The numerical range as a spectral set

Michel Crouzeix∗ and César Palencia†

February 3, 2017

Abstract

It is shown that the numerical range of a linear operator operator in a Hilbert space is a (complete) $(1+\sqrt{2})$-spectral set. The proof relies, among other things, in the behavior of the Cauchy transform of the conjugates of holomorphic functions.

2000 Mathematical subject classifications: 47A25 ; 47A30

Keywords: numerical range, spectral set

1 Introduction

Let us consider a smooth, bounded, convex domain $\Omega \subset \mathbb{C}$. In a seminal paper [11], Bernard and François Delyon showed that there exists a best constant $C_{\Omega}$ such that, for all rational functions $f$, there holds

$$\|f(A)\| \leq C_{\Omega} \sup_{z \in \Omega} |f(z)|,$$

whenever $A$ is a bounded linear operator in a complex Hilbert space $(H, \langle , , \rangle, \| \|)$ whose numerical range

$$W(A) := \{ (Av, v) : v \in H, \|v\| = 1 \}$$

satisfies $W(A) \subset \Omega$.

Their work has inspired the conjecture $Q := \sup_{\Omega} C_{\Omega} = 2$ and it has been shown in [5] that $2 \leq Q \leq 11.08$. Although there is numerical support to it [10], the conjecture $Q = 2$ remains to be an open problem and we refer to [2] [3] [10] [6] and the bibliography in [6] for the relevant background on the issue.

The aim of this paper is to present the improvement (Theorem 6 below)

$$2 \leq Q \leq 1+\sqrt{2}.$$

Note that, due to Mergelyan theorem, estimate (1) is valid, not only for rational functions $f$, but also for any $f$ belonging to the algebra

$$A(\Omega) := \{ f : f \text{ is holomorphic in } \Omega \text{ and continuous in } \overline{\Omega} \}.$$ 

Furthermore, by using a sequence of smooth convex domains $\Omega_n \supset W(A)$ converging to $W(A)$, from (2) we easily get

$$\|f(A)\| \leq (1+\sqrt{2}) \sup_{z \in W(A)} |f(z)|,$$

\[\text{2Feb2017}\]

∗Université de Rennes, email: Michel.Crouzeix@univ-rennes1.fr
†Universidad de Valladolid, email: cesar.palencia@tel.uva.es
which shows that the numerical range $W(A)$ is a $(1+\sqrt{2})$-spectral set for the operator $A$. Furthermore, since $C_\Omega$ is uniformly bounded, (1) is still valid for all convex domains, even for unbounded ones, which allows to extend (3) to unbounded operators under classical suitable conditions.

For the sake of simplicity, we work with complex-valued functions, but there is no difficulty in generalizing the proof we give of (2) to matrix-valued mappings $f$, without changing the constant. Therefore, the homomorphism $f \mapsto f(A)$, from the algebra $\mathcal{A}(W(A))$ into $\mathcal{B}(H)$, is completely bounded by $1+\sqrt{2}$. In other words, the numerical range $W(A)$ is a complete $(1+\sqrt{2})$-spectral set for the operator $A$.

Let us recall that the numerical radius $w(B)$ of a linear operator $B$ in the Hilbert space $H$ is the number

$$w(B) = \sup_{z \in W(B)} |z|.$$  

Then, after (2), the interesting result [7, Theorem 3.1] implies

$$w(f(A)) \leq \sqrt{2} \sup_{z \in W(A)} |f(z)|,$$

for all rational functions bounded in $W(A)$, an estimate which also holds in the complete version. In the terminology of [7], this means that $W(A)$ is a complete $\sqrt{2}$-radius set for the operator $A$.

Let us point out that our approach to (2) is based on the Cauchy transform and only uses elementary tools. In particular, we do not use dilation theory, which has shown its efficiency in the case where $\Omega$ is a disk.

The paper is organized in two sections. Section 2 is devoted to some auxiliary lemmata, one of them (Lemma 1) studies the behavior of the Cauchy transform $g$ of the conjugate of $f \in \mathcal{A}(\Omega)$ up to the boundary of $\Omega$, an interesting issue that is addressed in the maximum norm setting by using the double layer potential. This, combined with a representation for the balance $f(A) + g(A)^*$ (Lemma 3) are the tools for the proof of the main result (2), presented in Section 3. The proof of (2) also shows that

$$\|f(A)\| \leq 2 \|f\|_\infty,$$

in case $f$ takes values in some sector with vertex at the origin and angle $\pi/2$, as commented in final Remark 10.

