Multiplicity of closed characteristics on symmetric convex hypersurfaces in $\mathbb{R}^{2n}$

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Abstract

Let $\Sigma$ be a compact $C^2$ hypersurface in $\mathbb{R}^{2n}$ bounding a convex set with non-empty interior. In this paper it is proved that there always exist at least $n$ geometrically distinct closed characteristics on $\Sigma$ if $\Sigma$ is symmetric with respect to the origin.

1 Introduction and main results.

Our aim in this paper is to study the multiplicity of closed characteristics on any $C^2$-convex compact smooth hypersurface in $\mathbb{R}^{2n}$ which is symmetric with respect to the origin. Let $\Sigma$ be a $C^2$-compact hypersurface in $\mathbb{R}^{2n}$ bounding a convex compact set $C$ with non-empty interior, possess a non-vanishing Gaussian curvature, and $0 \in C$. We denote the set of all such hypersurfaces in $\mathbb{R}^{2n}$ by $\mathcal{H}(2n)$, and the set $\{\Sigma \in \mathcal{H}(2n) \mid \Sigma = -\Sigma\}$ by $\mathcal{SH}(2n)$, where $-\Sigma = \{x \in \mathbb{R}^{2n} \mid -x \in \Sigma\}$. For

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$x \in \Sigma$, let $N_{\Sigma}(x)$ be the outward unit normal vector at $x$ on $\Sigma$. We consider the given energy problem of finding $\tau > 0$ and an absolutely continuous curve $x: [0, \tau] \rightarrow \mathbb{R}^{2n}$ such that

$$
\begin{cases}
  \dot{x}(t) = JN_{\Sigma}(x(t)), & x(t) \in \Sigma, \\
  x(\tau) = x(0),
\end{cases}
$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $I_n$ is the identity matrix on $\mathbb{R}^n$. When there is no confusion we shall omit the subindex of the identity matrices. A solution $(\tau, x)$ of the problem (1) is called a closed characteristic on $\Sigma$. Two closed characteristics $(\tau, x)$ and $(\sigma, y)$ are geometrically distinct, if $x(R) \neq y(R)$. We denote by $J(\Sigma)$ and $\tilde{J}(\Sigma)$ the set of all closed characteristics $(\tau, x)$ on $\Sigma$ with $\tau$ being the minimal period of $x$ and the set of all geometrically distinct ones respectively. For $(\tau, x) \in J(\Sigma)$, we denote by $[(\tau, x)]$ the set of all elements in $J(\Sigma)$ which is geometrically the same as $(\tau, x)$. $\# A$ denotes the total number of elements in a set $A$.

The study on closed characteristics in the global sense started in 1978, when the existence of at least one closed characteristic on any $\Sigma \in \mathcal{H}(2n)$ was first established by P. Rabinowitz in [22] (for star-shaped hypersurfaces) and A. Weinstein in [25] independently. In [3] of I. Ekeland and L. Lassoued, [8] of I. Ekeland and H. Hofer in 1987, and [23] of A. Szulkin in 1988, $\# \tilde{J}(\Sigma) \geq 2$ was proved for any $\Sigma \in \mathcal{H}(2n)$ when $n \geq 2$. In the recent paper [20], Y. Long and C. Zhu proved $\# \tilde{J}(\Sigma) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$ for any $\Sigma \in \mathcal{H}(2n)$, where $\lfloor a \rfloor$ denotes the greatest integer which is not greater than $a$. On the other hand, $\# \tilde{J}(\Sigma) \geq n$ was proved by I. Ekeland and J. M. Lasry in [7] of 1980 for $\Sigma \in \mathcal{H}(2n)$ which is $\sqrt{2}$-pinched, by M. Girardi in [10] of 1984 for $\Sigma \in \mathcal{SH}(2n)$ which is $\sqrt{3}$-pinched. Other related significant progresses can be found in [24] and [21] for local results, and in [1], [3], and [11] for global results.

A typical example of $\Sigma \in \mathcal{H}(2n)$ is the ellipsoid $\mathcal{E}_n(r)$ defined as follows. Let $r = (r_1, \ldots, r_n)$ with $r_k > 0$ for $1 \leq k \leq n$. Define

$$
\mathcal{E}_n(r) = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^{2n} | \frac{1}{2} \sum_{k=1}^{n} \frac{|x_k|^2}{r_k^2} = 1 \}. 
$$

If $r_j/r_k$ is irrational whenever $j \neq k$, this $\mathcal{E}_n(r)$ is called a weakly non-resonant ellipsoid. In this case, it is well known that there are precisely $n$ geometrically distinct closed characteristics on $\mathcal{E}_n(r)$ (cf. §I.7 of [3]).

A long standing conjecture mentioned by I. Ekeland on Page 235 of [3] is

$$\# \tilde{J}(\Sigma) \geq n, \quad \forall \Sigma \in \mathcal{H}(2n).$$

Our following main result in this paper gives a positive answer to this conjecture for $\Sigma \in \mathcal{SH}(2n)$. 

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Theorem 1.1 For every $\Sigma \in \mathcal{SH}(2n)$, we have
\[ \# \tilde{J}(\Sigma) \geq n. \] (3)

For reader’s convenience, we briefly sketch the main idea of the proof below. Fix a $\Sigma \in \mathcal{SH}(2n)$ and suppose $\# \tilde{J}(\Sigma) < +\infty$. Our proof of this theorem is carried out in the following 7 steps.

