Büchi-Kamp Theorems for 1-clock ATA

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Abstract. This paper investigates Kamp-like and Büchi-like theorems for 1-clock Alternating Timed Automata (1-ATA) and its natural subclasses. A notion of 1-ATA with loop-free-resets is defined. This automaton class is shown to be expressively equivalent to the temporal logic \(\text{RatMTL} \) which is \(\text{MTL}[^{FI}]\) extended with a regular expression guarded modality. Moreover, a subclass of future timed MSO with \(k\)-variable-connectivity property is introduced as logic \(Q_k\text{MSO}\). In a Kamp-like result, it is shown that \(\text{RatMTL}\) is expressively equivalent to \(Q_k\text{MSO}\). As our second result, we define a notion of conjunctive-disjunctive 1-clock ATA (\(C \oplus D\) 1-ATA). We show that \(C \oplus D\) 1-ATA with loop-free-resets are expressively equivalent to the sublogic \(\text{FRatMTL}\) of \(\text{RatMTL}\). Moreover \(\text{FRatMTL}\) is expressively equivalent to \(Q_2\text{MSO}\), the two-variable connected fragment of \(Q_k\text{MSO}\). The full class of 1-ATA is shown to be expressively equivalent to \(\text{RatMTL}\) extended with fixed point operators.

1 Introduction

The celebrated Kamp theorem proves expressive equivalence between classical logic and temporal logic over words. Equally celebrated are Büchi theorems, which prove equivalence between classical/temporal logic and finite state automata. They constitute important results in the theory of logics, automata and their correspondences. Unfortunately, such correspondences have been hard to work out for timed automata and timed logics. This paper investigates Büchi-Kamp like theorems for an important class of timed languages, those accepted by 1-clock alternating timed automata (referred to as 1-ATA or 1-ATA from here on).

There are several different interpretations of timed logics in the literature. Notable variants are pointwise logics and continuous timed logics over finite and infinite timed words [19]. Of these, pointwise logic \(\text{MTL}[U]\) over finite words has a special place for having decidable satisfiability. In this paper, we focus on only pointwise logics over finite timed words.

1-ATA over finite words are perhaps the largest boolean closed class of timed languages for which emptiness is known to be decidable. Utilizing this fact, Ouaknine and Worrell showed in their seminal work that satisfiability of pointwise \(\text{MTL}[U]\) over finite words is decidable, by constructing a language equivalent 1-clock ATA [18], [17] for a formula of \(\text{MTL}[U]\). Unfortunately, the logic turns out to have much less expressive power than 1-ATA. Indeed \(\text{MTL}[U]\) can be reduced to partially ordered 1-ATA and is even weaker than the latter. In previous work [22], we presented a logic \(\text{SfrMTL}\) which is expressively equivalent for PO 1-clock ATA.

In a series of papers [21], [15] we have investigated decidable extensions of \(\text{MTL}[U]\) with increasing expressive power culminating in \(\text{RatMTL}\) [22], but they all turn out to be less expressive than 1-ATA. Strong Büchi-Kamp results have remained elusive. In this paper, we now attempt to present some Büchi-Kamp like theorems for 1-ATA and its several natural subclasses. Unfortunately, we do not yet have a full solution, although several structural restrictions on 1-ATA do allow such results.

Firstly, we define timed extensions of monadic second order logic which we call as \(Q_k\text{MSO}\) and its sublogics \(Q2\text{MSO}\), \(Qk\text{FO}\) and \(Q2\text{FO}\). These logics are inspired by the logic \(Q2\text{MLO}\) (over continuous time) defined by Hirshfeld and Rabinovich [10] as well as Hunter [12]. In \(Qk\text{MSO}\), logic MSO is extended with a metric quantifier block consisting of at most \(k\) quantifiers resulting
in a formula with exactly one free variable. This can be recursively used as an atomic predicate in MSO. A carefully defined syntax gives us a logic which allows only future time properties to be stated.

As our first main result, we define a subclass of 1-clock ATA called 1-clock ATA with loop-free resets (1-ATA-lfr). In these automata, on any run and for any location q, a reset transition leading to q (denoted x.q) must occur at most once. Equivalently there is no cycle involving x.q. We show that logic RatMTL, defined earlier in [22], is expressively equivalent to 1-ATA-lfr. Moreover, we also show that QkMSO is expressively equivalent to RatMTL. In proving this, we show a four variable property showing that QkMSO is expressively equivalent to Q4MSO. A variant of this result allows us to characterize a subclass of 1-ATA called partially ordered 1-ATA with star-free fragment of RatMTL (as shown in [22]) as well as the first order fragment QkFO. This in turn is expressively equivalent to Q4FO.

As our second main result, we introduce a notion of conjunctive-disjunctiveness in 1-clock ATA. Here, the ATA thread is either in conjunctive mode or in disjunctive mode at a time, and it can switch modes only on a reset transition. A restricted version of Rat modality called FRat was defined in [22]. A similar modality was also defined by Wilke earlier [24]. We show that conjunctive-disjunctive 1-ATA with loop-free resets has exactly the expressive power of FRatMTL which is the same as MTL extended with only FRat modality. Moreover, Q2MSO is expressively equivalent to FRatMTL. This result extends to other classes of 1-clock ATA.

However, the case of full 1-clock ATA needs to be investigated. Towards this, we show a Büchi-like theorem which says that µRatMTL, obtained by introducing fixed point operators to RatMTL, is expressively equivalent to full 1-clock ATA. However a Kamp theorem giving a classical logic equivalent to this remains under investigation.

## 2 Preliminaries

Let Σ be a finite set of propositions. A finite timed word over Σ is a tuple ρ = (σ, τ), where σ and τ are sequences σ₁σ₂...σₙ and τ₁τ₂...τₙ respectively, with σ₁ ∈ Γ = 2Σ\{∅}, and τᵢ ∈ ℝ₊ for 1 ≤ i ≤ n. For all i ∈ dom(ρ), we have τᵢ ≤ τᵢ₊₁, where dom(ρ) is the set of positions {1, 2, ..., n} in the timed word. For convenience, we assume τ₁ = 0. The σᵢ’s can be thought of as labeling positions i in dom(ρ). For example, given Σ = {a, b, c}, ρ = ({a, c}, 0)({a}, 0.7)({∅}, 1.1) is a timed word. ρ is strictly monotonic iff τᵢ < τᵢ₊₁ for all i, i + 1 ∈ dom(ρ). Otherwise, it is weakly monotonic. The set of finite timed words over Σ is denoted TΣ⁺. Given ρ = (σ, τ) with σ = σ₁...σₙ ∈ Γ⁺, σsingle denotes the set of all words w₁w₂...wₙ where each wᵢ ∈ σᵢ. ρsingle consists of all timed words (σsingle, τ). For the ρ as above, ρsingle consists of timed words
2.1 Temporal Logics

In this section, we define preliminaries pertaining to the temporal logics studied in the paper. Let $Iν$ be a set of open, half-open or closed time intervals. The end points of these intervals are in $N ∪ \{0, \infty\}$. For example, [1, 3), [2, \infty). For a time stamp $τ ∈ \mathbb{R}_{≥0}$ and an interval $(a, b)$, where $<$ is left-open or left-closed and $>$ is right-open or right-closed, $τ + (a, b)$ represents the interval $(τ + a, τ + b)$.

**Metric Temporal Logic (MTL).** Given a finite alphabet $Σ$, the formulae of logic MTL are built from $Σ$ using boolean connectives and time constrained version of the until modality $U$ as follows:

\[ ϕ := a(∈ Σ) | true | ϕ ∧ ϕ | ¬ ϕ | φ U_τ ϕ, \]

where $I ∈ Iν$. For a timed word $ρ = (σ, τ) ∈ TΣ^*$, a position $i ∈ dom(ρ)$, and an MTL formula $ϕ$, the satisfaction of $ϕ$ at a position $i$ of $ρ$ is denoted $ρ, i \models ϕ$, and is defined as follows: (i) $ρ, i \models a \leftrightarrow a σ_i$, (ii) $ρ, i \models ¬ ϕ \leftrightarrow ρ, i \not{\models} ϕ$, (iii) $ρ, i \models ϕ_1 ∧ ϕ_2 \leftrightarrow ρ, i \models ϕ_1$ and $ρ, i \models ϕ_2$, (iv) $ρ, i \models φ U_τ ϕ_2 \leftrightarrow ∃ j > i, ρ, j \models ϕ_2, τ_j - τ_i ∈ I$, and $ρ, k \models ϕ_1 \forall i < k < j$. The language of a MTL formula $ϕ$ is $L(ϕ) = \{ρ | ρ, 1 \models ϕ\}$. Two formulae $ϕ$ and $φ$ are said to be equivalent denoted as $ϕ \equiv φ$ if $L(ϕ) = L(φ)$. The subclass of MTL restricting the intervals $I$ in the until modality to non-punctual intervals is denoted MITL.

**Theorem 1** ([22]). MTL satisfiability is decidable over finite timed words and is non-primitive recursive.

**MTL with Rational Expressions (RatMTL)**

We first recall an extension of MTL with rational expressions (RatMTL), introduced in [22]. The modalities in RatMTL assert the truth of a rational expression (over subformulae) within a particular time interval with respect to the present point. For example, for an interval $I = (0, 1)$, the Rat$_I$ modality works as follows: the formula Rat$_I(0, 1)(ϕ_1,ϕ_2)$ is evaluated at point $i$ and asserts the existence of $2k$ points with time stamps $τ_i < τ_i+1 < τ_i+2 < \cdots < τ_i+k < τ_i+1$, $k > 0$, such that $ϕ_1$ evaluates to true at $τ_i+k+1$, and $ϕ_2$ evaluates to true at $τ_i+j+1$, for all $0 ≤ j < k$.

**RatMTL Syntax:** Formulae of RatMTL are built from a finite alphabet $Σ$ as:

\[ ϕ := a(∈ Σ) | true | ϕ ∧ ϕ | ¬ ϕ | Rat_I(ϕ), \]

where $I ∈ Iν$ and $Σ$ is a finite set of subformulae of $ϕ$, and $re(Σ)$ is defined as a rational expression over $Σ$. $re(Σ) := ε | ϕ(∈ Σ) | ϕ ∪ re(Σ) | ϕ + re(Σ) | ϕ ∗ re(Σ)$. Thus, RatMTL is MTL extended with modalities $Ure(Σ)$ and $at$. An atomic rational expression $re$ is any well-formed formula $ϕ$ in RatMTL.

**RatMTL Semantics:** For a timed word $ρ = (σ, τ) ∈ TΣ^*$, a position $i ∈ dom(ρ)$, a RatMTL formula $ϕ$, and a finite set $Σ$ of subformulae of $ϕ$, we define the satisfaction of $ϕ$ at a position $i$ as follows. For positions $i < j ∈ dom(ρ)$, let $Seg(S, i, j)$ denote the untimed word over $2^Σ$ obtained by marking the positions $k ∈ \{i + 1, \ldots, j - 1\}$ of $ρ$ with $ψ ∈ Σ$ if $ρ, k \models ψ$. For a position $i ∈ dom(ρ)$ and an interval $I$, let $TSeg(S, I, i)$ denote the untimed word over $2^Σ$ obtained by marking all the positions $k$ such that $τ_k - τ_i \in I$ of $ρ$ with $ψ ∈ Σ$ if $ρ, k \models ψ$.

- $ρ, i \models FQuant(Σ)(ϕ) ⇔ ∃ j > i, ρ, j \models ϕ, τ_j - τ_i \in I$ and $Seg(S, i, j)$ single $∩ L(re(Σ)) ≠ \emptyset$, where $L(re(Σ))$ is the language of the rational expression formed over the set $Σ$. In [22], the $Ure(Σ)$ modality was used instead of $FRat_I$; however, both have the same expressiveness. Note that $ϕ_1UQuant(Σ)(ϕ_2)$ is equivalent to $FQuant(Σ, ϕ_1)(ϕ_2)$ where $re'(Σ, ϕ_1) = re(Σ) ∩ ϕ_1$.
- $ρ, i \models Ure(Σ) (ϕ) ⇔ TSeg(S, I, i)$ single $∩ L(re(Σ)) ≠ \emptyset$. The language accepted by a RatMTL formula $ϕ$ is given by $L(ϕ) = \{ρ | ρ, 1 \models ϕ\}$. The subclass of RatMTL using only the $FRat_I$ modality is denoted FRatMTL. When only non-punctual intervals are used, then it is denoted FRatMITL.

Thus FRatMTL is MTL + FRat.

**Remark.** The classical $ϕ_1 Uϕ_2$ modality can be written in FRatMTL as $FQuant(Σ)(ϕ_1)(ϕ_2)$. Also, it can be shown [22] that the FRat modality can be expressed using the Rat modality.
**Modal depth:** We define the modal depth (md) of a RatMTL formula. An atomic RatMTL formula over $\Sigma$ is just a propositional logic formula over $\Sigma$ and has modal depth 0. A RatMTL formula $\varphi$ over $\Sigma$ having a single modality (Rat or FRat) has modal depth one and has the form $\text{Rat}_\mathcal{O} \varphi$ where $\mathcal{O}$ is a regular expression over propositional logic formulae over $\Sigma$ and $\varphi$ is a propositional logic formula over $\Sigma$. Inductively, we define modal depth as follows.

(i) $\text{md}(\varphi \land \psi) = \min\{\text{md}(\varphi), \text{md}(\psi)\}$.

(ii) Let $\varphi$ be a regular expression over the set of subformulae $\Sigma = \{\psi_1, \ldots, \psi_k\}$. Then $\text{md}(\varphi) = 1 + \max\{\text{md}(\psi_1), \ldots, \text{md}(\psi_k)\}$.

**Example 2.** Consider the formula $\varphi = \text{Rat}_{(0,1)}(a,b \cdot b)$. Then $\mathcal{O} = (aa)^*$. The subformulae of interest are $\Sigma = \{a, b\}$. For $\rho = \{(a), 0\} \{(a, b), 0, 3\} \{(a, b), 0, 7\} \{(b), 0, 9\}$, $\rho, 1 \models \varphi$, since $a \in \sigma_2$, $\sigma_5$, $b \in \sigma_4$, $\sigma_1 \in \{0, 1\}$ and $aa \in \text{Seg}(\{a, b\}, 1, 4)$. $\text{md}(\varphi) = 0$. Similarly, $\varphi \models \psi$ since even though $b \in \sigma_5$, $a \in \sigma_1$, and $i < 5$, $\text{Seg}(\{a, b\}, 1, 5)$. $\text{md}(\varphi) = 0$.

**Example 3.** Consider the formula $\varphi = \text{Rat}_{(0,2)}((a \mathcal{O}_{(0,1)} b, c \cdot e)^+ \cdot e)$. The subformulae of interest are $\Sigma = \{a \mathcal{O}_{(0,1)}, b, c, e\}$.

1. Let $\rho = \{(a), 0\} \{(a, b), 0, 3\} \{(a, b, c), 0, 4\} \{(a, b, c, e), 0, 9\} \{(b, c, e), 0, 9\} \{(b, c), 0, 9\} \{(b, e), 1, 3\} \{(e), 1, 9\}$. We want to check if $\text{TSeg}(\{0, 2\}) \cap L((a \mathcal{O}_{(0,1)}, b, c, e)^+)$. Note that we mark positions 2 to 7 with subformulae from $\Sigma$. Position 2 is marked $a \mathcal{O}_{(0,1), b, c}$ since $c \in \sigma_5$, $a \in \sigma_2$, $\sigma_4$ and $\text{Seg}(\{a, b\}, 2, 5) \cap L((a \mathcal{O}_{(0,1)}, b, c, e)^+)$. Similarly, positions 3 to 7 are marked $e$. Hence, $\text{TSeg}(\{0, 2\}) \cap L((a \mathcal{O}_{(0,1)}, b, c, e)^+)$. We thus have $\rho, 1 \models \varphi$.

2. For $\rho' = \{(b), 0\} \{(a, b), 0, 3\} \{(a, b), 0, 4\} \{(a, e), 0, 9\} \{(b, c), 0, 9\} \{(b, e), 1, 3\} \{(e), 1, 9\}$, $\rho, 1 \not\models \varphi$ since $\text{TSeg}(\{0, 2\}) \cap L((a \mathcal{O}_{(0,1)}, b, c, e)^+)$. Note that we cannot mark position 2 with $a \mathcal{O}_{(0,1), b, c}$, nor mark the remaining positions continuously with $e$. Even if we add $b$ to $\sigma_4$ or add $e$ to $\sigma_3$ (but not both), we still have non-satisfiability.

