Gauge theory of the Hubbard model on honeycomb lattice and its instanton effect

Abolhassan Vaezi* and Xiao-Gang Wen

Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

(Dated: January 11, 2011)

In this paper we investigate possible spin disordered phase in the Hubbard model on the honeycomb lattice. Using a slave-particle theory that include the charge fluctuations, we find a meanfield spin disordered phase in a range of on-site repulsion $U$. The spin disordered state is described by gapped fermions coupled to compact $U(1)$ gauge field. We study the confinement/deconfinement problem of the $U(1)$ gauge theory due to the instants proliferation. We calculate all allowed instanton terms and compute their quantum numbers. It is shown that the meanfield spin disordered phase is unstable. The instantons proliferation induce a translation symmetry breaking.

I. INTRODUCTION

Hubbard model[1, 2] is believed to describe the physics of many strongly correlated systems e.g., Mott insulator[3, 4] and high temperature superconductors[5–7]. Motivated by the experiments of high temperature superconductors, people have been look for spin liquid phase[8–10] that does not break any symmetry in in large $U$ Hubbard model and its generalization.

Recently, Meng et al. [11–13] have studied the Hubbard model on the honeycomb lattice using quantum Monte Carlo (QMC) method. They only consider the bipartite system in which the Hamiltonian only connects states on different sublattices. The reason is that the QMC does not have sign problem in this case. They report a gapped spin liquid phase for a range of $U/t$. This phase is an insulating that does not appear to break any symmetry. The true nature of the phase is still under debate.

Recently, in Ref. [14], we have studied the phase diagram of the Hubbard model on the honeycomb lattice (see Fig. 1 and 2) using the generalized slave-particle technique [15–18] which include charge fluctuations. Within meanfield approximation, we find a Mott phase transition to the insulating phase at $U_{c1} \approx 2.2t$ above which charge gap opens up and we obtain the gapped spin liquid phase (spin/charge gapped phase). There is also another phase transition between the spin liquid and the anti-ferromagnetic order phases at $U_{c2} \approx 3t$. In this phase, in contrast to the the gapped spin liquid phase, the mass of spinons is very small and negligible. In the case of nearest neighbor hopping Hubbard model, which is a bipartite system, the gauge theory of our meanfield spin liquid phase is the compact staggered $U(1)$. In this phase, all excitations are gapped except gauge fluctuations. These results are within mean-field and due to the compactness of the $U(1)$ lattice gauge theory, stability of such mean-field states are under question. In compact $U(1)$ gauge theories, instanton (anti-instanton) configurations are allowed and when they proliferate, spinons become confined and the results of the mean-field are no longer valid. Therefore studying the fate of this gapped spin liquid is necessary.

In this paper we find that instanton configurations are relevant. Instantons have nonzero fugacity and we do obtain a confined phase. More importantly, instanton operators carry a non-trivial crystal momentum. Also, under 60 degree lattice rotation and parity, an instanton is changed to an anti-instanton. However, the instantons carry trivial quantum numbers for other symmetries. Since a triple instanton carries trivial quantum numbers for all symmetries, so triple instanton can proliferate which leads to a confined phase. Since single instanton carries non-trivial crystal momentum, this allows us to conclude that the $U(1)$ confined phase is a phase that break translation symmetry but not spin rotation symmetry. Therefore we finally obtain an insulating phase at half filling that breaks the lattice translation symmetry! On the other hand, in the presence of second neighbor hopping in the Hubbard model the charge/spin gapped phase can be spin liquid that do not break translation, parity, 60 degree lattice rotation, and spin rotation symmetries.

II. SYMMETRY TRANSFORMATIONS ON THE HONEYCOMB LATTICE

Since the unit cell of the honeycomb lattice has two sites in it, and we can label them by $A$ and $B$ or $s = 0, 1$, any lattice point can be represented as $\bar{R} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + sy \equiv (x_1, x_2, s)$, where $\vec{a}_1 = \sqrt{3}a(1,0)$ ($a$ is the lattice spacing

