EXPANSIONS OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF MULTIPlicITIES 1 TO 4. COMBINED APPROACH BASED ON GENERALIZED MULTIPLE AND ITERATED FOURIER SERIES

DMITRIY F. KUZNETSOV

ABSTRACT. The article is devoted to the expansions of iterated Stratonovich stochastic integrals of multiplicities 1 to 4 on the base of the combined approach of generalized multiple and iterated Fourier series. We consider two different parts of the expansion of iterated Stratonovich stochastic integrals. The mean-square convergence of the first part is proved on the base of generalized multiple Fourier series that are converge in the sense of norm in Hilbert space \( L_2([t, T]^k) \), \( k = 1, 2, 3, 4 \). The mean-square convergence of the second part is proved on the base of generalized iterated Fourier series that are converge pointwise. At that, we do not use the iterated It\'o stochastic integrals as a tool of the proof and directly consider the iterated Stratonovich stochastic integrals. The cases of multiple Fourier–Legendre series and multiple trigonometric Fourier series are considered in detail. The considered expansions contain only one operation of the limit transition in contrast to its existing analogues. This property is very important for the mean-square approximation of iterated stochastic integrals. The results of the article can be applied to the numerical integration of It\'o stochastic differential equations.

CONTENTS

1. Introduction 2
2. Method of Expansion of Iterated It\'o Stochastic Integrals Based on Generalized Multiple Fourier Series 4
3. Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 2 to 4. Some Old Results 10
4. Auxiliary Lemmas 13
5. Proof of Theorem 4 Using the Combined Approach 16
6. Proof of Theorem 5 Using the Combined Approach 19
7. Proof of Theorem 6 Using the Combined Approach 24
8. Theorems 1–6 from Point of View of the Wong–Zakai Approximation 36
9. Recent Results on Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 3 to 6 40
References 43

MATHEMATICS SUBJECT CLASSIFICATION: 60H05, 60H10, 42B05.

KEYWORDS: ITERATED STRATONOVICH STOCHASTIC INTEGRAL, ITERATED ITO STOCHASTIC INTEGRAL, GENERALIZED MULTIPLE FOURIER SERIES, MULTIPLE FOURIER–LEGENDRE SERIES, MULTIPLE TRIGONOMETRIC FOURIER SERIES, EXPANSION, MEAN-SQUARE CONVERGENCE.
1. Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, let \(\{\mathcal{F}_t, t \in [0, T]\}\) be a nondecreasing right-continuous family of \(\sigma\)-algebras of \(\mathcal{F}\), and let \(f_t\) be a standard \(m\)-dimensional Wiener stochastic process, which is \(\mathcal{F}_t\)-measurable for any \(t \in [0, T]\). We assume that the components \(f_t^{(i)}\) \((i = 1, \ldots, m)\) of this process are independent. Consider an Ito stochastic differential equation (SDE) in the integral form

\[
(1) \quad x_t = x_0 + \int_0^t a(x_\tau, \tau) d\tau + \int_0^t B(x_\tau, \tau) d\mathbf{f}_\tau, \quad x_0 = x(0, \omega).
\]

Here \(x_t\) is some \(n\)-dimensional stochastic process satisfying the equation (1). The nonrandom functions \(a : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n\), \(B : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times m}\) guarantee the existence and uniqueness up to stochastic equivalence of a solution of (1). The second integral on the right-hand side of (1) is interpreted as an Ito stochastic integral. Let \(x_0\) be an \(n\)-dimensional random variable, which is \(\mathcal{F}_0\)-measurable and \(\mathbb{E}\{|x_0|^2\} < \infty\) (\(\mathbb{E}\) denotes a mathematical expectation). We assume that \(x_0\) and \(f_t - f_0\) are independent when \(t > 0\).

It is well known that one of the effective approaches to the numerical integration of Ito SDEs is an approach based on the Taylor–Ito and Taylor–Stratonovich expansions [2]–[5]. The most important feature of such expansions is a presence in them of the so-called iterated Ito and Stratonovich stochastic integrals (2) and (3). The only exception is connected with a narrow particular case, when \(i_1 = \ldots = i_k \neq 0\) and \(\psi_1(s), \ldots, \psi_k(s) \equiv \psi(s)\). This case allows the investigation with using of the Ito formula [2]–[5].

Consider a brief review of the mean-square approximation methods for the iterated stochastic integrals [2] and [3].
Seems that iterated stochastic integrals can be approximated by multiple integral sums of different types \([3, 5, 17]\). However, this approach implies partitioning of the interval of integration \([t, T]\) of iterated stochastic integrals (the length \(T - t\) of this interval is a small value, because it is a step of integration of numerical methods for Ito SDEs) and according to numerical experiments this additional partitioning leads to significant computational costs \([10]\).

In \([3]\) (also see \([2, 1, 5, 48, 49]\)) Milstein G.N. proposed to expand \((2)\) or \((3)\) into iterated series of products of standard Gaussian random variables by representing the Brownian bridge process as a trigonometric Fourier series with random coefficients (version of the so-called Karhunen–Loeve expansion). To obtain the Milstein expansion of \((3)\), the truncated Fourier expansions of components of the Wiener process \(f\) must be iteratively substituted in the single integrals, and the integrals must be calculated, starting from the innermost integral. This is a complicated procedure that does not lead to a general expansion of \((3)\) valid for an arbitrary multiplicity \(k\). For this reason, only expansions of single, double, and triple stochastic integrals \((2), (3)\) were presented in \([2, 4, 48, 49]\) \((k = 1, 2, 3)\) and in \([3, 5]\) \((k = 1, 2)\) for the case \(\psi_1(s), \psi_2(s), \psi_3(s) \equiv 1; i_1, i_2, i_3 = 0, 1, \ldots, m\).

Moreover, the authors of the works \([2]\) (Sect. 5.8, pp. 202–204), \([3]\) (pp. 82-84), \([15]\) (pp. 438-439), \([19]\) (pp. 263-264) use the Wong–Zakai approximation \([51]-[53]\) (without rigorous proof) within the frames of the Milstein approach \([3]\) based on the series expansion of the Brownian bridge process. See discussion in Sect. 7 of this paper for details.

Note that in \([50]\) the method (similar to the Milstein approach) of expansion of iterated (double) Ito stochastic integrals \((2)\) \((k = 2; \psi_1(s), \psi_2(s) \equiv 1; i_1, i_2 = 1, \ldots, m)\) based on expansion of the Wiener process using Haar functions and trigonometric functions has been considered.

It is necessary to note that the approach based on the Karhunen–Loeve expansion \([3]\) excelled in several times (or even in several orders) the methods of multiple integral sums \([3, 5, 17]\) considering computational costs in the sense of their diminishing.

An alternative strong approximation method was proposed for \((3)\) in \([6, 7]\) (also see \([14]-[19], 22, 24-27]\), where \(J^s[\psi^{(k)}]_{t,T}\) was represented as the multiple stochastic integral from the certain discontinuous nonrandom function of \(k\) variables, and the function was then expressed as the generalized iterated Fourier series by complete systems of continuously differentiable functions that are orthonormal in the space \(L_2([t,T])\). As a result, the general iterated series expansion of products of standard Gaussian random variables was obtained in \([6, 7]\) (also see \([14]-[19], 22, 24-27]\) for \((3)\) with an arbitrary multiplicity \(k\). Hereinafter, this method is referred to as the method of generalized iterated Fourier series. It was shown \([6, 7]\) (also see \([14]-[19], 22, 24-27]\) that the method of generalized iterated Fourier series leads to the Milstein approach based on the Karhunen–Loeve expansion \([3]\) in the case of trigonometric system of functions and to a substantially simpler expansions of \((3)\) in the case of Legendre polynomial system.

As we mentioned above, the Milstein approach based on the Karhunen–Loeve expansion \([3]\) and the method of generalized iterated Fourier series \([6, 7]\) (also see \([14]-[19], 22, 24-27]\) lead to iterated application of the operation of limit transition. So, these methods may not converge in the mean-square sense to the appropriate iterated Stratonovich stochastic integrals \([3]\) for some methods of series summation.

The mentioned problem (iterated application of the operation of limit transition) not appears in the method, which is proposed for \((2)\) in Theorems 1, 2 (see below) \([10, 22, 24-27]\). The idea of this method is as follows: the iterated Ito stochastic integral \((2)\) of multiplicity \(k\) is represented as the multiple stochastic integral from the certain discontinuous nonrandom function of \(k\) variables defined on the hypercube \([t,T]^k\), where \([t,T]\) is the interval of integration of the iterated Ito stochastic integral \((2)\). Then, the nonrandom function of \(k\) variables is expanded in the hypercube \([t,T]^k\) into the generalized multiple Fourier series converging in the mean-square sense in the space \(L_2([t,T]^k)\).

After a number of nontrivial transformations we obtain (see Theorems 1, 2 below) the mean-square converging expansion of the iterated Ito stochastic integral \((2)\) into the multiple series of products of standard Gaussian random variables. The coefficients of this series are the coefficients of generalized multiple Fourier series for the mentioned nonrandom function of \(k\) variables, which can be calculated.
using the explicit formula regardless of the multiplicity \( k \) of the iterated Ito stochastic integral \( 2 \).

We will call this method as the method of generalized multiple Fourier series.

2. Method of Expansion of Iterated Ito Stochastic Integrals Based on Generalized Multiple Fourier Series

Suppose that every \( \psi_l(\tau) \) \((l = 1, \ldots, k)\) is a nonrandom function from the space \( L_2([t, T]) \). Define the following function on the hypercube \([t, T]^k\)

\[
K(t_1, \ldots, t_k) = \begin{cases} 
\psi_1(t_1) \cdots \psi_k(t_k) & \text{for} \ t_1 < \ldots < t_k \\
0 & \text{otherwise}
\end{cases}, \quad t_1, \ldots, t_k \in [t, T], \quad k \geq 2,
\]

and \( K(t_1) \equiv \psi_1(t_1) \) for \( t_1 \in [t, T] \).

Suppose that \( \{ \phi_j(x) \}_{j=0}^\infty \) is a complete orthonormal system of functions in the space \( L_2([t, T]) \). The function \( K(t_1, \ldots, t_k) \) belongs to the space \( L_2([t, T]^k) \). At this situation it is well known that the generalized multiple Fourier series of \( K(t_1, \ldots, t_k) \in L_2([t, T]^k) \) is converging to \( K(t_1, \ldots, t_k) \) in the hypercube \([t, T]^k\) in the mean-square sense, i.e.

\[
\lim_{p_1, \ldots, p_k \to \infty} \left\| K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right\|_{L_2([t, T]^k)} = 0,
\]

where

\[
C_{j_k \ldots j_1} = \int_{[t, T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^k \phi_{j_l}(t_l) dt_1 \ldots dt_k,
\]

\[
\| f \|_{L_2([t, T]^k)} = \left( \int_{[t, T]^k} f^2(t_1, \ldots, t_k) dt_1 \ldots dt_k \right)^{1/2}.
\]

Consider the partition \( \{ \tau_j \}_{j=0}^N \) of \([t, T]\) such that

\[
t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta \tau_j \to 0 \quad \text{if} \quad N \to \infty, \quad \Delta \tau_j = \tau_{j+1} - \tau_j.
\]

**Theorem 1** \([10] \) (2006), \([11] - [22], [24] - [44] \). Suppose that every \( \psi_l(\tau) \) \((l = 1, \ldots, k)\) is a continuous nonrandom function on \([t, T]\) and \( \{ \phi_j(x) \}_{j=0}^\infty \) is a complete orthonormal system of continuous functions in the space \( L_2([t, T]) \). Then

\[
J[\psi^{(k)}]_{T,t} = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \left( \prod_{l=1}^k \phi_{j_l}^{(i_l)} \right) - \lim_{N \to \infty} \sum_{(l_1, \ldots, l_k) \in G_k} \phi_{j_1}(\tau_{l_1}) \Delta w_{\tau_{l_1}}^{(i_1)} \cdots \phi_{j_k}(\tau_{l_k}) \Delta w_{\tau_{l_k}}^{(i_k)},
\]

where \( G_k \) is the number of nonempty \( k \)-tuples of the natural numbers less than \( N \).
where $J[\psi^{(k)}]_{T,t}$ is defined by (2),

$$G_k = H_k |L_k, \quad H_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1\},$$

$$L_k = \{(l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N - 1; l_g \neq l_r \ (g \neq r); \ g, r = 1, \ldots, k\},$$

l.i.m. is a limit in the mean-square sense, $i_1, \ldots, i_k = 0, 1, \ldots, m$,

$$\zeta_j^{(i)} = \int_T^t \phi_j(s)dw_j^{(i)}$$

are independent standard Gaussian random variables for various $i$ or $j$ (if $i \neq 0$), $C_{j_k \ldots j_1}$ is the Fourier coefficient (3), $\Delta w_j^{(i)} = w_{T,j+1}^{(i)} - w_j^{(i)}$ $(i = 0, 1, \ldots, m)$, $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t, T]$, which satisfies the condition (4).

It was shown [12]-[19], [22], [24]-[27], [35] that Theorem 1 is valid for convergence in the mean of degree $2n$ ($n \in \mathbb{N}$) and for convergence with probability 1 [25]-[28].

Moreover, the complete orthonormal systems of Haar and Rademacher–Walsh functions in $L_2([t, T])$ can also be applied in Theorem 1 [12]-[19], [22], [24]-[27], [35]. The modification of Theorem 1 for complete orthonormal with weight $r(x) \geq 0$ systems of functions in the space $L_2([t, T])$ can be found in [24]-[27], [30].

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k = 1, \ldots, 6$ [10], [22], [24]-[41].
(12) \[ J[\xi^{(5)}]_{T,t} = \lim_{p_1, \ldots, p_5 \to \infty} \sum_{j_1 = 0}^{p_1} \cdots \sum_{j_5 = 0}^{p_5} C_{j_5 \cdots j_1} \left( \prod_{t=1}^{5} \zeta_{j_t}^{(i_t)} - \right. \]

\[ \left. - \{i_1 = i_2 \neq 0\} 1_{\{j_1 = j_2\}} 1_{\{i_3 = i_4 \neq 0\}} 1_{\{j_3 = j_4\}} 1_{\{j_5 = j_5\}} \right) \]

\[ + 1_{\{i_1 = i_4 \neq 0\}} 1_{\{j_1 = j_4\}} 1_{\{i_2 = i_3 \neq 0\}} 1_{\{j_2 = j_3\}} 1_{\{j_3 = j_5\}} 1_{\{j_4 = j_5\}} \]

(13) \[ J[\xi^{(6)}]_{T,t} = \lim_{p_1, \ldots, p_6 \to \infty} \sum_{j_1 = 0}^{p_1} \cdots \sum_{j_6 = 0}^{p_6} C_{j_6 \cdots j_1} \left( \prod_{t=1}^{6} \zeta_{j_t}^{(i_t)} - \right. \]

\[ \left. - \{i_1 = i_6 \neq 0\} 1_{\{j_1 = j_6\}} 1_{\{i_2 = i_6 \neq 0\}} 1_{\{j_2 = j_6\}} 1_{\{i_3 = i_6 \neq 0\}} 1_{\{j_3 = j_6\}} \right) \]

\[ + 1_{\{i_2 = i_5 \neq 0\}} 1_{\{j_2 = j_5\}} 1_{\{i_3 = i_4 \neq 0\}} 1_{\{j_3 = j_4\}} \]
\[ +1 \{ i_1 = i_4 \neq 0 \} 1_{\{ j_1 = j_4 \}} 1_{\{ i_2 = i_3 \neq 0 \} } 1_{\{ j_2 = j_3 \}} 1_{\{ t_1 = t_4 \}} \bigl( \zeta_{12} \bigl( \zeta_{14} \bigr) \bigl( \zeta_{123} \bigr) \bigl( \zeta_{124} \bigr) \bigl( \zeta_{13} \bigr) \bigr) - \bigl( \zeta_{12} \bigl( \zeta_{14} \bigr) \bigl( \zeta_{123} \bigr) \bigl( \zeta_{124} \bigr) \bigl( \zeta_{13} \bigr) \bigr) \]
where \(1_A\) is the indicator of the set \(A\).

Thus, we obtain the following useful possibilities of the method of generalized multiple Fourier series.

1. There is the explicit formula (see \(\text{[5]}\)) for calculation of expansion coefficients of the iterated Ito stochastic integral \(\text{(2)}\) with any fixed multiplicity \(k\).

2. We have new possibilities for exact calculation of the mean-square approximation error for the iterated Ito stochastic integrals \(\text{(2)}\) (see \(\text{[20], [22], [24]-[27], [34]}\)).

3. Since the used multiple Fourier series is a generalized in the sense that it is built using various complete orthonormal systems of functions in the space \(L_2([t, T])\), then we have new possibilities for approximation — we can use not only trigonometric functions as in \(\text{[21], [5] but Legendre polynomials.}\)

4. As it turned out (see \(\text{[0]-[22], [21]-[44]}\)), it is more convenient to work with the Legendre polynomials for constructing of approximations of the iterated Ito stochastic integrals \(\text{(2)}\). Approximations based on the Legendre polynomials essentially simpler than their analogues based on the trigonometric functions (see \(\text{[0]-[22], [21]-[44]}\)). Another advantages of the application of Legendre polynomials in the framework of the mentioned problem are considered in \(\text{[25]-[27], [39], [40]}\).

