CORRECTION TO: THE KUO CONDITION, AN INEQUALITY OF THOM’S TYPE AND (C)-REGULARITY

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Abstract. We correct a statement of a theorem on characterisation of the (c)-regularity we gave in Topology 37 (1998), 45–62. This theorem was used in the paper in the proof of two theorems on the (c)-regular stratification. In this note we give a weaker version of the theorem as an alternative lemma which ensures the (c)-regularity condition and turn to be sufficient for the proof of the theorems.

1. Introduction

We gave a characterisation of the (c)-regularity in Theorem 2.4 of [2]. In order to show the theorem, we used a statement that a one-dimensional subspace of $\mathbb{R}^{n+p}$

$$K_x := (\text{Ker } d\rho(x) \cap T_x X)^\perp \cap T_x X$$

is orthogonal to $\mathbb{R}^n := \mathbb{R}^n \times \{0\}$ at any $x \in X$. But this statement does not necessarily hold, namely $K_x$ is not always orthogonal to $\mathbb{R}^n$ as T. Gaffney pointed out to us in [3]. Therefore the proof of Theorem 2.4 in [2] is false. In addition, we proved in [2] Theorem 2.7 (resp. Theorem 2.8) that the stratification $\Sigma(\mathbb{R}^n \times J)$ (see §3 for the definition) is (c)-regular under the Kuo condition (resp. the second Kuo condition), using Theorem 2.4.

In this note we show an alternative lemma to Theorem 2.4, adding one more assumption to the original two assumptions. The (c)-regularity follows from these three assumptions. In addition, we show that the Kuo condition (resp. the second Kuo condition) implies not only the original assumptions but the new assumption of the lemma. In other words, Theorem 2.7 (resp. Theorem 2.8) in [2] follows from the lemma.

The authors would like to thank T. Gaffney for pointing out to us a mistake in the proof of Theorem 2.4 in [2].

2. Alternative lemma

Let $M$ be a smooth manifold, and let $X, Y$ be smooth submanifolds of $M$ such that $Y \subset \overline{X}$. We suppose now that $M$ is endowed with a Riemannian metric. Let $(T_Y, \pi, \rho)$ be a smooth tubular neighbourhood for $Y$ with the associated projection $\pi$
and a smooth non-negative control function \( \rho \) such that \( \rho^{-1}(0) = Y \) and \( \rho(x) \in \text{Ker } d\pi(x) \).

In this section we treat a lemma on regularity conditions in the stratification theory. Let us recall some regularity conditions.

**Definition 2.1.** (1) We say that the pair \((X,Y)\) is Whitney \((a)\)-regular at \(y_0 \in Y\), if for any sequence of points \(\{x_i\}\) of \(X\) which tends to \(y_0\) such that the sequence of tangent spaces \(\{T_{x_i}X\}\) tends to some plane \(\sigma\) in the Grassman space of \(\dim X\)-planes, then we have \(T_{y_0}Y \subset \sigma\). We say that \((X,Y)\) is \((a)\)-regular if it is \((a)\)-regular at any point \(y_0 \in Y\). (See \([3,6]\) for properties of the Whitney \((a)\)-regularity.)

(2) We say that the pair \((X,Y)\) is \((c)\)-regular at \(y_0 \in Y\) for the control function \(\rho\), if for any sequence of points \(\{x_i\}\) of \(X\) which tends to \(y_0\) such that the sequence of planes \(\{\text{Ker } d\rho(x_i) \cap T_{x_i}X\}\) tends to some plane \(\tau\) in the Grassman space of \(\dim X - 1\)-planes, then we have \(T_{y_0}Y \cap T_{y_0}X \subset \tau\). We say that \((X,Y)\) is \((c)\)-regular for the control function \(\rho\) if it is \((c)\)-regular at any point \(y_0 \in Y\) for the function \(\rho\). (See \([1]\) for properties of the \((c)\)-regularity.)

**Definition 2.2.** We say that the pair \((X,Y)\) satisfies condition \((m)\), if there exists some positive number \(\epsilon > 0\) such that \(\pi, \rho|_{X \cap T^*_Y} : X \cap T^*_Y \to Y \times \mathbb{R}\) is a submersion where \(T^*_Y := \{x \in T_Y \mid \rho(x) < \epsilon\}\).