## 2 Auxiliary lemmata

The boundary $\partial \Omega$ of the open, bounded, convex set $\Omega \subset \mathbb{C}$ is assumed to be smooth. In the following, the algebra $\mathcal{A}(\Omega)$ is provided with the norm

$$\|f\|_\infty = \max\{|f(z)| : z \in \overline{\Omega}\}, \quad f \in \mathcal{A}(\Omega).$$

Besides, $\mathcal{C}(\partial \Omega)$ stands for the set of the complex continuous functions on $\partial \Omega$, endowed with the norm

$$\|\varphi\|_{\partial \Omega} := \max\{ |\varphi(\sigma)| : \sigma \in \partial \Omega \}, \quad \varphi \in \mathcal{C}(\partial \Omega).$$

For $\sigma \in \partial \Omega$, the corresponding unit outward normal vector is denoted by $\nu = \nu(\sigma)$ and, for $\sigma \in \partial \Omega$ and $z \in \mathbb{C} \setminus \{\sigma\}$, we introduce the double layer potential

$$\mu(\sigma, z) = \frac{1}{2\pi} \left( \frac{\nu}{\sigma - z} + \frac{\overline{\nu}}{\overline{\sigma} - \overline{z}} \right),$$
where $\nu = \nu(\sigma)$. It is geometrically clear that the set
\[ \Pi_\sigma := \{ z \in \mathbb{C} : \text{Re}(\nu(\sigma - z)) > 0 \} = \{ z \in \mathbb{C} \setminus \{ \sigma \} : \mu(\sigma, z) > 0 \} \]
is the open half-plane tangent containing $\Omega$ which is tangent to $\partial \Omega$ at point $\sigma$. Therefore, since $\Omega$ is convex, the claim $z \in \Omega$ is equivalent to say that $\mu(\sigma, z) > 0$, for all $\sigma \in \partial \Omega$. Analogously, we also note that $\mu(\sigma, \sigma_0) \geq 0$, for $\sigma, \sigma_0 \in \partial \Omega$ such that $\sigma \neq \sigma_0$.

Furthermore, we use a counterclockwise oriented, arclength parametrization $\sigma(s)$ of $\partial \Omega$. Then, $\sigma(\cdot)$ is $L$ periodic, with $L$ the length of $\partial \Omega$. It is noteworthy that $\nu(\sigma(s)) = \sigma'(s)/i$ and
\[ \mu(\sigma(s), z) = \frac{1}{\pi} \frac{d \arg(\sigma(s) - z)}{ds}, \quad \forall z \neq \sigma(s), \quad (5) \]
where $\arg$ stands for any continuous branch of the argument function defined in some neighborhood of $\sigma(s) - z \neq 0$. In the light of this identity, it is also geometrically clear that
\[ \int_{\partial \Omega} \mu(\sigma, z) ds = 2, \quad \forall z \in \Omega, \quad \text{and} \quad \int_{\partial \Omega} \mu(\sigma, \sigma_0) ds = 1, \quad \forall \sigma_0 \in \partial \Omega, \quad (6) \]
and in particular $\mu(\sigma(\cdot), \sigma_0)$ is a density of probability for $\sigma_0 \in \partial \Omega$.

We define the Cauchy transform of a complex function $\varphi$, defined at least on the boundary of $\Omega$ and continuous on it, as
\[ C(\varphi, z) = \frac{1}{2\pi i} \int_{\partial \Omega} \varphi(\sigma) \frac{d\sigma}{\sigma - z}, \quad \text{for} \ z \in \Omega. \]
Whereas $C(\varphi, \cdot)$ is holomorphic in $\Omega$, the behavior of $C(\varphi, z)$ as $z \in \Omega$ approaches a boundary point, in general, is not clear. In Lemma 1 below, we address this issue when $\varphi$ is the boundary value of the conjugate of $f \in \mathcal{A}(\Omega)$, which is the situation of interest in the present paper.

