Morse theory in path space

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We consider the path space of a curved manifold on which a point particle is introduced in a conservative physical system with constant total energy to formulate its action functional and geodesic equation together with breaks on the path. The second variation of the action functional is exploited to yield the geodesic deviation equation and to discuss the Jacobi fields on the curved manifold. We investigate the topology of the path space using the action functional on it and its physical meaning by defining the gradient of the action functional, the space of bounded flow energy solutions and the moduli space associated with the critical points of the action functional. We also consider the particle motion on the $n$-sphere $S^n$ in the conservative physical system to discuss explicitly the moduli space of the path space and the corresponding homology groups.

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I. INTRODUCTION

In order to discuss the Morse inequalities [1, 2, 3], the supersymmetric quantum mechanics has been exploited by Witten [4]. Based on the spectral flow of the Hessian of the symplectic function [5], the Morse indices for pair of critical points of the symplectic action functional have been also investigated and on the Hilbert spaces the Morse homology [6] has been yielded by considering the flows of the critical points associated with the Morse index [7].

It has been noted [8, 9] in the hadron phenomenology using the chiral bag model that the quark phase spectrum is asymmetric about zero energy to yield the non-vanishing vacuum contribution to the baryon number. To obtain this vacuum contribution, the regularization has been exploited and it is closely related [10] to the eta invariant of Atiyah et al. [11]. The eta invariant has been later discussed by Witten in connection with the phase factor of the path integral in quantum field theory associated with the Jones polynomial and knot theory [12]. To relate the phase factor of the semiclassical partition function to the eta invariant [11], the Jacobi fields and their eigenvalues of the Sturm-Liouville operator associated with the particle geodesics on a curved manifold have been also investigated [13].

To yield geometric invariants of smooth four-manifolds, the moduli space of self-dual connections which are critical points of the Yang-Mills functional in SU(2) gauge theory [14, 15] and the moduli space of solutions of Seiberg-Witten monopole equations in U(1) gauge theory [16, 17] have been investigated. Recently, the Morse theoretical approach has been also applied to the Nambu-Goto string action functional to study the geodesic surface equation with the world sheet currents [18]. Constructing the second variation of the surface spanned by closed strings, the geodesic surface deviation equation has been discussed on the curved manifold, and the geodesic surface deviation equation in the orthonormal gauge has been derived to find the Jacobi field and the conjugate strings on the geodesic surface.

In this paper we will investigate the physical changes of the action by studying the geometry of the moduli space associated with the critical points of the action functional and the asymptotic boundary conditions in path space for point particles in a conservative physical system with constant total energy, after formulating the geodesic equation and geodesic deviation equation together with breaks on the path. Explicitly we will study the particle motion on the $n$-sphere $S^n$ in the conservative physical system to discuss the moduli space of the path space and the corresponding homology groups associated with the boundary homomorphism.

In Section II, the action functional for a point particle will be introduced in a conservative physical system with constant total energy to investigate the geodesic equation together with breaks on the path. By taking the second variation of the action functional generated by point particles, the geodesic deviation equation will be discussed in terms of the Jacobi fields on the curved manifold. In Section III, exploiting the gradient of the action functional, the space of bounded flow energy solutions will be investigated together with the moduli space associated with the critical points of the action functional and the asymptotic boundary conditions. The boundary homomorphism will be also introduced to define the homology group of the path space. In Section IV, the particle in a conservative

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We consider a particle of mass $m$ on a curved $n$-dimensional manifold with metric $\eta_{ab}$ ($a, b = 1, 2, \cdots, n$) in a conservative physical system with constant total energy $E = T + V$, where $T$ and $V$ are the kinetic and potential energies, respectively. We then define the line element $ds^2 = -dt^2 + g_{ab}dx^adx^b$ with a dressed metric $g_{ab}$ associated with $E$ and $V$ to yield

$$g_{ab} = \frac{m(E - 2V)^2}{2(E - V)}\eta_{ab}. \quad (2.1)$$

In order to define an action functional on the curved manifold, let $(M, g_{ab})$ be the complete Riemannian manifold of dimension $n$ associated with the metric $g_{ab}$. Given $g_{ab}$, we can have a unique covariant derivative $\nabla_a$ satisfying $[\nabla_a \nabla_b \eta_{bc} = 0$, $\nabla_k \omega^a = \partial_k \omega^a + \Gamma^a_{bc} \omega^b$ and

$$\left(\nabla_a \nabla_b - \nabla_b \nabla_a\right)\omega_c = R_{abc}^d \omega_d. \quad (2.2)$$

