Characterising pointsets in $\text{PG}(4, q)$ that correspond to conics

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Abstract

We consider a non-degenerate conic in $\text{PG}(2, q^2)$, $q$ odd, that is tangent to $\ell_\infty$ and look at its structure in the Bruck-Bose representation in $\text{PG}(4, q)$. We determine which combinatorial properties of this set of points in $\text{PG}(4, q)$ are needed to reconstruct the conic in $\text{PG}(2, q^2)$. That is, we define a set $\mathcal{C}$ in $\text{PG}(4, q)$ with $q^2$ points that satisfies certain combinatorial properties. We then show that if $q \geq 7$, we can use $\mathcal{C}$ to construct a regular spread $\mathcal{S}$ in the hyperplane at infinity of $\text{PG}(4, q)$, and that $\mathcal{C}$ corresponds to a conic in the Desarguesian plane $\mathcal{P}(\mathcal{S}) \cong \text{PG}(2, q^2)$ constructed via the Bruck-Bose correspondence.

1 Introduction

We begin by describing the Bruck-Bose representation of a translation plane of order $q^2$ with kernel containing $\text{GF}(q)$ in $\text{PG}(4, q)$, proved independently by André [11] and Bruck and Bose [6, 7]. Let $\Sigma_\infty$ be a hyperplane of $\text{PG}(4, q)$ and let $\mathcal{S}$ be a spread of $\Sigma_\infty$. Consider the incidence structure whose points are the points of $\text{PG}(4, q) \setminus \Sigma_\infty$, whose lines are the planes of $\text{PG}(4, q)$ which do not lie in $\Sigma_\infty$ but which meet $\Sigma_\infty$ in a line of $\mathcal{S}$ and where incidence is inclusion. This incidence structure is an affine translation plane and can be uniquely completed to a projective translation plane $\mathcal{P}(\mathcal{S})$ of order $q^2$ by adjoining the line at infinity $\ell_\infty$ whose points are the elements of the spread $\mathcal{S}$. The line $\ell_\infty$ is a translation line for $\mathcal{P}(\mathcal{S})$. The translation plane $\mathcal{P}(\mathcal{S})$ is Desarguesian if and only if the spread $\mathcal{S}$ is regular [6]. For more details on the Bruck-Bose representation, see [3], in particular, note that Baer subplanes of $\text{PG}(2, q^2)$ secant to $\ell_\infty$ are in one to one correspondence with affine planes of $\text{PG}(4, q)$ that do not contain a line of the regular spread $\mathcal{S}$. 
In PG(4, q), with Σ∞ being the hyperplane at infinity, we call the points of PG(4, q) \ Σ∞ affine points and the points in Σ∞ infinite points. The lines and planes of PG(4, q) that are not contained in Σ∞ are called affine lines and affine planes respectively.

Now consider a non-degenerate conic \( \mathcal{C} \) in PG(2, \( q^2 \)) that is tangent to \( \ell_\infty \) in the point \( P_\infty \). In the Bruck-Bose representation of PG(2, \( q^2 \)) in PG(4, q), the affine points \( \mathcal{C} = \mathcal{C} \setminus \{P_\infty\} \) correspond to a set of \( q^2 \) affine points in PG(4, q) also denoted by \( \mathcal{C} \). These points satisfy a variety of properties. In [5], algebraic properties of these points were determined, and it was shown they were caps. In this article we are interested in combinatorial properties the points satisfy, in particular, combinatorial properties relating to the planes of PG(4, q). Our aim was to find the smallest set of these properties that would allow us to reconstruct the projective plane and the conic \( \mathcal{C} \). The properties of the conic in PG(2, \( q^2 \)) that we are interested in are given in the next lemma, which is proved in Section 2.

**Lemma 1.1** Let \( \mathcal{C} \) be a non-degenerate conic of PG(2, \( q^2 \)), \( q \) odd, that is tangent to \( \ell_\infty \) in a point \( P_\infty \). Define a \( \mathcal{C} \)-plane to be a Baer subplane of PG(2, \( q^2 \)) that is secant to \( \ell_\infty \) and meets \( \mathcal{C} = \mathcal{C} \setminus \{P_\infty\} \) in \( q \) points. Then

1. Each \( \mathcal{C} \)-plane meets \( \mathcal{C} \) in a \( q \)-arc. Further, if a Baer subplane secant to \( \ell_\infty \) meets \( \mathcal{C} \) in more than four points, it is a \( \mathcal{C} \)-plane.

2. Every pair of points of \( \mathcal{C} \) lie in exactly one \( \mathcal{C} \)-plane.

3. The affine points of PG(2, \( q^2 \)) are of three types: points of \( \mathcal{C} \); points on no \( \mathcal{C} \)-plane (the interior points of \( \mathcal{C} \)); and points on exactly two \( \mathcal{C} \)-planes (the exterior points of \( \mathcal{C} \)).

We now consider what these properties correspond to in the Bruck-Bose representation in PG(4, q). The points of \( \mathcal{C} \) correspond to affine points of PG(4, q), and the \( \mathcal{C} \)-planes correspond to affine planes of PG(4, q) that contain \( q \) points of \( \mathcal{C} \). We suppose that we have a set of affine points and \( \mathcal{C} \)-planes satisfying the PG(4, q) equivalence of the combinatorial properties of Lemma 1.1 and show that we can reconstruct the conic in the Bruck-Bose plane. The main result of this paper is the characterisation given in the next theorem.

**Theorem 1.2** Let \( \Sigma_\infty \) be the hyperplane at infinity in PG(4, q), \( q \geq 7 \), \( q \) odd. Let \( \mathcal{C} \) be a set of \( q^2 \) affine points, called \( \mathcal{C} \)-points, and suppose there exists a set of affine planes called \( \mathcal{C} \)-planes satisfying the following properties:

(A1) Each \( \mathcal{C} \)-plane meets \( \mathcal{C} \) in a \( q \)-arc. Further, if a plane meets \( \mathcal{C} \) in more than four points, it is a \( \mathcal{C} \)-plane.

(A2) Every pair of points of \( \mathcal{C} \) lie in exactly one \( \mathcal{C} \)-plane.

(A3) The affine points of PG(4, q) are of three types: points of \( \mathcal{C} \), points on no \( \mathcal{C} \)-plane, and points on exactly two \( \mathcal{C} \)-planes.
Then there exists a unique spread $S$ in $\Sigma_\infty$ so that the $C$-points in the Bruck-Bose plane $\mathcal{P}(S)$ form a $q^2$-arc of $\mathcal{P}(S)$. Moreover, the spread $S$ is regular, and so $\mathcal{P}(S) \cong \text{PG}(2, q^2)$, and the $q^2$-arc can be completed to a conic of $\text{PG}(2, q^2)$.

We note that a similar characterisation when $q$ is even is given in [4].

The rest of this paper is devoted to proving this theorem. The main structure of the proof is as follows. We need a number of preliminary results, leading to Theorem 3.17 where we show how to construct a spread $S$ of $\Sigma_\infty$ from $C$. In Corollary 3.24, we show that $S$ is a regular spread. In Theorem 3.25, we show that the set $C$ corresponds to a $q^2$-arc in the Bruck-Bose plane $\mathcal{P}(S) \cong \text{PG}(2, q^2)$, and hence as $q$ is odd, $C$ is contained in a unique conic. In Theorem 3.26, we show that if $S'$ is any other spread of $\Sigma_\infty$, then the points of $C$ will not correspond to an arc in the associated Bruck-Bose plane $\mathcal{P}(S')$.

2 Proof of Lemma 1.1

In this section we prove that a conic in $\text{PG}(2, q^2)$ satisfies the combinatorial properties stated in Lemma 1.1.

Proof of Lemma 1.1 Let $C$ be a non-degenerate conic in $\text{PG}(2, q^2)$ tangent to $\ell_\infty$ in the point $P_\infty$, and let $C = \overline{C} \setminus \{P_\infty\}$. We begin with a note about subconics of $\overline{C}$ in Baer subplanes. Suppose $\pi$ is a Baer subplane that meets $\overline{C}$ in a subconic $\emptyset$. Note that for each point $P$ in $\emptyset \cap \pi$, the tangent line of $\overline{C}$ at $P$ is a line of $\pi$. Conversely, if a line $\ell$ of $\pi$ is a tangent line of $\overline{C}$, then the point of contact $\ell \cap \overline{C}$ lies in $\pi$.

We now prove part 1. The $q$ points of $C$ in a $C$-plane lie on the conic $\overline{C}$, so are clearly a $q$-arc. Suppose $\pi$ is a Baer subplane secant to $\ell_\infty$ which meets $C$ in five points, then those five points define a unique conic in $\text{PG}(2, q^2)$, namely $\overline{C}$. Further, the five points define a unique conic $\emptyset$ in $\pi$, which is necessarily a subconic of $\overline{C}$. As $\pi$ is secant to $\ell_\infty$, it contains the tangent $\ell_\infty$ to $\overline{C}$, hence by the above note, it contains the point $P_\infty$ of contact of the tangent $\ell_\infty$ to $\overline{C}$. Hence $\pi$ meets $C$ in $q$ points, and so is a $C$-plane.

For part 2, let $P, Q$ be two points of $C$ with tangents to $\overline{C}$ labeled $t_P, t_Q$ respectively. Suppose $\pi$ is any Baer subplane secant to $\ell_\infty$ containing $P, Q$. As $\pi$ contains $\ell_\infty$, it contains $P_\infty$ by the above note. Further, as $\pi$ contains $P, Q$, we have that $t_P, t_Q$ are lines of $\pi$. Thus $\pi$ contains the quadrangle $P, Q, t_P \cap \ell_\infty, P_\infty$ and hence there is exactly one such subplane. However, the quadrangle $P, Q, t_P \cap \ell_\infty, P_\infty$ defines a unique Baer subplane which contains five elements from the conic $\overline{C}$: namely the points $P, Q, P_\infty$ and the tangents $\ell_\infty, t_P$, and thus contains a subconic of $\overline{C}$ containing $P_\infty$. Thus every pair of points of $C$ lie on exactly one $C$-plane.