### 2.2 Classical Logics

In this section, we define all the preliminaries pertaining to classical logics needed for the paper. We introduce some quantitative variants of classical logics inspired by the logics in [10]. Let $\rho = (\sigma, \tau)$ be a timed word over a finite alphabet $\Sigma$, as before. We define a real-time logic forward QkMso (with parameter $k \in \mathbb{N}$) which is interpreted over such words. It includes $\text{MSO}[<]$ over words $\sigma$ relativized to specify only future properties. This is extended with a notion of time constraint formula $\psi(t)$. Let $t_0, t_1, \ldots$ be first order variables and $T_0, T_1, \ldots$ the monadic second-order variables. We have the two sorted logic consisting of MSO formulae $\phi$ and time constrained formulae $\psi$. Let $a \in \Sigma$, each $t_i$ range over first order variables and $T_i$ over second order variables. Each quantified first order variable in $\phi$ is relativized to the future of some variable, say $t_0$, called anchor variable, giving formula of $\text{MSO}^t$. The syntax of $\phi \in \text{MSO}^t$ is given by:

$t_\varphi= t_\varphi \mid t_\varphi < t_\varphi \mid Q_\varphi(t_\varphi)[T_i(t_i) \mid \varphi \land \varphi' \land \exists \varphi' \exists \varphi' [t_\varphi \varphi \exists T_i \varphi \psi(t_\varphi)]$.

Here, $\psi(t_\varphi)$ is a time constrained formula whose syntax and semantics are given later.

A formula in $\text{MSO}^t$ with first order free variables $t_0, t_1, \ldots t_k$ and second-order free variables $T_1, \ldots, T_m$ and which is relativized to the future of $t_0$ is denoted $\phi(\downarrow t_0, t_1, \ldots t_k, T_1, \ldots, T_m)$. (The $\downarrow$ is only to indicate the anchor variable. It has no other function.) The semantics of such formulae is as usual. Given $\rho$, positions $a_0, a_1, \ldots a_k$ in $\text{dom}(\rho)$, and sets of positions $A_1, \ldots, A_m$ with $A_k \subseteq \text{dom}(\rho)$, we define $\rho, (a_1, \ldots a_k, A_1, \ldots, A_m) = \phi(\downarrow t_0, t_1, \ldots, t_k, T_1, \ldots, T_m)$ inductively, as usual. For instance,

1. $(\rho, a_1, \ldots, a_k, A_1, \ldots, A_m) = t_i < t_j$ if $a_i < a_j$. 

2. \((\rho, a_1, \ldots, a_k, A_1, \ldots, A_m) \models Q_\phi(t_i)\) iff \(a \in \sigma(a_i)\).
3. \((\rho, a_1, \ldots, a_k, A_1, \ldots, A_m) \models T_j(t_i)\) iff \(a_i \in A_j\), and
4. \((\rho, a_1, \ldots, a_k, A_1, \ldots, A_m) \models \exists t \, t_0 < t \wedge \phi(\cup t_0, t_1, \ldots, t_k, T_1, \ldots, T_m)\) iff
\[
\rho, (a_0, a_1, \ldots, a_k, A_1, \ldots, A_m) = \phi(\cup t_0, t_1, \ldots, t_k, T_1, \ldots, T_m)\text{ for some } a_k \geq a_0.
\]
We omit other cases. The time constraint \(\psi(t_0)\) has the form \(Q_1 t_1 Q_2 t_2 \ldots Q_j t_j \phi(\cup t_0, t_1, \ldots, t_j)\) where \(\phi \in MSO^{10}\) and \(j < k\), the parameter of logic QkM\(S\). Each quantifier \(Q_i t_i\) has the form \(\exists t_i \in t_0 + I_i\) or \(\forall t_i \in t_0 + I_i\) for a time interval \(I_i\) as in MTL formulae. \(Q_i\) is called a metric quantifier. The semantics of such a formula is as follows: \((\rho, a_0) \models Q_1 t_1 Q_2 t_2 \ldots Q_j t_j \phi(\cup t_0, t_1, \ldots, t_j)\) iff for \(1 \leq i \leq j\), there exist for all \(a_i\) such that \(a_0 \leq a_i\) and \(\tau_{a_i} \in \tau_{a_0} + I_i\), we have \(\rho, (a_0, a_1, \ldots, a_j) = \phi(\cup t_0, t_1, \ldots, t_j)\). Note that each time constraint formula has exactly one free variable.

**Example 4.** Let \(\rho = (\{a\}, 0)\) \((\{b\}, 2.1)\) \((\{a, b\}, 2.75)\) \((\{b\}, 3.1)\) be a timed word. Consider the time constraint \(\psi(x) = \exists y \in x + (2, \infty) \exists z \in x + (3, \infty) (Q_b(y) \wedge Q_b(z))\). It can be seen that \(\rho, 1 \models Q_a(x) \wedge \psi(x)\).

**Metric Depth.** The metric depth of a formula \(\phi\) denoted \(md(\phi)\) gives the nesting depth of time constraint constructs. It is defined inductively as follows: For atomic formulae \(\phi, md(\phi) = 0\). All the constructs of \(MSO^{\tau}\) do not increase \(md\). For example \(md(\phi_1 \wedge \phi_2) = \max(md(\phi_1), md(\phi_2))\) and \(md(\exists t. \phi(t))\). However \(md\) is incremented for each application of metric quantifier block. \(md(\phi) = md(\phi) + 1\).

**Example 5.** The sentence \(\forall t_0 \exists t_1 \in t_0 + (1, 2) \{Q_a(t_1) \rightarrow (\exists t_2 \in t_1 + [1, 1] Q_b(t_2))\}\) accepts all timed words such that for each \(a\) which is at distance \((1, 2)\) from some time stamp \(t\), there is a \(b\) at distance 1 from it. This sentence has metric depth two.

**Special Cases of QkM\(S\).** The case when \(k = 2\) gives logic Q2M\(\phi\). The absence of second order variables and second order quantifiers gives logics QkFO and Q2FO. The formula in example 5 is a Q2FO formula. Note that our Q2FO is the pointwise counterpart of logic Q2M\(\phi\) studied in [10] in the continuous semantics.

### 2.3 1-clock Alternating Timed Automata

Let \(\Sigma\) be a finite alphabet and let \(\Gamma = 2^\Sigma \setminus \emptyset\). A 1-clock ATA or \(1\)-ATA [18] is a 5 tuple \(A = (\Gamma, S, s_0, F, \delta)\), where \(S\) is a finite set of locations, \(s_0 \in S\) is the initial location and \(F \subseteq S\) is the set of final locations. Let \(x\) denote the clock variable in the 1-clock ATA, and \(x \in I\) denotes a clock constraint where \(I\) is an interval. Let \(X\) denote a finite set of clock constraints of the form \(x \in I\). The transition function is defined as \(\delta : S \times \Gamma \rightarrow \Phi(S \cup X)\) where \(\Phi(S \cup X)\) is a set of formulae over \(S \cup X\) defined by the grammar \(\varphi ::= \top \mid \bot \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid x \in I \mid x \varphi \mid s \in S \mid s, x, \varphi\) where \(s \in S\), and \(x, \varphi\) is a binding construct resetting clock \(x\) to 0.

Without loss of generality, we assume that all transitions \(\delta(s, a)\) are in disjunctive normal form \(C_1 \vee C_2 \vee \cdots \vee C_n\) where each \(C_i\) is a conjunction of clock constraints and locations \(s, x, s\).

A configuration of a 1-clock ATA is a set consisting of locations along with their clock valuations. Given a configuration \(C = \{(s, \nu) \mid s \in S, \nu \in \mathbb{R}_{\geq 0}\}\), we denote by \(C + t\) the configuration \(\{(s, \nu + t) \mid s \in S, \nu + t \in \mathbb{R}_{\geq 0}\}\) obtained after a time elapse \(t\), when \(t\) is added to all valuations in \(C\). \(\delta(C, a)\) is the configuration obtained by applying \(\delta(s, a)\) to each location \(s\) such that \((s, \nu) \in C\). A run of the 1-clock ATA starts from the initial configuration \(C_0 = \{(s_0, 0)\}\) and has the form \(C_0 \xrightarrow{t_0} C_0 + t_0 \xrightarrow{C_1 + t_1} \cdots \rightarrow C_n\) and proceeds with alternating time elapse transitions and discrete transitions reading a symbol from \(\Sigma\). A configuration \(C\) is accepting iff for all \((s, \nu) \in C\), \(s \in F\). Note that the empty configuration is also an accepting configuration. The language
accepted by a 1-clock ATA $A$, denoted $L(A)$ is the set of all timed words $\rho$ such that starting
from $\{(s_0,0)\}$, reading $\rho$ leads to an accepting configuration.

We will define some terms which will be used in sections 4, 5. Consider a transition $\delta(s,a) = C_1 \vee \cdots \vee C_n$ in the 1-clock ATA. Each $C_i$ is a conjunction of $x \in I$, locations $p$ and $x.p$. We say that $p$ is free in $C_i$ if there is an occurrence of $p$ in $C_i$ and no occurrences of $x.p$ in $C_i$; if $C_i$ has an $x.p$, then we say that $p$ is bound in $C_i$. We say that $p$ is bound in $\delta(s,a)$ if it is bound in some $C_i$. A PO-1-ATA is one in which

• there is a partial order denoted $\prec$ on the locations, such that the locations appearing in any transition $\delta(s,a)$ are in $\{s\} \cup \downarrow s$ where $\downarrow s = \{p \mid p \prec s\}$.
• $x.s$ does not appear in $\delta(s,a)$ for any $s \in S, a \in \Gamma$.

It is known [22] that PO-1-ATA exactly characterize logic SfrMTL (this is a subclass of RatMTL where the regular expressions have an equivalent star-free expression).

Example 6. Consider the PO-1-ATA $A = (\Gamma, \{a,b\}, \{(t_0, t_1, t_2), (t_0, t_2), \delta_A\})$ with transitions $\delta_A(t_0, \{b\}) = t_0, \delta_A(t_0, \{a\}) = (t_0 \land x.t_1) \lor t_2, \delta_A(t_1, \{a\}) = (t_1 \land x < 1) \lor (x > 1) = \delta_A(t_1, \{b\})$, and $\delta_A(t_2, \{a\}) = t_2, \delta_A(t_2, \{a\}) = \bot$ for $t \in \{t_0, t_1, t_2\}$. The automaton accepts all strings where $\{a,b\}$ does not occur, and every non-last $\{a\}$ has no symbols at distance 1 from it, and has some symbol at distance $> 1$ from it.

2.4 Useful Tools

In this section, we introduce some notations and prove some lemmas which will be used several times in the paper. Let $A$ be a 1-clock ATA and let $c_{\max}$ be the maximum constant used in the transitions. The set $\text{reg} = \{0, (0,1), \ldots, c_{\max}, (c_{\max}, \infty)\}$ denotes the set of regions. Given a finite alphabet $\Sigma$, with $\Gamma = 2^\Sigma \setminus \emptyset$, a region word is a word over the alphabet $\Gamma \times \text{reg}$ called the interval alphabet. A region word $w = (a_1, I_1) (a_2, I_2) \ldots (a_m, I_m)$ is good iff $I_j \leq I_k \Rightarrow j < k$ and $I_1$ is initial region $0$. Here, $I_j \leq I_k$ represents that the upper bound of $I_j$ is at most the lower bound of $I_k$. A timed word $(a_1, \tau_1) \ldots (a_m, \tau_m)$ is consistent with a region word $(b_1, I_1)(b_2, I_2) \ldots (b_n, I_n)$ iff $n = m, a_j = b_j$, and $\tau_j \in I_j$ for all $j$. The set of timed words consistent with a good region word $w$ is denoted $T_w$. Likewise, given a timed word $\rho$, $\text{reg}(\rho)$ represents the good region word $w$ such that $\rho \in T_w$.

The following lemmas (proof of Lemma 7 in Appendix B) come in handy later in the paper.

Lemma 7. [Untiming $P \rightarrow A(P)$] Let $P$ be a 1-clock ATA over $\Gamma$ having no resets. We can construct an alternating finite automaton (AFA) $A(P)$ over the interval alphabet $\Gamma \times \text{reg}$ such that for any good region word $w = (a_1, I_1) \ldots (a_n, I_n)$, $w \in L(A(P))$ iff $T_w \subseteq L(P)$. Conversely, $\rho \in L(P)$ iff $\text{reg}(\rho) \in L(A(P))$, for any timed word $\rho$. Hence, $L(P) = \{T_w \mid w \in L(A(P))\}$.

Lemma 8. Let $A(P)$ be an AFA over the interval alphabet $\Gamma \times \text{reg}$ constructed from a reset-free 1-clock ATA $P$ as in Lemma 7. We can construct a RatMTL formula $\varphi$ such that $L(\varphi) = \{T_w \mid w \in L(A(P))\}$. Hence, by Lemma 4 $L(\varphi) = L(P)$. If the AFA is aperiodic, then $\varphi$ is a SfrMTL formula.

Proof. Let $\text{Det}_A(P)$ be the deterministic automaton which is language equivalent to $A(P)$. For any pair of states $p, q$ of $\text{Det}_A(P)$ and an interval (region) $I_i$, we can construct a regular expression $\text{re}(p,q, I_i)$ denoting the language $\{w \in (\Gamma \times \{I_i\})^* \mid \delta(p, w) = q\}$. Here $\delta(p, w)$ is the transition function of $\text{Det}_A(P)$ extended to words. To construct $\text{re}(p, q, I_i)$, let $\text{Det}_A[p, q, I_i]$ be the same DFA as $\text{Det}[A(P)]$ except that the initial location is $p$ and set of final locations is $\{q\}$. Let $\text{det}(A_i^* \{I_i\})$ denote the DFA accepting arbitrary words in $(\Gamma \times \{I_i\})^*$. Let $\text{Det}[A] = \text{Det}_A(P)[p, q, \cap \text{det}(A_i^* \{I_i\})]$ be the automaton which starts at $p$, accepts on $q$, and has only the $\Gamma \times \{I_i\}$ edges. Let $\text{re}(p, q, I_i)$ be the regular expression denoting the language of $\text{Det}[A_i]$.

Consider $\text{Det}[A_i]$. Let $s_0$ be its initial location, and $F$ be the set of its final locations. For any sequence of intervals $\text{iseq} = I_1 < I_2 < \ldots I_k$, where $I_1$ is the initial region 0 and any
sequence of locations \( sseq = q_0, q_1, q_2 \ldots q_k \) such that \( q_0 = s_0 \) and \( q_k \in F \), the regular expression \( \text{re}(q_0, q_1, I_1) \text{re}(q_1, q_2, I_2) \ldots \text{re}(q_{k-1}, q_k, I_k) \) denotes subsets of good region words accepted by \( \text{DetA}(P) \) where the control stays within \( I_i \) between locations \( q_{i-1} \) and \( q_i \). Define the \( \text{RatMTL} \) formula \( \varphi(\text{seq}, sseq) = \text{Rat}_1 \text{re}(q_0, q_1, I_1) \text{Rat}_2 \text{re}(q_1, q_2, I_2) \ldots \text{Rat}_t \text{re}(q_{t-1}, q_t, I_t) \) Last, where formula \( \text{Last} = \text{Rat}_{q_0, \infty} \) holds only for the last position of a word. Then \( L(\varphi(\text{seq}, sseq)) = (T_w \mid w \in L(\text{re}(q_0, q_1, I_1) \text{re}(q_1, q_2, I_2) \ldots \text{re}(q_{k-1}, q_k, I_k))) \), by construction.

Let \( \varphi = \lor_{\text{seq}} \lor_{\text{seq}} \varphi(\text{seq}, sseq) \). Then clearly, \( L(\varphi) = \{T_w \mid w \in L(\text{A}(P))\} \). Hence, by Lemma 9, \( L(\varphi) = L(P) \). Note that if we start with a PO 1-clock ATA \( P \) in Lemma 9 then the AFA \( \text{A}(P) \) obtained is aperiodic. In that case, the regular expressions above have a star-free equivalent, resulting in the \( \text{RatMTL} \) formula being an \( \text{SfrMTL} \) formula.

**Expressive Completeness and Equivalence.** Let \( F_i \) be a logic or automaton class i.e. a collection of formulae or automata describing/accepting finite timed words. For each \( \varphi \in F_i \) let \( L(F_i) \) denote the language of \( F_i \). We define \( F_1 \subseteq F_2 \) if for each \( \varphi \in F_1 \) there exists \( \psi \in F_2 \) such that \( L(\varphi) = L(\psi) \). Then, we say that \( F_2 \) is expressively complete for \( F_1 \). We also say that \( F_1 \) and \( F_2 \) are expressively equivalent, denoted \( F_1 \equiv e \ F_2 \), iff \( F_1 \subseteq e F_2 \) and \( F_2 \subseteq e F_1 \).

