REPRESENTATION OF STATIONARY AND STATIONARY INCREMENT PROCESSES VIA LANGEVIN EQUATION AND SELF-SIMILAR PROCESSES

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ABSTRACT. Let $W_t$ be a standard Brownian motion. It is well-known that the Langevin equation $dU_t = -\theta U_t dt + dW_t$ defines a stationary process called Ornstein-Uhlenbeck process. Furthermore, Langevin equation can be used to construct other stationary processes by replacing Brownian motion $W_t$ with some other process $G$ with stationary increments. In this article we prove that essentially all stationary processes arise from a Langevin equation with certain noise $G_{\theta}$. We also establish a one-to-one connection between self-similar processes and stationary increment processes which naturally extends the connection between self-similar processes and stationary processes established by Lamperti. Discrete analogies of our results are given and applications are discussed.

Keywords: Stationary processes, Stationary increment processes, self-similar processes, Lamperti transform, Langevin equation

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1. INTRODUCTION

Let $G$ be a continuous process with stationary increments. Then, by extending to $(-\infty, 0)$ if necessary, one can construct stationary process by exponential transform

$$U_t = e^{-\theta t} \int_{-\infty}^{t} e^{\theta s} dG_s.$$  

Such process can also be viewed as the unique stationary solution to Langevin equation

$$dU_t = -\theta U_t dt + dG_t.$$  

Furthermore, Langevin type equations have also important applications in statistical physics.

Another useful tool to construct stationary processes is via Lamperti theorem [17] which states that each $H$-self-similar process $X$ can be written as a Lamperti-transform $X = \mathcal{L}_H Y$, where $Y$ is a stationary process. Moreover, Lamperti-transform is invertible and hence stationary processes can be constructed from $H$-self-similar process via inverse transform $Y = \mathcal{L}_{H^{-1}} X$. Furthermore, $H$-self-similar processes and stationary processes are important classes of processes in many different fields. Especially, self-decomposable laws and Levy processes are important classes for both theory and applications, and both have drawn lot of attention in the literature (for details we refer to [3, 21] and references therein), and their connection to self-similar
processes is studied in [13, 24] to name few. Especially, Jeanblanc et al. [12] used $H$-self-similar processes to connect two different representations derived by [13, 24] and by [20] for self-decomposable laws where the first representation is in terms of Levy processes (so called background driving Levy process) and the second representation is in terms of $H$-self-similar processes. As another powerful application of $H$-self-similar processes we mention study by Kyprianou et al. [16] (see also references therein) where authors studied hitting distributions of stable processes. For more details on $H$-self-similar processes we refer to monographs [8, 9, 23] dedicated to the subject.

A process with special interest is the case of fractional Brownian motion with $H \in (0, 1)$ and it has been widely studied (see e.g. [4, 7, 14]). Recall that $B^H$ is the only Gaussian process which is $H$-self-similar and it has stationary increments. Consequently, both approaches can be used to construct stationary processes. However, it is also known that the resulting processes are the same (in law) only in the case $H = \frac{1}{2}$, i.e. in the case of standard Brownian motion. On the other hand, it was proved by Kaarakka and Salminen [14] that even the Lamperti transform of fBm can be defined as a solution to Langevin type equation with some additional driving noise $G$. Furthermore, statistical problems for fractional Ornstein-Uhlenbeck processes have been studied at least in [2, 3, 6, 11, 15, 25]. For research related to more general self-similar Gaussian processes, see also [19, 26].

In this article we study inverse Lamperti-transforms of arbitrary self-similar processes $X$ and prove that all resulting stationary processes $U = \mathcal{L}^{-1}_H X$ can be described as a unique stationary solution to Langevin equation (1.1) with some noise process $G$ belonging to a certain class $\mathcal{G}_H$. Furthermore, by applying Lamperti theorem we show that every continuous stationary process $U$ can be described as a solution to Langevin equation (1.1) with certain noise process $G$ from class $\mathcal{G}_H$ thus connecting two mentioned approaches in a natural way. Moreover, for fixed parameter $H$ the noise process $G$ is uniquely determined. As such, for fixed parameter $H$ we obtain natural one-to-one correspondence between stationary process $U$, $H$-self-similar process $X$ and processes $G \in \mathcal{G}_H$. We also present discrete analogies to our results. While our main results can be proved with surprisingly simple arguments, the results are not acknowledged in the literature and we believe that they might lead to significant applications as well as new theoretical results. To illustrate the power of the results, we list some straightforward consequences and possible applications. Firstly, we prove that all stationary processes arise uniquely from Langevin equations which gives strong motivation to study such equations while Langevin equations are already widely studied in the literature with different driving forces. Secondly, our results may have important applications in modelling. In particular, we provide new techniques to construct self-similar processes from stationary increment processes and vice versa. As a concrete example, we introduce a generalised AR(1) model which covers all stationary discrete models. Thirdly, our results can be used to obtain new interesting theoretical results. As an illustration of this, we show how all results presented in [12] follows from the results of present paper applied to a special case.
The rest of the paper is organised as follows. In section 2 we introduce our notation and preliminary results, and in section 3 we present and prove our main results. Section 4 is devoted to applications and examples, and we end the paper with a short discussion.

