The symplectic structure of general relativity in the double-null (2+2) formalism

RICHARD J. EPP
Department of Physics
University of California
Davis, CA 95616
USA

Abstract

In the (2+2) formulation of general relativity spacetime is foliated by a two-parameter family of spacelike 2-surfaces (instead of the more usual one-parameter family of spacelike 3-surfaces). In a partially gauge-fixed setting (double-null gauge), I write down the symplectic structure of general relativity in terms of intrinsic and extrinsic quantities associated with these 2-surfaces. This leads to an identification of the reduced phase space degrees of freedom. In particular, I show that the two physical degrees of freedom of general relativity are naturally encoded in a quantity closely related to the twist of the pair of null normals to the 2-surfaces. By considering the characteristic initial-value problem I establish a canonical transformation between these and the more usually quoted conformal 2-metric (or shear) degrees of freedom. (This paper is based on a talk given at the Fifth Midwest Relativity Conference, Milwaukee, USA.)

1email: repp@dirac.ucdavis.edu
The $(2+2)$ formulation of general relativity was first introduced by Sachs [1], and has since then attracted the interest of many researchers [2, 3, 4, 5, 6, 7, 8, 9]. The basic idea is to foliate spacetime by a two parameter family of spacelike 2-surfaces, rather than the more usual one parameter family of spacelike 3-surfaces. One can then construct 3-surfaces by stringing together one parameter families of these spatial 2-surfaces. Since the geometry of spacetime is encoded in intrinsic and extrinsic quantities associated with the spatial 2-surfaces only, the 3-surfaces can just as easily be spacelike, timelike, or null. In particular, one can avoid having to deal with the degenerate 3-metric on a null 3-surface.

The formalism is ideally suited to the study of spacetimes (or subsets thereof) with topology $R^{2} \times S$, where $S$ is a spacelike compact 2-surface without boundary. A good example is the extended Schwarzschild black hole. Furthermore, in the double-null gauge of the $(2+2)$ formalism spacetime is foliated by two sets of null 3-surfaces, which intersect in the spacelike 2-surfaces $S$. The bifurcate Killing horizon of the Schwarzschild black hole, for instance, can be identified with a pair of such null 3-surfaces, intersecting in the bifurcation 2-sphere. Also, a quantity such as the expansion of null geodesics normal to $S$ (used in the definitions of an apparent horizon and a trapped surface, for example [10]) is a natural quantity in the $(2+2)$ formalism. In short, this formalism provides a very convenient setting for the study of many questions in general relativity. For a recent, more comprehensive introduction see [9].

The main new result presented here is the expression of the symplectic structure of general relativity in terms of intrinsic and extrinsic quantities associated with the spatial 2-surfaces of the double-null $(2+2)$ formalism. I also introduce a new variable, closely related to the twist of the null normals to the 2-surfaces, which encodes the two physical gravitational degrees of freedom. This provides an interesting alternative to the more usual conformal 2-metric (or shear) degrees of freedom.

The paper is organized as follows. I begin with an introduction to the ideas and language of the double-null $(2+2)$ formalism, in particular the intrinsic and extrinsic quantities associated with the spatial 2-surfaces. Then I say a few words about gauge-fixing in order to understand geometrically how the twist encodes gravitational degrees of freedom. The symplectic structure is then presented, in two versions: one emphasizing the twist degrees of freedom, and the other the shear (or conformal 2-metric) degrees of freedom. In passing, I make a few comments about what is held fixed in the action principle when the spacetime boundary is null (at least for the Einstein-Hilbert action). Finally, I discuss the characteristic initial-value problem. Since the phase space can be identified with the space of initial data, this helps to illuminate the symplectic structure results. It also allows one to establish a canonical transformation between the twist and shear sets of degrees of freedom.
1. Double-null (2+2) formalism

