Superfluidity of Minkowskian Higgs vacuum with BPS monopoles quantized by Dirac may be described as Cauchy problem to Gribov ambiguity equation.

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Abstract

We show that manifest superfluid properties of the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac may be described in the framework of the Cauchy problem to the Gribov ambiguity equation.

The latter equation specifies the ambiguity in choosing the covariant Coulomb (transverse) gauge for Yang-Mills fields represented as topological Dirac variables, may be treated as solutions to the Gauss law constraint at the removal of temporal components of these fields.

We demonstrate that the above Cauchy problem comes just to fixing the covariant Coulomb gauge for topological Dirac variables in the given initial time instant $t_0$ and finding the solutions to the Gribov ambiguity equation in the shape of vacuum BPS monopoles and excitations over the BPS monopole vacuum referring to the class of multipoles.

The next goal of the present study will be specifying the look of Gribov topological multipliers entering Dirac variables in the Minkowskian Higgs model quantized by Dirac, especially at the spatial infinity, $|\mathbf{x}| \to \infty$ (that corresponds to the infrared region of the momentum space).

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1 Introduction.

In the recent paper [1] the Minkowskian Higgs model with vacuum BPS monopole solutions [2, 3, 4, 5, 6] was studied in the framework of the Faddeev-Popov (FP) "heuristic" quantization scheme [7].

The essence of the FP "heuristic" quantization approach is in fixing a gauge: say, $F(A) = 0$ in FP path integrals [7].

In particular, as it was demonstrated in [1] (repeating the arguments [2]), the FP "heuristic" quantization [7] of the Minkowskian Higgs model with vacuum BPS monopoles may be reduced to fixing the temporal (Weyl) gauge $A_0 = 0$ for temporal Yang-Mills (YM) components via the $\delta(A_0)$ multiplier in the appropriate FP path integral.

For stationary BPS monopole solutions it is equivalent to vanishing "electric" fields $F_{0i}$ in the above Minkowskian Higgs model.

There may be demonstrated [1] that the absence of "electric" fields $F_{0i}$ in the Minkowskian Higgs model with vacuum BPS monopole solutions prevents any (topologically) nontrivial dynamics in that model.

The same proves to be correct also for another Minkowskian (Higgs) models with vacuum monopoles: for instance, in the 't Hooft-Polyakov model [8, 9] or in the Wu-Yang one [10].

An essential point of all the Minkowskian (Higgs) models with vacuum monopoles quantized in the FP "heuristic" [7] wise is assuming the "continuous"

$$SU(2)/U(1) \sim S^2 \equiv R$$

vacuum geometry therein.

This implies point hedgehog topological defects in these Minkowskian Higgs models.

As it was explained in Ref. [2], the origin of point hedgehog topological defects in Minkowskian Higgs models with vacuum monopole solutions is in the isomorphism

$$\pi_2R = \pi_1U(1) \equiv \pi_1H,$$

taking place obviously for the residual $U(1)$ gauge symmetry group inherent in these models.

In the paper [1] there was also argued that the Minkowskian Higgs model with vacuum BPS monopoles is the unique Minkowskian model with monopoles in which the manifest superfluidity of the vacuum takes place.

It is induced by the Bogomol’nyi equation [2, 4, 5, 6]

$$B = \pm D\Phi,$$  \hspace{1cm} (1.1)

giving the relation between the vacuum "magnetic" field $B$ and the Higgs isomultiplet $\Phi$ (taking the shape of a BPS monopole).

Herewith the transparent parallel between the BPS monopole vacuum (quantized in the FP "heuristic" [7] wise) and the superfluid component in a liquid helium II specimen [11] was pointed out in [1].
This parallel comes to the treatment of the Bogomol’nyi equation as the potentiality condition for the BPS monopole vacuum.

In this case the vacuum ”magnetic” field $B$ plays the same role that the (critical) velocity $v_0$ [11] of the superfluid motion in a liquid helium II specimen.

On the other hand, to the end of the discussion in Ref. [1], there was noted that the Bogomol’nyi equation and manifest superfluid properties of the Minkowskian Higgs model with BPS monopoles (generated by this equation) prove to be compatible with the Dirac fundamental quantization [12] as well as with the FP ”heuristic” one [7] of that model.

A brief analysis of the Dirac fundamental quantization scheme [12] conformably to the Minkowskian Higgs model with BPS monopole solutions was performed recently in the paper [13].

This comes to the Gauss-shell reduction of the Minkowskian (YM-Higgs) action functional in terms of topological Dirac variables $\hat{A}^D_i$ ($i = 1, 2$) [14, 15, 16]: transverse and gauge invariant functionals of YM fields.

The transverse gauge for topological Dirac variables $\hat{A}^D_i$ may be written down as [1, 13, 14]

$$\partial_0 D_i \hat{A}^D_i(x, t) = 0.$$  \hspace{1cm} (1.2)

On the other hand, there is an ambiguity in choosing the transverse gauge (1.2) for topological Dirac variables $\hat{A}^D$: there are always two YM fields $A^D$ and $\hat{A}^D_1$ satisfying (1.2).

It is a purely non-Abelian effect discovered in the paper [17] by V. N. Gribov and called the Gribov ambiguity. The topological causes of the Gribov ambiguity were also enough good revealed in the monograph [2] (in §T26).

On the level of the Dirac fundamental quantization [12] of the Minkowskian Higgs model with BPS monopoles, the Gribov ambiguity in specifying the transverse gauge (1.2) for topological Dirac variables may be expressed [1, 5, 6, 18] with the aid of the Gribov ambiguity equation

$$D^2 \Phi = 0$$  \hspace{1cm} (1.3)

imposed onto the Higgs field $\Phi$ having the look of a vacuum BPS monopole.

This ”potentiality condition” comes to the Bogomol’nyi equation due to the Bianchi identity

$$DB = 0.$$  

Thus the Bogomol’nyi equation, derived [2, 3] in the temporal gauge $A_0 = 0$ for YM fields (and manifest superfluid properties of the Minkowskian BPS monopole vacuum induced by this equation), proves to be compatible with the FP ”heuristic” [7] as well as with the Dirac ”fundamental” quantization [12] schemes.

As a second-order differential equation in partial derivatives, the Gribov ambiguity equation (1.3) would involve two initial conditions in a given time instant $t_0$.

Just the Gribov ambiguity equation (1.3) with mentioned two initial conditions to this equation are responsible indeed for manifest superfluid properties of the Minkowskian
Higgs model with BPS monopoles quantized in the ”fundamental” \cite{12} wise, and our task in the present study therefore is to ascertain these initial conditions.

This will be the topic of Section 2.

It will be convenient to subdivide this Section into two subsections.

In Subsection 2.1 we recall some results regarding the Gauss-shell reduction of the Minkowskian Higgs model with vacuum BPS monopole solutions in terms of topological Dirac variables $\hat{A}^D$ that have been got in the papers \cite{5, 6, 16}.

Additionally, arguments in favour that these Dirac variables are gauge invariant will be adduced.

In Subsection 1.2 we proceed immediately to the analysis of the Gribov ambiguity equation \eqref{1.3} (responsible for manifest superfluid properties of the Minkowskian BPS monopole vacuum quantized by Dirac) and the initial conditions to this equation.

We demonstrate (this fact was noted as early as in Ref. \cite{5}) that the one of these initial conditions is just the fixed Coulomb covariant gauge for Dirac variables $\hat{A}^D$ in the zero topological sector of the Minkowskian Higgs model with vacuum BPS monopoles (with its ”Gribov copies” in other topological sectors of that model).

The second of these initial conditions comes to resolving the Gribov ambiguity equation \eqref{1.3} in terms of vacuum BPS monopole solutions in the YM sector and perturbation excitations over this BPS monopole vacuum.

As it was noted in \cite{5, 6}, mentioned excitations $\bar{A}_i(x, t)$ belong to the class of multipoles and have the behaviour $1/r^{l+1}$ ($l > 0$) at the spatial infinity.

Thus the Cauchy problem to the Gribov ambiguity equation \eqref{1.3} in the fixed time instant $t_0$ will be formulated.

The important property of the Gribov ambiguity equation \eqref{1.3} we shall encounter in Subsection 2.2 is that it characterizes the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization \cite{12} as an incompressible liquid possessing additionally the superfluidity.

In demonstrating this property of the Minkowskian BPS monopole vacuum we follow the arguments \cite{19}.

As it is well known from hydrodynamics \cite{19}, the continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div} \rho \mathbf{v} = 0$$

(with $\rho$ and $\mathbf{v}$ being, respectively, the density and velocity of the considered liquid) is simplified in a radical way as $\rho = \text{const}$ (i.e. when the density remains constant along the whole volume besetting by the liquid during the whole time of motion).

The condition $\rho = \text{const}$ means just that the liquid is incompressible \cite{19}.

In this case the continuity equation acquires the simplest look

$$\text{div} \mathbf{v} = 0$$

or

$$\Delta \phi = 0$$
if \( \mathbf{v} = \text{grad} \phi \) for a scalar field \( \phi \), i.e. if the considered liquid is potential.

Thereafter, it should be recalled [1] that the vacuum "magnetic" field \( \mathbf{B} \) plays the same role that the (critical) superfluidity velocity \( v_0 \) of the superfluid component in a liquid helium II [11].

