ALGEBRAIC STRUCTURE OF THE $L_2$ ANALYTIC FOURIER–FEYNMAN TRANSFORM ASSOCIATED WITH GAUSSIAN PATHS ON WIENER SPACE

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Abstract. In this paper we study algebraic structures of the classes of the $L_2$ analytic Fourier–Feynman transforms on Wiener space. To do this we first develop several rotation properties of the generalized Wiener integral associated with Gaussian paths. We then proceed to analyze the $L_2$ analytic Fourier–Feynman transforms associated with Gaussian paths. Our results show that these $L_2$ analytic Fourier–Feynman transforms are actually linear operator isomorphisms from a Hilbert space into itself. We finally investigate the algebraic structures of these classes of the transforms on Wiener space, and show that they indeed are group isomorphic.

1. Introduction. Let $C_0[0, T]$ denote one parameter Wiener space. Bearman’s rotation theorem [1] for Wiener measure has played an important role in various research areas in mathematics and physics involving Wiener integration theory. Bearman’s theorem was further developed by Cameron and Storvick [4] and by Johnson and Skoug [14] in their studies of Wiener integral equations.

The concept of the generalized Wiener integral and the generalized Feynman integral on $C_0[0, T]$ were introduced by Park and Skoug in [18], further studied by Chung, Park and Skoug in [9], extended by Park and Skoug in [19], and further studied by Huffman, Park and Skoug in [11]. In [9, 11, 18, 19], the generalized Wiener integral was defined by the Wiener integral

$$\int_{C_0[0, T]} F(Z_h(x, \cdot)) \, dm_w(x),$$

where $Z_h(x, \cdot)$ is the Gaussian path given by the stochastic integral $Z_h(x, t) = \int_0^t h(s) \, dx(s)$ with $h \in L_2[0, T]$ and $m_w$ denotes the Wiener measure.

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The concept of the analytic Fourier–Feynman transform (abbr. FFT) on the Wiener space \( C_0[0, T] \), initiated by Brue [2], has been developed in the literature. This transform and its properties are similar in many respects to the ordinary Fourier transform of functions on an Euclidean space. For an elementary introduction to the analytic FFT, see [21] and the references cited therein. In particular, in [11], Huffman, Park and Skoug introduced a generalized FFT associated with the Gaussian paths \( Z_h(x, \cdot) \). Since then, the generalized FFT was further developed by many mathematicians. For instance, see [6, 8].

In this paper we study algebraic structures of the classes of the \( L_2 \) analytic FFTs on Wiener space. To do this we first develop the Bearman-type theorems for the generalized Wiener integral given by (1.1). Using these results, we then take a closer look at the \( L_2 \) analytic FFT associated with the Gaussian paths \( Z_h(x, \cdot) \) (abbr. \( Z_h \)-FFT) introduced by Huffman, Park, and Skoug in [11]. Our results indicate that the \( L_2 \) analytic \( Z_h \)-FFTs are linear operator isomorphisms from a Hilbert space of cylinder functionals on Wiener space into itself. Furthermore the algebraic structures of these generalized transforms are examined. Based on this examination we know that these classes of generalized transforms are indeed group isomorphic.

2. Preliminaries. Given a positive real \( T > 0 \), let \( C_0[0, T] \) denote one-parameter Wiener space, that is, the space of all real-valued continuous functions \( x \) on the compact interval \([0, T]\) with \( x(0) = 0 \). Let \( \mathcal{M} \) denote the class of all Wiener measurable subsets of \( C_0[0, T] \) and let \( m_w \) denote Wiener measure which is a Gaussian measure on \( C_0[0, T] \) with mean zero and covariance function \( r(s, t) = \min\{s, t\} \). Then, as is well-known, \((C_0[0, T], \mathcal{M}, m_w)\) is a complete measure space.

A subset \( B \) of \( C_0[0, T] \) is said to be scale-invariant measurable [14] provided \( \rho B \in \mathcal{M} \) for all \( \rho > 0 \), and a scale-invariant measurable set \( N \) is said to be scale-invariant null provided \( m_w(\rho N) = 0 \) for all \( \rho > 0 \). A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (abbr. s-a.e.). A functional \( F \) is said to be scale-invariant measurable provided \( F \) is defined on a scale-invariant measurable set and \( F(\rho \cdot) \) is Wiener measurable for every \( \rho > 0 \).

If two functionals \( F \) and \( G \) are equal s-a.e., we write \( F \approx G \).

The Paley–Wiener–Zygmund (abbr. PWZ) stochastic integral [16] plays a key role throughout this paper. For \( v \) in \( L_2[0, T] \), the PWZ stochastic integral \( \langle v, x \rangle \) is given by the formula

\[
\langle v, x \rangle := \lim_{n \to \infty} \int_0^T \sum_{j=1}^n (v, \phi_j) \phi_j(t) \, dx(t)
\]

for \( m_w \)-a.e. \( x \in C_0[0, T] \), where \( \{\phi_n\} \) is a complete orthonormal set of functions of bounded variation on \([0, T]\) and \( \langle \cdot, \cdot \rangle_2 \) denotes the \( L_2 \)-inner product.

It is known that for each \( v \in L_2[0, T] \), the PWZ integral \( \langle v, x \rangle \) is essentially independent of the choice of the complete orthonormal set \( \{\phi_n\} \). If \( v \) is of bounded variation on \([0, T]\) then \( \langle v, x \rangle \) equals the Riemann–Stieltjes integral \( \int_0^T v(t) \, dx(t) \) for s-a.e. \( x \in C_0[0, T] \), and for all \( v \in L_2[0, T] \), \( \langle v, x \rangle \) is a Gaussian random variable on \( C_0[0, T] \) with mean zero and variance \( \|v\|_2^2 \). For a more detailed study of the PWZ stochastic integral, see [15, 17].

Given a function \( h \) in \( L_2[0, T] \) with \( \|h\|_2 > 0 \), let \( Z_h(x, t) \) be the PWZ stochastic integral

\[
Z_h(x, t) := \langle h \chi_{[0,t]}, x \rangle
\]

(2.1)
where $\chi_{[0,t]}$ denotes the indicator function of the set $[0,t]$. Next, let

$$\beta_h(t) := \int_0^t h^2(u)du.$$  

(2.2)

Then the stochastic process $Z_h$ on $C_0[0,T] \times [0,T]$, $(x,t) \mapsto Z_h(x,t)$, is a Gaussian process with mean zero and covariance function

$$\int_{C_0[0,T]} Z_h(x,s)Z_h(x,t)dm_w(x) = \beta_h(\min\{s,t\}).$$

In addition, by [22, Theorem 21.1], $Z_h(\cdot, t)$ is stochastically continuous in $t$ on $[0,T]$. If $h \in L_2[0,T]$ is of bounded variation on $[0,T]$, then for all $x \in C_0[0,T]$, $Z_h(x,t)$ is continuous in $t$. Also, for any $h_1, h_2 \in L_2[0,T]$,

$$\int_{C_0[0,T]} Z_{h_1}(x,s)Z_{h_2}(x,t)dm_w(x) = \int_0^{\min\{s,t\}} h_1(u)h_2(u)du.$$  

Of course if $h(t) \equiv 1$ on $[0,T]$, then the process $W$ on $C_0[0,T] \times [0,T]$ given by $(w,t) \mapsto W_t(x) = Z_t(x,t) = x(t)$ is a Wiener process. We note that the coordinate process $Z_t$ is stationary in time, whereas the stochastic process $Z_h$ generally is not. For more detailed studies on the stochastic process $Z_h$, see [9, 18].