*Electronic address: vaezi@mit.edu
between A and B atoms), $\tilde{a}_2 = \sqrt{3}a \left( \frac{1}{2}, \frac{-\sqrt{3}}{2} \right)$ and $s$ takes 0 and 1 values. The basis vectors of the reciprocal lattice are $\vec{G}_1 = \frac{4\pi}{3a} \left( \frac{\sqrt{3}}{2}, 1 \right)$ and $\vec{G}_2 = \frac{4\pi}{3a} \left( 0, 1 \right)$ (see Fig. 3). Type A atoms are connected to type B atoms by the following three vectors: $\vec{b}_1 = a(0, 1)$, $\vec{b}_2 = a \left( \frac{\sqrt{3}}{2}, \frac{-1}{2} \right)$ and $\vec{b}_3 = a \left( -\frac{\sqrt{3}}{2}, \frac{-1}{2} \right)$. Honeycomb lattice is invariant under five symmetry transformations: time reversal, parity ($\sigma : (x, y) \to (x, -y)$), 60 degree rotation ($C_6$), translation along $\vec{a}_1$ and $\vec{a}_2$ ($T_1$ and $T_2$). It is easy to check these symmetry operations act on the lattice as the following [19]:

$$T_1 : (x_1, x_2, s) \to (x_1 + 1, x_2, s) \quad (1)$$

$$T_2 : (x_1, x_2, s) \to (x_1, x_2 + 1, s) \quad (2)$$

$$T : (x_1, x_2, s) \to (x_1, x_2, s) \quad (3)$$

$$\sigma : (x_1, x_2, s) \to (x_1 + x_2, -x_2, 1 - s) \quad (4)$$

$$C_6 : (x_1, x_2, s) \to (1 - s - x_2, x_1 + x_2 + s - 1, 1 - s). \quad (5)$$

III. METHOD

In the Anderson-Zou slave particle method, electron operators are represented as:

$$C^\dagger_{i,\sigma} = f^\dagger_{i,\sigma} h_i + \sigma d^\dagger_{i,-\sigma} = \begin{bmatrix} f^\dagger_{i,\sigma} & d^\dagger_{i,-\sigma} \end{bmatrix} = \begin{bmatrix} f^\dagger_{i,\sigma} & d^\dagger_{i,-\sigma} \end{bmatrix} f^\sigma_{i,\sigma} = \begin{bmatrix} f^\dagger_{i,\sigma} & d^\dagger_{i,-\sigma} \end{bmatrix} \begin{bmatrix} f^\dagger_{i,\sigma} & d^\dagger_{i,-\sigma} \end{bmatrix}, \quad (6)$$

where $f^\dagger_{i,\sigma}$ creates a state with a single electron on it (a spinon), $h^\dagger_i$ creates a state with no charge on it (a holon), and $d^\dagger_i$ creates a state with two electron on site $i$ (a doublon). It should be mentioned that the physical Hilbert
space contains only four states: empty state (holon), one electron (spinon) and two electrons (doublon) on each site. So we always have one and only one slave particle on each site. So we conclude that we should put the local constraint $n^b_i + n^f_i + n^b_i + n^d_i = 1$, to get rid of redundant states. This is the physical constraint which should be satisfied on every site. We could also obtain this result by noting that the electron operators are fermion and should satisfy the anticommutation relations. From the definition of $C_i$, it is obvious that it is invariant under $U(1)$ gauge transformation (We require $h_i$ and $d_i$ to remain bosonic operators i.e., preserve their statistics after transformation, otherwise we would have SU(2) gauge invariance. However at $U = \infty$ we have only fermions and only in that case we have SU(2) gauge symmetry). It is worth noting that all the slave particles carry the same charge under the internal $U(1)$ gauge. Since the above constraint and as a result the Hubbard Hamiltonian are also gauge invariant, so is the action of the Hubbard model.

Using this slave technique the Hubbard Hamiltonian can be rewritten as the following:

$$H = \sum U d_i^\dagger d_i - t \sum_{\langle i,j \rangle} (\chi^\dagger_{i,j} \chi_{j,i}^b + \Delta^+_{i,j} \Delta^b_{i,j} + h.c.)
+ \lambda(f^\dagger_{i,t} f_{i,t} + f^\dagger_{i,i} f_{i,i} + h^b_i h_i + d^b_i d_i - 1),$$

in which we have used these notations $\chi^b_{i,j} = \sum_{\sigma} f_{i,\sigma}^\dagger f_{j,\sigma}$, $\chi^b_{i,j} = h^b_i h_i - d^b_i d_i$, $\Delta^b_{i,j} = \sum_{\sigma} \sigma f^\dagger_{-j,\sigma} f_{j,\sigma} + \Delta^b_{i,j} = d_i h_i + h_i d_i$. Within mean field and by using Hubbard-Stratonovic we can decouple spinons and bosons and obtain a mean field state. In our numerical studies we have obtained three phases. At small $U/t$ limit we obtain a semi-metallic phase. At large $U/t$ limit we obtain AF order and for moderate values of $U/t$ we obtain a spin liquid phase.

