Relations in the tautological ring of $\mathcal{M}_g$

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Abstract

Using a simple geometric argument, we obtain an infinite family of nontrivial relations in the tautological ring of $\mathcal{M}_g$ (and in fact of $\mathcal{M}_{g,2}$). One immediate consequence of these relations is that the classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate the tautological ring of $\mathcal{M}_g$, which has been conjectured by Faber in [F], and recently proven at the level of cohomology by Morita in [Mo].

Throughout this paper we will assume $g \geq 2$. Let $\mathcal{M}_g$ be the moduli space of nonsingular genus $g$ curves, $p: C_g \to \mathcal{M}_g$ its universal curve and let $\psi \in A^1(\mathcal{M}_g)$ be the first chern class of the relative dualizing sheaf $\omega_\pi$. Let $\kappa_a = \pi_*(\psi^{a+1}) \in A^a(\mathcal{M}_g)$. The tautological ring $R^*(\mathcal{M}_g) \subset A^*(\mathcal{M}_g)$ is the sub-ring of the Chow ring (taken with $\mathbb{Q}$ coefficients) generated by the $\kappa$ classes.

Mumford proved in [Mu] that the classes $\kappa_1, \ldots, \kappa_{g-2}$ generate the tautological ring $R^*(\mathcal{M}_g)$. The purpose of this paper is to show how to use similar methods to obtain other relations in the tautological ring $R^*(\mathcal{M}_g)$. As a consequence, Theorem 1.5 implies that the classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate the tautological ring $R^*(\mathcal{M}_g)$, a fact conjectured by Faber in [F].

Morita [Mo] recently used completely different methods to obtain polynomial relations among $\kappa$ classes in cohomology, which also prove Faber’s conjecture in cohomology. Morita’s methods are based on symplectic representation theory and don’t seem to extend to the Chow ring. In particular, he uses a “crushing” map to induce relations from a higher genus to a lower genus. Such crushing map is very natural in cohomology, but it does not have a correspondent in the Chow ring.

By contrast, our Theorem 1.1 uses a simple idea to obtain relations in the tautological ring $R^*(\mathcal{M}_g)$. Let $w_{d,g} \in A^*(\mathcal{M}_g)$ be the class corresponding to the moduli space of curves which can be written as degree $d$ covers of $\mathbb{P}^1$. Then for each $b \geq 0$, $\kappa_{b-1} w_{d,g}$ can be computed by first adding a marked point to the domain and then marking all the other $d-1$ points in the same fiber, while $w_{d,g}$ can be also computed by first marking one of the $r$ ramification points of the cover and then marking the remaining $d-2$ points in the same fiber. Porteous formula expresses each of these cycles in terms of tautological classes on $\mathcal{M}_g$, giving a relation in the Chow ring for each degree $d \geq 2$ and each $b \geq 0$.

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The geometric relations from Theorem 1.1 have a simple generating function expression. There is no reason, a priori, why all these relations would be given by one generating function, and especially one in the form of an exponential of a linear combination of \( \kappa \) classes. But in Theorem 1.4 we prove that that there are some coefficients \( c_{k,j} \) such that when \( b = 0 \) the geometric relation in \( R^{g+1-2d}(M_g) \) simply becomes

\[
\left[ \exp \left( - \sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j \right) \right]_{x^{g+1-2d}u^d} = 0
\]

Here, and throughout this paper, the notation \([f(t)]_{t^k}\) means the coefficient of \( t^k \) in a formal power series \( f(t) \).

Furthermore, for \( b \geq 0 \) the geometric relations become

\[
\left[ \exp \left( - \sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j \right) \cdot \left( \kappa_{b-1} - 2 \sum_{a=0}^{\infty} \kappa_{a+b} x^{a+1} \sum_{j=0}^{a} q_{a,j} u^{j+1} \right) \right]_{x^{g+2-2d}u^d} = 0 \quad (0.1)
\]

where the coefficients \( q_{k,j} \) are positive integers related to the coefficients \( c_{k,j} \) by the formula (1.8). Theorem 1.5 (part of Faber’s Conjecture) follows then as an easy consequence.

The paper is organized as follows. We begin the first section by proving the geometric relations mentioned above (Theorem 1.1). We next state the main Theorem 1.4 and discuss its consequences including Faber’s Conjecture (Theorem 1.5). The proof of Theorem 1.4 occupies the remaining sections of the paper. First, in Section 2 we express the geometric relations from Theorem 1.4 in terms of a universal generating function \( G \) which is the solution of a simple ODE (Theorem 2.2). Then in Section 3 we study the properties of the generating function \( G \), including a certain power series expansion for both \( G \) and its derivative. These are used in Theorem 1.4 to prove that the relations of Theorem 1.4 simplify to the form (0.1). We conclude Section 3 with the proof of Proposition 1.7 which gives another simple generating function for some of the above relations.

In fact, as explained in the Appendix, the relations discussed in this paper come from relations in the tautological ring of \( \mathcal{C}_g^2 \) (which in turn are restrictions of relations in the Chow ring \( A^*(\mathcal{M}_{g,2}) \)). Such relations are obtained by following the approach of [1]. Consider the moduli space of degree \( d \), genus \( g \) covers of \( \mathbb{P}^1 \) which have a fixed ramification pattern over four fixed points in \( \mathbb{P}^1 \). Just as in [1], one can split the target \( \mathbb{P}^1 \) into two rational components, each having two of the four fixed points. There are several ways to split, depending which points are together on the same bubble. When the ramification patterns are appropriately chosen, one obtains then relations in the Chow ring of the Deligne-Mumford moduli space.

In particular, the relation (0.1) is obtained from the relation in \( A^*(\mathcal{C}_g) \)

\[
\left[ \exp \left( - \sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j \right) \cdot \left( 1 - 2 \sum_{a=0}^{\infty} \psi_1^{a+1} x^{a+1} \sum_{j=0}^{a} q_{a,j} u^{j+1} \right) \right]_{x^{g+2-2d}u^d} = 0 \quad (0.2)
\]

after multiplying by \( \psi_1^b \) and pushing forward to \( \mathcal{M}_g \). Note that the \( \kappa \) classes in (0.2) are those pulled back from \( \mathcal{M}_g \). Relation (0.2) gives a nontrivial polynomial expression in \( \psi_1 \) and \( \kappa \) classes.
in degree $a = g + 2 - 2d$ for each $d \leq (g + 2)/3$. The coefficient of $\psi_1^a$ is $-2q_{a-1,d-1}$, so nonzero. As far as we know, this is the first time this type of polynomial relation appears in such low degree (compared to the genus).

1 Geometric relations

In this section we show how the simple geometric idea outlined in the introduction produces several families of nontrivial relations in the tautologic ring $R^*(M_g)$. In fact, as explained in the Appendix, these relations are coming from relations in $R^*(C_g)$, which in turn are restrictions of relations in the Chow ring $A^*(C_g)$.

For each $d \geq 2$, let $C_g^d$ be the $d$-fold fiber product of $C_g$ over $M_g$, and let $p : C_g^d \to M_g$ be the forgetful map. Denote by $D_{ij}$ the diagonal $x_i = x_j$, and let $\psi_j \in A^1(C_g^d)$ be the pullback of $\psi$ from $C_g$ along the map that forgets all the marked points except $x_j$. (We found it more convenient for this paper to use this pullback class rather then the customary first chern class of the relative cotangent bundle at the marked point $x_j$.) Consider

$$y_{d,g} = \{ (C, x_1, \ldots, x_d) \in C_g^d \mid h^0(O_C(x_1 + \ldots + x_d)) \geq 2 \},$$

the locus corresponding to degree $d$, genus $g$ covers of a non-rigid $\mathbb{P}^1$ which have all the points in a fiber marked.

**Theorem 1.1** For each $d \geq 2$, $g \geq 2$ and $b \geq 0$ we have the following relation in the tautological ring of $M_g$ in dimension $a = g + b + 1 - 2d$:

$$(2d + \kappa_0)p_*(y_{d,g}\psi_1^b) = (d - 1)\kappa_{b-1} \cdot p_*(D_{12} \cdot y_{d,g})$$

Furthermore, the class $y_{d,g}$ is given

$$y_{d,g} = \left[ \frac{c_t(E^*)}{c_t(F_d)} \right]_{g+1-d}$$

where $c_t(E^*) = c_t(E)^{-1}$ is the total chern class of the dual of the Hodge bundle $E$, and

$$c_t(F_d) = (1 - t\psi_1) \cdot (1 - t\psi_2 + tD_{2,1}) \cdot \ldots \cdot (1 - t\psi_d + tD_{d,1} + \ldots + tD_{d,d-1})$$

**Proof.** Denote by $W_{d,g}$ the moduli space of genus $g$, degree $d$ nonsingular covers of a nonrigid $\mathbb{P}^1$ (in other words, we also mod out by automorphisms of the target, not only of the domain). Its pushforward to $M_g$ defines a class $w_{d,g}$ in the Chow ring $A^{g+2-2d}(M_g)$. For each $b \geq 0$, $\kappa_{b-1} w_{d,g}$ can be computed by first adding a marked point to the domain and then marking all the other $d - 1$ points in the same fiber. Therefore

$$\frac{1}{(d-1)!} \cdot p_*(y_{d,g}\psi_1^b) = \kappa_{b-1} w_{d,g}$$
On the other hand, $w_{d,g}$ can be also computed by first marking one of the $r = 2d + \kappa_0$ ramification points of the cover and then marking the remaining $d - 2$ points in the same fiber.

$$\frac{1}{(d-2)!} p_* (y_{d,g} \cdot D_{12}) = (2d + \kappa_0) w_{d,g}$$

Combining the previous two displayed equations immediately gives (1.2).

