Probability representation of quantum mechanics and star product quantization

V N Chernega\textsuperscript{1}, S N Belolipetskiy\textsuperscript{2}, O V Man’ko\textsuperscript{1,2,5}, V I Man’ko\textsuperscript{1,3,4}

\textsuperscript{1} Lebedev Physical Institute, Russian Academy of Sciences, Leninskii Prospect 53, Moscow 119991, Russia
\textsuperscript{2} Bauman Moscow State Technical University, The 2nd Baumanskaya Str. 5, Moscow 105005, Russia
\textsuperscript{3} Moscow Institute of Physics and Technology (State University), Institutskii per. 9, Dolgoprudnyi, Moscow Region 141700, Russia
\textsuperscript{4} Tomsk State University, Department of Physics, Lenin Avenue 36, Tomsk 634050, Russia, Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia
\textsuperscript{5} E-mail: mankoov@lebedev.ru

Abstract. This paper presents a review of star-product formalism. This formalism provides a description for quantum states and observables by means of the functions called 'symbols of operators'. Those functions are obtained via bijective maps of the operators acting in Hilbert space. Examples of the Wigner-Weyl symbols (Wigner quasi-distributions) and tomographic probability distributions (symplectic, optical and photon-number tomograms) identified for the states of the quantum systems are discussed. Properties of quantizer-dequantizer operators required for construction of bijective maps of two operators (quantum observables) onto the symbols of the operators are studied. The relationship between structure constants of associative star-product of operator symbols and quantizer-dequantizer operators is reviewed.

1. Introduction

In classical mechanics, in classical statistical physics as well as in electromagnetic field theory the observables are described by functions of their position and momenta (e. g. energy of a particle or functions of spatial coordinates and time strength of electric or magnetic field). The product of these functions is standard point-wise product which is commutative and associative. It means that the product of two functions, e. g. \( f_1(q,p) \) and \( f_2(q,p) \) gives the function \( F_{12}(q,p) = f_1(q,p) f_2(q,p) \) and \( F_{21}(q,p) = F_{12}(q,p) \) as well as the product of three functions satisfies the associativity condition \( (f_1(q,p) f_2(q,p)) f_3(q,p) = f_1(q,p) (f_2(q,p) f_3(q,p)) \). For a function \( f(x) \) the product can be expressed by the following formula

\[
 f_1(x) f_2(x) = \int \delta(x - x_1) \delta(x - x_2) f_1(x_1) f_2(x_2) \, dx_1 \, dx_2,
\]

where \( \delta(y) \) is Dirac delta function with \( y = x - x_k, \) \( k = 1, 2.\) The formula can be rewritten in a generic form

\[
 (f_1 \ast f_2)(x) = \int K(x_1,x_2,x) f_1(x_1) f_2(x_2) \, dx_1 \, dx_2.
\]
where \( K(x_1, x_2, x) \) is a kernel of the form

\[
K(x_1, x_2, x) = \delta(x - x_1)\delta(x - x_2).
\]

This kernel provides the properties of the point-wise products commutativity and associativity. A generic star-product of the functions is not commutative but associative. It can be described by other kinds of kernels called structure constants of the associative product. If a variable \( x \) takes discrete values either in finite or infinite domains, the Dirac delta functions in (1), (2), (3) are replaced by Kronecker delta symbols, and the integral in (2) is replaced by a sum over discrete variables. In quantum mechanics and quantum field theory, states and observables are associated with operators. For example the quantum oscillator position is described by a position operator \( \hat{q} \) which acts on the wave function \( \psi(x) \) in position representation as \( \hat{q}\psi(x) = x\psi(x) \). The momentum of the system is described by the operator \( \hat{p} \) which acts on the wave function as \( \hat{p}\psi(x) = -i\hbar \frac{\partial \psi(x)}{\partial x} \), where \( \hbar \) is Planck constant. The product of position and momentum is non-commutative, and the property of commutator for these physical observables works as follows \([\hat{p}, \hat{q}] = \hat{p}\hat{q} - \hat{q}\hat{p} = -i\hbar \). This property provides the uncertainty relation for position and momentum as described by Heisenberg [1], Schrödinger [2], and Robertson [3]. The uncertainty relation points out that one cannot measure both the position and the momentum of an oscillator simultaneously. The formalism of star-product is based on introducing the bijective map of operators onto the functions. The product of operators is known to be associative, i.e. \((\hat{A}\hat{B})\hat{C} = \hat{A}(\hat{B}\hat{C})\). If one constructs a map of the functions associated with the operators, the functions must be multiplied according to (2) by some kernels having not the form (3) but another form which violates the commutativity but preserves the associativity conditions.

