BREUIL-KISIN MODULES AND INTEGRAL $p$-ADIC HODGE THEORY

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WITH AN APPENDIX BY YOSHIYASU OZEKI

Abstract. We show that finite height Galois representations (finite height in terms of Breuil-Kisin modules) are potentially semi-stable, answering a question of Liu. Previously, Caruso also provided a proof of this question which unfortunately contains a gap; the gap is observed by Ozeki and is discussed in the appendix. We furthermore construct a simplified version of Liu’s $(\varphi, \hat{\gamma})$-modules, using Breuil-Kisin modules and Breuil-Kisin-Fargues modules with Galois actions, to classify integral semi-stable representations; the theory can be regarded as the algebraic avatar of the integral $p$-adic cohomology theories of Bhatt-Morrow-Scholze and Bhatt-Scholze. A key step in the paper is to use the idea of locally analytic vectors to define a monodromy operator on the overconvergent étale $(\varphi, \tau)$-modules.

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1. Introduction

1.1. Overview and main theorems. In this paper, we study various aspects of the theory of $(\varphi, \tau)$-modules, which is a useful tool in $p$-adic Hodge theory. Let us first fix some notations.

Notation 1.1.1. Let $p$ be a prime, and let $k$ be a perfect field of characteristic $p$. Let $W(k)$ be the ring of Witt vectors, and let $K_0 := W(k)[1/p]$. Let $K$ be a totally ramified finite extension of $K_0$, and let $e := [K : K_0]$. Fix an algebraic closure $\overline{K}$ of $K$, and set $G_K := \text{Gal}(\overline{K}/K)$. Let $C_p$ be the $p$-adic completion of $\overline{K}$. Let $v_p$ be the valuation on $C_p$ such that $v_p(p) = 1$.

In $p$-adic Hodge theory, we use various “linear algebra” tools to study $p$-adic representations of $G_K$. A key idea in $p$-adic Hodge theory is to first restrict the Galois representations to some subgroup of $G_K$. In this paper, we study the $(\varphi, \tau)$-modules, which are constructed using the subgroup $G_{K_{\infty}} := \text{Gal}(\overline{K}/K_{\infty})$. The $(\varphi, \tau)$-modules are analogues of the more classical $(\varphi, \Gamma)$-modules, which are constructed using the subgroup $G_{p_{\infty}} := \text{Gal}(\overline{K}/K_{p_{\infty}})$. Here let us only quickly mention that the “$\Gamma$” is the group $\text{Gal}(K_{p_{\infty}}/K)$, and the “$\tau$” is a topological generator of the group $\text{Gal}(L/K_{p_{\infty}})$ (cf. Notation 1.4.3).
Similar to the \((\varphi, \Gamma)\)-modules, the \((\varphi, \tau)\)-modules also classify all \(p\)-adic representations of \(G_K\). Although these two theories are equivalent, they each have their own technical advantage, and both are indispensable. The \((\varphi, \Gamma)\)-modules are perhaps “easier” in the sense that both the \(\varphi\) and \(\Gamma\)-actions are defined over a same ring; whereas the \(\tau\)-action in \((\varphi, \tau)\)-modules can only be defined over a much bigger ring. However, the \(\varphi\)-action in \((\varphi, \tau)\)-modules stay tractable even when \(K\) has ramification; in contrast, the \(\varphi\)-action in \((\varphi, \Gamma)\)-modules become quite implicit when \(K\) has ramification. This dichotomy becomes much more substantial when we consider semi-stable Galois representations: in this situation, there exits very well-behaved Breuil-Kisin modules (also called Kisin modules, or \((\varphi, \hat{G})\)-modules in different contexts) which is a special type of \((\varphi, \tau)\)-modules; in contrast, such special type \((\varphi, \Gamma)\)-modules (called Wach modules) exist only if we consider crystalline representations and if \(K \subset \bigcup_{n \geq 1} K_0(\mu_n)\) (e.g., when \(K = K_0\) is unramified). To save space, we refer the readers to the introduction in [GP] for some discussion and comparison of the applications of these two theories in different contexts.

A main goal of this paper is to study the finite \(E(u)\)-height \((\varphi, \tau)\)-modules. Such modules first appeared in the study of \(p\)-divisible groups by Breuil (cf. e.g., [Bre98, Bre99]) and later are constructed for all semi-stable representations by Kisin (cf. [Kis06]). Such modules (and their variants) also naturally appear in non-semi-stable cases, and are useful to study the structure of potentially semi-stable deformation rings, cf. e.g. [LLHL18] for a recent progress. We also mention that in recent breakthroughs in integral \(p\)-adic cohomology theories (cf. [BMS18, BMS19, BS]), these finite \(E(u)\)-height modules play a major role and are realized as cohomology spaces. Furthermore, there is now a growing interest to study the moduli stack of these modules (cf. e.g., [PR09], [EGb], [EGA]); such moduli stacks are expected to be useful in modularity lifting problems and in the \(p\)-adic Langlands program. A main result of this paper is that we can characterize and classify all such finite \(E(u)\)-height \((\varphi, \tau)\)-modules, cf. Thm. 1.1.2 in the following.

Let us be more precise now. First, recall that the \(\varphi\)-action of a \((\varphi, \tau)\)-module is defined over the ring

\[
B_{K_{\infty}} := \{ \sum_{i=-\infty}^{+\infty} a_i u^i : a_i \in K_0, \lim_{i \to -\infty} v_p(a_i) = +\infty, \text{ and } \inf_{i \in \mathbb{Z}} v_p(a_i) > -\infty \}.
\]

The \(\tau\)-action is defined over a bigger ring \(\tilde{B}_L\) which we do not recall here, see §3. Indeed, roughly speaking, a \((\varphi, \tau)\)-module \(D\) is a finite free \(B_{K_{\infty}}\)-module equipped with certain commuting maps \(\varphi : D \to D\) and \(\tau : \tilde{B}_L \otimes B_{K_{\infty}} D \to \tilde{B}_L \otimes B_{K_{\infty}} D\). A \((\varphi, \tau)\)-module is called of finite \(E(u)\)-height (cf. Def. 5.4.3) if there exists some sort of “Breuil-Kisin lattice” inside it, i.e., if there exists some basis \(\vec{e}\) of the \((\varphi, \tau)\)-module such that \(\varphi(\vec{e}) = \vec{e} A\) with \(A \in \text{Mat}(\mathfrak{S})\) where

\[
\mathfrak{S} := W(k)[u] \subset B_{K_{\infty}},
\]

and such that there exists some \(B \in \text{Mat}(\mathfrak{S})\) satisfying \(AB = E(u)^h \cdot \text{Id}\) with \(h \in \mathbb{Z}^{\geq 0}\). Note that this definition has nothing to do with the \(\tau\)-action. One of the main goals of this paper is to prove the following theorem, which in particular answers a question of Tong Liu in [Liu10].

**Theorem 1.1.2.** (= Thm. 6.2.3) Let \(K^{ur} \subset \overline{K}\) be the maximal unramified extension of \(K\), let

\[
m := 1 + \max\{ i \geq 1, \mu_i \in K^{ur} \},
\]

and let \(K_m = K(\pi_{m-1})\). Let \(D\) be an étale \((\varphi, \tau)\)-module, and let \(V\) be the associated \(G_K\)-representation. Then \(D\) is of finite \(E(u)\)-height if and only if \(V|_{G_{K_m}}\) can be extended to a semi-stable \(G_K\)-representation with non-negative Hodge-Tate weights. In particular, if \(D\) is of finite \(E(u)\)-height, then \(V\) is potentially semi-stable.

**Remark 1.1.3.** The notion “of finite \(E(u)\)-height” indeed depends on the choices \(\{\pi_n\}_{n \geq 0}\). If \(V\) is of finite height with respect to all such choices, then we can use Thm. 1.1.2 to show that \(V\) is semi-stable; this result is due to Gee, cf. Thm. 6.2.5.

Using Thm. 1.1.2, and using earlier work of Liu and Ozeki, we can construct a simplified version of Liu’s \((\varphi, \hat{G})\)-modules. As the full definition involves too many notations, we only give a brief description of (a variant of) the simplified \((\varphi, \hat{G})\)-modules; cf. §7.2 for more details. Let \(R := \varprojlim \mathcal{O}_\pi/p\mathcal{O}_\pi\) where the transition maps are \(x \mapsto x^p\); \(R\) is a complete valuation ring. Let \(m_R\) be its maximal ideal, and let \(W(R)\) be the ring of Witt vectors (= \(A_{inf}\) in e.g. [BMS18]).
Definition 1.1.4. Let $\text{Mod}_{E,W(R)}^{\varphi, G_K}$ be the category consisting of triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, G_K)$ where

1. $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a Breuil-Kisin module;
2. $G_K$ is a continuous $\varphi_{\mathfrak{M}}$-commuting $W(R)$-semi-linear $G_K$-action on the Breuil-Kisin-Fargues module $\mathfrak{M} := W(R)\otimes_{E} \mathfrak{M}$, such that
   a. $\mathfrak{M} \subset \mathfrak{M}^{\text{Gal}(\overline{K}/K)}$ via the embedding $\mathfrak{M} \hookrightarrow \mathfrak{M};$
   b. $\mathfrak{M}/u\mathfrak{M} \subset (\mathfrak{M}/W(m_R)\mathfrak{M})^{G_K}$ via the embedding $\mathfrak{M}/u\mathfrak{M} \hookrightarrow \mathfrak{M}/W(m_R)\mathfrak{M}.$

Theorem 1.1.5. (= Thm. 7.2.12) The category $\text{Mod}_{E,W(R)}^{\varphi, G_K}$ is anti-equivalent to the category of $G_K$-stable $\mathbb{Z}_p$-lattices in semi-stable representations of $G_K$ with non-negative Hodge-Tate weights.

Remark 1.1.6. A key feature of Def. 1.1.4 is that the ring $\mathbb{R}$ (cf. §5.3), which is rather implicit yet extremely important in Liu’s theory, disappears in our simplified theory; this in particular makes it much easier to identify semi-stable representations. Another interesting observation is that the cohomology theories in [BMS18, BMS19, BS] “realize” the objects in Def. 1.1.4; see Rem. 7.2.14. In particular, the content of Def. 1.1.4(2)(b) is precisely a semi-stable comparison theorem.

Remark 1.1.7. In [Wac96], Wach studied some “finite height” $(\varphi, \Gamma)$-modules; it is shown in [Wac96, A.5] that these “finite height” $(\varphi, \Gamma)$-modules give rise to de Rham (indeed, potentially crystalline) Galois representations if there is some additional condition on the Lie algebra operator associated to the $\Gamma$-action. Note however that the “finite height” condition in loc. cit. is of a different type than the one in our paper. Indeed, the analogue of our “finite $E(u)$-height” condition in the setting of $(\varphi, \Gamma)$-modules should be the “finite $q$-height” condition, where $q := (1 + T)^{p-1}$ is a polynomial; cf. e.g., [Ber04, KR09]. In fact, we can use similar ideas in the current paper to study “finite $q$-height” $(\varphi, \Gamma)$-modules; in parallel, we can also study the “finite height” $(\varphi, \tau)$-modules (without $E(u)$ in the play) similar as in [Wac96]. All these will be discussed in another paper.

Remark 1.1.8. Caruso gave a proof of Thm. 1.1.2 in [Car13], which unfortunately contains a serious gap. The gap is first discovered by Yoshiyasu Ozeki, and is discussed in Appendix A. Indeed, the gap arises when Caruso tries to define a monodromy operator on the finite $E(u)$-height $(\varphi, \tau)$-modules, using the “truncated log” (cf. [Car13, §3.2.2]); very roughly speaking, Caruso tries to define a log-operator using $p$-adic approximation technique (using $p$-adic topology on various rings).

1.2. Strategy of proof. What we propose in the current paper is that one should use the Fréchet topologies (e.g., on various “Robba-style” rings) instead of the $p$-adic topologies mentioned in Rem. 1.1.8; and one can in fact get a monodromy operator directly (no approximation needed) using techniques of locally analytic vectors. Indeed, our monodromy operator will be defined for all (rigid-overconvergent, cf. Thm. 1.2.1 below) $(\varphi, \tau)$-modules (not just finite $E(u)$-height ones). Before we state the theorem concerning the monodromy operator, let us recall the overconvergence result of $(\varphi, \tau)$-modules.

Theorem 1.2.1 ([GL, GP]). The $(\varphi, \tau)$-modules (attached to any $p$-adic representations of $G_K$) are overconvergent. That is, (roughly speaking), the $\varphi$-action can be defined over the sub-ring:

\[ B_{K_{\infty}}^1 := \{ \sum_{i=0}^{+\infty} a_i u_i \in B_{K_{\infty}} \mid \lim_{i \to +\infty} (v_p(a_i) + i\alpha) = +\infty \text{ for some } \alpha > 0 \}; \]

also, the $\tau$-action can be defined over some sub-ring $\overline{B}_{L} \subset \overline{B}_{L}$.

Remark 1.2.2. (1) Thm. 1.2.1 is first conjectured by Caruso in [Car13] (as an analogue of the classical overconvergence theorem for $(\varphi, \Gamma)$-modules by Cherbonnier-Colmez in [CC98]). A first proof (which only works for $K/\mathbb{Q}_p$ a finite extension) is given in a joint work with Liu [GL], using a certain “crystalline approximation” technique; later a second proof (which works for all $K$) is given in a joint work with Poyeton in [GP], using the idea of locally analytic vectors.

(2) Let us mention that it is the second proof in [GP] that will be useful in the current paper. Not only because it works for all $K$ (which is a minor issue), but also more importantly, the idea of locally analytic vectors will be very critically used in the current paper to define the monodromy operator.
Let us introduce the following Robba ring (which contains $B_{K_{\infty}}^\dagger$),

\begin{equation}
B_{\text{rig},K_{\infty}}^\dagger := \{ f(u) = \sum_{i=-\infty}^{+\infty} a_i u^i, a_i \in K_0, f(u) \text{ converges for all } u \in \overline{K} \text{ with } 0 < v_{p}(u) < \rho(f) \text{ for some } \rho(f) > 0 \}.
\end{equation}

Let $V$ be a $p$-adic Galois representation of $G_K$, and let $D_{K_{\infty}}^\dagger(V)$ be the overconvergent $(\varphi, \tau)$-module associated to $V$ by Thm. 1.2.1. Define

$$D_{\text{rig},K_{\infty}}^\dagger(V) := B_{\text{rig},K_{\infty}}^\dagger \otimes B_{K_{\infty}}^\dagger D_{K_{\infty}}^\dagger(V),$$

which we call the rigid-overconvergent $(\varphi, \tau)$-module associated to $V$; as we will see in the following, it is the natural space where the monodromy operator lives in.

**Theorem 1.2.3.** (= Thm. 4.2.1) Let $\nabla_{\tau} := (\log \tau^n)/p^n$ for $n \gg 0$ be the Lie-algebra operator with respect to the $\tau$-action, and define $N_{\tau} := \Sigma_{t}^{1}$ where $t$ is a certain ("normalizing") element (cf. §4). Then

$$N_{\tau}(D_{\text{rig},K_{\infty}}^\dagger(V)) \subset D_{\text{rig},K_{\infty}}^\dagger(V).$$

Namely, $N_{\tau}$ is a well-defined monodromy operator on $D_{\text{rig},K_{\infty}}^\dagger(V)$.

**Remark 1.2.4.** (1) In comparison, if we use $D_{\text{rig},K_{\infty}}^\dagger(V)$ (denoted as "$D_{\text{rig},K}^\dagger(V)$" in [Ber02]) to denote the rigid-overconvergent $(\varphi, \Gamma)$-module associated to $V$ (which exists by [CC98]), then one can easily define a monodromy operator

$$\nabla_V : D_{\text{rig},K_{\infty}}^\dagger(V) \rightarrow D_{\text{rig},K_{\infty}}^\dagger(V)$$

as in [Ber02, §5.1]. Here "$\nabla_V$" (notation of [Ber02]) is precisely the Lie-algebra operator associated to $\Gamma$-action (i.e., it equals to $\frac{\log g}{\log \chi_{p}(g)}$ for $g \in \text{Gal}(L/K_{\infty})$ close enough to the identity where $\chi_{p}$ is the cyclotomic character.)

(2) The difficulty in defining $N_{\tau}$ for $(\varphi, \tau)$-modules is that $\tau$ (hence $\nabla_{\tau}$) does not act on $D_{\text{rig},K_{\infty}}^\dagger(V)$ itself (whereas $\Gamma$ acts directly on $D_{\text{rig},K_{\infty}}^\dagger(V)$); the action is defined only when we base change $D_{\text{rig},K_{\infty}}^\dagger(V)$ over a much bigger ring "$B_{\text{rig},L}^\dagger$" (cf. Def. 2.4.2). Fortunately, after dividing $\nabla_{\tau}$ by $\text{pt}$, and using ideas of locally analytic vectors, one gets back to the level of $D_{\text{rig},K_{\infty}}^\dagger(V)$.

Now, to prove Thm. 1.1.2, via results of Kisin (and some consideration of locally analytic vectors), it suffices to show the following "monodromy descent" result, which we achieve via a "Frobenius regularization" technique.

**Proposition 1.2.5.** (= Prop. 6.1.1) Given a finite $E(u)$-height $(\varphi, \tau)$-module, and let $\mathcal{M}$ denote the finite height Breuil-Kisin lattice inside it. Then

$$N_{\tau}(\mathcal{M}) \subset \mathcal{O} \otimes \mathcal{M}.$$ 

Here $\mathcal{O} \subset B_{\text{rig},K_{\infty}}^\dagger$ is the subring consisting of $f(u)$ that converges for all $u \in \overline{K}$ such that $0 < v_{p}(u) \leq +\infty$.

**Remark 1.2.6.** Indeed, the "road map" of our proof of Thm. 1.1.2 is roughly the same as in [Car13]. Namely, one first defines a certain monodromy operator, then one shows that (in the finite $E(u)$-height case) the operator can be indeed defined over the smaller ring $\mathcal{O}$. However, even though Thm. 1.2.3 fixes Caruso’s gap (or more precisely, provides a correct alternative) in defining the monodromy operator, the technical details in the latter half of our argument (Prop. 1.2.5, proved in §6.1) are also completely different from that of Caruso. Indeed, Caruso’s argument uses the rings of the rings used in $(\varphi, \Gamma)$-module theory, which have been substantially studied since their introduction in e.g., [Ber02]. Indeed, it seems that our argument is much easier and more natural.
1.2.6

Car13

BC16

Bre99

1.1.2

are proved in [K/K] Car13

1.2.3

A

Car13

GP

Car13

independently.

Let $\{\exists a sequence \forall c \}$ if there exists an open subgroup $H$ such that the representation is potentially semi-stable. In留守于四算子技术 to descend the monodromy operator to $O$; this implies that the attached representation is potentially semi-stable. In §7, we construct a simplified version of Liu’s $(\varphi, G)$-modules.

1.3. Structure of the paper. In §2, we review many period rings in $p$-adic Hodge theory; in particular we compute locally analytic vectors in some rings. In §3, we review the theory of $(\varphi, \tau)$-modules and the overconvergence theorem. Most results in §2 and §3 are proved in [GP] already. In §4, we define the monodromy operator on the rigid-overconvergent $(\varphi, \tau)$-modules. In §5, we review several theories used in the study of semi-stable Galois representation; in particular, we show that the monodromy operator defined in Kisin’s theory (for semi-stable representations) coincide with ours in §4. In §6, when the $(\varphi, \tau)$-module is of finite $E(w)$-height, we use a Frobenius regularization technique to descend the monodromy operator to $O$; this is known that the gap in [Bre99, §2]. We refer to Rem. 7.3.1 for more comments regarding the relation between the current paper and [Car13].

1.4. Some notations and conventions.

Notation 1.4.1. Recall that in Notation 1.1.1, we already defined:

$$K_\infty := \bigcup_{n=1}^{\infty} K(\pi_n), \quad K_p^\infty := \bigcup_{n=1}^{\infty} K(\mu_n), \quad L := \bigcup_{n=1}^{\infty} K(\pi_n, \mu_n).$$

Let

$$G_\infty := \text{Gal}(\overline{K}/K_\infty), \quad G_p^\infty := \text{Gal}(\overline{K}/K_p^\infty), \quad G_L := \text{Gal}(\overline{K}/L), \quad \hat{G} := \text{Gal}(L/K).$$

When $Y$ is a ring with a $G_K$-action, $X \subset \overline{K}$ is a subfield, we use $Y_X$ to denote the Gal$(\overline{K}/X)$-invariants of $Y$; we will use the cases when $X = L, K_\infty$.