Note that this allows us to count the number of $C$-planes. Let $\mathcal{A}$ be the incidence structure with points the points of $C$; lines the $C$-planes; and inherited incidence. Then $\mathcal{A}$ is a 2-$(q^2, q, 1)$ design, and so is an affine plane of order $q$. Hence there are $q^2 + q$ $C$-planes.
To prove part 3, let $\pi$ be a $C$-plane, and let $X$ be an affine point of $\pi$ that is not in $C$. Then $X$ lies on two tangents of $\pi \cap C^\perp$, which lie either in $\pi$ or in $\text{PG}(2, q^2) \setminus \pi$, depending on whether $X$ is an exterior point or an interior point of the conic $\pi \cap C^\perp$. In either case, $X$ lies on two tangents of $C^\perp$ in $\text{PG}(2, q^2)$, and so is an exterior point of $C^\perp$. Hence the affine points in a $C$-plane either lie in $C$, or are exterior points of $C^\perp$. That is, the interior points of $C$ lie on zero $C$-planes. It is straightforward to show that the group of homographies of $\text{PG}(2, q^2)$ fixing $C$ and $\ell_\infty$ is transitive on the affine exterior points of $C^\perp$. Hence all the affine exterior points of $C^\perp$ lie on a common number $x$ of $C$-planes. We now count incident pairs $(X, \pi)$ where $X$ is an affine exterior point of $C$ that lies on a $C$-plane $\pi$. We have $q^2(q^2 - 1)/2 \times x = (q^2 + q)(q^2 - q)$, hence $x = 2$ as required.

\section{Proof of Theorem 1.2}

\subsection{Properties of $C$-planes}

Let $C$ be a set of $q^2$ affine points in $\text{PG}(4, q) \setminus \Sigma_\infty$, $q \geq 7$, $q$ odd, satisfying assumptions (A1), (A2) and (A3) of Theorem 1.2. We begin by noting that the $C$-points and $C$-planes form an affine plane. This gives us a natural set of parallel classes which will be useful in our proof.

**Lemma 3.1** Let $A$ be the incidence structure with points the points of $C$; lines the $C$-planes; and inherited incidence. Then $A$ is an affine plane of order $q$, and so there are $q^2 + q$ $C$-planes. Hence there are $q + 1$ parallel classes of $C$-planes, each containing $q$ parallel $C$-planes.

**Proof** By (A1) and (A2), $A$ is a $2-(q^2, q, 1)$ design and so is an affine plane of order $q$. \hfill $\Box$

Note that in $\text{PG}(4, q)$, two $C$-planes $\pi, \alpha$ meet in a point or a line. The parallel classes of the affine plane $A$ tell us whether the intersection $\pi \cap \alpha$ contains a $C$-point. So we have: if $\pi$ and $\alpha$ belong to the same parallel class, then they have no common $C$-point; whereas if $\pi$ and $\alpha$ belong to different parallel classes, then they share exactly one $C$-point.

Let $\pi$ be a $C$-plane. By (A1), $\pi$ contains $q$ $C$-points that form a $q$-arc. As $q$ is odd, this arc uniquely completes to a conic by the addition of a point which we denote $\pi_\infty$ (see [9, Theorem 10.28]). We call $\pi_\infty$ the $\infty$-point of $\pi$ since we will show in Lemma 3.3 that $\pi_\infty$ is on the line $\pi \cap \Sigma_\infty$, and so is an infinite point of $\text{PG}(4, q)$. That is:

**Definition 3.2** Let $\pi$ be a $C$-plane, then the $\infty$-point $\pi_\infty$ of $\pi$ is the point that uniquely completes the $q$-arc $\pi \cap C$ to a conic.

To study the intersection of $C$-planes in more detail, we will need the next lemma involving three $C$-planes that all contain a fixed $\infty$-point.
Lemma 3.3 Let \( \pi \) be a \( \mathcal{C} \)-plane with \( \infty \)-point \( \pi_\infty \). Suppose two \( \mathcal{C} \)-planes \( \alpha, \beta \) both meet \( \pi \) in distinct lines through \( \pi_\infty \). Then \( \alpha \) and \( \beta \) share no point other than \( \pi_\infty \).

Proof Let \( \pi, \alpha, \beta \) be \( \mathcal{C} \)-planes such that \( \alpha \cap \pi, \beta \cap \pi \) are distinct lines through \( \pi_\infty \). Note that if \( \pi_\infty \notin \Sigma_\infty \), then \( \pi_\infty \) is on three \( \mathcal{C} \)-planes, namely \( \pi, \alpha \) and \( \beta \), contradicting (A3). So we have \( \pi_\infty \in \Sigma_\infty \). Suppose \( \alpha \) and \( \beta \) meet in a line \( \ell \) (so \( \ell \) contains \( \pi_\infty \)). The line \( \ell \) contains at most one \( \mathcal{C} \)-point by (A2). Note that the \( q + 1 \) lines of \( \pi \) through \( \pi_\infty \) consist of \( q \) 1-secants to \( \mathcal{C} \), and one 0-secant to \( \mathcal{C} \). Hence at least one of the lines \( \alpha \cap \pi, \beta \cap \pi \) is a 1-secant of \( \mathcal{C} \). Without loss of generality, suppose \( \pi \cap \beta \) meets \( \mathcal{C} \) in the point \( B \). As \( \mathcal{C} \) meets \( \pi \) in a \( q \)-arc, there are two 1-secants to \( \mathcal{C} \) through \( B \) in \( \pi \), one is \( B\pi_\infty \), let the other meet \( \pi \cap \alpha \) in the point \( R \). In the plane \( \beta \), there are two 1-secants to \( \mathcal{C} \) through \( B \), one is \( B\pi_\infty \), let the other meet \( \beta \cap \alpha \) in the point \( S \).

We now show that there exists a 2-secant \( m \) of \( \alpha \cap \mathcal{C} \) which does not contain the points \( S, R \) or \( \pi_\infty \), and which does not meet \( \ell \) or \( \alpha \cap \pi \) in a \( \mathcal{C} \)-point. As \( q \geq 7 \), the \( q \)-arc \( \alpha \cap \mathcal{C} \) has at least five points \( X, Y, Z, V, W \) which do not lie on \( \ell \) or on \( \alpha \cap \pi \) (each of which contain at most one \( \mathcal{C} \)-point). A 2-secant through two of these points will meet \( \ell \) and \( \alpha \cap \pi \) in points that are not in \( \mathcal{C} \). First consider the three 2-secants \( XY, XZ, YZ \). If none of these is suitable, then each contains one of \( R, S \), \( \pi_\infty \). So next consider the three 2-secants \( XW, YW, ZW \). If none of these is suitable, then each contains one of \( R, S, \pi_\infty \). Without loss of generality suppose \( R \in XW, S \in YW \) and \( \pi_\infty \in ZW \). Hence we also have \( R \in YZ, S \in XZ \), and \( \pi_\infty \in XY \). Now consider the 2-secant \( VX \). If \( S \in VX \), then \( X, Z, V \) would be collinear, contradicting \( \alpha \cap \mathcal{C} \) being an arc. So \( S \notin VX \). Similarly \( R \notin VX \) and \( \pi_\infty \notin VX \). So \( m = VX \) is a 2-secant of \( \mathcal{C} \) not containing \( R, S \) or \( \pi_\infty \), and meeting \( \ell \) and \( \alpha \cap \pi \) in points not in \( \mathcal{C} \).

So there exists a line \( m \) in \( \alpha \) that contains two \( \mathcal{C} \)-points, but does not contain the points \( S, R \) or \( \pi_\infty \), and does not meet \( \ell \) or \( \alpha \cap \pi \) in a \( \mathcal{C} \)-point. Consider the plane \( \tau = (m, B) \). Now, \( \tau \) meets \( \beta \) in a line through \( B \) that is not a 1-secant of \( \mathcal{C} \) as \( \pi_\infty, S \notin \tau \). Hence \( \tau \) contains two points of \( \beta \cap \mathcal{C} \) (one of which is \( B \)). Similarly, \( \tau \) meets \( \pi \) in two points of \( \pi \cap \mathcal{C} \), one of which is \( B \). Hence \( \tau \) contains five distinct points of \( \mathcal{C} \), and so is a \( \mathcal{C} \)-plane by (A1). Hence \( m \) contains two \( \mathcal{C} \)-points which lie on two \( \mathcal{C} \)-planes, namely \( \alpha \) and \( \tau \), contradicting (A2). Hence \( \alpha \) and \( \beta \) cannot meet in a line \( \ell \), so they share no point other than \( \pi_\infty \). \( \square \)

The next lemma is key to our proof. We first show that for each \( \mathcal{C} \)-plane \( \pi \), the \( \infty \)-point \( \pi_\infty \) of the \( q \)-arc \( \pi \cap \mathcal{C} \) lies in \( \Sigma_\infty \). Then we study how \( \mathcal{C} \)-planes can meet, and investigate properties of the parallel classes of the affine plane \( \mathcal{A} \) defined in Lemma 3.1. We use the following definition.

Definition 3.4 If \( \pi \) is a \( \mathcal{C} \)-plane, then we call the line \( \pi \cap \Sigma_\infty \) a \( \mathcal{C} \)-line.

Lemma 3.5

1. If \( \pi \) is a \( \mathcal{C} \)-plane, then its \( \infty \)-point \( \pi_\infty \) lies in \( \Sigma_\infty \).

2. Let \( \pi, \alpha \) be distinct \( \mathcal{C} \)-planes.
(a) If $\pi, \alpha$ lie in the same parallel class, then they meet in exactly one point $\pi_\infty$ (which is equal to $\alpha_\infty$).

(b) If $\pi, \alpha$ lie in different parallel classes, then they either meet in exactly one point of $\mathcal{C}$, or they meet in an affine line through $\pi_\infty$ (which is equal to $\alpha_\infty$) that contains one point of $\mathcal{C}$.

3. The $\mathcal{C}$-planes which are in a common parallel class all share the same $\infty$-point, and they pairwise intersect in only this $\infty$-point. Moreover, every $\infty$-point defines exactly two parallel classes.

**Proof** Let $\pi$ be a fixed $\mathcal{C}$-plane and let $\pi_\mathcal{C} = \pi \cap \mathcal{C}$. We say a $\mathcal{C}$-plane $\alpha$ (distinct from $\pi$) *covers* an affine point $P$ of $\pi \setminus \pi_\mathcal{C}$ if it contains $P$. Further an affine line of $\pi$ is called a *cover line* of $\pi$ if it is contained in a $\mathcal{C}$-plane distinct from $\pi$.

