THE CONFIGURATION SPACE OF EQUIDISTANT TRIPLES IN THE HEISENBERG GROUP

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ABSTRACT. We prove that the configuration space of equidistant triples on the Heisenberg group equipped with the Korányi metric, is isomorphic to a hypersurface of $\mathbb{R}^3$.

1. Introduction

Let $\mathfrak{H}$ be the first Heisenberg group equipped with the Korányi distance $d$. An equidistant triple is a triple of points $P = (p_1, p_2, p_3) \in \mathfrak{H}$ such that

$$d(p_1, p_2) = d(p_2, p_3) = d(p_3, p_1).$$

Denote by $\mathcal{ET}$ the space of equidistant triples in $\mathfrak{H}$. Then the similarity group

$$G = \text{Sim}(\mathfrak{H}, d) = \mathfrak{H} \times \mathbb{R} \times \mathbb{R}_+^*,$$

acts diagonally on $\mathcal{ET}$:

$$(g, (p_1, p_2, p_3)) \mapsto (g(p_1), g(p_2), g(p_3)).$$

We denote by $\mathcal{ETF}$ the quotient of this action; this is the configuration space of $G$-equivalent equidistant triples in $\mathfrak{H}$. In this paper we are dealing with the problem of parametrising $\mathcal{ETF}$. The problem is addressed and solved in a different manner in $\mathbb{H}$ (see Proposition 4.6 there). We prove here the following theorem:

**Theorem 1.1.** The configuration space $\mathcal{ETF}$ of $G$-equivalent equidistant triples is in bijection with the hypersurface $\mathcal{E}$ of $\mathbb{R}^3$ which is defined by

$$\mathcal{E} = \left\{ (a, b, c) \in [-2\pi/3, 2\pi/3]^3 \mid \cos a + \cos b + \cos c = \frac{3}{2} \right\}.$$ 

The proof of Theorem 1.1 relies upon the use of the cross-ratio variety $\mathfrak{X}$; this is a 4-dimensional variety parametrising the $\text{PU}(2,1)$-configuration space of pairwise distinct quadruples on the boundary $\partial H_\mathbb{C}^2$ of complex hyperbolic plane. In fact, $\mathcal{ETF}$ may be viewed as a 2-dimensional subvariety of $\mathfrak{X}$.

This paper is organised as follows: In Section 2 we review standard facts about complex hyperbolic plane and the Heisenberg group, as well as about cross-ratio variety. In Section 3 we prove Theorem 1.1 and discuss the particular case of equidistant triples lying in a $\mathbb{C}$-circle in Section 3.4.

Acknowledgements. I wish to thank Vassilis Chousionis for suggesting the problem to me, and also Viktor Schroeder for fruitful discussions.
2. Preliminaries

The material of this section is standard; a general reference is Goldman’s book, [5]. In Section 2.1 we review complex hyperbolic plane, its boundary and the Heisenberg group. Cartan’s angular invariant and complex cross-ratios are in Section 2.2. Finally, a brief overview of the cross-ratio variety and the $\text{PU}(2, 1)$-configuration of four pairwise distinct points in the boundary of complex hyperbolic plane is found in Section 2.3.

2.1. Complex hyperbolic plane and Heisenberg group. We consider the vector space $\mathbb{C}^{2, 1}$, that is, $\mathbb{C}^3$ with the Hermitian form of signature $(2, 1)$ given by

$$\langle z, w \rangle = z_1 \overline{w}_3 + z_2 \overline{w}_2 + z_3 \overline{w}_1.$$ 

We next consider the following subspaces of $\mathbb{C}^{2, 1}$:

$$V_-= \{ z \in \mathbb{C}^{2, 1} : \langle z, z \rangle < 0 \}, \quad V_0 = \{ z \in \mathbb{C}^{2, 1} \setminus \{0\} : \langle z, z \rangle = 0 \}.$$ 

Denote by $\mathbb{P} : \mathbb{C}^{2, 1} \setminus \{0\} \to \mathbb{C}P^2$ the canonical projection onto complex projective space. Then the complex hyperbolic plane $\mathbb{H}_C^2$ is defined to be $\mathbb{P}V_-$ and its boundary $\partial \mathbb{H}_C^2$ is $\mathbb{P}V_0$. Hence we have

$$\mathbb{H}_C^2 = \{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re(z_1) + |z_2|^2 < 0 \},$$ 

and in this manner, $\mathbb{H}_C^2$ is the Siegel domain in $\mathbb{C}^2$.