1. Using the method of I. Ekeland (cf. Section V.3 of [5]), this problem is transformed to a fixed energy problem, and then to a 1-periodic solution problem of a Hamiltonian system, whose Hamiltonian function is defined by $H_\alpha(x) = j_\Sigma(x)^\alpha$ for any $x \in \mathbb{R}^{2n}$ in terms of the gauge function $j_\Sigma(x)$ of $\Sigma$ and some $\alpha \in (1, 2)$. Solutions of this 1-periodic solution problem correspond to critical points of the Clarke-Ekeland dual action functional $f_\alpha : E_\alpha \to \mathbb{R}$ given by (20)-(21) below, and $f_\alpha$ possesses a strictly increasing infinite sequence of critical values $\{c_k\}$ with $c_k \to 0$ as $k \to +\infty$. These $c_k$’s are obtained by the Fadell-Rabinowitz $S^1$-cohomological index method as in Section V.3 of [5]. Note that here each $[(\tau, x)] \in \tilde{J}(\Sigma)$ and all of its iterations $\{x^m\}$ yield a strictly increasing infinite subsequence of $\{c_k\}$. Denote the Maslov-type index interval of the $m$-th iteration of $(\tau, x)$ by $I_m(\tau, x) = [i_{m\tau}(x^m), i_{m\tau}(x^m) + \nu_{m\tau}(x^m)]$.

Then $I_m(\tau, x) \cap I_{m'}(\tau, x) = \emptyset$ if $m \neq m'$, and $\{i_{m\tau}(x^m)\}$ is a strictly increasing sequence. The union of all the Maslov-type index intervals of all iterations of all the elements in $\tilde{J}(\Sigma)$ covers the integer set $2\mathbb{N} - 2 + n$, where $\mathbb{N}$ is the set of positive integers. Thus instead of studying the arrangements of the subsequences of $\{c_k\}$ of elements in $\tilde{J}(\Sigma)$ in the whole sequence $\{c_k\}$, we study how their Maslov-type index interval sequences cover the set $2\mathbb{N} - 2 + n$.

2. Let $O(x) = x(\mathbb{R})$ for any closed characteristic $(\tau, x) \in J(\Sigma)$. Our Lemma 4.2 below shows that every $(\tau, x) \in J(\Sigma)$ is either symmetric, i.e. $O(x) = O(-x)$, or asymmetric, i.e. $O(x) \cap O(-x) = \emptyset$. Moreover, any symmetric $(\tau, x)$ satisfies
\[ x(t + \frac{\tau}{2}) = -x(t), \quad \forall t \in \mathbb{R}. \] (4)

any asymmetric $(\tau, x)$ satisfies
\[ (i_{m\tau}(x^m), \nu_{\tau}(x^m)) = (i_{m\tau}((-x)^m), \nu_{\tau}((-x)^m)), \quad \forall m \in \mathbb{N}. \] (5)

Denote the numbers of symmetric and asymmetric elements in $\tilde{J}(\Sigma)$ by $q_1$ and $2q_2$. We can write
\[ \tilde{J}(\Sigma) = \{[(\tau_j, x_j)] \mid j = 1, \ldots, q_1\} \cup \{[(\tau_k, x_k)], [(\tau_k, -x_k)] \mid k = q_1 + 1, \ldots, q_1 + q_2\}. \] (6)
3° Applying the common index jump theorem proved by Y. Long and C. Zhu in [20] (cf. Theorem 4.1 below) to fundamental solutions of the linearized Hamiltonian systems at 

\((\tau_1, x_1), \ldots, (\tau_{q_1+q_2}, x_{q_1+q_2}), (2\tau_{q_1+1}, x_{q_1+1}^2), \ldots, (2\tau_{q_1+q_2}, x_{q_1+q_2}^2)\),

we obtain an integer \(N\) and iteration times \(m_1, \ldots, m_{q_1+2q_2}\) such that the rather precise information on the Maslov-type indices of iterations of \((\tau_j, x_j)\)'s are given in (51)-(58).

4° Combining above 1° and 3° together, by the property of Fadell-Rabinowitz \(S^1\)-cohomology index (cf. Lemma 3.1 below), we obtain an injection map \(p: \mathbb{N} \to \tilde{\mathcal{J}}(\Sigma) \times \mathbb{N}\) such that

\[p(N - s + 1) = \left(\left([\left(\tau_{k(s)}, x_{k(s)}\right)], m(s)\right), \quad \text{for } s = 1, \ldots, n,\right]\]

where \(m(s)\) is the iteration time of \((\tau_{k(s)}, x_{k(s)})\) such that

\[2(N - s + 1) - 2 + n \in \mathcal{I}_{m(s)}(\tau_{k(s)}, x_{k(s)}).\]

5° Let

\[S_1 = \{s \in \{1, \ldots, n\} | k(s) \leq q_1\}, \quad S_2 = \{1, \ldots, n\} \setminus S_1.\]

In Section 4, we prove

\[\# S_1 \leq q_1, \quad \text{and} \quad \# S_2 \leq 2q_2.\]

Then this implies

\[\# \tilde{\mathcal{J}}(\Sigma) = q_1 + 2q_2 \geq \# S_1 + \# S_2 = n.\]

6° To prove the first estimate in (9), using the property (4) of symmetric orbit \((\tau, x)\) and precise index information (51)-(58) of iterations of \((\tau_j, x_j)'s\), we conclude that the Maslov-type index, nullity, and splitting numbers of \(x|_{[0, \tau]}\) are the same as those of \((x|_{[0, \tau/2]}^2\). Therefore there holds

\[i_{\tau}(x) + 2S^+(x) - \nu_{\tau}(x) \geq n.\]

By this estimate, we get that the integer \(m(s)\) in (7) is uniquely determined by \(k(s)\) there provided \(k(s) \leq q_1\). Then the injection map \(p\) of (7) induces an injection map from \(S_1\) to \(\{\left([\tau_j, x_j]) | 1 \leq j \leq q_1\}\) and yields the first estimate in (9).

7° To prove the second estimate in (9), using the property (5) of asymmetric orbit \((\tau, x)\) and precise index information (51)-(58) of iterations of \((\tau_j, x_j)'s\), we conclude that the integer \(m(s)\) in (7) possesses at most two choices determined by the \(k(s)\) there, provided \(q_1 < k(s) \leq q_1 + q_2\). Then the injection map \(p\) of (7) induces a map from \(S_2\) to \(\Gamma = \{\left([\tau_j, x_j]) | q_1 < j \leq q_1 + q_2\}\) such that
any element in \( \Gamma \) is the image of at most two numbers in \( S_2 \). This yields the second estimate in (9), and completes the proof of Theorem 1.1.

