An approximation scheme for variational inequalities with convex and coercive Hamiltonians

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Abstract

We propose an approximation scheme for a class of semilinear variational inequalities whose Hamiltonian is convex and coercive. The proposed scheme is a natural extension of a previous splitting scheme proposed by Liang, Zariphopoulou and the author for semilinear parabolic PDEs. We establish the convergence of the scheme and determine the convergence rate by obtaining its error bounds. The bounds are obtained by Krylov’s shaking coefficients technique and Barles-Jakobsen’s optimal switching approximation, in which a key step is to introduce a variant switching system.

Keywords: Splitting, viscosity solutions, shaking coefficients technique, optimal switching approximation, variant switching system.

1 Introduction

This paper is an extension of a previous work started by Liang, Zariphopoulou and the author \[19\], in which they consider semilinear parabolic PDEs with convex and coercive Hamiltonians, and propose an approximation based on splitting the equation into a linear parabolic equation and a Hamilton-Jacobi equation. By the convexity property of Hamiltonians, the semilinear parabolic PDEs they considered can be written as HJB type of parabolic equations, which correspond to stochastic optimal control problems. Herein, we use the same setting and extend their work by treating optimal stopping as well as optimal control at the same time. This leads to obstacle problems with associated variational inequalities. To be more specific, we consider semilinear parabolic variational inequalities of the form

\[
\max\{-\partial_t u + g(t, x, \partial_x u, \partial_{xx} u), u - f(t, x)\} = 0 \quad \text{in} \ Q_T;
\]

\[
u(T, x) = U(x) \quad \text{in} \ \mathbb{R}^n,
\]

where

\[
g(t, x, p, X) := -\frac{1}{2} \text{tr} \left( \sigma \sigma^T (t, x) X \right) - b(t, x) \cdot p + H(t, x, p),
\]

and \( Q_T = [0, T) \times \mathbb{R}^n \). A key feature is that the Hamiltonian \( H(t, x, p) \) is convex and coercive in \( p \). In particular, this covers the case that \( H \) has quadratic growth in \( p \), a case that corresponds to a rich class of equations in mathematical finance arising, for example, in optimal investment with

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homothetic risk preferences (13), exponential indifference valuation (8, 15, 17) and entropic risk measures (9), just to name a few. Note that if \( u < f \) in \( Q_T \), equation (1.1) reduces exactly to the semilinear parabolic PDE considered in [19]:

\[-\partial_t u + g(t, x, \partial_x u, \partial_{xx} u) = 0 \text{ in } Q_T.\] (1.3)

Herein, we contribute to proposing an approximation scheme for variational inequalities of the type (1.1) using an approximation of (1.3) introduced in [19]. The key idea is to use in an essential way the convexity of the Hamiltonian. To the best of our knowledge, this property has not been adequately exploited in the existing approximation studies. The extension from the scheme for (1.3) to our scheme for (1.1) is natural. Suppose \( S \) denotes the approximation of (1.3), we propose an approximation scheme for (1.1):

\[
\max \left\{ S(\Delta, t, x, u^\Delta(t, x), \Delta(t + \Delta, t)), u^\Delta(t, x) - f(t, x) \right\} = 0,
\]

where \( \Delta \) is the discretisation time step and \( u^\Delta \) is the solution of the scheme used to approximate the (viscosity) solution of (1.1). We will formally introduce the approximation scheme in section 2; see (2.2) and (2.4) for details. We also refer to [19] for the construction and intuition of the approximation \( S \) of (1.3).

Next, we establish the convergence of the scheme solution to the unique (viscosity) solution of (1.1) and determine the rate of convergence. We do this by obtaining upper and lower bounds on the approximation error (Theorems 3.4 and 3.7, respectively). The main tools come from the shaking coefficients technique introduced by Krylov [23] [24] and the optimal switching approximation introduced by Barles and Jakobsen [1] [2].

While various arguments follow from adaptations of these techniques, a main difficulty is to derive the consistency error estimate. Fortunately, thanks to the previous work of [19] (Proposition 2.5, 2.8 therein), the consistency estimate follows immediately herein. Using this estimate and the comparison result for the approximation scheme (Proposition 2.6), we in turn derive an upper bound for the approximation error by perturbing the coefficients of (1.1).

The lower bound for the approximation error is obtained by another layer of approximation of (1.1) via an auxiliary optimal switching system. Barles and Jakobsen [1] [2] in their paper use standard optimal switching systems to approximate standard HJB type of equations. However, the variational inequality (1.1) considered herein cannot be written as a standard HJB equation but an HJB equation with obstacle. Thus, we modify the standard optimal switching system and introduce a variant type of switching system, which can be proved to approximate the HJB equation with obstacle. To the best of our knowledge, we are the first to introduce this variant switching system in the existing literature. We give the well-posedness, regularity as well as continuous dependence result for this variant switching system in Section 4.

A highly related work to this paper is Jakobsen [21], where he obtained error bounds for general monotone approximation schemes for Bellman equations arising in a stochastic optimal stopping and control problem. Due to the convex and coercive property of the Hamiltonian \( H(t, x, p) \), (1.1) can be written as the same type of Bellman equation, but with control set and coefficients unbounded. This is not the case in [21]. Furthermore, he used the same shaking coefficients technique to derive an error bound for one side, but for the other side he interchanged the roles of the approximation scheme and the original equation, based on an additional assumption (Assumption 2.5 therein) that the scheme solution has enough regularity. Unfortunately, our proposed scheme does not satisfy this assumption and thus, we need another layer of approximation of the original equation, which
decreases the convergence rate. Finally, a common point between his work and ours is that the scheme solution \( u^\Delta \) is defined at every point in \( Q_T \) rather than some certain time (and space) grids.

The paper is organized as follows. In the next section we introduce the approximation scheme for (1.1)-(1.2). In Section 3, we prove its convergence and establish the convergence rate by obtaining the upper and lower bounds for the approximation error, which are the main results of this paper. Section 4 is devoted to introducing the variant switching systems. We conclude in Section 5. Some technical proofs are provided in the appendix.

### 2 The approximation scheme for (1.1)

Let \( d \in \mathbb{Z}^+ \) and \( \delta > 0 \). For a function \( f : Q_T \to \mathbb{R}^d \), we introduce its (semi)norms

\[
|f|_0 := \sup_{(t,x) \in Q_T} |f(t,x)|,
\]

\[
[f]_{1,\delta} := \sup_{(t,x),(t',x') \in Q_T, t \neq t'} \frac{|f(t,x) - f(t',x')|}{|t - t'|^{\delta}},
\]

\[
[f]_{2,\delta} := \sup_{(t,x),(t',x') \in Q_T, x \neq x'} \frac{|f(t,x) - f(t,x')|}{|x - x'|^{\delta}}.
\]

Furthermore, \([f]_{\delta} := [f]_{1,\delta/2} + [f]_{2,\delta} \) and \([f]_{\delta} := |f|_0 + [f]_{\delta} \). Similarly, the (semi)norms of a function \( g : \mathbb{R}^n \to \mathbb{R}^d \) are defined as

\[
|g|_0 := \sup_{x \in \mathbb{R}^n} |g(x)|, \quad [g]_{\delta} := \sup_{x,x' \in \mathbb{R}^n, x \neq x'} \frac{|g(x) - g(x')|}{|x - x'|^{\delta}}, \quad |g|_{\delta} := |g|_0 + [g]_{\delta}.
\]

For \( S = Q_T, \mathbb{R}^n \) or \( Q_T \times \mathbb{R}^n \), we denote by \( C(S) \) the space of continuous real-valued functions on \( S \), and by \( C^0_b(S) \) the space of bounded and continuous real-valued functions on \( S \) with finite norm \([f]_{\delta} \), for any \( \delta \geq 0 \). Furthermore, we set \( C^0_b(S) = C_b(S) \) and also denote by \( C^\infty_b(S) \) the space of smooth real-valued functions on \( S \) with bounded derivatives of any order.

Throughout this paper we assume the following conditions for equations (1.1)-(1.2).

**Assumption 2.1**

(i) The \( n \times d \) matrix-valued diffusion coefficient \( \sigma \), the \( \mathbb{R}^n \)-valued drift coefficient \( b \), the real-valued obstacle \( f \) and terminal datum \( U \) have finite norms \( |\sigma|_1, |b|_1, |f|_1, |U|_1 \leq M \) for some \( M > 0 \). Moreover, \( f(T, \cdot) \geq U \) in \( \mathbb{R}^n \).

(ii) The Hamiltonian \( H(t,x,p) \in C(Q_T \times \mathbb{R}^n) \) is convex in \( p \), and satisfies the coercive condition

\[
\lim_{|p| \to \infty} \frac{H(t,x,p)}{|p|} = +\infty,
\]

uniformly in \((t,x) \in Q_T\). Moreover, for every \( p \), \([H(\cdot,\cdot,p)]_1 \leq M \) for the constant \( M \) in (i), and there exist two locally bounded functions \( H^+ \) and \( H^- : \mathbb{R}^n \to \mathbb{R} \) such that

\[
H^-(p) = \inf_{(t,x) \in Q_T} H(t,x,p), \quad H^+(p) = \sup_{(t,x) \in Q_T} H(t,x,p).
\]

Unless stated otherwise, we will then throughout this paper denote by \( C := C(T,M) \) some constant that depends only on \( T \) and \( M \). Then under the above assumptions, we have the following existence, uniqueness and regularity results for equation (1.1)-(1.2). Their proofs are provided in Appendix A.

**Proposition 2.2** Suppose that Assumption 2.1 is satisfied. Then, there exists a unique viscosity solution \( u \in C^1_b(Q_T) \) of (1.1)-(1.2), with \(|u|_1 \leq C \).
2.1 The backward operator $S_I(\Delta)$

Before introducing the approximation scheme, we introduce a backward operator $S_I(\Delta)$, which is defined in [19]. Herein, for the reader’s convenience, we repeat the definition. To this end, using the convexity and coerciveness of the Hamiltonian $H(t, x, p)$, we define its Legendre transform $L : Q_T \times \mathbb{R}^n \to \mathbb{R}$ by

$$L(t, x, q) := \sup_{p \in \mathbb{R}^n} \{ p \cdot q - H(t, x, p) \}. \quad (2.1)$$

Next, for any $t$ and $\Delta$ such that $0 \leq t < t + \Delta \leq T$ and any $\phi \in C_b(\mathbb{R}^n)$, the backward operator $S_I(\Delta) : C_b(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$ is defined by

$$\left\{ \begin{array}{ll}
S_I(\Delta)\phi(x) = \min_{y \in \mathbb{R}^n} \left\{ \Delta L \left( t, x, \frac{x - y}{\Delta} \right) + E[\phi(Y_{t+\Delta}^{t,y})|\mathcal{F}_t] \right\}, & x \in \mathbb{R}^n \\
y_s = y + b(t, y)(s - t) + \sigma(t, y)(W_s - W_t), & s \in [t, t + \Delta],
\end{array} \right. \quad (2.2)$$

on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $W$ is an $d$-dimensional Brownian motion with its augmented filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

Note that in the definition of the operator $S_I(\Delta)$, it is implied that for any $\phi \in C_b(\mathbb{R}^n)$, there always exists an associated minimizer $y^*$ and $S_I(\Delta)\phi$ is also in $C_b(\mathbb{R}^n)$. These along with other key properties of the backward operator $S_I(\Delta)$ are proved in [19].

2.2 The approximation scheme

We are now in a position to introduce the approximation scheme for the variational inequality (1.1)-(1.2). This approximation scheme is a natural extension of the scheme (2.11) in [19]: For $\Delta \in (0, T)$ and $(t, x) \in Q_{T-\Delta}$, we introduce the iterative algorithm

$$u^\Delta(t, x) = \min \{ S_I(\Delta)u^\Delta(t + \Delta, \cdot)(x), f(t, x) \} \quad (2.3)$$

with $u^\Delta(T, \cdot) = U(\cdot)$ and $S_I(\Delta)$ defined in (2.2). The values between $T - \Delta$ and $T$ are obtained by a standard linear interpolation.

Specifically, the approximation scheme is given by

$$\left\{ \begin{array}{ll}
\bar{S}(\Delta, t, x, u^\Delta(t, x), u^\Delta(t + \Delta, \cdot)) = 0 & \text{in } Q_{T-\Delta}, \\
u^\Delta(t, x) = g^\Delta(t, x) & \text{in } Q_T \setminus Q_{T-\Delta},
\end{array} \right. \quad (2.4)$$

where $\bar{S} : (0, T] \times Q_{T-\Delta} \times \mathbb{R} \times C_b(\mathbb{R}^n) \to \mathbb{R}$, and $g^\Delta : Q_T \setminus Q_{T-\Delta} \to \mathbb{R}$ are defined respectively by

$$\bar{S}(\Delta, t, x, p, v) = \max \{ S(\Delta, t, x, p, v), p - f(t, x) \}, \quad (2.5)$$

$$S(\Delta, t, x, p, v) = \frac{p - S_I(\Delta)v(x)}{\Delta}, \quad (2.6)$$

and

$$g^\Delta(t, x) = \omega_1(t)U(x) + \omega_2(t) \min \{ S_{T-\Delta}(\Delta)U(x), f(T - \Delta, x) \}, \quad (2.7)$$

with $\omega_1(t) = (t + \Delta - T)/\Delta$ and $\omega_2(t) = (T - t)/\Delta$ being the linear interpolation weights.

