Reduction formalism for Dirac fermions on de Sitter spacetime

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Abstract

The reduction formulas for Dirac fermions are derived, using the exact solutions of free Dirac equation on de Sitter spacetime. In the framework of the perturbation theory one studies the Green functions and derive the scattering amplitude in the first orders of perturbation theory.

1 Introduction

The Dirac equation on de Sitter spacetime has been in moving or static local charts leading to significant analytical solutions [2],[9],[10],[11]. The next step is to find solutions for the free electromagnetic field in moving local charts. The problems that arises are related to the fact that curved spacetimes have specific symmetries which, in general, differ from that of Minkowski spacetime. It is also known that form of the field equations and their solutions on curved spacetime are strongly dependent on the tetrad gauge and local chart in which one works and for that reason we don’t expect to find a general solution for field equations. Also is important to specify that the recent astrophysical observations shows that the expansion of the Universe is accelerating and the mathematical model that could describe the far future limit of our Universe is the de Sitter model.

Actually majority of investigations dedicated to Q.E.D on curved spacetimes don’t take into considerations scattering processes. This is because in the present we don’t have one scattering theory on curved spacetime. Our aim in this paper is to derive the reduction formulas for Dirac fermions and to use this result for developing the scattering theory on de Sitter spacetime. We will see that the scattering theory on de Sitter spacetime can be reproduced from that in Minkowski.

We continue in this paper our work related to the developing of perturbative Q.E.D on de Sitter spacetime. In what follows our attention will be focused on Dirac field and we shall discuss only this problem. In section 3 we will derive the equations of fields in interaction and we use this result for constructing
the reduction formalism for Dirac fermions, in section 4 we use this result to derive the elements of matrix for the amplitude of transition. Our conclusions is summarized in section 5. The results are presented in natural units $\hbar = c = 1$.

2 Plane wave

We start with the exact solutions of the free Dirac equation on de Sitter spacetime written in [2]. Let us write the de Sitter line element [1]

$$ds^2 = dt^2 - e^{2\omega t}d\vec{x}^2,$$

(1)

where $\omega$ is the expansion factor. We know that defining a spinor field on curved spacetime requires one to use the tetrad fields $e^\mu$, $e^\nu$, ..., which are labelled by the local indices $\mu, \nu, ... = 0, 1, 2, 3$. The form of the line element allows one to chose the simple Cartesian gauge with the non-vanishing tetrad components:

$$e^0 = e^{-\omega t}; \quad e^i = \delta^i_j e^{-\omega t},$$

(2)

so that $e^\mu = e^\nu e^\rho$ and have the orthonormalization properties $e^\mu e^\nu = \eta^\mu^\nu$.

In this gauge the Dirac operator reads [2]:

$$E_D = i\gamma^0 \partial_t + ie^{-\omega t}\gamma^i \partial_i + \frac{3i\omega}{2}\gamma^0 .$$

(3)

Now let us introduce normalized helicity spinors for an arbitrary vector $\vec{p}$ by notation:

$$\xi_\lambda(\vec{p}), \quad \vec{\sigma} \vec{p} \xi_\lambda(\vec{p}) = 2p\lambda \xi_\lambda(\vec{p}),$$

(4)

with $\lambda = \pm 1/2$ and where $\vec{\sigma}$ are the Pauli matrices and $p = |\vec{p}|$. For writing the solutions of Dirac equation on de Sitter spacetime we set:

$$k = \frac{m}{\omega}, \quad \nu_\pm = \frac{1}{2} \pm ik.$$

(5)

Then the positive frequency modes of momentum $\vec{p}$ and helicity $\lambda$ that were constructed in [2] as solution of Dirac equation $E_D \psi = m\psi$ using the gamma matrices in Dirac representation (with diagonal $\gamma^0$) are:

$$U_{\vec{p},\lambda}(t, \vec{x}) = \frac{\sqrt{\pi p/\omega}}{(2\pi)^{3/2}} \left( \frac{1}{2} e^{\pi k/2} H^{1}_\nu(\frac{p e^{-\omega t}}{\omega}) \xi_\lambda(\vec{p}) \right) e^{i\vec{p}\vec{x} - 2\omega t},$$

(6)

where $H^{1}_\nu(z)$ is the Hankel function of first kind.