1.4.2. Locally analytic vectors. Let us very quickly recall the theory of locally analytic vectors, see [BC16, §2.1] and [Ber16, §2] for more details. Indeed, all the explicit calculations of locally analytic vectors used in this paper are already carried out in [GP], hence the reader can refer to loc. cit. for more details.

Recall that a $Q_p$-Banach space $W$ is a $Q_p$-vector space with a complete non-Archimedean norm $\| \cdot \|$ such that $\|aw\| = \|a\|_p \|w\|, \forall a \in Q_p, w \in W$, where $\|a\|_p$ is the $p$-adic norm on $Q_p$. Recall the multi-index notations: if $c = (c_1, \ldots, c_d)$ and $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ (here $\mathbb{N} = \mathbb{Z}^0$), then we let $c^k = c_1^{k_1} \cdots c_d^{k_d}$.

Let $G$ be a $p$-adic Lie group, and let $(W, \| \cdot \|)$ be a $Q_p$-Banach representation of $G$. Let $H$ be an open subgroup of $G$ such that there exist coordinates $c_1, \ldots, c_d : H \to \mathbb{Z}_p$ giving rise to an analytic bijection $c : H \to \mathbb{Z}_p^d$. We say that an element $w \in W$ is an $H$-analytic vector if there exists a sequence $\{w_k\}_{k \in \mathbb{N}^d}$ with $w_k \to 0$ in $W$; such that

$$g(w) = \sum_{k \in \mathbb{N}^d} c(g)^k w_k, \quad \forall g \in H.$$ 

Let $W^{H-\text{an}}$ denote the space of $H$-analytic vectors. We say that a vector $w \in W$ is locally analytic if there exists an open subgroup $H$ as above such that $w \in W^{H-\text{an}}$. Let $W^{G-\text{la}}$ denote the space of such vectors. We have $W^{G-\text{la}} = \bigcup_H W^{H-\text{an}}$ where $H$ runs through open subgroups of $G$. We can naturally extend these definitions to the case when $W$ is a Fréchet- or LF- representation of $G$, and use $W^{G-pa}$ to denote the pro-analytic vectors, cf. [Ber16, §2]. We remark that we can naturally topologize $W^{H-\text{an}}$ (resp. $W^{G-\text{la}}$, resp. $W^{G-pa}$) to be a Banach (resp. LB, resp. LF) space.
Notation 1.4.3. Let $\hat{G} = \text{Gal}(L/K)$ be as in Notation 1.4.1, which is a $p$-adic Lie group of dimension 2. We recall the structure of this group in the following.

(1) Recall that:
- if $K_\infty \cap K_{p^\infty} = K$, then $\text{Gal}(L/K_{p^\infty})$ and $\text{Gal}(L/K_\infty)$ topologically generate $\hat{G}$ (cf. [Liu08, Lem. 5.1.2]);
- if $K_\infty \cap K_{p^\infty} \supseteq K$, then necessarily $p = 2$, and $\text{Gal}(L/K_{p^\infty})$ and $\text{Gal}(L/K_\infty)$ topologically generate an open subgroup (denoted as $G'$) of $\hat{G}$ of index 2 (cf. [Liu10, Prop. 4.1.5]).

(2) Note that:
- $\text{Gal}(L/K_{p^\infty}) \simeq \mathbb{Z}_p$, and let $\tau \in \text{Gal}(L/K_{p^\infty})$ be the topological generator such that $\tau(\pi_i) = \pi_i \mu_i, \forall i \geq 1$;
- $\text{Gal}(L/K_\infty) \subset (\text{Gal}(K_{p^\infty}/K) \subset \mathbb{Z}_p^\times)$ is not necessarily pro-cyclic when $p = 2$. If $\Delta \subset \text{Gal}(L/K_\infty)$ is the finite torsion subgroup, then $\text{Gal}(L/K_\infty)/\Delta$ is pro-cyclic.

Notation 1.4.4. We set up some notations with respect to representations of $\hat{G}$.

(1) Given a $\hat{G}$-representation $W$, we use
\[ W^{\tau=1}, \quad W^{\gamma=1} \]
to mean
\[ W^{\text{Gal}(L/K_{p^\infty})=1}, \quad W^{\text{Gal}(L/K_{\infty})=1}. \]
And we use
\[ W^{\tau-\text{la}}, \quad W^{\gamma-\text{la}} \]
to mean
\[ W^{\text{Gal}(L/K_{p^\infty})-\text{la}}, \quad W^{\text{Gal}(L/K_{\infty})-\text{la}}. \]

(2) Let $W^{\tau-\text{la}, \gamma=1} := W^{\tau-\text{la}} \cap W^{\gamma=1}$, then by [GP, Lem. 3.2.4]
\[ W^{\tau-\text{la}, \gamma=1} \subset W^{G-\text{la}}. \]

Remark 1.4.5. Note that we never define $\gamma$ to be an element of $\text{Gal}(L/K_{\infty})$; although when $p > 2$ (or in general, when $\text{Gal}(L/K_{\infty})$ is pro-cyclic), we could have defined it as a topological generator of $\text{Gal}(L/K_{\infty})$. In particular, although “$\gamma = 1$” might be slightly ambiguous (but only when $p = 2$), we use the notation for brevity.

1.4.6. Valuations. A non-Archimedean valuation of a ring $A$ is a map $v : A \to \mathbb{R} \cup \{+\infty\}$ such that $v(x) = +\infty \Leftrightarrow x = 0$ and $v(x + y) \geq \inf\{v(x), v(y)\}$. It is called sub-multiplicative (resp. multiplicative) if $v(xy) \geq v(x) + v(y)$ (resp. $v(xy) = v(x) + v(y)$), for all $x, y$. All the valuations in this paper are sub-multiplicative (some are multiplicative). Given a matrix $T = (t_{i,j})_{i,j}$ over $A$, let $v(T) := \min\{v(t_{i,j})\}$.

1.4.7. Some other notations. Throughout this paper, we reserve $\varphi$ to denote Frobenius operator. We sometimes add subscripts to indicate on which object Frobenius is defined. For example, $\varphi_{2\mathfrak{m}}$ is the Frobenius defined on $\mathfrak{m}$. We always drop these subscripts if no confusion arises. Given a homomorphism of rings $\varphi : A \to A$ and given an $A$-module $M$, denote $\varphi^* M := A \otimes_{\varphi, A} M$. We use Mat$(A)$ to denote the set of matrices with entries in $A$ (the size of the matrix is always obvious from context). Let $\gamma_i(x) := \frac{\varphi_i^k x}{k}$ be the usual divided power.

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2. RINGS AND LOCALLY ANALYTIC VECTORS

In this section, we review some period rings in $p$-adic Hodge theory. In particular, we compute the locally analytic vectors in some rings. In §2.1, we review some basic period rings; in §2.2 and §2.3, we define the rings $\tilde{B}^l$, $B^l$ and study their $G_{\infty}$-invariants; in §2.4, we study the relation of these rings via locally analytic vectors; finally in §2.5, we study a locally analytic element $t$, which plays a useful role in the definition of our monodromy operator.

2.1. Some basic period rings. Let $\tilde{E}^+ := \lim\inf \mathcal{O}_E/p\mathcal{O}_E$ where the transition maps are $x \mapsto x^p$, let $\tilde{E} := Fr\tilde{E}^+$. An element of $\tilde{E}$ can be uniquely represented by $(x(n))_{n \geq 0}$ where $x(n) \in C_p$ and $(x(n+1))^p = (x(n))$; let $v_E$ be the usual valuation where $v_E(x) := v_p(x(0))$. Let

$$\tilde{A}^+ := \tilde{W}(\tilde{E}^+), \quad \tilde{A} := \tilde{W}(\tilde{E}), \quad \tilde{B}^+ := \tilde{A}^+[1/p], \quad \tilde{B} := \tilde{A}[1/p],$$

where $W(\cdot)$ means the ring of Witt vectors. There is a natural Frobenius operator $x \mapsto x^p$ on $\tilde{E}$, which induces natural Frobenius operators (always denoted by $\varphi$) on all the rings defined above (and below); there are also natural $G_{\infty}$-actions on the rings defined above induced from that on $E$.

Let $[\pi] \in \tilde{A}^+$ be the Teichmüller lift of $\pi$. Let $A_{K_{\infty}}^+ := W(k[u])$ with Frobenius $\varphi$ extending the arithmetic Frobenius on $W(k)$ and $\varphi(u) = u^p$. There is a $W(k)$-linear Frobenius-equivariant embedding $A_{K_{\infty}}^+ \hookrightarrow \tilde{A}^+$ via $u \mapsto [\pi]$. Let $A_{K_{\infty}}$ be the $p$-adic completion of $A_{K_{\infty}}^+[1/u]$. Our fixed embedding $A_{K_{\infty}} \hookrightarrow \tilde{A}^+$ determined by $\pi$ uniquely extends to a $\varphi$-equivariant embedding $A_{K_{\infty}} \hookrightarrow \tilde{A}$, and we identify $A_{K_{\infty}}$ with its image in $\tilde{A}$. We note that $A_{K_{\infty}}$ is a complete discrete valuation ring with uniformizer $p$ and residue field $E_{K_{\infty}}$.

Let $B_{K_{\infty}} := A_{K_{\infty}}[1/p]$ (which is precisely the ring in (1.1.1)). Let $B$ be the $p$-adic completion of the maximal unramified extension of $B_{K_{\infty}}$ inside $\tilde{B}$, and let $A \subset B$ be the ring of integers. Let $A^+ := \tilde{A}^+ \cap A$. Then we have:

$$(A)^{G_{\infty}} = A_{K_{\infty}}, \quad (B)^{G_{\infty}} = B_{K_{\infty}}, \quad (A^+)^{!G_{\infty}} = A_{K_{\infty}}^+.$$
Indeed, these equations are used as definitions in [Proof. (called Frobenius)] ϕ
Throughout the paper, Convention 2.2.8.
If we can determine the explicit structure of these rings for
Lemma 2.2.9.
We have
Remark 2.2.7. For our purposes (indeed, also in other literature concerning these rings), it is only
necessary to study (the explicit structure of) these rings when
Furthermore, for any interval I such that \( \tilde{A}^I \) and \( \tilde{B}^I \) are defined, there is a natural bijection
(called Frobenius) \( \varphi : \tilde{A}^I \cong \tilde{A}^{pl} \) which is valuation-preserving. Hence in practice, it would suffice
if we can determine the explicit structure of these rings for
I_{\ell} := \{ [r_\ell, r_k], [r_\ell, +\infty], [0, r_k], [0, +\infty] \}, \text{ with } \ell \leq k \in \mathbb{Z}^\geq 0,
where \( r_n := (p-1)p^{n-1} \). The cases for \( I \) a general closed interval can be deduced using Frobenius
operation; the cases for \( I = [r, +\infty) \) can be deduced by taking Fréchet completion.
Convention 2.2.8. Throughout the paper, all the intervals \( I \) (over “\( \tilde{B} \)-rings,” “\( B \)-rings,” “\( D \)-modules”, etc.) satisfy
If they are not closed, then they are of the form \([0, +\infty) \) or \([r, +\infty) \). \( I \) is never allowed to be \([0, 0] \)
(or “\([+\infty, +\infty]\)”).
Lemma 2.2.9. We have
\[
\tilde{A}^{[0, r_k]} = \tilde{A}^+ \left\{ \frac{u^{ep^k}}{p} \right\},
\]
\[
\tilde{A}^{[r_\ell, +\infty]} = \tilde{A}^+ \left\{ \frac{p}{u^{ep^\ell}} \right\},
\]
\[
\tilde{A}^{[r_\ell, r_k]} = \tilde{A}^+ \left\{ \frac{p}{u^{ep^{k-1}}}, \frac{u^{ep^k}}{p} \right\}.
\]
Proof. Indeed, these equations are used as definitions in [GP, Def. 2.1.1]; these definitions are equivalent to our current Def. 2.2.5 by Lem. 2.2.6. See [GP, §2.1] for more details.

Proposition 2.2.10. [GP, Prop. 2.1.14] Recall that the subscripts \( K_\infty \) signifies \( G_\infty \)-invariants.
We have
\[
\tilde{B}^{[0, r_k]}_{K_\infty} = \tilde{A}^+_{K_\infty} \left\{ \frac{u^{ep^k}}{p} \right\}, \left\lfloor \frac{1}{p} \right\rfloor
\]
\[
\tilde{B}^{[r_\ell, +\infty]}_{K_\infty} = \tilde{A}^+_{K_\infty} \left\{ \frac{p}{u^{ep^\ell}} \right\}, \left\lfloor \frac{1}{p} \right\rfloor
\]
\[
\tilde{B}^{[r_\ell, r_k]}_{K_\infty} = \tilde{A}^+_{K_\infty} \left\{ \frac{p}{u^{ep^{k-1}}}, \frac{u^{ep^k}}{p} \right\}, \left\lfloor \frac{1}{p} \right\rfloor
\]
2.2.11. **Embedding into \( B_{\text{dR}}^+ \).** Let \( \theta : \tilde{A}^+ \to \mathcal{O}_{C_p} \) be the usual map where

\[
\theta(\sum_{n \geq 0} p^n [x_n]) = \sum_{n \geq 0} p^n x_n(0);
\]

it extends to \( \theta : \tilde{B}^+ \to C_p \). The kernels of both these maps are generated by \( E(u) \). For \( i \geq 1 \), equip \( \overline{B}^+/((E(u))^i) \) with the complete \( p \)-adic topology using the image of \( \tilde{A}^+ \) as ring of integers. Let \( B_{\text{dR}}^+ := \varprojlim_{i \geq 1} \overline{B}^+/((E(u))^i) \), and equip it with the Fréchet topology (also called the canonical topology or the natural topology on \( B_{\text{dR}}^+ \), which is weaker than the \( (E(u)) \)-adic topology). By construction, \( \theta \) extends continuously to

\[
\theta : B_{\text{dR}}^+ \to C_p.
\]

If \( x = \sum_{n \geq 0} p^n [x_n] \in \tilde{A}^{[r_0, +\infty]} \), then by [Col08, Lem. 5.18], the series \( \sum_{n \geq 0} p^n [x_n] \) also converges in \( B_{\text{dR}}^+ \) (with respect to the Fréchet topology). Indeed, this induces a **continuous** embedding

\[
t_0 : \tilde{A}^{[r_0, +\infty]} \hookrightarrow B_{\text{dR}}^+,
\]

where we use \( W^{[r_0, r_1]} \)-topology for \( \tilde{A}^{[r_0, +\infty]} \); see e.g., [Ber10, Prop. 28.1] for the proof of continuity. Hence \( t_0 \) induces a continuous embedding

\[
t_0 : \tilde{B}^{[r_0, r_1]} \hookrightarrow B_{\text{dR}}^+,
\]

which then induces the continuous embeddings

\[
(2.2.1) \qquad t_n := t_0 \circ \varphi^{-n} : \tilde{B}^I \hookrightarrow B_{\text{dR}}^+, \quad \forall I \subset [0, +\infty] \text{ such that } r_n \in I.
\]

2.2.12. **Embedding into \( B_{\text{cris}}^+ \).** Let

\[
A_{\text{cris}} := \{ \sum_{n=0}^{\infty} a_n \frac{E(u)^n}{n!} : a_n \in \tilde{A}^+, a_n \to 0 \text{ p-adically} \} \subset B_{\text{dR}}^+;
\]

and let \( B_{\text{cris}}^+ := A_{\text{cris}}[1/p] \). It is easy to check that the embedding

\[
(B_{\text{cris}}^+, \text{p-adic topology}) \hookrightarrow (B_{\text{dR}}^+, \text{Fréchet topology})
\]

is continuous. (As a side note, the embedding is not continuous if we use \( (E(u)) \)-adic topology on \( B_{\text{dR}}^+ \); for example consider the sequence \( \{p^n\}_{n \geq 1} \). Recall that we have

\[
\tilde{B}^{[0, r_1]} = \tilde{A}^+ \{ \frac{E(u)^n}{p^n} \} \{ \frac{1}{p^n} \} = \tilde{A}^+ \{ \frac{E(u)^n}{p^n} \} \{ \frac{1}{p^n} \},
\]

and hence the image of the embedding

\[
t_0 : \tilde{B}^{[0, r_1]} \hookrightarrow B_{\text{dR}}^+
\]

lands inside \( B_{\text{cris}}^+ \) (it is critical to use \( t_0 \) and not \( t_1 \) here); indeed, the image of \( \sum_{n \geq 0} a_n \frac{E(u)^n}{p^n} \) is precisely \( \sum_{n \geq 0} a_n (\frac{E(u)^n}{p^n})^n \). Furthermore, the induced embedding

\[
(2.2.2) \qquad t_0 : \tilde{B}^{[0, r_1]} \hookrightarrow B_{\text{cris}}^+
\]

is continuous because by Lem. 2.2.6, the \( W^{[0, r_1]} = W^{[r_1, r_1]} \)-topology on \( \tilde{B}^{[0, r_1]} \) is equivalent to the \( p \)-adic topology with respect to the \( p \)-adic integers \( \tilde{A}^{[0, r_1]} \), and \( t_0 \) maps \( \tilde{A}^{[0, r_1]} \) to \( A_{\text{cris}} \). The embedding (2.2.2) induces similar embeddings for \( \tilde{B}^I \) with \( I \supset [0, r_1] \); in particular, it induces the continuous embedding

\[
(2.2.3) \qquad t_0 : \tilde{B}^{[0, +\infty]} \hookrightarrow B_{\text{cris}}^+.
\]
2.3. The rings $\mathcal{B}^I$ and their $G_\infty$-invariants.

**Definition 2.3.1.** Let $r \in \mathbb{Z}^\geq 0[1/p]$.

1. Let 
   
   $$A^{[r, +\infty]} := A \cap \overline{\mathcal{A}}^{[r, +\infty]}, \quad B^{[r, +\infty]} := B \cap \overline{\mathcal{B}}^{[r, +\infty]}.$$ 

2. Suppose $[r, s) \subset [r, +\infty)$ is a non-empty closed interval such that $s \neq 0$. Let $\mathcal{B}^{[r, s]}$ be the closure of $\mathcal{B}^{[r, +\infty]}$ in $\overline{\mathcal{B}}^{[r, s]}$ with respect to $W^{[r, s]}$. Let $A^{[r, s]} := B^{[r, s]} \cap \overline{\mathcal{A}}^{[r, s]}$, which is the ring of integers in $B^{[r, s]}$.

3. Let 
   
   $$B^{[r, +\infty]} := \cap_{n \geq 0} B^{[r, s_n]}$$

   where $s_n \in \mathbb{Z}^\geq 0[1/p]$ is any sequence increasing to $+\infty$.

**Definition 2.3.2.**

1. For $r \in \mathbb{Z}^\geq 0[1/p]$, let $\mathcal{A}^{[r, +\infty]}(K_0)$ be the ring consisting of infinite series $f = \sum_{k \in \mathbb{Z}} a_k T^k$ where $a_k \in W(k)$ such that $f$ is a holomorphic function on the annulus defined by
   
   $$v_p(T) \in (0, \frac{p-1}{ep} \cdot \frac{1}{r}].$$

   (Note that when $r = 0$, it implies that $a_k = 0, \forall k < 0$). Let $\mathcal{B}^{[r, +\infty]}(K_0) := \mathcal{A}^{[r, +\infty]}(K_0)[1/p]$.

2. Suppose $f = \sum_{k \in \mathbb{Z}} a_k T^k \in \mathcal{B}^{[r, +\infty]}(K_0)$.
   
   (a) When $s \geq r, s > 0$, let
   
   $$W^{[s, s]}(f) := \inf_{k \in \mathbb{Z}} \{v_p(a_k) + \frac{p-1}{ps} \cdot \frac{k}{e}\}.$$ 

   (b) For $I \subset [r, +\infty)$ a non-empty closed interval, let
   
   $$W^I(f) := \inf_{a \in I, a \neq 0} W^{[a, a]}(f).$$

3. For $r \leq s \in \mathbb{Z}^\geq 0[1/p], s \neq 0$, let $\mathcal{B}^{[r, s]}(K_0)$ be the completion of $\mathcal{B}^{[r, +\infty]}(K_0)$ with respect to $W^{[r, s]}$. Let $\mathcal{A}^{[r, s]}(K_0)$ be the ring of integers in $\mathcal{B}^{[r, s]}(K_0)$ with respect to $W^{[r, s]}$.

