CHARACTERISTIC EQUATION FOR SYMPLECTIC GROUPOID AND CLUSTER ALGEBRAS

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Abstract. We use the Darboux coordinate representation found by two of the authors (L.Ch. and M.Sh.) for entries of general symplectic leaves of the $A_n$-grouppoid of upper-triangular matrices to express roots of the characteristic equation $\det(A - \lambda A^T) = 0$, with $A \in A_n$, in terms of Casimirs of this Darboux coordinate representation, which is based on cluster variables of Fock–Goncharov higher Teichmüller spaces for the algebra $sl_n$. We show that roots of the characteristic equation are simple monomials of cluster Casimir elements. This statement remains valid in the quantum case as well. We consider a generalization of $A_n$-grouppoid to a $A_{Sp_{2m}}$-grouppoid.

1. Introduction

1.1. Symplectic groupoid and induced Poisson structure on the unipotent upper triangular matrices. Let $A_n \subseteq gl_n$ be a subspace of unipotent upper-triangular $n \times n$ matrices. We identify elements $A$ of $A_n$ with matrices of bilinear forms on $\mathbb{C}^n$. The matrix $B \in GL_n$ of a change of a basis in $\mathbb{C}^n$ takes a matrix of bilinear form $A \in A_n$ to $BAB^T$ determining a dynamics of form transformations. A most interesting case is when the transformed form $BAB^T$ lies in the same class $A_n$. For a fixed $A \in A_n$ all $B \in GL_n$ such that $BAB^T \in A_n$ form a submanifold of $GL_n$, which is not a subspace and may have a very involved structure. We introduce the space of morphisms identified with admissible pairs of matrices $(B, A)$ such that

\begin{equation}
\mathcal{M} = \{(B, A) \mid B \in GL_n, A \in A_n, BAB^T \in A_n\}.
\end{equation}

In 2000, Bondal [1] obtained the Poisson structure on $A_n$ using the algebroid construction: For $B = e^g$, we first define the anchor map $D_A$ to the tangent space $T A_n$:

\begin{equation}
D_A : \mathfrak{g}_A \rightarrow TA_n, \quad g \mapsto A g + g^T A.
\end{equation}

where $\mathfrak{g}_A$ is the linear subspace

\begin{equation}
\mathfrak{g}_A := \{g \in \mathfrak{gl}_n(\mathbb{C}), | A + A g + g^T A \in A_n\}
\end{equation}

of elements $g$ leaving $A$ unipotent.

Lemma 1.1. [1] The map

\begin{equation}
P_A : T^* A_n \rightarrow \mathfrak{g}_A, \quad w \mapsto P_{-1/2}(w A) - P_{1/2}(w^T A^T),
\end{equation}

where $P_{\pm 1/2}$ are the projection operators:

\begin{equation}
P_{\pm 1/2}a_{i,j} := \frac{1 \pm \text{sign}(j - i)}{2}a_{i,j}, \quad i, j = 1, \ldots, n,
\end{equation}

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and \( w \in T^*A_n \) is a strictly lower triangular matrix, defines an isomorphism between the Lie algebroid \((\mathfrak{g}, D_\mathfrak{g})\) and the Lie algebroid \((T^*A_n, D_\mathfrak{g}P_\mathfrak{g})\).

The Poisson bi-vector \( \Pi \) on \( A_n \) is then obtained by the anchor map on the Lie algebroid \((T^*A_n, D_\mathfrak{g}P_\mathfrak{g})\) (see Proposition 10.1.4 in [21]) as:

\[
\Pi : T^*A_n \times T^*A_n \mapsto \mathcal{C}^\infty(A) \\
(\omega_1, \omega_2) \mapsto \text{Tr}(\omega_1 D_\mathfrak{g}P_\mathfrak{g}(\omega_2))
\]

(1.5)

It can be checked explicitly that the above bilinear form is in fact skew-symmetric and gives rise to the Poisson bracket

\[
\{a_{i,k}, a_{j,l}\} := \frac{\partial}{\partial d_{ik}} \wedge \frac{\partial}{\partial d_{jl}} \text{Tr}(d_{ik} D_\mathfrak{g}P_\mathfrak{g}(d_{jl})),
\]

having the following form:

\[
\begin{align*}
\{a_{ik}, a_{jl}\} &= 0, \quad \text{for } i < k < j < l, \text{ and } i < j < l < k, \\
\{a_{ik}, a_{jl}\} &= 2(a_{ij}a_{kl} - a_{il}a_{kj}), \quad \text{for } i < j < k < l, \\
\{a_{ik}, a_{kl}\} &= a_{ik}a_{kl} - 2a_{il}, \quad \text{for } i < k < l, \\
\{a_{ik}, a_{jk}\} &= -a_{ik}a_{jkl} + 2a_{ij}, \quad \text{for } i < j < k, \\
\{a_{ik}, a_{il}\} &= -a_{ik}a_{jkl} + a_{kl}, \quad \text{for } i < k < l.
\end{align*}
\]

(1.6)

This bracket is famous in mathematical physics and is known as Gavrilik–Klimyk–Nelson–Regge–Dubrovin–Ugaglia bracket [15, 23, 24, 10, 28]. It originally arose in the context of 2D quantum gravity; technically it governs from skein relations satisfied by a specially chosen [4] finite subset of geodesic functions \( a_{ik} \) (traces of monodromies of \( SL_2 \) Fuchsian systems), which are principal observables in the theory of 2D gravity; a log-canonical (Darboux) bracket on the space of Thurston shear coordinates \( z_\alpha \) on the Teichmüller space \( T_{g,s} \) of Riemann surfaces \( \Sigma_{g,s} \) of genus \( g \) with \( s = 1, 2 \) holes was shown [3] to induce the above bracket on \( a_{ik} \) for \( n = 2g + s \).

All geodesic functions on any \( \Sigma_{g,s} \) admit an explicit combinatorial description [11] and they are Laurent polynomials with positive integer coefficients of \( e^{z_\alpha/2} \). The Poisson bracket of \( z_\alpha \) spanning the Teichmüller space \( T_{g,s} \) has exactly \( s \) Casimirs, which are linear combinations of shear coordinates incident to the holes, so the subspace of \( z_\alpha \) orthogonal to the subspace of Casimirs parameterizes a symplectic leaf in the Teichmüller space called a geometric symplectic leaf.

