The SU($N$) Wilson Loop Average in 2 Dimensions

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Abstract

We solve explicitly a closed, linear loop equation for the SU(2) Wilson loop average on a two-dimensional plane and generalize the solution to the case of the SU($N$) Wilson loop average with an arbitrary closed contour. Furthermore, the flat space solution is generalized to any two-dimensional manifold for the SU(2) Wilson loop average and to any two-dimensional manifold of genus 0 for the SU($N$) Wilson loop average. The SU($N$) Wilson loop average follows an area law $W(C) = \sum_r P'_r e^{-\sum_i r_i^2 S_i}$ where $j^2 r_i$ is the quadratic Casimir operator for the window with area $S_i$. Only certain combinations of the Casimir operators are allowed in the sum over $i$. We give a physical interpretation of the constants $P'_r$ in the case of a non-self-intersecting composed path $C$ and of the constraints determining in which combinations the Casimir operators occur.

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1 Introduction

In a recent paper \cite{1} it was shown that the loop equation for the SU(2) Wilson loop average (WLA)

\[ W(C_1, \ldots, C_k) := \langle \prod_{i=1}^{k} \frac{1}{N} \text{Tr} P e^{\oint_{C_i} dx_\mu A_\mu} \rangle, \]  

where \( N = 2 \) and \( P \) stands for path ordering along a closed contour \( C_i \), is linear and closes for WLAs with the same number of traces \( k \). This raises hopes that the solution can be obtained. In earlier papers it was proved that it is possible to solve the U(\( N \)) (or SU(\( N \))) loop equation in two dimensions by iteration \cite{2} and in the large \( N \) limit the form of the solution was obtained showing modified area law behavior with area dependent polynomials multiplying the exponentials \cite{3}. The WLAs in two dimensions can also be calculated using an algorithm based on a non-Abelian version of Stokes theorem \cite{4} or by expanding a lattice theory in the characters and taking the continuum limit \cite{5, 6, 7}. During the preparation of this work we received a preprint \cite{8} which gives a general formula for the SU(\( N \)) WLA on an arbitrary two dimensional manifold in terms of maps of an open string world sheet onto the target space. In this paper, we solve the closed, linear SU(2) loop equation in two dimensional Euclidean space and generalize the solution to the case of the SU(\( N \)) (or U(\( N \))) WLA on a two-dimensional manifold of genus 0 and the SU(2) (or U(2)) WLA on any two-dimensional manifold. We also give a physical interpretation of the constants in the solution for a WLA with a non-self-intersecting composed contour. Because the WLA in the large distance limit follows the area law, confinement takes place \cite{9}. On the other hand if the WLA would follow the perimeter law as in four-dimensional QED the interaction would decrease sufficiently rapidly at large distances that there would be no confinement. Because confinement takes place all the physical information in two dimensional QCD can be derived from the WLAs by proper integration over the loop space.

This paper is organized as follows. In section 2, we briefly review the Migdal-Makeenko loop equation in any dimension and give the definitions of the path and area derivative. In section 3, we solve the loop equation for the SU(2) WLA on a two-dimensional plane. In section 4, the solution is generalized to the case of SU(\( N \)) WLA on a two-dimensional plane. In section
5, we discuss the U(N) WLA, consider the large N limit and generalize the
flat space solution to any two-dimensional manifold for the SU(2) (or U(2))
WLA and to any two-dimensional manifold of genus 0 for the SU(N) (or
U(N)) WLA. Finally, we summarize our results and discuss some interestin
g open questions.

2 The loop equations

For U(N) \((S = 0)\) or SU(N) \((S = 1)\) the Migdal-Makeenko loop equation is
linear (for finite \(N\)) and reads \[10,11\]

\[\partial_{\mu}(x) \frac{\delta}{\delta \sigma_{\mu}(x)} W(C) = g^2 \oint_{C_{xx}} dy \delta^d(x - y) \left[ NW(C_{xy}, C_{yx}) - \frac{S}{N} W(C) \right] \] (2)

where \(C_{xy}\) is a path from a point \(x\) to a point \(y\), \(C = C_{xy} C_{yx}\) is a closed path
and \(g\) is the coupling constant. An example of the paths \(C_{xy}, C_{yx}\) and \(C\) with
\(x = y\) is shown in Fig. 1. The path derivative \(\partial_{\mu}(x)\) is defined as follows

\[\lim_{z \to y}(z_{\mu} - y_{\mu}) \partial_{\mu}(y) F(C_{xy}) := \lim_{z \to y}[F(C_{xy} C_{yz}) - F(C_{xy})] \] (3)

for an open path \(C_{xy}\) and

\[\lim_{z \to y}(z_{\mu} - y_{\mu}) \partial_{\mu}(y) F(C_{yy}) := \lim_{z \to y}[F(C_{zy} C_{yy} C_{yz}) - F(C_{yy})] \] (4)

for a closed path \(C_{yy}\), where \(F\) is an arbitrary functional of its argument.
The area derivative \(\frac{\delta}{\delta \sigma_{\mu}(x)}\) is defined as

\[\Delta \sigma_{\mu\nu} \frac{\delta}{\delta \sigma_{\mu}(x)} F(C) := F(C \sigma_{\nu}) - F(C) \] (5)

where \(\Delta \sigma_{\mu\nu}\) is the oriented area of an infinitesimal loop \(C_{\sigma}\).

For large \(N\) the double trace WLA factorizes as

\[W(C_{xy}, C_{yx}) = W(C_{xy}) W(C_{yx}) + O\left(\frac{1}{N^2}\right)\] (6)

which yields a closed but non-linear loop equation \[10\].

For the SU(2) gauge field \(A_{\mu}\) \[12\]

\[Tr Pe^{\int_{C_{xy}} dx_{\mu} A_{\mu}} Tr Pe^{\int_{C_{yx}} dx_{\mu} A_{\mu}} = Tr Pe^{\int_{C} dx_{\mu} A_{\mu}} + Tr Pe^{\int_{C_{xy} C_{yx}} dx_{\mu} A_{\mu}} \] (7)
where the path $C_{yx}^{-1}$ is the path $C_{yx}$ travelled in the opposite direction. Thus we obtain the loop equation

$$\partial_\mu(x) \frac{\delta}{\delta \sigma_{\mu\nu}(x)} W(C) = g^2 \oint_{C_{xx}} dy_\nu \delta^d(x - y) \left[ \frac{1}{2} W(C) + W(C_{xy}C_{yx}^{-1}) \right]$$

which is both closed and linear.

