Disorder dependence of helical edge states in HgTe/CdTe quantum wells

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Abstract. In recent years, extensive attention has been focused on a new topological phase induced by nonmagnetic disorder, known as the topological Anderson insulator (TAI). In this work, we study the disorder strength dependence of the edge states in TAI phase in disordered HgTe/CdTe quantum wells. It is shown clearly that the disorder-induced edge states appear above a critical disorder strength after a gap-closing phase transition. These edge states are then found to decline with an increase of disorder strength in a stepwise pattern due to the finite-width effect, where the opposite edges couple to each other through the localized bulk states. This is in sharp contrast with the localization of the edge states themselves by time-reversal symmetry breaking. The size-independent phase boundaries are further obtained through scaling analysis, where a metallic phase is found separating two topologically distinct phases, which is due to the Fermi energy and mass renormalization.
1. Introduction

The disorder effect is one of the most important topics in the subject of topological insulators (TIs) (for reviews, see [1]). Recently, a nonmagnetic \textit{disorder-induced} TI state, the topological Anderson insulator (TAI), was found by computer simulations in both two-dimensional (2D) [2] and 3D [3] systems, and was further confirmed through independent simulations by studying the local currents [4]. This state is numerically characterized by precisely quantized conductance \( e^2/h \) per spin [2–4], and is theoretically understood as generated by the negative renormalized topological mass [3, 5] within the self-consistent Born approximation (SCBA). Phase diagrams of TAI on the energy-disorder plane at a fixed mass parameter were obtained [2, 3, 5], where the TAI phase appears as an island between the so-called weak- and strong-disorder boundaries [5]. The locations of these boundaries depend generally on the size of the system under simulation. Topological invariants in disordered systems were also discussed [6, 7]. In particular, it was argued that the 2D TAI phase is not a distinct one, but is instead part of the quantum spin Hall (QSH) phase [6] with the nontrivial spin-Chern number [8] in the presence of disorder if the phase diagram is extended to the mass axis.

Despite all these achievements on the understanding of the surprising disorder-induced topological phase, there still exist some questions. Firstly, it has been experimentally proved that the quantized conductance, \( 2e^2/h \), in clean 2D HgTe/CdTe quantum wells (QWs) [9] measured in transport studies [10] is a direct consequence of gapless conducting edge states. However, in the disorder-induced TI, the existence of such edge states in energy–momentum space, their dependence on disorder strength, as well as how the dependence corresponds to the wave-function behaviors in real space, have not been well investigated so far. Secondly, when departing from the TAI region, how these edge states terminate with an increase of the disorder
strength and how they differ from the usual bulk states are also currently unresolved questions. Finally, as mentioned in [5], the weak-disorder boundary is not an Anderson transition at all as the name TAI might suggest, in the sense that it is more of a band effect rather than the result of a mobility edge. Therefore the Anderson transition boundary, which is expected to be the true size-independent phase boundary of TAI in the strong-disorder region, is still absent for now.

Motivated by these issues, in this work we study the dependence of the disorder-induced conducting edge states on disorder strength, and the phase boundaries in the effective model of disordered HgTe/CdTe QWs. It is found that, firstly, the disorder-induced edge states are shown to appear only after a gap-closing phase transition when above a critical disorder strength, which symbolizes a change of the topological number [11]. Since we know that the system is initially a trivial insulator, this observation confirms in a straightforward way the topological origin of the reported quantized conductance plateau in previous transport studies [2–5] of TAI. Moreover, these disorder-induced edge states in the TAI phase possess similar spin–momentum locking properties to those of the helical liquid in QSH systems [12]. Secondly, the dependence of such disorder-induced edge states on disorder strength is focused on later. It is shown in particular that if the system width is finite, unlike the usual exponential decay of quasi-1D bulk states [13], the disorder-induced (1D) edge states decay in a stepwise pattern through the accidental coupling between the edge states at opposite boundaries assisted by the localized bulk states due to the competition between the sample width $L_y$ and the localization length $\xi$. This mechanism is insensitive to the details of boundary conditions. Finally, we have located the Anderson transition phase boundaries of the disordered HgTe/CdTe QWs on the energy-disorder plane by scaling analysis. It is seen that the so-obtained TAI plateau width is much wider than that of a finite-size system through transport study. Multiple phase transitions, from metal (or trivial insulator, the initial state depends on the Fermi energy) to TAI to metal and to Anderson insulator (AI), are identified. It is interesting to compare the origin of the metallic phase in between the TAI and AI, which separates two topologically distinct insulating phases, with the origin of the metallic phase discussed in the literature [11, 14–16]. In our case, the metallic phase appears as a consequence of the gap center renormalization by disorder, where the Fermi energy (even if it is in the gap region in the clean limit) extends to the band region at large disorder strength so that the system shows conducting behavior. This mechanism is quite different from the appearance of metallic phase relating to the spin $s_z$-nonconservation inversion asymmetry in both clean [11, 16] and disordered [14, 15] systems. However, the metallic phase in between the TAI and AI phases may still be observed in disordered HgTe/CdTe QWs in the experiments, even in the absence of $s_z$-nonconservation inversion asymmetry.

