Prominent interior GE-filters of GE-algebras

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Abstract: The concept of a prominent interior GE-filter (of type 1 and type 2) is introduced, and their properties are investigated. The relationship between a prominent GE-filter and a prominent interior GE-filter and the relationship between an interior GE-filter and a prominent interior GE-filter are discussed. Examples to show that any interior GE-filter is not a prominent interior GE-filter and any prominent GE-filter is not a prominent interior GE-filter are provided. Conditions for an interior GE-filter to be a prominent interior GE-filter are given. Also, conditions under which an internal GE-filter larger than a given internal GE filter can become a prominent internal GE-filter are considered, and an example describing it is given. The relationship between a prominent interior GE-filter and a prominent interior GE-filter of type 1 is discussed.

Keywords: (transitive) GE-algebra; GE-filter; interior GE-filter; prominent interior GE-filter (of type 1 and type 2)
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1. Introduction

Henkin and Skolem introduced Hilbert algebras in the fifties for investigations in intuitionistic and other non-classical logics. Diego [4] proved that Hilbert algebras form a variety which is locally finite. Bandaru et al. introduced the notion of GE-algebras which is a generalization of Hilbert algebras, and investigated several properties (see [1, 2, 7–9]). The notion of interior operator is introduced by Vorster [12] in an arbitrary category, and it is used in [3] to study the notions of connectedness and disconnectedness in topology. Interior algebras are a certain type of algebraic structure that encodes the idea of the topological interior of a set, and are a generalization of topological spaces defined by means of topological interior operators. Rachůnek and Svoboda [6]
studied interior operators on bounded residuated lattices, and Svrcek [11] studied multiplicative interior operators on GMV-algebras. Lee et al. [5] applied the interior operator theory to GE-algebras, and they introduced the concepts of (commutative, transitive, left exchangeable, belligerent, antisymmetric) interior GE-algebras and bordered interior GE-algebras, and investigated their relations and properties. Later, Song et al. [10] introduced the notions of an interior GE-filter, a weak interior GE-filter and a belligerent interior GE-filter, and investigate their relations and properties. They provided relations between a belligerent interior GE-filter and an interior GE-filter and conditions for an interior GE-filter to be a belligerent interior GE-filter is considered. Given a subset and an element, they established an interior GE-filter, and they considered conditions for a subset to be a bellergetent interior GE-filter. They studied the extensibility of the belligerent interior GE-filter and established relationships between weak interior GE-filter and belligerent interior GE-filter of type 1, type 2 and type 3. Rezaei et al. [7] studied prominent GE-filters in GE-algebras. The purpose of this paper is to study by applying interior operator theory to prominent GE-filters in GE-algebras. We introduce the concept of a prominent interior GE-filter, and investigate their properties. We discuss the relationship between a prominent GE-filter and a prominent interior GE-filter and the relationship between an interior GE-filter and a prominent interior GE-filter. We find and provide examples where any interior GE-filter is not a prominent interior GE-filter and any prominent GE-filter is not a prominent interior GE-filter. We provide conditions for an interior GE-filter to be a prominent interior GE-filter. We provide conditions under which an internal GE-filter larger than a given internal GE filter can become a prominent internal GE-filter, and give an example describing it. We also introduce the concept of a prominent interior GE-filter of type 1 and type 2, and investigate their properties. We discuss the relationship between a prominent interior GE-filter and a prominent interior GE-filter of type 1. We give examples to show that A and B are independent of each other, where A and B are:

\[
\begin{align*}
(1) & \quad \{ \begin{array}{c}
A: \text{prominent interior GE-filter of type 1.} \\
B: \text{prominent interior GE-filter of type 2.}
\end{array} \\
(2) & \quad \{ \begin{array}{c}
A: \text{prominent interior GE-filter.} \\
B: \text{prominent interior GE-filter of type 2.}
\end{array} \\
(3) & \quad \{ \begin{array}{c}
A: \text{interior GE-filter.} \\
B: \text{prominent interior GE-filter of type 1.}
\end{array} \\
(4) & \quad \{ \begin{array}{c}
A: \text{interior GE-filter.} \\
B: \text{prominent interior GE-filter of type 2.}
\end{array}
\end{align*}
\]

2. Preliminaries

Definition 2.1. [1] By a GE-algebra we mean a non-empty set $X$ with a constant $1$ and a binary operation $*$ satisfying the following axioms:

GE1) $u * u = 1,$

GE2) $1 * u = u,$

GE3) $u * (v * w) = u * (v * (u * w))$

for all $u, v, w \in X.$
In a GE-algebra $X$, a binary relation “≤” is defined by

$$(\forall x, y \in X) (x \leq y \iff x \ast y = 1). \quad (2.1)$$

**Definition 2.2.** [1, 2, 8] A GE-algebra $X$ is said to be transitive if it satisfies:

$$(\forall x, y, z \in X) (x \ast y \leq (z \ast x) \ast (z \ast y)). \quad (2.2)$$