We consider a manifold $M = R^2 \times S$, where $S$ is a compact 2-surface without boundary. For simplicity we restrict ourselves to the case $S = S^2$, which is relevant for black hole spacetimes. Introduce coordinates $x^A (A, B, \ldots = +, -)$ for $R^2$ and (local) coordinates $x^i (i = 1, 2)$ for the 2-sphere. Now equip $M$ with a (time-orientable) Lorentzian metric and apply a suitable (active) diffeomorphism such that the 3-surfaces $x^\pm = const$ (denoted by $\Sigma^\pm$) are null, and their intersecting 2-surfaces $S$ are spacelike. We shall work in this so-called double-null gauge, in which the spacetime metric takes the form

$$g_{ab} = h_{ab} - 2e^{-\lambda}\partial(a)x^+\partial_b x^-, \quad (1.1)$$

where $h_{ab}$ is the induced metric on $S$, and $a, b, \ldots$ denote abstract spacetime indices. At any point $p \in M$ the subspace of $T_p M$ orthogonal to $T_p S$ is spanned by null vector fields of the form

$$n^a_A := (\partial_A)^a - s^a_A, \quad (1.2)$$

where $\partial_A := \partial/\partial x^A$, and $s^a_A$ are shift vectors lying in $T_p S$. See Figure 1. It is assumed that both $n^a_A$ are future-pointing, and since

$$g_{ab} n^a_A n^b_A = -e^{-\lambda}, \quad (1.3)$$

we see that the scalar $\lambda$ is associated with their normalization.

Now we introduce the extrinsic fields, which measure how $S$ is imbedded into $M$. First there is the extrinsic curvature, which carries an index $A$ since there are now two normal directions to consider:

$$K_{Aab} := \perp \frac{1}{2} \mathcal{L}_A h_{ab}. \quad (1.4)$$

Here $\mathcal{L}_A$ denotes the Lie derivative with respect to $n^a_A$, and $\perp$ means spatial projection of all indices to the right by $h^a_{\ b}$. The trace of the extrinsic curvature is called the expansion:

$$\theta_A := h^{ab} K_{Aab}, \quad (1.5)$$

whose physical interpretation is clear from the relation

$$\perp \mathcal{L}_A \epsilon = e\theta_A, \quad (1.6)$$

However, surfaces of higher genus may introduce interesting topological degrees of freedom into the following analysis [19].
where $\epsilon$ denotes the volume form on $S$. In this relation $\perp$ acts on the volume form indices, which for convenience are suppressed. The trace-free part,

$$\sigma_{Aab} := K_{Aab} - \frac{1}{2} \theta_A h_{ab},$$

is called the shear. Defining the (inverse) conformal 2-metric

$$\tilde{h}^{ab} := \epsilon h^{ab},$$

its Lie derivative along the null normals is the shear density:

$$\tilde{\sigma}^{ab} A := \epsilon \sigma^{ab} A = - \perp \frac{1}{2} \mathcal{L}_A \tilde{h}^{ab}. \tag{1.9}$$

In mixed indices the shear has the standard form of a current, bilinear in the conformal 2-metric:

$$\sigma_{A c} = \perp \frac{1}{4} \tilde{h}^{ab} \mathcal{L}_A \tilde{h}_{bc}, \tag{1.10}$$

and can be interpreted as a gravitational wave current (see, for example, [4]). Here $\tilde{h}_{ab} := \epsilon^{-1} h_{ab}$ is the conformal 2-metric.

Another extrinsic quantity, one which will play an important role in our discussions, is the [normalized—cf (1.3)] twist

$$\omega^a := - \epsilon^\lambda [n_+, n_-]^a, \tag{1.11}$$

which is tangent to $S$ and measures the nonintegrability of the null normals. Finally, there is the “inaffinity” [1]

$$\nu_A := \mathcal{L}_A \lambda. \tag{1.12}$$

Its name derives from the fact that

$$n^b_A \nabla_b n^a_A = - \nu_A n^a_A \quad \text{(no sum on $A$)}, \tag{1.13}$$

so the $n^a_A$ generate null geodesics, but generally with non-affine parametrization. (Here $\nabla$ denotes the spacetime covariant derivative operator.)

2. Gauge-fixing and true degrees of freedom

It is well known that the gravitational field in general relativity has two degrees of freedom per space point. In the double-null $(2+2)$ formalism there are several quantities that have two independent components, each of which is therefore a good candidate for encoding the two gravitational degrees of freedom: the shear, being
symmetric and trace-free; the conformal 2-metric, which has unit determinant (in the dyad basis); and the twist, which is a vector tangent to \( S \). Of these three, the first two are closely related [see (1.9)], and usually it is one or the other of these which is used to represent the true gravitational degrees of freedom. In this paper I will emphasize instead the twist, so it is instructive at this point to say a few words about gauge-fixing, and elaborate on the physical interpretation of the twist.

