Many examples of non-cocompact Fuchsian groups sitting in \( \text{PSL}_2(\mathbb{Q}) \)

Mark Norfleet¹

Received: 26 July 2014 / Accepted: 23 April 2015 / Published online: 3 May 2015
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Abstract We will explicitly construct many non-arithmetic Fuchsian groups while controlling some geometric properties of the action on the boundary of the hyperbolic plane. With this construction, we produce infinitely many noncommensurable non-cocompact Fuchsian groups of finite covolume sitting in \( \text{PSL}_2(\mathbb{Q}) \) so that the set of hyperbolic fixed points of each group will contain a given finite collection of elements in the boundary of the hyperbolic plane.

Keywords Isometries of the hyperbolic plane · Non-arithmetic Fuchsian groups · Hyperbolic fixed points · Minkowski space

Mathematics Subject Classification (2010) 20H10 · 57M60

1 Introduction

Let \( \Gamma \) be a Fuchsian group, meaning a discrete subgroup of the group of orientation preserving isometries of \( H^2 \), the hyperbolic plane. A boundary point of \( H^2 \) fixed by a parabolic element of a Fuchsian group \( \Gamma \) is referred to as a cusp of \( \Gamma \), and a line fixed by a hyperbolic element is referred to as an axis with endpoints called hyperbolic fixed points. A prominent example of a Fuchsian group is the quotient \( \text{PSL}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})/\{\pm I\} \) of modular group \( \text{SL}_2(\mathbb{Z}) \).

¹ Department of Mathematics and Statistics, University of Nevada, Reno, 1664 N. Virginia Street, Reno, NV 89557-0084, USA
Recall that Fuchsian groups $\Gamma_1$ and $\Gamma_2$ are commensurable if $\Gamma_1$ has a subgroup of finite index which is conjugate to a subgroup of finite index in $\Gamma_2$. This work has been motivated by the following question:

**Question** If $\Gamma_1$ and $\Gamma_2$ are finite covolume Fuchsian groups with the same set of cusps (or same set of axes), when are they commensurable?

Long and Reid [2] exhibit four examples of mutually noncommensurable subgroups of $\text{PSL}_2(\mathbb{Q})$, which are not commensurable with the modular group, but each of them have cusp set exactly $\mathbb{Q} \cup \{\infty\}$; they call such groups pseudomodular. It is still unknown whether or not there are infinitely many pseudomodular groups up to commensurability. Any other possible candidate for pseudomodular groups are (non-arithmetic) discrete subgroups $\Delta \leq \text{PSL}_2(\mathbb{Q})$, since their cusp set is contained in $\mathbb{Q} \cup \{\infty\}$. For Fuchsian groups, a boundary point cannot both be a cusp and a hyperbolic fixed point. Hence arithmetic and pseudomodular groups cannot have rational hyperbolic fixed points. So if one can exhibit a hyperbolic element of $\Delta \leq \text{PSL}_2(\mathbb{Q})$ that has rational fixed points, then $\Delta$’s cusp set is properly contained in $\mathbb{Q} \cup \{\infty\}$; thus showing $\Delta$ to be neither arithmetic nor pseudomodular. So Long and Reid asked, how to predict when rational hyperbolic fixed points are present.

Another motivation arises from a question A. Rapinchuk asked: Are there infinitely many commensurability classes of finite covolume Fuchsian groups sitting in $\text{PSL}_2(\mathbb{Q})$? One construction which answer this question is supplied by Vinberg in a preprint [6]; he introduced a way to produce infinitely many noncommensurable finite covolume Fuchsian groups in $\text{SL}_2(\mathbb{Q})$. His examples arise as the even subgroup of a group generated by reflections in the sides of quadrilaterals. To establish that he constructed infinitely many groups up to commensurability, Vinberg uses results (from [5]) about the least ring of definition for his examples.

This paper provides a new solution to A. Rapinchuk’s question, which is different from Vinberg’s and other known constructions, and the results simultaneously address the presence of rational hyperbolic fixed points. Namely, we will construct Fuchsian groups sitting in $\text{PSL}_2(\mathbb{Q})$, all of which will possess a given finite set of rational hyperbolic fixed points. The main result is

**Theorem** Let $Y$ be a finite set of rational boundary points of the hyperbolic plane. Then there are infinitely many noncommensurable finite covolume Fuchsian groups sitting in $\text{PSL}_2(\mathbb{Q})$, whose set of hyperbolic fixed points contains $Y$.