The goal of our work is to give a review of different kinds of the star-product of the functions and to construct a generic scheme of quantizer-dequantizer operators which can be used to find kernels of star-products. In fact, different forms of star-product of the functions correspond to different representations of the operators identified with physical observables. Various aspects of mathematical formalism of quantum mechanics were discussed in [4] and [5]. In [4] it was shown that there exists a bijective relation between monotone quantum metrics associated with different operator monotone functions. In [5] various Euclidean, hyperbolic and elliptic analytic representations are introduced and relations among them are discussed.

The work is organized as follows. In Section 2 we review the scheme of quantization based on star-product of functions without concrete implementation of the map of operators onto functions and discuss the expression for the kernel of star-product in terms of quantizer-dequantizer operators, following [6–8]. In Section 3 we discuss the quantum observables and their connection with dual quantization scheme. In Section 4 we review Wigner function and the quantizer-dequantizer operators for Wigner-Weyl symbols of the operators. In Section 5 we discuss symplectic and optical tomographic state representations. In Section 6 we review the tomographic symbols of operators, dual tomographic symbols of operators, and mean values of quantum observables in the framework of star-product quantization scheme. In Section 7 we review the photon number tomography as an example of star-product quantization scheme. In Section 8 we discuss the relationship between an optical tomogram and a photon-number tomogram. In Section 9 we review classical tomographic symbols. In Section 10 we discuss the star-product kernel of classical tomographic symbols and its connection to quantum ones. In Sec. 11 we give the outlook of this research and its conclusions.

2. General scheme

In quantum mechanics, observables are described by operators acting in the Hilbert space of states. In classical mechanics they are described by \( c\)-number functions. In order to describe a quantum observable in a way adjacent to classical mechanics, we first introduce a map of
operators onto functions without concrete realization of it. Following [6], [7], [8], let us consider an operator $\hat{A}$ acting in a Hilbert space. We construct the function $f_A(x)$ of vector variables $x = (x_1, x_2, \ldots, x_n)$, supposing that we have a set of operators $\hat{U}(x)$ acting in Hilbert space such that
\[ f_A(x) = \text{Tr} \left[ \hat{A} \hat{U}(x) \right]. \tag{4} \]
The operator $\hat{U}(x)$ is called dequantizer [8] because it maps operator $\hat{A}$ onto a function. The function $f_A(x)$ is called symbol of the operator $\hat{A}$. Let us suppose that operators $\hat{D}(x)$ acting in Hilbert space exist and the relation (4) has an inverse. The map from function to operator is of the form
\[ \hat{A} = \int f_A(x)\hat{D}(x) \, dx. \tag{5} \]
The operator $\hat{D}(x)$ is called quantizer [8]. We have the bijective map
\[ f_A(x) \leftrightarrow \hat{A}. \tag{6} \]
Formulas (4) and (5) are self-consistent if the following property of the quantizer and dequantizer exists
\[ \text{Tr}[\hat{U}(x)\hat{D}(x')] = \delta(x - x'). \tag{7} \]
The delta-function in (7) is used in the case of continuous variables $x$, and the Kronecker symbol instead of Dirac delta-function is used in the case of discrete variable $x$. The introduced map provides the nonlocal associative product (star-product) of two functions (symbols of operators). The product of two symbols $f_A(x)$ and $f_B(x)$ corresponding to two operators $\hat{A}$ and $\hat{B}$ with the map (4), (5) can be introduced in the form (see, e.g. [6])
\[ f_{A\hat{B}}(x) = f_A(x) * f_B(x) = \int f_A(x')f_B(x'')K(x', x'', x)dx'dx'', \tag{8} \]
where the kernel of star product $K(x', x'', x)$ is of the form
\[ K(x', x'', x) = \text{Tr}[\hat{D}(x'')\hat{D}(x')\hat{U}(x)]. \tag{9} \]
One can see that the kernel of the star-product (9) is linear with respect to the dequantizer and nonlinear in the quantizer operator. Since the standard product of operators in a Hilbert space is an associative product $\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$, the product of functions (symbols of operators) has to be associative too
\[ f_A(x) * \left( f_B(x) * f_C(x) \right) = \left( f_A(x) * f_B(x) \right) * f_C(x). \tag{10} \]
The associativity condition for functions (symbols of operators) means that the kernel of star-product (9) satisfies the nonlinear equation (see e.g. [8])
\[ \int K(x_1, x_2, y)K(y, x_3, x_4)dy = \int K(x_1, y, x_4)K(x_2, x_3, y)dy. \tag{11} \]
If we take the density operator $\hat{\rho}$ determining some quantum state as operator $\hat{A}$, the function $f_{\hat{\rho}}(x)$ (symbol of operator $\hat{\rho}$) also determines this quantum state. In [9] the quantizer—dequantizer formalism was used to describe the evolution of a quantum system and it was shown that if the set of states is invariant with respect to some unitary evolution, the quantizer—dequantizer provides a classical-like realization of system dynamics.
3. The dual star–product scheme and quantum observable