Then one comes to the Gribov ambiguity equation \( \Delta \phi = 0 \) upon replacing

\[
\mathbf{v}_0 \Leftrightarrow \mathbf{B}; \quad \phi \Leftrightarrow \Phi.
\]

Indeed, the property of the Minkowskian BPS monopole vacuum quantized by Dirac to be an incompressible liquid is reduced to the existence of definite topological invariants characterized this vacuum.

There are the magnetic charge \( m \) and the degree of the map referring to the \( U(1) \subset SU(2) \) embedding.

These topological invariants will be us discussed in detail in Subsection 1.2.

In Section 2 we study the properties of stationary Gribov topological multipliers \( v^{(n)}(\mathbf{x}) \) entering topological Dirac variables \( A^D \) in the fixed time instant \( t_0 \).

Indeed, as it was demonstrated in Ref. [14],

\[
v^{(n)}(\mathbf{x}) = U(t, \mathbf{x})|_{t=t_0};
\]

herewith the matrices \( U(t, \mathbf{x}) \) are functions of the BPS monopole background [6].

As to Gribov topological multipliers \( v^{(n)}(\mathbf{x}) \), their outlined look was elucidated in Refs. [5, 6]:

\[
v^{(n)}(\mathbf{x}) = \exp(n\hat{\Phi}(\mathbf{x})),
\]

with \( \hat{\Phi}(\mathbf{x}) \) being the Gribov phase, a scalar value may be expressed through a combination of the Pauli matrices \( \tau^a \) \((a = 1, 2, 3)\) and a Higgs BPS monopole.

The principal result will be got in Section 2 is to show the spatial asymptotic

\[
v^{(n)}(\mathbf{x}) \to \pm 1 \quad \text{as} \; |\mathbf{x}| \to \infty
\]  

for Gribov topological multipliers \( v^{(n)}(\mathbf{x}) \).

We shall follow the paper [20] at grounding this fact.

More precisely, it will be shown that Gribov topological multipliers \( v^{(n)}(\mathbf{x}) \) may be rewritten to depend on Euler angles \( \phi_i \) \((i = 1, 2, 3)\). These, in turn, would be chosen in such a wise that the boundary condition \( \text{(1.4)} \) is satisfied.

As it was demonstrated in [20] (see also Ref. [13]), the spatial asymptotic \( \text{(1.4)} \) ensures the infrared (topological) confinement of Gribov multipliers \( v^{(n)}(\mathbf{x}) \) in fermionic and gluonic Green functions in all the orders of the perturbation theory (the author intend to return to this question in one of his future articles).
2 Cauchy problem to Gribov ambiguity equation and superfluidity of BPS monopole vacuum.

2.1 Constraint-shell reduction of Minkowskian Higgs model in terms of topological Dirac variables.

In this subsection, playing a rather auxiliary role in the present study, which principal goal is formulating the Cauchy problem to the Gribov ambiguity equation and revealing its importance for the superfluid effects occurring inside the BPS monopole vacuum, we recall the said in the papers [5, 6, 16] about the constraint-shell reduction of the Minkowskian Higgs model with vacuum BPS monopole solutions in terms of topological Dirac variables [14, 15].

The base of the Dirac fundamental approach [12] to the quantization of the Minkowskian Higgs model (with vacuum BPS monopoles) is solving the YM Gauss law constraint

$$\frac{\delta W}{\delta A_0^a} = 0 \iff [D^2(A)]^{ac} A_{0c} = D_i^{ac}(A) \partial_0 A_i^c$$

(2.1)

with the covariant (Coulomb) gauge

$$A^a|| \sim [D_i^{ac}(\Phi^{(0)}) A_i^c(0)] = 0|_{t=0}$$

(2.2)

(involving the YM BPS monopole background $\Phi^{(0)}$ in the zero topological sector of the Minkowskian Higgs model with vacuum BPS monopoles).

Eq. (2.2) permits the transparent treatment as the absence, in the initial time instant $t_0$, of longitudinal components of YM fields.

Meanwhile, temporal components $A_0^a$ of YM fields, standing on the left-hand side of the Gauss law constraint (2.1), are, indeed, nondynamical degrees of freedom, the quantization of which contradicts to the quantum principles [1], can and would be removed following [12].

At attempting [16] to solve the YM Gauss law constraint (2.1) in terms of nonzero stationary initial data:

$$\partial_0 A_i^c = 0 \implies A_i^c(t, x) = \Phi_i^{c(0)}(x) : (2.3)$$

the YM vacuum BPS monopole solutions in the Minkowskian Higgs model (in the zero topological sector of that model) is one of examples resolving (2.3) the YM Gauss law constraint (2.1), the former acquires the look

$$\partial_0 [D_i^{ac}(\Phi^{(0)}) A_i^c(0)] = 0$$

(2.4)

More precisely, the non-dynamic status of $A_0^a$ is not compatible with the quantization of these fields via their definite fixing (e.g. via the YM Gauss law constraint (2.1)), while the appropriate zero canonical momenta

$$E_0 \equiv \partial L/\partial(\partial_0 A_0) = 0$$

contradict the commutation relations and uncertainty principle [16].
upon the removal, ala [12], temporal YM components $A_0^a$ from the left-hand side of this Eq.

In this case the covariant Coulomb gauge (2.2) just will be [16] the solution to the YM Gauss law constraint (2.1) in the fixed time instant $t_0$ at resolving (2.3) this constraint in terms of YM vacuum BPS monopole solutions.

In particular, vacuum YM BPS monopole solutions (2.3) to the YM Gauss law constraint (2.1) may be chosen to be transverse and satisfy (2.2).

Note, in connection with the said, that such possibility for vacuum YM BPS monopole solutions to be transverse fields is the specific of rather the Dirac fundamental quantization method [12] (involving resolving the YM Gauss law constraint (2.1)) than the ”heuristic” one [7].

As it was discussed in [11, 13] (repeating the arguments [2]), in Minkowskian Higgs models with monopoles quantized in the ”heuristic” [7] wise, it is enough only to fix the temporal (Weyl) gauge $A_0 = 0$ via the $\delta(A_0)$ multiplier in appropriate FP path integrals. With taking account of the covariant Coulomb gauge (2.2), the YM Gauss law constraint (2.1) acquires the alternative look [6]

$$\partial_0 A^a[|A_i| + ((0), t, x)] = 0$$

over the set of (topologically trivial) vacuum YM BPS monopole solutions (2.3).

On the other hand, Eq. (2.5) may by treated also as the covariant Coulomb gauge. Just such treatment of (2.5) we shall utilize in the present study.

It is easy to see that Eqs. (2.4) and (2.5) are mathematically equivalent (more exactly, assuming (2.3), one comes to (2.4); as well as in another cases resolving the YM Gauss law constraint (2.1) though, since the covariant derivative $D$ and the time one, $\partial_0$, are commutative.). This remark will play the crucial role in the near future.

Indeed, the YM Gauss law constraint (2.1) refers to the ”pure” YM theory, without another (quantum) fields: for instance, Higgs and fermionic modes.

Meanwhile, if enumerated fields are present in a gauge non-Abelian model, it may be demonstrated (see e.g. §15 in [21]) that the Gauss law constraint would contain a current item $\rho^L$ that is, in the non-Abelian theory, the sum of two items: the non-Abelian and fermionic currents.

Regarding Higgs fields, also contributing to the current item $\rho^L$ [21], we should like to note the following.

When, at going over to the Minkowski space and violating the initial $SU(2)$ gauge symmetry group down to its (stationary) $U(1)$ subgroup, Higgs modes appear, having herewith look of stationary vacuum solutions, monopoles, and, perhaps, perturbation excitations over this monopole vacuum of higher orders (such that one can neglect them in the ”classical” YM Hamiltonian formalism), there are no an essential contribution from Higgs modes to the YM Gauss law constraint, at least in the lowest order of the perturbation theory.
The particular case resolving the YM Gauss law constraint (2.1) with the covariant Coulomb gauge is its resolving in terms of topological Dirac variables [14, 15, 16]: transverse and gauge invariant functionals of YM fields.

With account of the Gribov topological degeneration [17] of non-Abelian data, topological Dirac variables [14, 15, 16], satisfying the covariant Coulomb gauge (2.2) in the zero topological sector of the Minkowskian Higgs model quantized by Dirac [12] (and involving vacuum BPS monopole modes) and its "Gribov copies" [16]

\[ D^{ab}_{k}(\Phi^{(n)}_{k})A_b^{(n)} = 0, \tag{2.6} \]

in topologically nontrivial \((n \neq 0)\) sectors of that model, have the shape [6]

\[ \hat{A}^D_k = v^{(n)}(x)T \exp \left\{ \int_{t_0}^{t} d\tilde{t} \hat{A}_0(\tilde{t}, x) \right\} \left( \hat{A}_k^{(0)} + \partial_k \right) \left[ v^{(n)}(x)T \exp \left\{ \int_{t_0}^{t} d\tilde{t} \hat{A}_0(\tilde{t}, x) \right\} \right]^{-1} \tag{2.7} \]

with the symbol \(T\) standing for time ordering the matrices under the exponent sign.

In the initial time instant \(t_0\), the topological degeneration of initial (YM) data comes thus to "large" stationary matrices \(v^{(n)}(x) \ (n \neq 0)\) depending on topological numbers \(n \neq 0\) and called the factors of the Gribov topological degeneration or simply the Gribov multipliers.

One attempts [5, 6, 16] to find Gribov multipliers \(v^{(n)}(x)\), belonging to the \(U(1) \subset SU(2)\) embedding in the Minkowskian Higgs model, as

\[ \exp[n\hat{\Phi}_0(x)], \]

implicating the Gribov phase \(\hat{\Phi}_0(x)\).

