Surface vacuum energy in cutoff models: pressure anomaly and distributional gravitational limit

Ricardo Estrada\textsuperscript{1}, Stephen A Fulling\textsuperscript{2,3} and Fernando D Mera\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA
\textsuperscript{2} Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA
\textsuperscript{3} Department of Physics, Texas A&M University, College Station, TX 77843-4242, USA

E-mail: fulling@math.tamu.edu

Received 31 July 2012
Published 23 October 2012
Online at stacks.iop.org/JPhysA/45/455402

Abstract
Vacuum-energy calculations with ideal reflecting boundaries are plagued by boundary divergences, which presumably correspond to real (but finite) physical effects occurring near the boundary. Our working hypothesis is that the stress tensor for idealized boundary conditions with some finite cutoff should be a reasonable ad hoc model for the true situation. The theory will have a sensible renormalized limit when the cutoff is taken away; this requires making sense of the Einstein equation with a distributional source. Calculations with the standard ultraviolet cutoff reveal an inconsistency between energy and pressure similar to the one that arises in noncovariant regularizations of cosmological vacuum energy. The problem disappears, however, if the cutoff is a spatial point separation in a ‘neutral’ direction parallel to the boundary. Here we demonstrate these claims in detail, first for a single flat reflecting wall intersected by a test boundary, then more rigorously for a region of finite cross section surrounded by four reflecting walls. We also show how the moment-expansion theorem can be applied to the distributional limits of the source and the solution of the Einstein equation, resulting in a mathematically consistent differential equation where cutoff-dependent coefficients have been identified as renormalizations of properties of the boundary. A number of issues surrounding the interpretation of these results are aired.

PACS numbers: 03.70.+k, 04.20.Cv, 11.10.Gh
Mathematics Subject Classification: 81T55, 83C47

1. Energy density and pressure near a plane boundary

Introduction remarks. The effects of reflecting boundaries in quantum field theory continue to be of intense interest, from the mathematical to the experimental [1–4].
From a mathematical point of view there are two kinds of effects. First, there is the classic Casimir energy, which is distributed throughout space, associated semiclassically with periodic classical orbits, and finite after subtraction of the zero-point energy of infinite space. (Here we consider only space-time that is locally flat, and without any external potentials that would create additional distributed divergences.) Second, there is energy that is concentrated near the boundaries, associated with short paths, closed but not periodic, that bounce off the boundaries, and usually divergent in this sense: For a perfectly reflecting boundary (such as a perfect conductor in electromagnetism or a Dirichlet or Neumann boundary for a scalar field), the vacuum expectation value of the energy density has a nonintegrable behavior,

\[ \rho \sim \frac{c_1}{s^2} + \frac{c_2}{s^3} + \cdots, \tag{1} \]

where \( s \) is the distance from the boundary. (The leading exponent is \( d + 1 \) when the spatial dimension is \( d \).)

In the original Casimir scenario of the electromagnetic field between parallel flat conducting plates [5, 6], energy of the second type is absent. In the famous Boyer calculation [7] for the electromagnetic field inside and near a thin spherical shell of conductor, the total energy again comes out finite, though only because of a sequence of special cancellations [8] (between electric and magnetic modes for \( c_1 \), between interior and exterior for \( c_2 \) and \( c_4 \), and for a variety of complicated geometrical reasons [9, 10] for \( c_3 \)). In general the divergent boundary contributions are present and are playing a major role in the ongoing overhaul of the theoretical ideas forced by the huge improvements in experimental results since 1997.

Analytic regularization methods (dimensional regularization and zeta functions) usually yield finite values for total energy, and ultraviolet-cutoff calculations can reach the same values by discarding the divergent leading terms in an expansion in powers of the cutoff parameter; but much current opinion tends to limit the scientific validity of such methods to calculations of the forces between disjoint, rigid bodies. In studying deformable bodies or gravitational effects, the actual energy density near the boundary must be taken seriously, together with the energy of the boundary material itself, from which it cannot be cleanly separated. It is agreed that divergences result from overidealization of the physics of the boundary. For example, a real electromagnetic conductor is not perfectly conducting at arbitrarily high frequencies, and at truly small scales its atomic structure becomes relevant. For a realistic boundary the effects represented crudely by inverse powers in (1) are finite, but perhaps large.

In our present program we attempt to salvage some of the mathematical advantages of the old theory by hypothesizing that the vacuum stress tensor, \( T^{\mu\nu}(x) \), of a quantum field interacting with an idealized boundary with an exponential ultraviolet or similar cutoff kept finite is a reasonable ad hoc model for a realistic physical situation. We are particularly interested in the interpretation of the boundary divergences in the context of general relativity, an issue raised in the seminal paper of Deutsch and Candelas [8]. Since the actual gravitational effects of vacuum energy in the laboratory are surely very small, the problem is primarily one of establishing consistency of the properly interpreted theory, so this approach should be sufficient—certainly much more satisfactory than simply discarding divergent quantities that are integrals of perfectly finite and plausible local densities. The expectation is that the theory will have a sensible renormalized limit when the cutoff is taken away; this requires making sense of the Einstein equation with a distributional source, interpreting the cutoff as a ‘regularization’ of distributions in the mathematical sense [11]. (See also the discussions of mass renormalization in [12, 13].)

This program was initiated in [14]. Detailed calculations of the stress tensor in a model with two spatial dimensions were presented in [15], where complete consistency was claimed between force calculations based on the pressure and those based on differentiation of total
energy with respect to a geometrical parameter. However, close inspection later revealed a
sign discrepancy for one pair of terms, those related to the leading term in the cutoff boundary
energy near a boundary; in dimension 3 this problem persists, compounded by a discrepant
factor of 2 [16]. There is no evidence that this anomaly affects the finite terms constituting
the standard Casimir force. Nevertheless, since both methods of computing forces are widely
used and assumed to be equivalent [17], understanding the phenomenon is imperative. The
calculations in [15] and [16] used an exponential ultraviolet cutoff, roughly equivalent to
‘point-splitting’ regularization in the time direction. It is reasonable to think that the trouble
is related to the resulting violation of relativistic invariance. Therefore, a subprogram was
launched to consider point-splitting in other directions [18]. For rectangular configurations it
has been found that a cutoff in a ‘neutral’ direction (neither the time, nor the direction whose
pressure is being sought) yields consistent and physically plausible results. The purpose of this
paper is to present detailed results along this line, together with recapitulation and updating of
the material previously published only in conference proceedings [14, 16, 18].

In this section we review the basic theory of the vacuum stress tensor of a scalar field and
its coupling to the gravitational field in linear approximation, and we study the energy-pressure
relations and distributional limits associated to a plane Dirichlet boundary. The physical and
philosophical issues raised are discussed in section 2. The final section takes a more complete
look at the energy-pressure relations in a rectangular box (more precisely, a waveguide) when
the roles of all four boundaries are carefully taken into account.

**Basic equations: quantum field theory.** Generalized to curved space-time, the action,
Lagrangian function, and stress tensor of a scalar field in a cavity $\Omega$ are

$$
S = \int_{\Omega} L \sqrt{g} d^{d+1}x, \quad L = -\frac{1}{2} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi R \phi^2 \right], \quad T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} .
$$

We adopt the metric convention in which $g_{00} < 0$. The parameter $\xi$ labels different gravitational
couplings. In the flat-space limit the field equation and (classical) total energy are independent
of $\xi$, but the stress tensors are different. More specifically, that field equation is

$$
\frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi
$$

accompanied by boundary conditions on $\Omega$, which we shall take to be Dirichlet here ($\phi = 0$).
The simplest formulas for the stress tensor are obtained when $\xi = \frac{1}{4}$, which corresponds to
replacing the term $(\nabla \phi)^2$ in the energy density for minimal coupling ($\xi = 0$) by $-\phi \nabla^2 \phi$, to
which it is related by integration by parts:

$$
T_{00} \left( \frac{1}{4} \right) = \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial t} \right)^2 - \phi \nabla^2 \phi \right] ,
$$

$$
T_{jj} \left( \frac{1}{4} \right) = \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x_j} \right)^2 - \phi \frac{\partial^2 \phi}{\partial x_j^2} \right] \quad \text{for} \ j \neq 0.
$$

For other values of $\xi$ one has

$$
T_{\mu\nu} (\xi) = T_{\mu\nu} \left( \frac{1}{4} \right) + \Delta T_{\mu\nu} ,
$$

$$
\Delta T_{00} = -2 \left( \xi - \frac{1}{4} \right) \nabla \cdot (\phi \nabla \phi) ,
$$

$$
\Delta T_{jj} = -2 \left( \xi - \frac{1}{4} \right) \left[ \left( \frac{\partial \phi}{\partial t} \right)^2 - \sum_{k \neq j} \left( \frac{\partial \phi}{\partial x_k} \right)^2 + \phi \frac{\partial^2 \phi}{\partial x_j^2} \right] .
$$
Henceforth we consider $\xi = \frac{1}{4}$ unless otherwise indicated. The off-diagonal components of the tensor play no role in this paper.