Furthermore, in the case of non-arithmetic Fuchsian groups, it is not obvious how to explicitly construct them with different geometric properties; even though non-arithmetic Fuchsian groups are abundant. In addition to the construction controlling some geometric properties of the action on the hyperbolic plane, we show how to require that the action of the groups stabilize no vertex in Serre’s tree of $\text{SL}_2(\mathbb{Q}_p)$ for a given prime $p \equiv 3 \bmod 4$.

The main result will be proved in Sect. 4. As a brief outline, let $Y$ be a finite set of arbitrary chosen boundary points of the hyperbolic plane. We construct (with considerable freedom) examples of Fuchsian groups $\Gamma$ of signature $(0 : 2, \ldots, 2; 1; 0)$ such that the set of hyperbolic fixed points of $\Gamma$ contains $Y$ (see Sect. 2.1). When one choose $Y$ to be a set of rational boundary points and carries out the construction in Minkowski space, it is seen how to restrict some of the freedom so that one can guarantee $\Gamma \leq \text{PGL}_2(\mathbb{Q})$ (see Sect. 2.2). Then we address when the constructed groups are mutually noncommensurable, which relies on analyzing how they act on different trees (see Sect. 3). Specifically, we consider when $\Gamma$ will stabilize no vertex in the Serre’s tree of $\text{SL}_2(\mathbb{Q}_p)$ for a prime $p \equiv 3 \bmod 4$ (see Proposition 3). This
perspective allows us to construct an infinite family of mutually noncommensurable groups in $\text{PSL}_2(\mathbb{Q})$ while the set of hyperbolic fixed points is guaranteed to contain $Y$.

2 The construction

In this section, we will construct a fundamental domain for a Fuchsian group $\Gamma$ of signature $(0 : 2, \ldots, 2; 1; 0)$ so that the set of hyperbolic fixed points of $\Gamma$, denoted by $\text{HFix}(\Gamma)$, will contain a given finite set $Y$ of elements in the boundary of the hyperbolic plane. Before the construction begins, we introduce some notation and a lemma.

Let $H^2$ be the hyperbolic plane. We will write $xy$ for the closure of the geodesic line with endpoints $x$, $y$ in the closure of $H^2$. The isometry denoted by $\rho_f$ is rotation by $\pi$ with fixed point $f \in H^2$. Let $v_\infty$ be an element in $\partial H^2$; then we have an order $\leq$ on $\partial H^2 \setminus \{v_\infty\}$ from the order on the real line in the upper half plane model.

Lemma 2.1 Let $x < v < y < u$ in $\partial H^2 \setminus \{v_\infty\}$. For each $f$ in the interior of $\overline{xy}$, where $c = \overline{vu} \cap \overline{xy}$, one constructs $t = \rho_f(u)$ and $w = \rho_f(v)$. Then $x < v < t < y < w < u$ in $\partial H^2$ and $f = \overline{xy} \cap \overline{tu} \cap \overline{vw}$.

We omit the proof, but include Fig. 1 to help illustrate the lemma.

2.1 The construction of a fundamental domain for $\Gamma$

As an overview of the construction (Fig. 2 illustrates an example), we start with a finite set of points in the boundary of the hyperbolic plane (the set $Y = \{y_i\}$). We use Lemma 2.1 to sequentially construct the vertices and edges (the solid lines in Fig. 2) of an ideal convex polygon in $H^2$. We then show that the ideal convex polygon constructed is a fundamental domain for a discrete group $\Gamma$ generated by isometries that are rotations by $\pi$. Furthermore, $\Gamma$ is guaranteed to possess a set of hyperbolic elements (the dash lines are their axes in Fig. 2) whose fixed points contain the initially given set of boundary points.

Below we will have the notational convention: $f_i \in H^2$, $\rho_i = \rho_{f_i}$ is rotation by $\pi$ with fixed point $f_i$, and $y_i, x_i, v_i \in \partial H^2$.

We begin the construction. Let $Y$ be a finite set of $n - 1$ points in $\partial H^2 \setminus \{v_\infty\}$. Let $Y = \{y_i\}$ and $v_0 \neq v_\infty$ so that $v_0 < y_1 < \cdots < y_{n-1} \neq v_\infty$ in the boundary of the hyperbolic plane.

