ON THE RATIONALITY OF POINCARÉ SERIES
OF GORENSTEIN ALGEBRAS
VIA MACAULAY’S CORRESPONDENCE

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Abstract. Let $A$ be a local Artinian Gorenstein ring with algebraically closed residue field $A/\mathfrak{m} = k$ of characteristic 0, and let $P_A(z) := \sum_{p=0}^{\infty} \dim_k(\text{Tor}_p^A(k,k))z^p$ be its Poincaré series. We prove that $P_A(z)$ is rational if either $\dim_k(\mathfrak{m}^2/\mathfrak{m}^3) \leq 4$ and $\dim_k(A) \leq 16$, or there exist $m \leq 4$ and $c$ such that the Hilbert function $H_A(n)$ of $A$ is equal to $m$ for $n \in [2,c]$ and equal to 1 for $n > c$. The results are obtained thanks to a decomposition of the apolar ideal $\text{Ann}(F)$ when $F = G + H$ and $G$ and $H$ belong to polynomial rings in different variables.

1. Introduction and notation

Throughout this paper, by ring we mean a Noetherian, associative, commutative and unitary ring $A$ with maximal ideal $\mathfrak{m}$ and algebraically closed residue field $k := A/\mathfrak{m}$ of characteristic 0.

In [17] the author asked if the Poincaré series of the local ring $A$, i.e.

$$P_A(z) := \sum_{p=0}^{\infty} \dim_k(\text{Tor}_p^A(k,k))z^p,$$

is rational. Moreover he also proved its rationality when $A$ is a regular local ring. Despite many interesting results showing the rationality of the Poincaré series of some rings, in [1] the author gave an example of an Artinian local algebra $A$ with transcendental $P_A$. Later on the existence of an Artinian, Gorenstein, local ring with $\mathfrak{m}^4 = 0$ and transcendental $P_A$ was proved in [4].

Nevertheless, several results show that large classes of local rings $A$ have rational Poincaré series, e.g. complete intersections rings (see [19]), Gorenstein local rings with $\dim_k(\mathfrak{m}/\mathfrak{m}^3) \leq 4$ (see [2] and [15]), Gorenstein local rings with

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dim_k(\mathfrak{M}^2/\mathfrak{M}^3) \leq 2$ (see [16], [10]), Gorenstein local rings of multiplicity at most 10 (see [3]), Gorenstein local algebras with $\dim_k(\mathfrak{M}^2/\mathfrak{M}^3) = 4$ and $\mathfrak{M}^4 = 0$ (see [8]).

All the above results are based on the same smart combination of results on the Poincaré series from [3] and [13] first used in [16] combined with suitable structure results on Gorenstein rings and algebras. In this last case a fundamental role has been played by Macaulay’s correspondence.

In Section 2 we give a quick résumé of the main results that we need later on in the paper about Macaulay’s correspondence. In Section 3 we extend to arbitrary algebras a very helpful decomposition result already used in a simplified form in [9] and [8] for algebras with $\mathfrak{M}^4 = 0$. In Section 4 we explain how to relate the rationality of the Poincaré series of Gorenstein algebras with their representation in the setup of Macaulay’s correspondence making use of the aforementioned decomposition result. Finally, in Section 5 we use such relationship in order to prove the two following results.

**Theorem A.** Let $A$ be an Artinian, Gorenstein local $k$–algebra with maximal ideal $\mathfrak{M}$. If there are integers $m \leq 4$ and $c \geq 1$ such that

$$\dim_k(\mathfrak{M}^t/\mathfrak{M}^{t+1}) = \begin{cases} m & \text{if } t = 2, \ldots, c, \\ 1 & \text{if } t = c + 1, \end{cases}$$

then $P_A$ is rational.

**Theorem B.** Let $A$ be an Artinian, Gorenstein local $k$–algebra with maximal ideal $\mathfrak{M}$. If $\dim_k(\mathfrak{M}^2/\mathfrak{M}^3) \leq 4$ and $\dim_k(A) \leq 16$, then $P_A$ is rational.

The above theorems generalize the quoted results on stretched, almost–stretched and short algebras (see [16], [10], [6], [8]).

1.1. **Notation.** In what follows $k$ is an algebraically closed field of characteristic 0. A $k$–algebra is an associative, commutative and unitary algebra over $k$. For each $N \in \mathbb{N}$ we set $S[N] := k[x_1, \ldots, x_N]$ and $P[N] := k[y_1, \ldots, y_N]$. We denote by $S[N]_q$ (resp. $P[N]_q$) the homogeneous component of degree $q$ of such a graded $k$–algebra, and we set $S[N]_{\leq q} := \bigoplus_{i=1}^q S[N]_i$ (resp. $P[N]_{\leq q} := \bigoplus_{i=1}^q P[N]_i$). Finally, we set $S[n]_+ := (x_1, \ldots, x_n) \subseteq S[n]$. The ideal $S[n]_+$ is the unique maximal ideal of $S[n]$.

A local ring $R$ is Gorenstein if its injective dimension as $R$–module is finite.

If $\gamma := (\gamma_1, \ldots, \gamma_N) \in \mathbb{N}^N$ is a multi–index, then we set $t^\gamma := t_1^{\gamma_1} \cdots t_N^{\gamma_N} \in k[t_1, \ldots, t_N]$.

