ON $r$-NEIGHBORLY SUBMANIFOLDS IN $\mathbb{R}^N$

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Abstract. A submanifold $M \subset \mathbb{R}^N$ is $r$-neighborly if for any $r$ points in $M$ there is a hyperplane, supporting $M$ and touching it at exactly these $r$ points. We prove that the minimal dimension $\Delta(k, r)$ of the Euclidean space, containing a stably $r$-neighborly submanifold, is asymptotically no less than $2kr - k$.

1. Introduction

Let $M$ be a $k$-dimensional manifold, and $r$ a natural number.

Definition. A smooth embedding $M \to \mathbb{R}^N$ is $r$-neighborly if for any $r$ points in $M$ there is an affine hyperplane in $\mathbb{R}^N$, supporting $M$ and touching it at exactly these $r$ points.

Denote by $\delta(M, r)$ the minimal dimension $N$ of an Euclidean space, such that there exists an $r$-neighborly embedding $M \to \mathbb{R}^N$, and by $\delta(k, r)$ the maximum of numbers $\delta(M, r)$ over all connected $k$-dimensional manifolds $M$.

The problem of estimating the numbers $\delta(k, r)$ for all $k$ and $r$ was posed by M. Perles in the 1970-ies by analogy with similar problems of combinatorics, and was discussed in the Oberwolfach Combinatorics meetings in 1980-ies. Nontrivial examples of $r$-neighborly manifolds were constructed in [5]; as G. Kalai communicated to me, in their non-published work with A. Wigderson a polynomial upper estimate of type $\delta(k, r) = O(r^2k)$ was proved. However no nontrivial general lower estimates of these numbers are known.

We consider the similar problem concerning a slightly stronger condition.

Definition. A smooth embedding $M \to \mathbb{R}^N$ is stably $r$-neighborly, if it is $r$-neighborly, and any other embedding, sufficiently close to it in the $C^2$-topology, also is. The corresponding analogues of numbers $\delta(M, r)$ and $\delta(k, r)$ are denoted by $\Delta(M, r)$ and $\Delta(k, r)$, respectively.

It is more or less obvious that

$$\Delta(k, r) \geq (k + 1)r.$$  \hspace{1cm} (1)

Indeed, if $N < (k + 1)r$, then for any generic submanifold $M^k \subset \mathbb{R}^N$ and almost any set of $r$ points in $M^k$ the minimal affine plane in $\mathbb{R}^N$, touching $M^k$ at all points of this set, coincides with entire $\mathbb{R}^N$.

If $k = 1$, then this estimate is sharp: the moment embedding $S^1 \to \mathbb{R}^{2r}$, sending the point $\alpha$ to $(\sin \alpha, \cos \alpha, \ldots, \sin r\alpha, \cos r\alpha)$, is stably $r$-neighborly.

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We prove the following lower estimates of numbers $\Delta(M, r)$. Denote by $d(r)$ the number of ones in the binary representation of $r$.

**Theorem 1.** If $k$ is a power of 2, then
\begin{equation}
\Delta(R^k, r) > (k + 1)r + (k - 1)(r - d(r)) - 1.
\end{equation}
In particular, if $r$ also is a power of 2, then $\Delta(R^k, r) > 2kr - k$.

Of course, the same estimate of the number $\Delta(M, r)$ is true for any $k$-dimensional manifold $M$.

For an arbitrary fixed $r$ and growing $k$ we have a slightly better estimate of $\Delta(k, r)$.

**Theorem 2.** For any $r \leq k = 2^j$,
\begin{equation}
\Delta(RP^k, r) > (k + 1)r + kr - r(r + 1)/2 - 1.
\end{equation}
The estimate of Theorem 2 can be improved if $r = 1$.

**Proposition 1.** For any closed $M$, $\Delta(M, 1) \leq N$ if and only if $M$ can be smoothly embedded into $S^{N-1}$. In particular, for $k = 2^j$, $\Delta(RP^k, 1) = 2k + 1$.