1st Step: Choose $x_1$ such that $v_0 < x_1 < y_1$ in $\partial H^2$, and then choose $f_1 \in \overline{vx_1y_1}$. Define $v_1 = \rho_1(v_0)$; note $v_0 < x_1 < v_1 < y_1$ in $\partial H^2$, and $f_1 \in \overline{v_0v_1}$.

When $n > 2$, let $i \in \{2, \ldots, n - 1\}$.
\[ \begin{align*}
\text{\textit{i}th Step:} & \text{ Let } x_{i-1} < v_{i-1} < y_{i-1} < y_i \text{ in } \partial H^2. \text{ By Lemma 2.1, one can choose a } f_i \in \mathbb{H}_{x_{i-1}y_{i-1}} \text{ and construct } x_i = \rho_i(y_i) \text{ and } v_i = \rho_i(v_{i-1}), \text{ so that } v_{i-1} < x_i < y_{i-1} < v_i < y_i \text{ and } f_i \in \mathbb{H}_{x_{i-1}y_i}. \\
\text{\textit{n}th Step:} & \text{ Let } x_{n-1} < v_{n-1} < y_{n-1} \neq v_\infty \text{ in } \partial H^2. \text{ Now construct } f_n = \mathbb{H}_{x_{n-1}y_{n-1}} \cap \mathbb{H}_{v_{n-1}v_\infty}, \text{ and define } v_n = \rho_n(v_{n-1}). \\
\text{\textit{Last Step:}} & \text{ Given } \rho_n \ldots \rho_1(v_0) = v_n, \text{ construct } f_0 \in \mathbb{H}_{v_0v_n} \text{ and } \rho_0 \text{ so that } \rho_n \ldots \rho_1 \rho_0 \text{ is parabolic fixing } v_n. \\
\end{align*} \]

Remark The hyperboloid model of the hyperbolic plane in Minkowski space (see Sect. 2.2) can be utilized for constructing \( f_0 \) and \( \rho_0 \) in the last step. With a slight abuse of notation, \( f_i, v_i, x_i, y_i \) are vectors in Minkowski space \( \mathbb{R}^{2,1} \). One can see that the vectors \( v_n \) and \( v_\infty \) are linearly dependent light-like vectors. For \( f_0 \in \text{span} \{v_n + v_0\} \), the element \( \rho_0 \) as a Lorentz transformation maps \( v_n \) to \( v_0 \) (as vectors in \( \mathbb{R}^{2,1} \)), which shows \( \rho_n \ldots \rho_1 \rho_0 \) is parabolic fixing \( v_n \).

We have an ideal \( n + 1 \) sided convex polygon, \( P \), with vertices \( \{v_0, v_1, \ldots, v_{n-1}, v_n\} \) (as vertices \( v_n = v_\infty \); furthermore, \( f_i \) is on the edge \( \mathbb{H}_{v_{i-1}v_i} \), and \( f_0 \) is on the edge \( \mathbb{H}_{v_0v_n} \) (see Fig. 2). Since \( \rho_i \) is rotation by \( \pi \) with fixed point \( f_i \in H^2 \), \( \rho_i \) maps \( \mathbb{H}_{v_{i-1}v_i} \) to itself (likewise, \( \rho_0 \) maps \( \mathbb{H}_{v_0v_n} \) to itself); that is, \( \rho_i \) maps the directed edge \( f_i \overline{v_i v_{i-1}} \) to the directed edge \( f_i \overline{v_i v_i} \), and \( \rho_0 \) maps the directed edge \( f_0 \overline{v_0 v_0} \) to the directed edge \( f_0 \overline{v_0 v_n} \). By Poincaré’s Theorem (see section §9.8 in [1]), the group \( \Gamma \) generated by \{\( \rho_1, \ldots, \rho_n, \rho_0 \}\) is discrete, and \( P \) is a fundamental domain for \( \Gamma \). In the last step, we made \( \rho_n \ldots \rho_1 \rho_0 \) parabolic fixing \( v_n \); thus \( H^2 / \Gamma \) is a complete finite area once punctured 2-sphere with \( n + 1 \) cone points of order 2. So the signature of \( \Gamma \) is \( (0 : 2, \ldots, 2; 1 : 0) \) (as defined in section §10.4 in [1]).

