Hochschild Cohomology for Complex Spaces and Noetherian Schemes

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Abstract

The classical HKR-theorem gives an isomorphism of the n-th Hochschild cohomology of a smooth algebra and the n-th exterior power of its module of Kähler differentials. Here we generalize it for simplicial, graded and anticommutative objects in “good pairs of categories”. We apply this generalization to complex spaces and noetherian schemes and deduce two decomposition theorems for their (relative) Hochschild cohomology (special cases of those were recently shown by Buchweitz-Flenner and Yekutieli). The first one shows that Hochschild cohomology contains tangent cohomology: \( HH^n(X/Y, M) = \bigoplus_{i-j=n} \text{Ext}^i(L(X/Y), M) \). The left side is the n-th Hochschild cohomology of \( X \) over \( Y \) with values in \( M \). The right hand side contains the n-th relative tangent cohomology \( \text{Ext}^n(L(X/Y), M) \) as direct factor. The second consequence is a decomposition theorem for Hochschild cohomology of complex analytic manifolds and smooth schemes in characteristic zero: \( HH^n(X) = \bigoplus_{i-j=n} H^i(X, \wedge^j T_X) \). On the right hand-side we have the sheaf cohomology of the exterior powers of the tangent complex.

Keywords: admissible pair of categories, complex space, Hochschild cohomology, regular sequence, scheme

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Introduction

A better title for this paper would be: “Hochschild Cohomology for Admissible Pairs of Categories and Application to Complex Spaces and Noetherian Schemes”. Since this title would be too lengthy, and admissible pairs of categories seem not to be so well known, I did not mention them in the title. Admissible pairs of categories are pairs \((\mathcal{C}, \mathcal{M})\), where \(\mathcal{C}\) is a certain category of algebras (one should think of global sections of the structure sheaf of an “affine” space\(^1\)) and \(\mathcal{M}\) is a category of certain modules over objects in \(\mathcal{C}\) (one should think of global sections of coherent modules over an affine). They were invented by Bingener and Kosarew in [4] and are quite useful in deformation theory, since the PO-algebras and modules of Palamodov also fit into this definition. To describe spaces globally, one has to consider simplicial objects in \(\mathcal{C}\) and \(\mathcal{M}\) for an admissible pair \((\mathcal{C}, \mathcal{M})\), i.e. functors from the “nerf” of an affine covering of the space to \(\mathcal{C}\) and \(\mathcal{M}\). Each construction, using admissible pairs, which is canonical in a sense, can be generalized to the simplicial case. To generalize non-canonical constructions one has to do some work.

In this paper, we use admissible pairs of categories to unify the algebraic Hochschild theory, that can be found in several textbooks (for ex. [11]), and the geometrical approach, which in the analytical context is due to Buchweitz and Flenner. The central result will be a generalization of the classical Hochschild-Kostant-Rosenberg theorem for differential graded algebras. From this generalization we will deduce the following HKR-type theorem:

When \(X \rightarrow Y\) is a morphism of complex spaces (paracompact and separated) or a separated morphism of finite type of noetherian schemes in characteristic zero, then there is a quasi-isomorphism

\[
\mathbb{H}(X/Y) \approx \bigwedge L(X/Y)
\]

over \(\mathcal{O}_X\), where \(\mathbb{H}(X/Y)\) is the relative Hochschild complex of \(X\) over \(Y\) (see section 4) and \(L(X/Y)\) is the relative cotangent complex. This statement is also true, when \(Y\) is just a single point and \(X\) a smooth scheme in characteristic zero. From this main result we will deduce two nice decomposition theorems for Hochschild cohomology. The first one shows that Hochschild cohomology contains tangent cohomology:

\[
\mathbb{H}^n(X/Y, \mathcal{M}) = \prod_{i-j=n} \Ext^i(\bigwedge^j L(X/Y), \mathcal{M}).
\]

The left side is the \(n\)-th Hochschild cohomology of \(X\) over \(Y\) with values in \(\mathcal{M}\). The right hand-side contains the \(n\)-th relative tangent cohomology \(\Ext^n(L(X/Y), \mathcal{M})\) as direct factor. For complex spaces, this decomposition, as well as equation 0.1, was already shown in a different way by Buchweitz and Flenner [5].

The second consequence is a decomposition theorem for Hochschild cohomology of complex analytic manifolds and smooth schemes in characteristic zero:

\[
\mathbb{H}^n(X) = \prod_{i-j=n} H^i(X, \bigwedge^j T_X).
\]

On the right hand-side we have the sheaf cohomology of the exterior powers of the tangent complex. A proof of this result for complex analytic manifolds was announced (but not yet published) by Kontsevich. For schemes, the latter result is also proven by Yekutieli [14].

\(^1\)In the analytic context, by an “affine” space we mean a Stein compact.
To make this paper more independent, we will state all axioms and definitions that we need about admissible pairs of categories in section 1. One reason for this is, that the main reference [4] is not any more available.

To prove a global statement as equation 0.1, we need three steps: First, we have to prove it for good pairs of categories. Secondly, we have to generalize it to simplicial objects in $\mathcal{C}$ and $\mathcal{M}$. Third, by using a Cech-construction, we prove them for sheaves of algebras or modules.

This paper is organized as follows: In the beginning of section 1, we state the definitions for admissible pairs of categories. In 1.2, we will precise the notation of free objects and of good pairs of categories. In 1.3 and 1.6, we explain how to construct, starting from an admissible (resp. good) pair $(\mathcal{C}, \mathcal{M})$ of categories, the admissible (resp. good) pairs $(\text{gr}(\mathcal{C}), \text{gr}(\mathcal{M}))$ and $(\text{gr}^2(\mathcal{C}), \text{gr}^2(\mathcal{M}))$, containing (anti-commutative) graded and double graded objects. In 1.5, we define resolutions and resolvents and prove a first important statement, saying that two resolvents of a homomorphism of (simplicial) algebras are homotopy equivalent. This is an improvement of [4], proposition (8.4), saying that two resolvents are quasi-isomorphic. In 1.7, we generalize the definition of the (cyclic) bar complex for admissible pairs of categories and state their main properties. (In the classical literature, the cyclic bar complex is called Hochschild complex.) In 1.8, we adapt the definition of regular sequences to the graded anticommutative context. Here we prove the equivalence of four different conditions. In 1.9, we have to prove several statements, concerning the universal module of differentials. The reader who is only interested in the theory of schemes or complex spaces, i.e. in the examples $(\mathcal{C}(0), \mathcal{M}(0))$ and $(\mathcal{C}(1), \mathcal{M}(1))$ of good pairs (see example 1.1), can leave out the lecture of this subsection, since for these examples, all statements proven here, are well-known.

In section 2, we define Hochschild complexes and Hochschild cohomology for admissible pairs of categories. In 2.1, we show that the definitions in 2.2 generalize the classical definitions in the algebraic context. In section 3, we prove the main result, i.e. a generalization of the classical HKR-theorem and deduce a decomposition theorem for Hochschild cohomology. In section 4, this results are applied to schemes and complex spaces. For this, we have to introduce simplicial technics and to define (in 4.1) resolvents of homomorphisms of complex spaces and schemes. Then we can define Hochschild complexes and Hochschild cohomology in the same way as Buchweitz and Flenner [6], and the announced decomposition theorems will be deduced by the generalized HKR-theorem in the last two subsections.

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**Conventions:** For a ring $k$, we denote the category of $k$-modules by $k\text{-Mod}$. When we work with a morphism $f : A \to B$ in any category, we denote the kernel of $f$ in the categorial sense by $\text{kern } f$. So $\text{kern } f$ is a morphism $K \to A$, where $K$ is an object, determined up to a canonical isomorphism. By $\text{Kern } f$, we mean the object $K$. In the same manner we use the notations $\text{cokern}$, $\text{Cokern}$, $\text{im}$ and $\text{Im}$. So, for example, we have $\text{Im } f = \text{Kern}(\text{cokern } f)$. We will use $\cong$ to denote quasi-isomorphisms and $\simeq$ to denote homotopy-equivalences. We will use the letter $D$ to denote derived categories and $K$ to denote homotopy categories, i.e. the localization of categories by homotopy-equivalences.
1 Admissible pairs of categories

We fix a ground ring $\mathbb{K}$ (in our main reference \[4\] $\mathbb{K}$ is the field $\mathbb{Q}$, so here we start with a more general setting). Denote by $\mathcal{C}$ a category of commutative $\mathbb{K}$-algebras and by $\mathcal{CMod}$ the category of all modules over algebras in $\mathcal{C}$ and let $\mathcal{M}$ be a subcategory of $\mathcal{CMod}$. Then the pair $(\mathcal{C}, \mathcal{M})$ is called an admissible pair of categories if the following conditions hold:

(1) In $\mathcal{C}$ there exist finite fibered sums, that we denote as usual by $A \otimes_{\mathbb{K}} B$.

(2) When $\phi : A \rightarrow B$ is a homomorphism in $\mathcal{C}$ and $N$ a module in $\mathcal{M}(B)$, then $N$ is via $\phi$ an object of $\mathcal{M}(A)$ and for each module $M$ in $\mathcal{M}(A)$, $\text{Hom}_{\mathcal{M}(A)}(M, N)$ is the set of all $\phi$-homomorphisms $M \rightarrow N$ in $\mathcal{M}$.

(3) Let $A$ be an algebra in $\mathcal{C}$. Then $\mathcal{M}(A)$ is an additive category, in which kernels and cokernels exist. Further $\mathcal{C}_A$ is a subcategory of $\mathcal{M}(A)$ and the functor of $\mathcal{M}(A)$ in $\mathcal{AMod}$ commutes with kernels and finite direct sums.

(4) Let $\phi : A \rightarrow B$ a homomorphism in $\mathcal{C}$ and $u : M \rightarrow N$ a homomorphism in $\mathcal{M}(B)$. Let $L$ resp. $L'$ be the kernel of $u$ resp. $u|_{\phi}$ in $\mathcal{M}(B)$ resp. $\mathcal{M}(A)$. Then the canonical map $L' \rightarrow L|_{\phi}$ is an isomorphism in $\mathcal{M}(A)$.

(5) Let $A$ be an algebra in $\mathcal{C}$ and $N$ a module in $\mathcal{M}(A)$. Then for each finite family $M_i; i \in I$ of modules in $\mathcal{M}(A)$, there is a given $\mathbb{K}$-submodule

$$\text{Mult}_{\mathcal{M}(A)}(M_i, i \in I; N)$$

of the module $\text{Mult}_A(M_i, i \in I; N)$ of $A$-multilinear forms $\prod_{i \in I} M_i \rightarrow N$, which is functorial in each $M_i$ and $N$ and has the following properties:

(5.1) Let $i_0$ be an element of $I$ and $u : M'_0 \rightarrow M_{i_0}$ a homomorphism in $\mathcal{M}(A)$. Set $M''_0 := \text{Cokern}(u)$ and $M'_i := M''_i := M_i$ for $i \in I \setminus \{i_0\}$. Then the sequence

$$0 \rightarrow \text{Mult}_{\mathcal{M}(A)}(M''_i, i \in I; N) \rightarrow \text{Mult}_{\mathcal{M}(A)}(M_i, i \in I; N) \rightarrow \text{Mult}_{\mathcal{M}(A)}(M'_i, i \in I; N)$$

induced by $u$ is exact.

(5.2) For modules $M, N \in \mathcal{M}(A)$ there is a canonical isomorphism

$$\text{Mult}_{\mathcal{M}(A)}(M; N) \rightarrow \text{Hom}_{\mathcal{M}(A)}(M; N).$$

(5.3) For $M$ in $\mathcal{M}(A)$, the multiplication map $\mu_M : A \times M \rightarrow M$ is in $\text{Mult}_{\mathcal{M}(A)}(A \times M; M)$.

(5.4) If $\sigma : I \rightarrow J$ is a bijective map, then the restriction of the isomorphism

$$\text{Mult}_A(M_i, i \in I; N) \rightarrow \text{Mult}_A(M_{\sigma^{-1}(j)}, j \in J; N)$$

defined by $\sigma$, defines an isomorphism

$$\text{Mult}_{\mathcal{M}(A)}(M_i, i \in I; N) \rightarrow \text{Mult}_{\mathcal{M}(A)}(M_{\sigma^{-1}(j)}, j \in J; N).$$

(5.5) Each homomorphism $\phi : A \rightarrow B$ in $\mathcal{C}$ induces a cartesian diagram

$$\text{Mult}_{\mathcal{M}(B)}(M_i, i \in I; N) \longrightarrow \text{Mult}_{\mathcal{M}(A)}((M_i)_{[\phi]}, i \in I; N_{[\phi]})$$

$$\downarrow \hspace{2cm} \downarrow$$

$$\text{Mult}_B(M_i, i \in I; N) \longrightarrow \text{Mult}_A((M_i)_{[\phi]}, i \in I; N_{[\phi]})$$

(5.6) For each $i \in I$ let $L_j, j \in J_i$ a nonempty finite family of modules in $\mathcal{M}(A)$. Set $J := \Pi_{i \in I} J_i$. Then the canonical map

$$\prod_{i \in I} \text{Mult}_{\mathcal{M}(A)}(L_j, j \in J_i; M_i) \times \text{Mult}_{\mathcal{M}(A)}(M_i, i \in I; N) \rightarrow \text{Mult}_A(L_j, j \in J; N)$$

factorises through $\text{Mult}_{\mathcal{M}(A)}(L_j, j \in J; N)$. 


(5.7) The functor \( N \mapsto \text{Mult}_{\mathcal{M}(A)}(M_i, i \in I; N) \) on \( \mathcal{M}(A) \) is represented by a module \( \bigotimes_{i \in I_A} M_i \) in \( \mathcal{M}(A) \).

(5.8) If \( I \) is a disjoint union \( \bigcup_{j \in J} I_j \) with \( I_j \neq \emptyset \) for all \( j \), then the canonical homomorphism

\[
\bigotimes_{i \in I_A} M_i \rightarrow \bigotimes_{j \in J_A} \bigotimes_{i \in I_{j_A}} M_i
\]

is an isomorphism in \( \mathcal{M}(A) \).

(5.9) The canonical map \( A \otimes_A^M N \rightarrow M \) is an isomorphism in \( \mathcal{M}(A) \).

(6) Let \( \phi : A \rightarrow B \) be a homomorphism in \( C \) and \( M \) a module in \( \mathcal{M}(A) \) and \( N \) a module in \( \mathcal{M}(B) \). Then \( N \otimes_A^M M \) is via the canonical \( A \)-bilinear map

\[
B \times N \otimes_A^M M \rightarrow N \otimes_A^M M
\]

a module in \( \mathcal{M}(B) \). The analogue statement holds for \( M \otimes_A^M N \).

(7) Let \( k \rightarrow A \) and \( k \rightarrow B \) be two homomorphisms in \( C \) and \( \phi \) resp. \( \psi \) the canonical maps of \( A \) resp. \( B \) in \( C := A \otimes_k^B \). Let \( M \) be a module in \( \mathcal{M}(k) \) and \( \rho : C \times M \rightarrow M \) an element of \( \text{Mult}_{\mathcal{M}(k)}(C \times M; M) \) such that

(a) \( \rho \) extends the multiplication of \( k \) on \( M \).

(b) \( M \) is via \( \rho \) a \( C \)-module.

(c) \( M_\phi \) is in \( \mathcal{M}(A) \) and \( M_\psi \) in \( \mathcal{M}(B) \).

Then \( M \) is in \( \mathcal{M}(C) \).

(8) For algebras \( A \) and \( B \) in \( \mathcal{C}_k \), the canonical map \( A \otimes_k^B B \rightarrow A \otimes_k^B B \) is an isomorphism in \( \mathcal{M}(k) \).

We specify admissible pairs by the following axioms:

\textbf{Axioms} Let \( A \) be an algebra in \( C \).

(S1) When \( u : M \rightarrow N \) is a homomorphism of finite \( A \)-modules in \( \mathcal{M}(A) \), then the cokernel of \( u \) in \( \mathcal{M}(A) \) coincides with the cokernel of \( u \) in \( A\text{-}\mathfrak{Mod} \) and for \( N = A \) the cokernel of \( u \) is an algebra in \( \mathcal{C}_A \) with respect to the canonical projection \( A \rightarrow \text{Cokern}(u) \).

(S1') For any homomorphism \( u : M \rightarrow N \) of \( A \)-modules the cokernel of \( u \) in \( \mathcal{M}(A) \) coincides with the cokernel of \( u \) in \( A\text{-}\mathfrak{Mod} \) and for \( N = A \) the cokernel of \( u \) is an algebra in \( \mathcal{C}_A \) with respect to the canonical projection \( A \rightarrow \text{Cokern}(u) \).

(S2) Bijective homomorphisms in \( \mathcal{M}(A) \) are isomorphisms.

\textbf{Example 1.1}

(i) Let \( \mathcal{C}^{(0)} \) be the category of all commutative \( K \)-algebras and \( \mathcal{M}^{(0)} \) the category of modules over algebras in \( \mathcal{C}^{(0)} \). Then \( (\mathcal{C}^{(0)}, \mathcal{M}^{(0)}) \) is an admissible pair of categories that satisfies axioms (S1') and (S2).

(ii) In the first example, we can replace \( \mathcal{C}^{(0)} \) by the category of all noetherian, commutative \( K \)-algebras.

(iii) Let \( \mathcal{C}^{(1)} \) be the category of all analytic \( C \)-algebras, i.e. the category of all sections of the structure sheaf of a Stein compact. Then each algebra in \( \mathcal{C}^{(1)} \) is a DFN-algebra and each homomorphism of such algebras is continuous. Let \( \mathcal{M}^{(1)} \) be the category of all DFN-modules over algebras in \( \mathcal{C}^{(1)} \). For modules \( M \) and \( N \) in \( \mathcal{M}^{(1)} \), we set \( \text{Hom}_{\mathcal{M}^{(1)}}(M, N) \) to be the group of all continuous homomorphisms \( M \rightarrow N \) in \( \text{Hom}_{\mathcal{C}^{(1)}}(M, N) \). We set \( \text{Mult}_{\mathcal{M}^{(1)}}( \ ) \) to be the group of all continuous multilinear forms. Then \( (\mathcal{C}^{(1)}, \mathcal{M}^{(1)}) \) is an admissible pair of categories that satisfies axioms (S1) and (S2).

\textsuperscript{2}The existence of this map is a consequence of (2), (5.7) and (5.8).
(iv) In the last example, we can replace $\mathcal{C}^{(1)}$ by the category of local analytic algebras.

1.1 About the tensor product $\otimes_k^M$

Let $k$ be an algebra in $\mathcal{C}$. In the following considerations $A, B, M$ and $N$ are modules in $\mathcal{M}(k)$. By axiom (5.2), there is a natural isomorphism $\sim$: $\text{Mult}_{\mathcal{M}(k)}(A \times B, M) \rightarrow \text{Hom}_{\mathcal{M}(k)}(A \otimes_k^M B, M)$. This means that each morphism $f : M \rightarrow N$ in $\mathcal{M}(k)$ induces a commutative diagram

$$\text{Mult}_{\mathcal{M}(k)}(A \times B, M) \xrightarrow{\sim} \text{Hom}_{\mathcal{M}(k)}(A \otimes_k^M B, M)$$

$$\downarrow f^* \hspace{1cm} \downarrow f^*$$

$$\text{Mult}_{\mathcal{M}(k)}(A \times B, N) \xrightarrow{\sim} \text{Hom}_{\mathcal{M}(k)}(A \otimes_k^M B, N)$$

We denote the inverse map of $\sim$ also by $\sim$.

**Remark 1.1** For $h \in \text{Hom}_{\mathcal{M}(k)}(A \otimes_k^M B, N)$ we have $\bar{h} = h \circ \sim_{\mathcal{D} \otimes B}$.

**Proof:** In the diagram above, set $M := A \otimes_k^M B$ and $f := h$. We have $h = h^*(\sim_{\mathcal{D} \otimes B})$. So $\bar{h}$ is the image of $\sim_{\mathcal{D} \otimes B}$ by going through the diagram starting up right, going down and left. $h \circ \sim_{\mathcal{D} \otimes B}$ is the result, choosing the other way. $\square$

The following consequence reminds of the situation, where $A \otimes^M B$ is a topological tensor product in which the usual tensor product is a dense subset, and $\text{Hom}_{\mathcal{M}}$ stands for continuous mappings.

**Corollary 1.1** Each homomorphism $h \in \text{Hom}_{\mathcal{M}(k)}(A \otimes_k^M B, N)$ is uniquely determined by its values on the elements of the form $a \otimes b = \sim_{\mathcal{D} \otimes B}((a, b))$.

Now suppose that $A$ and $B$ are $k$-algebras in $\mathcal{C}$. We will explain that there are two ways to see the elements $a \otimes b$ in $A \otimes_k^M B = A \otimes_k^C B$: By the universal property of fibered products, there is a natural homomorphism of $k$-algebras $\alpha : A \otimes^\mathfrak{alg} B \rightarrow A \otimes^C B$.

**Remark 1.2** For elements $a \otimes b$ of $A \otimes_k^\mathfrak{alg} B$, we have $\alpha(a \otimes^\mathfrak{alg} b) = \sim_{\mathcal{D} \otimes^\mathfrak{alg} B}((a, b))$.

**Proof:** We see that $\alpha$ is just the image of $\sim_{\mathcal{D} \otimes^\mathfrak{alg} B}$ by the composition of the mappings

$$\text{Hom}_{\mathcal{M}}(A \otimes_k^M B, A \otimes_k^M B) \cong \text{Mult}_{\mathcal{M}}(A \times B, A \otimes_k^M B) \rightarrow \text{Mult}_{k-\mathfrak{alg}}(A \times B, A \otimes_k^M B) \cong \text{Hom}_{k-\mathfrak{alg}}(A \otimes_k^\mathfrak{alg} B, A \otimes_k^M B).$$

$\square$

**Corollary 1.2** For elements $a, a' \in A$ and $b, b' \in B$, we have $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

Remark that in the antisymmetrical graded context (see below) we will have $(a \otimes b)(a' \otimes b') = (-1)^{ba'} a a' \otimes b b'$ for homogeneous $a, b, a', b'$.

1.2 Free modules and algebras

In this subsection we remind the definitions of free objects in the categories $\mathcal{C}$ and $\mathcal{M}$, where $(\mathcal{C}, \mathcal{M})$ is an admissible pair of categories.