The expression (1.3) follows by Porteous formula as explained in [F] (see also Section 7 of [Mu]). Let $F_d = p_* (\mathcal{O}_{\mathcal{C}_{d+1}^g}(\Delta_{d+1})/\mathcal{O}_{\mathcal{C}_{d+1}^g})$ be the jet bundle at $d$ points, and $E^* = R^1 p_* \mathcal{O}_{\mathcal{C}_{d+1}^g}$ be the (pullback) of the dual of the Hodge bundle. Now look at the sequence

$$0 \to \mathcal{O}_{\mathcal{C}_{d+1}^g} \to \mathcal{O}_{\mathcal{C}_{d+1}^g}(\Delta_{d+1}) \to \mathcal{O}_{\mathcal{C}_{d+1}^g}(\Delta_{d+1})/\mathcal{O}_{\mathcal{C}_{d+1}^g} \to 0$$

which gives us the sequence

$$0 \to F_d \to E^* \to R^1 p_* (\mathcal{O}_{\mathcal{C}_{d+1}^g}(\Delta_{d+1})) \to 0$$

The first sheaf is locally free of rank $d$, while the second is locally free of rank $g$. Since

$$y_{d,g} = \{ (C,D) \mid h^0(C,D) \geq 2 \} = \{ (C,D) \mid h^1(C,D) \geq g - d + 1 \}$$

then $y_{d,g}$ is exactly the locus where the rank of $\alpha$ drops by one, so $y_{d,g}$ is the Chern class $c_{g+1-d}$ of the virtual bundle $E^* - F_d$, giving (1.3).

Furthermore, by the natural filtration of $F_d$,

$$c(F_d) = c(F_{d-1})(1 - \psi_d + D_{d,1} + \ldots + D_{d,d-1})$$

since the first chern class of the relative dualizing sheaf of $\mathcal{O}_{d+1}^g \to \mathcal{O}_{d-1}^g$ is $\psi_d - D_{d,1} - \ldots - D_{d,d-1}$. This gives (1.4).

**Remark 1.2** For each $b \geq 0$, relation (1.2) gives a (homogeneous) polynomial relation among $\kappa$ classes, whose structure we will study in this paper. For $b = 0, 1$ the relation is respectively

$$p_* y_{d,g} = 0$$

(1.5)

$$(2d + \kappa_0) \cdot p_*(\psi_1 \cdot y_{d,g}) = (d - 1) \cdot \kappa_0 \cdot p_*(D_{12} \cdot y_{d,g})$$

(1.6)

The relation (1.6) was studied in [R] for low degree $d = 2, 3$; the relations (1.5) and (1.6) were also mentioned in [F] as potential candidates for proving Theorem 1.5. Even though it is very plausible that the relations for $b = 0$ and 1 are enough to prove Theorem 1.5, our proof involves also the relations for $b \geq 2$.

Note that equation (1.2) from Theorem 1.1 gives a polynomial relation among $\kappa$ classes in the tautological ring $R^*(\mathcal{M}_g)$. But the specific form of this polynomial looks very complicated at first glance, and it is not clear, a priori, whether this relation is ever nontrivial.

We will show that in fact, the relations are encoded in a simple form by a generating function.
Definition 1.3 Consider the positive integers $q_{k,j}$ for $k \geq j \geq 0$ (and vanishing otherwise) defined recursively by the relation

$$q_{k,j} = (2k + 4j - 2)q_{k-1,j-1} + (j + 1)q_{k-1,j} + \sum_{m=0}^{k-1} \sum_{l=0}^{j-1} q_{m,l} q_{k-1-m,j-1-l}$$

(1.7)

with initial condition $q_{0,0} = 1$. Next, define recursively the coefficients $c_{k,j}$ for $k \geq 1$ and $k \geq j \geq 0$ (and vanishing otherwise), by the relation

$$q_{k,j} = (2k + 4j)c_{k,j} + (j + 1)c_{k,j+1}$$

(1.8)

for all $k \geq 1$ and $k \geq j \geq 0$.

The coefficients $c_{k,j}$ are not necessarily integers, nor are they always positive. Notice that, for example, (1.8) gives $c_{k,k} = q_{k,k}/(6k)$, and Proposition 1.7 gives a very simple generating function for $c_{k,k}$. Furthermore, we will see that $c_{k,0} = B_{k+1}/k(k+1)$ (where $B_k$ are the Bernoulli numbers), which is not at all obvious from their definition above.

With the notations of Definition 1.3

Theorem 1.4 For each $g, d \geq 2$ and each $b \geq 0$, relation (1.2) gives the following relation in $R^{g+1+b-2d}(M_g)$

$$\left[ \exp \left( -\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j \right) \cdot \left( \kappa_{b-1} - 2 \sum_{a=0}^{\infty} \kappa_{a+b} x^a \sum_{j=0}^{a} q_{a,j} u^{j+1} \right) \right]_{x^{g+2-2d}u^d} = 0$$

(1.9)

Furthermore, for $b = 0$, this relation simplifies to

$$\left[ \exp \left( -\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{a} c_{a,j} u^j \right) \right]_{x^{g+1-2d}u^d} = 0$$

(1.10)

The proof of this theorem essentially occupies the rest of the paper.

An immediate consequence of Theorem 1.4 is the following

Theorem 1.5 (Faber’s Conjecture) The $\lfloor g/3 \rfloor$ classes $\kappa_1, \ldots, \kappa_{\lfloor g/3 \rfloor}$ generate the tautological ring $R^*(M_g)$, with no relations in degrees less or equal then $\lfloor g/3 \rfloor$.

Proof. The fact that there are no relations in degree less or equal to $\lfloor g/3 \rfloor$ follows from Harer’s Stability result [H], as pointed out in [F]. So it is enough to show that for each $a \geq \lfloor g/3 \rfloor + 1$, Theorem 1.3 gives us a relation

$$\kappa_a = \text{polynomial in } \kappa_1, \ldots, \kappa_{a-1}$$

(1.11)

For $b \geq 2$, relation (1.9) gives in dimension $a = g + 1 + b - 2d$

$$q_{a-b,d-1}\kappa_a + \text{polynomial in } \kappa_1, \ldots, \kappa_{a-1} = 0$$
Since \( q_{a-b,d-1} \) is a positive integer as long as \( a - b \geq d - 1 \), this gives a relation of type \((1.11)\) for each \( a = g + 1 + b - 2d \geq b + d - 1 \). Assume \( a \geq (g + 5)/3 \). Then we can find a positive integer \( d \) in the range \((g + 3 - a)/2 \leq d \leq (g + 2)/3\) and choose \( b = a + 2d - g - 1 \geq 2 \). Therefore, we get a relation of type \((1.11)\) for each \( g \geq 2 \) and \( a \geq (g + 5)/3 \).

To cover the range \((g + 4)/3 \geq a \geq [g/3] + 1\), we will use the relations for \( b = 0 \) and \( 1 \). In dimension \( a = g + 1 + b - 2d \), relation \((1.10)\) for \( b = 0 \) and \((1.9)\) for \( b = 1 \) become respectively

\[
c_{a,d}\kappa_a = \text{polynomial in } \kappa_1, \ldots, \kappa_{a-1}
\]

\[
(\kappa_0 c_{a,d} + 2q_{a-1,d-1})\kappa_a = \text{polynomial in } \kappa_1, \ldots, \kappa_{a-1}
\]

By Lemma \(3.6\) the coefficient of \( \kappa_a \) in both these relations is positive for \( a = d, d + 1 \). Since \( a = g + 1 + b - 2d \) (with \( b = 0, 1 \)) then \( a = d \) gives \( 3a = g + 1 + b \), while \( a = d + 1 \) gives \( 3a = g + 3 + b \). Together, they cover the cases \( 3a = g + 1, \ldots, g + 4 \), exactly all missing cases in the range \((g + 4)/3 \geq a \geq [g/3] + 1\). \(\Box\)

**Remark 1.6** We believe that the relations \((1.9)\) just for \( b = 0 \) and \( 1 \) would be enough to prove Theorem \ref{thm:main}. For that, one needs to prove that both \( c_{a,d} \neq 0 \) and \((2a - 4d - 6)c_{a,d} + 2q_{a-1,d-1} \neq 0 \) for all \( a \geq d \geq 1 \). Then the previous two displayed equations give a relation of type \((1.11)\) for each \( g \geq 2 \) and \( a \geq [g/3] + 1 \). (We checked using a computer that this is the case for \( a \leq 60 \), but we don’t have a proof at this moment for why this would be true in general.)