However the Poisson dimensions of \( T_{g,s} \) (6\( g \)–6 +\( s \)) and those of \( A_n \) (\( n(n-1)/2 - [n/2] \)) become different for \( n \geq 6 \), so a novel Darboux coordinate construction for higher-dimensional symplectic leaves of \( A_n \) was found in [9] where elements of \( A_n \) were parameterized by cluster variables of the Fock–Goncharov framed moduli space \( X_{SL_n,\Sigma} \), where \( \Sigma \) is a disk with three marked points [12].

Note that algebra (1.7) is universal: any choice of the subspace \( A \subseteq sl_n \) for a \((B, A)\)-system subject to the form dynamics results in Poisson relations (1.7) provided \( A \) is a Lagrangian submanifold [6], so constructing cluster realizations of \( A \) outside the unipotent upper-triangular case is also important. We present this construction for the first natural extension of \( A_n \), which is \( A_{Sp_{2m}} \). The quantum analogue of (1.7) is known as the quantum reflection equation (see Theorem [31]), and the cluster coordinate realization of a general symplectic leaf of the quantum reflection equation was also constructed in [9].

An important problem is to construct Casimirs of algebras (1.7) in semiclassical and quantum cases. In the semiclassical limit, Bondal showed that the algebra \( A_n \) admits \([n/2]\) algebraically independent Casimir elements, which are coefficients of the reciprocal polynomial \( \det[A - \lambda A^T] \); in the geometric cases \( n = 3 \) and \( n = 4 \), these coefficients turn out to be simple functions of hole
perimeters $p_i = \sum z_{\alpha_i}$, which are always linear functions of shear coordinates. This statement remains valid in the quantum case as well. At the same time, being expressed in $a_{ij}$, the same coefficients become inhomogeneous functions producing the quantum Markov invariant for $n = 3$ and quantum invariants first obtained by Bullock and Przytycki “by brute force” in [2] for $n = 4$.

Although we do not exploit it in this text, note that the space $A_n$ admits a discrete braid-group action generated by special morphisms $\beta_{i,i+1} : A_n \rightarrow A_n$, $i = 1, \ldots, n - 1$, such that $\beta_{i,i+1}[A] := B_{i,i+1}A[B_{i,i+1}]^T \in A_n$; these transformations correspond to Dehn twists in the geometrical case, and matrices $B_{i,i+1}$ differ from the unity matrix only in their diagonal $2 \times 2$-blocks having the structure

|    | $i$    | $i + 1$ |
|----|--------|---------|
| $i$ | $q^{1/2}a_{i,i+1}^h$ | $-q$    |
| $i + 1$ | 1 | 0 |

in the quantum case constructed in [3]; the matrix $A$ then becomes a quantum matrix $A^h$:

$$
(1.8) \quad A^h := \begin{bmatrix}
q^{-1/2} & a_{1,2}^h & a_{1,3}^h & \cdots & a_{1,n}^h \\
0 & q^{-1/2} & a_{2,3}^h & \cdots & a_{2,n}^h \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & q^{-1/2}
\end{bmatrix},
$$

where $a_{i,j}^h$ are self-adjoint ($[a_{i,j}^h]^* = a_{i,j}^h$) operators enjoying quadratic–linear algebraic relations following from the quantum reflection equation (see Theorem 3.1). The quantum braid-group action is $\beta_{i,i+1}^h : A^h \rightarrow B_{i,i+1}^hA^h[B_{i,i+1}^h]^{\dagger}$ with $\dagger$ standing for the Hermitian conjugation. In [9], the braid-group action was realized as sequences of cluster mutations in the $A_n$-quiver obtained by a special amalgamation procedure (described below) from the quiver for $A_{\Sigma_{n,n}}$.

Our goal in this paper is to study the structure of resolvents of the quantum operators $A[A^{\dagger}]^{-1} := AA^{-\dagger}$. This combination plays a prominent role: first, assuming all operatorial expressions be invertible, for $A$ transforming as a form (1.1), this combination undergoes an adjoint transformation,

$$
(1.9) \quad AA^{-\dagger} \rightarrow BAA^{-\dagger}B^{-1},
$$

so we can address the problem of solving the resolvent equation: for which $\lambda \in \mathbb{C}$ the operator $AA^{-\dagger} - \lambda \mathbb{I}$ admits null vectors? We completely solve this problem in Theorem 4.1 expressing all $\lambda$ in question via Casimirs of the $A_n$-quiver.

With this solution in hands, we can address several intriguing problems: note that in the semiclassical case, the Jordan form of $AA^{-\dagger}$ determines the dimension of the corresponding Poisson leaf. Although our technique does not allow keeping control over the Jordan form structure in case of coinciding eigenvalues $\lambda_i$, we can fully treat the case where all $n$ eigenvalues $\lambda_i$ are different. Then the Jordan form is diagonal and, assuming the existence of natural $N \geq n$ for which $\lambda_i^N = 1$ for all $i$, we obtain that the $N$th power of the Jordan form is a unit matrix, and therefore $[AA^{-\dagger}]^N = \mathbb{I}$. This relation opens a way to constructing minimal-model representations (containing null vectors in the sense of Kac) of related non-Abelian Verma-type modules generated by the action of $a_{i,j}$ on a proper vacuum vector, where the action of canonical cluster variables plays a role of a Dotsenko–Fatteev free-field representation. We may also address finite-dimensional reductions of affine Lie–Poisson systems.
2. $b_n$-Algebras for the triangle $\Sigma_{0,1,3}$

Let $\Sigma_{g,s,p}$ denote a topological genus $g$ surface with $s$ boundary components and $p > 0$ marked points on the boundaries. Our main example is the disk with 3 marked points on the boundary, $\Sigma_{0,1,3}$, which corresponds to an ideal triangle in an ideal-triangle decomposition of a general $\Sigma_{g,s,p}$ (which comprises exactly $4g - 4 + 2s + p$ ideal triangles).