**Free algebras:** A marking on $\mathcal{C}$ is a family $(F_\tau)_{\tau \in \mathbb{T}}$ of subfunctors $F_\tau : \mathcal{C} \rightarrow (\text{sets})$ of the identity functor, such that $F_\tau(A)$ contains 0 for all $\tau$ and all objects $A$ in $\mathcal{C}$. For a given object $k$ of $\mathcal{C}$ and a family $(\tau_i)_{i \in I}$, we consider the functor $F_{I,k} : A \mapsto \prod_{i \in I} F_{\tau_i}(A)$ on the category $\mathcal{C}_k$. If $F_{I,k}$ is representable, i.e. there is a $k$-algebra $A$ and a canonical bijection

$$b : \text{Hom}_k^C(A, B) \rightarrow \prod_{i \in I} F_{\tau_i}(B)$$
for each algebra $B$ in $C_k$, then $A$ together with the family $(e_i)_{i \in I} = b(\text{Id}_A)$ is called the free algebra over $k$ with free algebra generators $e_i$, $i \in I$. We will write $A = k(e_i)_{i \in I}$. The marking $F$ is called representable, if $F_{I,k}$ is representable for each $k$ in $C$ and each finite family $(\tau_i)_{i \in I}$.

**Free modules:** A marking on $M$ is a family $(G_u)_{u \in U}$ of subfunctors $G_u : M \rightarrow \text{(sets)}$ of the identity functor, such that for each $u \in U$ the following condition holds: For each homomorphism $\phi : A \rightarrow B$ in $C$ and each module $N$ in $M(B)$ we have $G_u(N[\phi]) = G_u(N)$. For a given algebra $A$ in $C$ and a family $(u_i)_{i \in I}$, we consider the functor $G_{I,A} : M \rightarrow \prod_{i \in I} G_u(M)$ on the category $M(A)$. If $G_{I,A}$ is representable, i.e. there is an $A$-module $M$ and a canonical bijection

$$b : \text{Hom}_{M(A)}(M, N) \rightarrow \prod_{i \in I} G_u(N)$$

for each $A$-module $N$, then $M$ together with the family $(e_i)_{i \in I} = b(\text{Id}_M)$ is called the free module over $A$ with free module generators $e_i$, $i \in I$. We will write $M = \prod_{i \in I} A e_i$. The marking $G$ is called representable, if $G_{I,A}$ is representable for each $A$ in $C$ and each finite family $(u_i)_{i \in I}$.

**A marking** on $(C, M)$ is a pair $(F, G)$ of a marking $F = (F_\tau)_{\tau \in T}$ on $C$ and a marking $G = (G_u)_{u \in U}$ on $M$ together with a map $\eta : T \rightarrow U$, such that $F_\tau(A) \subseteq G_{\eta(\tau)}(A)$ for each $A$ in $C$ and each $\tau$ in $T$.

**Axioms** Let $(F, G)$ be a marking on $(C, M)$.

(F1) $F$ is representable.

(F2) Let $k$ be an algebra in $C$ and $A = k(e_i)_{i \in I}$ be a free $k$-algebra in $C$. Then the canonical homomorphism $k[e_i]_{i \in I} \rightarrow k(e_i)_{i \in I} \in k\text{-Mod}$ is flat and the functor $M \rightarrow A \otimes_k^\mathbb{L} M$ is exact on the category of finite modules in $M(k)$.

(F3) Let $A$ be like in (F2) and $A' = k(e_i')_{i \in I'}$ be another free $k$-algebra in $C$ with $I \subseteq I'$. Then $A'$ is flat over $A$ via the homomorphism $A \rightarrow A'$ with $e_i \mapsto e_i'$.

(F4) $G$ is representable.

(F5) For each $u \in U$ and each $A$ in $C$, $G_u$ is a right exact functor on $M(A)$.

(F6) Let $A$ be an algebra in $C$ and $E = \coprod_{i \in I} A e_i$ be a free $A$-module with respect to $G$ with finite basis $(e_i)_{i \in I}$ and let $M$ be a module in $M(A)$. Then the canonical homomorphism $M^I \rightarrow M \otimes_A E$ in $A\text{-Mod}$ is bijective.

(F7) Let $k$ be an algebra in $C$ and $A = k(e_i)_{i \in I}$ be a free $k$-algebra in $C$ with finite $I$. Then $\Omega_{A/k}$ is a free $A$-module with basis $de_i \in G_{\eta(\tau)}(\Omega_{A/k})$.

Remark that Axiom (F2) implies that free algebra generators (of degree 0) are no zero divisors.

**Definition 1.1** The marking $(F, G)$ is called good, if axioms (F1), (F4), (F5), (F6) and (F7) hold. An admissible pair of categories $(C, M)$ equipped with a good marking $(F, G)$ is called a **good pair of categories**, if it satisfies axioms (S1) and (S2).

**Example 1.2**

(i) On the admissible pair $(C^{(0)}, M^{(0)})$ of example [11] we work with the trivial marking, i.e. $F(A) = A$ for each algebra $A$ in $C$, $G(M) = M$ for each module $M$ in $M(A)$. With this marking, $(C^{(0)}, M^{(0)})$ is a good pair of categories, that satisfies additionally axioms (F2) and (F3).
(ii) Consider the admissible pair \((\mathcal{C}^{(1)}, \mathcal{M}^{(1)})\). For \(A \in \mathcal{C}^{(1)}\) and \(t \in T := (0, \infty)\) let \(F_t(A)\) be the set of all elements of \(A\), such that the transformation of Gelfand (see [3] for the definition) \(\chi(A) \to \mathbb{C}\) factorises through \(\{z \in \mathbb{C}: |z| \leq t\}\). Further, let \(G\) be the canonical marking on \(\mathcal{M}^{(1)}\). Then the pair \((\mathcal{C}^{(1)}, \mathcal{M}^{(1)})\), together with the marking \((F, G)\) is a good pair of categories, that satisfies axioms (F2) and (F3).

(iii) When \(\mathcal{C}\) is the category of local analytic algebras and \(\mathcal{M}\) the category of DFN-modules over \(\mathcal{C}\), then for \(G\) we use the trivial marking and for objects \(A\), we set \(F(A)\) to be the maximal ideal \(\mathfrak{m}_A\) of \(A\). Then \((\mathcal{C}, \mathcal{M})\) is a good pair of categories, that also satisfies axioms (F2) and (F3).

1.3 About graded objects

Let \((\mathcal{C}, \mathcal{M})\) be an admissible pair of categories. As in [4], we can construct a new admissible pair \((\text{gr}(\mathcal{C}), \text{gr}(\mathcal{M}))\) as follows:

Let \(\text{gr}(\mathcal{C})\) be the category of graded anti-commutative\(^3\) rings \(A = \bigoplus_{i \in \mathbb{Z}} A^i\) with \(A^0\) in \(\mathcal{C}\), all \(A^i\) in \(\mathcal{M}(A^0)\), such that the multiplication maps \(A^i \times A^j \to A^{i+j}\) belong to \(\text{Mult}_{\mathcal{M}(A^0)}(A^i \times A^j, A^{i+j})\). A homomorphism \(\phi : A \to B\) in \(\text{gr}(\mathcal{C})\) is a homomorphism of graded anti-commutative rings, such that \(\phi^0\) is a homomorphism in \(\mathcal{C}\) and all \(\phi^i : A^i \to B^i\) are \(\phi^0\)-linear homomorphisms in \(\mathcal{M}\).

Let \(\text{gr}(\mathcal{M})\) be the category of graded \(\text{gr}(\mathcal{C})\) whose objects over an algebra \(A\) in \(\text{gr}(\mathcal{C})\) are the graded \(A\)-modules \(M = \bigoplus_{i \in \mathbb{Z}} M^i\), with \(M^0 = 0\) for almost all \(i > 0\), such that each \(M^i\) is in \(\mathcal{M}(A^0)\) and the maps \(A^i \times M^j \to M^{i+j}\) belong to \(\text{Mult}_{\mathcal{M}(A^0)}(A^i \times M^j, M^{i+j})\). When \(B\) is another algebra in \(\text{gr}(\mathcal{C})\) and \(N\) a module in \(\text{gr}(\mathcal{M})(B)\), then \(\text{Hom}_{\text{gr}(\mathcal{M})}(M, N)\) is the set of all pairs \((\phi, f)\), where \(\phi : A \to B\) is a homomorphism in \(\text{gr}(\mathcal{M})\) and \(f : M \to N\) is a \(\phi\)-linear homomorphism of degree zero, such that all \(f^i : M^i \to N^i\) are homomorphisms in \(\mathcal{M}\) over \(\phi^0\).

For modules \(M_1, \ldots, M_n\) and \(N\) in \(\text{gr}(\mathcal{M})(A)\), let \(\text{Mult}_{\text{gr}(\mathcal{M})(A)}(M_1 \times \cdots \times M_n, N)\) be the \(\mathbb{K}\)-module of all maps \(f : M_1 \times \cdots \times M_n \to N\) with the following properties:

1. For \(k_1, \ldots, k_n\) in \(\mathbb{Z}\), the restriction \(f|_{M_1^{k_1} \times \cdots \times M_n^{k_n}}\) factorises over a map in \(\text{Mult}_{\mathcal{M}(A^0)}(M_1^{k_1} \times \cdots \times M_n^{k_n}, N^{k_1+\cdots+k_n})\).

2. For elements \(a \in A\) and \(m_\mu \in M_\mu\), we have
   \[
   f(m_1, \ldots, m_\mu a, m_{\mu+1}, \ldots, m_n) = f(m_1, \ldots, m_\mu, am_{\mu+1}, \ldots, m_n) \quad \text{for} \quad 1 \leq r < n
   \]
   \[
   f(m_1, \ldots, m_n a) = f(m_1, \ldots, n)a
   \]

We made use of the fact that we can make each \(M\) in \(\text{gr}(\mathcal{M})(A)\) an anti-symmetrical \(A\)-bimodule by setting \(m \cdot a := (-1)^{\text{deg}(m)\text{deg}(a)}a \cdot m\) for homogeneous elements \(a \in A\) and \(m \in M\).

To define free algebras in \(\text{gr}(\mathcal{C})\), we have to modify the definition in subsection [1.2] as follows:

There is a map \(g : T \to \mathbb{Z}_{\leq 0}\) and each functor \(F_T\) is a subfunctor of the functor \(A \mapsto A^{g(\tau)}\). In this graded context, when \(F\) is representable, then each functor \(F_{t,A}\) is representable, when for \(n \leq 0\), the set of all \(\tau_i\) with \(g(\tau_i) = n\) is finite. In this case the free algebra \(A(e_i)_{i \in I}\) is called \(g\)-finite free algebra. Of course, the degree \(g(e_i)\) of a free generator \(e_i\) is just \(g(\tau_i)\).

To define free modules in \(\text{gr}(\mathcal{M})\), we have to modify the definition in subsection [1.2] as follows:

There is a map \(g : U \to \mathbb{Z}\) and each functor \(G_u\) is a subfunctor of the functor \(M \mapsto M^{g(u)}\). In this graded context, when \(G\) is representable, then each functor \(G_{t,A}\) is representable, when for each \(n\), the set of all \(u_i\) with \(g(u_i) = n\) is finite. In this case the free module \(\bigoplus_{i \in I} A e_i\) is called \(g\)-finite free \(A\)-module. We have \(g(e_i) = g(u_i)\) for \(i \in I\).

---

\(^3\)i.e. \(ab = (-1)^{\text{deg}(a)\text{deg}(b)}ba\) for homogeneous \(a, b \in A\)
To define a marking on \((\text{gr}(\mathcal{C}), \text{gr}(\mathcal{M}))\), we have to add in subsection 1.2 the condition, that the map \(\eta : T \rightarrow U\) is compatible with \(g\).

**Example 1.3** When \(G\) is a marking on \(\mathcal{M}\), there is a marking \(\text{gr}(G) = (\text{gr}(G))_{u \in U'}\) on \(\text{gr}(\mathcal{M})\), defined in the following way: Set \(U' := U \times \mathbb{Z}\). For \(A\) in \(\text{gr}(\mathcal{C})\), \(M\) in \(\text{gr}(\mathcal{M})\) and \(u' = (u, n) \in U'\) set \(\text{gr}_u(G)(M) := G_u(M^n)\). Here we have \(g(u') = n\).

When \((F, G)\) is a marking on \((\mathcal{C}, \mathcal{M})\), there is a marking \(\text{gr}_G(F) = (\text{gr}_G(F))_{\tau' \in T'}\), defined in the following way: Let \(T'\) be the disjoint union of \(T \times \{0\}\) and \(U \times \mathbb{Z}_{<0}\). For \(A\) in \(\text{gr}(\mathcal{C})\) and \(\tau' = (\tau, n)\) we set \(\text{gr}_G(F)_{\tau'}(A) = F_{\tau}(A)\) if \(n = 0\) and \(\text{gr}_G(F)_{\tau'}(A) = G_{\tau}(A^n)\) if \(n < 0\).

When \((F, G)\) is a marking on \((\mathcal{C}, \mathcal{M})\), then \((\text{gr}_G(F), \text{gr}(G))\) is a marking on \((\text{gr}(\mathcal{C}), \text{gr}(\mathcal{M}))\) with the map \(\eta' : T' \rightarrow U'\) given by \((\tau, 0) \mapsto (\eta(\tau), 0)\) and \((u, n) \mapsto (u, n)\) for \(n < 0\).

Remark that by [4], lemma (7.6), free algebra generators of negative degree behave much like polynomial variables and when \((\mathcal{C}, \mathcal{M})\) is a good pair of categories, then \((\text{gr}(\mathcal{C}), \text{gr}(\mathcal{M}))\) is also a good pair of categories.

When \((\mathcal{C}, \mathcal{M})\) is an admissible pair of categories that satisfies axiom (S2), then by [4] proposition (6.9), the admissible pair \((\text{gr}(\mathcal{C}), \text{gr}(\mathcal{M}))\) also satisfies axiom (S2). In general this is not true for axiom (S1). But we have:

**Remark 1.3** Let \((\mathcal{C}, \mathcal{M})\) satisfy axiom (S1) and let \(A\) be an object of \(\text{gr}(\mathcal{C})\), such that all \(A^i\) are finite \(A^0\)-modules. Then for \(g\)-finite modules \(M, N\) in \(\text{gr}(\mathcal{M})(A)\) each homomorphism \(f : M \rightarrow N\) in \(\text{gr}(\mathcal{M})(A)\) is strict, i.e. the cokernel of \(f\) in \(\text{gr}(\mathcal{M})\) coincides with the set-theoretical cokernel.

**Remark 1.4** Let \((\mathcal{C}, \mathcal{M})\) be an admissible pair of categories with a marking \((F, G)\), where \(G\) is trivial. Suppose that axiom (S1) holds. Let \(k\) be an algebra in \(\mathcal{C}\) and let \(M_1, M_2\) and \(N\) be modules in \(\mathcal{M}(k)\), such that \(M_1\) and \(M_2\) are finite \(k\)-modules with \(M_1 \subseteq N\) and \(M_1 \cap M_2 = \{0\}\). Then we have

1. The inclusions \(M_i \hookrightarrow N\) are homomorphisms in \(\mathcal{M}(k)\).
2. \(M_1 + M_2\) is in \(\mathcal{M}(k)\).
3. The inclusions \(M_i \rightarrow M_1 + M_2\) are homomorphisms in \(\mathcal{M}(k)\).
4. The projections \(p_i : M_1 + M_2 \rightarrow M_i\) are homomorphisms in \(\mathcal{M}(k)\).
5. \(M_1 + M_2 = M_1 \oplus M_2\).

In \((\text{gr}(\mathcal{C}), \text{gr}(\mathcal{M}))\) the same statement is true when we suppose that all \(k^i\) are finite \(k^0\)-modules and \(M_1, M_2\) and \(N\) are \(g\)-finite.

**Proof:** (1) There are free finite \(k\)-modules \(L_i\) in \(\mathcal{M}(k)\) and homomorphisms \(\phi_i : L_i \rightarrow N\) such that the inclusions \(M_i \hookrightarrow N\) is \(\ker(\text{cokern}(\phi_i))\). (2) We have \(M_1 + M_2 \hookrightarrow N = \ker(\text{cokern}(\phi_1 + \phi_2))\). (3) \(M_i \rightarrow N\) factorises through \(\ker(\text{cokern}(\phi_1 + \phi_2))\). (4) The projection \(M_1 + M_2 \rightarrow M_1\) is the kernel of the inclusion \(M_2 \rightarrow M_1 + M_2\) in \(k\)-modules, so as well in \(\mathcal{M}(k)\). (5) Consider homomorphisms \(f_i : M_i \rightarrow P\) in \(\mathcal{M}(k)\). We define a homomorphism \(M + N \rightarrow P\) as \(f_1 \circ p_1 + f_2 \circ p_2\). Then the diagram

\[
\begin{array}{ccc}
M_1 + M_2 & \rightarrow & M_1 \\
\downarrow & & \downarrow f_1 \\
M_2 & \rightarrow & P \\
\end{array}
\]

\[\text{Proof:} \quad (1) \text{There are free finite } k\text{-modules } L_i \text{ in } \mathcal{M}(k) \text{ and homomorphisms } \phi_i : L_i \rightarrow N \text{ such that the inclusions } M_i \hookrightarrow N \text{ is } \ker(\text{cokern}(\phi_i)). (2) \text{We have } M_1 + M_2 \hookrightarrow N = \ker(\text{cokern}(\phi_1 + \phi_2)). (3) \text{M}_i \rightarrow N \text{ factorises through } \ker(\text{cokern}(\phi_1 + \phi_2)). (4) \text{The projection } M_1 + M_2 \rightarrow M_1 \text{ is the kernel of the inclusion } M_2 \rightarrow M_1 + M_2 \text{ in } k\text{-modules, so as well in } \mathcal{M}(k). (5) \text{Consider homomorphisms } f_i : M_i \rightarrow P \text{ in } \mathcal{M}(k). \text{ We define a homomorphism } M + N \rightarrow P \text{ as } f_1 \circ p_1 + f_2 \circ p_2. \text{ Then the diagram}
\]
in $\mathcal{M}(k)$ commutes. The graded case follows in the same manner. \qed

**Remark 1.5**

(i) Suppose that $(\mathcal{C}, \mathcal{M})$ is a good pair of categories and that $k$ is an algebra in $\text{gr}(\mathcal{C})$, such that each $k^i$ is a finite $k^0$-module. Let $R = k\langle T \rangle$ be a free $g$-finite algebra over $k$ in $\text{gr}(\mathcal{C})$. Then there is a decomposition

$$R = k \oplus \sum_{t \in T} R_t$$

in the category $\text{gr}(\mathcal{M})(k)$.

(ii) Suppose that additionally the marking $G$ on $\mathcal{M}$ is trivial and that axiom (F2) holds. Then, for each $n \geq 0$, there is a decomposition

$$R = k \oplus \sum_{t_1 \in T} t_1k \oplus \sum_{t_1, \ldots, t_n \in T} t_1 \cdot \ldots \cdot t_n k \oplus \sum_{t_1, \ldots, t_{n+1} \in T} t_1 \cdot \ldots \cdot t_{n+1} R.$$

**Proof:** (i) We can form the free $g$-finite $R$-module $M = \bigsqcup_{t \in T} R e(t)$, where to each free algebra generator $t \in \text{gr}_G(F)_r$ we have associated a free module generator $e(t) \in \text{gr}(G)_{g(t)}(M)$. Now we consider the homomorphism $M \rightarrow R$ in $\text{gr}(\mathcal{M})(R)$ with $e(t) \mapsto t$. By remark 1.4, the cokernel of this homomorphism coincides with the cokernel map in $R\text{-Mod}$, which is just the projection $p : R \rightarrow R/(T)$ and $R/(T)$ is an algebra in $\text{gr}(\mathcal{C})$. Now there is a diagram

$$\begin{array}{ccc}
R & \xrightarrow{\pi} & k \\
\downarrow p & & \downarrow \\
R/(T) & & \\
\end{array}$$

in $\text{gr}(\mathcal{C})$, where $\pi : R \rightarrow k$ is the homomorphism given by $t \mapsto 0$ for $t \in T$ and the homomorphism $k \rightarrow R/(T)$ is the canonical inclusion. The diagram commutes, since in both directions a $t \in T$ goes to $0$. So we get $\text{Kern}(\pi) = (T)$. But obviously, we have $R = k \oplus \text{Kern}(\pi)$.

(2) The case $n = 0$ is just (i). Instead of doing an induction step, we show the case $n = 1$: Iterating the decomposition in the axiom, we get

$$R = k \oplus \sum_{t_1 \in T} t_1k + \sum_{t_1, t_2 \in T} t_1 t_2 R.$$

We have to show that the sum on the right is a direct sum. With remark 1.4, it is enough to show that $\sum_{t_1 \in T} t_1k \cap \sum_{t_1, t_2 \in T} t_1 t_2 R = \{0\}$. We can restrict ourselves to the case where $T$ consists of a single element $t$. If the intersection was not trivial, then $t$ would be a zero divisor. In the case $g(t) = 0$, this contradicts (F2). In the case $g(t) < 0$ odd, the annihilator of $t$ is generated by $t$. In the case $g(t) < 0$ even, $t$ is no zero divisor. If the intersection was not trivial, we would find an annihilator of $t$, which is not in $(t)$. Contradiction! \qed

1.4 Simplicial objects

Let $I$ be an index set. Then a set $\mathcal{N}$ of subsets of $I$ is called simplicial scheme over $I$, if $\emptyset \notin \mathcal{N}$; if for all $i \in I$, we have $\{i\} \in \mathcal{N}$ and if every nonempty subset of an element in $\mathcal{N}$ is again in $\mathcal{N}$.

For an element $\alpha$ of a simplicial scheme $\mathcal{N}$ over $I$, containing $n$ elements, set $|\alpha| := n - 1$. Then for $n \geq 0$, the set $\mathcal{N}^{(n)}$ of all $\alpha \in \mathcal{N}$ with $|\alpha| \leq n$ is again a simplicial scheme over $I$.
A simplicial scheme \( \mathcal{N} \) can be seen as category, where \( \text{Hom}(\alpha, \beta) \) contains only the inclusion \( \alpha \subseteq \beta \), if \( \alpha \subseteq \beta \) and is empty in all other cases.

