Existence and approximate controllability of Hilfer fractional evolution equations with almost sectorial operators

Pallavi Bedi¹, Anoop Kumar¹, Thabet Abdeljawad²,³,⁴*, Zareen A. Khan⁵ and Aziz Khan²

Abstract
In this article, we are concerned with the existence of mild solutions and approximate controllability of Hilfer fractional evolution equations with almost sectorial operators and nonlocal conditions. The existence results are obtained by first defining Green's function and approximate controllability by specifying a suitable control function. These results are established with the help of Schauder’s fixed point theorem and theory of almost sectorial operators in a Banach space. An example is also presented for the demonstration of obtained results.

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1 Introduction
In current times, the rising interest of researchers in fractional calculus reflects the popularity of this branch [1–3, 5–13, 15, 19, 30–32]. Differential equations of fractional order are widely applicable in the areas of physics, chemistry, electromagnetics, and mechanics. Fractional differential equations (FDEs) have been used widely in the identification of different physical systems, in control theory, in simulating viscoelastic materials, and in the modeling of different complex phenomena [22–29, 39, 42]. The concept of exact controllability and approximate controllability of FDEs is an active field of investigation because of its major applications in physical sciences. Under some admissible control inputs, exact controllability steers the system to arbitrary final state, while approximate controllability steers the system to the small neighborhood of arbitrary final state. In the published works, there are numerous articles focussing on the exact or approximate controllability of systems represented by FDEs, neutral FDEs, FDEs with impulsive inclusions, and FDEs with delay functions [4, 18, 40, 44].

In particular, approximate controllability of Hilfer FDEs under different conditions has been discussed widely. Mahmudov et al. [36] investigated the exact controllability of Hilfer FDEs in a Hilbert space under the assumption that a linear system of the given equation is approximate controllable. In 2017, Yang et al. [41] discussed the approximate controllabil-
investigating proposed results: by the above-discussed works, we consider here the following system of Hilfer FDEs for Hilfer FDEs with almost sectorial operators has not been discussed yet. Thus, motivated torial operators. We found that in the available literature, approximate controllability of Jaiswal et al. [21] proved the existence of mild solutions of Hilfer FDEs with almost sectorial operators [16, 33, 43]. Recently, numerous works focussing on the existence of mild solutions and analytical solutions mild solutions and classical solutions of FDEs with almost sectorial operator. There are authors also mentioned the suitable assumptions required to establish the existence of operator generates the strongly continuous semigroup of bounded linear operators. In [37] Periago and Straub formulated theory to analyze almost sectorial operators. Here

\[ D_{0+}^{\nu,\mu}v(t) = A v(t) + \gamma w(t) + \psi(t, v(t), \int_{0}^{t} q(t, s) v(t, s) \, ds), \quad t \in \mathbb{J}, \]

\[ I_{0+}^{\nu(1-\mu)} v(0) = \sum_{k=1}^{m} c_{k} v(t_{k}), \]

where \( D_{0+}^{\nu,\mu} \) is the Hilfer fractional derivative of order \( 0 < \nu < 1 \) and type \( 0 \leq \mu \leq 1 \). \( A : D(A) \subset \mathcal{V} \rightarrow \mathcal{V} \) is an almost sectorial operator in Banach space \( \mathcal{V} \) and \( \mathcal{J} = [0, \beta], \beta > 0 \). The control function \( w(t) \) takes value in \( L^{2}(\mathcal{J}, \mathcal{W}) \). \( L^{2}(\mathcal{J}, \mathcal{W}) \) denotes the space of admissible control functions for Banach space \( \mathcal{W} \). \( \gamma \) is a bounded linear operator from \( \mathcal{V} \) into \( \mathcal{V} \). \( \psi : \mathcal{J} \times \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V} \) is a continuous linear mapping, \( 0 < t_{1} < t_{2} < \cdots < t_{m} < \beta, m \in \mathbb{N} \), and \( c_{k} \) are real numbers such that \( c_{k} \neq 0 \). The characteristics functions \( q(t, s) : \Delta \rightarrow \mathbb{R} \) and \( \varphi : \Delta \times \mathcal{V} \rightarrow \mathcal{V}, \Delta = (t, s) \in [0, \beta] \times [0, \beta] \) are specified in the next section.

2 Preliminaries

Let us consider \( \mathcal{V} \) and \( \mathcal{W} \) as real Banach spaces with respective norms \( \|v\|_{\mathcal{V}} \) and \( \|v\|_{\mathcal{W}} \). \( C(\mathcal{J}, \mathcal{V}) \) denotes the Banach space formed by all continuous functions from \( \mathcal{J} \) into \( \mathcal{V} \) with corresponding norm function \( \|w\| = \sup_{t \in \mathcal{J}} \|w(t)\| \). \( L^{2}(\mathcal{J}, \mathcal{W}) \) denotes the Banach space of all \( \mathcal{W} \)-valued Bochner square integrable functions defined on \( \mathcal{J} \) w.r.t. norm function

\[ \|w\| = \left( \int_{0}^{\beta} \|w(t)\|^{2} \, dt \right)^{\frac{1}{2}}, \quad w \in L^{2}(\mathcal{J}, \mathcal{W}). \]

**Definition 2.1** ([20]) Hilfer fractional derivative of a continuously differentiable function \( f \) of order \( 0 < \nu < 1 \) and \( 0 \leq \mu \leq 1 \) is defined as

\[ D_{0+}^{\nu,\mu} f(t) = I_{0+}^{\nu(1-\mu)} \frac{d}{dt} I_{0+}^{\nu(1-\mu)} f(t) = I_{0+}^{\nu(1-\mu)} D_{0+}^{\nu(1-\mu)} f(t), \]
where $I_0^{\alpha\nu}$ is a Riemann–Liouville fractional integral and $D_0^{\nu\alpha}$ is a Riemann–Liouville fractional derivative.

