Standard Subspaces of Hilbert Spaces of Holomorphic Functions on Tube Domains

Karl-Hermann Neeb\textsuperscript{1}, Bent Ørsted\textsuperscript{2}, Gestur Ólafsson\textsuperscript{3}

\textsuperscript{1} Department of Mathematics, Friedrich-Alexander-University of Erlangen-Nuremberg, Cauerstrasse 11, 91058 Erlangen, Germany. E-mail: neeb@math.fau.de
\textsuperscript{2} Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark. E-mail: orsted@math.au.dk
\textsuperscript{3} Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA. E-mail: olafsson@math.lsu.edu

Received: 29 July 2020 / Accepted: 9 June 2021
Published online: 26 July 2021 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract: In this article we study standard subspaces of Hilbert spaces of vector-valued holomorphic functions on tube domains $E + iC^0$, where $C \subseteq E$ is a pointed generating cone invariant under $e^{R h}$ for some endomorphism $h \in \text{End}(E)$, diagonalizable with the eigenvalues $1, 0, -1$ (generalizing a Lorentz boost). This data specifies a wedge domain $W(E, C, h) \subseteq E$ and one of our main results exhibits corresponding standard subspaces as being generated using test functions on these domains. We also investigate aspects of reflection positivity for the triple $(E, C, e^{\pi i h})$ and the support properties of distributions on $E$, arising as Fourier transforms of operator-valued measures defining the Hilbert spaces $\mathcal{H}$. For the imaginary part of these distributions, we find similarities to the well known Huygens’ principle, relating to wedge duality in the Minkowski context. Interesting examples are the Riesz distributions associated to euclidean Jordan algebras.

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The research of K.-H. Neeb was partially supported by DFG-Grant Ne 413/10-1. The research of G. Ólafsson was partially supported by Simons Grant 586106.
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1. Introduction

In mathematical physics and the theory of algebras of local observables, three fundamental concepts play a key role, namely (1) modular groups and conjugations (based on the Tomita-Takesaki Theorem); (2) KMS states of operator algebras, signifying the holomorphic nature of certain symmetries; and (3) Reflection Positivity, which classically connects euclidean and relativistic quantum field theories. In this paper, we shall revisit and elucidate some of the interesting connections between these three topics; in particular we shall give in a general framework of Hilbert spaces of holomorphic functions (already a ubiquitous category) the crucial construction of the relevant spaces in (3), and also make explicit the connections to (1) and (2). While the technical details are connected with tube domains, some of the methods are rather robust, and could be of interest in other geometric situations as well.

We start by reviewing some ideas stemming from physics, in particular quantum field theory and the Tomita-Takesaki theory. A closed real subspace $V$ of a complex Hilbert space $\mathcal{H}$ is called standard if $V \cap iV = \{0\}$ and $V + iV$ is dense in $\mathcal{H}$ ([Lo08]). A central goal of this paper is to provide explicit descriptions of standard subspaces in Hilbert spaces of holomorphic functions on tube domains.

Standard subspaces arise naturally in the modular theory of von Neumann algebras. If $\mathcal{M} \subseteq B(\mathcal{H})$ is a von Neumann algebra and $\Omega \in \mathcal{H}$ is a cyclic separating vector for $\mathcal{M}$, i.e., $\mathcal{M}\Omega$ is dense in $\mathcal{H}$ and the map $\mathcal{M} \to \mathcal{H}$, $m \mapsto m\Omega$, is injective, then

$$V_\mathcal{M} := \{m\Omega : m^* = m, m \in \mathcal{M}\}$$

is a standard subspace of $\mathcal{H}$. Conversely, one can use the functorial process provided by Second Quantization ([Si74]) to associate to each standard subspace $V \subseteq \mathcal{H}$ a von Neumann algebra $\mathcal{R}_\pm(V)$ on the bosonic/fermionic Fock space $\mathcal{F}_\pm(\mathcal{H})$, for which the vacuum vector $\Omega$ is cyclic and separating (see [NÓ17, §§ 4,6] and [Lo08] for details). This method has been developed by Araki and Woods in the context of free bosonic quantum fields ([AW63]); some of the corresponding fermionic results are more recent (cf. [EO73,Fo83,BJL02]). This establishes a direct connection between standard subspaces and pairs $(\mathcal{M}, \Omega)$ of von Neumann algebras with cyclic separating vectors. As such pairs play an important role in Algebraic Quantum Field Theory (AQFT) in the context of Haag–Kastler nets of local observables ([Ar99,Ha96,BW92]), we would like to understand standard subspaces and their geometric realizations. Here a crucial point is that $V_\mathcal{M}$ reflects many important properties of the von Neumann algebra $\mathcal{M}$, related to inclusions and symmetry groups quite faithfully, but in a much simpler environment ([Lo08], [NÓ17, § 4.2]). In AQFT, standard subspaces provide the basis for the technique of modular localization, developed by Brunetti, Guido and Longo in [BGL02], see [LL15] for more recent applications and [BDFS00] for geometric aspects of modular automorphism groups.

In this article we develop a method for the explicit identification of standard subspaces in Hilbert spaces $\mathcal{H}$ of holomorphic functions on tube domains, i.e., domains of the form $T = E + iC^0$, where $E$ is a finite dimensional real vector space, $C \subseteq E$ is a closed convex cone with non-empty interior $C^0$, and the additive group $(E, +)$ acts
unitarily by translations. Motivated by applications in AQFT, we are interested in those standard subspaces for which the modular group and the modular conjugation act naturally on $T$. Our construction provides in particular a direct conceptual way to Hilbert spaces of distributions on Minkowski space by taking boundary values of Hilbert spaces of holomorphic functions on suitable complex tube domains. In the subsequent papers [NÓ21a,NÓ21b,NÓ21c] and [Oeh21], some of our results will be used to construct nets of standard subspaces on Lie groups and causal homogeneous spaces.

To explain our key idea, let us describe a context which is much more general than needed here, but which may therefore exhibit the overall idea more clearly. Suppose that $/Xi$ is a complex manifold and that $H \subseteq \text{Hol}(/Xi)$ is a Hilbert space of holomorphic functions on $/Xi$. To implement the action of a modular group, we assume a group action $\sigma : R \times /Xi \rightarrow /Xi, (r, z) \mapsto \sigma_r(z) = r.z$ such that the maps $\sigma_r$ are holomorphic and $\tau /Xi := \sigma_{-1}$ is an antiholomorphic involution with non-trivial fixed point space $/Xi\tau /Xi$. We assume that $\sigma$ corresponds to an antiunitary representation $U : R \rightarrow \text{AU}(H)$, so that a standard subspace $V \subseteq H$ is specified by $\Delta V = U(e')$ and $J V = U(-1)$ (cf. Sect. 2). Here $\sigma$ provides a geometric implementation of the modular group and the modular conjugation on $/Xi$. We further assume that $/Xi$ sits in a larger manifold containing an $R \times$-invariant submanifold $M$ in the boundary of $/Xi$, such that there exists an injective boundary value map $b : H \rightarrow C^{-\infty}(M)$. Then $/Xi\tau /Xi$ is a totally real submanifold of $/Xi$, so that all elements of $\text{Hol}(/Xi)$ are determined by their values on $/Xi\tau /Xi$. We shall require that:

(C) For $m \in /Xi\tau /Xi$, the orbit map $\sigma^m : R \rightarrow /Xi, t \mapsto \exp(th).m$, extends holomorphically to a map $S_{-\pi/2,\pi/2} \rightarrow /Xi$ which further extends continuously to a map $S_{-\pi/2,\pi/2} \rightarrow /Xi \cup M$ satisfying $\sigma^m(\pm \pi i /2) \in M$.

This should lead to a realization of the standard subspace $V$ corresponding to $(\Delta V, J V)$ as a space of distributions on the boundary subsets $W_\pm := \{\sigma^m(\mp \pi i /2) : m \in /Xi\tau /Xi\}$, so-called wedge domains in $M$.

The concrete environment we study is specified in Sect. 3 in terms of the following geometric data:

(A1) $E$ is a finite dimensional real vector space.
(A2) $h \in \text{End}(E)$ is diagonalizable with eigenvalues $\{-1, 0, 1\}$ and $\tau := e^{\pi i h}$.
(A3) $C \subseteq E$ is a pointed, generating closed convex cone invariant under $e^{R h}$ and $-\tau$.

This geometric context includes in particular the case where $E$ is Minkowski space, $C$ is the closed forward light cone and $h$ generates a Lorentz boost. It also contains a series of interesting generalizations in the context of euclidean Jordan algebras discussed in Sect. 6 and Appendix B. Minkowski spaces are the simple euclidean Jordan algebras of rank 2. The other simple euclidean Jordan algebras of rank $r$ are

$$\text{Sym}_r(R), \text{Herm}_r(C), \text{Herm}_r(H) \text{ for } r \in N \text{ and } \text{Herm}_3(\mathcal{O}),$$

where $\mathcal{O}$ is the alternative algebra of octonions (see [JvNW34,FK94] for the classification) and for these the cone $C$ of squares consists of the positive hermitian matrices.
These configurations are not Lorentzian for $r \neq 2$, but they carry causal and conformal structures generalizing those of Minkowski space.

Besides the Lorentz boost, other triples $(E, C, h)$ that appear naturally in AQFT are $(\mathbb{R}, \mathbb{R}_+, \text{id}_{\mathbb{R}})$, which corresponds to positive energy representations of the $ax + b$-group and modular inclusions of standard subspaces, and the triple $(\mathbb{R}^{1,d-1}, V_+, \text{id}_{\mathbb{R}})$ appears in conformal field theory on Minkowski space, where the modular group acts by dilations and the corresponding wedge domain is the positive cone $V_+$ itself (cf. [Bu78]). Note that the triple $(E, C, h)$ does not encode any Lorentzian structure, only the causal structure specified by the cone and the “modular” structure specified by $h$. However, Lorentzian structures appear indirectly on low-dimensional subspaces (see Example 3.1(c)).

More such triples appear, where $E = g$ is a non-abelian Lie algebra with an invariant cone $C$ and $h \in g$ is such that $\text{ad} h$ defines a 3-grading of $g$ (in [MN21] these elements are called Euler elements), but then the associated unitary representations are more complicated than the ones we consider here (see [NÓ21a]). Here the Lie algebras arising in AQFT are $\mathfrak{sl}_2(\mathbb{R})$ (from the Möbius group in CFT), the Poincaré–Lie algebra and the conformal Lie algebra $\mathfrak{so}_{2,d}(\mathbb{R})$ of Minkowski space.

The above setup is supplemented by the following analytic data. We fix a complex Hilbert space $\mathcal{K}$ and a Herm $^+ (\mathcal{K})$-valued measure $\mu$ on the dual cone $E^* \subseteq E^*$ which defines a $\mathcal{K}$-valued $L^2$-space $\mathcal{H} := L^2(E^*, \mu; \mathcal{K})$ (cf. [Ne98]). In the special case $(E, C) = (\mathbb{R}^{1,d-1}, V_+)$ one identifies $E^*$ with $E$ and $C^*$ with $V_+$. Here the Lorentz invariant measures on the mass hyperboloids and the mantle of the light cone provide natural examples with $\mathcal{K} = \mathbb{C}$. Higher spin corresponds to non-trivial spaces $\mathcal{K}$ which carry a representation of the Lorentz group (cf. [SW64]).

To specify a standard subspace of $\mathcal{H}$, we fix a representation $\rho : \mathbb{R}^+ \to \text{GL}(\mathcal{K})$ by normal operators for which $J_\mathcal{K} := \rho(-1)$ is a conjugation and $\rho(e^t) = e^{t\Lambda}$ for a normal operator $\Lambda$ with bounded symmetric part. The compatibility between $\mu$ and $\rho$ consists in the relations

$$(\tau)_* \mu = J_\mathcal{K} \mu J_\mathcal{K} \quad \text{and} \quad \sigma(r)_* \mu = \rho(r)^* \cdot \mu \cdot \rho(r) \quad \text{for} \quad r > 0,$$

where the $\mathbb{R}^+_+$-action on $E^*$ is given by $\sigma(e^t) \lambda := \lambda \circ e^{-it}$. We specify a standard subspace $\mathcal{V} \subseteq \mathcal{H} = L^2(E^*, \mu; \mathcal{K})$ by

$$(\Delta^{(-it/2\pi)} f)(\lambda) = \rho(e^t) f(\lambda \circ e^{it}) \quad \text{and} \quad (J_\mathcal{V} f)(\lambda) := J_\mathcal{K} f(-\lambda \circ \tau), \quad \lambda \in C^*.$$

Then we have a realization

$$\Phi : L^2(C^*, \mu; \mathcal{K}) \to \text{Hol}(\mathcal{T}, \mathcal{K}), \quad \Phi(f)(z) = \int_{E^*} e^{i\lambda(z)} d\mu(\lambda) f(\lambda)$$

of $\mathcal{H}$ as a Hilbert space $\mathcal{H}_\mathcal{K}$ of $\mathcal{K}$-valued holomorphic functions on the tube $\mathcal{T} = E + i C^0$ whose reproducing kernel is given by

$$K : \mathcal{T} \times \mathcal{T} \to B(\mathcal{K}), \quad K(z, w) := \tilde{\mu}(z - w), \quad \text{where} \quad \tilde{\mu}(z) := \int_{C^*} e^{i\lambda(z)} d\mu(\lambda).$$

(Lemma 3.5). Passing to boundary values, we obtain a third realization of $\mathcal{H}$ as a subspace of $\mathcal{S}'(E, \mathcal{K})$, the space of $\mathcal{K}$-valued tempered distributions on $E$. Our first main result is the Standard Subspace Theorem (Theorem 3.14). To formulate it, we introduce the wedge domain

$$W = W(E, C, h) := E_0(h) + (C^0 \cap E_1(h)) - (C^0 \cap E_{-1}(h)) \subseteq E,$$
specified by the data in (A1-3), where $E_j(h) = \ker(h - j \text{id}_E)$ is the $j$-eigenspace of $h$ in $E$. Then the standard subspace $V \subseteq \mathcal{H}$ can be described in terms of $W$ by

$$V = \text{span}_\mathbb{R}\{\tilde{\varphi}\eta : \varphi \in C_c^\infty(W, \mathbb{R}), \eta \in V_K\},$$

where $V_K := \text{Fix}(e^{-\pi i \Lambda} J_K) \subseteq K$.

(1.1)

is a standard subspace and $	ilde{\varphi}(\lambda) = \int_E e^{i\lambda(x)} \varphi(x) d\mu_E(x)$ denotes the Fourier transform.

This theorem has several interesting consequences. First it exhibits an interesting relation with reflection positivity (Osterwalder–Schrader positivity) and its representation theoretic aspects ([NÓ14,NÓ18]). This starts with the observation that the triple $(\mathcal{H}_\mathbb{R}, V, J_\mathbb{R})$, where $\mathcal{H}_\mathbb{R}$ is the real Hilbert space underlying $\mathcal{H}$, is reflection positive, i.e., $\langle \xi, J\xi \rangle \geq 0$ for $\xi \in V$ ([Lo08]). Further, the multiplication representation $(U, \mathcal{H})$ of the additive group $(E, +)$ on $L^2(C^*, \mu; \tilde{K})$ is reflection positive for the triple $(E, W, \tau)$ in the sense of [NÓ18] (see Sect. 4 for details), so that the general theory of reflection positive distributions developed in [NÓ14] suggests that the restriction of the positive definite distribution $\tilde{\mu} \in S'(E, B(\tilde{K}))$ to the open cone $W$ should be positive definite with respect to the involution $x^\tau = -\tau(x)$. This is verified in Proposition 4.3, which even shows that it is represented by an operator-valued function.

Presently we do not see how this occurrence of reflection positivity is related to euclidean models, which was the original intention of this concept ([OS73]). This would require a better understanding of the group theoretic side of Schlingemann’s work [Sc99a, Sc99b].

From the description of $V$ in (1.1), we draw interesting conclusions on the support of the “imaginary part” of the operator-valued distribution $\tilde{\mu}$, provided that the infinitesimal generator $\Lambda$ of $\rho$ has integral spectrum (Proposition 5.4). This aspect is explored in detail in Sect. 6 for the Riesz measures $\mu_s$ of a simple euclidean Jordan algebra $E$. This is an important one-parameter family of scalar-valued measures on the dual $C^*$ of the positive cone $C$ of the Jordan algebra $E$. For their Fourier transforms $\tilde{\mu}_s$, we analyze the support of the imaginary part in detail. Using some tools developed in Appendix B, it is straightforward to decide which connected components of the set $E^\times$ of invertible elements are contained in the support (Proposition 6.3), but the description of the precise support requires closer inspection (Proposition 6.10). In Theorem 6.12 we characterize the situations when the support of $\text{Im}(\tilde{\mu}_s)$ is contained in the complement of the wedge domain $W(E, C, h)$. In the AQFT context on Minkowski space, this corresponds to wedge duality.

For the one-dimensional Jordan algebra $E = \mathbb{R}$, the Riesz measures $\mu_s$ are multiples of the measures $x^{s-1} dx$ ($s > 0$) on the positive half line. These measures arise naturally in the context of CFT from derivatives of the U(1)-current (see [Lo08, § 5.2] for details). The spaces $L^2(\mathbb{R}_+, \mu_s)$ carry a natural positive energy representation of the 2-dimensional group $\text{Aff}(\mathbb{R})$. These representations are mutually equivalent, but the realization by holomorphic functions on the half plane yields extensions to the simply connected covering $G$ of the Möbius group $\text{PGL}_2(\mathbb{R})$ which depend on $s$ (corresponding to the lowest weight). This leads to a one-parameter family of Möbius covariant nets of standard subspaces with rather different properties.

The contents of this paper are as follows: In Sect. 2 we collect some generalities on standard subspaces in Hilbert spaces of holomorphic functions. An important starting point is Proposition 2.1, characterizing elements of $V$ as those vectors $\xi \in \mathcal{H}$ for which the orbit map

$$\alpha^\xi : \mathbb{R} \to \mathcal{H}, \quad t \mapsto \Delta_{\mathcal{H}}^{-it/2\pi} \xi$$

...
extends to a continuous map on the closed strip \( \{ z \in \mathbb{C} : 0 \leq \text{Im} z \leq \pi \} \), holomorphic on the interior, and satisfying \( \alpha^{z}(\pi i) = J_{\xi} \). This condition is intimately related to (C) above. We also discuss the standard subspaces of \( \mathcal{K} \) naturally attached to the representation \( \rho \) and provide a direct description of the real subspaces \( \mathcal{H}^{/}_{\mathcal{K}} \) in terms of the reproducing kernel \( K \).

In Sect. 3 we turn to our subject proper, the spaces \( \mathcal{H} = L^{2}(\mathbb{C}^{*}, \mu; \mathcal{K}) \) and the corresponding standard subspace. After discussing the geometric implications of the axioms (A1-3) (Sect. 3.1), we introduce in Sect. 3.2 the Hilbert space \( \mathcal{H} \) and its realizations by holomorphic functions and tempered distributions. The Standard Subspace Theorem (Theorem 3.14) is proved in Sect. 3.3. We end this section with a discussion of the Riesz measures \( x^{s-1} \, dx (s > 0) \) on the positive half line as examples. The relation with reflection positivity is discussed in Sect. 4, and the support properties of the imaginary part of the distribution are investigated in Sect. 5 (for the general case), and in Sect. 6 for Riesz measures. Two appendices collect background material on standard subspaces (Appendix A) and structure theoretic results on Jordan algebras (Appendix B) that are used in Sect. 6 for the support analysis. They also find applications in forthcoming work ([NÓ21a, Oeh21]). For instance, in [NÓ21a] we construct isitone, covariant nets of standard subspaces with the Bisognano–Wichmann property on Jordan space-times, i.e., the universal covers of the conformal completion of a euclidean Jordan algebra \( E \). These are the Jordan space-times in the sense of Günaydin [Gu93], resp., the simple space-time manifolds in the sense of Mack–de Riese [MdR07].

We conclude this introduction with a discussion of some of the numerous connections of the techniques and the results of the present paper with the AQFT literature, where boundary values of holomorphic functions on various kinds of tube domains play a central role.

In [Bo95a, Bo95b], Borchers also considers analytic extensions of orbit maps to strips. His method to derive wedge duality from Lorentz covariance could be easily interpreted in terms of exhibiting a pair \((V, V')\) of mutually dual standard subspaces. This includes observations relating the standard right wedge and the future tube as in our Lemma 3.3. Here Lorentz covariance provides unitary one-parameter groups corresponding to boosts and the orbit maps of these boosts are extended analytically to a horizontal strip. On the level of operators, analytic continuation to the strip plays a key role in the identification of unitary endomorphisms of von Neumann algebras ([Bo95a, Prop. 2.1], [Bo95b, Thm. A]). This is related to the Araki–Zsidó Theorem [AZ05] characterizing operators between standard subspaces in the spirit of Proposition 2.1. We refer to [Ne19a] for applications of this result to the identification of endomorphism semigroup of standard subspaces.

In [Ep67] Epstein uses the action of the complex boost group \( \text{SO}_{1,1}(\mathbb{C}) \cong \mathbb{C}^{\times} \) on \( \mathbb{C}^{4} \) (acting in the first two components). He applies group elements to certain tube domains in \( \mathbb{C}^{4} \) to relate distributional boundary values from opposite tubes (a generalized retarded function and its CTP counterpart) by specializing to \( \pm 1 \). Epstein’s matrix-valued “functions” \( W^{+}(\rho) \) represent the spectral measure of the energy-momentum operator, resp., of the representation of the group of space-time translations. This corresponds to our \( L^{2} \)-picture in Sect. 3.2. He uses analytic extensions to implement a conjugation representing a CTP operator, whereas the geometric nature of the modular conjugation is built into our approach from the outset because it is implemented by the involution \( \tau \) on \( E \). Accordingly, the nature of the corresponding results on boundary values is different. Likewise, in [DHR74] similar analytic extension techniques are used to implement a
C-operator (charge inversion), which is an antiunitary conjugation, but in our context its existence is assumed.

Mund uses in [Mu01, Lemma 4] analytic continuation methods developed in [BE85] and derives the Bisognano–Wichmann property from localization in spacelike cones $C$, i.e., causal completions of pointed open cones in the time zero hyperplane. For fields with a positive mass $m$, he describes certain smooth functions on the mass shell

$$H_m = \{ k = (k_0, \mathbf{k}) \in \mathbb{R}^4 : k^2 = m^2, k_0 > 0 \}$$

as boundary values of holomorphic functions on the complexification of $H_m$, intersected with a tube over a spacelike cone. In this context no geometric realization of the modular group is assumed, but proved. All these results extend and refine the classical PCT and Spin-Statistics Theorems one finds in [SW64, Ch. 4].

These techniques are related to our paper as follows. For $E = \mathbb{R}^{1,d-1}$ and $C = V_+$, our setup covers all positive energy representations of the Poincaré group, restricted to the subgroup $E \times \mathbb{R}^\times$, consisting of space-time translations and the full $\text{SO}_{1,1}(\mathbb{R})$-group corresponding to a Lorentz boost. The vector-valued $L^2$-space then arises from the (energy-momentum) spectral measure which takes values in operators on multiplicity space. These measures are particularly simple for one-particle representations, but they also exist for representations defined by quantum fields, where the standard subspaces correspond to von Neumann algebras of local observables.

Our perspective is to start from a linear $\mathbb{R}^\times$-action and to determine the nature of the corresponding standard subspaces for general representations satisfying a natural spectral condition. Jordan algebras $E$ of rank $r (r = 2$ corresponds to Minkowski spaces) provide in particular interesting measures $\nu_m$ on the level sets of the Jordan determinant $\det_E$ on the positive cone. Their Fourier transforms $\hat{\nu}_m$ solve the generalized Klein–Gordon equation $(\det_E(\partial) + m^2)\hat{\mu}_m = 0$, where the operator $\det_E(\partial)$ is of order $r$. We expect that many of the established techniques used for the Klein–Gordon equation can be applied to this context as well. As the measures $\nu_m$ live on hypersurfaces, they can be projected to hyperplanes which resembles a “time-zero” realization. One may also write the measures $\nu_m$ in a suitable way as differences of boundary values of holomorphic functions (see [Re16, § 4.1]) and thus obtain decompositions of the imaginary part of Fourier transform $\hat{\nu}_m$ into an advanced and a retarded fundamental solution ([Ge19, § 2.2.2]) which can possibly be expressed in terms of Riesz distributions on $E$ (see [Gi09, § 3] for the Minkowski case $r = 2$). In conclusion, we have been strongly motivated by the deep structures in algebraic quantum field theory, and there are clear inspirations in these physical theories to pursue further in the context of Jordan algebras, not the least from the basic notions of causality and positive energy.

**Notation.** Here we collect some notation that we will use in the article.

- $\mathcal{H}$ is a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{H}}$ which is linear in the second argument.
- We write $\mathbb{R}_+ = [0, \infty)$ for the closed positive half line.
- If $M$ is a smooth manifold, we write $C^\infty_c(M)$ for the space of complex-valued test functions on $M$, endowed with the natural LF topology, and $C_c^{-\infty}(M)$ for the space of antilinear continuous functionals on this space, i.e., the space of distributions on $M$. 


