1 Introduction

The measurements of neutrino oscillation parameters have entered a precision era. The global fit to current data from neutrino oscillation experiments give the following best fit values and 2σ limits for the mixing parameters [1]:

\[
\begin{align*}
\sin^2 \theta_{12} &= 0.30 (0.25 - 0.34), \\
\sin^2 \theta_{23} &= 0.5 (0.38 - 0.64), \\
\sin^2 \theta_{13} &= 0 (< 0.028).
\end{align*}
\]

These values for the mixing parameters are very close to the values arising from the so-called “tri-bimaximal” mixing (TBM) matrix [2],

\[
U_{\text{TBM}} = \begin{pmatrix}
\sqrt{2/3} & 1/\sqrt{3} & 0 \\
-\sqrt{1/6} & 1/\sqrt{3} & -1/\sqrt{2} \\
\sqrt{1/6} & 1/\sqrt{3} & 1/\sqrt{2}
\end{pmatrix},
\]

which predicts \( \sin^2 \theta_{\text{atm,TBM}} = 1/2 \) and \( \sin \theta_{13,\text{TBM}} = 0 \). In addition, it predicts \( \sin^2 \theta_{\odot,\text{TBM}} = 1/3 \) for the solar mixing angle. Even though the predicted \( \theta_{\odot,\text{TBM}} \) is currently still allowed by the experimental data at 2σ, as it is very close to the upper bound at the 2σ limit, it may be ruled out once more precise measurements are made in the upcoming experiments.

It has been pointed out that the tri-bimaximal mixing matrix can arise from a family symmetry in the lepton sector based on \( A_4 \) [3]. However, due to its lack of doublet representations, CKM matrix is an identity in most \( A_4 \) models [4]. In addition, to explain the mass hierarchy among the charged fermions, one needs to resort to additional symmetry.

We consider in [5] a different finite group, the double tetrahedral group, \( (d)T \), which is a double covering of \( A_4 \). Because it has the same four inequivalent representations as in \( A_4 \), the tri-bimaximal mixing pattern can be reproduced. In addition, \( (d)T \) has three inequivalent doublets, 2, 2’, and 2”, which can be utilized to give the 2 + 1 representation assignments for the quarks [6], as having been known, in the context of SU(2) flavor group, to give realistic quark mixing matrix and mass hierarchy [7]. The \( SU(5) \) GUT symmetry in our model relates the down type quark and the charged lepton sectors, leading to a novel “quark-lepton complementarity” relation, a sum rule between the solar mixing angle and the Cabibbo angle. In addition, our model has an \( Z_{12} \times Z'_{12} \) symmetry, which gives rise to the mass hierarchy dynamically. As a result of the symmetries in our model, only nine operators are allowed up to dim-7, and thus our model is very predictive. This is the first GUT model combined
Table 1. Charge assignments. The parameter $\omega = e^{i \pi/6}$.

|   | $T_3$ | $T_a$ | $F$ | $H_5$ | $H_\phi$ | $\Delta_{15}$ |
|---|---|---|---|---|---|---|
| SU(5) | 10 | 10 | 5 | 5 | 5 | 45 |
| $^{(d)}T$ | 1 | 2 | 3 | 1 | 1 | 1 |
| $Z_{12}$ | $\omega^*$ | $\omega^*$ | $\omega^*$ | $\omega^*$ | $\omega^*$ | $\omega^*$ |
| $Z_{12}$ | $\omega$ | $\omega^*$ | $\omega^*$ | $\omega^*$ | $\omega^*$ | $\omega^*$ |
| $\phi$ | $\phi'$ | $\psi$ | $\psi'$ | $\zeta$ | $N$ | $\xi$ | $\eta$ |

with $^{(d)}T$ family symmetry in which the tri-bimaximal neutrino mixing and realistic CKM matrix arise.