**Definition 2.2 ([38])** For $\alpha > 0$, the Riemann–Liouville fractional integral of a continuously differentiable $f$ of order $\alpha$ is defined as follows:

$$I_0^{\alpha\nu}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds.$$  

**Almost sectorial operator** ([21, 37]) Let $0 < \beta < \pi$ and $-1 < \gamma < 0$. We define $S_\beta^\gamma = \{ \omega \in \mathbb{C} \setminus \{0\} : \arg \omega < \beta \}$ and $S_\beta = \tilde{S}_\beta = \{ \omega \in \mathbb{C} \setminus \{0\} : \arg \omega \leq \beta \} \cup \{0\}$. A closed linear operator $A : D(A) \subset V \rightarrow V$ is called an almost sectorial operator if the following hold:

1. $\sigma(A)$ is contained in $S_\beta$.
2. $\forall \beta \in (0, \pi)$ there exists a constant $M_\beta > 0$ such that $\| R(\beta, A) \|_{L(V)} \leq M_\beta |\beta|^{-1}$, where $R(\beta, A) = (\beta I - A)^{-1}$ is the resolvent operator of $A$ for $\beta \in \rho(A)$.

Let us define $\Theta_\nu^\gamma$ as a family of almost sectorial operators.

**Proposition 2.1** ([21]) Let $T(t)$ be the compact semigroup defined in [37] and $A \in \Theta_\nu^\gamma$ for $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. Then following holds for $T(t)$:

1. $T(t)$ is analytic and $\frac{d}{dt}T(t) = (\lambda A)^{\nu}T(t)$, $t \in S_\beta^\gamma$.
2. $T(t+s) = T(t)T(s)$, $t \in S_\beta^\gamma$.
3. $\| T(t) \|_{L(V)} \leq c_0 t^{-\gamma-1}$, $t > 0$; where $c_0 > 0$ is a constant.
4. Let $\Sigma_\theta = \{ \nu \in V : \lim_{t \rightarrow 0^+} \nu(t) = \nu \}$, then $D(A^\nu) \subset \Sigma_\theta$ if $\theta > 1 + \gamma$.

Let us define the operator families $\{ S_\nu(t) \}_{t \in S_\beta^\gamma}$ and $\{ Q_\nu(t) \}_{t \in S_\beta^\gamma}$ as follows:

$$S_\nu(t) = \int_0^\infty \mathcal{M}_\nu(\xi)T(t\xi) \, d\xi,$$

$$Q_\nu(t) = \int_0^\infty v\xi \mathcal{M}_\nu(\xi)T(t\xi) \, d\xi,$$

where $\mathcal{M}_\nu(\theta)$ is a Wright-type function defined as follows:

$$\mathcal{M}_\nu(\theta) = \sum_{n \in \mathbb{N}} \frac{(-\theta)^{\nu-1}}{\Gamma(1-vi(n-1))}, \quad \theta \in \mathbb{C}.$$  

**Proposition 2.2** ([21]) (i) For each fixed $t \in S_\beta^\gamma$, $S_\nu(t)$ and $Q_\nu(t)$ are bounded linear operators in $V$. Also

$$\| S_\nu(t) \| \leq C_\nu t^{\nu(1+\gamma)}, \quad \| Q_\nu(t) \| \leq C_\nu t^{\nu(1+\gamma)} \quad t > 0,$$

where $C_\nu$ and $C_\nu$ are constants depending only on $\nu$ and $\gamma$. (ii) $S_\nu(t)$ and $Q_\nu(t)$ are continuous in the uniform operator topology for $t > 0$. 


Lemma 2.1 ([2, 20]) The mild solution for the system of equations Eq. (1.1) is defined as follows:

\[ v(t) = S_{\nu, \mu}(t) \sum_{k=1}^{m} c_k v(t_k) + \int_0^t \mathcal{K}_0(t - s) \left\{ \int_0^s q(s, \zeta) \psi(s, \zeta) d\zeta \right\} ds, \]

where \( S_{\nu, \mu}(t) = \mathcal{T}_0^{\mu(1-\nu)} \mathcal{K}_0(t) \) and \( \mathcal{K}_0(t) = t^{\nu-1} \mathcal{Q}_0(t) \).

Proposition 2.3 ([21]) (i) For every fixed \( t \in \mathbb{S}_0^{\nu} \), \( S_{\nu, \mu}(t) \) and \( \mathcal{K}_0(t) \) are bounded linear operators on \( V \). For \( t > 0 \),

\[ \| \mathcal{K}_0(t)v \| \leq C_\nu t^{1-\nu} \| v \|, \quad \| S_{\nu, \mu}(t)v \| \leq \frac{\Gamma(-\nu)}{\Gamma(1-\nu)} C_\nu t^{\mu(1-\nu)-\nu-1} \| v \|. \]

(ii) \( S_{\nu, \mu}(t) \) and \( \mathcal{K}_0(t) \) are strongly continuous operators.

Let \( M_\xi = \sup_{t \in [0, \delta]} \| S_{\nu, \mu}(t) \|, \mathcal{K}_0(t) \mathcal{K}_0(t) = [0, \delta]. \) Assume that \( \sum_{k=1}^{m} |c_k| \leq \frac{1}{M_\xi} \).

