Heat Kernel Coefficients for Chern-Simons Boundary Conditions in QED

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Abstract

We consider the four dimensional Euclidean Maxwell theory with a Chern–Simons term on the boundary. The corresponding gauge invariant boundary conditions become dependent on tangential derivatives. Taking the four-sphere as a particular example, we calculate explicitly a number of the first heat kernel coefficients and obtain the general formulas that yields any desired coefficient. A remarkable observation is that the coefficient $a_2$, which defines the one-loop counterterm and the conformal anomaly, does not depend on the Chern–Simons coupling constant, while the heat kernel itself becomes singular at a certain (critical) value of the coupling. This could be a reflection of a general property of Chern–Simons theories.

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1 Introduction

The heat kernel expansion plays an important role in quantum field theory. In fact, the ultraviolet divergences and renormalization structure of the theory at the one-loop approximation are encoded in a few of the first heat kernel coefficients $a_n$. For massless theories in $d$ dimensions the only divergent term is proportional to $a_{d/2}$. The heat kernel expansion is closely related to the quantum anomalies, most notably, to the conformal and chiral anomalies. It has also numerous applications in mathematics. No wonder that much effort has been devoted to calculate the heat kernel expansion. For operators of the Laplace type, the coefficients $a_n$ are known fairly well, both for manifolds without a boundary and for Dirichlet and Neumann boundary conditions in manifolds with a boundary. More details on the physical applications of the heat kernel expansion can be found in the recent monographs [1]. The mathematical background is described in [2].

During the last couple of years, considerable interest has been attracted by the heat kernel expansion for boundary conditions depending on tangential derivatives [3]–[8], following the pioneering work [9]. In the mathematical literature such boundary conditions are called oblique. For physicists, the interest towards oblique boundary conditions has been primarily motivated by problems of quantum gravity [10]. It has been realized that the diffeomorphism invariant boundary conditions for the graviton inevitably contain tangential derivatives [11, 3, 12, 13]. Oblique boundary conditions may also appear in electrodynamics, when the properties of a physical boundary depend on the photon frequency. A more formal motivation for studying oblique boundary conditions can be found in [14], where tangential derivatives arise when one calculates vacuum expectation values of second order differential operators.

The usual strategy [3]–[8] employed in the calculation of the heat kernel coefficients for oblique boundary conditions is as follows. By means of the corresponding invariant theory, each coefficient can be expressed through several universal functions, which are calculated by using ordinary conformal variation techniques and some explicit examples. However, up to now, only a very few examples admitting calculation of the spectrum and an explicit evaluation of the heat kernel were known [3, 8]. In the present paper we study the Euclidean Maxwell theory with a Chern–Simons boundary term, which generates oblique boundary conditions. We find the spectrum of the Laplace operator on a ball and calculate the heat kernel expansion.

Apart from its application to the theory of the heat kernel expansion, our model may be interesting by itself. Tangential derivatives appear on the boundary conditions in a very natural way. An intriguing property of the model, established in the present paper, is that the conformal anomaly and the one-loop counterterm do not depend on the charge standing in front of the Chern–Simons action. Other heat kernel coefficients become singular for a certain value of this charge, what suggests the existence of a critical value of the Chern–Simons charge. Let us recall that the Chern–Simons gauge theory introduced in [15] exhibits some unusual and intriguing properties, as, for example, generation of states with fractional statistics first observed by Wilczek [16]. Later on this phenomenon was used in the theory of the Fractional Quantum Hall Effect (FQHE) (for a recent review see [17]). In spite of the fact that the geometry of our model is different from the ones usually considered in the FQHE, the results of our calculation may be important in order to gain insight into the general properties of Chern–Simons theories, as e.g. non-renormalization theorems [18].

The paper is organized as follows. In section 2 we formulate the model, discuss its general properties and define the eigenfunctions on the Euclidean ball. In section 3 we calculate the heat kernel asymptotics explicitly. A discussion on the possible significance of the results obtained (a
list of the first ten heat kernel coefficients is given in Table 1) is presented in section 4.

