On the existence of the “classical trajectories” of atoms in the Stern-Gerlach experiment

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Abstract: The widely accepted interpretation of the Stern-Gerlach experiment assumes the objective atomic trajectories (the “classical trajectories”) in front of the screen. Following this interpretation, we perform an ab initio analysis of the experiment and conclude that the objective trajectories do not physically exist. The alternative to our conclusion is substantially to change the model of the experiment.

1. Introduction

The Stern-Gerlach experiment is a paradigm of the quantum measurement of spin [1]. However, there is still some controversy in its physical interpretation. E.g., it is usually assumed (probably due originally to Bohr, Pauli and Mott) that the atoms in front of the screen bear the objective trajectories (the “classical trajectories”) [2, 3]. In this picture, the atomic center of mass serves as the “apparatus” for the spin measurement, due to the (classical) correlations between the spin-projection (system $S$) and the (objectively existing, semi-classical) trajectory of the atomic center of mass ($CM$).

Quantum mechanically, the state of the composite system $S + CM$ can be described by a “mixed” state $\hat{\rho}_{S+CM}$ e.g. of the following form:

$$\hat{\rho}_{CM+S} = \frac{1}{2} |\uparrow\rangle_S \langle \uparrow| \otimes |-\rangle_{CM} \langle -| + \frac{1}{2} |\downarrow\rangle_S \langle \downarrow| \otimes |+\rangle_{CM} \langle +|$$

where $|+\rangle_{CM}$ and $|-\rangle_{CM}$ are the states of the “center-of-mass” system (up and down trajectories, respectively) and $|\uparrow\rangle_S$ and $|\downarrow\rangle_S$ are the eigenstates of the spin projection along the axis of the external magnetic field.
This scenario actually assumes the passive role of the screen capturing the atoms. The screen is supposed passively to record the atomic trajectories, which objectively exist in front of the screen. However, this picture (scenario) is not the only one possible in the context of the quantum mechanical formalism. E.g., one may assume the active role of the screen, which actually assumes the entangled state of the $S + CM$ system in front of the screen of e.g. the following (simplified) form (cf. Ref. [4] for some details):

$$|\Psi\rangle_{CM+S} = \frac{1}{\sqrt{2}} (|\uparrow\rangle_S |\rightarrow\rangle_{CM} + |\downarrow\rangle_S |\rightarrow\rangle_{CM}). \quad (2)$$

The state eq. (2) elevates the active role of the screen: there are not the objectively present trajectories in front of the screen. Rather, the screen plays the active role in “collapsing” the state eq. (2) into the “reduced” state eq. (1), as described in the von Neumann’s quantum measurement theory [5].

The discrepancy between eq. (1) and eq. (2) is at the heart of the modern quantum mechanics, quantum measurement theory [5, 6, 7], the transition from quantum to classical [8, 9] as well as of the quantum information/computation theory [10]. E.g., the second scenario, as distinct from the first one, may imply some usefulness of entanglement in eq. (2) as a quantum information resource [4].

In order to solve this dilemma, we follow—as (in our opinion) the easier option—the interpretation formally described by eq. (1). Our starting point is the standard model of SG experiment [1], and we seek for a proper extension of the model that might account for the “objectively” existing trajectories. More precisely: we start an ab initio analysis of the atomic system in search for the possible physical origin of the proposed “objective trajectories”. We obtain the negative result: our conclusion is that the “classical trajectories” can not be considered to be physically realistic.

Given the $CM + S$ system is either in a mixed state eq. (1), or in an entangled state eq. (2), we conclude, that in the SG experiment, the screen is responsible for the appearance of the classical information about the atom “trajectory” and, consequently, about the spin projection. In other words: the screen unfolds the (probably irreversible) retrieval of a classical information from the quantum system.

In Section 2, we point out the necessity for the decoherence effect in providing the “classical trajectories”, and we go as much as possible in pursuing this idea in the sense of searching for the proper decoherence-based
model that could allow the “classical trajectories”. Interestingly enough, the model(s) obtained can not explain certain well-established experimental findings. Therefore, we are forced to consider the decoherence model(s) as physically unrealistic. Section 3 is discussion.