This paper is organized as follows. In §2, we briefly review the Maslov-type index theory. In §3, we discuss main variational properties of closed characteristics and give details of the above step 1°. In §4, we complete the above steps 2°-6° and prove Theorem 1.1.

2 The Maslov-type index theory and its iteration theory.

2.1 Maslov-type index theory.

In this subsection we give a brief review on the Maslov-type index theory for symplectic matrix paths.

For any \( n \in \mathbb{N} \) and \( \tau > 0 \), we define as usual
\[
\text{Sp}(2n) = \left\{ M \in \text{GL}(2n, \mathbb{R}) \mid M^TJM = J \right\},
\]
\[
\mathcal{P}_\tau(2n) = \left\{ \gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I \right\}.
\]

For any \( \gamma \in \mathcal{P}_\tau(2n) \), the Maslov-type index of \( \gamma \) is defined to be a pair of integers by C. Conley, E. Zehnder, and Y. Long in the works [4], [18], [12], and [13], which is denoted by
\[
(i_\tau(\gamma), \nu_\tau(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\},
\]
where \( \mathbb{Z} \) is the set of all integers. The iteration theory of this Maslov-type index theory was established in [15] by Y. Long via the \( \omega \)-index theory introduced there. Let \( U \equiv \{ z \in \mathbb{C} \mid |z| = 1 \} \).

For any \( \omega \in U \) and \( \gamma \in \mathcal{P}_\tau(2n) \), the \( \omega \)-index of \( \gamma \) is denoted by
\[
(i_{\tau,\omega}(\gamma), \nu_{\tau,\omega}(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\}.
\]
Note that there hold \( i_{\tau,1}(\gamma) = i_\tau(\gamma) \) and \( \nu_{\tau,1}(\gamma) = \nu_\tau(\gamma) \). The details of the definitions and properties of these index and their iteration theories can be found in the above mentioned works as well as [16] and [17]. We also refer to [19] for another approach.

For \( B \in C(\mathbb{R}/(\tau \mathbb{Z}), \text{gl}(2n, \mathbb{C})) \) with \( B(t) \) being self-adjoint for all \( t \), we consider the linear Hamiltonian system
\[
\dot{x} = JB(t)x, \quad x \in \mathbb{C}^{2n}.
\]

The following proposition gives the relationship between the Maslov-type index theory and the Ekeland index theory.
Proposition 2.1 (\cite{3}, Lemma 1.3 of \cite{14}, and Theorem 3.2 of \cite{19}) For $B \in C(R/\tau Z, gl(2n, R))$ with $B(t)$ being positively definite and symmetric for all $t$, let $\gamma_B \in P_\tau(2n)$ be the fundamental solution of the linear Hamiltonian system \cite{12}, $i^E_\tau(\gamma_B)$ and $\nu^E_\tau(\gamma_B)$ be the Ekeland index and nullitity given by Definition I.4.3 in \cite{5} with $J$ replaced by that in \cite{1}. Then we have
\begin{align}
i^E_\tau(\gamma_B) &= i_\tau(\gamma_B) - n, \\
\nu^E_\tau(\gamma_B) &= \nu_\tau(\gamma_B).
\end{align}

Corollary 2.1 For any positive definite symmetric path $B \in C(R/\tau Z, gl(2n, R))$, we have
\begin{align}i_\tau(\gamma_B) &\geq n. \tag{13}
\end{align}

2.2 Bott-type formulae for splitting numbers.

For $n \in \mathbb{N}$, $\tau > 0$ and a path $\gamma \in P_\tau(2n)$, we define the iteration path $\tilde{\gamma}$ of $\gamma$ by
\begin{equation}\tilde{\gamma}(t) = \gamma(t - j\tau)\gamma(\tau)^j, \quad \text{for } j\tau \leq t \leq (j+1)\tau \text{ and } j \in \{0\} \cup \mathbb{N}. \tag{14}\end{equation}

For $M \in Sp(2n)$, let $S^\pm_M(\omega)$ be the splitting numbers defined by Theorem 1.3 of \cite{15} for all $\omega \in U$. Then the following Bott-type formula follows.

Lemma 2.1 For any $M \in Sp(2n)$, $m \in \mathbb{N}$ and $z \in U$, there holds
\begin{equation}S^\pm_{M^m}(z) = \sum_{\omega^m = z} S^\pm_M(\omega). \tag{15}\end{equation}

**Proof.** Since $Sp(2n)$ is path-connected, we can choose a path $\gamma \in P_\tau(2n)$ with $\gamma(\tau) = M$. Define $\theta \in [0, 2\pi)$ by $z = e^{\sqrt{-1}\theta}$. Let $\tilde{\gamma}$ be the iteration path of $\gamma$ defined by (14). By the definition of the splitting numbers and Theorem 1.4 in \cite{13}, we obtain
\begin{align}S^+_{M^m}(z) &= \lim_{\epsilon \to 0^+} i_{\tau, e^{\sqrt{-1}(\theta + \epsilon)}}(\tilde{\gamma}) - i_{\tau, e^{\sqrt{-1}\theta}}(\tilde{\gamma}) \\
&= \lim_{\epsilon \to 0^+} \sum_{k=1}^{m} i_{\tau, e^{\sqrt{-1}(\theta + \epsilon + 2k\pi)/m}}(\gamma) - \sum_{k=1}^{m} i_{\tau, e^{\sqrt{-1}(\theta + 2k\pi)/m}}(\gamma) \\
&= \sum_{k=1}^{m} \left( \lim_{\epsilon \to 0^+} i_{\tau, e^{\sqrt{-1}(\theta + \epsilon + 2k\pi)/m}}(\gamma) - i_{\tau, e^{\sqrt{-1}(\theta + 2k\pi)/m}}(\gamma) \right) \\
&= \sum_{\omega^m = z} S^+_M(\omega).
\end{align}

Similarly we have
\begin{equation}S^-_{M^m}(z) = \sum_{\omega^m = z} S^-_M(\omega). \tag{15}\end{equation}

Q.E.D.