### 3 A Normal Form for 1-clock ATA

In this section, we establish a normal form for 1-clock ATA, which plays a crucial role in the rest of the paper. Let \( A = (\Gamma, S, s_0, F, \delta) \) be a 1-clock ATA. \( A \) is said to be in normal form iff

- The set of locations \( S \) is partitioned into two sets \( S_r \) and \( S_{nr} \). The initial state \( s_0 \in S_r \).
- The locations of \( S \) are partitioned into \( P_1, \ldots, P_k \) satisfying the following: Each \( P_i \) has a unique header location \( s_i^r \in S_r \). Also, \( P_i - \{s_i^r\} \subseteq S_{nr} \). Moreover, for any transition of \( A \) of the form \( \delta(s, a) = C_1 \lor C_2 \ldots C_k \) with \( C_i = x \in I \land p_1 \land \ldots \land p_m \land x.q_i \land \ldots \land x.q_k \) we have
  - (a) each \( q_i \in S_r \), and
  - (b) if \( s \in P_i \) then each \( p_j \in P_i - \{s_i^r\} \).

Each partition \( P_i \) can be thought of as an island of locations. Each island has a unique header (or reset) location \( s_i^r \). All transitions from outside into \( P_i \) occur only to this unique header location, and only with reset of clock \( x \). Moreover, all non-reset transitions stays in the same island until a clock is reset, at which point, the control extends to the header location of same or another island (this behaviour can be seen on each path of the run tree).

**Establishing the Normal Form**

The main result of this section is that every 1-clock ATA \( A \) can be normalized, obtaining a language equivalent 1-clock ATA \( \text{Norm}(A) \). The key idea behind this is to duplicate locations of \( A \) such that the conditions of normalization are satisfied. Let the set of locations of \( A \) be \( S = \{s_1, \ldots, s_n\} \). For each location \( s_i \), \( 1 \leq i \leq n \), create \( n+1 \) copies, \( s_i^r \) and \( s_i^{nr,j}, 1 \leq j \leq n \). If \( s_0 \) is the initial location of \( A \), then initial location of \( \text{Norm}(A) \) is \( s_0^r \). The partition \( P_i \) in \( \text{Norm}(A) \) will consist of locations \( s_i^r, s_i^{nr,j} \) for \( 1 \leq h \leq n \) and entry into \( P_i \) happens through \( s_i^r \). The superscript \( r \) on a location represents that all incoming transitions to it are on a clock reset, while \( nr, j \) represents that all incoming transitions to that location are on non-reset and it belongs to island \( P_j \). In a transition \( \delta(s_i, a) = \varphi \), all occurrences of locations \( x.s_j \) are replaced as \( x.s_j^r \) (leading into \( P_i \)), while occurrence of free locations \( s_h \) are replaced as \( s_h^{nr,j} \). The final locations of \( \text{Norm}(A) \) are \( s_i^r, s_i^{nr,j} \) for \( 1 \leq j \leq n \) whenever \( s_i \) is a final location in \( A \). Appendix B gives a formal proof for the following straightforward lemma.

**Lemma 9.** \( L(A) = L(\text{Norm}(A)) \).

**Remark** Due to above lemma, we assume without loss of generality, in the rest of the paper, that 1-ATA are in normal form.

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1 In section B we describe a subclass of 1-clock ATA where the transitions are sometimes restricted to be in CNF form. The equivalent restriction then is that each free location in the transition should be in \( P_i - \{s_i^r\} \) while each bound location should be in \( S_r \).
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Example 10. The 1-clock ATA $B = \{a, b\}, \{s_0, s_1, s_2\}, \{s_0, \delta\}$ with transitions $\delta(s_0, b) = x.s_2, \delta(s_0, a) = (s_0 \land x.s_1)$, $\delta(s_1, a) = (s_1 \land s_0)$ and $\delta(s_2, b) = x.s_0, \delta(s_2, a) = (s_2 \land x.s_1)$ is not in normal form. Following the normalization technique, we obtain $\text{Norm}(B)$ as follows.

$\text{Norm}(B)$ has locations $S = \{s_{0,i}, r_{i,j}, s_{1}, r_{i,j}, s_{2}\}$ and final locations $\{s_{1}, r_{i,j}, s_{2}\}$. The transitions $\delta'$ are as follows.

$\delta'(s_{0,0}, b) = x.s_2, \delta'(s_{0,0}, a) = (s_{0,0} \land x.s_1), \delta'(s_{0,1}, b) = x.s_2, \delta'(s_{0,1}, a) = (s_{0,1} \land x.s_1), \delta'(s_{1,0}, b) = x.s_2, \delta'(s_{1,0}, a) = (s_{1,0} \land x.s_1), \delta'(s_{2,0}, b) = x.s_2, \delta'(s_{2,0}, a) = (s_{2,0} \land x.s_1), \delta'(s_{2,1}, b) = x.s_2, \delta'(s_{2,1}, a) = (s_{2,1} \land x.s_1).$ (See Figure 1.) It is easy to see that $\text{Norm}(B)$ is in normal form: $S_0 = \{s_{0,i} \mid 0 \leq i \leq 2\}, S_{nr} = \{s_{nr,i} \mid 0 \leq i, j \leq 2\}$, and we have the disjoint sets $P_0 = \{s_{0,0}, s_{0,1}\}, P_1 = \{s_{1,0}, s_{1,1}\}$ and $P_2 = \{s_{2,0}, s_{2,1}\}$. See Figure 1.

![Figure 1 Norm(B) for B](image)

4 1-ATA-lfr and Logics

In this section, we show the first of our Büchi-Kamp like results connecting logics $\text{RatMTL}$, forward $\text{QkMSO}$ and a subclass of 1-clock ATA called 1-ATA with loop-free resets (1-ATA-lfr). We first introduce 1-ATA-lfr.

A 1-clock ATA $A$ (in normal form) is said to be a 1-ATA-lfr if it satisfies the following: There is a partial order $(S_r, \preceq)$ on the reset states (equivalently, islands $P_i$). Moreover, for any $p \in P_i$ and location $q$, if $x.q$ occurs in $\delta(p, a)$ for any $a$ (giving that $q = s_j^i$) then $s_j^i \prec s_j$. Thus, islands (which are only connected by reset transitions) form a DAG, and every reset transition goes to a lower level island (see Figure 2) where this phenomenon is called progressive island hopping. Semantically, this means that on any branch of run tree, a reset transition occurs at most once.

Example 11. The 1-clock ATA with locations $s, p, q$ and transitions $\delta(s, a) = (x.p \land x \leq 1) \lor (q \land x = 2), \delta(p, a) = x.q \land p$ and $\delta(q, a) = s \land (0 < x < 1)$ is not 1-ATA-lfr, since $q$ is bound in $\delta(p, a)$ and starting from $q$, we can reach $x.p$ via $s$.

4.1 Büchi Theorem for 1-ATA-lfr

In this section, we show the equivalence of 1-ATA-lfr and $\text{RatMTL}$.

Theorem 12. 1-ATA-lfr are expressively equivalent to $\text{RatMTL}$.

When restricting to logic $\text{SfrMTL}$, we obtain expressible equivalence with $\text{PO}-1$-ATA.
4.2 \text{1-ATA-lfr} \subseteq \text{RatMTL}

\textbf{Proof.} Let \( \mathcal{A} \) be a \text{1-ATA-lfr} in normal form. For each location \( s^*_i \in S_r \) (which is the header of partition \( P_i \)) let \( \mathcal{A}[s^*_i] \) denote the same automaton as \( \mathcal{A} \) except that the initial location is changed to \( s^*_i \). We can also delete all islands higher than \( P_i \) as their locations are not reachable. For each such automaton, we construct a language equivalent \text{RatMTL} formula \( \text{mtl}(\mathcal{A}[s^*_i]) \). Note that \((S_r, \preceq)\) is a partial order. The construction and proof of equivalence are by complete induction on the level of location \( s^*_i \) in the partial order.

Let \( s^*_i \) be the header of island \( P_i \). All \( x.s^*_j \) occurring in any transition of \( \mathcal{A}[s^*_i] \) are of lower level in the partial order \((S_r, \preceq)\). Hence, by induction hypothesis, there is a \text{RatMTL} formula \( \psi_j = \text{mtl}(\mathcal{A}[s^*_j]) \) language equivalent to \( \mathcal{A}[s^*_j] \).

Let \( w_j \) be a fresh witness variable for each \( x.s^*_j \) above, which also corresponds to \text{RatMTL} formula \( \psi_j \). Let the set of such witness variables be \( \{w_1, \ldots, w_k\} \). We construct a modified automaton \( \mathcal{A}^{\text{w}}[s^*_i] \) with transition function \( \delta' \) and set of locations \( P_i \), as follows. Its alphabet is \( \Gamma \times \{0, 1\}^k \) with \( j \)th component giving truth value of \( w_j \). Let \( \delta'(s, a, w_1, \ldots, w_k) = \delta(s, a)[w_j/x, s^*_j] \), i.e. in the transition formula each occurrence of \( x.s^*_j \) is replaced by truth value of \( w_j \) for \( 1 \leq j \leq k \).

Note that \( \mathcal{A}^{\text{w}}[s^*_i] \) is a reset-free 1-ATA. By Lemmas 7 and 8 we get a language equivalent \text{RatMTL} formula \( \phi^{\text{w}} \) over the variables \( \Sigma \cup \{w_1, \ldots, w_k\} \). Now we substitute each \( w_j \) by \( \psi_j \) (and hence \( \neg w_j \) by \( \neg \psi_j \)) in \( \phi^{\text{w}} \) to obtain the required formula \( \text{mtl}(\mathcal{A}[s^*_i]) \). It is clear from the substitution that \( L(\mathcal{A}[s^*_i]) = L(\text{mtl}(\mathcal{A}[s^*_i])) \).

An example illustrating this construction is in Appendix C. Notice that if \( \mathcal{A} \) (hence \( \text{Norm}(\mathcal{A}) \)) was \text{PO-1-ATA}, then each island \( P_i \) is a \text{PO-1-ATA}, and \( \mathcal{A}(P_i) \) will be an aperiodic automaton; hence, using Lemma 8 we obtain an equivalent \text{SfrMTL} formula.

4.3 \text{RatMTL} \subseteq_e \text{1-ATA-lfr}

\textbf{Proof.} Consider a formula \( \psi_1 = \text{Rat}_l(\text{re}_0) \) with \( I = [l, u] \). The case of other intervals are handled similarly. \( \psi_1 \) has modal depth 1 and has a single modality. As the formula is of modal depth 1, \( \text{re}_0 \) is an atomic regular expression over alphabet \( \Sigma \). Let \( D = (\Gamma, Q, q_0, Q_f, \delta') \) be a DFA such that \( L(D) = L(\text{re}_0) \), with \( \Gamma = 2^\Sigma \setminus \emptyset \). From \( D \), we construct the 1-clock ATA \( \mathcal{A} = (\Gamma, Q \cup \{q_{\text{init}}, q_{\text{timecheck}}, q_f\}, \{q_{\text{init}}, q_f\}, \delta) \) where \( q_{\text{init}}, q_{\text{timecheck}}, q_f \) are disjoint from \( Q \). The transitions \( \delta \) are as follows. Assume \( l > 0 \).

\begin{itemize}
  \item \( \delta(q_{\text{init}}, a) = x.q_{\text{timecheck}}, a \in \Gamma \),
  \item \( \delta(q_{\text{timecheck}}, a) = [(x \geq l \land \delta'(q_0, a) \lor (q_{\text{timecheck}})] \lor [x > u \land q_f] \) where the latter disjunct is added only when \( q_0 \in Q_f \),
  \item \( \delta(q, a) = (x \in [l, u]) \land \delta'(q, a), \) for all \( q \in Q \setminus Q_f \).
\end{itemize}
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\[ \delta(q, a) = (x \in [l, u) \land \delta'(q, a)) \lor (x > u \land q_f) \], for all \( q \in Q_f \), \( \delta(q_f, a) = q_f \).

It is easy to see that \( \mathcal{A} \) has the loop-free reset condition since \( q_f \) is the only location entered on resets, and control stays in \( q_f \) once it enters \( q_f \). The correctness of \( \mathcal{A} \) is easy to establish: the location \( q_{\text{timecheck}} \) is entered on the first symbol, resetting the clock; control stays in \( q_{\text{timecheck}} \) as long as \( x < l \), and when \( x \geq l \), the DFA is started. As long as \( x \in [l, u) \), we simulate the DFA. If \( x > u \) and we are in a final location of the DFA, the control switches to the final location \( q_f \) of \( \mathcal{A} \).

If \( q_0 \) is itself a final location of the DFA, then from \( q_{\text{timecheck}} \), we enter \( q_f \) when \( x > u \). It is clear that \( \mathcal{A} \) indeed checks that \( re_0 \) is true in the interval \([l, u)\). If \( l = 0 \), then the interval on which \( re_0 \) should hold is \([0, u)\). In this case, if \( q_0 \) is non-final, we have the transition \( \delta(q_0, a) = x.\delta'(q_0, a), a \in \Gamma \) (since our timed words start at time stamp 0, the first symbol is read at time 0, so \( x.\delta'(q_0, a) \) preserves the value of \( x \) after the transition \( \delta'(q_0, a) \)). The location \( q_{\text{timecheck}} \) is not used then.

The case when \( \psi_1 \) has modal depth 1 but has more than one \( \mathcal{R} \) modality is dealt as follows. Firstly, if \( \psi_1 = \neg R_{\mathcal{T}}(re_0) \), then the result follows since 1-ATA-lfr are closed under complementation (the fact that the resets are loop-free on a run does not change when one complements). For the case when we have a conjunction \( \psi_1 \land \psi_2 \) of formulae, having 1-ATA-lfr \( \mathcal{A}_1 = (\Gamma, Q_1, q_1, F_1, \delta_1) \) and \( \mathcal{A}_2 = (\Gamma, Q_2, q_2, F_2, \delta_2) \) such that \( L(A_1) = L(\psi_1) \) and \( L(A_2) = L(\psi_2) \), we construct \( \mathcal{A} = (\Gamma, Q_1 \cup Q_2 \cup \{ q_{\text{init}} \}, q_{\text{init}}, F, \delta) \) such that \( \delta(q_{\text{init}}, a) = x.\delta_1(q_1, a) \land \delta_2(q_2, a). \) Clearly, \( \mathcal{A} \) is a 1-ATA-lfr since \( \mathcal{A}_1, \mathcal{A}_2 \) are. It is easy to see that \( L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2) \). The case when \( \psi = \psi_1 \lor \psi_2 \) follows from the fact that we handle negation and conjunction.

Lifting to formulae of higher modal depth

Let us assume the result for formulae of modal depth \( \leq k \). Consider a formula of modal depth \( k + 1 \) of the form \( \psi_{k+1} = R_{\mathcal{T}}(re_k) \), where \( re_k \) is a regular expression over formulae of modal depth \( \leq k \). Let \( \psi_k \) be a formula of modal depth \( \leq k \). For each such occurrence of a smaller depth formula \( \psi_i \), let us allocate a witness variable \( Z_i \). Let \( Z = \{ Z_1, \ldots, Z_k \} \) be the set of all witness variables. Given a subset \( S \subseteq \Sigma \), let \( \Gamma_S \in S \times 2^S \). Any occurrence of an element \( S \) in \( re_k \) and \( \psi_k \) are replaced with \( \Gamma_S \). At the end of this replacement, \( re_k \) is a regular expression over \( \Gamma \times 2^S \) and \( \psi_k \) is a propositional logic formula over \( \Gamma \times 2^S \).

Since each \( Z_i \in Z \) is a witness for a smaller depth formula \( \psi_i \), by inductive hypothesis, there is a 1-ATA-lfr \( \mathcal{A}_Z \) that is equivalent to \( \psi_i \). Let \( \delta_{Z_i} \) be the transition function of \( \mathcal{A}_Z \) and let \( init_{Z_i} \) be the initial location of \( \mathcal{A}_Z \). We also construct the complement of each such automata \( \mathcal{A}_{\neg Z_i} \), which has as its transition function \( \delta_{\neg Z_i} \) and \( init_{\neg Z_i} \) as its initial location. The base case gives us a 1-ATA-lfr (call it \( \mathcal{C} \)) over the alphabet \( \Gamma \times 2^S \). Let \( \delta_C \) denote the transition function of \( \mathcal{C} \) and let \( \mathcal{S}_C \) be the set of locations of \( \mathcal{C} \). Consider a transition \( \delta_C(s, a) \) in \( \mathcal{C} \). For \( a \in \Gamma \times 2^S \) the transition \( \delta_C(s, a) \) is replaced with \( \delta'(s, S) = \bigvee_{T \subseteq Z} \delta_C(s, a) \land \bigwedge_{j \in T} \left[ x.\text{init}_{Z_j} \land \bigwedge_{j \in T} \left[ x.\text{init}_{\neg Z_j} \right] \right] \).