We work with the parallel classes of the affine plane $\mathcal{A}$ defined in Lemma 3.1. The $q$ points of $\pi_\mathcal{C}$ each lie on $q$ further $\mathcal{C}$-planes (one in each of the parallel classes not containing $\pi$). The $q^2 - q$ affine points of $\pi \setminus \pi_\mathcal{C}$ each lie in exactly one further $\mathcal{C}$-plane by (A3). That is, each of the $q^2 - q$ affine points of $\pi \setminus \pi_\mathcal{C}$ is covered by exactly one $\mathcal{C}$-plane. We will investigate how the $\mathcal{C}$-planes cover these points. First we look at how another $\mathcal{C}$-plane $\alpha$ meets $\pi$. Now $\alpha \cap \pi$ is either a point or a line of $\text{PG}(4, q)$. If $\alpha, \pi$ are in different parallel classes, then $\alpha, \pi$ contain exactly one common $\mathcal{C}$-point, so $\alpha \cap \pi$ is either a $\mathcal{C}$-point, or an affine 1-secant of $\mathcal{C}$. If $\alpha, \pi$ are in the same parallel class, then $\alpha, \pi$ contain no common $\mathcal{C}$-point. Let

$$\mathcal{P} = \{\beta_1, \ldots, \beta_{q-1}\}$$

be the $\mathcal{C}$-planes in the same parallel class as $\pi$. Then we have:

I. (a) If $\alpha \in \mathcal{P}$, then $\alpha$ meets $\pi$ in either a point of $\pi \setminus \pi_\mathcal{C}$ (possibly in $\Sigma_\infty$); or an affine 0-secant of $\pi_\mathcal{C}$; or an infinite line.

(b) If $\alpha \notin \mathcal{P}$, then $\alpha$ meets $\pi$ in either exactly one $\mathcal{C}$-point, or in an affine 1-secant of $\pi_\mathcal{C}$.

We now focus on I(a) and show that no $\mathcal{C}$-plane in $\mathcal{P}$ meets $\pi$ in an affine line. Suppose there is one such plane $\beta \in \mathcal{P}$ meeting $\pi$ in the affine line $b$, so $b$ is a 0-secant of $\pi_\mathcal{C}$ (we work towards a contradiction to show that $b$ cannot exist). No other cover line in $\pi$ can meet $b$ in an affine point, otherwise that point would be on three $\mathcal{C}$-planes, contradicting (A3). Hence every other cover line of $\pi$ (if any) meets $b$ in the point $X = b \cap \Sigma_\infty$.

Let $\mathcal{L}$ be the set of $q$ affine lines of $\pi$ through $X$. There are four possibilities for these lines, depending on the location of the $\infty$-point $\pi_\infty$. Firstly, suppose $\pi_\infty \in \Sigma_\infty$. Note that $\pi_\infty$ lies on exactly one 0-secant of $\pi_\mathcal{C}$, which in this case is $\pi \cap \Sigma_\infty$. Hence $\pi_\infty \neq X$, as otherwise $X$ lies on two 0-secants of $\pi_\mathcal{C}$, namely $\pi \cap \Sigma_\infty$ and $b$. Thus the $q$ lines of $\mathcal{L}$ consist of one 1-secant, $\frac{q-1}{2}$ 2-secants, and $\frac{q-1}{2}$ 0-secants of $\pi_\mathcal{C}$. Secondly, if $\pi_\infty \notin \Sigma_\infty$ and $X \pi_\infty$ is a 0-secant of $\pi_\mathcal{C}$, then $\mathcal{L}$ contains one 1-secant, $\frac{q-1}{2}$ 2-secants and $\frac{q-1}{2}$ 0-secants of $\pi_\mathcal{C}$. Thirdly, if $\pi_\infty \notin \Sigma_\infty$ and $X \pi_\infty$ is a 1-secant of $\pi_\mathcal{C}$, then either $\mathcal{L}$ contains three
1-secants, $\frac{q-3}{2}$ 2-secants and $\frac{q-3}{2}$ 0-secants of $\pi_C$, or $L$ contains one 1-secant, $\frac{q-1}{2}$ 2-secants and $\frac{q-1}{2}$ 0-secants of $\pi_C$.

Let $L' \subset L$ be the set of 2-secants of $\pi_C$ through $X$. In each of the four cases, $L'$ contains either $\frac{q-3}{2}$ or $\frac{q-3}{2}$ lines. As $q \geq 7$, $L'$ contains at least two lines $\ell, m$. By I, the affine cover lines of $\pi$ are either 0-secants or 1-secants of $\pi_C$, so $\ell$ and $m$ are not cover lines.

Further, we noted above that all cover lines must contain $X$. Hence the affine points of $\ell, m$ not in $C$ are not contained in any cover lines. Thus by I, the affine points of $\ell, m$ not in $C$ must be covered by planes in $P \setminus \{\beta\}$, with each plane covering at most one affine point. However, there are $2(q-2)$ such affine points and only $q-2$ $\pi_C$-planes in $P \setminus \{\beta\}$, a contradiction. Hence the line $b$ does not exist, so no $\pi_C$-plane in $P$ meets $\pi$ in an affine line. Hence the following modification of I(a) holds:

I(a)' If $\alpha \in P$, then $\alpha$ meets $\pi$ in either a point of $\pi \setminus \pi_C$, or in an infinite line.

We now show that $\pi_\infty \in \Sigma_\infty$. By I(a)', $\pi_C$-planes in $P$ meet $\pi$ in either 0 or 1 affine points. Hence the $\pi_C$-planes in $P$ cover at most $q-1$ of the $q^2 - q$ affine points of $\pi \setminus \pi_C$. Recall that each of these affine points is covered by exactly one $\pi_C$-plane. By I(b), $\pi_C$-planes not in $P$ either meet $\pi$ in exactly a $\pi_C$-point or in a 1-secant of $\pi_C$. So there are at least 
\[(q^2 - q) - (q - 1) = q - 1\]
$\pi_C$-planes not in $P$ that meet $\pi$ in a cover line, denote them by
\[Q = \{\alpha_1, \ldots, \alpha_q\}.\]

For each $i$, $\alpha_i$ and $\pi$ have exactly one common $\pi_C$-point, denoted $A_i$. As the cover line $\alpha_i \cap \pi$ is a 1-secant of the $q$-arc $\pi_C$ in $\pi$, it is either the line $A_i \pi_{\infty}$, or it is the unique tangent line at $A_i$ to the conic $\pi_C \cup \pi_{\infty}$, denote this tangent line by $\ell_i$. Suppose one of the cover lines $\alpha_1 \cap \pi$ is the unique tangent line $\ell_1$ (we work towards a contradiction to show this cannot happen). Note that two cover lines cannot meet in an affine point of $\pi \setminus \pi_C$ by (A3). As the cover lines $\alpha_2 \cap \pi, \ldots, \alpha_q \cap \pi$ are all 1-secants, they either meet $\ell_1$ in $A_1$ (there is at most one other such 1-secant, namely $A_1 \pi_{\infty}$) or in a 1-secant through $\ell_1 \cap \Sigma_\infty$ (there are at most two other such 1-secants through $\ell_1 \cap \Sigma_\infty$). This gives at most three other possibilities for cover lines, contradicting $|Q \setminus \alpha_1| = (q - 1) - 1 \geq 5$. Thus no tangent line $\ell_i$ is a cover line, so $A_i \pi_{\infty}$ is a cover line for $i = 1, \ldots, q - 1$. That is, $\pi_{\infty}$ lies on $q$ $\pi_C$-planes, so by (A3), $\pi_{\infty}$ lies in $\pi \cap \Sigma_\infty$. This completes part 1.

We now further improve I(a)' and show that the $\pi_C$-planes in $P$ either meet $\pi$ in the infinite point $\pi_{\infty}$, or in an infinite line. First note that we have shown that the $q-1$ $\pi_C$-planes in $Q$ all contain $\pi_{\infty}$. Hence by Lemma 3.33 these planes pairwise meet only in $\pi_{\infty}$. In particular, they pairwise have no common $\pi_C$-point. Hence the planes in $Q$ all belong to the same parallel class, of which there is one more member, $\alpha_q$ say. Denote by $X$ the set of $q-1$ affine points of $\pi \setminus \pi_C$ that are not covered by the $\pi_C$-planes in $Q$. The points of $X$ lie on a line through $\pi_{\infty}$. We will show that $\alpha_q$ meets $\pi$ in this line.

Consider a plane $\alpha_1 \in Q$. A similar argument to that of part 1 shows that the $\pi_{\infty}$-point $(\alpha_1)_\infty \in \alpha_1 \cap \Sigma_\infty$. If $\alpha_1$ and $\pi$ share $\pi_C$-lines, that is $\alpha_1 \cap \Sigma_\infty = \pi \cap \Sigma_\infty$, then $\alpha_1, \pi$ must coincide since they share two lines, namely their $\pi_C$-line and an affine line. Hence
their $C$-lines are distinct, and a similar argument to that of part 1 shows that $\pi$ meets $\alpha_1$ in a line through $(\alpha_1)_\infty$, hence $(\alpha_1)_\infty = \pi_\infty$. Next, similar to the set $\Omega$ related to the $C$-plane $\pi$, consider the set $\mathcal{T}$ of at least $q - 1$ $C$-planes that each meet $\alpha_1$ in a line through $(\alpha_1)_\infty$. By the above argument, they lie in a common parallel class. As $\pi \in \mathcal{T}$, it follows that the parallel class containing $\mathcal{T}$ is the parallel class $\mathcal{P} \cup \pi$ (since every $C$-plane belongs to exactly one parallel class). Hence the (at least) $q - 2$ planes $\beta_i \in \mathcal{P} \cap \pi$ satisfy $(\beta_i)_\infty = (\alpha_1)_\infty = \pi_\infty$. Thus at least $q - 2$ of the $C$-planes in $\mathcal{P}$ meet $\pi$ exactly in the point $\pi_\infty$. In particular, at least $q - 2$ of the $\beta_i$ contain no affine points of $\pi$. Hence there are at least $q - 2$ affine points in the set $X$ that are not covered by planes in $\mathcal{P}$. Hence by I, there is a $C$-plane $\alpha$ not in $\mathcal{P}$ that meets $\pi$ in a 1-secant through $\pi_\infty$. This line covers $q - 1$ affine points, so covers all the points of $X$. Note that by Lemma 3.3, $\alpha$ belongs to the parallel class containing $\Omega$, so $\alpha = \alpha_q$. In summary, we have:

II. • $C$-planes in the parallel class $\mathcal{P} \cup \pi$ all contain $\pi_\infty$, and pairwise contain no common affine point;
    • Each $C$-plane in the parallel class $\mathcal{Q} \cup \alpha_q$ meets $\pi$ in a 1-secant through $\pi_\infty$;
    • the remaining $C$-planes each meet $\pi$ in exactly one $C$-point.

