The sound damping constant
for generalized theories of gravity

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Abstract

The near-horizon metric for a black brane in Anti-de Sitter (AdS) space and the metric near the AdS boundary both exhibit hydrodynamic behavior. We demonstrate the equivalence of this pair of hydrodynamic systems for the sound mode of a conformal theory. This is first established for Einstein’s gravity, but we then show how the sound damping constant will be modified, from its Einstein form, for a generalized theory. The modified damping constant is expressible as the ratio of a pair of gravitational couplings that are indicative of the sound-channel class of gravitons. This ratio of couplings differs from
both that of the shear diffusion coefficient and the shear viscosity to entropy ratio. Our analysis is mostly limited to conformal theories but suggestions are made as to how this restriction might eventually be lifted.

1 Introduction

The near-horizon geometry of a black brane in an Anti-de Sitter (AdS) spacetime provides a translationally invariant and thermally equilibrated background; two of the characteristic features of any hydrodynamic theory. Indeed, the long-wavelength fluctuations of the near-horizon metric are known to satisfy equations of motion that are completely analogous to the hydrodynamic equations of a viscous fluid [1]. The very same statements can be made about the metric near the AdS boundary. It is, however, quite remarkable that this pair of effective theories appears to be described by an equivalently defined set of hydrodynamic parameters [1 2 3 4]; this, in spite of their obvious lack of proximity.

The relevant thermodynamic and hydrodynamic parameters — such as the entropy density or the various transport coefficients — are intrinsic properties of the black brane horizon. Consequently, such parameters should be determined by the near-horizon metric. This makes it all the more phenomenal that the very same parameters are employed in the AdS boundary theory. In the context of the gauge–gravity duality, this apparent non-locality becomes much more sensible. The duality relates the AdS boundary hydrodynamics to the hydrodynamics of strongly coupled gauge theories [5 6]. Mean-
while, the dual gauge theory is supposed to have its thermal properties ascribed to it, holographically, by the thermodynamic nature of the black brane.

Most calculations in this genre take place at the outer boundary, as this is the most convenient surface for relating the bulk gravitational theory to its gauge-theory dual. In many ways, however, the most natural setting is at the black brane horizon, where the various hydrodynamic parameters are actually defined. The graviton hydrodynamic “fluid” can be interpreted as “living” on the stretched horizon, and so it would be rather disturbing if the actual calculations could only be performed on a surface that is displaced a spacetime away. Our results will make it clear that there is really nothing particularly special about either the stretched horizon or the AdS outer boundary. Rather, all calculations might as well be done on any radial shell that is external to the horizon.

Strongly coupled gauge theories provide an intriguing theoretical laboratory to investigate the field of relativistic hydrodynamics. It is hoped that, by applying the duality to certain calculations on the gravity side, one would be able explain the experimental results of — for instance — heavy-ion collisions [6]. However, advancements in this direction has been somewhat impeded for the following reason: Studies on the hydrodynamics of the AdS boundary have, for the most part, been limited to Einstein’s theory of gravity, which — from the gauge-theory perspective — corresponds to infinitely strong ’t Hooft coupling. Insofar as the objective is to apply what can be learnt from the duality to physically real systems, one actually requires knowledge about gauge theories at finite values of ’t Hooft coupling.
As it so happens, a strong-coupling expansion on the gauge-theory side corresponds to an expansion in the number of derivatives on the gravity side. Since Einstein’s gravity is only a two-derivative theory, it should be clear that describing a finitely coupled gauge theory necessitates some sort of extension of Einstein’s theory. To put it another way, any discernible progress will depend upon our understanding of the boundary hydrodynamics for theories of generalized gravity.

In a previous paper [7], we were able to establish two relevant points. First, with the focus on the shear channel of fluctuating gravitational modes, it was shown that the AdS boundary hydrodynamics can be translated to and localized on any radial shell in the accessible spacetime; including at the (stretched) horizon of the black brane. Then, by following [8], we explicitly demonstrated how this formalism can be extended to any generalized (or Einstein-corrected) gravitational theory.