For all the other notations and results we refer to [12].

2. **Preliminary results**

In this section we list the main results on algebras we need in next sections. Let $A$ be a local, Artinian $k$–algebra with maximal ideal $\mathfrak{M}$. We denote by $H_A$ the Hilbert function of the graded associated algebra

$$\text{gr}(A) := \bigoplus_{t=0}^{+\infty} \mathfrak{M}^t/\mathfrak{M}^{t+1}.$$
We know that
\[ A \cong S[n]/J \]
for a suitable ideal \( J \subseteq S[n] \) \( \subseteq S[n] \), where \( n = \text{emdim}(A) := H_\alpha(1) \). Recall that the \textit{socle degree} \( \text{sdeg}(A) \) of \( A \) is the greatest integer \( s \) such that \( \mathfrak{M}^s \neq 0 \).

We have an action of \( S[n] \) over \( P[n] \) given by partial derivation defined by identifying \( x_i \) with \( \partial/\partial y_i \). Hence
\[ x^\alpha \circ y^\beta := \begin{cases} \alpha! (\beta) y^{\beta - \alpha} & \text{if } \beta \geq \alpha, \\ 0 & \text{if } \beta < \alpha. \end{cases} \]

Such an action endows \( P[n] \) with a structure of module over \( S[n] \). If \( J \subseteq S[n] \) is an ideal and \( M \subseteq P[n] \) is a \( S[n] \)-submodule we set
\[ J^\perp := \{ F \in P[n] \mid g \circ F = 0, \forall g \in J \}, \]
\[ \text{Ann}(M) := \{ g \in S[n] \mid g \circ F = 0, \forall F \in M \}. \]

For the following results see e.g. [11], [14] and the references therein. Macaulay’s theory of inverse system is based on the fact that constructions \( J \mapsto J^\perp \) and \( M \mapsto \text{Ann}(M) \) give rise to an inclusion–reversing bijection between ideals \( J \subseteq S[n] \) such that \( S[n]/J \) is a local Artinian \( k \)-algebra and finitely generated \( S[n] \)-submodules \( M \subseteq P[n] \). In this bijection Gorenstein algebras \( A \) with \( \text{sdeg}(A) = s \) correspond to cyclic \( S[n] \)-submodules \( \langle F \rangle_{S[n]} \subseteq P[n] \) generated by a polynomial \( F \) of degree \( s \). We simply write \( \text{Ann}(F) \) instead of \( \text{Ann}((\langle F \rangle_{S[n]} \).

On the one hand, given a \( S[n] \)-module \( M \), we define
\[ \text{tdf}(M)_q := \frac{M \cap P[n]_{\leq q} + P[n]_{\leq q-1}}{P[n]_{\leq q-1}} \]
where \( P[n]_{\leq q} := \bigoplus_{i=0}^q P[n]_i \), and \( \text{tdf}(M) := \bigoplus_{q=0}^\infty \text{tdf}(M)_q \). The module \( \text{tdf}(M) \) can be interpreted as the \( S[n] \)-submodule of \( P[n] \) generated by the top degree forms of all polynomials in \( M \).

On the other hand, for each \( f \in S[n] \), the lowest degree of monomials appearing with non–zero coefficient in the minimal representation of \( f \) is called the \textit{order of} \( f \) and it is denoted by \( \text{ord}(f) \). If \( f = \sum_{i=\text{ord}(f)}^\infty f_i, f_i \in S[n], \) then \( f_{\text{ord}(f)} \) is called the \textit{lower degree form} of \( f \). It will be denoted in what follows with \( \text{ldf}(f) \).

If \( f \in J \), then \( \text{ord}(f) \geq 2 \). The \textit{lower degree form ideal} \( \text{ldf}(J) \) associated to \( J \) is
\[ \text{ldf}(J) := (\text{ldf}(f) \mid f \in J) \subseteq S[n]. \]

We have \( \text{ldf}((\text{Ann}(M)) = \text{Ann}(\text{tdf}(M)) \) (see [11]; see also [9], Formulas (2) and (3)) whence
\[ \text{gr}(S[n]/\text{Ann}(M)) \cong S[n]/\text{ldf}(\text{Ann}(M)) \cong S[n]/\text{Ann}(\text{tdf}(M)). \]

Thus
\[ (1) \]
\[ H_{S[n]/\text{Ann}(M)}(q) = \dim_k(\text{tdf}(M)_q). \]

We say that \( M \) is \textit{non–degenerate} if \( H_{S[n]/\text{Ann}(M)}(1) = \dim_k(\text{tdf}(M)_1) = n \), i.e. if and only if the classes of \( y_1, \ldots, y_n \) are in \( \text{tdf}(M) \). If \( M = (\langle F \rangle_{S[n]} \), then we write \( \text{tdf}(F) \) instead of \( \text{tdf}(M) \).
Let $A$ be Gorenstein with $s := \text{sdeg}(A)$, so that $\text{Soc}(A) = \mathfrak{M}^s \cong k$. In particular $A \cong S[n]/\text{Ann}(F)$, where $F := \sum_{i=0}^s F_i$, $F_i \in P[n]_i$. For each $h \geq 0$ we set $F_{\geq h} := \sum_{i=h}^s F_i$ (hence $F_s = F_{\geq s}$). We have that $\text{tdf}(F_{\geq h})_i \subseteq \text{tdf}(F)_i$ and equality obviously holds if $i \geq h - 1$ (see Lemma 2.1 of [7]).

