An Induced Representation Method for Studying the Stability of Saturn’s Ring

Soumangsu Bhusan Chakraborty
Ecole Polytechnique, Palaiseau 91120, France

Siddhartha Sen
CRANN, Trinity College Dublin, Dublin 2, Ireland
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Using the method of induced representation for groups MacKey proved a set of formulas that could be used to calculate the sum of powers of the eigenvalues of a matrix which was symmetric under a finite group. We use MacKey’s results to derive the stability condition, \( m > C n^3 \), for a ring of Saturn model due to Maxwell where \( n \) is the number of unit mass particles in a ring and \( m \) the mass of Saturn. In Maxwell’s model a ring of Saturn is considered to be a symmetrically arranged collection of \( n \) identical unit mass particles revolving round Saturn in a circular orbit.

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1. INTRODUCTION

In this paper a group theoretical result of MacKey \cite{1} that relates the sum of powers of the eigenvalues of a perturbation matrix, symmetric under a finite group, to group characters is used to derive Maxwell’s stability condition \( m > C n^3 \) for a ring of Saturn. \( C \) is a constant, \( m \) is the mass of Saturn and \( n \) represents the number of identical symmetrically arranged unit mass particles that were assumed to be revolving round Saturn in a circular orbit by Maxwell.

Using group theoretical methods for this stability problem is reasonable as the \( n \) identical particles that form a ring have discrete rotational symmetry. The effect of linear perturbation on such a system can be described in terms of the geometric picture of a vector bundles. In a vector bundle two spaces are joined together in a precise way to form a larger space. For the Saturn’s ring perturbation problem these two spaces are: the initial symmetric location of the ring particles on a circle and the space of perturbation attached to each particle. The symmetry group of location induces a group action on the perturbation space. We will briefly outline the basic terminology of vector bundles and induced representations that we need in order to write down MacKey’s formulas and use them to get the stability result stated.

As stated already Maxwell assumed that each ring of Saturn was made up of \( n \) identical unit mass particles all revolving around a central massive planet (Saturn) of mass \( m \) in a planar circular orbit with the same angular velocity. He had earlier in the work established that a liquid ring could not be stable and that a solid ring could not be stable if the mass distribution was uniform. The stability condition for the orbiting ring particles he proved was \( m > 0.43n^3 \).

Recently Maxwell’s model has been revisited by many authors using different mathematical techniques \cite{4}\cite{5}\cite{3}\cite{7}\cite{8}\cite{9}\cite{6}. In these works, different ways of analyzing the linearized \( 4n \times 4n \) dimensional perturbation matrix appropriate for the Saturn’s ring problem are considered. Simplification of the problem follow once invariant subspace of the perturbation matrix are found which allow a projection of the perturbation matrix on to a particular invariant space. This is then shown to reduce the stability problem, for \( n \) even, to that of determining the eigenvalues of a \( 4 \times 4 \) matrix. However the invariant subspaces are found by ad hoc methods as there is no general approach available for finding them.

The virtue of our group theoretical approach is that the required stability conditions are found in terms of \( 4 \times 4 \) matrices, without determining invariant subspaces and the basic result that for linear stability \( m = C n^3 \) where \( C \) is a constant, is, as we will show easily obtained. The aim of this paper is to establish this result and not to precisely determine the constant \( C \).

We start by describing the set of MacKey’s character formulas for an induced representation.

2. THE CHARACTER FORMULAS

Analyzing the linear stability of a system requires knowledge of the eigenvalues of a perturbation matrix, which is a linear operator \( T \) acting on a finite dimensional vector space \( W \). For our problem the linear operator \( T \) of interest has a discrete symmetry. MacKey’s character formulas relates the sum of powers of the eigenvalues of \( T \) to group theoretical structures present in the problem. The proof of these formulas are given in Sternberg \cite{1}.

