The BNS invariants of the generalized solvable Baumslag-Solitar groups and of their finite index subgroups

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THE BNS INVARIANTS OF THE GENERALIZED SOLVABLE BAVMSLAG-SOLITAR GROUPS AND OF THEIR FINITE INDEX SUBGROUPS

WAGNER SGOBBI AND PETER WONG

ABSTRACT. We compute the Bieri-Neumann-Strebel invariants \( \Sigma^1 \) for the generalized solvable Baumslag-Solitar groups \( \Gamma_n \) and their finite index subgroups. Using \( \Sigma^1 \), we show that certain finite index subgroups of \( \Gamma_n \) cannot be isomorphic to \( \Gamma_k \) for any \( k \). In addition, we use the BNS-invariants to give a new proof of property \( R_\infty \) for the groups \( \Gamma_n \) and their finite index subgroups.

1. Introduction

The Bieri-Neumann-Strebel invariant \( \Sigma^1(G) \) \[1\] of a finitely generated group \( G \) is an important object of study in geometric group theory and has many connections to other areas of mathematics, especially with the Thurston norm in low dimensional topology. However, the computation of \( \Sigma^1 \) is very difficult in general and there are only few classes of groups for which \( \Sigma^1 \) is known (see e.g. \[7\] and the references therein).

A group \( G \) is said to have property \( R_\infty \) if \( R(\varphi) \) is infinite for every automorphism \( \varphi \in \text{Aut}(G) \). Here, \( R(\varphi) \) is the number of twisted conjugacy classes of \( \varphi \), that is, the number of equivalence classes in \( G \) given by the relation \( g \sim h \iff zg\varphi(z)^{-1} = h \) for some \( z \in G \). Twisted conjugacy classes are important in topological fixed point theory.

Let \( X \) be a space with universal covering \( \tilde{X} \) and \( f : X \to X \) be a homeomorphism with induced automorphism \( f_* : \pi_1(X) \to \pi_1(X) \). Then \( R(f_*) \) is actually the number of (topological) lifting classes of \( f \) in \( \tilde{X} \) given by a deck transformation conjugation, which also partitions the fixed points of \( f \) in \( X \). This number is an upper bound for the Nielsen number \( N(f) \), which is a sharp lower bound for the minimal number of fixed points in the homotopy class \([f]\) and one of the main objects of study in Nielsen Theory (see \[5\]). For instance in \[5\], property \( R_\infty \) was used to show that for any \( n \geq 5 \), there exists a \( n \)-dimensional nilmanifold \( M \) such that every self-homeomorphism \( f : M \to M \) is isotopic to be fixed point free.

The motivation for this work is \[11\] in which J. Taback and P. Wong showed property \( R_\infty \) for the generalized solvable Baumslag-Solitar groups \( \Gamma_n \) and for every group quasi-isometric to \( \Gamma_n \), using geometric group theoretic techniques. In \[4\], D. Gonçalves and D. Kochloukova used the Bieri-Neumann-Strebel (BNS or \( \Sigma^1 \)) invariant to deduce property \( R_\infty \) for certain classes of groups, including a new proof of the property \( R_\infty \) for the Thompson’s group \( F \). Since the

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Σ-invariants of the Baumslag-Solitar groups $BS(1, n)$ are sufficient to guarantee property $R_\infty$, it is natural to ask whether property $R_\infty$ for $\Gamma_n$ and for their finite index subgroups can also be deduced using $\Sigma^1$.

In this paper, we show that the property $R_\infty$ for $\Gamma_n$ and for their finite index subgroups can be deduced from their respective BNS-invariants. Here we compute the $\Sigma^1$ invariants of $\Gamma_n$ and of all its finite index subgroups $H$. We show that these invariants lie in an open hemisphere of the corresponding character spheres so that property $R_\infty$ follows from [4]. Furthermore, we extend the result to any finite direct product of these groups. Using $\Sigma^1$, we show that there exist finite index subgroups of $\Gamma_n$ that cannot be isomorphic to any $\Gamma_k$, in contrast to the fact that every finite index subgroup of a solvable Baumslag-Solitar group $BS(1, n)$ is again a $BS(1, k)$.

The paper is organized as follows. In section 2 we compute the $\Sigma^1$ for $\Gamma_n$ (Theorem 2.4). In section 3, we classify all the finite index subgroups $H$ of $\Gamma_n$ in terms of specific generators and index (Theorem 3.4), and give a presentation of $H$ (Theorem 3.5). Then we compute their $\Sigma^1$ invariant (Theorem 3.8) and use it to show that some $H$ cannot be a generalized solvable Baumslag-Solitar group (Theorem 3.9). In section 4 we use geometric arguments about the behavior of the induced homeomorphisms $\phi^*: S(G) \to S(G)$ to show that finding some special invariant convex polytopes in the character sphere of a finitely generated group $G$ is sufficient to guarantee property $R_\infty$ for $G$. In section 5, we give new proofs (Theorems 5.2 and 5.3) of property $R_\infty$ for the groups $\Gamma_n$ and $H$ above and also for any finite direct product of them (Theorem 5.4). Finally, in Proposition 5.6, we exhibit a family of groups $G$ where Theorem 4.8 can be used to guarantee property $R_\infty$ without complete information on $\Sigma^1(G)$.

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2. Computation of $\Sigma^1(\Gamma_n)$

In this section we compute the $\Sigma^1$ invariants of the generalized solvable Baumslag-Solitar groups $\Gamma_n$. First we recall the definition of the BNS-invariant $\Sigma^1(G)$ of a finitely generated group $G$. There are other equivalent definitions (see [1] and [9]) but we employ the following for our purposes.

Definition 2.1. Let $G$ be a finitely generated group. The character sphere of $G$ is the quotient space

$$S(G) = (Hom(G, \mathbb{R}) - \{0\})/\sim = \{[\chi] \mid \chi \in Hom(G, \mathbb{R}) - \{0\}\},$$

where $\chi \sim \chi' \Leftrightarrow r\chi = \chi'$ for some $r > 0$. 
It is well known that if the free rank of the abelianized group $G_{ab}$ is $n$ with generators $x_1, ..., x_n$, then $S(G) \cong S^{n-1}$ with homeomorphism
$$h : S(G) \to S^{n-1}$$
$$[\chi] \mapsto \frac{(\chi(x_1), ..., \chi(x_n))}{\| (\chi(x_1), ..., \chi(x_n)) \|}.$$  

Following [9], we have

**Definition 2.2.** Let $G$ be a finitely generated group with finite generating set $X \subset G$. Denote by $\Gamma = \Gamma(G, X)$ the Cayley graph of $G$ with respect to $X$. The first $\Sigma$-invariant (or BNS invariant) of $G$ is
$$\Sigma^1(G) = \{ [\chi] \in S(G) \mid \Gamma_{\chi} \text{ is connected} \},$$
where $\Gamma_{\chi}$ is the subgraph of $\Gamma$ whose vertices are the elements $g \in G$ with $\chi(g) \geq 0$ and whose edges are those of $\Gamma$ which connect two such vertices.

The solvable Baumslag-Solitar group $BS(1, n), n > 1$ is defined by the presentation
$$BS(1, n) = \langle a, t \mid tat^{-1} = a^n \rangle.$$  

We consider the following solvable generalization of $BS(1, n)$.

**Definition 2.3.** Let $n \geq 2$ be a positive integer with prime decomposition $n = p_1^{y_1}...p_r^{y_r}$, the $p_i$ being pairwise distinct. We define the solvable generalization of the Baumslag-Solitar group by
$$\Gamma_n = \langle a, t_1, ..., t_r \mid t_it_j = t_jt_i, \ i \neq j, \ t_ia^{-1} = a^{p_i^{y_i}}, \ i = 1, ..., r \rangle.$$  

More generally, let $S = \{n_1, ..., n_r\}$ be a set of pairwise coprime positive integers such that $n_i \geq 2$ for some $i$. Define
$$\Gamma(S) = \langle a, t_1, ..., t_r \mid t_it_j = t_jt_i, \ i \neq j, \ t_ia^{-1} = a^{n_i}, \ i = 1, ..., r \rangle.$$  

The group $\Gamma(S)$ is always torsion-free.

Note that $BS(1, n)$ is a metabelian group and it admits the following splitting
$$1 \to \mathbb{Z} \left[ \frac{1}{n} \right] \to BS(1, n) \xrightarrow{\psi} \mathbb{Z} \to 1.$$  

where $\mathbb{Z} \left[ \frac{1}{n} \right]$ denotes the $n$-adic rationals and contains the commutator subgroup $[BS(1, n), BS(1, n)]$.

Similarly, $\Gamma_n$ is characterized by the following short exact sequence
$$1 \to \mathbb{Z} \left[ \frac{1}{n} \right] \to \Gamma_n \xrightarrow{\varphi} \mathbb{Z}^r \to 1.$$  

Here, $\varphi$ is the canonical projection with $a \mapsto 1$, $\mathbb{Z} \left[ \frac{1}{n} \right] = \langle a_j, \ j \in \mathbb{Z} \mid a_j^n = a_{j+1}, \ j \in \mathbb{Z} \rangle$ and is generated by the elements
$$a_j = (t_1...t_r)^ja(t_1...t_r)^{-j} \in \Gamma_n.$$  

Using the presentation $\mathbb{Z}^r = \langle t_1, ..., t_r \mid t_it_j = t_jt_i, \ i \neq j \rangle$, the exact sequence (2.1) splits using the section $\mathbb{Z}^r \to \Gamma_n$ sending $t_i \mapsto t_i$. Thus, $\Gamma_n = \mathbb{Z} \left[ \frac{1}{n} \right] \rtimes \mathbb{Z}^r$ is the semidirect product of
these two subgroups, and every element \( w \in \Gamma_n \) can be uniquely written as \( w = t_{i_1}^{\alpha_1}...t_{i_r}^{\alpha_r} u \) for \( u \in \mathbb{Z} \left[ \frac{1}{n} \right] \) and \( \alpha_i \in \mathbb{Z} \) (we put \( u \) on the right side following the notation from Bogopolski in [3]). Observe that the “\( t \)-coordinates” in \( \Gamma_n \) are well behaved, that is, \((t_{i_1}^{\alpha_1}...t_{r}^{\alpha_r} u)(t_{1}^{\beta_1}...t_{r}^{\beta_r} u') = t_{1}^{\alpha_1+\beta_1}...t_{r}^{\alpha_r+\beta_r} u'' \) for some \( u'' \in \mathbb{Z} \left[ \frac{1}{n} \right] \). Secondly, because of the presentation of the subgroup \( \mathbb{Z} \left[ \frac{1}{n} \right] \), we see that any two generators \( a_i, a_j \) must be powers of the common generator \( a_{\text{min}(i,j)} \). Note that \( \mathbb{Z} \left[ \frac{1}{n} \right] \) is an infinitely generated abelian group and \( \Gamma_n \) is metabelian.