2. The decoherence model of SG experiment

The standard model of SG experiment reads (cf. e.g. [1]): a collimated beam of atoms (of the same chemical kind) traverse the external magnetic field, which should be considered as the external classical field not coupling with the atomic degrees of freedom. The dynamics generated by the strong magnetic field can be presented (approximately) by eq. (2), i.e. by the existence of the entanglement in the system $CM + S$ [1]. Observing the definite paths on the screen reveals the corresponding spin-projection of the atom.

However, bearing in mind eq. (2), the “objective trajectories” require an external action performed on the system—the action is supposed to be responsible for the appearance of the reduced, mixed state eq. (1). Needless to say, such an action should be of the quantum-measurement type, and the quantum decoherence process [2] seems not to have any alternative in this regard. In other words: the (semi-classically) objective trajectories require unfolding of the decoherence process in front of the screen; then, the screen is supposed passively to record the objectively existing trajectories. So, our task reduces to searching for a decoherence mechanism that might justify physical objectivity of the “classical trajectories”.

To this end, recently, a qualitative proposal has been made [11]. The core of this proposal reads: the SG experiment model generally discards the so-called “relative coordinates” system ($R$) from consideration, and probably this system $R$ might play the role of the (mesoscopic) internal environment for the $CM$ system, i.e. to induce the decoherence of the $CM$’s trajectories. The model stems an interesting and provoking physical picture of the internally induced decoherence [12]—which does not require any external environment. However, this is fully a qualitative proposal that does not offer a definite conclusion as to whether or not the corresponding model (the decoherence-based model) of SG experiment can actually be constructed. And this is the very issue of the present paper.

Generally, the center-of–mass ($CM$) system is defined by the canonical transformations of the position-variables $\hat{\vec{r}}_i$ of a system consisting of $K$ par-
ticles by:

\[ \hat{R}_{CM} = \sum_{i=1}^{K} m_i \hat{r}_i / \sum_{i=1}^{K} m_i. \]  

(3)

However, simultaneously and unavoidably are defined the “relative coordinates” (that formally define the “relative system” \( R \)) e.g. as:

\[ \hat{\rho}_{Ra} = \hat{r}_i - \hat{r}_j, \alpha = 1, 2, ..., K - 1. \]  

(4)

The system \( R \) is generally discarded from the standard model of SG experiment. And at this point appears the main idea of [11]: probably the system \( R \) might play the role of the environment for \( CM \) system—as the missing link to the “classical trajectories”.

Within the standard (and generally used) assumption that the SG magnet is not a dynamical system but the source of the external magnetic field for the atoms, it seems that the following operational models do not have any alternative. In other words: the following models seem to exhaust the models that might fit with the objective existence of the ”classical trajectories”.

2.1 The atomic center of mass

An atom is a collection of electrons (\( E \)), protons (\( P \)) and neutrons (\( N \)). Applying the transformations eqs. (3) and (4) to the composite system \( E + P + N \) introduces the center-of-mass and the “relative particles” system for the whole atom.

The ”atom” can be sufficiently-well defined by the following Hamiltonian:

\[ \hat{H} = Z \sum_{i=1}^{Z} \hat{T}_{Ei} + Z \sum_{j=1}^{Z} \hat{T}_{pj} + (A-Z) \sum_{k=1}^{A-Z} \hat{T}_{nk} + \hat{V}_{\text{Coul}}^{ee} + \hat{V}_{\text{Coul}}^{ep} + \hat{V}_{\text{Coul}}^{pp} + \hat{V}_{\text{nuc}} \]  

(5)

where \( \hat{T} \) stands for the kinetic terms, \( \hat{V}_{\text{Coul}} \) for the Coulomb interaction of the pairs of particles (\( ee \)-the electrons, \( ep \)-the electron-proton, \( pp \)-the protons pairs), and the nucleon interaction for a pair \( (n, n') \) of nucleons is given e.g. by [13]:

\[ \hat{V}_{nuc}^{nn'} \equiv -\gamma^2 \exp\left(-\mu |\hat{r}_n - \hat{r}_{n'}| \right) \]  

\[ |\hat{r}_n - \hat{r}_{n'}| \]  

