Intercept-resend attacks in the Bennett-Brassard 1984 quantum key distribution protocol with weak coherent pulses

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Unconditional security proofs of the Bennett-Brassard protocol of quantum key distribution have been obtained recently. These proofs cover also practical implementations that utilize weak coherent pulses in the four signal polarizations. Proven secure rates leave open the possibility that new proofs or new public discussion protocols obtain larger rates over increased distance. In this paper we investigate limits to error rate and signal losses that can be tolerated by future protocols and proofs.

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I. INTRODUCTION

Quantum key distribution (QKD) \cite{1,2} is a technique that exploits quantum effects to establish a secure secret key between two parties (usually called Alice and Bob). This secret key is the essential ingredient of the one-time-pad or Vernam cipher \cite{3}, the only known encryption method that can provide information-theoretic secure communications.

The first complete QKD scheme is that introduced by Bennett and Brassard in 1984 (BB84 for short) \cite{2}. In a quantum optical implementation of this protocol, Alice encodes each random bit into the polarization state of a single-photon: she chooses along one of two mutually unbiased bases, e.g. either a linear or a circular polarization basis. On the receiving side, Bob measures each photon by selecting at random between two polarization analyzers, one for each possible basis. As a result, Alice and Bob end up with some classical correlated data that can be described by a joint probability distribution \( P(A, B) \), where the random variables \( A \) and \( B \) represent the signal states prepared by Alice and the measurement results obtained by Bob, respectively. Next, Alice and Bob use an authenticated public channel to process these data in order to obtain a secret key. This second phase, usually called key distillation, involves, typically, postselection of data, error correction to reconcile the data, and privacy amplification to decouple the data from a possible eavesdropper (Eve) \cite{4}. A full proof of the unconditional security for the complete BB84 protocol has been obtained \cite{8}.

After the first experimental demonstration of the feasibility of the BB84 scheme \cite{5}, several experimental groups have realized long-distance implementations of QKD in the last years \cite{6}. However, these practical approaches differ in many important aspects from the original theoretical proposal, since that demands technologies that are beyond our present experimental capability. Especially, the signal states emitted by the source, instead of being single-photon signals, are usually weak coherent pulses (WCP) with a low probability of containing more than one photon (typical average photon-numbers are 0.1 or higher). The quantum channel (e.g. optical fiber) introduces considerable attenuation and errors that affect the signals even when Eve is not present. Finally, the detectors employed by the receiver have a low detection efficiency and are noisy due to dark counts. All these modifications from the ideal BB84 protocol to real implementations can jeopardize the security of the protocol, and lead to limitations of rate and distance that can be covered by these techniques \cite{7}. A positive security proof against all individual particle attacks, even with practical signals, has been given in \cite{10}. More recently, a complete proof of the unconditional security of the BB84 scheme in a realistic setting has also been achieved \cite{11}. This means that, despite of practical imperfections, it is still possible to obtain a secure secret key with the support of the classical information techniques used in the key distillation phase.

While all these positive security proofs are of great importance for QKD, they might be over-restrictive. This might be so either because of the particular mathematical techniques used in the security proofs, or because of the specific key distillation protocols considered. Different ideas to extend the proven secure regimes of Ref. \cite{8,11} have been found \cite{12,13}. More recently, it has been shown that the ultimate limit for secure QKD is given by the proven presence of quantum correlations in \( P(A, B) \) \cite{12}: whenever the joint probability distribution \( P(A, B) \), together with the knowledge of the corresponding signal states and detection methods employed by Alice and Bob, can be interpreted as coming from a separable state then no secret key can be obtained, whatever classical protocol may be used in the key distillation phase.

In this paper we are interested in this ultimate limit, given by the existence of quantum correlations in \( P(A, B) \), for practical QKD. In principle, to decide whether the joint probability distribution \( P(A, B) \), obtained after the first phase of QKD, contains quantum correlations, Alice and Bob can use all the information contained in \( P(A, B) \). However, we evaluate only those events where the signal preparation and detection methods employed by Alice and Bob use the same polariza-
tion basis. Moreover, in this paper, we consider that Alice and Bob analyze only two particular events: The expected click rate at Bob’s side, and the generalized error rate. For this, we investigate a simple eavesdropping strategy based on intercept-resend attacks: Eve measures out every signal emitted by Alice and prepares a new one, depending on the result obtained, that is given to Bob. Intercept-resend attacks correspond to entanglement breaking channels [13] and, therefore, they cannot lead to a secure key [12]. This kind of attacks has been already studied by Dušek et al. in Ref. [16] and by Félix et al. in Ref. [17]. In particular, Dušek et al. considered the case where Eve realizes unambiguous state discrimination (USD) [18, 19] of Alice’s signal states. Whenever Eve is successful with her USD measurement and she identifies the signal state sent by Alice, then she sends this information to a quantum source close to Bob via a classical channel. This source prepares the identified quantum state and subsequently forwards it to Bob. If the identification process does not succeed, then Eve sends the vacuum signal to Bob to avoid errors. This attack does not introduce errors in the signal states, but it requires high losses in the channel. Note that in order to unambiguously discriminate between the four BB84 signal states Eve requires that the signals contain, at least, three or more photons [16, 18]. In this paper, we investigate a different regime. Specifically, we propose a simple intercept-resend attack for the scenario where the attenuation introduced by the channel is not sufficiently high for Eve to perform the USD attack. This fact is compensated by allowing her to introduce some errors. As a result, we obtain an upper bound on the maximal distance achievable by the BB84 protocol as a function of the error rate and the mean photon-number of the signals sent by Alice. Beyond this upper bound no secure QKD is possible.

The paper is organized as follows. In Sec. II we describe in more detail the signal states and detection methods employed by Alice and Bob. Then, in Sec. III we introduce an intercept-resend attack for practical QKD and we derive an upper bound on the maximal distance achievable by the BB84 protocol. Finally, Sec. IV concludes the paper with a summary.