Let \(y_0 \in Y\), and let \(\{x_i\}\) be an arbitrary sequence of points of \(X\) which tends to \(y_0\) such that the sequence of planes \(\{\text{Ker } d\rho(x_i) \cap T_{x_i}X\}\) tends to some plane \(\tau\) in the Grassman space of \((\dim X - 1)\)-planes. We call such a sequence of points of \(X\) pre-regular. Taking a subsequence of \(\{x_i\}\) if necessary, we may assume that the sequence of planes \(\{\text{Ker } d\rho(x_i)\}\) and \(\{T_{x_i}X\}\) tend to some planes \(\mu\) and \(\sigma\) in the Grassman spaces of \((\dim M - 1)\)-planes and \((\dim X)\)-planes, respectively. We say that the pair \((X,Y)\) satisfies condition \((c_d)\) at \(y_0\), if \(\dim(\mu \cap \sigma) = \dim X - 1\) if and only if \(\mu\) and \(\sigma\) are transverse at \(y_0\) if and only if \(\sigma \not\subset \mu\) if and only if \(\mu^\perp \not\subset \sigma^\perp\).

**Remark 1.** Let \(M = \mathbb{R}^{n+p}\) and \(Y = \mathbb{R}^n \times \{0\}\). Then \(\dim(\mu \cap \sigma) = \dim X - 1\) if and only if \(\mu\) and \(\sigma\) are transverse at \(y_0\) if and only if \(\sigma \not\subset \mu\) if and only if \(\mu^\perp \not\subset \sigma^\perp\).

We show the following lemma, which gives a sufficient condition for the \((c)\)-regularity.

**Lemma 2.3.** Suppose that the pair \((X,Y)\) is \((a)\)-regular at \(y_0 \in Y\) and satisfies condition \((m)\) for a control function \(\rho\) and condition \((c_d)\) at \(y_0 \in Y\). Then \((X,Y)\) is \((c)\)-regular at \(y_0 \in Y\) for the function \(\rho\).

**Proof.** We first recall the setting of the proof of Theorem 2.4 in \([2]\). There is a chart \(\Phi : (U, U \cap Y, y_0) \to (\mathbb{R}^{n+p}, \mathbb{R}^n \times \{0\}, 0)\) for \(Y\) at \(y_0\) such that

1. \(\Phi \circ \pi \circ \Phi^{-1}\) is the projection \((y, t) \mapsto (y, 0)\) from \(\mathbb{R}^{n+p} \to \mathbb{R}^n \times \{0\}\).
2. \(\text{grad } (\rho \circ \Phi^{-1})(y, t) \in \{0\} \times \mathbb{R}^p\), i.e. is orthogonal to \(\mathbb{R}^n \times \{0\}\).
It follows that it is enough to prove the lemma in the case where $M = T^*_Y = \mathbb{R}^{n+p}$ and $Y = \mathbb{R}^n \times \{0\}$, $X$ is Whitney $(a)$-regular over $\mathbb{R}^n \times \{0\}$ at $0 \in \mathbb{R}^{n+p}$, $\pi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ defined by $\pi(y, t) = t$ and $\rho : \mathbb{R}^{n+p} \to \mathbb{R}$ satisfy

- $\rho^{-1}(0) = Y$,
- $(\pi, \rho)|_X : X \to \mathbb{R}^n \times \mathbb{R}$ is a submersion,
- $\text{grad } \rho(y, t) \in \{0\} \times \mathbb{R}^p$ if $(y, t) \in \mathbb{R}^{n+p} \setminus \mathbb{R}^n \times \{0\}$,

and $(X, Y)$ satisfies condition $(c_d)$ at $0 \in \mathbb{R}^{n+p}$.

Let us remark that condition $(m)$ guarantees

$$\dim(\text{Ker } d\rho(x) \cap T_x(X)) = \dim X - 1$$

at any point $x \in X$.

Let $\{x_i\}$ be a sequence of points of $X$ which tends to $0 \in \mathbb{R}^{n+p}$ such that the sequence of planes $\{\text{Ker } d\rho(x_i) \cap T_{x_i}X\}$ tends to some plane $\tau$ in the Grassmann space of $(\dim X - 1)$-planes. Taking a subsequence of $\{x_i\}$ if necessary, we may assume that the sequences of planes $\{\text{Ker } d\rho(x_i)\}$ and $\{T_{x_i}X\}$ tend to some planes $\mu$ and $\sigma$ in the Grassmann spaces of $(n+p-1)$-planes and $(\dim X)$-planes, respectively. Note that $\mathbb{R}^n \times \{0\} \subset \text{Ker } d\rho(x_i)$ for any $i \in \mathbb{N}$. Therefore we have

$$\mu = \lim_{i \to \infty} \text{Ker } d\rho(x_i) \supset \mathbb{R}^n \times \{0\}.$$ 

By the Whitney $(a)$-regularity, we have

$$\sigma = \lim_{i \to \infty} T_{x_i}X \supset \mathbb{R}^n \times \{0\}.$$ 

Therefore we have $\mu \cap \sigma \supset \mathbb{R}^n \times \{0\}$.

