Characterisation of observability and controllability for nonuniformly sampled discrete systems

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Abstract
A joint characterisation of the observability and controllability of a particular kind of discrete system has been developed. The key idea of the procedure can be reduced to a correct choice of the sampling sequence. This freedom, owing to the arbitrary choice of the sampling instants, is used to improve the sensitivity of system observability and controllability, by exploiting an adequate geometric structure. Some qualitative examples are presented for illustrative purposes.

1 Introduction
The concepts of observability and controllability, introduced first by Kalman [6], play an important role in modern control theory. In fact, these properties often govern the existence of a solution to an optimal control problem.

Observability and Controllability of discrete-time systems have also been treated in the literature in a generalised form. A survey of the main results for discrete system sampled nonuniformly can be found in [7].

In general, both concepts have been characterised by criteria mutually independent, but, in this work, it is demonstrated that, if certain conditions not very restrictive are imposed on the continuous system, then the characterisations of observability and controllability for the discrete system can be unified.

There is an important problem which arises in discrete-time systems but not in continuous-time systems. A linear system which is completely observable and controllable may lose these properties after the introduction of sampling. This stresses the importance of the sampling sequence to guarantee the above-mentioned internal properties.

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There are restrictions on the aperiodic sampling sequence, stronger than in the periodic case. At any rate, this partial freedom in the choice of the sampling instants can be conveniently used to obtain a rather well conditioned system, a reduction in the propagation of measuring and/or rounding errors etc.

Further interesting applications of these ideas, e.g. the sensitivity of system observability and controllability, are treated by exploiting an adequate geometric structure. In this way, the familiar analytical techniques [7] can be presented in a more intuitive form.

2 Basic assumptions and general considerations

This discussion is restricted to:

a) linear time-invariant single-input/single-output differential systems of finite order $n$

b) continuous-time systems completely controllable and observable in the sense given by Kalman [6]

c) systems whose transfer function is a strictly proper rational function.

Consequently, their impulse response $h(t)$ will be a particular solution of an $n$th-order homogeneous linear differential equation with constant coefficients of the form

$$h^{(n)}(t) + a_1h^{(n-1)}(t) + \ldots + a_nh(t) = 0 \quad t \geq 0 \quad (1)$$

Therefore

$$h(t) = \sum_{i=1}^{n} C_i \varphi_i(t) \quad (2)$$

where $C_i \in \mathbb{C}$ are constant coefficients. $\varphi_i: \mathbb{R} \to \mathbb{C} \quad (i = 1, \ldots, n)$ is the fundamental system of solutions of eqn. (1).

In the state space, the systems considered can be described by the following equations

$$\dot{X}(t) = AX(t) + bu(t) \quad (X_0 = X(0))$$

$$Y(t) = cX(t) \quad (3)$$

where $X \in \mathbb{R}^n$ denotes the state-vector and $u, y \in \mathbb{R}$ are the scalar input and output, respectively. The matrices $A, b, c$ are of appropriate orders and constant.

Canonical realisations in the state space obtained from the impulse response will be used. In particular,

(i) Observability canonical form $(A_{ob}, b_{ob}, c_{ob})$:

$A_{ob}$ is a $n \times n$ bottom-companion matrix with

$$(-a_n, -a_{n-1}, \ldots, -a_1) \quad (4)$$
in the last row,
\[ b_{ob} = (h_1, h_2, \ldots, h_n)' \] (5)
\( \) denotes the transpose\)

\[ h_{i+1} = \left. \frac{d^i h(t)}{dt^i} \right|_{t=0} \quad (i = 0, \ldots, n - 1) \] (6)

correspond to the \( n \)-first Markov parameters of the impulse response \( h(t) \)
and
\[ c_{ob} = (1, 0, \ldots, 0) \] (7)

(ii) **Controllability canonical form** \((A_{co}, b_{co}, c_{co})\):

\[ A_{co} = A_{ob}' \] (8)
\[ b_{co} = c_{ob}' \] (9)
\[ c_{co} = b_{ob}' \] (10)

All minimal realisations are related by similarity transformations and, in each particular problem, the most adequate one will be selected.

