Mean-Field Gauge Interactions in Five Dimensions I.
The Torus

Nikos Irges and Francesco Knechtli

Department of Physics, Bergische Universität Wuppertal
Gaussstr. 20, D-42119 Wuppertal, Germany
e-mail: irges, knechtli@physik.uni-wuppertal.de

Abstract

We consider the lattice regularization of a five dimensional \(SU(2)\) gauge theory with periodic boundary conditions. We determine a consistent mean-field background and perform computations of various observables originating from fluctuations around this background. Our aim is to extract the properties of the system in regimes of its phase diagram where it seems to be in a dimensionally reduced state. Within the mean-field theory we establish the existence of a second order phase transition at finite value of the gauge coupling for anisotropy parameter less than one, where there is evidence for dimensional reduction.
1 Introduction

It is possible that our world has more than four space-time dimensions. There are different ways that extra dimensions could leave a trace depending, among other things, on which of the fundamental forces we choose to look at. We will concentrate here on the gauge interactions. Leaving to the side for the moment the fermionic sector we present here our investigation of the dynamics of extra dimensional pure gauge theories with focus on dimensional reduction. More specifically, we would like to propose a scheme where this phenomenon can be analyzed analytically, far from the perturbative regime.

The phase diagram of a five dimensional $SU(N)$ gauge theory in infinite four dimensional spatial volume is parametrized by the two dimensionless couplings $\beta = 2N/(g_5^2 \Lambda)$ and $N_5 = 2\pi R \Lambda$, with $g_5$ the five dimensional bare coupling, $R$ the physical length parametrizing the size of the fifth dimension and $\Lambda$ a cut-off. The domain of weak coupling perturbation theory is the vicinity of the "trivial" point $\beta, N_5 \to \infty$ of the phase diagram which is approached as one removes the cutoff. From the formal point of view, up to now most analytical investigations of higher dimensional theories have been carried out in this domain, which however comes with two caveats: as one approaches the fixed point, physics is governed by the triviality of the coupling and as one tries to enter in the interior of the phase diagram cut-off effects dominate. Far from the trivial point, in the small $\beta$ regime, the system finds itself in a confined phase below a critical value $\beta_c$ of order one. The regime $\beta_c < \beta << \infty$ where neither perturbation theory nor the strong coupling expansion is useful, up to now, is analytically basically unexplored.

The success of the Standard Model (SM) in explaining experimental data requires, after introducing a new ingredient, a natural way to sufficiently hide it. As far as higher dimensional theories are concerned this would mean either that the extra dimensions are very small or that there is a four dimensional slice in the higher dimensional space where at least part of the physics is localized. From the phenomenological point of view the former situation has been also investigated by standard methods, mainly with a combination of Kaluza-Klein theory and weak coupling perturbation theory [1]. In this approach the size of the extra dimension(s) can be tuned to small values and the delicate issue is to make it small enough so that it does not lead to a contradiction with the data but large enough so that it can have a sizable effect. In the localization scheme, the extra dimension can be, in principle, as large as any other dimension (or even larger) but a dynamical mechanism is necessary to implement it. Unlike for gravity, for gauge fields a classical localization mechanism is not known and perturbation theory does not seem to help in this respect. Consequently in most model building approaches when a certain

---

1 Later we will be interested in anisotropic spaces where a third parameter, $\gamma$, will appear.
field needs to be localized (for some phenomenological reason) one just assumes that it is. Nevertheless, a few non-perturbative quantum mechanisms of gauge field localization were invented in the past [2,3]. In these, lattice Monte Carlo simulations have been proven a crucial tool. Despite all efforts, from the five dimensional point of view a quantum field theoretical approach to the localization of non-abelian gauge fields has been elusive.

This state of affairs calls for a complementary tool to the perturbative computations and the numerical simulations, one which would enable us to probe the system analytically up to its phase transition. The method known to work in the domain of $\beta$ of order one is the mean-field approximation [4,5]. This approach has been used before to locate the critical value of $\beta_c$ for various gauge theories on toroidal geometries (see [4] and references therein) and in [6] it was used to locate $\beta_c$ for an orbifold geometry. In the 80’s there was a considerable effort to define an expansion in the fluctuations around the saddle-point (or zero’th order) approximation [7]. The free energy was indeed computed to a high order in order to locate any phase transitions very precisely and hopefully predict their order. However, extensive observables other than the free energy were never computed systematically especially for theories of large dimensionality.

In fact, it is believed that the expansion around the mean-field background becomes a better and better approximation to the non-perturbative behavior of the system as the number of space-time dimensions increases so it seems that it is just tailored for our purposes. In addition, we need a regulator that maintains a finite cutoff while preserving gauge invariance. The computational scheme will be therefore a lattice regularization with lattice spacing $a = 1/\Lambda$ in the mean-field approximation to zero’th order. At higher orders we will perturb away from the mean-field background by corrections that indeed go as one over the number of space-time dimensions. For the basic definitions and conditions for a consistent formulation of a lattice non-abelian gauge theory in five dimensions as well as some related Monte Carlo results see [8,9].

In section 2 we describe the mean-field method applied to gauge theories and show how to compute local and global observables using fluctuations around the background. In section 3 we turn to the definition of lattice gauge theories around the mean-field background and concentrate on $SU(2)$. In section 4 we derive the free energy, the static potential and the mass formulae for the $SU(2)$ lattice gauge theory in five dimensions. In section 5 we discuss numerical results for these quantities and determine the phase diagram. In section 6 we give our conclusions.
2 Gauge theories in and around the mean-field background

2.1 The zero’th order approximation

The partition function of an $SU(N)$ gauge theory on a Euclidean lattice is

$$Z = \int D U e^{-S_G[U]}, \quad (2.1)$$

where $S_G[U]$ is the Wilson plaquette action defined in terms of oriented plaquettes $U(p)$. A plaquette is the product of four links around an elementary square on the lattice. Links are generically denoted by $U$. The mean-field computation starts by inserting an identity in the path integral, using the Fourier representation of the $\delta$ function

$$\delta(f(x)) = \int_{-i\infty}^{+i\infty} \frac{d\alpha(x)}{2\pi i} e^{-\alpha(x)f(x)}. \quad (2.2)$$

This allows to integrate out the $SU(N)$ variables $U$ in favor of a set of unconstrained variables $V$ and the set of Lagrange multipliers $H$ that implement the constraint $U = V$:

$$Z = \int D U \int D V \delta(V - U)e^{-S_G[U]} \quad (2.3)$$

$$= \int D U \int D V \int D H e^{(1/N)\text{Re tr}(HV)}e^{-S_G[V]} \quad (2.4)$$

An effective mean-field action $S_{\text{eff}}[V,H]$ is defined using

$$e^{-u(H)} = \int D U e^{(1/N)\text{Re tr}(UH)} \quad (2.5)$$

in terms of which we can express conveniently the partition function as

$$Z = \int D V \int D H e^{-S_{\text{eff}}[V,H]} \quad (2.6)$$

$$S_{\text{eff}} = S_G[V] + u(H) + (1/N)\text{Re tr}\{HV\}.$$

The mean-field or zero’th order approximation amounts to finding the minimum of the effective action when

$$H \rightarrow \bar{H}1, \quad V \rightarrow \bar{V}1, \quad S_{\text{eff}}[\bar{V},\bar{H}] = \text{minimal}. \quad (2.7)$$

Taking derivatives of eq. $(2.6)$ with respect to $V$ and $H$ one obtains the equations

$$\nabla V = -\frac{\partial u}{\partial H} \bigg|_{\bar{H}} \quad (2.8)$$

$$\bar{H} = -\frac{\partial S_G[V]}{\partial V} \bigg|_{\bar{V}}$$
which determine the mean-field background. It is a set of non-linear coupled algebraic equations that can be solved numerically by an iterative method.

The free energy per lattice site is defined as

$$F = -\frac{1}{N} \ln(Z),$$

where $N$ is the number of lattice sites. At 0th order it is simply

$$F^{(0)} = -\frac{1}{N} \ln(Z[V, H]) = S_{\text{eff}}[V, H].$$

### 2.2 First order corrections

Gaussian fluctuations are defined by setting

$$H = \bar{H} + h \quad \text{and} \quad V = \bar{V} + v$$

and Taylor expanding the effective action to second order in the derivatives

$$S_{\text{eff}} = S_{\text{eff}}[\bar{V}, \bar{H}] + \frac{1}{2} \left( \frac{\delta^2 S_{\text{eff}}}{\delta H^2} \right)_{\bar{V}, \bar{H}} h^2 + 2 \left( \frac{\delta^2 S_{\text{eff}}}{\delta H \delta V} \right)_{\bar{V}, \bar{H}} hv + \left( \frac{\delta^2 S_{\text{eff}}}{\delta V^2} \right)_{\bar{V}, \bar{H}} v^2.$$  \hfill (2.12)

We define the quadratic pieces of the propagator

$$\frac{\delta^2 S_{\text{eff}}}{\delta H^2} \bigg|_{\bar{V}, \bar{H}} h^2 = h_i K^{(hh)}_{ij} h_j = h^T K^{(hh)} h,$$

$$\frac{\delta^2 S_{\text{eff}}}{\delta V \delta H} \bigg|_{\bar{V}, \bar{H}} hv = v_i K^{(vh)}_{ij} h_j = v^T K^{(vh)} h,$$

$$\frac{\delta^2 S_{\text{eff}}}{\delta V^2} \bigg|_{\bar{V}, \bar{H}} v^2 = v_i K^{(vv)}_{ij} v_j = v^T K^{(vv)} v,$$

in terms of which we can express the part of the effective action quadratic in the fluctuations as

$$S^{(2)}[v, h] = \frac{1}{2} \left( h^T K^{(hh)} h + 2 v^T K^{(vh)} h + v^T K^{(vv)} v \right).$$

The integral of the fluctuations

$$z = \int Dv \int Dh \ e^{-S^{(2)}[v, h]}$$

is a Gaussian integral and it can be easily performed to give

$$z = \frac{(2\pi)^{|h|/2}(2\pi)^{|v|/2}}{\sqrt{\det(K^{(hh)} K)}},$$  \hfill (2.18)

The first order correction to the mean-field approximation in the fluctuations is second order with respect to the derivative expansion.
with $|h|$ and $|v|$ the dimensionalities of the $h$ and $v$ parameters respectively and
\[
K = -K^{(vh)} K^{(hh)^{-1}} K^{(vh)} + K^{(vv)} .
\] (2.19)
Therefore to first order
\[
Z^{(1)} = Z[V, H] \cdot z = e^{-S_{\text{eff}}[V,H]} \cdot z
\] (2.20)
The typical quantity of interest is
\[
\langle O \rangle = \frac{1}{Z} \int DU \mathcal{O}[U] e^{-S_G[U]}
\] (2.21)
\[
= \frac{1}{Z} \int DV \int DH \mathcal{O}[V] e^{-S_{\text{eff}}[V,H]},
\] (2.22)
with $\mathcal{O}$ a gauge invariant operator. Note that the $\delta$-function constraint in Eq. (2.3) has been used to replace $\mathcal{O}[U]$ with $\mathcal{O}[V]$. The purpose of this step is to replace a constrained $SU(N)$ path integral by a path integral over unconstrained matrix valued complex numbers. The number of degrees of freedom apparently increases but in fact the relevant constraint is encoded in $K^{(hh)}$.

The observable has a Taylor expansion of the form
\[
\mathcal{O}[V] = \mathcal{O}[\overline{V}] + \frac{\delta \mathcal{O}}{\delta \overline{V}} \bigg|_{\overline{V}} v + \frac{1}{2} \frac{\delta^2 \mathcal{O}}{\delta \overline{V}^2} \bigg|_{\overline{V}} v^2 + \ldots
\] (2.23)
The combined expansion of observable and action to the same order is
\[
\langle O \rangle = \frac{1}{Z} \int Dv \int Dh \left( \mathcal{O}[\overline{V}] + \frac{1}{2} \frac{\delta^2 \mathcal{O}}{\delta \overline{V}^2} \bigg|_{\overline{V}} v^2 \right) e^{-\left(S_{\text{eff}}[\overline{V},\overline{H}] + S^{(2)}[v,h]\right)}
\] (2.24)
\[
= \mathcal{O}[\overline{V}] + \frac{1}{2} \frac{\delta^2 \mathcal{O}}{\delta \overline{V}^2} \bigg|_{\overline{V}} \frac{1}{z} \int Dv \int Dh v^2 e^{-S^{(2)}[v,h]}.
\] (2.25)
The link two point function can be integrated to
\[
< v_i v_j > = \frac{1}{z} \int Dv \int Dh v_i v_j e^{-S^{(2)}[v,h]} = (K^{-1})_{ij},
\] (2.26)
where $K$ is the matrix defined in Eq. (2.19). The indices $i$ and $j$ denote collectively the indices on which the links depend. The physical observable expectation value to first order can be then expressed as
\[
\langle O \rangle = \mathcal{O}[\overline{V}] + \frac{1}{2} \text{tr} \left( \frac{\delta^2 \mathcal{O}}{\delta \overline{V}^2} K^{-1} \right).
\] (2.27)
At first order we define the free energy from Eq. (2.20) as
\[
F^{(1)} = F^{(0)} - \frac{1}{N} \ln(z) = F^{(0)} + \frac{1}{2N} \ln \left[ \det(K^{(hh)} K) \Delta_{FP}^{-2} \right],
\] (2.28)
where we have dropped an irrelevant additive constant and $\Delta_{FP}$ is the Faddeev-Popov determinant that appears after fixing the gauge.
2.2.1 Phase transitions

At zero’th order in the confined phase, the free energy $F$ has a minimum at $\mathcal{H} = 0$ (that typically implies $\mathcal{V} = 0$) which dominates over any other minima located at any other finite values of $\mathcal{H}$. As $\beta$ increases, a second local minimum develops and the critical value of $\beta$, which indicates when the system enters in the deconfined phase, is reached when this second local minimum becomes of equal height as the minimum at $\mathcal{H} = 0$. In the deconfined phase, the local minimum at $\mathcal{H} \neq 0$ turns into a global minimum. Typically, between the two minima the free energy develops a local maximum.

An alternative way to locate the phase transition is to look at the zero’th order mean-field conditions Eqs. (2.8) and define the deconfined phase wherever the numerical iterative method converges to a non-trivial solution. In general however we would like to emphasize that this is a safe method only as long as one is sure that the solution found corresponds to a global minimum of the free energy. Also we point out that the numerical value of $\beta_c$ depends quite a bit on whether we fix the gauge or not (this is an option at zero’th order), whether we use the iterative method or the direct minimization of the free energy, but not its existence. The numerical value has no physical meaning as far as we can tell. The only thing that has a physical meaning is if we are “near” or “far” from the phase transition and whether the phase we are sitting in is the global minum of the free energy or not.

To first order it is however necessary to fix the gauge. In gauges which require ghosts, a divergence in the free energy develops due to the presence of the Faddeev-Popov determinant that vanishes when the background is taken to zero. If the divergence is regulated (i.e. dropped) then the phase transition can be still located at the value of $\beta$ below which the global minimum is at $\mathcal{V} = 0$. If it is kept then the global minimum of the free energy has a discontinuous jump from a larger value to a much smaller but non-zero value. The value of $\beta$ at which this jump takes place signals the phase transition.

Our last comment on the issue of the phase transition concerns its degree. In [10] it was established and in [4] confirmed that on the isotropic lattice it is a strong, first order phase transition.

