Random Energy Model with complex replica number, complex temperatures and classification of the string’s phases

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The results by E. Gardner and B.Derrida have been enlarged for the complex temperatures and complex numbers of replicas. The phase structure is found. There is a connection with string models and their phase structure is analyzed from the REM’s point of view.

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I. INTRODUCTION

Random Energy Model (REM) [1-5] is connected with the many problems of modern physics. In [6-8] has been found, that correlators in the directed model are connected with the free energy in directed polymer. The last is equivalent to REM in thermodynamic limit.

Liouville model is closely connected with the bosonic string in the d-dimensional Euclidean space [9]. It is easy to check, that the connection of strings with REM is even stronger. If one considers the integration of string’s partition via area of closed surfaces, [10-13] then after integration via zero mode of a Laplacian an expression is obtained for the partition like to REM with finite replica numbers (solved for real temperatures in [5]):

\[ Z \sim \int \mathcal{D} \phi e^{\frac{1}{8\pi} \int d^2 w \sqrt{g} \phi \Delta \phi + QR \phi} \left( \int d^2 w \sqrt{g} e^{\alpha \phi} \right)^{-\frac{Q}{\alpha}} \]  

Here \( \phi(w) \) is a field on closed 2-d surface, \( \alpha, Q \) are parameters real for \( d < 1 \), \( R \) is a curvature, \( \mathcal{D} \phi \) is a measure. \( Q, \alpha \) are defined by \( d \) according to formulas of David-Distler-Kawai (see review [12]). The analytical continuation of parameters \( Q, \alpha \) at \( d > 1 \) gives complex value for parameters (see section 4).

One can understand the last expression as an average of the \( \mu = -\frac{Q}{\alpha} \)-th degree (replicas number) of the sum \( \sum_i e^{\alpha \phi(w_i)} \) via normal distribution of variables \( \phi_i \equiv \phi(w_i) \) with a quadratic form

\[ \frac{1}{8\pi} \int d^2 w \sqrt{g} \phi \Delta \phi + QR \phi \]  

The main idea of this work (following to [6-8]) is that the phase structure of the (1) can be mapped to other models with the simpler choice of quadratic form in the exponent of normal distribution that the one in Eq. (2). We are going to connect the system (1) with the chain of models (each one with the same phase structure as the previous one but simpler), where the last one in the chain is the Random Energy Model (REM). That’s why we decided to solve REM at complex temperatures [13-15] and complex replicas numbers.

REM is a model with

\[ P(E) = \frac{1}{\sqrt{2\pi N}} \exp\left[ \frac{E^2}{2N} \right] \]  

The total distribution of energies is factorized: for \( 1 \leq \alpha \neq \beta \leq M \)

\[ P(E_\alpha, E_\beta) = P(E_\alpha)P(E_\beta) \]  

The main our interest is connected with partition

\[ z = \sum_i \exp\{-\beta E_i\} \]

\[ Z = \langle z^\mu \rangle, \]  

for a general value of \( \mu \). One can observe the similarity of \( Z \) defined by (5) and (1), if identify \( w \) with \( i, \phi(w) \) with \( E_i, \alpha \) with \( -\beta, \mu = -\frac{Q}{\alpha} \) and \( z = \int d^2 w \sqrt{g} e^{\alpha \phi(w)} \) resembles \( \sum_i e^{-\beta E_i} \). In (1) there is a normal distribution like (5), the main difference -in (1) the normal distribution is non-diagonal. In case of REM we have \( 2^N \) physical degrees
of freedom, like \((L/a)^d\) degrees in a field theoretical model with ultraviolet \(a\) and infrared \(L\) cutoffs. The ensemble average (integration with a normal distribution of energies) of partition function’s \(\mu\)-th degree corresponds to our expression (1).

In the section 2 we are going to introduce directed polymer(DP) model on the hierarchic trees with branching number \(q\). The endpoints of the hierarchic tree correspond to the points \(w_i\) of the 2-d space in (1). The case \(q \to 1\) resembles the model (1) (for a field theoretical aspects see [17]), and the case \(q \to \infty\) is equivalent to the REM. There is a strict result [5] that at the case \(\mu > 0\) the thermodynamic limit of the introduced models are independent of \(q\).

In the section 3 we give a qualitative derivation of REM solution at complex temperature and replica numbers. In section 4 we give the classification of the phase structure of the model (1). In the appendix B. we prove, that in the opposite case \(q \to \infty\) DP is equivalent to REM the thermodynamic limit. In Appendix A there is a rigorous solution of REM at complex temperatures and replica numbers.