For \( 1 \leq i < n \), the element \( \rho_i \rho_{i+1} \) is hyperbolic with axis \( \overline{x_i y_i} \), since \( f_i \) and \( f_{i+1} \) both lie on the geodesic line \( \overline{x_i y_i} \) (by construction); therefore, \( y_i \) is a hyperbolic fixed point for \( \rho_i \rho_{i+1} \in \Gamma \).

By Lemma 2.1, one sees that there are infinitely many choices for each \( f_i \) (for \( 1 \leq i < n \)), producing infinitely many such Fuchsian groups \( \Gamma \); establishing the following:

**Proposition 1** Let \( Y \) be a finite set of \( n - 1 \) points in \( \partial H^2 \). Then there are infinitely many Fuchsian groups \( \Gamma \) of finite covolume of signature \( (0 : 2, \ldots, 2; 1 : 0) \) with \( n + 1 \) number of 2s such that \( Y \subset \text{HFix}(\Gamma) \).
2.2 $\Gamma$ in $O^+(2,1)$ and $PGL_2(\mathbb{R})$

In this section, we consider the action of the Fuchsian group $\Gamma$ on two standard models for the hyperbolic plane. For the development of the explicit metrics for the standard models for hyperbolic space and isometries between the different models, see [3]. We will consider the isometry $\rho_f$, rotation by $\pi$ with fixed point $f$, in the upper half plane model and the hyperboloid model. Using the hyperboloid model will ease how one restrict some of the freedom in constructing $\Gamma$ while clearly preserving infinitely many choices in the construction.

Let $M^3$ be a dimension 3 real vector space with a nondegenerate quadratic form $(\cdot, \cdot)$ of signature $(2,1)$. Choose a basis $\{e_0, e_1, e_2\}$ with $(e_i, e_j) = 0$ if $i \neq j$, $(e_i, e_i) = 1$ if $i \geq 1$, and $(e_0, e_0) = -1$. Such a basis is called a Lorentz orthonormal basis, and $M^3$ is denoted as $\mathbb{R}^{2,1}$ when such a basis is fixed; $\mathbb{R}^{2,1}$ is called Minkowski space. We will notate $L = \{v : (v, v) = 0\}$ (the set of light-like vectors) and $T = \{v : (v, v) < 0\}$ (the set of time-like vectors). Let $L^+$ and $T^+$ be the sets of vectors with positive $e_0$-coordinate in $L$ and $T$, respectively. We let $O^+(2,1)$ be the group of linear transformations of $M^3$ that preserve the quadratic form and upper sheet of the hyperboloid $\mathcal{H} = \{v : (v, v) = -1\} \cap T^+$. With a Lorentz orthonormal basis $\{e_0, e_1, e_2\}$, let the $\mathbb{Q}$-linear combination of $\{e_0, e_1, e_2\}$ be denoted by $\mathbb{Q}^2$; furthermore, let $L^+_Q = L^+ \cap \mathbb{Q}^{2,1}$ and $T^+_Q = T^+ \cap \mathbb{Q}^{2,1}$.

For every $w = (w_0, w_1, w_2) \in T^+$, there is a 2 by 2 real symmetric matrix whose determinant is $-\langle w, w \rangle$; namely,

$$w \mapsto \begin{pmatrix} w_0 + w_2 & w_1 \\ w_1 & w_0 - w_2 \end{pmatrix} = \begin{pmatrix} \langle w, e_0 + e_2 \rangle & \langle w, e_1 \rangle \\ \langle w, e_1 \rangle & \langle w, e_0 - e_2 \rangle \end{pmatrix}.$$ 

This is a one-to-one correspondence between $T^+$ and 2 by 2 real symmetric matrix with positive determinant. We have that $SL_2(\mathbb{R})$ acts on 2 by 2 real symmetric matrices by similarity and factors through $PSL_2(\mathbb{R})$; that is, $\Sigma \mapsto A^t \Sigma A$, where $A \in SL_2(\mathbb{R})$ and $\Sigma$ is a 2 by 2 real symmetric matrix. Since the action of $PSL_2(\mathbb{R})$ preserves the determinant of the real symmetric matrices, elements of $SL_2(\mathbb{R})$ map the upper sheet of the hyperboloid $\mathcal{H}$ to itself and preserves the quadratic form. By Theorem 3.2.3 of [3], each element of $PSL_2(\mathbb{R})$ extends to a unique element of $O^+(2,1)$.

