Two-Parameter Quantum Groups and Quantum Planes

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Usually the generators of a quantum group are assumed to be commutative with the noncommuting coordinates of a quantum plane. We have relaxed the assumption and investigated its consequences. Not only does a two-parameter quantum group arise naturally, but also the formulation leads us to many probable quantum planes associated with a quantum group. Several examples are presented.
I. INTRODUCTION

In recent years, the concept of a quantum group has extensively emerged in the physical and mathematical literature [1–3]. Quantum groups are nontrivial generalizations of ordinary Lie groups. Such generalizations are made in the framework of Hopf algebras [4–6]. A Hopf algebra is an algebra together with operations called the comultiplication, counit and antipode, which reflect the group structure. A quantum group is a non-commutative Hopf algebra consistent with these costructures. Usually, quantum groups are introduced as deformations of commutative Hopf algebras in the sense that they become commutative Hopf algebra as some parameters go to particular values [7,8]. Probably the most studied case of a quantum group is $GL_q(2)$ whose element $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfies the following nontrivial commutation relations:

$$AB = qBA, \quad AC = qCA,$$
$$BD = qDB, \quad CD = qDC,$$
$$BC = CB, \quad AD - DA = (q - q^{-1})BC.$$

On the other hand, quantum spaces or quantum planes may be introduced as representation spaces of quantum groups [1,3,9].

Corresponding to the quantum group $GL_q(2)$, Manin [1] has defined a quantum space as one generated by two noncommuting coordinates $x, y$ obeying

$$xy = qyx \quad (q \neq 0, 1).$$

Then the quantum group $GL_q(2)$ becomes a symmetry group of the quantum plane. In fact, the points $(x', y')$ and $(x'', y'')$, transformed respectively by means of the matrix $T$ and its transpose $T^t$, satisfy the relations $x'y' = qy'x'$ and $x''y'' = qy''x''$ where

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$
and

\[ T^t: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \]  

(4)

What we emphasize here is that the relation in Eq. (2) is invariant not only under the transformation \( T \) but also under its transpose \( T^t \). (In this sense, a one-parameter quantum group can be regarded as a symmetry group of a quantum plane) and that the generators of a quantum group are assumed to be commutative with the coordinates of a quantum plane.

In this work, we are naturally led to a two-parameter deformation of the group \( GL(2) \) and its corresponding quantum planes even though we do not put any restriction on the number of parameters at the outset. Thus even though the multi-parameter case has already been studied \([3][11]\), we shall concern ourselves with only the two-parameter case in this work. Two-parameter quantum planes have still attracted attention recently \([17][18]\).

Now let us recall two-parameter quantum groups. In fact, by solving the Yang-Baxter equation, one can get the universal \( R \)-matrix

\[ R_{p,q} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - \frac{1}{p} & \frac{2}{p} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \]

(5)

where \( p \) and \( q \) are free parameters \([12][14]\). From \( RTT \) relations, one has the commutation relations

\[ AB = pBA, \quad CD = pDC, \]
\[ AC = qCA, \quad BD = qDB, \]
\[ pBC = qCB, \quad AD - DA = (p - \frac{1}{q})BC. \]

(6)

We note that \( R_{p,q} \) and Eq. (5) become the well-known \( R_q \) solution and Eq. (1), respectively, in the limit \( p \to q \). However, Eq. (2) in the two-parameter case is not invariant under
the transformation in Eq. (4). It is only invariant under the transformation in Eq. (3).

Whenever one requires that Eq. (2) be invariant under the two transformations with the assumption that the generators of a quantum group and the coordinates of a quantum plane be commutative, one is led to a one-parameter quantum group.

Our observation is that even though there are no restrictions on the number of parameters at the outset, one is led naturally to a two-parameter quantum group $GL_{p,q}(2)$ in such a manner that the commutation relations in Eq. (5) come directly from the condition that $x y = q y x$ is preserved not only under the transformation in Eq. (3) but also under that in Eq. (4) as in the one-parameter case, if one relaxes the commutation relations between the generators of a quantum group and the noncommuting coordinates. Actually, the remarkable fact is that even in the case of one-parameter quantum groups, the generators of a quantum group do not commute with the coordinates of the quantum plane generically, as can be seen in the next section.

In Sec. II, we shall push this observation further in a more general fashion. This formulation leads us to many probable quantum planes associated with a quantum group. We shall discuss some special examples in Sec. III.