5. The approach based on the Karhunen–Loève expansion of the Brownian bridge process \(\text{[3]}\) (also see \(\text{[50]}\)) leads to iterated application of the operation of limit transition (the operation of limit transition is implemented only once in Theorem 1 and Theorem 2 (see below)) starting from the second multiplicity (in the general case) and third multiplicity (for the case \(\psi_1(s), \psi_2(s), \psi_3(s) \equiv 1; i_1, i_2, i_3 = 1, \ldots, m\) of iterated Ito stochastic integrals. Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as \(p_1, \ldots, p_k\)). For example, when \(p_1 = \ldots = p_k = p \rightarrow \infty\). For iterated series, the condition \(p_1 = \ldots = p_k = p \rightarrow \infty\) obviously does not guarantee the convergence of this series. However, in \(\text{[2]}\) (Sect. 5.8, pp. 202–204), \(\text{[4]}\) (pp. 82-84), \(\text{[15]}\) (pp. 438-439), \(\text{[19]}\) (pp. 263-264) the authors use (without rigorous proof) the condition \(p_1 = p_2 = p_3 = p \rightarrow \infty\) within the frames of the mentioned approach based on the Karhunen–Loève expansion of the Brownian bridge process \(\text{[3]}\) together with the Wong–Zakai approximation \(\text{[51]-[53]}\) (see discussion in Sect. 7 of this paper for details).

For further consideration, let us consider the generalization of formulas \(\text{[9]-[14]}\) for the case of an arbitrary multiplicity \(k\) (\(k \in \mathbb{N}\)) of the iterated Ito stochastic integral \(J^{(k)}_{t,T}[\psi]\) defined by \(\text{(2)}\). In order to do this, let us introduce some notations. Consider the unordered set \(\{1, 2, \ldots, k\}\) and separate it into two parts: the first part consists of \(r\) unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining \(k - 2r\) numbers. So, we have

\[
(\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}\}, \{q_1, \ldots, q_{k-2r}\}),
\]

where

\[
\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\},
\]

braces mean an unordered set, and parentheses mean an ordered set.

We will say that \((15)\) is a partition and consider the sum with respect to all possible partitions

\[
\sum_{\mathcal{A} \in \{\text{unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining } k-2r \text{ numbers. So, we have}}\}
\]

Below there are several examples of sums in the form \((16)\)
\[
\sum_{((s_1, s_2), (s_3, s_4)) \in \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4))\}} a_{g_1 g_2} = a_{12},
\]
\[
\sum_{((s_1, s_2), (s_3, s_4), (s_5, s_6)) \in \{(1,2,3), (1,2,4), (1,3,4), (2,3,4))\}} a_{g_1 g_2 g_3 g_4} = a_{1234} + a_{1324} + a_{2314} + a_{2314},
\]
\[
\sum_{((s_1, s_2), (s_3, s_4)) \in \{(1,2,3,4,5))\}} a_{g_1 g_2 g_1 g_2} =
\]
\[
= a_{12,34} + a_{13,24} + a_{14,23} + a_{23,14} + a_{24,13} + a_{34,12},
\]
\[
\sum_{((s_1, s_2), (s_3, s_4)) \in \{(1,2,3,4,5))\}} a_{g_1 g_2 g_3 g_4} =
\]
\[
= a_{12,34} + a_{13,24} + a_{14,23} + a_{15,23} + a_{23,14} + a_{24,13} + a_{25,14} + a_{34,12} + a_{35,12} + a_{45,13} + a_{45,13},
\]
\[
\sum_{((s_1, s_2), (s_3, s_4)) \in \{(1,2,3,4,5))\}} a_{g_1 g_2 g_3 g_4 g_1} =
\]
\[
= a_{12,34} + a_{13,24} + a_{14,23} + a_{15,23} + a_{12,35} + a_{13,25} + a_{15,24} + a_{13,24} + a_{15,54} + a_{14,53} + a_{14,53} + a_{15,24} + a_{15,24} + a_{52,34} + a_{53,24} + a_{54,23},
\]

Now we can write \eqref{eq:7} as

\[
J[\psi^{(k)}]_{T,t} = \lim_{p_1, \ldots, p_k \to \infty} \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} \cdots \sum_{j_k = 0}^{p_k} C_{j_k \cdots j_1} \left( \prod_{l=1}^{k} \left[ r \right] \sum_{r=1}^{[k/2]} (-1)^r \times \sum_{(s_1, s_2, \ldots, s_{k-2r}) \in \{(1,2, \ldots, k)\}} \prod_{s=1}^{r} \mathbf{1}_{i_{s_{2s-1}} = i_{s_{2s}} \neq 0} \mathbf{1}_{j_{s_{2s-1}} = j_{s_{2s}}} \prod_{l=1}^{k-2r} \Omega_{j_{s_l}}^{(1)} \right),
\]

where \([x]\) is an integer part of a real number \(x\); another notations are the same as in Theorem 1.

In particular, from \eqref{eq:14} for \(k = 5\) we obtain

\[
J[\psi^{(5)}]_{T,t} = \lim_{p_1, \ldots, p_5 \to \infty} \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} \cdots \sum_{j_5 = 0}^{p_5} C_{j_5 \cdots j_1} \left( \prod_{l=1}^{5} \Omega_{j_{s_l}}^{(1)} - \sum_{(s_1, s_2) \in \{(1,2, \ldots, 5)\}} \mathbf{1}_{i_s = i_{s_2} \neq 0} \mathbf{1}_{j_s = j_{s_2}} \prod_{l=1}^{3} \Omega_{j_{s_l}}^{(1)} \right) + \ldots
\]
converging in the mean-square sense is valid, where
is expanded into the multiple series

\[ (18) \]

\[ \sum_{\{i_1, i_2\} \ldots \{i_{k-2}, i_{k-1}\} \in \{1, \ldots, k\}} \prod_{l=1}^{r} 1_{\{i_{2l-1} = i_{2l} \neq 0\}} 1_{\{j_{2l} = j_{2l+1}\}} \prod_{l=1}^{r} c_{j_{2l} j_{2l+1}}^{(i_{2l})} + \sum_{r=1}^{[k/2]} (-1)^r \times \]

corresponding to \([9] \). Expansions of \([9] \)

Let us consider a generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space \( L_2([t, T]) \) and \( \psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T]) \).

**Theorem 2** \([25] \) (Sect. 1.11), \([35] \) (Sect. 15). Suppose that \( \psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T]) \) and \( \{ \phi_j(x) \}_{j=0}^{\infty} \) is an arbitrary complete orthonormal system of functions in the space \( L_2([t, T]) \). Then the following expansion

\[ J[\psi^{(k)}]_{t,T} = \lim_{p_1, \ldots, p_k \rightarrow \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \cdots j_1}^{(i_1)} \prod_{l=1}^{[k/2]} (-1)^r \times \]

\[ (18) \]

\[ \sum_{\{i_1, i_2\} \ldots \{i_{k-2}, i_{k-1}\} \in \{1, \ldots, k\}} \prod_{l=1}^{r} 1_{\{i_{2l-1} = i_{2l} \neq 0\}} 1_{\{j_{2l} = j_{2l+1}\}} \prod_{l=1}^{r} c_{j_{2l} j_{2l+1}}^{(i_{2l})} + \sum_{r=1}^{[k/2]} (-1)^r \times \]

corresponding in the mean-square sense is valid, where \([x] \) is an integer part of a real number \( x \); another notations are the same as in Theorem 1.

It should be noted that an analogue of Theorem 2 was considered in \([54] \). Note that we use another notations \([25] \) (Sect. 1.11), \([35] \) (Sect. 15) in comparison with \([54] \). Moreover, the proof of an analogue of Theorem 2 from \([54] \) is somewhat different from the proof given in \([25] \) (Sect. 1.11), \([35] \) (Sect. 15).

3. **Expansions of Iterated Stratonovich Stochastic Integrals of Multiplicities 2 to 4. Some Old Results**

As it turned out, Theorems 1, 2 can be adapted for the iterated Stratonovich stochastic integrals \([9] \) at least for multiplicities 1 to 6 (the case \( k = 1 \) obviously corresponds to \([9] \)). Expansions of the mentioned iterated Stratonovich stochastic integrals turned out simpler than the appropriate expansions for the iterated Ito stochastic integrals \([2] \) based on Theorems 1, 2. Let us formulate some theorems on expansions of the iterated Stratonovich stochastic integrals \([2] \) of multiplicities 2 to 4.

**Theorem 3** \([17]-[19], [22], [24]-[27], [43] \). Suppose that \( \{ \phi_j(x) \}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t, T]) \). At the same time \( \psi_2(\tau) \) is a continuously differentiable function on \([t, T]\) and \( \psi_1(\tau) \) is twice continuously differentiable function on \([t, T]\). Then, the iterated Stratonovich stochastic integral of second multiplicity

\[ J^*[\psi^{(2)}]_{t,T} = \int_t^T \psi_2(\tau) \int_t^\tau \psi_1(t_1) d\tilde{w}_{t_1}^{(i_1)} d\tilde{w}_{t_2}^{(i_2)} (i_1, i_2 = 0, 1, \ldots, m) \]

is expanded into the multiple series
\[
J^*[\psi^{(2)}]_{T,t} = \text{l.i.m.}_{p_1,p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2,j_1} \zeta_{j_2}^{(i_2)} \zeta_{j_1}^{(i_1)}
\]

converging in the mean-square sense, where

\[
\zeta_{j}^{(i)} = \int_{t}^{T} \phi_j(\tau) dw_{\tau}^{(i)}
\]

are independent standard Gaussian random variables for various \(i\) or \(j\) (if \(i \neq 0\)),

\[
C_{j_2,j_1} = \int_{t}^{T} \psi_2(t_2) \phi_{j_2}(t_2) \int_{t}^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2
\]

is the Fourier coefficient.

Note that in \([24]-[27], [41], [44]\) Theorem 3 is proved under weaker conditions.

**Theorem 4** \([24]-[27], [41], [44]\). Suppose that the following conditions are fulfilled:
1. The functions \(\psi_1(\tau)\) and \(\psi_2(\tau)\) are continuously differentiable at the interval \([t,T]\).
2. \(\{\phi_j(x)\}_{j=0}^{\infty}\) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \(L_2([t,T])\).

Then, the iterated Stratonovich stochastic integral of second multiplicity

\[
J^*[\psi^{(2)}]_{T,t} = \int_{t}^{*T} \psi_2(t_2) \int_{t}^{*t_2} \psi_1(t_1) dw_{t_1}^{(i_1)} dw_{t_2}^{(i_2)}
\]

(i_{1,2} = 0, 1, \ldots, m)

is expanded into the multiple series

\[
J^*[\psi^{(2)}]_{T,t} = \text{l.i.m.}_{p_1,p_2 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2,j_1} \zeta_{j_2}^{(i_2)} \zeta_{j_1}^{(i_1)}
\]

converging in the mean-square sense, where

\[
\zeta_{j}^{(i)} = \int_{t}^{T} \phi_j(\tau) dw_{\tau}^{(i)}
\]

are independent standard Gaussian random variables for various \(i\) or \(j\) (if \(i \neq 0\)),

\[
C_{j_2,j_1} = \int_{t}^{T} \psi_2(t_2) \phi_{j_2}(t_2) \int_{t}^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2
\]

is the Fourier coefficient.

**Theorem 5** \([18, 19, 22, 24]-[27], [43]\). Suppose that \(\{\phi_j(x)\}_{j=0}^{\infty}\) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \(L_2([t,T])\). At the same time \(\psi_2(\tau)\) is a continuously differentiable function at the interval \([t,T]\) and \(\psi_1(\tau), \psi_3(\tau)\) are twice continuously differentiable functions at the interval \([t,T]\). Then, for the iterated Stratonovich stochastic integral of third multiplicity
converging in the mean-square sense is valid, where

\[ J^{*}[\psi^{(3)}]_{T,t} = \int_{t}^{T} \psi_{3}(t_{3}) dt_{3} \int_{t}^{t_{3}} \psi_{2}(t_{2}) dt_{2} \int_{t}^{t_{2}} \psi_{1}(t_{1}) dt_{1} d\mathbf{f}_{t_{1}}^{(i_{1})} d\mathbf{f}_{t_{2}}^{(i_{2})} d\mathbf{f}_{t_{3}}^{(i_{3})} \quad (i_{1}, i_{2}, i_{3} = 1, \ldots, m) \]

the following expansion

\[ J^{*}[\psi^{(3)}]_{T,t} = \lim_{p \to \infty} \sum_{j_{1}, j_{2}, j_{3}=0}^{p} C_{j_{1}j_{2}j_{3}} \zeta_{i_{1}}^{(j_{1})} \zeta_{i_{2}}^{(j_{2})} \zeta_{i_{3}}^{(j_{3})} \]

converging in the mean-square sense is valid, where

\[ C_{j_{1}j_{2}j_{3}} = \int_{t}^{T} \psi_{3}(t_{3}) \phi_{j_{3}}(t_{3}) dt_{3} \int_{t}^{t_{3}} \psi_{2}(t_{2}) \phi_{j_{2}}(t_{2}) dt_{2} \int_{t}^{t_{2}} \psi_{1}(t_{1}) \phi_{j_{1}}(t_{1}) dt_{1} dt_{2} dt_{3}, \]

another notations are the same as in Theorems 1, 2.

**Theorem 6** [17]-[19], [22], [24]-[27], [35]. Suppose that \( \{\phi_{j}(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_{2}([t, T]) \). Then, for the iterated Stratonovich stochastic integral of fourth multiplicity

\[ I_{T,t}^{(i_{1}i_{2}i_{3}i_{4})} = \int_{t}^{T} \int_{t}^{t_{4}} \int_{t}^{t_{3}} \int_{t}^{t_{2}} d\mathbf{w}_{t_{1}}^{(i_{1})} d\mathbf{w}_{t_{2}}^{(i_{2})} d\mathbf{w}_{t_{3}}^{(i_{3})} d\mathbf{w}_{t_{4}}^{(i_{4})} \quad (i_{1}, i_{2}, i_{3}, i_{4} = 0, 1, \ldots, m) \]

the following expansion

\[ I_{T,t}^{(i_{1}i_{2}i_{3}i_{4})} = \lim_{p \to \infty} \sum_{j_{1}, j_{2}, j_{3}, j_{4}=0}^{p} C_{j_{1}j_{2}j_{3}j_{4}} \zeta_{i_{1}}^{(j_{1})} \zeta_{i_{2}}^{(j_{2})} \zeta_{i_{3}}^{(j_{3})} \zeta_{i_{4}}^{(j_{4})} \]

converging in the mean-square sense is valid, where

\[ \zeta_{i}^{(j)} = \int_{t}^{T} \phi_{j}(s) d\mathbf{w}_{s}^{(i)} \]

are independent standard Gaussian random variables for various \( i \) or \( j \) (if \( i \neq 0 \)),

\[ C_{j_{1}j_{2}j_{3}j_{4}} = \int_{t}^{T} \phi_{j_{1}}(t_{1}) dt_{1} \int_{t}^{t_{1}} \phi_{j_{2}}(t_{2}) dt_{2} \int_{t}^{t_{2}} \phi_{j_{3}}(t_{3}) dt_{3} \int_{t}^{t_{3}} \phi_{j_{4}}(t_{4}) dt_{4}, \]

\( \mathbf{w}_{t}^{(i)} = f_{t}^{(i)} \) for \( i = 1, \ldots, m \) and \( \mathbf{w}_{t}^{(0)} = \tau \).

Note that in [17]-[19], [22], [24]-[27], [35] the expansions [49]-[52] have been applied for the proof of Theorems 3–6. In this article, we will prove Theorems 4–6 by an another approach. This approach will be called as the combined approach. More precisely, we will use the scheme of the proof of Theorem 1 from this paper (see [10], [19], [22], [24]-[27], [35] for details) for the iterated Stratonovich stochastic integrals of multiplicities 2 to 4. As a result, we will obtain two different parts of the expansion of iterated Stratonovich stochastic integrals. The mean-square convergence of the
first part will be proved on the base of generalized multiple Fourier series converging in $L_2([t, T]^k)$ $(k = 2, 3, 4)$. At the same time, the mean-square convergence of the second part will be proved on the base of generalized iterated Fourier series converging pointwise. At that, we do not use the iterated Ito stochastic integrals (2) as a tool of the proof and directly consider the iterated Stratonovich stochastic integrals (3).

4. Auxiliary Lemmas

In this section, we collected several lemmas, which will be used for the proof of Theorems 4–6. Consider the partition $\{\tau_j\}_{j=0}^N$ of the interval $[t, T]$ such that

\begin{equation}
(20) \quad t = \tau_0 < \cdots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta \tau_j \to 0 \text{ if } N \to \infty, \quad \Delta \tau_j = \tau_{j+1} - \tau_j.
\end{equation}

Lemma 1 [10, 19, 22, 24–27, 35]. Suppose that every $\psi_l(\tau)$ $(l = 1, \ldots, k)$ is a continuous nonrandom function at the interval $[t, T]$. Then

\begin{equation}
(21) \quad J[\psi^{(k)}]_{T,t} = \text{l.i.m.}_{N \to \infty} \sum_{j=0}^{N-1} \cdots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^k \psi_l(\tau_{j_l}) \Delta w_{\tau_{j_l}}^{(i_l)} \text{ w. p. 1},
\end{equation}

where $J[\psi^{(k)}]_{T,t}$ has the form (2), $\Delta w_{\tau_{j_l}}^{(i_l)} = w_{\tau_{j_l+1}}^{(i_l)} - w_{\tau_{j_l}}^{(i_l)}$ $(i = 0, 1, \ldots, m)$, $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t, T]$ satisfying the condition (20); hereinafter w. p. 1 means with probability 1.

Remark 1. It is easy to see that if $\Delta w_{\tau_{j_l}}^{(i_l)}$ in (21) for some $l \in \{1, \ldots, k\}$ is replaced with $(\Delta w_{\tau_{j_l}}^{(i_l)})^P$ $(p = 2, i_l \neq 0)$, then the differential $dw_{\tau_{j_l}}^{(i_l)}$ in the integral $J[\psi^{(k)}]_{T,t}$ will be replaced with $dt_l$. If $p = 3, 4, \ldots$, then the right-hand side of the formula (21) will become zero w. p. 1. If we replace $\Delta w_{\tau_{j_l}}^{(i_l)}$ in (21) for some $l \in \{1, \ldots, k\}$ with $(\Delta \tau_{j_l})^P$ $(p = 2, 3, \ldots)$, then the right-hand side of the formula (21) will also be equal to zero w. p. 1.