**Theorem 2.4.** The complement \( \Sigma^1(\Gamma(S))^c \) of the \( \Sigma^1 \) of the group

\[
\Gamma(S) = \langle a, t_1, ..., t_r \mid t_it_j = t jt_i, i \neq j, \ t_iat_i^{-1} = a^{n_i}, \ i = 1, ..., r \rangle
\]

is given by

\[
\Sigma^1(\Gamma(S))^c = \{ [\chi_i] \mid \chi_i(t_i) = 1 \text{ and } \chi_i(t_j) = 0 \text{ for } j \neq i \},
\]

In particular, if \( n = p_1^{\alpha_1}...p_r^{\alpha_r} \) is a prime decomposition, then

\[
\Sigma^1(\Gamma_n)^c = \{ [\chi_1], ..., [\chi_r]\}.
\]

Furthermore, \( \Sigma^1(\Gamma(S))^c \) lies inside an open hemisphere in \( S(\Gamma(S)) \).

**Proof.** As pointed out in [1], the \( \Sigma^1 \) coincides with \( \Sigma_{G'} \) of [2]. For the metabelian group \( \Gamma(S) \), the quotient \( \mathbb{Z}^r \) is the torsion-free part of the abelianization so that \( S(\Gamma(S)) = S(\mathbb{Z}^r) \). It follows from Proposition 2.1 and formula (2.3) of [2] that

\[
\Sigma^1(\Gamma(S)) = \bigcup_{\lambda \in C(A)} \{ [\chi] \in S(\Gamma(S)) \mid \chi(\lambda) > 0 \}
\]

where \( A = \text{Ker} \varphi = \mathbb{Z} \left[ \frac{1}{n} \right] \) as a \( \mathbb{Z}[\mathbb{Z}^r] \)-module and \( C(A) = \{ \lambda \in \mathbb{Z}[\mathbb{Z}^r] \mid \lambda \cdot \alpha = \alpha, \text{ for all } \alpha \in A \} \) is the centralizer of \( A \). Let \([\chi] \in S(\Gamma(S)).\)

Case (1): If \( \chi(t_i) < 0 \) for some \( i, 1 \leq i \leq r \) then we let \( \lambda = n_it_i^{-1} \). Note that \( n_it_i^{-1} \cdot a = t_i^{-1}a^{n_i}t_i = a \). It follows that \( \lambda \in C(A) \) and \([\chi] \in \Sigma^1(\Gamma(S)).\)

Let \( I_k = \{ i_j \mid 1 \leq i_1 < ... < i_k \leq r \}, k \geq 2 \), be a subset of the set \( I = \{ 1, 2, ..., r \}. \)

Case (2): If \( \chi(t_{i_j}) > 0 \) for \( i_j \in I_k \) and \( \chi(s) = 0 \) for \( s \in I \setminus I_k \) then we let \( \lambda = \sum_{j=1}^{k} \alpha_j \cdot t_{i_j} \), where \( \alpha_j \) are integers such that \( \alpha_1n_1 + ... + \alpha_rn_r = 1 \) since \( n_i, ..., n_r \) are pairwise relatively prime. It is easy to see that \( \lambda \in C(A). \)

If \( \chi(\lambda) > 0 \) then \([\chi] \in \Sigma^1(\Gamma(S)).\)

Now suppose \( \kappa = \chi(\lambda) \leq 0 \). Without loss of generality, we may assume that \( \alpha_{i_1} > 0 \). Since \( n_1t_1^{-1} \in C(A) \), it follows that for any integer \( M, M\lambda - (M - 1)(n_i t_1^{-1}) \cdot a = a^{M-(M-1)} = a \) so that \( \lambda_M = M\lambda - (M - 1)(n_i t_1^{-1}) \in C(A). \) Now, it is straightforward to see that \( \chi(\lambda_M) = (M - 1)n_1 \cdot \chi(t_{i_1}) + M\kappa. \) There exists a positive integer \( M \) such that \( \chi(\lambda_M) > 0 \). In other words, \([\chi] \in \Sigma^1(\Gamma(S)).\)

Now, the set of characters that do not belong to Case (1) or Case (2) is \( \{ [\chi_i] \}, \) where \( \chi_i(t_i) = 1 \) and \( \chi_i(t_j) = 0 \) if \( j \neq i \). To see that this set is the complement of \( \Sigma^1(\Gamma(S)) \), it suffices to show that \([\chi_i] \in \Sigma^1(\Gamma(S))^c \) for each \( i \). Observe that if \( \gamma = \sum c_j t_j^\theta \in C(A) \) then either all \( q_j > 0 \) when \( c_j \neq 0 \) or for some \( j, c_j = n_j \) and \( q_j = -1 \) with \( q_i = 0 \) for \( i \neq j \). Thus, \( \chi_i(\gamma) = c_iq_i \) cannot be positive so each \([\chi_i] \notin \Sigma^1(\Gamma(S)).\) 

\[\square\]
Remark 2.5. In an earlier version of this paper, Theorem 2.4 was first proved using a general geometric argument [9, Theorem A3.1].

For the remaining of this paper, we focus on the groups $\Gamma_n$.

3. Finite index subgroups of $\Gamma_n$

In this section we study the finite index subgroups $H$ of $\Gamma_n$. First, in Theorem 3.4 we find a specific set of generators for $H$ using a generalization of an argument given by Bogopolski in [3]. We use these generators to compute the index of $H$ in $\Gamma_n$. Then, in Theorem 3.5 we give a presentation for $H$ and, in Theorem 3.8 we compute $\Sigma^1(H)$. We end the section by exhibiting finite index subgroups $H$ of $\Gamma_n$ which are not isomorphic to $\Gamma_k$ for any $k \geq 2$.

3.1. Generators, cosets and index. The following useful lemma has an elementary proof and was used by Bogopolski in [3].

Lemma 3.1. Let $n, s \geq 1$ be integers. Let $m$ be the biggest positive divisor of $s$ such that $\gcd(m, n) = 1$. Then $s$ divides $mn^s$.

To facilitate our computation, we aim to find a good set of generators of a finite index subgroup of $\Gamma_n$. To do so, we need the next two lemmas.

Lemma 3.2 (Replacing $j_0$ by any $j$). Suppose

\[(3.1) \quad H = \langle t_1^{k_1} ... t_r^{k_r} a_{q_1}^{l_1} t_2^{k_2} ... t_r^{k_r} a_{q_2}^{l_2} ... t_r^{k_r} a_{q_s}^{l_s} a_{l_j}^{l_j} \rangle \leq \Gamma_n \]

is a subgroup with arbitrary integers $k_i, l_i > 0$, $k_i j_i \geq 0$ and $q_i, l_i, j_0 \in \mathbb{Z}$. Then, for any chosen $j \in \mathbb{Z}$, we can replace $a_j^{l_j}$ above by $a_j^{l_j}$, up to modifying $l > 0$ by another positive integer (also called $l$), that is, $H = \langle t_1^{k_1} ... t_r^{k_r} a_{q_1}^{l_1} t_2^{k_2} ... t_r^{k_r} a_{q_2}^{l_2} ... t_r^{k_r} a_{q_s}^{l_s} a_j^{l_j} \rangle$.

Proof. If $j \leq j_0$ we know from the presentation of $\mathbb{Z} [\frac{1}{n}]$ that $a_j^{l_j}$ is a positive power of $a_j$, so $a_j^{l_j}$ is also a positive power of $a_j$ and the lemma is obviously true. Let us treat the case $j > j_0$.

Using that $\mathbb{Z} [\frac{1}{n}]$ is abelian and the relations of $\Gamma_n$, we can show that

\[(t_1^{k_1} ... t_r^{k_r} a_{q_1}^{l_1})^{-m_i} a_{l_0}^{l_0} (t_1^{k_1} ... t_r^{k_r} a_{q_1}^{l_1})^{-m_i} = a_j^{l_0 m_i n^i m_i n^i} \]

for every $i$ and every integer $m_i > 0$. Thus we can replace $a_j^{l_j}$ in the expression of $H$ by this element $a_j^{l_0 m_i n^i m_i n^i}$, that is, we can multiply the power $l$ of $a_j^{l_j}$ by $p_{m_i n^i m_i n^i}$ in (3.1), and since this new power is still positive we can repeat the process recursively. By doing this for $i = 1, ..., r$ we can replace the power $l$ of $a_j^{l_j}$ in (3.1) by any number of the form

\[l(p_1^{m_1 y_1 k_{i_1}} ... p_r^{m_1 y_r k_{i_r}})(p_2^{m_2 y_2 k_{i_2}} ... p_r^{m_2 y_r k_{i_r}})...(p_r^{m_r y_r k_{i_r}})\]

for any $m_1, ..., m_r > 0$. By putting together the first primes in the parentheses we rewrite this as

\[p_1^{m_1 y_1 k_{i_1}} p_2^{m_2 y_2 k_{i_2}} ... p_r^{m_r y_r k_{i_r}} l \lambda\]

for some integer $\lambda > 0$ depending on the $m_i$. In particular, for the integers $m_i = k_{i_1} ... k_{i_r} \lambda l_i$ we can replace the power $l$ of $a_j^{l_j}$ by

\[p_1 y_1 k p_2 y_2 k ... p_r y_r k l \lambda = n^k l \lambda,\]
where $k = k_{11} \ldots k_{rr}$. But $a_\lambda^{k_{11} a_{11}} = a_\lambda^{k_1} = 1$, which is a positive power of $a_\lambda^{j_{01} + 1}$. We repeat this process a finite number of times until we reach the index $j > j_0$ we wanted and the lemma is proved.

\[ \square \]

**Lemma 3.3** (Replacing $l$ by $m$). Let
\[ (3.2) \]
\[ H = \langle t_1^{k_{11}} \ldots t_r^{k_r} a_{11}^{l_1}, t_2^{k_{22}} \ldots t_r^{k_r} a_{22}^{l_2}, \ldots, t_r^{k_r} a_q^{l_q}, a_j^0 \rangle \leq \Gamma_n \]
be a subgroup with arbitrary integers $k_i, l > 0$, $k_i \geq 0$ and $q_i, t_i, j \in \mathbb{Z}$. Let $m$ be the biggest divisor of $l$ such that $\gcd(m, n) = 1$. Then we can replace $a_j^0$ by $a_j^m$ in the expression above, that is, $H = \langle t_1^{k_{11}} \ldots t_r^{k_r} a_{11}^{l_1}, t_2^{k_{22}} \ldots t_r^{k_r} a_{22}^{l_2}, \ldots, t_r^{k_r} a_q^{l_q}, a_j^m \rangle$.