II. TWO-PARAMETER QUANTUM GROUP AS A SYMMETRY GROUP

Let \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) be an element of a quantum group and let us assume that for some numbers, \( q_{ij} \)'s,

\[
\begin{align*}
  x A &= q_{11} A x, & y A &= q_{21} A y, \\
  x B &= q_{12} B x, & y B &= q_{22} B y, \\
  x C &= q_{13} C x, & y C &= q_{23} C y, \\
  x D &= q_{14} D x, & y D &= q_{24} D y.
\end{align*}
\] (7)
Also let us assume that, under the transformations in Eqs. (3) and (4), the relation \( x y = q y x \) is transformed, respectively, as

\[
x' y' = \bar{q} y' x'
\]

and

\[
x'' y'' = \bar{\bar{q}} y'' x''.
\]

Then, we have

\[
\left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in GL_{p,q}(2) \quad \text{for some nonzero } p, q \text{ with } pq' \neq -1,
\]

(2) \( \bar{q} = \bar{\bar{q}} \), and

(3) \[
q_{11} = 1, \quad q_{21} = q\bar{q}^{-1} q_{14} = qq'^{-1} k, \\
q_{12} = \bar{q} p^{-1}, \quad q_{22} = q\bar{q} p^{-1} (\bar{q} - (p - q'^{-1})k), \\
q_{13} = \bar{q} q'^{-1}, \quad q_{23} = q\bar{q} q'^{-1} (\bar{q} - (p - q'^{-1})k), \\
q_{14} = \bar{q} q'^{-1} k, \quad q_{24} = q\bar{q}^2 q'^{-1} p^{-1} (\bar{q} - (p - q'^{-1})k)
\]

where \( k \) is a complex number. In this section, we shall prove the above statement. The converse of the above statement is trivial. Also we note that if one requires that \( q_{ij} = 1 \), then \( \bar{q} = p = q' = q \) and \( \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in GL_q(2) \).

The proof is as follows: From the Eqs. (8) and (9), it follows that

\[
AC = q_{11} CA, \\
BD = q_{22} DB, \\
q q_{14} AD - \bar{q} q_{21} DA = q\bar{q} q_{12} CB - q_{23} BC,
\]

and
\[ AB = q_3 BA, \]
\[ CD = q_4 DC, \]
\[ q q_{14} AD - \tilde{q} q_{21} DA = q\tilde{q} q_{13} BC - q_{22} CB, \]

where \( q_1 = \tilde{q} q_{13}^{-1} q_{11}, q_2 = \tilde{q} q_{24}^{-1} q_{22}, q_3 = \tilde{q} q_{12}^{-1} q_{11}, \) and \( q_4 = \tilde{q} q_{24}^{-1} q_{23}. \)

We are now interested in those \( q_{ij} \)'s such that \( T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is an element of a quantum group. For the matrix \( T \) to be such a matrix, the entries \( A, B, C, \) and \( D \) should be consistent with the costructures of the Hopf algebra. We note that the comultiplication \( \Delta \) and the antipode \( S, \) among others, satisfy the following relations:

\[ \Delta \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]
\[ = \begin{pmatrix} A \otimes A + B \otimes C & A \otimes B + B \otimes D \\ C \otimes A + D \otimes C & C \otimes B + D \otimes D \end{pmatrix} \]  \hspace{1cm} (13)

and

\[ S \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}. \]  \hspace{1cm} (14)

From the consistent conditions \( \Delta(AC) = q_1 \Delta(CA) \) and \( \Delta(BD) = q_2 \Delta(DB), \) we can have \( q_1 = q_2 \equiv q' \) and

\[ AD - DA = q' CB - q'^{-1} BC. \]  \hspace{1cm} (15)

Also from the conditions \( \Delta(AB) = q_3 \Delta(BA) \) and \( \Delta(CD) = q_4 \Delta(DC), \) it follows that \( q_3 = q_4 \equiv p \) and

\[ AD - DA = p BC - p^{-1} CB. \]  \hspace{1cm} (16)

From Eqs. (13) and (16), it follows that
\[ p B C = q' C B, \]  

unless \( pq' = -1 \). Thus, we construct a two-parameter deformation of \( GL(2) \):

\[
egin{align*}
AB &= p BA, & CD &= p DC, \\
AC &= q' CA, & BD &= q' DB, \\
p BC &= q' CB, & AD - DA &= (p - \frac{1}{q'}) BC.
\end{align*}
\]

Hence \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{p,q'} \).

