Non-KPZ modes in two-species driven diffusive systems

V. Popkov, J. Schmidt

Institut für Theoretische Physik, Universität zu Köln, Zülpicher Str. 77, 50937 Cologne, Germany.

G.M. Schütz
Institute of Complex Systems II, Theoretical Soft Matter and Biophysics, Forschungszentrum Jülich, 52425 Jülich, Germany and Interdisziplinäres Zentrum für Komplexe Systeme, Universität Bonn, Brühlcr Str. 7, 53119 Bonn, Germany

(Dated: January 6, 2014)

Using mode coupling theory and dynamical Monte-Carlo simulations we investigate the scaling behaviour of the dynamical structure function of a two-species asymmetric simple exclusion process, consisting of two coupled single-lane asymmetric simple exclusion processes. We demonstrate the appearance of a superdiffusive mode with dynamical exponent \( z = 5/3 \) in the density fluctuations, along with a KPZ mode with \( z = 3/2 \) and argue that this phenomenon is generic for short-ranged driven diffusive systems with more than one conserved density. When the dynamics is symmetric under the interchange of the two lanes a diffusive mode with \( z = 2 \) appears instead of the non-KPZ superdiffusive mode.

PACS numbers: 05.60.Cd, 05.20.Jj, 05.70.Ln, 47.10.-g

Transport in one dimension has been known for a long time to be usually anomalous \([1,2]\). Signatures of this behaviour are a superdiffusive dynamical structure function and a power law divergence of transport coefficients with system size, characterized by universal critical exponents. Unfortunately, however, despite a vast body of work, analytical results for model systems have remained scarce and numerical results are often inconclusive. Therefore the exact calculation of the dynamic structure function for the universality class of the Kardar-Parisi-Zhang-equation \([3]\) with dynamical exponent \( z = 3/2 \) some ten years ago came as a major breakthrough. This function was obtained for a specific driven diffusive system, the asymmetric simple exclusion process which has a single conserved density and hence a single mode, the KPZ-mode. By virtue of universality many results for other types of systems such as growth models \([4]\), hard-core particle systems \([5]\) or anharmonic chains \([6,7]\) can thus be understood in terms of the KPZ universality class.

More recently it was established in the context of anharmonic chains \([9]\) and very general short-ranged one-dimensional Hamiltonian systems \([10]\) that in the presence of more than one conserved quantity the dynamics is richer and other modes have to be expected. In particular, in systems with three conservation laws a heat mode with \( z = 5/3 \) (corresponding to a divergent finite-size heat conductivity \( C \propto L^{1/3} \)) or a diffusive mode with \( z = 2 \) may be present besides two KPZ modes. The main assumption underlying these conclusions is that the relevant slow modes are given by the long wave length relaxation behaviour of the conserved quantities \([2,10]\).

Going back to driven diffusive systems we note that somewhat surprisingly, there is little information about the dynamical structure function in driven diffusive systems with more than one conservation law. In one dimension these systems are known to exhibit extremely rich stationary and dynamical behaviour and they serve widely as paradigmatic models for the detailed study of non-equilibrium phenomena. In view of this it is of interest to explore the transport properties of such systems, in particular which modes govern the fluctuations of the locally conserved slow modes.

In this spirit Ferrari et al. \([11]\) studied very recently a two-species exclusion process, using mode-coupling theory and Monte-Carlo simulations, and found two very clean KPZ-modes, but no other modes. For a similar model, exact finite size scaling analysis of the spectrum indicates a dynamical exponent \( z = 3/2 \) \([12]\). In older work on other lattice gas models with two conservation laws the presence of a KPZ mode and a diffusive mode was observed \([13,14]\). So far there has been no indication of the existence of a heat mode. In the light of the work \([9,10]\) on short-ranged Hamiltonian systems this is intriguing and raises the question whether a mode with \( z = 5/3 \) can exist in driven diffusive systems, and, if yes, how many conservation laws are required to generate it. In this letter we answer these questions by using the mode coupling theory developed in \([8,11]\) for non-linear fluctuating hydrodynamics and by confirming the analytical findings with Monte-Carlo simulations of a two-species asymmetric simple exclusion process. It will transpire that a superdiffusive \( z = 5/3 \) mode along with a KPZ mode exists and that two conservation laws are sufficient to generate the phenomenon. Also a KPZ mode along with a diffusive mode can occur on a line of higher symmetry, which is reminiscent of a similar behaviour in anharmonic chains \([3]\).