Since the charge conjugation in a curved background is point independent [8], as in Minkowski case, the negative frequency modes can be obtained using the charge conjugation,

$$U_{\vec{p},\lambda}(x) \rightarrow V_{\vec{p},\lambda}(x) = i\gamma^2 \gamma^0 (U_{\vec{p},\lambda}(x))^T$$

(7)
These spinors satisfy the orthonormalization relations [2]:

$$\int d^3x (-g)^{1/2} \bar{U}_{\vec{p},\lambda}(x) \gamma^0 U_{\vec{p},\lambda}(x) = (8)$$

$$\int d^3x (-g)^{1/2} \bar{V}_{\vec{p},\lambda}(x) \gamma^0 V_{\vec{p},\lambda}(x) = \delta_{\lambda\lambda'} \delta^3(\vec{p} - \vec{p}')$$

$$\int d^3x (-g)^{1/2} \bar{U}_{\vec{p},\lambda}(x) \gamma^0 V_{\vec{p},\lambda}(x) = 0,$$

where the integration extends on an arbitrary hypersurface $t = \text{const}$ and $(-g)^{1/2} = e^{3\omega t}$. They represent a complete system of solutions in the sense that [2]:

$$\int d^3p \sum_{\lambda} \left[ U_{\vec{p},\lambda}(t, \vec{x}) U_{\vec{p}',\lambda}(t, \vec{x}') + V_{\vec{p},\lambda}(t, \vec{x}) V_{\vec{p}',\lambda}(t, \vec{x}') \right] = e^{-3\omega t} \delta^3(\vec{x} - \vec{x}').$$

The quantization can be done considering the plane wave in momentum representation [2]:

$$\psi(t, \vec{x}) = \psi^{(+)}(t, \vec{x}) + \psi^{(-)}(t, \vec{x})$$

$$= \int d^3p \sum_{\lambda} \left[ U_{\vec{p},\lambda}(t, \vec{x}) a(\vec{p}, \lambda) + V_{\vec{p},\lambda}(t, \vec{x}) b^{(+)}(\vec{p}, \lambda) \right],$$

where the particle $(a, a^{+})$ and antiparticle $(b, b^{+})$ operators satisfy the standard anticommutation relations in momentum representation:

$$\{a(\vec{p}, \lambda), a^{(+)}(\vec{p}', \lambda')\} = \{b(\vec{p}, \lambda), b^{(+)}(\vec{p}', \lambda')\} = \delta_{\lambda\lambda'} \delta^3(\vec{p} - \vec{p}').$$

From Eq. (9) follows that the equal time anticommutator takes the canonical form [2]:

$$\{\psi(t, \vec{x}), \bar{\psi}(t, \vec{x}')\} = e^{-3\omega t} \gamma^0 \delta^3(\vec{x} - \vec{x}').$$

(12)

In any event, these are solutions of the Dirac equation and help one to write the Green functions in usual manner. Moreover, from the standard definition of the Feynman propagator:

$$S_F(t, t', \vec{x} - \vec{x}') = i\langle 0|T[\psi(x)\bar{\psi}(x')]|0 \rangle = \theta(t - t') S^{(+)}(t, t', \vec{x} - \vec{x}') - \theta(t' - t) S^{(-)}(t, t', \vec{x} - \vec{x}'),$$

(13)

in [2] was shown that:

$$[E_D - m] S_F(t, t', \vec{x} - \vec{x}') = -e^{-3\omega t} \delta^3(x - x').$$

(14)

Finally we specify that, in general, the partial anticommutator functions $S^{(+)}, S^{(-)}$ are rather complicated since for $t \neq t'$ their time-dependent parts are complicated [2]:

$$S^{(\pm)}(t, t', \vec{x} - \vec{x}') = i\{\psi^{(\pm)}(t, \vec{x}), \bar{\psi}^{(\pm)}(t', \vec{x}')\}.$$  

(15)

With these elements we can try to develop the reduction formalism and the scattering theory on de Sitter backgrounds.
We start with the interaction between spinor fields and electromagnetic field on de Sitter spacetime because a theory of free fields alone has no physical content. The nature of physical world is revealed to observers only through the interactions between fields. In this work we use for our calculation the same formalism as in \cite{3,5,6}. We adopt the minimal coupling corresponding to the classical interaction of a point charge as the prescription for introducing electrodynamic couplings. Also we will note the interacting fields with $\psi(x)$ and $A_{\alpha}(x)$, where the hated indices indicate label the components in the local Minkowski frames. This fields will satisfy one system of equations that can be obtained from an tetrad gauge invariant action of the free Dirac field, free electromagnetic field and an interaction term, all minimally coupled with gravitational field:

$$
S[e, \psi, A] = \int d^4x \sqrt{-g} \left\{ \frac{i}{2} \bar{\psi} \gamma^{\dot{\alpha}} D_{\dot{\alpha}} \psi - (\bar{D}_{\dot{\alpha}} \psi) \gamma^{\dot{\alpha}} \psi \right\} - m \bar{\psi} \psi - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - e \bar{\psi} \gamma^{\dot{\alpha}} A_{\dot{\alpha}} \psi,
$$

where $g = |\text{det}(g_{\mu\nu})|$, the Dirac matrices $\gamma^{\dot{\alpha}}$ satisfy $\{\gamma^{\dot{\alpha}}, \gamma^{\dot{\beta}}\} = 2 \eta^{\dot{\alpha}\dot{\beta}}$ and covariant derivatives in local frames, $D_{\dot{\alpha}} = e_{\dot{\alpha}}^\mu D_\mu = \partial_{\dot{\alpha}} + \Gamma_{\dot{\alpha}} \hat{\mu} D^\mu$, are expressed in terms of the spin connections.