Let us restrict our attention to the wedge of spacetime defined by \( x^A \geq 0 \). The null 3-surfaces \( x^\pm = 0 \) which bound this region are denoted by \( \Sigma^\pm \), and their intersection by \( S_0 \). The \( \Sigma^\pm \) 3-surfaces should be thought of as congruences of null geodesics, as shown in Figure 2a. Now consider an (active) diffeomorphism generated by the vector field

\[
\xi^a := \xi^a_\perp + \xi^a_\parallel := \xi^i_\perp (\partial_i)^a + \xi^A_\parallel n^a_A,
\]

where \( \partial_i := \partial/\partial x^i \). It turns out that the double-null gauge is preserved provided the \( \xi_\parallel \) diffeomorphisms satisfy the restrictions

\[
\mathcal{L}_{\pm} \xi^\pm_\parallel = 0.
\]

Geometrically, such \( \xi_\parallel \) diffeomorphisms correspond to moving null geodesics from any one \( \Sigma^A \) plane to other \( \Sigma^A \) planes, such that all these planes remain congruences of null geodesics. A gauge-fixing condition we shall find useful, and which is always reachable, is

\[
\mu_A := \nu_A + \kappa \theta_A = 0 \quad \text{on} \quad \Sigma^A_0,
\]

where \( \kappa \) is any constant (to be chosen later). Although this does not completely fix the \( \xi_\parallel \) gauge\(^2\) it is sufficient for our present purposes.

Turning attention to the remaining “\( S \)-diffeomorphisms”—those acting along the “fibers” \( S \)—it is clear from Figure 2a that one can always do a series of such \( \xi_\perp \) diffeomorphisms to “straighten-out” the null geodesics in, say, all of the \( \Sigma^+ \) planes. This corresponds to gauge-fixing one of the shift vectors to zero:

\[
s^a_+ = 0.
\]

Furthermore, one can still do \( S \)-diffeomorphisms which are the same for each \( S \) foliating a given \( \Sigma^+ \) (i.e. independent of \( x^+ \)), but which may differ from one \( \Sigma^+ \) to another. This freedom can be exactly used up by straightening-out the null geodesics on just one \( \Sigma^- \), say \( \Sigma^-_0 \). This corresponds to the gauge-fixing condition

\[
s^a_- = 0 \quad \text{on} \quad \Sigma^-_0.
\]

\(^2\)A further condition might be to restrict \( \xi^\pm = 0 \) on \( S_0 \) [and hence on \( \Sigma^\pm_0 \)—see (2.3)], i.e. to not allow the movement of null geodesics into or out of the wedge \( x^A \geq 0 \). But even in this case there remains a residual set of nontrivial \( \xi_\parallel \) diffeomorphisms which preserves (2.3)\(^3\).
Now, while this does not completely fix the $\xi_\perp$ gauge\footnote{One is still free to do $S$-diffeomorphisms which are the same for each $S$ foliating the spacetime wedge $x^A \geq 0$ (i.e. independent of $x^+$ and $x^-$). Up to a conformal Killing vector subtlety, this freedom can be exactly used up by fixing the inverse conformal 2-metric $\tilde{h}^{ab}$ on $S_0$ \cite{13}.} it is again sufficient for our purposes to stop here. See Figure 2b. Note that in this hierarchy of gauge-fixing conditions [\eqref{2.3}--\eqref{2.5}], reaching each successive condition preserves the previous ones.

The important point is that (almost) all of the gauge degrees of freedom have now been fixed, so any nontrivial degrees of freedom remaining must be physical: these are the two components of the twist, which encode, for example, the failure of points $p$ and $p'$ in Figure 2b to coincide. In fact, inspection of the symplectic structure (discussed in the next section) reveals that the relevant quantity is actually a slightly modified form of the twist:

$$\omega_\pm a := \pm \omega_a + D_a \lambda,$$

where $D$ is the covariant derivative operator in $S$. We remark, however, that this cannot be the whole story: the Schwarzschild solution, for example, has $\omega_\pm a = 0$ for all values of the mass. This subtlety is related to the word “almost” used in parenthesis at the beginning of this paragraph \cite{13}. The Kerr example will be analysed in detail elsewhere, where it is expected that the twist will be proportional to the black hole angular momentum \cite{19}.

3. Action principle and symplectic structure

It has long been known that the phase space of a classical system can be understood in a covariant way (i.e. with no preferred time slice) by considering it to be the space of classical solutions, $S$ \cite{16}. We shall briefly review the standard construction of the symplectic structure on $S$ (see, for example, \cite{11}), adapted here to the $(2+2)$ splitting of spacetime.