There are two distinguished points in $V_0$ which we denote by $o$ and $\infty$:

$$o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

Let $\mathbb{P}o = o$ and $\mathbb{P}\infty = \infty$. Then

$$\partial \mathbb{H}_C^2 \setminus \{\infty\} = \{ (z_1, z_2) \in \mathbb{C}^2 : 2\Re(z_1) + |z_2|^2 = 0 \},$$

and in particular, $o = (0, 0) \in \mathbb{C}^2$.

Conversely, if we are given a point $z = (z_1, z_2)$ of $\mathbb{C}^2$, then the point

$$z = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}.$$ 

is called the standard lift of $z$. Therefore the standard lifts of points of the complex hyperbolic plane and its boundary (except the point at infinity) are vectors of $V_-$ and $V_0$ respectively with the third inhomogeneous coordinate equal to 1.

Complex hyperbolic plane $\mathbb{H}_C^2$ is a Kähler manifold; its Kähler structure is given by the Bergman metric. The holomorphic sectional curvature equals to $-1$ and its real sectional curvature is pinched between $-1$ and $-1/4$. The full group of holomorphic isometries is the projective unitary group

$$\text{PU}(2, 1) = \text{SU}(2, 1)/\{I, \omega I, \omega^2 I\},$$

where $\omega$ is a non-real cube root of unity (that is $\text{SU}(2, 1)$ is a 3-fold covering of $\text{PU}(2, 1)$). There are two ways (up to $\text{PU}(2, 1)$ conjugacy) to embed real hyperbolic plane into complex hyperbolic plane; that is, as $\mathbb{H}^{1}_C$ as well as $\mathbb{H}^{2}_\mathbb{R}$. These embeddings give rise to complex lines,
i.e., isometric images of the embedding of $H^1_C$ into $H^2_C$ and Lagrangian planes, i.e., isometric images of $H^2_R$ into $H^2_C$, respectively.

There is an identification of the boundary of the Siegel domain with the one point compactification of $\mathbb{C} \times \mathbb{R}$: A finite point $z$ in the boundary of the Siegel domain has a standard lift of the form

$$ z = \begin{bmatrix} -|z|^2 + it \\ \sqrt{2}z \\ 1 \end{bmatrix}. $$

The unipotent stabiliser at infinity acts simply transitively and gives the set of these points the structure of a 2–step nilpotent Lie group, namely the Heisenberg group $\mathcal{H}$. This is $\mathbb{C} \times \mathbb{R}$ with group law:

$$(z, t) \ast (w, s) = (z + w, t + s + 2\Re(z\bar{w})).$$

The Heisenberg norm (Korányi gauge) is given by

$$|(z, t)|_{\mathcal{H}} = |A(z, t)|^{1/2},$$

where $A(z, t) = |z|^2 - it$. From this norm arises a metric, the Korányi-Cygan (K-C) metric, on $\mathcal{H}$ by the relation

$$d\left((z, t), (w, s)\right) = |(z, t)^{-1} \ast (w, s)|_{\mathcal{H}}.$$

The K-C metric is invariant under

a) left-actions $L_{(w, s)}$ of $\mathcal{H}$, $(z, t) \mapsto (w, s) \ast (z, t)$, $(w, s) \in \mathcal{H}$;

b) rotations $R_{\phi}$, $(z, t) \mapsto (ze^{i\phi}, t)$, $\phi \in \mathbb{R}$;

c) involution $j$, $j(z, t) = (\bar{z}, -t)$.