**Conjecture.** In both theorems 1 and 2, $\Delta$ can be replaced by $\delta$.

In fact, we prove this conjecture in a much more general situation than it follows from Theorems 1 and 2.

**Theorems 1!, 2!.** All $r$-neighborly embeddings $R^k \to R^N$ (respectively, $RP^k \to R^N$) with $N$ equal to the right-hand part of the inequality (2) (respectively, (3)), belong to a subset $\Sigma\Sigma$ of infinite codimension in the space $C^2(R^k, R^N)$ (respectively, $C^2(RP^k, R^N)$).

I believe that in fact such embeddings do not exist.

Of course, Theorems 1 and 2 follow from these ones and even from similar statements with “infinite codimension” replaced by “positive codimension”. The set $\Sigma\Sigma$ will be described in § 3.

I thank Professor Gil Kalai for stating the problem.

2. A topological lemma

**Convention and notation.** Using the parallel translations in $R^N$, we will identify the space $R^N$ with its tangent spaces at all points $y \in R^N$, and use the canonical projection $\pi : TR^N \to R^N$, sending any vector $v \in T_yR^N$ to the point $y + v$. In particular, any vector subspace $\Xi \subset T_yR^N$ defines the affine subspace $\pi\Xi \subset R^N$ of the same dimension. The sign $\oplus$ denotes the direct (Whitney) sum of two vector bundles. $w(\xi)$ is the total Stiefel–Whitney class of the vector bundle $\xi$, see [7], and $\bar{w}(\xi)$ is the total Stiefel–Whitney class of any vector bundle $\eta$ such that the bundle $\xi \oplus \eta$ is trivial. In particular, $w(\xi)\bar{w}(\xi) = 1 \in H^0(\text{the base of } \xi, \mathbb{Z}_2)$.

Let $L$ be a $l$-dimensional compact manifold without boundary (maybe not connected), and $\xi$ a $t$-dimensional vector bundle over $L$ ($l < N - t$).

**Definition.** A smooth homomorphism $i : \xi \to TR^N$ is a pair, consisting of a smooth map $i : L \to R^N$ and, for any $x \in L$, a homomorphism $\xi_x \to T_{i(x)}R^N$ smoothly depending on $x$. $i$ is a monomorphism if all these homomorphisms $\xi_x \to T_{i(x)}R^N$ are injective.
Let \( i : \xi \to TR^N \) be a smooth monomorphism such that the corresponding
map \( \iota : L \to R^N \) is a smooth embedding and for any \( x \in L \) two subspaces \( i(\xi_x) \) and
\( T_{i(x)}(\iota(L)) \equiv i_*(T_xL) \) of the tangent space \( T_{i(x)}R^N \) have no common nonzero vectors.
This monomorphism \( i \) induces the proper map \( i' \equiv \pi \circ i : E \to R^N \) of the total space
\( E \) of \( \xi \) into \( R^N \), sending any vector \( a \in \xi_x \) to the point \( \iota(x) + i(a) \). Suppose
that this map \( i' \) is transversal to \( \iota(L) \) everywhere in \( E \setminus L \) (where \( L \subset E \) denotes the
zero section of \( \xi \)). Then this intersection set \( \iota(L) \cap i'(E \setminus L) \) defines a \( Z_2 \)-cycle of
codimension \( N-l-t \) in \( \iota(L) \sim L \).

**Lemma 1.** The class in \( H^*(L, Z_2) \), Poincaré dual to this cycle \( \iota^{-1}(\iota(L) \cap i'(E \setminus L)) \),
is equal to
\[
\bar{w}_{N-l-t}(TL \oplus \xi),
\]
the \((N-l-t)\)-dimensional homogeneous component of \( \bar{w}(TL \oplus \xi) \).

In particular, if this class \([7]\) is nontrivial, then the set \( \iota(L) \cap i'(E \setminus L) \) is not empty;
moresover, the latter is true even if \( i' \) is not transversal to \( \iota(L) \) in \( E \setminus L \).

**Example.** If \( t = 0 \), we get the known obstruction to the existence of an embedding
\( L' \to R^N \), see \([7]\): if \( \bar{w}_{N-l}(TL) \neq 0 \), then the self-intersection set of any such
embedding is non-empty.