Next, Eq. (14) implies the existence of the inverse matrix \( T^{-1} \). From the ansatz

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D & \beta B \\ \gamma C & \alpha A \end{pmatrix} D^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

we can find \( \alpha = 1 \), \( \beta = -p^{-1} \), \( \gamma = -p \), and \( D = AD - p BC = DA - p^{-1} CB \), which is consistent with Eq. (16).

The quantum determinant \( D \) satisfies

\[
\begin{align*}
AD &= DA, & BD &= p^{-1} q' DB, \\
CD &= pq^{-1} DC, & DD &= DD.
\end{align*}
\]

This gives us

\[
\begin{align*}
D^{-1} A &= AD^{-1}, \\
D^{-1} B &= q' p^{-1} BD^{-1}, \\
D^{-1} C &= pq'^{-1} CD^{-1}, \\
D^{-1} D &= DD^{-1},
\end{align*}
\]

which is consistent with the requirement

\[
\begin{pmatrix} D & -\frac{1}{p} B \\ -p C & A \end{pmatrix} D^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

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The result in Eq. (21) is the same as the one in Ref 12.

Furthermore, the third equations in Eqs. (11) and (12) also should be identical to Eq. (16). If $qq_{14} \neq \bar{q}q_{21}$, the third eq. in Eq. (11) is $qq_{14}AD - \bar{q}q_{21}DA = \xi BC$ where $\xi = q\bar{q}_{12}q'_{-1}p - q_{23}$. In the case when $\xi = 0$, $qq_{14}AD = \bar{q}q_{21}DA$ which is of the form $AD = \epsilon DA$ with $\epsilon \neq 1$. However, the relation $\Delta(AD) = \epsilon\Delta(DA)$ leads us to $\epsilon = 1$, which is a contradiction. When $\xi \neq 0$, we have two cases: $p \neq q'_{-1}$ and $p = q'_{-1}$. The first case together with Eq. (14) gives $(qq_{14}(p - q'_{-1}) - \xi)AD = (\bar{q}q_{21}(p - q'_{-1}) - \xi)DA$. In every possible case, this equation contradicts either the fact that $D = AD - pBC$ is invertible or that $\epsilon = 1$ from $\Delta(AD) = \epsilon\Delta(DA)$ as in the above. In the second case when $p = q'_{-1}$, $AD = DA = \delta BC$ for some number $\delta$. However, from the relation $\Delta(AD) = \delta\Delta(BC)$, $\delta = p$, which is a contradiction to the existence of $D$. Thus, we conclude that $qq_{14} = \bar{q}q_{21}$.

The equation $qq_{14} = \bar{q}q_{21}$ follows from the third equation in Eq. (12) by a completely analogous method. Hence, we have

$$\bar{q} = \bar{q}.$$ (23)

Now let us summarize the relations between the $q_{ij}$’s:

$$q' = \bar{q}q_{12}^{-1}q_{11} = \bar{q}q_{24}^{-1}q_{22},$$
$$p = \bar{q}q_{12}^{-1}q_{11} = \bar{q}q_{24}^{-1}q_{23},$$
$$q q_{14} = \bar{q}q_{21},$$ (24)
$$p - q'_{-1} = \bar{q}q_{14}^{-1}q_{13} - q^{-1}q'_{-1}p q_{14}^{-1}q_{22}.$$}

The relation between $q, \bar{q},$ and $q'$ depends on the choice of $q_{ij}$’s. There may be (infinitely) many choices for $q_{ij}$’s consistent with the theory of quantum groups. In effect, there are two unknowns since there are six independent relationships between them, as can be seen in Eq. (24). Without loss of generality, we may assume that $q_{14} = kq_{13}$ for some number $k$. Then we can express all of the $q_{ij}$’s in one unknown $q_{11}$, which may be regarded as a proportional constant. Hence, if we put $q_{11} = 1$ for simplicity, we have
\[ q_{11} = 1, \quad q_{21} = qq^{-1}q_{14} = qq^{-1}k, \]
\[ q_{12} = \bar{q}p^{-1}, \quad q_{22} = q\bar{q}p^{-1}(\bar{q} - (p - q^{-1})k), \]
\[ q_{13} = \bar{q}q'^{-1}, \quad q_{23} = q\bar{q}q'^{-1}(\bar{q} - (p - q'^{-1})k), \]
\[ q_{14} = \bar{q}q'^{-1}k, \quad q_{24} = q\bar{q}q'^{-1}p^{-1}(\bar{q} - (p - q'^{-1})k) \]

where \( k \) is the only parameter to be determined. Thus, we prove the statement. As seen in the above, the choice \( q_{11} = 1 \) is arbitrary. In other words, the assumption that the generators of a one-parameter quantum group commute with the coordinates of the quantum plane is very special. They do not commute generically.

