$N = 2$ boundary supersymmetry in integrable models and perturbed boundary conformal field theory

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Abstract

Boundary integrable models with $N = 2$ supersymmetry are considered. For the simplest boundary $N = 2$ superconformal minimal model with a Chebyshev bulk perturbation we show explicitly how fermionic boundary degrees of freedom arise naturally in the boundary perturbation in order to maintain integrability and $N = 2$ supersymmetry. A new boundary reflection matrix is obtained for this model and $N = 2$ boundary superalgebra is studied. A factorized scattering theory is proposed for a $N = 2$ supersymmetric extension of the boundary sine-Gordon model with either (i) fermionic or (ii) bosonic and fermionic boundary degrees of freedom. Exact results are obtained for some quantum impurity problems: the boundary scaling Lee-Yang model, a massive deformation of the anisotropic Kondo model at the filling values $g = 2/(2n + 3)$ and the boundary Ashkin-Teller model.

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1 Introduction

In superstring theory, it has recently been shown that certain backgrounds are associated with two-dimensional integrable massive field theories on the worldsheet [1, 2]. For instance, the sine-Gordon model at its $N = 2$ supersymmetric point [3] corresponds to the simplest generalization of the pp-wave background [4] and the $N = 2$ supersymmetric sine-Gordon model [5] was suggested as a good string background [1]. Similarly, for open superstrings boundary integrable models with $N = 2$ supersymmetry arise naturally [6, 7]. Exact results for such integrable field theories are then desirable.

In statistical physics, several low-dimensional systems around their critical points possess a description in terms of two-dimensional integrable boundary quantum field theories. Among the famous examples, one finds the Kondo model which describes the s-wave scattering of electrons off a magnetic spin impurity. Another example

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corresponds to free fermions on the half-line which describes the scattering of fermions off a monopole (Callan-Rubakov effect). Also, minimal models and their extensions (with $N = 1, 2$ supersymmetry or $W$-symmetry) arise in quantum impurity problems. Although nonperturbative results for (pure) bulk perturbation can be derived using certain quantum group restrictions of (super)sine-Gordon, Bullough-Dodd or more generally affine Toda models, in case of an integrable boundary perturbation such a relation is still an open problem which needs further analysis.

As explained in [8, 9], a boundary perturbation of a massive integrable quantum field theory will preserve integrability if the perturbing boundary operator possesses a representation with respect to the underlying chiral algebra on the half-plane and provided it belongs to the same representation than the bulk potential. It is important to mention that [10] the choice of conformal boundary conditions plays a crucial role as it restricts the representations of the boundary operators.

Among the simplest known examples, one finds the critical Ising field theory (with central charge $c = 1/2$) perturbed by the energy operator in the bulk ($\Phi_{13}$ representation of the Virasoro algebra) [11]. Due to the argument [8] recalled above, it follows that a boundary operator in the $\Phi_{13}$ representation leads to a boundary integrable model. Indeed, from the fusion rule of the spin operator $\sigma \sigma \rightarrow I + \Phi_{13}$ a boundary integrable model can be constructed as an off-critical Ising model restricted on the half-line with free conformal boundary conditions perturbed by a boundary spin operator coupled with one fermionic boundary degree of freedom [8].

As mentioned in [9], it would be interesting to construct a nontrivial boundary integrable perturbation of a critical Ising field theory on the half-line perturbed by a relevant magnetic operator in the bulk ($\Phi_{12}$ representation). Indeed, if one starts with standard boundary conditions (free or fixed), one simply cannot obtain a boundary operator in the representation of the Virasoro algebra [10]. Starting from certain boundary conditions changing operators [10], it is however expected that one might be able to construct a well-defined boundary operator that preserves integrability. But in this case, the underlying boundary conformal field theory (BCFT) with these special boundary conditions has to be identified, which remains an open question. In order to solve this problem, it has been pointed out that introducing new degrees of freedom located at the boundary might provide a useful tool.

In both previous examples, a perturbed boundary minimal model (critical Ising field theory) is considered for which the representations of the Virasoro algebra (labeled by the conformal weights) are in finite number. Fusion rules then restrict drastically the number of models that can be constructed. It is probably one reason why the presence of boundary degrees of freedom seems to be sometimes necessary.

On the other hand, we may now wonder if boundary degrees of freedom occur in case of BCFTs with a continuous serie of representations like the Liouville or, more generally, Toda field theories perturbed simultaneously by bulk and boundary operators in the same representation. An interesting example is provided by the sine-Gordon (SG) field theory on the half-line without [14, 8] or with [15, 16] degrees of freedom at the boundary. In full generality, we define its Euclidean action as:

$$A_{b,SG} = \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx \left( \frac{1}{8\pi} (\partial_\nu \phi)^2 - 2\mu \cos(\tilde{\beta} \phi) \right) + \int_{-\infty}^{\infty} dy \Phi_{pert}^B(y) + A_{boundary}$$

where the interaction between the SG field and the boundary terms $E_\pm(y)$ with dimension $\sim \mu^{1/2}$ reads

$$\Phi_{pert}^B(y) = E_-(y)e^{i\tilde{\beta} \phi(0,y)/2} + E_+(y)e^{-i\tilde{\beta} \phi(0,y)/2}.$$  

Here we used the notation $\tilde{\beta} = \beta/\sqrt{4\pi}$ and introduced the UV mass parameter $\mu$. The last term only depends on the boundary degrees of freedom, and is only considered if necessary.

If there are no degrees of freedom at the boundary, the terms $E_\pm(y) \sim Const.$ are just $c$-numbers and we set $A_{boundary} \equiv 0$ in (1). This two-parameter family of boundary integrable models have been studied in

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3This can be shown in first order of conformal perturbation theory. If operators are very relevant, scaling arguments can be used to show that local conserved currents exists at all orders in conformal perturbation theory.

4For boundary conformal field theories with higher symmetries such as $W$-minimal models [12], $Z_n$ models [13]... the same situation occurs. For a discussion about the relation between boundary Toda models with imaginary coupling and perturbed boundary $W$-minimal models, see for instance [9].

5Only boundary parameters are mentioned here. The bulk mass is sometimes considered as a third (bulk) parameter.
details since many years, for which exact results [8,17,18,19,20,21,22,23,24,25,26,27,28,29,30] have been confirmed using various methods. This model can be considered as an integrable perturbation of the boundary Liouville field theory as suggested in [31]. Using vertex operators representations [31,32,33,34], it is not difficult to see in (1) that the boundary perturbing operators are indeed in the same representation than the bulk ones (for identical bulk and boundary background charges). Without boundary degrees of freedom, the UV limit of the boundary SG model defined in (1) can be identified either with the Gaussian field with Neumann boundary conditions or with the boundary Liouville field theory as considered in [31].