**Proposition 2.3.** [1] Every GE-algebra $X$ satisfies the following items:

$$(\forall u \in X) (u \ast 1 = 1). \quad (2.3)$$

$$(\forall u, v \in X) (u \ast (u \ast v) = u \ast v). \quad (2.4)$$

$$(\forall u, v \in X) (u \leq v \ast u). \quad (2.5)$$

$$(\forall u, v, w \in X) (u \ast (v \ast w) \leq v \ast (u \ast w)). \quad (2.6)$$

$$(\forall u \in X) (1 \leq u \Rightarrow u = 1). \quad (2.7)$$

$$(\forall u, v \in X) (u \leq (v \ast u) \ast u). \quad (2.8)$$

$$(\forall u, v \in X) (u \leq (u \ast v) \ast v). \quad (2.9)$$

$$(\forall u, v, w \in X) (u \leq v \ast w \iff v \leq u \ast w). \quad (2.10)$$

If $X$ is transitive, then

$$(\forall u, v, w \in X) (u \leq v \Rightarrow w \ast u \leq w \ast v, v \ast w \leq u \ast w). \quad (2.11)$$

$$(\forall u, v, w \in X) (u \ast v \leq (v \ast w) \ast (u \ast w)). \quad (2.12)$$

**Lemma 2.4.** [1] In a GE-algebra $X$, the following facts are equivalent each other.

$$(\forall x, y, z \in X) (x \ast y \leq (z \ast x) \ast (z \ast y)). \quad (2.13)$$

$$(\forall x, y, z \in X) (x \ast y \leq (y \ast z) \ast (x \ast z)). \quad (2.14)$$

**Definition 2.5.** [1] A subset $F$ of a GE-algebra $X$ is called a GE-filter of $X$ if it satisfies:

$$1 \in F,$$

$$(\forall x, y \in X) (x \ast y \in F, x \in F \Rightarrow y \in F). \quad (2.15)$$

**Lemma 2.6.** [1] In a GE-algebra $X$, every filter $F$ of $X$ satisfies:

$$(\forall x, y \in X) (x \leq y, x \in F \Rightarrow y \in F). \quad (2.16)$$

**Definition 2.7.** [7] A subset $F$ of a GE-algebra $X$ is called a prominent GE-filter of $X$ if it satisfies (2.15) and

$$(\forall x, y, z \in X) (x \ast (y \ast z) \in F, x \in F \Rightarrow ((z \ast y) \ast y) \ast z \in F). \quad (2.18)$$

Note that every prominent GE-filter is a GE-filter in a GE-algebra (see [7]).
Definition 2.8. [5] By an interior GE-algebra we mean a pair \((X, f)\) in which \(X\) is a GE-algebra and \(f : X \to X\) is a mapping such that
\[
\begin{align*}
(\forall x \in X)(x \leq f(x)), \\
(\forall x \in X)((f \circ f)(x) = f(x)), \\
(\forall x, y \in X)(x \leq y \implies f(x) \leq f(y)).
\end{align*}
\] (2.19) (2.20) (2.21)

Definition 2.9. [10] Let \((X, f)\) be an interior GE-algebra. A GE-filter \(F\) of \(X\) is said to be interior if it satisfies:
\[
(\forall x \in X)(f(x) \in F \implies x \in F). 
\] (2.22)

3. Prominent interior GE-filters

Definition 3.1. Let \((X, f)\) be an interior GE-algebra. Then a subset \(F\) of \(X\) is called a prominent interior GE-filter in \((X, f)\) if \(F\) is a prominent GE-filter of \(X\) which satisfies the condition (2.22).

Example 3.2. Let \(X = \{1, 2, 3, 4, 5\}\) be a set with the Cayley table which is given in Table 1.

Table 1. Cayley table for the binary operation “∗”.

|   | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 1 | 1 | 4 | 4 |
| 3 | 1 | 1 | 1 | 5 | 5 |
| 4 | 1 | 2 | 3 | 1 | 1 |
| 5 | 1 | 2 | 2 | 1 | 1 |

Then \(X\) is a GE-algebra. If we define a mapping \(f\) on \(X\) as follows:
\[
f : X \to X, \quad x \mapsto \begin{cases} 
1 & \text{if } x \in \{1, 4, 5\}, \\
2 & \text{if } x \in \{2, 3\}, 
\end{cases}
\]
then \((X, f)\) is an interior GE-algebra and \(F = \{1, 4, 5\}\) is a prominent interior GE-filter in \((X, f)\).