In [8, 7], we used insight from [9] to make a pertinent observation: Various hydrodynamic parameters of an AdS brane theory can be identified with the different components of a (generally) polarization-dependent gravitational coupling $\kappa_{\mu\nu}$. Meaning that, for a generic theory, differently polarized gravitons will effectively have differing Newton’s constants. As shown in [9], this distinction can be quantified at a remarkably rigorous level. With this prescription, the shear viscosity to entropy density ratio $\eta/s$ is generalized from its “standard” (Einstein) value of $1/4\pi$ according to [8] $\frac{\eta}{s} = \frac{1}{4\pi} \left( \frac{(\kappa_{rt})^2}{(\kappa_{xy})^2} \right)$, with the precise meaning of the subscripts to be clarified below. Moreover, the central finding of [7] was that the shear diffusion coefficient $D$ is modified from its usual expression $1/4\pi T$ into the form $D = \frac{\kappa_{zx}^2}{\kappa_{tx}^2} \frac{1}{4\pi T}$.
with $T$ being the temperature.

One should take note that the coupling ratios for $\eta/s$ and $D$ involve different polarization directions. This is a natural consequence of the class of gravitons that is implicated by each of the hydrodynamic parameters. In the so-called radial gauge, the non-vanishing gravitons separate into three decoupled classes or “channels”: scalar, shear and sound [10]. The shear viscosity $\eta$ is most directly associated with the first of these classes, whereas the shear diffusion coefficient $D$ is a characteristic of the second. As for the third class, one would analogously associate with it the sound damping constant $\Gamma$, as well as the sound velocity (squared) $c_s^2$.

The purpose of the current paper is to analyze the case of sound-mode fluctuations. A straightforward extension of previous analyses is inhibited by two technical issues that are intrinsic to almost any rigorous study of the sound channel. First, for a non-conformal gauge theory, the sound-channel analysis is highly model specific. Second, the same non-conformality induces would-be radial invariants to vary with radial position in the bulk. (See [11] for a discussion.) As a consequence, $\Gamma$ and $c_s^2$ are, even for Einstein’s gravity, model-dependent parameters that vary with radial position in a model-specific way.

We can still be quite definitive by restricting the immediate considerations to conformal theories. When conformality is protected by effectively “switching off” all massive fields, the above complications will no longer be of issue. At the same time, we will still be able to make statements about how deviations from conformality should influence the ensuing results. In this sense, the current study can be viewed as a significant first step towards a fully generic analysis.
Similarly to [7], we will begin here by establishing a direct connection between sound-mode (conformal) hydrodynamics on the AdS outer boundary and on any other radial shell up to the horizon of the black brane. This will be accomplished by examining the correlator of an appropriately defined graviton and verifying that its pole structure, which determines the associated dispersion relation, is a radial invariant.

Next, we will determine how this correlator pole is explicitly modified for a generalized (although still conformal) theory of gravity. This will enable us to extract the Einstein-corrected form of the damping constant $\Gamma$. Additionally, we will confirm that the sound velocity $c_s^2$ remains fixed at its conformal value. As also discussed, the very same outcomes can be deduced through an inspection of the conservation equation for the dissipative stress tensor.

The paper will conclude with a preliminary discussion of possible extensions of our analysis to the non-conformal case.

Note that, to avoid needless repetition, some salient points that are already covered thoroughly in [7] (also see [8]) will only be glossed over here.

## 2 Sound mode conformal hydrodynamics for Einstein’s gravity

Let us first introduce some notation and conventions, as well as establish the basic framework. We will be considering a black $p$–brane in a $d + 1$-dimensional (asymptotically) AdS spacetime. (Note that
Given translational invariance and spatial isotropy on the brane along with a static spacetime, the associated metric can always be expressed in the generic brane form

\[ ds^2 = -g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{xx}(r) \left( \sum_{i=1}^{p} dx_i^2 \right), \tag{1} \]

where \( g_{tt}(r) \) has a simple zero and \( g_{rr}(r) \) has a simple pole at the horizon \( r = r_h \), while \( g_{xx}(r_h) \) is finite and positive. For any \( r > r_h \), these metric components are all well-defined and strictly positive functions that go asymptotically to their respective AdS values \( (L^2/r^2) \) for \( g_{rr} \), otherwise \( r^2/L^2 \) as \( r \to \infty \). \( L \) is the AdS radius of curvature.

If the background theory is conformal, then one can be much more explicit. Assuming, for the sake of simplicity, that the brane is electromagnetically neutral, we obtain the Schwarzschild-like form such that \( g_{xx} = r^2/L^2 \) and \( g_{tt} = 1/g_{rr} = g_{xx} f(r) \), with \( f(r) = 1 - (r_h/r)^{p+1} \).