For certain degrees of freedom at the boundary and generic values of the coupling $\hat{\beta}$, the classical Hamiltonian associated with the model (1) above is integrable [15]. At quantum level, integrability is preserved [16] for

$$E_+(y) = \pm 2e_{UV} \frac{1 - \hat{\beta}^2}{\beta^2} \cosh \hat{p}(y) \quad \text{and} \quad E_-(y) = \pm 2e_{UV} \frac{1 - \hat{\beta}^2}{\beta^2} \cosh \hat{q}(y)$$

(2)

with $[\hat{p}(\pm \infty), \hat{q}(\pm \infty)] = \alpha \ \text{mod} \ (4i\pi)$ and $e_{UV}$ is a boundary mass parameter. Depending on each sign in (2), the boundary quantization length $\alpha/i$ associated with the boundary quantum mechanical system is fixed to

$$\alpha = i4\pi \left(\frac{\hat{\beta}^2 - 1}{\beta^2}\right) \quad (+) \quad \text{or} \quad \alpha = i2\pi \left(\frac{\hat{\beta}^2 - 2}{\beta^2}\right) \quad (-).$$

(3)

Introducing a “dynamical” extension [16] of the Cartan subalgebra (identified with the topological charge in the SG model) and using the boundary quantum group structure discovered in [35] 6 (coideal subalgebra of $U_q(\widehat{sl}_2)$), we obtained the soliton/antisoliton and breathers boundary reflection matrices. In particular, for generic values of the coupling $\hat{\beta}$, we found that the bulk and the boundary masses are locked together 7. We refer the reader to [16] for more details.

For generic values of the coupling $\hat{\beta}$, this model deserves certainly some interest, but taking some special values provides interesting insights as we are going to show. Indeed, for $\hat{\beta}^2 = 2p/p'$ with $0 < p < p'$ integers the bulk SG model is known to be related with the $\Phi_{13}$ bulk perturbation of minimal models. Furthermore, at the special point $\hat{\beta}^2 = 4/3$ (repulsive regime) it describes an integrable bulk perturbation (Chebyshev type) of $N = 2$ superconformal minimal model. The SG model with boundary degrees of freedom introduced and studied in [16] being one possible boundary integrable perturbation 8 of the bulk SG model, it is important to analyse in details these special points.

In this paper we will focus on $\hat{\beta}^2 = 4/(2n + 3), \ n \in \mathbb{N}$ in (1) for several reasons. First, it will provide new examples of perturbed conformal field theories with boundary degrees of freedom for which the factorized scattering theory is proposed. Secondly, for generic values of the coupling $\hat{\beta}$ in the model (1), free parameters can be introduced in the boundary operators $E_\pm(y)$ through a canonical transformation of $\hat{p}(y), \hat{q}(y)$. Then, it is important to see if different boundary operators $E_\pm(y)$ preserving integrability can be constructed at these special points for which the explicit dependence in terms of the free parameters might be changed. Third, for $n = 0$ a massive $N = 2$ boundary supersymmetry algebra appears explicitly.

The paper is organized as follows. In the next section, we exhibit an exact relation between the simplest bulk-boundary perturbed $N = 2$ superconformal minimal model and the sine-Gordon model at special point $\hat{\beta} = 4/3$ with fermionic boundary degrees of freedom. It shows that, similarly to the critical Ising model in a boundary magnetic field, boundary degrees of freedom appear naturally in this model. The boundary supersymmetric charges are explicitly constructed from the Lagrangean and differ from the ones proposed in [9,41]. Although their expressions are similar to the ones in [41] (which may probably be considered as “effective” supersymmetry charges in terms of the boundary structure) the main difference here is the fermionic boundary algebra that appears associated with the fermion number operator. These supersymmetry charges are used to construct a

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6This structure has been further studied in [36,37,38] for which new reflection matrices associated with non-dynamical boundary have been obtained.

7Up to a canonical transformation of the boundary degrees of freedom, the UV parameter $e_{UV}$ is fixed by the boundary non-local conserved charges algebraic structure in terms of the bulk UV mass parameter $\mu$. Although not shown explicitly, Warner predicted such phenomena [9].

8There, we did not discuss integrable massless models with boundary degrees of freedom considered for instance in [39,40].
new boundary reflection matrix with $N = 2$ supersymmetry which possesses two free parameters and is shown to satisfy the boundary Yang-Baxter equations. The corresponding massive $N = 2$ boundary superalgebra is constructed and the boundary free energy is proposed. In section 3, a boundary scattering associated with a $N = 2$ supersymmetric version of the boundary sine-Gordon model with purely fermionic (proposed in [42]) or a mixing of fermionic and bosonic boundary degrees of freedom is proposed. In any case, the supersymmetric part is different from the one suggested in [9, 41]. In section 4, we present applications of our results to quantum impurity problems. The boundary scaling Lee-Yang model is revisited in light of our results. The anisotropic Kondo model is shown to admit a simple massive integrable extension at special points $g = 2/(2n + 3)$ with $n \in \mathbb{N}$, for which scattering amplitudes are proposed. Finally, a boundary version of the Ashkin-Teller model is briefly presented.

2 $N = 2$ supersymmetry in the boundary sine-Gordon model

Among the class of relevant perturbation of $N = 2$ superconformal minimal model with central charge $c = 3\ell/\ell + 2$, the ones corresponding to integrable $N = 2$ supersymmetric field theories have been studied in details [43]. Three different types of relevant bulk perturbations can be considered. Here, we focus on the model associated with a perturbation of the form (the so-called Chebyshev perturbation):

$$\Phi_{\text{bulk}}^{\text{pert}}(z, \bar{z}) = \tilde{\lambda} G_{-\frac{1}{2}}^- G_{-\frac{1}{2}}^+ \Phi_k^+(z, \bar{z}) + \tilde{\lambda} G_{\frac{1}{2}}^+ G_{\frac{1}{2}}^- \Phi_k^-(z, \bar{z}),$$

where $\tilde{\lambda}$ is a complex parameter that characterizes the strength of the perturbation. The chiral primary fields $\Phi_k^\pm$ possess conformal dimensions $\Delta_k = \bar{\Delta}_k = k/2(\ell + 2)$ and $G_{\frac{1}{2}}^\pm (G_{-\frac{1}{2}}^\pm)$ are $N = 2$ supersymmetry generators. Actually, this perturbation is known to be the $N = 2$ superconformal analogue of the energy perturbation of the ordinary critical Ising model. Following the case of the off-critical Ising model in a boundary magnetic field [8] and in order to maintain supersymmetry (see details in [9]), it is then natural to consider the model with the bulk perturbation (4) on the half-line and a boundary perturbation of the form [9]:

$$\Phi_{\text{pert}}^{\text{boundary}}(y) = \mu Be^{-i\phi_0/\sqrt{3}} \nu^{uv}(y) G_{-\frac{1}{2}}^- \Phi_k^+(y) + \mu Be^{i\phi_0/\sqrt{3}} \nu^{uv}(y) G_{\frac{1}{2}}^+ \Phi_k^-(y).$$

Here $G_{\frac{1}{2}}^\pm \Phi_k^\pm(y)$ are boundary operators in the same $N = 2$ superconformal representation as the bulk operators, $\phi_0$ is a phase and $\mu_B$ is the boundary mass scale. It is important to stress that the resulting model is neither expected to be integrable nor $N = 2$ supersymmetric for any boundary degrees of freedom $\nu^{uv}(y)$. Also, both requirement must be simultaneously consistent. This can be checked from the structure of the Lagrangean, but also in the scattering amplitude. To proceed further, let us consider a concrete example: the sine-Gordon model at $\beta^2 = 4/3$.