(6)

where \( \gamma \) is a constant and \( r=\frac{1}{\mu} \) is the range of the nuclear forces. For simplicity, we omit the comparatively weak interactions, such as the spin-spin or spin-orbit interactions in the atom.
As apparent from eq. (5), the canonical transformations eqs. (3) and (4) give for the atomic Hamiltonian:

$$\hat{H} = \hat{T}_{CM} + \hat{H}_R + \hat{H}_{CM+S}.$$

(7)

where $\hat{H}_{CM+S} = \mu_B B(\hat{z}_{CM}) \otimes \hat{S}_z$ is the standard term [1, 11] coupling the center of mass ($\hat{z}_{CM}$) and the atomic spin ($\hat{S}_z$), while the $R$-system’s self-Hamiltonian reads:

$$\hat{H}_R = \sum_{a=1}^{Z+A-1} \hat{T}_{Ra} + \hat{V}_{\text{nucl}}^{(R)} + \hat{V}_{\text{Coul}}^{(R)} + \hat{M}_{\eta\nu}^{(R)},$$

(8)

where $Z$, $A$ are the atomic and the mass numbers, respectively, and $\hat{M}_{\eta\nu}^{(R)}$ is the internal interaction in $R$ [14].

Regarding eq. (7), it is important to note: being the distance-dependent, all the original interactions (the Coulomb interaction and the nuclear interaction in eq. (5)) transform into the “external fields” (the one-particle potentials $V(\hat{\rho}_R)$) for the “relative particles” system $R$. These effective potentials are the terms of the $R$’s self-Hamiltonian $\hat{H}_R$. Needless to say, this gives the exact separation of (non-interaction between) $CM$ and $R$ that does not leave a room for the desired decoherence of the $CM$ states [2, 15].

In the terms of the quantum states, the initial state, e.g.,

$$\frac{1}{\sqrt{2}}(|\uparrow_s + |\downarrow_s)|\Psi\rangle_{CM}|0\rangle_R$$

(9)

dynamically transforms as presented by the following simplified expression:

$$\hat{U} \frac{1}{\sqrt{2}}(|\uparrow_s + |\downarrow_s)|\Psi\rangle_{CM}|0\rangle_R = \frac{1}{\sqrt{2}}(|\uparrow_s\rangle_{CM} + |\downarrow_s\rangle_{CM})|0\rangle_R,$$

(10)

where $\hat{U}$ is generated by $\hat{H}$ eq. (7). After “tracing out” the environment $R$, one obtains the entangled state eq. (2)–there are not the “classical trajectories”, which require decoherence, i.e. the interaction in the $CM + R$ system.

However, in order to make our search for the desired interaction complete, we move a step further as presented in the next section.

2.2 The atomic-nucleus center of mass
More than 99.99 per-cents of the atomic mass is placed in the atomic nucleus. Practically, it is truly hard to distinguish between the atomic and the nucleus center-of-mass systems. So, we investigate another application of eqs. (3), (4): we introduce the center-of-mass system and the “relative system” for the atomic nucleus while leaving the electrons variables intact.

Introducing the collective degrees of freedom of the atomic nucleus is the standard procedure in nuclear physics [16]. On the other side, the similar idea appears in certain models of the quantum measurement theory, unfortunately not yet being fully elaborated [17]. So, introducing the center of mass of the atomic nucleus not yet involving the electrons is physically legitimate a procedure.

Then, “atom” is a composite system defined as \( E + CM + R + S \), where \( E \) stands for the electrons-system, \( CM \) and \( R \) for the nucleus center-of-mass and the “relative” systems, respectively, while \( S \) is the atomic spin.