From [9, Lemma 1], it follows that for each $v \in L_2[0,T]$ and $h \in L_\infty[0,T]$,

$$\langle v, Z_h(x, \cdot) \rangle = \langle vh, x \rangle$$  

(2.3)

for s.a.e. $x \in C_0[0,T]$.

Throughout the rest of this paper let $\mathbb{C}_+$ and $\tilde{\mathbb{C}}_+$ denote the set of complex numbers with positive real part and nonzero complex numbers with nonnegative real part, respectively.

Let $h$ be a function in $L_2[0,T]$ with $\|h\|_2 > 0$ and let $F$ be a complex-valued scale-invariant measurable functional on $C_0[0,T]$ such that

$$J(h; \lambda) := \int_{C_0[0,T]} F(\lambda^{-1/2}Z_h(x,\cdot))dm_w(x)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(h; \lambda)$ analytic on $\mathbb{C}_+$ such that $J^*(h; \lambda) = J(h; \lambda)$ for all $\lambda > 0$, then $J^*(h; \lambda)$ is defined to be the analytic Wiener integral (associated with Gaussian paths $Z(x,\cdot)$) of $F$ over $C_0[0,T]$ with parameter $\lambda$, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_0[0,T]}^\text{anw\lambda} F(Z_h(x,\cdot))dm_w(x) := J^*(h; \lambda).$$

Let $q$ be a nonzero real number and let $F$ be a functional such that

$$\int_{C_0[0,T]}^\text{anw\lambda} F(Z_h(x,\cdot))dm_w(x)$$

exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral (associated with Gaussian paths $Z_h(x,\cdot)$) of $F$ with parameter $q$, and we write

$$I_h^{\text{anf}}[F] = I_h^{\text{anf}}[F(Z_h(x,\cdot))] := \lim_{\lambda \to -iq} \int_{C_0[0,T]}^\text{anw\lambda} F(Z_h(x,\cdot))dm_w(x)$$

where $\lambda$ approaches $-iq$ through values in $\mathbb{C}_+$. We are now ready to state the definition of the $L_2$ analytic $Z_h$-FFT on Wiener space.
**Definition 2.1.** Let \( h \) be a function in \( L_2[0, T] \) with \( \|h\|_2 > 0 \) and let \( F \) be a scale-invariant measurable functional on \( C_0[0, T] \). For \( \lambda \in \mathbb{C}_+ \) and \( y \in C_0[0, T] \), let
\[
T_{\lambda, h}(F)(y) := \int_{C_0[0, T]} F(y + Z_h(x, \cdot)) dm_w(x).
\]
Also, let \( q \) be a nonzero real number. We define the \( L_2 \) analytic \( Z_h \text{-FFT} \), \( T^{(2)}_{q, h}(F) \) of \( F \), by the formula,
\[
T^{(2)}_{q, h}(F)(y) := 1 \text{. i. m. } T_{\lambda, h}(F)(y)
\]
if it exists; i.e., for each \( \rho > 0 \),
\[
\lim_{\lambda \rightarrow -iq, \lambda \in \mathbb{C}_+} \int_{C_0[0, T]} |T_{\lambda, h}(F)(\rho y) - T^{(2)}_{q, h}(F)(\rho y)|^2 dm_w(y) = 0.
\]
We note that \( T^{(2)}_{q, h}(F) \) is defined only s-a.e.. We also note that if \( T^{(2)}_{q, h}(F) \) exists and if \( F \approx G \), then \( T^{(2)}_{q, h}(G) \) exists and \( T^{(2)}_{q, h}(G) \approx T^{(2)}_{q, h}(F) \). One can also see that for each \( h \in L_2[0, T] \), \( T^{(2)}_{q, h}(F) \approx T^{(2)}_{q, -h}(F) \) since
\[
\int_{C_0[0, T]} F(x) dm_w(x) = \int_{C_0[0, T]} F(-x) dm_w(x).
\]

**Remark 2.2.** Frankly speaking, it follows that for any functions \( h_1 \) and \( h_2 \) in \( L_2[0, T] \) satisfying the equality \( h_1^2 = h_2^2 \) m_L-a.e. on \( [0, T] \) (m_L denotes the Lebesgue measure on \( [0, T] \) in this paper),
\[
\int_{C_0[0, T]} F(Z_{h_1}(x, \cdot)) dm_w(x) = \int_{C_0[0, T]} F(Z_{h_2}(x, \cdot)) dm_w(x)
\]
in the sense that if either side exists, both sides exist and equality holds. Thus, for any scale-invariant measurable functional \( F \) on \( C_0[0, T] \),
\[
T^{(2)}_{q, h_1}(F) \approx T^{(2)}_{q, h_2}(F),
\]
because the Gaussian processes \( Z_{h_1} \) and \( Z_{h_2} \) have the same distribution.

**Remark 2.3.** Note that if \( h \equiv 1 \) on \( [0, T] \), then the definition of the \( L_2 \) analytic \( Z_1 \text{-FFT} \) agrees with the previous definition of the \( L_2 \) analytic Fourier–Feynman transform \([3, 10, 13]\).

3. **Rotation of Wiener measures associated with Gaussian paths.** In this section we develop rotation theorems for the generalized Wiener integral of the functionals on \( C_0[0, T] \). To do this, we follow the exposition of \([5, 6, 7]\).

Throughout this section we will assume that each functional \( F : C_0[0, T] \rightarrow \mathbb{R} \) we consider is scale-invariant measurable and that
\[
\int_{C_0[0, T]} |F(\rho Z_h(x, \cdot))| dm_w(x) < +\infty
\]
for each \( \rho > 0 \) and for each \( h \in L_2[0, T] \).

Let \( v_1 \) and \( v_2 \) be functions in \( L_2[0, T] \) with \( \|v_1\|_2^2 = \|v_2\|_2^2 = \sigma^2 > 0 \). The random variables \( X_1(x) = \langle v_1, x \rangle \) and \( X_2(x) = \langle v_2, x \rangle \) will then have the same distribution, \( N(0, \sigma^2) \).

Next let \( h_1 \) and \( h_2 \) be functions in \( L_2[0, T] \). Then there exists a function \( s \in L_2[0, T] \) such that
\[
s^2(t) = h_1^2(t) + h_2^2(t)
\]
(3.1)
for \(\mathbb{m}_L\)-a.e. \(t \in [0, T]\). We note that for any nonzero function \(s_1\) ("nonzero function \(g\)" means that \(\|g\|_2 > 0\) throughout this paper) in \(L_2[0, T]\), there exists \(s_2 \in L_2[0, T]\) such that \(\|s_1\|_2 = \|s_2\|_2\), but, \(s_1 \neq s_2\) \(\mathbb{m}_L\)-a.e. on \([0, T]\) (more generally, \(\mathbb{m}_L(\{t \in [0, T] : s_1(t) \neq s_2(t)\}) > 0\)). Thus one can see that the function \(s\) satisfying (3.1) is not unique. Let \(s_1\) and \(s_2\) be functions in \(L_2[0, T]\) that satisfy equation (3.1). Then, in view of the observation above, \((s_1, x)\) and \((s_2, x)\) have the same distribution \(N(0, \|h_1\|_2^2 + \|h_2\|_2^2)\).

