SYZYGIES OF SECANT IDEALS OF PLÜCKER-EMBEDDED GRASSMANNIANS
ARE GENERATED IN BOUNDED DEGREE

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Abstract. Over a field of characteristic 0, we prove that for each \( r \geq 0 \) there exists a constant \( C(r) \) so that the prime ideal of the \( r \)th secant variety of any Plücker-embedded Grassmannian \( \text{Gr}(d, n) \) is generated by polynomials of degree at most \( C(r) \), where \( C(r) \) is independent of \( d \) and \( n \). This bounded generation ultimately reduces to proving a poset is noetherian, we develop a new method to do this. We then translate the structure we develop to the language of functor categories to prove the \( r \)th syzygy module of the coordinate ring of the \( r \)th secant variety of any Plücker-embedded Grassmannian \( \text{Gr}(d, n) \) is concentrated in degrees bounded by a constant \( C(i, r) \), which is again independent of \( d \) and \( n \).

1. Introduction

Given a vector space \( V \) of dimension \( n \) over a field \( k \) of characteristic 0, recall that \( \text{Gr}(d, V) \) is the space that parametrizes all dimension \( d \) subspaces of \( V \) called the Grassmannian. We will omit the choice of \( V \) and just write \( \text{Gr}(d, n) \). A classical result in algebraic geometry realizes \( \text{Gr}(d, n) \) as a projective variety via the Plücker embedding. Specifically, we can define a map \( \text{Gr}(d, n) \to P(\wedge^d k^n) \) as follows. Given a \( d \)-dimensional subspace spanned by \( v_1, \ldots, v_d \) in \( \text{Gr}(d, n) \) we send

\[
\text{span}(v_1, \ldots, v_d) \mapsto v_1 \wedge \cdots \wedge v_d.
\]

This choice of basis is not unique, but when we apply a change of basis we scale the wedge product by the determinant and so this map is well defined on projective space. The \( r \)th secant variety of the Plücker embedding of \( \text{Gr}(d, n) \) denoted \( \text{Sec}_r(\rho(\text{Gr}(d, n))) \), is the Zariski closure in \( P(\wedge^d k^n) \) of the set of expressions \( \sum_{i=0}^r x_i \) where \( x_i \) is in the embedded Grassmannian. Our convention is that the zeroth secant variety is the original variety.

Secant varieties have long been a topic of interest in algebraic geometry. Despite this, very little is known about their algebraic structure. Many results about secant varieties focus on the dimension of the space or finding bounds on the degrees of set theoretic generators [DE, DK]. Ideal-theoretic generators are hard to find [MM, LM, LO] and accordingly are not well understood.

Specifically for the Plücker embedding, a good amount is known about the dimensions of these secant varieties [CGG, BDdG]. Some set-theoretic results are also known. For example, in [KPRS] the authors prove that all Plücker embeddings are generated set theoretically by pullbacks of the Klein quadric.

Recently, in [DE] the authors greatly expand the scope of [KPRS] to show that for any fixed \( r \), the \( r \)th secant variety of the Plücker-embedded \( \text{Gr}(d, n) \) is defined set theoretically by polynomials of bounded degree independent of \( d \) and \( n \). They pose a question at the end of their paper about whether the ideal-theoretic version of their theorem holds. Furthermore, they mention that the ideas present in their paper will not suffice to address the ideal-theoretic version.

The purpose of this paper is to answer this question in the affirmative in characteristic 0. We ultimately prove the following:

**Theorem 1.1.** Assume \( \text{char}(k) = 0 \). For each \( r \geq 0 \), there is a constant \( C(r) \) such that the prime ideal of the \( r \)th secant variety of the Plücker-embedded \( \text{Gr}(d, n) \), is generated by polynomials of degree \( \leq C(r) \), where \( C(r) \) does not depend on the choice of \( d \) or \( n \).

This theorem has an immediate corollary resulting from the proof techniques. Exact descriptions of \( *_g \) and \( \cdot \), can be found in §3.
Corollary 1.2. Assume char(k) = 0. For r ≥ 0, the equations for the rth secant variety of the Plücker embedding of any Gr(d, n) can be built out of finitely many equations f₁, ..., fₙ of degree bounded by C(r) under the operations ⋆ₙ and ⋆—one of the key preparations in Theorem 1.3.

The main idea in proving Theorem 1.1 is to combine all of the ideals of the Plücker-embedded Grassmannians into a Hopf ring PΣ which we define in §3. We then prove noetherianity results with respect to the additional structure on this ring.

Once we define the Hopf ring PΣ and show it is noetherian, it is natural to ask if all finitely generated modules over PΣ are noetherian. To address this question, we must abstract the structure we develop in proving Theorem 1.1 to the language of functor categories as seen in [Sa2, CEF, SS1]. After we transition to this language, we use the new tools available to develop a syzygy theory for Plücker embedded Grassmannians analogous to the Δ-modules seen in [Sa2] and the Veronese theory in [Sa2], in particular we prove the following:

Theorem 1.3. There is a function C(i, r), depending on i, r, but independent of d, n, such that the ith syzygy module of the coordinate ring of the rth secant variety of the Plücker-embedded Gr(d, n) is concentrated in degrees bounded by C(i, r).

This theorem ultimately encapsulates Theorem 1.1 when we take i = 1, but the structure we develop to prove Theorem 1.1 is crucial in proving Theorem 1.3.

1.1 Outline of Argument

The proof of Theorem 1.1 breaks into the following steps:

1. For fixed r ≥ 0, we reduce to considering Gr(d, (r + 2)d) as d varies. This will allow us to bound the degrees of the ideal generators for the rth secant varieties of any Gr(d, n). For this we use [MM, Proposition 5.7] as explained in §5.

2. We now consider all values of d via the space PΣ = ⊕n,dSym^n((d^d)k+(r+2)d), where k is a field of characteristic 0. If V is a (r + 2)d dimensional vector space over k, we know V ∼= k^{(r+2)d}, so it suffices to consider PΣ. As in [Sa1], we observe that there are two products on this space: the usual “external” product that multiplies outside symmetric powers and a new “internal” product that multiplies inside exterior powers up to an increasing change of index. We show that subspaces of this space which are ideals for both products are finitely generated. The key insight is that in an infinite antichain of monomials in this space, both n and d cannot be unbounded, this is seen in §2. The internal product involves symmetrizations and so we must assume that the field k has characteristic 0. This step is done in §3 with the key preparations in §2.

3. Finally, we notice that the two products are compatible with the standard comultiplication on the symmetric algebra PΣ. We can define secant varieties in terms of comultiplication. Using this structure, we prove the essential fact the ideal of the rth secant variety of the direct sum of all the Plücker ideals corresponding to Gr(d, (r + 2)d) as d varies, is an ideal in PΣ with respect to both products. So using the above, because the (d, n)-bigraded component of this ideal corresponds to all degree n polynomials in the rth secant variety of the Plücker embedding of Gr(d, (r + 2)d), we can deduce finite generation. This result is stated in §4 with most of the preparation and work done in §3.

In the last two sections, the proof of Theorem 1.3 breaks into the following steps:

1. We translate the structure of P_M from §2 to the language of functor categories, by developing a category G_M whose principal projective generated in degree (0, 0) corresponds exactly to P_M and whose other morphisms (d, m) → (e, n) encapsulate multiplication from the (d, m) bigraded piece of P_M to the (e, n) bigraded piece. We then use the results from §2 to show G_M is a Gröbner category as defined in [SS1]. The bulk of this is done in §6.

2. We then define a symmetrized version of G_M called G_M whose principal projective generated in degree (0, 0) corresponds to (PΣ)_M as seen in §3. At the end of §6, we use the fact that G_M is Gröbner to prove that every finitely generated G_M-module is noetherian.

3. With this structure we can study free resolutions of secant ideals of Plücker embedded Grassmannians. In §7, we find a particular free resolution using the principal projectives in G_M which allows us to ultimately deduce Theorem 1.3.
1.2 Relation to previous work

- Sec$_0(\rho(\text{Gr}(d, n)))$ is just the Plücker-embedded Grassmannian. It is well known that its ideal is generated by quadratic polynomials (the Plücker equations), so $C(0) = 2$. The case for $d = 2$ is well known, in particular the ideal of Sec$_r(\rho(\text{Gr}(2, n)))$ is generated in degree $r + 2$ by sub-Phaffians of size $2r + 4$ [LO, §10]. This implies a lower bound, $C(r) \geq r + 2$, but outside of this we know very little about $C(r)$.
- As mentioned the Veronese case was addressed in [Sa1]. We address the Plücker case in this paper. Snowden developed $\Delta$-modules in [Su] to prove a boundedness result about the syzygies of the Segre embeddings. The question still remains, are the ideals of the secant varieties of the Segre embeddings defined in bounded degree? Can these techniques be used to address the Segre case and ultimately prove results about the syzygies of secant varieties as well?
- Rather than look at all Plücker embeddings of Grassmannians, one can consider all Segre embeddings of products of projective spaces or Veronese embeddings of projective space. If a Segre analogue of these methods can be developed, could it also apply to Segre-Veronese embeddings?
- In general, computing the ideals of secant varieties is difficult. We refer the reader to [MM, LO, LW] for references concerning these explicit computations for some cases of the Segre, Veronese and Plücker embeddings.
- The idea for showing ideals in $\mathcal{P}_\Sigma$ are finitely generated was motivated mainly by work in [Sa1]. The underlying idea in most of this work is noetherianity up to symmetry. For a nice introduction we recommend [D]. Ultimately, one works with a space or object on which a group acts and proves finite generation up to the action of this group. This idea is essential in [SS1, SS2, SS3, NSS], where the authors explore various manifestations of this idea to prove finite generation results for various representations of categories and twisted commutative algebras. These ideas are also present in [CEF, Sn, DE, DK, Hi, HS, To] where they were used to prove more surprising stability theorems.

1.3 Conventions

For the most part, $k$ will denote a field of characteristic 0. In §2, this assumption is not necessary and so we let $k$ be any commutative noetherian ring, but the assumption is needed in the following sections. We always tensor over $k$.

We always denote by $\Sigma_n$ the symmetric group on $n$ letters, and we denote the set $\{1, \ldots, n\}$ by $[n]$.

Given a vector space $V$, $\bigwedge^d V$ denotes its $d$th exterior power. Similarly, $\text{Sym}^d V$ denotes the $d$th symmetric power and $\text{Sym}(V) = \bigoplus_{d \geq 0} \text{Sym}^d V$.

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2. Shuffle-Star Algebra

Fix a commutative noetherian ring $k$ (for our purposes taking $k$ to be a field suffices, but the general case has the same proof and could be useful in the future).

For a fixed $M \in \mathbb{Z}_{\geq 0}$, consider the following algebra:

$$\mathcal{P}_M = \bigoplus_{d,n} \bigwedge^d k^{Md} \otimes n.$$

In general, we will suppress the subscript $M$, only using it when the value of $M$ will affect the definition or result. A monomial of $\mathcal{P}$ is an element of the form $w_1 \otimes \cdots \otimes w_n$ where each $w_i \in \bigwedge^d k^{Md}$ and $w_i = \alpha(v_{j_1} \wedge \cdots \wedge v_{j_d})$ where $v_{j_i}$ come from the standard basis for $k^{Md}$ and $\alpha \in k$.

We define two multiplication structures on $\mathcal{P}$. The first is the same as the shuffle product in [Sa1], we recall it for sake of completeness. Pick a subset $\{i_1 < \cdots < i_n\}$ of $[n + m]$ and let $\{j_1 < \cdots < j_m\}$ be its complement; denote this pair of subsets by $\sigma$ and call it a split of $[n + m]$. A split defines a shuffle product

$$\bigwedge^d k^{Md} \otimes n_{\sigma} \bigwedge^d k^{Md} \otimes m \rightarrow \bigwedge^d k^{Md} \otimes (n+m),$$

$$\bigwedge^d k^{Md} \otimes n_{\sigma} \bigwedge^d k^{Md} \otimes m \rightarrow \bigwedge^d k^{Md} \otimes (n+m),$$

$$\bigwedge^d k^{Md} \otimes n_{\sigma} \bigwedge^d k^{Md} \otimes m \rightarrow \bigwedge^d k^{Md} \otimes (n+m),$$
where \((u_1 \otimes \cdots \otimes u_n) \cdot _\sigma (v_1 \otimes \cdots \otimes v_m) = w_1 \otimes \cdots \otimes w_{n+m}\) with \(w_{ik} = u_k\) and \(w_{jk} = v_k\). Whenever we write \(f \cdot _\sigma g\), we are implicitly assuming that \(\sigma\) is a split of the correct format, otherwise define it to be 0.

Notice there is an action of \(\Sigma_\infty\) on \(\mathcal{P}\) by permuting all possible indices. We recall that for \(f \in \mathcal{P}\), the width of \(f\), denoted \(w(f)\), is the smallest integer \(n\) such that for every \(\sigma \in \Sigma_\infty\) that fixes \(\{1, \ldots, n\}\), \(\sigma\) also fixes \(f\). Every element of \(\mathcal{P}\) must be a finite linear combination of monomials, so only finite many indexes can appear. This means every \(f \in \mathcal{P}\) has finite width, or equivalently that \(\mathcal{P}\) satisfies the finite width condition.

We recall the definition of the monoid of increasing functions:

\[
\text{Inc}(\mathbb{N}) = \{ \rho : \mathbb{N} \to \mathbb{N} \mid \forall a < b, \rho(a) < \rho(b) \}.
\]

Since \(\mathcal{P}\) carries an action of \(\Sigma_\infty\), there is a natural action of Inc(\(\mathbb{N}\)) on \(\mathcal{P}\) as follows. Fix \(f \in \mathcal{P}\), for any \(\sigma \in \Sigma_\infty\), \(\sigma f\) only depends on \(\sigma|_{w(f)}\) Considering \(\sigma\) as a function \(\mathbb{N} \to \mathbb{N}\). For any \(\rho \in \text{Inc}(\mathbb{N})\), there exists some \(\sigma \in \Sigma_\infty\) such that \(\rho|_{w(f)} = \sigma|_{w(f)}\), define \(\rho f = \sigma f\). The same argument presented in [DK, Pages 6-7] shows that this gives a well defined action of Inc(\(\mathbb{N}\)) on \(\mathcal{P}\).

