Inverse beta-decay in magnetic fields

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Abstract

We calculate the cross section of the inverse beta decay process, $\nu_e + n \rightarrow p + e^-$, in a magnetic field which is much smaller than $m_p^2/e$. Using exact solutions of the Dirac equation in a constant magnetic field, we find that the cross section depends on the direction of the incident neutrino even when the initial neutron is assumed to be at rest. We discuss the implication of this result for pulsar kicks.

1 Introduction

It has been argued by various authors that asymmetric neutrino emission during the proto-neutron star phase can explain the observed recoil velocities of pulsars. One possible mechanism for this, first suggested by Kusenko and Segrè [1], relies on neutrino dispersion [2] and oscillation [3] in a background magnetic field. This requires non-zero masses for neutrinos. The other mechanism, which does not crucially depend on neutrino mass, relies on asymmetric neutrino cross section in a proto-neutron star. This can occur if the magnetic field in the proto-neutron star is asymmetric, as discussed by Bisnovatyi-Kogan [4]. His calculations were later checked and modified by Roulet [5], who calculated neutrino opacity in strong magnetic fields and concluded that sizeable modifications from the zero-field case can occur.

In this paper, we calculate the cross section of the inverse beta-decay process in a magnetic field. The process influences the neutrino opacity and hence the neutrino surface from which the neutrinos can escape freely from the stellar interior. The distorted ‘neutrino sphere’ produces a net momentum difference of the outgoing neutrinos which are emitted from the two hemispheres. Considerable work has been done on the URCA processes which have neutrinos in their final states [6, 7]. An angular dependence obtained in the differential cross section of these reactions imply that the neutrinos are created asymmetrically with respect to the magnetic field direction. In order to calculate the net momentum thrust for neutrino emission from a proto-neutron star, one should find out the angular dependence of the neutrino opacity owing to the presence of the magnetic field. This is the subject of the present paper.

The neutrinos are assumed to be strictly standard model neutrinos, without any mass and consequent properties. We show that asymmetric neutrino opacity can arise even in a uniform magnetic field. This is because the presence of the magnetic field breaks the isotropy of the background, and a careful calculation in this background reveals a dependence of the cross section on the incident neutrino direction with respect to the magnetic field.

The paper is organized as follows. In Sec 2, we provide some background for the calculation. Most of this section is not new, but we provide it for the sake of completeness, as well as for setting up the

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notation that would be used in the later sections. In Sec. 3, we define the fermion field operator and show how it acts on the states in the presence of a magnetic field. Sec. 4 contains the calculation of the cross section, which shows the asymmetric term mentioned above. In Sec. 5, we discuss the asymmetric term and estimate the magnitude of pulsar kick that can result from this term. Sec. 6 contains our conclusions.

2 Solutions of the Dirac equation in a uniform magnetic field

The Dirac equation in presence of a magnetic field is given by

$$i \frac{\partial \psi}{\partial t} = [\alpha \cdot (p - eA) + \beta m] \psi,$$  \hspace{1cm} (2.1)

where $\alpha$ and $\beta$ are the Dirac matrices, and $A$ is the vector potential. We will work with a constant magnetic field $B$ along the $z$-direction. The vector potential can be chosen in many equivalent ways. We take

$$A_0 = A_y = A_z = 0, \quad A_x = -yB.$$  \hspace{1cm} (2.2)

For stationary states, we can write

$$\psi = e^{-iEt} \begin{pmatrix} \phi \\ \chi \end{pmatrix},$$  \hspace{1cm} (2.3)

where $\phi$ and $\chi$ are 2-component objects. Using the Pauli-Dirac representation of the Dirac matrices, we can then write Eq. (2.1) as

$$(E - m)\phi = \sigma \cdot (p - eA)\chi,$$  \hspace{1cm} (2.4)

$$(E + m)\chi = \sigma \cdot (p - eA)\phi.$$  \hspace{1cm} (2.5)

Eliminating $\chi$, we obtain

$$(E^2 - m^2)\phi = [\sigma \cdot (p - eA)]^2 \phi$$

$$= \left[ p^2 + e^2y^2B^2 + eB(2yp_x - \sigma_z) \right] \phi,$$  \hspace{1cm} (2.6)

where in the last step we have used our choice of $A$ from Eq. (2.2). It is to be understood throughout that when we should write the spatial component of any vector with a lettered subscript, it should imply the corresponding contravariant component. Writing now $p = -i \nabla$ and noticing that the co-ordinates $x$ and $z$ do not appear in the equation except through the second derivative, we can write the solutions as

$$\phi = e^{ip \cdot X_y} f(y),$$  \hspace{1cm} (2.7)

where $f(y)$ is a 2-component matrix which depends only on the $y$-coordinate, and the eigenvalue of the operator $p_x$, which we also denote by the same symbol. We have also introduced the notation $X$ for the spatial co-ordinates (in order to distinguish it from $x$, which is one of the components of $X$), and $X_y$ for the vector $X$ with its $y$-component set equal to zero. In other words, $p \cdot X_y \equiv p_x x + p_z z$.