In the bulk and for $\ell = 1$ the simplest $N = 2$ superconformal minimal model possesses a Lagrangean representation in terms of a single free boson $\tilde{\phi}(x, y)$ compactified on the "supersymmetric" radius. In terms of its holomorphic and antiholomorphic parts it reads $\tilde{\phi}(x, y) = \tilde{\phi}(z) + \bar{\tilde{\phi}}(\bar{z})$ with $z = y + ix$ and $\bar{z} = y - ix$. If we denote the expectation value over the Fock vacuum space of massless fields $\langle \cdots \rangle_0$, then the holomorphic/antiholomorphic components are normalized such that $\langle \tilde{\phi}(z)\tilde{\phi}(w) \rangle_0 = -\ln(z - w), \langle \bar{\tilde{\phi}}(\bar{z})\bar{\tilde{\phi}}(\bar{w}) \rangle_0 = -\ln(\bar{z} - \bar{w}), \langle \tilde{\phi}(z)\bar{\tilde{\phi}}(\bar{w}) \rangle_0 = 0$.

The $N = 2$ supersymmetric generators correspond to the stress-energy tensor which ensures conformal invariance, the $U(1)$ current and the two supersymmetry generators. Their holomorphic parts read, respectively,

$$T(z) = -\frac{1}{2} (\partial_z \tilde{\phi})^2, \quad J(z) = \frac{i}{\sqrt{3}} \partial_z \tilde{\phi} \quad \text{and} \quad G^\pm(z) = \exp(\pm i \sqrt{3} \tilde{\phi}(z)).$$

The order parameter and its conjugate are defined as $\Phi_{k=1}^{\text{pert}}(z, \bar{z}) = \exp(\pm \frac{i}{\sqrt{3}} (\tilde{\phi}(z) + \bar{\tilde{\phi}}(\bar{z})))$. Using the definitions (6) above, it follows that the interacting terms in the bulk perturbation (4) become

$$G_{-\frac{1}{2}}^- G_{-\frac{1}{2}}^+ \Phi_k^+(z, \bar{z}) = \exp(\pm \frac{2i}{\sqrt{3}} (\tilde{\phi}(z) + \bar{\tilde{\phi}}(\bar{z}))).$$
On the half-line, one can use the method of mirror images for Neumann boundary conditions \( \partial_x \phi(x, y)|_{x=0} = 0 \) i.e. one defines \( \phi(x, y) = \bar{\phi}(x, y) + \hat{\phi}(-x, y) \). In terms of its holomorphic and antiholomorphic part restricted on the half-line it gives
\[
\varphi(z) = \hat{\varphi}(z) + \bar{\varphi}(z) \quad \text{and} \quad \bar{\varphi}(z) = \bar{\varphi}(\bar{z}) + \hat{\varphi}(\bar{z}) . \tag{8}
\]
from which one deduces the two-point functions in the BCFT with Neumann boundary conditions
\[
\langle \varphi(z) \varphi(w) \rangle_0 = -2 \ln(z - w) , \quad \langle \bar{\varphi}(z) \bar{\varphi}(w) \rangle_0 = -2 \ln(z - w) , \quad \langle \varphi(z) \bar{\varphi}(w) \rangle_0 = -2 \ln(z - w) .
\]
In particular, from these definitions\(^9\) and using the supersymmetric generators restricted on the half-line \( G^\pm(z) = \exp(\pm i \sqrt{3} \varphi(z)) \) the boundary perturbation becomes
\[
G^\pm_{\frac{1}{2}} \Phi^\pm_k (y) \equiv \exp(\mp \frac{i}{\sqrt{3}} \langle \varphi(z) + \bar{\varphi}(\bar{z}) \rangle)|_{x=0} = \exp(\mp \frac{i}{\sqrt{3}} (\phi(0, y))) . \tag{9}
\]
It follows that the action associated with the simplest \( N = 2 \) boundary superconformal minimal model corresponds to the boundary sine-Gordon model (1) at the special point \( \hat{\beta}^2 = 4/3 \) with the substitutions
\[
\mu \rightarrow -\hat{\lambda} = -\hat{\lambda} \quad \text{and} \quad E_\pm(y) \rightarrow \mu_B e^{\pm i \phi_0 / \sqrt{3} \nu^\pm_{uv}(y)} . \tag{10}
\]
At the supersymmetric point, the bulk SG model is in the repulsive regime (no breathers): the spectrum consists of one soliton and one antisoliton. In the following, they are denoted \( |u(\theta)\rangle \) and \( |v(\theta)\rangle \) with fermion number +1/2 and −1/2, respectively and form a two dimensional supermultiplet [44, 5, 45]. The corresponding bulk S-matrix is known for a long time [46]. For general values of the coupling in the SG model, four nonlocal charges with fractional spin and one topological charge generating the \( U_q(SL_2) \) affine quantum enveloping algebra exist [44]. However, at \( \hat{\beta}^2 = 4/3 \) they become local (spin 1/2) and together with the fermion number operator they generate the massive superalgebra. If \( \theta \) denotes the particle rapidity and \( M \) its mass, in the supermultiplet representation \( \pi_\theta \) they act as
\[
\pi_\theta(\mathbb{Q}_+)|u(\theta)\rangle = \sqrt{2Me^{\theta/2}}|u(\theta)\rangle , \quad \pi_\theta(\mathbb{Q}_-)|u(\theta)\rangle = \sqrt{2Me^{-\theta/2}}|v(\theta)\rangle , \quad \pi_\theta(\mathbb{Q}_+)|d(\theta)\rangle = \sqrt{2Me^{-\theta/2}}|u(\theta)\rangle \tag{11}
\]
and
\[
F|u(\theta)\rangle = \frac{1}{2}|u(\theta)\rangle , \quad F|d(\theta)\rangle = -\frac{1}{2}|d(\theta)\rangle . \tag{12}
\]
Using this representation, the massive superalgebra can be constructed [47, 5, 45] in terms of the Hamiltonian \( H \), momentum \( P \) and particle number \( \mathcal{N} \) with eigenvalues on one-particle states \( E = M \cosh(\theta) \), \( P = M \sinh(\theta) \) and 1 or 0, respectively. Using the definition of the coproduct [5, 45] which acts on multiparticle states one can check that the bulk S-matrix commutes with these supercharges and the fermion number operator.

