Induced actions for higher spin fields

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Abstract. We have calculated two-point correlators of higher spin currents, with spins up to six, in free massive QFT's with scalar and Dirac fields in $D = 3, 4, 5, 6$ spacetime dimensions. The infrared and ultraviolet limits are analysed, with the motivation for improving our understanding of the interacting higher spin theories.

1. Introduction

The construction of interacting quantum field theories with massless higher spin ($s > 2$) fields still poses an interesting theoretical problem. On one hand, there are different no-go theorems putting serious constraints on such theories, especially in flat spacetime. On the other hand, free fields can be constructed in the same manner as in lower spin cases, and in AdS background there is a fully consistent covariant HS theory [1]. In addition, free lower spin quantum field theories possess conserved higher spin currents which simply beg to be coupled to higher spin fields. To gain a deeper insight into the structure of QFT, it is important to understand what happens when higher spin fields are included, either by constructing and developing such theories or finding obstructions which forbid such couplings.

Differences between lower spin and higher spin theories emerge already at the level of classical free field theories. The simplest way to construct a free theory of a HS field is provided by the Fronsdal equation [2]:

$$ F \equiv \Box \phi - \partial \partial \cdot \phi + \partial^2 \phi = 0, \quad (1) $$

where spin-$s$ field is described by the completely symmetric rank-$s$ tensor field $\phi = \phi_{\mu_1 \ldots \mu_s}$. In this expression standard conventions from [3-4] are assumed. The Fronsdal equation (1) is invariant under local transformations parameterized by a traceless completely symmetric rank-$(s - 1)$ tensor fields $A = A_{\mu_1 \ldots \mu_{s-1}}$

$$ \delta_s \phi = \partial A, \quad (2) $$

when $A' = 0$. Moreover, the Fronsdal equation can be written in Lagrangian formulation only if one assumes $\phi'' = 0$ off-shell.
As constraining local symmetries in interacting theories usually have dramatic and unwanted consequences, one would like to have a formulation of HS theories which has: (i) unrestricted gauge symmetry, (ii) Lagrangian formulation with interaction with matter included. This implies that the linearised equations of motion (LEoM) must be of the form

\[ \mathbf{E}_{\mu_1 \cdots \mu_s} \left[\phi\right] = \mathbf{J}_{\mu_1 \cdots \mu_s} \cdot \partial \cdot \mathbf{E} = 0, \partial \cdot \mathbf{E} = 0, \]  

(3)

where symmetric LEoM-tensor \( \mathbf{E}_{\mu^r} \) is off-shell conserved (i.e., divergence-free) and linear in the HS field \( \phi \), and the current \( \mathbf{J}_{\mu^r} \) is the matter current which is on-shell conserved (i.e., when EoM for matter fields are assumed). It is important to understand that even if two LEoM tensors \( \mathbf{E}_{\mu^r} \) are equivalent in the free case (for \( J = 0 \)), this does not imply that they lead to equivalent EoM when coupled to the generic external current.

For \( s > 2 \) a LEoM-tensor \( \mathbf{E}_{\mu^r} \) which gives an EoM equivalent to the Fronsdal equation (1) in the case of a free HS field must either higher-derivative or nonlocal. This can be traced to the property that generalisation of field strength tensor to HS (g-Riemann tensor) is of \( s \)-derivative type, which means that it is a higher derivative object for \( s > 2 \). The least non-local LEoM tensor, which is second-order in derivatives, is given by [3]

\[ \mathbf{E}_{\mu_1 \cdots \mu_s} = \Box^{-n} \mathbf{G}_{\mu_1 \cdots \mu_s}, \quad s = \begin{cases} 2(n + 1), & s \text{ even} \\ 2n + 1, & s \text{ odd} \end{cases}, \]  

(4)

where \( \mathbf{G}_{\mu^r} \) is g-Einstein tensor, the lowest derivative fully gauge invariant divergence-free symmetric tensor of rank \( -s \).\(^1\) The non-local operator \( \Box^{-n} \) makes LEoM (3) 2nd order in derivatives, and at the same time makes the free LEoM equivalent to Fronsdal equation (1).

