Escape dynamics of a particle from a purely nonlinear truncated quartic potential well under harmonic excitation

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Abstract This paper focuses on the escape problem of a harmonically forced classical particle from a purely quartic truncated potential well. The latter corresponds to various engineering systems that involve purely cubic restoring force and absence of linear stiffness even under the assumption of small oscillations, such as pre-tensioned metal wires and springs, and compliant structural components made of polymer materials. This is in contrast to previous studies where the equivalent potential well could be treated as linear at first approximation under the assumption of small perturbations. Due to the strong nonlinearity of the current potential well, traditional analytical methods are inapplicable for describing the transient bounded and escape dynamics of the particle. The latter is analyzed in the framework of isolated resonance approximation by canonical transformation to action–angle variables and the corresponding reduced resonance manifold. The escape envelope is formulated analytically. Surprisingly, despite the essential nonlinearity of the well investigated, it exhibits a universal property of a sharp minimum due to the existence of multiple intersecting escape mechanisms. Unlike previous studies, three underlying mechanisms that govern the transient dynamics of the particle were identified: two maximum mechanisms and a saddle mechanism. The first two correspond to a gradual increase in the system’s response amplitude for a proportional increase in the excitation intensity, and the latter corresponds to an abrupt increase in the system’s response and therefore more potentially hazardous. The response of the particle is described in terms of energy-based response curves. The maximal transient energy is predicted analytically over the space of excitation parameters and described using iso-energy contours. All theoretical predictions are in complete agreement with numerical results.

Keywords Potential wells · Purely nonlinear systems · Cubic and quartic nonlinearities · Transient dynamics · Action–angle variables · Canonical formalism

Abbreviations

AA Action–angle (variables)
LPT Limiting phase trajectory
MM Maximum mechanism
MM0, MMπ Maximum mechanism at \( \nu = 0, \pi \), respectively
NES Nonlinear energy sink
PEA Passive energy absorber
RM Resonance manifold
SM Saddle mechanism

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List of symbols

\( (\cdot), (\cdot)' \quad \text{Differentiation with respect to timescale } \tau \) and to averaged energy \( \xi \), respectively

\( \hat{\Omega}, \hat{f} \quad \text{Excitation frequency and amplitude associated with the coexistence of the MM}_\pi \) and the SM.

\( F \quad \text{The incomplete elliptic integral of the first kind} \)

\( K \quad \text{The elliptic integral of the first kind} \)

\( v \quad \text{Phase variable} \)

\( \omega \quad \text{Response frequency of the particle} \)

\( \Omega^*, f^* \quad \text{Excitation frequency and amplitude associated with the coexistence of the MM}_0 \) and the SM

\( \tau \quad \text{Nondimensional timescale} \)

\( \text{sn}(z,k) \quad \text{The Jacobi elliptic function of module } k \)

\( \xi \quad \text{The maximal transient energy level} \)

\( \xi_s \quad \text{The transient energy level associates with the saddle point of the RM} \)

\( a_k \quad \text{The coefficient of the } k\text{th term in Fourier series of the solution } q(I, \theta) \)

\( C(v, \xi) \quad \text{The expression describes the resonance manifold} \)

\( E, \xi \quad \text{Instantaneous and averaged energy of the particle} \)

\( f, \Omega \quad \text{Nondimensional forcing amplitude and frequency} \)

\( f_m, f_s \quad \text{The critical forcing amplitudes associated with the occurrence of bifurcation through maximum and saddle mechanisms, respectively} \)

\( H \quad \text{Integral of motion/conservation law} \)

\( H_0 \quad \text{Initial conditions-related value of the integral of motion } H \)

\( i \quad \text{Unit imaginary number} \)

\( I, \theta \quad \text{Action and angle variables} \)

\( J \quad \text{Averaged action variable} \)

\( p \quad \text{Momentum of the particle} \)

\( q \quad \text{Nondimensional displacement of the particle} \)

\( U(q) \quad \text{Equivalent potential energy function} \)

1 Introduction

The profound physical problem of the escape of a classical particle from a potential well has numerous applications in classical physics, engineering, thermodynamics and chemistry [1–3]. Escape might take place due to external disturbances of various kinds, such as nonzero initial conditions, impact, periodic or stochastic excitations. For periodic excitation, the curve that represents the critical excitation parameters associated with escape is referred to as the escape envelope. Previous works reported on a universal V-shaped escape curve whose minimum is located near the natural frequency of the particle [4–6]. This universal pattern stems from the co-existence of two competing escape mechanisms that can be described using the assumption of 1:1 isolated resonance, allowing an efficient order reducing if the system’s dynamics onto a 2D phase portrait of a 1:1 resonance manifold (RM). The trajectory which corresponds to zero initial conditions, and therefore relevant for the current analysis, is called the limiting phase trajectory (LPT) [7,8]. In the phase portrait of the system, escape corresponds to the continuous passage of the LPT from zero initial energy to the critical energy level associated with escape. The first escape mechanism is referred to as the 'maximum mechanism' (MM). In this scenario, the LPT approaches directly from the bottom of the phase portrait to its upper bound, and therefore, from the well’s equilibrium point to its top edge. The second scenario is called the 'saddle mechanism' (SM) since it corresponds to the passage of the LPT through the saddle point of the RM followed by a sudden jump toward the upper bound of the well. In this case, the particle’s displacement increases abruptly, which makes this mechanism much more potentially hazardous when takes place in engineering systems. On the other hand, in previous works it was demonstrated that this mechanism is highly effective for energy absorption and can be utilized by nonlinear energy sinks (NESs) for efficient vibration mitigation purposes [9].