If we now consider the SG model on the half-line at the special value \( \hat{\beta}^2 = 4/3 \) as defined above (1), using the results of [16] together with a scale transformation it follows that the boundary supercharges
\[
\hat{\mathbb{Q}}_\pm = \mathbb{Q}_\pm + \nu^\pm_{ir}(-1)^F \quad \text{with} \quad \nu^\pm_{ir} = \frac{2\sqrt{M}}{k} \nu^\pm_{ir} \quad \tag{13}
\]
are conserved.\(^{10}\) Here \( k \) denotes a real parameter. To relate the IR and UV data (using the relation in the UV of the sine-Gordon non-local charges and the supercharges), asymptotically we have assumed the boundary conditions for the operators \( \nu^\pm_{uv}(y) \) at both “ends” of the boundary:
\[
\nu^\pm_{uv}(y = \pm \infty) \sim \nu^\pm_{ir} \quad \text{and} \quad \nu^\pm_{uv}(y = \pm \infty) \sim \nu^\pm_{ir} . \tag{14}
\]
\(^9\)One has \( \varphi(z)|_{x=0} = \bar{\varphi}(\bar{z})|_{x=0} = \phi(0, y)/2 \).
\(^{10}\)To do that, one uses the local expression of the supercharges in terms of the field and derives their conservation in first order of deformal perturbation theory. Using scaling arguments, it can be shown that this is true at all orders too, which allows to define them asymptotically i.e. in the IR limit.
Similarly to the boundary SG model with boundary degrees of freedom, the asymptotic boundary operators \( \nu^\text{ir}_\pm \) commute with the massive (bulk) \( N = 2 \) superalgebra. Thus we have

\[
\pi_\theta(\hat{Q}_\pm)_u^N = -i\nu^\text{ir}_\pm, \quad \pi_\theta(\hat{Q}_\pm)_d^N = \sqrt{2M}e^{\mp\theta/2}, \quad \pi_\theta(\hat{Q}_\pm)_u^N = \sqrt{2M}e^{\mp\theta/2}, \quad \pi_\theta(\hat{Q}_\pm)_d^N = i\nu^\text{ir}_\pm
\]

If we impose \( N = 2 \) boundary supersymmetry in the quantum model, the supercharges (13) must commute with the boundary reflection matrix. Then, we are looking for the “minimal” solution of the dynamical extension of the intertwining equation considered in [9, 41]:

\[
K_{\text{susy}}^\delta(\theta)\pi_\theta(\hat{Q}_\pm)_\nu^\zeta = \pi_{-\theta}(\hat{Q}_\pm)_\nu^\delta K_{\text{susy}}^\nu(\theta)
\]

(15)

where indices \( \{\delta, \nu, \zeta\} \in \{u, d\} \) refer to two-dimensional supermultiplet representation on asymptotic soliton states \( |u(\theta)|, |d(\theta)| \). Then, using this representation the solution \( K_{\text{susy}}(\theta) \) is written as a \( 2 \times 2 \) matrix with entries expressed in terms of the boundary operators. In the non-dynamical case [8, 9], the entries are just analytic functions of \( \theta \) as \( \nu^\text{ir}_\pm \) are \( c \)-numbers. Let us define

\[
K_{\text{susy}}^u(\theta) = A(\theta), \quad K_{\text{susy}}^d(\theta) = B(\theta), \\
K_{\text{susy}}^d(\theta) = D(\theta), \quad K_{\text{susy}}^d(\theta) = E(\theta).
\]

After some calculations, we find that the entries of the minimal solution \( K_{\text{susy}}(\theta) \) of the intertwining equation (15) takes the following form:

\[
A(\theta) = \left( e^{\theta/2}\nu^\text{ir}_+ + e^{-\theta/2}\nu^\text{ir}_- \right)/k\sqrt{2}, \quad B(\theta) = \left( -i\sinh(\theta) + \frac{i(\alpha + 1)}{k^2} [\nu^\text{ir}_+, \nu^\text{ir}_-] \right)/2, \\
E(\theta) = \left( e^{\theta/2}\nu^\text{ir}_- + e^{-\theta/2}\nu^\text{ir}_+ \right)/k\sqrt{2}, \quad D(\theta) = \left( -i\sinh(\theta) + \frac{i(\alpha - 1)}{k^2} [\nu^\text{ir}_-, \nu^\text{ir}_+] \right)/2,
\]

(17)

where the boundary operators satisfy

\[
\{\nu^\text{ir}_+, [\nu^\text{ir}_+, \nu^\text{ir}_-]\} = 0 \quad \text{and} \quad [\nu^\text{ir}_-^2, \nu^\text{ir}_+^2] = 0.
\]

(18)

Here \( \alpha \) is a real parameter. One obvious realization is \( \nu^\text{ir}_\pm = e^{\pm i\xi} \) in which case we recover the Ghoshal-Zamolodchikov-DeVega-Gonzalez-Ruiz minimal solution [8, 48]. An other solution to (18) is to introduce a dimensionless complex fermion at the boundary \( b(y), b^\dagger(y) \) such that using (14), asymptotically it satisfies

\[
\{b, b\} = \{b^\dagger, b^\dagger\} = 0 \quad \text{and} \quad \{b, b^\dagger\} = 1 \quad \text{with} \quad b(\pm \infty) \equiv 0.
\]

(19)

Together with the asymptotics (14) we choose the representation \( \nu^\text{ir}_\pm = \sigma_\pm \) in (17), where \( \sigma_\pm \) are Pauli matrices which act on the (pure) boundary space of states. In the Lagrangian (1) with (10), it leads to the choice

\[
\nu^\text{uv}_+(y) = b(y) \quad \text{and} \quad \nu^\text{uv}_-(y) = b^\dagger(y).
\]

(20)

We would like to stress that the minimal part (16) with (17) of the boundary reflection matrix associated with the SG model at its \( N = 2 \) supersymmetric point possesses two free parameters \( \{k, \alpha\} \). In full generality, one may choose to introduce a phase \( e^{\pm ir} \) in front of \( \nu^\text{ir}_\pm \) in (17). However this phase can be removed using a gauge transformation over the fermionic boundary degrees of freedom.