2.2.2 Extracting masses at first order

Let us take a generic, gauge invariant, time dependent operator or "observable" $\mathcal{O}(t)$ and its 2-point function $\mathcal{O}(t_0 + t)\mathcal{O}(t_0)$. In order to extract the mass associated with $\mathcal{O}(t)$, we
need the connected correlator
\[
C(t) = \langle \mathcal{O}(t_0 + t) \mathcal{O}(t_0) \rangle - \langle \mathcal{O}(t_0 + t) \rangle \langle \mathcal{O}(t_0) \rangle = C^{(0)}(t) + C^{(1)}(t) + \cdots
\] (2.29)
where the \(C^{(0)}(t)\) and \(C^{(1)}(t)\) can be identified from
\[
\langle \mathcal{O}(t_0 + t) \mathcal{O}(t_0) \rangle = \mathcal{O}^{(0)}(t_0 + t) \mathcal{O}^{(0)}(t_0) + \frac{1}{2} \text{tr} \left\{ \frac{\delta^2 \langle \mathcal{O}(t_0 + t) \mathcal{O}(t_0) \rangle}{\delta^2 v} K^{-1} \right\} + \cdots
\] (2.30)
Since \(C^{(0)}(t) = 0\) and because time independent contributions drop out from \(C^{(1)}(t)\), at first order we have
\[
C^{(1)}(t) = \frac{1}{2} \text{tr} \left\{ \frac{\delta^{(1,1)} \langle \mathcal{O}(t_0 + t) \mathcal{O}(t_0) \rangle}{\delta^2 v} K^{-1} \right\} = \frac{1}{2} \text{tr} \left\{ \frac{\tilde{\delta}^{(1,1)} \langle \mathcal{O}(t_0 + t) \mathcal{O}(t_0) \rangle}{\delta^2 v} \tilde{K}^{-1} \right\}
\] (2.31)
where the notation \(\delta^{(1,1)}\) means one derivative acting on each of the \(\mathcal{O}(t_0 + t)\) and \(\mathcal{O}(t_0)\) operators. In the second part of the equation tilded quantities are the Fourier transforms of the corresponding untilded quantities.

To extract the scalar mass we observe that a gauge invariant correlator admits an expansion in terms of the energy eigenvalues of the states it contains as
\[
C(t) = \sum_{\lambda} c_{\lambda} e^{-E_{\lambda} t},
\] (2.32)
where \(E_0 = m, E_1 = m^*, \cdots\) and therefore we have that
\[
m \simeq \lim_{t \to \infty} \ln \frac{C^{(1)}(t)}{C^{(1)}(t - 1)}.
\] (2.33)

2.2.3 The static potential

The static potential can be obtained from a Wilson loop \(\mathcal{O}_W\) extending in the time direction and in one spatial dimension. In the infinite time limit the expectation value of the loop is related to the static potential as
\[
t \to \infty : \quad e^{-Vt} \simeq \langle \mathcal{O}_W \rangle.
\] (2.34)
This immediately implies that in the mean-field background the potential is constant
\[
\mathcal{O}_W [\nabla] = N t_{V_{0}}^{2(r+1)}.
\] (2.35)
To obtain the first order correction, one must compute a gauge boson exchange between the two time-like legs of the loop plus the self energy and tadpole diagrams. All other
possible exchanges of gauge bosons vanish in the infinite time limit. The static potential is the quantity that will decide whether the system is in a dimensionally reduced state or not.

The static potential can be extracted to first order form

\[ V = -\lim_{t \to \infty} \frac{1}{t} \log \left( O_W[V] \right) - \lim_{t \to \infty} \frac{1}{t} \frac{\text{correction}}{O_W[V]} \]  

(2.36)

with the correction given by the sum of the exchange, self energy and tadpole diagrams.

### 2.3 Second order corrections

We will encounter cases where a physical observable (like a gauge boson mass for example) due to the choice of the operator that represents it, is identically vanishing to first order. We have to know therefore how to generalize our computations to second order in the expansion around the mean-field background.

To second order we have the expansion

\[ S_{\text{eff}} = S_{\text{eff}}[\nabla, H] + \frac{1}{2} \left( \frac{\delta^2 S_{\text{eff}}}{\delta H^2} h^2 + 2 \frac{\delta^2 S_{\text{eff}}}{\delta H \delta V} h v + \frac{\delta^2 S_{\text{eff}}}{\delta V^2} v^2 \right) \]

\[ + \frac{1}{6} \left( \frac{\delta^3 S_{\text{eff}}}{\delta H^3} h^3 + \frac{\delta^3 S_{\text{eff}}}{\delta V^3} v^3 \right) + \frac{1}{24} \left( \frac{\delta^4 S_{\text{eff}}}{\delta H^4} h^4 + \frac{\delta^4 S_{\text{eff}}}{\delta V^4} v^4 \right) + \cdots \]

(2.37)

The cross terms in the cubic and quartic terms vanish because of the special form of \( S_{\text{eff}} \).

We have a similar expansion for the observable

\[ O[V] = O[\nabla] + \frac{\delta O}{\delta V} v + \frac{1}{2} \frac{\delta^2 O}{\delta V^2} v^2 + \frac{1}{6} \frac{\delta^3 O}{\delta V^3} v^3 + \frac{1}{24} \frac{\delta^4 O}{\delta V^4} v^4 + \cdots \]

(2.38)

Straightforward algebra leads to the result that the tadpole-free contributions to the observable are

\[ < O > = O[\nabla] + \frac{1}{z} \int Dh Dv \left( \frac{1}{2} \frac{\delta^2 O}{\delta V^2} v^2 + \frac{1}{24} \frac{\delta^4 O}{\delta V^4} v^4 \right) e^{-S^{(2)}[v, h]} \]

(2.39)

where the second order correction involves the integral

\[ \frac{1}{z} \int Dv \int Dh \left( v_i v_j v_l v_m \right) e^{-S^{(2)}[v, h]} = \]

\[ (K^{-1})_{ij}(K^{-1})_{lm} + (K^{-1})_{il}(K^{-1})_{jm} + (K^{-1})_{im}(K^{-1})_{jl} \]

(2.40)

The contribution \( \frac{\delta O}{\delta V} \frac{\delta^4 S_{\text{eff}}}{\delta V^4} \) leads to a time independent tadpole contribution to the correlator Eq. (2.29) and can be dropped. We can finally express the physical expectation value
as

\[
\langle \mathcal{O} \rangle = \mathcal{O}[\mathcal{V}] + \frac{1}{2} \left( \frac{\delta^2 \mathcal{O}}{\delta \mathcal{V}^2} \right)_{ij} (K^{-1})_{ij} + \frac{1}{24} \sum_{i,j,l,m} \left( \frac{\delta^4 \mathcal{O}}{\delta \mathcal{V}^4} \right)_{ijlm} \left[ (K^{-1})_{ij}(K^{-1})_{lm} + (K^{-1})_{il}(K^{-1})_{jm} + (K^{-1})_{im}(K^{-1})_{jl} \right]
\]

(2.41)

To extract the masses to second order is straightforward. Again, all time independent self energies cancel from connected correlators and at the end the mass is obtained from

\[
m \simeq \lim_{t \to \infty} \ln \frac{C^{(1)}(t) + C^{(2)}(t)}{C^{(1)}(t-1) + C^{(2)}(t-1)}.
\]

(2.42)

with

\[
C^{(2)}(t) = \frac{1}{24} \sum_{i,j,l,m} \left( \frac{\delta^4 \mathcal{O}(t)}{\delta \mathcal{V}^4} \right)_{ijlm} \left[ (K^{-1})_{ij}(K^{-1})_{lm} + (K^{-1})_{il}(K^{-1})_{jm} + (K^{-1})_{im}(K^{-1})_{jl} \right].
\]

(2.43)

2.4 Gauge dependence and zero modes

Gauge dependence of the results turns out to be the most controversial issue in computing fluctuations around the mean-field background. It can be shown [11] that for any gauge invariant polynomial \( P(U) \) of the links it is true that

\[
\frac{1}{Z} \int DU e^{-S_W} P(U) = \frac{1}{Z} \int DU \delta(U_t, g) e^{-S_W} P(U)
\]

with \( S_W \) the gauge invariant (for example Wilson plaquette) action, and \( g \) an \( SU(N) \) group element. The inserted term \( \delta(U_t, g) \) allows the link \( U_t \) to be set equal to \( g \). This implies that one can set any particular link to a chosen element which is typically taken to be the unit element. Iterating this construction one can set to unity any number of links, as long as there are no closed loops forming in the process. The argument then is that since the lattice action, the measure and the operator ”observable” \( P(U) \) are explicitly gauge invariant, then any correlator is also gauge invariant. This allows one to compute the correlator in a fixed gauge and the previous argument guarantees that this will not affect the gauge invariant result, as long as one can compute exactly the left hand side of eq. (2.44) analytically. This is however not possible in general and one either computes the correlator via a Monte Carlo simulation or analytically in some approximation.

Higher dimensional gauge theories offer an excellent opportunity to combine successfully the Monte Carlo and mean-field methods because of the expectation that the expansion around the mean-field background becomes a better approximation as the number
of space time dimensions increases \cite{1}. Thus, in a five dimensional gauge theory one can compute the same quantities in two different ways, one numerical and gauge invariant and one analytical but gauge dependent and if these two agree to a satisfactory degree (as one would expect from general arguments) one can be confident that the calculations are not out of control. By "satisfactory degree" we mean that both methods should describe the same physics qualitatively (this is a necessary condition) and they must be in a good quantitative agreement. By good quantitative agreement we mean that of course the numbers should not be off by order of magnitudes and if the two methods agree on the physics, some quantitative disagreement can be perhaps tolerated. The next issue is which gauge to choose.

Taking maximal advantage of the gauge fixing possibility amounts to fixing the links to unity on a spanning (or maximal) tree. The disadvantage of this "maximal" gauge is that it is hard to perform explicit calculations with it for non-Abelian groups. Another simple choice is that of the axial gauge where one fixes to unity links along only a specific direction. An advantage of this gauge is that it is by construction ghost free. A great disadvantage is that it leaves part of the local gauge invariance unfixed and as a consequence most observables are plagued by zero modes of the propagator which correspond to local gauge transformations that are independent of the gauge fixed coordinate. In momentum space they appear whenever the momentum \( p_j = 0 \), where \( j \) is the gauge fixed direction. If one restricts the analysis to observables that do not see these zero modes (i.e. if they couple to sectors of the propagator with non-zero eigenvalues only), the axial gauge is actually fine. If the local zero modes do affect the observables, one must fix the residual gauge and then ghosts appear. A further subtle issue of the axial gauge concerns the \( j \)-coordinate dependence of the various quantities in the axial gauge on a finite lattice. In order that the axial gauge is performed, one has to assume that the gauge fixed direction is infinite, however one can compute in practice the propagator and observables only at a finite extent of the \( j \) direction. In order to minimize the ambiguities originating from this, one should take this direction as large as possible.

Finally, there are more conventional gauges such as the Lorentz gauge which is the one we will use here. It involves ghosts and one must take account of them. All gauge invariant correlators evaluated in the zero’th order background turn out to be gauge independent. In some cases, like for the static potential and the scalar field’s mass, we will be able to show this analytically and in others, like for the vector field’s mass, we will be able to show it only numerically. The only quantity which will have some gauge dependence is the first order correction to the free energy. We do not fix the gauge at zero’th order and we will make sure that the gauge dependence of the first order correction results only in quantitative changes (i.e. it moves around a bit the minimum) but does not change the qualitative properties of the system, for moderate varying of the gauge fixing parameter.
This of course means that numbers originating from the minimization of the free energy should not be given any physical meaning. If this had to be the case, like in a next to leading order calculation, a more careful treatment would be necessary. Since we will not use the first order free energy for anything else other than showing stability, we will not need to carry out any special regularization.

As already stated in [4], in any gauge with periodic boundary conditions there are additional zero modes, the global zero modes (torons) which appear when all lattice momenta vanish simultaneously. These zero modes are present and they must be taken into account as well [12]. In our case, they always show up as a zero of the inverse propagator at the same time with a zero in either the observable or in the Fadeev-Popov determinant. The result will be always a zero over zero contribution which can be regularized and a finite part can be extracted. This finite piece however is volume suppressed and contributes a negligible amount. Our regularization of the toron contributions will be thus to simply drop them. The only other regularization we do is drop an overall constant contribution from the free energy and which has no physical meaning.

3 The lattice model and its observables

3.1 The action

The discretized version of the torus is defined on a five-dimensional Euclidean lattice with lattice spacing $a$. The points are labeled by integer coordinates $n \equiv \{n_M\}$. The dimensionless lengths of the lattice are $L = l/a$ in the spatial directions ($M = \mu = 1, 2, 3$), $T$ in the time-like direction ($M = \mu = 0$) and $N_5 = 2\pi R/a$ in the extra dimension ($M = 5$). Periodic boundary conditions are used in all directions. The dimensionless length of a direction on the lattice is just the number of points in that direction. Occasionally, we will use the notation $\mathcal{N}$ for the total number of points in the lattice. The convention for the fifth coordinate labels is that $n_5 = 0, \cdots, N_5 - 1$. The lattice momenta are correspondingly

$$p_M = \frac{2\pi}{L_M}k, \quad k = 0, 1, \ldots, L_M - 1 \quad (3.1)$$

and $L_M$ is the dimensionless length of the lattice in direction $M$. The gauge field is the set of link variables $\{U(n, M) \in SU(N)\}$ and its action is taken to be the Wilson action

$$S_W[U] = \frac{\beta}{2N} \sum_p \text{tr} \left\{ 1 - U(p) \right\}, \quad (3.2)$$

\footnote{Unless otherwise stated, dimensionless coordinates and momenta are used.}
where the sum runs over all oriented plaquettes \( U(p) \). A plaquette at point \( n \) in directions \( M \) and \( K \) is defined as the product
\[
U(p) = \text{tr} \left\{ U(n, M)U(n + \hat{M}, K)U^\dagger(n + \hat{K}, M)U^\dagger(n, K) \right\}
\] (3.3)
of links. A gauge transformation \( \Omega \) acts on a link as
\[
U(n, M) \rightarrow \Omega(n)U(n, M)\Omega^\dagger(n + \hat{M})
\] (3.4)
The Wilson action reproduces the correct naive continuum gauge action of the fields \( A_M^B, B = 1, \ldots, N^2 - 1 \) of the five-dimensional gauge potential \( A_M \) defined through
\[
U(n, M) = \exp\{aA_M(n)\}.
\]

We will now apply the general formalism we described to a specific example: an \( SU(2) \) lattice gauge theory in 5 dimensions with fully periodic boundary conditions.

### 3.2 Ghosts

A general \( 2 \times 2 \) complex matrix can be represented by
\[
v_0 + i \sum_{A=1}^3 v_A^\sigma^A, \quad v_0, v_A \in \mathbb{C}.
\] (3.5)
We choose the Lorentz gauge, which amounts to adding the gauge fixing term
\[
S_{GF} = \frac{1}{2\xi} \sum_{A=1}^3 \sum_n [f_A(n)]^2
\] (3.6)
to the lattice action. We will now derive the corresponding Faddeev-Popov term. Notice that we are fixing the link fluctuations, not just the gauge field fluctuations. Also, we do not fix the link fluctuations along the Hermitian component \( v_0 \). Consistently, it will turn out that the propagator for \( v_0 \) does not have poles. A similar statement is implied in [4] for \( SU(N) \) in general.