Previous studies mainly focused on transient escape dynamics of weakly nonlinear potential wells [5,6,10,11]. There, the nonlinear features of the well could usually be omitted at first approximation under the assumption of small oscillations. In the current work, we explore analytically the effect of the absence of a linear term in the potential well, as well as the effect of strongly nonlinear quartic term and its effect on the escape mechanisms. Quartic potential corresponds to a cubic restoring force which is the most common nonlinearity in mechanical components and engineering systems that exhibit symmetry. Cubic stiffness can stem either from the system’s structural configuration...
or from its material properties. Examples include pretensioned wires, springs, polymers and foams [12]. Another engineering use-case of a cubic nonlinearity is passive energy absorption; the cubic NES utilizes its purely cubic nonlinearity in the form of an adaptive frequency that allows it to exhibit good vibration mitigation capabilities for a broad frequency range [13,14]. While previous studies focused merely on the particle’s escape from the well under harmonic excitation, in the current study we focus on both its bounded and escape dynamics. Two types of bifurcations are introduced: bifurcation of type I is referred to the escape from the well, while bifurcation of type II corresponds to reaching a critical maximal energy level. In other terms, the former is a particular case of the latter for the critical energy level of one. In engineering systems, crossing a given energy level can correspond to a critical/irreversible damage or reaching a target energy absorption rate by the PEA, i.e. a qualitative change in the system’s response. Therefore, this transition is referred to as a bifurcation, even though there is no change in the system’s dynamical regime per se. We use this distinction to describe both the escape envelope of the potential well over the space of excitation parameters (bifurcation of type I) and other contour lines associated with lower maximal transient energy levels. Those contours accumulate to a manifold over the excitation parameters space that maps sets of excitation parameters to their corresponding predicted maximal transient energy levels.

This paper is structured as follows: In Sect. 2 the dynamical model of the classical particle in the truncated quartic potential well is introduced. In Sect. 3, the dynamical response of the bounded particle is described analytically using an action–angle formalism in the perspective of the topology of an underlying resonance manifold. In Sect. 4, the maximal transient energy for any given set of excitation parameters is predicted analytically, and in Sect. 5 the escape envelope of the particle is constructed on the space of excitation parameters. In Sect. 6, the energy-based frequency response curve of the forced particle is obtained. In Sect. 7, all analytical results are validated numerically. Finally, Sect. 8 is dedicated to concluding remarks.

2 Model description

In order to capture the dynamical effect of a purely cubic restoring force, the following truncated quartic potential well is adopted, where \( q \) is a nondimensional displacement variable. A sketch of the potential well is shown in Fig. 1.

\[
U(q) = \begin{cases} 
q^4, & |q| < 1 \\
1, & \text{else}
\end{cases}
\]

(1)

The nondimensional equation of motion of the particle under a periodic mono-chromatic excitation is derived from Eq. (1) as follows:

\[
\ddot{q} + 4q^3 = f \sin(\Omega \tau)
\]

(2)

Here \( \Omega \) and \( f \) are the nondimensional frequency and amplitude of the external excitation and \( \tau \) is a nondimensional timescale. The particle escapes from the well when it reaches the upper bound of the well, i.e. when \( U(q) = 1 \). Hence, the following displacement-based escape criterion is adopted:

\[
\max_\tau |q(\tau)| = 1
\]

(3)

3 Topology exploration of the resonance manifold

The dynamical system in Eq. (2) can be derived from the following underlying Hamiltonian:

\[
H = H_0(p, q) - f q \sin(\Omega \tau), \quad H_0(p, q) = \frac{1}{2} p^2 + q^4
\]

(4)

Here, \( H_0 = E \) is the conservative component of the Hamiltonian, and \( E \) is the energy of the particle. It
corresponds to the free motion of the particle in absence of external excitation and is determined only by initial conditions. The momentum of the particle is denoted by $p = \dot{q}$. The canonical transformation to action-angle (AA) variables is performed using the following well-known formulas:

$$I(E) = \frac{1}{2\pi} \int p(q, E) dq,$$

$$\theta = \frac{\partial}{\partial E} \int_0^q p(q, I) dq$$  \hspace{1cm} (5)

Here $I$ and $\theta$ are the action and angle variables, respectively. Theoretically, expressions $p(I, \theta)$ and $q(I, \theta)$ can be obtained by inverting the relations in Eq. (5). The conservative component of the Hamiltonian depends on the action and angle variables as follows:

$$H = H_0(I) - f q(I, \theta) \sin (\Omega \tau)$$  \hspace{1cm} (6)

Due to the periodicity of the second term in Eq. (6) it can be expressed using a Fourier series as follows [15]:

$$H = H_0(I) + \frac{\alpha}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} q_n(I) \left( e^{i(n\theta+\Omega \tau)} - e^{-i(n\theta-\Omega \tau)} \right),$$

$$q_n = \bar{q}_{-n}$$  \hspace{1cm} (7)

Here bar stands for a complex conjugate. The following well-known relations between the action and angle variables and the Hamiltonian are used:

$$\dot{i} = -\frac{\partial H}{\partial \theta}, \quad \dot{\theta} = \frac{\partial H}{\partial I}$$  \hspace{1cm} (8)

Following Eqs. (7) and (8), the Hamilton equations take the following form:

$$\dot{i} = \frac{f}{2} \sum_{n=-\infty}^{\infty} q_n(I) \left( e^{i(n\theta+\Omega \tau)} - e^{-i(n\theta-\Omega \tau)} \right)$$

$$\dot{\theta} = \frac{\partial H_0}{\partial I} + \frac{f}{2} \sum_{n=-\infty}^{\infty} \frac{\partial q_n(I)}{\partial I} \left( e^{i(n\theta+\Omega \tau)} - e^{-i(n\theta-\Omega \tau)} \right) - \Omega$$  \hspace{1cm} (9)

In the current analysis, we assume that the excitation frequency is in the vicinity of one, i.e. $\Omega \approx 1$. In order to analyze this case, we also assume slow evolution of the phase variable $\nu = \theta - \Omega \tau$ with respect to all the other phase combinations in Eq. (9). Hence, averaging over these fast timescales yields the following slow-flow equations:

$$\dot{j} = -\frac{f}{2} \left( q_1(J) e^{i\nu} + \bar{q}_1(J) e^{-i\nu} \right)$$

$$\dot{\nu} = \frac{\partial H_0}{\partial J} - \frac{f}{2} \left( \frac{\partial q_1(J)}{\partial J} e^{i\nu} - \frac{\partial \bar{q}_1(J)}{\partial J} e^{-i\nu} \right) - \Omega$$  \hspace{1cm} (10)