Note that since \( \mathcal{C} \) as well as \( \mathcal{A}_Z \) and \( \mathcal{A}_{\neg Z_i} \) are 1-ATA-lfr, \( \delta' \) also respects the lfr condition. It is easy to see that the lfr condition is respected, since, once we enter the automata \( \mathcal{A}_Z \), \( \mathcal{A}_{\neg Z_k} \) on reset, we will not return to \( \mathcal{C} \), thereby preserving the linearity of resets. Call this 1-clock ATA \( B' \). Clearly, if \( a \in \Gamma \times T \) is read in \( \mathcal{C} \), such that \( T = \{ Z_i, Z_{i_1}, \ldots, Z_{i_k} \} \), then acceptance in \( B' \) is possible if \( \mathcal{C}, \mathcal{A}_{Z_j}, \mathcal{A}_{Z_{i_1}}, \ldots, \mathcal{A}_{Z_{i_k}} \) and \( \mathcal{A}_{\neg Z_j} \) for \( j \neq i, i_1, \ldots, i_k \) all reach accepting locations on reading the remaining suffix. An example illustrating this can be seen in Appendix C.1.

Note that if \( \psi = R_{\mathcal{T}}(re_0, \psi_0) \) is \( \mathcal{SFM}\text{TL} \), then \( re_0 \) is star-free, and all formulae in \( \psi_0 \) are \( \mathcal{SFM}\text{TL} \). For the base case, the DFA \( D \) obtained will be aperiodic, and the 1-clock ATA constructed will be PO. The inductive hypothesis guarantees this property, and for depth \( k + 1 \), we obtain PO 1-clock ATA since each of \( \delta_C, \delta_{Z_i}, \delta_{\neg Z_i} \) satisfy the PO condition, and once control shifts to some \( \mathcal{A}_{Z_i} \), it does not return back to \( \mathcal{C} \), preserving the PO condition. \( \square \)
4.4 Kamp Theorem for RatMTL and forward QkMSO

In this section, we establish the equivalence between RatMTL and forward QkMSO giving our first Kamp-like theorem.

Theorem 13. forward QkMSO is expressively equivalent to RatMTL.

A careful reading of the proof below also shows that when restricted to QkFO, we obtain expressive equivalence with logic SfrMTL which is RatMTL restricted to star-free regular expressions.

4.5 forward QkMSO ⊆ RatMTL

Proof is by Induction on the metric depth of the formula. For the base case, consider a formula \( \psi(t_0) = Q_{k_1}t_1 \ldots Q_{k_{m-1}}t_{k-1} \varphi(t_0, t_1, \ldots, t_{k-1}) \) of metric depth one. Let \( c_{\text{max}} \) be the maximal constant used in the metric quantifiers \( Q_k \). Define \( \text{CON}(I, t) = \forall \{R_j(t_i) \mid j \leq I\} \). We replace every quantifier \( \exists t_0 \in I \phi \) by \( \forall \{t_0 \leq t_i \land \text{CON}(I, t_i) \land \phi \} \). Every quantifier \( \forall t_0 \in I \phi \) is replaced by \( \forall \{t_0 \leq t_i \land \text{CON}(I, t_i) \rightarrow \phi \} \). To the resulting MSO formula we add a conjunct WELLREGION that states that (a) exactly one \( R_j(t) \) holds at any \( t \), and (b) \( \forall t, t' \left( [t < t' \land R_j(t) \land R_j(t')] \rightarrow j \leq j' \right) \) (asserting region order). Note that these are natural properties of region abstraction of time. This gives us the formula \( \psi_{t_0}(t) \). It has predicates \( R_j(t) \) for \( j \in \text{reg} \) and free variable \( t_0 \). Being MSO formula, we can construct a DFA \( A(\psi_{t_0}(0)) \) for it over the alphabet \( 2^\Sigma \times \{0, 1\}^{\text{reg}} \). Note that we have substituted 0 for \( t_0 \). This is isomorphic to automaton over the alphabet \( 2^\Sigma \times \text{reg} \). From the construction, it is clear that \( \rho = \psi(0) \) iff \( \text{reg}(\rho) \in L(A(\psi_{t_0}(0))) \). By Lemma 8 we then obtain an equivalent RatMTL formula \( \zeta \). It is easy to see that \( L(\psi(0)) = L(\zeta) \). Because \( \psi(0) \) and \( \zeta \) are purely future time formulae, this also gives us that \( \rho, i = \psi(t_0) \) iff \( \rho, i = \zeta \).

For the induction step, consider a metric depth \( n+1 \) formula \( \psi(t_0) \). We can replace every time constraint sub-formula \( \psi_i(t_k) \) occurring in it by a witness monadic predicate \( w_i(t_k) \). This gives a metric depth 1 formula and we can obtain a RatMTL formula, say \( \zeta \), over variables \( \Sigma \cup \{w_i\} \) exactly as in the base step. Notice that each \( w_i(t_k) \) was a formula of modal depth \( n \) or less. Hence by induction hypothesis we have an equivalent RatMTL formula \( \zeta_i \). Substituting \( \zeta_i \) for \( w_i \) in \( \zeta \) gives us a formula language equivalent to \( \psi(t_0) \).

4.6 RatMTL ⊆ forward QkMSO

Let \( \varphi \in \text{RatMTL} \). The proof is by induction on the modal depth of \( \varphi \). For the base case, let \( \varphi = \text{Rat}_j(\text{re}) \) where \( \text{re} \) is a regular expression over propositions. Let \( \zeta(x, y) \) be an MSO formula with the property that \( \sigma, i, j = \zeta(x, y) \) iff \( \sigma[x : y] \in L(\text{re}) \), where \( \sigma[x : y] \) denotes the substring \( \sigma(x+1) \ldots \sigma(y) \). Given that MSO has exactly the expressive power of regular languages, such a formula can always be constructed. Consider the time constraint formula \( \psi(t_0) \):

\[
\exists t_{\text{first}} \in t_0 + I, \exists t_{\text{last}} \in t_0 + I, \forall t' \in t_0 + I \left[ (t' = t_{\text{first}} \lor t' = t_{\text{last}}) \land \forall t_{\text{first}} < t' < t_{\text{last}} \land (t_{\text{first}}, t_{\text{last}}) \right]
\]

Then, it is clear that \( \rho, i = \varphi \) iff \( \rho, i = \psi(t_0) \). Note that \( \psi(t_0) \) is actually a formula of QkMSO with \( k = 4 \).

Atomic and boolean constructs can be straightforwardly translated. Now let \( \varphi = \text{Rat}_j(\text{re}) \) where \( \text{re} \) is over a set of subformulae \( S \). For each \( \zeta_i \in S \), substitute it by a witness proposition \( w_i \) to get a formula \( \varphi_{\text{flat}} \). This is a modal depth 1 formula and we can construct a language equivalent formula of QkMSO, say \( \Xi(t_0) \) over alphabet \( \Sigma \cup \{w_i\} \). By induction hypothesis, for each \( \zeta_i \) there exists a language equivalent time constrained QkMSO formula \( \kappa_i(t_0) \). Now substitute \( \kappa_i(t_j) \) for each occurrence of \( w_i(t_j) \) in \( \Xi(t_0) \) to get a formula \( \psi(t_0) \). Then \( \psi(t_0) \) is language equivalent to \( \varphi \). Also, by suitably reusing the variables, \( \psi(t_0) \) can be constructed to be in QkMSO with \( k = 4 \). □
5 \textbf{C\oplus D-1-ATA-1fr and Logics}

In this section, we show the second of our Büchi-Kamp like results connecting logics FRatMTL, forward Q2MSO and a subclass of 1-clock ATA called conjunctive-disjunctive (abbreviated C\oplus D) 1-clock ATA with loop-free resets.

Let \( A = (Q, q_0, F, \delta) \) be a 1-clock ATA. Let \( Q_x = \{x.q \mid q \in Q\} \) and let \( B(Q_x) ::= \text{true} \) if \( \alpha \in Q_x \), \( \alpha \land \alpha \lor \alpha \lor \alpha \). A 1-clock ATA which has both conditions of loop-free resets and conjunctive-disjunctiveness is called a 1-clock ATA with loop-free resets. Note that a 1-clock ATA which has both conditions of loop-free resets and conjunctive-disjunctiveness is called a \( \text{C\oplus D-1-ATA-1fr} \), while one which satisfies the properties is called a \( \text{C\oplus D-1-ATA-PO} \).

\textbf{Example 14.} We illustrate examples of ATA which are C\oplus D and which are not.

(a) The automaton \( A \) with \( \delta(s, \{a\}) = (x=1) \lor (x.p \land x.r) \), \( \delta(p, \{a\}) = x.s \lor x.q \lor p \), \( \delta(q, \{a\}) = x.r \), \( \delta(r, \{a\}) = x.q \lor r \) is a C\oplus D 1-clock ATA.

(b) For \( a \in \Sigma \), let \( S_a \) and \( S_{\neg a} \) denote any set containing \( a \) and not containing \( a \), respectively. Consider the automaton \( B \) with transitions \( \delta(s_0, S_a) = s_0 \lor s_1, \delta(s_2, \Gamma) = x \leq 1, \delta(s_0, S_{\neg a}) = s_0 \land s_2, \delta(s_1, \Gamma) = x > 1 \) with \( s_0 \) being initial and none of the locations being final satisfies 1fr but violates C\oplus D. The C\oplus D condition is violated since a clause contains more than one free location irrespective of \( s_0, s_1, s_2 \in Q_{\lor} \). This accepts the language of all words where the last symbol in (0, 1) has an \( a \).

(c) Let \( s_a, s_{\neg a} \) be as above. The automaton \( B \) with \( \delta(s_a, S_a) = s_0 \lor s_1, \delta(s_2, \Gamma) = x \leq 1, \delta(s_0, S_{\neg a}) = s_0 \land s_2, \delta(s_1, \Gamma) = x > 1 \) with \( s_0 \) being initial and none of the locations being final satisfies 1fr but violates C\oplus D. The C\oplus D condition is violated since the automata switches between conjunctive and disjunctive locations without any reset. Note that \( s_0 \in Q_{\lor} \) while \( s_1 \in Q_{\land} \). This accepts the language of all words where the second last symbol in (0, 1) has an \( a \).

5.1 \textbf{Büchi Theorem for C\oplus D-1-ATA-1fr}

The main result of this section is the expressive equivalence of C\oplus D-1-ATA-1fr and FRatMTL.

\textbf{Theorem 15.} \( \text{C\oplus D-1-ATA-1fr are expressively equivalent to FRatMTL} \).

5.2 \( \text{C\oplus D-1-ATA-1fr} \subseteq \text{FRatMTL} \)

The first thing is to convert C\oplus D 1-clock ATA with no resets to FRat formula of modal depth 1 as in Lemma 10.
Lemma 16. Given a \( \mathbb{C} \oplus D \) 1-clock ATA \( A \) over \( \Sigma \) with no resets, we can construct a FRat formula \( \varphi \) such that for any timed word \( p = (a_1, \tau_1) \ldots (a_m, \tau_m) \), \( p, i \models \varphi \) iff \( A \) accepts \( (a_i, \tau_i) \ldots (a_m, \tau_m) \).

Proof. Assuming \( q_0 \in Q_\varphi \), the key idea is to check how a word is accepted. The reset-freeness ensures that any transition \( \delta(q, a) = C_1 \lor \cdots \lor C_m \) is such that \( C_i \) is either a location or a clock constraint \( x \in I \). Assume acceptance happens through an empty configuration via a clock constraint \( x \in I_a \), from some location \( q \) on an \( a \), and \( q \) is reachable from \( q_0 \). Let \( \text{re}_{I_a} \) be the regular expression whose language is the set of all such words reaching some \( q \), from where acceptance happens via interval \( I_a \) on an \( a \). The formula \( \text{FRat}_{I_a, a} \) sums up all such words.

Disjuncting over all possible intervals and symbols, we have the result. The second case is when a final state \( q_f \) is reached from some \( q' \) reachable from \( q_0 \). If \( \text{re}_{q_f, a} \) is the regular expression whose language is all words reaching such a \( q' \), the formula \( \text{FRat}_{(0, \infty), \text{re}_{q_f, a}(a \land \Box \bot)} \) sums up all words accepted via \( q', a, q_f \). The \( \Box \bot \) ensures that no further symbols are read, and can be written as \( \neg \text{FRat}_{(0, \infty), a} \top \). Disjuncting over all possible final states \( q_f \) and \( a \in \Sigma \) gives us the formula. The case when \( q_0 \in Q_\varphi \) is handled by negating the automaton, obtaining \( \neg q_0 \in Q_\varphi \) and negating the resulting formula. Details in Appendix F.

The rest of the proof is very similar to Section 12 and omitted. Note that if we had started with a \( \mathbb{C} \oplus D \)-1-ATA-PO, then the regular expressions \( \text{re} \) in the FRatMTL formula obtained for the base case has an equivalent star-free expression, since the underlying automaton is aperiodic. For the inductive case with resets and PO, we obtain a FSfrMTL formula since plugging in witness variables with a FSfrMTL formula again yields a FSfrMTL formula.

5.3 FRatMTL \( \subseteq_c \mathbb{C} \oplus D \)-1-ATA-lfr

This is almost identical to the proof of section 13 and is provided in Appendix C for completeness. Finally, notice that FSfrMTL formulae correspond to \( \mathbb{C} \oplus D \)-1-ATA-PO.

5.4 Kamp Theorem for FRatMTL and forward Q2MSO

The expressive equivalence of forward Q2MSO and FRatMTL is stated in Theorem 17. If we restrict to logic forward Q2FO, then we obtain expressive equivalence with respect to FSfrMTL.

Theorem 17. FRatMTL is expressively equivalent to forward Q2MSO.

5.5 forward Q2MSO (Q2FO) \( \subseteq_c \) FRatMTL (FSfrMTL)

We first consider formulae of metric depth one. These have the form \( \psi(t_0) = Q_1 t_1 \varphi(\downarrow_t t_1, t_1) \) and \( \varphi(\downarrow_t t_1, t_1) \) is an MSO (FO) formula (bound first order variables \( t' \) in \( \varphi \) only have the comparison \( t' > t_0 \), and there are no free variables other than \( t_0, t_1 \), and hence no metric comparison exists in \( \varphi \)). Let \( \text{re}_\varphi \) be the regular expression equivalent to \( \varphi(\downarrow_t t_1, t_1) \). The presence of free variables \( t_0, t_1 \) implies that \( \text{re}_\varphi \) is over the alphabet \( 2^2 \times \{0, 1\}^2 \), where the last two bits are for \( t_0, t_1 \). As seen in the case of section 14, \( t_0 \) is assigned the first position of \( \text{re}_\varphi \) since all other variables take up a position to its right. Hence \( \text{re}_\varphi \) can be rewritten as \( (2^2, 1, 0) \text{re}_\varphi \). Since \( t_1 \) is assigned a unique position, there is exactly one occurrence of a symbol of the form \( (2^2, 0, 1) \) in \( \text{re}_\varphi \). Using (Lemma 7, page 16) [8], we can write \( \text{re}_\varphi \) as a finite union of disjoint expressions each of the form \( \text{re}_\varphi(\alpha, 0, 1) \text{re}_r \) where \( \alpha \in 2^2 \), and \( \text{re}_r, \text{re}_r \subseteq (2^2, 0, 1)^* \). \( \varphi(\downarrow_t t_0, t_1) \) is thus equivalent to having a symbol \( (\alpha, 0, 1) \) at a time point \( t \in t_0 + I \), and \( (2^2, 0, 1) \text{re}_r \) holds till \( t \), and beyond \( t \), \( \text{re}_r \) holds. This is captured by the formula \( \text{FRat}_{t, \text{re}}(\bigvee_{\alpha \in 2^2} (\langle(\alpha, 0, 1) \land \text{FRat}_{(0, \infty), \text{re}, \Box \bot}) \right) \). Here, \( \text{re}_\varphi = (2^2, 0, 1) \text{re}_r \), and the \( \Box \bot \) symbolizes the fact that we see \( \text{re}_r \) in the latter part after \( (\alpha, 0, 1) \) and no more symbols after that. If \( \psi(t_0) \in Q2FO \), then \( \text{re}_\varphi \) is a star-free expression, and so are \( \text{re}_r, \text{re}_r \). That gives us a FSfrMTL formula.
The inductive case for formulae of higher depth is in Appendix I. The case of going from FRatMTL to forward Q2MSO is similar to section 4.6 and is in Appendix I.

Remark C: In 1-ATA-fr with non-punctual guards gives expressiveness equivalence with FRatMTL. Likewise, FRatMTL is expressively equivalent to logic Q2MSO where none of the time constraints are punctual. Note that this is the case since the proof does not introduce punctual guards if there are none in the starting automaton/logic.

6 Temporal Logics with FixPoints

In this section, we look at the logics RatMTL and FRatMTL enhanced with fix point operators.