These form the group $\text{Isom}(\mathcal{H}, d)$ of Heisenberg isometries. Note that all the above are orientation-preserving.

The K-C metric is also scaled up to multiplicative constants by the action of
d) Heisenberg dilations $D_r$, $(z, t) \mapsto (rz, r^2 t)$, $r \in \mathbb{R}_+^\ast$.

and there is also an inversion, defined for each $p = (z, t) \in \mathcal{H}$, $p \neq o$, by

$$(z, t) \mapsto \left(-\frac{z}{-|z|^2 + it}, -\frac{t}{-|z|^2 + it}\frac{1}{|z|^2 + it}\right),$$

which satisfies

$$d_{\mathcal{H}}(R(p), R(p')) = \frac{d_{\mathcal{H}}(p, p')}{d_{\mathcal{H}}(p, o) d_{\mathcal{H}}(p', o)}.$$

The similarity group $G = \text{Sim}(\mathcal{H}, d)$ comprises compositions of maps of the form a), b), d).

Clearly, $G = \mathcal{H} \times \mathbb{R} \times \mathbb{R}_+^\ast$.

2.1.1. $\mathbb{R}$-circles and $\mathbb{C}$-circles. $\mathbb{R}$-circles are boundaries of Lagrangian planes and $\mathbb{C}$-circles are boundaries of complex lines. They come in two flavours, infinite ones (i.e., containing the point at infinity) and finite ones. We refer to [5] for more details about these curves.
2.2. **Cartan’s Angular Invariant.** Given a triple \((p_1, p_2, p_3)\) of points at the boundary \(\partial H^2_C\) the Cartan’s angular invariant \(\mathbb{A}(p_1, p_2, p_3)\) is defined by

\[
\mathbb{A}(p_1, p_2, p_3) = \arg\langle -\langle p_1, p_2 \rangle \langle p_2, p_3 \rangle \langle p_3, p_1 \rangle \rangle,
\]

where \(p_i\) are lifts of \(p_i\), \(i = 1, 2, 3\). The Cartan’s angular invariant lies in \([-\pi/2, \pi/2]\), is independent of the choice of the lifts and remains invariant under the diagonal action of \(\text{PU}(2, 1)\). Any other permutation of points produces angular invariants which differ from the above possibly up to sign. The following propositions are in [5] to which we also refer the reader for further details:

**Proposition 2.1.** Let \((p_1, p_2, p_3)\) be a triple of points lying in \(\partial H^2_C\) and let also \(\mathbb{A} = \mathbb{A}(p_1, p_2, p_3)\) be their Cartan’s angular invariant. Then:

1. All points lie in an \(\mathbb{R}\)-circle if and only if \(\mathbb{A} = 0\).
2. All points lie in a \(\mathbb{C}\)-circle if and only if \(\mathbb{A} = \pm \pi/2\).

**Proposition 2.2.** Suppose that \(p_i\) and \(p'_i\), \(i = 1, 2, 3\), are points in \(\partial H^2_C\). If there exists a holomorphic isometry \(g\) of \(H^2_C\) such that \(g(p_i) = p'_i\), \(i = 1, 2, 3\), then \(\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(p'_1, p'_2, p'_3)\). Conversely, if \(\mathbb{A}(p_1, p_2, p_3) = \mathbb{A}(p'_1, p'_2, p'_3)\), then there exists a holomorphic isometry \(g\) of \(H^2_C\) such that \(g(p_i) = p'_i\), \(i = 1, 2, 3\). This isometry is unique unless \(p_i\), \(i = 1, 2, 3\), lie in a \(\mathbb{C}\)-circle.

