HANKEL DETERMINANTS FOR CONVOLUTION POWERS OF CATALAN NUMBERS

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Abstract. The Hankel determinants \( \left( \frac{r}{2(i+j)+r} \left( \binom{2(i+j)+r}{i+j} \right) \right)_{0 \leq i,j \leq n-1} \) of the convolution powers of Catalan numbers were considered by Cigler and by Cigler and Krattenthaler. We evaluate these determinants for \( r \leq 31 \) by finding shifted periodic continued fractions, which arose in application of Sulanke and Xin’s continued fraction method. These include some of the conjectures of Cigler as special cases. We also conjectured a polynomial characterization of these determinants. The same technique is used to evaluate the Hankel determinants \( \left( \binom{2(i+j)+r}{i+j} \right)_{0 \leq i,j \leq n-1} \). Similar results are obtained.

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1. Introduction

Hankel determinants evaluation has a long history. This paper is along the line of using generating functions to deal with Hankel determinants. The Hankel determinants of a generating function \( A(x) = \sum_{n \geq 0} a_n x^n \) is defined by

\[ H_n(A(x)) = \det(a_{i+j})_{0 \leq i,j \leq n-1}, \quad H_0(A(x)) = 1. \]

In recent years, a considerable amount of work has been devoted to Hankel determinant evaluations of various counting coefficients. Many of such Hankel determinants have attractive compact closed formulas, such as binomial coefficients, Catalan numbers [23], Motzkin numbers [1, 10], large and little Shröder numbers [7]. See [2–7, 15, 16, 18, 20–22, 24–27] for further references.

Classical method of continued fractions, either by \( J \)-fractions (Krattenthaler [21] or Wall [28]), or by \( S \)-fractions (Jones and Thron [19, Theorem 7.2]), requires \( H_n(A(x)) \neq 0 \) for all \( n \). Gessel-Xin’s [17] continued fraction method allows \( H_n(A(x)) = 0 \) for some values of \( n \). Their method is based on three rules about two variable generating functions that can transform one set of determinants to another set of determinants of the same values. These rules corresponding to a sequence of elementary row or column operations. This method was systematically used by Sulanke-Xin [26] for evaluating Hankel determinants of quadratic generating functions, such as known results for Catalan numbers, Motzkin numbers, Shröder numbers, etc. They defined a quadratic transformation \( \tau \) such that \( H(F(x)) \) and \( H(\tau(F(x))) \) have simple connections. This method has many applications, including the Hankel determinants representation of Somos-4 sequence [8, 30], and is now called Sulanke-Xin’s continued fraction method.

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Recently, shifted periodic continued fractions (of order \( q \)) of the form
\[
F_0^{(p)}(x) \xrightarrow{\tau} F_1^{(p)}(x) \xrightarrow{\tau} \cdots \xrightarrow{\tau} F_0^{(p+1)}(x)
\]
was found in [29] to appear in Hankel determinants of many path counting numbers. Here \( p \) is an additional parameter. If one can guess an explicit formula of \( F_0^{(p)}(x) \), then their Hankel determinants can be easily computed.

One of our main objectives in this paper is to evaluate the Hankel determinants
\[
H_n(C(x)^r) = \det(C_n^{(r)})_{0 \leq i, j \leq n-1} = \det\left(\frac{r}{2n+r}\binom{2n+r}{n}\right)_{0 \leq i, j \leq n-1}
\]
of the \( r \)-th convolution power of the well-known Catalan numbers \( C_n = \frac{1}{n+1}\binom{2n}{n} \). The name comes after the generating function identity
\[
\sum_{n \geq 0} C_n^{(r)}x^n = C(x)^r = \left(\sum_{n \geq 0} C_n x^n\right)^r.
\]

These determinants were considered by Cigler in [9], where the cases \( r \leq 4 \) were evaluated by using the method of orthogonal polynomials. In what follows, we will denote by \( F(x, r) = C(x)^r \). The cases \( r = 1, 2 \) are well-known.
\[
H_n(F(x, 1)) = H_n(C(x)) = \det(C_{i+j})_{0 \leq i, j \leq n-1} = 1, \quad H_n(F(x, 2)) = H_n((C(x) - 1)/x) = \det(C_{i+j+1})_{0 \leq i, j \leq n-1} = 1.
\]

The cases \( r = 3, 4 \) are nice.

**Theorem 1.** [9, 11]
\[
H_{3n}(F(x, 3)) = H_{3n}(F(x, 3)) = (-1)^n, \quad H_{3n+2}(F(x, 3)) = 0.
\]

**Theorem 2.** [3]
\[
H_{2n}(F(x, 4)) = H_{2n+1}(F(x, 4)) = (-1)^n(n + 1).
\]

Besides some byproducts, Cigler observed that such determinants follow a modular pattern and are in some cases easy to guess. He made conjectures for \( 5 \leq r \leq 8 \) and find that the pattern for odd \( r \) is quite different from that of even \( r \). He thought \( D(n, 8) \) (equal to our \( H_n(F(x, 8)) \)) was already complicated.

**Theorem 3.**
\[
H_{5n}(F(x, 5)) = H_{5n+1}(F(x, 5)) = 1,
H_{5n+2}(F(x, 5)) = -H_{5n+4}(F(x, 5)) = -5(n + 1),
H_{5n+3}(F(x, 5)) = 0.
\]
**Theorem 4.**

\[ H_{7n}(F(x, 7)) = H_{7n+1}(F(x, 7)) = (-1)^n, \]
\[ H_{7n+2}(F(x, 7)) = (-1)^n \left( \frac{7}{6} (n + 1) (98 n^2 + 49 n - 12) \right), \]
\[ H_{7n+3}(F(x, 7)) = - H_{7n+5}(F(x, 7)) = (-1)^n (7(n + 1))^2, \]
\[ H_{7n+4}(F(x, 7)) = 0, \]
\[ H_{7n+6}(F(x, 7)) = (-1)^n \left( \frac{7}{6} (n + 1) (98 n^2 + 343 n + 282) \right). \]

**Theorem 5.**

\[ H_{3n}(F(x, 6)) = H_{3n+1}(F(x, 6)) = (-1)^n(n + 1)^2, \]
\[ H_{3n+2}(F(x, 6)) = (-1)^{n+1} \left( \frac{3}{2} (n + 2) (n + 1) (2 n + 3) \right). \]

**Theorem 6.**

\[ H_{4n}(F(x, 8)) = H_{4n+1}(F(x, 8)) = (n + 1)^3, \]
\[ H_{4n+2}(F(x, 8)) = \frac{2}{45} \cdot (n + 1)^2(n + 2)(2n + 3)(64n^2 + 32n - 75), \]
\[ H_{4n+3}(F(x, 8)) = - \frac{2}{45} \cdot (n + 1)(n + 2)^2(2n + 3)(64n^2 + 352n + 405). \]

For general \( r \), he only conjectured nice formulas for special values of \( n \).

**Conjecture 7.** For odd positive integer \( r = 2t + 1 \), we have
\[ H_{(2t+1)n}(F(x, 2t + 1)) = H_{(2t+1)n+1}(F(x, 2t + 1)) = (-1)^{tn}, \]
\[ H_{(2t+1)n+t+1}(F(x, 2t + 1)) = 0, \]
\[ H_{(2t+1)n+t}(F(x, 2t + 1)) = - H_{(2t+1)n+t+2}(F(x, 2t + 1)) = (-1)^{tn+\binom{t}{2}}((2t + 1)(n + 1)^{t-1}, \]
\[ H_{(2t+1)n-1}(F(x, 2t + 1)) + H_{(2t+1)n+2}(F(x, 2t)) = (-1)^{tn+1}(t - 1)(2t + 1). \]

**Conjecture 8.** For even positive integer \( r = 2t \), we have
\[ H_{tn}(F(x, 2t)) = H_{tn+1}(F(x, 2t)) = (-1)^{\binom{t}{2}}(n + 1)^{t-1}, \]
\[ H_{2tn-1}(F(x, 2t)) + H_{2tn+2}(F(x, 2t)) = -t(2t - 3)(2n + 1)^{t-1}. \]

We find shifted periodic continued fractions is perfect for evaluating \( H_n(C(x)^r) \) for \( r \) up to 31, thereby confirming some conjectured formulas of Cigler. Though we are not able to prove Conjectures 7 and 8 at this moment, we come up with the following conjectures.

**Conjecture 9.** For odd positive integer \( r = 2t + 1 \), we have, for \( 1 \leq j \leq t \), \((-1)^{tn} H_{rn+j}(F(x, r)) \) and \((-1)^{tn} H_{rn+r+1-j}(F(x, r)) \) are both polynomials in \( n \) of degree \((j-1)(r-2j)\).

**Conjecture 10.** For even positive integer \( r = 2t \), we have, for \( 1 \leq j \leq t \), \((-1)^{\binom{t}{2}} H_{tn+j}(F(x, r)) \) and \((-1)^{\binom{t}{2}} H_{tn+t-j+1}(F(x, r)) \) are both polynomials in \( n \) of degree \((2j-1)(t-j)\).

Our main result is the following.
Theorem 11. Conjectures 7, 8, 9 and 10 hold true for \( r \leq 31 \).

The paper is organized as follows. Section 2.1 introduces Sulanke-Xin’s continued fraction method, especially the quadratic transformation \( \tau \). In Section 2.2 we explain how to find the functional equation of \( F(x, r) = C(x)^r \) for specific \( r \). We can then guess explicit functional equations of \( F(x, r) \) for general \( r \). Details are put in Appendix 6.1. Section 3 devotes to evaluating the Hankel determinants of \( F(x, r) \) for odd \( r \) up to 9. The mail tool is Sulanke-Xin’s continued fraction method. We give detailed steps on the transformation \( \tau \) for small \( r \). Section 4 deals with the even \( r \) case for \( r \leq 8 \) in a similar way. For both odd \( r \) and even \( r \), the formulas suggest a polynomial characterization of the Hankel determinants, as stated in Conjectures 9 and 10. In Section 5, we consider the Hankel determinants \( \binom{2(i+j)+r}{i+j} \) for \( 0 \leq i, j \leq n-1 \), which have been studied in [12–14]. The corresponding generating function is \( G(x, r) = \frac{C(x)^r}{1-2xG(x)} \). We obtain similar results as for \( F(x, r) \). Finally, Appendix 6 includes the explicit functional equations of \( F(x, r) \) and \( G(x, r) \), as well as their proofs.

2. Preliminary

We have two goals in this section. Firstly we will introduce the continued fraction method of Sulanke and Xin, especially their quadratic transformation \( \tau \) in [26]. This is the main tool of this paper. Secondly we explain how to find the functional equation of \( F(x, r) = C(x)^r \), which is the starting point for applying the transformation \( \tau \).

2.1. Sulanke-Xin’s quadratic transformation \( \tau \). This subsection is copied from [29]. We include it here for reader’s convenience.

Suppose the generating function \( F(x) \) is the unique solution of a quadratic functional equation which can be written as

\[
F(x) = \frac{x^d}{u(x) + x^k v(x) F(x)},
\]

where \( u(x) \) and \( v(x) \) are rational power series with nonzero constants, \( d \) is a nonnegative integer, and \( k \) is a positive integer. We need the unique decomposition of \( u(x) \) with respect to \( d \): \( u(x) = u_L(x) + x^{d+2} u_H(x) \) where \( u_L(x) \) is a polynomial of degree at most \( d+1 \) and \( u_H(x) \) is a power series. Then Propositions 4.1 and 4.2 of [26] can be summarized as follows.