There will be two independent solutions for $f(y)$, which can be taken, without any loss of generality, to be the eigenstates of $\sigma_z$ with eigenvalues $s = \pm 1$. This means that we choose the two independent solutions in the form

$$f_+(y) = \begin{pmatrix} F_+(y) \\ 0 \end{pmatrix}, \quad f_-(y) = \begin{pmatrix} 0 \\ F_-(y) \end{pmatrix},$$  \hspace{1cm} (2.8)

Putting these back into Eq. (2.1) and using the dimensionless variable

$$\xi = \sqrt{eB} \left( y + \frac{p_x}{eB} \right),$$  \hspace{1cm} (2.9)

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we find that \( F_s \) satisfies the differential equation
\[
\left[ \frac{d^2}{d\xi^2} - \xi^2 + a_s \right] F_s = 0 ,
\]
where
\[
a_s = \frac{E^2 - m^2 - p_z^2 + eBs}{eB} .
\]
This is a special form of Hermite’s equation, and the solutions exist provided \( a_s = 2n' + 1 \) for \( n' = 0, 1, 2, \ldots \). This provides the energy eigenvalues
\[
E^2 = m^2 + p_z^2 + eB(2n' + 1 - s) ,
\]
and the solutions are
\[
N_n e^{-\xi^2/2} H_n(\xi) = I_{n'}(\xi) ,
\]
where \( H_n \) are Hermite polynomials of order \( n \), and \( N_n \) are normalizations which we take to be
\[
N_n = \left( \frac{\sqrt{eB}}{n! 2^n \sqrt{\pi}} \right)^{1/2} .
\]
We stress that the choice of normalization can be arbitrarily made, as will be clear later. With our choice, the functions \( I_n \) satisfy the completeness relation
\[
\sum_n I_n(\xi) I_n(\xi_*) = \sqrt{eB} \delta(\xi - \xi_*) = \delta(y - y_*) ,
\]
where \( \xi_* \) is obtained by replacing \( y \) by \( y_* \) in Eq. (2.9).

Let us now classify the solutions by the energy eigenvalues
\[
E_n^2 = m^2 + p_z^2 + 2neB ,
\]
which is the relativistic form of Landau energy levels. The solutions are two fold degenerate in general: for \( s = 1 \), \( n = n' \) and \( s = -1 \), \( n = n' - 1 \). In case of \( n = 0 \), only the first solution is available. The solutions can have positive or negative energies. We will denote the positive square root of the right side by \( E_n \). Representing the solution corresponding to this \( n \)-th Landau level by a superscript \( n \), we can then write for the positive energy solutions,
\[
f_{+}^{(n)}(y) = \begin{pmatrix} I_n(\xi) \\ 0 \end{pmatrix} , \quad f_{-}^{(n)}(y) = \begin{pmatrix} 0 \\ I_{n-1}(\xi) \end{pmatrix} .
\]
For \( n = 0 \), the solution \( f_- \) does not exist. We will consistently incorporate this fact by defining
\[
I_{-1}(y) = 0 ,
\]
in addition to the definition of \( I_n \) in Eq. (2.13) for non-negative integers \( n \).

The solutions in Eq. (2.17) determine the upper components of the spinor solutions through Eq. (2.7). The lower components, denoted by \( \chi \) earlier, can be solved using Eq. (2.14), and finally the positive energy solutions of the Dirac equation can be written as
\[
e^{-ip \cdot \mathbf{X}} U_s(y, n, p_z) ,
\]
where $X^\mu$ denotes the space-time coordinate. And $U_s$ are given by

$$
U_+(y, n, p) = \begin{pmatrix}
I_n(\hat{\xi}) \\
0 \\
\frac{p_z}{E_n + m} I_n(\hat{\xi}) \\
\frac{M_n}{E_n + m} I_{n-1}(\hat{\xi})
\end{pmatrix}, \quad U_-(y, n, p) = \begin{pmatrix}
0 \\
I_{n-1}(\hat{\xi}) \\
\frac{M_n}{E_n + m} I_n(\hat{\xi}) \\
-\frac{p_z}{E_n + m} I_{n-1}(\hat{\xi})
\end{pmatrix},
$$

(2.20)

where we have introduced the shorthand

$$M_n = \sqrt{2neB}.
$$

(2.21)

A similar procedure can be adopted for negative energy spinors which have energy eigenvalues $E = -E_n$. In this case, it is easier to start with the two lower components first and then find the upper components from Eq. (2.4). The solutions are

$$e^{ip\cdot X} V_s(y, n, p),
$$

(2.22)

where

$$V_+(y, n, p) = \begin{pmatrix}
-\frac{p_z}{E_n + m} I_n(\hat{\xi}) \\
-\frac{M_n}{E_n + m} I_{n-1}(\hat{\xi}) \\
I_n(\hat{\xi}) \\
0
\end{pmatrix}, \quad V_-(y, n, p) = \begin{pmatrix}
-\frac{M_n}{E_n + m} I_n(\hat{\xi}) \\
-\frac{p_z}{E_n + m} I_{n-1}(\hat{\xi}) \\
0 \\
I_{n-1}(\hat{\xi})
\end{pmatrix},
$$

(2.23)

with

$$\hat{\xi} = \sqrt{eB} \left( y - \frac{p_z}{cB} \right).
$$

(2.24)