Now, the standard model of the SG experiment is defined by the following form of the atomic Hamiltonian (in analogy with eq. (7)):

\[
\hat{H} = \hat{H}_E + \hat{T}_{CM} + \hat{H}_R + \hat{H}_{CM+S} + \hat{H}_{E+CM+R}.
\] (11)

Certainly, the Hamiltonian \( \hat{H} \) in eq. (11) and eq. (7) is the one and the same observable— it is just written in the different forms, yet in eq. (11) appearing the interaction term for \( E, CM \) and \( R \) systems:

\[
\hat{H}_{E+CM+R} = k \sum_{i=1}^{Z} \sum_{j=1}^{Z} \frac{1}{|\hat{r}_{Ei} - \tilde{R}_{CM} - \sum_{\alpha=1}^{A-1} \omega_{\alpha}^{(j)} \hat{\rho}_{R\alpha}^{(j)}|},
\] (12)

where \( \tilde{R}_{CM} + \sum_{\alpha=1}^{A-1} \omega_{\alpha}^{(j)} \hat{\rho}_{R\alpha}^{(j)} = \hat{r}_{pj} \), and \( \hat{r}_{pj} \) represents the \( j \)-th proton position. So, the tripartite interaction \( \hat{H}_{E+CM+R} \) is a particular form of the Coulomb interaction between the atomic electrons and the protons. Interestingly enough, this tripartite interaction can be reduced to a bipartite interaction coupling \( CM \) and \( R \) systems as follows.

The close inspection of the rhs of eq. (11) justifies the application of the adiabatic approximation that in its zeroth order separates the electrons system from the rest. More precisely (cf. Appendix 1): the electrons are too light relative to both the \( CM \)- and \( R \)-mass, thus allowing the standard procedure of the adiabatic approximation [18, 19, 20]. On the other side,
for the realistic atoms (not too large $Z$), the $CM$ and $R$ mass-ratio does not allow the application of the adiabatic approximation. So, we expect the approximate separation of the electrons-state from the rest, $CM + R + S$, as well as non-negligible entanglement between $CM$ and $R$. Formally, the state now reads:

$$|\chi_E\rangle \Phi_{CM+R+S} + |O(\kappa)\rangle_{E+CM+R+S},$$

where the small term (that bears entanglement, in general, of all of the subsystems) is of the norm $\kappa^{3/4}$, where $\kappa = \max\{\kappa_1, \kappa_2\}$, and $\kappa_i$ are the corresponding mass ratios, cf. Appendix 1.

In order to obtain the dynamics of the “slow” system $CM + R$ (i.e. of $CM + R + S$), one should discard the electrons system as (cf. Appendix 1):

$$\hat{H}_{CM+R+S} \equiv E \langle \chi | \hat{H} | \chi \rangle_E \approx \hat{T}_{CM} + \hat{H}_R + \hat{H}_{CM+S} + \hat{H}_{CM+R},$$

where

$$\hat{H}_{CM+R} \equiv E \langle \chi | \hat{H}_{E+CM+R} | \chi \rangle_E$$

represents the effective (the electrons–system mediated) interaction between $CM$ and $R$.

Now, due to the two interaction terms, $\hat{H}_{CM+S}$ and $\hat{H}_{CM+R}$ in eq. (14), it is straightforward dynamically to obtain entanglement in the dominant term of the state in eq. (13), $|\Phi\rangle_{CM+R+S}$. Actually, for the initial state eq. (9) and in analogy with eq. (10) one obtains:

$$\hat{U} \frac{1}{\sqrt{2}} (|\uparrow\rangle_S + |\downarrow\rangle_S) |\psi\rangle_{CM} |0\rangle_R \approx \frac{1}{\sqrt{2}} (|\uparrow\rangle_S - \langle CM | 1\rangle_R + |\downarrow\rangle_S + \langle CM | 2\rangle_R)$$

where $\hat{U}$ is generated by $\hat{H}$ represented in eq. (14). Now, assuming the orthogonality $R\langle 1|2\rangle_R \approx 0$, by tracing out the “environment” $R$ from the rhs of eq. (16) follows the mixed state eq. (1) for $CM + S$ system, as desired.

The interaction $\hat{H}_{CM+R}$ is analyzed in detail in [21] and is briefly presented in Appendix 2. This interaction provides the “minimal uncertainty states” as the good (approximate) pointer basis—in agreement with the standard model of SG experiment [1]—and for the larger atoms ($Z \sim 10$), the interaction scales approximately as $Z^2$. 