When \( \mathcal{A} \) is any category, by an \( \mathcal{N} \)-object in \( \mathcal{A} \), we mean a covariant functor \( \mathcal{N} \to \mathcal{A} \). The \( \mathcal{N} \)-objects in \( \mathcal{A} \) again form a category, denoted by \( \mathcal{A}^\mathcal{N} \). When \( (\mathcal{C}, \mathcal{N}) \) is an admissible pair of categories and \( A = (A_\alpha)_{\alpha \in \mathcal{N}} \) an object of \( \mathcal{C}^\mathcal{N} \), we denote the category of \( \mathcal{N} \)-objects \( M = (M_\alpha)_{\alpha \in \mathcal{N}} \) in \( \mathcal{M}^\mathcal{N} \) with \( M_\alpha \in \text{ob}(\mathcal{M}(A_\alpha)) \) by \( \mathcal{M}^\mathcal{N}(A) \).

Let \( (\mathcal{C}, \mathcal{M}) \) be an admissible pair of categories and \( \mathcal{N} \) a simplicial scheme. Let \( ((F_\tau)_{\tau \in T}, (G_u)_{u \in U}) \) be a marking on \( (\mathcal{C}, \mathcal{M}) \). Then for each pair \( (\alpha, \tau) \in \mathcal{N} \times T \), there is a functor \( F_{\alpha, \tau} : A \to F_{\tau}(A_\alpha) \).

For a family \( ((\alpha, \tau), i \in I \) of elements of \( \mathcal{N} \times T \) and \( A \in \mathcal{C}^\mathcal{N} \), there is a set-valued functor \( B \to \coprod_{i \in I} F_{\alpha_i, \tau}(B) \). We will denote it by \( F_{I, A} \).

**Remark 1.6** Suppose that for each \( \alpha \in \mathcal{N} \) the free \( \mathcal{A}_\alpha \)-algebra \( A'_\alpha = A_\alpha \langle e_i^{(\alpha)} \rangle_{\alpha \subseteq \alpha} \) exists. For \( \alpha \subseteq \beta \) let \( \rho_{\alpha \beta} : A'_\alpha \to A'_\beta \) be the homomorphism in \( \mathcal{C} \) over \( A_\alpha \), given by \( e_i^{(\alpha)} \mapsto e_i^{(\beta)} \). Then \( A' = (A'_\alpha)_{\alpha \in \mathcal{N}} \) is an algebra in \( \mathcal{C}^\mathcal{N} \) and together with the family \( (e_i^{(\alpha)})_{\alpha \in \mathcal{C}} \), it represents the functor \( F_{I, A} \). We call it the **free \( \mathcal{A} \)-algebra** in the free generators \( e_i := e_i^{(\alpha_i)} \in F_{\alpha_i, \tau}(A') \) and denote it by \( A(i) \).

For each pair \( (\alpha, u) \in \mathcal{N} \times U \), there is a functor \( G_{\alpha, u} : M \to G_u(M_\alpha) \). For a family \( ((\alpha, u), i \in I \) of elements of \( \mathcal{N} \times U \) and \( A \in \mathcal{C}^\mathcal{N} \), there is a set-valued functor \( N \to \coprod_{i \in I} G_{\alpha_i, u_i}(M) \). We will denote it by \( G_{I, A} \).

**Remark 1.7** Fix a family \( ((\alpha, u), i \in I \) of elements of \( \mathcal{N} \times U \) and an algebra \( A \in \mathcal{C}^\mathcal{N} \) Suppose that for each \( \alpha \in \mathcal{N} \) the free \( \mathcal{A}_\alpha \)-module \( M_\alpha = \prod_{\alpha \subseteq \alpha} A_\alpha e_i^{(\alpha)} \) in the free generators \( e_i^{(\alpha)} \in G_u(M_\alpha) \) exists. For \( \alpha \subseteq \beta \), let \( \rho_{\alpha \beta} : M_\alpha \to M_\beta \) be the homomorphism in \( \mathcal{C} \) over \( A_\alpha \), given by \( e_i^{(\alpha)} \mapsto e_i^{(\beta)} \). Then \( M = (M_\alpha)_{\alpha \in \mathcal{N}} \) is a module in \( \mathcal{M}^\mathcal{N} \) and together with the family \( (e_i^{(\alpha)})_{\alpha \in \mathcal{C}} \) it represents the functor \( G_{I, A} \). We call it the **free \( \mathcal{A} \)-module** with free generators \( e_i := e_i^{(\alpha_i)} \in G_{\alpha_i, u_i}(A') \) and denote it by \( \coprod_{i \in I} A(i) \).

To distinguish the nonsimplicial from the simplicial context, we call the first one **affine**.

### 1.5 Resolutions

For a DG-module \( K \) in \( \text{gr}(\mathcal{M})(R) \) with differential \( d \) for the instant- denote the image of \( d^{i-1} : K^{i-1} \to K^i \) in \( \mathcal{M}(R) \) by \( B^i_M \) and the kernel of \( d^i \) in \( \mathcal{M}(R) \) by \( Z^i_M \). We use the same letters without subscript to denote image and kernel in \( \text{R-Mod} \). In general, the quotient \( Z^i_M/B^i_M \) is an object of \( \mathcal{M}(R) \). So we define the \( i \)-th homology \( H^i_M(K) \) of \( K \) in \( \mathcal{M}(R) \) to be the cokernel of the map \( B^i_M \to Z^i_M \). When \( K \) is separated in the sense that the cokernels of the maps \( d : K^i \to K^{i+1} \), the induced maps \( K^i \to Z^{i+1}_M \) and the inclusions \( Z^i_M(K) \) coincide with their cokernels, formed in the category \( \text{R-Mod} \), then \( H^i_M(K) \) is as \( \text{R-module} \) isomorphic to \( H^i(K) = Z^i/B^i \). We call \( K \) **acyclic**, if \( H^i(K) = 0 \) for all \( i \). We call \( K \) **\( \mathcal{M} \)-acyclic**, if \( H^i_M(K) = 0 \) for all \( i \).

**Remark 1.8** Suppose that \( (\mathcal{C}, \mathcal{M}) \) satisfies axiom \((S1)\) and all \( K^i \) are finite \( R \)-modules. Then \( K \) is acyclic if and only if \( K \) is \( \mathcal{M} \)-acyclic.

By a **DG-resolution** of an object \( B \) in \( \mathcal{M} \), we mean a DG-module \( M \) in \( \text{gr}(\mathcal{M}) \), such that \( H^i_M(M) = 0 \) for \( i < 0 \) and \( H^i_M(M) = B \).

**Corollary 1.3** If the pair \( (\mathcal{C}, \mathcal{M}) \) satisfies \((S1)\) and \((S2)\), then for a DG \( R \)-module \( K \) in \( \text{gr}(\mathcal{M}) \) which is finite over \( R^0 \) in each degree, the following statements are equivalent:
(1) $K$ is a DG-resolution of an object $M \in \text{ob}(\mathcal{M})$.

(2) $K$ is a resolution of $M$ as differential graded $R$-module.

We remind definition (8.1) in [3]. Here we work in an admissible pair $(\mathcal{C}, \mathcal{M})$ with a fixed marking $(F, G)$.

**Definition 1.2** Let $A \to B$ be a homomorphism of DG-objects in $\text{gr}(\mathcal{C})^N$. When we talk of a resolvent of $B$ over $A$, we mean a free DG-algebra $R$ over $A$ in $\text{gr}(\mathcal{C})^N$ (with respect to the marking $\text{gr}_G(F)$) together with a morphism $R \to B$ of DG-Objects in $\text{gr}(\mathcal{C})^N$ which is a surjective quasi-isomorphism on each $\alpha \in N$.

In this paper, we will mostly work in a noetherian context, i.e. we will mostly assume that the following axiom is satisfied:

**Axiom** (N) Each algebra $A$ in $\mathcal{C}$ is noetherian and each finite module $M$ in $\mathcal{M}(A)$ is a quotient of a finite free $A$-module.

If the good pair $(\mathcal{C}, \mathcal{M})$ satisfies the axioms (N) and (F2) and if $A^i$ is a finite $A^0$-module for all $i$ and if $B^i$ is a finite $B^0$-module for all $i$ and if $A^0$ is a quotient of a $g$-finite $B^0$-module $C$ in $C^N$, such that each $C_\alpha$ is a finite free $B^0_\alpha$-algebra, then such resolvents exist by [3], prop. (8.7) and prop. (8.8). We can also deduce easily those results from remark II12. For the existence of resolvents in the non-noetherian case, see loc. cit.

The next proposition is of great importance for this work. Here we consider a good pair $(\mathcal{C}, \mathcal{M})$ of categories and suppose that the marking $G$ on $\mathcal{M}$ is trivial and that axiom (N) is satisfied.

**Proposition 1.1** Let $A \to B$ be a homomorphism of DG-objects in $\text{gr}(\mathcal{C})$. Then for two $g$-finite resolvents $R_1$ and $R_2$ of $B$ over $A$, there exist homomorphisms $R_1 \to R_2$ and $R_2 \to R_1$ in $\text{gr}(\mathcal{C})$, that are homotopy equivalences over $A$.

**Proof:** First case: Suppose that $R_0^1 = R_0^2$.

Set $A^i := A \otimes_{A_0} R_0^i$. Then $R_1$ and $R_2$ are resolvents of $B$ over $A^i$. With [3], prop. 8.2., there are quasi-isomorphisms $R_1 \to R_2$ and $R_2 \to R_1$ in $\text{gr}(\mathcal{C})$ over $A^i$. But since $R_0^1 = R_0^2 = A^0$, $R_1$ and $R_2$ are free $A^0$-modules in $\text{gr}(\mathcal{M})$. Hence the quasi-isomorphisms are already homotopy-equivalences.

Second case: Suppose that $R_0^2$ is a finite free algebra over $R_0^2$ in $\mathcal{C}$.

By induction, we can restrict ourselves to the case, where $R_0^2 = R_0^1(e)$ is just a free algebra in one generator. Consider the free $R_0^1$-algebra $R := R_0^1(e, f)$, in $\text{gr}(\mathcal{C})$ generated by a free generator $e$ of degree 0 and a free generator $f$ of degree $-1$. We define a differential on $R$ by setting $f \mapsto e$.

By remark II12(1), we have $R_0^1(e) = R_0 \oplus eR_0^1(e)$. So by axiom (S2), the differential gives an isomorphism $fR_0^1(e) \to eR_0^1(e)$. With this in mind, we can easily construct a contracting homotopy on $R$. Now let $R'_1 := R_1 \otimes_A R_0$ be homotopic over $R_1$ to $R_1$. More precisely, the inclusion $R_1 \to R'_1$ and the projection $R'_1 \to R'_{\alpha}$ are homotopy equivalences. By the first case, there are homotopy-equivalences $R'_1 \to R_2$ and $R_2 \to R'_1$.

**General case:** Let $R_3$ be a free $g$-finite resolution of $B$ over $R_1 \otimes_A R_2$. Now $R_3$ is free over $R_1$ and $R_2$ and by the second case, there are homotopy-equivalences $R_3 \to R_2$ and $R_2 \to R_3 \to R_1$ in $\text{gr}(\mathcal{C})$.

In the simplicial case\(^5\), there is one little difference. A free algebra in $\text{gr}(\mathcal{C})^N$ over an algebra $A$ in $\text{gr}(\mathcal{C})^N$ is not a free module in $\text{gr}(\mathcal{M})^N(A)$, even if all free algebra generators are of strictly negative degree. The point is, that even $A$ itself is not free as $A$-module. But we see that a free algebra over $A$ with free algebra generators of negative degree is as $A$-module in $\text{gr}(\mathcal{M})^N$ a direct sum $A \oplus M$ with a free $A$-module $M$.

\(^5\)Remember that we still assume that the marking $G$ on $\mathcal{M}$ is trivial.
We need two lemmas to prove a simplicial version of proposition\[14\] \[15\]. The first one is a simplicial version of the Comparison Theorem (for the affine case, see \[13\] theorem 2.2.6).

**Lemma 1.1** Let $A$ be a DG algebra in $\text{gr}(C)^N$. Let $P = \bigoplus_{i\in I} Ae_i$ be a free DG $A$-module in $\text{gr}(M)^N$ with a homomorphism $P^0 \rightarrow M$ of $A^0$-modules in $M^N(A^0)$. Let $N$ be an $A^0$-modules in $M^N$ and $Q$ in $\text{gr}(M)^N(A)$ a DG-resolution of $N$. Let $\phi : M \rightarrow N$ be an $A^0$-homomorphism in $M^N$. Then there exists a homomorphism $f : P \rightarrow Q$, lifting $\phi$ and it is unique up to a chain homotopy.

**Proof:** The existence of such an $f$ is not hard to prove. But we only make use of the uniqueness. So we only prove this part here: Let $f$ and $g$ two DG-homomorphisms, lifting $\phi$. Inductively we construct families $\{s_{\alpha} : |\alpha| \leq m\}$ of compatible homotopy maps $s_{\alpha} : P_{\alpha} \rightarrow Q_{\alpha}[{-1}]$ satisfying

$$g^n - f^n = d_Q \circ s^n_{\alpha} + (-1)^n s^{n+1}_{\alpha} \circ d_P.$$  

Suppose, that the free generator $e_i$ is associated to the pair $(\alpha_i, z_i)$ with $\alpha_i \in N$ and $z_i < 0$.

For $m = 0$ and each $\beta$ in $N$ with $|\beta| = 0$, we see that $P_{\beta}$ is free DG-module in $\text{gr}(M)(A_{\beta})$, and we can construct $s^i_{\alpha}$ just as in the affine case.

Now suppose that $\{s_{\alpha} : |\alpha| \leq m\}$ is already constructed. Then for each $\beta \in N$ with $|\beta| = m + 1$ we have

$$P_{\beta} = \bigoplus_{\alpha \subseteq \beta} A_{\beta} e_i.$$  

For $\alpha \subseteq \beta$, denote the restriction map $P_{\alpha} \rightarrow P_{\beta}$ by $\rho_{\alpha, \beta}$. For free generators $e_i$ with $\alpha_i \subseteq \beta$ but $\alpha \neq \beta$, set $s_{\beta}(e_i) := s_{\alpha, \beta}(s_{\alpha}(e_i))$. Then we get

$$(g_{\beta} - f_{\beta})(e_i) = \rho_{\alpha, \beta}((g_{\alpha} - f_{\alpha})(e_i)) = \rho_{\alpha, \beta}(|s_{\alpha}, d_{\alpha}|(e_i)) = |s_{\beta}, d_{\beta}|(e_i).$$  

For free algebra generators $e_i$ with $\alpha = \beta$ and say $n = z_i = g(e_i)$, exactly as in the affine case, by induction on $n$, we can find elements $m_i$ in $P^{z_i - 1}_{\beta}$ such that

$$(g_{\beta} - f_{\beta})(e_i) = s_{\beta}(d(e_i)) + (-1)^n d(m_i).$$  

Then we set $s_{\beta}(e_i) := m_i$.

In this manner, we get a family $(s_{\alpha})_{\alpha \in N}$ of compatible chain homotopies. \hfill \Box

**Lemma 1.2** Let $A$ be a DG algebra in $\text{gr}(C)^N$ such that each $A^i$ is a finite $A^0$-module. Let $M = \bigoplus_{i \in I} Ae_i$ and $N = \bigoplus_{j \in J} Ae_j$ two $g$-finite free DG $A$-modules in $\text{gr}(M)^N$, such that all generators $e_i$ and $e_j$ are of negative degree. Suppose that there is a quasi-isomorphism

$$f = \text{Id}_A \oplus f' : A \oplus M \rightarrow A \oplus N.$$  

Then $f$ is already a homotopy equivalence. More precisely, there is a homomorphism

$$g = \text{Id}_A \oplus g' : A \oplus N \rightarrow A \oplus N$$  

of DG-modules and a map $s_{\ast} : M \rightarrow M[{-1}]$ of graded modules, such that $s_0 = 0$ and $g \circ f - \text{Id} = [s, d]$.

**Proof:** Consider the following diagram, where the first line is just the mapping cone $\text{cone}(f) = N \oplus M[1]$ of $f$ and the vertical maps are the canonical inclusions:

$$\cdots \rightarrow M^{-1} \oplus N^{-2} \rightarrow M^0 \oplus N^{-1} \rightarrow N^0$$  

Since $f$ is a quasi-isomorphism, the mapping cone of $f$ is acyclic, so the first line is a resolution of the module $\{0\}$. The map $\iota$ of DG-modules is a lifting of the trivial map $0 \rightarrow 0$. The zero map
Remark 1.10 The pair $(\text{gr}(\mathcal{M}))^N$ is a second candidate for such a lifting. So we are almost in the situation of the uniqueness statement in the comparison theorem. The only difference is, that $N = A \oplus N'$ is not a free module in $\text{gr}(\mathcal{M})^N$. But to construct a chain homotopy $\sigma : N \to \text{cone}(f)[-1] = N[-1] \oplus M$ for $0 \simeq \iota$, we can set $\sigma|_A$ to be the composition of the inclusions $A \to A \oplus M' = M$ and $M \to \text{cone}(f)[-1]$. On the free generators of $N'$, the map $\sigma$ can be defined exactly as in the proof of the comparison theorem. So we can work with a family of maps

$$\sigma^n = (g^n, t^n) : N^n \to M^n \oplus N^{n-1}$$

for $n \leq 0$, satisfying the condition

$$t^n = \delta^{n-1}\sigma^n + (-1)^n\sigma^{n+1}d^n.$$ 

Here $d$ denotes the differential of $N$ and $\delta$ the differential of $\text{cone}(f)$. The evaluation of this conditions shows, that $g$ is a chain map $N \to M$ and that $t$ is a chain homotopy for $\text{Id}_N \simeq f \circ g$. In an analogue manner, we get a chain map $h : M \to N$ with $\text{Id}_M \simeq g \circ h$. We see easily that then we have $h \simeq f$, so we get $\text{Id}_M \simeq g \circ f$. \hfill $\square$

Of course, we can also show that two free module resolutions of a module in $\text{gr}(\mathcal{M})^N$ are homotopy equivalent over the base ring. Now we can state the announced simplicial version of proposition 1.1:

Suppose that the same assumptions hold.

**Proposition 1.2** Let $A \to B$ be a homomorphism of DG-objects in $\text{gr}(\mathcal{C})^N$. Then, for two $g$-finite resolvents $R_1$ and $R_2$ of $B$ over $A$, there exist homomorphisms $R_1 \to R_2$ and $R_2 \to R_1$ in $\text{gr}(\mathcal{C})^N$, that are homotopy equivalences over $A$.

**Proof:** We imitate the proof of prop. 1.1. For the first step we have to use lemma 1.2. The second and third step are easy to generalize. \hfill $\square$

### 1.6 Double graded objects

Let $(\mathcal{C}, \mathcal{M})$ be an admissible pair of categories. We define the pair $(\text{gr}^2(\mathcal{C}), \text{gr}^2(\mathcal{M}))$ as follows:

The objects of $\text{gr}^2(\mathcal{C})$ are the double graded rings $A = \coprod_{i,j \leq 0} A^{i,j}$ with $A^{0,0}$ in $\mathcal{C}$ and all $A^{i,j}$ in $\mathcal{M}(A^{0,0})$ such that

1. For $a \in A^{i,l}$ and $b \in A^{k,l}$ we have $ab = (-1)^{(i+j)(l+k)}ba$.
2. The multiplication maps $A^{i,j} \times A^{k,l} \to A^{i+j,k+k}$ belong to $\text{Mult}_{\text{M}(A^{0,0})}(A^{i,j} \times A^{k,l}, A^{i+j,k+k})$.

Following the ideas of subsection 1.3, we can define $\text{Hom}_{\text{gr}^2(\mathcal{C})}(A, B)$ for objects $A, B$ in $\text{gr}^2(\mathcal{C})$, the category $\text{gr}^2(\mathcal{M})$, $\text{Hom}_{\text{gr}^2(\mathcal{M})}(M, N)$ for modules $M, N$ of $\text{gr}^2(\mathcal{M})$ and $\text{Mult}_{\text{gr}^2(\mathcal{M})}(A)(M_1, ..., M_n, N)$ for modules $M_1, ..., M_n, N$ in $\text{gr}^2(\mathcal{M})(A)$. We don’t make this definitions explicit here.

**Remark 1.9** Let $A$ be an object of $\text{gr}^2(\mathcal{C})$ and $M, N$ objects of $\text{gr}^2(\mathcal{M})$. For $(p, q)$ in $\mathbb{Z} \times \mathbb{Z}$ set $T^{p,q} := \coprod_{i+j=p, k+l=q} A^{i,k} \otimes A^{j,l}. N^{j,l}$. Then $T := \coprod_{p,q} T^{p,q}$ is a tensor product of $A$ and $B$ in $\text{gr}^2(\mathcal{M})(A^{0,0})$. $T$ can be seen in two different ways as object of $\text{gr}^2(\mathcal{M})(A)$. Consider the homomorphism $u : A \otimes A^{0,0} T \to T$ in $\text{gr}^2(\mathcal{M})(A^{0,0})$, sending $a \otimes m \otimes n$ to $ma \otimes n - m \otimes an$. $u$ can be seen in two manners as homomorphism in $\text{gr}^2(\mathcal{M})(A)$. Both of them induce the same $A$-module structure on $\tilde{T} := \text{Coker}(u)$. We see that $\tilde{T}$ is a tensor product of $M$ and $N$ in $\text{gr}^2(\mathcal{M})(A)$.

**Remark 1.10** The pair $(\text{gr}^2(\mathcal{C}), \text{gr}^2(\mathcal{M}))$ is an admissible pair of categories.

**Proof:** Analogue to the proof of (6.9) in [4]. \hfill $\square$
Convention: When we consider an object $K$ of $\text{gr}(M)$ as object of $\text{gr}^2(M)$, we set $K^{i,0} = K^i$ and $K^{i,j} = 0$ for $j \neq 0$.

In the same manner as above, we can define a marking $(\text{gr}^2_G(F), \text{gr}^2(G))$ on the pair $(\text{gr}^2(C), \text{gr}^2(M))$:

Define the index set $T'$ as $T' = \{0\} \cup U \times (\mathbb{Z} \leq 0 \times \mathbb{Z} \leq 0) \setminus \{(0,0)\}$. For $\tau'' = (s,0,0) \in T''$ and $A \in \text{gr}^2(C)$ set $\text{gr}^2_G(F)_{\tau''}(A) := F_s(A^0,0)$ and for $\tau'' = (u,p,q) \in T''$ with $(p,q) \neq (0,0)$ set $\text{gr}^2_G(F)_{\tau''}(A) := G_u(A^{p,q})$. Define the index set $U'' = U \times \mathbb{Z}$. For $u'' = (u,p,q) \in U''$ and $M \in \text{gr}^2(M)$ set $\text{gr}(G)_{\nu''}(M) := G_u(M^{p,q})$.