Let $\Omega(M; p, q)$ be the set of piecewise smooth paths $\gamma(t)$ such that $\gamma : [0, 1] \to M$ and propagate from point $p$ to $q$ in $M$. On the manifold $M$ the action functional $S : \Omega(M; p, q) \to \mathbb{R}$ of the particle is given by

$$S = \int_0^1 d\tau \left(g_{ab}v^a v^b\right)^{1/2}, \quad (2.3)$$

with the proper time $\tau$ ($0 \leq \tau \leq 1$) and the vector field $v^a = (\partial / \partial \tau)^a \in T_\tau \Omega(M; p, q) \subset M$ be the deviation vector which comes from a variation $\tilde{\gamma} : (\epsilon, \epsilon) \times [0, 1] \to \Omega(M; p, q) \subset M$ such that $\tilde{\gamma}(0, \tau) = \gamma(t)$ and represents the displacement to an infinitesimally nearby path. Next let $\Sigma$ denote the two-dimensional submanifold spanned by the paths $\tilde{\gamma}(\alpha)$. We now may choose $\tau$ and $\alpha$ as coordinates of $\Sigma$ to yield the commutator relation,

$$\mathcal{L}_v \omega^a = v^b \nabla_b \omega^a - w^b \nabla_b v^a = 0. \quad (2.4)$$

The tangent space $T_\tau \Omega(M; p, q)$ of $\Omega(M; p, q)$ at a path $\gamma(t)$ will be then the vector space of all piecewise smooth vector fields $v^a$ along $\gamma(t)$ for which

$$v^a(0) = w^a(1) = 0. \quad (2.5)$$

Now we perform an infinitesimal variation of the paths $\gamma(t)$ traced by the particle during its evolution in order to find the geodesic equation from the least action principle. In the stationary phase approximation where $|v^a|$ is infinitesimally small, we find the first variation as follows

$$S^{(1)} = \frac{\partial S}{\partial \alpha} = \frac{1}{c} g^{ab} w_a |_{\alpha = 0} + \frac{1}{c} \int_0^1 d\tau \; \omega^a \nabla_a v^b, \quad (2.6)$$

where we have used that the Lagrangian is given by $L = \frac{1}{2} g_{ab} v^a v^b$ along the geodesic path $\gamma$. Without loss of generality, $w^a$ can be chosen orthogonal to $v^a$ and vanishes at end-points to yield the boundary conditions (2.5). Exploiting the boundary condition (2.5), the first term in (2.6) vanishes and the least action principle yields the geodesic equation

$$v^a \nabla_a v^b = 0. \quad (2.7)$$

If we have breaks $0 = \tau_0 < \cdots < \tau_k + 1 = 1$, and the restriction of $\gamma$ to each set $[\tau_{i-1}, \tau_i]$ is smooth, then the path $\gamma$ is piecewise smooth. However $v^a$ will generally have a discontinuity at each break $\tau_i$ ($1 \leq i \leq k$). This discontinuity is measured by

$$\Delta v^a(\tau_i) = v^a(\tau_i^+) - v^a(\tau_i^-), \quad (2.8)$$

where the first term derives from the restrictions $\gamma|_{[\tau_i, \tau_{i+1}]}$ and the second from $\gamma|_{[\tau_{i-1}, \tau_i]}$. If $\gamma$ and $v^a \in T_p M$, the breaks $\tau_1 < \cdots < \tau_k$ we have together with the conditions (2.5)

$$\sum_{i=0}^k v^a w_a |_{\tau_{i+1}} = -\sum_{i=1}^k w^a(\tau_i) \Delta v^a(\tau_i), \quad (2.9)$$

to yield

$$S^{(1)} = -\frac{1}{c} \sum_{i=1}^k w^a(\tau_i) \Delta v^a(\tau_i) - \frac{1}{c} \int_0^1 d\tau \; w_b v^a \nabla_a v^b. \quad (2.10)$$