The rest of this paper is organized as follows. In section 2, we introduce our simulating model and methods. In section 3, we present our numerical results and theoretical analyses. Finally, the conclusions are presented in section 4.

2. The model and method

Our starting point is the 2D effective Hamiltonian of HgTe/CdTe QWs [9] with on-site nonmagnetic disorders. In tight-binding representation on a square lattice, it reads [4]

$$
H = \sum_i \epsilon_i \psi_i^\dagger \psi_i + (t_x \psi_i^\dagger \psi_{i+1} + t_y \psi_i^\dagger \psi_{i+1,j} + \text{h.c.}),
$$

(1)
where $\psi_i = (\psi_{i\uparrow}, \psi_{i\downarrow}, \psi_{i\uparrow}, \psi_{i\downarrow})^T$ is the field for the four orbital states $|s, \uparrow\rangle$, $|p_x + ip_y, \uparrow\rangle$, $|s, \downarrow\rangle$, $|−p_x + ip_y, \downarrow\rangle$ on site $i = (m, n)$, and $\epsilon_i$ and $t_{i\langle y\rangle}$ are, respectively, the on-site energy and the overlap integral matrices along the $x(y)$ direction, which are explicitly

$$\begin{align*}
\epsilon_i &= \text{Diag} (\epsilon_s + \Delta_i, \epsilon_p + \Delta_i, \epsilon_s + \Delta_i, \epsilon_p + \Delta_i), \\
I_x &= \begin{pmatrix}
\frac{D+B}{a^2} & -\frac{iA}{2a} & 0 & 0 \\
-\frac{iA}{2a} & \frac{D-B}{a^2} & 0 & 0 \\
0 & 0 & \frac{D+B}{a^2} & \frac{iA}{2a} \\
0 & 0 & \frac{iA}{2a} & \frac{D-B}{a^2}
\end{pmatrix}, \\
I_y &= \begin{pmatrix}
\frac{D+B}{a^2} & \frac{A}{2a} & 0 & 0 \\
-\frac{A}{2a} & \frac{D-B}{a^2} & 0 & 0 \\
0 & 0 & \frac{D+B}{a^2} & \frac{A}{2a} \\
0 & 0 & \frac{-A}{2a} & \frac{D-B}{a^2}
\end{pmatrix}.
\end{align*}$$

In the above, $a$ is the lattice constant, $A$, $B$, $C$, $D$, $M$ are material parameters depending on the QW width [9], $\epsilon_s(p) = C \pm M - 4(D \pm B)/a^2$, and $\Delta_i$ is the on-site disorder energy for nonmagnetic impurities, which is identical for the four orbitals and uniformly distributed in $[-W/2, W/2]$, where $W$ is the disorder strength. To compare with the previous works, we use the same values for the parameters as in [2, 4], i.e. $A = 364.5$ meV nm, $B = -686$ meV nm$^2$, $C = 0$, $D = -512$ meV nm$^2$, $M = 1$ meV and $a = 5$ nm.

The method of twisted boundary conditions [17, 18] is used in this work to directly diagonalize the disordered system, and the transfer matrix method [13] is taken to obtain the longitudinal conductance as well as the wave-function distributions.