**Proof of Lemma 1.** There is a tubular \( \varepsilon \)-neighborhood \( U_\varepsilon \) of the submanifold \( \iota(L) \in R^N \)
such that the containing \( L \) component of \( (i')^{-1}(U_\varepsilon) \) is a tubular neighborhood
of \( L \) in \( E \), and the restriction of \( i' \) on the latter neighborhood (which we denote by
\( W_\varepsilon \)) is a diffeomorphism onto its image. Let us deform the submanifold \( \iota(L) \) in \( R^N \)
by a sufficiently \( C^1 \)-small generic diffeomorphism \( v \) of \( R^N \) so that
a) \( |v(y) - y| < \varepsilon/2 \) for any \( y \in \iota(L) \),
b) the image \( vU(L) \) of \( U(L) \) under this shift is a smooth manifold in \( U_\varepsilon \) transversal to
the manifold \( i'(W_\varepsilon) \);
c) the map \( i' \) remains to be transversal to \( vU(L) \) everywhere in \( E \setminus W_\varepsilon \).

The variety \( vU(L) \cap i'(E) \) consists of two disjoint parts. The first, \( vU(L) \cap i'(E \setminus W_\varepsilon) \),
defines in the group \( H_*(vU(L), Z_2) \cong H_*(L, Z_2) \) the same homology class as \( \iota(L) \cap i'(E \setminus W_\varepsilon) \), which we wish to calculate. The second, \( vU(L) \cap i'(W_\varepsilon) \), is Poincaré dual to the
first homological obstruction to the existence of a continuous non-zero section
of the quotient bundle \( R^N/(TL \oplus \xi) \) over \( L \), and thus is equal to \( \bar{w}_{N-l-t}(TL \oplus \xi) \),
see \([7]\).

Finally, the homology class in \( H_*(vU(L), Z_2) \) of the sum of these two cycles in
\( vU(L) \) is Poincaré dual to the restriction to \( vU(L) \) of the cohomology class \([E] \in H^{N-l-t}(R^N, Z_2) \) Poincaré dual to the direct image of the fundamental cycle of the
closed manifold \( E \) under the proper map \( i' \). The latter class belongs to a trivial
group, hence our two cycles in \( vU(L) \) are homologous mod 2. \( \Box \)

3. The General Topological Estimate

Let \( M \) be a \( k \)-dimensional manifold, and \( B(M, r) \) the \( r \)-th configuration space of
\( M \), i.e. the topological space, whose points are subsets of cardinality \( r \) in \( M \). Let
\( \psi \) (respectively, \( \tilde{\psi} \)) be the canonical \( r \)-dimensional (respectively, \((r-1)\)-dimensional)
vector bundle over \( B(M, r) \), whose fibre over the point \( x = \{x_1, \ldots, x_r\} \in B(M, r) \)
is the space of all functions \( f : \{x_1, \ldots, x_r\} \to R \) (respectively, of all such functions
with \( \sum f(x_i) = 0 \)). Obviously \( \psi \simeq \psi \oplus R^1 \).
Any embedding $I : M \to \mathbb{R}^N$ defines the map $I' : B(M, r) \to \mathbb{R}^N$, sending any point $x = \{x_1, \ldots, x_r\} \in B(M, r)$ to the mass center

$$\frac{1}{r} \sum I(x_i)$$

of points $I(x_1), \ldots, I(x_r)$. The image of the tangent space $T_x B(M, r)$ under the derivative map $I'_x$ of this map at the point $x$ is equal to the linear span of tangent spaces $T_{I(x_i)} I(M)$, $i = 1, \ldots, r$, translated to the point (5).

Also there is the natural homomorphism $\chi : \tilde{\psi} \to T\mathbb{R}^N$, sending any function $f : (x_1, \ldots, x_r) \to \mathbb{R}$ to the vector $\sum_{i=1}^r f(x_i) I(x_i) \in T_{I'(x)} \mathbb{R}^N$. The corresponding subset $\pi \circ \chi(\tilde{\psi}|_x) \subseteq \mathbb{R}^N$ is the minimal affine subspace in $\mathbb{R}^N$ containing all points $I(x_i)$.