Induced representations are representations of a group generated from a subgroup. We can also think of induced...
representations as representations of a group on the sections of a vector bundle from a knowledge of the representation of the group on the base space of the bundle. Let us briefly explain these terms. We recall that a vector bundle is a way of joining together two spaces where precise rules for joining the two spaces have to be given. Such a joining of two spaces is natural for linear perturbative stability problems. For example in the Saturn’s rings problem the two spaces to be joined are: the original set of equilibrium position and momenta coordinates (the base space) and the space \( R^2 \times R^2 \) (the fiber) that describes the planar perturbation from their equilibrium values of the position and momenta coordinates. Assigning positions and momenta for each perturbed particle in phase space is called a section of the bundle. To describe a section the location of a point in phase space and the corresponding perturbed position and momenta values have to be given. It represents a point of the vector bundle relevant for the Saturn’s ring problem. In this geometrical language a perturbation is a linear map between sections.

Let us now introduce the precise vocabulary needed to properly describe the character formulas that we will use. Suppose \( M \) is a finite set (the base space) with a vector space \( E_x \), the fiber, associated with each point \( x \in M \). The vector bundle \( E \) over \( M \) is then the disjoint union of all of these vector spaces \( E_x \). So we write \( E = \bigsqcup_{x \in M} E_x \). There is also a natural projection map \( \pi : E \to M \) given by \( \pi(v) = x \) if \( v \in E_x \). Its inverse \( E_x = \pi^{-1}(x) \) is the vector space associated with the point \( x \in M \). A section of \( E \) is the function \( f : M \to E \) which assigns a vector \( f(x) \in E_x \) to each \( x \in M \). So we can see that the map \( f \) satisfies \( \pi \circ f = \text{identity} \). For a fixed \( x \in M \), the space of all sections \( f(x) \) is given by \( \Gamma(E_x) \) and \( \Gamma(E) = \bigoplus_{x \in M} \Gamma(E_x) \). Thus the projection map tells us where a particular fiber is located while a section tells us about points in the vector bundle. The space of sections is not a vector space but it can be made into a vector space by introducing a suitable way of adding sections and multiplying them by scalars.

Let \( W \) be a vector space given by \( \Gamma(E) \) and \( E \) the vector bundle over the finite set \( M \) on which group \( G \) acts transitively. Let \( H \) be a subgroup of \( G \) and \( (\rho, W) \) be a representation of the group \( G \) with irreducible characters \( \chi_1, \chi_2, \ldots, \chi_s \). Let us consider that \( W = W_1 \oplus W_2 \oplus \cdots \oplus W_s \) be canonical decompositions of the space \( W \) where each \( W_i \) can be \( m_i \) copies of \( i^{th} \) irreducible representations of \( G \) of dimension \( d_i \). Let \( P_i \) be the projection operator on the space \( W_i \). Let us consider a linear map \( T \in \text{Hom}_G(W, W) \). Then we have: \( TP_i = P_i TP_i \). We would like to calculate the eigenvalues of \( TP_i \). If \( \lambda_1, \lambda_2, \ldots, \lambda_{d_i} \) are its eigenvalues then we can write: \( \frac{1}{d_i} \text{tr}(TP_i) = (\lambda_1 + \lambda_2 + \cdots + \lambda_{d_i})/d_i \). Hence knowing \( \text{tr}(T^k P_i) \) where \( k \) is an integer, for \( 1 \leq k \leq d_i \), we can in principle, determine all the eigenvalues. Let us consider a subgroup \( H \) of \( G \) such that \( M = G/H \). So \( E \) is the vector bundle over \( M = G/H \) induced from a representation \( (\sigma, V) \) of \( H \). It is shown in \( \cite{1} \) that,

\[
(Tf)(a) = \frac{1}{\#H} \sum_{b \in G} t(a, b)f(b), \quad \forall a \in G. \tag{1}
\]

where \( t(a, b) \in \text{Hom}(V, V) \) is a linear operator (a matrix) that sends each element of the vector space associated with the coset \( aH \) to that associated with coset \( bH \). It can be proved that for a given \( T \in \text{Hom}(\Gamma(E), \Gamma(E)) \) we can uniquely determine \( t \in \text{Hom}(V, V) \). We are now in a position to write the complete form of the character formula as: \( \cite{1} \)

\[
\text{tr}(T^k P_i) = \frac{1}{\#H} \sum_{a \in G} \chi_i(a) \text{tr}(t^k(a)), \quad \tag{2}
\]

where we denote \( t(e, a) \) by \( t(a) \) (a matrix) with \( e \) as the identity element of \( G \) and hence of \( H \). This is the result we use to discuss the stability of Saturn’s ring problem.