On the other hand, $\tau = \lim_{i \to \infty} (\text{Ker } d\rho(x_i) \cap T_{x_i}X)$. It follows that $\tau \subseteq \mu \cap \sigma$. Since $(X, Y)$ satisfies condition $(c_d)$ at $0 \in \mathbb{R}^{n+p}$, we have $\dim(\mu \cap \sigma) = \dim X - 1$. Therefore we have $\tau = \mu \cap \sigma \supset \mathbb{R}^n \times \{0\}$. Thus $(X, Y)$ is $(c)$-regular at $0 \in Y$ for the function $\rho$.

\[\square\]

3. Proofs of Theorems 2.7, 2.8 in [2]

Let $\mathcal{E}_{[\mathfrak{s}]}(n, p)$, $n \geq p$, denote the set of $C^s$ map-germs : $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, and let $j_r f(0)$ denote the $r$-jet of $f$ at $0 \in \mathbb{R}^n$ for $f \in \mathcal{E}_{[\mathfrak{s}]}(n, p)$, $s \geq r$. For $f \in \mathcal{E}_{[\mathfrak{r}]}(n, p)$, $\mathcal{H}_r(f; \mathfrak{w})$ denotes the horn-neighbourhood of $f^{-1}(0)$ of degree $r$ and width $\mathfrak{w}$,

$$\mathcal{H}_r(f; \mathfrak{w}) := \{x \in \mathbb{R}^n : |f(x)| \leq \mathfrak{w}|x|^r\}.$$ 

Let $v_1, \ldots, v_p$ be $p$ vectors in $\mathbb{R}^n$ where $n \geq p$. The Kuo distance $\kappa$ ([H]) is defined by $\kappa(v_1, \ldots, v_p) = \min \{\text{distance of } v_i \text{ to } V_i\}$, where $V_i$ is the span of the $v_j$'s, $j \neq i$.

In the case where $p = 1$, $\kappa(v) = \|v\|$.

**Definition 3.1.** A map-germ $f \in \mathcal{E}_{[\mathfrak{r}]}(n, p)$ satisfies the Kuo condition, if there are positive numbers $C, \alpha, \mathfrak{w} > 0$ such that

$$\kappa(\text{grad } f_1(x), \ldots, \text{grad } f_p(x)) \geq C|x|^{r-1}$$
Theorem 3.3. A map-germ \( f \in \mathcal{E}_{[r+1]}(n, p) \) satisfies the second Kuo condition, if for any map \( g \in \mathcal{E}_{[r]}(n, p) \) with \( j^+[g](0) = j^+[f](0) \), there are positive numbers \( C, \alpha, \underbar{w}, \delta > 0 \) (depending on \( g \)) such that
\[
\kappa(\text{grad } f_1(x), \cdots, \text{grad } f_p(x)) \geq C|x|^r - \delta
\]
in \( H_{r+1}(g; \underbar{w}) \cap \{|x| < \alpha\} \).

Definition 3.2. A map-germ \( f \in \mathcal{E}_{[r+1]}(n, p) \) satisfies the second Kuo condition, if for any map \( g \in \mathcal{E}_{[r]}(n, p) \) with \( j^+[g](0) = j^+[f](0) \), there are positive numbers \( C, \alpha, \underbar{w}, \delta > 0 \) (depending on \( g \)) such that
\[
\kappa(\text{grad } f_1(x), \cdots, \text{grad } f_p(x)) \geq C|x|^r - \delta
\]
in \( H_{r+1}(g; \underbar{w}) \cap \{|x| < \alpha\} \).

Let us recall Theorems 2.7 and 2.8 in [2]. Let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0), n \geq p \), be a \( C^r \) (resp. \( C^{r+1} \)) map, and let \( J \) be a bounded open interval containing \([0, 1]\). For arbitrary \( g \in \mathcal{E}_{[r]}(n, p) \) (resp. \( \mathcal{E}_{[r+1]}(n, p) \)) with \( j^+g(0) = j^+f(0) \), define a \( C^r \) (resp. \( C^{r+1} \)) map \( F : (\mathbb{R}^n \times J, \{0\} \times J) \to (\mathbb{R}^p, 0) \) by \( F(x, t) := f(x) + t(g(x) - f(x)) \).

Let us remark that the Kuo condition (resp. the second Kuo condition) guarantees that \( F^{-1}(0) \setminus \{0\} \times J \) is smooth around \( \{0\} \times J \) if it is not empty. Therefore, if \( F^{-1}(0) \neq \{0\} \times J \) as set-germs at \( \{0\} \times J \),
\[
\Sigma(\mathbb{R}^n \times J) := \{\mathbb{R}^n \times J \setminus F^{-1}(0), F^{-1}(0) \setminus \{0\} \times J, \{0\} \times J\}
\]
gives a stratification of \( \mathbb{R}^n \times J \) around \( \{0\} \times J \) under the assumption of the Kuo condition (resp. the second Kuo condition). In this case, \( \dim(F^{-1}(0) \setminus \{0\} \times J) = n + 1 - p \).