Here $J(\tau) = \langle I(\tau) \rangle$ is the averaged action variable and variable $q_1(J)$ is the first Fourier coefficient of $q(I, \theta)$. By direct integration of the system in Eq. (10), the following conservation rule is obtained:

$$C(J, \nu) = H_0(J) - \frac{f}{2} \left( q_1(J) e^{i\nu} - \bar{q}_1(J) e^{-i\nu} \right) - \Omega J$$  \hspace{1cm} (11)

Equation (11) represents a family of resonance manifolds (RMs), where variable $C$ is determined by initial conditions. In the current study, we consider the case in which the particle begins its motion from rest at the bottom of the well, i.e. with zero initial conditions. The contour on the RM which is associated with this case is referred to as the limiting phase trajectory (LPT) and corresponds to $H_0(J) = 0$ [6]. From Eqs. (4)–(5), the averaged action variable is calculated as follows:

$$J(\xi) = \beta \frac{\xi^{\frac{1}{2}}}{\sqrt{\pi}}, \quad \beta = \frac{4}{3\pi} K \left( \frac{1}{\sqrt{2}} \right)$$  \hspace{1cm} (12)

Here, the averaged energy of the particle is denoted by $\xi(\tau) = \langle E(\tau) \rangle$ and $K$ is the elliptic integral of the first kind. The frequency of the particle’s response is obtained as follows:

$$\omega(\xi) = \frac{\left( \frac{\partial J}{\partial \xi} \right)^{-1}}{K \left( \frac{1}{\sqrt{2}} \right)}, \quad \xi^{\frac{1}{2}}$$  \hspace{1cm} (13)

In Fig. 2, the monotonic increase of the response frequency $\omega(\xi)$ stems from hardening nonlinearity associated with the positive cubic term in the equation of motion of system (Eq. 2) [16,17].

Following Eq. (5), the solution of the system is given in terms of AA variables in the following form:

$$q(\theta, \xi) = \xi^{\frac{1}{2}} \sin (\theta, \xi)$$  \hspace{1cm} (14)

Here $\text{sn}(x, i)$ is the Jacobi elliptic function of module $i = \sqrt{-1}$. Recalling Eq. (11), the particle’s displacement can be approximated using the following Fourier series:

$$q(\theta, \xi) = a_0 + \sum_{n=1}^{\infty} a_n \sin(n\theta)$$  \hspace{1cm} (15)

Detailed derivation of the AA transformation is shown in Appendix A. By following Eqs. (11), (14), and (15), and recalling that $q_1 = -i a_1/2$, we calculate the coefficient of the first term of the Fourier series, i.e. $a_1$. The latter is given in Eq. (16) and plotted in Fig. 3.

$$a_1(\xi) = \alpha \xi^{\frac{1}{2}}, \quad \alpha = \frac{2\pi \eta}{K(i)(1 + \eta^2)}$$  \hspace{1cm} (16)
4 Maximal transient energy

The main goal of this section is to obtain an analytical prediction of the transient maximal energy reached by the particle for any given set of excitation parameters $\Omega$ and $f$. In engineering systems, the transient energy can serve as a measure of the vibration-induced damage. On the other hand, in PEAs the transient energy absorption rate usually serves as a measure for the vibration mitigation capabilities of the PEA [18–20]. The maximal transient energy is obtained through several underlying dynamical mechanisms that correspond to the changing topology of the RM. For the sake of clarity, let us distinguish between two scenarios: the first is an escape from the well that corresponds to maximal transient energy.
of one $\tilde{\xi} = 1$. The second scenario corresponds to the case of maximal transient energy that reaches an energy level of interest $\tilde{\xi}$. Both mechanisms are referred to as bifurcations of type I and type II, respectively. It can be easily seen that the type I bifurcation is a particular case of the type II bifurcation for $\tilde{\xi} = 1$. In the current section, we focus on the dynamical mechanisms that govern the increase of the transient energy of the particle, while creating the analytical infrastructure for exploring the escape phenomenon that will be treated separately in Sect. 5.

4.1 Saddle mechanism

Passage of the LPT through the RM’s stationary points at $\nu = \pi$ corresponds to the following conditions, obtained from Eqs. (17)–(18):

$$\begin{align*}
C(\nu = \pi, \xi_1 | f_s) &= \xi_1 + \frac{\tilde{C}}{2} a_1(\xi_1) - \Omega J(\xi_1) = 0 \\
\frac{\partial C}{\partial \xi}(\nu = \pi, \xi_1 | f_s) &= 1 + \frac{\tilde{C}}{2} a_1'(\xi_1) - \Omega J'(\xi_1) = 0
\end{align*}$$

(19)

Here $\xi_s$ is the energy level of the stationary points of the RM at $\nu = \pi$ and $f_s$ is the critical excitation amplitude associated with the passage of the LPT through them. Tag represents differentiation with respect to the averaged energy $\xi$. The system of equations in Eq. (19) yields the following relation between the excitation frequency and the energy level of the single stationary points of the RM on line $\nu = \pi$:

$$\xi_s = \left(\frac{2\beta}{3}\right)^4 \Omega^4$$

(20)

As one can learn by plotting $C(\nu, \xi)$, the single stationary point of the RM at $\nu = \pi$ is a saddle. The energy level of the saddle point $\xi_s$ vs. the forcing frequency $\Omega$ is plotted in Fig. 4. As one can see, the energy level of the saddle point increases monotonously with the forcing frequency. However, when $\xi_s$ reaches the upper boundary of the phase cylinder $(\nu, \xi) \in (0 - 2\pi, 0 - 1)$ the saddle point (and therefore the SM) ceases to exist. The critical forcing frequency associated with the disappearance of the saddle point is denoted by $\hat{\Omega}$. The latter is calculated in Eq. (21) and described by a vertical dashed line in Fig. 4.

$$\xi_s = 1 \rightarrow \hat{\Omega} = \frac{3}{2\beta} \approx 1.9062$$

(21)