Let us define the following multiple stochastic integral

\begin{equation}
(22) \quad \text{l.i.m.}_{N \to \infty} \sum_{j_1, \ldots, j_k=0}^{N-1} \Phi(\tau_{j_1}, \ldots, \tau_{j_k}) \prod_{l=1}^k \Delta w_{\tau_{j_l}}^{(i_l)} \overset{\text{def}}{=} J[\Phi^{(k)}]_{T,t},
\end{equation}

where $\Phi(t_1, \ldots, t_k) : [t, T]^k \to \mathbb{R}$ is a nonrandom function (the properties of this function will be specified further), $\Delta w_{\tau_{j_l}}^{(i_l)} = w_{\tau_{j_l+1}}^{(i_l)} - w_{\tau_{j_l}}^{(i_l)}$ $(i = 0, 1, \ldots, m)$, $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[t, T]$ satisfying the condition (20).

Denote

\begin{equation}
(23) \quad D_k = \{(t_1, \ldots, t_k) : t \leq t_1 < \cdots < t_k \leq T\}.
\end{equation}

We will use the same symbol $D_k$ to denote the open and closed domains corresponding to the domain $D_k$ defined by (23). However, we always specify what domain we consider (open or closed).

Also we will write $\Phi(t_1, \ldots, t_k) \in C(D_k)$ if $\Phi(t_1, \ldots, t_k)$ is a continuous nonrandom function of $k$ variables in the closed domain $D_k$.

Let us consider the iterated Ito stochastic integral
\[ I[\Phi]_{T,t}^{(k)} \overset{\text{def}}{=} \int_{t}^{T} \cdots \int_{t}^{t_2} \Phi(t_1, \ldots, t_k) dw_{t_1}^{(i_1)} \cdots dw_{t_k}^{(i_k)}, \]

where \( \Phi(t_1, \ldots, t_k) \in C(D_k) \).

**Lemma 2** \([10, 19, 22, 24-27, 35]\). Suppose that \( \Phi(t_1, \ldots, t_k) \in C(D_k) \) or \( \Phi(t_1, \ldots, t_k) \) is a continuous nonrandom function in the open domain \( D_k \) and bounded at its boundary. Then

\[ I[\Phi]_{T,t}^{(k)} = \lim_{N \to \infty} \sum_{j_k=0}^{N-1} \cdots \sum_{j_1=0}^{j_{k-1}-1} \Phi(t_{j_1}, \ldots, t_{j_k}) \prod_{l=1}^{k} \Delta w_{\tau_{j_l}}^{(i_l)} \text{ w. p. 1}, \]

where \( \Delta w_{\tau_{j_l}}^{(i_l)} = w_{\tau_{j_l+1}}^{(i_l)} - w_{\tau_{j_l}}^{(i_l)} \) (\( i = 0, 1, \ldots, m \)), \( \{\tau_{j}\}_{j=0}^{N} \) is a partition of the interval \([t, T]\) satisfying the condition \([20]\).

**Lemma 3** \([10, 19, 22, 24-27, 35]\). Suppose that every \( \varphi_i(\tau) \) (\( i = 1, \ldots, k \)) is a continuous nonrandom function at the interval \([t, T]\). Then

\[ \prod_{l=1}^{k} J[\varphi_l]_{T,t} = J[\Phi]_{T,t}^{(k)} \text{ w. p. 1}, \]

where

\[ J[\varphi_l]_{T,t} = \int_{t}^{T} \varphi_l(s) dw_s^{(i_l)}, \quad \Phi(t_1, \ldots, t_k) = \prod_{l=1}^{k} \varphi_l(t_l) \]

and the integral \( J[\Phi]_{T,t}^{(k)} \) is defined by the equality \([22]\).

Let us introduce the following notations

\[ J[\psi]_{T,t}^{[s_1, \ldots, s_l]} \overset{\text{def}}{=} \prod_{p=1}^{l} 1_{\{i_p = i_{p+1} \neq 0\}} \times \]

\[ \times \int_{t}^{T} \psi_k(t_k) \cdots \int_{t}^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int_{t}^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1}) \psi_{s_1+1}(t_{s_1+1}) \times \]

\[ \times \int_{t}^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1}) \cdots \int_{t}^{t_{s_1+3}} \psi_{s_1+2}(t_{s_1+2}) \int_{t}^{t_{s_1+2}} \psi_{s_1}(t_{s_1+1}) \psi_{s_1+1}(t_{s_1+1}) \times \]

\[ \times \int_{t}^{t_{s_1+1}} \psi_{s_1-1}(t_{s_1-1}) \cdots \int_{t}^{t_2} \psi_1(t_1) dw_{t_1}^{(i_1)} \cdots dw_{t_{s_1-1}}^{(i_{s_1-1})} dt_{s_1+1} dw_{t_{s_1+2}}^{(i_{s_1+2})} \cdots dw_{t_k}^{(i_k)}, \]

where

\[ A_{k,t} = \left\{(s_l, \ldots, s_1) : s_l > s_{l-1} + 1, \ldots, s_2 > s_1 + 1, s_1, \ldots, s_1 = 1, \ldots, k - 1\right\}, \]
(s_l, \ldots, s_1) \in A_{k,l}, \ l = 1, \ldots, [k/2], \ i_s = 0, 1, \ldots, m, \ s = 1, \ldots, k, \ \lfloor x \rfloor \text{ is an integer part of a real number } x, \ \mathbf{1}_A \text{ is the indicator of the set } A.

**Lemma 4** [6] (1997), [7], [10]-[19], [22], [24]-[27], [30]. Suppose that every \( \psi_l(\tau_l)(l = 1, \ldots, k) \) is a continuous nonrandom function at the interval \([t, T]\). Then, the following relation between iterated Stratonovich and Itô stochastic integrals is correct

\[
J^*[\psi^{(k)}]_{T,t} = J[\psi^{(k)}]_{T,t} + \frac{[k/2]}{2} \sum_{r=1}^{[k/2]} \sum_{(s_r, \ldots, s_1) \in A_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r, \ldots, s_1} \text{ w. p. 1},
\]

where \( \sum_{\emptyset} \) is supposed to be equal to zero.

Let us define the function \( K^*(t_1, \ldots, t_k) \) on the hypercube \([t,T]^k \) \((k \geq 2)\) by the following relation

\[
K^*(t_1, \ldots, t_k) = \prod_{l=1}^k \psi_l(t_l) \prod_{l=1}^{k-1} \left( 1_{\{t_1 < t_{l+1} \}} + \frac{1}{2} 1_{\{t_l = t_{l+1} \}} \right) =
\]

\[
= \prod_{l=1}^k \psi_l(t_l) \left( \prod_{l=1}^{k-1} 1_{\{t_1 < t_{l+1} \}} + \sum_{r=1}^{k-1} \frac{1}{2r} \sum_{(s_r, \ldots, s_1) \in A_{k,r}} \sum_{l=1}^{r-1} 1_{\{t_{s_l} = t_{s_{l+1}} \}} \prod_{l=1}^{k-1} 1_{\{t_l < t_{l+1} \}} \right),
\]

where \( \mathbf{1}_A \) is the indicator of the set \( A \).

**Lemma 5** [6], [7], [14]-[19], [22], [24]-[27], [30]. Under the conditions of Lemma 4 the following relation is correct

\[
J[K^*^{(k)}]_{T,t} = J^*[\psi^{(k)}]_{T,t} \text{ w. p. 1},
\]

where \( J[K^*^{(k)}]_{T,t} \) is defined by the equality \( (22) \).

**Proof.** Substituting \( (29) \) into \( (22) \) and using Lemmas 1, 2, 4 with Remark 1, it is easy to notice that w. p. 1

\[
J[K^*^{(k)}]_{T,t} = J[\psi^{(k)}]_{T,t} + \frac{[k/2]}{2} \sum_{r=1}^{[k/2]} \sum_{(s_r, \ldots, s_1) \in A_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r, \ldots, s_1} = J^*[\psi^{(k)}]_{T,t}.
\]

Let us consider the following generalized multiple Fourier sum

\[
\sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k, \ldots, j_1} \prod_{l=1}^k \phi_{j_l}(t_l),
\]

where \( C_{j_k, \ldots, j_1} \) is the Fourier coefficient of the form

\[
C_{j_k, \ldots, j_1} = \int_{[t,T]^k} K^*(t_1, \ldots, t_k) \prod_{l=1}^k \phi_{j_l}(t_l) dt_1 \ldots dt_k.
\]
Let us substitute the relation
\[ K^*(t_1, \ldots, t_k) = \sum_{j_1=0}^{p_1} \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) + K^*(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \]
into \( J[K^*]_{T,t}^{(k)} \) (here \( p_1, \ldots, p_k < \infty \)).

Then, using Lemma 3, we obtain
\[ J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{p_1} \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} + J[R_{p_1 \ldots p_k}]_{T,t}^{(k)} \text{ w. p. 1,} \]
where the stochastic integral \( J[R_{p_1 \ldots p_k}]_{T,t}^{(k)} \) is defined in accordance with (22) and
\[ R_{p_1 \ldots p_k}(t_1, \ldots, t_k) = K^*(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l), \]
where
\[ \zeta_{j_l}^{(i_l)} = \int_{T}^{t} \phi_{j_l}(s) d\mathbf{w}_{l}^{(i_l)}. \]

5. Proof of Theorem 4 Using the Combined Approach

From (23) we obtain
\[ J^*[\psi^{(2)}]_{T,t} = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \phi_{j_2}(t_2) + J[R_{p_1 p_2}]_{T,t}^{(2)} \text{ w. p. 1,} \]
where
\[ J[R_{p_1 p_2}]_{T,t}^{(2)} = \int_{t}^{T} \int_{t}^{T} R_{p_1 p_2}(t_1, t_2) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} + \int_{t}^{T} \int_{t}^{T} R_{p_1 p_2}(t_1, t_2) d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_1}^{(i_1)} + \\
+ \mathbf{1}_{\{t_1 = t_2 \neq 0\}} \int_{t}^{T} R_{p_1 p_2}(t_1, t_1) dt_1, \]
\[ R_{p_1 p_2}(t_1, t_2) = K^*(t_1, t_2) - \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_2) \quad (p_1, p_2 < \infty), \]
\[ K^*(t_1, t_2) = K(t_1, t_2) + \frac{1}{2} \mathbf{1}_{\{t_1 = t_2\}} \psi_1(t_1) \psi_2(t_1), \]
where

\[
K(t_1, t_2) = \begin{cases} 
\psi_1(t_1) \psi_2(t_2), & t_1 < t_2 \\
0, & \text{otherwise}
\end{cases}, \quad t_1, t_2 \in [t, T].
\]

Let us consider the case \(i_1, i_2 \neq 0\) (another cases can be considered absolutely analogously). Using standard estimates for moments of stochastic integrals [1], we get

\[
M \left\{ \left( J[R_{p_1p_2}^{(2)}]_{[t,t]} \right)^2 \right\} =
\]

\[
= M \left\{ \left( \int_t^T R_{p_1p_2}(t_1, t_2) dt_1 dt_2 \right)^2 \right\} + 
+ 1_{(i_1 = i_2 \neq 0)} \left( \int_t^T R_{p_1p_2}(t_1, t_1) dt_1 \right)^2
\leq 2 \left( \int_t^T (R_{p_1p_2}(t_1, t_2))^2 dt_1 dt_2 + \int_t^T (R_{p_1p_2}(t_1, t_1))^2 dt_1 dt_1 \right) + 
+ 1_{(i_1 = i_2 \neq 0)} \left( \int_t^T R_{p_1p_2}(t_1, t_1) dt_1 \right)^2
\]

\[
= 2 \int_{[t,T]^2} (R_{p_1p_2}(t_1, t_2))^2 dt_1 dt_2 + 1_{(i_1 = i_2 \neq 0)} \left( \int_t^T R_{p_1p_2}(t_1, t_1) dt_1 \right)^2.
\]

Moreover, we have

\[
\int_{[t,T]^2} (R_{p_1p_2}(t_1, t_2))^2 dt_1 dt_2 =
\]

\[
= \int_{[t,T]^2} \left( K^*(t_1, t_2) - \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_2) \right)^2 dt_1 dt_2 = 
\]

\[
= \int_{[t,T]^2} \left( K(t_1, t_2) - \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_2) \right)^2 dt_1 dt_2.
\]

The function \(K(t_1, t_2)\) is piecewise continuous in the square \([t,T]^2\). At this situation it is well known that the generalized multiple Fourier series of the function \(K(t_1, t_2) \in L_2([t,T]^2)\) is converging to this function in the square \([t,T]^2\) in the mean-square sense, i.e.
\[
\lim_{p_1, p_2 \to \infty} \left\| K(t_1, t_2) - \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2 j_1} \prod_{l=1}^{2} \phi_{j_l} (t_l) \right\|_{L_2([t, T]^2)} = 0,
\]

where
\[
\|f\|_{L_2([t, T]^2)} = \left( \int_{[t, T]^2} f^2(t_1, t_2) dt_1 dt_2 \right)^{1/2}.
\]

So, we obtain
\[
\lim_{p_1, p_2 \to \infty} \int_{[t, T]^2} (R_{p_1, p_2}(t_1, t_2))^2 dt_1 dt_2 = 0.
\]

Note that
\[
\int_{t}^{T} R_{p_1, p_2}(t_1, t_1) dt_1 =
\]
\[
= \int_{t}^{T} \left( \frac{1}{2} \psi_1(t_1) \psi_2(t_1) - \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2 j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_1) \right) dt_1 =
\]
\[
= \frac{1}{2} \int_{t}^{T} \psi_1(t_1) \psi_2(t_1) dt_1 - \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2 j_1} \int_{t}^{T} \phi_{j_1}(t_1) \phi_{j_2}(t_1) dt_1 =
\]
\[
= \frac{1}{2} \int_{t}^{T} \psi_1(t_1) \psi_2(t_1) dt_1 - \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} C_{j_2 j_1} 1_{j_1 = j_2} =
\]
\[
= \frac{1}{2} \int_{t}^{T} \psi_1(t_1) \psi_2(t_1) dt_1 - \sum_{j_1 = 0}^{\min(p_1, p_2)} C_{j_1 j_1}.
\]

In [19] (Theorem 3, p. A.59), [24] (Theorem 5.3, p. A.294), [25, 27] (Theorems 2.1, 2.2), [43] (Theorem 2), [44] (Theorem 6) the following equality
\[
\int_{t}^{T} \psi_1(t_1) \psi_2(t_1) dt_1 = \sum_{j_1 = 0}^{\infty} C_{j_1 j_1},
\]
is proved. Note that the existence of the limit on the right-hand side of (39) is proved in [25, 27], [43] for the polynomial and trigonometric cases.

From (36)–(39) it follows that
\[
\lim_{p_1, p_2 \to \infty} M \left\{ \left( J[R_{p_1, p_2}]_{T,t}^{(2)} \right)^2 \right\} = 0.
\]
Theorem 4 is proved.

6. Proof of Theorem 5 Using the Combined Approach

Let us consider (33) for $k = 3$ and $p_1 = p_2 = p_3 = p$

(40) \[ J^*[\psi(t)]_{t,t} = \sum_{j_1=0}^{p} \sum_{j_2=0}^{p} \sum_{j_3=0}^{p} C_{j_3,j_2,j_1} \phi_1^{(i_1)} \phi_2^{(i_2)} \phi_3^{(i_3)} + J[R_{ppp}]^{(3)}_{t,t} \text{ w. p. 1}, \]

where

\[ J[R_{ppp}]^{(3)}_{t,t} = \lim_{N \to \infty} \sum_{l_1=0}^{N-1} \sum_{l_2=0}^{N-1} \sum_{l_3=0}^{N-1} R_{ppp}(\tau_1, \tau_2, \tau_3) \Delta f^{(i_1)}_{\tau_1} \Delta f^{(i_2)}_{\tau_2} \Delta f^{(i_3)}_{\tau_3}, \]

\[ R_{ppp}(t_1, t_2, t_3) = \text{def} \{ t_1 \cup t_2 \cup t_3 \} = \frac{1}{2} 1_{\{t_1 = t_2\}} 1_{\{t_2 = t_3\}} + \frac{1}{4} 1_{\{t_1 = t_2 = t_3\}} \]

