Number of Vertices of the Polytope of Integer Partitions and Factorization of the Partitioned Number*

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ABSTRACT

The polytope of integer partitions of $n$ is the convex hull of the corresponding $n$-dimensional integer points. The graph of $v(n)$, the number of the polytope vertices, has a tooth-shaped form with the highest peaks at primes. We explain its shape by the large number of partitions of even $n$'s that were counted by N. Metropolis and P. R. Stein. We reveal that divisibility of $n$ by 3 also reduces $v(n)$ and characterize convex representations of integer points in arbitrary integral polytope via three other points. Using a specific classification of integers, we demonstrate that the graph of $v(n)$ is stratified into layers corresponding to resulting classes. Our main conjecture claims that the value of $v(n)$ depends on factorization of $n$. We also offer an argument for that the number of vertices of the master corner polyhedron on the cyclic group has similar features.

1. Introduction

Integer partitions are a classical object in mathematics. They are related to divergent problems in mathematics and statistical mechanics [1]. A partition of a positive integer $n$ is any finite non-decreasing sequence $\rho$ of positive integers $n_1, n_2, \ldots, n_r$ such that $\sum_{j=1}^r n_j = n$. The integers $n_1, n_2, \ldots, n_r$ are called parts of the partition $\rho$.

In this article, we develop the polyhedral approach to integer partitions proposed in [16]. It is based on the $n$-dimensional geometrical interpretation of partitions [22]. We refer to every partition $\rho$ as a nonnegative integer point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, a solution to the equation

$$x_1 + 2x_2 + \ldots + nx_n = n,$$

where $x_i$ is the number of parts $i$ in $\rho$, $i = 1, \ldots, n$. For example, the partition $8 = 4 + 2 + 1 + 1$ with three distinct parts 1, 2, 4 is identified with $x = (2, 1, 0, 1, 0, 0, 0) \in \mathbb{R}^8$. We keep on writing $x \vdash n$ to indicate that $x \in \mathbb{N}^n$ is a partition of $n$. Let $P(n)$ denote the set of partitions of $n$. The polytope of partitions of $n$, $P_n \subset \mathbb{R}^n$, is defined as the convex hull of $P(n)$:

$$P_n := \text{conv } P(n) = \text{conv } \{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x \vdash n \}. $$

The conversion from the set to a polytope reveals the geometrical structure of $P(n)$. As for every polytope, the key elements of $P_n$ are its facets and vertices. The facets were characterized in [16], the vertices were studied in [17, 18, 20]. The vertices of $P_n$ of special interest since, by Carathéodory’s theorem [4], every partition is a convex combination of some vertices. So, vertices of $P_n$ form a basis of the set of partitions of $n$ and the number of vertices can be viewed as a measure of complexity of its polyhedral structure. This fact strengthens interest in the number of vertices.

The problem of recognizing vertices of $P_n$ is proved to be decidable in polynomial time with the use of linear programming technique [13]. However no combinatorial characterization of vertices is available as yet. The only result in this direction is the criterion for a partition to be a convex combination of two others [18], see Theorem 1. Polynomial decidability by linear programming of three more problems regarding $P_n$ was proved in [13]: optimization, adjacency of vertices, and separation. Another polynomial algorithm for the optimization problem is proposed in [6]. We conclude this brief review of research related to $P_n$ by mentioning the work of Mano [10], in which he used the polyhedral structure of $P(n)$ in the study of certain types of hypergeometric distribution.

We computed vertices of $P_n$ for $n \leq 100$, see [21], and presented their numbers in the On-Line Encyclopedia of Integer Sequences (OEIS), sequence A203898. The number of vertices turned out to be much smaller than the number of partitions. Let $\text{Vert } P_n$ denote the set of vertices of $P_n$ and let $v(n) := |\text{Vert } P_n|$ be the number of vertices. The graph of $v(n)$ exhibits peculiar features, see Figure 1. In contrast to $p(n) = |P(n)|$, the number of partitions of $n$, the function $v(n)$ does not increase monotonically. It drops down at every even $n$ and its peaks at prime $n$’s seem to be higher than others. Inspired by these perplexing peculiarities, we concentrate on the asymptotic dependence of $v(n)$ on the multiplicative properties of $n$.

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For arbitrary polytope $P_n$, a point $x \in P$ is a vertex of $P$ if it cannot be expressed as a convex combination $x = \sum_{j=1}^{k} \lambda_j y_j$, $\sum_{j=1}^{k} \lambda_j = 1$, $\lambda_j > 0$, of some other points $y_j \in P$, $j = 1, \ldots, k$, $k > 1$, in particular, of vertices. So, every partition $x \in P(n) \setminus \text{Vert}P_n$ is a convex combination of some partitions of $n$. Denote by $\xi(x)$ the minimal number of partitions of $n$, which are needed for such a representation of an $x \vdash n$, $x \notin \text{Vert}P_n$, and let $C_\xi(n)$ be the set of partitions $x \vdash n$, for which $\xi(x) = \xi$. It is easy to see that the sets $C_\xi(n)$, $\xi = 2, 3, 4, \ldots$, are pairwise different and

$$\text{Vert}P_n = P(n) \setminus \bigcup_{\xi \geq 2} C_\xi(n). \quad (2)$$

While computing vertices of $P_n$, we saw that for all $n$ the most of $x \vdash n$, $x \notin \text{Vert}P_n$, are convex combinations of some two partitions of $n$, i. e., these $x$'s belong to $C_2(n)$. The following theorem gives a criterion for a partition $x \vdash n$ to belong to $C_2(n)$.
Theorem 1. ([17]) A partition $x \vdash n$ is a convex combination of two partitions of $n$ if and only if there exist two different collections of parts of $x$ with equal sums.