In [4] a surprising result was obtained - for \( s > 2 \) LEoM (3) with (4) is not consistent with propagation of only spin- \( s \) excitations for generic conserved currents. In the same paper it was shown that this problem can be circumvented in two ways. One is by realising that for every \( s > 2 \) there is a \([s/2]\) parameter family of possible HS kinetic terms, all giving LEoM equivalent to the Fronsdal equation for a free HS field. In our notation, the family can be written as

\[ \mathbf{E}_{\mu^r} = \Box^{-n} \left( \mathbf{G}_{\mu^r} + \sum_{j=1}^{[s/2]} \xi_j \left( \pi_{\mu^r} \right)^j \mathbf{G}^{[j]}_{\mu^{s+1}} \right), \]  

(5)

where \( \pi_{\mu^r} \) is a symmetric transverse projector defined in (9), \( \xi_j \) are arbitrary real numbers, and \( n \) is defined as in (4). The choice (4), which is obtained from (5) by taking \( \xi_j = 0 \), is special in having a non-locality of the lowest degree. The important result of [4] was that for every \( s \) there is a choice of parameters \( \xi_j \) for which equation (3) is consistent in the above mentioned sense. However, contrary to (4), this consistent choice for the kinetic term always has the highest degree of non-locality. The second way to cure the problem is to assume that the current in (3) is not generic. In spin-3 case it was shown in [4] that if the current can be written as

\[ \mathbf{J}_{\mu^r} = \mathbf{J}_{\mu^r} - \frac{3d^2 - 6d}{d(d+1)} \pi_{(\mu^r} \mathbf{J}_{\rho)} \cdot \partial \cdot \mathbf{J} = 0, \]  

(6)

then LEoM tensor (4) is viable as a kinetic term in (9). Note that this redefinition is nonlocal.

\(^1\) In \( d = 3 \) for odd spins there is a parity-odd tensor which has one derivative less then g-Einstein tensor. In the main part of the paper we assume \( d > 3 \) and we treat the \( d = 3 \) case separately.
In [5] we proposed to study HS actions by using the induced action method [6] as an inspiration and developed this idea further in [7–8]. In these papers we calculated two-point correlators of higher spin currents, with spins up to six, in free massive QFTs with scalar and Dirac fields in $D = 3, 4, 5, 6$ spacetime dimensions. In the following sections we present some of the main results of these calculations, and its implications for the linearised HS actions. In Appendix A we briefly review the HS covariant formalism we use in the main part.

2. Higher spin currents

All free relativistic theories possess conserved currents of all spins, represented by totally symmetric tensor fields $J_{\mu_{1}...\mu_{s}}$. We shall concentrate here on currents with integer spin $s$, and consider currents in free scalar field and free Dirac field (Majorana field in $d = 3$ spacetime dimensions) QFT’s, which we denote as bosonic and fermionic matter, respectively. It can be shown that conserved currents in these theories, in the simplest form, are given by

$$J_{\mu_{1}...\mu_{s}}^{(s)} = i \phi^{*} \not\partial_{\mu_{1}}...\not\partial_{\mu_{s}} \phi,$$

$$J_{\mu_{1}...\mu_{s}}^{(s/2)} = i \phi^{*} \not\partial_{\mu_{1}}...\not\partial_{\mu_{s/2}} \not\partial_{\mu_{s/2+1}}...\not\partial_{\mu_{s}} \phi.$$

(7)

Note that simple currents of the bosonic matter are $s$-th order in derivatives, while currents of the fermionic matter are of $(s - 1)$-th order. We shall call these form simple currents.

As is well known from the spin-2 case, there is a degeneracy present in the definition of conserved currents with $s > 1$. Once we have a full set of currents, and all relativistic free QFT’s have such full set, one can construct a $[s/2]$ parameter family in the following way

$$J_{\mu_{1}}^{(s)} \rightarrow J_{\mu_{1}}^{(s)} + \sum_{j=1}^{[s/2]} \zeta_{j} (\Box_{\mu_{1}}) J_{\mu_{1}}^{(s-2j)},$$

(8)

where $\zeta_{j}$ are arbitrary real dimensionless parameters, and $\pi_{\mu\nu}$ is the symmetric transverse projector defined by

$$\pi_{\mu\nu} = \eta_{\mu\nu} - \Box^{-1} \partial_{\mu} \partial_{\nu} \Rightarrow \nabla \cdot \pi = 0, \quad \pi^{2} = \pi, \quad \pi_{\mu} = (D - 1).$$

(9)

In this paper we shall consider also the traceless currents, defined by $J_{\mu_{1}}^{(s-2j)} = 0$ in the massless limit, for which the coefficients in (8) are given by

$$\zeta_{j}^{(s)} = \frac{(s - 1)! T\left(s + \frac{d - 3}{2} - j\right)}{4^{j} j!(s - 2j - 1)! T\left(s + \frac{d - 3}{2}\right)},$$

(10)

In view of (5) and (6) one should also consider the following nonlocal redefinitions of currents

$$J_{\mu_{1}}^{(s)} \rightarrow J_{\mu_{1}}^{(s)} + \sum_{j=1}^{n} \xi_{j} (\pi_{\mu\nu}) J_{\mu_{1}}^{(s-2j)},$$

(11)

where $\xi_{j}$ are arbitrary real numbers, and $n$ is defined as in (22).