Without disorder, the mass parameter $M$ characterizes the effect of band inversion by changing its sign from positive to negative, and results in a topological phase transition from a trivial insulating state ($M > 0$) to a QSH phase ($M < 0$) with a single pair of helical edge states at each boundary. This pair of helical edge states can be clearly seen in energy–momentum space by diagonalizing the clean Hamiltonian (setting $\Delta_i = 0$ in equation (1)). However, for disordered systems, the translational symmetry is generally broken and the momentum is no longer a good quantum number. But if we make a quasi-1D superlattice in $x$-direction with each cell being the disordered system of size $L_x \times L_y$, then the translational symmetry in the $x$-direction is restored, where a particle gains an extra phase factor $e^{i\phi_x}$ whenever it goes across the boundary of each cell. In such a case, $−\pi \leq \phi_x \leq \pi$ plays the role of the generalized momentum, and we could diagonalize the disordered system in the $E−\phi_x$-plane and study the edge properties as a function of $W$. It is noted that the choice of $L_x$ should be larger than the average decay length of the wave-functions. We find that $L_x = 15a$ is good enough for diagonalization, and we set $L_y = 100a$ (this width was taken in several previous transport studies [2, 5, 19], in which the size effect can be neglected [19]) throughout this paper unless mentioned otherwise. Moreover, the edge states obtained in such a way will span the entire $\phi_x$-space since all the states in the original one-cell system are folded into the Brillouin zone of the superlattice.

The study of wave-functions aims to gain insights into how a conventional insulating state transitions into a topological nontrivial phase and vice versa. To obtain the wave-functions in real space, we consider a two-terminal setup as shown in figure 1 and adopt the standard transfer
Figure 1. Schematic illustration of the two-terminal device. The central stripe is made of HgTe/CdTe QWs with length $L_x$ and width $L_y$ where nonmagnetic disorders exist. Two semi-infinite clean leads made of the same materials are fabricated, which are heavily doped to access the $|E_i| < M$ region [5] (see footnote 4).

The matrix method [13] with the iteration equation along the $x$-direction written as

$$
\begin{pmatrix}
\Psi(m + 1) \\
\Psi(m)
\end{pmatrix} = M(m)
\begin{pmatrix}
\Psi(m) \\
\Psi(m - 1)
\end{pmatrix},
$$

(5)

where $\Psi(m) = (\psi_{m1}, \psi_{m2}, \ldots, \psi_{mL_y})^T$ is the wave-function vector at slice $x = m$, and $M(m)$ is the corresponding transfer matrix. In the region of disordered HgTe/CdTe QWs, the transfer matrix reads explicitly

$$
M(m) = \begin{pmatrix}
T_x^{-1}[E_i I - H(m)] & -T_x^{-1}T_x^T \\
I & O
\end{pmatrix},
$$

(6)

where $E_i$ is the Fermi energy of the clean system, $I$ and $O$ are, respectively, the $4L_y \times 4L_y$ unit and null matrices, and

$$
T_x = \text{Diag}(t_x, t_x, \ldots, t_x)
$$

(7)

is the block diagonal overlap matrix with the same dimension, while

$$
H(m) = \begin{pmatrix}
\epsilon(m, 1) & t_y & & t_y \\
t_y^T & \epsilon(m, 2) & & \\ & \ddots & \ddots & \\
t_y^T & & \epsilon(m, n) & t_y \\
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3. Numerical results

3.1. Overview of the phase diagram

Let us first review the numerical and theoretical results of the TAI phase diagram [2, 5], which is reproduced [5] (see footnote 4) in figure 2(a).

Numerically, the TAI phase is identified by the suddenly emerged quantized conductance, $2e^2/h$, with the increasing of $W$ at a fixed $E_f$. See the red crescent-shaped region in figure 2(a), where the statistical fluctuations are vanishingly small. This region has a clear-cut weak-disorder boundary at which the TAI phase begins and a blurred strong-disorder boundary where the TAI phase terminates.