\section{IV. Instanton Proliferation and Confinement}

Now let us focus on the spin liquid phase. In this phase: $\chi_{i,j}^{f,h} = 0$. Therefore the effective Hamiltonian of spinons in this phase is of the following forms:

$$H_s = \lambda \sum_{i,\sigma,\tau} f_{i,\tau,\sigma}^\dagger f_{i,\tau,\sigma} - t \sum_{\langle i,j \rangle,\sigma,\tau} \Delta_b (i,j) \sigma f_{i,A,\sigma}^\dagger f_{j,B,-\sigma} + h.c..$$

Now let us use the following ansatz: $\Delta_b (\vec{k}) = \Delta_b$. So we have $\Delta_b (\vec{k}) = \Delta f \eta (\vec{k})$, where $\eta (\vec{k}) = e^{-ik_y} + 2e^{i\frac{\sqrt{2}}{2} \cos \frac{\sqrt{2}}{2} k_x}$. From the energy spectrum we see that spinons are gapped. But what if we include the effect of instantons? To answer this important question, we first study the gauge theory of this mean-field state.

In the above effective Hamiltonian we transform operators as: $f_{i,A,\sigma} \rightarrow e^{i\alpha} f_{i,A,\sigma}$ and $f_{i,B,\sigma} \rightarrow e^{-i\alpha} f_{i,B,\sigma}$ for any arbitrary phase $\alpha$, i.e. assuming a staggered global gauge transformation, then the effective Hubbard Hamiltonian does not change. Therefore the $IGG$ of the Hamiltonian is staggered $U(1)$. The reason is that there is no hopping term due
to the non-zero charge gap and the if we the gauge transformation of two neighboring sites have opposite phases, the total phase change of the pairing term becomes zero and therefore gauge fluctuations are described by staggered compact $U(1)$ instead of compact $U(1)$ gauge theory. This is equivalent to assuming have positive unit charge on sublattice $A$ and negative unit charge on sublattice $B$ for slave particles under the internal gauge transformation. So, at mean field level, the charge/spin gapped phase has a neutral spinless $U(1)$ gapless mode as its only low energy excitations. However, it is well known that $U(1)$ theory in 2+1D is confined due to instanton effects. So in the latter part of this paper, we will assume that the $U(1)$ fluctuations are weak and use the semiclassical approach to study the $U(1)$ confined phase where the $U(1)$ mode is gapped.

We like to remark that it is possible to break this staggered compact $U(1)$ down to a $Z_2$ [20, 24] one by Anderson-Higgs mechanism. If we add second neighbor hopping to the Hubbard model, within slave boson this term generate pairing terms of the form $f^\dagger_i \tau_i f^\dagger_j \tau_j - \sigma, i.e. it induces the same sublattice pairing and the Hamiltonian is no longer invariant under the staggered global $U(1)$ gauge transformation. In this case gauge fluctuations are gapped and thus our mean filed state is stable and we can trust our results. Therefore for this case spin liquid phase is physical. On the frustrated lattices like the triangular lattice the gauge theory is manifestly:

$$H_s = \sum_{i,\sigma,\tau} \lambda \left( f^\dagger_{i,A,\sigma} f^\dagger_{i,A,\sigma} - f^\dagger_{i,B,\sigma} f^\dagger_{i,B,\sigma} \right) - t \sum_{i,\delta,\sigma,\tau} \Delta_b \left( i, j \right) \sigma f^\dagger_{i,A,\sigma} f^\dagger_{j,B,\sigma} + h.c. \right).$$

(9)

In the absence of $\lambda$ energy band of spinons has two nodal points around $\vec{K} = \frac{4\pi}{3\sqrt{3}a} (1, 0)$ and $\vec{K}' = \frac{4\pi}{3\sqrt{3}a} \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right)$ (see Fig. 3). If we expand the $\eta_k$ around these two points we have:

$$\eta (K + \vec{q}) = \frac{3a}{2} (-q_x + iq_y)$$

$$\eta (K' + \vec{q}) = \frac{3a}{2} (q_x + iq_y),$$

(10)

(11)

where $\vec{q} = (q_x, q_y)$. So we have 8 flavors of spinons depending on their physical spin degree of freedom, sublattice index, and wether their momentum is around $K$ or $K'$. Therefore we define the 8 component spinor as: $\Psi^\dagger (x) = \left( f_{A,K,\uparrow} (x), f_{B,K',\uparrow} (x), f_{A,K,\downarrow} (x), f_{B,K',\downarrow} (x), f_{A,K',\uparrow} (x), f_{B,K',\uparrow} (x), f_{A,K',\downarrow} (x), f_{B,K',\downarrow} (x) \right)$. Using the linearized Hamiltonian around $K$ and $K'$ it is straightforward to show that the continuum model can be written as:

$$H = \int d^2x \Psi^\dagger (x) \left[ \lambda \mu^a \otimes \nu^0 \otimes \nu^0 - \frac{3a}{2} \Delta_b \partial_x \mu^1 \otimes \nu^0 \otimes \nu^0 - \frac{3a}{2} \Delta_b \partial_y \mu^2 \otimes \nu^0 \otimes \nu^0 \right] \Psi (x),$$

