), it is easy to obtain the following coefficient restriction

$$U''(x_0^b) = (W'_+(x_0^b)\tilde{W}_+(x_0^b))'' = W'_''(x_0^b)\tilde{W}_+(x_0^b) + 2W'_+(x_0^b)\tilde{W}_''(x_0^b) + W'_+(x_0^b)\tilde{W}_''(x_0^b) + 2W'_+(x_0^b)\tilde{W}_+(x_0^b) = 8\epsilon_0\epsilon_1.$$  (51)

Note, that existence of the fractional powers in the series expansion leads
to undesired poles of $\tilde{W}_0(x)$ at the points $x_0^b$

$$W_0(x)^{\pm}(x) = \pm \frac{1}{8} \sqrt{\frac{3U^{(3)}(x_0^b)}{\epsilon_0\epsilon_1(x - x_0^b)}} + o(x - x_0^b)^{1/2},$$

which can bring the singularity to the potential energy $V_-(x)$. It is easy to see,
that in the case of $U^{(3)}(x_0^b) = 0$ the fractional powers in the series expansions
disappear

$$\begin{cases}
W_+(x) = 2\epsilon_0(x - x_0^b) + o(x - x_0^b)^2 \\
\tilde{W}_+(x) = 2\epsilon_1(x - x_0^b) + o(x - x_0^b)^2
\end{cases}$$

The condition $U^{(3)}(x_0^b) = 0$ is satisfied in the simplest way if the point $x_0^b$
is a middle of the periodicity interval and $U(x)$ is even function with regard to
the $x_0^b$, which at the same time provides the fulfilment of (38).

Then superpotentials read

$$\begin{cases}
W_0(x)^{\pm}_0 = B_0^{(\pm)} + o(x - x_0^b) \\
W_1(x)^{\pm}_1 = B_1^{(\pm)} + o(x - x_0^b) \\
W_2(x)^{\pm}_2 = B_2^{(\pm)} + o(x - x_0^b)
\end{cases}$$

(54)
where
\[
\begin{cases}
B_0^{(+)} = B_1^{(-)} = B_2^{(+)} = B \\
B_0^{(-)} = B_1^{(+)} = B_2^{(-)} = -B \\
B = 1/4 \sqrt{32(\epsilon_0 - \epsilon_1) + U^{(4)}(x_0^b)/(2\epsilon_0\epsilon_1)}
\end{cases}
\]

Obtained potential \( V_-(x) \) will be continuous function
\[
\begin{cases}
V_-(x)^{(\pm)} = \beta^{(\pm)} + o(x - x_0^b) \\
\beta^{(\pm)} = \epsilon_0 + \frac{U^{(4)}(x_0^b)}{64\epsilon_0\epsilon_1} \pm \frac{U^{(5)}(x_0^b)}{320\epsilon_0\epsilon_1B}
\end{cases}
\]

Wave functions \( \psi^{-}_0(x) \), \( \psi^{-}_1(x) \), \( \psi^{-}_2(x) \) will read as follows
\[
\begin{cases}
\psi^{-}_0(x) = 1 + o(x - x_0^b) \\
\psi^{-}_1(x) = 2\epsilon_0(x - x_0^b) + o(x - x_0^b)^2 \\
\psi^{-}_2(x) = -2\epsilon_1 + o(x - x_0^b)
\end{cases}
\]

Thus, in the vicinity of the second-order zero of the \( U(x) \) potential energy \( V_-(x) \) and the wave functions \( \psi^{-}_0(x) \), \( \psi^{-}_1(x) \), \( \psi^{-}_2(x) \) will be continuous functions, if \( x_0^b = x_m \) is the middle of the periodicity interval and \( U(x) \) is even function with respect to \( x_0^b \). Wave function \( \psi^{-}_1(x) \) will have node at the points \( x_0^b \).