We define a new product \(*_g\) where \(g \in \text{Inc}(\mathbb{N})\). For monomials we define,

\[
(\bigwedge k^{M(d)} \otimes n) *_g (\bigwedge k^{M(c)} \otimes n) \to (\bigwedge k^{M(e+d)} \otimes n).
\]

as follows. We first require \(g([Md]) \subseteq [Me + d]\), otherwise we define the product to be zero. Whenever we use this product, we will always implicitly assume that \(g\) satisfies this property. Suppose \(g([Md]) = \{\alpha_1, \ldots, \alpha_{Md}\} \subseteq [Me + d]\) with \(g(i) = \alpha_i\), also let \(\{\beta_1, \ldots, \beta_{Me}\} = [Me + d]\ \backslash g([Md])\). For ease of notation, we define \(g^e\) in Inc(\(\mathbb{N}\)) to be \(g^e(i) = \beta_i\), we call this the complement of \(g\). Then for monomials,

\[
\left(\left(\bigotimes v_{i_1} \wedge \cdots \wedge v_{i_d}\right) \otimes \cdots \otimes \left(\bigotimes v_{j_{M(d-1)}} \wedge \cdots \wedge v_{j_{M(d)}}\right)\right) \cdot _g \left(\left(\bigotimes v_{j_1} \wedge \cdots \wedge v_{j_{Md}}\right) \otimes \cdots \otimes \left(\bigotimes v_{k_{M(d-1)}} \wedge \cdots \wedge v_{k_{M(d)}}\right)\right)
\]

\[
\quad = g\left(\left(\bigotimes v_{i_1} \wedge \cdots \wedge v_{i_d}\right) \cdot g^e\left(\left(\bigotimes v_{j_1} \wedge \cdots \wedge v_{j_{Md}}\right) \otimes \cdots \otimes \left(\bigotimes v_{k_{M(d-1)}} \wedge \cdots \wedge v_{k_{M(d)}}\right)\right)\right)
\]

\[
\quad = \left(\left(\bigotimes v_{i_1} \wedge \cdots \wedge v_{i_d}\right) \cdot g\left(\left(\bigotimes v_{j_1} \wedge \cdots \wedge v_{j_{Md}}\right) \otimes \cdots \otimes \left(\bigotimes v_{k_{M(d-1)}} \wedge \cdots \wedge v_{k_{M(d)}}\right)\right)\right)
\]

Where we view the \(v_{i_1}\) and \(v_{j_1}\) as the standard basis vectors in \(k^{M(d+e)}\). For general \(f \in \mathcal{P}_{d,n}\), \(h \in \mathcal{P}_{e,n}\) and \(g \in E\) extend bilinearly. Extend to the rest of \(\mathcal{P}\) by declaring all other products to be 0. We first notice a few properties of these products. There is a modified associativity.

**Lemma 2.1.** Given \(f \in \mathcal{P}_{d,n}\), \(b \in \mathcal{P}_{d,m}\) and \(a \in \mathcal{P}_{e,n+m}\), and a split \(\sigma\) of \([n+m]\) and \(g \in E\), there exist \(p_1, \ldots, p_r \in \mathcal{P}_{e,n}\) and \(h_1, \ldots, h_r \in \mathcal{P}_{d+e,m}\) so that,

\[
(f \cdot _\sigma b) \cdot _g a = \sum_{i=1}^r h_i \cdot _\sigma (f \cdot g p_i).
\]

**Proof.** Both \(\cdot _\sigma\) and \(*_g\) are bilinear, so assume without loss of generality that both \(a\) and \(b\) are monomials. Write \(a = a_1 \otimes \cdots \otimes a_{n+m}\) and \(b = \beta_1 \otimes \cdots \otimes \beta_{M(d)}\). Suppose \(\sigma\) is the split \(\{i_1, \ldots, i_m\}, \{j_1, \ldots, j_{Md}\}\) of \([n+m]\). Furthermore, suppose \([M(d+e)]\ \backslash g([Md]) = \{\gamma_1, \ldots, \gamma_{Me}\}\), let \(\alpha'_i\) be the image of \(\alpha_i\) under identifying \(k^{Me}\) with the subspace of \(k^{M(e+e)}\) spanned by the standard basis vectors \(\{v_{\gamma_i}\}\). So if \(\alpha_i = w_{j_1} \wedge \cdots \wedge w_{j_n}\), with \(\{w_i\}\) the standard basis for \(k^{Me}\), then \(\alpha'_i = v_{\gamma_{j_1}} \wedge \cdots \wedge v_{\gamma_{j_n}}\).

Taking \(h_i = g(\beta_1) \wedge \alpha'_1 \otimes \cdots \otimes g(\beta_m) \wedge \alpha'_{in}\) and \(p_i = \alpha_{j_1} \otimes \cdots \otimes \alpha_{j_n}\), makes the identity valid.

We can use the two products to define an ideal in \(\mathcal{P}\) as follows,

**Definition 2.2.** A homogeneous subspace \(I \subset \mathcal{P}\) is an ideal if \(f \in I\) implies that \(f \cdot _\sigma g \in I\) and \(h \cdot _\sigma f \in I\) for all \(h \in \mathcal{P}\). A subset of elements of \(\mathcal{P}\) generates an ideal \(I\) if \(I\) is the smallest ideal that contains the subset.

With this new language, we get an immediate corollary of Lemma 2.1:

**Corollary 2.3.** If \(f_1, f_2, \ldots\) generate an ideal \(I\), then every element of \(I\) can be written as a sum of elements of the form \(h \cdot _\sigma (f_i \cdot _g a)\) where \(a, h \in \mathcal{P}\).
**reading list** (RL), $S_{d,n} = (S^1, \ldots, S^n)$. Where each $S^i$ is an increasing word of length $d$ created from the finite alphabet $[Md]$. In particular, $|S^i| = d$ with $i = 1, \ldots, n$. $S^i$ records the indices in tensor position $i$. The subscript indicates which bigraded piece of $P$ the monomial is in. Let $\text{RL}$ denote the set of reading lists.

**Example 2.4.** To get an idea of what this looks like we give some examples encoding in both directions. A basic example would be the monomial $v_1 \wedge v_2 \otimes v_1 \vee v_3$ corresponds to,

$$\{(1,2), (1,3)\}.$$ 

As a more complicated example consider,

$$\{(1,5,6), (1,2,4), (1,2,6)\}.$$ 

This corresponds to the monomial $(v_1 \wedge v_5 \wedge v_6) \otimes (v_1 \wedge v_2 \wedge v_4) \otimes (v_1 \wedge v_2 \wedge v_6)$. □

Suppose we have a monomial ideal $J$ of $P$ that is not finitely generated. Then there is an infinite list of monomials $m_1, m_2, \ldots$ such that the ideal generated by $m_1, \ldots, m_i$ does not contain $m_{i+1}$. This list translates to an infinite list of RLs that are incomparable under the following order. Given a RL $S_{d,n} = (S^1, \ldots, S^n)$, this corresponds uniquely to a monomial $v_{S^1} \otimes \cdots \otimes v_{S^n}$.

We say that $S_{d,n} \leq T_{e,m}$ if and only if $v_T \otimes \cdots \otimes v_T$ is in the ideal generated by $v_{S^1} \otimes \cdots \otimes v_{S^n}$. Call this the **monomial order** on RLs. Notice the following:

**Lemma 2.5.** For RLs, $S_{d,n} \leq T_{e,m}$ in the monomial order is equivalent to the existence of a map $f : S_{d,n} \rightarrow T_{e,m}$ with the following properties. If $S_{d,n} = (S^1, \ldots, S^n)$ and $T_{e,m} = (T^1, \ldots, T^m)$, $f = (f_1, \ldots, f_n)$ where $f_i : S_i \rightarrow T_{k_i}$ with the following properties:

1. $k_1 < k_2 < \cdots < k_n$.
2. $f_i(S_i \cap S_j) = f_j(S_i \cap S_j)$, so they agree on overlap.
3. Each $f_i \in \text{Inc}(\mathbb{N})$.

**Proof.** This follows easily from definition. The first property means we must map the tensor positions in order. The last property is necessary because we have $g \in \text{Inc}(\mathbb{N})$. □

We will use this equivalence often. We define a new relation on the set $\text{RL} \leq_{RL}$, where $S_{d,n} \leq_{RL} T_{e,m}$ if a map as described in Lemma 2.5 exists from $S_{d,n} \rightarrow T_{e,m}$. This is easily seen to be a partial order. The above discussion can be summarized as follows:

**Corollary 2.6.** An infinite list of monomials $m_1, m_2, \ldots$ in $P$ such that $m_{i+1}$ is not contained in the ideal generated by $m_1, \ldots, m_i$ induces an infinite chain of incomparable RLs with respect to $\leq_{RL}$.

Given this infinite antichain of RLs from Corollary 2.6 we will show that either $n$ or $d$ must be bounded.

**Lemma 2.7.** Given an infinite antichain of reading lists under $\leq_{RL}$,

$$S_{d_1,n_1}, S_{d_2,n_2}, \ldots$$

either the $n_i$ or the $d_i$ must be bounded.

**Proof.** Suppose this is not the case. Fix $S_{d_1,n_1}$ in this antichain, i.e. the first element. If both $d_j$ and $n_j$ are unbounded, we can assume $d_j > 2Md_1$. For any RL $T_{d_j,n_j} = (T^1, T^2, \ldots, T^n)$ with $d_j > 2Md_1$ in our antichain, there are $(d_j \choose M_{d_j})$ unique sub-lists of size $M_{d_1}$ in each list $T^i$ of size $d_j$ in $T_{d_j,n_j}$. There are a total of $(M_{d_j} \choose M_{d_1})$ possible lists of this size that could occur in any $T^i$. Hence if

$$n_j > n_1 (M_{d_j} \choose M_{d_1})$$

by the pigeon hole principal we will have at least $n_1$ tensor positions (lists) whose intersection has size greater than $M_{d_1}$.

To see this, notice if $n_j > n_1 (M_{d_j} \choose M_{d_1})$, for each list of size $d_j$, we have $(d_j \choose M_{d_1})$ different sub-lists of size $M_{d_1}$. If we have a hole for each list of size $d_j$, there are $(M_{d_j} \choose M_{d_1})$ holes. For each list of side $d_j$, we place $(d_j \choose M_{d_1})$ pigeons into distinct holes. The holes are distinct because there are no repeated numbers. Each of these pigeons
represents a tensor position that the list appears in. If there are more than \( n_1 \left( \frac{M_{d_j}}{M_d} \right) \) lists, this implies we place more than \( n_1 \left( \frac{M_{d_j}}{M_d} \right) \) pigeons in the \( \left( \frac{M_{d_j}}{M_d} \right) \) holes. The pigeon hole principal implies that there is one hole with at least \( n_1 \) distinct elements in it.

Hence there is some list of size \( Md_1 \) that occurs in at least \( n_1 \) different tensor positions. Such a list cannot occur twice in the same tensor position because tensor positions cannot contain repeated numbers. As a result, there are at least \( n_1 \) tensor positions whose intersection contains at least \( Md_1 \) numbers.

We will show this lower bound is independent of \( d_j \). In particular,

\[
n_1M^{Md_1/2}M_d^{d_1-1} > n_1 \left( \frac{M_{d_j}}{M_d} \right).
\]

This is because

\[
\left( \frac{M_{d_j}}{M_d} \right) = \frac{(Md_1)(Md_1-1)\ldots(Md_1-Md_d+1)}{(d_1)(d_1-1)\ldots(d_1-Md_1+1)} = \frac{M^{Md_1}(d_1-1)\ldots(Md_1-Md_1+1)}{(d_1)(d_1-1)\ldots(d_1-Md_1+1)}.
\]

If we pair each of these terms as

\[
d_j - \frac{Md_1-1}{M} < \frac{Md_1-1}{M}.
\]

This can easily be checked because

\[
\frac{d_j - \frac{a}{M}}{d_j - \frac{a+1}{M}} < d_j - \frac{a+1}{d_j - (a+1)},
\]

reduces to the inequality

\[
\frac{d_j - a}{M} < d_j.
\]

Which is clearly true so long as \( d_j \) is positive, which it is. We now claim,

\[
1 \leq \frac{d_j - \frac{Md_1-1}{d_j-1} - 2}{d_j - (Md_1-1)} < 2.
\]

To see this we clear denominators since \( d_j > 2Md_1 \) we know the denominator is nonzero. So this inequality is equivalent to

\[
d_j - Md_1 + 1 \leq d_j - \frac{Md_1}{M} + 1 < 2d_j - 2Md_1 + 2.
\]

The first inequality reduces to

\[
\frac{M-1}{M} \leq (M-1)d_1,
\]

which is clearly true. The second inequality reduces to

\[
(2M-1)d_1 - \frac{2Md_1-1}{M} < d_j.
\]

However we took \( d_j > 2Md_1 \), so this is also true. This implies that every term in the product is in the interval \([1, 2]\). Clearly every term of the form

\[
\frac{d_j - \frac{a}{M}}{d_j - \frac{a+1}{M}},
\]

is greater than 1 for \( a = 1, \ldots, (d_1 - 1) \) and every term is less than \( \frac{d_j - \frac{a}{M}}{d_j - \frac{a+1}{M}} \) which we showed is less than 2. Hence

\[
1 \leq \frac{(d_1)(d_1-1)\ldots(Md_1-1)}{(d_1)(d_1-1)\ldots(Md_1-Md_1+1)} < 2Md_1-1.
\]

Notice this bound applies regardless of \( d_j \) so long as \( d_j > 2Md_1 \). This implies

\[
\frac{M^{Md_1}(d_1)(d_1-1)\ldots(Md_1-Md_1+1)}{(d_1)(d_1-1)\ldots(d_1-Md_1+1)} \leq M^{Md_1}2Md_1-1.
\]

So if \( n_j > n_1M^{Md_1}2Md_1-1 \) and \( d_j > 2Md_1 \), we can find \( n_1 \) tensor positions whose common intersection has size at least \( Md_1 \). If \( n_j \) and \( d_j \) are unbounded for any sequence if we fix some \( d_1 \) and \( n_1 \), this will always occur.

Say the \( n_1 \) tensor positions we find are \( i_1, \ldots, i_{n_1} \) and \( T^{i_1} \cap \cdots \cap T^{i_{n_1}} \) contains the \( Md_1 \) elements \( j_1 < \cdots < jMd_1 \). Then we have a clear map \( S_{d_1,n_1} \rightarrow T_{d_1,n_1} \), where we map \( S^n \rightarrow T^{i_1} \) and send \( i \mapsto j_i \). This is an order preserving injection that satisfies all the properties of Lemma 2.5 and implies \( S_{d_1,n_1} \leq T_{d_1,n_1} \), which is a contradiction.

\[\square\]
Remark 2.9. We believe the idea in Lemma 2.7 could have further applications in showing various posets are noetherian. In particular, the idea is to fix a small element in any given antichain and assume that the size of the elements in this antichain grow arbitrarily. With this assumption we prove the resulting elements are forced to eventually contain a structure that resembles the fixed element. We then deduce that the size of the elements must be bounded in some sense. This often drastically simplifies the problem as we will see below.

This leaves us with two cases. Either $d_i$ or $n_i$ is bounded. We will show that both lead to a contradiction.

Lemma 2.10. Given an infinite antichain in $(RL, \leq_{RL})$, 

$$S_{d_1,n_1}, S_{d_2,n_2}, \ldots$$

$n_i$ cannot be bounded.

Proof. Assume $n_i$ is bounded. Then since our antichain is infinite, this implies that there must be some infinite subchain of our antichain with $n_i$ constant. Restrict our attention to this sub-antichain with $n_i = n$.

We will now embed each $S_{d_i,n}$ of this antichain as a labeled tree and derive a contradiction via Kruskal’s tree theorem.

Send each $S_{d_i,n}$ to the tree $T_{S_{d_i,n}}$ with a root vertex labeled $(0,0,(n+1,n+1,\ldots,n+1))$. Define the function $\psi(j, S_{d_i,n})$, which takes as an input some RL $S_{d_i,n}$ and some element $j \in [Md_i]$ and returns the finite list of size $n$ with a $k$ in position $k$ if $j$ appears in $S^k$ and a zero if $j$ does not appear in $S^k$. This encodes which of the tensor positions $j$ appears in.