**Remark 3.1.** Consider the relation \(\sim\) on \(L_2[0, T]\) given by

\[
h \sim g \iff h^2 = g^2 \text{ \(\mathbb{m}_L\)-a.e.}.
\]  

(3.2)

Then \(\sim\) is an equivalence relation. The equivalence relation \(\sim\) is also well-defined on \(L_\infty[0, T]\).

We will use the symbol \(s(h_1, h_2)\) for the functions \(s\) that satisfy (3.1) above. Given functions \(h_1\) and \(h_2\) in \(L_2[0, T]\) (resp. \(L_\infty[0, T]\)), infinitely many functions, \(s(h_1, h_2)\), exist in \(L_2[0, T]\) (resp. \(L_\infty[0, T]\)). Thus the \(s(h_1, h_2)'s\) can be considered as equivalence classes of the relation (3.2), i.e., it allows us that

\[
s(h_1, h_2) = \left\{ s : s \sim \sqrt{h_1^2 + h_2^2} \right\}.
\]

But, by the observation above, it follows that for every function \(s\) in the equivalence class \(s(h_1, h_2)\), the Gaussian random variable \(\langle s, x \rangle\) has the normal distribution \(N(0, \|h_1\|_2^2 + \|h_2\|_2^2)\).

We also note that if the functions \(h_1\) and \(h_2\) are in \(L_\infty[0, T]\), then we can take \(s(h_1, h_2)\) to be in \(L_\infty[0, T]\).

Inductively, given a finite sequence \(\mathcal{H} = (h_1, \ldots, h_n)\) of functions in \(L_2[0, T]\) (resp. \(L_\infty[0, T]\)), let \(s(h_1, h_2, \ldots, h_n)\) be the functions \(s\) in \(L_2[0, T]\) (resp. \(L_\infty[0, T]\)) which satisfy the relation

\[
s^2(t) = h_1^2(t) + \cdots + h_n^2(t)
\]

(3.3)

for \(\mathbb{m}_L\)-a.e. \(t \in [0, T]\). Then for \(k \in \{1, \ldots, n - 1\}\),

\[
s^2(s(h_1, \ldots, h_k), h_{k+1})(t) = s^2(h_1, \ldots, h_k)(t) + h_{k+1}^2(t) = \sum_{j=1}^{k+1} h_j^2(t)
\]

(3.4)

for \(\mathbb{m}_L\)-a.e. \(t \in [0, T]\). For convenience, given a finite sequence \(\mathcal{H} = (h_1, \ldots, h_n)\) we denote \(s(h_1, \ldots, h_n)\) by \(s(\mathcal{H})\). From these arguments, we have the following properties:

(i) For any finite sequence \(\mathcal{H} = (h_1, \ldots, h_n)\) in \(L_2[0, T]\), \(s(\mathcal{H})\) has an consistency property. That is, for any permutation \(\pi\) of \(I_n = \{1, 2, \ldots, n\}\),

\[
s(h_1, h_2, \ldots, h_n) = s(h_{\pi(1)}, h_{\pi(2)}, \ldots, h_{\pi(n)})
\]

(3.5)

for \(\mathbb{m}_L\)-a.e. \(t \in [0, T]\).

(ii) Given two sequences \(\mathcal{H}_1 = (h_{11}, h_{12}, \ldots, h_{1n_1})\) and \(\mathcal{H}_2 = (h_{21}, h_{22}, \ldots, h_{2n_2})\) of functions in \(L_2[0, T]\), let \(\mathcal{H}_1 \wedge \mathcal{H}_2\) denote the cementation of \(\mathcal{H}_1\) and \(\mathcal{H}_2\), i.e.,

\[
\mathcal{H}_1 \wedge \mathcal{H}_2 := (h_{11}, h_{12}, \ldots, h_{1n_1}, h_{21}, h_{22}, \ldots, h_{2n_2}).
\]

Then

\[
s(\mathcal{H}_1 \wedge \mathcal{H}_2) = s(s(\mathcal{H}_1), s(\mathcal{H}_2)).
\]

(3.6)
For \( h_1, h_2 \in L_2[0,T] \) with \( \| h_j \|_2 > 0 \), \( j \in \{1, 2\} \), let \( Z_{h_1} \) and \( Z_{h_2} \) be the Gaussian processes given by (2.1) with \( h \) replaced with \( h_1 \) and \( h_2 \) respectively. Then the process
\[
\mathcal{F}_{h_1, h_2} : C_0[0,T] \times C_0[0,T] \times [0,T] \rightarrow \mathbb{R}
\]
given by
\[
\mathcal{F}_{h_1, h_2}(x_1, x_2, t) := Z_{h_1}(x_1, t) + Z_{h_2}(x_2, t)
\]
is also a Gaussian process with mean zero and covariance
\[
\int_{C_0^2[0,T]} \mathcal{F}_{h_1, h_2}(x_1, x_2, s) \mathcal{F}_{h_1, h_2}(x_1, x_2, t) dm_{w}^2(x_1, x_2) = \beta_{h_1}(\min\{s, t\}) + \beta_{h_2}(\min\{s, t\})
\]
where \( \beta_h \) is given by (2.2) above.

Next we consider the stochastic process \( Z_{\mathcal{H}(h_1, h_2)} : C_0[0,T] \times [0,T] \rightarrow \mathbb{R} \). As stated in Section 2 above, the process \( Z_{\mathcal{H}(h_1, h_2)} \) is Gaussian with mean zero and covariance
\[
\beta_{\mathcal{H}(h_1, h_2)}(\min\{s, t\}) = \int_0^{\min\{s, t\}} s^2(h_1, h_2)(u) du + \int_0^{\min\{s, t\}} (h_1^2(u) + h_2^2(u)) du
\]
\[
= \beta_{h_1}(\min\{s, t\}) + \beta_{h_2}(\min\{s, t\}).
\]

From these facts, one can see that \( \mathcal{F}_{h_1, h_2} \) and \( Z_{\mathcal{H}(h_1, h_2)} \) have the same distribution. Thus we obtain the following theorems and corollaries.

**Theorem 3.2** ([5]). Let \( F \) be a functional on \( C_0[0,T] \), and let \( h_1 \) and \( h_2 \) be nonzero functions in \( L_2[0,T] \). Then
\[
\int_{C_0^2[0,T]} F(Z_{h_1}(x_1, \cdot) + Z_{h_2}(x_2, \cdot)) dm_{w}^2(x_1, x_2) = \int_{C_0[0,T]} F(Z_{\mathcal{H}(h_1, h_2)}(x_\cdot)) dm_{w}(x). \tag{3.7}
\]

**Remark 3.3.** In [7], the authors established the rotation theorem, namely, equation (3.7), only in the case that the integrand functionals \( F \) are cylinder type on \( C_0[0,T] \). But the integrand functionals \( F \) in (3.7) are not restricted.