Trivially, if $s \geq 1$, we can always assume that the homogeneous part of $F$ of degree $0$ vanishes, i.e. $F = F_{\geq 1}$. Moreover, thanks to Lemma 2.2 of [7] we know that, if $s \geq 2$ and $\text{Ann}(F) \subseteq S[n]^2$, then we can also assume $F_1 = 0$, i.e. $F = F_{\geq 2}$: we will always make such an assumption in what follows.

We have a filtration with proper ideals (see [14]) of $\text{gr}(A) \cong S[n]/\text{ldf}(\text{Ann}(F))$

$$C_A(0) := \text{gr}(A) \supset C_A(1) \supset C_A(2) \supset \cdots \supset C_A(s - 2) \supset C_A(s - 1) := 0.$$  

Via the epimorphism $S[n] \twoheadrightarrow \text{gr}(A)$ we obtain an induced filtration

$$\hat{C}_A(0) := S[n] \supset \hat{C}_A(1) \supset \hat{C}_A(2) \supset \cdots \supset \hat{C}_A(s - 2) \supset \hat{C}_A(s - 1) := \text{ldf}(\text{Ann}(F)).$$

The quotients $Q_A(a) := C_A(a)/C_A(a + 1) \cong \hat{C}_A(a)/\hat{C}_A(a + 1)$ are reflexive graded $\text{gr}(A)$–modules whose Hilbert function is symmetric around $(s - a)/2$. In general $\text{gr}(A)$ is no more Gorenstein, but the first quotient

$$G(A) := Q_A(0) \cong S[n]/\text{Ann}(F_s)$$

is characterized by the property of being the unique (up to isomorphism) graded Gorenstein quotient $k$–algebra of $\text{gr}(A)$ with the same socle degree. Moreover, the Hilbert function of $A$ satisfies

$$H_A(i) = H_{G(A)}(i) = \sum_{a=0}^{s-2} H_{Q_A(a)}(i), \quad i \geq 0.$$  

Since $H_A(0) = H_{G(A)}(0) = 1$, it follows that if $a \geq 1$, then $Q_A(a)_0 = 0$, whence $Q_A(a)_i = 0$ when $i \geq s - a$ (see [14]) for the same values of $a$.

Moreover

$$H_{\text{gr}(A)/C_A(a+1)}(i) = H_{S[n]/\hat{C}_A(a+1)}(i) = \sum_{a=0}^{s} H_{Q_A(a)}(i), \quad i \geq 0.$$  

We set

$$f_h := \sum_{a=0}^{s-h} H_{Q_A(a)}(1) = H_{S[n]/\hat{C}_A(s-h+1)}(1) = H_{\text{gr}(A)/C_A(s-h+1)}(1)$$

so that $n = H_A(1) = f_2$.

Finally we introduce the following new invariant.

**Definition 2.1.** Let $A$ be a local, Artinian $k$–algebra with maximal ideal $\mathfrak{M}$ and $s := \text{sdeg}(A)$. The *capital degree*, $\text{cdeg}(A)$, of $A$ is defined as the maximum integer $i$, if any, such that $H_A(i) > 0$, $0$ otherwise. If $c = \text{cdeg}(A)$ we also say that $A$ is a $c$–stretched algebra (for short, stretched if $c \leq 1$).

By definition $\text{cdeg}(A) \geq 0$ and $\text{cdeg}(A) \leq \text{sdeg}(A)$: if $A$ is Gorenstein, then we also have $\text{cdeg}(A) < \text{sdeg}(A)$.
The rationality of the Poincaré series $P_A$ of every stretched ring $A$ is proved in [10]. The proof has been generalized to rings with $H_A(2) = 2$ in [10] and to rings with $H_A(2) = 3, H_A(3) = 1$ in [8]. The rationality of $P_A$ when $A$ is a 2-stretched algebra has been studied in [8] with the restriction $\text{sdeg}(A) = 3$.

3. Decomposition of the apolar ideal

In the present section we explain how to decompose the ideal $\text{Ann}(F)$ as the sum of two simpler ideals. Such a decomposition will be used in the next section in order to reduce the calculation of the Poincaré series of $A$ to the one of a simpler algebra.

**Lemma 3.1.** Let $m \leq n, G \in P[m], H \in k[y_{m+1}, \ldots, y_n]$ and $F = G + H$. Let us denote by $\text{Ann}(G)$ and $\text{Ann}(H)$ the annihilators of $G$ and $H$ inside $S[m]$ and $k[[x_{m+1}, \ldots, x_n]]$ respectively. Then

$$\text{Ann}(F) = \text{Ann}(G)S[n] + \text{Ann}(H)S[n] + (\sigma_G - \sigma_H, x_ix_j)_{1 \leq i \leq m, m+1 \leq j \leq n},$$

where $\sigma_G \in S[m]$ and $\sigma_H \in k[[x_{m+1}, \ldots, x_n]]$ are any series of order $\text{deg}(G)$ and $\text{deg}(H)$ such that $\sigma_G \circ G = \sigma_H \circ H = 1$.