Let us consider the following map

\[ f_A^{(d)}(x) = \text{Tr} \left[ \hat{A} \hat{D}(x) \right], \quad (12) \]

\[ \hat{A} = \int y f_A^{(d)}(y) \hat{U}(y) \, dy. \quad (13) \]

We permute the quantizer and the dequantizer. It is possible because the compatibility condition (9) is valid in both cases. We consider the new pair of quantizer–dequantizer as dual pair to the initial one

\[ \hat{U}'(x) \mapsto \hat{D}(x), \quad \hat{D}'(x) \mapsto \hat{U}(x). \]

This interchange is possible due to a specific symmetry of the equation (11) for associative star–product kernel. The star–product of dual symbols \( f_A^{(d)}(x) \hat{A}(x) \), \( f_B^{(d)}(x) \hat{B}(x) \) of two operators \( \hat{A} \) and \( \hat{B} \) is described by dual integral kernel

\[ K^{(d)}(x'', x', x) = \text{Tr} \left[ \hat{U}(x'') \hat{U}(x') \hat{D}(x) \right]. \quad (14) \]

The dual kernel (14) is another solution of nonlinear equation (11) [8]. Let us consider the mean value of quantum observable which is determined by the operator \( \hat{A} \)

\[ \langle \hat{A} \rangle = \text{Tr}(\hat{\rho}\hat{A}) = \int f_\rho(x) \text{Tr}(\hat{D}(x)\hat{A}) \, dx. \]

Using the expression for dual symbol of operator (12) we obtain the formula

\[ \langle \hat{A} \rangle = \int f_\rho(x) f_A^{(d)}(x) \, dx. \]

Therefore, the mean value of an observable \( \hat{A} \) can be calculated as overlap integral of the tomographic symbol of the density operator \( f_\rho(x) \) in a given quantization scheme and the symbol \( f_A^{(d)}(x) \) of the observable \( \hat{A} \) in the dual scheme.

4. Wigner-Weyl symbol

In this section we will consider an example of Heisenberg-Weyl representation. We introduce the operators (we assume Planck constant \( \hbar = 1 \))

\[ \hat{U}(q, p) = \int_{-\infty}^{\infty} \left| q + \frac{u}{2} \right\rangle \left\langle q - \frac{u}{2} \right| e^{-ipu} du, \quad (15) \]

and

\[ \hat{D}(q', p') = \frac{1}{2\pi} \hat{U}(q', p'), \quad (16) \]

where \( |q + u/2\rangle \) is the eigenvector of position operators \( \hat{q} \), i.e. \( \hat{q}|x\rangle = x|x\rangle \) and it satisfies the condition

\[ \langle x|x'\rangle = \delta(x - x'). \quad (17) \]

Operator (15) is a dequantizer operator according to general scheme (4). Operator (16) is a quantizer operator according to (5) in the Wigner-Weyl star-product quantization scheme. One can check that

\[ \text{Tr} \left( \hat{U}(q, p)\hat{D}(q', p') \right) = \delta(q - q')\delta(p - p'). \quad (18) \]
In view of this property according to general scheme for designing the symbols of the operators, the function

\[ W_A(q, p) = \text{Tr} \left( \hat{U}(q, p) \hat{A} \right) \]  

which is Wigner-Weyl symbol of the operator \( \hat{A} \), determines the operator \( \hat{A} \), i.e,

\[ \hat{A} = \frac{1}{2\pi} \int W_A(q, p) \hat{U}(q, p) \, dq \, dp. \]  

The kernel of star-product of the Wigner-Weyl symbols \[10\] reads

\[ \text{Tr} \left( \hat{D}(q_1, p_1)\hat{D}(q_2, p_2)\hat{U}(q_3, p_3) \right) = \frac{1}{2\pi^2} \exp \left[ \frac{2i}{\pi} (q_1 p_2 - q_2 p_1 + q_2 p_3 - q_3 p_2 + q_3 p_1 - q_1 p_3) \right]. \]  

For example, the symbol of the oscillator ground state is

\[ W_\rho(q, p) = W_\rho(q, p) = 2 \exp \left( -q^2 - p^2 \right). \]  

The operator \( \hat{U} \) can be given in another form

\[ \hat{U}(q, p) = 2 \hat{D}(2\alpha)\hat{I}, \]  

where \( \hat{I} \) is parity operator, \( \alpha = \frac{q + ip}{\sqrt{2}} \). \( \hat{D}(2\alpha) \) is displacement operator. The displacement operator is expressed through creation and annihilation operators in the form

\[ \hat{D}(\alpha) = \exp(\alpha \hat{a}^+ - \alpha^* \hat{a}). \]  

where creation and annihilation operators are expressed through operators of position \( \hat{q} \) and momentum \( \hat{p} \) of oscillator

\[ \hat{a} = \frac{\hat{q} + i\hat{p}}{\sqrt{2}}, \quad \hat{a}^+ = \frac{\hat{q} - i\hat{p}}{\sqrt{2}}. \]  

