MOTIVIC SERRE INVARIANTS MODULO THE SQUARE OF $L - 1$

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Abstract. Motivic Serre invariants defined by Loeser and Sebag are elements of the Grothendieck ring of varieties modulo $L - 1$. In this paper, we show that we can lift these invariants to modulo the square of $L - 1$ after tensoring the Grothendieck ring with $Q$, under certain assumptions.

1. Introduction

Let $K$ be a complete discrete valuation field with a perfect residue field $k$. For a smooth projective irreducible $K$-variety $X$, Loeser and Sebag [9] defined the motivic Serre invariant $S(X)$. This invariant belongs to the ring $K_0(\text{Var}_k)/(L - 1)$, where $K_0(\text{Var}_k)$ is the Grothendieck ring of $k$-varieties and $L := [A^1_k]$, the class of an affine line in this ring. Let $K_0(\text{Var}_k)_Q := K_0(\text{Var}_k) \otimes Z Q$. In this paper, we construct, under a certain assumption, an invariant

$$\tilde{S}(X) \in K_0(\text{Var}_k)_Q/(L - 1)^2$$

which coincides with $S(X)$ in $K_0(\text{Var}_k)_Q/(L - 1)$.

Remark 1.1. Loeser and Sebag defined the motivic Serre invariant more generally for smooth quasi-compact separated rigid $K$-spaces. For the sake of simplicity, we consider only the case where $X$ is a projective variety.

Let $\mathcal{O}$ be the valuation ring of $K$. The assumption we will make is that the desingularization theorem and the weak factorization theorem hold, their precise statements are as follows:

Assumption 1.2. (1) (Desingularization) There exists a regular projective flat $\mathcal{O}$-scheme $\mathcal{X}$ with the generic fiber $\mathcal{X}_K := \mathcal{X} \otimes_\mathcal{O} K = X$ such that the special fiber $\mathcal{X}_k := \mathcal{X} \otimes_\mathcal{O} k$ is a simple normal crossing divisor in $\mathcal{X}$. (We call such an $\mathcal{X}$ a regular snc model of $X$.)

(2) (Weak factorization) Let $\mathcal{X}$ and $\mathcal{X}'$ be regular snc models of $X$. Then there exist finitely many regular snc models of $X$,

$$\mathcal{X}_0 = \mathcal{X}, \mathcal{X}_1, \ldots, \mathcal{X}_n = \mathcal{X}',$$
such that for every $i$, either the birational map $X_i \dasharrow X_{i+1}$ is the blowup along a regular center $Z \subset X_{i+1,k}$ which has normal crossing\footnote{That $Z$ has normal crossings with $X_{i+1,k}$ means that for every closed point $x \in X_{i+1,k}$, there exist a regular system of parameters $x_1, \ldots, x_d \in O_{X_{i+1,k}}$ such that in an open neighborhood of $x$, the support of the special fiber $X_{i+1,k}$ is the zero locus of $\prod_{v \in V} x_v$ for some subset $V \subset \{1, \ldots, d\}$ and $Z$ is the common zero locus of $x_w$, $w \in W$ for some $W \subset \{1, \ldots, d\}$.} with $X_{i+1,k}$ or its inverse $X_{i+1} \dasharrow X_i$ has the same description with $X_{i+1,k}$ replaced with $X_{i,k}$.

When $X$ has dimension one, this assumption holds as is well-known. Indeed the above desingularization theorem in this case follows from the desingularization theorem for excellent surfaces by Abhyankar, Hironaka and Lipman (see [8]), while the weak factorization follows from the fact that every proper birational morphism of regular integral noetherian schemes of dimension two factors into a sequence of finitely many blowups at closed points. The last fact is well-known in the case of varieties over an algebraically closed field (for instance, [5, V, Cor. 5.4]) and is valid even in our situation as proved in [7, Th. 4.1] in a more general context. Assumption 1.2 holds also when $k$ has characteristic zero. This follows from the recent generalizations to excellent schemes respectively by Temkin [12, 13] and by Abramovich and Temkin [2] of the Hironaka desingularization theorem and the weak factorization theorem of Abramovich, Karu, Matsuki and Włodarczyk [1].