(12)

where $\mu^a (a = 0, 1, 2, 3)$ are Pauli matrices acting on the sublattice indices, $\tau^b$ are Pauli matrices acting on the physical spin, and $\nu^c$ are Pauli matrices that act on the valley indices. It is known that if filled band has a nontrivial total Chern number, then instanton effects can be ignored, since the nontrivial Chern number lead to a Chern-Simons term for the $U(1)$ gauge field. So we need to calculate the Chern number of the filled band.

We will calculate the Chern numbers through the Dirac nodes. Each Dirac node contribute $\pm 1/2$ Chern numbers, and each filled band have an even number of Dirac nodes. So adding the contribution from all the Dirac nodes, we obtain an integer Chern number for the filled band.

Let us consider the following two by two Dirac theory:

$$H = \int d^2x \psi^\dagger (x) \left[ m \sigma^3 + \frac{3a}{2} \Delta_b \psi^\dagger (x) \left( -i\sigma^1 \partial_x + i\sigma^2 \partial_y \right) \right] \psi (x),$$

(13)

where $\sigma^i$ are Pauli matrices. The mass of the above Hamiltonian is by definition $m$. It has been shown that each massive Dirac cone with mass $m$ has $C = \frac{m}{2m}$ nontrivial Chern number [23, 24]. Now let us consider following Hamiltonian:

$$H = \int d^2x \psi^\dagger (x) \left[ m \sigma^3 + \frac{3a}{2} \Delta_b \left( -i\sigma^1 \partial_x - i\sigma^2 \partial_y \right) \right] \psi (x).$$

(14)

If we us the following transformation: $\psi \to \psi' = \sigma^3 \psi$, then the above Hamiltonian can be rearranged as the following:

$$H = \int d^2x \psi'^\dagger (x) \left[ -m \sigma^3 + \frac{3a}{2} \Delta_b \left( -i\sigma^1 \partial_x + i\sigma^2 \partial_y \right) \right] \psi' (x).$$

(15)
Therefore the mass of this Hamiltonian is \(-m\), and therefore it has \(C = -\frac{m}{|m|}\) nontrivial Chern number.

Using the above arguments it can be shown that the mass of the two Dirac cones at \(\vec{k} = \vec{K}\) is \(\lambda < 0\) and they contribute \(C = -\frac{1}{2}\) Chern number. The mass of the two other Dirac cones at \(\vec{k} = \vec{K}'\) is \(-\lambda > 0\) and they contribute \(C = \frac{1}{2}\) Chern number. Therefore the total Chern number of our theory is \(2 \times -1/2 + 2 \times 1/2 = 0\). So the coefficient of the Chern-Simon action is zero, and it does not constraint the proliferation of instantons.

On the other hand since we have a massive Dirac theory, instant-instanton correlation function at large distances DOES NOT decay exponentially. Therefore nothing prevents instantons from proliferation. They will proliferate and gap out the gauge particles. So the \(U(1)\) gauge theory is in the confined phase. Now we should compute the quantum numbers \(\mathbb{Z}_{27-32}\) of instantons, in order to understand the symmetry properties of the \(U(1)\) confined phase. To do so let us first derive the instanton operators.

Since instantons (instantons) in the presence of the Chern-Simon action with chern number \(C\), create \(C\) fermions, therefore the instanton operator creates \(2 \times -1/2 = -1\) fermions at \(\vec{k} = \vec{K}\) (i.e. annihilates 1 fermion at \(\vec{K}\)), and creates \(2 \times 1/2 = 1\) fermions at \(\vec{k} = \vec{K}'\). Therefore, we obtain many different possibilities for the instanton operators, which include:

\[
\begin{align*}
\phi_1 &= f^\dagger_{A,\uparrow,K} f_{A,\uparrow,K} \\
\phi_2 &= F^\dagger_{B,\uparrow,K} f_{A,\uparrow,K} = f_{B,\downarrow,K} f_{A,\uparrow,K} \\
\phi_3 &= f^\dagger_{A,\downarrow,K} f_{A,\uparrow,K} \\
\phi_4 &= F^\dagger_{B,\downarrow,K} f_{A,\uparrow,K} = f_{B,\uparrow,K} f_{A,\uparrow,K} \\
\phi_5 &= f^\dagger_{A,\uparrow,K} F_{B,\uparrow,K} = f^\dagger_{A,\uparrow,K} f^\dagger_{B,\downarrow,K} f_{B,\downarrow,K} f_{A,\uparrow,K} \\
\phi_6 &= F^\dagger_{B,\uparrow,K} F_{B,\uparrow,K} = f_{B,\downarrow,K} f_{B,\downarrow,K} f_{B,\uparrow,K} f_{B,\uparrow,K} \\
\phi_7 &= f^\dagger_{A,\downarrow,K} F_{B,\uparrow,K} = f^\dagger_{A,\downarrow,K} f^\dagger_{B,\downarrow,K} f_{B,\downarrow,K} f_{B,\downarrow,K} \\
\phi_8 &= F^\dagger_{B,\downarrow,K} F_{B,\uparrow,K} = f_{B,\uparrow,K} f_{B,\uparrow,K} f_{B,\downarrow,K} f_{B,\downarrow,K} \\
\phi_9 &= f^\dagger_{A,\uparrow,K} f_{A,\downarrow,K} \\
\phi_{10} &= F^\dagger_{B,\uparrow,K} f_{A,\downarrow,K} = f_{B,\downarrow,K} f_{A,\downarrow,K} \\
\phi_{11} &= f^\dagger_{A,\downarrow,K} f_{A,\downarrow,K} \\
\phi_{12} &= F^\dagger_{B,\downarrow,K} f_{A,\downarrow,K} = f_{B,\uparrow,K} f_{A,\downarrow,K} \\
\phi_{13} &= f^\dagger_{A,\uparrow,K} F_{B,\downarrow,K} = f^\dagger_{A,\uparrow,K} f^\dagger_{B,\downarrow,K} f_{B,\downarrow,K} f_{B,\downarrow,K} \\
\phi_{14} &= F^\dagger_{B,\uparrow,K} F_{B,\downarrow,K} = f_{B,\downarrow,K} f_{B,\downarrow,K} f_{B,\uparrow,K} f_{B,\uparrow,K} \\
\phi_{15} &= f^\dagger_{A,\downarrow,K} F_{B,\downarrow,K} = f^\dagger_{A,\downarrow,K} f^\dagger_{B,\downarrow,K} f_{B,\downarrow,K} f_{B,\downarrow,K} \\
\phi_{16} &= F^\dagger_{B,\downarrow,K} F_{B,\downarrow,K} = f_{B,\uparrow,K} f_{B,\downarrow,K} f_{B,\uparrow,K} f_{B,\downarrow,K}.
\end{align*}
\]

It is obvious that all the above operators carry nonzero crystal momentum which is equal to \(\vec{K}' - \vec{K}\). Since the microscopic Hubbard Hamiltonian does not break translation symmetry, therefore the single instanton operator cannot appear in the path integral. On the other hand since \(3 (\vec{K}' - \vec{K}) = (0,0)\), triple-instanton is not forbidden and will appear in the path integral. So the the path integral contain a triple-instanton gas, which will cause a confinement of the \(U(1)\) theory.

V. QUANTUM NUMBER OF INSTANTONS

Now we like to compute other nontrivial quantum numbers of the above instanton operators. To do so, let us first comment on the transformation of the continuum wave-function. Using transformation rules in Table 1, we obtain following relations:
TABLE I: Symmetry transformation rules of spinon operators under translation, time reversal, parity and $\pi/3$ rotation.

| $f_{A,α,K}(x)$ | $T_1$ | $T_2$ | $T$ | $σ$ | $C_6$ |
|----------------|-------|-------|-----|-----|-------|
| $f_{A,α,K}(x)$ | $\exp\left(i\frac{2π}{3}\right)f_{B,α,K}(T_1 x)$ | $\exp\left(i\frac{2π}{3}\right)f_{B,α,K}(T_2 x)$ | $αf_{A,−α,K′}(x)$ | $f_{B,α,K}(σx)$ | $f_{B,α,K′}(C_6 x)$ |
| $f_{A,α,K}(x)$ | $\exp\left(i\frac{2π}{3}\right)f_{A,α,K}(T_1 x)$ | $\exp\left(i\frac{2π}{3}\right)f_{A,α,K}(T_2 x)$ | $αf_{B,−α,K}(x)$ | $f_{B,α,K′}(σx)$ | $f_{B,α,K′}(C_6 x)$ |
| $f_{A,α,K′}(x)$ | $\exp\left(i\frac{2π}{3}\right)f_{B,α,K′}(T_1 x)$ | $\exp\left(i\frac{2π}{3}\right)f_{B,α,K′}(T_2 x)$ | $αf_{A,−α,K}(x)$ | $f_{B,α,K′}(σx)$ | $f_{B,α,K′}(C_6 x)$ |
| $f_{B,α,K′}(x)$ | $\exp\left(i\frac{2π}{3}\right)f_{A,α,K′}(T_1 x)$ | $\exp\left(i\frac{2π}{3}\right)f_{A,α,K′}(T_2 x)$ | $αf_{B,−α,K}(x)$ | $f_{A,α,K′}(σx)$ | $f_{A,α,K′}(C_6 x)$ |

Physical spin rotation around z axis by angle $θ$: $S_x\Psi → e^{iθτ^3/2}\Psi(x)$.