Substituting Eq. (20) into system (19) yields the following expression of the critical excitation amplitude associated with bifurcation through the SM:

$$f_s(\Omega) = \frac{2}{a_1(\xi_s)} (\Omega J(\xi_s) - \xi_s) = \frac{8\beta^3}{27\alpha} \Omega^3$$

(22)

Substituting Eq. (22) into Eq. (20) yields the critical excitation amplitude associated with the disappearance of the saddle point and the SM:

$$\hat{f} = f_s(\hat{\Omega}) = \frac{1}{\alpha} \approx 1.0471$$

(23)

As one can learn from (22), the critical excitation amplitude associated with the SM does not depend on the critical transient energy level $\tilde{\xi}$. Bifurcation of type I (escape) thought the SM is demonstrated in Fig. 5 from the perspective of the phase portrait. There, both branches of the LPT intersect at $\xi_s$, creating a continuous connection between the bottom and top boundaries of the phase cylinder. Similar structure takes place in bifurcation of type II for $\tilde{\xi} < 1$.

4.2 Maximum mechanism at $\nu = 0$

For $f > f_s(\Omega)$, bifurcations are dominated by the maximum mechanism (MM) at $\nu = 0$. In this scenario, the LPT directly approaches the critical energy level $\tilde{\xi}$ at $\nu = 0$ (bifurcation of type I: $\tilde{\xi} = 1$, bifurcation of type II: $\tilde{\xi} < 1$), without passing through the saddle point at $\nu = \pi$. Thus, this MM is referred to as the MM0. The corresponding critical forcing amplitude is
Type I bifurcation (escape) through the saddle mechanism in phase portrait, defined by $C(ν, ξ)$. The LPT is marked by a red line. Dashed black line corresponds to the upper bound of the phase cylinder that corresponds to escape from the well, i.e. $\tilde{ξ} = 1$. Type II bifurcation has similar structure for $\tilde{ξ} < 1$.

For $Ω = 1.184$, and $f = 0.2509$, a 2D projection of $C(ν, ξ)$ on the $(ν, ξ)$ plane, b 3D plot of $C(ν, ξ)$

obtained from Eq. (18) using the following set of equations:

\[
\begin{aligned}
C(ν = 0, \tilde{ξ} | f_{m,0}) &= \tilde{ξ} - \frac{f_{m,0}}{2} a_1(\tilde{ξ}) - Ω J(\tilde{ξ}) = 0 \\
\frac{∂C}{∂ξ}(ν = 0, \tilde{ξ} | f_{m,0}) &= 1 - \frac{f_{m,0}}{2} \frac{∂a_1(\tilde{ξ})}{∂\tilde{ξ}} - Ω \frac{∂J(\tilde{ξ})}{∂\tilde{ξ}} = 0
\end{aligned}
\]  

(24)

The relations in Eq. (24) correspond to the fact that the LPT is tangent to the upper bound of the phase cylinder $\tilde{ξ}$ at $ν = 0$. Solving Eq. (24) yields the following relation between the excitation frequency to the critical forcing amplitude:

\[
f_{m,0}(Ω|\tilde{ξ}) = \frac{2}{a_1(\tilde{ξ})} \left( \tilde{ξ} - Ω J(\tilde{ξ}) \right)
\]

(25)

As mentioned above, the MM0 is dominant only above the curve described in Eq. (22), i.e. for $f > f_s(Ω)$. Bifurcation of type I (escape) thought the MM0 is demonstrated graphically in Fig. 6 from the perspective of the phase portrait.

4.3 Maximum mechanism at $ν = \pi$

In contrast to the SM, for $f < f_s(Ω)$, the upper and the lower branches of the LPT are yet to intersect since the forcing amplitude is insufficient for saddle bifurcation to take place. In this case, a sub-saddle MM takes place below the saddle point at $ν = \pi$, i.e. for $\tilde{ξ} < \xi_s(Ω)$. This mechanism is referred to as the MM$\pi$, and it satisfies the following set of equations:

\[
\begin{aligned}
C(ν = \pi, \tilde{ξ} | f_{m,\pi}) &= \tilde{ξ} + \frac{f_{m,\pi}}{2} a_1(\tilde{ξ}) - Ω J(\tilde{ξ}) = 0 \\
\frac{∂C}{∂ξ}(ν = \pi, \tilde{ξ} | f_{m,\pi}) &= 1 + \frac{f_{m,\pi}}{2} \frac{∂a_1(\tilde{ξ})}{∂\tilde{ξ}} - Ω \frac{∂J(\tilde{ξ})}{∂\tilde{ξ}} = 0
\end{aligned}
\]

(26)

The relations in Eq. (26) correspond to the fact that the LPT is tangent to the line $\tilde{ξ} < \xi_s(Ω)$ on the phase cylinder at $ν = \pi$. Solving Eq. (24) yields the following relation between the excitation frequency to the critical forcing amplitude:

\[
f_{m,\pi}(Ω|\tilde{ξ}) = -\frac{2}{a_1(\tilde{ξ})} \left( \tilde{ξ} - Ω J(\tilde{ξ}) \right)
\]

(27)
Fig. 6 Type I bifurcation through the maximum mechanism at \( \nu = 0 \) (MM0) in the phase portrait, defined by \( C(\nu, \xi) \). The LPT is marked by a red line. Dashed black line corresponds to the upper bound of the phase cylinder that corresponds to escape from the well, i.e. \( \tilde{\xi} = 1 \). For \( \Omega = 1.115, \ f = 0.255 \), and a 2D projection of \( C(\nu, \xi) \) on the \((\nu, \xi)\) plane, b 3D plot of \( C(\nu, \xi) \).