Proposition 12. Let \( F(x) \) be determined by (1). Then the quadratic transformation \( \tau(F) \) of \( F \) defined as follows gives close connections between \( H(F) \) and \( H(\tau(F)) \).

i) If \( u(0) \neq 1 \), then \( \tau(F) = G = u(0) F \) is determined by \( G(x) = \frac{x^d}{u(0)-1 u(x) + x^k u(0) - x^d v(x) G(x)} \), and \( H_n(\tau(F)) = u(0)^n H_n(F(x)) \);

ii) If \( u(0) = 1 \) and \( k = 1 \), then \( \tau(F) = x^{-1}(G(x) - G(0)) \), where \( G(x) \) is determined by

\[
G(x) = \frac{-v(x) - x u_L(x) u_H(x)}{u_L(x) - x^{d+2} u_H(x) - x^{d+1} G(x)},
\]

and we have

\[
H_{n-d-1}(\tau(F)) = (-1)^{d+1} H_n(F(x));
\]
iii) If \( u(0) = 1 \) and \( k \geq 2 \), then \( \tau(F) = G \), where \( G(x) \) is determined by

\[
G(x) = \frac{-x^{k-2}v(x) - u_L(x)u_H(x)}{u_L(x) - x^{d+2}u_H(x) - x^{d+2}G(x)},
\]

and we have

\[
H_{n-d-1}(\tau(F)) = (-1)^{\left(\frac{d+1}{2}\right)}H_n(F(x)).
\]

2.2. The functional equation of \( F(x, r) \). Our starting point is the well-known functional equation for Catalan generating function.

\[
C(x) = 1 + xC(x)^2 \iff xC(x)^2 - C(x) + 1 = 0 \iff C(x) = \frac{1}{1 - xC(x)}. \tag{5}
\]

A classical theory in Algebra asserts that for any rational function \( R(x) \), \( R(C(x)) \) can be uniquely written as \( \alpha(x) + \beta(x)C(x) \) for some rational functions \( \alpha(x) \) and \( \beta(x) \). Hence \( R(C(x)) \) satisfies a quadratic functional equation, so does \( F(x, r) = C(x)^r \).

Using the method of undetermined coefficients, one can easily obtain the functional equation of \( F(x, r) \) with the help of Maple. For odd integer \( r \), the first several functional equations are as follows.

\[
F(x, 3) = -\frac{1}{x^3 F(x, 3) + 3 x - 1}, \\
F(x, 5) = -\frac{1}{x^5 F(x, 5) - 5 x^2 + 5 x - 1}, \\
F(x, 7) = -\frac{1}{x^7 F(x, 7) - 7 x^3 - 14 x^2 + 7 x - 1}, \\
F(x, 9) = -\frac{1}{x^9 F(x, 9) - 9 x^4 + 30 x^3 - 27 x^2 + 9 x - 1}, \\
F(x, 11) = -\frac{1}{x^{11} F(x, 11) + 11 x^5 - 55 x^4 + 77 x^3 - 44 x^2 + 11 x - 1}.
\]

For even integer \( r \), the first several functional equations are as follows.

\[
F(x, 2) = -\frac{1}{x^2 F(x, 2) + 2 x - 1}, \\
F(x, 4) = -\frac{1}{x^4 F(x, 4) - 2 x^2 + 4 x - 1}, \\
F(x, 6) = -\frac{1}{x^6 F(x, 6) + 2 x^3 - 9 x^2 + 6 x - 1}, \\
F(x, 8) = -\frac{1}{x^8 F(x, 8) - 2 x^4 + 16 x^3 - 20 x^2 + 8 x - 1}, \\
F(x, 10) = -\frac{1}{x^{10} F(x, 10) + 2 x^5 - 25 x^4 + 50 x^3 - 35 x^2 + 10 x - 1}.
\]

Though these formula are sufficient for our purpose, the above formulas are so nice that we guess and prove explicit functional equation of \( F(x, r) \) for general \( r \). See Appendix 6.1 for details.
3. The results of $H_n(F(x,r))$ for odd $r = 2t + 1$

In this section, we evaluate the Hankel determinants of $F(x,r) = C(x)^r$ for odd $r = 2t+1$.

3.1. The results of $F(x,3)$. For $r = 3$, we have the functional equation

$$F(x,3) = \frac{1}{1 - 3x - x^3 F(x,3)}.$$  

Proof of Theorem 3. Let $F_0(x) = F(x,3)$. We apply Proposition 12 to obtain $F_1 = \tau(F_0)$. Firstly, $d = 0$. $u(x) = 1 - 3x$, thus $u_L(x) = 1 - 3x$ and $u_H(x) = 0$. Then by $u(0)^{-1} = 1$, we obtain

$$H_k(F_0) = H_{k-1}(F_1)$$

$$F_1(x) = \frac{x}{1 - 3x - x^3 F_1(x)}.$$  

Apply Proposition 12 to obtain $F_2 = \tau(F_1)$. This time $d = 1$. $u(x) = 1 - 3x$, thus $u_L(x) = 1 - 3x$ and $u_H(x) = 0$. Then by $u(0)^{-1} = 1$, we obtain

$$H_{k-1}(F_1) = (-1)^{(\frac{d}{2})} H_{k-3}(F_2) = -H_{k-3}(F_2)$$

$$F_2(x) = \frac{1}{1 - 3x - x^3 F_2(x)}.$$  

Now we see that $F_2 = F_0$. So the continued fractions is periodic of order 2: $F_0 \to F_1 \to F_2 = F_0$. By summarizing the above results, we obtain the recursion $H_k(F_0) = -H_{k-3}(F_0)$, which implies that

$$H_{3n+i}(F_0) = (-1)^{n} H_i(F_0), \text{ for } i = 0, 1, 2.$$  

The initial values are $H_0(F_0) = 1$, $H_1(F_0) = 1$, $H_2(F_0) = 0$. Theorem 3 then follows. 

3.2. The results of $F(x,5)$. For $r = 5$, we have the functional equation

$$F(x,5) = \frac{1}{1 - 5x + 5x^2 - x^5 F(x,5)}.$$  

Proof of Theorem 3. We apply Proposition 12 to $F_0 := F(x,5)$ by repeatedly using the transformation $\tau$. This results in a shifted periodic continued fractions of order 4:

$$F_0(x) \xrightarrow{\tau} F_1^{(1)}(x) \xrightarrow{\tau} F_2^{(1)}(x) \xrightarrow{\tau} F_3^{(1)}(x) \xrightarrow{\tau} F_4^{(1)}(x) \xrightarrow{\tau} F_5^{(1)}(x) = F_1^{(2)}(x) \cdots.$$  

We will carry out the details in this computation.

Apply Proposition 12 to obtain $F_1^{(1)} = \tau(F_0)$. We obtain

$$H_k(F_0) = H_{k-1}(F_1^{(1)}). \quad (6)$$  

Computer experiment suggests us to define, for $p \geq 1$,

$$F_1^{(p)} = -\frac{x^3 + 5^2 p (p - 1) x^2 + 5^2 p x - 5^2}{x^2 F_1^{(p)} + 5 (2p - 1) x^2 + 5 x - 1}.$$  

Apply Proposition 12 to obtain $F_2^{(p)} = \tau(F_1^{(p)})$. Firstly, $d = 0$. Next we need to decompose $u(x)$ with respect to $d$. We expand $u(x)$ as a power series and focus on (by displaying) those terms with small exponents ($\leq d + 1 = 1$):

$$u(x) = -\frac{5 (2 p - 1) x^2 + 5 x - 1}{x^3 + 5^2 p (p - 1) x^2 + 5^2 p x - 5^2} = -\frac{1}{5p} + x^2 + \frac{1}{25} \frac{125 p^2 - 1}{p^2} x^3 + \cdots.$$  

Thus \( u_L(x) = -\frac{1}{5p} \) is simple, and \( u_H(x) = -\frac{1}{5} \frac{25p^2 - x}{(x^3 + 25p(p-1)x^2 + 25px - 5p)p} \). Then by \( u(0)^{-1} = -5p \), we obtain
\[
H_{k-1}(F_1^{(p)}) = (-5p)^{k-1} H_{k-2}(F_2^{(p)}),
\]
\[
F_2^{(p)} = -\frac{x}{x^2 (x^3 + 5^2 p(p-1)x^2 + 5^2 px - 5p)p} F_1^{(p)} - 2x^3 + 5^2 px^2 - 5^2 xp + 5p.
\]

Apply Proposition 12 to obtain \( F_3^{(p)} = \tau(F_2^{(p)}) \), This time \( d = 1 \) and \( u(x) \) is indeed a polynomial:
\[
u(x) = 2x^3 - 5^2 px^2 + 5^2 xp - 5p.
\]
It then follows that \( u_L(x) = -5^2 px^2 + 5^2 xp - 5p \), \( u_H(x) = 2x \). Then by \( u(0)^{-1} = -\frac{1}{5p} \), we obtain
\[
H_{k-2}(F_2^{(p)}) = (-1)^{\binom{2}{1}} \left(-\frac{1}{5p}\right)^{k-2} H_{k-4}(F_3^{(p)}),
\]
\[
F_3^{(p)} = -\frac{1}{5} \frac{x^3 + 5^2 p(p+1)x^2 - 5^2 xp + 5p}{(5px^3 F_3^{(p)} - 2x^3 - 5^2 px^2 + 5^2 xp - 5p)p}.
\]

Apply Proposition 12 to obtain \( F_4^{(p)} = \tau(F_3^{(p)}) \), This time \( d = 0 \) and \( u(x) \) is expanded as:
\[
u(x) = -5\frac{-5^2 x^3 - 5^2 px^2 + 5^2 xp - 5p)p}{x^3 + 5^2 p(p+1)x^2 - 5^2 xp + 5p} = 5^2 - 25p^2 x^2 + \cdots.
\]
It then follows that \( u_L(x) = 5p \), \( u_H(x) = -5\frac{p (25p^2 - x)}{x^3 + 25p(p+1)x^2 - 25px + 5p} \). Thus by \( u(0)^{-1} = \frac{1}{5p} \), we obtain
\[
H_{k-4}(F_3^{(p)}) = \left(\frac{1}{5p}\right)^{k-4} H_{k-5}(F_3^{(p)}),
\]
\[
F_4^{(p)} = -\frac{5^2 p^2}{x^3 + 5^2 p(p+1)x^2 - 5^2 px + 5p) F_4^{(p)} - 5p(5^2(p+1)x^2 - 5x + 1)}.
\]

Apply Proposition 12 to obtain \( F_5^{(p)} = F_1^{(p+1)} = \tau(F_4^{(p)}) \), This time \( d = 0 \) and \( u(x) \) is indeed a polynomial:
\[
u(x) = \frac{5(2p+1)x^2 - 5x + 1}{5p}.
\]
It then follows that \( u_L(x) = -\frac{5x + 1}{5p} \), \( u_H(x) = \frac{2p+1}{p} \). Then by \( u(0)^{-1} = u(0)^{-1} = 5p \), we obtain
\[
H_{k-5}(F_4^{(p)}) = (5p)^{k-5} H_{k-6}(F_1^{(p+1)}).
\]

Combining the above formulas gives the recursion
\[
H_{k-1}(F_1^{(p)}) = H_{k-6}(F_1^{(p+1)}).
\]
Let $k - 1 = 5n + j$, where $0 \leq j < 5$. We then deduce that

$$H_{5n+j}(F_{1}^{(1)}) = H_{j}(F_{1}^{(n+1)}).$$  \hfill (7)

The initial values are

$$H_{0}(F_{1}^{(n+1)}) = 1, H_{1}(F_{1}^{(n+1)}) = -H_{3}(F_{1}^{(n+1)}) = -5(n + 1), H_{2}(F_{1}^{(n+1)}) = 0, H_{4}(F_{1}^{(n+1)}) = 1. \nonumber$$

