The Pendulum Arrangement: 
Maximizing the Escape Time of Heterogeneous Random Walks

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Abstract

We identify a fundamental phenomenon of heterogeneous one dimensional random walks: the escape (traversal) time is maximized when the heterogeneity in transition probabilities forms a pyramid-like potential barrier. This barrier corresponds to a distinct arrangement of transition probabilities, sometimes referred to as the pendulum arrangement. We reduce this problem to a sum over products, combinatorial optimization problem, proving that this unique structure always maximizes the escape time. This general property may influence studies in epidemiology, biology, and computer science to better understand escape time behavior and construct intruder-resilient networks.

1 Introduction

Estimating the escape behavior of random walks has been an important performance indicator in fields such as biology [22, 20], epidemiology [14, 9], cosmology [11], computer science [15], and more [21, 27, 3, 2]. Maximizing the escape time plays a crucial role in containing the spread of diseases or computer viruses [14, 18], where the probability of an epidemic outbreak is closely related to properties of the contact network [26]. In this work we identify a phenomenon related to the exact escape time of a heterogeneous random walk on the finite line. Specifically, we show that the escape time is always maximized by a unique structure of transition probabilities, also known as the “Pendulum Arrangement”.

The characteristics of escape times of random walks have been extensively studied under the names of first passage time, escape times, and hitting times. While analytical formulations of the escape time have been established [17, 1, 6], their analysis has mostly been based on mean-field theory, asymptotic characteristics, and approximations [17, 4, 1, 7, 25, 10, 13, 8]. Also related to our work, are studies on the speed of random walks in random environments [23, 24, 16, 19]. Specifically, [19] show that the speed is minimized asymptotically by equally spaced drifts on the line. In contrast, our work takes an exact, combinatorial view of the problem, revealing an intrinsic feature of the maximum escape time in the general setting of an arbitrary heterogeneous random walk.

We consider a heterogeneous random walk on a finite line [1]. Given a vector \( p = (p_1, \ldots, p_d) \) of \( d \) transition probabilities, the process, as depicted in Fig. 1, starts at position 0, moves backward with probability \( p_i \) (reflecting at 0), forward with probability \( 1 - p_i \), and ends once it reaches position \( d + 1 \). Our goal is to rearrange the elements of the vector \( p \) (corresponding to rearranging the transition probabilities of moving backward on the line), so as to maximize the expected escape time of the random walk, namely, the time to reach position \( d + 1 \) for the first time. Conceptually, we wish to form a potential barrier under a fixed budget, but are unsure where to place the barrier on the line.

It is not clear a-priori whether the structure of this barrier has a closed form solution as it may depend on delicate relationships between the given probabilities. Intuitively, one might choose to arrange the transition probabilities in decreasing or increasing order. Here, an increasing order of the probabilities corresponds to forming a potential barrier toward the end of the line, reinforcing nodes in the vicinity of the termination.

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node, whereas, a decreasing order corresponds to forming a barrier at the beginning of the line. Perhaps surprisingly, neither arrangement would maximize the escape time.

To obtain some intuition, consider an ascending order, where the highest probability is placed last. Notice that position $d$ is reached only after visiting position $d-1$, i.e., the second to last position is always visited more than the last position. It is thus unreasonable to place the highest probability last, as this would only decrease the expected escape time because it will be used less often. A similar argument can be made for a descending order, by switching between the first two probabilities. We will make this intuition precise in our complete derivation.

Our main result shows that there is a unique optimal order of the transition probabilities that does not depend on their absolute, but rather their relative value, i.e., their sorted order. This also implies that changing the probabilities in a way that does not change their sorted order does not change the optimal arrangement. More specifically, we prove that the optimal order of the probabilities is such that they form a special pyramid-like shape, sometimes referred to as the pendulum arrangement (see Fig. 2), where the highest probability is placed in the middle.

Finally, we formulate a continuous optimization variant of the problem, where the transition probabilities are optimized under limited budget constraints. We show that our main result can greatly diminish the complexity of finding an optimal solution. We also provide numerical experiments that illustrate the potential gains of using the pendulum arrangement, and discuss possible alternative statistics, including the minimum escape time.

2 Problem Statement

A vector $p = (p_1, \ldots, p_d) \in (0, 1)^d$ of transition probabilities defines a heterogeneous random walk on a finite line of $d+2$ states, as depicted in Fig. 1. Formally, this process is defined by the following Markov chain. Let $\mathcal{M}_p = \{X_t^p\}_{t=1}^\infty$ where $X_t^p \in \{0, 1, \ldots, d, d+1\}$ is a random process that satisfies

$$P(X_{t+1}^p = j | X_t^p = i) = \begin{cases} p_i, & 1 \leq i \leq d, j = i - 1 \\ 1 - p_i, & 1 \leq i \leq d, j = i + 1 \\ 1, & (j = 1 \land i = 0) \lor (j = i = d + 1) \\ 0, & \text{otherwise.} \end{cases}$$

We define the escape time $\tau\{p; k\}$ as the arrival time of $X_t^p$ to the termination state $d+1$ given that it started at state $k$, i.e.,

$$\tau\{p; k\} := \min\{t : X_t^p = d + 1; X_0^p = k\}.$$ 

Our goal is to find the arrangement of the elements of $p$ that maximizes the expected escape time starting at state $X_0 = 0$. Formally, let $\Sigma$ be the set of permutations on $\{1, \ldots, d\}$, i.e., $\sigma \in \Sigma$ is a bijective mapping of $\{1, \ldots, d\}$ onto itself. A vector $q = \sigma p$ is a permutation of the elements of $p$ defined as $q_i = p_{\sigma(i)}$. Our goal is to find a permutation $\sigma^* \in \Sigma$ such that

$$\sigma^* \in \arg \max_{\sigma \in \Sigma} \mathbb{E}\tau\{\sigma p; 0\}. \quad (P1)$$

In what follows we will show that $\sigma^*$ admits a unique solution that maps large transition values to the center, and small values to the edges of the line.
3 Main Result

This section states our main result, showing the optimal solution to Problem (P1) satisfies a unique symmetric arrangement, known as the pendulum arrangement, or its mirror. To that end, we define the mirror permutation $\sigma_{\text{mirror}}$, which reverses the vector it operates on.

**Definition 1 (Mirror Permutation).** The mirror permutation is defined by $\sigma_{\text{mirror}}(i) = d + 1 - i$.

Next, we define the pendulum arrangement.

**Definition 2 (Pendulum Arrangement).** We say $x \in \mathbb{R}^d$ satisfies the pendulum arrangement if

$$x_i \leq x_{d+1-i}, \forall 1 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor, x_{d+1-i} \leq x_{i+1}, \forall 1 \leq i \leq \left\lfloor \frac{d - 1}{2} \right\rfloor.$$  

We say $x_{\text{pend}}$ is a pendulum arrangement of $x$ if $\exists \sigma \in \Sigma$ such that $x_{\text{pend}} = \sigma x$ and $x_{\text{pend}}$ is a pendulum arrangement.

The pendulum arrangement has a special pyramid-like shape, as depicted in Fig. 2. Notice that traversing over its elements in descending order creates a pendulum-like motion hence explaining the name. Intuitively, the pendulum arrangement of a vector $x \in \mathbb{R}^d$ can be constructed by first sorting $x$ in decreasing order, and then placing the elements of the sorted array such that the largest element is in the middle, the next element to its left, the following element to its right, repeating this process until all elements have been placed in a pendulum-like ordering. This observation is made formal by the following lemma, which relates the pendulum arrangement to the sorted arrangement.

**Lemma 3.** For $x \in \mathbb{R}^d$ let $x_{\text{sort}}$ be the result of sorting the elements of $x$ in ascending order. Define

$$\theta(j) = \begin{cases} 2j - 1, & j \leq \frac{d+1}{2} \\ 2(d + 1 - j), & \text{otherwise} \end{cases},$$

then $x_{\text{pend}}$ is uniquely defined and satisfies $\theta x_{\text{sort}} = x_{\text{pend}}$.

The proof of the lemma is technical and deferred to Appendix B. We are now ready to state our main result.

**Theorem 4 (Main Result).** $\sigma^*p$ maximizes the expected escape time, i.e., solves Problem (P1), if and only if it is ordered according to the pendulum arrangement, $\sigma^*p = p_{\text{pend}}$, or its mirror $\sigma^*p = \sigma_{\text{mirror}}p_{\text{pend}}$.

In other words, solving Problem (P1) reduces to finding a pendulum arrangement of the elements of $p$, which is immediately obtained from their sorted order. Moreover, this solution is unique up to its mirror.
Theorem 4
(Main Result)
Proposition 5
(Closed Form Expression)
Theorem 6
(Optimal Sum of Products)
Lemma 3
(Pendulum Uniqueness)
Lemma 12
(Pendulum Sort)
Lemma 10
(Improving Permutation)

Figure 3: A flowchart for proving Theorem 4 (Main Result). In red are the theorems used to prove the final result, and in blue the main supporting lemmas.

4 Proof of Main Result

The proof of Theorem 4 consists of two parts, as seen in Fig. 3. In this section we focus on the right part of Fig. 3, showing a closed form expression for the expected escape time $\mathbb{E} \tau \{p; 0\}$, which reduces the problem to maximizing a sum over products. We then prove that the pendulum arrangement maximizes this sum of products, thus concluding the proof. This second part, which is used here as a tool, is the heart of the problem and we discuss and explain its main ideas in Section 5.

The following proposition derives a closed form expression for $\mathbb{E} \tau \{p; 0\}$. Its proof uses a direct inductive claim and is provided in Appendix A.