There will be $n$ branches off of the root. Branch $j$ will have $Md_i$ vertices. Vertex $k$ of branch $j$ will be labeled by $(k,j,\psi(k, S_{d_i,n}))$. Order the first label with the standard order on $\mathbb{Z}_{\geq 0}$. Order the last two labels with the componentwise order. In this case the quasi-well-order will be equality. This product is a quasi-well-order by Dickson’s lemma because the alphabets for the last two labels are finite. Hence, each component is a quasi-well-order and Dickson’s lemma tells us that the finite product of quasi-well orders compared componentwise is also a quasi-well-order.

Notice that this is an injective mapping from RLs to trees because we can easily recover $S_{d_i,n}$ from $T_S$ by reading off vertices.

Furthermore, using the order described in Kruskal’s tree theorem if $T_S \leq T_W$, this implies that $S \leq W$. We must send $S^i$ to $W^i$ because of the second label. Also, for any $k$, $\psi(k, S_{d_i,n})$ is fixed and we have that $T_S \leq T_W$ if every vertex $v$ maps to some vertex $F(v)$ with $v \leq F(v)$. In combination with the first label, this implies that every number $k$ maps to some number $m \geq k$ with $\psi(m, S_{d_i,n}) = \psi(m, S_{d_i,n})$. As a result, in each of the branches where $k$ occurs, there is some number $m \geq k$ that it can map to because $m$ will occur in all of the remaining branches in which $k$ occurs.

Additionally, we must map branches to branches. Hence if a vertex $v$ with first label $k$ maps to a vertex $F(v)$ with first label $m_k$, we have $m_1 < m_2 < \cdots < m_{Md_i}$, so the map on indices is in the monoid of increasing functions.

Define a mapping inductively form $S \rightarrow W$. Begin by sending 1 to the minimal first label $m_1$ that occurs for all the vertices corresponding to a vertex with first label 1. That is, let $\{v_1,\ldots,v_l\}$ be all the vertices in the tree $T_S$ with first entry 1. Each $v_i$ has an image $F(v_i)$ in $T_W$. Out of all the vertices $\{F(v_1),\ldots,F(v_l)\}$, send 1 to the minimum first entry that occurs, call it $\alpha_1$. By construction if 1 occurs in any branch, so must $\alpha_1$. Hence if 1 occurs in $S^1$ it has a well defined image in the corresponding $W^j$. Put $f(1) = \alpha_1$.

Now repeat the same procedure for 2. The element we send 2 to cannot be $\alpha_1$ because even if 1 and 2 occur in all the same branches, any vertex corresponding to a 1 occurs earlier in the branch that the vertex corresponding to a 2 and we preserve this order. Hence if $\alpha_1$ is the minimal element for 2, this would imply $\alpha_1$ is not minimal for 1. Continue in this way until we have a mapping of all the numbers occurring in $S$.

By construction, two numbers could map to the same $\beta_j$ if and only if they occur in exactly the same branches. So $F$ being well defined implies that $f$ is well defined, i.e., that every element has an image.

Furthermore, $f$ is a map of RLs because if a vertex with first entry $j$ is mapped to another vertex with first entry $m$, $m$ must occur in all of the branches that $j$ does. Hence when we map $j$ to $m$, $j$ has an image in each restriction. The map also satisfies property (4) in Lemma 2.5 because we must stay within a branch and we can only map a number to another number that is greater than or equal to it. Finally, $S^j$ must map into $W^j$ because the third label on each vertex must be equal.
The contrapositive implies that our infinite antichain of RLS yields an infinite antichain of trees. This contradicts Kruskal’s tree theorem.

**Remark 2.11.** We use Kruskal’s tree theorem because it provides a more intuitive picture, but Higman’s lemma [D, Theorem 1.3] would suffice. We could encode each branch as an element of a quasi-well-ordered set and then view the trees as words of length \( n \) over this quasi-well-ordered set. The mapping of trees in this context is equivalent to one of these words being a subsequence of the other.

**Example 2.12.** To see this proof in action, consider the following example. Suppose \( n = 3 \) is fixed and \( M = 2 \). Given the two trees

\[
(1, 1, (1, 0, 3)) \rightarrow (2, 1, (1, 2, 0)) \rightarrow (3, 1, (0, 2, 0)) \rightarrow (4, 1, (0, 0, 3)) \quad (2.13)
\]

\[
(0, 0, (4, 4, 4)) \rightarrow (1, 2, (1, 0, 3)) \rightarrow (2, 2, (1, 2, 0)) \rightarrow (3, 2, (0, 2, 0)) \rightarrow (4, 2, (0, 0, 3))
\]

\[
(1, 3, (1, 0, 3)) \rightarrow (2, 3, (1, 2, 0)) \rightarrow (3, 3, (0, 2, 0)) \rightarrow (4, 3, (0, 0, 3))
\]

and

\[
(1, 1, (1, 2, 0)) \rightarrow (2, 1, (1, 0, 3)) \rightarrow (3, 1, (1, 2, 0)) \rightarrow (4, 1, (0, 2, 0)) \rightarrow (5, 1, (0, 0, 3)) \rightarrow (6, 1, (0, 0, 3))
\]

\[
(0, 0, (4, 4, 4)) \rightarrow (1, 2, (1, 2, 0)) \rightarrow (2, 2, (1, 0, 3)) \rightarrow (3, 2, (1, 2, 0)) \rightarrow (4, 2, (0, 2, 0)) \rightarrow (5, 2, (0, 0, 3)) \rightarrow (6, 2, (0, 0, 3))
\]

\[
(1, 3, (1, 2, 0)) \rightarrow (2, 3, (1, 0, 3)) \rightarrow (3, 3, (1, 2, 0)) \rightarrow (4, 3, (0, 2, 0)) \rightarrow (5, 3, (0, 0, 3)) \rightarrow (6, 3, (0, 0, 3))
\]

Call these \( T_1 \) and \( T_2 \). It is easy to read off their corresponding RLS. \( T_1 \) has RL \( S_1 = ((1, 2), (2, 3), (1, 4)) \) and \( T_2 \) has RL \( S_2 = ((1, 2, 3), (1, 3, 4), (2, 5, 6)) \). Notice \( T_1 \leq T_2 \) in the order described in Kruskal’s tree theorem where we map the vertices as below

\[
(1, 1, (1, 0, 3)) \mapsto (2, 1, (1, 0, 3)) \mapsto (2, 1, (1, 2, 0)) \mapsto (3, 1, (0, 2, 0)) \mapsto (3, 1, (1, 2, 0)) \mapsto (4, 1, (0, 0, 3)) \mapsto (4, 1, (0, 0, 3)) \mapsto (5, 1, (0, 0, 3)) \mapsto (5, 1, (0, 0, 3)) \mapsto (6, 1, (0, 0, 3)) \mapsto (6, 1, (0, 0, 3))
\]

\[
(0, 0, (4, 4, 4)) \mapsto (1, 2, (1, 2, 0)) \mapsto (2, 2, (1, 0, 3)) \mapsto (3, 2, (1, 0, 3)) \mapsto (3, 2, (1, 2, 0)) \mapsto (4, 2, (0, 2, 0)) \mapsto (4, 2, (0, 2, 0)) \mapsto (5, 2, (0, 0, 3)) \mapsto (5, 2, (0, 0, 3)) \mapsto (6, 2, (0, 0, 3)) \mapsto (6, 2, (0, 0, 3))
\]

\[
(1, 3, (1, 2, 0)) \mapsto (2, 3, (1, 0, 3)) \mapsto (3, 3, (0, 2, 0)) \mapsto (4, 3, (0, 0, 3)) \mapsto (4, 3, (0, 0, 3)) \mapsto (5, 3, (0, 0, 3)) \mapsto (5, 3, (0, 0, 3))
\]

There can be multiple embeddings, but we choose one of them. This gives us a map from \( S_1 \to S_2 \) inductively as described in the proof. We see that 1 only maps to a vertex with first label 2. So we let \( f(1) = 2 \). Now we see that 2 only maps to vertices with first label 3. So \( f(2) = 3 \). Continuing we have \( f(3) = 4 \). Now 4 maps to vertices with different first labels, the set of first labels is \( \{5, 6\} \). We then map 4 to the minimal such label that has not yet been used, so \( f(4) = 5 \).

This map \( f \) induces a map from each component of \( S_1 \) to each component of \( S_2 \), so we can easily define \( f_i : S_1 \to S_2 \) by restriction. Clearly this satisfies property (1) of Lemma 2.5. Furthermore, \( f \) satisfies properties (2) and (3) of Lemma 2.5 because we defined it via restriction and \( f \) satisfies property (4) by construction.

As described above this gives us

\[
[(v_1 \land v_2) \otimes (v_2 \land v_3) \otimes (v_1 \land v_4)] \ast f (v_1 \otimes v_1 \otimes v_2) = (v_2 \land v_3 \land v_1) \otimes (v_3 \land v_4 \land v_1) \otimes (v_2 \land v_5 \land v_6)
\]

Where \( f \) is the map found above.

**Lemma 2.10** implies that in our infinite antichain we must have \( d_i \) bounded. Furthermore, because this is an infinite chain and only finitely many \( d_i \) can occur, we can find an infinite subchain with a fixed \( d \).

This implies that for this \( d \), we have an infinite antichain in

\[
\bigoplus_{n \geq 0} \left( \bigwedge_k M^d \right) \otimes n
\]
However there are $\beta_d = \binom{M_d}{d}$ basis vectors for $\bigwedge^d k^{M_d}$, order them in some way as $\{e_1, \ldots, e_{\beta_d}\}$. Then every monomial in this algebra is a word in the $e_i$. As there are finitely many basis vectors, Higman’s lemma implies that for any infinite sequence of such words, two are comparable. Hence we cannot possibly have an infinite antichain.

This discussion proves the following,

**Lemma 2.15.** Given an infinite antichain in $(RL, \leq_{RL})$,

$S_{d_1,n_1}, S_{d_2,n_2}, \ldots$

d_i cannot be bounded.

The results above cumulatively show,

**Theorem 2.16.** The poset $(RL, \leq_{RL})$ is noetherian.

*Proof.* Suppose there were an infinite anti-chain of RLs. From Lemma 2.7 we know that either the $n$ or $d$ that appear in this list must be bounded otherwise it could not be an antichain. However this cannot be the case via Lemmas 2.15 and 2.10. So no such antichain exists.

An immediate consequence of this theorem is the following,

**Theorem 2.17.** All monomial ideals of $P$ are finitely generated.

*Proof.* From Corollary 2.6 if there is an infinite anti-chain of monomials, this leads to an infinite anti-chain of RLs under the order described in Lemma 2.5. This contradicts Theorem 2.16. Hence all monomial ideals of $P$ are finitely generated.

Now place a total ordering $\preceq$ on monomials (ignoring coefficients) of the same bidegree $(d, n)$ as follows. Encode each monomial by $(\mathbb{Z}^d)^n$ via the $d$th index in the $n$th tensor position. First, define $\preceq$ on $\mathbb{Z}^d$ using lexicographic ordering, i.e. $(a_1, \ldots, a_d) \preceq (b_1, \ldots, b_d)$ if the first non-zero element of $(b_1 - a_1, \ldots, b_d - a_d)$ is positive. Then compare tensors using lexicographic ordering. We only work with bihomogeneous elements, so it is not necessary to find a way to compare elements of different bidegrees.

**Lemma 2.18.** Let $m, m', n$ be monomials. For any $g \in \text{Inc}(N)$ and $\sigma$ a splitting,

1. If $m \preceq m'$, then $m \ast_g n \preceq m' \ast_g n$.
2. For any $n$, if $m \preceq m'$ then $n \cdot_\sigma m \preceq n \cdot_\sigma m$.

*Proof.* The proof of the first claim follows because $g \in \text{Inc}(N)$. The second follows because we never change any indices.

Given $f \in P_{d,n}$ we define $\text{init}(f)$ as the largest monomial along with its coefficient with respect to $\preceq$ that has a non-zero coefficient in $f$. Given an ideal $I$, let $\text{init}(I)$ be the $k$-span of $\{\text{init}(f) \mid f \in I \text{ homogeneous}\}$.

**Lemma 2.19.** If $I$ is an ideal, then $\text{init}(I)$ is a monomial ideal.

*Proof.* If $m \in \text{init}(I)$, we have $m = \text{init}(f)$ for $f \in I$. For any monomial $n \in P$, we have $n \cdot_\sigma m = \text{init}(n \cdot_\sigma f)$ by Lemma 2.18 part (2). Furthermore, we claim that $m \ast_g n = \text{init}(f \ast_g n)$. Using the bilinearity of $\ast_g$, the result follows from Lemma 2.18 part (1).

So $m \ast_g n \cdot_\sigma m \in \text{init}(I)$. To generalize to any $h \in P$, because $\ast_g$ is bilinear it suffices to work with monomials and the above implies the desired result. Hence, $\text{init}(I)$ is a monomial ideal.

**Lemma 2.20.** If $I \subset J$ are ideals and $\text{init}(I) = \text{init}(J)$, then $I = J$. In particular, if $f_1, f_2, \ldots \in J$ and $\text{init}(f_1), \text{init}(f_2), \ldots$ generate $\text{init}(J)$ then $f_1, f_2, \ldots$ generate $J$.

*Proof.* Suppose $I$ does not equal $J$. Pick $f \in J \setminus I$ where $\text{init}(f)$ is non-minimal with respect to $\preceq$. Then because $\text{init}(I) = \text{init}(J)$, we have $\text{init}(f) = \text{init}(f')$ for some $f' \in I$. But $\text{init}(f - f')$ is strictly smaller than $\text{init}(f)$ and $f - f' \in J \setminus I$. This is a contradiction.

For the other statement, let $I$ be the ideal generated by $f_1, f_2, \ldots$.

**Corollary 2.21.** Every ideal of $P$ is finitely generated.

*Proof.* Combine Corollary 2.17, Lemma 2.19 and Lemma 2.20.
3. Symmetrizing

What we are really interested in is $\bigoplus_{d,n \geq 0} \text{Sym}^n(\wedge^d k^{Md})$ for fixed $M$. So we must symmetrize. Assume $k$ is a field of characteristic 0. Much of this section is translating the structure of [Sa1, §3] to suit $P$. Define

$$P^\Sigma_M = \bigoplus_{n,d \geq 0} \left( \bigotimes_{i=1}^d (k^{Md})^{\otimes n} \right)_{\Sigma_n},$$

$$(P^\Sigma)_M = \bigoplus_{n,d \geq 0} \left( \bigotimes_{i=1}^d (k^{Md})^{\otimes n} \right)_{\Sigma_n},$$

where $\Sigma_n$ acts by permuting the tensor factors, and the superscript and subscript respectively denote taking invariants and coinvariants. We generally suppress the additional $M$ subscript and when it matters explicitly mention which $M$ we are working with, for ease of notation writing just $P^\Sigma$ or $P^\Sigma_M$.