3 Joint characterisation of observability and controllability of nonuniformly sampled discrete systems

The observability (controllability) of a discrete system depends on the observability (controllability) of the continuous-time system, plus some additional conditions on the sampling sequence.

The problem of observing (controlling) the state of any realisation by means of the sampling can be reduced to the task of solving a system of linear equations. Consequently, an adequate choice of the sampling instants guarantees the compatibility of this system.

The question of the controllability will be discussed first. The corresponding results for the observability will be given later. Finally, a joint characterisation of both internal properties will be presented.

3.1 The controllability problem

Let \((A, b, c)\) be an arbitrary minimal realisation of order \( n \) for the kind of systems under study. The system considered will be completely \( n \)-controllable (controllable in \( n \) steps) \([1][7]\) if, for any initial state \( X_0 \) of eqn. \( \) the system can be directed to \( X = 0 \) by means of \( n \) impulse inputs applied at \( n \) consecutive sampling instants. The solution of the state-space equation for the system eqn. \( \) at time \( t_n \) can be written as

\[ X(t_n) = \exp(At_n)X_0 + \sum_{i=0}^{n-1} G_i u_i \]
\[ = \exp(A t_n) X_0 + [G_{n-1}, \ldots, G_0] \begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix} \] (11)

where

\[ G_i = \exp(A(t_n - t_{i+1})) \int_{t_i}^{t_{i+1}} \exp(A(t_{i+1} - r)) \times b \delta(r - t_i) dr \quad (i = 0, \ldots, n-1) \] (12)

\[ T_{i+1} = t_{i+1} - t_i \] (13)

is the length of the sampling interval between two consecutive sampling instants. The scalar input is

\[ u(t) = \delta(t - t_i) u_i \quad (i = 0, \ldots, n-1) \] (14)

\[ u_i \] being the value of the impulse input at the sampling instant \( t_i \). To consider the transference from an initial state to a final state in \( n \) steps, we study the rank of the matrix \([G_{n-1}, \ldots, G_0]\). Indeed,

\[ G_i = B \exp(J(t_n - t_{n-1})) \exp(J(t_{n-1} - t_i)) y_0 \] (15)

where \( J \) is the Jordan canonical form of matrix \( A \) and \( B \) is the invertible matrix of the change of basis:

\[ y_0 = B^{-1} b \] (16)

Therefore, we must compute the value of

\[ Det = [y_0, \exp(J(t_{n-1} - t_{n-2})) y_0, \ldots, \exp(J(t_{n-1} - t_0)) y_0] \] (17)

or briefly,

\[ Det = [\exp(J(\alpha_m)) y_0] \quad (m = 0, \ldots, n-1) \] (18)

with

\[ \alpha_m = t_{n-1} - t_{n-m-1} \quad (\alpha_0 = 0) \] (19)

By similarity transformations

\[ y_0 = B^{-1} b = B_{co}^{-1} b_{co} \] (20)

where \( B_{co} \) is the matrix of the change of basis of \( A_{co} \) to the Jordan canonical form. Then we denote the components of \( y_0 \) by

\[ y_0 = (y_1^1, \ldots, y_{m_1}^1, \ldots, y_1^r, \ldots, y_{m_r}^r)' \] (21)