We likewise consider tempered distributions $D \in \mathcal{S}'(E)$ on a real finite dimensional vector space $E$ as antilinear functionals on the Schwartz space $\mathcal{S}(E)$. The Fourier transform of an $L^1$-function $f$ on $E$ is defined by

$$\hat{f}(\lambda) := \int_E e^{-i\lambda(x)} f(x) \, d\mu_E(x), \quad \lambda \in E^*, \quad (1.2)$$

where $\mu_E$ denotes a Lebesgue measure on $E$. For tempered distributions $D \in \mathcal{S}'(E)$, we define the Fourier transform by

$$\hat{D}(\varphi) := D(\bar{\varphi}), \quad \text{where} \quad \bar{\varphi}(\lambda) := \bar{\varphi}(-\lambda) = \int_E e^{i\lambda(x)} \varphi(x) \, d\mu_E(x). \quad (1.3)$$

For the distribution $D_f(\varphi) := \int_E \varphi(x) f(x) \, d\mu_E(x)$ defined by an $L^1$-function, we then have $\hat{D}_f = D_{\hat{f}}$.

- $C^0$ is the interior of the convex cone $C$ and $C^* = \{ \alpha \in E^* : (\forall v \in C)\alpha(v) \geq 0 \}$ its dual cone.

2. Standard Subspaces in Hilbert Spaces of Holomorphic Functions

We have collected basic facts about standard subspaces in Appendix A. We refer to that section for notation, but let us recall that a standard subspace is a closed real subspace $V \subset H$ such that $V \cap iV = \{0\}$ and $V + iV$ is dense in $H$. Every standard subspace comes with a conjugation $J_V$ and positive, densely defined operator $T_V$ such that $T_V = J_V \Delta_V^{1/2}$ is the conjugation $u + iv \mapsto u - iv$, $u, v \in V$ and $J_V \Delta_V = \Delta_V^{-1} J_V$. The pair $(\Delta_V, J_V)$ is the pair of modular objects associated to $V$, and $T_V$ is the Tomita operator of $V$. In this section we study the situation where the Hilbert space is a reproducing kernel Hilbert space of vector-valued holomorphic functions on a complex manifold. In Proposition 2.1 we derive a fundamental connection between the standard subspace $V$ and the space $H^J_V$ of $J_V$-fixed vectors is described in Lemma 2.5. These results are needed in Sect. 3.3.

2.1. Standard Subspaces and $J$-Fixed Points. In this subsection we derive a characterization of the elements of a standard subspace $V$ specified by the pair $(\Delta, J)$ in terms of analytic continuation of orbit maps of the unitary one-parameter group $(\Delta^t)_{t \in \mathbb{R}}$ and the real space $H^J$.

Proposition 2.1. Let $V \subseteq H$ be a standard subspace with modular objects $(\Delta, J)$. For $\xi \in H$, we consider the orbit map $\alpha^\xi : \mathbb{R} \to H, t \mapsto \Delta^{-it/2\pi} \xi$. Then the following are equivalent:

(i) $\xi \in V$.
(ii) $\xi \in \mathcal{D}(\Delta^{1/2})$ with $\Delta^{1/2} \xi = J \xi$.
(iii) The orbit map $\alpha^\xi : \mathbb{R} \to H$ extends to a continuous map $S_\pi \to H$ which is holomorphic on $S_\pi$ and satisfies $J \xi = J_{\alpha^\xi(\pi i)} = J \xi$.
(iv) There exists an element $\eta \in H^J$ whose orbit map $\alpha^\eta$ extends to a continuous map $S_{-\pi/2, \pi/2} \to H$ which is holomorphic on the interior and satisfies $\alpha^\eta(-\pi i/2) = \xi$. 
Proof. The equivalence of (i) and (ii) follows from the definition of $\Delta$ and $J$.

(ii) $\Rightarrow$ (iii): If $\xi \in D(\Delta^{1/2})$, then $\xi \in D(\Delta^z)$ for $0 \leq \Re z \leq 1/2$, so that the map

$$f : \overline{S}_\pi \to \mathcal{H}, \quad f(z) := \Delta^{-iz/2\pi}\xi$$

is defined. Let $P$ denote the spectral measure of the selfadjoint operator

$$H := -\frac{1}{2\pi} \log \Delta \quad \text{and let} \quad P^\xi = (\xi, P(\cdot)\xi)$$

denote the corresponding positive measure on $\mathbb{R}$ defined by $\xi \in \mathcal{H}$. Then [NÓ18, Lemma A.2.5] shows that

$$L(P^\xi)(2\pi) = \int_{\mathbb{R}} e^{-2\pi \lambda} dP^\xi(\lambda) < \infty.$$ 

This implies that the kernel

$$(f(w), f(z)) = \langle \Delta^{-iw/2\pi}\xi, \Delta^{-iz/2\pi}\xi \rangle = \langle \xi, \Delta^{-i(z-w)/2\pi}\xi \rangle$$

is continuous on $\overline{S}_\pi \times \overline{S}_\pi$ by the Dominated Convergence Theorem, holomorphic in $z$, and antiholomorphic in $w$ on the interior ([Ne00, Prop. V.4.6]). This implies (iii) because it shows that $f$ is holomorphic on $S_\pi$ ([Ne00, Lemma A.III.1]) and continuous on $\overline{S}_\pi$.

(iii) $\Rightarrow$ (iv): For $\alpha^\xi : \overline{S}_\pi \to \mathcal{H}$ as in (iii), we have

$$J\alpha^\xi(z) = \alpha^\xi(\pi i + \overline{z}) \quad (2.1)$$

by analytic continuation, so that

$$\eta := \alpha^\xi(\pi i/2) \in \mathcal{H} \quad \text{with} \quad \alpha^\eta(z) = \alpha^\xi(z + \frac{\pi i}{2}).$$

(iv) $\Rightarrow$ (ii): We abbreviate $S := S_{-\pi/2, \pi/2}$. The kernel $K(z, w) := \langle \alpha^\eta(w), \alpha^\eta(z) \rangle$ is continuous on $\overline{S} \times \overline{S}$ and holomorphic in $z$ and antiholomorphic in $w$ on $S$. It also satisfies $K(z + t, w) = K(z, w - t)$ for $t \in \mathbb{R}$. Hence there exists a continuous function $\varphi$ on $\overline{S}$, holomorphic on $S$, such that

$$K(z, w) = \varphi\left(\frac{z - \overline{w}}{2}\right).$$

For $t \in \mathbb{R}$, we then have $\varphi(t) = \langle \eta, \alpha^\eta(2t) \rangle = \int_{\mathbb{R}} e^{2iti\lambda} dP^\eta(\lambda)$, so that [NÓ18, Lemma A.2.5] yields $L(P^\eta)(\pm \pi) < \infty$ and $\eta \in D(\Delta^{\pm 1/4})$. This implies that $\alpha^\eta(z) = \Delta^{-iz/2\pi}\eta$ for $z \in \overline{S}$.

From $\xi = \alpha^\eta(-\pi i/2) = \Delta^{-1/4}\eta$ we derive that

$$\alpha^\xi(z) = \alpha^\eta\left(z - \frac{\pi i}{2}\right) = \Delta^{-iz/2\pi}\xi \quad \text{for} \quad z \in \overline{S}_\pi.$$ 

Further, $J\eta = \eta$ implies

$$J\alpha^\xi(z) = J\alpha^\eta\left(z - \frac{\pi i}{2}\right) = \alpha^\eta\left(z + \frac{\pi i}{2}\right) = \alpha^\xi(\pi i + \overline{z}).$$

For $z = 0$, we obtain in particular $J\xi = \alpha^\xi(\pi i) = \Delta^{1/2}\xi$. □
2.2. The General Setting for Spaces of Holomorphic Functions. In Appendix A.3 we show that a standard subspace $V \subseteq \mathcal{H}$ always specifies a realization of the complex Hilbert space $\mathcal{H}$ as a vector-valued Hardy space on a strip, even if $\mathcal{H}$ has no specific geometric structure. In this subsection we consider an enriched geometric context. A key observation is Lemma 2.5 that will later be applied to the situation where the complex manifold is an open tube domain $T$.

On the geometric side, we consider a connected complex manifold $\Xi$, endowed with a smooth action

$$\sigma : \mathbb{R}^\times \times \Xi \to \Xi, \quad (r, m) \mapsto r.m =: \sigma_r(m) =: \sigma^m(r)$$

for which the diffeomorphisms $\sigma_r$ are holomorphic for $r > 0$ and antiholomorphic for $r < 0$. In particular, $\tau_\Xi := \sigma_{-1}$ is an antiholomorphic involution of $\Xi$. We further assume that $\Xi$ is an open domain in a larger complex manifold and that the boundary $\partial \Xi$ contains a real submanifold $M$ with the property that, for every fixed point $m \in \Xi^{\text{re}}$, the orbit map $\mathbb{R} \to \Xi, t \mapsto \sigma^m(t)$ extends to a holomorphic map $\sigma^m \overline{S}_{-\pi/2, \pi/2} \to \Xi$ which further extends to a continuous map

$$\sigma^m : \overline{S}_{-\pi/2, \pi/2} \to \Xi \cup M \quad \text{with} \quad \sigma^m(\pm i\pi/2) \in M. \quad (2.2)$$

Example 2.2. (Domains in $\mathbb{C}$) In one dimension we have the following standard examples of simply connected proper domains in $\mathbb{C}$ with their natural $\mathbb{R}^\times$-actions.

(a) (Strips) On the strip $S_\pi = \{ z \in \mathbb{C} : 0 < \text{Im} \; z < \pi \}$ we have the antiholomorphic involution $\tau_{S_\pi}(z) = \pi i + \overline{z}$ with fixed point set

$$S_{\pi}^{\tau_{S_\pi}} = \left\{ z \in S_\pi : \text{Im} \; z = \frac{\pi}{2} \right\}.$$

The group $\mathbb{R}^\times$ acts by translations via $\sigma_r(z) = z + t$, $M := \mathbb{R} \cup (\pi i + \mathbb{R}) = \partial S_\pi$ is a real submanifold, and for $\text{Im} \; z = \pi/2$, the orbit map $\sigma^z(t)$ extends to the closure of the strip $S_{-\pi/2, \pi/2}$ with $\sigma^z(\pm i/2) = z \pm i/2 \in M$.

(b) (Upper half plane) On the upper half plane $\mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Im} \; z > 0 \}$, we have the antiholomorphic involution $\tau_{\mathbb{C}_+}(z) = -\overline{z}$ and the action of $\mathbb{R}^\times$ by dilations $\sigma_r(z) = rz$. Here $M := \mathbb{R} = \partial \mathbb{C}_+$ is a real submanifold, and for $z = iy$, $y > 0$, the orbit map $\sigma^z(t) = e^t z$ extends to the closure of the strip $S_{-\pi/2, \pi/2}$ with $\sigma^z(\pm i/2) = \pm i(iz) = \mp iy$.

(c) (Unit disc) On the unit disc $D = \{ z \in \mathbb{C} : |z| < 1 \}$ we have the antiholomorphic involution $\tau_D(z) = \overline{z}$ and the action of $\mathbb{R}^\times \cong \text{SO}_{1,1}(\mathbb{R})_0$ by the maps

$$\sigma_r(z) = \frac{\cosh(t/2)z + \sinh(t/2)}{\sinh(t/2)z + \cosh(t/2)}. \quad (2.3)$$

Here $M := \mathbb{S}^1 = \partial D$ is a real submanifold, and for $z \in D \cap \mathbb{R}$, the orbit map $\sigma^z(t)$ extends to the closure of the strip $S_{-\pi/2, \pi/2}$ with

$$\sigma^z(\pm i/2) = \frac{\cos(\pi/4)z \pm i \sin(\pi/4)}{\pm i \sin(\pi/4)z + \cos(\pi/4)} = \frac{z \pm i}{\pm iz + 1} = \mp i \cdot \frac{z \pm i}{z \mp i}. \quad (2.4)$$

The biholomorphic maps

$$\text{Exp} : S_\pi \to \mathbb{C}_+, \quad z \mapsto e^z \quad \text{and} \quad \text{Cay} : \mathbb{C}_+ \to D, \quad z \mapsto \frac{z - i}{z + i}.$$
are equivariant for the described $\mathbb{R}^\times$-actions on the respective domains.

The Riemann mapping theorem implies that any antiholomorphic $\mathbb{R}^\times$-action on a proper simply connected domain $\mathcal{O} \subseteq \mathbb{C}$ is equivalent to the examples (a)–(c). In fact, we may w.l.o.g. assume that $\mathcal{O} = D$ and, since every isometric involution of the hyperbolic plane has a fixed point, we may also assume that $\sigma_{-1}(0) = 0$. Then $z \mapsto \sigma_{-1}(z)$ is biholomorphic on $D$ fixing $0$, hence of the form $z \mapsto e^{it}z$, and from that it follows that, up to conjugation with biholomorphic maps, we may assume that $\sigma_{-1}(z) = \overline{z}$. Now we simply observe that the centralizer of $\sigma_{-1}$ in the group $\text{PSU}_{1,1}(\mathbb{C}) \cong \text{Aut}(D)$ is $\text{PSO}_{1,1}(\mathbb{R})$, and this leads to the action in (2.3).

On the representation theoretic side, we consider a Hilbert space $\mathcal{H}$, realized on a connected complex manifold $\Xi$ for another complex Hilbert space $\mathcal{K}$ as subspace of the Fréchet space $\text{Hol}(\Xi, \mathcal{K})$ of holomorphic functions $f : \Xi \to \mathcal{K}$. Here we assume that the point evaluations $K_z : \mathcal{H} \to \mathcal{K}, \quad f \mapsto f(z)$

are continuous. Then

$$K : \Xi \times \Xi \to B(\mathcal{K}), \quad K(z, w) := K_z K_w^*$$

is a positive definite operator-valued kernel which determines $\mathcal{H}$ uniquely, so that we write $\mathcal{H}_K$ to emphasize the dependence of $\mathcal{H}$ from the kernel $K$ (see [Ne00, Ch. I] for details).

To connect this structure to the antiunitary representation $(U^\mathcal{V}, \mathcal{H})$ of $\mathbb{R}^\times$, corresponding to a standard subspace $\mathcal{V}$, we also need a representation of $\mathbb{R}^\times$ on $\mathcal{K}$. This is specified by a conjugation $J_\mathcal{K}$ on $\mathcal{K}$ and a strongly continuous homomorphism $\rho : \mathbb{R}^\times \to \text{GL}(\mathcal{K})$

whose range commutes with $J_\mathcal{K}$, so that it extends by $\rho(-1) := J_\mathcal{K}$ to a representation of $\mathbb{R}^\times$ on $\mathcal{K}$. We also assume that the operators $\rho(e^t)$ are normal and that the hermitian one-parameter group $t \mapsto \rho(e^t)\rho(e^t)^*$ is norm-continuous. This implies that

$$\rho(e^t) = e^{t\Lambda}, \quad \text{where} \quad \Lambda : \mathcal{D}(\Lambda) \to \mathcal{K}, \quad \Lambda \xi = \frac{d}{dt} \bigg|_{t=0} \rho(e^t)\xi,$$

is an unbounded operator of the form $\Lambda = \Lambda_- + \Lambda_+$, where $\Lambda_+$ is a bounded symmetric operator and $\Lambda_-$ is a skew-adjoint operator (possibly unbounded) that commutes with $\Lambda_+$. For $t \geq 0$, we have

$$\|e^{t\Lambda}\| = \|e^{t\Lambda_+}\| = e^{t \sup(\text{Spec}(\Lambda_+))},$$

showing that the boundedness of $\Lambda_+$ is required to obtain a one-parameter group of bounded operators.

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1 See “Appendix A.1” for this correspondence and in particular Theorem A.3.
2.2.1. Three Standard Subspaces of $\mathcal{K}$. Standard subspaces in $\mathcal{H}$ are closely related to standard subspaces in $\mathcal{K}$. In this section we associate three standard subspaces of $\mathcal{K}$ to the representation $(\rho, \mathcal{K})$. In particular, we construct a modular pair $(\Delta_\mathcal{K}, J_\mathcal{K})$ on $\mathcal{K}$ from the operator $\Lambda_-$, which specifies a standard subspace $\mathcal{V}_\mathcal{K}$ (cf. Theorem A.3). Further, the symmetric part $\Lambda_+$ of $\Lambda$ leads to twists $\mathcal{V}_\mathcal{K}^\sharp$ of $\mathcal{V}_\mathcal{K}$ and $\mathcal{V}_\mathcal{K}^\flat$ of $\mathcal{V}_\mathcal{K}$. These subspaces will be needed below in our description of the standard subspace $\mathcal{V} \subseteq \mathcal{H}$. We shall see in Sect. 5 that the case where $\mathcal{V}_\mathcal{K}^\sharp = \mathcal{V}_\mathcal{K}^\flat$ (Lemma 2.3) is of particular interest for the analysis of support properties.

With the measurable functional calculus of normal operators, we obtain the (possibly unbounded) normal operator

$$e^{\pi i \Lambda} := e^{\pi i \Lambda_+} e^{\pi i \Lambda_-} = e^{\pi i \Lambda_-} e^{\pi i \Lambda_+} \quad \text{with} \quad \mathcal{D}(e^{\pi i \Lambda}) = \mathcal{D}(e^{\pi i \Lambda_-}).$$

The operator $\Delta_\mathcal{K} := e^{2\pi i \Lambda_-}$ is strictly positive selfadjoint, and, as $J_\mathcal{K}$ commutes with $\Lambda_{\pm}$, we obtain the modular relation

$$J_\mathcal{K} \Delta_\mathcal{K} J_\mathcal{K} = \Delta_\mathcal{K}^{-1}.$$

Hence $T_\mathcal{K} := J_\mathcal{K} \Delta_\mathcal{K}^{1/2} = J_\mathcal{K} e^{\pi i \Lambda_-}$ is the Tomita operator of the standard subspace

$$\mathcal{V}_\mathcal{K} := \text{Fix}(T_\mathcal{K}) \subseteq \mathcal{K},$$

and its adjoint $T_\mathcal{K}^* = J_\mathcal{K} \Delta_\mathcal{K}^{-1/2}$ is the Tomita operator of $\mathcal{V}_\mathcal{K}$ (cf. Lemma A.2(iv)). If $\Lambda_+$ is non-zero, we have to deal with slight modifications. Then $e^{\pi i \Lambda_+}$ is a unitary operator, and

$$\mathcal{V}_\mathcal{K} := e^{-\pi i \Lambda_+} \mathcal{V}_\mathcal{K}^\flat$$

and

$$\mathcal{V}_\mathcal{K} := e^{\pi i \Lambda_+} \mathcal{V}_\mathcal{K}^\flat$$

are standard subspaces of $\mathcal{K}$. The corresponding Tomita operators are

$$T_\mathcal{K}^\sharp := e^{\pi i \Lambda_+} T_\mathcal{K} e^{\pi i \Lambda_+} = T_\mathcal{K} e^{\pi i \Lambda_+} = J_\mathcal{K} e^{\pi i \Lambda} = e^{\pi i \Lambda} J_\mathcal{K} \quad (2.5)$$

and

$$T_\mathcal{K}^\flat := e^{\pi i \Lambda_+} T_\mathcal{K}^* e^{-\pi i \Lambda_+} = T_\mathcal{K}^* e^{-\pi i \Lambda_+} = J_\mathcal{K} e^{-\pi i \Lambda} = e^{\pi i \Lambda} J_\mathcal{K} \quad (2.6).$$

We observe that

$$\mathcal{V}_\mathcal{K} = e^{-\pi i \Lambda_+} \mathcal{V}_\mathcal{K}^\flat \quad (2.7)$$

We also note that

$$J_\mathcal{K} T_\mathcal{K}^\sharp J_\mathcal{K} = e^{\pi i \Lambda} J_\mathcal{K} = J_\mathcal{K} e^{-\pi i \Lambda} = T_\mathcal{K}^\flat \quad \text{implies} \quad J_\mathcal{K} \mathcal{V}_\mathcal{K}^\sharp = \mathcal{V}_\mathcal{K}^\flat \quad (2.9).$$

We will need the following lemma in Proposition 5.4:

**Lemma 2.3.** The subspaces $\mathcal{V}_\mathcal{K}^\sharp$ and $\mathcal{V}_\mathcal{K}^\flat$ coincide if and only if $\Lambda = \Lambda_+$ and $\text{Spec}(\Lambda_+) \subseteq \mathbb{Z}$.

**Proof.** As $T_\mathcal{K}^\sharp = e^{-2\pi i \Lambda} T_\mathcal{K}^\flat$ by (2.6) and (2.7), the equality $\mathcal{V}_\mathcal{K}^\sharp = \mathcal{V}_\mathcal{K}^\flat$ is equivalent to $e^{2\pi i \Lambda} = 1$. By spectral calculus for unbounded normal operators, this is equivalent to $\text{Spec}(\Lambda) \subseteq \mathbb{Z}$, which immediately translates into the two conditions $\Lambda_- = 0$ and $\text{Spec}(\Lambda_+) \subseteq \mathbb{Z}$. □
Example 2.4. For $K = \mathbb{C}$, $J_K(z) = \bar{z}$, and $\Lambda = \Lambda_+ = \lambda 1$, $\lambda \in \mathbb{R}$, we have

$$V_K = \mathbb{R}, \quad V_K^\flat = e^{-\frac{\pi i}{2} \lambda} \mathbb{R} \quad \text{and} \quad V_K^\circ = e^{\frac{\pi i}{2} \lambda} \mathbb{R}.$$ 

Now assume that $\lambda \in \mathbb{Z}$. If $\lambda = 2\mu$ is even, then $e^{\pi i \lambda} = e^{\pi i \mu} 1$ so that $V_K^\circ = V_K^\flat = V_K$. But if $\lambda = 2\mu + 1$ is odd, then $V_K^\flat = e^{\pi i \lambda} \mathbb{R} = i \mathbb{R} = V_K^\circ$.

So the two subspaces $V_K^\circ$ and $V_K^\flat$ need not be equal to $V_K$, not even if they are equal.

2.2.2. The Real Space $\mathcal{H}^J$ On a reproducing kernel space $\mathcal{H}_K \subseteq \text{Hol}(\Xi_1, K)$, we now consider an antiunitary representation of the form

$$(U(r)F)(z) = \rho(r^{-1})^* F(r^{-1}z), \quad r \in \mathbb{R}^\times. \quad (2.10)$$

This means that $K_z U(r) = \rho(r^{-1})^* K_{r^{-1}}z$. Replacing $r$ by $r^{-1}$ and $F$ by $K_z^* \eta$ we get

$$U(r) K_z^* \eta = K_{r,z}^* \rho(r) \eta \quad \text{for} \quad r \in \mathbb{R}^\times, z \in \Xi, \eta \in K. \quad (2.11)$$

For the kernel $K$, this corresponds to the equivariance condition

$$K(rz, rw) = \rho(r^{-1})^* K(z, w) \rho(r^{-1}), \quad z, w \in \Xi, r \in \mathbb{R}^\times. \quad (2.12)$$

We are interested in a more concrete description of the standard subspace $V$ associated to the pair $(\Delta, J)$, specified by

$$J := U(-1) \quad \text{and} \quad \Delta^{-it/2\pi} = U(e^t), \quad t \in \mathbb{R},$$

in terms of an injective boundary value map

$$b : \mathcal{H}_K \to C^{-\infty}(M)$$

in distributions on $M \subseteq \partial \Xi$. In Sect. 3 we shall study the case where $M$ is a finite dimensional vector space $E$ and $\Xi = E + i C^0$ a tube domain specified by a convex cone in $E$ (see [NO21a, NO21b, NO21c] for more general situations).

To this end, we would like to use Proposition 2.1 which describes $V$ in terms of the real subspace $\mathcal{H}_K^J$ of $J$-fixed elements. This space is easily characterized by Lemma 2.5 below. The corresponding standard subspaces are harder to describe because they require information on analytic extensions of orbit maps of elements of $\mathcal{H}_K^J$ to the closure of the strip $S_{-\pi/2, \pi/2}$.

Lemma 2.5. Suppose that the submanifold $\Xi_\Xi^\Xi$ of $\tau_\Xi$-fixed points is not empty. Then, for every open subset $O \subseteq \Xi_\Xi$, we have

$$\mathcal{H}_K^J = \{ F \in \mathcal{H}_K : (\forall z \in \Xi_\Xi) F(z) \in K^J_K \} = \text{span}_\mathbb{R} \{ K_z^* \eta : z \in O, \eta \in K^J_K \}.$$
Proof. We have $JF(z) = J_KF(\tau z)$. Hence

$$\mathcal{H}_K^f = \{ F \in \mathcal{H}_K : (\forall z \in \mathbb{Z}^\tau) F(z) \in \mathcal{K}^J\mathcal{K} \},$$

as any holomorphic function on $\mathbb{Z}$ is uniquely determined by its restriction to the totally real submanifold $\mathbb{Z}^\tau$ because $\mathbb{Z}$ is connected.