It will be demonstrated in Section 2 (repeating the arguments [15]) that \(\hat{\Phi}_0(x)\) is a scalar constructed by contracting the Pauli matrices \(\tau^a\) and Higgs vacuum BPS monopole modes.

More exactly, in the initial time instant \(t_0\), the topological Dirac variables (2.7) acquire the look

\[ \hat{A}_k^{(n)} = v^{(n)}(x)(\hat{A}_k^{(0)} + \partial_k)v^{(n)}(x)^{-1}, \quad v^{(n)}(x) = \exp[n\Phi_0(x)]. \tag{2.8} \]

Regarding exponential multipliers (with the braces) in (2.7), it may be noted the following.

In the Minkowskian Higgs model with vacuum BPS monopole solutions, it is quite logical to assume that these would depend explicitly on YM BPS monopole modes (belonging to the zero topological sector of that Minkowskian Higgs model) [6].

Thus one can always consider the matrices

\[ U(t, x) = v(x)T \exp \left\{ \int_{t_0}^{t} \frac{1}{D_2^2(\Phi^{BPS})} \partial_0 D_k(\Phi^{BPS}) \hat{A}^k \right\} d\tilde{t} \right\}. \tag{2.9} \]
Following the work [6], let us denote as $U^D[A]$ the exponential expression in (2.9); this expression may be rewritten [6, 18] as

$$U^D[A] = \exp\left\{ \frac{1}{D^2(\Phi^{BPS})} D_k(\Phi^{BPS}) \hat{A}^k \right\}$$

(2.10)

over the stationary BPS monopole background.

We shall refer to the matrices $U^D[A]$ as to the *Dirac ’dressing’ of non-Abelian fields* (following [6, 16]).

Meanwhile, temporal component of topological Dirac variables $\hat{A}^D$ would be removed (following Dirac [12]) according to the grounds us stated above: these grounds come to the nondynamical status of temporal components of YM fields.

Thus [14]

$$U(t, x)(A_0^{(0)} + \partial_0)U^{-1}(t, x) = 0.$$  

(2.11)

Eq. (2.11) can serve for specifying *Dirac* matrices $U(t, x)$.

In order for Dirac variables (2.7) to be gauge invariant, it is necessary [14, 20] that exponential multipliers $U(t, x)$, (2.9), entering (2.7), cancel the action of YM gauge transformations

$$\hat{A}^u_i = u(t; x)(\hat{A}_i + \partial_i)u^{-1}(t; x).$$  

(2.12)

The said may be written down as the transformations law for $U(t, x)$:

$$U(t, x) \rightarrow U_u(t, x) = u^{-1}(t, x)U(t, x).$$  

(2.13)

If the transformations law (2.13) for matrices $U(t, x)$ takes place, it is easy to demonstrate that (topological) Dirac variables (2.7) are indeed gauge invariant performing the following computations proposed in Ref. [20].

Denoting matrices $U(t, x)$ as $v[A]$, one write [20]

$$\hat{A}^D_i[A^u] = v[A] u^{-1}(\hat{A}_i + \partial_i)u^{-1}u v[A]^{-1} = \hat{A}^D_i$$

(2.14)

(the fact that matrices $v[A]$ and $u$ are commute with each other was utilized at the computations (2.14); for the $U(1) \subset SU(2)$ embedding taking place in the Minkowskian Higgs theory in question, this is obviously and is not associated with additional difficulties).

In the light of Eqs. (2.7) and (2.14), the following can be concluded, and this is very important and interesting. Eq. (2.7) seems to be rather a kind of a topological map from the zero topological sector of the model we study now to its $n^{th}$ topological sector. Herewith its zero topological sector is represented by gauge fields $\hat{A}_k^{(0)}$.

There exists a simple mathematic model (see Lecture 2 in [23]) which describes correctly such a topological map. This is the *covering construction* where the set $B$ of gauge (covariant) fields $\hat{A}_k^{(0)}$ constitutes the base of such a covering while its discrete infinitely-valent fibre is the set of all fields $\hat{A}_k^{D(n)} \ (n \in \mathbb{Z})$, the (topological) Dirac variables (2.7).
Finishing this subsection, we should like make some concluding remarks regarding properties of topological Dirac variables.

1. In the majority of formulas we have encountered in the present subsection, YM fields $\hat{A}$ are present. These fields have indeed the following look in the Planckian $\hbar, c$ units [24]:

$$\hat{A}_\mu = g A^a_{\mu} \tau^a \frac{i \hbar c}{2i \hbar c}.$$ (2.15)

The account of $\hbar$ and $c$ in latter Eq. is closely connected with the actual value $g/(\hbar c)$ of the strong coupling constant.

2. The covariant Coulomb gauge (2.5) for YM fields acquires its look [14, 20, 24]

$$D_{i}^{\alpha\beta}(\hat{A}^{D})\partial_{0}(\hat{A}^{D}_{\alpha}) \equiv 0$$ (2.16)

in terms of topological Dirac variables $\hat{A}^{D}$.

Moreover, the Coulomb gauge (2.2), (2.6) for topological Dirac variables $\hat{A}^{D}$ [11, 13]:

$$D_{i}\hat{A}^{D}_{i} = 0,$$

and the removal (2.11) [14] of temporal components of these topological Dirac variables imply the generalized Lorentz gauge

$$D_{\mu}\hat{A}^{D}_{\mu} = 0$$ (2.17)

for them.

3. It is quite naturally to expand any (say, transverse) YM field $\hat{A}(x, t)$, (2.15), in the sum of a background stationary (vacuum) field $\hat{\Phi}(x)$ and a field $\hat{\Phi}(x, t)$ belonging to the excitation spectrum over the background $\hat{\Phi}$:

$$\hat{A}(x, t) = \hat{\Phi}(x) + \hat{\Phi}(x, t).$$ (2.18)

In the said is the essence of the postulate has been suggested in Ref. [18] for Minkowskian non-Abelian theories.

In Minkowskian non-Abelian theories, physical (transverse) YM variables always may be represented as sums of the (singular) stationary Bose condensate $\hat{b}(x)$ and dynamical regular fields $\hat{a}(x, t)$, treated as perturbation excitations over this (singular) stationary Bose condensate:

$$\hat{A}(x, t) = \hat{b}(x) + \hat{a}(x, t).$$ (2.19)

Generally speaking, this assumption is irrelevant to choosing the space where a non-Abelian gauge theory is considered (either the Minkowskian or Euclidian one), but in Minkowskian non-Abelian models it has by far interesting consequences than in Euclidian non-Abelian models.
Extracting the c-number field $\hat{b}(x)$, one should ensure herewith that energies of (stationary) quantum states are finite and that these quantum states are stable.

The decomposition (2.19), in the Minkowski space, shouldn’t be suppressed by factors of the $\exp(-S_E(\hat{b})/\hbar)$ type, always taking place in the Euclidian space $E_4$. And it is the one more argument in favour of going over to the Minkowski space from the Euclidian one at considering the non-Abelian vacuum (in particular, that quantized by Dirac [12]).

In the Minkowskian Higgs model quantized by Dirac, the postulate (2.19) [18] takes the shape

$$\hat{A}^{(n)}_i(t, x) = \hat{\Phi}^{(n)}_i(x) + \hat{A}^{(n)}_i(t, x)$$

(2.20)
in each topological class of that model.

If this model includes vacuum BPS monopole modes, $\hat{\Phi}^{(n)}_i(x)$ are just BPS monopole modes belonging to the YM sector and satisfying Eq. (2.3).

Eq. (2.20) is in a good agreement with Eqs. (2.7), (2.8) for topological Dirac variables $\hat{A}^D$. It is, actually, the look of $\hat{A}^D$ meeting the postulate (2.19) [18].

In particular, topological Dirac variables $\hat{A}^{(n)}_i(t, x)$, (2.20), are transverse and gauge invariant functionals of YM fields, satisfying herewith the covariant Coulomb gauge (2.6) in each topological class of the Minkowskian Higgs model quantized by Dirac [12].

Furthermore, there may be assumed (as it was done in [16]) that fields $\hat{A}^{(n)}_i(t, x)$ belong to the class of multipoles, with their $O(1/r^{l+1})$ ($l > 0$) behaviour at the spatial infinity.

Comparing Eqs. (2.20) and (2.8), one can single out [16] the topologically degenerated BPS monopole background

$$\hat{\Phi}^{(n)}_i := v^{(n)}(x)[\hat{\Phi}^{(0)}_i + \partial_i v^{(n)}(x)]^{-1}, \quad v^{(n)}(x) = \exp[n\hat{\Phi}_0(x)],$$

(2.21)
and topologically degenerated multipoles:

$$\hat{A}^{(n)} := v^{(n)}(x)[\hat{A}^{(0)} + \partial_i v^{(n)}(x)]^{-1},$$

(2.22)

Principal shortcomings of the Euclidian instanton non-Abelian model [25] were pointed out recently in the paper [13] (repeating the arguments of Refs. [16, 18, 26]).

Actually, as a principal shortcoming of the Euclidian instanton non-Abelian model [25], the purely imaginary values of the topological momentum [16, 18, 26]

$$P_N = \pm 8\pi i/g^2 \equiv 2\pi k + \theta,$$

referring to the Euclidian $\theta$-vacuum (with $\theta \in [-\pi, \pi]$ [18]), may be indicated.