From Eq. (18), it is obvious that
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{p,q'}(2) \text{ if and only if } \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in GL_{q',p}(2).
\]
On the other hand, \( GL_{p,q'}(2) = GL_{q',p}(2) \) in the sense that \( GL_{p,q'}(2) \) and \( GL_{q',p}(2) \) are the algebras freely generated by \( A, B, C, D \), and \( D^{-1} \) modulo the relations given by Eqs. (18) and (21) and by the equations \((AB - pBC)D^{-1} - 1 \) and \( D^{-1}(AB - pBC) - 1 \). Thus, Manin’s viewpoint that quantum groups are symmetry groups of quantum planes is recovered as in the one-parameter case under the commutation relation in Eq. (7) with \( q_{ij} \)'s given by Eq. (25) between quantum group generators and noncommuting coordinates.

**III. QUANTUM PLANES ASSOCIATED WITH A QUANTUM GROUP**

In this section, we shall discuss several interesting choices of \( q_{ij} \)'s. The diversity of the choices of \( q_{ij} \)'s means the diversity of quantum planes for a given quantum group.

**Case I: \( \bar{q} = q \)**

This case corresponds to the standard way of dealing with quantum planes. Then,
\[ q_{11} = 1, \quad q_{21} = q_{14} = qq^{-1}k, \]
\begin{align}
q_{12} &= q p^{-1}, & q_{22} &= q^2 p^{-1} (q - (p - q'^{-1})k), \\
q_{13} &= q q'^{-1}, & q_{23} &= q^2 q'^{-1} (q - (p - q'^{-1})k), \\
q_{14} &= q q'^{-1} k, & q_{24} &= q^3 q'^{-1} p^{-1} (q - (p - q'^{-1})k).
\end{align}

Now we introduce the exterior differential \( d \) as in Ref 15 and 16 except for the following:

\begin{align}
(dx) A &= q_{11} A dx, \\
(dy) A &= q_{21} A dy, \\
(dx) B &= q_{12} B dx, \\
(dy) B &= q_{22} B dy, \\
(dx) C &= q_{13} C dx, \\
(dy) C &= q_{23} C dy, \\
(dx) D &= q_{14} D dx, \\
(dy) D &= q_{24} D dy
\end{align}

where \( q_{ij} \)'s satisfy Eq. (26).

Now if we require that \( dx dy = -\frac{1}{p} dy dx \) is preserved under the transformation \( T \), it is easy to see that \( k = \frac{q'(qp-1)}{p(q'p-1)} \). Thus, we have, with \( q_{11} = 1 \),

\begin{align}
q_{12} &= q p^{-1}, & q_{21} = q_{14} &= \frac{q(qp-1)}{p(q'p-1)}, \\
q_{13} &= q q'^{-1}, & q_{22} &= q^2 p^{-2}, \\
q_{14} &= \frac{q(qp-1)}{p(q'p-1)}, & q_{23} &= q^2 q'^{-1} p^{-1}, \\
q_{24} &= q^3 q'^{-1} p^{-2}.
\end{align}

Now we may go further. In fact, it is natural to require that the two-parameter case become the one-parameter case in some limit. Therefore, if \( q_{ij} \longrightarrow 1 \) as \( p \longrightarrow q' \), then we must set \( q' = q \). Hence, Eq. (18) is the same as Eq. (3), and Eq. (28) becomes

\begin{align}
q_{11} = q_{13} = 1, \\
q_{12} = q_{14} = q_{21} = q_{23} = q p^{-1}, \\
q_{22} = q_{24} = q^2 p^{-2}.
\end{align}

The virtue of this formulation is that the relations for the differentials on a quantum plane are preserved not only under \( T \) but also under \( T^t \). According to Ref 16, one can define
the differential calculus on a quantum plane in the one-parameter case: For an exterior
differential \( d \) which is linear and satisfies \( d^2 = 0 \) and the Leibniz rule, one can choose

\[
\begin{align*}
    dx \, dy &= -\frac{1}{q} \, dy \, dx, \\
    x \, dx &= q^2 \, (dx) \, x, \\
    x \, dy &= q \, (dy) \, x + (q^2 - 1) \, (dx) \, y, \\
    y \, dx &= q \, (dx) \, y, \\
    y \, dy &= q^2 \, (dy) \, y .
\end{align*}
\]

(30)