RatMTL with fixed points (μRatMTL)

μRatMTL Syntax: Formulae of μRatMTL are built from a finite alphabet Σ and a finite set Z of recursion variables as:

\[
\varphi ::= \sigma(\in \Sigma)[\text{true}]Z(\in Z)[\varphi \land \varphi]\text{FRat(re(S))} \text{FRat}_{I_\varphi}(S)\varphi | \text{GRat}_{I_\varphi}(S)\varphi | \mu Z \circ \varphi \circ \nu Z \circ \varphi, \text{ where } I \in I_\nu, \text{ and } S, \text{ re(S)} \text{ are as before, and } \text{GRat}_{I_\varphi}(S)^{-}\ \varphi \text{ is equivalent to } \neg \text{FRat}_{I_\varphi}(S)^{-}\ \varphi.
\]

The subformulae S of ϕ now contain μRatMTL formulae. A μRatMTL formula is said to be sentence if every recursion variable Z is within the scope of a fix point operator. Otherwise the formula is open and we write it as \(\varphi(Z_1, \ldots, Z_i)\) where \(Z_1, \ldots, Z_i\) occur freely.

μRatMTL Semantics: To define the semantics, we first define a super structure. A super structure is a timed word over \([\mathcal{P}(Z) - \emptyset] \times \mathcal{P}(Z)\). The super structure is labelled with non-empty subsets of Z and with a possibly empty set of recursion variables at each position. For a super structure \(\rho = ((\sigma, Z), \tau)\), a position \(i \in \text{dom}(\rho)\), a μRatMTL formula \(\varphi\), and a finite set S of sub-formulae of \(\varphi\), we define the satisfaction of \(\varphi\) at a position \(i\) as follows. For \(Z \in Z, \rho, i \models Z\) iff \(Z \in \sigma(i)\). We use the notations \(\text{Seg}(\rho, S, i, j)\) and \(\text{TSeg}(\rho, S, I, i)\) as in the case of RatMTL.

The semantics of formulae which do not involve \(\mu\) are as defined earlier.

Two super structures \(\rho = (\sigma, \tau)\) and \(\rho' = (\sigma', \tau')\) agree except on \(Z\) iff \(w \in \sigma(i)\) iff \(w \in \sigma'(i)\) for all \(w \neq Z\) and all \(i \geq 1\). We say that a super structure \(\rho'\) is a fix point of \(Z \equiv \varphi(Z)\) with respect to \(\rho\) iff \(\rho'\) and \(\rho'\) agree except on \(Z\) and \(\rho', i \models Z\) iff \(\rho', i \models Z\). The formula \(\mu Z \circ \varphi(Z)\) (respectively \(\nu Z \circ \varphi(Z)\)) denotes the least (respectively greatest) fixpoint solution to the equation \(Z \equiv \varphi(Z)\). The super structure \(\rho'\) is a least fix point if whenever \(\beta\) is also a fixpoint, then for all \(i \geq 1\), \(\rho', i \models Z \Rightarrow \beta, i \models Z\). The super structure \(\rho'\) is a greatest fix point if whenever \(\beta\) is also a fixpoint, then for all \(i \geq 1\), \(\beta, i \models Z \Rightarrow \rho', i \models Z\). The semantics of fixed point formulae is as follows.

Semantics of Fix Point Formulae:

- \(\rho, i \models \mu Z \circ \varphi(Z)\) iff \(\rho'', i \models \mu Z \circ \varphi(Z)\) where \(\rho''\) is a least fix point for \(Z \equiv \varphi(Z)\) with respect to \(\rho\).
- \(\rho, i \models \nu Z \circ \varphi(Z)\) iff \(\rho'', i \models \nu Z \circ \varphi(Z)\) where \(\rho''\) is a greatest fix point for \(Z \equiv \varphi(Z)\) with respect to \(\rho\). For sentences \(\mu Z \circ \varphi(Z)\), the truth value of \(\varphi\) is determined using timed words \(\rho\) (super structures \(\rho = (\sigma, \tau)\) such that \(Z \notin \sigma(i)\) for all \(i\)). For a sentence \(\varphi\), a timed word \(\rho\), and \(i \geq 1\), we say that \(\rho, i \models \mu Z \circ \varphi(Z)\) if there is a least fix point \(\beta\) such that \(\beta, i \models \varphi\). For any sentence \(\varphi\), the language \(L(\varphi)\) is defined as set of all the timed words \(\rho\) such that \(\rho, 1 \models \varphi\). If we restrict ourselves to using only FRat, then the resultant logic is μRatMTL.

Example 18. Let \(\varphi = \mu Z,[a \rightarrow \text{Rat}_{(0,1)}([a + b]^{\ast}(b \lor Z))].\)

1. Let \(\rho' = (\{a\}, 0)((b, Z), 0.6)\{\{a\}, 0.9\}{\{b, Z\}, 1.7}\{\{a\}, 1.8\}\) and \(\rho = (\{a\}, 0)(\{b\}, 0.6)(\{a\}, 0.9)(\{b\}, 1.7)(\{a\}, 1.8)\) be super structures. Then \(\rho, \rho'\) agree except on \(Z\) and \(\rho'\) is a least fixed point of \(Z \equiv \varphi(Z)\) with respect to \(\rho\). It can be seen that \(\rho, 1 \not\models \mu Z,[a \rightarrow \text{Rat}_{(0,1)}([a + b]^{\ast}(b \lor Z))]\) since no super structure \(\beta\) which agrees with \(\rho\) except on \(Z\) can be such that \(\beta, 1 \not\models Z\).
2. Let \( \rho' = \{(a, Z), 0\}, \{(b, Z), 0.6\}, \{(a, Z), 0.9\}, \{(b, Z), 1.7\} \) and 
\[ \rho = \{(a, 0), \{(b, 0.6), \{(a, 0.9), \{(b, 1.7)\}\}\}\}\] 
be super structures. Then \( \rho, \rho' \) agree except on \( Z \) and 
\( \rho' \) is a least fixed point of \( \varphi(Z) \) with respect to \( \rho \). It can be seen that \( \rho, 1 \models \mu Z[a \rightarrow \text{Rat}_0, 0, 1 \models \nu Z] \) since \( \rho', 1 \models Z \).

Example 19. Let \( \varphi = \mu Z[Z \lor \text{FRat}_{(0, 1)}, (a, a + Z) + b] \).
1. Let \( \rho' = \{(a, Z), 0\}, \{(a, Z), 0.2\}, \{(a, Z), 0.7\}, \{(a, Z), 0.9\}, \{(a, Z), 1.1\}, \{(a, Z), 1.3\}, \{(a, Z), 1.7\} \) and 
\( \rho'' = \{(a, Z), 0\}, \{(a, 0.2), \{(a, 0.7), \{(b, 0.9), \{(a, 1.1), \{(a, 1.3), \{(b, 1.7)\}\}\}\}\}\}\] 
be two super structures. Then both \( \rho' \) and \( \rho'' \) are fixpoints of \( \varphi(Z) \) with respect to 
\( \rho = \{(a, 0), \{(a, 0.2), \{(a, 0.7), \{(b, 0.9), \{(a, 1.1), \{(a, 1.3), \{(b, 1.7)\}\}\}\}\}\}\] 
It can be seen that \( \rho'' \) is a least fix point and \( \rho', 1 \models Z \). Hence, \( \rho, 1 \models \varphi \).
2. The timed word \( \rho = \{(a, 0), \{(a, 0.7), \{(b, 0.9), \{(a, 1.1), \{(a, 1.3), \{(b, 1.7)\}\}\}\}\}\] 
is such that \( \rho, 1 \not\models \varphi \). Note that there does not exist a least fix point \( \beta \) that agrees with \( \rho \) except \( Z \) such that 
\( \beta, 1 \models \mu Z \).

Let \( Z \) denote a tuple of variables from \( \mathbb{Z} \).

Definition 20 (Guarded Fragment). We say that a recursion variable \( Z \) is guarded in a temporal \( \mu \) calculus formulae \( \psi(Z, Z) \) if and only variable \( Z \) is within the scope of a strict future modality. Any formulae is a guarded formulae if and only if in all its subformulae of the form \( \mu Z \circ \psi(Z, Z) \) (or \( \nu Z \circ \psi(Z, Z) \)), \( Z \) is guarded in \( \psi \).

It can be easily shown that the guarded restriction on temporal \( \mu \) calculus formulae does not affect the expressive power of the logic. Hence, we consider only guarded formulae. A proof of Lemma 21 is in Appendix J.

Lemma 21. Given any guarded formula \( \psi(Z) \), \( Z \equiv \psi(Z) \) has a unique solution if the models are finite timed words.

As a corollary, over finite timed words, \( \mu Z \circ \varphi(Z) \) is equivalent to \( \nu Z \circ \varphi(Z) \), provided that \( Z \) is guarded in \( \varphi(Z) \).

Definition 22 (Temporal Equation Systems). Consider a series of equations 
\( Z_1 \equiv \mu \psi_1(Z_1, \ldots, Z_m), \ldots, Z_m \equiv \mu \psi_m(Z_1, \ldots, Z_m) \), where \( \psi_1, \ldots, \psi_m \) are temporal logic formulae over \( \Sigma \cup \{Z_1, \ldots, Z_m\} \) and \( Z_i \equiv \nu \psi_i(Z_1, \ldots, Z_m) \) denotes that \( Z_i \) is the least fix point solution of \( \psi_i \). If the \( \psi \) are \( \text{RatMTL} \) or \( \text{FRatMTL} \) formulae then we call it as system of \( \text{RatMTL} \) equations or \( \text{FRatMTL} \) equations, respectively.

It can be shown (see [1], [2] and Appendix J for an example) that any \( \mu \text{RatMTL} \) and \( \mu \text{FRatMTL} \) can be equivalently reduced to their respective system of equations. By lemma 21, we know that the least and the greatest fix point operators have identical semantics over finite timed words. Hence, we will consider only \( \mu \) operators and will drop the superscript on \( \equiv \). Note that if this equation is true, \( Z_i \) is a witness for \( \psi_i \). The rest of the section establishes the expressive equivalence of 1-clock ATA (\( \mathbb{C} \equiv \mathbb{D} \) 1-clock ATA) with logic \( \mu \text{RatMTL} \) (\( \mu \text{FRatMTL} \)).

Theorem 23. (a) Given a 1-clock ATA \( \mathbb{A} \), there is a \( \mu \text{RatMTL} \) formula \( \psi \) s.t. \( L(\psi) = L(\mathbb{A}) \).

(b) Given a \( \mathbb{C} \equiv \mathbb{D} \) 1-clock ATA \( \mathbb{A} \), there is a \( \mu \text{FRatMTL} \) formula \( \psi \) s.t. \( L(\psi) = L(\mathbb{A}) \).

Proof Sketch: (a) For each island \( P_i \) of \( \text{Norm}(\mathbb{A}) \), we eliminate all the outgoing reset transitions using witnesses as done in section 1.2 resulting in a reset-free 1-clock ATA, which in turn is converted to \( \text{RatMTL} \) formulae \( \varphi_i \) over the extended alphabet consisting of witness variables \( w_j \) for island \( P_j \). (b) The islands \( P_i \) will be either conjunctive or disjunctive resulting in \( \text{FRat} \)
formulae $\varphi$, as in section 5.2. Solving the system $w_1 \equiv \varphi_1; \ldots; w_k \equiv \varphi_k$ of $\text{RatMTL}$ equations, (and $\text{FRatMTL}$ in case (b)) the set of words accepted by $A$ is given by the solution for $w_1$.

**Theorem 24.** (a) Given a $\mu\text{RatMTL}$ formula $\psi$, we can construct a 1-clock $\text{ATA}$ $A$ s.t. $L(\psi) = L(A)$. 
(b) Given a $\mu\text{FRatMTL}$ formula $\psi$, we can construct a $C \oplus D, 1$-clock $\text{ATA}$ $A$ s.t. $L(\psi) = L(A)$.

**Proof Sketch:** The proof is a generalization of sections 4.3, 5.3. Given any $\mu\text{RatMTL}$ or $\mu\text{FRatMTL}$ formula $\varphi$, we can convert it into a system of $\text{RatMTL}$ or $\text{FRatMTL}$ equations of the form $Z_1 \equiv \psi_1(Z_1, \ldots, Z_m); \ldots; Z_m \equiv \psi_m(Z_1, \ldots, Z_m)$. In the case of (a), for all $\psi_i$, we first construct an equivalent 1-clock ATA with loop free resets, $A_{Z_i}$. As each $A_{Z_i}$ is over $2^Z \times \mathbb{Z}$, where each $Z_i$ is a witness of $A_{Z_i}$, we can eliminate $Z_i$ from all $A_{Z_i}$ by adding reset transitions to $A_{Z_i}$ or $A_{-Z_i}$ appropriately as shown in section 4.3. For (b), we repeat similar construction obtaining $C \oplus D, 1$-ATA-lfr for each $\psi$. The only difference in (b) is to ensure that after eliminating witnesses, we retain the conjunctive-disjunctiveness of the automata. The formulae $\psi_i$ and $\psi_j$ can depend on each other; $\psi_i$ can contain witness $Z_j$ while $\psi_j$ can contain witness $Z_i$, unlike sections 4.3, 5.3. Due to this circular dependence, while eliminating witnesses, the resulting automaton may not have loop-free resets (lfr). As we need the solution to the first equation, the initial location of the constructed automaton will be the initial location of $A_{Z_i}$.

**Theorem 25.** Satisfiability of $\mu\text{FRatMITL}$ and reachability in $C \oplus D$ 1-clock $\text{ATA}$ with non-punctual guards have elementary decidability.

**Proof.** Any $\mu\text{FRatMITL}$ formula or $C \oplus D$ 1-clock $\text{ATA}$ can be reduced to an equivalent system of $\text{FRatMITL}$ equations with elementary blow up. Given any system of equations $Z_1 \equiv \psi_1; \ldots; Z_m \equiv \psi_m$, the $\text{FRatMITL}$ formula $\varphi \equiv Z_1 \land \bigwedge_{a \in \Sigma} (Z_1 \leftrightarrow \psi_1) \land \ldots \land (Z_m \leftrightarrow \psi_m) \land \bigwedge_{a \in \Sigma} (\bigvee (\varphi))$ over the extended alphabet $2^Z \cup \{Z_1, \ldots, Z_m\}$ is satisfiable iff there exists a solution to the above system of equations.

The blow up incurred in the construction of $\varphi$ is only linear compared to the size of the equations. Note that any $\text{FRatMITL}$ formula can be reduced to an MITL formula preserving satisfiability with a doubly exponential blow up (elementary) [22]. Using the elementary satisfiability [1] of MITL, we obtain an elementary upper bound for $\mu\text{FRatMITL}$. □

### 7 Discussion

We have proposed two new structural restrictions on 1-ATA:

1. Loop-Free-Resets, where there are no loops involving reset transitions, (1-ATA-lfr)
2. Conjunctive-disjunctive partitioning, where the automaton works in purely disjunctive or conjunctive mode between resets ($C \oplus D, 1$-ATA-lfr). Timing constraints only affect resets. In the disjunctive mode, the automaton behaves like an untimed NFA and the conjunctive mode is its dual. These structural restrictions were inspired by the quest for automata characterizations of some natural metric temporal logics.

One of the main contributions in this paper is the study of monadic second order logic with metric quantifiers QkMSO and its subclasses. We are able to obtain Kamp like theorems with our structural restrictions. It is interesting that we are able to prove a 4-variable property for QkMSO and QkFO. It is also noteworthy that conjunctive-disjunctive restriction on 1-ATA uniformly bring the expressiveness down to the two variable fragment.

Finally we give temporal fixpoint logics $\mu\text{RatMTL}$ and $\mu\text{FRatMTL}$ to characterize full 1-ATA and $C \oplus D, 1$-ATA. A proper temporal logic and classical logic characterizing the full 1-ATA is left open. We believe that $C \oplus D, 1$-ATA is strictly less expressive than the full 1-ATA, but a formal

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3 $\bigwedge_{a \in \Sigma} \varphi$ expands to $\varphi \land \bigwedge_{a \in \Sigma} \varphi$. 

4 $\bigvee (\varphi)$ expands to $\varphi \lor \bigwedge_{a \in \Sigma} \varphi$.
proof will appear in the full version of this work. The proof goes by extending EF games for MTL with threshold counting [21] to that for RatMTL.

One of the takeaways of this paper is the fact that both $C \oplus D$ 1-ATA and $\mu$FRatMTL enjoy the benefits of relaxing punctuality. That is, the reachability for $C \oplus D$-1-ATA and satisfiability checking for $\mu$FRatMTL restricted to non punctual timing constraints are decidable with elementary complexity. We believe this result is important since, $C \oplus D$ 1-ATA, to the best of our knowledge, is the first such class of timed automata which has alternations and yet the reachability is decidable with elementary complexity.