The theorem then follows by the above initial values, (6) and (7). \hfill \Box

3.3. **The results of** $H_{n}(F(x, 7))$. For $r = 7$, we have the functional equation

$$F(x, 7) = \frac{1}{1 - 7x + 14x^{2} - 7x^{3} - x^{4}F(x, 7)}. \nonumber$$

**Proof of Theorem 4.** We apply Proposition 12 to $F_{0} := F(x, 7)$, and repeat the transformation $\tau$. This results in a shifted periodic continued fractions of order 6:

$$F_{0}(x) \xrightarrow{\tau} F_{1}^{(1)}(x) \xrightarrow{\tau} F_{2}^{(1)}(x) \xrightarrow{\tau} F_{3}^{(1)}(x) \xrightarrow{\tau} F_{4}^{(1)}(x) \xrightarrow{\tau} F_{5}^{(1)}(x) \xrightarrow{\tau} F_{6}^{(1)}(x) \xrightarrow{\tau} F_{7}^{(1)}(x) = F_{1}^{(2)}(x) \cdots . \nonumber$$

If we define

$$F_{1}^{(p)} = -[x^{5} + 49 (p - 1) px^{4} + 49/6 (p - 1) p (196 p^{2} - 196 p + 25) x^{3} + 49/36 p(9604 p^{5} - 28812 p^{4} + 28861 p^{3} - 9702 p^{2} - 395 p + 408) x^{2} - 7/6 p (686 p^{2} - 1029 p + 253) x + 7/6 p(98 p^{2} - 147 p + 37)]/[F_{1}^{(p)} x^{2} - 7 (2 p - 1) x^{3} - 7/3 (2 p - 1) (49 p^{2} - 49 p - 6) x^{2} + 7 x - 1], \nonumber$$

Then the results can be summarized as follows:

$$H_{k}(F_{0}) = H_{k-1}(F_{1}^{(1)}),$$  \hfill (8)

and

$$H_{k-1}(F_{1}^{(p)}) = \left(\frac{7}{6} \cdot p \left(98 p^{2} - 147 p + 37\right)\right)^{k-1} H_{k-2}(F_{2}^{(p)}), \nonumber$$

$$H_{k-2}(F_{2}^{(p)}) = \left(-\frac{36}{(98 p^{2} - 147 p + 37)^2}\right)^{k-2} H_{k-3}(F_{3}^{(p)}), \nonumber$$

$$H_{k-3}(F_{3}^{(p)}) = (-1)^{\frac{p}{2}} \left(-\frac{1}{42} \cdot \frac{98 p^{2} - 147 p + 37}{p}\right)^{k-3} H_{k-5}(F_{4}^{(p)}), \nonumber$$

$$H_{k-5}(F_{4}^{(p)}) = \left(\frac{1}{42} \cdot \frac{98 p^{2} + 147 p + 37}{p}\right)^{k-5} H_{k-6}(F_{5}^{(p)}), \nonumber$$

$$H_{k-6}(F_{5}^{(p)}) = \left(-\frac{36}{(98 p^{2} + 147 p + 37)^2}\right)^{k-6} H_{k-7}(F_{6}^{(p)}), \nonumber$$

$$H_{k-7}(F_{6}^{(p)}) = \left(-\frac{7}{6} \cdot p \left(98 p^{2} + 147 p + 37\right)\right)^{k-7} H_{k-8}(F_{1}^{(p+1)}). \nonumber$$

Combining the above formulas gives the recursion

$$H_{k-1}(F_{1}^{(p)}) = -H_{k-8}(F_{1}^{(p+1)}).$$
Let \( k - 1 = 7n + j \), where \( 0 \leq j < 7 \). We then deduce that

\[
H_{7n+j}(F_1^{(1)}) = (-1)^n H_j(F_1^{(n+1)}).
\]  

(9)

The initial values are

\[
H_0(F_1^{(n+1)}) = 1, \quad H_1(F_1^{(n+1)}) = \frac{7}{6} (n + 1) (98 n^2 + 49 n - 12), \quad H_2(F_1^{(n+1)}) = -49 (n + 1)^2, \\
H_3(F_1^{(n+1)}) = 0, \quad H_4(F_1^{(n+1)}) = 49 (n + 1)^2, \quad H_5(F_1^{(n+1)}) = \frac{7}{6} (n + 1) (98 n^2 + 343 n + 282), \\
H_6(F_1^{(n+1)}) = 1.
\]

The theorem then follows by the above initial values, (8) and (9).

\[ \square \]

3.4. The results of \( H_n(F(x, 9)) \). For \( r = 9 \), we have the functional equation

\[
F(x, 9) = \frac{1}{1 - 9 x + 27 x^2 - 30 x^3 + 9 x^4 - x^9 F(x, 9)}.
\]

Theorem 13.

\[
H_{9n}(F(x, 9)) = H_{9n+1}(F(x, 9)) = 1, \\
H_{9n+2}(F(x, 9)) = -\frac{27}{10} (3 n + 2) (18 n + 1) (n + 1) (54 n^2 + 42 n + 5), \\
H_{9n+3}(F(x, 9)) = \frac{9}{20} (n + 1)^2 (26244 n^4 + 104976 n^3 + 108459 n^2 + 31266 n - 1460), \\
H_{9n+4}(F(x, 9)) = -H_{9n+6}(F(x, 9)) = (9(n + 1))^3, \quad H_{9n+5}(F(x, 9)) = 0, \\
H_{9n+7}(F(x, 9)) = \frac{9}{20} (n + 1)^2 (26244 n^4 + 104976 n^3 + 108459 n^2 + 17334 n - 50060), \\
H_{9n+8}(F(x, 9)) = \frac{27}{10} (18 n + 35) (3 n + 4) (n + 1) (54 n^2 + 174 n + 137).
\]

Proof. We apply Proposition 12 to \( F_0 := F(x, 9) \), and repeat the transformation \( \tau \). This results in a shifted periodic continued fractions of order 8:

\[
F_0(x) \xrightarrow{\tau} F_1^{(1)}(x) \xrightarrow{\tau} F_2^{(1)}(x) \xrightarrow{\tau} F_3^{(1)}(x) \xrightarrow{\tau} F_4^{(1)}(x) \xrightarrow{\tau} F_5^{(1)}(x) \xrightarrow{\tau} F_6^{(1)}(x) \\
\xrightarrow{\tau} F_7^{(1)}(x) \xrightarrow{\tau} F_8^{(1)}(x) \xrightarrow{\tau} F_9^{(1)}(x) = F_2^{(1)}(x) \cdots.
\]

The formula of \( F_1^{(p)}(x) \) is too complicated to print. Then the results can be summarized as follows:

\[
H_k(F_0) = H_{k-1}(F_1^{(1)}),
\]  

(10)

and

\[
H_{k-1}(F_1^{(p)}) = \left(-\frac{27}{10} (3p - 1)(18p - 17)p(54p^2 - 66p + 17)\right)^{k-1} H_{k-2}(F_2^{(p)}),
\]
\[
H_{k-2}(F_2^{(p)}) = \left(\frac{5}{81} (3p - 1)^2 (18p - 17)^2 (54p^2 - 66p + 17)^2\right)^{k-2} H_{k-3}(F_3^{(p)}),
\]
\[
H_{k-3}(F_3^{(p)}) = \left(-9720 (3p - 1)(18p - 17)(54p^2 - 66p + 17)\right)^{k-3} H_{k-4}(F_4^{(p)}),
\]
\[
H_{k-4}(F_4^{(p)}) = (-1)^{\frac{2}{3}} \left(\frac{1}{1620} 26244p^4 - 49005p^2 + 24300p - 2999\right)^{k-4} H_{k-5}(F_5^{(p)}),
\]
\[
H_{k-5}(F_5^{(p)}) = \left(-\frac{1}{1620} 26244p^4 - 49005p^2 + 24300p - 2999\right)^{k-6} H_{k-6}(F_6^{(p)}),
\]
\[
H_{k-6}(F_6^{(p)}) = \left(-9720 (18p + 17)(3p + 1)(54p^2 + 66p + 17)\right)^{k-7} H_{k-7}(F_7^{(p)}),
\]
\[
H_{k-7}(F_7^{(p)}) = \left(\frac{5}{81} (18p + 17)^2 (3p + 1)^2 (54p^2 + 66p + 17)^2\right)^{k-8} H_{k-8}(F_8^{(p)}),
\]
\[
H_{k-8}(F_8^{(p)}) = \left(\frac{27}{10} (18p + 17)(3p + 1)p(54p^2 + 66p + 17)\right)^{k-9} H_{k-9}(F_9^{(p+1)}).
\]

Combining the above formulas gives the recursion

\[
H_{k-1}(F_1^{(p)}) = H_{k-10}(F_1^{(p+1)}).
\]

Let \(k - 1 = 9n + j\), where \(0 \leq j < 9\). We then deduce that

\[
H_{9n+j}(F_1^{(1)}) = H_j(F_1^{(n+1)}).
\]  

The initial values are

\[
H_0(F_1^{(n+1)}) = 1, \quad H_1(F_1^{(n+1)}) = -\frac{27}{10} (3n + 2)(18n + 1)(n+1)(54n^2 + 42n + 5),
\]
\[
H_2(F_1^{(n+1)}) = \frac{9}{20} (n+1)^2 (26244n^4 + 104976n^3 + 108459n^2 + 31266n - 1460),
\]
\[
H_3(F_1^{(n+1)}) = (9(n+1))^3, \quad H_4(F_1^{(n+1)}) = 0, \quad H_5(F_1^{(n+1)}) = -(9(n+1))^3,
\]
\[
H_6(F_1^{(n+1)}) = \frac{9}{20} (n+1)^2 (26244n^4 + 104976n^3 + 108459n^2 - 17334n - 50060),
\]
\[
H_7(F_1^{(n+1)}) = \frac{27}{10} (18n + 35)(3n + 4)(n+1)(54n^2 + 174n + 137), \quad H_8(F_1^{(n+1)}) = 1.
\]

The theorem then follows by the above initial values, \([10]\) and \([11]\).

3.5. **A summary.** A computer program has been developed for calculating \(H_n(F(x, r))\) for \(r = 3, 5, 7, \ldots\). We succeed for \(r\) up to 31 because the coefficients appearing in the functional equation for \(F_1^{(p)}(x)\) are all polynomials in \(p\) of a reasonable small degree. This allows us to guess an explicit formula, and then the rest is straightforward.
From the obtained formulas, we find that \( \pm H_{rn+i}(F(x,r)) \) is a polynomial in \( n \) for fixed \( r \) and \( i \). The degrees are listed in Table 1.

| \( r \) | \( i \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-------|-------|---|---|---|---|---|---|---|---|---|---|-----|-----|-----|-----|-----|-----|
| 3     | 0     | 0 | 0 | 0 |   |   |   |   |   |   |   |     |     |     |     |     |     |
| 5     | 0     | 0 | 0 | 1 | 0 | 1 |   |   |   |   |   |     |     |     |     |     |     |
| 7     | 0     | 0 | 0 | 3 | 2 | 0 | 2 | 3 |   |   |   |     |     |     |     |     |     |
| 9     | 0     | 0 | 0 | 5 | 6 | 3 | 0 | 3 | 6 | 5 |   |     |     |     |     |     |     |
| 11    | 0     | 0 | 0 | 7 | 10| 9 | 4 | 0 | 9 | 10| 7 |   |     |     |     |     |     |
| 13    | 0     | 0 | 0 | 9 | 14| 15| 12| 5 | 0 | 5 | 12| 15| 14| 9 |   |     |     |
| 15    | 0     | 0 | 0 | 11| 18| 21| 20| 15| 6 | 0 | 6 | 15| 20| 21| 18| 11 |   |
| 17    | 0     | 0 | 0 | 13| 22| 27| 28| 25| 18| 7 | 0 | 7 | 18| 25| 28| 27| 22| 13 |

**Table 1.** Up to a sign, \( H_{rn+i}(F(x,r)) \) is a polynomial in \( n \), with the corresponding degree given in the table. (The degrees in each row are palindromic centering at the bold faced numbers.) For instance, the entry in the row indexed by \( (r =)11 \) and the column indexed by \( (i =)4 \) is 9. This means the degree in \( n \) of \( \pm H_{11n+4}(F(x,11)) \) is 9.