To verify the second equality, let $\mathcal{E} \subseteq \mathcal{H}_K$ denote the right hand side. Then (2.11), applied to $r = -1$, implies that $\mathcal{E} \subseteq \mathcal{H}_K^f$. It remains to show that $\mathcal{E}$ is total in $\mathcal{H}_K$. To verify this claim, suppose that $F \in \mathcal{H}_K$ is orthogonal to $\mathcal{E}$. Then $\langle K^*_\eta F \rangle = \langle \eta, F(z) \rangle = 0$ for $z \in \mathcal{O}$, $\eta \in \mathcal{K}^J\mathcal{K}$. As $\mathcal{K}^J\mathcal{K}$ generates $\mathcal{K}$ as a complex Hilbert space, it follows that $F|_{\mathcal{O}} = 0$. Since $\mathcal{O}$ is open in the totally real submanifold $\mathbb{Z}^\tau$ of the connected complex manifold $\mathbb{Z}$, it follows that $F = 0$. Therefore the closed real subspace $\mathcal{E}$ is total in $\mathcal{H}_K$, hence coincides with $\mathcal{H}_K^f$. $\square$

3. Standard Subspaces and Tube Domains

This is the core section of the article, culminating in Theorem 3.14, where we characterize the standard subspace $\mathcal{V}$ corresponding to an anti-unitary representation $U$ of $\mathbb{R}^\times$ in a vector-valued $L^2$-space. The setup is as follows. We consider tube domains $T := E + iC^0 \subset E_{\mathbb{C}}$ in the following environment:

(A1) $E$ is a finite dimensional real vector space.
(A2) $h \in \text{End}(E)$ is diagonalizable with eigenvalues $\{-1, 0, 1\}$ and $\tau := e^{\pi i h}$.
(A3) $C \subseteq E$ is a pointed, generating closed convex cone invariant under $e^{\mathbb{R}h}$ and $-\tau$.

After discussing these conditions in Sect. 3.1, we study in this section standard subspaces of vector-valued $L^2$-spaces $\mathcal{H} = L^2(E^*, \mu; \mathcal{K})$, where $\mathcal{K}$ is a Hilbert space and $\mu$ is a Herm+($\mathcal{K}$)-valued tempered measure supported in the dual cone $C^*$. Then we have a natural realization of $\mathcal{H}$ as a Hilbert space $\mathcal{H}_\mu \subseteq S'(E; \mathcal{K})$ of $\mathcal{K}$-valued tempered distributions and all these distributions extend to holomorphic functions $T \rightarrow \mathcal{K}$. Under suitable invariance conditions on the measure $\mu$, all these Hilbert spaces carry a natural antiunitary representation of $\mathbb{R}^\times$, corresponding to the geometric action on $E$ specified by the pair $(h, \tau)$ (Sect. 3.2). Our main results are obtained in Sect. 3.3, where we identify the standard subspace $\mathcal{V} \subseteq \mathcal{H}_\mu \subseteq S'(E; \mathcal{K})$ as the real subspace generated by acting with real-valued test functions supported on a certain wedge domain $W \subseteq E$ on a real subspace $V_\mathcal{K} \subseteq \mathcal{K}$ (Theorem 3.14).

Writing $E_{h, \tau} = E_{h, \tau}(h) := \ker(h - \lambda 1)$ for the $h$-eigenspaces and $E_{\pm} := \ker(\tau \mp 1)$ for the $\tau$-eigenspaces, (A2) implies

$$E = E_1 \oplus E_0 \oplus E_{-1}, \quad E^- = E_1 \oplus E_{-1}, \quad \text{and} \quad E^+ = E_0. \quad (3.1)$$

Accordingly, we write $x = x_1 + x_0 + x_{-1}$ for the decomposition of $x \in E$ into $h$-eigenvectors. As we shall see below, (A3) implies in particular that the wedge domain

$$W := W(E, C, h) := C^0_+ \oplus E_0 \oplus C^0_- \quad \text{for} \quad C_\pm := \pm C \cap E_{\pm1} \quad (3.2)$$

is nonempty. For generalizations of such configurations to non-abelian Lie groups and their properties, we refer to [Ne19a,Ne19b,NÓ21b,NÓ21c,Oeh21].
Example 3.1. (a) The simplest examples arise for $h = \text{id}_E$, $\tau = -\text{id}_E$, and a pointed generating closed convex cone $C \subseteq E$. Then $W = C^0 = C_+^0$.

(b) An example of importance in physics arises from $d$-dimensional Minkowski space $E = \mathbb{R}^d$ with the Lorentzian scalar product

$$(x_0, x)(y_0, y) = x_0 y_0 - \langle x, y \rangle$$

for $x_0, y_0 \in \mathbb{R}$, $x, y \in \mathbb{R}^{d-1}$.

Then the upper light cone

$$C := \{(x_0, x) : x_0 \geq 0, x^2_0 \geq x^2\}$$

is pointed and generating. We consider the generator $h \in \mathfrak{so}_{1,d-1}(\mathbb{R})$ of the Lorentz boost in the $(x_0, x_1)$-plane

$$h(x_0, x_1, \ldots, x_{d-1}) = (x_1, x_0, 0, \ldots, 0).$$

It is diagonalizable with the eigenvalues $0, -1, 1$, and the eigenspaces are

$$E_\pm = \mathbb{R}(e_0 \pm e_1) \quad \text{and} \quad E_0 = \{(0, 0)\} \times \mathbb{R}^{d-2}.$$

For $\tau = e^{\pi i h}$, we obtain

$$\tau (x_0, x_1, \ldots, x_{d-1}) = (-x_0, -x_1, x_2, \ldots, x_{d-1}) \quad \text{and} \quad E^{-} = \mathbb{R}^2 \times \{0\}.$$

The two cones $\pm C_\pm = C \cap E_\pm = \mathbb{R}_+(e_0 \pm e_1)$ are simply half-lines, so that

$$W = \mathbb{R}^\times_+(e_1 + e_0) \oplus \mathbb{R}^\times_+(e_1 - e_0) \oplus \mathbb{R}^{d-2},$$

is the standard open right wedge.

(c) On $E = \mathbb{R}^3$ with basis $e_1, e_2, e_3$, we consider the matrices

$$h = \text{diag}(1, -1, 0) \quad \text{and} \quad \tau = \text{diag}(-1, -1, 1).$$

We now describe all cones $C$ satisfying (A3). Up to sign choices, we may assume that

$$C_+ = \mathbb{R}_+ e_1 \quad \text{and} \quad C_- = -\mathbb{R}_+ e_2$$

(cf. Lemma 3.2 below). As $C$ is generating and $-\tau$-invariant, it is determined by the cone $\{x \in C : x_3 \geq 0\}$, which in turn is determined by the closed convex subset $D \subseteq \mathbb{R}^2$, given by

$$D = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1, x_2, 1) \in C\}.$$

This set has to be closed, convex, contained in $\mathbb{R}_+ e_1 \oplus \mathbb{R}_+ e_2$ (Lemma 3.2), not containing $(0, 0)$ (to ensure that $C$ is pointed), and invariant under $e^{\mathbb{R}h}$. This only leaves the sets

$$D_m := \{(x_1, x_2) : x_1, x_2 > 0, x_1 x_2 \geq m\}, \quad m > 0.$$

Then

$$C^m := \mathbb{R}_+(e_3 + D_m) \cup \mathbb{R}_+ (-e_3 + D_m)$$

is a closed convex pointed generating cone satisfying (A3). Up to sign changes, the cones satisfying (A3) are all of this form. They are Lorentzian with respect to the Lorentzian quadratic form

$$q(x_1, x_2, x_3) = x_1 x_2 - m x_3^2,$$

so they all arise from a 3-dimensional Minkowski space as in (b).
3.1. The Tube and Associated Wedge Domains. In this section we focus on the tube domain \( T = E + i \mathbb{C}^0 \), the wedge \( W = \mathbb{C}_0^+ \oplus E_0 \oplus \mathbb{C}_-^0 \) introduced in (3.2) and the holomorphic extension of the one-parameter group \((U(e^t))_{t \in \mathbb{R}}\). The main result is Lemma 3.3.

The tube domain \( T \) is obviously invariant under \( e^{\mathbb{R} \lambda} \) and \(-\tau\), where we use the same notation for the complex linear extensions to \( E_{\mathbb{C}} \). Denote by \( \bar{\tau} : E_{\mathbb{C}} \to E_{\mathbb{C}} \) the conjugate linear extension of \( \tau \) to \( E_{\mathbb{C}} \). Then

\[
E^c := (E_{\mathbb{C}})^\bar{\tau} = E^+ + i E^-.
\]

As \( \bar{\tau} \) acts on \( i E \) as \(-\tau\) and \( C \) is \(-\tau\) invariant, \( \bar{\tau}(T) = T \), and

\[
T^{\bar{\tau}} = T \cap E^c = E^+ + i (C^0 \cap E^-)
\]

is the cone of \( \bar{\tau} \)-fixed points in \( T \), a real tube domain.

**Lemma 3.2.** For the projections

\[
p_{\pm 1} : E \to E_{\pm 1}, x \mapsto x_{\pm 1}, \quad \text{and} \quad p^- : E \to E_1 \oplus E_{-1} = E^- , x \mapsto x_1 + x_{-1} = \frac{1}{2} (x - \tau x),
\]

the following assertions hold:

(i) \( p_{\pm 1}(C) = \pm C_{\pm} \) and \( p_{\pm 1}(C_0) = \pm C_{\pm}^0 \neq \emptyset \).

(ii) \( p^-(C) = C \cap E^- = C_+ \oplus -C_- \) and \( p^-(C_0) = C_0 \cap E^- = C_+^0 \oplus -C_-^0 \).

(iii) \( C \subseteq C_+ \oplus E_0 \oplus -C_- \).

**Proof.** (i) As \( \pm C_{\pm} \subseteq C \), we have \( \pm C_{\pm} \subseteq p_{\pm 1}(C) \). Using the \( e^{\mathbb{R} \lambda} \)-invariance of \( C \) and writing \( x = x_1 + x_0 + x_{-1} \) as before, \( e^{\mathbb{R} \lambda} x = e^{\mathbb{R} \lambda} x_1 + x_0 + e^{-\mathbb{R} \lambda} x_{-1} \). Now take the limit \( t \to \infty \) to see that

\[
C \ni e^{-t} e^{\mathbb{R} \lambda} x = x_1 + e^{-t} x_0 + e^{-2t} x_{-1} \to x_1 \quad \text{as} \quad t \to \infty.
\]

We likewise get \( x_{-1} = \lim_{t \to -\infty} e^{\mathbb{R} \lambda} e^{-t} x \in C \). It follows that \( x_{\pm} \in \pm C_{\pm} \), so that \( p_{\pm 1}(C) = \pm C_{\pm} \). As \( p_{\pm 1} \) are projections and \( C_0 \neq \emptyset \), it follows that \( p_{\pm 1}(C_0) \subseteq \pm C_{\pm}^0 \).

To obtain equality, it suffices to observe that \( C_+^0 \oplus -C_-^0 \subseteq (E^- \cap C)_0 \subseteq C_0 \) follows from \(-\tau(C) = C \).

(ii) The two leftmost equalities follow from \( \tau(C) = -C \), and the second two rightmost equalities from (i) and \( p^- = p_1 + p_{-1} \).

(iii) follows from (ii). \( \Box \)

We now describe the wedge \( W \) and its closure \( \overline{W} = C_+ \oplus E_0 \oplus C_- \) in terms of the tube domain \( T \):

**Lemma 3.3.** The wedge \( W \) and the tube \( T \) are related as follows:

(1) \( W = \{ x \in E : (\forall z \in S_\pi) e^{\mathbb{R} \lambda} x \in T \} \)

(2) \( \overline{W} = \{ x \in E : (\forall z \in S_\pi) e^{\mathbb{R} \lambda} x \in E + i C \} \).

**Proof.** For \( z = a + ib \in S_\pi \) and \( x = x_1 + x_0 + x_{-1} \), we have

\[
e^{\mathbb{R} \lambda} (x_1 + x_0 + x_{-1}) = e^{\mathbb{R} \lambda} x_1 + x_0 + e^{-\mathbb{R} \lambda} x_{-1}
\]

\[
= \cos(b)(e^{\mathbb{R} \lambda} x_1 + e^{-a} x_{-1}) + x_0 + i \sin(b)(e^{\mathbb{R} \lambda} x_1 - e^{-a} x_{-1}), \quad (3.3)
\]

with \( \sin(b) > 0 \). As the imaginary part determines whether this element is contained in \( T \), we see with Lemma 3.2(ii) that \( e^{\mathbb{R} \lambda} x \in T \) holds for every \( z \in S_\pi \) if and only if \( x_1 \in C_0^+ \) and \( x_{-1} \in C_0^- \). Likewise \( e^{\mathbb{R} \lambda} x \in \overline{T} = E + i C \) holds for every \( z \in S_\pi \) if and only if \( x_1 \in C_+ \) and \( x_{-1} \in C_- \). \( \Box \)
3.2. Operator-Valued Measures and the Corresponding Hilbert Spaces. In this subsection we construct the Hilbert spaces $\mathcal{H}$ that we are interested in, first as vector-valued $L^2$-spaces $L^2(E^*, \mu; K)$, defined by $\text{Herm}^+(K)$-valued measures $\mu$ on the dual cone $C^* \subseteq E^*$ (cf. [NÔ15, Thm. B.3], [Ne98]). Any such Hilbert space $\mathcal{H}$ carries a natural antiunitary representation of the group $G = E \rtimes_R \mathbb{R}^\times$ (Lemma 3.5), but this representation has several other interesting realizations. In Lemma 3.10 we realize it as a reproducing kernel Hilbert space of holomorphic functions on $T$ and use that realization in Lemma 3.11 to describe the space of $T$-fixed elements. We also describe a third realization of this representation in the space of distributions generated by the positive definite operator-valued distribution $\mu$ on $E$. The latter two realizations are connected by taking suitable boundary values. In particular, the Fourier transform $\hat{\mu}(z) = \int_{C^*} e^{i\lambda(z)} d\mu(\lambda)$ will play an important role in the proof of Theorem 3.14, both, as a holomorphic function of $z$ and as a distribution on $E$.

Let $K$ be a separable Hilbert space and $\mu$ be a tempered $\text{Herm}^+(K)$-valued Borel measure on $E^*$, supported in the dual cone

$$C^* = \{ \lambda \in E^* : \lambda(C) \subseteq [0, \infty) \}.$$

We then define the Hilbert space

$$\mathcal{H} := L^2(E^*, \mu; K) = L^2(C^*, \mu; K)$$

of measurable functions $f : E^* \to K$ such that the norm of $f$ with respect to the scalar product

$$\langle f, g \rangle = \int_{E^*} \langle f(\lambda), d\mu(\lambda)g(\lambda) \rangle_K$$

is finite. We refer to [Ne98] for more details on operator-valued measures and the corresponding $L^2$-spaces.

We let $G := E \rtimes_R \mathbb{R}^\times$ with Lie algebra $g = E \ltimes h \mathbb{R}$, where $\sigma(e^t)(v) = e^{th}v$ and $\sigma(-1) = \tau$. The involution $\tau$ extends naturally to $G$ by $\tau_G(v, r) = (\tau(v), r)$. We assume that

$$(-\tau)_*\mu = J_K \mu J_K \quad \text{and} \quad \sigma(r)_*\mu = \rho(r)^* \cdot \mu \cdot \rho(r) \quad \text{hold for} \quad r > 0. \quad (3.4)$$

Example 3.4. If $h = 0$ and $\tau = \text{id}_E$, then the assumption that $C$ is pointed and generating, combined with $-C = \tau(C) = C$, leads to $C = \{0\}$ and hence to $E = \{0\}$. Then we may assume that $\mu(\{0\}) = 1_K$, and (3.4) means that the operators $\rho(e^t)$, $t \in \mathbb{R}$, are unitary. So $(\rho, K)$ is an antiunitary representation of $\mathbb{R}^\times$ on $K$ which coincides with $U^{\sqrt{K}}$ (cf. Sect. 2.2.1).

Lemma 3.5. We obtain an antiunitary representation of $G$ on $\mathcal{H} = L^2(E^*, \mu; K)$ by

$$(U(x, 1)f)(\lambda) = e^{i\lambda(x)}f(\lambda), \quad (3.5)$$

$$(U(0, e^t)f)(\lambda) = \rho(e^t)f(\lambda \circ e^{th}) = \rho(e^t)((e^{th})_*f)(\lambda), \quad (3.6)$$

$$(U(0, -1)f)(\lambda) = J_Kf(-\lambda \circ \tau). \quad (3.7)$$

Proof. The relations (3.4) lead for $f, h \in L^2(E^*, \mu; K)$ to the transformation formulas

$$\int_{E^*} \langle f(\sigma(r)\lambda), d\mu(\lambda)h(\sigma(r)\lambda) \rangle = \int_{E^*} \langle \rho(r)f(\lambda), d\mu(\lambda)\rho(r)h(\lambda) \rangle$$

$$= \int_{E^*} \langle f(\lambda), \rho(r)^*d\mu(\lambda)\rho(r)h(\lambda) \rangle \quad (3.8)$$

and
\[ \int_{E^*} (f(-\tau \lambda), d\mu(\lambda)h(-\tau \lambda)) = \int_{E^*} (f(\lambda), J_{\mathcal{K}} d\mu(\lambda)J_{\mathcal{K}} h(\lambda)) \]
\[ = \int_{E^*} (J_{\mathcal{K}} h(\lambda), d\mu(\lambda)J_{\mathcal{K}} f(\lambda)). \]  

(3.9)

This implies that \( U(r) \) is (anti-)unitary for \( r \in \mathbb{R}^\times \). That \( U \) is a homomorphism is a standard calculation. □

**Remark 3.6.** The assumption \( \text{supp}(\mu) \subseteq C^* \) is equivalent to

\[ C \subseteq C_U := \{ x \in g : -i\partial U(x) \geq 0 \}, \]

where \( \partial U(x) = \frac{d}{dt}|_{t=0} U(e^{tx}) \) is the infinitesimal generator of the unitary one-parameter group \( (U(e^{tx}))_{t \in \mathbb{R}} \).

**Remark 3.7.** All Schwartz functions in \( S(E^*; \mathcal{K}) \) define elements of \( L^2(E^*, \mu; \mathcal{K}) \) because \( \mu \) is tempered. This leads to the embedding

\[ \Psi: L^2(E^*, \mu; \mathcal{K}) \to L^2(E^*; \mathcal{K}), \quad \Psi(f) = mf, \]

resp.,

\[ \Psi(f)(\varphi) = \int_{E^*} \varphi(\lambda) d\mu(\lambda) f(\lambda). \]

As the function \( e^{iz}(\alpha) = e^{i\alpha z} \) on \( C^* \) satisfies \( |e^{iz}| \leq 1 \) for \( z \in E + iC = \mathcal{T} \), the map

\[ \Gamma: E + iC \to L^2(E^*; B(\mathcal{K})), \quad z \mapsto e^{iz}\mu \]

is defined, weakly continuous (by the Dominated Convergence Theorem), and weakly holomorphic on the interior \( \mathcal{T} \). We further obtain with (3.4)

\[ \Psi(U(0, e^t)f) = \mu \cdot (\rho(e^t)e_\mu^t f) \]

(3.4)

\[ \rho(e^{-t})^*(e_\mu^t \mu) \cdot e_\mu^t f = \rho(e^{-t})^* e_\mu^t \Psi(f). \]

As

\[ (e_\mu^t e^{iz}(\lambda)) = e^{iz}(\lambda \circ e_\mu^t) = e^{i\lambda e^{iz}(\lambda)}, \]

we have the equivariance relation

\[ \Gamma(e^{iz} \lambda) = \rho(e^{-t})^* (e_\mu^t \lambda)_\# \Gamma(z). \]

(3.10)

Therefore Lemma 3.3(b) implies that \( \Gamma \) maps the closed wedge \( \overline{\mathcal{W}} \) into distributions which may produce elements of the standard subspace \( V \) when smeared with suitable test functions (cf. Proposition 2.1).

**Lemma 3.8.** The Fourier transform\(^2\)

\[ \tilde{\mu}(z) := \int_{E^*} e^{i\lambda(z)} d\mu(\lambda) = \int_{C^*} e^{i\lambda(z)} d\mu(\lambda) \in B(\mathcal{K}), \quad z \in \mathcal{T} = E + iC^0, \]

(3.11)

defines a holomorphic function on \( \mathcal{T} \) with the following properties:

\(^2\) Note that \( \tilde{\mu}(z) = \tilde{\mu}(-z) \), also on the level of distribution boundary values. In our context it minimizes the number of artificial minus signs to work with \( \tilde{\cdot} \) instead of \( \cdot \). As \( \tilde{\mu}(\varphi) = \int_{E^*} \varphi d\mu \) and \( \tilde{\mu}(\varphi) = \int_{E^*} \tilde{\varphi} d\mu \), we have on real-valued test functions \( \varphi \in C^\infty(E^*, \mathbb{R}) \) the relations \( \tilde{\mu}(\varphi) = \int_{E^*} \varphi d\mu \) and \( \tilde{\mu}(\varphi) = \int_{E^*} \tilde{\varphi} d\mu \). This means that \( \tilde{\mu} = \mu \) as distributions on \( E \).
The holomorphic function \( \tilde{\mu} \) on \( T \) has distributional boundary values in \( \mathcal{S}'(E; B(K)) \) in the sense that the tempered distribution \( \tilde{\mu} \), defined by \( \tilde{\mu}(\varphi) := \int_{E^*} \tilde{\varphi} \, d\mu \), satisfies
\[
\tilde{\mu}(\varphi) = \lim_{C^0 \ni y \to 0} \int_E \frac{\varphi(x)}{x^2} \tilde{\mu}(x + iy) \, d\mu_E(x) \quad \text{for} \quad \varphi \in \mathcal{S}(E). \tag{3.12}
\]

(b) For the antilinear extension \( \tau(x + iy) = \tau(x) - i \tau(y) \) of \( \tau \) to \( E_C \), we have
\[
\tilde{\mu}(e^{-ih}z) = \rho(e^i) \tilde{\mu}(z) \rho(e^i) \quad \text{and} \quad \tilde{\mu}(\tau z) = J_K \tilde{\mu}(z) J_K. \tag{3.13}
\]

Proof. (a) This follows easily from Fubini’s theorem and Lebesgue’s theorem on dominated convergence because \( e^{-\lambda y} \leq 1 \) for \( \lambda \in C^* \) and \( y \in C^* \):
\[
\int_E \varphi(x) \tilde{\mu}(x + iy) \, d\mu_E(x) = \int_E \varphi(x) \int_{C^*} e^{i\lambda(x+iy)} \, d\mu(\lambda) \, d\mu_E(x)
= \int_E \int_{C^*} e^{i\lambda(x)} \varphi(x) e^{-\lambda y} \, d\mu(\lambda) \, d\mu_E(x)
= \int_{C^*} \int_E \tilde{\varphi}(\lambda) \, d\mu(\lambda) \, d\mu_E(x) \quad \xrightarrow{y \to 0} \quad \int_{C^*} \int_E \tilde{\varphi}(\lambda) \, d\mu(\lambda) = \tilde{\mu}(\varphi).
\]

(b) The first formula is a direct consequence of the transformation properties (3.4). The second formula follows from
\[
\tilde{\mu}(\tau(z)) = \int_{E^*} e^{i\lambda(\tau z)} \, d\mu(\lambda) = \int_{E^*} e^{-i\lambda(z)} J_K \, d\mu(\lambda) J_K = J_K \left( \int_{E^*} e^{i\lambda(z)} \, d\mu(\lambda) \right) J_K.
\]

Remark 3.9. The covariance relation in Lemma 3.8 has an interesting consequence. For \( x = x_0 + x_1 + x_{-1} \in W \), we have \( e^{\frac{\pi}{2} h} x = e^{\frac{\pi}{2} h} x_0 + i(x_1 - x_{-1}) \in T \) by Lemma 3.3. In particular \( \iota(x) := e^{\frac{\pi}{2} h} x = x_0 + i(x_1 - x_{-1}) \in T \). If \( \Lambda \) is a bounded operator, we therefore expect for the boundary values of \( \tilde{\mu} \) on the wedge domain \( W \) a relation of the form
\[
\tilde{\mu}(x) = \tilde{\mu}(e^{-\frac{\pi}{2} h} \iota(x)) = e^{\frac{\pi}{2} \Lambda^*} \tilde{\mu}(\iota(x)) e^{\frac{\pi}{2} \Lambda} = e^{\frac{\pi}{2} \Lambda^*} e^{-\frac{\pi}{2} \Lambda} \tilde{\mu}(\iota(x)) e^{\frac{\pi}{2} \Lambda} = e^{rac{\pi}{2} \Lambda^*} e^{-\frac{\pi}{2} \Lambda} e^{rac{\pi}{2} \Lambda}.
\]

We shall see in Proposition 4.3 below that such a relation holds indeed in the sense that the distributional boundary values of \( \tilde{\mu} \) on \( E \) are represented on the open cone \( W \) by an operator-valued function.