**Lemma 1.** Assume that $f \in \mathcal{A}(\Omega)$. Then $g = C(\bar{f}, \cdot)$ belongs to $\mathcal{A}(\Omega)$ and satisfies
\[ \|g\|_{\infty} \leq \|f\|_{\infty}. \]
Furthermore, $g(\partial \Omega) = \{ g(\sigma) : \sigma \in \partial \Omega \}$ is contained in the convex hull $\text{conv}(\bar{f}(\partial \Omega))$ of $\bar{f}(\partial \Omega) = \{ \bar{f}(\sigma) : \sigma \in \partial \Omega \}$.

**Proof.** Clearly, $g$ is holomorphic in $\Omega$. Besides, conjugating in the Cauchy formula
\[ f(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(\sigma)}{\sigma - z} d\sigma = \frac{1}{2\pi} \int_{\partial \Omega} \frac{f(\sigma)\nu(\sigma)}{\sigma - z} ds, \quad z \in \Omega, \]
leads to
\[ g(z) = \int_{\partial \Omega} \bar{f}(\sigma) \mu(\sigma, z) ds - \bar{f}(z). \]
Along the well-known jump formula (discovered by C. Gauss around 1815), if $z$ tends to $\sigma_0 \in \partial \Omega$, then $g(z)$ tends to $g(\sigma_0)$ defined on the boundary by
\[ g(\sigma_0) = \int_{\partial \Omega \setminus \{ \sigma_0 \}} \bar{f}(\sigma) \mu(\sigma, \sigma_0) ds, \quad \text{for} \ \sigma_0 \in \partial \Omega. \quad (7) \]
It is known since Carl Neumann [14] that with this extension the function $g$ is continuous in $\overline{\Omega}$, see for instance [9] Theorem 3.22. (As noticed by Neumann, the radial continuity up to the boundary follows easily while the global continuity requires a careful analysis).

Since $\mu(\sigma, \sigma_0) ds$ is a probability measure on $\partial \Omega$, it follows from (2) that $g(\partial \Omega)$ is contained in the closed convex hull of $\bar{f}(\partial \Omega)$ which, by compactness, coincides with $\text{conv}(\bar{f}(\partial \Omega))$. Using the maximum principle, we get $\|g\|_{\infty} \leq \|f\|_{\infty}$. 

\[ \square \]
Remark 2. The first part of this lemma is still valid if $\Omega$ unbounded but, generally in this case, $\mu(\sigma, \sigma_0) d\sigma$ is no longer a probability measure, so that the convex hull part is not guarantee.

The rest of the section concerns the Hilbert space setting. Given a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$, both the norm in $H$ and the induced norm in the algebra $B(H)$ of bounded linear operators on $H$ are denoted by $\| \cdot \|$.

For $A \in B(H)$ and $\sigma \in \partial \Omega$ in the resolvent of $A$, we set

$$
\mu(\sigma, A) := \frac{1}{2\pi} \left( \nu(\sigma - A)^{-1} + \overline{\nu(\sigma - A^*)^{-1}} \right).
$$

It turns out that

$$
W(A) \subseteq \Omega \implies \nu(\sigma - A^*) + \nu(\sigma - A) > 0 \implies \mu(\sigma, A) > 0, \ \forall \sigma \in \partial \Omega. \quad (8)
$$

Under the assumption $W(A) \subseteq \Omega$, it is also meaningful to define

$$
C(\varphi, A) = \frac{1}{2\pi i} \int_{\partial \Omega} \varphi(\sigma)(\sigma - A)^{-1} d\sigma \in B(H).
$$

Notice that $C(\varphi, A)$ does not corresponds to $\varphi(A)$, unless $\varphi$ can be extended to a member of $A(\Omega)$.

Lemma 3. For $\varphi \in C(\Omega)$, let us set

$$
S(\varphi, A) = C(\varphi, A) + C(\overline{\varphi}, A^*) \in B(H).
$$

Then we have

$$
\|S(\varphi, A)\| \leq 2\|\varphi\|_{\partial \Omega}.
$$

Proof. Let $\varphi : \partial \Omega \to \mathbb{C}$ be continuous and let us assume, without loss of generality, that $\|\varphi\|_{\partial \Omega} = 1$. It follows from the definition of $C(\varphi, A)$ that

$$
S(\varphi, A) = \int_{\partial \Omega} \varphi(\sigma) \mu(\sigma, A) d\sigma. \quad (9)
$$

