Universal Prefactor of Activated Conductivity in the Quantum Hall Effect

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Abstract

The prefactor of the activated dissipative conductivity in a plateau range of the quantum Hall effect is studied in the case of a long-range random potential. It is shown that due to long time it takes for an electron to drift along the perimeter of a large percolation cluster, phonons are able to maintain quasi-equilibrium inside the cluster. The saddle points separating such clusters may then be viewed as ballistic point contacts between electron reservoirs with different electrochemical potentials. The network of ballistic conductances is shown to determine the conductivity. The prefactor is universal and equal to $2e^2/h$ at an integer filling factor $\nu$ and to $2e^2/q^2 h$ at $\nu = p/q$.

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The temperature dependence of the activated dissipative conductivity is widely used to study energy gaps in the quantum Hall effect. When the Fermi level lies in the middle between two Landau levels $\sigma_{xx}$ has a form [1–6]

$$\sigma_{xx} = \sigma_0 \exp(-\Delta/T),$$ (1)

provided the temperature $T$ is not too low (we use energy units for $T$). The prefactor $\sigma_0$ has attracted a great deal of interest since it was claimed [2,3] that $\sigma_0$ is equal to $e^2/h$ and does not depend on the Landau level number. This puzzling universality was reported to be valid in the fractional quantum Hall effect (FQHE) as well: at filling $\nu = p/q$ the prefactor $\sigma_0 = (e/q)^2/h$ [2,4]. On the other hand, both the numerical value of the prefactor [6] and the very independence of $\sigma_0$ on $T$ [5] have been questioned.

There have been several interesting attempts to calculate the prefactor universally, for any type of disorder, using the idea that at $T = 0$ the extended state exists at only one energy and phase-breaking processes are responsible for the delocalization of electrons within a narrow band of states near the Landau level center [3,5,7–10]. Nevertheless, it has not been shown yet that the range of temperatures exists where $\sigma_0 = e^2/h$.

We suggest a more specific approach and consider separately the cases of a short and a long range random potential. In our previous work [11] we evaluated explicitly two major contributions to the conductivity in the case of a white noise potential: the contribution of the narrow band of delocalized states which appear near the Landau level center at finite temperatures and the other one, related to the variable range hopping (VRH) between localized states in the tail of the density of states. It was shown that, in agreement with the conjecture made in Ref. [10], the interplay of these contributions is responsible for the inflection point in the dependence of $\ln \sigma_{xx}$ on $1/T$ (Fig. 1a). We demonstrated that approximation of $\sigma_{xx}$ by Eq. (1) with $T$-independent $\sigma_0$ in the vicinity of the inflection point yields $\sigma_0 \approx e^2/h$ being a very slow function of the rate of phase breaking processes. So we concluded that, strictly speaking, no universality can be obtained for the case of a short range potential.

In this paper, we calculate the prefactor $\sigma_0$ in the case of a long range random potential. Such a potential exists in high mobility heterojunctions where the two-dimensional electron gas (2DEG) is separated from randomly situated donors by an undoped layer of width $d$ which is much larger than the magnetic length $\lambda$. The potential harmonics with wavelengths smaller than $d$ do not reach the 2DEG. Therefore, the characteristic length of the potential fluctuations is equal to $d$. Most of experimental results for the prefactor $\sigma_0$ were obtained on the structures with large $d$. Thus the consideration of a long range potential is of special interest. We start from the integer quantum Hall effect but generalization to the FQHE is straightforward.

We show below that in the case of a smooth potential due to the suppression of tunneling, the conductivity may be described by a completely classical theory which yields a range of temperatures growing as a power of the ratio $d/\lambda$, where the prefactor is constant and universal (Fig. 1b). We obtain $\sigma_0 = e^2/h$ for the contribution of one Landau level to the conductivity in this range. The Plank constant appears in this expression only through the density of states of the Landau level.

We start with a demonstration of the universality of $\sigma_0$ for a simple model problem. Instead of a random potential we consider a periodic “chess-board” one
\[ W(x, y) = W \cos(\pi x/d) \cos(\pi y/d). \]  

(2)

It is assumed that the Fermi level \( \mu \) is located in the middle between two Landau levels separated by the gap \( 2\Delta \) and that \( W \ll \Delta \) and \( d \gg \lambda \). We consider the contribution to the conductivity of electrons activated to the upper Landau level where the potential energy oscillates around zero according to Eq. (2). At \( T \ll \Delta \) the concentration of activated electrons is exponentially small so that we can neglect the interaction between them. The inequality \( d \gg \lambda \) allows us to treat the electrons semiclassically. Electron guiding centers drift along equipotential lines \( W(x, y) = E \), all of which except one at \( E = 0 \) are closed loops (Fig. 2a). In principle, an electron can tunnel through a saddle point separating two loops of the same energy \( E \). However, the probability of tunneling falls off with increasing \( |E| \) as \( \exp(-|E|/T_1) \), where \( T_1 \sim W(\lambda/d)^2 \). Below we consider the wide temperature range \( T_1 \ll T \ll W \) in which tunneling can be neglected.