To complete the proof of part 2, we need to show that $C$-planes in $\mathcal{P}$ meet $\pi$ in exactly $\pi_\infty$. Suppose not, that is, by II, suppose that there is a $C$-plane $\beta_1$ in $\mathcal{P}$ that meets $\pi$ in an infinite line, so $\beta_1 \cap \pi = \pi \cap \Sigma_\infty$. Let $\alpha \in \mathcal{Q} \cup \alpha_q$, so $\alpha$ meets $\pi$ in a 1-secant through $\pi_\infty$. As $\pi, \beta_1$ are in the same parallel class, $\beta_1$ and $\alpha$ lie in different parallel classes, so have a common $C$-point, $A$ say. Then the line $A\pi_\infty$ lies in both $\beta$ and $\alpha$, contradicting Lemma 3.3. Hence $C$-planes in the parallel class $\mathcal{P} \cup \pi$ pairwise meet in exactly the point $\pi_\infty$.

Hence we have shown that $C$-planes in the same parallel class have the same $\infty$-point, and pairwise meet in exactly this point. To prove the second statement of part 3, we note that by II, the $C$-planes in the parallel classes $\mathcal{P} \cup \pi$ and $\mathcal{Q} \cup \alpha_q$ all contain the point $\pi_\infty$. Further, the remaining $C$-planes do not contain $\pi_\infty$. So $\pi_\infty$ lies on the $C$-planes of precisely two parallel classes. □

This lemma shows that two $C$-planes cannot meet in a $C$-line, so we have:

**Corollary 3.6** Each $C$-plane has a unique $C$-line, so there are $q^2 + q$ distinct $C$-lines in $\Sigma_\infty$.

We now characterise the points of $\Sigma_\infty$ in relation to the $C$-lines.

**Definition 3.7** A point of $\Sigma_\infty$ that lies on no $C$-line is called a 0-point.

**Lemma 3.8** There are three types of points in $\Sigma_\infty$: $\frac{q+1}{2}$ $\infty$-points, $\frac{q+1}{2}$ 0-points, and $q^3 + q^2$ points which lie on exactly one $C$-line each.
Proof Let \( \ell, m \) be two \( \mathcal{C} \)-lines in \( \Sigma_\infty \), and let \( \alpha, \beta \) be the unique \( \mathcal{C} \)-planes with \( \mathcal{C} \)-lines \( \ell, m \) respectively. By Lemma 3.5(2), \( \ell, m \) meet in exactly one point (namely \( \alpha_\infty = \beta_\infty \)), or not at all. So each point of \( \Sigma_\infty \) is either a 0-point, an \( \infty \)-point, or lies on exactly one \( \mathcal{C} \)-line. By Corollary 3.6, the number of points on exactly one \( \mathcal{C} \)-line is \( (q^3 + q^2)q = q^3 + q^2 \). By Lemma 3.5(3), each \( \infty \)-point lies in exactly two parallel classes, hence there are \( q + 1 \) \( \infty \)-points. Thus the number of 0-points is \( (q^3 + q^2 + q + 1) - (q + 1) - (q^3 + q^2) = \frac{q+1}{2} \). \( \square \)

As a direct consequence of this proof, we have the following corollary which tells us how \( \mathcal{C} \)-lines are positioned in \( \Sigma_\infty \).

Corollary 3.9 If two \( \mathcal{C} \)-lines meet, then they do so in an \( \infty \)-point. Further, each \( \infty \)-point lies on exactly \( 2q \) \( \mathcal{C} \)-lines.

3.2 Defining a spread in \( \Sigma_\infty \)

In this section we show how we can use the points of \( \mathcal{C} \) to construct a spread in \( \Sigma_\infty \). We show that each \( \mathcal{C} \)-point \( A \) defines a unique line \( t_A \) in \( \Sigma_\infty \). These resulting \( q^2 \) lines are mutually skew. The remaining \( q + 1 \) points in \( \Sigma_\infty \) are the 0-points and \( \infty \)-points, we show they lie on a line \( t_\infty \), and that the lines \( t_A \) and \( t_\infty \) form a spread in \( \Sigma_\infty \). We begin by defining for each \( \mathcal{C} \)-point \( A \), a set of \( q + 1 \) points \( t_A \).

Definition 3.10 Let \( A \) be \( \mathcal{C} \)-point and let \( \pi \) be a \( \mathcal{C} \)-plane through \( A \). The unique tangent to the conic \((\pi \cap \mathcal{C}) \cup \pi_\infty\) at the point \( A \) meets \( \Sigma_\infty \) in a point which we denote \( A_\pi \). Note that \( A_\pi \) is on the \( \mathcal{C} \)-line of \( \pi \). There are \( q + 1 \) \( \mathcal{C} \)-planes \( \pi_i \), \( i = 1, \ldots, q + 1 \), through \( A \). Define \( t_A \) to be the set of \( q + 1 \) points \( A_{\pi_i} \), \( i = 1, \ldots, q + 1 \).

We show in Theorem 3.14 that the set \( t_A \) is a line in \( \Sigma_\infty \). First we show that the \( \infty \)-points and the 0-points lie on a line of \( \Sigma_\infty \), denoted \( t_\infty \).

Lemma 3.11 The set of \( \infty \)-points and 0-points lie on a line \( t_\infty \).

Proof Let \( \ell \) denote the set of \( \infty \)-points and 0-points, so by Lemma 3.8, \( |\ell| = q + 1 \). Let \( \alpha \) be any plane of \( \Sigma_\infty \). If \( \alpha \) contains a \( \mathcal{C} \)-line, then it contains an \( \infty \)-point and so meets \( \ell \). If \( \alpha \) does not contain a \( \mathcal{C} \)-line, then \( \alpha \) contains at most one point of each of the \( q^2 + q \) \( \mathcal{C} \)-lines of \( \Sigma_\infty \). As \( \alpha \) has \( q^2 + q + 1 \) points, it follows that \( \alpha \) meets \( \ell \) in at least one point. Hence every plane of \( \Sigma_\infty \) meets \( \ell \), and so by \([9\) Theorem 3.5], \( \ell \) is a line. \( \square \)

The next two lemmas investigate the 3-space spanned by two \( \mathcal{C} \)-planes.

Lemma 3.12 A 3-space contains at most two \( \mathcal{C} \)-planes.
3.2 Defining a spread in $\Sigma_\infty$

**Proof** Let $\alpha$, $\beta$ be two $C$-planes that span a 3-space, so by Lemma 3.5(2), they are in different parallel classes, and $\alpha_\infty = \beta_\infty$. Thus, for a 3-space to contain three $C$-planes, they would have a common $\infty$-point, and the planes would be pairwise in different parallel class. This cannot occur as through any $\infty$-point there are at most two parallel classes by Lemma 3.5(3). \qed

**Lemma 3.13** Let $\Sigma$ be a 3-space containing two distinct $C$-planes $\alpha$ and $\beta$. Then the $C$-points in $\Sigma$ are exactly the $C$-points in $\alpha$ and $\beta$.

**Proof** If two $C$-planes $\alpha$ and $\beta$ generate a 3-space $\Sigma$, then they meet in a line $\ell$, which by Lemma 3.5(2) contains a unique $C$-point $A$. Further, $\alpha, \beta$ are in different parallel classes. Suppose $\Sigma$ contains a further $C$-point $B$ not in $\alpha$ or $\beta$. Of the $q + 1$ $C$-planes through $B$, one is parallel to $\alpha$, one is parallel to $\beta$ and one contains $A$. Let $\pi$ be any of the $q - 2 \geq 1$ remaining $C$-planes through $B$. Then $\pi$ meets $\alpha$ in a $C$-point $C$ and $\beta$ in a $C$-point $D$. As $B, C, D$ are $C$-points, they are not collinear, so $\pi = \langle B, C, D \rangle$. Hence $\pi, \alpha, \beta$ are three $C$-planes contained in a 3-space $\Sigma$, contradicting Lemma 3.12. \qed

We now show that the set of points $t_A$ defined in 3.10 form a line of $\Sigma_\infty$.

**Theorem 3.14** For each $C$-point $A$, the set $t_A$ is a line of $\Sigma_\infty$.

**Proof** By Lemma 3.8 there are $(q + 1)/2$ $\infty$-points, label these $T_1, \ldots, T_{q+1}/2$. Let $A$ be a $C$-point, each of the $q + 1$ $C$-planes through $A$ contains exactly one $\infty$-point. If three $C$-planes $\alpha, \beta, \gamma$ through $A$ contained the same $\infty$-point $T_1$ say, then by Lemma 3.5(2), they are all in different parallel classes. However, this contradicts Lemma 3.5(3) as $T_1$ defines exactly two parallel classes. Hence each of these $q + 1$ $\infty$-points lies in exactly two $C$-planes that contain $A$. Hence the $q + 1$ $C$-planes $\pi_1, \ldots, \pi_{q+1}$ through $A$ can be ordered so that the pair $(\pi_1, \pi_2)$ both contain $T_1$, the pair $(\pi_3, \pi_4)$ both contain $T_2$, and so on.

For each $C$-plane $\pi_i$, $i = 1, \ldots, q + 1$, through $A$, define the point $A_{\pi_i}$ as in Definition 3.10 and let $t_A = \{A_{\pi_1}, \ldots, A_{\pi_{q+1}}\}$. Now $\pi_1, \pi_2$ both contain the line $AT_1$, so $\Sigma = \langle \pi_1, \pi_2 \rangle$ is a 3-space. Consider the plane $\sigma_\infty = \Sigma \cap \Sigma_\infty$. By Lemma 3.12, $\pi_1$ and $\pi_2$ are the only $C$-planes contained in $\Sigma$, so the remaining $q^2 + q - 2$ $C$-planes each meet $\Sigma$ in a line. We now show that each of these $q^2 + q - 2$ lines meet the plane $\sigma_\infty$ in exactly one point. Suppose not, that is, suppose that some $C$-plane $\pi$ meets $\sigma_\infty$ in a line $\ell$, so $\ell$ is the $C$-line of $\pi$. Let $m, n$ be the $C$-lines of $\pi_1, \pi_2$ respectively. So the plane $\sigma_\infty$ contains three $C$-lines $\ell, m, n$ which pairwise meet. By Corollary 3.9 these $C$-lines contain a common $\infty$-point, namely $T_1$. As $\pi_1, \pi_2$ meet in a line, by Lemma 3.5(2), $\pi_1, \pi_2$ are in different parallel classes, and $\pi$ is in same parallel class as one of $\pi_1, \pi_2$. Suppose $\pi$ is in the same parallel class as $\pi_1$, then by Lemma 3.5 $\pi$ meets $\pi_2$ in a line. So as $\ell$ is also in $\sigma_\infty$, we conclude that $\pi$ lies in $\Sigma$, contradicting Lemma 3.12. Thus each of the $q^2 + q - 1$ $C$-planes distinct from $\pi_1, \pi_2$ contain exactly one point of $\sigma_\infty$.