3. STABILITY PROBLEM: ASSUMPTIONS

Let us now formulate the dynamical problem of stability that we consider. Our ring consists of \( n \) identical point particles of unit mass revolving in a plane around Saturn in a circular orbit of constant radius with constant angular velocity. We also assume that the \( n \) identical particles are symmetrically arranged about the central mass and that they all lie on a plane and use labels from 0 to \( n-1 \) in clockwise or anticlockwise sense to denote the \( n \) particles of the ring and the label \( n \) for Saturn. We set the mass of each particle \( m_i = 1, \forall i \in \{0, 1, 2, ..., n-1\} \) and set the mass of Saturn to be \( m \).

We consider the ringed system in isolation and thus only include inter-particle gravitational interaction and gravitational interaction of each identical particle with Saturn.

3.1. Relative equilibrium of the 1 + \( n \) body planet ring system

We now formulate the perturbation problem following Roberts \( \cite{3} \) and Moelck \( \cite{4} \) and then recast it in a form which highlights its symmetry. Let \( q_i \in \mathbb{R}^2 \) be the generalized coordinates of the planet and let \( p_i \in \mathbb{R}^2 \) be their generalized momentum. Let \( q = (q_0, q_1, \cdots, q_n) \in \mathbb{R}^{2(n+1)} \). The distance between the \( i^{th} \) and the \( j^{th} \) particle be \( r_{ij} = \|q_i - q_j\| \). Using Newton’s law of motion and the inverse square law of gravitation we get the following equation: \( \cite{4} \)

\[
m_i \ddot{q}_i = \sum_{i \neq j = 0}^n \frac{m_i m_j (q_i - q_j)}{r_{ij}^3} = \frac{\partial U}{\partial q_i}. \tag{3}
\]
Where \( U(q) \) is the Newtonian potential of the system given by \( U(q) = \sum_{i<j} \frac{m_i m_j}{r_{ij}} \). The generalized momentum can be written as \( p_i = m_i \dot{q}_i \) and let \( p = (p_0, p_1, \ldots, p_n) \in \mathbb{R}^{2(n+1)} \). Hence the equation of motions can be written as:

\[
\dot{q} = M^{-1} p = \frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p} = \nabla U(q) = -\frac{\partial H}{\partial q},
\]

where \( M = \text{diag}\{m_0, m_1, m_1, \ldots, m_n, m_n\} \) is a diagonal, \( 2(n+1) \times 2(n+1) \) matrix and \( H(q, p) \) is the Hamiltonian of the system.

Let us consider the ring isomorphism \( \mathbb{C} \to M_2(\mathbb{R}) \) given by: \((a + ib) \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\) with \((a, b) \in \mathbb{R}^2\). With the above isomorphism in mind we can make the following change of coordinates:

\[
\begin{align*}
\mathbf{x}_i &= e^{i\omega t} \mathbf{q}_i, \\
\mathbf{y}_i &= e^{i\omega t} \mathbf{p}_i.
\end{align*}
\]

Here \( \frac{2\pi}{\omega} \) is the common period of rotation of the identical particles about the central planet and 

\[
e^{i\omega t} \mapsto \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}.
\]

Hence in this new coordinates system the equation of motion becomes:

\[
\begin{align*}
\dot{\mathbf{x}} &= A \mathbf{x} + M^{-1} \mathbf{y}, \\
\dot{\mathbf{y}} &= \nabla U(\mathbf{x}) + A \mathbf{y},
\end{align*}
\]

where \( A = \begin{bmatrix} i\omega & 0 \\ 0 & 0 \end{bmatrix} \) with \( \mathbb{I} \) as the \( 2 \times 2 \) identity matrix.