**Proof.** It suffices to show that the inclusions $a_j^0 \in \langle t_1^{k_{11}} \ldots t_r^{k_r} a_{11}^{l_1}, t_2^{k_{22}} \ldots t_r^{k_r} a_{22}^{l_2}, \ldots, t_r^{k_r} a_q^{l_q}, a_j^m \rangle$ and $a_j^m \in H$ hold. The first inclusion is straightforward, because $l$ is a multiple of $m$ and so $a_j^0$ is a power of $a_j^m$. For the second inclusion first observe that by Lemma 3.1 $l$ must divide $mn^t$ and so it must also divide $ml^n$. This implies that the number
\[ \gamma = \frac{mn^{lk_{r1} \ldots p_{r1} y_{r_1} \ldots p_{r1} y_{r_1} (k_{r1} - 1)^{l_{k_{r1}}} \ldots p_{r1} y_{r_1} (k_{r1} - 1)^{l_{k_{r1}}} l_{k_{r1}}} l}{l} \]
is an integer. Let $A_1, \ldots, A_r$ be the first $r$ generators of $H$ in (3.2), that is, $H = \langle A_1, \ldots, A_r, a_j^0 \rangle$. It is straightforward to show that
\[ A_1^{-l_{k_{r1}}} \ldots A_r^{-l_{k_{r1}}} A_r^{-l} (a_j^0)^{\gamma} A_r^l \cdot A_r^{-l_{k_{r1}}} \ldots A_1^{-l_{k_{r1}}} = a_j^m, \]
then $a_j^m \in H$, as desired. \[ \square \]

**Theorem 3.4.** For any $\Gamma_n$, the following properties hold.

1) Every finite index subgroup $H$ of $\Gamma_n$ can be written as
\[ H = \langle t_1^{k_{11}} \ldots t_r^{k_r} a_{11}^{l_1}, t_2^{k_{22}} \ldots t_r^{k_r} a_{22}^{l_2}, \ldots, t_r^{k_r} a_q^{l_q}, a_j^0 \rangle \] (\*)
for $0 \leq k_{1i}, \ldots, k_{i-1,i} < k_{ii}, l_i \in \mathbb{Z}$ and $m > 0$ an integer such that $\gcd(m, n) = 1$ and $H \cap \langle a \rangle = \langle a^m \rangle$.

2) If $H$ is any subgroup of $\Gamma_n$ given by the expression (\*) for $0 \leq k_{1i}, \ldots, k_{i-1,i} < k_{ii}, l_i \in \mathbb{Z}$ and $m > 0$ such that $\gcd(m, n) = 1$ and $H \cap \langle a \rangle = \langle a^m \rangle$, then $T = \{ t_1^{\beta_1} \ldots t_r^{\beta_r} a^j \mid 0 \leq \beta_i < k_{ii}, 0 \leq j < m \}$ is a transversal of $H$ in $\Gamma_n$. In particular, the index of $H$ in $\Gamma_n$ is $k_{11} \ldots k_{rr} m$ and $H$ has finite index in $\Gamma_n$.

**Proof.** 1) First, since $\Gamma_n$ is finitely generated and $H$ is finite index, by the Reidemeister-Schreier theorem $H$ must be also finitely generated and we write
\[ H = \langle t_1^{a_{11}} \ldots t_r^{a_{1r} v_1}, \ldots, t_1^{a_{m1}} \ldots t_r^{a_{mr} v_m} \rangle \]
for $a_{ij} \in \mathbb{Z}$ and $v_i \in \mathbb{Z} \left[ \frac{1}{n} \right]$. Note that $m \geq r$. Otherwise, $\varphi(H)$ would be a subgroup of $\mathbb{Z}^r$ with rank $< r$ and then would have infinite index, a contradiction because $\varphi$ is surjective. With a similar projection argument, we see that there must be at least one $i$ such that $a_{i1} \neq 0$. Let $k_{11} = \gcd \{ a_{i1} \}$. Since $k_{11} > 0$ is the smallest positive integer combination of the $a_{i1} \neq 0$,
we can obtain inside \( H \) an element of the form \( t_1^{k_1} \ldots t_r^{k_r} u_1 \) for some \( k_{12}, \ldots, k_{1r} \in \mathbb{Z} \) and \( u_1 \in \mathbb{Z} \left[ \frac{1}{n} \right] \), so we can write

\[
(3.3) \quad H = \langle t_1^{\alpha_1} \ldots t_r^{\alpha_r} v_1, \ldots, t_1^{\alpha_{m_1}} \ldots t_r^{\alpha_{m_r}} v_m, t_1^{k_1} \ldots t_r^{k_r} u_1 \rangle.
\]

Now, since all the nonzero \( \alpha_i \) are multiples of \( k_{11} \), say, \( \alpha_{11} = d_1 k_{11} \), we can replace \( t_1^{\alpha_1} \ldots t_r^{\alpha_r} v_i \) by \( (t_1^{\alpha_{11}} \ldots t_r^{\alpha_{r1}} v_1)(t_1^{k_1} \ldots t_r^{k_r} u_1)^{-d_1} = t_2^{\alpha_{i2}} \ldots t_r^{\alpha_{ir} v_i'} \) in (3.3). Then, after relabeling these new generators, we can write

\[
H = \langle t_2^{\alpha_{i2}} \ldots t_r^{\alpha_{ir} v_1}, \ldots, t_2^{\alpha_{i_m}} \ldots t_r^{\alpha_{m_r} v_m}, t_1^{k_1} \ldots t_r^{k_r} u_1 \rangle.
\]

We added a new generator and “eliminated” all the \( t_1 \) coordinates of the first \( m \) generators of \( H \). This was the first step. In a similar way, we can do this for all the other \( t_2, \ldots, t_r \) coordinates.

After \( r \) steps, we added \( r \) new generators and eliminated all the \( t_1, \ldots, t_r \) letters from the first \( m \) generators from \( H \), so we have

\[
H = \langle v_1, \ldots, v_m, t_1^{k_1} \ldots t_r^{k_r} u_1, t_2^{k_22} \ldots t_r^{k_r} u_2, \ldots, t_r^{k_r} u_r \rangle
\]

with \( k_{ii} > 0 \) and \( v_i, u_i \in \mathbb{Z} \left[ \frac{1}{n} \right] \). But in \( \mathbb{Z} \left[ \frac{1}{n} \right] \) we have \( \langle v_1, \ldots, v_m \rangle = \langle u \rangle \) for some \( u \in \mathbb{Z} \left[ \frac{1}{n} \right] \) and

\[
(3.4) \quad H = \langle t_1^{k_1} \ldots t_r^{k_r} u_1, t_2^{k_22} \ldots t_r^{k_r} u_2, \ldots, t_r^{k_r} u_r, u \rangle
\]

By manipulating the generators above if necessary, we may suppose that \( 0 \leq k_{1r}, \ldots, k_{i-1,i} < k_{ii} \) (they could be also positive if we wanted) in (3.4). Finally, write \( u_i = a_{q_i}^l, u = a_q^l \) for \( q, q, l_i, l \in \mathbb{Z} \). Then

\[
(3.5) \quad H = \langle t_1^{k_1} \ldots t_r^{k_r} a_{q_1}^{l_1}, t_2^{k_22} \ldots t_r^{k_r} a_{q_2}^{l_2}, \ldots, t_r^{k_r} a_{q_r}^{l_r}, a_q^l \rangle.
\]

Let us show that we may assume \( l > 0 \) above. If \( l \neq 0 \) then, up to changing \( a_q^l \) by \( (a_q^l)^{-1} = a_q^{-l} \) if necessary, we are done. If \( l = 0 \), that is,

\[
(3.6) \quad H = \langle t_1^{k_1} \ldots t_r^{k_r} a_{q_1}^{l_1}, t_2^{k_22} \ldots t_r^{k_r} a_{q_2}^{l_2}, \ldots, t_r^{k_r} a_{q_r}^{l_r} \rangle,
\]

do the following: since \( \mathbb{Z}^r \) is abelian, every commutator of elements in \( H \) must be in \( Ker(\varphi) \) (and obviously in \( H \)). At least one of the commutators between the \( r \) generators of \( H \) in (3.6) must be non-trivial. Otherwise, \( H \) would be a finite index abelian subgroup of \( \Gamma_n \) and we would have \( \Sigma^1(\Gamma_n) = S(\Gamma_n) \) by using Proposition B1.11 in [9], a contradiction to Theorem 2.4. Then let \( a_{q_i}^{l'} (l' \neq 0) \) be a non-trivial commutator between two generators of \( H \). We can add it to (3.6) and up to changing \( a_j^{l'} \) by its inverse, we are done.

Our next steps will be eliminating the subindices \( q_i \) from the \( a \) letters in the generators of (3.5). Fix some \( 1 \leq i \leq r \). If \( q_i \geq 0 \), then \( a_{q_i}^{l_i} \) is a power of \( a \) and we are done by doing this replacement in (3.5). Suppose \( q_i < 0 \). By Lemma 3.2 we replace \( q \) by \( q_i \) in (3.5). Now, let \( m \) be the biggest divisor of \( l \) such that \( \gcd(m, n) = 1 \). By Lemma 3.3 we can also replace \( l \) by \( m \) above and obtain

\[
H = \langle t_1^{k_1} \ldots t_r^{k_r} a_{q_1}^{l_1}, t_2^{k_22} \ldots t_r^{k_r} a_{q_2}^{l_2}, \ldots, t_r^{k_r} a_{q_r}^{l_r}, a_q^m \rangle.
\]

Since \( \gcd(m, n) = 1 \) we also have \( \gcd(m, n^{-q_i}) = 1 \) and there must be \( \tilde{\alpha}, \tilde{\beta} \in \mathbb{Z} \) such that \( \tilde{\alpha} m + \tilde{\beta} n^{-q_i} = 1 \). Then for \( \alpha = l_i \tilde{\alpha} \) and \( \beta = l_i \tilde{\beta} \) we have \( \alpha m + \beta n^{-q_i} = l_i \), or

\[
l_i - m\alpha = n^{-q_i} \beta.
\]
Then, using the relations in $\Gamma_n$ we have

$$H = \langle t_1^{k_{11}}...t_r^{k_r}a_{q_1}^{l_1}, t_2^{k_{12}}...t_r^{k_r}a_{q_2}^{l_2}, ..., t_i^{k_{1i}}...t_r^{k_r}a_{q_i}^{l_i}, ..., t_r^{k_r}a_{q_r}^{l_r}, a_{q_i}^{m_i} \rangle$$

$$= \langle t_1^{k_{11}}...t_r^{k_r}a_{q_1}^{l_1}, t_2^{k_{12}}...t_r^{k_r}a_{q_2}^{l_2}, ..., t_i^{k_{1i}}...t_r^{k_r}a_{q_i}^{l_i}, ..., t_r^{k_r}a_{q_r}^{l_r}, a_{q_i}^{m_i} \rangle$$

$$= \langle t_1^{k_{11}}...t_r^{k_r}a_{q_1}^{l_1}, t_2^{k_{12}}...t_r^{k_r}a_{q_2}^{l_2}, ..., t_i^{k_{1i}}...t_r^{k_r}a_{q_i}^{l_i}, ..., t_r^{k_r}a_{q_r}^{l_r}, a_{q_i}^{m_i} \rangle$$

and relabeling $\beta$ by $l_i$, $m$ by $l$ and $q_i$ by $q$ again we have

$$H = \langle t_1^{k_{11}}...t_r^{k_r}a_{q_1}^{l_1}, t_2^{k_{12}}...t_r^{k_r}a_{q_2}^{l_2}, ..., t_i^{k_{1i}}...t_r^{k_r}a_{q_i}^{l_i}, ..., t_r^{k_r}a_{q_r}^{l_r}, a_{q_i}^{m_i} \rangle$$

that is, we removed the subindex $q_i$ from $a_{q_i}^{l_i}$ in $\mathbf{3.5}$. If we do this for all $i$ we remove all the subindices and obtain

$$H = \langle t_1^{k_{11}}...t_r^{k_r}a_1^{l_1}, t_2^{k_{12}}...t_r^{k_r}a_2^{l_2}, ..., t_r^{k_r}a_r^{l_r}, a_1^{l_1} \rangle$$

for some $q \in \mathbb{Z}$. We can use Lemma $\mathbf{3.2}$ to replace $q$ by 0 and we get the desired set of generators for $H$. To finish, let $m$ (a new one) be the biggest divisor of $l$ such that $\gcd(m, n) = 1$. By Lemma $\mathbf{3.3}$ we replace $a_i^l$ by $a_i^m$ in the expression above. If $H \cap \langle a \rangle = \langle a^m \rangle$, we are done. If not, let $m' = \min\{k \geq 1 | a^k \in H\}$. It’s easy to see that $H \cap \langle a \rangle = \langle a^{m'} \rangle$. Since $a^m \in H$, $m$ is a multiple of $m'$ and we have $\gcd(m', n) = 1$. Then, by adding $a^{m'}$ to the set of generators of $H$, the generator $a^m$ can be removed. By relabeling $m'$ by $m$, we obtain the desired result.