In order to compute the Faddeev-Popov determinant we have to compute the variation of the gauge fixing term
\[
f_A(n) = \sum_M [v_A(n, M) - v_A(n - \hat{M}, M)]
\] (3.7)
under infinitesimal gauge transformations. We parametrize gauge transformation functions as \( \Omega(n) = e^{i\omega^A(n)\sigma^A} = 1 + i\omega^A(n)\sigma^A \) where the second equality holds for infinitesimal transformations. Then, the gauge transformation rules of the components of a link are
\[
\begin{align*}
\delta v_0(n, M) &= v^A(n, M) \left( \omega^A(n + \hat{M}) - \omega^A(n) \right) \\
\delta v_C(n, M) &= -v_0(n, M) \left( \omega^C(n + \hat{M}) - \omega^C(n) \right) \\
&\quad - \epsilon^{ABC} v^B(n, M) \left( \omega^A(n + \hat{M}) + \omega^A(n) \right)
\end{align*}
\] (3.8)
and the transformation they induce on the gauge fixing function is

$$\delta f_C(n) = \sum_M \left[ \delta v_C(n, M) - \delta v_C(n - \hat{M}, M) \right] = \sum_{m,A} M_{Cn;Am} \omega^A(m) \quad (3.9)$$

with the ghost kernel defined as $M_{Cn;Am} = \sum_M M_{Cn;Am;Bm} C_{n;Am}^B$. (3.10)

The ghost action is then

$$S_{FP} = \sum_{n,m} \omega^A(n) M_{An;Bm} C_{n;Bm}^B(m). \quad (3.11)$$

The quadratic part of this defines the Faddeev-Popov kernel which in a general mean-field background $\mathbf{V} = \mathbf{V}(n, M) \cdot \mathbf{1}$ is

$$M_{An;Bm}(\mathbf{V}) = \delta^B \sum_M \left[ \delta_{n,m} \left( \mathbf{V}(n, M) + \mathbf{V}(n - \hat{M}, M) \right) - \delta_{m,n+\hat{A}} \mathbf{V}(n, M) - \delta_{m,n-\hat{A}} \mathbf{V}(n - \hat{M}, M) \right]. \quad (3.12)$$

Note that the gauge boson-gauge boson-ghost vertex is proportional to $\mathbf{V}$. In a constant background Eq. (3.12) is the lattice version of the Laplace operator, while in a non-constant background its appropriate generalization. Note also that $\delta \mathbf{v}_0(n)|_{\mathbf{V}} = 0$ is consistent with not gauge fixing $\mathbf{v}_0$.

### 3.3 Observables

We first define the auxiliary “lines”

$$l^{(m)}(m_0, \vec{m}) = \prod_{m_5=0}^{n_5-1} U((m_0, \vec{m}, m_5); 5) = \prod_{m_5=0}^{n_5-1} \left[ v_0 \mathbf{1} + v_\alpha ((m_0, \vec{m}, m_5); 5) \sigma^\alpha \right] \quad (3.13)$$

and

$$l^{(t)}(t_0, \vec{m}, m_5) = \prod_{m_0=t_0}^{t_0+t-1} U((m_0, \vec{m}, m_5); 0) = \prod_{m_0=t_0}^{t_0+t-1} \left[ v_0 \mathbf{1} + v_\alpha ((m_0, \vec{m}, m_5); 0) \sigma^\alpha \right], \quad (3.14)$$
Figure 1: Contributions to the static potential on the torus: from left to right, gauge boson exchange, self energy and tadpole.

where in the second parts of the equations we inserted the mean-field parametrization and we have introduced the matrices

$$\sigma^\alpha = \{1, i\sigma^A\}, \quad \bar{\sigma}^\alpha = \{1, -i\sigma^A\}, \quad \alpha = 0, A$$

(3.15)

The line along the extra dimension can start from any spatial point \(\vec{m}\) but will be always taken to start from \(m_5 = 0\) while the temporal line can start from any \(t_0, \vec{m}, m_5\). The superscript indicates the extent of the line. Finally, we will be using the label \(m_0\) or \(t, t', t''\) etc. for the temporal components according to convenience.

The static potential will be computed using the averaged version over the 4d starting position \((t_0, \vec{m})\)

$$\mathcal{O}_W^{(t-n_5)}(t_0, \vec{m}) = \frac{1}{TL^3} \sum_{t_0, \vec{m}} \mathcal{O}_W^{(t-n_5)}(t_0, \vec{m})$$

(3.16)

of the Wilson loop observable

$$\mathcal{O}_W^{(t-n_5)}(t_0, \vec{m}) = \text{tr}\left\{l^{(t)}(t_0, \vec{m}, 0) l^{(n_5)}(t + t_0, \vec{m}) l^{(t)\dagger}(t_0, \vec{m}, n_5) l^{(n_5)\dagger}(t_0, \vec{m})\right\}.$$  

(3.17)

Similarly to \(l^{(n_5)}\) one can define spatial lines \(l^{(n_k)}\) along the \(k = 1, 2, 3\) directions as well. In fig. 1 the diagrams contributing to the static potential can be seen.

For the masses we first define the Polyakov loop

$$P^{(0)}(t, \vec{m}) = l^{(N_5)}(t, \vec{m})$$

(3.18)

in terms of which we define the operator

$$\Phi^{(0)}(t, \vec{m}) = P^{(0)}(t, \vec{m}) - P^{(0)\dagger}(t, \vec{m}).$$

(3.19)

The scalar gauge invariant operator

$$\mathcal{O}_0(t, \vec{m}) = \text{tr}\{P^{(0)}(t, \vec{m})\}$$

(3.20)
is then used to determine the Scalar ("Higgs") observable, determined by the averaged over space and time location of the operator

\[ O_H(t) = \frac{1}{T} \sum_{t_0} \frac{1}{L^6} \sum_{\vec{m}', \vec{m}''} O_0(t_0, \vec{m}') O_0(t_0 + t, \vec{m}'') \]  

(3.21)

Next, we construct the displaced Polyakov loop

\[ W^{(0),A}_k(t, \vec{m}) = \sigma^A U((t, \vec{m}, 0); k) \Phi^{(0)\dagger}(t, \vec{m} + \hat{k}) U((t, \vec{m}, 0); k)^\dagger \Phi^{(0)}(t, \vec{m}) \]  

(3.22)

which is an operator with a vector and a gauge index. The gauge covariant quantity

\[ O^A_k(t, \vec{m}) = \text{tr} \left\{ W^{(0),A}_k(t, \vec{m}) \right\} \]  

(3.23)

will be used to define our Vector ("W") mass observable, via the 2-point function

\[ O_V(t) = \frac{1}{T} \sum_{t_0} \frac{1}{L^6} \sum_{\vec{m}', \vec{m}''} \sum_A \sum_{k,l} O^A_k(t_0, \vec{m}') \delta_{kl} O^A_l(t_0 + t, \vec{m}'') \]  

(3.24)

where \( A = 1, 2, 3 \) is a sum over the gauge index and \( k, l = 1, 2, 3 \) a sum over the spatial Euclidean index. We have set the lattice spacing \( a = 1 \) for simplicity. It will be reintroduced when we discuss the results, where it has a physical meaning.

The choice eq. (3.20) gives a non-trivial result for the Higgs mass already at first order. A consequence of this choice is that since it is essentially a single gauge boson exchange diagram, see Fig. 2 there is no \( L \)-dependence built in it by construction. Therefore, by measuring the mass using this observable, we conveniently obtain its infinite \( L \) limit value. If desired, the second order correction could be added which will bring in an intrinsic \( L \)-dependence. Here, we will restrict ourselves to the first order expression. In Fig. 3 the second order contribution to the gauge boson mass is shown.

These choices are of course not unique but not arbitrary either. Their general form is fixed by their transformation properties under charge conjugation (\( C \)), three dimensional coordinate inversion (\( P \)) and by their Lorentz (spin \( J \)) and gauge (isospin \( I \)) indices [13]. The basic quantity is the field \( \Phi^{(0)} \) which represents the extra dimensional component of the gauge field (which we have called Higgs for short). It is easy to see that its continuum limit is [9]

\[ \Phi^{(0)} = 4N_5 a A_5 + O(a^3). \]  

(3.25)

We will be interested in taking the continuum limit at finite \( R = \frac{N_5 a}{2\pi} \). In this limit, \( \Phi^{(0)} \) is fixed. The continuum limit of \( O^A_k \) is on the other hand [9]

\[ O^A_k = a \text{tr}\{\sigma^A (D_k \Phi^{(0)\dagger}) \Phi^{(0)}\}, \]  

(3.26)
Figure 2: Contributions to the mass of the scalar at first order in the mean-field. The blobs are arbitrary lattice points, identified on the torus.

Figure 3: Contribution to the gauge boson mass at second order in the mean-field. Points at a fixed $n_\mu$, at $n_5 = 0$ and $n_5 = N_5$ are identified.

where $D_k \Phi^{(0)} = \partial_k + [A_k, \Phi^{(0)}]$ is the covariant derivative.

The charge conjugation $C$ leaves the coordinates invariant and acts on links as

$$U^* = \sigma^2 U \sigma^2,$$

while parity $P$ acts on the coordinates as $Pn = n_P = (n_0, -\vec{n}, n_5)$ and on the links as

$$PU(n,k) = U^\dagger(n_P - \hat{k}, k),$$
$$PU(n,0) = U(n_P,0),$$
$$PU(n,5) = U(n_P,5).$$

By construction, $O_0$ of Eq. (3.20) has $C = P = 1$ (scalar) and $O_k^A$ of Eq. (3.23) has $C = 1, P = -1$ (vector). The isospin is defined by the transformation of the operators under global gauge transformations. For the definition of the spin of lattice operators we refer to [14]. It is easy to see that $O_0$ has $I = J = 0$ and $O_k^A$ has $I = J = 1$. 

17
4 Free energy, static potential and mass formulae in the SU(2) theory

In this section we continue our discussion on the SU(2) theory. In expressions valid though for general SU(N) we keep an explicit N.

4.1 At zero’th order

The SU(2) link matrices have the representation

\[ U(n, M) = u_0(n, M) \mathbf{1} + i \sum_{A=1}^{3} u_A(n, M) \sigma^A, \quad u_\alpha \in \mathbb{R}, \quad u_\alpha u_\alpha = 1, \quad (4.1) \]

where \( \alpha = 0, 1, 2, 3 \). The auxiliary matrices \( V \) and \( H \) are arbitrary complex \( 2 \times 2 \) matrices and can be represented by 4 complex numbers as

\[ V(n, M) = v_0(n, M) \mathbf{1} + i \sum_{A} v_A(n, M) \sigma^A, \quad v_\alpha \in \mathbb{C}, \quad (4.2) \]

\[ H(n, M) = h_0(n, M) \mathbf{1} - i \sum_{A} h_A(n, M) \sigma^A, \quad h_\alpha \in \mathbb{C}. \quad (4.3) \]

A standard computation gives for the effective action \( u \) the result

\[ u(H(n, M)) = -\ln \left( \frac{2}{\rho} I_1(\rho) \right) \quad \rho(n, M) = \sqrt{\sum_\alpha (\text{Re } h_\alpha(n, M))^2} \quad (4.4) \]

with \( I_1 \) the Bessel function of the \( I \) type and evidently it does not depend on the imaginary parts of the \( h_\alpha \)'s. The \( \delta \) function that we insert in the partition function is

\[ 1 = \prod_n \prod_M \prod_\alpha \int d\text{Re } v_\alpha d\text{Im } v_\alpha \delta[(\text{Re } v_\alpha) - u_\alpha] \delta(\text{Im } v_\alpha) \]

\[ = \prod_n \prod_M \prod_\alpha \int_{-i\infty}^{+i\infty} d\text{Re } h_\alpha d\text{Im } h_\alpha \frac{(2\pi i)^2}{(2\pi i)^2} d\text{Re } v_\alpha d\text{Im } v_\alpha \exp\{- (\text{Re } h_\alpha)(\text{Re } v_\alpha) - u_\alpha - (\text{Im } h_\alpha)(\text{Im } v_\alpha)\}. \quad (4.5) \]

Since \( u(H) \) does not depend on \( \text{Im } h_\alpha \) the integration over \( \text{Im } h_\alpha \) will simply give \( \delta(\text{Im } v_\alpha) \) and eliminate these degrees of freedom from the computation so finally we can take

\[ v_\alpha \in \mathbb{R} \quad \text{and} \quad h_\alpha \in \mathbb{R}. \quad (4.6) \]

The effective action is therefore

\[ S_{\text{eff}} = -\frac{\beta}{2} \sum_n \sum_{M<K} \text{Re tr} V(n; M, K) \]

\[ + \sum_n \sum_M \left( u(\rho(n, M)) + \sum_\alpha h_\alpha(n, M) v_\alpha(n, M) \right), \quad (4.7) \]
where $V(n; M, K)$ is the plaquette product at position $n$ in directions $M$ and $K$. When we calculate corrections

- we set

$$h_0(n, M) \rightarrow \bar{h}_0 + h_0(n, M),$$  \hspace{1cm} (4.8)

$$v_0(n, M) \rightarrow \bar{v}_0 + v_0(n, M),$$  \hspace{1cm} (4.9)

where $\bar{h}_0$ and $\bar{v}_0$ are the solution of the mean-field saddle-point equations

$$\bar{v}_0 = \frac{I_2(h_0)}{I_1(h_0)},$$

$$\bar{h}_0 = 2(d-1)\beta \bar{v}_0^3,$$ \hspace{1cm} (4.10)

where $d = 5$ in our case. If we insert the mean-field solution in the zero’th order effective action Eq. (4.7) we obtain the zero’th order expression for the free energy per lattice site

$$F^{(0)} = -\frac{\beta d(d-1)}{2} \bar{v}^4_0 + d\bar{h}_0 + d\bar{h}_0 \bar{v}_0.$$  \hspace{1cm} (4.11)

- we have to fix the gauge; we take the Lorentz gauge which in the continuum amounts to adding the gauge fixing term $1/(2\xi)\partial_M A^A_M \partial_N A^N_N$ to the Lagrangian. As mentioned, the lattice version of this is Eq. (3.6).

- the derivatives of the effective action are derivatives with respect to the components $h_\alpha(n, M)$ and $v_\alpha(n, M)$.

- the derivatives of the observables are derivatives with respect to the components $v_\alpha(n, M)$.

### 4.2 The Faddeev-Popov determinant

On the torus where the mean-field value of the links is universal, the ghost propagator is a simple application of Eq. (3.12):

$$M_{An,Bm}(\bar{V}) = i \delta^{AB} \bar{V} \sum_M \left(2\delta_{n,m} - \delta_{m,n+\hat{M}} - \delta_{m,n-\hat{M}}\right) \equiv -i \delta^{AB} \bar{V} (\partial^*_M \partial_M)_{n,m}.$$  \hspace{1cm} (4.12)

We then have (for $SU(N)$) the Faddeev-Popov term

$$(-i)^{N^2-1} (\det (\bar{V} \partial^*_M \partial_M))^{N^2-1}$$ \hspace{1cm} (4.13)

In momentum space we will use the form

$$\Delta_{FP} = \prod_p \left(\bar{v}_0 \sum_M \hat{p}_M^2\right)^{N^2-1}, \quad \hat{p}_M = 2 \sin(p_M/2).$$  \hspace{1cm} (4.14)
In the above, apart from a Fourier transformation we dropped a power of \(-i\) from \(\Delta_{\text{FP}}\) since it merely contribute an overall constant to the free energy. Instead, we choose to keep the factor of \(v_0\) with the analogous consequences, as discussed in sect. 2.2.1.