**Proposition 5 (Closed Form Expression).** We have that

$$\mathbb{E} \tau \{p; 0\} = (d + 1) + 2 \sum_{m=1}^{d} \sum_{i=1}^{d-m+1} \prod_{j=i}^{i+m-1} \frac{p_j}{1 - p_j}.$$  \hspace{1cm} (1)

One can immediately notice a symmetric property of $\mathbb{E} \tau \{p; 0\}$ in Eq. (1). Specifically, it is invariant to the mirror permutation, i.e.,

$$\mathbb{E} \tau \{p; 0\} = \mathbb{E} \tau \{\sigma_{\text{mirror}} p; 0\}, \quad \forall p \in (0, 1)^d.$$

This in turn implies that the pendulum arrangement and its mirror both achieve the same value, and thus proving that one of them is optimal will suffice to conclude Theorem 4 (Main Result). Also note that this implies that ascending and descending orderings of the elements of $p$ achieve identical (yet sub-optimal) values. This fact is indicative of a symmetric characteristic of $\mathbb{E} \tau \{p; 0\}$ that foreshadows the underlying pendulum arrangement.

Focusing on the sum over products term in Eq. (1), we have the following theorem, which states that the pendulum arrangement is its unique maximizer.

**Theorem 6 (Optimal Sum of Products).** For any $x \in \mathbb{R}_{++}^d$ we have that

$$\sigma^* \in \arg \max_{\sigma \in \Sigma} \sum_{m=1}^{d} \sum_{i=1}^{d-m+1} \prod_{j=i}^{i+m-1} x_{\sigma(i)}, \quad \iff \quad \sigma^* x \in \{x_{\text{pend}}, \sigma_{\text{mirror}} x_{\text{pend}}\}.$$  

As we will show next, combining Theorem 6 with Lemma 3 and Proposition 5 yields a straightforward proof for Theorem 4 (Main Result). The proof of Theorem 6 is the crux of this work and is outlined in the following section. Before diving into its details, we show how it can be used to prove Theorem 4 (Main Result).
Proof of Theorem 4 (Main Result). Consider the expression for $\mathbb{E}r\{p;0\}$ in Eq. (1). Denoting $x = p/(1-p)$, where the equality is element-wise, we have that

$$\sigma^* \in \arg \max_{\sigma \in \Sigma} \mathbb{E}r\{\sigma;0\} \iff \sigma^* \in \arg \max_{\sigma \in \Sigma} \sum_{m=1}^{d} \sum_{i=1}^{d-m+1} \prod_{j=i}^{i+m-1} x_{\sigma(i)}.$$  

Next, notice that $x \in \mathbb{R}^{d}_{++}$, and so using Theorem 6 we have that

$$\sigma^* \in \arg \max_{\sigma \in \Sigma} \sum_{m=1}^{d} \sum_{i=1}^{d-m+1} \prod_{j=i}^{i+m-1} x_{\sigma(i)} \iff \sigma^* x \in \{x_{\text{pend}}, \sigma_{\text{mirror}} x_{\text{pend}}\}.$$  

Finally, notice that the function $f(y) = y/(1-y)$ is strictly increasing in $[0,1)$ and thus the sorted order of $p$ and $f(p) = x$ are the same. Since the pendulum arrangement only depends on this order (see Lemma 3), we get that $\sigma^* x \in \{x_{\text{pend}}, \sigma_{\text{mirror}} x_{\text{pend}}\} \iff \sigma^* p \in \{p_{\text{pend}}, \sigma_{\text{mirror}} p_{\text{pend}}\}$, and combining these arguments concludes the proof.

5 Sum Over Products

In this section we will focus on proving Theorem 6. We do so by considering each of the inner summations in Theorem 6, reducing the problem further to individually maximizing each of the inner sums of products (Theorem 8). We then move to define the improving permutation (Definition 9), a uniquely designed permutation that: (1) always improves the sum over products (Lemma 10); and (2) converges after at most $d/2$ applications to the pendulum (optimal) arrangement (Lemma 12). These results will finalize the proof of Theorem 6, thereby concluding the proof of Theorem 4 (Main Result). To that end, we begin by focusing on the following construct.

Definition 7 (Sum Over Products Value). For any $x \in \mathbb{R}^{d}$ and $1 \leq m \leq d$ define the value function of $x$ for window size $m$ as

$$J(x; m) = \sum_{i=1}^{d-m+1} \prod_{j=i}^{i+m-1} x_{j}.$$  

The function $J(x; m)$ is a sum over all products of adjacent tuples of length $m$. For example, for $d = 5$ and $m = 3$ it can be explicitly written as $J(x; m) = x_{1}x_{2}x_{3} + x_{2}x_{3}x_{4} + x_{3}x_{4}x_{5}$. Notice that the expression in Theorem 6 is in fact a summation of the sum over products value, $J(x; m)$, for various window sizes $1 \leq m \leq d$. Theorem 6 is thus an immediate corollary of the following, more general result.

Theorem 8 (General Sum Over Products). For any $x \in \mathbb{R}^{d}_{++}$ we have that

1. (Sufficiency) $\forall 1 \leq m \leq d$, $\sigma^* x \in \{x_{\text{pend}}, \sigma_{\text{mirror}} x_{\text{pend}}\} \implies \sigma^* \in \arg \max_{\sigma \in \Sigma} J(\sigma x; m)$;

2. (Necessity) $\sigma^* \in \bigcap_{m=1}^{d} \arg \max_{\sigma \in \Sigma} J(\sigma x; m) \implies \sigma^* x \in \{x_{\text{pend}}, \sigma_{\text{mirror}} x_{\text{pend}}\}$.

5.1 Improving Permutation

The main tool for proving Theorem 8 is the following permutation.

Definition 9 (Improving Permutation). For $l = 1, \ldots, d$ define the $l^{th}$ improving permutation of a vector $x \in \mathbb{R}^{d}$ by

$$\sigma_{l}(i) = \begin{cases} 
  l + 1 - i, & \text{or } x_{i} > x_{l+1-i}, \; i \leq l/2 \\
  i, & x_{i} < x_{l+1-i}, \; l/2 < i \leq l \\
  \text{otherwise}. & \end{cases}$$  

(2)
Figure 4: An example of applying the improving permutations $\sigma_d$ and $\sigma_{\text{mirror}}\sigma_d\sigma_{\text{mirror}}$ iteratively on some given vector $x$. Plot shows direction in which switching of elements occur. Small values follow the circular arrows, whereas large values move in the reverse direction. Elements switch until reaching their final position in the pendulum arrangement.

We note that the vector $x$, with respect to which $\sigma_l$ is defined, is always the vector it permutes. While it is not denoted explicitly in $\sigma_l$, its identity will always be clear from context. The improving permutation, $\sigma_l$, compares elements across the symmetry axis $(l+1)/2$, and switches their positions such that the larger element is to the right of the symmetry axis (see Fig. 4). Notice that this may result in up to $l/2$ exchanges. While this may seem overly complicated, it is easy to give counter examples where any exchange of two elements will decrease the outcome (see Remark 11). As its name suggests, applying $\sigma_l$ to a vector increases its sum over products value, as shown by the following lemma. An exhaustive proof is provided Appendix C.

**Lemma 10** (Improving Permutation). For all $x \in \mathbb{R}^d_+$ and $m, l \in \{1, \ldots, d\}$ we have that

$$J(\sigma_l x; m) \geq J(x; m).$$

Moreover, if $\sigma_l x \notin \{x, \sigma_{\text{mirror}}x\}$ then there exists $m$ such that the inequality is strict.

**Proof sketch of Lemma 10.** We begin by denoting the product over a “window” of size $m$ starting at $i$ by $W_i^{(m)}(x) = \prod_{j=i}^{i+m-1} x_j$, i.e., $J(x; m) = \sum_{i=1}^d W_i^{(m)}(x)$. With some algebra, we then show that

$$J(x; m) = \sum_{i=1}^{\left\lceil \frac{l-1}{2} \right\rceil} [A_i(x) + B_i(x)] + C(x, m, l),$$

(3)

where $A_i(x) = W_i^{(m)}(x)$, $B_i(x) = W_i^{(m)}(li+2-m-i)(x)$, and $C(x, m, l) \approx \sum_{i=li+2-m}^{d-m+1} W_i^{(m)}(x)$. Fig. 5 depicts an example of how Eq. (3) reorganizes the elements of $J(x; m)$. 

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In Eq. (3), \(A_i\) and \(B_i\) were chosen such that if \(x_j\) participates in \(A_i(x)\) then \(x_{i+1-j}\) participates in \(B_i(x)\). Since these are the only kind of switches \(\sigma_i\) makes, we conclude that \(A_i(x)B_i(x) = A_i(\sigma_ix)B_i(\sigma_ix)\). Since \(\sigma_l\) puts the larger element in \(l+1-j\), i.e., in \(B_i\), we also have that \(B_i(\sigma_ix) \geq \max\{A_i(x), B_i(x)\}\), with strict inequality if only some but not all of the elements were switched. Combining the last two claims, it is immediate to conclude that

\[
A_i(\sigma_ix) + B_i(\sigma_ix) \geq A_i(x) + B_i(x).
\]

This is equivalent to saying that elongating the longer side of a rectangle while maintaining its area fixed (by shortening the other side) increases its circumference. The above holds for the relevant indices and thus showing that \(C(\sigma_lx, m, l) \geq C(x, m, l)\) concludes the proof. This is straightforward since \(\sigma_l\) essentially increases each of its terms individually.

**Remark 11.** When \(m = 2\), it is always possible to find a so called “improving” permutation that only exchanges two elements; however, this is not the case for \(m \geq 3\). To see this, take for example, the case of \(p = (0.17, 0.64, 0.85, 0.71)\). Exhaustive search shows that this is the second to best ordering and thus any improvement must lead to one of the optimal orderings (0.64, 0.85, 0.71, 0.17) or its mirror (0.17, 0.71, 0.85, 0.64). Notice that any such permutation must indeed exchange more than two elements. In other words, there exists an initialization vector \(p\) for which no “simple” permutation (i.e., one which exchanges only two elements) could iteratively converge to the optimal ordering. This motivates the use of more elaborate improving permutations as proposed in Definition 9.

### 5.2 Pendulum Sort

Having established that \(\sigma_l\) (1 \( \leq l \leq d\)) are always improving, we show that applying them consecutively converges to a pendulum arrangement. More specifically, the following lemma uses \(\sigma_d, \sigma_{d-1}\) and \(\sigma_{\text{mirror}}\) (see Definitions 1 and 9) to construct such a sequence. An exhaustive proof of the lemma is provided Appendix D.