2.3. **Cross-ratio variety and the configuration space.** Given a quadruple of pairwise distinct points \(p = (p_1, p_2, p_3, p_4)\) in \(\partial H^2_C\), we define their complex cross-ratio as follows:

\[
\mathbb{X}(p_1, p_2, p_3, p_4) = \frac{\langle p_3, p_1 \rangle \langle p_4, p_2 \rangle}{\langle p_4, p_1 \rangle \langle p_3, p_2 \rangle},
\]

where \(p_i\) are lifts of \(p_i\), \(i = 1, 2, 3, 4\), see also [6], [7], [8]. The cross-ratio is independent of the choice of lifts and remains invariant under the diagonal action of \(\text{PU}(2, 1)\). We stress here that for points in the Heisenberg group, the square root of its absolute value is

\[
|\mathbb{X}(p_1, p_2, p_3, p_4)|^{1/2} = \frac{d_5(p_4, p_2) \cdot d_5(p_3, p_1)}{d_5(p_4, p_1) \cdot d_5(p_3, p_2)}.
\]

Given a quadruple \(p = (p_1, p_2, p_3, p_4)\) of pairwise distinct points in the boundary \(\partial H^2_C\), all possible permutations of points gives us 24 complex cross-ratios corresponding to \(p\). Due to symmetries, see [3], Falbel showed that all cross-ratios corresponding to a quadruple of points depend on three cross-ratios which satisfy two real equations. Indeed, the following proposition holds; for its proof, see for instance [7].

**Proposition 2.3.** Let \(p = (p_1, p_2, p_3, p_4)\) be any quadruple of pairwise distinct points in \(\partial H^2_C\). Let

\[
\mathbb{X}_1(p) = \mathbb{X}(p_1, p_2, p_3, p_4), \quad \mathbb{X}_2(p) = \mathbb{X}(p_1, p_3, p_2, p_4), \quad \mathbb{X}_3(p) = \mathbb{X}(p_2, p_3, p_1, p_4).
\]

Then

\[
|\mathbb{X}_4|^2 = |\mathbb{X}_2|^2 / |\mathbb{X}_1|^2,
\]

\[
2|\mathbb{X}_1|^2 \Re(\mathbb{X}_3) = |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - 2\Re(\mathbb{X}_1) - 2\Re(\mathbb{X}_2) + 1.
\]
Equations (2.1) and (2.2) define a 4-dimensional real subvariety of \( \mathbb{C}^3 \) which we call the **cross-ratio variety** \( \mathcal{X} \). This variety is isomorphic to the subset \( \mathcal{Y}' \) of the \( \text{PU}(2, 1) \) configuration space \( \mathcal{Y} \) of pairwise disjoint quadruples of points in \( \partial \mathbb{H}^2_\mathbb{C} \), comprising quadruples whose points do not all lie in the same \( \mathbb{C} \)-circle. In the latter case, we have a 2–1 map between the subset \( \mathcal{Y}_\mathbb{R} \) comprising of quadruples whose points all lie in a \( \mathbb{C} \)-circle and the subvariety \( \mathcal{X}_\mathbb{R} \) of \( \mathcal{X} \) defined by

\[
\mathcal{X}_\mathbb{R} = \{(X_1, X_2) \in \mathbb{R}^2_+ \mid X_1 + X_2 = 1\},
\]

see [3], [2]. For further reference, we shall need the following:

**Remark 2.4.** Let \( p \) a quadruple of pairwise distinct points in \( \partial \mathbb{H}^2_\mathbb{C} \) which do not all lie in the same \( \mathbb{C} \)-circle and let also \( X_i(p) \), \( i = 1, 2, 3 \) be as above. Let also \( a = \text{arg}(X_1) \), \( b = \text{arg}(X_2) \), \( c = \text{arg}(X_3) \). We have

\[
a = A_1 - A_2, \quad b = -A_2 - A_4, \quad c = A_4 - A_1.
\]

Here,

\[
A_1 = A(p_2, p_3, p_4), \quad A_2 = A(p_1, p_3, p_4), \quad A_4 = A(p_1, p_2, p_3).
\]

As for \( A_3 = A(p_1, p_2, p_4) \) we have

\[
A_3 + A_1 = A_2 + A_4.
\]

### 3. The Configuration Space of Equidistant Triples

In this section we are going to prove Theorem 1.1. The proof will follow after a series of lemmas which follow below.