Fig. 7 Escape through the maximum mechanism at \( \nu = \pi \) (MM\( \pi \)) in the phase portrait, defined by \( C(\nu, \xi) \). The LPT is marked by a red line. For \( \Omega = 1.184, \ f = 0.25 \). Dashed black and blue lines correspond to the upper bound of the phase cylinder (\( \tilde{\xi} = 1 \)) and to the maximal energy level reached by the LPT, i.e. \( \tilde{\xi} = 0.13 \) (satisfies \( \tilde{\xi} < \xi_s(\Omega = 1.184) = 0.1488 \)); a 2D projection of \( C(\nu, \xi) \) on the \((\nu, \xi)\) plane, b 3D plot of \( C(\nu, \xi) \).
Type I bifurcation (escape) thought the MM$\pi$ is demonstrated graphically in Fig. 7.

Now, the maximal transient energy level reached for any set of $\Omega$ and $f$, i.e. $\bar{\xi}(\Omega, f)$, can be obtained by inverting the relations in Eqs. (25) and (27). Each of those cases corresponds to another region in the space of forcing parameters: the former (MM0) corresponds to $f > f_s(\Omega)$ and the latter (MM$\pi$) corresponds to $f < f_s(\Omega)$. Inversion of Eqs. (25) and (27) yields the following pair of equations:

$$\bar{\xi}^{\frac{1}{2}} - \beta \Omega \bar{\xi}^{\frac{1}{2}} \pm \frac{\alpha}{2} f = 0 \rightarrow \bar{\xi}(\Omega, f)$$  \hspace{1cm} (28)

Here, the plus and minus signs correspond to the ranges $f < f_s(\Omega)$ and $f > f_s(\Omega)$, respectively. The former yields two positive real (and thus physical) roots. The larger among them corresponds to $f > f_s(\Omega)$ and the latter to $f < f_s(\Omega)$ and, therefore, the lower root is chosen. The latter yields a single positive real root. Recalling that the maximal transient energy level is bounded by the upper bound of the well $\xi = 1$, an injective mapping between excitation parameters and the resulting maximal energy level is obtained, as shown in Fig. 8. The black curve that divides the plane into two regions corresponds to $f_s(\Omega)$. The descending and ascending lines correspond to $f_m,0(\Omega|\bar{\xi}) (f > f_s(\Omega))$ and $f_m,\pi(\Omega|\bar{\xi}) (f < f_s(\Omega))$, respectively. The yellow region ($\bar{\xi} = 1$) corresponds to escape from the well (bifurcation of type I). The black level lines are iso-energy lines that describe sets of excitation parameters that lead to identical maximal transient energy levels.

The escape envelope of the well corresponds to the top iso-energy line of $\bar{\xi} = 1$. As one can see in Fig. 8, the minimum of the iso-energy lines shifts to the right until the minimum of the escape curve is obtained in the vicinity of $\Omega = 1$. This similarity to weakly nonlinear potential wells with a linear term is somewhat surprising. Although the current well is purely nonlinear and absent a linear term, the shape of the escape envelope and the location of its minimum are still preserved.

The graphical representation in Fig. 8 gives a full perspective on the predicted response energy levels for any set of excitation parameters. In the perspective of equivalent engineering systems or PEAs, Fig. 8 gives the designer a complete understanding of the response of the system or energy absorption capabilities over the parameters space of monochromatic harmonic excitations.

5 Escape envelope

In the current section, the critical conditions for escape are discussed. The set of all critical excitation parameters that lead to escape are accumulated to a curve that is referred to as the escape envelope of the well. The latter separates the plane of excitation parameters into two basins: the escape basin and the safe basin. The former is characterized by transient energy levels that exceed the critical threshold $\bar{\xi} = 1$, while the latter is associated with lower energy levels $\bar{\xi} < 1$ and corresponds to the yellow basin in Fig. 8. The escape envelope of the well is the perimeter of the safe basin in the excitation parameters plane. The main goal of the current section is to obtain an analytical expression for the escape curve of the particle. This will be performed by leveraging the fact that a type I bifurcation (escape) is a particular case of the type II bifurcation analyzed in the previous section.

5.1 Maximum mechanism at $\nu = 0$

In this scenario, the LPT directly approaches the upper energy bound of the phase cylinder $\bar{\xi} = 1$ at $\nu = 0$, without passing through the saddle point at $\nu = \pi$. The critical excitation amplitude is obtained by substituting $\bar{\xi} = 1$ into Eq. (25) as follows:

$$f_{m,0}(\Omega) = \frac{2}{\alpha} (1 - \beta \Omega)$$  \hspace{1cm} (29)

5.2 Maximum mechanism at $\nu = \pi$

This escape scenario takes place when the saddle point vanishes, i.e. for $\Omega > \hat{\Omega}$. Then, the LPT directly approaches the upper energy bound of the phase cylinder $\bar{\xi} = 1$ at $\nu = \pi$, in absence of a saddle point at $\nu = \pi$. The critical forcing amplitude is obtained from Eq. (27) by substituting $\bar{\xi} = 1$ as follows:

$$f_{m,\pi}(\Omega) = -\frac{2}{\alpha} (1 - \beta \Omega)$$  \hspace{1cm} (30)

5.3 Saddle mechanism

Since the SM is independent on $\bar{\xi}$, it takes place as long as the saddle point exists, i.e. for $\Omega \in (0, \hat{\Omega})$, as described by Eq. (22). For $\Omega > \hat{\Omega}$ the SM is overruled.
by the MM$\pi$. In other words, for excitation frequency of $\hat{\Omega}$ (Eq. 21) there is a coexistence of both the SM and the MM$\pi$, as demonstrated in Fig. 9. Frequency $\hat{\Omega}$ and the corresponding excitation amplitude $\hat{f}$ are calculated in Eq. (21) and Eq. (23), respectively. Coexistence of both MM0 and the SM corresponds to the intersection of both mechanisms in Eqs. (29) and (22), i.e. $(\Omega^*, f^*)$, as demonstrated on the phase portrait in Fig. 10. In this case, the following relation holds:

$$f_{m,0}(\Omega) = f_s(\Omega) \rightarrow \frac{4\beta^3}{27} \Omega^3 + \beta \Omega^2 - 1 = 0$$

$$\rightarrow \Omega^* = \frac{3}{\beta} \left( \frac{\sqrt{1 + \sqrt{2}}}{2} - \frac{1}{2\sqrt{1 + \sqrt{2}}} \right) \approx 1.1362$$