**Corollary 3.4.** Let \( F \) be a functional on \( C_0[0,T] \), and let \( \mathcal{H} = (h_1, h_2, \ldots, h_n) \) be a finite sequence of nonzero functions in \( L_2[0,T] \). Then
\[
\int_{C_0^2[0,T]} F\left( \sum_{j=1}^{n} Z_{h_j}(x_j, \cdot) \right) dm_{w}^n(\vec{x}) = \int_{C_0[0,T]} F(Z_{\mathcal{H}(h_1,\ldots,h_n)}(x_\cdot)) dm_{w}(x).
\]

**Theorem 3.5.** Let \( F \) be a functional on \( C_0[0,T] \), and let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be finite sequences of nonzero functions in \( L_2[0,T] \). Then
\[
\int_{C_0^2[0,T]} F(Z_{\mathcal{H}_1}(x_1, \cdot) + Z_{\mathcal{H}_2}(x_2, \cdot)) dm_{w}^2(x_1, x_2) = \int_{C_0[0,T]} F(Z_{\mathcal{H}(\mathcal{H}_1, \mathcal{H}_2)}(x_\cdot)) dm_{w}(x) \tag{3.8}
\]
\[
= \int_{C_0[0,T]} F(Z_{\mathcal{H}_1 \cap \mathcal{H}_2}(x_\cdot)) dm_{w}(x).
\]

**Corollary 3.6.** Let \( F \) be a functional on \( C_0[0,T] \), and let \( \mathcal{H} = (h_1, h_2, \ldots, h_n) \) be a finite sequence of nonzero functions in \( L_2[0,T] \) and let \( h_{n+1} \) be a nonzero function in \( L_2[0,T] \). Then
\[
\int_{C_0^2[0,T]} F(Z_{\mathcal{H}}(x_1, \cdot) + Z_{h_{n+1}}(x_2, \cdot)) dm_{w}^2(x_1, x_2)
\]
4. Remark 3.8. From equation (2.4), we obtain equation (3.10) as a special case of (3.7).

Remark 3.8. Using seminal results by Bearman [1], Cameron and Storvick [4] established a rotation property of the Wiener measure \( m_w \). The result is summarized as follows: for a Wiener-integrable functional \( F \) and every nonzero real \( a \) and \( b \),

\[
\int_{C_0^2[0,T]} F(ax + by) d(m_w \times m_w)(x, y) = \int_{C_0^2[0,T]} F(\sqrt{a^2 + b^2} z) dm_w(z),
\]

where by \( \equiv \) we mean that if either side exists, both sides exist and equality holds.

Let \( h_1 \equiv a \) and \( h_2 \equiv b \) be constant functions on \([0,T] \) in (3.7) above. Then, it follows that

\[
\int_{C_0^2[0,T]} F(ax + by) dm_w^2(x, y) \equiv \int_{C_0^2[0,T]} F(Z_a(x, \cdot) + Z_b(y, \cdot)) dm_w^2(x, y)
\]

\[
= \int_{C_0^2[0,T]} F(Z_{s(a,b)}(z, \cdot)) dm_w(z).
\]

Furthermore, we can choose \( s(a, b) \) as a constant function. In this case, one can see that either \( s(a, b) = \sqrt{a^2 + b^2} \) or \( s(a, b) = -\sqrt{a^2 + b^2} \). Thus, in view of equation (2.4), we obtain equation (3.10) as a special case of (3.7).

Remark 3.9. For any finite sequence \( \mathcal{H} = (h_1, \ldots, h_n) \) of nonzero functions in \( L_2[0,T] \), let \( s_1 \) and \( s_2 \) be the functions satisfying equation (3.3). Then, in view of equation (2.5), it follows that for a functional \( F \) on \( C_0[0,T] \),

\[
T_{q,s_1}^{(2)}(F) \approx T_{q,s_2}^{(2)}(F).
\]

From this we see that \( T_{q,s_1}^{(2)}(F) \) is consistent with the representation of the function \( s(\mathcal{H}) \).

Remark 3.9. As mentioned in above, if the functions \( h_1 \) and \( h_2 \) are in \( L_\infty[0,T] \), then we can take \( s(h_1, h_2) \) to be in \( L_\infty[0,T] \). Thus equations (3.7) through (3.9) hold for the Gaussian processes whose kernel is in \( L_\infty[0,T] \).

4. \( L_2 \) analytic \( Z_h \)-Fourier–Feynman transform. In this section, we analyze the \( L_2 \) analytic \( Z_h \)-FFT of cylinder functionals. A functional \( F \) on \( C_0[0,T] \) is called a cylinder functional if there exists a linearly independent subset \( \{v_1, \ldots, v_m\} \) of functions in \( L_2[0,T] \) such that

\[
F(x) = \psi(\langle v_1, x \rangle, \ldots, \langle v_m, x \rangle), \quad x \in C_0[0,T],
\]

where \( \psi \) is a complex-valued Lebesgue measurable function on \( \mathbb{R}^m \).

It is easy to show that for the given cylinder functional \( F \) of the form (4.1), there exists an orthogonal subset \( \mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \) of \( L_2[0,T] \) such that \( F \) can be expressed as

\[
F(x) = f(\langle \alpha_1, x \rangle, \ldots, \langle \alpha_n, x \rangle), \quad x \in C_0[0,T],
\]

where \( f \) is a complex-valued Lebesgue measurable function on \( \mathbb{R}^n \). Thus, we loose no generality in assuming that every cylinder functional on \( C_0[0,T] \) is of the form (4.2).
For $h \in L_2[0, T]$ with $\|h\|_2 > 0$, let $Z_h$ be the Gaussian process given by (2.1) above and let $F$ be given by equation (4.2). Then by equation (2.3),

$$F(Z_h(x, \cdot)) = f((\alpha_1, Z_h(x, \cdot)), \ldots, (\alpha_n, Z_h(x, \cdot)))$$

$$= f((\alpha_1 h, x), \ldots, (\alpha_n h, x)).$$

Even though the set $A = \{\alpha_1, \ldots, \alpha_n\}$ of functions in $L_2[0, T]$ is orthogonal, the subset $Ah = \{\alpha_1 h, \ldots, \alpha_n h\}$ of $L_2[0, T]$ need not be orthogonal.

Let $A = \{\alpha_1, \ldots, \alpha_n\}$ be an orthogonal set of nonzero functions in $L_2[0, T]$, and let $O_\infty(A)$ denote the class of all nonzero functions $h$ in $L_\infty[0, T]$ such that $Ah$ is orthogonal in $L_2[0, T]$. Since $\dim L_2[0, T] = \infty$, infinitely many functions, $h$, exist in $O_\infty(A)$.

**Example 4.1.** For any orthogonal set $A = \{\alpha_1, \ldots, \alpha_n\}$ of nonzero functions in $L_2[0, T]$, each of whose element is of bounded variation on $[0, T]$, let $L(S)$ be the subspace of $L_2[0, T]$ which is spanned by $S = \{\alpha_i \alpha_j : 1 \leq i < j \leq n\}$, and let $L(S)\perp$ be the orthogonal complement of $L(S)$. Let

$$\mathcal{P}_\infty(A) := \{h \in L_\infty[0, T] : h^2 \in L(S)\perp \text{ and } \|h\|_2 > 0\}.$$ 

Since $\dim L(S)$ is finite, and $L_\infty[0, T]$ is dense in $L_2[0, T]$, $\dim(L(S)\perp \cap L_\infty[0, T]) = \infty$ and so $\mathcal{P}_\infty(A)$ has infinitely many elements.