For $n < 15$, all partitions $x \notin \text{Vert} P_n$ belong to $C_2(n)$. For $n = 15, 21, 24, 25, 27, 28$ and $n \geq 30$ there exist non-vertices of $P_n$ that belong to $C_3(n)$ (and hence do not belong to $C_2(n)$). The partition $x = (0, 0, 2, 1, 1, 3, 0^{10}) \vdash 15$ corresponding to $15 = 3 + 3 + 4 + 5$ is an example; here $0^{10}$ stands for 10 zeros. Indeed, $x = \frac{1}{3} (0, 0, 5, 0^{12}) + \frac{1}{3} (0, 0, 1, 3, 0^{11}) + \frac{1}{3} (0, 0, 0, 3, 0^{10})$, there are no other partitions of 15 with parts 3, 4, and 5, and $x$ is not a convex combination of any two of these partitions.

The minimal $n$ for which some $x \vdash n$ belongs to $C_4(n)$ is $n = 36$. This is the partition $36 = 7 + 8 + 9 + 12$, which is one quarter of the sum of partitions $7^4 + 8^4 + 12^4$. $C_4(n) \neq \emptyset$. Non-emptiness of $C_k(n)$ for $k \geq 5$ is not confirmed yet but we dare to suggest the following conjecture.

Conjecture 1. For every $\xi$, $C_\xi(n) \neq \emptyset$ for sufficiently large $n > n_0(\xi)$. For every $n$, $|C_\xi(n)|$ decreases when $\xi$ grows.

If this conjecture is true then the union in (2) can consist of arbitrarily large number of sets. The following theorem gives an upper bound for $\xi$ such that $C_\xi(n) \neq \emptyset$.

Theorem 2. If for some $n$ and $\xi > 2$, $C_\xi(n) \neq \emptyset$ then $\xi \leq \log_2 (n+1) + 1$.

Proof. Let a partition $x \in C_\xi(n)$, $\xi > 2$, have $m$ parts $\{n_1, n_2, \ldots, n_m\}$. It is proved in [18] that if $m > \log_2 (n+1)$ then $x \in C_2(n)$. Hence, $m \leq \log_2 (n+1)$ since $C_2(n) \cap C_\xi(n) = \emptyset$.

Let $x$ be a convex combination of $y^1, y^2, \ldots, y^\xi \vdash n$. Then $y^j = 0$ for $i \neq \{n_1, n_2, \ldots, n_m\}$, $j = 1, 2, \ldots, \xi$. Since $x \notin C_2(n)$ for $k < \xi$, then $y^1, y^2, \ldots, y^\xi$ are vertices of some $(\xi - 1)$-dimensional simplex in $\mathbb{R}^n$ and are affinely independent. Then the matrix with the rows $(y^j_1, y^j_2, \ldots, y^j_m)$, $j = 1, 2, \ldots, \xi - 1$, is of rank $\xi - 1$ and therefore $\xi - 1 \leq m$. The two inequalities imply that $\xi \leq \log_2 (n+1) + 1$.

Figure 2 shows the structure of the set $P(n)$ provided Conjecture 1 holds. The whole rectangle corresponds to all partitions of $n$. Vertices of $P_n$ form the utmost right rectangle. The inner rectangles in order from left to right correspond to $C_2(n)$, $C_3(n)$, $C_4(n)$, $C_5(n)$, where $k$ depends on $n$. The set $M_2(n)$, whose definition will be given in Section 3, consists of two parts: a subset of $C_2(n)$ depicted as the large rectangle from the left edge to the dashed line, and a small subset of vertices forming a tiny rectangle at the bottom right of the picture. The set $K(n) = P(n) \setminus C_2(n)$ will be considered in Section 4.

Remark 1. $\xi(x)$ can be defined for any integer point $x$ in any integral polytope $P$. It could be called “the index of convex embeddedness of $x$”. Then, in particular, vertices of $P$ would be of index 1. However, we refrain from coining a special term. The common state, for example, in combinatorial optimization, is that when a polytope is generated by a set of integral points, each of these points is a vertex. In particular this is true for the traveling salesman polyhedron and other $(0, 1)$-polytopes. Perhaps, this is a reason why the classes of points similar to $C_\xi$ were not considered earlier.

No criterion for $x \in C_3(n)$ is known but the computations show that such an $x$ always admits a representation $x = \sum_{j=1}^{3} \lambda_j y^j$, $y^j \vdash n$, $\lambda_j \geq 0$, $\sum_{j=1}^{3} \lambda_j = 1$, with all $\lambda_j = \frac{1}{3}$. The following theorem states that this holds for every integral polytope. Recall that a polytope is called integral if all its vertices are integer points.

Theorem 3. If $P \in \mathbb{R}^n$ is an integral polytope and an integer point $x \in P$ is a convex combination of three integer points in $P$ but is not a convex combination of any two integer points in $P$ then there exist integer points $y^1, y^2, y^3 \in P$ such that

$$x = \frac{1}{3} y^1 + \frac{1}{3} y^2 + \frac{1}{3} y^3. \quad (3)$$

Figure 2. Conjectured structure of the set of partitions of $n$. 
**Figure 3.** To the proof of Theorem 3.

**Proof.** We begin with the general case of an arbitrary integer $k > 2$ and an integer $x \in P$, which is a convex combination of $k$ integer points in $P$ and is not a convex combination of any less than $k$ integer points in $P$. Then $x$ is a strictly interior point in the $(k - 1)$-dimensional simplex $S$ with vertices in these $k$ points. Assume there is one more integer point $z \in S, z \neq x$.

If $z$ is strictly interior to $S$ then it divides $S$ to integral simplices $S_1, S_2, \ldots, S_k$ with vertices $z$ and any $k - 1$ vertices of $S$. Since $x$ is not a convex combination of any less than $k$ integer points in $S$, it does not lie in any facet of any $S_j$. Hence $x$ lies strictly inside one of these $(k - 1)$-dimensional simplices, say $x \in S_1$. In the other case, if $z$ lies on the border of $S$ let $q$ be the smallest number such that $z$ is strictly interior to some $q$-dimensional face $F$ of $S$. Then $z$ divides $F$ to $q + 1$ integral simplices $F_1, F_2, \ldots, F_{q+1}$. This implies that the simplex $S$ can be also divided to $q + 1$ integral simplices, each of whose vertices are vertices of some $F_j$ and the vertices of $S$ not belonging to $F$. As in the previous case, $x$ lies strictly inside one of these simplices, denote it again by $S_1$.