This involves [13] the bad behaviour of the $\theta$-vacuum plane wave function [18]

$$\Psi_0[A] = \exp(iP_N X[A])$$

implicating the winding number functional $X[A]$ taking integers) at the minus sign before $P_N$.

As a result, it is impossible to give the correct probability description of the instanton $\theta$-vacuum; that is why the latter one refers to unobservable, i.e. unphysical, values.

In the paper [26], the effect appearing the purely imaginary values of the topological momentum $P_N$ for the Euclidian $\theta$-vacuum was referred to as the so-called no-go theorem: the absence of physical solutions in the Euclidian instanton YM (non-Abelian) theory [25].
as excitations over this BPS monopole background.

Emphasise again that the fields $\Phi^{(n)}_i$ and $\bar{A}^{(n)}_i$ are transverse functionals of YM fields.

Topological Dirac variables (2.21), (2.22) permit the transparent physical interpretation as YM modes “dressed” in the Higgs Bose condensate given in the model [5, 6, 14, 15, 16] in the shape of Higgs BPS monopoles.

2.2 Cauchy problem to Gribov ambiguity equation and super-fluid properties of Minkowskian Higgs model with BPS monopoles quantized by Dirac.

As it was demonstrated in Ref. [5, 6, 16], the Coulomb constraint-shell gauge (2.6) keeps its look in each topological class of the Minkowskian Higgs model with vacuum BPS monopole solutions quantized by Dirac [12] if the Gribov phase $\Phi_0(x)$, entering Eqs. (2.7), (2.8) for topological Dirac variables in that model, satisfies the equation of the Gribov ambiguity (or simply the Gribov equation)

$$[D_i^2(\Phi_k^{(0)})]^{ab}\Phi_{(0)b} = 0.$$  (2.23)

The origin of latter Eq. is in the standard definition of a ”magnetic” field,

$$B^a_i = \epsilon_{ijk}(\partial^j A^{ak} + \frac{g}{2}\epsilon^{abc}A^j_b A^k_c).$$  (2.24)

Really, the values $D_i A^{ia}$ (in particular, $D_i A^{iD}$ if topological Dirac variables $A^D$ are in question) have the same dimension that a ”magnetic” YM field $B^a_i$, given via (2.24).

Then it is easy to see that the Gribov ambiguity equation (2.23) is the consequence of the Bogomol’nyi equation (1.1), implicating (topologically trivial) Higgs vacuum BPS monopole modes $\Phi_0$.

Speaking about the connection between the Bogomol’nyi and Gribov ambiguity equations, note that this connection may be given via the Bianchi identity

$$\epsilon^{ijk}\nabla_i F^b_{jk} = 0,$$

that is equivalent to

$$DB = 0$$

in terms of the (vacuum) ”magnetic” field $B$, (2.24).

Meanwhile, mathematically, the Gribov equation (2.23) implies that the FP determinant [21, 27]

$$\det(\tilde{\Delta}_a^b) \equiv \det(-D_{a_i}^b \partial^i) = \det(-[\partial_i^2 + \partial_i \text{ad}(A^i)])$$  (2.25)

with

$$\text{ad}(A)X \equiv [A, X]$$

for an element $X$ of the $SU(2)$ Lee algebra, becomes zero at setting the constraint-shell (Coulomb) transverse gauge (2.2), (2.6) in the Minkowskian Higgs model quantized by Dirac [12].
In the terminology of the paper [27] (see also [13]), there was shown that the $L^2$ (Lebesgue) norm $\|A\|^2$ of an YM potential $A$ along the fixed gauge orbit, i.e.

$$\|A\|^2 \equiv F_A(g) = - \int_M d^3x \text{tr}[(v^{-1}(x)A_i v(x) + v(x)\partial_i v^{-1}(x))^2],$$  \hspace{1cm} (2.26)

with $v(x) \in SU(2)$, attains its (local) minimum as this YM potential is transverse, $\partial_i A^i = 0$; in this case $\hat{\Delta}$, (2.25), becomes a positive defined operator $^4$.

The set of all such YM potentials is called the Gribov region. We shall denote it as $\Omega$ following [27], while its boundary $\partial \Omega$ is called the Gribov horizon [27].

At the Gribov horizon $\partial \Omega$, the lowest eigenvalue of the FP operator $\hat{\Delta}$, (2.25), vanishes,

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$^4$ In the paper [27], regarding the Euclidian instanton non-Abelian theory [25], the Weyl gauge $A_0 = 0$, has been fixed in the theory [25] and just resulting instanton stationary solutions, was utilized. This implied expressing the operator $\Delta$ in terms of only spatial indices.

The same result is achieved at the Gauss-shell reduction of the Minkowskian Higgs model (with vacuum BPS monopole solutions), as discussed in the previous subsection, with ruling out (2.11) [14] temporal components of topological Dirac variables $A^D$. 

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14
and points on $\partial \Omega$ are associated with coordinate singularities.

The next important definition regarding the FP operator $\hat{\Delta}$, (2.25), and its determinant $\text{det} \hat{\Delta}$ is the fundamental domain [27], we shall denote as $\Lambda$ henceforth. It is the set of absolute minima of the norm functional (2.26).

Upon our analysis of the FP operator $\hat{\Delta}$, (2.25), it becomes obvious that the Gribov horizon $\partial \Omega$ is just the set in the space of topological Dirac variables $A_i^D$ ($i = 1, 2$), (2.7).

In some physical literature, e.g. [17, 28], the equation

$$\hat{\Delta}(\Psi) \equiv (-\partial_i D^i(A)(\Psi)) = -(\partial^2 \Psi + \partial_i \text{ad}(A^i))\Psi = \epsilon(A)\Psi$$

is treated as a specific Schrödinger equation, with $A_i$ playing the role of a potential.

For small values of $A_i$ (that now don’t assumed to be transverse, $\partial_i A^i \neq 0$), this equation is solvable for positive $\epsilon$ only.

More precisely, denoting by $\epsilon_1(A), \epsilon_2(A), \epsilon_3(A), \ldots$ the eigenvalues corresponding to a given field configuration $A$, one has that, for small $A_i$, all the $\epsilon_i(A)$ are positive, $\epsilon_i(A) > 0$.

However, for a sufficiently large value of the field $A_\mu$, one of the eigenvalues, say $\epsilon_1(A)$, turns out to vanish, becoming then negative as the field increases further, and so on.

As in the case of the Schrödinger equation, this means that the field $A_\mu$ is large enough to ensure the existence of negative energy solutions, i.e. bound states.

For a greater magnitude of the field $A_\mu$, a second eigenvalue: say, $\epsilon_2(A)$, will vanish, becoming then negative as the field increases again.

Following Gribov [17], one can thus subdivide the functional space of gauge fields into the regions $C_0, C_1, C_2, \ldots, C_n$ over which the FP operator $\hat{\Delta}$, (2.25), has, respectively, 0, 1, 2, $\ldots$, $n$ negative eigenvalues.

These regions are separated by lines $l_1, l_2, l_3, \ldots, l_n$ on which the FP operator $\hat{\Delta}$ takes its zeros.

More exactly, in the region $C_0$ all the eigenvalues of the FP operator $\Delta$ are positive, i.e. $\Delta > 0$.

At the boundary $l_1$ of the region $C_0$, the first vanishing eigenvalue of the FP operator $\hat{\Delta}$ appears; namely on $l_1$ the FP operator $\hat{\Delta}$ possesses a normalizable zero mode $\chi$:

$$\hat{\Delta} \chi = 0.$$

In the region $C_1$, the FP operator $\hat{\Delta}$ has one bound state, i.e. one negative energy solution. At the boundary $l_2$, a zero eigenvalue reappears.

In the region $C_2$, the FP operator $\hat{\Delta}$ has two bound states, i.e. two negative energy solutions. On $l_3$ a zero eigenvalue shows up again, and so on.

Just the boundaries $l_1, l_2, l_3, \ldots, l_n$, on which the FP operator $\hat{\Delta}$, (2.25), has zero eigenvalues (the number of which coincides with the number of the boundary) are called the Gribov horizons in the terminology [17].

For instance, the boundary $l_1$, where the first vanishing eigenvalue appears, is called the first horizon [28]. The connection between this definition [17, 28] of the Gribov horizons and that given in Ref. [27] can be given with the relation

$$\partial \Omega = \bigcup_i l_i.$$

Note also [28] that the FP operator $\hat{\Delta}$, (2.25), is Hermitian one: $\hat{\Delta}^\dagger = \hat{\Delta}$.

Roughly, it is associated with the manifest Hermitian operator $i\partial$ (the momentum operator) entering Eq. (2.25) together with YM potentials $A$, chosen to be real.

The complete proof that the FP operator $\hat{\Delta}$, (2.25), is indeed Hermitian one is given in the paper [28], we recommend our readers for studying the question.
transverse functionals of gauge fields $A$, over which the Gribov ambiguity equation (2.23) is satisfied.

In the Minkowskian Higgs model quantized by Dirac [12] and involving vacuum BPS monopole modes, the Gribov horizon $\partial \Omega$ may be expressed alternatively (in comparison with the above definition given in Ref. [27]) as the set of (topologically degenerated) Higgs vacuum BPS monopole modes $\Phi_a^{(n)}$ belonging to the kernel of the FP operator $\Delta$, (2.25):

$$\Phi_a^{(n)} \in \ker \Delta, \quad n \in \mathbb{Z}.$$ (2.27)

Actually, it is the set of all the Higgs vacuum BPS monopole modes available in the quested Minkowskian Higgs model quantized by Dirac.