**Example** Consider the isometry $\rho_f$, rotation by $\pi$ with fixed point $f \in H^2$.

In the hyperboloid model, $\rho_f$ corresponds to an element in $O^+(2,1)$; let $f \in T^+$ be fixed by $\rho_f \in O^+(2,1)$. Let $\Sigma_f$ be the 2 by 2 real symmetric matrix associated to $f$,

$$\Sigma_f = \begin{pmatrix} \langle f, e_0 + e_2 \rangle & \langle f, e_1 \rangle \\ \langle f, e_1 \rangle & \langle f, e_0 - e_2 \rangle \end{pmatrix}.$$ 

In the upper half plane model of $H^2$, say $a + bi$ is fixed by $\rho_f$ as a matrix in $SL_2(\mathbb{R})$; that is,

$$\rho_f = \begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{b} \begin{pmatrix} a & -(b^2 + a^2) \\ 1 & -a \end{pmatrix}. \quad (*)$$

We have that $\rho_f$, as the matrix from $(*)$, acts by similarity on 2 by 2 real symmetric matrices and fixes $\Sigma_f$ when

$$b = -\sqrt{\langle f, f \rangle} \quad \text{and} \quad a = -\frac{\langle f, e_1 \rangle}{\langle f, e_0 + e_2 \rangle}.$$ 

**Remark** Let the group $GL^+_2(\mathbb{R})$ be invertible 2 by 2 matrices having positive determinant, and $PGL^+_2(\mathbb{R}) = GL^+_2(\mathbb{R})/Z(GL^+_2(\mathbb{R}))$, which is isomorphic to $PSL_2(\mathbb{R})$. By identifying
PGL$_2^+(\mathbb{R})$ with PSL$_2(\mathbb{R})$, we can view the group PGL$_2^+(\mathbb{Q}) \leq$ PGL$_2^+(\mathbb{R})$ as a subgroup of PSL$_2(\mathbb{R})$. Moreover, conjugating subgroups of PGL$_2^+(\mathbb{R})$ in PGL$_2(\mathbb{R})$ gives another subgroup of PGL$_2^+(\mathbb{R})$, since conjugation preserves the sign of the determinant. As an abuse of notation, $\rho_f$ will also denote its corresponding elements in $O^+(2,1)$ or PGL$_2(\mathbb{R})$. Considering the matrix from $(\ast)$ in PGL$_2(\mathbb{R})$, $\rho_f$ can be represented by a 2 by 2 matrix with entries in $\mathbb{Q}$ when $\langle f, f \rangle$, $\langle f, e_i \rangle$ are all in $\mathbb{Q}$.

**Proposition 2** Let $v_0 < y_1 < \cdots < y_{n-1} \neq v_\infty$ in $L_0^+$. Then there are infinitely many non-cocompact Fuchsian groups $\Delta$ of finite covolume sitting in PSL$_2(\mathbb{Q})$ such that \{ $y_1, \ldots, y_{n-1}$ \} $\subset$ HFix($\Delta$).

**Proof** We follow the construction of $\Gamma$ in Minkowski space. In the $i$th step (for $1 \leq i < n$) of Sect. 2.1, we additionally require the choice of $x_i$ and the $f_i$, as vectors in Minkowski space, to lie in $\mathbb{Q}^{2,1}$. Furthermore, $f_n$ and $f_0$ will be in $\mathbb{Q}^{2,1}$, since all $x_i, v_i, y_i$ will be in $\mathbb{Q}^{2,1}$. With all $f_i \in \mathbb{Q}^{2,1}$, $\rho_{f_i}$ $\in$ PGL$_2(\mathbb{Q})$; thus $\Gamma$ will sit in PGL$_2(\mathbb{Q})$. Even with these additional requirements in the construction, there are still infinitely many choices for each $f_i$ (for $1 \leq i < n$), producing infinitely many such Fuchsian groups $\Gamma$.

For each $\Gamma$, \{ $y_i$ \} $\subset$ HFix($\Gamma$), and let $\Delta$ be the kernel of $\Gamma \longrightarrow$ PGL$_2(\mathbb{Q})$/PSL$_2(\mathbb{Q})$, which is of finite index in $\Gamma$; therefore, it follows \{ $y_i$ \} $\subset$ HFix($\Delta$).

