Fast tests for probing the causal structure of quantum processes

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Abstract. The identification of causal relations is a cornerstone of the scientific method. Traditional approaches to this task are based on classical statistics. However, such classical approaches do not apply in the quantum domain, where a broader spectrum of causal relations becomes accessible. New approaches to quantum causal inference have been developed in recent years, and promising new features have been discovered. In this paper, we review and partly expand the framework and results of Ref. \cite{1}, which demonstrated quantum speedups in the identification of various types of causal relations induced by reversible processes.

Keywords: causal inference, quantum estimation theory, indefinite causal order

1 Introduction

Identifying cause-effect relations is a fundamental problem in science and engineering \cite{2–4}. In its simplest form, the problem can be described as follows: An experimenter has collected some amount of raw data about the values of a set of variables, and wants to determine whether a certain variable $X$ influences another variable $Y$ in the set. Since the data are typically subject to noise and fluctuations, the problem of identifying causal relations is ultimately a statistical inference problem.

Traditionally, causal inference methods have been designed for classical variables. Recent advances in quantum science and engineering motivate an extension of these methods to the quantum domain. Experimental techniques can now address individual quantum systems, initialise them in a given quantum state, and subject them to a variety of quantum measurements. In this context, the presence of causal dependencies between two quantum systems acquire a concrete practical relevance. For example, if the interaction of two quantum spins...
induces a causal dependence between them, then the state of one spin can be controlled by operating on the other spin, to some degree that is determined by the strength of the causal relation.

For quantum systems, classical methods of causal inference turn out to be inadequate. The reason is that such methods assume that randomness can always be reduced to ignorance about the initial conditions of some additional, latent variable. This assumption is at odds with the violation of Bell inequalities, which rules out local realistic models for quantum correlations. For this reason, classical causal models cannot be applied to the Bell scenario \[5\]. A similar conclusion also holds for more general scenarios, including more than two quantum systems, and/or timelike separations \[6,7\].

The breakdown of classical causal inference methods calls for a new, genuinely quantum formulation of the notions of cause and effect. Several frameworks have been proposed to date, with different features and sometimes different purposes. The framework of quantum combs, introduced by Chiribella, D’Ariano, and Perinotti in a series of works \[8–10\], describes networks of quantum processes connected with one another according to a given causal structure. In this framework, one can express the fact that a given process induces a causal dependence between a quantum system and another. More broadly, this notion can also be generalised to probabilistic theories beyond quantum mechanics \[11\], and to an even broader class of theories described by symmetric monoidal categories \[12,13\]. Frameworks for describing causal relations in quantum theory and beyond have been developed in \[14–16\]. More recently, a quantum framework for describing causal relations between quantum systems has been developed by Allen et al \[17\]. This framework, known as quantum causal models, can be regarded as an enrichment of the framework of quantum combs, with new conditions that allow one to express the conditional independences among quantum variables.

Given a framework for describing causal relationships among quantum variables, one can develop strategies for identifying such relations. Two interesting examples were presented in Refs \[18,19\]. In these works, the authors considered the problem of determining whether two quantum systems, say two photons, are in a spacelike or timelike configuration. Equivalently, this amounts to determining whether the states of the two systems are marginals of a bipartite quantum state (spacelike configuration, corresponding to a past common cause in the joint preparation of the two systems), or whether the state of one system is obtained from the state of the other by a quantum process (timelike configuration, corresponding to a cause-effect relation directed from one system to the other). Remarkably, the authors found out that certain quantum correlations can distinguish between these two situations, thus defying the classical motto “correlation does not imply causation”.

The fact that quantum correlations can be witnesses of causal relationships suggests that quantum measurements could offer more powerful ways to identify causal relations compared to their classical counterpart. In a recent work \[1\], Chiribella and Ebler showed that quantum features like entanglement and su-
perposion can lead to speedups in the identification of various types of causal
relations. In particular, they addressed the following question: given a variable,
which variable out of a list of candidate variables, is the effect of it? For sim-

clicity, they focussed on the basic scenario where the cause-effect relation is
induced by a reversible process. Classically, it turns out that the minimum prob-

ability of error is $p_{C_{\text{err}}}^N = 1/2d^N$, where $d$ is the dimension of the quantum
systems associated to the given variables, and $N$ is the number of times the
variables are probed. This means that the error probability decays exponentially
as $p_{C_{\text{err}}}^N \approx 2^{-R_{C}N}$, with decay rate $R_{C} = \log_2 d$. In stark contrast, Chiri-

bella and Ebler showed that the error probability for quantum strategies decays
quadratically faster, with an exponential decay rate equal to $2\log_2 d$. Similar
advantages arise in the task of identifying the presence of a causal link between
two variables, and in the task of identifying the cause of a given effect.