In analogy to [3], lemma (7,6), we get:

**Proposition 1.3**

(1) Let $A$ be an algebra in $\text{gr}^2(C)$ and $A' = A(e_i)_{i \in I}$ a free algebra over $A$, with respect to the marking $\text{gr}^2_G(F)$. Suppose that the bidegree of each $e_i$ is different to 0. Then the canonical homomorphism $A[e_i]_{i \in I} \longrightarrow A'$ in $A\text{-Alg}$ is bijective.

(2) If $(F,G)$ is good, then $(\text{gr}^2_G(F), \text{gr}^2(G))$ is also good.

**Definition 1.3** A DG-algebra $A$ in $\text{gr}^2(C)$ is an algebra $A$ in $\text{gr}^2(C)$ equipped with a (vertical) $A^{0,0}$-derivation $v : A \longrightarrow A$ of bidegree $(0,1)$ with $v^2 = 0$.

A DDG-algebra $A$ in $\text{gr}^2(C)$ is a DG-algebra $A$ in $\text{gr}^2(C)$ equipped with a (horizontal) derivation $h$ of bidegree $(1,0)$, that anti-commutes with $v$, such that $h^2 = 0$. A homomorphism between (D)DG-algebras is a morphism in $\text{gr}^2(C)$ that commutes with the vertical (and horizontal) differentials.

Now let $(A,s)$ be a DG-algebra in $\text{gr}^2(C)$. A DG-module in $\text{gr}^2(M)(A)$ is a module $M$ in $\text{gr}^2(M)(A)$ equipped with a (vertical) differential $v$ of bidegree $(0,1)$ such that for $a \in A$ and $m \in M^{k,l}$ the formula $v(ma) = v(m)a + (-1)^{k+l}mv(a)$ holds.

At least let $(A,v,h)$ be a DDG-algebra in $\text{gr}^2(C)$. A DDG-module in $\text{gr}^2(M)(A)$ is a DG-module $(M,v) \in \text{gr}^2(M)(A)$ equipped with a horizontal differential $h$, that commutes with $v$ and such that for $a \in A$ and $m \in M^{k,l}$ the formula $h(ma) = e(m)a + (-1)^{k+l}mv(a)$ holds.

A homomorphism between (D)DG-modules is a morphism in $\text{gr}^2(M)$ that anti-commutes with the vertical (and horizontal) differentials.

**Remark 1.11** Let $K = (K,h,v)$ be a DG-algebra in $\text{gr}^2(C)$. We consider a free algebra $K(E)$ over $K$ in $\text{gr}^2(C)$ with a set $E = \{e_i : i \in I\}$ of free algebra generators with $e_i \in \text{gr}^2_G(F)_{\nu''}(K(E))$ for a certain $\tau''' \in T''$. For each $i$, if $g(x_i) \neq (0,0)$ choose an element $h_i \in G_u(K(E)_{x_i}^{(0,0)})$ and an element $v_i \in G_u(K(E)_{g(x_i)}^{(0,0,0,1)})$, where $u_i$ is the first component of $\tau_i = (u_i,g(x_i))$. Then, setting $h(e_i) := h_i$ and $v(e_i) := v_i$, we get an extension of the horizontal and the vertical derivation $h$ and $v$ of $K$. This extensions make $K(E)$ a DDG-algebra, if and only if for each $i$, we have

1. $h(v_i) + v(h_i) = 0$
2. $h(h_i) = v(v_i) = 0$.

**Proof:** Inductively, we can reduce the proof to the case where $E$ consists of a single element $e$ of bidegree $(p,q)$. In this case, it is an easy calculation.

**Definition 1.4** By a DG-resolution of an algebra $B$ in $\text{gr}(C)$, we mean a DG-algebra $A$ in $\text{gr}^2(C)$, such that for all $i$ the $i$-th row is a DG-module resolution of $B^i$. By DDG-resolution of a DG-algebra $B$ in $\text{gr}(M)$, we mean a DDG-algebra $A$ in $\text{gr}^2(M)$ that is a DG-resolution of $B$ such that the map $A^{i,0} \longrightarrow B$ is a homomorphism of DG-Algebras in $\text{gr}(C)$.

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$^6$This means that for homogeneous $a,b \in A$ we have $v(ab) = v(a) + (-1)^a av(b)$. In the exponent, by $a$, we mean the total degree of $a$. 

---
Now when \( A \to B \) is a homomorphism of DG-Algebras in \( \text{gr}(C) \), to get a resolvent \( R \) of \( B \) over \( A \), it is enough to construct a DDG-resolution \( K \) of \( B \), that is free over \( A \) as object of \( \text{gr}^2(C) \). Then we can choose \( R \) to be the total complex \( \text{tot}(K) \). So the question of the existence of DDG-resolutions is quite natural. An answer to this question is given by the following remark:

**Remark 1.12** Suppose that for the pair \( (C, M) \) the axioms (N) and (F2) hold. Let \( K = (K, h, v) \) be a DDG-algebra in \( \text{gr}^2(C) \) and \( u : K^{*,0} \to A \) a homomorphism of DG-algebras in \( \text{gr}(C) \).

1. When \( A^l \) is a quotient of a free \( K^{0,0} \)-algebra, then there exists a free DDG-algebra \( L = K(F) \) over \( K \), where \( F \) is a \( g \)-finite set of generators of bidegree \((k,0); \ k \leq 0 \), and a surjective homomorphism \( L^{*,0} \to A \) over \( K^{*,0} \).

2. For a fixed \( p < 0 \), suppose that we have \( u^{p+1} = \text{cokern}(v^{p+1,-1}) \). Then there is a free DDG-Algebra \( L = \langle F \cup G \rangle \) over \( K \) with finite sets \( F \) resp. \( G \) of generators of bidegree \((p,-1) \) resp. \((p+1,-1) \), such that we still have \( u^{p+1} = \text{cokern}(v^{p+1,-1}) \) and additionally \( u^p = \text{cokern}(v^{p,-1}) \) holds.

3. Fix \( p \leq 0 \) and \( q \leq -1 \). Suppose that we have \( H^{q+1}(K^{p+1,*}) = 0 \). Then there is a free DDG-Algebra \( L = K(F \cup G) \) over \( K \) with finite sets \( F \) resp. \( G \) of generators of bidegree \((p,q) \) resp. \((p+1,q) \), such that we still have \( H^{q+1}(K^{p+1,*}) = 0 \) and additionally \( H^{q+1}(K^{p,*}) = 0 \) holds.

**Proof:** (i) is trivial. The proofs of (ii) and (iii) are very similar, so we only do the proof of (iii). We choose \( G \) such that there is an epimorphism \( \pi : \Pi_{g \in G} K^{0,0}g \to \text{Ker}(v^{p+1,q+1}) \cap \text{Ker}(h^{p+1,q+1}) \).

Set \( v(g) = \pi(g) \) and \( h(g) = 0 \).

We choose \( F \) such that there is an epimorphism \( \pi' : \Pi_{f \in F} K^{0,0}f \to \text{Ker}(v^{p,q+1}) \). Set \( v(f) = \pi'(f) \) and choose \( h(f) \) in \( \Pi IK^{0,0}g \) such that we get \( v(h(f)) = -h(v(f)) \).

**Definition 1.5** For a \( g \)-finite free DG-module \( M = \coprod_{i \in I} A f_i \) in \( \text{gr}(M) \) with differential \( d \) (this construction can be done more generally in \( \text{gr}(M)^N \)), we can define the **exterior algebra** \( \wedge_A M \), to be the free DDG-algebra \( A(E) = A(e_i)_{i \in I} \) in \( \text{gr}^2(C) \), where to each \( f_i \) we associate a free algebra generator \( e_i \) of bidegree \((g(i),-1) \). We wrote \( E \) for the set of all \( e_i \). The vertical differential of \( \wedge_A M \) is set to be trivial, and the horizontal differential \( h \) is defined as follows: Suppose that \( d(f_i) = \sum_j a_{ij} f_j \) for a finite family \( a_{ij} \) in \( A \). Then set \( h(e_i) := \sum_j a_{ij} e_j \).

The total complex of \( \wedge_A M \) has the structure of a DG-algebra in \( C \) and corresponds to the classical definition of an exterior algebra.

In this situation, let \( \wedge^j_A M \) be the DG-module in \( \text{gr}(M) \) with \( (\wedge^j_A M)^n = A(E)^{(n,j)} \) for all \( j \geq 0 \).

In particular we have \( \wedge^0_A M = A \) and \( \wedge^1_A M \cong M \) and

\[
\text{tot}(\wedge_A M) = \prod_{j \geq 0} \wedge^j_A M[j],
\]

(1.3)

**1.7 The (cyclic) bar complex**

Let \( (C, M) \) be an admissible pair of categories. Consider a homomorphism \( k \to A \) of DG-objects in \( \text{gr}(C) \).

By the universal property of fibered products, there are given two maps \( A \to A \otimes_C k A \); we denote them in the sequel by \( \iota_1 \) and \( \iota_2 \). We denote the multiplication map \( A \otimes_C k A \to A \) by \( \mu \). It is just the uniquely defined homomorphism such that the diagram
commutes. When \( k \to A \) is a homomorphism of DG-algebras and \( d \) is the differential of \( A \), then \( R := A \otimes_{\text{gr}(C)} A \) also is a DG-algebra, whose differential is given by \( s = d \otimes 1 + 1 \otimes d \) and the homomorphisms \( \iota_1, \iota_2 \) and \( \mu \) are morphisms of complexes.

Let \( M \) be a DG \( A \)-bimodule in \( \text{gr}(M) \), which is a symmetrical \( k \)-bimodule. Then we can consider \( M \) as DG object of \( \text{gr}(M)(R) \), where the scalar multiplication \( R \times M \to M \) satisfies \( (a \otimes a', m) \mapsto (-1)^{a'm}ama' \) for homogeneous elements\(^7\) \( a, a' \in A \) and \( m \in M \). To see this we have to apply axioms (5.3),(5.5) and (5.6). The same axioms must be used to define the mappings in the next paragraph.

We now define the **cyclic barcomplex** \( C^\text{cycl}_\bullet(A, M) = (C^\text{cycl}_n(A, M), b) \) as well as the (acyclic) **barcomplex** \( C^\text{bar}_\bullet(A, M) = (C^\text{bar}_n(A, M), b') \) of \( A \) with values in \( M \) as complex of DG-modules in \( \text{gr}(M)(A) \). For \( n = 0, 1, \ldots \) set \( C^\text{cycl}_n(A, M) := M \otimes A^\otimes n \) and \( C^\text{bar}_n(A, M) := M \otimes A^\otimes n \otimes A \).\(^8\) We can define homomorphisms

\[
d_i : M \otimes A \otimes \cdots \otimes A_n \to M \otimes A \otimes \cdots \otimes A_{n-1}
\]

sending elements \( a_0 \otimes \cdots \otimes a_n \) to \( a_0 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_n \) \((i = 0, \ldots, n-1)\) and a homomorphism \( d_n \), sending homogeneous elements \( a_0 \otimes \cdots \otimes a_n \) to \((-1)^{a_0(a_1 + \cdots + a_{n-1})}d_0 \cdot a_n \otimes a_1 \cdot \cdots \otimes a_{n-1} \). Those homomorphisms are homomorphisms in \( \text{gr}(M)(A) \), when we regard the tensor products \( M \otimes A \otimes \cdots \) as \( A \)-modules by left-multiplication on the first factor. Remark that when \( M \) is only a \( A \)-right module, we consider it as an antisymmetrical \( A \)-bimodule by setting \( m \cdot a := (-1)^{ma}a \cdot m \).

Set \( b'_{n-1} := d_0 - \cdots - (-1)^{n-1}d_{n-1} \) and \( b_n := b + (-1)^n d_n \).

Exactly as in the algebraic case (see \( [1] \), chapter III, (2.1)), \( b \) and \( b' \) define differentials, i.e. \( b^2 = 0 \) and \( b'^2 = 0 \). They are homomorphisms over \( A \). So \( C^\text{cycl}_\bullet(A, M) \) and \( C^\text{bar}_\bullet(A, M) \) are complexes in \( \text{gr}(M)(A) \). Remark that \( C^\text{bar}_\bullet(A, M) \) is even a complex in \( \text{gr}(M)(R) \), when we define the \( R \)-module structure on \( M \otimes A^\otimes n \otimes A \) by

\[
(a \otimes a') \cdot (m \otimes a \otimes a_{n+1}) = (-1)^{a(a'+m)}a'm \otimes a \cdot a_{n+1}
\]

for homogeneous elements \( a, a', \alpha \) and \( m \). In the sequel we will write \( C^\text{cycl}_\bullet(A) \) for \( C^\text{cycl}_\bullet(A, A) \) and \( C^\text{bar}_\bullet(A) \) for \( C^\text{bar}_\bullet(A, A) \).

**Remark 1.13** In \( \text{gr}(M)(k) \) there exist homomorphisms

\[
h_n : A \otimes_{k}^C \cdots \otimes_{k}^C A \to A \otimes_{k}^C \cdots \otimes_{k}^C A
\]

sending elements \( a_1 \otimes \cdots \otimes a_n \) to \( 1 \otimes a_1 \otimes \cdots \otimes a_n \). They define a contracting homotopy for the bar complex.

\(^7\)In the exponents we write sometimes just \( a \) instead of \( g(a) \) for homogeneous elements. \( ab \) then means \( g(a) \cdot g(b) \) and not \( g(ab) \), which is just \( g(a) + g(b) \).

\(^8\)Here all tensor products are formed in the category \( \text{gr}(M)(k) \).

''
As consequence we see that the acyclic bar complex $C_{\text{bar}}(A)$ complex is acyclic.

**Remark 1.14** The double complex $C_{\text{bar}}(A)$, equipped with the $*$-product, is a DDG-algebra in $\text{gr}(C)$ over $R := A \otimes_k A$. So its total complex can serve as a DG-resolution of $A$ over $R$.

**Proof:** [4], chapter III, theorem (2.2) \hfill $\square$

**Attention:** In the analytic case, $\text{tot}(C_{\text{bar}}(A))$ is not a free object in $\text{gr}(C)$.

Now we state two well-known relations between the cyclic and acyclic bar complexes. For this we consider $R$ as $A$-bimodule via $a(a_1 \otimes a_2) = aa_1 \otimes a_2$ and $(a_1 \otimes a_2)a = a_1 \otimes a_2a$.

**Proposition 1.4**

(i) There is an isomorphism $C^\text{cycl}_\cdot(A, R) \rightarrow C^\text{bar}_\cdot(A)$ of complexes in $\text{gr}(M)(A)$, which is in the $n$-th component given by

$$C^\text{cycl}_n(A, R) \rightarrow C^\text{bar}_n(A)$$

$$(a \otimes a') \otimes \alpha \mapsto (-1)^{a(a'+\alpha)}a' \otimes \alpha \otimes a$$

with $\alpha \in A \otimes A^n$.

(ii) There is an isomorphism $C.(A, M) \rightarrow M \otimes R C^\text{bar}_\cdot(A)$ of complexes in $\text{gr}(M)(A)$, where the differential of the second complex is given by $1 \otimes b'$. In the $n$-th component it is given by

$$C^\text{cycl}_n(A, M) \rightarrow M \otimes R C^\text{bar}_n(A)$$

$$m \otimes \alpha \mapsto m \otimes 1 \otimes \alpha \otimes 1.$$ 

In the algebraic Hochschild theory the cyclic bar complex is often called Hochschild chain complex and the Hochschild cochain complex is defined in the algebraic literature as the complex $C^\text{•}(A, M) = (C^\text{•}(A, M), \beta)$ where $C^0(A, M) = M$ and $C^n(A, M) = \text{Hom}_k(A^\otimes_n, M)$ for $n = 1, 2, \ldots$. The differential $\beta$ is given by:

$$\beta(f)(a_1, \ldots, a_{n+1}) = a_1 f(a_2, \ldots, a_{n+1}) - f(a_1 \cdot a_2, \ldots, a_{n+1}) + \ldots$$

$$+ (-1)^n f(a_1, \ldots, a_n a_{n+1}) + (-1)^{n+1} f(a_1, \ldots, a_n)a_{n+1}.$$ 

---

9For the definition of the $*$-product see [4]; we won’t use it here.
Proposition 1.5
(i) When $M$ is an anti-symmetrical $A$-bimodule, then there is an isomorphism of complexes

\[ \text{Hom}_k(A^\otimes n, M) \longrightarrow \text{Hom}_A(C_n^{\text{cycl}}(A), M), \]

where the differential on the left complex is $\beta$ and the differential on the right complex is the one induced by the differential $b$ on $C_n^{\text{cycl}}(A)$.

(ii) There is an isomorphism of complexes

\[ \text{Hom}_R(C_n^{\text{bar}}(A), M) \longrightarrow \text{Hom}_k(A^\otimes n, M), \]

sending an $f : C_n^{\text{bar}} \longrightarrow M$ to the mapping $a_1 \otimes \ldots \otimes a_n \mapsto f(1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1)$.

1.8 Regular sequences

In this section we want to define a regular sequence for the graded, anticommutative context. In our definition, the question if a sequence is regular won’t depend on the order of its elements. For this we suppose that the ground ring $\mathbb{K}$ contains the rational numbers.

Here we work with an admissible pair of categories $(\mathcal{C}, \mathcal{M})$, equipped with a marking $((F_t)_{t \in T}, (G_u)_{u \in U})$, that induces the marking $((\text{gr}_G(F))_{t \in T}, (\text{gr}(G))_{u \in U})$ on $(\text{gr}(\mathcal{C}), \text{gr}(\mathcal{M}))$ and the marking $((\text{gr}_G^2(F))_{t \in T}, (\text{gr}^2(G))_{u \in U})$ on $(\text{gr}^2(\mathcal{C}), \text{gr}^2(\mathcal{M}))$.

Definition 1.6 Let $R$ be an algebra in $\text{gr}(\mathcal{C})$. We call a g-finite subset $X$ of $R$ a handy sequence if for each $x$, there is an $u(x) \in U$ such that $x \in \text{gr}(G)_{u(x), g(x)}(R) = G_{u(x)}(R^{p(x)})$. When $R = (R, s)$ is a DG-algebra, then a handy sequence $X \subseteq s$ is called handy $s$-sequence if we have $s(X) \subseteq (X)$. For a handy sequence $X \subseteq R$, let $E$ be a set of free algebra generators, containing for each $x \in X$ a generator $e(x) \in \text{gr}_E^2(F)_{u(x), g(x), -(1)}(R(E))$ of bidegree $(g(x), -1)$. Then we call the free DG-algebra $10 \ K(X) := R(E)$ in $\text{gr}^2(\mathcal{C})$ over $R$, where the differential (of bidegree $(0, -1)$) is given by $e(x) \mapsto x$, the Koszul complex of $X$ over $R$.

For practical reasons, when we work with a handy sequence $X = \{x_i : i \in J\}$, we define an ordering on the index set $J$, subject to the condition $g(x_i) \leq g(x_j)$ for $i \leq j$. Remark that for a handy sequence $X \subseteq R$ and each subset $Y \subseteq X$, the quotient $11 \ R/(Y)$ exists in $\text{gr}(\mathcal{C})$. And when $R$ is a DG-Algebra $(R, s)$ and $X$ is $s$-handy, then the quotient $R/(X)$ is also a DG-Algebra.

Definition 1.7 (and Theorem) Suppose that $\mathbb{Q} \subseteq \mathbb{K}$.

Let $X \subseteq R$ be a handy sequence and let $I$ be the ideal $(X) \subseteq R$. Suppose that for each subset $Y \subseteq X$, we have $\cap_{n \geq 1} I^n R/(Y) = 0$. Then $X$ is called a regular sequence if one of the following equivalent conditions holds:

1. Let $T$ be a set of free algebra generators that contains for each $x \in X$, an element $t(x)$ with $g(t(x)) = g(x)$. Then the map $R/I[T] \longrightarrow \text{gr}_1(R) = R/I \oplus I/I^2 \oplus \ldots$ in $\text{gr}(\mathbb{Q} - \mathfrak{A}_{R/I})$, sending $t(x)$ to the class of $x$ in $I/I^2$ is an isomorphism of (differential) graded $R/I$-algebras.

2. For each $x \in X$ and for each ideal $J \subseteq R$ that is generated by a subset $Y \subseteq X$ with $x \not\in Y$ we have: If $g(x)$ is even, then $x$ is no zero divisor in $R/J$. If $g(x)$ is odd, then the annihilator of $x$ in $R/J$ is just the ideal, generated by the class of $x$.

3. The Koszul complex $K(X)$ is a DG-resolution of $R/(X)$ over $R$.

4. $H^{-1}(K(X)) = 0$.

10In the sense of definition 11
11By “quotient”, we mean the cokernel in $\text{gr}(\mathcal{M})$ of the embedding $(X) \hookrightarrow R$. 

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The implication (iii)$\Rightarrow$(iv) is trivial.  

**Proof of (i)$\Rightarrow$(ii)**  
For an element $r \in R$, let $n(r)$ be the greatest $n$ such that $r$ is contained in $I^n$ and let $\text{in}(r)$ be the element represented by $r$ in $I^n(r)/I^{n(r)+1} \subseteq \text{gr}_I(R)$. Then for elements $r, r' \in R$ we have  
\[
\text{in}(rr') = rr' + I^{n(r)+n(r')+1}.  
\]  
(1.4)

**Claim:** A subset $X \subseteq R$ satisfies condition (ii), if the subset $\{\text{in}(x): x \in X\} \subseteq \text{gr}_I(R)$ satisfies condition (ii).

Proof of the claim: First step: For $x \in X$, if $g(x)$ is even and $\text{in}(x)$ is no zero divisor, then $x$ is no zero divisor. If $g(x)$ is odd and the annulator of $\text{in}(x)$ in $\text{gr}_I(R)$ is the ideal, generated by $\text{in}(x)$, then the annulator of $x$ in $R$ is the ideal generated by $x$.

The even case follows immediately by (1.3). In the odd case, let $r$ be in the annulator of $x$, i.e. $rx = 0$. Then by (1.2), we get $\text{in}(x) \cdot \text{in}(r) = 0$. So by the assumption, there is an $y_1 \in R$, such that $\text{in}(r) = \text{in}(x) \cdot \text{in}(y_1)$. This implies that $r_1 := r - xy_1$ is in $I^{n(r)+1}$ and $n(r_1) \geq n(r) + 1$. Since $x^2 = 0$, we have $r_1x = rx = 0$, and in the same way, we find a $y_2 \in R$ with $r_2 := r_1 - xy_2 \in I^{n(r-1)+1}$. Inductively, for each $m \geq n(r)$, we find $y_1, ..., y_k$ such that $r_k := r - x(y_1 + ... + y_k) \in I^m$. So $r$ is an accumulation point of the ideal (x) in the $I$-adic topology of $R$. But the closure of (x) in this topology is just $\cap_{k \geq 0} (x + I^k)$, which, by the condition $\cap_{n \geq 1} I^nR/Y = 0$, is equal to (x). So r is an element of (x). The second inclusion is obvious.