The supersymmetric partner \( V_+(x) \) of the \( V_-(x) \) potential in the vicinity of \( x_0^b \) will have the following behavior
\[
\begin{cases}
V_+(x)^{(\pm)} = \beta^{(\pm)} + o(x - x_0^b) \\
\beta^{(\pm)} = \epsilon_0 - 2\epsilon_1 + \frac{U^{(4)}(x_0^b)}{64\epsilon_0\epsilon_1} \pm \frac{U^{(5)}(x_0^b)}{320\epsilon_0\epsilon_1B}
\end{cases}
\]

with the following wave functions
\[
\begin{cases}
\psi^{+}_1(x) = \sqrt{2}\epsilon_0 + o(x - x_0^b) \\
\psi^{+}_2(x) = 2\sqrt{2}\epsilon_1(\epsilon_0 + \epsilon_1)(x - x_0^b) + o(x - x_0^b)^2
\end{cases}
\]

Thus, in the vicinity of the second-order zeros \( x_0^b \) of the function \( U(x) \) potential energy \( V_+(x) \) and the corresponding wave functions \( \psi^{+}_1(x) \), \( \psi^{+}_2(x) \) will be continuous function and wave function \( \psi^{+}_2(x) \) will have nodes at the points \( x_0^b \).

Let us analyze the case when the function \( U(x) \) has the highest order of zeros at the points \( x_0^b \) using the particular case of the third-order zeros.
\[ U(x) = \frac{1}{6} U^{(3)}(x_0^c)(x - x_0^c)^3 + \frac{1}{24} U^{(4)}(x_0^c)(x - x_0^c)^4 + o(x - x_0^c)^5. \]  

(60)

Then the series expansion for the function \( W_+(x) \) in the vicinity of the points \( x_0^c \) will start from the terms which will be proportional to the \((x - x_0^c)^{3/2}\), thus, condition (35) will not be satisfied, and then obtained potential energy \( V_-(x) \) will have poles at the points \( x_0^c \). Consequently, function \( U(x) \) should not have zeros of the highest then second orders.

Singularities at the potential energy, except the zeros of \( U(x) \), can appear at the points where \( U'(x) = 0 \) or \( 1 - \sqrt{R(x)} = 0 \), that is

\[
\begin{bmatrix}
U'(x) &=& 0, \\
U(x) &=& 0, \\
U(x) &=& -2\varepsilon_0, \\
U(x) &=& 2\varepsilon_1.
\end{bmatrix}
\]  

(61)

Case of \( U(x) = 0 \) was considered in the details above. In the vicinity of the points \( a_0 \), where the derivative of \( U(x) \) is equal to zero, i.e. \( U'(a_0) = 0 \) and \( U(a_0) \neq 0 \), generating function \( U(x) \) can be written as

\[ U(x) = U(a_0) + \frac{1}{2} U''(a_0)(x - a_0)^2 + o(x - a_0)^3. \]  

(62)

Then behavior of function \( W_+(x) \) in the vicinity of \( a_0 \) will be the following

\[ W_+(x)^{(\pm)} = \pm \sqrt{\frac{U(a_0)(2\varepsilon_0 + U(a_0))}{U(a_0) - 2\varepsilon_1}} + \frac{U''(a_0)}{4\varepsilon_1 - 2U(a_0)}(x - a_0) + o(x - a_0)^2, \]  

(63)

in other words, in the vicinity of zeros of \( U'(x) \), which do not coincide with zeros of \( U(x) \), obtained solutions will be continuous functions.

In the vicinity of \( b_0 \), where \( U(b_0) = 2\varepsilon_1 \), function \( W_+(x) \) will behave as follows

\[
\begin{align*}
W_+(x)^{(\pm)} &= \frac{4\varepsilon_1(\varepsilon_0 + \varepsilon_1)}{U''(b_0)} + o(x - b_0), \\
W_+(x)^{(-)} &= -\frac{1}{x - b_0} + const + o(x - b_0).
\end{align*}
\]  

(64)