### 3 Acting on the tree of SL$_2(\mathbb{Q}_p)$

This section shows how the construction can be used to produce Fuchsian groups $\Delta$ in PSL$_2(\mathbb{Q})$ that will stabilize no vertex in Serre’s tree of SL$_2(\mathbb{Q}_p)$ for a given prime $p \equiv 3$ mod 4, in addition to HFix($\Delta$) containing the finite set $Y$ of rational boundary points.

As in Serre’s book [4], let $K$ denotes a field with a discrete valuation $v$, recall that $v$ is a homomorphism of $K^\times$ onto $\mathbb{Z}$, and $O_v$ denotes the valuation ring of $K$, i.e. the set of $x \in K$ such that $v(x) \geq 0$ or $x = 0$. Fix an element $\pi \in K$ with $v(\pi) = 1$, the uniformizer. If $K = \mathbb{Q}$, then most $v$ subscripts are replaced with the letter $p$ for the $p$-adic valuation $v_p$.

Let $V$ be a vector space of dimension 2 over $K$. A lattice in $V$ is any finitely generated $O_v$-submodule of $V$ which generates the $K$-vector space $V$; such a module is free of rank 2. The group $K^\times$ acts on the set of lattices; we call the orbit of a lattice $L$ under this action its class (at times notated $[L] = \Lambda$), and two lattices belonging to the same class are called equivalent. The set of lattice classes is denoted by $\mathcal{T}_v$, which is made into a combinatorial graph with edges between $\Lambda_0$ and $\Lambda_1$ when $[L_i] = \Lambda_i$ such that $L_0 \leq L_1$ and $L_1/L_0 \cong O_v/\pi O_v$. Serre proved that $\mathcal{T}_v$ is a tree (see Theorem 1 of II §1.1 in [4]). Given an element $s \in$ GL$(V) \cong$ GL$_2(K)$, the action is $s[L] = [sL]$. For example, let $\{ \xi_1, \xi_2 \}$ be a basis from $V$; consider $\rho = (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}) \in$ GL$_2(K)$ and the lattice $L$ with basis $\{ \xi_1, \pi \xi_2 \}$, and so $\rho L$ has a basis $\{ -1 \pi \xi_1, \xi_2 \}$.

The Lemmas below will explicitly look into the relationship between the group $\Gamma$’s action and that of the generators $\rho_j$ of $\Gamma$ on Serre’s trees of SL$_2$. As in $(\ast)$, let $\rho_j = C_j (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}) C_j^{-1}$, where $C_j = (\begin{smallmatrix} b & a \\ 0 & 1 \end{smallmatrix}) \in$ GL$_2(K)$. When $-1$ is not a square in $K$, the fixed set of the action of $\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}$ on $\mathcal{T}_v$ is easily calculated, and it is $\{ [O_v^2] \}$, where $O_v^2$ is the standard lattice. In this case, we will see (in Lemma 3.2) how the fixed set of $\rho_m \rho_k$ relates to the action of $C_k^{-1}C_m \in$ GL$_2(K)$ on the vertex $[O_v^2]$.

**Remark** This approach arises naturally from the geometric perspective of the action on $\mathcal{T}_v$ that one finds cultivated in Serre’s book [4]. When considering alternative approaches for
establishing an equivalency between statements like (1) and (3) of Lemma 3.2, one commonly works to algebraically simplify an inequality involving the valuation of the trace of $\rho_m \rho_k$. We focus more on a geometric perspective in the proof of Lemma 3.2.

**Lemma 3.1** Let $\Gamma$ be generated by a finite number of $\rho_j$, where $\rho_j = C_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C_j^{-1}$ with $C_j \in \text{GL}_2(K)$.

Then the following are equivalent statements about the action of $\Gamma$ on the tree $T_v$:

1. $\Gamma \leq \text{Stab}(\Lambda)$ for some $\Lambda \in T_v$;
2. $\bigcap C_j \text{Fix}_{T_v} \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \neq \emptyset$;
3. For each pair $(m, k)$, $\text{Fix}_{T_v}(\rho_m \rho_k) \neq \emptyset$.

**Proof** (1) $\Leftrightarrow$ (2): follows from the equality

$$\text{Fix}_{T_v} \left( C_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C_j^{-1} \right) = C_j \text{Fix}_{T_v} \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right).$$

(3) $\Leftrightarrow$ (1): see I§6.5 in [4].