The vacuum expectation values of components of the stress tensor can be expressed in terms of the Green function (called cylinder kernel or Poisson kernel)

$$T(t, r, r') = -\sum_{n=1}^{\infty} \frac{1}{\omega_n} \phi_n(r)\phi_n(r')^* e^{-\omega_n t},$$

(9)

where $\phi_n$ and $\omega_n$ are the eigenfunctions and eigenfrequencies of the cavity. The formulas are

$$\rho = \langle T_{00} \rangle = -\frac{1}{2} \frac{\partial^2 T}{\partial t^2},$$

(10)

$$p_j = \langle T_{jj} \rangle = \frac{1}{8} \left( \frac{\partial^2 T}{\partial x_j^2} + \frac{\partial^2 T}{\partial x'_j^2} - 2 \frac{\partial^2 T}{\partial x_j \partial x'_j} \right),$$

(11)

where it is understood, formally, that $r'$ is set equal to $r$ and $t$ to 0 at the end. Henceforth the angular brackets will usually be omitted from expectation values.

For more details of the foregoing formalism see [15].

In (9)–(11) $t$ is an ultraviolet cutoff parameter, not the physical time. However, it can be thought of as arising by a Wick rotation from the difference of two physical time coordinates: $t = -i(x^0 - x'^0)$. In the standard ultraviolet-cutoff approach, $t$ is kept different from 0 until the latest possible moment, so that (10)–(11) define a nonsingular function even when $r' = r$. On the other hand, if one thinks of $x^0$ as just another coordinate in parallel to the components $(x_j)$ of $r$, then it is natural to consider implementing a cutoff by separating the primed and unprimed points in some other direction in space-time. At the spectral level this maneuver does not lead to a classically convergent summation like (9), and if the separation is in a spatial direction, this technical problem persists even if the spatial coordinate in question is subjected to a Wick-like analytic continuation. The series will converge in a distributional sense, however. Moreover, when a closed-form expression for $T$ is available, as it is in all cases studied in this paper, the prescription can be applied to it without encountering any divergence.

In its original application to a theory in curved space-time without boundaries [19], the point separation could be in an arbitrary space-time direction, parametrized by a tangent vector $t u^\mu$. In the particular case of flat space-time the resulting formula of Christensen is

$$T_{\mu\nu} = \frac{1}{2\pi^2 t^4} \left( g_{\mu\nu} - \frac{u_\mu u_\nu}{u^\mu u^\nu} \right).$$

(12)

The usual interpretation is that the direction-dependent term averages to 0, while the scalar cutoff parameter $t$ is arbitrary; thus the stress tensor of flat space is

$$T_{00} = \Lambda \delta^{\mu\nu},$$

(13)

where $\Lambda$ is a cosmological constant to be determined by observation. (For an up-to-date discussion of this situation see [20].) Our present concern, however, is with a static boundary, and it seems that in this context the only sensible splitting directions are pure time (real or imaginary) and all the spatial directions parallel to the boundary surface$^4$.

$^4$ To foreshadow: from (16)–(18) it can be seen that if the separation mixes space and real time, the denominators will vanish somewhere in the physical region, which defeats the purpose of a cutoff. Splitting involving $z - z'$ is ambiguous at best and problematical to define when the point is closer to the boundary than the cutoff distance.
Figure 1. The reflecting plate is in the $x$-$y$ plane. The test wall is momentarily in the $z$-$y$ plane. The force on the test wall from the vacuum energy to its left is to be studied. See section 2 for discussion of the ignoring of the right side of the test wall, and see section 3 for discussion of the neglect of the finite size and other walls of the cavity on the left.

A reflecting wall intersected by a test wall. Consider a plane Dirichlet boundary at $z = 0$. We write $r_{\perp} = (x, y) = (x_1, x_2)$ for the coordinates parallel to the wall. The cylinder kernel for empty Minkowski space $\mathbb{R}^{3+1}$ is

$$T_0 = -\frac{1}{2\pi^2} \frac{1}{t^2 + |r - r'|^2}. \quad (14)$$

The kernel for the half-space $z > 0$ with the reflecting boundary is instantly obtained by the method of images, by adding to $T_0$ the reflected term

$$T = \frac{1}{2\pi^2} \frac{1}{t^2 + |r_{\perp} - r'_{\perp}|^2 + (z + z')^2}. \quad (15)$$

For determining the physically relevant stress tensor one needs only the term (15), because the contribution of $T_0$ is just (13), ubiquitous and already considered. Therefore, we now reserve the notation $\overline{T}$ for the reflection term alone (sometimes called the ‘renormalized’ cylinder kernel).

We can now calculate the energy density and pressure from (10) and (11). After the differentiation, without loss of generality we set $r'_{\perp} = 0$ (because of translation invariance) and $z' = z$ (because we will not consider point-splitting perpendicular to the boundary). Then $t$ and the components of $r_{\perp}$ are still available as cutoff parameters. We define

$$M = t^2 + x^2 + y^2 + 4z^2. \quad (16)$$

Then the results are

$$2\pi^2 \rho = M^{-3}[3t^2 + x^2 + y^2 + 4z^2], \quad (17)$$

$$2\pi^2 p_1 = M^{-3}[t^2 + 3x^2 - y^2 - 4z^2], \quad 2\pi^2 p_2 = M^{-3}[t^2 - x^2 + 3y^2 - 4z^2], \quad (18)$$

and $p_3 = 0$. Qualitatively, one can say that in the vacuum expectation there is a layer of energy against the wall, where $M$ is smallest. The vanishing of $p_3$ is understandable, because a rigid perpendicular displacement of the wall does not change the total energy of the configuration.

To understand the nonvanishing pressures parallel to the wall, imagine another planar boundary—a ‘test wall’—at $x = 0$ and find the force on it (from the left side only—see figure 1). The volume of space occupied by boundary energy increases with $x$, so the total energy on the left also changes linearly when the test wall moves.

In accordance with the principle of energy balance (also called principle of virtual work) one expects

$$F = \int_0^\infty p_1 dz = -\frac{\partial U}{\partial x} = -\int_0^\infty \rho \, dz \equiv -E. \quad (19)$$

Here a trivial transverse dimension is suppressed in the notation: $F$ is a force per unit length, $E$ is an energy per unit area, and $U$ is the total energy per unit length, equal to $E$ times the
length of the cavity (which may be arbitrarily large). If all the cutoffs are removed, one has
\[ \rho = \left( 32 \pi^2 z^4 \right)^{-1} = -p_1, \]
so energy balance is formally satisfied point-by-point in \( z \), but of course the integrals are divergent.

Consider now the traditional ultraviolet cutoff, \( t \neq 0 \) but \( r_\perp = 0 \). Evaluating the integrals in (19) with the integrands from (17) and (18), one finds that
\[ F = \frac{1}{2} E = \frac{1}{16 \pi t^3}, \]  
(20)
a clear discrepancy with (19) in both sign and magnitude. (For details see the appendix.) The \( E \) defined in this way is the same as obtained from the second (surface) term in the small-\( t \) expansion of
\[ E = -\frac{1}{2} \int_\Omega \frac{\partial^2 T}{\partial t^2} (t, r, r) \, d^3 x = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n e^{-\omega_n t}. \]  
(21)
But we shall argue that this \( E \) is wrong, whereas the \( F \) and (19) are correct (within the framework of the model).