Here a path $\gamma \in \Omega(M; p, q)$ is a critical point of $S$ if and only if the differential $dS_\gamma : T_\gamma \Omega(M; p, q) \to \mathbb{R}$ is zero, namely along the geodesic $dS_\gamma(w) = \frac{1}{c} \omega^a \nabla_a S(\gamma(\alpha)) = 0$ for every $w \in T_\gamma \Omega(M; p, q)$ or every variation $\gamma_\alpha$, if and only if $\gamma : [0, 1] \to M$ is a geodesic from $p$ to $q$ in $M$. For a given point $p \in M$ and a tangent vector $v^a \in T_p M$, there is a unique geodesic $\gamma_0 : \mathbb{R} \to M$ through $\gamma_0(0) = p$ whose tangent at $p$ is $v$. The exponential map $exp_p : T_p M \to M$ is defined by $exp_p(v) = \gamma_0(1)$. A point $q \in M$ is conjugate to $p$ if $q$ is a singular value of $exp_p : T_p M \to M$. The multiplicity of $p$ and $p$ as conjugate points is equal to the dimension of the null space of $d(exp_p)_v : T_v(T_p M) \to T_q M$.

In the stationary phase approximation where $|v^a|$ and $|w^a|$ are infinitesimally small for $w^1, w^2 \in T_\Omega(M; p, q)$, we find the second variation around the geodesic $\gamma$

$$S^{(2)} = \frac{\partial^2 S}{\partial \alpha_1 \partial \alpha_2} |_{\alpha_1 = \alpha_2 = 0} = -\frac{1}{c} \sum_{i=1}^k w^a \nabla_b (w_1a(\tau_i) \Delta v^a(\tau_i))$$

$$+ \frac{1}{c} \int_0^1 d\tau \; g_{ab} w_a^1 \Lambda_b^a w_c^2. \quad (2.11)$$

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1 In Ref. [33], a flat $n$-dimensional manifold with metric $\delta_{ab}$ ($a, b = 1, 2, \cdots, n$), instead of $\eta_{ab}$, was introduced in a conservative physical system with constant total energy.
where the Sturm-Liouville operator is given by
\[ A^u_b = -\delta^u_b v^c \nabla_c (v^d \nabla_d) - R_{bcd} a^c v^d, \quad (2.12) \]
If \( \gamma \) and \( v^a \) are in \( T_\gamma \) have no breaks, the second variation \( S(2) \) in \( (2.11) \) vanishes for all \( w^a_1 \in T_\gamma \Omega(M; p, q) \) if and only if
\[ v^b \nabla_b (v^c \nabla_c w^a_2) + R_{bcd} a^c v^d w^a_2 = 0. \quad (2.13) \]
We call then the \( w^a_2 \) a Jacobi field along \( \gamma \).