**Lemma 3.2** Let $\Gamma$ be generated by a finite number of $\rho_j$, where $\rho_j = C_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C_j^{-1}$ with $C_j = \begin{pmatrix} b_j & a_j \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(K)$.

When $-1$ is not a square in $K$, then the following are equivalent:

1. $\Gamma \leq \text{Stab}(\Lambda)$ for some $\Lambda \in T_v$;
2. For each pair $(m, k)$, $C_k^{-1} C_m \in \text{GL}_2(\mathcal{O}_v)$;
3. For each pair $(m, k)$,

$$v(a_m - a_k) \geq v(b_m) = v(b_k).$$

**Proof** When $-1$ is not a square in $K$, the $\text{Fix}_{T_v} \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) = \{[\mathcal{O}_v^2]\}$, where $\mathcal{O}_v^2$ is the standard lattice.

Let $\rho_m$ and $\rho_k$ be two generators of $\Gamma$. Then $\text{Fix}_{T_v}(\rho_m \rho_k) \neq \emptyset$ if and only if

$$C_m \text{Fix}_{T_v} \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \bigcap C_k \text{Fix}_{T_v} \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \neq \emptyset,$$

and that holds only when $C_k^{-1} C_m \in \text{GL}_2(\mathcal{O}_v)$. By Lemma 3.1, (1) and (2) are equivalent.

To complete the proof note that $C_k^{-1} C_m = \begin{pmatrix} b_m & am - a_k \\ b_k 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathcal{O}_v)$ if and only if

$$v \left( \begin{pmatrix} b_m \\ b_k \end{pmatrix} \right) = 0 \quad \text{and} \quad v \left( \begin{pmatrix} a_m - a_k \\ b_k \end{pmatrix} \right) \geq 0.$$

**Proposition 3** Let $v_0 < y_1 < y_2 < \cdots < y_{n-1} \neq v_\infty$ in $L^+_Q(n > 2)$, and let a prime $p \equiv 3 \mod 4$. Then there are non-cocompact Fuchsian groups $\Delta$ of finite covolume sitting in $\text{PSL}_2(\mathbb{Q})$ with $\{y_1, \ldots, y_{n-1}\} \subset \text{HFix}(\Delta)$, each of which stabilize no vertex in the tree $T_p$.

**Proof** Consider the construction of $\Gamma$ in Sect. 2.1 in Minkowski space; below we will describe additional requirements for choosing $x_1$ and the $f_i$.

In the 1st step, choose $x_1 \in L^+_Q$ so $v_0 < x_1 < y_1$. When choosing $f_1$, additionally require that $f_1 \in \text{span}_Q \{x_1, y_1\}$ and $|\langle f_1, f_1 \rangle|$ is in the rational square class $p(\mathbb{Q}^\times)^2 = \{p\alpha^2 : \alpha \in \mathbb{Q}\}$, which is possible because $\text{span}_Q \{x_1, y_1\}$ is isotropic. In the $i$th step (for
1 < i < n), one specifies a rational square class, say \( n_i \in (\mathbb{Q}^\times)^2 \in \mathbb{Q}^\times / (\mathbb{Q}^\times)^2 \) (\( n_i \) a square free integer) such that \( p \nmid n_i \). When choosing \( f_i \), additionally require that \( f_i \in \text{span}_\mathbb{Q} \{x_i, y_i\} \) and \( \langle (f_i, f_i) \rangle \in n_i(\mathbb{Q}^\times)^2 \), which is also possible because \( \text{span}_\mathbb{Q} \{x_i, y_i\} \) is isotropic.

Now note (as in the example in Sect. 2.2) that \( \rho_{f_i} \) as an element of \( \text{PGL}_2(\mathbb{Q}) \) is given by the matrix

\[
\rho_{f_i} = \begin{pmatrix} b_i & a_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_i & a_i \\ 0 & 1 \end{pmatrix}^{-1},
\]

where \( b_i = -\sqrt{\langle (f_i, f_i) \rangle} / (f_i, e_2 + e_0) \) and \( a_i = -\langle (f_i, e_1) \rangle / (f_i, e_2 + e_0) \).