This paper reviews the framework and the results of Ref. [1] in a way that
is suitable for non-specialists. The paper is organised as follows: in Section 2,
we provide preliminary notions that will be used later in the paper. In Section 3,
we review the problem of identifying of the effect of a given variable, and we
provide the minimum error probability and its decay rate for classical strategies.
Then, we provide quantum strategies that achieve a speedup over their classical
counterpart (Section 4). In Section 5, we generalize the above results for the case
of multiple hypotheses and provide some applications. Finally, we conclude in
Section 6 by discussing directions of future research.

2 Preliminaries

In this section we review and expand the framework of Ref. [1], providing some
additional definitions that help clarifying the different types of causal relations
induced by quantum processes.

2.1 Notation

For a given Hilbert space $\mathcal{H}$, we denote by $B(\mathcal{H})$ the space of bounded opera-
tors on $\mathcal{H}$, by $T(\mathcal{H})$ the space of trace-class operators, and by $\text{St}(\mathcal{H}) := \{\rho \in T(\mathcal{H}) \mid \rho \geq 0, \text{tr}[\rho] = 1\}$ the convex set of density operators. In the context of
causal inference, quantum systems are often called quantum variables. We will
denote quantum systems by Roman letters, such as $A, B, \ldots$, and the corre-
sponding Hilbert spaces as $\mathcal{H}_A, \mathcal{H}_B, \ldots$, respectively. We will use the shorthand
notation $\text{St}(A) := \text{St}(\mathcal{H}_A)$.

Let $A$ and $B$ be two quantum systems, and let $\mathcal{H}_A$ and $\mathcal{H}_B$ be the corre-
sponding Hilbert spaces. We will denote by $A \otimes B$ the composite system consist-
ing of subsystems $A$ and $B$, corresponding to the tensor product Hilbert space
$\mathcal{H}_{A \otimes B} = \mathcal{H}_A \otimes \mathcal{H}_B$. The partial trace over the Hilbert space $\mathcal{H}_A$ will be denoted
as $\text{tr}_A$.

A quantum process with input $A$ and output $B$ is described by a linear, completely positive, trace-preserving map $C : T(\mathcal{H}_A) \to T(\mathcal{H}_B)$, mapping input
states \( \rho \in \text{St}(A) \) into output states \( C(\rho) \in \text{St}(B) \). Linear, completely positive and trace-preserving maps will be sometimes abbreviated as CPTP maps. The set of CPTP maps from \( T(\mathcal{H}_A) \) to \( T(\mathcal{H}_B) \) will be denoted by \( \text{CPTP}(A \to B) \).

### 2.2 Cause-effect relations induced by quantum processes

Let \( A \) and \( B \) be two quantum systems.

**Definition 1** We say that a process \( C \in \text{CPTP}(A \to B) \) induces a cause-effect relation from \( A \) to \( B \) if and only if \( C \) is non-constant. If this is the case, then we say that \( A \) is a cause for \( B \), and that \( B \) is an effect of \( A \).

In other words, a process induces a causal relation from \( A \) to \( B \) if and only if changing the state of system \( A \) can induce changes in the state of system \( B \). The ability to induce changes serves as a witness of the fact that \( A \) is a cause for \( B \).

Definition (1) can be generalized to processes involving multiple inputs and outputs.

**Definition 2** We say that a bipartite process \( D \in \text{CPTP}((A \otimes A') \to (B \otimes B')) \) induces a cause-effect relation from \( A \) to \( B \) if and only if there exists at least one state \( \alpha \in \text{St}(\mathcal{H}_{A'}) \) such that the reduced process

\[
D_\alpha \in \text{CPTP}(A \to B), \quad \rho \mapsto \text{tr}_{B'}[D(\rho \otimes \alpha)] \tag{1}
\]

is non-constant. If this is the case, then we say that \( A \) is a cause for \( B \), and that \( B \) is an effect of \( A \).