A point \( q \) is conjugate to \( p \) along a geodesic \( \gamma \) if and only if there is a non-zero Jacobi field \( J \) along \( \gamma \) such that \( J(0) = J(1) = 0 \). Also if \( p \) and \( q \) are not conjugate along a geodesic \( \gamma \), then a Jacobi field \( J \) along \( \gamma \) is determined by its values at \( p \) and \( q \). A vector field \( J \in T_\gamma \Omega(M; p, q) \) is the null space of \( S(2) \) if and only if \( J \) is a Jacobi field. Thus \( S(2) \) is degenerate if and only if \( p \) and \( q \) are conjugate along \( \gamma \). The nullity of \( S(2) \) is equal to the multiplicity (or the dimension of the space of all Jacobi fields) of \( p \) and \( q \) as conjugate points. In the above geodesic deviation equation \( (2.13) \) we have a Jacobi field \( w^a_2 \) along \( \gamma \). A variation \( \tilde{\gamma} \) through geodesics \( \gamma \) produces a Jacobi field along \( \gamma \) and conversely every Jacobi field is obtained by a variation of \( \gamma \) through geodesics. Define an inner product on \( \Omega(M; p, q) \) for each \( \gamma \in \Omega(M; p, q) \) and \( w^a_1, w^a_2 \in T_\gamma \Omega(M; p, q) \),
\[ (w_1, w_2) = \int_0^1 d\tau \, g_{ab} w^a_1 w^b_2. \quad (2.14) \]
With this inner product \( T_\gamma \Omega(M; p, q) \) will then be a Hilbert space and \( \Omega(M; p, q) \) a Hilbert manifold. The critical point \( \gamma \in \Omega(M; p, q) \) of \( S \) is a geodesic from \( p \) to \( q \). Its index \( \text{ind} (\gamma) \) is defined by the number of points \( \gamma(\tau) \), with \( 0 < \tau < 1 \), such that \( \gamma(\tau) \) is conjugate to \( \gamma(0) \) along \( \gamma \), where each conjugate point is counted with its multiplicity. A geodesic segment \( \gamma : [0, 1] \to M \) contains only finitely many points which are conjugate to \( \gamma(0) \) along \( \gamma \), and the multiplicity of each conjugate point is less than \( \text{dim} M = n \). The space \( \Omega(M; p, q) \) of paths from point \( p \) to \( q \) in \( M \), with the inner product \( (, ) \) is a Hilbert manifold.

III. MODULI SPACE \( \mathcal{M}(\gamma_1, \gamma_2) \)

By definition the gradient of the action functional \( S : \Omega(M; p, q) \to \mathbb{R} \), a vector field \( \nabla S \) on \( \Omega(M; p, q) \) is given by for each \( w^a \in T_\gamma \Omega(M; p, q) \)
\[ ds(w^a) = w^a \nabla_a S = \frac{\partial S}{\partial \alpha} \quad (3.1) \]
which is equivalent to the first variation \( (2.6) \). We thus find \( ds(w^a) = 0 \) for all \( w^a \) if and only if \( \gamma \) is geodesic. If the path \( \gamma \) is smooth and \( \gamma \) and \( v^a \in T_\gamma \) have no breaks, then we obtain
\[ ds(w^a) = -\frac{1}{c} \int_0^1 d\tau \, w^a \nabla_b v^a. \quad (3.2) \]
to yield
\[ \nabla_a S = -\frac{1}{c} \int_0^1 d\tau \, v^b \nabla_b v^a. \quad (3.3) \]
Exploiting \( (3.3) \), we introduce the vector field \( u^a = (\partial/\partial \beta)^a \) associated with the gradient flow
\[ \frac{d}{d\tau} \frac{\partial S}{\partial \beta} = -\frac{\partial S}{\partial \beta} \frac{\partial S}{\partial \beta}, \quad \text{where the trajectory} \quad \tilde{\gamma}(\beta) \in \Omega(M; p, q) \quad \text{is identified with the map} \quad \tilde{\gamma} : \mathbb{R} \times [0, 1] \to M \text{ given by} \quad \tilde{\gamma}(\beta)(\tau) = \tilde{\gamma}(\beta, \tau) \quad \text{satisfying} \]
\[ u^a = \frac{1}{c} v^b \nabla_b v^a. \quad (3.4) \]
If for each \( \beta \in \mathbb{R} \), \( \tilde{\gamma}(\beta, \tau) \) is geodesic, then \( \frac{d\tilde{\gamma}(\beta, \tau)}{d\beta} \big|_{\tau=0} \) is a Jacobi field and \( \tilde{\gamma}(0, \tau) = \gamma(\tau) \) a geodesic path joining \( p \) and \( q \) in \( M \). If \( \gamma(\beta, \tau) \) satisfies the asymptotic boundary conditions
\[ \lim_{\beta \to -\infty} \gamma(\beta, \tau) = \gamma_1(\tau), \quad \lim_{\beta \to \infty} \gamma(\beta, \tau) = \gamma_2(\tau) \quad (3.5) \]
and \( \lim_{\beta \to \pm \infty} \frac{d\tilde{\gamma}(\beta, \tau)}{d\beta} \big|_{\tau=0} = 0 \) then \( \gamma(\beta) \) is a trajectory in \( \Omega(M; p, q) \) of the gradient flow joining the geodesic \( \gamma_1(\tau) \) and \( \gamma_2(\tau) \).