Thus for any $x \in B(M, r)$ we have two important subspaces in $T_{I'(x)} \mathbb{R}^N$: the image of $T_x B(M, r)$ under the derivative map $I'_x$ and $\chi(\tilde{\psi}|_x)$. Let $\tau(x) \subseteq T_{I'(x)} \mathbb{R}^N$ be the linear span of these two subspaces.

**Lemma 2.** Suppose that the embedding $I$ is $r$-neighborly. Then for any $x = (x_1, \ldots, x_r) \in B(M, r)$ the affine plane $\pi(\tau(x)) \subseteq \mathbb{R}^N$ intersects the set $I(B(M, r))$ only at the point $I(x)$.

**Proof.** The affine hyperplane in $\mathbb{R}^N$, touching $M$ at the points $x_1, \ldots, x_r$ and participating in the definition of a $r$-neighborly embedding, contains this plane $\pi(\tau(x))$ but cannot contain any point of $I'(B(M, r))$ other than $x$. □

**Definition.** The set $\Omega(I) \subseteq B(M, r)$ is the set of all points $x$ such that $\operatorname{dim} \tau(x) < kr + r - 1$. For $r > 1$ and $N \geq (k+1)r - 1$, the set $\operatorname{Reg}(r) \subseteq C^\infty(M, \mathbb{R}^N)$ consists of all maps $I : M \to \mathbb{R}^N$ such that the topological codimension of $\Omega(I)$ in $B(M, r)$ is greater than $N - (k+1)r + 1$ in the following exact sense: any compact $(N - (k+1)r + 1)$-dimensional submanifold in $B(M, r)$ is isotopic to one not intersecting $\Omega(I)$. For $r = 1$, set $\operatorname{Reg}(1) = C^\infty(M, \mathbb{R}^N)$.

In particular, for any such submanifold $L$, not intersecting $\Omega(I)$, the restriction on $L$ of the bundle $\tau(x)$ is isomorphic to $TB(M, r) \oplus \tilde{\psi}$, and the restriction of the map $I$ on $L$ is an immersion into $\mathbb{R}^N$ (and even an embedding if $r > 1$).

**Lemma 3.** For any $r > 1$, the set $\Sigma\Sigma(r) = C^\infty(M, \mathbb{R}^N) \setminus \operatorname{Reg}(r)$ is a subset of infinite codimension in $C^\infty(M, \mathbb{R}^N)$.

**Proof.** Any map $I : M \to \mathbb{R}^N$ defines its multijet extension $I^1_r : B(M, r) \to J^1_r(M, \mathbb{R}^N)$, sending any point $(x_1, \ldots, x_r) \in B(M, r)$ to the collection of 1-jets of $I$ at these points, see e.g. [4]. The set $\Omega(I)$ can be described as the pre-image under this map of a certain algebraic subset $\Sigma \subseteq J^1_r(M, \mathbb{R}^N)$, whose codimension is equal to $N - (k+1)r + 2$. If the map $I$ is of class $\Sigma\Sigma$, then this map $I^1_r$ is non-transversal to $\Sigma$ at infinitely many points, and our lemma follows from the Thom’s multijet transversality theorem, see [4], [9]. □

**Theorem 3.** Suppose that $N = (k+1)r + l - 1$ and there is a $l$-dimensional compact submanifold $L \subseteq B(M, r)$ such that

$$\langle [L], \bar{w}_l(TB(M, r) \oplus \tilde{\psi}) \rangle \neq 0,$$

where $[L]$ is the $\mathbb{Z}_2$-fundamental class of $L$. Then there are no $r$-neighborly embeddings $M \to \mathbb{R}^N$ of the class $\operatorname{Reg}(r)$. In particular, $\Delta(M, r) \geq (k+1)r + l$. 