**Proof.** The inclusions $\text{Ann}(G)S[n], \text{Ann}(H)S[n] \subseteq \text{Ann}(F)$ are completely trivial. Also the inclusion $(\sigma_G - \sigma_H, x_ix_j)_{1 \leq i \leq m, m+1 \leq j \leq n} \subseteq \text{Ann}(F)$ is easy to check. Thus

$$\text{Ann}(G)S[n] + \text{Ann}(H)S[n] + (\sigma_G - \sigma_H, x_ix_j)_{1 \leq i \leq m, m+1 \leq j \leq n} \subseteq \text{Ann}(F).$$

Conversely let $p \in \text{Ann}(F)$. Grouping the different monomials in $p$, we can write a decomposition $p = p_{\leq m} + p_{> m} + p_{\text{mix}}$, where $p_{\leq m} \in S[m], p_{> m} \in k[[x_{m+1}, \ldots, x_n]]$ and, finally, $p_{\text{mix}} \in (x_ix_j)_{1 \leq i \leq m, m+1 \leq j \leq n} \subseteq S[n]$.

It is clear that $p_{\text{mix}} \in \text{Ann}(G)S[n] + \text{Ann}(H)S[n] + (\sigma_G - \sigma_H, x_ix_j)_{1 \leq i \leq m, m+1 \leq j \leq n}$; hence it suffices to prove that

$$p_{\leq m} + p_{> m} \in \text{Ann}(G)S[n] + \text{Ann}(H)S[n] + (\sigma_G - \sigma_H, x_ix_j)_{1 \leq i \leq m, m+1 \leq j \leq n}.$$

To this purpose recall that $0 = p \circ F = p_{\leq m} \circ G + p_{> m} \circ H$, by definition. Hence $p_{\leq m} \circ G = -p_{> m} \circ H$. Since $p_{\leq m} \circ G \in P[m]$ and $p_{> m} \circ H \in k[y_{m+1}, \ldots, y_n]$, it follows that $u \in k$. So $p_{\leq m} - u(\sigma_G - \sigma_H) \in \text{Ann}(G)S[n]$, whence

$$p_{\leq m} \in (\sigma_G - \sigma_H) + \text{Ann}(G)S[n] \subseteq$$

$$\subseteq \text{Ann}(G)S[n] + \text{Ann}(H)S[n] + (\sigma_G - \sigma_H, x_ix_j)_{1 \leq i \leq m, m+1 \leq j \leq n}.$$

A similar argument shows that

$$p_{> m} \in (\sigma_G - \sigma_H) + \text{Ann}(H)S[n] \subseteq$$

$$\subseteq \text{Ann}(G)S[n] + \text{Ann}(H)S[n] + (\sigma_G - \sigma_H, x_ix_j)_{1 \leq i \leq m, m+1 \leq j \leq n},$$

and this concludes the proof. $\square$

Let $F$ be as in the statement above. Then Lemma 3.1 with $G := \sum_{i=2}^{s} F_i$ and $H := \sum_{j=m+1}^{n} y_j^2$ yield the following corollary.
Corollary 3.2. Let \( m \leq n \), \( G \in P[m] \) non-degenerate and \( F = G + \sum_{j=m+1}^{n} y_j^2 \). Let us denote by \( \text{Ann}(G) \) the annihilator of \( G \) inside \( S[m] \). Then
\[
\text{Ann}(F) = \text{Ann}(G)S[n] + (x_j^2 - 2\sigma, x_i x_j)_{1 \leq i < j \leq n, j \geq m+1}
\]
where \( \sigma \in S[m] \) has order \( \text{deg}(G) \) and \( \sigma \circ G = 1 \).

Proof. It suffices to apply Lemma 3.1 taking into account that \( \text{Ann}(H) = (x_j^2 - x_{m+1}^2, x_i x_j)_{m+1 \leq i < j \leq n, j \geq m+1} \) and that \( x_{m+1} \circ H = 2 \). \( \square \)

4. Rationality of Poincaré series

We now focus on the Poincaré series \( P_A(z) \) of the algebra \( A \) defined in the introduction: we will generalize some classical results (see [10], [11], [6]). Out of the decomposition results proved in the previous section, the main tools we use are the following ones:

- for each local Artinian, Gorenstein ring \( C \) with \( \text{emdim}(C) \geq 2 \)

\[
P_C(z) = \frac{P_{C/Soc(C)}(z)}{1 + z^2 P_{C/Soc(C)}(z)}
\]
(see [3]);

- for each local Artinian ring \( C \) with maximal ideal \( \mathfrak{m} \), if \( c_1, \ldots, c_h \in \mathfrak{m} \setminus \mathfrak{m}^2 \) are linearly independent elements of \( \text{Soc}(C) \), then

\[
P_C(z) = \frac{P_{C/(c_1, \ldots, c_h)}(z)}{1 - h z P_{C/(c_1, \ldots, c_h)}(z)}
\]
(see [13]).