Substituting Eq. (31) into Eq. (22) yields the corresponding excitation amplitude as follows:

$$f^* = \frac{8\beta^3}{27\alpha}\Omega^3 = \frac{8}{\alpha} \left( \frac{\sqrt{1 + \sqrt{2}}}{2} - \frac{1}{2\sqrt{1 + \sqrt{2}}} \right)^3 \approx 0.2217$$

All three mechanisms correspond to three curves on the forcing parameters plane. The curve that is defined by $f(\Omega) = \max\{f_{m,0}(\Omega), f_s(\Omega), f_{m,\pi}(\Omega)\}$ corresponds to the escape envelope of the well, as shown in Fig. 11. In other terms, for any excitation frequency, the upper branch among the triplet corresponds to the overruling escape mechanism that governs the transient escape process. Surprisingly, as one can see in Fig. 11, the universal property of a sharp minimum of the escape curve exists also in the current case, when the potential well lacks a linear term and the well is purely nonlinear. The right shift of the dip corresponds to the hardening effect of the quartic nonlinearity, i.e. the positive cubic term in the equation of motion (Eq. 2).

### 6 Frequency response

Due to the importance of the response energy to the escape phenomenon and due to the engineering interest of this quantity, the energy-based frequency response of the particle is analyzed. The frequency response curve is defined as the intersection curve between the manifold $\tilde{\xi}(\Omega, f)$ and a plane that corresponds to a desired forcing amplitude $f$. The response curve is calculated based on Eq. (28) as shown in Fig. 12. The left branch and the right branch of each frequency response curve correspond to type II bifurcation through the MM0 and the MM$\pi$, respectively. The horizontal dashed line corresponds to escape from the well (type I bifurcation, $\tilde{\xi} = 1$), and the vertical dashed line corresponds to the sudden energy jump associated with a type II bifurcation through the SM. The intersection points between the left branch and the horizontal dashed line, and between the right branch and the vertical dashed line correspond to type I bifurcation through the MM0 and MM$\pi$, respectively.
Fig. 9  Coexistence of both the saddle mechanism and the maximum mechanism at $v = \pi$, for $\Omega = \hat{\Omega} = 1.9062$ and $\hat{f} = \hat{f} = 1.0471$; a 2D projection of $C(v, \xi)$ on the $(v, \xi)$ plane, b 3D plot of $C(v, \xi)$

Fig. 10  Coexistence of both the saddle mechanism and the maximum mechanism at $v = 0$, for $\Omega = \Omega^* = 1.1362$ and $f = f^* = 0.2217$; a 2D projection of $C(v, \xi)$ on the $(v, \xi)$ plane, b 3D plot of $C(v, \xi)$
In this section, the analytical predictions of the escape envelope and frequency response curve are validated numerically. Since the analytical treatment is based on the assumption that the forcing frequency is in the vicinity $\Omega = 1$, we expect to have the validity of the approximate analysis near this frequency and degradation for significantly smaller/larger frequencies. Hence, we restrict the numerical analysis to the vicinity of the sharp dip of the escape curve, i.e. $(\Omega^*, f^*)$. Hence, the MM$\pi$ is not investigated herein and was mainly introduced for obtaining the inverse relation $\tilde{\xi}(\Omega, f)$ for $f < f_c(\Omega)$. All numerical results are obtained by integrating Eq. (2). Figures 13 and 14 demonstrate escape through the MM0 and SM, respectively. Without loss of generality, those simulations support the bifurcation of type I; however, similar simulations for type II bifurcation will have identical properties except the maximal transient energy reached ($\tilde{\xi} < 1$). In Fig. 13 one can see the gradual increase of the response displacement/energy until escape takes place. On the other hand, in Fig. 14 we demonstrate the characteristic sudden jump in the response energy associated with the SM.

In Fig. 15a the analytical prediction of the escape curve is compared with numerical results. As one can see and as expected, the analytical approximation is in good agreement with numerical results in the vicinity of the escape envelope’s minimum (smaller $f$ values). As shown in Fig. 15b, the numerical results are in good agreement with the predicted frequency response curve for $\Omega$ values that correspond to the bifurcation of type II through both the MM0 and the MM$\pi$. However, noisy results are observed on the branch associated with the MM$\pi$ near the occurrence of the SM. This chaotic-like nature stems from the sensitivity of the SM to small perturbation due to higher frequencies that are omitted in the analytical treatment. This sensitivity corresponds to the nature of the SM itself: a small divergence of the numerical results from the predicted LPT can cause the system to reach the saddle point and then to abruptly jump to a higher energy level. This chaotic-like nature of the escape through the SM becomes dominant also in the time required for an escape near the SM branch. The ‘wiggly’ look of the numerical results in Fig. 15a gives only a partial perspective on the noisy point cloud of critical forcing amplitude values that are obtained for other ter-
8 Contribution of environmental noise to escape

In the current work, we focused on the mathematical description of the escape mechanisms of a particle in a quartic potential well, and thus the simplest case of a purely sinusoidal excitation was examined. The main objective is to identify, understand and describe the underlying escape mechanisms and not to investigate the particle’s dynamics in a realistic environment or engineering applications. Nevertheless, in real physical and engineering systems, the presence of environmental noise in the form of stochastic perturbations is inevitable. The interaction between the particle and the noisy environment can give rise to rich nonlinear dynamical regimes, as can be learned in [21–25]. Additionally, environmental noise may have a positive role in transient dynamics and escape phenomena.
Fig. 15 Numerical validation of the analytical results by integration of the equation on motion Eq. (2); a escape curve (type I bifurcation, $\tilde{\xi} = 1$)—escape via MM0 and SM are marked by solid blue and red lines, respectively. A dashed line means that this escape mechanism is overruled by the other mechanism. Numerical simulations that correspond to escape through the MM and the SM are marked by blue and red points, respectively; b frequency response curves for $f = 0.1, 0.25, 0.5$, colored in green, red, and blue, respectively. Vertical dashed lines correspond to nonsmooth energy jumps associated with crossing the transition boundary that corresponds to a bifurcation of type II through the saddle mechanism. Horizontal black dashed line corresponds to escape (bifurcation of type I, $\tilde{\xi} = 1$). Numerical simulations for each case are marked in identical colors as their corresponding response curves.

in nonequilibrium systems [26–31]. Moreover, interesting noise-induced phenomena associated with the escape problem have been observed as well, such as noise-enhanced stability and resonant activation phenomena [24,25,32,33]. The latter effect is related to the work led by Noble Prize laureate Giorgio Parisi that studied the contribution of microfluctuations on both micro- and macro-dynamics of complex and multiscale systems. One can learn from the aforementioned works that stochastic fluctuations by themselves can lead to the escape of a classical particle from a given potential well beyond a certain noise magnitude, mean frequency and standard deviation that depends both on the potential well and noise characteristics.