### 3 The SU(2) Wilson loop average

In two dimensions the equation for SU(2) WLA can be written in the following form at a point of a self-intersection (at other points the right hand side vanishes) by integrating along a short path from $x - \Delta$ to $x + \Delta$ intersecting the loop at the self-intersection point $x = y$.

$$\frac{\delta W(C)}{\delta \sigma_{\mu\nu}(x + \Delta)} + \frac{\delta W(C)}{\delta \sigma_{\mu\nu}(x - \Delta)} = g^2 \left[ \frac{1}{2} W(C) + W(C_{xy}C_{yx}^{-1}) \right]$$

which can be further simplified to read

$$\frac{D}{DS_x} W_v(C) := \left( \frac{\delta}{\delta S_a} + \frac{\delta}{\delta S_b} - \frac{\delta}{\delta S_c} - \frac{\delta}{\delta S_d} - \frac{1}{2} \right) W_v(C) = W_{v-1}(C_{xy}C_{yx}^{-1})$$

where the areas of the windows $S$ of the loop touching the point of self-crossing are measured in units of $g^2$. The equation tells us the relation between the WLA $W_v(C)$ with $v$ self-intersections and the WLA $W_{v-1}(C_{xy}C_{yx}^{-1})$ with $v-1$ self-intersections where the self-intersection touching the areas $S_a$, $S_b$, $S_c$ and $S_d$ has been broken so that the areas corresponding to the operators multiplied by a minus sign are connected (Fig. 2).

By repeated use of the equation we can relate the WLA $W_v$ to $v$ WLAs $W_{v-1}$ by breaking all the self-crossings and each of the WLAs $W_{v-1}$ to $v-1$ WLAs $W_{v-2}$ etc. Thus we obtain a tree of WLAs related to each other by partial differential equations with $v!$ WLAs at the top of the tree.

In two dimensions the WLA $W(C)$ depends on the loop $C$ only through the areas of the windows formed by the loop i.e.

$$W(C) = W(S_a, S_b, ...)$$
which can be proved using diagram techniques (cf. 13) or by lattice theory 2, 3. From the definition of the WLA (1) we obtain the boundary conditions
\[ W(S_a, S_b, ...) \bigg|_{S_a=0, S_b=0} = 1. \] (12)
In addition, it can be seen (for example by an explicit calculation in the axial gauge \( A_1 = 0 \)) that the WLA does not depend on the infinite area \( S_0 \) external to the loop. Thus
\[ \frac{\delta W}{\delta S_0} = 0. \] (13)
Now we shall prove by induction that the WLA has the following form
\[ W_v(S_1, S_2, ...) = \sum_r P'_r e^{- \sum_i j_{ri}(j_{ri}+1)S_i} \] (14)
where \( P'_r \)s and \( j_{ri}(j_{ri}+1) \)s are constants. (The suggestive form of the latter will become clear shortly.) In the axial gauge \( A_1 = 0 \) the propagator
\[ G_{\mu\nu}(x - y) \propto |x_1 - y_1| \delta(x_2 - y_2) \] (15)
and because \([A_2(x), A_2(x)]\equiv 0\) there are no gluon self-interactions in two dimensions. Thus, for the non-nested WLA with one self-intersection shown in Fig. 1,
\[ W_1(S_a, S_b) = W_0(S_a + S_b). \] (16)
The loop equation (10) yields
\[ \left( -\frac{\delta}{\delta S_a} - \frac{\delta}{\delta S_b} - \frac{1}{2} \right) W_1(S_a, S_b) = W_0(S_a + S_b) \] (17)
which can be solved
\[ W_0(S) = e^{-\frac{3}{4}S}. \] (18)
Thus \( W_0(S) \) satisfies the assumption (14). Now we shall assume that \( W_{v-1} \) also satisfies the equation (14). The special solution \( W^*_v \) of the loop equation (10) is now
\[ W^*_v = \left( \frac{D}{DS} \right)^{-1} W_{v-1} = \frac{1}{C} W_{v-1} \] (19)
with \( C \neq 0 \) because vanishing of the constant \( C \) would imply that \( W_{v-1} \equiv 0 \) which is not possible according to the boundary condition (12). The special solution \( W^s \) has the right form and so does the homogeneous solution obtained by separating the variables. Thus we have proved equation (14).

Applying the operator \( \frac{\delta}{\delta S_c} - \frac{\delta}{\delta S_d} \) to equation (10) makes the right hand side vanish because \( W_{v-1} = W_{v-1}(S_a, \ldots, S_c+S_d) \). By operating on the tree of a WLA \( W_v \) with these operators it can be seen that \( j_{ri} \in \{0, \frac{1}{2}, 1, \ldots, Q\} \) subject to the condition that \( j_{ri} = j_{ri'} \pm \frac{1}{2} \) for neighbouring windows \( S \) and \( S' \). \( Q \) is the minimum number of times the contour \( C \) has to be crossed in order to reach the external area \( S_0 \) from the window \( S_i \). Thus the exponent is proportional to the time integral of the one-dimensional string potential between the spatially separated quark lines [5, 13].

\[
V(t) \propto g^2 |j(j+1)||x(t)|. \tag{20}
\]

The spin \( j \) is obtained by the following angular momentum addition rules. The SU(2) quarks transform in the fundamental spin \( s = \frac{1}{2} \) representation of the gauge group. Thus \( j = s = \frac{1}{2} \) for the areas \( S \) surrounded by a single quark line, \( j = |j-s|, \ldots, j+s = 0, 1 \) for the areas \( S' \) surrounded by two quark lines etc. Note that the limit \( \lim_{S \to \infty} W \neq 0 \) for windows \( S \) surrounded by pairs of quark lines, as expected. A quark and an adjacent antiquark, represented by a pair of antiparallel lines, can form a colorless \((j = 0)\) state i.e. a meson. Similarly, in the case of the gauge group SU(2) two adjacent quarks, represented by a pair of parallel lines, can form a colorless state i.e. a baryon.