It is clear that every prominent interior GE-filter is a prominent GE-filter. But any prominent GE-filter may not be a prominent interior GE-filter in an interior GE-algebra as seen in the following example.

Example 3.3. Let \(X = \{1, 2, 3, 4, 5\}\) be a set with the Cayley table which is given in Table 2.

Table 2. Cayley table for the binary operation “∗”.

|   | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 1 | 3 | 4 | 1 |
| 3 | 1 | 2 | 1 | 4 | 5 |
| 4 | 1 | 2 | 3 | 1 | 5 |
| 5 | 1 | 1 | 3 | 4 | 1 |
and define a mapping \( f \) on \( X \) as follows:

\[
f : X \to X, \quad x \mapsto \begin{cases} 1 & \text{if } x \in \{1, 2, 3, 5\}, \\ 4 & \text{if } x = 4. \end{cases}
\]

Then \((X, f)\) is an interior GE-algebra and \( F := \{1\} \) is a prominent GE-filter of \( X \). But it is not a prominent interior GE-filter in \((X, f)\) since \( f(2) = 1 \in F \) but \( 2 \not\in F \).

We discuss relationship between interior GE-filter and prominent interior GE-filter.

**Theorem 3.4.** In an interior GE-algebra, every prominent interior GE-filter is an interior GE-filter.

**Proof.** It is straightforward because every prominent GE-filter is a GE-filter in a GE-algebra. \( \square \)

In the next example, we can see that any interior GE-filter is not a prominent interior GE-filter in general.

**Example 3.5.** Let \( X = \{1, 2, 3, 4, 5\} \) be a set with the Cayley table which is given in Table 3.

**Table 3.** Cayley table for the binary operation “\(*\).”

| *  | 1 | 2 | 3 | 4 | 5 |
|----|---|---|---|---|---|
| 1  | 1 | 2 | 3 | 4 | 5 |
| 2  | 1 | 1 | 1 | 4 | 4 |
| 3  | 1 | 2 | 1 | 4 | 4 |
| 4  | 1 | 1 | 3 | 1 | 1 |
| 5  | 1 | 1 | 1 | 1 | 1 |

Then \( X \) is a GE-algebra. If we define a mapping \( f \) on \( X \) as follows:

\[
f : X \to X, \quad x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 2 & \text{if } x \in \{2, 4, 5\}, \\ 3 & \text{if } x = 3, \end{cases}
\]

then \((X, f)\) is an interior GE-algebra and \( F = \{1\} \) is an interior GE-filter in \((X, f)\). But it is not a prominent interior GE-filter in \((X, f)\) since \( 1 * (2 * 3) = 1 \in F \) but \( (3 * 2) * 3 = 3 \not\in F \).

**Proposition 3.6.** Every prominent interior GE-filter \( F \) in an interior GE-algebra \((X, f)\) satisfies:

\[
(\forall x, y \in X) \ (f(x * y) \in F \Rightarrow ((y * x) * x) * y \in F). \tag{3.1}
\]

**Proof.** Let \( F \) be a prominent interior GE-filter in \((X, f)\). Let \( x, y \in X \) be such that \( f(x * y) \in F \). Then \( x * y \in F \) by (2.22), and so \( 1 * (x * y) = x * y \in F \) by (GE2). Since \( 1 \in F \), it follows from (2.18) that \( ((y * x) * x) * y \in F \). \( \square \)

**Corollary 3.7.** Every prominent interior GE-filter \( F \) in an interior GE-algebra \((X, f)\) satisfies:

\[
(\forall x, y \in X) \ (x * y \in F \Rightarrow ((y * x) * x) * y \in F). \tag{3.2}
\]
Proof. Let $F$ be a prominent interior GE-filter in $(X, f)$. Then $F$ is an interior GE-filter in $(X, f)$ by Theorem 3.4. Let $x, y \in X$ be such that $x \ast y \in F$. Since $x \ast y \leq f(x \ast y)$ by (2.19), it follows from Lemma 2.6 that $f(x \ast y) \in F$. Hence $((y \ast x) \ast x) \ast y \in F$ by Proposition 3.6.

\Corollary 3.8. Every prominent interior GE-filter $F$ in an interior GE-algebra $(X, f)$ satisfies:

$$\forall x, y \in X \ (x \ast y \in F \implies f(((y \ast x) \ast x) \ast y) \in F).$$

\Proof Straightforward.

\Corollary 3.9. Every prominent interior GE-filter $F$ in an interior GE-algebra $(X, f)$ satisfies:

$$\forall x, y \in X \ (f(x \ast y) \in F \implies f(((y \ast x) \ast x) \ast y) \in F).$$

\Proof Straightforward.