2 The Model

The three generations of $\mathbf{5}$ are assigned into a triplet of $^{(d)}T$, in order to generate the tri-bimaximal mixing pattern in the lepton sector, and it is denoted by $\mathbf{5}$. On the other hand, to obtain realistic quark sector, the third generation of the 10-dim representation transforms as a singlet, so that the top quark mass is allowed by the family symmetry, while the first and the second generations form a doublet of $^{(d)}T$. These 10-dim representations are denoted by, respectively, $T_3$ and $T_a$, where $a = 1, 2$. The Yukawa interactions are mediated by a 5-dim Higgs, $H_5$, a 5-dim Higgs, $H_\phi$, as well as a 45-dim Higgs, $\Delta_{15}$, which is required for the Georgi-Jarlskog relations. We have summarized these quantum number assignment in Table 1. It is to be noted that $H_5$ and $H_\phi$ are not conjugate of each other as they have different $Z_{12}$ and $Z_{12}'$ charges.

The Lagrangian of the model is given as follows,

$$\mathcal{L}_{\nu} = \mathcal{L}_{TT} + \mathcal{L}_{TF} + \mathcal{L}_{FF}$$

$$\mathcal{L}_{TT} = y_t H_3 T_3 T_3 + \frac{1}{A^2} y_t H_3 T_3 T_3 \xi \zeta$$

$$\mathcal{L}_{TF} = \frac{1}{A^2} y_t H_3 T_3 T_3 + \frac{1}{A^2} y_t H_3 T_3 T_3 \phi^2$$

$$\mathcal{L}_{FF} = \frac{1}{A^2} y_t H_3 T_3 T_3 \zeta \xi$$

where $M_x$ is the cutoff scale at which the lepton number violation operator $H H H F$ is generated, while $A$ is the cutoff scale, above which the $^{(d)}T$ symmetry is exact. The parameters $y$s and $\lambda$s are the coupling constants. The vacuum expectation values (VEVs) of various SU(5) singlet scalar fields are,

$$^{(d)}T \rightarrow G_{TST^2} : \langle \xi \rangle = \xi_0 A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ,$$

$$(d)T \rightarrow G_T : \langle \phi \rangle = \phi_0 A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ,$$

$$(d)T \rightarrow G_{TST^2} : \langle \psi \rangle = \psi_0 A \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

$$(d)T \rightarrow G_{TS} : \langle \zeta \rangle = \zeta_0 , \langle N \rangle = N_0$$

$$(d)T - \text{invariant} : \langle \eta \rangle = u$$

where $G_{TST^2}$ denotes the subgroup generated by the elements $TST^2$. Realization of the vacuum alignment is currently under investigation.

The interactions in $\mathcal{L}_\nu$ give the following neutrino mass matrix,

$$M_\nu = \frac{\lambda_{12}^2}{M_x} \begin{pmatrix} 2\xi_0 + u & -\xi_0 & -\xi_0 \\ -\xi_0 & 2\xi_0 & u - \xi_0 \\ -\xi_0 & u - \xi_0 & 2\xi_0 \end{pmatrix} ,$$

and we have absorbed the Yukawa coupling constants by rescaling the VEV’s. This mass matrix $M_\nu$ is form diagonalizable, i.e. the orthogonal matrix that diagonalizes it does not depend on the eigenvalues. Its diagonal form is,

$$V_\nu^T M_\nu V_\nu = \text{diag}(u + 3\xi_0, u, -u + 3\xi_0, \frac{y_\nu}{M_x}) ,$$

where the matrix $V_\nu$ is the tri-bimaximal mixing matrix, $V_\nu = U_{TBM}$. Note that the three predict absolute masses for the neutrinos satisfy the sum rule,

$$m_1 - m_3 = 2m_2 .$$

The down type quark and charged lepton masses are generated by $\mathcal{L}_{TF}$. Because the renormalizable operator $H^*_3 FT^3$ is forbidden by the $^{(d)}T$ symmetry, the generation of $b$ quark mass requires the breaking of $^{(d)}T$, which naturally explains the hierarchy between $m_t$ and $m_b$. As $m_b$ and $m_\tau$ are generated by the same operator, $H^*_3 FT^3 \phi$, we obtain the successful $b - \tau$ unification relation. Upon the breaking of $^{(d)}T \rightarrow G_T$, the operator $H^*_3 FT^3 \phi N$ contributes to the $(22)$ element in $M_{dd}$, and thus gives rise to $m_d$ and $m_\tau$. As this operator involves $\Delta_{15}$, the GJ relation for the second family, $m_\tau \simeq 3m_d$ is obtained. If no further symmetry breaking takes place, the first generation masses, $m_u$ and $m_\tau$, vanishes. At this stage, the diagonalization mass matrix for the charged leptons (and down type quark) is identity, and hence the the tri-bimaximal mixing matrix is exact. To obtain the correct mass relation for the first generation, it inevitably calls for flavor mixing in the down quark sector, which then leads to corrections to the tri-bimaximal mixing pattern. The correction to the $\theta_{12}$ due to mixing in the charged lepton sector can account for the difference between $\sin^2 \theta_{12} = 1/3$ in
the tri-bimaximal mixing matrix and the experimentally observed best fit value, $\sin^2 \theta_{12} = 0.3$. The GJ relation for the first family, $m_2 \simeq 3 m_e$, is obtained due to the operator $H_\nu^T \mathbb{T}_a \phi \bar{\psi} \psi$, which further breaks the $(d)T$ symmetry down to nothing. The mass matrices for the down type quarks and charged leptons are thus given by,