2. Preliminaries

Throughout the paper we denote $X \overset{\text{law}}{=} Y$ if finite dimensional distributions of $X$ and $Y$ are the same. Furthermore, assume that $X$ is a continuous process. Consequently, integrals of form $\int_s^t f(u)dX_u$ are well-defined as a limit of Riemann-Stieltjes sums for every compact interval $[s, t]$ and every function $f$ which is of bounded variation. Moreover, we have integration by parts formula

$$\int_s^t f(u)dX_u = f(t)X_t - f(s)X_s - \int_s^t X_u df(u).$$

Note also that if $X$ has stationary increments it immediately follows that

$$\int_s^t f(u)dX_u \overset{\text{law}}{=} \int_0^0 f(u + t)dX_u.$$

We also consider indefinite integrals of type $\int_{-\infty}^t f(u)dX_u$ which are defined as

$$\int_{-\infty}^t f(u)dX_u := \lim_{n \to -\infty} \int_{-\infty}^n f(u)dX_u$$

provided that the limit exists along suitably chosen sequence. In particular, if $X$ has stationary increments we obtain

$$\int_{-\infty}^t f(u)dX_u \overset{\text{law}}{=} \int_{-\infty}^0 f(u + t)dX_u.$$

Similarly, if $X$ is cadlag process, i.e. right-continuous with left limits, then the integrals are well-defined and the above analysis is still valid. Thus we will assume for the rest of the paper that underlying processes are cadlag processes and hence integrals can be defined and integration by parts hold. Furthermore, all results presented in this paper are valid provided that corresponding integrals are well-defined and satisfies integration by parts formula.

Next we recall definition of Lamperti transform and its inverse together with the famous Lamperti theorem. First we recall the definition of self-similar processes.

**Definition 2.1.** Let $H > 0$. A process $X = (X_t)_{t \in [0,T]}$ is $H$-self-similar if

$$X_{at} \overset{\text{law}}{=} a^H X_t.$$

**Definition 2.2.** Let $X = (X_t)_{t \in [0,T]}$ and $U = (U_t)_{t \in (-\infty, \log T]}$ be stochastic processes. We define

$$(\mathcal{L}_H U)_t = t^H U_{t}^0, \quad t \in (0, T]$$

and its inverse

$$(\mathcal{L}_H^{-1} X)_t = e^{-Ht} X_{e^t}, \quad t \in (-\infty, \log T].$$

The result due to Lamperti [17] gives a one-to-one correspondence between stationary processes and $H$-self-similar processes.
Theorem 2.1 (Lamperti). Let $U = (U_t)_{t \in (-\infty, \log T]}$ be a stationary process. Then $X = \mathcal{L}_H U$ is $H$-self-similar. Conversely, if $X = (X_t)_{t \in [0, T]}$ is $H$-self-similar, then $U = \mathcal{L}_H^{-1} X$ is stationary.

Remark 2.1. Note that for a given $H$-self-similar process $X$ the process $U_t = e^{-\theta t} X_{e^{\theta t}}$ also defines a stationary process. However, for our purposes we only consider the case $\theta = 1$.

We also recall some basic facts on relation for cardinalities of sets $\mathcal{Y}$ and $\mathcal{Z}$ which measures the "number of elements" in the set. For details on the topic, we refer to [22].

Definition 3.1. Let $G = (G_t)_{t \geq 0}$ be a process with stationary increments. We define its two-sided version $\tilde{G} = (\tilde{G}_t)_{t \in \mathbb{R}}$ by setting $\tilde{G}_t = \tilde{G}_{-t}$ for $t < 0$, where $\tilde{G}$ is an independent copy of $G$. The class of such processes is denoted by $\mathcal{G}$.

Definition 3.2. Let $H > 0$ be fixed. We denote by $\mathcal{G}_H$ the class of processes $G \in \mathcal{G}$ which also satisfy

$$\lim_{t \to -\infty} e^{Ht} |G_t| = 0.$$  \hfill (3.1)

Remark 3.1. In general it is not clear which processes $G \in \mathcal{G}$ belongs also to $\mathcal{G}_H$. This is the topic of subsection 3.2.

