Quantum Fingerprinting with a Single Particle

S. Massar

Service de Physique Théorique, Université Libre de Bruxelles,
C.P. 225, Boîte du Triomphe, 1050 Bruxelles, Belgium

(Dated: April 1, 2022)

We show that the two slit experiment in which a single quantum particle interferes with itself can be interpreted as a quantum fingerprinting protocol: the interference pattern exhibited by the particle contains information about the environment it encountered in the slits which would require much more communication to learn classically than is required quantum mechanically. An extension to the case where the particle has many internal degrees of freedom is suggested and its interpretation is discussed in detail. A possible experimental realization is proposed.

PACS numbers: 03.65.Ta, 03.67.Hk, 42.50.Xa

I. INTRODUCTION

As first understood by de Broglie \cite{1}, a single quantum particle should exhibit both particle and wave-like properties. The simplest way to illustrate the wave-particle duality of quantum mechanics is the two slit experiment in which a single particle can follow one of two paths and then interfere with itself. When a single particle is sent through the setup a single click is registered, as is expected of individual particles. But upon accumulating statistics an interference pattern emerges, putting in evidence their wave-like behavior. The interference pattern only occurs if it is impossible to know which path the particle took. The two slit experiment is one of the fundamental Gedanken-Experimenten discussed by the fathers of quantum mechanics \cite{2}. It has by now been realized with a wide variety of physical systems, such as electrons \cite{3}, neutrons \cite{4}, photons \cite{5}, single atoms \cite{6} and even simple molecules \cite{7}.

The aim of this paper is to reexamine the two slit experiment in the light of recent advances in quantum information theory, and in particular the quantum fingerprinting protocol introduced in \cite{8}. We will show that to reproduce the interference pattern produced by a single particle would require much more classical communication than is necessary quantum mechanically. This allows one to put constraints on possible classical theories which could underly quantum mechanics.

The situation we will consider is the following (see figure 1). A single particle impinges on a beam splitter. If the particle is transmitted it goes to one party, Alice. If the particle is reflected it goes to another party, Bob. Thus the state is

\[
(|1_A⟩ + |1_B⟩) / √2
\]

where the subscript \(A\) or \(B\) denotes where the particle is. Alice and Bob also receive a classical input. Alice receives as input \(x\) and Bob receives as input \(y\). Alice and Bob then carry out a local unitary transformation on the particle which depends on their input. Thus the state becomes

\[
(U_A(x)|1_A⟩ + U_B(y)|1_B⟩) / √2
\]

We consider the possibilities that the particle either has or does not have internal degrees of freedom. If the particle does not have internal degrees of freedom, then the transformations \(U_A(x) = e^{iφ_A(x)}\) and \(U_B(x) = e^{iφ_B(x)}\) are simply phases. If the particle does have internal degrees of freedom, then the transformations \(U_A\) and \(U_B\) can act on the internal degrees of freedom. In general they transform the state to

\[
U_A(x)|1_A⟩ = \sum_{i=1}^{d} α_i(x)|i_A⟩, \quad U_B(y)|1_B⟩ = \sum_{i=1}^{d} β_i(y)|i_B⟩
\]

where \(d\) is the dimensionality of the internal Hilbert space. These internal degrees of freedom could for instance be the spin, momentum, or energy of the particle. Then, exactly as in the traditional two slit experiment, the particle goes to a third party, which we call the Referee, who makes the trajectories coming from Alice interfere with the trajectories coming from Bob. We will suppose that this is done using a beam splitter which combines the two incoming beams as:

\[
|i_A⟩ \rightarrow (|i_E⟩ + |i_N⟩) / √2, \quad |i_B⟩ \rightarrow (|i_E⟩ - |i_N⟩) / √2,
\]

to yield the state

\[
\frac{1}{2} \sum_{i=1}^{d} (α_i(x) + β_i(y))|i_E⟩ + (α_i(x) - β_i(y))|i_N⟩.
\]

The Referee then measures the particle in the basis \(|i_E⟩, |i_N⟩\). We have denoted the output ports of the final beam splitter by \(E\) (for “Equal”) and \(N\) (for “Not Equal”) because if \(U_A = U_B\) the particle exits by port \(E\), whereas if \(U_A = -U_B\) the particle exits by port \(N\). We shall show that if \(U_A\) and \(U_B\) are adequately chosen the interference pattern of the particle with itself allows

* Also at: Ecole Polytechnique, C.P. 165, Université Libre de Bruxelles, 1050 Brussels, Belgium
the Referee to learn information about the inputs \( x \) and \( y \) which he could only learn using much more classical communication than the amount of quantum communication that takes place in the two slit experiment.

