EXISTENCE AND REGULARITY
FOR THE GENERALIZED MEAN CURVATURE FLOW EQUATIONS

RONGLI HUANG$^{1,2}$ AND JIGUANG BAO$^3$

Abstract. By making use of the approximation method, we obtain the existence and regularity of the viscosity solutions for the generalized mean curvature flow. The asymptotic behavior of the flow is also considered. In particular, the Dirichlet problem of the degenerate elliptic equation

$$-|\nabla v|(div\left(\frac{\nabla v}{|\nabla v|}\right) + \nu) = 0$$

is solvable in viscosity sense.

1. Introduction

Let $n < 6$, and $D \subset \mathbb{R}^{n+1}$ a bounded domain with a $C^2$ boundary of mean curvature $H \geq 0$ with respect to its outer unit normal. For $\Omega = \mathbb{R}^{n+1} \setminus D$ and a nonnegative function $f \in C^{0,1}(\Omega)$, Bernhard Hein considered the viscosity solutions of the inverse mean curvature flow (cf. [1])

$$\begin{cases}
  u_t - \text{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u| = 0, & (x,t) \in \Omega \times (0, +\infty), \\
  u = 0, & (x,t) \in \partial D \times (0, +\infty), \\
  u = f(x), & (x,t) \in \Omega \times \{0\}.
\end{cases}
$$

Here $u_t = \frac{\partial u}{\partial t}$, $\nabla u = \text{grad} u$, div is the divergence operator in $\mathbb{R}^{n+1}$. He proved that there exists a unique nonnegative weak solution which satisfies (1.1). And there is a positive constant $C = C(n, D, f)$ such that for $x \in \Omega$ and all $t > 0$,

$$|\nabla u| \leq C, \quad -\frac{\sqrt{(n+1)C}}{\sqrt{\sqrt{t}}} - 2C \leq \frac{\partial u}{\partial t} \leq \frac{\sqrt{(n+1)C}}{\sqrt{\sqrt{t}}} + C.$$

Y.Giga, M.Ohuma and M.Sato studied the following Neumann problem (cf. [2])

$$\begin{cases}
  u_t - |\nabla u|\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0, & (x,t) \in D \times (0, +\infty), \\
  \frac{\partial u}{\partial \gamma} = 0, & (x,t) \in \partial D \times (0, +\infty), \\
  u = f(x), & x \in D \setminus \{0\},
\end{cases}
$$

where $\gamma$ is the outer unit normal of $\partial D$ and $f(x) \in C^2(\overline{D})$. They discovered some interesting properties of the solution $u(x,t)$ (see Theorem 1.1 in [2]) which satisfies (1.2) in viscosity sense.

2000 Mathematics Subject Classification. 35K55; 35K65.

Key words and phrases. Generalized mean curvature flow; Viscosity solutions; Maximum principle.
Inspired from [1] and [2], we investigate the global properties of solutions of the generalized mean curvature flow equations

\[
|u_t - |\nabla u|| \left( \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) + \nu \right) = 0, \quad (x,t) \in D \times (0, +\infty),
\]

where \(\nu\) is a constant. The equation (1.3) has a geometric significance because \(\gamma\)-level surface \(\Gamma(t)\) of \(u\) moves by its mean curvature when \(\nu = 0\) provided \(\nabla u\) does not vanish on \(\Gamma(t)\). Such a motion of surfaces has been studied by many authors in various conditions (cf. [3], [4], [5], [6], [7], [8]). However, the uniformly gradient estimates for solutions of (1.3) are little known and crucial for studying the global properties of viscosity solutions.

In the present paper we consider the initial and boundary value problem

\[
\begin{aligned}
\begin{cases}
|u_t - |\nabla u|| \left( \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) + \nu \right) = 0, & (x,t) \in D \times (0, +\infty), \\
u = h(x), & (x,t) \in \partial D \times [0, +\infty), \\
u = g(x), & x \in D \times \{0\}.
\end{cases}
\end{aligned}
\]

Here \(h(x)\) and \(g(x)\) are the given functions on \(D\).

Our main purposes are to show the existence and regularity of the viscosity solutions for (1.4), study their asymptotic behavior, and prove that \(u(x,t)\) converges to a solution of the Dirichlet problem of degenerate elliptic equation

\[
\begin{aligned}
\begin{cases}
-|\nabla v| \left( \text{div} \left( \frac{\nabla v}{|\nabla v|} \right) + \nu \right) = 0, & x \in D, \\
v = h(x), & x \in \partial D,
\end{cases}
\end{aligned}
\]

as \(t \to +\infty\). The solvability of (1.5) does not seem to be easily found in literature as far as we knew.

Quite naturally, we always use the following notations

\[
\varphi_i = \frac{\partial \varphi}{\partial x_i}, \quad \varphi_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}.
\]

Firstly we introduce the definition of viscosity solutions.

**Definition 1.1.** Suppose that \(u(x,t)\) is a function in \(C(\overline{D} \times [0, +\infty))\) and satisfies the initial and boundary conditions of (1.4). If \(\varphi \in C^\infty(D \times (0, +\infty))\), \((x,t) \in \Theta \subset D \times (0, +\infty)\) and \(\Theta\) is a bounded open set, which satisfies

\[
(u - \varphi)(x,t) = \max_{\Theta}(u - \varphi),
\]

and at \((x,t)\) that

\[
\varphi_t \leq \left( \delta_{ij} - \frac{\varphi_i \varphi_j}{|\nabla \varphi|^2} \right) \varphi_{ij} + \nu|\nabla \varphi|, \quad |\nabla \varphi| \neq 0.
\]

Or there exists \(\eta = (\eta_1, \eta_2, \cdots, \eta_{n+1})\) with \(|\eta| \leq 1\) at \((x,t)\) such that

\[
\varphi_t \leq (\delta_{ij} - \eta_i \eta_j) \varphi_{ij}, \quad |\nabla \varphi| = 0.
\]
Then $u(x,t)$ is viscosity sub-solution of (1.4).