Our internal product $\ast_g$ respects the structure of $P^\Sigma$ and so $P^\Sigma$ is a subalgebra of $P$ with respect to it. Unfortunately the shuffle product does not respect the symmetric invariance and so $P^\Sigma$ is not closed under it. We remedy this by defining

$$f \cdot g = \sum_{\sigma} f \cdot_{\sigma} g.$$

Averaging over all splits produces symmetric elements and so $P^\Sigma$ is naturally closed under this new $\cdot$. Additionally, $\cdot$ is both commutative and associative. Both of these algebras are naturally bigraded by $(d, n)$. We denote these bigraded pieces by $P^\Sigma_{d,n}$ and $(P^\Sigma)_d,n$. We will only consider bi-homogeneous subspaces of $P^\Sigma$ and $(P^\Sigma)_d,n$.

For each $d, n$ define a linear projection

$$\pi : P_{d,n} \to P^\Sigma_{d,n},$$

$$w_1 \otimes \cdots \otimes w_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}.$$

If $f \in P^\Sigma_{d,n}$, we have $\pi(f) = f$, so $\pi$ is surjective. We let $\pi' = n! \pi$. We also denote the direct sum of these maps by $\pi : P \to P^\Sigma$ and $\pi' : P \to P^\Sigma$. Next, define

$$\Theta : (P^\Sigma)_{d,n} \to P^\Sigma_{d,n},$$

$$w_1 \cdots w_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}.$$

Then $\Theta$ is a linear isomorphism, since $\Theta$ is the inverse of the composition

$$P^\Sigma_{d,n} \to P_{d,n} \to (P^\Sigma)_{d,n},$$

where the first map is the natural injection and the second is the natural projection. Denote the direct sum of these maps by $\Theta : P^\Sigma \to P^\Sigma$.

3.1 Properties of $P^\Sigma$

Lemma 3.1.1. (1) If $f \in P^\Sigma_{d,n}$ and $h \in P_{c,n}$ are homogeneous, then $\pi(f \ast_g h) = f \ast_g \pi(h)$ and $\pi(h \ast_g f) = \pi(h) \ast_g f$.

(2) If $f \in P_{d,n}$ and $g \in P_{d,m}$, then $\binom{n+m}{n} \pi(f \cdot_g h) = \pi(f) \cdot \pi(g)$ for any split of $[n+m]$.

Proof. (1) Both $\pi(f \ast_g h)$ and $f \ast_g \pi(h)$ are bilinear in $h$ and $f$, so without loss we assume $h = h_1 \otimes \cdots \otimes h_n$ for $h_i \in \wedge^c k^{Me}$ and that $f = \sum_{\sigma \in \Sigma_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}$ for $f_i \in \wedge^d k^{Md}$. Then

$$f \ast_g \pi(h) = \frac{1}{n!} \sum_{\sigma,\tau \in \Sigma_n} g f_{\sigma(1)} \wedge g^c h_{\tau(1)} \otimes \cdots \otimes g f_{\sigma(n)} \wedge g^c h_{\tau(n)}$$

$$\pi(f \ast_g h) = \frac{1}{n!} \sum_{\sigma,\tau \in \Sigma_n} g f_{\sigma(1)} \wedge g^c h_{\tau(1)} \otimes \cdots \otimes g f_{\sigma(n)} \wedge g^c h_{\tau(n)}.$$

These sums are identical. In the second, perform the change of variables $\sigma \mapsto \sigma^{-1}$. This shows $\pi(f \ast_g h) = f \ast_g \pi(h)$. 

We can apply similar reasoning to the second claim by considering
\[
\pi(h) * g f = \frac{1}{n!} \sum_{\sigma, \tau \in \Sigma_n} gh_{\sigma(1)} \wedge g^\sigma f_{\tau(1)} \otimes \cdots \otimes gh_{\sigma(n)} \wedge g^\sigma f_{\tau(n)}
\]
\[
\pi(h * g f) = \frac{1}{n!} \sum_{\sigma, \tau \in \Sigma_n} gh_{\sigma(1)} \wedge g^\sigma f_{\tau(1)} \otimes \cdots \otimes gh_{\sigma(n)} \wedge g^\sigma f_{\tau(n)}.
\]

(2) This is the same proof as in [Sa1, Lemma 3.1] because it is the same product. \qed

**Definition 3.1.2.** A homogeneous subspace \( I \subseteq \mathcal{P}^\Sigma \) is an ideal if \( f \in I \) implies that \( f \cdot h \in I \) and \( f \ast_g h \in I \) for all \( h \in \mathcal{P}^\sigma \) and \( g \in \text{Inc}(\mathcal{N}) \).

We now pass the noetherianity of \( \mathcal{P} \) to \( \mathcal{P}^\Sigma \).

**Proposition 3.1.3.** Every ideal of \( \mathcal{P}^\Sigma \) is finitely generated.

**Proof.** Let \( J \) be an ideal of \( \mathcal{P}^\Sigma \). As \( \mathcal{P}^\Sigma \) naturally lies inside \( \mathcal{P} \), we can consider the ideal in \( \mathcal{P} \) generated by \( J \), call it \( I \). By Corollary 2.21, \( I \) is finitely generated, say by \( f_1, \ldots, f_N \). As \( J \) generates \( I \), we may assume that each of the \( f_i \) belong to \( J \). We now claim that the \( f_i \) also generate \( J \) as an ideal in \( \mathcal{P}^\Sigma \). By Corollary 2.3, every \( f \in J \) can be written as a sum of terms of the form \( h \ast g (f \ast_g a) \) where \( h, a \in \mathcal{P} \), \( g \in \text{Inc}(\mathcal{N}) \). By Lemma 3.1.1 we have
\[
\pi(h \ast g (f \ast_g a)) = \left(\frac{n + m}{n}\right)^{-1} \pi(h) \cdot (f \ast_g \pi(a)),
\]
here \( h \in \mathcal{P}_{d, n} \) and \( f \ast_g a \in \mathcal{P}_{d, m} \). By the surjectivity of \( \pi \), we conclude that every element of \( J \) can be written as a sum of terms of the form \( h' \ast (f \ast_g a') \) where \( h', a' \in \mathcal{P}^\Sigma \) and \( g \in \text{Inc}(\mathcal{N}) \). \qed

For fixed \( d, \bigoplus_n \mathcal{P}_{d, n}^\Sigma \) is a free divided power algebra under \( \cdot \), and hence is freely generated in degree \( n = 1 \). So we can define a comultiplication \( \Delta: \mathcal{P}^\Sigma \to \mathcal{P}^\Sigma \otimes \mathcal{P}^\Sigma \) by \( w \mapsto 1 \otimes w + w \otimes 1 \) when \( w \in \mathcal{P}_{d, 1}^\Sigma \) and requiring that it is an algebra homomorphism for \( \cdot \).

We ultimately wish to show that the two products \( \cdot \) and \( \ast_g \) respect this comultiplication structure. To do this, we first extend the products to \( \mathcal{P}^\Sigma \otimes \mathcal{P}^\Sigma \) componentwise.

**Lemma 3.1.4.** Pick \( x \in \mathcal{P}_{d, n}^\Sigma \) and \( y \in \mathcal{P}_{e, m}^\Sigma \), and \( v \in \mathcal{P}_{e, n}^\Sigma \). Then
\[
\Delta(y \cdot v) = \Delta(y) \cdot \Delta(v)
\]
\[
\Delta(x \ast_g v) = \Delta(x) \ast_g \Delta(v)
\]

**Proof.** The first identity follows from the definition of \( \Delta \).

For the second identity, we once again notice \( \Delta(x \ast_g v) \) and \( \Delta(x) \ast_g \Delta(v) \) are both bilinear in \( x \) and \( v \). As a result, we may assume
\[
x = \sum_{\sigma \in \Sigma_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},
\]
for some \( x_1 \otimes \cdots \otimes x_n \in \mathcal{P}_{d, n} \) and that
\[
v = \sum_{\sigma \in \Sigma_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)},
\]
for some \( v_1 \otimes \cdots \otimes v_n \in \mathcal{P}_{e, n} \). Then by definition \( v = v_1 \cdots v_n \) and \( x = x_1 \cdots x_n \), here we are using the \( \cdot \) product. We defined \( \Delta \) so that it is an algebra homomorphism for \( \cdot \), so we have
\[
\Delta(v) = \Delta(v_1) \cdots \Delta(v_n) = \sum_{S \subseteq [n]} \pi'(v_S) \otimes \pi'(v_{[n] \setminus S})
\]
\[
\Delta(x) = \Delta(x_1) \cdots \Delta(x_n) = \sum_{S \subseteq [n]} \pi'(x_S) \otimes \pi'(x_{[n] \setminus S})
\]
where the sum is over all subsets $S = \{s_1 < \cdots < s_j\}$ of $[n]$ and $v_S = v_{s_1} \otimes \cdots \otimes v_{s_j}$ and $x_S = x_{s_1} \otimes \cdots \otimes x_{s_j}$.

This gives

\[
\Delta(x) \ast_g \Delta(v) = \left( \sum_{T \subseteq [n]} \pi'(x_T) \otimes \pi'(x_{[n]\setminus T}) \right) \ast_g \left( \sum_{S \subseteq [n]} \pi'(v_S) \otimes \pi'(v_{[n]\setminus S}) \right) \\
= \sum_{S,T \subseteq [n]} \pi'(x_T \ast_g \pi'(v_S)) \otimes \pi'(x_{[n]\setminus T} \ast_g \pi'(v_{[n]\setminus S})). \quad (3.1.5)
\]

Where in the second equality we use Lemma 3.1.1(1) to write $\pi'(x_T) \ast_g \pi'(v_S) = \pi'(x_T \ast_g \pi'(v_S))$. On the other hand,

\[
x \ast_g v = \sum_{\sigma,\tau \in \Sigma_n} gx_{\sigma(1)} \land g^\tau v_{\tau(1)} \otimes \cdots \otimes gx_{\sigma(n)} \land g^\tau v_{\tau(n)} \\
= \sum_{\sigma \in \Sigma_n} (gx_{\sigma(1)} \land g^\tau v_1) \cdots (gx_{\sigma(n)} \land g^\tau v_n),
\]

where in the second sum we are using the $\cdot$ product. In particular,

\[
\Delta(x \ast_g v) = \sum_{\sigma \in \Sigma_n} \sum_{S \subseteq [n]} \pi'(gx \land g^\tau v)_{\sigma,S} \otimes \pi'(gx \land g^\tau v)_{\sigma,[n]\setminus S} \quad (3.1.6)
\]

Where $(gx \land g^\tau v)_{\sigma,S} = gx_{\sigma(s_1)} \land g^\tau v_{s_1} \otimes \cdots \otimes gx_{\sigma(s_j)} \land g^\tau v_{s_j}$ if $S = \{s_1 < \cdots < s_j\}$. We also write $[n] \setminus S = \{s_{j+1} < \cdots < s_n\}$.

The same proof as in [Sa1, Lemma 3.4] shows that the expressions (3.1.5) and (3.1.6) for $\Delta(x) \ast_g \Delta(v)$ and $\Delta(x \ast_g v)$ are equal.

We can now present a symmetrized version of Lemma 2.1:

**Lemma 3.1.7.** Given $f \in \mathcal{P}_{d,n}^\Sigma$, $b \in \mathcal{P}_{d,m}^\Sigma$, and $a \in \mathcal{P}_{e,n+m}^\Sigma$, we have

\[
(f \otimes b) \ast_g a = (f \otimes b) \ast_g \Delta(a)
\]

**Proof.** The proof is essentially identical to the proof of Lemma 2.1. Assume that

\[
a = \sum_{\rho \in \Sigma_{n+m}} \alpha_{\rho(1)} \otimes \cdots \otimes \alpha_{\rho(n+m)},
\]

and

\[
b = \sum_{\tau \in \Sigma_m} \beta_{\tau(1)} \otimes \cdots \otimes \beta_{\tau(m)}.
\]

Write $a^\rho = \alpha_{\rho(1)} \otimes \cdots \otimes \alpha_{\rho(n+m)}$ and $\beta^\tau = \beta_{\tau(1)} \otimes \cdots \beta_{\tau(m)}$. Pick a split $\sigma = \{i_1, \ldots, i_n\}, \{j_1, \ldots, j_m\}$ of $[n+m]$. Then

\[
(f \cdot_{\sigma} b^\tau) \ast_g a^\rho = (f \ast_g (\alpha_{\rho(i_1)} \otimes \cdots \alpha_{\rho(i_n)})) \cdot_{\sigma} ((\beta_{\tau(1)} \otimes \cdots \beta_{\tau(m)}) \ast_g (\alpha_{\rho(j_1)} \otimes \cdots \alpha_{\rho(j_m)})),
\]

and $(f \otimes b) \ast_g a$ is the sum of these over all choices of $\rho, \tau, \sigma$. This is exactly $(f \otimes b) \ast_g \Delta(a)$.

**Remark 3.1.8.** Expanding on this componentwise definition of our products in $\mathcal{P}^\Sigma \otimes \mathcal{P}^\Sigma$, if we write $\Delta(a) = \sum a^{(1)} \otimes a^{(2)}$. Then

\[
(f \cdot b) \ast_g a = (f \otimes b) \ast_g \Delta(a) = \sum (f \ast_g a^{(1)}) \cdot (b \ast_g a^{(2)}).
\]
3.2 Properties of $\mathcal{P}_\Sigma$

We are ultimately interested in $\mathcal{P}_\Sigma$. It is much easier to work directly with $\mathcal{P}_\Sigma^2$ and most of this subsection will be devoted to transferring results from $\mathcal{P}_\Sigma^2$ to $\mathcal{P}_\Sigma$ via $\Phi$.

First we notice that $\mathcal{P}_\Sigma$ has an algebra structure. We can define $f \cdot h$ to be the image of $f \gamma_{\sigma} \overline{h}$ under $B \to B_{\Sigma}$ for any split $\sigma$ and lifts $f, h \in \mathcal{P}$ of $f, h \in \mathcal{P}_\Sigma$. It is not hard to see that this is independent of the choice of lists and the choice of split. To define a $*_{G}$-product on $\mathcal{P}_\Sigma$ we must rely heavily on $\mathcal{P}_\Sigma^2$ and the fact that $\Phi$ is a linear isomorphism. We define

$$f *_{G} h = \Phi^{-1}(\Phi(f) *_{G} \Phi(h)). \tag{3.2.1}$$

The reason for constructing this algebra is to prove a bounded generation result about the secant ideals of Plücker-embedded Grassmannians. We will show how to get this bounded generation from the finite generation in $\mathcal{P}_\Sigma$ for the sum of the Plücker ideals as a consequence of what we have developed. This example is complicated and skipping it will not detract from ones understanding of the paper. We include it to both explicitly illustrate the techniques we are developing and demonstrate the need for these more general techniques due to the difficulty of explicitly working through the easiest possible case, i.e., the 0th secant ideal.