**Lemma 12** (Pendulum Sort). For all \(x \in \mathbb{R}^d, k \geq \frac{d}{2}\), we have that \((\sigma_{\text{mirror}}\sigma_{d-1}\sigma_{\text{mirror}}\sigma_d)^k x = x_{\text{pend}}\).

**Proof sketch of Lemma 12.** Recall that \(\theta\) from Lemma 3 satisfies \(\theta x_{\text{sort}} = x_{\text{pend}}\). We define \(\tilde{\sigma}_d, \tilde{\sigma}_{d-1}\) as follows

\[
\tilde{\sigma}_d = \theta^{-1}\sigma_d\theta, \quad \tilde{\sigma}_{d-1} = \theta^{-1}(\sigma_{\text{mirror}}\sigma_{d-1}\sigma_{\text{mirror}})\theta.
\]

and a simple telescoping argument yields that

\[
(\sigma_{\text{mirror}}\sigma_{d-1}\sigma_{\text{mirror}}\sigma_d)^k = \theta(\tilde{\sigma}_{d-1}\tilde{\sigma}_d)^k\theta^{-1}.
\]
Figure 6: Pendulum Sort: An example of an application of the pendulum sort permutation $\theta(\tilde{\sigma}_{d-1}\tilde{\sigma}_d)^{d}\theta^{-1}$ iteratively. $\tilde{\sigma}_d$ and $\tilde{\sigma}_{d-1}$ compare element pairs, switching them whenever the left element is larger than the neighbor on its right. $\tilde{\sigma}_{d-1}$ compares pairs of elements in even indices, whereas $\tilde{\sigma}_d$ compares them at odd indices. Note that application of $\tilde{\sigma}_d$ or $\tilde{\sigma}_{d-1}$ on a sorted array is the identity permutation.

We then show that for any $y \in \mathbb{R}^d$, $(\tilde{\sigma}_{d-1}\tilde{\sigma}_d)^k y = y_{\text{sort}}$ for all $k \geq \frac{d}{2}$. Recalling Lemma 3 and choosing $y = \theta^{-1}x$ concludes the proof. To show that $(\tilde{\sigma}_{d-1}\tilde{\sigma}_d)^k y = y_{\text{sort}}$ we first find explicit expressions for $\tilde{\sigma}_d$, $\tilde{\sigma}_{d-1}$. These expressions are sorting procedures on the odd and even odd pairs of $x$ respectively. This means that applying them consecutively performs a sort of parallel bubble sort, which is depicted in Fig. 6. A simple analysis shows that this converges in $d^2$ steps and a more careful analysis gives the desired $d/2$ steps. ■

Proof of Theorem 8. First, recall that $J(x; m) = J(\sigma_{\text{mirror}} x; m)$ and so using Lemma 10 (Improving Permutation) recursively we get that

$$J((\sigma_{\text{mirror}} \sigma_{d-1} \sigma_{\text{mirror}} \sigma_d)^k x; m) \geq J(x; m), \forall k \geq 0.$$  

Taking $k \geq \frac{d}{2}$ and using Lemma 12 (Pendulum Sort) we then get that $J(x_{\text{pend}}; m) \geq J(x; m)$, and since this holds for any permutation of $x$, the first part of the proof is concluded. The uniqueness claim follows from the strict inequality condition of Lemma 10 (Improving Permutation). More concretely, let

$$\sigma^* \in \cap_{m=1}^d \arg\max_{\sigma \in \Sigma} J(\sigma x; m),$$

and assume in contradiction that $\sigma^* x \notin \{x_{\text{pend}}, \sigma_{\text{mirror}} x_{\text{pend}}\}$. However, from Lemma 12 (Pendulum Sort) we know that $(\sigma_{\text{mirror}} \sigma_{d-1} \sigma_{\text{mirror}} \sigma_d)^d \sigma^* x = x_{\text{pend}}$, and thus one of the terms composing $(\sigma_{\text{mirror}} \sigma_{d-1} \sigma_{\text{mirror}} \sigma_d)^d$ must change its input to something other than its mirror. The strict inequality condition of Lemma 10 (Improving Permutation) then implies that there exists $m$ such that $J(x_{\text{pend}}; m) > J(\sigma^* x; m)$, contradicting the optimality of $\sigma^*$. ■
Figure 7: Comparison of various arrangements and their escape times for a random walk on a line with $d = 8$ nodes in a random environment in which transition values were sampled i.i.d. from a uniform distribution in the interval $[0.5 - x, 0.5 + x]$ for various values of $x$. The random arrangement is the mean escape time taken w.r.t. the uniform measure over all possible permutations. The presented value for all statistics was averaged over 1000 different instantiations of the random environment.

6 Discussion and Future Work

In this section we demonstrate a continuous extension to our main result, conduct numerical experiments on random environments that illustrate the significance of our findings, and discuss alternate statistics of the escape time.

6.1 Continuous Weight Optimization

We consider the following continuous optimization variant of the combinatorial problem (P1):

$$p^* \in \arg \max_{p \in C} \mathbb{E} \tau \{p; 0\},$$

(P2)

where $C \subseteq [0, 1)^d$ is a set of budget constraints on the transition probabilities. The difficulty of (P2) strongly depends on the structure of the set $C$. Theorem 4 (Main Result) implies that for $C_p = \{\sigma p \mid \sigma \in \Sigma\}$, (P1) is efficiently solvable. The following proposition readily follows from Theorem 4 (Main Result), and extends it to a slightly more general class of constraints. For $A \subseteq C$ let $\text{ext}(A)$ denote the extreme points of the convex hull of $A$, and $A_{\text{pend}} = \{p_{\text{pend}} \mid p \in A\}$ (see Definition 2).

Proposition 13. For $C \subseteq [\frac{1}{2}, 1)^d$, if $(\text{ext}(C))_{\text{pend}} \subseteq \text{ext}(C)$ then $\exists p^* \in (\text{ext}(C))_{\text{pend}}$.

In other words, if the pendulum arrangement is always an element of the extreme points of $C$ then the optimal solution to Problem (P2) is an extreme point of $C$ which is ordered according to the pendulum arrangement. The proof of Proposition 13 is provided in Appendix E and uses the fact that $\mathbb{E} \tau \{p; 0\}$ is convex in $p$. This implies that there exists $p^* \in \text{ext}(C)$ and thus applying Theorem 4 (Main Result) with the assumed structure of $C$ concludes the proof. This result allows us to greatly reduce the search for an optimal solution. Particularly, it may reduce this search to a small constant number of possible candidates, as shown by the following example.
Figure 8: Escape time comparison of the maximal, sorted, and random arrangements as a function of the number of nodes $d$. (left) environment weights distributed $U([0, 0.513])$. (right) environment weights distributed $U([0, 0.5])$. Graphs display average result over 1000 environment instantiations where for each instantiation the random arrangement is calculated by averaging 1000 random permutations.

Example: Assume a linear budget constraint of the form

$$C_{a,b} = \left\{ p \in [0,a]^d \mid \|p\|_1 \leq b \right\},$$

where $a \in [0,1)$. Trivially, whenever $b \geq da$ the optimal solution is given by the uniform vector $p^* = [a, \ldots, a]$. Yet, when $b < da$, by Proposition 13, the optimal solution will be given by a pendulum arrangement over $\text{ext}(C_{a,b})$. This results in $b/a$ values of $a$ (up to a remainder term) placed in the center of the line. Concretely, $p^* = p_{\text{pend}}$ with

$$p = (a, \ldots, a, \mod(b,a), 0, \ldots, 0),$$

$$|\frac{b}{a}| \text{ times}$$

6.2 Random Environments

Theorem 4 (Main Result) shows that the pendulum arrangement yields the maximum expected escape time. In this section we perform several numerical experiments to give a more quantitative grasp of the behavior of the expected escape time under different arrangements: maximal (pendulum), minimal, sorted, and random. The minimal arrangement is the one that yields minimal expected escape time, and is found using exhaustive search. The sorted arrangement refers to sorting the weights (transition probabilities) in ascending order. The random arrangement refers to a random (uniform) arrangement of the given weights. For small values of $d$ this can be calculated exactly by averaging over all possible arrangements. When this becomes computationally infeasible, we use Monte-Carlo methods to estimate this quantity.

Our first experiment compares the maximum, random, and minimum arrangements. To do so, we consider a random walk in a random environment setting on a line with $d = 8$ nodes. We initialize the environment weights using a uniform distribution on $[0.5-x, 0.5+x]$ and perform a Monte-Carlo simulation (only on the initialization) to evaluate the expected escape time of each arrangement. The results are depicted in Fig. 7. Our choice of distribution keeps the expected value of the weights fixed while varying their variance. Unsurprisingly, the arrangement of the weights becomes more significant for higher variance weight initialization. Notice that the graph displays the logarithm of the escape time, and thus the increasing gaps between the arrangements imply a highly super-linear dependence on the variance.

Our second experiment examines the behavior of the escape time as a function of $d$ for the maximal, sorted and random arrangements (see Fig. 8). We observe two types of behaviors depending on the properties of the random environment. The first behavior occurs when all weights are smaller than 0.5, and yields a walk that is, in a sense, “strongly” transient, making the escape time grow slowly (linearly) in $d$ regardless of
Figure 9: Minimal arrangements of three vectors: (a) \([0.1, 0.2, 0.3, 0.4, 0.5, 0.6]\), (b) \([0.3, 0.4, 0.5, 0.6, 0.7, 0.8]\), and (c) \([0.4, 0.5, 0.6, 0.7, 0.8, 0.9]\). The arrangements are all unique to their values, suggesting that the minimal arrangement depends on the values of \(p\). Note that the arrangement in (c) is not the inverted pendulum arrangement, as the two largest values in the edges are flipped.

the arrangement. While there is a significant gain in using the maximal (pendulum) and sorted arrangements, which perform similarly here, the overall behavior of the escape time does not change compared to a random arrangement. The second case reveals an interesting phase transition. It considers a case where the random arrangement is transient but some proportion of the weights are greater than 0.5. In this case the random arrangement behaves as in the first environment (up to small factors). However, starting at some \(d_0\), the maximal and sorted arrangements grow exponentially, with a significant gap between them. We have tried various environment parameters and this behavior seems to persist with the only change being the critical value of \(d_0\) where the change in behavior occurs. We leave the formal investigation of this phenomenon to future work.