2) Let $H$ be such a subgroup. As shown in item 1), we may suppose that $k_{ij} > 0$ for all $i, j$. Let us first show that $\Gamma_n = \bigcup_{t_1^{\beta_1}...t_r^{\beta_r}a^l} Ht_1^{\beta_1}...t_r^{\beta_r}a^l$. Every element of $\Gamma_n$ is written as $t_1^{-\alpha_1}...t_r^{-\alpha_r} a^l t_1^{\gamma_1}...t_r^{\gamma_r}$ for $\alpha_i, \gamma_i \geq 0$ and $l \in \mathbb{Z}$. Since $k_{ij} > 0$ for all $i, j$, one can show that every coset of $\Gamma_n$ is of the form $Ha^l t_1^{\gamma_1}...t_r^{\gamma_r}$ for $l \in \mathbb{Z}$ and $\gamma_i \geq 0$. Now we claim that every such coset can also be written as $Ht_1^{\gamma_1}...t_r^{\gamma_r} a^k$ for some integer $l'$. In fact, because $1 = \gcd(m, n) = \gcd(m, p_1^{\gamma_1}...p_r^{\gamma_r})$, the prime decomposition of $m$ does not involve any of the $p_i$. Then it is also true that $\gcd(m, p_1^{\gamma_1}...p_r^{\gamma_r}) = 1$. Let $k, k'$ be integers such that $km + k'p_1^{\gamma_1}...p_r^{\gamma_r} = 1$. Then $l + (-lk)m = (lk')p_1^{\gamma_1}...p_r^{\gamma_r}$ and relabeling $-lk$ by $k$ and $lk'$ by $k'$ we get $l + km = k'p_1^{\gamma_1}...p_r^{\gamma_r}$. Now since $a^m \in H$ we do

$$Ha^l t_1^{\gamma_1}...t_r^{\gamma_r} = H(a^m)^k a^l t_1^{\gamma_1}...t_r^{\gamma_r}$$

$$= H a^{l+km} t_1^{\gamma_1}...t_r^{\gamma_r}$$

$$= H a^{k'p_1^{\gamma_1}...p_r^{\gamma_r}} t_1^{\gamma_1}...t_r^{\gamma_r}$$

$$= H t_1^{\gamma_1}...t_r^{\gamma_r} a^{k'}$$

and relabeling $k'$ by $l'$ we showed the claim. To transform this coset into one of the cosets in the theorem, we apply successive algorithms: choose some index $i$. If $\gamma_i < k_{ii}$ we stop the algorithm. If $\gamma_i \geq k_{ii}$, by manipulating this coset we show that

$$H t_1^{\gamma_1}...t_{i-1}^{\gamma_{i-1}} t_i^{\gamma_i-k_{ii}} t_{i+1}^{\gamma_{i+1}}...t_r^{\gamma_r} a^{k'}$$
for some integer \( l' \). If \( \gamma_i - k_{ii} < k_{ii} \) we stop the algorithm. If \( \gamma_i - k_{ii} \geq k_{ii} \) we do the above again. Then after finite steps our “i-algorithm” shows that

\[
H_t^1 \gamma_i^{l} \ldots t_r^{\gamma_r} a^l = H_t^1 \gamma_i^{l-1} t_i^{\beta_i} t_{i+1}^{\gamma_{i+1}} \ldots t_r^{\gamma_r} a^{l'}
\]

for some \( 0 \leq \beta_i < k_{ii} \). Now, starting with the coset \( H_t^1 \gamma_i^{l} \ldots t_r^{\gamma_r} a^l \), we successively apply the “i-algorithm” for \( i = 1, 2, \ldots, r \) and obtain exactly

\[
H_t^1 \gamma_i^{l} \ldots t_r^{\gamma_r} a^l = H_t^1 \beta_1^{j_1} \ldots t_r^{\beta_r} a^{j_r}
\]

for \( 0 \leq \beta_i < k_{ii} \) and \( l' \in \mathbb{Z} \). Finally, write \( l' = qm + j \) for \( 0 \leq j < m \). Then \( H_t^1 \beta_1^{j_1} \ldots t_r^{\beta_r} a^{j_r} = H_t^1 \beta_1^{j_1} \ldots t_r^{\beta_r} a^{j_r} \) because

\[
t_1^{\beta_1} \ldots t_r^{\beta_r} a^{l'} (t_1^{\beta_1} \ldots t_r^{\beta_r} a^{j_r})^{-1} = t_1^{\beta_1} \ldots t_r^{\beta_r} a^{l'} - j t_r^{\beta_r} \ldots t_1^{\beta_1} = t_1^{\beta_1} \ldots t_r^{\beta_r} a^{mq} t_r^{\beta_r} \ldots t_1^{\beta_1} = (a^m)^{p_1^{y_1} \ldots p_r^{y_r} a^j} \in H.
\]

This shows that \( \Gamma_n = \bigcup_{t_1^{\beta_1} \ldots t_r^{\beta_r} a^{j_r}} H_t^1 \beta_1^{j_1} \ldots t_r^{\beta_r} a^{j_r} \).

Now let us show that the cosets over \( T \) are all distinct. Let \( H_t^1 \beta_1^{j_1} \ldots t_r^{\beta_r} a^{j_r} = H_t^1 \beta_1^{j_1'} \ldots t_r^{\beta_r} a^{j_r'} \) for \( 0 \leq \beta_i, \beta_i' < k_{ii} \) and \( 0 \leq j, j' < m \). By definition,

\[
w = a^{p_1^{y_1} \ldots p_r^{y_r} (j - j')} t_1^{\beta_1 - \beta_1'} \ldots t_r^{\beta_r - \beta_r'} = t_1^{\beta_1 - j'} t_1^{\beta_1 - \beta_1'} \ldots t_r^{\beta_r - \beta_r'} \]

Then, projecting in \( \mathbb{Z}^r \),

\[
(\beta_1 - \beta_1', \ldots, \beta_r - \beta_r') = \varphi(w) \in \varphi(H) = \langle (k_{11}, k_{12}, \ldots, k_{1r}), (0, k_{22}, \ldots, k_{2r}), \ldots, (0, \ldots, 0, k_{rr}) \rangle.
\]

Write

\[
(\beta_1 - \beta_1', \ldots, \beta_r - \beta_r') = \lambda_1 (k_{11}, k_{12}, \ldots, k_{1r}) + \lambda_2 (0, k_{22}, \ldots, k_{2r}) + \ldots + \lambda_r (0, \ldots, 0, k_{rr})
\]

for integers \( \lambda_i \). Since the first vector \( (k_{11}, k_{12}, \ldots, k_{1r}) \) is the only one with non-vanishing first coordinate we have \( \beta_1 - \beta_1' = \lambda_1 k_{11} \). Since \( 0 \leq \beta_1, \beta_1' < k_{11} \) we must have \( \beta_1 = \beta_1' \) and therefore \( \lambda_1 = 0 \). By easy induction we can show that all the \( \lambda_i \) must vanish. Now, we just have to show that \( j = j' \). We already have \( a^{p_1^{y_1} \ldots p_r^{y_r} (j - j')} \in H \). Since \( H \cap \langle a \rangle = \langle a^m \rangle \) (by item 1)), we have

\[
p_1^{y_1} \ldots p_r^{y_r} (j - j') = qm
\]

for some \( q \in \mathbb{Z} \). So \( m \) divides \( p_1^{y_1} \ldots p_r^{y_r} (j - j') \). Since \( \gcd(n, m) = 1 \), \( m \) does not contain any of the \( p_i \) in its prime decomposition, and therefore \( m \) must divide \( j - j' \). Since \( 0 \leq j, j' < m \) we have \( j = j' \), as desired. This completes the proof. \( \square \)
3.2. A presentation. We now give a presentation for an arbitrary finite index subgroup $H$ of $\Gamma_n$.

**Theorem 3.5.** Let $H$ be any finite index subgroup of $\Gamma_n$ (see Theorem 3.4), say,

$$H = \langle t_1^{k_{i1}}...t_r^{k_{ir}}a_1^{l_1}, t_2^{k_{i2}}...t_r^{k_{ir}}a_2^{l_2}, ..., t_r^{k_{ir}}a_r^{l_r}, a^m \rangle \quad (\ast)$$

for $k_{ii} > 0$, $k_{ij} \geq 0$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that $\gcd(m, n) = 1$ and $H \cap \langle a \rangle = \langle a^m \rangle$.

Then $H$ has the following presentation:

$$H \simeq \langle \alpha, x_1, ..., x_r \mid x_i \alpha x_i^{-1} = \alpha^{P_i}, x_i x_j x_i^{-1} x_j^{-1} = \alpha^{R_{ij}} \rangle,$$

where $P_i = p_i^{y_{k_{ii}}}...p_i^{y_{k_{ir}}}$ ($i = 1, ..., r$) and $R_{ij} \in \mathbb{Z}$ characterized by

$$l_i P_i (1 - P_j) - l_j P_j (1 - P_i) = R_{ij} m.$$

*Proof.* It is easy to see that $(t_1^{k_{i1}}...t_r^{k_{ir}}a_1^{l_1})a^m(t_1^{k_{i1}}...t_r^{k_{ir}}a_1^{l_1})^{-1} = a^{m P_i}$ in $\Gamma_n$, for $i = 1, ..., r$. Also, since

$$(t_1^{k_{i1}}...t_r^{k_{ir}}a_1^{l_1})(t_1^{k_{ij}}...t_r^{k_{ir}}a_1^{l_1})^{-1}(t_1^{k_{ij}}...t_r^{k_{ir}}a_1^{l_1})^{-1} = a_i^{l_i P_i(1-P_j)-l_j P_j (1-P_i)} \in H \cap \langle a \rangle = \langle a^m \rangle,$$

we have $l_i P_i (1 - P_j) - l_j P_j (1 - P_i) = R_{ij} m$ for some integer $R_{ij}$.