Applying the same reasoning to $S_1$, if it contains an integer point $z_1 \neq x$, we come to a $(k - 1)$-dimensional integral simplex $S_2 \subset S_1$ with analogous condition on $x \in S_2$. After repeating this procedure a finite number of times, we obtain a $(k - 1)$-dimensional integral simplex $T \subset S_2 \subset S_1 \subset P$ with $x$ as its single strictly interior integer point satisfying conditions of the theorem and no integer points on the border of $T$.

From here on, we consider that $P$ is the triangle $T$ and $k = 3$, as in the theorem statement. The rest of the proof can be carried with the help of the Pick’s theorem as, for example, in [14]. We will continue using only elementary geometry. Figure 3 shows the triangle $T$ with vertices $A$, $B$, $C$ and the point $x$ denoted by $O$.

We will use the following property of the lattice $H_1$ of integer points in the plain $H$ that contains $T$: if for some $u_0 \in H_1$ and some $n$-dimensional vector $\bar{c}$, the points $u_0 + \bar{c}$ belong to $H_1$ then for every $u \in H_1$ the point $u \pm \bar{c}$ belongs to $H_1$.

Let $A_1, B_1, C_1$ be the midpoints of the sides of $T$ and $M$ be the barycenter of $T$. Assume $O \neq M$. Then $O$ lies strictly inside the triangle $A_1B_1C_1$ since otherwise, if for example $O \in \triangle A_1B_1C$, we could have the point $O + \overline{CO} \neq O$ in $T \cap H_1$. (Here $\overline{CO}$ is the vector from $C$ to $O$.) Hence $O$ lies in one of the triangles $A_1B_1M, A_1C_1M, B_1C_1M$ or on a common side of some two of them. Let $O \in \triangle A_1B_1M$. Draw the parallelogram $BOCO_1$ on the straight line segments $OB$ and $OC$. By the above property, $O_1 \in H_1$. The diagonal $OO_1$ of the parallelogram passes through $A_1$. Draw the ray $L$ parallel to $OO_1$ from $A$ inside the triangle $ABC$. Since $A_1O$ goes between $A_1M$ and $A_1B_1$, where $A_1M$ is allowed but $A_1B_1$ is not, $L$ goes between $AM$ and $AB$ and can contain $AM$ but not $AB$. Put the point $O' \in L$ at the distance $|A'O'| = |O_1O|$ from $A$. By the above property, $O'$ is an integer point. The triangles $ABM$ and $A_1B_1M$ are congruent with the congruence coefficient 2 and $|A'O'| = 2|A_1O|$. Hence $O \in \triangle A_1B_1M$ implies $O' \in \triangle ABM$. Note that $O'$ can lie on $AM$ or $BM$. In any case $O'$ is in $T$ and integrality of $O'$ implies $O' = O$.

Since $M$ is the single common point in $\triangle A_1B_1M \cap \triangle ABM$ the assumption $O \neq M$ implies $O' \neq O$. The contradiction proves that $O$ is the barycenter of $T$ and satisfies (3). 

Since all integer points in $P_n$ are partitions of $n$ [16], Theorem 3 implies the following corollary.

**Corollary 1.** Every partition $x \in C_3(n)$ is the barycenter of some partitions $y^1$, $y^2$, $y^3 \vdash n$ (so that the equality (3) holds).

All known partitions $x \in C_3(n)$ admit convex representations with coefficients $\frac{1}{3}$. However the analogue of Theorem 3 does not hold for such an $x$. This follows from the results of Reznik [14]: in case of an integral simplex $P$ with 4 vertices (a 3-dimensional tetrahedron) there are 7 variants for the values of coefficients in a convex representation of a single integer point in $P$ via its vertices. It is interesting that in each variant all denominators are simultaneously equal to one of the numbers 4, 5, 7, 11, 13, 17, 19. Nothing is known about the coefficients in convex representations via 5 integer points.

**3. Evenness of $n$ and Metropolis partitions**

Let us return to Figure 1 that presents the graph of the function $v(n)$ for $n \leq 100$. One immediately sees that the value of $v(n)$ depends on the evenness of $n$:

$$v(2r - 1) > v(2r)$$

(4)

except for small $r$. So, we can refer to the $v(n)$ graph as consisting of two subgraphs: for odd and even $n$'s, the latter lying below the former. This radically differs from the monotone increasing of $p(n)$, the number of partitions of $n$. 

Upon careful examination of Figure 1 we suspected that some points \((n, v(n))\) with \(n\) odd are disposed slightly higher than the main line. It turned out that they correspond to prime \(n\)'s. Comparison of their heights \(v(n)\) with the half-sums \(\frac{1}{2}(v(n-2) + v(n+2))\) confirmed this observation for all prime \(n \geq 43\) except \(n = 61\). The observed tooth-shaped form of the \(v(n)\) graph and special role of prime numbers raised the question of what multiplicative property of \(n\) affects the value of \(v(n)\).