In summary, when the particle does not have internal degrees of freedom this is the well known two slit experiment, although presented in a slightly different way. When the particle does have internal degrees of freedom then this is a straightforward generalization of the two slit experiment.

II. CLASSICAL AND QUANTUM FINGERPRINTING

Before proceeding further we first recall what is known about classical and quantum fingerprinting protocols. In the protocols we consider there are three parties, Alice, Bob and a Referee. Alice gets as input an \( n \) bit string \( x \in \{0, 1\}^n \). Bob gets as input an \( n \) bit string \( y \in \{0, 1\}^n \). The aim is for the Referee to decide whether \( x = y \) or \( x \neq y \). The Referee can make mistakes, but the probability of a mistake must be less than \( \epsilon \) for some \( 1/2 > \epsilon > 0 \). (By repeating a protocol a constant times, the error probability can be made exponentially small if \( \epsilon \) is not very close to 1/2 to start. For instance \( \epsilon = 1/3 \) is sufficient.) The aim is for the Referee to reach a decision while minimizing the amount of communication between the parties. There are several different protocols which depend on the resources used by the parties. Let us enumerate these different possibilities.

1. Alice, Bob and the Referee do not have access to any randomness and must use a deterministic protocol. Then \( O(n) \) bits of classical communication are necessary.

2. One way classical communication from Alice to the Referee and one way classical communication from Bob to the Referee is allowed (the "simultaneous message passing" model). Alice and Bob share \( O(\log n) \) random bits. Then a protocol exists in which Alice sends \( O(1) \) bits to the Referee and Bob sends \( O(1) \) bits to the Referee.

III. QUANTUM FINGERPRINTING USING A SINGLE PARTICLE WITH NO INTERNAL DEGREES OF FREEDOM

Let us first show that even when the particle has no internal degrees of freedom, the two slit experiment realizes a non trivial quantum communication protocol. The simplest case is when Alice’s input consists of a single bit \( x = 0, 1 \) and Bob's input consists of a single bit \( y = 0, 1 \). Then the quantum strategy is simply:

\[
|1_A \rangle \rightarrow e^{i\pi x}|1_A \rangle ; \quad |1_B \rangle \rightarrow e^{i\pi y}|1_B \rangle .
\]

This strategy solves the quantum fingerprinting problem with no error and with a single ebit of communication since the Hilbert space of the particle is two dimensional. Classically, in the simultaneous message passing model, it is of course necessary for both parties to send their input to the referee, i.e. two bits of classical communication are required. Note that this is the case irrespective of whether the parties have shared randomness. We have generalized the above protocol to the case where the inputs \( x, y \in \{0, 1, 2\} \) are trits. The quantum strategies are

\[
|1_A \rangle \rightarrow e^{i2\pi x/3}|1_A \rangle ; \quad |1_B \rangle \rightarrow e^{i2\pi y/3}|1_B \rangle .
\]

One easily checks that with such a strategy the Referee makes no error when \( x = y \) and will recognize that \( x \neq y \) with probability \( 3/4 \). Thus, if one averages over the inputs, the average error probability is \( 1/6 \).

In order to compare this with classical fingerprinting in the simultaneous message passing model, we suppose that Alice can send one trit of classical information to the Referee, whereas Bob is limited and can only send one bit to the Referee. We will show that this is not enough communication to reproduce the quantum protocol. Let us first consider deterministic strategies. If Alice sends one bit of information, she can tell the Referee what is her input. On the other hand with one bit of information Bob can only divide his inputs into two sets. One easily checks that there are necessarily at least two input pairs \( (x, y) \) (out of 9 possible pairs) for which the Referee will make a mistake. Thus averaged over the inputs the error probability is at least \( 2/9 \), which is greater than in the quantum protocol.

Note that in this case shared randomness does not allow the parties to do better than the quantum protocol. Indeed shared randomness can be viewed as allowing the parties to randomly choose a deterministic strategy. Since all deterministic strategies have an error probability of at least \( 2/9 \), the same is true of any probability distribution over deterministic strategies.
It is remarkable how economical the quantum protocol is compared to the classical protocol: it uses a single qubit of communication, and does better than any classical protocol in the simultaneous message passing model in which a bit and a trit are communicated. To do better than the quantum protocol two trits must be communicated.