For future use, we note down a few identities involving the spinors which can be obtained by direct substitutions of the solutions obtained above. The spin sum for the $U$-spinors is

$$P_U(y, y^*, n, p) = \sum_s U_s(y, n, p) \overline{U}_s(y^*, n, p)
$$

$$= \frac{1}{2(E_n + m)} \times \left\{ m(1 + \sigma_z) + \not{p} - \not{p \gamma_5} \right\} I_n(\xi) I_n(\xi^*)
$$

$$+ \left\{ m(1 - \sigma_z) + \not{p} + \not{p \gamma_5} \right\} I_{n-1}(\xi) I_{n-1}(\xi^*)
$$

$$+ M_n(\gamma_1 + i\gamma_2) I_n(\xi) I_{n-1}(\xi^*) + M_n(\gamma_1 - i\gamma_2) I_{n-1}(\xi) I_n(\xi^*),
$$

(2.25)

where we have introduced the following notations for any object $a$ carrying a Lorentz index:

$$a^\mu = (a_0, 0, a_z), \quad \bar{a}^\mu = (a_z, 0, a_0).
$$

(2.26)

For the sake of completeness, we give the spin sum for the $V$-spinors as well, although it will not be useful
for the rest of the present paper.

\[ P_V(y, y_s, n, p_s) = \sum_s V_s(y, n, p_s) V_s(y, n, p_s) \]

\[ = \frac{1}{2(E_n + m)} \times \left[ \left\{ -m(1 + \sigma_z) + \bar{p}_|| \bar{\gamma}_5 \right\} I_{n}(\xi) I_{n_s}(\xi_s) \right. \]

\[ + \left\{ -m(1 - \sigma_z) + \bar{p}_|| + \bar{p}_\gamma \gamma_5 \right\} I_{n-1}(\xi) I_{n-1}(\xi_s) \]

\[ - M_n(\gamma_1 + i\gamma_2) I_{n}(\xi) I_{n-1}(\xi_s) - M_n(\gamma_1 - i\gamma_2) I_{n-1}(\xi) I_{n_s}(\xi_s) \]  

\[ (2.27) \]

3 The fermion field operator

Since we have found the solutions to the Dirac equation, we can now use them to construct the fermion field operator in the second quantized version. For this, we write

\[ \psi(X) = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \int \frac{dp_x dp_z}{D} \left[ f_s(n, p_s) e^{-ip \cdot X} U_s(y, n, p_s) + f_s^\dagger(n, p_s) e^{ip \cdot X} V_s(y, n, p_s) \right]. \]  

\[ (3.1) \]

Here, \( f_s(n, p_s) \) is the annihilation operator for the fermion, and \( f_s^\dagger(n, p_s) \) is the creation operator for the antifermion in the \( n \)-th Landau level with given values of \( p_x \) and \( p_z \). The creation and annihilation operators satisfy the anticommutation relations

\[ [f_s(n, p_s), f_{s'}(n', p_{s'})]_+ = \delta_{ss'} \delta_{nn'} \delta(p_x - p'_x) \delta(p_z - p'_z), \]  

\[ (3.2) \]

and a similar one with the operators \( \tilde{f} \) and \( \tilde{f}^\dagger \), all other anticommutators being zero. The quantity \( D \) appearing in Eq. (3.1) depends on the normalization of the spinor solutions, and this is why \( N_n \) could have been chosen arbitrarily. It can be determined from the fermion field anticommutation relation, which is

\[ [\psi(X), \psi^\dagger(X_s)]_+ = \delta^3(X - X_s) \]  

\[ (3.3) \]

for \( X^0 = X_s^0 \). Plugging in the expression given in Eq. (3.1) to the left side of this equation and using the anticommutation relations of Eq. (3.2), we obtain

\[ [\psi(X), \psi^\dagger(X_s)]_+ = \sum_s \sum_n \int \frac{dp_x dp_z}{D^2} e^{-ip_x (x - x_s) + e^{-ip_x (z - z_s)}} U_s(y, n, p_s) U_s^\dagger(y_s, n, p_s) \]

\[ + e^{ip_x (x - x_s)} e^{-ip_x (z - z_s)} V_s(y, n, p_s) V_s^\dagger(y_s, n, p_s) \]. \]  

\[ (3.4) \]

Changing the signs of the dummy integration variables \( p_x \) and \( p_z \) in the second term, we can rewrite it as

\[ [\psi(X), \psi^\dagger(X_s)]_+ = \sum_s \sum_n \int \frac{dp_x dp_z}{D^2} e^{-ip_x (x - x_s) - e^{-ip_x (z - z_s)}} U_s(y, n, p_s) U_s^\dagger(y_s, n, p_s) \]

\[ + V_s(y, n, -p_s) V_s^\dagger(y_s, n, -p_s) \]. \]  

\[ (3.5) \]