We know from the computation that for every \(n\), the majority of partitions that are not vertices belong to \(C_2(n)\). By Theorem 2, these partitions have two collections of parts with equal sums. In particular, for even \(n\), \(C_2(n)\) contains partitions of the form

\[
\text{[partition}_1\text{ of } r] + \text{[partition}_2\text{ of } r],
\]

where

\[
\text{partition}_1 \neq \text{partition}_2.
\]

Denote the number of partitions (5), disregarding condition (6), by \(m_2(2r)\). It is not hard to see that

\[
m_2(2r) = \frac{1}{2} (p(r)^2 + p(r)) - \text{[number of duplicates in (5)]},
\]

but it is far from clear how to count the duplicates. Note that if a partition of the form (5) satisfies (6) it can be a vertex. The partition \((0, 2, 2, 0^7) + 10\) is an example. The number of such vertices is less than \(p(r)\), which is a rough estimate. Thus, when we are interested in the asymptotics of \(|C_2(n)|\) and \(v(n)\), we can ignore vertices of the form (5). Note that these vertices are shown in Figure 1 by the small rectangle in the Vert \(P_n\) area. The following conjecture may explain inequality (4) and the tooth-shaped form of the \(v(n)\) graph.

**Conjecture 2.** For \(n\) even, \(m_2(n)\) is large relative to \(v(n)\).

Having searched in the OEIS by the sequence of the first values of \(m_2(n)\) we encountered the sequence A002219 and the work of Metropolis and Stein [11], where the authors had counted partitions of \(n\) that can be obtained by joining \(r, r\) divides \(n\), not necessarily different partitions of \(\frac{n}{r}\) (for convenience, we slightly changed the original notation in [11]). We call these partitions Metropolis \(r\)-partitions. For \(n\) multiple of \(r\), denote the set of Metropolis \(r\)-partitions of \(n\) by \(M_r(n)\) and set \(m_r(n) := |M_r(n)|\). Note that Metropolis 2-partitions coincide with partitions (5). The main result of [11] is the formula for \(m_r(n)\) in the form of a finite series of binomial coefficients multiplied by certain integer coefficients, which depend only on \(r\). For \(m_2(n)\) this formula reads

\[
m_2(n) = \left(\frac{g + 2}{2}\right) + (g + 2)c_1 + c_2, \quad g = \left\lfloor \frac{n}{2} + 1 \right\rfloor, \quad \frac{n}{2} > 5,
\]

where \(c_1\) and \(c_2\) "must be determined by direct calculation" [11]. The sequence A002219 contains the values of \(m_2(n)\) for even \(n \leq 178\). Using (8), we obtain an upper bound \(b(n)\) for the number of vertices of \(P_n\).

**Theorem 4.**

\[
v(n) \leq b(n) := \begin{cases} p(n) - m_2(n), & n \text{ even}, \\ p(n) - m_2(n-1), & n \text{ odd}, \end{cases}
\]

where values of \(m_2(\cdot)\) are calculated with the use of (8).

**Proof.** The proof follows from the inclusion \(M_2(n) \subset C_2(n)\), if we ignore the small number of vertices belonging to \(M_2(n)\), and the fact that adding the part 1 to every partition in \(M_2(n-1)\), \(n\) odd, results in a partition in \(C_2(n)\).

Disregarding the duplicates in (5), one can obtain from (7) an upper bound on \(m_2(n)\). However, Metropolis and Stein pointed that, for large \(n\), much better is the bound \(p(n, \frac{n}{2})\), which is the number of partitions of \(n\) with no part greater than \(\frac{n}{2}\). It is not hard to show that \(p(n, \frac{n}{2})\) is asymptotically equal to \(p(n)\). An anonymous author under the nickname “joriki” presented the following proof of this fact in Stackexchange [2]. Every partition of \(n\) has at most one part \(m\) larger than \(\frac{n}{2}\), and the remaining parts form a partition of \(n - m\). Thus

\[
p \left( n, \frac{n}{2} \right) = p(n) - \sum_{i=0}^{\frac{n}{2}-1} p(i).
\]

For large \(n\), the terms in the sum are exponentially smaller than \(p(n)\), so asymptotically

\[
p \left( n, \frac{n}{2} \right) \sim p(n).
\]
Theorem 5. Let us consider arbitrary \( y \)'s. The expression (9) for \( b(n) \), the upper bound on the number of vertices of \( P_n \), may help to clarify, though not prove, the cause of the tooth-shaped form of the graph of \( v(n) \) under Conjecture 2. For \( n \) odd, it yields

\[
b(n) - \frac{1}{2} \left( b(n - 1) + b(n + 1) \right) = \left( p(n) - \frac{1}{2} p(n - 1) + p(n + 1) \right) + \frac{1}{2} \left( m(n + 1) - m(n - 1) \right),
\]

where the first term is asymptotically zero and the second term is positive. This means that \( b(n) \) has a peak at every large odd \( n \) and the graph of \( b(n) \) is of the tooth-shaped form, similar to that in Figure 1 for \( v(n) \).

Let us consider two examples to see what happens when \( n \) is even. For \( n = 78 \), we have \( b(78) = p(78) - m_2(78) = 281860 \), while \( b(77) = p(77) - m_2(77) = 1549719 \). So, the bound for \( v(78) \) is less than 0.19 \cdot b(77). In the same way we have \( b(100) < 0.09 \cdot b(99) \). Hence it is more than likely that, for \( n \) even, \( v(n) \) is not only less than \( \frac{1}{2} (v(n - 1) + v(n + 1)) \) but \( v(n) < v(n - 1) \). Thus, Conjecture 2 and the asymptotic equivalence (10), as its stronger form observed from the numerical data, reasonably justify the inequality (4) and the gap between the values of \( v(n) \) for even and odd \( n \).

The following theorem provides a supplemental indication of the importance of Metropolis 2-partitions for recognizing vertices of \( P_n \). Call a partition \( x \vdash n \) an extension of a partition \( y \vdash m \), \( m < n \), if every part of \( y \) is a part of \( x \).

**Theorem 5.** For every \( n \), every partition \( x \in C_2(n) \) is either a Metropolis 2-partition or an extension of some Metropolis 2-partition \( y \vdash m \), \( m < n \).