Theoretically, the transitions to the TAI phase at the weak-disorder boundary can be understood within the SCBA [3, 5]. It is demonstrated that the random potential renormalizes both the Fermi energy and the mass parameter as a function of disorder strength, $\bar{E}_f(W)$ and $\bar{M}(W)$, so that starting from a metallic phase, the mass changes its sign when it crosses some critical disorder strength, and the TAI phase is defined by the combination of conditions $\bar{M} < 0$ and $-|\bar{M}| < \bar{E}_f < |\bar{M}|$. In figure 2(a), the boundary $\bar{M}(W) = 0$, which separates the positive and negative effective masses, is plotted as a blue dashed line, and the band edges, $\bar{E}_f = \pm |\bar{M}|$, obtained from the SCBA are shown in cyan. It is seen that the results predicted from the SCBA...
agree very well with the weak-disorder boundary of the TAI phase obtained from our numerical simulations. However, the SCBA fails to give the correct strong-disorder boundaries, as seen by the dotted parts of the cyan lines.

Physically, the strong-disorder boundary is blurred because in this region there are impurity states that enter the renormalized bulk gap, which embodies the feature that the self-energy has a nonvanishing imaginary part [3]. In figure 2(a), the critical disorder strength is shown as a function of Fermi energy, \( W^*(E_f) \), see the green dashed line, where for \( W > W^* \) the self-energy becomes imaginary. This line distinguishes the TAI phase into two regions according to whether the band edges are effectively defined or not. If they are not, the mobility edges should play a role.

Finally, we note that the strong-disorder boundary is expected to be an Anderson transition boundary [3, 5, 7]; however, a direct scaling analysis of Anderson transitions is still absent, which we will present below.

3.2. Dependence of edge states on disorder strength

As mentioned above, the observed quantized conductance plateau in the TAI phase is attributed to the presence of conducting edge states induced by disorder. However, the existence of such edge states has never been shown directly in the TAI's energy spectrum. In this section, we first diagonalize the HgTe/CdTe QWs for a specific disorder realization with increasing disorder strength \( W \) (meV). The twisted and open boundary conditions are used respectively in the \( x \)- and \( y \)-directions. The obtained energy spectrum is in the \( E-\phi_x \)-plane. A complete dependence of the energy spectrum on disorder strength is presented in figure A.1 of appendix A, where five representatives with \( W = 0, 40, 80, 120 \) and 200 are shown explicitly in figure 2(b).

In the clean limit where \( W = 0 \), we see that the spectrum exhibits a topological-trivial feature with a full bulk gap \( E_g \simeq 2|M| \) (the derivations are due to the finite size of \( L_y \)). With the increasing of the disorder strength, say to \( W = 40 \), it is seen that although the system still behaves like a normal insulator, there are trivial edge states (see the dashed purple lines) grown from the lower band and the bulk gap becomes much narrower. In fact, we have observed the closing of the band gap around \( W_c = 50 \) (for our particular disorder realization) and its reopening immediately thereafter at slightly larger disorder strength, but in the presence of two pairs of gapless edge states. It is well known that transitions between ordinary and topological insulating states occur only when the gap is closed, which symbolizes a change in the topological number [11], bearing in mind that the system is initially a trivial insulator; therefore, we conclude that the system transitions into a topological nontrivial phase, TAI, when the gap is reopened for \( W > W_c \). A typical spectrum after this transition is exemplified at \( W = 80 \). The observation of the gapless edge states with nonvanishing disorders in the generalized momentum space provides the most direct evidence so far that it is the disorder-induced edge transport that leads to the quantized conductance in the TAI phase. Moreover, these gapless edge states persist to even stronger disorder strength; nevertheless, the low-energy states begin to localize and smear the band edges; see for example the \( W = 120 \) panel, which is in good agreement with the phase diagram. An interesting thing happens in the strong-disorder

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5 These purple dashed lines are marked as the extra states in the spectrum with the open boundary condition in the \( y \)-direction compared with the spectrum with the periodic boundary condition in the \( y \)-direction at the same parameters for a clearer view. See appendix B given therein for the explanation.
region. With strong disorder, naively we would expect that all states are localized and the system becomes an AI. However, the energy spectrum at \( W = 200 \) shows that although the bulk gap is completely smeared by the impurity states and the band edges are ill-defined, the edge states are still robust and winding around the projected Brillouin zone.