For the proof of the Standard Subspace Theorem 3.14 below we shall use the following realization of the unitary representation \( (U, \mathcal{H}) \) by holomorphic functions on the tube domain \( T = E + iC^0 \):

Lemma 3.10. The map
\[
\Phi : L^2(E^*, \mu; K) \to \text{Hol}(T, K), \quad \Phi(f)(z) = \int_{E^*} e^{i\lambda(z)} \, d\mu(\lambda) f(\lambda),
\]
\[
\langle \xi, \Phi(f)(z) \rangle = \langle e^{-i\bar{z} \xi}, \mu \cdot f \rangle \quad \text{for} \quad \xi \in K, z \in T,
\]
maps \( L^2(E^*, \mu; K) \) injectively onto a reproducing kernel Hilbert space \( \mathcal{H}_K \) with \( B(K) \)-valued kernel
\[
K(z, w) = \tilde{\mu}(z - \bar{w}).
\]
In particular, $\mathcal{H}_K$ is generated by the functions
\begin{equation}
(K_w^*\xi)(z) = K(z, w)\xi = \tilde{\mu}(z - \overline{w})\xi, \quad w \in T, \xi \in \mathcal{K},
\end{equation}
satisfying
\begin{equation}
\Phi(e^{-i\overline{w}\xi}) = K_w^*\xi.
\end{equation}
The antiunitary representation $(U, L^2(E^*, \mu; \mathcal{K}))$ of $G$ is intertwined by $\Phi$ with the antiunitary representation $U'$ on $\mathcal{H}_K$ given by
\begin{align}
(U'(x, 1)F)(z) &= F(z + x), \\
(U'(0, e')F)(z) &= \rho(e^{-t})^*F(e^{-t}z), \\
(U'(0, -1)F)(z) &= J_{K}F(\overline{T}z).
\end{align}

On the generators $K_w^*\eta$, this leads to
\begin{align}
U'(x, 1)K_w^*\eta &= K_{w-x}^*\eta, \\
U'(0, e')K_w^*\eta &= K_{e^{i\theta}w}^*\rho(e')\eta, \\
U'(0, -1)K_w^*\eta &= K_{\tau w}^*J_{K}\eta.
\end{align}

Proof. As $\text{supp}(\mu) \subseteq C^*$, for $z \in T$ we have $|e_{iz}(\lambda)| = |e^{i\lambda(z)}| \leq 1$ for $\mu$-almost all $\lambda \in E^*$ and the temperedness of $\mu$ implies that $e_{iz}\eta \in L^2(E^*, \mu; \mathcal{K})$ for $z \in T$ and $\eta \in \mathcal{K}$ ([HN01, Lemma B.1]). This implies that $\Phi$ is determined by the relation
\begin{equation}
\langle \eta, \Phi(f)(z) \rangle = \langle e^{-iz}\eta, f \rangle_{L^2} \quad \text{for} \quad f \in L^2(E^*, \mu; \mathcal{K}).
\end{equation}
Hence the point evaluations on $\mathcal{H}_K$ are continuous, and given by the scalar product with $\Phi(e^{-iz}\eta)$, so that the reproducing kernel is given by
\begin{equation}
K(z, w) = K_z K_w^* = \int_{E^*} e^{i\lambda(z-w)}d\mu(\lambda) = \tilde{\mu}(z - \overline{w}),
\end{equation}
resp.,
\begin{equation}
\langle \xi, K(z, w)\eta \rangle = \langle e^{-iz}\xi, e^{-i\overline{w}}\eta \rangle \quad \text{for} \quad z, w \in T, \xi, \eta \in \mathcal{K}.
\end{equation}
For $z, w \in T$ and $\xi, \eta \in \mathcal{K}$, we derive from (3.22) the relation
\begin{align}
\langle \eta, K_z \Phi(e^{-i\overline{w}\xi}) \rangle &= \langle \eta, \Phi(e^{-i\overline{w}\xi})(z) \rangle = \langle e^{-iz}\eta, e^{-i\overline{w}}\xi \rangle \\
&= \langle \eta, \tilde{\mu}(z - \overline{w})\xi \rangle = \langle \eta, K_z K_w^*\xi \rangle,
\end{align}
which implies (3.15). The remaining assertions are easily verified. We refer to [Ne98, Thm. III.9] for further details.  

From the general Lemma 2.5, we obtain in particular:

**Lemma 3.11.** For $J := U'(0, -1)$ as in (3.18) and $\emptyset \neq \mathcal{O} \subseteq E^* + i(E^- \cap C^0)$ open, we have
\begin{equation}
\mathcal{H}_K^J = \text{span}_{R}[K_z^*\eta : z \in \mathcal{O}, \eta \in \mathcal{K}^J]\mathcal{K}] = \{F \in \mathcal{H}_K : F(\mathcal{O}) \subseteq \mathcal{K}^J\mathcal{K}\}.
\end{equation}
**Definition 3.12.** For the positive definite tempered distribution \( D := \hat{\mu} \in S'(E, B(K)) \), defined by

\[
D(\varphi) := \int_{E^*} \overline{\varphi(\lambda)} \, d\mu(\lambda)
\]

(cf. (1.3)), we write \( \mathcal{H}_D \subseteq S'(E; K) \) for the corresponding reproducing kernel Hilbert space whose \( B(K) \)-valued kernel is given on \( S(E) \) by

\[
\langle \eta, K_D(\varphi, \psi) \xi \rangle := \langle \eta, D(\psi^* \varphi) \xi \rangle = \int_{E^*} \overline{\varphi(\lambda)} \tilde{\psi}(\lambda) \langle \eta, d\mu(\lambda) \xi \rangle = \langle \tilde{\varphi} \eta, \tilde{\psi} \xi \rangle_{L^2}.
\]

The Hilbert space \( \mathcal{H}_D \) is generated by the \( K \)-valued distributions \( (\psi \ast D \eta)(\varphi) := D(\psi^* \varphi) \eta \) for \( \psi \in C_c^\infty(E), \eta \in K \) satisfying

\[
\langle \varphi \ast D \eta, \psi \ast D \xi \rangle = \langle \eta, D(\psi^* \varphi) \xi \rangle = \langle \eta, K_D(\varphi, \psi) \xi \rangle
\]

(cf. [NÓ18, Def. 7.1.5]).

**Remark 3.13.** For the Hilbert space \( \mathcal{H} \) we now have four pictures:

(a) as the \( L^2 \)-space \( \mathcal{H} = L^2(E^*, \mu; K) \),
(b) as a subspace \( \Psi(L^2(E^*, \mu; K)) \subseteq S'(E^*, \mu; K) \) (distributions on \( E^* \)) (Remark 3.7), and
(c) as \( \mathcal{H}_D \subseteq S'(E; K) \) (distributions on \( E \)) (Definition 3.12), and
(d) as the reproducing kernel space \( \mathcal{H}_K = \Phi(L^2(E^*, \mu; K)) \subseteq \text{Hol}(T; K) \) (Lemma 3.10))

The realizations (b) and (c) are connected by the Fourier transform

\[ \mathcal{F} : S'(E^*; K) \rightarrow S'(E; K), \quad D \mapsto \hat{D}, \quad \hat{D}(\varphi) := D(\tilde{\varphi}). \]

For a Schwartz function \( \varphi \in S'(E^*) \) and \( \eta \in K \), we have

\[ \mathcal{F}(\Psi(\tilde{\varphi} \eta)) = \mathcal{F}(\mu \tilde{\varphi} \eta) = \varphi \ast \hat{\mu} \eta, \]

so that

\[ \mathcal{F} : S'(E^*; K) \supseteq \Psi(L^2(E^*, \mu; K)) \rightarrow \mathcal{H}_{\hat{\mu}} \subseteq S(E; K) \]

is unitary by (3.24) and (3.25).
3.3. Standard Subspaces from Wedge Domains. In this section we prove one of the main results of this paper (Theorem 3.14). It describes the standard subspace $V$ corresponding to $J = U(-1)$ and $\Delta^{-i/2\pi} = U(e')$ for the antiunitary representation of $\mathbb{R}^\times$ on $L^2(E^*, \mu; \mathcal{K})$, introduced in Lemma 3.5. In Corollary 3.16 we also describe its symplectic complement $V'$ in similar terms. Most of the section is devoted to the proof of Theorem 3.14.

Recall the notation from Sect. 2.2: We have $\rho(e') = e^{i\Lambda}$ with $\Lambda = \Lambda_+ + \Lambda_-$ with $\Lambda_+$ bounded and symmetric, $\Lambda_-$ skew-adjoint and possibly unbounded, and $[\Lambda_+, \Lambda_-] = 0$. On $\mathcal{K}$ we define the Tomita operator $T_\mathcal{K} \equiv J_\mathcal{K} \Delta^{-1/2}_\mathcal{K}$. The corresponding standard subspace is $V_\mathcal{K}$, and we also consider

$$V_\mathcal{K}^\circ = e^{-\frac{\pi i}{2} \Lambda_+} V_\mathcal{K} \quad \text{and} \quad V_\mathcal{K}^\circ = e^{\frac{\pi i}{2} \Lambda_-} V_\mathcal{K}.$$

The goal of this subsection is to prove Theorem 3.14 below. To formulate it, we introduce for a standard subspace $W \subset \mathcal{K}$ and an open subset $\emptyset \neq \mathcal{O} \subseteq E$, the real subspace

$$H_\mathcal{O}(W) := \text{span}_{\mathbb{R}} \{ \tilde{\varphi} \eta : \varphi \in C^\infty_c(\mathcal{O}, \mathbb{R}), \eta \in W \} \subseteq L^2(E^*, \mu; \mathcal{K}), \quad (3.26)$$

where $\tilde{\varphi}(\lambda) = \int_E e^{i\lambda(x)} \varphi(x) \, d\mu_E(x)$.

**Theorem 3.14.** (Standard Subspace Theorem) Let $U$ be as in (3.5)–(3.7) and recall the wedge domain

$$W = C^0_+ \oplus E_0 \oplus C^0_- \subseteq E.$$

Then the standard subspace $V \subseteq L^2(E^*, \mu; \mathcal{K})$, defined by $J_V = U(0, -1)$ and $\Delta^{-i/2\pi}_V = U(0, e')$ is

$$V = H_{V_\mathcal{K}^\circ}(W) = e^{-\frac{\pi i}{2} \Lambda_+} H_{V_\mathcal{K}}(W). \quad (3.27)$$

Combining Theorem 3.14 with Example 2.4, we obtain:

**Corollary 3.15.** If $\mathcal{K} = \mathbb{C}$, $J_{\mathcal{K}}(\eta) = \overline{\eta}$, and $\Lambda_- = 0$, then $V = e^{-\frac{\pi i}{2} \Lambda} H_{\mathcal{R}}(W)$.

**Corollary 3.16.** Let the assumptions and notation be as in Theorem 3.14. Then the symplectic complement of $V$ is $V' = H_{V_\mathcal{K}^\circ}(-W) = e^{\frac{\pi i}{2} \Lambda_+} H_{V_\mathcal{K}}(-W)$.

**Proof.** For $\varphi \in C^\infty_c(-W, \mathbb{R})$ we have $\psi := \varphi \circ \tau \in C^\infty_c(W, \mathbb{R})$, so that the functions $\tilde{\psi} \eta \in V$, $\eta \in V_\mathcal{K}^\circ$, generate $V$ by Theorem 3.14. We also recall from (2.9) that $J_{\mathcal{K}} V_\mathcal{K}^\circ = V_\mathcal{K}^\circ$. Now the assertion follows from $JV = V'$ and

$$(J \tilde{\psi} \eta)(\lambda) = J_{\mathcal{K}} \tilde{\psi}(-\lambda \circ \tau) \eta = \overline{\psi(-\lambda \circ \tau)} J_{\mathcal{K}} \eta = \overline{\tilde{\psi}(\lambda \circ \tau)} J_{\mathcal{K}} \eta = \overline{\tilde{\varphi}(\lambda)} J_{\mathcal{K}} \eta.$$

$\square$
We now prepare the notation and first steps for the proof of Theorem 3.14. Let $z \in T \cap E^c = E^+ \oplus i (C^0 \cap E^-)$ and

$$\eta \in \mathcal{K}^J \cap \mathcal{D}(\Delta_K^{1/4}) = \mathcal{K} \cap \mathcal{D}(\Delta_K^{-1/4}),$$

we have $JK^*_z = K^*zJ_K$ since $\bar{r}(z) = z$, and the orbit map $\alpha^K_z \eta(t) = K^*z \rho(e^{-it} \eta)$ extends holomorphically to $S\cdot [-\pi/2, \pi/2]$. However, in general this orbit map need not extend continuously to the boundary if $\mu$ is an infinite measure. So we have to use some regularization procedure to construct elements of the standard subspace $V$ by the characterization in Proposition 2.1(iv).

Instead of $z \in T \cap E^c$, which specifies the element $K^*_z \eta \in \mathcal{H}^J_K$, we consider the boundary value for $z = -\pi i/2$ in a smeared version. Recall the open wedge $W = C^0_+ \oplus E_0 \oplus C^0_0$. For $\varphi \in C^\infty_c(E)$ and $z \in T$, we define

$$K^*_\varphi(z) := \int_E \varphi(x) K^*_z \eta d\mu_E(x) \overset{(3.14)}{=} \int_E \varphi(x) \tilde{\mu} (z-x) \, d\mu_E(x)$$

$$= \int_{E^*} e^{i\lambda(z)} \tilde{\varphi}(\lambda) d\mu(\lambda) = (\varphi * \tilde{\mu})(z) \in B(K, \mathcal{H}_K). \tag{3.28}$$

Lebesgue’s Dominated Convergence Theorem and (3.28), see also Lemma 3.8, imply that this holomorphic operator-valued function has the distributional boundary values

$$\int_{E^*} e^{i\lambda(z)} \varphi(\lambda) \, d\mu(\lambda) = (\varphi * \tilde{\mu})(z) \in C^\infty (E, B(K)) \cap S'(E, B(K)).$$

We also note that the distributions $\varphi * \tilde{\mu} \eta, \varphi \in C^\infty_c(E), \eta \in \mathcal{K}$, are contained in the Hilbert space $\mathcal{H}_\mu \subseteq S'(E; K)$ (Definition 3.12). With the notation $\varphi^\vee(x) := \varphi(-x)$, we have by (3.28)

$$K^*_\varphi(\varphi^\vee)(z) = \int_{E^*} e^{i\lambda(z)^*} \bar{\varphi}(\lambda) \, d\mu(\lambda) \cdot \eta = \Phi(\bar{\varphi} \cdot \eta)(z)$$

(3.29)

**Lemma 3.17.** Let $\varphi \in C^\infty_c(W, \mathbb{C})$ and $y \in [0, \pi]$. Then the functions

$$\tilde{\varphi}_y(\lambda) := \tilde{\varphi}(\lambda \circ e^{yih}) = \int_E e^{i\lambda(e^{yih} x)} \varphi(x) \, d\mu_E(x)$$

have the following properties:

(i) $|\tilde{\varphi}_y(\lambda)| \leq \|\varphi\|_1$ for every $\lambda \in C^*$.

(ii) For every $k > 0$, there exists a constant $d_k$, such that

$$|\tilde{\varphi}_y(\lambda)| \leq \frac{d_k}{1 + \|\lambda\|^2k} \text{ for all } \lambda \in C^*, y \in [0, \pi].$$

(iii) For $\eta \in \mathcal{V}_K^\eta$, the map $[0, \pi] \to L^2(E^*, \mu; K), y \mapsto \tilde{\varphi}_y e^{iy\lambda} \eta$ is continuous.
(iv) For $\xi \in K, \eta \in \mathcal{V}_{K}^{\pi}$, the map $[0, \pi] \rightarrow \mathbb{C}$, $y \mapsto \int_{C^{*}}(\xi, \tilde{\varphi}_{y}(\lambda))d\mu(\lambda)e^{iy\Lambda}\eta$, is continuous.

**Proof.** (i) We clearly have for $y \in [0, \pi]$ the estimate
\[
|\tilde{\varphi}_{y}(\lambda)| \leq \int_{E}e^{-\text{Im}\lambda(e^{iyh}x)}|\varphi(x)|d\mu_{E}(x),
\]
so that (i) follows from $\text{Im}\lambda(e^{iyh}x) \geq 0$ for $x \in W$ and $\lambda \in C^{*}$, which in turn follows from Lemma 3.3.

(ii) If $P$ is a polynomial function on $E$ and $P(D)$ the corresponding constant coefficient differential operator on $E$, then (i) applies to $P(D)\varphi \in C_{c}^{\infty}(W, \mathbb{C})$. On the other hand,
\[
(P(D)\varphi)_{y}(\lambda) = (P(D)\varphi)_{y}(-\lambda) = P(-i\lambda \circ e^{iyh})\tilde{\varphi}_{y}(-\lambda) = P(-i\lambda \circ e^{iyh})\tilde{\varphi}_{y}(\lambda).
\]
Choosing coordinates on $E$ adapted to the $h$-eigenspaces and a scalar product for which $h$ is symmetric, we can choose the polynomial $P$ in such a way that
\[
|P(-i\lambda \circ e^{iyh})| \geq 1 + \|\lambda\|^{2k} \quad \text{for all} \quad y \in [0, \pi], \lambda \in E.
\]
With (i) and (3.30), this implies (ii).

(iii) First we observe that, for $\eta \in \mathcal{V}_{K}^{\pi}$, we have
\[
e^{iy\Lambda}\eta = e^{iy\Lambda_{+}}e^{iy\Lambda_{-}}\eta = e^{iy\Lambda_{+}}\Delta_{K}^{\pi/2\pi}\eta.
\]
Since $\Lambda_{+}$ is bounded by assumption (Sect. 2.2.1) and $\Delta_{K} = \Delta_{\mathcal{V}_{K}^{\pi}}$, this vector depends continuously on $y \in [0, \pi]$. Hence (iii) follows from (ii), the temperedness of $\mu$, and the Dominated Convergence Theorem.

(iv) As in (iii), this follows from the temperedness of $\mu$ and the Dominated Convergence Theorem.  

**Lemma 3.18.** For $\varphi \in C_{c}^{\infty}(W, \mathbb{R})$ and $\eta \in \mathcal{V}_{K}^{\pi}$, we have $\tilde{\varphi}\eta \in \mathcal{V} \subseteq L^{2}(E^{*}, \mu)$. Moreover, the analytic continuation of the map $\mathbb{R} \rightarrow \mathcal{H}, t \mapsto U(0, e^{t})\tilde{\varphi}\eta$, to $S_{\pi}$ is given by
\[
z \mapsto \tilde{\varphi}_{-iz}(\lambda)e^{z\Lambda}\eta.
\]

**Proof.** Let $\eta \in \mathcal{V}_{K}^{\pi}$. We consider the map
\[
\gamma_{\eta} : S_{\pi} \rightarrow \mathcal{H}_{K}, \quad \gamma_{\eta}(z) := \left(\int_{E}\varphi(-x)K_{e^{zh}x}^{*}d\mu_{E}(x)\right) \cdot e^{z\Lambda}\eta.
\]
To see that this map is defined, we first note that, for $z \in S_{\pi}$ and $x \in -W$, Lemma 3.3 implies that $\text{Im}(e^{zh}x) = \text{Im}(e^{zh}(-x)) \in C^{0}$, so that $e^{zh}x \in T$. We therefore obtain a continuous map
\[
S_{\pi} \times (-W) \rightarrow B(K, \mathcal{H}_{K}), \quad (z, x) \mapsto K_{e^{zh}x}^{*}
\]
which is holomorphic in $z$. Integrating over the compact support of $\varphi$, thus defines a holomorphic operator-valued function
\[
F : S_{\pi} \rightarrow B(K, \mathcal{H}_{K}), \quad F(z) = \int_{E}\varphi(-x)K_{e^{zh}x}^{*}d\mu_{E}(x)
\]
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(following [GN, Thm. 2.1.12, Ex. 2.1.7]). We now have \( \gamma_k(z) = F(z)e^{zA}k \), where

\[
\mathcal{S}_\pi \to \mathcal{K}, \ z \mapsto e^{zA}k = e^{zA} + \Delta_{\mathcal{K}}^{-iz/2\pi}k
\]

is a holomorphic \( \mathcal{K} \)-valued function (Proposition 2.1). As the evaluation map

\[
B(\mathcal{K}, \mathcal{H}_k) \times \mathcal{K} \to \mathcal{H}_k
\]

is complex bilinear and continuous, this implies the holomorphy of \( \gamma_k \) on \( \mathcal{S}_\pi \).

By (3.20), \( \gamma \) satisfies the equivariance relation

\[
\gamma_k(z + t) = U'(0, e^t)\gamma_k(z) \quad \text{for} \quad t \in \mathbb{R}, \ z \in \mathcal{S}_\pi. \tag{3.33}
\]

Next we observe that \( \eta = T^\mathcal{K}_{\pi} \eta = e^{-\pi i A}J_{\mathcal{K}} \eta \) leads to

\[
J_{\mathcal{K}}e^{zA}k = e^{zA}J_{\mathcal{K}} \eta \quad \text{(2.6)} \quad \Rightarrow \quad e^{zA}k \eta = e^{(\pi + \pi i)A} \eta, \tag{3.34}
\]

so that (3.21) further yields

\[
JK^*_e e^{zh}X e^{zA}k = K^*_e e^{zh}X e^{(\pi + \pi i)A} \eta. \tag{3.34}
\]

We thus arrive at the relation

\[
J \gamma_k(z) = \int_E \varphi(-x)J K^*_e e^{zh}X e^{zA}k \, d\mu_E(x)
\]

\[
= \int_E \varphi(-x)K^*_e e^{zh}X e^{(\pi + \pi i)A} \eta \, d\mu_E(x) = \gamma_k(\pi i + z). \tag{3.35}
\]

In view of (3.33), (3.35), and Proposition 2.1(iv), it remains to show that \( \gamma_k \) extends continuously to the closed strip \( \overline{\mathcal{S}_\pi} \) with \( \gamma_k(0) = \Phi(\varphi \eta) = K^*_e \eta \quad \text{(cf. (3.29))} \), to verify that \( \varphi \eta \in \mathcal{V} \). In view of the equivariance properties (3.33) and (3.35), this boils down to showing that

\[
\lim_{y \to 0^+} \gamma_k(y i) = K^*_e \eta. \tag{3.36}
\]

To this end, we consider the \( L^2 \)-realization and observe that, for \( z \in \mathcal{S}_\pi \):

\[
\Phi^{-1}(\gamma_k(z))(\lambda) = \int_E \varphi(-x) \Phi^{-1}(K^*_e e^{zh}X e^{zA}k)(\lambda) \, d\mu_E(x)
\]

\[
= \int_E \varphi(-x)e^{-i e^{zh}X \lambda} e^{zA}k \, d\mu_E(x)
\]

\[
= \varphi(\lambda) e^{zh}X e^{zA}k = \varphi(-iz)(\lambda) e^{zA}k.
\]

As \( \Phi^{-1}(K^*_e \eta) = \varphi \eta \) by (3.29), the assertion now follows from Lemma 3.17(iii).

We are now ready to prove Theorem 3.14.
Proof of Theorem 3.14. We first observe that \( \mathcal{V}_0 := \mathcal{H}_{\mathcal{K}}^{\perp} \) is a closed real subspace of \( \mathcal{H} = L^2(E^*, \mu; \mathcal{K}) \). It is invariant under the unitary one-parameter group \( \Delta^{-it/2\pi} = U(0, e^t) \) because the cone \( W \) is invariant under \( e^{Rh} \), and \( \mathcal{V}_0^\perp \) is invariant under \( \rho(\mathbb{R}_+^*) \).

We claim that \( \mathcal{V}_0 \) is a standard subspace. From \( \mathcal{V}_0 \subseteq \mathcal{V} \) (Lemma 3.18) it follows that \( \mathcal{V}_0 \cap i\mathcal{V}_0 \subseteq \mathcal{V} \cap i\mathcal{V} = \{0\} \).

So it remains to show that \( \mathcal{V}_0 \) is total in \( \mathcal{H} \), i.e., that \( \mathcal{V}_0 + i\mathcal{V}_0 \) is dense in \( \mathcal{H} \). Let \( \mathcal{V}_0 := \mathcal{V}_0 + i\mathcal{V}_0 \) and observe that this subspace is also invariant under the operators \( U_0(0, e^t) = \Delta^{-it/2\pi}, \ t \in \mathbb{R} \). This implies that, for \( \xi \in \mathcal{V}_0 \), the range of the extended orbit map \( \alpha^\xi : S_\pi \rightarrow \mathcal{H} \) is also contained in \( \mathcal{V}_0 \).