The Wigner-Weyl star-product scheme is self-dual due to condition (16). It is worthy to add that Weyl operators and symbols were investigated e.g. in [11]. A new quantum mechanical formalism based on probability representation of quantum states is used to investigate the special case of the measurement problem, known as Schrödinger’s cat paradox and the EPR-paradox in [12]. Using the discrete version of the star-product operation the evolution of the discrete Wigner function for prime and the power of prime dimensions was investigated in [13]. The explicit differential Moyal-like form of the star product is found and analyzed in the semi-classical limit in [14]. In [15] the associative star product of functions was investigated with the help of a square integrable representation of a locally compact group. The Wigner functions and tomographic probability distributions of two-qubit states were discussed in [16] where the kernel of the map, which provides the expression of the state tomogram in terms of the discrete Wigner function of the two-qubit state and the kernel of the inverse map and the connection of the constructed maps with the star-product quantization scheme is obtained in an explicit form. In [17] the evolution of Werner-like mixture was introduced by considering two correlated but different degrees of freedom and its tomographic characterization was provided. The Wigner functions for a harmonic oscillator are studied in [18] including corrections from generalized uncertainty principles and the corresponding marginal probability densities. The investigation of general quantum mechanical commutation relations consistent with the Heisenberg evolution equations was studied in [19]. The review of a family of non-commutative star-products based on a Weyl map is done in [20].
5. Notion of quantum state in symplectic and optical tomography approaches

The symplectic tomography was introduced in [21]. Tomographic approach to quantum states that leads to a probability representation of quantum states was discussed in [22], [23], [24], [25]. The state in symplectic tomography scheme is determined by probability distribution function \( w(X, \mu, \nu) \), which is called symplectic tomogram. The generic linear combination of quadratures which is a measurable observable \((\hbar = 1)\) is of the form

\[
\hat{X} = \mu \hat{q} + \nu \hat{p},
\]

where \( \hat{q} \) and \( \hat{p} \) are the position and momentum, respectively, and real parameters \( \mu \) and \( \nu \) determine the reference frame in classical phase space. The symplectic tomogram \( w(X, \mu, \nu) \) is nonnegative function which is normalized with respect to the variable \( X \) (position). The physical meaning of the parameters \( \mu \) and \( \nu \) is that they describe an ensemble of rotated and scaled reference frames in which the position \( X \) is measured. For \( \mu = \cos \theta \) and \( \nu = \sin \theta \), the symplectic tomogram coincides with distribution for the homodyne-output variable used in optical tomography [26], [27] and named optical tomogram

\[
w_{\text{opt}}(X, \theta) = w(X, \cos \theta, \sin \theta).
\]

The information contained in the symplectic tomogram \( w(X, \mu, \nu) \) is overcomplete. To determine the quantum state completely, it is sufficient to give the function for arguments with the constraints \((\mu^2 + \nu^2 = 1)\) which corresponds to the optical tomography scheme which is realized experimentally in [28], [29], i.e., \( \mu = \cos \theta \) and the rotation angle \( \theta \) labels the reference frame in classical phase space. Symplectic tomogram can be reconstructed from optical tomogram using the relation

\[
w(X, \mu, \nu) = \frac{1}{\sqrt{\mu^2 + \nu^2}} w_{\text{opt}} \left( \frac{X}{\sqrt{\mu^2 + \nu^2}}, \arctg \frac{\nu}{\mu} \right).
\]

The process of Stimulated Raman Scattering was investigated in the framework of the symplectic tomography representation in [30]. The quantum entanglement in Raman Scattering was investigated in [31], [32].

6. Symplectic tomography in the framework of star-product quantization

The tomographic symbol \( w_\hat{A}(x) \) of the operator \( \hat{A} \) is obtained by means of dequantizer operator of the form

\[
\hat{U}(X, \mu, \nu) = \delta(X \hat{1} - \mu \hat{q} - \nu \hat{p})
\]

where vector \( x = (X, \mu, \nu) \) has the real arguments, \( \hat{1} \) is identity operator. The quantizer operator in symplectic tomography is

\[
\hat{D}(X, \mu, \nu) = \frac{1}{2\pi} \exp \left( iX \hat{1} - i\nu \hat{p} - i\mu \hat{q} \right).
\]