It is often convenient to re-express this conformal metric by changing the radial coordinate to \( u = r_h^2/r^2 \); then

\[ ds^2 = -\frac{r_h^2}{L^2 u} f(u)dt^2 + \frac{L^2}{4u^2 f(u)} du^2 + \frac{r_h^2}{L^2 u} \left( \sum_{i=1}^{p} dx_i^2 \right), \tag{2} \]

with \( f(u) = 1 - u \frac{p+1}{2} \) and the horizon (outer boundary) now located at \( u = 1 \) \( (u = 0) \). When non-conformal theories are discussed, \( u \) will refer to a radial coordinate that is appropriately defined so as to extend over the same range of values.

Brane hydrodynamics entails expanding the metric: \( g_{\mu\nu} \to \bar{g}_{\mu\nu} + h_{\mu\nu} \), with \( h_{\mu\nu} \) representing the fluctuations or gravitons. Let us — without loss of generality — specify \( x_p \) to be the direction of graviton propagation on the brane and re-label it as \( z \). It follows that
\[ h_{\mu\nu} \sim \exp[-i\Omega t + iQz] \] (and, otherwise, depending only on \( u \)), where 
\( (\Omega, 0, \ldots, 0, Q) \) is the \( p + 1 \)-momentum of the graviton.

The choice of radial gauge, \( h_{u\alpha} = 0 \) for any \( \alpha \), is known to separate the non-vanishing fluctuations into three decoupled classes \([10]\). Our class of current interest — namely, the sound channel — includes the non-vanishing diagonal gravitons \( h_{\alpha\alpha} \) (\( \alpha \neq u \)) along with \( h_{tz} \).

Let us take note of the sound-mode dispersion relation \( \Omega = \pm c_s Q - i\Gamma Q^2 + O(Q^3) \) or, equivalently (given that the hydrodynamic or long-wavelength limit is in effect),

\[
\Omega^2 = c_s^2 Q^2 + i2\Gamma Q^2 + O(Q^4). \tag{3}
\]

Here, \( c_s^2 \) is the sound velocity (squared) and \( \Gamma \) is the sound damping constant. For a \( p + 1 \)-dimensional conformal theory, \( c_s^2 = 1/p \) and \( \Gamma \) is directly proportional to the shear viscosity to entropy density ratio times the inverse temperature: \( \Gamma = \frac{p-1}{p-2} \frac{n}{T} \). So that, for a \( p \)-brane theory of Einstein’s gravity, one can deduce that \( \Gamma = \frac{p-1}{4\pi T} \). \([12, 13]\), where \( T \) is the coordinate-invariant Hawking temperature of the brane. For this conformal case, \( T = (p + 1)\eta / 4\pi L^2 \). Meanwhile, for a non-conformal theory, both parameters can differ appreciably from their conformal values. For \( \Gamma \), this model-specific deviation is expressible in terms of the bulk viscosity \( \zeta \).

For a complete derivation of the sound-mode correlator (which is not needed here), one can follow the by-now standard prescription as documented in, for instance, \([14, 15, 16]\). The first step is to identify a gauge-invariant combination of the sound-mode fluctuations \( H_{tt} = (1/g_{tt})h_{tt}, \ H_{zz} = (1/g_{xx})h_{zz}, \ H_{tz} = (1/g_{xx})h_{tz} \) and
\[ H_X = \left( \frac{1}{g_{xx}} \right)^{\frac{1}{p-1}} \left( \sum_{i=1}^{p-1} h_{x_i x_i} \right): \]

\[ Z = q^2 \frac{g_{tt}}{g_{xx}} H_{tt} + 2q \omega H_{tz} + \omega^2 [H_{zz} - H_X] + q^2 \frac{g_{tt}'}{g_{xx}'} H_X. \tag{4} \]

Here, a prime indicates a differentiation with respect to \( u \); while \( \omega = \Omega / 2\pi T \) and \( q = Q / 2\pi T \) represent, respectively, a dimensionless frequency and wavenumber. In the hydrodynamic limit, \( \omega \) and \( q \) are both vanishing although not \textit{a priori} at the same rate.

The conformal version of the solution for \( Z \) can readily be extracted out of the existent literature — for instance, \[15, 16, 17\]. For the appropriately chosen boundary conditions (as discussed below), one finds that

\[ Z = C f(u)^{-\frac{1}{p-1}} \left[ Y(u) - \frac{\omega^2}{q^2} p - i \omega (p-1) f(u) + \mathcal{O}(q^2, \omega^2) \right], \tag{5} \]

where \( C \) is an integration constant (to be fixed by normalization considerations) and we have defined \( Y(u) \equiv g_{tt}' / g_{xx}' \). For this theory in particular, \( Y = (f / u)' / (1/u)' = f - uf' \), which is everywhere positive, non-vanishing and \( \mathcal{O}(1) \). Also note that \( Y = 1 \) on the outer boundary.