### 2.1 Boundary Yang-Baxter equations and boundary reflection matrix

In the bulk, the perturbed \( N = 2 \) superconformal minimal model (the SG model at \( \hat{\beta}^2 = 4/3 \)) is massive and integrable. Integrability imposes strong constraints on the system which implies that the \( S \)-matrix has to satisfy the quantum Yang-Baxter equations. The entries at \( \hat{\beta}^2 = 4/3 \) read for \( u = -i\theta \):

\[
S^\pm_{\text{susy}}(\theta) = a(\theta) = \cos(u/2)Z(u), \quad S^\pm_{\text{susy}}(\theta) = b(\theta) = \sin(u/2)Z(u), \quad S^\pm_{\text{susy}}(\theta) = c(\theta) = Z(u)
\]

(21)
and vanish otherwise. The factor $Z(u)$ ensures unitarity and crossing symmetry of the $S$-matrix. Its exact expression can be found in [45].

For boundary integrable field theories, the soliton/antisoliton reflection matrix is constrained by the boundary Yang-Baxter equations (the so-called reflection equations). In our case, it reads $R_{\text{susy}}(\theta)$ and obeys

$$S_{\text{susy}}(\theta - \theta') S_{\text{susy}}(\theta + \theta') = S_{\text{susy}}(\theta + \theta') S_{\text{susy}}(\theta - \theta')$$

where we used the notations $1 = M \otimes I$, $2 = I \otimes M$. If $N = 2$ supersymmetry is preserved, we write it in terms of the solution (16) as

$$R_{\text{susy}}(\theta) = K_{\text{susy}}(\theta) Y_{\text{susy}}(\theta) ,$$

where the scalar factor $Y_{\text{susy}}(\theta)$ has to be determined using some physical assumptions (see below). Recalling the definition of the entries (16) it leads to fourteen functional equations [16]:

(i) $a_c (BD' - B'D) + a_a [A, A'] = 0$

(ii) $b_{a} b_{+} [A, E'] + c_{a} c_{+} [E, E'] + c_{a} a_{+} [DB' - D'B] = 0$

(iii) $c_{b} + (EA' - E'A) + c_{a} [A, E'] + b_{a} [B, D'] = 0$

(iv) $b_{a} b_{+} A - c_{a} + ED' + c_{a} c_{+} DA - a_{a} A' - c_{a} a_{+} E'D = 0$

(v) $b_{a} b_{+} B'A + c_{a} c_{+} E'B + c_{a} a_{+} A' - a_{a} + AB' = - a_{a} A'B - c_{a} AB' = 0$

(vi) $b_{a} b_{+} D'E + c_{a} c_{+} D'A + c_{a} a_{+} ED' - a_{a} E'D = 0$

(vii) $b_{a} b_{+} E'B + c_{a} c_{+} AB' - a_{a} + B'E = 0$

(viii) $b_{a} b_{+} BE' + c_{a} c_{+} AB - a_{a} b_{+} E'B = 0$

(ix) $b_{a} b_{+} A'B + c_{a} c_{+} B'A + b_{a} c_{+} BA' = 0$

(x) $b_{a} b_{+} A'D + c_{a} c_{+} D'A - a_{a} + AD' = 0$

(xi) $b_{a} b_{+} DA + c_{a} c_{+} AD' + b_{a} c_{+} ED' = 0$

where we used the shorthand notations $a_{=} = a(\theta - \theta')$, $a_{+} = a(\theta + \theta')$ and similarly for $b$ and $c$ as well as $A = A(\theta)$ and $A' = A(\theta')$ and similarly for $B, D$ and $E$. The remaining three equations are obtained from (i), (ii), (iii) through the substitutions $A \leftrightarrow E$ and $B \leftrightarrow D$. Straightforward calculations show that $K_{\text{susy}}(\theta)$ given by (16) with (17) indeed satisfies all reflection equations above.

In [8], Ghoshal and Zamolodchikov proposed the use of the “boundary unitarity” and “boundary cross-unitarity” conditions to determine the overall factor $Y(\theta)$ associated with a non-dynamical boundary. In case of a dynamical boundary, it can be applied directly [16] and gives the following equations:

- Boundary unitarity:

$$R_{\text{susy}}(\theta) R_{\text{susy}}(\bar{\theta}) = \delta_0^n \mathbb{1} ;$$

- Boundary cross-unitarity:

$$R_{\text{susy}}(i \pi /2 - \theta) = S_{\text{susy}}(i \pi /2 + \theta) ,$$

where the operator $\mathbb{1}$ denotes the identity which acts trivially on the boundary ground state. We refer the reader to [8] for details. Let us now introduce two meromorphic functions $Y_{0}^{\text{susy}}(\theta)$ and $Y_{1}^{\text{susy}}(\theta)$ such that the prefactor in (23) is written

$$Y_{\text{susy}}(\theta) = Y_{0}^{\text{susy}}(\theta) Y_{1}^{\text{susy}}(\theta) .$$

Using the explicit expressions (17), they are chosen such that they solve the functional equations

$$Y_{0}^{\text{susy}}(\theta) Y_{0}^{\text{susy}}(-\theta) = 1 ,$$

$$Y_{0}^{\text{susy}}(i \pi /2 - \theta) = \cos(u) Z(2u) Y_{0}^{\text{susy}}(i \pi /2 + \theta) ,$$

$$Y_{1}^{\text{susy}}(\theta) Y_{1}^{\text{susy}}(-\theta) = \left[-\sin^2(u/2)\cos^2(u/2) - \frac{1}{k^2(1 - \alpha^2) + \frac{4k^2}{4k^4}}\right]^{-1} ,$$

$$Y_{1}^{\text{susy}}(i \pi /2 - \theta) = Y_{1}^{\text{susy}}(i \pi /2 + \theta)$$
in order to have (24) and (25). Using the results of [8] and [16] for \( \lambda = 2/\beta^2 - 1 \) we finally obtain

\[
Y_0^{\text{susy}}(\theta) = R_0(u)G_0(u)\big|_{\lambda=1/2}
\]

where we used [8]

\[
R_0(u) = \prod_{k=1}^{\infty} \left[ \frac{\Gamma(4\lambda k - 2\lambda u/\pi)\Gamma(1 + 4\lambda(k - 1) - 2\lambda u/\pi)}{\Gamma(\lambda(4k - 3) - 2\lambda u/\pi)\Gamma(1 + \lambda(4k - 1) - 2\lambda u/\pi)} \right] / (u \to -u)
\]

and [16]

\[
G_0(u) = \prod_{k=1}^{\infty} \left[ \frac{\Gamma(1 + (4k - 2)\lambda - 2\lambda u/\pi)\Gamma((4k - 2)\lambda - 2\lambda u/\pi)}{\Gamma((4k - 4)\lambda - 2\lambda u/\pi)\Gamma(1 + 4k\lambda - 2\lambda u/\pi)} \right] / (u \to -u).
\]