The system of equations we obtain from action (16) are:

$$
i \gamma^{\dot{\alpha}} D_{\dot{\alpha}} \psi - m \psi = e \gamma^{\dot{\alpha}} A_{\dot{\alpha}} \psi
\quad
\frac{1}{\sqrt{-g}} \partial_{\dot{\alpha}} \left( \sqrt{-g} F^{\dot{\alpha}\dot{\beta}} \right) = e \bar{\psi} \gamma^{\dot{\beta}} \psi.
$$

It is obvious from (17) that in discussing the coupling between three fields we are up against a nonlinear problem of vast complexity. This system of equations can be replaced with one system of integral equations which contain information about initial conditions. To do this one select two Green functions $S^G(x - y)$ and $D^G_{\dot{\alpha}\dot{\beta}}(x - y)$ corresponding to one initial condition, which help us to write the solutions of system (17) like follows:

$$
\psi(x) = \hat{\psi}(x) - e \int d^4y \sqrt{-g} S^G(x - y) \gamma^{\dot{\alpha}} A_{\dot{\alpha}}(y) \psi(y)
\quad
A_{\dot{\alpha}}(x) = \hat{A}_{\dot{\alpha}}(x) - e \int d^4y \sqrt{-g} D^G_{\dot{\alpha}\dot{\beta}}(x - y) \bar{\psi}(y) \gamma^{\dot{\beta}} \psi(y),
$$

where $\hat{\psi}(x)$ and $\hat{A}_{\dot{\alpha}}(x)$ are free fields. One can verify that the first solution from (18) are exact solution of the first equation from (17), applying $(E_D(x) - m)$ to $\psi(x)$, with the observation that $[E_D(x) - m] \bar{\psi}(x) = 0$, and using the fact that the Green function must satisfy:

$$
[E_D(x) - m] S^G(x - y) = -e^{-3\omega t} \delta^4(x - y).
$$
The Green function for Dirac field will satisfy one equation of the form (14) and for that reason, $\psi(x)$ is an exact solution of the system (17).

Equation (18) offers us the possibility of constructing free fields, which are asymptotic equal (at $t \to \pm \infty$) with solutions of system (17). Now we known that the retarded Green functions $S_R(x - y)$ vanishes at $t \to -\infty$ while the advanced one $S_A(x - y)$ vanishes for $t \to \infty$. If we write first relation from (18) with retarded and advanced functions,

$$\psi(x) = \hat{\psi}_{R/A}(x) - e \int d^4y \sqrt{-g} S_{R/A}(x - y) \gamma^{\hat{\alpha}} \hat{A}_{\hat{\alpha}}(y) \psi(y),$$

then they would define free fields, $\hat{\psi}_{R}(x)$ that satisfy:

$$\lim_{t \to \mp \infty} (\psi(x) - \hat{\psi}_{R/A}(x)) = 0.$$  (21)

The free fields $\hat{\psi}_{R}$ and $\hat{\psi}_{A}$ have mass $m$ and are equal at $t \pm \infty$ with exact solutions of the coupled equations and represent the fields before and after the interaction. Now we known from Minkowski theory that the mass of the interacting fields may differ from that of free fields because of the connection between mass and energy, and the energy of electromagnetic field. In our case we don’t know the relation between mass and energy but it is possible that this difference between mass of free and interacting fields to be due to the energy of electromagnetic field and to the coupling with gravitational field. With this observation the first equation from (17) can be rewritten:

$$i\gamma^\hat{\alpha} D_\hat{\alpha} \psi - m \psi = e \gamma^\hat{\alpha} \hat{A}_{\hat{\alpha}} \psi + \delta m \psi,$$  (22)

where the difference $\delta m$ will be calculated when one solve the system of coupled equations.