A specified pair of boundary conditions determines the solution for \( Z \). At the horizon \( u = 1 \), the solution should be that of an incoming plane wave, which determined the form of Eq. (5). In addition, the so-called Dirichlet boundary condition still needs to be imposed. It has become almost traditional to single out the AdS boundary and choose \( u^* = 0 \) as the radius at which this condition is enforced; however, one can freely impose this condition at any fixed radius \( u^* \) within \( 0 \leq u^* < 1 \). The Dirichlet boundary condition necessitates that \( Z(u) \) is, \textit{prior} to its normalization (see below), vanishing as \( u \to u^* \). Applying this condition to Eq. (5), we promptly obtain the associated dispersion
relation
\[\omega^2 = q^2 \frac{1}{p} Y(u^*) + i \omega q^2 p \frac{1}{p} f(u^*) + \mathcal{O}(q^4) , \quad (6)\]
where it is now clear that \( q \) and \( \omega \) are of the same order in the hydrodynamic limit. Let us choose, for instance, the “orthodox” boundary location of \( u^* = 0 \), so that Eq. (6) leads to
\[\omega^2 = \frac{q^2}{p} + i \omega q^2 (p - 1)/p + \mathcal{O}(q^4) . \]
Comparing this to the standard dispersion relation in Eq. (3), one can readily verify the expected identifications
\[c_s^2 = \frac{1}{p} \quad \text{and} \quad \Gamma = \frac{p - 1}{4\pi p T} \]
for a conformal theory.

One further normalization condition that complements the Dirichlet boundary condition is that \( Z \), rather than vanishing at \( u^* \), should ultimately be normalized to unity there. This can be achieved by the unique choice
\[C^{-1} = \left[ Y(u^*) - \frac{\omega^2}{q^2 p} - i \omega (p - 1) f(u^*) \right] . \quad (7)\]
Let us take notice that, given the associated dispersion relation, \( C^{-1} \) is a vanishing quantity as it must be to obtain a finite value of \( Z(u^*) \).

The normalized value of the field mode is simply 1, and so one might wonder as to the physical significance of the implied dispersion relation. However, we are simply using the standard “trick” of field-theoretic calculations to obtain the pole in the correlator. The properly normalized correlator \( G_{ZZ} \) for this gauge-invariant variable (up to an inconsequential numerical factor) is given by the boundary residue of the canonical term in the bulk action or \( G_{ZZ} \sim Z |_{u = u^*} \).

It should not be difficult to convince oneself that, at the leading hydrodynamic order, this quantity goes as \( G_{ZZ} \sim C \cdot \mathcal{O}(q^0) \), which is notably divergent and of finite hydrodynamic order. A proper accounting of the metrical factors in the action — namely, the product
The square root of $-g^{uu}$ — reveals that there are no other hidden zeros or infinities in this calculation at any permissible value of $u^*$.

What is really significant here is that — from the quasinormal-mode perspective of brane hydrodynamics [15] — the pole in the correlator assigns a clear physical credence to Eq. (6) as the spectrum for the dissipative modes of the black brane. However, one could (and should!) be rightfully concerned that this dispersion relation appears to vary as the Dirichlet-boundary surface is moved radially through the spacetime. This is not only in conflict with intuitive expectations but with the analysis of [11], where it is made evident that (inasmuch as the theory is conformal) both the sound mode and its correlator should be radial invariants.

We can readily account for the undesirable factor of $f(u^*)$ in the second term of Eq. (6): As detailed in [7], $\omega$ and $q$ should naturally be sensitive to the effects of a gravitational redshift. It was then argued — in the context of shear modes — that consistency of the hydrodynamic expansion along with protection of the incoming boundary condition necessitates that $\omega$ remains fixed while $q^2$ scales as $f^{-1}$. That is, $\omega(u) = \omega_b$ and $q^2(u) = q^2_b/f(u)$ (with the subscript $b$ indicating the outer-boundary value of a quantity). Then, since the gravitational redshift should not be able to discriminate between the different channels being probed, it follows that these same relations should persist for the current case.