The other part is given by

\[
Y_1^{\text{susy}}(\theta) = \frac{\sigma(\eta, u)\sigma(\imath\vartheta, u)}{\cos(\eta) \cosh(\vartheta)}\big|_{\lambda=1/2}
\]

with [8]

\[
\sigma(x, v) = \frac{\cos x}{\cos(x + \lambda v)} \prod_{i=1}^{\infty} \left[ \frac{\Gamma(1/2 + (2l - 1)\lambda - x/\pi - \lambda v/\pi)\Gamma(1/2 + (2l - 1)\lambda - x/\pi - \lambda v/\pi)}{\Gamma(1/2 + (2l - 2)\lambda - x/\pi - \lambda v/\pi)\Gamma(1/2 + 2\lambda + x/\pi - \lambda v/\pi)} \right] / (v \to -v). \tag{29}
\]

Here \( \eta \) and \( \vartheta \) are two real IR boundary parameters related with \( k \) and the parameter \( \xi \) by

\[
\cos(\eta) \cosh(\vartheta) = -\frac{1}{k} \cos \xi \quad \text{and} \quad \cos^2(\eta) + \cosh^2(\vartheta) = 1 + \frac{1}{k^2}. \tag{30}
\]

Using these definitions, the parameter \( \alpha \) entering in the boundary reflection matrix (23) with (16) and (26) becomes

\[
\alpha_\pm = \pm \sqrt{1 - 2k^2 \cos(2\xi)}. \tag{31}
\]

It should be stressed that apart from the commutative case \( \nu_{\text{ir}} \equiv e^{\pm i\xi} \), there are no values of the parameters for which the boundary reflection matrix (23) reduces to the Ghoshal-Zamolodchikov one. Furthermore, we also checked that our solution can not be obtained from the Ghoshal-Zamolodchikov one using the fusion procedure suggested in [9]. Notice that for the commutative case the exact relations between the IR boundary parameters \{\eta, \vartheta\} and the UV boundary parameters have been obtained by Al.B. Zamolodchikov [49] and are supported by truncated conformal space analysis [50].

### 2.2 The massive \( N = 2 \) boundary superalgebra and boundary free energy

Let us now see in which manner the presence of boundary fermionic degrees of freedom affects the \( N = 2 \) supersymmetry algebra. To see this, one can use the one-particle asymptotic states \( |u(\theta)\rangle, |d(\theta)\rangle \) representation (12) [45] as before. They provide a representation for the \( N = 2 \) massive superalgebra [47]

\[
\{Q_+, Q_-\} = 2(H + P), \quad \{\overline{Q}_+, \overline{Q}_-\} = 2(H - P), \quad \{Q_+, \overline{Q}_-\} = 2MN, \quad \{Q_+, Q_-\} = 2MN, \quad Q_\pm^2 = \overline{Q}_\pm^2 = 0 \quad \text{and} \quad \{Q_\pm, F\} = \{\overline{Q}_\pm, F\} = 0. \tag{32}
\]

Using these relations, the expressions for the boundary supercharges (13) and anticommutation relations for the asymptotic boundary degrees of freedom (19), it is straightforward to derive the relations

\[
\hat{Q}_\pm^2 = 2MN \quad \text{and} \quad \{\hat{Q}_+, \hat{Q}_-\} = 4(H - E_{\text{boundary}}^{\lambda=1/2}(-1)^{2F}) \quad \text{where} \quad E_{\text{boundary}}^{\lambda=1/2} = \frac{M}{k^2}. \tag{33}
\]
Here it is interesting to notice that the term associated with the boundary energy is nothing but the boundary energy of the non-dynamical boundary SG model\textsuperscript{11} at special point $\hat{\beta}^2 = 4/3$, given by [49] (see [50] for details)

$$E^\lambda_{\text{boundary}} = -\frac{M}{2\cos(\pi/2\lambda)} \left[ \cos(\eta/\lambda) + \cos(\theta/\lambda) - \frac{1}{2} \cos(\pi/2\lambda) + \frac{1}{2} \sin(\pi/2\lambda) - \frac{1}{2} \right]$$

(34)

together with (30). Consequently, both terms are positive definite for any value of the parameter $k$ and any boundary ground state, as required. As an example, we can consider the massless limit $M \to 0$ with $k$ finite. In this case, the expectation value in the second term of (33) takes its minimal value, independantly of $k$. Notice that this subalgebra is different from the one given in [41]. Although the second anticommutator is identical, the first differs by boundary contributions. Indeed, in [41] a term proportional to the operator $F^2$ arises in the r.h.s. of the first anticommutator. In other words (apart from the massless case), in order to avoid negative (or complex) expectation values of $\hat{Q}^2_{\pm}$ Bogomolnyi bounds are required in [41] for the boundary parameters $(k, \xi)$. It is however not the case here.

3 Comments about the boundary $N = 2$ supersymmetric sine-Gordon

In the bulk, the boundary $N = 2$ supersymmetric sine-Gordon model possesses local (nonlocal) conserved charges which commute with each other [5]. Together, they generate the quantum affine algebra $U_q(\widehat{sl}_2) \otimes U_{q^2 = -1}(\widehat{sl}_2)$. It follows that the soliton $S$-matrix has the factorized form [5]:

$$S_{SG}^{N=2}(\theta) = S_{SG, \hat{\beta}^2}(\theta) \otimes S_{SG, q^2 = -1}(\theta)$$

(35)

where the bosonic part $S_{SG, \hat{\beta}^2}(\theta)$ is the sine-Gordon scattering matrix. From the tensor product structure, in the boundary case the boundary reflection matrix also takes a factorized form:

$$R_{SG, \hat{\beta}^2}^{N=2}(\theta) = R_{SG, \hat{\beta}^2}(\theta) \otimes R_{\text{susy}}(\theta) .$$

(36)

In [42], an action for the $N = 2$ supersymmetric sine-Gordon model has been proposed. It is exact in case of massless bulk and an approximation to first order in the bulk mass. At the boundary, this action contains fermionic boundary degrees of freedom coupled with the fermionic and bosonic fields $\psi^\pm, \varphi^\pm$. Starting from this Lagrangean, we checked explicitly that the boundary supercharges take a form similar to (13) with (14) and (20). They are different from the ones proposed in [41], which do not contain fermionic boundary degrees of freedom in front of the term associated with the topological charge. Higher order corrections in the bulk mass term, if needed, would lead to the same form. Consequently, an exact action for the $N = 2$ supersymmetric boundary sine-Gordon model would lead to conserved boundary supercharges of the form (13) generating the $N = 2$ massive boundary superalgebra (33), instead of the ones proposed in [41]. Using previous analysis, it follows that the supersymmetric part $R_{\text{susy}}(\theta)$ of the reflection matrix (36) is given by (23).