IV. QUANTUM FINGERPRINTING USING A SINGLE PARTICLE WITH INTERNAL DEGREES OF FREEDOM

We now generalize the above protocols to the case where the particle has internal degrees of freedom. The inputs are taken to be $n$ bit strings. Let $E$ be an error correcting code which encodes words of length $n$ into words of length $m$, and such that if two words are distinct, then the Hamming distance between the encoded words is at least $t$. There exist such codes with the property that for sufficiently large $n$, $m \leq \mu n$ and $t \geq \nu m$ where $\mu > 1$ and $\nu > 0$ are independent of $n$. For instance in the case of Justesen codes $\nu = \frac{1}{\log n}$ for any $\mu > 2$ \cite{justesen}. We denote by $e(x)$ the encoded version of input $x$. The $i$’th bit of $e(x)$ is denoted $e_i(x)$. The encoding operations carried out by Alice and Bob are

$$U_A(x)|1_A\rangle = \frac{1}{\sqrt{m}}\sum_{i=1}^{m} (-1)^{e_i(x)}|i_A\rangle,$$

$$U_B(y)|1_B\rangle = \frac{1}{\sqrt{m}}\sum_{i=1}^{m} (-1)^{e_i(y)}|i_B\rangle.$$ 

When the particle is received by the referee, he lets the paths coming from Alice and the paths coming from Bob interfere as described above and measures whether the particle is in the subspace spanned by the $|i_E\rangle$ states or in the subspace spanned by the $|i_N\rangle$ states. If $x = y$ then the particle necessarily is in one of the $|i_E\rangle$ states. On the other hand if the inputs are different, then the probability that the particle is in one of the $|i_N\rangle$ states is at least $t/m \geq \nu > 0$. Thus the probability of error is less than $1 - \nu$. By repeating the protocol $k$ times, with $k \geq \log \epsilon/\log(1 - \nu)$, one can make the error probability less than $\epsilon$.

V. INTERPRETATION

We now turn to the interpretation of the above interference experiments. To this end we must compute the quantum and classical capacities of the communication channels used, and we must compute how much shared randomness exists between the parties. One of the aims of this discussion is to show that these experiments can constrain the properties of any classical theory which could underly quantum mechanics, but in the context of a single particle, in a similar way that EPR type experiments can constrain the properties of any classical theory which could underly quantum mechanics in the context of two entangled particles.

First we characterize the quantum experiment. To this end we need to compute how many qubits are transmitted. When Alice and Bob send the particle to the Referee, the particle belongs to a Hilbert space of dimension $2m$ (since there are $m$ paths from Alice to the Referee and $m$ paths from Bob to the Referee). The total capacity of the channels used to communicate to the referee is therefore $1 + \log m$ qubits.

Let us now compare this quantum protocol with classical protocols which achieve the same fingerprinting. First of all we note that since the communication from Alice and Bob to the Referee uses $1 + \log m$ qubits, Holevo’s theorem \cite{holevo} implies that at most $1 + \log m$ bits of classical communication could be transmitted during this phase. In order to carry out fingerprinting on $n$ bit inputs, the results stated above imply that classically either communication in all directions should be allowed (ie. one is not restricted to simultaneous message passing ), or the parties have prior shared randomness.

Concerning the first point, the only communication that takes place between the parties after the inputs is received is from Alice to the Referee and from Bob to the Referee. Hence it is natural to argue that no other communication between the parties is possible in the classical case either. It is possible to strengthen this argument by appealing to special relativity. Indeed let us suppose that the Referee is mid way between Alice and Bob in such a way that the distance between Alice and Bob is $L$, and the distance between Alice or Bob and the Referee is $L/2$. We can suppose that Alice and Bob receive their inputs $x$ and $y$ at time $t = 0$, and that the referee must provide an output before time $t = 3L/2c$. This implies that if no information can travel faster than the speed of light, then one must necessarily be in the simultaneous message passing model.

Let us now consider whether the parties have shared randomness. This shared randomness could presumably only be produced when the particle is sent to Alice and Bob. During this phase the state of the particle can be written as

$$\frac{(|1_A\rangle|vac_B\rangle + |vac_A\rangle|1_B\rangle)/\sqrt{2}}$$

where $|vac\rangle$ is the vacuum state. Hence this is an entangled state with one unit of entanglement (one ebit). By measuring whether or not they have the particle, Alice and Bob can generate one shared random bit. However a single shared random bit cannot decrease the amount of communication required for fingerprinting in the simultaneous message passing model by more than a factor $2\sqrt{2}$.

In summary quantum fingerprinting with a single particle seems to be incompatible with a classical description of the particle. If one insists on keeping a classical description then the underlying classical theory must have one of the following (surprising) properties.
1. A quantum particle following \( m \) different paths carries \( O(\sqrt{m}) \) bits of classical information (which is exponentially more than the amount of classical information obtained by measuring in which path is the particle).