The following lemma is evident and follows from integration by parts.
Lemma 3.1. Let \( G \in \mathcal{S}_H \). Then
\[
\int_{-\infty}^{0} e^{Hs} dG_s
\]
is a well-defined almost surely finite random variable.

The following simple lemma will be applied in the sequel.

Lemma 3.2. Let \( X \) be \( H \)-self-similar. Then for any \( N \) the process
\[
Y_t = \int_{N}^{t} e^{-Hs} dX_e^s
\]
has stationary increments.

Proof. Let \( t, s \geq N \) be arbitrary. We have
\[
Y_t - Y_s = \int_{s}^{t} e^{-Hu} dX_e^u
= \int_{s+h}^{t+h} e^{-H(v-h)} dX_{e^v-\langle h \rangle}
\]
and by self-similarity of \( X \) we obtain
\[
dX_{e^{-\langle h \rangle v}} \overset{\text{law}}{=} e^{-hH} dX_{e^v}.
\]
Hence
\[
Y_t - Y_s \overset{\text{law}}{=} Y_{t+h} - Y_{s+h}.
\]
\[\square\]

For the rest of the paper, all stationary processes \( U \) are defined on the whole real line \( \mathbb{R} \) unless otherwise specified, and all \( H \)-self-similar processes \( X \) are defined on the half line \([0, \infty)\). We also denote by \( \mathcal{U} \) and \( \mathcal{X}_H \) the sets of stationary processes and \( H \)-self-similar processes, respectively.

Consider now the Langevin equation
\[
dU_t = -HU_t dt + dG_t
\]
on \( t \in [0, T] \) with some \( G \in \mathcal{S}_H \) and initial condition
\[
U_0 = \int_{-\infty}^{0} e^{Hs} dG_s.
\]
Now the solution can be expressed as
\[
U_t = e^{-Ht} \int_{-\infty}^{t} e^{Hs} dG_s,
\]
and by the stationarity of the increments of \( G \) we obtain that the process \( U \) is stationary. Moreover, the process \( U_t \) can be extended to whole real line by equation (3.4). In other words, given any \( G \in \mathcal{S}_H \), there is a unique stationary process \( U_t \) which obeys dynamics (3.3). As our first main theorem we prove that every stationary \( U \) arises uniquely from a Langevin equation which is rather surprising since it means that every stationary process \( U \) has a representation (3.4) with some \( G \in \mathcal{S}_H \).
Theorem 3.1. Let $H > 0$ be fixed. Then

$$|U| = |\mathcal{G}_H|.$$  

Furthermore, $U \in \mathcal{U}$ can be expressed as the unique stationary solution to Langevin equation with noise $G \in \mathcal{G}_H$ and the bijection is given by relations

\begin{align}
(3.5) & \quad U_t = e^{-Ht} \int_{-\infty}^{t} e^{Hs} dG_s, \\
(3.6) & \quad G_t = U_t - U_0 + H \int_0^t U_v dv.
\end{align}

Proof. Define a process $X = \mathcal{L}_H U$. Now since $U$ is stationary, it follows by Lamperti theorem that $X$ is $H$-self-similar. Moreover, we obtain

$$dU_t = -HU_t dt + e^{-Ht} dX_e.$$  

Define now a process $Y$ by

$$Y_t = \int_0^t e^{-Hs} dX_e.$$  

Consequently, we have

$$dU_t = -HU_t dt + dY_t,$$

and $Y_t$ has stationary increments by Lemma 3.2. Furthermore, it is straightforward to check that $Y \in \mathcal{G}$. Moreover, we have

$$dY_t = e^{-Hs} dX_e$$

and

$$\int_{-\infty}^0 e^{Hs} dY_s = \int_{-\infty}^0 dX_e = X_1$$

which implies that $Y \in \mathcal{G}_H$. To show the uniqueness, let $H$ be fixed and assume there exists two processes $G^1 \in \mathcal{G}_H$ and $G^2 \in \mathcal{G}_H$ which yields same solution $U$ applied to (3.3). Hence

$$e^{Ht} U_t = \int_{-\infty}^{t} e^{Hu} dG^1_u = \int_{-\infty}^{t} e^{Hu} dG^2_u.$$  

In particular, for every $s < t$ we have

$$\int_s^t e^{Hu} dG^1_u = \int_s^t e^{Hu} dG^2_u$$

which together with integration by parts yields

$$e^{Ht}(G^1_t - G^2_t) - e^{Hs}(G^1_s - G^2_s) = H \int_s^t e^{Hu}(G^1_u - G^2_u) du.$$  