II. TOOLBOX FOR ALICE AND BOB

A. Alice

We consider that Alice uses WCP signal states described by coherent states with a small amplitude $\alpha$. Moreover, we assume the typical scenario where there is no phase reference available outside Alice’s setup. This results in effective signal states which are mixtures of Fock states with a Poissonian photon-number distribution of mean $\mu = |\alpha|^2$. Such states are described as

$$\rho_k = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} |n_k \rangle \langle n_k|,$$

where the states $|n_k\rangle$ denote Fock states with $n$ photons in one of the four BB84 polarization states, which are labeled with the index $k$, with $k = 0, ..., 3$. The symmetry imposed by the BB84 protocol on the signal states $|n_k\rangle$, with $n > 0$, guarantees that there exists a unitary transformation $U(n)$ such that

$$|n_k\rangle = U(n)|n_{k-1}\rangle = U^k(n)|n_0\rangle$$

$$|n_0\rangle = U(n)|n_3\rangle$$

$$U^k(n) = \mathbb{I}.$$  \hspace{1cm} (2)

For more details in our specific case see [16]. The unitary operator $U(n)$ can be expanded as

$$U(n) = \sum_{j=0}^{3} \exp \left( 2\pi i \frac{j^2}{4} \right) |\phi_j(n)\rangle \langle \phi_j(n)|,$$  \hspace{1cm} (3)

where the states $|\phi_j(n)\rangle$ represent a set of orthogonal states. This allows us to write the states $|n_k\rangle$, with $n > 0$, as

$$|n_k\rangle = \sum_{j=0}^{3} c_j(n) \exp \left( 2\pi i \frac{jk}{4} \right) |\phi_j(n)\rangle,$$  \hspace{1cm} (4)

where the coefficients $c_j(n)$ satisfy $\sum_{j=0}^{3} |c_j(n)|^2 = 1$. The exact values of the coefficients $|c_j(n)|$ can be obtained explicitly using the overlaps of the four states $|n_k\rangle$, with $k = 0, ..., 3$, according to the formula [16]

$$|c_j(n)|^2 = \frac{1}{16} \sum_{l,m} \exp \left( -2\pi i \frac{j(l-m)}{4} \right) \langle n_l|n_m\rangle.$$  \hspace{1cm} (5)

As a result, one finds the expressions [16]

$$|c_0(n)| = \sqrt{\frac{1}{4} + 2^{-1+n/2} \cos \left( \frac{\pi}{4} n \right)},$$ \hspace{1cm} (6)

$$|c_1(n)| = \sqrt{\frac{1}{4} + 2^{-1+n/2} \sin \left( \frac{\pi}{4} n \right)},$$

$$|c_2(n)| = \sqrt{\frac{1}{4} - 2^{-1+n/2} \cos \left( \frac{\pi}{4} n \right)},$$

$$|c_3(n)| = \sqrt{\frac{1}{4} - 2^{-1+n/2} \sin \left( \frac{\pi}{4} n \right)}.$$

B. Bob

Bob employs the active detection setup shown in Fig. II. It consists of a polarization analyzer and a polarization shifter that effectively changes the polarization basis of the subsequent measurement. The polarization analyzer
FIG. 1: The polarization shifter allows to change the polarization basis (+ and ) of the measurement as desired. The polarization analyzer consists of a polarizing beam splitter (PB) and two ideal detectors. The PB discriminates the two orthogonal polarized modes. Detection efficiencies are modeled by a beam splitter (BS) of transmittance \( \eta_{\text{det}} \).

has two detectors, each detector monitoring one output mode of a polarizing beam splitter. These detectors are characterized by their detection efficiency \( \eta_{\text{det}} \). They can be described by a single-loss beam splitter of transmittance \( \eta_{\text{det}} \) located after the transmission channel, together with ideal detectors \([21]\). We assume that the detectors cannot distinguish the photon-number of arrival signals, but they provide only two possible outcomes: “click” (at least one photon is detected), and “no click” (no photon is detected in the pulse).

We obtain that Bob’s detection device can be characterized by two positive operator value measures (POVM), one for each of the two polarization basis \( \beta \) used in the BB84 protocol \([22]\). Each POVM contains four elements \([23]\):

\[
F_{\text{vac}}^\beta = \sum_{n,m=0}^{\infty} \bar{\eta}^{n+m} |n,m\rangle_\beta \langle n,m|,
\]

\[
F_0^\beta = \sum_{n,m=0}^{\infty} (1 - \bar{\eta})^n \bar{\eta}^m |n,m\rangle_\beta \langle n,m|,
\]

\[
F_1^\beta = \sum_{n,m=0}^{\infty} (1 - \bar{\eta})^m \bar{\eta}^n |n,m\rangle_\beta \langle n,m|,
\]

\[
F_D^\beta = \sum_{n,m=0}^{\infty} (1 - \bar{\eta})(1 - \bar{\eta}^m) |n,m\rangle_\beta \langle n,m|,
\]

where \( \bar{\eta} = (1 - \eta_{\text{det}}) \), and \( |n,m\rangle_\beta \) denotes the Fock state which has \( n \) photons in one mode and \( m \) photons in the orthogonal polarization mode with respect to the polarization basis \( \beta \). The outcome of the operator \( F_{\text{vac}}^\beta \) corresponds to no click in the detectors, the following two POVM operators, \( F_0^\beta \) and \( F_1^\beta \), give precisely one detection click, and the last one, \( F_D^\beta \), gives rise to both detectors being triggered.

The detectors show also noise in the form of dark counts which are, to a good approximation, independent of the incoming signals. The observed errors can be though as coming from a two-step process: in the first step the signals are changed by Eve as they pass the quantum channel, in the second step random noise from the detector dark counts is added. If we assume that Eve cannot influence the second step, then only the error rate coming from the first step needs to be considered in the privacy amplification process of the key distillation phase.

In principle, to decide whether the joint probability distribution \( P(A,B) \) obtained after the first phase of QKD contains quantum correlations, Alice and Bob can use all the information contained in \( P(A,B) \). However, we evaluate only those events where the signal preparation and detection methods employed by Alice and Bob use the same polarization basis. Moreover, in this paper, we consider that Alice and Bob analyze only two particular events: The expected click rate at Bob’s side, and the generalized error rate \( e \). This generalized error rate takes into account also double clicks; these are not discarded, instead Bob decides at random a bit value for every double click \([4]\). Let us emphasize once more that we consider only errors generated in the channel while subtracting errors coming from trusted but imperfect detectors with known dark count rates.

### III. INTERCEPT-RESEND ATTACKS

Basically, an intercept-resend attack consists in Eve measuring out every signal emitted by Alice. Afterwards, she transmits the measurement results through a lossless classical channel to a source close to Bob, which prepares new signal states that are forwarded to Bob. These eavesdropping strategies transform the original quantum channel between Alice and Bob into an entanglement breaking channel \([15]\) and, therefore, they do not allow the distribution of a secret key \([12]\).