Like in Minkowski case, the free fields $\hat{\psi}_{R}$ and $\hat{\psi}_{A}$ are defined up to a normalization constant noted with $\sqrt{z^2}$. Following the same steps like in Minkowski case one could define in/out fields:

$$\sqrt{z^2} \psi_{in/out}(x) = \psi(x) + e \int d^4y \sqrt{-g} S_{R/A}(x - y) \gamma^{\hat{\alpha}} \hat{A}_{\hat{\alpha}}(y) \psi(y)$$

$$+ \delta m \int d^4y \sqrt{-g} S_{R/A}(x - y) \psi(y).$$  (23)

The in/out free fields defined above satisfy Dirac equation, can be written with the help of creation and annihilation operators and satisfy conditions:

$$\lim_{t \to \mp \infty} (\psi(x) - \sqrt{z^2} \psi_{in/out}(x)) = 0.$$  (24)

Using Eq.(22) one could write this fields as follows:

$$\sqrt{z^2} \psi_{in/out}(x) = \psi(x) + \int d^4y \sqrt{-g} S_{R/A}(x - y)[E_D(y) - m] \psi(y).$$  (25)
where the Dirac operator reads $E_D = i\gamma^\alpha D_\alpha$.

The above normalization rule allows us to write the definition for the creation and annihilations operators. From (10) if one uses the orthonormalization relations (8), one obtain:

$$a(p, \lambda)^{\text{in/out}} = \int d^3 x e^{i\vec{p}\cdot\vec{x}} U_{\vec{p}, \lambda}(x) \gamma^0 \psi^{\text{in/out}}(x)$$

$$b^+(\vec{p}, \lambda)^{\text{in/out}} = \int d^3 x e^{i\vec{p}\cdot\vec{x}} \bar{U}_{\vec{p}, \lambda}(x) \gamma^0 \psi^{\text{in/out}}(x). \quad (26)$$

The creation and annihilation operators defined above satisfy the anticommutation relations (11) and from that it follows that all the properties of free fields will be preserved.

Before starting our calculations we make a few remarks about the scattering operator. Denoting the vacuum state $|0\rangle$, the one particle states for Dirac fermions can be written:

$$a^+(\vec{p}, \lambda)^{\text{in/out}}|0\rangle = |\text{in/out}, 1(\vec{p}, \lambda)\rangle$$

$$b^+(\vec{p}, \lambda)^{\text{in/out}}|0\rangle = |\text{in/out}, \bar{1}(\vec{p}, \lambda)\rangle. \quad (27)$$

If we consider two states $|\text{in}, \alpha\rangle$ and $|\text{out}, \beta\rangle$ one define the probability amplitude of transition from state $\alpha$ to state $\beta$ as the scalar product of the two states: $\langle\text{out}, \beta|\text{in}, \alpha\rangle$, this is just the elements of matrix for scattering operator. This operator assures the stability of the vacuum state and one particle state, and in addition transform any out field in the equivalent in field.

The remaining task is to construct and studies, general matrix elements which describe the dynamical behavior of interacting particles. We are interested in the transitions amplitudes for interacting particles between different initial and final states, that is the $S$ matrix. As in Minkowski theory we can construct $n-$particle in and out states as in the free Dirac theory by repeated application to the vacuum of $a^+(\vec{p}, \lambda)^{\text{in/out}}$ and $b^+(\vec{p}, \lambda)^{\text{in/out}}$.

Let us consider the amplitude of one process in which particles from in states denoted with $\alpha$, together with an electron $1(\vec{p}, \lambda)$ pass in out state, in which we have one electron $1(\vec{p}', \lambda')$ and particles denoted by $\beta$. The amplitude of this process can be written as:

$$\langle\text{out}, \beta|1(\vec{p}', \lambda')\rangle|\text{in}, \alpha\rangle = \langle\text{out}, \beta|a_{\text{out}}(\vec{p}', \lambda')|\text{in}, \alpha\rangle + \langle\text{out}, \beta|a_{\text{in}}(\vec{p}', \lambda')|\text{in}, \alpha\rangle.$$  \quad (28)

The first term in (28) give if one use the anticommutation relations (11) the amplitude of one process where the electron passes from state in in out state without interacting with particles:

$$\delta_{\lambda\lambda'}(\vec{p} - \vec{p}') \langle\text{out}, \beta|\text{in}, \alpha\rangle.$$  \quad (29)
It remains to evolve the second terms, and for this we must evolve the difference:

\[ a_{\text{out}}(\vec{p}, \lambda') - a_{\text{in}}(\vec{p}, \lambda') = \int d^3x e^{3\omega t} \bar{U}_{\vec{p}, \lambda}(x) \gamma^0 (\psi_{\text{out}}(x) - \psi_{\text{in}}(x)). \quad (30) \]