**Related Work**: Büchi’s Theorem [14] showing expressive equivalence of MSO\[\langle \rangle\] and DFA, as well as Kamp’s Theorem [16] showing the expressive equivalence of FO\[\langle \rangle\] and LTL are classical results. Going on to timed languages and logics, enhancing regular expressions with quantitative timing properties was first done in [2]. Timed regular expressions defined in [2] are exactly equivalent to the class of languages definable by non-deterministic timed automata, and hence not closed under negations. Adding regular expressions to LTL was done in [7], [5], [9]. Addition of an automaton modality to MITL was done by Wilke [23]. Wilke’s modality is equivalent to our FRat modality but we also allow punctual intervals. In [11], pointwise MTL with “earlier” and “newer” modalities were introduced to obtain expressive completeness for FO\[\langle\rangle, +1\rangle\] over bounded timed words. The temporal logics RatMTL and FRatMTL studied in this paper were first defined in [22] where their decidability was established.

Expressive completeness for timed logics and languages aiming at Büchi-Kamp like theorems has been another prominent line of study. In the timed setting, continuous timed logics have been explored more. Hirshfeld and Rabinovich [10] showed expressive completeness for MITL and its counting extension with the subclasses QMLO and Q2MLO of FO\[\langle\rangle, +1\rangle\]. Their definition of Q2MLO has been adapted to pointwise setting and generalized to QkMSO in this paper. Ouaknine, Worrell and Hunter, in their seminal paper [13], showed expressive completeness for MTL with rational timing constants with FO \[\langle\rangle, +1\rangle\] (over timed signals). In a related work [12] proved that the expressive completeness carries over even by restricting to standard integer timing constants if MTL is extended by threshold counting modality. All these expressive completeness results were for continuous timed logics which are all undecidable. Our paper focuses on point-wise semantics and finite timed words. In this context, the notable result by Ouaknine and Worrell was the reduction of MTL\[U_I\] to partially ordered 1-clock ATA [18]. Unfortunately, the converse does not hold and MTL\[U_I\] is expressively weak. Going to full 1-clock ATA, Haase et al [8] extended 1-TPTL with fixpoints, which is a hybrid between first-order logic and temporal logic, featuring variables and quantification in addition to temporal modalities (quoting [13]). They established the expressive equivalence of the two. Raskin studied second order extensions of MITL in both continuous and pointwise time [20].

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Appendix

A Normal Form for 1-clock ATA

We start defining a homomorphism between ATA.

Homomorphism in 1 clock ATA

Let $A^1 = (\Sigma, S^1, s^1_0, F^1, \delta^1)$ and $A^2 = (\Sigma, S^2, s^2_0, F^2, \delta^2)$ be 1-clock ATA. We say that $A^2$ is homomorphic to $A^1$ (denoted $A^2 = h(A^1)$) if there is a map $h$ from $S^1$ to $S^2$ satisfying the following.

(i) The map preserves respective initial and final locations: $h(s^1_0) = s^2_0$, and for any $f^1 \in F^1$, $h(f^1) = f^2$ iff $f^2 \in F^2$.

(ii) The map $h$ extends in the usual way to transitions. Corresponding to any transition $\delta^1(s, a) = \varphi$ where $\varphi \in \Phi(S^1 \cup X)$, we obtain the transition $\delta^2(h(s), a) = h(\varphi)$, where $h(\varphi)$ is obtained by substituting all occurrences of locations $s \in S^1$ in $\varphi$ with $h(s)$.

Example 26. The 1-clock ATA $A$ in Example [8] is homomorphic to the 1-clock ATA $B = \{(a, b), \{s_0, s_1, s_2, s_3\}, s_0, \{s_0, s_2, s_3\}, \delta_B\}$ with transitions $\delta_B(s_0, b) = s_0, \delta_B(s_0, a) = (s_0 \land x, s_1) \lor s_2, \delta_B(s_1, a) = (s_1 \land x < 1) \lor (x > 1) = \delta_B(s_1, b), \text{ and } \delta_B(s_2, b) = s_3, \delta_B(s_2, a) = \bot, \delta_B(s_3, b) = s_2, \delta_B(s_3, a) = \bot$, under the map $h(s_0) = t_0, h(s_1) = t_1, h(s_2) = t_2 = h(s_3)$. $A = h(B)$.

Lemma 27. Let $A$ and $B$ be 1-clock ATA such that $B = h(A)$. Then $L(A) = L(B)$.

Proof. Let $A$ and $B$ be 1-clock ATA such that $B = h(A)$. Then we know that each location $s_0$ in $B$ is the map of some location $s_0$ in $A$; moreover, the initial and final locations of $A$ are mapped to initial, final locations respectively in $B$. Let $w \in L(A)$, and let $s^A_0$ be the initial location of $A$. Starting from the initial configuration $C_0 = \{(s^A_0, 0)\}$, there is a run on $w$ in $A$ which ends in an accepting configuration. In $B$, we start with $D_0 = \{(h(s^A_0), 0)\}$. Subsequent configurations obtained are such that $D_i = \{(h(s), t) \mid (s, t) \in C_i\}$. If $w$ was accepted in $A$ due to $C_n$ being accepting then we also have $D_n$ accepting due to the property of homomorphisms. The converse when $w \in L(B)$ is similar, since we can apply the inverse map of $h$ and draw the same conclusion. □

Normalization of 1-clock ATA

Next, we show that for every 1-clock ATA $A$, there exists a 1-clock ATA $\text{Norm}(A)$ in normal form such that $A$ is homomorphic to $\text{Norm}(A)$. Let $A = (\Sigma, S, s_0, F, \delta)$ with $S = \{s_0, s_1, \ldots, s_k\}$. The normalized ATA $\text{Norm}(A) = (\Sigma, S', s'_0, F', \delta')$ is as follows:

$S' = \{s'_i \mid s_i \in S\} \cup \{s^{nr,j}_i \mid s_i \in S, 0 \leq j \leq k\}$

For every $s_i \in S$ and $a \in \Sigma$, if $\delta(s_i, a) = \varphi$, then for all $0 \leq j \leq k$, $\delta'(s^{nr,j}_i, a) = \varphi'$ and $\delta'(s'_i, a) = \varphi''$ where $\varphi', \varphi''$ are obtained as follows.

- All locations $s_i$ occurring in $\varphi$ without the binding construct $x$ are replaced in $\varphi'$ with $s^{nr,j}_i$, and replaced in $\varphi''$ with $s^{nr,j}_i$.

- All locations $s_i$ occurring in $\varphi$ as $xs_h$, with the binding construct $x$ are replaced in $\varphi', \varphi''$ with $x.s^{nr,i}_h$.

- $s'_0 = s^A_0, F' = \{s'_i, s^{nr,j}_i \mid s_i \in F \land 0 \leq j \leq k\}$

As we will see, the intuition behind the normalization is that we can find disjoint sets $P_1, \ldots, P_n$ which partition the set of reachable locations where $P_i = \{s^{nr,i}_k, s'_i \mid 1 \leq i \leq n\}$. The initial location of a partition $P_i$ is $s'_i$.

Lemma 28. $A$ is homomorphic to $\text{Norm}(A)$, a 1-clock ATA in normal form.
Lemma 29. It can be seen that $A$ is homomorphic to $\text{Norm}(A)$ ($A = h(\text{Norm}(A))$) according to the map $h(s_i^j) = h(s_i^{nr,j}) = s_i$ for all $0 \leq i, j \leq k$. To see that $\text{Norm}(A)$ is in normal form, consider the partition $S^r = \{s_i^r | s_i \in S\}$ and $S^{nr} = \{s_i^{nr} | s_i \in S, 0 \leq i \leq k\}$. Clearly, locations of $S^r$ appear in transitions attached to the binding construct $x$. While locations of $S^{nr}$ always appear in transitions without the binding construct $x$. Further, the set of locations of $\text{Norm}(A)$ can be partitioned into $P_0 \cup \cdots \cup P_k$ where $P_j = \{s_i^j, s_i^{nr,j} | 0 \leq j \leq k\}$. For any $s \in P_j$, it is clear from the transitions $\delta(s,a) = \varphi$ that $s' \in P_j$ for any $s' \in \varphi$ if $s'$ is free in $\varphi$. Hence, the locations of $P_j$ are either $s_i^j$, (the initial location of $P_j$ which is obtained when $s_i^j$ occurs bound in some transition $\delta(s,a)$ with $s' \in P_k$ for some $k \neq j$), or $s_i^{nr,j}$ where $nr$ represents that location $s_i$ is free in the transition.

Example 30. The 1-clock ATA in Example 29 is in normal form. We have the partition $S_r = \{t_1\}$, $S_{nr} = \{t_0, t_2\}$ and also $S = P_1 \cup P_2$ with $P_1 = \{t_0, t_2\}$ and $P_2 = \{t_1\}$. The automaton in Example 29 is also in normal form.

B Proof of Lemma 7

First we describe the construction of $A(P)$. Consider any transition $\delta(s,a) = C_1 \lor \cdots \lor C_m$ in the 1-clock reset-free ATA $P$. Let $C_{a_1}, \ldots, C_{a_m}$ be clauses containing $x \in I$ for some interval $I$. We consider intervals $I$ in the region form $[0,0]((0,1), \ldots, (c_{max}, \infty))$, where $c_{max}$ is the maximal constant used in $A$. We rewrite $\delta(s,a)$ as $\delta(s,(a,I)) = C_{a_1}' \lor \cdots \lor C_{a_m}'$, where $C_{a_i}'$ is obtained from $C_{a_i}$ by removing the conjunct $x \in I$. Depending on the number of intervals $I$ that appear in $\delta(s,a)$, we obtain transitions $\delta(s,(a,I))$ by suitably combining clauses that share the same interval.

The above rewrite of transitions, expands the alphabet to $\Gamma \times \text{reg}$, where $\text{reg}$ is a set of intervals $[0,0], (0,1), \ldots, (c_{max}, \infty)$ and $\Gamma = 2^\Sigma \setminus \emptyset$. This rewrite results in making $P$ an untimed alternating finite automaton (AFA) over the interval alphabet $\Gamma \times \text{reg}$. Let $A(P)$ denote the AFA with initial location $s_0$ (same as the initial location of $P$).

The language of $P \downarrow (L(P))$ consists of all timed words that have a run from initial location $s_0$ to a final location $q$ of $P$. Let $w = (a_1,t_1) \ldots (a_m,t_m)$ be a timed word which has a run starting at $C_0 = \{(s,0)\}$, where $s_0$ is the initial location of $P$, to an accepting configuration $C_m$ in $P$. The run of $w$ on $P$ is $C_0 \xrightarrow{t_1} C_1 \xrightarrow{a_1} C_1^{t_2-t_1} \cdots \xrightarrow{t_m-t_{m-1}} C_{m-1} + (t_m-t_{m-1}) \xrightarrow{a_m} C_m$. Let $t_j \in I_j$. By construction of $A(P)$, each transition $\delta(s,a_j) = (x \in I_j \lor \psi) \lor C$ of $P$ has been translated into $\delta'(s,(a_j,I_j)) = \psi$ (wlg we assume that $C$ has no occurrence of $x \in I_j$). Let $D_0 = \{s_0\}$. The run in $P$ now translates into the run $D_0 \xrightarrow{(a_1,t_1)} D_1 \xrightarrow{(a_2,t_2)} D_2 \cdots \xrightarrow{(a_m,t_m)} D_m$, where $D_0 = \{s \mid (s,0) \in C_0\}$, and $D_j = \{s \mid (s,t) \in C_j, t \in I_j\}$, $j \geq 1$. Since all locations in $C_m$ are accepting, $D_m$ is an accepting configuration in $A(P)$ accepting $(a_1,1) \ldots (a_m,1)$.

Conversely, whenever a good word $(a_1,t_1) \ldots (a_m,t_m)$ is accepted by $A(P)$, we have an accepting run $D_0 \xrightarrow{(a_1,t_1)} D_1 \xrightarrow{(a_2,t_2)} D_2 \cdots \xrightarrow{(a_m,t_m)} D_m$ as above. By construction of $A(P)$, we obtain a run $C_0 \xrightarrow{t_1} C_1 \xrightarrow{a_1} C_1^{t_2-t_1} \cdots \xrightarrow{t_m-t_{m-1}} C_{m-1} + (t_m-t_{m-1}) \xrightarrow{a_m} C_m$ in $P$ on a word $(a_1,t_1) \ldots (a_m,t_m)$ with $t_j \in I_j$ for all $j$. Here, $C_0 = \{(s,0) \mid s \in D_0\}$, and $C_j = \{(s,t) \mid s \in D_j, t \in I_j\}$ for $j \geq 1$. All words $(a_1,t_1) \ldots (a_m,t_m)$ with $t_j \in I_j$ will be accepted by $P$, since $C_m = \{(s,t) \mid s \in D_m, t \in I_m\}$, and all locations $s$ in $D_m$ are final.

C An Example Illustrating Theorem 12

Example 31. We demonstrate the technique on an example. Consider the timed language consisting of all strings where every $a$ has an even number of $b$’s at a distance (1,2) from it. Let the alphabet be $\Sigma = \{a,b\}$. In this example, at any time point, exactly one symbol of $\Sigma$ is read.
This language is accepted by the 1-clock ATA $A = \{(a, b), \{s_0, s_1, s_2\}, \{s_0\}, \{s_0, s_1\}, \delta\}$ with transitions
1. $\delta(s_0, a) = s_0 \land x, s_1, \delta(s_0, b) = s_0$,
2. $\delta(s_1, a) = s_1, \delta(s_1, b) = (s_1 \land x \leq 1) \lor (s_2 \land x \in (1, 2)) \lor x \geq 2$,
3. $\delta(s_2, a) = s_2, \delta(s_2, b) = (s_1 \land x \in (1, 2)) \lor s_2$.

Note that each of the transitions can be easily made complete with respect to the clock constraints: that is, from each location, for each symbol $\delta$ transition leaves of untimed alternating automaton over the alphabet $\Sigma$.

Recall that we expand the alphabet $i = \{\{0, 1\}, \{0, 1\}, \{0, 1\}, \{0, 1\}\}$, and once a transition leaves $P_0$ and enters $P_1$, then it cannot come back to $P_0$. $P_1$ is thus a tail island. We rewrite the transitions as follows.

Recall that we expand the alphabet $\Sigma \times \{P_1, 0\} \times I$ for $A(P_0)$ and to $\Sigma \times I$ for $A(P_1)$. $A(P_1)$ represents the automaton consisting of locations of $P_1$, and transitions between locations of $P_1$. All transitions between locations of $A(P_0), A(P_1)$ are reset-free, and hence, $A(P_0)$ is an untimed alternating automaton over the alphabet $\Sigma \times \{P_1, 0\} \times I$, while $A(P_1)$ is an untimed alternating automaton over the alphabet $\Sigma \times I$. The initial location of $A(P_0)$ is $s_0$, while the initial location of $A(P_1)$ is $s_1$. The final locations of $A(P_0)$ are $s_0^0, s_0^1$, while the final locations of $A(P_1)$ are $s_1^0, s_1^1$.

The transitions in $A(P_0)$ are
1. $\delta'(s_0^0, (a, P_1, [0, \infty))) = s_0^r, \delta'(s_0^0, (b, [0, \infty))) = s_0^r$,
2. $\delta'(s_0^r, (a, P_1, [0, \infty))) = s_0^r, \delta'(s_0^r, (b, [0, \infty))) = s_0^r$.

Clearly, the language accepted by $A(P_0)$ is
\[(a, P_1, [0, 1] + (b, [0, 1]))^+(a, P_1, [1, 2])^+(b, [0, 1, 2])^+ + (b, [0, 2, \infty]))^+
\]

The transitions in $A(P_1)$ are
1. $\delta'(s_1^1, (a, [0, \infty))) = s_1^r, \delta'(s_1^1, (b, [0, \infty))) = s_1^r$,
2. $\delta'(s_1^r, (a, [1, 2])) = s_1^r, \delta'(s_1^r, (b, [2, \infty))) = s_1^r$,
3. $\delta'(s_1^r, (a, [0, \infty))) = s_1^r, \delta'(s_1^r, (b, [0, 1])) = s_1^r$,
4. $\delta'(s_1^r, (a, [1, 2])) = s_1^r, \delta'(s_1^r, (b, [2, \infty))) = s_1^r$,
5. $\delta'(s_2^r, (a, [0, \infty))) = s_2^r, \delta'(s_2^r, (b, [1, 2])) = s_2^r$,
6. $\delta'(s_2^r, (a, [0, 1])) = s_2^r, \delta'(s_2^r, (b, [2, \infty))) = s_2^r$.