Let $X$ be a regular snc model of $X$, let $X_{\text{sm}}$ be its $\mathcal{O}$-smooth locus and let $X_{\text{sm},k} := X_{\text{sm}} \otimes_{k} k$. Then $X_{\text{sm}}$ is a weak Neron model of $X$ in the sense of [3] and by definition,

$$S(X) = [X_{\text{sm},k}] \in K_0(\text{Var}_k)/(\mathbb{L} - 1).$$

To define our invariant $\tilde{S}(X)$, we also need information on the non-smooth locus of $X$. Let $X_k$ be a divisor and write it as $X_k = \sum_{i \in I} a_i D_i$, where $D_i$ are the irreducible components of $X_k$ and $a_i$ are the multiplicities of $D_i$ in $X$ respectively. For a subset $H \subset I$, we define

$$D_H^0 := \bigcap_{h \in H} D_h \setminus \bigcup_{i \in I \setminus H} D_i.$$

When $H = \{i\}$, we abbreviate it to $D_i^0$, and when $H = \{i, j\}$, to $D_{ij}^0$. These locally closed subsets give the stratification

$$X_k = \bigcup_{\emptyset \neq H \subset I} D_H^0$$

and the stratification

$$X_{\text{sm},k} = \bigcup_{i \in I : a_i = 1} D_i^0.$$

From the second stratification, we see

$$S(X) = \sum_{i \in I : a_i = 1} [D_i^0] \in K_0(\text{Var}_k)/(\mathbb{L} - 1).$$

Loeser and Sebag proved in the paper cited above that this is independent of the model $X$ and depends only on $X$. 
Definition 1.3. For a regular snc model $\mathcal{X}$ of $X$, we define
$$
\tilde{S}(\mathcal{X}) := \sum_{i \in I: a_i = 1} [D_i^\circ] + \sum_{\{i,j\} \subset I: (a_i, a_j) = 1} \frac{1}{a_i a_j} [D_{ij}^\circ](1 - L)
$$
as an element of $K_0(\text{Var}_k)_\mathbb{Q}/(L - 1)^2$. Here $(a, b)$ denotes the greatest common divisor of $a$ and $b$.

Obviously, the two invariants $S(X)$ and $\tilde{S}(\mathcal{X})$ coincide when they are sent to $K_0(\text{Var}_k)_\mathbb{Q}/(L - 1)$ by the natural maps.

The following is our main theorem:

**Theorem 1.4.** Let $X$ be a smooth projective $K$-variety. Under Assumption 1.2, the invariant $\tilde{S}(\mathcal{X})$ is independent of the chosen regular snc model $\mathcal{X}$ and depends only on $X$.

The theorem allows us to think of $\tilde{S}(\mathcal{X})$ as an invariant of $X$ and denote it by $\tilde{S}(X)$, which is what was mentioned at the beginning of this Introduction.

2. Preparatory reductions

We generalize the invariant $\tilde{S}(\mathcal{X})$ as follows. Let $\mathcal{X}$ be a regular flat $\mathcal{O}$-scheme of finite type such that $\mathcal{X}_K$ is smooth and $\mathcal{X}_k = \bigcup_{i \in I} D_i$ is a simple normal crossing divisor in $\mathcal{X}$. (We no longer suppose that $\mathcal{X}$ or $\mathcal{X}_K$ is projective.) For a constructible subset $C \subset \mathcal{X}_k$, we define
$$
\tilde{S}(\mathcal{X}, C) := \sum_{i \in I: a_i = 1} [D_i^\circ \cap C] + \sum_{\{i,j\} \subset I: (a_i, a_j) = 1} \frac{1}{a_i a_j} [D_{ij}^\circ \cap C](1 - L)
$$
as an element of $K_0(\text{Var}_k)_\mathbb{Q}/(L - 1)^2$.

Let $f: Y \to \mathcal{X}$ be the blowup along a smooth irreducible center $Z \subset \mathcal{X}_k$ which has normal crossings with $\mathcal{X}_k$. Then, $Y$ is an $\mathcal{O}$-scheme satisfying the same conditions as $\mathcal{X}$ does and we can similarly define $\tilde{S}(Y, C')$ for a constructible subset $C' \subset Y_k$.