Note that when $T - \Delta < t \leq T$, the approximate term $g^\Delta$ corresponds to the usual linear interpolation between $T - \Delta$ and $T$.  

4
The next proposition shows the well-posedness of the approximation scheme \((2.4)\). The proof is almost the same as Lemma 2.6 in [19] so we omit the detail here. We remark that, unlike the viscosity solution \(u\) of the variational inequality \([11] - [12]\), the solution \(u^\Delta\) of the approximation scheme does not in general have enough regularity.

**Proposition 2.3** Suppose that Assumption 2.1 is satisfied and let \(\Delta \in (0, T)\). Then, the approximation scheme \((2.4)\) admits a unique solution \(u^\Delta \in \mathcal{C}_b(\bar{Q}_T)\) with \(|u^\Delta|_0 \leq C\).

Thanks to the properties of \(S(\Delta, t, x, p, v)\) established in Proposition 2.8 of [19], we immediately obtain the following key properties of the approximation scheme \((2.4)\).

**Proposition 2.4** Suppose that Assumption 2.1 is satisfied and let \(\Delta \in (0, T), (t, x) \in \bar{Q}_T - \Delta\), \(p \in \mathbb{R}\) and \(v \in \mathcal{C}_b(\mathbb{R}^n)\). Then, the approximation scheme \(\bar{S}(\Delta, t, x, p, v)\) has the following properties:

(i) (Monotonicity) For any \(c_1, c_2 \in \mathbb{R}\), and any function \(u \in \mathcal{C}_b(\mathbb{R}^n)\) with \(u \leq v\),

\[
\bar{S}(\Delta, t, x, p + c_1, u + c_2) \geq \bar{S}(\Delta, t, x, p, v) + \min\left\{\frac{c_1 - c_2}{\Delta}, c_1\right\}.
\]

(ii) (Consistency) For any \(\phi \in \mathcal{C}_\infty_b(\bar{Q}_T)\),

\[
\left|\max\left\{-\partial_t \phi + g(t, x, \partial_x \phi, \partial_{xx} \phi), \phi - f(t, x)\right\}- \bar{S}(\Delta, t, x, \phi, \phi(t + \Delta, \cdot))\right| \\
\leq C\Delta (|\partial_{tt} \phi|_0 + |\partial_{xxxx} \phi|_0 + |\partial_{xxt} \phi|_0 + \mathcal{R}(\phi)) \quad \text{in} \quad \bar{Q}_T - \Delta,
\]

where the constant \(C\) depends only on \(\|\phi\|_{2,1}, M\) and \(T\), and \(\mathcal{R}(\phi)\) represents the “insignificant” terms containing the lower order derivatives of \(\phi\).

**Proof.** (i) follows immediately from the definition of \(\bar{S}\), \((2.5)-(2.6)\), and that of \(S_t(\Delta)\), \((2.2)\). (ii) follows from Proposition 2.8 (iv) in [19].

The monotonicity property (i) in Proposition 2.4 then implies the following comparison result for the approximation scheme \((2.4)\), which will be used throughout this paper. The proof is analogous to Proposition 2.9 of [19], with a slight difference to accommodate the extension to the variational inequality case.

**Proposition 2.5** Suppose that Assumption 2.1 is satisfied, and that \(u, v \in \mathcal{C}_b(\bar{Q}_T)\) are such that

\[
\bar{S}(\Delta, t, x, u, u(t + \Delta, \cdot)) \leq h_1 \quad \text{in} \quad \bar{Q}_T - \Delta,
\]

\[
\bar{S}(\Delta, t, x, v, v(t + \Delta, \cdot)) \geq h_2 \quad \text{in} \quad \bar{Q}_T - \Delta,
\]

for some \(h_1, h_2 \in \mathcal{C}_b(\bar{Q}_T - \Delta)\). Then,

\[
u - v \leq \sup_{\bar{Q}_T \setminus \bar{Q}_T - \Delta} (u - v)^+ + (T - t + 1) \sup_{\bar{Q}_T - \Delta} \frac{h_1 - h_2}{\Delta} \quad \text{in} \quad \bar{Q}_T.
\]

**Proof.** Without loss of generality, we assume that

\[
u - v \leq \bar{Q}_T \setminus \bar{Q}_T - \Delta \quad \text{and} \quad h_1 \leq h_2 \in \bar{Q}_T - \Delta,
\]

\[
u - v \leq \bar{Q}_T \setminus \bar{Q}_T - \Delta \quad \text{and} \quad h_1 \leq h_2 \in \bar{Q}_T - \Delta.
\]
since, otherwise, the function \( w := v + \sup_{\bar{Q}_T}Qu_{\bar{T} - \Delta} (u - v)^+ + (T - t + 1) \sup_{\bar{Q}_T}h_1 - h_2)^+ \) satisfies that \( u \leq w \) in \( \bar{Q}_T \) and by the monotonicity property (i) in Proposition 2.4
\[
\bar{S}(\Delta, t, x, w, w(t + \Delta, \cdot)) \geq \bar{S}(\Delta, t, x, v(t + \Delta, \cdot)) + \sup_{\bar{Q}_T - \Delta} (h_1 - h_2)^+ 
\]
\[
\geq h_2 + \sup_{\bar{Q}_T - \Delta} (h_1 - h_2)^+ \geq h_1 \text{ in } \bar{Q}_T - \Delta.
\]
Thus, it suffices to prove \( u \leq v \) in \( \bar{Q}_T \) when (2.10) holds.

To this end, for \( b \geq 0 \), let \( \psi_b(t) := b(T - t) \) and \( M(b) := \sup_{\bar{Q}_T} \{ u - v - \psi_b \} \). Then our goal becomes to prove \( M(0) \leq 0 \) and we prove by contradiction. Assume \( M(0) > 0 \), then by the continuity of \( M \), we must have \( M(b) > 0 \) for some \( b > 0 \). For such \( b \), take a sequence \( \{ (t_n, x_n) \} \) in \( Q_T \) such that \( \delta_n := M(b) - (u - v - \psi_b)(t_n, x_n) \downarrow 0 \), as \( n \to \infty \). Since \( M(b) > 0 \) but \( u - v - \psi_b \leq 0 \) in \( \bar{Q}_T \), we must have \( t_n \leq T - \Delta \) for sufficiently large \( n \). Then for such \( n \), we use the monotonicity property (i) in Proposition 2.4 again to obtain
\[
h_1(t_n, x_n) \geq \bar{S}(\Delta, t_n, x_n, u(t_n, x_n), u(t_n + \Delta, \cdot))
\]
\[
\geq \bar{S}(\Delta, t_n, x_n, v(t_n, x_n) + \psi_b(t_n + \Delta, \cdot) + \psi_b(t_n + \Delta, \cdot) + \psi_b(t_n + \Delta, \cdot) + M(b))
\]
\[
\geq \bar{S}(\Delta, t_n, x_n, v(t_n, x_n), v(t_n + \Delta, \cdot) + \min \{ b - \delta_n \Delta^{-1}, \psi_b(t_n) + M(b) - \delta_n \}
\]
\[
\geq h_2(t_n, x_n) + \min \{ 1, \Delta \} (b - \delta_n \Delta^{-1}),
\]
where the last inequality follows from \( M(b) > 0 \) and \( \psi_b(t_n) \geq \Delta b \). Since \( h_1 \leq h_2 \) in \( \bar{Q}_T - \Delta \), we then must have \( b - \delta_n \Delta^{-1} \leq 0 \). Thus, we deduce \( b \leq 0 \) by letting \( n \to \infty \), which is a contradiction. ■

3 Convergence rate of the approximation scheme

In this section, we establish the (uniform) convergence rate of the approximate solution \( u^\Delta \) to the viscosity solution \( u \) of the variational inequality (1.1)–(1.2), which is the main result of this paper. To this end, we shall derive a (uniform) bound for the approximation error \( u - u^\Delta \) in \( \bar{Q}_T \).

We start with the approximation error in the last time interval \( \bar{Q}_T \setminus \bar{Q}_T - \Delta \), where the value of \( u^\Delta \) involves only a one-time approximation and some linear interpolation. Therefore, the bound for the approximation error in this domain can be easily obtained by some properties of the backward operator \( S_t(\Delta) \) and some regularity results of \( u \). This is demonstrated in the following lemma.

Lemma 3.1 Suppose that Assumption [24] is satisfied. Let \( \Delta \in (0, T) \), \( u^\Delta \in C_b(\bar{Q}_T) \) be the unique solution of the approximation scheme (2.4) and \( u \in C_b^1(\bar{Q}_T) \) be the unique viscosity solution of equation (1.1)–(1.2). Then,
\[
\sup_{\bar{Q}_T \setminus \bar{Q}_T - \Delta} |u - u^\Delta| \leq C \sqrt{\Delta}.
\] (3.1)

Proof. From the property (v) of the operator \( S_t(\Delta) \) (cf. Proposition 2.5 in [19]), we have \( |U - S_{T - \Delta}(\Delta)|U|0 \leq C \sqrt{\Delta} \). On the other hand, by Assumption [24] (i), for any \( x \in \mathbb{R}^n \), \( f(T - \Delta, x) \geq \left[ f, 0 \right]_{1/2} \sqrt{\Delta} \geq U(x) \geq M \sqrt{\Delta} \). The above two inequalities then imply that for any \( x \in \mathbb{R}^n \),
\[
|U(x) - \min \{ S_{T - \Delta}(\Delta)U(x), f(T - \Delta, x) \}| \leq C \sqrt{\Delta}.
\] (3.2)
Then from (2.4), we have, for \((t, x) \in \bar{Q}_T \setminus \bar{Q}_{T-\Delta}\),

\[
|u(t, x) - u^\Delta(t, x)| = |u(t, x) - g^\Delta(t, x)|
\]

\[
= |u(t, x) - u(T, x) + \omega_2(t)(U(x) - \min\{S_{T-\Delta}(\Delta)U(x), f(T - \Delta, x)\})|
\]

\[
\leq |u(t, x) - u(T, x)| + |U(x) - \min\{S_{T-\Delta}(\Delta)U(x), f(T - \Delta, x)\}|
\]

\[
\leq C(\sqrt{T-t} + \sqrt{\Delta}) \leq C\sqrt{\Delta},
\]

where the second to last inequality follows from the regularity property of the solution \(u\) (cf. Proposition 2.2) and (3.2).

Next, we derive a bound of approximation error within the whole domain \(\bar{Q}_T\). We first consider a special case when (1.1)-(1.2) admits a unique smooth solution \(u\) with bounded derivatives of any order.

**Theorem 3.2** Suppose that Assumption 2.1 is satisfied. Let \(\Delta \in (0, T)\) and \(u^\Delta \in C_b(\bar{Q}_T)\) be the unique solution of the approximation scheme (2.4). Suppose that equation (1.1)-(1.2) admits a unique smooth solution \(u \in C^\infty_b(\bar{Q}_T)\). Then,

\[
|u - u^\Delta| \leq C\sqrt{\Delta} \text{ in } \bar{Q}_T.
\]

**Proof.** Using that \(u \in C^\infty_b(\bar{Q}_T)\), the consistency error estimate (2.8) yields

\[
|\bar{S}(\Delta, t, x, u(t, x), u(t + \Delta, \cdot))|
\]

\[
\leq C\Delta (|\partial_t u|_0 + |\partial_{xxxx} u|_0 + |\partial_{xxx} u|_0 + R(u)) \leq C\Delta,
\]

for \((t, x) \in \bar{Q}_{T-\Delta}\). On the other hand, from the definition of the approximation scheme (2.4), we have

\[
\bar{S}(\Delta, t, x, u^\Delta(t, x), u^\Delta(t + \Delta, \cdot)) = 0,
\]

for \((t, x) \in \bar{Q}_{T-\Delta}\). In turn, the comparison principle in Proposition 2.5 implies

\[
|u - u^\Delta| \leq \sup_{\bar{Q}_T \setminus \bar{Q}_{T-\Delta}} |u - u^\Delta| + (T-t+1)C\Delta \text{ in } \bar{Q}_T.
\]

The conclusion then follows by using the estimate (3.1) in Lemma 3.1. 

In general, since (1.1)-(1.2) only admits a viscosity solution \(u \in C^1_b(\bar{Q}_T)\) (cf. Proposition 2.2) due to the possible degeneracies of the equation, the above result does not hold. A natural idea is then to approximate the viscosity solution \(u\) by a sequence of smooth sub- and supersolutions \(u^\varepsilon\) and, in turn, compare them with \(u^\Delta\) using the comparison result for the approximation scheme (cf. Proposition 2.5) to obtain the upper and lower bound for the approximation error separately. We carry out this procedure next.