2. A single particle in the state eq. (1) carries with it \( O(\log m) \) bits of shared randomness along the two paths taken by the particle (whereas quantum mechanics predicts that only one bit of shared randomness could be obtained by measuring the state).

3. The classical theory allows \( O(\log m) \) bits of superluminal communication if the particle follows \( m \) different paths.

The best known classical theory which reproduces the behavior of a single quantum particle is Bohm’s theory\(^\text{[17]}\) in which the particle follows a well defined classical trajectory, but is accompanied by a "pilot wave" which determines the trajectory the particle must take. Bohm’s theory realizes the first possibility outlined above since the pilot wave is a classical description of the quantum wave function and therefore contains \( O(m) \) bits of classical information. The possibility of realizing quantum fingerprinting with a single particle shows that it is impossible to imagine a "compressed" version of Bohm’s theory in which the pilot wave carries less than \( O(\sqrt{m}) \) bits.

Since the above discussion makes appeal to causality, it is interesting to compare it to Bell’s appeal to causality when interpreting EPR type experiments\(^\text{[18]}\). Bell showed that "local realistic theories" are incompatible with quantum mechanics. These are classical theories which respect special relativity and which allow the entangled particles to carry unlimited shared randomness with them. Many experiments on Bell type correlations have been carried out over the years, but no completely conclusive experiment has been realized. Quantum fingerprinting with a single particle is weaker than Bell’s tests of quantum correlations since it does not allow one to disprove local realistic theories. However it does allow bounds to be put on the minimum amount of randomness which must be carried by a single quantum particle, or on the minimum amount of superluminal communication which must take place between the parties.

**VI. EXPERIMENTAL REALIZATION OF QUANTUM FINGERPRINTING WITH A SINGLE PARTICLE**

We will now show that realising a two slit experiment with a single particle with a large number (of order \( 10^6 \)) internal degrees of freedom seems possible with present technology, but that more theoretical work is necessary in order to establish whether non trivial fingerprinting takes place under these conditions. The setup is illustrated in figure 1. The particle is taken to be a photon produced by a mode-locked laser. Such lasers produce trains of light pulses with very long coherence lengths (up to \( 10^{16} \) pulses may be coherent\(^\text{[19]}\)). Note that it is essential that all the pulses in the train are coherent otherwise one is not dealing with a pure state. The light produced by the mode locked laser is attenuated so that on average less than a single photon is present in the pulse train. If a single photon is present, then the state of light is

\[
\psi = d^{-1/2} \sum_{i=1}^{d} |i \rangle .
\]

This train of pulses then impinges on a beam splitter. The transmitted beam is sent to Alice, the reflected beam to Bob. Thus the state becomes

\[
\psi = \frac{1}{\sqrt{2}} \sum_{i=1}^{d} (|i_A \rangle + |i_B \rangle).
\]

Alice and Bob can encode their input into the train of pulses by using a phase modulator. Note that the large value for \( d \) mentioned above is certainly not realistic, and imperfections, such as dark counts from detectors, will limit \( d \) to much smaller values.

By coupling the pulses into optical fibers, it should be possible to separate Alice and Bob by distances of several kilometers. If the pulses are separated by 1ns, then with Alice and Bob separated by 10 km, corresponding to a time separation of 3\( \mu s \), it should be possible to have \( d \) of order \( 10^3 \) while keeping Alice and Bob’s encoding operations spatially separated. This value for \( d \) seems much more realistic, but is still impressive if one notes that it corresponds to approximately 10 qubits.

Let us compare these figures with the theoretical predictions. In \(\text{[18]}\) it is proven that in the simultaneous message passing model, if Alice and Bob have no shared randomness, if the referee uses a deterministic decision strategy, and if the error probability is less than 1\%, then the number \( a \) of bits of communication sent by Alice to the referee, and the number \( b \) of bits of communication sent by Bob to the referee, must obey \( ab \geq n/400 \). This implies that one at least of \( a \) and \( b \) must be greater than \( \sqrt{n}/20 \). With a single bit of shared randomness, Alice and Bob must send at least half this amount of communication\(^\text{[12]}\), i.e. at least \( \sqrt{n}/40 \) bits.

Let us suppose that in the protocol described above Alice and Bob use Justesen codes with the parameter \( \mu = 2 \), which implies \( \nu = 1/5 \). Thus the number of internal degrees of freedom of the particle must be \( 2n \), and the number of qubits communicated by each party in a single run of the protocol is \( 1 + \log_2 n \). In order for the error probability to be less than 0.01, the parties must repeat the protocol \( k = 67 \geq \ln(0.01)/\ln(1 - \nu) \) times. The total amount of quantum communication sent by each party is \( k(1 + \log_2 n) \). This must be smaller than \( \sqrt{n}/40 \) which is satisfied if \( n \) is larger than about \( 10^{10} \).