In this new set of coordinate system we can write the Hamiltonian of the system as:

\[
H(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \mathbf{y}^T M^{-1} \mathbf{y} - U(\mathbf{x}) - \mathbf{x}^T A \mathbf{y}.
\]

For equilibrium we must have \((\dot{\mathbf{x}}, \dot{\mathbf{y}}) = (0, 0)\) This gives 

\[
\nabla U(\mathbf{x}) + \omega^2 M \mathbf{x} = 0.
\]

As all the revolving point masses are taken to have unit mass we set \( m_i = 1 \) for \( i \in \{0, 1, 2, \ldots, n-1\} \) and set the mass of Saturn, \( m_n = m \). We also scale the radius of the circular orbit to be equal to one. This gives \( \mathbf{x}_j = (\cos \theta_j, \sin \theta_j) \), where \( \theta_j = \frac{2\pi j}{n} \) for \( j \in \{0, 1, 2, \ldots, n-1\} \). A little bit of trigonometry gives \( r_{ij} = 2 \sin(\theta_j/2) \) and \( r_{ij}^2 = 2(1 - \cos(\theta_j)) \), where \( \theta_{ij} = \frac{2\pi(j-i)}{n} \) for \( i, j \in \{0, 1, 2, \ldots, n-1\} \). This will give \( \mathbf{r}_{ij} = 2 \sin(\theta_j/2) \) and \( r_{ij}^2 = 2(1 - \cos(\theta_j)) \).

Substituting these expressions in the equation \((8)\) we get \( \omega = \omega(m) = \frac{\cos^2 \theta_j}{2} + \frac{\omega^2}{m} \) where \( \sigma_n = \sum_{k=1}^{n-1} \frac{1}{r_{mk}} = \frac{1}{2} \sum_{k=1}^{n-1} \csc \frac{\pi k}{n} \). This formula shows that as the central mass increases, the period of rotation of the identical bodies decreases.

### 3.2. Linear stability matrix

Linearizing the system of equations \((4)\) we get the stability matrix: \( T = \begin{bmatrix} A & M^{-1} \\ D\nabla U(\mathbf{x}) & A \end{bmatrix} \) where \( D\nabla U(\mathbf{x}) \) denotes the derivative of the gradient of the potential and is a \( 2(n+1) \times 2(n+1) \) matrix. The matrix \( S \) can be written as: \( S = \left[ \begin{array}{ccc} S_{00} & \cdots & S_{0n} \\ \vdots & \ddots & \vdots \\ S_{n0} & \cdots & S_{nn} \end{array} \right] \). Here \( S_{ij} \) is a \( 2 \times 2 \) matrix given by \( S_{ij} = \frac{m_i m_j}{r_{ij}} \left[ x_{ij} x_{ij}^T - 3 x_{ij} x_{ij}^T \right] \) if \( i \neq j \) and \( S_{jj} = -\sum_{i\neq j} S_{ij} \), where \( x_{ij} = \frac{x_i - x_j}{r_{ij}} \). Note that \( S \) is a block symmetric matrix i.e. \( S_{ij} = S_{ji} \).

Now substituting the value of \( \mathbf{x}_i = (\cos \theta_i, \sin \theta_i) \) into the expression of \( S_{ij} \) we get

\[
S_{ij} = \frac{1}{2r_{ij}^3} \begin{bmatrix} -1 + 3 \cos(\theta_j) & 3 \sin(\theta_j) \\ 3 \sin(\theta_j) & -1 - 3 \cos(\theta_j) \end{bmatrix}
\]

for \( j \neq 0, n, \) and \( S_{0n} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \). Hence we have: \( S_{00} = -\sum_{i\neq 0} S_{0i} = -\sum_{j=1}^{n} S_{0j} - S_{0n} \). More generally we can write the other elements of the matrix \( S \) as:

\[
S_{ij} = \frac{1}{2r_{ij}^3} \begin{bmatrix} -1 + 3 \cos(\theta_i + \theta_j) & 3 \sin(\theta_i + \theta_j) \\ 3 \sin(\theta_i + \theta_j) & -1 - 3 \cos(\theta_i + \theta_j) \end{bmatrix}
\]

for \( i, j \neq n \) and \( i \neq j \),

\[
S_{jn} = m \begin{bmatrix} 1 - 3 \cos^2 \theta_j & -3 \cos \theta_j \sin \theta_j \\ -3 \cos \theta_j \sin \theta_j & 1 - \sin^2 \theta_j \end{bmatrix}
\]

for \( j \neq n \).