$$
M_d = \begin{pmatrix}
0 & (1 + i)\phi_0 \psi_0' & 0 \\
(1 - i)\phi_0 \psi_0' & \psi_0 N_0 & 0 \\
\phi_0 \psi_0' & \phi_0 \psi_0' & \zeta_0
\end{pmatrix} y_d v_d \phi_0,
$$

$$
M_e = \begin{pmatrix}
0 & 0 & (1 + i)\phi_0 \psi_0' \\
0 & -3 \psi_0 N_0 & \phi_0 \psi_0' \\
0 & 0 & \zeta_0
\end{pmatrix} y_e v_d \phi_0
$$

where we have absorbed the coupling constants $y_d$ and $y_e$ by re-scaling the VEV’s, $\phi_0$ and $\psi_0'$. Since the off diagonal elements in these mass matrices involve two VEV’s, $\phi_0 \psi_0'$, they are naturally smaller compared to $\psi_0$, assuming the VEV’s are naturally of the same order of magnitude. Besides explaining the mass hierarchy, it gives rise to the correct GJ relations in the first and the second families.

The up quark masses are generated by the following Yukawa interactions, $\mathcal{L}_{TTT}$. When the $(d)T$ symmetry is exact, the only operator that is allowed is $H_3 T_3 T_3$, thus only top quark mass is generated, which naturally explains why the top quark mass is much larger than all other fermion masses. When $(\psi)$ breaks $(d)T$ down to GT, the mass $m_3$, and $V_{td}$ is generated by the operators, $H_3 T_3 T_3 \phi \zeta$ and $H_3 T_3 T_3 \phi^2$. The breaking of $(d)T \rightarrow GT_{TST^2}$ gives rise to the up type quark mass through the operator $H_3 T_3 T_3 \phi \zeta^3$. These interactions give rise to the following mass matrix for the up type quarks,

$$
M_u = \begin{pmatrix}
i \frac{\phi_0^3}{2} & \phi_0^3 & \frac{1}{2} i \phi_0^3 \\
\phi_0^3 & 0 & (1 - i) \phi_0^3 \\
(1 - i) \phi_0^3 & y' \psi_0 \zeta_0 & 0
\end{pmatrix} y_t v_u \cdot
$$

The mixing angle $\theta_{12}^u$ from the up type quark mass matrix given in Eq. [2] is related to $m_e$ and $m_u$ as $\theta_{12}^u \simeq \sqrt{m_u/m_e}$, while the mixing angle $\theta_{12}^d$ arising from the down quark mass matrix $M_d$ given in Eq. [7] is related to the ratio of $m_d$ and $m_s$ as $\theta_{12}^d \simeq \sqrt{m_d/m_s}$, to the leading order. The Cabibbo angle, $\theta_c$, is therefore given by $\theta_c \simeq \left| \sqrt{\frac{m_d}{m_s}} - e^{i \alpha} \sqrt{\frac{m_u}{m_c}} \right| \sim \sqrt{\frac{m_d}{m_s}}$, where the relative phase $\alpha$ depends upon the coupling constants. Even though $\theta_{12}^d$ is of the size of the Cabibbo angle, the corresponding mixing angle in the charged lepton sector, $\theta_{12}^\ell$, is much suppressed due to the GJ relations,

$$
\theta_{12}^\ell \simeq \frac{1}{3} \frac{m_d}{m_{\mu}} \sim \frac{1}{3} \frac{m_d}{m_s} \sim \frac{1}{3} \theta_c .
$$