## 2 Chern–Simons boundary conditions

Consider the action for the Euclidean Maxwell theory on a manifold $M$

$$S = \frac{1}{4} \int_M d^4x \, g^{\frac{1}{2}} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (1)$$

We can add to the action (1) a boundary term. If we require gauge and coordinate invariance to hold and, if we do not want to introduce any dimensional parameter, the only choice available is the Chern–Simons action

$$S_{CS} = \frac{a}{2} \int_{\partial M} d^3x \, \varepsilon^{ijk} A_i \partial_j A_k, \quad (2)$$

where $\varepsilon^{ijk}$ is the Levi-Civita tensor and the $x^j, j = 1, 2, 3,$ are coordinates on the boundary $\partial M$, $a$ being a real parameter.

To calculate the path integral, it is convenient to write $S + S_{CS}$ in the form $\int A L A$, with $L$ a second order differential operator. To this end, we integrate by parts, obtaining

$$S + S_{CS} = \frac{1}{2} \int_M d^4x \, g^{\frac{1}{2}} A_\mu (-g^{\mu\nu} \Delta + \nabla^\nu \nabla^\mu) A_\nu + \frac{1}{2} \int_{\partial M} d^3x (h^\frac{1}{2} (\partial N A_i - \partial_i A_N) A_i + a \varepsilon^{ijk} A_i \partial_j A_k), \quad (3)$$

where $\Delta$ is the Laplace operator, $N$ the outward pointing normal vector, and $h$ is the determinant of the induced metric on $\partial M$. There are, at least, two sets of gauge invariant boundary conditions which ensure vanishing of the surface term in (3). The first set is ordinary relative (or magnetic) boundary conditions: $A_i|_{\partial M} = 0, i = 1, 2, 3, (\partial N + k) A_N|_{\partial M} = 0$. Here $k$ is the trace of the second fundamental form of the boundary. These boundary conditions have been extensively studied in the literature, but we shall not consider them here. Another possible set is the following:

$$A_N|_{\partial M} = 0, \quad (\partial N A_i + a h^{-\frac{1}{2}} \varepsilon^{ijk} \partial_j A_k)|_{\partial M} = 0 \quad i = 1, 2, 3. \quad (4)$$

For $a = 0$, Eqs. (4) become ordinary absolute (or electric) boundary conditions.

The boundary conditions (4) possess two properties which make them useful for quantum electrodynamics. First, they are gauge invariant. This means that if $A_\mu$ satisfies (4), then $A_\mu + \partial \phi$ also satisfies (4), provided that

$$\partial N \phi|_{\partial M} = 0. \quad (5)$$

After quantization, Eq. (3) becomes the boundary condition for the ghost field. Second, the Laplace operator is symmetric, i.e.,

$$\int_M d^4x \, g^{\frac{1}{2}} (A^{(1)\mu} \Delta A^{(2)}_\mu - A^{(2)\mu} \Delta A^{(1)}_\mu) = 0, \quad (6)$$

if both $A^{(1)}$ and $A^{(2)}$ satisfy (4).

It is natural to impose the Lorentz gauge condition

$$\nabla^\mu A_\mu = 0. \quad (7)$$
In this case the path integral is given by

\[ Z = \det_T^{-\frac{1}{2}}(-\Delta) \times \det_S^{\frac{1}{2}}(-\Delta), \]  

(8)

where the first determinant is calculated on the space of transversal vectors, with the boundary condition (4), and the second one on the space of scalar fields satisfying (5).

As an example, consider a ball with the metric

\[ ds^2 = (dx^0)^2 + (x^0)^2d\Omega^2, \quad 0 \leq x^0 \leq r, \]  

(9)

where \( d\Omega^2 \) is the metric on the unit sphere \( S^3 \).