We write $(t_1^{k_{i1}}...t_r^{k_{ir}}a_1^{l_1})(t_1^{k_{ij}}...t_r^{k_{ir}}a_1^{l_1})(t_1^{k_{ij}}...t_r^{k_{ir}}a_1^{l_1})^{-1}(t_1^{k_{ij}}...t_r^{k_{ir}}a_1^{l_1})^{-1} = a^{m R_{ij}}$. Now define a group

$$G = \langle \alpha, x_1, ..., x_r \mid x_i \alpha x_i^{-1} = \alpha^{P_i}, x_i x_j x_i^{-1} x_j^{-1} = \alpha^{R_{ij}} \rangle.$$

The group $G$ has the relations

$$x_i \alpha = \alpha^{P_i} x_i, \quad x_i \alpha^{-1} = \alpha^{-P_i} x_i, \quad x_i x_j = \alpha^{R_{ij}} x_i x_j, \quad x_i x_j^{-1} = x_j^{-1} \alpha^{-R_{ij}} x_i,$$

which shows that, for every fixed $i$, all the $x_i$-letters in a word with positive power can be pushed right as much as we want. Similarly, the relations

$$\alpha x_i^{-1} = x_i^{-1} \alpha^{P_i}, \quad \alpha^{-1} x_i^{-1} = x_i^{-1} \alpha^{-P_i}, \quad x_i x_i^{-1} = x_i^{-1} \alpha^{-R_{ij}} x_i, \quad x_i x_j^{-1} = x_j^{-1} x_i^{-1} \alpha^{-R_{ij}} x_i,$$

show that all the $x_i$-letters in a word with negative power can be pushed left as much as we want. Because of this, any element of $G$ is of the form $x_1^{\lambda_1}...x_r^{\lambda_r} \alpha^M x_r^{\delta_r}...x_1^{\delta_1}$ for $\lambda_i, \delta_i \geq 0$ and $M \in \mathbb{Z}$.

Now let us show that $G \simeq H$. Define $\theta : G \to \Gamma_n$ by putting $\theta(\alpha) = a^m$ and $\theta(x_i) = t_i^{k_{ii}}...t_r^{k_{ir}}a_1^{l_1}$ for $i = 1, ..., r$. It is easy to check that $\theta$ is a group homomorphism and surjective, so we only need to show that $\theta$ is also injective. Indeed, let $w = x_1^{\lambda_1}...x_r^{\lambda_r} \alpha^M x_r^{\delta_r}...x_1^{\delta_1} \in G$ such that $\theta(w) = 1$.

Then

$$(t_1^{k_{i1}}...t_r^{k_{ir}}a_1^{l_1})^{-\lambda_1}...(t_1^{k_{ir}}a_r^{l_r})^{-\lambda_r} \alpha^{m M}(t_1^{k_{ir}}a_r^{l_r})^{\delta_r}...(t_1^{k_{i1}}...t_r^{k_{ir}}a_1^{l_1})^{\delta_1} = 1.$$

By projecting both sides of equation above on the $t_1$-coordinate by the homomorphism $w \mapsto (w)^{1r}$, we get $k_{i1}(\delta_1 - \lambda_1) = 0$ and so $\delta_1 = \lambda_1$. Then by conjugating the above equation on both sides by $(t_1^{k_{i1}}...t_r^{k_{ir}}a_1^{l_1})^{\delta_1}$ we get

$$(t_2^{k_{i2}}...t_r^{k_{ir}}a_2^{l_2})^{-\lambda_2}...(t_r^{k_{ir}}a_r^{l_r})^{-\lambda_r} \alpha^{m M}(t_r^{k_{ir}}a_r^{l_r})^{\delta_r}...(t_2^{k_{i2}}...t_r^{k_{ir}}a_2^{l_2})^{\delta_2} = 1.$$

By doing this recursively we get $\delta_i = \lambda_i$ for $i = 1, ..., r$ and $a^{m M} = 1$. Then $M = 0$ (since $a$ is torsion free and $m > 0$). Thus $w = x_1^{\lambda_1}...x_r^{\lambda_r} \alpha^0 x_r^{\delta_r}...x_1^{\lambda_1} = 1$, as desired. This completes the proof. \qed
3.3. The $\Sigma^1$ invariant. Let $H$ be a finite index subgroup of $\Gamma_n$, say,

$$H = \langle t_{k_{11}} \ldots t_{r_{k_{rr}}} a_{l_1}, t_2^{k_{22}} \ldots t_{r_{k_{rr}}} a_{l_2}, \ldots, t_r^{k_{rr}} a_{l_r}, a^m \rangle \quad (*)$$

for $k_{ii} > 0$, $k_{ij} \geq 0$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that gcd($m, n$) = 1 and $H \cap \langle a \rangle = \langle a^m \rangle$. By Theorem 3.5, we write $H$ as

$$H = \langle \alpha, x_1, \ldots, x_r \mid x_i \alpha x_i^{-1} = \alpha^{P_i}, x_i x_j x_i^{-1} x_j^{-1} = \alpha^{R_{ij}} \rangle,$$

for $P_i = p_i^{y_{k_{ii}}} \ldots p_i^{y_{k_{rr}}} (i = 1, \ldots, r)$ and some $R_{ij} \in \mathbb{Z}$. Here, $\alpha = a^m$ and $x_i = t_i^{k_{ii}} \ldots t_r^{k_{rr}} a_{l_i}$. Since all the $p_i^{y_{k_{ii}}}$ are $\geq 2$, obviously the $P_i$ also are $\geq 2$ and so it is easy to see that $\alpha$ must have torsion in the abelianized group $H^{ab}$. The $x_i$ are torsion-free, though. So we have the homeomorphism

$$h : S(H) \rightarrow S^{r-1}$$

$$[\chi] \mapsto \frac{\langle \chi(x_1), \ldots, \chi(x_r) \rangle}{\|\langle \chi(x_1), \ldots, \chi(x_r) \rangle\|}.$$

To compute $\Sigma^1(H)$ inside this sphere, we will use the following fact.

**Proposition 3.6.** Let $G$ be a finitely generated group and $H \leq G$ a finite index subgroup with inclusion $i : H \rightarrow G$ and induced map $i^* : S(G) \rightarrow S(H)$, $i^*[\chi] = [\chi \circ i] = [\chi|_H]$. Suppose that any homomorphism $\chi : H \rightarrow \mathbb{R}$ can be extended to a homomorphism $\hat{\chi} : G \rightarrow \mathbb{R}$. Then

$$\Sigma^1(H) = i^*(\Sigma^1(G)) \text{ and } \Sigma^1(H)^c = i^*(\Sigma^1(G)^c).$$

**Proof.** By Proposition B1.11 in [9], for any $[\chi] \in S(G)$ we have $[\chi] \in \Sigma^1(G) \iff [\chi|_H] \in \Sigma^1(H)$. Then $i^*(\Sigma^1(G)) \subset \Sigma^1(H)$. On the other hand, let $[\chi] \in \Sigma^1(H)$ and let $\hat{\chi} : G \rightarrow \mathbb{R}$ be an extension of $\chi$. We have $[\hat{\chi}|_H] = [\chi] \in \Sigma^1(H)$, so again by Proposition B1.11 in [9] we have $[\hat{\chi}] \in \Sigma^1(G)$. Then $[\chi] = i^*[\hat{\chi}] \in i^*(\Sigma^1(G))$, as desired. The other equality is similar. $\square$

**Lemma 3.7.** Let $H$ be a finite index subgroup of $\Gamma_n$, say,

$$H = \langle t_{1}^{k_{11}} \ldots t_{r}^{k_{rr}} a_{l_1}, t_2^{k_{22}} \ldots t_{r}^{k_{rr}} a_{l_2}, \ldots, t_r^{k_{rr}} a_{l_r}, a^m \rangle \quad (*)$$

for $k_{ii} > 0$, $k_{ij} \geq 0$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that gcd($m, n$) = 1 and $H \cap \langle a \rangle = \langle a^m \rangle$. Then every homomorphism $\xi : H \rightarrow \mathbb{R}$ can be extended to a homomorphism $\chi : \Gamma_n \rightarrow \mathbb{R}$.

**Proof.** The equation $\chi|_H = \xi$ is equivalent to a system of $r$ equations

$$\begin{aligned}
\chi(t_1^{k_{11}} \ldots t_{r}^{k_{rr}} a_{l_1}) &= \xi(t_1^{k_{11}} \ldots t_{r}^{k_{rr}} a_{l_1}), \\
\chi(t_2^{k_{22}} \ldots t_{r}^{k_{rr}} a_{l_2}) &= \xi(t_2^{k_{22}} \ldots t_{r}^{k_{rr}} a_{l_2}), \\
&\vdots \\
\chi(t_r^{k_{rr}} a_{l_r}) &= \xi(t_r^{k_{rr}} a_{l_r}).
\end{aligned}$$

So, to create such an extension $\chi$ we just have to define $\chi(a) = 0$ and define the real numbers $\chi(t_i)$ satisfying equations (1) to (r) above. Equation (r) is equivalent to

$$k_{rr} \chi(t_r) = \xi(t_r^{k_{rr}} a_{l_r}).$$
so if we define $\chi(t_r) = \frac{1}{k_r-r} \xi(t_r^{k-r} a_r^{l_r})$, equation (r) is satisfied. Similarly, equation $(r-1)$ is equivalent to

$$k_{r-1,r-1} \chi(t_{r-1}) + k_{r-1,r} \chi(t_r) = \xi(t_{r-1}^{k_{r-1,r-1}} t_r^{k_{r-1,r}} a_r^{l_{r-1}}),$$

so if we define $\chi(t_{r-1}) = \frac{1}{k_{r-1,r-1}} \xi(t_{r-1}^{k_{r-1,r-1}} t_r^{k_{r-1,r}} a_r^{l_{r-1}}) - \frac{k_{r-1,r}}{k_{r-1,r-1}} \chi(t_r)$, equation $(r-1)$ is satisfied. By doing this recursively to all $i$, we are done. □

**Theorem 3.8.** Let $H$ be a finite index subgroup of $\Gamma_n$, say,

$$H = \langle t_1^{k_{11}} ... t_r^{k_{1r}} a_1^{l_1}, t_2^{k_{22}} ... t_r^{k_{2r}} a_2^{l_2}, ..., t_r^{k_{rr}} a_r^{l_r} \rangle$$

for $k_{ii} > 0$, $k_{ij} \geq 0$, $l_i \in \mathbb{Z}$ and $m > 0$ an integer such that gcd$(m, n) = 1$ and $H \cap \langle a \rangle = \langle a^m \rangle$, and let $\alpha = a^m$ and $x_i = t_i^{k_{ii}} ... t_r^{k_{rr}} a_i^{l_i}$ be its generators. Then $\Sigma^1(H)^c = \{[\xi_1], ..., [\xi_r]\}$, where $\xi_i(x_j) = k_{ji}$ if $j \leq i$ and $\xi_i(x_j) = 0$ if $j > i$.

