Re-entrant single particle mobility edge in quasiperiodic chain

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The single particle mobility edge (SPME) appears in the energy spectrum of a non-interacting quantum system as an energy separating the extended and the localized states. While the existence of the SPME is ruled out in the presence of a random disorder, its presence in certain types of one dimensional systems with quasiperiodic disorder has been established. The question is whether such an SPME is unique in a particular quantum system, or in other words, once the localization transition occurs whether the system remains localized forever? In our studies, we show that for a one dimensional dimerized lattice with staggered quasiperiodic disorder (having alternate signs across the lattice sites) there occurs a re-entrant localization phenomenon as a function of the disorder strength. We predict that for a range of the dimerization strength and beyond the localization transition, the SPME re-appears corresponding to larger values of the disorder strength, thereby resulting in more than one SPMEs in the system. By analyzing various physical quantities we concretely establish this re-entrant phenomena.

\textbf{Introduction.}- In condensed matter physics, random disorder plays an essential role in transport properties and leads to a fundamental phenomenon known as Anderson localization (AL) \cite{1}. Importantly, AL teaches us the role of dimensionality embedded therein, where a metal-insulator phase transition occurs for a finite critical disorder only for three dimensions. A related issue is the existence of the single particle mobility edge (SPME) which corresponds to the energy that separates the localized and the extended eigenstates of the system. The hallmark signature of the SPME is the coexistence of the extended and the localized states has also been reviewed with considerable attention \cite{2, 3}. The ramification of having a mobility edge is directly related to the metal-insulator phase transition occurring at a finite value of the disorder. Recent advancement in the experimental front has resulted in the seminal observations of the localization phenomena and the mobility edge in the context of quantum gas experiments \cite{4–8}.

On the other hand, similar physics can be obtained in one dimension by replacing the uncorrelated (random) disorder by a quasiperiodic potential which is exhibited by the non-interacting Aubry-André (AA) model \cite{9}. A phase transition from a delocalized phase to a localized phase occurs at a (finite) critical value of the quasiperiodic potential strength and the system does not exhibit a SPME at any value of the disorder potential. However, further analyses in the past several years show that the SPME can still appear in one dimension by further generalizing the AA Hamiltonian or in other quasiperiodic systems \cite{10–15}. The first experimental observation of the SPME in quasi-periodic system was made recently \cite{16} following the interesting theoretical proposals in the context of cold atoms in optical lattices \cite{17, 18}. Recently, the SPME has also been observed in a frustrated triangular ladder \cite{19}.

So far, it has been well established that in quasiperiodic systems the localization transition occurs either in the presence or absence of a SPME. Once the transition occurs, all the states of the system remain localized forever, and remain independent of the disorder strength thereafter. However, is it always true that the occurrence of SPME is unique to a particular system or is there a scenario, where the system undergoes another localization transition as a function of the disorder potential? In this work we show that it is indeed possible to engineer two different critical regions hosting the SPME in one dimension which arises as a consequence of the competition between dimerized hopping and disorder.

To this end, we investigate the localization phenomena in the paradigmatic Su-Schrieffer-Heeger (SSH) model \cite{20} when subjected to the AA type disorder given by the Hamiltonian;

\begin{equation}
H = -t_1 \sum_{i=1}^{N} (c_{i,B}^{\dagger} c_{i,A} + h.c.) - t_2 \sum_{i=1}^{N-1} (c_{i+1,A}^{\dagger} c_{i,B} + h.c.) + \sum_{i=1}^{L} (\lambda_A n_{i,A} + \lambda_B n_{i,B}) \cos(2\pi \beta i) \tag{1}
\end{equation}