Using (25) we obtain:

\[ \psi_{\text{out}}(x) - \psi_{\text{in}}(x) = -\frac{1}{\sqrt{2}} \int d^4y \sqrt{-g} S(x-y)[E_D(y) - m] \psi(y), \quad (31) \]

where \( S(x-y) = S_R(x-y) - S_A(x-y) \). Replacing (30) and (31) in (28) and in addition using:

\[ \int d^3x e^{3\omega t} \bar{U}_{\vec{p}, \lambda}(x) \gamma^0 S(x-y) = i \bar{U}_{\vec{p}, \lambda}(y), \quad (32) \]

we obtain:

\[ \langle \text{out}, 1(\vec{p}, \lambda')|\text{ina}, 1(\bar{p}, \lambda) \rangle = \delta_{\lambda \lambda'} \delta^3(\vec{p} - \vec{p}') \langle \text{out}|\text{ina} \rangle - i \frac{e}{\sqrt{2}} \int \sqrt{-g} \bar{U}_{\vec{p}, \lambda}(y)[E_D(y) - m]\langle \text{out}|\psi(y)\rangle |\text{ina}, 1(\bar{p}, \lambda) \rangle d^4y. \quad (33) \]

The above method can be used to any particle from \( \text{in} \) or \( \text{out} \) state. Using the same calculation we find the reduction formula for one positron from \( \text{out} \) state:

\[ \langle \text{out}, \bar{1}(\vec{p}, \lambda')|\text{ina}, 1(\bar{p}, \lambda) \rangle = \delta_{\lambda \lambda'} \delta^3(\vec{p} - \vec{p}') \langle \text{out}|\text{ina} \rangle + i \frac{e}{\sqrt{2}} \int \sqrt{-g} \langle \text{out}|\bar{\psi}(y)\rangle |\text{ina}, 1(\bar{p}, \lambda) \rangle [\bar{E}_D(y) - m] V_{\vec{p}, \lambda}(y) d^4y, \quad (34) \]

where we note \( \bar{E}_D = -i \gamma^0 \bar{D}_5 \). The calculations of reduction formulas for electron and positron from \( \text{in} \) state can be done using the above method obtaining for the transition amplitude:

\[ \langle \text{out}, \bar{1}(\vec{p}, \lambda')|\text{ina}, 1(\bar{p}, \lambda) \rangle = \delta_{\lambda \lambda'} \delta^3(\vec{p} - \vec{p}') \langle \text{out}|\text{ina} \rangle - i \frac{e}{\sqrt{2}} \int \sqrt{-g} \langle \text{out}|\bar{\psi}(y)\rangle |\text{ina}, 1(\bar{p}, \lambda) \rangle [\bar{E}_D(y) - m] U_{\vec{p}, \lambda}(y) d^4y, \]

\[ \langle \text{out}, \bar{1}(\vec{p}, \lambda')|\text{ina}, 1(\bar{p}, \lambda) \rangle = \delta_{\lambda \lambda'} \delta^3(\vec{p} - \vec{p}') \langle \text{out}|\text{ina} \rangle + i \frac{e}{\sqrt{2}} \int \sqrt{-g} V_{\vec{p}, \lambda}(y)[E_D(y) - m] \langle \text{out}|\bar{1}(\vec{p}, \lambda')|\psi(y)\rangle |\text{ina} \rangle d^4y. \quad (35) \]

Now we can proceed with the reduction of the second particle. For this we suppose that we already done the reduction of the first electron from \( \text{out} \) state and we obtain (33). Now if one suppose that in the particles denoted by \( \beta \) exist another electron then \( \beta = \beta' + 1(\vec{p}'', \lambda'') \), it follows to reduce this electron. The matrix element in which we are interested appears in (33):

\[ \langle \text{out} | \psi(y) | \text{ina}, 1(\bar{p}, \lambda) \rangle = \langle \text{out} | \bar{1}(\vec{p}, \lambda'') | \psi(y) | \text{ina}, 1(\bar{p}, \lambda) \rangle \]

\[ = \langle \text{out} | \bar{1}(\vec{p}, \lambda'') \psi(y) + \psi(y) a_{\text{in}}(\vec{p}, \lambda'') | \text{ina}, 1(\bar{p}, \lambda) \rangle \]

\[ - \langle \text{out} | \psi(y) a_{\text{in}}(\vec{p}, \lambda'') | \text{ina}, 1(\bar{p}, \lambda) \rangle. \quad (36) \]
The last terms is the form \( \langle \text{out} | \tilde{\beta} | \psi(y) \rangle | in\alpha \rangle \delta_{\lambda\lambda'} \delta^3(p\tilde{\rightarrow} - \tilde{p}) \), and corresponds to a process where electron don’t interact with other particles and for that reason this term is not interesting for us. The first term in (36) give the amplitude that interest us. We will start with the evaluation of the sum using (26):