\( m_j \) \((j = 1, \ldots, r)\) being the multiplicity of the \( r \) different eigenvalues of matrix \( A \) with \( r \leq n \). It is easy to see [5] that, for the controllability canonical form,
\[
B_{co}^{-1}b_{co} = (C_1, C_2, \ldots, C_n)'
\]

with \((C_i)\) defined as in eqn. 2; thus making use of the Laplace’s expansion by minors, the determinant eqn. 17 can be factorised as follows:

\[
Det[\exp(J\alpha_m)y_0] = N_1 N_2 Det[\varphi_i(\alpha_m)]
\]

\[
N_1 = \frac{1}{0!} \cdots \frac{1}{(m_1 - 1)!} \cdots \frac{1}{0!} \cdots \frac{1}{(m_r - 1)!}
\]

\[
N_2 = Det\left[\begin{array}{ccc}
y_1^1 & \cdots & y_1^{m_1} \\
\vdots & \ddots & \vdots \\
y_1^{m_1} & & \vdots \\
y_r^1 & \cdots & y_r^{m_r} \\
\vdots & \ddots & \vdots \\
y_r^{m_r} & & \vdots
\end{array}\right]
\]

Note that \(N_2\) will be non null if, and only if,

\[
y_{mj}^j \neq 0 \quad (j = 1, \ldots, r).
\]

which is guaranteed, according to the previous significance of the components of \(y_0\), because only minimal realisations are considered.

Finally, \([\varphi_i(\alpha_m)] (i = 1, \ldots, n; m = 0, \ldots, n - 1)\) is an \(n \times n\) matrix of the form

\[
\begin{bmatrix}
\varphi_1(\alpha_0) & \cdots & \varphi_n(\alpha_0) \\
\varphi_1(\alpha_1) & \cdots & \varphi_n(\alpha_1) \\
\vdots & \ddots & \vdots \\
\varphi_1(\alpha_{n-1}) & \cdots & \varphi_n(\alpha_{n-1})
\end{bmatrix}
\]

\((\varphi_i(\alpha))\) being the fundamental system of solutions of eqn. 1. The value of \(Det[\varphi_i(\alpha_m)]\) will depend on the choice of the sampling instants. Let us now consider one special aspect here, because it can be the source of some terminological problems.

The nonsingularity of the matrix \([G_{n-1}, \ldots, G_0]\) is a necessary and sufficient condition to ensure that any initial state \(X_0\) can be taken to an arbitrary state \(X_n\) in \(n\) steps: this is called \(n\)-reachability. If the state \(X_n\) coincides with the origin, then the above condition is sufficient to insure that any initial state \(X_0\) can be taken to the origin in \(n\) steps: this is called \(n\)-controllability.

The \(n\)-reachability is a more restrictive condition than the \(n\)-controllability. In fact, the \(n\)-reachability implies \(n\)-controllability, but the converse may not be true.
From eqn. 23 and the previous considerations, the following result is derived:

**Lemma 1:** The realisation \((A, b, c)\) is completely \(n\)-reachable (reachable in \(n\) steps) if, and only if, \(n\) consecutive sampling instants are chosen in such a way that

\[
\text{Det}[\varphi_i(\alpha_m)] \neq 0 \quad (i = 1, \ldots, n; \; m = 0, \ldots, n - 1) \quad (28)
\]

**Proof:** The proof is evident from the factorisation of the expression 23.

Note that, for this kind of system, the \(n\)-reachability depends on the characteristic modes of the continuous system and on the choice of the sampling instants.

We remark that the hypothesis \((b)\) in Section 2 is necessary, because, if this condition is not verified, then the canonical realizations obtained from the impulse response would not be actual realisations in the state space for the system considered. Consequently, the above result would not be true.

In eqn. 14, the scalar input is defined as impulse inputs at the sampling instants. In practice, a control of the form

\[
u(t) = u(t_i) = u_i \quad t_i \leq t < t_{i+1} \quad (29)
\]

is generally used.

For technical reasons, \(u(t)\) must be generated with the aid of a filter, so that the equations of the device which generates \(u(t)\) can be included in eqn. 1. The results are completely analogous.

### 3.2 The observability problem

The observability problem can be discussed without loss of generality putting \(u(t) = 0\). For the same arbitrary minimal realisation \((A, b, c)\) the observability question is studied. The system considered will be completely \(n\)-observable (observable in \(n\) steps) [1], [7], if any initial state \(X_0\) of eqn. 3 can be calculated from \(n\) values of the output taken at \(n\) consecutive sampling instants.