The remaining part of this proof is classical, we include it by the convenience of the reader. We first observe that, by Cauchy formula, there holds

$$
\int_{\partial \Omega} \mu(\sigma, A) d\sigma = 2.
$$

Besides, for $\sigma \in \partial \Omega$, the operator $\mu(\sigma, A)$ is self-adjoint and positive. Therefore, for $x, y \in H$, we have

$$
\langle (S(\varphi, A)x, y) \rangle = \left| \int_{\partial \Omega} \varphi(\sigma) \langle \mu(\sigma, A)x, y \rangle d\sigma \right| \leq \int_{\partial \Omega} \left| \langle \mu(\sigma, A)x, y \rangle \right| d\sigma
$$

$$
\leq \int_{\partial \Omega} \langle \mu(\sigma, A)x, x \rangle^{1/2} \langle \mu(\sigma, A)y, y \rangle^{1/2} d\sigma
$$

$$
\leq \left( \int_{\partial \Omega} \langle \mu(\sigma, A)x, x \rangle d\sigma \right)^{1/2} \left( \int_{\partial \Omega} \langle \mu(\sigma, A)y, y \rangle d\sigma \right)^{1/2}
$$

$$
= \left( \int_{\partial \Omega} \mu(\sigma, A) d\sigma \langle x, x \rangle \right)^{1/2} \left( \int_{\partial \Omega} \mu(\sigma, A) d\sigma \langle y, y \rangle \right)^{1/2}
$$

$$
= 2 \|x\| \|y\|,
$$
so that
\[ \|S(f, A)\| = \sup_{\|x\|=1,\|y\|=1} |\langle S(\varphi, A)x, y \rangle| \leq 2. \]

\[ \square \]

**Remark 4.** In the particular case of the unit disk \( \Omega = \mathbb{D} \) and for \( f \in \mathcal{A}(\Omega) \), it follows from Cauchy formula that \( C(\mathcal{F}, z) = \overline{f(0)} \) and \( C(\mathcal{F}, A)^* = f(0) \). Then, after the representation \( f(A) = S(f, A) - f(0) \), a direct application of Lemma 3 yields the famous Berger-Stampfl estimate [1]
\[ \|f(A)\| \leq 2 \|f\|_{\infty}, \quad \text{if} \quad f(0) = 0. \]

Let us point out that a further result of Okubo and Ando [13] shows that this estimate remains valid even if \( f(0) \neq 0 \). Note that the proof of Berger and Stampfl, as the ones of Okubo and Ando, are based on dilation theory.

**Remark 5.** If \( \Omega \) is unbounded, there exists a greater \( \alpha \geq 0 \) such that \( \Omega \) contains a sector of angle \( 2\alpha \). Then this lemma is still valid with the improvement
\[ \|S(\varphi, A)\| \leq 2 \frac{\pi - \alpha}{\pi} \|\varphi\|_{\partial \Omega}. \]

3 Main result

In this section, \( H \) will denote a complex Hilbert space, \( A \) a bounded operator on \( H \) and \( \Omega \) a bounded convex domain of \( \mathbb{C} \) with smooth boundary. As we commented in the Introduction, this smoothness assumption, convenient to avoid technical difficulties in the proofs, may be easily relaxed afterwards.

**Theorem 6.** The following uniform bounds holds: \( C_\Omega \leq 1 + \sqrt{2} \).

**Proof.** It suffices to look at the case \( C_\Omega > 1 \). Then, since \( C_\Omega \) is the best constant, given \( 0 < \varepsilon < C_\Omega - 1 \), there exist \( f \in \mathcal{A}(\Omega) \), with \( \|f\|_{\infty} = 1 \), a Hilbert space \( H \) and an operator \( A \in B(H) \) with \( \overline{W(A)} \subset \Omega \) and such that \( \lambda = \|f(A)\| \geq C_\Omega - \varepsilon > 1 \).