**Proof.** Consider arbitrary \( n \) and \( x \in C_2(n) \), \( x \notin M_2(n) \) if \( n \) is even. By Theorem 1, there exist two collections of parts of \( x \) with the same sum. Let \( s \) be the minimal value of such a sum. Clearly, \( s \leq \frac{1}{2} n \). The corresponding collections are disjoint and their union is a Metropolis 2-partition \( y \) of \( m = 2s \leq n \). Hence \( x \equiv y \) if \( m = n \) or \( x \) is an extension of \( y \) if \( m < n \).

## 4. The \( n \)'s multiple of 3 and knapsack partitions

Ehrenborg and Readdy [5] called a partition \( x \) a knapsack partition if for every integer, there is utmost one way to represent it as a sum of some parts of \( x \). Denote the set of knapsack partitions of \( n \) by \( K(n) \) and set \( k(n) := |K(n)| \). Theorem 1 implies relations

\[
K(n) = P(n) \setminus C_2(n),
\]

\[
C_2(n) \subset K(n), \quad \xi > 2,
\]

\[
\text{Vert } P_n \subset K(n).
\]

The smallness of \( |\text{Vert } P_n \cap M_2(n)| \) implies that for large \( n \),

\[
v(n) < k(n) < p(n) - m_2(n).
\]

Note that \( k(n) \) is a much better upper bound on \( v(n) \) than \( b(n) \) in (9) but no formula for \( k(n) \) is known. Ehrenborg and Readdy computed the values \( k(n) \) for \( n \leq 50 \) and exhibited them in the OEIS, sequence A108917. We extended this sequence till \( n = 165 \) as a by-product of our computation of vertices of \( P_n \). Table 2 enhances Table 1 by the \( k(n) \) values. Consider its first three rows with even \( n \). Looking at the columns \( v(n) / (p(n) - m_2(n)) \) and \( k(n) - v(n) \) and comparing the columns \( k(n) \) and \( p(n) - m_2(n) \), we see that many partitions of \( n \) that are neither vertices of \( P_n \) nor Metropolis 2-partitions are convex combinations of 2, 3, or more partitions of \( n \).

One can check that the graph of \( k(n) \), like that of \( v(n) \), disintegrates into two graphs, for \( n \) odd and \( n \) even. However we see that the ratio \( v(n) / k(n) \) does not increase monotonically and is approximately the same for odd \( n = 77 \) and even \( n = 100 \), which are

| \( n \) | \( p(n) \) | \( v(n) \) | \( m_2(n) \) | \( v(n) / p(n) \) | \( v(n) / m_2(n) \) | \( p(n) - m_2(n) \) | \( p(n) - m_2(n) / p(n) \) |
|---|---|---|---|---|---|---|---|
| 60 | 966467 | 5148 | 924522 | 0.005367 | 0.005568 | 41945 | 0.0434 |
| 78 | 12132164 | 17089 | 11850304 | 0.001409 | 0.001442 | 281860 | 0.0232 |
| 100 | 190569292 | 59294 | 188735609 | 0.00311 | 0.00314 | 1833683 | 0.0096 |

### Table 1. Relations between \( p(n) \), \( v(n) \), and \( m_2(n) \).
To examine the discovered dependence of the number of vertices of the polytope $P_n$ rather far from each other. Figure 4 presents the graph of $v(n)/k(n)$. We obviously see that the $n’s$ multiple of 3 are the local minima of $v(n)/k(n)$. This means that such $n’s$ have more partitions that are not vertices of $P_n$ and do not belong to $C_2(n)$ than the $n’s$ not multiple of 3.

The following conjecture naturally explains this phenomenon.

**Conjecture 3.** The majority of partitions in $K(n)$ that are not vertices of $P_n$ are convex combinations of three partitions of $n$.

Conjecture 3 is consistent with our computation experience. We know that for $n$ multiple of 3, most partitions in $C_3(n)$, that are not extensions of some partitions in $C_3(q)$, $q < n$, $q$ multiple of 3, with the additional part $n - q$, have a part $\frac{n}{3}$ and one of the partitions involved in the convex combination has three parts $\frac{n}{3}, \frac{n}{3}$, and $\frac{n}{3}$. For example, the non-vertex $x = (1^3, 9, 17, 22)\vdash 51$ has a part $17 = \frac{51}{3}$, and its convex representation is $\frac{1}{3}(1^7, 22^2) + \frac{1}{3}(1^2, 9^2, 22) + \frac{1}{3}(17^2)$.

As for the tendency of $v(n)/k(n)$ to decrease, we see its explanation in the increase of the number of partitions in $C_k(n)$, $k > 3$, with the growth of $n$.

### 5. Stratification of the numbers of vertices

To examine the discovered dependence of the number of vertices of the polytope $P_n$ on multiplicative properties of $n$ in more details, we consider the classes of integers

$$N_k := \{ n \mid n = kp, \ p \text{ is the largest prime divisor of } n \}, \ k = 1, 2, 3, ...$$

and the corresponding numbers of vertices

$$v_k(n) := v(n), \ n \in N_k.$$

**Figure 5** demonstrates the graphs of the functions $v_k(n)$ for $k = 1, 7, 5, 3, 2, 4, 6$ in order from top to bottom. They are generated with the use of the FindFit method of Wolfram Mathematica. We approximated the known values of $v_k(n)$ by the functions of the form $Ae^{B\sqrt{n}}$ with parameters $A$ and $B$.

The segment $n \in [60, 70]$ is chosen to split the graphs $v_k(n)$ visually. It also lies in the most interesting part of the segment $[1, 100]$, where we can expect our approximations to reveal a reliable picture of what happens. We do not consider the graphs of $v_k(n)$, $k > 7$, because they provide little information. We are interested in the behavior of $v_k(n)$ for large $n$ and there are too few primes $p \geq k$ such that $kp \in [1, 100]$ for these $k$. The graph of $v_7(n)$ is not fully reliable either since $7p \leq 100$ only for three prime $p \geq 7$.