In other words, if we identify $S(H) \simeq S^{r-1}$ as we did above, then

$$\Sigma^1(H)^c = \left\{ \frac{(k_{11}, 0, 0, ..., 0)}{|| (k_{11}, 0, 0, ..., 0) ||}, \frac{(k_{12}, k_{22}, 0, ..., 0)}{|| (k_{12}, k_{22}, 0, ..., 0) ||}, ..., \frac{(k_{rr}, k_{rr}, k_{rr}, ..., k_{rr})}{|| (k_{rr}, k_{rr}, k_{rr}, ..., k_{rr}) ||} \right\}.$$  

**Proof.** By Lemma 3.7, $\Sigma^1(H)^c = i^*(\Sigma^1(\Gamma_n)^c)$ so by Theorem 2.4, $\Sigma^1(H)^c = \{[\chi_1|H], ..., [\chi_r|H]\}$. Using that $\chi_i(t_j) = 1$ if $i = j$ and $\chi_i(t_j) = 0$, it is easy to see that the image of $[\chi_i|H]$ (which we denote by $[\xi_i]$) under the homeomorphism $S(H) \simeq S^{r-1}$ described above is $\frac{(k_{ii}, k_{ii}, 0, 0, ..., 0)}{|| (k_{ii}, k_{ii}, 0, 0, ..., 0) ||}$. This completes the proof. □

3.4. **Finite index subgroups that are not $\Gamma_k.$** In [3] it was shown that every finite index subgroup of a solvable Baumslag-Solitar group $BS(1, n)$ is also (isomorphic to) a solvable Baumslag-Solitar group $BS(1, n^k)$ for some $k \geq 1$. Since the groups $\Gamma_n$ are generalizations of $BS(1, n)$, it is natural to ask whether every finite index subgroup of $\Gamma_n$ is also (isomorphic to) another $\Gamma_k$ for some $k \geq 2$. In this section we show that this question has a negative answer. Below, we consider a specific class of finite index subgroups $H$ of $\Gamma_n$ for which we give necessary and sufficient conditions for $H$ to be isomorphic to $\Gamma_k$ for some $k \geq 2$.

**Theorem 3.9.** Let $H$ be a finite index subgroup of $\Gamma_n$ such that

$$H = \langle t_1^{k_{11}} ... t_r^{k_{1r}}, t_2^{k_{22}} ... t_r^{k_{2r}}, ..., t_r^{k_{rr}}, a_r^{l_r} \rangle$$

with $k_{ii} > 0$, $0 \leq k_{ij} < k_{ii}$ for all $1 \leq i < j \leq r$ and $m > 0$ such that gcd$(m, n) = 1$. Then $H \simeq \Gamma_k$ for some $k \geq 2$ if and only if $k_{ij} = 0$ for all $1 \leq i < j \leq r$.

**Proof.** Suppose first that $k_{ij} = 0$ for all $1 \leq i < j \leq r$. Then from Theorem 3.5, we immediately get that $H \simeq \Gamma_k$ for $k = p_1^{n_1} ... p_r^{n_r}$, $p_i^{n_i}$ is prime. Suppose now that $H \simeq \Gamma_k$ for some $k \geq 2$ and write $k = q_1^{z_1} ... q_s^{z_s}$, $q_1 < q_2 < ... < q_s$, $z_i \geq 1$ the prime decomposition of $k$. Then in particular $s = card(\Sigma^1(\Gamma_k)^c) = card(\Sigma^1(H)^c) = r$, so $k = q_1^{z_1} ... q_r^{z_r}$. By Theorem 3.5, $H$ has the presentation

$$H = \langle \alpha, x_1, ..., x_r \mid x_i \alpha x_i^{-1} = \alpha^{n_i}, x_i x_j = x_j x_i \text{ for all } i, j \rangle,$$

where $n_i = p_1^{n_i} ... p_r^{n_r}. k_r$. There is also a split exact sequence

$$1 \rightarrow ker(\pi) \rightarrow H \xrightarrow{\pi} \mathbb{Z}^r \rightarrow 1.$$
where \( \pi(x_i) = e_i, \pi(\alpha) = 0 \) and \( \ker(\pi) \) abelian. In particular, every element of \( H \) can be written as \( x_1^{\lambda_1} \ldots x_r^{\lambda_r} u \) for some \( \lambda_i \in \mathbb{Z} \) and \( u \in \ker(\pi) \). Since \( H \cong \Gamma_k \), then there must be \( r + 1 \) elements inside \( H \) (which are the images of the analogous \( r + 1 \) elements in \( \Gamma_k \)), say, 
\[
X_i = x_1^{k_{1i}} \ldots x_r^{k_{ri}} u_i, \quad 1 \leq i \leq r \quad \text{and} \quad A = x_1^{k_1} \ldots x_r^{k_r} \bar{u} \quad \text{for some } k_{ij}, \bar{k}_i \in \mathbb{Z} \text{ and } u_i, \bar{u} \in \ker(\pi),
\]
such that \( H = \langle X_1, \ldots, X_r, A \rangle \) and \( X_i A X_i^{-1} = A^u \) for all \( 1 \leq i \leq r \). By projecting any of these equations on \( \mathbb{Z}^r \) we obtain \( \bar{k}_1 = \ldots = \bar{k}_r = 0 \) and so \( A = \bar{u} = x_1^{-\lambda_1} \ldots x_r^{-\lambda_r} \alpha^M x_r^{\alpha} \ldots x_1^{\lambda_1} \) for some \( \lambda_i \geq 0 \) and \( M \neq 0 \). By replacing this in the \( r \) equations above and using that \( \ker(\pi) \) is abelian and the \( x_i \)'s commute with each other, we obtain the \( r \) equations in \( H \)
\[
(3.7) \quad x_1^{k_{1i}} \ldots x_r^{k_{ri}} \alpha^M x_r^{-k_{ri}} \ldots x_1^{-k_{1i}} = \alpha^M q_i
\]
for each \( 1 \leq i \leq r \). If a power \( k_{ij} \) is nonnegative we can use a relation of \( H \) to conjugate \( \alpha^M \). If it is negative, though, then since all the \( x_i \) commute we can push the two \( x_j \) from the left side to the right side of equation (3.7) and use the (now positive) power \( -k_{ij} \) to conjugate \( \alpha^M q_i \). Thus equation (3.7) will always imply an equality of a power of \( \alpha^M \) with a power of \( \alpha^M q_i \). Since \( H \) is torsion-free and \( M \neq 0 \), this yields an equation of prime decomposition which depends on the sign of the \( k_{ij} \). After a careful analysis of the possible prime decomposition equations we can conclude that \( k_{ij} = 1 \) if \( i = j \) and 0 otherwise. The equations (3.7) become
\[
x_i \alpha^M x_i^{-1} = \alpha^M q_i \quad \text{This implies } \quad p^u_{i+1} \ldots p^u_{i+1} p^u_{i+1} = p^u_{i+1}, \quad \text{which implies } \quad k_{i,i+1} = \ldots = k_{ir} = 0.
\]
Since \( i \) is arbitrary, we have that \( k_{ij} = 0 \) for any \( 1 \leq i < j \leq r \), as desired. \( \square \)

4. Convex polytopes and property \( R_\infty \)

In this section we show that finding a special kind of invariant convex polytope in the character sphere \( S(G) \) is enough to guarantee property \( R_\infty \) for a finitely generated group \( G \) (Theorem 4.8). We will use a slightly more general version of Theorem 3.3 in \([4]\), which we state below.

The proof is the same given there, just by observing that the authors didn’t use directly the definition of \( \Sigma^1(G)^c \) but only the fact that it is invariant in \( S(G) \) (that is, invariant under all permutations of the form \( [\chi] \mapsto [\chi \circ \varphi] \) for \( \varphi \in \text{Aut}(G) \)).

**Theorem 4.1.** Let \( G \) be a finitely generated group. Suppose there is a nonempty and finite subset \( A \subset S(G) \) which is invariant in \( S(G) \), consisting only of rational points and contained in an open hemisphere of \( S(G) \). Then \( G \) has property \( R_\infty \). \( \square \)

Let \( G \) be a finitely generated group whose abelianized group \( G^{ab} \) has free rank \( n \). Consider the homeomorphism

\[
\mathfrak{h} : S(G) \longrightarrow S^{n-1}
\]

\[
[\chi] \longmapsto \frac{(\chi(x_1), \ldots, \chi(x_n))}{\|(\chi(x_1), \ldots, \chi(x_n))\|},
\]
where the $x_i \in G$ are the free-abelian generators of $G^{ab}$. Given $\varphi \in \text{Aut}(G)$, we have the induced homeomorphism $\varphi^* : S(G) \to S(G)$ with $\varphi^* [\chi] = [\chi \circ \varphi]$. Let $\varphi^S : S^{n-1} \to S^{n-1}$ be the composition $\varphi^S = \vartheta \circ \varphi^* \circ \vartheta^{-1}$.

By the definition above, $K \subset S(G)$ is invariant in $S(G)$ if and only if $\vartheta(K)$ is invariant under $\varphi^S$ for all $\varphi \in \text{Aut}(G)$. From now on, we assume the standard definitions of convex subsets and convex hulls of euclidean spaces $\mathbb{R}^d$. For spherical objects, the definitions will be the following:

**Definition 4.2.** Let $A \subset S^n \subset \mathbb{R}^{n+1}$ and suppose $A$ is contained in an open hemisphere of $S^n$, say, $A \subset O(v) = \{x \in S^n \mid \langle x, v \rangle > 0\}$ for some $v \in S^n$. We say that $A$ is (spherically) convex if for any $a_1, a_2 \in A$, $\gamma_{a_1, a_2}(t) = \frac{(1-t)a_1+ta_2}{\| (1-t)a_1+ta_2 \|} \in A$ for all $t \in [0, 1]$. The convex hull of any subset $A \subset O(v)$ is the smallest convex subset of $O(v)$ which contains $A$ and is denoted by $\text{conv}(A)$.

It is an easy task to show that $\text{conv}(A)$ above can be described as

$$\text{conv}(A) = \left\{ \frac{t_1a_1 + \ldots + t_m a_m}{\| t_1a_1 + \ldots + t_m a_m \|} \mid m \geq 1, a_i \in A, t_i > 0 \right\}.$$ 

The following lemma shows a special property of the homeomorphisms $\varphi^S$.

**Lemma 4.3.** The homeomorphism $\varphi^S : S^{n-1} \to S^{n-1}$ maps convex hulls to convex hulls. Precisely, let $A \subset O(v)$ and suppose $\varphi^S(A) \subset O(w)$ for some $w$. Then $\varphi^S(\text{conv}(A)) = \text{conv}(\varphi^S(A))$.

**Proof.** Since $(\varphi^{-1})^S = (\varphi^S)^{-1}$, it is enough to show that $\varphi^S(\text{conv}(A)) \subset \text{conv}(\varphi^S(A))$. Let $P \in \text{conv}(A)$ and write $P = \frac{t_1a_1 + \ldots + t_m a_m}{\| t_1a_1 + \ldots + t_m a_m \|}$ for some $a_i \in A$ and $t_i > 0$. For each $a_i$, since $\vartheta : S(G) \to S^{n-1}$ is surjective we write $a_i = \vartheta[\chi_i]$ and by multiplying the representative $\chi_i$ by some $r > 0$ if necessary we can actually suppose $a_i = \vartheta[\chi_i] = (\chi_i(x_1), \ldots, \chi_i(x_n))$. Then, by definition, $\varphi^S(a_i) = \frac{1}{\lambda_i} (\chi_i \circ \varphi(x_1), \ldots, \chi_i \circ \varphi(x_n))$, where $\lambda_i = \| (\chi_i \circ \varphi(x_1), \ldots, \chi_i \circ \varphi(x_n)) \| > 0$.