### 3.3. Group Theoretical formulation of the stability problem

We are now ready to highlight symmetry features present in the problem. Our system consists of \( n \) identical particles, symmetrically placed on a circle, shifting the angular positions of the point particles on the circle by an angle \( 2\pi k/n \) for all integer \( k \) leaves the system unchanged. Thus our system possesses a \( \mathbb{Z}_n \) symmetry. We proceed to exploit this symmetry using the geometrical picture of vector bundles and sections.

Every particle is associated with a vector space \( V = \mathbb{R}^2 \oplus \mathbb{R}^2 \) one for the two positions and the other for the two momentum. Now we know that \( \mathbb{R}^2 \oplus \mathbb{R}^2 \) is isomorphic to \( \mathbb{C} \oplus \mathbb{C} = \mathbb{C}^2 \). Let us denote the vector spaces associated with the \( k^{th} \) particle as \( E_k = \mathbb{C} \oplus \mathbb{C} \) and for each \( \mathbb{C} \) we have a representation of the form \( \exp(2\pi ik/n) \).

So the representation \( \rho_k \) over the space \( E_k \) is given by the following matrix:

\[
\rho_k = \begin{bmatrix} \exp(2\pi ik/n) & 0 \\ 0 & \exp(2\pi ik/n) \end{bmatrix}.
\]

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Hence the representation \((\rho, W)\) of \(Z_n\) is given by:

\[
(\rho, W) = \bigoplus_{k \in \mathbb{Z}_n} (\rho_k, E_k).
\]

This is the decomposition of the representation of \(Z_n\) into irreducible representations. Here, we denote \(E = \bigcup_{k \in \mathbb{Z}_n} E_k\) (where \(\bigcup\) denotes disjoint union of vector spaces) as the vector bundle over \(\mathbb{Z}_n\). The space \(E_k\) is a section and we will denote the space of all sections by \(W\) i.e. \(W = \bigoplus_{k=0}^{k=n-1} E_k\).

There are altogether \(n+1\) particles but the underlying symmetry (for all \(n+1\) particles) is not \(\mathbb{Z}_n\). More importantly \(T \in \text{Hom}(W, W)\) and hence we have to use the formula

\[
\text{tr}(T) = \frac{1}{n} \sum_{k \in \mathbb{Z}_n} \chi_i(k) \text{tr}(t(0, k))
\]

with care. This is because in the character formula \(T\) is a homomorphism from \(W\) to \(W\) but the \(T\) matrix we have in this problem is a homomorphism from \(W \bigoplus E_n\) to \(W \bigoplus E_n\).

We thus need to reformulation the nature of the symmetry group. This is done by introducing a new group \(G = \{(n, j) \in \{0, 1, 2, \ldots, n-1\}\}\) which is isomorphic to \(I \otimes \mathbb{Z}_n\). The elements of \(G\) can be written as an ordered pairs \((n, j)\) where \(n\) comes from the trivial group \(I\) (containing only one element \(i\)) and the element \(j\) comes from the group \(\mathbb{Z}_n\). Now we attach with each element of \(G\) a space isomorphic to \(C^2 \otimes \mathbb{C}^2\) for one the \(j^{\text{th}}\) particle and another for the \(n^{\text{th}}\) particle. Under this new formulation \(E_k = C^2 \otimes \mathbb{C}^2 \sim C^4\) (i.e isomorphic to \(\mathbb{C}^4\)), \(W = \bigoplus_{k=0}^{n-1} E_k\) and \(T\) takes the following form;

\[
T = \begin{bmatrix}
\rho_0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \rho_0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \rho_0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \rho_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & \rho_0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\omega & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \omega & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \omega & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \omega & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & \omega \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

Equation \((1)\) tells us that a perturbation in the position and momentum of a particular particle can be expressed as a linear combination of the perturbed positions and momenta of the other particles.

Let us go back to equation \((2)\). We note that the left hand side of equation \((2)\) is the sum of different powers eigenvalues of the matrix \((TP_i)\). For our problem the value for \(k = 1, 2, 3, 4\) give conditions for stability that we need. This is because although we have taken into account the the presence of the center of mass in our formulation, it doesn’t contribute anything to the character formula since we have restricted ourselves to perturbations that do not change either the position or the momentum of the central mass. For this reason it is only the four eigenvalues that describe the motion from equilibrium of particles on the ring. There is no motion of Saturn in the perturbation. The symmetry of the problem also tells us that since all the particles revolving about Saturn are identical it follows that \(\text{tr}(TP_i)\) is the same for all \(i \in \{0, 1, \ldots, n-1\}\).