Also, by the same method as in the one-parameter case, we obtain the following relations
for the differentials in the two-parameter case:

\[
\begin{align*}
    dx \, dy &= -\frac{1}{p} \, dy \, dx, \\
    x \, dx &= p \, q \, (dx) \, x, \\
    x \, dy &= q \, (dy) \, x + (p \, q - 1) \, (dx) \, y, \\
    y \, dx &= p \, (dx) \, y, \\
    y \, dy &= p \, q \, (dy) \, y .
\end{align*}
\]

(31)

We note that Eq. (30) is invariant under the transformations \( T \) and \( T' \). Eq. (31) is also
invariant under the transformation \( T \), but it is easy to see that it is not invariant under
the transformation \( T' \) if the quantum group generators \( A, \, B, \, C, \) and \( D \) commute with
the noncommuting coordinates \( x, \, y \). However, if we choose the \( q_{ij} \)'s as in Eq. (29), then a
lengthy but straightforward calculation shows the nice property that Eq. (31) is invariant
not only under the transformation \( T \) but also under the transformation \( T' \). Moreover, we
have \( dx' \, dx' = dy' \, dy' = 0 \) and \( dx'' \, dx'' = dy'' \, dy'' = 0 .\)

**Case II:** \( p = q' \)
Let \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{q'}(2) \). If we put \( k = q_{12} \) (In effect, this choice of \( k \) gives the same equation, Eq. \((29)\), as case I above), then

\[
\begin{align*}
q_{11} &= 1, \\
q_{12} &= q_{1}p^{-1}, \\
q_{13} &= q_{1}q^{-1}, \\
q_{14} &= q^{2}q^{-1}p^{-1}, \\
q_{21} &= qq^{-1}p^{-1}, \\
q_{22} &= qq^{2}q^{-1}p^{-2}, \\
q_{23} &= qq^{2}q^{-2}p^{-1}, \\
q_{24} &= qq^{3}q^{-2}p^{-2}.
\end{align*}
\]

(32)

In order to see interesting aspects of quantum planes, it is enough only to consider the one-parameter case. Thus, if put \( p = q' \),

\[
\begin{align*}
q_{11} &= 1, \\
q_{12} &= q_{1}q'^{-1}, \\
q_{13} &= q_{1}q'^{-1}, \\
q_{14} &= q^{2}q'^{-1}p^{-1}, \\
q_{21} &= qq'^{-2}, \\
q_{22} &= qq^{2}q'^{-3}, \\
q_{23} &= qq^{2}q'^{-3}, \\
q_{24} &= qq^{3}q'^{-4}.
\end{align*}
\]

(33)

The quantum plane such that \( xy = qyx \) corresponding to these values of the \( q_{ij} \)'s is transformed into \( x'y' = qy'x' \) and \( x''y'' = qy''x'' \), respectively, under the action \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) and its transpose.

Now if we take a quantum plane for \( GL_{q'} \) such that \( q = 1 \) and \( \bar{q} = q' \), then

\[
\begin{align*}
q_{11} &= 1, \\
q_{2i} &= q'^{-1}, \\
q_{ij} &= 1.
\end{align*}
\]

(34)

for \( i = 1, \cdots, 4 \). This quantum plane is generated by \( x, y \) such that \( xy = yx \) and is transformed as \( x'y' = q'y'x' \). However, \( x', y' \) do not obey Eq. \((31)\). The case when \( q = \bar{q} = 1 \) is also interesting since the quantum plane looks like an ordinary plane in the sense that it is generated by commuting coordinates. If we take a quantum plane for \( GL_{q'} \) such that \( q = q' \) and \( \bar{q} = q' \), then \( q_{ij} = 1 \). This quantum plane is the original one \([11]\).
IV. CONCLUSIONS

In the one-parameter case, the condition that $xy = qyx$ is preserved under the transformation $T$ and its transpose $T^t$ gives the commutation relation between the generators of a quantum group $GL_q(2)$. Here, one assumes that the generators of a quantum group commute with the noncommuting coordinates of a quantum plane.

In this work, we have relaxed the assumption and investigated its consequences. We are naturally led to a two-parameter deformation of the group $GL(2)$ and its corresponding quantum planes even though we do not put any restrictions at the outset on the number of parameters. As a by-product, this formulation supports Manin’s viewpoint that quantum groups are symmetry groups of quantum planes, and the diversity of the choices of $q_{ij}$’s shows that there can be many quantum planes for a given quantum group $GL_{p,q}$. Associated with a given quantum group, there are some special quantum planes such as the original one in the literature. Especially, a quantum plane which looks like an ordinary plane attracts much attention and seems to be worthy of further research.

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