Example 3.2.2. First, we have the Pieri decomposition,

$$\bigwedge^{p} \otimes \bigwedge^{p} = \bigoplus_{i=0}^{p} S_{i, \{12(p-i)\}}. \tag{3.2.2}$$

The symmetric square has a simple description,

$$\text{Sym}^{2} \left( \bigwedge^{p} \right) = \bigoplus_{i \equiv p \mod 2} S_{i, \{12(p-i)\}}. \tag{3.2.3}$$

The Plücker equations of the Grassmannian $\text{Gr}(p, n)$ span the sum of the representations where $i < p$. Taking Hodge dual isomorphisms, we can generate all the Plücker equations from the basic ones $f_{1}, f_{2}, \ldots$, where $f_{i} \in \text{Sym}^{2}(\bigwedge^{2n})$ is defined as follows,

$$f_{n} = \frac{1}{2} \sum_{S \subseteq [4n], |S| = 2n} (-1)^{\text{sgn}(S)} x_{S} x_{[4n] \setminus S}.$$ 

To show the Plücker equations are all generated from finitely many equations it suffices to prove that the ideal generated by the $f_{i}$ in $\mathcal{P}_\Sigma$ is finitely generated. To see this explicitly we will show all the $f_{i}$ are generated by $f_{1}$ under our products. In particular, we claim

$$\sum_{g, \sigma} (-1)^{\text{sgn}(T)} f_{n} *_{G} (x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}) = \frac{\gamma(n)}{2} f_{n+1} \tag{3.2.4}$$

where

$$\gamma(n) = \frac{(4n) \cdot 6 \cdot (4n+1) \cdot 2}{(4n+4) \cdot 2n+1}.$$ 

The numerator is the total number of terms in the sum. The denominator is the number of terms in $f_{n+1}$. We then divide by $2$ because of $\Phi$. It is not hard to see $\gamma(n)$ is an even integer.

We sum over all $g \in \text{Inc}(N)$, $g_{[4n]} : [4n] \to [4n+4]$ such that $g(1) = 1$ and only one $a$ acts trivially on $f_{n}$. We also sum over all $\sigma \in \Sigma_{4}/(\Sigma_{2} \times \Sigma_{2})$ permutations such that $\sigma(1) < \sigma(2)$ and $\sigma(3) < \sigma(4)$. Let $(j_{1} < j_{2} < j_{3} < j_{4}) = [4n+4] \setminus g([4n])$ where $j_{k}$ is the image of $k$ in $[4n+4]$. In this sum, $T = (g(1), g(2), \ldots, g(2n), j_{\sigma(1)}, j_{\sigma(2)}, g(2n+1), g(n+2), \ldots, g(4n), j_{\sigma(3)}, j_{\sigma(4)})$ and $\text{sgn}(T)$ is the sign of the permutation that orders $T$. Tracing through definitions, we find that on monomials

$$x_{i_{1} i_{2}} x_{i_{3}} x_{i_{4}} *_{G} x_{j_{1}} x_{j_{2}} x_{j_{3}} x_{j_{4}} = \frac{1}{2} (x_{g(i_{1}) g(i_{2}) j_{1}} x_{g(i_{3}) j_{3}} x_{j_{4}} + x_{g(i_{1}) g(i_{2}) j_{2}} x_{j_{4}} x_{g(i_{3}) j_{3}}) \tag{3.2.5}$$

To see this we explicitly calculate what (3.2.1) does to monomials $f$ and $h$. First we know that

$$\Phi(x_{i_{1} i_{2}} x_{i_{3} i_{4}}) = \frac{1}{2} (x_{i_{1} i_{2}} \otimes x_{i_{3} i_{4}} + x_{i_{3} i_{4}} \otimes x_{i_{1} i_{2}}). \tag{3.2.6}$$
So

\[ \mathcal{S}(x_{i_1 i_2} x_{i_3 i_4}) \ast_g \mathcal{S}(x_{j_1 j_2} x_{j_3 j_4}) = \frac{1}{4}((x_{i_1 i_2} \otimes x_{i_3 i_4} + x_{i_3 i_4} \otimes x_{i_1 i_2}) \ast_g (x_{j_1 j_2} \otimes x_{j_3 j_4} + x_{j_3 j_4} \otimes x_{j_1 j_2})). \]

Expanding by linearity this becomes

\[ \frac{1}{4} \left( x_{g(i_1)}g(i_2)j_1 j_2 \otimes x_{g(i_3)}g(i_4)j_3 j_4 + x_{g(i_1)}g(i_3)j_1 j_3 \otimes x_{g(i_2)}g(i_4)j_2 j_4 \right) + \frac{1}{4} \left( x_{g(i_1)}g(i_2)j_1 j_3 \otimes x_{g(i_3)}g(i_4)j_2 j_4 + x_{g(i_1)}g(i_3)j_1 j_2 \otimes x_{g(i_2)}g(i_4)j_3 j_4 \right). \]

Now when we apply \( \mathcal{S}^{-1} \) we notice this is exactly

\[ \frac{1}{4}(x_{g(i_1)}g(i_2)j_1 j_2 x_{g(i_3)}g(i_4)j_3 j_4 + x_{g(i_1)}g(i_3)j_1 j_3 x_{g(i_2)}g(i_4)j_2 j_4). \]

Now to justify our claim, we will show explicitly that

\[ \sum_{g, \sigma} f_1 \ast_g (x_{i_1(1)} x_{i_2(2)} x_{i_3(3)} x_{i_4(4)}) = 9 f_2 \quad (3.2.4) \]

and the proof in the general case is exactly the same but with more indices. Referring to the definition,

\[ f_1 = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}. \]

We will also drop the \( \frac{1}{2} \) in the definition of \( f_1 \) without loss of generality because if we show

\[ 2 \cdot \sum_{g, \sigma} (-1)^{\text{sgn}(T)} f_n \ast_g (x_{\tau(1)} x_{\tau(2)} x_{\tau(3)} x_{\tau(4)}) = \frac{\gamma(n)}{2} (2 \cdot f_{n+1}) \]

this is equivalent to proving (3.2.3). We will compute an element of the sum in (3.2.4) for one choice of \( g \) to illustrate how we would proceed in general. We will then show that every term of \( f_2 \) appears. If we fix \( g = \text{id} \) and \( \sigma = \text{id} \). Then \( [8] \setminus g([4]) = (5,6,7,8) \). So this element of our summand is

\[ (x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}) \ast_g (x_{12}x_{34}). \]

From what we computed above we know this is

\[ \frac{1}{2}(x_{1256}x_{3478} + x_{1278}x_{3456} - x_{1356}x_{2478} - x_{1378}x_{2456} + x_{1456}x_{2378} + x_{1478}x_{2356}) \]

So we see that each term in our summand gives us 6 unique terms. In what follows we will compute without carrying through the \( \frac{1}{2} \) and add it back in at the end to simplify the exposition. That is, we will prove

\[ 2 \sum_{g, \sigma} f_1 \ast_g (x_{i_1(1)} x_{i_2(2)} x_{i_3(3)} x_{i_4(4)}) = 18 f_2 \]

and this implies (3.2.4). So every time we apply \( \ast_g \) we will not include the resulting \( \frac{1}{2} \). We will now show that we can get any term \( \beta = x_{i_1 i_2 i_3 i_4} x_{i_5 i_6 i_7 i_8} \) by choosing the correct summand of \( f_1 \), and choice of \( g \) and \( \sigma \). We can assume that \( i_1 < i_2 \) and \( i_3 < i_4 \). Under these conditions there are three possible orders that can occur if we fix \( i_1 = 1 \). We could have \( i_1 < i_2 < i_3 < i_4 \), \( i_1 < i_3 < i_2 < i_4 \) or \( i_1 < i_5 < i_6 < i_2 \). This choice will determine which of the monomials in \( f_1 \) we will use to get \( \beta \).

In particular, suppose we have \( i_1 < i_5 < i_2 < i_6 \), then we will use the term \( x_{13}x_{24} \) because we know \( g \) is order preserving and so this term that will have this order if we insert the numbers as the first two indices in each monomial. Choose \( g \) with

\[ g(1) = i_1, \quad g(2) = i_5, \quad g(3) = i_2, \quad g(4) = i_6. \]

Now we have four numbers remaining \( \{j_1 < j_2 < j_3 < j_4\} = [8] \setminus g([4]). \) Suppose \( \tau \) is the permutation with \( \tau(i_3) < \tau(i_4) < \tau(i_2) \). We realize \( \tau \) as an element of \( \Sigma_4/\Sigma_2 \times \Sigma_2 \) and let \( \sigma = \tau^{-1} \). After we apply \( \ast_g \) we know \( i \mapsto j_i \). We also have \( \tau(i_3) = j_3, \quad \tau(i_4) = j_2, \quad \tau(i_2) = j_3 \) and \( \tau(i_8) = j_4 \). So \( \sigma(i) \) maps to \( \sigma j_i = \sigma \tau(i_k) = i_k \), where the permutations act on the indices.

Then in this summand if we focus on the monomial coming from \( x_{13}x_{24} \) with our selected \( g \) and \( \sigma \), we have

\[ x_{13}x_{24} \ast_g x_{\tau(1)} x_{\tau(2)} x_{\tau(3)} x_{\tau(4)} = x_{i_1 i_2 i_3 i_4} x_{i_5 i_6 i_7 i_8}. \]

We only mention this once more, but here we recall that we have multiplied the entire sum by 2. Furthermore, notice this term has the correct sign. The monomial \( x_{13}x_{24} \) is negative in \( f_1 \) because the permutation ordering this sequence is odd. The monomial above should have sign \((-1)^{\text{sgn}(\tau)} \) where \( \tau \) orders the set
Similarly if we fix \( g(4) \), each paired with the appropriate \( \sigma \). Similarly if we fix \( g(3) = 5 \), there are once again 4 total options. If \( g(3) = 7 \), there are now 3 options for \( g(2) \) and only 1 for \( g(4) \) because \( g \) must be increasing. This gives us 11 total options.

Finally if we focus on the monomial \( x_{14}x_{24} \), when we fix \( g(4) = 4 \), then \( g(2) \) and \( g(3) \) must also be fixed. If \( g(4) = 5 \), once again we can only have \( g(2) = 2 \) and \( g(3) = 3 \). If \( g(4) = 7 \), then there are \( \binom{3}{2} \) options for where to send 2 and 3. Hence in this case there are 5 total ways to get \( \beta \).

In all, we found \( 2 + 11 + 15 = 18 \) ways \( \beta \) could appear in our sum. This shows that

\[
2 \sum_{g,\sigma} f_1 * g \left( x_{i_1}(1) x_{i_2}(2) x_{i_3}(3) i_{\sigma}(4) \right) = 18 f_2.
\]

Which is equivalent to (3.2.4). This same argument generalizes to any \( f_n \). We are essentially appending missing terms on in all possible ways to account for the symmetry present in \( f_n \).

In particular, this shows that if \( f_n \) is in our ideal, \( f_{n+1} \) is as well. So \( f_1 \) generates all the \( f_i \). By the above the Plücker ideal is finitely generated under the operations we have defined.

**Definition 3.2.5.** An ideal \( J \subseteq \mathcal{P}_\Sigma \) is a di-ideal if \( \mathcal{I}(J) \) is closed under both \( * g \) and \( \cdot \).

For ease of notation we make the following definition

**Definition 3.2.6.** For any fixed \( M \geq 2 \), call the sum of the Plücker ideals corresponding to \( \text{Gr}(d, M d) \) as \( d \) varies, \( \mathcal{S}_M \). So that \( (\mathcal{S}_M)_{d,n} \) is the space of degree \( n \) polynomials in the ideal of \( \rho(\text{Gr}(d, M d)) \) where \( \rho \) is the Plücker embedding.

We ultimately wish to show that for any \( r \geq 0 \), \( \mathcal{S}_r \) is a di-ideal. We will see this can be deduced from the following lemma:

**Lemma 3.2.7.** For any fixed \( M \geq 2 \), \( \mathcal{S}_M \), is closed under \( * g \) and \( \cdot \) in \( (\mathcal{P}_\Sigma)_M \).

**Proof.** Using Weyman’s construction of the Plücker equations in [We, Proposition 3.1.2], any Plücker equation \( f \) in the graded coordinate ring of the Plücker embedding of \( \text{Gr}(d, M d) \) which is contained in \( (\mathcal{P}_\Sigma)_M \) will be of the form

\[
f = \sum_{\beta} \text{sgn}(\beta) x_{i_1} \cdots x_{i_{j(1)}} \cdots x_{j(2d-1)} x_{j(2d-2) + 1} \cdots x_{j(2d-2) + v} i_1 \cdots i_v
\]
where we sum over all permutations \( \beta \) of \( \{1, \ldots, d - u - v\} \) such that \( \beta(1) < \beta(2) < \cdots < \beta(d - u) \) and \( \beta(d - u + 1) < \beta(d - u + 2) < \cdots < \beta(2d - u - v) \). And we choose \( i_1, \ldots, i_u, j_1, \ldots, j_{2d - u - v}, l_1, \ldots, l_v \) as distinct elements from \([Md]\).

Each term in \( f \) is a product \( x_I x_K \) with \( I \neq K \), \( |I| = |K| = d \) as seen in [We, Proposition 3.1.6]. An alternative way to see this is to refer back to the description of the image of the Grassmannian in Example 3.2.2. The Plücker equations span the sum of the representations

\[
\text{Sym}^2 \left( \bigwedge^d \right) = \bigoplus_{0 \leq i < d, i \equiv d \pmod{2}} S_{(i, 1^{(d-i)})},
\]

with \( i < d \). If \( I = K \), this would imply we only use \( d \) distinct numbers as indices in \( x_I x_K \). However, this only occurs in the representation \( S_{2d} \), which means the corresponding element could not be a Plücker equation.

Fix an element \( n \in (P_\Sigma)_{aM} \) of bidegree \( (a, m) \). For any Plücker equation \( f \), to get \( f \cdot g \) \( n \), from Example 3.2.2, we append additional distinct indices to the monomials appearing in \( f \). Hence, each term in \( f \cdot g \) \( n \) consists of \( x_I x_K \) where \( I \neq K \), \( |I| = |K| = d + a \).

As in Weyman, we may identify each \( x_I \) with the element \( e^*_i \in \wedge^{d+a}(k^{M(d+a)})^* \) where \( e^*_i \) denotes the dual basis element of the standard basis element \( e_i \) of \( k^{M(d+a)} \).

To show that \( f \cdot g \) \( n \) \( \not\in J \) it suffices to show that it vanishes on all decomposable elements of \( k^{M(d+a)} \). To do this consider all of the equations given by \( f \cdot g \) \( n \) for all \( f \) a Plücker equation, \( g \in \text{Inc}(N) \) and \( n \in (P_\Sigma)_{M} \).

The following computation is not that enlightening, but the result is very important so we will state it as a claim and skipping the proof will not take away from the proof of this lemma.

**Claim 3.2.8.** The collection of equations \( f \cdot g \) \( n \) with \( f \) a Plücker equation, \( g \in \text{Inc}(N) \) and \( n \in (P_\Sigma)_{M} \) of fixed bidegree \( (a, m) \) is \( \text{GL}(k^{M(d+a)}) \)-invariant.

**Proof.** Indeed, given \( h \in \text{GL}(k^{M(d+a)}) \), we have

\[
h(f \cdot g \) \( n \) = [g^{-1}hg] \cdot g \cdot (g')^{-1}hg\) \( n \).
\]

Where \( g' \in \text{Inc}(N) \) is the map induced by \( g \) which sends \([2a]\) to \([M(d+a)] \setminus g([Md])\) in increasing order. It is easy to check on indices that this is valid.