From the definition [27] of the Gribov region $\Omega$ it follows that this region is swept by the family of Coulomb covariant gauges (2.2) for topological Dirac variables $A^D$, in the zero topological sector of the Minkowskian Higgs model quantized by Dirac, and their Gribov copies (2.6) in other topological sectors of that model.

Moreover, in terms of the definitions of the Gribov region $\Omega$ and its boundary, the Gribov horizon $\partial \Omega$, given in the papers [27, 28], the said implies that the Gribov horizon $\partial \Omega$ becomes the space-like surface $\partial^2 \mu = 0$ when the constraint-shell transverse gauge (2.6) for topological Dirac variables $A^D$ is fixed course the Gauss-shell reduction of the Minkowskian Higgs model with vacuum BPS monopoles.

Further, the Dirac removal (2.11) [14] of temporal YM components implies formally that $\partial_{\mu} \hat{A}^D_{\mu} = 0$.

Then in terms of topological Dirac variables (2.7), (2.11), satisfying the constraint-shell transverse gauge (2.6), the Gribov horizon $\partial \Omega$ may be continued actually to the isotropic surface $\partial^2 \mu = 0$, and this means that gauge fields in the Minkowskian Higgs model quantized by Dirac [12], taking the look of (topological) Dirac variables (2.7), (2.11), upon the constraint-shell reduction of this theory, are, indeed, massless fields.

Eq. $\partial^2 \mu = 0$ for the Gribov horizon $\partial \Omega$ in the Minkowskian Higgs model quantized by Dirac implies (cf. [28]) that there is only a one Gribov region $C_0$ in that theory, where the FP operator (2.25) possesses 0 negative eigenvalues.

Respectively, instead of the set \{li\} [28] of Gribov horizons in a YM model where the transverse gauge of fields isn’t fixed, now we encounter only the one Gribov horizon $\partial \Omega$.

On the other hand, such Gribov horizon $\partial \Omega$ possesses, indeed, an infinite set of zero eigenvalues for the FP determinant $\det \Delta$ due to the nontrivial cohomological structure of YM fields, topological Dirac variables $\hat{A}^D$, has been revealed in the recent papers [5].

Indeed, the "gauge" $\partial_i A^i = 0$, utilized in Ref. [27] is "narrower" than the transverse gauge $D_i A^i = 0$ (the Coulomb covariant gauge (2.2), (2.6) for topological Dirac variables $A^D$, (2.7), in the Minkowskian Higgs model quantized by Dirac is the particular case of such transverse gauge).

But one can think that the transverse gauge $D_i A^i = 0$ comes to the gauge $\partial_i A^{i_a} = 0$ and to the condition

$$\epsilon_{ijk} A^{i_a} A^a_k = 0$$

(where the group indices $a$ are written down explicitly).

Just at latter two assuming, topological Dirac variables $A^D$, (2.7), in the Minkowskian Higgs model quantized by Dirac [12] may be treated as massless gauge fields, as it will be discussed below.

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(where the group indices $a$ are written down explicitly).

Just at latter two assuming, topological Dirac variables $A^D$, (2.7), in the Minkowskian Higgs model quantized by Dirac [12] may be treated as massless gauge fields, as it will be discussed below.
The said is also the specific of the Dirac fundamental quantization \[12\] of the Minkowskian Higgs model, involving its Gauss-shell reduction in terms of Dirac variables \((2.7), (2.11)\).

On the face of it, the zero FP determinant, \((2.25)\), can involve nontrivial Dirac dressing matrices \((2.10)\) at automatic fixing the Coulomb gauge \((2.2)\) for Dirac variables \((2.7)\).

But in the initial time instant \(t = t_0\) the integral in \((2.9)\) becomes zero; thus in this instant the Gribov ambiguity equation \((2.23)\) does not affect the gauge transformations \((2.8)\), i.e. the nature of Dirac variables, including the Gribov topological degeneration of initial YM data.

In other words, in each time instant \(t\) one can pick out a space-like surface \(\mathcal{H}(t)\) in the Minkowski space-time over which the topological degeneration of initial YM data occurs in the Minkowskian Higgs model quantized by Dirac \[12\] (and involving vacuum BPS monopole solutions).

This space-like surface is determined, in effect, by the Gribov ambiguity equation \((2.23)\).

Note that Eqs. \((2.4), (2.2)\) can be treated as Cauchy conditions for the Gribov ambiguity equation \((2.23)\) in the (initial) time instant \(t_0\).

Therefore to specify the space-like surface \(\mathcal{H}(t_0)\) over which the topological degeneration of initial YM data occurs in the Minkowskian Higgs model quantized by Dirac, one would solve the Cauchy problem \((2.23)\) with the initial conditions \((2.4), (2.2)\), i.e. in the class of vacuum YM BPS monopoles \((2.3)\) (and observable YM fields, multipoles \(\hat{\hat{A}}^{(n)}\), \((2.22)\), as perturbation excitations over this monopole vacuum with the same topological numbers that appropriate monopoles), satisfying herewith the Coulomb gauge \((2.2)\) at resolving the YM Gauss law constraint \((2.1)\) with removing a la Dirac \[12\] temporal components of YM fields.

In the light of the said above about the actual property of the Gribov horizon \(\partial \Omega\) in the Minkowskian Higgs model quantized by Dirac to be an isotropic surface \(p_0^2 = 0\) in the Minkowski space-time (in its momentum representation), it is easy to see that one can choose the space-like surface \(\mathcal{H}(t_0)\) in the Minkowski space-time (in its coordinate representation) in such a wise that it will cross the light cone in the coordinate Minkowski space (as, for instance, it is depicted in the monograph \[29\], in Fig. 1.2, i.e. it can be a three-sphere \(S^3\) set by Eq. \(t_0^2 = x^2 + y^2 + z^2\)).

The said implies the absence of the nonzero mass scale for gauge (YM) fields in the Minkowskian Higgs model quantized by Dirac \[12\].

In this is the principal distinction of that from the well-known (Minkowskian) Higgs model \[30, 31\], involving the choice of the Higgs complex \(SU(2)\) doublet \(\phi = (\phi_1, \phi_2)\) in the shape

\[
< \phi >_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix},
\]

where

\[
< \phi^\dagger \phi > = v^2/2 \quad \text{with} \quad v = \mu^2/\lambda.
\]
On the other hand, the YM model \[30, 31\] doesn’t assume fixing transverse gauge for YM fields.

Upon defining the new Higgs field \[30\]

\[\phi' = \phi - \langle \phi \rangle_0\]

(note that this "trick" is in a good agreement with the postulate \(2.19\) [18], so long as by that expansion there is extracted the vacuum \(\langle \phi \rangle_0\) and excitations \(\phi'\) over this vacuum in the Higgs field \(\phi\) ) and its substituting in the Lagrangian density of the Minkowskian Higgs model, after some maths, one gets the nonzero mass \[30\]

\[M_A = \frac{g v}{2}\]  \hspace{1cm} (2.29)

for YM fields.

Since the Bogomol’nyi equation \(1.1\) [1, 2, 5, 6] describes correctly (see e.g. [1]) manifest superfluid properties inherent in the Minkowskian Higgs model \[2, 3, 4\] with vacuum BPS monopole solutions, it is easy to guess that the Gribov ambiguity equation \(2.23\), got from the Bogomol’nyi equation \(1.1\) with the aid of the Bianchi identity, is responsible for superfluid properties of the Minkowskian Higgs model with vacuum BPS monopoles at its Dirac fundamental quantization \[12\].

Respectively, the superfluidity proper to the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \[12\] may be described in the framework of the Cauchy problem to the Gribov ambiguity equation \(2.23\).

As we have discussed above, this Cauchy problem to the Gribov ambiguity equation \(2.23\) comes in the fixed (initial) time instant \(t_0\) to laying down two initial conditions to this equation.

There are the Coulomb transverse gauge \(2.2\) for topological Dirac variables \(A^D_i\), \(2.7\), and finding solutions to the Gribov equation \(2.23\) in the shape \(2.3\) of stationary vacuum BPS monopoles and perturbation excitations over this BPS monopole vacuum: multipoles \(2.22\) possessing same topological numbers that appropriate vacuum BPS monopole solutions.

The latter condition is mathematically equivalent to Eq. \(2.4\).

In this case choosing a space-like surface \(\mathcal{H}(t_0)\) in the Minkowski space-time over which the Gribov topological degeneration of initial data occurs in the Minkowskian Higgs model quantized by Dirac (this surface is specified at solving the Cauchy problem \(2.23\), \(2.2\), \(2.3\) ) acquires the following highly transparent interpretation.

Any trajectory of the superfluid potential motion inside the Minkowskian non-Abelian vacuum crosses the total set \(\mathcal{H}^T(t)\) of such surfaces (may be set in the wise us pointed above \[29\]) in each fixed time instant \(t\).

In spite of the above mathematical compatibility of the Bogomol’nyi equation \(1.1\) and the Gribov ambiguity equation \(2.23\), there is the principal distinction between these equations from the point of view choosing the way how to quantized the Minkowskian Higgs model with vacuum BPS monopole solutions.
As it was discussed in the recent papers [1, 13], the Bogomol’nyi equation (1.1) is quite compatible with the "heuristic" FP [7] quantization scheme, coming for various Minkowskian Higgs models with monopoles to fixing the temporal (Weyl) gauge $A_0 = 0$ in appropriate FP path integrals.