To compare with the helical properties of the edge states in QSH phase in the inverted clean HgTe/CdTe QWs, we have studied in detail such disorder-induced edge states with disorder by analyzing their eigen-functions. The results are shown in the panel for \( W = 80 \). The edge states are labeled according to their density distributions \( \rho(y) = \int dx \Psi^\dagger(i) \Psi(i) \) as well as the spin polarization \( \vec{s}(y) = \int dx \vec{\sigma} \Psi^\dagger(i) \Psi(i) \), where the solid (empty) circles indicate the states exponentially localized near the \( y = 0(L_y) \) boundary, and red color (blue) represents the up (down) spin polarization; see figure C.1 of appendix C for details. The helical character of these edge states at a given energy is clearly seen, where there is a single Kramer’s pair at each edge with spin–momentum locking. The whole spectrum preserves the time-reversal symmetry by observing that \( E_{n\alpha}(\phi_x) = E_{n\bar{\alpha}}(-\phi_x) \), where \( n \) is the band index and \( \alpha = -\bar{\alpha} = \uparrow, \downarrow \), which manifests the nonmagnetic property of disorder. In contrast, the spin degeneracy is lifted due to the breaking of inversion symmetry (while \( s_z \) is still conserved) by disorder. At the time-reversal invariant points, \( \phi_x = 0, \pi \), the edge states switch their Kramer’s partners \([22]\), which leads to a nontrivial topology. This is also confirmed by calculating the \( Z_2 \) index \( \nu_0 \) of this 2D system with the twisted boundary conditions on both directions following the method in \([7]\). All these results strongly support the statement that the transition from a normal insulator at \( W = 0 \) to TAI is a topological one, and the quantized conductance plateau originates from the disorder-induced conducting helical edge states which are intrinsic to TAI.

### 3.3. Edge state destruction at strong disorder

The robustness of the disorder-induced edge states shown in figure 2(b) (the panel for \( W = 200 \)) naturally gives rise to the following questions: how will these edge states be destroyed in strong-disorder region and how do they differ from the exponential localization of bulk states? Unlike the case when the edge states themselves could be localized by magnetic perturbations due to the random backscattering, with nonmagnetic disorders, the time-reversal symmetry is preserved all the time, and no backscattering of edge states by disorder is allowed \([23]\); therefore the edge states are most likely to decay through coupling with each other. For a finite-width system, this generally depends on the competition between the localization length \( \xi(W) \) of bulk states and the system width \( L_y \). If \( \xi(W) \sim L_y \), even localized bulk states can couple the edge states at opposite boundaries and destroy the edge states; we may thus expect something interesting to happen. In contrast, for an infinite system where \( \xi(W) \ll L_y \) always, only extended states can couple the edge states far away from each other; therefore a localization–delocalization Anderson transition is expected, which is the true strong-disorder boundary of the TAI phase. In this section, we investigate the decay mechanism of the disorder-induced edge states for a finite-width system by studying its wave-functions in real space. The scaling analysis of Anderson transitions for an infinite system is discussed in the next section.

Let us first take \( L_x = 10^3 a \) and set \( E_f = 10 \) meV. The logarithmic wave-function distributions before, on, and at the tail of the conductance plateau are given, respectively, in figures 3(a)–(b), (c)–(d) and (e). Before the TAI plateau, we see that the bulk states are weakened with an enhancement of disorder strength. When on the TAI plateau, the bulk states disappear completely and two propagating 1D edge states are clearly seen, which contribute a quantized
Figure 3. Logarithmic wave-function intensities at $E_f = 10$ meV. The wave-function intensities before, on and at the tail of the TAI plateau are shown, respectively, in (a)–(b), (c)–(d) and (e). The edge state wave-function intensities in (e) at $y = 1a$ and $100a$ are given in (f), where the brown arrows indicate the positions where accidental couplings between the edge states occur due to the large fluctuations of the local decay length. For comparison, the edge states wave-function intensities of a wider stripe with $L_y = 300a$ are plotted in (g) with all the other parameters remaining the same as in (e). In all the figures, the first and the last 50 lattices form the leads region.