2.1. Notations. Let lattice $\Lambda = \mathbb{Z}^n$ be equipped with a skew-symmetric integer form $\langle \cdot , \cdot \rangle$. Introduce the $q$-multiplication operation in the vector space $\Upsilon = \text{Span}\{Z_{\lambda}\}_{\lambda \in \Lambda}$ as follows: $Z_{\lambda}Z_{\mu} = q^{\langle \lambda, \mu \rangle}Z_{\lambda+\mu}$. The algebra $\Upsilon$ is called a quantum torus. Fixing a basis $\{e_i\}$ in $\Lambda$, we consider $\Upsilon$ as a non-commutative algebra of Laurent polynomials in variables $Z_i := Z_{e_i}$, $i \in [1, N]$. For any sequence $a = (a_1, \ldots, a_t)$, $a_i \in [1, N]$, let $\Pi_a$ denote the monomial $\Pi_a = Z_{a_1}Z_{a_2} \ldots Z_{a_t}$. Let $\lambda_a = \sum_{j=1}^t e_{a_j}$. Element $Z_{\lambda_a}$ is called in physical literature the Weyl form of $\Pi_a$ and we denote it by two-sided colons $:\Pi_a:^*$ It is easy to see that $:\Pi_a:^* = Z_{\lambda_a} = q^{-\sum_{j<k} \langle e_{a_j}, e_{a_k} \rangle} \Pi_a$.

Every $k \times p$ (quantum) matrix $M$ lies in the direct product of $\text{Mat}_{k \times p} \otimes \mathbb{Q}$, where $\mathbb{Q}$ is the quantum operatorial space shared by all matrices and all matrix elements. We can interpret the matrix $M$ as a morphism from $\mathbb{C}^p \otimes W$ into a vector space $\mathbb{C}^k \otimes W$, where $W$ is an irreducible representation space for operators from the set $\mathbb{Q}$. In what follows, we assume that the order of product of quantum operators in $\mathbb{Q}$ always coincides with the order of the product of corresponding matrix elements. All classical operators act as the unit operators in $W$; in particular, all $R$-matrices are assumed to be classical. Since we assume the representation $W$ to be irreducible, all Casimirs are classical operators. To shorten the writing, we usually omit direct product symbols instead indicating by the index above the symbol of an operator the number of the space $\mathbb{C}^k$ in which this operator acts nontrivially. For example, $\frac{1}{2} \sum_{i,j} e_{ij} \otimes m_{ij}$ and $V = \sum_{k,l} e_{kl} \otimes v_{kl}$ with $m_{ij}$ and $v_{kl}$ from the set $\mathbb{Q}$ denotes the operator acting in $\mathbb{C}^k \otimes \mathbb{C}^k \otimes W$

$$\frac{1}{2} M V := \sum_{i,j,k,l} \left( e_{ij} \otimes e_{kl} \otimes m_{ij} v_{kl} \right),$$

where we use the standard notation $e_{ij}$ for an elementary matrix whose all elements vanish except the unit element at the intersection of the $i$th row and $j$th column.

In what follows, we let $I$ denote the unit matrix acting on $\mathbb{C}^k$, $I$ the unit operator acting in $W$, and $I = I \otimes I$ the unit operator acting in the direct product $\mathbb{C}^k \otimes W$. We imply that $I = I \otimes I$. The operation of matrix inversion, $M^{-1}$, acts in the both spaces: classical and quantum, so $M^{-1} M = I$ and the order of operators in the quantum space $\mathbb{Q}$ is inverted: e.g., $\left[ \frac{1}{2} M V \right]^{-1} = V^{-1} \frac{1}{2} M^{-1}$; on the contrary, transposition acts only in the classical space, so $\left[ \frac{1}{2} M V \right]^T = M^T V^T$. Finally, the Hermitian conjugation again permutes operators, $\left[ \frac{1}{2} M V \right]^\dagger = M^\dagger V^\dagger$.

2.2. Quantum transport matrices and Fock-Goncharov coordinates. We describe now how quantized transport matrices are expressed in terms of quantized Fock–Goncharov parameters.

In the quantization of $\mathcal{X}_{SL_n,\Sigma}$ for $\Sigma_{0,1,3}$, the quantized Fock–Goncharov variables form a quantum torus $\Upsilon$ with commutation relation described by the $b_n$-quiver ($b_n$ stands for a Borel subalgebra of $sl_n$) shown in Fig. 1 by solid and dashed lines. Vertices of the quiver label quantum Fock-Goncharov coordinates $Z_{\alpha}$ while arrows encode commutation relations: if there are $m$ arrows from vertex $\alpha$ to $\beta$ then $Z_{\beta} Z_{\alpha} = q^{-2m} Z_{\alpha} Z_{\beta}$. Dashed arrow counts as $m = 1/2$. In particular, a solid arrow from $Z_{\alpha}$ to $Z_{\beta}$ implies $Z_{\beta} Z_{\alpha} = q^{-2} Z_{\alpha} Z_{\beta}$, a dashed arrow from $Z_{\alpha}$ to
$Z_\beta$ implies $Z_\beta Z_\alpha = q^{-1} Z_\alpha Z_\beta$, and, say, a double arrow from $Z_\alpha$ to $Z_\beta$ means $Z_\beta Z_\alpha = q^{-4} Z_\alpha Z_\beta$. Vertices not connected by an arrow commute.

We next construct a directed network $N$ dual to the $b_n$-quiver. Dual directed network is a directed embedded graph in the disk (triangle) whose three-valent vertices are colored black and white and orientation is compatible with the coloring, i.e. each white vertex has exactly one incoming edge while every black vertex has exactly one outgoing edge. Vertices of the quiver that is dual to a directed network are located in the faces of the directed network (one vertex per each face) while directed edges of the quiver intersect black-white edges of the network in such a way that quiver’s arrows travel clockwise around white vertices of network and counterclockwise around black vertices. Since the quiver orientation depends only on a pattern of black and white vertices and not on orientations of network edges, several directed networks may correspond to the same quiver, but in the considered case orientation of the dual network is determined by fixing $n$ sources and $2n$ sinks among boundary vertices. In particular, choosing all sources of the network located along one side of the triangle Fig. 1 we fix the dual directed network uniquely.

We therefore have several possible choices of directed coloring-compatible networks for the $b_n$-quiver, two of which relevant for our studies are depicted in the figure: every solid arrow of a quiver has a white vertex on the right and a black vertex on the left; all dashed arrows have white vertices on the right. Directed edges of the network (double lines in the figure) are dual to edges of the quiver with sources and sinks dual to dashed arrows; it is easy to see that the choice of sources and sinks then determines the pattern of directed double lines inside the graph in a unique way.