Second step: For $x \in X$, if weather $g(x)$ is even and $\text{in}(x)$ is no zero divisor, or $g(x)$ is odd and the annulator of $\text{in}(x)$ in $\text{gr}_I(R)$ is (in(x)), then $(x) \cap I^{n(x)+n} = xI^n$ for each $n \geq 0$.

One inclusion and the even case are easy to see. Suppose that $g(x)$ is odd and that $rx$ is in $I^{n(x)+n}$. We have to find $r' \in I^n$ such that $xr = xr'$. If $r \in I^n$, we are done. Else, we have $n(r) < n$ and $\text{in}(r) \cdot \text{in}(x) = rx + I^{n(r)+n(x)+1} = 0$. So there is a $y \in R$ such that $\text{in}(r) = \langle x \rangle \cdot \text{in}(y)$. This means that $r_1 := r - xy$ is in $I^{n(r)+1}$ and we have $r_1x = rx$. Inductively, we find an $r' := r_{n-n(r)}$, such that $r' \in I^n$ and $r'x = rx$.

As consequence, when we set $\bar{R} := R/(x)$ and $\bar{I} := I/(x)$, we get an isomorphism  
\[
\text{gr}_I(R)/(\text{in}(x)) \cong \text{gr}_{\bar{I}}(\bar{R}).  
\]
We deduce inductively, that for $\bar{R} := R/(x_1, ..., x_s)$ and $\bar{I} := I/(x_1, ..., x_s)$, we get an isomorphism  
\[
\text{gr}_I(R)/(\text{in}(x_1), ..., \text{in}(x_s)) \cong \text{gr}_{\bar{I}}(\bar{R}).  
\]

Last step: When $g(x)$ is even, we have to show, that $x$ is no zero divisor in $R/(x_1, ..., x_s)$. We know that $\text{in}(x)$ is no zero divisor in $\text{gr}_I(R)/(\text{in}(x_1), ..., \text{in}(x_s)) \cong \text{gr}_{\bar{I}}(\bar{R})$. So by the first step, the assumption follows. For the odd case we use the analogue argument. So the claim is proven.

Now when (i) is true, it is clear that $\{\text{in}(x): x \in X\}$, which is just the set $T$, satisfies condition (ii) and by the claim, $X$ satisfies condition (ii).

**Proof of (iv)$\Rightarrow$(i)**

Without restriction, we can suppose that $C$ is the category of commutative $\mathbb{Q}$-algebras. For each $j \geq 0$, we have to show that the $j$-th homogeneous component $(R/I[T])_j$ in the $T$-grading of $R/I[T]$ maps isomorphically to $I^j/I^{j+1}$.

We will already make use of the implication (ii)$\Rightarrow$(iii). Set $S := \mathbb{Q}[T]$. We consider $R$ as $S$-algebra via the map $t(x) \mapsto x$. Obviously $T \subseteq S$ satisfies condition (ii), so by (iii), the Koszul complex $K_S(T)$ is a DG-resolution of $\mathbb{Q}$ over $S$.  

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We consider the exact sequence

$$0 \rightarrow (T)^i/(T)^{i+1} \rightarrow S/(T)^{i+1} \rightarrow S/(T)^i \rightarrow 0$$

of graded $S$-modules. $(T)^i/(T)^{i+1}$ is a free, graded $g$-finite $\mathbb{Q}$-vectorspace which is a $S$-module via the canonical map $S \rightarrow \mathbb{Q}$. We write $\prod_{i \in J} \mathbb{Q}e_i$ for it. Now $\prod_{i \in J} K_S(T)e_i$ is a free resolution of $\prod_{i \in J} \mathbb{Q}e_i$ over $S$. So we get

$$\text{Tor}^S_i((T)^i/(T)^{i+1}, R) = H^{-1}(\prod_{i \in J} (K_S(T)e_i \otimes_S R)) = \prod_{i \in J} H^{-1}(K(X)e_i) = 0.$$  

By the property of left derived functors, there is an exact sequence

$$0 \rightarrow \text{Tor}^S_1(S/(T)^{i+1}, R) \rightarrow \text{Tor}^S_1(S/(T)^i, R) \rightarrow (T)^i/(T)^{i+1} \otimes_S R \rightarrow S/(T)^{i+1} \otimes_S R \rightarrow S/(T)^i \otimes_S R \rightarrow 0$$

By induction on $j$ and the exactness of the first line, we see that $\text{Tor}^S_1(S/(T)^j, R) = 0$ for any $j \geq 0$. The second line gives rise to a short exact sequence

$$0 \rightarrow (R/I[T])_j \rightarrow R/I^j \rightarrow R/I^{j+1} \rightarrow 0,$$

which implies the desired isomorphism.

**Proof of (ii)$\Rightarrow$(iii)**

To prove this implication, we only have to show that for $p \leq 0$ the $p$-th row of the double complex $K(X)$ is a DG-resolution in $\mathcal{M}$ over $R^0$ of the $p$-th component of $R/(X)$. For this we can suppose that $X$ is finite with $g(x) \geq p$ for all $x \in X$. Say $X = \{x_1, \ldots, x_n\}$. Then we have $K(X) = K(x) \otimes \ldots \otimes K(x_n)$.

Each $K(X)^{(p,q)}$ is obviously a finite $R$-module, so with corollary 113 we only have to show that $K(X)^{(p,q)}$ is a resolution of $(R/(X))^p$ in the category of $R$-modules. To show this, we proceed by induction on $m$.

For $m=1$ we write $x$ instead of $x_1$ and $e$ instead of $e_1$. Set $m := g(x)$. Remark that if $m$ is even, then the total degree $m-1$ of $e$ is odd, so in this case we have $e^2 = 0$. If $m$ is odd, then the total degree of $e$ is even, so $e^2 \neq 0$. In the first case $K(x)$ is just the complex

$$0 \rightarrow R^0 e^2 \rightarrow \ldots \rightarrow R^m e \rightarrow \ldots$$

$s$ is injective since $x_1$ is no zero divisor in $R$, so the sequence is exact.

In the second case $K(x)$ is the complex

$$0 \rightarrow R^0 e^2 \rightarrow \ldots \rightarrow R^m e \rightarrow \ldots$$

In $R^i$, for $i < m$ there is no element that annihilates $x$, so up to the row $m - 1$, the situation is as above. In the m-th row the kernel of $R^m e \rightarrow R^m$ is just $R^0 xe$, so it coincides with the image of
the map $R^0e^2 \to R^mc$. Remark that here-fore we use that 2 is invertible in $R$. Inductively we see that all rows are exact. Here-fore we use that all naturals are invertible.

Now suppose that the statement is proven for $m$. We set $L := K(x_1,...,x_m) = R\langle e_1,...,e_m \rangle$ and $K := K(X) = K(x_1,...,x_{m+1})$. We write $x$ and $e$ instead of $x_m$ and $e_m$. $K(x)$ is (as object of $\text{gr}^2(\mathcal{M}(R))$ a direct product $K_0 \oplus K_{-1} \oplus K_{-2} \oplus \ldots$, where in the case where $x$ is even, we have $K_0 = R$, $K_1 = R[m, -1]$ and $K_s = 0$ for $s < -1$ and in the odd case we have $K_s = R[sm, -s]$ for all $s \leq 0$. The differential in $K(x)$ is given by the maps $d_q : K_q^{p,q} \to K_q^{p,q+1}$. Remark that we have $K_q^{p,q} = R^{p+qm}$ and $d_q : R^{p+qm} \to R^{p+(q+1)m}$ is just multiplication by $x$. Now $(K(x) \otimes L)^{p,q}$ is the class

$$\sum_{i+j=p} K_0^{i,j} \otimes R^0 L^{k,l} \oplus K_{-1}^{i,j} \otimes R^0 L^{k,l} + \ldots$$

which is equal to $L^{p,q} + L^{p-m,q+1} + L^{p-2m,q+2} + \ldots$. The differential of $K = K(x) \otimes L$ is given by the scheme

$$\sum_{i+j=p} K_0^{i,0} \otimes R^0 L^{k,q} + \sum_{i+j=p} K_{-1}^{i,-1} \otimes R^0 L^{k,q+1} + \ldots$$

But now in the even case the class $[\sum_{i+k=p} K_s^{i,s} \otimes R^0 L^{k,q-s}]$ is equal to $L[sm, -s]^{p,q}$ for $s = 0, -1$ and 0 for $s < -1$. In the odd case the class $[\sum_{i+k=p} K_s^{i,s} \otimes R^0 L^{k,q-s}]$ is equal to $L[sm, -s]^{p,q}$ for all $s \leq 0$. So in the even case, the complex $K(X)^{p,\ast}$ is the total complex of the double complex

In the odd case $K(X)^{p,\ast}$ is the total complex of the double complex
The first double complex is a DDG-resolution in \( \text{gr}^2(M)(R^0) \) of the DG-module
\[
(R/(x_1, \ldots, x_n))^p \leftarrow (R/(x_1, \ldots, x_n))^p \leftarrow 0 \leftarrow \ldots,
\]
where the left arrow stands for multiplication by \( x \). But this DG-module is a resolution of \( (R/(x_1, \ldots, x_n, x))^p \) over \( R^0 \), since \( g(x) \) is even. So \( K(X)^{p,*} \) is a resolution of \( (R/(x_1, \ldots, x_n, x))^p \).

The second double complex is a DDG-resolution in \( \text{gr}^2(M)(R^0) \) of the DG-module
\[
(R/(x_1, \ldots, x_n))^p \leftarrow (R/(x_1, \ldots, x_n))^p \leftarrow (R/(x_1, \ldots, x_n))^p \leftarrow \ldots,
\]
where the arrows stand for multiplication by \( x \). But this DG-module is a resolution of \( (R/(x_1, \ldots, x_n, x))^p \) over \( R^0 \), since \( g(x) \) is odd. So \( K(X)^{p,*} \) is a resolution of \( (R/(x_1, \ldots, x_n, x))^p \).

So for both cases the induction step is done. \( \square \)

**Remark 1.15** The assumption \( Q \subseteq \mathbb{K} \) is used only to prove the implications (ii)\( \Rightarrow \) (iii) and (iv)\( \Rightarrow \) (i). The assumption that for each subset \( Y \subseteq X \) we have \( \cap_{n \geq 1} I^n R/(Y) = 0 \) is used only to prove (i)\( \Rightarrow \) (ii). So if one wants to get rid of it, use condition (ii) for the definition of regular sequences. It can be stated in a slightly modified manner, which depends on the order of the elements of \( X \), then.

**Definition 1.8** Let \( R \) be a DG-algebra in \( \text{gr}(C)^N \). Let \( (\alpha_i, u_i, g_i)_{i \in J} \) be a family in \( N \times U' \) and \( X = \{ x_i : i \in J \} \) a family of elements with \( x_i \in G_u(R^0_{\alpha_i}) \) such that for \( \beta, \beta' \subseteq \alpha \) the sets \( \{ \rho_{\beta \alpha}(x_i) : \alpha_i = \beta \} \) and \( \{ \rho_{\beta' \alpha}(x_i) : \alpha_i = \beta' \} \) are disjoint. Suppose that
\[
X_\alpha := \bigcup_{\beta \subseteq \alpha} \{ \rho_{\beta \alpha}(x_i) : \alpha_i = \beta \}
\]
is a regular (resp. handy) \((s,\alpha)\)-sequence in \( R_\alpha \) for each \( \alpha \). Then \( X \) is called a regular \((s,\alpha)\)-sequence (resp. handy \((s,\alpha)\)sequence) in \( R \).

**Corollary 1.4** When \( R = (R, s) \) is a DG-algebra in \( \text{gr}(C)^N \) and \( X \) a handy \( s \)-sequence in \( R \), then \( K(X) \) is a DG-algebra in \( \text{gr}^2(C)^N \) and if \( X \) is regular, then \( K(X) \) is a DG-resolution of \( R/(X) \overline{\text{r}} \) over \( R \).

**Remarks:** When \( R \) carries the structure of a DG-algebra \((R, s)\), one would like the Koszul complex to carry the structure of a DG-module. In general this is not the case. When \( X \) is an \( s \)-handy sequence then, since \( I = (X) \) is \( s \)-stable, the algebra \( \text{gr}_i(R) \) has the structure of a DG algebra in \( \text{gr}(\mathfrak{A}_G) \), such that each \( I^n / I^{n+1} \) is a DG submodule of \( \text{gr}_i(R) \). If for example \( R \) is already a free DG-algebra in \( \text{gr}(\mathfrak{A} - \mathfrak{A}_G) \) with a set \( X \) of free algebra generators, i.e. \( R = R/I[X], \) then the differential of \( \text{gr}_i(R) = R \) differs in general from the differential \( s \). In this way we get a modified differential \( \tilde{s} \) on \( R \). In a similar way we get a modified differential, when \( R \) is a free DG-algebra in \( \text{gr}(C) \), for any good pair of categories \((C, M)\). This will play a role in section 3. In geometric language, going over from \( s \) to \( \tilde{s} \) is a deformation to the normal cone.

**1.9 The universal module of differentials**

Let \( k \rightarrow A \) be a morphism in \( \text{gr}(C) \). Set \( R := A \otimes_k^L A \) and denote the kernel of the multiplication map \( R \rightarrow A \) in the category \( \text{gr}(M)(R) \) by \( I \). In this subsection we use some notations of subsection 1.6.

Attention: In general \( A \) is the cokernel of the inclusion \( I \rightarrow R \) only in the category \( R-\text{Mod} \).

**Remark 1.16** Suppose that \((C, M)\) satisfies (S1), all \( A^i \) are finite \( A^0 \)-modules and \( I \) is a g-finite \( R \)-module. Then we have \( A = R/I := \text{Cokern}(I \rightarrow R) \). Here, of course, we mean the categorical cokernel.
In [4], (6.12) the universal module $\Omega_{A/k}$ of $k$-differentials is defined as the cokernel in $\text{gr}(\mathcal{M})(A)$ of the map $b_2 : A \otimes_k A \otimes_k A \to A \otimes_k A$, sending $a \otimes b \otimes c$ to $ab \otimes c - a \otimes bc + (-1)^{bc}ac \otimes b$ for homogeneous elements $a, b, c \in A$.

Here we consider all tensor powers as $A$-modules with respect to the first factor. Now when $(\mathcal{C}, \mathcal{M})$ is $(\mathcal{C}^{(0)}, \mathcal{M}^{(0)})$ or $(\mathcal{C}^{(1)}, \mathcal{M}^{(1)})$, there is a well-known canonical isomorphism $\Omega_{A/k} \cong I/I^2$ of DG-modules in $\text{gr}(\mathcal{M})(A)$. Here we want to show that under some weak assumptions this is true in good pairs of categories. So suppose that the pair $(\mathcal{C}, \mathcal{M})$ is good. For technical reasons, we also have to assume, that the marking $G$ is trivial. This is the case in all examples, when $\mathcal{M}$ is non-graded. First of all, we have to ask if we can consider the $R/I$-module $I/I^2$ as object of $\text{gr}(\mathcal{M})(A)$.

**Remark 1.17** Let $R = (R, s)$ be a DG-object in $\text{gr}(\mathcal{C})$ such that all $R^i$ are finite $R^0$-modules. Consider an ideal $I \subseteq R$ which is generated by a handy s-sequence $X = \{x_j : \ j \in J\}$ in $R$. Then $I$ is a DG-object of $\text{gr}(\mathcal{M})(R)$ and $I/I^2$ is isomorphic as $R/I$-module to a DG-object of $\text{gr}(\mathcal{M})(R/I)$.

**Proof:** For each $x \in X$ we choose a free module generator $e(x)$ with $g(e(x)) = g(x)$ and we see that $I$ is the image\(^{12}\) of the map from the free module $\prod_{x \in X} Re(x)$ to $R$, defined by $e(x) \mapsto x$. So $I$ is already an object of $\text{gr}(\mathcal{M})(R)$. But since $I$ is s-stable by assumption, $I$ is a DG-module. For each pair $i, j$ in $J$ with $i \leq j$ we choose a free module generator $e_{ij}$ with $g(e_{ij}) = g(x_i) + g(x_j)$.

We get a homomorphism $\prod_{i \leq j} Re_{ij} \to R$ of modules in $\text{gr}(\mathcal{M})(R)$ by sending $e_{ij}$ to the product $x_i x_j$. This homomorphism factorises through $I$, so there is a homomorphism $\pi : \prod_{i \leq j} Re_{ij} \to I$ and there is an isomorphism of $R$-modules $\text{Cokern} \pi \cong I/I^2$. It is easy to see, that the differential $s$ induces a differential on $\text{Cokern} \pi$, that makes it a DG-module in $\text{gr}(\mathcal{M})(R)$.

Now $I/I^2$ is also an $R/I$-module and in $R/I - 2\mathfrak{m}00$ the objects $\text{Cokern} \pi$ and $\text{Cokern} \pi \otimes_{R}^{\text{gr}(\mathcal{M})} R/I$ are isomorphic. And the latter is an object of $\text{gr}(\mathcal{M})(R/I)$.

In the sequel, by $I/I^2$ in fact we mean $\text{Cokern} \pi \otimes_{R}^{\text{gr}(\mathcal{M})} R/I$.

**Proposition 1.6** Let $k \to A$ be a homomorphism of DG-objects in $\text{gr}(\mathcal{C})$. Suppose that all $A^i$ are finite $A^0$-modules and that $I := \text{Kern}(\mu : A \otimes_k^C A \to A)$ is generated by an s-handy sequence $X$ in $R := A \otimes_k A \to A$. Here $s$ denotes the differential of $R$, induced by the differential of $A$. Then by $\bar{a} \to [a]$ we get an isomorphism $I/I^2 \to \Omega_{A/k}$ in $\text{gr}(\mathcal{M})(R)$, whose inverse is given by $[a] \mapsto \alpha = t_1(\mu(\alpha))$. Here $\bar{\pi}$ denotes the class in $I/I^2$ represented by $a$ and $[a]$ denotes the class in $\Omega_{A/k}$ represented by $a$.

**Proof:** First we have to show that the map $I/I^2 \to \Omega_{A/k}$, $\bar{a} \to [a]$ is well defined.

There is a homomorphism $\eta : I \otimes_R I \to I$ in $\text{gr}(\mathcal{M})(R)$ with $a \otimes b \mapsto ab$. Consider the homomorphism $\xi : I \to \Omega_{A/k}$, $a \to [a]$. For the well-definedness it is enough to prove that $\xi \circ \eta = 0$.

Since the barcomplex $C^b_{\text{bar}}(A/k)$ is acyclic, we see that $b'$ gives rise to an epimorphism $A^8 \to I$. Hence it is enough to show that the map\(^{12}\)

$A^8 \to \Omega_{A/k}$

$a \otimes b \otimes c \otimes d \otimes e \otimes f \mapsto [(ab \otimes c - a \otimes bc)(de \otimes f - d \otimes ef)]$

\(^{12}\)We remind that by image we mean the kernel of the cokernel map.
is zero. But the argument in the brace on the right hand-side is just the image of

\[ (-1)^{cd+ce+db} adbe(cf \otimes 1 \otimes 1 \otimes 1 \otimes c \otimes f) -
\]
\[ (-1)^{bd+cd} adh(cf \otimes 1 \otimes 1 \otimes c \otimes e f) -
\]
\[ (-1)^{bd+bc+cd+ce} ade(bc f \otimes 1 \otimes 1 \otimes bc \otimes f) +
\]
\[ (-1)^{bd+cd} adh(ce f \otimes 1 \otimes 1 \otimes bc \otimes e f)
\]

by the map \( b_2 \).

Secondly we have to show that the map \( \Omega_{A/k} \to I/I^2 \), \([a] \to \overline{a - s_1 \mu(a)}\) is well defined. But there is a derivation

\[ \delta : A \to I/I^2
\]
\[ a \mapsto \overline{a} \otimes a - a \otimes \overline{a}
\]

So by the universal property of \( \Omega_{A/k} \), see the proof of \([1]\) lemma (6.13), there is a map \( \Omega_{A/k} \to I/I^2 \) sending a class \([a \otimes b]\) to \( a\delta(b) = a \cdot \overline{1} \otimes b - b \otimes \overline{1} \) and we see that this map is just the map we want.

To see that the both given maps are inverse to each other, we remark that elements of the form \( a \otimes 1 \) in \( A \otimes A \) are in the image of \( b_2 \), so they represent the zero class.

\( I/I^2 \) has the structure of an \( A \)-module in \( \text{gr}(M)(A) \). The multiplication \( A \times I/I^2 \to I/I^2 \) is inherited by the multiplication \( a \cdot \alpha = \iota_1(a) \cdot \alpha \) on \( A \otimes A \). But on \( A \otimes A \) there is also a left multiplication \( \alpha \cdot a := \alpha \cdot \iota_2(a) \). Remark that the left- and right multiplication induced on \( I/I^2 \) make \( I/I^2 \) an antisymmetrical \( A \)-bimodule.

Now, let \( R = (R, s) \) be a DG-object of \( \text{gr}(C) \) and suppose that all \( R^i \) are finite \( R^0 \)-modules. Let \( I \subseteq R \) be an ideal which is generated by a regular \( s \)-sequence \( X \subseteq R \). Say \( X = \{x_i : i \in J\} \). \( s \) defines a differential \( \delta \) on \( R/I \), that we denote again by \( s \). Consider the free module \( \bigoplus_{i \in J} R/Ie_i \), where the \( e_i \) are free module generators of degree \( g(x_i) \).

Remark 1.18 We can make \( \bigoplus_{i \in J} R/Ie_i \) a DG-module, by defining a differential \( t \) in the following sense: For \( i \in J \), there is a finite family of elements \( x_{ij} \in X \) and \( a_{ij} \in R \) such that \( s(x_i) = \sum_j a_{ij} x_{ij} \). Now we set \( s(a) := s(a) \) for elements \( a \in R/I \) and \( s(e_i) := \sum_j a_{ij} e_{ij} \).