If \( F^{-1}(0) = \{0\} \times J \) as set-germs at \( \{0\} \times J \),
\[
\Sigma(\mathbb{R}^n \times J) := \{\mathbb{R}^n \times J \setminus \{0\} \times J, \{0\} \times J\}
\]
gives a stratification of \( \mathbb{R}^n \times J \) around \( \{0\} \times J \).

Theorem 3.3. (2, Theorem 2.7) If a \( C^r \) map \( f \in \mathcal{E}_r(n, p) \) satisfies the Kuo condition, then the stratification \( \Sigma(\mathbb{R}^n \times J) \) is \((c)\)-regular.

Theorem 3.4. (2, Theorem 2.8) Let \( f \in \mathcal{E}_{[r+1]}(n, p) \). If, for any polynomial map \( h \) of degree \( r + 1 \) such that \( j^+h(0) = j^+f(0) \), there are positive numbers \( C, \alpha, \underbar{w}, \delta > 0 \) (depending on \( h \)) such that
\[
\kappa(\text{grad } f_1(x), \cdots, \text{grad } f_p(x)) \geq C|x|^r - \delta
\]
in \( H_{r+1}(h; \underbar{w}) \cap \{|x| < \alpha\} \), then the stratification \( \Sigma(\mathbb{R}^n \times J) \) is \((c)\)-regular.

Remark. The condition which \( f \in \mathcal{E}_{[r+1]}(n, p) \) in Theorem 3.3 satisfies is equivalent to the second Kuo condition.

Proof of Theorem 3.3 Let us show Theorem 3.3 using Lemma 2.3. In the case where \( F^{-1}(0) = \{0\} \times J \) as set-germs at \( \{0\} \times J \), it is obvious that \( \Sigma(\mathbb{R}^n \times J) \) is \((c)\)-regular stratification. Therefore we consider only the case where \( F^{-1}(0) \neq \{0\} \times J \) as set-germs at \( \{0\} \times J \). We set \( X := \mathbb{R}^n \setminus F^{-1}(0), Y := F^{-1}(0) \setminus \{0\} \times J \) and \( Z := \{0\} \times J \). Then we can easily see that the pairs \((X, Y)\) and \((X, Z)\) are \((c)\)-regular (around \( Z \)). In order to show that the pair \((Y, Z)\) is \((c)\)-regular, we have to check the \((a)\)-regularity, condition \((m)\) and condition \((c_d)\). Here the control function is a non-negative function \( \hat{\rho} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) defined by \( \hat{\rho}(x, t) := x_1^2 + \cdots + x_n^2 \). We can
show the $(a)$-regularity and condition $(m)$ similarly to the proof of Theorem 2.7 in [2]. Therefore it remains to show that the pair $(Y, Z)$ satisfies condition $(c_d)$.

For $t \in J$, define a $C^r$ map $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ by $f_t(x) := F(x, t)$, namely $f_t(x) := f(x) + t(g(x) - f(x))$.

The $r$-jet of $f$ at $0 \in \mathbb{R}^n$, $j^r f(0)$, has a unique polynomial representative $z$ of degree not exceeding $r$. We do not distinguish the $r$-jet $j^r f(0)$ and the polynomial representative $z$ here. We set $q(x) := f(x) - z(x)$ and $r(x) := g(x) - z(x)$, and define $P_t(x) := q(x) + t(r(x) - q(x))$, $t \in J$. Then $f_t(x) = z(x) + P_t(x)$ where $P_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a $C^r$ map such that $j^r P_t(0) = 0$ for $t \in J$.

**Remark 3.**
(1) For $a_1 > 0$, there are positive numbers $b_1, \beta_1 > 0$ with $0 < b_1 \leq a_1$ such that

$$\mathcal{H}_r(f; b_1) \cap \{|x| < \beta_1\} \subset \mathcal{H}_r(f_t; a_1) \cap \{|x| < \beta_1\}$$

for any $t \in J$.

(2) For $a_2 > 0$, there are positive numbers $b_2, \beta_2 > 0$ with $0 < b_2 \leq a_2$ such that

$$\mathcal{H}_r(f; b_2) \cap \{|x| < \beta_2\} \subset \mathcal{H}_r(z; a_2) \cap \{|x| < \beta_2\}.$$

We denote by $V_x$ (resp. $V_{t,x}$) the $p$-dimensional subspace of $\mathbb{R}^n$ spanned by

$$\{\text{grad } z_1(x), \cdots, \text{grad } z_p(x)\} \quad (\text{resp. } \{\text{grad } f_{t,1}(x), \cdots, \text{grad } f_{t,p}(x)\})$$

for $x \in \mathcal{H}_r(f; \overline{w}) \cap \{|x| < \alpha\}$ and $t \in J$. Concerning $V_x$, we proved the following property in [2].