In the following example, we can see that any interior GE-filter $F$ in an interior GE-algebra $(X, f)$ does not satisfy the conditions (3.1) and (3.2).

\Example 3.10. Consider the interior GE-algebra $(X, f)$ in Example 3.5. The interior GE-filter $F := \{1\}$ does not satisfy conditions (3.1) and (3.2) since $f(2 \ast 3) = f(1) = 1 \in F$ and $2 \ast 3 = 1 \in F$ but $((3 \ast 2) \ast 2) \ast 3 = 3 \notin F$.

We provide conditions for an interior GE-filter to be a prominent interior GE-filter.

\Theorem 3.11. If an interior GE-filter $F$ in an interior GE-algebra $(X, f)$ satisfies the condition (3.1), then $F$ is a prominent interior GE-filter in $(X, f)$.

\Proof Let $F$ be an interior GE-filter in $(X, f)$ that satisfies the condition (3.1). Let $x, y, z \in X$ be such that $x \ast (y \ast z) \in F$ and $x \in F$. Then $y \ast z \in F$. Since $y \ast z \leq f(y \ast z)$ by (2.19), it follows from Lemma 2.6 that $f(y \ast z) \in F$. Hence $((z \ast y) \ast y) \ast z \in F$ by (3.1), and therefore $F$ is a prominent interior GE-filter in $(X, f)$.

\Lemma 3.12. [10] In an interior GE-algebra, the intersection of interior GE-filters is also an interior GE-filter.

\Theorem 3.13. In an interior GE-algebra, the intersection of prominent interior GE-filters is also a prominent interior GE-filter.

\Proof Let $\{F_i \mid i \in \Lambda\}$ be a set of prominent interior GE-filters in an interior GE-algebra $(X, f)$ where $\Lambda$ is an index set. Then $\bigcap \{F_i \mid i \in \Lambda\}$ is a set of interior GE-filters in $(X, f)$, and so $\cap \{F_i \mid i \in \Lambda\} \cap \{F_i \mid i \in \Lambda\}$ is an interior GE-filter in $(X, f)$ by Lemma 3.12. Let $x, y \in X$ be such that $f(x \ast y) \in \cap \{F_i \mid i \in \Lambda\}$. Then $f(x \ast y) \in F_i$ for all $i \in \Lambda$. It follows from Proposition 3.6 that $((y \ast x) \ast x) \ast y \in F_i$ for all $i \in \Lambda$. Hence $((y \ast x) \ast x) \ast y \in \cap \{F_i \mid i \in \Lambda\}$ and therefore $\cap \{F_i \mid i \in \Lambda\}$ is a prominent interior GE-filter in $(X, f)$ by Theorem 3.11.

\Theorem 3.14. If an interior GE-filter $F$ in an interior GE-algebra $(X, f)$ satisfies the condition (3.2), then $F$ is a prominent interior GE-filter in $(X, f)$.
Example 3.16. Let $F$ be an interior GE-filter in $(X, f)$ that satisfies the condition (3.2). Let $x, y, z \in X$ be such that $x * (y * z) \in F$ and $x \in F$. Then $y * z \in F$ and thus $((z * y) * y) * z \in F$. Therefore $F$ is a prominent interior GE-filter in $(X, f)$. □

Given an interior GE-filter $F$ in an interior GE-algebra $(X, f)$, we consider an interior GE-filter $G$ which is greater than $F$ in $(X, f)$, that is, we take two interior GE-filters $F$ and $G$ such that $F \subseteq G$ in an interior GE-algebra $(X, f)$. We are now trying to find the condition that $G$ can be a prominent interior GE-filter in $(X, f)$.

Theorem 3.15. Let $(X, f)$ be an interior GE-algebra in which $X$ is transitive. Let $F$ and $G$ be interior GE-filters in $(X, f)$. If $F \subseteq G$ and $F$ is a prominent interior GE-filter in $(X, f)$, then $G$ is also a prominent interior GE-filter in $(X, f)$.

**Proof.** Assume that $F$ is a prominent interior GE-filter in $(X, f)$. Then it is an interior GE-filter in $(X, f)$ by Theorem 3.4. Let $x, y \in X$ be such that $f(x * y) \in G$. Then $x * y \in G$ by (2.22), and so $1 = (x * y) * (x * y) \leq x * ((x * y) * y)$ by (GE1) and (2.6). Since $1 \in F$, it follows from Lemma 2.6 that $x * ((x * y) * y) \in F$. Hence $(((x * y) * y) * x) * y \in F$. Therefore $G$ is a prominent interior GE-filter in $(X, f)$.

The following example describes Theorem 3.15.