\[
a_{\text{out}}(p^\tilde{\rightarrow}, \lambda'') \psi(y) + \psi(y) a_{\text{in}}(p^\tilde{\rightarrow}, \lambda'') = \int e^{3\omega t} \hat{U}^\tilde{\rightarrow}_{\mu', \lambda''}(x) \gamma^0 \left[ \psi_{\text{out}}(x) \psi(y) \ight. + \psi(y) \psi_{\text{in}}(x) \right] dx, \tag{37}
\]

then using (25) one obtain:

\[
a_{\text{out}}(p^\tilde{\rightarrow}, \lambda'') \psi(y) + \psi(y) a_{\text{in}}(p^\tilde{\rightarrow}, \lambda'') = \frac{1}{\sqrt{2}} \int e^{3\omega t} \hat{U}^\tilde{\rightarrow}_{\mu', \lambda''}(x) \gamma^0 \left[ \psi(x) \psi(y) + \psi(y) \psi(x) \right] dx + \frac{1}{\sqrt{2}} \int \int e^{6\omega t} \hat{U}^\tilde{\rightarrow}_{\mu', \lambda''}(x) \gamma^0 \left[ -\theta(z^0 - x^0) S(x - z) [E_D(z) - m] \right. \\
\left. \times \psi(z) \psi(y) + \theta(x^0 - z^0) S(x - z) [E_D(z) - m] \psi(y) \psi(z) \right] dx dy dz. \tag{38}
\]

Using (32) and the explicit form of Dirac operator \( E_D = i\gamma^0 \partial_t + ie^{-\omega t} \gamma^i \partial_i + \frac{3\omega}{4} \gamma^0 \) one obtain:

\[
a_{\text{out}}(p^\tilde{\rightarrow}, \lambda'') \psi(y) + \psi(y) a_{\text{in}}(p^\tilde{\rightarrow}, \lambda'') = \frac{1}{\sqrt{2}} \int e^{3\omega t} \hat{U}^\tilde{\rightarrow}_{\mu', \lambda''}(x) \gamma^0 \left\{ \psi(x), \psi(y) \right\} dx \\
+ \frac{i}{\sqrt{2}} \int e^{3\omega t} \hat{U}^\tilde{\rightarrow}_{\mu', \lambda''}(z) [E_D(z) - m] \left[ -\theta(z^0 - x^0) \psi(z) \psi(y) + \theta(x^0 - z^0) \psi(y) \psi(z) \right] dx dy dz \\
+ \frac{i}{\sqrt{2}} \int e^{3\omega t} \hat{U}^\tilde{\rightarrow}_{\mu', \lambda''}(z) \left[ i\gamma^0 \delta(z^0 - x^0) \psi(z) \psi(y) + i\gamma^0 \delta(z^0 - x^0) \psi(y) \psi(z) \right] dx dy dz. \tag{39}
\]

The last term appears because we place Dirac equation in front of parenthesis, thus acting on distributions \( \theta \) and generating the last term with changed sign. Integrating after \( dz^0 \) in the last term from (39) and changing the spatial variables \( \tilde{z} \rightarrow \tilde{x} \), one obtain just the first term from (39) with opposite sign. Now observing that in (37) the spatial integral is done at the time \( x^0 \) which is arbitrary, one could choice \( x^0 = y^0 \) and obtain the chronological product \( T[\psi(z) \psi(y)] \). With this observation we finally obtain:

\[
a_{\text{out}}(p^\tilde{\rightarrow}, \lambda'') \psi(y) + \psi(y) a_{\text{in}}(p^\tilde{\rightarrow}, \lambda'') = \\
- \frac{i}{\sqrt{2}} \int \sqrt{-g} \hat{U}^\tilde{\rightarrow}_{\mu', \lambda''}(z) [E_D(z) - m] T[\psi(z) \psi(y)] dz. \tag{40}
\]

The matrix element will be:

\[
\langle \text{out} | \tilde{\beta} | \psi(y) \rangle + \psi(y) a_{\text{in}}(p^\tilde{\rightarrow}, \lambda'') | in\alpha, 1(p\tilde{\rightarrow}, \lambda) \rangle = \\
- \frac{i}{\sqrt{2}} \int \sqrt{-g} \hat{U}^\tilde{\rightarrow}_{\mu', \lambda''}(z) [E_D(z) - m] \langle \text{out} | \tilde{\beta} | T[\psi(z) \psi(y)] | in\alpha, 1(p\tilde{\rightarrow}, \lambda) \rangle dz. \tag{41}
\]

From the above calculations one sees that the reduction of the second particle is done using the same method as for the first particle. In the matrix
element that interest us (41), appears two field operators that are multiplied in chronological order. Repeating the calculation for other particles from in and out states one observe that the reduction calculus is the same, indifferent what type of particle is reduced.