For \( z = \pi i/2 \) and \( f = \tilde{\phi}\eta, \phi, \varphi \in C_c^\infty(W, \mathbb{R}) \), \( \eta \in \mathcal{V}_0^\perp \), we first recall from (3.29) that

\[
\Phi(\tilde{\phi}\eta) = K_{\phi^1}^\perp \eta = \gamma_\eta(0).
\]

We therefore obtain for \( \zeta = e^{\pi i h} \) by (3.35) in the proof of Lemma 3.18:

\[
\Phi(\alpha^f(\pi i/2)) = \gamma_\eta\left(\frac{\pi i}{2}\right) = \int_E \varphi(-x) K_{\zeta^{-1}(x)}^\perp e^{\pi i \Lambda} \eta d\mu_E(x).
\]

For a sequence of test functions \( \varphi_n \in C_c^\infty(W, \mathbb{R}) \) with total integral 1 whose supports converge to \( x_0 \in W \), we thus obtain

\[
\mathcal{V}_0 \ni \int_E \varphi_n(-x) K_{\zeta^{-1}}^\perp \cdot e^{\pi i \Lambda} \eta \rightarrow K_{\zeta^{-1}}^\perp e^{\pi i \Lambda} \eta,
\]

and thus \( K_{\zeta^{-1}(x_0)}^\perp e^{\pi i \Lambda} \mathcal{V}_0^\perp \subseteq \mathcal{V}_0 \) for every \( x_0 \in W \). As \( e^{\pi i \Lambda} \mathcal{V}_0^\perp \subseteq \mathcal{K}^{J\mathcal{K}} \) by (2.5) is a dense subspace, we obtain \( K_{\zeta^{-1}} e^{\pi i \Lambda} \mathcal{K}^{J\mathcal{K}} \subseteq \mathcal{V}_0 \). From

\[
\zeta^{-1}(-W) = E^+ \oplus (-i)(-C_0^0) \oplus i(-C_0^0) = E^+ \oplus i(C_0^0 \cap C_0^-) = E^+ \oplus i(C_0^- \cap E^-)
\]

and Lemma 3.11 it now follows that \( \mathcal{H}_f = \Phi^{-1}(\mathcal{H}_f^\perp) \subseteq \mathcal{V}_0 \), and this implies that \( \mathcal{V}_0 = \mathcal{H} \).

This shows that \( \mathcal{V}_0 \) is a standard subspace contained in the standard subspace \( \mathcal{V} \). As it is invariant under the modular group \( (\Delta_t^f)_{t \in \mathbb{R}} \), Lemma A.8 implies that \( \mathcal{V} = \mathcal{V}_0 \). \( \square \)

Our approach to the standard subspace \( \mathcal{V} \) also provides refined information on the tempered distribution \( \tilde{\mu} \in \mathcal{S}(E, B(\mathcal{K})) \), namely that its restriction to the wedge domain \( W \) is actually given by an operator-valued function.

Proposition 3.19. On the open wedge \( W \subseteq E \), the distribution \( \tilde{\mu} \in \mathcal{S}'(E, B(\mathcal{K})) \) is represented by the functions

\[
(\xi, \tilde{\mu}(x)\eta) = \langle e^{-\pi i \Lambda} \xi, \tilde{\mu}(i(x))e^{\pi i \Lambda} \eta \rangle \quad \text{for} \quad x \in W, \xi, \xi \in \mathcal{D}(\Delta^{-1/2}_\mathcal{K}), \eta \in \mathcal{D}(\Delta^{1/2}_\mathcal{K}).
\]

Proof. Let \( \varphi \in C_c^\infty(W, \mathbb{R}) \) and \( 0 < y < \pi \). Then Fubini’s Theorem implies that

\[
\int_W \varphi(x) \tilde{\mu}(e^{iyh} x) d\mu_E(x) = \int_W \varphi(x) \int_{C^*} e^{i\lambda(e^{iyh} x)} d\mu(\lambda) d\mu_E(x) = \int_{C^*} \int_W \varphi(x) e^{i\lambda(e^{iyh} x)} d\mu_E(x) d\mu(\lambda).
\]
Following function

This completes the proof. □

Therefore the restriction of the distribution $\tilde{\mu}_{\xi, v} := \langle \xi, \tilde{\mu} \cdot v \rangle$ to $W$ is represented by the following function

$$
\tilde{\mu}_{\xi, v}(x) = \lim_{y \to 0^+} \langle \xi, \tilde{\mu}(e^{iy}x)e^{iy}v \rangle.
$$

We now evaluate the right hand side using Lemma 3.8(b), which asserts that, for $t \in \mathbb{R}$ and $w \in T$, we have

$$
\tilde{\mu}(e^{-th} w) = e^{tA^*} \tilde{\mu}(w)e^{tA}. \tag{3.37}
$$

For $x \in W$, the element $t(x) = x_0 + it(x_1 - x_{-1})$ is contained in $T$ and $e^{-zh} t(x) \in T$ for $|\text{Im} z| < \frac{\pi}{2}$ (Lemma 3.3). Further, for $\eta \in \mathcal{D}(e^{\pi i A}) = \mathcal{D}(e^{\pi i A^-})$, the function $t \mapsto e^{tA} \eta$ extends analytically to $\mathcal{S}_N$ ([NÓ18, Lemma A.2.5]), so that, for $\xi \in \mathcal{D}(e^{-\pi i A})$, the function

$$
t \mapsto \langle \xi, \tilde{\mu}(e^{-th} t(x)) \rangle = \langle \xi, e^{tA^*} \tilde{\mu}(t(x))e^{tA} \rangle = (e^{tA} \xi, \tilde{\mu}(t(x))e^{tA} \eta)
$$

extends analytically to the function

$$
\left\{ z \in \mathbb{C} : |\text{Im} z| < \frac{\pi}{2} \right\} \to \mathbb{C}, \quad z \mapsto \langle \xi, \tilde{\mu}(e^{-zh} t(x)) \rangle = (e^{zA} \xi, \tilde{\mu}(t(x))e^{zA} \eta).
$$

From (3.37), we thus obtain for $0 < y < \frac{\pi}{2}$

$$
\langle \xi, \tilde{\mu}(e^{(y-i\frac{\pi}{2})h} t(x)) \rangle = e^{(y-i\frac{\pi}{2})A} \xi, \tilde{\mu}(t(x))e^{(y-i\frac{\pi}{2})A} \eta).
$$

Next we note that

$$
\tilde{\mu}(e^{iy}x) = \tilde{\mu}(e^{iy}x e^{i\frac{\pi}{2}} + x) = \tilde{\mu}(e^{iy}x e^{i\frac{\pi}{2}}),
$$

so that

$$
\langle \xi, \tilde{\mu}(x) \rangle = \lim_{y \to 0^+} \langle \xi, \tilde{\mu}(e^{iy}x)e^{iyA} \eta \rangle = \lim_{y \to 0^+} \langle \xi, \tilde{\mu}(e^{iy}x e^{i\frac{\pi}{2}}) \rangle
$$

$$
= \lim_{y \to 0^+} \langle e^{(y-i\frac{\pi}{2})A} \xi, \tilde{\mu}(t(x))e^{(y-i\frac{\pi}{2})A} \eta \rangle
$$

$$
= \lim_{y \to 0^+} \langle e^{(y-i\frac{\pi}{2})A} \xi, \tilde{\mu}(t(x))e^{\frac{\pi}{2}A} \eta \rangle
$$

$$
= \langle e^{-\frac{\pi}{2}A} \xi, \tilde{\mu}(t(x))e^{\frac{\pi}{2}A} \eta \rangle.
$$

This completes the proof. □
The Riesz measures on the half-line as examples. We consider the case $E = \mathbb{R}$, $C = \mathbb{R}_+ = [0, \infty)$, $h = \text{id}_E$, $\tau = -1$, and the open tube domain $\mathbb{C}_+ = T = \mathbb{R} + i(0, \infty)$. In this case we have $E = E_1$ and $W = C^0$.

The antiholomorphic extension $\overline{\tau}$ of $\tau$ to $T$ is given by $\overline{\tau}(z) = -\overline{z}$. It leaves $T$ invariant with $T^\tau = i\mathbb{R}_+^\times$. On $\mathbb{C}_+$ we have the positive definite kernels,

$$K_s(z, w) = \hat{\mu}_s(z - \overline{w}) = \mathcal{L}(\mu_s)\left(\frac{z - \overline{w}}{i}\right) = \left(\frac{z - \overline{w}}{i}\right)^{-s}, \quad s > 0,$$

(3.38)

where

$$d\mu_s(\lambda) = \Gamma(s)^{-1}\lambda^{s-1} d\lambda \quad \text{on} \quad (0, \infty) \subseteq \mathbb{C}^*.$$

see [NÓ14, Lem. 2.13].

From $h = \text{id}_E$ we derive for the action on the dual space $e^{th} = e^{-t} \lambda$, that

$$(e^{th})_*\mu_s = e^{st}\mu_s \quad \text{for} \quad t \in \mathbb{R}. \quad (3.39)$$

Comparing with (3.4), we may therefore consider this situation as arising from $K = \mathbb{C}$, $J_K(z) = \overline{z}$, and $\rho(e^t) = e^{st/2}$. This corresponds to $\Lambda = \Lambda_+ = \frac{\pi}{2}$. In particular we have, as in Example 2.4:

$$V_K = \mathbb{R}, \quad V_K^0 = e^{-\pi i s} \mathbb{R} \quad \text{and} \quad V_K^\circ = e^{\pi i s} \mathbb{R}.$$  

Finally $V^0 = V^\circ$ if and only if $s \in 2\mathbb{Z}$ (cf. Lemma 2.3).

For the Fourier–Laplace transform $\hat{\mu}_s$ we have $\hat{\mu}_s(z) = (-iz)^{-s}$ by (3.38). For the boundary values on $\mathbb{R}$, this leads for $\pm x > 0$ to

$$\hat{\mu}_s(x) = e^{-s \log(-ix)} = e^{-s \log |x| + \pi i/2} = e^{\pm s \pi i/2} |x|^{-s} = e^{\text{sgn}(x)s \pi i/2} |x|^{-s}. \quad (3.40)$$

The imaginary part is given on $\mathbb{R}^\times$ by

$$\hat{\mu}_{s,-}(x) = \pm \sin \left( s \frac{\pi}{2} \right) |x|^{-s}.$$

It vanishes if and only if $s \in 2\mathbb{Z}$. We will discuss a generalization of this phenomenon for more general Riesz measures in Sect. 6 below. For $s = 2k$, $k \in \mathbb{N}$, we have on $\mathbb{R}^\times$ the formula $\hat{\mu}_s(x) = (-1)^k x^{-2k}$, which is even and real. On $W = (0, \infty)$, we have by (3.40)

$$\hat{\mu}_s(x) = e^{s \pi i/2} x^{-s} = e^{s \pi i/2} \mathcal{L}(\mu_s)(x), \quad (3.41)$$

where $\mu_s$ is considered as a measure on $[0, \infty)$. In Proposition 4.3, we shall see a generalization of this relation to our general context.

We denote by $\mathcal{H}_s = \mathcal{H}_K$, the corresponding reproducing kernel Hilbert space. The conjugation on $\mathcal{H}_s$ is given by

$$(J f)(z) = \overline{f(-\overline{z})}$$

as in (3.18), so that

$$\mathcal{H}_s^J = \{ f \in \mathcal{H}_s : (\forall z \in \mathbb{C}_+) \ f(-\overline{z}) = f(z) \}$$

is the subspace of functions which are real-valued on $i\mathbb{R}_+^\times$. We also have

$$V = e^{-\pi i s} \{ \phi : \phi \in C^\infty_\circ((0, \infty), \mathbb{R}) \} \quad \text{and} \quad V' = e^{-\pi i s} \{ \phi : \phi \in C^\infty_\circ((-\infty, 0), \mathbb{R}) \}.$$

(3.42)
4. Reflection Positive Representations

Let us briefly recall the concept of a reflection positive representation, see [JÓl98, JÓl00, NÓl14, NÓl18, S86] for details.

A reflection positive Hilbert space is a triple \((\mathcal{E}, \mathcal{E}_+, \theta)\), where \(\mathcal{E}\) is a Hilbert space, \(\mathcal{E}_+\) is a closed subspace and \(\theta : \mathcal{E} \to \mathcal{E}\) is a unitary involution such that

\[
\|u\|_{\theta}^2 := \langle \theta u, u \rangle \geq 0 \quad \text{for all} \quad u \in \mathcal{E}_+.
\]

Then \(\mathcal{N} = \{u \in \mathcal{E}_+ : \|u\|_{\theta} = 0\}\) is a closed subspace of \(\mathcal{E}_+\), and we write \(\mathcal{E}\) for the Hilbert space completion of the quotient \(\mathcal{E}_+/\mathcal{N}\) with respect to \(\| \cdot \|_{\theta}\).

A symmetric semigroup is a triple \((G, S, \tau)\), consisting of a Lie group \(G\), an involutive automorphism \(\tau\) of \(G\), and a subsemigroup \(S \subseteq G\) invariant under the map \(g \mapsto g^\tau := \tau(g)^{-1}\). A reflection positive representation of \((G, S, \tau)\) on the reflection positive Hilbert space \((\mathcal{E}, \mathcal{E}_+, \theta)\) is a unitary representation \((U, \mathcal{H})\) of \(G\) on \(\mathcal{E}\), such that \(\theta U(g)\theta = U(\tau(g))\) for all \(g \in G\) and \(U(S)\mathcal{E}_+ \subseteq \mathcal{E}_+\). Then

\[
\langle \theta U(g)u, v \rangle = \langle \theta u, (g^\tau)v \rangle \quad \text{for} \quad g \in G, v \in \mathcal{E},
\]

and (4.1) leads to a \(*\)-representation of the involutive semigroup \((S, \#)\) by contractions on \(\mathcal{E}\) ([NÓl18, Prop. 3.3.3]). The passage from operators on \(\mathcal{E}_+\) to operators on \(\mathcal{E}\) is called the Osterwalder–Schrader transform.

In the articles cited above, \(\mathcal{E}\) and \(\mathcal{E}_+\) are complex Hilbert spaces and \(\theta\) is complex linear. In the context of standard subspaces, we encounter reflection positivity in the context of real Hilbert spaces. Any standard subspace \(\mathcal{V}\) of a complex Hilbert space \(\mathcal{H}\) satisfies by Lemma A.2 the reflection positivity condition \(\langle \xi, J\xi \rangle \geq 0\) for \(\xi \in \mathcal{V}\), so that we obtain the real reflection positive Hilbert space

\[(\mathcal{E}, \mathcal{E}_+, \theta) = (\mathcal{H}^R, \mathcal{V}, J).
\]

In this section we take a closer look at the reflection positivity of the unitary representation \(U\) of the translation group \((E, +)\) on \(\mathcal{H} = L^2(E^*, \mu; K)\), the additive subsemigroup \(S = (W, +)\) which is invariant under \(x \mapsto x^\tau = -\tau(x)\), and the standard subspace \(\mathcal{V} = H_{\mathcal{V}_K}(W)\). Here \(\mathcal{E} \cong \mathcal{H}^J\) by Proposition A.7, and we connect our description of \(\mathcal{V}\) with the Osterwalder–Schrader transform from the isometric representation of \((W, +)\) on \(\mathcal{V}\) to the \(*\)-representation of \((W, \#)\) on \(\mathcal{H}^J\).

Reflection positivity and the wedge semigroup \(W\). To see that the unitary representation \(U\) of the vector group \((E, +)\) on \(\mathcal{H} = L^2(E^*, \mu; K)\) defines a real reflection positive unitary representation of the triple \((E, W, \tau)\), we first observe that

- \(JU(v)J = U(\tau(v))\) for \(v \in E\) (Lemma 3.5),
- \(W\) is obviously invariant under the involution \(x^\tau = -\tau(x) = x_1 + x_{-1} - x_0\), hence inherits the structure of an open involutive semigroup \((W, \#)\),
- the subsemigroup \(W \subseteq E\) satisfies \(U(W)\mathcal{V} \subseteq \mathcal{V}\).

The last item easily follows from Theorem 3.14, where we have seen that

\[
\mathcal{V} = \text{span}_R \{ \tilde{\varphi} \eta : \varphi \in C_c^\infty(W, \mathbb{R}), \eta \in \mathcal{V}_{K}^\# \}.
\]
Now
\[(U(x, 1)\tilde{\varphi}\eta)(\lambda) = e^{{i\lambda(x)}}\tilde{\varphi}(\lambda)\eta = (\varphi(\cdot - x))(\lambda)\eta \tag{4.2}\]
for \(x \in E\) shows that \(\mathcal{V}\) is invariant under \(U(\hat{W})\).\(^3\) We thus obtain a reflection positive representation of \((E, W, \tau)\) on the real reflection positive Hilbert space \((\mathcal{H}_R^E, \mathcal{V}, J)\). Now the Osterwalder–Schrader transform of the representation of \(W\) on \(\mathcal{V}\) by isometries is a \(*\)-representation by contractions on the real Hilbert space \(\hat{\mathcal{E}} \cong \mathcal{H}^J\) (Proposition A.7; [NÓ18, Prop. 3.3.3]). From [Ne19a, Rem. 4.13, Prop. 3.6] we know already that the so obtained representation of \((W, \#)\) on \(\mathcal{H}^J\) coincides with the representation \(x \mapsto U(\iota(x))\), obtained from the embedding
\[
\iota : W \hookrightarrow \mathcal{T} = E + iC^0, \quad x = x_1 + x_0 + x_{-1} \mapsto ix_1 + x_0 - ix_{-1} \tag{4.3}
\]
and the holomorphic extension of \(U\) to \(\mathcal{T}\) by
\[
U(x + iy) := U(x)e^{i\partial U(y)} \quad \text{for} \quad x \in E, \quad y \in C^0, \quad \partial U(y) = \frac{d}{dt} \bigg|_{t=0} U(ty).
\]

**Remark 4.1.** From [NÓ14] we recall some of the background concerning bounded representations of the involutive semigroup \((W, \#)\). The involution \(\#\) defines on the convolution algebra \(C_c^{\infty}(W)\) the structure of a \(*\)-algebra by
\[
\varphi^{\#}(x) := \overline{\varphi(x^\#)} = (\tau_\ast \varphi^\#)(x). \tag{4.4}
\]
The corresponding Fourier–Laplace transform is
\[
\mathcal{FL}(\varphi)(\lambda) = \int_W e^{-\lambda(\cdot - x) + i\lambda^\ast(\cdot + x)} \varphi(x) \, dx \quad \text{for} \quad \varphi \in C_c^{\infty}(W). \tag{4.5}
\]
It defines a morphism of complex \(*\)-algebras \(C_c^{\infty}(W) \to C_0(\hat{W})\) (Gelfand-Transform), where
\[
\hat{W} \cong C^*_+ \times E^*_0 \times C^*_- \tag{4.6}
\]
is the cone of hermitian bounded characters of \(W\) and \(C_0(\hat{W})\) carries the canonical structure of a \(C^*\)-algebra.

For the unitary representation \((U, L^2(E^*; \mu; \mathcal{K}))\) and the positive definite distribution \(D = \hat{\mu} \in C^{-\infty}(E; B(\mathcal{K}))\), we therefore expect that the real subspace \(\mathcal{H}^J\) can be identified with an \(L^2\)-space on the character cone \(\hat{W}\) (see the Generalized Bochner Theorem, [NÓ14, Thm. 4.11]). To see which measure on the cone \(\hat{W}\) occurs here, we first observe that the projection of \(\mu\) to \(\hat{W}\) is closely related to \(\hat{\mu}\):

**Lemma 4.2.** The holomorphic function \(\tilde{\mu}(z) = \int_{C^*} e^{i\lambda(z)} \, d\mu(z)\) on \(\mathcal{T}\) defines by composition with \(\iota : W \hookrightarrow \mathcal{T}, \iota(x) = x_0 + ix_1 - ix_{-1}\), a positive definite analytic function on \((W, \#)\) which can be expressed as a Fourier–Laplace transform by
\[
\hat{\mu} \circ \iota = \mathcal{FL}(p_\ast \mu), \tag{4.7}
\]
where
\[
p : C^* \to \hat{W}, \quad p(\lambda_1 + \lambda_0 + \lambda_{-1}) = \lambda_1 + \lambda_0 - \lambda_{-1}
\]
is a slightly modified restriction map.

\(^3\) Alternatively, this can also be derived from the Borchers–Wiesbrock Theorem; see [NÓ17] and [Ne19a, Thm. 4.1].
\textbf{Proof.} This follows directly from the definition of $\tilde{\mu}$ in Lemma 3.8:

$$\tilde{\mu}(i(x_1 + x_0 + x_{-1})) = \tilde{\mu}(ix_1 + x_0 - ix_{-1}) = \int_{C^*} e^{-\lambda(x_1 + i\lambda(x_0) + \lambda(x_{-1})} d\mu(\lambda) \quad \text{for} \quad x \in W.$$\]

\hfill $\Box$

For $\varphi \in C_c^\infty(W, \mathbb{R})$ and $\eta \in V_K^\#$, we observe that

$$J(\tilde{\varphi}\eta) = ((-\tau)_* \tilde{\varphi}) J_K \eta = (\tau_* \varphi) \tilde{\mu} \cdot J_K \eta.$$

(4.8)

Accordingly, we have for the positive definite distribution $D = \hat{\mu}$ the relation:

$$J(\varphi * D \eta) = (\tau_* \varphi) * D J_K \eta.$$ \hfill (4.9)

This leads to

$$\langle \varphi * D \eta, J(\psi * D \xi) \rangle = \langle \eta, D(\tau_* \psi) * \varphi J_K \xi \rangle \quad \text{for} \quad \varphi, \psi \in C_c^\infty(W, \mathbb{R}), \xi, \eta \in V_K^\#.$$ \hfill (4.10)

Equation (4.10) expresses a reflection positivity condition for the distribution $D = \hat{\mu}$ with respect to the involutive semigroup $(W, \#)$.

We also know from Proposition 3.19 that

$$\langle \xi, \tilde{\mu}(x) \eta \rangle = \langle e^{-\pi i \Lambda / \sharp} \xi, \tilde{\mu}(i(x)) e^{\pi i \Lambda / \sharp} \eta \rangle \quad \text{for} \quad \xi \in V_K^\#, \eta \in V_K^\#.$$

For $\xi \in V_K^\#$ we have $J_K \xi \in V_K^\#$, so that this leads to

$$\langle J_K \xi, \tilde{\mu}(x) \eta \rangle = \langle e^{-\pi i \Lambda / \sharp} J_K \xi, \tilde{\mu}(i(x)) e^{\pi i \Lambda / \sharp} \eta \rangle. \quad \text{(4.11)}$$

Since $T_K^\# = e^{-\pi i \Lambda} J_K$ by (2.9), (2.7) implies

$$J_K \xi = e^{\pi i \Lambda} \xi \quad \text{for} \quad \xi \in V_K^\#.$$\hfill

Using

$$\tilde{\mu}(i(x)) = \mathcal{F} \mathcal{L}(p_\# \mu)(x) \quad \text{for} \quad x \in W$$\hfill

from Lemma 4.2, we obtain with (4.11):

\textbf{Proposition 4.3.} On the open cone $W \subseteq E$, the distribution $\tilde{\mu}$ is given by a density, an analytic operator-valued function determined by

$$\langle J_K \xi, \tilde{\mu}(x) \eta \rangle = \langle e^{\pi i \Lambda / \sharp} \xi, \tilde{\mu}(i(x)) e^{\pi i \Lambda / \sharp} \eta \rangle = \langle e^{\pi i \Lambda / \sharp} \xi, \mathcal{F} \mathcal{L}(p_\# \mu)(x) e^{\pi i \Lambda / \sharp} \eta \rangle \quad \text{for} \quad \xi, \eta \in V_K^\#, x \in W.$$
For \( \varphi, \psi \in C^\infty_c(W, \mathbb{R}) \) and \( \xi, \eta \in V^*_K \), the preceding proposition leads in particular to

\[
\langle \tilde{\psi}\xi, J(\tilde{\varphi}\eta) \rangle = \langle \tilde{\psi}\xi, (\tau_\alpha \psi)\tilde{J}_K\eta \rangle = \langle \tilde{\xi}, \psi(\tau_\alpha \varphi)\tilde{J}_K\eta \rangle = \langle \tilde{\xi}, \psi^*(\tau_\alpha \varphi)\tilde{J}_K\eta \rangle
\]

\[
= \langle \tilde{\xi}, (\psi^* \ast \tau_\alpha \varphi)\tilde{J}_K\eta \rangle = \langle \tilde{\xi}, \tilde{\mu}(\psi^* \ast \tau_\alpha \varphi)\tilde{J}_K\eta \rangle
\]

\[
= \langle \tilde{\xi}, J_K\tilde{\mu}(\psi^* \varphi)\eta \rangle = \langle J_K\xi, \tilde{\mu}(\psi^* \varphi)\eta \rangle
\]

\[
= \int_W (\tilde{\psi}^* \varphi)(x)\langle e^{\frac{\pi i}{2}\Lambda}x, J(x, \tilde{\mu}(\psi^* \varphi)\eta) \rangle. \tag{4.12}
\]

As \( V^*_K = e^{-\frac{\pi i}{4}\Lambda}V_K \), we have

\[
e^{\frac{\pi i}{2}\Lambda}V^*_K = e^{\frac{\pi i}{2}\Lambda}V_K = \Lambda^{1/4}V_K \subseteq K \cdot J\mathcal{K}.
\]

Therefore the above formula contains the main information of the Osterwalder–Schrader transform that passes from the \( J \)-twisted scalar product on \( V \) to the real scalar product on \( \mathcal{H}^J \).