We see that the graph of $v(n)$ is neither a single line nor a conjunction of two lines, for odd and even $n$, as in Figure 1. It is stratified into layers corresponding to the classes $N_k$ and resembles a layered cake. Its layers are of the same shape but are disposed at different levels. The topmost line corresponds to $N_1$, the class of primes. The graph of $v_5(n)$ goes below it. Between them, one below another, are disposed the graphs of $v_7(n)$ and $v_3(n)$, while for $k$ even, the graphs of $v_k(n)$ go below $v_3(n)$.

The levels of the graphs of $v_1(n)$, $v_2(n)$, $v_3(n)$ agree with Conjectures 2 and 3. The intermediate position of $v_7(n)$ and $v_5(n)$ and Conjecture 1 move us to suggest a more general conjecture that for prime $k$ dividing $n$, the determining influence on the level of $v_k(n)$ is exerted by the number of partitions of $n \in N_k$ that belong to $C_k(n)$. If $k_1, k_2$ are two primes, $k_1 > k_2$, then, in accordance

| $n$ | $p(n)$ | $v(n)$ | $m_2(n)$ | $p(n) - m_2(n)$ | $\frac{v(n)}{p(n) - m_2(n)}$ | $k(n)$ | $\frac{v(n)}{k(n)}$ | $k(n) - v(n)$ |
|-----|--------|--------|----------|-----------------|-----------------------------|--------|-----------------|-------------|
| 60  | 966467 | 5148   | 924522   | 41945           | 0.12                        | 5341   | 0.964           | 193         |
| 78  | 12132164 | 17089  | 11850304 | 281860          | 0.06                        | 17871  | 0.956           | 782         |
| 100 | 190569292 | 59294 | 188735609 | 1833683         | 0.03                        | 61692  | 0.967           | 2398        |
| 77  | 10619863 | 778    | 12132164 | 17089           | 0.06                        | 17871  | 0.956           | 782         |

Table 2. Relations between $p(n)$, $v(n)$, $m_2(n)$, and $k(n)$. 

![Figure 4](image-url) Ratio $v(n)/k(n)$ of the number of vertices to the number of knapsack partitions.

![Figure 5](image-url) The graphs of $v_k(n)$ for $k = 1, 7, 5, 3, 2, 4, 6$ in order from top to bottom. They are generated with the use of the FindFit method of Wolfram Mathematica. We approximated the known values of $v_k(n)$ by the functions of the form $Ae^{B\sqrt{n}}$ with parameters $A$ and $B$. The segment $n \in [60, 70]$ is chosen to split the graphs $v_k(n)$ visually. It also lies in the most interesting part of the segment $[1, 100]$, where we can expect our approximations to reveal a reliable picture of what happens. We do not consider the graphs of $v_k(n)$, $k > 7$, because they provide little information. We are interested in the behavior of $v_k(n)$ for large $n$ and there are too few primes $p \geq k$ such that $kp \in [1, 100]$ for these $k$. The graph of $v_7(n)$ is not fully reliable either since $7p \leq 100$ only for three prime $p \geq 7$.

We see that the graph of $v(n)$ is neither a single line nor a conjunction of two lines, for odd and even $n$, as in Figure 1. It is stratified into layers corresponding to the classes $N_k$ and resembles a layered cake. Its layers are of the same shape but are disposed at different levels. The topmost line corresponds to $N_1$, the class of primes. The graph of $v_5(n)$ goes below it. Between them, one below another, are disposed the graphs of $v_7(n)$ and $v_3(n)$, while for $k$ even, the graphs of $v_k(n)$ go below $v_3(n)$.

The levels of the graphs of $v_1(n)$, $v_2(n)$, $v_3(n)$ agree with Conjectures 2 and 3. The intermediate position of $v_7(n)$ and $v_5(n)$ and Conjecture 1 move us to suggest a more general conjecture that for prime $k$ dividing $n$, the determining influence on the level of $v_k(n)$ is exerted by the number of partitions of $n \in N_k$ that belong to $C_k(n)$. If $k_1, k_2$ are two primes, $k_1 > k_2$, then, in accordance
with Conjecture 1, for large and sufficiently close to each other \( n_1 \in N_k \) and \( n_2 \in N_k \), the inequality \(|C_{k_1}(n_1)| < |C_{k_2}(n_2)|\) holds and therefore the \( v_{k_1}(n) \) graph is disposed above the \( v_{k_2}(n) \) graph.

The case of \( k \), a composite divisor of \( n \), can be explained using the graphs of \( v_6(n) \) and \( v_4(n) \). \( v_6(n) \) is disposed below \( v_2(n) \) and \( v_3(n) \) because the level of \( v_6(n) \) is affected by partitions in \( C_2(n) \) and partitions in \( C_3(n) \). Similarly, \( v_4(n) \) goes between \( v_2(n) \) and \( v_6(n) \) because 4 is an additional (to 2) divisor of \( n \) and \(|C_4(n)| < |C_3(n)|\).

We summarize the above in the final conjecture.

**Conjecture 4.** The number of vertices of \( P_n \) depends on factorization of \( n \). The graph of \( v(n) \) is stratified into layers. The stratification is based on partitioning of integers to the classes \( N_k \). The topmost layer \( v_1(n) \) corresponds to prime numbers \( n \in N_1 \). For composite \( n = kp \in N_k \), \( k > 1 \), prime \( p \geq k \) varies, the level of \( v_k(n) \) is rendered by the least divisor of \( k \). Every successive divisor makes its additional contribution to lowering it. The smaller the divisor the more significant its effect.

Generalizing, we might say that the value of \( v(n) \) is determined by the proximity of \( n \) to its greatest prime divisor, which is defined by the lexicographic order on the set of increasing sequences of divisors of \( n \). For example, \( 38 = 2 \cdot 19 \) would be ”more prime” than \( 39 = 3 \cdot 13 \), hence the layer \( v_2(n) \), that contains \( v(38) \), is disposed lower than the layer \( v_3(n) \) containing \( v(39) \). The same would hold for \( 78 = 2 \cdot 3 \cdot 13 \) and \( 70 = 2 \cdot 5 \cdot 7 \). If we extend this speculation, we might come to a fractal structure of the graph of \( v(n) \). For example, the graph of \( v_5(n) \) together with \( v_{10}(n) \), \( v_{15}(n) \), \( v_{20}(n) \), \( v_{25}(n) \), \( v_{30}(n) \), \( v_{40}(n) \)...