Since \( b_1 \) has a factor \( \sqrt{p} \), and each \( b_i \) (\( 1 < i < n \)) does not,

\[ v_p(b_i) \neq v_p(b_1). \]

Moreover, \( -1 \) is not a square in \( \mathbb{Q}_p \) (since \( p \equiv 3 \mod 4 \)). By Lemma 3.2, \( \Gamma \) stabilizes no vertex in \( T_p \), and by construction \( \{y_i\} \subset \text{HFix}(\Gamma) \). Now let \( \Delta \) be the kernel of \( \Gamma \to \text{PGL}_2(\mathbb{Q})/\text{PSL}_2(\mathbb{Q}) \), which is of finite index in \( \Gamma \); then \( \Delta \) also stabilizes no vertex in \( T_p \) and \( \{y_i\} \subset \text{HFix}(\Delta) \).

**Remark** For each \( \Delta \) constructed in the proof of Proposition 2, there is an integer \( m \) such that \( \Delta \) stabilizes a vertex of \( T_q \) for all primes \( q > m \). To see this, choose \( m \) large enough so that \( m \) is greater than all the denominators of the entries of a matrix representing \( \rho_{f_i} \), for each \( i \), as an element of \( \text{PGL}_2(\mathbb{Q}) \).

### 4 Proof of the Theorem

**Theorem** Let \( Y \) be a finite set of rational points in the boundary of the hyperbolic plane. Then there are infinitely many noncommensurable non-cocompact Fuchsian groups \( \Delta \) of finite covolume sitting in \( \text{PSL}_2(\mathbb{Q}) \) so that \( Y \subset \text{HFix}(\Delta) \).

**Proof** We can let \( Y \) be a finite set of two or more rational points in the boundary of the hyperbolic plane (just add points if fewer than 2 are given). Let \( Y = \{y_i\} \), so that \( v_0 < y_1 < y_2 < \cdots < y_{n-1} \neq v_\infty \) in \( L_2^\times \mathbb{Q} \) (\( n > 2 \)). Now let the family \( \{\Delta\} \) be the set of non-cocompact Fuchsian groups of finite covolume sitting in \( \text{PSL}_2(\mathbb{Q}) \) such that \( \{y_1, \ldots, y_{n-1}\} \subset \text{HFix}(\Delta) \), which are constructed in the proof of Proposition 2.

Assume for the purpose of contradiction that there is a finite number \( k \) of commensurability classes in family \( \{\Delta\} \). Let \( \{\Delta_1, \ldots, \Delta_k\} \) be distinct representatives from the \( k \) commensurability classes.

From the remark just after Proposition 3, there is an integer \( m \) such that each \( \Delta_j \) (\( 1 \leq j \leq k \)) stabilizes a vertex in \( T_q \) for all \( q > m \). By Dirichlet’s theorem on arithmetic progressions, we can choose a prime \( p > m \) and \( p \equiv 3 \mod 4 \). By Proposition 3, there is \( \Delta_{k+1} \in \{\Delta\} \) with \( \{y_i\} \subset \text{HFix}(\Delta_{k+1}) \) and so that \( \Delta_{k+1} \) does not stabilize any vertex of \( T_p \). Therefore, each \( \Delta_j \) (\( 1 \leq j \leq k \)) stabilizes a vertex in \( T_p \) but \( \Delta_{k+1} \) does not. For subgroups of \( \text{PSL}_2(\mathbb{Q}) \), the presence or absence of fixed points in \( T_p \) descends to finite index subgroups, and is invariant under conjugation. Thus \( \Delta_{k+1} \) is not commensurable with any of the \( \Delta_j \) (\( 1 \leq j \leq k \)), which contradicts the assumption there are a finite number of commensurability classes in the family \( \{\Delta\} \).

**Remark** By using Proposition 3 and the remark just after it, one can inductively constructs an infinite family where the members lie in different commensurability classes.
As mentioned in the introduction, a boundary point cannot both be a cusp and a hyperbolic fixed point, for Fuchsian groups; thus a direct corollary of the theorem is

**Corollary** Let $Y$ be finite set of rationals. Then there are infinitely many noncommensurable non-cocompact Fuchsian groups of finite covolume sitting in $\mathrm{PSL}_2(\mathbb{Q})$ whose cusp set is properly contained in $(\mathbb{Q} \setminus Y) \cup \{\infty\}$.

**Acknowledgments** The author is grateful for all the conversations with his advisor Daniel Allcock and would also like to thank Alan Reid for helpful discussions about pseudomodular groups.

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