A quantum Chinos game with entanglement

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The Chinos game is a non-cooperative game between players who try to guess the total sum of coins drawn collectively. Semiclassical and quantum versions of this game were proposed by F. Guinea and M. A. Martín-Delgado, in J. Phys. A: Math. Gen. 36 L197 (2003), where the coins are replaced by a boson whose number occupancy is the aim of player’s guesses. Here, we propose other versions of the Chinos game using a hard-core boson, one qubit and two qubits. In the latter case, we find that using entangled states the second player has a stable winning strategy that becomes symmetric for non-entangled states. Finally, we use the IBM Quantum Experience to compute the basic quantities involved in the two-qubit version of the game.

I. INTRODUCTION

Game theory is a field that has fascinated mathematicians since the early 20th century. The idea is to be able to interpret complex problems in many different fields as a game of one or more players who, using logical reasoning, try to optimise their strategies in order to obtain the greatest possible profit. This area of study appeared with the aim of better understanding the economy, but it quickly extended to biology [1], politics [2], computation and computer science [3] (see review [4]).

The pioneers that formalise game theory were John von Neumann and Oskar Morgenstern [5], followed soon later by John Nash. A fundamental concept is that of the Nash equilibrium [6], in which it is assumed that every player knows and chooses their best possible strategy knowing the strategies of the other players. This implies that the Nash equilibrium situation is one in which no player would profit by changing her/his strategy if the other players maintain theirs. Most of multiplayer games tend to reach Nash equilibrium after a certain number of iterations.

In recent decades, physicists have begun to introduce features of the quantum world into game theory in order to gain advantages over classical strategies [7–11]. There are several reasons why quantifying games can be interesting. The first is simply because of the number of applications game theory has had in different fields. Moreover, its probabilistic nature makes one want to extend it to quantum probabilities. Another reason is the connection between game theory and quantum information theory. In fact, in the games themselves, players transmit information to other players and, since our world is quantum, it can be interpreted as quantum information [7].

There are several examples of well-known quantum game models. One is the prisoner’s dilemma where, by exploiting the peculiarities of quantum behaviour, both players can escape from the dilemma [7]. Another case is the PQ game (flipping or not flipping a coin a certain number of times each player), where it has been shown that if the first player can use quantum strategies (superposition of both options) she/he will always win no matter what action the second player takes [8]. Not only have quantum games been proposed, but they are already being used to model, for example, human decision-making behaviour [12].

This paper deals with the well-known (in Spain) Chinos game, which traditionally consists of a group of people who hide a certain number of coins in their hands. The aim of each of them is to guess the total number of coins hidden by all of them. This simple game shows a variety of behavioural patterns that have been used to model financial markets and information transmission [13].

This is a non-cooperative game, in which each player will seek to maximise their chances of victory and minimise those of the other players. For this reason, throughout the work we will always look for this situation in the analysis of the possible strategies. In other words, the Nash equilibrium of the game model will be pursued. We must stress the importance of entanglement in quantum games when looking for the Nash equilibrium [14–16].

Finally, measurements will be implemented on the quantum version of the game (for both, the game with maximally entangled qubits and non-entangled qubits) made with a quantum computer thanks to the IBM Quantum Experience. Another quantum games like the prisoner’s dilemma and the PQ game have been already implemented using IBM quantum computers [17, 18].

The organization of the paper is the following. In section II we review the classical Chinos game. In section III we review the semiclassical model of the Chinos game proposed in [19] and we propose two new semiclassical models. In section IV we propose a new quantum version observing the differences when the game is played with entagled and non-entagled states. In section V we state our conclusions and prospects. Finally, in the appendix the quantum version is simulated with a IBM quantum computer.

The following table summarizes the games considered in this work ordered in increasing quantumness according to the types of draws and guesses. The option of entanglement (ENT) is only possible for two qubits. SYM denotes the existence of a symmetry between the two players.

| SYSTEM | NATURE | DRAW | GUESS |
|--------|--------|------|-------|
|        | Coins  |      | Total coins |
| Coins  | Classical | Ladder operators | Number occupancy |
| 1 boson | No | Yes |
| 1 hard-core boson | No | Yes |
| 1 qubit | Semiclassical | Rotations | Number occupancy |
| 2 qubits | Quantum | 1 Bell operator | 2 Bell operators |
| 2 qubits | Quantum | 1 rotated Bell operator | 2 rotated Bell operators |

* * *

1 hard-core boson
2 qubits
Ladder operators
Ladder operators
Number occupancy
Number occupancy
No
No
2 Bell operators
Yes
No
Yes
Yes

1 Boson
Semiclassical
Fermion
Number occupancy
No
Yes

1 rotated Bell operator
2 rotated Bell operators
No
Yes

SYM

1 rotated Bell operator
2 rotated Bell operators
No
Yes
The model is defined by two parameters: $N_p$ that is the number of players and $N_c$ that is the number of coins that each player holds in her/his hand. In each round each player draws from 0 to $N_c$ coins, and guesses the total number of coins that they all draw, with the condition that the result predicted by the previous players cannot be repeated. For example for two players, $N_p=2$, and one coin, $N_c=1$, if player 1 guesses 1 coin, then player 2 can choose 0 or 2, but not 1. We will denote $d_{i,j}$ the number of coins that player $i = 1,...,N_p$ draws in round $j = 1,...,r$ and $g_{i,j}$ the guess of the $i^{th}$ player in the round $j$.

We shall next analyze the simple case of the example mentioned above of $N_p=2$ and $N_c=1$. In the classical case, the best strategy for player 1 (Alice) is to choose randomly $d_{1,j}$ and to always choose her prediction as $g_{1,j} = N_c - 1$, so as not to reveal information to player 2 (Bob) [19]. Also, this is because with the 4 possible tosses it is most likely that the sum of the coins will be 1 (Table I). As for Bob, the best strategy he can choose is to randomly choose $d_{2,j}$ and make his attempt $g_{2,j}$ in an “intelligent” way. By the word “intelligent” one means that, in case Bob chooses $d_{2,j} = 0$, he excludes the option of $g_{2,j} = 2$ and in case he chooses $d_{2,j} = 1$, he excludes the option of $g_{2,j} = 0$. Thus, Table I shows all the possible options of this game following the best strategy for each player. We denote $S = d_1 + d_2$ as the total sum of coins drawn in a round.

| $d_A$ | $g_A$ | $d_B$ | $g_B$ | $S$ | Winner |
|------|------|------|------|-----|--------|
| 0    | 1    | 0    | 0    | 0   | B      |
| 0    | 1    | 1    | 2    | 1   | A      |
| 1    | 1    | 0    | 1    | 2   | B      |
| 1    | 1    | 1    | 2    | 2   | A      |

TABLE I: All possible options for the game with $N_p=2$ and $N_c=1$. We use notation in which $d_A = d_1$ and $d_B = d_2$.