Herewith the Bogomol’nyi equation (1.1) is derived [2] issuing from the ordinary constrained Lagrangian of the Minkowskian Higgs model without performing the Gauss-shell reduction of the appropriate Hamiltonian.

In detail, upon fixing the temporal (Weyl) gauge $A_0 = 0$ in the Minkowskian Higgs model [2, 3, 4] with vacuum BPS monopole solutions, the Bogomol’nyi equation (1.1) is derived [2] at evaluating the Bogomol’nyi bound [1, 5, 6, 13]

$$E_{\text{min}} = 4\pi m/a, \quad a = \frac{m}{\sqrt{\lambda}}$$

(2.30)

(where $m$ denotes the magnetic charge) of the energy for the given configuration of YM and Higgs fields.

In this Eq., the "effective" Higgs mass $m/\sqrt{\lambda}$ appears.

It is taken in the so-called BPS (Bogomol’nyi-Prasad-Sommerfeld) limit [2, 3, 5, 6]

$$\lambda \to 0, \quad m \to 0 : \quad \frac{1}{\epsilon} \equiv \frac{gm}{\sqrt{\lambda}} \neq 0$$

(2.31)

for the Higgs mass $m$ and Higgs selfinteraction constant $\lambda$.

The parameter $\epsilon$, the typical size of BPS monopoles, arises in latter Eq.

On the other hand, the Bogomol’nyi equation (1.1) is compatible also with the Dirac fundamental quantization [12] of the Minkowskian Higgs model involving vacuum BPS monopole solutions.

This statement may be explained again by the actual independence the way deriving the Bogomol’nyi equation (1.1) on gauge fixing (indeed, upon removing temporal components of YM fields due to imposing the gauge condition $A_0 = 0$; in the framework of the FP "heuristic" quantization scheme [7] this can be performed by means the multiplier $\delta(A_0)$ in the appropriate path integral [1], while at the Dirac fundamental quantization [12] of the Minkowskian Higgs model with vacuum BPS monopoles temporal components of YM fields can be ruled out by the standard gauge transformation (2.11) [14]).

This justifies the logical and mathematical connection between the Bogomol’nyi and Gribov ambiguity Eqs. us discussed above.

Unlike the Bogomol’nyi equation (1.1), the Gribov ambiguity equation (2.23) is associated completely with the Dirac fundamental quantization [12] of the Minkowskian Higgs model with vacuum BPS monopole solutions, involving its Gauss-shell reduction in terms of topological Dirac variables (2.7).

It just specifies the ambiguity in the choice of these variables, gauge invariant and transverse functionals of YM fields.

The Gribov phase $\hat{\Phi}(x)$ (entering Gribov topological multipliers $v^{(n)}(x)$), whose explicit look [5, 6, 15] will be us given in the next section, is, indeed, an $U(1) \subset SU(2)$
isoscalar constructed by contracting (topologically trivial) Higgs vacuum BPS monopole
solutions $\Phi_{(0)a}$ and the Pauli matrices $\tau^a$ ($a = 1, 2, 3$).

The said just allows to assert that the Gribov ambiguity equation (2.23) affects the
Gribov phase $\hat{\Phi}(x)$.

In the paper [1] there was traced the transparent parallel between the vacuum ”magnetic” field $B$, given via the Bogomol’nyi equation (1.1), and the critical velocity $v_0$ of
the superfluid motion in a liquid helium II specimen.

Herewith there was cited [32] the enough simple relation

\[ v_0 = \frac{\hbar}{m} \nabla \Phi(t, r), \quad (2.32) \]

between this critical velocity $v_0$ and the phase $\Phi(t, r)$ of the helium Bose condensate wave function $\Xi(t, r) \in C$.

The latter one may be given as [32]

\[ \Xi(t, r) = \sqrt{n_0(t, r)} e^{i\Phi(t, r)}, \quad (2.33) \]

with $n_0(t, r)$ being the number of particles in the ground energy state $\epsilon = 0$ and serves as
a complex order parameter in the Bogolubov-Landau model [11] of the liquid helium.

In Eq. (2.32), $m$ is the mass of a helium atom.

We see that the Bogomol’nyi equation (1.1), specifying the vacuum ”magnetic” field $B$ in the Minkowskian Higgs model with vacuum BPS monopoles, and Eq. (2.32) [32],
specifying the critical velocity $v_0$ of the superfluid motion in the Bogolubov-Landau model
[11] of the liquid helium, have the similar look.

This explains the role of the Bogomol’nyi equation (1.1) as the potentiality condition
for the Minkowskian BPS monopole vacuum.

Really, any potentiality condition may be written down as [1]

\[ \text{rot grad } \Phi = 0 \quad (2.34) \]

for a scalar field $\Phi$.

Thus any potential field may be represented as grad $\Phi$ (to within a constant): in
particular, this is correctly for the vacuum ”magnetic” field $B$ in the Minkowskian Higgs
model with vacuum BPS monopoles.

The Gribov ambiguity equation (2.23), following from the Bogomol’nyi equation (1.1)
due to the Bianchi identity, also may be treated as the potentiality condition for the
Minkowskian BPS monopole vacuum, but now at the Dirac fundamental quantization
[12] of the Minkowskian Higgs model.

The next consequence of the Bianchi identity $DB = 0$ is that the Minkowskian BPS
monopole vacuum may be considered as an incompressible liquid (possessing additionally
the superfluidity) when the Dirac fundamental quantization scheme [12] is applied to the
to the Minkowskian Higgs model.
To ground the latter statement, it will be useful to recall hydrodynamics.
In the monograph [19] (in §9) there was analysed the potential motion in a liquid.
It turned out that it is necessary and sufficient that rot $\mathbf{v} = 0$ in the whole space (with $\mathbf{v}$ being the velocity of the liquid) in order for the motion of a liquid to be potential in the whole considered space.

As for each vector field possessing the zero curl, the velocity of a potentially moving liquid may be expressed [19] as the gradient of a scalar:

$$\mathbf{v} = \text{grad} \, \phi.$$ 

On the other hand, the Bianchi identity

$$D \, B = 0 \quad (2.35)$$

for the (vacuum) ”magnetic” tension $\mathbf{B}$ implies that the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization [12] is an incompressible liquid.

Really (see §10 in [19]), the well-known continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div} \, \rho \mathbf{v} = 0$$

(with $\rho$ being the density of the considered liquid) is simplified in a radical way as $\rho = \text{const}$ (i.e. when the density remains constant along the whole volume besetting by the liquid during the whole time of motion).

In this case the continuity equation acquires the simplest look

$$\text{div} \, \mathbf{v} = 0. \quad (2.36)$$

As a consequence of latter Eq., one gets

$$\Delta \Phi = 0 \quad (2.37)$$

for a Higgs scalar field $\Phi$ (in particular, for that taking the shape of a vacuum BPS monopole).

Comparing latter two Eqs. with Eqs. (2.35) and (2.23), we see distinctly a parallel between the Minkowskian BPS monopole vacuum (suffered the Dirac fundamental quantization [12]) and an incompressible liquid (possessing simultaneously manifest superfluid properties according the Gribov ambiguity equation (2.23)).

Meanwhile the Bogomol’nyi equation (1.1) describes only superfluid properties of the Minkowskian BPS monopole vacuum.

From the topological viewpoint, the property to be an incompressible liquid comes for the Minkowskian BPS monopole vacuum quantized by Dirac to the existence definite topological invariants in the appropriate Minkowskian Higgs model.

In a topologically nontrivial (non-Abelian) gauge theory, such values always exist, invariant with respect to continuous deformations of (non-Abelian) fields.
The one of most important topological invariants one encounters in a gauge theory is the winding number functional. In Ref. [16] it was defined as

\[ X[A] = -\frac{1}{8\pi^2} \int_V d^3x \epsilon^{ijk} \text{tr} \left[ \dot{A}_i \partial_j \dot{A}_k - \frac{2}{3} \dot{A}_i \dot{A}_j \dot{A}_k \right], \quad A_{\text{in, out}} = A(t_{\text{in, out}}, x). \]  (2.38)

Herewith, without loss the generality, it may be set \( t_{\text{in, out}} \to \pm \infty \) for appropriate time instants.

Knowing the winding number functional \( X[A] \), one can specify than the Chern-Simons functional (the Pontryagin index in the terminology [16, 24])

\[ \nu[A] = \frac{g^2}{16\pi^2} \int_{t_{\text{in}}}^{t_{\text{out}}} dt \int_V d^3x F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a = X[A_{\text{out}}] - X[A_{\text{in}}] = n(t_{\text{out}}) - n(t_{\text{in}}), \]  (2.39)

involving the (YM) tension tensor \( F_{\mu\nu}^a \) and its dual, \( \tilde{F}_{\mu\nu}^a \).

Alternatively to Eq. (2.38) for the winding number functional \( X[A] \), the (Pontryagin) degree of a map may be defined in a non-Abelian gauge theory as [2, 16]

\[ \mathcal{N}[n] = -\frac{1}{24\pi^2} \int_V d^3x \epsilon^{ijk} \text{tr} \left[ L_i^a L_j^a L_k^a \right] \in \mathbb{Z}, \]  (2.40)

with

\[ \dot{A}_i \Rightarrow L_i^n \equiv v^{(n)}(x) \partial_i v^{(n)}(x)^{-1} \quad \text{as} \quad |x| \to \infty \]  (2.41)

being the (classical) purely gauge vacuum configuration in this non-Abelian gauge theory.