### 3.1 Upper bound for the approximation error

We now derive an upper bound for the approximation error within the whole domain \(\bar{Q}_T\) for the general \(u \in C^1_b(\bar{Q}_T)\) case. We first construct a sequence of smooth subsolutions to (1.1) by perturbing its coefficients. This approach, known as the *shaking coefficients technique*, was initially
proposed by Krylov [23, 24], and further developed by Barles and Jakobsen [3, 21]. We apply this approach to obtain an upper bound for the approximation error \( u - u^\Delta \).

To this end, for small enough \( \varepsilon \geq 0 \), we extend the functions \( f \) and \( \eta := \sigma, b \) to \( Q_{T + \varepsilon^2}^+ := [-\varepsilon^2, T + \varepsilon^2) \times \mathbb{R}^n \) and \( H \) to \( Q_{T + \varepsilon^2}^+ \times \mathbb{R}^n \), so that Assumption 2.1 still holds. We then define \( \eta^\varepsilon(t, x) := \eta(t + \tau, x + e) \) and \( H^\varepsilon(t, x, p) := H(t + \tau, x + e, p) \), where \( \theta = (\tau, e) \) with \( \theta \in \Theta^\varepsilon := [-\varepsilon^2, 0] \times \varepsilon B(0, 1) \). We then consider the perturbed version of (1.1)-(1.2), namely,

\[
\max \{-\partial_t u^\varepsilon + \sup_{\theta \in \Theta^\varepsilon} g^\theta(t, x, \partial_x u^\varepsilon, \partial_{xx} u^\varepsilon), u^\varepsilon - f(t, x)\} = 0 \quad \text{in } Q_{T + \varepsilon^2};
\]

\[
u^\varepsilon(T + \varepsilon^2, x) = U(x) \quad \text{in } \mathbb{R}^n, \tag{3.4}
\]

where

\[
g^\theta(t, x, p, X) = -\frac{1}{2} \text{Trace} \left( \sigma^\theta \sigma^\theta^T(t, x)X \right) - b^\theta(t, x) \cdot p + H^\theta(t, x, p).
\]

Note that by letting the perturbation parameter \( \varepsilon = 0 \), we can retrieve our original variation inequality (1.1)-(1.2).

We establish existence, uniqueness and regularity results for the perturbed equation (3.3)-(3.4), and a comparison between \( u \) and \( u^\varepsilon \) in the next proposition, whose proof is provided in Appendix A.

**Proposition 3.3** Suppose that Assumption 2.1 is satisfied. Then, for small enough \( \varepsilon \geq 0 \), there exists a unique viscosity solution \( u^\varepsilon \in C^{1}_b(Q_{T + \varepsilon^2}) \) of (3.3)-(3.4), with \( |u^\varepsilon|_1 \leq C \). Moreover,

\[
|u - u^\varepsilon| \leq C \varepsilon \quad \text{in } \bar{Q}_T. \tag{3.5}
\]

Next, we regularize \( u^\varepsilon \) by a standard mollification procedure. For this, let \( \rho(t, x) \) be an \( \mathbb{R}_+ \)-valued smooth function with support in \((-1, 0) \times B(0, 1)\) and mass 1, and introduce the sequence of mollifiers \( \rho_\varepsilon \) for \( \varepsilon > 0 \),

\[
\rho_\varepsilon(t, x) := \frac{1}{\varepsilon^{n+2}} \rho \left( \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon} \right). \tag{3.6}
\]

For \( (t, x) \in \bar{Q}_T \), we then define

\[
u_\varepsilon(t, x) = u^\varepsilon \ast \rho_\varepsilon(t, x) = \int_{-\varepsilon^2 < \tau < 0} \int_{|x| < \varepsilon} u^\varepsilon(t - \tau, x - e) \rho_\varepsilon(\tau, e) \, de \, d\tau.
\]

Standard properties of mollifiers imply that \( u_\varepsilon \in C^\infty_c(Q_T) \),

\[
|u^\varepsilon - u_\varepsilon|_1 \leq C \varepsilon \tag{3.7}
\]

and, moreover, for positive integer \( i \) and multiindex \( j \),

\[
|\partial^j_i D^j u_\varepsilon|_1 \leq C \varepsilon^{1-2|j|}. \tag{3.8}
\]

We observe from (3.3) that the function \( v_\theta^\varepsilon(t, x) := u^\varepsilon(t - \tau, x - e) \) is a viscosity subsolution of

\[-\partial_t w(t, x) + g(t, x, \partial_x w(t, x), \partial_{xx} w(t, x)) = 0, \tag{3.9}
\]

in \( Q_T \) for any \( \theta \in \Theta^\varepsilon \). On the other hand, a Riemann sum approximation shows that \( u_\varepsilon(t, x) \) can be viewed as the limit of convex combinations of \( v_\theta^\varepsilon(t, x) \) for \( \theta \in \Theta^\varepsilon \). Since the nonlinear term
\( g(t, x, p, X) \) is convex in \( p \) and linear in \( X \), the convex combinations of \( v_0^\varepsilon(t, x) \) are also subsolutions of (3.9) in \( Q_T \). Using the stability of viscosity solutions, we deduce that \( u_\varepsilon(t, x) \) is still a subsolution of (3.9) in \( Q_T \), namely,

\[
- \partial_t u_\varepsilon + g(t, x, \partial_x u_\varepsilon, \partial_{xx} u_\varepsilon) \leq 0.
\]

(3.10)

We are now in a position to establish an upper bound for the approximation error.

**Theorem 3.4** Suppose that Assumption 2.1 is satisfied. Let \( \Delta \in (0, T) \), \( u^\Delta \in C^0_b(\bar{Q}_T) \) be the unique solution of the approximation scheme (2.4) and \( u \in C^1_b(\bar{Q}_T) \) be the unique viscosity solution of equation (1.1)-(1.2). Then,

\[
u - u^\Delta \leq C \Delta^{\frac{1}{4}} \text{ in } \bar{Q}_T.
\]

**Proof.** From (3.3), \( u^\varepsilon - f \leq 0 \) in \( Q_{T+\varepsilon^2} \). This yields that for \((t, x) \in \bar{Q}_{T-\Delta}\),

\[
u(t, x) - f(t, x) \leq \int_{-\varepsilon^2 < \tau < 0} \int_{|e| < \varepsilon} (f(t-\tau, x-e) - f(t, x)) \rho_e(\tau, e) \, de \, d\tau \leq [f]_1 \varepsilon \leq M \varepsilon.
\]

This together with (3.10) gives that in \( \bar{Q}_{T-\Delta}\),

\[
\max\{-\partial_t \nu_e + g(t, x, \partial_x \nu_e, \partial_{xx} \nu_e), \nu_e - f(t, x)\} \leq M \varepsilon.
\]

We then substitute \( \nu_e \) into the consistency error estimate (2.8) and use the estimate (3.8) to obtain

\[
S(\Delta, t, x, \nu_e(t, x), \nu_e(t + \Delta, \cdot)) \leq M \varepsilon + C \Delta \left( |\partial_t \nu_e|_0 + |\partial_x \nu_e|_0 + |\partial_{xx} \nu_e|_0 + |\partial_{xxt} \nu_e|_0 + |\partial_{xxtx} \nu_e|_0 + R(\nu_e) \right) \leq C(\varepsilon + \Delta \varepsilon^{-3}),
\]

for \((t, x) \in \bar{Q}_{T-\Delta}\). The comparison principle in Proposition 2.5 then implies

\[
\nu_e - u^\Delta \leq \sup_{\bar{Q}_T \setminus \bar{Q}_{T-\Delta}} (\nu_e - u^\Delta)^+ + C(T - t)(\varepsilon + \Delta \varepsilon^{-3}) \text{ in } \bar{Q}_T.
\]

Next, using the estimates (3.3) and (3.7), we obtain \( |u - u_e| \leq C \varepsilon \) and thus,

\[
u - u^\Delta = (u - u_e) + (u_e - u^\Delta) \leq C \varepsilon + \sup_{\bar{Q}_T \setminus \bar{Q}_{T-\Delta}} (u_e - u^\Delta)^+ + C(\varepsilon + \Delta \varepsilon^{-3}) \leq \sup_{\bar{Q}_T \setminus \bar{Q}_{T-\Delta}} (u - u^\Delta)^+ + C(\varepsilon + \Delta \varepsilon^{-3}) \text{ in } \bar{Q}_T.
\]

By choosing \( \varepsilon = \Delta^{\frac{1}{4}} \), we further obtain

\[
u - u^\Delta \leq \sup_{\bar{Q}_T \setminus \bar{Q}_{T-\Delta}} (u - u^\Delta)^+ + C \Delta^{\frac{1}{4}} \leq C \Delta^{\frac{1}{4}} \text{ in } \bar{Q}_T,
\]

where the last inequality follows from the estimate (3.1) in Lemma 3.1. 

\[9\]
3.2 Lower bound for the approximation error

To obtain a lower bound for the approximation error, we cannot follow the above perturbation procedure to construct approximate smooth supersolutions of (1.1). This is because if we perturb its coefficients in an opposite way to obtain a viscosity supersolution, its convolution with the mollifier may no longer be a supersolution due to the convexity of the function \( g \) in (1.1). One way to solve the convexity problem is to interchange the roles of equation (1.1)-(1.2) and its approximation scheme (2.4) (as in [17] and [21]), and perturb the scheme instead of the equation. This approach, however, does not work either, as \( u^\Delta \) does not in general have enough regularity (Assumption 2.5 in [21] fails).

To overcome these difficulties, in this paper we follow the idea of Barles and Jakobsen [2] to build approximate supersolutions which are smooth at the “right points” by introducing an appropriate optimal switching and stopping (with possible stochastic control) systems. Compared to the case in [2] where they use standard optimal switching systems to approximate an HJB equation, the systems herein are more difficult. This is because we need to add an extra region (equation) to the standard system in order to approximate a HJB equation with obstacle, i.e. a variational inequality. Thus, here we call the new system a variant switching system and give more details in Section 4.

To apply the above method to the problem herein, we first observe that, using the convex dual function \( L \) introduced in (2.1), we can write the equation (1.1) as

\[
\max \{-\partial_t u + \sup_{q \in \mathbb{R}^n} L_q(t, x, \partial_x u, \partial_{xx} u), u - f(t, x)\} = 0, \tag{3.11}
\]

with

\[
L_q(t, x, p, X) := -\frac{1}{2} \text{Trace} (\sigma \sigma^T (t, x) X) - (b(t, x) - q) \cdot p - L(t, x, q).
\]

It then follows from Proposition 2.3 (iv) in [19] and Proposition 2.2 that the supremum in (3.11) can be achieved at some point \( q^* \) with \( |q^*| \leq C \). Thus, we rewrite the equation (3.11) as

\[
\max \{-\partial_t u + \sup_{q \in K} L_q(t, x, \partial_x u, \partial_{xx} u), u - f(t, x)\} = 0, \tag{3.12}
\]

where \( K \subset \mathbb{R}^n \) is a compact set. Since \( K \) is separable, it has a countable dense subset, say \( K_\infty = \{q_1, q_2, q_3, \ldots\} \) and, in turn, the continuity of \( L_q \) in \( q \) implies that

\[
\sup_{q \in K} L_q(t, x, p, X) = \sup_{q \in K_\infty} L_q(t, x, p, X).
\]

Therefore, (3.11) further reduces to

\[
\max \{-\partial_t u + \sup_{q \in K_\infty} L_q(t, x, \partial_x u, \partial_{xx} u), u - f(t, x)\} = 0. \tag{3.13}
\]

Next, for \( m \in \mathbb{Z}^+ \), we consider the following equations to approximate the original equation \( 1.1/3.13/1.2 \),

\[
\max \{-\partial_t u^m + \sup_{q \in K_m} L_q(t, x, \partial_x u^m, \partial_{xx} u^m), u^m - f(t, x)\} = 0 \quad \text{in } Q_T; \tag{3.14}
\]

\[
u^m(T, x) = U(x) \quad \text{in } \mathbb{R}^n, \tag{3.15}
\]

where \( K_m := \{q_1, \ldots, q_m\} \subset K_\infty \) consisting the first \( m \) points in \( K_\infty \) and satisfying \( \bigcup_{m \geq 1} K_m = K_\infty \).