In particular, as the $C$-planes $\pi_3, \ldots, \pi_{q+1}$ all contain $A \in \Sigma$, they meet $\Sigma$ in affine lines labelled $\ell_3, \ldots, \ell_{q+1}$ respectively, so each line $\ell_i$ meets $\Sigma_\infty$ in a point. As $\pi_3, \ldots, \pi_{q+1}$
all contain the point $A$, they contain no further $C$-points of $\pi_1$ and $\pi_2$. In particular, $\ell_i$, $i = 3, \ldots, q + 1$ does not contain any $C$-points that lie in $\pi_1$ or $\pi_2$ (other than $A$). By Lemma 3.13, the only $C$-points in $\Sigma$ are those in $\pi_1$ and $\pi_2$. Hence each $\ell_i$ contains exactly one $C$-point, namely $A$. Recall that $\pi_1$ and $\pi_2$ have a common $\infty$-point $T_1$, and that $\pi_3, \ldots, \pi_{q+1}$ do not contain $T_1$. Thus in $\pi_i$, $\ell_i$ contains exactly one $C$-point that lies in $\pi_i$, and hence meets $\sigma_\infty$ in the point $A_{\pi_i}$.

From this we conclude that if $\sigma_\infty$ is the plane defined by the $C$-lines of $\pi_1$ and $\pi_2$, then $\sigma_\infty$ contains all the points $A_{\pi_i}$ for $i = 1, \ldots, q + 1$. Next we repeat the argument with the pair $(\pi_3, \pi_4)$ and define $\sigma'_\infty$ to be the plane defined by the $C$-lines of $\pi_3$ and $\pi_4$. A similar argument shows that $\sigma'_\infty$ contains all the points $A_{\pi_i}$ for $i = 1, \ldots, q + 1$. As $\sigma_\infty$ and $\sigma'_\infty$ meet $t_\infty$ in distinct points, it follows that they meet in a line (disjoint from $t_\infty$), and this line $t_A$ contains the $q + 1$ points $A_{\pi_i}$ for $i = 1, \ldots, q + 1$. □

The next two corollaries follow from the definition of $t_A$, and this result that $t_A$ is a line.

**Corollary 3.15** Any $C$-plane containing the point $A \in C$ meets the line $t_A$, and conversely, any $C$-plane that meets $t_A$ contains the point $A$.

**Corollary 3.16** Let $A$ be a $C$-point, then the affine plane $\langle t_A, A \rangle$ contains exactly one $C$-point.

We now show that we can use these lines to construct a spread in $\Sigma_\infty$.

**Theorem 3.17** The lines $t_A$, $A \in C$, together with $t_\infty$ form a spread $S$ of $\Sigma_\infty$.

**Proof** Let $\pi$ be a $C$-plane with $C$-line $\ell = \pi \cap \Sigma_\infty$, and consider the conic $D = (C \cap \pi) \cup \pi_\infty$ in $\pi$. The line $\ell$ is a tangent to $D$, hence every point of $\ell \setminus \pi_\infty$ lies on one further tangent of $D$. That is, each point of $\ell \setminus \pi_\infty$ lies in exactly one line $t_A$. By Lemma 3.8, each point of $\Sigma_\infty \setminus t_\infty$ lies in exactly one $C$-line, hence each point of $\Sigma_\infty \setminus t_\infty$ lies in exactly one of the $t_A$. Further, the number of points in $\Sigma_\infty \setminus t_\infty$ equals the number of points on the $q^2$ lines $t_A$. Hence the lines $t_A$, $A \in C$ and $t_\infty$ are mutually disjoint, and form a spread of $\Sigma_\infty$. □

### 3.3 The spread $S$ is regular

In this section we will show that the spread $S = \{t_A : A \in C\} \cup \{t_\infty\}$ is regular. We begin with two lemmas showing how certain affine planes meet $C$.

**Lemma 3.18** Let $\ell$ be a line of $\Sigma_\infty$ that meets $t_\infty$, but is not a $C$-line. Then

1. Every affine plane containing $\ell$ meets $C$ in at most two points.
2. If \( \ell \) meets a spread line \( t_A \) (with corresponding \( C \)-point \( A \)), then the plane \( \langle A, \ell \rangle \) contains exactly one \( C \)-point, namely \( A \).

**Proof** Let \( \ell \) be a line of \( \Sigma_\infty \) that meets \( t_\infty \) in a point, but is not a \( C \)-line. Suppose \( \pi \) is an affine plane through \( \ell \) that contains three \( C \)-points \( P, Q, R \). By (A2), the points \( P, Q \) lie on a unique \( C \)-plane, denoted \( \pi_{PQ} \). Similarly we have \( C \)-planes \( \pi_{PR}, \pi_{QR} \). As \( \ell \) is not a \( C \)-line, \( \pi \) is not a \( C \)-plane, hence the three \( C \)-planes \( \pi_{PQ}, \pi_{PR}, \pi_{QR} \) are distinct. Consider the 3-space \( \Sigma = \langle t_\infty, \pi \rangle \). Now \( P, Q, \pi_{PQ} \cap t_\infty \) are three non-collinear points in \( \pi_{PQ} \) and in \( \Sigma \), hence \( \pi_{PQ} \subset \Sigma \). Similarly \( \pi_{PR}, \pi_{QR} \subset \Sigma \). This contradicts Lemma 3.12. Hence every affine plane through \( \ell \) meets \( C \) in at most two points, proving part 1.

The line \( \ell \) meets \( t_\infty \) and \( q \) other lines of the spread, let \( t_A \) be one such line, and let \( A \) be the \( C \)-point corresponding to \( t_A \). Consider the plane \( \langle A, \ell \rangle \), and suppose it contains a second point \( R \) of \( C \). By (A2), there is a unique \( C \)-plane \( \pi_{AR} \) through \( A, R \). The line \( AR \) meets \( \ell \) in a point \( X \). By Corollary 3.10, \( X \notin t_A \), and the line through \( A \) and \( \ell \cap t_\infty \) is a 1-secant of \( C \), so \( X \) lies on a unique spread line \( t_B \), \( B \neq A \). Hence the \( C \)-plane \( \pi_{AR} \) meets \( \ell \) in this point \( X \), that is, the \( C \)-line \( m = \pi_{AR} \cap \Sigma_\infty \) meets \( \ell \) in the point \( X \) of \( t_B \). Further \( \ell \neq m \) as \( m \) is a \( C \)-line but \( \ell \) is not, so \( \ell \) and \( m \) meet \( t_\infty \) in distinct points. Also note that as \( A \in \pi_{AR} \), we have by Corollary 3.15 that \( \pi_{AR} \) meets \( t_A \), that is, \( m \) meets \( t_A \). Hence \( \langle \ell, m \rangle \) is a plane that contains the spread lines \( t_\infty, t_A \), a contradiction as \( t_\infty, t_A \) are skew by Theorem 3.17. Hence we have shown that \( \langle A, \ell \rangle \) contains only one \( C \)-point, namely \( A \). \( \square \)

The next two lemmas examine planes that contain \( t_\infty \).

**Lemma 3.19** Each affine plane through \( t_\infty \) meets \( C \) in exactly one point.

**Proof** Suppose \( \alpha \) is a plane through \( t_\infty \) that meets \( C \) in two points \( A, B \). By (A2), there is a unique \( C \)-plane \( \pi_{AB} \) containing \( A \) and \( B \). Let \( \pi_{AB} \cap t_\infty = Y \), then \( Y \) is the \( \infty \)-point of \( \pi_{AB} \), and so \( Y \) can be added to \( \pi_{AB} \cap \Sigma \) to form a conic. This is a contradiction as the line \( AB \) meets this conic in three points, namely \( A, B, Y \). Hence each affine plane through \( t_\infty \) meets \( C \) in at most one point. As there are \( q^2 \) affine planes through \( t_\infty \) and \( q^2 \) points of \( C \), we have exactly one \( C \)-point in each affine plane about \( t_\infty \). \( \square \)

**Lemma 3.20** Every plane in \( \Sigma_\infty \) containing \( t_\infty \) contains \( q \) \( C \)-lines through a common \( \infty \)-point, and the corresponding \( q \) \( C \)-planes are in one parallel class.

**Proof** Consider the \( q + 1 \) planes of \( \Sigma_\infty \) through \( t_\infty \). By Corollary 3.19, each plane about \( t_\infty \) contains at most \( q \) \( C \)-lines through a common \( \infty \)-point. There are \( q^2 + q \) \( C \)-lines, hence each of the \( q + 1 \) planes about \( t_\infty \) contains exactly \( q \) \( C \)-lines through a common \( \infty \)-point.

Let \( \ell, m \) be two \( C \)-lines in the same plane of \( \Sigma_\infty \) about \( t_\infty \). Let \( \alpha, \beta \) be the two \( C \)-planes with \( C \)-lines \( \ell, m \) respectively. Suppose \( \alpha, \beta \) are in different parallel classes, then
by Lemma 3.5(2) they meet in a line containing a $C$-point $A$, and span a 3-space. Consider any other $C$-line $n$ in the plane $(t_\infty, \ell, m)$. Now if $\delta$ is the $C$-plane with $C$-line $n$, then $\delta$ cannot be parallel to both $\alpha$ and $\beta$ as $\alpha$ and $\beta$ are not in the same parallel class. So without loss of generality say $\delta$ meets $\alpha$ in the $C$-point $B$. Consider the 3-space $(\alpha, \beta)$. It contains $\ell$ and $m$, and so contains $n$. Further, it contains $B$, so $\delta \subset (\alpha, \beta)$. This contradicts Lemma 3.12. Hence $\alpha$ and $\beta$ are not in different parallel classes, that is, the $C$-lines in a plane of $\Sigma_\infty$ through $t_\infty$ pass through a common $\infty$-point, and are contained in $C$-planes that belong to a common parallel class.  

We now consider a line of $\Sigma_\infty$ meeting $t_\infty$ and $q$ other spread lines, and show that the corresponding $C$-points lie in a common $C$-plane.

**Lemma 3.21** If $\ell$ is a line of $\Sigma_\infty$ meeting spread lines $t_\infty, t_{A_1}, \ldots, t_{A_q}$, then the corresponding $C$-points $A_1, \ldots, A_q$ lie on a common $C$-plane.