The following proposition gives the definition of the mean index.
Proposition 2.2 (Theorem 1.5 of [15]) For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, there holds
\[
\hat{\iota}_\tau(\gamma) \equiv \lim_{k \to +\infty} \frac{i_{k\tau}(\tilde{\gamma})}{k} = \frac{1}{2\pi} \int_\mathcal{U} i_{\tau, \omega}(\gamma) d\omega.
\] (16)

In particular, $\hat{\iota}_\tau(\gamma)$ is always a finite real number, which is called the \textbf{mean index} per $\tau$ of $\gamma$.

3 Variational properties of closed characteristics.

To solve the given energy problem (1), we follow § V.3 of [5]. Fix a given $\Sigma \in \mathcal{H}(2n)$ bounding a convex compact set $C$. Let $j_C : \mathbb{R}^{2n} \to [0, +\infty)$ be the gauge function of $C$ defined by
\[
j_C(0) = 0 \quad \text{and} \quad j_C(x) = \inf \{ \lambda > 0 \mid \frac{x}{\lambda} \in C \} \quad \text{for} \quad x \neq 0.
\] (17)

Fix a constant $\alpha$ satisfying $1 < \alpha < 2$ in this paper. As usual we define the Hamiltonian function $H_{\Sigma, \alpha} : \mathbb{R}^{2n} \to [0, +\infty)$ by
\[
H_{\Sigma, \alpha}(x) = j_C(x)^\alpha, \quad \forall x \in \mathbb{R}^{2n}.
\] (18)

Then $H_{\Sigma, \alpha} \in C^1(\mathbb{R}^{2n}, \mathbb{R}) \cap C^2(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R})$ is convex and $\Sigma = H_{\Sigma, \alpha}^{-1}(1)$. It is well-known that the problem (1) is equivalent to the following problem
\[
\begin{cases}
    \dot{z}(t) = JH_{\Sigma, \alpha}'(z(t)), & \forall t \in \mathbb{R}, \\
    z(1) = z(0).
\end{cases}
\] (19)

Denote by $\mathcal{J}(\Sigma, \alpha)$ the set of all solutions $(\tau, x)$ of the problem (19) where $\tau$ is the minimal period of $x$, and by $\tilde{\mathcal{J}}(\Sigma, \alpha)$ the set of all geometrically distinct elements in $\mathcal{J}(\Sigma, \alpha)$. Note that elements in $\mathcal{J}(\Sigma)$ and $\mathcal{J}(\Sigma, \alpha)$ are one to one correspondent to each other. The usual dual function $H_{\Sigma, \alpha}^*$ of $H_{\Sigma, \alpha}$ is defined by
\[
H_{\Sigma, \alpha}^*(x) = \sup_{y \in \mathbb{R}^{2n}} \{ (x, y) - H_{\Sigma, \alpha}(y) \},
\]
where $(\cdot, \cdot)$ denotes the standard inner product of $\mathbb{R}^{2n}$. For $1 < \alpha < 2$, let
\[
E_\alpha = \{ u \in L^{(\alpha-1)/\alpha}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \mid \int_0^1 u dt = 0 \}. \tag{20}
\]

The Clarke-Ekeland dual action functional $f_\alpha : E_\alpha \to \mathbb{R}$ is defined by
\[
f_\alpha(u) = \int_0^1 \left\{ \frac{1}{2} J(u, \Pi u) + H_{\Sigma, \alpha}^*(-J u) \right\} dt, \tag{21}
\]
where $\Pi u$ is defined by $\frac{d}{dt} \Pi u = u$ and $\int_0^1 \Pi u dt = 0$. Then $f_\alpha \in C^2(E_\alpha, \mathbb{R})$. Suppose $u \in E_\alpha \setminus \{0\}$ is a critical point of $f_\alpha$. By [5], there exists $\xi_u \in \mathbb{R}^{2n}$ such that $z_u(t) = \Pi u(t) + \xi_u$ is a 1-periodic
solution of the problem (19). Let \( h = H_{\Sigma, \alpha}(z_u(t)) \) and \( 1/m \) be the minimal period of \( z_u \) for some \( m \in \mathbb{N} \). Define

\[
x_u(t) = h^{-1/\alpha} z_u(h^{(2-\alpha)/\alpha} t) \quad \text{and} \quad \tau = \frac{1}{m} h^{(\alpha-2)/\alpha}.
\]

Then there hold \( x_u(t) \in \Sigma \) for all \( t \in \mathbb{R} \) and \( (\tau, x_u) \in \mathcal{J}(\Sigma, \alpha) \). Note that the period 1 of \( z_u \) corresponds to the period \( m\tau \) of the solution \((m\tau, x_u^{(m)})\) of (1) with minimal period \( \tau \).

On the other hand, every solution \((\tau, x) \in \mathcal{J}(\Sigma, \alpha)\) gives rise to a sequence \( \{z_m^x\}_{m \in \mathbb{N}} \) of solutions of the problem (19), and a sequence \( \{u_m^x\}_{m \in \mathbb{N}} \) of critical points of \( f_\alpha \) defined by

\[
z_m^x(t) = (m\tau)^{-1/(2-\alpha)} x(m\tau t),
\]

\[
u_m^x(t) = (m\tau)^{(\alpha-1)/(2-\alpha)} \hat{x}(m\tau t).
\]

For every \( m \in \mathbb{N} \) there holds

\[
f_\alpha(u_m^x) = -(1 - \frac{\alpha}{2})(\frac{2m}{\alpha} A(\tau, x))^{-\alpha/(2-\alpha)},
\]

where

\[
A(\tau, x) = \frac{1}{2} \int_0^\tau (-J \hat{x} \cdot x) dt.
\]

Following §V.3 of [5], we denote by "ind" the \( S_1 \)-cohomology index theory for \( S_1 \)-invariant subsets of \( E_\alpha \) defined in [1] (cf. also [2] of E. Fadell and P. Rabinowitz for the original definition).