It then follows from standard optimal stopping control problem (see [6, 15]) that (3.14)-(3.15) admits
a unique viscosity solution \( u^m \in C^1(\bar{Q}_T) \), with \( |u^m| \leq C \) independent of \( m \). Then, Arzela-Ascoli’s theorem yields that there exists a subsequence of \( \{u^m\} \), denoted as \( \{u^{m_n}\} \), such that, as \( n \to \infty \),

\[
u^{m_n} \to u \text{ locally uniformly in } \bar{Q}_T,
\]

(3.16)

and moreover, since the terminal condition for \( u \) and that for \( u^{m_n} \) (1.2 and 3.15) respectively) coincide, it follows from their regularity result that, for \( n \in \mathbb{Z}^+ \)

\[
|u(t, \cdot) - u^{m_n}(t, \cdot)|_0 \leq |u(t, \cdot) - u(T, \cdot)|_0 + |u^{m_n}(t, \cdot) - u^{m_n}(T, \cdot)|_0 \leq C\sqrt{T - t}.
\]

(3.17)

For \( m \in \mathbb{Z}^+ \), we then construct a sequence of (local) smooth supersolutions to approximate the solution \( u^m \) of (3.14)-(3.19). To this end, we consider the following existence, uniqueness and regularity result of the viscosity solution (3.21)-(3.22) admits a unique viscosity solution appropriately. It then follows from Theorem 4.5 and Theorem 4.4 that (3.21)-(3.22) admits a unique viscosity solution \( v \).

Theorem 4.5 in the next section. To get the estimates (3.20), we first check that

\[
\begin{align*}
\max \left\{ -\partial_i v_i + L^v(t, x, \partial_x v_i, \partial_{xx} v_i), v_i - M_i^k v \right\} &= 0, & i \in \mathcal{I} := \{1, \ldots, m\}, \quad \text{in } Q_T, \\
\max \{v_{m+1} - f, v_{m+1} - M_{m+1}^k v\} &= 0, \\
v_i(T, x) &= U(x), & i \in \bar{\mathcal{I}} := \mathcal{I} \cup \{m + 1\}
\end{align*}
\]

(3.18)

with

\[
v_i(T, x) = U(x), & i \in \bar{\mathcal{I}}
\]

(3.19)

where \( M_i^k v := \min_{j \neq i, j \in \bar{\mathcal{I}}} \{v_j + k\} \), for some constant \( k > 0 \) representing the switching cost.

By the results from the next section, we then have the following existence, uniqueness and regularity results for the solution \( v \) of variant switching system (3.18)-(3.19), and a comparison between \( v \) and \( u^m \).

Proposition 3.5 Suppose that Assumption \( (2.4) \) is satisfied. Then, for any \( m \in \mathbb{Z}^+ \), there exists a unique viscosity solution \( v = (v_1, \ldots, v_{m+1}) \) of (3.18)-(3.19), with \( |v_i| \leq C \). Moreover, for \( k \) small enough,

\[
0 \leq v_i - u^m \leq Ck^{\frac{1}{\gamma}} \text{ in } \bar{Q}_T, \quad i \in \bar{\mathcal{I}}.
\]

(3.20)

Proof. The existence, uniqueness and regularity result of the viscosity solution \( v \) is given by Theorem 4.5 in the next section. To get the estimates (3.20), we first check that \( w = (u^m, \ldots, u^m) \) is a subsolution of (3.18), then comparison result for the variant switching system (which is implied by Theorem 4.5) yields \( u^m \leq v_i \) for \( i \in \bar{\mathcal{I}} \).

To derive the other bound, we follow the same regularization procedure as that in Section 3.1.

For small enough \( \varepsilon > 0 \), we consider the following perturbed system of (3.18):

\[
\begin{align*}
\max \left\{ -\partial_i v_i^\varepsilon + \sup_{(\tau, e) \in \Theta^\varepsilon} L^v(t + \tau, x + e, \partial_x v_i^\varepsilon, \partial_{xx} v_i^\varepsilon), v_i^\varepsilon - M_i^k v^\varepsilon \right\} &= 0, & i \in \mathcal{I}, \quad \text{in } Q_{T + \varepsilon}, \\
\max \{v_{m+1}^\varepsilon - f, v_{m+1}^\varepsilon - M_{m+1}^k v^\varepsilon\} &= 0, \\
v_i^\varepsilon(T + \varepsilon^2, x) &= U(x), & i \in \bar{\mathcal{I}}
\end{align*}
\]

(3.21)

with

\[
v_i^\varepsilon(T + \varepsilon^2, x) = U(x), & i \in \bar{\mathcal{I}}
\]

(3.22)

where \( \Theta^\varepsilon = [-\varepsilon^2, 0] \times \varepsilon B(0, 1) \). Note that here we also extend the coefficients \( \sigma, b, f \) and \( L \) appropriately. It then follows from Theorem 4.5 and Theorem 4.4 that (3.21)-(3.22) admits a unique viscosity solution \( v^\varepsilon = (v_1^\varepsilon, \ldots, v_{m+1}^\varepsilon) \) such that for \( i \in \bar{\mathcal{I}} \),

\[
|v_i^\varepsilon|_1 \leq C \text{ and } |v_i^\varepsilon - v_i| \leq C\varepsilon \text{ in } \bar{Q}_T.
\]

(3.23)
It then follows from (3.21) that for any \((\tau, e) \in \Theta^\varepsilon\),

\[-\partial_t v^\varepsilon_i + \mathcal{L}^\varepsilon(t + \tau, x + e, \partial_x v^\varepsilon_i, \partial_{xx} v^\varepsilon_i) \leq 0 \quad \text{in} \quad Q_{T+\varepsilon^2}, \quad i \in I.\]

For each \(i \in I\), define \(v_{i,e} := v^\varepsilon_i * \rho_\varepsilon\), where \(\rho_\varepsilon\) is defined in (3.6). By the same argument as in Section 3.1, we have

\[-\partial_t v_{i,e} + \mathcal{L}^\varepsilon(t, x, \partial_x v_{i,e}, \partial_{xx} v_{i,e}) \leq 0 \quad \text{in} \quad Q_T, \quad i \in I. \tag{3.24}\]

On the other hand, it follows again from (3.21) that for any \((\tau, e) \in \bar{I}\), \(v^\varepsilon_i \leq M^\varepsilon_i v^\varepsilon = \min_{j \neq i, j \in \bar{I}} v^\varepsilon_j + k\) in \(Q_{T+\varepsilon^2}\), which implies that

\[|v^\varepsilon_i - v^\varepsilon_j|_0 \leq k, \quad i, j \in \bar{I}.\]

Then by standard properties of mollifiers, we have

\[|\partial_t v_{i,e} - \partial_t v_{j,e}|_0 \leq Ck\varepsilon^{-2}; \quad |D^*_1 v_{i,e} - D^*_2 v_{j,e}|_0 \leq Ck\varepsilon^{-n}, \quad n \in \mathbb{N}, \quad i, j \in \bar{I},\]

These estimates then yield that

\[|\partial_t v_{i,e} + \mathcal{L}^\varepsilon(t, x, \partial_x v_{i,e}, \partial_{xx} v_{i,e})| \leq Ck(\varepsilon^{-2} + \varepsilon^{-1}) \quad \text{in} \quad Q_T, \quad i, j \in \bar{I}.\]

This together with (3.24) gives that

\[-\partial_t v_{i,e} + \mathcal{L}^\varepsilon(t, x, \partial_x v_{i,e}, \partial_{xx} v_{i,e}) \leq Ck(\varepsilon^{-2} + \varepsilon^{-1}) \quad \text{in} \quad Q_T, \quad i, j \in \bar{I},\]

which means

\[-\partial_t v_{i,e} + \sup_{q \in K_m} \mathcal{L}^\varepsilon(t, x, \partial_x v_{i,e}, \partial_{xx} v_{i,e}) \leq Ck(\varepsilon^{-2} + \varepsilon^{-1}) \quad \text{in} \quad Q_T, \quad i \in \bar{I}.\]

Next, since for any \(i \in \bar{I}\), \(v^\varepsilon_i \leq M^\varepsilon_i v^\varepsilon \leq v_{m+1}^\varepsilon + k \leq f + k\) in \(Q_{T+\varepsilon^2}\), we obtain that for \((t, x) \in Q_T\),

\[v_{i,e}(t, x) - f(t, x) \leq k + \int_{-\varepsilon^2 < \tau < 0} \int_{|e| < \varepsilon} (f(t - \tau, x - e) - f(t, x)) \rho_\varepsilon(\tau, e) d\tau \leq k + M\varepsilon.\]

From the above two inequalities we can see that for any \(i \in \bar{I}\), \(v_{i,e}(T - t)Ck(\varepsilon^{-2} + \varepsilon^{-1}) - k - M\varepsilon\) is a subsolution of (3.14) in \(Q_T\) with terminal value \(v_{i,e}(T, \cdot) - k - M\varepsilon\). Then by standard continuous dependence result for (3.1.) we have for \(i \in \bar{I}\),

\[v_{i,e} - u^m \leq |(v_{i,e}(T, \cdot) - k - M\varepsilon - u^m(T, \cdot))^+|_0 + (T - t)Ck(\varepsilon^{-2} + \varepsilon^{-1}) + k + M\varepsilon\]

\[ \leq |v_{i,e}(T, \cdot) - v_i^\varepsilon(T + \varepsilon^2, \cdot)|_0 + C(\varepsilon + k\varepsilon^{-2} + k\varepsilon^{-1} + k)\]

\[ \leq |v_{i,e}(T, \cdot) - v_i^\varepsilon(T + \varepsilon^2, \cdot)|_0 + C(\varepsilon + k\varepsilon^{-2}) \quad \text{in} \quad Q_T.\]

Hence by the properties of mollifiers and regularity of \(v^\varepsilon\), we have for \(i \in \bar{I}\),

\[v^\varepsilon_i - u^m = v^\varepsilon_i - v_{i,e} + v_{i,e} - u^m\]

\[ \leq C\varepsilon + |v_{i,e}(T, \cdot) - v_i^\varepsilon(T, \cdot)|_0 + |v_i^\varepsilon(T, \cdot) - v_i^\varepsilon(T + \varepsilon^2, \cdot)|_0 + C(\varepsilon + k\varepsilon^{-2}) \quad \text{in} \quad Q_T.\]
Furthermore, since \( v^m_{m+1} \leq \min_{i \in T} v^\tau_i + k \), the above inequality holds for all \( i \in \tilde{T} \). It finally follows from (3.23) that for \( i \in \tilde{T} \),

\[
v_i - u^m = v_i - v^\tau_i + v^\tau_i - u^m \\
\leq C\varepsilon + C(\varepsilon + k\varepsilon^{-2}) \\
\leq C(\varepsilon + k\varepsilon^{-2}) \quad \text{in} \ Q_T.
\]

We then choose \( \varepsilon = k^{\frac{1}{2}} \) and finish the proof. ■

Next, still following the approach of [2], we construct smooth approximations of \( v_i \). Since when \( i \in I \), in the continuation region of (3.18), the solution \( v_i \) satisfies the linear equation, namely,

\[
\begin{align*}
-\partial_t v_i + \mathcal{L}^\varepsilon (t, x, \partial_x v_i, \partial_{xx} v_i) &= 0 \quad \text{in} \ \{(t, x) \in Q_T : v_i(t, x) < M^k_i v(t, x)\}, \quad i \in I,
\end{align*}
\]

we may perturb its coefficients to obtain a sequence of smooth supersolutions. This will in turn give a lower bound of the error \( v_i - u^\Delta \), and thus a lower bound of \( u^m - u^\Delta \) by the estimate (3.20). A subtle point herein is how to identify the continuation region by appropriately choosing the switching cost \( k \). For this, we follow the idea used in Lemma 3.4 of [2].