**Example 3.16.** Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 4.

|   | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 1 | 1 | 5 | 5 |
| 3 | 1 | 1 | 1 | 5 | 5 |
| 4 | 1 | 3 | 3 | 1 | 1 |
| 5 | 1 | 3 | 3 | 1 | 1 |

and define a mapping $f$ on $X$ as follows:

$$f : X \rightarrow X, \quad x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 3 & \text{if } x \in \{2, 3\}, \\ 5 & \text{if } x \in \{4, 5\}.
\end{cases}$$
Then \((X, f)\) is an interior GE-algebra in which \(X\) is transitive, and \(F := \{1\}\) and \(G := \{1, 4, 5\}\) are interior GE-filters in \((X, f)\) with \(F \subseteq G\). Also we can observe that \(F\) and \(G\) are prominent interior GE-filters in \((X, f)\).

In Theorem 3.15, if \(F\) is an interior GE-filter which is not prominent, then \(G\) is also not a prominent interior GE-filter in \((X, f)\) as shown in the next example.

**Example 3.17.** Let \(X = \{1, 2, 3, 4, 5\}\) be a set with the Cayley table which is given in Table 5,

|   | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 1 | 1 | 4 | 1 |
| 3 | 1 | 5 | 1 | 4 | 5 |
| 4 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 4 | 1 |

and define a mapping \(f\) on \(X\) as follows:

\[
f : X \rightarrow X, \quad x \mapsto \begin{cases} 
1 & \text{if } x = 1, \\
3 & \text{if } x = 3, \\
4 & \text{if } x = 4, \\
2 & \text{if } x \in \{2, 5\}.
\end{cases}
\]

Then \((X, f)\) is an interior GE-algebra in which \(X\) is transitive, and \(F := \{1\}\) and \(G := \{1, 3\}\) are interior GE-filters in \((X, f)\) with \(F \subseteq G\). We can observe that \(F\) and \(G\) are not prominent interior GE-filters in \((X, f)\) since \(2 \ast 3 = 1 \in F\) but \((3 \ast 2) \ast 2 = (5 \ast 2) \ast 3 = 1 \ast 3 = 3 \notin F\), and \(4 \ast 2 = 1 \in G\) but \((2 \ast 4) \ast 4 \ast 2 = (4 \ast 4) \ast 2 = 1 \ast 2 = 2 \notin G\).

In Theorem 3.15, if \(X\) is not transitive, then Theorem 3.15 is false as seen in the following example.

**Example 3.18.** Let \(X = \{1, 2, 3, 4, 5, 6\}\) be a set with the Cayley table which is given in Table 6.

|   | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 1 | 1 | 1 | 6 | 6 | 6 |
| 3 | 1 | 1 | 1 | 5 | 5 | 5 |
| 4 | 1 | 1 | 3 | 1 | 1 | 1 |
| 5 | 1 | 2 | 3 | 2 | 1 | 1 |
| 6 | 1 | 2 | 3 | 2 | 1 | 1 |
If we define a mapping $f$ on $X$ as follows:

$$f : X \to X, \ x \mapsto \begin{cases} 1 & \text{if } x = 1, \\ 4 & \text{if } x = 4, \\ 5 & \text{if } x = 5, \\ 6 & \text{if } x = 6, \\ 2 & \text{if } x \in \{2, 3\}, \end{cases}$$

then $(X, f)$ is an interior GE-algebra in which $X$ is not transitive. Let $F := \{1\}$ and $G := \{1, 5, 6\}$. Then $F$ is a prominent interior GE-filter in $(X, f)$ and $G$ is an interior GE-filter in $(X, f)$ with $F \subseteq G$. But $G$ is not prominent interior GE-filter since $5 \ast (3 \ast 4) = 5 \ast 5 = 1 \in G$ and $5 \in G$ but $((4 \ast 3) \ast 3) \ast 4 = (3 \ast 3) \ast 4 = 1 \ast 4 = 4 \notin G$.

**Definition 3.19.** Let $(X, f)$ be an interior GE-algebra and let $F$ be a subset of $X$ which satisfies (2.15). Then $F$ is called:

- A **prominent interior GE-filter of type 1** in $(X, f)$ if it satisfies:

$$\forall x, y, z \in X (x \ast (y \ast f(z)) \in F, f(x) \in F \Rightarrow ((f(z) \ast y) \ast y) \ast f(z) \in F). \quad (3.3)$$

- A **prominent interior GE-filter of type 2** in $(X, f)$ if it satisfies:

$$\forall x, y, z \in X (x \ast (y \ast f(z)) \in F, f(x) \in F \Rightarrow ((z \ast f(y)) \ast f(y)) \ast z \in F). \quad (3.4)$$

**Example 3.20.** (1). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 7,

|  | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 2 | 1 | 2 | 2 |
| 4 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 |

and define a mapping $f$ on $X$ as follows:

$$f : X \to X, \ x \mapsto \begin{cases} 1 & \text{if } x \in \{1, 3\} \\ 2 & \text{if } x = 2, \\ 4 & \text{if } x = 4, \\ 5 & \text{if } x = 5. \end{cases}$$

Then $(X, f)$ is an interior GE-algebra and $F := \{1, 3\}$ is a prominent interior GE-filter of type 1 in $(X, f)$.