**Proof** Let $\ell$ be a line meeting spread lines $t_\infty, t_{A_1}, \ldots, t_{A_q}$. Let $V = \ell \cap t_\infty$. Let $\pi$ be a $C$-plane not through $V$, let $A$ be a $C$-point not in $\pi$ and let $\Sigma = (A, \pi)$. We first show that $V \notin \Sigma$. By Lemma 3.19, the plane $(t_\infty, A)$ contains exactly $C$-point, namely $A$. As each affine line of $\pi$ through $\pi_\infty = \pi \cap t_\infty$ meets $C$ in a point, $(t_\infty, A)$ meets the plane $\pi$ in exactly the point $\pi_\infty$. Hence $(t_\infty, A)$ is not a plane of $\Sigma$, and so $t_\infty$ is not contained in $\Sigma$, thus $V \notin \Sigma$. Hence we can project the points of $\Sigma$ from $V$ onto $\Sigma$ to obtain a set $C'$. By Lemma 3.19 no line through $V$ can contain 2 points of $C$, hence $C'$ contains $q^2$ distinct points. We consider this projection in the two cases: when $V$ is a 0-point, and when $V$ is an $\infty$-point.

**Case 1:** Suppose $V$ is a 0-point, we will show that the set $\mathcal{E} = C' \cup \{\pi_\infty\}$ is an elliptic quadric in $\Sigma$. Further, each $C$-plane projects to a unique affine plane of $\Sigma$ through $\pi_\infty$, and conversely, each affine plane of $\Sigma$ through $\pi_\infty$ is the image of a unique $C$-plane.

We first show that $C' \cup \{\pi_\infty\}$ is a cap of $\Sigma$. By Lemma 3.18 a plane meeting $t_\infty$ in the 0-point $V$ meets $C$ in at most 2 points. Hence no three points of $C'$ are collinear (otherwise their preimages would be coplanar with $V$). Further, if the point $\pi_\infty$ were collinear with two points $B', C'$ of $C'$, then their preimages $B, C \in C$ would lie in a plane about $t_\infty$, which is not possible by Lemma 3.19. So $\mathcal{E} = C' \cup \{\pi_\infty\}$ is a set of $q^2 + 1$ points, no three collinear in a 3-space of order $q$, $q$ odd. Hence $\mathcal{E}$ is an elliptic quadric, see [2] or [10]. Now let $\alpha$ be a $C$-plane. As $V$ is a 0-point, $\alpha$ meets $t_\infty$ in a point distinct from $V$, and so $\alpha$ projects to an affine plane $\alpha'$ through $\pi_\infty$. Let $\alpha, \beta$ be two distinct $C$-planes with images $\alpha', \beta'$ respectively. If $\alpha' = \beta'$, then as $\alpha, \beta$ together contain at least $2q - 1$ $C$-points, the plane $\alpha'$ contains $2q - 1$ points of the elliptic quadric $\mathcal{E}$, a contradiction. As there are $q^2 + q$ $C$-planes and $q^2 + q$ affine planes in $\Sigma$ through $\pi_\infty$ (that is, planes not contained in $\Sigma_\infty$), each $C$-plane projects to a unique affine plane of $\Sigma$ through $\pi_\infty$, and conversely. This completes the proof of the statement at Case 1.

Recall that $\ell$ meets the spread lines $t_{A_1}, \ldots, t_{A_q}$, we now consider the corresponding $C$-points $A_i$, and their images $A'_i$ under the projection from $V$. By Lemma 3.18 the plane
\[\pi_i = \langle \ell, A_i \rangle\] contains exactly one \(C\)-point, namely \(A_i\). Let \(L = \ell \cap \Sigma\), so \(L \in \Sigma_{\infty}\) as \(\ell \subseteq \Sigma_{\infty}\). The plane \(\pi_i = \langle \ell, A_i \rangle\) projects to the line \(LA_i\) which contains exactly one point of \(C\), namely \(A_i\), hence these lines are distinct for distinct \(i\). As \(\Sigma_{\infty}\) contains no \(C\)-points, the projection \(\Sigma \cap \Sigma_{\infty}\) of \(\Sigma_{\infty}\) from \(V\) onto \(\Sigma\) is a tangent plane to \(E\), so \(L\pi_{\infty}\) is a tangent line to \(E\). Thus from a point \(L \in \Sigma\) we have a set \(\{L\pi_{\infty}, LA'_1, \ldots, LA'_q\}\) of \(q + 1\) tangent lines of \(E\). As \(E\) is an elliptic quadric, the points \(\pi_{\infty}, A'_1, \ldots, A'_q\) all lie on a plane \(\beta'\), namely the polar plane of \(L\), see \[8, Theorem 15.3.10\]. Hence by the above argument, \(\beta'\) is the image of a \(C\)-plane \(\beta\), and hence the points \(A_i\) lie on a common \(C\)-plane, namely \(\beta\). So the lemma holds in the case \(V = \ell \cap t_{\infty}\) is a 0-point.

Case 2: Suppose \(V\) is an \(\infty\)-point. We will show that the set \(C'\) can be completed to a hyperbolic quadric \(H\) with the addition of two lines through \(\pi_{\infty}\) in \(\Sigma \cap \Sigma_{\infty}\). Further, the \(C\)-planes are of two types: the \(C\)-planes through \(V\) project to the generator lines of \(H\) not through \(\pi_{\infty}\); and the \(C\)-planes not through \(V\) project to planes through \(\pi_{\infty}\) which meet \(H\) in a conic.

A \(C\)-plane through \(V\) is projected onto a line of \(\Sigma\) not through \(\pi_{\infty}\). By Lemma \[3.5, (2)\], there are two parallel classes \(P_1, P_2\) of \(C\)-planes through \(V\), and \(C\)-planes in the same parallel class pairwise meet in exactly \(V\). Hence one parallel class \(P_1\) through \(V\) is mapped to a set \(T_1\) of \(q\) mutually skew lines (each line containing \(q\) points of \(C'\)), and the other parallel class \(P_2\) through \(V\) is mapped to a set \(T_2\) of \(q\) mutually skew lines (each line containing \(q\) points of \(C'\)), with each line from \(T_1\) meeting every line from \(T_2\) in a point of \(C'\). We complete these line sets into a regulus and its opposite regulus as follows. Consider the parallel class \(P_1\) of \(C\)-planes through \(V\). By Lemma \[3.20\] the \(C\)-lines of these \(C\)-planes lie in a common plane of \(\Sigma_{\infty}\) through \(t_{\infty}\). Hence they all meet \(\Sigma \cap \Sigma_{\infty}\) in collinear points on a line \(\ell_1\) through \(\pi_{\infty}\). The line \(\ell_1\) meets every \(C\)-plane in \(P_1\), and so meets every line in \(T_1\). Similarly we have a line \(\ell_2\) through \(\pi_{\infty}\) corresponding to the parallel class \(P_2\), and \(\ell_2\) meets every line in \(T_2\). Thus the lines \(T_1 \cup \ell_2\) form a regulus with opposite regulus \(T_2 \cup \ell_1\). Hence \(C' \cup \ell_1 \cup \ell_2\) is a hyperbolic quadric in \(\Sigma\). So we have shown that the \(2q\) \(C\)-planes through \(V\) project to \(2q\) lines of the hyperbolic quadric \(H\). The remaining \(q^2 - q\) \(C\)-planes not through \(V\) project to planes through \(\pi_{\infty}\) that contain \(q\) points of \(C'\), hence meet \(H\) in a conic. Note that the remaining \(2q + 1\) planes through \(\pi_{\infty}\) meet \(H\) in two lines of \(H\), with one of the lines necessarily a line through \(\pi_{\infty}\). This proves the statement for Case 2.

Recall that \(\ell\) meets the spread lines \(t_{A_1}, \ldots, t_{A_q}\), we now consider the corresponding \(C\)-points \(A_i\), and their images \(A'_i\) under the projection from \(V\). If \(\ell\) is a \(C\)-line, then the \(C\)-points \(A_1, \ldots, A_q\) lie on a common \(C\)-plane by Corollary \[3.13\]. So suppose \(\ell\) is not a \(C\)-line. By Lemma \[3.18\] the plane \(\pi_i = \langle \ell, A_i \rangle\) contains exactly one \(C\)-point, namely \(A_i\). So \(\pi_i\) is mapped to a line through the points \(A'_i\) and \(L = \ell \cap (\Sigma \cap \Sigma_{\infty})\). Note that \(LA'_i\) meets \(C\) in exactly one point, namely \(A'_i\). The lines \(LA'_1, \ldots, LA'_q\) are distinct tangent lines to \(H\). Further, \(L\pi_{\infty}\) is a tangent line to \(H\). Thus from a point \(L \notin H\) we have a set of \(q + 1\) tangent lines of \(H\). Hence the points \(\pi_{\infty}, A'_1, \ldots, A'_q\) all lie on a plane through \(\pi_{\infty}\) which meets \(H\) in a conic, namely the polar plane of \(L\), see \[8, Theorem 15.3.16\]. Hence by the above argument, this plane is the image of a \(C\)-plane, and hence the points \(A_i\) lie on a common \(C\)-plane. That is, the lemma also holds in the case when \(V\) is an \(\infty\)-point. \(\Box\)
We now show that any regulus containing \( t_\infty \) and two other lines of the spread \( S \) is contained in \( S \). Next we will use the Klein quadric to show that a spread with this property is regular.

**Lemma 3.22** Let \( t_A, t_B \) be two elements of the spread \( S \). Then the unique regulus determined by the three lines \( t_\infty, t_A, t_B \) is contained in \( S \).

**Proof** Let \( t_A, t_B \) be two elements of the spread \( S \). Through each point \( V_i \in t_\infty, i = 0, \ldots, q \), there is a unique line \( L_i \) that meets both \( t_A \) and \( t_B \). Further, the lines \( L_i \) form a regulus \( R' \). The opposite regulus \( R \) is the unique regulus containing \( t_\infty, t_A, t_B \). We want to show that \( R \subset S \).

Consider the \( C \)-points \( A, B \) corresponding to \( t_A, t_B \) respectively. By (A2), they lie in a unique \( C \)-plane \( \pi \). Now \( \pi \) meets \( \Sigma_\infty \) in a \( C \)-line \( \ell \), and \( \ell \) meets \( t_\infty \) by Lemmas 3.5(1) and 3.11. Further \( \ell \) meets \( t_A \) and \( t_B \) by Corollary 3.15. Hence \( \ell \) is one of the lines \( L_i \in R' \). Now \( \ell \) meets \( q + 1 \) lines of the spread \( S \), denote them \( t_\infty, t_A, t_B, t_{C_3}, \ldots, t_{C_q} \). We want to show that these are the lines of \( R \). Note that the corresponding \( C \)-points \( A, B, C_3, \ldots, C_q \) lie on the \( C \)-plane \( \pi \) by Corollary 3.15.

