A conjecture that the roots of a univariate polynomial lie in a union of annuli
(Interim Revised Version)*

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March 29, 2022

Abstract

We conjecture that the roots of a degree-\(n\) univariate complex polynomial are located in a union of \(n - 1\) annuli, each of which is centered at a root of the derivative and whose radii depend on higher derivatives. We prove the conjecture for the cases of degrees 2 and 3, and we report on tests with randomly generated polynomials of higher degree.

We state two other closely related conjectures concerning Newton’s method. If true, these conjectures imply the existence of a simple, rapidly convergent algorithm for finding all roots of a polynomial.

1 Conjecture concerning annuli

Let \(p(z)\) be a univariate polynomial with coefficients in \(\mathbb{C}\). Let \(z_1, \ldots, z_n\) be its roots. Let \(\zeta_1, \ldots, \zeta_{n-1}\) be the roots \(p'\). This paper proposes the conjecture that \(z_1, \ldots, z_n\) lie in a union of \(n - 1\) annuli, one for each of \(\zeta_1, \ldots, \zeta_{n-1}\). The two radii of each annulus are determined from higher derivative values, and the inner radius is a constant fraction of the outer radius. The formal statement is as follows.

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Conjecture 1 There exist two universal constants $0 < \iota_1 \leq 1 \leq \iota_2$ with the following property. Let $z_1, \ldots, z_n$ be the roots of a degree-$n$ complex univariate polynomial $p(z)$. Let $\zeta_1, \ldots, \zeta_{n-1}$ be the roots of its derivative. Let $\rho_1, \ldots, \rho_{n-1}$ be defined by

$$\rho_j = \min_{k=2, \ldots, n} \left| \frac{p(\zeta_j)k!}{p^{(k)}(\zeta_j)} \right|^{1/k}. \quad (1)$$

Define annulus

$$A_j = \{ z : \iota_1 \rho_j \leq |z - \zeta_j| \leq \iota_2 \rho_j \} \quad (2)$$

for $j = 1, \ldots, n-1$. Then for each $i = 1, \ldots, n$,

$$z_i \in A_1 \cup \cdots \cup A_{n-1}. \quad (3)$$

An example of the conjecture for a particular polynomial is illustrated in Fig. 1.

NOTE ADDED IN REVISION. Recently, M. Giusti, J. Heintz, G. Lecerf, L. Pardo, B. Salvy and J.-C. Yakoubsohn have shown (unpublished communication) that Conjecture 1 is partly true and partly false. In particular, they have shown that $\iota_1$ exists and may be taken to be $(\sqrt{5} - 1)/2$. They have also constructed a family of polynomials showing that $\iota_2$ cannot exist (i.e., $|z - \zeta_j|/\rho_j \to \infty$) as the degree tends to infinity for this family. Their counterexample apparently does not invalidate either Conjecture 2 or Conjecture 3 below. A more detailed revision of this paper is forthcoming.

This conjecture, if true, suggests a simple algorithm based on Newton’s method for finding all the roots of a complex univariate polynomial. The algorithm (as well as two other related conjectures) is described in Section 4.

Observe that some of the terms in the min in (1) could be infinite, but they cannot all be infinite since $p^{(n)}$ is a nonzero constant. Observe also that the conjecture is invariant under change of variables of the form $z' = az + b$ for any nonzero $a \in \mathbb{C}$ and any $b \in \mathbb{C}$. It is also invariant under conjugation and rescaling of the polynomial (i.e., replacing $p$ by $cp$ or $\bar{c}p$ for a nonzero $c \in \mathbb{C}$).
Figure 1: The ‘×’ marks shown in the figure are the 12 roots of a degree-12 polynomial plotted in the complex plane. There is a cluster of two very close roots and another cluster of three. The ‘.’ markers show the 11 roots of the derivative. The $j$th derivative root is surrounded by an annulus with inner radius $0.66 \rho_j$ and outer radius $1.33 \rho_j$, where $\rho_j$ is given by (1) and the scalars $0.66$ and $1.33$ are chosen based on experiments in Section 3 as approximations to $\nu_1$ and $\nu_2$. 