**Proposition 3.6** Suppose that Assumption 2.7 is satisfied. Let \( \Delta \in (0, T) \), \( m \in \mathbb{Z}^+ \), \( u^\Delta \in C_0(\bar{Q}_T) \) be the unique solution of the approximation scheme (2.4) and \( u^m \in C_0(\bar{Q}_T) \) be the unique viscosity solution of equation (3.14)-(3.15). Then,

\[
u^\Delta - u^m \leq \sup_{\bar{Q}_T \setminus \partial_{Q_T} \Delta} (u^\Delta - u^m)^+ + C\Delta \frac{M^k_i}{T} \quad \text{in} \ \bar{Q}_T.
\]

**Proof.** In analogy to (3.21) but in the opposite direction, for small enough \( \varepsilon > 0 \), we perturb the coefficients of the system (3.18)-(3.19), and consider the following variant switching system:

\[
\begin{align*}
\max \left\{ -\partial_t v^\varepsilon_i + \inf_{(\tau, \varepsilon) \in \Theta} \mathcal{L}^{\varepsilon} (t + \tau, x + \varepsilon, \partial_x v^\varepsilon_i, \partial_{xx} v^\varepsilon_i), v^\varepsilon_i - M^k_i v^\varepsilon \right\} &= 0, \quad i \in \tilde{I} , \quad \text{in} \ Q_{T+\varepsilon^{-2}}, (3.25) \\
\max \left\{ v^\varepsilon_{m+1} - f, v^\varepsilon_{m+1} - M^k_{m+1} v^\varepsilon \right\} &= 0,
\end{align*}
\]

with

\[
v^\varepsilon_i (T + \varepsilon^2, x) = U(x), \quad i \in \tilde{I} , \quad (3.26)
\]

It then follows again from Theorem 4.5 and Theorem 4.4 that (3.25) in (3.26) admits a unique viscosity solution \( v^\varepsilon = (v^\varepsilon_1, \ldots, v^\varepsilon_{m+1}) \), with \( |v^\varepsilon_i| \leq C \) and, moreover, for each \( i \in \tilde{I} \),

\[
|v^\varepsilon_i - v_i| \leq C\varepsilon \quad \text{in} \ Q_T.
\]

In turn, this and inequality (3.20) imply that, for each \( i \in \tilde{T} \),

\[
|v^\varepsilon_i - u^m| \leq |v^\varepsilon_i - v_i| + |v_i - u^m| \leq C(\varepsilon + k^{\frac{1}{2}}) \quad \text{in} \ Q_T.
\]

Next, we regularize \( v^\varepsilon \) and define \( v_{i, \varepsilon} := v^\varepsilon_i + \rho \varepsilon \) in \( \bar{Q}_T \) for \( i \in \tilde{T} \), where \( \rho \varepsilon \) is the mollifier defined in (3.6). Then, \( v_{i, \varepsilon} \in C_0^\infty(\bar{Q}_T) \),

\[
|v_{i, \varepsilon} - v_i|_0 \leq C\varepsilon, \quad (3.28)
\]
and moreover, for positive integer $m$ and multiindex $n$,

$$|\partial^m_{\bar{\tau}} D^*_{\bar{\tau}} \nu_{\bar{\tau}}|_0 \leq C\varepsilon^{1-2m-|n|}.$$  \hfill (3.29)

Define $w_{\varepsilon} := \min_{i \in \mathbb{I}} \nu_{i,\varepsilon}$ in $Q_T$. Then the function $w_{\varepsilon} \in C_b(Q_T)$ and is smooth in $Q_T$ except for finitely many points. Then, \hfill (3.27) and \hfill (3.28) yield

$$|u^m - w_{\varepsilon}| \leq C(\varepsilon + k^{\frac{1}{2}}) \text{ in } Q_T. \hfill (3.30)$$

Now we fix any $(t,x) \in \bar{Q}_{T-\Delta}$ and let $j = \text{arg} \min_{i \in \mathbb{I}} \nu_{i,\varepsilon}(t,x)$. Then we obtain that

$$v_{j,\varepsilon}(t,x) - M_j^{\varepsilon} v_{\varepsilon}(t,x) = \max_{i \neq j, i \in \mathbb{I}} \{v_{j,\varepsilon}(t,x) - v_{i,\varepsilon}(t,x) - k\} \leq -k.$$

In turn, inequality \hfill (3.28) implies that

$$v_j^\varepsilon(t,x) - M_j^{\varepsilon} v^\varepsilon(t,x) \leq v_{j,\varepsilon}(t,x) - M_j^{\varepsilon} v_{\varepsilon}(t,x) + C \varepsilon \leq -k + C \varepsilon.$$

Furthermore, since $|v_j^\varepsilon|_1 \leq C$, we also have for any $(\tau,e) \in \Theta^\varepsilon$,

$$v_j^\varepsilon(t-\tau,x-e) - M_j^{\varepsilon} v^\varepsilon(t-\tau,x-e) \leq v_j^\varepsilon(t,x) - M_j^{\varepsilon} v^\varepsilon(t,x) + C(|\tau|^{\frac{1}{2}} + |e|) \leq -k + C \varepsilon + 2C \varepsilon.$$

By choosing $k = 4C \varepsilon$, we then obtain that, for any $(\tau,e) \in \Theta^\varepsilon$,

$$v_j^\varepsilon(t-\tau,x-e) - M_j^{\varepsilon} v^\varepsilon(t-\tau,x-e) < 0.$$

Therefore, the points $(t-\tau,x-e)$, for $(\tau,e) \in \Theta^\varepsilon$, are in the continuation region of \hfill (3.26). Now we consider two cases when $j \in \mathbb{I}$ and when $j = m+1$ separately.

(i) If $j \in \mathbb{I}$, then we have that, for $(\tau,e) \in \Theta^\varepsilon$,

$$-\partial_t v_j^\varepsilon(t-\tau,x-e) + \inf_{(\tau',e')} L^{\varepsilon}(t-\tau + \tau', x - e + e', \partial \tau v_j^\varepsilon(t-\tau,x-e), \partial_{xx} v_j^\varepsilon(t-\tau,x-e)) = 0,$$

and therefore,

$$-\partial_t v_j^\varepsilon(t-\tau,x-e) + L^{\varepsilon}(t,x, \partial_t v_j^\varepsilon(t-\tau,x-e), \partial_{xx} v_j^\varepsilon(t-\tau,x-e)) \geq 0.$$  \hfill (3.31)

Using the definition of $v_{j,\varepsilon}$ and that $L^{\varepsilon}$ is linear in $\partial_t v_j^\varepsilon$ and $\partial_{xx} v_j^\varepsilon$, we further have

$$-\partial_t v_{j,\varepsilon}(t,x) + L^{\varepsilon}(t,x, \partial_t v_{j,\varepsilon}(t,x), \partial_{xx} v_{j,\varepsilon}(t,x))$$

$$= \int_{-\varepsilon^2 < \tau < 0} \int_{|e| < \varepsilon} \left( -\partial_t v_j^\varepsilon(t-\tau,x-e) + L^{\varepsilon}(t,x, \partial_t v_j^\varepsilon(t-\tau,x-e), \partial_{xx} v_j^\varepsilon(t-\tau,x-e)) \right) \rho_{\varepsilon}(\tau,e) d\tau \geq 0.$$  \hfill (3.31)

Since $w_{\varepsilon}(t,x) = v_{j,\varepsilon}(t,x)$ and $w_{\varepsilon}(\tau + \Delta, \cdot) \leq v_{j,\varepsilon}(t+\Delta, \cdot)$, we apply Proposition \hfill (2.4) and use the estimate \hfill (3.29) and \hfill (3.31) to obtain that

$$\bar{S}(\Delta, t, w_{\varepsilon}(t,x), w_{\varepsilon}(t+\Delta, \cdot))$$

$$\geq \bar{S}(\Delta, t, v_{j,\varepsilon}(t,x), v_{j,\varepsilon}(t+\Delta, \cdot))$$

$$\geq - \partial_t v_{j,\varepsilon}(t,x) + g(t,x, \partial_t v_{j,\varepsilon}(t,x), \partial_{xx} v_{j,\varepsilon}(t,x)) - C \Delta \varepsilon^{-3}$$

$$= - \partial_t v_{j,\varepsilon}(t,x) + \sup_{q \in \mathbb{R}^n} L_q(t,x, \partial_t v_{j,\varepsilon}(t,x), \partial_{xx} v_{j,\varepsilon}(t,x)) - C \Delta \varepsilon^{-3}$$

$$\geq - \partial_t v_{j,\varepsilon}(t,x) + L_q(t,x, \partial_t v_{j,\varepsilon}(t,x), \partial_{xx} v_{j,\varepsilon}(t,x)) - C \Delta \varepsilon^{-3}$$

$$\geq - C \Delta \varepsilon^{-3}.$$  \hfill (3.31)
(ii) If \( j = m + 1 \), then we have that, for \((\tau, \varepsilon) \in \Theta^\varepsilon\),
\[
v_j^\varepsilon(t - \tau, x - e) - f(t - \tau, x - e) = 0.
\]
Since \( w_\varepsilon(t, x) = v_{j, \varepsilon}(t, x) \), we then obtain that
\[
S(\Delta, t, x, w_\varepsilon(t, x), w_\varepsilon(t + \Delta, \cdot)) \geq w_\varepsilon(t, x) - f(t, x)
\]
\[
= \int_{\tau^2 < \tau < 0} \int_{|e| < \varepsilon} (f(t - \tau, x - e) - f(t, x)) \rho_\varepsilon(\tau, e) \, de \, d\tau
\]
\[
\geq - M\varepsilon.
\]
Thus, in any case, we always have
\[
S(\Delta, t, x, w_\varepsilon(t, x), w_\varepsilon(t + \Delta, \cdot)) \geq - C(\Delta \varepsilon^{-3} + \varepsilon).
\]
Note that the right hand side of the above inequality does not depend on \((t, x)\), thus this inequality holds in \( \bar{Q}_{T-\Delta} \). In turn, the comparison principle in Proposition \ref{prop:comparison} implies that
\[
u^\Delta - w_\varepsilon \leq \sup_{\bar{Q}_T \setminus \bar{Q}_{T-\Delta}} (u^\Delta - w_\varepsilon)^+ + C'(T - t + 1)(\Delta \varepsilon^{-3} + \varepsilon) \text{ in } \bar{Q}_T.
\]
Combining the above inequality with \eqref{eq:3.34}, we further get
\[
u^\Delta - u^m = (u^\Delta - w_\varepsilon) + (w_\varepsilon - u^m)
\]
\[
\leq \sup_{\bar{Q}_T \setminus \bar{Q}_{T-\Delta}} (u^\Delta - w_\varepsilon)^+ + C(T - t + 1)(\Delta \varepsilon^{-3} + \varepsilon) + C(\varepsilon + k^{\frac{1}{2}})
\]
\[
\leq \sup_{\bar{Q}_T \setminus \bar{Q}_{T-\Delta}} (u^\Delta - u^m)^+ + C(\varepsilon^{\frac{1}{4}} + \Delta \varepsilon^{-3})
\]
\[
\leq \sup_{\bar{Q}_T \setminus \bar{Q}_{T-\Delta}} (u^\Delta - u^m)^+ + C\Delta^\frac{1}{m} \text{ in } \bar{Q}_T,
\]
where we used \( k = 4C\varepsilon \) in the second to last inequality, and choose \( \varepsilon = \Delta \frac{1}{m} \) in the last inequality. ■

We are now ready to obtain the lower bound for the approximation error.

**Theorem 3.7** Suppose that Assumption \ref{assumption:2.1} is satisfied. Let \( \Delta \in (0, T) \), \( u^\Delta \in C_b(\bar{Q}_T) \) be the unique solution of the approximation scheme \eqref{eq:2.4} and \( u \in C_b(\bar{Q}_T) \) be the unique viscosity solution of equation \eqref{eq:1.1} - \eqref{eq:1.2}. Then,
\[
u^\Delta - u \geq - C\Delta^\frac{1}{m} \text{ in } \bar{Q}_T.
\]

**Proof.** Applying Proposition \ref{prop:3.6} to the sequence \( \{u^m_n\} \), we get for \( n \in \mathbb{Z}^+ \),
\[
u^\Delta - u = (u^\Delta - u^m_n) + (u^m_n - u)
\]
\[
\leq \sup_{\bar{Q}_T \setminus \bar{Q}_{T-\Delta}} (u^\Delta - u^m_n)^+ + C\Delta^\frac{1}{m} + (u^m_n - u)
\]
\[
\leq \sup_{\bar{Q}_T \setminus \bar{Q}_{T-\Delta}} (u^\Delta - u)^+ + \sup_{\bar{Q}_T \setminus \bar{Q}_{T-\Delta}} (u - u^m_n)^+ + C\Delta^\frac{1}{m} + u^m_n - u
\]
\[
\leq C\Delta^\frac{1}{m} + u^m_n - u \text{ in } \bar{Q}_T,
\]
where we applied estimate \eqref{eq:3.1} in Lemma \ref{lem:3.1} and estimate \eqref{eq:3.17} to the last inequality. The conclusion then follows by letting \( n \to \infty \) and using \eqref{eq:3.16}. ■

15
4 Well-posedness, Regularity, and Continuous Dependence for a Variant Switching System

In this section we construct the well-posedness, regularity, and continuous dependence results for general variant switching systems, which include equation (3.18), (3.21) and (3.25) as special cases. These results are vitally important in the previous section when obtaining the lower bound.

For $T > 0$ and $m \in \mathbb{Z}^+$, we consider the following $m+1$ dimensional general variant switching system in $Q_T$:

$$
\left\{ \begin{array}{l}
\max \left\{ -\partial_t u_i + \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} L_i^{\alpha,\beta}(t, x, \partial_x u_i, \partial_{xx} u_i), u_i - M_k^i u \right\} = 0, \quad i \in \mathcal{I} := \{1, \ldots, m\}, \\
\max\{u_{m+1} - f, u_{m+1} - M_{m+1}^k u\} = 0,
\end{array} \right. 
$$

with terminal condition

$$u_i(T, x) = U(x), \quad i \in \mathcal{I}, 
$$

where

$$L_i^{\alpha,\beta}(t, x, p, X) = -\frac{1}{2} \text{Trace} \left( \sigma_i^{\alpha,\beta} \sigma_i^{\alpha,\beta \top} (t, x) X - b_i^{\alpha,\beta}(t, x) \cdot p - L_i^{\alpha,\beta}(t, x) \right),$$

$\mathcal{A}$ and $\mathcal{B}$ are compact metric spaces. Note that this variant switching system is a general version of (3.18), (3.21), and (3.25). We then make the following assumption:

**Assumption 4.1** There exists a constant $\tilde{C} > 0$ independent of $\alpha, \beta, i$ and $t$ such that for any $\alpha, \beta, i$, and $t$,

$$|\sigma_i^{\alpha,\beta}(t, \cdot)|_1, |b_i^{\alpha,\beta}(t, \cdot)|_1, |L_i^{\alpha,\beta}(t, \cdot)|_1, |f|_1, |U|_1 \leq \tilde{C}.$$ 

We firstly give the existence of the solution of (4.1)-(4.2), then proceed to the continuous dependence on the coefficients, which implies the standard comparison principle and uniqueness, and finally obtain the regularity for the solution based on continuous dependence results.