Note that a process \( C \in \text{CPTP}(A \to B) \) may not induce a cause-effect relation from \( A \) to \( B \), but may still be the reduced process of some other process \( D \in \text{CPTP}((A \otimes A') \to (B \otimes B')) \) that does induce a cause-effect relation from \( A \) to \( B \). This observation shows that the presence of a cause-effect relation, as defined in Definitions (1) and (2), is a property of the process under consideration, and not just of the variables \( A \) and \( B \). In other words, a cause-effect relation that is actually present may not be detected by inspecting the process from \( A \) to \( B \) alone.

The notion of “process inducing a cause-effect relation” provided in Definitions (1) and (2) is rather weak, because it allows the influence of the cause on the effect to be arbitrarily small. A stronger notion arises when the cause-effect relation is **faithful**, in the following sense:

**Definition 3** We say that a process \( C \in \text{CPTP}(A \to B) \) induces a faithful cause-effect relation from \( A \) to \( B \) if and only if \( C \) is correctable, meaning that there exists another process \( R \in \text{CPTP}(B \to A) \) such that

\[
R \circ C = \mathcal{I}_A, \tag{2}
\]

where \( \mathcal{I}_A \in \text{CPTP}(A \to A) \) is the identity process. When this is the case, we say that \( B \) is a causal intermediary of \( A \).
Intuitively, variable $B$ is a causal intermediary for variable $A$ if all the possible causal influences of $A$ can be reconstructed from $B$. In other words, every process from $A$ to a third variable $B'$ must factorise into the process from $A$ to $B$, followed by some process from $B$ to $B'$. This intuition is formalised by the following proposition:

**Proposition 1** The process $C \in \text{CPTP}(A \rightarrow B)$ induces a faithful cause-effect relation from $A$ to $B$ if and only if, for every quantum system $B'$, and for every process $E \in \text{CPTP}(A \rightarrow B')$ there exists a process $D \in \text{CPTP}(B \rightarrow B')$ such that

$$E = D \circ C.$$  

(3)

**Proof.** Suppose that $C$ induces a faithful causal-effect relation. Then, let $R \in \text{CPTP}(B \rightarrow A)$ be the process that inverts $C$, namely $R \circ C = I_A$. Then, for every system $B'$ and every process $E \in \text{CPTP}(A \rightarrow B')$, one has $E = D \circ C$, with $D := E \circ R$.

Conversely, suppose that, for every quantum system $B'$ and for every process $E \in \text{CPTP}(A \rightarrow B')$, there exists a process $D \in \text{CPTP}(B \rightarrow B')$ such that $E = D \circ C$. In particular, one can pick $B' \equiv A$ and $E \equiv I_A$, in which case the condition $E = D \circ C$ implies that $C$ is correctable, and therefore induces a faithful cause-effect relation.

In general, the presence of a faithful cause-effect relation from $A$ to $B$ does not imply that the causal influences of $A$ propagate exclusively through $B$. For example, quantum secret sharing protocols, such as those presented in Ref. [20], provide examples of processes where a given cause can have multiple causal intermediaries. The situation is much simpler when the cause-effect relation is reversible, in the following sense:

**Definition 4** We say that a process $C \in \text{CPTP}(A \rightarrow B)$ induces a reversible cause-effect relation between $A$ and $B$ if and only if $C$ is reversible, meaning that there exists another process $R \in \text{CPTP}(B \rightarrow A)$ such that

$$R \circ C = I_A \quad \text{and} \quad C \circ R = I_A,$$

(4)

where $I_X \in \text{CPTP}(X \rightarrow X)$ is the identity process on system $X \in \{A, B\}$.