**Assertion 3.5.** ([2], Claim I) Let $\epsilon_1$ be an arbitrary positive number. Then there are positive numbers $\alpha_1, \overline{w}_1$ with $0 < \alpha_1 \leq \alpha$ and $0 < \overline{w}_1 \leq \overline{w}$ such that

$$d(x, V_x) \geq (1 - \epsilon_1)|x| \quad \text{in} \quad \mathcal{H}_r(z; \overline{w}_1) \cap \{|x| < \alpha_1\}.$$

We first recall the proof of Claim III in [2] (since the details are not mentioned in the paper). Let us denote by $v(x)$ and $v_t(x)$ the projections of $x$ on $V_x$ and $V_{t,x}$, respectively. For $x \in \mathcal{H}_r(f; \overline{w}) \cap \{|x| < \alpha\}$ (and $t \in J$), we consider $\{N_{t,1}(x), \cdots, N_{t,p}(x)\}$ the basis of $V_x$ constructed as follows:

$$N_j(x) := \text{grad } z_j(x) - \tilde{N}_j(x) \quad (1 \leq j \leq p),$$

where $\tilde{N}_j(x)$ is the projection of $\text{grad } z_j(x)$ to the subspace $V_{x,j}$ spanned by the $\text{grad } z_k(x)$, $k \neq j$, and $\{N_{t,1}(x), \cdots, N_{t,p}(x)\}$ the corresponding basis of $V_{t,x}$. Then we have the following assertion.

**Assertion 3.6.** ([2], Claim IV) For any $\epsilon_2 > 0$, there are positive numbers $\alpha_2, \overline{w}_2$ with $0 < \alpha_2 \leq \alpha$ and $0 < \overline{w}_2 \leq \overline{w}$ such that the following inequality holds:

$$(1 + \epsilon_2)|N_j(x)| \geq |N_{t,j}(x)| \geq (1 - \epsilon_2)|N_j(x)| \quad (1 \leq j \leq p)$$

for $x \in \mathcal{H}_r(f; \overline{w}_2) \cap \{|x| < \alpha_2\}$ and $t \in J$.

By Lemma 3.2 in T.-C. Kuo [4], we have

$$v(x) = \sum_{j=1}^p x, \text{grad } z_j(x) > \frac{N_j(x)}{|N_j(x)|^2}, \quad v_t(x) = \sum_{j=1}^p x, \text{grad } f_{t,j}(x) > \frac{N_{t,j}(x)}{|N_{t,j}(x)|^2}.$$
for \( x \in \mathcal{H}_r(f; \overline{w}) \cap \{|x| < \alpha \} \) and \( t \in J \). From the proof of Claim I in \([2]\), we can assume that for any \( \epsilon_3 > 0 \), there are positive numbers \( \alpha_3, \overline{w}_3 \) with \( 0 < \alpha_3 \leq \alpha \) and \( 0 < \overline{w}_3 \leq \overline{w} \) such that

\[
(3.1) \quad |v(x)| \leq \sum_{j=1}^{p} \frac{|< x, \text{grad} z_j(x) >|}{|N_j(x)|} \leq \epsilon_3 |x| \quad \text{in} \quad \mathcal{H}_r(z; \overline{w}_3) \cap \{|x| < \alpha_3\}.
\]

Since \( f_t(x) = z(x) + P_t(x), t \in J \), there are positive numbers \( \alpha_4, \overline{w}_4 \) with \( 0 < \alpha_4 \leq \min\{\alpha_2, \alpha_3\} \) and \( 0 < \overline{w}_4 \leq \min\{\overline{w}_2, \overline{w}_3\} \) such that for \( x \in \mathcal{H}_r(f; \overline{w}_4) \cap \{|x| < \alpha_4\} \) and \( t \in J \), the following inequalities hold:

\[
|v_t(x)| \leq \sum_{j=1}^{p} \frac{|< x, \text{grad} f_{t,j}(x) >|}{|N_{t,j}(x)|} \leq \sum_{j=1}^{p} \frac{|< x, \text{grad} z_j(x) >|}{|N_{t,j}(x)|} + \frac{\sum_{j=1}^{p} |< x, \text{grad} P_{t,j}(x) >|}{|N_{t,j}(x)|} \leq \frac{\epsilon_3}{1 - \epsilon_2} |x| + \sum_{j=1}^{p} \frac{|< x, \text{grad} P_{t,j}(x) >|}{|N_{t,j}(x)|}.
\]