### 4.3 The propagator

Next we compute the matrix Eq. (2.19) in Fourier space. The kernels which define the propagators in coordinate space are matrices

\[
K(n', M', \alpha'; n'', M'', \alpha'') .
\]

In Fourier space they are

\[
\tilde{K}(p', M', \alpha'; p'', M'', \alpha'') = \frac{1}{N} \sum_{n',n''} e^{i p' n'} e^{i p' M' / 2} e^{i p'' n''} e^{i p'' M'' / 2} K(n', M', \alpha'; n'', M'', \alpha'') ,
\]

where the factor \(1/N\) has been inserted to guarantee that the Fourier transform of the identity \(\delta_{n'n''}\) is the identity \(\delta_{p'p''}\) and \(\text{tr}\{\tilde{K}\} = \text{tr}\{K\} \).}

The matrix \(K^{(\text{eh})}\) is equal to the unit matrix. A straightforward calculation yields the result

\[
\tilde{K}(p', M', \alpha'; p'', M'', \alpha'') = \delta_{p'p''} \delta_{\alpha'\alpha''} C_{M'M''}(p', \alpha') ,
\]

with the notation

\[
C_{M'M''}(p', \alpha') = [A \delta_{M'M''} + B_{M'M''}(1 - \delta_{M'M''})] ,
\]

and \((b_1\) and \(b_2\) are defined in Eq. (4.21) below)

\[
A = \left[ -\frac{1}{b_2} (1 - \delta_{\alpha'0}) + \frac{1}{b_1} \delta_{\alpha'0} \right] - 2\beta v_0^2 \left[ \sum_{N \neq M'} \cos (p'_N) + (1 - \delta_{\alpha'0}) \frac{1}{\xi} \sin^2 (p'_M / 2) \right] ,
\]

and

\[
B_{M'M''} = -4\beta v_0^2 \left[ \delta_{\alpha'0} \cos \left( \frac{p'_M}{2} \right) \cos \left( \frac{p'_M}{2} \right) + \frac{y}{1 - \delta_{\alpha'0}} \sin \left( \frac{p'_M}{2} \right) \sin \left( \frac{p'_M}{2} \right) \right] ,
\]

where

\[
b_1 = -\frac{1}{h_0 I_1(h_0)} \left( I_2(h_0) - h_0 \left( \frac{I_2(h_0)^2}{I_1(h_0)} - I_3(h_0) \right) \right) ,
\]

\[
b_2 = -\frac{v_0}{h_0} ,
\]

and \(y = 2 - 1/\xi\). For convenience we rescaled the gauge fixing parameter \(\xi\). This propagator was presented in [4] in the axial gauge.
4.4 The Free Energy

If we write Eq. (2.16) in momentum space, the free energy Eq. (2.28) can be evaluated as

\[ F^{(1)} = F^{(0)} + \frac{1}{2N} \ln \left[ \prod_{p \neq 0} \det \left( -1 + \tilde{K}_{00}^{(hh)} \tilde{K}_{00}^{(vv)} \right) \det \left( -1 + \tilde{K}_{00}^{(hh)} \tilde{K}_{00}^{(vv)} \right)^3 (p^2)^{-6} \right]. \]  

(4.22)

The toron contribution \( p = 0 \) is dropped according to our regularization scheme.

The determinants in Eq. (4.22) are invariant under the flip of the sign of any of the components of the momentum \( p \). We compute their contributions to the free energy by summing the logarithms of the absolute value of their eigenvalues and we keep track of their signs. The flip symmetry then implies that the overall sign of the determinant is decided by the momenta whose components are 0 or \( \pi \).

4.5 The Static Potential

The observable we will use to compute the static potential is the Wilson loop extending \( n_5 \) points along the extra dimension and \( t \) points in time. We have defined the observable in eq. (3.16). The objective here is to obtain the correction as it appears in Eq. (2.36). The calculation of Fig. 1 yields the first order correction to the tree level result

\[ \mathcal{O} = 2(n_5)2^{(t+n_5)}, \]  

(4.23)

which can be brought in the form

\[ \frac{1}{2} \sum_{p_1', p_2', p_3', \ldots, p_0'} \left\{ 2 \cos(p_5'n_5) + 2[C_{00}^{-1}(p', 0) + 3[2 \cos(p_5'n_5) - 2] \frac{1}{C_{00}(p', 1)} \right\}. \]  

(4.24)

Here we have used the fact that for \( \alpha' \neq 0 \) and \( p_0' = 0 \) the matrix \( \tilde{K} \) is block diagonal, in particular \( C_{00}^{-1} = 1/C_{00} \). This can be seen by inspection of the matrix \( \tilde{K}_{00}^{(hh)} \)

\[ -2\beta v_0^2 \begin{pmatrix} -4 + \sum' c_{M'} - \frac{1}{s} \sin^2 \left( \frac{p_0'}{2} \right) \ y_0 s_0/2 s_1/2 & y_0 s_0/2 s_2/2 & y_0 s_0/2 s_3/2 & y_0 s_0/2 s_5/2 \\ y_0 s_1/2 s_0/2 & -4 + \sum' c_{M'} - \frac{1}{s} \sin^2 \left( \frac{p_0'}{2} \right) \ y_0 s_1/2 s_2/2 & y_0 s_1/2 s_3/2 & y_0 s_1/2 s_5/2 \\ y_0 s_2/2 s_0/2 & y_0 s_2/2 s_1/2 & -4 + \sum' c_{M'} - \frac{1}{s} \sin^2 \left( \frac{p_0'}{2} \right) \ y_0 s_2/2 s_3/2 & y_0 s_2/2 s_5/2 \\ y_0 s_3/2 s_0/2 & y_0 s_3/2 s_1/2 & y_0 s_3/2 s_2/2 & -4 + \sum' c_{M'} - \frac{1}{s} \sin^2 \left( \frac{p_0'}{2} \right) \ y_0 s_3/2 s_5/2 \\ y_0 s_4/2 s_0/2 & y_0 s_4/2 s_1/2 & y_0 s_4/2 s_2/2 & y_0 s_4/2 s_3/2 & -4 + \sum' c_{M'} - \frac{1}{s} \sin^2 \left( \frac{p_0'}{2} \right) \end{pmatrix} \delta_{p'p'}, \]  

\[ \sum' \text{ means sum over all values of } M' \text{ except the one that corresponds to its column/row index (the index ordering is } M = 0, 1, 2, 3, 5) \text{ and } s_{M/2} \equiv \sin(p_{M/2}'/2) \text{ (similarly, } c_{M/2} \equiv \cos(p_{M/2}'/2)). \]
A crucial cancellation happens for $\alpha' \neq 0$. As can be seen directly from Eq. (4.19) in this case, if $p' = 0$ then $A = 0$ and the matrix $C$ has a zero mode. This "toron" is precisely cancelled by the factor $[2 \cos(p'_5 n_5) - 2]$, where the $-2$ comes from adding the self-energy and tadpole contributions. Another important observation is that the static potential which selects the value $p'_0 = 0$ is $\xi$ independent. Also, it is easy to check that in the limit

$$\beta \to \infty, \quad \bar{v}_0 \to 1$$

one recovers the proper form of the perturbative static potential. This is because the $\alpha' = 0$ part of the propagator is of order $1/\beta$ with respect to its $\alpha' \neq 0$ part (which itself is of order $1/\beta$), for large $\beta$.

For the static potential we arrive at the final expression

$$V(r) = -2 \log(\bar{v}_0) - \frac{1}{2\bar{v}_0} \frac{1}{L^3 N_5} \times \left\{ \sum_{p'_M \neq 0, p'_0 = 0} \left[ \frac{1}{4} \sum_{N \neq 0} (2 \cos(p'_N r) + 2) \right] C^{-1}_{00}(p', 0) \right. \\
+ 3 \sum_{p'_M \neq 0, p'_0 = 0} \left[ \frac{1}{4} \sum_{N \neq 0} (2 \cos(p'_N r) - 2) \right] \frac{1}{C_{00}(p', 1)} \right\}. \quad (4.26)$$

In the above we have used translational invariance to average over Wilson loops of spatial size $r$ along the $M = 1, 2, 3, 5$ directions.

### 4.6 The Scalar and Gauge Boson masses

In this section we compute the scalar and vector masses to the order illustrated in Fig. 2 and Fig. 3 respectively. At this order there are no ghost contributions. The computation of the mass of the scalar observable on the torus starts from Eq. (2.31), the contraction of the second derivative of the observable with the propagator. The observable is computed using Eq. (3.21) and Eq. (3.20) Fourier transformed and the propagator given by Eq. (4.17).

We define the very useful quantity

$$\Delta^{(m_s)}(n_5) = \sum_{r=0}^{m_5-1} \frac{\delta_{n_5 r}}{\bar{v}_0(\hat{r})}, \quad \hat{r} = r + 1/2, \quad (4.27)$$

and its Fourier transform

$$\tilde{\Delta}^{(m_s)}(p) = \sum_{r=0}^{m_5-1} \frac{e^{ip\hat{r}}}{\bar{v}_0(\hat{r})}. \quad (4.28)$$

---

5 The argument of $\bar{v}_0(m)$ refers to a more general situation where the background can depend on the position along the extra dimension, like on the orbifold $[6]$. Integer values for $m$ label four dimensional links and half-integer values the links along the extra dimension. On the torus $\bar{v}_0(\hat{r}) = \bar{v}_{05}$. 

22
The final expression for the scalar mass observable in Fig. 2 turns out to be

$$C_{H}^{(1)}(t) = \frac{4}{N}(P_{0}^{(0)})^{2} \sum_{p_0'} \cos(p_0't) \sum_{p_5'} \mid \Delta^{(N_5)}(p_0') \mid^2 \tilde{K}^{-1}((p_0', \bar{0}, p_5'), 5, 0; (p_0', 0, p_5'), 5, 0),$$

(4.29)

where $P_{0}^{(0)}$ is the Polyakov loop evaluated on the zero'th order background. It is interesting to note that the component of the propagator that contributes to Eq. (4.29) in gauge space does not have a pole at the toron $p' = 0$. The toron contribution is in fact the dominant one. Also, because the only component of the propagator that gives a contribution to the mass is along the 0-direction and the latter is not gauge fixed, it is exactly gauge independent.

For the $W$ gauge boson mass we define the contraction

$$\overline{K}^{-1}((p_0', \hat{p}'), 5, \alpha) = \sum_{p_5', p_5''} \tilde{\Delta}^{(N_5)}(p_0') \tilde{\Delta}^{(N_5)}(-p_0'') K^{-1}((p_0', 5, \alpha; p', 5, \alpha))$$

(4.30)

with the property

$$\overline{K}^{-1}((p_0', \hat{p}'), 5, \alpha) = \overline{K}^{-1}((p_0', -\hat{p}'), 5, \alpha).$$

(4.31)

In addition, we define a second useful contraction as

$$\overline{K}^{-1}(t, \hat{p'}, \alpha) = \sum_{p_0'} e^{ip_0't} \overline{K}^{-1}((p_0', \hat{p}'), 5, \alpha).$$

(4.32)

A calculation similar to the one for the scalar, gives for Fig. 3 the result

$$C_{V}^{(2)}(t) = \frac{768}{N^2} (P_{0}^{(0)})^4 (\bar{v}_0(0))^4 \sum_{\hat{p}} \sum_{k} \sin^2(p_k') \left( \overline{K}^{-1}(t, \hat{p'}, 1) \right)^2.$$  

(4.33)

The toron here is cancelled due to the $\sin^2(p_k')$ factor. Here gauge independence is not obvious because it is gauge fixed components of the propagator that contribute. A careful numerical investigation shows that the vector mass is actually gauge independent to an accuracy of $10^{-11}$.

In Appendix A we give more details about the calculations leading to Eq. (4.29) and Eq. (4.33).

4.7 Anisotropy

The only regime of parameters where dimensional reduction is guaranteed is when the physical size of the extra dimension becomes much smaller than the spatial dimensions, $R << l$. On the lattice, such a scenario can be realized when an anisotropy parameter
is introduced along the extra dimension. To this effect, following \[15\], we define the anisotropy factor

$$\gamma = \sqrt{\frac{\beta_5}{\beta_4}}$$

(4.34)

whose zero’th order value is \(\gamma = a_4/a_5\), the ratio of the lattice spacings along four dimensional hyperplanes and along the extra dimension and

$$\beta_4 = \frac{2Na_5}{g_4^2}, \quad \beta_5 = \frac{2Na_4^2}{g_5^2a_5}.$$  

(4.35)

Then,

$$\beta_5 = \beta \gamma, \quad \beta_4 = \frac{\beta}{\gamma}.$$  

(4.36)

A quantity that is worth keeping in mind is the four dimensional effective bare coupling which rescales to

$$g_4^2 = \frac{g_5^2}{2\pi R} = \frac{2N\gamma}{N_5\beta}$$

(4.37)

and it can in principle become non-perturbative for large enough anisotropy factor if the other parameters are kept fixed. We are also interested in the ratio

$$\frac{N_5}{L} = \frac{2\pi Ra_4}{a_5 l} = \frac{2\pi R}{l} \gamma$$

(4.38)

which controls dimensional reduction via compactification, provided that it is kept fixed while \(\gamma\) is increased. We distinguish three regimes of the anisotropy parameter when \(N_5/L = 1\):

- \(\gamma = 1\). This defines the isotropic lattice which, in the limit of an infinite lattice, should represent the "non-compact phase" of the continuum gauge theory.

- \(\gamma \gg 1\). This is a regime where the size of the extra dimension is small with respect to the spatial length and in the limit \(L \to \infty\) it represents the "compact phase" of the continuum gauge theory.

- \(\gamma < 1\). This is a situation where in the \(L \to \infty\) limit the continuum gauge theory has a large extra dimension. If dimensional reduction is realized in this phase, it must be due to a localization effect.

The \(\gamma \neq 1\) cases are interesting in case there is dimensional reduction. Then one expects the four dimensional effective coupling to behave as a confining/asymptotically free coupling. If this is indeed the case, such a property should be reflected by the static potential.
The parameters we choose to parametrize anisotropy are the coupling $\beta$ and the anisotropy factor $\gamma$. In terms of these, we rewrite the Wilson action as

$$S_W = \frac{\beta}{2N} \left[ \frac{1}{\gamma} \sum_{4d-p} w(p) \left( 1 - \text{tr}\{U_p\} \right) + \gamma \sum_{5d-p} \left( 1 - \text{tr}\{U_p\} \right) \right],$$

(4.39)

where the first term contains the effect of all plaquettes along the four dimensional slices of the five dimensional space and the second term contains the effect of plaquettes having two of their sides along the extra dimension.

The partition function in the presence of anisotropy gets modified. All derivatives of plaquettes containing a link pointing in the extra dimension will be now different. There will be two mean values for the links, $v_0$ and $v_{05}$ determined by the extremization of

$$\frac{S_{\text{eff}}[V,H]}{N} = -\beta_4 (d-1)(d-2) \frac{1}{2} \tau_0^4 - \beta_5 (d-1) \tau_0^2 + (d-1)u(\tau_0) + u(\tau_{05}) + (d-1)\tau_0 \tau_0 + \tau_{05} \tau_{05}$$

(4.40)

which yields the conditions

$$\tau_0 = -u(\tau_0)', \quad \tau_{05} = -u(\tau_{05})',$$

$$\tau_0 = \frac{6\beta}{\gamma} \tau_0^3 + 2\beta \gamma \tau_0 \tau_{05},$$

$$\tau_{05} = 8 \beta \gamma \tau_0 \tau_{05},$$

(4.41)

for $d = 5$.

The anisotropy factor cancels out in scalar products when written in terms of dimensionless momenta and coordinates

$$pn = p_5 n_5 + \sum_{\mu} p_\mu n_\mu = (p_5 a_5)(x_5/a_5) + \sum_{\mu} (p_\mu a_4)(x_\mu/a_4),$$

(4.42)

where in the second equality we explicitly put the lattice spacings.