Next, we propose a simple intercept-resend attack for the BB84 protocol that is specially suited for the signal states and detection methods employed by Alice and Bob, together with the attenuation introduced by the channel. Let us mention already here that this attack might not be optimal, but, as we will show below, it already imposes strong restrictions on the maximal distance achievable by the BB84 protocol with realistic means. But before addressing the whole analysis for this attack, let us start by describing it from a qualitative point of view.

We begin by introducing Eve’s measurement device. For simplicity, we decompose it into three steps: in the first step, Eve obtains the total photon-number, \( n \), of each signal state sent by Alice. This information can be acquired via a quantum-non-demolition (QND) measurement that does not introduce any errors in the signal states. Thanks to this information, the problem of identifying which polarization state \( k \) was used by Alice is reduced to the problem of distinguishing between four pure symmetric states \( |n_k\rangle \), with \( k = 0, ..., 3 \). Once the photon-number \( n \) is known, Eve uses the value of the losses in the channel expected by Alice and Bob to discard some signals. For that, she performs a filter op-
eration on the states $|n_k\rangle$ with the intention to make them, with some finite probability, more “ distinguishable”, while keeping the symmetry property of the set. The signal states that has to be discarded are those for which the filter operation does not succeed. The natural restriction here is that the probability of the filter operation to discard signals has to mimic the losses in the channel. Finally, in the third step, Eve measures out each filtered state with the so-called square-root-measurement (SRM) \cite{20,24}. This measurement gives her the minimum value of the error probability when distinguishing symmetric states. After deciding which polarization state was used by Alice, Eve needs to prepare a new signal in the state identified and give it to Bob. For that, we assume that she uses the signal states described by Eq. (1), but without the vacuum component.

Next, we study each of these steps in more detail. The objective is to find the error rate introduced by Eve with the intercept-resend attack described above for a given value of the losses in the channel and of the mean photon-number of the signal states.

A. Filter Operation

The purpose of this operation is to make, with some finite probability, the four possible input states $|n_k\rangle$, with $k = 0,...,3$ and $n > 0$, more “distinguishable”. For that, we assume that Eve uses the filter operation defined by the following two Kraus operators \cite{26}:

$$A_{\text{succ}}(n) = \sum_{j=0}^{3} \alpha_j(n) |\phi_j(n)\rangle\langle\phi_j(n)|,$$

$$A_{\text{fail}}(n) = \sum_{j=0}^{3} \sqrt{1 - |\alpha_j(n)|^2} |\phi_j(n)\rangle\langle\phi_j(n)|,$$

where the states $|\phi_j(n)\rangle$ represent the set of orthogonal states introduced in Eq. (9). The coefficients $\alpha_j(n)$ satisfy $|\alpha_j(n)|^2 \leq 1$ for all $j = 0,...,3$ and for all $n > 0$.

Suppose now that the filter operation receives as input the state $|n_k\rangle$. The probability of getting a successful result, $P_{r\text{succ}}(n)$, can be calculated as $P_{r\text{succ}}(n) = Tr(|n_k\rangle\langle n_k| A_{\text{succ}}(n) A_{\text{succ}}(n)^\dagger)$. This quantity is given by

$$P_{r\text{succ}}(n) = \sum_{j=0}^{3} |\gamma_j(n)|^2,$$

where $\gamma_j(n) = \alpha_j(n) \epsilon_j(n)$. If the filter operation succeeded, the resulting normalized filtered state, that we shall denote as $|n_k^{\text{succ}}\rangle$, can be calculated as $|n_k^{\text{succ}}\rangle = (1/\sqrt{P_{r\text{succ}}(n)}) A_{\text{succ}}(n)|n_k\rangle$. We obtain

$$|n_k^{\text{succ}}\rangle = \frac{1}{\sqrt{P_{r\text{succ}}(n)}} \sum_{j=0}^{3} \gamma_j(n) \exp \left(2\pi i \frac{k_j}{4}\right) |\phi_j(n)\rangle,$$

Note that, as desired, the set of states $|n_k^{\text{succ}}\rangle$, with $k = 0,...,3$ and $n > 0$, forms still a symmetric set \cite{11,20}.

As mentioned previously, the filter operation has to be designed such as $e^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^n}{n!} P_{r\text{succ}}(n)$ equals the probability of a signal state to arrive at Bob’s detection device, i.e.

$$e^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^n}{n!} P_{r\text{succ}}(n) = 1 - e^{-\mu \eta},$$

where $\eta$ is the transmission efficiency of the quantum channel \cite{26}.

B. Square-Root-Measurement

In order to decide which polarization state $k$ was used by Alice, we consider that Eve follows the approach of minimum error discrimination. That is, her aim is to find for each signal state sent by Alice a measurement strategy that guesses the value of $k$ with the minimum probability of making an error. As introduced previously, for the case of pure symmetric states $|n_k^{\text{succ}}\rangle$ with equal a priori probabilities, like it is the case that we have here, the optimal minimum error discrimination measurement is given by the so-called square-root-measurement (SRM) \cite{20,24}.

This measurement can be characterized with four POVM elements: $F_k(n) = |\omega_k(n)\rangle\langle \omega_k(n)|$, with $k = 0,...,3$. The states $|\omega_k(n)\rangle$ are not necessarily normalized or orthogonal, and they can be obtained from the states $|n_k^{\text{succ}}\rangle$ as $|\omega_k(n)\rangle = \Phi(n)^{-1/2}|n_k^{\text{succ}}\rangle$, where the operator $\Phi(n)$ is defined as $\Phi(n) = \sum_{k=0}^{3} |n_k^{\text{succ}}\rangle\langle n_k^{\text{succ}}|$. Using the symmetric representation of $|n_k^{\text{succ}}\rangle$ given by Eq. (11) we obtain

$$|\omega_k(n)\rangle = \frac{1}{2} \sum_{j=0}^{3} \exp \left(2\pi i \frac{k_j}{4} + i \psi_j(n)\right) |\phi_j(n)\rangle,$$

where the angles $\psi_j(n)$ denote the complex arguments of the coefficients $\gamma_j(n)$, i.e., $\psi_j(n) = \arg(\gamma_j(n))$.