Using now the solutions for the \( U \) and the \( V \) spinors from Eqs. (2.21) and (2.22), it is straightforward to verify that

\[ \sum_s \left( U_s(y, n, p_s) U_s^\dagger(y_s, n, p_s) + V_s(y, n, -p_s) V_s^\dagger(y_s, n, -p_s) \right) \]

\[ = \left( 1 + \frac{\bar{p}_\gamma^2 + 2neB}{E_n + m} \right) \times \text{diag} \left[ I_n(\xi) I_{n-1}(\xi_s), I_{n-1}(\xi) I_{n-1}(\xi_s), I_n(\xi) I_{n-1}(\xi_s), I_{n-1}(\xi) I_{n-1}(\xi_s) \right], \]  

\[ (3.6) \]
where ‘diag’ indicates a diagonal matrix with the specified entries, and $\xi$ and $\xi^*$ involve the same value of $p_x$. At this stage, we can perform the sum over $n$ in Eq. (3.5) using the completeness relation of Eq. (2.15), which gives the $\delta$-function of the $y$-coordinate that should appear in the anticommutator. Finally, performing the integrations over $p_x$ and $p_z$, we can recover the $\delta$-functions for the other two coordinates as well, provided

$$
\frac{2E_n}{E_n + m} \frac{1}{D^2} = \frac{1}{(2\pi)^2},
$$

(3.7)

using the expression for the energy eigenvalues from Eq. (2.16) to rewrite the prefactor appearing on the right side of Eq. (3.6). Putting the solution for $D$, we can rewrite Eq. (3.1) as

$$
\psi(X) = \sum_{s=\pm} \sum_{n=0}^{\infty} \int \frac{dp_x dp_z}{2\pi} \sqrt{\frac{E_n + m}{2E_n}}
$$

$$
\times \left[ f_s(n, p_x) e^{-ip \cdot X} U_s(y, n, p_x) + \tilde{f}_s(n, p_x) e^{ip \cdot X} V_s(y, n, p_x) \right].
$$

(3.8)

The one-fermion states are defined as

$$
|n, p_y\rangle = C f_s(n, p_x) |0\rangle.
$$

(3.9)

The normalization constant $C$ is determined by the condition that the one-particle states should be orthonormal. For this, we need to define the theory in a finite but large region in the $x$ and $z$-directions, of lengths $L_x$ and $L_z$ respectively. This gives

$$
C = \frac{2\pi}{\sqrt{L_x L_z}}.
$$

(3.10)

Then

$$
\psi_U(X) |n, p_x\rangle = \sqrt{\frac{E_n + m}{2E_n L_x L_z}} e^{-ip \cdot X} U_s(y, n, p_x) |0\rangle,
$$

(3.11)

where $\psi_U$ denotes the term in Eq. (3.8) that contains the $U$-spinors. Similarly,

$$
\langle n, p_x | \psi_U(X) = \sqrt{\frac{E_n + m}{2E_n L_x L_z}} e^{ip \cdot X} \bar{U}_s(y, n, p_x) \langle 0 | .
$$

(3.12)

## 4 Inverse beta-decay

In this section, we calculate the cross section for the inverse beta-decay process $\nu_e + n \rightarrow p + e^-$ in a background magnetic field. The magnitude of the field is assumed to be much smaller than $m_n^2/e$ or $m_p^2/e$, where $m_n$ and $m_p$ are the masses of the neutron and the proton. For this reason, we can ignore the magnetic field effects on the proton and neutron spinors. The electron spinors, on the other hand, are the ones appropriate for the Landau levels. Thus, we can write the process as

$$
\nu_e(k) + n(P) \rightarrow p(P') + e(p', n').
$$

(4.1)

### 4.1 The $S$-matrix element

The interaction Lagrangian for this process is

$$
\mathcal{L}_{\text{int}} = \sqrt{2} G \bar{\psi}_{(e)} \gamma^\mu L \psi_{(\nu_e)} \bar{\psi}_{(p)} \gamma^\mu (g_V + g_A \gamma_5) \psi_{(n)} ,
$$

(4.2)
Putting these back into Eq. (4.3) and performing the integrations over all co-ordinates except y-coordinates except y using the notations $E$ spinors for the electron. Using Eq. (3.12), we obtain

\[ S_{fi} = \sqrt{2} G_\beta \int d^4x \left\langle \bar{\psi}(e) \gamma^\mu L\psi(e) \right| \nu_e(k) \rangle \left\langle p(P) \left| \bar{\psi}(p) \gamma_\mu (gV + gA\gamma_5)\psi(n) \right| n(P) \right\rangle . \tag{4.3} \]

For the hadronic part, we should use the usual solutions of the Dirac field which are normalized within a box of volume $V$, and this gives

\[ \left\langle p(P) \left| \bar{\psi}(p) \gamma_\mu (gV + gA\gamma_5)\psi(n) \right| n(P) \right\rangle = \frac{e^{i(p' - P) \cdot x}}{\sqrt{2E'V\sqrt{2E'V}}} \left[ \bar{\pi}(p') \gamma_\mu (gV + gA\gamma_5)\psi(n)(P) \right] , \tag{4.4} \]

using the notations $E = P_0$ and $E' = P'_0$. For the leptonic part, we need to take into account the magnetic spinors for the electron. Using Eq. (3.12), we obtain

\[ \left\langle e(p'_e, n') \left| \bar{\psi}(e) \gamma^\mu L\psi(e) \right| \nu_e(k) \rangle = \frac{e^{-ik \cdot X + ip'_e \cdot X}}{\sqrt{2\omega'V}} \sqrt{\frac{E_{n'} + m}{2E_{n'}L_xL_z}} \left[ \bar{U}(e)(y, n', p'_i) \gamma^\mu L\psi(n_e)(k) \right] , \tag{4.5} \]