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2.3 Inconsistency of the decoherence-based model with certain experiments

The model of Section 2.2 bears certain straightforward consequences. Here, we give only those of importance for our conclusion; for more detailed discussion see Ref. [21].

First, as obvious from eq. (12), without $R$, there is not the basis for the decoherence effect. So, for the particles not bearing $R$ (e.g. electrons, neutrons, protons), likewise for the hydrogen atom, for which $A = 1$ and $R$ simply does not exist, one can not expect the occurrence of decoherence. Second, existence of $R$ does not suffice for the occurrence of decoherence. According to eq. (12), the presence of the electrons is necessary in order to obtain the effective (the electrons-mediated) interaction $\hat{H}_{CM+R}$; otherwise, the model reduces to the scenario of Section 2.1. So, the model proposes nonappearance of the SG effect also for the bare atomic nuclei.

While certain predictions of the model of Section 2.2 fit with some experimental findings, there is a couple of the experimental observations/results being yet in some inconsistency (and probably in contradiction) with the model.

First, the model of Section 2.2 predicts the absence of the SG effect for the hydrogen atom— in contradiction with the well-known experiments performed first by Phipps and Taylor [22]. Second, this model does not allow a room for explanation of certain atomic interference experiments [23, 24].

As to the later, it is worth stressing: the existence of the environment $R$ i.e. of the decoherence of the center-of-mass trajectories, makes some interference-procedures for the atoms impossible. Actually, in the notation of Section 2.2, the standard equality of “quantum erasure” reads [24]:

$$2^{-1/2}(| \uparrow \rangle_S | - \rangle_{CM} + | \downarrow \rangle_S | + \rangle_{CM}) = 2^{-1}(| \rightarrow \rangle_S (| - \rangle_{CM} + | + \rangle_{CM}) + | \leftarrow \rangle_S (| - \rangle_{CM} - | + \rangle_{CM})$$

where appear the (spin $x$-projection) $\hat{S}_x$ eigenstates on the rhs of eq. (17), $| \rightarrow \rangle_S$ and $| \leftarrow \rangle_S$. Of course, the measurement of $\hat{S}_x$ can give the value +1 with the probability 1/2 and with the final state:

$$2^{-1/2} | \rightarrow \rangle_S (| - \rangle_{CM} + | + \rangle_{CM})$$

providing the reunion (interference) of the initial coherence of the different trajectories— as experimentally verified [23, 24].
The presence of the environment $R$ does not allow the trajectories-reunion by the quantum measurement of $\hat{S}_x$. Actually, the inclusion of $R$ gives the rhs of eq. (10) and the measurement of $\hat{S}_x$ gives rise to the final state:

$$2^{-1/2} \rightarrow \left< S \left| - \right>_{CM} \langle - | 1 \rangle_R + \left| + \right>_{CM} \langle + | 2 \rangle_R \right)$$

i.e. to entanglement in the $CM + R$ system. Effectively, the $CM$ system is in the mixed state $1/2(\left| - \right>_{CM} \langle - | + \rangle_{CM} \left| + \right>_{CM})$, instead of the coherent state eq. (18).

### 3. Discussion

If the magnetic field is not a dynamical system and the “center-of-mass” should bring the information about the atomic spin, then we do not see any alternative to our conclusion that the “classical trajectories”-based interpretation of SG experiment should be refuted.

So, it is of interest to answer the following question: which assumptions about the experiment could question our conclusion. The following list in this regard is of interest. Actually, (i) one may assume that the decoherence process is not of interest, e.g. that there exists an alternative to the decoherence process in providing the “classical trajectories”. On the other side, if decoherence can not be circumvented, then one may (ii) assume that the magnetic field plays a role of the environment, or (iii) that some external, not yet recognized environment is effective, or (iv) that another internal environment should be recognized. One may also speculate (v) that the center-of-mass is not of interest (e.g. the screen monitors another collective observable of the atom). In answer to these remarks, respectively, we want to emphasize: (i) to the best of our knowledge, the decoherence effect is currently the only candidate for providing the (approximately) classical behaviour of a genuinely (yet open) quantum system, i.e. to “produce” the “classical trajectories” [2]. Bearing this in mind, (ii) considering the magnetic field as a dynamical system in the SGE-like situations could be in contradiction e.g. with the neutron interferometry experiments [25]. The points (iii)-(v) seem virtually intractable to us as requiring a substantial reconsideration/modeling of SG experiment from the very beginning. So, it is fair to say, that, as yet, we do not see any reasonable alternative to our conclusion on nonexistence of the “classical trajectories” in the Stern-Gerlach experiment.