In this paper, we will focus on situations where the cause-effect relation between two variables is reversible. In this case, the presence of a cause-effect relation between variables $A$ and $B$ rules out the possibility of any cause-effect relation from $A$ to any other variable $B'$ that is independent of $B$. More precisely, if a process $D \in \text{CPTP}(A \rightarrow B \otimes B')$ is such that its reduced process $C = (I_B \otimes \text{tr}_{B'}) \circ D$ is reversible, then $D$ must be of the form $D = C \otimes \beta$, where $\beta \in \text{St}(B')$ is some fixed state of system $B'$. 
2.3 Discrimination of causal hypotheses

Consider the situation where an experimenter has access to a black box, implementing a quantum process with a given set of input systems and a given set of output systems. The goal of the experimenter is to figure out the causal relations among the systems involved in the process. For example, the black box could implement a process with input system $A$ and output system $B$, and the experimenter may want to figure out whether the process induces a cause-effect relation.

In general, we will assume that the black box is guaranteed to satisfy one, and only one, of $k$ possible hypotheses $(H_i)_{i=1}^k$ about the cause-effect relations occurring between its inputs and outputs. No further information about the process is available to the experimenter. To figure out which hypothesis is correct, the experimenter will set up a test designed to probe the causal relations. We will use the term causal hypotheses for hypotheses on the causal relations induced by a given process. The problem of distinguishing between alternative causal hypotheses will be called discrimination of causal hypotheses. To illustrate the problem, we will focus on one basic instance: discover which of two variables $B$ and $C$ is the effect of a given variable $A$. More specifically, we consider the following alternative hypotheses:

$H_1$: $B$ is a causal intermediary for $A$, and $C$ is uniformly random,

$H_2$: $C$ is a causal intermediary for $A$, and $B$ is uniformly random.

For simplicity, we will assume that systems $A, B,$ and $C$ have the same dimension, equal to $d < \infty$. In terms of the process $C \in \text{CPTP}(A \rightarrow (B \otimes C))$ the two hypotheses correspond to the following statements:

$H_1$: $C$ has the form $C(\cdot) = U \cdot U^\dagger \otimes I/d$, for some unitary operator $U \in B(\mathbb{C}^d)$, $U^\dagger U = I$,

$H_2$: $C$ has the form $C(\cdot) = I/d \otimes V \cdot V^\dagger$, for some unitary operator $V \in B(\mathbb{C}^d)$. $V^\dagger V = I$.

In either case, the unitary operators $U$ and $V$ are unknown to the experimenter.

In general, every hypothesis on the causal relations between inputs and outputs is in one-to-one correspondence with a subset of CPTP maps. In the above example, the two hypotheses $H_1$ and $H_2$ correspond to the sets

$H_1 = \{ U \cdot U^\dagger \otimes I/d \mid U \in B(\mathbb{C}^d), U^\dagger U = I \}$ \hspace{1cm} (5)

$H_2 = \{ I/d \otimes V \cdot V^\dagger \mid V \in B(\mathbb{C}^d), V^\dagger V = I \}$ . \hspace{1cm} (6)

The problem is to determine whether the given process $C$ belongs to $H_1$ or to $H_2$.

For the discrimination between these two hypotheses we will consider setups that probe the unknown process $C$ for $N$ times, by inserting them in a sequential quantum circuit, as illustrated in Figure 1. As a figure of merit, we will consider the minimization of the probability of error, in the worst case over all possible processes $C$ that are compatible with the causal hypotheses (5) and (6).

In general, the error probability in distinguishing two known quantum processes decays exponentially with the number of interrogations [21]. Informally,
Fast tests of causal structure

This means that the error probability scales asymptotically as $p_{\text{err}}(N) \approx 2^{-NR}$ for some exponent $R > 0$, called the decay rate. More formally, the decay rate is defined as $R := \lim \inf_{N \to \infty} -\log[p_{\text{err}}(N)]/N$, where we assume base 2 for the logarithm here and in the rest of the paper. In the following, we will use the decay rate to assess the performance of various strategies for distinguishing causal hypotheses.

3 Optimal classical strategy for finding a causal intermediary

Let us consider the classical version of the problem in Subsection 2.3. The classical version involves three classical random variables $A, B, C$, each with sample space of cardinality $d < \infty$. The two causal hypotheses are:

$H_1$: $B$ is a permutation of $A$, and $C$ is uniformly random,

$H_2$: $C$ is a permutation of $A$, and $B$ is uniformly random.

In either case, the exact form of the permutation is unknown to the experimenter.