Putting these back into Eq. (4.3) and performing the integrations over all co-ordinates except y, we obtain

\[ S_{fi} = (2\pi)^3 \delta_y^3 (P + k - P' - p') \left[ \frac{E_{n'} + m}{2\omega'V\sqrt{2E'V}} \frac{1}{2E_{n'}L_xL_z} \right]^{1/2} M_{fi} . \tag{4.6} \]

Here, $\delta_y^3$ implies, in accordance with the notation introduced earlier, the $\delta$-function for all space-time co-ordinates except y. Contrary to the field-free case, we do not get 4-momentum conservation because the y-component of momentum is not a good quantum number in this problem. The quantity $M_{fi}$ is the Feynman amplitude, given by

\[ M_{fi} = \sqrt{2} G_\beta \left[ \bar{\pi}(p')(P') \gamma_\mu (gV + gA\gamma_5)\psi(n)(P) \right] \int dy e^{iq_yy} \left[ \bar{U}(e)(y, n', p'_i) \gamma_\mu L\psi(n_e)(k) \right] , \tag{4.7} \]

using the shorthand

\[ q_y = P_y + k_y - P'_y . \tag{4.8} \]

The transition rate is given by $|S_{fi}|^2/T$, where the quantization is done in a large time $T$. From Eq. (4.6), using the usual rules like

\[ \left\langle \delta(E + \omega - E - E_{n'}) \right\rangle^2 = \frac{T}{2\pi} \delta(E + \omega - E - E_{n'}), \]

\[ \left\langle \delta(P_{x'} + k_x - P'_x - p'_x) \right\rangle^2 = \frac{L_z}{2\pi} \delta(P_{x'} + k_x - P'_x - p'_x), \]

\[ \left\langle \delta(P_{x} + k_z - P'_x - p'_x) \right\rangle^2 = \frac{L_z}{2\pi} \delta(P_{x} + k_z - P'_x - p'_x), \tag{4.9} \]

we obtain

\[ |S_{fi}|^2/T = \frac{1}{16} (2\pi)^3 \delta_y^3 (P + k - P' - p') \frac{E_{n'} + m}{V^3\omega'\epsilon'\epsilon_{n'}} |M_{fi}|^2 . \tag{4.10} \]

### 4.2 The scattering cross section

Using unit flux $1/V$ for the incident particle as usual, we can write the differential cross section as

\[ d\sigma = V \left| \frac{S_{fi}}{T} \right|^2 d\rho , \tag{4.11} \]
where \( d\rho \), the differential phase space for final particles, is given in our case by

\[
d\rho = \frac{L_z^2}{2\pi} \frac{dp_z'}{2\pi} \frac{L_z^2}{2\pi} \frac{dp_z'}{2\pi} \frac{V}{(2\pi)^3} d^3P'.
\] (4.12)

Therefore

\[
d\sigma = V \frac{|M_{fi}|^2}{T} \frac{L_z^2L_z}{(2\pi)^2} \frac{dp_z'}{2\pi} \frac{V}{(2\pi)^3} d^3P' = \frac{1}{64\pi^2} \delta^3(P + k - P' - p') \frac{E_{n'} + m}{\omega E E_{n'}} |M_{fi}|^2 \frac{L_z^2L_z}{V} \frac{dp_z'}{2\pi} \frac{dp_z'}{2\pi} d^3P'.
\] (4.13)

The square of the matrix element, averaged over the initial neutron spin and summed up over the final spins, is

\[
|M_{fi}|^2 = G_\beta^2 \epsilon_{\mu\nu} H^{\mu\nu},
\] (4.14)

where \( H^{\mu\nu} \) is the hadronic part, which is unaffected by the magnetic field because of our assumption stated earlier:

\[
H^{\mu\nu} = 4(g_V^2 + g_A^2)(P^\mu P'^{\nu} + P'^{\mu} P^\nu - g^{\mu\nu} P \cdot P') + 4(g_V^2 - g_A^2)m_pm_pg^{\mu\nu} + 8g_Vg_A\epsilon^{\mu\nu\lambda\rho} P_\lambda P_\rho.
\] (4.15)

The leptonic part \( \epsilon_{\mu\nu} \), on the other hand, is affected by the magnetic field. It is given by

\[
\epsilon_{\mu\nu} = \int dy \int dy' e^{iq_y(y-y')} Tr\left[ P_U(y,y',n',p')\gamma_\mu k_\nu L \right].
\] (4.16)

To perform the \( y \) integrations, we use the result \[3\]

\[
\int dy e^{iq_y y} I_n(\xi) = i^n \frac{2\pi}{eB} e^{-iq_y p_\parallel /eB} I_n\left(\frac{q_y}{\sqrt{eB}}\right),
\] (4.17)

which gives

\[
\epsilon_{\mu\nu} = \frac{2\pi}{eB} \frac{1}{(E_{n'} + m)} (\Lambda_\mu k_\nu + \Lambda_\nu k_\mu - k \cdot \Lambda g_{\mu\nu} - i\epsilon_{\mu\nu\alpha\beta} \Lambda^\alpha k^\beta),
\] (4.18)

where

\[
\Lambda^\alpha = \left[I_n\left(\frac{q_y}{\sqrt{eB}}\right)\right]^2 (p_\parallel p_\parallel - p_\parallel^2) + \left[I_n-1\left(\frac{q_y}{\sqrt{eB}}\right)\right]^2 (p_\parallel p_\parallel + p_\parallel^2) + 2M_n g_A^2 I_n\left(\frac{q_y}{\sqrt{eB}}\right) I_n-1\left(\frac{q_y}{\sqrt{eB}}\right).
\] (4.19)