II. HIERARCHIC TREES WITH CONSTANT BRANCHING NUMBER \(Q\)

Let us consider the model on the hierarchic tree [2-3],[5]. Originally one has a point (origin of the tree). At the first level of hierarchy there are \(q\) branches. At the \(i\)-th level of hierarchy there are \(q\) new branches from the every branch of the \(i−1\)-th level. At the last \(K\)-th level we have \(q^K\) end points. Let us consider field \(\phi(x)\) at the endpoints. Every point \(x\) is connected with the origin of the tree with a single path. For the any pair of points \(x\) and \(x'\) at the level \(K\) it is possible introduce a hierarchic distance

\[
v(x, x') = \frac{(K - i)V}{K},
\]

where their paths to the origin meet at the \(i\)-th level of hierarchy, \(V\) is a parameter (the maximal hierarchic distance between points on the tree).

We define random variables \(f_{il}\) on the branches at the \(i\)-th level of the tree with distribution

\[
\sqrt{\frac{K}{2\pi}} \exp\left\{-\frac{K}{2V} f_{il}^2\right\}
\]

We define fields \(\phi(x)\) as a sum of \(f_{il}\) along the path \(il(x)\) connecting the point \(x\) with the origin:

\[
\phi(x) = \sum_{il(x)} f_{il}
\]

One can check, that

\[
<\phi(x)\phi(x')> = V - v(x, x')
\]

If one defines the distance between two points \(x, x'\) as

\[
r(x, x')^2 = \exp(v(x, x'))
\]

then Eq.(10) coincides with the ordinary expression of the 2d free field with the action (2)

\[
<\phi(x)\phi(x')> = \ln \frac{L^2}{r^2}
\]

with ultraviolet cutoff \(L = \exp(V/2)\) and infrared one 1.

Actually we are interested only in the distance \(r(x, x')\) for \(x \neq x'\) (for the \(x' = x\) we can take \(r(x, x) = 0\)).

We can construct a model connected with the one defined by Eq. (1). Let us consider a partition

\[
<\exp\left(\sum_x \phi(x)\right)^\mu >.
\]

At the limit \(\mu \to 0\) this model has been considered rigorously in [5]. At the thermodynamic limit the model is equivalent to REM (with the same total number of configurations \(q^K\) and variance \(<E^2> = <\phi(x)^2>\) for all values of \(q\).
We suggest the first hypothesis of this work (it can be checked numerically), the thermodynamic limit of Eq. (12) is independent of parameter \( q \) like the case \( \mu \to 0 \).

We can find the phase structure of the model at the limit \( q \to \infty \), it is again equivalent to REM. While the system (6)-(7),(12) is similar to the system (1) at every value of \( q \) due to property (9)-(11), there is a serious difference between different choice of \( q \). In the case of finite \( q \) one should consider a combinatorial problem. Only the limiting case \( q \to 1 \) is similar to 2d Euclidean space, as one can use a small parameter \( q - 1 \) and construct measure. Thus we formulate the second hypothesis of the work: at the limit \( q \to 1 \) model on hierarchic tree is equivalent to the system (1).

Let us define expression (12) for the general case of \( q \), then take the limit \( q \to 1 \). First we introduce the \( \delta \) function in integral representation for the partition \( z \equiv \exp(\alpha \sum_x \phi(x)) \) and Eq. (12) transforms into:

\[
< z^\mu > = \int_\infty^\infty \int_\infty^\infty dudv\delta(Rez - u)\delta(Imz - v)(u + iv)^\mu = \\
\frac{1}{(2\pi)^2} \int_{-\infty}^\infty dk_1dk_2 \int_{-\infty}^\infty dudv(u + iv)^\mu \exp(-ik_1u - ik_2v)G(k_1,k_2)
\]

\[
G(k_1,k_2) = \exp[ik_1Re(\alpha x) + ik_2Im(\alpha x)]
\]

Now the problem is to calculate the generating function \( G(k_1,k_2) \). It can be done by means of recurrence equations:

\[
I_1(x) = \sqrt{\frac{K}{2V\pi}} \int_{-\infty}^\infty \exp\{-\frac{K}{2V}y^2 + U(x + y)\} dy
\]

\[
I_{t+1}(x) = \sqrt{\frac{K}{2V\pi}} \int_{-\infty}^\infty \exp\{-\frac{K}{2V}y^2\} [I_t(x + y)]^q dy
\]

\[
U(y) = ik_1Re(\alpha y) + ik_2Im(\alpha y)
\]

\[
G(k_1,k_2) = [I_K(0)]^q
\]

Let us consider the limit

\[
q \to 1 \quad K \to \infty \quad q^K = \exp(V)
\]

Then one can express \( G(k_1,k_2) \) by means of a function \( W(t, x) \):

\[
\frac{dW}{dt} = W ln W + \frac{1}{2} \frac{d^2W}{dx^2}
\]

\[
0 < t < V, -\infty < x < \infty
\]

\[
W(0, x) = e^{ik_1Re(\alpha x) + ik_2Im(\alpha x)}
\]

\[
G(k_1,k_2) = W(V, 0)
\]

The case of \( q \to 1 \) trees (13)-(16) with real potential has been considered recently in [17]. Using Eq. (16) it is possible to found the phase structure of corresponding model in 2d Euclidean space (see. [17]). The suggested method gives an exact phase structure (mean field approach gives a correct list of phases but approximate borders between phases), as well as correct two point correlators and three point corellators for isosceles triangles [17]. In this work we are restricted only by phase structure.