Therefore, according to eqn. 3 and for the same values \(\alpha_0, \alpha_1, \ldots, \alpha_n\) as before, the following system of linear equations is set:

\[
y(\alpha_m) = c \exp(A\alpha_m)X_0 \quad (m = 0, \ldots, n - 1) \quad (30)
\]

In mathematical terms, the condition of \(n\)-observability means that

\[
\text{rank}[c \exp(A\alpha_m)] = n \quad (m = 0, \ldots, n - 1) \quad (31)
\]

The linear system eqn. 30 can be rewritten as

\[
y(\alpha_m) = cB \exp(J\alpha_m)z_0 \quad (m = 0, \ldots, n - 1) \quad (32)
\]

where \(J\) and \(B\) are defined as before

\[
z_0 = B^{-1}X_0 \quad (33)
\]
Now, we study the value of
\[ \text{Det} [cB \exp(J\alpha_m)] \quad (m = 0, \ldots, n - 1) \] (34)

By similarity transformations
\[ cB = c_{ob}B_{ob} \] (35)

where \( B_{ob} \) is the matrix of the change of basis of \( A_{ob} \) to the Jordan canonical form. It is known [5] that, for the observability canonical form,
\[ c_{ob}B_{ob} = \begin{pmatrix} m_1 & \cdots & m_r \\ 1,0,\ldots,0 & \cdots & 1,0,\ldots,0 \end{pmatrix} \] (36)

where \( m_j \) (\( j = 1,\ldots,r \)) is the multiplicity of the eigenvalues of matrix \( A \), with \( r \leq n \).

Thus, making use of the Laplace’s expansion by minors, the determinant eqn. 14 can be factorised as follows:
\[ \text{Det} [cB \exp(J\alpha_m)] = M_1 M_2 \text{Det} [\varphi_i(\alpha_m)] \] (37)

where
\[ M_1 = \frac{1}{0!} \cdots \frac{1}{(m_1 - 1)!} \cdots \frac{1}{0!} \cdots \frac{1}{(m_r - 1)!} \] (38)

\[ M_2 = \text{Det} \begin{bmatrix} I_1 & & \\ & I_2 & \\ & & \ddots \\ & & & I_r \end{bmatrix} \] (39)

with \( I_j \) (\( j = 1,\ldots,r \)) identity matrix of dimension \( m_j \times m_j \). Finally, \([\varphi_i(\alpha_m)] \) (\( i = 1,\ldots,n \), \( m = 0,\ldots,n - 1 \)) is the same matrix as the one defined in eqn. 27. Now, from the expression 37, the following result is derived:

**Lemma 2:** The realisation \((A,b,c)\) is completely \( n \)-observable (observable in \( n \) steps) if, and only if, \( n \) consecutive sampling instants are chosen in such a way that
\[ \text{Det} [\varphi_i(\alpha_m)] \neq 0 \quad (i = 1,\ldots,n; \ m = 0,\ldots,n - 1) \] (40)

**Proof:** The proof is evident from the factorisation of the expression 37.

Note that, for this kind of system, the \( n \)-observability depends on the characteristic modes of the continuous system and on the choice of the sampling instants. We remark that the result is analogous to that of the preceding subsection.
3.3 Main result

The results obtained in the two preceding subsections can be summarised as follows:

Theorem: A system verifying the conditions (a), (b) and (c) in Section 2, is jointly \( n \)-reachable and \( n \)-observable, if, and only if, \( n \) consecutive sampling instants are chosen in such a way that

\[
\text{Det} [\varphi_i(\alpha_m)] \neq 0 \quad (i = 1, \ldots, n; \quad m = 0, \ldots, n - 1)
\]  

(41)

Proof: The proof is evident from the use of the results obtained in Sections 3.1 and 3.2.

The condition 41 imposes a rather weak restriction for the choice of the sampling instants. In fact, intervals can be specified \[3\], \[7\] so that complete reachability (observability) is preserved.