One of our goals is to understand the differences among forms of currents, in context of building proper couplings of matter to HS fields.
3. Two-point correlators

3.1. Definitions
We are interested in two-point time ordered correlators of HS currents discussed in previous section. We have calculated also mixed correlators, i.e., correlators of two currents with different spins $s_1$ and $s_2$. We use interchangeably coordinate and momentum space representations

$$\langle 0 | \mathcal{J}_{\mu_{1} \ldots \mu_{s_1}} (x) \mathcal{J}_{\eta_{1} \ldots \eta_{s_2}} (0) | 0 \rangle. \tag{12}$$

We divide correlators into conserved (i.e., divergence-free) part and non-conserved parts. Conserved part is of the form

$$\langle 0 | \mathcal{J}^{(c)}_{\mu_{1} \ldots \mu_{s_1}} (x) | 0 \rangle = m^{d+4} R \int \frac{d^{d} k}{(2 \pi)^{d}} \frac{k^2}{m^2} \langle \Phi (k) \rangle. \tag{13}$$

where $r_{s_1 s_2} (z)$ are the form factors and $\Gamma$ are the complete sets of divergence-free polarization tensors. Form factors are analytic functions for $|z| < 1$, with branch-cuts extending from $z = \pm 4$ to infinity. In $d \neq 3$ there is only a parity even sector, and conserved part is spanned by properly symmetrized products of transverse projectors (9). For diagonal correlators, $s_i = s$, there are $\left[ \frac{s+1}{2} \right]$ independent polarisation tensors.

In $d = 3$ one has, in addition, a parity-odd contribution to the 2-point correlators. In the case of diagonal correlators, in this sector there are $\left[ \frac{s}{2} \right]$ independent polarisation tensors.

The full expressions for spins $s \leq 5$ in $d = 3, 4, 5, 6$ together with a detailed analysis of the behaviours in different limits can be found in [8]. Here we present some generic properties and relations we found particularly interesting.

An application of these correlators can be seen as follows. One couples the matter quantum fields to background higher spin fields, represented by completely symmetric rank-$s$ tensor fields $\phi_{\mu \ldots \nu} (x)$, by linear coupling to the corresponding conserved currents

$$L_s = \sum_{s_1} J^{(c)}_{\mu_{1} \ldots \mu_{s_1}} \phi_{\mu_{1} \ldots \mu_{s_1}}. \tag{14}$$

Then from 2-point correlators one calculates the quadratic part of an effective action (or induced action, in Sakharov’s language) for higher spin field

$$S_{2}^{\text{eff}} [\phi] = \sum_{s_1} \int d^{d} k \tilde{\phi} (-k) \mu_{\mu_{1} \ldots \mu_{s_2}} \tilde{J} (k) \tilde{\phi} (k). \tag{15}$$

or, in other words, the linear part of the corresponding induced equations of motion (LEoM)

$$\left\langle \left( J^{(s)}_{\mu \ldots \nu} \right) \right\rangle_{\phi, \lambda} = \sum_{s_1} \tilde{J}^{(s)}_{\mu_{1} \ldots \mu_{s_1}} \tilde{\phi}^{s_1 \ldots s_2}. \tag{16}$$

Let us mention that in the even spin case 1-point correlators are generically non-vanishing. They contribute to the constant part of the effective EoM

$$\left\langle \left( J^{(s)}_{\mu} \right) \right\rangle_{\phi, 0} = \left\langle J^{(s)}_{\mu} \right\rangle = m^{d+4} c (s, d) \left( \eta_{\mu} \right)^{s_1 \ldots s_2},. \tag{17}$$
where \( c(s,d) \) are dimensionless constants. We see that this can be interpreted as (the 0th-order part of) a generalised cosmological constant term. For odd spins there is no such term.

3.2. Non-conserved part of the 2-point correlators

In this paper we mainly ignore non-transverse parts of correlators. We can always choose them to be local, so they could in principle be subtracted out from the effective action by local counter-terms. Also, we extract the non-conserved part in the “minimal way”. We expect that non-conserved terms play a major role in the analysis of the seagull (quadratic coupling of HS fields to matter fields) and tadpole (generalised cosmological constant) terms, which we plan to do in the near future. For now, we ignore these terms, and leave the detailed analysis of the non-covariant part to our future work in which we plan to investigate non-linear HS structures.

3.3. Non-diagonal terms in LEoM

For the currents we considered in this paper, 2-point correlators of currents with different spins are generally non-vanishing. The exceptions are only correlators of traceless currents of massless matter. This introduces non-diagonal terms in LEoM’s, which significantly complicates analysis of the propagating degrees of freedom.