The probability, $P_{r_{kk}}(n)$, of obtaining the result $k$ when the output state from Eve’s QND measurement is $|n_k\rangle$, with $n > 0$, and the filter succeed is given by $P_{r_{kk}}(n) = Tr(|n_k^{\text{succ}}\rangle|n_k^{\text{succ}}\rangle |F_k(n)\rangle) = |\langle n_k^{\text{succ}}| \omega_k(n)\rangle|^2$. According to Eq. (11) and Eq. (13) we find that this probability is of the form

$$P_{r_{kk}}(n) = \sum_{l,m=0}^{3} |\gamma_l(n)| |\gamma_m(n)| \exp \left(2\pi i \frac{k(l-m)}{4}\right) \frac{4 P_{r\text{succ}}(n)}{4 P_{r\text{succ}}(n)}.$$
C. Signal preparation

Once Eve has measured all the signal states sent by Alice, she needs to prepare new signal states and give them to Bob. The objective of Eve is to minimize the error rate while reproducing the expected click rate at Bob’s side. One possible solution for Eve in order to fulfill this condition is the following: Whenever the signal states sent by Alice contain, at least, one photon \( (n > 0) \) and the subsequently filter operations were successful, Eve gives Bob the signal states

\[
\rho_{k\text{succ}}^k = \frac{e^{-\mu_{\eta}}}{1 - e^{-\mu_{\eta}}} \sum_{n=1}^{\infty} \frac{(\mu_{\eta})^n}{n!} |n_k\rangle\langle n_k|, \quad (15)
\]

where \( k \) denotes the polarization states identified with the SRM. Otherwise, Eve sends Bob the vacuum state.

Let us emphasize that for each state \( \rho_k \) sent by Alice, Eve can prepare five possible states: the four states \( \rho_{k\text{succ}} \) given by Eq. \( (15) \), with \( k = 0, \ldots, 3 \), and the vacuum state.

Finally, in the next section, we obtain the error rate \( e \) introduced by this intercept-resend attack as a function of the losses in the channel and the mean photon-number of the signal states sent by Alice.

D. Error-rate in the sifted key

As introduced previously, here the error rate \( e \) refers only to those events where Alice and Bob use the same polarization basis for the signal preparation and for the detection method, respectively. Moreover, we consider that double clicks contribute also to the error rate with probability \( \frac{1}{2} \). It turns out that the intercept-resend attack presented above introduces the same error rate \( e \) for all the four possible polarizations states \( k \) used by Alice, and the resulting joint probability distribution \( P(A, B) \) after Eve’s attack has some expected symmetries. (See Appendix A.) From Eq. \( (7) \) we obtain: whenever Eve identifies the right polarization state, \( k = k \); then no errors are introduced; if Eve obtains a polarization state orthogonal to the one sent by Alice, \( |k - k| = 2 \), then \( e = 1 \); finally if Eve obtains one of the other two possible polarization states, \( |k - k| = 1 \) or \( |k - k| = 3 \), then \( e = \frac{1}{2} \). When Eve sends to Bob the vacuum state then no errors are introduced.

Let us start our analysis by obtaining an expression for the partial error rate \( e(n) \) coming from a signal state containing \( n \) photons, with \( n > 0 \). According to the previous paragraph, we have that \( e(n) \) is the same for all \( k \). This means that we can calculate \( e(n) \) for the case where Alice used, for instance, \( k = 0 \). For this case we obtain

\[
e(n) = Pr_{02}(n) + \frac{1}{2}(Pr_{01}(n) + Pr_{03}(n)).
\]

The probabilities \( Pr_{kk}(n) \) given by Eq. \( (14) \) satisfy \( Pr_{01}(n) = Pr_{03}(n) \) for all \( n > 0 \). This means that the partial error rate \( e(n) \) can be expressed as a function of only two probabilities

\[
e(n) = \frac{1}{2} \sum_{j=1}^{2} Pr_{0j}(n), \quad (16)
\]

for all \( k \). If we insert into this expression the value of the probabilities \( Pr_{kk}(n) \) given by Eq. \( (14) \), we obtain

\[
e(n) = \frac{1}{2} - \frac{\sum_{l,m=0}^{1} |\gamma_{2l}(n)\gamma_{2m+1}(n)|}{2 Pr_{\text{succ}}(n)}. \quad (17)
\]

The final error rate \( e \) can be calculated from the partial error rates \( e(n) \), just by taking into account the \textit{a priori} probabilities of the signal states to contain \( n \) photons, together with the probability of the filter operation to generate a successful result. This way we obtain that \( e \) is given by

\[
e = \frac{-\mu}{1 - e^{-\mu}} \sum_{n=1}^{\infty} \frac{\mu^n}{n!} Pr_{\text{succ}}(n)e(n). \quad (18)
\]

Using the values of \( Pr_{\text{succ}}(n) \) and \( e(n) \) given by Eq. \( (10) \) and Eq. \( (17) \), respectively, and employing Eq. \( (12) \) we finally obtain

\[
e = \frac{1}{2} \left( 1 - \frac{-\mu}{1 - e^{-\mu}} \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \sum_{l,m=0}^{1} |\gamma_{2l}(n)\gamma_{2m+1}(n)| \right). \quad (19)
\]

Next, we calculate explicitly \( e \) for different values of the losses in the channel and of the mean photon-number of the signal states sent by Alice. We distinguish two cases. First we analyze the case of low losses in the channel. Then we analyze the situation of high losses in the channel. The reason to study these two scenarios separately arises from the fact that, as we will show below, for signals states containing \( 0 < n \leq 2 \) photons the filter operation reduces to a multiple of the identity operator.

1. Case 1: Low losses in the channel

Let us start with the case where the channel does not introduce any attenuation at all, i.e. \( \eta_k = 1 \). From Eq. \( (12) \) we have that the probabilities \( Pr_{\text{succ}}(n) \) must satisfy: \( Pr_{\text{succ}}(n) = 1 \) for all \( n > 0 \). That is, the filter operation is reduced as expected to the identity operation for all \( n > 0 \). According to Eq. \( (19) \) we find, in this case, that the error rate \( e \) is given by

\[
e = \frac{1}{2} \left( 1 - \frac{-\mu}{1 - e^{-\mu}} \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \sum_{l,m=0}^{1} |c_{2l}(n)c_{2m+1}(n)| \right). \quad (20)
\]
As soon as the channel begins to introduce some losses, i.e. $\eta < 1$, Eve can start to use her filter operation to decrease the value of the error rate $e$. In this scenario, we can distinguish two regimes where the calculations follow quite straightforward.