Proof. Suppose that \( I \) is a \( r \)-neighborly embedding \( M \to \mathbb{R}^N \) of class \( \text{Reg}(r) \), \( r > 1 \). Denote by \( \nu \) the \((kr-l)\)-dimensional vector subbundle in the restriction of \( TB(M, r) \) to \( L \), orthogonal to the tangent bundle \( TL \); let be \( \xi = \nu \oplus \tilde{\psi}_L \), so that \( TL \oplus \xi = T|_L B(M, r) \oplus \tilde{\psi}_L \). Since \( I \in \text{Reg}(r) \), we can assume that \( L \) does not meet \( \Omega(I) \), in particular the homomorphism \( \xi \oplus \chi : \xi \to TR^N \) is a monomorphism satisfying conditions of Lemma 1. Then conclusions of Lemmas 1 and 2 contradict to one another. Finally, in the case \( r = 1 \) our condition (6) prevents the existence of any embedding \( M \to \mathbb{R}^N \), see [7] or Example after Lemma 1, and Theorem 3 is completely proved. \( \square \)

4. Proof of main theorems

4.1. Proof of Proposition 1. Any embedding \( M^k \to S^{N-1} \) obviously is a stable 1-neighborly embedding \( M \to \mathbb{R}^N \). Conversely, suppose that \( M \) is a 1-neighborly submanifold in \( \mathbb{R}^N \). The Gauss map establishes the natural homeomorphism between \( S^{N-1} \) and the set of all supporting hyperplanes of \( M \). For any point \( x \in M \) consider the set \( H(x) \) of all oriented hyperplanes, supporting \( M \) only at this point. This is a convex semialgebraic subset of the sphere \( S^{N-1-k} \subset S^{N-1} \), consisting of all oriented hyperplanes parallel to \( T_x M \). If \( M \) is generic, then the set of interior points of \( H(x) \) in \( S^{N-1-k} \) is non-empty, and the union of such sets forms a smooth fiber bundle over \( M \) with fibers homeomorphic to \( \mathbb{R}^{N-1-k} \). Any smooth section of this bundle is the desired embedding. \( \square \)

Remark. The same considerations prove that if \( M \) is a (non-stably) 1-neighborly submanifold of \( \mathbb{R}^N \), then it is homeomorphic to a subset of \( S^{N-1} \). However, the example of the curve \( t \to (t, t^3, t^4) \) shows that for non-generic embeddings the set \( H(t) \) can consist of unique point, and our fiber bundle \( \{H(x) \to x\} \) can be not smooth and even not locally trivial.

4.2. Proof of Theorem 1. First suppose that \( r = 2^s \). The corresponding submanifold \( L(r) \subset B(\mathbb{R}^k, r) \), satisfying the conditions of Theorem 3, is constructed as follows (cf. [3], [9]).

Let us fix some \( \varepsilon \in (0, 1/2] \). Consider in \( \mathbb{R}^k \) a sphere of radius 1 and mark on it some two opposite points \( A_1, A_2 \). Then consider two spheres of radius \( \varepsilon \) with centers in these two points and mark on any of them two opposite points: \( A_{11}, A_{12} \) on the first and \( A_{21}, A_{22} \) on the second. Consider four spheres of radius \( \varepsilon^2 \) with centers at all these four points and mark on any of them two opposite points: \( A_{111}, \ldots, A_{222} \). After the \( s \)-th step we obtain \( 2^s = r \) pairwise different points in \( \mathbb{R}^k \), i.e. a point of the space \( B(\mathbb{R}^k, r) \). \( L(r) \) is defined as the union of all points of the latter space, which can be obtained in this way; this is a \((k-1)(r-1)\)-dimensional smooth compact manifold.

Proposition 2. 1. For any \( k \) and \( r \), the tangent bundle \( TB(\mathbb{R}^k, r) \) is isomorphic to the direct sum of \( k \) copies of the bundle \( \psi \).

2. If \( k \) is a power of 2, then the \( k \)-th power of the total Stiefel–Whitney class of the bundle \( \tilde{\psi} \) (or \( \psi \)) over \( B(\mathbb{R}^k, r) \) is equal to \( 1 \in H^0(B(\mathbb{R}^k, r), \mathbb{Z}_2) \). In particular, \( \tilde{\psi}(\tilde{\psi}) = (w(\tilde{\psi}))^{k-1} \).