From Table 1, we guess a formula for the degrees. This leads to Conjecture 9. If the conjecture holds true, then we can use Lagrange’s interpolation formula to obtain the formula of \( H_{rn+j}(F(x,r)) \). For instance, if \( r = 31 \), then the max degree is 190 attained at \( j = 11 \). Thus to obtain a formula for \( H_{31n+11}(F(x,31)) \), it is sufficient to evaluate these determinants for \( 0 \leq n \leq 190 \). For \( n = 190 \), we need to evaluate a determinant of size 5901. At the beginning, we thought Maple can compute these determinants quickly, but it is not the case, even if we modulo a large prime. The computation time is given in Table 3.5, where we can see that the computation of \( H_{800}(F(x,31)) \) (mod 274177) already takes about 117 seconds.

| \( r \) | \( n \) | 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 |
|-------|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| 1     |       | 0.093| 0.639| 2.652| 7.160| 16.458| 32.479| 59.639| 103.615|
| 6     |       | 0.093| 0.655| 2.558| 7.160| 16.411| 31.995| 60.949| 103.584|
| 11    |       | 0.093| 0.655| 2.542| 7.082| 16.337| 31.793| 57.860| 97.485 |
| 16    |       | 0.093| 0.655| 2.698| 7.269| 16.551| 32.198| 58.843| 97.001 |
| 21    |       | 0.093| 0.639| 2.636| 7.098| 16.348| 31.715| 57.813| 99.252 |
| 26    |       | 0.093| 0.639| 2.605| 7.051| 16.333| 31.793| 58.188| 99.060 |
| 31    |       | 0.093| 0.655| 2.558| 7.441| 16.567| 32.229| 58.016| 98.124 |
| 36    |       | 0.078| 0.670| 2.558| 7.207| 16.395| 32.105| 58.328| 99.419 |
| 41    |       | 0.093| 0.670| 2.636| 7.316| 16.239| 31.917| 57.455| 96.751 |

**Table 2.** Time (in seconds) spent for calculating \( H_n(F(x,r)) \) (mod 274177).

Indeed, fast evaluation of \( H_{rn+i}(F(x,r)) \) is still by using our transformation \( \tau \). For instance when \( r = 31 \), we start with the functional equation of \( F_0 = F(x,31) \) and successively compute \( F_i = \tau(F_{i-1}) \). The computer can produce \( F_i \) for \( i = 1 \) to 6000 in 198.480 seconds. Then we
need to record the \((u_0, d)\)'s for each \(F_i\) and obtain \(H_n(F(x, 31))\) for \(n \leq 6200\) (some of the determinants are 0). Using this approach we are able to verify Conjecture 9 for \(r \leq 41\).

4. The case of even \(r = 2t\)

The computation of the Hankel determinants for even \(r\) is similar, but the formulas have a quite different pattern.

4.1. The results of \(H_n(F(x, 4))\). For \(r = 4\), we have the functional equation

\[
F(x, 4) = \frac{1}{1 - 4x + 2x^2 - x^4 F(x, 4)}.
\]

**Proof of Theorem 2.** We apply Proposition 12 to \(F_0 := F(x, 4)\) by repeatedly using the transformation \(\tau\). This results in a shifted periodic continued fractions of order 2:

\[
F_0(x) \rightarrow F_1^{(1)}(x) \rightarrow F_2^{(1)}(x) \rightarrow F_3^{(1)}(x) = F_1^{(2)}(x) \cdots.
\]

If we define

\[
F_1^{(p)} = \frac{x^2 + 4p(p + 1)x - p(p + 1)}{p^2 - 4px^2 - 2px^2 - p^2 F_1^{(p)}},
\]

Then the results can be summarized as follows:

\[
H_k(F_0) = H_{k-1}(F_1^{(1)}),
\]

and

\[
H_{k-1}(F_1^{(p)}) = \left( -\frac{p + 1}{p} \right)^{k-1} H_{k-2}(F_2^{(p)}),
\]

\[
H_{k-2}(F_2^{(p)}) = \left( -\frac{p}{p + 1} \right)^{k-2} H_{k-3}(F_1^{(p+1)}).
\]

Combining the above formulas gives the recursion

\[
H_{k-1}(F_1^{(p)}) = - \left( \frac{p + 1}{p} \right)^{k-2} H_{k-3}(F_1^{(p+1)}).
\]

Let \(k - 1 = 2n + j\), where \(j = 0, 1\). We then deduce that

\[
H_{2n+j}(F_1^{(1)}) = (-1)^n (n + 1) H_j(F_1^{(n+1)}).
\]

The initial values are

\[
H_0(F_1^{(n+1)}) = 1, \quad H_1(F_1^{(n+1)}) = -\left( \frac{n + 2}{n + 1} \right).
\]

The theorem then follows by the above initial values, (12) and (13). \(\square\)
4.2. The results of \( H_n(F(x, 6)) \). For \( r = 6 \), we have the functional equation

\[
F(x, 6) = \frac{1}{1 - 6x + 9x^2 - 2x^3 - x^4 F(x, 6)}.
\]

Proof of Theorem \([5]\). We apply Proposition \([12]\) to \( F_0 := F(x, 6) \) by repeatedly using the transformation \( \tau \). This results in a shifted periodic continued fractions of order 3:

\[
F_0(x) \xrightarrow{\tau} F_1^{(1)}(x) \xrightarrow{\tau} F_2^{(1)}(x) \xrightarrow{\tau} F_3^{(1)}(x) \xrightarrow{\tau} F_4^{(1)}(x) = F_2^{(1)}(x) \cdots .
\]

We obtain

\[
H_k(F_0) = H_{k-1}(F_1^{(1)}).
\]

For \( p \geq 1 \). Computer experiment suggests us to define

\[
F_1^{(p)} = \left( -\frac{2}{3} \frac{x^4}{(p+1) (p+1) p} - 2 \frac{x^3 (p-1)}{(p+1) p} - \frac{1}{2} \frac{x^2 (2p-3)(6p^2+3p-1)}{(p+1) p} - \frac{4}{3} \frac{x(9p+5)}{(p+1)(2p+1)} + 1 \right) + \left(-\frac{x^3}{3}\right) + \left( \frac{2}{3} \frac{x^2 F_1^{(p)}}{(p+1)(2p+1)} \right).
\]

We shall mention a difficulty in guessing the above formula. At the beginning, we failed because we assumed the coefficients to be polynomials in \( p \), just like the odd \( r \) case, but there is a cancellation for those coefficients. For instance, the expected \( F_7 \) is \(-\frac{1}{12} \frac{4x^4+96x^3+2232x^2+3072x-504}{3x^2 F_7^2-2x^4+768x^2+18x-3} \), but Maple will give the cancelled form \( F_7 = -\frac{1}{3} \frac{x^4+24x^3+558x^2+768x-126}{3x^2 F_7^2-2x^4+357x^2+18x-3} \). After a little thought, we tried rational functions instead and succeed.

Then the results can be summarized as follows:

\[
H_{k-1}(F_1^{(p)}) = \left( -\frac{3}{2} \frac{(p+1)(2p+1)}{p} \right)^{k-1} H_{k-2}(F_2^{(p)}),
\]

\[
H_{k-2}(F_2^{(p)}) = \left( -\frac{4}{9} \frac{1}{(2p+1)^2} \right)^{k-2} H_{k-3}(F_3^{(p)}),
\]

\[
H_{k-3}(F_3^{(p)}) = \left( -\frac{3}{2} \frac{(2p+1)p}{p+1} \right)^{k-3} H_{k-4}(F_1^{(p+1)}).
\]

Combining the above formulas gives the recursion

\[
H_{k-1}(F_1^{(p)}) = -\left( \frac{p+1}{p} \right)^2 H_{k-4}(F_1^{(p+1)}).
\]

Let \( k - 1 = 3n + j \), where \( 0 \leq j < 3 \). We then deduce that

\[
H_{3n+j}(F_1^{(1)}) = (-1)^n (n+1)^2 H_j(F_1^{(n+1)}).
\]

The initial values are

\[
H_0(F_1^{(n+1)}) = 1, \quad H_1(F_1^{(n+1)}) = -\frac{3}{2} \frac{(n+2)(2n+3)}{n+1}, \quad H_2(F_1^{(n+1)}) = -\frac{(n+2)^2}{(n+1)^2}.
\]

By the above initial values, \([14]\) and \([15]\), we obtain the theorem. \(\square\)
4.3. The results of $H_n(F(x, 8))$. For $r = 8$, we have the functional equation

$$F(x, 8) = \frac{1}{1 - 8x + 9x^2 - 2x^3 - x^4}.$$

Proof of Theorem 6. We apply Proposition 12 to $F_0 := F(x, 8)$ by repeatedly using the transformation $\tau$. This results in a shifted periodic continued fractions of order 4:

$$F_0(x) \xrightarrow{\tau} F_1^{(1)}(x) \xrightarrow{\tau} F_2^{(1)}(x) \xrightarrow{\tau} F_3^{(1)}(x) \xrightarrow{\tau} F_4^{(1)}(x) \xrightarrow{\tau} F_5^{(1)}(x) = F_2^{(1)}(x) \cdots.$$

We obtain

$$H_k(F_0) = H_{k-1}(F_1^{(1)}).$$

(16)

For $p \geq 1$. Computer experiment suggests us to define

$$F_1^{(p)} = -\frac{1}{45} (2025 x^6 + 10800 x^5 (p - 1) (p + 1) + 540 x^4 (64 p^2 - 41) (p - 1) (p + 1) + 120 x^3 (1024 p^5 - 1024 p^4 - 944 p^3 + 944 p^2 + 307 p - 172) (p + 1) + x^2 (p + 1)
+ (65536 p^7 - 65536 p^6 - 251904 p^5 + 251904 p^4 + 88464 p^3 - 174864 p^2 - 52621 p + 7396) - 360 x (128 p^2 - 192 p - 101) (2 p + 1) (p + 1) + 90 (2 p + 1) (p + 1) p (64 p^2 - 96 p - 43)) / \left[ (45 F_1^{(p)} x^2 p + 90 x^4 + (-480 p^2 - 240) x^3 + (-512 p^4 + 1240 p^2 + 172) x^2 + 360 x p - 45 p) p \right].$$

Then the results can be summarized as follows:

$$H_{k-1}(F_1^{(p)}) = \left( -\frac{2}{45} \frac{(1 + 2 p) (1 + p) (43 + 96 p - 64 p^2)}{p} \right)^{k-1} H_{k-2}(F_2^{(p)}),$$

$$H_{k-2}(F_2^{(p)}) = \left( -\frac{45}{2} \frac{117 + 224 p + 64 p^2}{(1 + 2 p) (43 + 96 p - 64 p^2)^2} \right)^{k-2} H_{k-3}(F_3^{(p)}),$$

$$H_{k-3}(F_3^{(p)}) = \left( -\frac{45}{2} \frac{43 + 96 p - 64 p^2}{(1 + 2 p) (117 + 224 p + 64 p^2)^2} \right)^{k-3} H_{k-4}(F_4^{(p)}),$$

$$H_{k-4}(F_4^{(p)}) = \left( -\frac{2}{45} \frac{(1 + 2 p) p (117 + 224 p + 64 p^2)}{1 + p} \right)^{k-4} H_{k-5}(F_1^{(p+1)}).$$

Combining the above formulas gives the recursion

$$H_{k-1}(F_1^{(p)}) = \frac{(1 + p)^3}{p^3} H_{k-5}(F_1^{(p+1)}).$$

Let $k - 1 = 4n + j$, where $0 \leq j < 4$. We then deduce that

$$H_{4n+j}(F_1^{(1)}) = (1 + n)^3 H_j(F_1^{(n+1)}).$$

(17)

The initial values are

$$H_0(F_1^{(n+1)}) = 1, \quad H_1(F_1^{(n+1)}) = -\frac{2}{45} \frac{(2 + n) (3 + 2 n) (75 - 32 n - 64 n^2)}{1 + n},$$

$$H_2(F_1^{(n+1)}) = -\frac{2}{45} \frac{(2 + n)^2 (3 + 2 n) (405 + 352 n + 64 n^2)}{(1 + n)^2}, \quad H_3(F_1^{(n+1)}) = \frac{(2 + n)^3}{(1 + n)^4}.$$

Then the theorem follows by the above initial values, (16) and (17). □
4.4. **A summary.** Similarly, we have computed $H_n(f(x, r))$ for even $r$ up to 30, and found a polynomial characterization as in Table 3. This leads to Conjecture 10.