The linear stability of a system is analyzed by determining the eigenvalues of its perturbation matrix. In our problem stability requires that all the eigenvalues \(\lambda_i\) of the perturbation matrix have to be purely imaginary. The system, after perturbation, will then oscillates about its original equilibrium configuration with a finite amplitude. This form of stability is appropriate for systems where dissipation of energy is not allowed. From the theory of equations we know that the purely imaginary roots of an algebraic equation of even order with real coefficients, must appear as complex conjugate pairs. It is easily checked that the perturbation matrix \(T\) of our problem is of even order and there are at least four stability conditions, which are

1. \(B^k = \sum \lambda_i^k = 0\) (for \(k = 1, 3\))
2. \(B^2 < 0\)

From now on, by \(T\) we mean the perturbation matrix

\[
\begin{bmatrix}
A & M^{-1} \\
S & A
\end{bmatrix}
\]

and \(T\) as the newly formulated one.

In this new basis the first order linear perturbation equation takes the form:

\[
\begin{bmatrix}
\delta X_0 \\
\delta X_1 \\
\vdots \\
\delta X_{n-1}
\end{bmatrix} = T
\begin{bmatrix}
\delta Y_0 \\
\delta Y_1 \\
\vdots \\
\delta Y_{n-1}
\end{bmatrix}
\]

where \(n' = n - 1\) and \(I\) is the \(2 \times 2\) identity matrix.
3. $B^4 > 0$.

These results hold when the four eigenvalues of the perturbation matrix are all purely imaginary. It is easily checked that $B^k = 0, k = 1, 3$. Hence a necessary condition for the stability is satisfied. Next we determine $\text{tr}(TP_b)$ and check if the stability conditions holds for each irreducible subspace.

From equation (11) it is clear that $t(0,0) = t(0) = \begin{bmatrix} i\omega & 0 & I & 0 \\ 0 & 0 & 0 & 1/mI \\ S_{00} & i\omega & 0 \\ S_{0n} & 0 & 0 & 0 \end{bmatrix}$.

and $t(0,j) = t(j) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ S_{01} & 0 & 0 & 0 \\ S_{jn} & 0 & 0 & 0 \end{bmatrix}$ for $j \neq n$ and $j \neq 0$.

This shows $\text{tr}(TP_b) = 0$ since $\text{tr}(t(0)) = \text{tr}(t(j)) = 0$ i.e. $\sum \lambda_i = 0$ over the space $E_0$ (remember in our notation $i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$). This is a much stronger stability condition. The most general element $t(i,j)$ is given by $t(i,j) = \begin{bmatrix} (C)_{ij} & (D)_{ij} \\ (P)_{ij} & (Q)_{ij} \end{bmatrix}$. We have thus shown that the eigenvalues are of the matrix $TP_b$ are purely imaginary and from this it follows, as stated before, that $\sum \lambda^2 < 0$. We next examine the consequence of this condition.

As discussed in the section on the character formula, $\text{tr}(T^2P)$ gives the sum of the squares of the eigenvalues. Squaring $T$ we get: $T^2 = \begin{bmatrix} A^2 + M^{-1}S & AM^{-1} + M^{-1}A \\ AS + SA & SM^{-1} + A^2 \end{bmatrix}$. Next we now calculate $\sum \lambda^2$.

First we reformulate $T^2$ in the new basis

$$\begin{bmatrix} \delta x_0 \\ \delta x_{n-1} \\ \delta y_0 \\ \delta y_{n-1} \end{bmatrix}^T$$

and then use the method discussed earlier to calculate the matrix $t(i,j)$ and extract from it the matrix $t(0,j)$, where the identity of the group has the label 0. Substituting $t(0,j)$ in the character formula gives the relation $\sum \lambda^2 \propto \sum_{k=0}^{n} \chi(k) t(0,0)$, we get:

$$\sum \lambda^2 \propto \left[ -4\omega^2 + 2m + \frac{1}{4} \sum_{k=1}^{n-1} \csc^3(\frac{k\pi}{2}) \right] + \sum_{k=1}^{n-1} 4\cos(\theta_k) \left(\frac{1}{4} \csc^3(\frac{k\pi}{2}) \right).$$