Despite singularity of the function \( W_+(x)^{(-)} \), potential energy and wave functions will be continuous functions, because pole of \( W_+(x) \) satisfies the condition (36).
In the vicinity of \( c_0 \), where \( U(c_0) = -2\epsilon_0 \), function \( W_+(x) \) can be figured out as follows
\[
\begin{align*}
W_+(x)^{(+)} &= -2\epsilon_0 (x - c_0) + o(x - c_0)^2, \\
W_+(x)^{(-)} &= \frac{U''(c_0)}{2(\epsilon_0 + \epsilon_1)} + o(x - c_0),
\end{align*}
\]
thus, potential energy and wave functions will be continuous functions again. Consequently, at the all points, where denominator of \( W_+(x) \) can turn into zero, potential energy \( V_-(x) \) and wave functions \( \psi_0^-(x), \psi_1^-(x), \psi_2^-(x) \) will be free of singularities.

Similar analysis, which we shall omit due to its inconvenience, with respect to the potential \( V_+(x) \) shows, that potential \( V_+(x) \) and corresponding wave functions will be free of singularities at the all considered points except the points \( c_0 \), where potential \( V_+(x)^{(+)\,} \) will have pole with the following behavior
\[
V_+(x)^{(+)\,} = \frac{1}{(x - c_0)^2} + \text{const} + o(x - c_0).
\]
Fortunately, this singularity can be avoided if within the parts of periodicity interval which contains \( c_0 \) we apply solution \( V_+(x)^{(-)} \) instead of \( V_+(x)^{(+)\,} \).

Another way to avoid singularities in the potential \( V_+(x) \) is to exclude zeros in the denominator of \( W_+(x) \) by picking up the amplitude of the function \( U(x) \) in such a manner that equations \( U(x) + 2\epsilon_0 = 0 \) and \( U(x) - 2\epsilon_1 = 0 \) not be fulfilled. Indeed, since energy levels \( \epsilon_0, \epsilon_1 \) are positively defined values and \( U(x) \) is a periodic bounded function, we can always fit the amplitude of generating function \( U(x) \) using the following rule
\[
\begin{align*}
\epsilon_0 < 1/2 \min U(x), \\
\epsilon_1 > 1/2 \max U(x),
\end{align*}
\]
where \( \min U(x) \) and \( \max U(x) \) - minimal and maximal values of the \( U(x) \) at the periodicity interval respectively.

Thus, periodic function \( U(x) \) generates quasi exactly solvable potential \( V_-(x) \) with three known eigenfunctions \( \psi_0^-(x), \psi_1^-(x), \psi_2^-(x) \) for the energy values \( \epsilon_0 > 0 \) and \( \epsilon_1 > 0 \), if \( R(x) \geq 0 \) for all periodicity interval. Simultaneously, function \( U(x) \) generates quasi exactly solvable potential \( V_+(x) \) with two known eigenfunctions \( \psi_1^+(x), \psi_2^+(x) \) in the case of \( U(x) \in (-2\epsilon_0; 2\epsilon_1) \) and \( R(x) \geq 0 \) for all periodicity interval.

To provide free of singularities potential energy and extended bounded wave functions, \( U(x) \) must be even function with respect to the middle of
the periodicity interval \( x_m \) and must have second order zero at this point. Generating function \( U(x) \) may have first-order zeros at the other points of the periodicity interval and should not have zeros of the highest order. The derivative of the \( U''(x) \) at the point \( x_m \) should satisfy the condition \( U''(x_m) = 8\epsilon_0\epsilon_1 \). It is necessary to use solutions with opposite signs from the left and right sides with regard to point \( x_m \).

To illustrate the above described method we give a short example.

**Trigonometric extension of the Razavy potential.** Let us start from the generating function

\[
U(x) = 4\epsilon_0\epsilon_1 \sin^2 x. \tag{68}
\]

Similar generating function \( U(x) = 4\epsilon_0\epsilon_1 \sinh^2 x \) at \( \epsilon_1 = \epsilon_0 + 1/2 \) gives well known quasi exactly solvable Razavy potential [22]. Than \( \mathcal{R}(x) \) can be rewritten in the following form

\[
\mathcal{R}(x) = (-1 + 2\epsilon_0 - 2\epsilon_1 + 4\epsilon_0\epsilon_1 \sin^2 x) \tan^2 x. \tag{69}
\]

We shall omit the general expression for the superpotentials and potential energy as it is huge and rather useless. There are at least three sets of \( \epsilon_0, \epsilon_1 \), which allow us to resolve the root in the function \( R(x) \) and therefore to significantly simplify the final results.