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|---|----|
| 4   | 1 | 1 |   |   |   |   |   |   |   |   |    |
| 6   | 2 | 2 | 3 |   |   |   |   |   |   |   |    |
| 8   | 3 | 3 | 6 | 6 |   |   |   |   |   |   |    |
| 10  | 4 | 4 | 9 | 10| 9 |   |   |   |   |   |    |
| 12  | 5 | 5 | 12| 15| 15| 12|   |   |   |   |    |
| 14  | 6 | 6 | 15| 20| 21| 20| 15|   |   |   |    |
| 16  | 7 | 7 | 18| 25| 28| 25| 18|   |   |   |    |
| 18  | 8 | 8 | 21| 30| 35| 36| 30| 21|   |   |    |
| 20  | 9 | 9 | 24| 35| 42| 45| 42| 35| 24|   |    |
| 22  | 10| 10| 27| 40| 49| 54| 55| 49| 40| 27|    |

Table 3. Up to a sign, $H_{tn+i}(F(x, r))$ is a polynomial in $n$, with the corresponding degree given in the table.

5. **On Hankel determinants** $\det\left(\binom{2i+2j+r}{i+j}\right)_{0 \leq i,j \leq n-1}$

Our main objective here is to evaluate the Hankel determinants $\det\left(\binom{2i+2j+r}{i+j}\right)_{0 \leq i,j \leq n-1}$.

5.1. **Some formulas.** The generating function of $\binom{2n+r}{n}$ is

$$G(x, r) = \sum_{n \geq 0} \binom{2n+r}{n} x^n = \frac{C(x)^r}{1 - 2xC(x)} = \frac{C(x)^r}{\sqrt{1 - 4x}}. \quad (18)$$

The cases $r = 0, 1$ are well-known.

$$H_n(G(x, 0)) = 2^{n-1}, \quad H_n(G(x, 1)) = 1. \quad (19)$$

Egecioglu, Redmond, and Ryavec computed $H_n(G(x, 2))$ and $H_n(G(x, 3))$ and stated some conjectures for $r > 3$. See [13, 14]. We list the formulas for $r \leq 8$ as follows. The readers will find that the patterns differ for even and odd $r$'s.

**Theorem 14.** [L3]

$$H_{3n}(G(x, 3)) = H_{3n+1}(G(x, 3)) = 2n + 1, \quad H_{3n+2}(G(x, 3)) = -4(n + 1).$$

**Theorem 15.**

$$H_{5n}(G(x, 5)) = H_{5n+1}(G(x, 5)) = (2n + 1)^2,$$
$$H_{5n+2}(G(x, 5)) = -\frac{1}{3} \left( 50n^2 + 89n + 39 \right) (2n + 1),$$
$$H_{5n+3}(G(x, 5)) = -16(n + 1)^2,$$
$$H_{5n+4}(G(x, 5)) = \frac{1}{3} \left( 100n^2 + 272n + 183 \right) (n + 1).$$
Theorem 16 (Explicitly conjectured in [13]).
\[ H_{7n}(G(x, 7)) = H_{7n+1}(G(x, 7)) = (2n + 1)^3, \]
\[ H_{7n+2}(G(x, 7)) = \frac{1}{90} (n + 1) (9604 n^3 + 9604 n^2 - 1323 n - 2340)(2n + 1)^2, \]
\[ H_{7n+3}(G(x, 7)) = -\frac{1}{45} (19208 n^3 + 67228 n^2 + 70854 n + 23445) (n + 1)^2 (2n + 1), \]
\[ H_{7n+4}(G(x, 7)) = 64 (n + 1)^3, \]
\[ H_{7n+5}(G(x, 7)) = \frac{1}{45} (n + 1)^2 (38416 n^4 + 153664 n^3 + 208936 n^2 + 103344 n + 9045), \]
\[ H_{7n+6}(G(x, 7)) = -\frac{1}{90} (9604 n^3 + 48020 n^2 + 75509 n + 38110) (3 + 2n)^2 (n + 1). \]

Theorem 17. [14]
\[ H_{2n}(G(x, 2)) = H_{2n+1}(G(x, 2)) = (-1)^n. \]

Theorem 18. [12]
\[ H_{4n}(G(x, 4)) = H_{4n+1}(G(x, 4)) = 1, \]
\[ H_{4n+2}(G(x, 4)) = -H_{4n+3}(G(x, 4)) = -8(n + 1). \]

Theorem 19.
\[ H_{6n}(G(x, 6)) = H_{6n+1}(G(x, 6)) = (-1)^n, \]
\[ H_{6n+2}(G(x, 6)) = (-1)^n (n + 1) (144 n^2 + 72 n - 19), \]
\[ H_{6n+3}(G(x, 6)) = -H_{6n+4}(G(x, 6)) = (-1)^{n+1} 144(n + 1)^2, \]
\[ H_{6n+5}(G(x, 6)) = (-1)^n (n + 1) (144 n^2 + 504 n + 413). \]

Theorem 20.
\[ H_{8n}(G(x, 8)) = H_{8n+1} = 1, \]
\[ H_{8n+2}(G(x, 8)) = -\frac{2}{15} (n + 1) (256 n^2 + 192 n + 15) (256 n^2 + 192 n + 17), \]
\[ H_{8n+3}(G(x, 8)) = \frac{16}{45} (65536 n^4 + 262144 n^3 + 272896 n^2 + 79104 n - 3915) (n + 1)^2, \]
\[ H_{8n+4}(G(x, 8)) = -H_{8n+5}(G(x, 8)) = 4096 (n + 1)^3 = (4 \times 4)^3 (n + 1)^3, \]
\[ H_{8n+6}(G(x, 8)) = \frac{16}{45} (65536 n^4 + 262144 n^3 + 272896 n^2 - 36096 n - 119115) (n + 1)^2, \]
\[ H_{8n+7}(G(x, 8)) = \frac{2}{15} (n + 1) (256 n^2 + 832 n + 655) (256 n^2 + 832 n + 657). \]

For general \( r \), the following partial results were proved in [12].

Theorem 21. [12] Theorems 7.3 and 7.6] For odd positive integer \( r = 2t + 1 \), we have
\[ H_{(2t+1)n}(G(x, r)) = H_{(2t+1)n+1}(G(x, r)) = (2n + 1)^t, \]
\[ H_{(2t+1)n+t+1}(G(x, r)) = (-1)^{(t+1)/2} 4^t (n + 1)^t. \]
For even positive integer \( r = 2t \), we have
\[
H_{2tn}(G(x, r)) = H_{2n+1}(G(x, r)) = (-1)^n t^n,
\]
\[
H_{2n+t}(G(x, r)) = H_{2n+t+1}(G(x, r)) = (-1)^n \binom{2t}{2}(4t)^{t-1}(n + 1)^{t-1}.
\]

Note that this theorem includes \( H_{5n+i}(F(x, 5)) \) for \( 0 \leq i \leq 3 \) as special cases.

Our method works for these Hankel determinants for specific \( r \). Firstly, for odd \( r \), the first several functional equations are as follows.
\[
G(x, 1) = \frac{1}{(1 - 4x)(xG(x, 1) + 1)},
\]
\[
G(x, 3) = \frac{1}{(1 - 4x)(x^3G(x, 3) - (x - 1))},
\]
\[
G(x, 5) = \frac{1}{(1 - 4x)(x^5G(x, 5) + (x^2 - 3x + 1))},
\]
\[
G(x, 7) = \frac{1}{(1 - 4x)(x^7G(x, 7) - (x^3 - 6x^2 + 5x - 1))},
\]
\[
G(x, 9) = \frac{1}{(1 - 4x)(x^9G(x, 9) + (x^4 - 10x^3 + 15x^2 - 7x + 1))},
\]
\[
G(x, 11) = \frac{1}{(1 - 4x)(x^{11}G(x, 11) - (x^5 - 15x^4 + 35x^3 - 28x^2 + 9x - 1))},
\]
\[
G(x, 13) = \frac{1}{(1 - 4x)(x^{13}G(x, 13) + (x^6 - 21x^5 + 70x^4 - 84x^3 + 45x^2 - 11x + 1))}.
\]

For even \( r \), the first several functional equations are as follows.
\[
G(x, 2) = \frac{1}{(1 - 4x)(x^2G(x, 2) + 1)},
\]
\[
G(x, 4) = \frac{1}{(1 - 4x)(x^4G(x, 4) - (2x - 1))},
\]
\[
G(x, 6) = \frac{1}{(1 - 4x)(x^6G(x, 6) + (3x^2 - 4x + 1))},
\]
\[
G(x, 8) = \frac{1}{(1 - 4x)(x^8G(x, 8) - (4x^3 - 10x^2 + 6x - 1))},
\]
\[
G(x, 10) = \frac{1}{(1 - 4x)(x^{10}G(x, 10) + (5x^4 - 20x^3 + 21x^2 - 8x + 1))},
\]
\[
G(x, 12) = \frac{1}{(1 - 4x)(x^{12}G(x, 12) - (6x^5 - 35x^4 + 56x^3 - 36x^2 + 10x - 1))}.
\]

Just like that of \( F(x, r) \), we guess and prove explicit functional equations of \( G(x, r) \) for general \( r \). See Appendix 6.2 for details.

5.2. **Proof for odd** \( r = 2t + 1 \). For \( r = 3 \), we have the functional equation
\[
G(x, 3) = -\frac{1}{(4x^4 - x^3)G(x, 3) - 4x^2 + 5x - 1}.
\]
Then the results can be summarized as follows:

\[
G_0(x) \xrightarrow{\tau} G_1^{(1)}(x) \xrightarrow{\tau} G_2^{(1)}(x) \xrightarrow{\tau} G_3^{(1)}(x) \xrightarrow{\tau} G_4^{(1)}(x) = G_2^{(1)}(x) \cdots .
\]

We apply Proposition 12 to \(G_0 := G(x, 3)\) by repeatedly using the transformation \(\tau\). This results in a shifted periodic continued fractions of order 3:

\[
G_0(x) \xrightarrow{\tau} G_1^{(1)}(x) \xrightarrow{\tau} G_2^{(1)}(x) \xrightarrow{\tau} G_3^{(1)}(x) \xrightarrow{\tau} G_4^{(1)}(x) = G_2^{(1)}(x) \cdots .
\]

We obtain

\[
H_k(G_0) = H_{k-1}(G_1^{(1)}).
\]  \hspace{1cm} (20)

For \(p \geq 1\). Computer experiment suggests us to define

\[
G_1^{(p)} = -\frac{4 x^2 + (18 p + 1) x (2 p - 1) - 4 p (2 p - 1)}{(G_1^{(p)})^2 (2 p - 1) + 4 x^2 + (10 p - 5) x - 2 p + 1}.
\]

Then the results can be summarized as follows:

\[
\begin{align*}
H_{k-1}(G_1^{(p)}) &= \left( -\frac{4 p}{2p - 1} \right)^{k-1} H_{k-2}(G_2^{(p)}), \\
H_{k-2}(G_2^{(p)}) &= \left( \frac{4 p^3 - 1}{16 p^2} \right)^{k-2} H_{k-3}(G_3^{(p)}), \\
H_{k-3}(G_3^{(p)}) &= \left( -\frac{4 p}{2p + 1} \right)^{k-3} H_{k-4}(G_1^{(p+1)}).
\end{align*}
\]