Denoting $g(t) = e^{Ht}(G^1_t - G^2_t)$ we obtain that for each fixed $s$ we have

$$g(t) - g(s) = \int_s^t Hg(u) du.$$  

Now it is well-known that the only solution to such equation is

$$g(t) = Ce^{Ht}$$
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which implies that there exists a constant \( c \) such that \( G^1_t - G^2_t = c \) for every \( t \). Applying this with \( t = 0 \) we obtain \( c = G^1_0 - G^2_0 = 0 \). To conclude we note that representation (3.6) is direct consequence from dynamics (3.3). □

Next we give some representations which are straightforward consequences of Theorem 3.1.

**Proposition 3.1.** Let \( H > 0 \) be fixed. A process \( X_t = (X_t)_{t \geq 0} \) is \( H \)-self-similar if and only if

\[
X_t = \int_{-\infty}^{\log t} e^{Hu} dG_u,
\]

for some \( G \in \mathcal{S}_H \). Furthermore, the process \( G \) in the representation is unique.

**Proof.** "if": First note that \( X_t \) defined by (3.7) is well-defined since \( G \in \mathcal{S}_H \). Moreover,

\[
X_{\alpha t} = \int_{-\infty}^{\log(\alpha t)} e^{Hs} dG_s
\]

\[=
\int_{-\infty}^{\log t + \log \alpha} e^{Hs} dG_s
\]

\[=
\int_{-\infty}^{\log t} e^{Hu + H \log \alpha} dG_{u + \log \alpha}
\]

\[\overset{\text{law}}{=} \alpha H \int_{-\infty}^{\log t} e^{Hu} dG_u
\]

\[=
\alpha^H X_t.
\]

Hence \( X \) is \( H \)-self-similar.

"only if": Assume that \( X \) is \( H \)-self-similar and define \( U = \mathcal{L}_H^{-1} X \). Hence \( U \) is stationary, and by Theorem 3.1 there exists an unique \( G \in \mathcal{S}_H \) such that

\[U_t = e^{-Ht} \int_{-\infty}^{t} e^{Hu} dG_u.
\]

On the other hand,

\[X_t = t^H U_{t \log t}
\]

from which the statement follows. □

Conversely, the following theorem gives representation of process \( G \in \mathcal{S}_H \) in terms of self-similar process.

**Proposition 3.2.** Let \( H > 0 \) be fixed and \( G \) be a process. Then \( G \in \mathcal{S}_H \) if and only if it admits a representation

\[
G_t = \int_{0}^{t} e^{-Hu} dX_s, \quad t \in \mathbb{R}
\]

for some \( H \)-self-similar process \( X \). Furthermore, the process \( X \) in the representation is unique.
Proof. "if": Assume (3.8) holds. Then the increments of $G$ are stationary by Lemma 3.2. Moreover,

$$\int_{-\infty}^{0} e^{Hs} dG_s = X_1$$

which implies that $G \in \mathcal{G}_H$.

"only if": Assume that $G \in \mathcal{G}_H$. Consequently, by Theorem 3.1 we obtain that the process $U_t$ as a solution to Langevin equation is stationary. Now set $X = \mathcal{L}_H U$ which yields

$$U_t = (\mathcal{L}_H^{-1} X)_t = e^{-Ht} X_{e^t}.$$  

Consequently, we get

$$dU_t = -HU_t dt + e^{-Ht} dX_{e^t}.$$  

Hence by defining

$$Y_t = \int_{0}^{t} e^{-Hu} dX_{e^u}$$

we obtain

$$dU_t = -HU_t dt + dY_t.$$  

On the other hand, $U_t$ is a solution to

$$dU_t = -HU_t dt + dG_t.$$  

Clearly this implies $G = Y$ almost surely. □

Let now $X = (X_t)_{t \geq 0}$ and $G = (G_t)_{t \in \mathbb{R}}$ be processes. We introduce operators $A_H G$ and its inverse $A_H^{-1}$ by

$$(A_H G)_t = \int_{-\infty}^{t} e^{Hs} dG_s, \quad t > 0$$

and

$$(A^{-1}_{H} X)_t = \int_{0}^{t} e^{-Hs} dX_{e^s}, \quad t \in \mathbb{R}.$$  

Clearly the operators $A_H$ and $A_H^{-1}$ are well-defined. Furthermore, for every $G$ we have

$$A_H^{-1} A_H G = G$$

and for every $X$ we have

$$A_H A_H^{-1} X = X.$$  

With these operators we are able to summarize Propositions 3.1 and 3.2 to our second main theorem which is close in spirit to Lamperti Theorem 2.1.