The first set is

\[
\begin{cases}
4\epsilon_0\epsilon_1 = 0 \\
-1 + 2\epsilon_0 - 2\epsilon_1 \geq 0
\end{cases} \tag{70}
\]

for which we obtain trivial solution \( \epsilon_0 = 0 \) or \( \epsilon_1 = 0 \), what leads to the \( U(x) = 0 \).

In the case of the second set

\[
\begin{cases}
-1 + 2\epsilon_0 - 2\epsilon_1 = -4\epsilon_0\epsilon_1 \\
-1 + 2\epsilon_0 - 2\epsilon_1 \geq 0
\end{cases} \tag{71}
\]

we obtain \( \epsilon_1 = -1/2 \). Then

\[
W_+(x) = \frac{\epsilon_0 \sin 2x}{1 + \sqrt{2\epsilon_0} \sin x}. \tag{72}
\]

Function \( W_+(x) \) has zeros at the points \( x_k = \pi n/2, n = 0, \pm 1, \ldots \). The derivations \( W'_+(x) \) at this points are \(-2\epsilon_0/(1 + \sqrt{2\epsilon_0})\) or \(2\epsilon_0\) and condition (35) is not fulfilled.
The last set
\[
\begin{aligned}
-1 + 2\epsilon_0 - 2\epsilon_1 &= 0 \\
4\epsilon_0\epsilon_1 &\geq 0
\end{aligned}
\] (73)
gives \(\epsilon_1 = \epsilon_0 - 1/2\), then square root can be rewritten in the following form
\[
\begin{aligned}
R(x) &= 2\sqrt{\epsilon_0\epsilon_1}\sin x \tan x \\
\epsilon_0 &\geq 1/2
\end{aligned}
\] (74)

Function \(W_+(x)\) reads as follows
\[
W_+(x) = \frac{2\epsilon_0(\cos^2 x + 2\epsilon_0\sin^2 x)\tan x}{1 + 2\sqrt{\epsilon_0\epsilon_1}\sin x \tan x}.
\] (75)

Function \(W_+(x)\) has singularities at the points \(x_k^{(1)} = \pm \arccos(\sqrt{\epsilon_0/\epsilon_1} + 2\pi n, n = 0, \pm 1, \ldots\) and \(x_k^{(2)} = \pm \arccos(-\sqrt{\epsilon_1/\epsilon_0} + 2\pi n, n = 0, \pm 1, \ldots\). Due to the limitation \(\epsilon_0 \geq 1/2\), solutions \(x_k^{(1)}\) belong to the complex space and thus, can be dismissed. At the points \(x_k^{(2)}\) function \(W_+(x)\) has simple poles with the pole coefficient \(-1\), thus potential energy \(V_-(x)\) will be regular function at points \(x_k^{(2)}\) for any \(\epsilon_0\). Additionally, function \(W_+(x)\) has simple zeros at the points \(x_k = \pi n, n = 0, \pm 1, \ldots\). The derivations \(W_+(x)\) at all these points are equal to \(2\epsilon_0\), so all conditions imposed on generating function \(U(x)\) to provide non-singular real potential energy \(V_-(x)\) are satisfied for any \(\epsilon_0 > 1/2\).

