Brownian Bridge and Self-Avoiding Random Walk.

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Abstract

We derive the Brownian bridge asymptotics for a scaled self-avoiding walk conditioned on arriving to a far away point $na$ for $a \in (Z, 0, ..., 0)$, and outline the proof for all other $a$ in $Z^d$.

1 Introduction.

1.1 Self-Avoiding Walks and Brownian Bridge.

First we briefly introduce the notion of a self-avoiding walk based on the material rigorously presented in [14], and the notion of the Brownian Bridge followed by the history of the problem solved as well as that of the tools and ideas used in producing the results. We will conclude the introduction with the word description of the results of this research paper given in light of the chronological perspective, with the brief outline of the proofs that follow.

Self-Avoiding Random Walk (SARW): An $N$-step self-avoiding walk (path) $\omega$ on $Z^d$, beginning at 0 is a sequence of sites: $\omega(0) = 0, \omega(1), ..., \omega(N)$ with $|\omega(j + 1) - \omega(j)| = 1$ and $\omega(i) \neq \omega(j)$ for all $i \neq j$. We let $c_N$ denote the number of $N$-step self-avoiding walks beginning at zero. It had been established that the limit representing the connective constant $\mu = \lim_{N \to \infty} c_N^{1/N}$ exists due to a subadditivity property of log $c_N$ (see [14]). We also let $c_N(x, y)$ to be the number of $N$-step self-avoiding walks $\omega$ with $\omega(0) = x$ and $\omega(N) = y$.

The two-point function $g_\beta(x, y)$ (as defined below) is an important tool in the theory of self-avoiding walks:

$$g_\beta(x, y) \equiv \sum_{N=0}^{\infty} c_N(x, y) e^{-\beta N} = \sum_{\omega : x \to y} e^{-\beta |\omega|},$$

where the second sum is taken only over all self-avoiding walks $\omega : x \to y$ on the lattice. For the simplicity of notation (due to the shift-invariance property of $c_N(x, y)$ ) we denote $g_\beta(x) \equiv g_\beta(0, x)$. The supercritical $\beta > \beta_c(d)$ is the one for which the equivalent sums above are finite. It was shown (see [14]) that for the supercritical $\beta$, the ”bubble diagram”

$$B_d(\beta) \equiv \sum_x g_\beta(x)^2$$
is finite. The significance of the bubble diagram is discussed in Section 1.5 of [14].

Since the radius of convergence $e^{-\beta_c(d)} = \frac{1}{\mu}$, it is apparent that the two-point function decays exponentially:

$$g_\beta(0, x) \leq C_\beta e^{-c_\beta \|x\|}$$  \hspace{1cm} (1)

for all $\beta > \beta_c(d)$ and some corresponding $C_\beta, c_\beta > 0$.

The notion of a "mass" of a two-point function applies here as well. The mass $m(\beta)$ is the rate of exponential decay of $g_\beta(x, y)$ in the direction of the first coordinate vector:

$$m(\beta) = \liminf_{n \to \infty} -\frac{\log g_\beta(0, (n, 0, ..., 0))}{n}.$$  

It can be shown that the lim inf above can be replaced by the limit.

**Brownian Bridge:** defined as a sample-continuous Gaussian process $B^0$ on $[0, 1]$ with mean 0 and $\mathbb{E}B^0_s B^0_t = s(1-t)$ for $0 \leq s \leq t \leq 1$. So, $B^0_0 = B^0_1 = 0$ a.s. Also, if $B$ is a Brownian motion, then the process $B_t - tB_1$ $(0 \leq t \leq 1)$ is a Brownian Bridge. For more details see [2], [7] and [8]. In a more general setting, we call the process $B^0_t \tilde{a} = B^0_t + t\tilde{a}$ "a Brownian Bridge connecting points zero and $\tilde{a}$".

### 1.2 History of the Problem, the Results and Strategy of the Proof.

The main goal of this paper is to show the derivation of the weak Brownian bridge asymptotics for a scaled self-avoiding walk conditioned on arriving to a far away point $n\tilde{a}$ $(\tilde{a} \in \mathbb{Z}^d)$.