The modifications to the propagators are slightly more involved. The propagators in the anisotropic vacuum become

$$\tilde{K}^{(hh)} = -\delta_{p'p''} \delta_{\alpha'\alpha''} \frac{I_2(\tau_0)}{\tau_0 I_1(\tau_0)} \left[ 1 - \epsilon \cdot \frac{\tau_0}{I_2(\tau_0)} \left( \frac{I_2^2(\tau_0)}{I_1(\tau_0)} - I_3(\tau_0) \right) \right] \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$-\delta_{p'p''} \delta_{\alpha'\alpha''} \frac{I_2(\tau_{05})}{\tau_{05} I_1(\tau_{05})} \left[ 1 - \epsilon \cdot \frac{\tau_{05}}{I_2(\tau_{05})} \left( \frac{I_2^2(\tau_{05})}{I_1(\tau_{05})} - I_3(\tau_{05}) \right) \right] \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(4.43)
where \( \epsilon = 1 \) for \( \alpha' = 0 \) and \( \epsilon = 0 \) for \( \alpha' \neq 0 \). For later reference, we note that the limit of the factor in the second line of the above expression for \( \overline{r}_{05} \to 0 \) is \(-1/4\). Also,

\[
K_{\alpha' \neq 0}^{(uv)} = \delta_{\rho' \rho''} \delta_{\alpha' \alpha''} \left(-2\frac{\beta}{\gamma} \overline{r}_{05}^{2}\right) \cdot
\left(\begin{array}{cccccc}
\sum' c_{M'} - \frac{1}{\xi} s_{0/2}^{2} & y_{s_{0/2}} s_{1/2} & y_{s_{0/2}} s_{2/2} & y_{s_{0/2}} s_{3/2} & y_{5 s_{0/2}} s_{5/2} \\
y_{s_{1/2}} s_{0/2} & \sum' c_{M'} - \frac{1}{\xi} s_{1/2}^{2} & y_{s_{1/2}} s_{2/2} & y_{s_{1/2}} s_{3/2} & y_{5 s_{1/2}} s_{5/2} \\
y_{s_{2/2}} s_{0/2} & y_{s_{2/2}} s_{1/2} & \sum' c_{M'} - \frac{1}{\xi} s_{2/2}^{2} & y_{s_{2/2}} s_{3/2} & y_{5 s_{2/2}} s_{5/2} \\
y_{s_{3/2}} s_{0/2} & y_{s_{3/2}} s_{1/2} & y_{s_{3/2}} s_{2/2} & \sum' c_{M'} - \frac{1}{\xi} s_{3/2}^{2} & y_{5 s_{3/2}} s_{5/2} \\
y_{5 s_{5/2}} s_{0/2} & y_{5 s_{5/2}} s_{1/2} & y_{5 s_{5/2}} s_{2/2} & y_{5 s_{5/2}} s_{3/2} & \gamma^2 \left(\sum' c_{M'} - \frac{1}{\xi} s_{5/2}^{2}\right)
\end{array}\right)
\]

where \( y = 2 - \frac{1}{\xi} \), \( y_5 = 2 \gamma^2 \overline{r}_{05}^{2} - \frac{\gamma}{\xi} \quad (4.44) \)

and

\[
K_{\alpha' = 0}^{(uv)} = \delta_{\rho' \rho''} \delta_{\alpha' \alpha''} \left(-2\frac{\beta}{\gamma} \overline{r}_{05}^{2}\right) \cdot
\left(\begin{array}{cccccc}
\sum' c_{M'} & 2 c_{0/2} c_{1/2} & 2 c_{0/2} c_{2/2} & 2 c_{0/2} c_{3/2} & 2 c_{0/2} c_{5/2} \overline{r}_{05}^{2} \\
2 c_{1/2} c_{0/2} & \sum' c_{M'} & 2 c_{1/2} c_{2/2} & 2 c_{1/2} c_{3/2} & 2 c_{1/2} c_{5/2} \overline{r}_{05}^{2} \\
2 c_{2/2} c_{0/2} & 2 c_{2/2} c_{1/2} & \sum' c_{M'} & 2 c_{2/2} c_{3/2} & 2 c_{2/2} c_{5/2} \overline{r}_{05}^{2} \\
2 c_{3/2} c_{0/2} & 2 c_{3/2} c_{1/2} & 2 c_{3/2} c_{2/2} & \sum' c_{M'} & 2 c_{3/2} c_{5/2} \overline{r}_{05}^{2} \\
2 c_{5/2} c_{0/2} \gamma^2 \overline{r}_{05}^{2} & 2 c_{5/2} c_{1/2} \gamma^2 \overline{r}_{05}^{2} & 2 c_{5/2} c_{2/2} \gamma^2 \overline{r}_{05}^{2} & 2 c_{5/2} c_{3/2} \gamma^2 \overline{r}_{05}^{2} & \gamma^2 \sum' c_{M'}
\end{array}\right)
\quad (4.45)
\]

The only point not explicitly shown in these formulas is that in the diagonal elements, \( c_5 = \gamma^2 \overline{r}_{05}^{2} \cos (p_5^0) \).

In Eq. (4.44) we have already implemented the anisotropic version of the gauge fixing term

\[
S_{GF} = \frac{1}{2\xi} \sum_{A=1}^{3} \sum_{n} \left[ \sum_{\mu} \left( v_{A}(n, \mu) - v_{A}(n - \hat{\mu}, \mu) \right) + \gamma \left( v_{A}(n, 5) - v_{A}(n - 5, 5) \right) \right]^2 \quad (4.46)
\]

(up to a redefinition of \( \xi \)). The resulting Faddeev-Popov determinant splits accordingly to two parts:

\[
\Delta_{FP} = \left[ \prod_{p} \frac{4}{p} \sum_{\mu} \sin^2 \left( \frac{a_{4 p \mu}}{2} \right) + \gamma \overline{r}_{05} \sin^2 \left( \frac{a_{5 p_{5}}}{2} \right) \right]^{N^2-1} . \quad (4.47)
\]

As far as the observables are concerned, the modifications to the free energy and the mass formulae are simple. For the free energy Eq. (4.22) all the information about the anisotropy is contained in the propagator and the Faddeev-Popov determinant. The correlators Eq. (4.29) and Eq. (4.33) for the masses do not change. On an anisotropic lattice
there are two inequivalent Wilson loops. One along the four dimensional hyperplanes for which the static potential is given by

\[
V(r) = -2 \log(\tau_0) - \frac{1}{2\tau_0^2} \frac{1}{L^3 N_5} \sum_{p'_{M=0} \neq 0, p'_0 = 0} \left\{ \frac{1}{3} \sum_k (2 \cos(p'_k r) + 2) \right\} C^{-1}_{00}(p', 0) \]

\[
+3 \sum_{p'_{M=0} \neq 0, p'_0 = 0} \left[ \frac{1}{3} \sum_k (2 \cos(p'_k r) - 2) \right] \frac{1}{C_{00}(p', 1)} \right\}.
\]

The other Wilson loop is along the extra dimension and gives the potential

\[
V(r) = -2 \log(\tau_0) - \frac{1}{2\tau_0^2} \frac{1}{L^3 N_5} \sum_{p'_{M=0} \neq 0, p'_0 = 0} \left\{ [2 \cos(p'_5 r) + 2] C^{-1}_{00}(p', 0) + 3 [2 \cos(p'_5 r) - 2] \frac{1}{C_{00}(p', 1)} \right\}.
\]

The matrix \( C \) is defined as in Eq. (4.17).

For completeness and as a check, the perturbative limit \( \beta \to \infty, \tau_0 \to 1, \tau_{05} \to 1 \) of the static potential at distance \( r \) in direction \( M \) is easily computed to be

\[
a_4 V_{\text{pert}}(r) \sim \frac{g_5^2}{a_4} \sum_{p'_M \neq 0, p'_0 = 0} \frac{\cos(p'_M r) - 1}{\sum_k (1 - \cos(p'_k a_4)) + \gamma^2 (1 - \cos(p'_5 a_5))}
\]

and for \( \gamma \to \infty \) one is left with a pure Coulomb potential at \( p'_5 = 0 \) (for clarity we have explicitly put in the lattice spacings).

5 The phase diagram

In the anisotropic background the phase diagram can be split in three regimes according to the solution to which the numerical iteration Eq. (4.41) converges. Where it converges to \( \tau_0 = \tau_{05} = 0 \) we define the confined phase. Where it gives a solution of the form \( \tau_0, \tau_{05} \neq 0 \) we define the deconfined phase and where it gives a solution of the form \( \tau_0 \neq 0, \tau_{05} = 0 \) we define the "layered phase" \([2]\). The latter is a regime where the four dimensional coupling is deconfined and the five dimensional coupling confined. This is an approach that gives a first approximation to the phase diagram. One then has to check whether the solution of Eq. (4.41) indeed corresponds to the global minimum of the free energy Eq. (4.22).

The mean-field method is blind to the confined phase of the five dimensional theory so we do not have to say anything about it. According to the first order free energy
Figure 4: Left: The free energy at first order Eq. (4.22) computed on isotropic $T = L = N_5 = 10$ lattices as a function of $\beta$ and the background value $v$. For $\beta \geq 1.55$ the minimum is at $v \geq 0.80$ (at 0th order we found a non-trivial minimum for $\beta > 2.12$). For $\beta < 1.55$ the minimum jumps to values $0.42 \leq v \leq 0.48$. Right: The free energy at first order in the layered phase for $\beta = 0.905$ and $\gamma = 1/3.8$ computed on a $T = L = N_5 = 10$ lattice. The minimum found at 0th order ($\tau_0 = 0.9030$, $\tau_{50} = 0$) is stable. Where the free energy is complex, it is not plotted.

Figure 5: Left: The free energy at first order Eq. (4.22) in the d-compact phase for $\beta = 2.136$ and $\gamma = 0.25$ computed on a $T = L = N_5 = 10$ lattice as a function of the background values $v$ and $v_5$. The minimum found at 0th order ($\tau_0 = 0.9679$ and $\tau_{50} = 0.0288$) is stable. Right: The free energy at first order in the compact phase for $\beta = 1$ and $\gamma = 10$ computed on a $T = L = N_5 = 10$ lattice. The minimum found at 0th order ($\tau_0 = 0.9179$, $\tau_{50} = 0.9773$) is not stable, it moves to a “phase” with $v_5 = 1$ (and $v = 0.86$). Where the free energy is complex, it is not plotted.
the layered phase is stable. The 0th order background (or saddle-point) solution $v_0 = 0.9030, v_{05} = 0$ remains as a minimum when 1st order corrections are included, see right of Fig. 4.

For now, the deconfined phase will be the regime of our interest. The isotropic lattice is stable as the left of Fig. 4 shows. Clearly this is a case not interesting from the point of view of dimensional reduction so we will not have to say much about it. The right plot of Fig. 5 shows the free energy for $\beta = 1$ and $\gamma = 10$ (compact phase) computed on a $T = L = N_5 = 10$ lattice as a function of the background values $v$ and $v_5$. The component of the 0th order background along the four dimensional hyperplanes $v_0 = 0.9179$ moves to $v = 0.86$ (precision 0.01). The background along the extra dimension however, which at zero'th order is $v_{05} = 0.9773$ is unstable since the minimum moves to $v_5 = 1.00$ (with an uncertainty less than 0.01) when the 1st order correction is included. In fact, the same instability is observed everywhere inside the compact phase at small $\beta$: the $v_{05}$ component of the background runs to 1, the value that is the solution to the background equations at $\beta = \infty$. This instability could be an inherent property of the system or just a property of the first order approximation, we can not tell for sure until a higher order correction to the free energy is computed.

The left plot of Fig. 5 shows the free energy for $\beta = 2.136$ and $\gamma = 0.25$ computed on a $T = L = N_5 = 10$ lattice. As it will turn out this is in an especially interesting regime of the deconfined phase which we call the 'd-compact' phase. The 0th order background $v_0 = 0.9679, v_{05} = 0.0288$ is stable, including the first order correction we get a minimum at essentially the same values. Furthermore, this result is quantitatively unchanged on a $T = L = N_5 = 6$ lattice. In general, the d-compact phase, for $\gamma$ not too small, seems to have a deeper from the zero'th order minimum at $v = 0$ and $v_5 = 1$. This global minimum however gradually disappears as $\gamma$ is lowered, as it is the case on the left of Fig. 5. The local, first order minimum on the other hand is consistent with the zero’th order result and is separated from the global minimum by a regime where the free energy is complex. We conclude this part of the discussion by saying that all the caveats regarding the interpretation of the $\xi$-dependent free energy mentioned before, remain.

Next, using our local (Wilson loop) and global (Higgs and $W$ masses) quantities we will characterize more precisely the various regimes of the phase diagram. We keep always $N_5 = L$. The first observable we use is the $W$ gauge boson mass from Eq. (4.33). In Fig. 6 we present a scaling study of the behavior of the mass in lattice units as a function of $1/L$ for the point $\beta = 2.136, \gamma = 0.25$. The errors represent the uncertainty in the plateau value of Eq. (2.42). A linear fit in $1/L$ gives an extrapolation $a_tm_w = 0.0006(3)$ for $L \rightarrow \infty$ with $\chi^2/d.o.f. = 0.32$. The slope is 12.51(1). Adding a quadratic term to the fit confirms the conclusion that the mass seems to be consistent with zero within
errors. We can not exclude of course a tiny (possibly exponentially suppressed) mass. We will come back to this issue when we analyze the static potential. A scan of the phase diagram reveals that the $W$ mass is actually nearly independent of any parameter other than $L$. There is a very mild $\beta$ and $\gamma$ dependence but it does not seem easy at the moment to attribute any quantitative physical significance to it. Therefore, we conclude that the deconfined phase, at least in regimes where the system is five dimensional (i.e. not dimensionally reduced), is in a Coulomb phase. Note that since the background is along the Hermitian component of the links it can not possibly play the role of a symmetry breaking vev. That is, in the absence of an explicit vev we can not probe the Higgs phase, unlike in a Monte Carlo simulation where choosing the appropriate input parameters they system can be in a Higgs phase or not.

The second of our observables that we will use is the static potential derived from Wilson loops oriented along the four dimensional slices of the lattice which are orthogonal to the extra dimension \footnote{Note that on the anisotropic lattice there are two types of inequivalent Wilson loops. One along the four dimensional slices and one along the extra dimension. We discuss here the former.}. The points obtained will be fitted to both four dimensional ($c_0 + c/r$, where $c_0$ and $c$ are the fitted parameters) and five dimensional ($d_0 + d/r^2$, where $d_0$ and $d$ are the fitted parameters) Coulomb forms. This will decide whether the theory is in a dimensionally reduced phase or not, as already mentioned. Let us fix $\beta = 2.136$. In
order to obtain a first, qualitative picture of dimensional reduction we perform global fits to four dimensional or five dimensional Coulomb potentials and assign to the potential points fictitious relative errors\footnote{This is justified if we want to compare fits at different values of $\gamma$.} and perform a comparative study of the $\chi^2$ of the fits. A much more careful study of the local curvature of the potential will be performed later. A scan in $\gamma$ of the global fits shows that for $0.35 < \gamma < 4$ there is no dimensional reduction. In particular on a lattice with $L = N_5 = 100$ for $\gamma = 1$ one sees a perfect global five dimensional Coulomb fit, see Fig. 7. For $\gamma > 4$, as expected, we start seeing a four dimensional Coulomb law which becomes better as $\gamma$ increases when only the points corresponding to small $r$ are fitted. For large $r$ we see deviations. We call this phase the compact phase. According to the free energy analysis the compact phase for small $\beta$ is unstable to this order in the mean-field expansion, therefore we do not analyze further its properties. The somewhat surprising fact is that for $\gamma < 0.35$ the potential turns again four dimensional Coulomb, until the layered phase is hit, at around $\gamma = 0.25$. A similar picture can be obtained for other values of $\beta$ in the deconfined phase. The narrow band for $\gamma < 1$ which, at least according to our mean-field approach, apparently describes a phase where the extra dimension is large but the system is dimensionally reduced, we call the "d-compact phase".

We now turn to the Higgs observable. By construction it is $L$-independent so we do not

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{The static potential on the isotropic lattice. It is fitted by a five dimensional Coulomb form, as expected from the corresponding continuum theory.}
\end{figure}
Figure 8: The scalar mass in lattice units on the isotropic lattice shows a ferromagnet type of behavior with exponent $\nu = 1/4$.

Figure 9: The scalar mass in lattice units $a_4m_H$ in the d-compact phase shows a ferromagnet type of behavior with critical exponent $\nu = 1/2$.

have to scale it with $L$. It turns out to be also $N_5$ independent and to have a strong $\beta$ and $\gamma$ dependence. At a generic point in the interior of the phase diagram the scalar is heavy
in lattice units. It starts loosing weight as boundaries between phases are approached. In Fig. 8 we show its behavior as the isotropic phase transition is approached. The way the scalar mass approaches zero reminds us of the magnetization of a ferromagnet as a function of the temperature as the Curie temperature is approached where a second order phase transition occurs. Even though in our case the phase transition is definitely known to be of first order, the resemblance of the mass dependence is close enough so that we attempt to fit the data points to a critical law of the form

$$am_H \sim (1 - \beta_c / \beta)^\nu$$

which yields $\nu = 0.2470$ when we take $\beta_c = 1.6762017$. When we take $\beta_c = 1.6762016$ we obtain $\nu = 0.2546$ instead. The average value from the two fits $(0.2546 + 0.2470)/2 = 0.251 \pm 0.004$ allows us to safely approximate the exponent to be

$$\nu = \frac{1}{4}$$

That this can not be a second order phase transition is also supported by the fact that the mass does never really go to zero. It reaches a small value and it stops there. The same behavior can be verified along the whole line that separates the confined and deconfined phases.