For parallel strategies where the unknown process is probed on $N$ independent inputs, the optimal error probability $p_{\text{err}}^C(N)$ is [1]

$$p_{\text{err}}^C(N) = \frac{1}{2^{dN-1}}.$$ (7)

This implies the optimal decay rate is

$$R_C = \log d.$$ (8)

A result by Hayashi [22] implies that the decay rate cannot be improved by considering more general sequential strategies such as those in Figure 1.
4 Quantum advantages

Classical random variables can be regarded as quantum systems that have lost coherence with respect to a privileged orthonormal basis of “classical states”. But what if the coherence is preserved? In the following we will show that the possibility of probing processes through coherent superpositions of classical inputs can enhance our ability to identify the correct causal hypothesis.

4.1 A benefit of quantum coherence

The three classical random variables $A, B, C$ considered in the previous section can be considered as the decohered versions of three quantum systems $A, B, C$ with $d$-dimensional Hilbert spaces. Similarly, a permutation of the values of the random variable $A$ according to an element $\pi$ of the symmetric group $S(d)$ can be regarded as the decohered version of the unitary permutation operator $U_\pi = \sum_{i=1}^{d} |\pi(i)\rangle\langle i|$, where $\{|i\rangle\}_{i=1}^{d}$ is the orthonormal basis representing the classical states.

In this scenario, the two “classical” causal hypotheses become

$H_1$: $C$ has the form $C(\cdot) = U_\pi \cdot U_\pi^\dagger \otimes I/d$, for some permutation $\pi \in S(d)$,

$H_2$: $C$ has the form $C(\cdot) = I/d \otimes U_\sigma \cdot U_\sigma^\dagger$, for some permutation $\sigma \in S(d)$.

To distinguish between these two hypotheses, the experimenter could prepare $N$ probes in the superposition state

$$|e_0\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle,$$  \hspace{1cm} (9)

which is invariant under permutations. Thus, the unknown process $C$ yields either $(|e_0\rangle \langle e_0| \otimes \frac{1}{d})^{\otimes N}$ or $(\frac{1}{d} \otimes |e_0\rangle \langle e_0|)^{\otimes N}$ depending on which causal hypothesis is correct.

The probability of error can be computed by taking advantage of the symmetry of the problem, which implies that the worst case error probability is equal to the average error probability when the prior probability for the two hypotheses $H_1$ and $H_2$ is uniform. In this case, Helstrom’s theorem [23] states that the minimum error probability in distinguishing between the states $\rho_1$ and $\rho_2$, given with uniform a priori probability, is

$$p_{\text{err, ave}}(\rho_1, \rho_2) = 1 - \frac{1}{2} \|\rho_1 - \rho_2\|_1,$$  \hspace{1cm} (10)

where $\|G\|_1 := \text{tr} \sqrt{G^\dagger G}$ denotes the trace norm of a generic trace class operator $G \in T(\mathcal{H})$. Applied to our problem, Helstrom’s theorem yields the error probability

$$p_{\text{err}}^{\text{coh}} = \frac{1}{2dN}. \hspace{1cm} (11)$$
By comparison with the classical error probability (7), we can see that coherence provides a reduction of the error probability by a factor $d$. This is a relatively minor improvement, as it does not affect the decay rate. Still, it is an interesting first illustration of how quantum effects can affect the discrimination of causal hypotheses.

4.2 General quantum scenario

Let us move on now to the general quantum scenario, where the relation between cause and effect is described by an arbitrary unitary operator (not necessarily a permutation operator, as in the previous subsection).

At first, this problem appears to be much more challenging for our experimenter, since the dependence between cause and effect can be any arbitrary element of the special unitary group $SU(d)$. Surprisingly, however, the same error probability as in Eq. (11) can be achieved.

Let us consider the case where $N$ is an integer multiple of $d$, say $N = dt$ for some non-negative integer $t$. In this case, a universal quantum strategy with error probability (11) is to divide the $N$ probes into groups of $d$, and to prepare each group of probes in $SU(d)$ singlet state

$$|S_d⟩ = \frac{1}{\sqrt{d!}} \sum_{k_1, \ldots, k_d} \epsilon_{k_1 \ldots k_d} |k_1, \ldots, k_d⟩,$$

(12)

where $\epsilon_{k_1 \ldots k_d}$ represents the totally antisymmetric tensor and the summation extends over all the vectors in the computational basis.