Proof: To show that this defines a differential on \( \bigoplus_{i \in J} R/Ie_i \), we only have to show, that \( \delta^2(e_i) = 0 \) for \( i \in J \). But since \( s \) is a differential on \( R \), we have

\[ 0 = \delta^2(x_i) = s(\sum_j a_{ij} x_{ij}) = \sum_{j,k} (-1)^{a_{ij} a_{jk}} a_{jk} x_k + \sum_j s(a_{ij}) x_k.
\]

We can reorganize the coefficients and get a sum \( \sum_{k=1}^m b_k x_{ik} = 0 \) where the \( x_{ik} \) are pairwise different. Remark that \( \sum_{k=1}^m \overline{b_k e_{ik}} = 0 \) is just \( \delta^2(e_i) \). To show that this sum is zero, we have to show that each \( b_k \) belongs to \( I \). But assume that one \( b_k \), say \( b_m \) does not belong to \( I \), then \( b_m \) is a nonzero annulator of \( x_m \) in \( R/(x_1, \ldots, x_{m-1}) \) and it doesn’t belong to \( Rx_m \). This contradicts the hypothesis that \( X \) is regular.

\( \square \)

In the algebraic case, the following proposition is an immediate consequence of condition (i) in definition and theorem \([17]\).

Proposition 1.7 In this situation, the assignment

\[ \prod_{x \in X} R/Ie(x) \to I/I^2, \quad e(x) \mapsto \overline{x}
\]

gives rise to an isomorphism of DG-objects in \( \text{gr}(M)(R/I) \).
Proof: It is clear that the map commutes with the differentials. Obviously the map is well defined and surjective. With axiom (S2), we only have to show that the map is injective. Let \( \sum_{i=1}^{m} e_i \alpha_i \) be an element of the kernel of this map. Then we have \( \sum_{i=1}^{m} a_i x_i = 0 \), i.e. \( \sum_{i=1}^{m} a_i x_i \in I^2 \). We must show that all \( a_i \) are elements of \( I \). Let \( Y \) be a finite subset of \( X \) such that \( \sum a_i x_i \) is a sum \( \sum_{y,y' \in Y} a(y,y')yy' \) with \( a(y,y') \in R \). Now as in the well-known non-graded case, when we assume that one \( a_i \), say \( a_m \), is not in \( I \), we can deduce that \( a_m \) is a zero divisor in \( R/J \), where \( J \subseteq R \) is the ideal generated by \( Y \setminus x_m \). This leads to a contradiction! \( \square \)

The condition on \( A \) in the following corollary is something like a smoothness condition.

Corollary 1.5 Suppose that all \( A_i \) are finite \( A^0 \)-modules. If the kernel of the multiplication map \( R := A \otimes_k A \rightarrow A \) is generated by a regular s-sequence \( X \) in \( R \) then there is a natural isomorphism of DG-modules in \( \text{gr}(M)(A) \)

\[
\Omega_{A/k} \rightarrow \prod_{x \in X} Ae(x).
\]

Here \( X \) denotes the regular s-sequence in \( R \) and to \( x \in X \) we have associated a free module generator \( e(x) \) with \( g(e(x)) = g(x) \). The differential on the left is given by the rule \( e(x) \rightarrow \sum a_y e(x_y) \), where for \( x \in X \) the family \( a_y \) is chosen in a way, such that \( s(x) = \sum a_y y \) and \( a \) denotes the residue class in \( R/(X) \cong A \) of an element \( a \in R \).

From this statement, we can deduce the corresponding simplicial statement:

Proposition 1.8 Suppose that \( A \) is a DG-algebra in \( \text{gr}(C)^N \) over \( k \) and set \( R := A \otimes_k A \). Denote the differential on \( R \), induced by the differential of \( A \) by \( s \). Suppose that the kernel of the map \( R \rightarrow A \) is generated by a regular s-sequence \( X \) in \( R \) in the sense that for each \( x \in N \), the kernel of \( R_x \rightarrow A_x \) is generated by \( X_x := \cup_{y \in \alpha} \{ a_y(x) \} \). Then there is an isomorphism in the category of DG-modules in \( \text{gr}(M)^N \)

\[
\Omega_{A/k} \rightarrow \prod_{x \in X} Ae(x).
\]

2 Hochschild complexes and Hochschild cohomology

2.1 The algebraic context

In this subsection we want to generalize the algebraic definition of Hochschild complexes in such a way that it is also useful in analytical contexts. Here let \( C \) be the category of commutative rings and \( M \) the category of modules over objects of \( C \).

Let \( k \rightarrow a \) be a homomorphism in \( C \). Let \( A \) be a resolvent of \( a \) over \( k \). Set \( R := A \otimes_k A \). As in subsection 1.7, we denote the cyclic bar complex and the bar complex of a DG-algebra over \( k \) by \( C^{\text{cyc}}( ) \) and \( C^{\text{bar}}( ) \).

The double complex \( C^{\text{cyc}}(A) \) is a DDG-resolution of \( C^{\text{cyc}}(a) \). Hence the free \( R \)-algebra \( \text{tot}(C^{\text{cyc}}(A)) \) is quasi-isomorphic over \( R \) to \( C^{\text{cyc}}(a) \). Since \( \text{tot}(C^{\text{cyc}}(a)) \otimes_R A \cong C^{\text{cyc}}(A) \), there are quasi-isomorphisms:

\[
C^{\text{cyc}}(a) \approx \text{tot}(C^{\text{cyc}}(A)) \approx \text{tot}(C^{\text{bar}}(A)) \otimes_R a.
\]

Now since in the algebraic case, \( \text{tot}(C^{\text{bar}}(A)) \) is a resolvent of \( A \) over \( R \), and two such resolvents are homotopy-equivalent, we get the following result:

Proposition 2.1 Let \( S \) be a g-finite resolvent of \( A \) over \( R \). Then there is a quasi-isomorphism

\[
S \otimes_R a \rightarrow C^{\text{cyc}}(a)
\]

over \( a \).
Remember that on the right hand-side we have the classical Hochschild complex. This shall justify the definition in the next subsection. Remark that proposition \(2.1\) keeps true in the simplicial context.

### 2.2 The noetherian context

Fix a good pair of categories \((\mathcal{C}, \mathcal{M})\) with marking \((F, G)\), where \(G\) is the trivial marking of \(\mathcal{M}\). Suppose that the axioms (N) and (F2) are satisfied.

Let \(k \to a\) be a finite morphism of \(\mathcal{N}\)-objects in \(\mathcal{C}\), i.e. \(a\) is a quotient of a free \(k\)-algebra \(b\) in \(\mathcal{C}^N\), such that for each \(\alpha \in \mathcal{N}\) the algebra \(a_{\alpha}\) is a free finite \(k_{\alpha}\)-algebra. Then, with \([2]\), proposition (8.8), there exists a g-finite resolvent of \(a\) over \(k\). Fix such a resolvent \(\hat{A}\). Set \(R := \hat{A} \otimes^{gr(\mathcal{C})}_{k} A\) and consider \(A\) as algebra over \(R\) by the multiplication map \(\mu : R \to A\). Let \(S\) be a free g-finite resolvent\(^{13}\) of \(A\) over \(R\).

**Definition 2.1** We define the simplicial Hochschild complex \(\mathbb{H}_\ast(a/k)\) of \(a\) over \(k\) to be the object represented by the complex \(S \otimes_R a\) in the homotopy category \(K^-(\mathcal{M}^N(a))\).

**Proposition 2.2** \(\mathbb{H}_\ast(a/k)\) is a well defined object in \(K^-(\mathcal{M}^N(a))\).

**Proof:** For \(i = 1, 2\) let \(A_i\) be a g-finite resolvent of \(a\) over \(k\), \(R_i := A_i \otimes_k A_i\) and let \(S_i\) be a g-finite resolvent of \(A_i\) over \(R_i\). We have to show the existence of a homotopy equivalence \(S_1 \otimes_{R_1} a \simeq S_2 \otimes_{R_2} a\) over \(a\).

First remark that \(R_i\) is a g-finite resolvent of \(a \otimes_k a\) over \(k\). Hence by proposition \([12]\) there is a homomorphism \(R_1 \to R_2\) in \(gr(\mathcal{C})^N\) which is a homotopy equivalence over \(k\). Hence we get a quasi-isomorphism

\[ S_1 = S_1 \otimes_{R_1} R_1 \to S_1 \otimes_{R_1} R_2 \]

over \(R_1\). \(S_1' := S_1 \otimes_{R_1} R_2\) is a free algebra over \(R_2\) and as \(S_2\) it is a resolution of \(a\) over \(R_2\). Hence by proposition \([12]\) there is a homomorphism \(S_1' \to S_2\) in \(gr(\mathcal{C})^N\), which is a homotopy equivalence over \(R_2\). We can tensorise both sides over \(R_2\) with \(a\) and still get a homotopy equivalence \(S_1 \otimes_{R_1} a \to S_2 \otimes_{R_2} a\). \(\square\)

We consider \(a\) as object of \(\text{gr}(\mathcal{M})^N(A)\) via the surjection \(\alpha : A \to a\). In this sense the Hochschild complex can be seen as DG-module over \(A\) in \(\text{gr}(\mathcal{M})^N\).

**Remark 2.1** The map \(\tilde{\alpha} : S \otimes_R A \to S \otimes_R a\) induced by \(\alpha\) is a quasi-isomorphism over \(A\).

**Definition 2.2** Let \(M\) be an object of \(\mathcal{M}^N\) over \(a\). We define the Hochschild cochain complex of \(a\) over \(k\) with values in \(M\) to be the complex

\[ \text{Hom}^N_{\mathcal{A}}(\mathbb{H}_\ast(a/k), M), \]

with the differential induced by the differential of \(\mathbb{H}_\ast(a/k)\). We define the Hochschild cohomology \(\text{HH}(a/k, M)\) of \(a\) over \(k\) with values in \(M\) to be the cohomology of the Hochschild cochain complex.

**Proposition 2.3** The Hochschild cochain complex is well defined up to homotopy equivalence.

**Proof:** This is a consequence of proposition \([2.2]\) and \([3]\), chapter I, lemma (3.7). \(\square\)

**Lemme 2.1** The Hochschild cohomology of \(a\) over \(k\) with values in \(M\) is equal to the cohomology of the complex \(\text{Hom}^N_{\mathcal{A}}(S \otimes_R A, M)\), where \(M\) is considered as object over \(A\) via the map \(A \to a\).

\(^{13}\)Again with loc. cit., such a resolvent exists. We can even construct it in such a way that \(S^0 = R^0\).
Example 3.1 isomorphism from this free algebra in $R$. Suppose that the marking $gr$ is in $R$. The first example is trivial. For the second example, we show that if a free generator $A$ is a homomorphism of DG-objects in $gr(C)$. For the construction of the Hochschild complexes of $a$ over $k$, we are interested in resolvents $S$ of $A$ as DG-object in $gr(C)(R)$. Proof: With lemma 6.4, we have

$$\text{Hom}_a(a \otimes R S, M) = \text{Hom}_R(S, M) = \text{Hom}_A(A \otimes_R S, M).$$

□

3 A decomposition theorem for Hochschild cohomology

Let $A = k(T)$ be a $g$-finite free DG-object of $gr(C)$ over $k$, with differential $d$. We think of $A$ as the resolvent of a $k$-algebra $a$ in $gr(C)$. Further we consider the free algebra $R := A \otimes_k A$ in $gr(C)_k$ with its differential $s = d \otimes 1 + 1 \otimes d$. Then the multiplication map $\mu : R \rightarrow A$ is a homomorphism of DG-objects in $gr(C)$. First, we will not construct such an $S$ but something very similar. For this we modify the differential $s$ on $R$ as sketched in subsection 1.8, which geometrically corresponds to a deformation to the normal cone. We call the modified differential $\tilde{s}$, which modification doesn’t touch the structure of $A$ as DG-algebra over $R$, since $s(r) - \tilde{s}(r)$ will be in the kernel of $\mu$ for each $r \in R$. Then we construct a resolvent $S$ of $A$ over $\tilde{R} := (R, \tilde{s})$ with the help of a Koszul complex. As we will see in subsection 5.2, this is enough for the construction of the Hochschild complex.

If $B$ is an object of $gr(C)$ and $R := B(T)$ is a free algebra over $B$ in $gr(C)$ with a $g$-finite set $T$ of free generators $t$ with $t \in F_r(t)(R^g(t))$. Then $R \otimes_B R$ is a free algebra over $B$ with two free algebra generators $t_1 = t \otimes 1$ and $t_2 = 1 \otimes t$ for each $t \in T$. For $t \in T$ set $t^+ := \frac{1}{2}(t_1 + t_2)$ and $t^- := \frac{1}{2}(t_1 - t_2)$. Let $T^+$ be the set of all $t^+$ and $T^-$ be the set of all $t^-$. We say that the marking $F$ on $C$ is balanced if for each $\tau \in T$ and each $A$ in $C$ and each $t \in F_r(A)$ we have $-t \in F_r(A)$. We say that the marking $F$ is convex if for each $\tau \in T$, each $A$ in $C$, each $t_1, t_2 \in F_r(A)$ and each $a, b \in \mathbb{K}$ with $a + b = 1$ we have $at_1 + bt_2 \in F_r(A)$.

Remark 3.1 Suppose that the marking $gr_C(F)$ on $gr(C)$ is balanced and convex, then we have $R \otimes_B R \cong B(T^+ \cup T^-)$. More precisely this means that there is a free algebra over $B$ and an isomorphism from this free algebra in $R \otimes_B R$, sending the free generators on the elements $t^+$ and $t^-$. Example 3.1

(i) The trivial marking on $C$ is balanced and convex, so when $C$ is the category $C'(0)$ of (noetherian) rings and $M$ is the category $M'(0)$ of modules over $C'(0)$, then remark 3.1 is true.

(ii) When $C$ is the category $C'(1)$ of (local) analytic algebras and $M$ the category $M'(1)$ of DFN-modules over $C'(1)$, then the marking $F$ on $C$ (see example 1.2) is balanced and convex.

Proof: The first example is trivial. For the second example, we show that if a free generator $t$ is in $F_r(R)$, then $T^+$ and $T^-$ are in $F_r(R \otimes_B R)$: Here $\tau$ stands for a positive real number and $F_r(R)$ is the set of all $r \in R$, such that for each character $\xi \in \text{X}(R)$, we have $|\xi(r)| \leq \tau$. Now $t_1 = \nu_1(t) = t \otimes 1$ and $t_2 = \nu_2(t) = 1 \otimes t$ belong to $F_r(R \otimes_B R)$, so for each character $\xi \in \text{X}(R \otimes_B R)$, we have $|\xi(t_1)| \leq \tau$ and $|\xi(t_2)| \leq \tau$. Hence $|\xi(t^+)| = |\frac{1}{2}(\xi(t_1) + \xi(t_2))| \leq \tau$ and $|\xi(t^-)| = |\frac{1}{2}(\xi(t_1) - \xi(t_2))| \leq \tau$. The case of local analytic algebras is clear, since maximal ideals are additively closed. □

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3.1 Deformation to the normal cone

For the rest of this section suppose that the marking \( F \) on \( \mathcal{C} \) is balanced and convex and that the marking \( G \) on \( \mathcal{M} \) is trivial. Further suppose that (N) and (F2) are satisfied. Then we have \( \hat{R} = k(T^+ \cup T^-) \). Since \( T^- \) is s-stable and g-finite, it is a regular s-sequence. (In the algebraic case we have \( \hat{R} = \text{gr}_{t^-(T^-)}(R) \), so as we have already mentioned in section 1.8 there is a deformation \( \hat{s} \) of \( s \), that respects the submodules \( (T^- \cap T^-)^{+1} \). This is what we mean by “deformation to the normal cone”. Here we do a similar construction for the general case.) Each element \( r \) of \( R \) has a unique decomposition \( r = \hat{r} + \hat{r} + \hat{r} \) with \( \hat{r} \in \hat{R} := k(T^+) \). \( \hat{r} \in \hat{R} := \sum_{t \in T^-} tk(T^+) \) and \( \hat{r} \in \hat{R} := \sum_{t,v \in T^-} tt'k(T^+) \). Now we define a \( R^0 \)-derivation \( \hat{s} \) on \( R \) setting \( \hat{s}(t) := (s(t))^\vee \) for \( t \in T^+ \) and \( \hat{s}(t) := (s(t))^\vee \) for \( t \in T^- \). The philosophy of this modification is that roughly speaking \( \hat{s} \) preserves the \( T^- \)-degree of homogeneous elements in \( R \). More precisely we have \( \hat{s}(R) \subseteq \hat{R}, \hat{s}(R) \subseteq \hat{R} \) and \( \hat{s}(R) \subseteq \hat{R} \) in contrast to \( s(R) \subseteq R \Pi \hat{R} \) and \( s(R) \subseteq \hat{R} \).

**Proposition 3.1** \( \hat{s} \) is a differential, i.e. \( \hat{s}^2 = 0 \).

**Proof:** First remark that for \( a \in k(T^+) \) we have

\[
(s(a))^\vee = \hat{s}(a).
\]

To prove this we can suppose that \( a \) is of the form \( a_0t_1 \cdots t_n \) with \( a_0 \in k(T^{+,0}) \) and \( t_i \in T^{+,<0} \). In this case it is easy to see.

Now suppose that \( t \) is in \( T^+ \). Then \( s(t) = \hat{s}(t) + \text{rest} \), where rest is in \( \hat{R} \Pi \hat{R} \). So \( \hat{s}^2(t) = \hat{s}^2(t) + \text{rest}' \) where rest' is in \( \hat{R} \Pi \hat{R} \). Since \( \hat{s}^2(t) \) is in \( \hat{R} \) and \( \hat{s}^2(t) = 0 \), we get \( \hat{s}^2(t) = 0 \). Similarly, we see that \( \hat{s}^2(t) = 0 \) for \( t \in T^- \), which proves the proposition. \( \square \)

We write \( X \) for the regular \( \hat{s} \)-sequence \( T^- \). Next we will see that for \( \hat{R} = (R, \hat{s}) \), the Koszul complex \( (K(X), v) \) has the structure \( (K(X), h, v) \) of a DG-algebra, so its total complex is a resolution of \( A = \hat{R}/(X) \) over \( \hat{R} \). Again we denote by \( E \) the set of free algebra generators, containing for each \( x_i \) in \( X \) an element \( e_i \) of bidegree \( (g(x_i), -1) \). Here \( \hat{s}(x_i) \) is a sum of the form \( \sum a_jx_j \), where no \( a_j \) belongs to the ideal \( (X) \). So in fact all \( a_j \) belong to \( B = k(T^+) \). Now there is exactly one choice for the element \( h_i \), which shall be the image of \( e_i \) by the horizontal differential \( h \) of \( R(E) \). The choice is \( h_i = \sum a_jc_j \). Now we have \( 0 = \hat{s}^2(x_i) = \hat{s}(\sum a_jx_j) = \sum_{j,k} a_ja_{jk}x_{jk} \) and the coefficients \( a_ja_{jk} \) belong to \( k(T^+) \). So we have \( h^2(e_i) = h(\sum a_jc_j) = \sum_{j,k} a_ja_{jk}c_{jk} = 0 \). I.e. the hypothesis of remark 3.1 is satisfied. So the Koszul complex \( (K(X), v) \), equipped with the horizontal differential \( h \), is a DG-resolution in \( \text{gr}^2(C) \) of \( A = \hat{R}/(X) \) over \( \hat{R} = (R, \hat{s}) \). And the total complex \( \hat{S} \) of \( K(X) \) is a free algebra resolution of \( A \) over \( \hat{R} \).

The reason for this construction is that there is a nice description of the tensor product

\[
\hat{S} \otimes_{\hat{R}} A = \text{tot}(K(X)) \otimes_{\hat{R}} A = \text{tot}(K(X) \otimes_{\hat{R}} A).
\]

**Namely:**

**Remark 3.2** The double complex \( K(X) \otimes_{\hat{R}} A \) is just \( \wedge_A \Omega_{A/k} \).

**Proof:** This follows imidiately by proposition 3.1 and definition 3.1 \( \square \)

The going over from \( s \) to \( \hat{s} \) is natural, i.e. when \( (R, s) = (r_\alpha, s_\alpha)_{\alpha \in N} \) is a DG-algebra in \( \text{gr}(C)^N \), then \( \hat{R} := (R_\alpha, \hat{s}_\alpha)_{\alpha \in N} \) is again a DG-algebra in \( \text{gr}(C)^N \) and remark 3.2 keeps true in the simplicial case.

3.2 Construction of the resolution \( S \)

To get a resolution \( S \) of \( A \) over \( R \), which is good for our purpose, we have to work harder. The strategy is to construct again a DG-resolution \( K \) in \( \text{gr}^2(C)^N \), that is free over the double graded object \( K(X) \) and such that the projection \( K \longrightarrow K(X) \) is a morphism of DG-objects (of course it
will not be a morphism of DDG-objects) and such that the induced map $K \otimes_R A \to K(X) \otimes_R A$ is a morphism of DDG-objects in $gr^2(C)^N$, that induces an isomorphism on vertical homology. When we have managed to realize this, with $S := \text{tot}(K)$ we get a resolution of $A$ over $R$ and a quasi-isomorphism

$$S \otimes_R A \to \tilde{S} \otimes_R A$$

(3.6)

over $A$. Since both complexes are free objects over $A$, the quasi-isomorphism is even a homotopy-equivalence.

First we explain heuristically the construction of $K$: We take the Koszul-complex $K(X)$ with its vertical differential $v$. The problem is to define a horizontal differential $h$ on it, since here in general there are no good candidates for the values $h(e)$ of $h$ on the free generators $e \in E$. When we have $s(x) = \sum a_{xy}y$, to get commutative diagrams, $h(e(x))$ must be something like $\sum a_{xy}e(y)$. The problem is that $h(\sum a_{xy}e(y))$ won’t be zero. So, inductively, for $e$ we add free algebra-generators $f$ of bidegree $(g(x) + 1, -1)$ with $v(f) = 0$, in such a way that we can find candidates for $h(e)$ in $K(X)^{g(x)+1,-1} + R^0f$. When this is done for all $e \in E$, we get a DDG-structure on the extension $K(X)\langle F \rangle = R\langle E \cup F \rangle$. Now this extension is not any more a resolution. To get a resolution again, we apply the construction of remark[112]

Now we begin with the construction of $K$. First we place ourselves in the affine situation. For each $x \in X$ we fix a finite family $a_{xy}, y \in Y \subseteq X$ such that $s(x) = \sum a_{xy}y$. Set $K(X) := R(E)$ as double graded algebra.