We can also remark that, for this kind of system, the \( n \)-reachability and \( n \)-observability are inseparable concepts. These systems are either \( n \)-reachable and \( n \)-observable or, if not, they are neither \( n \)-reachable nor \( n \)-observable.

It must be noticed that the condition 41 depends exclusively on the characteristic modes and the sampling sequence. Thus, the \( n \)-reachability (\( n \)-observability) for these systems is independent of the chosen realisation.

If the system does not verify the condition (b), then the results obtained will still be valid for the subsystem controllable and observable.

Note how reachability involves statements about inputs and states, while observability about outputs and states. Nevertheless, both aspects are related to the same intrinsic system properties. Consequently, the algebraic characterization for reachability and observability can be unified.

4 Problems of sensitivity of system observability and controllability

In stability theory, it is not enough to know whether a system is stable or unstable but also the degree of stability. In a similar way, it is important to know the degree of observability (controllability) of the system; in the sense that there are different levels of certitude in the process of resolution of the corresponding linear equations.

From a heuristic viewpoint, the degree of observability (controllability) depends on the spatial relationship among the column vectors of the observability (controllability) matrix. In particular, maximum observability (controllability) is obtained when these vectors are mutually orthogonal \[7\].

Different geometric structures, which show the evolution in time of the above mentioned vectors, can be found in \[8\]. Thus, the question is to choose conveniently the sampling instants to obtain maximum orthogonality.

We start from an arbitrary minimal realisation \((A, b, c)\) of the given system eqn. 3. The controllability matrix
can be easily reduced to the matrix

\[ [Y_0, Y_1, \ldots, Y_{n-1}] \] (43)

\[ Y_i = \exp(J(t_{n-1} - t_{n-1-i}))y_0 \quad (i = 0, \ldots, n-1) \] (44)

\( J, y_0 \) defined as before) more convenient for the geometric interpretation; as the factor \( B \exp(J(t_n - t_{n-1})) \) in eqn. 15 affects only the module of the column vectors but not their spatial position. Analogous results can be obtained making use of the transpose observability matrix.

Note that matrix 43 depends exclusively on the system characteristic modes and the sampling sequence; thus the degree of observability and controllability is independent of the chosen realisation. Now the problem is reduced to a right choice of the sampling instants, in such a way that the vectors \( Y_i \) are mutually orthogonal. Some qualitative examples are presented:

**Example 1:** We are going to consider a 2nd-order model with a pair of complex eigenvalues

\[ a + jb \in \mathbb{C} \quad (b > 0) \] (45)

then

\[ [Y_0, Y_1] = [\exp(J\alpha_0)y_0, \exp(J\alpha_1)y_0] \] (46)

\[ J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad (b > 0) \] (47)

\( \alpha_i, y_0 \) defined as before.
As we are in $\mathbb{R}^2$, the geometric interpretation is very simple. The generic operator $\exp(J\alpha)$ applied to the vector $y_0$ can be viewed as follows. It is a counterclockwise rotation through $b\alpha$ radians, followed by a stretching (or shrinking) of the length of $y_0$ by a factor $\exp(a\alpha)$ [4].

From this interpretation, it is easy to see (Fig. 1) that the vectors $Y_0, Y_1$, will be mutually orthogonal if, and only if,

$$b\alpha_1 = b(t_1 - t_0) = (2m + 1)\frac{\pi}{2} \quad (m = 0, 1, \ldots) \quad (48)$$

In this case, maximum observability (controllability) is achieved. In this situation, expression 48 coincides with the results obtained by Troch [7] for the same model, when the problem of a minimum transmission of the measuring errors is discussed.