As long as we adopt the standard model of SG experiment, that assumes the magnet (i.e. the magnetic field) not to represent a dynamical system,
we consider our analysis to be complete and therefore conclusive. Our logic is as follows. Assuming that there is not any external environment in SG experiment, we are forced to look for another environment among the internal degrees of freedom. Of course, of interest is the center-of-mass system, and the application of the canonical transformations eqs. (3), (4) seem essentially to be without alternative. Certainly, there exist the alternatives to the definition eq. (4) of the “relative coordinates” and therefore the different formal definitions of the “relative system” \( R \). Nevertheless, and this is the point, existence of the (no matter how formally defined) system \( R \) is unavoidable. So, the system \( R \) is the only candidate for playing the role of the CM’s (internal) environment. Now, due to nonexistence of the interaction between the atomic subsystems \( CM \) and \( R \), there does not seem to appear any alternative to the model described in Section 2.2. Finally, as described in Section 2.3, the model fails to describe certain well-established experimental findings. E.g., as long as the trajectories reunion is performed by measuring the proper observable of \( S \), and not of a composite system \((S + R)\) observable, the effect eq. (18) can not in principle be obtained.

While the different formal definitions of the “relative coordinates” i.e. of the system \( R \) are possible, the variations in this regard seem nothing to change in our conclusion: by discarding the decoherence effect as a physical basis of the "classical trajectories", we promote the screen as the “quantum apparatus” responsible for acquiring a classical information from the atoms impinging on the screen in SG experiment.

Of course, this does not mean that the decoherence model of Section 2.2 is formally wrong, or that it generally gives the wrong predictions. The model is derived from the first principles and nicely reproduces certain well-known experimental findings (cf. Ref. [21] for some details). The model is just in inconsistency with certain experimental findings as emphasized above—that is the reason we are forced to consider the model not to be physically realistic. The physical reasons for this might be [21]: (a) that the screen observes the atomic- not yet the nucleus- center-of-mass–in agreement with the model of Section 2.1), or (b) that the (formally possible) decoherence due to \( \hat{H}_{CM+R} \) eq. (15) is not physically efficient due to the fact that \( R \) is a small environment. In any case, we conclude that the "classical trajectories"-based interpretation of the Stern-Gerlach experiment should be refuted—which is our conclusion.

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Appendix 1

Let us focus on the atomic nucleus with the simplification of the equal masses of the protons and the neutrons, $m$. Then, eqs. (3), (4) define the total (the $CM$ system) mass $M = Am$ and the “relative mass” $\mu$ (for all the “relative particles” enumerated by $\alpha$ in eq. (4)) as:

$$\mu = (1 - A^{-1})m.$$  

(20)

The kinetic terms for the electrons, the $CM$ system and the $R$ system, as implicit in eq. (11) read, respectively, as follows:

$$\hat{T}_E = \frac{\hat{P}_E^2}{2m_E}, \quad \hat{T}_{CM} = \frac{\hat{P}_{CM}^2}{2M}, \quad \hat{T}_{R\alpha} = \frac{\hat{P}_{R\alpha}^2}{2\mu_\alpha}, \forall \alpha = 1, 2, ..., K - 1.$$  

(21)

With eqs. (20), (21) in mind, there appear the three parameters,

$$\kappa_1 \equiv \frac{m_e}{M}, \quad \kappa_2 \equiv \frac{m_e}{\mu}, \quad \kappa_3 \equiv \frac{\mu}{M},$$  