The technique used in the proofs originates from the methodology developed in the process of establishing a precise Ornstein-Zernike decay for a variety of spin systems and lattice field theories and the development of the renewal theory. It turned out that the technique developed by Ornstein and Zernike in 1914 for the case of the classical fluid can be implemented in many classical models of statistical mechanics (self-avoiding walks, percolation, 2D Ising model and many other spin systems) for all noncritical temperatures. For this, for the given two-point function, one needs to construct a "direct correlation function" with a strictly greater rate of decay. This approach was implemented in the case of the $d$-dimensional self-avoiding walks [4] giving the precise Ornstein-Zernike behavior of the two-point function $g_\beta(0, n\tilde{a})$ connecting the origin to a point on an axis (the case $\tilde{a} = (\|\tilde{a}\|, 0, ..., 0)$) for all noncritical $\beta$. There the so called "mass gap" condition (or separation of mass)is proved. In that case, the two-point function with a different rate of decay is the generating function corresponding to the self-avoiding walks with all non-trivial (more than one) intersections with the hyper spaces $\{x_1 = c\}$ situated in between the origin and the destination point.

The work of proving the Ornstein-Zernike behavior (the coefficient of order $\|x\|^{-\frac{d-1}{2}}$ near the decay exponent of the two-point function) was completed in [11] for any supercritical value of the parameter $\beta > \beta_c(d)$. There the complete precise asymptotics (1) of the decay was derived in any direction $\tilde{a}$ as the result of an extensive study of the geometric properties of corresponding equi-decay level sets, broadening the methodology of [4].
The corresponding developments in subcritical bond percolation model followed the advances in the theory of self-avoiding walks. In [1], [3] and [4] some similar equi-decay level sets are studied, and corresponding Ornstein-Zernike asymptotics is produced. This technique was used in [13] together with the technical result of section 1.3 to produce a Brownian bridge asymptotics of a scaled percolation cluster conditioned on reaching a far away point, and also proving the shrinking of such clusters. In this paper, we follow up on the result of [13]. We prove the weak convergence of a scaled interpolation "skeleton" going through the regeneration points (see definition 4) of a self-avoiding walk, and terminating at a far away point \( n\vec{a} \) to Time\( \times (d - 1) \)-dimensional Brownian bridge as \( n \to \infty \). Later, the shrinking of the self-avoiding walk to the above interpolation skeleton is proved (see section 2.5). We prove the result for \( \vec{a} = (\|\vec{a}\|, 0, ..., 0) \) given an appropriate measure on such self-avoiding walks (see (6)). We outline the proof of the result for all other \( \vec{a} \) in \( \mathbb{Z}^d \).

1.3 Asymptotic Convergence to Brownian Bridge.

The following technical result was proved in [13]. Let \( X_1, X_2, ... \) be i.i.d. random variables on \( \mathbb{Z}^d \) with the span of the lattice distribution equal to one (see [8], section 2.5), and let there be a \( \bar{\lambda} > 0 \) such that the moment-generating function

\[
E(e^{\theta \cdot X_1}) < \infty
\]

for all \( \theta \in B_{\bar{\lambda}} \).

Now, for a given vector \( \vec{a} \in \mathbb{Z}^d \), let \( X_1 + ... + X_i = [t_i, Y_i]_f \in \mathbb{Z}^d \) when written in the new orthonormal basis such that \( \vec{a} = (\|\vec{a}\|, 0)_f \) (in the new basis \([\cdot, \cdot]_f \in \mathbb{R} \times \mathbb{R}^{d-1}\)). Also let \( P[\vec{a} \cdot X_i > 0] = 1 \). We define the process \([t, Y_{n,k}(t)]_f\) to be the interpolation of 0 and \([\frac{1}{n\|\vec{a}\|}t_i, \frac{1}{\sqrt{n}} Y_i]_{i=0,1,...,k}\), in Section 2.2 we will show that

**Technical Theorem.** The process

\[
\{Y_{n,k}^* \text{ for some } k \text{ such that } [t_k, Y_k]_f = n\vec{a}\}
\]

conditioned on the existence of such \( k \) converges weakly to the Brownian Bridge (of variance that depends only on the law of \( X_1 \)).

2 The Main Result in SARW.

In this section we work only with supercritical SARW (\( \beta > \beta_c(d) \)).