3.4. IR analysis - general \( d \)

The infrared (IR) expansion of the conserved part of the correlators, valid for \( \frac{|k|}{m} < 1 \), can generally be written as

\[
\bar{J}^{(c)(d)}_{\mu\nu'}(k) = m^{d-2-\frac{2|s|}{|s|}} k^{-\frac{|s|}{|s|}} \sum_{j=1}^{\infty} a_j(s,d) \left( \frac{k^2}{m^2} \right)^j \sum_{\pi} b_{\pi}(s,d) \Pi(k)_{\mu\nu'},
\]

where \( a_j(s,d) \) and \( b_{\pi}(s,d) \) are dimensionless coefficients. When plugged into (15) the conserved part contributes weakly non-local contribution to the effective action.\(^3\)

It can be shown that for the simple conserved currents, and all currents whose definition do not include \( d \) explicitly (e.g., lowest derivative ones), the form of the polarisation tensors at each order \( n \) is independent of \( d \) when expressed in terms of \( \pi \)-projectors, which means that the coefficients \( b_{\pi}(s,d) \) do not depend on \( d \).

Let us take the spin-3 case as an example. One has

\[
\bar{J}^{e:IR}_{\mu\rho\nu\rho'} = \sum_{j=1}^{\infty} a_j(3,d) m^{d-2} k^{2+j} \pi_{\mu\nu} \left( \pi_{\mu\nu} + b_j(3,d) \pi_{\mu\nu} \pi_{\nu\nu} \right),
\]

where for the simple form of the currents we obtain

\[
b_j^l(3,d) = \frac{j-3}{2j}; \quad b_j^h(3,d) = -1; \quad b_j^f(3,d) = \frac{3}{2}, \quad j > 1.
\]

The leading order term of the IR expansion of the conserved part is universal for all spins. The simplest way to present it is in the language of LEoM. We obtain

\[
\langle J_{\mu\nu'} \rangle_{d=3} = a_l(s,d) m^{d-2-\frac{2|s|}{|s|}} G_{\mu\nu'} + (\text{higher - derivative terms}),
\]

2 For the sake of economy, we take diagonal case \( s_1 = s_2 = s \).

3 Weakly non-local means that the action is an infinite sum of local terms.
where $G_{\mu\nu}(x)$ is the g-Einstein tensor (see Appendix A for definition). Going to next-to-leading term in IR expansions, we find no universal behaviour.

Considering that the effective actions are weakly nonlocal in the IR region, and that g-Einstein is the lowest-derivative LEoM-tensor, this is a natural result. Note that in the lowest order in IR, Eq. (21) is almost identical to the LEoM (3)-(4), proposed in [3]. The difference is in the presence of nonlocal operators $\Box^{-n}$ in (4). This difference may be avoided if, instead of (14), we define the coupling as

$$L_{\Delta} = \sum_{s} J_{\mu_{1}...\mu_{s}}^{(s)} \Box^{n} g^{\mu_{1}...\mu_{s}}.$$  

(22)

Redefinitions of current (11) are not expected to change the form of the right hand side of (21), because it is what one generically expects for the conserved part. This means that instead of (21) one expects to obtain in the leading order in IR

$$\Box^{n}\left\langle J_{\mu_{1}...\mu_{s}}^{(s)}\right\rangle_{P-odd} \propto m^{d+2(n-1)-2[\nu/2]} G_{\mu_{1}...\mu_{s}}.$$  

(23)

which is the same as (3)-(4). As noted in Sec. 1, for generic currents and $s > 2$ solutions of this LEoM do not yield only physical spin-$s$ excitations expected from the analysis of the Poincare group. In [4] it was shown that, at least in the spin-3 case (see Eq. (6)), the resolution can be found through a proper choice of the conserved current, by using redefinition of the type (11). We plan to investigate this point in the future.

Let us summarise our findings from the IR regime. Based on our analysis, the following procedure has the potential to reconcile our results, based on the induced action, with the analyses from [4]:

(i) The linear coupling of matter to HS fields should be of the higher-derivative form (22).

(ii) One should use HS current which satisfies proper current exchange equation, as described in [4]. Based on the spin-3 example, we assume that this can be achieved by a proper choice of coefficients $\zeta_{j}$, for all spins. We emphasize that the coupling is local.

(iii) The remaining freedom in the choice of the currents should be used to achieve vanishing of the non-diagonal ($s_{1} \neq s_{2}$) correlators.

We plan to investigate this program in the future.