3. If both \( k \) and \( r \) are powers of 2, then the \((k-1)(r-1)\)-dimensional homogeneous component \( \tilde{\psi}(k-1)(r-1)(\tilde{\psi}) \in H^{(k-1)(r-1)}(B(\mathbb{R}^k, r), \mathbb{Z}_2) \) of the total inverse
Stiefel–Whitney class \( \bar{w}_s(\psi) \) is nontrivial and its value on the fundamental cycle of the manifold \( L(r) \) is equal to 1.

Proof. Statement 1 is obvious (and remains true if we replace \( \mathbb{R}^k \) by any parallelizable manifold). Statement 2 (and, moreover, similar assertion concerning any class of the form \( 1 + \{ \text{terms of positive dimension} \} \in H^*(B(\mathbb{R}^k, r), \mathbb{Z}_2) \)) is proved in [6]. Statement 3 is proved in [10], § I.3.7 (and no doubt was known to the author of [6]). □

Thus the submanifold \( L(r) \subset B(\mathbb{R}^k, r), k = 2^j \), satisfies the conditions of Theorem 3, and Theorem 1! is proved in the case when \( r \) is a power of 2. For an arbitrary \( r \) the similar submanifold \( L(r) \) is constructed as follows. Suppose that \( r = 2^{t_1} + \cdots + 2^{t_d}, t_1 > \cdots > t_d, d = d(r) \). Then the manifold \( L(r) \) consists of all collections of \( r \) points in \( \mathbb{R}^k \), such that the collection of first \( 2^{t_1} \) of them belongs to the manifold \( L(2^{t_1}) \), the next \( 2^{t_2} \) are obtained from some collection \( x_2 \in L(2^{t_2}) \) by the translation along the vector \( (3, 0, \ldots, 0) \), the next \( 2^{t_3} \) are obtained from some collection \( x_3 \in L(2^{t_3}) \) by the translation along the vector \( (6, 0, \ldots, 0) \), etc. In particular, \( L(r) \sim L(2^{t_1}) \times \cdots \times L(2^{t_d}) \). (If \( t_d = 1 \), then we set \( L(t_d) = \{ \text{the point 0} \}) \). In restriction to \( L(r) \), the bundle \( \tilde{\psi} \) is obviously isomorphic to the direct sum of \( d \) bundles, induced from similar \( (2^{t_d} - 1) \)-dimensional bundles over the factors \( L(2^{t_i}) \), and the \( (r - 1) \)-dimensional trivial bundle. Thus the manifold \( L(r) \) for an arbitrary \( r \) also satisfies conditions of Theorem 3 (with \( l = (k - 1)(r - d) \)). □

4.3. Proof of Theorem 2!. Suppose that \( k = 2^j \). Define the submanifold \( L(k, r) \subset B(\mathbb{R}P^k, r) \) as the set of all unordered collections of \( r \) pairwise orthogonal points in \( \mathbb{R}P^k \) (with respect to any Euclidean metrics in \( \mathbb{R}^{k+1} \)) lying in some fixed subspace \( \mathbb{R}P^{k-1} \subset \mathbb{R}P^k \). This is a smooth \( (kr - \binom{k}{2}) \)-dimensional manifold.

Consider also the manifold \( \Lambda(k, r) \), consisting of similar ordered collections; it is a submanifold of the space \((\mathbb{R}P^k)^\tau\) and also the space of a \( r! \)-fold covering \( \theta : \Lambda(k, r) \to L(k, r) \).

**Proposition 3. 1.** There is a ring isomorphism

\[
H^*(\Lambda(k, r), \mathbb{Z}_2) \simeq H^*(\mathbb{R}P^{k-1}, \mathbb{Z}_2) \otimes H^*(\mathbb{R}P^{k-2}, \mathbb{Z}_2) \otimes \cdots \otimes H^*(\mathbb{R}P^{k-r}, \mathbb{Z}_2).
\]

2. The vector bundle over \( \Lambda(k, r) \), induced by the map \( \theta \) from the bundle \( \psi \) or \( \tilde{\psi} \) on \( L(k, r) \), is equivalent to the trivial one.