**Lemma 2.3.3.**

1. For $r > 0$, $\mathcal{B}^{[r, +\infty]}(K_0)$ is complete with respect to $W^{[r, r]}$, and $\mathcal{A}^{[r, +\infty]}(K_0)$ is the ring of integers with respect to this valuation.

2. For $s > 0$, we have $W^{[0, s]}(x) = W^{[s, s]}(x)$. Furthermore, $\mathcal{B}^{[0, s]}(K_0)$ is the ring of integers of infinite series $f = \sum_{k \in \mathbb{Z}} a_k T^k$ where $a_k \in K_0$ such that $f$ is a holomorphic function on the closed disk defined by
   
   $$v_p(T) \in [-\frac{p-1}{ep} \cdot \frac{1}{s}, +\infty].$$

3. For $I = [r, s) \subset (0, +\infty)$, we have $W^I(x) = \inf \{W^{[r, r]}(x), W^{[s, s]}(x)\}$. Furthermore, $\mathcal{B}^{[r, s]}(K_0)$ is the ring of integers of infinite series $f = \sum_{k \in \mathbb{Z}} a_k T^k$ where $a_k \in K_0$ such that $f$ is a holomorphic function on the annulus defined by
   
   $$v_p(T) \in [-\frac{p-1}{ep} \cdot \frac{1}{s}, -\frac{p-1}{ep} \cdot \frac{1}{r}].$$

**Proof.** In [GP, Lem. 2.2.5], we stated the results with $[r, s] = [r_k, r_k]$; but they are true for general intervals. \hfill $\square$

**Lemma 2.3.4.** [GP, Lem. 2.2.7] Let $A^I_{K_\infty}$ be the $G_\infty$-invariants of $A^I$. The map $f(T) \mapsto f(u)$ induces isometric isomorphisms

\[
A^{[0, +\infty]}(K_0) \simeq A^{[0, +\infty]}_{K_\infty};
\]

\[
A^{[r, +\infty]}(K_0) \simeq A^{[r, +\infty]}_{K_\infty}[1/u], \quad \text{when } r > 0;
\]

\[
A^I(K_0) \simeq A^I_{K_\infty}, \quad \text{when } I \subset [0, +\infty) \text{ is a closed interval.}
\]

We record an easy corollary of the above explicit description of the rings $\mathcal{B}^I_{K_\infty}$.

**Corollary 2.3.5.** Let $I \subset [0, +\infty]$ be an interval. Suppose $x \in \mathcal{B}^I_{K_\infty}$ such that $\varphi(x) \in \mathcal{B}^I_{K_\infty}$, then we have $x \in \mathcal{B}_{K_\infty}^{I/p}$. 

Proof. Indeed, by Lem. 2.3.4, \( x = \sum_{i \in \mathbb{Z}} a_i u^i \) with \( a_i \in K_0 \) satisfying certain convergence conditions related with the interval \( I \) as described in Lem. 2.3.3. We have \( \varphi(x) = \sum_{i \in \mathbb{Z}} \varphi(a_i) u^{pi} \), hence using the explicit convergence condition, it is easy to see that \( \varphi(x) \in \mathcal{B}_{K_\infty}^\lambda \) if and only if \( x \in \mathcal{B}_{K_\infty}^{\lambda/I_p} \).

Finally, we write out the explicit structures of some of these rings.

**Proposition 2.3.6.** [GP, Prop. 2.2.10] We have

\[
\begin{align*}
\mathbb{A}_{K_\infty}^{[0, +\infty]} &= \mathbb{A}_{K_\infty}^+, \\
\mathbb{A}_{K_\infty}^{[0, r_k]} &= \mathbb{A}_{K_\infty}^+ \left( \frac{u^{ep^k}}{p} \right), \\
\mathbb{A}_{K_\infty}^{[r_k, +\infty]} &= \mathbb{A}_{K_\infty}^+ \left( \frac{p}{u^{ep^k}} \right), \\
\mathbb{A}_{K_\infty}^{[r_k, r_{k+1}]} &= \mathbb{A}_{K_\infty}^+ \left( \frac{p}{u^{ep^k}} \right).
\end{align*}
\]

**Remark 2.3.7.** Note that \( \varphi : \mathbb{B}_L \to \mathbb{B}_L^{\lambda/I_p} \) is always a bijection; however the map \( \varphi : \mathbb{B}_L \to \mathbb{B}_L^{\lambda/I_p} \) is only an injection. Indeed, \( \mathbb{B}_L / \varphi(\mathbb{B}_L) \) is a degree \( p \) field extension. However, one can always find explicit expressions for \( \mathbb{B}_{K_\infty}^{\lambda/I_p} \) using Lem. 2.3.3.

2.4. Locally analytic vectors in rings. Recall that we use the subscript \( L \) to indicate the Gal\((\overline{K}/L)\)-invariants. Recall for a \( \hat{G} = \text{Gal}(L/K)\)-representation \( W \), we denote \( W^{\tau_{\lambda,a},\gamma=1} := W^{\tau_{\lambda,a} \cap W^{\gamma=1}} \). The following theorem in [GP] is the main result concerning calculation of locally analytic vectors in period rings.

**Theorem 2.4.1.** [GP, Thm. 3.4.4]

1. For \( I = [r_k, r_k] \) or \( [0, r_k] \), we have \( (\mathbb{A}_L^\lambda)^{\tau_{\lambda,a},\gamma=1} = \bigcup_{m \geq 0} \varphi^{-m}(\mathbb{A}_{K_\infty}^{\lambda/I_p}) \).

2. For any \( r \geq 0 \), \( (\mathbb{B}_L^{[r, +\infty]})^{\tau_{\lambda,a},\gamma=1} = \bigcup_{m \geq 0} \varphi^{-m}(\mathbb{B}_{K_\infty}^{[r, +\infty]}) \).

**Definition 2.4.2.**

1. Define the following rings (which are LB spaces):

\[
\mathbb{B}_L^{\lambda/I_p} := \bigcup_{r \geq 0} \mathbb{B}_L^{[r, +\infty]}, \quad \mathbb{B}_L^{\lambda} := \bigcup_{r \geq 0} \mathbb{B}_L^{[r, +\infty]}, \quad \mathbb{B}_{K_\infty}^{\lambda/I_p} := \bigcup_{r \geq 0} \mathbb{B}_{K_\infty}^{[r, +\infty]}.
\]

2. Define the following rings (which are LF spaces):

\[
\mathbb{B}_L^{\lambda/I_p \text{ rig}} := \bigcup_{r \geq 0} \mathbb{B}_L^{[r, +\infty]}, \quad \mathbb{B}_L^{\lambda \text{ rig}} := \bigcup_{r \geq 0} \mathbb{B}_L^{[r, +\infty]}, \quad \mathbb{B}_{K_\infty}^{\lambda/I_p \text{ rig}} := \bigcup_{r \geq 0} \mathbb{B}_{K_\infty}^{[r, +\infty]}.
\]

Note that \( \mathbb{B}_{K_\infty}^{\lambda/I_p} \) (resp. \( \mathbb{B}_{K_\infty}^{\lambda/I_p \text{ rig}} \)) is precisely the explicitly defined ring in (1.2.1) (resp. (1.2.2)).

**Corollary 2.4.3.** [GP, Cor. 3.4.7]

\[
(\mathbb{B}_L^{\lambda/I_p \text{ rig}})^{\tau_{\lambda,a},\gamma=1} = \bigcup_{m \geq 0} \varphi^{-m}(\mathbb{B}_{K_\infty}^{\lambda/I_p \text{ rig}}).
\]

2.5. The element \( \mathfrak{t} \). In this subsection, we study a locally analytic element \( \mathfrak{t} \), which plays a useful role in the definition of our monodromy operators.

Recall we defined the element \( [\varepsilon] \in \mathfrak{A}^+ \) in the beginning of §2.2. Let \( \mathfrak{t} = \log([\varepsilon]) \in \mathbb{B}_{\text{cris}}^+ \) be the usual element. Define the element

\[
\lambda := \prod_{n \geq 0} (\varphi^n(\frac{E(u)}{E(0)})) \in \mathbb{B}_{K_\infty}^{[0, +\infty]} \subset \mathbb{B}_{\text{cris}}^+.
\]

The equation \( \varphi(x) = \frac{E(u)}{E(0)} \cdot x \) over \( \mathfrak{A}^+ \) has solutions in \( \mathfrak{A}^+ + p\mathfrak{A}^+ \), which is unique up to units in \( \mathbb{Z}_p \). By [Liu07, Example 5.3.3], there exists a unique solution \( \mathfrak{t} \in \mathfrak{A}^+ \) such that

\[
p\lambda \mathfrak{t} = \mathfrak{t},
\]

which holds as an equation in \( \mathbb{B}_{\text{cris}}^+ \). Since \( \mathfrak{t} \in \mathfrak{A}^+ \subset \mathbb{B}_L^{\lambda/I_p} \), and since \( \mathbb{B}_L^{\lambda/I_p} \) is a field ([Col08, Prop. 5.12]), there exists some \( r(\mathfrak{t}) > 0 \) such that \( 1/\mathfrak{t} \in \mathbb{B}_L^{[r(\mathfrak{t}), +\infty]} \).

**Lemma 2.5.1.** [GP, Lem. 5.1.1] We have \( \mathfrak{t}, 1/\mathfrak{t} \in (\mathbb{B}_L^{[r(\mathfrak{t}), +\infty]} \hat{G}\text{-pa}) \).
3. Modules and locally analytic vectors

In this section, we recall the theory of étale $(\varphi, \tau)$-modules and their overconvergence property. In particular, we discuss the relation between locally analytic vectors and the overconvergence property.

In this section and in §5, we will introduce several categories of modules with structures. We will always omit the definition of morphisms for these categories, which are always obvious (i.e., module homomorphisms compatible with various structures).

3.1. Étale $(\varphi, \tau)$-modules.

**Definition 3.1.1.** Objects in the following are called étale $\varphi$-modules.

1. Let $\text{Mod}_{A_{K_\infty}}^\varphi$ denote the category of finite free $A_{K_\infty}$-modules $M$ equipped with a $\varphi_{A_{K_\infty}}$-semi-linear endomorphism $\varphi_M : M \to M$ such that $1 \otimes \varphi : \varphi^* M \to M$ is an isomorphism.
2. Let $\text{Mod}_{B_{K_\infty}}^\varphi$ denote the category of finite free $B_{K_\infty}$-modules $D$ equipped with a $\varphi_{B_{K_\infty}}$-semi-linear endomorphism $\varphi_D : D \to D$ such that there exists a finite free $A_{K_\infty}$-lattice $M$ such that $M[1/p] = D$, $\varphi_D(M) \subset M$, and $(M, \varphi_D) \in \text{Mod}_{A_{K_\infty}}^\varphi$.

**Definition 3.1.2.** Objects in the following are called étale $(\varphi, \tau)$-modules.

1. Let $\text{Mod}_{A_{K_\infty}}^{\varphi, \hat{G}}$ denote the category consisting of triples $(M, \varphi_M, \hat{G})$ where
   - $(M, \varphi_M) \in \text{Mod}_{A_{K_\infty}}^\varphi$;
   - $\hat{G}$ is a continuous $A_L$-semi-linear $\hat{G}$-action on $\hat{M} := \hat{A}_L \otimes_{A_{K_\infty}} M$, and $\hat{G}$ commutes with $\varphi_M$ on $\hat{M}$;
   - regarding $M$ as an $A_{K_\infty}$-submodule in $\hat{M}$, then $M \subset \hat{M}^{\text{Gal}(L/K_\infty)}$.
2. Let $\text{Mod}_{B_{K_\infty}}^{\varphi, \hat{G}}$ denote the category consisting of triples $(D, \varphi_D, \hat{G})$ which contains a lattice (in the obvious fashion) $(M, \varphi_M, \hat{G}) \in \text{Mod}_{A_{K_\infty}}^{\varphi, \hat{G}}$.

3.1.3. Let $\text{Rep}_{\mathbb{Q}_p}(G_\infty)$ (resp. $\text{Rep}_{\mathbb{Q}_p}(G_K)$) denote the category of finite dimensional $\mathbb{Q}_p$-vector spaces $V$ with continuous $\mathbb{Q}_p$-linear $G_\infty$ (resp. $G_K$)-actions.

- For $D \in \text{Mod}_{B_{K_\infty}}^\varphi$, let
  $$V(D) := (\tilde{B} \otimes_{B_{K_\infty}} D)^{\varphi=1},$$
  then $V(D) \in \text{Rep}_{\mathbb{Q}_p}(G_\infty)$. If furthermore $(D, \varphi_D, \hat{G}) \in \text{Mod}_{B_{K_\infty}}^{\varphi, \hat{G}}$, then $V(D) \in \text{Rep}_{\mathbb{Q}_p}(G_K)$.
- For $V \in \text{Rep}_{\mathbb{Q}_p}(G_\infty)$, let
  $$D_{K_\infty}(V) := (B \otimes_{\mathbb{Q}_p} V)^{G_\infty},$$
  then $D_{K_\infty}(V) \in \text{Mod}_{B_{K_\infty}}^\varphi$. If furthermore $V \in \text{Rep}_{\mathbb{Q}_p}(G_K)$, let
  $$\tilde{D}_L(V) := (\tilde{B} \otimes_{\mathbb{Q}_p} V)^{G_L},$$
  then $\tilde{D}_L(V) = \tilde{B} \otimes_{B_{K_\infty}} D_{K_\infty}(V)$ has a $\hat{G}$-action, making $(D_{K_\infty}(V), \varphi, \hat{G})$ an étale $(\varphi, \tau)$-module.

As we already mentioned in Rem. 1.2.9, Thm. 3.1.4 below is the only place we use results from [Car13].

**Theorem 3.1.4.**

1. [Fon90, Prop. A 1.2.6] The functors $V$ and $D_{K_\infty}$ induce an exact tensor equivalence between the categories $\text{Mod}_{B_{K_\infty}}^\varphi$ and $\text{Rep}_{\mathbb{Q}_p}(G_\infty)$.
2. [Car13, Thm. 1] The functors $V$ and $(D_{K_\infty}, \tilde{D}_L)$ induce an exact tensor equivalence between the categories $\text{Mod}_{B_{K_\infty}}^{\varphi, \hat{G}}$ and $\text{Rep}_{\mathbb{Q}_p}(G_K)$.

3.2. Overconvergence and locally analytic vectors. Let $V \in \text{Rep}_{\mathbb{Q}_p}(G_K)$. Given $I \subset [0, +\infty)$, let

$$D_{K_\infty}^I(V) := (B^I \otimes_{\mathbb{Q}_p} V)^{G_\infty},$$

$$\tilde{D}_L^I(V) := (\tilde{B}^I \otimes_{\mathbb{Q}_p} V)^{G_L}.$$
**Definition 3.2.1.** Let $V \in \text{Rep}_{\mathbb{Q}_p}(G_K)$, and let $\hat{D} = (D_{K_{\infty}}(V), \varphi, \hat{G})$ be the étale $(\varphi, \tau)$-module associated to it. We say that $\hat{D}$ is _overconvergent_ if there exists $r > 0$, such that for $I' = [r, +\infty]$, 

1. $D_{K_{\infty}}^{I'}(V)$ is finite free over $\mathcal{B}_{K_{\infty}}$, and $\mathcal{B}_{K_{\infty}} \otimes_{\mathcal{B}_{K_{\infty}}} D_{K_{\infty}}^{I'}(V) \simeq D_{K_{\infty}}(V)$;

2. $\tilde{D}_{L}(V)$ is finite free over $\mathcal{B}_L$ and $\mathcal{B}_L \otimes_{\mathcal{B}^+_L} \tilde{D}_{L}(V) \simeq \tilde{D}_{L}(V)$.

**Theorem 3.2.2.** [GL, GP] For any $V \in \text{Rep}_{\mathbb{Q}_p}(G_K)$, its associated étale $(\varphi, \tau)$-module is overconvergent.

**Remark 3.2.3.** As we already mentioned in Rem. 1.2.2, a first proof of Thm. 3.2.2 is given in [GL] (which only works for $K/\mathbb{Q}_p$ a finite extension), and a second proof is given in [GP]; it is the second proof that will be useful for the current paper, e.g., see Cor. 3.2.4 below.

Let $V \in \text{Rep}_{\mathbb{Q}_p}(G_K)$ of dimension $d$, let $D_{K_{\infty}}^{I'}(V)$ be as in Def. 3.2.1 for $I' = [r(V), +\infty]$ (which exists by Thm. 3.2.2), and denote

\begin{equation}
D_{\text{rig}, K_{\infty}}^{I}(V) := D_{K_{\infty}}^{I'}(V) \otimes_{\mathcal{B}_{K_{\infty}}} \mathcal{B}_{\text{rig}, K_{\infty}},
\end{equation}

which is finite free over $\mathcal{B}_{\text{rig}, K_{\infty}}$ of rank $d$; we call it the _rigid-overconvergent_ $(\varphi, \tau)$-module associated to $V$.

**Corollary 3.2.4.** $D_{\text{rig}, K_{\infty}}^{I}(V)$ forms a $(\mathcal{B}_{\text{rig}, L})^{\tau, \text{pa}, \gamma = 1}$-basis of $((\mathcal{B}_{\text{rig}, \mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V)^{G_L})^{\tau, \text{pa}, \gamma = 1}$. Namely,

\begin{equation}
((\mathcal{B}_{\text{rig}, \mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V)^{G_L})^{\tau, \text{pa}, \gamma = 1} = D_{\text{rig}, K_{\infty}}^{I}(V) \otimes_{\mathcal{B}_{\text{rig}, K_{\infty}}} (\mathcal{B}_{\text{rig}, L})^{\tau, \text{pa}, \gamma = 1}.
\end{equation}

**Proof.** This is extracted from the proof of Thm. 3.2.2 in [GP, Thm. 6.2.6]; indeed, it easily follows from [GP, Eqn. (6.2.5)].

4. Monodromy operator for $(\varphi, \tau)$-modules

In this section, we define a natural monodromy operator on the rigid-overconvergent $(\varphi, \tau)$-modules.

Let $\nabla_\tau := \frac{\log(\tau^n)}{p^n}$ for $n \gg 0$ be the Lie-algebra operator acting on $\hat{G}$-locally analytic representations.

**4.1. Monodromy operator over rings.** Recall that by Cor. 2.4.3,

\begin{equation}
(\mathcal{B}_{\text{rig}, L})^{\tau, \text{pa}, \gamma = 1} = \cup_{m \geq 0} \varphi^{-m}(\mathcal{B}^I_{\text{rig}, K_{\infty}}).
\end{equation}

Hence $\nabla_\tau$ induces a map:

\begin{equation}
\nabla_\tau : \mathcal{B}^I_{\text{rig}, K_{\infty}} \to (\mathcal{B}_{\text{rig}, L})^{\hat{G}_{\text{pa}}}.
\end{equation}

Since by Lem. 2.5.1, we have $1/t \in (\mathcal{B}_{L}^{[r(t), +\infty)})^{\hat{G}_{\text{pa}}}$, we further have

\begin{equation}
\frac{\nabla_\tau}{pt} : \mathcal{B}^I_{\text{rig}, K_{\infty}} \to (\mathcal{B}_{\text{rig}, L})^{\hat{G}_{\text{pa}}}.
\end{equation}

**Remark 4.1.1.** The $p$ in the denominator of (4.1.1) makes our monodromy operator compatible with earlier theory of Kisin in [Kis06].

**Lemma 4.1.2.** Denote the operator $N_{\nabla} := \frac{\nabla_\tau}{pt}$, then the map in Eqn. (4.1.1) induces

\begin{equation}
N_{\nabla} : \mathcal{B}^I_{\text{rig}, K_{\infty}} \to \mathcal{B}^I_{\text{rig}, K_{\infty}};
\end{equation}

which sends $x$ to $-u\lambda\frac{d}{dx}(x)$. Indeed, for any $I \subset [0, +\infty)$, $N_{\nabla}$ induces

\begin{equation}
N_{\nabla} : \mathcal{B}^I_{K_{\infty}} \to \mathcal{B}^I_{K_{\infty}}.
\end{equation}

**Proof.** By linearity and density, it suffices to check the cases $x = u^i$ with $i \in \mathbb{Z}$, which is obvious. The map (4.1.3) is well-defined because $\lambda \in \mathcal{B}^{[0, +\infty)}_{K_{\infty}}$ and because $x \in \mathcal{B}^I_{K_{\infty}}$ implies that $\frac{d}{dx}(x) \in \mathcal{B}^{I}_{K_{\infty}}$ (e.g., by Lem. 2.3.3). Note that as we mentioned in Rem. 4.1.1, this operator is the same as the one in [Kis06] when $I = [0, +\infty)$. \qed
4.2. Monodromy operator over modules. By Cor. 3.2.4, we have
(4.2.1) \((\tilde{B}_{\text{rig}} \otimes \mathbb{Q}_p V)^{G_L})^{G_{\text{pa}}} = D^\dagger_{\text{rig}, K_{\infty}}(V) \otimes_{B^\dagger_{\text{rig}, K_{\infty}}} (\tilde{B}_{\text{rig}, L})^{G_{\text{pa}}}\).