For \([f_\alpha]_c \equiv \{u \in E_\alpha \ | \ f_\alpha(u) \leq c\} \) we define

\[
c_k = \inf\{c < 0 \ | \ \text{ind}([f_\alpha]_c) \geq k\}.
\]

Then all these \( c_k \)'s are critical values of \( f_\alpha \) and there hold

\[
-\infty < \min_{u \in E_\alpha} f_\alpha(u) = c_1 \leq c_2 \leq \cdots \leq c_k \leq c_{k+1} \leq \cdots < 0,
\]

\[
c_k \to 0 \quad \text{as} \quad k \to +\infty.
\]

By Theorem V.3.4 of [5], for each \( k \in \mathbb{N} \) there exists a function \( u_k \in E_\alpha \) such that there hold

\[
f'_\alpha(u_k) = 0 \quad \text{and} \quad f_\alpha(u_k) = c_k,
\]

\[
i_1^E(u_k) \leq 2k - 2 \leq i_1^E(u_k) + \nu_1^E(u_k) - 1.
\]

Based upon this result and Proposition 2.1, we have the following lemma.
Lemma 3.1 (cf. Lemma 3.1 of [20]) Suppose \( \# \tilde{\mathcal{J}}(\Sigma) < +\infty \), there exist an injection map \( p = p(\Sigma, \alpha): \mathbb{N} \to \tilde{\mathcal{J}}(\Sigma, \alpha) \times \mathbb{N} \) such that for any \( k \in \mathbb{N} \), \((\tau, x) \in \mathcal{J}(\Sigma, \alpha)\) and \( m \in \mathbb{N} \) satisfying \( p(k) = ((\tau, x), m) \), there hold

\[
\begin{align*}
 f'_\alpha(u^x_m) &= 0 \quad \text{and} \quad f_\alpha(u^x_m) = c_k, \\
 i_{m\tau}(x^m) &\leq 2k - 2 + n \leq i_{m\tau}(x^m) + \nu_{m\tau}(x^m) - 1,
\end{align*}
\]

where \( u^x_m \) is defined by (24).

Lemma 3.2 (cf. Corollary and 3.1 of [20]) Fix \( \Sigma \in \mathcal{H}(2n) \) and \( \alpha \in (1, 2) \). For any \((\tau, x) \in \mathcal{J}(\Sigma, \alpha)\) and \( m \in \mathbb{N} \), there hold

\[
\begin{align*}
 i_{(m+1)\tau}(x) - i_{m\tau}(x) &\geq 2, \\
 i_{(m+1)\tau}(x) + \nu_{(m+1)\tau}(x) - 1 &\geq i_{(m+1)\tau}(x) + \nu_{m\tau}(x) - 1, \\
 i_{\tau}(x) &\geq 2.
\end{align*}
\]

4 Proof of the main results.

Firstly we recall the common index jump theorem of [20].

Theorem 4.1 (Theorem 4.3 of [20]) Let \( \gamma_k \in \mathcal{P}_{\tau_k}(2n) \), \( k = 1, \ldots, q \) be a finite collection of symplectic paths. Let \( M_k = \gamma(\tau_k) \) be the end points and \( \tilde{\gamma}_k \) be the iteration path of \( \gamma_k \) defined by (14) for all \( k = 1, \ldots, q \). Denote by

\[
\begin{align*}
i^m_k &= i_{m\tau_k}(\tilde{\gamma}_k) \quad \text{and} \quad \nu^m_k = \nu_{m\tau_k}(\tilde{\gamma}_k).
\end{align*}
\]

If \( i_{\tau_k}(x_k) > 0 \) for every \( k = 1, \ldots, q \), there exist infinitely many \((N, m_1, \ldots, m_q) \in \mathbb{N}^{q+1}\) such that

\[
\begin{align*}
 \nu^{2m_k-1}_k &= \nu^1_k, \\
 \nu^{2m_k+1}_k &= \nu^1_k, \\
 i^{2m_k-1}_k + \nu^{2m_k-1}_k &= 2N - (i^1_k + 2S_{M_k}^+(1) - \nu^1_k), \\
 i^{2m_k+1}_k &= 2N + i^1_k, \\
 i^{2m_k}_k &\geq 2N - n, \\
 i^{2m_k}_k + \nu^{2m_k}_k &\leq 2N + n
\end{align*}
\]

for all \( k = 1, \ldots, q \). Moreover, \( M_k \) is elliptic if the equality in one of the inequalities of (41) and (42) holds.
We need a lemma.

**Lemma 4.1** Let $\gamma \in \mathcal{P}_r(2n)$ be a symplectic paths. Let $M = \gamma(\tau)$ be the end point and $\tilde{\gamma}$ be the iteration path of $\gamma$ defined by (14). If $i_\tau(\gamma) \geq n$, we have

$$i_{2\tau}(\tilde{\gamma}) + 2S_+^{+}(1) - \nu_{2\tau}(\tilde{\gamma}) \geq n. \quad (43)$$

**Proof.** For $\theta \in [0, 2\pi)$, we define the splitting number $S_{\tau, M}^{+}(e^{\sqrt{-1}\theta})$ by

$$S_{\tau, M}^{+}(e^{\sqrt{-1}\theta}) = \nu_{\tau,e^{\sqrt{-1}\theta}}(\gamma) - S_{M}^{+}(e^{\sqrt{-1}\theta}). \quad (44)$$

By Corollary 4.13 of [15], we obtain

$$S_{M}^{+}(e^{\sqrt{-1}\theta}) \geq 0. \quad (45)$$

By Lemma 6.3 of [15] and Lemma 4.2 of [19], we obtain

$$\sum_{\theta \in (0, \pi)} S_{M}^{+}(e^{\sqrt{-1}\theta}) + S_{\tau, M}^{-}(1) + S_{\tau, M}^{-}(-1) = \sum_{\theta \in (0, \pi)} S_{M}^{+}(e^{\sqrt{-1}\theta}) + \frac{m(1)}{2} - r^{-}(1) + \frac{m(-1)}{2} - r^{+}(-1) \leq \sum_{\theta \in (0, \pi)} q(e^{\sqrt{-1}\theta}) + \frac{m(1)}{2} + \frac{m(-1)}{2} \leq n, \quad (46)$$

where $q(e^{\sqrt{-1}\theta})$ is the negative Krein number of $e^{\sqrt{-1}\theta}$ for $M$, $m(\pm 1)$ is the total multiplicity of $\pm 1 \in \sigma(M)$, $2r^{-}(\pm 1)$ is the number of Floquet multipliers which arrive on the unit circle at $\pm 1$ for $M$.