The constants \( P'_r \) can be solved from the boundary conditions (12) for the WLAs in the tree of \( W_v \) but this approach would require us to draw the tree with of order \( v! \) WLAs for every WLA \( W_v \) we want to solve. Instead we can relate each term in the sum of equation (14) to the term with the lowest possible exponents of a nested WLA \( W^{\text{nested}} \) (Fig. 3) by breaking and adding self-intersections. Next, we calculate the constant \( P^{\text{nested}}_{\frac{1}{2}, \ldots, \frac{1}{2}} \) multiplying the exponential with the most negative exponent of a nested WLA. We know that

\[
W^{\text{nested}}(S_1, S_2, \ldots, S_{v+1}) = \sum_{j_0=0, \frac{1}{2}, \ldots, \frac{Q}{2}, \ldots} P^{\text{nested}}_{j_1, j_2, \ldots, j_{v+1}} e^{-\sum_{Q=1}^{v+1} j_Q(j_Q+1)S_Q} \tag{21}
\]

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W^{\text{nested}}(S_1, S_2, \ldots, S_{v+1}) = \sum_{j_0=0, \frac{1}{2}, \ldots, \frac{Q}{2}, \ldots} P^{\text{nested}}_{j_1, j_2, \ldots, j_{v+1}} e^{-\sum_{Q=1}^{v+1} j_Q(j_Q+1)S_Q} \tag{21}
\]
Also
\[ W_{v-1}^{\text{nested}} = \lim_{S_{v+1} \to 0} W_v^{\text{nested}}. \] (22)

If \( j_v = \frac{v}{2} \) then \( j_{v+1} = \frac{v}{2} \pm \frac{1}{2} \) thus
\[ P_{\frac{1}{2} ... \frac{v}{2}}^{\text{nested}} = P_{\frac{1}{2} ... \frac{v+1}{2}}^{\text{nested}} + P_{\frac{1}{2} ... \frac{v}{2}}^{\text{nested}}. \] (23)

On the other hand operating with \( \frac{D}{DS} \) on the innermost vertex of \( W_v^{\text{nested}} \) yields
\[ P_{\frac{1}{2} ... \frac{v}{2}}^{\text{nested}} = -(v + 1) P_{\frac{1}{2} ... \frac{v+1}{2}}^{\text{nested}}. \] (24)

Thus
\[ P_{\frac{1}{2} ... \frac{v}{2}}^{\text{nested}} = \frac{v + 2}{v + 1} P_{\frac{1}{2} ... \frac{v}{2}}^{\text{nested}}. \] (25)

which yields
\[ P_{\frac{1}{2} ... \frac{v}{2}}^{\text{nested}} = \frac{v + 1}{2} P_{\frac{1}{2}}^{\text{nested}} = \frac{v + 1}{2}. \] (26)

Now we can write the explicit solution of the loop equation for the SU(2) WLA in two dimensions
\[
W_v(C) = W_v(S_{11}, \ldots, S_{QI}, \ldots) = \\
\sum_{j_0 = 0; \ldots; j_{QI} = 0; \ldots; j_{Q+2} = \ldots}^{2\text{max}} \prod_{Q,I} B(\ldots, j_{QI}, \ldots) \times \\
\prod_{Q,I} J(\ldots, j_{QI}, \ldots) e^{-\sum_{Q,I} j_Q(j_{QI} + 1)S_{QI}} \] (27)

where \( I \) is a label which distinguishes between different windows with the same value of \( Q \) and the constraint \( j_{Q+1,I'} = j_{QI} \pm \frac{1}{2} \) in the sum applies for neighbouring windows \( S_{QI} \) and \( S_{Q+1,I'} \). We have related a generic term of the WLA to the term of the nested WLA \( W_{2\text{max}}^{\text{nested}}(S_1, \ldots, S_{2\text{max}}) \) with the smallest possible values of \( j \). There is a breaking factor \( B(\ldots, j_{QI}, \ldots) \) for every self-intersection one has to break to get the nested WLA from the original WLA \( W_v(C) \). The total number of factors \( B \) in a term is \( v - (2\text{max}j_{QI} - 1) \).

From equation (28) we see that connecting windows \( S_c \) and \( S_d \) (with \( j_c = j_d \)) touching the same vertex yields a breaking factor
\[
B(j_a, j_b, j_c) = \frac{1}{-j_a(j_a + 1) - j_b(j_b + 1) + 2j_c(j_c + 1) - \frac{1}{2}}. \] (28)
In addition we have joining factors \( J(\ldots, jQ, \ldots) \) for the remaining sets of windows with the same value of \( j \) which cannot be connected by breaking vertices. There are \( w - 1 \) joining factors for a set of \( w \) windows. From equation (10) it follows that connecting windows \( S_c \) and \( S_d \) separated by a neighbouring area \( S_a \) (with \( j_a = j_c \pm \frac{1}{2} \)) yields a joining factor

\[
J(j_a, j_c) = \frac{-2j_c(j_c + 1) + 2j_a(j_a + 1) - \frac{1}{2}}{-2j_a(j_a + 1) + 2j_c(j_c + 1) - \frac{1}{2}}.
\]

(29)

There can be more than one area separating the windows \( S_c \) and \( S_d \) meaning only that there are other areas to be joined first. Note that we do not have to keep track of the orientation of the loop at the point where a vertex is to be broken or added because we can always break and add vertices in a particular order so that the windows with the same value of \( j \) become connected.