Combining the above formulas gives the recursion

\[
H_{k-1}(G_1^{(p)}) = \frac{2p + 1}{2p - 1} H_{k-4}(G_1^{(p+1)}).
\]

Let \(k - 1 = 3n + j\), where \(0 \leq j < 3\). We then deduce that

\[
H_{3n+j}(G_1^{(1)}) = (2n + 1) H_j(G_1^{(n+1)}).
\]  \hspace{1cm} (21)

The initial values are

\[
H_0(G_1^{(n+1)}) = 1, \quad H_1(G_1^{(n+1)}) = -\frac{4 (n + 1)}{2n + 1}, \quad H_2(G_1^{(n+1)}) = \frac{2n + 3}{2n + 1}.
\]

Then the theorem follows by the above initial values, (20) and (21). \(\square\)

For \(r = 5\), we have the functional equation

\[
G(x, 5) = -\frac{1}{(4 x^6 - x^5) G(x, 5) + 4 x^3 - 13 x^2 + 7 x - 1}.
\]

Proof of Theorem \([13]\) We apply Proposition \([12]\) to \(G_0 := G(x, 5)\) by repeatedly using the transformation \(\tau\). This results in a shifted periodic continued fractions of order 5:

\[
G_0(x) \xrightarrow{\tau} G_1^{(1)}(x) \xrightarrow{\tau} G_2^{(1)}(x) \xrightarrow{\tau} G_3^{(1)}(x) \xrightarrow{\tau} G_4^{(1)}(x) \xrightarrow{\tau} G_5^{(1)}(x) = G_1^{(2)}(x) \cdots .
\]

We obtain

\[
H_k(G_0) = H_{k-1}(G_1^{(1)}).
\]  \hspace{1cm} (22)

For \(p \geq 1\). Computer experiment suggests us to define

\[
G_1^{(p)} = -\frac{1}{3} \frac{36 x^4 + p_2(p) x^3 + p_4(p) x^2 + p_3(p) x - p_3(p)}{(6 p - 3) x^2 G_1^{(p)} - 12 x^3 + p_2(p) x^2 + (42 p - 21) x - 6 p + 3}.
\]
Then the theorem follows by the above initial values, (22) and (23). □

The initial values are

\[
H_{k-1}(G_1^{(p)}) = \left(-\frac{1}{3} \frac{p(50p - 11)}{2p - 1}\right)^{k-1} H_{k-2}(G_2^{(p)}),
\]

\[
H_{k-2}(G_2^{(p)}) = \left(-\frac{144}{2500 p^2 - 1100 p + 121}\right)^{k-2} H_{k-3}(G_3^{(p)}),
\]

\[
H_{k-3}(G_3^{(p)}) = \left(-\frac{1}{2304} \frac{100 p^2 + 72 p + 11}{p^2}\right) H_{k-4}(G_4^{(p)}),
\]

\[
H_{k-4}(G_4^{(p)}) = \left(-\frac{144}{2500 p^2 + 1100 p + 121}\right)^{k-4} H_{k-5}(G_5^{(p)}),
\]

\[
H_{k-5}(G_5^{(p)}) = \left(-\frac{1}{3} \frac{50 p + 11}{2p + 1}\right)^{k-5} H_{k-6}(G_1^{(p+1)}).
\]

Combining the above formulas gives the recursion

\[
H_{k-1}(G_1^{(p)}) = \left(\frac{2p + 1}{2p - 1}\right)^2 H_{k-6}(G_1^{(p+1)}).
\]

Note: Here and what follows, we use \(P_d(p)\) to denote a polynomial in \(p\) of degree \(d\). The contents of these polynomials are not important enough to be printed. Then the results can be summarized as follows:

Let \(k \in \mathbb{N}\) be summarized as follows:

\[
\begin{align*}
H_{k-1}(G_1^{(p)}) &= \left(-\frac{1}{3} \frac{p(50p - 11)}{2p - 1}\right)^{k-1} H_{k-2}(G_2^{(p)}), \\
H_{k-2}(G_2^{(p)}) &= \left(-\frac{144}{2500 p^2 - 1100 p + 121}\right)^{k-2} H_{k-3}(G_3^{(p)}), \\
H_{k-3}(G_3^{(p)}) &= \left(-\frac{1}{2304} \frac{100 p^2 + 72 p + 11}{p^2}\right) H_{k-4}(G_4^{(p)}), \\
H_{k-4}(G_4^{(p)}) &= \left(-\frac{144}{2500 p^2 + 1100 p + 121}\right)^{k-4} H_{k-5}(G_5^{(p)}), \\
H_{k-5}(G_5^{(p)}) &= \left(-\frac{1}{3} \frac{50 p + 11}{2p + 1}\right)^{k-5} H_{k-6}(G_1^{(p+1)}).
\end{align*}
\]

Combining the above formulas gives the recursion

\[
H_{k-1}(G_1^{(p)}) = \left(\frac{2p + 1}{2p - 1}\right)^2 H_{k-6}(G_1^{(p+1)}).
\]

Let \(k - 1 = 5n + j\), where \(0 \leq j < 5\). We then deduce that

\[
H_{5n+j}(G_1^{(1)}) = (2n + 1)^2 H_j(G_1^{(n+1)}). \quad (23)
\]

The initial values are

\[
\begin{align*}
H_0(G_1^{(n+1)}) &= 1, & H_1(G_1^{(n+1)}) &= -\frac{1}{3} \frac{50 n^2 + 89 n + 39}{2n + 1}, & H_2(G_1^{(n+1)}) &= -16 \frac{(n + 1)^2}{(2n + 1)^2}, \\
H_3(G_1^{(n+1)}) &= \frac{1}{3} \frac{(n + 1)(100 n^2 + 272 n + 183)}{(2n + 1)^2}, & H_4(G_1^{(n+1)}) &= \frac{(2n + 3)^2}{(2n + 1)^2}.
\end{align*}
\]

Then the theorem follows by the above initial values, (22) and (23).

For \(r = 7\), we have the functional equation

\[
G(x, 7) = \frac{1}{(1 - 4 x) (x^7 G(x, 7) - (x^3 - 6 x^2 + 5 x - 1))}.
\]

Proof of Theorem 16: We apply Proposition 12 to \(G_0 := G(x, 7)\) by repeatedly using the transformation \(\tau\). This results in a shifted periodic continued fractions of order 7:

\[
\begin{align*}
G_0(x) &\xrightarrow{\tau} G_1^{(1)}(x) \xrightarrow{\tau} G_2^{(1)}(x) \xrightarrow{\tau} G_3^{(1)}(x) \xrightarrow{\tau} G_4^{(1)}(x) \xrightarrow{\tau} G_5^{(1)}(x) \\
&\xrightarrow{\tau} G_6^{(1)}(x) \xrightarrow{\tau} G_7^{(1)}(x) \xrightarrow{\tau} G_8^{(1)}(x) = G_1^{(2)}(x) \cdots.
\end{align*}
\]

We obtain

\[
H_k(G_0) = H_{k-1}(G_1^{(1)}). \quad (24)
\]
For $p \geq 1$. Computer experiment suggests us to define

$$G_1^{(p)} = -\frac{1}{180} \frac{32400 x^6 + \mathcal{P}_2(p) x^5 + \mathcal{P}_4(p) x^4 + \mathcal{P}_6(p) x^3 + \mathcal{P}_8(p) x^2 - \mathcal{P}_5(p) x + \mathcal{P}_5(p)}{(90p - 45)x^2G_1^{(p)} + 180 x^4 + \mathcal{P}_2(p) x^3 + \mathcal{P}_4(p) x^2 + \mathcal{P}_1(p) x - 90p + 45} (2p - 1).$$

Then the results can be summarized as follows:

$$H_{k-1}(G_1^{(p)}) = \left(\frac{1}{90} \frac{p (9604 p^3 - 19208 p^2 + 8281 p - 1017)}{2p - 1}\right)^{k-1} H_{k-2}(G_2^{(p)}),$$

$$H_{k-2}(G_2^{(p)}) = \left(-180 \frac{19208 p^3 + 9604 p^2 - 5978 p + 611}{(9604 p^3 - 19208 p^2 + 8281 p - 1017)^2}\right)^{k-2} H_{k-3}(G_3^{(p)}),$$

$$H_{k-3}(G_3^{(p)}) = \left(\frac{1}{1440} \frac{9604 p^3 - 19208 p^2 + 8281 p - 1017}{(19208 p^3 + 9604 p^2 - 5978 p + 611)^2}\right)^{k-3} H_{k-4}(G_4^{(p)}),$$

$$H_{k-4}(G_4^{(p)}) = \left(-\frac{1}{(2)^4 (3)^4 (5)^2} (2p + 1) \frac{19208 p^3 - 9604 p^2 - 5978 p - 611}{(2p - 1)} (2p - 1)\right)^{k-4} H_{k-5}(G_5^{(p)}),$$

$$H_{k-5}(G_5^{(p)}) = \left(-\frac{1440}{180} \frac{9604 p^3 + 19208 p^2 + 8281 p + 1017}{(19208 p^3 - 9604 p^2 - 5978 p - 611)^2}\right)^{k-5} H_{k-6}(G_6^{(p)}),$$

$$H_{k-6}(G_6^{(p)}) = \left(\frac{1}{180} \frac{19208 p^3 - 9604 p^2 - 5978 p - 611}{(19208 p^3 + 19208 p^2 + 8281 p + 1017)^2}\right)^{k-6} H_{k-7}(G_7^{(p)}),$$

$$H_{k-7}(G_7^{(p)}) = \left(-\frac{1}{90} \frac{p (9604 p^3 + 19208 p^2 + 8281 p + 1017)}{2p + 1}\right)^{k-7} H_{k-8}(G_8^{(p+1)}).$$

Combining the above formulas gives the recursion

$$H_{k-1}(G_1^{(p)}) = \left(\frac{2p + 1}{2p - 1}\right)^3 H_{k-8}(G_1^{(p+1)}).$$

Let $k - 1 = 7n + j$, where $0 \leq j < 7$. We then deduce that

$$H_{7n+j}(G_1^{(1)}) = (2n+1)^3 H_j(G_1^{(n+1)}).$$

The initial values are

$$H_0(G_1^{(n+1)}) = 1, \quad H_1(G_1^{(n+1)}) = \frac{1}{90} \frac{(n+1) (9604 n^3 + 9604 n^2 - 1323 n - 2340)}{2n + 1},$$

$$H_2(G_1^{(n+1)}) = -\frac{1}{45} \frac{(19208 n^3 + 67228 n^2 + 70854 n + 23445) (n + 1)^2}{(2n + 1)^2},$$

$$H_3(G_1^{(n+1)}) = 64 \frac{(n + 1)^3}{(2n + 1)^3},$$

$$H_4(G_1^{(n+1)}) = \frac{1}{45} \frac{(n + 1)^2 (38416 n^4 + 153664 n^3 + 208936 n^2 + 103344 n + 9045)}{(2n + 1)^3},$$

$$H_5(G_1^{(n+1)}) = -\frac{1}{90} \frac{(9604 n^3 + 48020 n^2 + 75509 n + 38110) (3 + 2n)^2 (n + 1)}{(2n + 1)^3}.$$
Then the theorem follows by the above initial values, (24) and (26). □

5.3. The result of $H_n(G(x, r))$ for $r = 2t$. For $r = 2$, we have the functional equation

$$G(x, 2) = \frac{1}{(1 - 4x)(x^2G(2x) + 1)}.$$ 

Proof of Theorem 17. Let $G_0 := G(x, 2)$, we repeatedly apply Proposition 12 to obtain the following periodic continued fractions.