5. Support Properties of the Imaginary Part of \( \tilde{\mu} \)

For \( D = \tilde{\mu} \in S'(E; B(K)) \), we consider the Hilbert space \( \mathcal{H} = \mathcal{H}_D \subseteq S'(E; K) \) (Definition 3.12). Then, as before, for an open subset \( \mathcal{O} \subseteq E \) and a closed real subspace \( K \subseteq \mathcal{K} \), we define the closed real subspaces

\[
\mathcal{H}_K(\mathcal{O}) := \overline{\text{span}}_{\mathbb{R}} [\varphi \ast D\eta : \varphi \in C^\infty_c(\mathcal{O}, \mathbb{R}), \eta \in K]. \tag{5.1}
\]

In Theorem 3.14, we have seen that, for the open wedge \( W \subseteq E \), the subspace \( \mathcal{H} := \mathcal{H}_{V^*_W}(W) \) is standard. By Corollary 3.16 its symplectic complement is \( \mathcal{H}' = \mathcal{H}_{V^*_W}(-W) \).

Furthermore, by Lemma 2.3 we have \( V^*_K = V^*_K \) if and only if \( \Lambda = \Lambda_+ \) has spectrum contained in \( \mathbb{Z} \).

For real-valued test functions \( \varphi \), we have

\[
D(\varphi) = \int_{E^*} \tilde{\varphi} d\mu = \int_{E^*} \tilde{\varphi} d\mu
\]

and thus

\[
D(\varphi)^* = \left( \int_{E^*} \varphi(\alpha) d\mu(\alpha) \right)^* = \int_{E^*} \varphi(\alpha) d\mu(\alpha) = \int_{E^*} \varphi(-\alpha) d\mu(\alpha) = \int_{E^*} \varphi(\alpha) d\mu(-\alpha). \tag{5.2}
\]

Hence the operators \( D(\varphi), \varphi \in C^\infty_c(E, \mathbb{R}) \), are all hermitian if and only if \( \mu \) is invariant under the reflection \( r(x) = -x \) in the sense that \( r_\ast \mu = \mu \). As \( \text{supp}(\mu) \subseteq \mathcal{C}^* \) and \( C \) is generating, this condition implies \( \text{supp}(\mu) = \{0\} \).

In general, the distribution \( D \) decomposes as

\[
D = D_+ + i D_-,
\]

where \( D_+ \) is the Fourier transform of the measure \( \frac{1}{2} (\mu + r_\ast \mu) \), and \( D_- \) is the Fourier transform of \( \frac{1}{2i} (\mu - r_\ast \mu) \).
Lemma 5.1. The distributions $D_{\pm}$ are hermitian in the sense that
\[ D_{\pm}(\varphi) \in \text{Herm}(K) \quad \text{for} \quad \varphi \in S(E, \mathbb{R}). \] (5.3)

Furthermore, if $\varphi \in C_c^\infty(E, \mathbb{R})$, then
\[ D(\varphi \circ (-\tau)) J_K = J_K D(\varphi)^* \quad \text{and} \quad D_{\pm}(\varphi \circ (-\tau)) J_K = J_K D_{\pm}(\varphi). \]

Proof. That $D_{\pm}(\varphi)^* = D_{\pm}(\varphi)$ follows from (5.2) and the relation $r_\ast \mu_{\pm} = \pm \mu_{\pm}$.

For $\varphi \in C_c^\infty(E, \mathbb{R})$, we have
\[ D(\varphi \circ (-\tau)) = \int_{E^*} \overline{\varphi} \, d((\tau)_* \mu) = \int_{E^*} \overline{\varphi} \, J_K d\mu J_K = J_K \left( \int_{E^*} \overline{\varphi} \, d\mu \right) J_K \] which by the first part implies that $D_{\pm}(\varphi \circ (-\tau)) = J_K D_{\pm}(\varphi) J_K$.

Corollary 5.2. If $\mu$ is invariant under the involution $-\tau$ and $\varphi = \varphi^\pm$, then the hermitian operators $D_{\pm}(\varphi)$ commute with $J_K$.

Remark 5.3. By (3.4), $\mu$ is invariant under $-\tau$ if and only if all values of the measure $\mu$ commute with $J_K$.

To explore the support properties of $D_-$, we observe that, for $\varphi, \psi \in C_c^\infty(E, \mathbb{R})$, by (5.3)
\[ \omega(\varphi * D_\xi, \psi * D_\xi) = \text{Im} \langle \xi, D(\psi^* \varphi)\xi \rangle = \langle \xi, D_-(\psi^* \varphi)\xi \rangle = \langle \xi, (\psi * D_-)(\varphi)\xi \rangle. \] (5.4)

Proposition 5.4. Assume that $\mathbb{V}_\mathcal{K}^\pm = \mathbb{V}_\mathcal{K}^\pm$. For $\xi \in \mathbb{V}_\mathcal{K}^\pm$, define $D_\xi(-\varphi) = \langle \xi, D_-(\varphi)\xi \rangle$. Then $\text{supp} D_\xi \subseteq W^c$.

In particular, if $\mathcal{K} = \mathbb{C}$ with $J_K(z) = \overline{z}$ and $\Lambda \in \mathbb{Z}i$, we have $\text{supp} D_- \subseteq W^c$. As the examples in Sect. 6 (with $\mathcal{K} = \mathbb{C}$) show, we do not expect substantial restrictions on the support of the distributions $D_-$ if $\mathbb{V}_\mathcal{K}^\pm = \mathbb{V}_\mathcal{K}^\pm$, i.e., if $\text{Spec}(\Lambda) \not\subseteq \mathbb{Z}$.

Proof. By (5.4) and the fact that $H_D(W, \mathbb{V}_\mathcal{K}^\pm)^{1,\omega} = H_D(-W, \mathbb{V}_\mathcal{K}^\pm)$ under our assumption $\mathbb{V}_\mathcal{K}^\pm = \mathbb{V}_\mathcal{K}^\pm$, it follows that the real-valued distribution $D_\xi$ satisfies
\[ \text{supp}(\psi * D_\xi) \subseteq W^c := E \setminus W \quad \text{for} \quad \psi \in C_c^\infty(-W). \]

As $C_c^\infty(-W, \mathbb{R})$ contains an approximate identity $\psi_n \rightarrow \delta_0$, we obtain from
\[ \text{supp}(\psi_n * D_\xi) \subseteq W^c \quad \text{for every} \quad n, \]
that $\text{supp}(D_\xi) \subseteq W^c$. □

Example 5.5. In the Lorentz context $E = \mathbb{R}^{1,d-1}$, where $\mu$ is a Lorentz invariant scalar-valued measure ($\mathcal{K} = \mathbb{C}$), the distribution $D_-$ is also Lorentz invariant, hence in particular invariant under the rotation group $\text{SO}_{d-1}(\mathbb{R})$. Therefore
\[ \text{supp}(D_-) \subseteq \bigcap_{g \in \text{SO}_{d-1}(\mathbb{R})} gW^c = C \cup -C, \]
so that $\text{supp}(D_-)$ is contained in the closed double light cone. We refer to [RS75, §X.7, p. 215] for a different derivation of this result from concrete information on the nature of the distribution $\hat{\mu}$. 


6. The Fourier Transform of Riesz Measures

In this section we specialize the setting of Sect. 3 to simple euclidean Jordan algebras. We briefly recall the relevant concepts. A Jordan algebra is a, not necessarily associative, algebra $E$ such that the product satisfies

$$xy = yx \quad \text{and} \quad x(x^2y) = x^2(xy) \quad \text{for all} \quad x, y \in E.$$ 

We then define

$$L(x)y = xy \quad \text{and} \quad P(x) = 2L(x)^2 - L(x^2).$$

$P$ is called the quadratic representation of $E$. We always assume that $E$ has an identity $e$, which means that $L(e) = \text{id}_E$.

A Jordan algebra $E$ over $\mathbb{R}$ is called euclidean if there exists an inner product $(\cdot, \cdot)$ on $E$ such that $L(x)$ is symmetric for all $x \in E$. If $E$ is euclidean, then the interior $C^0$ of the closed convex cone $C = \{x^2 : x \in E\}$ of squares in $E$ is an open symmetric cone. It is the connected component of $e$ in the set $E^\times$ of invertible elements in $E$, as well as the set of all $x \in E$ such that $L(x)$ is strictly positive ([FK94, Thm. III.2.1]).

An element $c$ in $E$ is idempotent if $c^2 = c$. The idempotent $c$ is primitive if it cannot be written as a sum of two non-zero idempotents. The idempotents $c_1, \ldots, c_r$ form a Jordan frame if each $c_j$ is primitive, $c_i c_j = 0$ if $i \neq j$, and $e = c_1 + \cdots + c_r$. Jordan frames always exist and the group $\text{Aut}(E)$ of unital automorphisms of $E$ acts transitively on the set of Jordan frames ([FK94, Cor. IV.2.7]). In particular, the number $r$ of elements in a Jordan frame is independent of the frame. It is called the rank of $E$.

Example 6.1. (a) Minkowski space $E = \mathbb{R}^{1,d-1}$ carries the structure of a euclidean Jordan algebra of rank $r = 2$ which is simple for $d \neq 2$. The product is given by

$$(x_0, x)(y_0, y) := (x_0y_0 + xy, x_0y + y_0x).$$

Here $e = (1, 0)$ is a unit element and

$$c_1 = \frac{1}{2}(1, 1, 0, \ldots, 0), \quad c_2 = \frac{1}{2}(1, -1, 0, \ldots, 0)$$

form a Jordan frame.

(b) The other simple euclidean Jordan algebras of rank $r$ are

$$\text{Sym}_r(\mathbb{R}), \quad \text{Herm}_r(\mathbb{C}), \quad \text{Herm}_r(\mathbb{H}),$$

for $r \in \mathbb{N}$ and $\text{Herm}_3(\mathbb{O})$,

where $\mathbb{O}$ is the alternative algebra of octonions (see [JvNW34,FK94] for the classification). Here the Jordan product is given by

$$x * y := \frac{xy + yx}{2},$$

the euclidean form is $(x, y) = \text{tr}(xy)$, the identity matrix $e = 1$ is the unit, and the diagonal matrices $c_j := E_{jj}$ form a Jordan frame.
In this section \( E \) is a simple euclidean Jordan algebra of rank \( r \) whose Pierce subspaces (B.2) are of dimension \( d \). For \( z \in E_{\mathbb{C}} \) (the complexified Jordan algebra), we define the Jordan determinant by

\[
\Delta(z) = \det(L(z)|_{\mathbb{C}[z]}),
\]

where \( \mathbb{C}[z] \subseteq E \) is the unital subalgebra generated by \( z \).

For

\[
s \in \left\{ 0, \ldots, (r - 1)\frac{d}{2} \right\} \cup \left( (r - 1)\frac{d}{2}, \infty \right),
\]

we consider the corresponding Riesz measure \( \mu_{s} \) whose Fourier (Laplace) transform satisfies

\[
\tilde{\mu}_{s}(z) = \Delta(-iz)^{-s} \quad \text{for} \quad z \in E + iC^{0}
\]

([FK94, Thm. VII.3.1]).

**Remark 6.2.** The structure group \( \text{Str}(E) \) is the group of all \( g \in \text{GL}(E) \) such that for \( x \in E \), we have \( P(gx) = gP(x)g^{\top} \) where \( P(x) = 2L(x)^{2} - L(x^{2}) \) is the quadratic representation of \( E \). The structure group contains the automorphism group

\[
G(C^{0}) = \{ g \in \text{GL}(E) : gC^{0} = C^{0} \}
\]

of the open cone \( C^{0} \) as a subgroup of index 2.

For \( g \in \text{Str}(E) \), and \( x, y \in E \), we have

\[
\Delta(gx) = \det(g)^{r/n} \Delta(x) \quad \text{and} \quad \Delta(P(y)x) = \Delta(y)^{2}\Delta(x) \quad \text{for} \quad g \in \text{Str}(E), x, y \in E
\]

([FK94, Prop. III.4.2]). It follows that \( \mu_{s} \) and its Fourier transform are semi-invariant under the identity component \( \text{Str}(E)_{0} \) of the structure group, so that the support of real and imaginary part are closed unions of orbits of this group. More concretely, \( \mathcal{L}(g_{\ast}\mu_{s}) = g_{\ast}\mathcal{L}(\mu_{s}) \) and

\[
(g_{\ast}\mathcal{L}(\mu_{s}))(z) = \mathcal{L}(\mu_{s})(g^{-1}z) = \Delta(g^{-1}z)^{-s} = |\det(g)|^{rs/n}\Delta(z)^{-s} = |\det(g)|^{rs/n}\mathcal{L}(\mu_{s})(z)
\]

imply that

\[
g_{\ast}\mu_{s} = |\det(g)|^{rs/n}\mu_{s}. \quad (6.2)
\]

**Proposition 6.3.** The imaginary part of the tempered distribution \( \tilde{\mu}_{s} \) vanishes on the connected component \( E_{j}^{\times} = \{ x \in E^{\times} : \text{ind}(x) = j \} \) of the set \( E^{\times} \) of invertible elements of \( E \) if and only if \( sj \in 2\mathbb{Z} \).

**Proof.** As a distribution on \( E \), the Fourier transform \( \tilde{\mu}_{s} \) is given on the open subset \( E^{\times} \) of invertible elements by the limit

\[
\tilde{\mu}_{s}(x) = \lim_{y \to 0, y > 0} \Delta(-ix + y)^{-s}.
\]

For \( s \geq 0 \) and \( x \in \mathbb{R}^{\times} \) with \( \pm x > 0 \), we have

\[
\lim_{y \to 0^{+}} (-ix + y)^{-s} = e^{-s\log(-ix)} = e^{-s\log|x| \mp \frac{\pi i}{2}} = |x|^{-s} e^{\pm \frac{i\pi}{2}}.
\]

(6.3)
As $\Delta(x) = \prod_{j=1}^{r} x_j$ is the product of the spectral values of $x$, this leads to

$$\tilde{\mu}_s(x) = |\Delta(x)|^{-s} e^{i \text{ind}(x) \frac{rs}{2}} \quad \text{for} \quad x \in E^\times. \quad (6.4)$$

It follows in particular that

$$\tilde{\mu}_s(x) \in \mathbb{R} \iff s \, \text{ind}(x) \in 2\mathbb{Z}. \quad (6.5)$$

This completes the proof. \qed

**Remark 6.4.** (a) (Analogy with Huygens principle) Proposition 6.3 shows that the support properties of $\tilde{\mu}_s$ depend crucially on the parity of the rank $r$. If $r$ is even, then there exist invertible elements of index 0 and

$$\text{supp}(\text{Im}\tilde{\mu}_s) \cap E_0^\times = \emptyset.$$ 

If $r$ is odd, then $\text{ind}(x)$ is odd for every invertible element, so that there exist parameters $s$ as in (6.1), for which $\text{Im}(\tilde{\mu}_s)$ has full support.

(b) The Riesz measures $\mu_s$ satisfy the differential equation

$$\Delta(\partial) \mu_s = \mu_{s-1} \quad ((FK94, \text{Thm. VII.2.2})],$$

so that $s \in \mathbb{N}$ implies

$$\Delta(\partial)^s \mu_s = \delta_0,$$

i.e., $\mu_s$ is a fundamental solution of the differential operator $\Delta(\partial)^s$ of order $rs$. This relation also provides information on the Fourier transform:

$$\Delta(-ix)^s \tilde{\mu}_s(x) = 1.$$ 

As $\Delta$ is homogeneous of degree $r$, this can also be written as

$$(-i)^{rs} \Delta(\partial)^s \tilde{\mu}_s = 1. \quad (6.6)$$

As a consequence, $\text{supp}(\tilde{\mu}_s) = E$ for $s \in \mathbb{N}_0$.

**Example 6.5.** (a) $r = 2$ (Minkowski space of dimension $n = d + 2$). Then the admissible positive values of $s$ are given by $s \geq \frac{d}{2} = \frac{n+2}{2}$ and the possible values of the index are 2, 0, $-2$.

As $\text{supp}(\text{Im}\tilde{\mu}_s) \cap E_0^\times = \emptyset$ (Proposition 6.3), the support is always contained in the closed double cone $C \cup -C$. It is contained in the boundary of the closed double cone if and only if $s \in \mathbb{Z}$.

For $n = 4$, resp., $d = 2$, we have $s \geq 1$. For $s = 1$ the distribution $\text{Im}(\tilde{\mu}_s)$ is supported in the boundary of the double cone.

(b) $\text{Im}(\tilde{\mu}_s)$ is supported in the complement of the open double cone $C^0 \cup -C^0$ if and only if $rs \in 2\mathbb{Z}$ (Proposition 6.3).

(c) We consider the case $s = \frac{d}{2}$ (the minimal positive value). Then $\text{Im}(\tilde{\mu}_s)$ vanishes on $E_j^\times$ if and only if $jd \in 4\mathbb{Z}$ (Proposition 6.3). For $d = 1$, this means that $j \in 4\mathbb{Z}$ which can only happen for $r \in 2\mathbb{Z}$. Here the Hilbert space is the even part of the Fock space, carrying the metaplectic representation of the 2-fold covering group $Mp_{2r}(\mathbb{R})$ of $\text{Sp}_{2r}(\mathbb{R})$ ([HNØ96, Sect. V]).
**Remark 6.6 (Locality condition).** The closed convex cone $C \subseteq E$ defines an order structure on $E$. In terms of this order, the requirement on the distribution $\text{Im} \tilde{\mu}_s$ that corresponds in the case of Minkowski space (Example 6.5(a)) to the locality condition of corresponding quantum fields is

$$\text{supp}(\text{Im} \tilde{\mu}_s) \subseteq C \cup -C. \quad (6.7)$$

If this condition is satisfied, then $\Delta(-ix)^j$ is real on the components $E_j^\times$, for $j \neq \pm r$. This is equivalent to $sj \in 2\mathbb{Z}$ for $j = r - 2, r - 4, \ldots, 2 - r$ (Proposition 6.3). For $r \geq 3$ odd, this implies for $j = 1$ that $s \in 2\mathbb{N}_0$, and if $r \geq 4$ is even, we obtain for $j = 2$ that $s \in \mathbb{N}$. In both cases we obtain with Proposition 6.3 that $\text{supp}(\text{Im} \tilde{\mu}_s) \cap E_j^\times = \emptyset$. In Proposition 6.10 below we show that $\text{supp}(\text{Im} \tilde{\mu}_s) = E \setminus E_j^\times$ in this case. In particular, the locality condition (6.7) is never satisfied for $r \geq 3$ because there exist non-invertible elements $x \notin \pm C$, i.e., at least one spectral value is positive and another one negative.

### 6.1. The Case $E = \mathbb{R}$. **Proof.** 

We now specialize the result from the previous section to the case $E = \mathbb{R}$ considered in Sect. 3.3. We have already seen in the introduction how this related to the $U(1)$-current in CFT. In this case the Riesz measures are given by

$$d\mu_s(x) = \Gamma(s)^{-1} x^{s-1} \, dx \quad \text{on} \quad C^* = [0, \infty) \quad \text{for} \quad s > 0$$

and by $\mu_0 := \delta_0$ (Dirac measure in 0). For the Fourier–Laplace transform $\tilde{\mu}_s$ we have $\tilde{\mu}_s(z) = (-iz)^{-s}$ for $\text{Im} z > 0$. For the boundary values on $\mathbb{R}^\times$, we obtain with (6.3)

$$\tilde{\mu}_s(x) = e^{\pm s \frac{\pi i}{2}} |x|^{-s} \quad \text{and} \quad \text{Im} \tilde{\mu}_s(x) = \pm \sin \left( s \frac{\pi}{2} \right) |x|^{-s}. \quad (6.8)$$

This shows that $\text{Im} \tilde{\mu}_s$ vanishes on $\mathbb{R}^\times$ if and only if $s \in 2\mathbb{Z}$.

**Lemma 6.7.** For $E = \mathbb{R}$, the following assertions hold:

(i) $\text{supp}(\text{Im} \tilde{\mu}_s) = \mathbb{R}$ for $s \notin 2\mathbb{Z}$.

(ii) $\text{supp}(\text{Im} \tilde{\mu}_s) = \mathbb{R}$ and $\text{Re} \tilde{\mu}_s = c_s \delta_0^{(s-1)}$ with $c_s \in \mathbb{R}^\times$ if $s \in \mathbb{N}$ is odd.

(iii) $\text{supp}(\text{Re} \tilde{\mu}_s) = \mathbb{R}$ and $\text{Im} \tilde{\mu}_s = c_s \delta_0^{(s-1)}$ with $c_s \in \mathbb{R}^\times$ if $s \in \mathbb{N}$ is even.

**Proof.** (i) follows from (6.8).

(ii), (iii) For $z \in \mathbb{C}_+$ we have $\tilde{\mu}_s(z) = (-iz)^{-s}$. Taking derivatives, we get

$$\tilde{\mu}_s(z) = is(-iz)^{-s-1} = is \cdot \tilde{\mu}_{s+1}(z),$$

so that we obtain for the boundary values in $S'(\mathbb{R})$

$$\tilde{\mu}_{s+1} = \frac{1}{is} \tilde{\mu}'_s \quad \text{for} \quad s > 0. \quad (6.9)$$

As $s \in \mathbb{N}$, we have to take a closer look at $\mu_1$. We know already that $\text{supp}(\text{Im} \tilde{\mu}_1) = \mathbb{R}$. We claim that $\text{Re} \tilde{\mu}_1 = \pi \delta_0$. In fact, if the real test function $\varphi$ vanishes in 0, then $\tilde{\mu}_1(z) = iz^{-1}$ for $\text{Im} z > 0$ yields

$$\tilde{\mu}_1(\varphi) = \lim_{\varepsilon \to 0} i \int_{\mathbb{R}} \frac{\varphi(x)}{x + i\varepsilon} \, dx = i \int_{\mathbb{R}} \frac{\varphi(x)}{x} \, dx \in i\mathbb{R},$$
so that $\Re \tilde{\mu}_1(\varphi) = 0$. We conclude that $\Re \tilde{\mu}_1 = c_1 \delta_0$ for some $c_1 \in \mathbb{R}$. To evaluate this constant, we consider a test function $\varphi$ constant on an interval $[-\delta, \delta]$ for $\delta > 0$. Then

$$c_1 \varphi(0) = \Re \lim_{\varepsilon \to 0^+} i \int_{\mathbb{R}} \frac{\varphi(x)}{x + i \varepsilon} \, dx = \Re \lim_{\varepsilon \to 0^+} i \int_{-\delta}^{\delta} \frac{\varphi(x)}{x + i \varepsilon} \, dx$$

$$= \Re \left( i \varphi(0) \lim_{\varepsilon \to 0} \int_{-\delta}^{\delta} \frac{1}{1 + i \varepsilon} \, dx \right).$$

This integral is easily evaluated using the holomorphic logarithm on $\mathbb{C} \setminus (-\infty, 0]$: 

$$\int_{-\delta}^{\delta} \frac{1}{1 + i \varepsilon} \, dx = \log(\delta + i \varepsilon) - \log(-\delta + i \varepsilon) = \log(\delta + i \varepsilon) - \pi i - \log(\delta - i \varepsilon)$$

which tends to $-\pi i$ for $\varepsilon \to 0$. We thus obtain $c_1 = \pi$. This shows that $\Re \tilde{\mu}_1 = \pi \delta_0$.

With the recursion formula (6.9), we obtain

$$\Re \tilde{\mu}_s = \frac{(-1)^k \pi}{2k(2k - 1) \cdots 1} \delta_0^{(2k)} = \frac{(-1)^k \pi}{(2k)!} \delta_0^{(2k)}$$

for $s = 1 + 2k \in 1 + 2\mathbb{N}_0$, and

$$\Im \tilde{\mu}_s = \frac{(-1)^{k+1} \pi}{(2k - 1)(2k - 2) \cdots 2} \delta_0^{(2k+1)} = \frac{(-1)^{k+1} \pi}{(2k - 1)!} \delta_0^{(2k+1)}$$

for $s = 2k, k \in \mathbb{N}$.

The remaining assertions now follow immediately from the recursion formula (6.9). 

\[ \square \]

6.2. The Generalization to $E = \mathbb{R}^r$. In this subsection we extend the results from the last section to the euclidean space $E = \mathbb{R}^r$. It turns out that, as for the Huygens principle, there is a fundamental difference between $r$ even and odd. We have already noted above that, for $r = 1$, this case relates to the U(1)-current in CFT. For $r = 2$, it relates to 2-dimensional Minkowski case, considered as a Jordan algebra, where coordinates refer to a Jordan frame $(c_1, c_2)$, consisting of future pointing lightlike vectors. For $r > 2$, it corresponds to the diagonal matrices in the Jordan algebras $\text{Herm}_r(\mathbb{K})$.