The first regime arises from the fact that, whenever the signal states sent by Alice contain only one photon, $n = 1$, the filter operation does not help to make the states $|1_k\rangle$, with $k = 0, \ldots, 3$, more distinguishable. (See Appendix B.) For these states the partial error rate $e(1)$ is fixed and given by $e(1) = \frac{1}{\mu}$, independently of the losses in the channel. (See Appendix B) Since $e(1) > e(n)$ and $Pr(1) > Pr(n)$ for all $n > 1$, where $Pr(n)$ denotes the probability of Alice’s signal states to contain $n$ photons, we obtain that the best strategy for Eve is to start employing the losses in the channel to discard first all the single-photon signals, before she begins to filter the multi-photon pulses. From Eq. (12), and imposing $Pr_{\text{suc}}(n) = 1$ for all $n \geq 2$, we find that the probability of discarding single-photon signals, $Pr_{\text{fail}}(1) = 1 - Pr_{\text{suc}}(1)$, is given by

$$Pr_{\text{fail}}(1) = \frac{e^{\mu(1-\eta)}}{\mu}.$$  

(21)

All the single-photon signals are discarded as soon as $Pr_{\text{fail}}(1) = 1$, i.e. when $\eta_t = 1 - \frac{\ln(1+\mu)}{\mu}$. Inserting these values of $Pr_{\text{suc}}(n)$ and $e(n)$ in Eq. (18) we find, therefore, that in the interval

$$1 > \eta_t \geq 1 - \frac{\ln(1+\mu)}{\mu}$$  

(22)

the error rate $e$ is given by

$$e = \frac{1}{2} \left\{ 1 - \frac{e^{-\mu}}{1-e^{-\mu\eta_t}} \left[ \frac{1}{2} (1+\mu - e^{\mu(1-\eta_t)}) \right] + \sum_{n=2}^{\infty} \frac{\mu^n}{n!} \frac{1}{\sqrt{n+1}} \sum_{l,m=0}^{n} |c_{2l}(n)c_{2m+1}(n)| \right\}.$$  

(23)

It turns out that the optimal filter operation reduces also to a multiple of the identity operator for the case $n = 2$. (See Appendix B) That is, the filter operation does not help either to make the states $|2_k\rangle$, with $k = 0, \ldots, 3$, more distinguishable and $e(2) = \frac{2^{1/2}}{\mu}$ independently of the losses in the channel. (See Appendix B) The same argumentation used above for the case $n = 1$ can also be employed here to find that in the regime given by

$$1 - \frac{\ln(1+\mu)}{\mu} > \eta_t \geq 1 - \frac{\ln(1+\mu + \frac{\mu^2}{2})}{\mu}.$$  

(24)

Eve needs to discard first the two-photon signals, before she starts to use the filter operation with the remaining $n$-photon signals, with $n \geq 3$. From Eq. (18) we find, therefore, that the error rate $e$ in this regime is of the form

$$e = \frac{1}{2} \left\{ 1 - \frac{e^{-\mu}}{1-e^{-\mu\eta_t}} \left[ \frac{1}{2} (1+\mu + \frac{\mu^2}{2} - e^{\mu(1-\eta_t)}) \right] + \sum_{n=3}^{\infty} \frac{\mu^n}{n!} \frac{1}{\sqrt{n+1}} \sum_{l,m=0}^{\infty} |c_{2l}(n)c_{2m+1}(n)| \right\}.$$  

(25)

where we have omitted the intermediate calculations since they are analogous to the previous case $n = 1$.

The value of $e$ for these two scenarios, i.e. for the case $\eta_t \geq 1 - \ln(1+\mu + \frac{\mu^2}{2})/\mu$, is illustrated in Fig. 2 as a function of the losses in dB of the quantum channel [20], and for different values of the mean photon-number $\mu$. We also include in Fig. 2a bound for tolerable error rate arising from established security proofs [11, 27]. Note that these proven secure regions will be extended thanks to recently proposed key distillation methods [14].

2. Case 2: High losses in the channel

In this section we consider the final case of $\eta_t$ satisfying

$$1 - \frac{\ln(1+\mu + \frac{\mu^2}{2})}{\mu} > \eta_t > 0.$$  

(26)
This scenario corresponds to the case where Eve has already discarded all the single-photon and all the two-photon signals, and she can start to use her filter operation with the remaining \( n \)-photon pulses, with \( n \geq 3 \). In this case, the error rate \( e \) is given by Eq. (19) but with the first summation starting at \( n = 3 \).

The objective of Eve is to find, for a given value of the mean photon-number \( \mu \) and for a given value of \( \eta_t \) inside the interval given by Eq. (20), the coefficients \( \alpha_j(n) \) of the filter operation, with \( n \geq 3 \), that minimize \( e \). Moreover, this has to be done in a way that Eq. (12) is fulfilled, with \( Pr_{\text{succ}}(1) = Pr_{\text{succ}}(2) = 0 \). This problem can be reduced to solving the following non-linear optimization problem with equality and inequality constraints

\[
\text{minimize} \quad - \sum_{n=3}^{\infty} \frac{\mu^n}{n!} \sum_{m=0}^{1} |\gamma_2(n)\gamma_{2n+1}(n)|, \\
\text{subject to} \quad \sum_{n=3}^{\infty} \frac{\mu^n}{n!} \sum_{j=0}^{3} |\gamma_j(n)|^2 = \frac{1 - e^{-\mu \eta_t}}{e^\mu}, \\
0 \leq |\gamma_j(n)| \leq |c_j(n)| \quad \forall j, \quad \text{and} \quad \forall n \geq 3,
\]

(27)

where the infinitely many inequality constraints \( 0 \leq |\gamma_j(n)| \leq |c_j(n)| \) guarantee that \( |\alpha_j(n)|^2 \leq 1 \) for all \( j = 0, \ldots, 3 \), and for all \( n \geq 3 \).