Each of the $d$ particles in each group is then fed into one use of the unknown process. Since the singlet state is invariant under unitary transformations, the problem becomes to distinguish between the state $|S_d⟩⟨S_d| \otimes (I/d)^\otimes N$ (hypothesis $H_1$) and $(I/d)^\otimes N \otimes |S_d⟩⟨S_d| \otimes (I/d)^\otimes t$ (hypothesis $H_2$). In this case, Helstrom’s theorem again gives error probability $1/(2d^N)$, which is equal to the value $p_{coh}$ obtained in the previous subsection.

If $N$ is not a multiple of $d$, a rough strategy is to use only the first $\tilde{N} := d \lfloor N/d \rfloor$ probes, and to apply the above procedure. In this way, one obtains error probability $1/(2d^\tilde{N})$, which is suboptimal but still has the same decay rate as the coherent strategy we saw in the previous subsection.

4.3 Parallel strategies with an external reference system

We now show that the decay rate of the minimal error probability can be doubled if the experimenter uses an additional reference system entangled with $N$ input probes in the protocol. For simplicity, we will assume that $N$ is a multiple of $d$.

The basic idea for constructing the quantum-enhanced strategy is the following. In the absence of a reference system, the strategy was to divide the $N$ probes into $N/d$ subgroups of size $d$, and to put the particles in each group in the singlet state. However, there are many ways of dividing $N$ particles into groups
of $d$. Each of these ways leads to the error probability $p_{\text{err}}^{\text{coh}} = 1/2d^N$. What about trying all possible ways in a coherent superposition?

Consider an external reference system with an orthonormal basis $\{|i\rangle\}_{i=1}^{G_{N,d}}$, where the index $i$ labels the possible ways in which $N$ particles can be divided into groups of $d$, and $G_{N,d}$ denotes the number of such ways. Then, one can construct the superposition state

$$|\psi\rangle = \frac{1}{\sqrt{G_{N,d}}} \sum_{i=1}^{G_{N,d}} (|S_d\rangle^\otimes N/d)_i \otimes |i\rangle,$$

where $(|S_d\rangle^\otimes N/d)_i$ denotes the product of $N/d$ singlet states distributed in the $i$-th way.

While a classical randomization over all the possible ways to group the $N$ particles is useless, the quantum superposition (13) turns out to be very useful. After some algebra [1], it is possible to show that the optimal setup using the state (13) has error probability

$$p_{\text{err}}^{Q}(N) = \frac{m(N,d)}{2d^N} \left( 1 - \sqrt{1 - \frac{1}{m(N,d)^2}} \right),$$

where

$$m(N,d) := N! \prod_{i=1}^{d} \frac{(d-i)!}{(N/d + d - 1)!}$$

is the multiplicity of the trivial representation of $\text{SU}(d)$ in the $N$-fold tensor representation $U \mapsto U^\otimes N$. For large $N$, the above expression can be approximated as

$$p_{\text{err}}^{Q}(N) \approx \frac{1}{4m(N,d)d^N}.$$

Using the explicit expression (15), we then obtain the decay rate

$$R_Q = -\lim_{N \to \infty} \frac{\log[p_{\text{err}}^{Q}(N)]}{N} = 2\log(d) = 2R_C.$$

Thus, the presence of entanglement between the probes and the external reference system allows one to double the decay rate of the error probability.

The asymptotic limit can be achieved for sufficiently small number of interrogations of the unknown process. For example, an error probability less than $10^{-6}$ can be achieved by using 12 interrogations for discrimination of a causal relation between two quantum bits, while 20 interrogations are necessary for classical binary variables in order to achieve the same error threshold.
4.4 The ultimate quantum limit

In the previous subsections (4.2) and (4.3), we considered strategies where the unknown process was applied for \( N \) times in parallel on \( N \) input systems. These strategies are a special case of the sequential strategies shown in Figure 1. Can our experimenter further reduce the error probability by using these more general strategies?