To demonstrate the method we will write the answer for the WLA shown in Fig. 4

\[
W(S_1, S_{21}, S_{22}, S_3) = \sum_{j_{21}=0,1; j_{22}=0,1; j_3=\frac{1}{2}} C_{j_{21}j_{22}j_3} \times e^{-\frac{1}{4}S_1-j_{21}(j_{21}+1)S_{21}-j_{22}(j_{22}+1)S_{22}-j_3(j_3+1)S_3}
\]

(30)

where

\[
C_{00}\frac{1}{2} = \frac{2j_3 + 1}{2} B(j_1, j_1, j_2 = j_0) B(j_1, j_1, j_2 = j_2) \times B(j_2, j_2, j_3 = j_1) \bigg|_{j_2 = j_2 = j_0 \equiv 0; j_3 = j_1 \equiv \frac{1}{2}},
\]

\[
C_{01}\frac{1}{2} = \frac{2j_0 + 1}{2} B(j_1, j_1, j_2 = j_0) B(j_2, j_2, j_3 = j_1) \bigg|_{j_2 = j_0 \equiv 0; j_3 = j_1 \equiv 0; j_2 = j_2 = 1},
\]

\[
C_{10}\frac{1}{2} = \frac{2j_2 + 1}{2} B(j_1, j_1, j_3 = j_1) J(j_1, j_2 = j_0) \times B(j_2, j_2, j_1 = j_1) \bigg|_{j_2 = j_0 \equiv 0; j_3 = j_1 \equiv \frac{1}{2}; j_2 = j_2 = 1},
\]

\[
C_{11}\frac{1}{2} = \frac{2j_2 + 1}{2} B(j_1, j_2, j_3 = j_1) B(j_1, j_1, j_2 = j_2) \bigg|_{j_3 = j_1 \equiv \frac{1}{2}; j_2 = j_2 = 1},
\]

\[
C_{10}\frac{1}{2} = \frac{2j_3 + 1}{2} J(j_1, j_2 = j_0) B(j_2, j_2, j_1 = j_1) \bigg|_{j_2 = j_0 \equiv 0; j_3 = j_1 \equiv \frac{1}{2}; j_2 = j_2 = 1; j_3 = j_3 \equiv \frac{1}{2}}.
\]
\[
C_{1\frac{3}{2}} = \frac{2j_3 + 1}{2} B(j_1, j_1, j_{22} = j_{21}) \Bigg|_{j_1 = \frac{1}{2} ; j_{22} = j_{21} = 1 ; j_3 = j_3 = \frac{3}{2}}.
\]

The right orientations in the terms having the joining factor $J$ can be maintained by first adding a new vertex (which yields the numerator of $J$) then breaking the vertex between the windows $S_{21}$ and $S_{22}$ and finally breaking the new vertex (which yields the denominator of $J$). The solution satisfies the boundary condition (12), as it should. Namely

\[
W(0, 0, 0, 0) = \frac{1}{4} - \frac{3}{4} - \frac{1}{4} - \frac{1}{4} + 1 + 1 = 1.
\] (31)

To this point, we have discussed WLAs $W(C)$ with only a single trace ($k = 1$). Equation (11) implies that a general WLA $W(C_1, ..., C_k)$, for which the path is composed of any number of closed contours $C_i$, can be written in terms of the single loop WLAs i.e.

\[
W(C_1, C_2, ..., C_k) = W(C_1, \tilde{C}_2, ..., \tilde{C}_k) = \frac{1}{2^{k-1}} \sum_{p_2, ..., p_k = \pm 1} W(C_1 \tilde{C}_2^{p_2} ... \tilde{C}_k^{p_k})
\] (32)

where the loops $\tilde{C}_i$ are formed from the loops $C_i$ so that the areas of the windows and the self-crossings do not change but a smallest possible number of self-touchings are added to make the composite contour connected (cf. Fig. 5a and 5b). The equality of the original WLA and the WLA with the modified path follows from equation (11). Equation (32) implies that

\[
W(C_1^{p_1}, C_2^{p_2}, ..., C_k^{p_k}) = W(C_1, C_2, ..., C_k)
\] (33)

for any combination of directions $p_i = \pm 1$. Later it will become clear that the reason why the relative orientation of loops makes no difference is that $N - 1 = 1$ for SU(2). Furthermore, it can be seen from equations (11) and (15) that

\[
W(C_1, ..., C_i, C_{i+1}, ..., C_k) = W(C_1, ..., C_i) W(C_{i+1}, ..., C_k)
\] (34)

for two (composite) contours $C_1, ..., C_i$ and $C_{i+1}, ..., C_k$ windows of which do not overlap. The value of $j$ in the exponents for a general WLA follows the same angular momentum addition rules as for the single loop WLA.
4 The SU(N) Wilson loop average

In order to be able to generalize the result to SU(N) we rewrite the answer for a WLA with a non-self-intersecting composed path. The WLA reads

\[
W(S_{11}, ..., S_{QI}, ...) = \sum_{j_0=0; \ldots; j_{QI}=0, \frac{1}{2}, \ldots, \frac{QI-1}{2}} P_{j_{11} \ldots j_{QI}} \cdots e^{-\sum_{Q,I} j_{QI}(j_{QI}+1)S_{QI}}. \tag{35}
\]

The constant \( P_{j_{11} \ldots j_{QI}} \) is the probability of the combination \( j_{11}, \ldots, j_{QI}, \ldots \) for the spins \( j \) and it can be calculated as follows. The multiplicity of a spin \( j \) state is \( D(j) = 2j + 1 \). If the spin for the window \( S_{QI} \) is \( j_{QI} \) then the probability, that the spin \( j_{QI+1,I}' \) for the neighbouring window \( S_{QI+1,I}' \) equals \( j_{QI} \pm \frac{1}{2} \), is

\[
P(j_{QI+1,I}' = j_{QI} \pm \frac{1}{2}) = \frac{D(j_{QI} \pm \frac{1}{2})}{D(j_{QI} - \frac{1}{2}) + D(j_{QI} + \frac{1}{2})} = \frac{D(j_{QI+1,I}')}{2D(j_{QI})}. \tag{36}
\]

Note that the above equation is also valid for \( j_{QI} = 0 \) because \( D(-\frac{1}{2}) = 0 \). Thus we obtain

\[
P_{j_{11} \ldots j_{QI}} = \prod_{j_{QI} \in \{ j' | S' \text{ simply connected} \}} \frac{D(j)}{2^L \prod_{j_{QI} \in \{ j' | S' \text{ not simply connected} \}} [D(j)]^{w_{QI}-1}} \tag{37}
\]

where \( L \) is the number of loops and \( w_{QI} \) is the number of windows \( S_{QI+,I}' \) surrounded by the window \( S_{QI} \).