**Proposition 4.2** Suppose that Assumption 4.1 is satisfied. Then there exists a bounded viscosity solution $u$ of (4.1)-(4.2), with $|u|_0 \leq C$ depending only on $T$ and $\tilde{C}$.

**Proof.** Here we only give an optimal switching representation which can be proved to be a viscosity solution of (4.1)-(4.2) using a similar procedure given in [31], and skip the technical proof. Under the same setting as in [31], the viscosity solution of (4.1)-(4.2) can be represented on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ by

$$u_i(t, x) = \inf_{\theta \in \Theta_i[t, T], \alpha \in \mathcal{A}[t, T]} \inf_{\nu \in \mathcal{B}[t, T]} \sup_{\beta \in \mathcal{B}[t, T]} \mathbb{E}^{\nu, x} \left[ \int_{t}^{\tau_{\nu}} L_{\theta_s}^{\alpha,\beta}(s, X_{s}^{\alpha,\beta}) \, ds + k(\nu - 1_{\{\tau_{\nu} = T\}}) ight]$$

$$+ f(\tau_{\nu}, X_{\tau_{\nu}}^{\alpha,\beta}) 1_{\{\tau_{\nu} < T\}} + U \left( X_{\tau_{\nu}}^{\alpha,\beta} \right) 1_{\{\tau_{\nu} = T\}} |\mathcal{F}_t|,$$

with the state equation

$$dX_{s}^{\alpha,\beta} = b_{\theta_s}^{\alpha,\beta}(s, X_{s}^{\alpha,\beta}) 1_{\{\theta_s \neq m+1\}} \, ds + \sigma_{\theta_s}^{\alpha,\beta}(s, X_{s}^{\alpha,\beta}) 1_{\{\theta_s \neq m+1\}} \, dW_s,$$

where $\mathcal{A}[t, T]$ and $\mathcal{B}[t, T]$ are $\mathcal{A}$ and $\mathcal{B}$-valued progressive-measurable processes respectively, and $\Theta_i[t, T]$ is the space of all admissible continuous switching control processes on $[t, T]$ starting from
i. Specifically, for any admissible switching control process, there is a pair of corresponding sequence \( \{\xi_n, \tau_n\}_{n \geq 0} \) such that \( \{\tau_n\}_{n \geq 0} \) is a sequence of nondecreasing stopping time with 
\[
t = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq T, \quad \text{a.s.}
\]
and that each \( \xi_n \) is a \( \mathcal{F}_n \)-measurable random variable valued in \( \tilde{I} := I \cup \{m + 1\} \), with \( \xi_0 = i \) and \( \xi_n \neq \xi_{n+1} \) a.s. Then an admissible switching control process \( \theta \) in \( \Theta_i[t,T] \) is identified as
\[
\theta_t = \Sigma_{n \geq 1} \xi_n \mathbf{1}_{(\tau_{n-1},\tau_n)}(t).
\]
Finally, for each \( \theta \in \Theta_i[t,T] \), \( T^\theta_i[t,T] := \{\nu : \xi_\nu = m + 1, \tau_\nu < T\} \cup \{\nu : \tau_\nu = T\} \), which is valued in \( \mathbb{N} \) and represents how many times (or minus 1 if \( \tau_\nu = T \)) the agent will switch among different regions before she stops.

Loosely speaking, the agent in this optimal switching system has to firstly choose a switching control \( \theta \) (i.e. a sequence \( \{\xi_n, \tau_n\}_{n \geq 0} \)), and then based on it choose how many times to switch among regions before stopping. However, the stop time must be chosen either such that she is in the last region (\( i = m + 1 \)), or to be terminal time \( T \).

Remark 4.3 A standard optimal switching system consists only the first \( m \) equations in (4.1).
In this case, for each switching control \( \theta \), \( T^\theta_i[t,T] := \inf\{\nu : \tau_\nu = T\} \) consists only one element. Representation (4.3) then reduces to a standard optimal switching representation.

The additional equation for \( u_{m+1} \) in (4.1) means that there is an additional region in the variant switching system, and more importantly, the agent in this variant optimal switching problem can not stay in this region: once entering into this region (or be in this region from the start), she only has two options, either stop immediately (to pay a cost of \( f(t,x) \)) or switch immediately to another region.

We now give the continuous dependence result for the general variant switching system (4.1).

Theorem 4.4 For any \( s \in (0,T] \), let \( u \in USC(Q_s) \) be a bounded from above viscosity subsolution of (4.1) with coefficients \( \{\sigma_i^{\alpha,\beta}, b_i^{\alpha,\beta}, L_i^{\alpha,\beta}, f\} \), and \( \bar{u} \in LSC(Q_s) \) be a bounded from below viscosity supersolution of (4.1) with coefficients \( \{\bar{\sigma}_i^{\alpha,\beta}, \bar{b}_i^{\alpha,\beta}, \bar{L}_i^{\alpha,\beta}, \bar{f}\} \). Suppose that Assumption (4.1) holds for both sets of coefficients, and that \( |u(s,\cdot)|_1 \leq M \) or \( |\bar{u}(s,\cdot)|_1 \leq M \) for some constant \( M \), then there exists a constant \( C \) depending only on \( M, C \), and \( s \) such that for each \( i \in \tilde{I} \),
\[
u_i - \bar{u}_i \leq C \left( \sup_{s} |(u_i(s,\cdot) - \bar{u}_i(s,\cdot))^{+}|_0 + \sup_{s,\alpha,\beta} |\sigma - \bar{\sigma}|_0 + |b - \bar{b}|_0 \right) \leq C \left( |L - \bar{L}|_0 + |f - \bar{f}|_0 \right) \quad \text{in } Q_s.
\] (4.4)

Proof. The proof is mainly based on the proofs of Theorem A.1 in [21] and Theorem A.3 in [24], and can be regarded as a combination of these two proofs. We give the details here for reader’s convenience.

Fix \( 0 < s \leq T \) and in \( Q_s \), we define functions \( v(t,x) := e^t u(t,x) \), \( \bar{v}(t,x) := e^t \bar{u}(t,x) \) and \( g(t,x) := e^t g(t,x) \) for \( g = L, L, f, \bar{f} \). It is then follows that \( v \in USC(Q_s) \) and \( \bar{v} \in LSC(Q_s) \) are
bounded above viscosity subsolution and bounded below supersolution of the following system with coefficients \( \{\sigma, b, L', f'\} \) and \( \{\bar{\sigma}, \bar{b}, \bar{L}', \bar{f}'\} \) respectively:

\[
\begin{align*}
&\max \left\{ -\partial_t v_i + v_i + \sup_{\alpha \in A} \inf_{\beta \in B} C^{\alpha,\beta}_i (t,x,\partial_x v_i, \partial_{xx} v_i), v_i - M^k v_i \right\} = 0, \quad i \in \mathcal{I}, \\
&\max \{ v_{m+1} - f', v_{m+1} - M^k v_{m+1} \} = 0.
\end{align*}
\]

(4.5)

We now use a doubling variables argument to derive a upper bound for \( v_i - \bar{v}_i \) and then will derive for \( u_i - \bar{u}_i \) by using back-substitution.

To continue, we define in \([0,s] \times \mathbb{R}^n \times \mathbb{R}^n\) that \( \psi(t,x,y) := e^{\lambda(s-t)}(x-y)^2 + \varepsilon |(x-y)|^2 \) and for any \( i \in \mathcal{I}, \psi_i(t,x,y) := v_i(t,x) - \bar{v}_i(t,y) - \phi(t,x,y), \) where \( \lambda, \delta, \varepsilon > 0 \) are positive constants. Then we let \( m_{i}^s = \sup_{i,t,x,y} \psi_i(t,x,y)^+ \) and \( m_{\lambda,\delta,\varepsilon} = \sup_{i,t,x,y} \psi_i(t,x,y) - m_{i}^s \). Since \( v_i \) and \( \bar{v}_i \) are bounded from above and below respectively, by classical arguments there exists \( \lambda, \delta, \varepsilon = \sup_{i,t,x,y} \psi_i(t,x,y) - m_{i}^s \). Since \( v_i \) and \( \bar{v}_i \) by using back-substitution.

\[
\max_{t} \{ v(t,x,y) - \bar{v}(t,x,y) \} = \sup_{i,t,x,y} \psi_i(t,x,y)^+ \]

(4.6)

for any \((t,x) \in [0,s] \times \mathbb{R}^n\). We now try to derive the upper bound for \( m_{i}^s \) and \( m_{\lambda,\delta,\varepsilon} \).

Since \( [u(s,\cdot)]_1 \leq M \) or \( [\bar{u}(s,\cdot)]_1 \leq M \), without loss of generality we assume \( [\bar{u}(s,\cdot)]_1 \leq M \), then \( [\bar{v}(s,\cdot)]_1 \leq Me^s \) and for any \( x, y \in \mathbb{R}^n \)

\[
\psi_x(s,x,y) \leq v_i(s,x) - \bar{v}_i(s,y) - \delta |x-y|^2 \\
\leq \sup_{i} |(v_i(s,x) - \bar{v}_i(s,y))|_0 + Me^s |x-y| - \delta |x-y|^2 \\
\leq \sup_{i} |(v_i(s,x) - \bar{v}_i(s,y))|_0 + Me^s |x-y| - \delta |x-y|^2.
\]

where the last second inequality follows from that \( \sup_{t \geq 0} (Ct^2 - \delta t^2) = C^2/4\delta \) for any \( C, \delta > 0 \). Thus, we get the upper bound for \( m_{i}^s \):

\[
m_{i}^s \leq \sup_{i} |(v_i(s,x) - \bar{v}_i(s,y))|_0 + C_1 t^{-1},
\]

(4.7)

where \( C_1 = M^2e^{2s}/4 \).

On the other hand, we assume that \( m_{\lambda,\delta,\varepsilon} > 0 \) and derive its (positive) upper bound. Of course this upper bound still holds for \( m_{\lambda,\delta,\varepsilon} \leq 0 \). Follow this assumption, we have \( t_0 < s \), since otherwise, \( m_{\lambda,\delta,\varepsilon} = \sup_{t_0 \leq x \leq \mathbb{R}^n} \psi(s,x,y) - m_{i}^s \leq 0 \). Then we can apply the parabolic maximum principle for semicontinuous functions, Theorem 8.3 in [18] to get that there are \( a, b \in \mathbb{R} \) and \( X, Y \in \mathcal{S}^n \) such that \( (a, D_x \phi(t_0,x_0,y)), X) \in \mathcal{P}^{2,+} \) \( v_{t_0}(t_0,x_0) \) and \( (b, -D_y \phi(t_0,x_0,y), Y) \in \mathcal{P}^{2,-} \) \( \bar{v}_{t_0}(t_0,y_0) \) with \( a - b = \phi(t_0,x_0,y_0) \) and the following inequality holds

\[
\left( \begin{array}{cc}
X & 0 \\
0 & -Y
\end{array} \right) \leq 3e^{\lambda(s-t_0)} \left( \begin{array}{cc}
I & -I \\
-I & I
\end{array} \right) + 3\varepsilon \left( \begin{array}{cc}
I & 0 \\
0 & I
\end{array} \right).
\]

(4.8)

We now discuss two different situations where \( i_0 \in \mathcal{I} \) or \( i_0 = m+1 \) and derive the upper bounds for \( m_{\lambda,\delta,\varepsilon} \) in both situations, then we add them to obtain an upper bound that holds in all situations.
(i) If \( t_0 \in \mathcal{I} \), then by the definitions of viscosity sub- and supersolutions, as well as the fact that \( \bar{v}_{i_0}(t_0, y_0) < \mathcal{M}_{i_0} \bar{v}(t_0, y_0) \) and \( \sup \inf(\cdots) - \sup \inf (\cdots) \leq \sup \sup (\cdots - \cdots) \), we have

\[
0 \leq -\lambda e^{\lambda(s-t_0)} \delta |x_0 - y_0|^2 - \bar{v}_{i_0}(t_0, x_0) + \bar{v}_{i_0}(t_0, y_0) \\
+ \sup_{\alpha, \beta} \left\{ \frac{1}{2} \text{tr}[\sigma_{i_0}\sigma_{i_0}^T(t_0, x_0)X - \tilde{\sigma}_{i_0}\tilde{\sigma}_{i_0}^T(t_0, y_0)Y] \\
- \bar{b}_{i_0}(t_0, y_0) \cdot (2e^{\lambda(s-t_0)} \delta (x_0 - y_0) - 2\varepsilon y_0) \\
+ \bar{b}_{i_0}(t_0, x_0) \cdot (2e^{\lambda(s-t_0)} \delta (x_0 - y_0) + 2\varepsilon x_0) \\
- \tilde{L}_{i_0}'(t_0, y_0) + L_{i_0}'(t_0, x_0) \right\}.
\]