**Definition 1.2.** Suppose that $u(x,t)$ is a function in $C(\overline{D} \times [0, +\infty))$ and satisfies the initial and boundary conditions of (1.4). If $\varphi \in C^\infty(D \times (0, +\infty))$, $(x,t) \in \Theta \subset D \times (0, +\infty)$ and $\Theta$ is a bounded open set, which satisfies

$$(u - \varphi)(x,t) = \min_{\Theta}(u - \varphi),$$

and at $(x,t)$ that

$$\varphi_t \geq \left(\delta_{ij} - \frac{\varphi_i \varphi_j}{|\nabla \varphi|^2}\right)\varphi_{ij} + \nu|\nabla \varphi|, \quad |\nabla \varphi| \neq 0.$$

Or there exists $\eta = (\eta_1, \eta_2, \ldots, \eta_{n+1})$ with $|\eta| \leq 1$ at $(x,t)$ such that

$$\varphi_t \geq (\delta_{ij} - \eta_i \eta_j)\varphi_{ij}, \quad |\nabla \varphi| = 0.$$

Then $u(x,t)$ is viscosity super-solution of (1.4).

**Definition 1.3.** If $u(x,t)$ is a viscosity sub-solution and also is a viscosity super-solution of (1.4), then $u(x,t)$ is a viscosity solution of (1.4).

Let us fix $h(x) = g(x)$ on $\partial D$, $h(x) \in C^2(\partial D)$, $g(x) \in C^2(\overline{D})$ and suppose that $D$ is a smooth strictly convex bounded domain in $\mathbb{R}^{n+1}$, and $|\nu| < \frac{H_0}{n+1}$, where $H_0$ is the positive lower bound of the mean curvature of $\partial D$.

One of the main results in this paper is the existence and regularity of viscosity solutions of (1.4).

**Theorem 1.4.** There exists a unique function $u(x,t)$ which satisfies (1.4) in viscosity sense and

\begin{align*}
(1.6) \quad & u \in C(\overline{D} \times [0, +\infty)), \quad u_t \in L^\infty(D \times [0, +\infty)), \quad \nabla u \in L^\infty(D \times [0, +\infty)), \\
(1.7) \quad & \|u\|_{L^\infty(D \times [0, +\infty))} + \|\nabla u\|_{L^\infty(D \times [0, +\infty))} + \|u_t\|_{L^\infty(D \times [0, +\infty))} \leq C, \\
(1.8) \quad & \int_0^{+\infty} \int_D |u_t|^2 \, dx \, dt \leq C,
\end{align*}

where the constant $C$ depends only on $n$, $\nu$, $D$, $\|h\|_{C^2(\partial D)}$ and $\|g\|_{C^2(\overline{D})}$.

As an application of Theorem 1.4 we have

**Corollary 1.5.** Suppose $u(x,t)$ is the viscosity solution of (1.4). Then there exists a function $v(x)$ which satisfies $v(x) \in C(\overline{D}), \nabla v \in L^\infty(D)$, such that

\begin{align*}
(1.9) \quad & \lim_{t \to +\infty} u(x,t) = v(x), \quad \text{in} \quad C(\overline{D}),
\end{align*}

and $v(x)$ satisfies (1.4) in viscosity sense.

**Remark 1.6.** By Corollary 1.5 the Dirichlet problem (1.3) is solvable. But we don’t know whether the viscosity solutions of (1.3) is unique.
The second result of this paper is the Liouville-type property of the viscosity solutions. Suppose that $D'$ is a smooth convex bounded domain in $\mathbb{R}^n$, such that

$$D \cap \{(x', x_{n+1}) \in \mathbb{R}^{n+1} | |x_{n+1}| < m + 1\} = D' \times (-m - 1, m + 1),$$

where $x' = (x_1, \ldots, x_n)$ and $m$ is a positive constant.

**Theorem 1.7.** Let $\nu \geq 0$ and suppose that $g(x', x_{n+1})$ is a non-decreasing function of $x_{n+1}$ which satisfies

$$g(x', x_{n+1}) = \lambda, \quad x_{n+1} \geq m,$$

where $\lambda$ is a constant. Then the viscosity solution $u(x, t)$ in $C(D \times [0, +\infty))$ of (1.4) satisfies

$$u(x', x_{n+1}, t) = \lambda, \quad x_{n+1} \geq m.$$

In the next section we construct an approximation problem of (1.4) and establish uniform estimates for their classical solutions. In the last section we present the proof of the main results.

### 2. Preliminary Estimates

Consider the approximate problem of (1.4) (with $\epsilon \in (0, 1)$) in a smooth strictly convex bounded domain in $\mathbb{R}^{n+1}$

$$\begin{cases}
  u_t - \sqrt{\epsilon^2 + |\nabla u|^2} \cdot \left(\text{div} \left(\frac{\nabla u}{\sqrt{\epsilon^2 + |\nabla u|^2}}\right) + \nu\right) = 0, & (x, t) \in D \times (0, +\infty), \\
  u = h(x), & (x, t) \in \partial D \times [0, +\infty), \\
  u = g(x), & x \in D \times \{0\}.
\end{cases}$$

We want to use the continuity method to prove the solvability of (2.1) and then obtain the estimates similarly to (1.7) and (1.8).