**3.1. The lemmas.** Throughout this section we will have the following notation: We shall denote by \( P \) a triple \((p_1, p_2, p_3)\) of pairwise distinct points in the Heisenberg group \( \mathcal{H} \). We will consider also the quadruple \( p = (p_1, \infty, p_2, p_3) \); let \( X_i(p) \), \( i = 1, 2, 3 \) be the corresponding point on the cross-ratio variety \( \mathcal{X} \) and let

\[
a = \text{arg}(X_1(p)), \quad b = \text{arg}(X_2(p)), \quad c = \text{arg}(X_3(p)).
\]

There is an important note here: points of \( p \) **cannot** all lie in the same (infinite) \( \mathbb{C} \)-circle. To see this, normalise so that

\[
p_1 = (0, 0), \quad p_2 = (0, t), \quad p_3 = (0, s), \quad t, s \in \mathbb{R}.
\]

Then conditions \( d(p_1, p_2) = d(p_1, p_3) = d(p_2, p_3) \) deduce

\[
|t|^{1/2} = |s|^{1/2} = |t - s|^{1/2},
\]

which cannot happen.

**Lemma 3.1.** The triple \( P = (p_1, p_2, p_3) \) is equidistant if and only if

\[
|X_1(p)| = |X_2(p)| = 1.
\]

**Proof.** Since \( P = (p_1, p_2, p_3) \) is an equidistant triple, we have

\[
d(p_1, p_2) = d(p_2, p_3) = d(p_3, p_1),
\]

where \( d \) is the Korányi distance. The result follows from the formulae

\[
|X_1(p)|^2 = \frac{d(p_2, p_1)}{d(p_3, p_1)}, \quad |X_2(p)|^2 = \frac{d(p_3, p_2)}{d(p_3, p_1)}.
\]
Note that the above imply as well $|X_3(p)| = 1$. □

**Lemma 3.2.** If $P = (p_1, p_2, p_3)$ is an equidistant triple then $a, b, c$ satisfy

\[
(3.1) \quad \cos a + \cos b + \cos c = \frac{3}{2}.
\]

**Proof.** Consider the cross-ratio variety equations (2.1) and (2.2) as in the previous section. We may rewrite (2.2) equivalently as

\[
(3.2) \quad 2|X_1||X_2| \cos c = |X_1|^2 + |X_2|^2 - 2|X_1| \cos a - 2|X_2| \cos b + 1.
\]

If $P$ is an equidistant triple then $|X_i(p)| = 1, \ i = 1, 2, 3$ and thus (3.2) reduces to (3.1). □

**Lemma 3.3.** If $P = (p_1, p_2, p_3)$ is an equidistant triple, $p = (p_1, \infty, p_2, p_3)$ and $(a, b, c)$ as above. If $g \in G = \text{Sim}(\mathfrak{h})$, we set

\[
P' = g(P) = (g(p_1), g(p_2), g(p_3)) = (p_1', p_2', p_3'), \quad p' = (p_1', \infty, p_2', p_3').
\]

Then

\[
a' = \arg(X_1(p')) = a, \quad b' = \arg(X_2(p')) = b, \quad c' = \arg(X_3(p')) = c.
\]

**Proof.** The proof is immediate by invariance of cross-ratios. □

**Lemma 3.4.** Let $(a, b, c) \in [-2\pi/3, 2\pi/3]^3$ such that it satisfies

\[
\cos a + \cos b + \cos c = \frac{3}{2}.
\]