The conductance $e^2/h$ per edge. However, with further increase of disorder strength, the bulk states reappear and begin to localize but coexist with the conducting edge states, as seen in figure 3(d). This is equivalent to the $W = 120$ panel in figure 2(b) in the generalized momentum space where the band edges are ill-defined but the mobility edge starts to count. We see that the wave-function behaviors in real space agree very well with the pictures presented in figure 2(b) if we trace the tendency of the energy spectrum at $E_f = 10$ meV with disorder strength.

An interesting thing happens when we move to the tail of the TAI plateau, as seen in figure 3(e) where a longer length $L_x = 5 \times 10^3a$ is set. With periodic boundary conditions on both directions, only bulk states exist, which decay exponentially in the propagating direction [13]. Differently here, with the open boundary condition in the $y$-direction, we see that the edge states fade in three segments as indicated by the brown arrows and decay much more slowly with localized bulk states in between. To check the interplay between these localized bulk states and the edge states, we choose two lines at $n = 1$ and $100$ and plot the logarithmic intensities of their wave-functions in figure 3(f). It is striking to find that the edge states decay in a stepwise pattern in contrast to the exponential way. This unusual behavior can be understood by the accidental coupling between the edge states at opposite boundaries assisted by the localized bulk states due to the fluctuations of the decay length $L_d$. It is estimated that $\xi \simeq \langle L_d \rangle \sim 10a$ at this disorder strength, which is comparable to $L_y$, and the intensities...
of edge state wave-functions drop approximately at the interfaces of each segment, $m = 1600$ and 3800, where the local decay length, $L_d(m)$, is large. To check this, we enlarge the system width to $L_y = 300a$ but keep all other parameters the same, and the corresponding edge state wave-function intensities at $n = 1$ and 300 are shown in figure 3(g). In this case, we have $\xi \approx \langle L_d \rangle \ll L_y$; therefore a longer system length is needed for accidental coupling between the edge state to happen. For the current simulation, it is seen that the edge states persist throughout the system, and no coupling occurs at the much larger scale of $5 \times 10^3 a$. We have also checked the insensitivity of this decay mechanism to the details of boundary conditions; by changing the open boundary condition, for example, to the parabolic and finite square potentials in the $y$-direction, similar stepwise decay patterns are observed.

3.4. Scaling analysis of Anderson transitions

The above wave-function analysis shown in figures 3(f)–(g) implies that the width of the TAI plateau is system size dependent. For a fixed length $L_x$ at a given energy, the wider the width of the system is, the broader the conductance plateau will be. This is indeed true as seen in figure 4 where the conductance plateaux for $L_y = 100a$, 200a and 300a at $L_x = 500a$ are presented. Then what will happen when $L_y$ tends to infinity? For an infinite system, if the bulk states remain localized, where the localization length $\xi$ has a finite value, then the edge transport and so the conductance plateaux will not be destroyed no matter how large the $\xi$ is. Thus the TAI phase disappears in an infinite system only when the bulk states become delocalized and $\xi$ is divergent. Therefore a localization–delocalization Anderson transition is expected, which kills the TAI phase in the strong-disorder region.

To see the existence of such an Anderson transition, we carry out the scaling analysis [24–26] of the bulk conductance using periodic boundary conditions in both directions. The average logarithmic conductance, $\langle \ln G \rangle$, on the $E_f$–$W$-plane is calculated for different system sizes $L_x = L_y = L$, and the average is performed under $2 \times 10^3$ disorder realizations. For each $E_f$, the Anderson transitions are recognized as the crossings of the $\langle \ln G(W) \rangle$ lines with different $Ls$, as seen in figure 5(a). In each region between two successive crossings, the system is judged to be in insulating (metallic) phase if $\langle \ln G \rangle$ increases with a decrease (increase) of $L$ at a fixed $W$. For example, there are three Anderson transitions at $E_f = 60$ meV, as indicated by

![Figure 4](http://www.njp.org/)
Figure 5. (a) Scaling behavior of the average logarithmic conductance as a function of $W$ at different energies $E_f = 60$ and 0 meV. The red and blue curves are calculated at the parameter $D = -512$ meV nm$^2$, while the green curves are calculated by setting $D = 0$ at $E_f = 0$. The blue and red curves are scaled by the left axis, and the green curves are scaled by the right axis. Inset: the Anderson transitions obtained from the scaling analysis are marked by white symbols in the phase diagram. (b) The average conductance $\langle G \rangle$ versus system width $L_y$ at different energies with $W = 260$ meV. Inset: rough estimation of the strong-disorder boundary of TAI phase at $L_y = 100a$, 200a and 300a shown by dashed, dotted and solid lines, respectively.