We adopt a barycentric enumeration of vertices and cluster variables $Z_{(i,j,k)}$ of the $b_n$-quiver: $i, j, k$ are nonnegative integers such that $i + j + k = n$ (one can think about these vertices as integer points of intersection of the lattice $\mathbb{Z}_{+0} \otimes \mathbb{Z}_{+0} \otimes \mathbb{Z}_{+0}$ with the plane $x + y + z = n$).

**Figure 1.** Two directed networks $N$ dual to the Fock–Goncharov $b_6$-quiver. Double arrows are edges of the directed network, cluster variables correspond to faces of $N$ (vertices of the $b_n$-quiver dual to $N$). On the left-hand side we present the directed network defining the transport matrices $M_1$ (from the side $\{1 − 6\}$ to the side $\{1′ − 6′\}$) and $M_2$ (from the side $\{1 − 6\}$ to the side $\{1″ − 6″\}$) and on the right-hand side we present the directed network defining the transport matrix $M_3$ (from the side $\{1′ − 6′\}$ to the side $\{1″ − 6″\}$). We also indicate three barycentrically enumerated cluster variables at three corners of the quiver.
We assign to every oriented path $P : j \rightsquigarrow i'$ from the right side to the left side or to the bottom side $P : j \rightsquigarrow i''$ the quantum weight
\[
(2.1) \quad w(P) = \prod_{\text{faces } \alpha \text{ lie to the right of the path } P} Z_\alpha.
\]

**Definition 2.1.** We define two $n \times n$ non-normalized quantum transition matrices
\[
(M_1)_{i,j} = \sum_{\text{directed path } P : j \rightsquigarrow i'} w(P) \quad \text{and} \quad (M_2)_{i,j} = \sum_{\text{directed path } P : j \rightsquigarrow i''} w(P).
\]

Note that $M_1$ is a lower-triangular matrix and $M_2$ is an upper-triangular matrix.

Theorem 9.3 of [9] states that for any planar network with separated $n$ sources and $m$ sinks, defining entries of $m \times n$ quantum transition matrix as in Definition 2.1, we obtain that these entries enjoy the quantum $R$-matrix relation
\[
(2.2) \quad R_m(q)M \otimes \tilde{M} = \tilde{M} \otimes M R_n(q),
\]
where $R_k(q)$ denotes a $k^2 \times k^2$ matrix
\[
(2.3) \quad R_k(q) = \sum_{1 \leq i,j \leq k} e_{ii} \otimes e_{jj} + (q - 1) \sum_{1 \leq i,j \leq k} e_{ii} \otimes e_{ji} + (q - q^{-1}) \sum_{1 \leq j < i \leq k} e_{ij} \otimes e_{ji}
\]
having the properties that $R_k^{-1}(q) = R_k(q^{-1})$ and $R_k(q) - R_k(q^{-1}) = (q - q^{-1})P_k$, with $P_k$ the standard permutation matrix $P_k := \sum_{1 \leq i,j \leq k} e_{ij} \otimes e_{ji}$. Note that the total transposition of $R_k(q)$ results in interchanging the space labels $1 \leftrightarrow 2$.

In Fig. 1 we have an example of a directed network with $n = 6$ sources and $2n = 12$ sinks; for any $n$ we present the corresponding quantum transition matrix in the block form $M = (M_1 \ M_2)$ with $M_1$ and $M_2$ denoting the respective upper and lower $n \times n$ blocks. Relations (2.2) then imply the $R$-matrix commutation relations between these blocks:
\[
(2.4) \quad R_n(q)M_i \otimes M_i = \tilde{M}_i \otimes M_i R_n(q), \quad i = 1, 2,
\]
and
\[
(2.5) \quad M_1 \otimes M_2 = M_2 \otimes M_1 R_n(q).
\]

We also introduce the third transition matrix $M_3$ for paths going from the side $\{1' - 6'\}$ to the side $\{1'' - 6''\}$ in the transformed network in the right side of Fig. 1.

We now define the quantum transport matrices of the standard $b_n$-quiver:

**Definition 2.2.** Quantum transport matrices read
\[
M_1 = QSM_1, \quad M_2 = QSM_2, \quad \text{and} \quad M_3 = QSM_3,
\]
where $Q := \sum_{i=1}^{n} q^{-i+1/2} e_{i,i} \otimes I$ is a diagonal matrix and $S = \sum_{i=1}^{n} (-1)^{i+1} e_{i,n+1-i} \otimes I$ is an antidiagonal matrix.

Note that
\[
\frac{1}{S} \otimes \frac{1}{S} R_n(q) = R_n(q)^T \frac{1}{S} \otimes \frac{1}{S} \quad \text{for any antidiagonal classical matrix } S.
\]
and
\[
\frac{1}{Q} \otimes \frac{1}{Q} R_n(q) = R_n(q)^T \frac{1}{Q} \otimes \frac{1}{Q} \quad \text{for any diagonal classical matrix } Q;\]
in particular, setting $Q = AB$ with two diagonal classical invertible matrices $A$ and $B$, we have

\begin{equation}
A \otimes B R_n(q) A^{-1} \otimes A^{-1} = B^{-1} \otimes A^{-1} R_n(q) A \otimes B.
\end{equation}

**Remark 2.3.** Multiplying a transition matrix by the matrix $S$ is standard in the Fock–Goncharov approach in order to make orientations of boundaries of adjacent triangles compatible and hence to “invert” the corresponding flags upon reaching a boundary of a network; additional $q$-factors collected in the matrix $Q$ are necessary for ensuring the groupoid property below; we do not normalize transport matrices (and, correspondingly, the $R$-matrix) as in [9] because this normalization does not affect our consideration below. We also omit the sign of the direct product in the subsequent text.

We have the following theorem.

**Theorem 2.4.** [9]. The quantum transport matrices $M_1$, $M_2$, and $M_3$ satisfy the relations

\begin{equation}
R_n(q) M_i M_i = M_i M_i R_n(q), \quad i = 1, 2, 3,
\end{equation}

\begin{equation}
M_1 M_2 = M_2 M_1 R_n(q), \quad M_1 M_3 = M_3 R_n(q) M_1, \quad M_2 M_3 = R_n(q) M_3 M_2
\end{equation}

with the quantum trigonometric $R$-matrix (2.3).