We have

\[ \left\| \sum_{k=1}^{m} c_k S_{\nu, \mu}(t_k) \right\| \leq M_\xi \frac{1}{M_\xi} < 1. \]

By the operator spectrum theorem, \( O = (I - \sum_{k=1}^{m} c_k S_{\nu, \mu}(t_k))^{-1} \) exists bounded and \( D(O) = V \).

By the Neumann series expression, \( O \) can be expressed as \( O = \sum_{n=0}^{\infty} [\sum_{k=1}^{m} c_k S_{\nu, \mu}(t_k)]^n \).

Therefore

\[ \| O \| \leq \frac{1}{1 - \sum_{k=1}^{m} |c_k| M_\xi}. \]

By Lemma 2.1 mild solutions of Eq. (1.1) are given by

\[ v(t) = S_{\nu, \mu}(t) \hat{v} + \int_0^t \mathcal{K}_0(t - s) \left\{ \int_0^s q(s, \zeta) \psi(s, \zeta) d\zeta \right\} ds, \]

where \( \hat{v} = \sum_{k=1}^{m} c_k v(t_k) \).

At \( t = t_k \),

\[ v(t_k) = S_{\nu, \mu}(t_k) \hat{v} + \int_0^{t_k} \mathcal{K}_0(t_k - s) \left\{ \int_0^s q(s, \zeta) \psi(s, \zeta) d\zeta \right\} ds, \]

\[ \sum_{k=1}^{m} c_k v(t_k) = \sum_{k=1}^{m} c_k S_{\nu, \mu}(t_k) \hat{v} + \sum_{k=1}^{m} c_k \int_0^{t_k} \mathcal{K}_0(t_k - s) \left\{ \int_0^s q(s, \zeta) \psi(s, \zeta) d\zeta \right\} ds \]

\[ \times \left\{ \int_0^s q(s, \zeta) \psi(s, \zeta) d\zeta \right\} ds \]
Thusthe mild solutions of Eq. (1.1) are defined as follows:

\[ \begin{align*}
\hat{v}(t) &= \sum_{k=1}^{m} c_k \hat{S}_{k,l}(t) \int_{0}^{t} \mathcal{K}_0(t - s) \\
&\quad \times \left\{ \mathcal{T} w(s) + \Psi \left( s, v(s), \int_{0}^{s} q(s, \zeta) \varphi(s, \zeta) \, d\zeta \right) \right\} \, ds
\end{align*} \]

Thus the mild solutions of Eq. (1.1) are defined as follows:

\[ \begin{align*}
v(t) &= \sum_{k=1}^{m} c_k S_{k,l}(t) \int_{0}^{t} \mathcal{K}_0(t - s) \\
&\quad \times \left\{ \mathcal{T} w(s) + \Psi \left( s, v(s), \int_{0}^{s} q(s, \zeta) \varphi(s, \zeta) \, d\zeta \right) \right\} \, ds
\end{align*} \]

Now we introduce the Green's function \( G(t, s) \) as follows:

\[ G(t, s) = \sum_{k=1}^{m} \chi_{t_k}(s) S_{k,l}(t) \mathcal{O} \mathcal{K}_0(t_k - s) + \chi_{t_0}(s) \mathcal{K}_0(t - s), \]

where

\[ \chi_{t_k}(s) = \begin{cases} 
0, & s \in [0, t_k), \\
1, & s \in [t_k, \delta), \\
0, & s \in [\delta, \bar{\delta}).
\end{cases} \]

The mild solutions of Eq. (1.1) in terms of the Green's function are expressed as

\[ v(t) = \int_{0}^{t} G(t, s) \left\{ \mathcal{T} w(s) + \Psi \left( s, v(s), \int_{0}^{s} q(s, \zeta) \varphi(s, \zeta) \, d\zeta \right) \right\} \, ds, \quad t \in \mathcal{T}. \]

In addition to the above-mentioned propositions and lemma, we assume here the following assumptions to establish the proposed results:

(A1) For each \( t \in \mathcal{T} \), \( \Psi : \mathcal{T} \times \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V} \) is a Caratheodory function.

(A2) There exist \( \psi_1 \in L^1(\mathcal{T}, \mathbb{R}^*) \) and a continuous function \( \psi_2 \) such that, for \( \psi_1, \psi_2 \in \mathcal{V} \) and \( t \in \mathcal{T} \), \( \Psi(t, \psi_1, \psi_2) \leq \psi_1(t) \| \psi_1 \| + \psi_2(t) \| \psi_2 \| \).

(A3) \( \varphi(t,s) : \mathcal{V} \rightarrow \mathcal{V} \) is a Caratheodory function and there exists \( \bar{\psi} : \Delta \rightarrow \mathbb{R} \) with

\[ \bar{\psi} = \sup_{\tau \in \mathcal{T}} \int_{0}^{t} w(t, s) \, ds < \infty \quad \text{such that} \quad \| \varphi(t, s, \psi) \| \leq \bar{\psi}(t) \| \psi \|, \quad \psi \in \mathcal{V}. \]
(A4) \( q(t, s) \) is bounded on \( J = [0, \delta] \) with \( q^* = \sup \{q(t, s) : 0 \leq s \leq t \} < \infty \).

(A5) There exists \( \kappa \in L^1(J, \mathbb{R}^+) \) such that \( \| Y w(t) \| \leq \kappa(t), \forall t \in J \) and \( w \in \mathcal{L}^2(J, \mathcal{W}) \).