Then there exists an equidistant triple $P = (p_1, p_2, p_3)$ such that if $p = (p_1, \infty, p_2, p_3)$ then

\[
\arg(X_1(p)) = a, \quad \arg(X_2(p)) = b, \quad \arg(X_3(p)) = c.
\]

**Proof.** Set $2\eta = \arg(1 - e^{ia} - e^{ib})$. We have

\[
|1 - e^{ia} - e^{ib}|^2 = 3 - 2 \cos a - 2 \cos b + 2 \cos(a - b)
\]

\[
= 3 - 4 \cos \left(\frac{a + b}{2}\right) \cos \left(\frac{a - b}{2}\right) + 4 \cos^2 \left(\frac{a - b}{2}\right) - 2
\]

\[
= 4 \left(\cos \left(\frac{a - b}{2}\right) - \frac{1}{2} \cos \left(\frac{a + b}{2}\right)\right)^2 + \sin^2 \left(\frac{a + b}{2}\right).
\]

This is strictly positive unless

\[
(a, b, c) \in B = \{(\pi/3, -\pi/3, \pm \pi/3), (-\pi/3, \pi/3, \pm \pi/3)\}.
\]

Assume first that $(a, b, c) \notin B$ and set

\[
A_1 = \frac{a - b - c}{2}, \quad A_4 = \frac{a - b + c}{2},
\]

with $A_1, A_4 \in (-\pi/2, \pi/2)$. Notice that we have

\[
4 \cos(A_1) \cos(A_4) = 2 \cos(a - b) + 2 \cos c
\]

\[
= 2 \cos(a - b) + 3 - 2 \cos a - 2 \cos b
\]

\[
= |1 - e^{ia} - e^{ib}|^2 > 0.
\]
Consider the triple \( P = (p_1, p_2, p_3) \) of points in \( \mathcal{F} \) where
\[
p_1 = \left( \sqrt{\cos(A_4)} e^{i(b+c)/2}, \sin(A_4) \right), \quad p_2 = (0, 0), \quad p_3 = \left( -\sqrt{\cos(A_1)} e^{i(\eta - \vartheta)}, \sin(A_4) \right),
\]
and the quadruple \( p = (p_1, \infty, p_2, p_3) \), with lifts:
\[
p_1 = \begin{bmatrix} -e^{-ia/2} \\ \sqrt{2 \cos(A_4)} e^{-i\eta} \\ e^{i(c-b)/2} \end{bmatrix}, \quad \infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_3 = \begin{bmatrix} e^{i(b+c)/2} \\ \sqrt{2 \cos(A_1)} e^{i\eta} \\ -e^{ia/2} \end{bmatrix}.
\]
Now,
\[
\langle p_1, \infty \rangle = e^{i(c-b)/2}, \quad \langle p_1, p_2 \rangle = -e^{-ia/2}, \quad \langle \infty, p_2 \rangle = 1,
\]
\[
\langle \infty, p_3 \rangle = -e^{-ia/2}, \quad \langle p_2, p_3 \rangle = e^{-i(b+c)/2},
\]
and
\[
\langle p_1, p_3 \rangle = e^{-ia} + e^{-ib} + \sqrt{2 \cos(A_1)} \cos(A_4) \cdot e^{-2i\eta} = e^{-ia} + e^{-ib} + \left| 1 - e^{-ia} - e^{-ib} \right| \cdot \left| 1 - e^{-ia} - e^{-ib} \right| = 1.
\]
This gives
\[
X_1 = e^{ia}, \quad X_2 = e^{ib}, \quad X_3 = e^{ic},
\]
which proves our claim.

Finally, we consider \((a, b, c) \in B\); we shall only treat the case where \( a = \pi/3, b = -\pi/3 \) and \( c = \pi/3 \), in other words when \( A_4 = \pi/2 \) and \( A_1 = \pi/6 \). Then we set
\[
p_1 = \begin{bmatrix} -\sqrt{3+i} \\ 0 \\ 1+i\sqrt{3} \end{bmatrix}, \quad \infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 1 \\ 3^{1/4} \\ -\sqrt{3+i} \end{bmatrix}.
\]
All other cases can be treated in a similar manner.

