SOME INEQUALITIES FOR KUREPA’S FUNCTION

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Abstract. In this paper we consider Kurepa’s function $K(z)$ [3]. We give some recurrent relations for Kurepa’s function via appropriate sequences of rational functions and gamma function. Also, we give some inequalities for Kurepa’s function $K(x)$ for positive values of $x$.

1. Kurepa’s function $K(z)$

Duro Kurepa considered, in article [3], the function of left factorial $!n$ as a sum of factorials $!n = 0! + 1! + 2! + \ldots + (n-1)!$. Let us use the standard notation:

$$K(n) = \sum_{i=0}^{n-1} i!.$$  

(1.1)

Sum (1.1) corresponds to the sequence A003422 in [3]. An analytical extension of the function (1.1) over the set of complex numbers is determined by the integral:

$$K(z) = \int_{0}^{\infty} e^{-t} \frac{t^z - 1}{t-1} dt,$$  

(1.2)

which converges for Re $z > 0$ [4]. For function $K(z)$ we use the term Kurepa’s function. It is easily verified that Kurepa’s function $K(z)$ is a solution of the functional equation:

$$K(z) - K(z - 1) = \Gamma(z).$$  

(1.3)

Let us observe that since $K(z - 1) = K(z) - \Gamma(z)$, it is possible to make the analytic continuation of Kurepa’s function $K(z)$ for Re $z \leq 0$. In that way, the Kurepa’s function $K(z)$ is a meromorphic function with simple poles at $z = -1$ and $z = -n$ ($n \geq 3$) [4]. Let us emphasize that in the following consideration, in sections 2 and 3 it is sufficient to use only the fact that function $K(z)$ is a solution of the functional equation (1.3).

2. Representation of the Kurepa’s function via sequences of polynomials and gamma function

Duro Kurepa considered, in article [4], the sequences of following polynomials:

$$P_n(z) = (z - n)P_{n-1}(z) + 1,$$  

(2.1)

with an initial member $P_0(z) = 1$. On the basis of [4] we can conclude that the following statements are true:

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1
Lemma 2.1. For each \( n \in \mathbb{N} \) and \( z \in \mathbb{C} \) we have explicitly:

\[
(2.2) \quad P_n(z) = 1 + \sum_{j=0}^{n-1} \prod_{i=0}^{j} (z - n + i).
\]

Theorem 2.2. For each \( n \in \mathbb{N} \) and \( z \in \mathbb{C} \setminus (\mathbb{Z}^- \cup \{0, 1, \ldots, n\}) \) is valid:

\[
(2.3) \quad K(z) = K(z - n) + (P_n(z) - 1) \cdot \Gamma(z - n).
\]

3. Representation of the Kurepa’s function via sequences of rational functions and gamma function

Let us observe that on the basis of a functional equation for gamma function \( \Gamma(z+1) = z\Gamma(z) \), it follows that the Kurepa’s function is the solution of the following functional equation:

\[
(3.1) \quad K(z + 1) - (z + 1)K(z) + zK(z - 1) = 0.
\]

For \( z \in \mathbb{C} \setminus \{0\} \), based on (3.1), it is valid:

\[
(3.2) \quad K(z - 1) = \frac{z+1}{z}K(z) - \frac{1}{z}K(z + 1) = Q_1(z)K(z) - R_1(z)K(z + 1),
\]

for rational functions \( Q_1(z) = \frac{z+1}{z}, R_1(z) = \frac{1}{z} \) over \( \mathbb{C} \setminus \{0\} \). Next, for \( z \in \mathbb{C} \setminus \{0, 1\} \), based on (3.1), it is valid:

\[
(3.3) \quad K(z - 2) = \frac{z}{z-1}K(z - 1) - \frac{1}{z-1}K(z) = \frac{z}{z-1}K(z - 1) - \frac{1}{z-1}\left(\frac{z+1}{z}K(z) - \frac{1}{z}K(z + 1)\right) = \frac{z}{z-1}K(z) - \frac{1}{z-1}K(z + 1) = Q_2(z)K(z) - R_2(z)K(z + 1),
\]

for rational functions \( Q_2(z) = \frac{z}{z-1}, R_2(z) = \frac{1}{z-1} \) over \( \mathbb{C} \setminus \{0, 1\} \). Thus, for values \( z \in \mathbb{C} \setminus \{0, 1, \ldots, n-1\} \), based on (3.1), by mathematical induction it is true:

\[
(3.4) \quad K(z - n) = Q_n(z)K(z) - R_n(z)K(z + 1),
\]

for rational functions \( Q_n(z), R_n(z) \) over \( \mathbb{C} \setminus \{0, 1, \ldots, n-1\} \), which fulfill the same recurrent relations:

\[
(3.5) \quad Q_n(z) = \frac{z-n+2}{z-n+1}Q_{n-1}(z) - \frac{1}{z-n+1}Q_{n-2}(z)
\]

and

\[
(3.6) \quad R_n(z) = \frac{z-n+2}{z-n+1}R_{n-1}(z) - \frac{1}{z-n+1}R_{n-2}(z),
\]

with different initial functions \( Q_{1,2}(z) \) and \( R_{1,2}(z) \).