The language accepted by $A(P_1)$ is
\[(a + b, [0, 1])^+ + (a + b, [0, 1])^+[(ba^*h, (1, 2))]^+ + (a + b, [0, 1])^+[(ba^*h, (1, 2))]^+[(b + a, [0, 2, \infty)]^+\]

We obtain the RatMTL formula $\psi_1$ for $A(P_1)$ as
\[\text{Rat}_{[0,1]}(a+b)^* \lor \text{Rat}_{[0,1]}(a+b)^*\text{Rat}_{(1,2]}[ba^*b]^* \lor \text{Rat}_{[0,1]}(a+b)^*\text{Rat}_{(1,2]}[ba^*b]^*\text{Rat}_{[2,\infty]}(a+b)^*\]

while the formula $\psi_0$ for $A(P_0)$ is
\[\text{Rat}_{[0,1]}((a, P_1) + b)^* \lor \text{Rat}_{[0,1]}((a, P_1) + b)^*\text{Rat}_{(1,2]}(((a, P_1) + b)^*\lor\]

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To obtain the correct formula $\psi_0$, we replace the symbols $(a, P_1)$ with $a \land \psi_1$, giving the formula

$$\text{Rat}_{[0,1]}((a, P_1) + b)^* \text{Rat}_{[1,2]}((a, P_1) + b)^* \text{Rat}_{[2,\infty]}((a, P_1) + b)^*$$

Note that wherever we encounter $b$, we strike it out, the subformula $\text{Rat}_{[1,2]}[ba^*b]^*$ of $\psi_1$ ensures that if there are non-zero $b$'s at distance (1,2) from $a$, it will be even.

### C.1 Example Illustrating Section 4.3

Let $\psi_3 = \text{Rat}_{[0,1]}((a + \text{Rat}_{[1,1]}[b^+])^*)$ over $\Sigma = \{a, b, c\}$. We rewrite $\psi_3$ as $\psi_3 = \text{Rat}_{[0,1]}((a + \psi_1)^*)$, and $\psi_1 = \text{Rat}_{[1,1]}[a^+]$. We have 3 formulae here $\psi_1, \psi_2, \psi_3$, and hence 3 witness variables $Z_1, Z_2$ and $Z_3$. Let $Z_i$ be any subset of $Z = \{Z_1, Z_2, Z_3\}$ containing $Z_i$ for $i = 1, 2, 3$. Let $\Gamma = 2^Z \otimes Z \times Z$. Since we only have single letters from $\Sigma$ true at any point in the formula $\psi_3$, we define for $a \in \Sigma, \Gamma \subseteq a \times Z$. For each $i = 1, 2, 3$, let $\Gamma_i \subseteq \Sigma \times Z_i$. Let $\Sigma_a$ denote any subset of $\Sigma$ containing $a \in \Sigma$.

1. The base case applies to $\psi_1$ and one can construct a 1-clock ATA $A_{Z_1}$ equivalent to $\psi_1$, and use $Z_1$ as a witness variable for $\psi_1$, wherever it appears. The DFA $D_1$ accepting $\Sigma_1 = b^+$ is as follows. $\delta_1(q_0, \Sigma_a) = q_1$, $\delta_1(q_1, \Sigma_b) = q_1$, and $q_1$ is a final location of $D_1$, and $q_0$ is its initial location.

The automaton $A_{Z_1}$ has locations $(\{\text{init}, \text{check}, q_0, q_1, q_f\})$ with $q_f$ as its final location and $q_{\text{init}}$ as its initial location. The transitions $\delta_{Z_1}$ are as follows.

a. $\delta_{Z_1}(\text{init}, \alpha) = x.\text{check}$, for $\alpha \in \Sigma_a \cup \Sigma_b \cup \Sigma_c$,

b. $\delta_{Z_1}(\text{check}, \Sigma_b) = ((x = 1) \land q_1) \lor \text{check}$,

c. $\delta_{Z_1}(\text{check}, \alpha) = x < 1 \lor \text{check}$, for $\alpha \in \Sigma_a \cup \Sigma_b \cup \Sigma_c$,

d. $\delta_{Z_1}(q_1, \Sigma_b) = (x = 1 \land q_1) \lor (x > 1 \land q_f)$

e. $\delta_{Z_1}(q_f, \alpha) = q_f$, for $\alpha \in \Sigma_a \cup \Sigma_b \cup \Sigma_c$.

2. We can rewrite $\psi_2$ as $\text{Rat}_{[0,1]}((\Gamma_a + \Gamma_1)^*)$. This makes $\psi_2$ a formula of depth one over the extended alphabet $\Gamma$. The DFA $D_2$ accepting $(\Gamma_a + \Gamma_1)^*$ is as follows. There are two locations, $s_0$ the initial as well as accepting location, and $s_1$, a dead location. $\delta_2(s_0, \alpha) = s_0$ for $\alpha \in \{\Gamma_a, \Gamma_1\}$, $\delta_2(s_0, \alpha) = s_1$ for $\alpha \notin \{\Gamma_a, \Gamma_1\}$, and finally $\delta_2(s_1, \alpha) = s_1$ for all $\alpha \in \Gamma$. The automaton $A_{Z_2}$ has locations $(s_0, s_1, s_{\text{init}}, s_{\text{check}}, s_f)$ with $s_f$ as its final location and $s_{\text{init}}$ as its initial location. The transitions $\delta_{Z_2}$ are as follows.

a. $\delta_{Z_2}(s_{\text{init}}, \alpha) = x.s_{\text{check}}$ for all $\alpha \in \Gamma$,

b. $\delta_{Z_2}(s_{\text{check}}, \alpha) = [x \geq 1 \land s_f \land x.\delta_{Z_1}(\text{init}, \alpha)]$, if $\alpha \in \Gamma_1$,

c. $\delta_{Z_2}(s_{\text{check}}, \alpha) = [x \geq 1 \land s_f \land x.\delta_{Z_1}(\text{init}, \alpha)]$, if $\alpha \in \Gamma_a, \alpha \notin \Gamma_1$,

d. $\delta_{Z_2}(s_f, \alpha) = s_f \land x.\delta_{Z_1}(\text{init}, \alpha)$ if $\alpha \in \Gamma_1$,

e. $\delta_{Z_2}(s_f, \alpha) = s_f \land x.\delta_{Z_1}(\text{init}, \alpha)$ if $\alpha \notin \Gamma_1$.

Note that wherever we encounter $\Gamma_1$, we parallelly start checking $A_{Z_1}$, and at places where we do not have $\Gamma_1$, we start checking $A_{Z_1}$.

3. Replacing $\psi_2$, we obtain $\psi_3 = \text{Rat}_{[0,1]}(\Gamma_d \Gamma_2)$ as a modal depth 1 formula over the alphabet $\Gamma$. This results in a modal depth 1 formula over the extended alphabet consisting of witness variables. We now obtain the 1-clock lfr ATA for $\psi_3 = \text{Rat}_{[0,1]}(\Gamma_d \Gamma_2)$. The DFA $D_3$ accepting $\Gamma_d \Gamma_2$ has locations $r_0, r_1, r_2, r_3$ where $r_0$ is initial, $r_2$ is accepting and has transitions $\delta_3(r_0, \Gamma_d) = r_1, \delta_3(r_1, \Gamma_2) = r_2$ and $\delta_3(r_0, \alpha) = r_3$ for $\alpha \notin \Gamma_d, \beta \notin \Gamma_2$ and $\delta_3(r_2, \alpha) = r_3 = \delta_3(r_3, \alpha)$ for all $\alpha$.

The automaton $A_{Z_3}$ has as locations $\{r_0, r_1, r_2, r_{\text{init}}, r_f, r_{\text{check}}\}$ with $r_{\text{init}}$ as the initial location and $r_f$ as the final location. The transitions $\delta_{Z_3}$ are as follows.

a. $\delta_{Z_3}(r_{\text{init}}, \alpha) = x.r_{\text{check}}$, for $\alpha \in \Gamma$,
compute the automaton equivalent to Krishna, Khushraj and Paritosh Pandya 23

Using the extra predicates Example 32. 1. Let us apply to the formula $\delta_{3.\forall}$ the DFA for $\delta_{n.\forall}$.

$\delta_{\Gamma}\left[x.\delta_{Z_{1}}\left(s_{\text{init}},\alpha\right)\land x.\delta_{Z_{1}}\left(q_{\text{init}},\alpha\right)\right], \alpha \in \Gamma_2, \alpha \notin \Gamma_1$.

$\delta_{\Gamma}\left[r_{1.\forall} \land \left[x.\delta_{Z_{2}}\left(s_{\text{init}},\alpha\right)\land x.\delta_{Z_{2}}\left(q_{\text{init}},\alpha\right)\right], \alpha \in \Gamma_2, \alpha \notin \Gamma_1$.

$\delta_{\Gamma}\left[r_{2.\forall} \land \left[x.\delta_{Z_{2}}\left(s_{\text{init}},\alpha\right)\land x.\delta_{Z_{2}}\left(q_{\text{init}},\alpha\right)\right], \alpha \in \Gamma_2, \alpha \notin \Gamma_1$.

$\delta_{\Gamma}\left[r_{f.\forall} \land \left[x.\delta_{Z_{2}}\left(s_{\text{init}},\alpha\right)\land x.\delta_{Z_{2}}\left(q_{\text{init}},\alpha\right)\right], \alpha \notin \Gamma_2, \alpha \notin \Gamma_1$.

Note that if at a point, the symbol read is in both $\Gamma_2$ and $\Gamma_1$, then we start $A_{Z_{1}}$ and $A_{Z_{2}}$ in parallel. Likewise, if the symbol read is in $\Gamma_1$ but not in $\Gamma_2$, then we start $A_{Z_{1}}$ and $\neg A_{Z_{2}}$ in parallel.

D Example Illustrating Section 4.5

In this section, we show how to convert a QkMSO formula into a RatMTL formula, as described in Section 4.5. Let us apply to the formula $\psi(t_0)$ in Example 5. We also show here, how to compute the automaton equivalent to $\psi(t_0)$.

Example 32. 1. Using the extra predicates $R_{1.\forall}(t_1), R_{1.\forall}(t_2)$ we obtain $\forall t_1\{[Q_a(t_1) \land R_{1.\forall}(t_1)] \rightarrow \exists t_2\{Q_a(t_2) \land R_{1.\forall}(t_2)\}\}$. Note that we can draw the automaton corresponding to the predicate $R_{1.\forall}(t_1)$ as follows. The free variable $t_0$ is assigned the first position.

![Figure 3](image-url)

Figure 3 $\Gamma = 2^\mathbb{Z}\setminus\emptyset$. The alphabet for the DFA is $\Gamma \times \text{reg} \times \{0, 1\}^2$. reg is the set consisting of 0, (0, 1), 1, (1, 2), 2 and (2, \infty). On the right, the corresponding timed automata obtained by removing elements $\alpha \in \text{reg}$ with the clock constraint $x \in \alpha$. The self loops can have constraints $x \in I$ compatible with $x \in (1, 2)$. 2. Substituting a witness $W$ for $\exists t_2\{Q_a(t_2) \land R_{1.\forall}(t_2)\}$ we obtain $\forall t_1\{[Q_a(t_1) \land R_{1.\forall}(t_1)] \rightarrow W\}$, which is rewritten as $\chi = \neg \exists t_1\{Q_a(t_1) \land R_{1.\forall}(t_1) \land \neg W\}$. The timed automata for $W, \neg W$ can be seen in Figure 4. This is obtained by first constructing the DFA for $W$, and then putting in the time constraint in the same way we dealt with Figure 5.
12. To get the formula directly, first we notice that plugging in the formula for RatMTL:

\[ \text{Consider the } q \text{ that for any timed word } t, \text{ acceptance is possible only when there is a } b \text{ at distance } 1. \]

Figure 4: \( S_i \) stands for any subset of \( \Sigma \) containing \( b \). \( S_{a-W} \) represents any set containing \( a \) and \( \neg W \).

One way to obtain the formula now is to convert this automaton to logic, as done in Theorem 12. To get the formula directly, first we notice that \( \neg S_{a-W} \) is a short hand for \( \neg a \lor (a \land W) \). The RatMTL formula corresponding to the DFA over \( 2^{\Sigma \cup W} \times \text{reg} \) is \( \text{Rat}_{1(2)}[\neg S_{a-W}^*] \) which is same as \( \text{Rat}_{1(2)}[(\neg a \lor (a \land W))^*] \). Now the RatMTL corresponding to the DFA for \( W \) is \( \text{Rat}_{[1,1]}(-b)^*b \).

12. Plugging in the formula for \( W \), we obtain the formula \( \text{Rat}_{(1,2)}[\neg a \lor (a \land \text{Rat}_{[1,1]}(-b)^*b)] \), which can be rewritten as \( \text{Rat}_{(1,2)}[a \rightarrow (a \land \text{Rat}_{[1,1]}(-b)^*b)] \).

### E Example Illustrating Section 4.6

Consider the RatMTL formula \( \varphi = \text{Rat}_{(1,2)}[a \rightarrow \text{Rat}_{(0,1)}b^+] \).

1. As a first step, we write the QkMSO formula for \( \text{Rat}_{(0,1)}b^+ \). This is given by

\[ \zeta_1(t) = \exists t_{\text{first}}[t < t_{\text{first}} < t + 1] \exists t_{\text{last}}[t < t_{\text{last}} < t + 1] \forall t'(t < t' < t + 1) \varphi(\downarrow t, t_{\text{first}}, t_{\text{last}}, t') \]

where \( \varphi(\downarrow t, t_{\text{first}}, t_{\text{last}}, t') \) is given by

\[ (t' = t_{\text{first}}) \lor (t' = t_{\text{last}}) \lor (t_{\text{first}} < t' < t_{\text{last}}) \land \{Q_b(t_{\text{first}}) \land Q_b(t_{\text{last}}) \land \forall u''[t_{\text{first}} < t'' < t_{\text{last}} \rightarrow Q_b(t'')] \} \]

2. Next, we rewrite \( \varphi \) as \( \psi = \text{Rat}_{(1,2)}[a \rightarrow Z_1] \), where \( Z_1 \) is the witness for \( \text{Rat}_{(0,1)}b^+ \).

3. Since \( \psi \) is now a RatMTL formula of modal depth one, we have the QkMSO formula \( \zeta_0(u) = \exists u_1[u + 1 < u_1 < u + 2] \varphi(\downarrow u, u_1) \) equivalent to it.

\[ \varphi(\downarrow u, u_1) = \neg Q_a(u_1) \lor (Q_a(u_1) \land Q_{Z_1}(u_1)) \land \forall u''[u + 1 < u'' < u + 2 \rightarrow (u'' = u_1)] \]

4. It now remains to plug-in \( \zeta_1(u) \) in place of \( Q_{Z_1}(u_1) \) in \( \zeta_0(u) \). Doing gives \( \exists u_1[u + 1 < u_1 < u + 2] \{[\neg Q_a(u_1) \lor (Q_a(u_1) \land \zeta_1(u_1))] \land \forall u''[u + 1 < u'' < u + 2 \rightarrow (u'' = u_1)]\} \).

Note that on plugging in \( \zeta_1(u_1) \), the formula obtained is in QkMSO; further all the bound first order variables \( v \) are respectively ahead of anchors \( u, u_1 \).

### F Proof of Lemma 16

Given a C□D 1-clock ATA \( A \) over \( \Sigma \) with no resets, we can construct a FRat formula, \( \varphi \) such that for any timed word \( \rho, \rho, i = \varphi \) iff \( \rho \), starting from position \( i \) has an accepting run in \( A \). Let \( q_0 \) be the initial location of \( A \). Let us consider the case when \( q_0 \in Q_v \). Since \( A \) has no reset transitions, \( Q = Q_v, \) and a transition looks like \( \delta(q, a) = C_1 \lor \cdots \lor C_w \) where each \( C_i \) is either a...
location \( q' \in Q \) or a clock constraint \( x \in I \). A word \( w \) is accepted in \( \mathcal{A} \) when one of the following is true.

1. Starting from \( q_0 \), there is a run which reaches some final location \( q_f \). Let \( \text{re}_{q_f,a} \) denote the regular expression (untimed since no clock constraint has been checked this far) that leads us from \( q_0 \) to \( q_f \), and assume that we enter \( q_f \) on reading \( a \). Then the resultant set of words \( w'a \) is accepted where \( w' \in L(\text{re}_{q_f,a}) \). Disjuncting over all possibilities of final locations \( q_f \) and symbols \( a \in \Sigma \) we obtain the formula \( \psi_1 = \bigvee_{q_f \in F, a \in \Sigma} \text{FRat}_{(0,\infty)} \text{re}_{q_f,a}(a \land \square \bot) \), where each disjunct captures all regular expressions \( \text{re}_{q_f,a} \) that guarantee acceptance through \( q_f \) when reached on \( a \). The \( \square \bot \) ensures that no further symbols are read, and can be written as \( -\text{FRat}_{(0,\infty)} \Sigma \top \).

2. The second case is when there is a run which reaches some location \( q \) from where, on reading \( a \), we choose the disjunct \( x \in I_a \) in \( \delta(q,a) \), and enter an empty configuration. Let \( \text{re}_{I_a} \) signify the regular expression that collects all words \( w \) that reach some location \( q \) from \( q_0 \), such that on reading \( a \) from \( q \), the clock constraint \( x \in I_a \) is satisfied. Disjuncting over all combinations of intervals and symbols, we obtain the formula \( \psi_1 = \bigvee_{a,I} \text{FRat}_{I_a,a} \). Each disjunct in the formula says that we see an \( a \) in the interval \( I_a \), and the regular expression \( \text{re}_{I_a} \) holds good till that point. Since \( \text{re}_{I_a} \) exhaustively collects all words that can reach some location \( q \) from where \( a \) is read when \( x \in I_a \), the formula \( \psi_1 \) captures all possible ways to accept on enabling a clock constraint. Notice that any suffix can be appended to these set of words, since from the empty configuration, there is no restriction on what can be read.