3.5. IR analysis for $d = 3$

The specialty of $d = 3$ dimensions is that beside parity-even sector there is also a parity-odd sector of 2-point correlators, present for Majorana fermion matter QFT. The parity-even sector has been already described above, so we turn our attention here to the parity-odd sector. There is an important difference in the IR behaviour between odd spin and even spin cases.

For odd spins the LEoM term produced in the IR limit is

$$\left\langle J_{\mu_{1}...\mu_{s}}^{(s)}\right\rangle_{P-odd} \sim \text{sign}(m)m^{s-1} B_{\mu_{1}...\mu_{s}},$$  

(24)

where the tensor $B_{\mu_{1}...\mu_{s}}$ is defined in (A.10). The $B_{\mu_{1}...\mu_{s}}$ tensor is by definition the lowest derivative local LEoM-tensor in $d = 3$.

In the case of even spins we obtain

$$\left\langle J_{\mu_{1}...\mu_{s}}^{(s)}\right\rangle_{P-odd} \sim \text{sign}(m)m^{s-2} e_{\mu_{1}...\mu_{s}} \partial_{\nu} G_{\mu_{1}...\mu_{s}}.$$  

(25)

We see that in the spin-odd case the leading IR term is $s$-th order in derivatives, while in spin even case it is of $(s + 1)$-th order. We can compare this with the parity even part (18). We see that the parity-odd sector is dominant for odd spins and subdominant for even spins, which generalises the known behaviour for spins 1 and 2. Note that for the Majorana field in $d = 3$, the sign of the mass term is relevant, and the action of parity transformation in the free Lagrangian amounts to $m \rightarrow -m$. 


Note that while in spin 1 and 2 cases the tensor $B_{\mu s}$ is equal to the generalised Cotton tensor, this is not so for $s > 2$. The reason is that the number of independent parity-odd LEOM tensors is $\left[\frac{s + 1}{2}\right]$, which for $s > 1$ is larger than one. By definition the g-Cotton tensor is of the highest order type in derivatives, so it is not natural to expect that it appears as the lowest order parity-odd term in the IR expansion for $s > 2$.

3.6. UV limit and massless matter

For massless matter it is natural to consider traceless currents. The contribution to the induced LEoM for a HS field is

$$\left\langle\left\langle j^{(1)}_{\mu}\right\rangle\right\rangle \propto \Box^{-2} E^{(w)}_{\mu\nu},$$

where $E^{(w)}_{\mu\nu}$ is the lowest-derivative local traceless spin-$s$ LEoM tensor (A.9). Note that the right hand side in (26) is non-local for odd $d$ and $d = 2$. As it is of $(2s)$-derivative type, it is easy to see that the corresponding effective action for even $d \geq 4$ should be

$$S^{(m=0)}(\phi) \propto \int d^d x W \cdot \Box^{-2} W,$$

where $W_{\mu_1...\mu_d}$ is the spin-$s$ g-Weyl tensor, the generalisation of the linearised Weyl tensor to spin-$s$ (see Appendix A.1 for definition and properties). This result suggests that in $d = 4$ dimensions there should be an action term of the form

$$S_w = \int W^2,$$

which is g-Weyl invariant not only in linearised theory, but also in some hypothetical full nonlinear HS theory. Indeed, in the spin-2 case we know that this is true.

If one considers the conserved part of 2-point correlators of the traceless currents for massive matter in the UV limit $\left[\frac{k^2}{m^2} \gg 1\right]$, then the leading term is the same as in the massless case. However, in $d = 3$ for fermionic matter there is an interesting sub-leading contribution in the parity-odd sector

$$j^{(UV)}_{\mu s_{1}s_{2}}(k) = (-1)^{s_{1}+s_{2}} \frac{imk^{s_{1}+s_{2}-3}}{2^{s_{2}+1}} \epsilon_{s_{1}k} k^\sigma \times \bar{\pi}^{s_{1}-1}_w \bar{\pi}^{s_{2}}_{\nu\mu} F_1\left(1-s_1,\frac{s_1}{2},1-s_1,1,\frac{\bar{\pi}_w\bar{\pi}_w}{\bar{\pi}^2_{\mu\nu}}\right),$$

where spins $s_1$ and $s_2$ have the same parity, and $s_2 > s_1$. The hypergeometric function is a polynomial in its last argument. The polarisation tensor is a unique traceless and conserved one in the parity-odd sector. From (29) we obtain the following leading order contribution to LEoM in the UV limit contribution for the diagonal case $s_1 = s_2 = s$:

$$\left\langle\left\langle j^{(1)}_{\mu}\right\rangle\right\rangle_{P\text{-odd,UV}} \propto m^{-3/2} C_{\mu s},$$