Based on the previous consideration we can conclude:

Lemma 3.1. For each \( n \in \mathbb{N} \) and \( z \in \mathbb{C} \setminus \{0, 1, \ldots, n-1\} \) let the rational function \( Q_n(z) \) be determined by the recurrent relation (3.5) with initial functions \( Q_1(z) = \frac{z+1}{z} \) and \( Q_2(z) = \frac{z}{z-1} \). Thus the sequence \( Q_n(z) \) has an explicit form:

\[
(3.7) \quad Q_n(z) = 1 + \sum_{j=0}^{n-1} \prod_{i=0}^{j} \frac{1}{z - i}.
\]
Lemma 3.2. For each \( n \in \mathbb{N} \) and \( z \in \mathbb{C} \setminus \{0, 1, \ldots, n-1\} \) let the rational function \( R_n(z) \) be determined by the recurrent relation (3.6) with initial functions \( R_1(z) = \frac{1}{z} \) and \( R_2(z) = \frac{1}{z-1} \). Thus the sequence \( R_n(z) \) has an explicit form:

\[
R_n(z) = \sum_{j=0}^{n-1} \prod_{i=0}^{j} \frac{1}{z-i}.
\]

Theorem 3.3. For each \( n \in \mathbb{N} \) and \( z \in \mathbb{C} \setminus \{0, 1, \ldots, n-1\} \) it is valid:

\[
K(z) = K(z-n) + (Q_n(z) - 1) \cdot \Gamma(z+1)
\]

and

\[
K(z) = K(z-n) + R_n(z) \cdot \Gamma(z+1).
\]

4. Some inequalities for Kurepa’s function

In this section we consider the Kurepa’s function \( K(x) \), given by an integral representation (1.2), for positive values of \( x \). Thus the Kurepa’s function is positive and in the following consideration we give some inequalities for the Kurepa’s function.

Lemma 4.1. For \( x \in [0, 1] \) the following inequalities are true:

\[
\Gamma(x+1/2) < x^2 - \frac{7}{4}x + \frac{9}{5}
\]

and

\[
(x+2)\Gamma(x+1) > \frac{9}{5}.
\]

Proof. It is sufficient to use an approximation formula for the function \( \Gamma(x+1) \) with polynomial of the fifth degree: \( P_5(x) = -0.1010678x^5 + 0.4245549x^4 - 0.6998588x^3 + 0.9512363x^2 - 0.5748646x + 1 \) which has an absolute error \( |\varepsilon(x)| < 5 \cdot 10^{-5} \) for values of argument \( x \in [0, 1] \) [12] (formula 6.1.35., page 257.). To prove the first inequality, for values \( x \in [0, 1/2] \), it is necessary to consider an equivalent inequality obtained by the following substitution \( t = x + 1/2 \) (thus \( \Gamma(x+1/2) = \Gamma(t+1)/t \)). To prove the first inequality, for values \( x \in (1/2, 1] \), it is necessary to consider an equivalent inequality by the following substitution \( t = x - 1/2 \) (thus \( \Gamma(x+1/2) = \Gamma(t+1) \)).

Remark 4.1. Let us notice that for a proof of the previous inequalities it is possible to use other polynomial approximations (of a lower degree) of functions \( \Gamma(x+1/2) \) and \( \Gamma(x+1) \) for values \( x \in [0, 1] \).

Lemma 4.2. For \( x \in [0, 1] \) the following inequality is true:

\[
K(x) \leq \frac{9}{5}x.
\]

Proof. Let us notice that the first derivation of the Kurepa’s function \( K(x) \), for values \( x \in [0, 1] \), is given by the following integral [14]:

\[
K'(x) = \int_0^\infty e^{-t}t^x \frac{\log t}{t-1} \, dt.
\]
For \( t \in (0, \infty) \backslash \{1\} \) Karamata’s inequality is true: \( \frac{\log t}{t-1} \leq \frac{1}{\sqrt{t}} \) \([2]\). Hence, for \( x \in [0, 1] \) the following inequality is true:

\[
K'(x) = \int_0^\infty e^{-tx} \frac{\log t}{t-1} \, dt \leq \int_0^\infty e^{-tx^{1/2}} \, dt = \Gamma(x + 1/2).
\]

Next, on the basis of the Lemma 4.1 and inequality (4.5), for \( x \in [0, 1] \), the following inequalities are true:

\[
K(x) \leq \int_0^x \Gamma(t + 1/2) \, dt \leq \int_0^x \left( t^2 - \frac{7}{4}t + \frac{9}{5} \right) \, dt \leq \frac{9}{5}x.
\]

\[\square\]

**Theorem 4.3.** For \( x \geq 3 \) the following inequality is true:

\[
K(x - 1) \leq \Gamma(x),
\]

while the equality is true for \( x = 3 \).