In order to solve (2.1) we use the following form of fixed point theorem (cf. [9]).

**Lemma 2.1.** (Leray – Schauder) Suppose that $\mathcal{B}$ is a Banach space, $\chi(b, \sigma)$ is a map from $\mathcal{B} \times [0, 1]$ to $\mathcal{B}$. If $\chi$ satisfies

1. $\chi$ is continuous and compact.
2. $\chi(b, 0) = 0, \forall b \in \mathcal{B}$.
3. There exists constant $C > 0$ such that

$$\|b_0\|_{\mathcal{B}} \leq C, \quad \forall b_0 \in \{b \in \mathcal{B}| \exists \sigma \in [0, 1], \ b = \chi(b, \sigma)\}.$$

Then there exists $b_0 \in \mathcal{B}$ such that $\chi(b_0, 1) = b_0$.

For any $T > 0$, we define

$$\mathcal{B} = \{u|u \in C(D \times [0, T)), \quad \nabla u \in C(D \times [0, T])\}, \quad D_T = D \times [0, T).$$
From the theory on linear parabolic equation (cf. [10]) and for any \( \tilde{u} \in \mathcal{B} \), \( \sigma \in [0, 1] \), there exists a unique function \( u \), where \( u \in \mathcal{B} \), \( u \in W^{2,1}_p(D_T) \) with any \( p > 0 \) such that \( u \) satisfies

\[
\begin{aligned}
& u_t - \left( \delta_{ij} - \frac{\epsilon^2}{\epsilon^2 + \sigma^2|\nabla u|^2} \right) u_{ij} = \sigma \nu \sqrt{\epsilon^2 + \sigma^2|\nabla u|^2}, \quad (x, t) \in D \times (0, T), \\
& u = \sigma h(x), \quad (x, t) \in \partial D \times [0, T), \\
& u = \sigma g(x), \quad x \in D \times \{0\}.
\end{aligned}
\]  

(2.2)

By (2.2) we can define a map from \( \mathcal{B} \) to \( \mathcal{B} \) and denote \( u = \chi(\tilde{u}, \sigma) \). The main step in our argument is to validate the three conditions of Lemma 2.1 one by one.

It is obvious that \( \chi(\tilde{u}, 0) = 0 \) for every \( \tilde{u} \in \mathcal{B} \) by the uniqueness of the initial and boundary value problem (2.2). We see that the map \( \chi \) is compact by Schauder estimates and Sobolev embedding theorem (cf. [10]). Consequently we claim that \( \chi \) is continuous. In deed, this fact follows from the compactness of \( \chi \) and the uniqueness of the mapping \( \chi(\tilde{u}, \sigma) \).

So it remains to verify the third condition for applying Lemma 2.1 to the problem (2.2). Suppose \( \chi(u, \sigma) = u \). It follows from (2.2) that \( u \) satisfies

\[
\begin{aligned}
& u_t - \sqrt{\epsilon^2 + \sigma^2|\nabla u|^2} \cdot \left( \text{div} \left( \frac{\nabla u}{\sqrt{\epsilon^2 + \sigma^2|\nabla u|^2}} \right) + \sigma \nu \right) = 0, \quad (x, t) \in D \times (0, T), \\
& u = \sigma h(x), \quad (x, t) \in \partial D \times [0, T), \\
& u = \sigma g(x), \quad x \in D \times \{0\}.
\end{aligned}
\]  

(3.2)

By using regularity theory, \( u \in C^\infty(D_T) \cap C^{2,1}(\overline{D}_T) \). Then the condition (3) in Lemma 2.1 is equivalence to the boundness of \( u \) and \( \nabla u \) in the \( L^\infty \) norm which is independence of \( \sigma \) if \( u \in C^\infty(D_T) \cap C^{2,1}(\overline{D}_T) \) and \( u \) satisfies (2.3).

In this section we derive \( W^{1,\infty} \) estimates for the classical solutions of (2.3) in which the bound is not only independent of \( \sigma \), but also independent of \( \epsilon \) and \( T \).

Set

\[
L_\sigma u = u_t - \sqrt{\epsilon^2 + \sigma^2|\nabla u|^2} \cdot \left( \text{div} \left( \frac{\nabla u}{\sqrt{\epsilon^2 + \sigma^2|\nabla u|^2}} \right) + \sigma \nu \right),
\]  

(2.4)

and

\[
\partial_t D_T = (\partial D \times [0, T)) \cup (D \times \{t = 0\}).
\]

The estimates follow from the next three lemmas. The following comparison principle is by Theorem 14.1 in [10].

**Lemma 2.2.** Suppose that \( u_1, u_2 \in C^{2,1}(D \times (0, T)) \cap C(\overline{D} \times [0, T)) \). If

\[
L_\sigma u_1 \geq L_\sigma u_2, \quad u_1|_{\partial_t D_T} \geq u_2|_{\partial_t D_T},
\]

then

\[
u_1|_{D_T} \geq u_2|_{D_T}.
\]
The estimates of the maximum norm for the solutions of (2.3) is the following:

**Lemma 2.3.** If $u \in C^\infty(D \times (0, T)) \cap C(\overline{D} \times [0, T))$ is a solution of (2.3). Then

\begin{equation}
\|u\|_{L^\infty(D \times [0, T))} \leq C,
\end{equation}

where $C$ is depending only on $\|h\|_{C(\partial D)}$, $\|g\|_{C(D)}$, and $D$.