Now we compute $\varphi^S(P)$. It is easy to see that $\vartheta[t_1\chi_1 + \ldots + t_m \chi_m] = P$, since $a_i = \vartheta[\chi_i]$. By denoting

$$\lambda = \| (t_1(\chi_1 \circ \varphi)(x_1) + \ldots + t_m(\chi_m \circ \varphi)(x_1), \ldots, t_1(\chi_1 \circ \varphi)(x_n) + \ldots + t_m(\chi_m \circ \varphi)(x_n))\|,$$

we have

$$\varphi^S(P) = \frac{t_1}{\lambda} ((\chi_1 \circ \varphi)(x_1), \ldots, (\chi_1 \circ \varphi)(x_n)) + \ldots + \frac{t_m}{\lambda} ((\chi_m \circ \varphi)(x_1), \ldots, (\chi_m \circ \varphi)(x_n))$$

$$= \frac{\lambda t_1}{\lambda} \varphi^S(a_1) + \ldots + \frac{\lambda t_m}{\lambda} \varphi^S(a_m)$$

$$= \frac{\lambda t_1}{\lambda^2} \varphi^S(a_1) + \ldots + \frac{\lambda t_m}{\lambda^2} \varphi^S(a_m)$$

(since the above vector is already unitary)

$$\in \text{conv}(\varphi^S(A)),$$

as desired. \qed

Given an open hemisphere $O(v) = \{x \in S^n \mid \langle x, v \rangle > 0\}$ of $S^n$ for some $v \in S^n$, consider the affine $n$-space $v + \{v\}^\perp = \{w + v \mid \langle w, v \rangle = 0\} \subset \mathbb{R}^{n+1}$. One can show that there is a
homeomorphism \( \theta_v : v + \{v\}^\perp \to O(v) \) with \( \theta_v(P) = \frac{P}{\|P\|} \), the inverse map given by \( P \mapsto \frac{\|v\|^2}{\langle P,v \rangle} P \) (see next figure). From now on we identify \( \mathbb{R}^n = v + \{v\}^\perp \).

\[ \mathbb{R}^n = v + \{v\}^\perp \]

It is straightforward to show that \( \theta_v : \mathbb{R}^n \to O(v) \) maps convex hulls of \( \mathbb{R}^n \) to convex hulls of \( O(v) \). Now we will define the convex polytopes in our context.

**Definition 4.4** (Euclidean convex polytopes). A closed halfspace in \( \mathbb{R}^d \) is a set of the form \( H = \{x \in \mathbb{R}^d \mid \langle x,v \rangle \geq \beta \} \) for some \( 0 \neq v \in \mathbb{R}^d \) and \( \beta \in \mathbb{R} \). A convex polytope \( K \) in \( \mathbb{R}^d \) is a finite intersection \( K = \cap_{i=1}^n H_i \) of closed halfspaces \( H_i \) which is also a bounded subset. Thinking of \( K \) as a submanifold of \( \mathbb{R}^d \) (with boundary), there is a well defined dimension \( r = \dim(K) \), so we say that \( K \) is an \( r \)-polytope.

We can always suppose that the family \( \{H_i\} \) of closed halfspaces defining \( K \) is irredundant, that is, is the minimal family necessary to define \( K \).

**Definition 4.5** (Spherical convex polytopes). For any \( n \geq 0 \), a closed hemisphere in \( S^n \) is a set having the form \( C(w) = \{p \in S^n \mid \langle p,w \rangle \geq 0\} \) for some \( w \in S^n \). A convex polytope \( K \subset S^n \) is a finite intersection of closed hemispheres in \( S^n \). Given a finitely generated group \( G \) with \( S(G) \simeq S^{n-1} \), we say that \( K \subset S(G) \) is a convex polytope if \( \mathfrak{h}(K) \) is a convex polytope in \( S^{n-1} \).

The next lemma uses some known facts about Euclidean polytopes with which we will assume the reader is familiar.

**Lemma 4.6.** Let \( K \subset \mathbb{R}^d \) be a (Euclidean) \( d \)-polytope (maximal dimension) and \( f : K \to K \) a homeomorphism. If \( f \) maps segments to segments, that is, for any \( P,Q \in K \), \( f(\text{conv}(P,Q)) = \text{conv}(f(P),f(Q)) \), then \( f \) maps vertices to vertices.

**Proof.** Let \( K = \cap_{i=1}^n H_i \) for an irredundant family \( \{H_i\} \) and let \( F_i = K \cap H_i \) be its facets. It is known that \( n \geq d+1 \), that \( \partial K = F_1 \cup \ldots \cup F_n \) and that a point of \( K \) is a vertex if and only if it belongs to at least \( d \) different facets. Since \( f \) is a homeomorphism, it must map the boundary \( \partial K \) to itself, and so \( f(F_1 \cup \ldots \cup F_n) = F_1 \cup \ldots \cup F_n \). Suppose by contradiction that a vertex \( P \in K \) is mapped to a non-vertex point \( f(P) \in K \) (but obviously \( P,f(P) \in \partial K \)). If a point \( Q \in K \) belongs to any facet of \( K \) containing \( P \) (say, \( F \)), then \( \text{conv}(Q,P) \subset F \), since every facet is convex. Then \( \text{conv}(f(Q),f(P)) \subset f(F) \subset \partial K \) by hypothesis, so the whole straight path joining \( f(Q) \) and \( f(P) \) is contained in the boundary \( \partial K \). Then one can show that \( f(Q) \) must
be in a facet which also contains \( f(P) \). This argument shows that all the facets containing \( P \) must be mapped into the facets containing \( f(P) \). But there are at least \( d \) facets containing \( P \), say, \( F_1, \ldots, F_d \) and at most \( d-1 \) facets containing \( f(P) \), say, \( F_{i_1}, \ldots, F_{i_{d-1}} \). Then

\[
f(F_1 \cup \ldots \cup F_d) \subset F_{i_1} \cup \ldots \cup F_{i_{d-1}}.
\]

We continue: since there are at least \( d+1 \) facets, let \( Z \in \partial K \) be a point outside \( F_{i_1} \cup \ldots \cup F_{i_{d-1}} \), say, \( Z \in F_{i_d} \), and we can suppose \( F_{i_d} \) is the only facet containing \( Z \). Since \( f \) is surjective, \( Z = f(W) \), so \( W \) must be a boundary point outside \( F_1 \cup \ldots \cup F_d \), say, \( W \in F_{d+1} \). By the same argument above, we have \( f(F_{d+1}) \subset F_{i_d} \) and so \( f(F_1 \cup \ldots \cup F_{d+1}) \subset F_{i_1} \cup \ldots \cup F_{i_d} \). If \( d+1 = n \), we stop. If not, we follow these same steps. After a finite number of steps we will have

\[
f(F_1 \cup \ldots \cup F_n) \subset F_{i_1} \cup \ldots \cup F_{i_{n-1}},
\]

so \( f(\partial K) \not\subset \partial K \), contradiction. \( \square \)

**Theorem 4.7.** Let \( G \) be a finitely generated group and \( K \subset S(G) \) a convex polytope contained in an open hemisphere of \( S(G) \). Then \( K \) is invariant in \( S(G) \) if and only if \( V(K) \) is invariant in \( S(G) \).

**Proof.** The convex polytope \( \mathfrak{h}(K) \) is contained in some open hemisphere \( O(v) \) of \( S^{n-1} \). Let \( \theta_v : \mathbb{R}^{n-1} \rightarrow O(v) \) be the homeomorphism previously defined. One can verify from the definition of \( \theta_v \) that the preimage of a closed hemisphere in \( S^{n-1} \) under \( \theta_v \) is a closed halfspace in \( \mathbb{R}^{n-1} \). Then to see that the preimage \( K' = \theta_v^{-1}(\mathfrak{h}(K)) \) is a polytope it suffices to see that it is bounded. Since \( \mathfrak{h}(K) \) is closed in the compact \( S^{n-1} \), it is compact. Since \( \theta_v \) is a homeomorphism, \( K' \) is also compact in \( \mathbb{R}^{n-1} \) and therefore bounded, so it is in fact a \( r \)-polytope for some \( 0 \leq r \leq n-1 \).

To show the theorem, let \( \varphi \in \text{Aut}(G) \). It is enough to show that \( \mathfrak{h}(K) \) is invariant under \( \varphi^S \) if and only if \( V(\mathfrak{h}(K)) \) is. Suppose first that \( V(\mathfrak{h}(K)) \) is invariant under \( \varphi^S \). In Euclidean space, every convex polytope is the convex hull of its vertices. Since \( \theta_v \) maps convex hulls to spherical convex hulls, it follows that \( \mathfrak{h}(K) \) is also the convex hull of its vertices. Using Lemma 4.3 we have

\[
\varphi^S(\mathfrak{h}(K)) = \varphi^S(\text{conv}(V(\mathfrak{h}(K)))) = \text{conv}(\varphi^S(V(\mathfrak{h}(K)))) = \text{conv}(V(\mathfrak{h}(K))) = \mathfrak{h}(K),
\]

as desired. Now, suppose \( \varphi^S(\mathfrak{h}(K)) = \mathfrak{h}(K) \). If \( r < n-1 \), then \( K' \) is contained in a proper \( r \)-hyperspace of \( \mathbb{R}^{n-1} \), say, \( E^r \). There is a linear isomorphism and isometry \( T : \mathbb{R}^r \rightarrow E^r \) and a \( r \)-polytope \( \tilde{K} \subset \mathbb{R}^r \) such that \( K' = T(\tilde{K}) \). Consider the composition of homeomorphisms

\[
\tilde{K} \xrightarrow{T} K' \xrightarrow{\theta_v} \mathfrak{h}(K) \xrightarrow{\varphi^S} \mathfrak{h}(K) \xrightarrow{\theta_v^{-1}} K' \xrightarrow{T^{-1}} \tilde{K}.
\]

Since \( T \) maps straight paths to straight paths, \( \theta_v \) maps straight paths to geodesic paths and \( \varphi_S \) maps geodesic paths to geodesic paths, this composition is a homeomorphism which maps straight paths to straight paths. Since \( \tilde{K} \) has maximal dimension in \( \mathbb{R}^r \), by Lemma 4.6 this composition must map the vertices of \( \tilde{K} \) to themselves. Since the vertices of \( \mathfrak{h}(K) \) are the image of the ones from \( K' \), it follows that \( \varphi^S \) must map the vertices of \( \mathfrak{h}(K) \) to themselves, as desired. If \( K' \) already had maximal dimension \( r = n-1 \), the proof is the same, but we don’t even need to use \( \tilde{K} \) and \( T \). \( \square \)
Theorem 4.8. Let $G$ be a finitely generated group. If there is a convex polytope $K \subset S(G)$ contained in an open hemisphere of $S(G)$ and with rational vertices such that it is invariant under all homeomorphisms induced by automorphisms of $G$, then $G$ has property $R_\infty$. In particular, if $\Sigma^1(G)^c$ is one such polytope, then $G$ has property $R_\infty$.

Proof. By the previous theorem, $V(K) \subset S(G)$ is finite, invariant and by definition contained in an open half-space of $S(G)$. Then the result follows directly from Theorem 4.1.