Let \( A \) be a local, Artinian, Gorenstein, \( k \)-algebra with \( s = \text{sdeg}(A) \) and \( n = H_A(1) \). Assume \( A = S[n]/\text{Ann}(F) \) where \( F = G + \sum_{j=m+1}^{n} y_j^2 \in P[n] \) with \( G \in P[m] \). Thanks to Corollary 3.2 we have
\[
\text{Ann}(F) + (\sigma, x_{m+1}, \ldots, x_n) = \text{Ann}(G)S[n] + (\sigma, x_{m+1}, \ldots, x_n),
\]
thus
\[
\frac{S[n]}{\text{Ann}(F) + (\sigma, x_{m+1}, \ldots, x_n)} \cong \frac{S[m]}{\text{Ann}(G) + (\sigma)}.
\]
Trivially \( S[m]/\text{Ann}(G) \) is a local, Artinian, Gorenstein, \( k \)-algebra.

Since \( \text{Soc}(A) \) is generated by the class of \( \sigma \), it follows from formula (4) that
\[
P_A(z) = \frac{P_{S[n]/\text{Ann}(F) + (\sigma)}(z)}{1 + z^2 P_{S[n]/\text{Ann}(F) + (\sigma)}(z)}.
\]
Notice that \( x_i x_j \in \text{Ann}(F) + (\sigma), i = 1, \ldots, n, j = m+1, \ldots, n, i \leq j \). In particular \( x_{m+1}, \ldots, x_n \in \text{Soc}(S[n]/\text{Ann}(F) + (\sigma)) \). It follows from formula (5) that
\[
P_{S[n]/\text{Ann}(F) + (\sigma)}(z) = \frac{P_{S[n]/\text{Ann}(F) + (\sigma, x_{m+1}, \ldots, x_n)}(z)}{1 - (n - m) z P_{S[n]/\text{Ann}(F) + (\sigma, x_{m+1}, \ldots, x_n)}(z)}.
\]
The inverse formula of (4) finally yields

\[ P_{S[n]/\text{Ann}(F)}(z) = P_{S[m]/\text{Ann}(G)}(z) = \frac{P_{S[m]/\text{Ann}(G)}(z)}{1 - z^2 P_{S[m]/\text{Ann}(G)}(z)}. \]

Combining the above equalities we finally obtain the following

**Proposition 4.1.** Let \( G \in P[m] \), \( F := G + \sum_{j=m+1}^{n} y_j^2 \) and define \( A := S[n]/\text{Ann}(F) \) and \( B := S[m]/\text{Ann}(G) \). Then

\[ P_A(z) = \frac{P_B(z)}{1 - (H_A(1) - H_A(2))zP_B(z)}. \]

A first immediate consequence of the above Proposition is the following corollary.

**Corollary 4.2.** Let \( G \in P[m] \), \( F := G + \sum_{j=m+1}^{n} y_j^2 \) and define \( A := S[n]/\text{Ann}(F) \) and \( B := S[m]/\text{Ann}(G) \). The series \( P_B(z) \) is rational if and only if the same is true for \( P_A(z) \).

Now assume that \( m \leq 4 \). Since the Poincaré series of each local Artinian, Gorenstein ring with embedding dimension at most four is rational (see [17], [19], [20], [15]) we also obtain the following corollary.

**Corollary 4.3.** Let \( G \in P[4] \), \( F := G + \sum_{j=5}^{n} y_j^2 \) and define \( A := S[n]/\text{Ann}(F) \).

Then \( P_A(z) \) is rational.

Let \( A \) be a local, Artinian, Gorenstein \( k \)-algebra with \( n := H_A(1) \).

**Corollary 4.4.** Let \( A \) be a local, Artinian, Gorenstein \( k \)-algebra such that \( f_3 \leq 4 \). Then \( P_A(z) \) is rational.

**Proof.** If \( s := \text{sdeg}(A) \), then

\[ A \cong S[n]/\text{Ann}(F) \]

where \( F := \sum_{i=2}^{s} F_i + \sum_{j=f_3+1}^{n} y_j^2 \), \( F_i \in P[f_i], i \geq 3 \) and \( F_2 \in P[f_2] \) (see [7], Remark 4.2). Thus the statement follows from Corollary 4.3.

5. **Examples of algebras with rational Poincaré series**

In this section we give some examples of local, Artinian, Gorenstein \( k \)-algebras \( A \) with rational \( P_A \) using the results proved in the previous section.

We start with the following Lemma generalizing a result in [18].

**Lemma 5.1.** Let \( A \) be a local, Artinian, Gorenstein, 3-stretched \( k \)-algebra. If \( H_A(3) \leq 5 \), then \( \sum_{a=0}^{s-4} H_{Q_{A(a)}}(2) \geq H_A(3) \).

**Proof.** We set \( m := H_A(3) \) and \( p := \sum_{a=0}^{s-4} H_{Q_{A(a)}}(2) \). We have to show that \( p \geq m \):

Assume \( p \leq m - 1 \).