In principle one can find numerical solution of the last system (16) and compare it with the REM’s analytical results for the free energy to check the first hypothesis of the work (\( q \) independence of thermodynamic limit for the model on a \( q \)-tree). In the next section we find the phase structure of REM, then in the appendix we prove, that system (14) at large \( q \) is equivalent to REM.

### III. QUALITATIVE DERIVATION OF 4 REM PHASES.

Our goal is to calculate

\[
Z = < z^{\mu_1+i\mu_2} >, \quad z = \sum_i e^{-(\beta_1+i\beta_2)E_i}
\]
where energies are distributed via (3). Let us consider these expressions for positive integer values of $\mu$, where the average is over the distribution (3) for each $E_i$. There are two competing terms in expression of $z^\mu$ (after series expansion).

The paramagnetic (PM) phase is originated from the cross terms in the $z^\mu$ series expansion expansion:

$$Z = M^\mu < e^{-\beta E_1} e^{-\beta E_2} \cdots e^{-\beta E_n} >$$

$$\ln Z = \mu \ln M + N \beta^2 \mu \frac{(\beta_1^2 + \beta_2^2)\mu}{2}$$

$$N\beta_c^2 = 2 \ln M$$  \hspace{1cm} (18)

The second one is the correlated paramagnetic (CPM) \cite{4} (in Parisi’s picture there is a correlation between different replicas), it is originated from the diagonal terms in the $z^\mu$ series expansion like to $e^{-\beta \mu E_i}$:

$$Z = < \left( \sum_{i=1}^{M} e^{-\mu \beta E_i} \right) >$$

$$\ln Z = \ln M + \frac{N \beta^2 \mu^2}{2} = \frac{N(\beta_1^2 + \beta_2^2 \mu^2)}{2}.$$  \hspace{1cm} (19)

Let us consider continuation of (18) to the region $\mu < 1$. At critical temperature $\beta_c$ it’s entropy $\ln Z - \beta \frac{\partial \ln Z}{\partial \mu}$ disappears. We assume that in this region $\ln Z$ is proportional to $\beta$ (it is natural for a system with zero entropy) and $\mu$. The continuity of $\ln Z$ gives for spin-glass (SG) phase

$$\ln Z = N \mu \beta_c \mu$$  \hspace{1cm} (20)

If one goes to complex temperatures \cite{11-12}, then (18) transforms to (it is easy check directly for integer $\mu$)

$$\ln Z = N(\beta_1^2 + \beta_2^2 - \beta_1^2 \mu)$$  \hspace{1cm} (21)

For the SG phase one has to replace $\beta$ by $\beta_1$ in (20):

$$\ln Z = N \mu \beta_c \beta_1$$  \hspace{1cm} (22)

For complex temperatures there is a fourth, Lee-Yang-Fisher (LYF) phase. The derivation is not direct. The point is, that for noninteger values of $\mu$

$$Z \sim < |z|^{\mu} >$$  \hspace{1cm} (23)

After this trick it is easy to derive the LYF expression. The principal terms are $e^{-2\beta_1 E_1}$:

$$\ln Z = \frac{N(\beta_1^2 + 4 \beta_1^2 \mu)}{4}$$  \hspace{1cm} (24)

Let us now continue our four expressions to complex values of $\mu$. For PM phase an analytical continuation of Eq. (18) gives

$$\ln Z = N(\beta_1^2 + \beta_2^2 - \beta_1^2 \mu_1 - 2 \beta_1 \beta_2 \mu_2)$$  \hspace{1cm} (25)

For SG phase we have

$$\ln Z = N \mu_1 \beta_c \beta_1$$  \hspace{1cm} (26)

For LYF phase:

$$\ln Z = \frac{N(\beta_1^2 + 4 \beta_1^2 \mu_1)}{4}$$  \hspace{1cm} (27)

For CPM an analytical continuation of Eq. (19) gives

$$\ln Z = \frac{N[\beta_1^2 + (\beta_1^2 - \beta_2^2)(\mu_1^2 - \mu_2^2) - 4 \beta_1 \beta_2 \mu_1 \mu_2)]}{2}$$  \hspace{1cm} (28)
To find the borders between four phases one should first find the correct phase at $\mu \to 0$ limit, then compare its finite $\mu$ expression for $|\ln Z|$ with the corresponding one given by CPM phase. It is known, that LYF phase exists at [14,15]:

$$\beta < \frac{\beta_c}{2}$$  (29)

and PM one at $\beta < \beta_c$. For a complex temperatures one has a condition for SG phase

$$\beta_1 > \beta_c + \beta_2.$$  (30)

The last point. Strict derivation gives, that LYF for noninteger $\mu_1$ exists only at

$$\mu_1 > -2.$$  (31)

The paramagnetic phase is the most symmetric one, there are local symmetries in the model.