2.1 Preliminaries.

Here we briefly go over the definitions that one can find in [14]. We start with the decay rate \( \tau_{\beta}(\vec{x}) \):

\[
\tau_{\beta}(\vec{x}) \equiv - \lim_{n \to \infty} \frac{1}{n} \log g_{\beta}([n\vec{x}]),
\]
where the limit is always defined since
\[ \frac{g_\beta(\vec{x} + \vec{y})}{B_d(\beta)} \geq \frac{g_\beta(\vec{x})}{B_d(\beta)} \frac{g_\beta(\vec{y})}{B_d(\beta)}. \]

Now, \( \tau_\beta(\vec{x}) \) is the support function of the compact convex set
\[ \mathbf{K}^\beta \equiv \bigcap_{\vec{n} \in \mathbb{S}^{d-1}} \{ \vec{r} \in \mathbb{R}^d : \vec{r} \cdot \vec{n} \leq \tau_\beta(\vec{n}) \}, \]
with non-empty interior \( \text{int} \{ \mathbf{K}^\beta \} \) containing point zero.

Let \( \omega(j) = (\omega_1(j), ..., \omega_d(j)) \) be a self-avoiding path defined for \( j \in [a, b] \cap \mathbb{N}, a \leq b \in \mathbb{Z}^+ \).

**Definition 1.** We call \( \omega \) a bridge if
\[ \omega_1(a) < \omega_1(j) \leq \omega_1(b) \]
for all \( a < j \leq b \). If \( x = \omega(a) \) is the initial point and \( y = \omega(b) \) is the final point, we write \( \omega : x \rightarrow b \rightarrow y \).

For \( \vec{x} \in \mathbb{Z}^d \), we define the cylindrical two-point function
\[ h(\vec{x}) \equiv \sum_{\omega : 0 \rightarrow b \rightarrow \vec{x}} e^{-\beta |\omega|}, \]
where \( h(\vec{x}) = \delta_0(\vec{x}) \) for all \( \vec{x} \in \{0\} \times \mathbb{Z}^{d-1} \).

**Definition 2.** We say that \( k \in \mathbb{N} \) \( (\omega_1(a) < k < \omega_1(b)) \) is a break point of \( \omega \) if there exists \( r \in [a, b] \) such that \( \omega_1(j) \leq k \) whenever \( j \leq r \) and \( \omega_1(j) > k \) whenever \( j > r \).

**Definition 3.** A bridge \( \omega : x \rightarrow b \rightarrow y \) (where, as before, \( x = \omega(a) \) and \( y = \omega(b) \)) is called irreducible if it has no break points. In that case we write \( \omega : x \rightarrow b \rightarrow y \).

Now, for \( \vec{x} \in \mathbb{Z}^d \), we define the irreducible two-point function
\[ f(\vec{x}) \equiv \sum_{\omega : 0 \rightarrow b \rightarrow \vec{x}} e^{-\beta |\omega|}, \]
with \( f(\vec{x}) = \delta_0(\vec{x}) \) for all \( \vec{x} \in \{0\} \times \mathbb{Z}^{d-1} \).

### 2.2 SARW and Regeneration Structures.

It turned out that if counting the bridges between the origin and a point \( \vec{k} = (k_x, k_y) \in \mathbb{N} \times \mathbb{Z}^{d-1} \), that \( f \) and \( h \) satisfy the recurrence equation (see [4]):
\[ h(\vec{k}) = \sum_{i=1}^{k_x} \sum_{l \in \mathbb{Z}^{d-1}} f(i, l) h(k_x - i, k_y - l), \quad (3) \]
which together with \( h(0, \tilde{k}) = \delta_0(\tilde{k}) \) (for \( \tilde{k} \in \mathbb{Z}^d \)) are called the Ornstein-Zernike equations. Now, for any \( \tilde{r} \in \mathbb{Z}^{d-1} \), we define
\[
H_n(\tilde{r}) \equiv \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} e^{\tilde{r} \cdot \tilde{k}} h(n, \tilde{k}) \quad \text{and} \quad F_n(\tilde{r}) \equiv \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} e^{\tilde{r} \cdot \tilde{k}} f(n, \tilde{k}),
\]

as well as the corresponding mass
\[
m_H(\tilde{r}) \equiv \lim_{n \to +\infty} \frac{1}{n} \log H_n(\tilde{r}) \quad \text{and} \quad m_F(\tilde{r}) \equiv \lim_{n \to +\infty} \frac{1}{n} \log F_n(\tilde{r}).
\]