The formula (1) is motivated by considering the polynomial \( p(z) = z^n - c \). The roots of this polynomial are the \( n \) roots of \( c \). The derivative roots are all at 0. For this polynomial, all the terms in (1) are infinite except the \( k = n \) term, and therefore \( \rho_j = |c|^{1/n} \) for all \( j \). Thus, we could take \( \iota_1 = \iota_2 = 1 \) for this restricted class of polynomials. Some additional background is given in Section 4. In Section 2 we establish the conjecture for degree 2 and degree 3 polynomials. In Section 3 we describe a computational experiment to test the conjecture for high-degree polynomials.

Aside from the application to Newton’s method, the issue of location of roots with respect to the derivative roots is an inherently interesting matter that has attracted quite a bit of attention in the literature. A classic result in this regard is the Gauss-Lucas theorem, which states that the roots of \( p' \) lie in the convex hull of the roots of \( p \). The most comprehensive treatment of the relationships between the roots of \( p' \) and roots of \( p \) appears to be the monograph of [4]. The present conjecture, however, appears to provide more precise information about the location of the roots of \( p \) in terms of derivative roots than any of the known theorems in [4].

## 2 Polynomials of degree 2 or 3

It follows immediately from the observations in the previous section that the conjecture is true for polynomials of degree 2 since any nondegenerate quadratic, after suitable change of variables and rescaling, can be transformed to \( z^2 - 1 \). Thus, for quadratics, the conjecture is true with \( \iota_1 = \iota_2 = 1 \). The conjecture is trivially true in the degenerate case of a quadratic with a double root since \( \rho_1 = 0 \) and \( z_i - \zeta_1 = 0 \).

In the case of degree-3 polynomials, first consider the cases of a root with multiplicity 2 or 3. For the multiplicity-3 case, the conjecture is true for the same reason as in the last paragraph. For the multiplicity-2 case, by rescaling and changing variables, we may assume the polynomial is \( p(z) = z^2(z-1) \). In this case, \( z_1 = z_2 = 0 \) and \( z_3 = 1 \), while \( \zeta_1 = 0 \) and \( \zeta_2 = 2/3 \). Also, one checks that \( \rho_1 = 0 \) and \( \rho_2 = \min(\sqrt{4/27}, (4/27)^{1/3}) \approx .3849 \). Since \( |z_3 - \zeta_2| = 1/3 \), for the conjecture to be true in this case requires an \( \iota_1 \leq (1/3)/\sqrt{4/27} \approx 0.866 \) and \( \iota_2 = 1 \).

The remainder of this section considers the case of a cubic with three distinct roots. Again by rescaling, changing variables, and taking complex
conjugates, we may assume that the polynomial is monic and that its closest pair of roots lie at $-1$ and $1$, and that the third root $a$ lies in Quadrant I. Thus, $p(z) = (z - 1)(z + 1)(z - a)$ where $\text{Re } a \geq 0$, $\text{Im } a \geq 0$, and $|a - 1| \geq 2$. (The latter inequality follows by the assumption that $-1$ and $1$ are closer to each other than to $a$.) One checks that $p'(z) = 3z^2 - 2az - 1$ hence

$$
\zeta_1, \zeta_2 = \frac{a \pm \sqrt{a^2 + 3}}{3}.
$$

Since $a^2 + 3$ is in the upper half-plane by assumptions made about $a$, we will assume that the branch of square-root appearing in the preceding formula is chosen so that $\sqrt{a^2 + 3}$ lies in Quadrant I. We will let $\zeta_1$ be the ‘$-$’ branch and $\zeta_2$ the ‘$+$’ branch.

Let $\zeta_\nu$ stand for either $\zeta_1$ or $\zeta_2$. Solving $p'(\zeta_\nu) = 0$ yields the identity $a = (3\zeta_\nu^2 - 1)/(2\zeta_\nu)$ and hence $\zeta_\nu - a = (-\zeta_\nu^2 + 1)/(2\zeta_\nu)$.