There is a natural finite dimensional approximation of the full path space \( \Omega(M; p, q) \), namely for \( 0 < a_1 < \cdots < a_i < \cdots < a_\infty \), let \( \Omega^{a_i} = S^{-1}[0, a_i] \), then \( \Omega^{\infty} = \Omega(M; p, q) \). We choose a subdivision \( 0 = \tau_0 < \tau_1 < \cdots < \tau_\lambda = 1 \) of the unit interval \([0, 1]\). Let \( \Omega(\tau_0, \cdots, \tau_\lambda) = \{ \gamma \in \Omega(M; p, q) \mid \gamma(0) = p, \gamma(1) = q, \text{geodesic for each} \ i = 1, \cdots, \lambda \} \), then in fact \( \Omega(\tau_0, \cdots, \tau_\lambda) \subset \Omega(\tau_0, \cdots, \tau_\lambda) \cap \Omega^{a_i} \) is a finite dimensional space and \( \Omega(M; p, q) \) has a homotopy type of a countable CW-complex \([2, 19]\) which contains one cell of dimension \( \text{ind} (\gamma) \) for each geodesic \( \gamma \) (critical point of \( S \)) in \( \Omega(M; p, q) \). Suppose \( \gamma_1 \in \Omega(M; p, q) \) are critical points of the action functional \( S \) with index \( k \) and \( k-1 \), respectively. By Sard’s theorem, for \( p \in M \), almost all \( q \in M \) are not conjugate to \( p \) along any geodesic.

We denote by \( \mathcal{M}(M; p, q) \) the space of bounded flow energy solutions of \( (3.4) \), namely
\[ \mathcal{M}(M; p, q) = \{ \tilde{\gamma} : \mathbb{R} \times [0, 1] \to M \mid u^a = \frac{1}{c} v^b \nabla_b v^a = 0 \} \quad (3.6) \]
associated with the bounded flow energy
\[ \Phi(\tilde{\gamma}) = \int_{-\infty}^{\infty} \int_0^1 d\beta \, d\tau \left( |u^a|^2 + \frac{1}{c^2} |v^b \nabla_b v^a|^2 \right). \quad (3.7) \]
The space \( \mathcal{M}(M; p, q) \) may not be compact in the topology of uniform convergence with all derivative. Assume the flow is of Morse-Smale type \([20]\): for every pair \( (\gamma_1, \gamma_2) \) of critical points the unstable submanifold \( W^u(\gamma_1) \) and the stable submanifold \( W^s(\gamma_2) \) intersect transversely.
\[ \mathcal{M}(\gamma_1, \gamma_2) = \{ \tilde{\gamma} : \mathbb{R} \times [0, 1] \to M \mid u^a - \frac{1}{c} v^b \nabla_b v^a = 0, \quad \lim_{\beta \to -\infty} \tilde{\gamma}(\beta, \tau) = \gamma_1(\tau), \lim_{\beta \to \infty} \tilde{\gamma}(\beta, \tau) = \gamma_2(\tau) \} = W^u(\gamma_1) \cap W^s(\gamma_2). \quad (3.8) \]
and the dimension of \( \mathcal{M}(\gamma_1, \gamma_2) \) is given by
\[
dim \mathcal{M}(\gamma_1, \gamma_2) = \text{ind} \gamma_1 - \text{ind} \gamma_2.
\]
Moreover if \( \dim \mathcal{M}(\gamma_1, \gamma_2) = 1 \), then the manifold of unparametrized trajectories from \( \gamma_1 \) to \( \gamma_2 \), \( \mathcal{M}(\gamma_1, \gamma_2) = \mathcal{M}(\gamma_1, \gamma_2)/R \) has dimension 0 and is compact and orientable. Here note that
\[
\int_{-\infty}^{\infty} \int_{0}^{1} d\beta \, d\tau \left| u^a - \frac{1}{c} v^b \nabla_b v^a \right|^2 = \Phi(\gamma) + 2S(\gamma_2) - 2S(\gamma_1),
\] (3.9)
where we have used (3.3). If \( \gamma \in \mathcal{M}(\gamma_1, \gamma_2) \), then \( \Phi(\gamma) = 2S(\gamma_1) - 2S(\gamma_2) \). Thus \( \mathcal{M}(\gamma_1, \gamma_2) \) is the set of absolute minima of the function \( \Phi \) subject to the asymptotic boundary conditions (3.5).