Furthermore, we have w. p. 1

\[ J[R_{ppp}]^{(3)}_{t,t} = \lim_{N \to \infty} \sum_{l_1=0}^{N-1} \sum_{l_2=0}^{N-1} \sum_{l_3=0}^{N-1} R_{ppp}(\tau_1, \tau_2, \tau_3) \Delta f^{(i_1)}_{\tau_1} \Delta f^{(i_2)}_{\tau_2} \Delta f^{(i_3)}_{\tau_3} = \]

\[ = \lim_{N \to \infty} \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{N-1} \sum_{i_3=0}^{N-1} \left( R_{ppp}(\tau_1, \tau_2, \tau_3) \Delta f^{(i_1)}_{\tau_1} \Delta f^{(i_2)}_{\tau_2} \Delta f^{(i_3)}_{\tau_3} + \right. \]

\[ + R_{ppp}(\tau_1, \tau_2, \tau_3) \Delta f^{(i_1)}_{\tau_1} \Delta f^{(i_2)}_{\tau_2} \Delta f^{(i_3)}_{\tau_3} + \]

\[ + R_{ppp}(\tau_1, \tau_2, \tau_3) \Delta f^{(i_1)}_{\tau_1} \Delta f^{(i_2)}_{\tau_2} \Delta f^{(i_3)}_{\tau_3} + \]

\[ + R_{ppp}(\tau_1, \tau_2, \tau_3) \Delta f^{(i_1)}_{\tau_1} \Delta f^{(i_2)}_{\tau_2} \Delta f^{(i_3)}_{\tau_3} + \]

\[ + R_{ppp}(\tau_1, \tau_2, \tau_3) \Delta f^{(i_1)}_{\tau_1} \Delta f^{(i_2)}_{\tau_2} \Delta f^{(i_3)}_{\tau_3} + \]
\[ + R_{ppp}(\tau_{t_2}, \tau_{t_2}, \tau_{t_2}) \Delta f_{\tau_{t_3}}^{(i_1)} \Delta f_{\tau_{t_2}}^{(i_2)} \Delta f_{\tau_{t_2}}^{(i_3)} \] + \\
+ \text{l.i.m.}_{N \to \infty} \sum_{i_3=0}^{N-1} \sum_{i_2=0}^{N-1} \left( R_{ppp}(\tau_{t_2}, \tau_{t_2}, \tau_{t_3}) \Delta f_{\tau_{t_3}}^{(i_1)} \Delta f_{\tau_{t_2}}^{(i_2)} \Delta f_{\tau_{t_2}}^{(i_3)} \right) + \\
\text{l.i.m.}_{N \to \infty} \sum_{i_3=0}^{N-1} \left( R_{ppp}(\tau_{t_2}, \tau_{t_2}, \tau_{t_3}) \Delta f_{\tau_{t_3}}^{(i_1)} \Delta f_{\tau_{t_2}}^{(i_2)} \right) + \\
\text{l.i.m.}_{N \to \infty} \sum_{i_3=0}^{N-1} \left( R_{ppp}(\tau_{t_2}, \tau_{t_2}, \tau_{t_3}) \Delta f_{\tau_{t_3}}^{(i_1)} \right) + \\
\text{l.i.m.}_{N \to \infty} \sum_{i_3=0}^{N-1} R_{ppp}(\tau_{t_2}, \tau_{t_2}, \tau_{t_3}) \Delta f_{\tau_{t_3}}^{(i_1)} \Delta f_{\tau_{t_3}}^{(i_2)} \Delta f_{\tau_{t_3}}^{(i_3)} = \\
= R_{T,t}^{(1)ppp} + R_{T,t}^{(2)ppp},
\]

where

\[ R_{T,t}^{(1)ppp} = \]

\[ = \int_{t}^{t_3} \int_{t}^{t_2} \int_{t}^{t_2} R_{ppp}(t_1, t_2, t_3) \, df_{t_1}^{(i_1)} \, df_{t_2}^{(i_2)} \, df_{t_3}^{(i_3)} + \int_{t}^{t_3} \int_{t}^{t_2} \int_{t}^{t_2} R_{ppp}(t_1, t_3, t_2) \, df_{t_1}^{(i_1)} \, df_{t_2}^{(i_2)} \, df_{t_3}^{(i_3)} + \]

\[ + \int_{t}^{t_3} \int_{t}^{t_2} \int_{t}^{t_2} R_{ppp}(t_2, t_1, t_3) \, df_{t_2}^{(i_1)} \, df_{t_2}^{(i_2)} \, df_{t_3}^{(i_3)} + \int_{t}^{t_3} \int_{t}^{t_2} \int_{t}^{t_2} R_{ppp}(t_2, t_3, t_1) \, df_{t_2}^{(i_1)} \, df_{t_2}^{(i_2)} \, df_{t_3}^{(i_3)} + \]

\[ + \int_{t}^{t_3} \int_{t}^{t_2} \int_{t}^{t_2} R_{ppp}(t_3, t_2, t_1) \, df_{t_3}^{(i_1)} \, df_{t_2}^{(i_2)} \, df_{t_1}^{(i_3)} + \int_{t}^{t_3} \int_{t}^{t_2} \int_{t}^{t_2} R_{ppp}(t_3, t_1, t_2) \, df_{t_3}^{(i_1)} \, df_{t_2}^{(i_2)} \, df_{t_1}^{(i_3)} , \]

\[ R_{T,t}^{(2)ppp} = \]
We have

\[
\int_t^T \int_t^{t_2} \int_t^{t_3} R_{ppp}(t_2, t_2, t_3) dt_2 dt_3 dt_4^{(i)} + 1_{\{i_1 = i_2 \neq 0\}} \int_t^T \int_t^{t_3} R_{ppp}(t_2, t_2, t_3) dt_2 dt_3^{(i_2)} + \\
+ 1_{\{i_2 = i_3 \neq 0\}} \int_t^T \int_t^{t_3} R_{ppp}(t_3, t_2, t_2) dt_2 dt_3^{(i_1)} + 1_{\{i_2 = i_3 \neq 0\}} \int_t^T \int_t^{t_3} R_{ppp}(t_1, t_3, t_3) dt_1 dt_3^{(i_1)} dt_3 + \\
+ 1_{\{i_1 = i_3 \neq 0\}} \int_t^T \int_t^{t_3} R_{ppp}(t_3, t_1, t_3) dt_1 dt_3^{(i_2)} + 1_{\{i_1 = i_2 \neq 0\}} \int_t^T \int_t^{t_3} R_{ppp}(t_3, t_3, t_1) dt_1 dt_3^{(i_3)} dt_3.
\]

Moreover, we obtain

\[
M \left\{ \left( J[R_{ppp}]_{T,t}^{(3)} \right)^2 \right\} \leq 2M \left\{ \left( R_{T,t}^{(1)} \right)^2 \right\} + 2M \left\{ \left( R_{T,t}^{(2)} \right)^2 \right\}.
\]

Now, using standard estimates for moments of stochastic integrals [1], we obtain the following inequality

\[
M \left\{ \left( R_{T,t}^{(1)} \right)^2 \right\} \leq \\
\leq 6 \int_t^T \int_t^{t_2} \int_t^{t_3} \left( R_{p_1p_2p_3}(t_1, t_2, t_3) \right)^2 + \left( R_{p_1p_2p_3}(t_1, t_3, t_2) \right)^2 + \left( R_{p_1p_2p_3}(t_2, t_1, t_3) \right)^2 + \\
+ \left( R_{p_1p_2p_3}(t_2, t_3, t_1) \right)^2 + \left( R_{p_1p_2p_3}(t_3, t_2, t_1) \right)^2 + \left( R_{p_1p_2p_3}(t_3, t_1, t_2) \right)^2 dt_1 dt_2 dt_3 = \\
= 6 \int_{[t,T]^3} \left( R_{ppp}(t_1, t_2, t_3) \right)^2 dt_1 dt_2 dt_3.
\]

We have

\[
\int_{[t,T]^3} \left( R_{ppp}(t_1, t_2, t_3) \right)^2 dt_1 dt_2 dt_3 = \\
\int_{[t,T]^3} \left( K^*(t_1, t_2, t_3) - \sum_{j_1=0}^p \sum_{j_2=0}^p \sum_{j_3=0}^p C_{j_3j_2j_1}(t_1) \phi_{j_1}(t_2) \phi_{j_2}(t_3) \right)^2 dt_1 dt_2 dt_3 = \\
= \int_{[t,T]^3} \left( K(t_1, t_2, t_3) - \sum_{j_1=0}^p \sum_{j_2=0}^p \sum_{j_3=0}^p C_{j_3j_2j_1}(t_1) \phi_{j_1}(t_2) \phi_{j_2}(t_3) \right)^2 dt_1 dt_2 dt_3,
\]
where
\[
K(t_1, t_2, t_3) = \begin{cases} 
\psi_1(t_1)\psi_2(t_2)\psi_3(t_3), & t_1 < t_2 < t_3 \\
0, & \text{otherwise}
\end{cases}, \quad t_1, t_2, t_3 \in [t, T].
\]

So, we get
\[
\lim_{p \to \infty} M\left\{\left(R_{T,t}^{(1)ppp}\right)^2\right\} \leq 6 \lim_{p \to \infty} \int_{[t,T]^3} \left(K(t_1, t_2, t_3) - \sum_{j_1=0}^p \sum_{j_2=0}^p \sum_{j_3=0}^p C_{j_1 j_2 j_3} \phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3)\right)^2 dt_1 dt_2 dt_3 = 0,
\]

where \(K(t_1, t_2, t_3) \in L_2([t,T]^3)\).

After the integration order replacement in iterated Ito stochastic integrals \[16\] (also see \[19, 24\] or Chapter 3 in \[25, 27\]) from \(R_{T,t}^{(2)ppp}\) we obtain w. p. 1
\[
R_{T,t}^{(2)ppp} =
\]
\[
= 1_{\{i_1 = i_2 \neq 0\}} \left( \int_t^T R_{ppp}(t_2, t_2, t_3) dt_2 dt_3 \right) + \int_t^T R_{ppp}(t_3, t_3, t_1) dt_3 \right) + 
\]
\[
+ 1_{\{i_2 = i_3 \neq 0\}} \left( \int_t^T R_{ppp}(t_3, t_2, t_2) dt_2 dt_3 \right) + \int_t^T R_{ppp}(t_3, t_3, t_1) dt_3 \right) + 
\]
\[
+ 1_{\{i_1 = i_3 \neq 0\}} \left( \int_t^T R_{ppp}(t_2, t_2, t_3) dt_2 dt_3 \right) + \int_t^T R_{ppp}(t_3, t_3, t_1) dt_3 \right) = 
\]
\[
= 1_{\{i_1 = i_2 \neq 0\}} \left( \int_t^T R_{ppp}(t_2, t_2, t_1) dt_2 dt_3 \right) + \int_t^T R_{ppp}(t_2, t_2, t_1) dt_2 dt_3 = 
\]
\[
+ 1_{\{i_2 = i_3 \neq 0\}} \left( \int_t^T R_{ppp}(t_1, t_2, t_2) dt_2 dt_3 \right) + \int_t^T R_{ppp}(t_2, t_2, t_1) dt_2 dt_3 = 
\]
\[
+ 1_{\{i_1 = i_3 \neq 0\}} \left( \int_t^T R_{ppp}(t_2, t_1, t_2) dt_2 dt_3 \right) + \int_t^T R_{ppp}(t_2, t_2, t_1) dt_2 dt_3 = 
\]
\[ \begin{align*}
&= 1_{\{i_1 = i_2 \neq 0\}} \int_t^T \left( \int_t^T R_{ppp}(t_2, t_2, t_3) dt_2 \right) df_t^{(i_3)} + \\
&+ 1_{\{i_2 = i_3 \neq 0\}} \int_t^T \left( \int_t^T R_{ppp}(t_1, t_2, t_2) dt_2 \right) df_t^{(i_1)} + \\
&+ 1_{\{i_1 = i_3 \neq 0\}} \int_t^T \left( \int_t^T R_{ppp}(t_3, t_2, t_3) dt_3 \right) df_t^{(i_2)} = \\
&= 1_{\{i_1 = i_2 \neq 0\}} \int_t^T \int_t^T \left( \left( \frac{1}{2} 1_{\{t_2 < t_3\}} + \frac{1}{4} 1_{\{t_2 = t_3\}} \right) \psi_1(t_2) \psi_2(t_2) \psi_3(t_3) - \\
&- \sum_{j_1=0}^p \sum_{j_2=0}^p \sum_{j_3=0}^p C_{j_3,j_2,j_1} \phi_{j_1}(t_2) \phi_{j_2}(t_2) \phi_{j_3}(t_3) \right) dt_2 dt_3 df_t^{(i_3)} + \\
&+ 1_{\{i_2 = i_3 \neq 0\}} \int_t^T \int_t^T \left( \left( \frac{1}{2} 1_{\{t_2 < t_2\}} + \frac{1}{4} 1_{\{t_1 = t_2\}} \right) \psi_1(t_1) \psi_2(t_2) \psi_3(t_2) - \\
&- \sum_{j_1=0}^p \sum_{j_2=0}^p \sum_{j_3=0}^p C_{j_3,j_2,j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_2) \phi_{j_3}(t_2) \right) dt_2 df_t^{(i_1)} + \\
&+ 1_{\{i_1 = i_3 \neq 0\}} \int_t^T \int_t^T \left( \frac{1}{4} 1_{\{t_2 = t_3\}} \psi_1(t_3) \psi_2(t_2) \psi_3(t_3) - \\
&- \sum_{j_1=0}^p \sum_{j_2=0}^p \sum_{j_3=0}^p C_{j_3,j_2,j_1} \phi_{j_1}(t_3) \phi_{j_2}(t_2) \phi_{j_3}(t_3) \right) dt_3 df_t^{(i_2)} = \\
&= 1_{\{i_1 = i_2 \neq 0\}} \int_t^T \left( \frac{1}{2} \psi_3(t_3) \int_t^T \psi_1(t_2) \psi_2(t_2) dt_2 - \sum_{j_1=0}^p \sum_{j_3=0}^p C_{j_3,j_2,j_1} \phi_{j_3}(t_3) \right) df_t^{(i_3)} + \\
&+ 1_{\{i_2 = i_3 \neq 0\}} \int_t^T \left( \frac{1}{2} \psi_1(t_1) \int_t^T \psi_2(t_2) \psi_3(t_2) dt_2 - \sum_{j_1=0}^p \sum_{j_3=0}^p C_{j_3,j_2,j_1} \phi_{j_1}(t_1) \right) df_t^{(i_1)} + \\
&+ 1_{\{i_1 = i_3 \neq 0\}} \int_t^T (-1) \sum_{j_1=0}^p \sum_{j_2=0}^p C_{j_1,j_2,j_3} \phi_{j_1}(t_2) df_t^{(i_2)} =
\end{align*}\]
(Theorem 3) we obtain

\[ \begin{align*}
&= 1_{\{i_1 = i_2 \neq 0\}} \left( \frac{1}{2} \int_t^T \psi_3(t_3) \int_t^{t_3} \psi_1(t_2) \psi_2(t_2) dt_2 df_{i_3}^{(i_3)} - \sum_{j_1 = 0}^p \sum_{j_3 = 0}^p C_{j_3, j_1} \zeta_{i_3}^{(i_3)} \right) + \\
&+ 1_{\{i_2 = i_3 \neq 0\}} \left( \frac{1}{2} \int_t^T \psi_1(t_1) \int_t^{t_1} \psi_2(t_2) \psi_3(t_2) dt_2 df_{i_1}^{(i_1)} - \sum_{j_1 = 0}^p \sum_{j_3 = 0}^p C_{j_3, j_1} \zeta_{i_1}^{(i_1)} \right) - \\
&- 1_{\{i_1 = i_3 \neq 0\}} \sum_{j_1 = 0}^p \sum_{j_3 = 0}^p C_{j_1, j_3, j_1} \zeta_{i_2}^{(i_2)}. \end{align*} \]

From [19] (Theorem 6, pp. A.116–A.117), [24] (Theorem 5.5', p. A.371), [25–27] (Chapter 2), [43] (Theorem 3) we obtain

\[ M \left\{ \left( R_{T, t}^{(2)pp} \right)^2 \right\} \leq \]

\[ \leq 3 \left( 1_{\{i_1 = i_2 \neq 0\}} M \left\{ \left( \frac{1}{2} \int_t^T \psi_3(t_3) \int_t^{t_3} \psi_1(t_2) \psi_2(t_2) dt_2 df_{i_3}^{(i_3)} - \sum_{j_1 = 0}^p \sum_{j_3 = 0}^p C_{j_3, j_1} \zeta_{i_3}^{(i_3)} \right) \right\}^2 + \right. \]

\[ + 1_{\{i_1 = i_3 \neq 0\}} M \left\{ \sum_{j_1 = 0}^p \sum_{j_3 = 0}^p C_{j_1, j_3, j_1} \zeta_{i_2}^{(i_2)} \right\}^2 \] +

\[ + 1_{\{i_2 = i_3 \neq 0\}} M \left\{ \left( \frac{1}{2} \int_t^T \psi_1(t_1) \int_t^{t_1} \psi_2(t_2) \psi_3(t_2) dt_2 df_{i_1}^{(i_1)} - \sum_{j_1 = 0}^p \sum_{j_3 = 0}^p C_{j_3, j_1} \zeta_{i_1}^{(i_1)} \right) \right\}^2 \] \rightarrow 0 \]

if \( p \rightarrow \infty \). Using [49–53], we obtain the expansion [19]. Theorem 5 is proved.