This brings us to the first term in Eq. (6), which has the awkward appearance of $Y(u^*)$ to be dealt with. Clearly, this will require new inputs. The key here is the association of this term with $c_S^2$; cf. Eq. (3). Normally, the sound velocity of a hydrodynamic fluid is presented as
the variation of the pressure with respect to the energy density or 
\( c_s^2 = \frac{\delta P}{\delta \epsilon} \). This cannot, however, be a universally accurate account.

A closer look at the derivation of the sound dispersion relation (see, e.g., [13]) reveals that the actual variation which enters under the guise of the sound velocity comes packaged in the term \((\delta T^{zz}/\delta T^{tt}) \partial_z^2 T^{tt}\), where \( T^{\alpha\beta} \) is the stress tensor for the brane theory. For a flat or an effectively flat brane, such as at the AdS outer boundary, this distinction is of no consequence, but this is not a general truism. On a “warped” brane, rather, \( T^{zz} = g^{zz} P \) and \( T^{tt} = -g^{tt} \epsilon \). Hence, the correct statement about the relevant term is (by way of the chain rule)

\[
\frac{\delta T^{zz}}{\delta T^{tt}} \partial_z^2 T^{tt} = g^{zz} \frac{\delta P}{\delta \epsilon} \partial_z^2 \epsilon + g^{tt} \frac{P}{\epsilon} \frac{\partial_u g^{zz}}{\partial_e g^{tt}} \partial_z^2 \epsilon .
\]  

(8)

We will now argue that the first term on the right-hand side of Eq. (8) is parametrically smaller than the second and, thus, the former can be disregarded for current purposes. It follows from the thermodynamic relation \( sT = \epsilon + P \) and the infinite transverse volume of the brane that \( \partial P/\partial \epsilon = 0 \). So, to the leading non-vanishing order, \( \delta P = \frac{1}{2} \frac{\partial^2 P}{\partial \epsilon^2} (\delta \epsilon)^2 \) or \( \frac{\delta P}{\delta \epsilon} \sim \frac{\delta \epsilon}{\epsilon} \ll 1 \). On the other hand, \( P/\epsilon = \frac{1}{p} \) is of the order of unity. Now, comparing \( \frac{\partial}{\partial_u} \frac{\partial g^{zz}}{\partial u} \) with \( g^{zz} \), one will find that the ratio of these quantities is of \( \mathcal{O}(1) \).

Having deemed the first term in Eq. (8) as inconsequential, we need only to evaluate the second. Since the brane metric is diagonal (so that \( g^{\alpha\alpha} = g_{\alpha\alpha}^{-1} \)), the right-hand side reduces to

\[
\frac{\delta T^{zz}}{\delta T^{tt}} \partial_z^2 T^{tt} = \frac{P}{\epsilon} \frac{g_{tt}}{g_{xx}^2} Y^{-1} \partial_z^2 \epsilon + \cdots ,
\]  

(9)

where we have returned to the brane notation of Eq. (1) (so that \( g_{tt} > 0 \)) and recalled the definition of \( Y(u) \) beneath Eq. (5).
Actually, all other terms in the dispersion relation contain a spatial component of the brane stress tensor, and so share a common factor of $g_{xx}^{-1}$. Hence, we can strip off one factor of this from the right side of Eq. (9). Next, let us identify $P/\epsilon$ as the sound velocity as measured on the outer AdS boundary and everything to the left of $\partial_z^2 \epsilon$ as the sound velocity as measured on a radial shell of arbitrary radius. Then it follows that the sound velocity scales relative to the outer boundary as

$$
[c_s^2]_u = Y^{-1}(u) g_{tt}(u) g_{xx}(u) [c_s^2]_b,
$$

where a subscript of $u$ denotes the value of a parameter at that radius. Calling again on our conformal-theory notation, let us take note that, by definition, $f = g_{tt}/g_{xx}$, and so the sound velocity equivalently scales as $f/Y$.