It is seen that, classically, this game is symmetrical, as each player will win half of the time by applying her/his or her best strategy.

Starting from the simple case, we are going to verify that, in general, the Chinos game is symmetrical for 2 players and $N_c$ coins. This time the best strategy for Alice will be to randomly choose $d_A$ and always guess $g_A = N_c$ in order not to reveal information and because it will be the most probable option among all the possible tosses. Again, for Bob the best strategy will be to choose randomly $d_B$ and make his guess in an “intelligent” way. This can be shown by calculating the probabilities of winning denoted as $p_A$ and $p_B$. Given the conditions of the game we have that the probability of Bob winning is [19]:

$$P_B = \frac{1 - p_A}{N_c}.$$  

Therefore, if we take the normalised probabilities $P_i = p_i / \sum_{i=A,B} p_i$ we have:

$$P_B = \frac{1 - p_A}{1 + p_A(N_c - 1)}. \quad \text{(2)}$$

In view of the eq. (2) Bob will try to minimise $p_A$ but Alice will set a lower limit by randomly choosing $d_A$ among the number of coins $p_{A,\inf} = \frac{1}{N_c}$. Therefore, $p_A$ does not exceed this limit, Bob will also randomly choose $d_B$, reaching a symmetrical situation since in this case [19]:

$$P_A = P_B = \frac{1}{2}. \quad \text{(3)}$$

This shows that the classic two-player Chinos game is completely symmetrical.

$$A \leftrightarrow B. \quad \text{(4)}$$

Moreover, this strategy for both players is stable so that classically the symmetry will never be broken. From now on, several quantum versions of the game will be proposed to test whether this classical symmetry is broken.

III. SEMICLASSICAL MODELS OF THE CHINOS GAME

The first semiclassical model presented is a summary of the results obtained in [19] and the same idea will be used to propose two new semiclassical variants. The first step in quantizing the game is to make a quantum extension of what each player can draw, leaving the range of classical predictions intact. The most natural thing is to replace coins with qubits but, given that in the game the goal is to guess the total number of coins, it is proposed to use a single boson degree of freedom model.

For this, we use the bosonic creation and annihilation operators which follow the canonical commutation rule $[b, b^\dagger] = 1$ and which act on the bosonic states in the usual way: $b|0\rangle = 0, b^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle$ where $|n\rangle = (b^\dagger)^n|0\rangle/\sqrt{n!}$.

We will study the case of two players. For this purpose, we introduce a set of operators for each player $i = 1,2$ which are defined by two angles as follows

$$O_i(\theta_i, \phi_i) = \cos \frac{\theta_i}{2} + e^{i\phi_i} \sin \frac{\theta_i}{2} b^\dagger,$$
\begin{equation} \quad \text{(5)} \end{equation}

where $\theta_i \in [0,\pi]$ and $\phi_i \in [0,2\pi]$.

Now each player selects one of these operators, which is the quantum analogue of drawing a coin, and apply them to the boson vacuum to generate the joint state,

$$|\Psi_{i,j}\rangle = N_{i,j}^{-1/2} O_i^A O_j^B |0\rangle = \sum_{n=0}^{c_{i,j}} c_{i,j}(n)|n\rangle,$$
\begin{equation} \quad \text{(6)} \end{equation}

where $N_{i,j}$ is a normalization constant and $c_{i,j}(n)$ the coefficients of the expansion. This state exhibits the property of the game that what is relevant is the total sum of coins since each player contributes to the expansion coefficients. Given this joint state, the players will make their predictions in the classical way.
i.e. by choosing a number \( n \in \{0, 1, 2\} \). We define the probability that a player wins by selecting the value \( n \)

\[
p_{i,j}(n) = |\langle n | \psi_{i,j} \rangle|^2 = c_{i,j}^2(n). \tag{7}
\]

With this formulation for quantifying what each player can draw, one has infinitely many possible choices, so it is reasonable to reduce this set of operators to just a few. In particular, we consider the case where the set of possible operators is reduced to two quantum superpositions among them at

\[
O_1 = I, O_2 = \frac{I + b^\dagger}{\sqrt{2}}, O_3 = \frac{I - b^\dagger}{\sqrt{2}}, O_4 = b^\dagger. \tag{8}
\]

By choosing these operators we are leaving the purely classical case at \( O_1 \) and \( O_2 \), which are equivalent to taking 0 and 1 coins respectively, and introducing two quantum superpositions among them at \( O_2 \) and \( O_3 \).

With this set of operators, all the options of the game are analysed and the probabilities \( p_{i,j}(n) \) of each move are obtained, which is shown in Table II.

| \( O^A_i \) | \( O^B_i \) | \( O^A_j \) | \( O^B_j \) |
|---|---|---|---|
| \( p(0) = 1 \) | \( p(0) = 1/2 \) | \( p(0) = 1/2 \) | \( p(0) = 0 \) |
| \( p(1) = 0 \) | \( p(1) = 1/2 \) | \( p(1) = 1/2 \) | \( p(1) = 1 \) |
| \( p(2) = 0 \) | \( p(2) = 0 \) | \( p(2) = 0 \) | \( p(2) = 0 \) |

\begin{align*}
\text{TABLE II: Odds of getting 0, 1 or 2 for all possible plays.}
\end{align*}

For the analysis of the results it is assumed that the two players know the strategy of the classical case, so they try to carry it out by choosing randomly between the 4 operators \( O_i \), \( i = 1, 2, 3, 4 \). In this way, Alice has the probabilities of 0, 1 or 2 coins shown in Table III depending on the operator she chooses. These probabilities are obtained by averaging over all Bob’s possible moves from Table II.

\[
\langle p_i(n) \rangle = \frac{1}{4} \sum_{j=1}^{4} p_{i,j}(n). \tag{9}
\]

It is found that if both make their choice randomly, the best choices for Alice are that on drawing \( O^A_0 \) her guess is 0 (or 1) and on drawing \( O^B_2, O^A_3 \) and \( O^B_4 \) her guess is 2. With this, her total probability of winning is:

\[
P_A = \frac{1}{4} \left( 1 + \frac{1}{2} \cdot \frac{1}{128} + \frac{1}{4} \cdot \frac{7}{12} \right) = \frac{53}{112} < \frac{1}{2}. \tag{10}
\]

Therefore, following the same strategy as in the classical case breaks the symmetry between the two players. After many rounds, Alice realises that, drawing randomly, her probability of winning is lower than Bob’s, so she decides to improve her probability by choosing randomly only between the equivalent operators to the classical case, \( O^A_1 \) and \( O^A_4 \). With this, her probability of victory is:

\[
P_A = \frac{1}{2} \left( 1 + \frac{1}{2} \cdot \frac{1}{12} \right) = \frac{13}{24} > \frac{1}{2}. \tag{11}
\]

After this change, Bob will notice this imbalance, so he will decide to choose randomly between the operators of the classical case, \( O^B_1 \) and \( O^B_2 \). By doing this, we arrive at exactly the classical game in which the existing symmetry is restored, \( P_A = P_B = \frac{1}{2} \).