The Ornstein-Zernike asymptotics has been proved for the cylindrical two-point function \( h(\cdot) \) (see [6] and [11]), using the "mass gap" condition, e.g. existence of a point \( \tilde{r}_o \in \mathbb{Z}^{d-1} \), inside a neighborhood of points with finite mass \( m_H \), such that \( m_H(\tilde{r}_o) > m_F(\tilde{r}_o) \). It was also shown (see [11], Section 2) that the mass gap condition with the renewal theorem ([14], Appendix B) imply that \( \exp\{-nm_H(\tilde{r}_o)\}F_n(\tilde{r}_o) \) is a probability distribution (where \( \tilde{r}_o \) is as above):
\[
\sum_{n \in \mathbb{N}} \exp\{-nm_H(\tilde{r}_o)\}F_n(\tilde{r}_o) = 1.
\]

As it was mentioned in the introduction, the mass gap condition was crucial in obtaining the Ornstein-Zernike decay (see [11]):

**Theorem 1.** For all \( d \geq 2 \) and \( \beta > \beta_c(d) \),
\[
g_\beta(x) = \psi_\beta(\frac{x}{\|x\|}) \frac{e^{-\tau_\beta(x)}}{\|x\|^d} (1 + o(1))
\]
uniformly in \( \|x\| \), where \( \psi_\beta(\cdot) \) is analytic on the unit circle.

### 2.3 Measure \( Q_{r_0}(x) \).

We notice that substituting the sum \( F_n(\tilde{r}_o) \), as defined in (4), into (5) we obtain (after some simple manipulations) an enhanced version of (5):
\[
\sum_{x \in \mathbb{N} \times \mathbb{Z}^{d-1}} f(x)e^{\tilde{r}_o \cdot (-m_H(\tilde{r}_o), \tilde{r}_o)} = 1,
\]
where \( \tilde{r}_o \equiv (-m_H(\tilde{r}_o), \tilde{r}_o) \in \partial K^\beta \) as it was shown in [11], Section 3.

Now, let for \( \tilde{x} \in \mathbb{N} \times \mathbb{Z}^{d-1} \),
\[
Q_{r_0}(\tilde{x}) \equiv f(\tilde{x})e^{\tilde{x} \cdot \tilde{r}_0}.
\]

Due to the equation above, \( Q_{r_0}(\cdot) \) is a probability measure on \( \mathbb{N} \times \mathbb{Z}^{d-1} \). It is similar to the regeneration measure, defined for the subcritical bond percolation model in Section 4 of [11], and later used in [13] for derivation of Brownian Bridge asymptotics for that model.

The mass gap condition implies the exponential decay of \( Q_{r_0}(\tilde{x}) \).
2.4 The Result For $\mathbf{a} = (1,0,\ldots,0)$.

We fix $\mathbf{a} \in \mathbb{Z}^d$. We let for a supercritical constant $\beta$ and all $n \in \mathbb{N}$, $P_n(\cdot)$ to be a law on a set of self-avoiding random paths $\omega$, conditioned on $\omega$ being a bridge between 0 and $n\mathbf{a}$ ($\omega : 0^{-b} \to n\mathbf{a}$). More precisely, we define $P_n$ as

$$P_n(\omega : 0^{-b} \to n\mathbf{a}) \equiv \frac{\exp (-\beta|\omega|)}{\sum_{\omega:0^{-b} \to n\mathbf{a}} \exp (-\beta|\omega|)} \frac{\exp (-\beta|\omega|)}{h(n\mathbf{a})}. \quad (6)$$

For now, we let $\mathbf{a} = [1,0] \equiv (1,0,\ldots,0)$ and $\bar{r}_0 = (\mathbb{Z}^+,0,\ldots,0) \cap \partial \mathbb{K}^\beta$. Here, we define the regeneration points in a way, similar to that, used to define the regeneration points for the case of Bernoulli bond percolation model:

**Definition 4.** Given a self-avoiding path $\omega$, and a break point $b$. We say that $\omega(T_b)$ is the corresponding regeneration point if $T_b = \max\{t : \omega_1(t) = b\}$.