(2). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in Table 8,
Table 8. Cayley table for the binary operation “∗”.

| *  | 1 | 2 | 3 | 4 | 5 |
|----|---|---|---|---|---|
| 1  | 1 | 2 | 3 | 4 | 5 |
| 2  | 1 | 1 | 1 | 1 | 1 |
| 3  | 1 | 1 | 1 | 4 | 1 |
| 4  | 1 | 1 | 1 | 1 | 5 |
| 5  | 1 | 1 | 3 | 4 | 1 |

and define a mapping \( f \) on \( X \) as follows:

\[
f : X \rightarrow X, \ x \mapsto \begin{cases} 
1 & \text{if } x = 1, \\
2 & \text{if } x \in \{2, 3, 4, 5\}.
\end{cases}
\]

Then \((X, f)\) is an interior GE-algebra and \( F := \{1, 3\} \) is a prominent interior GE-filter of type 2 in \((X, f)\).

**Theorem 3.21.** In an interior GE-algebra, every prominent interior GE-filter is of type 1.

**Proof.** Let \( F \) be a prominent interior GE-filter in an interior GE-algebra \((X, f)\). Let \( x, y, z \in X \) be such that \( x * (y * f(z)) \in F \) and \( f(x) \in F \). Then \( x \in F \) by (2.22). It follows from (2.18) that \( ((f(z) * y) * y) * f(z) \in F \). Hence \( F \) is a prominent interior GE-filter of type 1 in \((X, f)\). \( \square \)

The following example shows that the converse of Theorem 3.21 may not be true.

**Example 3.22.** Let \( X = \{1, 2, 3, 4, 5\} \) be a set with the Cayley table which is given in Table 9,

Table 9. Cayley table for the binary operation “∗”.

| *  | 1 | 2 | 3 | 4 | 5 |
|----|---|---|---|---|---|
| 1  | 1 | 2 | 3 | 4 | 5 |
| 2  | 1 | 1 | 1 | 1 | 1 |
| 3  | 1 | 1 | 1 | 1 | 5 |
| 4  | 1 | 1 | 3 | 1 | 1 |
| 5  | 1 | 1 | 1 | 1 | 1 |

and define a mapping \( f \) on \( X \) as follows:

\[
f : X \rightarrow X, \ x \mapsto \begin{cases} 
1 & \text{if } x = 1, \\
2 & \text{if } x \in \{2, 3\}, \\
5 & \text{if } x \in \{4, 5\}.
\end{cases}
\]

Then \((X, f)\) is an interior GE-algebra and \( F := \{1\} \) is a prominent interior GE-filter of type 1 in \((X, f)\). But it is not a prominent interior GE-filter in \((X, f)\) since \( 1 * (3 * 4) = 1 \in F \) but \( (4 * 3) * 3 * 4 = 4 \notin F \).

The following example shows that prominent interior GE-filter and prominent interior GE-filter of type 2 are independent of each other, i.e., prominent interior GE-filter is not prominent interior GE-filter of type 2 and neither is the inverse.
Example 3.23. (1). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 10,

**Table 10.** Cayley table for the binary operation “*”.

| $*$ | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|
| 1   | 1 | 2 | 3 | 4 | 5 |
| 2   | 1 | 1 | 1 | 1 | 1 |
| 3   | 1 | 5 | 1 | 1 | 5 |
| 4   | 1 | 1 | 1 | 1 | 1 |
| 5   | 1 | 3 | 3 | 1 | 1 |

and define a mapping $f$ on $X$ as follows:

$$f : X \to X, x \mapsto \begin{cases} 
1 & \text{if } x = 1, \\
4 & \text{if } x \in \{3, 4\} \\
5 & \text{if } x \in \{2, 5\}.
\end{cases}$$

Then $(X, f)$ is an interior GE-algebra and $F := \{1\}$ is a prominent interior GE-filter in $(X, f)$. But it is not a prominent interior GE-filter of type 2 since $1 * (5 * f(2)) = 5 * 5 = 1 \in F$ and $f(1) = 1 \in F$ but $(2 * f(5)) * 2 = (2 * 5) * 2 = 1 * 5 = 2 \notin F$.