**Theorem 2.5.** [9]. The quantum transport matrices satisfy the quantum groupoid condition

\begin{equation}
M_3 M_1 = M_2.
\end{equation}

This condition is consistent with the quantum commutation relations in Theorem 2.4.

Using relations (2.6) we obtain

**Lemma 2.6.** The commutation relations in Theorem 2.4, as well as the groupoid condition in Theorem 2.3 are invariant under transformations

\begin{equation}
M_1 \to AM_1C, \quad M_2 \to BM_2C, \quad M_3 \to BM_3A^{-1},
\end{equation}

where $A$, $B$, and $C$ are arbitrary nondegenerate classical diagonal matrices.

3. The Groupoid of Upper Triangular Matrices

3.1. Representing an upper-triangular $A$. Consider a special combination of $M_1$ and $M_2$:

\begin{equation}
A := M_1^T D M_2 = M_1^T D M_3 M_1,
\end{equation}

where $D$ is any classical diagonal matrix. (In notations of Lemma 2.6 $D = A^T B$.) We can use the freedom in choice of $D$ to attain the canonical form (1.8) of the quantum $A$-matrix with the diagonal entries equal to $q^{-1/2}$. Then the properly normalized expression for this matrix reads

\begin{equation}
A^\hbar = M_1^T M_3 Q S M_1,
\end{equation}

i.e., $D^{-1} = S^T Q^T Q S$, and we use expression (3.2) when solving the problem of singular values of $\lambda$ in Sec. 4.

Note that since $M_1$ and $M_1^T$ are upper-anti-diagonal matrices and $M_2$ is a lower-anti-diagonal matrix in the case of $\Sigma_{0,1,3}$, the matrix $A$ is automatically upper-triangular.

The following theorem was proved in [9] for $A$ having the form $M_1^T M_2$. We slightly modify this proof to adjust it to $A$ of the form $M_1^T D M_3 M_1$. 
Theorem 3.1. For the matrices $M_1$ and $M_3$ enjoying commutation relations in Theorem 2.4 and for any diagonal matrix $D$, the matrix $A$ given by (3.1) enjoys the quantum reflection equation

$$R_n(q)A R_n^t(q)^2 = ^t A R_n(q)$$

with the trigonometric $R$-matrix (2.3), where $R_n^t(q)$ is a partially transposed (w.r.t. the first space) $R$-matrix.

The proof uses only $R$-matrix relations. Transposing relations in Theorem 2.4 with respect to different spaces, we obtain

$$M_i^T R_n^t(q) M_i = M_i R_n^t(q) M_i^T, \quad R_n(q) M_i^T M_i = M_i^T R_n^t(q) M_i, \quad \text{and} \quad M_i^T M_i = M_i^T R_n^t(q) M_i$$

Note that $R_n^t(q)$ retains its form if we interchange the space indices $1 \leftrightarrow 2$; note also that $D$ is a classical matrix, so, say, $D$ commutes with all $M_i$. We have

$$R_n(q) M_i^T M_i D M_3 M_1 R_n^t(q) M_i = R_n(q) M_i^T D M_3 M_1 R_n^t(q) M_i M_i^T M_i$$

$\text{and} \quad D M_3 M_1 R_n^t(q) M_i R_n(q) = D M_3 M_1 R_n^t(q) M_i R_n(q) M_i^T M_i$.

which completes the proof.

We have therefore a Darboux coordinate representation for operators satisfying the reflection equation. Moreover, all matrix elements of $A$ are Laurent polynomials with positive coefficients of $Z_\alpha$ and $q$.

By construction of quantum transport matrices in Sec. 2.2 all matrix elements of $M_1$ and $M_2$ are Weyl-ordered. For $[A]_{i,j} = \sum_{k=i}^j [M_1]_{k,i}[M_2]_{k,j}$ we obtain that for $i < j$, $[M_1]_{k,i}$ commutes with $[M_2]_{k,j}$ (all paths contributing to $[M_1]_{k,i}$ are disjoint from all paths contributing to $[M_2]_{k,j}$ for $i < j$), so the corresponding products are automatically Weyl-ordered.

$[A]_{i,j} = \sum_{k=i}^j [M_1]_{k,i}[M_2]_{k,j}$. For $i = j$, the corresponding two paths share the common starting edge, and then $[A]_{i,i} = q^{-1/2} [M_1]_{i,i}[M_2]_{i,i}$. This explains the appearance of $q^{-1/2}$ factors on the diagonal of the quantum matrix $A^h$ (see [1.8, 5]). Note that all Weyl-ordered products of $Z_\alpha$ are self-adjoint and we assume that all Casimirs are also self-adjoint.

To obtain a full-dimensional (not upper-triangular) form of the matrix $A$ let us consider adjoint action by any transport matrix:

Theorem 3.2. Any matrix $A' := M_1^T S^T A S M_1$ where $M_1$ is a (transport) matrix satisfying commutation relations of Theorem 2.4 and commuting with $A = M_1^T M_2$ satisfies the quantum reflection equation of Theorem 3.1.

3.2. Casimirs of $b_n$-quiver. For the full-rank $b_n$-quiver we have exactly $\binom{n}{2} + 1$ Casimirs depicted in Fig 2 for the example of $b_6$: numbers at vertices indicate the power with which the corresponding variables come into the product; all nonnumbered variables have power zero. All Casimirs correspond to closed broken-line paths in the $b_n$ quiver with reflections at the boundaries (the “frozen” variables at boundaries enter the product with powers two, powers of non-frozen variables can be $0, 1, 2$, and $3$, and they count how many times the path...
goes through the corresponding site). The total Poisson dimension of the full-rank quiver is therefore \[
\frac{(n+2)(n+1)}{2} - \left\lfloor \frac{n}{2} \right\rfloor - 1.
\]

Figure 2. Four central elements of the full-rank \(b_0\)-quiver.

**Notation.** Let

\[
T_i := \prod_{j=0}^{n-i} Z_{(j,n-j-i,i)};
\]

denote products of cluster variables along SE-diagonals of the \(b_n\)-quiver.