The kernel of star–product given by (9) of two tomographic symbols of operators \( \hat{A} \) and \( \hat{B} \) has the following form [6], [7]

\[
K(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2, X, \mu, \nu) = \frac{\delta \left( \mu (\nu_1 + \nu_2) - \nu (\mu_1 + \mu_2) \right)}{4\pi^2} \times \exp \left( \frac{i}{2} \left( \nu_1 \mu_2 - \nu_2 \mu_1 \right) + 2X_1 + 2X_2 - \frac{2(\nu_1 + \nu_2)X}{\nu} \right).
\]
Let us consider the mean value of quantum observable $\hat{A}$

$$\langle \hat{A} \rangle = \text{Tr}(\hat{\rho} \hat{A}) = \text{Tr} \int w(X, \mu, \nu) \hat{D}(X, \mu, \nu) \hat{A} \, dX \, d\mu \, d\nu = \int w(X, \mu, \nu) \text{Tr}(\hat{D}(X, \mu, \nu) \hat{A}) \, dX \, d\mu \, d\nu.$$ 

For this purpose we introduce the dual tomographic symbol (12) in symplectic tomography scheme

$$w^{(d)}_\hat{A}(X, \mu, \nu) = \text{Tr} \left[ \hat{A} \hat{D}(X, \mu, \nu) \right].$$

Then for the mean value of observable $\langle \hat{A} \rangle$

$$\langle \hat{A} \rangle = \int w(X, \mu, \nu) w^{(d)}_\hat{A}(X, \mu, \nu) \, dX \, d\mu \, d\nu.$$ 

The mean value of the observable $\hat{A}$ in symplectic tomography scheme is given by the overlap integral of the symplectic tomogram (the tomographic symbol of the density operator) $w(X, \mu, \nu)$ in the given quantization scheme and the symbol $w^{(d)}_\hat{A}(X, \mu, \nu)$ of the observable $\hat{A}$ in the dual scheme. The tomograms and eigenvalues of energy computed in terms of tomographic symbols are shown in [33]. The dynamic of quantum particles was described by Kolmogorov equations for non-negative propagators in tomography representation in [34]. The symmetrized product of quantum observables is defined in [35].

7. Photon–number tomography as example of star–product quantization

The photon-number tomography was introduced in [36], [37], [38] and developed in [39], [40], [41], [42]. It is a method to reconstruct density operator of quantum state using measurable probability distribution function (photon statistics) called photon-number tomogram. In photon–number tomography, the discrete random variable is measured for reconstructing quantum state. The photon–number tomogram

$$\omega(n, \alpha) = \langle n \mid \hat{D}(\alpha)\hat{\rho}\hat{D}^{-1}(\alpha) \mid n \rangle$$

is the function of integer photon number $n$ and complex number $\alpha$, $\hat{\rho}$ is the state density operator. The photon–number tomogram is the photon distribution function (the probability to have $n$ photons) in the state described by the displaced density operator. For example, the photon–number tomograms of excited oscillator states with density operators $\hat{\rho}_m = \vert m \rangle \langle m \vert$ are

$$w^{(m)}(n, \alpha) = \frac{n!}{m!} \left| \alpha \right|^{2(m-n)} e^{-\vert \alpha \vert^2} \left( L^{m-n}_n(\vert \alpha \vert^2) \right)^2, \quad m \geq n,$$

$$w^{(m)}(n, \alpha) = \frac{m!}{n!} \left| \alpha \right|^{2(n-m)} e^{-\vert \alpha \vert^2} \left( L^{n-m}_m(\vert \alpha \vert^2) \right)^2, \quad m \leq n,$$

where $L^n_m(x)$ are Laguerre polynomials. In [43] state reconstruction from the measurement statistics of phase space observables generated by photon number states was considered.

Let us consider photon–number tomogram in the framework of star–product quantization following [44], [45], [46], [47]. In the given photon–number tomography quantization scheme the dequantizer operator is of the form

$$\hat{U}(x) = \hat{D}(\alpha)\hat{\rho}\hat{D}^{-1}(\alpha), \quad x = (n, \alpha).$$
The quantizer operator in photon number tomography scheme is

\[ \hat{D}(x) = \frac{4}{\pi(1-s^2)} \left( \frac{s}{s+1} \right)^{(a_1 + a_2 + \ldots - n)} \]

where \( s \) is ordering parameter \([48]\), \( a \) is complex number \((a = \Re a + \Im a)\), \( D(a) \) is the Weyl displacement operator \((24)\). The kernel \((9)\) of star–product of photon–number tomograms in the photon–number tomography quantization scheme is

\[ K(n_1, \alpha_1, n_2, \alpha_2, n_3, \alpha_3) = \text{Tr}[\hat{D}(n_1, \alpha_1)\hat{D}(n_2, \alpha_2)\hat{D}(n_3, \alpha_3)]. \] (33)