We can use the basis of Ref. [19] in the space of transversal vector fields:

\[ \{ A^T \} = \{ A^\perp, A(\psi) \}, \]

where

\[ A^\perp_0 = 0, \quad (3)\nabla^i A^\perp_i = 0, \]

\[ A_0(\psi) = (3)\Delta r \psi, \quad A_i(\psi) = -(3)\nabla_i (\partial_0 + \frac{1}{r}) r \psi, \quad i = 1, 2, 3, \]  

(10)

\( \psi \) being a scalar field, and \((3)\nabla\) and \((3)\Delta\) the covariant derivative and the Laplacian on \( S^3 \), respectively.

The boundary conditions for the field \( \psi \) do not depend on \( a \). Hence, the contribution of \( \psi \) to the heat kernel and functional determinant is the same as for the absolute boundary condition [19, 20]. Therefore, let us concentrate on the \( A^\perp \) contribution. The operator \( h^{-\frac{1}{2}}\varepsilon^{ijk}\partial_j \) can be diagonalized on the unit sphere \( S^3 \) (see, e.g., [21]). It has for eigenvalues \( \pm(l + 1) \), with degeneracies given by

\[ D^\pm_l = l(l + 2), \quad l = 1, 2, \ldots \]  

(11)

The corresponding eigenvalues of the Hodge–de Rham Laplacian on \( S^3 \) are \( -(l + 1)^2 \). The eigenfunctions of the vector Laplace operator on the unit ball are found to be

\[ J_{l+1}(\lambda x^0) Y^\pm_l(x^i), \]  

(12)

where \( J_{l+1} \) are ordinary Bessel functions and \( Y^\pm_l \) are vector spherical harmonics. The eigenvalues of the Laplacian are \( -\lambda^2 \), where the \( \lambda \)'s are defined, implicitly, through the equation for the boundary condition:

\[ \lambda^\pm_l J'_{l+1}(\lambda^\pm_l) \pm \frac{a(l + 1)}{r} J_{l+1}(\lambda^\pm_l) = 0. \]  

(13)

The prime denotes here differentiation with respect to the argument. The contribution of \( A^\perp \) to the path integral reads

\[ Z^\perp = \prod_{\lambda^\pm_l} (\lambda^\pm_l)^{(l+2)}. \]  

(14)

If one proceeds, as usually, by splitting the calculation of the determinant into two parts, one has to take into account a possibly non-vanishing determinant anomaly (see, e.g., [25]).

3 The zeta function: calculation of the singularities

In the following, we will use the connection between the zeta function and the heat kernel:

\[ \zeta^\pm(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \sum_k e^{-(\lambda_k^\pm)^2 t}. \]  

(15)
in order to obtain the heat kernel coefficients $C^\pm_j$ or $a^{\pm}_{n/2}$:

$$
\sum_k e^{-(\lambda_k^\pm)^2}t = \frac{1}{(4\pi)^2} \sum_{n=0}^\infty t^{n-2}a^{\pm}_{n/2},
$$

$$
\frac{1}{(4\pi)^2}a^{\pm}_{n/2} = \frac{1}{(4\pi)^{3/2}}C^{\pm}_{(n-1)/2} = \text{Res} \left[ \zeta^{\pm}(s) \Gamma(s) \right]_{s=2-\frac{n}{2}}.
$$

(16)

The shifted heat kernel coefficients $C^{tot}_j = C^+_j + C^-_j$ are commonly used in Casimir energy calculations. Calling $\Phi^\pm$ the functions that determine implicitly the spectral values corresponding to the given boundary conditions (13), the zeta function is obtained as

$$
\zeta^{\pm}(s) = \sum_{l=0}^\infty D^\pm_l(\lambda^\pm_l)^{-2s} = \sum_{l=0}^\infty l(l+2) \int_0^\gamma \frac{dk}{2\pi t} k^{-2s} \frac{\partial}{\partial k} \ln \Phi^\pm_l(kr),
$$

with

$$
\Phi^\pm_l(kr) \equiv J'_{l+1}(kr) \pm \frac{a(l+1)}{kr} J_{l+1}(kr),
$$

(17)

where we have transformed the sum over the spectrum into a contour integral which goes around all the real zeros of the function $\Phi$ defining the boundary conditions.