Theorem 1.4 follows from:

**Proposition 2.1.** Let $\mathcal{X}$ be as above. For any constructible subset $C \subset \mathcal{X}_k$, we have
$$
\tilde{S}(\mathcal{X}, C) = \tilde{S}(Y, f^{-1}(C)).
$$

Indeed, Theorem 1.4 is a direct consequence of this proposition with $C = \mathcal{X}_k$ and Assumption 1.2.

In what follows, we will prove this proposition. First we will reduce it to the local situation by using:

**Lemma 2.2.** (1) If $C$ is the disjoint union $\bigcup_{s=1}^l C_s$ of constructible subsets $C_s$, then
$$
\tilde{S}(\mathcal{X}, C) = \sum_{s=1}^l \tilde{S}(\mathcal{X}, C_s).
$$
Proof. The first assertion is obvious. To show the second one, we first claim that there exists a stratification \( C \) in some \( U \), then construct \( C \). Indeed we can take \( C_1 \) applying the same procedure to \( C \). From the first assertion of the above lemma, since we obviously have \( X \) and that the special fiber \( \{ \} \) we may also assume that \( k \). We denote \( \tilde{S}(\mathcal{X}, C) = \tilde{S}(\mathcal{Y}, f^{-1}(C)) \).

(2) Let \( \mathcal{X} = \bigcup_{\lambda \in \Lambda} U_{\lambda} \) be an open covering. Suppose that for every constructible subset \( C \subset \mathcal{X} \) and for every \( \lambda \in \Lambda \),

\[
\tilde{S}(\mathcal{X}, C \cap U_{\lambda}) = \tilde{S}(\mathcal{Y}, f^{-1}(C \cap U_{\lambda})).
\]

Then, for every constructible subset \( C \subset \mathcal{X} \), we have

\[
\tilde{S}(\mathcal{X}, C) = \tilde{S}(\mathcal{Y}, f^{-1}(C)).
\]

\[\square\]

Let \( x \in \mathcal{X}_k \) be a closed point and take a local coordinate system \( x_1, \ldots, x_d \in \mathcal{O}_{\mathcal{X}_k} \). By shrinking \( \mathcal{X} \) if necessary, we may suppose that \( x_1, \ldots, x_d \) are global sections of \( \mathcal{O}_X \) and that the special fiber \( \mathcal{X}_k \) is the zero locus of \( \prod_{i=1}^{d'} x_i, d' \leq d \) (thus we identify \( I \) with \( \{1, \ldots, d'\} \)) and \( Z \) is the common zero locus of \( x_j, j \in J \) for some subset \( J \subset \{1, \ldots, d\} \). From the first assertion of the above lemma, since we obviously have

\[
\tilde{S}(\mathcal{X}, C \setminus Z) = \tilde{S}(\mathcal{Y}, f^{-1}(C \setminus Z)),
\]

we may also assume that

\[C \subset Z.\]

(2.1)

In a few following sections, we will prove Proposition \( \ref{2.1} \) in this situation, discussing separately in the cases \( (\sharp I =) d' = 1, d' = 2 \) and \( d' \geq 3 \). Before that, we prepare some notation and a lemma.

Notation 2.3. For \( i \in I \), let \( D_i \) be the prime divisor of \( \mathcal{X} \) given by \( x_i = 0 \) and let \( E_i \subset \mathcal{Y}_k \) be its strict transform. Let \( E_0 \subset \mathcal{Y}_k \) be the exceptional divisor of the blowup \( f: \mathcal{Y} \to \mathcal{X} \). We denote \( f^{-1}(C) \) by \( \tilde{C} \).

The multiplicity of \( E_i \) in \( \mathcal{Y}_k \) is \( a_i \) for \( i \in I \) and

\[a_0 := \sum_{Z \subset D_i} a_i \]

for \( i = 0 \). We will use the following lemma several times.

Lemma 2.4. For \( i \in I \setminus J \), if \( C \subset Z \cap D_i \), then we have \( \tilde{C} \subset E_i \).