If \( s = 4 \), then \( H_{Q_{A(0)}} = \sum_{a=0}^{s-4} H_{Q_{A(a)}}(1, m, p, m, 1) \). If \( s \geq 5 \), then we have

\[ H_{Q_{A(a)}} = \begin{cases} (1, 1, 1, 1, 1, \ldots, 1) & \text{if } a = 0, \\ (0, 0, 0, 0, 0, \ldots, 0) & \text{if } a = 1, \ldots, s - 5, \\ (0, m - 1, p - 1, m - 1, 0, \ldots, 0) & \text{if } a = s - 4. \end{cases} \]
In particular $\sum_{a=0}^{s-4} H_{QA(a)} = H_{QA(0)} + H_{QA(s-4)}$. Notice that $f_4 = m$.

Macaulay’s growth theorem (see [5], Theorem 4.2.10) and the restriction $m \leq 5$ imply that $3 \leq p = m - 1$ necessarily. Thus we can restrict our attention to the two cases $p = 3, 4$. We examine the second case, the first one being analogous.

Let $n := H_{A}(1)$, take a polynomial $F := y^n_1 + F_3 + F_4$, $F_i \in P[F_i]$ for $i = 1, 2$, $x_1^3 \circ F_4 = 0$ such that $A \cong S[n]/\text{Ann}(F)$ (see Remark 4.2 of [7]) and set $B := S[n]/\text{Ann}(F_{\geq 4})$.

We first check that $H_B = \sum_{a=0}^{s-4} H_{QA(a)} = (1, 5, 4, 5, 1, \ldots, 1)$. On the one hand, Lemma 1.10 of [14] implies that $\widehat{C}_A(a) = \widehat{C}_B(a)$, $a \leq s - 3$, whence

$$H_B(1) \geq \sum_{a=0}^{s-4} H_{QA(a)}(1) = \sum_{a=0}^{s-4} H_{QA(a)}(1) = 5.$$ 

On the other hand, $F_{\geq 4} \in P[f_4] = P[5]$, whence $5 = H_B(1) \leq 5$. It follows that equality holds, thus $H_{QA(s-2)}(1) = H_{QA(s-3)}(1) = 0$. By symmetry we finally obtain $H_{QA(s-2)} = H_{QA(s-3)} = 0$. This last vanishing completes the proof of the equality $H_B = \sum_{a=0}^{s-4} H_{QA(a)} = (1, 5, 4, 5, 1, \ldots, 1)$.

Let $I \subseteq k[x_1, \ldots, x_n] \subseteq S[n]$ be the ideal generated by the forms of degree at most 2 inside $\text{Ann}(tdf(F_{\geq 4})) = \text{ldf}(\text{Ann}(F_{\geq 4}))$. We obviously have $x_6, \ldots, x_n \in I$, because $F_{\geq 4} \in P[5]$. Denote by $I^{\text{sat}}$ the saturation of $I$ and set $R := k[x_1, \ldots, x_n]/I$, $R^{\text{sat}} := k[x_1, \ldots, x_n]/I^{\text{sat}}$. Due to the definition of $I$ we know that $H_R(t) \geq H_B(t)$ for each $t \geq 0$, and equality holds true for $t \leq 2$. Moreover, we know that

$$H_B(2) = H_B(3) \leq H_R(3) \leq H_R(2) = H_B(2),$$

hence

$$H_R(3) = \binom{4}{3} + \binom{2}{2} = H_R(2).$$

Gotzmann Persistence Theorem (see [3], Theorem 4.3.3) implies that

$$H_R(t) = \binom{t+1}{t} + \binom{t-1}{t-1} = t + 2, \quad t \geq 2.$$ 

We infer $H_{R^{\text{sat}}}(t) = t + 2$, $t \gg 0$.

When saturating, the ideal can only increase its size in each degree, hence $H_{R^{\text{sat}}}(t) \leq H_R(t)$ for each $t \geq 0$. Again Macaulay’s bound thus forces $H_{R^{\text{sat}}}(t) = H_R(t) = t + 2$ for $t \geq 2$. In particular the components $I_t$ and $I_t^{\text{sat}}$ of degree $t \geq 2$ of $I$ and $I^{\text{sat}}$ coincide.

Since $H_{R^{\text{sat}}}$ is non-decreasing, it follows that

$$H_{R^{\text{sat}}}(1) \leq H_{R^{\text{sat}}}(2) = 4 < 5 = H_B(1) = H_R(1).$$

In particular there exists a linear form $\ell \in I^{\text{sat}} \setminus I$. The equality $I_2 = I_2^{\text{sat}}$ forces $\ell x_j \in I_2 \subseteq \text{Ann}(tdf(F_{\geq 4})), j = 1, \ldots, n$. Since $x_6, \ldots, x_n \in I$, it follows that we can assume $\ell \in S[5] \subseteq S[n]$. Moreover we also know that $y^1_1 \in tdf(F_{\geq 4})$, hence $\ell$ cannot be a multiple of $x_1$. In particular we can change linearly coordinates in such a way that $\ell = x_5$.