We consider the following stochastic lattice gas model. Particles hop randomly on two parallel chains with \( N \) sites each, without exchanging the lane, unidirectionally
leads to stationary currents and with a hard core exclusion and periodic boundary conditions. We denote the particle occupation number on site \( k \) in the first (upper) lane by \( n_k \), and on the other lane by \( m_k \). A hopping event from site \( k \) to site \( k + 1 \) on the same lane may happen if site \( k \) is occupied and site \( k + 1 \) is empty. The rate of hopping depends on the sum of particle numbers at sites \( k, k + 1 \) on the adjacent lane as follows (Fig. 1): Let us denote the sum of particles on the sites \( k, k + 1 \) at the second lane as \( m := m_k + m_{k+1} \). Then the rates \( r_m \) of hopping from site \( k \) to site \( k + 1 \) on the first lane are given by

\[
r_m = 1 + \frac{\gamma m}{2},
\]

where \( \gamma \) is a coupling parameter. Analogously, the rates \( d_n \) of hopping from site \( k \) to site \( k + 1 \) on the second (lower) lane are given by

\[
d_n = b + \frac{\gamma n}{2}
\]

with the sum of particles \( n := n_k + n_{k+1} \) on the sites \( k, k + 1 \) at the first lane. Since there is only hopping within lanes, the total number of particles \( M_i \) in each lane is conserved. The parameter \( b \) makes the lanes inequivalent. Since the rates have to be nonnegative, if follows that \( \gamma \geq \min(1, b) \). For \( b = 1 \) we recover the two-lane model of \[13\].

The model in a more general multilane geometry was introduced in \[10\]. It was shown that the choice of rates \( b, \gamma \) results in a stationary distribution which is a product measure, both between lanes and between the sites. For the two-lane system that we study here this leads to stationary currents

\[
\begin{align*}
    j_1(\rho_1, \rho_2) &= \rho_1(1 - \rho_1)(1 + \gamma \rho_2) \\
    j_2(\rho_1, \rho_2) &= \rho_2(1 - \rho_2)(b + \gamma \rho_1)
\end{align*}
\]

where \( \rho_{1,2} = M_{1,2}/N \) are the densities of particles in the first and second lane respectively. Notice that a product measure corresponds to a grandcanonical ensemble with a fluctuating particle number. These fluctuations are described by the symmetric compressibility matrix \( C \) with matrix elements

\[
C_{ij} = \frac{1}{N} < (M_i - \rho_i N)(M_j - \rho_j N) >= \rho_i(1 - \rho_i)\delta_{i,j}.
\]

The starting point for investigating the large-scale dynamics of this microscopic model is the system of conservation laws \( \partial_t \rho_i(x,t) + \partial_x j_i(x,t) = 0 \) where \( \rho_i(x,t) \) is the coarse-grained local density of component \( i \), and \( j_i(x,t) \) is the associated current given as a function of the local densities by \[3\]. These equations can be written in vector form as

\[
\frac{\partial}{\partial t} \bar{\rho} + A \frac{\partial}{\partial x} \bar{\rho} = 0
\]

where \( \bar{\rho} \) is the column vector with the densities as entries and \( A \) is the current Jacobian with matrix elements \( A_{ij} = \partial j_i / \partial \rho_j \). Its eigenvalues \( c_i \) are the characteristic velocities which on microscopic scale are the speeds of local perturbations \[13\]. The matrix \( S = AC \) is symmetric \[15\] which guarantees that the system \[5\] is hyperbolic \[19\].

Eq. \( 5 \) describes the deterministic time evolution of the density under Eulerian scaling. The effect of fluctuations, which occur on finer space-time scales, can be captured by adding phenomenological white noise terms \( \xi \) and taking the non-linear fluctuating hydrodynamics approach together with a mode-coupling analysis of the non-linear equation \[11\]. In this framework one expands the local densities around their long-time stationary values \( \rho_i(x,t) = \rho_i + \eta_i(x,t) \) and transforms to normal modes \( \bar{\phi} = R\bar{\eta} \) where \( A \) is diagonal. The transformation matrix \( R \) uniquely defined by \( RAR^{-1} = \text{diag}(c_i) \) and the normalization condition \( RCR^T = 1 \). Keeping terms to first non-linear order yields

\[
\partial_t \phi_i = -\partial_x \left( c_i \phi_i + \frac{1}{2} \langle \bar{\phi}, G^{(i)} \bar{\phi} \rangle - \partial_x (D\bar{\phi})_i + (B\xi)_i \right).
\]