Note that

\[
\frac{\partial P_{t,j}}{\partial x_i}(0) = 0 \quad (1 \leq i \leq n, \ 1 \leq j \leq p) \quad \text{for} \ t \in J.
\]

Therefore, for any \( \epsilon_4 > 0 \), there are positive numbers \( \alpha_5, \overline{w}_5 \) with \( 0 < \alpha_5 \leq \alpha_4 \) and \( 0 < \overline{w}_5 \leq \overline{w}_4 \) such that

\[
(3.2) \quad |v_t(x)| \leq \epsilon_4 |x| \quad \text{for} \ x \in \mathcal{H}_r(f; \overline{w}_5) \cap \{|x| < \alpha_5\} \quad \text{and} \ t \in J
\]

under the assumption of the Kuo condition. Then, by (3.1) and (3.2), we have the following assertion.

**Assertion 3.7.** \([2], \text{Claim III}\) For any \( \epsilon_5 > 0 \), there are positive numbers \( \alpha_6, \overline{w}_6 \) with \( 0 < \alpha_6 \leq \alpha_5 \) and \( 0 < \overline{w}_6 \leq \overline{w}_5 \) such that

\[
|d(x, V_{t,x}) - d(x, V_x)| \leq |v_t(x) - v(x)| \leq |v_t(x)| + |v(x)| \leq \epsilon_5 |x|
\]

for \( x \in \mathcal{H}_r(f; \overline{w}_6) \cap \{|x| < \alpha_6\} \) and \( t \in J \).

The most important result of §3 in \([2]\) follows from Assertions 3.5 and 3.7.

**Lemma 3.8.** \([2], \text{Claim II}\) There are positive numbers \( \alpha_0, \overline{w}_0 \) with \( 0 < \alpha_0 \leq \min\{\alpha_1, \alpha_6\} \) and \( 0 < \overline{w}_0 \leq \min\{\overline{w}_1, \overline{w}_6\} \) such that

\[
d(x, V_{t,x}) \geq \frac{1}{2} |x| \quad \text{for} \ x \in \mathcal{H}_r(f; \overline{w}_0) \cap \{|x| < \alpha_0\} \quad \text{and} \ t \in J.
\]

We next denote by \( W(x,t) \) the \( p \)-dimensional subspace of \( \mathbb{R}^n \times \mathbb{R} \) spanned by \{\text{grad} \( F_i(x,t), \cdots, \text{grad} \( F_p(x,t) \)\} for \( x \in \mathcal{H}_r(f; \overline{w}) \cap \{|x| < \alpha\} \) and \( t \in J \).

Then we can show the following lemma.