7. PROOF OF THEOREM 6 USING THE COMBINED APPROACH

Let us consider [63] for the case \( k = 4, p_1 = p_2 = p_3 = p_4 = p \), and \( \psi_1(\tau), \psi_2(\tau), \psi_3(\tau), \psi_4(\tau) \equiv 1 \)

\[ \int \int \int \int dw_{t_1}^{(i_1)} dw_{t_2}^{(i_2)} dw_{t_3}^{(i_3)} dw_{t_4}^{(i_4)} = \]

\[ = \sum_{j_1 = 0}^p \sum_{j_2 = 0}^p \sum_{j_3 = 0}^p \sum_{j_4 = 0}^p C_{j_4, j_3, j_2, j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} + J[R_{pppp}]_{T, t} \] w. p. 1,
where

\[ J[R_{pppp}]_{T,t}^{(4)} = \lim_{N \to \infty} \sum_{l_4=0}^{N-1} \sum_{l_3=0}^{N-1} \sum_{l_2=0}^{N-1} \sum_{l_1=0}^{N-1} R_{pppp}(\tau_{l_1}, \tau_{l_2}, \tau_{l_3}, \tau_{l_4}) \Delta w_{\tau_{l_1}} \Delta w_{\tau_{l_2}} \Delta w_{\tau_{l_3}} \Delta w_{\tau_{l_4}}, \]

\[ R_{pppp}(t_1, t_2, t_3, t_4) \overset{\text{def}}{=} K^*(t_1, t_2, t_3, t_4) - \]

\[ - \sum_{j_1=0}^{P} \sum_{j_2=0}^{P} \sum_{j_3=0}^{P} \sum_{j_4=0}^{P} C_{j_4j_3j_2j_1} \phi_{j_1} \phi_{j_2} \phi_{j_3} \phi_{j_4}(t_4), \]

\[ K^*(t_1, t_2, t_3, t_4) \overset{\text{def}}{=} \prod_{l=1}^{3} \left( 1_{\{t_l < t_{l+1}\}} + \frac{1}{2} 1_{\{t_l = t_{l+1}\}} \right) = \]

\[ = 1_{\{t_1 < t_2 < t_3 < t_4\}} + \frac{1}{2} 1_{\{t_1 = t_2 < t_3 < t_4\}} + \frac{1}{2} 1_{\{t_1 < t_2 = t_3 < t_4\}} + \]

\[ + \frac{1}{4} 1_{\{t_1 = t_2 = t_3 < t_4\}} + \frac{1}{2} 1_{\{t_1 < t_2 < t_3 = t_4\}} + \frac{1}{4} 1_{\{t_1 = t_2 < t_3 = t_4\}} + \]

\[ + \frac{1}{4} 1_{\{t_1 < t_2 = t_3 = t_4\}} + \frac{1}{8} 1_{\{t_1 = t_2 = t_3 = t_4\}}. \]

Moreover, we have

\[ J[R_{pppp}]_{T,t}^{(4)} = \sum_{i=0}^{7} R^{(i)}_{pppp} \quad \text{w. p. 1}, \]

where

\[ R^{(0)}_{pppp} = \lim_{N \to \infty} \sum_{l_4=0}^{N-1} \sum_{l_3=0}^{N-1} \sum_{l_2=0}^{N-1} \sum_{l_1=0}^{N-1} \sum_{(l_1,l_2,l_3,l_4)} \left( R_{pppp}(\tau_{l_1}, \tau_{l_2}, \tau_{l_3}, \tau_{l_4}) \times \right) \]

\[ \times \Delta w_{\tau_{l_1}} \Delta w_{\tau_{l_2}} \Delta w_{\tau_{l_3}} \Delta w_{\tau_{l_4}}, \]

where permutations \((l_1, l_2, l_3, l_4)\) when summing are performed only in the expression, which is enclosed in parentheses,

\[ R^{(1)}_{pppp} = 1_{\{i_1 = i_2 \neq 0\}} \lim_{N \to \infty} \sum_{l_4=0}^{N-1} \sum_{l_3=0}^{N-1} \sum_{l_2=0}^{N-1} \sum_{l_1=0}^{N-1} \sum_{(l_1,l_2,l_3,l_4)} \left( R_{pppp}(\tau_{l_1}, \tau_{l_2}, \tau_{l_3}, \tau_{l_4}) \Delta n_{l_1} \Delta w_{\tau_{l_3}} \Delta w_{\tau_{l_4}}, \right) \]

\[ R^{(2)}_{pppp} = 1_{\{i_1 = i_3 \neq 0\}} \lim_{N \to \infty} \sum_{l_4=0}^{N-1} \sum_{l_3=0}^{N-1} \sum_{l_2=0}^{N-1} \sum_{l_1=0}^{N-1} \sum_{(l_1,l_2,l_3,l_4)} \left( R_{pppp}(\tau_{l_1}, \tau_{l_2}, \tau_{l_3}, \tau_{l_4}) \Delta n_{l_1} \Delta w_{\tau_{l_2}} \Delta w_{\tau_{l_4}}, \right). \]
The relations (44) and (45) imply that Theorem 6 will be proved if

\[\lim_{p \to \infty} M \left\{ \left( R_{T,t}^{(i)} \right)^2 \right\} = 0, \quad i = 0, 1, \ldots, 7.\]

We have (see 241)

\[R_{T,t}^{(0)} = \int_t^T \int_t^{t_3} \int_t^{t_2} \int_t^{t_4} \sum_{i=1}^{N-1} R_{ppp}(\tau_{i_1}, \tau_{i_2}, \tau_{i_3}, \tau_{i_4}) \Delta \tau_{i_1} \Delta \tau_{i_2} \Delta \tau_{i_3} \Delta \tau_{i_4},\]

where permutations \((t_1, t_2, t_3, t_4)\) when summing are performed only in the expression, which is enclosed in parentheses.

From the other hand \([19, 241-271, 35]\)

\[R_{T,t}^{(0)} = \sum_{(i_1, i_2, i_3, i_4)} \int_t^T \int_t^{t_3} \int_t^{t_2} \int_t^{t_4} R_{ppp}(t_1, t_2, t_3, t_4) d\tau_{i_1} d\tau_{i_2} d\tau_{i_3} d\tau_{i_4}.\]
where permutations \((t_1, \ldots, t_4)\) when summing are performed only in the values \(dw_{t_1} \ldots dw_{t_4}\). At the same time, the indexes near upper limits of integration in the iterated stochastic integrals are changed correspondently and if \(t_r\) swapped with \(t_q\) in the permutation \((t_1, \ldots, t_4)\), then \(t_r\) swapped with \(t_q\) in the permutations \((t_1, \ldots, t_4)\).

So, we obtain

\[
M \left\{ \left( K_{T,t}^{(0)pppp} \right)^2 \right\} \leq 24 \sum_{(t_1,t_2,t_3,t_4)} \int_t^T \int_t^T \int_t^T \int_t^T (R_{pppp}(t_1, t_2, t_3, t_4))^2 \, dt_1 dt_2 dt_3 dt_4 =
\]

\[
= 24 \int_{[t,T]^4} (R_{pppp}(t_1, t_2, t_3, t_4))^2 \, dt_1 dt_2 dt_3 dt_4 \to 0
\]

if \(p \to \infty\), where \(K^*_{1}(t_1, t_2, t_3, t_4) \in L_2([t, T]^4)\).

Let us consider \(R_{T,t}^{(1)pppp}\)

\[
R_{T,t}^{(1)pppp} = 1_{\{i_1=i_2\neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{i_4 \neq i_3 \neq 0} \sum_{N-1} R_{pppp}(\tau_1, \tau_1, \tau_3, \tau_4) \Delta \tau_1 \Delta \tau_3 \Delta w_{\tau_3}^{(i_3)} \Delta w_{\tau_4}^{(i_4)} =
\]

\[
= 1_{\{i_1=i_2\neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{i_4 \neq i_3 \neq 0} \sum_{N-1} \left( \frac{1}{2} 1_{\{\tau_1 < \tau_3 < \tau_4\}} + \frac{1}{4} 1_{\{\tau_1 < \tau_3 = \tau_4\}} + \frac{1}{8} 1_{\{\tau_1 = \tau_3 = \tau_4\}} \right) -
\]

\[
- \sum_{j_4,j_3,j_2,j_1=0}^P C_{j_4,j_3,j_2,j_1} \phi_{j_1}(\tau_1) \phi_{j_2}(\tau_1) \phi_{j_3}(\tau_3) \phi_{j_4}(\tau_4) \Delta \tau_1 \Delta w_{\tau_3}^{(i_3)} \Delta w_{\tau_4}^{(i_4)} =
\]

\[
= 1_{\{i_1=i_2\neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{i_4 \neq i_3 \neq 0} \sum_{N-1} \left( \frac{1}{2} 1_{\{\tau_1 < \tau_3 < \tau_4\}} \right) -
\]

\[
- \sum_{j_4,j_3,j_2,j_1=0}^P C_{j_4,j_3,j_2,j_1} \phi_{j_1}(\tau_1) \phi_{j_2}(\tau_1) \phi_{j_3}(\tau_3) \phi_{j_4}(\tau_4) \Delta \tau_1 \Delta w_{\tau_3}^{(i_3)} \Delta w_{\tau_4}^{(i_4)} =
\]
\[-1\{i_1 = i_2 \neq 0\} \{i_3 = i_4 \neq 0\} \lim_{N \to \infty} \sum_{i_4 = 0}^{N-2} \sum_{i_1 = 0}^{N-2} \begin{pmatrix} 0 - \\
\end{pmatrix} \]

\[- \sum_{j_4, j_3, j_2, j_1 = 0}^{p} C_{j_4 j_3 j_2 j_1} \phi_{j_1}(\tau_{i_1}) \phi_{j_2}(\tau_{i_1}) \phi_{j_3}(\tau_{i_4}) \phi_{j_4}(\tau_{i_4}) \Delta \tau_{i_1} \Delta \tau_{i_4} = \\
= 1\{i_1 = i_2 \neq 0\} \left( \frac{1}{2} \int_{t}^{T} \int_{t}^{T} dt_1 d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} - \sum_{j_4, j_3, j_1 = 0}^{p} C_{j_4 j_3 j_1} \xi_{j_3}^{(i_3)} \xi_{j_4}^{(i_4)} \right) + \\
+ 1\{i_1 = i_2 \neq 0\} \{i_3 = i_4 \neq 0\} \sum_{j_4, j_1 = 0}^{p} C_{j_4 j_1} \text{ w. p. 1.} \]

In [19] (see the proof of Theorem 8, p. A.135), [24] (see the proof of Theorem 5.7, p. A.388), [25] [27] (see Chapter 2), [32] (see the proof of Theorem 4) we have proved that

\[
\lim_{p \to \infty} \sum_{j_4, j_3, j_1 = 0}^{p} C_{j_4 j_3 j_1} \Delta \tau_{i_1} \Delta \tau_{i_4} = \\
= \frac{1}{4} \int_{t}^{T} \int_{t}^{T} dt_1 dt_2, \]

\[
\lim_{p \to \infty} \sum_{j_4, j_3, j_1 = 0}^{p} C_{j_4 j_3 j_1} \xi_{j_3}^{(i_3)} \xi_{j_4}^{(i_4)} = \\
= \frac{1}{4} \int_{t}^{T} \int_{t}^{T} dt_1 d\mathbf{w}_{t_3}^{(i_3)} d\mathbf{w}_{t_4}^{(i_4)} + \\
+ 1\{i_1 = i_2 \neq 0\} \frac{1}{4} \int_{t}^{T} \int_{t}^{T} dt_1 dt_2 \text{ w. p. 1.} \]

Then

\[
\lim_{p \to \infty} M \left\{ \left( R_{T, t}^{(1)pppp} \right)^2 \right\} = 0. \]

Let us consider $R_{T, t}^{(2)pppp}$

\[
R_{T, t}^{(2)pppp} = 1\{i_1 = i_3 \neq 0\} \lim_{N \to \infty} \sum_{i_4 = 0}^{N-1} \sum_{i_1 \neq i_2 \neq i_3 \neq i_4}^{N-1} G_{pppp}(\tau_{i_1}, \tau_{i_2}, \tau_{i_3}, \tau_{i_4}) \Delta \tau_{i_2} \Delta \tau_{i_4} = \\
= 1\{i_1 = i_3 \neq 0\} \lim_{N \to \infty} \sum_{i_4 = 0}^{N-1} \sum_{i_1 \neq i_2 \neq i_3 \neq i_4}^{N-1} G_{pppp}(\tau_{i_1}, \tau_{i_2}, \tau_{i_3}, \tau_{i_4}) \Delta \tau_{i_2} \Delta \tau_{i_4} = \\
= 1\{i_1 = i_3 \neq 0\} \lim_{N \to \infty} \sum_{i_4 = 0}^{N-1} \sum_{i_1 \neq i_2 \neq i_3 \neq i_4}^{N-1} \left( \frac{1}{4} 1\{\tau_{i_1} < \tau_{i_2} < \tau_{i_4}\} + \frac{1}{8} 1\{\tau_{i_1} = \tau_{i_2} = \tau_{i_4}\} \right) - 
\]
\[- \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1} \phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_4}) \Delta \tau_{l_1} \Delta w^{(i_2)}_{\tau_{l_2}} \Delta w^{(i_4)}_{\tau_{l_4}} = \]

\[= 1\{i_1=i_3 \neq 0\} \lim_{N \to \infty} \sum_{l_4=0}^{N-1} \sum_{l_2=0}^{N-1} \sum_{l_1=0}^{N-1} (-1)^{l_1} \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1} \times \]

\[\times \phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_4}) \Delta \tau_{l_1} \Delta w^{(i_2)}_{\tau_{l_2}} \Delta w^{(i_4)}_{\tau_{l_4}} - \]

\[-1\{i_1=i_3 \neq 0\} 1\{i_2=i_4 \neq 0\} \lim_{N \to \infty} \sum_{l_4=0}^{N-1} \sum_{l_2=0}^{N-1} \sum_{l_1=0}^{N-1} (-1)^{l_1} \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1} \times \]

\[\times \phi_{j_1}(\tau_{l_1})\phi_{j_2}(\tau_{l_2})\phi_{j_3}(\tau_{l_3})\phi_{j_4}(\tau_{l_4}) \Delta \tau_{l_1} \Delta \tau_{l_4} = \]

\[= -1\{i_1=i_3 \neq 0\} \sum_{j_4,j_2,j_1=0}^{p} C_{j_4j_2j_1} \phi_{j_2}^{(i_2)} \phi_{j_4}^{(i_4)} + \]

\[+1\{i_1=i_3 \neq 0\} 1\{i_2=i_4 \neq 0\} \sum_{j_4,j_1=0}^{p} C_{j_4j_1} \text{ w. p. } 1. \]

In [19] (see the proof of Theorem 8, p. A.135), [24] (see the proof of Theorem 5.7, p. A.388), [25-27] (see Chapter 2), [33] (see the proof of Theorem 4) we have proved that

\[\lim_{p \to \infty} \sum_{j_4,j_1=0}^{p} C_{j_4j_1} = 0. \]

Then

\[\lim_{p \to \infty} \mathbf{M} \left\{ \left( R^{(2)pppp}_{T,t} \right)^2 \right\} = 0. \]

Let us consider $R^{(3)pppp}_{T,t}$

\[R^{(3)pppp}_{T,t} = 1\{i_1=i_4 \neq 0\} \lim_{N \to \infty} \sum_{l_3,l_2,l_1=0}^{N-1} G_{pppp}(\tau_{l_1}, \tau_{l_2}, \tau_{l_3}, \tau_{l_4}) \Delta \tau_{l_1} \Delta w^{(i_2)}_{\tau_{l_2}} \Delta w^{(i_4)}_{\tau_{l_4}} = \]

\[= 1\{i_1=i_4 \neq 0\} \lim_{N \to \infty} \sum_{l_3,l_2,l_1=0}^{N-1} G_{pppp}(\tau_{l_1}, \tau_{l_2}, \tau_{l_3}, \tau_{l_4}) \Delta \tau_{l_1} \Delta w^{(i_2)}_{\tau_{l_2}} \Delta w^{(i_4)}_{\tau_{l_4}} = \]
Then let us consider \( R \)

In [19] (see the proof of Theorem 8, p. A.135), [24] (see the proof of Theorem 5.7, p. A.388), [25] [27] (see Chapter 2), [38] (see the proof of Theorem 4) we have proved that

\[
\lim_{p \to \infty} \sum_{j_4, j_3, j_2 = 0}^p C_{j_4 j_3 j_2} = 0 \text{ w. p. 1,}
\]

Then

\[
\lim_{p \to \infty} \mathcal{M} \left\{ \left( R_{T,t}^{(3)pppp} \right)^2 \right\} = 0.
\]

Let us consider \( R_{T,t}^{(4)pppp} \)

\[
R_{T,t}^{(4)pppp} = 1_{\{i_2 = i_3 \neq 0\}} \text{l.i.m.} \sum_{N \to \infty}^{N-1} G_{pppp}(T_{i_1}, T_{i_2}, T_{i_3}, T_{i_4}) \Delta w_{T_{i_1}} \Delta T_{i_2} \Delta w_{T_{i_4}} =
\]
\[= 1_{\{i_2=i_3\neq 0\}} \lim_{N \to \infty} \sum_{\substack{i_4, i_2, i_1 = 0 \\ i_1 \neq i_4}}^{N-1} G_{pppp}(\tau_1, \tau_2, \tau_2, \tau_4) \Delta w_{\tau_1}^{(i_1)} \Delta w_{\tau_2}^{(i_2)} \Delta w_{\tau_4}^{(i_4)} =
\]

\[= 1_{\{i_2=i_3\neq 0\}} \lim_{N \to \infty} \sum_{\substack{i_4, i_2, i_1 = 0 \\ i_1 \neq i_4}}^{N-1} \left( \frac{1}{2} 1_{\{\tau_1 < \tau_2 < \tau_4\}} + \frac{1}{4} 1_{\{\tau_1 = \tau_2 < \tau_4\}} + \frac{1}{8} 1_{\{\tau_1 = \tau_2 = \tau_4\}} \right)
\]

\[-\sum_{j_4, j_3, j_2, j_1 = 0}^{p} C_{j_4, j_3, j_2, j_1} \phi_{j_1}(\tau_1) \phi_{j_2}(\tau_2) \phi_{j_3}(\tau_2) \phi_{j_4}(\tau_4) \Delta w_{\tau_1}^{(i_1)} \Delta w_{\tau_2}^{(i_2)} \Delta w_{\tau_4}^{(i_4)} =
\]

\[= 1_{\{i_2=i_3\neq 0\}} \lim_{N \to \infty} \sum_{\substack{i_4, i_2, i_1 = 0 \\ i_1 \neq i_4}}^{N-1} \left( \frac{1}{2} 1_{\{\tau_1 < \tau_2 < \tau_4\}} \right)
\]

\[-\sum_{j_4, j_3, j_2, j_1 = 0}^{p} C_{j_4, j_3, j_2, j_1} \phi_{j_1}(\tau_1) \phi_{j_2}(\tau_2) \phi_{j_3}(\tau_2) \phi_{j_4}(\tau_4) \Delta w_{\tau_1}^{(i_1)} \Delta w_{\tau_2}^{(i_2)} \Delta w_{\tau_4}^{(i_4)} =
\]

\[-1_{\{i_2=i_3\neq 0\}} \sum_{i_4=0}^{p} \sum_{i_2=0}^{p} (-1)^i C_{j_4, j_3, j_2, j_1} \phi_{j_1}(\tau_1) \phi_{j_2}(\tau_2) \phi_{j_3}(\tau_2) \phi_{j_4}(\tau_4) \Delta w_{\tau_1}^{(i_1)} \Delta w_{\tau_2}^{(i_2)} \Delta w_{\tau_4}^{(i_4)} =
\]

\[= 1_{\{i_2=i_3\neq 0\}} \left( \frac{1}{2} \int_{t}^{T} \int_{t}^{t_{i_2}} \int_{t}^{t_{i_4}} dw_{i_1}^{(i_1)} dt_{i_2} dw_{i_4}^{(i_4)} - \sum_{j_4, j_2, j_1 = 0}^{p} C_{j_4, j_2, j_1} \phi_{j_1}(\tau_1) \phi_{j_2}(\tau_2) \phi_{j_4}(\tau_4) \right) +
\]

\[+ 1_{\{i_2=i_3\neq 0\}} \sum_{j_4, j_2 = 0}^{p} C_{j_4, j_2} \phi_{j_4}(\tau_4) \text{ w. p. 1.}
\]