When exposed to Gaussian environmental noise, a realistic noise model for various engineering and physical systems, the escape envelope can be considered as a two-dimensional manifold over the space of the three characterizing parameters: magnitude, mean frequency and standard deviation. The latter can be considered as a generalization of the escape envelope obtained in the current work since it embodies not only the frequency and amplitude of the monochromatic excitation but also considers a bandwidth of a frequency set. The aforementioned escape manifold separates the noise parameter sets that correspond to escape and those that correspond to locked motion. The resulting escape manifold is characteristic of the potential well under investigation. Further study will explore the contribution of the latter three-parameter Gaussian noise to the escape problem of the potential well described in the current paper and study the spatial features of the corresponding escape manifold.

9 Concluding remarks

In the current study, the transient escape problem of a harmonically excited particle from a purely quartic potential well was investigated. This system can be observed as an equivalent model for physical and engineering systems with a cubic restoring force such as metal wires and springs, polymer structural components, and the cubic nonlinear energy sink. While previous works pointed out two underlying dynamical mechanisms, here three distinct mechanisms were identified and described analytically, giving a full understanding...
Appendix A: Detailed derivation of action–angle transformation

In this appendix, a detailed derivation of the action–angle transformation is described. The action variable is calculated by definition as follows:

\[
I(E) = \frac{4}{2\pi} \int_{0}^{\sqrt{E}} 2(E - q^4) dq = \frac{4}{\sqrt{2\pi}} E^{\frac{1}{2}} \int_{0}^{1} \frac{1}{\sqrt{1-u^4}} du = \frac{4}{3\pi} K \left( \frac{1}{\sqrt{2}} \right) E^{\frac{1}{2}} = \beta E^{\frac{1}{2}}.
\]

Here \( u = q/\sqrt{E} \) and \( K \) is the elliptic integral of the first kind. Now, the angle variable is derived according to Eq. (5):

\[
\theta = \frac{\partial}{\partial t} \int_{0}^{q} \sqrt{2(E - U(x))} dx = \omega(E) \frac{\partial}{\partial E} \int_{0}^{q} \sqrt{2(E - U(x))} dx = \frac{\omega(E)}{\sqrt{2E}} \int_{0}^{q} \frac{1}{\sqrt{1-U(x)/E}} dx
\]

Using Eq. (A.2), the angle variable is obtained as follows:

\[
\theta = \frac{\omega(E)}{\sqrt{2}} E^{-\frac{1}{2}} \int_{0}^{q/\sqrt{E}} \frac{1}{\sqrt{1-u^4}} du = \frac{\omega(E)}{\sqrt{2}} E^{-\frac{1}{2}} \left[ F(u, i) \right]_{u=0}^{q/\sqrt{E}}
\]

\[
= \frac{\omega(E)}{\sqrt{2}} E^{-\frac{1}{2}} F(q/\sqrt{E}, i)
\]

Here \( i = \sqrt{-1} \), and \( F \) is the incomplete elliptic integral of the first kind. By inverting Eq. (A.3) we obtain the following expression for the displacement:

\[
u(E, \theta) = \text{sn} \left( \frac{\sqrt{2\theta} E^{\frac{1}{2}}}{\omega(E)}, i \right) \rightarrow q(E, \theta) = E^{\frac{1}{2}} \text{sn} \left( \frac{\sqrt{2\theta} E^{\frac{1}{2}}}{\omega(E)}, i \right)
\]
Here \( k' = \sqrt{1 - k^2} \). Hence, the first term of the series is as follows:

\[
\sin(z, k) \approx \frac{2\pi}{K(k)} \sqrt{Q(k)} \sin(\zeta) \tag{A.6}
\]

The first-order approximation of the displacement can be written as follows:

\[
q(E, \theta) = \gamma(E) \sin(\theta) = \sum_{n=1}^{\infty} a_n \sin(n\theta),
\]

\[
\gamma(E) = \frac{2\pi \eta}{K(i)(1 + \eta^2)} E^{1/2} = \alpha E^{1/2}
\]

Here \( \eta = \sqrt{|Q(i)|} = e^{-\pi/2} \). The coefficient of the first term in the series is obtained as follows:

\[
a_1(E) = \frac{1}{\pi} \int_{-\pi}^{\pi} \gamma(E) \sin(\theta) \sin(\theta) d\theta = \gamma(E) = \alpha E^{1/2}
\]