By the inequality (4.8) and the fact that \((s + t)^2 \leq 2(s^2 + t^2)\) for \( s, t \in \mathbb{R} \), we obtain

\[
\text{tr}[\sigma_{i_0}\sigma_{i_0}^T(t_0, x_0)X - \tilde{\sigma}_{i_0}\tilde{\sigma}_{i_0}^T(t_0, y_0)Y] \leq 6e^{\lambda(s-t_0)} \delta \left\{ |\sigma_{i_0}(t_0, x_0) - \tilde{\sigma}_{i_0}(t_0, x_0)|^2 + |\tilde{\sigma}_{i_0}(t_0, x_0) - \tilde{\sigma}_{i_0}(t_0, y_0)|^2 \right\} \\
+ 3\varepsilon \left\{ |\sigma_{i_0}(t_0, x_0)|^2 + |\tilde{\sigma}_{i_0}(t_0, y_0)|^2 \right\} \leq 6e^{\lambda(s-t_0)} \delta \left\{ |\sigma_{i_0} - \tilde{\sigma}_{i_0}|_0^2 + |\tilde{\sigma}_{i_0}|_{2,1}|x_0 - y_0|^2 \right\} \\
+ 3\varepsilon \left\{ |\sigma_{i_0}|_0^2 + |\tilde{\sigma}_{i_0}|_{2,1}^2 \right\}.
\]

Furthermore, we have the following estimates

\[
(b_{i_0}(t_0, x_0) \cdot (x_0 - y_0)) \cdot (x_0 - y_0) \leq \frac{1}{2} (|b_{i_0}(t_0, x_0) - \bar{b}_{i_0}(t_0, x_0)|^2 + |x_0 - y_0|^2) \\
+ |\bar{b}_{i_0}(t_0, x_0) - \bar{b}_{i_0}(t_0, y_0)||x_0 - y_0| \\
\leq \frac{1}{2} (|b_{i_0} - \bar{b}_{i_0}|_0^2 + |x_0 - y_0|^2) + |\bar{b}_{i_0}|_{2,1}|x_0 - y_0|^2,
\]

\[
(b_{i_0}(t_0, x_0) \cdot x_0) \leq |b_{i_0}(t_0, x_0)||x_0| \leq (|b_{i_0}|_{2,1}|x_0| + |b_{i_0}(t_0, 0)||x_0| \\
\leq \tilde{C}(1 + |x_0|^2) \leq 2\tilde{C}(1 + |x_0|^2),
\]

and similarly

\[
\bar{v}_{i_0}(t_0, y_0) \leq 2\tilde{C}(1 + |y_0|^2),
\]

\[
\bar{v}_{i_0}(t_0, y_0) - \bar{v}_{i_0}(t_0, x_0) \leq -\psi(t_0, x_0, y_0) \leq -m_{\lambda, \delta, \varepsilon},
\]

and

\[
L_{i_0}'(t_0, x_0) - \tilde{L}_{i_0}'(t_0, y_0) \leq |L_{i_0}' - \tilde{L}_{i_0}'|_0 + |L_{i_0}'|_{2,1}|x_0 - y_0|.
\]

Plugging in all these estimates into inequality (4.9) yields

\[
m_{\lambda, \delta, \varepsilon} \leq 3e^{\lambda(s-t_0)} \delta \sup_{i, \alpha, \beta} \left\{ |\sigma - \tilde{\sigma}|_0^2 + |b - \bar{b}|_0^2 \right\} + e^{s} \sup_{i, \alpha, \beta} |L - \tilde{L}|_0 \\
+ (C_2 - \lambda)e^{\lambda(s-t_0)} \delta |x_0 - y_0|^2 + e^{s}(L_{i_0}|_{2,1}|x_0 - y_0| + C_3(1 + |x_0|^2 + |y_0|^2)\varepsilon. \tag{4.10}
\]

where \( C_2 = 3\tilde{C}^2 + 2\tilde{C} + 1 \), \( C_3 = 3\tilde{C}^2 + 4\tilde{C} \) are some positive constants.

(ii) If \( i_0 = m + 1 \), then we can easily obtain that

\[
m_{\lambda, \delta, \varepsilon} \leq v_{i_0}(t_0, x_0) - \bar{v}_{i_0}(t_0, y_0) \leq \tilde{f}'(t_0, x_0) - f'(t_0, y_0) \leq e^{s}|f - \tilde{f}|_0 + e^{s}|f|_{2,1}|x_0 - y_0|. \tag{4.11}
\]
Combine (4.10) and (4.11), we get an upper bound for $m_{\lambda, \delta, \varepsilon}$ in all situations. Plug this upper bound as well as (4.6) into (4.7), we have
\[
v_i(t, x) - \bar{v}_i(t, x) \leq \sup_i |(v_i(s, \cdot) - \bar{v}_i(s, \cdot))|_0 + 3\varepsilon^{\lambda(s-t_0)} \delta \sup_{i, \alpha, \beta} \{|\sigma - \bar{\sigma}|^2 + |b - \bar{b}|^2\} \\
+ \varepsilon^{s} \{ \sup_{i, \alpha, \beta} |L - \bar{L}|_0 + |f - \bar{f}|_0 \} + (C_2 - \lambda) \varepsilon^{\lambda(s-t_0)} \delta |x_0 - y_0|^2 + 2\varepsilon^{2} \bar{C} |x_0 - y_0| \\
+ C_1 \delta^{-1} + C_3 (1 + |x_0|^2 + |y_0|^2) \varepsilon + 2\varepsilon^2 + 2\varepsilon |x|^2.
\]
Note that this estimate holds for any $\lambda, \delta, \varepsilon > 0$. We then try to select appropriate value for them (or take limit) to draw our conclusion. Firstly we may choose $\lambda = C_2 + 1$ and follow again that $\sup_{\varepsilon \geq 0} (C r - \delta r^2) = C^2/4\delta$ to get rid of the $|x_0 - y_0|$ term. Then, by standard arguments, we know that for any fixed $\lambda$ and $\delta$, $\lim_{\varepsilon \to 0} \varepsilon (|x_0|^2 + |y_0|^2) = 0$. By letting $\varepsilon \to 0$, we further get
\[
v_i(t, x) - \bar{v}_i(t, x) \leq \sup_i |(v_i(s, \cdot) - \bar{v}_i(s, \cdot))|_0 + \varepsilon^s \{ \sup_{i, \alpha, \beta} |L - \bar{L}|_0 + |f - \bar{f}|_0 \} \\
+ 3\varepsilon (C_{2} + 1)(s-t_0)^{-1} \delta \sup_{i, \alpha, \beta} \{|\sigma - \bar{\sigma}|^2 + |b - \bar{b}|^2\} + (C_1 + \varepsilon^{2} \bar{C}^2 e^{-(C_{2} + 1)(s-t_0)})^{-1} \delta^{-1}.
\]
Note that $\min_{r>0} (ar + br^{-1}) = 2(ab)^{1/2}$ for any $a, b > 0$, we can choose the $\delta$ minimising the right hand side to get
\[
v_i(t, x) - \bar{v}_i(t, x) \leq \sup_i |(v_i(s, \cdot) - \bar{v}_i(s, \cdot))|_0 + \varepsilon^s \{ \sup_{i, \alpha, \beta} |L - \bar{L}|_0 + |f - \bar{f}|_0 \} \\
+ C_4 \sup_{i, \alpha, \beta} \{|\sigma - \bar{\sigma}|_0 + |b - \bar{b}|_0\}.
\]
where $C_4 = 2(3C_1 e^{(C_{2} + 1)s} + 3\varepsilon^{2} \bar{C}^2)^{1/2}$ and we used that $(s^2 + t^2)^{1/2} \leq |s| + |t|$ for any $s, t \in \mathbb{R}$ in the last inequality. Finally, the conclusion follows by back-substituting $v$ and $\bar{v}$ by $u$ and $\bar{u}$. \]

Finally, by using the above continuous dependence result, we show that the bounded viscosity solution $u$ of (4.1)-(4.2) is the unique bounded solution, and moreover, it admits some regularity results.

**Theorem 4.5** Suppose that Assumption (4.7) is satisfied. Then, there exists a unique viscosity solution $u$ of (4.1)-(4.2), with $|u|_1 \leq C$ depending only on $T$ and $\bar{C}$.

**Proof.** In this proof, we denote by $C$ some constant depending only on $T$ and $\bar{C}$. Proposition 4.2 gives the existence, and the continuous dependence result (4.1) implies uniqueness and the $x$-regularity. To proof the $t$-regularity, we follow the idea in Appendix A. Fix any $(t, s)$ such that $0 \leq t < s \leq T$ and define the functions $U_{i, s}^\varepsilon := u_i(s, \cdot) * \rho_{s-t}$ in $\mathbb{R}^n$ for $i \in \bar{I}$ and some $\varepsilon > 0$ that shall be decided later, where $\rho_{s-t}$ is the same mollifiers defined in Appendix A. Then similarly we have
\[
|U_{i, s}^\varepsilon - u_i(s, \cdot)|_0 \leq C\varepsilon \quad \text{and} \quad |D_{i}^2 U_{i, s}^\varepsilon|_0 \leq C\varepsilon^{1-|j|}.
\]
Let $u^\varepsilon$ be the unique bounded solution of (4.1) in $Q_s$ with terminal condition $U^\varepsilon$, for some $\varepsilon > 0$ that shall be decided later. The continuous dependence results (4.6) then implies that for any $i \in \bar{I}$,
\[
|u_i^\varepsilon - u_i| \leq \sup_i |U_i^\varepsilon - u_i(s, \cdot)|_0 \leq C\varepsilon \quad \text{in} \quad \bar{Q}_s.
\]

20
Next, for each $i \in I$, define two functions $w_{C_1}^+(t, x) := U_1^+(x) + C_1(s - t)$ in $Q_s$, for some $C_1 = C(\varepsilon^{-1} + 1)$. It then can be easily checked that, for $i \in I$, the functions $w_{C_1}^+$ and $w_{C_1}^-$ are bounded subsolution and bounded supersolution of
\[- \partial_t w + \sup \inf_{\alpha, \beta \in A} L_{\alpha, \beta}^i(t, x, \partial_x w, \partial_{xx} w) = 0, \text{ in } Q_s.\]

Thus, the function $\bar{v} = (\bar{v}_1, \bar{v}_2, ..., \bar{v}_M)$ such that $\bar{v}_i = \bar{w}_i$ for $i \in I$ and that $\bar{v}_{M+1} = f$, is a bounded supersolution of (4.1) in $Q_s$. Applying (4.4) for $w^\varepsilon$ and $\bar{v}$ yields that for $i \in I$,
\[u_i^\varepsilon(t, x) - w_{C_1}^+(t, x) \leq \sup_i \|(U_i^\varepsilon - \bar{v}(s, \cdot))^{+}\|_0 = \|(U_i^\varepsilon + f(s, \cdot))^{+}\|_0 \leq \|(w^\varepsilon_{M+1}(s, \cdot) - f(s, \cdot))^{+}\|_0 + C\varepsilon = C\varepsilon,\]

which implies that for $i \in I$,
\[u_i^\varepsilon(t, x) - U_i^\varepsilon(x) \leq C\varepsilon(s - t) + C\varepsilon.\]

Now we construct a bounded subsolution of (4.1) in $Q_s$ based on $w_{C_1}^-$. Note that since $M_i^k u$ is concave in $u$, we have for $i \in I$,
\[U_i^\varepsilon = u_i(s, \cdot) \rho \leq (M_i^k u(s, \cdot)) \rho \leq M_i^k (u(s, \cdot) \rho) = M_i^k U^\varepsilon,\]

and thus $w_{C_1}^- - M_i^k w_{C_1}^- \leq 0$ in $Q_s$. Furthermore, similar to the arguments in Appendix A we have that in $Q_s$, $w_{C_1, M+1}^--f \leq C\varepsilon$. Thus, the function $\underline{v} = w_{C_1}^- - C\varepsilon$ is a bounded subsolution of (4.4) in $Q_s$. Applying (4.4) for $\underline{v}$ and $w^\varepsilon$ yields that for $i \in I$,
\[w_{C_1, i}^-(t, x) - C\varepsilon - u_i^\varepsilon(t, x) \leq 0,\]

which implies that for $i \in I$,
\[U_i^\varepsilon(x) - u_i^\varepsilon(t, x) \leq C\varepsilon(s - t) + C\varepsilon.\]

In turn, we obtain that for $i \in I$,
\[|u_i(t, x) - u_i(s, x)| \leq |u_i(t, x) - u_i^\varepsilon(t, x)| + |u_i^\varepsilon(t, x) - U_i^\varepsilon(x)| + |U_i^\varepsilon(x) - u_i(s, x)| \leq 2C\varepsilon + C\varepsilon(s - t) + C\varepsilon \leq C\left(\varepsilon + \frac{(s - t)}{\varepsilon} + (s - t)\right).
\]

We choose $\varepsilon = \sqrt{s - t}$ to minimize the right hand side and obtain that for $i \in I$,
\[|u_i(t, x) - u_i(s, x)| \leq C\sqrt{s - t}.
\]

Note that from (4.1), we have that $u_{M+1} = \min\{f, u_1 + k, u_2 + k, ..., u_M + k\}$. Therefore, the above inequality also holds for $i = m + 1$. This, together with the boundedness and the $x$-regularity, implies that $|u_i| \leq C$. ■
5 Conclusion

We propose an approximation scheme for a class of semilinear parabolic variational inequalities whose Hamiltonian is convex and coercive. The proposed scheme is a natural extension of a previous splitting scheme proposed by Liang, Zariphopoulou and the author [19] for semilinear parabolic PDEs with the same Hamiltonians. We establish the convergence of the scheme and determine the convergence rate by obtaining its error bounds. The bounds are obtained by Krylov’s shaking coefficients technique and Barles-Jakobsen’s optimal switching approximation, while the switching system we use is a variant switching system. Compared to the results in [19], the convergence rates of our proposed scheme remain the same: the upper convergence rate is 1/4 (Theorem 3.4) and the lower convergence rate is 1/10 (Theorem 3.7).