Then, using the definition of function \(W_+(x)\) (27), solution for superpotentials \(W_0(x), W_1(x), W_2(x)\) (22), and relation between superpotential \(W_0(x)\) and potential energy \(V_-(x)\) (11), we can find three eigenstates of the potential
\[
V_-(x) = \epsilon_0 - \frac{1}{2} + \frac{1}{4}\left(\epsilon_0\epsilon_1 - 6\sqrt{\epsilon_0\epsilon_1}\cos x - \epsilon_0\epsilon_1\cos 2x\right),
\] (76)
where \(\epsilon_1 = \epsilon_0 - 1/2\). The energy values of this eigenstates are \(E_0^- = 0\), \(E_1^- = \epsilon_0\), \(E_2^- = \epsilon_0 + \epsilon_1\) and wave functions are given by (30)
\[
\begin{aligned}
\psi_0^- (x) &= C^0_- e^{\sqrt{4\epsilon_0\epsilon_1}\cos^2 \frac{x}{2}}\left(1 + 4(\sqrt{\epsilon_0\epsilon_1} + \epsilon_1)\cos^2 \frac{x}{2}\right) \\
\psi_1^- (x) &= C^1_- e^{\sqrt{4\epsilon_0\epsilon_1}\cos^2 \frac{x}{2}}\epsilon_0 \sin x \\
\psi_2^- (x) &= C^2_- e^{\sqrt{4\epsilon_0\epsilon_1}\cos^2 \frac{x}{2}} 2\epsilon_1 \left(1 + 4(\sqrt{\epsilon_0\epsilon_1} - \epsilon_0)\cos^2 \frac{x}{2}\right)
\end{aligned}
\] (77)
Potential $V_-(x)$ and the wave functions $\psi_0^-(x)$, $\psi_1^-(x)$, $\psi_2^-(x)$ are presented at the figure 1. Here $\epsilon_0 = 1$, $C_0^- = 0.05$, $C_1^- = 0.3$, $C_2^- = 1.3$ are used.

Because wave function $\psi_0^-(x)$ does not have nodes, eigenstate with energy $E_0^- = 0$ is a ground state of this potential. The wave functions $\psi_1^-(x)$ and $\psi_2^-(x)$ have two nodes per interval of periodicity, then eigenstates with energies $E_1^-$ and $E_2^-$ describe the limits of the second forbidden energy band.

This quasi exactly solvable potential belongs to the class of QES potentials presented by Turbiner in his paper [5] in the following form

$$V(x) = \frac{1}{2} \left( -a^2 \cos^2(2\alpha x) - 2\alpha a(2n + 1) \cos(2\alpha x) \right),$$

(78)
in the case of $n = 1$, $\alpha = 1/2$; $a$ is a free parameter of quantum mechanics problem.

Now let us consider supersymmetric partner of the potential $V_-(x)$:

$$V_+(x) = \frac{1}{2} \left[ \epsilon_0^2 + \frac{3}{2} \epsilon_0 - 1 - \sqrt{\epsilon_0 \epsilon_1} \cos x - \epsilon_0 \epsilon_1 \cos^2 x \right] + \frac{\sum_{i=0}^{7} a_i \cos^i x}{2 \sum_{i=0}^{8} b_i \cos^i x},$$

(79)
\[ \begin{align*}
\begin{array}{l}
  a_0 = 16\epsilon_0^3 \\
  a_1 = -8\sqrt{\epsilon_0}\epsilon_0(2 - 5\epsilon_0 + 2\epsilon_0^2) \\
  a_2 = -12\epsilon_0(1 - 2\epsilon_0 - 2\epsilon_0^2 + 4\epsilon_0^3) \\
  a_3 = 8\sqrt{\epsilon_0}\epsilon_1(1 + 3\epsilon_0 - 12\epsilon_0^2 + 6\epsilon_0^3) \\
  a_4 = 1 + 16\epsilon_0 - 48\epsilon_0^2(1 - \epsilon_0^2) \\
  a_5 = -6\sqrt{\epsilon_0}\epsilon_1(1 + 2\epsilon_0 - 12\epsilon_0^3 + 8\epsilon_0^4) \\
  a_6 = -8\epsilon_0^2(3 + 2\epsilon_0) \\
  a_7 = 16\epsilon_0^3\epsilon_0\sqrt{\epsilon_0}\epsilon_1 \\
 \end{array}
\end{align*} \]