Hence similarly as in the §4.1, we have a map:
(4.2.2) \(N_\nabla := \frac{N}{\text{pt}} : D^\dagger_{\text{rig}, K_{\infty}}(V) \to D^\dagger_{\text{rig}, K_{\infty}}(V) \otimes_{B^\dagger_{\text{rig}, K_{\infty}}} (\tilde{B}_{\text{rig}, L})^{G_{\text{pa}}}\).

**Theorem 4.2.1.** The map (4.2.2) induces a map
(4.2.3) \(N_\nabla : D^\dagger_{\text{rig}, K_{\infty}}(V) \to D^\dagger_{\text{rig}, K_{\infty}}(V)\).

Namely, \(N_\nabla\) is a well-defined operator on \(D^\dagger_{\text{rig}, K_{\infty}}(V)\). Furthermore, there exists some \(r' \geq r(V)\) (cf. the notation above (3.2.1)) such that if \(I \subset [r', +\infty)\), then
(4.2.4) \(N_\nabla : D^\dagger_{K_{\infty}}(V) \to D^\dagger_{K_{\infty}}(V),\)

where recall \(D^\dagger_{K_{\infty}}(V) = D^\dagger_{K_{\infty}}^{[r(V), +\infty]}(V) \otimes_{B^\dagger_{K_{\infty}}} B^\dagger_{K_{\infty}}\).

**Proof.** Note that we have the relation
\[g\tau g^{-1} = \tau^{\chi_p}(g), \quad \text{for } g \in \text{Gal}(L/K_{\infty}),\]
where \(\chi_p\) is the cyclotomic character. Note also \(g(t) = \chi_p(g)t\) for \(g \in \text{Gal}(L/K_{\infty})\). Hence we have the relation
(4.2.5) \(gN_\nabla = N_\nabla g \text{ for } g \in \text{Gal}(L/K_{\infty}).\)

Since \(\text{Gal}(L/K_{\infty})\) acts trivially on \(D^\dagger_{\text{rig}, K_{\infty}}(V)\), hence \(\text{Gal}(L/K_{\infty})\) also acts trivially on \(N_\nabla(D^\dagger_{\text{rig}, K_{\infty}}(V))\) using (4.2.5). Thus, we have
\[N_\nabla(D^\dagger_{\text{rig}, K_{\infty}}(V)) \subset \left( D^\dagger_{\text{rig}, K_{\infty}}(V) \otimes_{B^\dagger_{\text{rig}, K_{\infty}}} (\tilde{B}_{\text{rig}, L})^{G_{\text{pa}}} \right)^{G_\nabla} = D^\dagger_{\text{rig}, K_{\infty}}(V) \otimes_{B^\dagger_{\text{rig}, K_{\infty}}} (\tilde{B}_{\text{rig}, L})^{G_{\text{pa}}},\]
by Cor. 2.4.3. Choose a basis \(\varepsilon\) of \(D^\dagger_{\text{rig}, K_{\infty}}(V)\), then
\[N_\nabla(\varepsilon) \subset D^\dagger_{\text{rig}, K_{\infty}}(V) \otimes_{B^\dagger_{\text{rig}, K_{\infty}}} \varphi^{-m}(B^\dagger_{\text{rig}, K_{\infty}}) \text{ for some } m \gg 0.\]

Recall \(\varphi(t) = \frac{pE(u)}{E(0)}t\). Note that \(\varphi \tau = \tau \varphi\) over \(D^\dagger_{\text{rig}, K_{\infty}}(V) \otimes_{B^\dagger_{\text{rig}, K_{\infty}}} (\tilde{B}_{\text{rig}, L})^{G_{\text{pa}}}\), hence we have the relation
(4.2.6) \(N_\nabla \varphi = \frac{pE(u)}{E(0)} \varphi N_\nabla.\)

Using (4.2.6), we can easily deduce that
\[N_\nabla(\varphi^m(\varepsilon)) \subset D^\dagger_{\text{rig}, K_{\infty}}(V),\]
and hence we can conclude the proof of (4.2.3). Finally, we can use \(r' = p^m r(V)\) to derive (4.2.4). \(\square\)

5. Monodromy Operator for Semi-Stable Representations

In this section, we will recall Kisin’s definition of a monodromy operator on certain modules associated to semi-stable representations. The main aim of this section is to show that Kisin’s monodromy operator coincides with the one defined by us in §4 (in the semi-stable case). The task might seem innocuous, however the proof turns out to be rather non-trivial, and makes substantial use of almost all tools from integral p-adic Hodge theory, particularly the \((\varphi, G)\)-modules developed by Liu. Indeed, it seems to us that Kisin’s monodromy operator is defined in an “algebraic” fashion (cf. §5.2.3), whereas our monodromy operator in Thm. 4.2.1 is defined in an “analytic” fashion. It turns out we will need to pass both these operators to a ring \(S_{K_0}\) in order to show that they coincide.
In §5.1, we review Fontaine’s filtered \((\varphi, N)\)-modules over \(K_0\), and Breuil’s filtered \((\varphi, N)\)-modules over \(S_{K_0}\). In §5.2, we review Kisin’s construction of \(\mathcal{O}\)-modules, and recall its relation with modules in §5.1. In §5.3, we review Liu’s construction of \((\varphi, \hat{G})\)-modules. In §5.4, we review a result of Cherbonnier on maximal overconvergent submodules and use it to study finite height modules. In §5.5, we prove the main result of this section, namely the coincidence of monodromy operators.

As we will use several tools from integral \(p\)-adic Hodge theory, we first recall some notations commonly used therein. Let
\[
R := \mathbb{E}^+, \quad \text{Fr} R := \mathbb{E}, \quad W(R) = \mathbb{A}^+, \quad \mathfrak{S} := A^+_K.
\]

5.1. **Filtered \((\varphi, N)\)-modules over \(K_0\) and \(S_{K_0}\).**

**Definition 5.1.1.** Let \(\text{MF}^\varphi_N\) be the category of (positive) filtered \((\varphi, N)\)-modules over \(K_0\) which consists of finite dimensional \(K_0\)-vector spaces \(D\) equipped with

1. a Frobenius \(\varphi : D \to D\) such that \(\varphi(ax) = \varphi(a)\varphi(x)\) for all \(a \in K_0, x \in D\);
2. a monodromy \(N : D \to D\), which is a \(K_0\)-linear map such that \(N\varphi = p\varphi N\);
3. a filtration \((\text{Fil}^i D_K)_{i \in \mathbb{Z}}\) on \(D_K = D \otimes_{K_0} K\), by decreasing \(K\)-vector subspaces such that \(\text{Fil}^i D_K = D_K\) and \(\text{Fil}^i D_K = 0\) for \(i \gg 0\).

Let \(\text{MF}^\varphi_N,wa\) denote the usual subcategory of \(\text{MF}^\varphi_N\) consisting of weakly admissible objects.

5.1.2. **The ring \(S\).** Let
\[
(5.1.1) \quad S := \left\{ \sum_{n=0}^{\infty} a_n \frac{E(u)^n}{n!}, a_n \in \mathfrak{S}, a_n \to 0 \text{ p-adically} \right\} \subset A^\text{cris}.
\]
Let \(S_{K_0} := S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \subset B^+\text{cris}\). There is a Frobenius operator \(\varphi : S_{K_0} \to S_{K_0}\) which extends the absolute Frobenius on \(K_0\), and \(\varphi(u) = u^p\). The map \(\iota_0 : B^{(0,+\infty)}(0) \hookrightarrow B^{+\text{cris}}\) induces a continuous embedding
\[
\iota_0 : B^{(0,+\infty)}_{S_{K_0}} \to S_{K_0},
\]
which is \(\varphi\)-equivariant. We can (forcibly) define an operator \(N_{\varphi} : S_{K_0} \to S_{K_0}\) to be \(-u\lambda \frac{d\varphi}{du}\), which extends the one on \(B^{(0,+\infty)}_{S_{K_0}}\). We further define \(N_S : S_{K_0} \to S_{K_0}\) to be the operator \(-u\lambda \frac{d\varphi}{du}\). We can define a filtration on \(S\): for \(i \geq 0\), let \(\text{Fil}^i S \subset S\) consist of elements with \(a_n = 0\) for \(n < i\) in (5.1.1), and let \(\text{Fil}^i S_{K_0} := \text{Fil}^i S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\).

**Definition 5.1.3.** Let \(\text{MF}^{\varphi,N}_{S_{K_0}}\) be the category of (positive) filtered \((\varphi, N)\)-modules over \(S_{K_0}\) which consists of finite free \(S_{K_0}\)-modules \(D\) equipped with:

1. a \(\varphi_{S_{K_0}}\)-semi-linear morphism \(\varphi_D : D \to D\) such that the determinant of \(\varphi_D\) is invertible in \(S_{K_0}\);
2. a decreasing filtration \(\{\text{Fil}^i D\}_{i \in \mathbb{Z}}\) of \(S_{K_0}\)-submodules of \(D\) such that \(\text{Fil}^0 D = D\) and \(\text{Fil}^i S_{K_0}\) are \(\text{Fil}^i D \subseteq \text{Fil}^{i+1} D\);
3. a \(K_0\)-linear map \(N : D \to D\) such that \(N(fm) = N(f)m + fN(m)\) for all \(f \in S_{K_0}\) and \(m \in D\), \(N\varphi = p\varphi N\) and \(N(\text{Fil}^i D) \subseteq \text{Fil}^{i-1} D\).

5.1.4. **Functor from \(\text{MF}^{\varphi,N}_{S_{K_0}}\) to \(\text{MF}^{\varphi,N\otimes_{W(k)}}\).** For \(D \in \text{MF}^{\varphi,N}_{S_{K_0}}\), we can associate an object in \(\text{MF}^{\varphi,N}_{S_{K_0}}\) by \(D = D(D) = S \otimes_{W(k)} D\) and define on \(D\)

1. \(\varphi := \varphi_S \otimes \varphi_D\);
2. \(N := N_S \otimes 1 + 1 \otimes N_D\);
3. \(\text{Fil}^0 D := D\) and inductively,
\[
\text{Fil}^{i+1} D := \{x \in D | N(x) \in \text{Fil}^i D\text{ and } f_p(x) \in \text{Fil}^{i+1} D_K\},
\]
where \(f_p : D \to D_K\) is induced by \(s(u) \otimes x \mapsto s(\pi)x\).

**Theorem 5.1.5** ([Bre97]). The functor in §5.1.4 induces an equivalence between \(\text{MF}^{\varphi,N}_{K_0}\) and \(\text{MF}^{\varphi,N}_{S_{K_0}}\).
5.1.6. In fact, for $D \in \text{MF}_{K_0}^{\varphi,N}$ and $\mathcal{D} \in \text{MF}_{S_{K_0}}^{\varphi,N}$, we can associate $G_K$-representations via
\[ V_{st}(D) := \text{Hom}_{\varphi,N}(D, B_{st}) \cap \text{Hom}_{\text{Fil}^\bullet}(D, K \otimes_{K_0} B_{st}), \]
\[ V_{st}(\mathcal{D}) := \text{Hom}_{S,\text{Fil},\varphi,N}(\mathcal{D}, A_{st}[1/p]), \]
where $B_{st}$ is the usual period ring and $A_{st}$ is a more complicated ring, both of which are equipped with various structures; we omit all the details since we will not need them; see [Bre97] for more details. The functor in §5.1.4 is compatible with Galois representations, i.e., $V_{st}(D) \simeq V_{st}(\mathcal{D}(D))$ as $\mathbb{Q}_p[G_K]$-modules.

**Definition 5.1.7.** Let $\text{MF}_{K_0}^{\varphi,N,\text{wa}}$ as the essential image of the functor $\mathcal{D}$ restricted to $\text{MF}_{K_0}^{\varphi,N,\text{wa}}$.

Recall that $\text{MF}_{K_0}^{\varphi,N,\text{wa}}$ classifies semi-stable $p$-adic representations of $G_K$ with non-negative Hodge-Tate weights (via $V_{st}$), and hence $\text{MF}_{S_{K_0}}^{\varphi,N,\text{wa}}$ does too. (Note that using our convention, the cyclotomic character has Hodge-Tate weight $\{1\}$).

5.2. Kisin’s construction of $\mathcal{O}$-modules. Let $\mathcal{O} := B_{K_\infty}^{[0, +\infty)}$ (also denoted as $\mathcal{O}^{(0,1)}$ in [Kis06]). Explicitly,
\[ \mathcal{O} = B_{K_\infty}^{[0, +\infty)} = \{ f(u) = \sum_{i=0}^{+\infty} a_i u^i, a_i \in K_0, f(u) \text{ converges for all } u \in \overline{K} \text{ with } 0 < v_p(u) \leq +\infty \}; \]
i.e., it consists of series that converge on the entire open unit disk. Recall that $N_{\mathcal{V}}$ is the operator $-u \lambda \frac{d}{du}$ on $\mathcal{O}$.

**Definition 5.2.1.** Let $\text{Mod}_{\mathcal{O}}^{\varphi,N_{\mathcal{V}}}$ denote the category which consists of finite free $\mathcal{O}$-modules $M$ equipped with
\begin{enumerate}
  \item a $\varphi_{\mathcal{O}}$-semi-linear morphism $\varphi : M \to M$ such that the cokernel of $1 \otimes \varphi : \varphi^* M \to M$ is killed by $E(u)^h$ for some $h \in \mathbb{Z}^{\geq 0}$;
  \item $N_{\mathcal{V}} : M \to M$ is a map such that $N_{\mathcal{V}}(f m) = N_{\mathcal{V}}(f) m + f N_{\mathcal{V}}(m)$ for all $f \in \mathcal{O}$ and $m \in M$, and $N_{\mathcal{V}} \varphi = \frac{pE(u)}{E(0)} \varphi N_{\mathcal{V}}$.
\end{enumerate}

5.2.2. For $M \in \text{Mod}_{\mathcal{O}}^{\varphi,N_{\mathcal{V}}}$, Let $\mathcal{D}_{\mathcal{O}}(M) := S_{K_0} \otimes_{\varphi, \mathcal{O}} M$ (i.e., base change along $\varphi := \varphi \circ \iota_0 : \mathcal{O} \to S_{K_0} \to S_{K_0}$) and equip it with:
\begin{enumerate}
  \item $\varphi := \varphi_{S_0} \otimes \varphi_M$;
  \item $N := N_S \otimes 1 + \frac{1}{\varphi(M)} \otimes N_{\mathcal{V}}$;
  \item $\text{Fil}^i(\mathcal{D}_{\mathcal{O}}(M)) := \{ m \in \mathcal{D}_{\mathcal{O}}(M) \mid (1 \otimes \varphi)(m) \in \text{Fil}^i S_{K_0} \otimes_{\mathcal{O}} M \}$
\end{enumerate}
By [Liu08, Prop. 3.2.1], $\mathcal{D}_{\mathcal{O}}(M)$ is indeed an object in $\text{MF}_{S_{K_0}}^{\varphi,N_{\mathcal{V}}}$.

5.2.3. For $D \in \text{MF}_{K_0}^{\varphi,N_{\mathcal{V}}}$, we can associate an object $M \in \text{Mod}_{\mathcal{O}}^{\varphi,N_{\mathcal{V}}}$ by [Kis06]. The construction is rather complicated, and we only give a very brief sketch. (We want to give a sketch here, since we care about the construction of the $N_{\mathcal{V}}$-operator in this section).

For $n \geq 0$, let $K_{n+1} = K(n)$ (hence $K_1 = K$), and let $\widehat{\mathcal{S}}_n$ be the completion of $K_{n+1} \otimes W(k) \mathcal{G}$ at the maximal ideal $(u - \pi_n)$; $\widehat{\mathcal{S}}_n$ is equipped with its $(u - \pi_n)$-adic filtration, which extends to a filtration on the quotient field $\widehat{\mathcal{S}}_n[1/(u - \pi_n)]$.

There is a natural $K_0$-linear map $\mathcal{O} \to \widehat{\mathcal{S}}_n$ simply by sending $u$ to $u$. Recall that we have maps $\varphi : \mathcal{G} \to \mathcal{G}$ and $\varphi : \mathcal{O} \to \mathcal{O}$ which extends the absolute Frobenius on $W(k)$ and sends $u$ to $u^p$. Denote $\varphi_W : \mathcal{G} \to \mathcal{G}$ and $\varphi_W : \mathcal{O} \to \mathcal{O}$ which only acts as absolute Frobenius on $W(k)$ and sends $u$ to $u$.

We adjoin a formal variable $\ell_u$ to $\mathcal{O}$ which behaves like “$\log u$”. We extend the map $\mathcal{O} \to \widehat{\mathcal{S}}_n$ to $\mathcal{O}[\ell_u] \to \widehat{\mathcal{S}}_n$ which sends $\ell_u$ to
\[ \log([\frac{u - \pi_n}{\pi_n}] + 1) := \sum_{i=1}^{\infty} (-1)^{i-1} \left( \frac{u - \pi_n}{\pi_n} \right)^i \in \widehat{\mathcal{S}}_n. \]
We also extend $\varphi$ to $\mathcal{O}[\ell_u]$ by setting $\varphi(\ell_u) = p\ell_u$, and extend $N_\varphi$ to $\mathcal{O}[\ell_u]$ by setting $N_\varphi(\ell_u) = -\lambda$. Finally, write $N$ for the derivation on $\mathcal{O}[\ell_u]$ which acts as $\mathcal{O}$-derivation with respect to the formal variable $\ell_u$.

Given $D \in \text{MF}_{K_0}^{\varphi,N}$, write $\iota_n$ for the following composite map:

\[(5.2.2) \quad \mathcal{O}[\ell_u] \otimes_{K_0} D \xrightarrow{\varphi^w \otimes \varphi - n} \mathcal{O}[\ell_u] \otimes_{K_0} D \to \widehat{\mathcal{S}}_n \otimes_{K_0} D = \widehat{\mathcal{S}}_n \otimes K D_K\]

where the second map is induced from the map $\mathcal{O}[\ell_u] \to \widehat{\mathcal{S}}_n$. The composite map extends to

\[(5.2.3) \quad \iota_n : \mathcal{O}[\ell_u, 1/\lambda] \otimes_{K_0} D \to \widehat{\mathcal{S}}_n[1/(u - \pi_n)] \otimes K D_K\]

Now, set

\[(5.2.4) \quad M(D) := \{x \in (\mathcal{O}[\ell_u, 1/\lambda] \otimes_{K_0} D)^{N=0} : \iota_n(x) \in \text{Fil}^0 \left( \widehat{\mathcal{S}}_n[1/(u - \pi_n)] \otimes K D_K \right), \forall n \geq 1\},\]

where the $\text{Fil}^0$ in (5.2.4) comes from tensor product of two filtrations. Then Kisin shows that $M(D)$ is in fact a finite free $\mathcal{O}$-module. We can equip $\varphi$ on $M(D)$ induced from $\varphi$ on $\mathcal{O}[\ell_u, 1/\lambda]$ and $D$; we can also equip $N_\varphi$ on $M(D)$ induced by $N_\varphi \otimes 1$ on $\mathcal{O}[\ell_u, 1/\lambda] \otimes_{K_0} D$. Kisin shows that this makes $M(D)$ into an object in $\text{Mod}_{\mathcal{O}}^{\varphi,N_\varphi}$.

**Theorem 5.2.4.** [Liu08, Cor. 3.2.3] We have the following commutative diagram of equivalences of categories:

\[
\begin{array}{ccc}
\text{MF}^{\varphi,N}_{K_0} & \longrightarrow & \text{MF}^{\varphi,N}_{S_{K_0}} \\
\downarrow & & \downarrow \\
\text{Mod}_{\mathcal{O}}^{\varphi,N_\varphi} & \longrightarrow & \text{Mod}_{\mathcal{O}}^{\varphi,N_\varphi,0}
\end{array}
\]

using functors constructed in §5.1.4, §5.2.2, and §5.2.3.

