The local Morrey-type space Associated with Ball Quasi-Banach Function Spaces and Application

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Abstract: In this paper, we define for the first time the local Morrey-type space associated with ball quasi-Banach function spaces and show the related series of properties. In addition, Hardy-Littlewood maximal operator’s boundedness is proved. We investigate nonsmooth decomposition of the local Morrey-type space associated with ball quasi-Banach function spaces via the Hardy local Morrey-type spaces associated with ball quasi-Banach function spaces. And we consider Hardy operator’s boundedness.

Key Words: the local Morrey-type spaces, ball quasi-Banach function spaces, Hardy-Littlewood maximal operator, nonsmooth decompositions, Hardy operator

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1 Introduction

In 1938, Morrey [1] introduced the first study of Morrey spaces $M^q_p(\mathbb{R}^n)$ for partial differential equations. In 1975, D. Adams [2] established that Morrey spaces $L^p_\lambda(\mathbb{R}^n)$ can describe the boundedness property of Riesz potentials. In 2009, Samko [3] examined the boundedness of Hardy operator on Morrey spaces $L^p_\lambda(\mathbb{R}^n)$. In 2014, Iida and Sawano [4] studied atomic decomposition for Morrey spaces $M^q_p(\mathbb{R}^n)$.

In 2004, Burenkov and Guliyev [5] proposed local Morrey-type spaces $LM^q_{\rho\theta,w}(\mathbb{R}^n)$, a sufficient and necessary condition for the boundedness of Hardy-littlewood maximal operator was shown on spaces $LM^q_{\rho\theta,w}(\mathbb{R}^n)$. In 2010, Burenkov and Nursultanov [6] description of interpolation for local Morrey spaces $LM^p_{\rho\theta,w}(\mathbb{R}^n)$. In 2011, Burenkov et.al. [7] studied the boundedness of the Hardy operator on spaces $LM^p_{\rho\theta,w}(\mathbb{R}^n)$. And in 2014, Batbold and Sawano [8] obtained the decomposition of spaces $LM^p_{\rho\theta,w}(\mathbb{R}^n)$. In 2017, Guliyev [9] researched the decomposition of spaces $LM^p_{\rho\theta,w}(\mathbb{R}^n)$. Other related references [10–15].

The variable exponential Lebesgue spaces $L^{p(\cdot)}(\mathbb{R})$ first appeared in 1931 by Orlicz [16]. In 2008, Kokilashvili and Meskhi [17] introduced Variable Morrey Spaces $M^{p(\cdot)}_{\rho(\cdot)}(X)$ In 2020, the local variable exponential Morrey space $LM^{p(\cdot)}_{u}(\mathbb{R}^n)$ was proposed by Yee et.al. [18]. Other related references [19, 20].

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Benedek and Panzone [21] proposed mixed Lebesgue spaces $L^p_{\vec{p}}(\mathbb{R}^n)$ in 1961. Mixed Morrey spaces $M^p_{\vec{q}}(\mathbb{R}^n)$ were introduced by Nogayama [22] in 2019. In 2021, Zhang and Zhou [23] proposed local mixed Morrey spaces $LM^p_{\vec{q},w}(\mathbb{R}^n)$. In 2022, Shi and Zhou [24] obtained a sufficient and necessary condition for the Hardy-littlewood maximal operator on spaces $LM^p_{\vec{q},w}(\mathbb{R}^n)$. In 2022, Shi and Zhou [25] obtained nonsmooth decompositions of spaces $LM^p_{\vec{q},w}(\mathbb{R}^n)$. Other related references [26].