For each pair \((\gamma_1, \gamma_2)\) of critical points of the action functional \( S : \Omega(M; p, q) \to \mathbb{R} \), we have the space of trajectories of the gradient flow of Morse-Smale type connecting \( \gamma_1 \) and \( \gamma_2 \), namely (3.8). Every trajectory \( \gamma : \mathbb{R} \times [0, 1] \to M \) has a one-dimensional family of reparametrization \( \gamma(\lambda + \beta, \tau) \), \( \lambda \in \mathbb{R} \). Denote the space of unparametrized trajectories from \( \gamma_1 \) to \( \gamma_2 \) by \( \mathcal{M}(\gamma_1, \gamma_2) \).

For a trajectory \( \gamma : \mathbb{R} \times [0, 1] \to M \), we have the asymptotic boundary conditions (3.5) and the end point conditions \( \gamma(0, \beta, \tau) = \gamma(3.1) = q \) and the trajectory \( \gamma(\beta, \tau) \) satisfies (3.4).

For each nonnegative integer \( k \), let \( C_k \) be the free abelian group generated by the set of all critical points \( \gamma \) with index \( k \) of the action functional \( S : \Omega(M; p, q) \to \mathbb{R} \). If \( \gamma \in C_k \) and \( \gamma \in C_{k-1} \), then \( \mathcal{M}(\gamma_1, \gamma_2) \) is a zero-dimensional compact oriented manifold. Let \( n(\gamma_1, \gamma_2) \) be the number of points of \( \mathcal{M}(\gamma_1, \gamma_2) \) counted with the sign of \( M \).

We define, as usual, the boundary homomorphism
\[
\partial_k : C_k \to C_{k-1}
\] (3.10)
by
\[
\partial_k(\gamma_1) = \sum_{\gamma_2 \in C_{k-1}} n(\gamma_1, \gamma_2) \gamma_2, \quad (3.11)
\]
then the composition of consecutive homomorphisms is zero, namely \( \partial_k \circ \partial_k = 0 \) for all \( k \) [21] and the homology group of \( \Omega(M; p, q) \) is
\[
H_k(\Omega(M; p, q), \mathbb{Z}) = \ker \partial_k : C_k \to C_{k-1}/ \text{im} \partial_k : C_{k+1} \to C_k. \quad (3.12)
\]

For homology theory, see Refs. [19, 22].

We denote by \( M(S) \) the space of smooth functions \( \gamma : \mathbb{R} \times [0, 1] \to M \) which satisfy (3.4) and have finite flow energy \( \Phi(\gamma) \) in (3.7). Then \( \mathcal{M}(\gamma_1, \gamma_2) \subseteq M(S) \) is the set of absolute minima of the energy functional \( \Phi(\gamma) \) subject to the asymptotic boundary conditions (3.5). We consider a vector field \( F : M(S) \to TM(S) \) on \( M(S) \) given by for \( \gamma \in M(S) \)
\[
F^a(\gamma) = u^a - \frac{1}{c} v^b \nabla_b v^a. \quad (3.13)
\]