Let $h$ be an element of $\mathcal{P}_\infty(A)$. It is easy to show that $\|\alpha_j h\|_2 > 0$ for all $j \in \{1, \ldots, n\}$. From the definition of the $\mathcal{P}_\infty(A)$, we see that for $i, j \in \{1, \ldots, n\}$ with $i \neq j$,

$$(\alpha_i h, \alpha_j h)_2 = \int_0^T \alpha_i(t) \alpha_j(t) h^2(t) dt = 0.$$ 

From these, we see that $Ah$ is an orthogonal set in $L_2[0, T]$ for any $h$ in $\mathcal{P}_\infty(A)$, i.e., $\mathcal{P}_\infty(A) \subset O_\infty(A)$. Other examples can be found in [7].

Given $h_1$ and $h_2$ in $O_\infty(A)$, let $s(h_1, h_2)$ be an element of $L_\infty[0, T]$ which satisfies equation (3.1) above. Then we observe that for all $j, l \in \{1, \ldots, n\}$ with $j \neq l$,

$$(\alpha_j s(h_1, h_2), \alpha_l s(h_1, h_2))_2 = \int_0^T \alpha_j(t) \alpha_l(t) s^2(h_1, h_2)(t) dt$$

$$= \int_0^T \alpha_j(t) \alpha_l(t) (h_1^2(t) + h_2^2(t)) dt$$

$$= \int_0^T \alpha_j(t) \alpha_l(t) h_1^2(t) dt + \int_0^T \alpha_j(t) \alpha_l(t) h_2^2(t) dt$$

$$= (\alpha_j h_1, \alpha_l h_1)_2 + (\alpha_j h_2, \alpha_l h_2)_2 = 0$$

and that for each $j \in \{1, \ldots, n\}$,

$$\|\alpha_j s(h_1, h_2)\|_2^2 = \int_0^T \alpha_j^2(t) h_1^2(t) dt + \int_0^T \alpha_j^2(t) h_2^2(t) dt = \|\alpha_j h_1\|_2^2 + \|\alpha_j h_2\|_2^2.$$ 

Hence, from (4.3) and (4.4), we see that $As(h_1, h_2) = \{\alpha_1 s(h_1, h_2), \ldots, \alpha_n s(h_1, h_2)\}$ is an orthogonal set of functions in $L_2[0, T]$ and that the PWZ stochastic integrals

$$\langle \alpha_j, Z_s(h_1, h_2)(x, \cdot) \rangle = \langle \alpha_j s(h_1, h_2), x \rangle, \ j \in \{1, \ldots, n\}$$

form a set of independent Gaussian random variables on $C_0[0, T]$.

Given an orthogonal set $A$ of nonzero functions in $L_2[0, T]$, let $\mathcal{A}(g)$ be the linear space of all functionals $F : C_0[0, T] \to \mathbb{C}$ of the form (4.2) for s-a.e. $x \in C_0[0, T]$.  

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where \( f : \mathbb{R}^n \to \mathbb{C} \) is in the Hilbert space \( L_2(\mathbb{R}^n) \). Note that \( F \in \mathfrak{A}^{(2)} \) implies that \( F \) is scale-invariant measurable.

**Remark 4.2.** In this paper the Hilbert space \( L_2(\mathbb{R}^n) \) has the complex scalar field \( \mathbb{C} \). Thus the inner product \( \langle \cdot, \cdot \rangle_{L_2(\mathbb{R}^n)} \) on \( L_2(\mathbb{R}^n) \) is given by

\[
(f_1, f_2)_{L_2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f_1(\vec{u})\overline{f_2(\vec{u})}d\vec{u}
\]

for \( f_1, f_2 \in L_2(\mathbb{R}^n) \).

We define a sesquilinear form \( \langle \langle \cdot, \cdot \rangle \rangle_{\mathfrak{A}^{(2)}} \) on the linear space \( \mathfrak{A}^{(2)} \) as follows: for \( F_1 \) and \( F_2 \) in \( \mathfrak{A}^{(2)} \), let

\[
\langle \langle F_1, F_2 \rangle \rangle_{\mathfrak{A}^{(2)}} := (f_1, f_2)_{L_2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f_1(\vec{u})\overline{f_2(\vec{u})}d\vec{u},
\]

where \( f_j, j \in \{1, 2\} \), is the corresponding function of \( F_j \) by equation (4.2). Then one can see that the sesquilinear form (4.5) is well-defined and it is an inner product on the linear space \( \mathfrak{A}^{(2)} \). Also, one can see that the correspondence (4.2), \( F \mapsto f \), between \( \mathfrak{A}^{(2)} \) and \( L_2(\mathbb{R}^n) \) is one-to-one and onto. Thus \( (\mathfrak{A}^{(2)}, \| \cdot \|_{\mathfrak{A}^{(2)}}) \) forms a complex Hilbert space, where \( \| \cdot \|_{\mathfrak{A}^{(2)}} := \sqrt{\langle \langle \cdot, \cdot \rangle \rangle_{\mathfrak{A}^{(2)}}} \).

Note that \( (\mathfrak{A}^{(2)}, \| \cdot \|_{\mathfrak{A}^{(2)}}) \) is a rich class of functionals on \( C_0[0,T] \), because it contains many unbounded functionals. It is well known that the Schwartz class \( \mathcal{S}(\mathbb{R}^n) \) is a dense subspace of \( L_2(\mathbb{R}^n) \). The functionals \( F \in \mathfrak{A}^{(2)} \) whose corresponding function \( f \) by equation (4.2) is of class \( \mathcal{S}(\mathbb{R}^n) \) are of interest in Feynman integration theory and quantum mechanics.

Throughout this and next sections, for convenience, we use the following notation: given an orthogonal set \( A = \{\alpha_1, \ldots, \alpha_n\} \) of nonzero functions in \( L_2[0,T], f \in L_2(\mathbb{R}^n), q \in \mathbb{R} \setminus \{0\} \) and \( h \in \mathcal{O}_\infty(A) \), let

\[
\psi^q_{f,Ah}(\vec{r}) \equiv \psi^q_{f,Ah}(r_1, \ldots, r_n) \equiv \left( \prod_{j=1}^n \frac{-iq}{2\pi\|\alpha_j h\|_2^2} \right)^{1/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp \left\{ \frac{iq}{2} \sum_{j=1}^n \frac{(u_j - r_j)^2}{\|\alpha_j h\|_2^2} \right\} d\vec{u}.
\]

In [10], Huffman, Park, and Skoug established the existence of the \( L_2 \) analytic \( \mathcal{Z}_1 \)-FFT for cylinder functionals having the form (4.2). The following theorem is a restatement of Theorem 2.2 in [10].