In 2017, Sawano et.al. [27] established Hardy spaces for ball quasi-Banach function spaces $H_X(\mathbb{R}^n)$. In 2019, Zhang et.al. [28] and Wang et.al. [29] established the weak Hardy-type space associated with ball quasi-Banach function space $WH_X(\mathbb{R}^n)$. In 2019, Ho [30] proposed Morrey-Banach spaces $M^p_X(\mathbb{R}^n)$. In 2019, Ho [31] investigated Weak Type Estimates of Singular Integral Operators on spaces $M^p_X(\mathbb{R}^n)$. In 2021, Zhang et.al. [32] established Ball Campanato-Type function space $L^p_{\vec{q},d,s}(\mathbb{R}^n)$. In 2022, Shi et.al. [33] obtained the boundedness of some type of maximal operators on spaces $M^p_X(\mathbb{R}^n)$. In 2022, Wang and Zhou [34] by means of ball quasi-Banach spaces, it is proved that the Calderón-Zygmund singular integral operator is bounded on the generalized Orlicz space $L^p_{\vec{q}}(\mathbb{R}^n)$ via a new atomic decomposition. Currently, more and more researchers are solving problems with the help of ball quasi-Banach function spaces. Other more related references [35–39].

In this paper, we define the local Morrey-type space associated with ball quasi-Banach function spaces. In Section 2, some properties of the local Morrey-type space associated with ball quasi-Banach function spaces are derived. And the relationship of local Morrey-type spaces associated with ball quasi-Banach function spaces to some of the other function spaces is discussed. In Section 3, the boundedness of Hardy-littlewood maximal operators is obtained on the local Morrey-type space associated with ball quasi-Banach function spaces. In Section 4, an interpolation theorem is proved. In Section 5, vector valued maximal inequalities are obtained on the local Morrey-type space associated with ball quasi-Banach function spaces. In Section 6, predual spaces of the local Morrey-type space associated with ball quasi-Banach function spaces are proved. In section 7, the Hardy local Morrey-type space associated with ball quasi-Banach function spaces are characterized. In section 8, we attain nonsmooth decompositions on local Morrey-type spaces associated with ball quasi-Banach function spaces. In Section 9, the boundedness of Hardy operator is obtained.

2 Definition and properties

Definition 2.1. [40] A quasi-Banach space $X \subset \mathcal{M}$ is called a ball quasi-Banach function space if it satisfies

(i) $\|f\|_X = 0$ implies that $f = 0$ almost everywhere;

(ii) $|g| \leq |f|$ almost everywhere implies that $\|g\|_X \leq \|f\|_X$;

(iii) $0 \leq f_m \uparrow f$ almost everywhere implies that $\|f_m\|_X \uparrow \|f\|_X$;

(iv) $B \in \mathcal{B}$ implies that $\chi_B \in X$, where $\mathcal{B} := \{B(x, r) : x \in \mathbb{R}^n \text{ and } r \in (0, \infty)\}$.

Moreover, a ball quasi-Banach function space $X$ is called a ball Banach function space if the norm of $X$ satisfies the triangle inequality: for all $f, g \in X$,

$$\|f + g\|_X \leq \|f\|_X + \|g\|_X,$$

and, for any $B \in \mathcal{B}$, there exists a positive constant $C(B)$, depending on $B$, such that, for all $f \in X$,

$$\int_B |f(x)|dx \leq C(B) \|f\|_X$$
(iii) Let $\text{Lemma 2.1.}$

By $K$ set

$\text{Definition 2.2.}$

(iii) Consider the partition of unity $\{\text{inclusions, the set of all polynomials of degree less than or equal to } d \text{ is denoted by such that the following properties hold.}\}$

(ii) Define $O = \{\text{translations} \}$

$\text{Lemma 2.1.}$

Let $f \in S'(\mathbb{R}^n)$, the grand maximal operator $Mf$ of $f$ is defined by

$\mathcal{M}f(x) := \mathcal{M}_N f(x) := \sup_{t > 0, \varphi \in \mathcal{F}_N} \{ |t^{-n}\varphi(t^{-1} \ast f(x))| \} \text{ for } x \in \mathbb{R}^n.$

$C_c^\infty(\mathbb{R}^n)$ denotes the set of all compactly supported infinitely continuously differentiable functions, the set of all polynomials of degree less than or equal to $d$ is denoted by $\mathcal{P}_d(\mathbb{R}^n)$. [42]

Let $f \in S'(\mathbb{R}^n) \cap L^1_{\text{loc}}(\mathbb{R}^n), d \in \mathbb{N} \cup \{0\}$ and $j \in \mathbb{Z}$. Then there exist an index set $K_j$, collections of cubes $\{Q_{j,k}\}_{k \in K_j}$ and functions $\{\eta_{j,k}\}_{k \in K_j} \subset C_c^\infty(\mathbb{R}^n)$, which are all indexed by $K_j$ for every $j$, and a decomposition

\[ f = g_j + b_j, \quad b_j = \sum_{k \in K_j} b_{j,k}, \]

such that the following properties hold.

