A note on the non-Markovianity of quantum semi-Markov processes

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The non-Markovianity of the stochastic process called the quantum semi-Markov (QSM) process is studied using a recently proposed quantification of memory based on the deviation from semigroup evolution and thus providing a unified description of divisible and indivisible channels. This is shown to bring out the property of QSM process to exhibit memory effects in the CP-divisible regime. An operational meaning to the non-Markovian nature of semi-Markov processes is also provided.

I. INTRODUCTION

Open quantum systems is the study of the evolution of the system of interest taking into account the effect of its environment [1, 2]. The effect of the environment is encoded in the nature of its interaction with the system of interest [3–5]. Traditionally, weak system environment interaction is associated with Markovian (memoryless) regime, where the environmental time scale is much smaller than the system time scale. A more involved scenario occurs when the effects of memory are considered, and is broadly called non-Markovian physics. Defining, characterizing, detecting and quantifying non-Markovianity of open system dynamics has been an intense research activity in the last decade [6–11]. This has impacted a wide range of applications, ranging from quantum cryptography [12–14], quantum walks [15, 16], quantum thermodynamics [17], quantum coherence and correlations [18–21].

Historically speaking, whenever a process deviates from being a semigroup, one talks of non-Markovian processes [7]. Later, quantum dynamical maps approach discarded this notion in favour of a more strict definition of completely positive (CP-) divisibility [22]. This approach was revisited in [10] where a quantitative notion of non-Markovianity was developed by identifying it with deviation from “temporal self-similarity”, the property of a system dynamics whereby the propagator between two intermediate states is independent of the initial time. A number of approaches devoted to the study of open system dynamics have been developed, such as the GKSL master equation approach [23, 24], collisional models [25, 26], and process tensor formalism [27], to name a few. In each of these approaches, the way (non-)Markovianity is defined varies. Another approach was made in [28] by introducing a post-Markovian master equation.

In many of the models in the frameworks mentioned above, a process that is non-Markovian according to CP-divisibility condition [22] may not be non-Markovian according to the distinguishability or information back-flow criterion [29]. A class of processes that are CP-divisible but deviate from having a semigroup structure are the so-called semi-Markov processes [30]. These processes are classically non-Markovian while being CP-divisible, hence the name. Recently, it was shown [15] that a class of noises called power law (PL) and Ornstein-Uhlenbeck (OU) noises that have memory kernel with colored spectral density are Markovian according to CP-divisibility condition. Also, for instance, OU noise has been shown to delay the entanglement sudden death in an entangled two-spin system evolution [31]. A definition based on temporal self-similarity of map points out that even CP-divisible processes, such as PL and OU noises, possess memory of weak form which may be thought of as slowing down the decay process [10]. In this work, we quantify non-Markovianity of semi-Markov processes and argue that they too fall in a similar category.

As mentioned earlier, a universal definition of non-Markovianity has remained elusive and in this light there have been recent proposals of reinstating a definition that accounts for memory effects present even in CP-divisible processes. See for example, a definition of non-Markovianity introduced based on conditional past-future (CPF) correlations [32, 33] and also the one based on deviation from temporal self-similarity of map [10]. Also, one may define a hierarchy of divisibility of quantum dynamical maps, namely P/divisibility [34], CP-divisibility [22] and L-divisibility [35]. The latter has been shown to be equivalent to the semigroup evolution. In this light, one may argue that L-divisibility, lack of CPF correlations, and temporal self-similarity of dynamical maps are equivalent definitions pertaining to the semigroup evolution, while deviation from any of these notions signals non-Markovian evolution of the weakest form, even when the dynamics is CP-divisible.

The plan of the paper is as follows. In Section II, we briefly review classical and quantum semi-Markov processes, particularly focusing on dephasing quantum semi-Markov processes. In Section III, we point out the possible usefulness of quantum semi-Markov processes from an information theoretic perspective. In Section IV, we argue that quantum semi-Markovian processes that are CP-divisible can be non-Markovian and quantify the non-Markovianity of these processes using a recently introduced measure [10]. Then we conclude in Section V.

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II. CLASSICAL AND QUANTUM SEMI-MARKOV PROCESS

A classical stochastic process is one which takes a valid probability distribution to another valid probability distribution. For example, consider a classical stochastic process [36] in which a system, characterized by a set of states, jumps from one state to another with a jump probability \( \pi \), such that the dynamics of the system is given by

\[
Q(\tau) = \left( \begin{array}{cc} 1 - \pi & \pi \\ \pi & 1 - \pi \end{array} \right) f(\tau) \equiv \Pi f(\tau),
\]

where \( \Pi \) is the transition matrix and \( f(\tau) \) is called the waiting time distribution. If the probability distribution which characterizes the time distribution of the system to stay in a particular state is given by an exponential probability distribution \( f(\tau) = \lambda e^{-\lambda \tau} \), then the process is classically Markovian and non-Markovian otherwise. The survival probability is \( g(t) = 1 - \int_0^t d\tau f(\tau) \).