**Theorem 4.3.** Let \( F \in \mathfrak{A}^{(2)} \) be given by equation (4.2) and let \( h \) be an element of \( \mathcal{O}_\infty(A) \). Then

(i) for all nonzero real \( q \), the \( L_2 \) analytic \( \mathcal{Z}_h \)-FFT of \( F \), \( T_{q,h}^{(2)}(F) \), exists, belongs to \( \mathfrak{A}^{(2)} \) and is given by the formula

\[
T_{q,h}^{(2)}(F)(y) = \psi^q_{f,Ah}(\langle \alpha_1 h, x \rangle, \ldots, \langle \alpha_n h, x \rangle)
\]

for s-a.e. \( y \in C_0[0,T] \), where \( \psi^q_{f,Ah} \) is given by (4.6); and

(ii) for all nonzero real \( q \), \( T_{-q,h}^{(2)}(T_{q,h}^{(2)}(F)) \approx F \). In other words, the \( L_2 \) analytic \( \mathcal{Z}_h \)-FFT, \( T_{q,h}^{(2)} \), has the inverse transform

\[
\{T_{q,h}^{(2)}\}^{-1} = T_{-q,h}^{(2)}.
\]
Theorem 4.4. Let \( F \in \mathfrak{A}(2) \) be given by equation (4.2), and let \( h_1 \) and \( h_2 \) be elements of \( \mathcal{O}_\infty(\mathcal{A}) \). Then for all nonzero real \( q \),
\[
T_{q,h_2}^{(2)}(T_{q,h_1}^{(2)}(F)) = T_{q,s(h_1,h_2)}^{(2)}(F)(y)
\] (4.7)
for s-a.e. \( y \in C_0[0,T] \).

Proof. In view of Theorem 4.3, the two FFTs in equation (4.7) exist. Thus equality is what needs to be shown. Hence, to establish equation (4.7), it will suffice to show that for each \( \lambda > 0 \),
\[
T_{\lambda,h_2}(T_{\lambda,h_1}(F))(y) = T_{\lambda,s(h_1,h_2)}(F)(y)
\]
for s-a.e. \( y \in C_0[0,T] \). But, using equation (3.7), for each \( \lambda > 0 \) and s-a.e. \( y \in C_0[0,T] \), we obtain
\[
T_{\lambda,h_2}(T_{\lambda,h_1}(F))(y)
\]
\[
= \int_{C_0[0,T]} F(y + \lambda^{-1/2}Z_{h_1}(x_1,\cdot) + \lambda^{-1/2}Z_{h_2}(x_2,\cdot))dm_w(x_1,x_2)
\]
\[
= \int_{C_0[0,T]} F(y + Z_{h_1/\sqrt{\lambda}}(x_1,\cdot) + Z_{h_2/\sqrt{\lambda}}(x_2,\cdot))dm_w(x_1,x_2)
\]
\[
= \int_{C_0[0,T]} F(y + Z_{s(h_1,h_2)/\sqrt{\lambda}}(x,\cdot))dm_w(x)
\]
\[
= \int_{C_0[0,T]} F(y + \lambda^{-1/2}Z_{s(h_1,h_2)}(x,\cdot))dm_w(x)
\]
\[
= T_{\lambda,s(h_1,h_2)}(F)(y).
\]
Thus, by the definition of the FFT, we obtain the desired result. \( \square \)

For a detailed proof of the theorem, we refer to the reference [5]. Using mathematical induction and equation (3.9), we obtain the following corollary.

Corollary 4.5. Let \( F \in \mathfrak{A}(2) \) be given by equation (4.2), and let \( \mathcal{H} = (h_1,\ldots,h_n) \) be a finite sequence in \( \mathcal{O}_\infty(\mathcal{A}) \). Then for all nonzero real \( q \),
\[
T_{q,h_n}(\cdots(T_{q,h_1}(F))\cdots)(y) = T_{q,s(\mathcal{H})}(F)(y)
\] (4.8)
for s-a.e. \( y \in C_0[0,T] \).

Applying equation (3.8), we also obtain the following corollary.

Corollary 4.6. Let \( F \in \mathfrak{A}(2) \) be given by equation (4.2), and let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be finite sequences in \( \mathcal{O}_\infty(\mathcal{A}) \). Then for all nonzero real \( q \),
\[
T_{q,s(\mathcal{H}_2)}(T_{q,s(\mathcal{H}_1)}(F))(y) = T_{q,s(\mathcal{H}_1 \cup \mathcal{H}_2)}(F)(y)
\] (4.9)
for s-a.e. \( y \in C_0[0,T] \).

5. Algebraic structures of \( Z_h \)-Fourier–Feynman transforms. Given \( q \in \mathbb{R} \setminus \{0\} \), let
\[
T_{q,\mathcal{O}_\infty(\mathcal{A})} \equiv T_{q,\mathcal{O}_\infty(\mathcal{A})}[\mathfrak{A}(2)] := \{ T_{q,h}^{(2)} : h \in \mathcal{O}_\infty(\mathcal{A}) \cup \{0\} \}
\]
denote the class of \( L_2 \) analytic \( Z_h \)-FFTs acting on \( \mathfrak{A}(2) \). In the case that \( h \equiv 0 \), i.e., \( \|h\|_2 = 0 \), it follows that \( T_{q,h}^{(2)} \equiv T_{q,0}^{(2)} \) is the identity transform for all \( q \in \mathbb{R} \). For notational convenience, let \( s(h) \equiv h \) for \( h \in L_\infty[0,T] \).
By Theorems 4.3 and 4.4, we see that for all \( h_1, h_2 \in \mathcal{O}_\infty(\mathcal{A}) \cup \{0\} \) and all \( F \in \mathfrak{A}^{(2)} \),
\[
(T_{q,h_2}^{(2)} \circ T_{q,h_1}^{(2)})(F) \equiv (T_{q,h_2}^{(2)} (T_{q,h_1}^{(2)}(F))) = T_{q,T_{q,h_1}(h_2)}^{(2)}(F)
\]
is in \( \mathfrak{A}^{(2)} \). Because
\[
s(s(h_3, h_2), h_1) = s(h_3, h_2, h_1) = s(h_3, s(h_2, h_1)),
\]
for all \( h_1, h_2, h_3 \in \mathcal{O}_\infty(\mathcal{A}) \cup \{0\} \), we see that the composition \( \circ \) of \( L_2 \) analytic FFTs is associative. Also, because \( s(h_1, h_2) = s(h_2, h_1) \), we see that \( (T_{q,h_2}^{(2)} \circ T_{q,h_1}^{(2)})(F) = (T_{q,h_1}^{(2)} \circ T_{q,h_2}^{(2)})(F) \), and clearly \( (T_{q,0}^{(2)} \circ T_{q,h}^{(2)})(F) = T_{q,h}^{(2)}(F) \) for any \( h_1, h_2, h \in \mathcal{O}_\infty(\mathcal{A}) \cup \{0\} \) and every \( F \in \mathfrak{A}^{(2)} \). Thus, we have the following assertion.

**Theorem 5.1.** The space \( (T_{q,\mathcal{O}_\infty(\mathcal{A})}, \circ) \) is a commutative monoid. Furthermore, the monoid \( T_{q,\mathcal{O}_\infty(\mathcal{A})} \) acts on the space \( \mathfrak{A}^{(2)} \) in the sense that \( (T_{q,h}^{(2)}, F) \mapsto T_{q,h}^{(2)}(F) \).