**Theorem 3.2.** Let $H > 0$ be fixed and let $G \in \mathcal{G}_H$. Then $A_H G$ is $H$-self-similar. Conversely, let $X$ be $H$-self-similar. Then $A_H^{-1} X \in \mathcal{G}_H$. In particular, we have

$$|G_H| = |X_H|$$

and the bijection is given by relations $G = A_H^{-1} X$ and $X = A_H G$.  

3.1. **Analogy in discrete time.** In this section we prove analogous results in discrete time case. Not surprisingly, the proofs and results are quite similar and the only problematic part is to find discrete analogies to different concepts. We also use same notation with additional superindex \(d\) to emphasize the discrete aspect of the analysis. For example, \(U^d\) denotes the class of discrete time stationary processes indexed on some subset \(T\) of integer numbers \(\mathbb{Z}\). We also use short notation \(\Delta_k G = G_k - G_{k-1}\).

The discrete analogy to class \(\mathcal{G}_H\) is straightforward.

**Definition 3.3.** Let \(G = (G_n)_{n\geq 0}\) be a process with stationary increments. We define its two-sided version \(G^d = (G_n)_{n\in \mathbb{Z}}\) by setting \(G_n = \tilde{G}_{-n}\) for \(n < 0\), where \(\tilde{G}\) is an independent copy of \(G\). The class of such processes is denoted by \(\mathcal{G}^d\).

**Definition 3.4.** Let \(H > 0\) be fixed. We denote by \(\mathcal{G}^d_H\) the class of processes \(G \in \mathcal{G}^d\) which also satisfy

\[
\left| \sum_{k=-\infty}^{0} e^{Hk} \Delta G_k \right| < \infty.
\]

(3.9)

Note that if \(G\) belongs to \(\mathcal{G}^d_H\), then the process defined by

\[
U_n = e^{-Hn} \sum_{k=-\infty}^{n} e^{Hk} \Delta_k G
\]

(3.10)

is well-defined and stationary process. Furthermore, it is straightforward to see that \(U_n\) satisfies difference equation

\[
\Delta_n U = (e^{-H} - 1) U_{n-1} + \Delta_n G,
\]

(3.11)

and hence difference equation (3.11) is a natural analogy to Langevin equation (3.3). We call a stationary process satisfying (3.11) a *generalised AR(1)-model*. Finally, we need analogy to the Lamperti transform \(L_H U_t = t^H U_{\log t}\).

**Theorem 3.3.** Let \(H > 0\) be fixed. Then

\[
|U^d| = |\mathcal{G}^d_H|.
\]
Furthermore, the bijection is given by relations \((5.10)\) and

\[ G_n = U_n - U_0 - (e^{-H} - 1) \sum_{k=0}^{n-1} U_k. \]

### 3.2. On the class \( \mathcal{G}_H \).

In this subsection we analyse the class \( \mathcal{G}_H \). In particular, we give simple sufficient condition for a process \( G \in \mathcal{G} \) to also satisfy \( G \in \mathcal{G}_H \) for any \( H > 0 \). We also show that there exists a bijection between classes \( \mathcal{G} \) and \( \mathcal{G}_H \) while there exists processes \( G \in \mathcal{G} \) such that \( G \notin \mathcal{G}_H \) for every \( H > 0 \).

**Theorem 3.4.** Let \( G = (G_t)_{t \geq 0} \) be a process with stationary increments such that

\[ \mathbb{E}|G_1|^p < \infty \]

for some \( p > 0 \). Then \( G \in \mathcal{G}_H \) for every \( H > 0 \).

**Proof.** We have to prove that

\[ \int_{-\infty}^{0} e^{Hs} dG_s \]

is well-defined random variable or, equivalently,

\[ \int_{0}^{\infty} e^{-Hs} dG_s. \]

Let now \( N \) be fixed. Applying integration by parts we have

\[ \int_{0}^{N} e^{-Hs} dG_s = e^{-HN} G_N + H \int_{0}^{N} e^{-Hs} G_s ds. \]

Hence it is sufficient to prove that for every \( H > 0 \) we have

\[ \lim_{N \to \infty} e^{-HN} |G_N| \to 0 \tag{3.12} \]

almost surely. Note first that without loss of generality we can assume that \( N \) is an integer. Moreover, we have

\[ |G_N| \leq \sum_{k=1}^{N} |G_k - G_{k-1}|. \]