Example 2: Let us now study a 3-order model with a real pole and a complex pair:

$$\lambda \in \mathbb{R}, \quad a + jb \in \mathbb{C} \quad (b > 0) \quad (49)$$

Therefore, then

$$[Y_0, Y_1, Y_2] = [\exp(J\alpha_0)y_0, \exp(J\alpha_1)y_0, \exp(J\alpha_2)y_0] \quad (50)$$

with

$$J = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & \lambda \end{bmatrix} \quad (b > 0) \quad (51)$$

$\alpha, y_0$ as usual. The problem is treated in the 3-dimensional space and the geometric structure can be described as follows: the generic vector $Y(\alpha)$ is written as

$$Y(\alpha) = (\exp(aa) \cos(b\alpha), \exp(aa) \sin(b\alpha), \exp(\lambda\alpha))^\prime \quad (52)$$

thus it is a spiral on the surface revolution:

$$z = (x^2 + y^2)^{\lambda/2a} \quad (53)$$

whose form is determined by the real part of the system eigenvalues.

The vectors $Y_0, Y_1, Y_2$ have their origin at the point $(0, 0, 0)$ and their end at the points $Y(\alpha_0), Y(\alpha_1), Y(\alpha_2)$ of the parametric curve (Fig. 2). Heuristically we can imagine the same vectors as before (2-dimensional example) pointing upwards from the $XY$-plane as they have a third component on the $Z$-axis.

Given $Y_0$ and $Y_1$, the question is now to choose the sampling instant such that the vector $Y_2$ obeys its motion law and, simultaneously, is the most orthogonal vector to the other two.

In fact, let $P_0, P_1, P_2$ be the projection on the $XY$-plane of the vectors $Y_0, Y_1, Y_0 \times Y_1$, respectively:

$$P_0 = (x_0, y_0) = (\exp(aa_0) \cos(b\alpha_0), \exp(aa_0) \sin(b\alpha_0)) \quad (54)$$
Figure 2: Relationships among the vectors $Y_0, Y_1$ and $Y_2$

\[ P_1 = (x_1, y_1) = (\exp(a \alpha_1) \cos(b \alpha_1), \exp(a \alpha_1) \sin(b \alpha_1)) \]  

\[ P_2 = (x_2, y_2) = (y_0 z_1 - y_1 z_0, x_1 z_0 - x_0 z_1) \]  

with

\[ z_i = \exp(\lambda \alpha_i) \quad (i = 0, 1) \]

The sampling sequence must be selected such that

\[ b \alpha_2 = \arccos \frac{\langle P_0, P_2 \rangle}{|P_0| |P_2|} = M \]

then

\[ b(t_2 - t_0) = M + 2\pi m \quad (m = 0, 1, \ldots) \]

This number can be understood as the angle of a counterclockwise rotation. To obtain the more adequate value for $m$, we analyse the geometry of this situation. Let $Q_2$ be a vector orthogonal to $Y_0$ and $Y_1$ with its end on the revolution surface. Indeed,

\[ Q_2 = \mu(Y_0 \times Y_1) \]
with $\mu$ easily computable according to eqn. 53; then the best value of $m$ will be the integer for which the expression
\[
\left| \left( \mu^2(x_2^2 + y_2^2) \right)^{\lambda/2a} - \exp\left( \frac{\lambda}{6} (M + 2\pi m) \right) \right|
\]
(61)
is minimum. The process is computed again for each new sampling instant.

These examples show that the optimal sampling sequences involve only differences between sampling instants, but not absolute positions of such instants on the time axis. It perfectly agrees with the kind of time invariant systems we are considering.

5 Conclusions

It has been shown that, for completely observable and controllable systems described by eqns. 3, a joint characterisation of the $n$-reachability and $n$-observability of the corresponding discrete systems can be given. The formulation considered stresses the importance of the sampling instants to preserve the aforementioned internal properties. In this way, such systems have an additional element for analysis and manipulation.

There are no overrestrictive conditions on the choice of the sampling sequence. Moreover, it has also been possible to give strategies for some special cases. At the same time, different geometric structures have been used to improve the sensitivity of system observability and controllability, according to a correct selection of the sampling instants. These geometric structures can also be used with other performance criterions to give some insight on the system evolution in terms of familiar concepts.

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