**Proposition 6.8.** Let $E = \mathbb{R}^r$ with pointwise multiplication and $s \in \mathbb{N}$. If either $r$ is even, or if $r$ is odd and $s$ is even, then

$$\text{supp}(\text{Im} \tilde{\mu}_s) = E \setminus E^\times.$$

**Proof.** $r$ even:

Proposition 6.3 implies that $\Im \tilde{\mu}_s$ vanishes on $E^\times$ because ind$(x) \in 2\mathbb{Z}$. It remains to show that any element of the form $x = (x, 0)$ with $x$ invertible in $\mathbb{R}^{r-1}$ is contained in $\text{supp}(\text{Im} \tilde{\mu}_s)$. We use the relation $\tilde{\mu}_s(x, x_r) = \tilde{\mu}_s^{\mathbb{R}^{r-1}}(x)\tilde{\mu}_1^R(x_r)$ in the sense of distributions. If $\varphi_1 \in C^\infty_c((\mathbb{R}^{r-1})^\times)$ and $\varphi_2 \in C^\infty_c(\mathbb{R})$, then

$$\tilde{\mu}_s(\varphi_1 \otimes \varphi_2) = \tilde{\mu}_s^{\mathbb{R}^{r-1}}(\varphi_1) \tilde{\mu}_1^R(\varphi_2) \in i^s \mathbb{R}$$

(6.10)
by (6.4) because \( \text{ind}(x) \in r - 1 + 2\mathbb{Z} \) is odd. If \( s \) is even, this leads to

\[
\text{Im} \tilde{\mu}_s(\varphi_1 \otimes \varphi_2) = \tilde{\mu}_s^{r-1}(\varphi_1) \cdot \text{Im} \tilde{\mu}_s^{\mathbb{R}}(\varphi_2) = \tilde{\mu}_s^{r-1}(\varphi_1) c_s(-1)^{r-1} \varphi_2^{(s-1)}(0)
\]

(Lemma 6.7(iii)), so that \((x, 0) \in \text{supp}(\text{Im} \tilde{\mu}_s)\). If \( s \) is odd, (6.10) leads to

\[
\text{Im} \tilde{\mu}_s(\varphi_1 \otimes \varphi_2) = \text{Im} \tilde{\mu}_s^{\mathbb{R}}(\varphi_1) \text{Re} \tilde{\mu}_s^{\mathbb{R}}(\varphi_2) = \tilde{\mu}_s^{r-1}(\varphi_1) c_s(-1)^{r-1} \varphi_2^{(s-1)}(0)
\]

(Lemma 6.7(ii)), so that \((x, 0) \in \text{supp}(\text{Im} \tilde{\mu}_s)\). Using the invariance of \( \tilde{\mu}_s \) under permutations of the coordinates, the assertion follows.

**r odd:** Then \( s \) is even by assumption, so that Proposition 6.3 implies that \( \text{Im} \tilde{\mu}_s \) vanishes on \( E^\times \). The same argument as in the case where \( r \) is even now shows that \( \text{supp}(\text{Im} \tilde{\mu}_s) = E \setminus E^\times \). □

6.3. The Support of the Fourier Transform of Riesz Measures. Let \((c_1, \ldots, c_r)\) be a Jordan frame in \( E \) and \( E_0 := \text{span}\{c_1, \ldots, c_r\} \) denote the corresponding euclidean Jordan subalgebra isomorphic to \( \mathbb{R}^r \), endowed with the componentwise multiplication. We write \( p : E^* \to E_0^* \) for the restriction map and observe that this map is proper on \( C^* \) with \( p(C^*) = (C_0)^g \cong (\mathbb{R}_+)^r \), where \( C_0 := E_0 \cap C \) is the closed positive cone in \( E_0 \). For a Riesz measure \( \mu_s \), the measure

\[
\mu_s^0 := p_\ast \mu_s
\]

then satisfies

\[
\mathcal{L}(\mu_s)(z) = \Delta(z)^{-s} = \Delta_0(z)^{-s} = (z_1 \cdots z_r)^{-s} = \mathcal{L}(\mu_s^0)(z) \quad \text{for} \quad z = z_j c_j \in E_0 + i C_0^0.
\]

To transfer information on the support of \( \text{Im} (\tilde{\mu}_s) \) from \( E_0 \) to \( E \), we need the following lemma.

**Lemma 6.9.** Let \( E_0 \subseteq E \) be a real subspace, let \( p : E^* \to E_0^* \) be the restriction map, and \( \mu \) be a tempered measure on \( E^* \) for which \( \mu_0 := p_\ast \mu \) is also tempered. Then

\[
\text{supp}(\text{Im} \tilde{\mu}_0) \subseteq \text{supp}(\text{Im} \tilde{\mu}).
\]

**Proof.** Pick a vector space complement \( E_1 \subseteq E \) for \( E_0 \), so that \( E = E_0 \oplus E_1 \). For \( x_0 \in \text{supp}(\text{Im} \tilde{\mu}_0) \) and an open neighborhood \( U_0 \) of \( x_0 \), there exists a real test function \( \varphi_0 \) on \( E_0 \) with

\[
0 \neq \text{Im} \tilde{\mu}_0(\varphi_0) = \text{Im} \int_{E_0^*} \overline{\varphi_0(-\lambda_0)} d\mu_0(\lambda_0) = \text{Im} \int_{E_0^*} \varphi_0(\lambda_0) d\mu_0(\lambda_0) = \text{Im} \int_{E_1} \varphi_0(\lambda_0) d\mu(\lambda_0, \lambda_1),
\]

where the existence of the integral follows from the temperedness of \( \mu_0 = p_\ast \mu \).

Now let \( (\delta_n)_{n \in \mathbb{N}} \) be a \( \delta \)-sequence in \( C_c^\infty(E_1, \mathbb{R}) \), i.e.,

\[
supp(\delta_n) \to [0], \quad 0 \leq \delta_n, \quad \text{and} \quad \int_{E_1} \delta_n(x) \, dx = 1.
\]
Then $|\tilde{\delta}_n| \leq 1$ and the sequence $\tilde{\delta}_n$ converges pointwise to 1. This shows that

$$0 \neq \Im \int_E \tilde{\varphi}_0(\lambda_0) \, d\mu(\lambda) = \lim_{n \to \infty} \Im \int_E \tilde{\varphi}_0(\lambda_0) \tilde{\delta}_n(\lambda_1) \, d\mu(\lambda) = \lim_{n \to \infty} \tilde{\mu}(\varphi_0 \otimes \delta_n).$$

This shows that $(x_1, 0) \in \text{supp}(\text{Im} \tilde{\mu})$. □

**Proposition 6.10.** Let $E$ be a simple euclidean Jordan algebra or rank $r$ and $s \in \mathbb{N}$. If either $r$ is even, or if $r$ is odd and $s$ is even, then

$$\text{supp}(\text{Im} \tilde{\mu}_s) = E \setminus E^\times.$$

**Proof.** As the measure $\mu_s^0$ is a tensor product of tempered measures, it is tempered. Therefore Lemma 6.9 implies that

$$\text{supp}(\text{Im} \tilde{\mu}_s^0) \subseteq \text{supp}(\text{Im} \mu_s). \quad (6.11)$$

Since $\mu_s$ is semi-invariant with respect to $\text{Str}(E)_0$, the support of its imaginary part is a closed union of orbits of this group. Any such orbit meets the Jordan subalgebra $E_0$. Therefore the support of $\text{Im} \tilde{\mu}_s$ can is determined completely by the support of $\text{Im} \tilde{\mu}_s^0$, which corresponds to Riesz measures on the associative Jordan algebra $E_0 \cong \mathbb{R}^r$. □

**Example 6.11.** For $r = 3$, the possible indices of invertible elements are $\pm 1, \pm 3$. Hence $\text{Im} \tilde{\mu}_s$ vanishes on some $E_j^\times$ if and only if vanishes on $E_3^\times$, which is equivalent to $s \in \frac{2}{3} \mathbb{Z}$ (Proposition 6.3). It vanishes on all of $E^\times$ if and only if $s \in 2\mathbb{N}_0$. In the latter case,

$$\text{supp}(\text{Im} \tilde{\mu}_s) = E \setminus E^\times$$

by Proposition 6.10. If $s \in \frac{2}{3} \mathbb{Z} \setminus \mathbb{Z}$, we immediately obtain

$$\text{supp}(\text{Im} \tilde{\mu}_s) = E \setminus (E_3^\times \cup E_{-3}^\times) = E \setminus (C^0 \cup -C^0)$$

because $E_1^\times \cup E_{-1}^\times$ is dense in this set.

### 6.4. Jordan Wedges.

For $k \in \{0, \ldots, r\}$, we consider the endomorphisms

$$h_k := L(c_1 + \cdots + c_k) - L(c_{k+1} + \cdots + c_r) \in \text{End}(E)$$

(cf. Appendix B for the notation). For the Riesz measures $\mu_s$, we obtain with Lemma B.1

$$(e^t h_k)_s \mu_s = e^{t \text{tr}(h_k) \frac{s}{r}} \mu_s = e^{ts(2k-r)} \mu_s,$$

which leads to

$$\rho(e^t) = e^{t v_k} \quad \text{with} \quad v_k = s \left(k - \frac{r}{2}\right).$$

Assume that $s \in \mathbb{N}_0$. The factor $v_k$ is either integral for each $k \in \{0, \ldots, r\}$ (if $r$ or $s$ is even) or never (if $r$ and $s$ are odd). If $v_k$ is integral, then

$$\text{supp}(\text{Im} \tilde{\mu}_s) = E \setminus E^\times$$

by Proposition 6.10. We now relate this to the support conditions derived from Proposition 5.4.
Theorem 6.12. The following assertions are equivalent for the wedge domains $W(h_k) \subseteq E$:

(i) $v_k \in \mathbb{Z}$.
(ii) $\text{supp}(\text{Im} \tilde{\mu}_s) \subseteq W(h_k)^c$.
(iii) $\text{supp}(\text{Im} \tilde{\mu}_s) \cap E_{2k-r}^\times = \emptyset$.

Proof. “(i) $\Rightarrow$ (ii)” follows from Proposition 5.4.
(ii) $\Rightarrow$ (iii): As $W(h_k) \subseteq E_{2k-r}^\times$ by Corollary B.5, condition (ii) implies that $\text{Im} \tilde{\mu}_s$ vanishes on $E_{2k-r}^\times$.
(iii) $\Rightarrow$ (i): By Proposition 6.3, (iii) implies that $2v_k = (2k - r)s \in 2\mathbb{Z}$, i.e., that $v_k \in \mathbb{Z}$.

The preceding theorem shows that Proposition 5.4 does not provide any information on non-invertible elements in the support of $\text{Im} \tilde{\mu}_s$. In particular, if every $v_k$ is integral, it only shows that $\text{Im}(\tilde{\mu}_s) \cap E^\times = \emptyset$, so that Proposition 5.4 provides strictly finer information if $r > 2$.

Theorem 6.12 also shows that, if $\nu^r_{K} \neq \nu^b_{K}$, i.e., if $v_k$ is not integral (Lemma 2.3), then we do not expect restrictions on the support of $\text{Im} \tilde{\mu}_s$.

Acknowledgement. We are most grateful to the two referees for their constructive criticism and for pointing out several interesting references to the AQFT literature.

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A Standard subspaces

In this appendix we collect some facts about standard subspaces $\mathbb{V} \subseteq \mathbb{H}$. In particular we describe the connection to antiunitary representations of the multiplicative group $\mathbb{R}^\times$, and the connection to KMS conditions and modular objects. Most of the material in this section is standard and well known. We refer to [Lo08] for the basic theory of standard subspaces, other references are [NÓ17, NÓ19]. Proofs are sometimes included for the sake of clarity of exposition.

A.1 Standard subspaces and antiunitary representations.

Definition A.1. A closed real subspace $\mathbb{V}$ of a complex Hilbert space $\mathbb{H}$ is called standard if

$$\mathbb{V} \cap i\mathbb{V} = \{0\} \quad \text{and} \quad \mathbb{H} = \mathbb{V} + i\mathbb{V}. \quad (A.1)$$

If $\mathbb{V} \subseteq \mathbb{H}$ is a standard subspace, then

$$T_\mathbb{V} : D(T_\mathbb{V}) := \mathbb{V} + i\mathbb{V} \to \mathbb{H}, \quad x + iy \mapsto x - iy \quad (A.2)$$

defines a closed operator with $\mathbb{V} = \text{Fix}(T_\mathbb{V})$. It is called the Tomita operator of $\mathbb{V}$. Its polar decomposition can be written as $T_\mathbb{V} = J_\mathbb{V} \Delta_\mathbb{V}^{1/2}$, where $J_\mathbb{V}$ is a conjugation (an antiunitary involution) and $\Delta_\mathbb{V}$ is a positive selfadjoint operator such that the modular relation

$$J_\mathbb{V} \Delta_\mathbb{V} J_\mathbb{V} = \Delta_\mathbb{V}^{-1} \quad (A.3)$$
holds. We call \((\Delta_V, J_V)\) the pair of modular objects associated to \(V\).

Denote the inner product on \(H\) by \(\langle \cdot, \cdot \rangle\) and let \(\omega(u, v) = \text{Im}\langle u, v \rangle\). Then \(\omega\) is a symplectic form on \(H\). For a real subspace \(W \subset H\) let

\[
W' = \{ v \in H : (\forall w \in W) \omega(w, v) = 0 \}.
\]

Then \(W'\) is also a real subspace and \(W'' = W\), the closure of \(W\).

In the following lemma we collect several properties of standard subspaces that will be used in this article:

**Lemma A.2.** Let \(V, V_1, V_2\) be standard subspaces. Then the following assertions hold:

1. \(V_1 \subseteq V_2\) implies \(V'_2 \subseteq V'_1\).
2. \(\langle \xi, J_V \xi \rangle \geq 0\) for all \(\xi \in V\).
3. \(V\) is standard if and only if \(V'\) is standard.
4. \(J_V = J_{V'}\) and \(\Delta_{V'} = \Delta_V^{-1}\).
5. \(V = V'\) if and only if \(\Delta_V = 1\).
6. \(J_V V = V'\).
7. \((V')' = V\).

**Proof.** (i) is obvious.

(ii) Let \(\xi \in V\). Then \(\xi = T_V(\xi) = J_V \Delta_V^{1/2} \xi\) implies that \(\Delta_V^{1/2} \xi = J_V \xi\). As \(\Delta_V^{1/2}\) is positive selfadjoint, it follows that \(\langle \xi, J_V \xi \rangle \geq 0\).

(iii) follows from [Lo08, § 3.1] and (iv) is contained in [Lo08, Prop. 3.3].

(v) follows from (iii), the fact that the pair \((\Delta_V, J_V)\) determines \(V\) and the observation that \(\Delta_V = \Delta_V^{-1}\) is equivalent to \(\Delta_V = 1\).

(vi) As \(\langle \xi, J_V \xi \rangle\) is real by (ii), it follows that \(J_V V \subseteq V'\). Applying this argument to \(V'\) and using (iii), we also obtain \(J_V V' \subseteq V\), so that (v) follows from the fact that \(J_V\) is an involution.

(vii) follows from (iv) which entails \(J_{V''} = J_V\) and \(\Delta_{V''} = \Delta_V\). \(\square\)

We have already seen that every standard subspace \(V\) determines a pair \((\Delta_V, J_V)\) of modular objects and that \(V\) can be recovered from this pair by \(V = \text{Fix}(J_V \Delta_V^{1/2})\). This observation can be used to obtain a representation theoretic parametrization of the set of standard subspaces of \(H\) (cf. [BGL02, NO17]): Each standard subspace \(V\) specifies a homomorphism \(U^V : \mathbb{R}^\times \to \text{AU}(H)\) by

\[
U^V(e^t) := \Delta_V^{-it/2\pi} = e^{itH_V}, \quad U^V(-1) := J_V, \quad \text{where } H_V = -\frac{1}{2\pi} \log \Delta_V. \tag{A.4}
\]

**Theorem A.3.** The map \(V \mapsto U^V\) defines a bijection between standard subspaces and antiunitary representations of the graded group \((\mathbb{R}^\times, \varepsilon_{\mathbb{R}^\times})\). The inverse is given by assigning to the antiunitary representation \(U : \mathbb{R}^\times \to \text{AU}(H)\) the operators

\[
H = -i \left. \frac{d}{dt} \right|_{t=0} U(e^t), \quad \Delta := e^{-2\pi H}, \quad \text{and } J := U(-1).
\]

**Lemma A.4.** Let \(V\) be a standard subspace. Then the following assertions hold:

(a) \(U^V(e^t)V = V\) for all \(t \in \mathbb{R}\).
(b) \(U^V(r) = U^V(r^{-1})\) for \(r \in \mathbb{R}^\times\).
(c) \(V \cap V' = H^V\).
Proof. (a) Let \( \xi \in \mathbb{V} \) and \( t \in \mathbb{R} \). Then

\[
T_{\mathbb{V}}(U^{\mathbb{V}}(e^{i})\xi) = J_{\mathbb{V}}\Delta_{\mathbb{V}}^{-it/2\pi} \xi = \Delta_{\mathbb{V}}^{-it/2\pi} (J_{\mathbb{V}}\Delta_{\mathbb{V}}^{1/2} \xi) = U^{\mathbb{V}}(e^{i})T_{\mathbb{V}}\xi = U^{\mathbb{V}}(e^{i})\xi.
\]

(b), (c) follow from \([\text{NÓ}17, \text{Lemma 3.7}]\).

**Definition A.5.** Let \( \mathbb{V} \subseteq \mathcal{H} \) be a real subspace and \( J \) be a conjugation on \( \mathcal{H} \). We say that \( \mathbb{V} \) is \( J \)-positive if \( \langle \xi, J\xi \rangle \geq 0 \) for \( \xi \in \mathbb{V} \).

Recall that a conjugation on \( \mathcal{H} \) is an antiunitary involution. The following lemma explores the question when the positivity of a conjugation \( J \) on a real subspace \( \mathbb{V} \) implies that \( \mathbb{V} \) is standard with \( J = J_{\mathbb{V}} \).

**Lemma A.6.** For a closed real subspace \( \mathbb{V} \subseteq \mathcal{H} \) and a conjugation \( J \), the following assertions hold:

(i) If \( \mathbb{V} \) is \( J \)-positive, then \( J\mathbb{V} \subseteq \mathbb{V}' \).

(ii) If \( \mathbb{V} + i\mathbb{V} \) is dense in \( \mathcal{H} \) and \( J\mathbb{V} \subseteq \mathbb{V}' \), then \( \mathbb{V} \cap i\mathbb{V} = \{0\} \).

(iii) Assume that \( \mathbb{V} \) is standard. Then the following are equivalent:

(a) \( J = J_{\mathbb{V}} \).

(b) \( \mathbb{V}' \) is \( J \)-positive and \( J\mathbb{V} \subseteq \mathbb{V}' \).

(c) \( \mathbb{V} \) and \( \mathbb{V}' \) are both \( J \)-positive.

**Proof.** (i) The form \( \beta(\xi, \eta) := \langle J\xi, \eta \rangle \) on \( \mathcal{H} \) is complex bilinear and symmetric. That \( \mathbb{V} \) is \( J \)-positive implies that \( \beta \) is real on all pairs \( (\xi, \xi) \), \( \xi \in \mathbb{V} \), hence by polarization also on \( \mathbb{V} \times \mathbb{V} \). This means that \( J\mathbb{V} \subseteq \mathbb{V}' \).

(ii) The subspace \( \mathbb{V}_{0} := \mathbb{V} \cap i\mathbb{V} \) of \( \mathcal{H} \) is complex and satisfies \( J\mathbb{V}_{0} \subseteq \mathbb{V}' \). Since \( J\mathbb{V}_{0} \) is also a complex subspace, it follows that \( J\mathbb{V}_{0} \) is orthogonal to the total subset \( \mathbb{V} \), hence trivial.

(iii) That (a) implies (b),(c) follows from Lemma A.2(ii),(iv),(vi). Further, (b) implies \( J\mathbb{V}' \supseteq JJ\mathbb{V} \mathbb{V} = \mathbb{V} \), so that the \( J \)-positivity of \( \mathbb{V}' \) implies by \([\text{Lo}08, \text{Prop. 3.9}]\) that \( J = J_{\mathbb{V}} = J_{\mathbb{V}'} \), hence (a). If (c) holds, then (i) shows that the \( J \)-positivity of \( \mathbb{V} \) implies \( J\mathbb{V} \subseteq \mathbb{V}' \). Hence (c) implies (b). This proves (iii). \( \square \)

**Proposition A.7.** (Reflection positivity and standard subspaces) Let \( \mathbb{V} \subseteq \mathcal{H} \) be a standard subspace with modular objects \( (\Delta, J) \). Then the following assertions hold:

(i) \( (\mathcal{E}, \mathcal{E}_{+}, \theta) := (\mathcal{H}^{\mathbb{R}}, \mathbb{V}, J_{\mathbb{V}}) \) is a real reflection positive Hilbert space.\(^4\)

(ii) The map \( \Delta^{1/4} : \mathbb{V} \rightarrow \mathcal{H}^{J} \) extends to an isometric isomorphism \( \overline{\mathbb{V}} \rightarrow \mathcal{H}^{J} \), where \( \overline{\mathbb{V}} \) is the completion of \( \mathbb{V} \) with respect to scalar product \( \langle v, w \rangle_{J} := \langle v, Jw \rangle \) for \( v, w \in \mathbb{V} \).

**Proof.** (i) follows directly from \( \langle v, Jv \rangle = \langle v, \Delta^{1/2}v \rangle = \| \Delta^{1/4}v \|^{2} \).

(ii) Next we note that \( \mathbb{V} \subseteq \mathcal{D}(\Delta^{1/2}) \subseteq \mathcal{D}(\Delta^{1/4}) \) implies that \( \Delta^{1/4} \) is defined on \( \mathbb{V} \). For \( v \in \mathbb{V} \), we have

\[
J\Delta^{1/4}v = \Delta^{-1/4}Jv = \Delta^{-1/4}\Delta^{1/2}v = \Delta^{1/4}v,
\]

so that \( \Delta^{1/4}\mathbb{V} \subseteq \mathcal{H}^{J} \). Using the spectral decomposition of \( \Delta \) it follows easily that \( \Delta^{1/4}\mathbb{V} \) is dense in \( \mathcal{H}^{J} \). This implies (ii). \( \square \)

The following simple observation is taken from \([\text{MN}21]\). It slightly extends \([\text{Lo}08, \text{Prop. 3.10}]\).

\(^4\) See the introduction to Sect. 4 for details.
Proposition A.8. Suppose that $V_1 \subseteq V_2$ are standard subspaces of $\mathcal{H}$. If

(a) $\Delta_{V_2}^{it} V_1 = V_1$ for $t \in \mathbb{R}$, or
(b) $\Delta_{V_1}^{it} V_2 = V_2$ for $t \in \mathbb{R}$,

then $V_1 = V_2$.

Proof. That (a) implies $V_1 = V_2$ follows from [Lo08, Prop. 3.10]. From (b) we obtain by dualization $V'_2 \subseteq V'_1$ with $\Delta_{V'_1}^{it} V'_2 = V'_2$ for $t \in \mathbb{R}$, so that we obtain $V'_1 = V'_2$ with (a), hence $V_1 = V_2$ also holds in this case. $\square$

A.2 Standard subspaces and the KMS condition. As mentioned above, the bijection in Theorem A.3 is closely related to the characterization of standard subspaces and their modular objects in terms of a KMS condition ([Lo08], [NÓ19]).

Definition A.9. Let $V$ be a real vector space and $\text{Bil}(V)$ be the space of real bilinear maps $V \times V \rightarrow \mathbb{C}$. A function $\psi : \mathbb{R} \rightarrow \text{Bil}(V)$ is said to be positive definite if the kernel $\psi(t-s)(v, w)$ on the product set $\mathbb{R} \times V$ is positive definite.

We say that a positive definite function $\psi : \mathbb{R} \rightarrow \text{Bil}(V)$ satisfies the KMS condition for $\beta > 0$ if $\psi$ extends to a function $\overline{S}_\beta \rightarrow \text{Bil}(V)$ which is pointwise continuous and pointwise holomorphic on the interior $S_\beta$, and satisfies

$$\psi(i\beta + t) = \overline{\psi(t)} \quad \text{for} \quad t \in \mathbb{R}.$$  \hfill (A.5)

In a similar fashion as Lemma A.6(iv) characterizes the conjugation $J_V$ of a standard subspace $V$ in terms of the $J$-positivity of $V$ and $V'$, the following proposition characterizes the corresponding modular group in terms of a KMS condition.

Proposition A.10. Let $V \subseteq \mathcal{H}$ be a standard subspace and $U : \mathbb{R} \rightarrow \text{U}(\mathcal{H})$ be a continuous unitary one-parameter group. Then $U(t) = \Delta_{V}^{-it/2\pi}$ holds for all $t \in \mathbb{R}$ if and only if the positive definite function

$\varphi : \mathbb{R} \rightarrow \text{Bil}(V), \quad \varphi(t)(\xi, \eta) := \langle \xi, U(t)\eta \rangle$

satisfies the KMS condition for $\beta = 2\pi$.