In [19] (see the proof of Theorem 8, p. A.135), [23] (see the proof of Theorem 5.7, p. A.388), [25]–[27] (see Chapter 2), [33] (see the proof of Theorem 4) we have proved that

\[\lim_{p \to \infty} \sum_{j_4, j_2 = 0}^{p} C_{j_4, j_2} = 0,
\]
Let us consider \( R_{t,t} \)

\[
\lim_{p \to \infty} \sum_{j_1, j_2, j_3 = 0}^{N-1} C_{j_1 j_2 j_3} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} = \frac{1}{2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} dw_{t_1}^{(i_1)} dt_2 dw_{t_4}^{(i_4)} \text{ w. p. 1.}
\]

Then

\[
\lim_{p \to \infty} M \left\{ \left( R_{t,t}^{(5)} \right)^{pppp} \right\} = 0.
\]

Let us consider \( R_{t,t}^{(5)} \)

\[
R_{t,t}^{(5)} = \sum_{i_2 = i_4 \neq 0} \lim_{N \to \infty} \sum_{l_3, l_2, l_1 = 0}^{N-1} G_{pppp}(\tau_1, \tau_2, \tau_3, \tau_4) \Delta w_{\tau_1}^{(i_1)} \Delta w_{\tau_2}^{(i_2)} \Delta w_{\tau_3}^{(i_3)} =
\]

\[
= \sum_{i_2 = i_4 \neq 0} \lim_{N \to \infty} \sum_{l_3, l_2, l_1 = 0}^{N-1} \left( \frac{1}{4} \mathbb{1}_{\tau_1 < \tau_2 = \tau_3} + \frac{1}{8} \mathbb{1}_{\tau_1 = \tau_2 = \tau_3} \right) -
\]

\[
- \sum_{j_1, j_2, j_3, j_4 = 0}^{P} C_{j_1 j_2 j_3} \phi_{j_1} (\tau_1) \phi_{j_2} (\tau_2) \phi_{j_3} (\tau_3) \phi_{j_4} (\tau_4) \Delta w_{\tau_1}^{(i_1)} \Delta w_{\tau_2}^{(i_2)} \Delta w_{\tau_3}^{(i_3)} =
\]

\[
= \sum_{i_2 = i_4 \neq 0} \lim_{N \to \infty} \sum_{l_3, l_2, l_1 = 0}^{N-1} (-1) \sum_{j_4, j_3, j_2 = 0}^{P} C_{j_1 j_2 j_3} \times
\]

\[
\phi_{j_1} (\tau_1) \phi_{j_2} (\tau_2) \phi_{j_3} (\tau_3) \phi_{j_4} (\tau_4) \Delta w_{\tau_1}^{(i_1)} \Delta w_{\tau_2}^{(i_2)} \Delta w_{\tau_3}^{(i_3)} =
\]

\[
= \sum_{i_2 = i_4 \neq 0} \lim_{N \to \infty} \sum_{l_3, l_2, l_1 = 0}^{N-1} (-1) \sum_{j_4, j_3, j_2 = 0}^{P} C_{j_1 j_2 j_3} \times
\]

\[
\phi_{j_1} (\tau_1) \phi_{j_2} (\tau_2) \phi_{j_3} (\tau_3) \phi_{j_4} (\tau_4) \Delta w_{\tau_1}^{(i_1)} \Delta w_{\tau_2}^{(i_2)} \Delta w_{\tau_3}^{(i_3)} =
\]

\[
= -\sum_{i_2 = i_4 \neq 0} \lim_{N \to \infty} \sum_{l_3, l_2, l_1 = 0}^{N-1} (-1) \sum_{j_4, j_3, j_2 = 0}^{P} C_{j_1 j_2 j_3} \times
\]

\[
\phi_{j_1} (\tau_1) \phi_{j_2} (\tau_2) \phi_{j_3} (\tau_3) \phi_{j_4} (\tau_4) \Delta w_{\tau_1}^{(i_1)} \Delta w_{\tau_2}^{(i_2)} \Delta w_{\tau_3}^{(i_3)} =
\]
\[ +1_{\{i_2=i_4 \neq 0\}} \cdot 1_{\{i_1=i_3 \neq 0\}} \sum_{j_4,j_1=0}^{p} C_{j_4j_3j_4j_1} \quad w. \ p. 1. \]

In [19] (see the proof of Theorem 8, p. A.135), [24] (see the proof of Theorem 5.7, p. A.388), [25-27] (see Chapter 2), [43] (see the proof of Theorem 4) we have proved that

\[
\lim_{p \to \infty} \sum_{j_4,j_1=0}^{p} C_{j_4j_3j_4j_1} = 0.
\]

Then

\[
\lim_{p \to \infty} M \left\{ \left( R_{T,t}^{(6)pppp} \right)^2 \right\} = 0.
\]

Let us consider \( R_{T,t}^{(6)pppp} \)

\[
R_{T,t}^{(6)pppp} = 1_{\{i_3=i_4 \neq 0\}} \cdot \text{l.i.m.} \sum_{N \to \infty}^{N-1} G_{pppp}(\tau_1, \tau_2, \tau_3, \tau_4) \Delta w^{(i_1)}_{\tau_1} \Delta w^{(i_2)}_{\tau_2} \Delta \tau_3 =
\]

\[
= 1_{\{i_3=i_4 \neq 0\}} \cdot \text{l.i.m.} \sum_{N \to \infty}^{N-1} G_{pppp}(\tau_1, \tau_2, \tau_3, \tau_4) \Delta w^{(i_1)}_{\tau_1} \Delta w^{(i_2)}_{\tau_2} \Delta \tau_3 =
\]

\[
= 1_{\{i_3=i_4 \neq 0\}} \cdot \text{l.i.m.} \sum_{N \to \infty}^{N-1} \left( \frac{1}{2} 1_{\{\tau_1 < \tau_2 < \tau_3\}} + \right.
\]

\[
+ \frac{1}{4} 1_{\{\tau_1 = \tau_2 < \tau_3\}} + \frac{1}{4} 1_{\{\tau_1 < \tau_2 = \tau_3\}} + \frac{1}{8} 1_{\{\tau_1 = \tau_2 = \tau_3\}} -
\]

\[
- \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_4j_1} \phi_{j_1}(\tau_1) \phi_{j_2}(\tau_2) \phi_{j_3}(\tau_3) \phi_{j_4}(\tau_4) \Delta w^{(i_1)}_{\tau_1} \Delta w^{(i_2)}_{\tau_2} \Delta \tau_3 =
\]

\[
= 1_{\{i_3=i_4 \neq 0\}} \cdot \text{l.i.m.} \sum_{N \to \infty}^{N-1} \left( \frac{1}{2} 1_{\{\tau_1 < \tau_2 < \tau_3\}} -
\]

\[
- \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_4j_1} \phi_{j_1}(\tau_1) \phi_{j_2}(\tau_2) \phi_{j_3}(\tau_3) \phi_{j_4}(\tau_4) \Delta w^{(i_1)}_{\tau_1} \Delta w^{(i_2)}_{\tau_2} \Delta \tau_3 =
\]

\[
= 1_{\{i_3=i_4 \neq 0\}} \left( \frac{1}{2} \int_{\frac{T}{t}}^{\frac{T}{t}} \int_{\frac{T}{t}}^{\frac{T}{t}} dw^{(i_1)}_{\tau_1} dw^{(i_2)}_{\tau_2} dt_{3} - \sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_4j_1} \phi_{j_1}(\tau_1) \phi_{j_2}(\tau_2) \phi_{j_3}(\tau_3) \phi_{j_4}(\tau_4) \right) -
\]
\[-1_{\{i_3=i_4\neq 0\}}1_{\{i_1=i_2\neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{i_3=0}^{N-1} \sum_{i_1=0}^{N-1} (-1)^i \sum_{j_4, j_3, j_2, j_1=0}^{P} C_{j_4, j_3, j_2, j_1} \times \]

\[\times \phi_{j_1}(T_{i_1}) \phi_{j_2}(T_{i_2}) \phi_{j_3}(T_{i_3}) \phi_{j_4}(T_{i_4}) \Delta T_{i_1} \Delta T_{i_3} = \]

\[= 1_{\{i_3=i_4\neq 0\}} \left( \frac{1}{2} \int_t^T \int_t^{t_3} \int_t^{t_2} d\omega_{t_1}^{(i_1)} d\omega_{t_2}^{(i_2)} dt_3 - \sum_{j_4, j_2, j_1=0}^{P} C_{j_4, j_2, j_1} \gamma_{j_1}^{(i_1)} \gamma_{j_2}^{(i_2)} \right) + \]

\[+ 1_{\{i_1=i_2\neq 0\}}1_{\{i_3=i_4\neq 0\}} \sum_{j_4, j_1=0}^{P} C_{j_4, j_1} \]

\[= 1_{\{i_3=i_4\neq 0\}} \left( \frac{1}{2} \int_t^T \int_t^{t_3} \int_t^{t_2} d\omega_{t_1}^{(i_1)} d\omega_{t_2}^{(i_2)} dt_3 + \frac{1}{4} 1_{\{i_1=i_2\neq 0\}} \int_t^T \int_t^{t_3} dt_1 dt_3 - \sum_{j_4, j_2, j_1=0}^{P} C_{j_4, j_2, j_1} \gamma_{j_1}^{(i_1)} \gamma_{j_2}^{(i_2)} \right) + \]

\[+ 1_{\{i_1=i_2\neq 0\}}1_{\{i_3=i_4\neq 0\}} \left( \sum_{j_4, j_1=0}^{P} C_{j_4, j_1} - \frac{1}{4} \int_t^T \int_t^{t_3} dt_1 dt_3 \right) \text{ w. p. 1.} \]

In \cite{19} (see the proof of Theorem 8, p. A.135), \cite{24} (see the proof of Theorem 5.7, p. A.388), \cite{25, 27} (see Chapter 2), \cite{43} (see the proof of Theorem 4) we have proved that

\[\lim_{p \to \infty} \sum_{j_4, j_1=0}^{P} C_{j_4, j_1} = \frac{1}{4} \int_t^T \int_t^{t_3} dt_1 dt_3,\]

\[\text{l.i.m.}_{p \to \infty} \sum_{j_4, j_2, j_1=0}^{P} C_{j_4, j_2, j_1} \gamma_{j_1}^{(i_1)} \gamma_{j_2}^{(i_2)} = \frac{1}{2} \int_t^T \int_t^{t_3} \int_t^{t_2} \omega_{t_1}^{(i_1)} d\omega_{t_2}^{(i_2)} dt_3 + \]

\[+ 1_{\{i_1=i_2\neq 0\}} \frac{1}{4} \int_t^T \int_t^{t_3} dt_1 dt_3 \text{ w. p. 1.} \]

Then

\[\lim_{p \to \infty} M \left\{ \left( R_{T, t}^{(6) pppp} \right)^2 \right\} = 0.\]

Let us consider \( R_{T, t}^{(7) pppp} \).
\[ R_{T,t}^{(7)pppp} = 1_{\{i_1 = i_2 \neq 0\}} 1_{\{i_3 = i_4 \neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{\substack{i_4,i_2 = 0 \atop i_2 \neq i_4}}^{N-1} G_{pppp}(\tau_{i_2}, \tau_{i_4}, \tau_{i_2}, \tau_{i_4}) \Delta \tau_{i_2} \Delta \tau_{i_4} + \\
+ 1_{\{i_1 = i_3 \neq 0\}} 1_{\{i_2 = i_4 \neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{\substack{i_4,i_2 = 0 \atop i_2 \neq i_4}}^{N-1} G_{pppp}(\tau_{i_2}, \tau_{i_4}, \tau_{i_2}, \tau_{i_4}) \Delta \tau_{i_2} \Delta \tau_{i_4} + \\
+ 1_{\{i_1 = i_4 \neq 0\}} 1_{\{i_2 = i_3 \neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{\substack{i_4,i_2 = 0 \atop i_2 \neq i_4}}^{N-1} G_{pppp}(\tau_{i_2}, \tau_{i_4}, \tau_{i_2}, \tau_{i_4}) \Delta \tau_{i_2} \Delta \tau_{i_4} = \\
= 1_{\{i_1 = i_2 \neq 0\}} 1_{\{i_3 = i_4 \neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{i_4 = 0}^{N-1} \sum_{i_2 = 0}^{N-1} G_{pppp}(\tau_{i_2}, \tau_{i_4}, \tau_{i_4}, \tau_{i_4}) \Delta \tau_{i_2} \Delta \tau_{i_4} + \\
+ 1_{\{i_1 = i_3 \neq 0\}} 1_{\{i_2 = i_4 \neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{i_4 = 0}^{N-1} \sum_{i_2 = 0}^{N-1} G_{pppp}(\tau_{i_2}, \tau_{i_4}, \tau_{i_2}, \tau_{i_4}) \Delta \tau_{i_2} \Delta \tau_{i_4} + \\
+ 1_{\{i_1 = i_4 \neq 0\}} 1_{\{i_2 = i_3 \neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{i_4 = 0}^{N-1} \sum_{i_2 = 0}^{N-1} G_{pppp}(\tau_{i_2}, \tau_{i_4}, \tau_{i_4}, \tau_{i_4}) \Delta \tau_{i_2} \Delta \tau_{i_4} = \\
= 1_{\{i_1 = i_2 \neq 0\}} 1_{\{i_3 = i_4 \neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{i_4 = 0}^{N-1} \sum_{i_2 = 0}^{N-1} \left( \frac{1}{4} \delta_{\{\tau_{i_2} < \tau_{i_4}\}} + \frac{1}{8} \delta_{\{\tau_{i_2} = \tau_{i_4}\}} \right) \Delta \tau_{i_2} \Delta \tau_{i_4} + \\
- \sum_{j_4,j_3,j_2,j_1 = 0}^{p} C_{j_4,j_3,j_2,j_1} \phi_{j_1}(\tau_{i_2}) \phi_{j_2}(\tau_{i_4}) \phi_{j_3}(\tau_{i_4}) \phi_{j_4}(\tau_{i_4}) \Delta \tau_{i_2} \Delta \tau_{i_4} + \\
+ 1_{\{i_1 = i_3 \neq 0\}} 1_{\{i_2 = i_4 \neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{i_4 = 0}^{N-1} \sum_{i_2 = 0}^{N-1} \left( \frac{1}{8} \delta_{\{\tau_{i_2} = \tau_{i_4}\}} \right) - \sum_{j_4,j_3,j_2,j_1 = 0}^{p} C_{j_4,j_3,j_2,j_1} \times \\
\phi_{j_1}(\tau_{i_2}) \phi_{j_2}(\tau_{i_4}) \phi_{j_3}(\tau_{i_4}) \phi_{j_4}(\tau_{i_4}) \Delta \tau_{i_2} \Delta \tau_{i_4} + \\
+ 1_{\{i_1 = i_4 \neq 0\}} 1_{\{i_2 = i_3 \neq 0\}} \text{l.i.m.}_{N \to \infty} \sum_{i_4 = 0}^{N-1} \sum_{i_2 = 0}^{N-1} \left( \frac{1}{8} \delta_{\{\tau_{i_2} = \tau_{i_4}\}} \right) - \sum_{j_4,j_3,j_2,j_1 = 0}^{p} C_{j_4,j_3,j_2,j_1} \times \\
\phi_{j_1}(\tau_{i_2}) \phi_{j_2}(\tau_{i_4}) \phi_{j_3}(\tau_{i_4}) \phi_{j_4}(\tau_{i_4}) \Delta \tau_{i_2} \Delta \tau_{i_4} = \\
= 1_{\{i_1 = i_2 \neq 0\}} 1_{\{i_3 = i_4 \neq 0\}} \left( \frac{1}{4} \int_{t}^{T} dt_2 \int_{t}^{T} dt_4 - \sum_{j_4,j_1 = 0}^{p} C_{j_4,j_4,j_1,j_1} \right) - \\
- 1_{\{i_1 = i_3 \neq 0\}} 1_{\{i_2 = i_4 \neq 0\}} \sum_{j_4,j_1 = 0}^{p} C_{j_4,j_4,j_1,j_1} - 
\]
\[-1_{\{i_1=i_4 \neq 0\}} 1_{\{i_2=i_3 \neq 0\}} \sum_{j_4,j_2=0}^p C_{j_4,j_2,j_4}.\]

In [19] (see the proof of Theorem 8, p. A.135), [24] (see the proof of Theorem 5.7, p. A.388), [25]-[27] (see Chapter 2), [43] (see the proof of Theorem 4) we have proved that

\[
\lim_{p \to \infty} \sum_{j_4,j_1=0}^p C_{j_4,j_4,j_1j_1} = \frac{1}{4} \int_{t}^{T} \int_{t}^{T} dt_2 dt_4,
\]

\[
\lim_{p \to \infty} \sum_{j_4,j_1=0}^p C_{j_4,j_4,j_1j_1} = 0,
\]

\[
\lim_{p \to \infty} \sum_{j_4,j_2=0}^p C_{j_4,j_2,j_4} = 0.
\]

Then

\[
\lim_{p \to \infty} R_{T,t}^{(7)pppp} = 0.
\]

Theorem 6 is proved.