A useful tool to address this question is the notion of fidelity divergence between two processes, introduced in Ref. [1], and later generalized to a broader set of channel divergences in Ref. [24]. The fidelity divergence between processes \( C_1 \) and \( C_2 \) is defined as

\[
\partial F(C_1, C_2) := \inf_R \inf_{\rho_1, \rho_2} \frac{F(\rho_1', \rho_2')}{F(\rho_1, \rho_2)},
\]

where \( \rho_1 \) and \( \rho_2 \) are states of the process input and of a reference system \( R \), \( \rho_1' := (C_1 \otimes I_R)(\rho_1) \), and \( \rho_2' := (C_2 \otimes I_R)(\rho_2) \). In order for the above expression to be well defined, the infimum is taken over all the states \( (\rho_1, \rho_2) \) such that \( F(\rho_1, \rho_2) \neq 0 \).

Let us denote by \( p_{\text{err}}^{\text{seq}}(C_1, C_2; N) \) the error probability in distinguishing between the two processes \( C_1 \) and \( C_2 \) using a sequence of \( N \) interrogations, as in Figure 1. It can be shown that the error probability satisfies the bound [1]

\[
p_{\text{err}}^{\text{seq}}(C_1, C_2; N) \geq \frac{\partial F(C_1, C_2)^N}{4}.
\]

In the special case where \( C_1 = U \otimes \mathbb{I}/d \) and \( C_2 = \mathbb{I}/d \otimes V \) with \( U(\cdot) = U \cdot U^\dagger \) and \( V(\cdot) = V \cdot V^\dagger \) being fixed unitary processes, the fidelity divergence is \( 1/d^2 \), and the error probability is lower bounded as

\[
p_{\text{err}}^{\text{seq}}(C_1, C_2; N) \geq \frac{1}{4d^2 N}.
\]

Hence, the decay rate cannot be larger than \( 2 \log d \), even if the unitaries \( U \) and \( V \) are known! If the unitaries \( U \) and \( V \) are unknown, as in the causal discrimination scenario, then the decay rate can only be \( 2 \log d \) or smaller. This observation proves that the decay rate \( 2 \log d \), achievable with a parallel strategy, is optimal among all decay rates achievable by arbitrary sequential strategies.

As a further curiosity, one may ask whether the rate could be improved in some exotic scenario where the order of the \( N \) interrogations is indefinite, unlike in the scenario of Figure 1, where the \( N \) interrogations happen in a well-defined sequential order. In principle, quantum probability theory is logically compatible with scenarios where quantum processes are combined in an indefinite causal order [25–27]. Physically, these scenarios may arise in exotic quantum gravity regimes, although research on such realizations is still in its infancy (and, of course, no complete theory of quantum gravity has been formulated yet). Still, as a theoretical possibility, one can already investigate the question of whether the ability to test a process for \( N \) times in an indefinite causal order could help identifying the causal relations occurring between its inputs and outputs.
For the identification of the causal intermediary, the answer turns out to be negative. The proof strategy is to bound the discrimination error for two simple processes, namely, $C_1 = I \otimes I/d$ and $C_2 = I/d \otimes I$ using arbitrary setups with indefinite order. Using semidefinite programming, Ref. [1] showed that the minimum error probability over all setups that place $N$ uses of the unknown process in an indefinite order is

$$p_{\text{err}}^{\text{ind}}(C_1, C_2; N) \geq \frac{1}{2} \left( 1 - \sqrt{1 - \frac{1}{d^2 N}} \right).$$

This result establishes the decay rate $2 \log d$ as the ultimate limit placed by quantum mechanics to the identification of a causal intermediary. In addition, the strong duality of semidefinite programming guarantees that there exists a suitable setup (possibly requiring indefinite causal order) that achieves the above error probability exactly.

5 Other examples of speedups in the identification of causal hypotheses

5.1 Multiple candidates for the causal intermediary

In the previous sections, we have seen how to identify the causal intermediary of a variable $A$ among two possible candidates $B$ and $C$. What about more than two candidates? For $k > 2$ candidates, the derivation is technically more involved, but the main results remain unchanged:

- In the classical case, when causal relations are described by arbitrary permutations, the minimal error probability is given by

$$p_{\text{err},k}^C = \frac{(k-1)}{2d^{N-1}} + O(d^{-2N}),$$

and the decay rate is $\log d$.