Following this analysis, it is concluded that, by extending the strategy from the classical game to the semiclassical game, it becomes a winning strategy for Bob although it is unstable. And, the symmetric stable strategy for both players is reached by randomly selecting from only the operators equivalent to the classical case [19].

### A. Hard-core bosons

In order to approximate a quantum computer simulation, hard-core bosons have been used to have only the \( |0> \) and \( |1> \) states. That is, the bosonic creation operator now has the condition:

\[
\langle b^\dagger \rangle^2 = 0. \tag{12}
\]

When introducing this constraint, the operator \( O_4 \) is necessarily eliminated from the set of operators that had been selected at the beginning, eq. (8), to avoid the possibility of a null move. It should also be noted that this case is a version in which the classical game cannot be recovered, since \( O_4 \) is eliminated from the options to be chosen by Alice and Bob.

Analogously to the previous case, for the study of the strategies it is necessary to know the probabilities of obtaining 0 and 1 for each of the possible moves. This is shown in Table IV.

| \( O^A_i \) | \( O^B_i \) | \( O^A_j \) | \( O^B_j \) |
|---|---|---|---|
| \( p(0) = 1/2 \) | \( 1/2 \) | \( 41/168 \) | \( 41/168 \) |
| \( p(1) = 1/2 \) | \( 59/168 \) | \( 59/168 \) | \( 5/12 \) |
| \( p(2) = 0 \) | \( 68/168 \) | \( 68/168 \) | \( 7/12 \) |

\begin{align*}
\text{TABLE III: Average odds of getting 0, 1 or 2 for Alice.}
\end{align*}
Similarly, it is assumed that both players know the classical strategy and try to extend it to this semiclassical version of the game. Both choose what to draw randomly among $O_i$, $i=1,2,3$. Thus, we find from Table IV the probabilities, averaged over Bob’s possible choices, that Alice has of getting 0 or 1 coins depending on which operator she chooses. This is shown in Table V.

| $O_1$ | $O_2$ | $O_3$ |
|-------|-------|-------|
| $p(0) = 1$ | $p(0) = 1/2$ | $p(0) = 1/2$ |
| $p(1) = 0$ | $p(1) = 1/2$ | $p(1) = 1/2$ |

The probability of Alice winning is to choose randomly between 2 players. By doing this, what Alice chooses operators $O_3$ and $O_4$. With this it turns out that Alice has a winning strategy if they both randomly choose which operator to draw. Moreover, this winning strategy is stable since Bob, realising this, will try to reverse the situation, but the best he can do to maximise his chances of winning is to choose randomly between the $O_2$ and $O_3$ operators. By doing this, what Bob achieves is to slightly reduce Alice’s probability of winning, but even with this, Alice still has a probability of more than a half:

$$P_A = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9} > \frac{1}{2}.$$  \hspace{1cm} (13)

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$$P_A = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9} > \frac{1}{2}.$$  \hspace{1cm} (14)

Therefore, Alice has a winning and stable strategy since Bob has no choice to restore the classical symmetry of the game.

Observe this, in order to improve the chances of Bob we shall generalise the problem again by introducing a free parameter $\theta \in [0, \pi/2]$ so that the operators are defined as follows:

$$O_1 = I, \quad O_2 = cI + sb^4, \quad O_3 = cI - sb^4,$$  \hspace{1cm} (15)

where $c = \cos(\theta)$ and $s = \sin(\theta)$. The previous case corresponds to taking $\theta = \pi/4$. The parameter $\theta$ takes these values since in $\theta = \pi/2$ we would again have the null move that was avoided by initially eliminating the $O_4$ operator and the values $(\pi/2, \pi)$ show results symmetrical to the previous ones so they are not studied. In making this generalisation, the previous analysis is repeated in terms of $\theta$. The probabilities of obtaining 0 or 1 coins as a function of $\theta$ for all the moves are shown in Table VI.

| $O_1$ | $O_2$ | $O_3$ |
|-------|-------|-------|
| $p(0) = 1$ | $p(0) = c^2$ | $p(0) = c^2$ |
| $p(1) = 0$ | $p(1) = s^2$ | $p(1) = s^2$ |

Similarly, it is assumed that both players know the classical strategy and try to extend it to this semiclassical version of the game. Both choose what to draw randomly among $O_i$, $i=1,2,3$. Thus, we find from Table IV the probabilities, averaged over Bob’s possible choices, that Alice has of getting 0 or 1 coins depending on which operator she chooses. This is shown in Table V.

| $O_1$ | $O_2$ | $O_3$ |
|-------|-------|-------|
| $p(0) = 1/2$ | $p(0) = c^2$ | $p(0) = c^2$ |
| $p(1) = 1/2$ | $p(1) = s^2$ | $p(1) = s^2$ |

We go back to the classic strategy and calculate the probabilities of obtaining the states $|0\rangle$ or $|1\rangle$ for Alice averaged over Bob’s moves as a function of $\theta$. They are shown in Table VII.

| $O_1$ | $O_2$ | $O_3$ |
|-------|-------|-------|
| $p(0) = 1/2$ | $p(0) = c^2$ | $p(0) = c^2$ |
| $p(1) = 1/2$ | $p(1) = s^2$ | $p(1) = s^2$ |

![FIG. 1: Probabilities as a function of $\theta$ for the game with hard-core bosons.](image-url)
\[
(p(0))A_1 = (p(1))A_1 = \frac{1}{2} \implies (p(0))A_2 = (p(1))A_2 = \frac{1}{2} \implies \theta_2 = \frac{\pi}{3}.
\]

\[
\theta_1 = 2 \tan^{-1}\left(\sqrt{\frac{3}{2}}(1 + \sqrt{7}) - 2\sqrt{17 - \sqrt{73}}\right) = 0.291\pi,
\]

\[
\theta_1 \leq \theta \leq \theta_2,
\]

• 0 < \theta < \theta_1 : Alice will guess 0 regardless of her choice of operator between \(O_1^A, O_2^A\) and \(O_3^A\).