For the case of SG phase there is some order and no local symmetry. For the case of LYF phase there is some correlation between couples of replicas. Those two phases (SG and LYF) resemble non-unitary models in field theory. For the case of CPM phase there is a correlation between all the replicas, but local symmetries are conserved. This phase has not any pathology. Thus together with PM phase it can be connected with unitary models.

### IV. STRING PHASES

What can we say about quantum field theory and strings on the basis of our results? One should understand that, in some sense, there is a hierarchy of requirements physical theory to be mathematically rigorous.

On the top level in quantum field theory one demands the unitarity of the theory as the main constraint. In the lower level of statistical field theory in Euclidean space the genesis of two probabilities: in ensemble and Boltzmann one is not too complicated. Therefore one can consider non-unitary model also, connected with some physical situations [18]. We have another physical constraint: finite number of primary fields. In the lowest level of hierarchy is an ordinary statistical mechanics. Here one is happy, when can construct a thermodynamic limit ($N \to \infty$) for the free energy, entropy.

We see, that our results are quite restrictive. First, we should forbid a situation like Lee-Yang-Fisher singularity with too negative replica numbers $\mu_1 < -2$. Here it is impossible construct a thermodynamic limit. Other situations with Lee-Yang-Fisher phases as well as with spin-glass are a bit interesting, but here could not be any unitary theory. The most interesting are paramagnetic and correlated paramagnetic phases. In this area there is a some chance for unitarity.

Let us return to the partition of bosonic d-dimensional string (1). For the ultraviolet cutoff $L$ and infrared one $a$ the number of degrees is

$$M = \frac{L^2}{a^2}. \quad (32)$$

configurations. Let us define distribution of $\phi(w)$ over all points $w$, using the free field action from (1):

$$\rho(\phi_0) \equiv \delta(\phi_0 - \phi(w)) \sim \exp\left(-\frac{\phi_0^2}{2G(0)}\right),$$  (33)

where $G$ is correlator of $\phi(w)$ fields, the average is over the distribution

$$\rho(\phi(w)) \sim e^{\frac{1}{\pi \alpha} \int d^2w \sqrt{g_\phi} \Delta \phi + Q R \phi},$$  (34)

and

$$G(0) = 2 \ln \frac{L}{a}. \quad (35)$$

We replace our system (1) with a REM model having the same number $M$ independent variables $E_i \sim \phi(w)$ with the same distribution (32):

$$N = G(0), \alpha_c = \sqrt{\frac{2 \ln M}{G(0)}} = \sqrt{2}$$

$$\ln Z_{PM} = 2 \frac{\alpha^2 + \alpha^2}{2} \ln \frac{L}{a}.$$  (36)
Here $Z_{PM}$ is the partition connected with the Eq. (1) for the PM phase at real $\alpha$. We rescale the temperature:

$$\frac{\alpha}{\sqrt{2}} = \beta, \beta_c = 1, \mu = \frac{Q}{\alpha}$$  \hspace{1cm} (37)

DDK formulas give for $d \equiv c$ dimensions:

$$Q = \sqrt{\frac{25 - c}{3}}, \alpha = -\frac{1}{\sqrt{12}}(\sqrt{25 - c} - \sqrt{1 - c})$$  \hspace{1cm} (38)

For the sphere topology: For the $1 < d < 25$:

$$\beta_1 = \sqrt{\frac{25 - c}{24}}, \beta_2 = -\sqrt{\frac{c - 1}{24}}, \mu_1 = \frac{1}{12}(25 - c), \mu_2 = \sqrt{(25 - c)(c - 1)}\frac{1}{12}$$  \hspace{1cm} (39)

For $25 < d < 26$ we have:

$$\beta_2 = \sqrt{\frac{c - 25}{24}} - \sqrt{\frac{c - 1}{24}}, \mu_1 = \frac{1}{12}(25 - c) + \sqrt{(25 - c)(c - 1)}\frac{1}{12}$$  \hspace{1cm} (40)

For other string topologies one should rescale the $\mu$ expressions in Eqs. (39)-(40):

$$\mu \rightarrow (1 - g)\mu$$  \hspace{1cm} (41)