Physical spin rotation around y axis by angle $θ$: $S_y\Psi → e^{iθτ^3/2}\Psi(x)$.

Translation $T_1$: $T_1\Psi → e^{-i\frac{2π}{3}\Psi}(x')$.

Translation $T_2$: $T_2\Psi → e^{+i\frac{2π}{3}\Psi}(x')$.

Time reversal: $T\Psi → iτ^2 \otimes ν^1\Psi(x')$.

Parity: $σ\Psi → μ^1 \otimes τ^1 \otimes ν^0\Psi^\dagger(x')$.

$C_6$: $C_6\Psi → μ^1 \otimes τ^1 \otimes ν^1\Psi^\dagger(x')$.

where $x'$ is the transformed $x$ under each symmetry transformations.

A. Symmetry transformations on instanton operators

Using symmetry transformation of continuum wavefunction, we can read the corresponding transformation of monopole operators. Under $π/3$ rotation ($C_6$) we have:

$$C_6 : \phi_i → −\phi^\dagger_{17−i}. \quad (32)$$

Parity operator $σ$ where takes $y$ to $−y$ acts on monopole operators as following:

$$\phi_1 → −\phi_{16}, \quad \phi_2 → −\phi_{12}, \quad \phi_3 → −\phi_8, \quad \phi_4 → −\phi_4 \quad (33)$$
$$\phi_5 → −\phi_{15}, \quad \phi_6 → −\phi_{11}, \quad \phi_7 → −\phi_7, \quad \phi_8 → −\phi_3 \quad (34)$$
$$\phi_9 → −\phi_{14}, \quad \phi_{10} → −\phi_{10}, \quad \phi_{11} → −\phi_6, \quad \phi_{12} → −\phi_2 \quad (35)$$
$$\phi_{13} → −\phi_{13}, \quad \phi_{14} → −\phi_9, \quad \phi_{15} → −\phi_5, \quad \phi_{16} → −\phi_1. \quad (36)$$

Under time reversal $T$ we have

$$\phi_1 → +\phi^\dagger_{11}, \quad \phi_2 → −\phi^\dagger_{15}, \quad \phi_3 → −\phi^\dagger_3, \quad \phi_4 → +\phi^\dagger_7 \quad (37)$$
$$\phi_5 → −\phi^\dagger_{12}, \quad \phi_6 → +\phi^\dagger_{16}, \quad \phi_7 → +\phi^\dagger_4, \quad \phi_8 → −\phi^\dagger_8 \quad (38)$$
$$\phi_9 → −\phi^\dagger_6, \quad \phi_{10} → +\phi^\dagger_{13}, \quad \phi_{11} → +\phi^\dagger_1, \quad \phi_{12} → −\phi^\dagger_5 \quad (39)$$
$$\phi_{13} → +\phi^\dagger_{10}, \quad \phi_{14} → −\phi^\dagger_{14}, \quad \phi_{15} → −\phi^\dagger_2, \quad \phi_{16} → +\phi^\dagger_6. \quad (40)$$

Monopole operators transform under translation along $R_1$ as

$$T_1 : \phi_k → \exp\left(\frac{2π}{3}\right)\phi_k. \quad (41)$$

Similarly for translation along $R_2$ we have

$$T_2 : \phi_k → \exp\left(-\frac{2π}{3}\right)\phi_k. \quad (42)$$
And finally rotation around z axis by \( \theta \) angle:

\[
\begin{align*}
\phi_1 &\rightarrow \phi_1, & \phi_2 &\rightarrow \phi_2, & \phi_3 &\rightarrow \exp(i\theta) \phi_3, & \phi_4 &\rightarrow \exp(i\theta) \phi_4 \\
\phi_5 &\rightarrow \phi_5, & \phi_6 &\rightarrow \phi_6, & \phi_7 &\rightarrow \exp(i\theta) \phi_7, & \phi_8 &\rightarrow \exp(i\theta) \phi_8 \\
\phi_9 &\rightarrow \exp(-i\theta) \phi_9, & \phi_{10} &\rightarrow \exp(-i\theta) \phi_{10}, & \phi_{11} &\rightarrow \phi_{11}, & \phi_{12} &\rightarrow \phi_{12} \\
\phi_{13} &\rightarrow \exp(-i\theta) \phi_{13}, & \phi_{14} &\rightarrow \exp(-i\theta) \phi_{14}, & \phi_{15} &\rightarrow \phi_{15}, & \phi_{16} &\rightarrow \phi_{16}.
\end{align*}
\]