Let us now turn to the bosonic part. Actually, there are two kinds of Lagrangean that can be constructed. On one hand, with a boundary interaction of the form [42] which only contains fermionic boundary degrees of freedom (here denoted $\mathbf{b}(y), \mathbf{b}^\dagger(y)$). In this case, the pure bosonic boundary reflection scattering matrix $R_{SG, \hat{\beta}^2}(\theta)$ contains two free parameters and follows from the one proposed by Ghoshal-Zamolodchikov [8]. On the other hand, it is well expected that a boundary interaction including bosonic (denoted $\mathbf{p}(y), \mathbf{q}(y)$) and fermionic boundary degrees of freedom can be constructed as well. In such case, the bosonic boundary reflection matrix follows from the results of [16], i.e. it reads

$$R_{SG, \hat{\beta}^2}(\theta) = K_0(\theta)Y(\theta)$$

(37)

where $Y(\theta)$ ensures boundary unitarity and boundary cross-unitarity symmetry. Its “minimal” part $K_0(\theta)$ takes the same form as in (16) with entries

$$A(\theta) = \pm (q^{-1} e^{\theta/\hat{\beta}^2} \cosh(p) - q e^{-\theta/\hat{\beta}^2} \cosh(q)) (q - q^{-1})/2c$$

\textsuperscript{11}Whereas we checked explicitly that it doesn’t coincide with the boundary free energy calculated for general values of the coupling, i.e. in case of boundary degrees of freedom of the form (2).
\[ E(\theta) = \pm (q^{-1} e^{\theta/\beta^2} \cosh(q) - q e^{-\theta/\beta^2} \cosh(p)) (q - q^{-1})/2c, \]
\[ B(\theta) = \left( -c^2 q^{-1} e^{2\theta/\beta^2} - c^2 q e^{-2\theta/\beta^2} + \frac{q - q^{-1}}{q + q^{-1}} (q^{-1} \cosh(q) \cosh(p) - q \cosh(p) \cosh(q)) \right)/2c^2, \]
\[ D(\theta) = \left( -c^2 q^{-1} e^{2\theta/\beta^2} - c^2 q e^{-2\theta/\beta^2} + \frac{q - q^{-1}}{q + q^{-1}} (-q \cosh(q) \cosh(p) + q^{-1} \cosh(p) \cosh(q)) \right)/2c^2. \] (38)

Here, the deformation parameter \( q = e^{i\pi/\beta^2} \) and \( c = \sin(\pi/\beta^2) \). Depending on the sign in (38), the boundary quantization condition is fixed to
\[ [p, q] = -\frac{2i\pi}{\beta^2} \mod (2i\pi). \] (39)

In total, the boundary reflection matrix (36) of each model contains four or two boundary parameters (up to a canonical transformation of the boundary degrees of freedom \( p \) and \( q \)), respectively. Probably there exists some relation between them, following the analysis of pole structure in [41]. Details as well as further study of the second model will be considered elsewhere.

To conclude, notice that considering the analytic continuation \( \beta^2 = -b^2 \) would provide the factorized scattering theory for the (second) \( N = 2 \) supersymmetric boundary sinh-Gordon model. In this case, up to the sign, the boundary quantization condition for boundary bosonic degrees of freedom becomes \( [p, q] = 2i\pi/b^2 \). Due to the recent conjecture [51] about the dual representation of \( N = 2 \) supersymmetric Liouville field theory, it is worth interesting to understand whether a dual boundary action with \( [p, q] = 2i\pi b^2 \) can be explicitly constructed.

## 4 Relations with quantum impurity problems

Several quantum impurity problems can be analysed using analytical methods. For strongly interacting systems, nonperturbative ones are crucial in order to study bosonized versions of gapless (critical) quantum systems (for a review, see for instance [52]). Among the boundary quantum field theories that can describe such systems, the boundary massless SG model has been proposed and some exact results obtained from its nonperturbative analysis [53]. Also, massive theories have applications in 1D impurity systems with an excitation gap. Let’s see how our model provides exact results for such systems.

- **The boundary scaling Lee-Yang model**

The scaling limit of the Ising model with a purely (bulk) imaginary magnetic field is described by the Lee-Yang model. Its UV limit leads to the non-unitary \( M_{2/5} \) minimal conformal field theory (with central charge \( c = -22/5 \)) which possesses only one primary field with conformal dimension \( \Delta = -1/5 \). The bulk perturbation is identified with this field. In the IR limit, there is only one specie of particles associated with the Faddeev-Zamolodchikov creation operator “\( A(\theta) \)” whose scattering is described by the bulk S-matrix [54]
\[ A(\theta_1)A(\theta_2) = S(\theta_1 - \theta_2)A(\theta_2)A(\theta_1) \quad \text{with} \quad S(\theta) = -(1/3)(2/3) \] (40)
and where we used the notations of the previous sections. In particular, the Lee-Yang model can be thought as the SG model at coupling constant \( \beta^2 = 4/5 \) (\( \lambda = 3/2 \)). Then, the bulk S-matrix factor (40) can be obtained from the first SG breather calculated in [46] after projecting out the soliton sector.

On the half-line, there is a single relevant boundary perturbation of the BCFT with certain conformal boundary conditions [55]. The scattering of the fundamental particle on the boundary (with creation operator \( B \)) is described by the reflection factor \( R(\theta) \) which satisfies
\[ R(\theta)R(-\theta) = 1, \quad R(i\pi/2 - \theta) = S(2\theta)R(i\pi/2 + \theta), \quad \text{and} \quad R(\theta) = S(2\theta)R(\theta + i\pi/3)R(\theta - i\pi/3). \] (41)

Without degrees of freedom at the boundary, the boundary scaling Lee-Yang model has been studied in details in [55] from both analytical and numerical approach. In case of boundary degrees of freedom, it is thus natural to expect some changes in the scattering data. Taking \( \beta^2 = 4/5 \) (\( \lambda = 3/2 \)) in the model (1), it is not difficult to
show that the corresponding minimal part of the soliton/antisoliton reflection matrix can be written in terms of (16) with (17) as \(^{12}\)

\[ K(\theta)|_{\lambda=3/2} = \sigma_3 K_{\text{susy}}(3\theta)\sigma_3 \quad \text{where} \quad \sigma_3 = \text{diag}(1,-1). \]  

(42)

For \( \lambda = 3/2 \) there is only one breather \( n = 1 \). Using the boundary bootstrap equation [18, 17], it is straightforward to obtain the first breather reflection amplitude

\[ R_B^{(1)}(\theta) = -R_0^{(1)}(u)S^{(1)}(0,u)S^{(1)}(\pi/2,u)S^{(1)}(\eta,u)S^{(1)}(i\theta,u)|_{\lambda=3/2} \]  