Next look at point-splitting perpendicular to the test wall \( (x \neq 0, t = 0 \neq y) \). It is easy to see that the calculation is identical to the previous one except that \((t, \rho) \) change places with \((x, -p_1) \). Thus the result is
\[ F = +2E. \]  
(22)
Finally, consider point-splitting in the ‘neutral’ direction, \( y \neq 0, t = 0 = x \). One gets
\[ F = -E, \]  
(23)
as should happen, according to (19). In fact, the balance relation holds pointwise for the integrands; their values are
\[ \rho = \frac{1}{2 \pi^2 (y^2 + 4z^2)^2}, \quad p_1 = -\frac{1}{2 \pi^2 (y^2 + 4z^2)^2}. \]  
(24)
The quantities in (23) and (24) agree with (20) for the pressure and with (22) for the energy. We therefore propose to adopt them as ‘correct’. Note that for the Dirichlet wall, this \( E \) is positive, unlike the traditional boundary energy from (21).

A similar calculation for the ‘conformal correction’ terms (7)–(8) is entirely uneventful. They do not exhibit the anomaly: \( \Delta p_1 = -\Delta \rho \) always. Furthermore, as long as the cutoff is finite, their integrals over all \( z \) are 0 anyway, as befits their origin in an integration by parts in a theory where the field vanishes on the boundary. (That this consistency relation fails without the cutoff was pointed out by Ford and Svaiter [21].)

It may appear strange that the pressure and the energy have opposite sign. This behavior is standard, however, for an energy density that remains constant as a geometrical parameter is varied, so that the total energy is proportional to the volume occupied. It holds, for instance, for the surface tension between two fluid media [22, pp 84–7], and also for cosmological ‘dark energy’ (cf (13)).

**Basic equations: general relativity.** In the notation of [23], the linearized Einstein equation is
\[ \Box \tilde{h}_{\mu \nu} = -16 \pi T_{\mu \nu}, \]  
(25)
where
\[ \tilde{h}_{\mu \nu} = h_{\mu \nu} - \frac{1}{2} (\text{Tr} \, h) \eta_{\mu \nu}, \quad h_{\mu \nu} = g_{\mu \nu} - \eta_{\mu \nu}. \]  
(26)
We apply (25) to our model of a quantized scalar field with a single plane boundary, the boundary condition being one of the point-splitting cutoffs discussed above, approaching the Dirichlet condition as a limit. Gravitational effects in the lab, although formally infinite in the theory without cutoff, are presumably actually tiny. Therefore, the linearized Einstein equation (with a flat background) should be physically adequate. In a slightly better approximation one might use a curved background to represent the effect of the mass of the boundary.

We assume a static solution. Then, tentatively adopting the usual ultraviolet cutoff, we have the basic equation of [14],

$$\Delta \bar{h}_{00} = 16\pi \rho = \frac{8}{\pi} \frac{4\varepsilon^2 - 3t^2}{(t^2 + 4\varepsilon^2)^3} \theta(\varepsilon) + (\cdots)\theta(-\varepsilon).$$

Because we consider only the region $\varepsilon > 0$ and the equation is linear, the term describing matter sources at negative $\varepsilon$ decouples and can be ignored. However, it is essential to include the factor $\theta(\varepsilon)$ to study the singular limiting behavior of $\bar{h}_{00}$ and $\rho$ at $\varepsilon = 0$. For an infinite plane wall, $\bar{h}_{00}$ should depend only on $\varepsilon$, so

$$\Delta \bar{h}_{00} = \bar{h}_{00}''(\varepsilon).$$

The solution of (27) that vanishes for $\varepsilon < 0$ is

$$\bar{h}_{00}(\varepsilon) = \frac{\theta(\varepsilon)}{\pi} \left[ \frac{4\varepsilon}{t^3} \tan^{-1}\left(\frac{2\varepsilon}{t}\right) - \frac{1}{t^2 + 4\varepsilon^2} + \frac{1}{t^2} \right].$$

For a complete solution one should add a similar term with support in $\varepsilon \leq 0$ representing the effect of the matter in that region, and also a possible global solution (affine linear in $\varepsilon$) of the homogeneous equation. (If the coefficient of $\varepsilon$ in this homogeneous solution is made different on the two sides, a delta function representing the energy of the wall itself will be added to $\rho$.)

Taking the limit $t \downarrow 0$ in (27) will yield a differential equation with a distribution as its source. Taking that limit in the solution will yield a singular distribution. Both limits involve somewhat arbitrary regularizations (Hadamard finite parts). Before presenting the details, we review some needed distribution theory in the next subsection.

**Distributions and the moment expansion theorem.** Our analysis requires the study of the behavior of several quantities of physical interest as a parameter, $t$, approaches 0. The singularity of the quantity suggests that instead of an ordinary limit one needs to consider a distributional limit. Fortunately, a distributional theory of asymptotic expansions is available [24].

The key result in the distributional theory of asymptotic expansions is the *moment asymptotic expansion*, that says that if $f$ is a distribution of one variable, $f \in D'(\mathbb{R})$, that decays rapidly at $\pm \infty$, then the asymptotic behavior of $f(\lambda x)$ as the parameter $\lambda$ becomes large is

$$f(\lambda x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n! \lambda^{n+1}}.$$

Here the $\{\mu_n\}$ are the moments of the distribution $f$, namely,

$$\mu_n = \langle f(x), x^n \rangle,$$

which become $\mu_n = \int_{-\infty}^{\infty} f(x)x^n dx$ if $f$ is a locally integrable function or become $\mu_n = \sum_{k=-\infty}^{\infty} a_k(b_k)^n$ if $f(x) = \sum_{k=-\infty}^{\infty} a_k \delta(x - b_k)$ is a train of delta functions.

Observe that the behavior of $f$ at $\pm \infty$ is given by considering its *parametric* behavior; this reflects the fact that distributions do not have point values, in general, as ordinary functions
do. An alternative approach that uses the ideas of Cesàro summability is possible [25], but in this paper it is exactly the parametric behavior that we need to consider.

We used the term ‘of rapid decay at $\pm \infty$’ to describe the distributions for which the moment asymptotic expansion holds. In technical terms, $f$ is of rapid decay at $\pm \infty$ if it belongs to the distribution space $\mathcal{K}'(\mathbb{R})$, dual of $\mathcal{K}(\mathbb{R})$. The space $\mathcal{K}$ is formed by the so-called GLS symbols [26]; a smooth function $\phi$ belongs to $\mathcal{K}$ if there is a constant $\gamma$ such that $\phi^{(k)}(x) = O(|x|^{-\gamma-k})$ as $|x| \to \infty$ for $k = 0, 1, 2, \ldots$, that is, if $\phi(x) = O(|x|^{-\gamma})$ strongly; the topology of $\mathcal{K}$ is given by the canonical seminorms.

Notice that if $\mathcal{E}'(\mathbb{R})$ is the space of distributions with compact support, then $\mathcal{E}'(\mathbb{R}) \subset \mathcal{K}'(\mathbb{R})$, so that the moment asymptotic expansion holds for distributions of compact support.