• \(\theta_1 < \theta < \theta_2\)

• \(\theta_2 < \theta < \pi/2\) : Alice will guess 1 regardless of her choice of operator between \(O_1^A, O_2^A\) and \(O_3^A\).

We check what happens at the boundaries of the regions:

• \(\theta = \theta_1\) : At this point it might seem that if Bob chooses between the operators \(O_2^B\) and \(O_3^B\), it would even the game again since as seen in Fig. 1 the value of the averaged probability when drawing these operators is 1/2 (although probability is obtained for Alice, for Bob it is exactly the same if averaged over Alice’s moves because of the symmetry of Table VI). However, after a few rounds Alice would notice this and in order to reverse the situation she would decide to draw randomly between the operators \(O_2^A\) and \(O_3^A\). With this, evaluating at \(\theta = \theta_1\) on the values in Table VI we arrive at new probabilities for Alice averaged over Bob’s moves. These are shown in Table VIII.

| \(O_2^A\) | \(O_3^A\) |
|---|---|
| \(p(0)\) | 0.564 |
| \(p(1)\) | 0.436 |

TABLE VIII: Average odds of getting 0 or 1 for Alice in the case \(\theta = \theta_1\).

Therefore, if Alice chooses 0 as her prediction, she will have a probability of victory of more than half: \(P_A = 0.564 > 1/2\).

• \(\theta = \theta_2\) : This case is simpler, since Bob would try to match the game by choosing only \(O_2^B\), but after a certain number of rounds, Alice would notice it and would decide to choose only \(O_1^A\) and make 0 as her guess (this can be seen from Table VI), winning then with total probability.

With this analysis, it is shown that, even if the parameter \(\theta \in [0, \pi/2]\) is left free, Alice always has a winning and stable strategy for the Chinos game with hard-core bosons.

### B. Unitary operators

In the previous section we have been using the operators in Eq. (15) to create states of a hard-core boson. However, we shall next propose a realization where the states \(|0\rangle\) and \(|1\rangle\) are the computational states of a qubit, that leave more freedom to act on them with quantum gates. It is proposed then to use unitary operators acting on the computational basis of a qubit, \(|0\rangle, |1\rangle\). In order to acquire some intuition about the results, it is decided to use rotations, leaving again a free parameter \(\theta \in [0, \pi]\). Three operators are used, from which Alice and Bob can choose.

\[
O_1 = I, \quad O_2 = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad O_3 = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.
\]

The first operator is the identity. The second and third are rotations of angle \(\theta/2\) and \(-\theta/2\) respectively. With this choice of operators, the analysis of the previous version is repeated. The probabilities of obtaining the states \(|0\rangle\) or \(|1\rangle\) are shown in Table IX which is similar to Table VI but not equal.

| \(O_1^A\) | \(O_2^A\) | \(O_3^A\) |
|---|---|---|
| \(p(0)\) = 1 |
| \(p(1)\) = 0 |
| \(p(0)\) = \cos^2(\theta/2) |
| \(p(1)\) = \sin^2(\theta/2) |
| \(p(0)\) = \sin^2(\theta/2) |
| \(p(1)\) = \cos^2(\theta/2) |
| \(p(0)\) = \sin(\theta/2) |
| \(p(1)\) = \cos(\theta/2) |
| \(p(0)\) = \cos(\theta/2) |
| \(p(1)\) = \sin(\theta/2) |
| \(p(0)\) = \cos(\theta/2) |
| \(p(1)\) = \sin(\theta/2) |

TABLE IX: Probabilities of obtaining 0 or 1 for all possible plays in the game with unitary operators leaving \(\theta\) free.

We return to the assumption that both players know the classical strategy, i.e. both of them choose their draw randomly among the three operators. We study the probabilities of 0 and 1 for Alice averaged over Bob’s moves. This is shown in Table X.

| \(O_1^A\) | \(O_2^A\) | \(O_3^A\) |
|---|---|---|
| \(p(0)\) = \frac{1}{2}(1 + 2\cos^2(\theta/2)) |
| \(p(1)\) = \frac{1}{2}(2\sin^2(\theta/2)) |
| \(p(0)\) = \frac{1}{2}(1 + \cos^2(\theta/2) + \cos^2(\theta/2)) |
| \(p(1)\) = \frac{1}{2}(\sin^2(\theta/2) + \sin^2(\theta/2)) |
| \(p(0)\) = \frac{1}{2}(1 + \cos^2(\theta/2) + \cos^2(\theta/2)) |
| \(p(1)\) = \frac{1}{2}(\sin^2(\theta/2) + \sin^2(\theta/2)) |

TABLE X: Average odds of getting 0 or 1 for Alice in the game with unitary operators leaving \(\theta\) free.

In order to study the strategies as a function of the \(\theta\) parameter, we plot in Fig. 2 the functions obtained in Table X. Again we find 3 differentiated regions where Alice’s strategy is winning and stable. However, this time at the values for \(\theta\) at the boundary points, Bob is able to balance the game and restore the classical symmetry. The values for \(\theta\) at the boundary points are obtained by matching the probabilities in Table X to 1/2, as in the previous model.
From Table XI, one can easily see that the game is totally symmetric. Both will decide to choose randomly, Alice among the 3 operators and Bob only among the last two. With this, it is shown that for $\theta = \theta'_1$ Alice has a winning but unstable strategy as Bob after certain rounds will notice that he can even the game.

- $\theta = \theta'_0$: This point is even more unique because by crossing all the curves in Fig. 2 at the value of $1/2$ Bob does not even have to change his strategy to make the game symmetrical again. Moreover, Alice can do nothing to prevent this. This is observed by evaluating the probabilities shown in Table IX in $\theta'_0$. It is concluded that for the semiclassical Chinos game with unitary operators, Alice will have a winning and stable strategy for any angle $\theta \in [0, \pi]$ except at the values $\theta'_1$ and $\theta'_2$. In the former, Bob can choose a strategy that symmetrizes the game, and in the latter, the game is symmetrized without the need for Bob to change his strategy.

IV. A QUANTUM GAME WITH ENTANGLEMENT

In this section we begin explaining the idea of [19] to fully quantized the Chinos game and afterwards we propose a different quantization scheme.

Pushing the idea of semi-classical models further, a fully quantum version is proposed in which the guess space is also quantized. Thus, Alice and Bob will make as prediction a quantum state, $|g^i\rangle$ with $i = A, B$. The classical condition that the players cannot repeat the guess of the previous players will be translated into an orthogonality condition between the corresponding states. In the case under discussion, this condition is that the state predicted by Bob, $|g^B\rangle$, must be orthogonal to the state predicted by Alice, $|g^A\rangle$. That is:

$$\langle g^A | g^B \rangle = 0. \quad (21)$$

In this new quantum version of the game, one needs to define a payoff function, which evaluates how successful each player’s guess was.

$$f^A = |\langle g^A | \Psi_{i,j} \rangle|^2,$$

$$f^B = |\langle g^B | \Psi_{i,j} \rangle|^2,$$

where $f^A$ and $f^B$ are Alice’s and Bob’s payoffs, respectively. The winner of the game is the one with the highest payoff.