6.3 Alternate Statistics

In this work we focused on the maximization of the expected escape time. While maximizing the expected value is a highly accepted notion, one could also consider other criteria that, for example, consider some notion of risk. For instance, one might wish to find a permutation for which \(f(\mathbb{E}\tau\{p\sigma; 0\}, \text{var}(\tau\{p\sigma; 0\}))\) is maximized. Some classical examples include the Sharpe Ratio \(f(x, y) = \frac{x}{\sqrt{y}}\), and Mean-Variance criterion \(f(x, y) = x - \lambda y\).

An alternative notion that is of separate interest is minimizing the expected escape time. This problem was studied in a simplified setting where weights are constrained to one of two values, showing that the asymptotic optimal order requires equal spacing between the larger weights [19, 12]. In Fig. 9 we depict three instantiations of general weight assignments for a line of \(d = 6\) nodes. Contrary to the maximal expected escape time, the minimal optimal permutation is value dependent, suggesting that understanding the structure of the minimal permutation is more involved. Extensive simulations lead us to the conjecture that “large” values are indeed spaced more or less evenly, but it remains unclear how to characterize this notion formally. We leave the topic of alternate statistics as an open question for future work.

6.4 Conclusion

In this work we conducted exact analysis of a newly discovered phenomenon of heterogeneous random walks. We showed that the maximum escape time is established when the transition probabilities relating to the slowdown drift of the process are ordered in a unique arrangement, known as the pendulum arrangement (see Fig. 2). Our result follows careful inspection of a sum over products combinatorial optimization problem, which may be of broader interest in fields out of the scope of this paper.

Finally, our work lays the foundations for Markov chain Design, through careful design of the topology and weights of Markov chains. This may enable the construction of networks that are insusceptible to cyber-attacks, resilient to the spread of infectious diseases, and control the flow of perilous processes (e.g., harmful ideas) on social networks and the web.
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Appendix: Missing Proofs

A Proof of Proposition 5

The proof follows standard induction analysis, (see e.g., Proposition 2 of [1]), and is provided here for completeness. For 1 ≤ k ≤ d, due to the Markov property,

\[ \mathbb{E} \tau \{ p; k \} = p_k \mathbb{E} \tau \{ p; k - 1 \} + (1 - p_k) \mathbb{E} \tau \{ p; k + 1 \} + 1. \]

Rearranging the above yields

\[ \mathbb{E} \tau \{ p; k \} - \mathbb{E} \tau \{ p; k + 1 \} = \frac{p_k}{1 - p_k} (\mathbb{E} \tau \{ p; k - 1 \} - \mathbb{E} \tau \{ p; k \}) + \frac{1}{1 - p_k}. \]

Denoting \( D_k = \mathbb{E} \tau \{ p; k \} - \mathbb{E} \tau \{ p; k + 1 \} \) we get

\[ D_k = \frac{p_k}{1 - p_k} D_{k-1} + \frac{1}{1 - p_k}. \]

Solving this equation by iteration yields

\[ D_k = \frac{1}{1 - p_k} + D_0 \prod_{i=1}^{k} \frac{p_i}{1 - p_i} + \sum_{m=1}^{k-1} \frac{1}{1 - p_m} \prod_{i=m+1}^{k} \frac{p_i}{1 - p_i}. \]

Furthermore we have that

\[ \tau \{ p; d + 1 \} = 0 \]
\[ \tau \{ p; 0 \} = \tau \{ p; 1 \} + 1 \Rightarrow D_0 = 1. \]

Then, combining the above we get that

\[ \mathbb{E} \tau \{ p; 0 \} = 1 + \mathbb{E} \tau \{ p; 1 \} \]
\[ = 1 + \sum_{k=1}^{d} (\mathbb{E} \tau \{ p; k \} - \mathbb{E} \tau \{ p; k + 1 \}) \]
\[ = 1 + \sum_{k=1}^{d} \left( \frac{1}{1 - p_k} + \prod_{i=1}^{k} \frac{p_i}{1 - p_i} + \sum_{m=1}^{k-1} \frac{1}{1 - p_m} \prod_{i=m+1}^{k} \frac{p_i}{1 - p_i} \right). \]

Finally Lemma 14 below shows how the final expression can be technically derived from the above, using simple algebraic manipulations.

**Lemma 14.** It holds that

\[ \sum_{k=1}^{d} \left( \frac{1}{1 - p_k} + \prod_{i=1}^{k} \frac{p_i}{1 - p_i} + \sum_{m=1}^{k-1} \frac{1}{1 - p_m} \prod_{i=m+1}^{k} \frac{p_i}{1 - p_i} \right) = d + 2 \sum_{m=1}^{d} \sum_{i=1}^{d-m+1} \prod_{j=i}^{d-m+1} \frac{p_j}{1 - p_j}. \]

**Proof.** For \( z \in (0, 1)^d, 1 \leq x \leq y \leq d \) denote

\[ G_{x,y}(z) = \prod_{i=x}^{y} z_i, \quad M_{x,y}(z) = G_{1,x-1}(1 - z)G_{x,y}(z)G_{y+1,d}(1 - z). \]

Recalling that \( p \in (0, 1)^d \) denotes the vector of probabilities \( (p_1, \ldots, p_d) \), we have that

\[ \sum_{k=1}^{d} \left( \frac{1}{1 - p_k} + \prod_{i=1}^{k} \frac{p_i}{1 - p_i} + \sum_{m=1}^{k-1} \frac{1}{1 - p_m} \prod_{i=m+1}^{k} \frac{p_i}{1 - p_i} \right) = \sum_{k=1}^{d} \frac{1}{1 - p_k} + \sum_{k=1}^{d} \frac{G_{1,k}(p)}{G_{1,k}(1 - p)} + \sum_{k=1}^{d} \sum_{m=1}^{k-1} \frac{G_{m+1,k}(p)}{G_{m,k}(1 - p)} \]
\[ = \sum_{k=1}^{d} \frac{1}{1 - p_k} + \frac{1}{G_{1,d}(1 - p)} \sum_{k=1}^{d} \left( M_{1,k}(p) + \sum_{m=1}^{k-1} (M_{m+1,k}(p) + M_{m,k}(p)) \right), \]

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where in the last two steps we use the definition of \( G, M \) and the fact that

\[
M_{m+1,k}(p) + M_{m,k}(p)
= G_{1,m}(1-p)G_{m+1,k}(p)G_{k+1,d}(1-p) + G_{1,m-1}(1-p)G_{m,k}(p)G_{k+1,d}(1-p)
= (1-p_m + p_m)G_{1,m-1}(1-p)G_{m+1,k}(p)G_{k+1,d}(1-p)
= G_{1,m-1}(1-p)G_{m+1,k}(p)G_{k+1,d}(1-p).
\]

Next, denote

\[
W_{x,y}(z) = \prod_{i=x}^{y} \frac{p_i}{1-p_i},
\]

and notice that

\[
W_{x,y}(p) = \frac{M_{x,y}(p)}{G_{1,d}(1-p)}
\]

Then, we have that

\[
\mathbb{E} \tau \{ p; 0 \} = \sum_{k=1}^{d} \frac{1}{1-p_k} + \sum_{k=1}^{d} \left( W_{1,k}(p) + \sum_{m=1}^{k-1} (W_{m+1,k}(p) + W_{m,k}(p)) \right)
= d + 2 \sum_{m=1}^{d} \sum_{i=1}^{d-m+1} W_{i,i+m-1}(p),
\]

where the last step is proven by induction on \( d \). Substituting for \( W \) completes the proof.

**Induction** We show that

\[
\sum_{k=1}^{d} \frac{1}{1-p_k} + \sum_{k=1}^{d} \left( W_{1,k}(p) + \sum_{m=1}^{k-1} (W_{m+1,k}(p) + W_{m,k}(p)) \right) = d + 2 \sum_{m=1}^{d} \sum_{i=1}^{d-m+1} W_{i,i+m-1}(p),
\]

by induction on \( d \).

**Base case:** \( d = 1 \). We have that

\[
\frac{1}{1-p_1} + W_{1,1}(p) = \frac{1-p_1 + p_1}{1-p_1} + \frac{p_1}{1-p_1} = 1 + 2 \frac{p_1}{1-p_1} = 1 + 2W_{1,1}(p).
\]
\textbf{Induction step.} Assume Eq. (4) holds for some \(d = n\). We will show it holds for \(n + 1\) as well. Indeed,}

\[
\begin{align*}
\sum_{k=1}^{n+1} \frac{1}{1-p_k} + \sum_{k=1}^{n+1} & \left( W_{1,k}(p) + \sum_{m=1}^{k-1} (W_{m+1,k}(p) + W_{m,k}(p)) \right) \\
= & \frac{1}{1-p_{n+1}} + W_{1,n+1}(p) + \sum_{m=1}^{n} (W_{m+1,n+1}(p) + W_{m,n+1}(p)) \\
+ & \sum_{k=1}^{n} \frac{1}{1-p_k} + \sum_{k=1}^{n} \left( W_{1,k}(p) + \sum_{m=1}^{k-1} (W_{m+1,k}(p) + W_{m,k}(p)) \right) \\
= & \frac{1}{1-p_{n+1}} + W_{1,n+1}(p) + \sum_{m=1}^{n} (W_{m+1,n+1}(p) + W_{m,n+1}(p)) + n + 2 \sum_{m=1}^{n} \sum_{i=1}^{n-m+1} W_{i,i+m-1}(p) \\
= & 1 + W_{n+1,n+1} + W_{1,n+1}(p) + \sum_{m=1}^{n} (W_{m+1,n+1}(p) + W_{m,n+1}(p)) + n + 2 \sum_{m=1}^{n} \sum_{i=1}^{n-m+1} W_{i,i+m-1}(p) \\
= & n + 1 + 2 \left( W_{1,n+1}(p) + \sum_{m=1}^{n} W_{n-m+2,n+1}(p) + \sum_{m=1}^{n} \sum_{i=1}^{n-m+1} W_{i,i+m-1}(p) \right) \\
= & n + 1 + 2 \sum_{m=1}^{n+1} W_{n-m+2,n+1}(p) + \sum_{i=1}^{n-m+1} W_{i,i+m-1}(p) \\
= & n + 1 + 2 \sum_{m=1}^{n+1} \sum_{i=1}^{n-m+2} W_{i,i+m-1}(p).
\end{align*}
\]