Next, let us re-express Eq. (6) in a way that makes the scaling properties of the parameters explicit:

$$
\omega^2 u^* = q^2 u^* Y(u^*) [c_s^2]_u^* + i\omega u^* q^2 u^* \frac{p-1}{p} f(u^*) + O(q^4 u^*).
$$

Here, we have made the identification $1/p \to [c_s^2]_u^*$ on the basis that the Dirichlet-boundary surface is where the sound velocity should be calibrated to its conformal value — just like it is the Dirichlet surface that defines where the field $Z$ is exactly unity. We can now apply the previously discussed scalings ($q^2 \sim 1/f$, $c_s^2 \sim f/Y$ and an invariant $\omega$) to convert the above expression into one that involves only the outer-boundary values of the parameters. Also recalling that $f = Y = 1$ at the AdS boundary $u = 0$, we then have

$$
\omega_b^2 = q_b^2 Y(0) [c_s^2]_b + i\omega_b q_b^2 \frac{p-1}{p} f(0) + O(q_b^4).
$$
But this is precisely what would have been obtained had we made the choice of $u^* = 0$ in the first place. Hence, the dispersion relation is indeed a radial invariant and, by direct implication, the correlator is as well.

3 Sound mode conformal hydrodynamics for generalized theories of gravity

Next on the agenda, we will investigate as to how the scenario changes when the theory is extended from Einstein’s gravity. It will be shown that, for a quite general (although still conformal) gravity theory, the damping constant is modified in a very precise way. Meanwhile, the sound velocity is shown to be unmodified, as must be the case for a conformal theory. These tasks will be accomplished by examining the (modified) pole of the just-discussed correlator. These generalizations will be further supported by a simple argument that is based upon inspecting the conservation equation that gives rise to the sound dispersion relation.

By a generalized gravity theory, we have in mind a Lagrangian that can be expressed as Einstein’s form plus higher-derivatives terms. If Einstein’s gravity is “non-trivially” modified by these corrections — meaning that the general Lagrangian can not be converted into Einstein’s form by a field redefinition — then the gravitational coupling is no longer as simple as $\kappa_E^2 = \text{constant}$. Rather, the coupling (or effective Newton’s constant) can be expected to depend on the polarization of the gravitons being probed. We will denote this dependence
by expressing the general couplings as $\kappa_{\mu\nu}$.

It is now well understood as to how one should calculate these couplings for a given theory \[9, 8, 7\]. These formalities need not concern the present discussion, although a schematic understanding of how the couplings come about should prove useful. One begins by writing the Lagrangian as a perturbative expansion in powers of the metric fluctuations or $h$’s. Of particular significance are the terms that are quadratic in $h$ and contain exactly two derivatives. For such terms, the gravitational couplings are identified on the premise that $h_{\mu\nu} \rightarrow \kappa_{\mu\nu} h_{\mu\nu}$ leads to a canonical kinetic term for the $\mu\nu$-polarized graviton.

As it turns out, the gravitational couplings are expressible strictly in terms of the metric at the horizon. Like the metric, they are typically radial functions; however, at the level of a two-derivative expansion of the Lagrangian, the couplings can safely be treated as (polarization-dependent) constants. Moreover, since the horizon is the true arena for black brane hydrodynamics, this locality is quite natural and falls in line with other parameters, such as the entropy and shear viscosity, being intrinsic properties of this special surface.

Let us re-emphasize that any given hydrodynamic parameter should be modified according to the class of gravitons that it probes. By working in the radial gauge and then restricting to the decoupled set of modes that defines the sound channel, we are limited to a select class. Namely, the $zz$, $tt$, and $tz$-polarized gravitons, as well as the “trace mode”, which can be identified with $H_X$ in Eq. (4).

When the theory is conformal, we can anticipate a further limitation. To elaborate, in obtaining the solution for $Z$ (see, e.g., \[15\]...
one finds that the $H_{tt}$ mode makes no direct contribution to Eq. (5). (This is not at all true when conformality is broken.) Recalling that the gauge-gravity duality identifies the $tt$-polarized gravitons with fluctuations in the energy density, we suspect that this null contribution is another manifestation of the suppression of the variation $\delta P/\delta \epsilon$ (as discussed in the previous section). On this basis, it seems reasonable to suggest that the $tt$ fluctuations can be excluded from a conformal theory in the hydrodynamic limit.

As implied above, the modifications of interest can be extracted from the pole structure of the (generalized) correlator $G_{ZZ}$. Critical to this procedure is the identification of the gravitational coupling $h_{\mu\nu} \to \kappa_{\mu\nu} h_{\mu\nu}$, which persuades us to adapt the gauge-invariant variable $Z$ of Eq. (4) as follows:

$$Z = 2q \omega \kappa_{tz} H_{tz} + \omega^2 \kappa_{zz} H_Z + q^2 Y \kappa_{zz} H_X,$$

(13)

where $H_Z \equiv H_{zz} - H_X$, the non-contributing mode $H_{tt}$ has been dropped and, as before, $Y = g_{tt}'/g_{xx}'$. Also, the spatial isotropy of the brane has enabled us to make the convenient substitution $\frac{1}{p-1} \sum_{i=1}^{p-1} \kappa_{xixi} \to \kappa_{zz}$.