The non-linear optimization problem given by Eq. (27) is not easy to handle, since it involves infinitely many variables and infinitely many constraints. However, as we show in Appendix C, one can easily obtain a good upper bound on the error rate \( e \) by considering only an optimization problem with a finite number of variables and constraints parametrized by a parameter \( s < \infty \). This finite-variable optimization problem can then be solved numerically. (See Appendix C.) This approximation corresponds to considering only a restricted class of eavesdropping attacks. In Fig. 3 we plot the result for the case \( s = 5 \), and for different values of the mean photon-number \( \mu \) and of the parameter \( \eta_t \). (See Appendix C.) The existing gap between the minimum error rate \( e \) achieved by each curve in Fig. 3 and the value \( e = 0 \) is due to the fact that for each restricted class of attacks there is a minimum value of \( e \) that can be obtained. When \( s \) increases then the minimum value of the error rate \( e \) that one can reach with these restricted strategies approaches zero for sufficiently high losses. In the limit \( s \to \infty \) then \( e = 0 \) for the transmission efficiency \( \eta_t \) plotted with a point in Fig. 3. This case correspond to the USD attack. (See Appendix C.) Moreover, in this limit \( s \to \infty \) the upper bound for \( e \) represented in Fig. 3 coincides with the optimal value of \( e \).

**FIG. 3:** Upper bound for the error rate \( e \) for the case \( s = 5 \) (See Appendix C), \( \eta_t < 1 - \ln(1 + \mu + \frac{\mu^2}{2})/\mu \), and for different values of the mean photon-number: \( \mu = 0.1 \) (solid), \( \mu = 0.2 \) (dashed), \( \mu = 0.3 \) (dashdot), \( \mu = 0.4 \) (dotted). The points with \( e = 0 \) correspond to the USD case when \( s \to \infty \).

**IV. CONCLUSION**

In this paper we have investigated the ultimate limit for secure practical QKD given by the proven existence of quantum correlations between Alice and Bob. In principle, to decide whether the joint probability distribution \( P(A,B) \) obtained after the first phase of QKD contains quantum correlations, Alice and Bob can use all the information contained in \( P(A,B) \). However, we evaluate only those events where the signal preparation and detection methods employed by Alice and Bob use the same polarization basis. Moreover, in this paper, we consider that Alice and Bob analyze only two particular events: The expected click rate at Bob's side, and the generalized error rate. For that, we have analyzed a simple eavesdropping strategy based on intercept-resend attacks: Eve measures out every signal emitted by Alice and prepares a new one, depending on the result obtained, that is given to Bob. These kind of attacks correspond to entanglement breaking channels and, therefore, the resulting correlations cannot lead to a secure key. Specifically, we have proposed an intercept-resend attack for the scenario where the attenuation introduced by the channel is not sufficiently high for Eve to perform unambiguous state discrimination of the signal states sent by Alice, but now she is allowed to introduce some errors. As a result, we obtained an upper bound on the maximal distance achievable by the BB84 set-up with realistic means. This upper bound depends on the error rate in the sifted key, and on the mean photon-number of the signals sent by Alice. It states that no key distillation protocol can provide a secret key from the correlations established by the users.
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APPENDIX A: JOINT PROBABILITY DISTRIBUTION $P(A, B)$

In this Appendix we provide the resulting joint probability distribution $P(A, B)$ obtained by Alice and Bob after Eve’s attack, and we show that $P(A, B)$ has some expected symmetries.

In order to do that, let us start by introducing some notation: we shall denote each element of $P(A, B)$ as $P(A_k, F_i^β)$, where the index $k = 0, ..., 3$ labels the four possible BB84 polarization states $ρ_k$ prepared by Alice, and the two indexes $β = \{+, −\}$, and $i = \{\text{vac}, 0, 1, D\}$, represent the two POVMs $\{F^0_{\text{vac}}, F^0_{0}, F^0_{1}, F^0_{D}\}$ characterizing Bob’s detection setup, and the four possible outcomes for each of these two POVMs, respectively.

The probabilities $P(A_k, F_i^β)$ can be calculated as

$$P(A_k, F_i^β) = Pr(ρ_k) \sum_{k=0}^4 Pr(ρ_k|ρ_k) Pr(F_i^β|ρ_k), \quad (A1)$$

where $Pr(ρ_k)$ denotes the a priori probability that Alice sends to Bob the quantum state $ρ_k$, $Pr(ρ_k|ρ_k)$ refers to the conditional probability that Eve forwards to Bob the quantum state $ρ_k$ after receiving from Alice $ρ_k$, and $Pr(F_i^β|ρ_k)$ denotes the conditional probability that Bob detects the event characterized by $F_i^β$ after obtaining from Eve $ρ_k$. Note that here we decide to use the states $ρ_k$, with $k = 0, ..., 3$, to denote the four possible states $ρ_k^{\text{succ}}$ given by Eq. (15), while the state $ρ_k$, with $k = 4$, refers to the vacuum state.

In the standard BB84 protocol we have that $Pr(ρ_k) = \frac{1}{4}$ for all $k$. The conditional probabilities $Pr(F_i^β|ρ_k)$ can be calculated as $Pr(F_i^β|ρ_k) = \frac{1}{2}Tr(F_i^β ρ_k)$. We illustrated the result for $Pr(F_i^β|ρ_k)$ in Table 1 where the parameters $a, b, c, d$ and $e$ are given by

$$
\begin{align*}
a &= \frac{1}{2(1-e^{-µt})}
(\frac{1}{2}e^{-µt}e^{2iθ} - e^{-µt}e^{iθ}), \\
b &= \frac{1}{2(1-e^{-µt})}
(\frac{1}{2}1 - e^{-µt}e^{2iθ}), \\
c &= \frac{1}{2(1-e^{-µt})}
(\frac{1}{2}e^{-µt}e^{2iθ} - e^{-µt}e^{iθ}), \\
d &= \frac{1}{2(1-e^{-µt})}
(\frac{1}{2}1 + e^{-µt}e^{2iθ} - 2e^{-µt}e^{iθ}).
\end{align*}
$$

TABLE I: Conditional probability $Pr(F_i^β|ρ_k)$.

The conditional probabilities $Pr(ρ_k|ρ_k)$ can be calculated as

$$Pr(ρ_k|ρ_k) = e^{-µ} \sum_{n=1}^{∞} \frac{µ^n}{n!} Pr_{\text{succ}}(n) Pr_{ρ_k}(n), \quad (A3)$$

for the case $k = 0, ..., 3$, with $Pr_{ρ_k}(n)$ being the probability that Eve obtains the result $k$ when the output of her filter operation is $|n_{ρ_k}^{\text{succ}}⟩$. Including the value of $Pr_{ρ_k}(n)$ given by Eq. (14) into Eq. (A3) we find, therefore, that for the case for $k = 0, ..., 3$, $Pr(ρ_k|ρ_k)$ is given by

$$Pr(ρ_k|ρ_k) = e^{-µ} + \sum_{n=1}^{∞} \frac{µ^n}{n!}(1-Pr_{\text{succ}}(n)), \quad (A5)$$

According to Eq. (12), this last conditional probability reduces to $Pr(ρ_k|ρ_k) = e^{-µt}$ for $k = 4$ and for all $k$.