As mentioned above it is probably impossible to reach such large values of \( n \) in a realistic experiment and the
The value of $d = 10^3$ suggested above seems much more realistic. However in this case it is unclear whether one is doing non trivial fingerprinting. Nevertheless it may be possible, by optimising the proofs, to decrease the required values of $d$ significantly. In fact it is not unreasonable to conjecture that non trivial quantum fingerprinting is possible for all values of $d$, since we know that it is possible when $n = 1$ (particle with no internal degrees of freedom), and it is possible for sufficiently large $d > 10^{10}$.

VII. CONCLUSION

In summary we have shown that recent theoretical advances in quantum communication allow a new interpretation of one of the most famous experiments of quantum mechanics, namely the interference of a single particle with itself. Indeed the two slit experiment can be interpreted as a quantum fingerprinting experiment which requires more classical communication than is used quantum mechanically. A generalisation of quantum fingerprinting with a single particle to the case where the particle has internal degrees of freedom is proposed. The possibility of realising such an experiment using a photon in a train of many coherent pulses is also discussed. The interpretation of these experiments is discussed in detail.

Acknowledgements: I would like to thank Harry Burhman, Hein Roërig and Louis-Philippe Lamoureux for helpful discussions. Financial support from the Communauté Française de Belgique under grant ARC 00/05-251, from the IUAP programme of the Belgian government under grant V-18 and from the EU through project RESQ (IST-2001-37559) is gratefully acknowledged.

[1] L. de Broglie, Nature 112, 540 (1923).
[2] J. A. Wheeler and W. H. Zurek (eds), Quantum theory and measurement, Princeton University Press, 1983
[3] C. J. Davis and L. H. Germer, Nature 119, 558-560 (1927).
[4] H. v. Jr. Halban and P. Preiswerk, C. R. Acad. Sci. 203, 73-75 (1936).
[5] Ph. Grangier et al., Europhys. Lett 1 173(1986).
[6] P. Berman (ed.) Atom Interferometry (Academic, 1997).
[7] M. Arndt, O. Nairz, J. Vos-Andreae, C. Keller, G. Van der Zouw and A. Zeilinger, Nature 401, 680 (1999)
[8] H. Buhrman, R. Cleve, J. Watrous, R. de Wolf, Phys. Rev. Lett. 87, 167902 (2001)
[9] E. Kushilevitz and N. Nisan, Communication Complexity, Cambridge University Press, 1997
[10] A. C.-C. Yao, in Proceedings of the 11th ACM STOC, pages 209-213, 1979
[11] A. Ambainis, Algorithmica, 16 (3): 298-301, 1996
[12] I. Newman and M. Szegedy, in Proceedings of the 28th ACM STOC, pages 561-570, 1996
[13] L. Babai and P. G. Kimmel, in Proceedings of the 12th Annual IEEE Conference on Computational Complexity, pages 239-246, 1997
[14] J. Justesen, IEEE Trans. Inf. Theory 18, 652 (1972)
[15] Suppose that Alice and Bob do not have the shared random bit. Then Alice could send the information she would send if the bit had value 0 as well as the information she would send if the bit had value 1. Similarly for Bob. This protocol is at least as good as the protocol with the shared random bit, but uses twice as much conversation.
[16] A. S. Holevo, Probl. Peredachi Inf. 9 (1973) 3 [Probl. Inf. Transm. (USSR) 9 (1973) 177]
[17] D. Bohm, Phys. Rev. 85 (1952) 166
[18] J. S. Bell, Speakable and unspeakable in quantum mechanics, Cambridge University Press, 1993
[19] Th. Udem et al., Opt. Lett. 24 (1999) 881
FIG. 1: Quantum Fingerprinting using a single photon with many internal degrees of freedom. The mode locked laser (M L Laser) produces a train of pulses. The train is coupled into an optical fiber. An attenuator (A) decreases the intensity so that the train contains less than a single photon. A coupler (C) sends with amplitude $\frac{1}{\sqrt{2}}$ the pulses to Alice, and with amplitude $\frac{1}{\sqrt{2}}$ the pulses to Bob. Alice and Bob put phases on the successive pulses using their phase modulators $\phi_A$ and $\phi_B$. The pulses are then sent back to the referee who recombines them using a coupler. If the phases put by Alice and Bob are equal, the photon (if one is present) exits by the Equal port. If they are opposite, they exit by the Not Equal port. Finally the photon is detected by single photon detectors (D).