3 Existence result

For \( J = [0, \delta] \) and \( J' = [0, \delta] \), we introduce

\[
\mathcal{Y} = \left\{ y \in C(J', \mathcal{V}) : \lim_{t \to 0} \{1^{(1+\nu)}(1-\mu)\} v(t) \text{ exists and is finite} \right\}
\]

with the norm defined as \( \| v \|_{\mathcal{Y}} = \sup_{t \in J'} \{1^{(1+\nu)}(1-\mu)\} \| v(t) \| \).

Let \( y(t) = \{1^{(1+\nu)}(1-\mu)\} v(t) \). Then \( v \in \mathcal{Y} \) iff \( y \in C(J, \mathcal{V}) \) and \( \| v \|_{\mathcal{Y}} = \| y \| \).

We define

\[
B_{\delta}(J) = \left\{ y \in C(J, \mathcal{V}) : \| y \| = \delta \right\} \quad \text{and} \quad B_{\delta}'(J') = \left\{ v \in \mathcal{Y} : \| v \|_{\mathcal{Y}} = \delta \right\}.
\]

Define an operator \( \mathcal{P} : B_{\delta}(J) \to B_{\delta}(J) \) such that

\[
\mathcal{P} y(t) = \begin{cases} 0, & t = 0, \\ \{1^{(1+\nu)}(1-\mu)\} \Pi v(t), & t \in J' = (0, \delta], \end{cases}
\]

where \( \Pi : C(J', \mathcal{V}) \to C(J', \mathcal{V}) \) is defined as

\[
\Pi v(t) = \int_0^\delta G(t, s) \left\{ Y w(s) + \Psi \left( s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds.
\]

Let us denote \( \mathcal{N}_1 = \sup_{t \in J'} \{\psi_1(t)\} \), \( \mathcal{N}_2 = \sup_{t \in J'} \{\psi_2(t)\} \), \( \tilde{m} = \sup_{t \in J'} \{\kappa(t)\} \), and \( \wedge_0 = \frac{\sum_{k=1}^m |c_k|}{\sqrt{\sum_{k=1}^m |c_k|^2}} \).

Lemma 3.1 Let \( A \in \Theta_1^\nu \) for \( -1 < \nu < 0 \) and \( 0 < \omega < \frac{\nu}{2} \). Assume that assumptions (A1)–(A5) and Proposition 2.3 hold. Then the operator \( \{\mathcal{P} y : y \in B_{\delta}(J)\} \) is bounded and continuous if \( C_\nu (\mathcal{N}_1 + \mathcal{N}_2 \tilde{w} q^*) (1 + M_1 \wedge_0) \approx 1 < 1 \).

Proof For \( y \in B_{\delta}(J) \), we have

\[
\| \mathcal{P} y \| = \| \{1^{(1+\nu)}(1-\mu)\} \Pi v(t) \| = \| \{1^{(1+\nu)}(1-\mu)\} \Pi v(t) \|
\]

\[
\leq \mathcal{M}_1 \mathcal{M}_2 \{1^{(1+\nu)}(1-\mu)\} \frac{\sum_{k=1}^m |c_k|}{\sqrt{\sum_{k=1}^m |c_k|^2}} \int_0^{1/k} \mathcal{N}_1 (t - s) \times \left\{ Y w(s) + \Psi \left( s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \]

\[
+ \| \{1^{(1+\nu)}(1-\mu)\} \int_0^t \mathcal{N}_2 (t - s) \times \left\{ Y w(s) + \Psi \left( s, v(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, v(\zeta)) d\zeta \right) \right\} ds \]

\[
\leq \mathcal{M}_1 \mathcal{M}_2 \{1^{(1+\nu)}(1-\mu)\} \int_0^{1/k} (t - s)^{-1-\nu} \left\{ \psi_1(s) \| v \| + \psi_2(s) \tilde{w} q^* \| v \| + \kappa(s) \right\} ds
\]
If possible, suppose \( \| \mathcal{P}g \| > \delta \).

This implies

\[
C_F (\mathcal{N}_1 + \mathcal{N}_2 \tilde{w}q^1) (1 + \mathcal{M}_\delta \wedge_\alpha) \delta^{-\gamma} + \frac{C_F \hat{M}(1 + \mathcal{M}_\delta \wedge_\alpha) \delta^{(1+\gamma)/(1-\mu)+\gamma}}{\delta} > 1.
\]

Applying \( \lim_{n \to \infty} \) on both sides, we have

\[
C_F (\mathcal{N}_1 + \mathcal{N}_2 \tilde{w}q^1) (1 + \mathcal{M}_\delta \wedge_\alpha) \delta^{-\gamma} > 1,
\]

which is a contradiction.

So our supposition is wrong. Therefore \( \| \mathcal{P}g \| \leq \delta \). This implies the boundedness of operator \( \mathcal{P} \).

Now, to show that \( \mathcal{P} : B_3(\mathcal{J}) \to B_3(\mathcal{J}) \) is a continuous operator, let \( y_n \) and \( y \in B_3(\mathcal{J}) \) with \( \lim_{n \to \infty} y_n = y, \ n \in \mathbb{N} \).