Note that $S_{M}^{+}(\pm 1) = S_{M}^{-}(\pm 1)$ and $i_\tau(\gamma) \geq n$. By Theorem 1.4 of [13] and Lemma 2.1, we have

$$i_{2\tau}(\tilde{\gamma}) + 2S_{M}^{+}(1) - \nu_{2\tau}(\tilde{\gamma}) = i_\tau(\gamma) + i_{\tau,-1}(\gamma) + 2S_{M}^{+}(1) + 2S_{M}^{-}(-1) - \nu_\tau(\gamma) - \nu_{\tau,-1}(\gamma) \geq 2n - n = n. \quad \text{Q.E.D.}$$
Now we can give the proof of our main result. Fix $\Sigma \in \mathcal{H}(2n)$ and $\alpha \in (1, 2)$. For any $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$, let $\gamma_x \in \mathcal{P}_\tau(2n)$ be the fundamental solution of the linearized system

$$\dot{y}(t) = JH^\prime_{\Sigma, \alpha}(x(t))y(t).$$

We call $\gamma_x$ the associated symplectic path of $(\tau, x)$. Then the Maslov-type index $(i_{\tau}(x), \nu_{\tau}(x))$ and the splitting number $S^+(x)$ of $x$ at 1 are defined respectively by

$$i_{\tau}(x) = i_{\tau}(\gamma), \quad \nu_{\tau}(x) = \nu_{\tau}(\gamma),$$

$$S^+(x) = S^+_{\gamma_{\tau}}(1).$$

**Lemma 4.2** Fix $\Sigma \in \mathcal{SH}(2n)$. For any $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$, there holds $(\tau, -x) \in \mathcal{J}(\Sigma, \alpha)$, and either $\mathcal{O}(x) = \mathcal{O}(-x)$ or $\mathcal{O}(x) \cap \mathcal{O}(-x) = \emptyset$. If $\mathcal{O}(x) \cap \mathcal{O}(-x) \neq \emptyset$, we have

$$x(t) = -x(t + \frac{\tau}{2}), \quad \forall t \in \mathbb{R}.$$  

**Proof.** Since $\Sigma = -\Sigma$, there hold

$$H_{\Sigma, \alpha}(x) = H_{\Sigma, \alpha}(-x), \quad (47)$$

$$H'_{\Sigma, \alpha}(x) = -H'_{\Sigma, \alpha}(-x), \quad (48)$$

$$H''_{\Sigma, \alpha}(x) = H''_{\Sigma, \alpha}(-x). \quad (49)$$

So for any $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$, there holds $(\tau, -x) \in \mathcal{J}(\Sigma, \alpha)$.

If $\mathcal{O}(x) \cap \mathcal{O}(-x) \neq \emptyset$, there exist $s_1, s_2 \in [0, \tau]$ such that $x(s_1) = -x(s_2)$. By the fact $x(\tau) = x(0) \neq 0$ and $x(t) \neq 0$ for any $t \in \mathbb{R}$, we have $s_2 - s_1 \neq 0, \pm \tau$. Since $x(s_1 + t)$ and $-x(s_2 + t)$ satisfies the same system

$$\dot{y} = JH'_{\Sigma, \alpha}(y),$$

we have $x(s_1 + t) = -x(s_2 + t)$ and hence $x(t) = x(2(s_2 - s_1) + t)$ for $t \in \mathbb{R}$. Since $\tau$ is the minimal period of $x$, we have $2(s_2 - s_1) = \pm \tau$. Therefore $x(t) = -x(t + \frac{\tau}{2})$. Specially there holds $\mathcal{O}(x) = \mathcal{O}(-x)$. Q.E.D.

From $x(t) = -x(t + \frac{\tau}{2})$ we obtain $H''_{\Sigma, \alpha}(x(t)) = H''_{\Sigma, \alpha}(x(t + \frac{\tau}{2}))$. Let $\gamma_x$ be the associated symplectic path of $(\tau, x)$. Then we have

$$\gamma_x(t + \frac{\tau}{2}) = \gamma_x(t)\gamma_x(\frac{\tau}{2}), \quad \forall t \in [0, \frac{\tau}{2}], \quad (50)$$

**Proof of Theorem 1.1** It suffices to consider the case $\# \tilde{\mathcal{J}}(\Sigma) < +\infty$. By (17) and (18), $(\tau, -x) \in \mathcal{J}(\Sigma, \alpha)$ if $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$. Note that we have

$$(i_{m\tau_k}(x^m_k), \nu_{m\tau_k}(x^m_k)) = (i_{m\tau_k}(-x^m_k), \nu_{m\tau_k}(-x^m_k)), \quad \forall m \in \mathbb{N}.$$
By Lemma 4.2, we denote the elements in $\tilde{J}(\Sigma, \alpha)$ by

$$\tilde{J}(\Sigma, \alpha) = \{[(\tau_j, x_j) | j = 1, \ldots, q_1] \cup \{[(\tau_k, x_k)], [(\tau_k, -x_k)] | k = q_1 + 1, \ldots, q_1 + q_2\},$$

where $O(x_j) = O(-x_j)$ for $j = 1, \ldots, q_1$, and $O(x_k) \cap O(-x_k) = \emptyset$ for $k = q_1 + 1, \ldots, q_1 + q_2$. By Lemma 3.1, we get an injection map $p = p(\Sigma, \alpha): \mathbb{N} \to \tilde{J}(\Sigma, \alpha) \times \mathbb{N}$. Note that $(\tau_k, x_k)$ and $(\tau_k, -x_k)$ have the same index intervals. Thus by Lemma 3.1 we can further require that

$$\text{im}(p) \subset \{[(\tau_k, x_k)] | k = 1, \ldots, q_1 + q_2\} \times \mathbb{N}.$$