Let us consider first the sphere topology. We denote $y = \frac{25 - d}{24}$. For the $1 < d < 25$ we have:

$$\beta_1 = \sqrt{7}, \beta_2 = -\sqrt{1 - y}, \mu_1 = 2y, \mu_2 = 2\sqrt{y(1 - y)}.$$  We derive the following 4 expressions for the $\ln Z$:

$$\frac{\mu_1[1 + (\beta_1^2 - \beta_2^2)] - 2\mu_2\beta_1\beta_2}{2} = \frac{4y^2 + 4y(1 - y)}{2} = 2y, PM$$

$$\frac{1 + (\mu_1^2 - \mu_2^2)(\beta_1^2 - \beta_2^2) - 4\mu_1\mu_2\beta_1\beta_2}{2} = \frac{1 + 4y(2y - 1)^2 + 16y^2(1 - y)}{2} = \frac{1 + 4y}{2}, CPM$$

$$\mu_1\beta_1 = \frac{2y^{3/2}, SG}{4} = \frac{y(1 + 4y)}{2}, LYF$$

We see, that CPM phase is preferable in the region $1 \leq d \leq 19$ ($0 \leq y \leq 1$). For the $19 \leq d \leq 26$ we should compare $\ln Z$ expressions for the CPM and LYF phases, as $\beta_1 = 0, \mu_1 > 0$. Now we denote $y = \frac{25 - d}{24}, \mu_1 = -2y + 2\sqrt{y(y + 1)}, \beta_2 = y - \sqrt{1 + y}$. For the CPM phase we have $\ln Z = \frac{1 - \mu_1^2\beta_2^2}{2} = (1 - 4(y - \sqrt{y(y + 1)})^2)(1 - y)^2/4$ and for the LYF phase $\ln Z = \frac{y^{3/2}}{4} = \frac{-2y + 2\sqrt{y(1 + y)}}{4}$. We see that for spherical topology for the whole region $1 \leq d \leq 26$ string is in CPM phase.

Let us consider the torus topology case. We have for PM phase $\ln Z = \frac{1 + (\beta_1^2 - \beta_2^2)}{2} = y$, for SG phase $\ln Z = \beta_1 = \sqrt{7}$ and for LYF phase $\ln Z = \frac{1 + \beta_1^2}{4} = \frac{1 + 4y}{4}$, where $y = \frac{25 - d}{24}$. At $1 \leq d \leq 19$ system with torus topology is in the SG phase. LYF phase exists at $19 \leq d \leq 26$.

Let us consider now now higher topologies $g \leq 2$. Again system is in SG phase for $1 \leq d \leq 19$. At $19 < d < 25$ still exists a thermodynamic limit and system is in LYF phase, if $\frac{25 - d}{24}(g - 1) < 2$, therefore $g = 5, d = 19$ is a multicritical point. For the $d = 26$ there is a thermodynamic limit with LYF phase at $(g - 1)/3 < 2$.

Let us consider now a case of superstring. Now one has:

$$Q = \sqrt{\frac{9 - d}{2}}, -\alpha = \frac{\sqrt{9 - d - i\sqrt{c - 1}}}{2\sqrt{2}}$$

Let us denote $u = \frac{9 - d}{8}$

$$\beta_1 = \frac{\sqrt{9 - c}}{2\sqrt{2}} = u^{1/2}, \beta_2 = -\frac{\sqrt{d - 1}}{2\sqrt{2}} = -(1 - u)^{1/2}$$
\[ \mu_1 = \frac{9-d}{4} = 2u, \mu_2 = \frac{\sqrt{(9-d)(1-d)}}{4} = 2\sqrt{u(1-u)} \]

We see a mapping \( y \to u \). Now the transition to the LYF phase is at \( d = 7 \). According the [19] interesting dimension is \( d = 5 \), connected with QCD interpretation as strings [20].

What one can say about string’s physics on the ground of the REM picture? The most interesting case is the sphere case. When one climbs over the \( d = 1 \) barrier, nothing happens in REM picture, system is still in CPM phase, as for the \( d < 1 \). The free energy has not any singularity (might be there are singularities in some correlators). To reveal interesting (unitary) theories explicitly one should solve the directed polymer at finite replica number including finite size corrections and correlators. But at least for the sphere case the REM analysis seems to be quite reliable.

V. CONCLUSIONS

In sections 1,2 and Appendix B we gave an arguments for the connection of string’s partition with a finite replica number REM. In section 3 and in Appendix A we solved Random Energy Modela at complex temperatures and replica numbers. In section 4 we take string model with an analytical continuation of the David-Kawai-Distler formulas at \( d > 1 \) and mapped it to REM. The validity of DDK formulas at \( d > 1 \) is still under question, but we hope that the analytical continuation could reveal singularities of the system. It is a typical situation in statistical physics, when there is a singularity in free energy expression, when it is analytically continued from one of phases to the border between phases.

We obtained a bit strange result about difference of phases for the different topologies of string surfaces at \( d > 1 \). For the spherical case there is no any barrier at \( d = 1 \), at least for the free energy. For the other topologies at \( d > 1 \) system is in SG or LYF phase, sometimes the model is so pathological that there is no any thermodynamic limit.

There have been early attempts to connect strings with spin glasses. I have several discussions with V. Knizhnik in Alma-Ata conference in 1985, later in Yerevan before his death. He was highly intrigued with a ultrametry property of spin-glasses and trying to connect them with string.