In a very important step, we notice that given the Ornstein-Zernike equations (3) above and the definition of probability distribution $P_n(\cdot)$, we can explicitly write (in terms of measure $Q_{r_0}$) the probability of the walk passing through the particular regeneration points $s_1 \equiv x_1$, $(s_2 \equiv x_1 + x_2),\ldots,$ $(s_{k-1} \equiv x_1 + \ldots + x_{k-1})$, where are all $x_i \in \mathbb{Z}^+ \times \mathbb{Z}^{d-1}$:

$$P_n[s_1,\ldots,s_{k-1} \text{ are reg. pts.}] = \frac{1}{h(n\mathbf{a})} \left( \sum_{\omega:0^{-b} \to s_1} e^{-\beta|\omega|} \right) \ldots \left( \sum_{\omega:s_{k-1}^{-b} \to s_k} e^{-\beta|\omega|} \right)
= \frac{1}{h(n\mathbf{a})} f(x_1)\ldots f(x_k)
= \frac{Q_{r_0}(x_1)\ldots Q_{r_0}(x_k)}{\sum_{\kappa} \sum_{s_1=\ldots=\kappa=n\mathbf{a}} Q_{r_0}(s_1)\ldots Q_{r_0}(s_k)}, \quad (7)$$

where $s_0 \equiv 0$ and $s_k \equiv x_1 + \ldots + x_{k-1} + x_k = n\mathbf{a}$.

We recall that the moment generating function (the Laplace transform) under the measure $Q_{r_0}(\cdot)$ is finite in a neighborhood of zero:

$$\mathbb{E}_{r_0}(e^{\theta X_1}) < \infty$$
for all $\theta \in B_\lambda(0)$. We use the brackets $[\cdot,\cdot] \in \mathbb{R} \times \mathbb{R}^{d-1}$ to denote the coordinates of $\mathbb{R}^d$ vectors for the simplicity of notation. Obviously $\mathbf{a} = [1,0]$. We want to prove that the process corresponding to the last $d-1$ coordinates in the new basis of the scaled ($1/n$ times along $\mathbf{a}$ and $1/\sqrt{n}$ times in the orthogonal $d-1$ dimensions) interpolation of regeneration points of the self-avoiding path $\omega$ conditioned on $\omega : 0^{-b} \to n\mathbf{a}$ converges weakly to the Brownian Bridge $B^a(t)$ (with variance that depends only on measure $Q_{r_0}$) where $t$ represents the scaled down first coordinate.

Let $X_1, X_2, \ldots$ be i.i.d. random variables distributed according to $Q_{r_0}$ law. We interpolate $0, X_1, (X_1 + X_2),\ldots,(X_1 + \ldots + X_k)$ and scale by $1/n$ and $1/\sqrt{n}$ along $<\mathbf{a}> \times <\mathbf{a}>\perp$ to get the process $[t,Y_{n,k}(t)]$. The technical theorem (2) implies the following
Theorem 2. The process

\[ \{Y_{n,k} \text{ for some } k \text{ such that } X_1 + \ldots + X_k = n\bar{a}\} \]

conditioned on the existence of such \( k \) converges weakly to the Brownian Bridge (with variance that depends only on measure \( Q_{r_0} \)).

Now, let for \( y_1, \ldots, y_k \in \mathbb{Z}^d \) with positive increasing first coordinates, \( \gamma(y_1, \ldots, y_k) \) be the last \( (d - 1) \) coordinates in the new basis of the scaled \((1/n, 1/\sqrt{n})\) interpolation of points \( 0, y_1, \ldots, y_k \) (where the first coordinate is time). Notice that \( \gamma(y_1, \ldots, y_k) \in C_0[0, 1]^{d-1} \) as a function of scaled first coordinate whenever \( y_k = n\bar{a} \).