In (a) we used the induction step, in (b) we used the fact that \(\frac{1}{1-p_{n+1}} = \frac{1-p_{n+1}+p_{n+1}}{1-p_{n+1}} = 1 + W_{n+1,n+1}(p_{n+1})\), and in (c) reorganization of the summands.

\[\blacksquare\]

\section*{B Proof of Lemma 3}

\textbf{Proof.} Recall that

\[
\theta(j) = \begin{cases} 
2j - 1 & , j \leq \frac{d+1}{2} \\
2(d+1-j) & , \text{otherwise.}
\end{cases}
\]

It is easy to verify that the inverse of this permutation, i.e., \(\theta^{-1}\), has the following form

\[
\theta^{-1}(j) = \begin{cases} 
\frac{j+1}{2} & , j \text{ is odd} \\
\frac{d+1-j}{2} & , j \text{ is even.}
\end{cases}
\]

Assume that \(x_{\text{sort}} = \theta^{-1}x_{\text{pend}}\) and so by the uniqueness of the sorted order and the permutation \(\theta^{-1}\) we conclude the uniqueness of \(x_{\text{pend}}\). Since both sides are now uniquely defined, we can apply \(\theta\) to both sides to obtain the other part of the lemma.

We show that \(x_{\text{sort}} = \theta^{-1}x_{\text{pend}}\) thus concluding the proof. Let \(y = x_{\text{pend}}\) and \(z = \theta^{-1}y\). Let \(1 \leq i \leq d-1\) be odd, then \(\theta^{-1}(i) = (i+1)/2\) and \(\theta^{-1}(i+1) = d+1 - ((i+1)/2)\), and so we have that

\[
z_i = y_{\theta^{-1}(i)} = y_{(i+1)/2} \leq y_{d+1-((i+1)/2)} = y_{\theta^{-1}(i+1)} = z_{i+1},
\]

where the inequality used the first part of Definition 2 (pendulum arrangement). Now, for let \(2 \leq i \leq d-1\) be even, then \(\theta^{-1}(i) = d+1 - (i/2)\) and \(\theta^{-1}(i+1) = (i/2)\), and so we have that

\[
z_i = y_{\theta^{-1}(i)} = y_{d+1-(i/2)} \leq y_{(i+2)/2} = y_{\theta^{-1}(i+1)} = z_{i+1},
\]

where the inequality used the second part of Definition 2 (pendulum arrangement). Overall we conclude that \(z_i \leq z_{i+1}\) for all \(1 \leq i \leq d-1\), i.e., \(z = x_{\text{sort}}\), as desired. \[\blacksquare\]
C Proof of Lemma 10

The proof of Lemma 10 is an immediate corollary of the three following results. To ease notation, we make the following definition. For $i \in \mathbb{Z}, x \in \mathbb{R}^d$, and $1 \leq m \leq d$ let

$$W_i^{(m)}(x) = \begin{cases} 
\prod_{j=i}^{i+m-1} x_j, & 1 \leq i \leq d+1-m \\
0, & \text{otherwise,}
\end{cases} \quad (5)$$

where the otherwise case serves to avoid some edge cases in what follows. When $x$ is clear from context, we will only write $W_i^{(m)}$. The first result, whose proof may be found in Appendix C.1, decomposes the value.

Lemma 15 (Value decomposition). We have that

$$J(x; m) = \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} \left[ W_i^{(m)}(x) + W_{i+2-m}^{(m)}(x) \right] + \sum_{i=i+2-m}^{d+1-m} \left[ W_i^{(m)}(x) \right] + W_{i+2-m}^{(m)}(x) \mathbb{1}\{l - m \in 2\mathbb{N}\}$$

The second result, whose proof may be found in Appendix C.2, shows that the terms in the first sum of the decomposition, as well as the last term, increase as a result of applying $\sigma_i$.

Lemma 16 (Improving Window Pairs). For all $m > 0$ and $1 \leq i \leq \frac{1}{2}(l + 2 - m)$, we have that

$$W_i^{(m)}(x) + W_{i+2-m}^{(m)}(x) \leq W_i^{(m)}(\sigma_i x) + W_{i+2-m}^{(m)}(\sigma_i x).$$

Moreover, if $l = d$ and $\sigma_d x \notin \{x, \sigma_{\text{mirror}} x\}$ then there exist $i, m$ such that the inequality is strict.

The third and final result, whose proof may be found in Appendix C.3, shows that the terms in the second sum of the decomposition increase as a result of applying $\sigma_i$.

Lemma 17 (Improving Single Windows). For all $m > 0$ and $(l + 2 - m) \leq i \leq d+1-m$, we have that

$$W_i^{(m)}(x) \leq W_i^{(m)}(\sigma_i x).$$

Moreover, if $l \leq d - 1$ and $\sigma_i x \neq x$ then there exist $i, m$ such that the inequality is strict.

Proof of Lemma 10. Combining Lemmas 15 to 17 we get that

$$J(x; m) \leq \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} \left[ W_i^{(m)}(\sigma_i x) + W_{i+2-m}^{(m)}(\sigma_i x) \right] + \sum_{i=i+2-m}^{d+1-m} \left[ W_i^{(m)}(\sigma_i x) \right] + W_{i+2-m}^{(m)}(\sigma_i x) \mathbb{1}\{l - m \in 2\mathbb{N}\}$$

The strict inequality condition follows by combining those of Lemmas 16 and 17. ■
C.1 Proof of Lemma 15

Proof. We have that

\[ J(x; m) = \sum_{i=1}^{d-m+1} \prod_{j=i}^{x} x_j \]

\[ = \sum_{i=1}^{d-m+1} W^{(m)}_i(x) \]

\[ = \left\lceil \frac{l-m}{2} \right\rceil W^{(m)}_i(x) + \sum_{i=\left\lceil \frac{l-m}{2} \right\rceil +1}^{l+1-m} W^{(m)}_i(x) + \sum_{i=l+2-m}^{d-m+1} W^{(m)}_i(x) \]

\[ = \left\lceil \frac{l-m}{2} \right\rceil W^{(m)}_i(x) + \sum_{i=1}^{l+2-m} W^{(m)}_{l+i-2-m} - 1(x) + \sum_{i=l+2-m}^{d-m+1} W^{(m)}_i(x) + W^{(m)}_{\frac{l+2-m}{2}}(x) 1 \{ l - m \in 2\mathbb{N} \} , \]

where the last two transitions used the change of variables \( i = (l + 2 - m) - j \) and the fact that

\[ (l + 1 - m) - \left\lceil \frac{l-m}{2} \right\rceil = \left\lceil \frac{l-m}{2} \right\rceil + 1 = \left\lceil \frac{l-m}{2} \right\rceil + 1 \{ l - m \in 2\mathbb{N} \}. \]

\[ \blacksquare \]

C.2 Proof of Lemma 16

To prove this lemma, we need a few intermediate results. The first is a simple and well known claim, whose geometric interpretation is that for equal area rectangles, the one with the longest side has a larger circumference. See proof in Appendix C.4.

**Lemma 18.** Let \( x_1, x_2, y_1, y_2 \geq 0 \) such that \( x_1 x_2 = y_1 y_2 \) then if \( \max\{y_1, y_2\} < \max\{x_1, x_2\} \) then

\[ y_1 + y_2 < x_1 + x_2. \]

The following lemma will imply the condition \( x_1 x_2 = y_1 y_2 \) of the previous lemma. See proof in Appendix C.5.

**Lemma 19** (Permutation invariant window pairs). For all \( m > 0 \) and \( 1 \leq i \leq \frac{l}{2}(l+2-m) \), we have that

\[ W^{(m)}_i(x) W^{(m)}_{\frac{l}{2}(l+2-m)-i}(x) = W^{(m)}_i(\sigma(x) W^{(m)}_{\frac{l}{2}(l+2-m)-i}(\sigma(x)). \]

Finally, the following lemma will imply the condition \( \max\{y_1, y_2\} < \max\{x_1, x_2\} \) in Lemma 18. See proof in Appendix C.6.