Set $i(k, m) = i_m \tau_k(x_k^m)$ and $\nu(k, m) = \nu_m \tau_k(x_k^m)$. By Lemma 3.2 we have $\hat{i}_\tau(x_k) \geq 2$ for $k = 1, \ldots, q_1 + q_2$. Applying Theorem 4.1 to the associated symplectic path of

$$(\tau_1, x_1), \ldots, (\tau_{q_1+q_2}, x_{q_1+q_2}), (2\tau_{q_1+q_2}, x_{q_1+q_2}^2), \ldots, (2\tau_{q_1+q_2}, x_{q_1+q_2}^2),$$

we get infinitely many $(N, m_1, \ldots, m_{q_1+q_2}) \in \mathbb{N}^{q_1+2q_2+1}$ such that

$$i(k, 2m_k + 1) = 2N + i(k, 1),$$

$$i(k, 2m_k - 1) + \nu(k, 2m_k - 1) = 2N - (i(k, 1) + 2S^+(x_k) - \nu(k, 1)),\quad i(k, 2m_k) \geq 2N - n,\quad i(k, 2m_k) + \nu(k, 2m_k) \leq 2N + n$$

for $k = 1, \ldots, q_1 + q_2$ and

$$i(k - q_2, 4m_k + 2) = 2N + i(k - q_2, 2),$$

$$i(k - q_2, 4m_k - 2) + \nu(k - q_2, 4m_k - 2) = 2N - (i(k - q_2, 2) + 2S^+(x_{k-q_2}^2) - \nu(k - q_2, 2)),\quad i(k - q_2, 4m_k) \geq 2N - n,\quad i(k - q_2, 4m_k) + \nu(k - q_2, 4m_k) \leq 2N + n$$

for $k = q_1 + q_2 + 1, \ldots, q_1 + 2q_2$.

By (53), (54) and Lemma 3.1, we have

$$i(k, 2m_k) \geq 2N - n$$

$$\geq 2N - (i(k, 2) + 2S^+(x_k^2) - \nu(k, 2)),$$

$$= i(k, 4m_{k+q_2} - 2) + \nu(k, 4m_{k+q_2} - 2)$$

$$> i(k, 4m_{k+q_2} - 2).$$
By (54), (55) and Lemma 4.1, we have
\begin{align*}
i(k, 2m_k) &< i(k, 2m_k) + \nu(k, 2m_k) \\
&\leq 2N + n \\
&\leq 2N + i(k, 2) \\
&= i(k, 4m_{k+1} + 2)
\end{align*}
for \( k = q_1 + 1, \ldots, q_1 + q_2 \). By Lemma 3.2 we have
\[ 4m_{k+q_2} - 2 < 2m_k < 4m_{k+q_2} + 2. \]
Thus
\[ m_k = 2m_{k+q_2}, \quad \forall k = q_1 + 1, \ldots, q_1 + q_2. \] (59)

Denote by \((\tau_{k(s)}, x_{k(s)})], m(s) = p(N - s + 1), where \( s = 1, \ldots, n \), \( k(s) \in \{1, \ldots, q_1 + q_2\} \) and \( m(s) \in \mathbb{N} \). By the definition of \( p \) and (33) we have
\begin{align*}
i(k(s), m(s)) &\leq 2(N - s + 1) - 2 + n = 2N - 2s + n \\
&\leq i(k(s), m(s)) + \nu(k(s), m(s)) - 1.
\end{align*} (60)

So we have
\begin{align*}
i(k(s), m(s)) &\leq 2N - 2s + n \\
&< 2N + n \\
&\leq 2N + i(k(s), 1) \\
&= i(k(s), 2m_{k(s)} + 1),
\end{align*} (61)
for every \( s = 1, \ldots, n \), where we have used (51) in the last equality. When \( k(s) \leq q_1 \), by Lemma 4.1, (50), and (52), we have
\begin{align*}
i(k(s), 2m_{k(s)} - 1) + \nu(k(s), 2m_{k(s)} - 1) &\leq 2N - (i(k(s), 1) + 2S^+(x_{k(s)}) - \nu(k(s), 1)) \\
&\leq 2N - n \\
&\leq 2N - 2s + n \\
&\leq i(k(s), m(s)) + \nu(k(s), m(s)) - 1.
\end{align*} (62)

When \( q_1 < k(s) \leq q_1 + q_2 \), by (53) we have \( m_{k(s)} = 2m_{k(s)+q_2} \). Then by (54), Lemma 4.1, and (50), we obtain
\begin{align*}
i(k(s), 2m_{k(s)} - 2) + \nu(k(s), 2m_{k(s)} - 2) &\leq 2N - (i(k(s), 2) + 2S^+(x_{k(s)}^2) - \nu(k(s), 2)) \\
&\leq 2N - n \\
&\leq 2N - 2s + n \\
&\leq i(k(s), m(s)) + \nu(k(s), m(s)) - 1.
\end{align*} (63)
By Lemma 3.2, (61) and (62) we have \(2m_{k(s)} - 1 < m(s) < 2m_{k(s)} + 1\) for \(k(s) \leq q_1\). By Lemma 3.2, (61) and (63) we have \(2m_{k(s)} - 2 < m(s) < 2m_{k(s)} + 1\) for \(q_1 < k(s) \leq q_1 + q_2\). Hence

\[
m(s) = 2m_{k(s)}, \quad \text{if} \quad k(s) \leq q_1 \tag{64}
\]

\[
m(s) \in \{2m_{k(s)} - 1, 2m_{k(s)}\} \quad \text{if} \quad q_1 < k(s) \leq q_1 + q_2. \tag{65}
\]