By doing this we have reduced the problem to the evaluation of certain contour integrals, which can be solved in an explicit —though technically quite involved— way. The interested reader is addressed to the Refs. [23, 24], where this powerful method is described in great detail —in the particular case $a = 0$. Having understood this case, however, the situation here is not difficult to grasp, the main differences being in Eqs. (23) and (24) to follow, which will lead, in the end, to more involved hypergeometric functions. In any case, the final expressions will be still explicit, allowing for the straightforward use of very quick algebraic computation machinery, to obtain the final expressions of the heat kernel coefficients to any order. This will demonstrate, once more, the power (and adaptability) of the zeta-function procedure to perform these sort of calculations (which are extremely cumbersome, by any means).

By deforming the integration contour around the imaginary axis, in the usual way [23, 24], we obtain

$$
\zeta^{\pm}(s) = \frac{\sin(\pi s)}{\pi} \sum_{l=0}^\infty l(l+2) \int_0^\infty dk k^{-2s} \left( I'_{l+1}(kr) \pm \frac{a(l+1)}{kr} I_{l+1}(kr) \right) \ln \left( k^{-(l+1)} \right).
$$

(19)

Introducing into the equation the asymptotic expansions corresponding to $I$ and $I'$ when $k,l \to \infty$, with $z = kr/(l+1)$ fixed, we can write the zeta function under the form

$$
\zeta^{\pm}(s) = \frac{\sin(\pi s)}{\pi} \sum_{l=0}^\infty l(l+2) \left( \frac{l+1}{r} \right)^{-2s} \int_0^\infty dz z^{-2s} \frac{\partial}{\partial z} \left( (l+1) \ln \left( \frac{e^{1/t}}{zt} \sqrt{\frac{1-t}{1+t}} \right) \right) + \ln \left[ 1 + \sum_{j=1}^\infty \frac{v_j(t)}{(l+1)^j} \pm at \left( 1 + \sum_{j=1}^\infty \frac{u_j(t)}{(l+1)^j} \right) \right].
$$

(20)
where the functions $u_j(t)$ and $v_j(t)$ give the asymptotics of the Bessel functions in the way

\[ I_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi \nu}} \frac{e^{\nu z}}{(1+z^2)^{\frac{\nu}{2}}} \left[ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right], \]

\[ I'_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi \nu}} \frac{e^{\nu z}(1+z^2)^{\frac{\nu}{2}}}{z} \left[ 1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{\nu^k} \right], \]

with $t = 1/\sqrt{1+z^2}$ and $\eta = \sqrt{1+z^2} + \ln[z/(1+\sqrt{1+z^2})]$. The first few coefficients are given in [23], higher coefficients are easy to obtain using the recursions

\[ u_{k+1}(t) = \frac{1}{2} t^2 (1-t^2) u_k'(t) + \frac{1}{8} \int_0^t d\tau \ (1-5\tau^2) u_k(\tau), \]

\[ v_k(t) = u_k(t) + t(t^2-1) \left( \frac{1}{2} u_{k-1}(t) + tu_{k-1}'(t) \right) \]  

(see [23, 24]). It is convenient to write the last logarithmic function in the expression of $\zeta^\pm(s)$ above as a series

\[ \ln \left[ 1 + \sum_{j=1}^{\infty} \frac{v_j(t)}{(l+1)^{j}} \pm at \left( 1 + \sum_{j=1}^{\infty} \frac{u_j(t)}{(l+1)^{j}} \right) \right] = \ln(1 \pm at) + \sum_{n=1}^{\infty} F^\pm_n(t) (1+\gamma)^{-n}, \]

\[ F^\pm_n(t) = \frac{\hat{p}^\pm_{4n}(t)}{(1 \pm at)^n}, \]

with $\hat{p}^\pm_{4n}(t)$ a polynomial in $t$ of degree $4n$, which has the form

\[ \hat{p}^\pm_{4n}(t) = \sum_{j=0}^{3n} z^j_{n,j} t^{n+j}. \]