(43)

where [17]

\[ R_0^{(n)}(u) = \left( \frac{1}{2} \right) \left( \frac{1 + n^2}{4 + n^2} \right) \prod_{l=1}^{n-1} \left( \frac{1}{2} + \frac{1}{2x} \right)^2 \]  

and

\[ S^{(n)}(x,u) = \prod_{l=0}^{n-1} \left( \frac{x - \frac{1}{2} + n^{2l-1}}{x^2 + \frac{1}{2} + \frac{n^{2l-1}}{2}} \right) \]  

(44)

with the shorthand notation \( (x) = \sin(u/2 + x\pi/2)/\sin(u/2 - x\pi/2) \). However, in order to be a solution of (41) it is necessary to restrict the values of the boundary parameters to \( 2\eta/\pi = \pm(b/2+2), \ 2\theta/\pi = \pm(b/2+1) \). The final result for the reflection factor reads

\[ R(\theta) = \left( \frac{3}{6} \right)^{n-1} \left( \frac{5}{6} \right)^{n-1} \left( \frac{1 + b}{6} \right)^{n-1} \left( 1 - \frac{b}{6} \right)^{n-1} \left( \frac{5 + b}{6} \right)^{n-1} \left( \frac{5 - b}{6} \right)^{n-1} \]  

(45)

Following the notations of [55], for \( b = 0, 1 \) and \( 2 \) one recovers the minimal solutions \( R_{(2)}, R_{(4)} \) and \( R_{(1)} \) of (41), respectively. In particular, in [55] the boundary condition \( \Pi \) was identified with the reflection factor \( R_{(1)} \).

In case of fermionic boundary degrees of freedom added, together with

\[ R(i\pi/2 - \theta) = \frac{S(\theta - i(b - 2)/6)}{S(\theta + i(b - 2)/6)} R(i\pi/2 + \theta) \]  

(46)

we conclude that the fixed point \( b = 2 \) of the symmetry transformation (46) is associated with the conformal boundary condition \( \Pi \). Without fermionic boundary degrees of freedom, it was not the case in [55]. This phenomenon clearly needs further investigation which however goes beyond the scope of this paper.

- **The anisotropic Kondo model and its massive extension**

In [56] two integrable massive versions of the anisotropic spin 1/2 Kondo model have been proposed at the reflectionless points. For general values of the coupling \( \beta \), the model (1) is an other one [16], although boundary degrees of freedom do not belong anymore to \( su_q(2) \). From the results of the previous section, we can now consider the Hamiltonian

\[ H_{MK} = \frac{1}{4\pi g} \int_{-\infty}^{0} dx \left( (\pi(x))^2 + (\partial_x \phi(x))^2 + 8\pi g \mu \cos(2\phi(x)) \right) - \mu_B (S_+ e^{i\phi(0)} + S_- e^{-i\phi(0)}) \quad \text{for} \quad g = \frac{2}{2n + 3} \]  

(47)

where \( n \in \mathbb{N} \) and \( S_\pm \) are Pauli matrices. In the limit \( \mu = 0 \), it corresponds to the usual anisotropic (at zero voltage) Kondo model. Taking \( \phi_0 = 0 \) and \( \nu^{uv}(y) = S_\pm(y) \) in (5), the transformation \( \phi \leftrightarrow -\phi \) with \( S_+ \leftrightarrow S_- \) leaves the Hamiltonian invariant. Then, the corresponding soliton/antisoliton reflection matrix reflection follows from (23) setting \( \alpha = 0 \) and \( \theta \rightarrow (2n + 1)\theta \) in (17), i.e. it reads

\[ \mathcal{R}^{(n)}(\theta) = K_{\text{susy}}((2n + 1)\theta)|_{\alpha=0} \left[ R_0(u)G_0(u) \frac{\sigma(\eta,u)\sigma(i\theta,u)}{\cos(\eta) \cosh(\theta)} \right]_{\lambda=n+1/2}. \]  

(48)

Notice that the massive deformation for \( n = 0 \) in (47) preserves the known \( N = 2 \) supersymmetry of the massless Kondo model at that point [57]. Breathers boundary reflection amplitudes are calculated directly and given by

\[ R_B^{(n)}(\theta) = (-1)^n R_0^{(n)}(u)S^{(n)}(0,u)S^{(n)}(\pi/2,u)S^{(n)}(\eta,u)S^{(n)}(i\theta,u)|_{\lambda=1/2+n} \]  

(49)

\(^{12}\)This solution is obvious as the boundary Yang-Baxter equations are invariant under the simultaneous change of sign in front of \( B(\theta) \) and \( D(\theta) \).
together with (44). It shows that the massive extension proposed in [56] is also integrable at the special points \( g = 2/(2n+3) \). If one considers a non-zero voltage, an interesting question would be to check whether the duality \( g \leftrightarrow 1/g \) [58, 59] still exists at these points in the massive case, or not.

- **The boundary critical Ashkin-Teller model**

  In two dimensions, the Ashkin-Teller model [60] corresponds to two Ising models coupled by a local four spin interaction. The action associated with the critical line of the Ashkin-Teller \((\mathbb{Z}_4)\) model in the bulk corresponds to a conformal field theory with central charge \( c = 1 \) (a free massless scalar field \( \phi \)). This critical line can be parametrized by the conformal dimension \( \Delta = \beta^2/2 \) of the thermal operator \( \epsilon \equiv \sqrt{2} \cos(\beta \phi) \). Then, its \( \epsilon \)-perturbation coincides with the sine-Gordon model. The order parameters are the fields \( \sigma \), \( \sigma^\dagger \) and the field \( \Sigma \sim \sigma^2 \) with conformal dimensions \( \Delta = 1/16 \) (independently of \( \beta \)) and \( \Delta^\Sigma = \beta^2/8 \), respectively. The field \( \Sigma \) which is local with respect to \( \phi \) can be realized in terms of \( \exp(\pm \beta \phi/2) \). At the special point \( \beta^2 = 4/3 \), the Ashkin-Teller model enjoys \( N = 2 \) supersymmetry.

  For generic values of the coupling \( \beta \), a natural boundary version of the Ashkin-Teller model can be associated with action (1) and (2). At the special points \( \beta^2 = 4/(2n+3) \), it admits a two-parameter family of boundary Lagrangean representation of the form

\[
A_{bAT} = \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx \left( \frac{1}{8\pi} (\partial_x \phi)^2 - 2\kappa \epsilon(x,y) \right) + \int_{-\infty}^{\infty} dy \left( \nu_0 b(y) + \nu_0^* b^\dagger(y) \right) \Sigma(y) + A_{\text{boundary}} \tag{50}
\]

where \( \nu_0 \) is a complex parameter and the boundary dynamics are described by

\[
A_{\text{boundary}} = i \int_{-\infty}^{\infty} b^\dagger \frac{d}{dy} b \, dy \tag{51}
\]

The factorized scattering theory associated with these two phases follows from the results of [16] and those presented here. In particular, for \( n = 0 \) this model exhibits \( N = 2 \) boundary supersymmetry defined in section 2.

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