Thus,

\[
|M_{fi}|^2 = 8G_\beta^2 \left[(g_V^2 + g_A^2)(P \cdot \Lambda P' \cdot k + P' \cdot \Lambda P \cdot k) - (g_V^2 - g_A^2)m_pm_pk \cdot \Lambda + 2g_Vg_A(P \cdot \Lambda P' \cdot k - P' \cdot \Lambda P \cdot k)\right] \times \frac{2\pi}{eB} \frac{1}{(E_{n'} + m)}.
\] (4.20)

We now adopt the rest frame of the neutron and choose the axes such that the 3-momentum of the incoming neutrino is in the \( x-z \) plane. We will also assume that \( |P'| \ll m_p \) for the range of energies of interest to us. In that case,

\[
|M_{fi}|^2 = 8G_\beta^2 \times \frac{2\pi}{eB} \frac{m_pm_p}{E_{n'} + m} \left[(g_V^2 + 3g_A^2)\Lambda_0\omega + (g_V^2 - g_A^2)k_z\Lambda\right].
\] (4.21)
The electron momentum $p'$ enters the expression only through $\Lambda^0$, which has only the 0 and 3 components non-vanishing. The components of $P'$ does not appear in this expression because of our assumption made earlier, except $p'_y$, which appears in $\Lambda$. Thus the integrations over $k'_z, p'_y$ and $p'_x$ can be done easily in Eq. (4.13). The integrations over $p'_y$ and $p'_x$ get rid of the $\delta$-functions over the corresponding components of momenta. As for $k'_x$, we refer to Eq. (2.9). Since the center of the oscillator has to lie between $-\frac{1}{2}L_y$ and $\frac{1}{2}L_y$, we conclude that $-\frac{1}{2}L_y eB \leq p'_x \leq \frac{1}{2}L_y eB$. Thus the integration over $k'_x$ gives a factor $L_y eB$. Putting back in Eq. (4.13) and using $V = L_x L_y L_z$, we obtain

$$d\sigma = \frac{G^2}{4\pi} \frac{\delta(Q + \omega - E_{n'})}{\omega E_{n'}} \left[ (g_V^2 + 3g_A^2) \omega \Lambda_0 + (g_V^2 - g_A^2) k_z \Lambda \right] dP'_y dP'_z, \quad (4.22)$$

where $Q$ is the neutron-proton mass difference, $m_n - m_p$.

We next perform the integration over $P'_y$. In the integrand, it occurs only as the argument of the functions $I_n$ and $I_{n-1}$. The functions $I_n$ are orthogonal in the sense that

$$\int da I_{n}(a) I_{n'}(a) = \sqrt{eB} \delta_{nn'}. \quad (4.23)$$

First of all, this shows that the term containing $M_n$ in the expression for $\Lambda^0$ in Eq. (4.19) does not contribute to the cross section. The other two terms do contribute for $n' > 0$, and we obtain

$$d\sigma_{n' > 0} = \frac{eBG^2}{2\pi} \frac{\delta(Q + \omega - E_{n'})}{\sqrt{(Q + \omega)^2 - m^2 - M_{n'}}} \left[ (g_V^2 + 3g_A^2) + (g_V^2 - g_A^2) k_z \right] dP'_z. \quad (4.24)$$

Writing the argument of the remaining $\delta$-function in terms of $p'_z$, we find that the zeros occur when

$$p'_z = p'_z \equiv \pm \sqrt{(Q + \omega)^2 - m^2 - M_{n'}}. \quad (4.25)$$

Thus, after integrating over $p'_z$, we obtain the total cross section to be

$$\sigma_{n' > 0} = \frac{eBG^2}{\pi} (g_V^2 + 3g_A^2) \frac{Q + \omega}{\sqrt{(Q + \omega)^2 - m^2 - M_{n'}}}. \quad (4.26)$$

For $n' = 0$, however, the result is different, because in this case even the second term of Eq. (4.19) vanishes owing to the definition in Eq. (2.18). Thus, for this case, performing the integral over $P'_y$ in Eq. (4.22), we obtain

$$d\sigma_0 = \frac{eBG^2}{4\pi} \frac{\delta(Q + \omega - E_0)}{\sqrt{(Q + \omega)^2 - m^2}} \left[ (g_V^2 + 3g_A^2) - (g_V^2 - g_A^2) k_z \right] \frac{E_0 - p'_z}{E_0} dP'_z. \quad (4.27)$$

This gives

$$\sigma_0 = \frac{eBG^2}{2\pi} \frac{1}{\sqrt{(Q + \omega)^2 - m^2}} \left[ (g_V^2 + 3g_A^2) - (g_V^2 - g_A^2) k_z \right] \frac{Q + \omega}{\sqrt{(Q + \omega)^2 - m^2 - M_{n'}}}. \quad (4.28)$$