**Lemma 3.9.** There are positive numbers \( \alpha_{11}, \overline{w}_{11} \) such that

\[
d((x,0), W(x,t)) \geq \frac{1}{4} |(x,0)| \quad \text{for} \ x \in \mathcal{H}_r(f; \overline{w}_{11}) \cap \{|x| < \alpha_{11}\} \quad \text{and} \ t \in J.
\]
Proof. Let us remark that
\[
\text{grad } F_j(x, t) = (\text{grad } f_{t,j}(x), \frac{\partial F_j}{\partial t}(x, t)), \quad 1 \leq j \leq p.
\]
We denote by \(U_{(x, t)}\) the \(p\)-dimensional subspace of \(\mathbb{R}^n \times \mathbb{R}\) spanned by
\[
\{(\text{grad } f_{t,1}(x), 0), \ldots, (\text{grad } f_{t,p}(x), 0)\} \quad \text{for } x \in \mathcal{H}_r(f; \overline{\omega}) \cap \{|x| < \alpha\} \text{ and } t \in J.
\]
By Lemma 3.8 we have
\[
d((0, x), U_{(x, t)}) = d(x, V_{(t, x)}) \geq \frac{1}{2} |x| \quad \text{for } x \in \mathcal{H}_r(f; \overline{\omega}_0) \cap \{|x| < \alpha_0\} \text{ and } t \in J.
\]
Let \(u(x, t)\) and \(\omega(x, t)\) be the projections of \((x, 0)\) on \(U_{(x, t)}\) and \(W_{(x, t)}\), respectively.
Then we have
\[
d((0, x), U_{(x, t)}) = |(x, 0) - u(x, t)|, \quad d((0, x), W_{(x, t)}) = |x, 0 - \omega(x, t)|
\]
for \(x \in \mathcal{H}_r(f, \overline{\omega}) \cap \{|x| < \alpha\} \text{ and } t \in J.\) Therefore we have
\[
|d((0, x), U_{(x, t)}) - d((0, x), W_{(x, t)})| \leq |u(x, t) - \omega(x, t)| \leq |u(x, t)| + |\omega(x, t)|
\]
for \(x \in \mathcal{H}_r(f, \overline{\omega}) \cap \{|x| < \alpha\} \text{ and } t \in J.
\]
For any \(x \in \mathcal{H}_r(f; \overline{\omega}) \cap \{|x| < \alpha\} \text{ and } t \in J,\) let us consider \(\{M_1(x, t), \ldots, M_p(x, t)\}\) the basis of \(U_{(x, t)}\) constructed as follows:
\[
M_j(x, t) := (\text{grad } f_{t,j}(x), 0) - \tilde{M}_j(x, t) \quad (1 \leq j \leq p),
\]
where \(\tilde{M}_j(x, t)\) is the projection of \((\text{grad } f_{t,j}(x), 0)\) to the subspace \(U_{(x, t), j}\) spanned by \((\text{grad } f_{t,k}(x), 0), k \neq j,\) and let \(\{L_1(x, t), \ldots, L_p(x, t)\}\) be the corresponding basis of \(W_{(x, t)}\). By definition, we have
\[
|\text{grad } F_j(x, t) - (\text{grad } f_{t,j}(x), 0)| = |\frac{\partial F_j}{\partial t}(x, t)| = |g_j(x) - f_j(x)|
\]
where \(j^*(g_j - f_j)(0) = 0 \quad (1 \leq j \leq p).\) Therefore there are positive numbers \(\alpha_\tau, \overline{\omega}_\tau\) with \(0 < \alpha_\tau \leq \alpha\) and \(0 < \overline{\omega}_\tau \leq \overline{\omega}\) such that for any \(\lambda_j,
\[
\frac{|\sum_j \lambda_j (\text{grad } F_j(x, t) - (\text{grad } f_{t,j}(x), 0))|}{|\sum_j \lambda_j (\text{grad } f_{t,j}(x), 0)|} \to 0 \quad \text{as } x \to 0
\]
in \(x \in \mathcal{H}_r(f, \overline{\omega}_\tau) \cap \{|x| < \alpha_\tau\}\) (uniformly for \(t \in J\)) under the assumption of the Kuo condition. Then, using a similar argument to the proof of Claim IV in [2] (see Assertion 3.6 above), we can show the following assertion.

Assertion 3.10. For any \(\epsilon_6 > 0,\) there are positive numbers \(\alpha_8, \overline{\omega}_8\) with \(0 < \alpha_8 \leq \alpha_\tau\) and \(0 < \overline{\omega}_8 \leq \overline{\omega}_\tau\) such that the following inequality holds:
\[
(1 + \epsilon_6)|M_j(x, t)| \geq |L_j(x, t)| \geq (1 - \epsilon_6)|M_j(x, t)| \quad (1 \leq j \leq p)
\]
for \(x \in \mathcal{H}_r(f; \overline{\omega}_8) \cap \{|x| < \alpha_8\} \text{ and } t \in J.
\]
For any \(x \in \mathcal{H}_r(f; \overline{\omega}) \cap \{|x| < \alpha\} \text{ and } t \in J,\)
\[
u(x, t) = \sum_{j=1}^{p} \left< (x, 0), (\text{grad } f_{t,j}(x), 0) \right> \frac{M_j(x, t)}{|M_j(x, t)|^2},
\]
\( \omega(x, t) = \sum_{j=1}^{p} < (x, 0), \text{grad } F_j(x, t) > \frac{L_j(x, t)}{|L_j(x, t)|^2}. \)

By construction, \( < (x, 0), (\text{grad } f_{i,j}(x), 0) >= < x, \text{grad } f_{i,j}(x) > \) and \( |M_j(x, t)| = |N_{i,j}(x)|, j = 1, \cdots, p, \) for \( t \in J. \) It follows from (3.2) that

\[
(3.5) \quad |u(x, t)| \leq \sum_{j=1}^{p} \frac{| < (x, 0), (\text{grad } f_{i,j}(x), 0) > |}{|M_j(x, t)|} \leq \epsilon_4 |x|
\]

for \( x \in \mathcal{H}_r(f; \overline{w}_5) \cap \{|x| < \alpha_3\} \) and \( t \in J. \) On the other hand,

\[
|\omega(x, t)| \leq \sum_{j=1}^{p} \frac{| < (x, 0), F(x, t) > |}{|L_j(x, t)|} = \sum_{j=1}^{p} \frac{| < (x, 0), (\text{grad } f_{i,j}(x), 0) > |}{|L_j(x, t)|}
\]

for \( x \in \mathcal{H}_r(f; \overline{w}) \cap \{|x| < \alpha\} \) and \( t \in J. \) By Assertion 3.10, we have

\[
(3.6) \quad |\omega(x, t)| \leq \frac{\epsilon_4}{1 - \epsilon_6} |x| \quad \text{for } x \in \mathcal{H}_r(f; \overline{w}_9) \cap \{|x| < \alpha_9\} \quad \text{and } t \in J,
\]

where \( \alpha_9 = \min\{\alpha_5, \alpha_8\} \) and \( \overline{w}_9 = \min\{\overline{w}_5, \overline{w}_8\}. \)

By (3.4), (3.5) and (3.6), we have the following assertion.