The sufficient condition for stability is $\sum \lambda^2 < 0$ which gives:

$$-4\omega^2 + 2m + \sum_{k=1}^{n-1} \csc^3(\frac{k\pi}{2}) + \sum_{k=1}^{n-1} 4\cos(\theta_k) \left(\frac{1}{4} \csc^3(\frac{k\pi}{2}) \right) < 0.$$
where $\tilde{T} = \begin{bmatrix} \hat{P}^{-1} A \hat{P} & \hat{P}^{-1} M^{-1} \hat{P} \\ \hat{P}^{-1} S \hat{P} & \hat{P}^{-1} A \hat{P} \end{bmatrix}$. Thus we now have a different perturbation matrix $\tilde{T}$ for our group $G$. Again we can see that $\sum \lambda_i = 0$ because $\text{tr}(t(j)) = 0 \forall j \in \{0, 1, \cdots, n-1\}$ hence the necessary condition for stability is satisfied. Since the eigenvalues are all purely imaginary we also have the conditions $\sum \lambda_i^2 < 0$ and $\sum \lambda_i > 0$ and so on. We will comment on the quartic constraint and possible higher order constraints later on. Now we show that the quadratic constraint leads to a mass constraint.

Substituting the relevant matrices in the character formula we get;

$$\sum \lambda_i^2 \propto \left[ -4a^2 + 2m + 4\sum_{k=0}^{\infty} \cos(\pi k) \cos(\theta_k) csc^3\left(\frac{\theta_k}{2}\right) \right] - \sum_{k=1}^{n-1} \cos(\theta_k) \cos(\theta_k) csc^3\left(\frac{\theta_k}{2}\right) < 0$$

i.e.

$$m > \frac{1}{2} \sum_{k=1}^{n-1} \cos(\theta_k) \cos(\theta_k) csc^3\left(\frac{\theta_k}{2}\right) - \frac{1}{2} \sum_{k=1}^{n-1} \cos(\theta_k) \cos(\theta_k) csc^3\left(\frac{\theta_k}{2}\right) - 2 \sum_{k=1}^{n-1} \cos(\theta_k) csc^3\left(\frac{\theta_k}{2}\right)$$

(13)

Thus we need to calculate $\sum_{k=1}^{n-1} \cos(\theta_k) \cos(\theta_k) csc^3\left(\frac{\theta_k}{2}\right)$. So we have:

$$\sum_{k=1}^{n-1} \cos(\theta_k) \cos(\theta_k) csc^3\left(\frac{\theta_k}{2}\right) = \frac{1}{2} \sum_{k=1}^{n-1} \left[ \cos\left(l + 1\right) \cos\left(l - 1\right) + \cos\left(l - 1\right) \cos\left(l + 1\right) \right] csc^3\left(\theta_k/2\right)$$

$$= \frac{1}{2} \sum_{k=1}^{n-1} \left[ \cos\left(l + 1\right) \theta_k + \cos\left(l - 1\right) \theta_k \right] csc^3\left(\theta_k/2\right)$$

$$= \frac{1}{2} \sum_{k=1}^{n-1} \left[ \cos\left(l + 1\right) \frac{\pi k}{n} + \cos\left(l - 1\right) \frac{\pi k}{n} \right] csc^3\left(\frac{\pi k}{n}\right)$$

Substituting $l = (n-1)/2$ for the strongest perturbation and taking the large $n$ limit we get;

$$\sum_{k=1}^{n-1} \cos(\theta_k) \cos(\theta_k) csc^3\left(\frac{\theta_k}{2}\right) \approx 2 \sum_{k=1}^{n-1} \cos(\pi k) csc^3\left(\frac{\pi k}{n}\right)$$

$$= 2 \sum_{k=1}^{n-1} \left(\frac{\pi k}{n}\right) csc^3\left(\frac{\pi k}{n}\right)$$

$$\approx 2 \pi \sum_{k=1}^{n-1} \left(\frac{\pi k}{n}\right) csc^3\left(\frac{\pi k}{n}\right)$$