Then \( M(\gamma_1, \gamma_2) \subseteq F^{-1}(0) \). Moreover if \( \gamma \in M(\gamma_1, \gamma_2) \), then the projection to the fiber of the differential of \( F \) at \( \gamma \), \( dF_\gamma : T_\gamma M(S) \to T_\gamma M(S) \) is given by along the geodesic \( dF_\gamma(w^a) = \frac{\partial}{\partial \alpha} F(\tilde{\gamma}) \) where \( \tilde{\gamma} : (-\varepsilon, \varepsilon) \times \mathbb{R} \to \Omega([0, 1] \to M \) defined by \( \tilde{\gamma}(0, \beta, \tau) = \gamma(\beta, \tau) \). The map \( \tilde{\gamma}(\alpha, \beta, \tau) \) satisfies \( \tilde{\gamma}(0, \beta, 0) = p, \tilde{\gamma}(0, \beta, 1) = q \) and the asymptotic boundary conditions \( \lim_{\beta \to -\infty} \tilde{\gamma}(0, \beta, \tau) = \gamma_1(\tau) \) and \( \lim_{\beta \to \infty} \tilde{\gamma}(0, \beta, \tau) = \gamma_2(\tau) \). Moreover along the geodesic we find
\[
\frac{\partial F^a}{\partial \alpha} = u^b \nabla_b w^a + 1/c(\Lambda w) - a \quad (3.14)
\]
where \( \Lambda \) is the Sturm-Liouville operator in (2.12). Here we have used the commutator relations (2.4) and
\[
\mathcal{L}_u w^a = u^b \nabla_b w^a - w^b \nabla_b u^a = 0. \quad (3.15)
\]
For \( w^1, w^2 \in T_M(S) \), we have \( L^2 \)-inner product on \( T_M(S) \)
\[
(w_1, w_2) = \int_{-\infty}^{\infty} \int_{0}^{1} d\beta \, d\tau \sum_{i=1}^{n} \frac{\partial F^a}{\partial \alpha} u^b \nabla_b w^a + 1/c(\Lambda w) - a \quad (3.16)
\]
where \( \Lambda \) is the Sturm-Liouville operator in (2.12), and \( \tilde{\gamma}_i = \gamma^a w^a \) \( (i = 1, 2) \) and \( \tilde{\gamma}_i = \gamma^a w^a \) is constant and the potential energy \( \Phi(\gamma) \) is given by along the geodesic \( \tilde{\gamma}(\alpha, \beta, \tau) \) where \( \Lambda \) is the Sturm-Liouville operator in (2.12).

IV. EXAMPLE ON THE SPHERES

We consider a particle of mass \( m \) on the \( n \)-sphere \( S^n \) in a conservative physical system where the total energy \( E \) is constant and the potential energy \( V(r) \) depends only on the radial distance \( r \) from the center of the sphere \( S^n \), such as a particle on \( S^n \) in an attractive gravitational potential of \( V(r) \). In this case both the metrics \( g_{ab} \) and \( g_{ab} \) in (2.1) are just metrics for \( S^n \). Suppose that two points \( p \) and \( q \) in \( S^n \) are neither identical nor antipodal. Then there are \( 2n \) geodesics connecting \( \gamma_1 \), \( \gamma_2 \), \( \cdots \) from \( p \) to \( q \) in \( S^n \). Here let \( \gamma_0 \) be the shortest great circle arc \( pq \) from \( p \) to \( q \), let \( \gamma_1 \) be the long circle arc \( (p(-q))(-q)q \), let \( \gamma_2 \) be the arc \( pq(-p)(-q)\)q, and so on. The set \( C(S) = \{ \gamma_0, \gamma_1, \cdots \} \) is the critical points of the action functional \( S : \Omega(S^n; p, q) \to \mathbb{R} \).

The subscript \( k \) of \( \gamma_k \) is the number of times that \( p \) or \( (-p) \) lies in the interior of \( \gamma_k \). Each of the points \( p \) or \( (-p) \) in the interior of \( \gamma_k \) is conjugate to \( p \) with multiplicity \( n-1 \). That is the path space \( \Omega(S^n; p, q) \) has the homotopy type of a CW-complex structure [2, 19] with one cell each in the dimension \( 0, n-1, 2(n-1), \cdots \). Using the CW-complex structure and the trajectory of gradient flow of \( S \), we may compute the homology groups of \( \Omega(S^n; p, q) \). For \( n \geq 3 \), since all boundary map of the complex \( C_1(\Omega(S^n; p, q)) \) is zero, the homology group \( H_k(\Omega(S^n; p, q)) = \mathbb{Z} \), \( k = 0, 1, 2, \cdots \). For \( n = 1 \), the path space \( \Omega(S^1; p, q) \)
has countably many connected components. Each component of them is contractible and has minimum action functional at the unique geodesic in its component. The path space $\Omega(S^1; p, q)$ has the homotopy type of a CW-complex structure with countably many zero-cells. Thus the homology group of $\Omega(S^1; p, q)$ is the group of countable direct sum of $\mathbb{Z}$ at zero dimension; $H_0(\Omega(S^1; p, q)) = \bigoplus \mathbb{Z}$. For $n = 2$ the path space $\Omega(S^2; p, q)$ has the homotopy type of a CW-complex structure with one cell in each dimension. The singular complex of $\Omega(S^2; p, q)$ is $\ldots \rightarrow C_k = \langle \gamma_k \rangle = \mathbb{Z} \xrightarrow{\partial} C_{k-1} = \langle \gamma_{k-1} \rangle = \mathbb{Z} \rightarrow \ldots$ (4.1)