Proof. The morphism \( \tilde{C} \to C \) is a \( \mathbb{P}^{d-1} \)-bundle. The divisor \( E_i \) is the blowup of \( D_i \) along \( Z \cap D_i \), which has codimension \( \sharp J \) in \( D_i \). It follows that \( E_i \cap \tilde{C} \to C \) is also a \( \mathbb{P}^{d-1} \)-bundle. Hence \( \tilde{C} \) and \( E_i \cap \tilde{C} \) coincide and the lemma follows. \[\square\]
3. The case \( d' = 1 \).

We now begin the proof of Proposition 2.1 in the situation described just before Notation 2.3. In this section, we consider the case \( d' = 1 \).

Since \( Z \subset X \), recalling \( I = \{1, \ldots, d'\} \), we see that \( 1 \in J \). Then

\[
\tilde{S}(\mathcal{X}, C) = \begin{cases} 
[C] & (a_1 = 1) \\
0 & \text{(otherwise)}
\end{cases}.
\]

From (2.2), \( a_0 = a_1 \), and \((a_0, a_1) = a_1 \). Hence, if \( a_1 \neq 1 \), then

\[
\tilde{S}(\mathcal{Y}, \tilde{C}) = 0 = \tilde{S}(\mathcal{X}, C).
\]

If \( a_1 = 1 \), then recalling that \( C \subset Z \), we see that \( \tilde{C} \subset E_0 = f^{-1}(Z) \) and that

\[
\tilde{S}(\mathcal{Y}, \tilde{C}) = [\tilde{C} \setminus E_1] + [E_1 \cap \tilde{C}](1 - L).
\]

To compute the right hand side of this equality, we first observe that \( \tilde{C} \) is a \( \mathbb{P}^{d-1} \)-bundle over \( C \). The divisor \( E_1 \) is the blowup of \( D_1 \) along \( Z \). Therefore \( E_1 \cap \tilde{C} \) is a \( \mathbb{P}^{d-2} \)-bundle over \( C \). Hence

\[
\tilde{S}(\mathcal{Y}, \tilde{C}) = [\tilde{C} \setminus E_1] + [E_1 \cap \tilde{C}](1 - L).
\]

We conclude that if \( d' = 1 \), then \( \tilde{S}(\mathcal{X}, C) = \tilde{S}(\mathcal{Y}, \tilde{C}) \).

4. The case \( d' = 2 \).

Next we consider the case \( d' = 2 \). We have

\[
C = (C \cap D_1^2) \sqcup (C \cap D_2^2) \sqcup (C \cap D_{12}^2).
\]

From the case \( I = 1 \) treated in the last section, we have

\[
\tilde{S}(\mathcal{X}, C \cap D_i^\circ) = \tilde{S}(\mathcal{Y}, f^{-1}(C \cap D_i^\circ)) \quad (i = 1, 2).
\]

Therefore, from Lemma 2.2 replacing \( C \) with \( C \cap D_{12}^\circ \), we may suppose that

\[
(4.1) \quad C \subset D_{12}^\circ = D_1 \cap D_2.
\]

Then we have

\[
\tilde{S}(\mathcal{X}, C) = \begin{cases} 
\frac{1}{a_1 a_2}[C](1 - L) & ((a_1, a_2) = 1) \\
0 & \text{(otherwise)}
\end{cases}.
\]

We next compute \( \tilde{S}(\mathcal{Y}, \tilde{C}) \) separately in the case \( Z \subset D_1 \cap D_2 \) and in the case \( Z \not\subset D_1 \cap D_2 \).
In the former case, we have \( a_0 = a_1 + a_2 \neq 1 \) and
\[
\tilde{S}(\mathcal{Y}, \tilde{C}) = \sum_{i \in \{1,2\}, (a_0, a_i) = 1} \frac{1}{a_0 a_i} [\tilde{C} \cap E_{0i}] (1 - \mathbb{L}).
\]

If \((a_1, a_2) \neq 1\), then \((a_0, a_1) \neq 1\) and \((a_0, a_2) \neq 1\), which show \( \tilde{S}(\mathcal{Y}, \tilde{C}) = 0 = \tilde{S}(\mathcal{X}, C) \).