M. Virassoro also informed me about his and G. Parisi’s attempts to connect strings with spin glasses.

We could succeed due to work done in [4], [6-8] and a simple observation that string’s partition is similar to finite replica REM just after zero mode integration. String theory is too mathematized. In this work we tried to catch some narrow but crucial aspect of the theory using more physics and less complicated mathematical tools.

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To calculate expression (5) we introduce an identity
\[
\int dU_1 \delta(U_1 - \text{Re} \sum_i e^{(\beta_1 + i\beta_2)E_i}) \int dU_2 \delta(U_1 - \text{Im} \sum_i e^{(\beta_1 + i\beta_2)E_i}) = 1
\]
and an integral representation for \( \delta(z - u) \equiv \frac{1}{2\pi} \int dk e^{ik(z-u)}; \)
\[
\frac{1}{\sqrt{N\pi}} \int_{-\infty}^{\infty} dx \exp[-\frac{x^2}{2N}] \exp(ik_1 e^{\beta_1 x} \cos(\beta_2 x) + i k_2 e^{\beta_1 x} \sin(\beta_2 x))]^M
\]
Having an expression for the function \( f(k_1, k_2) \) we can define the partition \( Z \):
\[
Z = \frac{1}{4\pi} \int_{-\infty}^{\infty} dk_1 dk_2 dU_1 dU_2 e^{-ik_1 U_1 - ik_2 U_2} (U_1 + iU_2)^\mu f(k_1, k_2)
\]
This is an exact expression. In thermodynamic limit we will consider four different asymptotics for the function \( f(k_1, k_2) \).

In the paramagnetic phase we expand an exponent via degrees of \( k_1, k_2 \):
\[
g(k_1, k_2) \approx 1 + i k_1 \text{Re} e^{N \frac{(\beta_1^2 - \beta_2^2) + 2\beta_1 \beta_2}{2}} + i k_2 \text{Im} e^{N \frac{(\beta_1^2 - \beta_2^2) + 2\beta_1 \beta_2}{2}} \]
Integration via \( dk_1, dk_2 \) gives
\[
\delta(U_1 - \text{Re} e^{N \frac{(\beta_1^2 - \beta_2^2) + 2\beta_1 \beta_2}{2}}) \delta(U_2 - \text{Im} e^{N \frac{(\beta_1^2 - \beta_2^2) + 2\beta_1 \beta_2}{2}})
\]
Eventually we derive for the PM phase:
\[
f(k_1, k_2) \approx \exp[i k_1 M \text{Re} e^{N \frac{(\beta_1^2 - \beta_2^2) + 2\beta_1 \beta_2}{2}} + i k_2 M \text{Im} e^{N \frac{(\beta_1^2 - \beta_2^2) + 2\beta_1 \beta_2}{2}}]
ln < z^\mu > = N \mu_1 (\beta_1^2 + \beta_2^2 - \beta_2^2) - 2 \mu_2 \beta_1 \beta_2
\]
We miss the imaginary part in the expression of \( \ln < Z > \).

For the Lee-Yang-Fisher (LYF) phase we take the second terms in the expansion of the exponent :
\[
g(k_1, k_2) \approx 1 - \frac{1}{\sqrt{2N\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2N}} \frac{(k_1 \cos(\beta_2 x) + k_2 \sin(\beta_2 x))^2}{2}
= 1 - \frac{k_1^2 + k_2^2}{4} e^{2N\beta_1^2}
\]
We obtain:
\[
f(k_1, k_2) \approx \exp[-M \frac{k_1^2 + k_2^2}{4} e^{2N\beta_1^2}]
\]
\[
Z = \frac{1}{\pi M e^{2N\beta_1^2}} \int dU_1 dU_2 \exp[-\frac{(U_1^2 + U_2^2)}{Me^{2N\beta_1^2}}] (U_1 + iU_2)^{\mu_1 + i\mu_2}
= e^{\mu_1 N \frac{\beta_1^2 + \beta_2^2}{4}} \frac{1}{\pi} \int_0^{2\pi} d\phi \int_0^\infty dr r^2 \exp[-r^2] r^{\mu_1 + 1 + i\mu_2} e^{(\mu_1 + i\mu_2)i\phi}
= \frac{1}{\pi} e^{\mu_1 N \frac{\beta_1^2 + \beta_2^2}{4}} \Gamma(\mu_1 + 1 + i\mu_2) \exp(2\pi(\mu_1 + i\mu_2)) - 1
\]
Then after integration by parts:

$$\delta$$

The result is

$$\mu$$

Therefore we can expand the exponent in the $f(k_1, k_2)$ expression:

$$f(k_1, k_2) = \left[ \frac{1}{\sqrt{\pi N}} \int_{-\infty}^{\infty} dx \exp\left[ -\frac{x^2}{N} + ik_1 \cos(\beta_2 x) + ik_2 \sin(\beta_2 x) \right] \right]^M$$

$$\approx 1 + M \{ \int_{-\infty}^{\infty} \frac{dk_1 dk_2}{\sqrt{\pi N}} dx \exp\left[ -\frac{x^2}{N} + i e^{\beta_1 x} (k_1 \cos(\beta_2 x) + k_2 \sin(\beta_2 x)) \right] - 1 \}