Now we can show that the joint probability distribution $P(A, B)$ exhibits the symmetry property expected in typical situations in QKD. That is, if Alice and Bob employ the same polarization bases to prepare and measure the signal states, then the probability that Bob identifies the correct state has to be the same for all $k$. This symmetry applies also to the situation where Bob obtains as a result a state orthogonal to the one sent by Alice. Moreover, if Alice and Bob do not use the same bases in the preparation and detection methods, then the probabilities to obtain the results characterized by the two operators $F_i^β$, with $i = 0, 1$, has to be the same for all $k$ and for all $β$.

Note that the probabilities $P_{ρ_k}(n)$ given by Eq. (14) satisfy the following symmetry property: $P_{ρ_k}(n)$ is the same for all $k$, and $P_{ρ_k}(n)$ has the same value for those events satisfying: $|k-k| = 1$ and $|k-k| = 3$. According to Eq. (A5) we find that these properties are also fulfilled by $Pr(ρ_k|ρ_k)$. Combining this result with the symmetry
present in the conditional probability $P_r(F_i^\alpha | \rho_k)$ given by Table II it turns out that $P(A, B)$ satisfies the desired symmetry property.

We illustrate this result in Table II where we plot the resulting $P(A, B)$ after Eve’s attack for the case: $\mu = 0.1$, $\eta_k \approx 7.001 (\approx 29.7 dB)$, and $\eta_{det} = 0.2$. The error rate for this example is $\epsilon \approx 0.00$. In order to solve Eq. (A4) we used $s = 5$.

APPENDIX B: FILTER OPERATION ($n = 1, n = 2$)

In this Appendix we show that if the signal stated sent by Alice contain only $0 < n \leq 2$ photons, then the filter operation reduces to a multiple of the identity operator, and $e(1) = \frac{1}{4}$ and $e(2) = \frac{2 - \sqrt{2}}{2}$ independently of the losses introduced by the channel.

Let us start with the case $n = 1$. From Eq. (B1) we have that the coefficients $c_j(1)$ of the symmetric representation of the states $|1_k\rangle$, with $k = 0, \ldots, 3$, satisfy: $|c_5(1)| = |c_1(1)| = \frac{1}{\sqrt{2}}$ and $|c_2(1)| = |c_3(1)| = 0$, respectively. From Eq. (B2) we obtain that, in this case, the partial error rate $e(1)$ is given by

$$e(1) = \frac{1 - 1}{2} - \frac{1}{2} |\alpha_0(1)\alpha_1(1)|^2.$$

It is easy to see that this quantity fulfills $e(1) \geq \frac{1}{4}$ for all $|\alpha_j(1)|^2 \leq 1$. Its minimum value, $e(3) = \frac{1}{4}$, is attained for $|\alpha_0(1)| = |\alpha_1(1)|$. That is, independently of the losses in the channel, the filter operation reduces to a multiple of the identity operator.

The analysis for the case $n = 2$ is similar. Now the coefficients $c_j(2)$ of the signal states $|2_k\rangle$ are of the form $|c_0(2)| = |c_5(2)| = \frac{1}{2}$, $|c_2(2)| = \frac{1}{\sqrt{2}}$, and $|c_3(2)| = 0$, respectively. The partial error rate $e(2)$ is given by

$$e(2) = \frac{1 - 1}{2} - \frac{1}{2} |\alpha_0(2)\alpha_1(2)| + |\alpha_6(2)| \alpha_2(2)|^2.$$

This quantity is bounded from below by $e(2) \geq \frac{2 - \sqrt{2}}{2} = 0.15$ for all $|\alpha_j(2)|^2 \leq 1$. Its minimum value, $e(2) = 2 - \sqrt{2}$, can be achieved again directly with a filter operation that is a multiple of the identity operator, i.e. by imposing $|\alpha_0(2)| = |\alpha_4(8)| = |\alpha_2(2)|$ independently of the losses introduced by the channel.

APPENDIX C: OPTIMIZATION PROBLEM

In this Appendix we obtain an upper bound for the error rate $e$ for the case of high losses in the channel, i.e. $\eta_k < 1 - \ln(1 + \mu + \frac{\mu}{\sqrt{2}})/\mu$.

In order to do that, we well consider that Eve employs her filter operation to filter first only those $n$-photon signals that satisfy $3 \leq n \leq s$, for some $s \geq 0$. The remaining $n$-photon signals, with $n > s$, are measured directly by Eve with the SRM, without filtering these signals. That is, we consider that $P_{succ}(n) = 1$ for all $n > s$.

From the limit given by USD [13, 19] we know that Eve filters completely all the $n$-photon signals with $3 \leq n \leq s$, i.e. $e(n) = 0$ for all $n \in [3, s]$, when $P_{succ}(n) = 4|c_{min}(n)|^2$, where $|c_{min}(n)| = \min_c|c_j(n)|$. This fact corresponds to the case where all the coefficients $\alpha_j(n)$ of the filter operation are of the form $\alpha_j(n) = \frac{c_{min}(n)}{c_j(n)}$, i.e. $|\gamma_j(n)| = |c_{min}(n)|$ for all $n \in [3, s]$. According to Eq. (C2), we obtain that this limit is given by the following transmission efficiency of the quantum channel:

$$\eta^*_t = -\ln(1 - P_s)/\mu,$$
where the probability $P_m$ is of the form

$$ P_s = e^{-\mu} \left( \sum_{n=3}^{s} \frac{\mu^n}{n!} |c_{\min}(n)|^2 + \sum_{n=s+8}^{\infty} \frac{\mu^n}{n!} \right). \quad (C2) $$