The solution for a WLA with a non-self-intersecting composed path can be generalized to the case of the gauge group SU(N) and it reads

\[
W(..., S_{QRI}, ...) = \sum_{n_1, n_2, ..., n_{QRI}} P_{n_1, n_2, ..., n_{QRI}} e^{-\sum_{Q,R,I} j^2[n_1, n_2, ..., n_{QRI}]S_{QRI}}. \tag{38}
\]

The quadratic Casimir operator \( j^2[n_1, n_2, ...] \), where \( n_i \) is the number of boxes in the \( i \)th column of the Young tableau for the representation \( [n_1, n_2, ...] \), is given by the well known formula

\[
j^2[n_1, n_2, ...] = \frac{1}{2}[N \sum_i n_i - \sum_i n_i(n_i + 1 - 2i) - \frac{S}{N}(\sum_i n_i)^2] \tag{39}
\]
where $S = 1$ for SU($N$) (and $S = 0$ for U($N$)). For SU(3) (with $n_i = 2$ if $1 \leq i \leq r$ and $n_i = 1$ if $r < i \leq r + q$) the expression for the quadratic Casimir operator simplifies to

$$j^2[n_1, n_2, ...] = \frac{1}{3}(q^2 + qr + r^2) + q + r. \quad (40)$$

Let us connect the areas $S_{Q0}$ and $S_{QRI}$ with a directed open path $L_{QRI}$ from the external infinite area $S_{Q0}$ to the window $S_{QRI}$ so that the contour crosses the path as few times as possible. Then $Q$ ($R$) is the number of times the contour crosses the path from left to right (from right to left). Examples of the paths $L_{QRI}$ and the indices $Q$ and $R$ are shown in Fig. 6. The following constraints apply to the sum over representations. The first constraint on the representation $[n_1, n_2, ...]_{QRI}$ is that it must be one of the elements of the direct sum obtained by expanding $([1] \otimes ([N - 1] \otimes R)^Q = \ldots \oplus [m_1, m_2, ...] \oplus \ldots$. 

$$[n_1, n_2, ...]_{QRI} \in \{[m_1, m_2, ...] | ([1] \otimes ([N - 1] \otimes R)^Q = \ldots \oplus [m_1, m_2, ...] \oplus \ldots \}. \quad (41)$$

In addition, the representations $[n_1, n_2, ...]_{Q+1,R'}$ and $[l_1, l_2, ...]_{QRI}$ for neighbouring windows $S_{Q+1,R'}$ and $S_{QRI}$ are related. $[n_1, n_2, ...]_{Q+1,R'}$ is an element of the direct sum obtained by expanding $[l_1, l_2, ...]_{QRI} \otimes [1]$. i.e.

$$[n_1, n_2, ...]_{Q+1,R'} \in \mathcal{A} := \{[m_1, m_2, ...] | [l_1, l_2, ...]_{QRI} \otimes [1] = \ldots \oplus [m_1, m_2, ...] \oplus \ldots \}. \quad (42)$$

Finally, the representations $[n_1, n_2, ...]_{Q,R+1,I'}$ and $[l_1, l_2, ...]_{QRI}$ for neighbouring windows $S_{Q,R+1,I'}$ and $S_{QRI}$ are related. $[n_1, n_2, ...]_{Q,R+1,I'}$ is an element of the direct sum obtained by expanding $[l_1, l_2, ...]_{QRI} \otimes [N - 1]$. i.e.

$$[n_1, n_2, ...]_{Q,R+1,I'} \in \mathcal{B} := \{[m_1, m_2, ...] | [l_1, l_2, ...]_{QRI} \otimes [N - 1] = \ldots \oplus [m_1, m_2, ...] \oplus \ldots \}. \quad (43)$$

As before a quark and an adjacent antiquark can form a colorless ($[N] = [0]$) state, i.e. a meson (because $[N - 1] \otimes [1] = [N] \oplus \ldots$), but in the case of SU($N$) gauge group $N$ adjacent quarks are needed to form a colorless state, i.e. a baryon, as expected. If the representation for the window $S_{QRI}$ is $[m_1, m_2, ...]_{QRI}$ then the probability, that the representation is $[n_1, n_2, ...]_{Q+1,R'}$ for the neighbouring window $S_{Q+1,R'}$, reads

$$P([n_1, n_2, ...]_{Q+1,R'} | [m_1, m_2, ...]_{QRI}) =$$
because \( D([1]) = N \). The dimension \( D \) of a representation \([n_1, n_2, \ldots] \) is given by Weyl’s formula

\[
D([n_1, n_2, \ldots]) = \prod_{1 \leq i < k \leq N} \frac{r_i - r_k + k - i}{k - i}
\]  

(45)

where \( r_i \) is the number of boxes in the \( i \)th row of the Young tableau. For \( SU(3) \) the expression simplifies to

\[
D([n_1, n_2, \ldots]) = \frac{1}{2} (q + 1)(q + r + 2)(r + 1)
\]  

(46)

Similarly,

\[
\frac{D([n_1, n_2, \ldots]_{Q,R+1})}{\sum_{[l_1, l_2, \ldots]_{Q,R+1} \in A} D([l_1, l_2, \ldots]_{Q,R+1})} = \frac{D([n_1, n_2, \ldots]_{Q,R+1})}{N D([m_1, m_2, \ldots]_{QRI})}
\]

(47)

because \( D([N-1]) = N \). Thus we obtain

\[
P_{[n_1, n_2, \ldots]_{QRI}} = \prod_{[n_1, n_2, \ldots]_{QRI}} \frac{D([n_1, n_2, \ldots]_{QRI})}{D([m_1, m_2, \ldots]_{QRI})}
\]  

(48)

where \( w_{QRI} \) is the number of windows \( S_{Q+1,R} \) and \( S_{Q,R+1} \) surrounded by the window \( S_{QRI} \).