In the absence of inelastic processes \( \sigma_0 = 0 \). To find the conductivity of electrons due to inelastic collisions, we should know their energy distribution in the presence of an external electric field. Electrons with energies \( E < 0 \) within a given well are in equilibrium with each other because they circulate many times around the well and experience many inelastic collisions before leaving the well due to activation to trajectories with \( E > 0 \). Therefore, their distribution is characterized by an electrochemical potential which is constant inside the well. We show below that electrons with energies \( 0 < E \lesssim T \) play a very important role in transport. They occupy “T-strips” surrounding each potential hill (Fig. 2a). The distribution of these electrons depends on the parameter \( v\tau/d \), where \( v \) is the typical drift velocity and \( \tau \) is the time necessary to change the energy by \( T \). Our crucial assumption is that \( v\tau \ll d \). This means that electrons of T-strips “tune” their electrochemical potential so that it equals the electrochemical potential in the adjacent well.

Let us imagine that an electric field \( F \) is applied along a diagonal \( MOQ \) of the chess-board (Fig. 2a). Then the electrochemical potential drops along the line \( QOM \) only in the vicinity of the saddle point \( O \). The value of this drop is \( eU = \sqrt{2}eFd \), where \( U \) is the voltage drop between \( M \) and \( Q \). The net current from the well \( M \) to the well \( Q \) appears as the difference between the two opposite drift flows in T-strips near the saddle point \( O \). They are shown by in Fig. 2a. In the crossection \( AOD \) electrons going from \( Q \) to \( M \) and back have chemical potentials \( \mu_Q = -\Delta + eU/2 \) and \( \mu_M = -\Delta - eU/2 \) respectively. When electrons arrive from one well to the other and continue to drift along the T-strips, their chemical potential very quickly relaxes to the new value.

Thus we have arrived at the concept of a network of reservoirs (potential wells) connected via ballistic contacts (saddle points) shown in Fig. 2b. Let us show that the conductance \( G \) of each of these contacts has the form

\[ G = \frac{e^2}{h} \exp(-\Delta/T). \]  

(3)

If the \( x \)-axis is directed along \( AOD \) with the reference point at \( O \), the current from \( Q \) to \( M \) can be calculated as

\[ I_{QM} = -e \int_0^\infty dx n(x)v(x), \]  

(4)

where \( n(x) = (1/2\pi \lambda^2) \exp(-\Delta/T + eU/2T - W(x)/T) \) is the two-dimensional concentration of electrons and \( v(x) = (c/eB)\partial W/\partial x \) is their drift velocity. Calculation of the integral gives
Similarly, for the current in the opposite direction we get $I_{MQ} = -(eT/h) \exp(-\Delta/T - eU/2T)$. Calculating the net current in the Ohmic conditions ($eU/T \ll 1$) we arrive at Eq. (3). It agrees with general expression $G = \nu e^2/h$, where $\nu$ is the filling factor at the saddle point [14]. In our case $\nu = \exp(-\Delta/T)$.

The conductivity of the whole resistor network shown in Fig. 2b is

$$\sigma_{xx} = G = \frac{e^2}{h} \exp(-\Delta/T).$$  \hspace{1cm} (6)

Note that this calculation gives contribution to the conductivity only of electrons on the upper Landau level. Holes on the lower Landau level form their own resistor network. It can be considered independent of the electron network so long as the electron-hole recombination is, as usual, slow enough. Therefore, the total conductivity of both parallel networks is of the form of Eq. (1) with $\sigma_0 = 2e^2/h$.

Let us now turn to the random long range potential. The simplest way to imagine it, departing from the chess-board potential, is to assume that heights of the saddle points $W_i$ are randomly distributed in an interval of energies ($-W_1, W_1$), where $W_1$ is comparable with $W$. Assuming then that $v \tau \ll d$ and repeating the same arguments as for the chess-board case, we arrive at the network of random conductances

$$G_i = \frac{e^2}{h} \exp(-\Delta/T - W_i/T).$$ \hspace{1cm} (7)

To calculate the conductivity of such a network, we use the Dykhne theorem [15]. According to this theorem a two-dimensional network of random conductances with a symmetrical distribution of $\ln G_i$ around the average value $<\ln G_i>$ has the conductivity

$$\sigma_{xx} = \exp(<\ln G_i>).$$ \hspace{1cm} (8)