References

1. Virgin, L.N.: Approximate criterion for capsize based on deterministic dynamics. Dyn. Stab. Syst. 4(1), 56–70 (1989)
2. Virgin, L.N., Plaut, R.H., Cheng, C.-C.: Prediction of escape from a potential well under harmonic excitation. Int. J. Nonlinear Mech. 27(3), 357–365 (1992)
3. Mann, B.: Energy criterion for potential well escapes in a bistable magnetic pendulum. J. Sound Vib. 323(3–5), 864–876 (2009)
4. Rega, G., Lenci, S.: Dynamical integrity and control of nonlinear mechanical oscillators. J. Vib. Control 14(1–2), 159–179 (2008)
5. Naiger, D., Gendelman, O.: Escape dynamics of a forced-damped classical particle in an infinite-range potential well. ZAMM J. Appl. Math. Mech. 101, e201800298 (2019)
6. Farid, M.: Escape of a harmonically forced classical particle from asymmetric potential well. Commun. Nonlinear Sci. Numer. Simul. 84, 105182 (2020)
7. Manevitch, L.I., Gendelman, O.V.: Tractable Models of Solid Mechanics: Formulation, Analysis and Interpretation. Springer, Berlin (2011)
8. Manevitch, L., Kovaleva, A., Shepelev, D.: Non-smooth approximations of the limiting phase trajectories for the duffing oscillator near 1:1 resonance. Phys. D Nonlinear Phenom. 240(1), 1–12 (2011). https://doi.org/10.1016/j.physd.2010.08.001
9. Farid, M.: Dynamics of a hybrid vibro-impact oscillator: canonical formalism, arXiv preprint arXiv:2104.07757
10. Gendelman, O.: Escape of a harmonically forced particle from an infinite-range potential well: a transient resonance. Nonlinear Dyn. 93(1), 79–88 (2018)
11. Farid, M., Gendelman, O.V.: Escape of a forced-damped particle from weakly nonlinear truncated potential well. Nonlinear Dyn. 103(1), 63–78 (2021)
12. Qiu, D., Seguy, S., Paredes, M.: Tuned nonlinear energy sink with conical spring: design theory and sensitivity analysis. J. Mech. Des. 140(1), 011404 (2018)
13. Vakakis, A.F., Gendelman, O.V., Bergman, L.A., McFarland, D.M., Kerschen, G., Lee, Y.S.: Nonlinear Targeted Energy Transfer in Mechanical and Structural Systems, vol. 156. Springer, Berlin (2008)
14. Li, T., Seguy, S., Berljoz, A.: Dynamics of cubic and vibro-impact nonlinear energy sink: analytical, numerical, and experimental analysis. J. Vib. Acoust. 138(3), 031010 (2016)
15. Zaslavsky, G.M.: The Physics of Chaos in Hamiltonian Systems. World Scientific, Singapore (2007)
16. Nayfeh, A.H., Mook, D.T., Holmes, P.: Nonlinear oscillations. John Wiley & Sons (2008)
17. Nayfeh, A.H., Balachandran, B.: Applied Nonlinear Dynamics: Analytical, Computational, and Experimental Methods. Wiley, Hoboken (2008)
18. Farid, M., Gendelman, O.V.: Tuned pendulum as nonlinear energy sink for broad energy range. J. Vib. Control 23(3), 373–388 (2017)
19. Farid, M., Levy, N., Gendelman, O.: Vibration mitigation in partially liquid-filled vessel using passive energy absorbers. J. Sound Vib. 406, 51–73 (2017)
20. Farid, M., Gendelman, O., Babitsky, V.: Dynamics of a hybrid vibro-impact nonlinear energy sink. ZAMM J. Appl. Math. Mech. 101, e201800341 (2019)
21. Gardiner, C.: Springer series in synergetics. Stochastic Methods. Springer Berlin, Heidelberg (2009)
22. Mikhailov, A.S., Loskutov, A.Y.: Foundations of Synergetics II: Chaos and Noise, vol. 52. Springer, Berlin (2013)
23. Horsthemke, W., Lefever, R.: Noise-induced transitions in physics, chemistry, and biology. Noise-induced Trans.: Theor. Appl. Phys. Chem. Biol. 164–200 (1984)
24. Pizzolato, N., Fiasconaro, A., Adorno, D.P., Spagnolo, B.: Resonant activation in polymer translocation: new insights into the escape dynamics of molecules driven by an oscillating field. Phys. Biol. 7(3), 034001 (2010)
25. Guarcello, C., Valenti, D., Spagnolo, B.: Phase dynamics in graphene-based Josephson junctions in the presence of thermal and correlated fluctuations. Phys. Rev. B 92(17), 174519 (2015)
26. Kramers, H.A.: Brownian motion in a field of force and the diffusion model of chemical reactions. Physica 7(4), 284–304 (1940)
27. Denaro, G., Valenti, D., La Cognata, A., Spagnolo, B., Bonanno, A., Basilone, G., Mazzola, S., Zgozi, S., Aronica, S., Brunet, C.: Spatio-temporal behaviour of the deep chlorophyll maximum in mediterranean sea: development of a stochastic model for picophytoplankton dynamics. Ecol. Complex. 13, 21–34 (2013)
28. Mikhaylov, A., Gryaznov, E., Belov, A., Korolev, D., Sharapov, A., Guseinov, D., Tetelbaum, D., Tikhov, S., Malekhonova, N., Bobrov, A., et al.: Field- and irradiation-induced phenomena in memristive nanomaterials. Physica Status Solidi (c) 13(10–12), 870–881 (2016)
29. Giuffrida, A., Valenti, D., Ziino, G., Spagnolo, B., Panebianco, A.: A stochastic interspecific competition model to predict the behaviour of lysteria monocytogenes in the fermentation process of a traditional sicilian salami. Eur. Food Res. Technol. 228(5), 767–775 (2009)
30. Denaro, G., Valenti, D., Spagnolo, B., Basilone, G., Mazzola, S., Zgozi, S.W., Aronica, S., Bonanno, A.: Dynamics of two picophytoplankton groups in mediterranean sea: analysis of the deep chlorophyll maximum by a stochastic advection–reaction–diffusion model. PLoS ONE 8(6), e66765 (2013)

31. Carollo, A., Valenti, D., Spagnolo, B.: Geometry of quantum phase transitions. Phys. Rep. 838, 1–72 (2020)

32. Guarcello, C., Valenti, D., Carollo, A., Spagnolo, B.: Effects of Lévy noise on the dynamics of sine-Gordon solitons in long Josephson junctions. J. Stat. Mech. Theory Exp. 2016(5), 054012 (2016)

33. Guarcello, C., Valenti, D., Carollo, A., Spagnolo, B.: Stabilization effects of dichotomous noise on the lifetime of the superconducting state in a long Josephson junction. Entropy 17(5), 2862–2875 (2015)

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