**Proof.** Based on the functional equation (1.3) the inequality (4.7), for \( x \geq 3 \), is equivalent to the following inequality:

\[
K(x) \leq 2\Gamma(x).
\]

Let us represent \([3, \infty) = \bigcup_{n=3}^\infty [n, n+1] \). Then, we prove that the inequality (4.8) is true, by mathematical induction over intervals \([n, n+1] \) \((n \geq 3)\).

**(i)** Let \( x \in [3, 4) \). Then following decomposition is true: \( K(x) = K(x - 3) + \Gamma(x - 2) + \Gamma(x - 1) + \Gamma(x) \). Hence, by Lemma 4.2 the following inequality is true:

\[
K(x) \leq \frac{9}{5}(x - 3) + \Gamma(x - 2) + \Gamma(x - 1) + \Gamma(x),
\]

because \( x - 3 \in [0, 1] \). Next, by Lemma 4.1 the following inequality is true:

\[
\frac{9}{5}(x - 3) \leq (x - 1)(x - 3)\Gamma(x - 2),
\]

because \( x - 3 \in [0, 1] \). Now, based on (4.9) and (4.10) we conclude that the inequality is true:

\[
K(x) \leq (x - 1)(x - 3)\Gamma(x - 2) + \Gamma(x - 2) + \Gamma(x - 1) + \Gamma(x) = 2\Gamma(x).
\]

**(ii)** Let the inequality (4.8) be true for \( x \in [n, n+1] \) \((n \geq 3)\).

**(iii)** For \( x \in [n+1, n+2] \) \((n \geq 3)\), based on the inductive hypothesis, the following inequality is true:

\[
K(x) = K(x - 1) + \Gamma(x) \leq 2\Gamma(x - 1) + \Gamma(x) \leq 2\Gamma(x).
\]

\[\square\]

**Remark 4.2.** The inequality (4.8) is an improvement of inequalities of Arandelović: \( K(x) \leq 1 + 2\Gamma(x) \), given in [3], with respect to the interval \([3, \infty)\).

**Corollary 4.4.** For each \( k \in \mathbb{N} \) and \( x \geq k + 2 \) the following inequality is true:

\[
\frac{K(x-k)}{\Gamma(x-k+1)} \leq 1,
\]

while the equality is true for \( x = k + 2 \).
Theorem 4.5. For each $k \in \mathbb{N}$ and $x \geq k+2$ the following double inequality is true:

$$R_k(x) < \frac{K(x)}{\Gamma(x+1)} \leq \frac{P_{k-1}(x)+1}{P_{k-1}(x)} \cdot R_k(x),$$

while the equality is true for $x = k+2$.

Proof. For each $k \in \mathbb{N}$ and $x > k$ let’s introduce the following function $G_k(x) = \sum_{i=0}^{k-1} \Gamma(x-i)$.

Thus, the following relations:

$$G_k(x) = \Gamma(x+1) \cdot R_k(x)$$

and

$$G_k(x) = \Gamma(x-k) \cdot (P_k(x) - 1)$$

are true. The inequality $G_k(x) < K(x)$ is true for $x > k$. Hence, based on (4.15), the left inequality in (4.14) is true for all $x \geq k+2$. On the other hand, based on (4.16) and (4.13), for $x \geq k+2$, the following inequality is true:

$$\frac{K(x)}{G_k(x)} = 1 + \frac{K(x-k)}{G_k(x)} = 1 + \frac{K(x-k)}{\Gamma(x-k)(P_k(x)-1)}$$

$$= 1 + \frac{K(x-k)/\Gamma(x-k+1)}{P_{k-1}(x)} \leq 1 + \frac{1}{P_{k-1}(x)} = \frac{P_{k-1}(x)+1}{P_{k-1}(x)}.$$

Hence, based on (4.15), the right inequality in (4.14) is true for all $x \geq k+2$. □

Corollary 4.6. If for each $k \in \mathbb{N}$ we mark:

$$A_k(x) = R_k(x) \quad \text{and} \quad B_k(x) = \frac{P_{k-1}(x)+1}{P_{k-1}(x)} \cdot R_k(x),$$

thus, the following is true:

$$A_k(x) < A_{k+1}(x) < \frac{K(x)}{\Gamma(x+1)} \leq B_{k+1}(x) < B_k(x) \quad (x \geq k+3)$$

and

$$A_k(x), B_k(x) \sim \frac{1}{x} \quad \text{and} \quad B_k(x) - A_k(x) = \frac{R_k(x)}{P_{k-1}(x)} \sim \frac{1}{x^k} \quad (x \to \infty).$$

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