A. Bell states

Specifying the model, a 2-qubit system is considered. The computational basis is given by $|i_0, i_1\rangle$ with $i_0, i_1 = 0, 1$.

Bell states are defined as follows and represent a 2-qubit Hilbert space basis of maximally entangled states:
$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}; \quad |\phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}};$$

$$|\psi^+\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}; \quad |\psi^-\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}.$$ (23)

To construct these Bell states, the 1-qubit quantum gates of Pauli, X, and Hadamard, H, will be used:

$$X|0\rangle = |1\rangle; \quad X|1\rangle = |0\rangle;$$
$$H|0\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}; \quad H|1\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}},$$ (24)

and also the 2-qubit quantum gate, CNOT_{0,1}.

$$CNOT_{0,1}|i_1, i_0\rangle = |i_1 \oplus i_0, i_0\rangle,$$ (25)

where \( i_1 \oplus i_0 \) (where \( \oplus \) is the sum modulo 2) is the equivalent to the classical XOR gate. The \( i_0 \) qubit acts as the control qubit and the \( i_1 \) qubit acts as the target qubit. We will call CNOT_{0,1} as CNOT for simplicity.

Bell states are constructed based on these gates acting on the \( |00\rangle \) state:

$$Bell(i_1, i_0) = CNOT(id \otimes H)(X^{i_1} \otimes X^{i_2}).$$ (26)

With this, Bell states can be constructed as:

$$|\phi^+\rangle = Bell(0,0)|00\rangle; \quad |\phi^-\rangle = Bell(0,1)|00\rangle;$$
$$|\psi^+\rangle = Bell(1,0)|00\rangle; \quad |\psi^-\rangle = Bell(1,1)|00\rangle.$$ (27)

For simplicity of notation they will be denoted:

$$O_0 = Bell(0,0); \quad O_1 = Bell(0,1);$$
$$O_2 = Bell(1,0); \quad O_3 = Bell(1,1).$$ (28)

Note that the subscripts are given by \( n = i_0 + 2i_1 \).

Using this set of operators, eq. (28), the quantum version of the Chinos game is proposed. Alice and Bob will start by choosing which operator to draw, \( O_{a_0}^A \) and \( O_{b_0}^B \) being \( a_0, b_0 = 0, 1, 2, 3 \) to form a joint quantum state. For simplicity, we choose to act first with the operator draw by Bob and then the one draw by Alice, but notice that the order is important in this version.

$$|\psi_{a_0, b_0}\rangle = O_{a_0}^A O_{b_0}^B |00\rangle.$$ (29)

Notice that there is no need for normalization factors, as in eq. (6), since the operators, eq. (28), are unitary, so the states, eq. (29), are automatically normalized. This joint state can be expressed as

$$|\psi_{a_0, b_0}\rangle = \sum_{i_0, i_1} |\psi_{a_0, i_1, i_0}| i_1 i_0\rangle.$$ (30)

One obtains that \( |\psi_{i_0, i_1}\rangle = \pm 1/2 \) for all states where the signs of the coefficients are given in Table XII.

![Table XII: Signs of the coefficients ψ_{i_0, i_1} of all joint states.](image)

The parameter \( k \) can only take the values of 0 and ±1. In the case of \( k = 0 \) one has a maximally entangled state, since the density matrix is diagonal with eigenvalues 1/2:

$$|k = 0\rangle \quad \rho = \frac{1}{2} \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix}, \quad k = \frac{a_{0,0}a_{1,1} + a_{1,0}a_{0,1}}{2}. \quad (34)$$

In the case \( k = \pm 1 \) one has non-entangled states, as the density matrix is:

$$|k = \pm 1\rangle \quad \rho = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}. \quad (36)$$

Using Table XII, it is observed that for all states \( k = 0 \), i.e. the joint states are maximally entangled. After this, Alice and Bob will make their guesses, this time being a quantum state. In order to compare it with the joint state they have formed, the guesses will be given by 2 operators of eq. (28) as follows:

First, Alice makes her guess

$$|g_{a_1, a_2}\rangle = O_{a_1}^A O_{a_2}^A |00\rangle; \quad a_1, a_2 = 0, 1, 2, 3. \quad (37)$$
Next, Bob makes his guess
\[ |g_{b_1,b_2}^B \rangle = O_{b_1}^BO_{b_2}^B|00\rangle; \quad b_1, b_2 = 0, 1, 2, 3, \quad (38) \]

but Bob’s predicted state must be orthogonal to Alice’s one, that is

\[ \langle g_{a_1,a_2}^A | g_{b_1,b_2}^B \rangle = 0. \quad (39) \]

The payoff for each of them is given by:

\[
\begin{align*}
  f^A &= |\langle g_{a_1,a_2}^A | \Psi_{a_0,b_0} \rangle|^2, \\
  f^B &= |\langle g_{b_1,b_2}^B | \Psi_{a_0,b_0} \rangle|^2. 
\end{align*}
\quad (40)\]

In view of this, it is convenient to define the following metric in the space expanded by the operators \( O_iO_j \) acting on \(|00\rangle\) as it is done in the model presented in [19],

\[
G_{(i_1,j_1),(i_2,j_2)} = \langle 00 | O_{i_1}^JO_{j_1}^1O_{i_2}O_{j_2}|00\rangle; \quad i_1,j_1,i_2,j_2 = 0,1,2,3, \quad (41)
\]

in terms of which the condition eq. (39) becomes

\[ G_{(a_1,a_2),(b_1,b_2)} = 0, \quad (42) \]

and the payoffs of each player,

\[
\begin{align*}
  f^A &= |G_{(a_1,a_2),(a_0,b_0)}|^2, \\
  f^B &= |G_{(b_1,b_2),(a_0,b_0)}|^2. 
\end{align*}
\quad (43)\]

In order to find the best strategies for Alice and Bob, we will analyse the metric. Note that it is not necessary to take into account the signs of its coefficients as they are irrelevant for the uses of the metric, eq. (42) and eq. (43).

We begin by searching the symmetries of the metric (41). It has the block diagonal structure shown in Table XIII, where the only non-vanishing entries involve the states in the same subsets,

| Set 1 | Set 2 | Set 3 | Set 4 |
|-------|-------|-------|-------|
| \{00, 12, 22, 30\} | \{01, 13, 23, 31\} | \{02, 10, 20, 32\} | \{03, 11, 21, 33\} |

Moreover, the 4 sets are orthogonal to each other, so that if Alice chooses a state belonging to one set, Bob will not be able to choose a state belonging to the same set. With this, one can speak of choosing the set of states instead of choosing a single state.