If $f$ is a distribution of one variable that does not belong to $\mathcal{K}'(\mathbb{R})$ then it does not satisfy the moment asymptotic expansion, and, in general, the behavior of $f(\lambda x)$ as $\lambda \to \infty$ is very complicated. There is one situation, however, when $f(\lambda x)$ has a simple expansion [27]. Indeed, suppose that $f$ has support bounded on the left, supp $f \subset [a, \infty)$ for some $a$, and suppose that $f$ is an ordinary locally integrable function for $x$ large and that the ordinary expansion

$$f(x) = b_1 x^{\beta_1} + \cdots + b_n x^{\beta_n} + O(x^\beta), \quad \text{as } x \to \infty, \quad (31)$$

holds, where $\beta_1 > \beta_2 > \cdots > \beta_n > \beta$ and where $-(k+1) > \beta > -(k+2)$ for some integer $k$. Then $f(\lambda x)$ has the distributional development

$$f(\lambda x) = \sum_{j=1}^{n} b_j g_j(\lambda x) + \sum_{j=0}^{k} \frac{(-1)^j \mu_j \beta^{(j)}(x)}{j! \lambda^{j+1}} + O(\lambda^\beta), \quad (32)$$

as $\lambda \to \infty$. Here the distributions $g_j(x)$ are suitable regularizations of the functions $\theta(x)^{\beta_j}$, namely $\mathcal{P} f(\theta(x)^{\beta_j})$, as we explain below, while the $\mu_j$ are still the moments, but understood in a generalized sense, since the integrals $\int_a^\infty f(x)x^j \, dx$ will be divergent at infinity, in general, and thus we need to consider their Hadamard finite part, F.P. $\int_a^\infty f(x)x^j \, dx$.

Interestingly, the asymptotic expansion (32) considers the behavior of $f$ at infinity, but it is written in terms of distributions that could be singular at the origin. The process of associating a distribution to a singular, not locally integrable, function is considered in [11]; in the mathematical literature one says that the distribution is a regularization of the singular function. There are many regularization procedures, and, in general, regularizations are not unique [28]; for our purposes we need to consider the Hadamard finite part of integrals and the corresponding regularization method.

Let us first recall that the finite part of the limit of $F(\varepsilon)$ as $\varepsilon \to 0^+$ exists and equals $A$ if we can write $F(\varepsilon) = F_{\text{fin}}(\varepsilon) + F_{\text{min}}(\varepsilon)$, where the infinite part $F_{\text{fin}}(\varepsilon)$ is a linear combination of functions of the type $\varepsilon^{-p} \ln^q \varepsilon$, where $p > 0$ or $p = 0$ and $q > 0$, and where the finite part $F_{\text{min}}(\varepsilon)$ is a function whose limit as $\varepsilon \to 0^+$ is $A$ [24]. We then write

$$\text{F.P. } \lim_{\varepsilon \to 0^+} F(\varepsilon) = A. \quad (33)$$

A similar idea can be applied to limits at infinity, which turns out to be equivalent to a change of variables,

$$\text{F.P. } \lim_{\lambda \to \infty} G(\lambda) = \text{F.P. } \lim_{\varepsilon \to 0^+} G(1/\varepsilon). \quad (34)$$

Suppose now that we have an integral $\int_a^b f(x) \, dx$ such that $\int_a^b f(x) \, dx$ is well defined for $c > a$, but for which $\int_a^b f(x) \, dx$ is divergent. Then one considers the function $F(\varepsilon) = \int_{a+c}^{b+c} f(x) \, dx$ for $\varepsilon > 0$, and if the finite part of the limit exists as $\varepsilon \to 0^+$, then we call this value the Hadamard finite part of the integral, and write

$$\text{F.P. } \int_a^b f(x) \, dx = \text{F.P. } \lim_{\varepsilon \to 0^+} \int_{a+c}^{b+c} f(x) \, dx. \quad (35)$$
Let \( f \) now be a function of one variable that vanishes for \( x < a \), that is locally integrable in the open interval \((a, \infty)\), but for which the integral \( \int_a^b f(x) \, dx \) is divergent for \( b > a \). Then, in general, one cannot associate a distribution in \( \mathcal{D}'(\mathbb{R}) \) to \( f \). Nevertheless, if the Hadamard finite part of the integral, \( \text{F.P.} \int_a^b f(x) \, dx \), exists for \( b > a \), then one can define a distribution \( \mathcal{P} f(f) \), its ‘partie finie’, by putting
\[
(\mathcal{P} f(f), \phi(x)) = \text{F.P.} \int_a^b f(x) \phi(x) \, dx, \quad \phi \in \mathcal{D}(\mathbb{R}).
\]

In the case when \( f(x) = \theta(x)x^\beta \) for some \( \beta \in \mathbb{C} \) we obtain the distribution \( \mathcal{P} f(\theta(x)x^\beta) \). When \( \text{Re} \beta > -1 \), however, the function \( \theta(x)x^\beta \) is locally integrable even at \( x = 0 \) and thus \( \mathcal{P}f(\theta(x)x^\beta) \) reduces to the standard distribution associated to \( \theta(x)x^\beta \), which is usually denoted by \( x_+^\beta \); one can perform analytic continuation of \( x_+^\beta \), and obtain a distribution for \( \beta \neq -1, -2, -3, \ldots \), and it can be shown [24] that \( \mathcal{P} f(\theta(x)x^\beta) = x_+^\beta \) for such values of \( \beta \). On the other hand, the symbol \( x_+^\beta \) does not make sense for \( \beta = -1, -2, -3, \ldots \), while \( \mathcal{P} f(\theta(x)x^{-k}) \) is well defined for \( k = 1, 2, 3, \ldots \).

The formulas
\[
(x_+^\beta)' = \beta x_+^{\beta-1},
\]
\[
(\lambda x_+^\beta) = \lambda^\beta x_+^\beta,
\]
hold for \( \beta \neq -1, -2, -3, \ldots \) since they hold for \( \text{Re} \beta \) large, by the principle of analytic continuation. However, we have the modified formulas
\[
\frac{d}{dx} (\theta(x) \ln x) = \mathcal{P} f \left( \frac{\theta(x)}{x} \right),
\]
\[
\frac{d}{dx} \left( \mathcal{P} f \left( \frac{\theta(x)}{x^k} \right) \right) = -k \mathcal{P} f \left( \frac{\theta(x)}{x^{k+1}} \right) + \frac{(-1)^k \delta^{(k)}(x)}{k!}
\]
and
\[
\mathcal{P} f \left( \frac{\theta(\lambda x)}{(\lambda x)^2} \right) = \frac{1}{\lambda^2} \mathcal{P} f \left( \frac{\theta(x)}{x^{\lambda+1}} \right) + \frac{(-1)^k \ln \lambda \delta^{(k)}(x)}{k! \lambda^k}.
\]

**Distributional limits of energy and pressure.** In [14] the foregoing mathematical theory was applied to (27) and (28). Let \( \lambda = 1/t \). The limit of (27) is
\[
- \frac{d^2 \tilde{h}_{00}(z)}{dz^2} = \frac{1}{2\pi} \text{F.P.} \left( \frac{\theta(z)}{z^2} \right) - 2\lambda \delta(z) + \frac{1}{\pi} \lambda^2 \delta'(z) + \frac{1}{8\pi} \delta''(z) - \frac{1}{12\pi} \ln(2\lambda \delta''(z)) + \mathcal{O} \left( \frac{1}{\lambda} \right),
\]
and the limit of (28) is
\[
\tilde{h}_{00}(z) = 2\lambda \delta(z) z - \frac{1}{\pi} \lambda^2 \theta(z) - \frac{1}{12\pi} \text{F.P.} \left( \frac{\theta(z)}{z^2} \right) - \frac{1}{18\pi} \delta'(z) + \frac{1}{12\pi} \ln(2\lambda \delta'(z)) + \mathcal{O} \left( \frac{1}{\lambda} \right).
\]
(The apparent dimensional incoherence of the \( \delta' \ln(2\lambda) \) term is attributable to the violation of scale invariance by the finite-part operation.)

One quickly verifies that (43) is a solution of (42). The limit of the solution is the solution of the limit. This also follows from the definitions of the distributional operations.
A consequence of the moment expansion theorem is that, when one uses a singular function to define a distribution, divergent leading powers can be replaced by derivatives of \( \delta \) with arbitrary finite coefficients [11]. These terms can be interpreted as renormalizations of properties of the boundary. The previous observation thus shows that our toy Einstein equation survives the renormalization process as a mathematically consistent differential equation.