Let us consider the quantity of action in an “evolution region” $\mathcal{E}$ of $M$, defined by $0 \leq x^A \leq 1$ (see Figure 3):\footnote{Note that there is no loss of generality in taking endpoints at $x^A = 1$ since we can always effectively rescale the $x^A$-axes by (active) diffeomorphisms of the fields.}

$$I_\mathcal{E} = \int_\mathcal{E} dx^+ dx^- L(\varphi).$$

The Lagrangian $L(\varphi)$ is a functional of fields, collectively denoted as $\varphi$, and takes the form of an integral over $S$. Variation of the action results in a term proportional to
to the Euler-Lagrange equations, which vanishes when $\phi \in \mathcal{S}$, leaving the surface term:

$$
\delta I_E = \int_{\partial \mathcal{E}} \left\{ J^+(\varphi, \delta \varphi) \, dx^- - J^-(\varphi, \delta \varphi) \, dx^+ \right\}.
$$

(3.2)

This defines the current components $J^A(\varphi, \delta \varphi)$, at least up to the ambiguity

$$
J^\pm(\varphi, \delta \varphi) \mapsto J^\pm(\varphi, \delta \varphi) \pm \partial Z(\varphi, \delta \varphi),
$$

(3.3)

for arbitrary $Z = Z(\varphi, \delta \varphi)$. This ambiguity will be exploited in section 3.2 below.

Now consider any 3-surface $\Sigma$ consisting of a one parameter family of spatial 2-surfaces $S$ stretching between $S_L$ and $S_R$, as shown in Figure 3. The presymplectic potential is defined as

$$
\Theta_\Sigma := \int_\Sigma \left\{ J^+(\varphi, \delta \varphi) \, dx^- - J^-(\varphi, \delta \varphi) \, dx^+ \right\},
$$

(3.4)

and the presymplectic structure as its second antisymmetrized variation:

$$
\Omega = \int_\Sigma \left\{ \Omega^+(\varphi, \delta_1 \varphi, \delta_2 \varphi) \, dx^- - \Omega^-(\varphi, \delta_1 \varphi, \delta_2 \varphi) \, dx^+ \right\},
$$

(3.5)

where

$$
\Omega^A(\varphi, \delta_1 \varphi, \delta_2 \varphi) := \delta_1 J^A(\varphi, \delta_2 \varphi) - (1 \leftrightarrow 2).
$$

(3.6)

Here $\delta_\mu \varphi \in T_\varphi \mathcal{S}$, $\mu = 1, 2$, and can be thought of as the partial derivative of $\varphi$ with respect to some (suppressed) solution space coordinates. $\Omega$ is the presymplectic structure evaluated at the phase space point $\varphi \in \mathcal{S}$, contracted with the two “vector fields” $\delta_1 \varphi$ and $\delta_2 \varphi$, which are solutions to the linearized Euler-Lagrange equations.

It is easy to show that, while $\Theta_\Sigma$ in general depends on the choice of $\Sigma$ interpolating between $S_L$ and $S_R$, $\Omega$ does not. For instance, we may equally well use either the Cauchy surface $\Sigma$ or the characteristic surface $\Sigma'$ in Figure 3 to evaluate $\Omega$. This is what is meant by the covariance of the phase space. It should also be noted that $\Omega$ is invariant under the addition of boundary terms to the action.

### 3.1. Action principle

For the Einstein-Hilbert action we obtain

$$
\begin{align*}
J^\pm(\varphi, \delta \varphi) &= \int_S \epsilon \left\{ -\sigma^{ab}(\delta h)^{\mu b}_T - \omega^{a}_{\pm a} s^a_{\pm} + \delta(\theta_{\pm} - 2\nu_{\pm}) \right\} + \partial_\pm \int_S \epsilon \delta(\ln \sqrt{h} + \lambda) \\
&= \int_S \epsilon \left\{ -\sigma^{ab}(\delta h)^{\mu b}_T - \omega^{a}_{\pm a} s^a_{\pm} + \delta(\theta_{\pm} - 2\nu_{\pm}) \right\} + \partial_\pm \int_S \epsilon \delta(\ln \sqrt{h} + \lambda).
\end{align*}
$$

(3.7)