where $C_{\mu s}$ is the g-Cotton tensor (see Appendix A.3 for definition and properties). Observe that the above expression is non-local. However, if we assume that there are $N$ free Majorana fermion fields
with the same mass \( m \), and take particular large-\( N \) UV limit defined by \( N \to \infty \) and \( \frac{m}{k} \to 0 \) with 

\[
\lambda \equiv N \frac{m}{k} = \text{fixed}, \quad \text{then from (30) follows}
\]

\[
\left\{ \left\{ \frac{\mathbf{F}^{(i)}}{\mu} \right\} \right\}_{\mu = \text{UV}} \propto \lambda C^\mu + \left( \text{higher – order terms} \right), \tag{31}
\]

which is a local LEOm term. As for \( s = 2 \) this term follows from the (gravitational) Chern-Simons Lagrangian term, we expect that the same is true for all spins. Note that if the same limit is applied to the non-diagonal case, one obtains Chern-Simons-like terms of the mixed kind.

**Appendix A. Linearised covariant formalism for higher-spin fields**

In this section we first introduce well-known definitions and properties about higher spin tensors, their linearized EoM's and their possible geometrical formulations. We complete and review the covariant formalism for higher spin (HS) gauge fields in the linear approximation in the flat Minkowski background, originally introduced in [9] and further developed in spin-3 case in [10].

**Appendix A.1. Generalized curvature tensors**

A spin-\( s \) field is described by a totally symmetric rank-\( s \) tensor field \( \varphi_{\mu_1...\mu_s} \). We assume there is the gauge symmetry, which we call \( \text{g-diff} \), with transformations defined in (2).\(^4\) The idea is to generalise the covariant formalism of Maxwell theory (spin-1) and linearised General Relativity (spin-2) to arbitrary spin.

While one usually starts with generalised connections, as we shall not use them here we skip directly to the generalisation of Riemann (g-Riemann) tensor. The g-Riemann tensor is

\[
R_{\alpha \mu_1...\alpha \mu_s} \equiv \partial_{\alpha \mu_1}...\partial_{\alpha \mu_s} \varphi_{\mu_1...\mu_s} \bigg|_{\mu_1...\mu_s}, \tag{A.1}
\]

where anti-symmetrization (without weight factors) in all pairs of indices \( \{ \alpha_j, \mu_j \} \), \( j = 1,...,s \) is understood. Symmetries of such defined g-Riemann tensor straightforwardly generalise those of the ordinary Riemann tensor - it is antisymmetric under the one exchange \( \mu_j \leftrightarrow \nu_j \), symmetric under the exchange of any two pairs \( (\alpha_j, \mu_j) \leftrightarrow (\alpha_k, \mu_k) \), and satisfies cyclic and Bianchi identities.

Note that the g-Riemann tensor is of \( s \)-th order in derivatives. It is easy to see that in the spin-1 case (A.1) gives the field strength tensor \( F_{\alpha \mu} \equiv \partial_{\alpha} \varphi_{\mu} - \partial_{\mu} \varphi_{\alpha} \), while in the spin-2 case (A.1) it gives the linearised Riemann tensor. The important property of the g-Riemann tensor is encoded in the theorem stating that if it vanishes in some region then the field \( \varphi \) is a pure gauge in that region.

For even spin one can construct a symmetric rank-\( s \) tensor by tracing all \( \alpha \) indices

\[
R_{\mu_1...\mu_s} = \eta^\alpha_{\nu_1...\nu_s} R_{\alpha \mu_1...\alpha \mu_s}, \tag{A.2}
\]

which we call the g-Ricci tensor. By repeatedly tracing this tensor one obtains the tower of symmetric g-Ricci tensors, \( R_{\nu_1...\nu_s} \), with even ranks ranging from \( s \) to 0.

For odd spin \( s = 2n+1 \), the g-Ricci tensor is obtained by tracing g-Riemann tensor \( n \) times.

\(^4\) In \( d > 4 \) one also need to take into account fields with mixed symmetries. Also, keep in mind that the formalism is linearised both in the HS field \( \varphi \) and g-diff parameter field \( A \). Symmetrizations (…) and antisymmetrisations [...] are in the Appendix defined with the weight 1, if not mentioned otherwise.
Here $g$-Ricci tensor is antisymmetric in the first pair of indices $(\alpha, \mu)$, symmetric in the remaining indices $(\nu)$ and satisfies $R_{\mu
u_1...\nu_{s-1}}$. By successive tracings over $\nu$ indices one can define a tower of $g$-Ricci tensors, $R_{\mu
u_1...\nu_{s-1}}$, with even ranks ranging from $(s+1)$ to $2$.$^5$

For $s > 2$ one can introduce a generalisation of the Weyl tensor, as the completely traceless part of the Riemann tensor. The $g$-Weyl tensor is invariant on the $g$-Weyl transformations

\[
\delta_{\omega_{\mu_1...\nu_{s-1}}} = \frac{s(s-1)}{2} \epsilon_{\mu_1...\nu_{s-1}} \eta_{\omega_{\mu_1...\nu_{s-1}}}.
\]

where the gauge parameter $\omega_{\mu_1...\nu_{s-1}}$ is an arbitrary symmetric rank-$(s-2)$ tensor field. It can be shown that vanishing of the $g$-Weyl tensor implies that the HS field is a pure $g$-Weyl gauge.