From Eq. (19) we obtain, therefore, that in the interval

$$ 1 - \ln \left( 1 + \mu + \frac{\mu^2}{\mu} \right) > \eta_t \geq \eta_t^l, \quad (C3) $$

an upper bound for the error rate $e$, that we shall denote as $e^s$, is given by

$$ e^s = \frac{1}{T} \left[ 1 - e^{-\mu} e^{-\mu_{s+1}} - x_{\min} \right] + \sum_{n=s+1}^{\infty} \frac{\mu^n}{n!} \sum_{l,m=0}^{1} |c_2\left(n\right)c_{2m+1}\left(n\right)|, \quad (C4) $$

where $x_{\min}$ represents the solution to the following non-linear optimization problem

$$ \begin{align*}
\text{minimize} & \quad -\sum_{n=3}^{s} \frac{\mu^n}{n!} \sum_{l,m=0}^{1} |\gamma_2(n)\gamma_{2m+1}(n)|, \\
\text{subject to} & \quad \sum_{n=4}^{s} \frac{\mu^n}{n!} \sum_{j=0}^{3} |\gamma_j(n)|^2 = \frac{1 - e^{-\mu_{s+1}}}{e^{-\mu}} \\
& \quad - \sum_{n=s+1}^{\infty} \frac{\mu^n}{n!}, \\
& \quad 0 \leq |\gamma_j(n)| \leq |c_j(n)| \forall j, \text{ and } \forall n \in [3, s]. \quad (C5)
\end{align*} $$

This optimization problem involves $4(s-2)$ variables and $8(s-4) + 1$ constraints, and it can be solved numerically.

In the limit $s \to \infty$ the optimization problem given by Eq. (C5) reduces to a linear problem given by Eq. (27). To see how tight is the upper bound $e^s$ in general, note that $e^s - e$ is given by

$$ e^s - e = \frac{e^{-\mu}}{1 - e^{-\mu_{s+1}}} \left[ \sum_{n=3}^{s} \frac{\mu^n}{n!} P_{r_{s+1}}(n) e^s(n) \right. $$

$$ - P_{r_{s+1}}(e^{opt}(n)) + \sum_{n=s+1}^{\infty} \frac{\mu^n}{n!} (\hat{e}(n)) \left. - P_{r^{opt}_{s+1}}(n) e^{opt}(n) \right], \quad (C6) $$

where $P_{r_{s+1}}(n)$ and $e^{opt}(n)$, with $3 \leq n \leq s$, are obtained from solving the optimization problem given by Eq. (C3). $P_{r_{s+1}}(n)$ and $e^{opt}(n)$, with $n \geq 3$, refer to the optimal values of $P_{r_{s+1}}(n)$ and $e(n)$ obtained from solving Eq. (27). Moreover, $\hat{e}(n)$ denotes the value of $e(n)$ without using the filter operation, i.e. for the case $|\gamma_j(n)| = |c_j(n)|$ for all $j = 0, ..., 3$, and for all $n > s$, and is given by

$$ \hat{e}(n) = \frac{1}{2} \left( 1 - \sum_{l,m=0}^{1} |c_2\left(n\right)c_{2m+1}\left(n\right)| \right). \quad (C7) $$

We find that $P_{r_{s+1}}(n) e^s(n) - P_{r^{opt}_{s+1}}(n) e^{opt}(n) \leq 0$ for all $3 \leq n \leq s$. This is the case since for a given value of the losses in the channel $\eta_t$, the problem given by Eq. (C3) uses $\eta_t$ to filter only a finite number of signals. Note that in Eq. (27) $\eta_t$ is distributed to filter all the signals. Moreover, it is evident that $\hat{e}(n) - P_{r^{opt}_{s+1}}(n) e^{opt}(n) \leq \hat{e}(n)$ for all $n > m$, since we have that $P_{r^{opt}_{s+1}}(n) e^{opt}(n) \geq 0$. This way, we obtain that

$$ e^s - e \leq \frac{e^{-\mu}}{1 - e^{-\mu_{s+1}}} \sum_{n=s+1}^{\infty} \frac{\mu^n}{n!} (\hat{e}(n)) $$

$$ < \frac{e^{-\mu}}{1 - e^{-\mu_{s+1}}} \sum_{n=s+1}^{\infty} \frac{\mu^n}{n!}, \quad (C8) $$

where in the last inequality we have used the fact that $\hat{e}(n) < 1$ for all $n$. To conclude, note that in the interval of $\eta_t$ given by Eq. (C3) the expected click rate at Bob’s side, $P_{s+1} = 1 - e^{-\mu_{s+1}}$, is much bigger than the probability $Pr(n > s) = e^{-\mu} \sum_{n=s+1}^{\infty} n \frac{\mu^n}{n!}$ of Alice’s signal states to contain more than $s$ photons. Most importantly, this happens already for quite small values of $s$, e.g. $s = 5$.

In Fig. 3 we plot $e^s$ for the case $s = 5$, and for different values of the mean photon-number $\mu$ and of the parameter $\eta_t$. In order to solve the minimization problem given by Eq. (C3) we use the package Gloptipoly [28] based on ScDuMi [29], which is freely available. This package has a number of desirable features, in particular, it provides a certificate for global optimality. It is based on the method introduced by Lasserre [30] (see also [31]) to solve a global optimization problem with a multivariable polynomial objective function subject to polynomial equality and inequality constraints. Lasserre’s method finds hierarchies of solutions to the original problem in a way that each step in the hierarchy provides a better approximation to the global optimum than the previous one. Each step itself amounts to solving an efficiently implementable semi-definite program [32]. Moreover, the hierarchy is asymptotically complete, in the sense that the exact solution is asymptotically attained. For the problem given by Eq. (C3) we find that the global optimum is already obtained after the first relaxation step.

We include also in Fig. 3 the minimum value of $\eta_t$ that satisfies $e = 0$. This corresponds to the USD case where $Pr_{s+1}(n) = 4|c_{\min}(n)|^2$ for all $n \geq 3 (s \to \infty)$ [13, 19]. According to Eq. (19), we obtain that this limit is given
where the probability $P_D$ is of the form \[ P_D = e^{-\mu} \sum_{n=3}^{\infty} \frac{\mu^n}{n!} 4|c_{\min}(n)|^2 \] (C10)

\[ = 1 - e^{-\mu} \left( \sqrt{2} \sinh \frac{\mu}{\sqrt{2}} + 2 \cosh \frac{\mu}{\sqrt{2}} - 1 \right). \] (C11)

\[ \eta_t = \frac{-\ln(1 - P_D)}{\mu}, \] (C9)

by

\[ \eta_t = \frac{-\ln(1 - P_D)}{\mu}, \] (C9)

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