As a result, the correction to the tri-bimaximal mixing pattern due to the mixing in the charged lepton sector is small, and is given, to the leading order, by,

$$
\tan^2 \theta_c \simeq \tan^2 \theta_{\text{TBM}} + \frac{1}{2} \theta_c \cos \beta ,
$$

where the relative phase $\beta$ is determined by strengths and phases of the VEV’s, $\phi_0$ and $\psi_0'$, and is identified as the Dirac leptonic CP phase $[3]$. This deviation could account for the difference between the prediction of the TBM matrix, which gives $\tan^2 \theta_{\text{GJ}} = 1/2$, and the experimental best fit value, $\tan^2 \theta_{\text{exp}} = 0.429$, if $\cos \beta \simeq 2/3$ (with $\theta_c \simeq 0.22$). The off-diagonal matrix elements in $M_e$ also generate a non-zero value for the neutrino mixing angle $\theta_{13} \simeq \theta_{\text{GJ}}/\sqrt{2} \sim 0.05$. We note that a more precise measurement of $\tan \theta_c$ will pin down the phase of $\phi_0 \psi_0'$, and thus the three leptonic CP phases. These phases could have interesting implications for lepton flavor violating charged lepton decays and leptogenesis $[9]$.

3 Numerical Results

The observed quark masses respect the following relation,

$$
m_u : m_c : m_t = c^2 : \epsilon_u : 1 ,$$

$$
m_d : m_s : m_b = c^2 : \epsilon_d : 1 ,$$

where $\epsilon_u \simeq (1/200) = 0.005$ and $\epsilon_d \simeq (1/20) = 0.05$.

In our model, the mass matrices for the down type quarks and charged leptons can be parametrized as,

$$
\frac{M_d}{y_d v_d \phi_0 \zeta_0} = \begin{pmatrix}
0 & (1 + i)b & 0 \\
(1 - i)b & c & 0 \\
b & b & 1
\end{pmatrix} ,
$$

$$
\frac{M_e}{y_e v_d \phi_0 \zeta_0} = \begin{pmatrix}
0 & -3c & b \\
0 & -3c & b \\
0 & 0 & 1
\end{pmatrix} ,
$$

and with the choice of $b \equiv \phi_0 \psi_0 / \zeta_0 = 0.00789$ and $c \equiv \psi_0 N_0 / \zeta_0 = 0.0474$, the mass ratios for the down type quarks and for the charged leptons are given by,

$$
m_d : m_s : m_b = 0.00250 : 0.0499 : 1.00 ,
$$

$$
m_e : m_\mu : m_\tau = 0.000870 : 0.143 : 1.00 .
$$

These predictions are consistent with the observed values and are in good agreement with the GJ relations. The overall scale factor is $y_d \phi_0 \zeta_0 \simeq m_u / m_t \simeq 0.011$ at the GUT scale, assuming the top Yukawa coupling is 1.

For the up type quarks, the mass matrix can be written as,

$$
M_u = \begin{pmatrix}
i g & \frac{1}{2} i g & 0 \\
\frac{1}{2} g & g + h & k \\
0 & k & 1
\end{pmatrix} y_t v_u ,
$$

and with the choice of $k \equiv y' \psi_0 \zeta_0 = -0.032$, $h \equiv \psi_0' = 0.0053$ and $g \equiv \phi_0^3 = -2.25 \times 10^{-5}$, the ratio among the three up type quarks is given by,

$$
m_u : m_c : m_t = 0.0000252 : 0.005 : 1.00 .
$$
which is consistent with the observed values. The absolute values of the CKM matrix elements are given by,

\[
|V_{\text{CKM}}| = \begin{pmatrix} 0.976 & 0.217 & 0.0778 \\ 0.216 & 0.975 & 0.040 \\ 0.015 & 0.0378 & 0.999 \end{pmatrix}.
\]

(15)