**Lemma 20** (Improving disjoint windows). For all \( m > 0 \) and \( 1 \leq i \leq \frac{l}{2}(l+2-2m) \) we have that

\[ \max\{W^{(m)}_i(x), W^{(m)}_{\frac{l}{2}(l+2-m)-i}(x)\} \leq W^{(m)}_{\frac{l}{2}(l+2-m)-i}(\sigma(x)). \]

Moreover, for \( l = d \) if \( \sigma(x) \notin \{x, \sigma_{\text{mirror}}x\} \) then there exist \( i, m \) such that the inequality is strict.
**Proof of Lemma 16.** First, notice that the strict inequality condition follows directly from that of Lemma 20. Now, Denote

\[ y_1 = W_i^{(m)}(x) \]
\[ y_2 = W_{(l+2-m)-i}^{(m)}(x) \]
\[ x_1 = W_i^{(m)}(\sigma_lx) \]
\[ x_2 = W_{(l+2-m)-i}^{(m)}(\sigma_lx). \]

Then, by Lemma 19, \( x_1x_2 = y_1y_2 \). We show that \( \max\{y_1, y_2\} \leq x_2 \), thus satisfying the requirements of Lemma 18 and concluding the proof. If \( i \leq \frac{1}{2}(l+2-2m) \) then Lemma 20 immediately implies the desired. Otherwise, if \( i > \frac{1}{2}(l+2-2m) \) then

\[
\max\{y_1, y_2\} = \max\left\{ W_i^{(m)}(x), W_{(l+2-m)-i}^{(m)}(x) \right\} \\
= \max\left\{ \prod_{j=i}^{i+m-1} x_j, \prod_{j=(l+2-m)-i}^{l+1-i} x_j \right\} \\
= \max\left\{ \prod_{j=i}^{(l+1-m)-i} x_j, \prod_{j=(l+2-m)-i}^{l+1-i} x_j \right\} \\
= \max\left\{ W_i^{(m)}(x), W_{(l+1)+2-m-i}^{(m)}(x) \right\} W_{\frac{m+1}{2}(l+2-m,0)}^{(m)}(x),
\]

where \( m_0 = 2i + l - 2 \), and \( m_1 = m - m_0 = l + 2 - m - 2i \). Next, notice that

\[
m_0 > 2m + (l + 2 - 2m) - l - 2 = 0,
\]

and thus taking Lemma 19 with \( m = m_0 \) and \( i = \frac{l + 2 - m}{2} \) we get that

\[
W_{\frac{m+1}{2}(l+2-m,0)}^{(m)}(x) = W_{\frac{m+1}{2}(l+2-m,0)}^{(m)}(\sigma_lx).
\]

Next, notice that

\[
m_1 > l + 2 - m - (l + 2 - m) = 0,
\]
\[\frac{1}{2}(l + 2 - 2m_1) = 2i - \frac{1}{2}(l + 2 - 2m) > i,\]

and thus taking Lemma 20 with \( m = m_1 \) we get that

\[
\max\left\{ W_{(l+1)+2-m-i}^{(m)}(x), W_{\frac{m+1}{2}(l+2-m,0)}^{(m)}(x) \right\} \leq W_{(l+1)+2-m-i}^{(m)}(\sigma_lx).
\]

Plugging Eqs. (7) and (8) into Eq. (6) we finally get that

\[
\max\{y_1, y_2\} \leq W_{\frac{m+1}{2}(l+2-m,0)}^{(m)}(\sigma_lx) W_{\frac{m+1}{2}(l+2-m,0)}^{(m)}(l_{(l+2-m)-i}(\sigma_lx) = x_2,
\]

as desired. \( \blacksquare \)

**C.3 Proof of Lemma 17**

**Proof.** We split the proof into three cases according to the value of \( i \). First, if \( i < 1 \) then the claim holds trivially since \( W_i^{(m)}(x) = 0 \) for all \( x \). Second, if \( (l+1)/2 \leq i \leq d+1-m \) we have that for all \( j \geq i x_{\sigma(j)} \geq x_j \) (by definition of \( \sigma_j \)) and thus

\[
W_i^{(m)}(x) = \prod_{j=i}^{i+m-1} x_j \leq \prod_{j=i}^{i+m-1} x_{\sigma(j)} = W_i^{(m)}(\sigma_lx),
\]

where \( \sigma \) is the permutation obtained by mapping \( 1 \) to \( i \) and \( (l+1) \) to \( l \). Thus, the desired inequality follows.

Finally, if \( i > (l+1)/2 \) then \( \sigma_lx \) is obtained by mapping \( 1 \) to \( l \) and \( (l+1) \) to \( 1 \), so

\[
W_i^{(m)}(x) = \prod_{j=i}^{i+m-1} x_j \leq \prod_{j=i}^{i+m-1} x_{\sigma(j)} = W_i^{(m)}(\sigma_lx),
\]

as desired. \( \blacksquare \)
Third, if \( \max\{1, l + 2 - m\} \leq i \leq \min\{(l + 1)/2, d + 1 - m\} \) then letting \( m_2 = l + 2 - 2i > 0 \), we notice that \( m - m_2 \geq i > 0 \) and so we get that

\[
W_i^{(m)}(x) = W_i^{(m_2)}(x)W_{i+m_2}^{(m-m_2)}(x)
\]

\[
= W_{\frac{i}{2}(l+2-m_2)}^{(m_2)}(x)W_{i+m_2}^{(m-m_2)}(x)
\]

\[
= W_{\frac{i}{2}(l+2-m_2)}^{(m_2)}(\sigma_i x)W_{i+m_2}^{(m-m_2)}(\sigma_i x)
\]

(by Lemma 19 with \( i = (l + 2 - m_2)/2 \))

\[
\leq W_{\frac{i}{2}(l+2-m_2)}^{(m_2)}(\sigma_i x)W_{i+m_2}^{(m-m_2)}(\sigma_i x)
\]

\[
= W_{i}^{(m)}(\sigma_i x),
\]

(reversing initial equalities)

where \((*)\) follows from Eq. (9) since \((l+1)/2 < i + m_2 \leq d + 1 - (m - m_2)\). We covered all the desired values of \( i \) thus proving the weak inequality.

Finally, we show the strict inequality condition. If \( \sigma_i x \neq x \) then there exists \( i > (l + 1)/2 \) such that \( x_{\sigma_i(i)} > x_i \) (as in Lemma 20). Taking \( i, m = d + 1 - i \), it is trivial to see that the weak inequality in Eq. (9) becomes strict. Notice that \( m > 0, i \leq d + 1 - m \), and since \( l \leq d - 1 \) we have that \( i \geq l + 2 - m \). We conclude that \( i, m \) satisfy the conditions of the lemma and the desired strict inequality.

\[\blacksquare\]

C.4 Proof of Lemma 18

Proof. If any of \( x_1, x_2, y_1, y_2 \) are equal to zero then the claim follows trivially. For the remainder of the proof we assume that \( x_1, x_2, y_1, y_2 > 0 \). Without loss of generality, let \( x_2 = \max\{x_1, x_2\} \) and \( y_2 = \max\{y_1, y_2\} \). By the assumptions of the lemma, this implies that \( x_2 > y_2 > y_1 > x_1 > 0 \). Then there exists \( \varepsilon > 0 \) such that

\[
x_2 = y_2 + \varepsilon.
\]

We then also have that

\[
x_1 = \frac{x_1 x_2}{x_2} = y_1 \frac{y_2}{x_2} = y_1 \left(1 - \frac{\varepsilon}{x_2}\right)
\]

(by Eq. (10))

\[
> y_1 - \varepsilon,
\]

and adding up both results yields the desired.

\[\blacksquare\]

C.5 Proof of Lemma 19

Proof. Denote the following two sets of indices

\[
I_1 = \{i, \ldots, i + m - 1\}
\]

\[
I_2 = \{(l + 2 - m) - i, \ldots, (l + 1) - i\}.
\]
We will show that \( \sigma \) is also a permutation on \( I_1 \cup I_2 \) and \( I_1 \cap I_2 \), i.e., \( I_1 \cup I_2 = \sigma(I_1 \cup I_2) \) and \( I_1 \cap I_2 = \sigma(I_1 \cap I_2) \), where \( \sigma(I) \) is the result of applying \( \sigma \) to each element of \( I \). The proof follows immediately since

\[
W_i^{(m)}(x)W_{(l+2-m)-i}(x) = \prod_{j=1}^{i+m-1} x_j \prod_{j=(l+2-m)-i}^{l+1-i} x_j \\
= \prod_{j \in I_1 \cup I_2} x_j \prod_{j \in I_1 \cap I_2} x_j \\
= \prod_{j \in \sigma(I_1 \cup I_2)} x_j \prod_{j \in \sigma(I_1 \cap I_2)} x_j \\
= \prod_{j=1}^{i+m-1} x_{\sigma(j)} \prod_{j=(l+2-m)-i}^{l+1-i} x_{\sigma(j)} = W_i^{(m)}(\sigma x)W_{(l+2-m)-i}(\sigma x).
\]

Since \( \sigma \) is a permutation and thus injective, it suffices to show that

\[
\sigma(I_1 \cap I_2) \subseteq I_1 \cap I_2; \quad \sigma(I_1 \cup I_2) \subseteq I_1 \cup I_2.
\]

Indeed, if \( I_1 \cap I_2 = \emptyset \) then Eq. (11) is trivial. Otherwise, let \( j \in I_1 \cap I_2 = \{ (l+2-m)-i, \ldots, i+m-1 \} \). If \( \sigma(j) = j \) then clearly \( \sigma(j) \in I_1 \cap I_2 \). Otherwise \( \sigma(j) = l+1-j \) and we have that

\[
l+1-j \geq l+1-(i+m-1) = (l+2-m)-i, \\
l+1-j \leq l+1-(l+2-m)+i = i+m-1,
\]

thus showing Eq. (11). Now for Eq. (12), let \( j \in I_1 \cup I_2 \). If \( \sigma(j) = j \) then clearly \( \sigma(j) \in I_1 \cup I_2 \). Otherwise \( \sigma(j) = l+1-j \) and we have the following. If \( j \in I_1 \) then

\[
l+1-j \geq l+1-i, \\
l+1-j \leq l+1-(i+m-1) = (l+2-m)-i,
\]

meaning \( \sigma(j) \in I_2 \). On the other hand, if \( j \in I_2 \) then

\[
l+1-j \geq l+1-(l+1-i) = i, \\
l+1-j \leq l+1-(l+2-m)+i = i+m-1,
\]

meaning \( \sigma(j) \in I_1 \) thus showing Eq. (12) and completing the proof.

**C.6 Proof of Lemma 20**

**Proof.** Recalling the definition of \( W_i^{(m)}(x) \) in Eq. (5), we have that

\[
\max \left\{ W_i^{(m)}(x), W_{(l+2-m)-i}(x) \right\} = \max \left\{ \prod_{j=1}^{i+m-1} x_j, \prod_{j=(l+2-m)-i}^{l+1-i} x_j \right\} \\
= \max \left\{ \prod_{j=(l+2-m)-i}^{l+1-i} x_{l+1-j}, \prod_{j=(l+2-m)-i}^{l+1-i} x_j \right\} \\
\leq \prod_{j=(l+2-m)-i}^{l+1-i} \max \{ x_{l+1-j}, x_j \} \\
= \prod_{j=(l+2-m)-i}^{l+1-i} x_{\sigma(j)} \\
= W_i^{(m)}_{(l+2-m)-i}(\sigma x),
\]

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where the second to last equality follows from the definition of \( \sigma_l \). To see this, notice that for \( j > \frac{1}{2} \) (which is indeed our case since \( j \geq (l + 2 - m) - i \) and \( i \leq \frac{1}{2}(l + 2 - 2m) \)), if \( x_j < x_{l+1-j} \), then \( x_{\sigma_l(j)} = x_{l+1-j} = \max\{x_{l+1-j}, x_j\} \). Otherwise, \( x_j \geq x_{l+1-j} \) and then \( x_{\sigma_l(j)} = x_j = \max\{x_{l+1-j}, x_j\} \), giving the desired equality.