Let us set \( g = C(\mathcal{F}, \cdot) \in \mathcal{A}(\Omega) \) and \( S = C(f, A) + C(g, A)^* \in B(H) \). We note that for the multiplication \( fg \in \mathcal{A}(\Omega) \) there holds \( fg(A) = g(A)f(A) \), so that
\[ \lambda^2 - f(A)^*f(A) = \lambda^2 - S^*f(A) + fg(A). \]

Moreover, by Lemma 1 we have \( \|fg\|_{\infty} \leq 1 \), and since
\[ |\lambda^2 + fg| \geq \lambda^2 - 1 > 0, \quad (10) \]
we deduce that the mapping \( \lambda^2 + fg \in \mathcal{A}(\Omega) \) never vanishes. Therefore, the operator \( \lambda^2 + fg(A) \) is invertible and we can write
\[ \lambda^2 - f(A)^*f(A) = (I - S^*h(A))(\lambda^2 + fg(A)), \]
where \( h = f/(\lambda^2 + fg) \in \mathcal{A}(\Omega) \). Next, we observe that the operator \( \lambda^2 - f(A)^*f(A) \) is singular, whence the factor \( (I - S^*h(A)) \) is also singular. Therefore, \( 1 \leq \|S^*h(A)\| \) and then, in view of Lemma 3 and (10), we obtain
\[ 1 \leq 2\|h(A)\| \leq 2C_\Omega \|h\|_{\infty} \leq \frac{2C_\Omega}{\lambda^2 - 1}. \]
This shows that \((C_\Omega - \varepsilon)^2 = \lambda^2 \leq 2C_\Omega + 1\), which, by letting \(\varepsilon \to 0_+\), readily yields
\[
C_\Omega^2 \leq 2C_\Omega + 1, \quad \text{and thus} \quad C_\Omega \leq 1 + \sqrt{2}.
\]

\[\square\]

As it is easily checked, Theorem 6 remains valid in the complete version. Therefore, since
\[
\frac{1}{2} \left( 1 + \sqrt{2} + \frac{1}{1 + \sqrt{2}} \right) = \sqrt{2},
\]
application of Theorem 6 and [7, Theorem 3.1] readily leads to the next

**Corollary 7.** We assume that \(W(A) \subset \overline{\Omega}\). Then there holds
\[
w(f(A)) \leq \sqrt{2} \|f\|_\infty, \quad \forall f \in \mathcal{A}(\Omega).
\]

**Remark 8.** In principle, estimating \(\|f(A)\|\) from the Corollary, in a direct way, would give
\[
\|f(A)\| \leq 2w(f(A)) \leq 2\sqrt{2}\|f\|_\infty. \quad \text{However, the Corollary holds in its complete version indeed}
\]
and then [7, Theorem 3.1] shows that the statements in Theorem 6 and in Corollary 7 are equivalents.

**Remark 9.** In the particular case of the unit disk \(\Omega = \mathbb{D}\), Drury [8] has obtained a more accurate estimate
\[
w(f(A)) \leq \frac{5}{4}\|f\|_\infty \quad \text{(see also [13])}; \quad \text{this constant} \ \frac{5}{4} \quad \text{is optimal. Previously, it was known}
\]
that, if furthermore \(f(0) = 0\), then \(w(f(A)) \leq \|f\|_\infty\); this result was obtained independently by Kato [12] and by Berger and Stampfli [1].

**Remark 10.** For \(\Sigma \subset \mathbb{C}\), set
\[
\mathcal{A}(\Omega, \Sigma) = \{ f \in \mathcal{A}(\Omega) : f(\Omega) \subset \Sigma \}.
\]

With very few changes, the proof of Theorem 6 also shows that
\[
\|f(A)\| \leq 2\|f\|_\infty, \quad \forall f \in \mathcal{A}(\Omega, \Sigma),
\]
whenever \(\Sigma \subset \mathbb{C}\) is a sector with vertex at the origin and angle \(\pi/2\). To see this, we first fix \(A\) and set
\[
C_{\Omega, \Sigma} = \sup\{ \|f(A)\| : f \in \mathcal{A}(\Omega, \Sigma), \|f\|_\infty = 1 \} < C_\Omega,
\]
so that we must prove that \(C_{\Omega, \Sigma} \leq 2\). It suffices to consider the case \(1 < C_{\Omega, \Sigma}\) and we also note that there is no loss of generality in assuming that \(\Sigma\) is the sector \(|\arg(z)| \leq \pi/4\).