**Assertion 3.11.** For any \( \epsilon_7 > 0, \) there are positive numbers \( \alpha_{10}, \overline{w}_{10} \) with \( 0 < \alpha_{10} \leq \alpha_9 \) and \( 0 < \overline{w}_{10} \leq \overline{w}_9 \) such that

\[
|d((x, 0), U(x, t)) - d((x, 0), W(x, t))| \leq \epsilon_7 |x| \quad \text{for } x \in \mathcal{H}_r(f; \overline{w}_{10}) \cap \{|x| < \alpha_{10}\} \quad \text{and } t \in J.
\]

By (3.3) and Assertion 3.11, there are positive numbers \( \alpha_{11}, \overline{w}_{11} \) with \( 0 < \alpha_{11} \leq \alpha_{10} \) and \( 0 < \overline{w}_{11} \leq \overline{w}_{10} \) such that

\[
d((x, 0), W(x, t)) \geq \frac{1}{4} |(x, 0)| \quad \text{for } x \in \mathcal{H}_r(f; \overline{w}_{11}) \cap \{|x| < \alpha_{11}\} \quad \text{and } t \in J.
\]

By Lemma 3.9, we have

\[
d\left(\frac{(x, 0)}{|(x, 0)|}, W(x, t)\right) \geq \frac{1}{4} \quad \text{for } x \in \mathcal{H}_r(f; \overline{w}_{11}) \cap \{0 < |x| < \alpha_{11}\} \quad \text{and } t \in J.
\]

Let \( \ell_{(x,t)} \) be the the 1-dimensional subspace of \( \mathbb{R}^n \times \mathbb{R} \) spanned by \( \text{grad } \hat{\rho}(x, t) \) for \( x \neq 0 \) and \( t \in \mathbb{R}. \) Here \( \text{grad } \hat{\rho}(x, t) = (2x_1, \cdots, 2x_n, 0). \) Therefore we have

\[
(3.7) \quad \overline{d}(\ell_{(x,t)}, W(x, t)) \geq \frac{1}{4} \quad \text{for } x \in \mathcal{H}_r(f; \overline{w}_{11}) \cap \{0 < |x| < \alpha_{11}\} \quad \text{and } t \in J.
\]

Here

\[
\overline{d}(\ell, W) := \max_{||v||=1} \{d(v, W) \mid v \in \ell\}
\]

for subspaces \( \ell, W \) of \( \mathbb{R}^m \) with \( \dim \ell \leq \dim W. \) Note that \( \ell \subset W \) if and only if \( \overline{d}(\ell, W) = 0. \)

Let \( (0,t_0) \in Z = \{0\} \times J, \) and let \( \{(x_i, t_i)\} \) be any pre-regular sequence of points of \( Y = F^{-1}(0) \setminus \{0\} \times J \) which tends to \( (0, t_0). \) Namely, the sequence of planes \( \{\text{Ker } \text{grad } \hat{\rho}((x_i, t_i)) \cap T_{(x_i, t_i)} Y\} \) tends to some plane in the Grassmann space of \((n - p)\)-planes. Taking a subsequence of \( \{(x_i, t_i)\} \) if necessary, we may assume that the sequence of planes \( \{\text{Ker } \text{grad } \hat{\rho}((x_i, t_i))\} \) and \( \{T_{(x_i, t_i)} Y\} \) tend to some planes \( \mu \)
and $\sigma$ in the Grassmann spaces of $n$-planes and $(n + 1 - p)$-planes, respectively. By (3.7), we have

$$d(\ell(x_i, t_i), W(x_i, t_i)) \geq \frac{1}{4} \quad \text{for} \quad (x_i, t_i) \in Y \cap \{0 < |x| < \alpha_1\}.$$

Since $W(x_i, t_i) = (T_{(x_i, t_i)}Y)\perp$ and $\ell(x_i, t_i) = (\ker \text{grad} \hat{\rho}((x_i, t_i)))\perp$, it follows that $\mu \perp \not\subset \sigma \perp$. By Remark 1, this implies that $(Y, Z)$ is $(c_d)$-regular at $(0, t_0)$.

This completes the proof of Theorem 3.3.

The proof of Theorem 3.4 goes almost in the same way as the above argument.

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