Now, the weak inequality above becomes strict if and only if there exist \( j_1, j_2 \in [(l + 2 - m) - i, i + 2 - m] \) such that \( x_{j_1} < x_{l+1-j_1} \) and \( x_{j_2} > x_{l+1-j_2} \). We show that the strict inequality condition implies the existence of such \( j_1, j_2 \) thus concluding the proof. Let \( l = d \) and recall that for \( j > (d+1)/2, \sigma_d \) exchanges \( x_j \) and \( x_{d+1-j} \) if and only if \( x_j < x_{d+1-j} \). Since \( \sigma_d \neq x \), i.e., \( \sigma_d \) makes an exchange, there exists \( j_1 > (d + 1)/2 \) such that \( x_{j_1} < x_{l+1-j_1} \). Since \( \sigma_d \neq \sigma_{\text{mirror}} \), there exists \( j \) such that \( x_{\sigma_d(j)} \neq x_{d+1-j} \) and since \( \sigma_d(j) \in \{j, d+1-j\} \) we have that \( \sigma_d(j) = j \). If \( j > (d + 1)/2 \) this implies that \( x_j > x_{d+1-j} \) and so we take \( j_2 = j \). If \( j \leq d/2 \) then \( x_j < x_{d+1-j} \) and so we take \( j_2 = d + 1 - j > (d+1)/2 \). Assume without loss of generality that \( j_1 < j_2 \) and take \( m = 1 + (j_2 - j_1) \geq 2 \) and \( i = (d + 2 - m) - j_1 = d + 1 - j_2 \).

\[
\begin{align*}
  j_1, j_2 &\in [(d + 2 - m) - i, i + 2 - m] = [j_1, j_2],
\end{align*}
\]

and since \( j_1 > (d + 1)/2 \) we also have that \( i < (d + 2 - 2m)/2 \). We conclude that the chosen \( i, m \) satisfy the condition for strict inequality, as desired.

\[\square\]

### D Proof of Lemma 12

We first need the following lemma whose proof may be found in Appendix D.1.

**Lemma 21.** Let \( \sigma_d = \theta^{-1} \sigma_d \theta \), \( \sigma_{d-1} = \theta^{-1} (\sigma_{\text{mirror}} \sigma_d^{-1} \sigma_{\text{mirror}}) \theta \), where \( \theta \) is from Lemma 3 and \( \sigma_l \) is from Definition 9, and for \( z \in \mathbb{R}^d \), let \( N_i(z) \) be the number of elements in \( \{z_1, \ldots, z_{i-1}\} \) that are strictly greater than \( z_i \), i.e.,

\[
N_i(z) = |\{j \mid j < i \land z_j > z_i\}|.
\]

We have that

\[
N_i(\sigma_{d} z) \leq \begin{cases} 
N_{\max\{1, i-1\}}(z), & \text{i even}, \\
\max\{N_i(z), N_{i+1}(z) - 1\}, & \text{i odd and } i < d, \\
N_d(z), & \text{i = d odd},
\end{cases}
\]

\[
N_i(\sigma_{d-1} z) \leq \begin{cases} 
N_{\max\{1, i-1\}}(z), & \text{i odd}, \\
\max\{N_i(z), N_{i+1}(z) - 1\}, & \text{i even and } i < d, \\
N_d(z), & \text{i = d even}.
\end{cases}
\]

**Proof of Lemma 12.** Let \( \sigma_d, \sigma_{d-1} \) be defined as in Lemma 21, and notice that

\[
\theta(\sigma_{d-1} \sigma_d)^k \theta^{-1} = \theta(\theta^{-1} \sigma_{\text{mirror}} \sigma_d^{-1} \sigma_{\text{mirror}} \sigma_d \theta)^k \theta^{-1} = (\sigma_{\text{mirror}} \sigma_{d-1} \sigma_{\text{mirror}} \sigma_d)^k.
\]

Recall that by Lemma 3 we have that \( \theta x_{\text{sort}} = x_{\text{pend}} \). We show that \( (\sigma_{d-1} \sigma_d)^k z = z_{\text{sort}} \) for all \( z \in \mathbb{R}^d \), \( k \geq d/2 \), and then choosing \( z = \theta^{-1} x \) concludes the proof.

To prove the desired we need the following definition. For \( z \in \mathbb{R}^d \), let \( N_i(z) \) be the number of elements in \( \{z_1, \ldots, z_{i-1}\} \) that are strictly greater than \( z_i \). Formally

\[
N_i(z) = |\{j \mid j < i \land z_j > z_i\}|.
\]

Notice that

\[
z = z_{\text{sort}} \iff N_i(z) = 0, \forall 1 \leq i \leq d,
\]

and also that \( N_i(z) \leq i - 1 \) for all \( z \in \mathbb{R}^d \). Now, let \( i \in \{1, \ldots, d\} \) be odd, then using Lemma 21 we have that

\[
N_i(\sigma_{d-1} \sigma_d z) \leq N_{\max\{1, i-1\}}(\sigma_d z) \leq N_{\max\{1, i-2\}}(z),
\]

\[\tag{13}
\]
where the second transition used the fact that \( i - 1 \) is even. Applying this recursively, we get that for \( i \) odd and \( k \geq 0 \)

\[
N_i((\tilde{\sigma}_{d-1}\tilde{\sigma})^k z) \leq N_{\max\{1,i-2k\}}(z) \leq \max\{0,i-2k-1\}.
\]  

(14)

Now, let \( i \in \{1,\ldots,d\} \) be even, and split into three cases. In the first case, \( i = d \) and thus \( d \) is even. Then using Lemma 21 we have that

\[
N_i(\tilde{\sigma}_{d-1}\tilde{\sigma}z) = N_d(\tilde{\sigma}_{d-1}\tilde{\sigma}z) \leq N_d(\tilde{\sigma}z) \leq N_{\max\{1,d-1\}}(z),
\]

and since \( d - 1 \) is odd, we use Eq. (14) we get that for \( i = d \) even and \( k \geq 0 \)

\[
N_i((\tilde{\sigma}_{d-1}\tilde{\sigma})^k z) \leq N_{\max\{1,d-1\}}((\tilde{\sigma}_{d-1}\tilde{\sigma})^{k-1} z) \leq \max\{0,d-2k\}.
\]

(15)

In the second case, \( i = d - 1 \) and thus \( d \) is odd. Then using Lemma 21 we have that

\[
N_i(\tilde{\sigma}_{d-1}\tilde{\sigma}z) = N_{d-1}(\tilde{\sigma}_{d-1}\tilde{\sigma}z) \leq \max\{N_{d-1}(\tilde{\sigma}z),N_{d}(\tilde{\sigma}z) - 1\}
\]

\[\leq \max\{N_{\max\{1,d-2\}}(z),N_d(z) - 1\},\]

and since \( d,d - 2 \) are odd, we can use Eq. (14) to get that for \( i = d - 1, i \) even and \( k \geq 0 \)

\[
N_i((\tilde{\sigma}_{d-1}\tilde{\sigma})^k z) \leq \max\{N_{\max\{1,i-1\}}(z),N_{i+1}(z) - 1,N_{i+2}(z) - 2\}
\]

Replacing \( z \) with \((\tilde{\sigma}_{d-1}\tilde{\sigma})^{k-1} z\) and applying Eq. (14) we get that for \( k \geq 0 \)

\[
N_i((\tilde{\sigma}_{d-1}\tilde{\sigma})^k z) \leq \max\{0,i - 1 - 2k + 1,i + 1 - 2k,N_{i+2}((\tilde{\sigma}_{d-1}\tilde{\sigma})^{k-1} z) - 2\}
\]

\[\leq \max\{0,d - 2k,N_{i+2}((\tilde{\sigma}_{d-1}\tilde{\sigma})^{k-1} z) - 2\}.
\]

Now, let \( k \geq d/2 \) and let \( k_i = \lfloor (d - i - 1)/2 \rfloor \). We open the recursion above \( k_i \) times to get that for \( k \geq d/2 \)

\[
N_i((\tilde{\sigma}_{d-1}\tilde{\sigma})^k z) \leq \max\{0,d - 2k,N_{i+2k_i}((\tilde{\sigma}_{d-1}\tilde{\sigma})^{k-k_i} z) - 2k_i\}
\]

\[\leq \max\{0,d - 2k,d - 2(k - k_i) - 2k_i\} = \max\{0,d - 2k_i\},
\]

(17)

where the second to last transition follows using Eqs. (15) and (16) since \( i + 2k_i \in \{d - 1,d\} \). Combining Eqs. (14) to (17) with \( k \geq d/2 \) we conclude that

\[
N_i((\tilde{\sigma}_{d-1}\tilde{\sigma})^k z) = 0 \text{ for all } 1 \leq i \leq d \text{ and thus by Eq. (13)}
\]

that \((\tilde{\sigma}_{d-1}\tilde{\sigma})^k z = z_{\text{sort}}\).

\[\blacksquare\]

**D.1 Proof of Lemma 21**

We first need the following lemma whose proof may be found in Appendix D.2.