**Proof.** Step 1. By $|\nu| < \frac{nH_0}{n+1}$ and Theorem 16.10 in [9], there exists $\alpha > 0, \nu \in C^2(\alpha) D$, such that

\begin{equation}
\begin{cases}
-\sqrt{\varepsilon^2 + \sigma^2 |\nabla \nu|^2} \cdot \left( \frac{\nabla \nu}{\sqrt{\varepsilon^2 + \sigma^2 |\nabla \nu|^2}} \right) + \sigma \nu = 0, & x \in D, \\
v = 1, & x \in \partial D.
\end{cases}
\end{equation}

Set $w = \frac{\sigma}{\varepsilon} \nu$. Then $w$ is a classical solution of the following Dirichlet problem:

\begin{equation}
\begin{cases}
\text{div} \left( \frac{\nabla w}{\sqrt{1 + |\nabla w|^2}} \right) + \sigma^2 \nu = 0, & x \in D, \\
w = \frac{\sigma}{\varepsilon}, & x \in \partial D.
\end{cases}
\end{equation}

It follows from Theorem 6.1 in [11] that there exists a constant $C$ depending only on $n$ and $\text{diam} D$ such that

\begin{equation}
\max_{\overline{D}} |w| \leq \frac{\sigma}{\varepsilon} + C \sigma^2 \nu.
\end{equation}

So

\begin{equation}
\max_{\overline{D}} |\nu| \leq 1 + \epsilon \sigma \nu C \leq C.
\end{equation}

Step 2. Suppose that $\kappa$ is a positive constant which will be determined later. Set $\nu_1 = \nu + \kappa$.

Then $\nu_1$ satisfies

\begin{equation}
\begin{cases}
-\sqrt{\varepsilon^2 + \sigma^2 |\nabla \nu_1|^2} \cdot \left( \frac{\nabla \nu_1}{\sqrt{\varepsilon^2 + \sigma^2 |\nabla \nu_1|^2}} \right) + \sigma \nu = 0, & x \in D, \\
\nu_1 = 1 + \kappa, & x \in \partial D.
\end{cases}
\end{equation}

By (2.6) we can choose $\kappa$ depending only on $\|h\|_{C(\partial D)}$, $\|g\|_{C(D)}$, and $D$ such that

\begin{equation}
\nu_1(x) \geq g(x), \quad \nu_1(x) \geq h(x), \quad x \in \overline{D}.
\end{equation}

By applying Lemma 2.2 we arrive at

\begin{equation}
u_1(x) \leq \nu_1(x) \leq C + \kappa \leq C, \quad (x, t) \in \overline{D} \times [0, T).
\end{equation}

For the same reason we obtain

\begin{equation}
u_1(x) \leq -C, \quad (x, t) \in \overline{D} \times [0, T).
\end{equation}

This yields the desired results.

The following is the gradient estimates for the solutions of (2.3).
Lemma 2.4. If \( u \in C^\infty(D \times (0, T)) \cap C(\overline{D} \times [0, T]) \) and is a solution of (2.3). Then

\[
(2.7) \quad \| \nabla u \|_{L^\infty(D \times [0, T])} \leq C.
\]

where \( C \) is depending only on \( \| h \|_{C^2(\partial D)}, \| g \|_{C^1(\overline{D})}, \) and \( D \).

Proof. Step 1. We derive the gradient estimates of \( u \) at the boundary and the methods comes from [4]. Set \( w = u - h \). Then by (2.3) \( w \) satisfies the following equations on \( D \times (0, T) \):

\[
\mathbf{L} w \triangleq w_t - \left( \delta_{ij} - \alpha \frac{\bar{\nabla} w + \nabla h}{\sqrt{\bar{\nabla}^2 w + \nabla h^2}} \right) (w_{ij} + h_{ij}) - \beta \nabla \sqrt{\bar{\nabla}^2 w + \nabla h^2} = 0.
\]

In the neighborhood \( \Theta \) of \( \partial D \times [0, T] \) we will construct the functions \( \psi^\pm \) which are independent of \( t \) and satisfy

\[
(2.8) \quad \pm \mathbf{L} \psi^\pm \geq 0, \quad (x, t) \in \Theta \cap (D \times (0, T)),
\]

\[
(2.9) \quad \psi^\pm = w = 0, \quad (x, t) \in \Theta \cap (\partial D \times [0, T]),
\]

\[
(2.10) \quad \psi^- \leq w \leq \psi^+, \quad (x, t) \in (\partial \Theta \cap (D \times [0, T])) \cup (\Theta \cap (D \times \{0\})).
\]

Consequently by Lemma 2.2 we have

\[
(2.11) \quad \psi^- \leq w \leq \psi^+, \quad (x, t) \in \tilde{\Theta} \cap (D \times [0, T]).
\]

For \( (x, t) \in \partial D \times [0, T] \), if \( \bar{a} \) is the normal vector of \( \partial D \) such that

\[
x + s \bar{a} \in \Theta \cap (D \times [0, T]), \quad \text{when} \quad 0 < s \leq 1.
\]

Then by (2.9) and (2.11) we obtain

\[
\frac{\psi^- (x + s \bar{a}) - \psi^- (x)}{s} \leq \frac{w(x + s \bar{a}, t) - w(x, t)}{s} \leq \frac{\psi^+ (x + s \bar{a}) - \psi^+ (x)}{s}.
\]