(22)

that allow the standard adiabatic-approximation considerations [18, 19, 20]. For the realistic atoms, $Z \lesssim 10^2$, one may state the following estimates:

$$\kappa_1 \lesssim 10^{-4}, \quad \kappa_2 \lesssim 10^{-3}, \quad \kappa_3 \lesssim 10^{-2}.$$  

(23)

Then the values eq. (23) justify the applicability of the adiabatic approximation [2, 13, 14] for $E + CM + R$ as follows: the small values of $\kappa_{1,2}$ justify
the adiabatic cut of the electronic system \((E)\) from both \(CM\) and \(R\) systems, while \(CM\) and \(R\) can not be properly mutually separated.

Now, the standard adiabatic approximation stems \([18, 19, 20]\): (a) the exact state of \(E + CM + R + S\) system reads

\[
|\chi\rangle_E|\Phi\rangle_{CM+R+S} + |O(\kappa)\rangle_{E+CM+R+S},
\]

where \(\kappa = \max\{\kappa_1, \kappa_2\}\), while (b) the “slow” system \(CM + R + S\) is described by the following effective Hamiltonian

\[
\hat{H}_{CM+R+S} \simeq E \langle \chi | \hat{H} | \chi \rangle_E.
\]

**Appendix 2**

For \(|\chi\rangle_E\) in eq. (15), we take the \(Z\)-electrons Slater determinant constructed from the hydrogen-atom states. This simplification ease our task yet without introducing a significant quantitative error. Then, formally, our task reduces to calculating the following expression \([21]\):

\[
\hat{H}_{CM+R} = kZ \sum_{i=1}^{Z} \int \frac{|\phi_{i}(\hat{\xi})|^2}{|\hat{\xi} - \hat{\xi}_{CM+R}|} d^3\hat{\xi}.
\]

where \(\hat{\Omega}_{CM+R} \equiv -\hat{r}_{CM}\hat{I}_E + \hat{R}_{CM} + \sum_{a=1}^{A-1} \omega_{a}\hat{\rho}_{Ra}\). Taking the point-like nucleus gives rise to the shift \(\hat{r}_{Ei} \rightarrow \hat{\xi}_{Ei} = \hat{r}_{Ei} - \hat{r}_{CM}\hat{I}_E\) as explicit in eq. (26).

The details of calculating the rhs of eq. (26) are given in Ref. [21], and the result for the atoms with the “closed shells” reads:

\[
\hat{H}_{CM+R} = kZ \sum_{n} \sum_{\ell=0}^{n-1} \sum_{g=0}^{n-\ell-1} \sum_{t=0}^{2g} \frac{2\ell + 1}{2n2^{2(n-\ell-1)}} \left(\frac{2(n - \ell - 1) - 2g}{n - \ell - 1 - g}\right) \times
\]

\[
\frac{(2g)!}{g!(2\ell + 1 + g)!} \left(\frac{2g + 2(2\ell + 1)}{2g - t}\right) \frac{(-2)^t}{t!} \left\{(2\ell + t + 2)! \left(1 - \exp\left(-\frac{2Z\hat{\Omega}}{na_{\mu}}\right) \sum_{f=0}^{2\ell+\ell+2} \frac{(2Z\hat{\Omega})^f}{f!}\right) \hat{\Omega}^{-1}
\right.
\]

\[
\left. + \frac{2Z}{na_{\mu}}(2\ell + t + 1)! \exp\left(-\frac{2Z\hat{\Omega}}{na_{\mu}}\right) \sum_{f=0}^{2\ell+\ell+1} \frac{(2Z\hat{\Omega})^f}{f!}\right\}. \tag{27}
\]
The notation is as follows: the big brackets indicate, as usual, the binomial coefficients and the sign "!" stands for the factorial.

For the atoms for which $Z \sim 10$, one can simplify eq. (27) and to estimate that the interaction scales as $Z^2$. Finally, as it can be easily shown, the minimal-uncertainty states (the “coherent states”) appear as the approximate “pointer basis” [2, 15] for the model eq. (27)–in accordance with the standard model of SG experiment [1].