The instanton operators should have the same symmetry as the microscopic Hamiltonian and therefore they carry trivial quantum numbers. Using the above transformations, it is easy to see The following term is invariant under all transformations:

\[
L = g \int \, d^2x \left( \phi^3(x) + \phi^{13}(x) \right),
\]

where \( \phi \) is defined as the following:

\[
\begin{align*}
\phi &= \phi_2 - \phi_{12} + \phi_5 - \phi_{15} \\
&= f_{B,\downarrow, K} f_{A,\uparrow, K} - f_{B,\uparrow, K} f_{A,\downarrow, K} + f_{A,\uparrow, K'} f_{B,\downarrow, K'} - f_{A,\downarrow, K'} f_{B,\uparrow, K'}.
\end{align*}
\]

This operator has the following symmetry properties:

\[
\begin{align*}
\sigma : & \quad \phi \rightarrow \phi & (50) \\
T : & \quad \phi \rightarrow \phi^\dagger & (51) \\
C_6 : & \quad \phi \rightarrow \phi^\dagger & (52) \\
T_1 : & \quad \phi \rightarrow \exp \left(-i \frac{2\pi}{3} \right) \phi & (53) \\
T_2 : & \quad \phi \rightarrow \exp \left(+i \frac{2\pi}{3} \right) \phi. & (54)
\end{align*}
\]

It can be easily checked that \( \phi \) operator is also invariant under rotation \( x \) and \( y \) axes.

VI. DISCUSSION AND CONCLUSION

We have found a triple-instanton operator that has all symmetries of the microscopic Hamiltonian. Therefore this term is relevant and has a non-zero fugacity. Because of instanton proliferation, the \( U(1) \) gauge fluctuations are now gapped out. On the other hand, since single instanton operator carries nonzero crystal momentum, the translation symmetry breaks spontaneously. To have a better insight of the situation, we can use the duality between \( U(1) \) gauge theory and the nonlinear sigma model. If we identify \( \phi \) operator with \( \exp(i\theta) \), then \( g \left( \phi^3(x) + \phi^{13}(x) \right) = 2g \cos(3\theta) \).

The \( U(1) \) gauge theory with triple-instantons can be described by the following dual theory:

\[
L = \frac{m}{2} \dot{\theta}^2 - \frac{e}{2} \left( \nabla \theta \right)^2 + 2g \cos(3\theta).
\]

Therefore \( T \) and \( C_6 \) transformation are equivalent to \( \theta \rightarrow -\theta \), \( \sigma \) is trivial and \( T_1 \) and \( T_2 \) are equivalent to \( \theta \rightarrow \theta - \frac{2\pi}{3} \) and \( \theta \rightarrow \theta + \frac{2\pi}{3} \) respectively. This model has three inequivalent ground states determined by \( \theta \in \left\{ 0, \frac{2\pi}{3}, \frac{4\pi}{3} \right\} \) (see Fig. 4). Therefore the ground-state degeneracy of our model is also three. This happens because we lose the translation symmetry along \( \vec{a}_1 \) and \( \vec{a}_2 \) direction. The reason is that \( \phi \) operator carries nonzero crystal momentum. But it is easy to show that \( \phi \) is invariant under \( T_1^3 \) and \( T_1 T_2 \) transformation. So the basis vectors of the new lattice are \( \vec{R}_1' = (3, 0) = 3 \vec{R}_1 \) and \( \vec{R}_2' = (1, 1) = \vec{R}_1 + \vec{R}_2 \). The area of the unit cell is three times bigger and it contains six atoms in it.