Given \(\epsilon > 0\), we can select \(f \in \mathcal{A}(\Omega, \Sigma)\) such that \(\|f\|_\infty = 1\) and
\[
\lambda = \|f(A)\| \geq C_{\Omega, \Sigma} - \epsilon > 1.
\]

Following the same steps and notations than in the proof of Theorem 6, we first observe that, in view of Lemma 3, the mapping \(g = C(f, \cdot)\) also takes values in \(\Sigma\). Therefore, at points \(z \in \Omega\) where \(f(z) \neq 0\), we have
\[
h(z) = \frac{f(z)}{\lambda^2 + f(z)g(z)} = \frac{|f(z)|^2}{\lambda^2 |f(z)| + |f(z)|^2 g(z)}
\]
and, since \(|f(z)| + |f(z)|^2 g(z)\) \(\in \Sigma\), we deduce that \(h(z) \in \Sigma\). We thus conclude that \(h \in \mathcal{A}(\Omega, \Sigma)\). Furthermore, it is also clear that \(\text{Re}(fg) \geq 0\), so that \(|\lambda^2 + fg| \geq \lambda^2\), which gives \(\|h\|_\infty \leq 1/\lambda^2\).
Finally, since $h \in A(\Omega, \Sigma)$, we have $\|h(A)\| \leq C_{\Omega, \Sigma}\|h\|_{\infty} \leq C_{\Omega, \Sigma}/\lambda^2$ and, recalling that $1 \leq 2\|h(A)\|$, we obtain

\[ 1 \leq 2\|h(A)\| \leq C_{\Omega, \Sigma}/\lambda^2 \Rightarrow \lambda^2 \leq 2C_{\Omega, \Sigma}, \]

whence $(C_{\Omega, \Sigma} - \epsilon)^2 \leq \lambda^2 \leq 2C_{\Omega, \Sigma}$.

ACKNOWLEDGEMENTS: Second author has been financed by the Spanish Ministerio de Economía y Competitividad under project MTM2014-54710-P.

References

[1] C.A. Berger and J.G. Stampfli, Mapping theorems for the numerical range, Am. J. Math., 89 (1967), pp. 1047–1055.

[2] D. Choi, A proof of Crouzeix’s conjecture for a class of matrices, Linear Algebra and its Applications, 438, no. 8 (2013), pp. 3247–3257.

[3] D. Choi, A. Greenbaum, Roots of matrices in the study of GMRES convergence and Crouzeix’s conjecture. SIAM J. Matrix Anal. Appl. 36 (2015), no. 1, pp. 289–301.

[4] M. Crouzeix, Bounds for analytic functions of matrices, Int. Equ. Op. Th., 48, (2004), pp. 461–477.

[5] M. Crouzeix, Numerical range and functional calculus in Hilbert space, J. Funct. Anal., 244 (2007), pp. 668–690.

[6] M. Crouzeix, Some constants related to numerical ranges, SIAM Journal on Matrix Analysis and Applications, 37 (2016), pp. 420–442.

[7] K.R. Davidson, V.I. Paulsen and H.J. Woerdeman, Complete spectral sets and Numerical Range, ArXiv:1612.05633v1.

[8] S.W. Drury, Symbolic calculus of operators with unit numerical radius, Lin. Alg. Appl. 428 (2008), pp. 2061–2069.

[9] G.B. Folland, Introduction to Partial Differential Equations, Princeton Univ. Press, 1995.

[10] A. Greenbaum, A.S. Lewis, M.L. Overton, Variational Analysis of the Crouzeix Ratio, Mathematical Programming (2016), to appear.

[11] B. & F. Delyon, Generalization of Von Neumann’s spectral sets and integral representation of operators, Bull. Soc. Math. France, 1 (1999), pp. 25–42.

[12] T. Kato, Some mapping theorems for the numerical range, Proc. Japan Acad. 41 (1965), pp. 65–655.

[13] H. Klaja, J. Mashreghi, and Th. Ransford, On mapping theorems for numerical range. Proc. Amer. Math. Soc. 144 (2016), no. 7, pp. 3009–3018.

[14] C. Neumann, Untersuchungen über das logarithmische und Newton’sche Potential, Leipzig, (1877).
[15] K. Okubo and T. Ando, Constants related to operators of class $C_ρ$, *Manuscripta Math.* **16**, no 4, (1975), pp. 385–394.

[16] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Univ. Press, 2002.