In explicit form

\[
K(n_1, \alpha_1, n_2, \alpha_2, n_3, \alpha_3) = \left( \frac{4}{\pi(1-s^2)} \right)^2 \exp(it(n_1 + n_2 - 2n_3)) \exp[-\alpha_1 - \alpha_1 e^{-it} + \alpha_2 e^{-2it} + \alpha_3 e^{-2it} + \frac{1}{2}(-\alpha_3 \alpha_1^2 + \alpha_3 \alpha_1 - \alpha_1 \alpha_2^2 + \alpha_1^2 \alpha_2 - \alpha_2 \alpha_3^2 + \alpha_2^2 \alpha_3 + \alpha_3 \alpha_3^2 \alpha_2 e^{iit}) \]

\[ -|\alpha_1|^3 e^{it \alpha_1^2} + \alpha_1 \alpha_2 e^{iit} - \alpha_1 \alpha_3 e^{iit} + |\alpha_1|^2 e^{it \alpha_2^2} + \alpha_2 \alpha_3 e^{iit} - \alpha_1 \alpha_3 e^{-it} + \alpha_3 \alpha_3^2 e^{2iit} \]

\[ -|\alpha_1|^2 e^{it \alpha_2^2} + \alpha_2 \alpha_3 e^{-it} + \alpha_2 \alpha_3 e^{-2it} - |\alpha_2|^2 e^{it \alpha_3^2} + \alpha_2 \alpha_3 e^{-2it} - \alpha_1 \alpha_2 e^{-it} + \alpha_2 \alpha_3 e^{-2it} \]

\[ +|\alpha_2|^3 e^{iit \alpha_2^2} - \alpha_2 \alpha_3 e^{-2it} - \alpha_2 \alpha_3 e^{-2it} + \alpha_1 \alpha_2 \alpha_3 e^{-2it} - |\alpha_2|^2 e^{2iit} + |\alpha_2|^3 e^{-2it} - \alpha_1 \alpha_3 e^{-2it} \]

\[ +|\alpha_1| \alpha_2 \alpha_3 e^{-2it} - |\alpha_2| \alpha_3 e^{2iit} - |\alpha_2| \alpha_3 e^{-2it} - |\alpha_2| \alpha_3 e^{-2it} - |\alpha_2|^2 e^{2iit} + |\alpha_2|^3 e^{-2it} - \alpha_1 \alpha_2 e^{-2it} + \alpha_2 e^{-2it} + \alpha_3 e^{-2it} \]

\[ L_n(x) \text{ is Laguerre polynomial. The kernel (34) is the solution of equation (11).} \]

8. The relationship between optical tomogram and photon number tomogram

In this section we find the relations between optical, symplectic and photon numbers tomograms of quantum state following \([44]\). The photon number tomogram can be expressed in terms of symplectic tomogram by using the integral transform \([45]\)

\[ \omega(n, \alpha) = \int w(X, \mu, \nu)K(X, \mu, \nu, n, \alpha)\, dX\, d\mu\, d\nu. \] (35)

Here the kernel of the integral transform is expressed in terms of matrix elements of the displacement operator

\[ K(X, \mu, \nu, n, \alpha) = \frac{1}{2\pi}(n | \hat{D}(-\alpha) e^{i(X-\mu \hat{q} - \nu \hat{p})} \hat{D}(\alpha) | n). \] (36)

The explicit dependence of the kernel on real parameters of symplectic transform is given by the expression

\[ K(X, \mu, \nu, n, \alpha) = \frac{1}{2\pi} \exp \left( iX + \frac{\nu - i\mu}{\sqrt{2}} \alpha - \frac{\nu + i\mu}{\sqrt{2}} \alpha \right) \langle n | \hat{D} \left( \frac{\nu - i\mu}{\sqrt{2}} \right) n \rangle. \] (37)
Using the known formula for diagonal elements of the displacement operator

\[ D_{nn}(\gamma) = \langle n \mid \hat{D}(\gamma) \mid n \rangle = e^{-\frac{1}{4}|\gamma|^2} L_n(|\gamma|^2), \tag{38} \]

where \( \gamma = (\nu + i\mu)/\sqrt{2} \), one has the kernel expressed in terms of Laguerre polynomial

\[ K(X, \mu, \nu, n, \alpha) = \frac{1}{2\pi} \exp \left[ iX + \frac{\nu - i\mu}{\sqrt{2}} \alpha^* - \frac{\nu + i\mu}{\sqrt{2}} \alpha \right] L_n \left( \frac{\nu^2 + \mu^2}{2} \right). \tag{39} \]

In view of this, the photon number tomogram is expressed in terms of optical tomogram as follows

\[
w(n, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \int_{-\infty}^\infty k \exp \left[ ik \left( X - \sqrt{2}(\alpha_1 \cos \theta + \alpha_2 \sin \theta) \right) - \frac{k^2}{4} \right] \\
\times L_n \left( \frac{k^2}{2} \right) w_0(X, \theta) \, d\theta \, dk \, dX. \tag{40} \]