5. Property $R_\infty$ for $\Gamma_n$, its finite index subgroups, and direct products

In this section we use all the information previously gathered to guarantee property $R_\infty$ for $\Gamma_n$ (Corollary 5.2), its finite index subgroups $H$ (Corollary 5.3) and also for any (finite) direct product involving these groups (Corollary 5.4). Note that property $R_\infty$ is already known for $\Gamma_n$ and its finite index subgroups (see [11]). However, by using sigma theory, we obtain the same results with new and easier proofs. Corollary 5.4 for the direct product was not considered in [11]. In Proposition 5.6, we exhibit a group $G$ where Theorem 4.8 can be used to guarantee property $R_\infty$ without the need of completely computing the $\Sigma^1$ invariant.

We will make use of the following theorem.

Theorem 5.1 ([4], Theorem 3.3). Let $G$ be a finitely generated group such that

$$\Sigma^1(G)^c = \{[\chi_1],\ldots,[\chi_m]\}$$

is a (nonempty) finite set of rational points. If $\{[\chi_1],\ldots,[\chi_m]\}$ is contained in an open hemisphere of $S(G)$, then $G$ has property $R_\infty$.

Corollary 5.2. The generalized solvable Baumslag-Solitar groups $\Gamma_n$ have property $R_\infty$.

Proof. Observe that, by Theorem 2.4, $\Sigma^1(\Gamma_n)^c$ is a finite set of rational points and is contained in the open hemisphere $O\left(\frac{(1,\ldots,1)}{\| (1,\ldots,1) \|}\right)$ of $S(\Gamma_n)$. The result follows from Theorem 5.1.

Corollary 5.3. All finite index subgroups of $\Gamma_n$ have property $R_\infty$.

Proof. Let $H$ be such finite index subgroup. As above, just observe that, by Theorem 3.8, $\Sigma^1(H)^c$ is a finite set of rational points and is contained in the open hemisphere $O\left(\frac{(1,\ldots,1)}{\| (1,\ldots,1) \|}\right)$ of $S(H)$. The result follows from Theorem 5.1.

Now we show property $R_\infty$ for any (finite) direct product between the groups $\Gamma_n$ and its finite index subgroups.

Corollary 5.4. Let $G = G_1 \times \ldots \times G_m$, where each $G_i$ is some $\Gamma_n$ or some finite index subgroup $H$ of $\Gamma_n$. Then $G$ has $R_\infty$ property.

Proof. By Theorems 2.4 and 3.8, and by the known formula for the $\Sigma^1$ invariant of a direct product of groups (Proposition A2.1 of [9], for example), we easily see that $\Sigma^1(G)^c$ is a finite set of rational points of $S(G)$. Furthermore, by Theorems 2.4 and 3.8, we know that $\Sigma^1(G_i)^c$ is contained in an open hemisphere $O(v_i)$ of $S(G_i)$, for every $i$. From that, it is easy to see that $\Sigma^1(G)^c$ is contained in the open hemisphere $O(v_1,\ldots,v_m)$ of $S(G)$. The result follows from Theorem 5.1.
Let \( G \) be a finitely generated group and \( X \) a finite set of generators for \( G \). A path in the Cayley graph \( \Gamma = \Gamma(G, X) \) of \( G \) is denoted by \( p = (g, y_1, ..., y_n) \). The path \( p \) starts at \( g \), walks through the edge \((g, y_1)\) until the vertex \( g y_1 \), walks through \((g y_1, y_2)\) until \( g y_1 y_2 \) and so on, until its terminus \( g y_1 ... y_n \). Given \( \chi \in \text{Hom}(G, \mathbb{R}) \), the evaluation function \( \nu_\chi \) is given by

\[
\nu_\chi(p) = \min \{ \chi(g), \chi(g y_1), ..., \chi(g y_1 ... y_n) \}.
\]

We are going to use the following geometric \( \Sigma^1 \)-criterion given by R. Strebel (Theorem A3.1) in [9] in Proposition 5.6 to illustrate a situation where we can use Theorem 4.8 to guarantee property \( R_\infty \) for a finitely generated group \( G \) without having to completely compute \( \Sigma^1(G) \).

**Theorem 5.5 (Geometric Criterion for \( \Sigma^1 \)).** Let \( G \) be a finitely generated group with finite generating set \( X \) and denote \( Y = X^\pm \). Let \( [\chi] \in S(G) \) and choose \( t \in Y \) such that \( \chi(t) > 0 \). Then the following are equivalent:

1. \( \Gamma_\chi \) is connected (or \( [\chi] \in \Sigma^1(G) \));
2. For every \( y \in Y \), there exists a path \( p_y \) from \( t \) to \( yt \) in \( \Gamma \) such that \( \nu_\chi(p_y) > \nu_\chi((1, y)) \).

**Proposition 5.6.** Let

\[
G = \langle a, t, s \mid tat^{-1} = a^n, \ sas^{-1} = a^m, \ tst^{-1}s^{-1} = a^r \rangle
\]

for some coprime numbers \( n, m \geq 2 \) and some \( r \in \mathbb{Z} \). Then \( G \) has property \( R_\infty \).

**Proof.** We have the homeomorphism \( h : S(G) \to S^1 \), sending \([\chi]\) to the normalized of \((\chi(t), \chi(s))\). Let us compute \( \Sigma^1(G) \) by the geometric criterion. Fix \( X = \{a, t, s\} \) and \( Y = \{a, a^{-1}, t, t^{-1}, s, s^{-1}\} \).

1. if \( \chi(t) < 0 \) then \([\chi] \in \Sigma^1(G) \). Fix \( t^{-1} \) such that \( \chi(t^{-1}) > 0 \). By using the relations on \( G \), one can see that the paths \( p_a = (t^{-1}, a^n), p_{a^{-1}} = (t^{-1}, a^{-n}), p_t = (t^{-1}, t), p_{t^{-1}} = (t^{-1}, t^{-1}), p_s = (t^{-1}, a^r s) \) and \( p_{s^{-1}} = (t^{-1}, s^{-1} a^{-r}) \) satisfy 2) of [5.5] so \([\chi] \in \Sigma^1(G) \).
2. if \( \chi(s) < 0 \) then \([\chi] \in \Sigma^1(G) \). Similar to item 1).
3. if \( \chi(t) = 1 \) and \( \chi(s) = 0 \) then \([\chi] \notin \Sigma^1(G) \).

Suppose by contradiction that \([\chi] \in \Sigma^1(G) \). Then, in particular, there is a path \( p = (1, w) \) in \( \Gamma_\chi \) from 1 to \( t^{-1}at \). Write

\[
w = t^{k_{11}} s^{k_{12}} a^{r_1} ... t^{k_{c1}} s^{k_{c2}} a^{r_c}.
\]

Since \( p \) is contained in \( \Gamma_\chi \), \( \chi(t) = 1 \) and \( \chi(s) = 0 \) we must have

\[
k_{11} \geq 0, \ k_{11} + k_{21} \geq 0, \ ..., \ k_{11} + ... + k_{c-1,1} \geq 0 \text{ and } k_{11} + ... + k_{c1} = 0.
\]

By using the relations on \( G \), we push right \( t^{k_{11}} \) until \( t^{k_{21}} \), then we push right \( t^{k_{11}+k_{21}} \) until \( t^{k_{31}} \), and so on. Since \( k_{11} + ... + k_{c1} = 0 \), we eliminate from \( w \) all the \( t \)-letters and (after relabeling the \( s \) and \( a \) powers) we can write \( w = s^{k_1} a^{r_1} ... s^{k_{c}} a^{r_{c}} \) in \( G \). But, as a vertex, \( w \) must be the end of the path \( p \). So we have \( w = t^{-1}at \) and therefore

\[
a = twt^{-1} = t(s^{k_1} a^{r_1} ... s^{k_{c}} a^{r_{c}})t^{-1} = (a^r s)^{k_1} a^{nr_1} ... (a^r s)^{k_{c-1}} a^{nr_{c-1}} (a^r s)^{k_{c}} a^{nr_{c}} = 1.
\]

or

\[
w' = (a^r s)^{k_1} a^{nr_1} ... (a^r s)^{k_{c-1}} a^{nr_{c-1}} (a^r s)^{k_{c}} a^{nr_{c-1}} = 1.
\]
in $G$. By projecting this equation onto the $s$-coordinate, we have $k_1 + \ldots + k_c = 0$. Also, 
$(a^r s) a^M = a^m M (a^r s)$ and $a^M (a^r s)^{-1} = (a^r s)^{-1} a^m M$ for every $M \in \mathbb{Z}$. This means that, in $\omega'$, the entire positive pieces $(a^r s)^{k_i}$ can be pushed right and the negative ones can be pushed left. After doing this, we obtain an expression of the form

$$(a^r s)^{-\lambda} a^m a_1^{nr_1} + \ldots + a_c^{nr_c-1} + \alpha (nr_c - 1) = 1,$$

where each $\alpha_i$ is either 1 or a positive power of $m$. This easily implies

$$\alpha_1^{nr_1} + \ldots + \alpha_c^{nr_c-1} + \alpha (nr_c - 1) = 0.$$

By putting all the multiples of $n$ above to the left and only $\alpha_c$ on the right, we get either $Mn = 1$ (contradiction with the fact $n \geq 2$) or $Mn = m^Q$ for $Q \geq 1$ (contradiction with the fact $\gcd(n, m) = 1$). This shows item 3).

4) if $\chi(t) = 0$ and $\chi(s) = 1$ then $[\chi] \not\in \Sigma^1(G)$. Similar to item 3).

Now identify $S(G)$ with $S^1$ by the homeomorphism $\mathcal{H}$ and let $[\chi_1]$ and $[\chi_2]$ be the points of items 3) and 4), respectively. Items 1) and 2) showed that the geodesic $\gamma$ in $S(G)$ between these points contains $\Sigma^1(G)^c$. We claim that $\gamma$ is invariant in $S(G)$. In fact, if $\varphi \in \text{Aut}(G)$ and $p \in \gamma$, then by Lemma 4.3, $\varphi^\ast(p)$ must be in the geodesic between $\varphi^\ast[\chi_1]$ and $\varphi^\ast[\chi_2]$. By the $\Sigma$ invariance and by items 3) and 4), $\varphi^\ast[\chi_1]$ and $\varphi^\ast[\chi_2]$ are in $\Sigma^1(G)^c$; therefore, by items 1) and 2), they must be in $\gamma$. Since $\gamma$ is a convex subset we have $\varphi^\ast(p) \in \gamma$, which shows our claim. Thus, in $S(G)$ we have $\gamma$ an invariant, convex 1-dimensional polytope with the two rational vertices $[\chi_i]$ and the proposition follows from Theorem 4.8. \hfill $\square$

**Remark 5.7.** In Proposition 5.6, if $r \neq 0$, we do not know whether the group $G$ is metabelian in general. While in such cases the proof of Theorem 2.4 does not necessarily apply, the geometric criterion does apply. Of course, if $r = 0$, we have $G = \Gamma(S)$ for $S = \{n, m\}$, so $G$ is metabelian. Therefore, Proposition 5.6 illustrates an alternative way to derive property $R_\infty$ besides using the BNS invariant $\Sigma^1$.

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