Next let \( S_l(\mathcal{O}_\infty(\mathcal{A})) \) be the set of all finite sequences of functions in \( \mathcal{O}_\infty(\mathcal{A}) \cup \{0\} \), and let
\[
T_{q,S_l(\mathcal{O}_\infty(\mathcal{A}))}^{(2)} \equiv T_{q,S_l(\mathcal{O}_\infty(\mathcal{A}))}[\mathfrak{A}^{(2)}] := \{ T_{q,s(H)}^{(2)} : H \in S_l(\mathcal{O}_\infty(\mathcal{A})) \}.
\]
From the fact that for any \( H \in S_l(\mathcal{O}_\infty(\mathcal{A})) \), \( s(H) \in \mathcal{O}_\infty(\mathcal{A}) \cup \{0\} \subset S_l(\mathcal{O}_\infty(\mathcal{A})) \), we know that the classes \( T_{q,\mathcal{O}_\infty(\mathcal{A})} \) and \( T_{q,S_l(\mathcal{O}_\infty(\mathcal{A}))} \) coincide as sets. However, we will consider another operation, \( \sim \), on \( T_{q,S_l(\mathcal{O}_\infty(\mathcal{A}))} \), defined as follows: for \( T_{q,s(H_1)}^{(2)} \) and \( T_{q,s(H_2)}^{(2)} \) in \( T_{q,S_l(\mathcal{O}_\infty(\mathcal{A}))} \), let
\[
T_{q,s(H_1)}^{(2)} \sim T_{q,s(H_2)}^{(2)} := T_{q,s(H_1 \wedge H_2)}^{(2)}
\]
where \( H_1 \wedge H_2 \) is defined by (3.5) above. By equation (3.6), one can see that the operation \( \sim \) is well defined.

**Remark 5.2.** By equation (4.9), we observe
\[
T_{q,s(H_1)}^{(2)} \sim T_{q,s(H_2)}^{(2)} = T_{q,s(H_1 \wedge H_2)}^{(2)} = T_{q,s(H_1)}^{(2)} \circ T_{q,s(H_2)}^{(2)}.
\]

**Theorem 5.3.** The space \( (T_{q,S_l(\mathcal{O}_\infty(\mathcal{A}))}, \sim) \) is a commutative monoid. Furthermore, the monoid \( T_{q,S_l(\mathcal{O}_\infty(\mathcal{A}))} \) acts on the space \( \mathfrak{A}^{(2)} \) in the sense that \( (T_{q,s,H_1}^{(2)}, F) \mapsto T_{q,s,H_1}^{(2)}(F) \).

**Remark 5.4.** The operation \( \sim \) is a semigroup action of \( T_{q,S_l(\mathcal{O}_\infty(\mathcal{A}))} \) on \( \mathfrak{A}^{(2)} \).

The sequence space \( S_l(\mathcal{O}_\infty(\mathcal{A})) \) is a monoid under the operation \( \wedge \) given by (3.5). Define an equivalence relation \( \sim \) on \( S_l(\mathcal{O}_\infty(\mathcal{A})) \) as follows: for \( H_1 \) and \( H_2 \) in \( S_l(\mathcal{O}_\infty(\mathcal{A})) \),
\[
H_1 \sim H_2 \iff s(H_1) = s(H_2).
\]
Also, let
\[
S_l^{\sim} \equiv S_l(\mathcal{O}_\infty(\mathcal{A})) / \sim := \{ [H]_{s} : H \in S_l(\mathcal{O}_\infty(\mathcal{A})) \}
\]
be the quotient set of \( S_l(\mathcal{O}_\infty(\mathcal{A})) \) by \( \sim \). Then from (3.3) and (3.4), we see that \( S_l^{\sim} \) is the quotient monoid under the operation \( \wedge \) on \( S_l^{\sim} \), given by
\[
[H_1]_{s} \wedge [H_2]_{s} := [H_1 \wedge H_2]_{s}.
\]
Define a relation on \( T_{q,S_l(\mathcal{O}_\infty(\mathcal{A}))} \) as follows: for \( T_{q,s(H_1)}^{(2)} \) and \( T_{q,s(H_2)}^{(2)} \) in \( T_{q,S_l(\mathcal{O}_\infty(\mathcal{A}))} \),
\[
T_{q,s(H_1)}^{(2)} \overset{1}{\sim} T_{q,s(H_2)}^{(2)} \iff H_1 \sim H_2.
\]
From (4.8) and (3.4), we see that for every \((h_1, \ldots, h_n) \in S_l(O_{\infty}(A))\) and any permutation \(\pi\) of \(\{1, \ldots, n\}\),
\[
T_{q,\pi(h_1, \ldots, h_n)}^{(2)}(F) = T_{q,\pi(h_{\pi(1)}, \ldots, h_{\pi(n)})}^{(2)}(F)
\]
for all \(F\) in \(\mathfrak{A}^{(2)}\). Thus, the relation \(\sim\) is a well-defined equivalence relation, and so we can obtain the quotient monoid
\[
T_{q, S_l(O_{\infty}(A))} \equiv T_{q, S_l(O_{\infty}(A))}/\sim := \{ [T_{q,\pi}(H)]_s : T_{q,\pi}(H) \in T_{q, S_l(O_{\infty}(A))} \}
\]
with the operation \(\bar{\wedge}\) given by
\[
[T_{q,\pi}(H)]_s \bar{\wedge} [T_{q,\pi}(H')]_s := [T_{q,\pi}(H \wedge H')]_s.
\]

**Theorem 5.5.** The map \(\Xi : (T_{q, S_l(O_{\infty}(A)))}/\sim) \rightarrow (S^\sim, \wedge)\) given by
\[
\Xi([T_{q,\pi}(H)]_s) = [H]_s.
\]
is a monoid isomorphism.

**Proof.** It follows from (5.2) and (5.1) that
\[
\Xi([T_{q,\pi}(H)]_s \bar{\wedge} [T_{q,\pi}(H')]_s) = \Xi([T_{q,\pi}(H \wedge H')]_s) = [H_1 \wedge H_2]_s = [H_1]_s \wedge [H_2]_s
\]
\[
= \Xi([T_{q,\pi}(H)]_s) \bar{\wedge} \Xi([T_{q,\pi}(H')]_s).
\]
Clearly, the map given by equation (5.3) is bijective. \(\Box\)

6. **Free group** \(F(T_{q, S_l(O_{\infty}(A))})\). In this section, we describe a transformation group that is the free group generated by \(T_{q, S_l(O_{\infty}(A))}\).

Given an orthogonal set \(\mathcal{A}\) of functions in \(L_2[0, T]\), let \(O^n(\mathcal{A})\) be the class of all nonzero functions \(h\) in \(L_\infty[0, T]\), such that \(Ah\) is orthonormal in \(L_2[0, T]\). For a detailed example for the class \(O^n(\mathcal{A})\), see [7, Example 2.2].

The following lemma is due to Cameron and Storvick in [3, Lemma H].