which is a one dimensional chain of \(N\) unit cells comprising of two sublattice sites \(A\) and \(B\). \(i\) represents the unit cell index and \(L = 2N\) is the length of the chain. \(c_{i,A}^{\dagger} (c_{i,A})\) and \(c_{i,B}^{\dagger} (c_{i,B})\) are the creation (annihilation) operators corresponding to sites in the \(A\) and \(B\) sublattices which we denote by \((i, A)\) and \((i, B)\) and the site number operators are denoted as \(n_{i,A}\) and \(n_{i,B}\). The intra- and inter-cell hopping strengths are represented by \(t_1\) and \(t_2\) respectively. The strength of the onsite potential at the sublattice site \(A\) (\(B\)) is represented by \(\lambda_A\) (\(\lambda_B\)) and \(\beta\) determines the period of quasiperiodic potential. The model Eq. 1 in the limit of vanishing disorder is the pure SSH model which demonstrates a topological phase transition. The topological phases are known to be protected by the underlying symmetries of
the system. A variety of studies are focused on investigating the effect of disorder on the topological properties of the SSH model, where both the diagonal (onsite) and the off-diagonal (hopping) disorder are considered separately \([21–23]\). Note that while the chiral symmetry of the SSH model is preserved in the case of finite hopping disorder, it is explicitly broken at any finite value of the onsite disorder strength. Therefore, in the latter case, the zero-energy edge modes of the clean system become energetic in the presence of any finite disorder. Efforts have been made to understand such interesting scenarios in the context of the nature of chirality \([24]\), the interplay between long-range hopping and disorder \([25]\), and also the possible existence of the mobility edge \([26]\).

In our analysis, we consider two different types of disorder, namely, (i) uniform disorder, that is, \(\lambda_A = \lambda_B = \lambda\) and (ii) staggered disorder i.e. \(\lambda_A = -\lambda_B = \lambda\) in Eq. 1 and explore their effects in both the trivial \((t_1 > t_2)\) and the topological \((t_1 < t_2)\) limits of the SSH model. We show that for both types of disorder, the system undergoes a localization transition as a function of \(\lambda\) exhibiting the SPME when \(t_1 \neq t_2\). However, an interesting scenario happens in the case of staggered disorder where the system returns back to a critical regime after the first localization transition hosting a second SPME at higher values of \(\lambda\). This interesting and counter-intuitive result reveals the re-entrant behaviour of the localization transition and the SPME which we shall elaborately discuss in the following. We choose \(\beta = (\sqrt{5} - 1)/2\), a Diophantine number \([27]\) in our work and fix the intra-cell hopping, \(t_1 = 1\) as the energy scale. For convenience, we define a quantity \(\delta = t_2/t_1\) which controls the hopping dimerization in Eq. 1. The system size considered in our simulations is \(L = 1220\), that is, \(N = 610\).

**Uniform disorder** \((\lambda_A = \lambda_B = \lambda)\). To analyze the physics of the model shown in Eq. 1, we rely on the inverse participation ratio (IPR) and the normalized participation ratio (NPR) \([28, 29]\), which are the two most significant diagnostic tools to characterize the localization transition. For the \(n\)-th eigenstate, \(\phi_n^i\), the IPR and the NPR are defined as,

\[
\text{IPR}_n = \sum_{i=1}^{L} |\phi_n^i|^4, \quad \text{NPR}_n = \left( L \sum_{i=1}^{L} |\phi_n^i|^4 \right)^{-1}.
\]

As mentioned in the previous section, in the limit of \(\delta = 1\), there is no SPME associated with the localization transition and Eq. 1 denotes the AA limit. However, moving away from the AA limit, that is, going into both the trivial and the topological regimes, we predict that the localization transition occurs through an intermediate/critical regime hosting the SPME. In Fig. 1(a) and (b) we plot the (IPR) and the (NPR) as a function of \(\lambda\) for the two exemplary points, namely, \(\delta = 0.5\) and \(\delta = 3\) corresponding to the trivial and the topological cases respectively of the pure SSH model. Here the (IPR) and the (NPR) denote the averages of the IPR and NPR computed by considering all the eigenstates for a particular value of \(\lambda\). It can be seen that contrary to the simple AA model \((\delta = 1)\), corresponding to either of the limits of dimerization, the plots for the (IPR) and the (NPR) do not sharply cross each other at the duality point \(\lambda = 2\)\([30]\). Rather they cross each other at very different values of \(\lambda\), thereby creating a noticeable coexisting region where both the (IPR) and the (NPR) are finite (shaded regions). This signifies the presence of both the localized and the extended states for a range of \(0.7 < \lambda < 1.4\) when \(\delta = 0.5\) and \(1.6 < \lambda < 3.4\) when \(\delta = 3\) which is denoted as the critical phase. As mentioned earlier, the appearance of such critical phases is an indicator of the existence of the SPME \([29]\) which is absent in the case of the pure AA model \((\delta = 1)\).