If \((a_1, a_2) = 1\), then we have \((a_0, a_1) = (a_0, a_2) = 1\), and
\[
\tilde{S}(\mathcal{Y}, \tilde{C}) = \sum_{i = 1}^{2} \frac{1}{a_0 a_i} [\tilde{C} \cap E_{0i}] (1 - \mathbb{L}).
\]

Since \( E_1 \cap \tilde{C} = E_0 \cap E_1 \cap \tilde{C} \rightarrow C \) is a trivial \( \mathbb{P}^{d-2}_J \)-bundle and \( E_1 \cap E_2 \cap \tilde{C} \rightarrow C \) is a hyperplane in it, \( E_0 \cap \tilde{C} \rightarrow C \) is a trivial \( \mathbb{A}^{d-2}_J \)-bundle. (Note that if \( J = 2 \), then \( E_1 \cap E_2 = \emptyset \) and \( E_1 \cap \tilde{C} = E_0 \cap \tilde{C} \rightarrow C \) is an isomorphism and still a trivial \( \mathbb{A}^{d-2}_J \)-bundle.) Similarly for \( E_0 \cap \tilde{C} \rightarrow C \). Hence
\[
\tilde{S}(\mathcal{Y}, \tilde{C}) = \left( \frac{1}{(a_1 + a_2)a_1} + \frac{1}{(a_1 + a_2)a_2} \right) [C] \mathbb{L}^{d-2}_J (1 - \mathbb{L})
\]
\[
= \frac{1}{a_1 a_2} [C] \mathbb{L}^{d-2}_J (1 - \mathbb{L})
\]
\[
= \frac{\star}{a_1 a_2} [C] (1 - \mathbb{L})
\]
\[
= \tilde{S}(\mathcal{X}, C).
\]

Here the equality marked with \( \star \) follows from
\[
\mathbb{L}(1 - \mathbb{L}) = (\mathbb{L} - 1)(1 - \mathbb{L}) + 1 - \mathbb{L} = 1 - \mathbb{L} \mod (\mathbb{L} - 1)^2.
\]

In the case \( Z \not\subset D_1 \cap D_2 \), we have either \( Z \subset D_1 \) or \( Z \subset D_2 \). Since the two cases are similar, we only discuss the former case. Since \( 2 \in I \setminus J \), from assumptions \((2.1)\) and \((4.1)\) and Lemma \((2.4)\) we have \( \tilde{C} \subset E_0 \cap E_2 \). Since \( a_0 = a_1 \), \( \tilde{C} \rightarrow C \) is a \( \mathbb{P}^{d-1}_J \)-bundle and \( \tilde{C} \cap E_1 \rightarrow C \) is a \( \mathbb{P}^{d-2}_J \)-bundle, we have
\[
\tilde{S}(\mathcal{Y}, \tilde{C}) = \frac{1}{a_0 a_2} [\tilde{C} \cap E_{02}] (1 - \mathbb{L})
\]
\[
= \frac{1}{a_1 a_2} [\tilde{C} \setminus E_1] (1 - \mathbb{L})
\]
\[
= \frac{1}{a_1 a_2} [C] [\mathbb{P}^{d-1}_J \setminus \mathbb{P}^{d-2}_J] (1 - \mathbb{L})
\]
\[
= \frac{1}{a_1 a_2} [C] \mathbb{L}^{d-1}_J (1 - \mathbb{L})
\]
\[
= \frac{1}{a_1 a_2} [C] (1 - \mathbb{L})
\]
\[
= \tilde{S}(\mathcal{X}, C).
\]

We have completed the proof that \( \tilde{S}(\mathcal{Y}, \tilde{C}) = \tilde{S}(\mathcal{X}, C) \), when \( d' = 2 \).
5. The case $d' \geq 3$.

As in the last section, by induction on $\sharp I$, we may suppose that

\[(5.1)\quad C \subset \bigcap_{i \in I} D_i.\]

Then $\tilde{S}(\mathcal{X}, C) = 0$. On the other hand, $\tilde{S}(\mathcal{Y}, \tilde{C})$ is a $\mathbb{Q}$-linear combination of

\[A_i := \left[\tilde{C} \cap E_{0i}^0\right](1 - L), \quad i \in I,\]

and

\[B := \delta_{1,a_0} \left[\tilde{C} \cap E_0^0\right],\]

with $\delta_{1,a_0}$ being the Kronecker delta. Thus it suffices to show that $A_i = 0$, $i \in I$ and that $B = 0$.