Letting \( s \to 0 \), we have

\[
\frac{\partial \psi^-}{\partial \bar{a}} (x, t) \leq \frac{\partial w}{\partial \bar{a}} (x, t) \leq \frac{\partial \psi^+}{\partial \bar{a}} (x).
\]

A direct calculation yields on \( \Theta \cap (\partial D \times [0, T]) \)

\[
(2.12) \quad |\nabla u| \leq \| \nabla w \|_{C(\Theta \cap (\partial D \times [0, T]))} + \| \nabla h \|_{C(\Theta \cap (\partial D \times [0, T]))}
\]

\[
\leq \| \nabla \psi^+ \|_{C(\Theta \cap (\partial D \times [0, T]))} + \| \nabla \psi^- \|_{C(\Theta \cap (\partial D \times [0, T]))} + \| \nabla h \|_{C(\Theta \cap (\partial D \times [0, T]))} \leq C.
\]

In the following we constitute \( \psi^+ \) and \( \psi^- \) which satisfy (2.8)-(2.10) in detail. Firstly set

\[
\psi^+ (x) = \lambda d(x), \quad x \in \overline{D}, \quad N = \{ x \in D | d(x) < \rho \},
\]

where \( d(x) \) is the distance from \( x \) to \( \partial D \), \( \rho \) and \( \lambda \) are positive constant which will be determined later. Selecting the positive constant \( \rho \) to be small enough such that \( d(x) \) satisfies

(a) \( d(x) \in C^2(N) \).
(b) In $N$, $|\nabla d| = 1$, and
\[
\sum_{i=1}^{n+1} d_i d_{ij} = 0, \quad j = 1, 2, \ldots, n + 1.
\]

(c) If $x \in N$, then there exists $x_0 \in \partial D$ such that $d(x) = |x - x_0|$. By Lemma 14.17 in [9] we can present the formula
\[
-\Delta d(x) = \sum_{i=1}^{n} \frac{k_i}{1 - k_d(x)},
\]
where $k_1, k_2, \ldots, k_n$ are the principle curvature of $\partial D$ at $x_0$.

Now we verify $\psi^+$ satisfying (2.8)–(2.10) one by one.

(1) By the definition of $d(x)$, $\psi^+$ satisfies (2.9).

(2) If $x \in N$, then we can choose $x_0 \in \partial D$ such that $d(x) = |x - x_0|$. And by $w(x_0, 0) = 0$ we obtain
\[
w(x, 0) = g(x) - h(x) - [g(x_0) - h(x_0)] \leq \beta |x - x_0| = \beta d(x),
\]
where $\beta$ is depending only on $\|h\|_{C^1(\partial D)}$ and $\|g\|_{C^1(\overline{D})}$. On the other hand, if $x \in \partial N \cap D$, then $d(x) = \rho$. So we can select the positive constant $\lambda$, such that $\psi^+$ satisfies (2.10).

(3) $\psi^+$ satisfies (2.8). In fact,
\[
\mathcal{L} \psi^+ = -\lambda \Delta d - \Delta h + \sigma^2 \left( \frac{\lambda^2 d_i d_j + \lambda d_i h_j + \lambda h_i d_j + h_i h_j}{\epsilon^2 + \sigma^2 \lambda \nabla d + \nabla h^2} \right) (\lambda d_{ij} + h_{ij})
\]
\[
-\sigma \nu \sqrt{\epsilon^2 + \sigma^2 \nabla h^2} + 2 \lambda \sigma^2 \nabla \cdot \nabla h + \sigma^2 \lambda^2.
\]

Then by (2.14) and $\sum_{i=1}^{n+1} d_i d_{ij} = 0$ we have
\[
\mathcal{L} \psi^+ \geq n \lambda H_0 - \|h\|_{C^2(\overline{D})} + \sigma^2 \left( \frac{\lambda^2 d_i d_j h_{ij} + 2 \lambda h_i d_j h_{ij} + h_i h_j h_{ij} + \lambda d_i h_j h_{ij}}{\epsilon^2 + \sigma^2 \lambda \nabla d \cdot \nabla h + \sigma^2 \nabla h^2} \right)
\]
\[
-\sigma \nu \sqrt{\epsilon^2 + \sigma^2 \nabla h^2} + 2 \sigma^2 \lambda \nabla d \cdot \nabla h + \sigma^2 \lambda^2.
\]

By Lemma 14.17 in [9], $|d_{ij}|$ have an upper bound depending only on $\partial D$. Let positive constant $\lambda$ to be large enough then we obtain
\[
(2.15) \quad \mathcal{L} \psi^+ \geq n \lambda H_0 - \lambda \sigma^2 |\nu| - C \geq n \lambda H_0 - \lambda |\nu| - C,
\]
where $C$ is depending only on $\partial D, \|h\|_{C^2(\partial D)}$. From (2.15) and $|\nu| < n H_0$ let $\lambda$ to be large enough which is depending only on $\partial D$ and $\|h\|_{C^2(\partial D)}$ then we have
\[
\mathcal{L} \psi^+ \geq 0.
\]

For the same reason we can construct $\psi^-$ which satisfies (2.8)–(2.10). So we have obtained the desired results of step 1 by (2.12).
Step 2. For $i \in \{1, 2, \cdots, n+1\}$, let $\varpi = u_i$. Differentiating (2.3) with respect to $x_i$ we get

$$\varpi_t - a^{kl} \varpi_{kl} - b^l \varpi_l = 0, \quad (x, t) \in D \times (0, T),$$

where

$$a^{kl} = \delta_{kl} - \frac{\sigma^2 u_k u_l}{\epsilon^2 + \sigma^2 |\nabla u|^2},$$

$$b^l = \frac{2\sigma^4 u^c_k u^c_m u^c_n u_l}{(\epsilon^2 + \sigma^2 |\nabla u|^2)^2} - \frac{2\sigma^2 u_k u_{kl}}{\epsilon^2 + \sigma^2 |\nabla u|^2} - \frac{\nu \sigma^3 u_l}{\epsilon^2 + \sigma^2 |\nabla u|^2}.$$