Here the angular brackets denote the inner product in component space and

\[
G^{(i)} = \frac{1}{2} \sum_j R_{ij}(R^{-1})^T H^{(j)} R^{-1}.
\]
out the scenarios relevant to our model, as predicted by mode-coupling theory. (i) If both \(G^{(1)}_{11}\) and \(G^{(2)}_{22}\) are non-zero we expect two KPZ modes with \(z = 3/2\). (ii) On the other hand, if e.g. \(G^{(1)}_{11} = 0\), but \(G^{(2)}_{22} \neq 0\) and \(G^{(2)}_{22} \neq 0\), then mode coupling theory predicts mode 1 to be a superdiffusive mode with \(z = 5/3\) and mode 2 to be KPZ. (iii) Finally, if both \(G^{(1)}_{11} = G^{(2)}_{22} = 0\) but \(G^{(2)}_{22} \neq 0\) then mode 1 becomes diffusive, while mode 2 is KPZ.

For our system, the explicit forms of \(A\) and \(H^{(i)}\) are

\[
A = \begin{pmatrix}
(1 + \gamma \rho_2)(1 - 2 \rho_1) & \gamma \rho_1(1 - \rho_1) \\
\gamma \rho_2(1 - \rho_2) & (b + \gamma \rho_1)(1 - 2 \rho_2)
\end{pmatrix}
\]

\[
H^{(1)} = \begin{pmatrix}
-2(1 + \gamma \rho_2) & \gamma(1 - 2 \rho_1) \\
\gamma(1 - 2 \rho_1) & 0
\end{pmatrix}
\]

\[
H^{(2)} = \begin{pmatrix}
0 & \gamma(1 - 2 \rho_2) \\
\gamma(1 - 2 \rho_2) & -2(b + \gamma \rho_1)
\end{pmatrix}
\]

To prove that all three scenarios (i) - (iii) can be realized, we choose \(\rho_1 = \rho_2 =: \rho\) and for convenience we set \(\gamma = 1\). Consider first \(b = 2\). Then

\[
R = R^0 \begin{pmatrix} 1 - \rho & -\rho \\ \rho & 1 - \rho \end{pmatrix}
\]

where \(R^0 = \sqrt{\rho(1 - \rho)(\rho^2 + (1 - \rho)^2)}\). The characteristic velocities are

\[
c_1 = 1 - \rho - 3\rho^2, \quad c_2 = 2 - 3\rho - \rho^2
\]

The matrices \(G^{(1)}, G^{(2)}\) are symmetric and have matrix elements \(G^{(1)}_{11} = -2g_0(6\rho^4 - 8\rho^3 + 5\rho^2 + \rho - 1), \quad G^{(1)}_{12} = G^{(1)}_{21} = g_0(4\rho^3 - 10\rho^2 + 8\rho - 1), \quad G^{(1)}_{22} = -2g_0\rho(1 - \rho)(2\rho^2 - 6\rho + 3)\) and \(G^{(2)}_{11} = 4g_0\rho(1 - \rho)\), \(G^{(2)}_{12} = G^{(2)}_{21} = -g_0(1 - 2\rho^2)^2, \quad G^{(2)}_{22} = 4g_0(1 - 3\rho(1 - \rho))\). with \(g_0 = -1/2 \rho(1 - \rho)/(1 - 2\rho(1 - \rho))^{3/2}\). Therefore, generically condition (i) for the presence of two KPZ modes is satisfied. However, while \(G^{(2)}_{11} \neq 0\) and \(G^{(2)}_{22} \neq 0\) \(\forall \rho \in (0, 1)\), the self coupling coefficient \(G^{(1)}_{11}\) changes sign at \(\rho^* = 0.45721 \ldots\). Since \(G^{(2)}_{22}(\rho^*) \neq 0\), the condition for case (ii), KPZ mode plus superdiffusive non-KPZ mode, is thus satisfied at density \(\rho = \rho^*\). In fact, diagonalizing \(A\) for arbitrary densities \(\rho_1, \rho_2\) one can show that for \(b \neq 1\) there is a curve in the space of densities where condition (ii) is satisfied. On the other hand, there is no density where condition (iii), \(G^{(1)}_{11} = G^{(2)}_{22} = 0\), is satisfied. Indeed, numerical inspection of the mode coupling matrices for several other parameter choices of \(\gamma\) and \(b\) suggests that condition (iii) cannot be satisfied when \(b \neq 1\).