**Lemma 3.5.** Let \((a, b, c) \in \mathcal{E}\) and consider from Lemma 3.4 the equidistant triple \( P = (p_1, p_2, p_3) \) of points in \( \mathcal{F} \) which is such that
\[
a = \arg(X_1(p)), \quad b = \arg(X_2(p)), \quad c = \arg(X_3(p)).
\]
Then any other equidistant triple \( P' = (p'_1, p'_2, p'_3) \) such that
\[
a = \arg(X_1(p')), \quad b = \arg(X_2(p')), \quad c = \arg(X_3(p'))
\]
is of the form \( P' = g(P) \), where \( g \in G = \text{Sim}(\mathcal{F}) \).

**Proof.** We consider \( P, P' \) and \( p, p' \), respectively. Since \( X_i(p) = X_i(p') \), \( i = 1, 2, 3 \), we have from Proposition 5.10 in \([7]\) and Lemma 5.5 in \([3]\) that since not all points in \( p \) lie in the same \( \mathbb{C} \)-circle, there exists a \( g \in \text{PU}(2, 1) \) such that \( g(p_i) = p'_i \), \( i = 1, 2, 3 \) and \( g(\infty) = \infty \). That is, \( g \in G \) and the proof is complete. \( \square \)
3.2. The Equidistant surface. Equation (3.1) is the equation of a hypersurface in $\mathbb{R}^3$ which we shall call **equidistant hypersurface** and denote it by $\mathcal{E}$. This hypersurface comprises of infinitely many connected components. Notice that we have

$$(a, b, c) \in \mathcal{E} \implies (a + 2n_1\pi, b + 2n_2\pi, c + 2n_3\pi) \in \mathcal{E}, \quad n_1, n_2, n_3 \in \mathbb{Z}.$$ 

On the other hand, since for instance

$$\cos a = \frac{3}{2} - \cos b - \cos c \geq -\frac{1}{2},$$

we have that $\cos a$, and in the same manner $\cos b$ and $\cos c$, are $\geq -1/2$. We deduce that the connected components of $\mathcal{E}$ may be taken by transporting the central component where

$$(a, b, c) \in [-2\pi/3, 2\pi/3]^3,$$

by multiples of $2\pi$ in all possible directions, see the figure where the central component of $\mathcal{E}$ is clearly shown.

3.3. **Proof of Theorem 1.1**. Given an equidistant triple $P = (p_1, p_2, p_3)$ we consider the quadruple $p = (p_1, \infty, p_2, p_3)$ in the $\text{PU}(2, 1)$-configuration space of pairwise distinct points on the boundary $\partial \mathbb{H}^2_\mathbb{C}$ of complex hyperbolic plane. Let $X_i(p)$, $i = 1, 2, 3$ be the cross-ratios associated to $p$; the triple $(X_1(p), X_2(p), X_3(p))$ defines a point in the cross-ratio variety $\mathfrak{X}$. In particular, in this case we have by Lemma 3.1 that $|X_i(p)| = 1$, $i = 1, 2, 3$ and moreover, if

$$a = \arg(X_1(p)), \quad b = \arg(X_1(p)), \quad c = \arg(X_3(p)),$$

then from Lemma 3.2

$$\cos a + \cos b + \cos c = \frac{3}{2}.$$ 

By Lemma 3.3 this equation is invariant by the diagonal action of $G$ on the space of equidistant triples $\mathcal{E}T$: This proves that the map

$$\mathcal{E} \mathfrak{X} \to \mathcal{E}, \quad \{P\} \mapsto (a, b, c),$$
is well-defined.