$$H_k(G_0) = H_{k-1}(G_1)$$
$$H_{k-1}(G_1) = -H_{k-2}(G_2)$$

Now we see that $G_2 = G_0$. So the continued fractions is periodic of order 2: $G_0 \rightarrow G_1 \rightarrow G_0$. By summarizing the above results, we obtain the recursion $H_k(F_0) = -H_{k-2}(F_0)$, which implies that $H_{2n+i}(G_0) = (-1)^n H_i(G_0)$, for $i = 0, 1$. The initial values are $H_0(G_0) = 1$, $H_1(G_0) = 1$. Then we can get $H_n(G(x, 2))$. □

For $r = 4$, we have the functional equation

$$G(x, 4) = -\frac{1}{(4x^5 - x^4) G(x, 4) - 8x^2 + 6x - 1}.$$ 

Proof of Theorem 18. We apply Proposition 12 to $G_0 := G(x, 4)$ by repeatedly using the transformation $\tau$. This results in a shifted periodic continued fractions of order 4:

$$G_0(x) \xrightarrow{\tau} G_1^{(1)}(x) \xrightarrow{\tau} G_2^{(1)}(x) \xrightarrow{\tau} G_3^{(1)}(x) \xrightarrow{\tau} G_4^{(1)}(x) = G_1^{(2)}(x) \cdots.$$ 

We obtain

$$H_k(G_0) = H_{k-1}(G_1^{(1)}). \quad (27)$$

For $p \geq 1$. Computer experiment suggests us to define

$$G_1^{(p)} = -\frac{4x^3 + (64p^2 - 64p - 1)x^2 + 48px - 8p}{x^2G_1^{(p)} - 1 + (16p - 8)x^2 + 6x}.$$ 

Then the results can be summarized as follows:

$$H_{k-1}(G_1^{(p)}) = (-8p)^{k-1} H_{k-2}(G_2^{(p)}),$$
$$H_{k-2}(G_2^{(p)}) = \left(\frac{1}{8p}\right)^{k-2} H_{k-3}(G_4^{(p)}),$$
$$H_{k-3}(G_3^{(p)}) = \left(-\frac{1}{8p}\right)^{k-3} H_{k-4}(G_4^{(p)}),$$
$$H_{k-4}(G_4^{(p)}) = (8p)^{k-4} H_{k-5}(G_1^{(p+1)}).$$
Combining the above formulas gives the recursion

\[ H_{k-1}(G_1^{(p)}) = H_{k-5}(G_1^{(p+1)}). \]

Let \( k - 1 = 4n + j \), where \( 0 \leq j < 4 \). We then deduce that

\[ H_{4n+j}(G_1^{(1)}) = H_j(G_1^{(n+1)}). \]

The initial values are

\[ H_0(G_1^{(n+1)}) = 1, \quad H_1(G_1^{(n+1)}) = -8(n + 1), \quad H_2(G_1^{(n+1)}) = 8(n + 1), \quad H_3(G_1^{(n+1)}) = 1. \]

Then the theorem follows by the above initial values, (27) and (28). \( \square \)

For \( r = 6 \), we have the functional equation

\[ G(x, 6) = \frac{1}{(4x^7 - x^6)G(x, 6) + 4x^3 - 13x^2 + 7x - 1}. \]

**Proof of Theorem 19.** We apply Proposition 12 to \( G_0 := G(x, 6) \) by repeatedly using the transformation \( \tau \). This results in a shifted periodic continued fractions of order 6:

\[ G_0(x) \xrightarrow{\tau} G_1^{(1)}(x) \xrightarrow{\tau} G_2^{(1)}(x) \xrightarrow{\tau} G_3^{(1)}(x) \xrightarrow{\tau} G_4^{(1)}(x) \xrightarrow{\tau} G_5^{(1)}(x) \xrightarrow{\tau} G_6^{(1)}(x) \]

\[ \xrightarrow{\tau} G_7^{(1)}(x) = G_1^{(2)}(x) \cdots . \]

We obtain

\[ H_k(G_0) = H_{k-1}(G_1^{(1)}). \]

For \( p \geq 1 \). Computer experiment suggests us to define

\[ G_1^{(p)} = -\frac{4x^5 + p_2(p)x^4 + p_4(p)x^3 + p_6(p)x^2 - p_3(p)x + p_3(p)}{G_1^{(p)}x^2 - 1 + p_1(p)x^3 - p_3(p)x^2 + 8x}. \]

Then the results can be summarized as follows.

\[ H_{k-1}(G_1^{(p)}) = (p(144p^2 - 216p + 53))^{k-1} H_{k-2}(G_2^{(p)}), \]

\[ H_{k-2}(G_2^{(p)}) = \left(-\frac{144}{(144p^2 - 216p + 53)^2}\right)^{k-2} H_{k-3}(G_3^{(p)}), \]

\[ H_{k-3}(G_3^{(p)}) = \left(\frac{1}{144} \frac{144p^2 - 216p + 53}{p}\right)^{k-3} H_{k-4}(G_4^{(p)}), \]

\[ H_{k-4}(G_4^{(p)}) = \left(-\frac{1}{144} \frac{144p^2 + 216p + 53}{p}\right)^{k-4} H_{k-5}(G_5^{(p)}), \]

\[ H_{k-5}(G_5^{(p)}) = \left(-\frac{144}{(144p^2 + 216p + 53)^2}\right)^{k-5} H_{k-6}(G_6^{(p+1)}), \]

\[ H_{k-6}(G_6^{(p)}) = (-p(144p^2 + 216p + 53))^{k-6} H_{k-7}(G_7^{(p+1)}). \]

Combining the above formulas gives the recursion

\[ H_{k-1}(G_1^{(p)}) = -H_{k-7}(G_1^{(p+1)}). \]
Let \( k - 1 = 6n + j \), where \( 0 \leq j < 6 \). We then deduce that
\[
H_{6n+j}(G_1^{(1)}) = (-1)^n H_j(G_1^{(n+1)}).
\] (30)

The initial values are
\[
\begin{align*}
H_0(G_1^{(n+1)}) &= 1, \quad H_1(G_1^{(n+1)}) = (n + 1) \left(144n^2 + 72n - 19\right), \quad H_2(G_1^{(n+1)}) = -144(n + 1)^2, \\
H_3(G_1^{(n+1)}) &= 144(n + 1)^2, \quad H_4(G_1^{(n+1)}) = (n + 1) \left(144n^2 + 504n + 413\right), \quad H_5(G_1^{(n+1)}) = -1.
\end{align*}
\]

Then the theorem follows by the above initial values, (29) and (30).
\[\square\]

For \( r = 8 \), we have the functional equation
\[
G(x, 8) = -\frac{1}{(4x^9 - x^8)G(x, 8) + 4x^3 - 13x^2 + 7x - 1}.
\]

**Proof of Theorem 20** We apply Proposition 12 to \( G_0 := G(x, 8) \) by repeatedly using the transformation \( \tau \). This results in a shifted periodic continued fractions of order 8:
\[
\begin{align*}
G_0(x) &\xrightarrow{\tau} G_1^{(1)}(x) \xrightarrow{\tau} G_2^{(1)}(x) \xrightarrow{\tau} G_3^{(1)}(x) \xrightarrow{\tau} G_4^{(1)}(x) \xrightarrow{\tau} G_5^{(1)}(x) \xrightarrow{\tau} G_6^{(1)}(x) \xrightarrow{\tau} G_7^{(1)}(x) \xrightarrow{\tau} G_8^{(1)}(x) \xrightarrow{\tau} G_9^{(1)}(x) = G_1^{(2)}(x) \cdots .
\end{align*}
\]

We obtain
\[
H_k(G_0) = H_{k-1}(G_1^{(1)}).
\] (31)

For \( p \geq 1 \). Computer experiment suggests us to define
\[
G_1^{(p)} = -\frac{900x^7 + \mathcal{P}_2(p)x^6 + \mathcal{P}_4(p)x^5 + \mathcal{P}_6(p)x^4 + \mathcal{P}_8(p)x^3 + \mathcal{P}_{10}(p)x^2 + \mathcal{P}_5(p)x - \mathcal{P}_3(p)}{15G_1^{(p)}x^2 - 15 + \mathcal{P}_1(p)x^4 + \mathcal{P}_3(p)x^3 + \mathcal{P}_5(p)x^2 + 150x}.
\]

Then the results can be summarized as follows:
\[
\begin{align*}
H_{k-1}(G_1^{(p)}) &= \left(-\frac{2}{15} p \left(256p^2 - 320p + 79\right) \left(256p^2 - 320p + 81\right)\right)^{k-1} H_{k-2}(G_2^{(p)}), \\
H_{k-2}(G_2^{(p)}) &= \left(20 \frac{65536p^4 - 120320p^2 + 57600p - 6731}{(256p^2 - 320p + 79)^2 (256p^2 - 320p + 81)^2}\right)^{k-2} H_{k-3}(G_3^{(p)}), \\
H_{k-3}(G_3^{(p)}) &= \left(-4320 \frac{256p^2 - 320p + 79}{(65536p^4 - 120320p^2 + 57600p - 6731)^2}\right)^{k-3} H_{k-4}(G_4^{(p)}), \\
H_{k-4}(G_4^{(p)}) &= \left(-\frac{1}{11520} \frac{65536p^4 - 120320p^2 + 57600p - 6731}{p}\right)^{k-4} H_{k-5}(G_5^{(p)}), \\
H_{k-5}(G_5^{(p)}) &= \left(\frac{1}{11520} \frac{65536p^4 - 120320p^2 - 57600p - 6731}{p}\right)^{k-5} H_{k-6}(G_6^{(p)}), \\
H_{k-6}(G_6^{(p)}) &= \left(-4320 \frac{256p^2 + 320p + 79}{(65536p^4 - 120320p^2 - 57600p - 6731)^2}\right)^{k-6} H_{k-7}(G_7^{(p)}), \\
H_{k-7}(G_7^{(p)}) &= \left(20 \frac{65536p^4 - 120320p^2 - 57600p - 6731}{(256p^2 + 320p + 79)^2 (256p^2 + 320p + 79)^2}\right)^{k-7} H_{k-8}(G_8^{(p)}),
\end{align*}
\]
Then the theorem follows by the above initial values, (31) and (32).

For odd positive integer

\[ r \]

Conjecture 22.

Let \( k \) be an even positive integer. The initial values are

\[ H_0(G_1(n)) = 1, \quad H_1(G_1(n)) = -\frac{2}{15} (n + 1) (256 n^2 + 192 n + 15) (256 n^2 + 192 n + 17), \]

\[ H_2(G_1(n)) = \frac{16}{45} (65536 n^4 + 262144 n^3 + 272896 n^2 + 79104 n - 3915) (n + 1)^2, \]

\[ H_3(G_1(n)) = 4096 (n + 1)^3, \quad H_4(G_1(n)) = -4096 (n + 1)^3, \]

\[ H_5(G_1(n)) = \frac{16}{45} (65536 n^4 + 262144 n^3 + 272896 n^2 - 36096 n - 119115) (n + 1)^2, \]

\[ H_6(G_1(n)) = \frac{2}{15} (n + 1) (256 n^2 + 832 n + 655) (256 n^2 + 832 n + 657), \quad H_7(G_1(n)) = 1. \]

Then the theorem follows by the above initial values, (31) and (32).

We arrive at the following conjecture.

Conjecture 22. For odd positive integer \( r \) with \( r = 2t + 1 \), we have, for \( 1 \leq j \leq \left\lfloor \frac{r+3}{4} \right\rfloor \),

\[ H_{rn+j}(G(x, r)), H_{rn+t+j}(G(x, r)) \text{ and } H_{rn+t+2-j}(G(x, r)), H_{rn+r+1-j}(G(x, r)) \]

are all polynomials in \( n \) of degree \( (2j-1)(t+1-j) \).

For even positive integer \( r \) with \( r = 2t \), we have, for \( 1 \leq j \leq t \), \((-1)^t H_{rn+j}(G(x, r))\) and \((-1)^t H_{rn+r+1-j}(G(x, r))\) are both polynomials in \( n \) of degree \( (j-1)(r+1-2j) \).