By the maximum principle for linear parabolic equation (cf. [10]) and (2.12) we obtain (2.8). \(\Box\)

From Lemma 2.1–2.4 and the Schauder estimates we conclude that

**Theorem 2.5.** For any $\epsilon > 0$, there exists $u^\epsilon$ which satisfies

$$u^\epsilon \in C^\infty(D \times (0, +\infty)), \quad u^\epsilon \in C(\overline{D} \times [0, +\infty)), \quad \nabla u^\epsilon \in C(\overline{D} \times [0, +\infty)),$$

and $u^\epsilon$ is a classical solution of (2.7). And there holds

$$\|u^\epsilon\|_{L^\infty(D \times [0, +\infty))} \leq C, \quad \|\nabla u^\epsilon\|_{L^\infty(D \times [0, +\infty))} \leq C,$$

where $C$ depends only on $\|h\|_{C^2(\partial D)}$, $\|g\|_{C^1(\overline{D})}$, $H_0$ and $D$.

**Corollary 2.6.** Suppose $u^\epsilon$ is a classical solution of (2.7). Then there holds

$$\int_0^{+\infty} \int_D |u^\epsilon_t|^2 dx dt \leq C,$$

where $C$ depends only on $\|h\|_{C^2(\partial D)}$, $\|g\|_{C^1(\overline{D})}$, $H_0$ and $D$.

**Proof.** Set

$$J(t) = \int_D \sqrt{|\nabla u^\epsilon|^2 + \epsilon^2} dx.$$

Then

$$J'(t) = \int_D \frac{\nabla u^\epsilon \cdot \nabla u^\epsilon_t}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} dx = -\int_D \text{div} \frac{\nabla u^\epsilon}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} u^\epsilon_t dx.$$

From (2.1) we see that

$$\text{div} \frac{\nabla u^\epsilon}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} = \frac{u^\epsilon_t}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} - \nu.$$

Substituting (2.18) into (2.17) we obtain

$$J'(t) + \int_D \frac{|u^\epsilon_t|^2}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} dx = \nu \int_D u^\epsilon_t dx.$$

For (2.19) integrating from 0 to $T$ and using (2.5) we have

$$\int_0^T \int_D \frac{|u^\epsilon_t|^2}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} dx dt = J(0) - J(t) + \nu \int_D u^\epsilon_t|_{t=T} dx - \nu \int_D u^\epsilon_t|_{t=0} dx \leq J(0) + C,$$
where $C$ is a constant which is independent of $\epsilon$. Taking $T \to +\infty$ we get
\[(2.20)\]
\[
\int_0^{+\infty} \int_D \frac{|u_i|^2}{\sqrt{|\nabla u_i|^2 + \epsilon^2}} \, dx \, dt \leq C.
\]
Combining (2.7) with (2.20) we arrive at
\[
\int_0^{+\infty} \int_D |u_\epsilon t|^2 \, dx \, dt = \int_0^{+\infty} \int_D |u_\epsilon t|^2 \sqrt{|\nabla u_\epsilon|^2 + \epsilon^2} \, dx \, dt
\]
\[
\leq (\|\nabla u\|_{L^\infty(D \times [0, +\infty))} + \epsilon) \int_0^{+\infty} \int_D |u_\epsilon t|^2 \sqrt{|\nabla u_\epsilon|^2 + \epsilon^2} \, dx \, dt
\]
\[
\leq C,
\]
where $C$ depends only on $\|h\|_{C^2(\partial D)}$, $\|g\|_{C^1(\overline{D})}$, $H_0$ and $D$. 

**Corollary 2.7.** Suppose $u_\epsilon$ is a classical solution of (2.1). Then there holds
\[(2.21)\]
\[
\|u_\epsilon t\|_{L^\infty(D \times [0, +\infty))} \leq C.
\]
where $C$ depends only on $\|g\|_{C^2(D)}$.

**Proof.** Set $\omega = u_\epsilon t$. Differentiating (2.1) with respect to $t$ we get
\[
\omega_t - a^{kl} \omega_{kl} - b^l \omega_t = 0, \quad (x, t) \in D \times (0, +\infty),
\]
where
\[
a^{kl} = \delta_{kl} - \frac{u_\epsilon^k u_\epsilon^l}{\epsilon^2 + |\nabla u_\epsilon|^2},
\]
\[
b^l = 2u_\epsilon^k u_\epsilon^m u_\epsilon^m u_\epsilon^l \left( \frac{1}{\epsilon^2 + |\nabla u_\epsilon|^2} \right) - 2u_\epsilon^k u_\epsilon^m u_\epsilon^l \left( \frac{1}{\epsilon^2 + |\nabla u_\epsilon|^2} \right) \sqrt{\epsilon^2 + |\nabla u_\epsilon|^2} - \frac{\nu u_\epsilon^l}{\epsilon^2 + |\nabla u_\epsilon|^2}.
\]
From $u_\epsilon|_{\partial D \times [0, T)} = h(x)$, there holds
\[
\omega = 0, \quad (x, t) \in \partial D \times [0, +\infty).
\]
From (2.1) we see that
\[
\omega = \sqrt{\epsilon^2 + |\nabla g|^2} \cdot \left( \text{div} \left( \frac{\nabla g}{\sqrt{\epsilon^2 + |\nabla g|^2}} \right) + \nu \right), \quad (x, t) \in D \times \{0\}.
\]
This yields (2.21) by using maximum principle. 