Appendix A.2. LEoM-tensors

For the purpose of constructing linearised equations of motion which are gauge-invariant under unrestricted $g$-diffs, one needs to classify divergence-free symmetric tensors of rank $s$ which are linear in $\varphi$, which we call LEoM-tensors.$^6$ Using transverse projector (9) one can classify LEoM-tensors, as they must be expressible as linear combinations of the terms which can be written in the form $\Box^s \pi_{\mu_1...\mu_\nu}$, where for spin $s$ there are $s$ projectors in the product, and there are $s$ contractions which include all indices belonging to HS field $\varphi$. In [7] we showed that there are $\left[ s/2 \right] + 1$ independent LEoM-tensors (i.e., independent polarisation structures).

The $d = 3$ spacetime dimensions presents a special case in which there are in addition parity-odd LEoM-tensors. This is developed in Appendix A.3.

The lowest derivative local parity-even LEoM-tensor we call $g$-Einstein tensor. Using the formalism of $\pi$-projectors it is easy to show that this tensor is unique, and it is given by

\[
G_{\mu_1} = \left[ \frac{s+1}{2} \right] \pi_{\mu_1}^{mod 2} \left( \pi_{\mu_1} - \pi_{\mu_1} \right)^{\left[ s/2 \right]} \phi^\nu,
\]

where $[ ]$ denotes integer part and a symmetrisation over $\mu_1 \equiv \{ \mu_1, \ldots, \mu_s \}$ indices is assumed. Obviously, the $g$-Einstein tensor is of order $s + (s \mod 2)$ in derivatives. We want to express the $g$-Einstein tensor using the covariant formalism. As is visible from the number of derivatives in (A.5), there is a difference between even and odd spin cases.

For even spin the $g$-Einstein tensor has the form

\[
G_{\mu_1} = R_{\mu_1} + a^{(s)}_{(1)} \eta_{\mu_1} R_{\mu_2} + \ldots + a^{(s)}_{(s)} \eta_{\mu_1}^n R, \quad s = 2n. \tag{A.6}
\]

The coefficients $a_j$ are uniquely fixed by the condition $\partial \cdot G = 0$.

For odd spin one needs an extra derivative aside those present in the Riemann tensor

\[
G_{\mu_1} = \partial^{\alpha} \left( R_{\mu_\alpha} + a^{(s)}_{(1)} \eta_{\mu_\alpha} R_{\mu_{s-2}} + \ldots + a^{(s)}_{(s)} \eta_{\mu_1}^n R_{\mu_1} \right), \quad s = 2n + 1, \tag{A.7}
\]

where for $s \geq 3$ symmetrisation over $\mu$ indices is understood. Again, the coefficients $a_j$ are uniquely fixed by the condition $\partial \cdot G = 0$.

$^5$ As $R_{\mu_1}$ is antisymmetric, there is no $g$-Ricci scalar in spin-odd cases.

$^6$ This follows if the EoM is generated from an action invariant on gauge transformations (2).
Once the Einstein tensor is constructed, a full basis for LEoM-tensors can be defined by the following set of tensors

\[ E^{(r)}_{\mu} = \Box^r \pi^{\mu}_{\nu \rho \ldots \rho} G^{H}_{\mu \nu} \quad 0 \leq r \leq \left[ \frac{s}{2} \right] \]  

(A.8)

Every LEoM-tensor can be written as a linear combination of tensors \( \Box^r E^{(r)}_{\mu} \).

If the linearised equations of motion are obtained from an action invariant on g-Weyl transformations (A.4), then LEoM-tensors are in addition traceless. Again, the \( \pi \)-projectors formalism offers a practical way to study and classify such Weyl-LEoM tensors. In fact, there is only one such tensor for given \( s \) in \( d \) dimensions, which is given by

\[ E^{(W)}_{\mu \nu} = \Box^s \pi^{\mu \nu}_{\rho ... \rho} F_1 \left( \frac{1-s}{2}, \frac{s}{2} - \frac{3-d}{2}, -s, \pm \frac{\pi^{\mu \nu}_{\rho ... \rho}}{\pi^{2}} \right) \phi^s, \]  

(A.9)

where \( F_1 \) is the hypergeometric function which is a polynomial of the order \( \left[ \frac{s}{2} \right] \) in its last argument. Observe that Weyl-LEoM tensors are of \( (2s) \)-th order in derivatives, i.e., they are at the top level in the tower (A.8).