By the continuity of function \( \Psi \) w.r.t. to the second and third variable

\[
\Psi \left( t, v_n(t), \int_0^t q(t, s) \varphi(t, s, v_n(t)) \, ds \right) \to \Psi \left( t, v(t), \int_0^t q(t, s) \psi(t, s, v(s)) \, ds \right) \quad \text{as} \quad n \to \infty.
\]

For \( t \in \mathcal{J} \),

\[
\mathcal{P}y_n(t) - \mathcal{P}y(t) = \left| \mathcal{N}_1 (t_k - s) \right| \mathcal{M}_\beta \wedge_\alpha \left( t^{(1+\gamma)/(1-\mu)} + \int_{t_k}^t \left\| \mathcal{N}_2 (t_k - s) \right\| \left\| \Psi \left( s, v_n(s), \int_0^s q(s, \xi) \psi(s, \xi, v_n(\xi)) \, d\xi \right) \right. \\
- \left. \Psi \left( s, v(s), \int_0^s q(s, \xi) \psi(s, \xi, v(\xi)) \, d\xi \right) \right\| ds \right.
\]

\[
+ \left( t^{(1+\gamma)/(1-\mu)} \right) \mathcal{M}_\beta \wedge_\alpha \left( t_k - s \right) \left\| \Psi \left( s, v_n(s), \int_0^s q(s, \xi) \psi(s, \xi, v_n(\xi)) \, d\xi \right) \right.
\]

\[
- \left. \Psi \left( s, v(s), \int_0^s q(s, \xi) \psi(s, \xi, v(\xi)) \, d\xi \right) \right\| ds
\]

\[
\leq \mathcal{M}_\beta \wedge_\alpha \left( t^{(1+\gamma)/(1-\mu)} \right) C_F \int_{t_k}^t \left( t_k - s \right)^{-1-\gamma} \left\| \Psi \left( s, v_n(s), \int_0^s q(s, \xi) \psi(s, \xi, v_n(\xi)) \, d\xi \right) \right.
\]

\[
- \left. \Psi \left( s, v(s), \int_0^s q(s, \xi) \psi(s, \xi, v(\xi)) \, d\xi \right) \right\| ds
\]
\[ + t^{(1+\gamma)(1-\mu)} \int_0^t (t-s)^{-1-\gamma} \left\| \psi \left( s, \nu(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, \nu(\zeta)) \, d\zeta \right) \right\| ds. \]

Since
\[ t^{(1+\gamma)(1-\mu)} \left\| \psi \left( s, \nu(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, \nu(\zeta)) \, d\zeta \right) \right\| \leq 2 \delta (\mathcal{X}_1 + \mathcal{X}_2 \tilde{w} q^*), \]
which is finite,

by the Lebesgue dominated convergence theorem and the continuity of function \( \psi \),

\[ \left\| \mathcal{P}_{y_n}(t) - \mathcal{P}_y(t) \right\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

This completes the proof of Lemma 3.1. \( \square \)

**Lemma 3.2** Let \( A \in \Theta' \) for \(-1 < \gamma < 0\) and \( 0 < \omega < \frac{1}{\gamma} \). If assumptions (A1)–(A5) and Proposition 2.3 are satisfied, the operator \( \{ \mathcal{P}_y : y \in B_s(\mathcal{A}) \} \) is equicontinuous for \( t \in \mathcal{A} \).

**Proof** For \( y \in B_s(\mathcal{A}) \) and \( 0 \leq t_1 < t_2 < b \),

\[ \left\| \mathcal{P}_y(t_2) - \mathcal{P}_y(t_1) \right\| \]
\[ = t_2^{(1+\gamma)(1-\mu)} \left\| \Psi(t_2) - t_1^{(1+\gamma)(1-\mu)} \int_0^t \mathcal{A}(t_2, s) \left\{ \mathcal{A}(s, \nu(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, \nu(\zeta)) \, d\zeta) \right\} \right\| ds \]
\[ - t_1^{(1+\gamma)(1-\mu)} \left\| \Psi(t_1) - t_1^{(1+\gamma)(1-\mu)} \int_0^t \mathcal{A}(t_1, s) \left\{ \mathcal{A}(s, \nu(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, \nu(\zeta)) \, d\zeta) \right\} \right\| ds \]
\[ \leq t_2^{(1+\gamma)(1-\mu)} \sum_{k=1}^m \epsilon_k \mathcal{S}_{n, \nu}(t_2) \int_0^{t_k} \mathcal{A}(t_k - s) \]
\[ \times \left\{ \mathcal{A}(s, \nu(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, \nu(\zeta)) \, d\zeta) \right\} \right\| ds \]
\[ - t_1^{(1+\gamma)(1-\mu)} \sum_{k=1}^m \epsilon_k \mathcal{S}_{n, \nu}(t_1) \int_0^{t_k} \mathcal{A}(t_k - s) \]
\[ \times \left\{ \mathcal{A}(s, \nu(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, \nu(\zeta)) \, d\zeta) \right\} \right\| ds \]
\[ + t_2^{(1+\gamma)(1-\mu)} \int_0^{t_1} \mathcal{A}(t_2 - s) \]
\[ \times \left\{ \mathcal{A}(s, \nu(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, \nu(\zeta)) \, d\zeta) \right\} \right\| ds \]
\[ - t_1^{(1+\gamma)(1-\mu)} \int_0^{t_1} \mathcal{A}(t_1 - s) \]
\[ \times \left\{ \mathcal{A}(s, \nu(s), \int_0^s q(s, \zeta) \varphi(s, \zeta, \nu(\zeta)) \, d\zeta) \right\} \right\| ds. \]
Clearly, here, the strong continuity of operator $\mathcal{S}_{\nu,\mu}(t)$ yields $I_1 \to 0$ as $t_1 \to t_2$.