Figure 3. Six mutually commuting elements \(K_l\) \[(3.5)\] of the \(b_0\)-quiver; all these elements commute with all variables of the \(A_6\)-quiver in Fig. 4 and setting all \(K_l = 1\) results in the \(A_6\) matrix of form \[(1.8)\].

3.3. \(A_n\)-quiver. We now construct the quiver corresponding to the system described by the reflection equation. Note that transposition results, in particular, in amalgamation of variables \(Z_{i,0,n-i}\) and \(Z_{(0,n-i,i)}\) for \(1 \leq i \leq n-1\). We indicate these amalgamations by dashed arrows in Fig. 4. A more precise statement reads

**Lemma 3.3.**

- Matrix entries of \(A\) depend only on the variables \(Z_{(i,j,k)}\) with \(i, j, k > 0\) and amalgamated variables \(Z_{(i,0,n-i)} Z_{(0,n-i,i)}\), which are the variables of the reduced quiver, or \(A_n\)-quiver, in Fig. 4 and on \(n\) variables \(K_l\), \(1 \leq l \leq n\), depicted in Fig. 4.

\[
K_l = \prod_{i=1}^{l} Z_{(i-1,n-l,l-i+1)} Z_{(l,n-l,0)}^2 \prod_{j=1}^{n-l} Z_{(l-1,n-l-j+1)}.
\]
elements $K_l$ commute mutually and commute with all variables of the reduced quiver;

upon setting all $K_l = 1$ the matrix $\hat{A}$ takes the form (1.8).

We now unfreeze all amalgamated variables $\cdot Z_{(i,0,n-i)} Z_{(0,n-i,i)} \cdot$. From now on, we set all elements $K_l$ defined in (3.5) equal the unity. This eliminates the dependence on all remaining frozen variables and we remain with the reduced, or $A_n$-quiver in Fig. 4 in which we indicate additional amalgamations by dashed lines.

**Figure 4.** The reduced quiver (the $A_n$-quiver) obtained by amalgamating variables of the $b_n$-quiver and removing all remaining frozen variables. (The example in the figure corresponds to $b_6$) Variables of the same color constitute Casimirs (with unit powers for all variables in every Casimir) depicted in Fig. 5.

**Lemma 3.4.** [9] Casimirs of $A_n$-quiver are $\left[ \frac{n}{2} \right]$ elements $C_i$ given by the formula:

\[ (3.6) \quad C_i = \cdot T_i T_{n-i} \cdot \text{ for } 1 \leq i < n/2; \quad C_{n/2} = T_{n/2} \text{ if } n/2 \text{ is an integer,} \]

where $T_i$ are products (3.4) of variables of $b_n$ quiver; note that $C_i$ depend only on amalgamated variables. The example for $n = 6$ is depicted in Fig. 5.

Since all Casimirs of $A_n$ are generated by $\lambda$-power expansion terms for $\det(\hat{A} - \lambda \hat{A}^T)$, we automatically obtain a semiclassical statement that $\det(\hat{A} - \lambda \hat{A}^T) = P(C_1, \ldots, C_{[n/2]})$, where $C_i$ are Casimirs of the $A_n$-quiver. In the next section we explore this dependence.

**Figure 5.** Three central elements for $A_6$-quiver.
4. Solving characteristic equation

The main theorem follows

**Theorem 4.1.** Consider the $A_n$-quiver in Fig. 4. Let $\lambda \in \mathbb{C}$ be an eigenvalue of the quantum operator $AA^{-1}$, i.e., the number at which the equation $(AA^{-1} - \lambda I)\psi = 0$ has a nontrivial null vector $\psi \in V \otimes W$, where $A$ is $A^h$ from (4.3). Then $n$ admissible values of $\lambda_i$ are

$$
\lambda_i = (-1)^{n-1}q^{-n} \times \left\{ \begin{array}{cl}
\prod_{k=i}^{[n/2]} C_k & \text{for } 1 \leq i \leq [n/2]; \\
\prod_{k=n+1-i}^{[n/2]} C_k & \text{for } i = (n+1)/2 \text{ for odd } n; \\
1 & \text{for } n - [n/2] + 1 \leq i \leq n
\end{array} \right.
$$

**Proof.** We use representation (3.2) for the quantum matrix $A^h$. Since all entries of all matrices $M_i$ are self-adjoint, $M_i^\dagger = M_i^T$, $Q^\dagger = Q^{-1}$ and $S^\dagger = S^T = (-1)^{n+1}S$. Then

$$
\begin{align*}
[A^h] & = [M_i^T S^T Q^{-1} M_3^T M_1^T, \\
\lambda S & = [M_i^T (M_3 Q S - \lambda S^T Q^{-1} M_3^T) M_1]
\end{align*}
$$

so only the central block $S^T Q^{-1} M_3^T M_1^T$ sandwiched between $M_i^T$ and $M_1$ is changed, and

$$
\begin{align*}
A^h - \lambda [A^h] & = \lambda M_i^T (M_3 Q S - \lambda S^T Q^{-1} M_3^T) M_1
\end{align*}
$$

whereas the singularity equation becomes

$$
(M_3 Q S - \lambda S^T Q^{-1} M_3^T) \psi = 0
$$

for some nonzero vector $\psi$.

The crucial observation is that both matrices: $M_3 Q S$ and $S^T Q^{-1} M_3^T$ in (4.3) are upper-anti-diagonal matrices! All solutions to the singularity equation therefore correspond to the situation where the combination of these matrices in (4.3) has zero element on the anti-diagonal (then the determinant of this combination becomes zero). The matrix $M_3$ is lower-triangular with the diagonal elements

$$
m_1 = Z_{(n,0,0)}, \quad m_i = \cdot Z_{(n,0,0)} \prod_{j=1}^{i-1} T_i, \quad 2 \leq i \leq n,
$$

where $T_i$ are defined in (3.4). The matrices

$$
S^T Q^{-1} M_3^T = \begin{bmatrix} \ast & \ast & a_n \\ \ast & \ddots & 0 \\ a_1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_3 Q S = \begin{bmatrix} \ast & \ast & b_n \\ \ast & \ddots & 0 \\ b_1 & 0 & 0 \end{bmatrix}
$$

then both are upper-anti-diagonal with

$$
\begin{align*}
a_i = (-1)^{i+1} q^{-i-1/2} m_i, \\
b_i = (-1)^{n-i} q^{-n+i-1/2} m_{n+1-i}
\end{align*}
$$

and solutions of the singularity equation are

$$
\lambda_i = b_i/a_i = (-1)^{n-1} q^{-n} m_{n+1-i}^{-1} (m_i)^{-1}
$$

Note that $m_i$ themselves are not Casimirs, but their quotients $m_{n+1-i}^{-1}$ are: these quotients are just expressions in cases in Theorem 4.1. This completes the proof.