For a WLA with an arbitrary path \( C \) we find

\[
W_v(C) = W_v(..., S_{QRI}, ...) = \sum_{[n_1, n_2, \ldots]_{QRI}, \ldots} P'_{[n_1, n_2, \ldots]_{QRI}, \ldots} e^{-\sum_{Q,R,I} j^2 [n_1, n_2, \ldots]_{QRI} S_{QRI}}
\]  

(49)

with the same constraints for the sum over representations as before. Recall that the constants \( P' \) can be interpreted as probabilities (\( P \)) only when the loop is non-self-intersecting (\( v = 0 \)). The expression for the constants \( P' \) can be easily derived for the case in which \( S_a \equiv S_b \). In this case we can break every vertex using the Migdal-Makeenko loop equations which yield

\[
P'_{[n_1, n_2, \ldots]_{QRI}, \ldots} = P_{[n_1, n_2, \ldots]_{QRI}, \ldots} \prod_{i=1}^{n} N^{r_i} B_{[\ldots, [n_1, n_2, \ldots]_{QRI}, \ldots]}
\]  

(50)
where $\epsilon_i = 1$ ($\epsilon_i = -1$) if the number of loops increases (decreases) when breaking a vertex and the general breaking factor $B_+$ reads ($j_a^2 = j_b^2$)

$$B_+([l_1, l_2, \ldots]_c, [m_1, m_2, \ldots]_d, [n_1, n_2, \ldots]_a) = \frac{1}{-2j^2[n_1, n_2, \ldots]_a + j^2[l_1, l_2, \ldots]_c + j^2[m_1, m_2, \ldots]_d + \frac{\pi}{N}}. \quad (51)$$

Note that the general SU($N$) breaking operation connects the areas $S_a$ and $S_b$ in Fig. 2a and it does not change the orientation of any part of the loop [4]. For a general self-intersecting loop, in addition to breaking vertices, we need to take the limit $S_{QRI} \to 0$ for various areas $S_{QRI}$ to relate a term of the WLA $W$ to a term of a WLA with a non-self-intersecting path. The limit $S_b \to 0$ gives

$$\sum_{[m_1, m_2, \ldots]_b} P'_{[l_1, l_2, \ldots]_a[m_1, m_2, \ldots]_b[n_1, n_2, \ldots]_c \ldots} = P'_{[l_1, l_2, \ldots]_a[n_1, n_2, \ldots]_c \ldots} \quad (52)$$

where the sum is over all the allowed representations for the Casimir operator $j^2[m_1, m_2, \ldots]_b$ when all the other representations are kept fixed. The bullet means that $P'_{[l_1, l_2, \ldots]_a[n_1, n_2, \ldots]_c \ldots}$ is calculated for $\lim S_b \to 0 W$.

To demonstrate how the method works we determine the constant $P'_{\alpha\beta_1\gamma\delta\epsilon\phi\gamma_1}$ for the WLA $W(C) = W(\ldots, S_{60}, S_{701}, S_{702}, S_{703}, S_{704}, S_{80}, S_{90})$ where $\alpha = [3, 2, 1]_{60}$, $\beta_1 = [3, 2, 2]_{701}$, $\gamma = [3, 3, 1]_{702}$, $\delta = [3, 2, 2]_{703}$, $\epsilon = [3, 3, 1]_{704}$, $\phi = [3, 3, 2]_{80}$ and $\varphi_1 = [3, 3, 2, 1]_{90}$. The loop $C$ is shown in Fig. 7a. If we keep the representation $\phi$ fixed the allowed representations for $j^2_{90}$ are $\varphi_1 = [3, 3, 2, 1]$, $\varphi_2 = [3, 3, 3]$ and $\varphi_3 = [4, 3, 2]$. Because the border of the window $S_{90}$ is a non-self-intersecting loop we obtain

$$P'_{\alpha\beta_1\gamma\delta\epsilon\phi\gamma_1} = \frac{D(\varphi_1)}{\sum_{i=1}^{3} D(\varphi_i)} P'_{\alpha\beta_1\gamma\delta\epsilon\phi\phi} \quad (53)$$

If we keep the representations $\alpha$ and $\phi$ fixed the allowed representations for $j^2_{701}$ are $\beta_1 = [3, 2, 2]$ and $\beta_2 = [3, 3, 1]$ which yields

$$P'_{\alpha\beta_1\gamma\delta\epsilon\phi\phi} = P'_{\alpha\phi\gamma\delta\epsilon\phi} - P'_{\alpha\beta_2\gamma\delta\epsilon\phi} \quad (54)$$

Notice that $\gamma$ and $\delta$ fix the value of $\phi$. Thus

$$P'_{\alpha\phi\gamma\delta\epsilon\phi} = P_{\alpha\phi\phi\delta\epsilon\phi} \frac{D(\delta)D(\epsilon)}{N^8D(\alpha)} \quad (55)$$
where the index $\gamma$ has also disappeared because areas $S_{702}$ and $S_{704}$ are joined in the limit $S_{80} \to 0$. Because $\beta_2 = \gamma = \epsilon$, and $\gamma$ and $\delta$ fix the value of $\phi$

$$P'_{\alpha, \beta, \gamma, \delta, \phi} = [B_+ (\gamma, \alpha, \phi)]^2 P_{\alpha, \beta, \gamma, \delta, \phi}.$$  

(56)

We have written the constant $P'_{\alpha, \beta, \gamma, \delta, \phi}$ in terms of the dimensions $D$ of the representations, the breaking factors $B_+$ and the probability $P_{\alpha, \beta, \gamma, \delta, \phi}$ for a WLA with a non-self-intersecting contour shown in Fig. 7b.

Notice that first we have to calculate the probabilities for the non-self-intersecting parts surrounded by a self-intersecting part to be able to contract the non-self-intersecting parts (cf. equation (53) of the example above). Also note that the representation $[n_1, n_2, ...]_{QR}$ is limited to one possibility if the representations $[l_1, l_2, ...]_{Q+1, R_I}$ and $[m_1, m_2, ...]_{Q+1, R_{I'}}$ (or $[l_1, l_2, ...]_{Q, R+1, I}$ and $[m_1, m_2, ...]_{Q, R+1, I'}$), for windows $S_{Q+1, R_I}$ and $S_{Q+1, R_{I'}}$ (or $S_{Q, R+1, I}$ and $S_{Q, R+1, I'}$) adjacent to the window $S_{QR}$, are fixed and not the same. Namely for $[l_1, l_2, ...]_{Q+1, R_I} \neq [m_1, m_2, ...]_{Q+1, R_{I'}}$ (or $[l_1, l_2, ...]_{Q, R+1, I} \neq [m_1, m_2, ...]_{Q, R+1, I'}$)

$$[n_1, n_2, ...]_{QR} = \min(l_1, m_1), \min(l_2, m_2), ...$$  

(57)