Also we can obtain a generalization of the above formulas, supposing that we have \( n - \text{out} \) and \( m - \text{in} \), Dirac particles, after we apply the reduction formalism finally obtain:

\[
\left(-\frac{i}{\sqrt{2}}\right)^{m+n} \prod_{i=1}^{n} \int d^4 x_i \sqrt{-g(x_i)} \prod_{j=1}^{m} \int d^4 y_j \sqrt{-g(y_j)} \bar{U}_{\vec{p}_i,\lambda_i}(x_i) [E_D(x_i) - m] \\
\times \langle 0 [\psi(x_1)\ldots\psi(x_n)\bar{\psi}(y_1)\ldots\bar{\psi}(y_m)]|0 [E_D(y_j) - m]U_{\vec{p}_j,\lambda_j}(y_j) \rangle.
\]

The sign \((-)^{m+n}\) is governed by the number of sign changes dictated by the definition of time ordering for fermion fields.

Now we are in the position of making important observations about the above reduction method. One can show that all particles, will be replaced by formulas that was obtain in the reduction of one particle. Also when more particles are reduced, in matrix element appears the time ordered products of corresponding field operators. Every particle will be replaced after reduction with expressions which depend on field operator \(\psi(x)\) ((33),(34),(35)). After we reduce all particles from in and out states we arrive at a vacuum expectation value of time ordered product of fields. We don’t write explicitly the spinorial indices because is obviously that \(E_D(y)\) will act as \((4 \times 4)\) matrix and as differential operator just on spinor \(\psi(y)\). One observe that thought reduction method the amplitudes was written as function of fundamental solutions of the free Dirac equation on de Sitter spacetime, and as function of vacuum expectation value of time ordered product of fields. The vacuum expectation value of time ordered product of fields, together with normalization constants \((\sqrt{2})\) define the Green functions of the interacting fields. One can associate one Green function to any process of interaction. After we apply the reduction formalism, the Green functions of interacting fields must be calculated.

At the end of this section we write the reduction rules for particles and antiparticles, denoting one electron by \(1(\vec{p}, \lambda)\) and one positron by \(\bar{1}(\vec{p}, \lambda)\), after reduction of particles from in and out states one obtain:

\[
1(\vec{p}, \lambda) \quad \text{out} \quad \rightarrow \quad -\frac{i}{\sqrt{2}} \int (-g)^{1/2} \bar{U}_{\vec{p},\lambda}(x) [E_D(x) - m] \psi(x) d^4 x, \\
\bar{1}(\vec{p}, \lambda) \quad \text{out} \quad \rightarrow \quad \frac{i}{\sqrt{2}} \int (-g)^{1/2} \bar{\psi}(x) [E_D(x) - m] V_{\vec{p},\lambda}(x) d^4 x, \\
1(\vec{p}, \lambda) \quad \text{in} \quad \rightarrow \quad -\frac{i}{\sqrt{2}} \int (-g)^{1/2} \bar{\psi}(x) [E_D(x) - m] U_{\vec{p},\lambda}(x) d^4 x, \\
\bar{1}(\vec{p}, \lambda) \quad \text{in} \quad \rightarrow \quad \frac{i}{\sqrt{2}} \int (-g)^{1/2} \bar{\psi}(x) [E_D(x) - m] \psi(x) d^4 x. \quad (43)
\]
4 The perturbation theory

The Green functions of the interacting fields can’t be calculated exact and for that reason we will use perturbation methods. The form of the amplitudes obtained from reduction formalism allows one to use perturbation calculus.

It is clear now that the entirely perturbation theory on de Sitter spacetime can be reproduced from Minkowski theory \[3\], \[4\], \[5\], \[6\]. For calculating the Green functions we must write then as functions of free fields, because we know their form and properties. We write the Green function in generally as follows:

\[
G(y_1, y_2, ..., y_n) = \frac{1}{\langle 0 | S | 0 \rangle} \langle 0 | T[\hat{\psi}(y_1) \hat{\psi}(y_2) ... \hat{\psi}(y_n), \tilde{S}] | 0 \rangle ,
\]

(44)

where \(\tilde{S}\) is a unitary operator and have a closer form with the same operator from Minkowski theory. Like in Minkowski case the operator \(\tilde{S}\) must be correlated with scattering operator \(S\). One can show that this two operators are equal up to a phase factor.