\[ \begin{align*}
\begin{array}{l}
  b_0 = 8\epsilon_1^3 \\
  b_1 = 8\epsilon_0^2\sqrt{\epsilon_0}\epsilon_1 \\
  b_2 = -2\epsilon_0^2(1 - 12\epsilon_0 + 16\epsilon_0^2) \\
  b_3 = -8\epsilon_0\sqrt{\epsilon_0}\epsilon_1(3\epsilon_0 - 1) \\
  b_4 = \epsilon_0(1 + 10\epsilon_0 - 48\epsilon_0^3)(1 - \epsilon_0^2) \\
  b_5 = 2\epsilon_0^2\epsilon_1(1 - 8\epsilon_0 + 12\epsilon_0^2) \\
  b_6 = -2\epsilon_0^2(-1 - 4\epsilon_0 + 16\epsilon_0^2) \\
  b_7 = -8\epsilon_0^3\sqrt{\epsilon_0}\epsilon_1 \\
  b_8 = 8\epsilon_0^3\epsilon_1 \\
 \end{array}
\end{align*} \]

Since we know eigenfunctions \( \psi^-_1(x) \) and \( \psi^-_2(x) \) of Hamiltonian \( H_- \), using supersymmetric relations (7) we can find the wave functions \( \psi^+_1(x) \) and \( \psi^+_2(x) \), which are eigenfunctions of the Hamiltonian \( H_+ \) with the corresponding energy values \( E^+_1 = \epsilon_0 \) and \( E^+_2 = \epsilon_0 + \epsilon_1 \):

\[ \begin{align*}
\psi^+_1(x) &= C^+_1 \epsilon_0 e^{\sqrt{4\epsilon_0}\epsilon_1 \cos^2 x} \frac{\sum_{i=0}^{4} k_i \cos^i x}{2 \sum_{i=0}^{2} l_i \cos^i x} \\
\psi^+_2(x) &= C^+_2 e^{\sqrt{4\epsilon_0}\epsilon_1 \cos^2 x} \frac{\sum_{i=0}^{3} m_i \cos^i x}{2 \sum_{i=0}^{3} n_i \cos^i x}
\end{align*} \] (82)

where

\[ \begin{align*}
\begin{array}{l}
  k_0 = 4\sqrt{2}\epsilon_0\epsilon_1 \\
  k_1 = 4\sqrt{2}\epsilon_0\epsilon_1 \\
  k_2 = -\sqrt{2}(8\epsilon_0\epsilon_1 - 1) \\
  k_3 = -4\sqrt{2}\epsilon_0\epsilon_1 \\
  k_4 = 4\sqrt{2}\epsilon_0\epsilon_1 \\
  l_0 = 4\epsilon_0\sqrt{\epsilon_0}\epsilon_1 \\
  l_1 = 2\epsilon_0 \\
  l_2 = 2(1 - 4\epsilon_0)\sqrt{\epsilon_0}\epsilon_1 \\
  l_3 = 2\epsilon_1 \\
  l_4 = 4\epsilon_1\sqrt{\epsilon_0}\epsilon_1 \\
 \end{array}
\end{align*} \]

(83)
Figure 2: Potential $V_+(x)$ (thick line) and the wave functions $\psi_1^+(x)$, $\psi_2^+(x)$ (solid line and dashed line respectively) at the interval $x \in [0, 2\pi]$. Here $\epsilon_0 = 1$, $C_1^+ = 0.2$, $C_2^+ = 0.7$ are used.

\[
\begin{align*}
    m_0 &= -4\sqrt{2}\epsilon_0\epsilon_1(4\epsilon_0 - 1)(\epsilon_1 - \sqrt{\epsilon_0\epsilon_1}) \\
    m_1 &= 2\sqrt{2}\epsilon_0(\sqrt{\epsilon_0} - \sqrt{\epsilon_1})(8\epsilon_0^3 - 14\epsilon_0^2 + 7\epsilon_0 - 1) \\
    m_2 &= -\sqrt{2}(\sqrt{\epsilon_0\epsilon_1} - \epsilon_1)(1 - 4\epsilon_0 - 4\epsilon_0^2 + 16\epsilon_0^3) \\
    m_3 &= -4\epsilon_0^2\sqrt{2}\epsilon_0(\sqrt{\epsilon_0} - \sqrt{\epsilon_0})(4\epsilon_0 - 1)
\end{align*}
\]