Appendix A.3. Specifics in \( d = 3 \)

It is well-known that for spins 1 and 2 there are some specific properties present in three dimensional spacetime. In particular: (1) the Riemann tensor is reducible, i.e., it can be expressed in algebraic fashion using the Ricci tensor \( R^{(l)} \) and the metric tensor, (2) the Weyl tensor vanishes, and the Cotton tensor assumes its role, (3) there are parity-odd Chern-Simons terms which may contribute to LEoM. In [10] it was shown that these properties straightforwardly generalise to spin-3, and later in [11] to all integer spins.

For a start, let us analyse the parity-odd sector of LEoM-tensors in 3D using of \( \pi \)-projectors. It is generated by the tensors of the form \( \Box^r \pi^{(e \cdot e) \pi ... \pi} \phi \), where \( \pi^{\mu \nu}_{\rho ... \rho} \) is the Levi-Civita tensor. For spin \( s \) there are \( (s-1) \) projectors \( \pi \) in the product, and there are \( s \) contractions which include all indices belonging to HS field \( \phi \). In [7] we showed that there are \( \left[ \frac{s+1}{2} \right] \) independent tensors of this type, so the total number of independent LEoM-tensors, of both parities, is \( (s+1) \).

A particularity of \( D = 3 \) is that one can easily construct the full set of LEoM-tensors in covariant formalism, by using Levi-Civita tensor. One starts by observing that

\[ B_{\mu_1 \ldots \mu_s} \equiv \varepsilon_{\mu_1 \nu_1} \ldots \varepsilon_{\mu_s \nu_s} R^{\nu_1 \ldots \nu_s}_{\mu_1 \ldots \mu_s} \]  

(A.10)

is a LEoM-tensor.\(^7\) This tensor is \( s \)-derivative, and it is the lowest derivative local LEoM-tensor.

For even spins the \( B \) LEoM-tensor must be proportional to the g-Einstein tensor, due to uniqueness of the latter. However for odd spins it has one derivative less than the g-Einstein tensor, and, having the odd number of Levi-Civita's, it is parity-odd. Its interpretation becomes clearer after observing that in spin-1 case it is the standard gauge Chern-Simons LEoM-term \( \varepsilon_{\mu \nu \rho} F^{\mu \nu} \).

It is easy to understand the existence of \( B \) tensor for odd spins using transverse projectors. For odd \( s \) a local parity-odd rank- \( s \) tensor of \( s \)-th order in derivatives can be constructed as

\(^7\) That \( B_{\mu_1 \ldots \mu_s} \) is divergence-free follows by applying Bianchi identity.
By using the $B$ LEoM-tensor one can construct a tower of LEoM-tensors in the following way

$$B^{(n)}_{\mu_1...\mu_n} \equiv \partial_{\mu_1} ... \partial_{\mu_n} B_{\mu_1...\mu_n} e_{\rho_1}^{\mu_1} ... e_{\rho_n}^{\mu_n}, \quad 1 \leq n \leq s. \quad (A.12)$$

The tensor $B(n)$ is $(s + n)$-th order in derivatives. Together with $B$, we have constructed $(s + 1)$ independent LEoM-tensors, which means it is a complete set (both parities).

In spin-2 case there is one parity-odd LEoM-tensor, the (linearised) Cotton tensor generated by gravitational CS Lagrangian term, which has some important properties including Weyl covariance. It can be generalised to all spins, and it sits at the top of the tower (A.12). As shown in [11], g-Cotton tensor is the unique tensor satisfying the following properties: (i) rank-$s$ completely symmetric traceless tensor, (ii) divergence-free, (iii) invariant under g-Weyl transformations, (iv) replaces g-Weyl tensor in controlling g-Weyl invariance (vanishing of g-Cotton tensor is necessary and sufficient condition for gauge field to be pure g-Weyl gauge).

The properties (i) and (ii) indicate that g-Cotton tensor is an LEoM-term. g-Cotton tensor can be expressed in different ways, the most useful for our purposes is

$$C_{\rho \sigma} = e_{\rho \mu \nu} \partial_{\sigma}^{\mu} \square^{s-1} 2 F_{1} 1 - s \pi^{2} \pi_{\mu \nu} \phi^{\rho \sigma}. \quad (A.13)$$

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