Figure 10: Tuning of the bare coupling $\beta$ as a function of $L$ when the phase transition is approached at constant $\rho = m_W/m_H = 1.44333$ and $\gamma = 0.25$, $N_5 = L$. The solid line is a fit according to Eq. (5.4).
A qualitative difference appears on the boundary between the d-compact and layered phases. To begin, the scalar mass in lattice units goes all the way down to zero as the layered phase is approached. Moreover, in Fig. 9 we measure a critical exponent
\[ \nu = \frac{1}{2} \]  
when we set \( \beta_c = 2.1349 \), indicating that the d-compact phase describes different physics from the rest of the deconfined phase. We note that this is the critical exponent of the four dimensional Ising model and it is believed that if the \( SU(N) \) theory should have a second order phase transition in \( d+1 \) dimensions, its order parameter is the \( Z_N \) values of the Polyakov loop and its critical exponent should be \( \nu = 1/N \) [17]. Furthermore, if there is a tricritical point then above it the phase transition is expected to turn into a first order one with "exponent" 1/4. We have also checked that the free energy, as the phase transition is approached looses its local maximum and it becomes flatter around its global minimum, both facts consistent with a second order phase transition [5]. In our case the second derivative of the free energy at the minimum approaches zero as the phase transition is reached. Whether this is the real physical picture or it is just an artificial effect that the mean-field method produces, is not clear. Until a Monte Carlo simulation study is performed we can not be sure. Finally, we have examined the passing from the layered phase into the confined phase. As the scalar mass [4] gets stuck at a large value, this can not be a second order phase transition. In this case the indication is that it is a first order phase transition, like in the \( U(1) \) theory [18].

For now we assume that what we observe is a second order phase transition and we study the scaling behavior as we approach it. First, we tune the values of \( \beta \) as the phase transition is approached keeping \( \rho = m_W/m_H \) fixed (we keep \( N_5 = L \) and \( \gamma \) fixed as usual) as a function of the lattice size, see Fig. 10. The behavior fits well to the form
\[ \beta = c_0 + \frac{c_1}{L^2} = c_0 + c_1 \frac{a_4}{l^2}, \]  
which implies that the bare coupling squared \( g_0^2 = 4/\beta \) approaches its critical value as a quadratic function of the lattice spacing.

Second, for the values of \( \beta(L) \) at constant \( \rho = 1.44333 \) we calculate the static potential \( V(r) \) in the four-dimensional slices orthogonal to the extra dimension and fit it to the four dimensional Coulomb form
\[ V(r) = -\frac{q^2}{r} + \text{const}. \]  

---

8 The Polyakov loop Eq. (3.18) is zero in the layered phase. In order to measure the scalar mass there, we rotate the Polyakov loop in the \( k = 3 \) direction.

9 It is important to notice that the ratio \( \rho \) is independent on \( N_5 \) since both \( m_W \) and \( m_H \) turn out to be independent on \( N_5 \). This means that as the lattice spacing goes to zero as the phase transition is approached one can keep \( R \) fixed and therefore, when \( L = N_5 \), also \( l \) fixed.
Figure 11: Scaling of the charge $q^2$ defined in Eq. (5.5) as the phase transition is approached along the values $\beta(L)$ of Fig. 10 at constant $\rho = 1.44333$.

The fits become better as $L$ increases. The fitted parameter $q^2$ can be interpreted as a charge. Since the static potential is renormalized up to an additive divergent constant [19], $q^2$ is a physical quantity and Fig. 11 shows its scaling behavior. The points $L = 16 \ldots 48$ closest to the phase transition can be very well fitted by

$$q^2 = -\frac{0.30}{L^2} + 0.0342$$

with $\chi^2$/d.o.f = 3.7/4. The fit is shown in Fig. 11. Adding a $1/L$ term to the fits give $q^2 = -0.37/L^2 + 0.0054/L + 0.0341$ with $\chi^2$/d.o.f = 3.0/3\(^{10}\) so we conclude that the continuum limit of $q^2$ is approached up to quadratic cut-off effects as expected [20].

5.1 The static potential

We now turn to a closer examination of the static potential for $\gamma \neq 1$. We concentrate on the d-compact phase and we choose to approach the 2nd order phase transition keeping $\gamma = 0.25$

fixed. We choose the values $\beta = 2.13495, 2.136$ and 2.3, which correspond to Higgs masses of $a_4m_H = 0.020, 0.094$ and 1.12 respectively. In the d-compact phase, the relations (for

\(^{10}\) We assign artificial relative errors of 0.001 to the potential in order to compare the fits.)
Potential along the extra dimension, $\beta=2.136$, $\gamma=0.25$

**Figure 12:** The static potential derived from the Wilson loop along the extra dimension is independent on $N_5$ and has a strong $L$ dependence. As $L$ increases, its slope decreases.

$$N_5 = L$$

$$a_5 = a_4/\gamma \quad \text{and} \quad R/l = 1/(2\pi\gamma) \quad (5.7)$$

indicate that the extra dimension is large compared to the size of the four-dimensional space. We take the continuum limit at fixed anisotropy, as indicated by the arrow in Fig. 21. It is interesting that only if $\gamma < 1$ a continuum limit exists, that means that the anisotropy survives in the limit and has physical consequences. We measure the static potentials along four-dimensional hyperplanes orthogonal to the extra dimension and along the extra dimension.

We start by looking at the potential along the extra dimension. In Fig. 12 we plot the static potential at $\beta = 2.136$ for various values of $N_5$ and $L$ equal to 32 or 64. Due to the periodicity, we measure the static potential up to distances $L/2$ and we can therefore compare distances up to 16 in lattice units. One can see that it is essentially independent\(^{11}\) on $N_5$ and that its slope strongly decreases as $L$ increases. These two facts give rise to the suspicion that it may not represent a physical interaction. To check this, we plot in Fig. 13 the physical dimensionless force

$$\tau^2 F(\tau) = \tau^2 [V(r) - V(r - a_5)]/a_5, \quad \tau = r - \frac{1}{2}a_5 + O(a_5^2) \quad (5.8)$$

derived from the potential, keeping $N_5 = L$ and increasing $L = 24, 32, 48, 64, 100$ (left) and

\(^{11}\)Barring small effects due to the periodicity of the lattice.
Figure 13: The dimensionless force between two static charges separated in the extra dimension. Apparently it is a pure finite size effect that approaches zero in the infinite volume limit.

Figure 14: The static potential derived from four dimensional Wilson loops is $N_5$ and $L$ independent.

$L = 100, 125, 150, 200$ (right). The plots show that the force goes to zero as the $L \to \infty$ is approached suggesting that the four dimensional hyperplanes decouple in this limit and the system reduces to an array of non-interacting “branes” where gauge interactions are localized.

We continue by looking at the potentials along four-dimensional hyperplanes. First, in Fig. 14 we check $N_5$ and $L$ dependence. As the figure shows, these potentials are essentially independent of $N_5$ and $L$. Thus, it is likely that they reflect physical interactions. Recall
that the global fits of the potential along the four dimensional hyperplanes and the analysis of the extra dimensional force suggested that in the large $L$ limit the potential along the four dimensional planes should describe four dimensional physics to a good approximation. In order to show this we analyze the potentials on $L = 200$ lattices for $\beta = 2.13495$, $\beta = 2.136$ and $\beta = 2.3$, shown in Fig. 15. We perform local fits to the form

$$V(r) = c_0 + \sigma r + \frac{c}{r} + \frac{d}{r^2} + l \ln(r).$$  \hspace{1cm} (5.9)

The inclusion of a logarithmic correction is motivated for example in [24]. From the dimensionless potential $a_4 V(x)$, $x = r/a_4$, we derive the auxiliary quantities

$$Z_1(x) = x^3 [V(x+1) - V(x-1)]/2,$$  \hspace{1cm} (5.10)

$$Z_2(x) = Z_1(x+1) + Z_1(x-1) - 2Z_1(x),$$  \hspace{1cm} (5.11)

$$Y_i(x) = x^i [V(x+1) + V(x-1) - 2V(x)], \ i = 2, 3, 4,$$  \hspace{1cm} (5.12)

$$Y_1(x) = x^3 [Y_2(x+1) - Y_2(x-1)]/2,$$  \hspace{1cm} (5.13)

in terms of which we can estimate the coefficients in Eq. (5.9) to be

$$\sigma(x + 0.5) = \sigma(x + 0.5)a_4^2 = [Z_2(x+1) - Z_2(x)]/6,$$  \hspace{1cm} (5.14)

$$c(x + 0.5) = -[Y_1(x+1) - Y_1(x)]/2,$$  \hspace{1cm} (5.15)

$$d(x) = d(x)/a_4 = x^3 [Y_3(x+1) + Y_3(x-1) - 2Y_3(x)]/12,$$  \hspace{1cm} (5.16)

$$l(x) = l(x)a_4 = -[Y_4(x+1) + Y_4(x-1) - 2Y_4(x)]/2.$$  \hspace{1cm} (5.17)
Figure 16: Analysis of the static potential along four-dimensional hyperplanes in the d-compact phase on lattices of size $L = 200$ at fixed anisotropy $\gamma = 0.25$. Left: The coefficient $c(r)$ of a four dimensional Coulomb term. Right: The coefficient $d(r)/a_4$ of a five dimensional Coulomb term. These coefficients are determined from local fits to the form in Eq. (5.9).

Figure 17: Analysis of the static potential along four-dimensional hyperplanes in the d-compact phase on lattices of size $L = 200$ at fixed anisotropy $\gamma = 0.25$. Left: The string tension $\sigma(r)a_4^2$. Right: The coefficient $l(r)a_4$ of a logarithm term. These coefficients are determined from local fits to the form in Eq. (5.9).

In Fig. 16 we plot the coefficients $c$ and $d$ of the four dimensional and five dimensional Coulomb term respectively. Away from the phase transition, at $\beta = 2.3$, we observe almost plateaus for the coefficients $d \in [0.15, 0.18]$ and $c \in [0.004, 0.007]$ in the range of distances $x = 40 \ldots 60$. As can be seen in Fig. 17, the coefficients $\sigma$ and $l$ are basically zero in this same range of distances. Therefore, the potential is a balance between a four-dimensional and a five-dimensional Coulomb term and we interpret this as the onset of dimensional reduction from five to four dimensions.

In fact, as we go closer to the phase transition at $\beta = 2.136$ and $\beta = 2.13495$ the
potential becomes four dimensional. In the range of distances $x = 15 \ldots 35$ we see in Fig. 16 a plateau for the four dimensional Coulomb coefficient $c \in [-0.031, -0.029]$, where the coefficient of the five-dimensional Coulomb term is much smaller $d \in [-0.01, -0.005]$. In the same range, see Fig. 17, the coefficient of the logarithmic term $\bar{7}$ is basically zero and the string tension has a plateau at a very small but negative value $\bar{\sigma} \in [-3 \times 10^{-6}, -2 \times 10^{-6}]$.

In Fig. 18 we fit the potential at $\beta = 2.136$ in the range $x = 10 \ldots 50$ globally to purely four or five dimensional Coulomb forms (i.e. we set $\sigma = d = l = 0$ in Eq. (5.9)). Clearly, the four dimensional Coulomb is an excellent fit while the five dimensional Coulomb law is excluded. Also, these plots exclude a possible large contribution from the presence of a light excited vector state which would contribute an extra Yukawa term in the short (and therefore also in the long) distance part of the potential. As already briefly mentioned in the beginning of this section, this was our first indication for dimensional reduction in the $d$-compact phase. Together with the vanishing of the extra dimensional force, it can be taken as a strong evidence for dimensional reduction via localization.
5.2 Discussion

We have now to compare these results with the behavior that we expect based on the underlying physical picture. If conventional Kaluza-Klein dimensional reduction works and the low-energy theory contains a massless gauge boson and a Higgs particle (the assumption here is that higher excitations are separated by a large mass gap), then this low-energy theory corresponds to the four-dimensional Georgi-Glashow ($SU(2)$ gauge theory with adjoint Higgs) model. In [21] the phase diagram of this model is discussed, it has a confined phase and a Higgs phase where the gauge symmetry is broken to $U(1)$ [22]. Note that in this model string breaking cannot happen when the static charges are in the fundamental representation and the matter in the adjoint. In order to attempt an interpretation of the long distance part of the static potential that is consistent with such a physical picture we will assume that the data in this regime is not unphysical. Since we have no guarantee that this is the case, the following discussion must be read with great caution. Also, in the following we will take seriously the short distance analysis of this section which lead to the conclusion that the system in the d-compact phase is truly dimensionally reduced.

On one hand, the fact that we do not see a non-vanishing mass for the gauge bosons (no Higgs mechanism) implies that our four dimensional effective theory should land in a confined phase. On the other hand one would expect a mass gap to appear in the confined phase which means that the mass of a gauge invariant vector operator like Eq. (3.24), representing a vector glueball state, should be non-zero. To reconcile this apparent contradiction we recall the argument of the first reference of [3] where the dimensional reduction of a five dimensional gauge theory\(^{12}\) in a Coulomb phase is discussed. There, in the dimensionally reduced state, the non-perturbatively generated mass gap behaves like

\[
m_{G\alpha} \sim e^{-k_{\text{eff}} / a}
\]

(\(k\) is a constant) when \(\hat{g}_4 \to 0\) and \(a \to 0\). We have denoted the asymptotically free coupling by \(\hat{g}_4\) to distinguish it from our effective coupling \(g_4\) defined in Eq. (4.37) and which approaches a non-zero critical value as \(a \to 0\). The mass gap in lattice units is exponentially suppressed near the continuum limit. Thus, in order to interpret our results in the context of dimensional reduction we must assume that also in our case such a mechanism is at work. This is an assumption since our data is not able to distinguish a zero from an exponentially small mass.

Based on these arguments we expect to see in the large distance behavior of the potential a string tension contribution, which rises linearly with the distance. Instead

\(^{12}\) In the context of D-theory the scalar field is decoupled, this can be achieved by choosing orbifold boundary conditions.
the local fits based on the assumed form of the potential give a very small but negative string tension, see Fig. 17. This could signal a breakdown of the meanfield at large distances, absence of dimensional reduction (these go against our working assumptions though) or that there are important terms in the confining string potential which we have neglected. In order to circumvent this ambiguity we have assumed the presence of the universal L"uscher term $-\pi/(8r)$ in $d = 5$ dimensions [23], subtracted it from the data and performed global fits of only the long distance part of the rest, varying $L$. In this case we obtained a positive string tension. The same analysis applied to the long distance part of the isotropic potential shown in Fig. 7 gave a similar result though, with almost the same value for the string tension. This means that the ambiguity persists.

In order to check whether it is possible to find a unambiguous signal for a positive string tension we computed the potential along the line of phase transitions separating the d-compact from the layered phase. We choose three points in the $(\beta, \gamma)$ parameter space, represented by blobs in Fig. 21: (1.358, 0.5), (2.136, 0.25) and (2.998, 0.172). These points are at the “same” distance from the phase transition if we measure it by the inverse correlation length $a_4 m_H$ (the Higgs mass in lattice units), which for the chosen points is 0.101, 0.094 and 0.103 respectively. We present the results for the coefficients $c, \bar{d}$ in Fig. 19 and for the coefficients $\sigma, \bar{t}$ in Fig. 20. While not much difference is seen between the points (2.136, 0.25) and (2.998, 0.172), for the point (1.358, 0.5) closest to the tricritical point of the phase diagram (the point where the compact, layered and d-compact phases meet) we get a positive string tension in the range of distances $r/a_4 = 27 \ldots 43$ with average value $1.2 \times 10^{-6}$. The value of the string tension increases as $L$ increases. We also
Figure 20: Analysis of the static potential along four-dimensional hyperplanes in the d-compact phase on lattices of size $L = 200$ at fixed inverse correlation length $a_4 m_H = 0.1$. Left: The string tension $\sigma(r)a_4^2$. Right: The coefficient $l(r)a_4$ of a logarithm term. These coefficients are determined from local fits to the form in Eq. (5.9).

note that the coefficient $c$ of the $1/r$ term increases by almost an order of magnitude as we get closer to the tricritical point. These observations deserve to be studied further.