We mention that the approaches and results herein rely heavily on the Lipschitz continuity of (viscosity) solutions of the equation (1.1) with respect to the space variable. A possible extension is to consider a case when solutions are $\beta$-Hölder continuous for some $\beta \in (0, 1)$. This is challenging in the sense that in this case, (1.1) can not be written as (3.12) where the control set $K$ is compact, which is a key step when obtaining the lower bound. This will be left as future work. One may also consider another version of variational inequalities where the gradient of the solution is constrained rather than the solution itself. These are naturally related to singular stochastic optimization problems. An early result in this direction but in elliptic case can be bound in [11].

A Proof of Proposition 2.2 and 3.3

We note that equation (1.1)-(1.2) is a special case (choosing $\varepsilon = 0$) of the equation (3.3)-(3.4). Therefore, we omit the proof of Proposition 2.2 and only prove Proposition 3.3.

We first show that there exists a bounded solution to (3.3)-(3.4). To this end, using the convex coefficients technique and Barles-Jakobsen’s optimal switching approximation, while the switching system we use is a variant switching system. Compared to the results in [19], the convergence rates of our proposed scheme remain the same: the upper convergence rate is 1/4 (Theorem 3.4) and the lower convergence rate is 1/10 (Theorem 3.7).

We mention that the approaches and results herein rely heavily on the Lipschitz continuity of (viscosity) solutions of the equation (1.1) with respect to the space variable. A possible extension is to consider a case when solutions are $\beta$-Hölder continuous for some $\beta \in (0, 1)$. This is challenging in the sense that in this case, (1.1) can not be written as (3.12) where the control set $K$ is compact, which is a key step when obtaining the lower bound. This will be left as future work. One may also consider another version of variational inequalities where the gradient of the solution is constrained rather than the solution itself. These are naturally related to singular stochastic optimization problems. An early result in this direction but in elliptic case can be bound in [11].
the collection of all $\mathbb{F}$-stopping times with values in $[t, T + \varepsilon^2]$, and $W$ is an $d$-dimensional Brownian motion with its augmented filtration $\mathbb{F}$.

Next, we identify its value function with a bounded viscosity solution to (A.1)-(A.2). For this, we only need to establish upper and lower bounds for the value function $u^\varepsilon(t, x)$ and, in turn, use standard arguments as in Theorem A.1 of [21], and is similarly, we also have $u(t, x) = u^\varepsilon(t, x)$, uniformly in $\theta, \varepsilon$.

To find an upper bound of $u^\varepsilon$, we choose an arbitrary perturbation parameter process $\theta \in \Theta^s[t, T + \varepsilon^2]$, an arbitrary stopping time $\nu \in T[t, T + \varepsilon^2]$, and a particular control $\tilde{q}$ for $\theta_s \equiv 0$. Then, Proposition 2.3 (ii) in [19] yields

$$u^\varepsilon(t, x) \leq \mathbb{E}\left[\int_t^\nu L^\varepsilon(s, X_s^\varepsilon)ds + f(\nu, X^\varepsilon)1_{\{\nu < T + \varepsilon^2\}} + U(X^\varepsilon)1_{\{\nu = T + \varepsilon^2\}}\right]$$

$$\leq (T + \varepsilon^2 - t)|L^\varepsilon(0)| + M \leq C.$$

For the lower bound, we use again Proposition 2.3 (ii) in [19] to obtain that $L^\varepsilon(\theta) = -H^\varepsilon(0) \geq -|H^\varepsilon(0)|$, for any $\theta \in \Theta^s$. In turn, for any $(\theta, q, \nu) \in \Theta^s[t, T + \varepsilon^2] \times H^2[t, T + \varepsilon^2] \times T[t, T + \varepsilon^2]$, $\mathbb{E}\left[\int_t^\nu L^\varepsilon(s, X_s^\varepsilon)ds + f(\nu, X^\varepsilon)1_{\{\nu < T + \varepsilon^2\}} + U(X^\varepsilon)1_{\{\nu = T + \varepsilon^2\}}\right]$

$$\geq \mathbb{E}\left[\int_t^\nu L^\varepsilon(s, q_s)ds|F_t\right] - M \geq -(T + \varepsilon^2 - t)|H^\varepsilon(0)| - M \geq C,$$

and, thus, $u^\varepsilon(t, x) \geq C$ and $|u^\varepsilon|_0 \leq C$.

The uniqueness of the viscosity solution is a direct consequence of the following continuous dependence result, whose proof follows along similar arguments as in Theorem A.1 of [21], and is thus omitted.

**Lemma A.1** For any $s \in (0, T + \varepsilon^2]$, let $u \in USC(Q_s)$ be a bounded from above viscosity sub-solution of (3.3) with coefficients $\{\sigma^\theta, \bar{B}^\theta, H^\theta, f\}$, and $\bar{u} \in LSC(Q_s)$ be a bounded from below viscosity super-solution of (3.3) with coefficients $\{\bar{\sigma}^\theta, \bar{B}^\theta, \bar{H}^\theta, \bar{f}\}$. Suppose that Assumption 2.4 holds for both sets of coefficients with respective constants $M$ and $\bar{M}$, uniformly in $\theta \in \Theta^s$, and that either $u(s, \cdot) \in C^1_b(\mathbb{R}^n)$ or $\bar{u}(s, \cdot) \in C^1_b(\mathbb{R}^n)$. Then, there exists a constant $C$, depending only on $M, \bar{M}, |u(s, \cdot)|_1, |\bar{u}(s, \cdot)|_1$, and $s$, such that in $Q_s$, $u - \bar{u} \leq C\left(|u(s, \cdot) - \bar{u}(s, \cdot)|_0 + \sup_{\theta \in \Theta^s}|\sigma^\theta - \bar{\sigma}^\theta|_0 + |B^\theta - \bar{B}^\theta|_0 + |H^\theta - \bar{H}^\theta|_0 + |f - \bar{f}|_0\right)$.

**A.3**

We now continue the proof of Proposition 2.3. The $x$-regularity of $u^\varepsilon$ then follows easily from (A.3) by choosing $u = u^\varepsilon$, $\bar{u} = u^\varepsilon(\cdot + \varepsilon)$ and $s = T + \varepsilon^2$. To get the $t$-regularity, let $\rho(x)$ be a $\mathbb{R}_+\times \mathbb{R}$-valued smooth function with compact support $B(0, 1)$ and mass 1, and for $\varepsilon > 0$, let $\rho_\varepsilon(x) := \varepsilon^{-1}\rho\left(\frac{x}{\varepsilon}\right)$ be a sequence of mollifiers. Next, fix any $(t, s)$ such that $0 \leq t < s \leq T + \varepsilon^2$ and let $u_{\varepsilon^t}$ be the unique bounded solution of (3.3) in $Q_s$ with terminal condition $u_{\varepsilon^t}(s, x) = u^\varepsilon(s, \cdot)\rho_\varepsilon(x)$, for some $\varepsilon^t > 0$ that shall be decided later. It then follows from (A.3) that, in $Q_s$, $u^\varepsilon - u_{\varepsilon^t} \leq C\left(|u^\varepsilon(s, \cdot) - u_{\varepsilon^t}(s, \cdot)|_0 + C|u^\varepsilon(s, \cdot)|_1\varepsilon \leq C\varepsilon^t.\right.$

Similarly, we also have $u_{\varepsilon^t} - u^\varepsilon \leq C\varepsilon^t.$

23
On the other hand, standard properties of mollifiers imply that $|D_x^j u_{c'}(s,\cdot)|_0 \leq C\varepsilon^{j-1}|b|$. Thus, for any $(\xi, x) \in Q_s$, we have

$$|\sup_{\theta \in \Theta} g^\theta(\xi, x, \partial_x u_{c'}(s, x), \partial_{xx} u_{c'}(s, x))| \leq C(\frac{1}{\varepsilon} + 1) =: C_{c'}.$$ 

Define two functions $w_{c'}^\pm(t, x) := u_{c'}(s, x) \pm C_{c'}(s-t)$ in $\bar{Q}_s$. It then can be easily checked that the function $w_{c'}^+(t, x)$ is a bounded supersolution of (3.3) in $Q_s$. Thus, by (A.3), we have in $\bar{Q}_s$,

$$u_{c'}(t, x) - u_{c'}(s, x) \leq C_{c'}(s-t).$$

We then construct a bounded subsolution of (3.3) in $Q_s$ based on $w_{c'}^-$. Since $u_{c'}(s, x) \leq u^\varepsilon(s, x) + C\varepsilon' \leq f(s, x) + C\varepsilon'$, we obtain that for any $(\xi, x) \in Q_s$,

$$w_{c'}^-(\xi, x) - f(\xi, x) = u_{c'}(s, x) - C_{c'}(s-\xi) - f(\xi, x)$$

$$\leq f(s, x) - f(\xi, x) - C_{c'}(s-\xi) + C\varepsilon'$$

$$\leq M\sqrt{s-\xi} - C_{c'}(s-\xi) + C\varepsilon'$$

$$\leq \frac{M^2}{4C_{c'}} + C\varepsilon' \leq C\varepsilon',$$

where we used $\sup_{r \geq 0}(C_1 r - C_2 r^2) = C_1^2/4C_2$ for any $C_1, C_2 > 0$, and $\frac{1}{C_{c'}} \leq C\varepsilon'$. This implies that $w_{c'}^- - C\varepsilon'$ is a bounded subsolution of (3.3) in $Q_s$. By (A.3), we then have in $\bar{Q}_s$, $w_{c'}^- - C\varepsilon' - u_{c'} \leq 0$, which implies that

$$u_{c'}(s, x) - u_{c'}(t, x) \leq C_{c'}(s-t) + C\varepsilon'.$$

In turn, we obtain that

$$|u^\varepsilon(t, x) - u^\varepsilon(s, x)| \leq |u^\varepsilon(t, x) - u_{c'}(t, x)| + |u_{c'}(t, x) - u_{c'}(s, x)| + |u_{c'}(s, x) - u^\varepsilon(s, x)|$$

$$\leq 2C\varepsilon' + C_{c'}(s-t) + C\varepsilon'$$

$$\leq C\left(\varepsilon' + \frac{(s-t)}{\varepsilon'} + (s-t)\right).$$

We choose $\varepsilon' = \sqrt{s-t}$ to minimize the right hand side and conclude $|u^\varepsilon(t, x) - u^\varepsilon(s, x)| \leq C\sqrt{s-t}$, which, together with the boundedness and the $x$-regularity, implies that $|u^\varepsilon|_1 \leq C$.

Finally, note that $u(t, x)$ is also the bounded viscosity solution of (3.3) when $\sigma^b \equiv \sigma$, $b^\theta \equiv b$ and $H^\theta \equiv H$. Applying (A.3) once more and the regularity of $\sigma$, $b$, $H$ and $u^\varepsilon$, we deduce that in $\bar{Q}_T$,

$$u^\varepsilon - u \leq C \left(|u^\varepsilon(T, \cdot) - u(T, \cdot)|_0 + \sup_{\theta \in \Theta^*} \{|\sigma^b - \sigma|_0 + |b^\theta - b|_0\} + \sup_{\theta \in \Theta^*} |H^\theta - H|_0\right)$$

$$\leq C \left(|u^\varepsilon(T, \cdot) - u^\varepsilon(T + \varepsilon^2, \cdot)|_0 + \varepsilon\right) \leq C\varepsilon,$$

Similarly, we also have $u - u^\varepsilon \leq C\varepsilon$, and we conclude.

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