The use of this expansion can be based on the corresponding one for the case $a = 0$ (see Refs. [23, 24]) and will be fully justified by the subsequent calculations. For any heat kernel coefficient only a limited number of terms in (23) are needed. The integrations are readily performed using the basic result (Re $s < 1$):

\[ \int_0^\infty dz \ z^{-2s} \frac{\partial}{\partial z} \left[ \frac{t^k}{(1+\gamma)^n} \right] = -\Gamma(1-s) \frac{\Gamma(n+h)}{\Gamma(n)} \sum_{h=0}^{\infty} (0^h a)^h \frac{\Gamma(s+k+h)}{h! \Gamma \left( \frac{k+h}{2} \right)} \]

\[ = \Gamma(1-s) \left[ \pm a \ n \ \frac{\Gamma \left( \frac{s+k+1}{2} \right)}{\Gamma \left( \frac{k+1}{2} \right)} H \left( \left\{ \frac{n+1}{2}, s+\frac{k+1}{2} \right\}, \left\{ \frac{3}{2}, s+\frac{k+1}{2} \right\}; a^2 \right) \]

\[ - \frac{\Gamma \left( s+\frac{k}{2} \right)}{\Gamma \left( \frac{s+k}{2} \right)} H \left( \left\{ \frac{n+1}{2}, s+\frac{k}{2} \right\}, \left\{ \frac{1}{2}, s+\frac{k}{2} \right\}; a^2 \right), \]

where $H$ is the generalized hypergeometric function. Observe that for the sum over the two signs of $a$ the result simplifies to twice the second term on the r.h.s. of Eq. (25).

However, one has to take into account that, as it stands, $\zeta^\pm(s)$ is a divergent quantity. We shall extract the singular parts of the zeta function according to the prescription in [23]...
[24], namely, by splitting the zeta function into a regular and a singular part and adding and subtracting a number, say $M$, of leading terms of the asymptotic expansion:

$$\zeta^\pm(s) = Z^\pm(s) + \sum_{i=-1}^M A^\pm_{i,a}(s).$$

(26)

The functions $A^\pm_{i,a}(s)$ are found to be:

$$A^\pm_{1}(s) = \frac{r^{2s} \Gamma(s-1/2)}{4\sqrt{\pi} \Gamma(s+1)} [\zeta(2s-3) - \zeta(2s-1)],$$

$$A^\pm_{0}(s) = \frac{r^{2s}}{4} [\zeta(2s-2) - \zeta(2s)],$$

$$A^\pm_{0,a}(s) = \frac{r^{2s}}{2 \Gamma(s)} [\zeta(2s-2) - \zeta(2s)] \sum_{h=1}^{\infty} (\pm a)^h \frac{\Gamma(s+h/2)}{\Gamma(1+h/2)},$$

$$A^\pm_{n,a}(s) = -\frac{r^{2s}}{\Gamma(s)} [\zeta(2s+n-2) - \zeta(2s+n)] \times \sum_{j=0}^{3n} \frac{z^\pm_{n,j}}{\Gamma(n)} \sum_{h=0}^{\infty} (\pm a)^h \frac{\Gamma(n+h) \Gamma(s+j+h)}{h! \Gamma(\frac{j+n+h}{2})},$$

where $\zeta(s)$ is the Riemann zeta function. Notice that all the series that appear here are in fact of the form corresponding to Eq. (25) and thus give rise to generalized hypergeometric functions (the result simplifies, when adding the ± contributions). Let us study the poles of $A^\pm_n$ at half integer numbers, $s = 3/2 - k$, $k \in \mathbb{N}$. For even $n = 2p$ the poles of the gamma functions at $s = 3/2 - k$ on the r.h.s. of [24] do not contribute to $A^{\text{tot}} = A^+ + A^-$ due to the property $z^+_n = (-1)^j z^-_{n,j}$. For odd $n = 2p - 1$ the Riemann zeta functions on the r.h.s. have poles at integer values of $s$ only. From those expressions, the following residua are obtained ($k \in \mathbb{N}$):