We first show that $B = 0$. If $\sharp(I \cap J) \geq 2$, then $a_0 = \sum_{i \in I \cap J} a_i > 1$.

Hence $B = 0$. If $\sharp(I \cap J) < 2$, then $I \setminus J$ is non-empty. Assumptions (2.1) and (5.1) and Lemma 2.4 show that $\tilde{C} \cap E_0^0$ is empty, hence $B = 0$.

Next we show that $A_i = 0$. If $\sharp(I \setminus J) \geq 2$, then from Lemma 2.4 for every $i \in I$, there exists $i' \in I \setminus \{i\}$ such that $\tilde{C} \subseteq E_{i'}$. Hence $\tilde{C} \cap E_{0i}^0 = \emptyset$ and $A_i = 0$.

If $\sharp(I \setminus J) = 1$, then by the same reasoning as above, $A_i = 0$ for $i \in I \cap J$. For $i \in I \setminus J$,

\[\tilde{C} \cap E_{0i}^0 = \mathbb{P}^{d-1}_C \setminus \bigcup_{j \in I \cap J} H_j,\]

where $\mathbb{P}^{d-1}_C$ denotes the trivial $\mathbb{P}^{d-1}$-bundle $\mathbb{P}^{d-1}_C \times C$ over $C$ and $H_j$ are coordinate hyperplanes of $\mathbb{P}^{d-1}_C$. Since $\sharp(I \cap J) \geq 2$,

\[A_i = [C][G^m_{(I \cap J)} - 1] \times A^{d-1}_{(I \cap J)} \cap (1 - L) = -[C][L]^{I_{(I \cap J)}}(L - 1)^{(I \cap J)} = 0 \mod (L - 1)^2.\]

If $\sharp(I \setminus J) = 0$, equivalently if $Z \subset D_i$ for every $i \in I$, then for every $i \in I$,

\[\tilde{C} \cap E_{0i}^0 = \mathbb{P}^{d-2}_C \setminus \bigcup_{j \in I \setminus \{i\}} H_j,\]

where $H_j$ are coordinate hyperplanes of $\mathbb{P}^{d-2}_C$. We have

\[A_i = [C][G^m_{(I - 1)} - 1] \times A^{d-1}_{(I - 1)} \cap (1 - L) = -[C][L]^{I_{(I - 1)}}(L - 1)^{(I - 1)} = 0 \mod (L - 1)^2.\]

We thus have proved that $\tilde{S}(\mathcal{X}, C) = \tilde{S}(\mathcal{Y}, \tilde{C}) = 0$ also when $d' \geq 3$, which completes the proofs of Proposition 2.1 and Theorem 1.4.
6. Closing Comments

It is natural to try to refine \( \tilde{S}(X) \) further by lifting it to \( K_0(\text{Var}_k)_{Q}/(L - 1)^n \) for \( n > 2 \) and by adding extra terms of the form

\[
c[D'_H](1 - L)^{2H - 1}
\]

with \( c \in Q, \ H \subset I, \ H \geq 3 \). However the author did not manage to find such a refinement.

The original invariant considered by Serre \cite{11} and denoted by \( i(X) \) was defined for a \( K \)-analytic manifold when the residue field \( k \) is finite, and lives in \( Z/(\sharp k - 1) \). There seems to be no counterpart of \( \tilde{S}(X) \) in this context, at least in a naïve way, because \( Z \otimes Z Q = Q \) is a field and the ideal generated by \( (\sharp k - 1)^2 \) in it is the entire field.

The author has no convincing explanation of the meaning of fractional coefficients appearing in the definition of \( S(X) \). However, as a possibly related work, we note that also Denef and Loeser \cite{4} previously considered motivic invariants with coefficients in \( Q \).

Nicaise and Sebag \cite[Th. 5.4]{10} gave a nice interpretation of the Euler characteristic representation of \( S(X) \) in terms of cohomology of the generic fiber (see also \cite{6} for another proof). It would be interesting to look for a similar interpretation of representations of \( \tilde{S}(X) \) or \( S(X) \) itself.

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