if $\sum_i l_i = \sum_i m_i$ and

$$[n_1, n_2, ...]_{QR} = \min(N, m_1), \min(l_1, m_2), \min(l_2, m_3), ...$$  

(58)

if $N + \sum_i l_i = \sum_i m_i$. For $[l_1, l_2, ...]_{Q-1, R_I} \neq [m_1, m_2, ...]_{Q-1, R_{I'}}$ (or $[l_1, l_2, ...]_{Q, R+1, I} \neq [m_1, m_2, ...]_{Q, R+1, I'}$)

$$[n_1, n_2, ...]_{QR} = \max(l_1, m_1), \max(l_2, m_2), ...$$  

(59)

if $\sum_i l_i = \sum_i m_i$ and

$$[n_1, n_2, ...]_{QR} = \max(N, m_1), \max(l_1, m_2), \max(l_2, m_3), ...$$  

(60)

if $N + \sum_i l_i = \sum_i m_i$. There are at most two possible representations $[n_1, n_2, ...]_{QR}$ if the representations $[l_1, l_2, ...]_{Q-1, R}$ (or $[l_1, l_2, ...]_{Q, R+1}$) and $[m_1, m_2, ...]_{Q+1, R}$ (or $[m_1, m_2, ...]_{Q, R-1}$) for windows adjacent to the window $S_{QR}$ are fixed because there are at most two different orders in which the two boxes can be added to make the Young tableau of the representation $[m_1, m_2, ...]_{Q+1, R}$ from the one for $[l_1, l_2, ...]_{Q-1, R}$. Thus we can break vertices to join windows with the same representation, change representations so that windows can be joined or take limits $S_{QRI} \to 0$ until there are no vertices.
5 Extensions to U(N), the large N limit and curved manifolds

Thus, we have shown that all SU(N) (and in particular SU(3)) Wilson loop averages obey the area law. Therefore, two-dimensional QCD exhibits confinement. In two dimensions even QED has confinement because in one spatial dimension the electromagnetic field cannot spread out and thus the field strength is constant. In four-dimensional space-time the electromagnetic field decreases sufficiently at large distances that there is no confinement but the gluon field strength is conjectured to remain constant because gluon self-interactions keep quarks confined. The well known expression for the U(1) WLA showing the area law behaviour can be derived to all orders in perturbation theory using the axial gauge $A_1 = 0$. It reads

$$W^{U(1)}(C) = W^{U(1)}(..., S_{QRI}, ...) = e^{-\frac{1}{2} \sum_{Q,R,I} (Q-R)^2 S_{QRI}}$$

(61)

where $Q - R$ is the winding number of the loop around the window $S_i$. The winding number can only have a single value for each window of a loop which reflects the abelian character of the gauge group U(1). Thus the U(1) WLA has only one term for any loop (the SU(N) WLA has a term corresponding to each possible combination of the representations of the non-abelian gauge field). An electron and an adjacent positron, represented by antiparallel lines, form a chargeless ($Q = R$) state i.e. a positronium atom. The U(N) WLA in two dimensions is a product of the U(1) and the SU(N) WLAs because the U(1) and the SU(N) parts of the action decouple. Thus in general ($S = 0$ for U(N) and $S = 1$ for SU(N))

$$W(C) = \left[W^{U(1)}(C) \right]^{\frac{1}{1 - S}} W^{SU(N)}(C) = \sum_{...,[n_1,n_2,...]QRI,...} P'_{...,[n_1,n_2,...]QRI,...} e^{-\sum_{Q,R,I} \left(\frac{1 - S}{2N}(Q-R)^2 + j_1^2[n_1,n_2,...]QRI S_{QRI}\right)}$$

(62)

where $j_1^2$ is the SU(N) Casimir operator.

In the large $N$ limit (with fixed $g^2 N$) the WLAs follow a modified area law with area dependent “constants” $P'$ [4]. The reason for the modified behaviour can be easily seen by considering the SU(N) WLA for Fig. 5(c) [4]. It reads

$$W(C_1 \tilde{C}_2) = W(S_1, S_2) = e^{-\frac{N^2 - 1}{2N} g^2 S_1} e^{-\frac{2N^2 - 1}{2N} g^2 S_2} \left[\frac{N + 1}{2} e^{-g^2 S_2} - \frac{N - 1}{2} e^{-g^2 S_2}\right].$$

(63)
In the large $N$ limit it reduces to
\[
\lim_{N \to \infty} W(S_1, S_2) = e^{-\frac{N}{2}g^2S_1 - Ng^2S_2(1 - Ng^2S_2)} \tag{64}
\]
where the second term arises because $P' \propto N^E$ for constant $E \geq 1$. It can be shown by induction (by adding loops) that for a WLA with a non-self-intersecting contour $P \propto N^{-2E}$ where $E \geq 0$. Thus the WLAs with a non-self-intersecting contour do not have area dependent "constants" in the large $N$ limit. But the breaking factor $B_\perp$ increases the exponent $E$ by one if its denominator is proportional to $N^0$. Thus modified area dependence is possible for WLAs with a self-intersecting path.

Finally, we will consider the WLAs on an arbitrary two-dimensional manifold of genus $G$ and generalize the solution in the case of non-self-intersecting loops found in [6] to the case of arbitrary loops. A WLA on a non-orientable manifold coincides with a WLA on a non-compact manifold. Thus we can concentrate only on orientable manifolds [6]. A WLA with a non-self-intersecting composed path has the same form as the WLA in equation (62) except that the constant $P$ is replaced by a more general constant $P^G$ and the constraint (41) does not apply because there is no external infinite area. We can assign the values $Q = R = 0$ to an arbitrary window to calculate the indices $Q$ and $R$ for the other windows. The constant $P^G$ is given by
\[
P_{...[n_1,n_2,...]QRI...}^G = \frac{\prod_{Q,R,I} [D([n_1,n_2,...]_{QRI})]^{2-H_{QRI}}}{N^LZ_G} \tag{65}
\]
where $H_{QRI}$ is the number of disconnected loops that form the border of $S_{QRI}$ and the partition function $Z_G$ reads
\[
Z_G = \sum_{[n_1,n_2,...]} [D([n_1,n_2,...])]^{2-2G}e^{-j_2[n_1,n_2,...] \sum_{Q,R,I} S_{QRI}}. \tag{66}
\]
The partition function depends on the total area of the manifold and the sum is over all the irreducible representations of SU($N$) (or U($N$)). In the limit $S_{00} \to \infty$ the only term that survives is the one with the one-dimensional trivial representation ($j^2[0]_{00} = 0$) giving the constraint (111) and making $Z_G = 1$. In this limit on a manifold with no handles the WLA reduces to the one given by equations (38) and (48).
For a SU(2) (or U(2)) WLA with a self-intersecting path the constant $P^G$ can be calculated by breaking and adding vertices, as on the flat manifold, because the breaking operation is local and the WLA depends on the path only through the areas of the windows (which can be seen by using the technique in [6, 7]). Note that