With this metric, each player’s payoff can only have a value of 0 or 1 as there is no position with a coefficient other than 0 or ±1. Therefore, it will be said that Alice and Bob can either win or lose.

Note that Alice will be able to make her guess, choosing a set of eq. (44) without revealing any information to Bob about the operator she has chosen to draw, \( O_{a_0} \), since, whatever it is, she can win with any set depending on the operator Bob chooses to draw, \( O_{b_0} \). This is proved by observing that for any \( a_0 = 0,1,2,3 \) Alice’s payoff can take value 1 for any \( a_1, a_2 \) depending on the value of \( b_0 \).

An example is studied using the symmetries described above:

Alice, without knowing \( b_0 \), will choose randomly among the 4 sets of eq. (44). Let us suppose that Bob chooses \( b_0 = 0 \), for example, then he will guess a state in the set \( \text{Set}_1 \) or \( \text{Set}_3 \). The reason is that, observing the metric, these are the only 2 sets by which he can win for the value of \( b_0 = 0 \). The different scenarios are checked:

- Alice selects \( \text{Set}_1 \) as her guess: Bob, applying intelligence and obeying the orthogonality rule, will choose \( \text{Set}_3 \) as his guess. With this, half the time Alice will win and half the time Bob will win, since, if Alice rolls \( a_0 = 0 \) (or 3), she will win, but if she rolls \( a_0 = 1 \) (or 2), Bob will win.
- Alice selects \( \text{Set}_2 \) as her guess: Bob, applying intelligence, will choose randomly between \( \text{Set}_1 \) and \( \text{Set}_3 \), since he has no restriction on either of them as they are both orthogonal to \( \text{Set}_2 \). With this, Alice will lose whatever she draws as \( a_0 \) and Bob will be right half the time.
- Alice selects \( \text{Set}_3 \) as her guess: this case is completely analogous to the case where Alice chooses \( \text{Set}_1 \).
- Alice selects \( \text{Set}_4 \) as her guess: this case is completely analogous to the case where Alice chooses \( \text{Set}_2 \).
This leaves us with a situation where, half the time, both have a 50% chance of winning and, in the other half of the situations, Alice has a 0% chance of winning and Bob has a 50% chance of winning. The total winning probabilities of each are therefore

\[ P_A = \frac{1}{4}; \quad P_B = \frac{3}{4} \tag{45} \]

which shows that the game is not symmetric.

In view of this, Alice will try to reverse the situation, but it will be impossible since, only knowing \(a_0\), she cannot reject any set because she could win with every set whatever the value of \(a_0\) is, depending on the value of \(b_0\).

Hence, unlike the other versions, it is Bob who has a winning and stable strategy for the Chinos quantum game. In other quantum versions of the game with bosons \([19]\) it is found that it is Alice who has a winning and stable strategy, contrary to what is found in the model presented here.

### B. Rotations. States with different entanglement

We shall next extend the previous model replacing the CNOT gate by a CU gate which acts as follows

\[ CU(|i_1\rangle \otimes |i_0\rangle) = (U^{i_0}|i_1\rangle) \otimes |i_0\rangle, \tag{46} \]

where the unitary matrix used is a rotation:

\[ U(\theta) = e^{-i\frac{\theta}{2} \sigma^x}, \tag{47} \]

being \(\sigma^x\) the Pauli matrix.

With this, it can be said that rotated Bell states are constructed by applying the above gates on the state \(|00\rangle\)

\[ \text{RBell}(i_1, i_0) = CU(i \otimes H)(X^{i_1} \otimes X^{i_0}). \tag{48} \]

Therefore, the rotated Bell states are:

\[ |\phi^+_{RB}\rangle = \text{RBell}(0, 0)|00\rangle = \frac{|00\rangle + e^{-i\theta/2}|11\rangle}{\sqrt{2}}, \]

\[ |\phi^-_{RB}\rangle = \text{RBell}(0, 1)|00\rangle = \frac{|00\rangle - e^{-i\theta/2}|11\rangle}{\sqrt{2}}, \tag{49} \]

\[ |\psi^+_{RB}\rangle = \text{RBell}(1, 0)|00\rangle = \frac{|10\rangle + e^{-i\theta/2}|01\rangle}{\sqrt{2}}, \]

\[ |\psi^-_{RB}\rangle = \text{RBell}(1, 1)|00\rangle = \frac{|10\rangle - e^{-i\theta/2}|01\rangle}{\sqrt{2}}. \]

For simplicity the operators of eq. (48) will be denoted:

\[ O_0' = \text{RBell}(0, 0); \quad O'_1 = \text{RBell}(0, 1); \quad O_2' = \text{RBell}(1, 0); \quad O_3' = \text{RBell}(1, 1). \tag{50} \]

Following the same procedure, the joint quantum states will be given by,

\[ |\psi_{a_0, b_0}\rangle = O_{a_0}^A O_{b_0}^B |00\rangle, \tag{51} \]

which can be rewritten with the coefficients, as in eq. (30).

We obtain that \(|\psi_{i_1, i_0}\rangle = 1/2\) for all joint states as before with the CNOT gate. However, now we not only find signs, but also phases given in Table XIV.

| \(i_1\) | \(i_2\) |
|---|---|
| 00 | 00 |
| 00 | 01 |
| 00 | 02 |
| 00 | 03 |

TABLE XIV: The signs and phases of the coefficients \(\psi_{i_1, i_2}\) of all possible joint states where \(z = e^{-i\theta}\).

It is shown that the entanglement of the states depends on the value of the parameter \(\theta\). Writing the state wave function as a \(2 \times 2\) matrix, as in eq. (31), we can compute the density matrix by tracing over qubit 0, as in eq. (32).

The quantum states can still be rewritten as in eq. (33), but in this case \(a_{i,j}\) correspond to the phases shown in Table XIV for each state. The following density matrix is obtained:

\[ \rho = \frac{1}{2} \begin{pmatrix} 1 & k \\ k^* & 1 \end{pmatrix}, \quad k = \frac{a_{0,0}a_{1,0}^* + a_{1,1}^*a_{0,1}}{2}. \tag{52} \]

Taking the phases of each state, it is observed that the parameter \(k\) can only take two values \(k = \pm i \sin(\theta/2)\). This fact implies that the entanglement of the qubits depends on the parameter \(\theta\), having a maximum at \(\theta = 0\) and a minimum at \(\theta = \pi\). This shows that the CNOT gate case is included in this one when \(\theta = 0\).