In our case $<\ln G_i> = -\Delta/T$ because $<W_i> = 0$, and thus we again arrive at Eq. (6). It is important that the limits of applicability of Eq. (6) are much broader for a random potential than for the periodic one with the same values of $W$ and $d$. This happens because, according to percolation theory, randomness of the saddle point heights generates two new large scales at $|E| \ll W$:

$$\xi = d \left( \frac{W}{|E|} \right)^{\nu_p}, \quad p = d \left( \frac{W}{|E|} \right)^{\gamma}.$$ \hspace{1cm} (9)

Here $\xi$ and $p$ are the “diameter” and the perimeter of critical equipotential loops at a given energy $E$. By “critical” we mean that the probability of finding a loop with a size smaller than the critical one at a given $E$ decays as a power law function of the size, while it decays exponentially at sizes larger than critical. It is known that $\nu_p = 4/3$ and $\gamma = \nu_p + 1 = 7/3$ [16]. In the above derivation of Eq. (3) based on Eq. (8), we considered all of the saddle points of the random chess-board potential. Actually, the conductivity is determined only by those of them for which $|W_i| \sim T$. At $T \ll W$ these saddle points separate critical loops corresponding to $|E| \sim T$ which have very long perimeters.
\[ p_T \sim d \left( \frac{W}{T} \right)^\gamma. \] (10)

Such “T-loops” together with saddle points with heights \( |W_i| \sim T \) form a network which is topologically similar to the chess-board. We can consider it as a network of ballistic conductances given by Eq. (7) and arrive at Eq. (6) if the electrons of a T-loop are in equilibrium with the cluster of potential wells they circulate alongside, i.e. if \( v \tau \ll p_T \). According to Eq. (10) this inequality puts a much weaker condition on \( \tau \) than \( v \tau \ll d \), which is required for the periodic chess-board.

Let us discuss now the width of the interval of \( T \) where \( v \tau \ll p_T \) and \( \sigma_0 = \frac{2e^2}{h} \).

This requires an explicit discussion of the nature of inelastic processes. As we mentioned above, activated electrons are far from each other and thus interaction between them is negligible. We can neglect also the interaction with electrons on the Fermi level: they are concentrated in droplets only in the rare places where the Fermi level touches the Landau level. Therefore, we assume below that inelastic processes are only due to electron-phonon interactions. (Note that any additional process is able only to expand the range of validity of the universal prefactor.)

To discuss the electron-phonon scattering, we first estimate the typical drift velocity:

\[ v \sim cW/eBd \sim W \lambda^2 / \hbar d. \]

One can easily verify that under realistic conditions \( v > s \), where \( s \) is the sound velocity. This means that one-phonon processes are permitted by the conservation laws. To evaluate the corresponding electron-phonon collision time \( \tau_c \), one may assume that an electron is in a uniform electric field \( \sim W/ed \). Then using Fermi’s golden rule, one obtains \( \tau_c \sim \hbar / \alpha T \) for \( T \gtrsim W \lambda / d \) and \( \tau_c \sim \hbar d / \alpha W \) for \( T \lesssim W \lambda / d \). Here \( \alpha = \hbar C^2 / 2 \rho s^3 \lambda^2 \) is the electron-phonon coupling constant, \( C \) and \( \rho \) are the deformation potential constant and the crystal density respectively. For GaAs \( \alpha \approx 0.1(100 \text{Å}/\lambda)^2 \) [12].

The characteristic energy \( W \lambda / d \) appears above because typical hops occur between two trajectories a distance \( \lambda \) from each other and \( W \lambda / d \) is the typical phonon energy. In order to get \( \tau \) from \( \tau_c \) one should again consider separately the two cases when \( W \lambda / d \) is larger or smaller than \( T \). While in the former case the energy changes by \( T \) via one scattering, in the latter one an electron slowly diffuses along the energy axis. This yields

\[ \tau \sim \frac{\hbar}{\alpha T} \left( \frac{T d}{W \lambda} \right)^2 \] (11)

for \( T \gtrsim W \lambda / d \) and \( \tau \sim \hbar d / \alpha W \) for \( T \lesssim W \lambda / d \). Using these results together with Eq. (10), we find that for large enough spacer \( d \gg d_c \), the inequality \( v \tau \ll p_T \) is valid for T-loops and, correspondingly, \( \sigma_0 = \frac{2e^2}{h} \) in a wide range of temperatures \( T_1 \ll T \ll T_2 \). Here \( d_c = \lambda / \alpha^{1/(\gamma+1)} \) and \( T_2 = W \alpha^{1/(\gamma+1)} \). For \( \lambda \approx 10 \text{nm} \) and \( \alpha \approx 0.1 \) one gets \( d_c \approx 20 \text{nm} \). Thus for samples with \( d >> 20 \text{nm} \) we arrive at the universal prefactor \( \sigma_0 = \frac{2e^2}{h} \). Below we deal only with the case \( d \gg d_c \). We will consider elsewhere the narrow range \( \lambda < d < d_c \), where the crossover between the short and long range cases takes place.