$^{5}$Presymplectic (as opposed to symplectic) refers to the fact that $\Omega$ has degenerate directions on $T \mathcal{S}$; these are tangent to the gauge orbits of the theory $[16]$. An example will be given in section 3.2 below.
First observe that $J^\pm$ consists of a “bulk” term and an “edge” term—the latter makes contributions to (3.2) only at the “edges” $S_0$, $S_1$, $S_L$, and $S_R$ (see Figure 3). Next,

$$\delta h^{ab} = -h^{ab}\delta \ln \sqrt{h} + (\delta h)^{ab}_T$$

has been split into its trace and trace-free parts, respectively, where $h$ is the determinant of the spatial 2-metric in the dyad basis. Following the standard decomposition of a trace-free symmetric tensor \[12\] we can write

$$(\delta h)^{ab}_T = (\delta h)^{ab}_T + (Lv)^{ab}.$$  

Here $(\delta h)^{ab}_T$ is the transverse trace-free part of $\delta h^{ab}$, i.e. it is trace-free and satisfies $D_a(\delta h)^{ab}_T = 0$, where, as noted above, $D$ is the covariant derivative operator in $S$. In general, the transverse trace-free sector is spanned by the Teichmüller parameters associated with the topology of $S$ \[13\], but for our choice ($S = S^2$) this space is empty. All that remains is the “exact” part:

$$(Lv)^{ab} : = D^a v^b + D^b v^a - h^{ab} D^c v^c,$$  

which defines the vector $v^a \in TS$ uniquely up to conformal Killing vectors (CKV)—if any exist.

The form of $J^A$ given in (3.7) has the conformal metric and shear playing a dominant role. However, in this paper we wish to emphasize the twist degrees of freedom. To achieve this we integrate the $(Lv)^{ab}$ term by parts and use the following Euler-Lagrange equation:

$$\perp L^\pm \tilde{\omega}^\pm_a = \epsilon 2D^b \sigma^\pm_{ab} - \epsilon D(a(\theta^\pm - 2\nu^\pm),$$  

where $\tilde{\omega}^\pm_a : = \epsilon \omega^\pm_a$. After also integrating $\perp L^\pm \tilde{\omega}^\pm_a$ by parts we get

$$J^\pm(\varphi, \delta \varphi) = \int_S \left\{ \tilde{\omega}^\pm_a \Delta n^a_\pm + \epsilon \Delta(\theta^\pm - 2\nu^\pm) \right\} + \partial_\mp \int_S \left\{ \tilde{\omega}^\pm_a v^a + \epsilon \delta(\lambda + \ln \sqrt{h}) \right\}.$$  

Here we used $\delta s^a_\pm = -\delta n^a_\pm$, which is obvious from (1.2). Note that

$$\Delta := \delta + L_v,$$  

is a manifestly $S$-diffeomorphism invariant variation. This is because under an $S$-diffeomorphism generated by $\xi^a$ we have $\delta \varphi = -L_{\xi^a} \varphi$ (at least for the fields $\varphi$ that $\Delta$ will be applied to—for example, it is not true for the shift vectors). On the other hand, the following statement is not true:

$$(\theta^\pm - 2\nu^\pm)$$

appears naturally also in the Euler-Lagrange equation (3.11).

\[6\] The distinction between bulk and edge terms is not unambiguous; the form written down here is based partly on experience working with the equations. Note in particular that the combination $(\theta^\pm - 2\nu^\pm)$ appears naturally also in the Euler-Lagrange equation (3.11).
hand, it turns out that $v^a = \xi^a \perp \text{ (up to CKV)}$, so $\Delta \varphi = 0 \text{ (up to a possible CKV term)}$. Note that the ambiguity of $v^a \text{ up to CKV}$ appears to be a problem because $v^a \text{— not just } (Lv)^{ab} \text{— appears in the above formula for } J^A$. But in fact, $J^A \text{ (bulk plus edge terms)}$ is invariant under $v^a \mapsto v^a + v^a_{\text{CKV}}$, which is actually obvious since the original form [see (3.7)] depends only on $(Lv)^{ab}$. So in the case of $S$-diffeomorphisms the only contribution to $J^A$ comes from the edge terms, the result being

$$J^\pm(\varphi, \delta \varphi) = \mp \partial_\mp \int_S \tilde{\omega}_a \xi^a \perp \text{.} \quad (3.14)$$

Notice that this produces an exact differential in the integrand of (3.2), and so, since $\partial \partial E \equiv 0$, the action is indeed invariant under $S$-diffeomorphisms, as it should be.