Let us now consider the three roots $\{z_1, z_2, z_3\} = \{1, -1, a\}$ in order. For $z_1 = 1$, consider the annulus about $\zeta_1$. By definition, $\iota_1$ and $\iota_2$ must be universal lower and upper bounds on the quantity

$$
\min \left( \frac{|2p(\zeta_1)/p''(\zeta_1)|^{1/2}}{|\zeta_1 - 1|}, \frac{|p(\zeta_1)|^{1/3}}{|\zeta_1 - 1|} \right),
$$

which, after using the facts that $p(\zeta_1) = (\zeta_1 - 1)(\zeta_1 + 1)(\zeta_1 - a)$ and $\zeta_1 - a = (-\zeta_1^2 + 1)/(2\zeta_1)$, simplifies to

$$
\min \left( \frac{|(\zeta_1 + 1)^2|^{1/2}}{3\zeta_1^2 + 1}, \frac{|(\zeta_1 + 1)^2|^{1/3}}{2\zeta_1(\zeta_1 - 1)} \right). \tag{4}
$$

To establish an upper bound on (4), observe that $(3\zeta_1^2 + 1) - 2\zeta_1(\zeta_1 - 1) = (\zeta_1 + 1)^2$, hence

$$
|\zeta_1 + 1|^2 \leq |3\zeta_1^2 + 1| + |2\zeta_1(\zeta_1 - 1)|. \tag{5}
$$

Let $\alpha^*$ be chosen as the real root of

$$
(\alpha^*)^3 - \alpha^* - 1 = 0 \tag{6}
$$

This $\alpha^*$ is close to $1.3247$. It follows from (6) that $(1 + 1/\alpha^*)^{1/2}$ and $(1 + \alpha^*)^{1/3}$ are both equal to $\alpha^*$.

Turning back to (5), take two subcases depending on the relative sizes of the two terms on the right-hand side of (5). Subcase 1 is that $|3\zeta_1^2 +
\[ | \zeta_1 + 1 | \leq \alpha^* | 2 \zeta_1 (\zeta_1 - 1) |. \] In this case, the right-hand side of (5) is at most 
\[ (1 + \alpha^*) \cdot | 2 \zeta_1 (\zeta_1 - 1) |, \] and hence the second term of (4) is at most \( (1 + \alpha^*)^{1/3} \), which is equal to \( \alpha^* \).

Subcase 2 is that \[ | 3 \zeta_1^2 + 1 | \geq \alpha^* | 2 \zeta_1 (\zeta_1 - 1) |. \] In this case, the right-hand side of (5) is at most \( (1 + 1/\alpha^*) \cdot | 3 \zeta_1^2 + 1 | \) and hence the first term of (4) is at most \( (1 + 1/\alpha^*)^{1/2} \), which is equal to \( \alpha^* \). This establishes the upper bound of \( \alpha^* \) on (4).

For the lower bound, we use the following cruder argument. Let \( Q_I \) denote the first quadrant. Observe that the function of \( a \) given by 
\[ \zeta_1(a) = (a - \sqrt{a^2 + 3})/3 \] is analytic in the interior of \( Q_I \) with a singularity on the boundary (at \( \sqrt{3i} \)), and therefore its real part is harmonic. This means that the minimum value of the real part of \( \zeta_1 \) is attained either for \( a \) on the boundary of \( Q_I \) or in the limit for infinitely large \( a \). Observe that \( \zeta_1 = a(1 - \sqrt{1 - 3/a^2})/3 \), and for very large \( |a| \), \( \sqrt{1 - 3/a^2} \approx 1 - 1.5/a^2 \), hence \( \zeta_1 \approx 1/(2a) \) which tends to 0 for large \( |a| \). Thus, the minimum real part of \( \zeta_1 \) is attained on the boundary rather than \( \infty \). For \( a \) on the positive imaginary axis (one boundary of \( Q_I \), say \( a = ti \), we compute that 
\[ \zeta_1 = (-\sqrt{3 - t^2} + ti)/3, \] which has real part equal to 0 if \( t \geq \sqrt{3} \) else real part equal to \( -\sqrt{3}/3 \) or greater for \( t \in [0, \sqrt{3}] \). Along the positive real axis (the other boundary of \( Q_I \)), \( \zeta_1 \) is increasing, as one can check from the derivative, hence the minimum value of the real part is again at 0 and is equal to \( -\sqrt{3}/3 \).

Therefore, \( \text{Re}(\zeta_1 + 1) \geq 1 - \sqrt{3}/3 \) hence
\[ | \zeta_1 + 1 | \geq 1 - \sqrt{3}/3 \approx 0.423. \] (7)

(This bound would be improved if we also accounted for the constraint that \( |a - 1| \geq 2 \).)