However, having relations for \( b \geq 2 \) makes the proof of Theorem \ref{thm:main} much easier. Furthermore, notice that for each \( g \geq 2 \) and \( a \geq [g/3] + 1 \) fixed, the relations \((1.9)\) obtained for various \( d \) and \( b \) (with \( a = g + 1 - 2d + b \)) are linearly independent. This is because in \((1.9)\) the coefficients of \( \kappa_1^{a-j}\kappa_j \) are zero for \( j \leq b - 2 \), and nonzero for \( j = b - 1 \) (when \( b \geq 1 \)). More generally, note that there are no monomials in only the variables \( \kappa_1, \ldots, \kappa_{b-2} \) appearing in \((1.9)\) for \( b \geq 3 \).

We conclude this section with a simple generating function for some of the relations \((1.9)\):

**Proposition 1.7** The coefficients \( c_{k,k} \) of Definition \ref{def:coefficients} are also given by the equality

\[
\exp \left( \sum_{k=1}^{\infty} c_{k,k} t^k \right) = \sum_{a=0}^{\infty} \frac{(6k)!}{(2k)!(3k)!} \left( \frac{t}{72} \right)^a
\]

Furthermore, we have the following relations in \( A^e(\mathcal{M}_g) \):

\[
\left[ \exp \left( - \sum_{j=1}^{\infty} c_{j,j} \kappa_j t^j \right) \right]_{t^a} = 0 \tag{1.12}
\]

for \( a = g/3 + 1 \) or \((g + 1)/3\), and

\[
\left[ \exp \left( - \sum_{j=1}^{\infty} c_{j,j} \kappa_j t^j \right) \left( \kappa_{b-1} t^{b-1} - 2\kappa_b t^b - 12 \sum_{j=1}^{\infty} j c_{j,j} \kappa_j + b t^j + b \right) \right]_{t^a} = 0 \tag{1.13}
\]

for \( a = (g - 1)/3 + b \) or \((g + 1)/3 + b \) when \( b \geq 1 \).
The proof of this Proposition is in the last section of this paper.

Remark 1.8 Faber mentioned in [F] that the relation (1.2) is the unique (up to a scalar multiple) relation in degree $a = (g + 1)/3$. He also claimed that there is a unique relation in degree $a = (g + 2)/3$; relation (1.3) for $b = 1$ gives then its explicit form.

2 Generating functions

It is obvious that Theorem 1.1 gives polynomial relations in $\kappa$ classes, but the specific form of these polynomials looks very complicated at first glance. We begin by showing that the relations are encoded in a simple form by a universal generating function $G$.

Definition 2.1 Let $G(x, w) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} x^k \alpha_{k,j} w^j$ be the unique formal power series in $x$ and $w$ which solves the recursive equation

$$xwG_{ww} = w(G_w)^2 + (1 - x)G_w - 1 \quad (2.1)$$

$$G(x, 0) = -\sum_{a=2}^{\infty} \frac{B_a}{a(a-1)} x^a \quad (2.2)$$

where $B_a$ are the Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{a=0}^{\infty} B_a \frac{x^a}{a!} = 1 - \frac{x}{2} + \sum_{i=1}^{\infty} B_{2i} \frac{x^{2i}}{(2i)!} \quad (2.3)$$

We next show that the relations from Theorem 1.1 take a simple form in terms of this generating function $G$.

Theorem 2.2 For each $g \geq 2$, $d \geq 2$ and $b \geq 0$, the relation (1.3) is equivalent to

$$\left[ \exp \left( \frac{1}{t} p_* G(t\psi, w) \right) \right] \cdot p_* \left( (2wG_w(t\psi, w) + 1)\psi^b \right) \bigg|_{t^{g+1 - 2d}w^d} = 0 \quad (2.4)$$

where $p : C_g \to M_g$ is the forgetful map.

Note that in terms of the power series expansion of $G$

$$\frac{1}{t} p_* G(t\psi, w) = \sum_{a=0}^{\infty} \sum_{j=0}^{\infty} t^{a-1} \kappa_{a-1} \alpha_{a,j} w^j$$

$$p_* \left( (2G_w(t\psi, w) + 1)\psi^b \right) = \kappa_{b-1} + 2 \sum_{a=0}^{\infty} \sum_{j=1}^{\infty} t^a \kappa_{a+b} \alpha_{a,j} w^j$$

where $\kappa_{-1} = 0$, $\kappa_0 = 2g - 2$. Furthermore, for $b = 0$ relation (2.4) simplifies to

$$\left[ \exp \left( \frac{1}{t} p_* G(t\psi, w) \right) \right]_{t^{g+1 - 2d}w^d} = 0 \quad (2.5)$$
2.1 Proof of Theorem 2.2

The proof of Theorem 2.2 is done is several steps. The first step is to expand (1.3). We begin with some notations. If \(1 \leq i_1 < \ldots < i_k \leq d\) is a sequence of integers, we denote by

\[D_{i_1, \ldots, i_k}\]

the (closed) stratum of \(C^d_g\) where all the points \(x_{i_l}\) are equal for \(l = 1, \ldots, k\). More generally, the strata of \(C^d_g\) are in one to one correspondence with partitions of the set \(\{x_1, \ldots, x_d\}\). Given an unordered partition \(\{J_1, \ldots, J_r\}\) of \(\{x_1, \ldots, x_d\}\) we denote by

\[\Delta_{J_1, \ldots, J_r}\]

the codimension \(d - k\) multi diagonal in \(C^d_g\) where all points in each \(J_i\) are equal. Given such a strata \(\Delta_{J_1, \ldots, J_r}\) we will denote by \(x^*_{i}\) any one of the points of \(J_i\) and by \(l_i = \ell(J_i)\) the number of points in \(J_i\).

In what follows, we will denote by \(X_d(t) = (c_t(F_d))^{-1}\) and

\[Y_d(t) = \frac{p^* c_t(E^*)}{c_t(F_d)} = p^* c_t(E^*) \cdot X_d(t)\]  

(2.6)

both thought as elements of \(A^*(C^d_g, \mathbb{Q}[t])\), where \(F_d\) and \(E\) are defined in Theorem 1.1.

**Proposition 2.3** Using the notations above,

\[
p^* c_t(E^*) = p^* \exp \left( \sum_{a=1}^{\infty} \frac{B_{a+1} \kappa_a t_a}{a(a+1)} \right) \cdot \sum_{r=0}^{\infty} \sum_{\{J_1, \ldots, J_r\}} t^{d-r} \Delta_{J_1, \ldots, J_r} \prod_{i=1}^{r} G_{\ell(J_i)}(t \psi_{si}) \]  

(2.7)

where the last sum is over all partitions \(\{J_1, \ldots, J_r\}\) of \(\{x_1, \ldots, x_d\}\), and the formal power series

\[G(x, w) = \sum_{d=0}^{\infty} G_d(x) \frac{w^d}{d!} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{kj} x^k w^j\]  

(2.8)

satisfies the relation (2.1) with initial condition (2.2).

**Proof.** Mumford proved in [Mu] that on \(\overline{M}_g\) the Chern character of the Hodge bundle is

\[ch(E_g) = g + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} \left( \kappa_{2i-1} + \frac{1}{2} \xi_i \sum_{m=0}^{2i-2} (-1)^m \varphi_1^m \varphi_2^{2i-2-m} \right)\]  

(2.9)

But then the general formula expressing the total chern class in terms of the components of the chern character

\[
\sum_{n=0}^{\infty} c_n t^n = \exp \left( \sum_{j=0}^{\infty} (-1)^{j-1}(j-1)! \cdot ch_j t^j \right) 
\]
(see for example [3]) implies that on $\mathcal{M}_g$

$$c_t(E_g^*) = \exp\left(-\sum_{i=1}^{\infty} \frac{B_{2i} \cdot \kappa_{2i-1} t^{2i-1}}{2i(2i-1)}\right)$$ \hspace{1cm} (2.10)

This accounts for the first factor in (2.7).

Next, for each $i = 1, \ldots, d$ we focus on the factor $(1 - t \psi_i + t D_{i,1} + \ldots + t D_{i,i-1})^{-1}$. Expanding in power series, this is equal to

$$\sum_{k=0}^{\infty} t^k (\psi_i - D_{i,1} - \ldots - D_{i,i-1})^k = \sum_{k=0}^{\infty} (-t)^k \binom{k}{a, a_1, \ldots, a_{i-1}} (-\psi_i)^a (D_{i,1})^{a_1} \ldots (D_{i,i-1})^{a_{i-1}}$$

But on $C_g^d D_{ij}^2 = -\psi_i D_{ij}$ so if $a > 0$ $D_{ij}^a = (-\psi_i)^{a-1} D_{ij}$ while if $1 \leq n_1 < \ldots < n_l < i$ then $D_{i,n_1} \cdot \ldots \cdot D_{i,n_l} = D_{i,n_1,\ldots,n_l}$. Thus separating $a_n > 0$ from $a_n = 0$ for $n = 1, \ldots, i-1$ in the previous displayed equation we get

$$\sum_{k=0}^{\infty} (-t)^k \sum \binom{k}{a, a_1, \ldots, a_{i-1}} (-\psi_i)^{a + \sum_{m=1}^{i} (a_{nm}-1)} D_{i,n_1,n_2,\ldots,n_l}$$

where the second sum is over all subsets $N = \{n_1, \ldots, n_l\} \subset \{1, \ldots, i-1\}$ (including the empty subset) and all $a \geq 0$, $a_{n_1}, \ldots, a_{n_l} \geq 1$. Let $F_i$ be the power series in $x$:

$$F_i(x) = \sum_{a \geq 0} \sum_{m \geq 1} \binom{k}{a, a_1, \ldots, a_l} x^{k-l} (-1)^l$$

Then

$$\frac{1}{1 - t \psi_i + t D_{i,1} + \ldots + t D_{i,i-1}} = \sum_N F_{\ell(N)}(t \psi_i) \cdot D_{i,N}^{\ell - \ell(N) - 1}$$ \hspace{1cm} (2.11)

where the sum is over all subsets $N$ of $\{1, \ldots, i-1\}$ and $\ell(N)$ is the number of elements of $N$.