The total cross section is then given as a sum over all possible values of $n'$, i.e.,

$$\sigma = \sum_{n'=0}^{n'_{\text{max}}} \sigma_{n'} = \frac{eBG^2}{2\pi} \sum_{n'=0}^{n'_{\text{max}}} \left[ g_{n'}(g_V^2 + 3g_A^2) - \delta_{n',0}(g_V^2 - g_A^2) k_z \right] \frac{Q + \omega}{\sqrt{(Q + \omega)^2 - m^2 - M_{n'}}}. \quad (4.29)$$

where $g_{n'}$ is the degeneracy of the $n'$-th Landau level, 1 for $n' = 0$ and 2 for all other values of $n'$. And $n'_{\text{max}}$ is the maximum Landau level allowed at the given energy, given by the fact that the quantity under the square root sign in Eq. (4.26) must be non-negative, i.e.,

$$n'_{\text{max}} = \text{int} \left\{ \frac{1}{2eB} [(Q + \omega)^2 - m^2] \right\}. \quad (4.30)$$
In Fig. 1, we have plotted the total cross section as a function of the magnetic field. The spikes in this plot appear at values of the magnetic field for which the denominator of Eq. (4.29) vanishes for some \( n' \). For field values larger than this, that particular Landau level does not contribute to the cross section. The spikes in fact go all the way up to infinity, and their finite heights in the figure is an artifact of the finite step size taken in plotting it. In a real situation, where the initial neutrinos are not exactly monochromatic, these spikes are smeared out.

![Figure 1: Total cross section as a function of the magnetic field, normalized to the cross section in the field-free case. The initial neutrino energy is 10 MeV. The solid and the dashed lines are for the initial neutrino momentum parallel and antiparallel to the magnetic field.](image)

It is instructive to check that the results obtained above reduce to the known results for the field-free case. The contribution proportional to \( g_V^2 - g_A^2 \) appears only in \( \sigma_0 \), and vanishes in the limit \( B \to 0 \) owing to the overall factor of \( eB \) present in Eq. (4.28). Contributions proportional to \( g_V^2 + 3 g_A^2 \) also have the factor \( eB \) with them, but in this case we also need to sum over infinitely many states. This gives

\[
\sigma = \frac{eBG_\beta^2}{\pi} (g_V^2 + 3g_A^2) \left( \sum_{n'=0}^{n'_{\text{max}}} \frac{Q + \omega}{\sqrt{(Q + \omega)^2 - m^2 - 2n'eB}} - \frac{Q + \omega}{2\sqrt{(Q + \omega)^2 - m^2}} \right),
\]

where \( n'_{\text{max}} \) is the largest integer for which the quantity under the square root sign is non-negative. For \( B \to 0 \), the last term vanishes, and we can identify \( n'_{\text{max}} \) as the integer for which the denominator of the summand vanishes. Thus we obtain

\[
\sigma \to \frac{eBG_\beta^2}{\pi} (g_V^2 + 3g_A^2) \int_0^{n'_{\text{max}}} dn' \frac{Q + \omega}{\sqrt{(Q + \omega)^2 - m^2 - 2n'eB}} = \frac{G_\beta^2}{\pi} (g_V^2 + 3g_A^2)(Q + \omega)\sqrt{(Q + \omega)^2 - m^2},
\]

which is the correct result in the field-free case.

5 The asymmetric cross-section

The calculation of the cross section for inverse beta decay process has been performed earlier by Roulet. He assumed that the matrix element remains unaffected by the magnetic field, only the modified
Figure 2: The cross section as a function of $\cos \theta$, where $\theta$ is the angle the initial neutrino makes with the magnetic field. From bottom to top, the values of the maximum Landau level are 0, 1, 2 and 10.

phase space integral makes the difference in the cross section. The results he obtained is the same as the term proportional to $g_\ell^2 + 3g_A^2$ that we obtained.

It is the other term, not found by earlier calculations, that introduces the most important feature of the cross section obtained in Eq. (4.29), viz., its anisotropy. This comes only from the $n' = 0$ contribution, which depends on the direction of the incoming neutrino momentum $\mathbf{p}$. However, for $n' \neq 0$, there is a cancellation between the two possible states in a Landau level which washes out all angular dependence. In a real situation, then, the asymmetry will come only from the $n' = 0$ state and its amount will depend on the relative contribution of this state to the total cross section. If the magnetic field is so high that only the $n' = 0$ state can be obtained for the electron, the asymmetry will be large, about 18%. For smaller and smaller magnetic fields, the asymmetry decreases with new Landau levels contributing. This is shown in Fig. 2.

This fact can have far reaching consequences for neutrino emission from a proto-neutron star. It has been discussed in the literature that the presence of asymmetric magnetic fields can cause asymmetric neutrino emission from a proto-neutron star, thereby explaining the pulsar kicks. However, our calculations show that such an asymmetry in the magnetic field may not be necessary in producing the asymmetry in neutrino emission because of the $k_z$-dependent term in the cross section. Below, we make a rough estimate of the momentum asymmetry produced by this term.