The existence of the SPME can be easily inferred from the energy spectrum and the associated IPR of the individual states. We plot the energy spectra, \(E\) corresponding to the Hamiltonian in Eq. 1 for \(\delta = 0.5\) and \(3\) in Fig. 1(c) and (d) respectively. Here, the eigenergies are color coded with the corresponding IPR values. Due to the nature of the SSH model, we get two distinct energy bands separated by an energy gap at \(\lambda = 0\) and the energy levels are completely extended for both the trivial (Fig. 1(c)) and the topological (Fig. 1(d)) cases in this limit. As the value of \(\lambda\) increases, the gaps in both the dimerized limits tend to vanish beyond a critical \(\lambda\). However, other minibands with some states in the gaps between them appear due to the disorder potential. For the present analysis, these minibands and associated physics are irrelevant. In both cases, as a function of \(\lambda\),
FIG. 2. (a) and (b) show the $\langle \text{IPR} \rangle$ and the $\langle \text{NPR} \rangle$ for $\delta = 1.5$ and 2.2 respectively. The shaded regions represent the critical phases. (c) Half of the energy eigenvalue spectrum superimposed with their respective IPR values is shown demonstrating the extended, critical and localized states. (d) The eigenstate indices and their associated IPR are plotted as function $\lambda$ for $\delta = 2.2$ for a system of size $L = 466$. The region between the vertical lines in (c) and (d) represents the second critical region of (b) and the color bars on the top represent the IPR.

the fully extended (blue) and the localized regions (red) are separated by a critical phase of $\lambda$, as inferred from the analysis made earlier. It is evident from the energy spectrum that the mobility edges exist for both $\delta < 1$ and $\delta > 1$ regimes. Note that similar signature is visible from the eigenstates of the system. We also confirm the existence of the SPME in the density of states (DOS) for different values of $\lambda$. Quite expectedly, in Fig. 1(d), the appearance of the localized states at $\lambda = 0$ are the topological edge modes present in the middle of the gap. We shall discuss about the fate of these edge modes later.

Staggered disorder ($\lambda_A = -\lambda_B = \lambda$). Following the analysis similar to the case of uniform disorder we compute the (IPR) and the (NPR) by moving away from the AA limit, namely, $\delta = 1$. Interestingly, even in this case, we find the signature of the SPME in both the dimerization limits which is evident from the existence of the critical phase where both the (IPR) and the (NPR) are finite. As it is well known and already mentioned before, in quasi-periodic lattices hosting the SPME, for the values of $\lambda$ prior to (beyond) the critical phase, all the states of the system are extended (localized). Once the system is in the localized phase, it remains localized as a function of the strength of the potential, $\lambda$. As a result, one gets $\langle \text{IPR} \rangle \neq 0$ and $\langle \text{NPR} \rangle = 0$ for all values of $\lambda$ after the critical regime. However, surprisingly in presence of the staggered disorder, we find that for a range of $\delta$, the system undergoes two localization transitions through two critical phases as a function of $\lambda$. This re-entrant feature can be very well discerned by analyzing the (IPR) and the (NPR). In Figs. 2(a) and (b) we show the $\langle \text{IPR} \rangle$ and the $\langle \text{NPR} \rangle$ corresponding to the topological phase of the SSH model, namely, for $\delta = 1.5$ and 2.2 respectively. Clearly for $\delta = 1.5$ (Fig. 2(a)), there is a transition to the localized phase through a critical region for the range of $\lambda$, namely, $0.9 < \lambda < 2.5$ and hence certifies the existence of an SPME. For $\lambda > 2.5$, all the states are localized. On the other hand, for $\delta = 2.2$ (Fig. 2(b)), there are two critical regions in the range $0.9 < \lambda < 1.8$ and $2.3 < \lambda < 2.9$ where both the (IPR) and the (NPR) are finite. In the region between the two critical phases and again beyond the second critical phase, the system is fully localized. This indicates that the system can host two SPMEs as a function of $\lambda$. Note that the extent of the second critical region occurs for a small range of $\lambda$. This feature is clearly visible in the energy spectrum encoded with the corresponding IPR as shown in the Fig. 2(c). For clarity we depict only the upper band of the spectrum which clearly shows a series of the extended-critical-localized-critical-localized regions as a function of $\lambda$. This feature is also clearly seen by plotting the IPR of the individual eigenstates as shown in Fig. 2(d). The critical regions in Fig. 2(c) and (d) are denoted by two vertical lines. We further confirm the existence of the SPME by plotting the IPR and the NPR for the individual eigenstates of the system at the critical regime. In Fig. 3(a) we plot the IPR and NPR for all the eigenmodes for $\delta = 2.2$ at