Finally, by [4],

$$X_{d}(t) = \left\{ \frac{1}{1 - t \psi_1} \cdot \frac{1}{1 - t \psi_2 + t D_{2,1}} \cdot \ldots \cdot \frac{1}{1 - t \psi_d + t D_{d,1} + \ldots + t D_{d,d-1}} \right\}$$ \hspace{1cm} (2.12)

But relation (2.11), applied for each $i = 1, \ldots, d$, then gives

$$X_{d}(t) = \sum_{N} D_{1,D_{2,N_2}} \ldots D_{d,N_d} \prod_{i=1}^{d} F_{\ell(N)}(t \psi_i)$$

where the sum is over all subsets $N_k$ of $\{1, \ldots, k-1\}$ for each $k = 1, \ldots, d$. Now, by definition, $F_{d}$ is symmetric in $x_{1}, \ldots, x_{d}$, which combines with the previous relation to give an expression

$$X_{d}(t) = \sum_{r=1}^{d} \sum_{J_1,\ldots,J_r} \Delta_{J_1,\ldots,J_r} \prod_{j=1}^{r} G_{i_j}(t \psi_{x_j})$$ \hspace{1cm} (2.13)
where $J_1, \ldots, J_r$ is now a partition of $\{1, \ldots, d\}$; the function $G_l(x)$, to be determined, is a power series in $x$ which depends only on the length $l \geq 1$. Note that the codimension of $\Delta_{J_1, \ldots, J_r}$ is $d - r$, which accounts for the power of $t$.

Combining relations (2.10) and (2.13) gives (2.7).

The final step in the proof of Proposition 2.3 is to prove the recursive formula (2.1) for the generating function $G = G_0(x) + \sum_{l=1}^{\infty} G_l(x)w^l/l!$, where $G_0$ is chosen so that the initial condition (2.2) is satisfied. Let $\pi : C^d_g \to C^{d-1}_g$ be the map that forgets $x_d$. Then by the definition (2.12) of $X_d$, for any $d \geq 1$ we have the relation

$$\pi^*(X_{d-1}(t)) = X_d(t) \cdot (1 - t\psi_d + tD_{d,1} + \ldots + tD_{d,d-1})$$

(2.14)

Note that if $J'$ is a partition of $\{1, \ldots, d - 1\}$ then the pullback $\pi^*(\Delta_{J'}) = \Delta_{J', \{d\}}$. Moreover, if $J_1, \ldots, J_r$ is a partition of $\{1, \ldots, d\}$ such that $d \in J_r$ then

$$\Delta_{J_1, \ldots, J_r} \cdot D_{d,i} = \begin{cases} -\psi_{sr} \Delta_{J_1, \ldots, J_r} & \text{if } i \in J_r \\ \Delta_{J_1, \ldots, \hat{J}_m, \ldots, J_r \cup J_m} & \text{if } i \in J_m, \text{ for } m \neq r \end{cases}$$

so

$$\Delta_{J_1, \ldots, J_r} \left(1 - \psi_d + \sum_{j=1}^{d-1} D_{d,j}\right) = \Delta_{J_1, \ldots, J_r} - l_1 \Delta_{J_1, \ldots, J_r} \psi_{sr} + \sum_{m=1}^{r-1} l_m \Delta_{J_1, \ldots, \hat{J}_m, \ldots, J_r \cup J_m \cup J_r}$$

We plug this and relation (2.13) into (2.14) and collect the coefficient of the stratum $\Delta_{J_0}$ on both sides, where $J_0 = \{1, \ldots, d\}$. We get

$$\delta_{d,1} = G_d(x) - dxG_d(x) + \sum_{l_1=1}^{d-1} \left(\frac{d - 1}{l_1}\right) l_1 G_{l_1}(x) G_{d-l_1}(x)$$

(2.15)

for all $d \geq 1$, where $x = \psi_{1,1} = \psi$ and $\delta_{d,1}$ is the Kronecker symbol. Multiplying with $w^{d-1}/(d-1)!$ and summing over $d$ gives

$$1 = G_w - x(wG_w)_w + w(G_w)^2$$

which is equivalent to (2.1), thus completing the proof of Proposition 2.3.

The final step in the proof of Theorem 2.2 is to use Proposition 2.3 to express the relations (1.2) in terms of the universal generating function $G$.

Lemma 2.4 The relation (1.2) is equivalent to the following

$$\left[ \exp \left( \frac{1}{t} p_s G(t\psi, w) \right) \cdot p_s \left( (2wG_w(t\psi, w) + 1)\psi^b \right) \right]_{t \psi + 2 - 2w^d} = 0$$

(2.16)
Lemma 2.5 below: Plugging this back in (2.17) gives (2.18). The push-forward $s$ of the terms of (2.18) are given by $b$

On the other hand, relation (2.17) for

In particular,

Substituting in (2.18) and dividing by $d$

We first show that after some algebraic manipulation, (2.17) is equivalent to

On one hand, relation (2.14) implies that for $d \geq 2$

In particular,

On the other hand, relation (2.17) for $b = 1$ gives

We can then solve for $[ p_*(D_{d,1} \cdot Y_d ) ]_{t^g + 1 - d}$ out of the previous two relations

Plugging this back in (2.17) gives (2.18). The push-forwards of the terms of (2.18) are given by Lemma 2.5 below:

Substituting in (2.18) and dividing by $d!$ gives (2.19).

Lemma 2.5 In terms of the generating function $G$,

Proof. The expressions for both $Y_d(t)$ and $X_d(t)$ are given by Proposition 2.3 For $X_d(t)$, the right hand side of (2.13), after being pushed forward by $p$, depends only on the lengths $l_j$ of sets $J_j$. So

Proof. With the notations in this section, relation (2.2) becomes

We have shown that after some algebraic manipulation, (2.17) is equivalent to

We first show that after some algebraic manipulation, (2.17) is equivalent to

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Plugging this back in (2.17) gives (2.18). The push-forwards of the terms of (2.18) are given by Lemma 2.5 below:

Substituting in (2.18) and dividing by $d!$ gives (2.19).
where the last sum is over all partitions $m = (m_1, m_2, \ldots)$ of $d$ into $m_l$ sets of order $l \geq 1$, so $\sum_{l=1}^{\infty} lm_l = d$. Multiplying both sides with $w^d$ and summing over $d \geq 1$ we get

$$\sum_{d=1}^{\infty} \frac{w^d t^{-d}}{d!} p_* X_d(t) = \exp \left( \sum_{l=1}^{\infty} \frac{w^l}{l!} p_* G_l(t) \right)$$

But $Y_d(t) = p^* c_t(E^*) \cdot X_d(t)$ and (2.10) combined with (2.2) implies that

$$c_t(E^*) = \exp \left( \frac{p_* G_0(t)}{t} \right)$$

Therefore

$$\sum_{d=1}^{\infty} \frac{w^d t^{-d}}{d!} p_* Y_d(t) = \exp \left( \frac{1}{t} \sum_{l=0}^{\infty} \frac{w^l}{l!} p_* G_l(t) \right)$$

which gives (2.19).