$$= \frac{2\pi}{2} \sum_{k=1}^{n-1} \left(\frac{\pi k}{n}\right) csc^3\left(\frac{\pi k}{n}\right)$$

$$= - \sum_{k=1}^{n-1} \pi k csc^3\left(\frac{\pi k}{n}\right)$$

Substituting these results in (13) we get;

$$m > \frac{7n^3}{4\pi^3} \zeta(3) = 0.068n^3$$

Thus the introduction of a perturbation that changes both the radial positions and the angular coordinates in the system leads to a bigger mass value for stability. However, it is still smaller than the result found by Maxwell $(m > 0.3n^3)$. But we have yet to consider other constraints on the eigenvalues identified that are present in the problem.

Let us summarize the complete set of eigenvalue constraints necessary for linear stability and hence justify the list of constraints given above. We note that $TP_i$ is a $8 \times 8$ matrix but the effective rank of the matrix is 4 as the coupling between the two major $4 \times 4$ matrices of the system is very small. Using this fact reduces the rank of the matrix $TP_i$ to 4. So the characteristic polynomial of the matrix $TP_i$ is going to be of order four in $\lambda$ and for such an equation there are four conditions for stability which have to be imposed on the sums; $\sum \lambda_i$, $\sum \lambda_i^2$, $\sum \lambda_i^3$, and $\sum \lambda_i^4$. Out of these the sums of odd powers of the eigenvalues have to vanish. These two conditions we checked hold. Then we have conditions on the sum of the square of the eigenvalues, which we have examined and finally we have the condition: $\sum \lambda_i > 0$ as a condition for stability. Besides these conditions there is a further condition that follow from the conditions listed. As the eigenvalues are purely imaginary and complex conjugate of each other we can write $\lambda_1 = i\lambda$, $\lambda_2 = -i\lambda$, $\lambda_3 = i\mu$ and $\lambda_4 = -i\mu$ where $\lambda$ and $\mu$ are positive constants. This gives;

$$\sum_{i=1}^{4} \lambda_i = 0, \quad \sum_{i=1}^{4} \lambda_i^2 = 2(\lambda^2 + \mu^2), \quad \sum_{i=1}^{4} \lambda_i^3 = 0$$

and $\sum_{i=1}^{4} \lambda_i^4 = 2(\lambda^4 + \mu^4)$.

There no higher order independent constraints as the perturbation matrix is $4 \times 4$. Now we write $-2\lambda^2 = x$, $-2\mu^2 = y$ and $x + y = -\lambda$. Then, $2(\lambda^2 + \mu^2) = \frac{1}{2}(x^2 + y^2) = \beta > 0$.

Thus we have $x + y = -\alpha$ and $x^2 + y^2 = 2\beta$. Hence we get:

$$x^2 - (\alpha - x)^2 = 2\beta$$

$$\implies x = -\alpha \pm \sqrt{2\beta}.$$

Now the condition that $x < 0$ gives us a new condition to be satisfied: $2\beta < \alpha^2$. We do not analyze these conditions further as our aim was to show how easily the stability result $m > Cn^3$ follows from group theory and not to determine $C$.

4. CONCLUSION

We have described an induced representation character formula method for studying the stability of a system with a discrete group symmetry. The important point of the approach is that it does not require knowledge of the invariant subspaces of the system. The entire procedure is group theoretical. We saw that the reason for the emergence of induced representations was due to the fact that perturbations, in this framework, are linear maps between sections of a discrete vector bundle. Hence in order to exploit the symmetry properties of the bundle one needed to use induced representation. The value of the constant $C$ in the stability condition $m > Cn^3$ we found by the group theoretical method was less than that found by Maxwell. This is because we stopped at the level of the quadratic trace of eigenvalue constraints, for instance, the quartic constraint and other constraint identified, were not considered. The quartic constraint and other constraints when used lead to a quadratic equation in the square of one of the two imaginary eigenvalues of the system as shown in the other approaches and was thus not analyzed. Our aim was to show the power of the group theoretical approach to give the general structure of the mass constraint in a very simple way. We believe the group theoretical method described is a useful and
powerful tool for analyzing linear stability problems that have symmetry.

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