**Proposition 3.2** There is a $g$-finite family $F = \bigcup_{p \leq 0}F^p$ of free algebra generators with $g(f) = (p, -1)$ for $F \in F^p$ and a DDG-algebra structure $(L, h, v)$ on $L := R(E \cup F)$ such that

(i) $v(e(x)) = x$

(ii) $v(f) = 0$

(iii) $h(f)$ is in the ideal $(X \cup F)$, generated by $X$ and $F$.

(iv) $h(e(x)) = \sum_y a_{xy}e(y) + \gamma(x)$ with a $\gamma(x) \in (F)$.

**Proof:** We construct a sequence $L_k = (L_k, h_k, v_k)$ of free DDG-algebras over $R$ as well as a family \{$\gamma(x) : x \in X \wedge g(x) \geq k - 1$\} with $\gamma(x) \in L_k^{k-1}$ for $x \in X^{k-1}$ such that the following conditions hold:

(a) $L_0 = R(E^0)$.

(b) $L_k^{-1} = L_k(E^{k-1} \cup F^{k-1})$ is a free DDG-algebra over $L_k$, where $F^{k-1}$ is a finite set of algebra generators of bidegree $(k-1, -1)$ and we have the rules $v_{k-1}(e(x)) = x$ (as in the Koszul-construction) and $v_{k-1}(f) = 0$.

(c) $h_{k-1}$ maps the submodule $\pi_{f \in F^{k-1}}R^0f$ of $L_k^{k-1}$ surjectively onto $\text{Kern}(h_k^{k-1}) \cap \text{Kern}(v_k^{k-1}) \subseteq L_k^{k-1}$.

(d) For all $i \geq k - 1$ the sequence $L_{k-1}^i \to R^i \to A^i$ in $\mathcal{M}(R^0)$ is exact.

(e) For $x \in X^{k-1}$ we have $\gamma(x) \in \pi_{f \in F^k}R^0f$ and $h_k^{k-1}(\gamma(x)) = h_k^{k-1}(\sum_y a_{xy}e(y))$.

For $L_0 = R(E^0)$ we have already seen that by setting $v(e(x)) := x$ we get an exact sequence $L_0^0 \to R^0 \to A^0$. Now suppose that $L_k$ and \{$\gamma(x) : g(x) \geq k$\} is already constructed. We choose finitely many free algebra generators $f$ of bidegree $(k - 1, -1)$ such that there exists an epimorphism

$$\pi : \pi R^0f \to \text{Kern}(h_k^{k-1}) \cap \text{Kern}(v_k^{k-1}) \subseteq (X \cup F).$$

To explain the inclusion: The vertical differential on the subalgebra $K(X)$ is exact. So a homogeneous element of $K(X)\langle F \rangle$, which is in the kernel of $v$, is a sum of an element in the image of $v$
Properties of $L$

(i) $L^0,^\ast = K(\mathbb{X})^0,^\ast$, hence this is a resolution of $A^0$ over $R^0$.

(ii) $L^0,^p = R^p$ for all $p \leq 0$.

(iii) The sequence $L^{-1},^p \rightarrow L^0,^p \rightarrow A^p \rightarrow 0$ is exact for all $p \leq 0$.

(iv) The inclusion $K(\mathbb{X}) \hookrightarrow L$ and the projection $L \rightarrow K(\mathbb{X})$ are homomorphisms of DG-algebras over $(R,0)$, so in the category of DG-modules in gr$(\mathbb{M})$ there is a decomposition $L = K(\mathbb{X}) \amalg L'$. The (vertical) homology of $L$ is contained in $L'$.

Proposition 3.3 There is a g-finite family $G = \cup_{p\leq = q \leq -2} G^{p,q}$ of free algebra generators with $g(g) = (p,q)$ for $g \in G^{p,q}$ and extensions of $h$ and $v$ on $K := L(\mathbb{G})$, such that

(i) The $i$-th row of $K$ is a $R^0$-module resolution of $A^i$.

(ii) $v(g) \in (F \cup G)$

(iii) $h(g) \in (G)$

Proof: We can construct the free $\text{DDG}$-resolution $K$ of $A$ over $R$ with the method of remark 1.12. □

Comparing the values of $h$ on the free generators $e$ with its values by the differential $\hat{h}$ of the Koszul-complex over $R$ with the modified differential, we see that $\hat{h}(e) - h(e) \in \Pi R^0 g + \sum_{x \in X} x K$.

Consequence: Consider the projection $\pi : K = R(E \cup F \cup G) \rightarrow R(E) = K(\mathbb{X})$ (a priori only as map of algebras in gr$^2(\mathbb{C})$). With proposition 3.2 (ii) and proposition 3.3 (ii), we see that $\pi$ respects the vertical differential. By the construction of $\hat{s}$ and proposition 3.2 (iv), we see that

$\pi \otimes 1 : K \otimes_R A \rightarrow K(\mathbb{X}) \otimes_R A$

is a homomorphism of $\text{DDG}$-algebras in gr$^2(\mathbb{C})$ over $A$.

Now we can prove the (affine case of the) crucial result of this chapter. It says that to construct the Hochschild complex it is enough to work with a resolvent of $A$ over $R$.

Theorem 3.1 With $S := \text{tot}(K)$ and $\tilde{S} := \text{tot}(K(\mathbb{X}))$, there is a homotopy-equivalence

$S \otimes_R A \rightarrow \tilde{S} \otimes_R A$

over $A$.

Proof: First we have seen that the projection $\pi : K \rightarrow K(\mathbb{X})$ is a homomorphism of $\text{DG}$-Algebras in gr$^2(\mathbb{C})$ over $(R,0)$. Since both double complexes are free resolutions, for each $p$ the restriction $K^p,^\ast \rightarrow K(\mathbb{X})^p,^\ast$ is a homotopy equivalence over $R^0$. So we see that there is a well defined map of $\text{DG}$-algebras $\pi \otimes 1 : K \otimes_R A \rightarrow K(\mathbb{X}) \otimes_R A$, which is by the properties (c) and (d) even a homomorphism of $\text{DDG}$-algebras. Further we see that for each $p$ the restriction

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\((K \otimes R A)^{p+} \to (K(X) \otimes R A)^{p+}\) is a homotopy equivalence. Hence \(\pi \otimes 1\) induces a quasi-isomorphism 

\[
S \otimes R A = \text{tot}(K) \otimes R A = \text{tot}(K \otimes R A) \to \text{tot}(K(X) \otimes R A) = \tilde{S} \otimes R A.
\]

But a quasi-isomorphism of free algebras is already a homotopy-equivalence. \(\square\)

Again, we have to explain that the same construction also works in the simplicial case: So suppose that \(k\) is an object of \(\text{gr}(C)^N\) and \(A\) is a free algebra over \(k\) in \(\text{gr}(C)^N\). Say \(A = k\langle T \rangle\), where each \(t \in T\) is associated to a pair \((\alpha_t, \tau_t, g_t)\) \(N \times T \times \mathbb{Z}_{\leq 0}\). Now \(T^+\) and \(X := T^-\) are sets of free generators in the simplicial sense. Write \(X = \{x_i : i \in I\}\) and \((\alpha_i, \tau_i, g_i)\) for the triple associated to \(x_i\). For \(\alpha \in N\) set 

\[
X_\alpha := \{\rho_{\alpha i}(x_i) : \alpha i \subseteq \alpha\}.
\]

In the sequel we will simply write \(x_i\) for the element \(\rho_{\alpha i}(x_i)\) of \(R_\alpha\). Let \(E = \{e_i : i \in I\}\) be a family of free algebra generators containing for each \(x_i \in X\) an \(e_i\) of degree \((g(x), -1)\), belonging to the simplex \(\alpha_i\). We form the free algebra \(K(X) = R(E)\) in \(\text{gr}^2(C)^N\). Set \(E_\alpha := \{\rho_{\alpha i}(e_i) : \alpha i \subseteq \alpha\}\). Then we have \(K(X)_\alpha = R_\alpha(E_\alpha)\). For each \(x = x_i\) in \(X\) we fix a family \(a_{xy}, y \in Y\) with \(Y \subseteq X_\alpha\), and \(a_{xy} \in R_\alpha\), such that \(s_a(x) = \sum_y a_{xy}y\).

**Proposition 3.4** There is a g-finite family \(F = \{f_j : j \in J\}\) of free algebra generators, where \(f_j\) belongs to \(\alpha_j\) and is of bidegree \((g_j, -1)\) and a DDG-algebra structure \((L, h, v)\) on \(L = R(E \cup F)\) over \(R\), such that for all \(\alpha\) and all \(x \in X_\alpha\) and all \(f \in F_\alpha\) := \{f_j : \alpha j \subseteq \alpha\} the following conditions hold:

(i) \(v_\alpha(e(x)) = x\)

(ii) \(v_\alpha(f) = 0\)

(iii) \(h_\alpha(f) \in (X_\alpha \cup F_\alpha)\)

(iv) \(h_\alpha(e(x)) = \sum_y a_{xy}e(y) + \gamma(x)\) for a \(\gamma(x)\) in \((F_\alpha)\).

**Proof:** We reduce the proposition by induction on the following statement: Suppose that there is a family \(F^{(n)}\) and a DDG-algebra structure on \((R(E)_{\alpha}|_{|\alpha| \leq n}\{F^{(n)}\})\), such that the conditions (i)-(iv) hold for all \(\alpha \in N^{(n)}\). Then there is a family \(F^{(n+1)}\) and a DDG-algebra structure on \((R(E)_{\alpha}|_{|\alpha| \leq n+1}\{F^{(n+1)}\})\) such that the conditions (i)-(iv) hold for all \(\alpha \in N^{(n+1)}\). The case \(n = 0\) as well as the induction step can be done easily as in the affine case. \(\square\)

**Proposition 3.5** There is a g-finite family \(G = \{g_j : j \in J\}\) of free algebra generators, where \(g_j\) belongs to \(\alpha_j\) and is of bidegree \(g_j\) and a DDG-algebra structure on \(K = L(G)\) over \(L\), such that for all \(\alpha \in N\) and all \(x \in X_\alpha\) and all \(g \in G_\alpha\) the following conditions hold:

(i) \(v_\alpha(g) \in (G_\alpha \cup F_\alpha)\)

(ii) \(h_\alpha(g) \in (G_\alpha)\).

**Proof:** With the same method as above, we reduce the statement to the affine case. \(\square\)

Now we see that theorem 3.4 holds as well in the simplicial context.

### 3.3 A HKR-type theorem

When we now resume what we know about the resolution \(K(X) \otimes R A\), we get the following result. It generalizes in a sense the classical Hochschild-Kostant-Rosenberg theorem. We use the same
assumptions as in the last subsection.

**Theorem 3.2** Consider a homomorphism \( k \rightarrow a \) in \( \mathcal{C}^N \). Suppose that \( \mathcal{Q} \subseteq a \). Let \( A \) be a resolvent of \( a \) in \( \text{gr} \mathcal{C}^N(k) \). Then there is a quasi-isomorphism
\[
\text{tot}(\wedge \Omega_{A/k}) \rightarrow \mathbb{H}(a/k)
\]
of DG-algebras in \( \text{gr} \mathcal{C}^N \) over \( A \) and a quasi-isomorphism
\[
\text{tot}(\wedge \mathbb{L}_{a/k}) \rightarrow \mathbb{H}(a/k)
\]
in \( \text{gr} \mathcal{C}^N \) over \( a \).

**Corollary 3.1** When \( a \) is already free over \( k \) (in this case there is no need to assume that \( \mathcal{Q} \subseteq a \)) and \( A = a \), then \( \Omega_{a/k} \) is an object of \( \mathcal{C}^N_a \) and we get isomorphisms
\[
\wedge^n \Omega_{a/k} \cong H_n(\mathbb{H}(a/k))
\]
Dually, with \( T_{A/k} := \text{Hom}_A(\Omega_{A/k}, A) \) we get
\[
H^n(\text{Hom}_A(\mathbb{H}(a/k), a)) \cong \wedge^n T_{a/k}.
\]

**Proof:** The first statement follows directly from the theorem. For the second statement, remark that in the case, where \( A = a \), we have a quasi-isomorphism of free DG-algebras in \( \text{gr} \mathcal{C}^N \) over \( A \):
\[
\wedge^n \Omega_{a/k} \rightarrow \mathbb{H}(a/k),
\]
where the differential on the left side is trivial. With proposition \[12\] it is even a homotopy equivalence. So with \[7\] lemma (3.7), the dual homomorphism
\[
\text{Hom}_a(\mathbb{H}(a/k), a) \rightarrow \text{Hom}_a(\wedge \Omega_{a/k}, a) = \wedge T_{a/k}
\]
is also a homotopy equivalence, which proves the statement. \( \Box \)

### 3.4 The decomposition theorem

**Theorem 3.3** We have the following decomposition of Hochschild cohomology:
\[
\text{HH}^n(a/k, M) = \prod_{i-j=n} H^i(\text{Hom}_A(\wedge^j \Omega_{A/k}, M))
\]

**Proof:**
\[
\text{HH}^n(a/k, M) = H^n(\text{Hom}_a(\mathbb{H}(a/k), M)) = H^n(\text{Hom}_A(S \otimes_R A, M)) = H^n(\text{Hom}_A(\text{tot}(K(X) \otimes_R A), M)) = H^n(\text{Hom}_A(\bigwedge^j \Omega_{A/k}[j], M)) = H^n(\text{Hom}_A(\bigwedge^j \Omega_{A/k}[j], M)) = \prod_{i-j=n} H^i(\text{Hom}_A(\wedge^j \Omega_{A/k}, M)).
\]
The first equality holds by definition. The second one follows by equation \[2.5\]. The third one follows by the simplicial version of theorem \[3.1\]. The fourth equality holds by remark \[3.2\]. The fifth follows by equation \[1.3\]. The other equalities are elementary. \( \Box \)
4 Application to complex spaces and noetherian schemes

In this section, all schemes and complex spaces are supposed to be paracompact and separated. For details on many of the constructions we refer to [5] and [6].

First we will sketch the correlation between the theory of coherent sheaves on schemes or complex spaces and the theory of $\mathcal{N}$-objects in good pairs of categories. The main tools that we need here are:

1. Instead of considering a space $X$, we consider the simplicial scheme, associated to an affine covering of $X$. By an affine subspace, we mean an open affine subscheme in the case of schemes and a Stein compact in the case of complex spaces. There are functors that make simplicial schemes out of sheaves of modules and functors that do the inverse.

2. For affine subsets $U \subseteq X$ we use the equivalence of categories, of coherent $\mathcal{O}_X$-modules and finite modules over the ring $\Gamma(U, \mathcal{O}_X)$. (Remember that $\Gamma(U, \mathcal{O}_X)$ is noetherian, when $X$ is an analytic space.) This equivalence is given by Cartan's theorem A in the analytic case and by [9] chapter 2, exc. 2.4 in the algebraic case.

Now, more generally, let $X$ be a ringed space and $(X_i)_{i \in I}$ a covering of $X$. The nerve $\mathcal{N}$ of this covering is the set of all subsets $\alpha \subseteq I$, such that $\bigcap_{i \in \alpha} X_i \neq \emptyset$. $\mathcal{N}$ is a simplicial scheme in the sense of subsection 1.4. Further there is a contravariant functor from $\mathcal{N}$ in the category of ringed spaces, mapping an object $\alpha$ to the object $X_\alpha := \bigcap_{i \in \alpha} X_i$. For $\alpha \subseteq \beta$ denote the inclusion $X_\beta \rightarrow X_\alpha$ by $p_{\alpha\beta}$. Such a functor is called simplicial scheme of ringed spaces. Let $X_* = (X_\alpha)_{\alpha \in \mathcal{N}}$ be a simplicial scheme of ringed spaces. Now we define the category of $\mathcal{O}_{X_*}$-modules. Its objects are the families $\mathcal{F}_* = (\mathcal{F}_\alpha)_{\alpha \in \mathcal{N}}$ with $\mathcal{F}_\alpha$ in $\text{Mod}(X_\alpha)$ together with compatible maps $p_{\alpha\beta}^* \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$.

For $\mathcal{O}_{X_*}$-modules $\mathcal{F}, \mathcal{G}$, we set $\text{Hom}_{X_*}(\mathcal{F}, \mathcal{G})$ to be the set of families $f_\alpha : \mathcal{F}_\alpha \rightarrow \mathcal{G}_\alpha$, that are compatible. We denote this category by $\text{Mod}(X_*)$. The full subcategory of those $\mathcal{F}_*$, where each $\mathcal{F}_\alpha$ is coherent is denoted by $\text{Coh}(X_*)$.

**Definition 4.1** Let $A$ and $B$ be simplicial schemes over the index sets $A_0$ and $B_0$. Suppose that $X_* = (X_\alpha)_{\alpha \in A}$ and $Y_* = (Y_\beta)_{\beta \in B}$ are simplicial schemes of ringed spaces. A **morphism** $f : X_* \rightarrow Y_*$ consists of a mapping $\tau : A_0 \rightarrow B_0$, such that for $\alpha \in A$ we get $\tau(\alpha) \in B$, and a family of compatible maps $f_\alpha : X_\alpha \rightarrow Y_{\tau(\alpha)}$.

As in [7], we can form the adjoint functors

$$f^* : \text{Mod}(Y_*) \rightarrow \text{Mod}(X_*) \quad \text{and} \quad f_* : \text{Mod}(X_*) \rightarrow \text{Mod}(Y_*).$$

For $\mathcal{F}$ in $\text{Mod}(Y_*)$ and $\alpha \in A$, we have $(f^* \mathcal{F})_\alpha := f_\alpha^* \mathcal{F}_\tau(\alpha)$. The construction of $f_*$ is more complicated. For the general case, we refer to *loc.cit.* But we need only the following special case:

**Remark 4.1** Let $\mathcal{F}_*$ be an object of $\text{Mod}(X_*)$. Then for elements $\beta \in B$ of the form $\beta = \tau(\alpha)$, we have

$$(f_* \mathcal{F})_\beta = f_{\alpha*} \mathcal{F}_\alpha.$$

Hence, if the map $\tau : A_0 \rightarrow B_0$ is surjective, then the construction of $f_*$ becomes very simple.

**Example 4.1**

(i) When $X$ is a scheme or a complex space and $(X_i)_{i \in I}$ is a covering by affine subspaces, then by the separated condition, all $X_i$ are affine. Now let $(\mathcal{C}, \mathcal{M})$ be the good pair $(\mathcal{C}^{(0)}, \mathcal{M}^{(0)})$ or $(\mathcal{C}^{(1)}, \mathcal{M}^{(1)})$ (see [11]). Then $a_* := (\Gamma(X_\alpha, \mathcal{O}_{X_i}))_{\alpha \in \mathcal{N}}$ is an $\mathcal{N}$-object in $\mathcal{C}$ and there is a 1:1-correspondence between the objects of $\text{Coh}(X_*)$ and the $\mathcal{N}$-objects $M_*$ in $\mathcal{M}$ over $a_*$, such that each $M_\alpha$ is finite over $a_\alpha$. 

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(ii) When $X$ is a complex space, and the covering $(X_i)_{i \in I}$ is locally finite and chosen in such a way that each $X_i$ admits a closed embedding into a polydisc $P_\alpha$, then we get another simplicial scheme of Stein compacts: Set $P_\alpha := \prod_{i \in I} P_i$. Then for $\alpha \leq \beta$, we have the projection $P_\beta \to P_\alpha$. This makes $P_\alpha = (P_\alpha)_{\alpha \in N}$ a simplicial scheme of Stein compacts and there is a closed embedding $X_\alpha \to P_\alpha$.

(iii) Let $X$ be a scheme of finite type over a Ring $K$ and $(X_i)_{i \in I}$ an open affine covering of $X$. Again, we can construct a new simplicial scheme: Set $a_\alpha := \Gamma(X_\alpha, O_{X_\alpha})$ for $\alpha \in N$. For each $\alpha$, there is a free, finitely generated algebra $K[T]$, that maps surjectively onto $a_\alpha$. So we get a closed embedding $X_\alpha \to \text{Spec}(K[X]) := P_\alpha$. As above, we get a simplicial scheme $P_\alpha$ and a closed embedding $X_\alpha \to P_\alpha$.

The inclusions $j_\alpha : X_\alpha \to X$ give rise to a map $J : X \to X$ of simplicial schemes of ringed spaces. Next we will study the adjoint functors $j_\ast$ and $j^\ast$: $j^\ast$ is just the exact functor, mapping an $O_X$-module $F$ to the $O_{X_\alpha}$-module $(F|_{X_\alpha})_{\alpha \in N}$. To describe $j_\ast$, we consider the Cech-functor: For an $O_{X_\alpha}$-module $M_\alpha$ set

$$\check{\mathcal{C}}^p(F) := \prod_{|\alpha| = p} j_\ast F_\alpha$$

and define a differential on $\check{\mathcal{C}}^\bullet(F_\alpha)$ in the usual sense. Then $j_\ast F_\alpha$ is just $H^0(\check{\mathcal{C}}^\bullet(F_\alpha))$.

$j_\ast j^\ast$ is the identity functor. One can prove the adjointness of $j^\ast$ and $j_\ast$ directly by a gluing argument. Since $j^\ast$ is an exact functor and $j_\ast$ is right adjoint to $j^\ast$, we see that $j_\ast$ transforms injective objects in $\text{Mod}(X)$ into injective objects in $\text{Mod}(X)$.

By [5], proposition 2.26, each $O_{X_\alpha}$-module admits an injective resolution by modules of the form $\prod_{\alpha \in N} P_\alpha I_\alpha$ with injective $O_{X_\alpha}$-modules $I_\alpha$. We will use the following properties of the functor $\check{\mathcal{C}}^\bullet$.

**Remark 4.2**

(i) For $p \geq 0$, the functor $\check{\mathcal{C}}^p$ is exact.

(ii) If $F_\alpha$ is an $O_{X_\alpha}$-module, then $\check{\mathcal{C}}^\bullet(p_\alpha F_\alpha)$ is a resolution of $j_\ast(p_\alpha F_\alpha)$.

(iii) If $F$ is an $O_X$-module, then $\check{\mathcal{C}}^\bullet(j^\ast F))$ is a resolution of $F$.