The typical size of a neutron star is about $R \approx 10$ km. Thus, when a neutrino arrives at a density where its mean free path is about $R$, it escapes from the star. Therefore the condition for the neutrino to escape can be written as

$$n_n \sigma = \frac{1}{R},$$  \hspace{1cm} (5.1)

where $n_n$ is the neutron number density. We already observed that $\sigma$ is direction dependent. Therefore, the value of $n_n$ on the “neutrino sphere” depends on the direction as well, and the surface is no longer a sphere. Different values of $n_n$ will correspond to different temperatures. Thus, neutrinos will be emitted with different momenta in different directions. This can result in a kick to the star.

To estimate the magnitude of the kick, let us abbreviate Eq. (4.29) as

$$\sigma = eBG^2(\alpha + b \cos \theta),$$  \hspace{1cm} (5.2)

where $\theta$ is the angle between the magnetic field and the neutrino momentum. If we now consider the
directions $\theta = 0$ and $\theta = \pi$, the difference in neutron density on the corresponding points on the neutrino surface is given by

$$\Delta n = \frac{2b}{\epsilon BG^2 a^2 R},$$

(5.3)
neglecting corrections of order $b/a$.

The neutron gas in a typical proto-neutron star can be considered to be non-relativistic and degenerate. The number density of neutrons is thus given by

$$n_n = \frac{p_F^3}{3\pi^2} \left[ 1 + \frac{\pi^2 m_n^2 T^2}{2p_F^4} + \cdots \right],$$

(5.4)

where $p_F$ is the Fermi momentum, and we have neglected higher order terms in the temperature. This gives

$$\frac{dn_n}{dT} = \frac{m_n^2}{3} \left( \frac{T}{p_F} \right).$$

(5.5)

So the temperature difference between the points on the neutrino surface in the $\theta = 0$ and $\theta = \pi$ directions is

$$\Delta T = \frac{3}{m_n^2} \frac{p_F}{T} \frac{2b}{eBG^2 a^2 R}.$$ 

(5.6)

The momentum asymmetry can now be written as

$$\frac{\Delta k}{k} = \frac{4}{6} \Delta T,$$

(5.7)

where we have assumed a black body radiation luminosity ($\propto T^4$) for the effective neutrino surface. The factor $1/6$ comes in because the asymmetry pertains only to $\nu_e$, whereas 6 types of neutrinos and antineutrinos contribute to the energy emitted. This gives

$$\frac{\Delta k}{k} = \frac{4}{m_n^2} \frac{p_F}{T} \frac{b}{eBG^2 a^2 R},$$

(5.8)

To find $p_F$, we use the leading term in Eq. (5.4) and estimate $n_n$ from the equation

$$n_n = \frac{\rho(1 - Y_e)}{m_p},$$

(5.9)

where $Y_e$ is the electron fraction and $\rho$ is the mass density. Taking $Y_e = 1/10$, we obtain

$$p_F = 24 \rho_{11}^{1/3} \text{ MeV},$$

(5.10)

where $\rho_{11}$ is the mass density in units of $10^{11}$ g cm$^{-3}$. Putting this back in Eq. (5.8), we obtain

$$\frac{\Delta k}{k} = 27 \rho_{11}^{1/3} B_{14}^{-1} T_{\text{MeV}}^{-2} \frac{b}{a^2},$$

(5.11)

where $B_{14} = B/(10^{14}$ Gauss) and $T_{\text{MeV}} = T/(1 \text{ MeV})$. For $\omega \gg m_e$ which is the relevant case,

$$b = \frac{g_A^2 - 9g_V^2}{2\pi} = 9.3 \times 10^{-2},$$

(5.12)

using $g_V = 1$ and $g_A = 1.26$. The value of $a$ will depend on $n'_\text{max}$. If only the $n' = 0$ level contributes, we obtain

$$a = \frac{g_V^2 + 3g_A^2}{2\pi} = 9.2 \times 10^{-1}.$$

(5.13)

This gives

$$\frac{\Delta k}{k} = 3 \rho_{11}^{1/3} B_{14}^{-1} T_{\text{MeV}}^{-2}.$$ 

(5.14)

Obviously, with reasonable choices of $\rho$, $B$ and $T$, it is possible to obtain a fractional momentum imbalance of the order of 1% which is necessary for explaining the pulsar kicks.
We have calculated the cross section of the inverse beta decay process in the background of a uniform magnetic field of arbitrary magnitude, using exact spinor solutions in a magnetic field. Our result shows that the cross section is asymmetric, viz., it depends on the direction of the incident neutrino. We have shown how this might explain the pulsar kicks.

In the calculation, we have not taken the Pauli blocking effect into account. However, Roulet [5] has pointed out that this effect is not very important in the range of densities in question. This can be seen from Fig. 3, where we have plotted the critical density beyond which the final electron is totally blocked at zero temperature because the Fermi energy is higher than $Q + \omega$. At low but finite temperatures, the effect will be somewhat smeared. But it has been shown [5] that in the relevant temperature range, the distortion of the Fermi distribution from the zero-temperature one is small.

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[9] After the calculation of the cross section was presented at the COSMO-99 conference held at ICTP Trieste in September 1999, it was pointed out to us that another recent calculation has also found this asymmetric term — A. A. Gvozdev and I. S. Ognev: astro-ph/9909154. They have, however, calculated only the transition to the lowest Landau level.

[10] See, e.g., §58 of E. M. Lifshitz and L. P. Pitaevskii, Statistical Physics, 3rd edition, Part 1 (Pergamon Press 1980).