### 3. The Proof of Main Results

In the third section, we give the proof of Theorem 1.4, Corollary 1.5, and Theorem 1.7.

**Proof of Theorem 1.4.** Consider the classical solution of approximate problem (2.1). From Theorem 2.5 and Corollary 2.7, we see there exists $\{\epsilon_i\}_{i=1}^{+\infty}$ satisfying $\lim_{i \to +\infty} \epsilon_i = 0$ such that there holds
\[
\begin{align*}
  u_{\epsilon_i} &\to u, \quad \text{in } C(\overline{D} \times [0, +\infty)), \\
  \nabla u_{\epsilon_i} &\to \nabla u, \quad \text{in } L^\infty(D \times [0, +\infty)), \\
  u_{\epsilon_i} t &\to u_t, \quad \text{in } L^\infty(D \times [0, +\infty)).
\end{align*}
\]
Combining Corollary 2.6 with Fatou’s Lemma we verify that $u$ satisfies (1.6)–(1.8). On the other hand, by the stability theorem of viscosity solutions (cf. Theorem 2.4 in [3]) $u$ is a viscosity solution of (1.4). This completes the proof of Theorem 1.4.

**Proof of Corollary 1.5** The main idea comes from Y. Giga, M. Ohnuma and M. Sato (cf. [2]).

Consider the viscosity solution $u$ of (1.4). For $(x, t) \in \overline{D_1} \triangleq \overline{D} \times [0, 1]$, set

$$u_k(x, t) = u(x, k + t), \quad k = 1, 2, \ldots$$

From (1.7) and the Ascoli-Arzela’s Theorem, there exists a subsequence of $\{u_k\}$ (still denote the subsequence by $\{u_k\}$) and the function $v(x, t)$, such that

$$\lim_{k \to +\infty} u_k(x, t) = v(x, t), \quad \text{in } C(\overline{D_1}).$$

By (1.8) we obtain

$$\lim_{k \to +\infty} \int_0^1 \int_D |u_{kt}|^2\,dx\,dt = \lim_{k \to +\infty} \int_k^{k+1} \int_D |u_t|^2\,dx\,dt = 0.$$

Letting $k \to +\infty$ we have

$$u_{kt} \to 0, \quad \text{in } L^2(D_1).$$

It follows from (3.1) and (3.2) that for any $\phi \in C_0^\infty(D)$ and $\chi \in C_0^\infty(0, 1)$,

$$\int_D \int_0^1 v\phi\chi_t\,dtdx = 0.$$

Then

$$v_t = 0, \quad (x, t) \in D_1.$$

By (1.4) $u_k$ satisfies the following equation in viscosity sense

$$\begin{cases}
    u_{kt} - |\nabla u_k| \left( \text{div} \left( \frac{\nabla u_k}{|\nabla u_k|} \right) + \nu \right) = 0, & (x, t) \in D_1 \times (0, 1), \\
    u_k = h(x), & (x, t) \in \partial D_1 \times [0, 1].
\end{cases}$$

From (3.4) taking $k \to +\infty$ and using (1.7), (3.1), (3.3) and applying Theorem 2.4 in [3] we deduce that $v$ satisfies

$$\begin{cases}
    -|\nabla v| \left( \text{div} \left( \frac{\nabla v}{|\nabla v|} \right) + \nu \right) = 0, & x \in D, \\
    v = h(x), & x \in \partial D,
\end{cases}$$

in viscosity sense. This completes the proof of Corollary 1.5.

**Proof of Theorem 1.7** Firstly taking positive constant $\delta$ to be small enough we can construct a pair of non-decreasing $C^2$ functions $g^+(\tau)$ and $g^-(\tau)$ such that

$$\begin{cases}
    g^-(x_{n+1}) = g^+(x_{n+1}) = \lambda, & x_{n+1} \geq m + \delta, \\
    g^-(x_{n+1}) \leq \max_{x' \in D} g(x', x_{n+1}) \leq g^+(x_{n+1}), & x_{n+1} \leq m + \delta.
\end{cases}$$
In fact, by the hypothesis of $g$ we can choose $g^+(\tau) \equiv \lambda$. Set

$$g_\varepsilon(\tau) = \begin{cases} 
\lambda, & \text{if } \tau \geq m + \delta, \\
\frac{\lambda}{\varepsilon}(\tau - m - \delta + \varepsilon), & \text{if } \tau \leq m + \delta.
\end{cases}$$

By smoothing the point $(m + \delta, \lambda)$ and letting $\varepsilon = \varepsilon(\delta)$ to be small enough we obtain $g^-(\tau)$ which satisfies (3.5).

Let

$$u^+(x', x_{n+1}, t) = g^+(x_{n+1} + \nu t),$$

$$u^-(x', x_{n+1}, t) = g^-(x_{n+1}).$$

We claim that $u^+(x', x_{n+1}, t)$ and $u^-(x', x_{n+1}, t)$ are viscosity sub-solution and viscosity super-solution of (1.4) respectively. Then using Theorem 1.7 in [12], i.e., the comparison principle for the viscosity solution of (1.4) which is likely to Lemma 2.2 we get

$$u^+(x', x_{n+1}, t) \leq u(x', x_{n+1}, t) \leq u^-(x', x_{n+1}, t), \quad (x', x_{n+1}, t) \in \overline{D} \times [0, +\infty).$$

In particular, if $x_{n+1} \geq m + \delta$ for any $\delta > 0$, then $u^+(x', x_{n+1}, t) = u^-(x', x_{n+1}, t) \equiv \lambda$ by making use of (3.5). Taking $\delta \to 0$ we obtain (1.11).