**Lemma 6.1.** For \(f \in L_2(\mathbb{R}^n)\) and \(h \in O^n(\mathcal{A})\), let \(\psi^q_{f,Ah}(\vec{r})\) be given by equation (4.6). Then \(\psi^q_{f,Ah} \in L_2(\mathbb{R}^n)\). The integral in the right side of (4.6) is to be interpreted as an \(L_2\)-limiting integral in the sense that

\[
\lim_{\delta \to \infty} \int_{\mathbb{R}^n} \left| \frac{-iq}{2\pi} \right|^{n/2} \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} f(\vec{u}) \exp \left\{ \frac{iql}{2} \sum_{j=1}^{n} (u_j - r_j)^2 \right\} d\vec{u} - \psi^q_{f,Ah}(\vec{r}) \right| d\vec{r} = 0.
\]

In this case, we have \(\|\psi^q_{f,Ah}\|_2 = \|f\|_2\).

In view of Theorem 4.3 and Lemma 6.1, we obtain the following theorem.

**Theorem 6.2.** Let \(q \in \mathbb{R} \setminus \{0\}\) and let \(h \in O^n(\mathcal{A})\). Then \(L_2\)-analytic \(Z_{h,\text{FFT}}\),
\[
T_{q,h}^{(2)} : \mathfrak{A}^{(2)} \to \mathfrak{A}^{(2)}
\]
is a linear operator isomorphism. That is, \(\|F\|_{\mathfrak{A}^{(2)}} = \|T_{q,h}^{(2)}(F)\|_{\mathfrak{A}^{(2)}}\) for all \(F \in \mathfrak{A}^{(2)}\).

Thus we have \(\|T_{q,h}^{(2)}\|_o = 1\), where \(\| \cdot \|_o\) denotes the operator norm.

For any nonzero real \(q\), let \(T_{q,S_l(O_{\infty}(A))}^\sim := T_{q,S_l(O_{\infty}(A))} \setminus \{[T_{q,\pi}^{(2)}]\}_s\). Given \(q \in \mathbb{R} \setminus \{0\}\), define a map
\[
\mathcal{W} : T_{q,S_l(O_{\infty}(A))}^\sim \longrightarrow T_{q,S_l(O_{\infty}(A))}^\sim
\]
by $\mathcal{W}([T_{q,s}^{(2)}_{q,s(\mathcal{H}_1)}]) = [T_{q,s}^{(2)}_{q,s(\mathcal{H}_2)}])$. Then, $\mathcal{W}$ is one-to-one correspondence. Thus, by the usual argument in the free group theory, one can obtain the group $F(T_{q,s}^{*}(O_{\infty}^{*}(A)))$ freely generated by $T_{q,s}^{*}(O_{\infty}^{*}(A))$.

Note that

$$[T_{q,s}^{(2)}_{q,s(\mathcal{H}_1)}]_{t} \sim [T_{q,s}^{(2)}_{q,s(\mathcal{H}_2)}]_{t} = [T_{q,s}^{(2)}_{q,s(\mathcal{H}_1 \land \mathcal{H}_2)}]_{t} = [T_{q,s}^{(2)}_{q,s(\mathcal{H}_1)} \circ T_{q,s}^{(2)}_{q,s(\mathcal{H}_2)}]_{t},$$

by equation (4.9). Given two transforms $T_1$ and $T_2$ in $F(T_{q,s}^{*}(O_{\infty}^{*}(A)))$, let the group operation between $T_1$ and $T_2$ be given by

$$(T_1 \circ T_2)(F) \equiv T_1(T_2(F)), \quad F \in \mathfrak{A}^{(2)}.$$

For an element $T$ of $F(T_{q,s}^{*}(O_{\infty}^{*}(A)))$, let $l_w(T)$ denote the length of the word $T$. Given $T \in F(T_{q,s}^{*}(O_{\infty}^{*}(A)))$, assume that $T$ is not the empty word (i.e., it is not the identity transform $[T_{q,0}^{(2)}]$). If $l_w(T) = 1$, then $T$ is an element of the set

$$T_{q,s}^{*}(O_{\infty}^{*}(A)) \subseteq T_{q,s}^{*}(O_{\infty}^{*}(A)).$$

Alternatively, if $l_w(T) > 1$, then $T$ cannot be expressed as (an equivalence class of) a single FFT by the concept of the reduced word in the free group theory. But, in view of the assertion (ii) of Theorem 4.3 and Lemma 6.1, we see that for any $T \in F(T_{q,s}^{*}(O_{\infty}^{*}(A)))$, $T$ is a linear operator isomorphism from $\mathfrak{A}^{(2)}$ into $\mathfrak{A}^{(2)}$.

On the other hand, we consider other algebraic structure of transforms as follows: given $h \in L_{\infty}[0,T]$, let $T_{0,h}^{(2)}$ denote the identity transform, i.e., $T_{0,h}^{(2)}(F) = F$, and let

$$T_{R,h} = \{T_{q,h}^{(2)} : q \in \mathbb{R}\}.$$

Now, by using [12, equation (2.14)] (clearly, equations (2.10) and (2.14) in [12] hold for $L_2$ analytic FFTs $T_{q,h}^{(2)}$), we obtain that for all $q_1, q_2 \in \mathbb{R}$ with $q_1 + q_2 \neq 0$ and all $F \in \mathfrak{A}^{(2)}$,

$$(T_{q_2,h}^{(2)} \circ T_{q_1,h}^{(2)})(F) \equiv T_{q_2,h}^{(2)}(T_{q_1,h}^{(2)}(F)) \approx T_{q_2,h}^{(2)} \circ T_{q_1,h}^{(2)}(F),$$

and that

$$(T_{q_2,h}^{(2)} \circ T_{q_1,h}^{(2)})(F) \approx (T_{q_1,h}^{(2)} \circ T_{q_2,h}^{(2)})(F).$$

If $q_1 + q_2 = 0$, then by Theorem 4.3, we obtain $T_{q_2,h}^{(2)} = T_{-q_2,h}^{(2)} = \{T_{q_1,h}^{(2)}\}^{-1}$.

**Theorem 6.3.** For each $h \in L_{\infty}[0,T]$, the space $(T_{R,h}, \circ)$ forms a commutative group. Clearly, $T_{R,h}$ with $\|h\|_2 = 0$ is a trivial group.

Given a function $h$ in $L_{\infty}[0,T]$ with $\|h\|_2 > 0$, we consider the free group $F(T_{R,h})$ generated by the transformation group $T_{R,h}$. Then one can easily see that $F(T_{R,h})$ has the free basis $\{T_{q,h}^{(2)} : q > 0\}$. Because

$$|\{T_{q,h}^{(2)} : q > 0\}| = \aleph_1 = \aleph_0^{|S|} = |S| \equiv |T_{q,s}^{*}(O_{\infty}^{*}(A))|,$$

the free groups $F(T_{q,s}^{*}(O_{\infty}^{*}(A)))$ and $F(T_{R,h})$ have the same rank. Thus, by the concept of rank of free groups [20, Proposition 2.1.4, p.47], we conclude that

$$F(T_{q,s}^{*}(O_{\infty}^{*}(A))) \cong F(T_{R,h}),$$

where $\cong$ is a group isomorphism.
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