The set of Plücker equations is \( \text{GL} \)-invariant [We, Proposition 3.1.2] and we can view \( g^{-1}hg \in \text{GL}(k^M) \), so \( (g')^{-1}hg\) \( n \) \( \in (P_\Sigma)_{M} \) and has the same bidegree \( (a, m) \). \( \square \)

Furthermore, \( \text{GL}(k^{M(d+a)}) \) acts transitively on \( \text{Gr}(d + a, M(d + a)) \) and so acts transitively on the decomposable elements of \( \wedge^{d+a} k^{M(d+a)} \). As a result, it suffices to show that all the equations \( f \cdot g \) \( n \) as \( f, g \) \( n \) vary as described in Claim 3.2.8 vanish on one decomposable element, say \( e_{i_1} \wedge \cdots \wedge e_{i_{d+a}} \), where \( i_1 < \cdots < i_{d+a} \) are chosen from \([M(d+a)]\).

Indeed, any decomposable element is in the \( \text{GL} \) -orbit of this element, but the set of \( f \cdot g \) \( n \) with \( f, g \) \( n \) varying is \( \text{GL} \)-invariant. Accordingly, if every equation in this set vanishes on \( e_{i_1} \wedge \cdots \wedge e_{i_{d+a}} \), they will vanish on every decomposable element and so will be in \( S_M \).

However, it must be the case that the set of equations \( f \cdot g \) \( n \) for any \( f, g \) \( n \) vanish on \( e_{i_1} \wedge \cdots \wedge e_{i_{d+a}} \), because only \( e^*_{i_1} \wedge \cdots \wedge e^*_{i_{d+a}} \) does not vanish on \( e_{i_1} \wedge \cdots \wedge e_{i_{d+a}} \). As mentioned above, \( f \cdot g \) \( n \) is a sum of products of terms of the form \( x_I x_K \) where \( I \neq K \), so one of \( x_I \) or \( x_K \) will vanish. This means every term in the sum vanishes, i.e. \( f \cdot g \) \( n \) for any choice of \( f \) \( n \).

The case for \( f \cdot n \) is clearer because we just multiply \( f \) and \( n \) in \( P_\Sigma \). By definition \( f \cdot n \) will vanish on any decomposable element and so \( f \cdot n \) will vanish as well. \( \square \)

We also notice that \( P_\Sigma \) has a natural comultiplication \( \Delta \) defined in the following way. If \( v_1 \cdots v_n \in \text{Sym}^n(\wedge^d k^{Md}) \), then

\[
\Delta(v_1 \cdots v_n) = \sum_{S \subseteq [n]} v_S \otimes v_{[n]\setminus S},
\]

where \( v_S = \prod_{i \in S} v_i \in \text{Sym}^{|S|}(\wedge^d k^{Md}) \). Less explicitly, but still importantly, this is defined in the usual way by letting \( w \mapsto w \otimes 1 + 1 \otimes w \) when \( w \in \text{Sym}^1(\wedge^d k^{Md}) \) and extending uniquely while requiring that \( \Delta : P_\Sigma \to P_\Sigma \otimes P_\Sigma \) be an algebra homomorphism.

We will now show that Lemma 3.2.7 implies \( J \) is a di-ideal. To do this, we need the following results.
Proposition 3.2.9. The symmetrization map $\mathcal{G} : \mathcal{P}_\Sigma \to \mathcal{P}_\Sigma$ is an isomorphism of bigraded bialgebras under the · product. More precisely, the following two diagrams commute:

\[
\begin{array}{ccc}
\mathcal{P}_\Sigma \otimes \mathcal{P}_\Sigma & \to & \mathcal{P}_\Sigma \\
\mathcal{G} \otimes \mathcal{G} \downarrow & & \downarrow \mathcal{G} \\
\mathcal{P}_\Sigma & \to & \mathcal{P}_\Sigma \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{P}_\Sigma \otimes \mathcal{P}_\Sigma & \to & \mathcal{P}_\Sigma \\
\mathcal{G} \otimes \mathcal{G} \downarrow & & \downarrow \mathcal{G} \\
\mathcal{P}_\Sigma & \to & \mathcal{P}_\Sigma \\
\end{array}
\]

Proof. This is the same proof as in [Sa1, Proposition 3.8].

With this proposition, we can prove the following important fact:

Proposition 3.2.10. If $\mathcal{J} \subseteq \mathcal{P}_\Sigma$ is closed under $*$ and · in $\mathcal{P}_\Sigma$, then $\mathcal{J}$ is a di-ideal.

Proof. Proposition 3.2.9 shows that if $\mathcal{J} \subseteq \mathcal{B}_\Sigma$ is an ideal under ·, the same is true for $\mathcal{G}(\mathcal{J}) \subseteq \mathcal{P}_\Sigma$.

It remains to show $\mathcal{G}(\mathcal{J})$ is closed under the $*$ product in $\mathcal{P}_\Sigma$. We have $f * g n \in \mathcal{J}$ for all monomials $n$ and $f, g \in \mathcal{J}$. In $\mathcal{P}_\Sigma$, we defined

$$ f * g n = \mathcal{G}^{-1}(\mathcal{G}(f) * \mathcal{G}(g)(n)),$$

so

$$ \mathcal{G}(f * g n) = \mathcal{G}(f) * \mathcal{G}(g)(n).$$

As $\mathcal{G}$ is a linear isomorphism, we can write any $n' \in \mathcal{P}_\Sigma$ as $\mathcal{G}(n')$ for some $n' \in \mathcal{P}_\Sigma$. To see $\mathcal{G}(\mathcal{J})$ is closed under $*$, we just need to check $\mathcal{G}(f) * g n' \in \mathcal{G}(\mathcal{J})$ for any $n' \in \mathcal{P}_\Sigma$. Find $n \in \mathcal{P}_\Sigma$ such that $\mathcal{G}(n) = n'$, then we know

$$ f * g n \in \mathcal{J},$$

hence

$$ \mathcal{G}(f) * g \mathcal{G}(n) \in \mathcal{G}(\mathcal{J}),$$

but $\mathcal{G}(n) = n'$.

Theorem 3.2.11. For any fixed $M \geq 2$, $S_M$, is a di-ideal.

Proof. Combine Lemma 3.2.7 with Propositions 3.2.9 and 3.2.10.

4. Joins and Secants

Let $V$ be a vector space and $\text{Sym}(V)$ be its symmetric algebra. Given ideals $I, J \subseteq \text{Sym}(V)$, their join $I \star J$ is the kernel of

$$ \text{Sym}(V) \xrightarrow{\Delta} \text{Sym}(V) \otimes \text{Sym}(V) \to \text{Sym}(V)/I \otimes \text{Sym}(V)/J,$$

where the first map is the standard comultiplication. Note that $\star$ is an associative and commutative operation since $\Delta$ is coassociative and cocommutative. Set $I^{*0} = I$ and $I^{*r} = I \star I^{*(r-1)}$ for $r > 0$.

Proposition 4.1. [Sa1, Proposition 4.1] Assume $k$ is an algebraically closed field. If $I$ and $J$ are radical ideals, then $I \star J$ is a radical ideal. If $I$ and $J$ are prime ideals, then $I \star J$ is a prime ideal.

These definitions make sense for ideals $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}_\Sigma$, so we can define the join $\mathcal{I} \star \mathcal{J}$. To be precise, $(\mathcal{I} \star \mathcal{J})_{d,n}$ is the kernel of the map

$$ (\mathcal{P}_\Sigma)_{d,n} \xrightarrow{\Delta} \bigoplus_{i=0}^{n} (\mathcal{P}_\Sigma/\mathcal{I})_{d,i} \otimes (\mathcal{P}_\Sigma/\mathcal{J})_{d,n-i}.$$ 

Since $\mathcal{G}$ is compatible with $\Delta$ (Proposition 3.2.9), we deduce that

$$ \mathcal{G}(\mathcal{I} \star \mathcal{J}) = \mathcal{G}(\mathcal{I}) \star \mathcal{G}(\mathcal{J}).$$

Proposition 4.2. If $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}_\Sigma$ are di-ideals, then $\mathcal{I} \star \mathcal{J}$ is a di-ideal.

Proof. Pick $v \in \mathcal{G}(\mathcal{I} \star \mathcal{J})$. By definition, $v$ is in the kernel of the map

$$ \Delta : \mathcal{P}_\Sigma \to \mathcal{P}_\Sigma/\mathcal{G}(\mathcal{I}) \otimes \mathcal{P}_\Sigma/\mathcal{G}(\mathcal{J}).$$

Since both $\mathcal{G}(\mathcal{I})$ and $\mathcal{G}(\mathcal{J})$ are ideals under $*$ via Proposition 3.2.10, it gives a well-defined multiplication on $\mathcal{P}_\Sigma/\mathcal{G}(\mathcal{I}) \otimes \mathcal{P}_\Sigma/\mathcal{G}(\mathcal{J})$. By Lemma 3.1.4, given $x \in \mathcal{P}_\Sigma$, we have $\Delta(x *_g v) = \Delta(x) *_g \Delta(v)$. But $\Delta(v) = 0$, so $x *_g v \in \mathcal{G}(\mathcal{I} \star \mathcal{J})$. \qed
Let \( V \) be a vector space. A subscheme \( X \subseteq V \) is **conical** if its defining ideal \( I_X \) is homogeneous. The \( r \)th **secant scheme** of \( X \) is the subscheme of \( V \) defined by the ideal \( I_X^r \). We wish to consider secant varieties of Plücker embeddings, we make the following definition,

**Definition 4.3.** For any fixed \( r \geq 0 \) and \( M \geq 2 \), let \( S_M(r) = (S_M)^r \). Where \( S_M \) is defined in Definition 3.2.6.

An immediate corollary of what we have just shown

**Corollary 4.4.** \( S_M(r) \) is a di-ideal for any \( r \geq 0 \) and \( M \geq 2 \).

**Proof.** This follows immediately from combining Proposition 4.2 and Theorem 3.2.11. \( \square \)

5. **Proof of Theorem 1.1**

Before we can get to the main result, we need the following lemma

**Lemma 5.1.** For any fixed \( r \geq 0 \), to bound the degrees of the ideal generators of \( S_M(r) \) for any \( M \), it suffices to consider \( M = (r + 2) \).

**Proof.** This follows immediately from [MM, Proposition 5.7]. Rephrasing, in our case for a given Grassmannian \( \text{Gr}(d, V) \), the number of nonzero rows in \( \lambda \) with \( S_k(V) = \wedge^d V \) is \( d \). So from the quoted result, to bound the degrees of the ideal generators of the \( r \)th secant variety of a given Grassmannian it suffices to consider a vector space of dimension \( (r + 2)d \). It is then clear that to bound the degrees of the ideal generators of \( S_M(r) \) it suffices to consider \( S_{r+2}(r) \) in \( P_{r+2} \).

**Remark 5.2.** The result, [MM, Proposition 5.7], specifically says that if the degrees of the ideal generators of the \( r \)th secant variety of a given Grassmannian for a vector space of dimension \( (r + 2)d \) are bounded by \( C \), then the same is true for the degrees of the ideal generators of the \( r \)th secant variety of a given Grassmannian for a vector space of higher dimension. One might be concerned that we are not considering vector spaces of dimension less than \( (r + 2)d \). However if we work with any lower dimensional vector space say of dimension \( c \leq (r + 2)d \), the ideal of the \( r \)th secant variety of \( \text{Gr}(d, c) \) is contained in the ideal of the \( r \)th secant variety of \( \text{Gr}(d, (r + 2)d) \) by the exact argument seen in the proof of [MM, Proposition 5.7]. So for any given \( d \), we can fix the dimension of the vector spaces as \((r + 2)d\) to bound the ideal generators of the \( r \)th secant variety of \( \text{Gr}(d, n) \) for any \( n \).

**Proof of Theorem 1.1 and Corollary 1.2.** From Lemma 5.1 it suffices to consider \( \text{Gr}(d, (r + 2)d) \). The ideal of \( \text{Sec}_r(\rho(\text{Gr}(d, (r + 2)d))) \) is contained in \( S_{r+2}(r) \), where again \( \rho \) is the Plücker embedding. In particular, \( (S_{r+2}(r))_{d,n} \) is precisely the space of degree \( n \) polynomials in the ideal of \( \text{Sec}_r(\rho(\text{Gr}(d, (r + 2)d))) \).

Corollary 4.4 implies \( S_{r+2}(r) \) is a di-ideal. By Proposition 3.1.3, \( G(S_{r+2}(r)) \) is generated by finitely many elements \( f_1, \ldots, f_N \) under \( \cdot \) and \( \ast_g \). Every element of \( G(S_{r+2}(r)) \) can be written as a linear combination of elements of the form \( h \cdot (f_i \ast_g a) \) and so for fixed \( d \) a set of generators for \( \text{Sec}_r(\rho(\text{Gr}(d, (r + 2)d))) \) can be taken to be the set of all \( G^{-1}(f_i \ast_g a) \) such that \( f_i \ast_g a \in P_{d,n}^{\Sigma} \) for some \( n \). The degree of \( f_i \ast_g a \) is the same as that of \( f_i \) (if \( f \in P_{d,n}^{\Sigma} \) then its degree is \( n \)). So we can take \( C(r) = \max(\deg(f_1), \ldots, \deg(f_N)) \).

The proof of Corollary 1.2 follows immediately from the above.

**Remark 5.3.** In the case where \( r = 0 \), this theorem tells us that the homogeneous ideal of the Plücker image of all Grassmannians can be generated by finitely many polynomials of a finite degree bounded by \( C(0) \) under the operations \( \ast_g \) and \( \cdot \). In this case we know \( C(0) = 2 \) and Example 3.2.2 shows that all Plücker equations can be obtained from the Klein quadric. \( \square \)

6. **The Plücker Category**

In this section, we will translate the above work into the language of functor categories in the spirit of [Sa2]. Once we transition to this language, we can study free resolutions of secant ideals of Plücker embedded Grassmannians. In particular, we will show that the \( i \)th syzygy module of the coordinate ring of the \( r \)th secant variety of the Plücker embedded \( \text{Gr}(d, n) \) (whose space of generators is the \( r \)th Tor group with the residue field) is generated in bounded degree with bound independent of \( d \) and \( n \). The case \( i = 1 \) corresponds to the above results.
Let \( k \) be a commutative ring and fix \( M \geq 0 \). Recall that \( \mathcal{P}_{d,n} = \text{Sym}^n(\wedge^d k^M) \). We will now encode the morphisms from \( \mathcal{P}_{d,m} \) to \( \mathcal{P}_{e,n} \) as the space of morphisms from an objects \((d, m)\) to another object \((e, n)\) in the abstract category \( \textbf{k} \mathcal{U}_M \). The operations \( \cdot \) and \( * \), \( \cdot \) and \( * \) tell us how to do this when \( d = e \) or \( m = n \) respectively. More explicitly, when \( d = e \) an operation \( \mathcal{P}_{d,m} \) to \( \mathcal{P}_{d,n} \) is given by a partition \( \sigma \) of \([n]\) and an element of \( \mathcal{P}_{d,n-m} \). A basis for these operations can be encoded by an order-preserving injection \([m] \rightarrow [n]\) together with a list of monomials.