**Remark 4.2.** Note that representation (3.2) remains valid for any, not necessarily lower-triangular matrix $M_1$. This means that Theorem 4.1 remains valid in the case of general quantum $A$-matrices, which may be not upper-triangular, provided they are presented in the cluster-variable decomposition form $B^T M_3 Q S B$ with $B$ having the same commutation relations with itself and with $M_3$ as $M_1$. In particular, it holds for any (amalgamated) $B = M_1 Q S F$ with $F$ commuting with all $M_i$ and enjoying Lie–Poisson relations for its entries.
The factors $(-1)^{n-1}q^n$ in $\lambda_i$ in Theorem 4.1 are inessential and can be removed by a proper normalization. A very interesting case is however when all $\lambda_i$ are different: in this case we have a full control over the Jordan form of $AA^{-\dagger}$, which becomes diagonal. Note also that since $\lambda_i$ are combinations of Casimirs, they are unit operators $I$ in the quantum space, so we conclude that

$$A^h = U^{-1}\Lambda U,$$

where $\Lambda := \sum_{i=1}^{n} \lambda_i e_{i,i} \otimes I$.

**Corollary 4.3.** Assuming all $\lambda_i$ are distinct and we have a natural $N$ such that $\lambda_i^N = \lambda_j^N$ for all $i,j$ (so all $\lambda_i$ are proportional to distinct $N$th roots of the unity), we have that $\left[A^h [A^h]^{-\dagger}\right]^N = \text{const} \cdot I$.

5. $Sp_{2m}$ **systems**

In this section, we find Casimirs for the matrix $A$ having $2 \times 2$-block matrix form. These systems correspond to $Sp_{2m}$ twisted Yangians (see [22]). Transport matrices for this system must have block-triangular form. A directed network corresponding to the case of $2 \times 2$ blocks is presented in the left part of Fig. 6. Note that we have to eliminate $m$ redundant sinks in this picture and amalgamate variables of faces separated by the corresponding sinks. The resulting quiver is presented in the right part of the figure.

![Figure 6](image)

*Figure 6.* In the left picture we present the $Sp_{2m}$ directed network for $m = 3$. We remove three sinks indicated in the light color and located at the NE side of the triangle and amalgamate cluster variables of faces separated by the corresponding directed edges (these amalgamations are denoted by dashed arcs). Note the appearance of (three) double arrows directed upward: the transport matrix ceases to be upper triangular and becomes only block-upper triangular. The resulting $Sp_{2m}$-quiver is presented in the right picture.

Constructing transport matrices $M_1$ and $M_2$ by the same rules as in Sec. 2.2 and composing the block-upper triangular matrix $A := M_1^T M_2$ as in Sec. 3 we have to amalgamate frozen variables on two sides of the $Sp_{2m}$ quiver thus obtaining the quiver depicted in Fig. 7 which we call the $A_{Sp_{2m}}$-quiver.

Note that $A_{Sp_{2m}}$-quiver resembles $A_n$-quiver except for two rows on the right. We label variables of these rows by $S_i$ and $Q_{2r}$; each variable $S_i$ has a corresponding variable $Z_{(i,n-i,0)}$ in the $b_n$-quiver and $Q_{2r}$ are new variables.
We now find Casimirs for the $A_{Sp_{2m}}$-quiver. Note, first, that Casimirs $C_i$ (3.6) remain Casimirs of the $A_{Sp_{2m}}$-quiver. It was proved in [6] that for $mk \times mk$ block-triangular matrix $A$ with blocks of size $k \times k$ we have $\left[\frac{mk}{2}\right]$ Casimirs $C_i$ and exactly $m\left[\frac{k+1}{2}\right]$ new Casimirs depending only on matrix entries $a_{i,j}$ inside diagonal $k \times k$ blocks. For $k = 2$ we have exactly one such Casimir per each diagonal block, and this Casimir is just the determinant of the corresponding $2 \times 2$ block. We now find the corresponding Casimirs for variables of $A_{Sp_{2m}}$-quiver.

We first introduce quasi-Casimirs $K_l$, $l = 1, \ldots, 2m$ having the same content as Casimirs $K_i$ (3.5) in $b_n$-quiver thus denoting by the same letter; we just replace $Z_{(l,n-l,0)}$ by $S_l$. An example of $K_4$ is in the left picture in Fig. 7. Quasi-Casimirs $K_l$ commute with all cluster variables except $S_i$ and $Q_{2r}$. Substituting now $K_l$ for all $S_i$, we detach the part of the quiver containing only variables $K_l$ and $Q_{2r}$; this part looks as

It is easy to observe that, say, $Q_2^2K_6K_7^2K_2Q_2^2$ is a Casimir (numbers in the figure above indicate powers of the corresponding cluster variables). We therefore have the following statement.

**Lemma 5.1.** Casimirs of $A_{Sp_{2m}}$-quiver are $m$ Casimirs $C_i$ (3.6) and $m$ Casimirs $R_j$,

$$R_j := Q_2^2K_{2j}K_{2j+1}^2K_{2j+2}Q_{2j+2}^2,$$

with $K_l$ given by the same combinations (3.5) of cluster variables as in the case of $b_n$-quiver.
6. Concluding remarks

In this paper, we have solved the problem of finding solutions to the eigenvalue problem \((A - \lambda A^\dagger)\psi = 0\) for the quantum matrix \(A^h\) from (1.8). Our solution was based on the cluster-variable realization of entries of \(A^h\) found in [9]. We had also performed an extensive verification of our results (in the semiclassical limit) using Mathematica computer code [30].