$$\text{Res } A^{\text{tot}}_{-1}(1/2) = \frac{r}{2 \pi},$$

$$\text{Res } A^{\text{tot}}_{0,a}(3/2) = \frac{r^3}{2} \left[ (1 - a^2)^{-3/2} - 1 \right],$$

$$\text{Res } A^{\text{tot}}_{0,a}(1/2) = -\frac{r}{2} \left[ (1 - a^2)^{-1/2} - 1 \right],$$

$$\text{Res } A^{\text{tot}}_{2n,a}(3/2 - n) = -\frac{r^{3-2n}}{\Gamma(3/2 - n)} \left\{ \sum_{j=0}^{3n} \frac{z^+_{2n,2j}}{\Gamma(2n)} \sum_{h=0}^{\infty} a^{2h} \Gamma(2n+2h) \Gamma(j+h+3/2) \right\}, \quad n \geq 1,$n \geq 1,$

$$\text{Res } A^{\text{tot}}_{2n,a}(1/2 - n) = \frac{r^{1-2n}}{\Gamma(1/2 - n)} \left\{ \sum_{j=0}^{3n} \frac{z^+_{2n,2j+1}}{\Gamma(2n)} \sum_{h=0}^{\infty} a^{2h-1} \Gamma(2n+2h-1) \Gamma(j+h+3/2) \right\}, \quad n \geq 1,$$n \geq 1,$

$$\text{Res } A^{\text{tot}}_{2n-1,a}(3/2 - k) = \frac{(-1)^k r^{3-2k}}{\Gamma(3/2 - k)} [\zeta(2n-2k) - \zeta(2n-2k+2)].$$
given in Table 1. They are given as power series on generalized hypergeometric functions of the type of Eq. (25) (what is quite clear from the coefficients involved than that for the case of the Laplacian on the sphere with Robin boundary conditions. The formula can be immediately turned into a computer code to perform the final part of the calculation. It is important to observe that we have obtained, in fact, a residue. In fact we can calculate the zeta functions \( \zeta(n) \) are needed and one has \( \frac{1}{2} \pi^2 \). The relevant \( C_{k+1/2}^{\text{tot}} \) for integer values of \( k \) up to any (reasonable) order. The first few of them are given in Table 1. They are given as power series on \( a \) with rational coefficients.

Recall that the Riemann zeta function at the negative integers is given by Bernoulli numbers: \( \zeta(-n) = -B_{n+1}/(n+1) \). Using then the relation between these residua and the heat kernel coefficients

\[
\text{Res } A_{2n-1, a}^{\text{tot}}(3/2 - k) = \frac{(-1)^k 4r^{3-2k}}{\Gamma(3/2 - k)} \left[ \zeta(2n - 2k) - \zeta(2n - 2k + 2) \right]
\]

we can calculate the \( C_{k}^{\text{tot}} \), for which an explicit, closed expression can be given, in terms of generalized hypergeometric functions of the type of Eq. (25) (what is quite clear from the consideration above). The formula can be immediately turned into a computer code to perform the final part of the calculation. It is important to observe that we have obtained, in fact, a closed, explicit formula for the heat-kernel coefficients, even if the actual calculation is much more involved than that for the case of the Laplacian on the sphere with Robin boundary conditions. With the help of a fast computer we can construct (in a couple of hours) a table of heat kernel coefficients \( C_{k}^{\text{tot}} \) for integer values of \( k \) up to any (reasonable) order. The first few of them are given in Table 1. They are given as power series on \( a \) with rational coefficients.