FIG. 3. (a) IPR (red squares) and NPR (blue circles) of different eigenstates for parameters $\delta = 2.2$, $\lambda = 1.2$ (upper panel) and $\delta = 2.2$, $\lambda = 2.7$ (lower panel). The scattered states with finite IPR in the extended regime are the emerging edge modes in the fractal gaps. (b) Shows the DOS for $\delta = 2.2$, $\lambda = 1.2$ (upper panel) and $\delta = 2.2$, $\lambda = 2.7$ (lower panel) and the vertical lines separate the extended and localized regions. (c) and (d) shows the edge states and the corresponding IPRs for uniform and staggered disorder respectively for $\delta = 1.5$ (left panel) and $\delta = 5$ (right panel). $E^-$ (blue dot-dashed) and $E^+$ (red dashed) corresponding to the two edge states along with their IPR i.e. $\langle \text{IPR} \rangle^-$ (blue solid) and $\langle \text{IPR} \rangle^+$ (red dotted).
\(\lambda = 1.2\) and \(2.7\) in the upper and lower panels respectively. The plot shows a clear distinction between the extended states (finite NPR) from the localized states (finite IPR) of the spectrum. Similar signature is also seen in the density of states (DOS) by analyzing it with the IPR of the individual states indicating the existence of the mobility edge as shown in Fig. 3(b) (see figure caption for detail).

**Phase diagram:** Finally, we present the key results of our simulation in the form of two phase diagrams as displayed in Fig. 4(a) and (b) for the uniform and the staggered disorder respectively in the \(\delta - \lambda\) plane. The phase diagrams are obtained by computing a quantity \(\eta\) introduced in Ref. [29] as;

\[
\eta = \log_{10}(\langle\text{IPR}\rangle \times \langle\text{NPR}\rangle)
\]

Clearly, the intermediate region (red region bounded by the symbols) is distinguished from the fully extended or the fully localized regions (blue regions) in the phase diagram for both the cases. Note that in either of the phase diagrams there exists two intermediate regions separated by the a narrow passage at \(\delta = 1\) (AA model) where a sharp localization transition occurs. While in Fig. 4(a) the intermediate regime indicates the presence of an SPME, in Fig. 4(b), the re-entrant feature and consequently two SPMEs can be inferred for a range of \(\delta\) on either side of \(\delta = 1\). We complement the above finding by directly locating the critical region boundaries by examining the values of (IPR) and (NPR) which are shown via the filled squares and they match well with the boundaries demarcating the critical regions. This non-trivial features in the phase diagram due to the re-entrant localization transition and SPME can be attributed to the extended nature of a few low energy states in the spectrum. It is worth mentioning that there also exists a region of re-entrant localization phenomena and the mobility edge in the \(\delta < 1\) (trivial) regime which is narrower compared to the \(\delta > 1\) scenario. Hence, an important conclusion can be drawn at this point is that the underlying topological properties has no role in establishing the second critical region and the re-entrant mobility edge.