Now we prove that $u^+(x', x_{n+1}, t)$ is viscosity super-solution of (1.4). In a similar way we can prove that $u^-(x', x_{n+1}, t)$ is a viscosity sub-solution of (1.4).

In fact, for any $(x, t) \in D \times [0, +\infty)$, if $\varphi \in C^\infty(D \times [0, +\infty)$) and there exists a neighborhood $\Theta$ of $(x, t)$ in $D \times [0, +\infty)$ such that

$$(u^+ - \varphi)(x, t) = \min_{\Theta} (u^+ - \varphi).$$

Then at $(x, t)$ we have

$$(3.7) \quad u^+_t - \varphi_t \leq 0, \quad \nabla u^+ = \nabla \varphi, \quad D^2 u^+ \geq D^2 \varphi.$$ 

If $\nabla \varphi = 0$. Then by taking $\eta = (\eta_1, \eta_2, \cdots, \eta_n, \eta_{n+1}) = (0, 0, \cdots, 0, 1)$ and using (3.7) we obtain

$$(3.8) \quad (\delta_{ij} - \eta_i \eta_j) \varphi_{ij} \leq (\delta_{ij} - \eta_i \eta_j) u^+_{ij} = (1 - \eta_n \eta_{n+1}) u^+_{n+1,n+1} = 0.$$ 

By (3.7) and (3.8) there holds

$$\varphi_t \geq u^+_t = \nu (g^+)' = \nu \varphi_{n+1} = 0 \geq (\delta_{ij} - \eta_i \eta_j) \varphi_{ij}.$$ 

On the other hand, if $\nabla \varphi \neq 0$. Then by (3.7) we get

$$(3.9) \quad \varphi_i = u^+_i = 0, \quad u^+_i = 0, \quad i = 1, 2, \cdots, n, \quad \varphi_{n+1} = u^+_{n+1} \neq 0.$$ 

Combining (3.7) with (3.9), we obtain

$$(3.10) \quad \left(\frac{\varphi^2}{|\nabla \varphi|^2}\right) \varphi_{ij} = \sum_{i=1}^n \varphi_{ii} \leq \sum_{i=1}^n u^+_{ii} = 0,$$

$$(3.11) \quad \varphi_t \geq u^+_t = \nu \varphi_{n+1}.$$
It follows from (3.10) and (3.11) that
\[
\left( \delta_{ij} - \frac{\varphi_i \varphi_j}{|\nabla \varphi|^2} \right) \varphi_{ij} + \nu |\nabla \varphi| \leq \varphi_t.
\]

So we conclude that \( u^+ (x', x_{n+1}, t) \) is viscosity super-solution of (1.4). This completes the proof of Theorem 1.7.

**Acknowledgements.** This work is supported by the National Natural Science Foundation of China (10671022) and Doctoral Programme Foundation of Institute of Higher Education of China (20060027023).

**References**

[1] B. Hein: A homotopy approach to solving the inverse mean curvature flow, Cal.Var. and PDE’s. 28(2007), 249C273.
[2] Y.Giga, M.Ohnuma and M.Sato: On the strong maximum principle and the large time behavior of generalized mean curvature flow with the neumann boundary condition, J. Diff. Equa. 154(1999), 107–131.
[3] Y.G. Chen, Y.G. Ga and S.Goto: Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J. Diff. Geom. 33(1991), 749–786.
[4] P.Sternberg and W.P.Ziemer: Generalized motion by curvature with a Dirichlet condition, J. Diff. Equa. 114(1994), 580–600.
[5] L.Evans and J.Spruck: Motion of level sets by mean curvature, I, J. Diff. Geom. 33(1991), 635–681.
[6] L.Evans and J.Spruck: Motion of level sets by mean curvature, II, Trans. Amer. Math. Soc. 330(1992), 321–331.
[7] L.Evans and J.Spruck: Motion of level sets by mean curvature, III, J. Geom. Anal. 2(1992), 121–150.
[8] L.Evans and J.Spruck: Motion of level sets by mean curvature, IV, J. Geom. Anal. 5(1995), 77–114.
[9] D.Gilbarg and N.Trudinger: Elliptic partial differential equations of second order, Second edition, Grundlehren der Mathematischen Wissenschaften, 224, Berlin: Springer-Verlag, 1998.
[10] Gary M.Lieberman: Second order parabolic differential equations, World Scientific, 1996.
[11] Y.Z.Chen and L.C.Wu: Second order elliptic equations and elliptic systems (B. Hu, Trans.), Science Press, Beijing, 1997 (Original work published 1991, in Chinese): Translations of Mathematical Monographs, vol. 174. American Mathematical Society, Providence, RI, 1998.
[12] H.Ishii and P.Sougandis: Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor, Tôhoku Math J. 47(1995), 227–250.

1. School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People’s Republic of China
   
   *E-mail address: hrl602@mail.bnu.edu.cn*

2. Institute of Mathematics, Fudan University, Shanghai 200433, People’s Republic of China
   
   *E-mail address: huangronglijane@yahoo.cn*

3. Corresponding author. School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People’s Republic of China
   
   *E-mail address: jgbao@bnu.edu.cn*