Note that Remark 4.2 indicates that our results are by no means confined to a “triangle” \(\Sigma_{0, 1, 3}\): they remain valid for any system undergoing the “form dynamics” described in the introduction. However, in the case of matrices \(A\) of a more general form, besides \([n/2]\) Casimirs generated by the same expression \(\text{det}(A - \lambda A^T)\) as in the upper-triangular case, we have additional Casimir operators related to ratios of minors located in the lower-left corner of the matrix \(A\) (see, e.g., [6]). We constructed the corresponding Casimirs for the \(A_{Sp_{2m}}\)-quiver corresponding to matrices \(A\) of block-upper triangular form with blocks of size \(2 \times 2\). This however does not change the results concerning the Jordan form of \(AA^\dagger\) and the corresponding symmetries because for every such system, \(A = M_1^T M_3 M_1\) with the same matrix \(M_3\). Passing to more general \(A\) also does not change the conclusion that \([AA^\dagger]_{\mathbb{N}} = I\) for any such matrix provided \(\lambda_i\) constructed in Theorem 4.1 are distinct \(\mathbb{N}\)th roots of unity.

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References

[1] A. Bondal, A symplectic groupoid of triangular bilinear forms and the braid groups, preprint IHES/M/00/02 (Jan. 2000).
[2] D. Bullock and J.H. Przytycki, Multiplicative structure of Kauffman skein module quantization, Proceedings Amer. Math. Soc. 128(3) (1999) 923–931
[3] Chekhov L., Fock V., Quantum mapping class group, pentagon relation, and geodesics, Proc. Steklov Math. Inst. 226 (1999), 149–163.
[4] Chekhov L., Fock V., Observables in 3d gravity and geodesic algebras, Czech. J. Phys. 50 (2000) 1201–1208.
[5] Chekhov L.O., Mazzocco M., Isomonodromic deformations and twisted Yangians arising in Teichmüller theory, Advances Math. 226(6) (2011) 4731-4775, arXiv:0909.5350
[6] L. Chekhov and M. Mazzocco, Block triangular bilinear forms and braid group action, Comm. Math. Phys. 322 (2013) 49–71.
[7] L. Chekhov and M. Mazzocco, On a Poisson space of bilinear forms with a Poisson Lie action, Russ. Math. Surveys 72(6) (2017) 1109–1156; arXiv:1404.0988v2.
[8] Chekhov L.O., Mazzocco M. and Rubtsov V., Algebras of quantum monodromy data and decorated character varieties, arXiv:1705.01417 (2017).
[9] L.O. Chekhov and M. Shapiro, Darboux coordinates for symplectic groupoid and cluster algebras, arXiv:2003.07499, 43pp
[10] Dubrovin B., Geometry of 2D topological field theories, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., 1620, Springer, Berlin, (1996) 120–348.
[11] Fock V.V., Dual Teichmüller spaces, arXiv:dg-ga/9702018v3, (1997).
[12] V. V. Fock and A. B. Goncharov, Moduli spaces of local systems and higher Teichmüller theory, Publ. Math. Inst. Hautes Études Sci. 103 (2006), 1-211, math.AG/0311149v4.
[13] V. V. Fock and A. A. Rosly, Moduli space of flat connections as a Poisson manifold, Advances in quantum field theory and statistical mechanics: 2nd Italian-Russian collaboration (Como, 1996), Internat. J. Modern Phys. B 11 (1997), no. 26-27, 3195–3206.
[14] Fomin S., Shapiro M., and Thurston D., \textit{Cluster algebras and triangulated surfaces. Part I: Cluster complexes}, Acta Math. \textbf{201} (2008), no. 1, 83–146.

[15] A. M. Gavriliuk and A. U. Klimyk, \textit{q-Deformed orthogonal and pseudo-orthogonal algebras and their representations} Lett. Math. Phys., \textbf{21} (1991) 215–220.

[16] M. Gekhtman, M. Shapiro, and A. Vainshtein, \textit{Cluster algebra and Poisson geometry}, Moscow Math. J. \textbf{3}(3) (2003) 899-934.

[17] Goldman W.M., \textit{Invariant functions on Lie groups and Hamiltonian flows of surface group representations}, Invent. Math. \textbf{85} (1986) 263–302.

[18] Henneaux M. and Teitelboim Cl., \textit{Quantization of Gauge Systems}, Princeton University Press, 1992.

[19] M. V. Karasev, \textit{Analogues of objects of Lie group theory by nonlinear Poisson brackets}, Math. USSR Izvestia \textbf{28} (1987) 497–527.

[20] Korotkin D. and Samtleben H., \textit{Quantization of coset space \(\sigma\)-models coupled to two-dimensional gravity}, Comm. Math. Phys. \textbf{190}(2) (1997) 411–457.

[21] Kirill Mackenzie \textit{General Theory of Lie Groupoids and Lie Algebroids}, LMS Lect. Note Series \textbf{213} (2005).

[22] A. Molev, E. Ragoucy, P. Sorba, \textit{Coideal subalgebras in quantum affine algebras}, Rev. Math. Phys., \textbf{15} (2003) 789–822.

[23] Nelson J.E., Regge T., \textit{Homotopy groups and (2+1)-dimensional quantum gravity}, Nucl. Phys. B \textbf{328} (1989), 190–199.

[24] Nelson J.E., Regge T., Zertuche F., \textit{Homotopy groups and (2+1)-dimensional quantum de Sitter gravity}, Nucl. Phys. B \textbf{339} (1990), 516–532.

[25] Postnikov A., \textit{Total positivity, Grassmannians, and networks}, arXiv:math/0609764

[26] Postnikov, A., Speyer, D., Williams, L., \textit{Matching polytopes, toric geometry, and the non-negative part of the Grassmannian}, Journal of Algebraic Combinatorics 30 (2009), no. 2, 173-191.

[27] G. Schrader and A. Shapiro, \textit{Continuous tensor categories from quantum groups I: algebraic aspects}, arXiv:1708.08107

[28] Ugaglia M., \textit{On a Poisson structure on the space of Stokes matrices}, Int. Math. Res. Not. \textbf{1999} (1999), no. 9, 473–493, http://arxiv.org/abs/math.AG/9902045math.AG/9902045.

[29] A. Weinstein, \textit{Coisotropic calculus and Poisson groupoids}, J. Math. Soc. Japan \textbf{40}(4) (1988) 705–727.

[30] S. Wolfram Mathematica, wolfram.com/mathematica/.

[31] M. Yakimov, \textit{Symplectic leaves of complex reductive Poisson–Lie groups}, Duke Math. J. \textbf{112} (2002) 453–509.