Similarly, if we assume that $d \in J_1$ then

$$t^{-d} p_*(X_d(t) \cdot \psi_d^b) = \sum_r \sum_{l_1, \ldots, l_r \geq 1} t^{-r} \left( l_1 - 1, \ldots, l_r \right) p_*(G_{l_1}(t \psi_{\psi_1}) \cdot \psi_d^b) \prod_{j=2}^{r} p_* G_{l_j}(t)$$

Therefore after multiplying by $c_t(E^*) w^{d-1}/(d-1)!$ and summing over $d$ we similarly get

$$\sum_{d=1}^{\infty} \frac{w^{d-1} t^{-d}}{(d-1)!} p_*(\psi_d^b \cdot Y_d) = \exp \left( \frac{p_* G(t \psi, w)}{t} \right) \frac{p_*(G_w(t \psi, w) \psi_b^b)}{t}$$

completing the proof of (2.20). □

### 3 Properties of the generating function $G$

In order to prove Theorem 1.4 we need to better understand the structure of the generating function $G$ appearing in Theorem 2.2. We begin with the following

**Lemma 3.1** Equation (2.1) determines the function $G_w$ recursively as a power series in $x$:

$$G_w(x, w) = \frac{-1 + \sqrt{1 + 4w}}{2w} + \frac{x}{1 + 4w} + \sum_{k=1}^{\infty} \sum_{j=0}^{k} x^{k+1} q_{k,j} (-w)^j (1 + 4w)^{-j - \frac{k}{2} - 1}$$

where the coefficients $q_{k,j}$ are those appearing in Definition 1.3.

\[ \text{12} \]
\textbf{Proof.} Denote
\[ G_w(x, w) = \sum_{k=0}^{\infty} x^k A_k(w) \quad (3.2) \]
where \( A_k(w) \) is a function of \( w \). If we let \( x = 0 \) in the equation (2.1), we get a quadratic equation
\[ w A_0(w)^2 + A_0(w) - 1 = 0 \]
with solution \( A_0(w) = \frac{-1 \pm \sqrt{1 + 4w}}{2w} \). Since \( A_0(w) \) is a power series in \( w \) then
\[ G_w(0, w) = A_0(w) = -1 + \sqrt{1 + 4w} \quad (3.3) \]
which accounts for the first term in (3.1).

We next separate \( A_0(w) \) from \( G_w \). For that we write
\[ G_w(x, w) = A_0(w) + xZ(x, w) \quad \text{where} \quad Z(x, w) = \sum_{k=0}^{\infty} x^k A_{k+1}(w) \quad (3.4) \]
and plug in equation (2.1), using the fact that \( A_0(w) \) is a solution for \( x = 0 \) (so there will be no free term in \( x \)):
\[ x^2 w Z_w + xw A'_0 = x^2 w Z^2 + 2xw A_0 Z + (1 - x)x Z - x A_0 \]
Dividing by \( x \) and rearranging we get
\[ (w A'_0) + xw Z_w - xw Z^2 + x Z = Z(1 + 2w A_0) \]
But \( 1 + 2w A_0 = (1 + 4w)^{1/2} \) while \( (w A'_0) = (1 + 4w)^{-1/2} \) so after multiplying with \( (1 + 4w)^{-1/2} \) the equation becomes
\[ Z = (1 + 4w)^{-1} + x(1 + 4w)^{-1/2}(wZ_w - w Z^2 + Z) \quad (3.5) \]
In particular we see that \( Z(0, w) = (1 + 4w)^{-1} \) which combined with (3.4) explains the second term in (3.1).

To prove the remaining part of the expansion (3.1), we let \( Q = (1 + 4w)Z \) and make a change of variable
\[ u = \frac{-w}{1 + 4w} \quad \text{so} \quad (1 + 4w)^{-1} = 1 + 4u \quad \text{and} \quad w = \frac{-u}{1 + 4u} \quad (3.6) \]
Then \( Z = (1 + 4u)Q \) while \( (wZ)_w = -(1 + 4u)^2 \cdot (wZ)_u = (1 + 4u)^2 (uQ)_u \). Plugging into (3.5) and dividing both sides by \( 1 + 4u \) we get
\[ Q = 1 + x(1 + 4u)^{1/2} \left( (1 + 4u)(uQ)_u + uQ^2 \right) \quad (3.7) \]
It is clear then by induction on powers of \( x \) that \( Q \) has a formal power series expansion
\[ Q = \sum_{k=0}^{\infty} \sum_{j=0}^{k} x^k (1 + 4u)^{k/2} q_{k,j} u^j \quad (3.8) \]
with \( q_{k,j} \) positive integers (and \( q_{0,0} = 1 \)). Since in terms of the original variable \( w \) the function \( Q \) is given by

\[
G_w(x, w) = G_w(0, w) + x(1 + 4w)^{-1}Q(x, w)
\]

then the expansion (3.8) implies the last part of the expansion (3.1).

It only remains to show that the coefficients \( q_{k,j} \) satisfy the relation (1.7) from Definition 1.3. But

\[
(1 + 4u)(uQ)_u = \sum_{k=0}^{\infty} \sum_{j=0}^{k} x^k(1 + 4u)^{k/2} 2kq_{k,j}u^{j+1} + \sum_{k=0}^{\infty} \sum_{j=0}^{k} x^k(1 + 4u)^{k/2+1}q_{k,j}(j+1)u^j
\]

after expanding \((1 + 4u)^{-1} Q(x, u) = 1 / (1 + 4u) + \sum_{k=0}^{\infty} x^k(1 + 4u)^{k/2} \tilde{c}_{k,j} u^j + \alpha_k \). Therefore integrating

\[
(1 + 4u)^{-1} Q(x, u) = \frac{1}{1 + 4u} + \sum_{k=1}^{\infty} x^k(1 + 4u)^{k/2-1} \sum_{j=0}^{k} q_{k,j} u^j
\]

we get an expansion

\[
-G(x, u) = -G(0,u) + \frac{x}{4} \ln(1 + 4u) + \sum_{k=1}^{\infty} x^{k+1} \left( \sum_{j=0}^{k} (1 + 4u)^{k/2} \tilde{c}_{k,j} u^j + \alpha_k \right)
\]

Next we focus on the expansion for \( G \)

**Lemma 3.2** The solution \( G(x, w) \) of the equations (2.1) and (2.2) has the form:

\[
G(x, w) = G(0, w) + \frac{x}{4} \ln(1 + 4w) - \sum_{k=1}^{\infty} \sum_{j=0}^{k} x^{k+1} c_{k,j} (-w)^j (1 + 4w)^{-j-\frac{k}{2}}
\]

where the coefficients \( c_{k,j} \) are related to the coefficients \( q_{k,j} \) via the formula (1.8).

**Proof.** After the change of variable (3.6) relation (3.9) combined with the fact that \( G_w = -(1 + 4u)^2 G_u \) implies that

\[
-G_u(x, u) = -G_u(0,u) + x(1 + 4w)^{-1}Q(x, u)
\]

where \( Q \) has the expansion (3.8). Therefore integrating

\[
(1 + 4u)^{-1} Q(x, u) = \frac{1}{1 + 4u} + \sum_{k=1}^{\infty} x^k(1 + 4u)^{k/2-1} \sum_{j=0}^{k} q_{k,j} u^j
\]

we get an expansion

\[
-G(x, u) = -G(0,u) + \frac{x}{4} \ln(1 + 4u) + \sum_{k=1}^{\infty} x^{k+1} (1 + 4u)^{k/2} \sum_{j=0}^{\infty} c_{k,j} u^j
\]
The coefficients \( \alpha_k \) are determined from the initial condition (2.2). In fact, this initial condition implies that all \( \alpha_k = 0 \) (i.e. \( c_k,j = 0 \) for all \( j > k \)). We were not able to show this directly from the equations, but Lemma 3.4 provides an indirect argument based on Harer’s stability result [H]. In any event, the expansion (3.12), after changing back to the \( w \) variable becomes (3.10).

Furthermore, differentiating relation (3.12) with respect to \( u \) gives

\[
-G_u(x, u) = \frac{x}{1+4u} + \sum_{k=1}^{\infty} x^{k+1} \sum_{j=0}^{\infty} (1+4u)^{k/2-1} 2k c_{k,j} u^j + \sum_{k=1}^{\infty} x^{k+1} \sum_{j=1}^{\infty} (1+4u)^{k/2} j c_{k,j} u^{j-1}
\]

\[
= x(1+4u)^{-1} \sum_{k=1}^{\infty} x^k \sum_{j=0}^{\infty} (1+4u)^{k/2} j^2 (2k c_{k,j} + 4jc_{k,j} + (j+1)c_{k,j+1})
\]

The last equality follows by expanding the factor \( 1+4u \) in the second sum of the previously displayed equation and then rearranging the terms. Comparison with (3.11) immediately implies (3.12).