(2). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 11,

**Table 11.** Cayley table for the binary operation “*”.

| $*$ | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|
| 1   | 1 | 2 | 3 | 4 | 5 |
| 2   | 1 | 1 | 1 | 1 | 1 |
| 3   | 1 | 2 | 1 | 1 | 1 |
| 4   | 1 | 1 | 1 | 1 | 1 |
| 5   | 1 | 1 | 1 | 1 | 1 |

and define a mapping $f$ on $X$ as follows:

$$f : X \to X, x \mapsto \begin{cases} 
1 & \text{if } x = 1, \\
5 & \text{if } x \in \{2, 3, 4, 5\}.
\end{cases}$$

Then $(X, f)$ is an interior GE-algebra and $F := \{1\}$ is a prominent interior GE-filter of type 2 in $(X, f)$. But it is not a prominent interior GE-filter in $(X, f)$ since $1 * (2 * 3) = 1 * 1 = 1 \in F$ and $1 \in F$ but $(3 * 2) * 3 = (2 * 2) * 3 = 1 * 3 = 3 \notin F$.

The following example shows that prominent interior GE-filter of type 1 and prominent interior GE-filter of type 2 are independent of each other.

Example 3.24. (1). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 12,
Table 12. Cayley table for the binary operation “∗”.

|   | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 1 | 1 | 5 | 5 |
| 3 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 |

and define a mapping \( f \) on \( X \) as follows:

\[
f : X \to X, \quad x \mapsto \begin{cases} 
1 & \text{if } x = 1, \\
3 & \text{if } x \in \{2, 3\}, \\
5 & \text{if } x \in \{4, 5\}.
\end{cases}
\]

Then \((X, f)\) is an interior GE-algebra and \( F := \{1, 2, 4\} \) is a prominent interior GE-filter of type 1 in \((X, f)\). But it is not a prominent interior GE-filter of type 2 since \(1 \ast (5 \ast f(2)) = 1 \ast (5 \ast 3) = 1 \ast 1 = 1 \in F\) and \(f(1) = 1 \in F\) but \((2 \ast f(5)) \ast f(5)) \ast 2 = (2 \ast 5) \ast f(5)) \ast 2 = (5 \ast 5) \ast 2 = 1 \ast 2 = 2 \not\in F\).

(2). Let \(X = \{1, 2, 3, 4, 5\}\) be a set with the Cayley table which is given in the following Table 13,

Table 13. Cayley table for the binary operation “∗”.

|   | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 1 | 4 | 4 | 5 |
| 3 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 2 | 2 | 1 | 5 |
| 5 | 1 | 1 | 1 | 1 | 1 |

and define a mapping \( f \) on \( X \) as follows:

\[
f : X \to X, \quad x \mapsto \begin{cases} 
1 & \text{if } x = 1, \\
2 & \text{if } x = 2, \\
4 & \text{if } x = 4, \\
3 & \text{if } x \in \{3, 5\}.
\end{cases}
\]

Then \((X, f)\) is an interior GE-algebra and \( F := \{1\} \) is a prominent interior GE-filter of type 2 in \((X, f)\). But it is not a prominent interior GE-filter of type 1 in \((X, f)\) since \(1 \ast (5 \ast f(2)) = 1 \ast (5 \ast 2) = 1 \ast 1 = 1 \in F\) and \(f(1) \in F\) but \(((f(2) \ast 5) \ast 5) \ast f(2) = ((2 \ast 5) \ast 5) \ast 2 = (5 \ast 5) \ast 2 = 1 \ast 2 = 2 \not\in F\).

The following example shows that interior GE-filter and prominent interior GE-filter of type 1 are independent of each other.

Example 3.25. (1). Let \(X = \{1, 2, 3, 4, 5\}\) be a set with the Cayley table which is given in the following Table 14,
and define a mapping $f$ on $X$ as follows:

$$f : X \rightarrow X, \ x \mapsto \begin{cases} 
1 & \text{if } x = 1, \\
2 & \text{if } x = 2, \\
5 & \text{if } x \in \{3, 4, 5\}.
\end{cases}$$

Then $(X, f)$ is an interior GE-algebra and $F := \{1\}$ is an interior GE-filter in $(X, f)$. But $F$ is not prominent interior GE-filter of type 1 since $1 \ast (5 \ast f(2)) = 1 \ast (5 \ast 2) = 1 \ast 1 = 1 \in F$ and $f(1) = 1 \in F$ but $((f(2) \ast 5) \ast 2 = (2 \ast 5) \ast 2 = (5 \ast 5) \ast 2 = 1 \ast 2 = 2 \notin F$.