By the important observation \((\ell)\) we’ve made before, for any function \( F(\cdot) \) on \( C[0, 1]^{d-1} \),

\[ \sum_k \sum_{x_1 + \ldots + x_k = n\bar{a}} F(\gamma(x_1, x_1 + x_2, \ldots, \sum_{i=1}^k x_i)) \]

\[ \times P_n[0 \leftarrow h_{r_0} \rightarrow x \mid \text{regeneration points: } x_1, x_1 + x_2, \ldots, \sum_{i=1}^k x_i = x] \]

\[ = \sum_k \sum_{x_1 + \ldots + x_k = n\bar{a}} F(\gamma(x_1, x_1 + x_2, \ldots, \sum_{i=1}^k x_i)) f(x_1) \ldots f(x_k) \]

\[ = e^{-r_0 \cdot n\bar{a}} \sum_k \sum_{x_1 + \ldots + x_k = n\bar{a}} F(\gamma(x_1, x_1 + x_2, \ldots, \sum_{i=1}^k x_i)) Q_{r_0}(x_1) \ldots Q_{r_0}(x_k). \]

Therefore, for any \( A \subset C[0, 1]^{d-1} \)

\[ P_p[\gamma(\text{regeneration points of } \omega) \in A \mid \omega : 0 \rightarrow ^b \rightarrow n\bar{a}] \]

\[ = \frac{\sum_k \sum_{x_1 + \ldots + x_k = n\bar{a}} I_A(\gamma(x_1, x_1 + x_2, \ldots, \sum_{i=1}^k x_i)) f(x_1) \ldots f(x_k)}{\sum_k \sum_{x_1 + \ldots + x_k = n\bar{a}} f(x_1) \ldots f(x_k)} \]

\[ = \frac{\sum_k \sum_{x_1 + \ldots + x_k = n\bar{a}} I_A(\gamma(x_1, x_1 + x_2, \ldots, \sum_{i=1}^k x_i)) Q_{r_0}(x_1) \ldots Q_{r_0}(x_k)}{\sum_k \sum_{x_1 + \ldots + x_k = n\bar{a}} Q_{r_0}(x_1) \ldots Q_{r_0}(x_k)} \]

\[ = P[Y_{n,k} \in A \text{ for the } k \text{ such that } X_1 + \ldots + X_k = n\bar{a} \mid \exists k \text{ such that } X_1 + \ldots + X_k = n\bar{a}]. \]

Hence, we have proved the following

Corollary. The process corresponding to the last \( d - 1 \) coordinates of the scaled \((1/n, 1/\sqrt{n})\) interpolation of regeneration points of the self-avoiding path \( \omega \) (with the scaled first coordinate denoting the time interval) conditioned on \( \omega : 0 \rightarrow ^b \rightarrow n\bar{a} \) converges weakly to the Brownian Bridge (with variance that depends only on measure \( Q_{r_0} \)).
2.5 Shrinking of the Self-Avoiding Walks.

Here we again let \( \vec{a} = [1, 0] \equiv (1, 0, ..., 0) \) and \( \vec{r}_o = [Z^+, 0] \cap \partial K^\beta \). In the way of proving that the scaled walk \( \omega : 0^-b \to n\vec{a} \) shrinks, we shall need to show that the consequent regeneration points are situated relatively close to each other:

**Lemma.**

\[
P_p[\max_i |x_i - x_{i-1}| > n^{1/3}, \text{ } x_i\text{- reg. points } | 0^-b \to n\vec{a}] < \frac{1}{n}
\]

for \( n \) large enough.

**Proof.** Since \( \vec{r}_o = [\|\vec{r}_o\|, 0] \in \partial K^\beta \) and therefore

\[
\vec{r}_o \cdot [v_1, v_2] = \vec{r}_o \cdot [v_1, 0] \leq \tau_\beta([v_1, 0]) \leq \tau_\beta([v_1, v_2])
\]

for all \([v_1, v_2] \in \mathbb{Z}^d\),

\[
\vec{r}_o \cdot [1, 0] = \tau_\beta([1, 0]) \quad \text{and} \quad \|\vec{r}_o\|^2 = \tau_\beta(\vec{r}_o).
\]

Hence, by the pseudo-linearity of \( \tau_\beta(\cdot) \),

\[
\nabla \tau_\beta(\vec{a}) = [\tau_\beta(1), 0] = [\|\vec{r}_o\|, 0] = \vec{r}_o.
\]