Furthermore, it is clear that for each \( \epsilon > 0 \) we have

\[ \mathbb{P} \left( \sum_{k=1}^{N} |G_k - G_{k-1}| > \epsilon \right) \leq \sum_{k=1}^{N} \mathbb{P} \left( |G_k - G_{k-1}| > \frac{\epsilon}{N} \right) \]

and by stationarity of the increments we have

\[ \sum_{k=1}^{N} \mathbb{P} \left( |G_k - G_{k-1}| > \frac{\epsilon}{N} \right) = N \mathbb{P} \left( |G_1| > \frac{\epsilon}{N} \right). \]

Consequently, we obtained

\[ \mathbb{P} \left( e^{-HN} |G_n| > \epsilon \right) \leq N \mathbb{P} \left( |G_1| > \frac{e^{HN} \epsilon}{N} \right). \]
Hence for any $c > 0$ we have

$$N 1_{|G_1| > c} \leq \frac{N}{c^p} |G_1|^p$$

which yields

$$N \mathbb{P} \left( |G_1| > \frac{e^{HN} \epsilon}{N} \right) \leq e^{-p e^{-HN} N^{1+p} E|G_1|^p}.$$ 

 Choosing $\epsilon = \epsilon_N = N^{-1}$ we obtain that

$$\sum_{N=1}^{\infty} \mathbb{P} \left( e^{-HN} |G_N| > \frac{1}{N} \right) < \infty.$$ 

Hence the result follows by Borel-Cantelli Lemma. \qed

**Remark 3.2.** Clearly we have analogous result for discrete processes.

**Remark 3.3.** Note that by careful examinations of the above proof the result could easily be strengthened to processes which satisfies weaker assumptions on integrability. For example, certain kind of logarithmic integrability would be sufficient. On the other hand, we believe that characterisation of the class $\mathcal{G}_H$ in terms of integrability cannot be obtained since the dependence structure of increments of $G$ can be arbitrary.

By Theorem 3.4 any kind of integrability is sufficient to guarantee that a process $G \in \mathcal{G}$ also belongs to $\mathcal{G}_H$. It turns out that even more is true; for every $H > 0$ there exists a bijection between classes $\mathcal{G}$ and $\mathcal{G}_H$. This is the topic of the next theorem.

**Theorem 3.5.** Let $H > 0$ be fixed. Then $|\mathcal{G}| = |\mathcal{G}_H|$. 

**Proof.** Clearly we have $|\mathcal{G}_H| \leq |\mathcal{G}|$ since $\mathcal{G}_H \subset \mathcal{G}$. To prove the opposite direction, fix $H, \epsilon > 0$ and define a process $U_t = G_{t+\epsilon} - G_t$. Consequently, $U_t$ is stationary and we obtain $|\mathcal{G}| \leq |\mathcal{U}|$. On the other hand, by Theorems 3.1 and 3.2 we have $|\mathcal{U}| = |\mathcal{G}_H| = |\mathcal{X}_H|$. Hence $|\mathcal{G}| = |\mathcal{G}_H|$ by Cantor-Bernstein-Schroeder Theorem 2.2. \qed

**Remark 3.4.** Again, it is straightforward to give discrete analogy.

As a simple corollary we obtain the existence of bijection between stationary increment processes and stationary processes. A result which is intuitively quite clear but not, to the best of our knowledge, explicitly presented in the literature.

**Corollary 3.1.** We have $|\mathcal{G}| = |\mathcal{U}|$.

It would be tempting to claim that, by Theorem 3.5, every process $G \in \mathcal{G}$ also satisfies $G \in \mathcal{G}_H$. In other words, we have that $\mathcal{G} = \mathcal{G}_H$. However, this is not the case and we will give a discrete counterexample in next proposition. Analogous continuous version is left to the reader.

**Proposition 3.3.** There exists a process $G \in \mathcal{G}^d$ which is not $\mathcal{G}_H^d$ process for any $H > 0$. 


Proof. Let \( \alpha \in (0, 1) \) and let \( \xi_k \) be i.i.d. sequence of Pareto(\( \alpha \)) random variables with tail function
\[
P(\xi_k > x) = x^{-\alpha}.
\]
for \( x > 1 \). Set
\[
Z_k = e^{\xi_k}
\]
and define
\[
G_n = \sum_{k=1}^{n} Z_k.
\]
Clearly, \( G_n \) has stationary increments and satisfies
\[
G_n \geq Z_n.
\]
In particular, to prove that \( e^{-H_n} G_n \) does not converge it is sufficient to prove that
\[
e^{-H_n} Z_n
\]
does not converge. However, this follows immediately from the second Borel-Cantelli Lemma together with the fact that events \( \{e^{-H_n} Z_n > 1\} \) are independent and
\[
\sum_{n=1}^{\infty} P(e^{-H_n} Z_n > 1) = \infty.
\]
\( \Box \)

4. Examples and applications

4.1. Wide-sense stationary processes. Clearly all the results remains valid if we consider only wide-sense stationary processes. On the other hand, in this particular case the only interesting objects are covariance function \( r(t) \) of the process \( U_t \) and the variance function \( V(t) \) of the corresponding process \( G_t \) together with their means. As an application of our results we obtain the following which states that the mean of an integrable stationary increment process \( G \) is linear. This fact is known for stationary independent increment processes [18] but, to the best of our knowledge, it is not widely acknowledgement in the literature to hold for all stationary increment processes.