8. THEOREMS 1–6 FROM POINT OF VIEW OF THE WONG–ZAKAI APPROXIMATION

The iterated Ito stochastic integrals and solutions of Ito SDEs are complex and important functionals from the independent components \(f_i(s), i = 1, \ldots, m\) of the multidimensional Wiener process \(f_s, s \in [0, T]\). Let \(f_s^{(i)}, p \in \mathbb{N}\) be some approximation of \(f_s^{(i)}, i = 1, \ldots, m\). Suppose that \(f_s^{(i)p}\) converges to \(f_s^{(i)}, i = 1, \ldots, m\) if \(p \to \infty\) in some sense and has differentiable sample trajectories.

A natural question arises: if we replace \(f_s^{(i)}\) by \(f_s^{(i)p}\), \(i = 1, \ldots, m\) in the functionals mentioned above, will the resulting functionals converge to the original functionals from the components \(f_s^{(i)}, i = 1, \ldots, m\) of the multidimensional Wiener process \(f_s\)? The answer to this question is negative in the general case. However, in the pioneering works of Wong E. and Zakai M. [51], [52], it was shown that under the special conditions and for some types of approximations of the Wiener process the answer is affirmative with one peculiarity: the convergence takes place to the iterated Stratonovich stochastic integrals and solutions of Stratonovich SDEs and not to iterated Ito stochastic integrals and solutions of Ito SDEs. The piecewise linear approximation as well as the regularization by convolution [51]-[53] relate the mentioned types of approximations of the Wiener process. The above approximation of stochastic integrals and solutions of SDEs is often called the Wong-Zakai approximation.

Let \(f_s, s \in [0, T]\) be an \(m\)-dimensional standard Wiener process with independent components \(f_s^{(i)}, i = 1, \ldots, m\). It is well known that the following representation takes place [55], [56]

\[
(46) \quad f_t^{(i)} - f_t^{(i)} = \sum_{j=0}^{\infty} \int_{t}^{T} \phi_j(s) ds \, \zeta_j^{(i)}; \quad \zeta_j^{(i)} = \int_{t}^{T} \phi_j(s) df_s^{(i)},
\]
where \( \tau \in [t, T] \), \( t \geq 0 \), \( \{\phi_j(x)\}_{j=0}^{\infty} \) is an arbitrary complete orthonormal system of functions in the space \( L_2([t, T]) \), and \( \xi_j^{(i)} \) are independent standard Gaussian random variables for various \( i \) or \( j \). Moreover, the series (46) converges for any \( \tau \in [t, T] \) in the mean-square sense.

Let \( f^{(i)}_\tau - f^{(i)}_t \) be the mean-square approximation of the process \( f^{(i)}_\tau - f^{(i)}_t \), which has the following form

\[
(47) \quad f^{(i)}_\tau - f^{(i)}_t = \sum_{j=0}^{p} \int_t^\tau \phi_j(s) ds \zeta_j^{(i)}.
\]

From (47) we obtain

\[
(48) \quad df^{(i)}_\tau = \sum_{j=0}^{p} \phi_j(\tau) \zeta_j^{(i)} d\tau.
\]

Consider the following iterated Riemann–Stieltjes integral

\[
(49) \quad \int_t^\tau \int_{t_1}^{t_2} \cdots \int_{t_{i-1}}^{t_i} \psi_1(t_1) \psi_2(t_2) \cdots \psi_k(t_k) \prod_{l=1}^{k} \zeta_j^{(i_l)} dt_1 \cdots dt_k,
\]

where \( i_1, \ldots, i_k = 0, 1, \ldots, m, \quad p_1, \ldots, p_k \in \mathbb{N} \),

\[
(50) \quad dw^{(i)}_\tau = \begin{cases} 
    df^{(i)}_\tau & \text{for } i = 1, \ldots, m \\
    d\tau & \text{for } i = 0
\end{cases}
\]

and \( df^{(i)}_\tau \) in defined by the relation (48).

Let us substitute (48) into (49)

\[
(51) \quad \int_t^\tau \int_{t_1}^{t_2} \cdots \int_{t_{i-1}}^{t_i} \psi_1(t_1) \psi_2(t_2) \cdots \psi_k(t_k) \prod_{l=1}^{k} \zeta_j^{(i_l)} dt_1 \cdots dt_k = \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} \sum_{l=1}^{k} C_{j_k \ldots j_1} \zeta_j^{(i_l)},
\]

where

\[
\zeta_j^{(i)} = \int_t^\tau \phi_j(s) dw^{(i)}_s
\]

are independent standard Gaussian random variables for various \( i \) or \( j \) (in the case when \( i \neq 0 \)), \( w_s^{(i)} = f_s^{(i)} \) for \( i = 1, \ldots, m \) and \( w_s^{(0)} = s \),

\[
C_{j_k \ldots j_1} = \int_t^\tau \int_{t_1}^{t_2} \cdots \int_{t_{i-1}}^{t_i} \psi_1(t_1) \psi_2(t_2) \cdots \psi_k(t_k) dt_1 \cdots dt_k
\]

is the Fourier coefficient.
To best of our knowledge \[51\]-\[53\] the approximations of the Wiener process in the Wong–Zakai approximation must satisfy fairly strong restrictions \[53\] (see Definition 7.1, pp. 480–481). Moreover, approximations of the Wiener process that are similar to (47) were not considered in \[51\], \[52\] (also see \[53\], Theorems 7.1, 7.2). Therefore, the proof of analogs of Theorems 7.1 and 7.2 \[53\] for approximations of the Wiener process based on its series expansion \[46\] should be carried out separately. Thus, the mean-square convergence of the right-hand side of (51) to the iterated Stratonovich stochastic integral (3) does not follow from the results of the papers \[51\], \[52\] (also see \[53\], Theorems 7.1, 7.2).

From the other hand, Theorems 1–6 and Theorems 7–10 (see below) from this paper can be considered as the proof of the Wong–Zakai approximation for the iterated Stratonovich stochastic integrals (3) of multiplicities 1 to 6 based on the Riemann–Stieltjes integrals \[49\] and approximation (47) of the Wiener process. At that, the mentioned Riemann–Stieltjes integrals converge (according to Theorems 1–6 and Theorems 7–10 (see below)) to the appropriate Stratonovich stochastic integrals \[3\]. Recall that \[\{\phi_j(x)\}_{j=0}^{\infty}\] (see (46), (47), and Theorems 3–10) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \[L^2([t,T])\].

To illustrate the above reasoning, consider two examples for the case \(k = 2\), \(\psi_1(s), \psi_2(s) \equiv 1; i_1, i_2 = 1, \ldots, m\).

The first example relates to the piecewise linear approximation of the multidimensional Wiener process (these approximations were considered in \[51\]-\[53\]). Let \(b_{\Delta}^{(i)}(t), t \in [0,T]\) be the piecewise linear approximation of the \(i\)th component \(f_{\Delta}^{(i)}(t)\) of the multidimensional standard Wiener process \(f_t, t \in [0,T]\) with independent components \(f_{\Delta}^{(i)}(t), i = 1, \ldots, m\), i.e.

\[
b_{\Delta}^{(i)}(t) = f_{k\Delta}^{(i)} + \frac{t - k\Delta}{\Delta} f_{k\Delta}^{(i)}.
\]

where

\[
\Delta f_{k\Delta}^{(i)} = f_{(k+1)\Delta}^{(i)} - f_{k\Delta}^{(i)}, \quad t \in [k\Delta, (k+1)\Delta), \quad k = 0, 1, \ldots, N - 1.
\]

Note that w. p. 1

\[
\frac{db_{\Delta}^{(i)}}{dt}(t) = \frac{\Delta f_{\Delta}^{(i)}}{\Delta}, \quad t \in [k\Delta, (k+1)\Delta), \quad k = 0, 1, \ldots, N - 1.
\]

Consider the following iterated Riemann–Stieltjes integral

\[
\int_0^T \int_0^s db_{\Delta}^{(i_1)}(\tau) db_{\Delta}^{(i_2)}(s), \quad i_1, i_2 = 1, \ldots, m.
\]

Using (52) and additive property of the Riemann–Stieltjes integral, we can write w. p. 1

\[
\int_0^T \int_0^s db_{\Delta}^{(i_1)}(\tau) db_{\Delta}^{(i_2)}(s) = \int_0^T \int_0^s \frac{db_{\Delta}^{(i_1)}}{d\tau}(\tau) \frac{db_{\Delta}^{(i_2)}}{ds}(s) ds =
\]

\[
= \sum_{l=0}^{N-1} \int_{l\Delta}^{(l+1)\Delta} \left( \sum_{q=0}^{(l-1)\Delta} \int_{q\Delta}^{(q+1)\Delta} \frac{\Delta f_{\Delta}^{(i_1)}}{\Delta} d\tau + \int_{l\Delta}^s \frac{\Delta f_{\Delta}^{(i_1)}}{\Delta} d\tau \right) \frac{\Delta f_{\Delta}^{(i_2)}}{\Delta} ds =
\]
\[
\sum_{l=0}^{N-1} \sum_{q=0}^{l-1} \Delta f_{q}^{(i_1)} \Delta f_{\Delta}^{(i_2)} + \frac{1}{\Delta^2} \sum_{l=0}^{N-1} \Delta f_{l}^{(i_1)} \Delta f_{\Delta}^{(i_2)} \int \frac{d\tau ds}{i_{\Delta}} = \\
\sum_{l=0}^{N-1} \sum_{q=0}^{l-1} \Delta f_{q}^{(i_1)} \Delta f_{\Delta}^{(i_2)} + \frac{1}{2} \sum_{l=0}^{N-1} \Delta f_{l}^{(i_1)} \Delta f_{\Delta}^{(i_2)}.
\]

Using (53), it is not difficult to show that

\[
\lim_{N \to \infty} \int_{0}^{T} \int_{0}^{s} dB_{\Delta}^{(i_1)}(\tau) dB_{\Delta}^{(i_2)}(s) = \int_{0}^{s} \int_{0}^{S} df_{\tau}^{(i_1)} df_{s}^{(i_2)} + \frac{1}{2} \sum_{i_1=i_2} \int_{0}^{T} ds = \\
= \int_{0}^{sT} \int_{0}^{s} df_{\tau}^{(i_1)} df_{s}^{(i_2)},
\]

where \(\Delta \to 0\) if \(N \to \infty\) (\(N\Delta = T\)).

Obviously, (54) agrees with Theorem 7.1 (see [53], p. 486).

The next example relates to the approximation of the Wiener process based on its series expansion (46) for \(t = 0\), where \(\{\phi_j(x)\}_{j=0}^{\infty}\) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \(L_2([0,T])\).

Consider the following iterated Riemann–Stieltjes integral

\[
\int_{0}^{T} \int_{0}^{s} df_{\tau}^{(i_1)p} df_{s}^{(i_2)p}, \quad i_1, i_2 = 1, \ldots, m,
\]

where \(df_{\tau}^{(i_1)p}\) is defined by the relation (48).

Let us substitute (48) into (55)

\[
\int_{0}^{T} \int_{0}^{s} df_{\tau}^{(i_1)p} df_{s}^{(i_2)p} = \sum_{j_1, j_2=0}^{P} C_{j_2 j_1} \phi_{j_2}^{(i_1)} \phi_{j_2}^{(i_2)},
\]

where

\[
C_{j_2 j_1} = \int_{0}^{T} \phi_{j_2}^{(i)}(s) \int_{0}^{s} \phi_{j_1}(\tau) d\tau ds
\]

is the Fourier coefficient; another notations are the same as in (53).

As we noted above, approximations of the Wiener process that are similar to (47) were not considered in [51], [52] (also see Theorems 7.1, 7.2 in [53]). Furthermore, the extension of the results of Theorems 7.1 and 7.2 [53] to the case under consideration is not obvious.

On the other hand, we can apply the theory built in Chapters 1 and 2 of the monographs [25]-[27]. More precisely, using Theorems 3, 4 from this paper, we obtain from (53) the desired result
$$\lim_{p \to \infty} \int_0^s \int_0^T df_r^{(i_1)} df_s^{(i_2)} = \lim_{p \to \infty} \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1} c_{j_1}^{(i_1)} c_{j_2}^{(i_2)} =$$

$$= \int_0^T \int_0^s df_r^{(i_1)} df_s^{(i_2)}.$$

(57)

From the other hand, by Theorems 1, 2 (see (11)) for the case \( k = 2 \) we obtain from (56) the following relation

$$\lim_{p \to \infty} \int_0^s \int_0^T df_r^{(i_1)} df_s^{(i_2)} = \lim_{p \to \infty} \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1} c_{j_1}^{(i_1)} c_{j_2}^{(i_2)} =$$

$$= \lim_{p \to \infty} \sum_{j_1, j_2 = 0}^{p} C_{j_2 j_1} \left( c_{j_1}^{(i_1)} c_{j_2}^{(i_2)} - 1_{j_1 = j_2} 1_{j_1 = j_2} \right) + 1_{j_1 = j_2} \sum_{j_1 = 0}^{\infty} C_{j_1 j_1} =$$

$$= \int_0^T \int_0^s df_r^{(i_1)} df_s^{(i_2)} + 1_{j_1 = j_2} \sum_{j_1 = 0}^{\infty} C_{j_1 j_1}.$$  

(58)

Since

$$\sum_{j_1 = 0}^{\infty} C_{j_1 j_1} = \frac{1}{2} \sum_{j_1 = 0}^{\infty} \left( \int_0^T \phi_j(\tau) d\tau \right)^2 = \frac{1}{2} \left( \int_0^T \phi_0(\tau) d\tau \right)^2 = \frac{1}{2} \int_0^T ds,$$

then from standard relation between Ito and Stratonovich stochastic integrals and (58) we obtain (57).

9. RECENT RESULTS ON EXPANSIONS OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS OF MULTIPlicities 3 TO 6

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained [25] (Sect. 2.10–2.16), [32] (Sect. 5–11), [33] (Sect. 7–13), [43] (Sect. 13–19), [63] (Sect. 4–9), [64]. Let us formulate four theorems that were obtained using this approach.

Theorem 7 [25], [32], [33], [43], [63]. Suppose that \( \{\phi_j(x)\}_{j=0}^{\infty} \) is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \( L_2([t, T]) \). Furthermore, let \( \psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \) are continuously differentiable nonrandom functions on \([t, T]\). Then, for the iterated Stratonovich stochastic integral of third multiplicity

$$J^* \left[ \psi^{(i_3)} \right]_{T, t} = \int_t^{T} \int_t^{t_3} \int_t^{t_2} \int_t^{t_1} d\psi_3(t_3) d\psi_2(t_2) d\psi_1(t_1) d\omega_{t_1}^{(i_1)} d\omega_{t_2}^{(i_2)} d\omega_{t_3}^{(i_3)} (i_1, i_2, i_3 = 0, 1, \ldots, m)$$
the following relations

\begin{equation}
J^*\left[\psi^{(3)}\right]_{T,t} = 1.\ i.m. \sum_{j_1,j_2,j_3=0}^p C_{j_3j_2j_1} \xi^{(i_1)}_{j_1} \xi^{(i_2)}_{j_2} \xi^{(i_3)}_{j_3},
\end{equation}

\begin{equation}
M \left\{ \left( J^*\left[\psi^{(3)}\right]_{T,t} - \sum_{j_1,j_2,j_3=0}^p C_{j_3j_2j_1} \xi^{(i_1)}_{j_1} \xi^{(i_2)}_{j_2} \xi^{(i_3)}_{j_3} \right)^2 \right\} \leq \frac{C}{p}
\end{equation}

are fulfilled, where \(i_1,i_2,i_3 = 0,1,\ldots,m\) in (59) and \(i_1,i_2,i_3 = 1,\ldots,m\) in (60), constant \(C\) is independent of \(p\),

\[ C_{j_3j_2j_1} = \int_t^T \psi_3(t_3) \phi_{j_3}(t_3) \int_{t_3}^{t_2} \psi_2(t_2) \phi_{j_2}(t_2) \int_{t_1}^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3 \]

and

\[ \xi^{(i)}_j = \int_t^T \phi_j(\tau) d\tau^{(i)} \]

are independent standard Gaussian random variables for various \(i\) or \(j\) (in the case when \(i \neq 0\)); another notations are the same as in Theorems 1, 2.