Now, by the convexity of \( \tau_\beta(\cdot) \),

\[
\frac{\tau_\beta(\vec{a}) - \tau_\beta(\vec{a} - \frac{\vec{x}}{n})}{(\|\vec{x}\|/n)} \leq \frac{\vec{x}}{\|\vec{x}\|} \cdot \nabla \tau_\beta(\vec{a})
\]

for \( \vec{x} \in \mathbb{Z}^d (\vec{x} \neq 0) \), and therefore

\[
\tau_\beta(n\vec{a}) - \tau_\beta(n\vec{a} - \vec{x}) = \|\vec{x}\| \frac{\tau_\beta(\vec{a}) - \tau_\beta(\vec{a} - \frac{\vec{x}}{n})}{(\|\vec{x}\|/n)} \leq \vec{x} \cdot \nabla \tau_\beta(\vec{a}) = \vec{r} \cdot \vec{x}.
\]

Thus, since \( Q_{\tau_0}(\vec{x}) \) decays exponentially and therefore

\[
f(\vec{x})e^{\tau_\beta(n\vec{a}) - \tau_\beta(n\vec{a} - \vec{x})} \leq Q_{\tau_0}(\vec{x})
\]

and also decays exponentially. Hence by Ornstein-Zernike result (Theorem [4]),

\[
P_p[n^{1/3} < |\vec{x}|, \text{ } \vec{x}\text{-first reg. point } | 0^-b \to n\vec{a}] = \sum_{n^{1/3} < |\vec{x}|} f(\vec{x}) \frac{h(n\vec{a} - \vec{x})}{h(n\vec{a})} < \frac{1}{n^2}
\]

for \( n \) large enough. So, since the number of the regeneration points is no greater than \( n \),

\[
P_p[\max_i |x_i - x_{i-1}| > n^{1/3}, \text{ } x_i\text{- reg. points } | 0^-b \to n\vec{a}] < \frac{1}{n}
\]

for \( n \) large enough.
Now, it is really easy to check that there is a constant \( \lambda_f > 0 \) such that
\[
f(\vec{x}) > e^{-\lambda_f \|\vec{x}\|}
\]
for all \( \vec{x} \) such that \( f(\vec{x}) \neq 0 \) (here we only need to connect points zero and \( \vec{x} \) with an “S”-shaped irreducible bridge). Hence, due to the exponential decay (1) of the two point function \( g_\beta(x, y) \), for a given \( \epsilon > 0 \),
\[
P_p[ \text{the walk } \{\omega(i)\}_{i=0,...,|\omega(\vec{x})|} \not\subset [\mathbb{R}, B_{\epsilon\sqrt{n}}(0)] | 0^{-b} \to \vec{x}] < C_\beta e^{\lambda_f \|\vec{x}\|} - c_\beta \epsilon \sqrt{n},
\]
and therefore, summing over the regeneration points, we get
\[
P_p[ \text{the scaled walk } \{\omega(i)\}_{i=0,...,|\omega(\vec{x})|} \not\subset \epsilon\text{-neighbd. of } [0, 1] \times \gamma(\text{ reg. points }) | 0^{-b} \to n\vec{a}] < \frac{1}{n} + nC_\beta e^{\lambda_f \|\vec{x}\|} - c_\beta \epsilon \sqrt{n}
\]
for \( n \) large enough due to the lemma above.

We can now state the main result for \( \vec{a} = [1, 0] \):

**Main Theorem.** The process corresponding to the last \( d-1 \) coordinates of the scaled \( (\frac{1}{n} \times \frac{1}{\sqrt{n}}) \) interpolation of regeneration points of the self-avoiding path \( \omega \) (with the scaled first coordinate denoting the time interval) conditioned on \( \omega : 0^{-b} \to n\vec{a} \) converges weakly to the Brownian Bridge (with variance that depends only on measure \( Q_{\text{mea}} \)).

Also for a given \( \epsilon > 0 \)
\[
P_p[ \text{the scaled walk } \{\omega(i)\}_{i=0,...,|\omega(\vec{x})|} \not\subset \epsilon\text{-neighbd. of } [0, 1] \times \gamma(\text{ reg. points }) | 0^{-b} \to n\vec{a}] \to 0
\]
as \( n \to \infty \).