Except for the element $V_{ub}$, which is slightly higher than the current experimental upper bound of $\sim 0.005$, all other elements are in good agreement with current data. This discrepancy can be alleviated by allowing additional operators to be present in the model. In this way, we can improve the model's ability to reproduce the observed neutrino mass hierarchy without invoking additional $U(1)$ symmetry. Due to the $(d)^T$ transformation property of the matter fields, the $b$-quark mass can be generated only when the $(d)^T$ symmetry is broken, which naturally explains the hierarchy between $m_b$ and $m_t$. The $Z_{12} \times Z_{12}$ symmetry, to a very high order, also forbids operators that lead to nucleon decays. We obtain the Georgi-Jarlskog relations for three generations. This inevitably requires non-vanishing mixing in the charged lepton sector, leading to correction to the tri-bimaximal mixing pattern. The model predicts non-vanishing $\theta_{13}$, which is related to the Cabibbo angle as, $\theta_{13} \sim \theta_c/3\sqrt{2}$. In addition, it gives rise to a sum rule, $\tan^2 \theta_\odot \sim \tan^2 \theta_{\odot,\text{TBM}} - \frac{1}{2} \theta_c \cos \beta$, which is a consequence of the Georgi-Jarlskog relations in the quark sector. This deviation could account for the difference between the experimental best fit value for the solar mixing angle and the value predicted by the tri-bimaximal mixing matrix.

\[
|U_{\text{MNS}}| = |V_{eL}^T U_{\text{TBM}}| = \begin{pmatrix} 0.838 & 0.545 & 0.0550 \\ 0.364 & 0.608 & 0.706 \\ 0.409 & 0.578 & 0.706 \end{pmatrix},
\]

which gives $\sin^2 \theta_{\text{atm}} = 1$, $\tan^2 \theta_\odot = 0.424$ and $|U_{e3}| = 0.055$. This value for $|U_{e3}|$ lies in the lower end of the reach in reactor experiments and it is relatively small compared to typical GUT model predictions [10].

The absolute masses of the neutrinos are determined by two parameters, $u$ and $\xi_0$. With

\[
u = -0.0593, \quad \xi_0 = 0.0369
\]

the experimental best fit values

\[
\begin{align*}
\Delta m_{\text{atm}}^2 &= 2.4 \times 10^{-3} \text{ eV}^2 \\
\Delta m_{\odot}^2 &= 8.1 \times 10^{-5} \text{ eV}^2
\end{align*}
\]

are accommodated, and the three absolute masses of the neutrinos are predicted to be,

\[
\begin{align*}
m_1 &= 0.0156 \text{ eV} \\
m_2 &= 0.0179 \text{ eV} \\
m_3 &= 0.0514 \text{ eV}.
\end{align*}
\]

Note that the total number of parameters in our model is seven in the charged fermion sectors and two in the neutrino sector.

4 Conclusion

In this talk, we present a grand unified model based on SU(5) combined with the double tetrahedral group, $(d)^T$, which successfully, for the first time, gives rise to near tri-bimaximal leptonic mixing as well as realistic CKM matrix elements for the quarks. Due to the presence of the $Z_{12} \times Z_{12}$ symmetry, only nine operators are allowed in the model, and hence the model is very predictive, the total number of parameters being nine in the Yukawa sector for the charged fermions and the neutrinos. In addition, it provides a dynamical origin for the mass hierarchy without invoking additional $U(1)$ symmetry. Due to the $(d)^T$ transformation property of the matter fields, the $b$-quark mass can be generated only when the $(d)^T$ symmetry is broken, which naturally explains the hierarchy between $m_b$ and $m_t$. The $Z_{12} \times Z_{12}$ symmetry, to a very high order, also forbids operators that lead to nucleon decays. We obtain the Georgi-Jarlskog relations for three generations. This inevitably requires non-vanishing mixing in the charged lepton sector, leading to correction to the tri-bimaximal mixing pattern. The model predicts non-vanishing $\theta_{13}$, which is related to the Cabibbo angle as, $\theta_{13} \sim \theta_c/3\sqrt{2}$. In addition, it gives rise to a sum rule, $\tan^2 \theta_\odot \sim \tan^2 \theta_{\odot,\text{TBM}} - \frac{1}{2} \theta_c \cos \beta$, which is a consequence of the Georgi-Jarlskog relations in the quark sector. This deviation could account for the difference between the experimental best fit value for the solar mixing angle and the value predicted by the tri-bimaximal mixing matrix.

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