the red lines in figure 5(a). The system first transitions from metallic to topological insulating phase, which corresponds to the weak-disorder boundary of the TAI phase obtained by the SCBA and then transitions from TAI to metallic again, which is the localization–delocalization strong-disorder boundary of the TAI phase we expect. After the third transition, the whole system is totally localized and becomes an AI. The interesting thing is that similar multiple phase transitions also occur when the Fermi energy is initially in the clean gap, with the only difference being that the system starts as a trivial insulator. The complete result is given in the inset of figure 5(a), where the Anderson transitions are marked as white symbols on the phase diagram. The region enclosed by the white crosses is the TAI island, and we see that this region is much wider than the simulation result for a finite-width system.

We have also further traced how the strong-disorder boundary of the TAI phase in a finite-width system approaches the Anderson transitions as the system width is increases. As shown in the inset of figure 5(b), strong-disorder boundaries for three different system widths, $L_y = 100a$ (dashed), 200a (dotted) and 300a (solid), are roughly estimated. It is seen that the

The method of scaling analysis we perform here is to treat the system as 2D and is well established in the literature [24–26]. We have also used an alternative method of scaling analysis with accuracy to within less than 1% by treating the system as a quasi-1D one where $\xi / L_y$ is scaled with $L_y$ [14]. The results of both methods support a metallic phase in between TAI and AI and agree well with each other.
strong-disorder boundary goes nearer to the Anderson transitions as the system becomes wider. Therefore, within the TAI region, the higher the energy is, the faster the system will be lifted onto the quantized conductance plateau predicted by the scaling analysis at a given disorder strength. As visioned in figure 5(b), the average conductance $\langle G \rangle$ versus system width $L_y$ at four different energies $E_f = 20$, 30, 40 and 50 meV is plotted.

Finally, it is interesting to compare our results with those in the clean systems [11, 16], where a metallic phase which separates two topologically distinct insulating phases is identified as the signature of the presence of $s_z$-nonconservation inversion-asymmetry in the gap-closing processes for both 2D and 3D spin–orbit coupling systems. Surprisingly, such a signature also appears in disordered spin–orbit coupling systems [14, 15]. In particular, the authors of [14] observe that if a Rashba-type interaction is included in the disordered HgTe/CdTe QWs, a finite metallic region exists on both sides of the TAI phase and partitions $Z_2$-nontrivial and $Z_2$-trivial insulating phases. However, the presence of the $s_z$-nonconservation inversion-asymmetry mechanism for the existence of such a metallic phase as discussed in [14] is quite different from ours. In our case, spin-$s_z$ is conserved, and the emergence of metallic phase in between the TAI and the AI phases is due to the Fermi energy renormalization by disorder, which is excluded in the discussions of [14]. The Fermi energy is renormalized only when the kinetic term, $Dk^2$, is nonvanishing. The combination of Fermi energy and mass renormalization leads to the renormalization of the gap center in the upward direction, as seen in the phase diagram in figure 2(a), as well as the complete dependence on disorder strength of energy spectrum as shown in figure A.1 of appendix A. Therefore if we focus on the energy $E_f = 0$ which is initially in the gap, then this energy will extend to the band region at large disorder strength so that the system shows conducting behavior. This picture is supported by our scaling analysis (see footnote 6) shown by blue lines in figure 5(a). But if the kinetic term vanishes, as considered in [14] where a particle–hole symmetry is constrained, then the Fermi energy will not be renormalized and neither will the gap center. So, starting at $E_f = 0$, the finite intervening metallic region shrinks into a point due to the lack of $s_z$-conservation inversion-asymmetry [14], see the green lines in figure 5(a), which is consistent with the universal phase diagrams in TIs [11, 14, 15]. However, since the kinetic term of the HgTe/CdTe QWs exists in general, such a metallic phase may still be observed in experiments even in the absence of $s_z$-nonconservation inversion-asymmetry.