We shall now go beyond [14] in two ways. First, keeping \( t \) as the cutoff parameter, we examine the differential equation for the pressure component, using (18) with \( x = 0 = y \):

\[
- \frac{d^2 \tilde{h}_{11}(z)}{dz^2} = 16\pi p_1 = -\frac{8}{\pi} \frac{1}{(t^2 + 4z^2)^2} \theta(z).
\] (44)

The appropriate solution analogous to (28) is

\[
\tilde{h}_{11}(z) = \frac{\theta(z)}{\pi} \left[ \frac{2\pi}{t^3} \tan^{-1} \left( \frac{2z}{t} \right) \right].
\] (45)

For the evaluation of the asymptotic behavior of (44), the relevant distribution to use in the moment expansion theorem is

\[
f_1(z) = \frac{1}{(1 + 4z^2)^2} \theta(z) = \frac{1}{16} \frac{1}{z^4} + O \left( \frac{1}{z^5} \right) \quad \text{as} \quad z \to \infty.
\] (46)

The moment expansion theorem states, up to the relevant order, that the asymptotic expansion of \( f_1(\lambda z) \) is

\[
f_1(\lambda z) \sim \frac{1}{16} \mathcal{P} \int \frac{\theta(\lambda z)}{(\lambda z)^4} + \sum_{j=0}^{3} (-1)^j \mu_j(f_1) \frac{\delta^{(j)}(\lambda z)}{j!} + O \left( \frac{1}{\lambda^3} \right)
\]

\[
= \frac{1}{16} \left\{ \frac{1}{\lambda^4} \mathcal{P} \int \frac{\theta(z)}{z^4} - \frac{\ln \lambda}{3! \lambda^4} \theta^3(z) \right\} + \sum_{j=0}^{3} (-1)^j \mu_j(f_1) \frac{\delta^{(j)}(z)}{j! \lambda^{j+1}} + O \left( \frac{1}{\lambda^5} \right),
\] (47)

where the moments \( \mu_j(f_1) \) of the function \( f_1 \) are

\[
\mu_0(f_1) = \int_0^\infty \frac{1}{(1 + 4z^2)^2} dz = \frac{\pi}{8},
\] (48)

\[
\mu_1(f_1) = \int_0^\infty \frac{1}{(1 + 4z^2)^2} z dz = \frac{1}{8},
\] (49)

\[
\mu_2(f_1) = \int_0^\infty \frac{1}{(1 + 4z^2)^2} z^2 dz = \frac{\pi}{32},
\] (50)

\[
\mu_3(f_1) = \text{F.P.} \int_0^\infty \frac{1}{(1 + 4z^2)^2} z^3 dz = -\frac{1}{32} + \frac{1}{16} \ln 2.
\] (51)

Therefore, the distributional limit of the differential equation (44) is

\[
- \frac{d^2 \tilde{h}_{11}(z)}{dz^2} = -\frac{1}{2\pi} \mathcal{P} \int \frac{\theta(z)}{z^4} - \lambda^3 \delta(z) + \frac{1}{\pi} \lambda^2 \delta'(z) - \frac{1}{8} \lambda \delta''(z)
\]

\[
- \frac{1}{24\pi} \frac{\delta'''}(z) + \frac{1}{12\pi} \ln(2\lambda) \delta''(z) + O \left( \frac{1}{\lambda} \right).
\] (52)

Similarly, the relevant function for the analysis of (45) is

\[
f_2(z) = z \tan^{-1} (2z) \theta(x) = \frac{\pi}{2} z - \frac{1}{2} + \frac{1}{24} \frac{1}{z^4} + O \left( \frac{1}{z^4} \right).
\] (53)
The moment expansion theorem says that
\[ f_2(\lambda z) \sim \frac{\pi}{2} \theta(\lambda z) - \frac{1}{2} \theta(\lambda z) + \frac{1}{24} \mathcal{P} \left( \frac{\theta(\lambda z)}{\lambda z} \right) \]
\[ + \sum_{j=0}^{\infty} (-1)^j \mu_j(f_2) \frac{\delta^{(j)}(\lambda z)}{j!} + \mathcal{O}\left(\frac{1}{\lambda}\right) \]
\[ = \frac{\pi}{2} \lambda \theta(z) - \frac{1}{2} \theta(z) + \frac{1}{24 \lambda^2} \mathcal{P} f\left( \frac{\theta(z)}{\lambda^2} \right) - \frac{1}{24 \lambda^2} \ln \lambda \delta(z) \]
\[ + \sum_{j=0}^{\infty} (-1)^j \mu_j(f_2) \frac{\delta^{(j)}(z)}{j! \lambda^{j+1}} + \mathcal{O}\left(\frac{1}{\lambda}\right). \tag{54} \]

Finally, the relevant moment expansion coefficients this time are
\[ \mu_0(f_2) = \int_0^\infty z \tan^{-1}(2z) \, dz = \frac{\pi}{16}, \tag{55} \]
\[ \mu_1(f_2) = \int_0^\infty z^2 \tan^{-1}(2z) \, dz = \frac{1}{72} + \frac{1}{24} \ln 2. \tag{56} \]

Forming the correct linear combination of these terms, we have
\[ \tilde{h}_{11}(z) = \lambda^3 \theta(z) - \frac{1}{2} \lambda^2 \theta(z) + \frac{1}{12 \pi} \mathcal{P} f\left( \frac{\theta(z)}{\lambda^2} \right) + \frac{1}{8} \lambda \delta(z) \]
\[ - \frac{2}{\pi} \left[ \frac{1}{72} \ln 2 + \frac{1}{24} \ln 2 \right] \delta(z) - \frac{1}{12 \pi} \ln \lambda \delta(z) + \mathcal{O}\left(\frac{1}{\lambda}\right) \]
\[ = \lambda^3 \theta(z) - \frac{1}{2} \lambda^2 \theta(z) + \frac{1}{12 \pi} \mathcal{P} f\left( \frac{\theta(z)}{\lambda^2} \right) + \frac{1}{8} \lambda \delta(z) - \frac{1}{36 \pi} \delta(z) \]
\[ - \frac{1}{12 \pi} \ln(2\lambda) \delta(z) + \mathcal{O}\left(\frac{1}{\lambda}\right). \tag{57} \]

By taking the second derivative of the solution (57), according to the Hadamard \( \mathcal{P} f \) formulas, we verify that the differential equation (52) is satisfied. This completes the counterpart for the pressure of the energy calculation in [14].

Next we want to see what happens when we use \( y \) as the cutoff parameter. It is neutral with respect to both the time–energy and the \( x–p_1 \) dimensions. The pertinent formulas are (24). One immediately sees that the \( p_1 \) calculation is identical to what we just did, with \( y \) in the role of \( t \), and also that \( \rho \) in this case is just the negative of \( p_1 \). So we do not need to do any more calculating, and we have three new, agreeing results (two for \( p_1 \) and one for \( \rho \)) that outvote the old formula for \( \rho \) in [14] and (27). Similarly, we have formulas for \( \tilde{h}_{11} \) and \( \tilde{h}_{00} \) that now seem more trustworthy than the old formula for \( h_{00} \) in [14] and (28).

