A New Look at the Arcsine Law and “Quantum-Classical Correspondence”

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Abstract
We prove that the arcsine law as the time-averaged distribution for classical harmonic oscillators emerges from the distributions for quantum harmonic oscillators in terms of noncommutative algebraic probability. This is nothing but a simple and rigorous realization of “Quantum-Classical Correspondence” for harmonic oscillators.

1 Introduction
The normalized arcsine law \( \mu_{As} \) is the probability distribution on \( \mathbb{R} \) with support \([−\sqrt{2}, \sqrt{2}]\) defined as

\[
\mu_{As}(dx) = \frac{1}{\pi} \frac{dx}{\sqrt{2 - x^2}}
\]

whose \( n \)-th moment \( M_n := \int_\mathbb{R} x^n \mu_{As}(dx) \) is given by

\[
M_{2m+1} = 0, \quad M_{2m} = \frac{1}{2m} \binom{2m}{m}.
\]

In this case the moment problem is determinate, that is, the moment sequence \( \{M_n\} \) characterizes \( \mu_{As} \).

The distribution \( \mu_{As} \) often appears in classical probability theory. In the noncommutative context, it is also known as the limit distribution for “monotone central limit theorem” ([3], a simple proof is found in [4]). Here we

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discuss another aspect of this distribution: the relationship with the classical harmonic oscillator.

Let $x(t) = A \sin t$ be a classical harmonic oscillator with the amplitude $A$. Then it is easy to see that the time-averaged distribution $\mu$ of position $x$ has the form

$$\mu(dx) = C \frac{dx}{\sqrt{A^2 - x^2}}$$

where $C$ denotes the normalizing constant. In $A = \sqrt{2}$ case, $\mu = \mu_{A^*}$.

Then a question arises: Is it possible to see whether and in what meaning the “Quantum-Classical Correspondence” holds for harmonic oscillators? This question, which is related to fundamental problems in Quantum theory and asymptotic analysis [2], is analyzed and generalized from the viewpoint of noncommutative algebraic probability with quite a simple combinatorial argument.

2 Basic notions

Let $\mathcal{A}$ be a $\ast$-algebra. We call a linear map $\varphi : \mathcal{A} \to \mathbb{C}$ a state on $\mathcal{A}$ if it satisfies

$$\varphi(1) = 1, \quad \varphi(a^*a) \geq 0.$$ 

A pair $(\mathcal{A}, \varphi)$ of a $\ast$-algebra and a state on it is called an algebraic probability space. Here we adopt a notation for a state $\varphi : \mathcal{A} \to \mathbb{C}$, an element $X \in \mathcal{A}$ and a probability distribution $\mu$ on $\mathbb{R}$.

Notation 2.1. We use the notation $X \sim_{\varphi} \mu$ when $\varphi(X^m) = \int_{\mathbb{R}} x^m \mu(dx)$ for all $m \in \mathbb{N}$.

Remark 2.2. Existence of $\mu$ for $X$ which satisfies $X \sim_{\varphi} \mu$ always holds. The uniqueness of such $\mu$ holds if the moment problem is determinate.

Definition 2.3 (Quantum harmonic oscillator). A quantum harmonic oscillator is a triple $(\Gamma(\mathbb{C}), a, a^*)$ where $\Gamma(\mathbb{C})$ is a Hilbert space $\Gamma(\mathbb{C}) := \oplus_{n=0}^{\infty} \mathbb{C} \Phi_n$ with inner product given by $\langle \Phi_n, \Phi_m \rangle = \delta_{n,m}$, and $a, a^*$ are operators defined as follows:

$$a \Phi_0 = 0, \quad a \Phi_n = \sqrt{n} \Phi_{n-1} (n \geq 1)$$

$$a^* \Phi_n = \sqrt{n+1} \Phi_{n+1}$$

Let $\mathcal{A}$ be the $\ast$-algebra generated by $a$, and $\varphi_n$ be the state defined as $\varphi(\cdot) := \langle \Phi_n, (\cdot) \Phi_n \rangle$. Then $(\mathcal{A}, \varphi_n)$ is an algebraic probability space. It is well known that

$$X := \frac{1}{\sqrt{2}} (a + a^*)$$
represents the “position” and that
\[ X \sim \varphi_0 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx. \]
That is, in \( n = 0 \) case, the distribution of position is Gaussian.
On the other hand, the asymptotic behavior of the distributions of position as \( n \) tends to infinity is quite nontrivial.