Next we study \(b = 1\). In this case the system is symmetric under interchange of the two lanes, which is reflected in the relation \(J_2(\rho_1, \rho_2) = J_1(\rho_2, \rho_1)\) for the currents \(J_i\). Calculating the mode coupling matrices for \(\rho_1 = \rho_2 =: \rho\) and \(\gamma = 1\) yields

\[
G^{(1)} = \gamma_0(1 + \rho) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3\rho \end{pmatrix}, \quad G^{(2)} = \gamma_0(2 - \rho) \begin{pmatrix} 2 - \rho & 0 \\ 0 & 3\rho \end{pmatrix}
\]

with \(\gamma_0 = -2\rho(1 - \rho)\). Interestingly, in this case condition (iii) is satisfied only for \(\rho = 1\), i.e., mode 1 is expected to be diffusive and mode 2 is KPZ. The occurrence of a diffusive mode is somewhat counter-intuitive as both particle species interact and hop in a totally asymmetric fashion which in the case of the AHR-model prevents the existence of a diffusive mode \([1]\).

In order to check the predictions of mode coupling theory we performed dynamical Monte-Carlo simulations, using a random sequential update where in each step a random site is chosen uniformly and jumps are performed with probabilities determined by normalizing the jump rates \([1, 2]\) by the largest jump rate, provided the target site is empty. A full Monte Carlo time step then corresponds to \(2N\) such update steps. The initial distribution is sampled from the uniform distribution, except for the occupation number at site \(N/2\) which is determined according to the normal modes given by the transformation matrix \(R\). Averages are performed over up to \(10^8\) realizations of the process and \(N = 200 \ldots 300\). In order to measure the dynamical exponent, we compute the first and second moment of the dynamical structure function, from which we obtain the variance \(\sigma^2(t) = \langle X^2(t) \rangle > \langle X(t) \rangle^2 \propto t^{2/\zeta}\). of the density distribution as a function of time.

In order to test the existence of a superdiffusive non-KPZ mode we have chosen \(\gamma = -0.52588\) and \(b = 1.3\). This yields \(G^{(2)}_{22} = 0\) at \(\rho_1^* = \rho_2^* = 0.5500003 \approx 55/100\). The matrices \(G^{(1)}, G^{(2)}\) become

\[
G^{(1)} = \begin{pmatrix} 0.2950 & 0.0717 \\ 0.0717 & 0.3157 \end{pmatrix}, \quad G^{(2)} = \begin{pmatrix} 0.0706 & 0.2972 \\ 0.2972 & 0 \end{pmatrix}
\]

which means that mode 2 is expected to be a non-KPZ mode and mode 1 is KPZ. The corresponding characteristic velocities are \(c_1(\rho^*) = -0.2171, c_2(\rho^*) = 0.0449\) and the eigenvectors are \((-0.7465, 0.6654)^T\) for \(c_2\) (non-KPZ mode) and \((0.6654, 0.7465)^T\) for \(c_1\) (KPZ mode).

The simulations confirm the predictions, see Figs. 2 and 3. For both modes the measured velocity differs from the theoretical prediction by less than 0.003. A linear least-square fit on log-log scale of the simulation results for the variance of the non-KPZ mode 2 yields \(\zeta_{22}^2 = 1.19 \pm 0.02\), very close to the mode-coupling value \(2z_{22}^2 = 6/5 = 1.2\). For the amplitude \(c\) \(\xi^{-1/2}\) at the maximum as a function of time we find \(1/z_{22}^2 MC = 0.58\), also in good agreement with \(1/z_{22} = 0.6\). The fitted exponent \(2\zeta_{22}^2 MC = 1.302\) of the KPZ mode 1 deviates slightly from 4/3, which is consistent with the strong coupling to the non-KPZ mode: The matrix element \(G^{(1)}_{22} \approx 0.63\) is larger than the KPZ self-coupling constant \(G^{(1)}_{11} \approx 0.59\).
Figure 2: (Colour online) Case (ii): Dynamical structure functions for particles on chain 1, for the KPZ mode (top) and the non-KPZ mode (bottom) at different times from Monte Carlo simulations, averaged over $10^7$ histories for the KPZ (non-KPZ) mode for $N = 300$ ($N = 200$). Statistical errors are smaller than symbol size.