Next we have the following chain of inequalities to analyze the first term of (4):
\[
| 3 \zeta_1^2 + 1 | = | 3 \zeta_1^2 + 6 \zeta_1 + 3 - 6 \zeta_1 - 6 + 4 | \\
\leq | 3 \zeta_1^2 + 6 \zeta_1 + 3 | + | 6 \zeta_1 + 6 | + 4 \\
= 3 | \zeta_1 + 1 |^2 + 6 | \zeta_1 + 1 | + 4 \\
= 3 | \zeta_1 + 1 |^2 + 6 \left| \frac{\zeta_1 + 1}{| \zeta_1 + 1 |} \right| + 4 \\
\leq [3 + 6/(1 - \sqrt{3}/3) + 4/(1 - \sqrt{3}/3)^2] \cdot | \zeta_1 + 1 |^2, \tag{8}
\]
where we have used the inequality (7) to obtain the last line. The quantity in square brackets is approximately 39.6.
The second term can be similarly analyzed:

\[
|2\zeta_1(\zeta_1 - 1)| = |2\zeta_1^2 + 4\zeta_1 + 2 - 6\zeta_1 - 6 + 4| \\
\leq 2|\zeta_1 + 1|^2 + 6|\zeta_1 + 1| + 4 \\
\leq [2 + 6/(1 - \sqrt{3}/3) + 4/(1 - \sqrt{3}/3)^2] \cdot |\zeta_1 + 1|^2. \tag{9}
\]

Combining (8) and (9) establishes a rather poor lower bound of \(\sqrt{1/39.6}\) on (4).

For \(z_2 = -1\), we must obtain lower and upper bounds on

\[
\frac{\min(|2p(\zeta_1)/p''(\zeta_1)|^{1/2}, |p(\zeta_1)|^{1/3})}{|\zeta_1 + 1|},
\]

which simplifies to

\[
\min \left( \left| \frac{(\zeta_1 - 1)^2}{3\zeta_1^2 + 1} \right|^{1/2}, \left| \frac{(\zeta_1 - 1)^2}{2\zeta_1(\zeta_1 + 1)} \right|^{1/3} \right). \tag{10}
\]

To establish a upper bound on (10), use an argument analogous to the preceding analysis of \(z_1\): \((3\zeta_1^2 + 1) - 2\zeta_1(\zeta_1 + 1) = (\zeta_1 - 1)^2\), hence

\[
|\zeta_1 - 1|^2 \leq |3\zeta_1^2 + 1| + |2\zeta_1(\zeta_1 + 1)|.
\]

Depending on the relative sizes of the terms on the right-hand side, either \(|\zeta_1 - 1|^2 \leq (1 + 1/\alpha^*) \cdot |3\zeta_1^2 + 1|\), implying that the first term of (10) is at most \(\alpha^*\), or else \(|\zeta_1 - 1|^2 \leq (1 + \alpha^*) \cdot |2\zeta_1(\zeta_1 + 1)|\), implying that the second term of (4) is at most \(\alpha^*\). Here, \(\alpha^*\) was defined by (6).

To establish a lower bound, again similar arguments are used. We first claim that \(\text{Re} \, \zeta_1 \leq 0\); this follows again by considering the extremal cases for \(\text{Re} \, \zeta_1\) as a function of \(\alpha\) as above. This implies that \(|\zeta_1 - 1| \geq 1\).

Then we have the following chain of inequalities to analyze the first term of (10):

\[
|3\zeta_1^2 + 1| = |3\zeta_1^2 - 6\zeta_1 + 3 + 6\zeta_1 - 6 + 4| \\
\leq |3\zeta_1^2 - 6\zeta_1 + 3| + |6\zeta_1 - 6| + 4 \\
= 3|\zeta_1 - 1|^2 + 6|\zeta_1 - 1| + 4 \\
= 3|\zeta_1 - 1|^2 + 6\frac{|\zeta_1 - 1|^2}{|\zeta_1 - 1|} + 4 \\
\leq (3 + 6 + 4)|\zeta_1 - 1|^2,
\]
where we have used the inequality $|\zeta_1 - 1| \geq 1$ to obtain the last line.