**Comparison of old and new energy formulas.** The old formula (43) for the distributional limit of \( h_{00} \), based on the \( t \) cutoff, was
\[ \tilde{h}_{00}^{\text{old}}(z) = 2\lambda^3 \theta(z)z - \frac{1}{\pi} \lambda^2 \theta(z) - \frac{1}{12 \pi} \mathcal{P} f\left( \frac{\theta(z)}{z^2} \right) - \frac{1}{18 \pi} \delta(z) + \frac{1}{12 \pi} \ln(2\lambda) \delta(z). \tag{58} \]
The new one, based on the \( y \) cutoff, is the negative of (57):
\[ \tilde{h}_{00}^{\text{new}}(z) = -\lambda^3 \theta(z)z + \frac{1}{\pi} \lambda^2 \theta(z) - \frac{1}{12 \pi} \mathcal{P} f\left( \frac{\theta(z)}{z^2} \right) - \frac{1}{8} \lambda \delta(z) \]
\[ + \frac{1}{36 \pi} \delta(z) + \frac{1}{12 \pi} \ln(2\lambda) \delta(z). \tag{59} \]
Table 1 Ratio $\rho_{\text{old}} / \rho_{\text{new}}$.

| Term                        | Ratio of coefficients |
|-----------------------------|-----------------------|
| $\lambda^3 \theta(z)$      | -2                    |
| $\lambda^2 \theta(z)$      | -1                    |
| $\lambda \delta(z)$        | 0                     |
| $\ln(2\lambda) \delta'(z)$ | 1                     |
| $\delta'(z)$               | -2                    |
| $\mathcal{P} \int \left( \frac{\theta(z)}{z^2} \right)$ | 1                     |

The structure is exactly the same (except for the accidental absence of a $\delta(z)$ term from (58)). Only the numerical coefficients are different; their ratios are simple integers, shown in Table 1. The coefficients are the same for the regularized $\theta(z)/z^2$ term, which is the only term describing a spatially extended vacuum effect of the quantum field theory. All the other terms (apart from the bare $\delta'(z)$ terms, which could in each case be absorbed by rescaling the argument of the logarithm) involve powers or logarithm of the large cutoff parameter $\lambda$. The spirit of renormalization is to regard these terms as part of the wall, not the field, and to replace the $\lambda$-dependent factors by arbitrary finite constants. Note that the two terms involving $\theta(z)$ can be regarded as part of the solution of the homogeneous Einstein equation, and the other two terms are localized on the wall. A full discussion of the homogeneous solution is beyond the scope of this paper.

2. Caveats and implications: a dialogue

Salviati: Friends, I am perplexed by the paradoxical behavior of the energy and pressure in my model. The change in energy as the side of a box moves does not match the force on the side. Even the sign is wrong!

Simplicio: I do not understand why you are so upset. We know that the divergent terms in vacuum energy are unphysical and must be discarded. The divergences you are looking at are not logarithmic, so in an analytic regularization scheme (dimensional regularization or zeta functions) they do not arise in the first place. Also, those methods are Lorentz-covariant, whereas point-splitting in any particular direction is not; it is not surprising that the latter gives noncovariant results for the cutoff-dependent terms.

Sagredo: I disagree with Simplicio about the correct way to think about divergences in Casimir calculations. (With a realistic boundary the ‘divergences’ will be finite but nonzero, and they have physical content.) But I, too, think that your ‘paradox’ is a nonproblem, for a whole slew of reasons. First, in the calculation symbolized by figure 1 you have completely ignored the space to the right of the test wall. Surely it is obvious that the force $F$ is canceled by an equal and opposite force from the other side. Second, your scalar field is a toy that does not exist in the real world. Nature’s massless field is electromagnetism, and one of the most famous properties of electromagnetic Casimir energy is that the surface divergences you are studying vanish there, because the electric and magnetic contributions cancel. Also, there is no surface divergence for a flat boundary and a scalar field if $\xi = \frac{1}{2}$. Third, your cutoff is not a physical model of a realistic boundary; it is just an ad hoc procedure. If you use a noncovariant or otherwise unphysical cutoff, you must be prepared to insert unphysical counterterms; they are not a physical pathology but an artifact of a suboptimal formulation of the problem.
Salviati: Give me a few minutes to respond to this barrage! First, this anomaly was first noticed in calculations for a thin reflecting spherical shell [31]. In that case the force in question acts on the sphere itself, not a test surface perpendicular to it. The leading term in the surface energy per unit area is the same for convex and concave surfaces, and the area (hence the volume occupied by the surface energy) depends on radius in the same way. So from the energy point of view, the inside and outside forces should be in the same direction! And indeed, the direct calculation of force from pressure gave that result, but with that same discrepant factor, $-\frac{1}{2}$ [16].

Simplicio: Why not just show us the sphere calculations?

Salviati: They are rather complicated (and computer-assisted) and not yet available in publishable form. Instead, I have shown you that the same phenomenon occurs in the much simpler rectangular configuration. My belief is that energy-pressure balance ought to hold for each side of the test wall separately, since they are effectively independent systems (especially if the test wall is itself another perfectly reflecting wall). I do not know or care what exists on the other side of the wall. But if you insist that I analyze a thin partition, taking both sides into account, I can imagine scenarios where the two sides do not balance. Recall that the surface energy for a Neumann plate has the opposite sign from a Dirichlet plate. So if the plate in figure 1 has the Neumann property for $x > 0$, the resulting force on the test wall will be in the same direction as that from the Dirichlet plate on the other side.

Sagredo: Whoa! Does your plate magically change from one material to another as the test wall moves?

Salviati: At the instant portrayed in figure 1, the force is as I have described. What happens after the test wall has moved a finite distance is not really relevant. But anyway, I said I can 'imagine' these situations, not create them in a lab. Perhaps the reflecting media stretch and shrink as the test wall is moved, but in such a way that their reflective properties do not change. I see no reason why such a situation could not exist in principle; we're talking here about preserving a very general physical principle. Or perhaps the Neumann material is a thin strip that winds around a spool as the test wall moves to positive $x$.

Sagredo: But the boundary of the Neumann substance still exists even if it is wound into a spiral. This reminds me of your somewhat ill-fated pistol design [15]. Leaving aside the fact that the pistol sucked in its bullet instead of firing it, you had to admit that two conductors in close contact are not the same thing as a solid block of conducting material; rather, one has a limiting case of a small gap between conductors, and in the gap the Casimir energy is very important.

Salviati: Yes, but our point was that the idealized theory simply must break down when the gap is very small (atomic dimensions), even if we do not know exactly how. I admit that the spool scenario has complicating factors. How about this one: there is no boundary surface at all along the $z$ axis at positive $x$. Instead, (to use electrical terminology) the right side of the apparatus is filled with a liquid conductor—mercury—which is pushed out of the way in the $z$ direction into a large tank as the test wall moves. Then there is no boundary vacuum energy in the right half. My point is that there is a vast variety of things that could exist on the right of the wall, and no way to prove that they all give the same force on the wall. Anyway, this rectangular scenario is just a simple model of a more general problem that cannot be argued away. I have already reported our result about the sphere. My reply to the point about electromagnetic surface energy is similar: The leading surface terms cancel there, but for curved surfaces there are higher-order terms that do not, and the same issue will probably arise.

Simplicio: You said something that bothered me. You want to make your test wall into a perfect reflector in its own right. So does not its vacuum energy need to be counted? Also,
you started by calling your apparatus a ‘box’, implying that it has walls on all sides instead of extending to infinity as it does in your figure. But then your argument for the correctness of the equation $p_3 = 0$ breaks down. You argued that moving the boundary at $z = 0$ would merely push the vacuum energy around instead of changing its total value. That is not true if the cavity’s length is changing. And your argument for (19) explicitly assumed that the cavity’s length in the $x$ direction is finite and changing! You cannot have it both ways.

Salviati: You are right. For a convincing and complete treatment we should consider a bounded system—a rectangular box—and study the energy and pressure associated with all those walls. In fact, that has been done (see section 3). The analogue of the $p_3$ argument does work out, but contributions from infinitely many image sources must be combined to get energy–pressure balance.

Sagredo: How seriously do you take your model physically?

Salviati: I do not know. There are two philosophies one could take. First, one might hope that the finite-cutoff theory (possibly including some finite, physically motivated counterterms) could be a physically plausible model of a real boundary, including its gravitational effects, at least at distances not extremely close to the wall. The prototype I have in mind is the ‘hard core’ in the Lennard–Jones potential, which nuclear physicists regard as adequate when they do not need to probe the chromodynamical structure of nucleons at short distances. Surely in that situation a theory that violates energy-pressure balance is unacceptable. Neutral point-splitting gives a theory that is acceptable in this respect, but I cannot regard it as a logically sound, long-term solution, since it amounts to changing the rules of the game in each situation until I get a result that I like. Its only justification is that, unlike less contrived alternatives, it does not immediately produce results that are obviously wrong. In the long run, detailed study of the physics of real boundaries will be necessary.