There may demonstrated (see Refs. [14, 16] and the article by Jackiw "Topological investigations of quantized gauge theories" in [33]) that upon performing gauge transformations (2.41), the winding number functional \( X[A] \), (2.38), takes the look

\[ X[A_i^{(n)}] = X[A_i^{(0)}] + \mathcal{N}(n) + \frac{1}{8\pi^2} \int d^3x \epsilon^{ijk} \text{tr} \left[ \partial_i (\dot{A}_j^{(0)} L_k^n) \right]. \]  (2.42)

Let us denote, following [14], as

\[ \mathcal{N}(n) + \frac{1}{8\pi^2} \int d^3x \epsilon^{ijk} \text{tr} \left[ \partial_i (\dot{A}_j^{(0)} L_k^n) \right] \equiv N(\mathcal{N}, A) \]  (2.43)

two latter items in (2.42).

On the other hand, according to (2.38), the winding number functional \( X[A_i^{(n)}] \) would take integers.

Thus we come to the so-called self-consistency condition [14]

\[ N(\mathcal{N}, A) = N, \quad N \in \mathbb{Z}. \]  (2.44)
To satisfy (2.42), we may choose the third item in (2.42) to be

\[ N - \frac{\sin(2\pi N)}{2\pi}; \]

(2.45)
then the second item therein should be equal to

\[ \frac{\sin(2\pi N)}{2\pi}. \]

(2.46)

In the Minkowskian Higgs model, implying always the spontaneous breakdown of the initial (for instance, SU(2)) gauge symmetry, one encounters some more topological invariant: the magnetic charge \( m \).

Generally, it may be specified over a (Higgs-YM) field configuration \((\Phi, A)\) as

\[ m(\Phi, A) = C \zeta(\Phi, A), \quad \zeta(\Phi, A) \in Z; \]

(2.47)
with \( C \) being a constant that doesn’t depend on the (Higgs-YM) field configuration \((\Phi, A)\).

Thus any magnetic charge \( m \) proves to be a topological invariant due to this independence of \( C \) on \((\Phi, A)\), i.e. on fluctuations (deformations) of these fields.

Indeed, latter Eq. is not obvious, and we recommend our readers the monograph where it was derived in §Φ7.

In particular, in the Minkowskian Higgs model involving vacuum BPS monopole solutions, the magnetic charge \( m \) determines the Bogomol’nyi lowest bound (2.30) of the YMH energy.

If this model is suffered the Dirac fundamental quantization [12], of the YMH (vacuum) energy is like the density of an incompressible liquid [19]: herewith the properties of an incompressible (superfluid) liquid come, in the Minkowskian YM theory involving vacuum BPS monopoles, to the properties of topological (deformation) invariants: magnetic charges \( m \), (2.47), degrees of the map [1, 6]

\[ \pi_2 S^2 = \pi_3(SU(2)) = \pi_1(U(1)) = \pi_1 S^1 = Z, \]

(2.48)
referring to the \( U(1) \subset SU(2) \) imbedding, and the winding number functional (2.38).

Of course, enumerated topological invariants are present in each Minkowskian Higgs model involving BPS monopole modes, but the incompressibility property may remain paid no heed in these models considered in the heuristic quantization scheme [7].

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7 As a rule, one considers linear continuous deformations of fields: for example [2],

\[ A_{\mu}^n(x) \rightarrow tA_{\mu}^n(x) + (1 - t)A_{\mu}^n(x), \]

with \( t \in [0, 1] \).
3 Properties of Gribov topological multipliers.

There may shown [5, 6, 14, 15, 16] that the Gribov ambiguity equation (2.23) together with the topological condition

\[ X[A^{(n)}_k] = n; \quad n \in \mathbb{Z}; \]  

(3.1)

are compatible with the unique solution \( \Phi_0 \) to the classical YM equations of motion.

The nontrivial solution to the equation for the Gribov phase \( \hat{\Phi}_0(r) \) is well-known in this case [5, 6, 16]:

\[ \hat{\Phi}_0(r) = -i\pi \frac{\tau^a x^a}{r} f_{01}^{BPS}(r), \quad f_{01}^{BPS}(r) = \left[ \frac{1}{\tanh(r/\epsilon)} - \frac{\epsilon}{r} \right]. \]  

(3.2)

It is just a \( U(1) \subset SU(2) \) isoscalar ”made” of Higgs vacuum BPS monopole modes.

As a definite linear combination of these vacuum BPS monopole modes, the Gribov phase (3.2) satisfies actually the Gribov ambiguity equation (2.23).

In Ref. [15] there was shown that the function \( f_{01}^{BPS}(r) \), entering Eq. (3.2) for the Gribov phase \( \Phi_0(r) \), has the asymptotic

\[ f_{01}^{BPS}(0) = 0; \quad f_{01}^{BPS}(\infty) = 1. \]  

(3.3)

In the series of papers (for instance, [18, 20, 24, 26]), there was demonstrated that the Gribov exponential topological multipliers \( v^{(n)}(x) \) (entering Dirac variables \( A^D \)) satisfy the boundary condition

\[ v^{(n)}(x) \to \pm 1, \quad |x| \to \infty. \]  

(3.4)

The one way to prove latter Eq. was pointed out in Ref. [24].

Ibid it was proposed the following representation for Gribov multipliers \( v^{(n)}(x) \):

\[ v^{(n)}(x) = \cos(\pi n f_{01}^{BPS}(r)) - i n^a \tau_a \sin(\pi n f_{01}^{BPS}(r)); \quad n^a = x^a / r. \]  

(3.5)

The ”technology” deriving latter Eq. is enough simple. It was applied, for instance, in the monograph [33], in §7.1, at the analysis of \( SU(2) \) (global) rotations of a spinor \( \varphi \).

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\[ \text{8} \] Indeed, it is unique to within the whole family of gauge fields cohomologial to each other [2, 6].

This family consists of such gauge fields that two 1-forms \( \omega \equiv A_\mu d\mu' \) and \( \omega' \equiv A'_\mu d\mu' \), involving (transverse) YM fields \( A_\mu \) and \( A'_\mu \), respectively, belonging to a one class of cohomologies differ on the exact 1-form \( d\sigma: \omega - \omega' = d\sigma \) [2].

Latter Eq. may be rewritten approximately as

\[ \partial_\mu (\omega - \omega') = \partial_\mu d\sigma = 0, \]

since \( d \cdot d\sigma = 0 \) due do the Poincare lemma [2].

The analysis of the cohomological structure of gauge fields in the Minkowskian Higgs model quantized by Dirac was started in the paper [6].

The author intends to continue this analysis in one of the next works.
As in that case, one would take account of the relations
\[(n\tau)^{2k}=1; \quad (n\tau)^{2k+1}=n\tau; \quad k\in\mathbb{Z}\]
and expand \(\cos(x)\) and \(\sin(x)\) in the series.

Then the boundary condition \((3.4)\) for Gribov exponential topological multipliers \(v^{(n)}(x)\) follows immediately from the spatial asymptotic \((3.3)\) \cite{15} for \(f_{01}^{BPS}(r)\) (when \(|x| \to \infty\)).

The alternative way to demonstrate the spatial asymptotic \((3.4)\) for \(v^{(n)}(x)\) is recasting them, following \cite{20}, to the look
\[v^{(n)}(x) = \exp(\hat{\lambda}_{n,\phi}(x)), \quad (3.6)\]
with
\[\hat{\lambda}_{n,\phi}(x) \equiv i\tau^a \Omega_{ab}(\phi_i) \frac{x^b}{r} f_{01}^{BPS}(r) \pi n \quad (3.7)\]
and
\[(\tau^a)_{\beta}^{\alpha} \Omega_{ab}(\phi_i) = (u(\phi_i))_{\beta}^{\alpha} (\tau^a)_{\delta}^{\gamma} (u^{-1}(\phi_i))_{\delta}^{\gamma}, \quad (3.8)\]
\[(u(\phi_i))_{\beta}^{\alpha} = (e^{i\tau_1\phi_1/2})_{\gamma}^{\alpha} (e^{i\tau_2\phi_2/2})_{\delta}^{\gamma} (e^{i\tau_3\phi_3/2})_{\beta}^{\delta}.\]

Here \(\phi_i\) \((i = 1, 2, 3)\) are three Euler angles fixing the position of the coordinate system in the \(SU(2)\) group space \cite{4}.

To achieve the necessary asymptotic \((3.4)\) for Gribov exponential topological multipliers \(v^{(n)}(x)\) at the spatial infinity one would impose the appropriate conditions onto the Gribov phase in Eq. \((3.6)\).

More precisely, such conditions may be imposed onto the Euler angles \(\phi_i\): Gribov topological factors \(v^{(n)}(x)\) become \(\pm 1\) at the spatial infinity when
\[\tau_i\phi_i = 4\pi n \quad (\text{respectively}, \quad \tau_i\phi_i = 2\pi n); \quad n\in\mathbb{Z} \quad (3.9)\]

Indeed, one would take account of the fundamental constants \(\hbar\) and \(c\) at writing down Gribov exponential topological multipliers \(v^{(n)}(x)\), would be dimensionless in their definition.

In Ref. \cite{24} it was shown how to take account of \(\hbar\) and \(c\) in Gribov topological multipliers \(v^{(n)}(x)\), \((2.8)\), and now we shall stick to the arguments \cite{24} at statement the problem.

First of all, the mentioned fundamental constants enter various strong interaction models (YM and QCD) as \(g/(\hbar c)\) in the lowest order of the perturbation theory.