**Theorem 8** 26, 32, 33, 34, 33. Let \{\phi_j(x)\}_{j=0}^\infty be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space \(L_2([t,T])\). Furthermore, let \(\psi_1(\tau),\ldots,\psi_4(\tau)\) be continuously differentiable nonrandom functions on \([t,T]\). Then, for the iterated Stratonovich stochastic integral of fourth multiplicity

\begin{equation}
J^*\left[\psi^{(4)}\right]_{T,t} = \int_t^T \psi_1(t_4) \int_t^{t_4} \psi_3(t_3) \int_t^{t_3} \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) dw_{t_1}^{(i_1)} dw_{t_2}^{(i_2)} dw_{t_3}^{(i_3)} dw_{t_4}^{(i_4)}
\end{equation}

the following relations

\begin{equation}
J^*\left[\psi^{(4)}\right]_{T,t} = 1.\ i.m. \sum_{j_1,j_2,j_3,j_4=0}^p C_{j_4j_3j_2j_1} \xi^{(i_1)}_{j_1} \xi^{(i_2)}_{j_2} \xi^{(i_3)}_{j_3} \xi^{(i_4)}_{j_4},
\end{equation}

\begin{equation}
M \left\{ \left( J^*\left[\psi^{(4)}\right]_{T,t} - \sum_{j_1,j_2,j_3,j_4=0}^p C_{j_4j_3j_2j_1} \xi^{(i_1)}_{j_1} \xi^{(i_2)}_{j_2} \xi^{(i_3)}_{j_3} \xi^{(i_4)}_{j_4} \right)^2 \right\} \leq \frac{C}{p^{1-\varepsilon}}
\end{equation}

are fulfilled, where \(i_1,\ldots,i_4 = 0,1,\ldots,m\) in (61), (62) and \(i_1,\ldots,i_4 = 1,\ldots,m\) in (63), constant \(C\) does not depend on \(p\), \(\varepsilon\) is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space \(L_2([t,T])\) and \(\varepsilon = 0\) for the case of complete orthonormal system of trigonometric functions in the space \(L_2([t,T])\).
$C_{j_4j_3j_2j_1} =$
\[ = \int_t^T \psi_4(t_4) \phi_{j_4}(t_4) \int_t^{t_4} \psi_3(t_3) \phi_{j_3}(t_3) \int_t^{t_2} \psi_2(t_2) \phi_{j_2}(t_2) \int_t^{t_1} \psi_1(t_1) \phi_{j_1}(t_1) dt_1 dt_2 dt_3 dt_4; \]

another notations are the same as in Theorem 7.

**Theorem 9** [25, 32, 33, 43, 63]. Assume that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$ and $\psi_1(\tau), \ldots, \psi_5(\tau)$ are continuously differentiable nonrandom functions on $[t, T]$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity

\[ J^*[\psi^{(5)}]_{T,t} = \int_t^T \psi_5(t_5) \cdots \psi_1(t_1) dw_{i_1}^{(i_1)} \cdots dw_{i_5}^{(i_5)} \]

the following relations

\[ J^*[\psi^{(5)}]_{T,t} = \text{l.i.m.}_{p \to \infty} \sum_{j_1, \ldots, j_5 = 0}^{p} C_{j_5 \ldots j_1} \phi_{j_1} \phi_{j_2} \phi_{j_3} \phi_{j_4} \phi_{j_5} ; \]

\[ \mathbb{M} \left\{ \left( J^*[\psi^{(5)}]_{T,t} - \sum_{j_1, \ldots, j_5 = 0}^{p} C_{j_5 \ldots j_1} \phi_{j_1} \phi_{j_2} \phi_{j_3} \phi_{j_4} \phi_{j_5} \right)^2 \right\} \leq \frac{C}{p^{1-\varepsilon}} \]

are fulfilled, where $i_1, \ldots, i_5 = 0, 1, \ldots, m$ in [64], [65] and $i_1, \ldots, i_5 = 1, \ldots, m$ in [66], constant $C$ is independent of $p$, $\varepsilon$ is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space $L_2([t, T])$ and $\varepsilon = 0$ for the case of complete orthonormal system of trigonometric functions in the space $L_2([t, T])$,

\[ C_{j_5 \ldots j_1} = \int_t^T \psi_5(t_5) \phi_{j_5}(t_5) \cdots \psi_1(t_1) \phi_{j_1}(t_1) dt_1 \cdots dt_5; \]

another notations are the same as in Theorems 7, 8.

**Theorem 10** [25, 32, 33, 43, 63]. Suppose that $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$. Then, for the iterated Stratonovich stochastic integral of sixth multiplicity

\[ J^*_{T,t} = \int_t^T \cdots \int_t^{t_2} dw_{i_1}^{(i_1)} \cdots dw_{i_6}^{(i_6)} \]

the following expansion
that converges in the mean-square sense is valid, where $i_1, \ldots, i_6 = 0, 1, \ldots, m$,

$$C_{j_6 \ldots j_1} = \int_t^{t_2} \phi_{j_6}(t_6) \cdots \int_t^{t_1} \phi_{j_1}(t_1) dt_1 \cdots dt_6;$$

another notations are the same as in Theorems 7–9.

REFERENCES

[1] Gihman I.I., Skorohod A.V. Stochastic Differential Equations and its Applications. Naukova Dumka, Kiev, 1982, 612 pp.
[2] Kloeden P.E., Platen E. Numerical Solution of Stochastic Differential Equations. Springer, Berlin, 1995, 632 pp.
[3] Milstein G.N. Numerical Integration of Stochastic Differential Equations. Ural University Press, Sverdlovsk, 1988, 225 pp.
[4] Kloeden P.E., Platen E., Schurz H. Numerical Solution of SDE Through Computer Experiments. Springer, Berlin, 1994, 292 pp.
[5] Milstein G.N., Tretyakov M.V. Stochastic Numerics for Mathematical Physics. Springer, Berlin, 2004, 616 pp.
[6] Kuznetsov D.F. A method of expansion and approximation of repeated stochastic Stratonovich integrals based on multiple Fourier series on full orthonormal systems. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1997), 18-77. Available at: [http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html](http://diffjournal.spbu.ru/EN/numbers/1997.1/article.1.2.html)
[7] Kuznetsov D.F. Problems of the Numerical Analysis of Itô Stochastic Differential Equations. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (1998), 66-367. Available at: [http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html](http://diffjournal.spbu.ru/EN/numbers/1998.1/article.1.3.html) Hard Cover Edition: 1998, SPbGTU Publishing House, 204 pp. (ISBN 5-7422-0645-5)
[8] Kuznetsov D.F. Mean square approximation of solutions of stochastic differential equations using Legendres polynomials. [In English]. Journal of Automation and Information Sciences (Begell House), 2000, 32 (Issue 12), 69-86. DOI: [http://doi.org/10.1615/JAutomatInfSci.v32.i12.80](http://doi.org/10.1615/JAutomatInfSci.v32.i12.80)
[9] Kuznetsov D.F. New representations of explicit one-step numerical methods for jump-diffusion stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 41, 6 (2001), 874-888. Available at: [http://www.sde-kuznetsov.spb.ru/01b.pdf](http://www.sde-kuznetsov.spb.ru/01b.pdf)
[10] Kuznetsov D.F. Numerical Integration of Stochastic Differential Equations. 2. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2006, 764 pp. DOI: [http://doi.org/10.18720/SPBPU/2/s17-227](http://doi.org/10.18720/SPBPU/2/s17-227) Available at: [http://www.sde-kuznetsov.spb.ru/06.pdf](http://www.sde-kuznetsov.spb.ru/06.pdf) (ISBN 5-7422-1191-0)
[11] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 1st Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, 778 pp. DOI: [http://doi.org/10.18720/SPBPU/2/s17-228](http://doi.org/10.18720/SPBPU/2/s17-228) Available at: [http://www.sde-kuznetsov.spb.ru/07b.pdf](http://www.sde-kuznetsov.spb.ru/07b.pdf) (ISBN 5-7422-1394-8)
[12] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 2nd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2007, XXXII+770 pp. DOI: [http://doi.org/10.18720/SPBPU/2/s17-229](http://doi.org/10.18720/SPBPU/2/s17-229) Available at: [http://www.sde-kuznetsov.spb.ru/07a.pdf](http://www.sde-kuznetsov.spb.ru/07a.pdf) (ISBN 5-7422-1439-1)
[13] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs, 3rd Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2009, XXXIV+768 pp. DOI: [http://doi.org/10.18720/SPBPU/2/s17-230](http://doi.org/10.18720/SPBPU/2/s17-230) Available at: [http://www.sde-kuznetsov.spb.ru/09.pdf](http://www.sde-kuznetsov.spb.ru/09.pdf) (ISBN 978-5-7422-2132-6)
Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. With MatLab Programs. 4th Edition. [In Russian]. Polytechnical University Publishing House, Saint-Petersburg, 2010, XXX+786 pp. DOI: http://doi.org/10.18720/SPBPB/2/s17-231
Available at: http://www.sde-kuznetsov.spb.ru/10.pdf (ISBN 978-5-7422-2448-8)

Kuznetsov D.F. Multiple Stochastic Ito and Stratonovich Integrals and Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 6 (2010), 5586-5596. DOI: http://doi.org/10.18720/SPBPB/2/s17-211
Available at: http://www.sde-kuznetsov.spb.ru/11b.pdf (ISBN 978-5-7422-2988-9)

Kuznetsov D.F. Strong Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Solution of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2020), 1058-1070. DOI: http://doi.org/10.1134/S0965542518070059

Kuznetsov D.F. Mean-Square Approximation of Iterated Ito and Stratonovich Stochastic Integrals Based on Generalized Multiple Fourier Series. Application to Numerical Integration of Itô SDEs and Semilinear SPDEs. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2021), A.1-A.788. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.4/article.1.8.html

Kuznetsov D.F. The proof of convergence with probability 1 in the method of expansion of iterated Ito stochastic integrals based on generalized multiple Fourier series. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2020), 89-117. Available at: http://diffjournal.spbu.ru/RU/numbers/2020.2/article.1.6.html

Kuznetsov D.F. Mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 6 from the Taylor-Ito and Taylor-Stratonovich expansions using Legendre polynomials. [In English]. arXiv:1801.00231 [math.PR], 2019, 166 pp.

Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity based on generalized iterated Fourier series converging pointwise. [In English]. arXiv:1801.00784 [math.PR], 2018, 77 pp.
[31] Kuznetsov D.F. Strong numerical methods of orders 2.0, 2.5, and 3.0 for Ito stochastic differential equations based on the unified stochastic Taylor expansions and multiple Fourier-Legendre series [In English]. arXiv:1807.02190 [math.PR]. 2018, 44 pp.

[32] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of fifth and sixth multiplicity based on generalized multiple Fourier series. [In English]. arXiv:1802.06043 [math.PR]. 2022, 129 pp.

[33] Kuznetsov D.F. The hypotheses on expansions of iterated Stratonovich stochastic integrals of arbitrary multiplicity and their partial proof. [In English]. arXiv:1801.03495 [math.PR]. 2022, 138 pp.

[34] Kuznetsov D.F. Exact calculation of the mean-square error in the method of approximation of iterated Ito stochastic based on generalized multiple Fourier series. [In English]. arXiv:1801.01079 [math.PR]. 2018, 68 pp.

[35] Kuznetsov D.F. Expansion of iterated Ito stochastic integrals of arbitrary multiplicity based on generalized multiple Fourier series converging in the mean. [In English]. arXiv:1712.09746 [math.PR]. 2022, 111 pp.

[36] Kuznetsov D.F. Expansion of iterated stochastic integrals with respect to martingale Poisson measures and with respect to martingales based on generalized multiple Fourier series. [In English]. arXiv:1801.06501 [math.PR]. 2018, 40 pp.

[37] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series. Ufa Mathematical Journal, 11, 4 (2019), 49-77. DOI: http://doi.org/10.13108/2019-11-4-49

Available at: http://matem.anrb.ru/en/article?artid=604

[38] Kuznetsov D.F. On numerical modeling of the multidimensional dynamic systems under random perturbations with the 2.5 order of strong convergence. [In English]. Automation and Remote Control, 80, 5 (2019), 867-881. DOI: http://doi.org/10.1134/S0005117919050060

[39] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. Computational Mathematics and Mathematical Physics, 59, 8 (2019), 1236-1250. DOI: http://doi.org/10.1134/S0965542519080116

[40] Kuznetsov D.F. Comparative analysis of the efficiency of application of Legendre polynomials and trigonometric functions to the numerical integration of Ito stochastic differential equations. [In English]. arXiv:1901.02545 [math.GM]. 2019, 40 pp.

[41] Kuznetsov D.F. Expansion of multiple Stratonovich stochastic integrals of second multiplicity based on double Fourier-Legendre series summarized by Pringsheim method [In Russian]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 1 (2018), 1-34. Available at: http://diffjournal.spbu.ru/EN/numbers/2018.1/article.1.1.html

[42] Kuznetsov D.F. Application of the method of approximation of iterated Ito stochastic integrals based on generalized multiple Fourier series to the high-order strong numerical methods for non-commutative semilinear stochastic partial differential equations. [In English]. arXiv:1905.03724 [math.GM]. 2019, 41 pp.

[43] Kuznetsov D.F. Expansions of iterated Stratonovich stochastic integrals based on generalized multiple Fourier series: multiplicities 1 to 6 and beyond. [In English]. arXiv:1712.09516 [math.PR]. 2022, 204 pp.

[44] Kuznetsov D.F. Expansion of iterated Stratonovich stochastic integrals of multiplicity 2 based on double Fourier-Legendre series summarized by Pringsheim method. [In English]. arXiv:1801.01962 [math.PR]. 2018, 49 pp.

[45] Kuznetsov D.F. Explicit one-step numerical method with the strong convergence order of 2.5 for Ito stochastic differential equations with a multi-dimensional nonadditive noise based on the Taylor-Stratonovich expansion. Computational Mathematics and Mathematical Physics, 60, 3 (2020), 379-389. DOI: http://doi.org/10.1134/S0965542519080116

[46] Kuznetsov D.F. Integration replacement technique for iterated Ito stochastic integrals and iterated stochastic integrals with respect to martingales. [In English]. arXiv:1801.04634 [math.PR]. 2018, 28 pp.

[47] Allen E. Approximation of triple stochastic integrals through region subdivision. Communications in Applied Analysis (Special Tribute Issue to Professor V. Lakshmikantham). 17 (2013), 355-366.

[48] Kloeden P.E., Platen E., Wright I.W. The approximation of multiple stochastic integrals. Stochastic Analysis and Applications. 10, 4 (1992), 431-441.

[49] Platen E., Bruti-Liberati N. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer, Berlin-Heidelberg, 2010. 868 pp.

[50] Prigarin S.M., Belov S.M. On one application of the Wiener process decomposition into series. Preprint 1107. Novosibirsk, Siberian Branch of the Russian Academy of Sciences, 1998, 16 pp. [In Russian].

[51] Wong E., Zakai M. On the relation between ordinary and stochastic differential equations. Int. J. Eng. Sci., 3 (1965), 213-229.

[52] Wong E., Zakai M. On the convergence of ordinary integrals to stochastic integrals. Ann. Math. Stat., 5, 36 (1965), 1560-1564.

[53] Ikeda N., Watanabe S. Stochastic Differential Equations and Diffusion Processes. 2nd Edition. North-Holland Publishing Company, Amsterdam, Oxford, New-York, 1989. 555 pp.
[54] Rybakov K.A. Orthogonal expansion of multiple Itô stochastic integrals. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 3 (2021), 109-140. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.3/article.1.8.html

[55] Liptser R.Sh., Shiryaev A.N. Statistics of Stochastic Processes: Nonlinear Filtering and Related Problems. [In Russian]. Moscow, Nauka, 1974. 696 pp.

[56] Luo W. Wiener chaos expansion and numerical solutions of stochastic partial differential equations. PhD thesis, California Inst. of Technology, 2006, 225 pp.

[57] Kuznetsov M.D., Kuznetsov D.F. SDE-MATH: A software package for the implementation of strong high-order numerical methods for Itô SDEs with multidimensional non-commutative noise based on multiple Fourier–Legendre series. [In English]. Differential Equations and Control Processes, 1 (2021), 93-422. Available at: http://diffjournal.spbu.ru/EN/numbers/2021.1/article.1.5.html

[58] Kuznetsov M.D., Kuznetsov D.F. Implementation of strong numerical methods of orders 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0 for Itô SDEs with non-commutative noise based on the unified Taylor-Itô and Taylor-Stratonovich expansions and multiple Fourier-Legendre series. [In English]. arXiv:2009.14011 [math.PR], 2020, 343 pp.

[59] Kuznetsov M.D., Kuznetsov D.F. Optimization of the mean-square approximation procedures for iterated Itô stochastic integrals of multiplicities 1 to 5 from the unified Taylor-Itô expansion based on multiple Fourier-Legendre series arXiv:2010.13564 [math.PR], 2020, 63 pp. [In English].

[60] Kuznetsov D.F., Kuznetsov M.D. Optimization of the mean-square approximation procedures for iterated Itô stochastic integrals based on multiple Fourier–Legendre series. [In English]. Journal of Physics: Conference Series, Vol. 1925 (2021), article id: 012010, 12 pp. DOI: http://doi.org/10.1088/1742-6596/1925/1/012010

[61] Kuznetsov D.F. Application of multiple Fourier-Legendre series to the implementation of strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs. [In English]. Proceedings of the XIII International Conference on Applied Mathematics and Mechanics in the Aerospace Industry AMMAI-2020 (Crimea, Alushta, 6-13 September, 2020), MAI, Moscow, 2020, pp. 451-453. Available at: http://www.sde-kuznetsov.spb.ru/20e.pdf

[62] Kuznetsov D.F., Kuznetsov M.D. Mean-square approximation of iterated stochastic integrals from strong exponential Milstein and Wagner-Platen methods for non-commutative semilinear SPDEs based on multiple Fourier–Legendre series. Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, Eds. Shiryaev, A.N., Samouylov, K.E., Kozyrev, D.V. Springer, Cham, 2021, pp. 17-32. DOI: http://doi.org/10.1007/978-3-030-83266-7_2

[63] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 2 (2022), 83-186. Available at: http://diffjournal.spbu.ru/EN/numbers/2022.2/article.1.6.html

[64] Kuznetsov D.F. A new approach to the series expansion of iterated Stratonovich stochastic integrals of arbitrary multiplicity with respect to components of the multidimensional Wiener process. II. [In English]. Electronic Journal "Differential Equations and Control Processes" ISSN 1817-2172 (online), 4 (2022). To appear. Available at: http://diffjournal.spbu.ru/EN/collection.html

Dmitriy Feliksovich Kuznetsov

PETER THE GREAT SAINT-PETERSBURG POLYTECHNIC UNIVERSITY,
POLYTECHNICHESKAYA UL., 29,
195251, SAINT-PETERSBURG, RUSSIA

Email address: sde_kuznetsov@inbox.ru