As we have seen in the proof of the previous lemmas, it is more convenient to make the following change of variables

\[
u = \frac{-w}{1+4w} \quad \text{and} \quad y = \frac{x}{\sqrt{1+4w}} = x(1+4u)^{1/2}
\]

(3.13)

**Lemma 3.3** Let \( P(x,w) \) be a formal power series in \( x \) and \( w \). Denote by \( \hat{P}(y,u) \) the formal power series in \( y,u \) obtained from \( P(x,w) \) after the change of variables (3.13). Then

\[
[P(x,w)]_{x^a w^d} = (-1)^d \left[ (1+4u)^{a+2d-2} \hat{P}(y,u) \right]_{y^a u^d}
\]

(3.14)

**Proof.** To begin with,

\[
[P(x,w)]_{x^a} = (1+4w)^{-a/2} [P(y,w)]_{y^a}
\]

But if we make the change of variable (3.13), \( du = -(1+4w)^{-2} dw \) then:

\[
(1+4w)^{-a/2} f(w)_{w^d} = \int (1+4w)^{-a/2} f(w)_{w^{d+1}} dw = (-1)^d \int (1+4w)^{(a+2d-2)/2} \hat{f}(u)_{u^{d+1}} du
\]

which gives (3.14). □

**Lemma 3.4** In terms of the expansions of Lemmas 3.1 and 3.2, after the change of variables (3.13), the relations (2.5) and (2.4) become respectively

\[
\exp \left( -\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{\infty} c_{a,j} u^j \right)_{x^{g+1-2d} u^d} = 0 \quad (3.15)
\]

\[
\exp \left( -\sum_{a=1}^{\infty} x^a \kappa_a \sum_{j=0}^{\infty} c_{a,j} u^j \right) \cdot \left( \kappa_{b-1} - 2 \sum_{a=0}^{\infty} \kappa_a + b x^{a+1} \sum_{j=0}^{a} q_{a,j} u^{j+1} \right)_{x^{g+2-2d} u^d} = 0 \quad (3.16)
\]

Furthermore, \( c_{k,j} = 0 \) for all \( j > k \).
Proof. With the expansion (3.10),

$$\frac{1}{t} p_\ast G(t\psi, w) = \kappa_0 4 \ln(1 + 4w) - \sum_{a=1}^\infty t^a \kappa_a \sum_{j=0}^\infty c_{aj} (-w)^j (1 + 4w)^{-j-a/2}$$

so after changing the variables

$$\exp \left( \frac{1}{t} p_\ast G(t\psi, w) \right) = (1 + 4w)^{-\kappa_0/4} \exp \left( \sum_{a=1}^\infty y^a \kappa_a \sum_{j=0}^\infty c_{aj} u^j \right)$$

(3.17)

Similarly, with the expansion (3.1),

$$p_\ast ((2wG_w(t\psi, w) + 1)\psi^b)$$

becomes

$$\kappa_{b-1} \sqrt{1 + 4w} - 2 \sum_{a=0}^\infty t^{a+1} \kappa_{a+b} \sum_{j=0}^a q_{aj} (-w)^{j+1} (1 + 4w)^{-j-a/2}$$

$$= (1 + 4w)^{1/2} \left( \kappa_{b-1} - 2 \sum_{a=0}^\infty t^{a+1} \kappa_{a+b} \sum_{j=0}^a q_{aj} (-w)^{j+1} (1 + 4w)^{-j-a/2} \right)$$

so after changing the variables

$$p_\ast ((2wG_w(t\psi, w) + 1)\psi^b) = (1 + 4w)^{-1/2} \left( \kappa_{b-1} - 2 \sum_{a=0}^\infty y^{a+1} \kappa_{a+b} \sum_{j=0}^a q_{aj} u^{j+1} \right)$$

(3.18)

Next we use the change of variables (3.14) in both (2.5) and (2.4). By Lemma 3.3 we pick up a factor of $(1 + 4u)^{a+2d-2}$, where $a = g + 1 - 2d$ for the first relation, and $a = g + 2 - 2d$ for the second one. But then in the first case

$$\frac{a + 2d - 2}{2} = \frac{g + 1 - 2d}{2} = \frac{\kappa_0}{4}$$

while for the second $(a + 2d - 2)/2 = \frac{\kappa_0}{4} + 1/2$. This means that the powers of $(1 + 4u)$ cancel out in both cases, giving precisely (3.15) and (3.16) respectively.

Finally, we use Harer’s stability result [H] to show that $c_k, j = 0$ whenever $j > k$. With the notations in the proof of Lemma 3.2 it is enough to show that $\alpha_k = 0$ for all $k$. Assume by contradiction that $\alpha_k \neq 0$ for some $k \geq 1$. Then for any $g > 3a$ relation (3.15) provides a relation between $\kappa$ classes in degree $a$ in which the coefficient of $\kappa_a$ is a nonzero multiple of $\alpha_a$, thus nonzero. But this contradicts Harer stability theorem, which implies that for each $a$ the ring

$$\bigoplus_{i=0}^{2a} H^i(M_g, \mathbb{Q})$$

stabilizes for $g \geq 3a$ to $\mathbb{Q}[\kappa_1, \ldots, \kappa_a]$ with no relations. □

Proof of Theorem 1.4. Together, the previous Lemmas completely prove Theorem 1.4. More precisely, since $c_k, j = 0$ for $j > k$ by Lemma 3.3, then the relation (3.16) becomes exactly (1.9). Furthermore, Lemma 3.1 and Lemma 3.2 show that the coefficients in the relation (1.9) are exactly those appearing in Definition 1.3.
Remark 3.5  Note that the relation (1.8) combined with the initial condition for $G$ implies the following combinatorial relation for the Bernoulli numbers

$$
\sum_{k=1}^{\infty} \frac{B_{k+1}}{k(k+1)} t^k = \sum_{k=1}^{\infty} c_{k0} t^k = \frac{1}{4} \sum_{k=1}^{\infty} t^k \sum_{l=0}^{k} (-4)^{-l} q_{kl} \cdot \frac{l!(k/2 - 1)!}{(k/2 + l)!}
$$

We conclude this section with the

**Proof of Proposition 1.7.**  The formula for the generating function for the coefficients $c_{k,k}$ is given by Lemma 3.6 below.

Next, if we make the change of variable $t = xu$ and $y = u^{-1}$ in relations (1.10) and (1.9) they respectively become after re-indexing the sums

\[
\left[ \exp \left( -\sum_{a=0}^{\infty} \sum_{j=0}^{a} \kappa_a c_{a,a-j} t^a y^j \right) \right] \cdot \left( \kappa_{b-1} - 2 \sum_{a=0}^{\infty} \sum_{j=0}^{a} \kappa_{a+b} q_{a,a-j} t^{a+1} y^j \right)_{t^{g+2-2d} y^{3d-g-2}} = 0
\]

We are interested in the coefficients of $y^0$ and $y^1$. Since by Lemma 3.6 $10c_{a,a-1} = ac_{a,a}$ while $10q_{a,a-1} = (a+1)q_{a,a}$ for $a \geq 1$ and $q_{0,0} = 1$ then the previous two displayed equations become respectively

\[
\left[ \exp \left( f(t) + tf'(t) \frac{y}{10} + O(y^2) \right) \left( g(t) + (tg'(t) + 2\kappa_b) \frac{y}{10} + O(y^2) \right) \right]_{t^{g+2-2d} y^{3d-g-2}} = 0 \quad (3.19)
\]

\[
\left[ \exp \left( f(t) + tf'(t) \frac{y}{10} + O(y^2) \right) \right]_{t^{g+1-2d} y^{3d-g-1}} = 0 \quad (3.20)
\]

where

\[
f(t) = -\sum_{j=0}^{\infty} \kappa_a c_{a,a} t^a \]

\[
g(t) = \kappa_{b-1} - 2t\kappa_b - 2 \sum_{a=1}^{\infty} \kappa_{a+b} q_{a,a} t^{a+1} = \kappa_{b-1} - 2t\kappa_b - 12 \sum_{a=1}^{\infty} \kappa_{a+b} q_{a,a} t^{a+1}
\]

Then for $3d - g - 1 = 0$ and $a = g + 1 - 2d$, relation (3.20) becomes

\[
\left[ \exp \left( f(t) \right) \right]_{t^a} = 0 \quad (3.21)
\]

while for $3d - g - 1 = 1$ and $a = g + 1 - 2d$ it becomes

\[
\left[ \exp \left( f(t) \right) tf'(t) \right]_{t^a} = 0
\]

which is of course the same as (3.21) (up to a scalar factor of $a$). Together, they give (1.12).
Similarly, after subtracting $\kappa_b/5$ times $[3.20]$ from $[3.19]$, we see that in the new relation $[3.19]$ the coefficient of $x^1$ is up to a scalar multiple $t$ times the derivative in $t$ of the coefficient of $x^0$. Therefore, for both $g + 2 - 3d = 0$ and $g + 2 - 3d = 1$, relation $[3.19]$ implies

$$[ \exp(f(t)) g(t)]_{t^2 + 2 - 2d} = 0$$

which gives $[1.13]$. □

**Lemma 3.6** We have the following equality

$$\exp \left( \sum_{k=1}^{\infty} c_{kk} z^k \right) = \sum_{k=1}^{\infty} \frac{(6k)!}{(2k)! (3k)!} \left( \frac{z}{72} \right)^k$$

(3.22)

Moreover, each $k \geq 1$

$$0 < q_{kk} = 6kc_{kk}, \quad 0 < q_{k,k} = 60c_{k,k-1}$$

(3.23)

**Proof.** Let

$$Q_0(z) = \sum_{k=1}^{\infty} q_{k,k-i} z^{k+1-i} \quad \text{and} \quad Q_i(z) = \sum_{k=i}^{\infty} q_{k,k-i} z^{k+1-i} \quad \text{for } i \geq 1$$

(3.24)

With this notation,

$$uQ = \sum_{k \geq i \geq 0} q_{k,k-i} x^k (1 + 4u)^{k/2} u^{k-i+1} = u + \sum_{i=0}^{\infty} x^{i-1} (1 + 4u)^{i-1/2} Q_i(xu \sqrt{1 + 4u})$$