The remaining case is when \( q_0 \in Q_f \). In this case \( Q = Q_f \). If we negate \( \mathcal{A} \), then we obtain \( q_0 \in Q_f \), and we can apply the case discussed above, obtaining a formula in \( \text{FRatMTL} \) equivalent to the negation of \( \mathcal{A} \). If we negate this formula, we obtain the formula \( \psi_2 \) equivalent to \( \mathcal{A} \). Any transition to an accepting configuration has to pass through one of the two cases above. Thus the formula that we are interested is one of \( \psi_1 \) or \( \psi_2 \) depending on whether \( q_0 \notin Q_f \) or \( q_0 \in Q_f \).

**G Proof of Section 5.3**

We prove for formulae of modal depth 1 first. To give an idea, consider \( \psi = \text{FRat}_{I,\text{re}_0} \psi_0 \), a formula of modal depth 1, and having only one modality (the \( \text{FRat} \) modality). We have a DFA \( D \) that accepts the regular expression \( \text{re}_0 \) over some alphabet \( \Gamma = 2^I \setminus \emptyset \). The one-clock \( \text{ATA} \mathcal{A} \) we construct is such that, on reading the first symbol of a timed word, we reset \( x \) and go to the initial location of the DFA \( D \). \( D \) continues to run until we reach a final location of \( D \). While in a final location of \( D \) (hence we have witnessed \( \text{re}_0 \)), we check if \( x \in I \), and if the symbol read currently satisfies \( \psi_0 \). If so, we accept. Otherwise, we continue running \( D \), looking for this combination. It is easy to see that the \( \mathcal{A} \) described here indeed captures \( \psi \).

**Lemma 33** (\( \text{FRatMTL} \) modal depth 1 to \( \text{C} \oplus \text{D}-1-\text{ATA-lfr} \)). Given a \( \text{FRatMTL} \) formula \( \psi \) of modal depth 1, one can construct a \( \text{C} \oplus \text{D}-1-\text{ATA-lfr} \) \( \mathcal{A} \) such that for any timed word \( \rho = (a_1, \tau_1) (a_2, \tau_2) \ldots (a_n, \tau_n) \), \( \rho, i \models \psi \) iff \( \mathcal{A} \) accepts \( (a_i, \tau_i) \ldots (a_n, \tau_n) \).

**Proof.** Consider a formula \( \psi_1 = \text{FRat}_{I,\text{re}_0} \psi_0 \) of modal depth 1, and having only one modality. Clearly, \( \text{re}_0 \) is an atomic regular expression over some alphabet \( \Gamma \) and \( \psi_0 \) is a propositional logic formula over \( \Gamma \). Let \( D = (\Sigma, Q, q_0, Q_f, \delta) \) be a DFA such that \( L(D) = L(\text{re}_0) \), and \( \Sigma = 2^I \setminus \emptyset \). Given \( D \), we now construct the 1-clock \( \text{ATA} \mathcal{A} = (\Sigma, Q \cup \{ q_{\text{init}}, q_f \}, q_{\text{init}}, \{ q_f \}, \delta) \) where \( q_{\text{init}}, q_f \) are respectively the initial and final locations of \( \mathcal{A} \), and are disjoint from \( Q \). The transitions are as follows.

1. \( \delta(q_{\text{init}}, \alpha) = x \cdot q_0 \), for all \( \alpha \in \Sigma \),
2. \( \delta(q, \alpha) = \delta'(q, \alpha) \), for all \( q \in Q \setminus Q_f \),
Thus, we have proved the claim for $\Gamma = 2$. It is easy to see that the combination of (i) reaching an accepting location of $A$ and (ii) the time stamp of the next symbol read is in interval $I$. If so, we reset $x$ and enter the accepting location $q_f$ of $A$. Once in $q_f$, we always stay in $q_f$. If the time stamp of $\alpha$ is not in $I$, then we continue running $D$, until we reach again an accepting location of $D$. The transitions of $D$ are used until we reach the combination of (i) reaching an accepting location of $D$ along with (ii) the time stamp of the next symbol read is in interval $I$. At this point, we let go of $D$ and accept, by entering $q_f$. It is easy to see that $A$ is a $C \oplus D$-1-ATA-lfr. Now, we explain how to handle a boolean combination of FRat formulae of modal depth 1 (here, the number of modalities are $> 1$ though the depth is 1).

1. It is easy to see that $C \oplus D$-1-ATA-lfr are closed under complement. On complementation, the locations $Q_{\lor}$ and $Q_{\land}$ are interchanged, and so are final and non-final locations. The argument for correctness of complementation follows as in the general case of 1-clock ATA. This takes care of formulae of the form $\neg \psi$ when $\psi \in$ FRat.

2. Consider the case when we start with a formula $\psi_1 \land \psi_2$ with $\psi_1, \psi_2$ being formulae of modal depth 1 and having only one modality. The above construction gives us a $C \oplus D$-1-ATA-lfr $A_1 = (\Gamma, Q_1, q_0, q_{11}, \delta_1), A_2 = (\Gamma, Q_1, q_0, q_{12}, \delta_1)$, that are equivalent to $\psi_1, \psi_2$ respectively. We construct the 1-clock ATA $A$ with locations $Q_1 \cup Q_2 \cup \{q_{init}, \bot\}$, having as initial location $q_{init}$ disjoint from $Q_1 \cup Q_2$, and having transitions $\delta(q_{init}, a) = (x, \delta_1(q_{01}, a) \land x, \delta_2(q_{02}, a)) \lor \bot$, $\delta(\bot, a) = \bot$ where $\bot$ is a rejecting location. Clearly, when we start in $A$, we move on to the locations as dictated by $A_1, A_2$ respectively. The remaining transitions of $A$ are obtained from $\delta_1, \delta_2$. Since the initial transition respects the $C \oplus D$ condition, and since $A_1, A_2$ are $C \oplus D$-1-ATA-lfr, we see that $A$ is also a $C \oplus D$-1-ATA-lfr. Acceptance is possible in $A$ only when $A_1, A_2$ simultaneously accept. Thus $L(A) = L(A_1) \cap L(A_2)$.

3. The case of $\psi_1 \lor \psi_2$ with $\psi_1, \psi_2$ being formulae of modal depth 1 and having only one modality follows from the fact that we handle complementation and conjunction. Thus, we have proved the claim for FRatMTL formula $\psi$ of modal depth 1. 

**Lifting to formulae of higher modal depth**

We will induct on the modal depth of the formulae. For the base case, we have the result thanks to Lemma 33.

Let us assume the result for formulae of modal depth $\leq k$. Consider a formula of modal depth $k + 1$ of the form $\psi_{k+1} = FRat_{lfr_k} (\psi_k)$ where $lfr_k$ is a regular expression over formulae of modal depth $\leq k$ and $\psi_k$ is a formula of modal depth $\leq k$. For each such occurrence of a smaller depth formula $\psi_k$, let us allocate a witness variable $Z_k$. Let $Z = \{Z_1, \ldots, Z_k\}$ be the set of all witness variables. Let $\Gamma = 2 \sqrt{\Sigma} \setminus \emptyset$. Given a subset $S \subseteq \Sigma$ let $\Gamma_S \in 2 \times 2^Z$. Any occurrence of an element $S \in 2^Z$ in $lfr_k$, $\psi_k$ is replaced with $\Gamma_S$. At the end of this replacement, $lfr_k$ is a regular expression over $\Gamma \times 2^Z$ and $\psi_k$ is a propositional logic formula over $\Gamma \times 2^Z$.

Since each $Z_i$ is a witness for a smaller depth formula $\psi_i$, by inductive hypothesis, there is a $C \oplus D$-1-ATA-lfr $A_{Z_i}$ that is equivalent to $\psi_i$. Let $\delta_{Z_i}$ be the transition function of $A_{Z_i}$, and let $init_{Z_i}$ be the initial location of $A_{Z_i}$. We also construct the complement of each such automata $A_{\sim Z_i}$ which has as its transition function $\delta_{\sim Z_i}$ and $init_{\sim Z_i}$ as its initial location.

Lemma 33 gives us $C \oplus D$-1-ATA-lfr (call it $C$) over the alphabet $\Gamma \times 2^Z$. Let $\delta_C$ denote the transition function of $C$ and let $S_C$ be the set of locations of $C$. Consider a transition $\delta_C(s, \alpha)$ in $C$. If $s \in S \times 2^Z \subseteq \Gamma_S$, for some $S \in 2^Z$, then the transition $\delta_C(s, \alpha)$ is replaced with $\delta(s, S) = \bigvee_{T \subseteq \Gamma_S} \delta_C(s, \alpha) \land \bigwedge_{(u, Z \in T)} [x. init_{Z}] \land \bigwedge_{(u, Z \in T)} [x. init_{\sim Z}]$. Note that $C$ as well as $A_{Z_i}$ and $A_{\sim Z_i}$ are $C \oplus D$-1-ATA-lfr. To see why the $C \oplus D$ condition is respected, let $C$ be $C \oplus D$ with $S_C$ partitioned into $S_\lor$ and $S_\land$. If $s \in S_\lor$, then $\delta_C(s, \alpha)$ has the form $C_1 \lor \cdots \lor C_m$, and
conjoining the reset locations still preserves the form, since these reset locations can be pulled into each $C_i$. In case of $s \in S_\lambda$ the above procedure does not seem to preserve the conjunctive property of the island. Note that the 1-clock ATA $C$ is a conjunctive island. In this case, take the negation of $C$, call it $C'$ resulting in a 1-clock ATA which is disjunctive. We then eliminate witnesses using reset transitions as shown above on $C'$, obtaining an automaton over $\Gamma$. This automaton is then again complemented to get an automaton equivalent to $C$.

In both cases, let us call the resultant 1-clock ATA $B'$ over $\Gamma$. Clearly, if $\alpha \in S \times T$ is read in $C$, such that $T = \{Z_i, Z_{i_1}, \ldots, Z_{i_k}\}$, then acceptance in $B'$ is possible iff $C, A_{Z_i}, A_{Z_{i_1}}, \ldots, A_{Z_{i_k}}$ and $A_{-Z_j}$ for $j \neq i, i_1, \ldots, i_k$ all reach accepting locations on reading the remaining suffix.

**H The case of higher depth formulae in Section 5.5**

Consider a formula $\psi(t_0)$ of metric depth $k + 1$. $\psi(t_0) = Q_1 t_1 \varphi(\downarrow t_0, t_1)$, such that the metric depth of $\varphi(\downarrow t_0, t_1)$ is at most $k$. We can replace every time constraint sub-formula $\psi_i(t_k)$ occurring in it by a witness monadic predicate $w_i(t_k)$. This gives a metric depth 1 formula and we can obtain a FRatMTL formula, say $\zeta$, over variables $\Sigma \cup \{w_i\}$ exactly as in the base step. Notice that each $\psi_i(t_k)$ was a formula of modal depth $k$ or less. Hence by induction hypothesis we have an equivalent FRatMTL formula $\zeta_i$. Substituting $\zeta_i$ for $w_i$ in $\zeta$ gives us a formula language equivalent to $\psi(t_0)$. Since plugging in an FRatMTL formula inside another FRatMTL formula results in FRatMTL, we obtain the result.

For the case when we start with a Q2FO formula of higher depth $k$, we obtain by inductive hypothesis, FSfrMTL formulae corresponding to each $\psi_i$; secondly, corresponding to the FO formula obtained over the extended alphabet containing $w_i$, we obtain a FSfrMTL formula using the base case. Plugging in a FSfrMTL formula in place of the $w_i$ in an FSfrMTL formula will continue to give a FSfrMTL formula.

**I FRatMTL to forward Q2MSO**

Consider a formula $\varphi = \text{FRat}_{t, \text{re} \psi}$ of modal depth 1. Hence, $\psi$ as well as re are atomic. Let $\phi_1$ be an MSO sentence equivalent to re. The formula $\zeta(t) = \exists t' \in t + I(\phi_1' \wedge Q_\varphi(t'))$ where $\phi_1'$ is same as $\phi_1$ except that all quantified first order variables $t''$ in $\phi_1'$ lie strictly between $t, t'$ (by semantics of FRat, the regular expression is asserted strictly in between), and $Q_\varphi$ is obtained by replacing all occurrences of $a \in \Sigma$ in $\psi$ with $Q_a$ (if $\psi = a \wedge \sim b$, then $Q_\varphi(t') = Q_a(t') \wedge \sim Q_b(t')$). It can be seen that $\zeta(t)$ is forward, Q2MSO. If we induct on the modal depth, and proceed exactly as in section 1.6, we obtain a forward, Q2MSO formula equivalent to $\varphi$. Note that the formulae we obtain at each level of FRat will be forward, Q2MSO, and plugging in retains this structure.

If we start with a FSfrMTL formula, then $\phi_1$ will be a FO sentence, and hence $\zeta(t)$ will be forward Q2FO. The inductive hypothesis will continue to yield Q2FO formulae $\zeta(u_1)$ equivalent to $Q_{Z_i}(u_1)$; plugging $\zeta(u_1)$ in place of $Q_{Z_i}(u_1)$ in the bigger Q2FO formula will hence give rise to a Q2FO formula.

**J Proofs from Section 6**

**J.1 Proof of Lemma 21**

**Proof.** We prove this using contradiction. Assume that there are two distinct solutions $\alpha$ and $\beta$ for the equation $Z \equiv \varphi(Z)$ with respect to some $\rho$. Thus $\alpha$ and $\beta$ will agree on the truth value of all other propositions except $Z$. Without loss of generality, let $i$ be the last point in the domain of $\alpha$ where $\alpha$ and $\beta$ disagree on the truth value of $Z$. Without loss of generality, we assume $\alpha, i \models Z$ while $\beta, i \not\models Z$. As both $\alpha$ and $\beta$ are fix point solutions of $Z = \varphi(Z)$, $\alpha, i \models \varphi(Z)$ and $\beta, i \models \neg \varphi(Z)$. Note that as the formulae are guarded, the $Z$ in $\varphi$ will only occur within the
scope of a Rat or FRat modality. Both the modalities reason about strict future. That is, the truth value of these modalities depend only on the truth values of propositions at points which are in strict future. Thus the disagreement of $\varphi(Z)$ at point $i$ should imply that the future of $\alpha$ and $\beta$ from the point $i$ is not the same. That is $\alpha[i + 1...] \neq \beta[i + 1......]$. This is a contradiction as we assumed that the last point where $\alpha$ and $\beta$ disagree is $i$. □

### J.2 Equivalence of $\mu$.RatMTL and System of RatMTL Equations [3], [4]

We start with an example. Consider the formula $\mu Z_1 \circ (Rat_{(1,2)}(a.Z_1.(\nu Z_2 \circ (Rat_{(3,3)}(a.Z_2.Z_1.b))))).b)$. This could be written as $Z_1 \equiv_a Rat_{(1,2)}(a.Z_1.Z_2.b); Z_2 \equiv_\nu Rat_{(3,3)}(a.Z_2.Z_1.b)$. Thus for any $\mu$ temporal logic formulae $\varphi$, the equivalent system of equations contains as many equations as there are fix point operators in $\varphi$. The simple algorithm of conversion to a system of equations for a given formula $\psi$ of the form $\sigma Z_i \circ (\psi(Z_1, \ldots, Z_i))$ will be to reduce it to the equation $Z_i \equiv' \psi'(Z_1, \ldots, Z_m)$ where $\psi'$ is obtained from $\psi$ by replacing all its subformulae of the form $\sigma Z_j \circ (\psi(Z_1, \ldots, Z_j))$ with $Z_j$. The set of models accepted by the starting formulae reduced in this way is therefore the solution of $Z_1 \equiv \psi_1$ (that is the solution to the outer most fix point operator).

Similarly, one can also show that any system of such equations can be reduced to the $\mu$ temporal logic formulae. For example, consider the system of equations $Z_1 \equiv_\mu Rat_{(1,2)}(Z_1.Z_2.b^*); Z_2 \equiv_\nu Rat_{(3,3)}(Z_2.Z_1.a^*)$. The solution of the first equation is thus equivalent to $\mu Z_1 \circ [Rat_{(1,2)}(Z_1.(\nu Z_2 \circ (Rat_{(3,3)}(Z_2.Z_1.a^*))).b^*)]$. There is an equivalent formalism with a slightly different syntax in the literature for fixpoints called vectorial fixpoints. The system of equations can also be thought of as a vector of fix point variables simultaneously recursing over the models. The reduction from the system of equations (or vectorial fixpoints) to formulae is possible due to Bekić Identity [3]. The blow up is at most exponential.