When \( m = n \), an operation \( \mathcal{P}_{d,m} \) to \( \mathcal{P}_{e,n} \) consists of a choice of an element of \( \mathcal{P}_{e-d,n} \) as well as a map \( g \in \text{inc}(\mathbb{N}) \) with \( g([d]) \subset [e] \). It has a basis given by the monomials, which are represented by an ordered list of \( n \) monomials in \( \wedge^{e-d} k^M(e-d) \). Once again, we prefer to represent these lists of monomials by reading lists, denote the poset of readings lists by \( \textbf{RL} \), and the poset of readings lists with \( n \) entries of size \( e \) by \( \text{RL}_{n,e} \). Explicitly, if \( S \in \text{RL}_{n,e} \), then \( S = (S^1, \ldots, S^n) \) with \( |S^i| = e \). Where the readings lists are defined above in \( \S 2 \), below Corollary \ref{cor:corollary}. In particular, each \( S^i \) consists of distinct numbers selected from \([Me]\).

When \( d \neq e \) and \( m \neq n \), it is harder to describe a basis for the space of operations. Given a map \( \alpha: [d] \rightarrow [e] \), suppose \([e] \setminus \alpha([d]) = \{a_1, \ldots, a_{e-d}\} \), define \( \alpha^c: [e-d] \rightarrow [e] \) as \( \alpha^c(i) = a_i \). We call this the complement of \( \alpha \).

**Definition 6.1.** Define the **Plücker category** \( \mathcal{G}_M \) as follows. The objects of \( \mathcal{G}_M \) are pairs \((d, m) \in \mathbb{Z}^2 \geq 0 \) and a morphism \( \alpha: (d, m) \rightarrow (e, n) \) consists of the following data:

- An order-preserving injection \( \alpha_1: [m] \rightarrow [n] \).
- A function \( \alpha_2: [n] \setminus \alpha_1([m]) \rightarrow \text{RL}_{1,e} \).
- A function \( \alpha_3: [m] \rightarrow \text{RL}_{1,e-d} \).
- An order-preserving injection \( \alpha_4: [d] \rightarrow [e] \).

In particular, \( \text{Hom}_{\mathcal{G}_M}((d, m), (e, n)) = \emptyset \) if \( d > e \). Given another morphism \( \beta: (e, n) \rightarrow (f, p) \), the composition \( \beta \circ \alpha = \gamma: (d, m) \rightarrow (f, p) \) is defined by

- \( \gamma_1 = \beta_1 \circ \alpha_1 \).
- \( \gamma_2: [p] \setminus \gamma_1([m]) \rightarrow \text{RL}_{1,f} \) is defined by:
  - if \( i \in [p] \setminus \beta_1([n]) \), then \( \gamma_2(i) = \beta_2(i) \), and
  - if \( i \in \beta_1([n] \setminus \alpha_1([m])) \), then \( \gamma_2(i) = \beta_4(\alpha_2(i')) + \beta_4(\beta_3(i')) \) where \( i' \) is the unique preimage of \( i \) under \( \beta_1 \).
- \( \gamma_3: [m] \rightarrow \text{RL}_{1,f-d} \) is defined by \( \gamma_3(i) = (\gamma_4)^{-1}(\beta_4 \alpha_4 \alpha_3(i) + \beta_4(\beta_3(\alpha_1(i)))) \).
- \( \gamma_4 = \beta_4 \circ \alpha_4 \).

When \( d = e \), the functions \( \alpha_3 \) and \( \alpha_4 \) are superfluous and the pair \((\alpha_1, \alpha_2)\) encodes an operation as discussed above. Similarly, when \( n = m \), the functions \( \alpha_1 \) and \( \alpha_2 \) are superfluous, and \( \alpha_3 \) also encodes an operation as discussed above.

**Remark 6.2.** Each of these maps \( \alpha_2 \) and \( \alpha_3 \) can be represented by reading lists in \( \text{RL}_{n-m,e} \) and \( \text{RL}_{m,e-d} \) respectively. To explicitly see this, \( \alpha_3 \) can be represented as \( S_{n-d,e} \) with \( S^i \) exactly the image of \( i \). We do not take this perspective for ease of composition in the above definition. However, taking the reading list perspective will be important in Proposition \ref{prop:proposition}. \hfill \( \square \)

**Lemma 6.3.** Composition as defined above is associative.

**Proof.** Suppose we are give three morphisms

\[
(d, m) \xrightarrow{\alpha} (e, n) \xrightarrow{\beta} (f, p) \xrightarrow{\gamma} (g, q).
\]

We will verify that all three components of both ways of interpreting \( \gamma \beta \alpha \) are the same.

- The associativity of the first and fourth maps follows by the associativity of function composition.
- Consider \([q] \setminus (\gamma \beta \alpha([m])) \rightarrow \text{RL}_{1,g} \).
  - If \( i \in [q] \setminus \gamma_1([p]) \) then \( i \mapsto \gamma_2(i) \) under both compositions.
  - If \( i \in \gamma_1([p] \setminus \beta_1([n])) \), let \( i'' \) be the unique preimage of \( i \) under \( \gamma_1 \) and let \( i''' \) be the unique preimage of \( i'' \) under \( \beta_1 \).

Under \( \gamma \beta \alpha \), we have \( i \mapsto \gamma_4((\beta_4 \alpha_4)(i')) + \gamma_4((\gamma_3)(i'')) = \gamma_4 \beta_2(i''') + \gamma_3(i'''') \)

Under \( (\gamma \beta) \alpha \), we have \( i \mapsto (\gamma \beta)(i'') = \gamma_4 \beta_2(i''') + \gamma_3(i'''') \). This is because \( i \in [q] \setminus \gamma_1([m]) \) by assumption.

\( \square \)
If \( i \in \gamma_1\beta_1([n] \setminus \alpha_1([m])) \), let \( i' \) be the unique preimage of \( i \) under \( \gamma_1 \) and let \( i'' \) be the unique preimage of \( i' \) under \( \beta_1 \).

Under \( \gamma(\beta_1) \), we have \( i \mapsto \gamma_4((\beta_1\alpha_2(i')) + \gamma^2_3(i') = \gamma_4\beta_4\alpha_2(i'') + \alpha_4\beta_4\beta_3(i'') + \gamma^2_3(i') \).

Under \( (\gamma \beta_1) \alpha_2 \), we have \( i \mapsto (\gamma\beta_1\alpha_2(i'')) + (\gamma\beta_1\alpha_2(i'')) = \gamma_4\beta_4\alpha_2(i'') + \alpha_4\beta_4\beta_3(i'') + \gamma^2_3(\beta_1(i'')) \).

But \( \beta_1(i'') = i' \), so these are equal.

We can recover \( \alpha \leq \beta \) fixed and \( (\gamma \beta_1) \alpha_2 \) with the lexicographic order on \( \mathbb{R}L \). To do this first put the lexicographic order on \( \mathbb{R}L \). That is, \( (d,m) \leq (e,n) \) if and only if \( d \leq e \) and \( m \leq n \).

Using this order, we can define \( P_{d,m} \) by

\[ P_{d,m}(e,n) = \text{Hom}_{\mathcal{G}_M}(((d,m), (e,n))) \]

This is the principal projective \( kG_M \)-module generated in bidegree \( (d,m) \), and they give a set of projective generators for the category of \( kG_M \)-modules. That is, every \( kG_M \)-module is a quotient of a direct sum of projective modules.

For further exposition on principal projectives we refer the reader to [SS1, §3.1]. Then \( P_{d,m}(e,n) \) is the space of operations from \( P_{d,m} \) to \( P_{a,n} \) which we discussed at the beginning of the section, so \( P_{d,m} \) is a \( P \)-module freely generated in bidegree \( (d,m) \).

To emphasize the category we may sometimes write \( P_{d,m}^{\mathcal{G}_M} \). With these definition we can now make sense of what it means for modules to be finitely generated. A \( kG_M \)-module \( N \) is finitely generated if there is a surjection

\[ \bigoplus_{i=1}^g P_{d,m_i} \to N \to 0, \]

with \( g \) finite. A \( kG_M \)-module is noetherian if all of its submodules are finitely generated. For a definition of a Gröbner category, see [SS1, Definition 4.3]. We only need this definition for the next result and it is lengthy, so we choose to omit it so as not to distract.

**Proposition 6.4.** \( \mathcal{G}_M \) is a Gröbner category. In particular, if \( k \) is noetherian, then every finitely generated \( kG_M \) module is noetherian.

**Proof.** We will use [SS1, Theorem 4.3.2]. Fix \( (d,m) \in \mathbb{Z}_{\geq 0}^2 \). Let

\[ \Sigma = \mathbb{R}L \times \mathbb{R}L \times \mathbb{Z}^m_{\geq 0} \times \mathbb{Z}^d_{\geq 0} \]

By Dickson’s Lemma and Theorem 2.16, the finite product of noetherian posets is also noetherian with the componentwise order. Hence \( \Sigma \) is noetherian.

From Remark 6.2 we can associate to each \( \alpha_2 \) and \( \alpha_3 \) a \( \mathbb{R}L \), \( S_{\alpha_i} \), for \( i = 2,3 \). Given a morphism \( \alpha : (d,m) \to (e,n) \) encode it as \( w(\alpha) \in \Sigma \)

\[ w(\alpha) = (S_{\alpha_2}, S_{\alpha_3}, \text{im}(\alpha_1), \text{im}(\alpha_4)). \]

We can recover \( \alpha \) from \( w(\alpha) \), so this is an injection. Define \( \alpha \leq \gamma \) if there exists some \( \beta \) such that \( \gamma = \beta \circ \alpha \).

Then, it follows from the definition of composition that the set of morphisms \( \alpha : (d,m) \to (e,n) \) with \( (d,m) \) fixed and \( (e,n) \) varying is naturally a subset of \( \Sigma \), i.e., \( \alpha \leq \alpha' \) if and only if \( w(\alpha) \leq w(\alpha') \). Since noetherian is inherited by subposets, we conclude that this partial order on morphisms with source \( (d,m) \) is noetherian.

It remains to prove that the set of morphisms with source \( (d,m) \) is orderable, i.e., for each \( (e,n) \) there exists a total ordering on the set of morphisms \( (d,m) \to (e,n) \) so that for any \( \beta : (e,n) \to (f,p) \), we have \( \alpha < \alpha' \) implies that \( \beta \alpha < \beta \alpha' \). To do this first put the lexicographic order on \( \mathbb{R}L \). That is, \( (d,m) < (e,n) \) if and only if \( d < e \) and \( m < n \).
$S_{d,n} = (S^1, \ldots, S^n)$ and $T_{e,m} = (T^1, \ldots, T^m)$ we say $S_{d,n} \preceq T_{e,m}$ if $n < m$ or if $n = m$ and $d < e$, if $m = n$ and $d = e$ we compare the lists in $(\mathbb{Z}^d)^n$ using the natural lexicographic order described in §2. In particular, we first compare $S^1$ and $T^1$ lexicographically, if they are equal we consider $S^2$ and $T^2$, etc.

This defines a total order on $RL$. Now put a lexicographic order on $\mathbb{Z}^m_{\geq 0}$ and $\mathbb{Z}^d_{\geq 0}$ in the natural way. Totally order $\Sigma$ by declaring all of the elements of the first $RL$ to be larger than the second $RL$ which is larger than $\mathbb{Z}^m_{\preceq 0}$ which is larger than $\mathbb{Z}^d_{\geq 0}$. This is just another lexicographic order. This orders $\Sigma$, which in turn gives the desired ordering. □

This proves $G_M$ is Gröbner, in particular this also implies Theorem 2.17.

6.1 Symmetrized versions.

In $kG_M$, the space of morphisms $(0,0) \to (d,m)$ is identified with the tensor power $(\wedge^d k^M)^{\otimes m}$. For our applications, we need symmetric powers, $\text{Sym}^m(\wedge^d k^M)$, so we now define symmetrized versions of the Gröbner category $kG_M$.

Definition 6.1.1. Given $\alpha : (d,m) \to (e,n)$ and $\sigma \in \Sigma_n$, there is a unique $\tau \in \Sigma_m$ so that $\sigma \alpha_1 \tau^{-1}$ is order-preserving; we refer to $\tau$ as the permutation induced by $\sigma$ with respect to $\alpha_1$. Define $\sigma(\alpha)$ by

- $\sigma(\alpha)_1 = \sigma \alpha_1 \tau^{-1}$,
- $\sigma(\alpha)_2 = \alpha_2 \sigma^{-1}$,
- $\sigma(\alpha)_3 = \alpha_3 \tau^{-1}$,
- $\sigma(\alpha)_4 = \alpha_4$.

This defines an action of $\Sigma_n$ on $\text{Hom}_{G_M}((d,m),(e,n))$, and we set

$$\text{Hom}_{kG_M}((d,m),(e,n)) = k[\text{Hom}_{G_M}((d,m),(e,n))]^{\Sigma_n}$$

where the superscript denotes taking invariants. □

Lemma 6.1.2. Given $\alpha : (d,m) \to (e,n)$ and $\beta : (e,n) \to (f,p)$, and $\sigma \in \Sigma_p$, we have $\sigma(\beta \circ \alpha) = \sigma(\beta) \circ \tau(\alpha)$ where $\tau \in \Sigma_n$ is the permutation induced by $\sigma$ with respect to $\beta_1$. In particular, $kG_M$ is a $k$-linear subcategory of $\Sigma_\alpha$.

Proof. Let $\rho \in \Sigma_d$ be the permutation induced by $\tau$ with respect to $\alpha_1$. Then $(\sigma \beta_1 \tau^{-1})(\tau \alpha_1 \rho^{-1})$ is order-preserving, so $\rho$ is also the permutation induced by $\sigma$ with respect to $\beta_1 \alpha_1$. Hence $\sigma(\beta_1) = \sigma(\beta_1 \tau(\alpha))_1$.