Now consider the line \( L_j \in R', \ell_j \neq \ell \), it meets \( q + 1 \) lines of the spread \( S \), denote these by \( t_\infty, t_A, t_B, t_{D_3}, \ldots, t_{D_q} \). By Lemma 3.21, the corresponding \( C \)-points \( A, B, D_3, \ldots, D_q \) lie on a common \( C \)-plane, \( \alpha \) say. As there is a unique \( C \)-plane containing \( A, B \), we have \( \alpha = \pi \), and so \( \{C_3, \ldots, C_q\} = \{D_3, \ldots, D_q\} \). Hence each line \( L_i \) in the regulus \( R' \) meets the spread lines \( t_\infty, t_A, t_B, t_{C_3}, \ldots, t_{C_q} \), and so \( R = \{t_\infty, t_A, t_B, t_{C_3}, \ldots, t_{C_q}\} \), and so \( R \subset S \). That is, the unique regulus containing \( t_\infty, t_A, t_B \) is contained in \( S \).

To prove that \( S \) is a regular spread we now show that a spread satisfying the conditions of Lemma 3.22 is regular. We will use the Klein correspondence from the set of lines in \( \text{PG}(3,q) \) to the set of points of the Klein quadric \( \mathcal{H}_5 \), a hyperbolic quadric in \( \text{PG}(5,q) \). For details of this correspondence, see [8, Section 15.4]. We note that a regular spread of \( \text{PG}(3,q) \) corresponds under the Klein correspondence to a 3-dimensional elliptic quadric contained in \( \mathcal{H}_5 \), that is, the intersection of \( \mathcal{H}_5 \) and a 3-space that forms an elliptic quadric. Further, a regulus of \( \text{PG}(3,q) \) corresponds to a conic in \( \mathcal{H}_5 \).

**Theorem 3.23** Let \( S \) be a spread of \( \text{PG}(3,q) \) with a special line \( t_\infty \) with the property that the regulus containing \( t_\infty \) and any further two lines of \( S \) is contained in \( S \). Then \( S \) is a regular spread.

**Proof** We begin with some notation. Let \( S' \) be the points on the Klein quadric \( \mathcal{H}_5 \) corresponding to the lines of the spread \( S \). For each line \( t_A \) of \( S \), let \( t'_A \) denote the corresponding point of \( \mathcal{H}_5 \). Similarly, if \( R \) is a regulus of \( \text{PG}(3,q) \), let \( R' \) denote the set of points of \( \mathcal{H}_5 \) corresponding to the lines of \( R \), and note that the points in \( R' \) lie on a conic. For three mutually skew lines \( \ell, m, n \) in \( \text{PG}(3,q) \), let \( R(\ell, m, n) \) denote the unique regulus containing them.
3.3 The spread \( S \) is regular

We first show that all the points of \( S' \) are contained in a common 3-space. We will repeatedly use the assumption that the regulus determined by \( t_\infty \) and two other lines of \( S \) is contained in \( S \).

Fix a spread element \( t_A \neq t_\infty \) and consider any two distinct reguli \( R_1, R_2 \) of \( S \) containing \( t_\infty \) and \( t_A \). Suppose that

\[
R_1 = R(t_\infty, t_A, t_B_1) = \{t_\infty, t_A, t_B_1, \ldots, t_{B_q-1}\}
\]

\[
R_2 = R(t_\infty, t_A, t_C_1) = \{t_\infty, t_A, t_{C_1}, \ldots, t_{C_q-1}\}.
\]

Two distinct reguli have at most two common lines, so \( R_1 \) and \( R_2 \) intersect in exactly \( t_\infty, t_A \). Hence in the Klein quadric, we have two conics \( R'_1, R'_2 \), they lie in two distinct planes which meet in the line \( t'_\infty t'_A \) and hence span a 3-space denoted \( \Sigma \). Now consider the regulus \( T \) of \( S \) determined by \( t_\infty, t_{B_1}, t_{C_1} \):

\[
T = R(t_\infty, t_{B_1}, t_{C_1}) = \{t_\infty, t_{B_1}, t_{C_1}, t_{D_3}, \ldots, t_{D_q}\}.
\]

Note that the lines \( t_{B_1}, t_{C_1}, t_{D_k} \) are all distinct. In \( PG(5, q) \), \( T' \) is a conic that contains three points \( t'_\infty, t'_{B_1}, t'_{C_1} \) of \( \Sigma \), hence \( T' \subset \Sigma \).

We now use the lines \( t_{D_3}, \ldots, t_{D_q} \) of \( T \) to construct \( q - 3 + 1 \) more reguli of \( S \) through \( t_\infty, t_A \):

\[
R_i = R(t_\infty, t_A, t_{D_i}), \quad i = 3, \ldots, q.
\]

We have a set \( \{R_1, \ldots, R_q\} \) of \( q \) reguli of \( S \) that pairwise intersect in exactly the lines \( t_\infty, t_A \), so they cover \( 2 + q(q - 1) = q^2 - q + 2 \) elements of \( S \). The remaining \( q - 1 \) spread elements \( t_{E_1}, \ldots, t_{E_{q-1}} \) of \( S \) lie on a common regulus \( U \) through \( t_\infty, t_A \) (as every three elements determine a unique regulus), that is,

\[
U = \{t_\infty, t_A, t_{E_1}, \ldots, t_{E_{q-1}}\}.
\]

Now each reguli \( R_i, i = 3, \ldots, q, \) is mapped to a conic \( R'_i \) of \( H_5 \). Further each conic \( R'_i, \)

\( i = 3, \ldots, q \) contains three points \( t'_\infty, t'_{A}, t'_{D_i} \) of \( \Sigma \). Hence \( R'_i \subset \Sigma, i = 1, \ldots, q \). To show that \( S' \subset \Sigma \), it remains to show that \( t'_{E_1}, \ldots, t'_{E_{q-1}} \in \Sigma \).

Now consider the two reguli \( R_1, U \) of \( S \). They map to two conics \( R'_1, U' \) of \( H_5 \) that span a 3-space denoted by \( \Sigma' \). Consider another regulus of \( S \):

\[
V = R(t_\infty, t_{B_1}, t_{E_1}) = \{t_\infty, t_{B_1}, t_{E_1}, t_{F_3}, \ldots, t_{F_q}\}.
\]

Then \( V' \) is a conic of \( H_5 \) with three points \( t'_\infty, t'_{B_1}, t'_{E_1} \) in \( \Sigma' \), and so \( V' \) is contained in \( \Sigma' \). As \( U, V \) meet exactly in \( t_\infty, t_{E_1} \), the lines \( t_{F_i}, i = 3, \ldots, q \) are distinct from the lines \( t_{E_i}, i = 1, \ldots, q-1 \) and so belong to the \( q^2 - q + 2 \) elements of \( S \) in \( \Sigma \). Hence we have \( t_{F_i} \subset \Sigma \) and so \( t'_{F_i} \in \Sigma' \cap \Sigma, i = 3, \ldots, q \). Thus \( \Sigma \cap \Sigma' \) contains \( R'_1 \) and \( V' \) which is more than a plane, and so \( \Sigma = \Sigma' \). That is, \( t'_{E_1}, \ldots, t'_{E_{q-1}} \in \Sigma \). Thus the lines of \( S \) are mapped into points of \( H_5 \) that lie in a 3-space \( \Sigma \).
The intersection $Q$ of the 3-space $\Sigma$ with $H_5$ is a quadric of $\Sigma$, hence is either an elliptic, hyperbolic or a degenerate quadric. Note that in all cases other than the elliptic quadric, $Q$ does not correspond to a spread (or a set containing a spread) of $\text{PG}(3,q)$. Hence $Q$ is an elliptic quadric, and so $S$ is a regular spread. 

As an immediate consequence of Lemma 3.22 and Theorem 3.23, we have that the spread $S$ constructed in Theorem 3.17 is regular.

**Corollary 3.24** The spread $S = \{t_A : A \in C\} \cup \{t_\infty\}$ is regular.

### 3.4 $C$ gives rise to an arc in $\text{PG}(2,q^2)$

By Corollary 3.24, we have a regular spread $S = \{t_A : A \in C\} \cup \{t_\infty\}$ in $\Sigma_\infty$ from which we can construct a Desarguesian plane $P(S) \cong \text{PG}(2,q^2)$ via the Bruck-Bose correspondence. Let $O$ be the set of points in $P(S)$ corresponding to the affine points of $C$ together with the point $T_\infty$ on $\ell_\infty$ corresponding to the spread line $t_\infty$.

**Theorem 3.25** $O$ is a conic in $P(S) \cong \text{PG}(2,q^2)$.

**Proof** We show that $\emptyset = C \cup T_\infty$ is a $(q^2+1)$-arc in $\text{PG}(2,q^2)$, and hence a conic. A line through $T_\infty$ in $\text{PG}(2,q^2)$ corresponds to an affine plane of $\text{PG}(4,q)$ that contains the spread line $t_\infty$. By Lemma 3.19, the affine planes of $\text{PG}(4,q)$ through $t_\infty$ meet $C$ in exactly one point. Hence in $\text{PG}(2,q^2)$, a line through $T_\infty$ meets $O$ in at exactly one further point.

Let $t_A$ be a line of the spread $S$ in $\text{PG}(4,q)$ with corresponding $C$-point $A$. Let $\alpha$ be an affine plane of $\text{PG}(4,q)$ through the spread line $t_A$ that contains three $C$-points $P, Q, R$. We obtain a contradiction to show that this is not possible. Note that $\alpha \neq \langle A, t_A \rangle$ since by Corollary 3.19, the plane $\langle A, t_A \rangle$ contains exactly one $C$-point. By (A2), $P, Q$ lie in a unique $C$-plane $\pi_{PQ}$ which meets $t_A$ in the point $PQ \cap t_A$. Hence by Corollary 3.15, $\pi_{PQ}$ contains the point $A$. Similarly, $P, R$ lie on a unique $C$-plane $\pi_{PR}$ that contains $A$. Hence we have two distinct $C$-planes $\pi_{PQ}, \pi_{PR}$ that both contain the two distinct points $A, P$ of $C$, contradicting (A2). Hence any affine plane that contains a spread line $t_A$ contains at most two points of $C$. Thus in the Bruck-Bose plane $\text{PG}(2,q^2)$, a line through a point of $\ell_\infty \setminus T_\infty$ meets $O$ in at most two points. Further, $\ell_\infty$ meets $O$ in one point, so we have shown that $O$ is a $(q^2+1)$-arc in $\text{PG}(2,q^2)$. As $q$ is odd, by Segre [12], $O$ is a conic in $\text{PG}(2,q^2)$.