Theorem 4.1. Let \( G \) be integrable stationary increment process. Then there exists a constant \( c \) such that
\[
\mathbb{E} G_t = ct.
\]

Proof. By Theorems 3.1 and 3.4 each such \( G \) has representation
\[
G_t = U_t - U_0 + H \int_0^t U_v \, dv
\]
from which the statement follows. \( \Box \)

Similarly, it is straightforward to compute the covariance of \( U_t \) in terms of variance function of \( G \) and vice versa. The following result gives the other direction while the other is left to the reader.
Proposition 4.1. Let \( G \in \mathcal{S} \) be square integrable and centered. Then its variance is given by

\[
\mathbb{E}G_t^2 = 2r(0) - 2r(t) + H^2 \int_0^t r(v)(t-v)dv.
\]

where \( r(t) \) is the covariance function of the associated stationary process \( U_t \). Conversely, for any square integrable stationary process \( U_t \) with covariance function \( r(t) \) equation \((4.2)\) determines the variance function of some process \( G \in \mathcal{S} \).

Proof. The result follows from representation \((3.6)\) together with change of variable. \(\square\)

4.2. Analysis on positive half line and extensions of processes. In this section we briefly explain how our results could be applied to study stationary processes on \([0, \infty)\). First note that for given stationary increment process \( G = (G_t)_{t \geq 0} \), its extension to whole \( \mathbb{R} \) is defined by two-sided process. Moreover, for a given stationary increment process \( G = (G_t)_{t \leq 0} \) on \((-\infty, 0]\) we define its extension to whole \( \mathbb{R} \) similarly. Now for \( H \)-self-similar process \( X = (X_t)_{t \geq 0} \), thanks to our main theorems, there is one-to-one correspondence between \( X \) and processes \( U \in \mathcal{U} \) and \( G \in \mathcal{S}_H \). Assume next that we are given a stationary process \( U = (U_t)_{t \geq 0} \). Then its Lamperti transform \( \tilde{X}_t = (\mathcal{L}_H U)_t \) defines \( H \)-self-similar process \( X = (X_t)_{t \geq 1} \), and it is not clear in general how to extend \( H \)-self-similar process on \([1, \infty)\) to an \( H \)-self-similar process on \([0, \infty)\). For that purpose, we first define a stationary process \( \tilde{U} = (\tilde{U}_t)_{t \leq 0} \) by time change \( \tilde{U}_t = U_{-t} \), and consequently its Lamperti transform \( X = \mathcal{L}_H \) defines \( H \)-self-similar process on \([0, 1]\). Furthermore, we can define a stationary increment process \( G \) on \((-\infty, 0] \) by representation \((3.3)\). Now extending such \( G \) to whole \( \mathbb{R} \) we obtain a process \( G \in \mathcal{S}_H \), and by representation \((3.5)\) we obtain \( H \)-self-similar process \( X = (X_t)_{t \geq 0} \). With the use of this process \( X \), \( \tilde{U} \) can be uniquely extended to whole \( \mathbb{R} \) by defining \( U = \mathcal{L}^{-1} X \). Furthermore, by defining new \( H \)-self-similar process \( \tilde{X} = (\tilde{X}_t)_{t \geq 0} \) by setting \( \tilde{X}_0 = 0 \) and

\[
\tilde{X}_t = t^{2H-1} \tilde{X}_{\frac{t}{t}}, \quad t > 0
\]

it is straightforward to check that original \( U = (U_t)_{t \geq 0} \) corresponds to the inverse Lamperti transform \( \mathcal{L}_H^{-1} \tilde{X} \) for \( t \geq 0 \). Consequently, \( U \) can be extended to whole real line simply by defining \( U = \mathcal{L}_H^{-1} \tilde{X} \), \( t \in \mathbb{R} \). Consequently, \( U \) can be expressed with representation \((6.6)\). We also note that similar analysis can be applied for processes on compact intervals \([0, T]\) provided that one of the processes \( X, U \) or \( G \) can be extended to whole \([0, \infty)\).