(84)

\[
\begin{align*}
    n_0 &= 2\epsilon_0\sqrt{\epsilon_0\epsilon_1} \\
    n_1 &= \epsilon_0 \\
    n_2 &= -2(\epsilon_0 + \epsilon_1)\sqrt{\epsilon_0\epsilon_1} \\
    n_3 &= -\epsilon_1 \\
    n_4 &= 2\epsilon_1\sqrt{\epsilon_0\epsilon_1}.
\end{align*}
\]

(85)

Thus we obtain QES potential $V_+(x)$ (79) with two exactly know eigenstates $E_1^+ = \epsilon_0$, $\psi_1^+(x)$ and $E_2^+ = \epsilon_0 + \epsilon_2$, $\psi_2^+(x)$ given by (82). Potential $V_+(x)$ and the wave functions $\psi_1^+(x)$, $\psi_2^+(x)$ are presented at the figure 2.

Because the wave functions $\psi_1^+(x)$ and $\psi_2^+(x)$ have two nodes per periodicity interval, the eigenstates with energy values $E_1^+$ and $E_2^+$ describe limits of the second forbidden energy band. Note, that QES potential (79) does not belong to the general Turbiner’s case [5] and is completely new.
4 Conclusions

In the present paper we have extended the SUSY method of constructing well-like QES potentials with three known eigenstates potentials for the case of periodic potentials.

Thus, periodic function $U(x)$ generates quasi exactly solvable potential $V_-(x)$ with three known eigenstates $\psi_0^-(x)$, $\psi_1^-(x)$, $\psi_2^-(x)$ and quasi exactly solvable potential $V_+(x)$ with two known eigenstates $\psi_1^+(x)$, $\psi_2^+(x)$.

Since we are interested in the real potential energy, condition $R(x) \geq 0$ should be satisfied. To provide free of singularities potential energy and extended bounded wave functions, generating function $U(x)$ must have second order zero at the middle of the periodicity interval $x_m$ and must be even function with respect to this point. $U(x)$ may have first-order zeros at the other points of the periodicity interval and should not have zeros of the highest order. The derivative of the $U''(x)$ at the point $x_m$ should satisfy the condition $U''(x_m) = 8\epsilon_0\epsilon_1$. It is necessary to use solutions for the superpotentials, potentials and wave functions with opposite signs from the left and right sides with regard to point $x_m$ to obtain continuous extended wave functions.

As an example of the above described method starting from the generating functions $U(x) = 4\epsilon_0\epsilon_1 \sin^2 x$ we have obtained QES periodic potential $V_-(x) = \epsilon_0 - 1/2 + 1/4(\epsilon_0\epsilon_1 - 6\sqrt{\epsilon_0\epsilon_1}\cos x - \epsilon_0\epsilon_1\cos 2x)$, which is trigonometric extension of the well known Razavy QES potential, with three known eigenstates $E_0^- = 0$, $E_1^- = \epsilon_0$ and $E_2^- = \epsilon_0 + \epsilon_1$, where $\epsilon_1 = \epsilon_0 - 1/2$ and $\epsilon_0$ is a free parameter. This potential belongs to the class of QES potentials presented by Turbiner at [5]. Eigenstate with energy $E_0^-$ = 0 is the ground state of this potential. Eigenstates with energies $E_1^-$ and $E_2^-$ describes the limits of the second forbidden energy band.

The supersymmetric partner $V_+(x)$ of potential $V_-(x)$ gives us a new QES periodic potential for which we know two eigenstates $E_1^+ = \epsilon_0$ and $E_2^+ = \epsilon_0 + \epsilon_2$ in the explicit form, where $\epsilon_1 = \epsilon_0 - 1/2$ and $\epsilon_0$ is a free parameter. This eigenstates describe the limits of the second forbidden energy band.

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