5.2.5. Let $\mathcal{R}$ be the Robba ring as in [Kis06, §1.3], which is precisely $B^\dagger_{\text{rig},K_\infty}$ in our Def. 2.4.2. Recall that

\[\mathcal{O} = B^{0,+\infty}_{K_\infty} \subset B^\dagger_{\text{rig},K_\infty} = \mathcal{R}.\]

Let $\text{Mod}^{\varphi,N_\varphi,0}_{\mathcal{O}}$ be the subcategory of $\text{Mod}^{\varphi,N_\varphi}_{\mathcal{O}}$ consisting of objects $M$ such that $\mathcal{R} \otimes \mathcal{O} M$ is pure of slope 0 in the sense of Kedlaya (cf. [Ked04, Ked05] or [Kis06, §1.3]).

**Theorem 5.2.6.** [Kis06, Thm. 1.3.8] Let $D \in \text{MF}^{\varphi,N}_{K_0}$, and let $M = M(D) \in \text{Mod}^{\varphi,N_\varphi}_{\mathcal{O}}$ as in §5.2.3. Then $D \in \text{MF}^{\varphi,N,\text{wa}}_{K_0}$ if and only if $M \in \text{Mod}^{\varphi,N_\varphi,0}_{\mathcal{O}}$.

**Corollary 5.2.7.** The diagram in Thm. 5.2.4 induces the following commutative diagram of equivalences of categories:

\[
\begin{array}{ccc}
\text{MF}^{\varphi,N,\text{wa}}_{K_0} & \longrightarrow & \text{MF}^{\varphi,N,\text{wa}}_{S_{K_0}} \\
\downarrow & & \downarrow \\
\text{Mod}^{\varphi,N_\varphi,0}_{\mathcal{O}} & \longrightarrow & \text{Mod}^{\varphi,N_\varphi,0}_{\mathcal{O}}
\end{array}
\]

5.3. *Liu’s construction of $(\varphi,G)$-modules.* Define a subring inside $B^+_{\text{cris}}$:

\[\mathcal{R}_{K_0} := \left\{ x = \sum_{i=0}^{\infty} f_i t^{(i)} : f_i \in S_{K_0} \text{ and } f_i \to 0 \text{ as } i \to +\infty \right\},\]

where $t^{(i)} = \frac{\anti{t}^{(i)}}{\vartheta^{(i)}}$ and $\anti{t}$ satisfies $i = \anti{t}(p - 1) + r(i)$ with $0 \leq r(i) < p - 1$. (Let us mention that $\mathcal{R}_{K_0}$ here has nothing to do with $\mathcal{R}$ in §5.2.5; we are only following the notation of [Liu10].) Define $\hat{\mathcal{R}} := W(R) \cap \mathcal{R}_{K_0}$. One can show that $\mathcal{R}_{K_0}$ and $\hat{\mathcal{R}}$ are stable under the $G_K$-action and the $G_K$-action factors through $\hat{G}$. Let $m_\mathfrak{p}$ be the maximal ideal of $R$ and $I_+ \hat{\mathcal{R}} = W(m_R) \cap \hat{\mathcal{R}}$, then one has $\hat{\mathcal{R}}/I_+ \hat{\mathcal{R}} \cong \mathcal{S}/u\mathcal{S} = W(k)$. See [Liu10, §2.2] for all these facts.

**Definition 5.3.1.** Let $\text{Mod}^\varphi_{\mathcal{O}}$ denote the category of finite free $\mathcal{S}$-modules $\mathcal{M}$ equipped with a $\varphi_\mathcal{O}$-semi-linear endomorphism $\mathcal{M} \to \mathcal{M}$ such that the co-kernel of $1 \otimes \varphi : \mathcal{M} \to \mathcal{M}$ is killed by $E(u)^h$ for some $h \in \mathbb{Z}^\geq$. We say that $\mathcal{M}$ is of $E(u)$-height $h$ (to be precise, $\leq h$). Objects in this category are called Breuil-Kisin modules.
For $\mathfrak{M} \in \text{Mod}^G_{\hat{\mathcal{O}}}$, define
\begin{equation}
(5.3.1) \quad T_{\hat{\mathcal{O}}}(\mathfrak{M}) := \text{Hom}_{\hat{\mathcal{O}},\varphi}(\mathfrak{M}, W(R)).
\end{equation}
$T_{\hat{\mathcal{O}}}(\mathfrak{M})$ is a finite free $\mathbb{Z}_p$-representation of $G_{\infty}$ and $\text{rank}_{\mathbb{Z}_p} T_{\hat{\mathcal{O}}}(\mathfrak{M}) = \text{rank}_{\mathcal{O}}(\mathfrak{M})$.

**Definition 5.3.2.** Let $\text{Mod}^{\varphi,\hat{G}}_{\hat{\mathcal{O}},\hat{\mathcal{R}}}$ be the category consisting of triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ which are called $(\varphi, \hat{G})$-modules, where
1. $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in \text{Mod}^G_{\hat{\mathcal{O}}}$;
2. $\hat{G}$ is a continuous $\hat{\mathcal{R}}$-semi-linear $\hat{G}$-action on $\hat{\mathfrak{M}} := \hat{\mathcal{R}} \otimes_{\varphi, \hat{\mathcal{O}}} \mathfrak{M}$;
3. $\hat{G}$ commutes with $\varphi_{\mathfrak{M}}$ on $\hat{\mathfrak{M}}$;
4. regarding $\mathfrak{M}$ as a $\varphi(\hat{\mathcal{O}})$-submodule in $\hat{\mathfrak{M}}$, then $\mathfrak{M} \subset \hat{\mathfrak{M}}_{\text{Gal}(L/K_{\infty})}$;
5. $\hat{G}$ acts on the $W(k)$-module $\hat{\mathfrak{M}}/I_{\mathfrak{M}} \hat{\mathfrak{M}}$ trivially.

**5.3.3.** Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$ be a finite free $(\varphi, \hat{G})$-module. By the proof in [Liu10, Prop. 3.1.3] we can associate a $\mathbb{Z}_p[G_K]$-module:
\begin{equation}
(5.3.2) \quad \hat{T}(\hat{\mathfrak{M}}) := \text{Hom}_{\hat{\mathcal{R}}, \varphi}(\hat{\mathcal{R}} \otimes_{\varphi, \hat{\mathcal{O}}} \mathfrak{M}, W(R)) \simeq \text{Hom}_{W(R), \varphi}(W(R) \otimes_{\hat{\mathcal{R}}} \hat{\mathfrak{M}}, W(R)),
\end{equation}
where $G_K$ acts on $\hat{T}(\hat{\mathfrak{M}})$ via $g(f)(x) = g(f(g^{-1}(x)))$ for any $g \in G_K$ and $f \in \hat{T}(\hat{\mathfrak{M}})$. Recall that we regard $\mathfrak{M}$ as a $\varphi(\hat{\mathcal{O}})$-submodule of $\hat{\mathfrak{M}}$. By [Liu10, Lem. 3.1.1], there is an isomorphism of $\mathbb{Z}_p[G_{\infty}]$-modules $\theta : T_{\hat{\mathcal{O}}}(\mathfrak{M}) \rightarrow \hat{T}(\hat{\mathfrak{M}})$ where for $f \in T_{\hat{\mathcal{O}}}(\mathfrak{M})$, we have
\begin{equation}
\theta(f)(a \otimes x) = a \varphi(f(x)), \forall a \in \hat{\mathcal{R}}, \forall x \in \mathfrak{M}.
\end{equation}

**Theorem 5.3.4.** [Liu10] The functor $\hat{T}$ induces an anti-equivalence between $\text{Mod}^{\varphi,\hat{G}}_{\hat{\mathcal{O}},\hat{\mathcal{R}}}$ and $\text{Rep}_{p,\geq 0}^{st} (G_K)$, where the latter is the category of $G_K$-stable $\mathbb{Z}_p$-lattices in semi-stable representations of $G_K$ with non-negative Hodge-Tate weights.

**5.3.5. Construction of $(\varphi, \hat{G})$-modules.** It is important that we recall the construction of $(\varphi, \hat{G})$-modules starting from an integral semi-stable Galois representation.

**Step 1:** construction of the $\varphi$-module. (See [Kis06] for more details.) Let $T \in \text{Rep}_{p,\geq 0}^{st} (G_K)$, and let $V = T[1/p]$. Let $D \in \text{MF}_{K_0}^{\varphi,N,\text{wa}}$ be the module associated to $V$, and let $M = M(D) \in \text{Mod}_{\hat{\mathcal{O}}}^{G, N, 0}$. Let $M_R := \mathcal{R} \otimes M$. The slope filtration theory of Kedlaya implies that a $\varphi$-module over $\mathcal{R}$ of pure slope 0 can be descended to the “bounded Robba ring” $\mathcal{R}^b := \mathbb{B}_{K_\infty}^+$ in our Def. 2.4.2), in the sense that there exists some $\varphi$-module $M_{R^b}$ over $\mathcal{R}^b$ such that
\begin{equation}
\mathcal{R} \otimes M_{R^b} = M_R.
\end{equation}
Then very roughly speaking, we can define some module over $\mathcal{O}[1/p]$ by $M^b := M \cap M_{R^b}$ by noting the fact that $\mathcal{O}[1/p] = \mathcal{O} \cap \mathcal{R}^b$. One then get some integral $\mathfrak{M}$ inside $M^b$ using the integral structure of $T$ (indeed, the integral structure of the $(\varphi, \tau)$-module associated to $T$; cf [Kis06, Lem. 2.1.15]). Finally, $\mathfrak{M}$ is finite height because $M$ is so by [Kis06, Lem. 1.2.2].

**Step 2:** equipping the $\tau$-action. (Step 2 of the construction is more important for our purpose; see [Liu10] for more details.) Let $D := D_{\mathcal{O}}(M) \in \text{MF}_{K_0}^{\varphi,N}$ as in §5.2.2. Then we can define a $G_K$-action on $\mathbb{B}_{\text{cris}}^+ \otimes_{\mathcal{O}} D$ such that:
\begin{equation}
(5.3.3) \quad \sigma(a \otimes x) = \sum_{i=0}^{\infty} \sigma(a) \gamma_i(- \log(\sigma)) \otimes N_D^i(x), \quad \forall \sigma \in G_K, a \in \mathbb{B}_{\text{cris}}^+, x \in D
\end{equation}
where $\gamma_i(\sigma) = \frac{\sigma^i(\sigma)}{\sigma}$. Let us recall that (5.3.3) is well-defined and converges with respect to the $p$-adic topology (induced from that on $\mathbb{B}_{\text{cris}}^+$), by e.g., [Liu08, Lem. 5.1.1]. By the construction of $\mathfrak{M}$, we obviously have
\begin{equation}
S_{K_0} \otimes_{\varphi, \hat{\mathcal{O}}} \mathfrak{M} = D.
\end{equation}
Hence for $x \in \mathfrak{M}$, we have
\begin{equation}
(5.3.4) \quad (\tau - 1)(1 \otimes x) = \sum_{m=1}^{\infty} \gamma_m(t) \otimes N_D^m(1 \otimes x).\]
This shows that we have indeed defined a $\tau$-action (hence $\hat{G}$-action) on $R_{K_0} \otimes_{\varphi, \hat{G}} M$. One then needs to use some rather non-trivial argument to show that the $\tau$-action is indeed defined over $W(R)$, i.e.,

$$(5.3.5) \quad W(R) \otimes_{\varphi, \hat{G}} M \subset B_{\text{cris}}^+ \otimes_{\varphi, \hat{G}} M = B_{\text{cris}}^+ \otimes_S D$$

Since we do not use the ingredient in the proof of $(5.3.5)$, let us only mention that the proof makes use of the fact that $M$ is of finite height; see [Liu10, Prop. 3.2.1] for more details. This finishes the proof that the $\tau$-action is defined over $\hat{R}$. Finally, one easily checks that $\hat{G}$ acts on $\hat{M} / I_+ \hat{M}$ trivially by using $(5.3.4)$. This finishes the construction of the $(\varphi, \hat{G})$-module.

5.4. Finite height modules and overconvergent modules. In this subsection, we review a result of Cherbonnier, which says that a finite free étale $\varphi$-modules (cf. Def. 3.1.1) always contains a maximal finite free “overconvergent submodule” (possibly of a smaller rank). If the étale $\varphi$-module is of finite height, we show that the $\mathcal{S}$-module inside it generates the maximal overconvergent submodule.

Let

$$O_{\epsilon} := A_{K_\infty}, \quad \mathcal{O}_{\epsilon} := A_{K_\infty}^! := A_{K_\infty} \cap B_{K_\infty}^!.$$

Recall $\text{Mod}_{\mathcal{O}_{\epsilon}}$ is the category of finite free étale $\varphi$-modules (cf. Def. 3.1.1). Define $\text{Mod}^{\mathcal{G}}_{\mathcal{O}_{\epsilon}}$ analogously; indeed, it consists of finite free $\mathcal{O}_{\epsilon}^!$-modules $M$ equipped with $\varphi_{\mathcal{O}_{\epsilon}}$-semi-linear $\varphi : M \to M$ such that $1 \otimes \varphi : \varphi^* M \to M$ is an isomorphism.

**Definition 5.4.1.** (1) Let

$$j^* : \text{Mod}^{\mathcal{G}}_{\mathcal{O}_{\epsilon}} \to \text{Mod}^{\mathcal{G}}_{\mathcal{O}_{\epsilon}}$$

denote the functor where $\mathcal{N} \mapsto \mathcal{N} \otimes_{\mathcal{O}_{\epsilon}} \mathcal{O}_{\epsilon}$. (2) Let $\mathcal{M} \in \text{Mod}^{\mathcal{G}}_{\mathcal{O}_{\epsilon}}$. Let $F_1(\mathcal{M})$ be the set consisting of $\mathcal{O}_{\epsilon}^!$-submodules $\mathcal{N} \subset \mathcal{M}$ of finite type such that $\varphi(\mathcal{N}) \subset \mathcal{N}$. Let $j^!(\mathcal{M})$ be the union of elements in $F_1(\mathcal{M})$.

**Proposition 5.4.2.** Let $\mathcal{M} \in \text{Mod}^{\mathcal{G}}_{\mathcal{O}_{\epsilon}}$.

1. We have $j^!(\mathcal{M}) \in \text{Mod}^{\mathcal{G}}_{\mathcal{O}_{\epsilon}}$, and hence $j^*$ defines a functor

$$j_* : \text{Mod}^{\mathcal{G}}_{\mathcal{O}_{\epsilon}} \to \text{Mod}^{\mathcal{G}}_{\mathcal{O}_{\epsilon}}.$$

Furthermore, we have

$$\text{rk}_{\mathcal{O}_{\epsilon}^!} j^!(\mathcal{M}) \leq \text{rk}_{\mathcal{O}_{\epsilon}} \mathcal{M}.$$

2. The functor $j^*$ is a left adjoint to $j^*$, namely, if $\mathcal{N}_1 \in \text{Mod}^{\mathcal{G}}_{\mathcal{O}_{\epsilon}}$, $\mathcal{N}_2 \in \text{Mod}^{\mathcal{G}}_{\mathcal{O}_{\epsilon}}$, then

$$\text{Hom}(\mathcal{N}_1, j^!(\mathcal{N}_2)) = \text{Hom}(j^*(\mathcal{N}_1), \mathcal{N}_2),$$

where $\text{Hom}$ denotes the set of morphisms in each category.

**Proof.** This is contained in [Che96, §3.2, Prop. 2]. Let us mention that the “$\varphi$-operator” in loc. cit. can be any lift of the Frobenius $x \mapsto x^p$ on the ring $\mathcal{O}_{\epsilon} / \mathcal{O}_{\epsilon}$ (cf. the beginning of [Che96, §2.2]); hence [Che96, §3.2, Prop. 2] applies to the different $\varphi$-actions in both the $(\varphi, \Gamma)$-module setting and $(\varphi, \tau)$-module setting.

**Definition 5.4.3.** Let $T \in \text{Rep}_{\mathcal{G}}(G_\infty)$. We say $T$ is of finite $E(u)$-height with respect to $\bar{\tau} = \{\pi_n\}_{n \geq 0}$ if there exists some $\mathcal{M} \in \text{Mod}^{\mathcal{G}}_\infty$ such that $T \otimes_{\mathcal{O}_{\epsilon}} \mathcal{M} \simeq T$. We say that $V \in \text{Rep}_{\mathcal{G}}(G_\infty)$ is of finite $E(u)$-height with respect to $\bar{\tau}$ if there exist some $G_\infty$-stable $\mathcal{Z}_p$-lattice $T$ (equivalently, any $G_\infty$-stable lattice by [Kis06, Lem. 2.1.15]) which is so. We say that $V \in \text{Rep}_{\mathcal{G}}(G_K)$ is of finite $E(u)$-height with respect to $\bar{\tau}$ if $V_{|G_\infty}$ is so. Throughout the paper, when $E(u)$ and $\bar{\tau}$ are unambiguous, we just say of finite height for short.

**Remark 5.4.4.** (1) Let $V \in \text{Rep}_{\mathcal{G}}(G_K)$, and let $(D, \varphi_D, \hat{G})$ be the $(\varphi, \tau)$-module attached to $V$. Then clearly $V$ is of finite height if and only there exists some $\mathcal{M} \in \text{Mod}^{\mathcal{G}}_\infty$ such that $B_{K_\infty} \otimes \mathcal{M} \simeq D$ as $\varphi$-modules. Indeed, note that Def. 5.4.3 has nothing to do with $\tau$-part of the $(\varphi, \tau)$-modules.
(2) Let \( m \geq 1 \). Let \( \pi_{K_m} := \{ \pi_i \}_{i \geq m-1} \in R \), let \( u_{K_m} := [\pi_{K_m}] \in W(R) \), and let \( G_{K_m} := W(k)[u_{K_m}] \). We have \( W(k) \)-linear embeddings

\[
\mathcal{G} \to G_{K_m} \to W(R), \quad \text{where } u \mapsto u^{m-1}_{K_m};
\]

note this implies \( E(u) = E_{K_m}(u_{K_m}) \in W(R) \) where \( E_{K_m}(X) = E(X^{m-1}) \) is the minimal polynomial of \( \pi_m \) over \( K_0 \). Let \( V \in \text{Rep}_{\mathcal{O}_\mathbb{Q}_p}(G_K) \). If \( V \) is of finite \( E(u) \)-height with respect to \( \{ \pi_i \}_{i \geq m-1} \), then \( V |_{G_{K_m}} \) is of finite \( E_{K_m}(u_{K_m}) \)-height with respect to \( \{ \pi_i \}_{i \geq m-1} \), but not vice versa!

**Proposition 5.4.5.** Let \( T \in \text{Rep}_{\mathcal{O}_p}(G_{\mathbb{Q}}) \), and let \( M \in \text{Mod}_{\mathcal{O}_p} \) be its associated étale \( \varphi \)-module. Suppose \( T \) is of finite height, and let \( \mathfrak{M} \in \text{Mod}_{\mathcal{O}_p} \) be the associated module. Then \( \mathfrak{M} \otimes_{\mathcal{O}} \mathcal{O}_{\mathbb{E}}^1 = j_{\mathbb{E}}^!(\mathcal{M}) \).

**Proof.** Clearly \( \mathfrak{M} \subset j_{\mathbb{E}}^!(\mathcal{M}) \) by Def. 5.4.1. By Prop. 5.4.2(2),

\[
\text{Hom}(\mathfrak{M} \otimes_{\mathcal{O}} \mathcal{O}_{\mathbb{E}}^1, j_{\mathbb{E}}^!(\mathcal{M})) = \text{Hom}(\mathfrak{M} \otimes_{\mathcal{O}} \mathcal{O}_{\mathbb{E}}, \mathcal{M})
\]

One can check that the morphism on the LHS of (5.4.1) corresponding to the isomorphism on the RHS has to be an isomorphism itself, using the fact that

\[
\mathcal{O}_{\mathbb{E}}^1 \cap (\mathcal{O}_{\mathbb{E}})^x = (\mathcal{O}_{\mathbb{E}}^1)^x.
\]

(See e.g. the paragraph above [Che96, §3.1, Def. 5] for a proof of (5.4.2)).

\( \Box \)

5.5. **Coincidence of monodromy operators.** In this subsection, we show that the monodromy operators in Kisin’s construction and in our construction *coincide* in the case of semi-stable representations. We first recall some useful notations. For \( n \geq 1 \), let

\[
I[n] B_{\text{cris}}^+ := \{ x \in B_{\text{cris}}^+ : \varphi^k(x) \in \text{Fil}^n B_{\text{cris}}^+, \text{ for all } k > 0 \};
\]

and for any subring \( A \subset B_{\text{cris}}^+ \), write \( I[n] A := A \cap I[n] B_{\text{cris}}^+ \). As shown in [Liu10, Lem. 3.2.2], \( I[n] W(R) \) is a principle ideal generated by \( (\varphi(t))^n \).