Currently, the conjecture is not hard to be verified for \( r \leq 26 \) by our method.

6. Appendix: Proof of functional equations

6.1. Functional equations of \( F(x, r) = C(x)^r \). We have the following result.

Proposition 23. Let \( F(x, r) = C(x)^r \). For odd positive integer \( r = 2t + 1 \), we have

\[ F(x, 2t + 1) = -\frac{1}{\left( \sum_{i=0}^{t} (-1)^{i+1} \frac{(2t+1)(2t+1)}{2t-2i+1} x^i \right)} + x^{2t+1} F(x, r). \]

For even positive integer \( r = 2t \), we have

\[ F(x, 2t) = -\frac{1}{\left( \sum_{i=0}^{t} (-1)^{i+1} \frac{(2t)(2t-1)}{2t-i} \cdot x^i \right)} + x^t F(x, r). \]
Proof. We find it sufficient to prove the following Lemma 24. We only prove the odd case formula (33). The even case formula (34) is similar.

By setting $y = -C(x)$ in (35) below, we obtain

$$-(C(x) - 1)^{2t+1} - 1 = -\sum_{i=0}^{t} (-1)^i \frac{2t+1}{2t-2i+1} \binom{2t-i}{i} (C(x) - 1)^i C(x)^{2t-2i+1},$$

$$(xC(x)^2)^{2t+1} + 1 = \sum_{i=0}^{t} (-1)^i \frac{2t+1}{2t-2i+1} \binom{2t-i}{i} (xC(x)^2)^i C(x)^{2t-2i+1},$$

$$x^{2t+1} F(x, 2t+1)^2 + 1 = \sum_{i=0}^{t} (-1)^i \frac{2t+1}{2t-2i+1} \binom{2t-i}{i} x^i F(x, 2t+1).$$

This is clearly equivalent to (33). 

\[\square\]

**Lemma 24.** For odd positive integer $r = 2t+1$, we have

$$(y+1)^{2t+1} - 1 = \sum_{i=0}^{t} \frac{2t+1}{2t-2i+1} \binom{2t-i}{i} (y+1)^i y^{2t-2i+1}.$$  

(35)

For even positive integer $r = 2t$, we have

$$(y+1)^{2t} + 1 = \sum_{i=0}^{t} \frac{2t+1}{2t-2i+1} \binom{2t-i}{i} (y+1)^i y^{2t-2i}.$$  

(36)

**Remark 25.** The coefficients $\frac{2t+1}{2t-2i+1} \binom{2t-i}{i}$ are positive integers, since it is easy to verify that

$$\frac{2t+1}{2t-2i+1} \binom{2t-i}{i} = \binom{2t-i}{i} + 2 \binom{2t-i}{i-1}.$$ 

**Proof of Lemma 24.** By making the substitution $y \to y-1$, (35) becomes

$$y^{2t+1} - 1 = \sum_{i=0}^{t} \frac{2t+1}{2t-2i+1} \binom{2t-i}{i} y^i (y-1)^{2t-2i+1}
= \sum_{i=0}^{t} \frac{2t+1}{2t-2i+1} \binom{2t-i}{i} y^i \sum_{j=0}^{2t-2i+1} (-1)^{2t-2i+1-j} \binom{2t-2i+1}{j} y^j
= \sum_{m=0}^{2t+1} y^m \sum_{i+j=m, 0 \leq i \leq t, 0 \leq j \leq 2t+1-2i} (-1)^{j-1} \frac{2t+1}{2t-2i+1} \binom{2t-i}{i} \binom{2t-2i+1}{j}
= \sum_{m=0}^{2t+1} y^m \sum_{0 \leq i \leq \min(t,m,2t+1-m)} (-1)^{m-i} \frac{2t+1}{2t-2i+1} \binom{2t-i}{i} \binom{2t-2i+1}{m-i}.$$ 

By equating coefficients, we need to show that

$$Z(t, m) = 0, \quad \text{for } 1 \leq m \leq 2t \quad \text{and } Z(t, 0) = -1, \ Z(t, 2t+1) = 1,$$ 

(37)

$$Z(t, m) = \sum_{0 \leq i \leq \min(t,m,2t+1-m)} (-1)^{m-i} \frac{2t+1}{2t-2i+1} \binom{2t-i}{i} \binom{2t-2i+1}{m-i}.$$ 

(38)
Equation (37) is a hypergeometric sum identity. By using the EKHAD Maple package of Zeilberger’s creative telescoping [31], we find

\[(2t + 1)(m - 2t - 3)Z(t + 1, m) - (m^2 - 2mt - 3m + 2t + 3)(m - 2t - 1)Z(t, m) = 0.\]  

(39)

Now we are ready to prove (37) by induction on \( t \).

The boundary case are easy to verify:

\[Z(t, 0)_{i=0, m=0} = (-1)^{-1} \frac{2t + 1}{2t + 1} \binom{2t}{0} \binom{2t + 1}{0} = -1,\]

\[Z(t, 2t + 1)_{i=0, m=2t+1} = (-1)^{2t} \frac{2t + 1}{2t + 1} \binom{2t}{2t + 1} \binom{2t + 1}{0} = 1.\]

The base case when \( t = 1 \) is easy, so we omit it. Assume (37) holds for \( t \) by the induction hypothesis, i.e., \( Z(t, m) = 0 \) for \( 1 \leq m \leq 2t \). We need to show that \( Z(t + 1, m) = 0 \) for \( 1 \leq m \leq 2t + 2 \).

The case \( 1 \leq m \leq 2t \) follows from (39) and \( Z(t, m) = 0 \). The case \( m = 2t + 1 \) directly follows from (39). The case \( m = 2t + 2 \) follows by direct computation:

\[Z(t + 1, 2t + 2) = (-1)^{2t+1} \frac{2t + 3}{2t + 3} \binom{2t + 2}{0} \binom{2t + 3}{2t + 2} + (-1)^{2t+2} \frac{2t + 3}{2t + 1} \binom{2t + 1}{1} \binom{2t + 1}{2t + 1} = 0.\]

This completes the proof. \( \square \)

6.2. Functional equations of \( G(x, r) = \frac{C(x)^r}{1 - 2xC(x)} \). We have the following result.

Proposition 26. Let \( G(x, r) = \frac{C(x)^r}{1 - 2xC(x)} = \frac{C(x)^r}{\sqrt{1 - 4x}} \). For odd integer \( r = 2t + 1 \), we have

\[G(x, 2t + 1) = \frac{1}{1 - 4x} \cdot x^{2t+1} G(x, r) + \frac{1}{1 - 4x} \cdot \sum_{i=0}^{t} (-1)^{i} \binom{2t-i}{i} x^i.\]  

(40)

For even integer \( r = 2t \), we have

\[G(x, 2t) = \frac{1}{1 - 4x} \cdot x^{2t} G(x, r) + \frac{1}{1 - 4x} \cdot \sum_{i=0}^{t-1} (-1)^{i} \binom{2t-i-1}{i} x^i.\]  

(41)

Proof. The proof is similar to that of Proposition 23. We also only prove the odd case formula (40).

We find it sufficient to prove the following Lemma 27.
By setting \( y = -C(x) \) in (42), we obtain

\[
(1 - C)^{2t+1} + 1 = \sum_{i=0}^{t} \binom{2t - i}{i} (1 - C)^i C^{2t+1-2i} + 2 \sum_{i=0}^{t} \binom{2t - i}{i} (1 - C)^{i+1} C^{2t-2i}.
\]

\[
-(xC^2)^{2t+1} + 1 = \sum_{i=0}^{t} \binom{2t - i}{i} (-1)^i (xC^2)^i C^{2t+1-2i} + 2 \sum_{i=0}^{t} \binom{2t - i}{i} (-1)^{i+1} (xC^2)^{i+1} C^{2t-2i},
\]

\[
1 - x^{2t+1} C^{4t+2} = \sum_{i=0}^{t} \binom{2t - i}{i} (-1)^i x^i C^{2t+1} - 2 \sum_{i=0}^{t} \binom{2t - i}{i} (-1)^i x^{i+1} C^{2t+2},
\]

\[
= \sum_{i=0}^{t} (-1)^i \binom{2t - i}{i} x^i C^{2t+1}(1 - 2xC).
\]

By dividing both sides of the equation by \((1 - 4x)\), we obtain:

\[
\frac{1}{1 - 4x} - x^{2t+1} \left( \frac{C^{2t+1}}{\sqrt{1 - 4x}} \right)^2 = \sum_{i=0}^{t} (-1)^i \binom{2t - i}{i} x^i \frac{C^{2t+1}}{\sqrt{1 - 4x}},
\]

\[
\Leftrightarrow \frac{1}{1 - 4x} - x^{2t+1} G(x, 2t + 1)^2 = \sum_{i=0}^{t} (-1)^i \binom{2t - i}{i} x^i G(x, 2t + 1).
\]

This is clearly equivalent to (40). \( \square \)

**Lemma 27.** For odd positive integer \( r = 2t + 1 \), we have

\[
(y + 1)^{2t+1} + 1 = - \sum_{i=0}^{t} \binom{2t - i}{i} (1 + y)^i y^{2t+1-2i} + 2 \sum_{i=0}^{t} \binom{2t - i}{i} (1 + y)^{i+1} y^{2t-2i}.
\]

(42)

For even positive integer \( r = 2t \), we have

\[
(y + 1)^{2t} - 1 = - \sum_{i=0}^{t-1} \binom{2t - i - 1}{i} (1 + y)^i y^{2t-2i} + 2 \sum_{i=0}^{t-1} \binom{2t - i - 1}{i} (1 + y)^{i+1} y^{2t-2i-1}.
\]

(43)

**Proof.** By making the substitution \( y \to y - 1 \), (42) becomes

\[
y^{2t+1} + 1 = - \sum_{i=0}^{t} \binom{2t - i}{i} y^i (y - 1)^i y^{2t+1-2i} + 2 \sum_{i=0}^{t} \binom{2t - i}{i} y^{i+1} (y - 1)^{2t-2i}
\]

\[
= \sum_{i=0}^{t} \binom{2t - i}{i} y^i \sum_{j=0}^{2t+1-2i} (-1)^j \binom{2t + 1 - 2i}{j} y^j + 2 \sum_{i=0}^{t} \binom{2t - i}{i} y^{i+1} \sum_{j=0}^{2t-2i} (-1)^j \binom{2t - 2i}{j} y^j
\]

\[
= \sum_{m=0}^{2t+1} y^m \sum_{i+j=m, 0 \leq i \leq t, 0 \leq j \leq 2t + 1 - 2i} (-1)^{m-i} \binom{2t - i}{i} \binom{2t + 1 - 2i}{j} - 2 \binom{2t - 2i}{j}
\]

\[
= \sum_{m=0}^{2t+1} y^m \sum_{0 \leq i \leq \min(t, m, 2t+1-m)} (-1)^{m-i} \binom{2t - i}{i} \binom{2t + 1 - 2i}{m - i} - 2 \binom{2t - 2i}{m - i - 1}.
\]
By equating coefficients, we need to show that

\[ Z(t,m) = 0, \quad \text{for} \quad 1 \leq m \leq 2t \quad \text{and} \quad Z(t,0) = 1, \quad Z(t,2t+1) = 1, \quad \text{where}, \]

\[
Z(t,m) = \sum_{0 \leq i \leq \min(t,m,2t+1-m)} (-1)^{m-i} \binom{2t-i}{i} \left( \binom{2t+1-2i}{m-i} - 2 \binom{2t-2i}{m-i-1} \right). \tag{44}
\]

Equation (44) is a hypergeometric sum identity. By using the EKHAD Maple package of Zeilber’s creative telescoping, we find

\[
(2m - 2t - 1)(m - 2t - 3)Z(t+1,m) - (m^2 - 2tm - m - 2t - 3)(m - 2t - 1)Z(t,m) = 0.
\]

The proof is similar to that of Lemma 24 and we omit it. \hfill \Box

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