Here \( \alpha \) is a complex number \((\alpha = \alpha_1 + i\alpha_2)\). Let us introduce the characteristic function

\[ F(k, \theta) = \int e^{ikX} w_0(X, \theta) \, dX = \langle e^{ikX} \rangle, \quad k \geq 0. \tag{41} \]

The photon number tomogram reads

\[
w(n, \alpha) = \frac{1}{2\pi} \sum_m \int_0^{2\pi} \int_0^\infty k \exp \left[ -ik\sqrt{2}(\alpha_1 \cos \theta + \alpha_2 \sin \theta) - \frac{k^2}{4} \right] \\
\times L_n \left( \frac{k^2}{2} \right) (i)^m \frac{k^m}{m!} \langle X^m \rangle_{\theta} \, dk \, d\theta. \tag{42} \]

Here we introduce moments of optical tomogram \( \langle X^m \rangle_{\theta} = \int w(X, \theta) X^m \, dX \). Let us consider an example of excited oscillator state \( \hat{\rho}_m = \ket{m} \bra{m} \). One can show that the tomogram of this state reads

\[ w_m(X, \mu, \nu) = \frac{e^{-\frac{X^2}{\nu^2 + \mu^2}}}{\sqrt{\pi^2 + 2n}} \frac{1}{m!} \frac{1}{2^n} H_n \left( \frac{X}{\sqrt{\mu^2 + \nu^2}} \right). \tag{43} \]

Applying general relation (35), we get for \( m \leq n \)

\[ m! \, \gamma \, |2(n-m)| \, e^{-|\gamma|^2} L_{m-n}^m(|\gamma|^2) = \frac{1}{2\pi} \exp \left[ iX + \frac{\nu - i\mu}{\sqrt{2}} \gamma^* - \frac{\nu + i\mu}{\sqrt{2}} \gamma \right] \\
\times e^{-\frac{\nu^2 + \mu^2}{4} L_n \left( \frac{\nu^2 + \mu^2}{2} \right)} \frac{e^{-\frac{\nu^2 + \mu^2}{2}}}{\sqrt{\pi^2 + 2n}} \frac{1}{m!} \frac{1}{2^n} H_n \left( \frac{X}{\sqrt{\mu^2 + \nu^2}} \right) \, dX \, d\mu \, d\nu \tag{44} \]

which provides a new integral relation of Hermite and Laguerre polynomials [44], [45]. For ground state with optical tomogram

\[ w_0(X, \theta) = \frac{-e^{X^2}}{\sqrt{\pi}} \]

the photon number tomogram

\[ w_0(n, \alpha) = \frac{e^{-|\alpha|^2}}{n!} |\alpha|^{2n}. \]

is distribution of Poisson. It is necessary to add that optical tomograms of the time-evolved states generated by evolution of different kinds of initial wave packets in a Kerr medium and optical tomograms of maximally entangled states generated at the output modes of a beam splitter were theoretically studied in [49], [50]. In [51] a review of the Radon transform and the instability of tomographic reconstruction process were discussed.
9. Classical Tomographic Symbols

In this section we review following [24], [52], [53], [54], [55] the consideration of classical mechanics within the framework of the tomographic representation. The reversible relationship between the tomogram \( w_f(X, \mu, \nu) \) of the probability distribution \( f(q, p) \) in classical mechanics was defined in [55] as follows:

\[
\begin{align*}
\text{Here we assumed the normalization condition for } f(q, p) \text{ to be } & \\
\frac{1}{2\pi} \int f(q, p) \delta(X - \mu q - \nu p) \, dq \, dp = 1 & \text{analogously to the normalization condition of the Wigner function } W(q, p) \text{ in quantum mechanics [56].}
\end{align*}
\]

According to [52] in classical mechanics one can introduce the operators \( \hat{A}_{cl} \) for which their formal Weyl symbol \( W_{A_{cl}}(q, p) \) coincides with a classical observable \( A(q, p) \), i.e.,

\[
W_{A_{cl}}(q, p) = 2 \text{Tr} \hat{A}_{cl} \hat{D}(2\alpha) \hat{I} = A(q, p).
\]

Thus we interpret a function in the phase space as Wigner–Weyl symbol of an operator \( \hat{A}_{cl} \) acting in Hilbert space. Consequently, we can consider the phase-space function \( A(q, p) \) as the classical Weyl symbol of the corresponding observable in classical mechanics. The quantum tomographic symbol for a unity operator was found in [57]. Due to mentioned above, we have the same result for the classical tomographic symbol, i.e.,

\[
w_1(X, \mu, \nu) = -\pi |X| \delta(\mu) \delta(\nu).
\]