**Theorem 5.5.1.** Let \( T \in \text{Rep}_{\mathcal{O}_p}^{\text{st} \geq 0}(G_K) \), and let \( V = T[1/p] \). Let \( (\mathfrak{M}, \varphi, \mathcal{G}) \) be the \((\varphi, \mathcal{G})\)-module associated to \( T \).

- Let \( D_{\text{rig}, K_\infty}^! (V) \) be the rigid-overconvergent \((\varphi, \tau)\)-module attached to \( V \), and let \( N_{\text{la}}^V \) denote the monodromy operator defined in Thm. 4.2.1.
- Let \( (M = \mathcal{O} \otimes_{\mathfrak{M}} \mathcal{M}, \varphi, N_{\text{Kis}}^V) \in \text{Mod}_{\mathcal{O}_p}^{\mathcal{O}_{\mathcal{E}}, \mathfrak{M}} \) be the module corresponding to \( V \) constructed by Kisin. Recall by Prop. 5.4.5, we have

\[
M \otimes_{\mathcal{O}} B_{\text{rig}, K_\infty}^! = \mathfrak{M} \otimes_{\mathfrak{M}} B_{\text{rig}, K_\infty}^! = D_{\text{rig}, K_\infty}^! (V);
\]

hence we can extend \( N_{\text{Kis}}^V \) to \( D_{\text{rig}, K_\infty}^! (V) \) by \( N_{\text{Kis}}^V \otimes 1 + 1 \otimes N_{\mathfrak{M}}^V B_{\text{rig}, K_\infty}^! \), which we still denote as \( N_{\text{Kis}}^V \).

Then on \( D_{\text{rig}, K_\infty}^! (V) \), we have

\[
N_{\text{Kis}}^V = N_{\text{la}}^V.
\]

**Proof.** **Step 1:** We first review some known results concerning the interplay between the \( \tau \)-operator and \( N_{\mathcal{D}} \). All contents in Step 1 are known results; cf. in particular [Liu12, §2.4] for more details.

Let \( x \in \mathfrak{M} \). By (5.3.3), we have

\[
(\tau - 1)^n (1 \otimes x) = \sum_{m=n}^{\infty} \left( \sum_{i_1 + \ldots + i_m = m, i_j \geq 1} \frac{m!}{i_1! \ldots i_m!} \right) \gamma_m(t) \otimes N_{\mathcal{D}}^m (1 \otimes x).
\]

Hence \((\tau - 1)^n (1 \otimes x) \in I[n] B_{\text{cris}}^+ \otimes_{\mathfrak{M}} \mathfrak{M} \). By (5.3.5), \( W(R) \otimes_{\mathfrak{M}} \mathfrak{M} \) is \( G_K \)-stable, and hence

\[
(\tau - 1)^n (1 \otimes x) \in I[n] W(R) \otimes_{\mathfrak{M}} \mathfrak{M}.
\]
Using the fact that \( \frac{\varphi(t)^n}{nt} \in A_{\text{cris}} \) and that they converge to 0 as \( n \to +\infty \), we can conclude that \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\tau - 1)^n}{nt} (1 \otimes x) \) converges in \( A_{\text{cris}} \otimes \varphi, \in \mathfrak{M} \), and we have
\[
N_D(1 \otimes x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\tau - 1)^n}{nt} (1 \otimes x).
\]

Note that if we replace \( \tau \) by \( \tau^a \) with any \( a \in \mathbb{Z}_p - \{0\} \), the above argument still works (change \( t \) in (5.5.1) to \( at \)), and we have
\[
(\tau^a - 1)^n (1 \otimes x) \in I^{[n]}W(R) \otimes \varphi, \in \mathfrak{M},
\]
and hence
\[
1 \otimes N_D(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\tau^a - 1)^n}{ant} (1 \otimes x).
\]

Note that the LHS of (5.5.3) is independent of \( a \).

**Step 2:** By theory of \((\varphi, \hat{G})\)-modules (cf. [Liu10, Prop. 3.1.3]), we have a \( G_K \)-equivariant embedding
\[
W(R) \otimes \varphi, \in \mathfrak{M} \hookrightarrow W(R) \otimes \mathbb{Z}_p T,
\]
which induces a \( G_K \)-equivariant embedding
\[
W(R) \otimes \mathfrak{M} = W(R) \otimes \varphi^{-1}, W(R) W(R) \otimes \varphi, \in \mathfrak{M} \hookrightarrow W(R) \otimes \mathbb{Z}_p T.
\]
Indeed, this together with Prop. 5.4.5 shows that we have a \( \varphi \)-equivariant and \( G_K \)-equivariant isomorphism
\[
\hat{B}_{\text{rig}, L}^* \otimes \mathfrak{M} \simeq \hat{B}_{\text{rig}, L}^* \otimes \hat{B}_{\text{rig}, K^\infty}^* D_{\text{rig}, K^\infty}^* (V);
\]
this says that the \( \tau \)-action from the theory of \((\varphi, \hat{G})\)-modules (LHS of (5.5.5)) is the same as the \( \tau \)-action from \((\varphi, \tau)\)-modules (RHS of (5.5.5)).

For \( x \in \mathfrak{M} \), using the \( G_K \)-action from (5.5.4), Eqn. (5.5.2) implies that
\[
(\tau^a - 1)^n (x) \in t^n W(R) \otimes \mathfrak{M}.
\]
Recall that \( t \in W(R) - pW(R) \) and satisfies \( \varphi(t) = \frac{pE(a)}{E(0)} \cdot t \). Hence it is easy to deduce that \( \mathfrak{t} \in \mathfrak{m}_R - \{0\} \), where \( \mathfrak{t} \in R \) denotes the mod \( p \) reduction. It implies that
\[
\frac{t^n}{n} \in \hat{B}^{[0, +\infty)},
\]
and they converge to 0 as \( n \to +\infty \). Thus (5.5.6) implies that
\[
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\tau^a - 1)^n}{n \cdot p} (x) \text{ converges in } \hat{B}^{[0, +\infty]} \otimes \mathfrak{M}.
\]

By the construction, when \( a \in p^a \mathbb{Z}_p \) with \( a \gg 0 \), the operator in (5.5.7) is precisely \( N_{\mathfrak{V}}^\text{la} \); furthermore, using the continuous embedding \( i_0 : \hat{B}^{[0, +\infty]} \hookrightarrow \hat{B}_{\text{cris}}^+ \otimes \mathfrak{M} \), the series in (5.5.7) converges in \( \hat{B}_{\text{cris}}^+ \otimes \mathfrak{M} \) to the same operator. Hence by constructions in Step 1, we must have
\[
\frac{p}{\varphi(\lambda)} \otimes \varphi N_{\mathfrak{V}}^\text{la} (x) = N_D(1 \otimes \varphi, x) \text{ for } x \in \mathfrak{M}.
\]
(Note that this also shows that the expression in (5.5.7) is independent for any \( a \in \mathbb{Z}_p \).) However, by the construction in [Liu08] recalled in our §5.2.2, we also have
\[
\frac{p}{\varphi(\lambda)} \otimes \varphi N_{\mathfrak{V}} \text{Kis} (x) = N_D(1 \otimes \varphi, x) \text{ for } x \in \mathfrak{M}.
\]
Thus we have \( \nabla^\text{la} = \nabla \text{Kis} \). \( \square \)

6. Frobenius regularization and finite height representations

In this section, we use our monodromy operator to study finite \( E(u) \)-height representations. In §6.1, we show that the monodromy operator can be descended to the ring \( \mathcal{O} \) for a finite \( E(u) \)-height representation; in §6.2, we show such representations are potentially semi-stable. The results will be used in §7 to construct the simplified \((\varphi, \hat{G})\)-modules.
6.1. Frobenius regularization of the monodromy operator.

Proposition 6.1.1. Suppose $T \in \text{Rep}_{\mathbb{Z}_p}(G_K)$ is of finite $E(u)$-height (with respect to the fixed choice of $\pi = \{\pi_n\}_{n \geq 0}$), and let $\mathfrak{M} \in \text{Mod}_{\mathbb{Z}_p}^E$ be the corresponding Breuil-Kisin module. Let $N_T$ be the monodromy operator constructed in Thm. 4.2.1, then $N_T(\mathfrak{M}) \subset \mathfrak{M} \otimes_{\mathbb{Z}_p} \mathcal{O}$.

We will use a “Frobenius regularization” technique to prove Prop. 6.1.1. Roughly, by Thm. 4.2.1, we already know that the coefficients of the matrix for $N_T$ live near the boundary of the open unit disk; to show that they indeed live on the entire open unit disk in the finite height case, we use the Frobenius operator to “extend” their range of convergence: this is where we critically use the finite height condition for the Frobenius operator. Indeed, the proof relies on the following key lemma.

Lemma 6.1.2. Let $h \in \mathbb{R}_{>0}$ and $r \in \mathbb{Z}_{>0}$. For $s > 0$, let $\tilde{A}^{(s, +\infty)}$ be the subset consisting of $x \in \tilde{B}^{(s, +\infty)}$ such that $W[x, s] \geq 0$, and let $A^{[s, +\infty)} := \tilde{A}^{(s, +\infty)} \cap B^{[s, +\infty)}$. Then we have:

\begin{align*}
(6.1.1) & \quad \bigcap_{n \geq 0} p^{-hn} \tilde{A}^{[r/p^n, +\infty)} = \tilde{A}^{[0, +\infty]} = \tilde{A}^+ \\
(6.1.2) & \quad \bigcap_{n \geq 0} p^{-hn} \tilde{A}^{[r/p^n, +\infty)} \subset \tilde{B}^{[0, +\infty)} \\
(6.1.3) & \quad \bigcap_{n \geq 0} p^{-hn} A^{\infty}_{K_{\infty}} = A^{[0, +\infty]} = \mathcal{S} \\
(6.1.4) & \quad \bigcap_{n \geq 0} p^{-hn} A^{\infty}_{K_{\infty}} \subset B^{[0, +\infty)} = \mathcal{O}
\end{align*}

Proof. The relations (6.1.1) and (6.1.2) are from [Ber02, Lem. 3.1]. We can intersect $B_{K_{\infty}}$ with (6.1.1) to get (6.1.3). For (6.1.4), it follows from similar argument as in [Ber02, Lem. 3.1]. Indeed, suppose $x \in \text{LHS}$, then for each $n \geq 0$, we can write $x = a_n + b_n$ with $a_n \in p^{-hn} A^{\infty}_{K_{\infty}}$ and $b_n \in B^{[0, +\infty)}$. Then

$$a_n - a_{n+1} \subset \bigcap_{n \geq 0} p^{-h(n+1)} \tilde{A}^{[r/p^n, +\infty)} \cap B^{[0, +\infty)} = p^{-h(n+1)} A^{[0, +\infty)}_{K_{\infty}} \cap B^{[0, +\infty)} = \mathcal{O}.$$

By modifying $a_{n+1}$, we can assume $a_n = a$ for all $n \geq 0$, and hence $a \in \mathcal{S}$ by (6.1.3). Thus, $x = a + b_0 \in \mathcal{O}$. \hfill $\square$

Remark 6.1.3. We use an example to illustrate the idea of (6.1.4). Consider the element $1/u \in \bigcap_{n \geq 0} \tilde{B}^{[r/p^n, +\infty)}$, it does not belong to LHS of (6.1.4) because the valuations $W[r/p^n, r/p^n]/u$ converge to $-\infty$ in an exponential rate, rather than the linear rate on LHS of (6.1.4).

By (6.1.4), the proof of Prop. 6.1.1 will rely on some calculations of valuations and ranges of convergence; we first list two lemmas. For the reader’s convenience, recall that for $x = \sum_{i \geq 0} p^i [x_i] \in \tilde{B}^{[r/\infty, +\infty]}$ and for $s \geq r$, $s > 0$ we have the formula

$$W[x, s] := \inf_{k \geq 0} \{k + \frac{p-1}{ps} \cdot v_E(x_k)\} = \inf_{k \geq 0} \{k + \frac{p-1}{ps} \cdot w_k(x)\}.$$

Lemma 6.1.4. (1) Suppose $x \in \tilde{B}^+$, then

\begin{align*}
(6.1.5) & \quad W[0, r] > W[s, s] \quad \forall 0 < r < s < +\infty. \\
(6.1.6) & \quad W[s, s](\varphi(x)) > W[s, s](x) \quad \forall s \in (0, +\infty)
\end{align*}

(2) We have

$$W[s, s](E(u)) \in (0, 1], \quad \forall s \in (0, +\infty)$$

(3) Recall $\lambda = \prod_{n \geq 0} \varphi^n(\frac{E(u)}{E(0)})$. Let $\ell \in \mathbb{Z}_{\geq 0}$. Then

$$W[s, s](\lambda) > -\ell, \quad \forall s \in (0, r_\ell].$$

Proof. Item (1) follows from definition.

For Item (2), first recall that $W[s, s]$ is multiplicative. Note that $E(u) = \sum a_i u^i$ where $a_\infty = 1$, $p|a_i$ for $i > 0$ and $p|a_0$. It is easy to see that $W[s, s](E(u)) > 0$ for all $s > 0$. When
\( s \leq r_0, \) \( W^{[s,s]}(u^c + a_0) = 1 \) and \( W^{[s,s]}(\sum_{i=1}^{c} a_i u^i) > 1, \) and hence \( W^{[s,s]}(E(u)) = 1. \) Hence \( W^{[s,s]}(E(u)) \leq 1 \) if \( s > r_0 \) by Item (1).

For Item (3), we have \( W^{[s,s]}(\lambda) = \sum_{i \geq 0} W^{[s,s]}(\varphi^i(E(u)/E(0))). \) Using Item (1), it suffices to treat the case \( s = r_k. \) We have

\[
W^{[r_r,r_l]}(\lambda) = W^{[r_r,r_l]}(\varphi^l(\lambda)) + W^{[r_r,r_l]}(\prod_{i=0}^{l-1} \varphi^i(\frac{E(u)}{E(0)}))
\]

\[
> W^{[r_0,r_0]}(\lambda) + W^{[r_r,r_l]}(\prod_{i=0}^{l-1} \varphi^i(\frac{1}{E(0)}))
\]

\[
> -\ell
\]

where the last row uses that \( W^{[r_0,r_0]}(\lambda) > 0 \) (note \( W^{[r_0,r_0]}(\frac{E(u)}{E(0)}) = 0 \) and apply (6.1.6)).

**Lemma 6.1.5.** Suppose \( x \in \mathcal{B}_{K_0}^h. \) Let \( g(u) \in K_0[u] \) be an irreducible polynomial such that \( g(u) \notin K_0[u^p] = \varphi(K_0[u]). \) Suppose \( g(u)^h \varphi(x) \in \mathcal{B}_{K_0}^h \) for some \( h \geq 1, \) then we have \( x \in \mathcal{B}_{K_0}^{h/p}. \)

**Proof.** We can suppose \( \inf I \neq 0 \) (otherwise the lemma is trivial). Then we can further suppose \( g(u) \neq u \) (otherwise the lemma follows easily from Cor. 2.3.5). First we treat the case \( h = 1. \) Suppose \( g = \sum_{i=0}^{N} b_i u^i. \) Let \( g_1 := \sum_{i=0}^{N} b_i u^i \) (i.e., \( g_1 \) contains all the \( u \)-powers with \( p \)-divisible exponents, including the non-zero constant term) and let \( g_2 := \sum_{i=0}^{N} b_i u^i; \) hence \( g_2 \neq 0 \) but \( g_2 \neq g. \) Recall that if we write \( x = \sum_{i \in \mathbb{Z}} a_i u^i \) with \( a_i \in K_0, \) then it satisfies certain convergence conditions with respect to \( I \) as described in Lem. 2.3.3; similarly, the expansion of the product \( g(u) \varphi(x) = (g_1 + g_2) \cdot (\sum_{i \in \mathbb{Z}} \varphi(a_i) u^{p^i}) \) satisfies these convergence conditions. However, there is no “intersection” between the two parts \( g_1 \varphi(x) \) and \( g_2 \varphi(x) \) as the former part contains exactly all the \( u \)-powers with \( p \)-divisible exponents. Hence both \( g_1 \varphi(x) \) and \( g_2 \varphi(x) \) satisfy the aforementioned convergence conditions, and hence both \( g_1 \varphi(x), g_2 \varphi(x) \in \mathcal{B}_{K_0}^h. \) Since \( g \) and \( g_2 \) are co-prime in \( K_0[u], \) we must have \( \varphi(x) \in \mathcal{B}_{K_0}^h. \) Thus \( x \in \mathcal{B}_{K_0}^{h/p} \) by Cor. 2.3.5.

Suppose now \( h \geq 2, \) and write \( g^h = g_1 + g_2 \) similarly as above. Similarly we have \( g_2 \varphi(x) \) in \( \mathcal{B}_{K_0}^h. \) If \( (g^h, g_2) = g^k \) with \( k < h, \) then \( g^k \varphi(x) \in \mathcal{B}_{K_0}^h. \) This reduces the proof to an induction argument.

**Remark 6.1.6.** The condition \( g(u) \notin K_0[u^p] \) in Lem. 6.1.5 is necessary. For example, suppose \( e = p, \) let \( g(u) = u^p - p, \) and let \( x = \frac{1}{u^p} = \frac{1}{u} \left( \sum_{i \geq 0} \frac{(E(0))^i}{u^i} \right) \in \mathcal{B}_{K_0}^{[r_0, +\infty)}. \) We have \( g(u) \varphi(x) = 1, \) but \( x \notin \mathcal{B}_{K_0}^{[r_0/2, +\infty)}. \)

**Proof of Prop. 6.1.1.** Let \( \vec{e} \) be an \( \mathcal{G} \)-basis of \( \mathcal{M}, \) and suppose

\[
\varphi(\vec{e}) = \vec{e} A, N_\varphi(\vec{e}) = \vec{e} M, \quad A \in \text{Mat}(\mathcal{G}), M \in \text{Mat}(\mathcal{B}_{rig,K_0}^1).
\]

Since \( N_\varphi \varphi = \frac{p}{E(0)} E(u) \varphi N_\varphi, \) we have

\[
(6.1.7) \quad MA + N_\varphi(A) = \frac{p}{E(0)} E(u) A \varphi(M).
\]

Let \( B \in \text{Mat}(\mathcal{G}) \) such that \( AB = E(u)^h \cdot \text{Id}, \) then we get

\[
(6.1.8) \quad BMA + BN_\varphi(A) = \frac{p}{E(0)} E(u)^{h+1} \varphi(M).
\]

Suppose \( N \geq 0 \) is the maximal number such that \( E(u) \in K_0[u^{p^N}] \). Denote

\[
D_i(u) := \prod_{k=1}^{i} \varphi^{-k}(E(u)), \quad \forall 0 \leq i \leq N;
\]

where we always let \( D_0(u) := 1. \) For brevity, we denote \( E := E(u) \) and \( D_i := D_i(u). \)

Let

\[
\vec{M} := D_N^{h+1} M,
\]

then from (6.1.8), we have

\[
(6.1.9) \quad B\vec{M}A + D_N^{h+1} BN_\varphi(A) = \frac{p}{E(0)} \varphi^{-N}(E^{h+1}) \varphi(N).
\]
Let $\ell \gg 0$ so that 
$$\tilde{M} \in \text{Mat}(\mathcal{B}_{K_\infty}^{[r, +\infty]}).$$

Note that $A, B, D, \in \text{Mat}(\mathcal{S})$ and $N_{\mathcal{X}}(A) \in \lambda \cdot \text{Mat}(\mathcal{S})$, hence

$$\text{LHS of (6.1.9)} \in \text{Mat}(\mathcal{B}_{K_\infty}^{[r, +\infty]}).$$

Since $\varphi^{-N}(E)$ satisfies the conditions in Lem. 6.1.5, we can iteratively use the lemma and (6.1.9) to conclude that

$$(6.1.10) \quad \tilde{M} \in \text{Mat}\left(\bigcap_{n \geq 0} \mathcal{B}_{K_\infty}^{[r/n, +\infty]}\right).$$

To show that $M \in \text{Mat}(\mathcal{O})$, our proof proceeds in 2 steps: we first show that $\tilde{M} \in \text{Mat}(\mathcal{O})$ using Lem. 6.1.2; then we show that $M \in \text{Mat}(\mathcal{O})$ using a trick of Caruso.