**Edge modes.** Having analyzed the physics of the bulk spectrum, we discuss about the fate of the topological zero energy edge modes as a function of the disorder strength, \(\lambda\). We note that the initially \((\lambda = 0)\) localized zero modes become energetic and finally hybridize into the bulk bands with increase in \(\lambda\) corresponding to both the uniform and the staggered disorder as already shown in Fig. 1(d) and Fig. 2(c) respectively. To explicitly understand the behavior of these modes, we separately plot the edge modes as a function of \(\lambda\) in Fig. 3 along with their IPR values. We consider two different values of \(\delta\), namely, \(\delta = 1.5\) and \(\delta = 5\) which represent weak and strong dimerization limits pertaining to the topological regime. As mentioned earlier, owing to the breaking of the chiral symmetry induced by the quasi-periodic potential, both the edge modes, namely, the particle mode \((E^+\) shown by a dashed red line) and the hole mode \((E^-\) shown by a dot-dashed blue line) asymmetrically separate out from each other towards the opposite bands as \(\lambda\) increases (Fig. 3(c)) for the case of uniform potential. However, in the case of the staggered disorder, both the edge modes move differently towards the lower band (Figs. 3(d)) [31]. Eventually for all the cases, beyond certain critical values of \(\lambda\), \(E^+\) and \(E^-\) tend to merge with each other. We also plot the corresponding IPR for both the modes as IPR\(^+\) (dotted red) and IPR\(^-\) (solid blue). It can be seen that in all the four cases the IPR initially decreases and then increases as a function of \(\lambda\). In the case of weak dimerization (\(\delta = 1.5\)), initially the states are not fully localized. As the value of \(\lambda\) increases, the states tend to become delocalized and then become fully localized. On the other hand, in the case of strong dimerization, the states which are fully localized (IPR \(\sim 1\)) at the beginning (small \(\lambda\)) tend to be delocalized and then become localized again. This re-entrant localization of the edge states is slow as a function of \(\lambda\) in the case for larger \(\delta\).

**Conclusions.** We have studied the localization transition in an SSH model superimposed with quasi-periodic disorder. By considering both the uniform and the staggered disorder, we analyze the existence of an SPME in both the trivial and the topological regimes corresponding to two dimerization limits of the SSH model. We show that for the case of uniform disorder, the SPME exists in both the limits of dimerization for a range of values of disorder strengths. However, interesting things happen in presence of staggered disorder potential. In this case, although the SPME appears in both the limits of dimerization, there exists a finite region in terms of the dimerization strength where the system exhibits two
SPMEs as a function of disorder strengths. This feature shows a phenomenon of re-entrant localization which is an unusual scenario in the context of localization physics in presence of quasi-periodic disorder. We confirm these findings by examining the participation ratios, the single particle spectrum and the behavior of the individual eigenstates and present a phase diagram depicting all the above findings for both the limits of dimerization, namely, \( \delta < 1 \) and \( \delta > 1 \). In the end we discuss the fate of the zero energy edge modes as a function of disorder which were initially localized in the absence of any disorder.

Our findings will certainly open up new directions to study the localization phenomena in quasi-periodic systems. The re-entrant feature may reveal interesting physics in transport properties and also in dynamical systems. It is worth noting that recently several interesting investigations are underway to study the topological phases in interacting SSH model. Therefore, an immediate extension could be to study the stability of the re-entrant phenomenon in the context of many-body localization. Due to the recent experimental progress in systems of ultracold atoms in optical lattices to simulate SSH model and quasi-periodic systems and the recent experiment on disorder induced topological phase transition using \(^{171}\text{Yb}\), our findings can in principle be simulated in the state-of-the art quantum gas experiments.

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