3 Emergence of the Arcsine law

Theorem 3.1. Let \( \mu_N \) be a probability distribution on \( \mathbb{R} \) such that
\[ \frac{X}{\sqrt{N}} \sim \varphi \mu_N. \]
Then \( \mu_N \) weakly converges to \( \mu_{\text{As}} \).

Proof. We only have to prove moment convergence because it is known that moment convergence implies weak convergence when the moment problem for the limit distribution is determinate.

First we can easily prove that
\[ \langle \varphi_N((\frac{X}{\sqrt{N}})^{2m+1}) = \langle \Phi_N, (\frac{a+a^*}{\sqrt{2N}})^{2m+1} \Phi_N \rangle = 0 \]
since \( \langle \Phi_N, \Phi_M \rangle = 0 \) when \( N \neq M \).

To consider the moments of even degrees, we introduce the following notations:
- \( \Lambda^{2m} := \{ \text{maps from } \{1, 2, \ldots, 2m\} \text{ to } \{1, *\} \} \),
- \( \Lambda^m := \{ \lambda \in \Lambda^{2m}; |\lambda^{-1}(1)| = |\lambda^{-1}(*)| = m \} \).

Note that the cardinality \( |\Lambda^m| \) equals to \( \binom{2m}{m} \) because the choice of \( \lambda \) is equivalent to the choice of \( m \) elements which consist the subset \( \lambda^{-1}(1) \) from \( 2m \) elements in \( \{1, 2, \ldots, 2m\} \).

It is clear that for any \( \lambda \notin \Lambda^m \)
\[ \langle \Phi_N, a^{\lambda_1}a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle = 0 \]
since \( \langle \Phi_N, \Phi_M \rangle = 0 \) when \( N \neq M \).

On the other hand, for any \( \lambda \in \Lambda^m \) the inequality
\[ N \cdots (N - m + 1) \leq \langle \Phi_N, a^{\lambda_1}a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle \leq (N + 1) \cdots (N + m) \]
holds when $N$ is sufficiently large, because the minimum is achieved when
\[
\lambda_i = \begin{cases} 
1, & (1 \leq i \leq m) \\
*, & (m + 1 \leq i \leq 2m)
\end{cases}
\]
and the maximum is achieved when
\[
\lambda_i = \begin{cases} 
*, & (1 \leq i \leq m) \\
1, & (m + 1 \leq i \leq 2m)
\end{cases}
\]
by the definition of $a, a^*$. Using the inequality above we have
\[
\frac{1}{N^m} \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle \to 1 \quad (N \to \infty).
\]
and then
\[
\varphi_N \left( \frac{X}{\sqrt{N}} \right)^{2m} = \langle \Phi_N, (\frac{a + a^*}{\sqrt{2N}})^{2m} \Phi_N \rangle
\]
\[
= \frac{1}{2^m} \sum_{\lambda \in \Lambda_{2m}} \frac{1}{N^m} \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle
\]
\[
= \frac{1}{2^m} \sum_{\lambda \in \Lambda_{2m}} \frac{1}{N^m} \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle
\]
\[
\to \frac{1}{2^m} \left| \lambda_{2m} \right| = \frac{1}{2^m} \binom{2m}{m} \quad (N \to \infty).
\]

\textbf{Remark 3.2.} The theorem above can be extended to the cases for $q$-Fock spaces ($0 \leq q \leq 1$), which are typical example of “interacting Fock spaces $[1]$”, if we just replace integer $N$ by
\[
N_q := 1 + q + q^2 + \cdots + q^{N-1}.
\]
The proof is quite similar and we omit it here.

\section{Summary and prospects}

As we have stated, the Arcsine law as the time-averaged distribution for classical harmonic oscillator emerges from the distributions for quantum harmonic oscillators. This is nothing but a noncommutative probabilistic realization of Quantum-Classical Correspondence for harmonic oscillators. The
“time averaged” nature is deeply related to the notion of Bohr’s “complementarity” for energy and time. Starting from energy eigenstates, one cannot obtain the classical harmonic oscillator itself but time averaged distribution of it.

Mathematically, the result above also shows an important aspect of the Arcsine law as the universal distribution for many kinds of “interacting Fock spaces”\([1]\) (not only \(q\)-Fock spaces) which are deeply connected to the theory of orthogonal polynomials (This point of view is due to Professor Bożejko). The condition for interacting Fock spaces (orthogonal polynomials) from which the Arcsine law emerges as the high-energy limit distribution should be discovered.

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