Next we make another change of variables

$$z = xu \sqrt{1 + 4u} \quad \text{and} \quad y = x \sqrt{1 + 4u}$$

Then using logarithmic differentiation

$$y_u = \frac{2y}{1 + 4u} \quad \text{and} \quad z_u = \frac{z}{u} + \frac{2z}{1 + 4u} = \frac{z(1 + 6u)}{u(1 + 4u)} = \frac{y + 6z}{1 + 4u}$$

In terms of this new variables,

$$uQ = u + \sum_{i=0}^{\infty} y^{i-1} Q_i(z)$$

$$(1 + 4u)(uQ)_u = 1 + \sum_{i=0}^{\infty} y^{i-1} \left[ 2(i-1) Q_i(z) + Q'_i(z)(y + 6z) \right]$$

Multiplying both sides of $[3.7]$ by $u$ gives $uQ = u + [z(1 + 4u)(uQ)_u + y(uQ)^2]$; substituting the previous two displayed equations we get

$$\sum_{i=0}^{\infty} y^{i-1} Q_i(z) = z \sum_{i=0}^{\infty} y^{i-1} \left[ 2(i-1) Q_i(z) + Q'_i(z)(y + 6z) \right] + y[z/y + \sum_{i=0}^{\infty} y^{i-1} Q_i(z)]^2$$

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Multiplying both sides with $y$ gives

$$ \sum_{i=0}^{\infty} y^i Q_i(z) = z \sum_{i=0}^{\infty} y^i \left[ 2(i-1)Q_i(z) + Q'_i(z)(y+6z) \right] + \left[ z + \sum_{i=0}^{\infty} y^i Q_i(z) \right]^2 \quad (3.25) $$

Now we can collect the coefficient of $y^0$ to get

$$ Q_0 = 6z^2Q'_0 + Q_0^2 \quad (3.26) $$

This equation can be easily integrated, using the integrating factor

$$ P(z) = \exp \left( \frac{1}{6} \int_0^z \frac{Q_0(\zeta)}{\zeta^2} d\zeta \right) = \sum_{k=0}^{\infty} p_k z^k $$

Then $P' = \frac{Q_0}{6z^2}$ while

$$ P'' = P \frac{Q'_0}{36z^4} + P' \frac{Q'_0}{6z^2} - 2P \frac{Q_0}{6z^3} = P \frac{Q^2_0 + 6z^2Q'_0 - 12zQ_0}{36z^4}.$$ 

Multiplying both sides of (3.26) by $\frac{P}{6z^2}$ gives

$$ 6P' = 5P + 36z^2P'' + 72zP' $$

This means

$$ 6(k+1)p_{k+1} = p_k(5 + 36k(k-1) + 72k) = (6k + 1)(6k + 5)p_k $$

which gives inductively

$$ p_k = \frac{(6k)!}{(3k)!(2k)!} \frac{72^{-k}}{k} $$

But of course since $Q_0(z) = \sum_{k=1}^{\infty} q_{kk} z^{k+1}$ then

$$ P(z) = \exp \left( \frac{1}{6} \sum_{k \geq 1} q_{kk} \frac{z^k}{k} \right) = \exp \left( \sum_{k \geq 1} c_{k,k} z^k \right) $$

which gives (3.22).

For $i \geq 1$, the coefficient of $y^i$ in (3.25) gives

$$ Q_i = 6z^2Q'_i + 2izQ_i + z^2Q'_{i-1} + 2Q_0Q_i + \sum_{l=1}^{i-1} Q_l Q_{i-l} $$

When $i = 1$ this becomes

$$ Q_1 = 6z^2Q'_1 + 2zQ_1 + zQ'_0 + 2Q_1Q_0 $$

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It follows then that $10 Q_1 = Q'_0$: differentiating (3.26) gives

$$Q'_0 = 6z^2 Q''_0 + 12zQ'_0 + 2Q_0 Q'_0$$

So if $F = 10 Q_1 - Q'_0$ then subtracting the previous two displayed equations we get

$$F = 6z^2 F' + 2z F + 2Q_0 F$$

which by induction on the powers of $z$ implies $F = 0$. This means that $10q_{k,k-1} = (k+1)q_{k,k}$ for each $k \geq 1$.

Taking $k = j$ in (1.8) gives $6k c_{k,k} = q_{k,k}$, while $k = j + 1$ gives $q_{k,k-1} = (6k - 4)c_{k,k-1} + kc_{k,k}$. Therefore, when $k \geq 1$,

$$(6k - 4)c_{k,k-1} = q_{k,k-1} - \frac{1}{6}q_{kk} = q_{kk} \left( \frac{k + 1}{10} - \frac{1}{6} \right) = q_{kk} \frac{3k - 2}{30}$$

which together complete the proof of (3.23). □

4 Appendix

The relations of Theorem 1.1 are actually coming from relations in the tautological ring of $C^2_g$, which in turn are restrictions of relations in the Chow ring $A^*(\overline{M}_{g,2})$, as explained below.

We will work with the moduli space $Y_{d,g}$ of degree $d$, genus $g$ covers of $\mathbb{P}^1$ with a fixed ramification pattern over some points of the target, as defined in [I]. The arguments used in this appendix are very similar to those in [I], to which we refer the reader for more detailed explanations.

Fix 4 distinct points $p_1, \ldots, p_4$ on $\mathbb{P}^1$ and consider the moduli space

$$Y_{d,g}(B_1^d(y_1)B_1^d(y_2)B_{2,1}^{d-2}B_{2,1}^{d-2})$$

of genus $g$, degree $d$ relatively stable covers of this $\mathbb{P}^1$ which are simply ramified over $p_3$ and $p_4$ and have a marked point $y_1$ and $y_2$ in each fiber over $p_1$ and respectively $p_2$. The push forward via the stabilization map $st$ defines a class in Chow ring $A^{g+3-2d}(\overline{M}_{g,2})$. (Notice that this class is not the same as the relative virtual fundamental cycle described in [GV]; however, we next restrict to covers whose domain contains a smooth genus $g$ component, where the two definitions agree)

Now, as in Proposition 2.8 of [I], we split the target $\mathbb{P}^1$ in two ways. We use the fact that on $\overline{M}_{0,4}$ the divisor corresponding to fixing the location of the marked points $p_1, \ldots, p_4$ is linear equivalent to the boundary divisor $(p_1,p_2|p_3,p_4)$ where $p_1,p_2$ are on a bubble and $p_3,p_4$ on the other. So the class $st_* Y_{d,g}(B_1^d(y_1)B_1^d(y_2)B_{2,1}^{d-2}B_{2,1}^{d-2})$ can be computed by splitting the target either as $(p_1,p_2|p_3,p_4)$ or as $(p_1,p_3|p_2,p_4)$ and using the degeneration formula (1.23) of [I]. This gives a relation in the Chow ring of $\overline{M}_{g,2}$. 20
Here we are only interested in the restriction of the above relation to the Chow ring of \(\mathbb{P}^g\), so we are interested only in those covers of \(\mathbb{P}^1 \vee \mathbb{P}^1\) whose domain has a smooth genus \(g\) component, and the rest are rational components (the “symbol” of the relation, using the terminology of \([I]\)).

By dimension count, in the symbol, the only possibility is that on one side we have a genus \(g\), degree \(d\) cover, while the cover on the other side has only rational components, each totally ramified over the node of the target. (Of course, the ramification pattern of the two covers has to match over the double point of the target.) If the genus \(g\) component had degree less then \(d\) then its push forward by \(st\) would give a smaller dimensional class in the Chow ring. If one of the rational components was not totally ramified over the node of the target, it would then have at least two points in common with the degree \(d\), genus \(g\) component, which is impossible.

So, for the splitting of \(\mathbb{P}^1\) with \(p_1, p_2\) on the right bubble and \(p_3, p_4\) on the left bubble, in the symbol, either

1. the rational components are on the left, each of degree 1; the two marked points \(y_1\) and \(y_2\) are either on different components, or on the same component. After pushforward, this gives

\[
st_\ast \mathcal{Y}_{d,g}(B_{1^d}(y_1; y_2)B_{2,1^d-2}B_{2,1^d-2}) + st_\ast \mathcal{Y}_{d,g}(B_{1^d}(y_1 = y_2)B_{2,1^d-2}B_{2,1^d-2})
\]

where the notation \(B_{1^d}(y_1 = y_2)\) means the strata where \(y_1, y_2\) are on a bubble attached to a point where the cover has ramification pattern \(B_{1^d}\). Forgetting the marking of the two branch points, the last displayed equation becomes

\[
r(r - 1) \left( st_\ast \mathcal{Y}_{d,g}(B_{1^d}(y_1; y_2)) + st_\ast \mathcal{Y}_{d,g}(B_{1^d}(y_1 = y_2)) \right) \tag{A.1}
\]

where \(r = 2d + \kappa_0\) is the total number of branch points;

2. or the rational components are on the right, in which case the two marked branch points must land on rational components. There are two cases: they either land on the same rational component, which then forces that component to have degree 3, or they land on two different degree 2 components. After pushforward, this gives respectively

\[
3st_\ast \mathcal{Y}_{d,g}(B_{1^d}(y_1)B_{1^d}(y_2)B_{3,1^d-3}) + \frac{2 \cdot 2}{2!} st_\ast \mathcal{Y}_{d,g}(B_{1^d}(y_1)B_{1^d}(y_2)B_{2,2,1^d-4})
\]