The other point of view is that the cutoff is a mathematical means to an end, which is a limiting theory with the cutoff removed. At the intermediate stage a violation of energy-pressure balance may be tolerated, so long as the final theory is physically acceptable. As you both remarked, an unphysical regularization can necessitate unphysical counterterms, so that the undesirable terms appear in the final equations with coefficient 0 (i.e., not at all). I could live with that, but it is certainly highly unaesthetic.

Simplicio: So, you have left us all with several questions to think about.

• Are both of those philosophies physically tenable? Or one, or neither?
• What is the meaning of terms proportional to $\delta$ and $\delta'$ in a component of a metric tensor? Do we learn anything more than if we had just discarded them?
• When you put in numbers, is the vacuum stress sufficiently small to justify your use of linearized Einstein equations? Must you add counterterms to assure that?
• In particular, I am concerned by the terms in $\tilde{h}_{\mu\nu}$ that are linear in $\lambda$. What is the significance of the linear and constant terms (solutions of the homogeneous equation)? To what extent can one exclude them by boundary conditions or remove them by gauge (coordinate) transformations?
• Naïvely I would expect a term that is independent of $\lambda$ to be ‘physical’ and hence well defined. But there is one such term in table 1 for which the ratio is not unity. Should we be worried about that? I can see that that term could be regarded as part of the logarithmic term, although the coefficient ratio for the logarithmic term itself is unity.
• What will you do near the corner of a parallelepiped, where there is no neutral direction?
3. Energy density and pressure inside a rectangular box

In this section we give a more careful accounting of the total energy associated with two perpendicular boundaries as in figure 1. The test wall itself is made into a perfectly reflecting boundary, and to make all integrals proper two other such walls are introduced parallel to the original ones. The cavity is thus a finite box as shown in figure 2. More precisely, the cavity is an infinite rectangular waveguide, because we have no need to introduce boundaries parallel to the $x$–$z$ plane. Therefore, in the following, ‘energy’ should be read as ‘energy per unit length’ (in the $y$ direction), and similarly for ‘force’.

The wave equation in this space can be solved by a sum over infinitely many images. The analysis follows rather closely the two-dimensional theory in [15, 14], but the details of the expressions are different. As shown in figure 2, there are four classes of images. Points periodically displaced (with periods twice the length, $a$, and width, $b$, of the box) from the source point at $r' = (x', 0, z')$ contribute to the cylinder kernel $T(t, r, r')$ the terms

$$T_{Pmn} = -\frac{1}{2\pi} \frac{1}{t^2 + y^2 + (2ma + x - x')^2 + (2nb + z - z')^2}. \quad (60)$$

The term $T_{P00}$ is the free kernel $T_0$ and hence should be omitted as it was in (15). Points reflected horizontally (i.e., reflected through a vertical line in the lattice of figure 2), together possibly with a periodic displacement in the vertical direction, contribute

$$T_{Hmm} = +\frac{1}{2\pi} \frac{1}{t^2 + y^2 + (2ma - x - x')^2 + (2nb + z - z')^2}. \quad (61)$$

Similarly, vertical reflections produce

$$T_{Vnn} = +\frac{1}{2\pi} \frac{1}{t^2 + y^2 + (2ma + x - x')^2 + (2nb - z - z')^2}. \quad (62)$$

Finally, reflections in both dimensions give rise to terms

$$T_{Cnn} = -\frac{1}{2\pi} \frac{1}{t^2 + y^2 + (2ma - x - x')^2 + (2nb - z - z')^2}. \quad (63)$$

(Such a point is connected to the source point by a straight line that passes through an intersection of the horizontal and vertical lines of the lattice. When folded back into the box, this line can be characterized as a classical path or optical ray that is reflected from a corner of the box. All the other terms have similar classical-path interpretations.)

On the plates at $x = 0$ and $x = a$ each term contributes a pressure in the $x$ direction according to (11) (with $j = 1$, which will be henceforth understood). One finds that (for any $x$)
\[ p_{\text{Hmn}} = 0 = p_{\text{Cmn}}, \]  

\[ p_{\text{Pmn}} = \frac{8m^2a^2}{\pi^2(t^2 + y^2 + 4m^2a^2 + 4n^2b^2)^3} + \frac{1}{2\pi^2(t^2 + y^2 + 4m^2a^2 + 4n^2b^2)^2}, \]  

\[ p_{\text{Vmn}} = \frac{8m^2a^2}{\pi^2(t^2 + y^2 + 4m^2a^2 + 4(z - nb)^2)^3} - \frac{1}{2\pi^2(t^2 + y^2 + 4m^2a^2 + 4(z - nb)^2)^2}. \]  

Note that \( p_{\text{Pmn}} \) and \( p_{\text{Vmn}} \) are independent of \( x \).

Similarly, the energy density due to each term can be computed by (10). For certain purposes, however, it is convenient to take only one \( t \) derivative before evaluating one of the sums or integrals in closed form. In particular, consider (at \( x' = x, z' = z \))

\[ T_{\text{Hmn}} \equiv \frac{\partial T_{\text{Hmn}}}{\partial t} = -\frac{t}{\pi^2(t^2 + y^2 + 4(x - ma)^2 + 4n^2b^2)^2}. \]  

One has

\[ \sum_{m=-\infty}^{\infty} \int_{-a}^{a} T_{\text{Hmn}} \, dx = -\sum_{m=-\infty}^{\infty} \int_{m}^{(m+1)a} \frac{t \, dx}{\pi^2(t^2 + y^2 + 4x^2 + 4n^2b^2)^2}, \]

\[ = -\int_{-\infty}^{\infty} \frac{t \, dx}{\pi^2(t^2 + y^2 + 4x^2 + 4n^2b^2)^2}. \]  

which is independent of \( a \). Therefore, the \( H \)-energy in the box is unchanged whenever one of the vertical walls moves, which is precisely consistent with (64) (summed over \( m \)). Note, however, that it was essential to consider all values of \( m \) at once; for \( m = 0 \), for instance, the energy in the box is not exactly constant because a part of the surface energy distribution created by one wall is pushed through the opposite wall.

The \( V \)-energy density is easily seen to be independent of \( x \), since, in analogy to (67),

\[ T_{\text{Vmn}} = -\frac{t}{\pi^2(t^2 + y^2 + 4m^2a^2 + 4(z - nb)^2)^2}. \]  

Therefore, the total \( V \)-energy is equal to that density times \( a \); so its derivative with respect to \( a \) has two terms, one just equal to the energy density,

\[ \rho_{\text{Vmn}} = -\frac{1}{2} \frac{\partial T_{\text{Vmn}}}{\partial t}, \]  

and the other equal to

\[ a \frac{\partial}{\partial a} \rho_{\text{Vmn}}. \]  

One calculates

\[ \rho_{\text{Vmn}} = -\frac{2t^2}{\pi^2(t^2 + y^2 + 4m^2a^2 + 4(z - nb)^2)^3} + \frac{1}{2\pi^2(t^2 + y^2 + 4m^2a^2 + 4(z - nb)^2)^2}. \]  

If \( t = 0 \) (the neutral cutoff), then (71) equals the negative of the first term of (66), and (70) is the negative of the second term. Thus the principle of energy balance (19) holds. If \( t \neq 0 \) and \( y = 0 \) (the ultraviolet cutoff), the first term in (72) survives and creates the pressure anomaly (specifically, from the terms with \( m = 0 \)).

A calculation similar to (68) yields the formula

\[ \frac{t^2 - y^2}{8\pi(t^2 + y^2)^2}. \]  

(73)
for the total C-energy. It is independent of \( a \), hence consistent with (64). In dimension 2 the corresponding term vanished entirely. The difference is that in dimension 3 the ‘corner’ is really an edge of a parallelepiped, whose true corners would indeed give no energy in a cutoff theory.

The P terms are nondivergent, and of course for \( \rho \) they give the standard (finite and constant) bulk Casimir energy density of the waveguide.

Acknowledgments

This research was supported by National Science Foundation Grants PHY-0554849 and PHY-0968269. We thank Lev Kaplan, Kim Milton, Martin Schaden, and the participants in the 2012 Oklahoma–Texas–Louisiana Quantum Vacuum Meeting for helpful discussions that raised many of the issues aired in section 2. Klaus Kirsten as coauthor of [14] contributed greatly to formulas (42)–(43), thus setting the pattern for the calculations that follow them.