The space $\Omega(S^2; p, q)$ has only one connected component since $S^2$ is simply connected. (In fact, so is $\Omega(S^n; p, q)$ if $n > 2$.) The moduli space $\mathcal{M}(\gamma_k, \gamma_{k-1})$ has two distinct unparameterized trajectories with opposite orientations. Thus $\partial_k = 0$ for all $k \geq 0$ and the homology groups of $\Omega(S^2; p, q)$ are

$$H_k(\Omega(S^2; p, q)) = \mathbb{Z}, \quad \text{for all } k \geq 0. \quad (4.2)$$

| Table I: Homologies of $S^n$ and $\Omega(S^n; p, q)$ |
|-----------------------------------------------|
| $\text{dim}$ | $\text{space}$ | $\text{dim}$ | $\text{space}$ |
|------------|----------------|-------------|----------------|
| 0          | $\mathbb{Z} \oplus \mathbb{Z}$ | 0           | $\mathbb{Z}$  |
| 1          | $\mathbb{Z} \oplus \mathbb{Z}$ | 1           | $\mathbb{Z}$  |
| 2          | $\mathbb{Z} \oplus \mathbb{Z}$ | 2           | $\mathbb{Z}$  |
| 3          | $\mathbb{Z} \oplus \mathbb{Z}$ | 3           | $\mathbb{Z}$  |
| 4          | $\mathbb{Z} \oplus \mathbb{Z}$ | 4           | $\mathbb{Z}$  |

The real projective space $\mathbb{R}P^\infty$ of dimension $\infty$ has also a CW-complex structure with one cell in each dimension. However the homology group $H_k(\mathbb{R}P^\infty)$ of $\mathbb{R}P^\infty$ is different with the one of $\Omega(S^2; p, q)$. Thus $\Omega(S^2; p, q)$ and $\mathbb{R}P^\infty$ are not homotopically equivalent. The homology of real projective space $\mathbb{R}P^\infty$ is

$$H_k(\mathbb{R}P^\infty) = \begin{cases} 
\mathbb{Z} & \text{if } k = 0 \\
\mathbb{Z}_2 & \text{if } k > 0 \text{ is odd} \\
0 & \text{if } k > 0 \text{ is even}.
\end{cases} \quad (4.3)$$

V. CONCLUSIONS

In conclusion, the action functional for a point particle has been introduced in a conservative physical system with constant total energy to formulate the geodesic equation together with breaks on the path. By taking the second variation of the action functional, the geodesic deviation equation has been derived and discussed in terms of the Jacobi fields on the curved manifold.

Defining the gradient of the action functional, the space of bounded flow energy solutions has been investigated to construct the moduli space associated with the critical points of the action functional and the asymptotic boundary conditions. The boundary homomorphism has been also introduced to define the homology group of the path space. We have considered the particle motion on the $n$-sphere $S^n$ in the conservative physical system to discuss the moduli space of the path space and the corresponding homology groups.

Applying the Morse theoretic approach developed for the point particle to the string theory, one could consider the gradient of the string action functional and the moduli space associated with the critical strings of the string action functional. It would be also desirable if the homology group of the stringy tube space can be studied in the framework of the Morse theory. These works are in progress and will be reported elsewhere.

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