... may have a structure similar to that of \( v(n) \). However, it is too early to foresee so far ahead — more numerical data of \( v(n) \) is needed. Then Conjecture 4 might be further detalized.

### 6. Remark on the Gomory’s corner polyhedron

Let \( G \) be a finite Abelian group, \( G^+ \) be the set of its nonzero elements, and \( g_0 \in G \). The master corner polyhedron \( P(G,g_0) \) was defined by R. E. Gomory [7] as the convex hull of solutions

\[
t = (t(g) : g \in G^+) \in \mathbb{R}^{|G^+|}, \ t(g) \text{ integer, } t(g) \geq 0,
\]

to the equation

\[
\sum_{g \in G^+} t(g)g = g_0.
\]  

(11)

For \( G_{n+1} := \mathbb{Z}/(n + 1)\mathbb{Z} \), the cyclic group of order \( n + 1 \), and \( g_0 = n \), equation (11) reads

\[
t_1 + 2t_2 + \ldots + nt_n \equiv n \mod (n + 1),
\]

which differs from (1) only in that the addition here is modulo \( n + 1 \). That is why we call \( P(G,g_0) \) the elder brother of \( P_n \).

Our experience in studying both polyhedra shows that the vertex structure of \( P_n \) is more transparent and easy for understanding than that of the \( P(G,g_0) \), even in the case of the cyclic group. In our opinion, this is because the standard addition on the segment of integers \([1, n]\), albeit defined only partially, is much easier to comprehend than the group addition. Most results on vertices of \( P_n \) were successfully transferred to vertices of the master corner polyhedron [19].

Statistics on vertices of \( P(G,g_0) \) is unbelievably poor. For many years, all that we knew about their numbers could be found in the R. E. Gomory’s seminal paper [7]. The researchers concentrated their efforts on studying facets of \( P(G,g_0) \) since they induce the most efficient cuts for the integer linear programs. In contrast, vertices — though they are no less important for understanding the structure of \( P(G,g_0) \) — fell out of research. In [7] Gomory computed vertices of \( P(G,g_0) \) for all groups \( G \) of the order up to 11 and all
The numbers of vertices of the corner polyhedra \( P(G_{n+1}, n) \), \( n = 1, 2, \ldots, 21 \), constitute the sequence A300795 in the OEIS. Figure 6 exhibits the graph of this sequence.

We perceive this picture as a forerunner of a graph similar to that depicted in Figure 1. The tooth-shaped form of the \(|\text{Vert } P(G_{n+1}, n)|\) graph is obvious even in this initial part. Some of the above features of \( v(n) \) may also become visible when the sequence of numbers of vertices of \( P(G_{n+1}, n) \) will be extended.

### 7. Concluding remarks

Vertices of the polytope \( P_n \) of integer partitions of \( n \) are of importance because they form a kind of basis for the set of all partitions of \( n \) since every partition is a convex combination of some vertices. Computations show that \( v(n) \), the number of vertices of \( P_n \), is much smaller than the number of all partitions. In order to study irregular character of the function \( v(n) \) we investigated the structure of the set of partitions that are not vertices. We divided it to disjoint subsets \( C_\xi(n) \) according to the minimum number \( \xi \) of partitions needed to represent a partition as their convex combination. Using the available numerical data, we demonstrated that the set of Metropolis 2-partitions of \( n \) constitutes a larger part of partitions that are not vertices of \( P_n \). As a consequence, vertices of \( P_n \) form a small subset of partitions of \( n \). We proved that an integer point in an arbitrary integral polytope \( P \), which belongs to the subset of integer points in \( P \) analogous to \( C_\xi(n) \), admits a convex representation via three integer points with all coefficients equal to \( \frac{1}{3} \).

Thorough analysis of the computed values of \( v(n) \) revealed intriguing properties of this function. Comparing this data with available numbers of knapsack and Metropolis 2-partitions moved us to suggest several conjectures that explain observed peculiarities. The main conjecture claims that \( v(n) \) depends on factorization of \( n \). We presented visual but convincing arguments in its favor. We showed that the graph of \( v(n) \) is stratified into layers, the subgraphs corresponding to the classes of integers that are determined by factorization of \( n \). The upper layer corresponds to prime numbers and the others correspond to collections of small divisors of \( n \). Every prime divisor makes its own contribution to lowering the level of the layer. The smaller the divisor the more significant its effect.

We provided an argument in favor of a similar dependence for the number of vertices of the master corner polyhedron on the cyclic group on factorization of the group order. It is based on the limited data set, but we believe that this argument deserves further investigation because of the closeness of the master corner polyhedron and the \( P_n \) revealed in [19]. If the computation of vertices of this polyhedron is continued and the new data confirm the expected dependence, important consequences may follow. It is a general rule that the more vertices a polyhedron has, the more facets it has. Thus, most likely, the master corner polyhedron on a group of prime order has the largest number of facets. In particular, it has many small facets that are hard to find. Then, for algorithms to solve integer linear programs that use cuts generated from the facets of a corner polyhedron, the hardest program may be that for which the basic group of the polyhedron is of prime order. We hope that this work will give an impetus to further study of vertices of the corner polyhedron.