Appendix. The pressure anomaly in dimension \( d \)

When the spatial dimension is \( d (\geq 2) \), the free cylinder kernel is, up to a constant factor,

\[
T_0 \propto (t^2 + |r - r'|^2)^{-(d-1)/2}.
\]

Thus the reflected kernel generalizing (15) is

\[
T \propto (t^2 + |r_\perp - r'_\perp|^2 + (z + z')^2)^{-(d-1)/2}.
\]

Therefore, following (10)–(11) and (16)–(18), one finds

\[
\rho \propto -M(d)^{(d+3)/2}(-t^2d + r_\perp^2 + 4z^2), \tag{A.3}
\]

\[
p_1 \propto M(d)^{(d+3)/2}(t^2 - x^2d + \sum_{j\neq 1} x_j^2 + 4z^2), \tag{A.4}
\]

with \( z = x_d, x = x_1 \), and

\[
M(d) = t^2 + r_\perp^2 + 4z^2. \tag{A.5}
\]

The missing numerical constants in (A.3) and (A.4) are the same, namely [30]

\[
\frac{1}{2}C(d) = \frac{1}{2}\pi^{-(d+1)/2}\Gamma \left( \frac{d + 1}{2} \right). \tag{A.6}
\]

We are to choose one of the coordinates, \( t \) or \( x_j (j \neq d) \), to serve as the cutoff and to set the others to 0. Call that coordinate \( w \). Then we are confronted with integrals of the type

\[
I_w = \int_0^\infty \frac{dz}{w^2 + 4z^2} \frac{-w^2d + 4z^2}{(w^2 + 4z^2)^{(d+3)/2}} \tag{A.7}
\]

and

\[
I = \int_0^\infty \frac{dz}{w^2 + 4z^2} \frac{1}{(w^2 + 4z^2)^{(d+1)/2}} = \int_0^\infty \frac{dz}{(w^2 + 4z^2)^{(d+3)/2}}. \tag{A.8}
\]

The form (A.8) applies if \( w \) is ‘neutral’ with respect to the component of energy or pressure concerned; (A.7) applies if \( w \) is not neutral (\( w = t \) for energy density, \( w = x \) for \( p_1 \), etc).

Note that

\[
I_w = I - (d + 1)w^2 \int_0^\infty \frac{1}{(w^2 + 4z^2)^{(d+3)/2}}. \tag{A.9}
\]
Because
\[
\frac{d}{dz} \left( (w^2 + 4z^2)^{(d+1)/2} \right) = \frac{(d+1)w^2}{(w^2 + 4z^2)^{(d+3)/2}} - \frac{d}{(w^2 + 4z^2)^{(d+1)/2}},
\]
the second term in (A.9) equals
\[
- \int_{0}^{\infty} dz \frac{d}{(w^2 + 4z^2)^{(d+1)/2}} = -d \cdot I.
\]
Thus
\[
I_w = -(d - 1) I. \tag{A.10}
\]
When \(d = 3\), (A.10) yields the anomalies in (20) and (22), and (A.8) yields the value in (20).
When \(d = 2\), the anomaly reduces to a sign change, which was overlooked in the discussion near the top of p 18 of [15].

References

[1] Lamoreaux S K 1999 Resource letter CF-1: Casimir force Am. J. Phys. 67 850–61
[2] Milton K A 2011 Resource letter VWCPF-1: van der Waals and Casimir–Polder forces Am. J. Phys. 79 697–711
[3] Dalvit D, Milonni P, Roberts D and Rosa F (ed) 2011 Casimir Physics (Lecture Notes in Physics vol 834) (Berlin: Springer)
[4] Asorey M, Bordag M and Elizalde E (ed) 2012 Int. J. Mod. Phys. A 27 (Special Issue—Selected Papers from 10th Conf. on Quantum Field Theory Under the Influence of External Conditions (QFEXT11) (Spain, 18–24 Sep. 2011))
[5] Casimir H B G 1948 On the attraction between two perfectly conducting plates K. Nederl. Akad. Wet., Proc. Sec. Sci. 51 793–5
[6] Brown L S and Maclay G J 1969 Vacuum stress between conducting plates: an image solution Phys. Rev. 184 1272–9
[7] Boyer T H 1968 Quantum electromagnetic zero-point energy of a conducting spherical shell and the Casimir model for a charged particle Phys. Rev. 174 1764–76
[8] Deutsch D and Candelas P 1979 Boundary effects in quantum field theory Phys. Rev. D 20 3063–80
[9] Fulling S A 2003 Systematics of the relationship between vacuum energy calculations and heat-kernel coefficients J. Phys. A: Math. Gen. 36 6857–73
[10] Bernasconi F, Graf G M and Hasler D 2003 The heat kernel expansion for the electromagnetic field in a cavity Ann. Henri Poincaré 4 1001–13
[11] Estrada R and Fulling S A 2002 How singular functions define distributions J. Phys. A: Math. Gen. 35 3079–89
[12] Milton K A, Fulling S A, Parashar P, Romeo A, Shahesh K V and Wagner J A 2008 Gravitational and inertial mass of Casimir energy J. Phys. A: Math. Theor. 41 164052
[13] Milton K A, Parashar P, Shahesh K V and Wagner J 2007 How does Casimir energy fall? II J. Phys. A: Math. Theor. 40 10935–43
[14] Estrada R, Fulling S A, Kaplan L, Kirsten K, Liu Z H and Milton K A 2008 Vacuum stress-energy density and its gravitational implications J. Phys. A: Math. Theor. 41 164055
[15] Fulling S A, Kaplan L, Kirsten K, Liu Z H and Milton K A 2009 Vacuum stress and closed paths in rectangles, pistons, and pistoles J. Phys. A: Math. Theor. 42 155042
[16] Fulling S A 2010 Vacuum energy density and pressure near boundaries Int. J. Mod. Phys. A 25 2364–72 (QFEXT09)
[17] Milton K A 2011 Local and global Casimir energies: divergences, renormalization, and the coupling to gravity Casimir Physics (Lecture Notes in Physics vol 834) (Berlin: Springer) pp 39–95
[18] Fulling S A, Milton K A and Wagner J 2012 Energy density and pressure in power-wall models Int. J. Mod. Phys. A 27 1260009 (QFEXT11)
[19] Christensen S M 1976 Vacuum expectation value of the stress tensor in an arbitrary curved background: the covariant point-separation method Phys. Rev. D 14 2490–301
[20] Hollenstein L, Jaccard M, Maggiore M and Mitsou E 2012 Zero-point quantum fluctuations in cosmology Phys. Rev. D 85 124031
[21] Ford I H and Svaiter N F 1998 Vacuum energy density near fluctuating boundaries Phys. Rev. D 58 065007
[22] Pippard A B 1964 The Elements of Classical Thermodynamics (Cambridge: Cambridge University Press)
[23] Schutz B F 1985 A First Course in General Relativity (Cambridge: Cambridge University Press)
[24] Estrada R and Kanwal R P 2002 A Distributional Approach to Asymptotics: Theory and Applications (Boston, MA: Birkhäuser)

[25] Estrada R 1998 The Cesàro behavior of distributions Proc. R. Soc. A 454 2425–43

[26] Grossman A, Loupias G and Stein E M 1968 An algebra of pseudodifferential operators and quantum mechanics in phase space Ann. Inst. Fourier 18 343–68

[27] Estrada R and Kanwal R P 1990 A distributional theory of asymptotic expansions Proc. R. Soc. A 428 399–430

[28] Estrada R 2003 The non-existence of regularization operators J. Math. Anal. Appl. 286 1–10

[29] Balian R and Bloch C 1972 Distribution of eigenfrequencies for the wave equation in a finite domain: III Ann. Phys. 69 76–160

[30] Liu Z H and Fulling S A 2006 Casimir energy with a Robin boundary: the multiple-reflection cylinder-kernel expansion New J. Phys. 8 234

[31] Fulling S and Schaden M 2012 unpublished