- In the quantum case, parallel strategies without a reference system achieve error probability

$$p_{\text{err},k}^Q = \frac{(k-1)}{2d^N} + O(d^{-2N}),$$

and the decay rate is still $\log d$. In contrast, parallel strategies using entanglement with an external reference system can achieve a doubled decay rate $2 \log d$.

5.2 Detection of causal link between two variables

A basic example of identification of causal hypotheses is to determine whether there is a causal link between two variables $A$ and $B$. In this case, the task is
to determine whether $B$ is a causal intermediary for $A$, or whether $B$ fluctuates at random independently of $A$. As it turns out, entanglement with a reference system can once again double the decay rate of the error probability: the classical error probability decays with rate $\log d$, while the quantum error probability with reference systems decays with doubled rate $2\log d$.

5.3 Identification of the cause of a variable

Another interesting problem is to identify which variable in a given set \( \{A_1, \ldots, A_m\} \) is the cause for a given variable $B$ (assuming that one and only one variable can be the cause). Again, we assume that causal relation is induced by a reversible process (a permutation in the classical case, or a unitary gate in the quantum case) and that all the variables have sample space of cardinality $d$ in the classical case, or Hilbert space of dimension $d$ in the quantum case.

Classically, the problem is to find the random variable $A_i$ such that $B$ is a permutation of $A_i$, with $i \in \{1, \ldots, m\}$. In the simplest case, when the permutation is known, the cause can be determined without error by interrogating the unknown process $\lceil \log_d(m) \rceil$ times.

In the quantum case, Ref. [1] showed that, if the unitary operator inducing the causal relation is known, then there exists a tests that achieves error probability

$$p_{\text{err}}(N) = \frac{m - 1}{d^{2N} + m - 1}$$

This means that one can get an error probability smaller than any desired $\epsilon > 0$ by using $N = \lceil (1 + \epsilon)\log_d(m)/2 \rceil$ interrogations, which is approximately half of the number of classical interrogations.

In the case where the dependency between cause and effect is unknown (arbitrary permutation in the classical case, or arbitrary unitary operator in the quantum case), the analysis is more complex. However, it is still possible to show that quantum strategies can identify the cause using only $N = \lceil (1 + \epsilon)\log_d(m)/2 \rceil$ interrogations.

6 Discussion and conclusions

In this paper, we reviewed the framework and the results of Ref. [1], which showed that quantum features such as coherence and entanglement offer advantages in detecting cause-effect relations induced by reversible processes.

In the problem of identifying the effect of a given variable, it was shown that entanglement between the probes and an external reference system can double the rate at which the error probability decays. For classical random variables with $d$ possible values, the decay rate is $\log d$. For quantum systems of dimension $d$, the decay rate is $2\log d$. As it turns out, the value $2\log d$ is the ultimate limit posed by quantum theory to the problem of identifying the causal intermediary of a given variable.
Interestingly, both the classical and quantum decay rates can be expressed as 
\[ \log \dim \mathcal{S}(A) \] where \( \mathcal{S}(A) \) is the vector space spanned by linear combinations of states of system \( A \). For classical systems, the states are probability distributions over the sample space, and the dimension of the corresponding vector space is \( d \). For quantum systems, the states are density matrices, and the dimension of the vector space is \( d^2 \). It would be interesting to study the problem of causal hypothesis discrimination in toy theories with higher dimensional state spaces, such as quantum theory on quaternionic Hilbert spaces [28] or the quartic toy theory proposed by Życzkowski in Ref. [29].

On a more practical side, it is important to extend the analysis from the idealized scenario where the cause-effect dependencies are induced by reversible processes, to the more realistic scenario where they are induced by general noisy processes. Preliminary results in Ref. [1] indicate that quantum advantages may still persist for sufficiently low noise levels. However, a fully general treatment of noise is still lacking and will be important for future applications.

Given the success of causal discovery algorithms in classical statistics and machine learning, it is natural to expect that the development of quantum causal discovery algorithms may have applications to the burgeoning field of quantum machine learning [30–32]. This connection is largely unexplored and represents an exciting direction of future research.

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