In a similar way we can proceed with the calculation of the coefficients for half-integer index \( C_{k+1/2}^{\text{tot}} \), \( k \in \mathbb{N} \). The zeta functions \( \zeta^{\pm}(s) \) have only one pole at integer \( s \), with an \( a \)-dependent residue. In fact

\[
\text{Res } A_{1}^{\text{tot}}(1) = \frac{r^2}{a^2 - 1}
\]

The values \( \zeta^{\pm}(1 - k) \) are needed and one has \( M = 2k \). Here the \( A_{1}^{\pm}(s) \) for \( i \) odd, \( i = 2j - 1, j \in \mathbb{N}_0 \), do not contribute. The relevant \( A_{1}^{\text{tot}}(s) \) read now

\[
A_{0}^{\text{tot}}(1 - k) = \frac{r^{2-2k}}{2} \left[ \zeta(-2k) - \zeta(2 - 2k) \right],
\]

\[
A_{0,a}^{\text{tot}}(1) = -\frac{r^2}{2} \left( 1 + \frac{\pi^2}{3} \right) \frac{a^2}{1 - a^2},
\]
\[
A_{2k+1}^{\text{tot}}(1 - k) = r^{2 - 2k}(-1)^k(k - 1)! \sum_{j=0}^{3k+1} \left\{ \frac{z_{2k+1,2j}^+}{(2k)!} \sum_{h=0}^\infty a^{2h}(2k + 2h)! \Gamma(j + h + 3/2) - \frac{z_{2k+1,2j+1}^+}{(2k)!} \sum_{h=1}^\infty a^{2h-1}(2k + 2h - 1)! \Gamma(j + h + 3/2) \right\},
\]

\[
A_{2k-1}^{\text{tot}}(1 - k) = -r^{2 - 2k}(-1)^k(k - 1)! \sum_{j=0}^{3k-2} \left\{ \frac{z_{2k-1,2j}^+}{(2k - 2)!} \sum_{h=0}^\infty a^{2h}(2k + 2h - 2)! \Gamma(j + h + 1/2) - \frac{z_{2k-1,2j+1}^+}{(2k - 2)!} \sum_{h=1}^\infty a^{2h-1}(2k + 2h - 3)! \Gamma(j + h + 1/2) \right\},
\]

And, for \( n \in \mathbb{N}, n \leq k - 1 \),

\[
A_{2n}^{\text{tot}}(1 - k) = -4r^{2 - 2k}(k - 1)! \left[ \zeta(2n - 2k) - \zeta(2n - 2k + 2) \right] \times \left\{ \sum_{j=0}^{\min(k-n-1,3n)} \frac{z_{2n,2j}^+}{(2n - 1)!} \sum_{h=0}^\infty \frac{(-1)^{n+j+h}(2n + 2h - 1)! a^{2h}}{(2h)! (k - n - j - h - 1)! (n + j + h - 1)!} \right. \\

\left. - \sum_{j=0}^{\min(k-n-2,3n-1)} \frac{z_{2n,2j+1}^+}{(2n - 1)!} \sum_{h=1}^\infty \frac{(-1)^{n+j+h}(2n + 2h - 1)! a^{2h-1}}{(2h - 1)! (k - n - j - h - 1)! (n + j + h - 1)!} \right\}.
\]

Notice again that the summation ranges are different depending now on the fact that \( k \leq 4n + 1 \) or \( k \geq 4n + 1 \). The corresponding heat-kernel coefficients are readily calculated from

\[
\zeta^\pm(1 - k) = \sum_{n=0}^{k-1} A_{2n}^\pm(1 - k) + A_{2k-1}^\pm(1 - k) + A_{2k+1}^\pm(1 - k) = \frac{(-1)^{k-1}(k - 1)!}{(4\pi)^{\frac{3}{2}}} C_{k + \frac{1}{2}}^\pm.
\]

A number of the coefficients \( C_{k + \frac{1}{2}}^{\text{tot}} \) are listed in Table 1. If desired, the coefficients \( a_n \) can be then obtained by using Eq. (19). Let us finish this section by noting that here we have calculated the \( a \)-dependent part of the full heat kernel expansion for the Euclidean Maxwell theory with the boundary conditions (10) only. Complete result can be immediately recovered by taking \( C_{k + \frac{1}{2}}^{\text{tot}} \) from Table 1, subtracting the \( a^0 \) part, and adding this to the known results [19, 26, 27, 20] for absolute boundary conditions. All coefficients, including \( C_1 \), are non-singular at \( a = 0 \).