$$I_2 = \left\| t_2^{(1+\nu)(1-\mu)} \int_0^{t_2} \mathcal{K}_c(t_2 - s) \times \left\{ \mathcal{T} \omega(s) + \Psi \left( s, \nu(s), \int_0^s q(s, \zeta) \varphi(s, \nu(s)) d\zeta \right) \right\} ds \right\|$$

Following Lemma 3.1 and assumptions (A1)–(A5), it is easy to see that

$$\left\| \int_0^{t_1} \mathcal{K}_c(t_1 - s) \left\{ \mathcal{T} \omega(s) + \Psi \left( s, \nu(s), \int_0^s q(s, \zeta) \varphi(s, \nu(s)) d\zeta \right) \right\} ds \right\|, \quad i = 1, 2,$$

exists and is bounded. Thus $I_2 \to 0$ as $t_2 \to t_1$.

$$I_3 = \left\| t_2^{(1+\nu)(1-\mu)} \int_{t_1}^{t_2} \mathcal{K}_c(t_2 - s) \times \left\{ \mathcal{T} \omega(s) + \Psi \left( s, \nu(s), \int_0^s q(s, \zeta) \varphi(s, \nu(s)) d\zeta \right) \right\} ds \right\|$$

After following the given assumptions and performing some steps of calculation, we have

$$I_3 \leq t_2^{(1+\nu)(1-\mu)} C_\varphi \delta (\mathcal{N}_0 + \mathcal{N}_\varphi \bar{w} q^*) (t_2 - t_1)^{-\nu} + t_2^{(1+\nu)(1-\mu)} C_\varphi \bar{M} (t_2 - t_1)^{-\nu}.$$
So, we have
\[
\|P_y(t_2) - P_y(t_1)\| \rightarrow 0 \quad \text{as} \quad t_2 \rightarrow t_1.
\]
This proves the equicontinuity of operator \(P_y(t)\). \(\square\)

**Theorem 3.1** Let \(-1 < \gamma < 0, 0 < \omega < \frac{\pi}{\gamma}\), and \(A \in \Theta^\omega_{\gamma}\). The system of equations Eq. (1.1) has at least one mild solution in \(B_1(\bar{3})\) if assumptions (A1)–(A5) hold along with Lemma 3.1 and Lemma 3.2.

**Proof** The mild solution of Eq. (1.1) is equivalent to the fixed point of operator \(P : B_1(\bar{3}) \rightarrow B_1(\bar{3})\). Here we prove that the operator \(P\) has at least one fixed point.

In the following, we show that the operator \(P\) is relatively compact in \(\mathcal{V}\) for every \(t \in \bar{3}\).

We prove this by showing that there is a relatively compact set \(\{P^\varepsilon y(t) : y \in B_1(\bar{3})\}\) arbitrarily close to the set \(\{P_y(t) : y \in B_1(\bar{3})\}\) in \(\mathcal{V}\) for every \(0 < \varepsilon \leq \delta\).

Define the operator \(P^\varepsilon y\) by
\[
P^\varepsilon y(t) = t^{\nu^\gamma(1 - \mu)} \sum_{k=1}^{m} c_k \mathcal{S}(t \xi - s) \int_{0}^{t \xi} \mathcal{K}^\varepsilon(t - s) \times \left\{ F \omega(s) + \psi(s, \mu(s), \int_{0}^{s} q(s, \xi) \varphi(s, \xi) \, d\xi) \right\} \, ds
\]
\[
+ t^{\nu^\gamma(1 - \mu)} \mathcal{K}_0(\varepsilon) \int_{0}^{t - \varepsilon} \mathcal{K}_0(t - s)
\]
\[
\times \left\{ F \omega(s) + \psi(s, \mu(s), \int_{0}^{s} q(s, \xi) \varphi(s, \xi) \, d\xi) \right\} \, ds.
\]

Since \(\mathcal{S}(t \xi)\) and \(\mathcal{K}_0(t)\) are compact for \(t > 0\), the set \(P^\varepsilon y(t)\) is compact for \(t \in \bar{3}\). In the following
\[
\|P^\varepsilon y(t) - P_y(t)\|
\]
\[
\leq t^{\nu^\gamma(1 - \mu)} \int_{0}^{t - \varepsilon} \left\{ \mathcal{K}_0(t - s) \mathcal{K}_0(\varepsilon) - \mathcal{K}_0(t - s) \right\}
\]
\[
\times \left\{ F \omega(s) + \psi(s, \mu(s), \int_{0}^{s} q(s, \xi) \varphi(s, \xi) \, d\xi) \right\} \, ds
\]
\[
+ t^{\nu^\gamma(1 - \mu)} \mathcal{K}_0(\varepsilon) \int_{0}^{t - \varepsilon} \mathcal{K}_0(t - s)
\]
\[
\times \left\{ F \omega(s) + \psi(s, \mu(s), \int_{0}^{s} q(s, \xi) \varphi(s, \xi) \, d\xi) \right\} \, ds
\]
\[
\leq \int_{0}^{t - \varepsilon} \left\| \mathcal{K}_0(t - s) \mathcal{K}_0(\varepsilon) - \mathcal{K}_0(t - s) \right\| (\mathcal{N}_1 + \mathcal{N}_2 \omega^\delta) \, ds
\]
\[
+ \delta^{\nu^\gamma(1 - \mu)} \hat{M} \int_{0}^{t - \varepsilon} \left\| \mathcal{K}_0(t - s) \mathcal{K}_0(\varepsilon) - \mathcal{K}_0(t - s) \right\| \, ds
\]
\[
+ C_\delta (\mathcal{N}_1 + \mathcal{N}_2 \omega^\delta) \frac{\varepsilon^{\nu^\gamma}}{(-\nu^\gamma)} + \hat{M} C_\delta \delta^{\nu^\gamma(1 - \mu)} \frac{\varepsilon^{\nu^\gamma}}{(-\nu^\gamma)}, \quad (\nu^\gamma) > 0.
\]
Following this, we first introduce the following two operators:

\[
\Gamma_0^\delta = \int_0^\delta G(\delta, s) T T^* G^*(\delta, s) ds,
\]

\[
R(\lambda, \Gamma_0^\delta) = (\lambda I + \Gamma_0^\delta)^{-1}, \quad \lambda > 0,
\]

where \( T^* \), \( O^* \), \( \mathcal{A}_c^* \), and \( \mathcal{S}_{c,\mu}^* \) characterize the adjoint operators of \( T \), \( O \), \( \mathcal{A}_c \), and \( \mathcal{S}_{c,\mu} \) respectively,

\[
G^*(\delta, s) = \sum_{k=1}^m \chi_k(s) \mathcal{S}_{c,\mu}^* (\delta) O^* \mathcal{A}_c^* (t_k - s) + \chi_k(s) \mathcal{A}_c^* (\delta - s), \quad s \in [0, \delta].
\]

**Theorem 4.1** Let \( A \in \Theta_0^\mu \) for \(-1 < \gamma < 0\) and \( 0 < \omega < \frac{\gamma}{2} \). Assume that assumptions (A1)–(A5) and Proposition 2.3 hold. The system of equations Eq. (1.1) is approximate controllable on \( J \) if \( \lambda R(\lambda, \Gamma_0^\delta) \rightarrow 0 \) as \( \lambda \rightarrow 0^+ \) in the strong operator topology.

**Proof** By Theorem 3.1, the system of equations Eq. (1.1) has at least one mild solution \( v_b \in B_3 \) given by

\[
v_b(t) = \int_0^s G(t, s) \left\{ T \omega(s) + \mathcal{A}_c \int_0^s q(s, \zeta) \psi(s, \zeta, v, (\zeta)) d\zeta \right\} ds, \quad t \in J,
\]

where we define the control function \( \omega \) as

\[
\omega(t) = T^* \mathcal{A}^* (\delta, t) R(\lambda, \Gamma_0^\delta) \partial \left( v(\cdot) \right)
\]
with
\[ \vartheta(v(\cdot)) = v_0 - \int_0^b \mathcal{G}(b,s) \Psi\left(s, v(s), \int_0^s q(s, \zeta)v(s, \zeta) d\zeta \right) ds. \] (4.4)

At \( t = b \),
\[ v_b(\cdot) = \int_0^b \mathcal{G}(b,s) \{ R v_b(s) + \Psi\left( s, v_b(s), \int_0^s q(s, \zeta)v(s, \zeta) d\zeta \right) \} ds \]
\[ = v_0 - \vartheta(v_b(\cdot)) + \int_0^b \mathcal{G}(b,s) R v_b(s) ds \]
\[ = v_0 - \vartheta(v_b(\cdot)) + \int_0^b \mathcal{G}(b,s) R \mathcal{G}^*(b,s) R(\lambda, \Gamma_0^b) \vartheta(v_b(\cdot)) ds \]
\[ = v_0 - (\lambda I + \Gamma_0^b) \frac{1}{(\lambda I + \Gamma_0^b)} \vartheta(v_b(\cdot)) + \Gamma_0^b R(\lambda, \Gamma_0^b) \vartheta(v_b(\cdot)) \]
\[ = v_0 - (\lambda I + \Gamma_0^b) R(\lambda, \Gamma_0^b) \vartheta(v_b(\cdot)) + \Gamma_0^b R(\lambda, \Gamma_0^b) \vartheta(v_b(\cdot)) \]
\[ \implies v_b(\cdot) = v_0 - \lambda R(\lambda, \Gamma_0^b) \vartheta(v_b(\cdot)), \] (4.5)
\[ \left( \int_0^b \| \Psi(s, v_b(s), \int_0^s q(s, \zeta)v(s, \zeta) d\zeta \|^2 \right)^{\frac{1}{2}} \]
\[ \leq \left( \int_0^b (\psi_1(s) v_b(s) + \psi_2(s) v_b(s) q^* s^2) ds \right)^{\frac{1}{2}} \]
\[ \leq \left( \int_0^b (\psi_1(s) s^2 + \psi_2(s) v_b(s) q^* s^2) ds \right)^{\frac{1}{2}} \]
\[ \leq \delta^2 \delta (\mathcal{A}_0^2 + \mathcal{A}_0^2 v^2 + 2 \mathcal{A}_0^2 \tilde{w} q^* s^2)^{\frac{1}{2}}, \]

which implies that the sequence \( \{ \Psi(\cdot, v_b(\cdot), \cdot) : \lambda > 0 \} \) is bounded in the Hilbert space \( L^2(\mathcal{J}, \mathcal{W}) \). Therefore, there exists a subsequence of the sequence \( \{ \Psi(\cdot, v_b(\cdot), \cdot) : \lambda > 0 \} \) converging weakly to some point \( \varrho(\cdot) \in L^2(\mathcal{J}, \mathcal{W}) \).