The said allows redefine gauge fields \(A\) in terms of the strong interaction coupling constant \(g/(\hbar c)\), as it was done, for instance, in \((2.15)\) \cite{24}.

9 Indeed, we should always remember that "large" matrices \(v^{(n)}(x)\) belong to the residual \(U(1)\) symmetry group, embedded in the initial \(SU(2)\) group.

25
That is why it is quite reasonable to recast Gribov exponential topological multipliers $v^{(n)}(x)$, (2.8), in such a wise that the dimension of the Gribov phase $\hat{\Phi}(x)$, (3.2), compensates the dimension of the coupling constant $g/(\hbar c)$, will now enter Gribov exponential topological multipliers $v^{(n)}(x)$: thus these remain dimensionless upon recasting.

On the other hand, such look of Gribov topological multipliers $v^{(n)}(x)$ ensures necessary properties of topological Dirac variables (2.7), (2.8): in particular, that they are manifestly transverse and gauge invariant.

Additionally, the spatial asymptotic (3.4) for $v^{(n)}(x)$ would be taken into account at this recasting.

Following [24], we write

$$v^{(n)}(x) = \exp[n\hat{\Phi}(x)g/(\hbar c)].$$

(3.10)

It is easy to see [24] that the Gribov phase $\hat{\Phi}(x)$, given by (3.2), may be recast to the look $2\tau^a\lambda_a$, with

$$\lambda_a = -i(\pi/2)n_a\Theta_{BPS}(r).$$

(3.11)

In this case, it is necessary to multiply $\lambda_a$ by $\hbar c/g$ (the value got in this way we shall denote as $\lambda_a' \equiv (\lambda_a\hbar c)/g$) and substitute then in (3.10) in order for Gribov topological multipliers $v^{(n)}(x)$, given through (3.10), to be, indeed, dimensionless.

By that we come back to Eq. (3.2) for the Gribov phase $\hat{\Phi}(x)$, that is free from $g$, $\hbar$, $c$. Respectively, Gribov topological multipliers $v^{(n)}(x)$ become dimensionless.

Note that the value [24]

$$\hat{\lambda} = \lambda_a'\tau^a\frac{g}{\hbar c}$$

(3.12)

may be read from (3.10), (3.11).

In this case Gribov exponential multipliers $v^{(n)}(x)$ acquire the look [24]

$$v^{(n)}(x) = \exp(2n\hat{\lambda}); \quad n \in \mathbb{Z};$$

(3.13)

in terms $\hat{\lambda}$ (free from $\hbar$ and $c$, as it should be indeed).

The said is in a good agreement with the definition (2.15) [24] of gauge fields $\hat{A}$ and with the property to be gauge invariant and transverse for the topological Dirac variables (2.7).

4 Discussion.

Finishing this study, we should like tell our readers about our further plans in developing the fundamental quantization formalism [12] for the Minkowskian Higgs model with vacuum BPS monopoles.

The outlines of this fundamental quantization were stated recently in Ref. [13], and our readers can obtain the general idea, at reading this paper, how to quantize the Minkowskian Higgs model with vacuum BPS monopoles and which consequences implies such fundamental quantization.
Now we shall narrate about our nearest investigations in this direction. They will concern studying the nontrivial topological dynamics inherent in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \[12\].

Herewith the arguments \[5, 6, 13, 14, 15, 16\] will be repeated and extended.

It will be argued that the origin of the mentioned nontrivial topological dynamics proves to be in resolving the YM Gauss law constraint (2.1) in the covariant Coulomb gauge (2.2), (2.6).

In particular, topological Dirac variables \(A^D\), (2.7), satisfy the Coulomb gauge (2.2), (2.6).

Thus resolving the YM Gauss law constraint (2.1) in terms of topological Dirac variables (2.7) turns the YM Gauss law constraint (2.1) into the second-order homogeneous differential equation

\[
[D_i^2(\Phi^{(0)})]^{ac}A_{0c} = 0,
\]

permitting the family of so-called zero mode solutions \[16, 18\]

\[
A^c_0(t, x) = \dot{N}(t)\Phi^{(0)}_c(x) \equiv Z^c,
\]

implicating the topological variable \(\dot{N}(t)\) and Higgs (topologically trivial) vacuum Higgs BPS monopole modes \(\Phi^{(0)}_c(x)\).

\(A_0\), specified in such a wise, may be treated as temporal components of gauge fields additional to those equal to zero \[14, 2.11\], got course the Dirac removal.

It is also the merit of the Dirac fundamental quantization \[12\] of the Minkowskian Higgs model (with vacuum BPS monopole solutions).

Recall in this context \[13\] that at the FP "heuristic" quantization \[7\] of Minkowskian Higgs models with monopoles, \(A_0\) components of gauge fields are ruled out via fixing the Weyl gauge \(A_0 = 0\).

YM potentials \(A_0\), (4.1), referring actually to the BPS monopole vacuum, induce specific \(F^a_{0i}\) components of the YM tension tensor, taking the shape of so-called vacuum "electric" monopoles \[5, 6\]

\[
F^a_{0i} = \dot{N}(t)D_i^{ac}(\Phi^{(0)}_k)\Phi_{0c}(x).
\]

Issuing from vacuum "electric" monopoles \(F^a_{0i}\), one can construct \[5, 6, 13, 14\] the action functional

\[
W_N = \int d^4x \frac{1}{2}(F^c_{0i})^2 = \int dt \frac{\dot{N}^2I}{2},
\]

involving the rotary momentum \[14\]

\[
I = \int d^3x(D_i^{ac}(\Phi^{(0)}_k)\Phi_{0c})^2 = \frac{4\pi^2\epsilon}{\alpha_s} = \frac{4\pi^2}{\alpha_s^2} \frac{1}{V < B^2>}.\]

The YM coupling constant

\[
\alpha_s = \frac{g^2}{4\pi(hc)^2}
\]

\[4\]

\[27\]
enters this expression for $I$.

The action functional $W_N$ describes correctly collective solid rotations of the BPS monopole vacuum (suffered the Dirac fundamental quantization [12]) with angular velocities $\dot{N}(t)$, proves to be constant (as the author intend to demonstrate).

In demonstrating constancy of $\dot{N}(t)$, the arguments [24] will be repeated. The principal result will be got that

$$\dot{N}(t) = \text{const} = (n_{\text{out}} - n_{\text{in}})/T \equiv \nu/T \quad (4.2)$$

where $n_{\text{out}}, n_{\text{in}} \in \mathbb{Z}$ refer to the fixed time instants $t = \pm T/2$, respectively.

It will be argued herewith (due to general QFT reasoning) that it would be set $T \to \infty$.

This implies actual approaching zero by angular velocities $\dot{N}(t)$ for collective solid rotations inside the BPS monopole vacuum.

The crucial point in grounding Eq. (4.2) is investigating the explicit look of $N(t)$, the noninteger degree of the map referring to the $U(1) \subset SU(2)$ embedding [24]:

$$\nu[A_0, \Phi^{(0)}] = \frac{g^2}{16\pi^2} \int_{t_{\text{in}}}^{t_{\text{out}}} dt \int d^3x F^a_{\mu \nu} \tilde{F}^{a\mu\nu} = \frac{\alpha_s}{2\pi} \int d^3x F^b_{\mu 0} B^b_i (\Phi^{(0)}) [N(t_{\text{out}}) - N(t_{\text{in}})]$$

$$= N(t_{\text{out}}) - N(t_{\text{in}}). \quad (4.3)$$

The topological variable $N(t)$, specified via (4.3), determines the purely real (i.e. unambiguous physical) energy-momentum spectrum of the free rotator $W_N$:

$$P_N = \dot{N} I = 2\pi k + \theta; \quad \theta \in [-\pi, \pi];$$

accompanied by the wave function

$$\Psi_N = \exp(iP_N N).$$

The general origin of collective solid rotations inside the BPS monopole vacuum suffered the Dirac fundamental quantization [12] is in the Josephson effect [24], coming to persistent circular motions of material points (quantum fields may be considered as a particular case of such material points) without (outward) sources.

These persistent circular motions of quantum fields are characterized [13, 24] by never vanishing (until $\theta \neq 0$) momenta

$$P = \hbar \frac{2\pi k + \theta}{L},$$

with $L$ being the length of the whole closed line along which the given quantum field moves.

Such momenta $P$ attain their nonzero minima $p = \hbar \theta / L$ as $k = 0$ and if $\theta \neq 0$.  

28
The investigations about the Josephson effect are also planned for the future.

Repeating the arguments \[24\], it will be shown \[14\] that besides collective solid rotations inside the BPS monopole vacuum suffered the Dirac fundamental quantization \[12\] (these rotations are characterized constant angular velocities $\dot{N}(t)$), the Josephson effect in the appropriate Minkowskian Higgs model comes to never vanishing (until $\theta \neq 0$) vacuum "electric" fields ("electric" monopoles)

$$(E_i^a)_{\text{min}} = \theta \frac{\alpha_s}{4\pi^2\epsilon} B_i^a; \quad -\pi \leq \theta \leq \pi.$$  

Such minimum value of the vacuum "electric" field $E$ corresponds to trivial topologies $k = 0$, while generally \[14\],

$$F_{i0} = E_i^a = \dot{N}(t) (D_i(\Phi_k^{(0)}) \Phi(0))^a = P_N \frac{\alpha_s}{4\pi^2\epsilon} B_i^a(\Phi(0)) = (2\pi k + \theta) \frac{\alpha_s}{4\pi^2\epsilon} B_i^a(\Phi(0)).$$

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