Note that this is the pullback by the projection \(\pi_{1,2}\) that forgets both \(y_1, y_2\) of

\[
3st_\ast \mathcal{Y}_{d,g}(B_{3,1^d-3}) + 2st_\ast \mathcal{Y}_{d,g}(B_{2,2,1^d-4}) \tag{A.2}
\]

On the other hand, when we split the target \(\mathbb{P}^1\) with \(p_1, p_3\) on the right bubble and \(p_2, p_4\) on the left bubble, we get something symmetric in \(y_1, y_2\). So it is enough to consider the case where the rational components are on the left; the other case is obtained by switching \(y_1\) with \(y_2\). In the first case, the rational component containing the branch point has degree 2, and the others have degree 1. The marked point \(y_1\) can be either on the degree 2 component or on a degree 1 component. After pushforward, this gives

\[
2st_\ast \mathcal{Y}_{d,g}(B_{2,1^d-2}(y_1)B_{1^d}(y_2)B_{2,1^d-2}) + st_\ast \mathcal{Y}_{d,g}(B_{1,2,1^d-3}(y_1)B_{1^d}(y_2)B_{2,1^d-2})
\]
Note that this is equal to the pulled back by $\pi_2$ (the map that forgets $y_2$) of

$$(r - 1) \left( 2 s_\pi \mathcal{Y}_{d,g}(B_{2,1^{d-2}}(y_1)) + s_\pi \mathcal{Y}_{d,g}(B_{1,2,1^{d-3}}(y_1)) \right) \quad (A.3)$$

Assembling (A.1, A.2, A.3) as well as A.3 in which $y_1$ is switched with $y_2$ we get the following relation in $R^{g+3-2d}(\mathcal{C}_2^g)$:

$$r(r - 1) \left( s_\pi \mathcal{Y}_{d,g}(B_{1^{d}}(y_1; y_2)) + s_\pi \mathcal{Y}_{d,g}(B_{1^{d}}(y_1 = y_2)) \right) + \pi_{1,2}^* \left( 3 s_\pi \mathcal{Y}_{d,g}(B_{3,1^{d-3}}) + 2 s_\pi \mathcal{Y}_{d,g}(B_{2,2,1^{d-4}}) \right) = (r - 1) \pi_2^* \left( 2 s_\pi \mathcal{Y}_{d,g}(B_{2,1^{d-2}}(y_1)) + s_\pi \mathcal{Y}_{d,g}(B_{1,2,1^{d-3}}(y_1)) \right)$$

$$+ (r - 1) \pi_1^* \left( 2 s_\pi \mathcal{Y}_{d,g}(B_{2,1^{d-2}}(y_2)) + s_\pi \mathcal{Y}_{d,g}(B_{1,2,1^{d-3}}(y_2)) \right) \quad (A.4)$$

This relation can be rewritten in terms of the class $y_{d,g}$ of Theorem 1.1. For that, we introduce $d$ more marked points $x_1, \ldots, x_d$, and let $p: \mathcal{C}_g^{d+2} \to \mathcal{C}_g^2$ be the map that forgets these $d$ points (we use the letter $\pi$ for the map that forgets the points $y_1$ or $y_2$). Let $y_{d,g}$ denote the class defined by (1.1), as well as its pullback to $\mathcal{C}_g^{d+2}$.

Each term of relation (A.4) corresponds to covers of a nonrigid $\mathbb{P}^1$ with a fixed ramification pattern over a point in $\mathbb{P}^1$. Such class can be expressed in terms of $y_{d,g}$ by marking all the points in that fiber. For example,

$$s_\pi \mathcal{Y}_{d,g}(B_{1^{d}}(y_1)) = \frac{1}{(d-1)!} s_\pi \mathcal{Y}_{d,g}(B_{1^{d}}(y_1; x_2, \ldots, x_d)) = \frac{1}{(d-1)!} p_\pi(y_{d,g} \cdot D_{y_1,x_1})$$

where $D_{y_1,x_2}$ denotes the diagonal $y_1 = x_1$. So (A.4) gives

**Proposition 4.1** With the notations above, the following relation holds in $R^{g+3-2d}(\mathcal{C}_2^g)$:

$$r(r - 1) \left( \frac{1}{(d-2)!} p_\pi(y_{d,g} \cdot D_{y_1,x_1} D_{y_2,x_2}) + \frac{1}{(d-1)!} p_\pi(y_{d,g} \cdot D_{y_1,y_1;y_2}) \right) + \pi_{1,2}^* \left( 3 \frac{1}{(d-3)!} p_\pi(y_{d,g} \cdot D_{x_1,x_2,x_3}) + 2 \frac{1}{(d-4)!} p_\pi(y_{d,g} \cdot D_{x_1,x_2} D_{x_3,x_4}) \right)$$

$$= (r - 1) \pi_2^* \left( 2 \frac{1}{(d-2)!} p_\pi(y_{d,g} \cdot D_{x_1,x_2};y_1) + \frac{1}{(d-3)!} p_\pi(y_{d,g} \cdot D_{x_1,y_1} D_{x_2,x_3}) \right)$$

$$+ (r - 1) \pi_1^* \left( 2 \frac{1}{(d-2)!} p_\pi(y_{d,g} \cdot D_{x_1,x_2};y_2) + \frac{1}{(d-3)!} p_\pi(y_{d,g} \cdot D_{x_1,y_2} D_{x_2,x_3}) \right) \quad (A.5)$$

for all $g, d \geq 2$.

Each term in the equation above can be expressed in terms of the generating function $G$ just as in the proof of Lemma 2.5. In fact, if for each fixed $j$ we denote by $p_j$ the map that forgets all the marked points except $x_1, \ldots, x_j$, then the generating function of $p_j y_{d,g}$ is

$$\sum_d p_j y_{d,g} \frac{w^{d-j} t^{-d}}{(d-j)!} = \exp \left( \frac{1}{t} p_\psi G(\psi, w) \right) \sum_{J_1, \ldots, J_r} \Delta_{J_1, \ldots, J_r} t^{-r} \prod_{i=1}^r (\frac{\partial}{\partial w})^{l_i} G(\psi_{x_i}, w) \quad (A.6)$$
where the last sum is over all partitions \( \{J_1, \ldots, J_r\} \) of \( \{x_1, \ldots, x_j\} \). For example, when \( j = 1 \), this simply becomes
\[
\exp \left( \frac{1}{t} p_* G(t \psi, w) \right) t^{-1} G_w(t \psi_1, w)
\]
while for \( j = 2 \) it becomes
\[
\exp \left( \frac{1}{t} p_* G(t \psi, w) \right) \left( t^{-2} G_w(t \psi_1, w) G_w(t \psi_2, w) + t^{-1} G_{ww}(t \psi_1, w) D_{1,2} \right)
\] (A.7)

Note that in such expressions, the factor \( \exp \left( \frac{1}{t} p_* G(t \psi, w) \right) \) is pulled back from \( \mathcal{M}_g \).

Thus for example the first term of (A.5) is nothing but \( r (r - 1) \) times the coefficient of \( t^{g+2-2d} w^d \) in the expression (A.7). Using (A.6), relation (A.5) then becomes a fairly complicated, but explicit relation in \( A^*(\mathcal{C}_g^*) \) involving up to 4 derivatives in \( w \) of the generating function \( G \). One could use the ODE (2.1) satisfied by \( G \) to reduce the relation above to a not much simpler one involving, besides the factor \( \exp \left( \frac{1}{t} p_* G(t \psi, w) \right) \), an explicit polynomial in \( G_w(t \psi_1, w) \), \( G_w(t \psi_2, w) \) and \( D_{1,2} \).

**Remark 4.2** If we pushforward relation (A.4) by \( \pi_2 \) and divide by \( (r - 1)d \) we simply get
\[
r \cdot st_* Y_{d,g}(B_1 d(y_1)) = \pi_1^* st_* Y_{d,g}(B_{2,1d-2})
\]
which as before is equivalent to
\[
\frac{1}{(d - 1)!} p_*(y_{d,g} \cdot D_{y_1,x_1}) = \frac{1}{(d - 2)!} p_*(y_{d,g} \cdot D_{x_1,x_2})
\]
Thus in \( R^{g+2-2d}(\mathcal{C}_g) \) we have the relation
\[
r \cdot p_*(y_{d,g} \cdot D_{y_1,x_1}) = (d - 1)p_*(y_{d,g} \cdot D_{x_1,x_2})
\] (A.8)
where \( y_{d,g} \) defined by (1.1). Multiplying by \( \psi_1^b \) and the pushing forward by \( \pi_1 \) gives exactly the relations (1.2) of Theorem 1.1. It is easy to modify the arguments in Section 2 to see that in terms of the generating function \( G \), (A.8) becomes
\[
\left[ \exp \left( \frac{1}{t} p_* G(t \psi, w) \right) (2w G_w(t \psi_1, w) - 1) \right]_{g+2-2d} = 0
\]
which in turn becomes relation (0.2) mentioned in the introduction.

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