4.3. Additive self-similar processes. A process \( X \) is called additive if \( X \) is stochastically continuous with cadlag paths, \( X \) has independent increments and \( X_0 = 0 \). If in addition \( X \) has also stationary increments, then \( X \) is a Levy process which is particularly important and widely applied class of processes. In [12] the authors studied \( H \)-self-similar additive processes \( X \) and proved that such \( X \) can be represented as an indefinite integral with respect to a Levy process. Furthermore, as a corollary (Corollary 2 in [12])
the authors obtained that the inverse Lamperti transform of an additive $H$-self-similar process $X$ can be described as a solution to a Langevin equation driven by a Levy process. Compared to our results, we obtain that all the results in [12] arises from results presented in this paper as a special case. Indeed, if $X$ is $H$-self-similar process additive process it is clear from representation (3.8) that the corresponding process $G$ is also an additive process with independent increments and thus a Levy process. Conversely, given a Levy process $G$ it is straightforward to see from representation (3.7) that the corresponding process $X$ defines an $H$-self-similar additive process. Furthermore, representation of of $L_H^{-1}X$ as a solution to a Levy-driven Langevin equation (Corollary 2 of [12]) follows immediately from Theorem 3.1.

4.4. Gaussian processes. It is known that a continuous time stationary Gaussian process $U_t$ is either continuous or unbounded on every interval $[a,b]$ (see, e.g. [1]) and the latter case is hardly interesting. Furthermore, every continuous Gaussian process $G \in \mathcal{G}$ is also an element of $\mathcal{G}_H$ for every $H > 0$. Consequently, every continuous time Gaussian process $U_t$ can be represented as the unique solution to Langevin equation with some noise term $G \in \mathcal{G}_H$. Conversely, given any Gaussian process $G \in \mathcal{G}$ and given any fixed $H > 0$, the Langevin equation determines unique Gaussian stationary process.

Example 4.1. Consider Lamperti transform of standard Brownian motion defined by 

$$U_t = \frac{1}{\sqrt{2\theta}} e^{-\theta t} W_{e^{2\theta t}}.$$ 

Now $U_t$ can be described as solution to equation

$$dU_t = -\theta U_t dt + dG_t$$

with noise term

$$G_t = \frac{1}{\sqrt{2\theta}} \int_0^t e^{-\theta u} dW_{e^{2\theta u}}.$$ 

Consequently, the Lamperti transform of standard Brownian motion is a pathwise solution to Langevin equation with noise given by (4.3). On the other hand, it is straightforward to see that $G_t - G_s \xrightarrow{law} W_t - W_s$.

4.5. ARMA$(p, q)$-models. As a discrete time example, consider a general ARMA$(p, q)$-model (for details on time series, we refer to [10]) defined by

$$X_n = c + \sum_{k=1}^{p} \alpha_k X_{n-k} + \sum_{k=1}^{q} \beta_k \xi_{n-k},$$

where the sequence $\xi_{n-k}$ is a white noise. Consequently, stationary ARMA$(p, q)$ process $X_n$ can be uniquely represented as a generalised AR(1) model as

$$X_n = e^{-H} X_{n-1} + \Delta_k G$$

with some parameter $H$ and stationary increment noise $G$ which do not have independent increments. Similarly, by Wold’s representation Theorem every
discrete time wide-sense stationary process can be represented as $MA(\infty)$ model with representation

$$X_n = \sum_{j=0}^{\infty} b_j \xi_{n-j} + \eta_n,$$

where $\xi_{n-j}$ is the stochastic innovation process and $\eta_n$ is deterministic. Now, thanks to Theorem 3.3, such process can equivalently be written as *generalised AR(1) model* with representation (3.10). In a similar way, all stationary generalisations of ARMA$(p, q)$ reduce back to *generalised AR(1) model*. Hence it might be more useful for applications to estimate the properties of the noise $G$ itself rather than parameters of the original model.

5. Discussions

In this paper we have proved that (essentially) all stationary processes $U_t$ can be described in a natural way as a solution to a Langevin equation. Moreover, we have established a natural connection between stationary processes, stationary increment processes and self-similar processes which is close in spirit with the work of Lamperti. Discrete analogies have been established. As such our results can be applied to further analyse the properties of stationary processes, self-similar processes and processes with stationary increments. Furthermore, our results can be applied in modelling to construct self-similar processes from stationary increment noise and vice versa. We have also introduced a new class of models, *generalised AR(1) models*, which covers all stationary discrete time models. Consequently, with such a result it would be more natural to estimate the noise term $G$ directly instead of considering some other general model and estimate multiple parameters.

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