The second term can be similarly analyzed:

$$|2\zeta_1(\zeta_1 + 1)| = |2\zeta_1^2 - 4\zeta_1 + 2 + 6\zeta_1 - 6 + 4| \leq 2|\zeta_1 - 1|^2 + 6|\zeta_1 - 1| + 4 \leq (2 + 6 + 4)|\zeta - 1|^2.$$ 

The last root to analyze is $z_3 = a$; for this root we will consider the annulus about derivative root $\zeta_2$ instead of $\zeta_1$. The quantity to analyze is

$$\min\left(\frac{|2p(\zeta_2)/p''(\zeta_2)|^{1/2}}{|\zeta_2 - a|}, \frac{|p(\zeta_2)|^{1/3}}{|\zeta_2 - a|}\right),$$

which, after simplification, is equal to

$$\min \left( \frac{4\zeta_2^2}{3\zeta_2^2 + 1} \right)^{1/2}, \frac{4\zeta_2^2}{\zeta_2^2 - 1} \right)^{1/3}. \tag{11}$$

An upper bound on (11) is obtained by observing that $4\zeta_2^2 = 3\zeta_2^2 + 1 + \zeta_2^2 - 1$, hence

$$|4\zeta_2^2| \leq |3\zeta_2^2 + 1| + |\zeta_2^2 - 1|.$$ 

Then, using the same logic as in the previous two cases, we conclude that (11) has $\alpha^*$ as an upper bound.

For the lower bound, observe that $\sqrt{a^3 + 3}$ lies in Quadrant I (denoted $Q_I$) provided that $a \in Q_I$. If $w_1, w_2$ are any two complex numbers both lying in $Q_I$, then $|w_1 + w_2| \geq |w_1|$. Therefore, since $\zeta_2 = (a + \sqrt{a^3 + 3})/3$, we conclude that $|\zeta_2| \geq |a|/3$. The assumptions $|a - 1| \geq 2$ and $a \in Q_I$ together imply that $|a| \geq \sqrt{3}$, and therefore $|\zeta_2| \geq \sqrt{3}/3$, hence $|\zeta_2^2| \geq 1/3$.

The inequality derived in the last paragraph yields a lower bound on both terms of (11). For the first term, $|3\zeta_2^2 + 1| \leq 3|\zeta_2^2| + 1 \leq 6|\zeta_2^2|$. Therefore, the first term of (11) is at least $(2/3)^{1/2}$. For the second term, a similar use of the previous paragraph shows $|\zeta_2^2 - 1| \leq 4|\zeta_2^2|$, and hence the second term is at least 1.

This concludes the analysis of the $n = 3$ case. We have shown an upper bound of $\alpha^*$ for $\iota_2$ and a lower bound of $\sqrt{1/39.6}$ for $\iota_1$. We have written a Matlab program that computes $\iota_1$ and $\iota_2$ for each cubic polynomial with roots at $-1, 1, a$, where $a$ ranges over a fairly dense grid lying in the set $\{a \in Q_I : |a - 1| \geq 2\}$. We found that $\iota_1$ appears to be approximately 0.82 while $\iota_2$ appears to be exactly $\alpha^*$, and in particular, the upper bound of $\alpha^*$ on (11) appears to be tight.
3 Computational experiment with higher degree polynomials

In this section we describe our Matlab computational experiment with higher degree polynomials.

We experiment with three degrees: \( n = 10, 20, 40 \). For each \( n \), we generate 3000 random polynomials. Each polynomial is chosen by selecting its \( n \) roots uniformly at random on the unit circle. The rationale for this choice (as opposed to a distribution over a 2-dimensional domain) is to greatly increase the likelihood of nearby or clustered roots, which is a more difficult case for root-finding.

The roots of the derivative polynomial are then computed, as are the parameters \( \rho_1, \ldots, \rho_{n-1} \) given by (1). For each polynomial root \( z_i, i = 1, \ldots, n \), the program seeks the \( j \) in \( \{1, \ldots, n-1\} \) such that \( |z_i - \zeta_j|/\rho_j \) is closest to 1. Call this quotient \( \iota_{z_i} \). The program then tabulates the minimum and maximum \( \iota_{z_i} \) encountered among all 3000 trials; these are taken to be estimates for \( \iota_1 \) and \( \iota_2 \).