Proof (see also [NÓ19, Thm. 2.6]). In [Lo08, Prop. 3.7], this characterization is stated for the function $\langle U(t)\xi, \eta \rangle$, but this should be $\langle \xi, U(t)\eta \rangle$ if the scalar product is linear in the second argument. $\square$

A.3 Hardy space and graph realizations. Let $\Delta > 0$ be a positive selfadjoint operator on $\mathcal{H}$. Then $\mathcal{D}(\Delta^{1/2})$ is a dense subspace of $\mathcal{H}$, and the map

$$\Psi : \mathcal{D}(\Delta^{1/2}) \rightarrow \Gamma(\Delta^{1/2}), \quad \xi \mapsto (\xi, \Delta^{1/2}\xi)$$

is a complex linear bijection onto the closed graph of the selfadjoint operator $\Delta^{1/2}$ in the Hilbert space $\mathcal{H} \oplus \mathcal{H}$. We thus obtain on $\mathcal{D}(\Delta^{1/2})$ the structure of a complex Hilbert space for which $\Psi$ is unitary.

The operator $\Delta$ defines a unitary one-parameter group $(\Delta^{it})_{t \in \mathbb{R}}$, and we consider the $\mathcal{H}$-valued Hardy space
of equivariant bounded holomorphic maps $S_\pi \to \mathcal{H}$. For $\Delta^{-it/2\pi} = e^{itH}$, i.e., $H = -\frac{1}{2\pi} \log \Delta$, and the spectral measure $P_H$ of $H$, we have

$$\|\Delta^{y/2\pi} \xi\|^2 = \|e^{-yH} \xi\|^2 = \int_{\mathbb{R}} e^{-2\lambda y} dP_H^\xi(\lambda),$$

so that the Monotone Convergence Theorem implies for $f \in H^2(S_\pi, \mathcal{H})^\Delta$ and $\xi := f(\pi i/2)$

$$\int_{\mathbb{R}} e^{\pm\lambda y} dP_H^\xi(\lambda) < \infty, \quad \text{so that} \quad \xi \in \mathcal{D}(\Delta^{\pm1/4}).$$

Thus [NÓ18, Lemma A.2.5] implies that $f$ extends to a continuous function on $\overline{S_\pi}$, also denoted $f$. It satisfies

$$\sup_{0 < y < \pi} \|f(iy)\| = \max(\|f(0)\|, \|f(\pi i)\|).$$

In particular, the map

$$\Phi : H^2(S_\pi, \mathcal{H})^\Delta \to \mathcal{H} \oplus \mathcal{H}, \quad \Phi(f) := (f(0), f(\pi i))$$

is defined. To identify the range of $\Phi$, we use [NÓ18, Lemma A.2.5] to see that $\xi = f(0)$ for some $f \in H^2(S_\pi, \mathcal{H})^\Delta$ if and only if $\xi \in \mathcal{D}(\Delta^{1/2})$. Then $f(\pi i) = \Delta^{1/2} \xi$, and we conclude that

$$\Phi(H^2(S_\pi, \mathcal{H})^\Delta) = \Gamma(\Delta^{1/2})$$

(cf. [LLQR18, Prop. 3.4]). As $\Phi$ is injective with closed range, it is an isomorphism of Banach spaces but not necessarily isometric.

[LLQR18, Prop. 3.2] also contains observations which are very similar to the following lemma.

**Lemma A.11.** If $J$ is a conjugation on $\mathcal{H}$, then

$$\tilde{\Phi}(\xi, \eta) := (J\eta, J\xi)$$

defines a conjugation on $\mathcal{H} \oplus \mathcal{H}$, and $\tilde{\Phi}$ maps $\Gamma(\Delta^{1/2})$ into itself if and only if the modularity relation $J\Delta J = \Delta^{-1}$ holds.

**Proof.** If the modularity relation holds, then we also have $\Delta^{-1/2}J = J\Delta^{1/2}$, so that $J\mathcal{D}(\Delta^{1/2}) = \mathcal{D}(\Delta^{-1/2}) = \mathcal{R}(\Delta^{-1/2})$, and therefore

$$\tilde{\Phi}(\xi, \Delta^{1/2} \xi) = (J\Delta^{1/2} \xi, J\xi) = (\Delta^{-1/2} J\xi, J\xi) \in \Gamma(\Delta^{1/2}) \quad \text{for} \quad \xi \in \mathcal{D}(\Delta^{1/2}).$$

If, conversely, $\tilde{\Phi}$ preserves $\Gamma(\Delta^{1/2})$, then

$$J\Delta^{1/2} \xi \in \mathcal{D}(\Delta^{1/2}) \quad \text{and} \quad \Delta^{1/2} J\Delta^{1/2} \xi = J\xi \quad \text{for} \quad \xi \in \mathcal{D}(\Delta^{1/2}).$$

This means that $J\xi \in \mathcal{D}(\Delta^{-1/2})$ with $J\Delta^{1/2} \xi = \Delta^{-1/2} J\xi$. As $J$ is an involution, $J\mathcal{D}(\Delta^{1/2}) = \mathcal{D}(\Delta^{-1/2})$ and $J\Delta^{1/2} J = \Delta^{-1/2}$. This implies $J\Delta J = \Delta^{-1}$. \qed
If \( J \Delta J = \Delta^{-1} \), the preceding lemma shows that the closed subspace \( \Gamma(\Delta^{1/2}) \) of \( \mathcal{H} \oplus \mathcal{H} \) is invariant under \( \tilde{T} \). We also observe that the antilinear operator

\[
T := J \Delta^{1/2} = \Delta^{-1/2} J : \mathcal{D}(\Delta^{1/2}) \to \mathcal{D}(\Delta^{1/2})
\]
satisfies

\[
\tilde{T} \Psi(\xi) = (J \Delta^{1/2} \xi, J \xi) = (T \xi, J \xi) = (T \xi, \Delta^{1/2} T \xi) = \Psi(T \xi) \quad \text{for} \quad \xi \in \mathcal{D}(\Delta^{1/2}).
\]

For the standard subspace \( \mathcal{V} \) with \( J \mathcal{V} = \mathcal{J} \) and \( \Delta \mathcal{V} = \Delta \), \( T \mathcal{V} := \mathcal{T} \) is the corresponding Tomita operator (Sect. A.1), and the relation

\[
\tilde{T} \circ \Psi = \Psi \circ T \mathcal{V}
\]

implies that \( \Psi(\mathcal{V}) = \Gamma(\Delta^{1/2}) \tilde{T} \). In particular, \( T \mathcal{V} \) is a conjugation for the complex Hilbert space structure on \( \mathcal{D}(\Delta^{1/2}) \cong \Gamma(\Delta^{1/2}) \), whose fixed point space is \( \mathcal{V} \).

Next we observe that, as \( J \) commutes with the unitary operators \( \Delta^{it}, t \in \mathbb{R}, \)

\[
(\mathcal{J} f)(z) = J f(\pi i + \overline{z})
\]
defines an isometric involution on the Hardy space \( H^2(\mathcal{S}_\pi, \mathcal{H})^\Delta \). For \( f \in H^2(\mathcal{S}_\pi, \mathcal{H})^\Delta \) and \( \xi := f(0) \in \mathcal{D}(\Delta^{1/2}) \), we have

\[
\tilde{T} \Phi(f) = (J f(\pi i), J f(0)) = \Phi(\tilde{J} f),
\]
so that \( \Phi \) intertwines the conjugations \( \tilde{T} \) and \( \tilde{J} \). We conclude in particular that

\[
\Phi^{-1} \Psi(\mathcal{V}) = \{ f \in H^2(\mathcal{S}_\pi, \mathcal{H})^\Delta : \tilde{J}(f) = f \}. \tag{A.6}
\]

**Lemma A.12.** For \( f \in H^2(\mathcal{S}_\pi, \mathcal{H})^\Delta \), the following conditions are equivalent:

\begin{enumerate}[(a)]
  \item \( \tilde{J}(f) = f \), i.e., \( f(\pi i + \overline{z}) = J f(z) \) for \( z \in \mathcal{S}_\pi \).
  \item \( f(z) \in \mathcal{H}^I \) for \( \text{Im} z = \frac{\pi}{2} \).
  \item \( f(0) \in \mathcal{V} \).
  \item \( f(\pi i) \in \mathcal{V}' \).
  \item \( f(\pi i) = J f(0) \).
\end{enumerate}

**Proof.** The equivalence of (a) and (b) follows by uniqueness of analytic continuation from the line \( \frac{\pi}{2} i + \mathbb{R} \subseteq \mathcal{S}_\pi \). The equivalence of (a) and (c) follows from \( \Psi^{-1} \Phi(f) = f(0) \) and (A.6). As \( f(\pi i) = \Delta^{1/2} f(0) \) is contained in \( \mathcal{V}' = J \mathcal{V} \) if and only if \( J \Delta^{1/2} f(0) \in \mathcal{V} \), which in turn is equivalent to \( f(0) \in \mathcal{V} \), conditions (c) and (d) are also equivalent. The equivalence of (c) and (e) follows from Proposition 2.1.

The map \( \text{ev}_0 = \Psi^{-1} \Phi : \text{Fix}(\tilde{J}) \to \mathcal{V} \) is an isometry of real Hilbert spaces because \( \tilde{J}(f) = f \) implies \( \| f(0) \| = \| f(\pi i) \| \). In this sense every standard subspace can be realized in a natural way as a “real form” of a Hardy space on the strip \( \mathcal{S}_\pi \).
B Wedges in euclidean Jordan algebras

We expect that the reader is familiar with the basic theory of simple euclidean Jordan algebras. We use [FK94] as a standard reference. For the basic definitions we refer to Sect. 6. From now on \( E \) is always a simple euclidean Jordan algebra with

\[
\dim(E) = n \quad \text{and} \quad \text{rank}(E) = r,
\]
and \( c_1, \ldots, c_r \) is a Jordan frame. We then obtain the Pierce decomposition

\[
E = \bigoplus_{j=1}^{r} \mathbb{R} c_j \oplus \bigoplus_{i<j} E_{ij} \quad \text{with} \quad E_{ij} = \{ v \in E : c_i v = \frac{1}{2} v, c_j v = \frac{1}{2} v \} \quad \text{(B.2)}
\]

([FK94, § IV.1]). The set \( E^\times \) of invertible elements of \( E \) has \( r+1 \) connected components that can be described as follows. We fix a spectral decomposition \( x = \sum_{j=1}^{r} x_j \tilde{c}_j \) of an element \( x \in E \), where \((\tilde{c}_1, \ldots, \tilde{c}_r)\) is a Jordan frame ([FK94, Thm. III.1.1]). This means that, under the automorphism group \( \text{Aut}(E) \), the element \( x \) is conjugate to \( \sum_{j=1}^{r} x_j c_j \).

For \( E = \text{Herm}_r(\mathbb{K}) \), this corresponds to the conjugation of a hermitian matrix \( x \) by \( U_r(\mathbb{K}) \) to a diagonal matrix, and for Minkowski space, it corresponds to conjugation of an element \( x \in \mathbb{R}^{1,d-1} \) under \( O_{d-1}(\mathbb{R}) \) to one with \( x_2 = \cdots = x_{d-1} = 0 \).

We define (cf. [FK94, p. 29]):

- the index of \( x \) by \( \text{ind}(x) := \sum_{j=1}^{r} \sgn(x_j) \in \{ r, r-2, \ldots, -r \} \),
- the determinant of \( x \) by \( \Delta(x) = \prod_{j=1}^{r} x_j \), and
- the trace of \( x \) by \( \text{tr}(x) = \sum_{j=1}^{r} x_j \).

Then the connected components of \( E^\times \) are the subsets

\[
E^\times_j := \{ x \in E^\times : \text{ind}(x) = j \}, \quad j = r, r-2, \ldots, -r.
\]

For a multiplication operator

\[
h := \sum_{j=1}^{r} a_j L(c_j),
\]

the Pierce decomposition (B.2) shows that the eigenvalues are \( a_1, \ldots, a_r \) and \( \frac{a_i + a_j}{2} \) for \( i \neq j \). For \( h \neq 0 \), it follows that the eigenvalues of \( h \) are contained in \( \{-1, 0, 1\} \) if and only if \( a_j \in \{ \pm 1 \} \). Reordering the Jordan frame, we see that, up to applying an automorphism of \( E \), any such element is conjugate to one of the form

\[
h := h_k := L(c_1 + \cdots + c_k) - L(c_{k+1} + \cdots + c_r) \quad \text{for some} \quad k \in \{ 0, \ldots, r \}.
\]

Then

\[
E_1(h) = \bigoplus_{j=1}^{k} \mathbb{R} c_j \oplus \bigoplus_{i<j} E_{ij}, \quad E_0(h) = \bigoplus_{i<k<j} E_{ij} \quad \text{and} \quad E_{-1}(h) = \bigoplus_{j=k+1}^{r} \mathbb{R} c_j \oplus \bigoplus_{k<i<j} E_{ij},
\]

where \( E_{\pm 1}(h) \) are Jordan subalgebras of \( E \). Note that \( E_1(-h) = E_{-1}(h) \).
We now observe that the quadruple \((E, C, h, \tau := e^{\pi i h})\) satisfies the assumptions (A1-3) in Sect. 3. Here (A1) and (A2) are obvious. To verify (A3), note that \(e^{\mathbb{R}h} C = C\) follows from \(e^{L(x)} C = C\) for every \(x \in E\) ([FK94, p. 48]). Moreover, \(\tau = e^{\pi i h} \in \text{Str}(E)\) ([FK94, Prop. VIII.2.8]) satisfies \(\tau(e) = -e\), so that \(\tau(C) = -C\). This proves (A3).

In this context, the constructions of Sect. 3 have a Jordan theoretic interpretation. The cones \(C_+ := C \cap E_1(h)\) is the positive cone in the Jordan algebras \(E_1(h)\) and \(C_- = -C \cap E_{-1}(h)\) is the negative cone in \(E_{-1}(h)\). The corresponding wedge is

\[
W := W(h) := C^0_+ \oplus C^0_- \oplus E_0(h). \tag{B.4}
\]

Note that \(x \in W^c = E \setminus W\) if and only if \(x_1 \notin C^0_+\) or \(x_{-1} \notin C^0_-\). For the extremal situations \(k = 0, r\), we obtain \(W(h_r) = C^0\) and \(W(h_0) = -C^0\).

**Lemma B.1.** \(\text{tr}(h_k) = (2k - r) \frac{n}{r}\) for \(n = \dim(E)\) and \(r = \text{rank}(E)\).

**Proof.** From the Pierce decomposition (B.2) it follows that

\[
n = \dim E = r + \frac{r(r-1)}{2}d = r \left(1 + (r-1)\frac{d}{2}\right) \tag{B.5}
\]

and

\[
\text{tr}(h_k) = k - (r-k) + d \left(\frac{k(k-1)}{2} - \frac{(r-k)(r-k-1)}{2}\right)
= 2k - r + \frac{d}{2}(k^2 - k + (r-k) - (r^2 - 2rk + k^2))
= 2k - r + \frac{d}{2}(r - 2k + 2rk - r^2)
= 2k - r + \frac{d}{2}(2k - r)(r - 1) = (2k - r) \left(1 + (r-1)\frac{d}{2}\right) = (2k - r) \frac{n}{r}.
\]

\(\square\)

**Remark B.2.** We are interested in the parity of the numbers \(\text{tr}(h_k)\). First, we observe that \(\frac{n}{r} \in \frac{1}{2}\mathbb{Z}\) by (B.5) and that this is an integer if and only if \((r-1)d\) is even. This is equivalent to \(d\) even or \(r\) odd. In this case \(\text{tr}(h_k)\) is even if and only if \(n\) is even. In the other case the parity of \(\text{tr}(h_k)\) depends on \(k\) if \(\frac{2k}{r}\) is odd.

**Lemma B.3.** Let \(p_1 : E \twoheadrightarrow E_1(h), x \mapsto x_1\), denote the projection map. Then \(p_1(C) \subseteq C\) and \(\text{rank} p_1(x) \leq \text{rank} x\) for \(x \in C\).

**Proof.** Let \(m := \text{rank} x\) and \(x \in C\). The subset \(C_{\leq m} := \{w \in C: \text{rank} w \leq m\}\) is a closed (non-convex) cone invariant under \(e^{\mathbb{R}h}\). Therefore \(x_1 = \lim_{t \to \infty} e^{-t} e^{ih} x \in C_{\leq m}\).

For an element \(x \in E\), we write \(x = x_+ - x_-\) with \(x_{\pm} \in C\) and \(x_+ x_- = 0\) for the canonical decomposition of \(x\) into positive and negative part which can be obtained from the spectral decomposition ([FK94, Thm. III.1.1]).

The following proposition and its corollary constitute the main result of this appendix. They are the key tool for the finer analysis of the support properties of the Fourier transforms \(\hat{\mu}_s\) of the Riesz measures \(\mu_s\).

**Proposition B.4.** Let \(v = v_+ - v_-\) be the canonical decomposition of \(v \in E\) into positive and negative part. Then \(\text{Aut}(E) v \subseteq W(h_k)\) if and only if

\[
\text{rank} v_+ < k \quad \text{or} \quad \text{rank} v_- < r - k.
\]
Proof. For \( v \in W(h_k) \) we have \( p_1(v) \in C_+^0 \), so that \( p_1(v_\pm) \in C \) yields
\[
p_1(v_+) = p_1(v) + p_1(v_-) \in C_+^0 + C_+ \subseteq C_+^0
\]
is invertible in \( E_1(h) \). Lemma B.3 thus implies that \( \text{rank } v_+ \geq \text{rank } p_1(v_+) = k \). Therefore \( \text{rank } v_+ < k \) entails \( v \in W(h_k)^c \). For \( g \in \text{Aut}(E) \), we have \( \text{rank}(gv_+) = \text{rank}(v_+) \), so that \( \text{rank}(v_-) < k \) implies \( \text{Aut}(E)v \subseteq W(h_k)^c \). Likewise \( \text{rank } v_- < r - k \) implies that \( \text{Aut}(E)v \subseteq W(h_k)^c \).

Suppose, conversely, that \( \text{Aut}(E)v \subseteq W(h_k)^c \). Then there exists a \( g \in \text{Aut}(E) \) with
\[
gv = \sum_{j=1}^r v_jc_j \in \sum_{j=1}^r \mathbb{R}c_j \subseteq E_1(h) \oplus E_{-1}(h) \quad \text{and } \quad v_1 \geq \cdots \geq v_r.
\]
As \( gv \not\in W(h_k) \), we have \( (gv)_1 = \sum_{j \leq k} v_jc_j \not\in C_+^0 \) or \( (gv)_{-1} = \sum_{j > k} v_jc_j \not\in C_-^0 \). In the first case \( v_k \leq 0 \), so that \( \text{rank } v_+ < k \), and in the second case \( v_{k+1} \geq 0 \), so that \( \text{rank } v_- < r - k \).

By negation we immediately obtain:

Corollary B.5. For \( v = v_+ - v_- \) as in Proposition B.4, the following are equivalent:

(i) \( \text{Aut}(E)v \cap W(h_k) \neq \emptyset \) for \( W(h_k) \) as in (B.4).
(ii) \( \text{rank}(v_+) = k \) and \( \text{rank}(v_-) = r - k \).
(iii) \( v \) is invertible of index \( \text{ind}(v) = 2k - r \).

In particular, \( W(h_k) \subseteq E_{2k-r}^\times \).

Proof. The equivalence of (i) and (ii) follows from Proposition B.4. For the equivalence of (ii) and (iii), we note that \( v = v_+ - v_- \) is invertible if and only if \( \text{rank}(v_+) + \text{rank}(v_-) = r \). Then \( \text{ind}(v) = \text{rank}(v_+) - \text{rank}(v_-) = 2k - r \) is equivalent to \( \text{rank}(v_+) = k \) and \( \text{rank}(v_-) = r - k \). \( \square \)

Example B.6. (a) For \( k = r \) we have \( E = E_1(h) \) and \( W(h_r) = C_0 \). Therefore \( \text{Aut}(E)v \subseteq W(h_r)^c \) is equivalent to \( \text{rank } v_+ < r \), which is equivalent to \( v \not\in C_0 \).
(b) For \( r = 2 \) and \( k = 1 \) (Lorentz boost on Minkowski space), we obtain by \( W(h_1) \) a wedge domain in the Minkowski space \( E \). Then \( \text{Aut}(E)W(h_1) = E_0^\times \) is the open subset of spacelike vectors whose complement is the closed double cone \( \bigcap_{g \in \text{Aut}(E)} gW(h_1)^c = C \cup -C \) (cf. Example 5.5).

References

[Ar99] Araki, H.: Mathematical Theory of Quantum Fields, Series of Monographs on Physics. Oxford University Press, Oxford (1999)
[AZ05] Araki, H., Zsidó, L.: Extension of the structure theorem of Borchers and its application to half-sided modular inclusions. Rev. Math. Phys. 17(5), 491–543 (2005)
[AW63] Araki, H., Woods, E.J.: Representations of the canonical commutation relations describing a nonrelativistic infinite free Bose gas. J. Math. Phys. 4, 637–662 (1963)
[BJL02] Baumgärtel, H., Jurke, M., Lledo, F.: Twisted duality of the CAR-algebra. J. Math. Phys. 43(8), 4158–4179 (2002)
[BW92] Baumgärtel, H., Wollenberg, M.: Causal Nets of Operator Algebras. Akademie Verlag, Berlin (1992)
[Bo95a] Borchers, H.-J.: When does Lorentz invariance imply wedge duality. Lett. Math. Phys. 35, 39–60 (1995)
[Ne19b] Neeb, K.-H.: Semigroups in 3-graded Lie groups and endomorphisms of standard subspaces, Kyoto Math. J., to appear; arXiv:1912.13367

[NÓ14] Neeb, K.-H., Ólafsson, G.: Reflection positivity and conformal symmetry. J. Funct. Anal. 266, 2174–2224 (2014)

[NÓ15] Neeb, K.-H., Ólafsson, G.: Reflection positivity for the circle group, In: “Proceedings of the 30th International Colloquium on Group Theoretical Methods,” Journal of Physics: Conference Series 597 (2015), 012004; arXiv:1411.2439

[NÓ17] Neeb, K.-H., Ólafsson, G.: Antiunitary representations and modular theory. In: Grabowska, K., et al, Grabowski, J., Fialowski, A., Neeb, K.-H. (eds) “50th Sophus Lie Seminar”, Banach Center Publications, vol. 113, pp. 291–362 (2017). arXiv:1704.01336

[NÓ18] Neeb, K.-H., Ólafsson, G.: Reflection Positivity. A Representation Theoretic Perspective, Springer Briefs in Mathematical Physics 32, (2018)

[NÓ19] Neeb, K.-H., Ólafsson, G.: KMS conditions, standard real subspaces and reflection positivity on the circle group, Pac. J. Math. 299:1 (2019), 117–169; arXiv:1611.00080

[NÓ21a] Neeb, K.-H., Ólafsson, G.: Nets of standard subspaces on Lie groups, Advances in Math. 384 (2021) 107715; arXiv:2006.09832

[NÓ21b] Neeb, K.-H., Ólafsson, G.: Wedge domains in non-compactly causal symmetric spaces, in preparation

[NÓ21c] Neeb, K.-H., Ólafsson, G.: Wedge domains in compactly causal symmetric spaces, in: preparation

[Oeh21] Oeh, D.: Nets of standard subspaces induced by unitary representations of admissible Lie groups, arXiv:2104.02465

[OS73] Osterwalder, K., Schrader, R.: Axioms for Euclidean Green’s functions. I. Commun. Math. Phys. 31, 83–112 (1973)

[RS75] Reed, S., Simon, B.: Methods of Mathematical Physics II: Fourier Analysis. Self-adjointness. Academic Press, New York (1975)

[Re16] Rejzner, K.: Perturbative Algebraic Quantum Field Theory. An Introduction for Mathematicians. Mathematical Physics Studies. Springer, Cham (2016)

[Sc99a] Schlingemann, D.: From Euclidean field theory to quantum field theory. Rev. Math. Phys. 11(9), 1151–1178 (1999)

[Sc99b] Schlingemann, D.: Application of Tomita–Takesaki theory in algebraic euclidean field theories, Preprint, arXiv:hep-th/9912219 22 (Dec 1999)

[S86] Schrader, R.: Reflection positivity for the complementary series of SL(2n, C). Publ. Res. Inst. Math. Sci. 22, 119–141 (1986)

[Si74] Simon, B.: The $P(\Phi)^2$ Euclidean (Quantum) Field Theory. Princeton University Press, Princeton (1974)

[SW64] Streeter, R.F., Wightmann, A.S.: PCT, Spin, Statistics and All that. Physics Monograph Series. Math. W. A. Benjamin Inc, New York (1964)

Communicated by C. Schweigert