**Step 1:** Write $r = r_\ell$. Choose $c \gg 0$ (in particular, we ask that $c > \ell$) such that

$$\tilde{M} \in \text{Mat}(p^{-c}A_{K_\infty}^{[r, +\infty]}).$$

Then we have

$$W^{[r,r]}(\text{LHS of (6.1.9)}) \geq \min\{W^{[r,r]}(\tilde{M}), W^{[r,r]}(\lambda)\}, \text{ since } W^{[r,r]} \text{ is multiplicative}$$

$$\geq \min\{-c, -\ell\}, \text{ by Lem. 6.1.4(3)}$$

$$= -c$$

By Lem. 6.1.4(2), we have

$$W^{[r,r]}(\varphi^{-N}(E)) = W^{[p^N r, p^N r]}(E) \leq 1.$$ 

Hence using (6.1.9), we have

$$W^{[r/p^r / p]}(\tilde{M}) = W^{[r,r]}(\varphi(\tilde{M})) \geq -c - (h + 1).$$

Iterate the above argument, we get that for all $n$,

$$W^{[r/p^n r / p^n]}(\tilde{M}) \geq -c - n(h + 1).$$

By Lem. 2.2.6, we have

$$\tilde{M} \in p^{-c-n(h+1)}A_{K_\infty}^{[r/p^n, +\infty]}, \quad \forall h.$$ 

Using Eqn. (6.1.4) in Lem. 6.1.2, we can conclude that

$$\tilde{M} \in \text{Mat}(\mathcal{O}).$$

**Step 2:** In this step, we show that $M \in \text{Mat}(\mathcal{O})$. If $N = 0$, then there is nothing to prove. Suppose now $N \geq 1$. So we have

$$(6.1.11) \quad \tilde{M} = D_{N}^{h+1} \cdot M = \varphi^{-N}(E^{h+1}) \cdot D_{N-1}^{h+1} \cdot M \in \text{Mat}(\mathcal{O}).$$

From (6.1.7), we also have

$$ME^{h} + N_{\mathcal{X}}(A)B = \frac{p}{E(0)}EA\varphi(M)B,$$

and hence

$$E^{h+1}D_{N-1}^{h+1} \cdot (ME^{h} + N_{\mathcal{X}}(A)B) = \frac{p}{E(0)}EA\varphi(\tilde{M})B.$$ 

So we have

$$(6.1.12) \quad E^{h+1}D_{N-1}^{h+1} \cdot ME^{h} = E^{2h+1} \cdot D_{N-1}^{h+1} \cdot M \in \text{Mat}(\mathcal{O}).$$

Note that both $\varphi^{-N}(E)$ and $E$ are irreducible in $K_0[u]$ and hence they are co-prime. Hence we can use (6.1.11) and (6.1.12) to conclude that

$$D_{N-1}^{h+1} \cdot M \in \text{Mat}(\mathcal{O}).$$

If $N - 1 \geq 1$, we can repeat the above argument. (Note that in the argument of this Step 2, we do not use the fact $\varphi^{-N}(E) \notin K_0[u^p]$; indeed, this condition is used only to conclude (6.1.10)). Hence in the end we must have

$$M \in \text{Mat}(\mathcal{O}).$$

$\square$
Remark 6.1.7. The argument in Step 2 above is taken from the paragraph containing [Car13, p. 2595, Eqn. (3.15)]; in particular, the use of $D_{N}(u)$ is inspired by the argument in loc. cit.. However, the argument before Step 2 are completely different from those in [Car13].

6.2. Potential semi-stability of finite height representations. In this subsection, we show that finite height representations are potentially semi-stable; in fact, our result is slightly stronger. Let us first recall two useful lemmas.

For any $K \subset X \subset \mathcal{K}$, let

$$m(X) := 1 + \max\{i \geq 1, \mu_{i} \in X\}.$$ 

Recall for each $n \geq 1$, we let $K_{n} = K(\pi_{n-1})$ (hence $K_{1} = K$). Note that

- (for $n \geq 2$), $K_{n}$ here = “$K_{n}$” in [Kis06] = “$K_{n-1}$” in [Liu10] and [Oze17].

Lemma 6.2.1. Let $m := m(K_{ur})$ where $K_{ur}$ is the maximal unramified extension of $K$ (contained in $\mathcal{K}$). Suppose $V \in \text{Rep}_{\mathbb{Q}_{p}}(G_{K})$ is semi-stable over $K_{n}$ for some $n \geq 1$, then $V$ is semi-stable over $K_{m}$.

Proof. This is proved in [Oze17, Rem. 2.5]. Note that this fixes a gap in [Liu10, Thm. 4.2.2], where Liu claims a similar statement using “$m := m(K)$” instead. Note that in general, it is possible that $m(K_{ur}) > m(K)$, cf. [Oze17, §5.4].

Lemma 6.2.2. Let $K \subset K^{(1)} \subset K^{(2)}$ be finite extensions such that $K^{(2)}/K^{(1)}$ is totally ramified, then the restriction functor from semi-stable $G_{K^{(1)}}$-representations to semi-stable $G_{K^{(2)}}$-representations is fully faithful.

Proof. This is [Oze17, Lem. 4.11]. (It is basically [Car13, Prop. 3.4], but it is completely elementary).

Theorem 6.2.3. Let $V \in \text{Rep}_{\mathbb{Q}_{p}}(G_{K})$. Then $V$ is of finite $E(u)$-height (with respect to the fixed choice of $\{\pi_{n}\}_{n \geq 0}$) if and only if there exists $W \in \text{Rep}_{\mathbb{Q}_{p}}^{st \geq 0}(G_{K})$ such that

$$V|_{G_{K}} = W|_{G_{K}},$$

with $m = m(K_{ur})$. (Equivalently, the condition says that $V|_{G_{K}}$ is semi-stable and can be extended to a $G_{K}$-semi-stable representation with non-negative Hodge-Tate weights.)

Proof. Sufficiency is obvious since $W$ is of finite height. We now prove necessity. By Prop. 6.1.1, starting from the finite height $V$, we can construct a triple $(\mathfrak{M} \otimes_{\mathfrak{O}} \mathfrak{O}, \varphi, N_{V}) \in \text{Mod}_{\mathfrak{O}}^{st \geq 0}$, which gives us a semi-stable $G_{K}$-representation $W$. Here $(\mathfrak{M} \otimes_{\mathfrak{O}} \mathfrak{O}, \varphi)$ is pure of slope zero because $(\mathfrak{M} \otimes_{\mathfrak{O}} \mathfrak{B}_{K_{\infty}}, \varphi)$ is part of the étale $(\varphi, \tau)$-module associated to $V$.

Let $(D_{K_{\infty}}(V), \varphi, \acute{G}_{V})$ be the étale $(\varphi, \tau)$-module associated to $V$, and let $D_{K_{\infty}}^{\dagger}(V)$ (resp. $D_{\text{rig}, K_{\infty}}(V)$) be the overconvergent (resp. rigid-overconvergent) module equipped with the induced $\varphi$ and $\acute{G}_{V}$. Let $(D_{K_{\infty}}(W), \varphi, \acute{G}_{W}), D_{K_{\infty}}^{\dagger}(W), D_{\text{rig}, K_{\infty}}^{\dagger}(W)$ be similarly defined. It is clear that

$$
\begin{align*}
D_{K_{\infty}}(V) &= \mathfrak{M} \otimes_{\mathfrak{O}} \mathfrak{B}_{K_{\infty}} = D_{K_{\infty}}(W) \\
D_{K_{\infty}}^{\dagger}(V) &= \mathfrak{M} \otimes_{\mathfrak{O}} \mathfrak{B}_{K_{\infty}}^{\dagger} = D_{K_{\infty}}^{\dagger}(W) \\
D_{\text{rig}, K_{\infty}}^{\dagger}(V) &= \mathfrak{M} \otimes_{\mathfrak{O}} \mathfrak{B}_{\text{rig}, K_{\infty}}^{\dagger} = D_{\text{rig}, K_{\infty}}^{\dagger}(W)
\end{align*}
$$

and they carry the same $\varphi$-action.

By the definition of locally analytic actions, the $\tau_{V/W}$-action (which means $\tau_{V}$ or $\tau_{W}$) over $D_{\text{rig}, K_{\infty}}^{\dagger}(V/W)$ can be “locally” recovered by $\nabla_{\tau, V/W} = pt \cdot N_{V/W}^{(1)}$ where $N_{V/W}^{(1)}$ is the monodromy operator defined in Thm. 4.2.1. Indeed, let $\bar{e}$ be an $\mathfrak{O}$-basis of $\mathfrak{M}$, then there exists some $a \gg 0$ such that

$$
(6.2.1) \quad \tau_{V/W}^{a}(\bar{e}) = \sum_{i=0}^{\infty} a^{i} \cdot \frac{(\nabla_{\tau, V/W})^{i}(\bar{e})}{i!}, \quad \forall \alpha \in p^{\nu} \mathbb{Z}_{p}.
$$
Note that a priori, the series in (6.2.1) converges to some element in $\widetilde{B}_l^{t} \otimes_{B_{\text{rig} K_{\infty}}} D_{\text{rig} K_{\infty}}^{t}(V/W)$; however since $c'$ is also basis for $D_{K_{\infty}}(V/W)$ and $D_{K_{\infty}}^{t}(V/W)$, the converged element has to fall into

$$\widetilde{B}_l^{t} \otimes_{B_{\text{rig} K_{\infty}}} D_{K_{\infty}}^{t}(V/W) \subset \widetilde{B}_l \otimes_{B_{K_{\infty}}} D_{K_{\infty}}(V/W).$$

Now, $N_{\forall, V}^{la} = N_{\forall}$ by construction, and $N_{\forall, W}^{la} = N_{\forall}$ by Thm. 5.5.1. Thus we have

$$N_{\forall, V}^{la} = N_{\forall, W}^{la}. $$

This implies that the $\text{Gal}(L/K_{p^m}((\pi_a)))$-action on the two $(\varphi, \tau)$-modules are the same. Since $K_{p^m}((\pi_a)) \cap K_{\infty} \supseteq K((\pi_{a+1}))$ (possible equality only when $p = 2$, cf. Notation 1.4.3(1)). Hence we must have

$$V|_{G_{K_{a+2}}} = W|_{G_{K_{a+2}}}.$$

We can always first choose $a \geq m(K_{ur}) = m$, and hence by Lem. 6.2.1, $V$ is semi-stable over $K_{m}$. Thus by Lem. 6.2.2 (using the totally ramified extension $K_{a+2}/K_{m}$), we have:

$$V|_{G_{K_{m}}} = W|_{G_{K_{m}}}.$$ 

□

We record some consequences of Thm. 6.2.3.

**Corollary 6.2.4.** Let $T \in \text{Rep}_{\mathbb{Z}_p}(G_K)$, and fix $\bar{\pi}$.

1. Suppose $m(K_{ur}) = m(K)$ (which only sometimes happens, cf. [Oze17, §5.4]). If $T$ is finite height with respect to $\bar{\pi}$, then it is also finite height with respect to $\bar{\mu} \cdot \bar{\pi} = \{\mu_n \pi_n\}_{n\geq 0}$.

2. Suppose $m(K_{ur}) = 1$ (hence necessarily $p > 2$). Then $T \in \text{Rep}_{\mathbb{Z}_p}^{st, \geq 0}(G_K)$ if and only if it is of finite $E(u)$-height (for the fixed $\bar{\pi}$).

*Proof.* This is obvious. □

The statement and idea of proof of the following Thm. 6.2.5 is due to Toby Gee. Furthermore, Toby Gee and Tong Liu obtained some similar results (under stronger assumptions, but using completely different approach) before this paper was written; cf. the upcoming [EGa].

**Theorem 6.2.5.** (Gee) Let $T \in \text{Rep}_{\mathbb{Z}_p}(G_K)$. Then $T \in \text{Rep}_{\mathbb{Z}_p}^{st, \geq 0}(G_K)$ if and only if $T$ is of finite height with respect to all choices of $\bar{\pi}$.

We first introduce an elementary lemma.

**Lemma 6.2.6.** Let $K \subset M_1, M_2$ be finite extensions, and let $M = M_1 M_2$. Suppose $M/K$ is Galois and totally ramified, and suppose $M_1 \cap M_2 = K$. Let $V$ be a $G_K$-representation and suppose it is semi-stable over both $M_1$ and $M_2$, then $V$ is semi-stable over $K$.

*Proof.* It is easy to see that $\text{Gal}(M/K)$ acts trivially on $D_{st}^M(V) = (V \otimes_{Q_p} B_{st})^G_M$, and hence $D_{st}^M(V) = D_{st}^K(V)$. □

*Proof of Thm. 6.2.5.* Necessity is proved in [Kis06]. We prove sufficiency. Let $V := T[1/p]$, let $U := V|_{G_{K_{ur}}}$, and let $K_{ur}$ denote the completion of $K_{ur}$. It suffices to show that the $G_{K_{ur}}$-representation $U$ is semi-stable. First fix one $\bar{\pi}$. Choose another unifomizer $\pi' \in K_{ur}$ (hence of $K$) such that $\pi/\pi' \in O_{K_{ur}}^{x}(\mathcal{O}_{K_{ur}}^{x})^p$, and choose any compatible system $\bar{\pi'} = \{\bar{\pi}'_n\}_{n \geq 0}$. By [Bir67, p.90, §2, Lem. 3], for each $i \geq 1$, the two fields $K_{ur}(\pi_i)$ and $K_{ur}(\pi'_i)$ are different (indeed, the lemma even implies that $K_{ur}(\pi_i, \mu_i) \neq K_{ur}(\pi'_i, \mu_i)$). Combined with [Liu10, Lem. 4.1.3], it is easy to show that

$$K_{ur}(\pi_i) \cap K_{ur}(\pi'_i) = K_{ur}, \forall i \geq 1.$$ 

Let $m := m(K_{ur})$. Consider the 3-step extensions

$$K_{ur} \subset K_{ur}(\pi_{m-1}), K_{ur}(\pi'_{m-1}) \subset M = K_{ur}(\pi_{m-1}, \pi'_{m-1}).$$

Since $\mu_{m-1} \in K_{ur}$, $M/K_{ur}$ is Galois and totally ramified. By Thm. 6.2.3, $V$ is semi-stable over $K(\pi_{m-1})$, hence $U$ is semi-stable over $K_{ur}(\pi_{m-1})$; similarly $U$ is semi-stable over $K_{ur}(\pi'_{m-1})$. Thus $U$ is semi-stable over $K_{ur}$ by Lem. 6.2.6. □
7. Simplified \((\varphi, \hat{G})\)-modules

In this section, we construct a simplified version of Liu’s \((\varphi, \hat{G})\)-modules. In §7.1, we review some results of Ozeki on weak \((\varphi, \hat{G})\)-modules. In §7.2, we construct our simplified \((\varphi, \hat{G})\)-modules; using results in this paper and earlier results of Liu and Ozeki, we show that they classify integral semi-stable Galois representations.

7.1. Weak \((\varphi, \hat{G})\)-modules. In this subsection, we recall Ozeki’s result on weak \((\varphi, \hat{G})\)-modules.

Definition 7.1.1. Fix a choice of \(\pi\). Let \(n \geq 1\). We denote by \(C_n(\pi)\) the category of finite free \(\mathbb{Z}_p\)-representations \(T\) of \(G_K\) such that we have an \(G_{K_n}\)-equivariant isomorphism
\[
T[1/p]|_{G_{K_n}} \cong W|_{G_{K_n}}
\]
for some \(W \in \text{Rep}^{st}_{\mathbb{Z}_p}(G_K)\).

Definition 7.1.2. A triple \((\mathcal{M}, \varphi_{\mathcal{M}}, \hat{G})\) that satisfies Items (1)-(4) (but not necessarily Item (5)) in Def. 5.3.2 is called a weak \((\varphi, \hat{G})\)-module. Denote the category of such objects as \(\text{wMod}^{\varphi, \hat{G}}\).

For a weak \((\varphi, \hat{G})\)-module, the functor
\[
\hat{T}(\mathcal{M}) := \text{Hom}_{W(R), \varphi}(W(R) \otimes_{\hat{G}} \mathcal{M}, W(R)),
\]
still gives a \(\mathbb{Z}_p[G_K]\)-module as in §5.3.3; and we still have \(\text{rank}_{\mathbb{Z}_p} \hat{T}(\mathcal{M}) = \text{rank}_R \mathcal{M}\). The following is the main theorem of [Oze17].

Theorem 7.1.3. [Oze17, Thm. 1.2] The functor \(\hat{T}\) induces an anti-equivalence between \(\text{wMod}^{\varphi, \hat{G}}\) and \(C_{m(K)}(\hat{\pi})\).

We also review another useful result in [Oze17].

Notation 7.1.4. Let \(E\) be a complete discrete valuation field of characteristic 0 with a perfect residue field of characteristic \(p\). We use \(\mathfrak{S}_E, \hat{R}_E\) (resp. \(\hat{G}_E\)) to denote the rings (resp. the group) constructed using the field \(E\) (and some fixed chosen \(\pi_E\)) corresponding to \(\mathfrak{S}, \hat{R}\) (resp. \(\hat{G}\)) which are constructed using \(K\) and \(\hat{\pi}\).

Definition 7.1.5. [Oze17, Def. 3.1] Let \(E/K\) be a finite extension. Let \(\text{Mod}^{\varphi, \hat{G}_E, G_K}_{\mathfrak{S}_E, \hat{R}_E, W(R)}\) be the category consisting of \((\hat{\mathcal{M}}_E, G_K)\) which are called \((\varphi, \hat{G}_E, K)\)-modules, where:

1. \(\hat{\mathcal{M}}_E = (\mathcal{M}_E, \varphi, \hat{G}_E) \in \text{Mod}^{\varphi, \hat{G}_E}_{\mathfrak{S}_E, \hat{R}_E}\) is a \((\varphi, \hat{G}_E)\)-module;
2. \(G_K\) is a continuous \(W(R)\)-semi-linear \(\varphi\)-commuting \(G_K\)-action on \(W(R) \otimes_{\varphi, \hat{G}_E} \mathcal{M}_E\);
3. The two \(G_E\)-actions on \(W(R) \otimes_{\varphi, \hat{G}_E} \mathcal{M}_E\), respectively induced from Item(1) and restricted from Item (2), coincide.

Let \(\hat{\mathcal{M}} = (\hat{\mathcal{M}}_E, G_K)\) be a module as in Def. 7.1.5, let
\[
\hat{T}_{E/K}(\hat{\mathcal{M}}) := \text{Hom}_{W(R), \varphi}(W(R) \otimes_{\varphi, \hat{G}_E} \mathcal{M}_E, W(R)).
\]

Theorem 7.1.6. [Oze17, Thm. 3.2] The functor \(\hat{T}_{E/K}\) induces an anti-equivalence between \(\text{Mod}^{\varphi, \hat{G}_E, G_K}_{\mathfrak{S}_E, \hat{R}_E, W(R)}\) and \(\text{Rep}^{\text{st}, \geq 0}(G_K)\), where the latter is the category of \(\mathbb{Z}_p\)-representations \(T\) of \(G_K\) such that \(T[1/p]|_{G_{E^\text{st}}}\) is semi-stable with non-negative Hodge-Tate weights.

7.2. Simplified \((\varphi, \hat{G})\)-modules. In this subsection, we construct a simplified version of \((\varphi, \hat{G})\)-modules. Recall that \(G_L = \text{Gal}(L/K)\) and \(\mathfrak{m}_R\) is the maximal ideal of \(R\). Denote
\[
W(R)_L := (W(R))^{G_L}, \quad W(\mathfrak{m}_R)_L := (W(\mathfrak{m}_R))^{G_L}.
\]

Definition 7.2.1. Let \(\text{Mod}^{\varphi, \hat{G}}_{\mathfrak{S}, W(R)_L}\) be the category consisting of triples \((\mathcal{M}, \varphi_{\mathcal{M}}, \hat{G})\), which we call the simplified \((\varphi, \hat{G})\)-modules, where

(1) \((\mathcal{M}, \varphi_{\mathcal{M}}) \in \text{Mod}^{\varphi}_{\mathfrak{S}}\);
(2) \(\hat{G}\) is a continuous \(W(R)_L\)-semi-linear \(\hat{G}\)-action on \(\hat{\mathcal{M}} := W(R)_L \otimes_{\mathfrak{S}} \mathcal{M}\);
Remark 7.2.2. (1) Note that not only \( \hat{R} \) disappears in Def. 7.2.1, there is also no more \( \varphi \)-twist in \( \hat{\mathcal{M}} = W(R) \otimes_{\mathcal{O}} \mathcal{M} \).