The results are as follows. For \( n = 10 \), \( \iota_1 = 0.67 \) and \( \iota_2 = 1.32 \); for \( n = 20 \), \( \iota_1 = 0.66 \) and \( \iota_2 = 1.33 \); for \( n = 40 \), \( \iota_1 = 0.66 \) and \( \iota_2 = 1.33 \). Thus, there seems to be little appreciable change in the experimental values of \( \iota_1 \) or \( \iota_2 \) as the degree increases.

We mention two subtleties concerning the implementation of this computational experiment. As mentioned above, the program selects the roots of the polynomial at random on the unit circle and then computes the derivative roots. The naive method to compute derivative roots, namely, form the standard monomial basis for the polynomial, differentiate it term by term, and then use the Matlab \texttt{roots} function on the derivative, is unstable for polynomials with clusters of roots. This naive implementation gave incorrect experimental results. We found that a better method for finding derivative roots is to compute the eigenvalues of the \((n-1) \times (n-1)\) matrix

\[
M = \text{diag}(z_2, \ldots, z_n) - \mathbf{ev}^T/n
\]

where \( \mathbf{e} \) is the vector of all 1’s and \( \mathbf{v} \) is the vector whose \( i \)th entry is \( z_{i+1} - z_1 \). A brief explanation of why these eigenvalues are derivative roots is as follows. If \( M\mathbf{x} = \zeta\mathbf{x} \), then \( z_ix_i - d = \zeta x_i \) for each \( i = 2, \ldots, n \), where \( d = \mathbf{v}^T\mathbf{x}/n \). Solving yields \( x_i = d/(z_i - \zeta) \). Substituting this formula for \( x_i \) into \( d = \mathbf{v}^T\mathbf{x}/n \) and simplifying yields \( 1/(\zeta - z_1) + \cdots + 1/(\zeta - z_n) = 0 \), which is the same
as \( p'(\zeta) = 0 \).

It is easy to check that this method for derivative roots works much better than the naive method for contrived examples of polynomials with root clusters, e.g., the polynomial \((z - 2)^{10} - .01^{10} = 0\), which has 10 roots lying on a circle of radius 0.01 about the point 2. We have not, however, attempted a formal proof of stability of this method.

A second stability subtlety is the computation of \( \rho_j \). The naive method, namely, forming \( p \) in the standard monomial basis and differentiating term by term to obtain all the derivatives in (1), is unstable. Our implementation uses the following method, which appears to be stabler. Form the standard monomial representation of the polynomial \( q(z) = p(z + \zeta_j) \) by multiplying together its degree-1 factors \((z - z_1 + \zeta_j), \ldots (z - z_n + \zeta_j)\) (or equivalently, by applying the Matlab \texttt{poly} function to the \( n \) shifted roots \( z_1 - \zeta_j, \ldots, z_n - \zeta_j \)). Then derivatives of the form \( p^{(k)}(\zeta_j) \) are directly obtained from the coefficients of \( q \).

## 4 Application to Newton’s method

Recall that Newton’s method for finding a root of a complex function \( \phi \) is given by the iteration

\[
x^{k+1} = x^k - \frac{\phi(x^k)}{\phi'(x^k)}.
\]

Newton’s method is known to converge quadratically to a nondegenerate root \( x^* \) if the starting point \( x^0 \) is sufficiently close to \( x^* \). The \textit{basin of attraction} for \( x^* \) is the set of starting points \( x^0 \) such that Newton’s method will converge to \( x^* \) for that starting point. Note that although Newton’s method is asymptotically quadratic, a point in the basin of attraction could lead to a sequence of iterates that meanders far away from \( x^* \) for an arbitrary number of iterations before asymptotic quadratic convergence takes hold.

This difficulty leads us to define the \textit{basin of fast convergence} for \( x^* \) to be the set of \( x^0 \) such that the sequence of iterates generated by Newton’s method starting from \( x^0 \) converges quadratically immediately (rather than asymptotically) according to the following inequality:

\[
|x^k - x^*| \leq \left( \sqrt{\frac{1}{2}} \right)^{2k-1} |x^0 - x^*|
\]

for all \( k \geq 0 \). This definition is similar to one from Blum et al. [1].
For a degree-$n$ polynomial $p$ such that the roots of $p'$ are $\zeta_1, \ldots, \zeta_{n-1}$, let us define the $DR$-circles to be the set of circles about $\zeta_1, \ldots, \zeta_{n-1}$ of radius $\rho_1, \ldots, \rho_{n-1}$ respectively. (Here, DR stands for “derivative root.”) By Conjecture 1, the roots of $p$ apparently all lie close to the union of its DR-circles. This suggests that many points on DR-circles lie in the basins of fast convergence of the roots of $p$.