(2). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 15,

Table 15. Cayley table for the binary operation “$\ast$”.

| $\ast$ | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|
| 1     | 1 | 2 | 3 | 4 | 5 |
| 2     | 1 | 1 | 5 | 1 | 5 |
| 3     | 1 | 2 | 1 | 1 | 1 |
| 4     | 1 | 1 | 3 | 1 | 5 |
| 5     | 1 | 1 | 1 | 1 | 1 |

and define a mapping $f$ on $X$ as follows:

$$f : X \rightarrow X, \ x \mapsto \begin{cases} 
1 & \text{if } x \in \{1, 2, 4\}, \\
5 & \text{if } x \in \{3, 5\}.
\end{cases}$$

Then $(X, f)$ is an interior GE-algebra and $F := \{1, 2\}$ is a prominent interior GE-filter of type 1 in $(X, f)$. But it is not an interior GE-filter in $(X, f)$ since $2 \ast 4 = 1$ and $2 \in F$ but $4 \notin F$.

The following example shows that interior GE-filter and prominent interior GE-filter of type 2 are independent of each other.

**Example 3.26.** (1). Let $X = \{1, 2, 3, 4, 5\}$ be a set with the Cayley table which is given in the following Table 16,
Table 16. Cayley table for the binary operation “∗”.

|   | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 2 | 1 | 1 | 2 |
| 4 | 1 | 2 | 3 | 1 | 5 |
| 5 | 1 | 1 | 1 | 1 | 1 |

and define a mapping \( f \) on \( X \) as follows:

\[
f : X \rightarrow X, \ x \mapsto \begin{cases} 
1 & \text{if } x \in \{1, 4\} \\
2 & \text{if } x = 2, \\
3 & \text{if } x = 3, \\
5 & \text{if } x = 5.
\end{cases}
\]

Then \((X, f)\) is an interior GE-algebra and \( F := \{1, 4\} \) is an interior GE-filter in \((X, f)\). But \( F \) is not prominent interior GE-filter of type 2 since \( 4 \ast (2 \ast f(3)) = 4 \ast (2 \ast 3) = 4 \ast 1 = 1 \in F \) and \( f(4) = 1 \in F \) but \((3 \ast f(2)) \ast f(2)) \ast 3 = (3 \ast 2) \ast 3 = (2 \ast 2) \ast 3 = 1 \ast 3 = 3 \notin F \).

(2). Let \( X = \{1, 2, 3, 4, 5\} \) be a set with the Cayley table which is given in the following Table 17,

Table 17. Cayley table for the binary operation “∗”.

|   | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 5 |
| 5 | 1 | 1 | 1 | 1 | 1 |

and define a mapping \( f \) on \( X \) as follows:

\[
f : X \rightarrow X, \ x \mapsto \begin{cases} 
1 & \text{if } x = 1, \\
3 & \text{if } x \in \{2, 3, 4, 5\}.
\end{cases}
\]

Then \((X, f)\) is an interior GE-algebra and \( F := \{1, 2, 5\} \) is a prominent interior GE-filter of type 2 in \((X, f)\). But it is not an interior GE-filter in \((X, f)\) since \( 5 \ast 4 = 1 \) and \( 5 \in F \) but \( 4 \notin F \).

Before we conclude this paper, we raise the following question.

**Question.** Let \((X, f)\) be an interior GE-algebra. Let \( F \) and \( G \) be interior GE-filters in \((X, f)\). If \( F \subseteq G \) and \( F \) is a prominent interior GE-filter of type 1 (resp., type 2) in \((X, f)\), then is \( G \) also a prominent interior GE-filter of type 1 (resp., type 2) in \((X, f)\)?

4. Conclusions

We have introduced the concept of a prominent interior GE-filter (of type 1 and type 2), and have investigated their properties. We have discussed the relationship between a prominent GE-filter and a
prominent interior GE-filter and the relationship between an interior GE-filter and a prominent interior GE-filter. We have found and provide examples where any interior GE-filter is not a prominent interior GE-filter and any prominent GE-filter is not a prominent interior GE-filter. We have provided conditions for an interior GE-filter to be a prominent interior GE-filter. We have given conditions under which an internal GE-filter larger than a given internal GE filter can become a prominent internal GE-filter, and have provided an example describing it. We have considered the relationship between a prominent interior GE-filter and a prominent interior GE-filter of type 1. We have found and provide examples to verify that a prominent interior GE-filter of type 1 and a prominent interior GE-filter of type 2, a prominent interior GE-filter and a prominent interior GE-filter of type 2, an interior GE-filter and a prominent interior GE-filter of type 1, and an interior GE-filter and a prominent interior GE-filter of type 2 are independent each other. In future, we will study the prime and maximal prominent interior GE-filters and their topological properties. Moreover, based on the ideas and results obtained in this paper, we will study the interior operator theory in related algebraic systems such as MV-algebra, BL-algebra, EQ-algebra, etc. It will also be used for pseudo algebra systems and further to conduct research related to the very true operator theory.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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