In summary, if we treat the \( U(1) \) gauge field as a semi-classical field (i.e. Gaussian approximation), by analogy to the nonlinear sigma model, instantons proliferate, gauge field gaps out and lattice symmetry spontaneously breaks. What we finally obtain is a band insulator instead of a spin liquid phase. We want to mention that if gauge fluctuations are strong, other possibilities may happen. Among them, the \( Z_2 \) spin liquid is of more interest. This phase can be obtained if strong gauge fluctuations generate hopping term to the nearest site or a nonzero pairing amplitude to the second neighboring site. The presence of any of these two terms breaks \( U(1) \) down to \( Z_2 \) gauge theory which is stable.
FIG. 4: Two possible valence bond solid (VBS) states in honeycomb lattice that break the translation symmetry. Bold line indicates the stronger bonds and narrow line the weaker links. The bond operator in our case corresponds to the exchange energy i.e. \( \Delta_b (i,j) \). \( \vec{R}_1 = 3 \vec{R}_2 \) and \( \vec{R}_2 = \vec{R}_1 + \vec{R}_2 \) are new basis vectors of the lattice. The area of the unit cell is three times bigger than the translation symmetric case and contains six atoms in it. (a) Honeycomb lattice with broken translation symmetry, while \( C_6 \) and time reversal symmetries are unbroken. This phase corresponds to \( \theta = 0 \). \( \theta \) can correspond to the flux of the hexagon. (b) Honeycomb lattice with broken translation, \( C_6 \) rotation and time reversal symmetry (center of rotation is yellow hexagons). This state corresponds to \( \theta = 2 \pi / 3 \). The phase with \( \theta = -2 \pi / 3 \) is related to this by \( C_6 \) or \( T \). From this figure it is clear that in this case \( C_6 \) breaks down to \( C_3 \).

Acknowledgement

We thank P.A. Lee, T. Senthil, B. Swingle and F. Wang for their useful comments and helpful discussions. This research is supported by NSF Grant No. DMR-1005541 and NSFC 11074140.

---

[1] J. Hubbard, Royal Society of London Proceedings Series A 276, 238 (1963).
[2] J. E. Hirsch, Phys. Rev. B 31, 4403 (1985).
[3] N. F. Mott, Proceedings of the Physical Society A 62, 416 (1949).
[4] M. Imada, A. Fujimori, and Y. Tokura, Rev. Mod. Phys. 70, 1039 (1998).
[5] J. G. Bednorz and K. A. Mueller, Z. Phys. B64 189 (1986).
[6] P. W. Anderson, Science 235, 1196 (1987).
[7] P. A. Lee, N. Nagaosa, and X. Wen, Reviews of Modern Physics 78, 17 (2006).
[8] N. Read and S. Sachdev, Phys. Rev. Lett. 66, 1773 (1991).
[9] X.-G. Wen, Phys. Rev. B 44, 2664 (1991).
[10] X.-G. Wen, Phys. Rev. B 65, 165113 (2002).
[11] Z. Y. Meng, T. C. Lang, F. F. Wessel, S. Assaad, and A. Muramatsu, Nature 464, 847 (2010).
[12] N. Furukawa and M. Imada, Journal of the Physical Society of Japan 61, 3331 (1992).
[13] L. Lilly, A. Muramatsu, and W. Hanke, Phys. Rev. Lett. 65, 1379 (1990).
[14] A. Vaezi and X. Wen, arXiv e-prints (2010), arXiv:1010.5744.
[15] Z. Zou and P. W. Anderson, Phys. Rev. B 37, 627 (1988).
[16] S. Florens and A. Georges, Phys. Rev. B 70, 035114 (2004), arXiv:cond-mat/0404334.
[17] S.-S. Lee and P. A. Lee, Phys. Rev. Lett. 95, 036403 (2005).
[18] T. Senthil, Phys. Rev. B 78, 045109 (2008).
[19] Y. Lu and Y. Ran, ArXiv e-prints (2010), 1005.4229.
[20] Y. Lu and Y. Ran, ArXiv e-prints (2010), arXiv:1007.3266.
[21] C. Xu, ArXiv e-prints (2010), arXiv:1010.0455.
[22] M. Tran and K. Kim, ArXiv e-prints (2010), arXiv:1011.1700.
[23] S. Kou, L. Liu, J. He, and Y. Wu, ArXiv e-prints (2009), arXiv:0910.2070.
[24] S. Sachdev, ArXiv e-prints (2010), arXiv:1012.0299.
[25] R. Jackiw, Phys. Rev. D 29, 2375 (1984).
[26] R. Jackiw and S. Pi, Physics Letters B 423, 364 (1998), arXiv:hep-th/9712087.
[27] J. Goldstone and F. Wilczek, Phys. Rev. Lett. 47, 986 (1981).
[28] J. Alicea, Phys. Rev. B 78, 035126 (2008).
[29] C. Xu, Phys. Rev. B 78, 054432 (2008).
[30] L. Balents, L. Bartosch, A. Burkov, S. Sachdev, and K. Sengupta, Phys. Rev. B 71, 144508 (2005).
[31] L. Balents, L. Bartosch, A. Burkov, S. Sachdev, and K. Sengupta, Phys. Rev. B 71, 144509 (2005).
[32] W.-H. Ko, P. A. Lee, and X.-G. Wen, Phys. Rev. B 79, 214502 (2009).