There are several different possible conjectures that could be made about DR-circles and basins of convergence. The first conjecture is that the basin of fast convergence of each root of $p$ contains a subsegment of at least one DR circle. Consideration of the polynomial $z^n - c$ indicates that the length of this segment could be as small as $O(1/n)$ radians. This is our conjecture:

**Conjecture 2** There is a universal constant $\eta_1 > 0$ with the following property. Let $p$ be a degree-$n$ univariate complex polynomial whose roots are $z_1, \ldots, z_n$. Let $C_1, \ldots, C_{n-1}$ be the DR-circles of $p$, that is, circles centered about the roots $\zeta_1, \ldots, \zeta_{n-1}$ of $p'$ such that the radius of $C_j$ is $\rho_j$ as defined by (1). Let $z_i$ be any root of $p$ of multiplicity 1. Then there exists a segment of a DR-circle of length $\eta_1/n$ radians lying in the basin of fast convergence of $z_i$.

To test this conjecture, we used the same set-up (3000 polynomials of degrees 10, 20 and 40) as in Section 3. We discretized each DR-circle with $10n$ evenly spaced points, and for each root of $p$ and each DR-circle we counted the number of such DR-circle points in the root’s basin of fast convergence. If the conjecture were true, this number would always be greater than a positive constant $10\eta_1/(2\pi)$ for at least one DR-circle. In fact the minimum number for $n = 10$ was 7, for $n = 20$ was 10, and for $n = 40$ was 13. This gives some evidence in favor of the conjecture.

If this conjecture were true, it would imply a very simple algorithm based solely on Newton’s method for finding all the roots of a degree-$n$ polynomial. First, find the unique root of the linear polynomial $p^{(n-1)}$. From this root of $p^{(n-1)}$, find the two roots of $p^{(n-2)}$ by starting Newton’s method from a sufficient number of sample points on the DR-circle of $p^{(n-2)}$. Once the two roots of $p^{(n-2)}$ are found, construct the DR-circles of $p^{(n-3)}$, sample them with points, and carry out Newton’s method to find roots of $p^{(n-3)}$, etc., until finally we find the roots of $p$ from those of $p'$. The conjecture suggests that the number of sample points per DR-circle ought to be $O(n)$.

The complexity of this algorithm may be estimated as follows. Suppose the roots lie in a disk of radius $R$ and root accuracy of $\epsilon$ is desired. Starting
from the basin of fast convergence, Newton’s method requires $O(\log \log(R/\epsilon))$ iterations to achieve the desired accuracy. See Renegar [5] for a more careful explanation of the factor $\log \log(R/\epsilon)$ as well as a matching lower bound.

For finding the roots of $p^{(n-k)}$, we require Newton’s method to be started on $k-1$ circles, with $O(k)$ points per circle. Each iteration of Newton’s method requires $O(k)$ arithmetic operations. Thus, the number of operations for the roots of $p^{(n-k)}$ is $O(k^3 \log \log(R/\epsilon))$. This is summed from $k = 1, \ldots, n$, yielding a bound of $O(n^4 \log \log(R/\epsilon))$ operations.

The previous literature has many algorithms for finding all roots of a univariate polynomial; see e.g., the survey of Pan [3]. Our complexity bound is worse than published bounds for rootfinding in terms of its dependence on $n$, although it is much simpler than most algorithms.

Another rootfinding algorithm that uses only Newton’s method is due to Hubbard et al. [2] and is even simpler than ours in that it uses a fixed set of Newton starting points that depends only on the degree $n$. The drawback of the algorithm of Hubbard et al. is that Newton’s method is in general not necessarily initiated in the basin of fast convergence, so complexity estimates are far from optimal.