Conversely, if \((a, b, c) \in \mathcal{E}\) where \((a, b, c) \in [-2\pi/3, 2\pi/3]\), by Lemma 3.4 there exists a \(P = (p_1, p_2, p_3) \in \mathcal{ET}\) such that if \(p = (p_1, \infty, p_2, p_3)\) then \(a = \arg(X_1(p))\), \(b = \arg(X_2(p))\) and \(c = \arg(X_3(p))\); therefore \(\mathcal{ET} \to \mathcal{E}\) is onto. Finally, by Lemma 3.5 \(\mathcal{ET} \to \mathcal{E}\) is 1–1 when the points in \(p\) do not all lie in the same \(C\)-circle and 2–1 when all points in \(p\) lie in the same \(C\)-circle. This concludes the proof of Theorem 1.1. □

3.4. The \(C\)-circle case. The subset \(\mathcal{ET}_C\) of \(\mathcal{ET}\) comprising equivalent equidistant triples of points lying on a \(C\)-circle is of special interest. We can show in an elementary way that \(\mathcal{ET}_C\) is just two points on the equidistant hypersurface \(\mathcal{E}\). We start with a lemma:

**Lemma 3.6.** With the assumptions of Section 3.1 let \(P = (p_1, p_2, p_3)\) and \(p = (p_1, \infty, p_2, p_3)\). Then

\[
a + b + c = -2\mathcal{A}_2 = -2\mathcal{A}(p_1, p_2, p_3) = -2\mathcal{A}.
\]

**Proof.** We have

\[
a + b + c = \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_4 + \mathcal{A}_4 - \mathcal{A}_1 = -2\mathcal{A}_2.
\]

We now prove

**Proposition 3.7.** The subset \(\mathcal{ET}_C\) of \(\mathcal{ET}\) comprising equivalence classes of equidistant triples of points lying on the same \(C\)-circle is on bijection with the points \((\pi/3, \pi/3, \pi/3)\) or \((-\pi/3, -\pi/3, -\pi/3)\) of the equidistant surface \(\mathcal{E}\).

**Proof.** We will show that three equidistant points lie on a \(C\)-circle if and only if the equidistant hypersurface reduces to the point \((\pi/3, \pi/3, \pi/3)\) or \((-\pi/3, -\pi/3, -\pi/3)\). We start by assuming that the three points lie on a \(C\)-circle, that is, \(\mathcal{A} = \pm \pi/2\). Here, \(\mathcal{A} = \mathcal{A}(p_1, p_2, p_3)\). Since from Lemma 3.6 we have

\[
a + b + c = -2\mathcal{A},
\]

Equation (3.1) becomes

\[
\cos a + \cos b - \cos(a + b) = \frac{3}{2}.
\]

This is written equivalently as

\[
2(\cos a + \cos b) - 2 \cos a \cos b + 2 \sin a \sin b = \cos^2 a + \sin^2 a + \cos^2 b + \sin^2 b + 1,
\]

or,

\[
(\cos a + \cos b)^2 - 2(\cos a + \cos b) + 1 + (\sin a - \sin b)^2 = 0,
\]

that is,

\[
(\cos a + \cos b - 1)^2 + (\sin a - \sin b)^2 = 0.
\]

Therefore we obtain,

\[
\cos a + \cos b = 1 \quad \text{and} \quad \sin a = \sin b.
\]

This gives

\[
a = b = c = \pm \frac{\pi}{3}.
\]

Conversely, suppose that three points lie on a \(C\)-circle and \(\arg(X_i) = \pm \frac{\pi}{3}\). Then \(p_i\) are equidistant. Indeed, from equation (3.2) we have

\[
|X_1| |X_2| = |X_1|^2 + |X_2|^2 - |X_1| - |X_2| + 1.
\]
Factoring out we may write equivalently
\[
\left( |X_1| - \frac{|X_2|}{2} - \frac{1}{2} \right)^2 + \frac{3}{4} (|X_2| - 1)^2 = 0.
\]
This gives $|X_1| = |X_2| = 1$ and therefore the points are equidistant. \qed

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