Analogously, the metric \(G'_{(i_1, j_1), (i_2, j_2)}\) is calculated:

\[ G'_{(i_1, j_1), (i_2, j_2)} = \langle 00 | O^\dagger_{i_1} O^\dagger_{i_2} O_{j_1} O_{j_2} |00\rangle; \]

\[ i_1, j_1, i_2, j_2 = 0, 1, 2, 3. \tag{53} \]

With this, it is observed that, if we take the states without entanglement between qubits (\(\theta = \pi\)), new sets of states appear, that allow an analysis of the strategies can be made (the metric \(G'\) for \(\theta = \pi\) ordered according to these new sets can be found in the subsection B of the appendix). It should be remembered that the sign and phase of the elements of the metric have no relevance in the game, since from the metric we extract the orthogonality information of the states and the payoffs, which are the modulus squared. We have now the following sets:

\[ Set'_2 = \{00, 13, 22, 31\} \quad \text{Set}'_2 = \{01, 12, 23, 30\} \]

\[ Set'_3 = \{02, 11, 20, 33\} \quad \text{Set}'_3 = \{03, 10, 21, 32\}. \tag{54} \]
since they could be winners too. This is so because, the 3 sets orthogonal to the one Alice has chosen, applying intelligence, in this case will choose among strategy. The strategy of the players, in particular it changes takes two values among 0,1,2,3. This detail changes values from 0 to 3 while in the previous one it only be winners for any value of among the 4 sets of eq. (54) since all of them could

Let’s look at an example explicitly:

First, we study the case of 2 players with one coin each. It is concluded that the best strategy for Alice is to draw or not the coin randomly and that her guess is 1, while Bob will also draw or not randomly and his guess should be the “intelligent” option. With this strategy, it is observed that it is a symmetric game, i.e. each player will win half of the time. After this, it was generalised to the case of two players and \(N_c\) coins reaching the same conclusion, the classical symmetry of the game.

In order to quantize the game, we follow the same steps as in [19], we start with a semiclassical version in which the classical guess space is left intact while the space of draws is quantized. The first idea is to opt for a boson model. After analysing all the possible moves, it is concluded that Bob has a winning but unstable strategy, as they both follow the classical strategy of drawing randomly among the 4 operators. On the other hand, Alice will notice this and will reduce her space of guesses.

Indeed, we have verified that the game restores classical symmetry for the case \(\theta = \pi\), i.e., in the case of non-entangled states. Moreover, this is stable since, with the information available to each of them, it is not possible to reduce the space of guesses.

It would be interesting to do an analysis for intermediate values of \(\theta\) but this is rather more complex and beyond the scope of this paper.

In order to complete the research, these two studied cases of the quantum version of the game have been simulated on a real quantum computer. This has been made possible thanks to the IBM Quantum experience [20]. The results of the simulations are shown in subsection A of the appendix.

V. CONCLUSIONS

This paper begins with an analysis of the classical Chinos game, reproducing the results obtained in [19]. First, we study the case of 2 players with one coin each. It is concluded that the best strategy for Alice is to draw or not the coin randomly and that her guess is 1, while Bob will also draw or not randomly and his guess should be the “intelligent” option. With this strategy, it is observed that it is a symmetric game, i.e. each player will win half of the time. After this, it was generalised to the case of two players and \(N_c\) coins reaching the same conclusion, the classical symmetry of the game.

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We propose a new version of the latter model by replacing the bosons by hard-core bosons in which only the \(|0\rangle\) and \(|1\rangle\) states are allowed. Starting both from the classical strategy, it is observed that there is a winning and stable strategy for Alice, since Bob has no possible choice that achieves \(P_A \leq 1/2\). Having seen this, we seek to generalise the set of 3 operators

\[
P_A = \frac{1}{4} \left( 0 + 0 + 0 + 1 \right) = \frac{1}{4}.
\]

\[
P_B = \frac{1}{4} \left( 1 \frac{1}{3} + \frac{1}{3} + 1 + 0 \right) = \frac{1}{4}.
\]

Alice selects \(\text{Set}_1\): in this case Bob will choose randomly between \(\text{Set}_{1}'\), \(\text{Set}_{2}'\) and \(\text{Set}_{3}'\). With this it will be Alice who wins with full probability while Bob has no choice at all.

Once we have the 4 scenarios, which will all have the same probability since Alice could win with any set depending on \(b_0\) regardless of the value of \(a_0\), we calculate the total winning probabilities of both:

\[
\begin{align*}
p_A &= \frac{1}{4} \left( 0 + 0 + 0 + 1 \right) = \frac{1}{4}, \\
p_B &= \frac{1}{4} \left( \frac{1}{3} + \frac{1}{3} + 1 + 0 \right) = \frac{1}{4}.
\end{align*}
\]
by leaving the parameter $\theta \in [0, \pi/2)$ free and redo the analysis in terms of this parameter. We study 3 different regions of $\theta$ values and their boundary points. Analysing the strategies in each region and at the two boundary points, a winning and stable strategy for Alice is found in all cases.

Another original semiclassical version is based on the use of unitary operators, as the quantum logic gates are made up of them. Rotations are considered, leaving a parameter $\theta \in [0, \pi)$ free. Three regions of $\theta$ values appear again in which there is still a winning and stable strategy for Alice. However, this time at both boundary points, it is found that Bob can restore the classical symmetry of the game.

After the previous semiclassical versions, we go one step further by quantizing the guess space. In order to maintain the condition of the game that the guess made by previous players cannot be repeated, it is imposed that Bob’s quantum prediction state must be orthogonal to Alice’s quantum guess state (same condition as in [19]). An original model with 2 qubits is proposed in which the 4 operators they can choose are the 4 operators that create the four Bell states. It is observed that in this model all quantum states that can be formed have maximum entanglement. To analyse the possible strategies, a metric $G_{i_1,j_1,i_2,j_2} (i_1,j_1,i_2,j_2 = 0,1,2,3)$ is constructed. It is concluded that Bob has a winning and stable strategy if both follow the classical strategy, namely $P_A = 1/4$ and $P_B = 3/4$. This is a new result as compared to the one found in [19], where Alice has always a winning strategy.

Observing this imbalance, the quantum version is generalised by replacing the CNOT gate in the operators by a CU gate where U is a rotation of angle $\theta$ around the $x$ axis. The parameter $\theta$ controls the entanglement of the states created by Alice and Bob. The previous case (maximum entanglement) being included in $\theta = 0$. We have also studied the case $\theta = \pi$, where there is no entanglement, and found quite interestingly that the classical symmetry is restored in a stable way. This result is quite interesting, since in the version of the game with 2 maximally entangled qubits there is a winning and stable strategy for Bob while in the model with 2 non-entangled qubits the classical symmetry is maintained. We leave open the problem of at which value of $\theta$ occurs this transition.