This work draws forth new questions regarding the vertices of the integer partition polytope. Finding a formal proof of the conjectured dependence of the number of vertices of \( P_n \) on factorization of \( n \) remains an open problem for the future research. Further computation of \( v(n) \) would be of great help for its detailed study. One of the most challenging problems is to find a combinatorial criterion for vertices of \( P_n \). More specific problems are concerned with the structure of partitions in \( C_\xi(n) \). Counting knapsack partitions does not look unworkable. This problem looks easier than enumerating vertices and its solution will provide a rather good

---

1. One extra point \( t \in P(G_{11}, 10) \), with \( t(5) = 1, t(9) = 3 \) and all other \( t(i) = 0 \), indicated in [7] as a vertex was excluded in [19].
estimate for \(v(n)\). It may well turn out that the growth order of \(v(n)\) is less than the growth order of \(k(n)\), which in turn is less than that of \(p(n)\).

In recent decades, the number of works in which polyhedral representation was used in the study of sets of integer partitions has noticeably increased. Can we advance further in studying vertices of \(P_n\) using the methods of polyhedral combinatorics developed so far? Let us consider a few typical examples of successful research.

A deep understanding of the polytope of lecture hall partitions and related polyhedral sets has been achieved, see reviews [12, 15]. However the vertices of this polytope are surprisingly simple. Breuer et al. [3] obtained a bijective proof of congruences for the number of partitions into three parts, similar to the Ramanujan congruences, by considering these partitions as integer points in a triangle, a section of a 3-dimensional polyhedral cone, and their clever rearrangement. Konvalinka and Pak [9] proved an extension of the Cayley theorem on compositions and partitions with parts equal to powers of 2. For this, they constructed a bijection between the sets of these objects and showed that the Ehrhart polynomials of the corresponding polytopes coincide. The mentioned papers are characterized by a deep insight into the structure of integer points in polyhedral sets and application of the Ehrhart theory.

The idea of applying the Ehrhart theory to the study of the polytope \(P_n\) looks very tempting. If successful, it could lead to a breakthrough in the research. However, it is not yet clear how to implement this powerful theory. The same refers to the idea of constructing a vertex-preserving bijection between the integer points in \(P_n\), in fact, partitions, and the integer points in some other polytope. The main obstacle, in our opinion, is the lack of a satisfactory description of the vertices of the partition polytope.

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**Declaration of Interest**

No potential conflict of interest was reported by the author.

**References**

[1] Andrews, G. E. (1976). *The Theory of Partitions* (Encyclopedia of Mathematics and Its Applications, Vol. 2). Reading, MA: Addison-Wesley.

[2] Anonymous (joriki). Number of partitions of \(2n\) with no element greater than \(n\). Stackexchange, mathematics. Available at: https://math.stackexchange.com/questions/96085/number-of-partitions-of-2n-with-no-element-greater-than-\(n\).

[3] Breuer, F., Eichhorn, D., Kronholm, B. (2017). Polyhedral geometry, supercranks, and combinatorial witnesses of congruences for partitions into three parts. *Eur. J. Combinat.* 65: 230–252.

[4] Carathéodory, C. (1911). Über den Variabilitätsbereich der Fourier’schen Konstanten von positiven harmonischen Funktionen. *Rendiconti del Circolo Matematico di Palermo* 32: 193–217.

[5] Ehrenborg, R., Readdy, M. A. (2007). The Möbius function of partitions with restricted block sizes. *Adv. Appl. Math.* 39: 283–292.

[6] Engel, K., Radzik, T., Schlage-Puchta, J.-C. (2014). Optimal integer partitions. *Eur. J. Combinat.* 36: 425–436.

[7] Gomory, R. E. (1969). Some polyhedra related to combinatorial problems. *Linear Algebra Appl.* 2: 451–558.

[8] Gomory, R. E. (2007). The atoms of integer programming. *Ann. Oper. Res.* 149: 99–102.

[9] Konvalinka, M., Pak, I. (2014). Cayley compositions, partitions, polytopes, and geometric bijections. *J. Combinat. Theory. Ser. A.* 123: 86–91.

[10] Mano, S. (2017). Partition structure and the A-hypergeometric distribution associated with the rational normal curve. *Electron. J. Stat.* 11: 4452–4487.

[11] Metroplis, N., Stein, P. R. (1970). An elementary solution to a problem in restricted partitions. *J. Combinat. Theory* 9: 365–376.

[12] Olsen, M.C. (2019). Polyhedral geometry for lecture hall partitions. In: Hibi, T., Tsuchiya A., eds. *Algebraic and Geometric Combinatorics on Lattice Polytopes. Proceedings of the Summer Workshop on Lattice Polytopes.* Singapore: World Scientific, pp. 330–353.

[13] Onn, S., Shlyk, V. A. (2015). Some efficiently solvable problems over integer partition polytopes. *Discret. Appl. Math.* 180: 135–140.

[14] Reznik, B. (1986). Lattice point simplices. *Discret. Math.* 60: 219–242.

[15] Savage, C. D. (2016). The mathematics of lecture hall partitions. *J. Combinat. Theory. Ser. A* 144: 443–475.

[16] Shlyk, V. A. (2005). Polytopes of partitions of numbers. *Eur. J. Combinat.* 26: 1139–1153.

[17] Shlyk, V. A. (2008). On the vertices of the polytopes of partitions of numbers. *Doklady Natsional’noi Akademii Nauk Belarusi* 52: 5–10 (in Russian).

[18] Shlyk, V. A. (2013). Integer partitions from the polyhedral point of view. *Electron. Notes Discret. Math.* 43: 319–327.

[19] Shlyk, V. A. (2013). Master corner polyhedron: Vertices. *Eur. J. Oper. Res.* 226: 203–210.

[20] Shlyk, V. A. (2014). Polyhedral approach to integer partitions. *J. Combinat. Math. Combinat. Comput.* 89: 113–128.

[21] Vroublevski, A. S., Shlyk, V. A. (2015). Computing vertices of integer partition polytopes. *Informatics* 4: 34–48 (in Russian).

[22] Weisstein, E. W. Partition. From *Math World* – A Wolfram Web Resource. Available at: http://mathworld.wolfram.com/Partition.html.

[23] Yang, D. (2018). University of California, Los Angeles. Personal communication.