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Some Properties of Yao $Y_4$ Subgraphs

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Abstract

The Yao graph for $k = 4$, $Y_4$, is naturally partitioned into four subgraphs, one per quadrant. We show that the subgraphs for one quadrant differ from the subgraphs for two adjacent quadrants in three properties: planarity, connectedness, and whether the directed graphs are spanners.

1 Introduction

The Yao graph is defined for an integer parameter $k$; here we study only $k = 4$, and call $\vec{Y}_4$ the directed Yao graph, and $Y_4$ the undirected version. For a set of points $P$, $\vec{Y}_4$ connects each point to its closest neighbor in each of the four quadrants surrounding it, defined as in Figure 1. Ties are broken arbitrarily. The undirected graph $Y_4$ simply ignores the direction.

Figure 1: Definition of quadrants. Solid lines are closed, dotted lines are open.

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The question of whether $Y_4$ is a spanner was raised in [DMP09]. A $t$-spanner has the property that the path between $a$ and $b$ in the graph is no longer than $t|ab|$, for a constant $t$. In this note, we do not further motivate the study of $Y_4$, but rather investigate some properties of subgraphs of $Y_4$, which may ultimately have some bearing on whether it is a spanner.

We make two “general position” assumptions:

1. No two pair of points determine the same distance (so there are no ties).
2. No two points share a vertical or horizontal coordinate.

These assumptions simplify the presentation. In this note, we will not explore whether the assumptions can be removed while retaining all the results.

Notation. $Q_i(a, b)$ is the circular quadrant whose origin is at $a$ and which reaches out to $b$. Often the subscript $i$ will be dropped, as it is determined by $a$ and $b$. $Q_i(a)$ is the unbounded quadrant with corner at $a$. Thus, $Q_i(a, b) = Q_i(a) \cap \text{disk}(a, |ab|)$. $R(a, b)$ is the closed rectangle with opposite corners $a$ and $b$.

We focus on two adjacent quadrants, $Q_0$ and $Q_1$. Let $Y_4^\lambda$ be the $Y_4$ graph restricted to the quadrants in the list $\lambda$. See Figure 2 for examples.

Our results are summarized in Table 1.

| Property     | $Y_4^{(i)}$ | $Y_4^{(i,i+1)}$ |
|--------------|-------------|-----------------|
| Planarity    | planar      | not planar      |
| Connectedness| not connected| connected       |
| Undirected spanner | not a spanner | not a spanner |
| Directed spanner | spanner      | not a spanner   |

Table 1: Summary of Results

2 Planarity

It is known that $Y_4^{(i)}$ is a planar forest, in general disconnected; see Figure 2(a,b). This is folklore but we offer a proof of planarity.

Lemma 1 No two edges of $Y_4^{(i)}$ properly cross.

Proof: Let both $ab$ and $cd$ be in $Y_4^{(0)}$, and suppose $ab$ and $cd$ properly cross. see Figure 3. The quadrants $Q(a, b)$ and $Q(c, d)$ must be empty of points. We consider three cases, depending on the location of $c$ w.r.t. $a$.

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1 Mirela Damian [private communication, Feb. 2009].
Figure 2: $Y_4^{\{0\}}$, $Y_4^{\{1\}}$, and $Y_4^{\{0,1\}}$, for the same 40-point set.
1. $c \in Q_3(a)$. Then $cd$ crosses $ab$ from below. We analyze just this case in detail. Because $b \not\in Q(c,d)$, the circular boundary of $Q(c,d)$ must cut $ab$, say at $x$. Consider two further cases

   (a) The slope of the arc of $Q(c,d)$ at $x$ is shallower than the slope of the arc of $Q(a,b)$ at $b$; see Figure 3(a). Then $d \in Q(a,b)$.

   (b) The slope at $x$ is equal to or steeper than that at $b$. Then, because $c$ is strictly below $a$, the radius $|cd|$ is greater than $|ab|$. But then $c$ cannot be in $Q_3(a)$.

2. $c \in Q_2(a)$. Then $cd$ could cross $ab$ from below, Figure 3(b), or from above, Figure 3(c). In both cases, a quadrant that must be empty is not.

3. $c \in Q_1(a)$. This case is the same as the first case, with the roles of $a$ and $c$ interchanged.

   In contrast, $Y_4^{(i,i+1)}$ may be nonplanar. Figure 4(a) shows two crossing edges; (b) shows the full graph $Y_4^{(0,1)}$.

   As should be evident from Figure 2(c), crossing edges are rare, requiring precise placement of four points. Although it would be difficult to quantify, a “typical” $Y_4^{(i,i+1)}$ graph is planar.