Since the map \(p\) is injective, if there exist \(s_1 \neq s_2\) such that \(k(s_1) = k(s_2) \leq q_1\), we must have \(m(s_1) \neq m(s_2)\). Thus \(m_{k(s_1)} \neq m_{k(s_2)}\) by (64). This contradicts to \(k(s_1) = k(s_2)\). Similarly, if there exist \(s_1 \neq s_2\) such that \(k(s_1) = k(s_2) > q_1\), by the above arguments, we must have \(m(s_1) \neq m(s_2)\) and \(\{m(s_1), m(s_2)\} = \{2m_{k(s_1)} - 1, 2m_{k(s_1)}\}\). So there hold

\[
\#\{s \in \{1, \ldots, n\}|k(s) \leq q_1\} \leq q_1
\]

and

\[
\#\{s \in \{1, \ldots, n\}|k(s) > q_1\} \leq 2q_2.
\]

Therefore we have

\[
\#\tilde{J}(\Sigma) = \#\tilde{J}(\Sigma, \alpha) = q_1 + 2q_2 \geq n.
\]

This completes our proof. Q.E.D.

References

[1] H. Berestycki, J. M. Lasry, G. Mancini and B. Ruf, Existence of multiple periodic orbits on starshaped Hamiltonian systems. Comm. Pure Appl. Math. 38. (1985). 253-289.

[2] R. Bott, On the iteration of closed geodesics and the Sturm intersection theory. Comm. Pure Appl. Math. 9. (1956). 171-206.

[3] V. Brousseau, L’index d’un système hamiltonien linéaire. C. R. Acad. Sc. Paris, t.303. Série 1. (1986). 351-354.

[4] C. Conley and E. Zehnder, Morse-type index theory for flows and periodic solutions for Hamiltonian equations. Comm. Pure Appl. Math. 37. (1984). 207-253.

[5] I. Ekeland, Convexity Methods in Hamiltonian Mechanics. Springer. Berlin. (1990).
[6] I. Ekeland and H. Hofer, Convex Hamiltonian energy surfaces and their closed trajectories. *Comm. Math. Phys.* 113 (1987), 419-467.

[7] I. Ekeland and L. M. Lasry, On the number of periodic trajectories for a Hamiltonian flow on a convex energy surface. *Ann. of Math.* 112. (1980). 283-319.

[8] I. Ekeland and L. Lassoued, Multiplicité des trajectoires formées d’un système hamiltonien sur une hypersurface d’énergie convexe. *Ann. INP. Analyse non lineairé*. 4 (1987), 1-29.

[9] E. Fadell and P. Rabinowitz, Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. *Invent. Math.* 45. (1978) 139-174.

[10] M. Girardi, Multiple orbits for Hamiltonian systems on starshaped surfaces with symmetries. *Ann. INP. Analyse non lineairé*. 1. (1984). 285-294.

[11] H. Hofer, K. Wysocki, and E. Zehnder, The dynamics on three-dimensional strictly convex energy surfaces. *Ann. of Math.* 148 (1998) 197-289.

[12] Y. Long, Maslov-type index, degenerate critical points, and asymptotically linear Hamiltonian systems. *Science in China ( Scientia Sinica). Series A.* 7. (1990). 673-682. (Chinese edition), 33. (1990). 1409-1419. (English edition).

[13] Y. Long, A Maslov-type index theory for symplectic paths. *Top. Meth. Nonl. Anal.* 10 (1997) 47-78.

[14] Y. Long, Hyperbolic closed characteristics on compact convex smooth hypersurfaces in $\mathbb{R}^{2n}$. *J. Diff. Equa.* 150 (1998), 227-249.

[15] Y. Long, Bott formula of the Maslov-type index theory. Research Report. Nankai Inst. of Math. Nankai Univ. (1995). *Pacific J. Math.* 187 (1999), 113-149.

[16] Y. Long, Precise iteration formulae of the Maslov-type index theory and ellipticity of closed characteristics. *Advances in Math.* 154 (2000) 76-131.

[17] Y. Long, The Maslov-type index and its iteration theory with applications to Hamiltonian systems. *Third School on Nonlinear Analysis and Applications to Differential Equations. (10.12-30,1998). ICTP Lecture Notes. SMR 1071/2. . Minimax Theory, Morse Theory and the Applications to Differential Equations.* H. Brezis, S. Li, J.-Q. Liu, P. Rabinowitz ed. International Press. to appear.
[18] Y. Long and E. Zehnder, Morse theory for forced oscillations of asymptotically linear Hamiltonian systems. In *Stoc. Proc. Phys. and Geom.*, S. Albeverio et al. ed. World Sci. (1990). 528-563.

[19] Y. Long and C. Zhu, Maslov-type index theory for symplectic paths and spectral flow (II). *Chinese Ann. of Math.* 21 B: 1 (2000) 89-108.

[20] Y. Long and C. Zhu, Closed characteristics on compact convex hypersurfaces in $\mathbb{R}^{2n}$. Nankai Inst. of Math. Nankai Univ. Preprint. (1999). Revised March 2001. *Annals of Math.* to appear.

[21] J. K. Moser, Periodic orbits near an equilibrium and a theorem by A. Weinstein. *Comm. Pure Appl. Math.* 29 (1976), 727-747.

[22] P. Rabinowitz, Periodic solutions of Hamiltonian systems. *Comm. Pure Appl. Math.* 31. (1978). 157-184.

[23] A. Szulkin, Morse theory and existence of periodic solutions of convex Hamiltonian systems. *Bull. Soc. Math. France.* 116 (1988), 171-197.

[24] A. Weinstein, Normal modes for nonlinear Hamiltonian systems. *Inven. Math.* 20. (1973). 47-57.

[25] A. Weinstein, Periodic orbits for convex Hamiltonian systems. *Ann. of Math.* 108. (1978). 507-518.

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