We conclude this section by summarizing our results for the phase diagram of the anisotropic five dimensional $SU(2)$ gauge theory in Fig. 21 and by making a couple of remarks regarding the validity of our mean-field results.

Firstly, the results we presented for our physical quantities (static potential, masses) are independent of the gauge fixing parameter $\xi$. This is an immediate consequence of the fact that these quantities are gauge invariant and provides a check of our programs. The free energy depends\textsuperscript{13} on $\xi$ but not the conclusions we derive from it. We checked that the instability of the compact phase and the stability of the d-compact phase are independent on the choice of $\xi$. The precise location of the minimum of $F^{(1)}$ depends on $\xi$: for example at $\beta = 2.136$ and $\gamma = 0.25$ on a $T = L = N_5 = 10$ lattice, the minimum is at $v = \overline{v}_0$ and $v_5 = 0.48$ ($\xi = 1$) or $v = \overline{v}_0$ and $v_5 = 0.42$ ($\xi = 10$). Thus we would not base a serious criticism of our results on that.

Secondly, it is difficult to predict the effects of higher order corrections and/or the general domain of validity of the mean-field approximation. The mean-field is known to produce fake physics in certain cases, so it may generate consistent but unphysical phase transitions. This criticism concerns mainly the line of second order phase transitions separating the d-compact from the layered phase, where we have carried out most of our analysis. This is indeed possible but our working assumption has been that whenever \textsuperscript{13} The gauge dependence of the free energy is inherent in the meanfield approximation. Already at 0th order its value depends whether we fix or not the gauge.
Figure 21: The phase diagram for lattice $SU(2)$ gauge theory on an anisotropic torus, according to the mean-field method. In bold, the line of second order phase transition of critical exponent $1/2$ is plotted. The bold dashed line that separates the layered phase from the confined phase is a line of first order phase transition. The normal line indicates also first order phase transitions. The dashed lines indicate cross-over regions. The gray shaded area is unstable according to the first order free energy. The arrow shows the approach to the continuum limit for the discussion of the static potential at fixed anisotropy in Sect. 5.1. The blobs represents three points at fixed inverse correlation length $a_4m_H$ discussed in Sect. 5.2.

the mean-field is non-trivial, it describes a physical property. The existence of such an ultraviolet fixed point is suggested by the epsilon expansion [27] but has been elusive so far in Monte Carlo simulations. This also deserves further study.

6 Conclusions

Using the mean-field approach we explored the phase diagram of a five dimensional $SU(2)$ gauge theory with periodic boundary conditions. On the anisotropic lattice, we found two regimes where dimensional reduction seems to be at work.

One, the compact phase at large values of the anisotropy parameter is found to be unstable for moderately small values of $\beta$ down to the phase transition into the confined
phase when the free energy is computed at first order. In the limit $\beta \to \infty$ the mean-field reduces to standard perturbation theory where the compact phase does not have an instability but the theory becomes trivial.

The other regime, the d-compact phase, is at small values of the anisotropy parameter as approaching the phase where the coupling along the fifth dimension becomes confined – the layered phase. The latter phase is metastable to first order in the free energy. The transition between the d-compact and the layered phase is found to be a second order phase transition. This allows us to take the continuum limit of physical quantities. We explicitly computed the continuum limit of a physical charge. In a dimensionally reduced state we are entitled to view the spectrum in a basis of four dimensional quantum numbers. In this basis, ours is an $SU(2)$ gauge-Higgs system with the latter in the adjoint representation. Approaching the critical line, the Higgs mass remains finite while we are removing the cut-off. This is the first example of a four dimensional theory with this property as far as we know and is an independent check of results from Monte Carlo simulations [25] and effective field theory approaches [26]. The gauge boson mass vanishes within errors in the limit of infinite volume at any fixed lattice spacing. We cannot exclude an exponentially small mass expected if the system reduces to an effective four dimensional theory in a confined phase.

We investigated dimensional reduction in the d-compact phase by computing the static potential. The static potential along the extra dimension seems to vanish in the infinite volume limit, a fact which would support a scenario of dimensional reduction via localization in the d-compact phase. Moreover we found that as we get closer to the phase transition, the potential along the four dimensional hyperplanes turns into a four dimensional Coulomb form at short to intermediate distances. The unambiguous observation of a positive string tension contribution to the mean-field potential along the four dimensional planes at large distances remains elusive. But as the tricritical point is approached, local fits indicate a small but clear plateau were the string tension is positive.

It remains to see what a fully non-perturbative Monte Carlo computation will tell, in particular concerning the existence of a second order phase transition at small values of the anisotropy parameter. Our next analytical step following this work is to change the boundary conditions along the fifth dimension from periodic to orbifold [16].

Acknowledgments. We wish to thank Burkhard Bunk for sending his notes on $U(1)$ mean-field gauge theory, Peter Hasenfratz for comments, Martin Lüscher for an important clarification, Antonio Rago for discussions, Rainer Sommer for correspondence on the perturbative static potential and Peter Weisz for discussions and for his comments on a draft of our paper prior to publication. N. I. acknowledges the support of the Alexander von Humboldt Foundation via a Fellowship for Experienced Researcher.
A Meanfield corrections to Scalar and Gauge Boson masses

In this Appendix we describe in detail the computations of the diagram in Fig. 2 and Fig. 3 leading to the formulae given by Eq. (4.29) and Eq. (4.33), respectively.

A.1 The Scalar mass to first order

The first order correction to the Higgs mass is given by Eq. (2.31) with the operator \( O \) defined by Eq. (3.20). The second derivative to be computed is

\[
\frac{\delta \text{tr} \{ P^{(0)}(t_0, \vec{m}') \} \delta \text{tr} \{ P^{(0)}(t_0 + t, \vec{m}'') \} }{\delta v_{\alpha_2}(n_2, M_2) \delta v_{\alpha_1}(n_1, M_1)} = \Delta^{(N_5)}((n_5)_1) \Delta^{(N_5)}((n_5)_2)(P_0^{(0)})^2
\]

where \( P_0^{(0)} \) is the background value of the Polyakov loop and the symbol \( \Delta^{(N_5)}(n_5) \) is defined in Eq. (4.27). After averaging over starting time \( t_0 \) and space positions \( \vec{m}', \vec{m}'' \), the Fourier transform is

\[
\frac{4}{\mathcal{N}} \sum_{n_1, n_2} e^{i p'_n t_0} e^{-i p''_n t_0} \frac{1}{T} \sum_{t_0} \frac{1}{L^6} \sum_{\vec{m}', \vec{m}''} \Delta^{(N_5)}((n_5)_1) \Delta^{(N_5)}((n_5)_2)
\]

\[
\left[ \delta_{(n_0)_1, t_0 + t} \delta_{(n_0)_2, t_0} \delta_{\vec{n}, \vec{m}'} \delta_{\vec{n}', \vec{m}''} + \delta_{(n_0)_1, t_0} \delta_{(n_0)_2, t_0 + t} \delta_{\vec{n}, \vec{m}'} \delta_{\vec{n}', \vec{m}''} \right]
\]

\[
= \frac{4}{\mathcal{N}} \hat{\Delta}^{(N_5)}(p_5') \hat{\Delta}^{(N_5)}(-p_5'') \left[ \frac{1}{T} \sum_{t_0} e^{i p'_0 t_0} e^{-i p''_0 t_0} \frac{1}{L^6} \sum_{\vec{m}', \vec{m}''} e^{i p'' \vec{m}' e^{-i p'' \vec{m}''}} 
\right.
\]

\[
\left. + \frac{1}{T} \sum_{t_0} e^{i p'_0 t_0 + t} e^{-i p''_0 t_0} \frac{1}{L^6} \sum_{\vec{m}', \vec{m}''} e^{i p'' \vec{m}' e^{-i p'' \vec{m}''}} \right]
\]

\[
= \frac{8}{\mathcal{N}} \delta_{p'_0 p''_0} \cos(p'_0 t_0) \delta_{\vec{p}'_0} \delta_{\vec{p}_0} \hat{\Delta}^{(N_5)}(p_5') \hat{\Delta}^{(N_5)}(-p_5'') . \quad (A.2)
\]

We have also added the term that corresponds to interchanging the index 1 \( \leftrightarrow 2 \). Putting everything together and contracting with the propagator, we have the final expression for the first order scalar mass observable Eq. (4.29).

A.2 The Gauge Boson mass to second order

Next we turn to the \( W \) bosons. As discussed, the contribution to the gauge boson’s masses comes only at second order and is given in Eq. (2.43) with the operator \( O \) defined by Eq. (3.24). What we will need is the derivative of the displaced Polyakov loop \( W^{(0),A}_k \) defined in Eq. (3.22), which is essentially determined by the single derivative of \( \Phi^{(0)} \), Eq. (3.19). All terms with derivatives acting on the links \( U \) and any terms with more than one derivative acting on \( \Phi^{(0)} \) vanish identically because of \( \Phi^{(0)} = 0 \) when evaluated
on the background. The single derivative is simply
\[
\frac{\delta \Phi^{(0)}(t, \vec{m})}{\delta \nu_\alpha(n, M)} = 2 \Sigma_0^\alpha P_0^{(0)} \delta_{\nu_\alpha \nu_\beta} \delta_{\vec{m}, \vec{m}} \Delta^{(N_5)}(n_5) .
\] (A.3)

On the torus, we have to take \( \Sigma^0 = 0 \) and \( \Sigma^4 = i \sigma^4 \). The non-vanishing contributions to the fourth derivative in Eq. (2.43) are of the form
\[
\frac{\delta W_{(0), A}^{(0)}(t_0, \vec{m}')} {\delta \nu_\alpha_1(n_1, M_1) \delta \nu_\alpha_2(n_2, M_2) \delta \nu_\alpha_3(n_3, M_3) \delta \nu_\alpha_4(n_4, M_4)} = 16(P_0^{(0)})^4 \delta_{(\nu_\alpha_1),(\nu_\alpha_2),(\nu_\alpha_3),(\nu_\alpha_4)} \delta_{\nu_\alpha_4,t_0} \Delta^{(N_5)}((n_5)_1) \Delta^{(N_5)}((n_5)_2) \Delta^{(N_5)}((n_5)_3) \Delta^{(N_5)}((n_5)_4)
\]
\[
\left( \text{tr} \left\{ \sigma^4 \Sigma_{\alpha_1} \Sigma_{\alpha_2} \Sigma_{\alpha_3} \Sigma_{\alpha_4} \right\} \right) \delta_{\vec{m}, \vec{m}''} = 0 \] (A.4)

where \( (1 \leftrightarrow 2) \) means that \( (\alpha_1, n_1) \) in the second last line of Eq. (A.4) has to be exchanged with \( (\alpha_2, n_2) \) and similarly for the other exchanges of indices. According to Eq. (2.43) we have to compute the schematic expression
\[
\frac{1}{24} \sum_{(n_1, M_1, \alpha_1) \cdots (n_4, M_4, \alpha_4)} \frac{1}{T} \sum_{t_0} \frac{1}{L^6} \sum_{\vec{m}', \vec{m}''} 6 \frac{\delta^2 W}{\delta v_1 \delta v_2 \delta v_3 \delta v_4}
\]
\[
\left[ K^{-1}(1; 2) K^{-1}(3; 4) + K^{-1}(1; 3) K^{-1}(2; 4) + K^{-1}(1; 4) K^{-1}(2; 3) \right] .
\] (A.5)

in momentum space. The factor 6 is a symmetry factor that accounts for the different choices of pairs of indices out of \( v_1, v_2, v_3 \) and \( v_4 \) when taking the double derivatives. The first term in the bracket corresponds to time independent (self energy) contributions so it can be dropped, since it cancels out from the connected correlator. We will rewrite each of the two remaining terms as
\[
\sum_{m,n} O_1(m; n) K^{-1}(n; m) \times \sum_{r,s} O_2(r; s) K^{-1}(s; r)
\]
\[
= \text{tr}(O_1 K^{-1}) \times \text{tr}(O_2 K^{-1}) = \text{tr}(\tilde{O}_1 \tilde{K}^{-1}) \times \text{tr}(\tilde{O}_2 \tilde{K}^{-1}) .
\] (A.6)

For easier reference, we call the two relevant contractions 1 and 2 (the second and third term in the bracket of Eq. (A.5) respectively). Also, we label the four terms in the last part of eq. (A.4) with labels from \( I \) to \( IV \).

In order to keep our formulae as general as possible we will not assume momentum conservation in the fifth direction (like on the orbifold) and we will consider a propagator

\footnote{By introducing the matrices \( \Sigma^n \), Eq. (A.3) is applicable to the orbifold case as well.}
that can have off-diagonal terms in gauge space (like it can happen when the mean-field background is not simply proportional to the unit matrix).

A term of the form

\[ O(m; n) = O(m_0, n_0)O(m_5, n_5)O(m, n) \]  \hspace{1cm} (A.7)

will transform in momentum space into

\[ \tilde{O}(p'; p'') = \tilde{O}(p'_5, p''_5) \frac{1}{T} \sum_{m_0, n_0} e^{i\vec{p}'_0 \cdot \vec{m}_0} e^{-i\vec{p}''_0 \cdot \vec{n}_0} O(m_0, n_0) \frac{1}{L^3} \sum_{\tilde{m}, \tilde{n}} e^{i\vec{\tilde{m}} \cdot \tilde{\vec{m}}} e^{-i\vec{\tilde{n}} \cdot \tilde{\vec{n}}} O(\tilde{m}, \tilde{n}) , \]  \hspace{1cm} (A.8)

(and analogously using \( q'; q'' \) for the operator in the other factor of Eq. (A.6)) where the part along the extra dimension is the universal term

\[ \tilde{O}(p'_5, p''_5) = \frac{1}{N^2} \tilde{\Delta}^{(N_2)}(p'_5) \tilde{\Delta}^{(N_3)}(-p''_5) . \]  \hspace{1cm} (A.9)

The temporal Fourier transform gives for the first contraction

\[ O(3; 1) : \frac{1}{T} e^{i\vec{p}_0 \cdot (t_0 + t)} e^{-i\vec{q}'_0 \cdot t_0}, \quad O(4; 2) : \frac{1}{T} e^{i\vec{p}_0 \cdot (t_0 + t)} e^{-i\vec{q}''_0 \cdot t_0}, \] \hspace{1cm} (A.10)

and for the second contraction

\[ O(4; 1) : \frac{1}{T} e^{i\vec{p}_0 \cdot (t_0 + t)} e^{-i\vec{q}'_0 \cdot t_0}, \quad O(3; 2) : \frac{1}{T} e^{i\vec{p}_0 \cdot (t_0 + t)} e^{-i\vec{q}''_0 \cdot t_0}. \]

Both products result into

\[ \frac{1}{T} \sum_{t_0} \frac{1}{T^2} e^{i\omega(p'_0 - p''_0 + q'_0 - q''_0)} e^{i\eta(p'_0 + q'_0)} \delta_{p'_0 q'_0} \delta_{p''_0 q''_0} = \frac{1}{T^2} e^{i\omega(p'_0 + q'_0)} \delta_{p'_0 q'_0} \delta_{p''_0 q''_0}, \] \hspace{1cm} (A.11)

where we made conservation of temporal momentum, enforced upon taking the trace with the propagator, explicit. Next we turn to the spatial Fourier transforms. Each of the two propagator contractions contains four terms, which we list here inserting conservation of spatial momentum.