Let us briefly discuss what happens away from the temperature range \( T_1 \ll T \ll T_2 \). If \( T \lesssim T_1 \) tunneling becomes important and an additional VRH contribution to the conductivity appears, similar to the case of a short range potential [14]. VRH conductivity is determined by hops at a so called transport energy which is a result of an interplay between the probabilities of tunneling and of activation [17]. The transport energy is negative and
its absolute value grows with decreasing $T$. Thus the plot of $\ln \sigma(1/T)$ eventually deviates upwards from the straight line of the universal regime. Such deviations are seen in some of the experimental data [3,4] but they seem to occur at higher $T$ than we predict.

On the other side, when $T \gtrsim T_2$ the perimeter $p_T$ of the $T$-loops becomes smaller than $v\tau$. In other words the $T$-strip becomes so wide that only the low energy part of it is still in equilibrium with the adjacent potential well. In order to find the width $\Gamma$ of the band $0 < E \lesssim \Gamma$, where equilibrium is still supported, one should solve the equation

$$v\tau(\Gamma) \sim p(\Gamma).$$

Here $\tau(\Gamma)$ is the time it takes to change the energy by $\Gamma$ via diffusion in the energy space (for $T \gg T_2$ and $d \gg d_c$ only the diffusion regime is relevant). This time can be found by replacing $T^2$ in the numerator of Eq. (11) with $\Gamma^2$. We obtain the perimeter $p(\Gamma)$ from Eq. (9) by substituting $\Gamma$ for $|E|$. The solution of Eq. (12) yields $\Gamma \sim T_2^\beta T^{1-\beta}$, where $\beta = (\gamma+1)/(\gamma+2) = 10/13$. Repeating the calculation of the ballistic conductance of a saddle point similar to that using Eq. (4), we have to restrict the integration to the strip in which $0 < W(x) \lesssim \Gamma$. (Electrons with $\Gamma \gtrsim E \lesssim T$ circulating around the hills $A$ and $D$ have the same the electrochemical potentials and therefore do not contribute to the ballistic current at the saddle point $O$.) This leads to the replacement $T \rightarrow \Gamma$ in the prefactor of Eq. (5). As a result, we get for the prefactor of conductivity at $T \gg T_2$

$$\sigma_0 \sim \frac{e^2}{h} \left(\frac{T_2}{T}\right)^\beta.$$

Note that Eq. (13) is equivalent to the result of calculation of the effective diffusion coefficient for a classical advective-diffusive motion [10].

Eq. (13) predicts deviation downwards from the straight line in the plot of $\ln \sigma$ vs $1/T$ when $T$ becomes larger than $T_2$ (Fig. 1b). Strong deviations of this type were observed in Refs. [3,4]. In some cases the dependence of $\ln \sigma$ on $1/T$ even saturates. We believe that such behavior can be explained using Eq. (13), only if the screening of the long range potential by activated electrons and holes, which reduces $W$ and $T_2$, is taken into account. Screening becomes important when the concentration of activated carriers $(1/2\pi\lambda^2) \exp(-\Delta/T)$ is comparable to the fluctuations of the charge donor concentration which have length scale $d$. For the case of large $d$, the latter concentration may be much smaller than $1/(2\pi\lambda^2)$, and therefore screening may become important even at $T \ll \Delta$.

In conclusion, we have shown that in a heterostructure with a large spacer there is a range of temperatures at which the prefactor $\sigma_0$ is universal. It is equal to $2e^2/h$ if the Fermi level is in the middle between two Landau levels. In the case of the FQHE our theory can be applied to the conductivity of excitations with fractional charges (quasielectrons and quasiholes). As a result $\sigma_0 = 2e^2/q^2h$ at $\nu = p/q$. Our values of $\sigma_0$ differ by the factor 2 from the values claimed in Refs. [2,3]. Note, however, that larger values of $\sigma_0$ were reported by another group [4]. Moreover, recently $\sigma_0$ has been found to be proportional to $1/T$ [5]. We do not know how to resolve these contradictions. It seems that more experimental evidence is needed here.

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FIGURES

FIG. 1. Schematical plot of $\ln \sigma$ vs $1/T$ when the Fermi level is in the middle between two Landau levels: (a) as calculated for a short range potential [1], (b) as obtained in this work for a long range potential.

FIG. 2. (a) Equipotential lines $W(x, y) = 0$ (square lattice) and $W(x, y) = T$ for a chess-board potential. Saddle point $O$ separates two hills $A$, $D$ and two wells $Q$, $M$. Arrows show directions of the drift current. (b) Equivalent circuit. Each conductance $G$ is given by Eq. (3).