3 Connectedness

We can see in Figure 2(a,b) that $Y_4^{(i)}$ is, in general, disconnected. In contrast, $Y_4^{(i,i+1)}$ is connected. See again Figure 2(c).

**Lemma 2** $Y_4^{(i,i+1)}$ is a connected graph.
Proof: We choose \(i=0\) w.l.o.g. So we are concerned with upward +y-connections, in \(Q_0\) and \(Q_1\). The proof is by induction on the number of points \(n\) in the set \(P\). The basis of the induction is trivial, for an \(n=1\) point set is connected. Let \(P\) have \(n > 1\) points, and let \(a\) be the point with the lowest \(y\)-coordinate. By Assumption (2), \(a\) is unique.

Delete this from \(P\), reducing to a point set \(P'\) with \(|P'| = n-1\). Then the set of points \(P' = P\setminus\{a\}\) satisfies the induction hypothesis, and so is connected into a graph \(G'\). See Figure ??.

Put back point \(a\). Because all the quadrants determining edges \(bc \in G'\) are \(Q_0\) or \(Q_1\), they lie at or above \(b_y\), the \(y\)-coordinate of the lowest point in \(P\), \(b\). Thus \(a\) cannot lie in any quadrant, and so adding \(a\) to \(P'\) does not break any edge of \(G'\). Finally, \(a\) itself must have at least one outgoing edge upward, for \(Q_0\) and \(Q_1\) cover the half-plane above \(a_y\), which contains at least one point of \(P'\).

\(\square\)

4 Undirected Spanners

It is clear that \(Y_4^{\{i\}}\) is not a spanner, because it may be disconnected. Points on a negatively sloped line result in a completely disconnected graph of isolated points. Neither is \(Y_4^{\{i,i+1\}}\) a spanner. Points uniformly spaced on two lines forming a ‘Λ’ shape both have directed paths up to the apex in \(Y_4^{\{0,1\}}\), but the leftmost and rightmost lowest points can be arbitrarily far apart in the graph.

5 Directed Spanners

We turn then to directed versions of these questions. Call a directed graph a directed spanner if every directed path is no more than \(t\) times the path’s

\(^2\) Note that if the induction instead removed the topmost point from \(P\), this claim would no longer hold.
end-to-end Euclidean distance, for \( t \) a constant.

**Lemma 3** \( \overrightarrow{Y_{4}^{i,j}} \) is a directed spanner: no directed path is more than \( \sqrt{2} \) times the end-to-end Euclidean distance.

**Proof:** Let \( a \) and \( b \) be the endpoints of the path. Then the path is an \( xy \)-monotone path remaining inside \( R(a,b) \). Therefore its length is at most half the perimeter of this rectangle, which is at most \( \sqrt{2} \) times the diagonal length. \( \square \)

**Lemma 4** \( \overrightarrow{Y_{4}^{i,i+1}} \) is not a directed spanner: directed paths can be arbitrarily long: more than any constant \( t > 1 \) times the end-to-end Euclidean distance.

**Proof:** Consider the path \((a, b, c, d)\) in Figure 6(a). It is clear that this path can be made arbitrarily long with respect to \(|ad|\), by lowering the vertical coordinates of \( c \) and \( d \). Now we show how to avoid any other directed connection between \( a \) and \( d \).

Let the other outgoing edge from \( a \) go to \( e \) as shown. We now direct paths from \( d \) and from \( e \) that do not connect. The idea is depicted in Figure 6(b). We create a series of nearly vertical paths from \( d \), and from \( e \). Above \( d = (d_x, d_y) \), two points are placed at \((d_x \pm \epsilon, d_y + 1)\), \( 0 < \epsilon \ll 1 \). The two outgoing edges from \( d \) will terminate on these. Then above those we place two more points at \((d_x \pm 2\epsilon, d_y + 2)\). Now we get both upward and diagonal connections among the four points, with one “diagonal” being horizontal. The point is that all the outgoing edges are accounted for.

Repeating this construction, we can make a nearly vertical tower of points, connected by vertical paths, but otherwise insulated from one another. So the only path from \( a \) to \( d \) is \((a, b, c, d)\). \( \square \)

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\[3\] The definition in Figure 1 shows that \((d_x - \epsilon, d_y + 1)\) will connect horizontally to \((d_x + \epsilon, d_y + 1)\).
Figure 6: An arbitrarily long path in $Y_{4}^{(0,1)}$.

6 Future Work

The obvious next step is to examine properties of three quadrants, $Y_{4}^{(i,i+1,i+2)}$, before finally tackling $Y_{4}$ itself.

References

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