This almost completes the proof of Theorem 1.2. It remains to show that the spread $S$ is unique, which we do in the next section.
3.5 The spread $S$ is unique

We now show that the spread $S = \{t_A : A \in C\} \cup \{t_{\infty}\}$ constructed in Theorem 3.17 is the only spread in $\Sigma_{\infty}$ for which the $C$-points give rise to an arc in the Bruck-Bose plane $P(S)$.

**Theorem 3.26** Let $S'$ be a spread of $\Sigma_{\infty}$ distinct from the spread $S = \{t_A : A \in C\} \cup \{t_{\infty}\}$. Then in the associated Bruck-Bose plane $P(S')$, the set of points corresponding to $C$ do not form an arc.

**Proof** Let $\ell$ be a line in $\Sigma_{\infty}$ disjoint from $t_{\infty}$ and assume that all the affine planes through $\ell$ meet $C$ in at most two points. We will show that $\ell$ must be one of the spread lines $t_A$ for some point $A \in C$. Let $X_1$ be a point on $\ell$, by Lemma 3.8 and 3.11 there exists a unique $C$-line $\ell_1$ through $X_1$. By Corollary 3.3, $\ell_1$ is contained in a unique $C$-plane $\pi_1$. Let $E_1$ be the conic in $\pi_1$, that is, $E_1 = (C \cap \pi_1) \cup (\pi_1 \cap t_{\infty})$. Now $X_1$ is on one tangent to $E_1$, namely $\ell_1$, hence $X_1$ is on a second tangent $m_1$ to $E_1$. There are $(q-1)/2$ 2-secants to $E_1$ through $X_1$, each together with the line $\ell$ determines an affine plane, we label these planes $\alpha_1, \ldots, \alpha_{(q-1)/2}$. As each plane contains a 2-secant to $E_1$ through $X_1$, each contains at least two $C$-points, and hence by our assumption about $\ell$, contains exactly two $C$-points.

Similarly, let $X_2$ be a point on $\ell$, lying on a unique $C$-line $\ell_2$, defining the unique $C$-plane $\pi_2$, and construct planes $\beta_1, \ldots, \beta_{(q-1)/2}$ which contain $\ell$ and exactly two $C$-points. Suppose $\alpha = \alpha_i = \beta_j$ for some $i, j$. Then $\alpha$ contains at least three $C$-points (as $\alpha_i$ contains a 2-secant of $C$ through $X_1$, $\beta_j$ contains a 2-secant of $C$ through $X_2$, and $X_1 \neq X_2$). This contradicts our assumption that planes about $\ell$ contain at most two $C$-points. Hence the planes $\alpha_1, \ldots, \alpha_{(q-1)/2}, \beta_1, \ldots, \beta_{(q-1)/2}$ are distinct, and cover $(q-1) \times 2$ distinct $C$-points.

Repeating this for the remaining points $X_3, \ldots, X_{q+1}$ of $\ell$, we obtain a set $K$ of distinct planes about $\ell$ (including the $\alpha_i, \beta_j$) of size $(q+1) \times (q-1)/2$. Each plane in $K$ contains exactly two $C$-points, accounting for $q^2 - 1$ points of $C$ that lie on planes through $\ell$.

Now for each $i$, consider the unique affine tangent line $m_i$ through $X_i$ (distinct from $\ell_i$) to the conic $E_i$, in the plane $\pi_i$. We have a set of $q+1$ planes $\{\ell, m_i\}$, not necessarily distinct, but distinct from the planes of $K$, each meeting $C$ in at least one point. However, there is only one $C$-point unaccounted for by the planes of $K$, so the planes $\{\ell, m_i\}$ are all the same plane $\alpha$ which contains exactly one point $Z$ of $C$. The $q + 1$ lines $ZX_1, \ldots, ZX_{q+1}$ through $Z$ in $\alpha$ are respectively the tangents $m_1, \ldots, m_{q+1}$ to the conic $E_1, \ldots, E_{q+1}$ in $\pi_1, \ldots, \pi_{q+1}$ at $Z$. Further, the tangents $m_1, \ldots, m_{q+1}$ meet $\Sigma_{\infty}$ at points $X_1, \ldots, X_{q+1}$ of $\ell$, so by Definition 3.10 $\ell = t_Z$. That is, $\ell$ is one of the spread lines of $S$. We conclude that every line in $\Sigma_{\infty}$ which is disjoint from $t_{\infty}$ and not in $S$ lies on an affine plane that contains at least three $C$-points.

Suppose $S'$ is a spread of $\Sigma_{\infty}$ distinct from $S$ such that in the Bruck-Bose plane $P(S')$ the $C$-points form an arc. Let $S' \setminus S$ denote the set of lines in $S'$ that are not in $S$. If $S' \setminus S$ contains a line $\ell$ disjoint from $t_{\infty}$, then by the above argument, $\ell$ lies on some affine plane
α that contains at least three C-points. In the Bruck-Bose plane \( \mathcal{P}(S') \), \( \alpha \) corresponds to a line that contains three C-points, so \( C \) is not an arc, contradicting our assumption. Hence \( S' \setminus S \) cannot contain a line \( \ell \) disjoint from \( t_\infty \), thus the lines in \( S' \setminus S \) must meet \( t_\infty \). So if \( S' \neq S \), then \( S' \setminus S \) contains \( q+1 \) lines that meet \( t_\infty \) in a point. Let \( \ell \in S' \setminus S \), so \( \ell \) meets \( t_\infty \) and another \( q \) spread lines \( t_{A_1}, \ldots, t_{A_q} \) of \( S \). Let \( \mathcal{R} = \{ t_\infty, t_{A_1}, \ldots, t_{A_q} \} \), then the lines of \( S' \setminus S \) cover the same points as the lines of \( \mathcal{R} \) do. Let \( S' \setminus S = \{ \ell, \ell_1, \ldots, \ell_q \} \), then each line \( \ell_i \) meets each line in \( \mathcal{R} \).

Let \( t_A, t_B \) be any two lines of \( \mathcal{R} \) distinct from \( t_\infty \). There are exactly \( q + 1 \) lines meeting \( t_\infty, t_A, t_B \), and since \( \ell, \ell_1, \ldots, \ell_q \) meet each of \( t_\infty, t_A, t_B \), they form a regulus \( \mathcal{R}' \). A similar argument with three elements of \( \mathcal{R}' \) shows that \( \mathcal{R} \) is the opposite regulus of \( \mathcal{R}' \). By Corollary 3.15 there exists a unique C-line \( m \) meeting \( t_\infty, t_A, t_B \), namely the C-line corresponding to the unique C-plane \( \pi_{AB} \) through \( A \) and \( B \). The line \( m \) meets three lines of \( \mathcal{R} \) and hence it is a line of \( \mathcal{R}' \). That is, \( m \) is a line of \( S' \). However, the C-plane \( \pi_{AB} \) through \( m \) corresponds to a line of \( \mathcal{P}(S') \) that contains \( q \) points of \( C \). Hence \( C \) is not an arc of \( \mathcal{P}(S') \). Thus the only spread which gives rise to an arc in the corresponding Bruck-Bose plane is the regular spread \( S \) constructed in Theorem 3.17. □

This completes the proof of Theorem 1.2.

### 4 Conclusion

In this paper we characterised sets in \( \text{PG}(4, q) \), \( q \) odd, \( q \geq 7 \) satisfying the combinatorial properties given in Theorem 1.2 as corresponding via the Bruck-Bose correspondence to conics in \( \text{PG}(2, q^2) \). We note that a similar characterisation when \( q \) is even is given in [4]. The cases when \( q = 3 \) or \( 5 \) are still open.

An interesting geometric question arises from the properties of a conic given in Lemma 1.1. Let \( C \) be a conic in \( \text{PG}(2, q^2) \), \( q \) odd, tangent to \( \ell_\infty \), and let \( \pi \) be a C-plane. By property 3 of Lemma 1.1 the points of \( \pi \) that are not in \( C \) lie on exactly one more C-plane. Let \( P \) be a point of \( \pi \setminus C \). If \( P \) is an interior point of the subconic \( \pi_C = \pi \cap C \), then we can use the polarity of \( \pi_C \) to construct the second C-plane containing \( P \). If \( P \) is an exterior point of \( \pi_C \), then it would be interesting to have a geometric construction of the second C-plane containing \( P \).

### References

[1] J. André. Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe, Math. Z. 60 (1954), 156–186.

[2] A. Barlotti. Un’estensione del teorema di Segre-Kustaaheim. Boll. U.M.I., 10 (1955) 498-506.
[3] S.G. Barwick and G.L. Ebert. Unitals in projective planes. Springer Monographs in Mathematics. Springer, New York, 2008.

[4] S.G. Barwick and W.A. Jackson. A characterisation of translation ovals in finite even order planes. http://arxiv.org/abs/1305.6673

[5] S.G. Barwick, W.A. Jackson and C.T. Quinn. Conics and caps. J. Geom. 100 (2011) 15–28.

[6] R.H. Bruck and R.C. Bose. The construction of translation planes from projective spaces, J. Algebra 1 (1964) 85–102.

[7] R.H. Bruck and R.C. Bose. Linear representations of projective planes in projective spaces, J. Algebra 4 (1966) 117–172.

[8] J.W.P. Hirschfeld. Finite Projective Spaces of Three Dimensions. Oxford University Press, 1985.

[9] J.W.P. Hirschfeld. Projective Geometry over Finite Fields, Second Edition. Oxford University Press, 1998.

[10] G. Panella. Caratterizzazione delle quadriche di uno spazio (tri-dimensionale) lineare sopra un corpo finito. Boll. Un. Mat. Ital., 10 (1955) 507–513.

[11] C.T. Quinn. The André/Bruck and Bose representation of conics in Baer subplanes of PG(2, q^2). J. Geom 74 (2002) 123-138.

[12] B. Segre. Ovals in a finite projective plane. Can. J. Math., 7 (1955) 414–416.

[13] B. Segre. Introduction to Galois geometries. Atti Accad. Naz. Lincei Mem., 8 (1967) 133–236. (edited by J.W.P. Hirschfeld.)