Then entire perturbation calculus is based on development of operator \(\tilde{S}\):

\[
\tilde{S} = Te^{-i \int \sqrt{-g}L_I(x)dx} = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int (-g)^{n/2}T[L_I(x_1)...L_I(x_n)]dx_1...dx_n,
\]

(45)

where the density lagrangian of interaction reads:

\[
L_I(x) = -e : \bar{\psi}(x)\gamma^\alpha A_\alpha(x)\psi(x) : -\delta \mu : \bar{\psi}(x)\psi(x) : .
\]

Each term from (45) corresponds to a rang from perturbation theory. Replacing (45) in the expression of Green function (44) one obtain perturbation series which allows one to calculate the amplitude in any order. The term of rang \(n\) of this development is:

\[
\frac{(-i)^n}{n! \langle 0 | S | 0 \rangle} \int (-g)^{n/2} \langle 0 | T[\hat{\psi}(y_1) \hat{\psi}(y_2) ... \hat{\psi}(y_n) , \hat{L}_I(x_1)...\hat{L}_I(x_n)] | 0 \rangle dx_1...dx_n.
\]

(46)

The evaluation of the integrant from (46) is the same as in Minkowski case, we have a cinematic part which is obtained from reduction formalism and a dynamic one represented by operator \(\tilde{S}\). Following the same steps as in Minkowski theory we will make the \(T\) contractions between cinematic and dynamic part, with the observation that the \(T\) contractions between fields from cinematic part will not give contributions to the scattering amplitude. Also in the case of \(T\) contractions between fields from dynamic part we have two possibilities. One is that all fields from dynamic part coupled fields from cinematic one and the second is that one part of the fields from dynamic part contract between them. The second possibility will give one term of the form \(\langle 0 | S | 0 \rangle\), which will simplify the nominator.

As an application to our formalism we can obtain the scattering amplitudes in first orders in perturbation theory, thus completing the framework that one needs for calculating scattering processes in the first order of perturbation.
theory, on de Sitter expanding universe. Now using the reduction formalism developed in section 3 for one amplitude of the form \( \langle \text{out}, 1 (\vec{p}', \lambda') | \text{in}, 1 (\vec{p}, \lambda) \rangle \), one obtain the development:

\[
\langle \text{out}, 1 (\vec{p}', \lambda') | \text{in}, 1 (\vec{p}, \lambda) \rangle = \delta_{\lambda \lambda'} \delta^3 (\vec{p} - \vec{p}') - \frac{1}{z_2} \int \int (-g) \bar{\Upsilon}_{\vec{p}', \lambda'} (y) [E_D (y) - m] [0 | T | \psi (y) \bar{\psi} (z)] [0 | \bar{E}_D (z) - m] U_{\vec{p}, \lambda} (z) d^4 y d^4 z. \tag{47}
\]

Using (46) and neglecting the first term in (47), one obtain the for scattering amplitude in first order of perturbation theory:

\[
A_{i \rightarrow f} = -ie \int \int \int (-g)^{3/2} \bar{\Upsilon}_{\vec{p}', \lambda'} (x) [E_D (y) - m] [0 | T | \psi (y) \bar{\psi} (z) : \psi (x) \gamma_\mu A_\mu (x) \psi (x) :] [0 | \bar{E}_D (z) - m] U_{\vec{p}, \lambda} (x) d^4 x d^4 y d^4 z. \tag{48}
\]

After we make the \( T \) contractions between cinematic and dynamic part using the method from Minkowski theory, and use (13) and (14), finally obtain:

\[
A_{i \rightarrow f} = -ie \int \sqrt{-g} \bar{\Upsilon}_{\vec{p}', \lambda'} (x) \gamma_\mu A_\mu (x) U_{\vec{p}, \lambda} (x) d^4 x. \tag{49}
\]

The above expression is just the scattering amplitude that we use in our previous work [7], for calculate the Coulomb scattering on de Sitter expanding universe. Following the same steps as above one can obtain the scattering amplitude in superior orders in perturbation theory. With the above calculations we establish the general rules of calculation which can be used in the language of Feynman graphs.

5 Conclusion

In this paper we investigate the reduction formalism for solution of the free Dirac equation on de Sitter background. We obtain that the reduction formalism can be calculated using the same method as in Minkowski case. Also we show that after reduction of particles from \( \text{in} \) and \( \text{out} \) states one obtain the vacuum average of fields written in chronological order. As in the Minkowski case to any interaction one can associate one Green function, that can be evolved using perturbation theory. From our formalism of reduction and using perturbation theory, we deduce the correct definition for the scattering amplitudes.

From our point of view is important to do the same calculations for the electromagnetic field, thus completing this theory and the framework that one needs for developing perturbative Q.E.D on de Sitter spacetime.

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