The answer to the question of what is fixed on the boundary in the action principle of general relativity is well understood, at least for boundaries composed of spacelike and timelike sections [14]. The double-null (2+2) formalism appears to be ideally suited to extending our understanding to the case of null boundaries. Let us make one comment in this regard. Ignoring all but the “$\tilde{\omega} \Delta n$” term in (3.12) we learn that what is fixed on a null boundary are the generators of the null geodesics, up to $S$-diffeomorphisms. This represents two “$q$”s—half of the reduced phase space degrees of freedom. This simple answer is complicated by the other bulk term (which is associated with the parametrization of the null geodesics), as well as the edge terms. Also, of course, the question of what is held fixed is sensitive to boundary terms added to the Einstein-Hilbert action (to make it first order, for instance) [18].

It is instructive at this point to draw an analogy with York’s classic results [14] on the conformal 3-metric degrees of freedom of the gravitational field. For spacetime $M = R \times \Sigma$, $\Sigma$ spacelike and closed, the variation of the “cosmological action” $S_K$ is

$$\delta S_K = \int_{\Sigma_2 - \Sigma_1} d^3 x \left( \tilde{p}^{ij} \delta \tilde{\gamma}_{ij} + p_K \delta K \right). \quad (3.15)$$

Like the “$\tilde{\omega} \Delta n$” term in (3.12), the “$\tilde{p} \delta \tilde{\gamma}$” term represents two independent “$q$”s: five in the conformal 3-metric $\tilde{\gamma}_{ij}$, minus three because $\delta S_K = 0$ for spatial diffeomorphisms. The other bulk term, $p_K \delta K$, where $K$ is the trace of the extrinsic curvature of $\Sigma$, is “trivialized” by the gauge choice $K = \text{const}$ on the spatial 3-surfaces. The (2+2) analogue of $K$ appears to be $(\theta - 2 \nu)$, and the corresponding gauge choice would be $(\theta_\mp - 2 \nu_\mp) = \text{const}$ (or zero) on the spatial 2-surfaces foliating $\Sigma_0 \mp$. This gauge choice is discussed more in the following subsection.

**3.2. Symplectic structure**

Finally, let us calculate the symplectic structure. Equation (3.7) instructs us to take the second variation of $J^A$ and antisymmetrize. Using the fact that partials
commute (so $\delta_1 \delta_2 \varphi = \delta_2 \delta_1 \varphi$), the only further bit of information we need is

$$\delta_2 v_1^a - \delta_1 v_2^a = [v_1, v_2]^a,$$

(3.16)

which follows from the fact that the second antisymmetrized variation of both sides of (3.8) must vanish. We find

$$\Omega^\pm (\varphi, \delta_1 \varphi, \delta_2 \varphi) = \int_S \left\{ \Delta_1 \tilde{\omega}_a \Delta_2 n_+^a + \Delta_1 \epsilon \Delta_2 (\theta_+ - 2 \nu_+) \right\}$$

$$+ \partial_\pm \int_S \left\{ \Delta_1 \tilde{\omega}_a v_2^a + \frac{1}{2} \tilde{\omega}_a [v_1, v_2]^a + \delta_1 \epsilon \delta_2 \lambda \right\} - (1 \leftrightarrow 2).$$

(3.17)

The bulk term has the standard $\Delta p \wedge \Delta q$ form, which allows us to immediately identify canonically conjugate pairs of phase space variables. Note the appearance again of the $S$-diffeomorphism invariant variation, $\Delta$.

The edge term in (3.17) can be simplified as follows. As pointed out earlier, the choice of $J^\pm (\varphi, \delta \varphi)$ has the arbitrariness displayed in equation (3.3). Equation (3.14) suggests we chose

$$Z(\varphi, \delta \varphi) = \int_S \tilde{\omega}_a v^a,$$

(3.18)

in which case it can be shown that

$$\Omega^\pm (\varphi, \delta_1 \varphi, \delta_2 \varphi) \mapsto \ldots + \partial_\pm \int_S \Delta_1 \epsilon \Delta_2 \lambda - (1 \leftrightarrow 2).$$

(3.19)

Thus, in this form at least, $S$-diffeomorphisms are manifestly associated with degenerate directions of the symplectic structure, i.e. they are gauge transformations. This is not surprising, of course, but on the other hand, the situation is not nearly so clear for the $\xi_\parallel$ diffeomorphisms [satisfying (2.2)]. These will be analyzed elsewhere [18], and are expected to be associated with so-called edge degrees of freedom. (For a recent review of edge degrees of freedom and their possible role in explaining black hole entropy, see [17].)