Let us write
\[ \sigma = v_0 - \int_0^b \mathcal{G}(b,s) \varrho(s) ds. \] (4.6)

Hence, by Eqs. (4.4) and (4.6), we have
\[ \| \vartheta(v_b) - \sigma \| \leq \left\| \int_0^b \mathcal{G}(b,s) \left[ \Psi\left(s, v_b(s), \int_0^s q(s, \zeta)v(s, \zeta) d\zeta \right) - \varrho(s) \right] ds \right\|. \] (4.7)

By the compactness of operators \( \mathcal{A}_0(t) \) and \( \mathcal{S}_{v_b}(t) \) for \( t > 0 \), one gets the compactness of Green's function \( \mathcal{G}(t,s) \) for \( t, s > 0 \), which implies that
\[ \left\| \int_0^b \mathcal{G}(b,s) \left[ \Psi\left(s, v_b(s), \int_0^s q(s, \zeta)v(s, \zeta) d\zeta \right) - \varrho(s) \right] ds \right\| \]
\[ \to 0 \quad \text{as } \lambda \to 0^+. \] (4.8)
Thus from Eq. (4.7) and Eq. (4.8) we get

$$\| \vartheta(v_\lambda) - \sigma \| \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0^*.$$  \hfill (4.9)

Equation (4.5) implies that

$$\| v_\lambda(\vartheta) - v_\lambda \| \leq \| \lambda \mathcal{R}(\lambda, \Gamma_0^\lambda) \vartheta(v_\lambda) \| + \| \lambda \mathcal{R}(\lambda, \Gamma_0^\lambda) \sigma \| \cdot \| \vartheta(v_\lambda) - \sigma \| \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0^* \quad (\text{by the stated condition and Eq. (4.9)}).$$

Hence the approximate controllability of Eq. (1.1).

\section{5 Applications}

Here we investigate the proposed results for the following Cauchy problem:

$$\left\{ \begin{array}{l}
H D^{\nu, \mu}_0 v(t, x) = -\frac{\partial^2 v(t, x)}{\partial t^2} + \varphi w(t, x) + \frac{\cos^2 \nu v(t, x)}{(20 + e^t)} + \frac{e^{-\frac{\nu}{15}}}{15} \int_0^1 e^{-s} \sin(\nu(s, x)) \, ds, \quad x \in [0, \pi], t \in J, \\
v(t, 0) = v(t, \pi) = 0, \quad t \in J, \\
\mathcal{I}^{(1-\nu)(1-\mu)} v(0, x) = \sum_{k=1}^m c_k v(t_k, x), \quad x \in [0, \pi],
\end{array} \right.$$  \hfill (5.1)

in the Banach space $\mathcal{V} = \mathcal{C}^\nu[0, \pi]$, $0 < \eta < 1$, where $\nu = \frac{1}{4}$, $\mu = \frac{1}{2}$, $\varphi > 0$ and $c_k \in \mathbb{R}$, $k = 1, 2, \ldots, m$ are such that $\sum_{k=1}^m |c_k| < \frac{1}{\lambda_\nu}$. On substituting $v(t, x) = v(t)$, Eq. (5.1) reduces to Eq. (1.1) with $A = \frac{\partial^2}{\partial x^2}$ and $D(A) = \{ v \in \mathcal{C}^{\nu+2}[0, \pi] \text{ such that } v(t, 0) = v(t, \pi) = 0 \}$. It follows from article [37] that there exist constants $\rho, \epsilon > 0$ such that $A + \rho \epsilon \Theta_{\nu}^{\frac{\nu}{2} - 1} v$. The compactness of semigroup $\{ \mathcal{I}(t) \}$ follows from (Lemma 4.66) [42]. Since $D(A) \subset \mathcal{C}^{\nu+2}[0, \pi]$ and $\mathcal{C}^{\nu+2}[0, \pi]$ is embedded in $\mathcal{C}^\nu[0, \pi]$, the compactness of resolvent operators follows for every $\eta > 0$.

$$\Psi \left( t, v(t), \int_0^t q(t, s) \varphi(t, s, v(s)) \, ds \right) = \frac{\cos^2 \nu v(t)}{(20 + e^t)} + \frac{e^{-\frac{\nu}{15}}}{15} \int_0^1 e^{-s} \sin(v(s)) \, ds.$$  \hfill (4.10)

The bounded linear operator $\mathcal{N} : \mathcal{W} = \mathcal{V} \longrightarrow \mathcal{V}$ is defined as $\mathcal{N} w(t) = \varphi w(t)$.

Following this discussion and the definition of function $\Psi$ and bounded operator $\mathcal{N}$, it is easy to verify that assumptions (A1)--(A5) hold with

$$\mathcal{N}_1 = \frac{1}{20 + e^\delta}, \quad \mathcal{N}_2 = \frac{e^{\frac{\nu}{15}}}{15}, \quad \tilde{w} = (1 - e^{-\delta}), \quad \text{and} \quad q^* = 1.$$  \hfill (4.11)

Hence the existence and approximate controllability of Eq. (5.1) follow from Theorem 3.1 and Theorem 4.1 respectively.

\section{6 Conclusion}

In this paper, we discussed the approximate controllability of Hilfer fractional differential equations with almost sectorial operators. We first prove the existence of mild solutions for similar equations by applying fixed point theory. We will try to investigate the exact controllability and stability of a similar problem in our future research work.
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Author details
1Department of Mathematics and Statistics, Central University of Punjab, Bathinda, 151001 Punjab, India. 2Department of Mathematics and General Sciences, Prince Sultan University, 66833, 11586 Riyadh, Saudi Arabia. 3Department of Medical Research, China Medical University, 40402, Taichung, Taiwan. 4Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan. 5Department of Mathematics, College of Science, Princess Nourah Bint Abdulrahman University, Riyadh, Saudi Arabia.

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