### 4 Conclusions

The conditions (4) we have considered in this paper are mixed and contain tangential derivatives. For boundary conditions of this type only the first non-trivial coefficient \( C_0 \) was known analytically, up to now (4). Our result for \( C_0 \) agrees with the one in Ref. (4). The rest of the coefficients have been calculated here explicitly for the first time. Our calculations put restrictions on the universal functions (see (3)–(8)) entering in higher heat kernel coefficients. Finally, we must point out that, even if we only list a few number of them in Table 1, in fact our program allows for the calculation of any coefficient to any degree of approximation in the \( a \) dependence (notice that the coefficients are given by rational numbers, as it should be). However, to obtain each one of them as a closed function of \( a \) is outside the scope of our purposes here.

Several of the first heat kernel coefficients are singular at \( a = 1 \). This is a manifestation of the lack of strong ellipticity in the corresponding boundary value problem (4, 7).
A very interesting feature of our results is the vanishing of the \( a \)-dependent part of \( C_{3/2} \) (or \( a_2 \)), while the heat kernel itself becomes singular at a certain (critical) value of the coupling. This means that the conformal anomaly and the one-loop counterterms are not modified by the presence of the Chern–Simons boundary action, what could be a reflection of a general property of Chern–Simons theories. Since the fundamental explanation of very important physical phenomena of lower-dimensional QED, such as the fractional quantum Hall effect and high-temperature superconductivity, seem to rely very heavily on the fundamental structures provided by Chern–Simons theories, we conclude that the property we have here found (by looking to a particular model) could have important physical consequences for those subjects. This issue is presently under investigation.

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Table 1: The first ten heat kernel coefficients.

| n  | Coefficient                                                                 |
|-----|----------------------------------------------------------------------------|
| 0   | $2\pi^2 r^3 \left[ (1 - a^2)^{-3/2} - 1 \right]$                           |
| 1/2 | $8\pi^{3/2} (a^2 - 1)^{-1}$                                                |
| 1   | $\pi r \left[ \frac{1}{2a^4} (1 - a^2)^{-1/2} (2 + a^2 - 8a^4) + \frac{117}{32} - \frac{1}{a^2} - \frac{1}{a^4} \right]$ |
| 3/2 | $\frac{59}{45} \pi^{3/2}$                                                  |
| 2   | $\frac{3}{r} \left( \frac{3}{\pi} - \frac{1339}{4096} + \frac{11 a^2}{1024} + \frac{115 a^4}{4096} + \frac{435 a^6}{16384} + \cdots \right)$ |
| 5/2 | $\pi^{3/2} r^2 \left( \frac{117919}{45045} + \frac{2048 a^2}{15015} + \frac{10112 a^4}{109395} + \frac{816896 a^6}{14549535} + \cdots \right)$ |
| 3   | $\frac{21}{8\pi} - \frac{57455}{393216} + \frac{6787 a^2}{65536} + \frac{43053 a^4}{1048576} + \frac{14431 a^6}{1048576} + \cdots$ |
| 7/2 | $\frac{371148101}{116396280} + \frac{571904 a^2}{2909907} + \frac{1259008 a^4}{37182145} - \frac{310784 a^6}{42902475} + \cdots$ |
| 4   | $\frac{633}{160\pi} - \frac{16417555}{201326592} + \frac{468537 a^2}{4194304} - \frac{64091 a^4}{16777216} - \frac{515873 a^6}{33554432} + \cdots$ |
| 9/2 | $\frac{399265868279}{60235074900} + \frac{997830656 a^2}{5019589575} - \frac{386748416 a^4}{8562829275}$ |
|     | $- \frac{68230270976 a^6}{1933976154825} + \cdots$ |