Let us now remark on the repeated appearance of the combination of variables $(\theta_+ - 2 \nu_\pm)$ throughout the foregoing analysis. Recalling the gauge-fixing condition (2.3), we see that the symplectic structure provides strong support for the choice $\kappa = -1/2$, which would then eliminate an “unwanted” bulk term in (3.17). Comparing with the literature (for example [4]) we find instead that the usual choice is $\kappa = 0$: affine parametrization of the null geodesics on $\Sigma^A_0$. Alternatively, Hayward [5] points out that the choice $\kappa = +1/2$ simplifies (linearizes) one of the Euler-Lagrange equations, namely the Raychaudhuri, or “focusing”, equation. But this is opposite in sign to the $\kappa$ required to simplify the symplectic structure! It appears that there is some fundamental complexity here which shows up in one place or the another—it cannot be gauge-fixed away.
For completeness let us record also the “shear form” of the symplectic structure, obtained straightforwardly from (3.7), and the fact that \( \epsilon (\delta h)^{ab} = \perp \delta \tilde{h}^{ab} \):

\[
\Omega^\pm (\varphi, \delta_1 \varphi, \delta_2 \varphi) = \int_S \left\{ -\delta_1 \sigma_{\mp ab} \delta_2 \tilde{h}^{ab} - \delta_1 \tilde{\omega}_{\mp a} \delta_2 s^a_{\mp} + \delta_1 \epsilon \delta_2 (\theta_{\mp} - 2 \nu_{\mp}) \right\} + \partial_{\mp} \int_S \delta_1 \epsilon \delta_2 \lambda - (1 \leftrightarrow 2).
\]

(3.20)

If one imposes the gauge-fixing conditions discussed in section 2 [equations (2.3-2.5)] the only bulk term that survives (on \( \Sigma_0^A \) at least) is the shear term. Like the twist term in (3.17) it also encodes the two true degrees of freedom of the theory. However, ignoring for a moment the \( (\theta - 2\nu) \) term common to both descriptions, the twist term does so covariantly—without having to gauge-fix the shift vectors. The advantage is that the twist form is thus applicable even when \( \Sigma \) is not a characteristic 3-surface, in particular it can be taken to be Cauchy (see Figure 3). This may have important implications in a quantization programme. In this sense the (modified) twist is a more natural choice than the shear to describe the true degrees of freedom.

To help better understand both forms of the symplectic structure it is instructive to now examine the characteristic initial-value problem, since the phase space can also be thought of as the set of initial data.

4. Characteristic initial-value problem

The earliest discussion of the characteristic initial-value problem for general relativity is due to Sachs [1], followed by others. We shall not repeat this analysis here, but merely quote the main result: the initial data required to obtain a unique solution in the wedge \( x^A \geq 0 \) is indicated in Figure 4a [1]. The placement of the various fields in the figure has the following meaning: \( s^a_{\pm} \) is specified everywhere in the wedge; \( s^a_{\mp} \) is specified only on \( \Sigma_0^- \); \( \tilde{h}^{ab} \) is specified only on \( S_0 \); etc. This might be called the “shear version” of the initial-value problem, where \( \tilde{\sigma}_{\pm}^{ab} \) on \( \Sigma_0^\mp \) is the main physical data. In Figure 4b we introduce a “twist version”, which emphasizes the alternative data \( \tilde{\omega}_{\pm a} \) on \( \Sigma_0^\pm \). It is instructive to at least outline the proof that these two sets of initial data are equivalent.