Figure 4: (Colour online) Case (iii): Dynamical structure function of the diffusive mode for particles on chain 2 with $N = 300$, with $c_2 = 0.2$ (bottom) at different times $t$, from Monte Carlo simulations, averaged over $10^7$ histories. Statistical errors are smaller than symbol size.

Figure 5: (Colour online) Case (iii). Variance of the dynamical structure function shown in Fig. 4, as function of time. The line with the predicted universal slope $2/\alpha = 1$ for the diffusive mode (bottom) are guides to the eye. Error bars (not shown) are approximately symbol size.

In order to test case (iii) (KPZ and diffusive mode), we choose $\gamma = -0.8, b = 1, \rho_1 = \rho_2 = 0.5$. The characteristic velocities are $c_1 = -0.2$ (eigenvector $(1, 1)^T/\sqrt{2}$) and $c_1 = 0.2$ (eigenvector $(1, -1)^T/\sqrt{2}$). The mode coupling matrices are given by

$$G_1 = 0.2121 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G_2 = 0.2121 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In summary, we have shown that the two-lane asymmetric simple exclusion process with two conservation laws exhibits anomalous transport and has a superdiffusive non-KPZ mode with dynamical exponent $\alpha = 5/3$ on a line in the space of conserved densities $(\rho_1, \rho_2)$, provided that the model is not symmetric under lane change. In the latter case of higher symmetry a diffusive mode can occur instead of the non-KPZ mode. This is surprising as the hopping of both particle species is totally asymmetric. We did not find any point in parameter space where the KPZ mode would be completely absent. We argue that the existence of a superdiffusive non-KPZ mode is generic for driven diffusive systems with more than one conservation law and will generally occur at some specific manifold in the space of conserved densities $\rho_i$. This new
universality class for anomalous transport in driven diffusive systems is expected to result in a novel exponent for the stationary density profile in open systems [20]. An interesting open problem that is raised by our findings is the role of symmetries for the suppression of the non-KPZ mode and the occurrence of a diffusive mode.

V.P. acknowledges financial support by DFG. G.M.S. thanks H. Spohn and H. van Beijeren for most illuminating discussions at TU Munich and at the Oberwolfach workshop Large Scale Stochastic Dynamics. We also thank them and C. Mendl for useful comments on the manuscript.

[1] B.J. Alder and T.E. Wainwright, Phys. Rev. Lett. 18, 988 (1967).
[2] M.H. Ernst, E.H. Hauge, and J.M.J. van Leeuwen, J. Stat. Phys. 15, 7 (1976).
[3] M. Prähofer and H. Spohn, in: In and Out of Equilibrium, edited by V. Sidoravicius, Vol. 51 of Progress in Probability (Birkhauser, Boston, 2002).
[4] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. 56, 889 (1986).
[5] M. Prähofer and H. Spohn, J. Stat. Phys. 115, 255 (2004).
[6] P. Grassberger, W. Nadler, and L. Yang, Phys. Rev. Lett. 89, 180601 (2002).
[7] L. Delfini, S. Lepri, R. Livi and A. Politi, J. Stat. Mech. P02007 (2007).
[8] C. Bernardin and P. Gonçalves, [arXiv:1205.1879v3 (2013).
[9] C.B. Mendl and H. Spohn, Phys. Rev. Lett. 111, 230601 (2013).
[10] H. van Beijeren, Phys. Rev. Lett. 108, 108601 (2012).
[11] P.L. Ferrari, T. Sasamoto and H. Spohn, J. Stat. Phys. 153, 377–399 (2013).
[12] C. Arita, A. Kuniba, K. Sakai and T. Sawabe, J. Phys. A: Math. Theor. 42 345002 (2009).
[13] D. Das, A. Basu, M. Barma, and S. Ramaswamy, Phys. Rev. E 64, 021402 (2001).
[14] A. Rákos and G.M. Schütz, J. Stat. Phys. 117, 55–76 (2004).
[15] V. Popkov and G.M. Schütz, J. Stat. Phys. 112, 523-540 (2003).
[16] V. Popkov and M. Salerno, Phys. Rev. E 69, 046103 (2004).
[17] C. Kipnis and C. Landim, Scaling limits of interacting particle systems (Springer, Berlin, 1999)
[18] R. Grisi and G.M. Schütz, J. Stat. Phys. 145, 1499–1512 (2011).
[19] B. Tóth and B. Valkó, J. Stat. Phys. 112, 497–521 (2003).
[20] J. Krug, Phys. Rev. Lett. 67 1882 (1991).