If $j \geq 2$, then $x_j \circ F_{\geq 4} = x_j \circ F_4$, thus the condition $x_j x_5 \in I_2 \subseteq \text{Ann}(tdf(F_{\geq 4})), j = 2, \ldots, 5$, and $x_5^2 \circ F_4 = 0$ imply that $x_5 \circ F_4 = 0$. Such a vanishing contradicts
the linear independence of the derivatives

\[ x_2 \circ F_{\geq 4}, \quad x_3 \circ F_{\geq 4}, \quad x_4 \circ F_{\geq 4}, \quad x_5 \circ F_{\geq 4}. \]

Indeed \(5 = H_B(1) = \dim_k(tdf(F_{\geq 4}))\) and \(x_j \circ F_{\geq 4} = 0, \ j \geq 6. \)

Using the results proved in the previous section and the Lemma above we are able to handle the first example of this section, proving the following theorem generalizing Corollary 2.2 of [8].

**Theorem 5.2.** Let \(A\) be a local, Artinian, Gorenstein \(k\)-algebra with \(H_A(2) \leq 4\) and \(cdeg(A) \leq 3\). Then \(P_A\) is rational.

**Proof.** Let us examine the case \(cdeg(A) = 3\), the other ones being similar. Lemma 5.1 yields

\[
H_A(2) \geq \sum_{a=0}^{s-4} H_{Q(a)}(2) \geq H_A(3).
\]

If \(sdeg(A) \geq 5\), then Decomposition (3) is

\[ (1, 1, \ldots, 1) + (0, a_1, a_2, a_1, 0) + (0, b_1, b_1, 0) + (0, c_1, 0) \]

for some integers \(a_1, a_2, b_1, c_1\). Inequality (6) is equivalent to \(a_1 \leq a_2\). We know that \(H_A(2) = a_2 + b_1 + 1 \leq 4\), so \(f_4 = a_1 + b_1 + 1 \leq 4\) and the argument follows from Corollary 4.4. In the case \(sdeg(A) = 4\), the decomposition (3) changes, but the argument stays the same. \(\square\)

Now we skip the condition \(cdeg(A) = 3\) but we impose a restriction on the shape of \(H_A\). The following theorem generalizes a well-known result proved when either \(m = 1, 2\) (see [16] and [10] respectively) or \(m \leq 4\) and \(s = 3\) (see again [8]).

**Theorem 5.3.** Let \(A\) be a local, Artinian, Gorenstein \(k\)-algebra such that \(H_A(i) = m, \ 2 \leq i \leq cdeg(A)\). If \(m \leq 4\), then \(P_A\) is rational.

**Proof.** Let \(c := cdeg(A), \ n := H_A(1)\), take a polynomial \(F := y_1^c + F_{c+1} + \ldots, F_{c+1} \in P[f_{c+1}]_{c+1} = P[m]_{c+1}\) such that \(A \cong S[n]/Ann(F)\) (see Remark 4.2 of [7]) and set \(B := S[n]/Ann(F_{\geq c+1})\) so that \(Q_A(a) = Q_B(a)\) for \(a \leq s - c - 1\) (again by Lemma 1.10 of [14]). In particular \(H_B(c) = m\), thus Decomposition (3) implies \(H_B(1) \geq m\). Since we know that \(F_{\geq c+1} \in P[m]\), it follows that \(H_B(1) \leq m\), hence equality must hold.

As in the proof of the previous lemma one immediately checks that either \(s = c+1\), and \(H_{Q_A(0)} = (1, m, \ldots, m, 1)\), or \(s \geq c+2\), and

\[ H_{Q_A(a)} = \begin{cases} 
(1, 1, \ldots, 1, 1, \ldots, 1) & \text{if } a = 0, \\
(0, \ldots, 0, 0, \ldots, 0) & \text{if } a = 1, \ldots, s - c - 2, \\
(0, m - 1, \ldots, m - 1, 0, \ldots, 0) & \text{if } a = s - c - 1.
\end{cases} \]

Assume that \(H_B(i) \leq m - 1 \leq 3\) for some \(i = 2, \ldots, c - 1\). Let \(i_0\) be the maximal of such \(i\)'s. We know that there are \(k(i_0) > k(i_0 - 1) > k(i_0 - 2) > \ldots\) such that

\[ H_B(i_0) = \binom{k(i_0)}{i_0} + \binom{k(i_0 - 1)}{i_0 - 1} + \binom{k(i_0 - 2)}{i_0 - 2} + \ldots \leq m - 1 \leq 3. \]
If \( i_0 \geq 3 \), it would follow \( k(i_0) \leq i_0 \), thus Macaulay’s bound implies
\[
H_B(i_0 + 1) \leq H_B(i_0)^{(i_0)} = \left( \frac{k(i_0) + 1}{i_0 + 1} \right) + \left( \frac{k(i_0 - 1) + 1}{i_0} \right) + \left( \frac{k(i_0 - 2) + 1}{i_0 - 1} \right) + \ldots =
\]
a contradiction. We conclude that \( i_0 = 2 \).

Due to the symmetry of \( H_{Q_B(s-c-1)} \) we deduce that \( c = 3 \). If \( H_{Q_B(s-3)}(2) = q \), the symmetry of \( H_{Q_B(s-3)} \) implies \( H_{Q_B(s-3)}(1) = q \), hence Decomposition (3) implies
\[
m = H_B(1) = \sum_{a=0}^{s-2} H_{Q_B(a)}(1) = m + q + H_{Q_B(s-2)}(1).
\]
It follows that \( q = H_{Q_B(s-2)}(1) = 0 \), whence \( H_B = (1, m, p, m, 1, \ldots, 1) \) where \( p \leq m - 1 \) which cannot occur by Lemma 5.1.