Renegar also has an algorithm [5] for all roots of a univariate polynomial based primarily on Newton’s method. Renegar’s algorithm always initiates Newton’s method in the basin of fast convergence and hence also has a running time proportional to $\log \log(R/\epsilon)$ but has a better dependence on $n$ than ours. Renegar’s algorithm, which to some extent motivated the present work, is based on the following key idea. A root $z$ of a polynomial $p$ has a large basin of fast convergence unless $z$ is part of a cluster of closely spaced roots. Suppose, for example, that three roots of $p$ are clustered, and no other root is nearby. In this case, all three will have small basins of fast convergence. On the other hand, it is guaranteed in this case that there is a root of $p''$ close to the three clustered roots, and this root of $p''$ will have a large basin of fast convergence. More generally, an isolated cluster of $k$ roots must be near a root of $p^{(k-1)}$ that has a large basin of fast convergence. Thus, Renegar’s algorithm consists of zooming in on root clusters (possibly recursively, since clusters can be nested inside other clusters) by using Newton’s method for roots of derivatives. Once a point near the cluster is found, subdivision is used to find the basin of fast convergence for each individual root in the cluster. The drawback of Renegar’s method is that, in addition to Newton’s method, it involves some other operations such as computation of approximate winding numbers that might be difficult to implement in practice.
A proof of Conjecture 2 might lead to further insight that would reduce the $n^4$ factor in the complexity bound. For example, suppose it were possible to predict which DR-circle would have at least one constant-sized segment in the union of basins of fast convergence. In this case, we could modify the above procedure by tracking only one DR-circle per derivative and sampling that circle with a constant number of points. This yields an $O(n^2 \log \log(R/\epsilon))$ algorithm for finding a single root of $p$. Then this root could be used to deflate the polynomial, and the process could be repeated, yielding an $O(n^3 \log \log(R/\epsilon))$ algorithm to find all roots.

We check this latter possibility by computing, for each polynomial in our test set, what is the minimum number of DR-circles that have at least 1/10 of their sample points (i.e., a total segment length of $2\pi/10$ radians) in a basin of fast convergence. This number appears to grow linearly: for $n = 10$, there were always at least 6 such DR-circles, for $n = 20$ there were always at least 13, and for $n = 40$ there were at always at least 25. Let us state this as another conjecture.

**Conjecture 3** There exist two universal constants $\eta_2 > 0$, $\eta_3 > 0$ as follows. For any degree-$n$ polynomial $p$, at least $\eta_2 n$ of its DR-circles contain segments of length $\eta_3$ radians that lie in the union of the basins of fast convergence of roots of $p$.

A final possible conjecture concerns the total length of basins of convergence. The total length of DR-circles in radians is $2\pi(n - 1)$, and $p$ has $n$ roots, so one might conjecture that the basin of fast convergence of a particular root meets DR-circles in at least a constant number of total radians. Our computational experiment, however, did not support this conjecture—in fact, our experiments suggests that it is more likely that the minimum number of total radians in the basin of convergence for a particular root is $O(1/n)$, i.e., no better than what Conjecture 2 implies for a single DR-circle.

Again, there is one subtle numerical stability issue concerning the tests in this section. If the Newton update term $p(z)/p'(z)$ is computed by writing down $p$ and $p'$ in standard monomial form and substituting the current iterate, then an unstable procedure results and the computational test yields false results. Instead, the computational test uses the mathematically equivalent formula

$$p(z)/p'(z) = \frac{1}{(z - z_1)^{-1} + \cdots + (z - z_n)^{-1}}.$$
This formula is applicable only in the case that the roots of $p$ are already known, which obviously would not occur in practical application of Newton’s method.

5 Concluding remarks

This paper raises three conjectures concerning the location of roots of a polynomial. Clearly, the main topic for future work would be to prove one of them. Another interesting question for practical work concerns a numerically stable implementation of the root-finding procedure outlined in the previous section. We imagine that for numerical stability, Newton’s method should always be applied to shifted polynomials (i.e., polynomials of the form $q(z) = p(z - s)$, where $p$ is the original polynomial), where the shift is close to the sought-after root.

A final interesting question is whether the conjecture generalizes to arbitrary entire functions whose roots have finite multiplicities. In this case, $k$ appearing in (1) ought to run from 2 to $\infty$.

6 Acknowledgment

The author thanks James Renegar and Alex Vladimirsky of Cornell for helpful discussions about this work.

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