The scaling properties of the damping constant can now be determined with a methodology akin to dimensional analysis: First, redefine the wavenumber and the frequency (and other parameters as necessary) with a scaling operation, second, re-express the solution in terms of these revised parameters and, third, interpret the modified pole structure. With regard to the first step, it is actually necessary to fix $\omega$, otherwise the incoming boundary condition at the horizon would be jeopardized. We are, however, free at this level of analysis
to change the normalization of $Z$. On this basis, we arrive at

$$Z = 2q\omega \frac{\kappa_{tz}}{\kappa_{zz}} H_{ tz} + \omega^2 H_Z + q^2 Y H_X$$

(14)

or

$$Z = 2\tilde{q}\omega H_{ tz} + \omega^2 H_Z + \tilde{q}^2 \tilde{Y} H_X,$$

(15)

with

$$\tilde{q} \equiv q \frac{\kappa_{tz}}{\kappa_{zz}},$$

$$\tilde{Y} \equiv Y \frac{\kappa_{zz}}{\kappa_{tz}}.$$  

(16)

By invoking $Y \rightarrow \tilde{Y}$, we do not mean to suggest that this function actually gets rescaled. Rather, the presence of $Y$ in Eq. (6) for $Z$ represents a direct contribution from $q^2 H_X$, which — after rescaling $q^2$ — picks up the extra factor $\kappa_{zz}^2/\kappa_{tz}^2$.

Since the couplings can be regarded as constants, the solution in Eq. (5) is formally unchanged and need only be rewritten in terms of the rescaled parameter. By this logic, the same can be said about the dispersion relation in Eq. (6), which takes on the modified form

$$\omega^2 = \tilde{q}^2 \frac{1}{p} \tilde{Y} + i \omega \tilde{q}^2 \frac{p - 1}{2p} \frac{\kappa_{zz}^2}{\kappa_{tz}^2}.$$  

(17)

Taking $u^a = 0$ and then comparing directly to Eq. (3), we can promptly extract the damping constant for a generalized (but conformal) theory of gravity:

$$\Gamma = \frac{\kappa_{zz}^2 p - 1}{\kappa_{tz}^2 \frac{p - 1}{4\pi T}}.$$  

(18)

and, as advertised, the sound velocity is clearly unmodified.

Let us briefly comment upon the significance of this result. It is commonplace, for a conformal theory, to relate the sound damping
constant directly to the shear diffusion coefficient or (equivalently) the shear viscosity to entropy ratio: \( \Gamma = \frac{p}{\rho} D \) and \( \Gamma = \frac{p}{\rho} \frac{1}{\eta} \frac{\eta}{s} \) respectively. This is all indisputably true for an Einstein theory of gravity; however, as we have now shown, these relations can not be taken verbatim for a generalized theory. To be clear, let us compare Eq. (18) to our prior results from [7] \( D = \frac{\kappa_{xz}^2}{4 \pi} \frac{1}{\kappa_{zz}} \) and from [8] \( \eta/s = \frac{1}{4 \pi} \frac{\kappa_{zz}^2}{\kappa_{xz}} \) (where \( x \) and \( z \) could be any pair of orthogonal directions on the brane and note that, in general, \( \kappa_{xz} \neq \kappa_{zz} \)). It should now be evident that both of the above relations for \( \Gamma \) will generally be modified for an Einstein-corrected theory.

The very same outcome as in Eq. (18) can be surmised from the \( z \)-component of the conservation equation for the dissipative stress tensor; with this being the equation that gives rise to the sound-mode dispersion relation (see, e.g., [13]). An inspection of this conservation equation \( \partial_t T^{tz} + \partial_z T^{zz} = 0 \) and the steps leading up to the dispersion relation (3) is quite revealing. It is the \( tz \) component of the stress tensor that accounts for the \( \Omega^2 \) term in Eq. (3), whereas the \( zz \) component gives rise to the \( Q^2 \Omega \) term. Now, given a gravitational pedigree for the hydrodynamic modes, it is natural to associate a coupling of \( \kappa_{\mu\nu}^2 \) with the \( \mu\nu \) component of the stress tensor. Hence, we anticipate that, for a generalized gravity theory, the conservation equation should really be \( \kappa_{xz}^2 \partial_t T^{tz} + \kappa_{zz}^2 \partial_z T^{zz} = 0 \). Similarly, we can expect the dispersion relation to take on the modified form \( \Omega^2 = \frac{\kappa_{xz}^2}{\kappa_{zz}} \left[ i2\Gamma Q^2 \right] + \ldots \) (with the dots referring to the sound-velocity and higher-order terms). Absorbing this ratio of couplings into the damping constant, we have precisely the same generalized form as obtained in Eq. (18).