This classical concept, when transported to the quantum context in its weakest sense, gives the quantum semi-Markov (QSM) process, whose time-local and time-nonlocal master equations are studied in Ref. [37].

One may define a QSM process as follows. The generator of an open quantum system dynamics may be written as

\[
\mathcal{L}(t) = \gamma(t)[\mathcal{P} - I],
\]

where \( \mathcal{P} \) is the CPTP projector and \( \gamma(t) \) is the time-dependent decay rate. Equivalently, noting that the map \( \mathcal{E}(t) = \mathcal{T} \exp\{\int_0^t \mathcal{L}(\tau)d\tau\} \) itself obeys a non-local master equation

\[
\frac{d\mathcal{E}}{dt} = \int_0^t \mathcal{K}(t - \tau)\mathcal{E}(\tau)d\tau,
\]

the non-local generator \( \mathcal{K}(t) \) may be given an alternate representation via the memory kernel function \( k(t) \) as

\[
\mathcal{K}(t) = k(t)[\mathcal{P} - I],
\]

where the nature of the memory kernel function \( k(t) \) determines if the dynamics is non-Markovian or not. The non-local equation for the map gives rise to the solution \( \mathcal{E}(t) = g(t)I + (1 - g(t))\mathcal{P} \), with \( g(t) = 1 - \int_0^t d\tau f(\tau) \), where \( f(t) \) is associated with the memory kernel \( k(t) \) via the Laplace transform \( \tilde{f}(t) = \int_0^\infty d\tau e^{-\tau t} f(\tau) \) such that

\[
\tilde{k}(t) = \frac{u \tilde{f}(t)}{1 - f(t)}.
\]

A quantum process of the above type (4) is said to be semi-Markov if \( f(t) \geq 0 \) and \( \int_0^\infty d\tau f(\tau) \leq 1 \).

One the other hand, if the two-time correlation function of the environment spectral density is delta correlated then the Born-Markov approximation holds with the \( \mathcal{K}(t) = \delta(t - \tau)\mathcal{K} \), giving rise to semigroup evolution.

In this work, we consider a dephasing quantum semi-Markov process given by the operator-sum representation

\[
\mathcal{E}(t)[\rho] = \sum_j C_j(t)\rho C_j^\dagger(t),
\]

with the Kraus operators

\[
C_1 = \sqrt{\frac{1 + q(t)}{2}} I; \quad C_2 = \sqrt{\frac{1 - q(t)}{2}} Z,
\]

where

\[
q(t) = e^{-st/2}\left( \cosh\left(\frac{\eta st}{2}\right) + \frac{1}{\eta} \sinh\left(\frac{\eta st}{2}\right) \right),
\]

The function \( q(t) \) is obtained by considering a convolution \( f(t) = f_1(t) * f_2(t) \) of two exponential waiting time distributions of the form \( f_1(t) = \lambda_1 e^{-\lambda_1 t} \) and \( f_2(t) = \lambda_2 e^{-\lambda_2 t} \), where \( \lambda_i \) have the physical interpretation of rates determining the speed of the process and the parameters are set to \( s = \lambda_1 + \lambda_2 \) and \( p = \lambda_1 \lambda_2 \) [36].

For \( p < \frac{s^2}{8} \), the process is said to be Markovian according to divisibility and distinguishability criteria even though it is non-Markovian classically. It is CP-indivisible when \( p > \frac{s^2}{8} \). When \( p \to 0 \), we have a QDS limit, that is \( q(t) = 1 \).

In the time-local master equation

\[
\frac{d\rho}{dt} = \gamma(t)[\sigma_z \rho \sigma_z - \rho],
\]

describing the dephasing process as given above, the decay rates given by [11, 30]

\[
\gamma(t) = -\frac{1}{2q(t)} \frac{dq(t)}{dt},
\]

where \( q(t) \) is given in Eq. (7).

With this example, we show that semi-Markov processes are non-Markovian according to a measure introduced in [10].

III. DECOHERENCE MITIGATION IN SEMI-MARKOV PROCESSES

Given the quantum states \( \rho_j \), occurring with probability \( p_j \), the maximum amount of classical information that can be encoded using them is upper bounded by the Holevo \( \chi \) quantity \( \chi := S(\rho) - \sum_k p_k S(\rho_k) \). The quantity \( S(\rho) = -\text{Tr}(\rho \log \rho) \) is the von Neumann entropy.