Then after integration by parts:

$$Z = \frac{M}{4 \pi^2} \int_{-\infty}^{\infty} dk_1 dk_2 e^{-ik_1 U_1 - ik_2 U_2} (U_1 + i U_2)^\nu \left( -\frac{id}{dk_1} + \frac{d}{dk_2} \right)^n f(k_1, k_2)$$

$$= \frac{M}{4 \pi^2} \int_{-\infty}^{\infty} dk_1 dk_2 \int_{-\infty}^{\infty} dU_1 dU_2 e^{-ik_1 U_1 - ik_2 U_2} (U_1 + i U_2)^\nu \frac{1}{\sqrt{\pi N}} dx$$

$$\exp\left[ -\frac{x^2}{N} + i Re(k_1 - ik_2) e^{(\beta_1 + i \beta_2) x} \right] e^{(\beta_1 + i \beta_2) \nu x}$$

We miss the term 1 in the expression of $f(k_1, k_2)$, because its contribution is equal to 0 after integration by parts. Let us denote $E = \exp((\beta_1 + i \beta_2) x)$, $K = k \exp(i \varphi) = k_1 + ik_2, U = U_1 + i U_2$. First we take the integration via $dk_1, dk_2$. The result is $\delta(E - U)$. Then we calculate Gaussian integral via $dx$ and derive an expression for the correlated paramagnetic phase (CPM):

$$Z \equiv \gamma^M = \frac{M}{\sqrt{\pi N}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2} + \mu x (\beta_1 + i \beta_2) x} =$$

$$\exp\left[ N \left( \mu_1^2 - \mu_2^2 \right) + 4 \beta_1 \beta_2 \mu_1 \mu_2 + \beta_2^2 \right]$$

Let us calculate SG phase. It is convinient to use another representation of function $f(k_1, k_2)$ [16]. Using the Stratanovich transformation for the energy density term

$$\exp\left( -\frac{\beta^2}{2} \right) = \frac{\beta_1 \sqrt{N}}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} dy e^{\frac{-y^2}{2} + \beta_1 \sqrt{N} \beta_2 x}$$

we derive

$$g(k_1, k_2) = \frac{\sqrt{N}}{2 \pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \exp\left( \frac{N \beta_2 y^2}{2} + \beta_1 \sqrt{N} x \right) + ik_1 \text{Re} e^{(\beta_1 + i \beta_2) \sqrt{N} x} + ik_2 \text{Im} e^{(\beta_1 + i \beta_2) \sqrt{N} x}$$

(A.11)

After transformation $v = k \exp(\sqrt{N} \beta_1 x)$ we have

$$f(k_1, k_2) = \left( \frac{1}{2 \pi^2} \right) \int_{-\infty}^{\infty} dy e^{-y \ln k + \frac{N \beta_2 y^2}{2}} G(y, k, \varphi)$$

$$G(y, k, \varphi) = \int_{0}^{\infty} dv e^{iy \cos(\beta_2 / \beta_1 \ln(\nu \ln k - \varphi))} \nu(y-1).$$

(A.12)
We are interested in Eq.(12) for the $|\ln k| \sim N$, therefore we can calculate the asymptotic of the function $f(k_1,k_2)$ via the saddle point method. There is a pole of function $G(y)$ at $y = 0$ with residue equal to 1 (it can be derived putting a small low integration limit). Let us shift the integration loop to the saddle point. We have:

$$f(k,\varphi) = M(1 + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dy e^{-y\ln|k|} e^{N\frac{\beta^2 y^2}{2}} G(y,k,\varphi))$$ \hspace{1cm} (A.13)

For the saddle point we have

$$y_0 = \frac{\ln k}{N\beta_1^2}$$ \hspace{1cm} (A.14)

We move the integration loop via $dy$ to catch the saddle point. For the analytical continuation to the region $[-1 < \Re y < 0]$ we transform expression of $G(y)$ from (A12) using the integration by parts:

$$G(y,k,\varphi) = \frac{-1}{\sqrt{\pi N\beta_1}} \int_0^\infty dv v^y \exp\{iv\cos[\beta_2/\beta_1(\ln \frac{v}{k} - \varphi)] \cos[\beta_2/\beta_1(\ln \frac{v}{k} - \varphi)] - \beta_2/\beta_1 \sin[\beta_2/\beta_1(\ln \frac{v}{k} - \varphi)]\}$$ \hspace{1cm} (A.15)

We have an asymptotics:

$$g(k,\varphi) = 1 - \frac{1}{\sqrt{\pi N\beta_1}} \chi\{G(\frac{2\ln k}{N\beta_1^2},k,\varphi)\} e^{-\frac{\ln k^2}{N\beta_1}}$$

$$f(k_1,k_2) = \exp[-M e^{-\frac{\ln k^2}{N\beta_1^2}} A]$$

$$A = -G(y_0,k,\varphi)$$ \hspace{1cm} (A.16)