Naively, this latter argument would also suggest that \( c_s^2 \) scales in
the same way as $\Gamma$, given that both are associated with the same $zz$ component of the stress tensor. However, this is not really correct: The sound velocity is associated with the variation of the pressure; with the pressure having originated from the non-dissipative background part of the stress tensor. Meanwhile, the other terms in the dispersion relation are strictly associated with the fluctuations or leading-order dissipative part. On this basis, we would not anticipate the sound velocity to be scaled for a generalized (conformal) theory; again in compliance with the previous analysis.

4 Discussion: Some aspects of the non-conformal case

To summarize, we have demonstrated two important outcomes for the sound-mode conformal hydrodynamics of an AdS brane theory. First, we have confirmed, for Einstein’s theory, that the hydrodynamics at the outer boundary is equivalent to that of any other radial shell up to (and including at) the stretched horizon. Second, we have shown — quite precisely — how the sound velocity and damping coefficient will be modified for a generalized (but conformal) theory of gravity. More specifically, $c_s^2$ is unaffected, whereas $\Gamma$ is scaled by a particular ratio of (generalized) gravitational couplings. Further note that, inasmuch as the couplings can be treated as constants, the former outcome will carry through unfettered for any Einstein-corrected conformal theory.

It is also of some interest to reflect upon how a non-conformal theory would impact upon our findings. Let us first consider the is-
sue of radial invariance for Einstein’s theory. Clearly, this invariance for the correlator depended, in large part, on being able to disregard the first term in Eq. (8). However, the introduction of a massive field into the spacetime (a prerequisite for breaking conformality) would be tantamount to the inclusion of a chemical potential into the thermodynamics. Such an inclusion would then negate our previous argument for the suppression of the scrutinized term; in particular, \( sT = \epsilon + P \) could no longer be true. Hence, there could no longer be any reason to expect that \( \frac{\delta P}{\delta \epsilon} \) is a parametrically small quantity for a non-conformal theory — meaning that the radial scaling of the sound velocity would certainly be more complicated. However, that this deviation from the conformal calculation is seemingly encapsulated in the single variation \( \frac{\delta P}{\delta \epsilon} \) gives one hope of being able to describe even the fully general situation by way of a radial “flow” equation. Although, it should be kept in mind that a further breach of radial invariance is possible (if not probable) from additional terms that would (almost inevitably) appear in the \( \mathcal{O}(q^1) \) solution for \( Z \).

For the case of generalized gravity, the state of affairs can become significantly more convoluted for a non-conformal theory. Here, the first order of business is to re-incorporate the previously disregarded \( tt \) mode — but then what? Well, at a first glance, the situation does not appear to look too bad. For the reason discussed at the end of the prior section, we would not expect the sound velocity to be modified irrespective of the generalized gravitational couplings. As for the damping coefficient, one can show that \( H_{tt} \) makes no contribution to this particular term, so it seems reasonable to suggest that \( \Gamma \) maintains its modified form of Eq. (18).
It is, however, a nearly certain likelihood that the situation can not be as simple as so far discussed. For a non-conformal theory, there is an inevitable mixing between $H_X$ and the massive bulk fields, and it is not yet clear as to how this mixing might effect the scaling relations for either $\Gamma$ or $c_s^2$ (with both of these being directly implicated with the “polluted” $H_X$ mode). Certainly, a mode formed out of $H_X$ and some, for instance, massive scalar field, could no longer have an effective coupling as trivial as $\kappa_{zz}$.

The main issue of non-conformal treatments is that, due to the high degree of model dependence in the formalism, very little can be said in a generic sense. There has, however, been some recent progress in such a direction [18, 19]. These papers indicate that a better starting point might be to look at certain classes of non-conformal theories, as opposed to the “extreme limiting cases” of a specific model or completely generality. Work along this line is only at a preliminary stage.

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