**Lemma 22.** Define \( \tilde{\sigma}_d = \theta^{-1}\sigma_d\theta, \tilde{\sigma}_{d-1} = \theta^{-1}(\sigma_{\text{mirror}}\sigma_{d-1}\sigma_{\text{mirror}})\theta \), where \( \theta \) is from Lemma 3 and \( \sigma_i \) is from Definition 9. Then we have that

\[
\tilde{\sigma}_d(i) = \begin{cases} 
  i - 1 & , i > 1 \text{ even } \land x_i < x_{i-1} \\
  i + 1 & , i < d \text{ even } \land x_i > x_{i+1} \\
  i & , \text{otherwise}
\end{cases}
\]

\[
\tilde{\sigma}_{d-1}(i) = \begin{cases} 
  i - 1 & , i > 1 \text{ odd } \land x_i < x_{i-1} \\
  i + 1 & , i < d \text{ odd } \land x_i > x_{i+1} \\
  i & , \text{otherwise}
\end{cases}
\]

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\textbf{Proof of Lemma 21.} We prove the expression for $\tilde{\sigma}_d$. The proof for $\tilde{\sigma}_{d-1}$ is identical. Throughout the proof we treat $\tilde{\sigma}_d$ as the expression derived for it in Lemma 22. First, notice that for $j < i$ we have that $\tilde{\sigma}_d(j) \leq i$. Moreover, if $i$ is odd then $\tilde{\sigma}_d(i-1) \leq i-1$ and thus $\tilde{\sigma}_d(j) < i$. We conclude that

$$\{\tilde{\sigma}_d(j) \mid j < i\} \subseteq \begin{cases} \{j \mid j < i\}, & i \text{ odd} \\ \{j \mid j \leq i\}, & \text{otherwise}. \end{cases} \quad (18)$$

Using the above, we have that for any $z \in \mathbb{R}^d$

$$N_i(\tilde{\sigma}_d z) = |\{j \mid j < i \wedge z_{\tilde{\sigma}_d(j)} > z_{\tilde{\sigma}_d(i)}\}|$$

$$= |\{\tilde{\sigma}_d(j) \mid j < i \wedge z_{\tilde{\sigma}_d(j)} > z_{\tilde{\sigma}_d(i)}\}|$$

$$\leq \begin{cases} |\{j \mid j < i \wedge z_j > z_{\tilde{\sigma}_d(i)}\}|, & i \text{ odd} \\ |\{j \mid j \leq i \wedge z_j > z_{\tilde{\sigma}_d(i)}\}|, & \text{otherwise} \end{cases} \quad \text{(by Eq. (18))}$$

$$\leq \begin{cases} |\{j \mid j < i \wedge z_j > z_{i-1}\}|, & i > 1 \wedge i \text{ even} \wedge z_i < z_{i-1} \\ |\{j \mid j < i \wedge z_j > z_{i+1}\}|, & i < d \wedge i \text{ odd} \wedge z_i > z_{i+1} \end{cases} \quad \text{(by Lemma 22)}$$

Notice that if $i > 1$ and $z_i < z_{i-1}$ then

$$|\{j \mid j \leq i \wedge z_j > z_{i-1}\}| = |\{j \mid j < i-1 \wedge z_j > z_{i-1}\}| = N_{i-1}(z),$$

and if $i < d$ and $z_i > z_{i+1}$ then

$$|\{j \mid j < i \wedge z_j > z_{i+1}\}| = |\{j \mid j < i+1 \wedge z_j > z_{i+1}\}| - 1 = N_{i+1}(z) - 1,$$

and finally that

$$|\{j \mid j < i \wedge z_j > z_i\}| = |\{j \mid j < i \wedge z_j > z_i\}| = N_i(z).$$

Plugging these back into the above inequality we get that

$$N_i(\tilde{\sigma}_d z) \leq \begin{cases} N_{i-1}(z), & i > 1 \wedge i \text{ even} \wedge z_i < z_{i-1} \\ N_{i+1}(z) - 1, & i < d \wedge i \text{ odd} \wedge z_i > z_{i+1} \\ N_i(z), & \text{otherwise}. \end{cases}$$

Now, if $z_i \geq z_{i-1}$ then

$$N_i(z) = |\{j \mid j < i \wedge z_j > z_i\}| = |\{j \mid j < i-1 \wedge z_j > z_i\}|$$

$$\leq |\{j \mid j < i-1 \wedge z_j > z_{i-1}\}| = N_{i-1}(z),$$

and using this fact, and some manipulations on the cases of the previous inequality, we conclude that

$$N_i(\tilde{\sigma}_d z) \leq \begin{cases} N_{i-1}(z), & i \text{ even} \\ \max\{N_i(z), N_{i+1}(z) - 1\}, & i < d \wedge i \text{ odd} \\ N_d(z), & i = d \text{ odd}. \end{cases}$$

Since for $i$ even we have that $i-1 = \max\{1, i-1\}$, the proof is concluded. \hfill \blacksquare

\textbf{D.2 Proof of Lemma 22}

\textbf{Proof.} Recall that $\sigma_i$ is defined w.r.t. the vector it permutes.

Specifically, we have that $(\sigma_d \partial x)(i) = x_{\theta_{\sigma_d(i)}^\partial}$, where we used $\sigma_d^\partial$ to denote $\sigma_d$ w.r.t. the vector it permutes,
i.e., w.r.t. $\theta x$. We have that

$$
\sigma^\theta_x = \begin{cases}
  d + 1 - i, & \text{or } x_{\theta(i)} > x_{\theta(d+1-i)}, \ i \leq d/2 \\
  i, & x_{\theta(i)} < x_{\theta(d+1-i)}, \ d/2 < i \leq d \\
end{cases}
$$

$$= \begin{cases}
  d + 1 - i, & x_{2i-1} > x_{2i}, \ i \leq d/2 \\
  i, & x_{2(d+1-i)} < x_{2(d+1-i)-1}, \ d/2 < i \leq d \\
end{cases}
$$

To prove the lemma, we will show that $\theta \tilde{\sigma}_d = \sigma_d \theta$, i.e., $\theta \tilde{\sigma}_d^\theta = \sigma_d^\theta \theta$.

Indeed,

$$
\sigma_d \theta(i) = \begin{cases}
  2\sigma_d(i) - 1, & \sigma_d(i) \leq \frac{d+1}{2} \\
  2(d + 1 - \sigma_d(i)), & \sigma_d(i) > \frac{d+1}{2} \\
end{cases} \text{ or } \begin{cases}
  2(d + 1 - i) - 1, & x_{2(d+1-i)} < x_{2(d+1-i)-1}, \ d/2 < i \leq d \\
  2i - 1, & x_{2i-1} > x_{2i}, \ i \leq d/2 \\
  \theta(i), & \text{otherwise}
end{cases}
$$

and

$$
\theta \tilde{\sigma}_d(i) = \begin{cases}
  \theta(i) - 1, & \theta(i) > 1 \wedge \theta(i) \text{ even } \wedge x_{\theta(i)} < x_{\theta(i)-1} \\
  \theta(i) + 1, & \theta(i) < d \wedge \theta(i) \text{ odd } \wedge x_{\theta(i)} > x_{\theta(i)+1} \\
  \theta(i), & \text{otherwise}
end{cases}
$$

$$= \begin{cases}
  2(d + 1 - i) - 1, & 2(d + 1 - i) > 1 \wedge i > \frac{d+1}{2} \wedge x_{2(d+1-i)} < x_{2(d+1-i)-1} \\
  2i - 1, & 2i - 1 < d \wedge i > \frac{d+1}{2} \wedge x_{2i-1} > x_{2i-1+1} \\
  \theta(i), & \text{otherwise}
end{cases}
$$

$$= \begin{cases}
  2(d + 1 - i) - 1, & x_{2(d+1-i)} < x_{2(d+1-i)-1}, \ d/2 < i \leq d \\
  2i, & x_{2i-1} > x_{2i}, \ i \leq d/2 \\
  \theta(i), & \text{otherwise}
end{cases}
$$

thus $\theta \tilde{\sigma}_d = \sigma_d \theta$.

\section*{E Proof of Proposition 13}

\textbf{Proof.} The proof follows by the convexity of $\mathbb{E} \tau \{ p; 0 \}$ on $C \subseteq [\frac{1}{2}, 1]^d$, as shown in Lemma 23 below. Since $\mathbb{E} \tau \{ p; 0 \}$ is convex, there exists a maximizer $p^* \in \text{ext}(C)$, and by assumption we also have that $p^*_{\text{pend}} \in \text{ext}(C)$. By Theorem 4 we have that $\mathbb{E} \tau \{ p^*_{\text{pend}}; 0 \} \geq \mathbb{E} \tau \{ p^*; 0 \}$ and thus $p^*_{\text{pend}} \in (\text{ext}(C))_{\text{pend}}$ is also a maximizer.

\textbf{Lemma 23 (Escape Time Convexity).} $\mathbb{E} \tau \{ p; 0 \}$ is convex for $p \in [\frac{1}{2}, 1]^d$.

\textbf{Proof.} Define $f(a) = \mathbb{E} \tau \{ \frac{1}{2} + a; 0 \}$; then by Proposition 5 we have that

$$
f(a) = (d + 1) + \sum_{m=1}^{d} \sum_{i=1}^{d-m+1} \prod_{j=i}^{d-m+1} \frac{1}{2} + a_j.
$$

Since this is a linear variable exchange, it suffices to show that $f$ is convex over $[0, 1]^d$. Denote

$$
g_{i,m}(x) = \prod_{j=i}^{i+m-1} \frac{1}{2} + x_j - \frac{1}{2} x_j.
$$
Then
\[ f(a) = (d + 1) + \sum_{m=1}^{d} \sum_{i=1}^{d-m+1} g_{i,m}(a). \]

It is thus enough to show that \( g_{i,m} \) are convex in \([0, \frac{1}{2})^d\). We use Theorem 3.2 of [5] which states that \( g_{i,m} \) is convex if and only if \( \frac{1}{2} + x \) is log-convex for \( x \in [0, \frac{1}{2}) \). Indeed,
\[
\frac{\partial^2}{\partial x^2} \left( \log \frac{1}{2} + x \right) = \frac{2x}{(\frac{1}{2} - x)(\frac{1}{2} + x)^2} \geq 0, \quad \forall x \in \left[0, \frac{1}{2}\right).
\]
\[ \blacksquare \]