We generalize a part of [5], proposition 2.28 for the case where $X$ is just a ringed space and $X_\ast$ is the simplicial scheme of ringed spaces associated to some covering $(X_i)_{i \in I}$ of $X$:

**Proposition 4.1** The functor $j^\ast : \mathcal{D}(X) \to \mathcal{D}(X_\ast)$ embeds $\mathcal{D}(X)$ as a full and exact subcategory into $\mathcal{D}(X_\ast)$ and $\check{\mathcal{C}}^\bullet = Rj_\ast$ is an exact right adjoint. In particular, for $F, G \in \mathcal{D}(X)$ and $M_{\ast} \in \mathcal{D}(X_\ast)$, there are functorial isomorphisms

$$\text{Ext}^k_X(F, G) \cong \text{Ext}^k_X(j^\ast F, j^\ast G) \quad \text{and}$$

$$\text{Ext}^k_X(j^\ast F, M_{\ast}) \cong \text{Ext}^k_X(F, \check{\mathcal{C}}^\bullet(M_{\ast})).$$

When all the maps $p^\ast_\alpha(M_{\ast}) \to M_{\ast}$ for $\alpha \leq \beta$ in $N$ are quasi-isomorphisms, then for all $n$, there are isomorphisms

$$\text{Ext}^n_{X_\ast}(M_{\ast}, j^\ast F) \cong \text{Ext}^n_X(\check{\mathcal{C}}^\bullet(M_{\ast}), F).$$

**Proof:** For the proof, that $\check{\mathcal{C}}^\bullet$ is the right derived functor of $j_\ast$, we use an injective resolution $I_{\ast}$ of an $O_{X_\ast}$-module $F_{\ast}$ from the same form as above. Then we have

$$(Rj_\ast)(F_{\ast}) = (j_\ast I_{\ast})^\bullet = \prod j_\ast(p_{\alpha \ast} I_{\alpha})^\bullet = \prod \check{\mathcal{C}}^\bullet(p_{\alpha \ast} I_{\alpha}) = \check{\mathcal{C}}^\bullet(I^\bullet_{\ast}) = \check{\mathcal{C}}^\bullet(F_{\ast}).$$
We only prove the first formula for $\text{Ext}$. Here $I^*_X$ denotes an injective resolution of $j^*\mathcal{G}$.

$$\begin{align*}
\text{Ext}^n_X(j^*\mathcal{F}, j^*\mathcal{G}) &= H^n(\text{Hom}_X, (j^*\mathcal{F}, I^*_X)) = H^n(\text{Hom}_X (\mathcal{F}, j_I^*)) = \\
\text{Ext}^n_X(\mathcal{F}, j^*I^*_X) &= \text{Ext}^n_X(\mathcal{F}, (Rj_!)^*(j^*\mathcal{G})) = \\
\text{Ext}^n_X(\mathcal{F}, C^*(j^*\mathcal{G})) &= \text{Ext}^n_X(\mathcal{F}, \mathcal{G}).
\end{align*}$$

In the sequel, let $X$ be a complex space or a scheme of finite type over a noetherian ring. Now the structure sheaf $\mathcal{O}_X$ defines an $\mathcal{N}$-object $\mathcal{F}$ in $\mathcal{M}$ over $a$. In the analytic case each coherent $\mathcal{O}_X$-module $\mathcal{F}$ defines an $\mathcal{N}$-object $\mathcal{F} = \mathcal{F}_a$ in $\mathcal{M}$ over $a$. Here $(\mathcal{C}, \mathcal{M})$ stands for $(\mathcal{C}^{(0)}, \mathcal{M}^{(0)})$ in the algebraic case and for $(\mathcal{C}^{(1)}, \mathcal{M}^{(1)})$ in the analytic case.

We make the following convention to avoid the distinction between analytic and algebraic tensor products:

**Convention**: Let $f : X_\ast \rightarrow Y$ be a morphism of simplicial schemes of Stein compacts and let $\mathcal{F}, \mathcal{G}$ be graded objects in $\text{Mod}(X_\ast)$, coherent in each degree. Then by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$, we mean the object in $\text{Mod}(X_\ast)$, which is given by the sheafification of the object $T_\ast$ in $\text{gr}(\mathcal{C}^N)$ given as follows:

For $\alpha \in \mathcal{N}$ set $B_\alpha := \Gamma(Y_\alpha, \mathcal{O}_{Y_\alpha})$, $F_\alpha := \Gamma(X_\alpha, \mathcal{F}_\alpha)$ and $G_\alpha := \Gamma(X_\alpha, \mathcal{G}_\alpha)$. Then $F_\alpha$ and $G_\alpha$ are modules over $B_\alpha$ via the comorphism of $f_\alpha$. Set $T_\alpha := F_\alpha \otimes_{B_\alpha} G_\alpha$ This defines a simplicial DG algebra $T_\ast$.

In the same manner, we define the tensor product $\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G}$, when $\mathcal{F}$ and $\mathcal{G}$ are modules over a sheaf of $\mathcal{O}_X$-modules $\mathcal{R}$, coherent in each degree.

### 4.1 Hochschild-cohomology for complex spaces and schemes

Let $f : X \rightarrow Y$ be a morphism of complex spaces or a morphism of finite type of noetherian schemes.

By a **resolvent** of $X$ over $Y$, we understand a collection of the following things:

1. The simplicial scheme $X_\ast$ associated to a local finite affine covering $(Y_j)_{j \in J}$ of $Y$; (2) the simplicial scheme $X_\ast = (X_\alpha)_{\alpha \in \mathcal{N}}$ associated to a local finite affine covering $(X_{ji})_{j \in J, i \in I_j}$ of $X$.

This covering is chosen in a way such that for a fixed $j \in J$ the family $(X_{ji})_{i \in I_j}$ is a covering of $f^{-1}(Y_j)$; (3) a simplicial scheme $P_\ast = (P_\alpha)_{\alpha \in \mathcal{N}}$ with the same index category; (4) a commutative diagram

$$
\begin{array}{ccc}
X_\ast & \xrightarrow{\iota} & P_\ast \\
\downarrow f & & \downarrow g \\
Y_\ast & \xrightarrow{\tau} & \\
\end{array}
$$

Here $\bar{f} = (\bar{f}, \tau)$ is the induced map of simplicial schemes, $\iota$ is a closed embedding and $g$ is a smooth map\textsuperscript{14}; (5) a free resolution $\mathcal{A}_\ast$ of $\mathcal{O}_{X_\ast}$ as sheaf of DG-algebras over $P_\ast$ with $\mathcal{A}_0^0 = \mathcal{O}_{P_\ast}$, such in each degree there are only a finite number of free algebra generators.

If $\mathcal{A}_\ast \rightarrow \mathcal{B}_\ast$ is a morphism of sheaves of DG-algebras, coherent in each degree, on a simplicial space $X_\ast$, where each $X_\alpha$ is affine, then going over to global sections, we can construct a free resolution $S_\ast$ of $\mathcal{B}_\ast := (\Gamma(X_\alpha, \mathcal{B}_\alpha))_{\alpha \in \mathcal{N}}$ over $\mathcal{A}_\ast := (\Gamma(X_\alpha, \mathcal{A}_\alpha))_{\alpha \in \mathcal{N}}$, at least when $\mathcal{B}_0^0$ is a quotient of a free algebra over $\mathcal{A}_0^0$ in $\text{gr}(\mathcal{C})^N$. This follows by [3], prop. 8.8. Sheafifying $S_\ast$, we get a free resolution $S_\ast$ of $\mathcal{B}$ over $\mathcal{A}$. Using this remark, it is easy to deduce the existence of free resolutions in the cases we are going to consider.

\textsuperscript{14}This means that for each $\alpha \in \mathcal{N}$ and each $p \in P_\ast$ the stalk $\mathcal{O}_{P_\ast, p}$ is free (in the analytic case as local analytic algebra) over $\mathcal{O}_{Y_{\iota(p)}, Y}$. 

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Example 4.2 Consider the case where \( X \) is smooth and \( Y \) is just the single point \( \text{Spec}(\mathbb{C}) \). Here we can choose \( P_i = X_i \) for \( i \) in the index set \( I \). Then \( X_\alpha \) is a diagonal in \( P_\alpha \) and \( A \) can be chosen to be a Koszul resolution of \( a = (\Gamma(X_\alpha, \mathcal{O}_{X_\alpha}))_{\alpha \in N} \) over \( A^0 = (\Gamma(P_\alpha, \mathcal{O}_{P_\alpha}))_{\alpha \in N} \). In this case one can prove that for each \( \alpha \), \( \Omega_{A_\alpha} \) is a module resolution of \( \Omega_{\alpha} \). It follows, that for \( \alpha \subseteq \beta \), the restriction maps \( L_\alpha(a/J) \to L_\beta(a/J) \) are quasi-isomorphisms. Consequently, the canonical map \( \mathbb{L}(X) \to \Omega_X \) is a quasi-isomorphism.

Now, suppose that there is a given resolution \( (X_*,Y_*,P_*,A_*) \) of the morphism \( f : X \to Y \). Furthermore, set \( \mathcal{R} := A \otimes_{\mathcal{O}_X} A \) and let \( \mathcal{S} \) be a free resolution of \( A \) over \( \mathcal{R} \).

The following definition coincides for complex spaces with the one, given in [6]:

**Definition 4.2** The simplicial Hochschild complex of \( X \) over \( Y \) is the object in the derived category \( D(X_*) \) of \( \mathcal{O}_{X_*} \)-modules, represented by

\[
\mathbb{H}_X(X/Y) := \mathcal{S} \otimes_{\mathcal{R}} \mathcal{O}_{X_*}.
\]

The Hochschild complex of \( X \) over \( Y \) is defined as the object in \( D(X) \), represented by

\[
\mathbb{H}(X/Y) := \mathcal{C}^*(\mathbb{H}_X(X/Y)).
\]

When \( Y \) is just the simple point, we will write \( \mathbb{H}(X) \) instead of \( \mathbb{H}(X/Y) \).

To show the independence of the Hochschild complex of the choice of the resolvent, we have to use the following version of [3], lemma (13.7):

**Lemma 4.1** Let \( f : X \to X' \) be a flat homomorphism of complex spaces resp. schemes and \( (X_i)_{i \in I} \) and \( (X'_i)_{i \in I} \) be compact locally finite coverings of \( X \) and \( X' \) by Stein compacts resp. open affine subsets. Let \( \tau : I \to I' \) be a mapping, such that \( f(X_i) \subseteq X'_i \) for all \( i \in I \). Denote the associated simplicial schemes by \( X_* \) and \( X'_* \). Then \( f \) defines a homomorphism \( (\bar{f}, \tau) \) of simplicial schemes of ringed spaces. Let \( \mathcal{G}^* \) be a complex in \( \text{Coh}(X') \) such that for \( \alpha \subseteq \beta \) the restriction map \( p'_\alpha_* \mathcal{G}^*_\alpha \to \mathcal{G}^*_\beta \) is a quasi-isomorphism. Then the canonical homomorphism

\[
f^* \mathcal{C}(\mathcal{G}^*) \to \mathcal{C}(f^* \mathcal{G}^*)
\]

is a quasi-isomorphism.

**Proposition 4.2** The definition of \( \mathbb{H}(X/Y) \) depends neither on the resolvent \( (Y_*,X_*,P_*,A_*) \) nor on the choice of the resolvent \( \mathcal{S} \).

**Proof:** Let \( (Y_*,X_*,P_*,A_*) \) and \( (\tilde{Y}_*,\tilde{X}_*,\tilde{P}_*,\tilde{A}_*) \) be two resolvents, \( \mathcal{S} \) a free resolution of \( A \) over \( A \otimes A \) and \( \tilde{\mathcal{S}} \) a resolvent of \( \tilde{A} \otimes \tilde{A} \). We have to show that there is a quasi-isomorphism

\[
\tilde{\mathcal{C}}(\tilde{\mathcal{S}} \otimes_{\mathcal{R}} \mathcal{O}_{\tilde{X}_*}) \to \mathcal{C}(\mathcal{S} \otimes_{\mathcal{R}} \mathcal{O}_{X_*}).
\]

**First case:** Suppose that \( Y_* = \tilde{Y}_* \), \( X_* = \tilde{X}_* \) and \( P_* = \tilde{P}_* \). Then it follows with proposition 2.2 that there is a quasi-isomorphism

\[
\mathcal{S} \otimes_{\mathcal{R}} \mathcal{O}_{X_*} \to \mathcal{S} \otimes_{\mathcal{R}} \mathcal{O}_{X_*}
\]
in \( \text{Mod}(X_*) \). Applying the Cech functor, this case is proven.

**General case:** Let \( Y'_* \) be the simplicial scheme associated to the covering \( \{Y_j\} \cup \{Y'_j\} \) and \( X'_* \) be the simplicial scheme associated to the covering \( \{X_{ij}\} \cup \{X'_{ij}\} \). We construct \( P'_* \) in the canonical way and can find a resolvent \( \mathcal{A}' \), such that \( (Y'_*,X'_*,P'_*,\mathcal{A}'_*) \) forms another resolvent of \( f : X \to Y \).

There is a commutative diagram

\[
\begin{array}{ccc}
X_* & \xrightarrow{h} & X'_* \\
\downarrow{f} & & \downarrow{f'_*} \\
Y_* & \to & Y'_*
\end{array}
\]

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By the first case, there is a quasi-isomorphism

\[ h^*(S' \otimes_R O_{X'}) \approx S \otimes_R O_X. \]

With lemma [14] there is a quasi-isomorphism

\[ \tilde{C}(S' \otimes_R O_{X'}) \approx \tilde{C}(h^*(S' \otimes_R O_{X'})). \]

Hence we get \( \tilde{C}(S \otimes_R O_X) \approx \tilde{C}(S' \otimes_R O_X) \) and in the same way we get \( \tilde{C}(S \otimes_R O_X) \approx \tilde{C}(S' \otimes_R O_X) \). □

As in [8], we define the **Hochschild cohomology of** \( X \) **over** \( Y \) with values in the sheaf \( F \) as \( \text{Ext}_X(\mathbb{H}(X/Y), F) \). At least in the case where \( F \) is coherent, we want to show that this definition is equal to the following one, which seems to be more natural, from the viewpoint of good pairs of categories:

**Definition 4.3 [alternative]**

Suppose that \( M_* := \mathbb{H}_*(X/Y) \), the assumption of the second part of proposition [11] is satisfied, i.e. for \( \alpha \subseteq \beta \) the maps \( p^*_{\alpha \beta}(M_\alpha) \rightarrow M_\beta \) are quasi-isomorphisms.

**Proof:** [8], lemma 1.7. □

**Corollary 4.1** For coherent \( O_X \)-modules \( F \), the two definitions of Hochschild cohomology coincide, i.e.

\[ \text{HH}^n(X/Y, F) := H^n(\text{Hom}_a(\mathbb{H}_*(a/k), F)). \]

**Proof:** Since \( \mathbb{H}_*(X/Y) \) is a complex of free \( O_X \)-modules, with proposition [11] we get

\[ \text{Ext}_X^1(\mathbb{H}_*(X/Y), F) = \text{Ext}_X^1(\mathbb{H}_*(X/Y), J^* F) = H^1(\text{Hom}_a(\mathbb{H}_*(a/k), F_*)) = \text{HH}^1(X/Y, F). \] □

For a (noetherian) scheme \( X \), we want to show that the definition of the Hochschild complex \( \mathbb{H}(X) \) coincides with one of the definitions given in [12] or [14]:

**Proposition 4.3** Let \( \mathbb{C}^{\text{cycl}}(X) \) be the complex of sheaves in \( \text{Mod}(X) \), associated to the presheaf \( U \rightarrow \mathbb{C}^{\text{cycl}}(\Gamma(U, O_X)) \). Then in \( D(X) \), there is an isomorphism

\[ \mathbb{C}^{\text{cycl}}(X) \cong \mathbb{H}(X). \]

**Proof:** Choose a resolvent \((X_*, P_*, A_*)\) of \( X \) over the base ring \( \mathbb{K} \). Let \( S \) be a resolvent of \( A \) over \( \mathcal{R} = A \otimes A \). Let \( a, A, R \) and \( S \) be the simplicial algebras in \( \text{gr}(C)^N \) corresponding to \( O_{X_*}, A, \mathcal{R} \) and \( S \). There is an isomorphism in \( D(X) \):

\[ \tilde{C}(j^* \mathbb{C}^{\text{cycl}}(O_X)) \cong \mathbb{C}^{\text{cycl}}(O_X). \]

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But $j^*C^\text{cycl}(\mathcal{O}_X)$ corresponds to $C^\text{cycl}(a)$, which is quasi-isomorphic to $C^\text{bar}(A) \otimes_R a$. $C^\text{bar}(A)$ is a resolution of $A$ over $R$, so by [3], proposition (8.4), there is a quasi-isomorphism $f : S \to C^\text{bar}(A)$. Now each $C^\text{bar}(A)_\alpha = C^\text{bar}(A_\alpha)$ is, as well as $S_\alpha$, a free module resolution of $A_\alpha$ over $R_\alpha$. So each $f_\alpha$ is a homotopy equivalence. Hence for each $\alpha$

$$f_\alpha \otimes 1 : S_\alpha \otimes_{R_\alpha} a_\alpha \to C^\text{bar}(A_\alpha) \otimes_{R_\alpha} a_\alpha$$

is a homotopy equivalence. So $f \otimes 1$ is a quasi-isomorphism. In $D(X_*)$ it induces an isomorphism

$$j^*(C^\text{cycl}(\mathcal{O}_X)) \cong S \otimes_R \mathcal{O}_X.$$

Forming the Cech complex gives the desired result. $\square$

### 4.2 The decomposition Theorem

The quasi-isomorphism $\text{tot}(\wedge L_{a/k}) \to H(a/k)$ in $\text{gr}(\mathcal{M})^N$ over $a$ in theorem [3] defines a quasi-isomorphism

$$\text{tot}(\wedge L_*(X/Y)) \to H_*(X/Y)$$

in $\mathcal{M}od(X_*)$. Since the Cech-functor is exact, we get the following HKR-type theorem:

**Theorem 4.1** There is an isomorphism

$$\text{tot}(\wedge L(X/Y)) \to H(X/Y)$$

in the derived category $D(X)$.

From this we deduce easily the announced decomposition theorem:

**Corollary 4.2** There is a natural decomposition

$$\text{HH}^n(X/Y, \mathcal{M}) = \bigoplus_{p+q=n} \text{Ext}^p_X(\text{tot}(\wedge^q L(X/Y)), \mathcal{M})$$

For complex spaces, this is just theorem 4.2 of [3]. There is another nice description of Hochschild cohomology of complex spaces or noetherian schemes over a field $K$ in any characteristic:

**Remark 4.4** $H^n(X) = \text{Ext}^n_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$.

**Proof:** We will use the letter $K$ for the field $K$ or for the complex numbers, depending on the context. With the notations as above, we get:

$$H^n(X) = H^n(\text{Hom}_A(S \otimes_R a, a)) = H^n(\text{Hom}_R(S, a)) = H^n(\text{Hom}_{\mathcal{O}_{X_2}^2}(S \otimes_R a, a)) = \text{Ext}^n_{\mathcal{O}_{X_2}^2}(\mathcal{O}_{X_2}, \mathcal{O}_{X_2}) = \text{Ext}^n_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X).$$

Here we have used that $S \otimes_R \mathcal{O}_{X_2}^2$ is a resolution of $\mathcal{O}_{X_2}$ and the fact that it is free over $\mathcal{O}_{X_2}$. $\square$
4.3 Hochschild-Cohomology for manifolds and smooth varieties

At last we will see that the decomposition theorem for Hochschild cohomology of complex manifolds, announced in [10], follows easily from theorem 4.1. It holds as well for smooth schemes of finite type over a field \( \mathbb{K} \) of characteristic zero. This case was proven in a different way by Yekutieli [14].

**Theorem 4.2** Let \( X \) be a complex analytic manifold or a smooth scheme of finite type over a field \( \mathbb{K} \) of characteristic zero. Then there is a decomposition of Hochschild cohomology:

\[
\text{HH}^n(X) = \bigoplus_{i-j=n} H^i(X, \wedge^j T_X).
\]

**Proof:** For complex analytic manifolds, we work with a fixed covering by Stein compacts and its associated simplicial scheme \( X^* \). For the case of smooth schemes of finite type over \( \mathbb{K} \), we work with an open affine covering by schemes of the form \( \text{Spec}(A) \), where \( A \) is a finitely generated \( \mathbb{K} \)-algebra. Denote the associated simplicial scheme also by \( X^* \).

By proposition [10], theorem [12] and example [12] there are quasi-isomorphisms

\[
j^*(\mathbb{H}(X)) = j^* C(\mathbb{H}_*(X)) \approx \mathbb{H}_*(X) \approx \wedge_* \Omega_X = j^*(\wedge_\mathcal{O}_X \Omega_X)
\]

of \( \mathcal{O}_X \)-modules. \( j_* j^* \) is the identity functor, so there is a quasi-isomorphism of \( \mathcal{O}_X \)-modules

\[
\mathbb{H}(X) \approx \wedge_\mathcal{O}_X \Omega_X.
\]

We consider \( \wedge \Omega_X \) as complex in negative degrees, so \( \wedge \Omega_X = \bigoplus_{j \geq 0} \wedge^j \Omega_X[j] \) and

\[
\text{HH}^n(X) = \text{Ext}^n_X(\mathbb{H}(X), \mathcal{O}_X) \cong \bigoplus_{j \geq 0} \text{Ext}^{n+j}_X(\wedge^j \Omega_X, \mathcal{O}_X).
\]

By [5], theorem (7.3.3), there is a (bounded) spectral sequence with \( E_2^{p,q} = H^p(X, \text{Ext}^q_X(\wedge^j \Omega_X, \mathcal{O}_X)) \), converging to \( \text{Ext}_X(\wedge^j \Omega_X, \mathcal{O}_X) \). But \( \wedge \Omega_X \) is a locally free \( \mathcal{O}_X \)-module, so \( \text{Ext}^q_X(\wedge^j \Omega_X, \mathcal{O}_X) \) is zero for \( q > 0 \) and \( \text{Hom}_X(\wedge^0 \Omega_X, \mathcal{O}_X) \) for \( q = 0 \). So the spectral sequence degenerates at once and we get

\[
\text{Ext}^q_X(\wedge^j \Omega_X, \mathcal{O}_X) = H^q(X, \text{Hom}_X(\wedge^j \Omega_X, \mathcal{O}_X)).
\]

Now there is a natural isomorphism of sheaves

\[
\wedge^j T_X = \wedge^j \text{Hom}_X(\Omega_X, \mathcal{O}_X) \to \text{Hom}_X(\wedge^j \Omega_X, \mathcal{O}_X),
\]

which by [11], prop. 7, p. 154 is an isomorphism. So we get \( \text{HH}^n(X) = \bigoplus_{j \geq 0} H^{n+j}(X, \wedge^j T_X) = \bigoplus_{i-j=n} H^i(X, \wedge^j T_X). \) □
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