Start with the data in Figure 4b, which provides, in particular, \( \perp \mathcal{L}_{\pm} \tilde{\omega}_{\pm a} \) and \( (\theta_{\pm} - 2\nu_{\pm}) \) on \( \Sigma_0^\pm \). However, we know \( \sqrt{h} \)—and hence \( \epsilon \)—only on \( S_0 \). Thus, the Euler-Lagrange equation (3.11) gives us \( D^b \sigma_{\pm ab} \) on \( S_0 \) (only). Now, using the fact

\[\text{Note that we do not gauge-fix any of the initial data; we shall elaborate elsewhere on the splitting of this data into physical and gauge parts [15]. Also, we are not concerned here with the question of existence of solutions, or caustics they may develop.}\]
that $\sigma_{\pm ab}$ is trace-free (and $S = S^2$) we can write

$$\sigma_{\pm ab} = (L\tau_\pm)_{ab} := D_a \tau_{\pm b} + D_b \tau_{\pm a} - h_{ab} D^c \tau_{\pm c},$$

(4.1)

which defines $\tau^a_\pm \in TS$ uniquely up to CKV. (Note that we know the full spatial metric on $S_0$.) Thus,

$$D^b \sigma_{\pm ab} = (D^2 + \frac{1}{2} R) \tau_{\pm a},$$

(4.2)

where $D^2$ is the Laplacian and $R$ the Ricci scalar on $S$. Given the left hand side, this equation can be solved for $\tau^a_\pm$, unique up to CKV [12]. We can then uniquely determine $\tilde{\sigma}_{\pm ab}$ on $S_0$. From here we use various Euler-Lagrange equations to iterate our way up $\Sigma^\pm_0$, at each step using the nonlocal transformation (4.2)—the heart of the canonical transformation from Figure 4b to 4a initial data. It is easy to show also the converse, from Figure 4a to 4b.

Finally, let us compare the initial data in Figures 4a and 4b with the corresponding symplectic structures: (3.20) and (3.17). The latter have the general form $\delta p \wedge \delta q$. Now normally $p$ and $q$ are independent, and can be identified with the initial data, but here the characteristic initial data tells us that only $p$ is independent, and $q$ is to be determined by integrating the Euler-Lagrange equations up $\Sigma^\pm_0$ from $S_0$. This is because essentially $p = \partial_\pm q$ on $\Sigma^\pm_0$; the independent data can be thought of as $p$ on $\Sigma^\pm_0$ and $q$ on $S_0$. In the symplectic structure, $q$ on $\Sigma^\pm_0$ is then determined nonlocally from this independent data by integration. Only when $\Sigma$ is a Cauchy surface do we recover the usual local symplectic structure, with $p$ and $q$ independent, and so on.

5. Concluding remarks

The (2+2) formulation of general relativity has resurfaced time and again since 1962 [1, 2, 3, 4, 5, 6, 7, 8, 9]. The main contribution of this paper is two formulas for the symplectic structure of general relativity in the double-null (2+2) formalism. These formulas immediately suggest two results: (i) an alternative, and in some respects more natural set of physical degrees of freedom based on the twist, rather than the shear or conformal 2-metric used in most of the previous (2+2) work, and (ii) an alternative non-affine parametrization of the null geodesics generating the characteristic initial-value 3-surface—in particular, “opposite” to Hayward’s suggestion [6]. By looking at the characteristic initial-value problem we also establish

\[\text{Notice that in order for solutions to exist the left hand side, or source term, must be orthogonal to the kernel of } (D^2 + \frac{1}{2} R) \text{. This amounts to restrictions on the twist initial data such that, for example, } J^A \text{ is invariant under } v^a \mapsto v^a + v^a_{\text{CKV}}, \text{ as discussed in the paragraph containing equation (3.12) [12].}\]
a (nonlocal) canonical transformation between the twist and shear type degrees of freedom. Finally, we also learn that in the action principle (based on the Einstein-Hilbert action) what is held fixed on a null spacetime boundary are the generators of the null geodesics, up to diffeomorphisms of the spatial 2-surfaces (and certain other subtleties which will be addressed elsewhere [18]).

It should be emphasized that the symplectic structure is a very powerful tool: classical mechanics is contained in the statement

\[ i_{X_f} \Omega = -df, \]

where \( \Omega \) is the symplectic structure and \( X_f \) is the Hamiltonian vector field canonically generated by the observable \( f \). In general relativity, for example, if \( X_f \) is associated with an asymptotic time translation of an asymptotically flat spacetime then \( f \) is the ADM mass [16]. In a related, but more general context, certain diffeomorphisms are quite nontrivial in a spacetime with boundary (such as a black hole), and result in so-called edge degrees of freedom. When quantized, the latter give rise to quantum gravitational boundary states, which may be important to providing a microscopic understanding of black hole entropy [17]. I hope to pursue these and other related issues elsewhere [19].

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[18] These and other details will be discussed more fully in forthcoming papers by the author.

[19] I expect to pursue this issue elsewhere.
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