Contraction \( 1I \):

\[ \frac{1}{L^6} \sum_{\vec{m}, \vec{m}''} \frac{1}{L^6} \delta_{\vec{p}', \vec{p}'} \delta_{\vec{q}', \vec{q}'} e^{i\vec{p}' \cdot (\vec{m}' + \vec{m}'' - \vec{i} - (\vec{m}' + \vec{k}))} e^{i\vec{q}' \cdot (\vec{m}'' - \vec{i} - \vec{m}')} = \frac{1}{L^6} \delta_{\vec{p}', \vec{q}'} \delta_{\vec{q}', \vec{p}'} \delta_{\vec{p}', \vec{q}'} e^{i\vec{p}' \cdot (\vec{q}' - \vec{q})}. \]

Contraction \( 1II \):

\[ \frac{1}{L^6} \sum_{\vec{m}, \vec{m}''} \frac{1}{L^6} \delta_{\vec{p}', \vec{p}'} \delta_{\vec{q}', \vec{q}'} e^{i\vec{p}' \cdot (\vec{m}' + \vec{m}'' - \vec{i} - \vec{m}')} e^{i\vec{q}' \cdot (\vec{m}'' + \vec{i} - \vec{m}')} = \frac{1}{L^6} \delta_{\vec{p}', \vec{p}'} \delta_{\vec{q}', \vec{q}'} \delta_{\vec{p}', \vec{q}'} e^{i\vec{p}' \cdot (\vec{q}' + \vec{q})}. \]

Contraction \( 1III \):

\[ \frac{1}{L^6} \sum_{\vec{m}, \vec{m}''} \frac{1}{L^6} \delta_{\vec{p}', \vec{p}'} \delta_{\vec{q}', \vec{q}'} e^{i\vec{p}' \cdot (\vec{m}'' - (\vec{m}' + \vec{k}))} e^{i\vec{q}' \cdot ((\vec{m}'' + \vec{l}) - \vec{m}'')} = \frac{1}{L^6} \delta_{\vec{p}', \vec{q}'} \delta_{\vec{q}', \vec{p}'} \delta_{\vec{p}', \vec{q}'} e^{-i\vec{p}' \cdot (\vec{q}' + \vec{q})}. \]
Contraction 1IV:

\[
\frac{1}{L_6} \sum_{\vec{m}', \vec{m}''} \frac{1}{L_6} \delta_{\vec{p}', \vec{p}''} \delta_{\vec{q}', \vec{q}''} e^{i\vec{p}'(\vec{m}'' - \vec{m}')} e^{i\vec{q}'((\vec{m}'' + \vec{l}) - (\vec{m}' + \vec{k}))} = \frac{1}{L_6} \delta_{\vec{p}', \vec{p}''} \delta_{\vec{q}', \vec{q}''} \delta_{\vec{p}, -\vec{q}} e^{-i(\vec{p}_1' - \vec{p}_k')} .
\]

Contraction 2I:

\[
\frac{1}{L_6} \sum_{\vec{m}', \vec{m}''} \frac{1}{L_6} \delta_{\vec{p}', \vec{p}''} \delta_{\vec{q}', \vec{q}''} e^{i\vec{p}'(\vec{m}'' - (\vec{m}' + \vec{k}))} e^{i\vec{q}'((\vec{m}'' + \vec{l}) - \vec{m}')} = \frac{1}{L_6} \delta_{\vec{p}', \vec{p}''} \delta_{\vec{q}', \vec{q}''} \delta_{\vec{p}, -\vec{q}} e^{-i(\vec{p}_1' + \vec{p}_k')} .
\]

Contraction 2II:

\[
\frac{1}{L_6} \sum_{\vec{m}', \vec{m}''} \frac{1}{L_6} \delta_{\vec{p}', \vec{p}''} \delta_{\vec{q}', \vec{q}''} e^{i\vec{p}'((\vec{m}'' + \vec{l}) - (\vec{m}' + \vec{k}))} e^{i\vec{q}'(\vec{m}'' - \vec{m}')} = \frac{1}{L_6} \delta_{\vec{p}', \vec{p}''} \delta_{\vec{q}', \vec{q}''} \delta_{\vec{p}, -\vec{q}} e^{i(\vec{p}_1' - \vec{p}_k')} .
\]

Contraction 2III:

\[
\frac{1}{L_6} \sum_{\vec{m}', \vec{m}''} \frac{1}{L_6} \delta_{\vec{p}', \vec{p}''} \delta_{\vec{q}', \vec{q}''} e^{i\vec{p}'((\vec{m}'' + \vec{l}) - (\vec{m}' + \vec{k}))} e^{i\vec{q}'(\vec{m}'' - \vec{m}')} = \frac{1}{L_6} \delta_{\vec{p}', \vec{p}''} \delta_{\vec{q}', \vec{q}''} \delta_{\vec{p}, -\vec{q}} e^{i(\vec{p}_1' + \vec{p}_k')} .
\]

Contraction 2IV:

\[
\frac{1}{L_6} \sum_{\vec{m}', \vec{m}''} \frac{1}{L_6} \delta_{\vec{p}', \vec{p}''} \delta_{\vec{q}', \vec{q}''} e^{i\vec{p}'(\vec{m}'' + \vec{l} - \vec{m}') e^{i\vec{q}'(\vec{m}'' - \vec{m}')}} = \frac{1}{L_6} \delta_{\vec{p}', \vec{p}''} \delta_{\vec{q}', \vec{q}''} \delta_{\vec{p}, -\vec{q}} e^{i(\vec{p}_1' + \vec{p}_k')} .
\]

We can simplify the expressions if we define

\[
\overline{K}^{-1}(p_0', p_0''), 5, \alpha_1, \alpha_2) = \sum_{p_5, p_5''} \Delta^{(N_5)}(p_5') \Delta^{(N_5)}(-p_5'') K^{-1}(p_5'', 5, \alpha_1; p_5', 5, \alpha_2) .
\]  

(A.12)

The components of the propagator diagonal in the Euclidean index are even under reflecting the spatial momenta, which means

\[
\overline{K}^{-1}(p_0', -p_0''), 5, \alpha_1, \alpha_2) = \overline{K}^{-1}(p_0', p_0''), 5, \alpha_1, \alpha_2) .
\]  

(A.13)

If we define in addition

\[
\overline{K}^{-1}(t, p_0', \alpha_1, \alpha_2) = \sum_{p_0'} e^{i\vec{p}_0' t} \overline{K}^{-1}(p_0', p_0''), 5, \alpha_1, \alpha_2) ,
\]  

(A.14)

and use the fact that \( \Sigma^\dagger = -\Sigma \) we can write out the whole expression Eq. (A.4) as

\[
\frac{6 \cdot 16 \cdot 4}{24} \frac{1}{N^2} \left( P_0^{(0)} \right)^4 \left( \overline{u}_0(0) \right)^4 \sum_{\vec{p}} \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \text{tr} \left\{ \overline{\sigma}^A \Sigma^{\alpha_1} \Sigma^{\alpha_2} \right\} \text{tr} \left\{ \sigma^A \Sigma^{\alpha_3} \Sigma^{\alpha_4} \right\} \]

\[
\times \left[ \cos (p_1' - p_k') \overline{K}^{-1}(t, p_0', \alpha_1, \alpha_3) \overline{K}^{-1}(t, p_0', \alpha_2, \alpha_4) \\
+ \cos (p_1' + p_k') \overline{K}^{-1}(t, p_0', \alpha_2, \alpha_3) \overline{K}^{-1}(t, p_0', \alpha_1, \alpha_4) \right] .
\]  

(A.15)
Finally we simplify this expression in some special cases. If the propagator is diagonal in the gauge index, Eq. (A.15) becomes

$$\frac{16}{N^2} (P_0^{(0)})^4 (\bar{\sigma}_0(0))^4 \sum_{\tilde{p}' \alpha_1, \alpha_2} \text{tr}\left\{ \sigma^A \Sigma^{\alpha_1} \Sigma^{\alpha_2} \right\} \cdot \left[ \text{tr}\left\{ \sigma^A \{ \Sigma^{\alpha_1}, \Sigma^{\alpha_2} \} \right\} \cos(p'_l) \cos(p'_k) + \text{tr}\left\{ \sigma^A \{ \Sigma^{\alpha_1}, \Sigma^{\alpha_2} \} \right\} \sin(p'_l) \sin(p'_k) \right] \cdot \frac{1}{K^{-1}(t, \tilde{p}', \alpha_1)} \frac{1}{K^{-1}(t, \tilde{p}', \alpha_2)}. \tag{A.16}$$

On the torus, $\Sigma^0 = 0$ and $\Sigma^A = i\sigma^A$ imply that the term with the cosines in Eq. (A.16) is absent and so the toron does not contribute to the correlator. Using $\text{tr}\{\sigma^A \sigma^A \} = 2i\epsilon^{A_1 A_2}$ and summing over the gauge index $A$ and the Euclidean indices with $\delta_{kl}$ gives the result in Eq. (4.33).

References

[1] M. Quiros,
   
   New ideas in symmetry breaking, hep-ph/0302189 and references therein.

[2] Y. K. Fu and H. B. Nielsen,
   
   A Layer phase in a nonisotropic $U(1)$ lattice gauge theory: dimensional reduction, a new way, Nucl. Phys. B236 (1984) 167.

D. Berman and E. Rabinovici,
   
   Layer phases in anisotropic lattice gauge theories, Phys. Lett. B66 (1985) 292.

A. Huselbos, C. P. Korthals-Altes and S. Nicolis,
   
   Gauge theories with a layered phase, Nucl. Phys. B450 (1995) 437.

P. Dimopoulos, K. Farakos, A. Kehagias and G. Koutsoumbas,
   
   Lattice evidence for gauge field localization on a brane, Nucl. Phys. B617 (2001) 237.

P. Dimopoulos, K. Farakos and G. Koutsoumbas,
   
   The phase diagram for the anisotropic SU(2) adjoint Higgs model in 5D: Lattice evidence for layered structure, Phys. Rev. D65 (2002) 074505.

[3] S. Chandrasekharan and U. J. Wiese,
   
   Quantum link models: A discrete approach to gauge theories, Nucl. Phys. B492 (1997) 455.

S. Ejiri, J. Kubo and M. Murata,
   
   A study on the nonperturbative existence of Yang-Mills theories with large extra dimensions, Phys. Rev. D62 (2000) 105025.
M. Laine, H. B. Meyer, K. Rummukainen and M. Shaposhnikov,
Effective gauge theories on domain walls via bulk confinement?, JHEP 04 (2004) 027.

[4] J. M. Drouffe and J. B. Zuber,
Strong Coupling And Mean Field Methods In Lattice Gauge Theories, Phys. Rept. 102 (1983) 1.

[5] J. Zinn-Justin,
Quantum Field Theory and Critical Phenomena, International Series of Monographs on Physics, Vol. 113, Clarendon Press, Oxford, 2002.

[6] F. Knechtli, B. Bunk and N. Irges,
Gauge theories on a five-dimensional orbifold, PoS LAT2005 (2006) 280.

[7] W. Ruhl,
The mean field perturbation theory of lattice gauge models with covariant gauge fixing, Z. Phys. C18 (1983) 207.
B. Lautrup and W. Ruhl,
Higher order mean field expansions for lattice gauge theories, Z. Phys. C23 (1984) 49.
N. Kawamoto and K. Shigemoto,
Systematic expansion of lattice mean field calculation, NBI-HE-83-02.
H. Flyvbjerg, P. Mansfeld and B. Soderberg,
High precision mean-field results for lattice gauge theories, Nucl. Phys. B240 (1984) 171.
B. Bunk, Unpublished

[8] N. Irges and F. Knechtli,
Non-perturbative definition of five-dimensional gauge theories on the $\mathbb{R}^4 \times S^1/\mathbb{Z}_2$ orbifold, Nucl. Phys. B719 (2005) 121.

[9] N. Irges and F. Knechtli,
Lattice Gauge Theory Approach to Spontaneous Symmetry Breaking from an Extra Dimension, Nucl. Phys. B775 (2007) 283.

[10] M. Creutz,
Confinement and the criticality of space-time, Phys. Rev. Lett. 43 (1979) 553.

[11] M. Creutz,
Gauge fixing, the Transfer Matrix and Confinement on a Lattice, Phys. Rev. D15 (1977) 118.
[12] A. Coste, A. Gonzales-Arroyo, J. Jurkiewicz and C. P. Korthals-Altes,
Zero momentum contribution to Wilson loops in periodic boxes, Nucl. Phys. B262
(1985) 67,
B. Petersson and T. Reisz,
Polyakov loop correlations at finite temperature, Nucl. Phys. B353 (1991) 757.

[13] I. Montvay,
Correlations in the SU(2) fundamental Higgs model, Phys. Lett. B150 (1985) 441.

[14] I. Montvay and G. Munster,
Quantum fields on a lattice, Cambridge, UK: Univ. Pr. (1994) 491 p. (Cambridge
monographs on mathematical physics)

[15] The second of ref. [3].

[16] N. Irges and F. Knechtli,
Mean-Field Gauge Interactions in Five Dimensions II. The orbifold; in preparation.

[17] B. Svetitsky and L. G. Yaffe,
Critical Behavior at Finite Temperature Confinement Transitions,
Nucl. Phys. B210 (1982) 423.

[18] P. Dimopoulos, K. Farakos and S. Vrentzos,
The 4-D Layer Phase as a Gauge Field Localization: Extensive Study of the 5-D
Anisotropic U(1) Gauge Model on the Lattice, Phys. Rev. D74 (2006) 094506.

[19] V. S. Dotsenko and S. N. Vergeles,
Renormalizability Of Phase Factors In The Nonabelian Gauge Theory, Nucl. Phys.
B 169 (1980) 527.
R. A. Brandt, F. Neri and M. Sato,
Renormalization Of Loop Functions For All Loops, Phys. Rev. D 24 (1981) 879.

[20] S. Necco and R. Sommer,
The N(f) = 0 heavy quark potential from short to intermediate distances, Nucl. Phys.
B622 (2002) 328.

[21] R. E. Shrock,
Lattice Higgs models, Nucl. Phys. Proc. Suppl. 4 (1988) 373.

[22] I. H. Lee and J. Shigemitsu,
Spectrum Calculations In The Lattice Georgi-Glashow Model, Nucl. Phys. B 263
(1986) 280.
[23] M. Lüscher, K. Symanzik and P. Weisz,  
*Anomalies Of The Free Loop Wave Equation In The Wkb Approximation*, Nucl. Phys. B 173, 365 (1980). 
M. Lüscher,  
*Symmetry Breaking Aspects Of The Roughening Transition In Gauge Theories*, Nucl. Phys. B 180, 317 (1981). 
M. Lüscher and P. Weisz,  
*Quark confinement and the bosonic string*, JHEP 0207 (2002) 049 [arXiv:hep-lat/0207003].

[24] S. L. Adler and T. Piran,  
*Flux confinement in the leading logarithm model*, Phys. Lett. B 113 (1982) 405 [Erratum-ibid. B 121 (1983) 455]. 
G. Perez-Nadal and J. Soto,  
*Effective string theory constraints on the long distance behavior of the subleading potentials*, arXiv:0811.2762 [hep-ph].

[25] N. Irges and F. Knechtli,  
*Non-perturbative mass spectrum of an extra-dimensional orbifold*, arXiv:hep-lat/0604006.

[26] L. Del Debbio, E. Kerrane and R. Russo,  
*Mass corrections in string theory and lattice field theory*, arXiv:0812.3129 [hep-th].

[27] H. Gies,  
*Renormalizability of gauge theories in extra dimensions*, Phys. Rev. D 68, 085015 (2003) [arXiv:hep-th/0305208]. 
T. R. Morris,  
*Renormalizable extra-dimensional models*, JHEP 0501 (2005) 002 [arXiv:hep-ph/0410142].