(2) The embedding \( \mathcal{M}/u\mathcal{M} \hookrightarrow \hat{\mathcal{M}}/W(m_R)\hat{\mathcal{M}} \) induces an \( \varphi \)-equivariant isomorphism

\[
W(k_L) \otimes_{W(k)} \mathcal{M}/u\mathcal{M} \cong \hat{\mathcal{M}}/W(m_R)\hat{\mathcal{M}},
\]

where \( k_L \) is the residue field of \( L \). Note that \( \hat{G} \) acts on \( W(k_L) \) (in general non-trivially). If we only suppose:

\[
(7.2.1) \quad \hat{\mathcal{M}}/W(m_R)\hat{\mathcal{M}} \text{ is trivial as a } W(k_L)-\text{representation of } \hat{G};
\]

we do not know if we can show \( \hat{G} \) acts on \( \mathcal{M}/u\mathcal{M} \) trivially. Hence we cannot use (7.2.1) to replace Def. 7.2.1(5).

The following Def. 7.2.3 is inspired by discussions with Heng Du, Toby Gee and Tong Liu; it is obviously equivalent to Def. 7.2.1.

Definition 7.2.3. Let \( \text{Mod}^{\varphi,G_K}_{\mathcal{O},W(R)} \) be the category consisting of triples \( (\mathcal{M}, \varphi_{\mathcal{M}}, G_K) \) where

1. \( (\mathcal{M}, \varphi_{\mathcal{M}}) \in \text{Mod}^{\varphi}_{\mathcal{O}} \);
2. \( G_K \) is a continuous \( W(R) \)-semi-linear \( G_K \)-action on \( \hat{\mathcal{M}} := W(R) \otimes_{\mathcal{O}} \mathcal{M} \);
3. \( G_K \) commutes with \( \varphi_{\mathcal{M}} \) on \( \hat{\mathcal{M}} \);
4. \( \mathcal{M} \subset \hat{\mathcal{M}}^{\text{Gal}(\overline{K}/K_{\infty})} \) via the embedding \( \mathcal{M} \hookrightarrow \hat{\mathcal{M}} \);
5. \( \mathcal{M}/u\mathcal{M} \subset (\hat{\mathcal{M}}/W(m_R)\hat{\mathcal{M}))^{G_K} \) via the embedding \( \mathcal{M}/u\mathcal{M} \hookrightarrow \hat{\mathcal{M}}/W(m_R)\hat{\mathcal{M}} \).

Remark 7.2.4. The module \( \hat{\mathcal{M}} = W(R) \otimes_{\mathcal{O}} \mathcal{M} \) above is a Breuil-Kisin-Fargues module, cf. [BMS18, Def. 1.5].

We record an equivalent condition for Def. 7.2.3(5).

Lemma 7.2.5. Let \( (\mathcal{M}, \varphi_{\mathcal{M}}, G_K) \) be a triple satisfying Items (1)-(4) of Def. 7.2.3, then the condition in Def. 7.2.3(5) is satisfied if and only if \( \hat{\mathcal{M}}/W(m_R)\hat{\mathcal{M}} \) is fixed by \( G_{K^w} \).

Proof. Necessity is obvious, and we prove sufficiency. If \( \hat{\mathcal{M}}/W(m_R)\hat{\mathcal{M}} \) is fixed by \( G_{K^w} \), then so is \( \mathcal{M}/u\mathcal{M} \). But \( \mathcal{M}/u\mathcal{M} \) is also fixed by \( G_{K_{\infty}} \) by Def. 7.2.3(4), hence it is fixed by \( G_K \) as \( K^w \cap K_{\infty} = K \). \( \square \)

Definition 7.2.6. (1) Let \( \text{wMod}^{\varphi,G}_{\mathcal{O},W(R)L} \) denote the category consisting of triples \( (\mathcal{M}, \varphi_{\mathcal{M}}, \hat{G}) \) satisfying Items (1)-(4) of Def. 7.2.1

(2) Let \( \text{wMod}^{G_{\mathcal{O},W(R)}} \) denote the category consisting of triples \( (\mathcal{M}, \varphi_{\mathcal{M}}, G_K) \) satisfying Items (1)-(4) of Def. 7.2.3.

Remark 7.2.7. It is tempting to call objects in Def. 7.2.6 “simplified weak \( (\varphi, \hat{G}) \)-modules”; however we perhaps should not do so. As we shall see Thm. 7.2.9, they are equivalent to the category of finite \( E(u) \)-height representations of \( G_K \); namely, the \( \hat{G} \)-stability (resp. \( G_K \)-stability) of \( \hat{\mathcal{M}} \) is automatic for finite \( E(u) \)-height representations.

For \( \hat{\mathcal{M}} \in \text{wMod}^{\varphi,G}_{\mathcal{O},W(R)L} \), resp. \( \hat{\mathcal{M}} \in \text{wMod}^{G_{\mathcal{O},W(R)}} \), the functor

\[
\hat{T}(\hat{\mathcal{M}}) := \text{Hom}_{W(R)_{\phi}}(W(R) \otimes_{W(R)L} \hat{\mathcal{M}}, W(R))
\]

resp.

\[
\hat{T}(\hat{\mathcal{M}}) := \text{Hom}_{W(R)_{\phi}}(\hat{\mathcal{M}}, W(R))
\]

still gives a \( \mathbb{Z}_p[G_K] \)-module as in §5.3.3.
7.2. The tensor map $\mathfrak{M} \mapsto W(R)_L \otimes_{\varphi^{-1}} W(R)_L \otimes_R \mathfrak{M}$ induces a functor:

$$\mathfrak{M}_{W(R)_L} : \text{wMod}^{\varphi, \hat{G}}_{\mathfrak{E}, R} \rightarrow \text{wMod}^{\varphi, \hat{G}}_{\mathfrak{E}, W(R)_L}.$$ 

Similarly the tensor map $\mathfrak{M} \mapsto W(R) \otimes_{W(R)_L} \mathfrak{M}$ induces a functor:

$$\mathfrak{M}_{W(R)} : \text{wMod}^{\varphi, \hat{G}}_{\mathfrak{E}, W(R)_L} \rightarrow \text{wMod}^{\varphi, \hat{G}_K}_{\mathfrak{E}, W(R)}.$$

**Theorem 7.2.9.** We have equivalences of categories

$$(7.2.2) \quad \text{wMod}^{\varphi, \hat{G}}_{\mathfrak{E}, W(R)_L} \xrightarrow{\mathfrak{M}_{W(R)_L}} \text{wMod}^{\varphi, \hat{G}_K}_{\mathfrak{E}, W(R)_L} \xrightarrow{\hat{T}} \text{Rep}_{Z_p}^{E(u)-\text{ht}}(G_K) \xrightarrow{\pi} C_{m(K^u)}(\bar{\pi}),$$

where $\text{Rep}_{Z_p}^{E(u)-\text{ht}}(G_K)$ denotes the category of finite $E(u)$-height $G_K$-representations.

**Proof.** It is obvious that $\mathfrak{M}_{W(R)}$ induces an equivalence. The last equivalence is Thm. 6.2.3. The full faithfulness of $\hat{T}$ can be proved using exactly the same argument as the proof for the full faithfulness of $\bar{T}$ : $\text{wMod}^{\varphi, \hat{G}}_{\mathfrak{E}, R} \rightarrow \text{Rep}_{Z_p}(G_K)$ in the end of [Liu10, §3.1]; note that in loc. cit., Item (5) of Def. 5.3.2 is not needed to show full faithfulness. Hence it suffices to show that $\hat{T}$ is essentially surjective.

Indeed, suppose $T \in \text{Rep}_{Z_p}^{E(u)-\text{ht}}(G_K)$, let $\mathfrak{M}$ be the corresponding module, and let $V = T[1/p]$. By Thm. 6.2.3, $V|_{G_{K^u}}$ is semi-stable where $m = m(K^u)$. Hence by Thm. 7.1.6, there exists a unique module in $\text{Mod}^{\varphi, \hat{G}_E}_{\mathfrak{E}, E, W(R)}$ which corresponds to $T$ where $E = K_m$ and we use the sequence $\{\pi_i\}_{i \geq m-1}$ in as Rem. 5.4.4(2). In particular, one easily sees that $\mathfrak{M}_E = \mathfrak{S}_E \otimes_{\mathfrak{E}} \mathfrak{M}$. We can identify $W(R) \otimes_{\varphi, E} \mathfrak{M} = W(R) \otimes_{\varphi, E} \mathfrak{M}_E$ together with $\varphi$-structures. Since $W(R) \otimes_{\varphi, E} \mathfrak{M}_E$ is $G_K$-stable, we can equip a $G_K$-action on $W(R) \otimes_{\varphi, E} \mathfrak{M}$ through the aforementioned identification; this induces a $G_K$-action on $W(R) \otimes_{\varphi, E} \mathfrak{M}$ and gives the desired pre-image of $T$. \hfill $\square$

**Remark 7.2.10.** The essentially surjectivity of $\hat{T}$ is also proved in [Car13, Prop. 3.3]; the proof there utilizes a (rather complicated and implicit) ring “$\mathfrak{S}_{u-dp, \tau}$”, but it does not require knowing the potential-semistability of finite height representations a priori. However, the proof in Thm. 7.2.9 perhaps fits better in our studies, as the underlying strategy (e.g., in the proof of Thm. 7.1.6, cf. [Oze17]) is quite common in the study of $(\varphi, \hat{G})$-modules.

**Corollary 7.2.11.** If $m(K) = m(K^u)$, then we have equivalences of categories

$$(7.2.3) \quad \text{wMod}^{\varphi, \hat{G}}_{\mathfrak{E}, R} \xrightarrow{\mathfrak{M}_{W(R)_L}} \text{wMod}^{\varphi, \hat{G}}_{\mathfrak{E}, W(R)_L} \xrightarrow{\mathfrak{M}_{W(R)}} \text{wMod}^{\varphi, \hat{G}_K}_{\mathfrak{E}, W(R)_L} \xrightarrow{\hat{T}} \text{Rep}_{Z_p}^{E(u)-\text{ht}}(G_K) \xrightarrow{\pi} C_{m(K^u)}(\bar{\pi}) \xrightarrow{\pi} C_{m(K)}(\bar{\pi});$$

$$(7.2.4) \quad \text{Mod}^{\varphi, \hat{G}}_{\mathfrak{E}, R} \xrightarrow{\mathfrak{M}_{W(R)_L}} \text{Mod}^{\varphi, \hat{G}}_{\mathfrak{E}, W(R)_L} \xrightarrow{\mathfrak{M}_{W(R)}} \text{Mod}^{\varphi, \hat{G}_K}_{\mathfrak{E}, W(R)_L} \xrightarrow{\hat{T}} \text{Rep}_{Z_p}^{\leq 0}(G_K).$$

**Proof.** The equivalences in (7.2.3) follow from Thm. 7.1.3 and Thm. 7.2.9. The equivalences in (7.2.4) easily follow from (7.2.3) and Thm. 5.3.4. \hfill $\square$

**Theorem 7.2.12.** We have equivalences of categories (which strengthen (7.2.4))

$$(7.2.5) \quad \text{Mod}^{\varphi, \hat{G}}_{\mathfrak{E}, R} \xrightarrow{\mathfrak{M}_{W(R)_L}} \text{Mod}^{\varphi, \hat{G}}_{\mathfrak{E}, W(R)_L} \xrightarrow{\mathfrak{M}_{W(R)}} \text{Mod}^{\varphi, \hat{G}_K}_{\mathfrak{E}, W(R)_L} \xrightarrow{\hat{T}} \text{Rep}_{Z_p}^{\leq 0}(G_K).$$

**Proof.** It is obvious $\mathfrak{M}_{W(R)}$ induces an equivalence. To prove the theorem, it suffices to check that if $\mathfrak{M} \in \text{Mod}^{\varphi, \hat{G}_K}_{\mathfrak{E}, W(R)_L}$, then $\hat{T}(\mathfrak{M})$ is semi-stable. The object $\mathfrak{M}$ induces an object $\mathfrak{M}_{K^u} \in \text{Mod}^{\varphi, \hat{G}_K}_{\mathfrak{E}, W(R)_L}$ using $\mathfrak{M}_{K^u} := W(k)[u] \otimes \mathfrak{M}$ (and using Lem. 7.2.5). It suffices to show that $\hat{T}(\mathfrak{M})|_{G_{K^u}}$ is semi-stable as a $G_{K^u}$-representation; but it is isomorphic to $\hat{T}(\mathfrak{M}_{K^u})$, which is semistable by Cor. 7.2.11. \hfill $\square$
Remark 7.2.13. The content of Thm. 7.2.12 is surprising, as there is no appearance of \( \hat{\mathcal{R}} \) in the simplified (\( \varphi, \hat{G} \))-modules (although the proof of the theorem depends on it!). The ring \( \hat{\mathcal{R}} \) is a rather implicit ring (although \( \mathcal{R}_{K_0} \) is very explicit), but it plays a huge role in Liu’s theory of (\( \varphi, \hat{G} \))-modules. It seems very likely we can actually give a direct proof of the equivalence between \( \text{Mod}_{\mathcal{S}, W(R)}^{\varphi, \hat{G}} \) and \( \text{Rep}_{Z_p}^{\varphi, \geq 0}(G_K) \) without using the ring \( \hat{\mathcal{R}} \), using the idea of locally analytic vectors. We postpone this investigation to another occasion.

Remark 7.2.14. (I thank Peter Scholze for useful discussions on this remark.) Let \( \mathfrak{X} \) be a proper smooth (formal) scheme over \( \mathcal{O}_K \) as in the beginning of [BMS19]. Recall \( C \) resp. \( A_{\inf} \) there is precisely our \( C_p \) resp. \( W(R) \). Then the cohomology theories \( \Gamma_{\mathcal{S}}(\mathfrak{X}) \) and \( \Gamma_{A_{\inf}}(\mathfrak{X}_{\mathcal{O}_C}) \) “satisfy” the conditions in Def. 7.2.3 (although the cohomology groups are not necessarily finite free). Indeed, the \( \varphi \)-commuting \( G_K \)-action on \( \Gamma_{A_{\inf}}(\mathfrak{X}_{\mathcal{O}_C}) \) comes from functoriality of the constructions (e.g., in [BMS18]) with respect to \( \mathfrak{X} \to \text{Spf} \mathcal{O}_C \). Def. 7.2.3(2)(4) follow from the comparison theorem [BMS19, Thm. 1.2(1)]. Def. 7.2.3(5) is precisely the crystalline comparison theorem [BMS19, Thm. 1.2(3)].

Let us summarize the relations between the various module categories. Here “\( \simeq \)” signifies an equivalence, and a hooked arrow “\( \implies \)” means “is a subcategory of”.

\[
\begin{array}{c}
\text{Rep}_{Z_p}^{E(u_{\inf})}(G_K) \simeq \text{wMod}_{\mathcal{S}, W(R)}^{\varphi, \hat{G}} \simeq \text{wMod}_{\mathcal{S}, W(R)}^{\varphi, G_K} \simeq \text{C}_{m(K^{ur})}(\bar{\pi}) \\
\text{wMod}_{\mathcal{S}, \hat{\mathcal{R}}}^{\varphi, \hat{G}} \simeq \text{Mod}_{\mathcal{S}, W(R)}^{\varphi, \hat{G}} \simeq \text{Mod}_{\mathcal{S}, W(R)}^{\varphi, G_K} \simeq \text{Rep}_{Z_p}^{\varphi, \geq 0}(G_K)
\end{array}
\]

In the end, we give a crystallinity criterion for the (simplified) (\( \varphi, \hat{G} \))-modules.

**Proposition 7.2.15.** Suppose the following assumption is satisfied (which is automatic when \( p > 2 \), and can be made to happen when \( p = 2 \); cf. Rem. 7.2.16 below):

\[ (7.2.6) \quad K_{\infty} \cap K_p = K. \]

1. Let \( \hat{\mathfrak{M}} \) be an object in \( \text{Mod}_{\mathcal{S}, \hat{\mathcal{R}}}^{\varphi, \hat{G}} \), then the following is equivalent:
   a. \( \hat{T}(\hat{\mathfrak{M}}) \) is a crystalline representation;
   b. \( (\tau - 1)(\hat{\mathfrak{M}}) \subset \varphi(t) W(m_R) \otimes_{\mathcal{S}, \hat{\mathcal{R}}} \hat{\mathfrak{M}} \);
   c. \( (\tau - 1)(\hat{\mathfrak{M}}) \subset w_\varphi(t) W(R) \otimes_{\mathcal{S}, \mathcal{R}} \mathfrak{M} \).

2. Let \( \hat{\mathfrak{M}} \) be an object in \( \text{Mod}_{\mathcal{S}, W(R)}^{\varphi, \hat{G}} \) or in \( \text{Mod}_{\mathcal{S}, W(R)}^{\varphi, G_K} \), then the following is equivalent:
   a. \( \hat{T}(\hat{\mathfrak{M}}) \) is a crystalline representation;
   b. \( (\tau - 1)(\hat{\mathfrak{M}}) \subset t W(m_R) \otimes_{\mathcal{S}} \mathfrak{M} \);
   c. \( (\tau - 1)(\hat{\mathfrak{M}}) \subset u t W(R) \otimes_{\mathcal{S}} \mathfrak{M} \).

**Proof.** Item (1) is [Oze13, Thm. 21]. Note that there is a running assumption \( p > 2 \) in [Oze13]; however it is only to guarantee that (7.2.6) holds. (Recall that assuming (7.2.6) is equivalent to assuming \( \tau \) and \( \text{Gal}(L/K_{\infty}) \) topologically generate \( \hat{G} = \text{Gal}(L/K) \); cf. Notation. 1.4.3) Indeed, the implication from (1)(a) to (1)(c) is already proved in [GLS14, Prop. 5.9], where \( p > 2 \) is also assumed only to guarantee that (7.2.6) holds. When \( p = 2 \) and (7.2.6) holds, the argument in [GLS14, Prop. 5.9] and [Oze13, Thm. 21] work through unchanged.

Item (2) follows easily from Item (1), using the equivalence of categories in Thm. 7.2.12. \( \square \)

**Remark 7.2.16.** When \( p = 2 \), (7.2.6) does not always happen. However, by [Wan, Lem. 2.1], we can make some choice of \( \bar{\pi} = (\pi_n)_{n \geq 0} \) (indeed, a specific choice of \( \pi = \pi_0 \) already suffices) so that (7.2.6) holds. This choice is a bit artificial (although only when \( p = 2 \)); however what we care most are the representations (attached to these (\( \varphi, \hat{G} \))-modules), hence the sacrifice seems minimal for applications.
7.3. Independence from Caruso’s work.

Remark 7.3.1. (1) The only place in the current paper where we actually use results from [Car13] is in Thm. 3.1.4 (also Def. 3.1.2) which is [Car13, Thm. 1]. (Lem. 6.2.2 is also from [Car13], but it is completely elementary).

(2) Besides those in §1 and in Thm. 3.1.4, the only places where we make references to [Car13] are in Rem. 6.1.7 and Rem. 7.2.10, where we mention some argument and results from [Car13].

(3) The results in [Car13] are not used in any of the cited papers in our bibliography, except [Car13, Thm. 1], which is used in [GL, GP].

(4) Hence, the current paper is independent of the work [Car13] (except [Car13, Thm. 1]).

Appendix A. A gap in Caruso’s work, by Yoshiyasu Ozeki

The statement of [Car13, Prop. 3.7] is false. Here is a counter-example:

- Let $A = \mathbb{Q}_p$ with $p = 3$, let $m = 4$ and let $\Lambda = p^{-4} \cdot \mathbb{Z}_p$. Let $a = b = 0$.

The mistake in the proof of [Car13, Prop. 3.7] is rather hidden. Indeed, on [Car13, p. 2580], Caruso obtains the formula:

$$\log_m(ab) - (\log_m a + \log_m b) \in \sum_{j=1}^{p^{m-1}-1} \frac{p^{(p^m - j) + t(j)}}{j} \cdot \Lambda.$$

As far as we understand, Caruso is implicitly using that $\Lambda$ is a ring (namely, a $\mathbb{Z}_p$-algebra). But in fact, $\Lambda$ is only a $\mathbb{Z}_p$-module.

Prop. 3.7 of [Car13] is a key proposition in the paper. Indeed, it is used to prove Prop. 3.8, Prop. 3.9 in loc. cit.. These propositions are then repeatedly used in later argument of that paper.

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