Next, we show that $\sigma(\beta \alpha)_2 = (\sigma(\beta) \tau(\alpha))_2$. If $i \in [p] \setminus \sigma(\beta)_1([n])$, then

$$\sigma(\beta \alpha)_2(i) = (\beta \alpha)_2 \sigma^{-1}(i) = (\beta \alpha)_2(i) = (\sigma(\beta) \tau(\alpha))_2(i).$$

Else, $i \in \sigma(\beta)_1([n] \setminus \tau(\alpha)_1([m]))$, let $i'$ be the unique preimage of $i$ under $\beta \tau^{-1}$. Then $\tau^{-1}(i')$ is the unique preimage of $\tau^{-1}(i)$ under $\beta_1 \tau^{-1}$. We have

$$\sigma(\beta \alpha)_2(i) = (\beta \alpha)_2 \sigma^{-1}(i) = \beta_4(\alpha_2(\tau^{-1}(i'))) + \beta_5(\alpha_3(\tau^{-1}(i'))))$$

$$= \beta_4(\tau(\alpha)_2(i')) + \beta_5(\tau(\alpha)_3(i')) = (\sigma(\beta) \tau(\alpha))_2(i).$$

Now, we show that $\sigma(\beta \alpha)_3 = (\sigma(\beta) \tau(\alpha))_3$. For $i \in [l]$, we have

$$(\sigma(\beta) \tau(\alpha))_3(i) = ((\sigma(\beta) \tau(\alpha) \alpha_3(i))^\alpha \sigma(\beta) \tau(\alpha)_3(i) + (\sigma(\beta) \tau(\alpha)_3(i)))$$

$$= ((\sigma(\beta) \tau(\alpha)_4)^\alpha(\beta_4 \alpha_3 \tau^{-1}(\beta_3 \tau^{-1}(\alpha_1 \rho^{-1}(i))))$$

$$= ((\sigma(\beta) \tau(\alpha)_4)^\alpha(\beta_4 \alpha_3 \tau^{-1}(\beta_3 \tau^{-1}(\rho^{-1}(i))))$$

$$= (\sigma(\beta \alpha)_3 \rho^{-1}(i))$$

Finally, we show that $\sigma(\beta \alpha)_4 = (\sigma(\beta) \tau(\alpha))_4$. This is clear because $\sigma$ acts trivially on this map, so

$$\sigma(\beta \alpha)_4 = \beta_4 \alpha_4 = \sigma(\beta) \tau(\alpha)_4 = (\sigma(\beta) \tau(\alpha))_4.$$

□
A $kG^\Sigma_{M}$-module is a $k$-linear functor from $kG^\Sigma_{M}$ to the category of $k$-modules. For each $(d,m)$, the principal projective $kG^\Sigma_{M}$-module is defined by

$$P_{d,m}^{kG^\Sigma_{M}}(e,n) = \text{Hom}_{kG^\Sigma_{M}}((d,m),(e,n)),$$

and we say that a $kG^\Sigma_{M}$-module $N$ is finitely generated if there is a surjection

$$\bigoplus_{i=1}^{g} P_{d,m}^{kG^\Sigma_{M}} \to N \to 0$$

with $g$ finite.

**Proposition 6.1.3.** If $k$ contains a field of characteristic 0, then every finitely generated $kG^\Sigma_{M}$-module is noetherian.

**Proof.** Set $P_{d,m} = P_{d,m}^{G_{d,m}}$ and $Q_{d,m} = P_{d,m}^{kG^\Sigma_{M}}$; we have a natural inclusion $Q_{d,m}(e,n) \subseteq P_{d,m}(e,n)$ for all $(e,n)$. Given a $kG^\Sigma_{M}$-submodule $M$ of $Q_{d,m}$, let $N$ be the $G_{M}$-submodule of $P_{d,m}$ that it generates. Given a list of generators of $N$ coming from $M$, Proposition 6.4 shows $P_{d,m}$ is noetherian and so some finite subset $\gamma_{1}, \ldots, \gamma_{g}$ of them already generates $N$. Let $\pi$ be the symmetrization map

$$k[\text{Hom}_{G_{M}}((d,m),(e,n))] \to k[\text{Hom}_{G_{M}}((d,m),(e,n))]^{\Sigma_{n}}$$

$$\alpha \mapsto \frac{1}{m!} \sum_{\sigma \in \Sigma_{n}} \sigma(\alpha).$$

If $\alpha \in k[\text{Hom}_{G_{M}}((d,m),(e,n))]^{\Sigma_{n}}$, then $\pi(\alpha) = \alpha$; given $\beta \in k[\text{Hom}_{G_{M}}((e,n),(f,p))]$, then $\pi(\beta \alpha) = \pi(\beta)\alpha$ by Lemma 6.1.2.

Now assume that $k$ contains a field of characteristic 0. We define the **symmetrized Plücker category** $\mathcal{G}_{M} = (kG_{M})_{\Sigma}$ as follows. First, set

$$\text{Hom}_{G_{M}}((d,m),(e,n)) = k[\text{Hom}_{G_{M}}((d,m),(e,n))]^{\Sigma_{n}}$$

where the subscript denotes coinvariants under $\Sigma_{n}$. As in §3, we have an isomorphism

$$k[\text{Hom}_{G_{M}}((d,m),(e,n))]^{\Sigma_{n}} \overset{\cong}{\to} k[\text{Hom}_{G_{M}}((d,m),(e,n))]^{\Sigma_{n}}$$

$$\alpha \mapsto \frac{1}{m!} \sum_{\sigma \in \Sigma_{n}} \sigma(\alpha),$$

and as above we use this to transfer the $k$-linear category structure from $kG^\Sigma_{M}$ to $\mathcal{G}_{M}$. Now we notice that $\text{Hom}_{G_{M}}((0,0),(d,m))$ is identified with $\text{Sym}^{m}(\bigwedge^{d} k^{(r+2)d})$, which was our goal. This isomorphism in combination with Proposition 6.1.3 give us the following:

**Proposition 6.1.4.** Suppose $k$ is a field of characteristic 0. Every finitely generated $\mathcal{G}_{M}$-module is noetherian.

**Remark 6.1.5.** These definitions parallel the constructions in §3. In particular, we can identify $P^{\Sigma}$ and $P_{\Sigma}$ from this section with the principal projectives generated in degree $(0,0)$ in $kG^\Sigma_{M}$ and $\mathcal{G}_{M}$ respectively. Furthermore, the notions of ideal and di-ideal translate to submodules in both cases. So Proposition 3.1.3 is a special case of Proposition 6.1.3.

## 7. Syzygies of Secant Ideals

In this section, $k$ is a field of characteristic 0. For this section fix some $M \geq 0$. The principal projective $P_{0,0}$ in $\mathcal{G}_{M}$ is the algebra $\mathcal{P}_{\Sigma}$ from §3 and each principal projective $P_{d,m}$ is a module over it. We use $\mathcal{P}_{\Sigma}(-d,-m)$ to denote this module; by Proposition 6.1.4 these are all noetherian modules.

In Definition 3.2.6, we defined $\mathcal{S}_{M}$ to be the sum of the Plücker ideals corresponding to $\text{Gr}(d,Md)$ as $d \geq 0$ varies and $\mathcal{S}_{M}(r) = (\mathcal{S}_{M})^{*r}$. By Corollary 4.4, $\mathcal{S}_{M}(r)$ is a $\mathcal{G}_{M}$-submodule of $P_{0,0}$ for all $r$. 


For $d$ fixed, $\bigoplus_m S_M(r)_{d,m}$ is an ideal in $\text{Sym}(\bigwedge^d k^{Md})$. So we can define an algebra

$$\text{Sec}_{d,r}(M) = \bigoplus_{m \geq 0} \text{Sym}^m(\bigwedge^d k^{Md})/S_M(r)_{d,m}$$

which is a quotient of $\text{Sym}(\bigwedge^d k^{Md})$. Notice this is exactly the $r$th secant ideal of the Plücker-embedded Grassmannian $\text{Gr}(d, Md)$. More generally, if $M$ is a $G_M$-module, then for $d$ fixed, $\bigoplus_m M_{d,m}$ is a $\text{Sym}(\bigwedge^d k^{Md})$-module.

**Lemma 7.1.** Fix $d, e, n$. Then

$$\bigoplus_{m \geq 0} \mathcal{P}_\Sigma(-e, -n)_{d,m}$$

is a free $\text{Sym}(\bigwedge^d k^{Md})$-module generated in degree $n$ whose rank is $\dim_k(\text{Sym}^n(\bigwedge^{d-e} k^{M(d-e)}))$.

**Proof.** We have

$$\bigoplus_{m \geq 0} \mathcal{P}_\Sigma(-e, -n)_{d,m} = \bigoplus_{m \geq 0} (\text{Sym}^n(\bigwedge^{d-e} k^{M(d-e)}) \otimes \text{Sym}^{m-n}(\bigwedge^d k^{Md}))$$

$$= \text{Sym}^n(\bigwedge^{d-e} k^{M(d-e)}) \otimes \text{Sym}^d(\bigwedge^d k^{Md})(-n).$$

As follows from the definitions, the action of $\text{Sym}(\bigwedge^d k^{Md})$ on this space corresponds to the usual multiplication on $\text{Sym}(\bigwedge^d k^{Md})(-n)$.

**Theorem 7.2.** There is a function $C_M(i, r)$, depending on $i, r, M$, but independent of $d$, such that

$$\text{Tor}_i^{\text{Sym}(\bigwedge^d k^{Md})}(\text{Sec}_{d,r}(M), k)$$

is concentrated in degrees $\leq C_M(i, r)$.

**Proof.** We know $S_M(r)$ is a finitely generated submodule of $\mathcal{P}_\Sigma$, and hence has a projective resolution

$$\cdots \to F_i \to F_{i-1} \to \cdots \to F_0,$$

such that each $F_i$ is a finite direct sum of principal projective modules by Proposition 6.1.4. For $d$ fixed, we get an exact complex of $\text{Sym}(\bigwedge^d k^{Md})$-modules

$$\cdots \to \bigoplus_m (F_i)_{d,m} \to \bigoplus_m (F_{i-1})_{d,m} \to \cdots \to \bigoplus_m (F_0)_{d,m} \to \text{Sec}_{d,r}(M) \to 0.$$ 

If $F_i = \bigoplus_{j \geq 0} \mathcal{P}_\Sigma(-d_j, -m_j)$, then set $C_M(i, r) = \max(m_1, \ldots, m_k)$. In particular, by Lemma 7.1, this gives a free resolution which can be used to compute $\text{Tor}_i^{\text{Sym}(\bigwedge^d k^{Md})}(\text{Sec}_{d,r}(M), k)$ which we conclude is concentrated in degrees $\leq C_M(i, r)$.

**Remark 7.3.** If we write $T_{i,d,r}(M) = \text{Tor}_i^{\text{Sym}(\bigwedge^d k^{Md})}(\text{Sec}_{d,r}(M), k)$. As used above, this is $\mathbb{Z}$-graded, so we denote the $m$th graded component by $T_{i,d,r}(M)_m$. For fixed $i, m, r$, we get a functor on the full subcategory $kG_M$ on objects of the form $(d, m)$ by

$$(d, m) \mapsto T_{i,d,r}(M)_m.$$ 

From the results above, we conclude that this is a finitely generated functor. In particular, as we allow $d$ to vary, this means that $T_{i,d,r}(M)_m$ is “built out” of $T_{i,d',r}(M)_m$ where the $d'$ range over some finite list of integers. This can be thought of as the Plücker analogue of $\Delta$-modules from [Sn].

As above, we would like to find a bound independent of $M$, i.e., independent of the chosen vector space for the Plücker embedded Grassmannian.

**Theorem 7.4.** The function $C_M(i, r)$ is independent of $M$ once $M \geq r + 1 + i$. In particular, there is a bound $C(i, r)$ that works for all $M$ simultaneously.
5.1. Theorem. The case i \(\leq \ell(k+1)\) can be treated by the subadditivity of \(\ell\). In particular, no information is lost by specializing to the case \(Md = d(r+1+i)\) [SS2, Corollary 9.1.3]. So it suffices to take \(M = r+1+i\).

\[\text{□}\]

Remark 7.5. This is a special case of Lemma 5.1. There, \(i = 1\) and we can take \(M = r+2\), so that \(C_{r+2}(1,r) = C(r)\) from Theorem 1.1.

\[\text{□}\]

Proof of Theorem 1.3. Combine Theorem 7.2 and Theorem 7.4.

References

[BDdG] Karin Baur, Jan Draisma, Willem A. de Graaf, Secant dimensions of minimal orbits: computations and conjectures, Exp. Math. 16 (2007), no. 2, 239-250.

[CEF] Thomas Church, Jordan Ellenberg, Benson Farb, FI-modules and stability for representations of symmetric groups, Duke Math. J. 164 (2015), no. 9, 1833-1910, arXiv:1006.5248v4

[CGG] Maria V. Vatalisano, Anthony V. Geramita, Alessandro Gimigliano, Secant varieties of Grassmann varieties, Proc. Am. Math. Soc. 133 (2005), no. 3, 633-642.

[D] Jan Draisma, Noetherianity up to symmetry, Combinatorial algebraic geometry, Lecture Notes in Math. 2108, Springer, 2014, arXiv:1310.1705v2

[DE] Jan Draisma, Rob H. Eggermont, Plücker varieties and higher secants of Sato’s Grassmannian, J. Reine Angew. Math., to appear, arXiv:1402.1667v2.

[DK] Jan Draisma, Jochen Kuttler, Bounded-rank tensors are defined in bounded degree, Duke Math. J. 163 (2014), no. 1, 35-63, arXiv:1103.5336v2.

[Hi] Graham Higman, Ordering by divisibility in abstract algebras, Proc. London Math. Soc. (3) 2 (1952), 326-336.

[HS] Christopher J. Hillar, Seth Sullivant, Finite Gröbner bases in infinite dimensional polynomial rings and applications, Adv. Math. 221 (2012), 1-25, arXiv:0908.1777v2.

[KPRS] Alex Kasman, Kathryn Pedings, Amy Reiszl and Takahiro Shiota, Universality of Rank 6 Plücker Relations and Grassmann Cone Preserving Maps, The Proceedings of the American Mathematical Society, 136 (2008), 77-87.

[LM] J.M. Landsberg, L. Manivel, On the ideals of secant varieties of Segre varieties, Found. Comput. Math. 4 (2004), no. 4, 397-422, arXiv:math/0311388v1.

[LO] J.M. Landsberg, Giorgio Ottoviani, Equations for secant varieties of Veronese and other varieties, Ann. Mat. Para Appl. (4) 192 (2013), no. 4, 596-606, arXiv:1111.4567v1.

[LW] J.M. Landsberg, Jerzy Weyman, On the ideals and singularities of secant varieties of Segre varieties, Bull. Lond. Math. Soc. 39 (2007), no. 4, 685-697, arXiv:math/0601452v2.

[MM] Laurent Manivel, Mateusz Michałek, Secants of minuscule and cominuscule minimal orbits, Linear Algebra Appl. 481 (2015), 288-312.

[NSS] Rohit Nagpal, Steven V Sam, Andrew Snowden, Noetherianity of some degree two twisted commutative algebras, Selecta Math. (N.S.), 22 (2016), no.2, 913-937.

[Sa1] Steven Sam, Ideals of bounded rank symmetric tensors are generated in bounded degree, Invent. Math. 207 (2017), no. 1, 1-21, arXiv:1501.04904.

[Sa2] Steven Sam, Syzygies of bounded rank symmetric tensors are generated in bounded degree, Math. Ann. 368 (2017), no. 3, 1095-1108, arXiv:1608.01722

[SS1] Steven Sam, Andrew Snowden, Gröbner methods for representations of combinatorial categories, J. Amer. Math. Soc. 30 (2017), 159-203, arXiv:1409.1670

[SS2] Steven V Sam, Andrew Snowden, Introduction to twisted commutative algebras, arXiv:1209.5122v1
[SS3] Steven V Sam, Andrew Snowden, Stability patterns in representation theory, Forum. Math. Sigma 3 (2015), e11, 108pp., arXiv:1302.5859v2

[Sn] Andrew Snowden, Syzygies of Segre embeddings and ∆-modules, Duke Math. J. 162 (2013), no.2, 225-277, arXiv:1006.5248v4.

[To] Philip Tosteson, Stability in the homology of Deligne-Mumford compactifications, arXiv:1801.03894

[We] Jerzy Weyman, Cohomology of vector bundles and syzygies, Cambridge Tracts in Mathematics 149, Cambridge University Press, Cambridge, 2003.

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