Since in quantum mechanics Weyl symbols for the position operator \( \hat{q} \) and momentum operator \( \hat{p} \) are \( c \)-numbers \( q \) and \( p \), in view of Eqs. (45), we have for both classical and quantum tomographic symbols:

\[
\begin{align*}
w_q(X, \mu, \nu) &= \frac{\pi}{2} X |X| \delta'(\mu) \delta(\nu), \\
w_p(X, \mu, \nu) &= \frac{\pi}{2} X |X| \delta(\mu) \delta'(\nu).
\end{align*}
\]

10. Star-product kernel of classical tomographic symbols

In this Section we present the kernel of star-product of functions-symbols of operators in classical mechanics and show its connection to star-product of quantum tomographic symbols of operators. The star-product kernel \( K(X, \mu, \nu, X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) \) for two tomographic symbols of observables \( w_1(X_1, \mu_1, \nu_1) \) and \( w_2(X_2, \mu_2, \nu_2) \) was defined in [52] as follows:

\[
w(X, \mu, \nu) = \int K(X, \mu, \nu, X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) w_1(X_1, \mu_1, \nu_1) \times w_2(X_2, \mu_2, \nu_2) \, dX_1 \, d\mu_1 \, d\nu_1 \, dX_2 \, d\mu_2 \, d\nu_2,
\]

where by definition

\[
w(X, \mu, \nu) = w_1(X, \mu, \nu) \ast w_2(X, \mu, \nu).
\]
Explicit form of commutative star-product kernel of the tomographic symbols is

$$K(X, \mu, \nu, X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) = \frac{1}{(2\pi)^2} e^{i(X_1 + X_2 - X(\nu_1 + \nu_2)/\nu)} \delta\left(\nu(\mu_1 + \mu_2) - \mu(\nu_1 + \nu_2)\right). \quad (48)$$

Similarly, we can find the star-product kernel for quantum tomographic symbols [6] if we consider $W_A(q, p)$ and $W_B(q, p)$ as common quantum Weyl symbols of the operators $\hat{A}$ and $\hat{B}$ with the Weyl noncommutative star-product. The relationship between tomographic star-product kernels in quantum and classical mechanics reads

$$K_{\text{quant}}(X, \mu, \nu, X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2) = K_{\text{classic}}(X, \mu, \nu, X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2)e^{i[\nu_2(\mu_1) - \mu_1(\nu_2)/2]}.$$

The relation between the quantum state description and the classical state description is elucidated in [58]. In [59] Brownian motion was considered in the space of fields and was shown that the corresponding probability distribution can be approximately described by the same mathematical formalism as the one is used in quantum mechanics and in the theory of Hermitian operators in complex Hilbert space. In [60] the review of classical probability representations of quantum states and observables is given and it is shown that the correlations of the observables involved in the Bohm–Bell type experiments can be expressed as correlations of classical random variables.

11. Conclusion

We reiterate the main results of our paper. We presented a review of the map which provides the correspondence rule between the operators acting in a Hilbert space and the functions of some variables. The construction of the map is based on using a pair of operators depending on these variables called quantizer and dequantizer operators. These operators give possibility to consider the operators acting in Hilbert space as vectors and functions (symbols of operators) as components of these vectors, i.e. coordinates in a specific basis in the linear space. Then, a pair of quantizer and dequantizer operators correspond to the bases in the linear space providing the possibility to consider any operator in Hilbert space as a vector in the linear space which is given if the components of the vector in this basis are known. Several examples of such construction were considered. Wigner-Weyl symbols of operators, symplectic tomographic symbols, photon number tomographic symbols were studied using the pairs of corresponding quantizer-dequantizer operators. In this formalism, the evolution equation for quantum states can be written in the form of kinetic equation for the probability distribution [67]. The evolution of tomograms for different quantum systems, both of finite and infinite dimensions was considered in [68]. A method based on tomography representation to simulate the quantum dynamics was applied to the wave packet tunneling of one and two interacting particle in [69]. Non-orthogonal bases of projectors on coherent states were introduced to expand Hermitean operators acting on the Hilbert space of a spin $s$ in [70].
In our work we discuss the kernels of star-product of symbols for the mentioned examples. These kernels generalized the known Gröenewold kernel which provides the associative product of the Wigner–Weyl symbols [10]. In this paper the examples of infinite dimensional Hilbert spaces were considered. A similar construction exists for finite dimensional Hilbert spaces corresponding to the states of spins and qudits [71], [72], [73], [74], [75], [76], [77], [78]. In [79] a class of deformed products for spin observables is presented. A review of different quantum-optical states defined in finite-dimensional Hilbert space of operators which have a discrete spectrum was considered in [80]. The methods of star-product, tomography and probability representation of quantum mechanics applied to different problems of quantum phenomena were considred in [81]- [96]. The quantization method based on star-products of functions can be applied for studying the process in different areas of physics.

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