4. Conclusions

In summary, we have studied the disordered HgTe/CdTe QWs in detail. A complete dependence of its energy spectrum in the generalized momentum space on disorder strength is obtained. The disorder-induced edge states are found to appear only after a gap-closing phase transition, which symbolizes a change in topological number. These edge states are further shown to possess the same spin–momentum locking properties as the helical liquid in QSH systems. With nonmagnetic perturbations which preserve the time-reversal symmetry, the edge states are shown to be destroyed through coupling with each other. For a finite-width system, the edge states decay in a stepwise pattern assisted by the localized bulk states due to the fluctuations of the decay length, which is in extraordinary contrast with the exponential decay of bulk states. In contrast, for an infinite system the edge states can become decoherent only through

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coupling with extended states; therefore a localization–delocalization Anderson transition is expected in the strong-disorder region, which is confirmed by the scaling analysis. Moreover, multiple Anderson transitions are also obtained, where in particular a metallic region that separates two topologically distinct insulating phases is found. The emergence of such a metallic phase is quite different from what is seen in the universal phase diagrams in TIs where the $s_z$-nonconservation inversion-asymmetry is essential. In our case, the metallic phase results from the renormalization of the gap center, which is absent if the particle–hole symmetry appears. However, since this symmetry is in general lacking in disordered HgTe/CdTe QWs, we may expect to observe the metallic phase in between the TAI and AI phases even in the absence of $s_z$-nonconservation inversion-asymmetry.

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Appendix A. The complete dependence of the energy spectrum on disorder strength in disordered HgTe/CdTe quantum wells (QWs)

In figure A.1, we have presented the complete dependence of the energy spectrum on disorder strength in disordered HgTe/CdTe QWs, where the disorder strength ranges from $W = 0$ to 200 meV every 10 meV. In these figures, five representatives at $W = 20, 40, 80, 120$ and 200 meV are shown in figure 2(b) in the main text. Here twisted and open boundary conditions are used in the $x$- and $y$-directions, respectively. The system size for the disordered HgTe/CdTe QWs is chosen as $L_x = 15a$ and $L_y = 100a$. We see that the band edges, and so the gap center shift upward with an increase of disorder strength because of the combination of Fermi energy and mass renormalization.

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Figure A.1. Complete dependence of the energy spectrum on disorder strength in disordered HgTe/CdTe QWs. All the energies are in units of meV, and the system size is the same as in figure 2.
Figure B.1. Comparison of the dependence of the energy spectrum on disorder strength between OBC and PBC in the y-direction with $W \in [80, 200]$. The twisted boundary condition is used in the x-direction in both cases. The edge states are identified as the extra states in the spectrum with OBC relative to the spectrum with PBC. We have marked the edge states with purple dashed lines in figure 2(b) as a guide to the eye. The system size is the same as in figure 2.

Appendix B. Comparison of the energy spectrum in disordered HgTe/CdTe QWs between periodic and open boundary conditions in the y-direction

In figure B.1, we explain how the purple dashed lines are marked in figure 2(b) of the main text, and why they are edge states. These purple dashed lines are identified as the extra states appearing in the energy spectrum with open boundary condition (OBC) in the y-direction by comparing with the energy spectrum using the periodic boundary condition (PBC) in the y-direction. The twisted boundary condition is used in both cases in the x-direction. Since edge states exist only when the system has boundaries, these extra states in the spectrum with OBC are safely recognized as the edge states compared with that using PBC.

Appendix C. Spin–momentum locking of the edge states in disordered HgTe/CdTe QWs in the topological Anderson insulator phase

In figure B.1, we have analyzed the spin–momentum locking properties of the edge states in TAI plateaux by studying their wave-functions density and the spin polarization distributions at a given energy for different disorder strengths. It is clearly seen that the left movers are correlated with spin-up and the right movers are correlated with spin-down at the $y = 0$ boundary of the $W = 120$ panel, which is similar to the helical liquid in the QSH systems. In contrast, this spin–momentum correlation is reversed at the same boundary for the $W = 200$ panel.
Figure C.1. Examples of the wave-function density and spin polarization distributions on the purple dashed lines of the last two panels of figure 2(b). The energies are $E = 9.149$ meV for $W = 120$ and $E = 31.4$ meV for $W = 200$.

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