Acknowledgements

GS acknowledges financial support through the Spanish MINECO grant PGC2018-095862-B-C21, the Comunidad de Madrid grant No. S2018/TCS-4342, the Centro de Excelencia Severo Ochoa Program SEV-2016-0597 and the CSIC Research Platform on Quantum Technologies PTI-001.

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Appendix

A. Simulation of the quantum game with a quantum computer

As we have mentioned before, the two studied cases of the quantum version of the game have been simulated on an IBM’s quantum computer.

The idea of the simulations is that the whole strategy of the game is based on the metric $G$. Therefore, we have obtained the inputs of the metric for each case.

For the case of maximally entangled qubits, this is done from the eq. (41) in which 4 operators are applied and projected onto the $\ket{00}$ state. In order to implement the operators $O_i$ ($i = 0, 1, 2, 3$) in a quantum circuit, we use the definition eq. (26) and for the operators $O_i^\dagger$, we use

$$Bell(i_1, i_0) = (X^{i_1} \otimes X^{i_0})(id \otimes H)CNOT.$$  (56)

**Fig. 3** shows the circuit to compute $G_{2,2}(3,0)$ for the model with maximal entanglement as an example.

![FIG. 3: Circuit designed to obtain the input of the $G_{2,2}(3,0)$ metric in the case of maximum entanglement.](image)

We used the qubits $q_0$ and $q_1$ of the quantum computer ibmq-Manila whose topology is shown in Fig. 4. The number of shots was set to its maximum, 8192, obtaining the results shown in Fig. 5. From these results, we compute $|G|$ because $|G|^2$ is the percentage of shots that the state $\ket{00}$ has been measured. However, there are certain entries with a negative sign, or in other words, a phase. To observe this phase we have used the IBM Quantum Experience tool that provides the ideal state generated by the circuit and its phase. Since the phase has no relevance to the strategies of the game, we have not performed quantum tomography to measure it.

![FIG. 4: ibmq-Manila topology](image)

![FIG. 5: Histogram of the probabilities of measure each state after 8192 shots for the example of $G_{2,2}(3,0)$ in the case of maximum entanglement.](image)

We see that in some shots the quantum computer measures states that we do not expect theoretically. This is due to the fact that current quantum computers have noise that alters the state of the qubits. This can be seen in the metric of the model with maximal entanglement, Table XVI, which should be compared with Table XIII, where the theoretical result is shown. It can be seen that in the entries that theoretically have modulus 1, we now have values very close to 1 but never exact. Similarly, in the rest of the inputs that are theoretically null, low values are observed, although never null. In the light of Tables XIII and ?? it can be said that the ibmq-Manila has an error of 2-3%. For the strategies of the game, we should now readjust the condition that the state chosen by Bob has to be orthogonal to the one chosen by Alice. A threshold could be set below which states would be considered eligible. In any case, the strategies will vary significantly since, each time the metric is measured on a quantum computer, different results will be obtained since the noise is unpredictable.

| $i_1$ | $i_2$ | $i_3$ | $i_4$ | $i_5$ | $i_6$ | $i_7$ | $i_8$ | $i_9$ | $i_{10}$ | $i_{11}$ | $i_{12}$ | $i_{13}$ | $i_{14}$ | $i_{15}$ | $i_{16}$ | $i_{17}$ | $i_{18}$ |
|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 |
| 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 |

**TABLE XVI:** Metric $G_{2,2}(3,0)$ measured on a quantum computer, reordered by sets for the case of maximum entanglement.
An analogous process is used in the case of non-entangled qubits ($\theta = \pi$). To calculate the metric, we start from the eq. (53) in which 4 operators are applied and projected onto the |00⟩. In order to implement the operators $O_i$ and $O_i^\dagger$ ($i = 0, 1, 2, 3$), we start from the definition of the first ones, eq. (48), characterising the gate $U(\theta, \phi, \lambda) = U(\pi, -\pi/2, \pi/2)$.

With this, the circuits for each input are constructed, which only differ from the maximum entanglement case in the CU gates instead of the CNOT gates. Fig. 6 shows the circuit to compute the entry $G'_{(2,2),(3,0)}$ for the non-entangled case as an example.

**FIG. 6:** Circuit designed to obtain the input of the $G'_{(2,2),(3,0)}$ metric in the non-entangled case.

Fig. 7 shows the histogram for the example of the $G'_{(2,2),(3,0)}$ input of the non-entangled case. The same process is followed to obtain the modulus and phase of each entry.

**FIG. 7:** Histogram of the probabilities of measure each state after 8192 shots for the example of $G'_{(2,2),(3,0)}$ in the case without entanglement.

Very similar results are found as in the previous case. This is shown in Table XVII, which should be compared with Table XVIII in the section B of the appendix, which shows the theoretical result. For the inputs that theoretically have modulus 1, values very close to unity are measured, but in no case are they exact. For the rest of the inputs, null according to the theory, small values are measured but never null. The cause is the same, the noise in today's quantum computers. Regarding the strategies, the same happens as in the previous case, it would be necessary to re-adjust the condition that the state which Bob can choose must be orthogonal to Alice's. In the same way, the strategies would vary from the previous case. Similarly, the strategies would vary each time the metric is measured on a quantum computer, since each time slightly different results would be obtained.

**TABLE XVII:** Metric $G'_{(i_1,j_1),(i_2,j_2)}$ measured on a quantum computer for the null entanglement case.

| 00 | 13 | 22 | 31 | 01 | 12 | 23 | 30 | 02 | 11 | 20 | 33 | 03 | 10 | 21 | 32 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 00 | i | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | -i | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 22 | i | 1 | i | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 31 | -i | -1 | i | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 01 | 0 | 0 | 0 | 0 | 1 | -i | 1 | -i | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | i | 1 | i | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 23 | 0 | 0 | 0 | 0 | 1 | -i | 1 | -i | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 30 | 0 | 0 | 0 | 0 | i | 1 | i | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 02 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | i | i | 0 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -i | -1 | -1 | 0 | 0 | 0 | 0 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | i | i | 0 | 0 | 0 | 0 |
| 33 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -i | -1 | -1 | 0 | 0 | 0 | 0 |
| 03 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -i | 1 | -1 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | i | i |
| 21 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -i | 1 | -1 |
| 32 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | i | i | 1 |

**TABLE XVIII:** Metric $G'_{(i_1,j_1),(i_2,j_2)}$ reordered by sets for the case $\theta = \pi$. 