We conclude that \( H_{Q_A(s-c-1)}(i) = H_{Q_B(s-c-1)}(i) = m - 1 \) for each \( i = 2, \ldots, c \), then the hypothesis on \( H_A(i) \) and Decomposition (3) yield
\[
H_{Q_A(a)} = \begin{cases} (0, 0, 0, \ldots, 0, 0, \ldots, 0) & \text{if } a = s-c, \ldots, s-3, \\ (0, n-m, 0, \ldots, 0, 0, \ldots, 0) & \text{if } a = s-2,
\end{cases}
\]
whence \( f_3 = \sum_{a=1}^{s-3} H_{Q_A(a)}(1) = m \leq 4. \square \)

As third example we skip the condition on the shape of \( H_A \) but we put a limit on \( \dim_k(A) \), slightly extending the result proved in [7].

**Theorem 5.4.** Let \( A \) be a local, Artinian, Gorenstein \( k \)-algebra with \( \dim_k(A) \leq 16 \) and \( H_A(2) \leq 4 \). Then \( P_A \) is rational.

**Proof.** Thanks to [15] we can restrict our attention to algebras \( A \) with \( H_A(1) \geq 5 \).

The rationality of the Poincaré series of stretched algebras is proved in [10]. For almost stretched algebras see [10]. For the case of algebras \( A \) with \( \sdeg(A) = 3 \) and \( H_A(2) \leq 4 \) see [8]. Finally the case \( H_A(i) = m, 2 \leq i \leq \cdeg(A) \) with \( m \leq 4 \) is covered by Theorem 5.3 above.

There are several cases which are not covered by the aforementioned results. In each of these cases one can check that the condition \( f_3 \leq 4 \) of Corollary 4.4 is fulfilled. We know that necessarily \( H_A(2) \geq 3 \), otherwise \( A \) is almost stretched by Macaulay’s bound. The restriction \( H_A(2) \leq 4 \) implies \( H_A(3) \leq 5 \) again by Macaulay’s bound.

Theorem 5.2 deals with the case \( \sdeg(A) = 4 \). Let us analyze the case \( \sdeg(A) = 5 \) and \( \dim_k(A) \leq 16 \). The decomposition is
\[
(1, a_1, a_2, a_2, a_1, 1) + (0, b_1, b_2, b_1, 0) + (0, c_1, c_1, 0) + (0, d_1, 0)
\]
for some integers \( a_1, a_2, b_1, b_2, c_1, d_1 \). If \( a_1 = 1 \) then the algebra is 3-stretched, so we may suppose \( a_1 \geq 2 \). We know that \( H_A(2) = a_2 + b_2 + c_1 \leq 4 \) and we would like to prove \( a_1 + b_1 + c_1 \leq 4 \). Suppose \( a_1 + b_1 + c_1 \geq 5 \), then the inequality on the dimension of \( A \) shows that \( 2 \cdot a_2 + b_2 \leq 4 \), in particular \( a_2 \leq 2 \) and from Macaulay’s bound it follows that \( a_1 = a_2 = 2 \). It follows that \( b_2 = 0 \) and once again from Macaulay’s bound \( b_1 = 0 \). This forces \( a_1 + b_1 + c_1 = 2 + c_1 = a_2 + b_2 + c_1 \leq 4 \), a contradiction.
Let us now suppose that $s\deg(A) = 6$. Look at the first row of the symmetric decomposition \( (1, a_1, a_2, a_3, a_2, a_1) \).

- If $a_1 \geq 3$, then $a_2, a_3 \geq 3$ and the sum of the row is at least 17.
- If $a_1 = 2$ then $a_2 = a_3 = 2$ and the sum of the row is 12. If we suppose that $f_3 \geq 5$, then the sum of the first column of the remaining part of the decomposition will be at least three, so the sum of whole remaining part will be at least $2 \cdot 3 = 6$ and the dimension will be at least $12 + 6 > 16$.
- Suppose $a_1 = 1$ and look at the second row \( (0, b_1, b_2, b_2, b_1, 0) \). If $b_1 = 0$ then the algebra is 3-stretched so the result follows from Theorem 5.2. From $H_A(2) \leq 4$ it follows that $b_2 \leq 3$. If $b_2 = 3$, then $b_1 \geq 2$ so the dimension is at least $7 + 10 > 16$. If $b_2 \leq 2$ then $b_1 \leq b_2$ from Macaulay’s bound. Hence, the same argument as before applies.

Let us finally suppose that $s\deg(A) \geq 7$. Take the first row, beginning with \( (1, a_1, a_2, \ldots) \). If $a_1 \geq 3$ then its sum is at least $3 \cdot s\deg(A) - 1 > 16$. If $a_1 = 2$, the sum of this row is $2 \cdot s\deg(A) > 14$. Then one can argue as in the case $s\deg(A) = 6, a_1 = 2$. A similar reasoning shows that when $a_1 = 1$ the algebra has decomposition \( (1, 1, \ldots, 1) + (0, 4, 4, 0) \) and so $H_A(2) \geq 5$. □

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