### 2.6 General Case.

Now, we turn our attention to all \( \vec{a} \in \mathbb{Z}^d \) not on the axis. It turned out that the main theorem of section 2.5 holds for all \( \vec{a} \) in \( \mathbb{Z}^d \). In a more direct approach used in the corresponding developments in percolation (see Section 4 of [4]) and finite range Ising models (see [5]), we can replicate the same recurrence structures, as those in section 2.5, in a given direction (say \( \vec{a} \)), yielding the same renewal relations (as in section 2.5). The technique is simpler than that used in percolation and finite range Ising models. We choose a direction vector \( \vec{r} \in \partial K^\beta \), we define the corresponding notions of “a bridge” in the direction \( \vec{r} \) and the \( \vec{r} \)-regeneration points:

**Definition 5.** We call \( \omega \) an \( \vec{r} \)-bridge if
\[
\omega(a) \cdot \vec{r} < \omega(j) \cdot \vec{r} \leq \omega(b) \cdot \vec{r}
\]
for all \( a < j \leq b \). If \( x = \omega(a) \) is the initial point and \( y = \omega(b) \) is the final point, we write \( \omega : x^{-b(\vec{r})} \to y \).
Similarly, we define the \textit{cylindrical} two-point function

\[ h_\vec{r}(\vec{x}) \equiv \sum_{\omega:0 \rightarrow \vec{x} \rightarrow \vec{r}} e^{-\beta|\omega|}, \]

where \( h_\vec{r}(\vec{x}) = \delta_0(\vec{x}) \) for all \( \vec{x} \in <\vec{r}>^\perp \).

**Definition 6.** We say that \( \omega(k) \in \mathbb{Z}^d \) \( (a < k < b) \) is an \( \vec{r}\)-\textit{regeneration point} of \( \omega \) if there exists \( N \in [a,b] \) such that \( \omega(j) \cdot \vec{r} \leq \omega(k) \cdot \vec{r} \) whenever \( j \leq N \) and \( \omega(j) \cdot \vec{r} > \omega(k) \cdot \vec{r} \) whenever \( j > N \).

**Definition 7.** An \( \vec{r}\)-\textit{bridge} \( \omega: x \to y \) (where, as before, \( x = \omega(a) \) and \( y = \omega(b) \)) is called \( \omega(k) \cdot \vec{r}\)-\textit{irreducible} if it has no \( \vec{r}\)-regeneration points. In that case we write \( \omega: x \to ib(\vec{r}) \to y \).

We again redefine the corresponding \textit{irreducible} two-point function

\[ f_\vec{r}(\vec{x}) \equiv \sum_{\omega:0 \rightarrow \vec{x} \rightarrow \vec{r}} e^{-\beta|\omega|}, \]

with \( f_\vec{r}(\vec{x}) = \delta_0(\vec{x}) \) for all \( \vec{x} \in <\vec{r}>^\perp \).

The generalized Ornstein-Zernike recurrence equations also hold here: by counting the \( \vec{r}\)-bridges between the origin and a lattice point \( \vec{k} \in <\vec{r}> \times <\vec{r}>^\perp \), we establish

\[ h_\vec{r}(\vec{k}) = \sum_{0 < \vec{m} \cdot \vec{r} \leq \vec{k} \cdot \vec{r}} f_\vec{r}(\vec{m}) h_\vec{r}(\vec{k} - \vec{m}), \]

where, in the sum, all \( \vec{m} \in \mathbb{Z}^d \).

As in [4], we can replicate all the regeneration structures, and in particular show the existence of a positive \( \bar{\lambda} \) such that

\[ Q_{\tau_0}(\vec{x}) = f_\vec{r}(\vec{x}) e^{\vec{x} \cdot \vec{r}_0} \]

is a probability measure whenever \( \vec{r}_0 \in B_\bar{\lambda}(\vec{r}) \cap \partial K^\beta \). Taking an appropriate \( \vec{r} \) (say \( \vec{r} = \nabla \tau_\beta(\vec{a}) \)), we can show, as it was done in [13] for percolation clusters in subcritical phase, the skeleton convergence and shrinking of the scaled self-avoiding walks, conditioned on arriving to \( n\vec{a} \). Whence the main theorem of section 2.5 would hold if we scale the walks by \( \frac{1}{n \|\vec{a}\|} \) along \( <\vec{a}> \) and by \( \frac{1}{\sqrt{n}} \) in all orthogonal directions (along \( <\vec{a}>^\perp \)).

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