One should take only this asymptotic instead of (A3),(A5) if, while shifting the integration loop, we don’t intersect the pole at $y = -1$,

$$\frac{|\ln k|}{N\beta_1^2} < 1$$ \hspace{1cm} (A.17)

Otherwise, at $\frac{|\ln k|}{N\beta_1^2} > 1$ we should consider all three different asymptotics (A3),(A5),(A16) and choose the largest one. From the Eq. (a16) we derive immediately the bulk value of partition:

$$Z \sim \exp[\mu_1 N\beta_1\beta_2]$$ \hspace{1cm} (A.18)

We derived accurate expressions for the PM phase (A4),LYF phase (A6), CPM phase (A10) and bulk expression for the SG phase (A17). We see, that LYF phase can be constructed only at $\mu_1 > -2$. Thus a situation, when bulk expression for the $< Z >$ is given by LYF phase and $\mu_1 < -2$, model is to pathalogic and there is no thermodynamic limit.

To find the borders between phases one should solve the model at the limit $\mu \to 0$, then compare the corresponding expression of the largest free energy with the one in CPM for the finite $\mu_1$ (it exists only at $\mu_1 > 0$). One should choose a phase, having larger value of $|\ln Z|$. Let us remember also that SG phase exists at $\beta_1 + \beta_2 > \beta_c$, the CPM phase at $\beta_2 > \frac{\beta_1}{2}$, $\beta_1 < \frac{\beta_2}{2}$.

**APPENDIX B: EQUIVALENCE OF REM AND DIRECTED POLYMER IN THERMODYNAMIC LIMIT**

We consider a hierarchic tree with $K$ levels and large $Q$. We have, that along any path connecting endpoint with the origin:

$$\sum_{\alpha} \langle \epsilon^2_{\alpha} \rangle = N$$ \hspace{1cm} (B.1)

Let us first consider the PM phase. We should define the generating function

$$\langle f(k_1,k_2) \rangle \equiv \sum \langle \exp[iRe(k_1 - ik_2)e^{\beta_1+i\beta_2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots + \epsilon_K)] \rangle$$ \hspace{1cm} (B.2)
Here the sum is over all the paths, connecting endpoints with the origin. Let us consider first the integration via the last level of hierarchy. We expand an exponent and after integration via $d\varepsilon_{iK}$:

$$< G(k_1, k_2) > = \exp[iRe(k_1 - ik_2)e^{(\beta_1+i\beta_2)(\varepsilon_1+\varepsilon_2+...\varepsilon_{K-1})}]Q\varepsilon^{\frac{N\beta_1^2-\beta_2^2+i2\beta_1\beta_2}{2}}$$

(B.3)

Repeating this procedure $K$ times, we obtain

$$< f(k_1, k_2) > = \exp[iRe(k_1 - ik_2)Q^K e^{N\frac{\beta_1^2-\beta_2^2+i2\beta_1\beta_2}{2}}]$$

(B.4)

In principle the expression in the exponent could be large. We recover the REM result for the PM phase with accuracy $o(1)$.

For the LYF the integration via the last level of hierarchy gives

$$< f(k_1, k_2) > \equiv < 1 - \frac{k_1^2 + k_2^2}{4}e^{2\beta_1(\varepsilon_1+\varepsilon_2+...\varepsilon_{K-1})}Q\frac{k_1^2 + k_2^2}{4}e^{\frac{N\beta_1^2}{2}} >$$

(B.5)

Repeating the integration $K$ times (we expand the exponent all the time via $(k_1^2 + k_2^2)$) gives the REM result for LYF phase

$$< f(k_1, k_2) > = \exp[-\frac{k_1^2 + k_2^2}{4}Q^K e^{N\beta_1^2}]$$

(B.6)

Again we have equivalence with accuracy $O(1)$.

The case of SG phase is a bit complicated. Now the integration via the last level of hierarchy gives

$$< f(k_1, k_2) > \equiv < 1 - \frac{k_1^2 + k_2^2}{4}Q\frac{\ln k + \beta_1(\varepsilon_1+\varepsilon_2+...\varepsilon_{K-1})}{N\beta_1^2} >$$

where $c \sim O(1)$. This expression resembles the case of real temperatures, where the generating function is calculated [16]. Using those results for the real temperature directed polymer, we derive again the REM expression for the SG phase.

Let us consider now the case of CPM phase. Here we use again formulas (a7), (a8). Now all the integrations decouple and we recover the result of a simple REM:

$$< z^\mu > = M < e^{-(\mu_1+i\mu_2)(\beta_1+i\beta_2)x} >= \exp[\frac{N(\mu_1^2 - \mu_2^2)(\beta_1^2 - \beta_2^2) - 4\beta_1\beta_2\mu_1\mu_2 + \beta_1^2}{2}]$$

(B.8)