3. The vector bundle over \( \Lambda(k, r) \), induced by the map \( \theta \) from the tangent bundle \( TB(\mathbb{R}P^k, r) \), coincides with the restriction on \( \Lambda(k, r) \) of the tangent bundle of the manifold \( (\mathbb{R}P^k)^\tau \).

4. The inverse Stiefel–Whitney class \( \bar{w}_s \equiv (w_s)^{-1} \) of the latter tangent bundle satisfies the inequality

\[
\langle [\Lambda(k, r)], \bar{w}_{kr-\binom{k}{2}}(T(\mathbb{R}P^k)^\tau) \rangle \neq 0.
\]

Proof. Statement 1 is a standard exercise on homology of fiber bundles, see e.g. [2]. Namely, consider the fiber bundle \( p : \Lambda(k, r) \to \Lambda(k, r - 1) \), sending any ordered collection \( (x_1, \ldots, x_r) \) to \( (x_1, \ldots, x_{r-1}) \). Its fiber \( F \) is equal to \( \mathbb{R}P^{k-r} \), hence the fundamental group of the base acts trivially on \( H^*(F, \mathbb{Z}_2) \), and the term \( E_2^{p,q} \) of the \( \mathbb{Z}_2 \)-spectral sequence of this bundle is naturally isomorphic to \( H^p(\Lambda(k, r - 1), \mathbb{Z}_2) \otimes H^q(F, \mathbb{Z}_2) \). All further differentials \( d^i, i \geq 2 \), of the spectral sequence act trivially on all elements of the column \( E_0^{0,*} \simeq H^*(F, \mathbb{Z}_2) \), because the embedding
homomorphism $H^*(\Lambda(k,r),\mathbb{Z}_2) \to H^*(\mathcal{F},\mathbb{Z}_2)$ is epimorphic: indeed, its composition with the map $H^*(\mathbb{RP}^k,\mathbb{Z}_2) \to H^*(\Lambda(k,r),\mathbb{Z}_2)$, defined by the projection of the space $\Lambda(k,r) \subset (\mathbb{RP}^k)^r$ onto the $r$-th copy of $\mathbb{RP}^k$, is just the epimorphism $H^*(\mathbb{RP}^k,\mathbb{Z}_2) \to H^*(\mathbb{RP}^{k-r},\mathbb{Z}_2)$, induced by the embedding. Since the spectral sequence is multiplicative, it degenerates at the term $E_2$.

Statements 2 and 3 of Proposition 3 are obvious, and statement 4 follows from the induction conjecture for $\Lambda(k,r-1)$ and the fact that for $k = 2^j$ the inverse Stiefel–Whitney class $\tilde{w}_s(\mathbb{RP}^k)$ is equal to $1 + \alpha + \alpha^2 + \cdots + \alpha^{k-1}$, where $\alpha$ is the multiplicative generator of the ring $H^*(\mathbb{RP}^k,\mathbb{Z}_2)$, see e.g. [7]. □

Using the functoriality of Stiefel–Whitney classes and the Whitney multiplication formula for these classes of a Whitney sum of bundles, we obtain from this proposition that $\tilde{w}_{kr-\binom{r}{2}}(TB(\mathbb{RP}^k, r) \oplus \tilde{\psi})$ is a non-trivial element of $H^{kr-\binom{r}{2}}(L(k,r),\mathbb{Z}_2)$.

Theorem 2 is now reduced to Theorem 3. □

Remark. The number $\Delta(M,k)$ is a measure of the “topological complexity” of the configuration space $B(M,r)$. This complexity appears from two issues: the obvious free action of the symmetric group $S(r)$ and the topological complexity of the manifold $M$ itself. In Theorem 1 we essentially exploit only the first issue, and in Theorem 2 only the second. The simultaneous consideration of these two interacting components should give us more precise estimates of $\Delta(M,r)$.

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