$$P^\text{nested} = 2^{v-1} P^G = \frac{\prod_{Q=1}^n (Q + 1)^{2-HQ}}{2Z_G}$$

which reduces to $\frac{v+1}{2}$ on Euclidean space. The first equality above follows from equation (50). A general constant $P^G_{jQ_1\ldots}$ reads

$$P^G_{jQ_1\ldots} = P^\text{nested} G B(..., jQ_1\ldots) \prod J(..., jQ_1\ldots).$$

As before there is a breaking factor $B$ for every self-intersection one has to break (and a joining factor $J$ for the remaining sets of windows with the same value of $j$ which cannot be connected by breaking vertices) to get the nested WLA with the constant $P^\text{nested} G$ from the original WLA with the constant $P^G_{jQ_1\ldots}$. We chose to use $B$ and $J$ (which is a product of $B(..., jQ_1\ldots)$ and $[B(..., jQ'_1\ldots)]^{-1}$) rather than the two different breaking factors $B$ and $B_+$ to relate a generic term of the WLA to a term of a WLA with a non-self-intersecting contour so that we did not have to keep track of the orientation of the contour.

The constant $P^G$ for a SU($N$) (or U($N$)) WLA can be calculated (by breaking vertices and by letting $S_{QRI} \to 0$ for various windows $S_{QRI}$) only when the boundary of the window $S_{QRI}$ can be shrunk to an empty set of points because only then can the vertices on the boundary be eliminated. This is always possible on a manifold with no handles ($G = 0$) and we can write $P^G$ in terms of the dimensions $D$ of the representations, the breaking factors $B_+$ and the probabilities $P^G$. Note that also on a compact manifold $P^0_{[n_1,n_2,...]_{QRI}\ldots}$ can be interpreted as the relative probability of the combination $[n_1, n_2, ...]_{QRI\ldots}$ for the representations.

6 Conclusions

We have solved explicitly the closed, linear loop equation for the SU(2) WLAs on a two-dimensional plane and generalized the solution to case of the SU($N$)
(or U(N)) WLA on a two-dimensional manifold of genus 0 and the SU(2) (or U(2)) WLA on any two-dimensional manifold. The WLA follows an area law
\[ W(C) = \sum_r P'_r e^{-\sum_C r S_i} \] where \( C_{r_i} \) is the quadratic Casimir operator for SU\( (N) \) (plus a U(1) term proportional to the winding number squared of the loop around the window \( S_i \) for U(N)). Only certain combinations of the Casimir operators are allowed in the sum over \( i \). Namely, the representations of the Casimir operators differ by one box (or \( N - 1 \) boxes) in the Young tableau for neighbouring windows. This means that \( N \) quarks or a quark and an antiquark can form a particle i.e. a baryon or a meson. In the case of a non-self-intersecting composed path on a manifold with no handles the constant \( P_r \) can be interpreted as the probability of the combination \( r \) for the representations.

It would be interesting to calculate the meson spectrum in two dimensions for SU(3) or SU(2) QCD and to compare it to the large \( N \) spectrum [14]. This could be done along the lines of [8, 13] in a single sector of the theory. The modified area law for the large \( N \) WLAs suggests that there might be important differences between the case with infinite \( N \) and the one with finite \( N \). On the other hand, the WLAs with the modified area law behaviour are not needed to calculate the large \( N \) meson spectrum to first order in \( g^2 N \) [13]. The ultimate challenge is to solve the closed, linear loop equation for the SU(2) WLA in four dimensions and to generalize the solution to the case of the SU(3) WLA to solve the riddle of confinement in QCD.

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Figure Captions

Fig. 1. Graphical representation of the non-nested WLA \( W_1(C) = W_1(C_{xy}C_{yx}) \) \( = W_1(S_a, S_b) \) where the WLA is represented by a loop equivalent to \( C \) (having the same topology and areas of the windows as \( C \)) and where the path \( C_{xy} (C_{yx}) \) with \( x = y \) is the boundary of the area \( S_a \) (\( S_b \)).

Fig. 2. (a) A self-intersection and (b) a broken self-intersection at a point \( x \).

Fig. 3. Graphical representation of a nested WLA \( W_{3\text{\scriptsize nested}} \).

Fig. 4. Graphical representation of the WLA \( W(S_1, S_{21}, S_{22}, S_3) \).

Fig. 5. Graphical representation of
(a) a WLA \( W(C_1, C_2) = \frac{1}{2}[W(C_1\tilde{C}_2) + W(C_1\tilde{C}_2^{-1})] \) with a composite contour,
(b) a WLA \( W(C_1, \tilde{C}_2) = W(C_1, C_2) \) with a self-touching contour,
(c) a WLA \( W(C_1\tilde{C}_2) \) with a self-crossing contour, and
(d) a WLA \( W(C_1\tilde{C}_2^{-1}) \) with a simple loop.

Fig. 6. An example of the calculation of the indices \( Q \) and \( R \). \( L_{QR} \) is a directed open path from the external infinite area \( S_{00} \) to the window \( S_{QR} \) where \( Q \) (\( R \)) is the number of times the contour \( C_1C_2 \) crosses the path \( L_{QR} \) from left to right (from right to left).

Fig. 7. Graphical representation of
(a) the WLA \( W(S_{10}, S_{20}, ..., S_{60}, S_{701}, S_{702}, S_{703}, S_{704}, S_{80}, S_{90}) \) and
(b) the WLA \( W(S_{10}, S_{20}, ..., S_{60}, S_{702} + S_{704}, S_{703}) \).