Keeping in mind that information contained in the quantum states decays under decoherence, one may analyze the effect of semi-Markov processes on the Holevo bound which gives an upper bound on the accessible information. In [38], accessible information was used...
to re-instate the interpretation of information back-flow with non-Markovianity of the quantum evolution. As is known, information back-flow is associated with the recurrence of distinguishability of two initially orthogonal states under non-Markovian CPTP maps. Figure (1) depicts the decay of Holevo information under processes that are semigroup, semi-Markov and CP-indivisible. It is worth noting that, semi-Markov processes can mitigate the loss of information by slowing down the rate of information leakage into the environment, i.e., slowing down the rate of decrease of distinguishability, while CP-indivisible processes can, for certain intervals, revive the same. This provides an operational meaning to the non-Markovian behavior of semi-Markov processes. It is important to note that semigroup evolution corresponds to weaker system-environment coupling, hence it is natural that the decay of information is very slow. In contrast to this, one expects that the decay is rapid when the coupling is strong enough. In this light, semi-Markov processes may be thought to be intermediate between the two extreme cases of semigroup and strong non-Markovian behavior, which manifests itself by mitigating the loss of information. This could be envisaged to have potential applications in quantum information processing. In order to harness the advantages due to semi-Markov nature of the noise, one may consider a possibility of simulating the noisy evolution of the qubit. See for example, [39] where an all-optical simulation of random telegraph noise is given.

IV. QUANTIFYING NON-MARKOVIANITY OF QUANTUM SEMI-MARKOV PROCESSES

A qualitative measure that quantifies the non-Markovianity in a CP-divisible processes as a function of the magnitude of deviation from semigroup evolution was developed in [10], which for brevity may be called the SSS measure. It may be noted here that this quantification of non-markovianity is applicable quite generally, even to maps that are non-invertible, whereas one based on regions where a decoherence rate is negative [40], can overestimate the quantity, because positivity of the decoherence rates is only sufficient but not necessary in case of non-invertible maps [41].

We now quantify the non-Markovian content of semi-Markov processes using the the SSS measure

\[ \zeta = \min_{\mathcal{L}} \frac{1}{\mathcal{T}} \int_0^\mathcal{T} ||\dot{\mathcal{L}}(t)| - \dot{\mathcal{L}}^*||dt, \]

where \( \mathcal{L}(t) \) is the time-dependent generator \( (\mathcal{E}(t) = T e^{\int_0^t \mathcal{L}(\gamma)dt}) \) and \( \mathcal{L}^* \) is the generator of semigroup evolution. Here, \( \mathcal{T} \) is the time-ordering operator. For a dephasing process, the above measure (10) reduces to

\[ \zeta = \min_{\gamma^*} \frac{1}{\mathcal{T}} \int_0^\mathcal{T} |\gamma(t) - \gamma^*|dt, \]

where \( \gamma(t) \) is the decay rate of the process in the region beyond the semigroup and \( \gamma^* \) is the constant rate characterizing the semigroup evolution. And for the particular process that we consider in this work, the rate in the semigroup limit turns out to be \( \gamma^* = 0 \).

Note that the dephasing process in Eq. (6) in the CP-indivisible region, i.e., \( p > \frac{s^2}{\pi} \), corresponds to a singular generator at the master equation level. The decay rate for the process (6) is obtained as

\[ \gamma(t) = \frac{2p}{s \sqrt{1 - 2p} \coth \left( \frac{1}{2} s \sqrt{1 - 2p} \right) + s}, \]

For \( p < \frac{s^2}{\pi} \), the process is CP-divisible. Considering the case of \( s = 1 \), for \( p = 3 > \frac{1}{8} \) the process is CP-indivisible and the decay rate is plotted in Figure (2). Since the range of \( p \) is unbounded after \( \frac{s^2}{\pi} \), the measure (11) goes up to infinity as \( p \) goes to infinity. Therefore, the measure may be normalized to fit the range \( (0, 1) \) by considering \( \zeta = \frac{M}{1+M} \). The figure (3) shows the deviation of the semi-Markov process from QDS structure for \( s = 1 \), for the range \( p \in \left[ 0, \frac{1}{2} \right] \). This brings out the non-Markovian behavior of semi-Markov processes, even in the CP-divisible regime.

V. CONCLUSIONS

Here, we have studied an interesting stochastic process called the semi-Markov process. This belongs to the
intriguing category of processes that exhibit memory effects in the CP-divisible regime. The principal tool used to quantify this memory, in the quantum regime, is the recently introduced SSS measure [10], that quantifies the non-Markovianity as a function of the magnitude of deviation from semigroup evolution. Further, the behavior of the Holevo bound under the influence of semi-Markov processes brings out the effect of decoherence mitigation, that is, the loss of information leakage into the environment is slowed down. This provides an operational meaning to the non-Markovian nature of semi-Markov processes.

As was noted earlier [10, 15], PL and OU noises are also non-Markovian but CP-divisible. It may be an interesting future direction to explore the equivalence between semi-Markov and CP-divisible non-Markovian processes such as PL and OU noises.

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