(i) $g_j, b_j, b_{j,k} \in S'(\mathbb{R}^n)$.

(ii) Define $\mathcal{O}_j := \{ y \in \mathbb{R}^n : Mf(y) > 2^j \}$ and consider its Whitney decomposition. Then the cubes $\{Q_{j,k}\}_{k \in K_j}$ have the bounded intersection property, and

\[ \mathcal{O}_j = \bigcup_{k \in K_j} Q_{j,k} \]  \hspace{1cm} (1)

(iii) Consider the partition of unity $\{\eta_{j,k}\}_{k \in K_j}$ with respect to $\{Q_{j,k}\}_{k \in K_j}$. Then each function $\eta_{j,k}$ is supported in $Q_{j,k}$ and

\[ \sum_{k \in K_j} \eta_{j,k} = \chi_{\{ y \in \mathbb{R}^n : Mf(y) > 2^j \}}, \quad 0 \leq \eta_{j,k} \leq 1. \]

(iv) $g_j$ is an $L^\infty(\mathbb{R}^n)$-function satisfying $\|g_j\|_{L^\infty} \leq 2^{-j}$.

(v) Each distribution $b_{j,k}$ is given by $b_{j,k} = (f - c_{j,k}) \eta_{j,k}$ with a certain polynomial $c_{j,k} \in \mathcal{P}_d(\mathbb{R}^n)$ satisfying

\[ \langle f - c_{j,k}, \eta_{j,k} \cdot P \rangle = 0 \text{ for all } q \in \mathcal{P}_d(\mathbb{R}^n) \]

and

\[ \mathcal{M}b_{j,k}(x) \lesssim \mathcal{M}f(x) \chi_{Q_{j,k}}(x) + 2^j \frac{\rho_{j,k}^{n+d+1}}{|x - x_{j,k}|^{n+d+1}} \chi_{\mathbb{R}^n \setminus Q_{j,k}}(x) \]

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for all $x \in \mathbb{R}^n$.

In the above, $x_{j,k}$ and $\ell_{j,k}$ denote the center and the edge-length of $Q_{j,k}$.

**Definition 2.4.** [44] (Lebesgue spaces) Let $0 < p < \infty$, the Lebesgue space $L^p$ is defined to be the set of all measurable function $f$ on $\mathbb{R}^n$ such that

$$
\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}
$$

**Definition 2.5.** [21] (Mixed Lebesgue spaces) Let $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty]^n$. Then define the mixed Lebesgue norm $\| f \|_{L^\vec{p}}$ by

$$
\|f\|_{L^\vec{p}} := \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} |f(x_1, x_2, \ldots, x_n)|^{p_1} dx_1\right)^{\frac{p_2}{p_1}} dx_2 \cdots dx_n\right)^{1/p_n}
$$

**Definition 2.6.** [16] (Variable Lebesgue spaces) Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a measurable function. Then the variable Lebesgue space $L^{p(\cdot)} (\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$
\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} |f(x)|^\lambda dx \leq 1 \right\} < \infty
$$

**Definition 2.7.** (the local Morrey-type space associated with ball quasi-Banach function spaces) Let $\lambda \geq 0$ and $0 < q \leq \infty$. We denote the local Morrey space associated with ball quasi-Banach function spaces $LM^\lambda_{X,q}$, where $X$ is the ball quasi-Banach function space. For any functions $f \in L^1_{1,\text{loc}}$, we say $f \in LM^\lambda_{X,q}$ when the quasi-norms

$$
\|f\|_{LM^\lambda_{X,q}} = \left(\int_0^{\infty} \left(r^{-\lambda} \|f\chi_{B(0,r)}\|_X\right)^q \frac{dr}{r}\right)^{\frac{1}{q}} < \infty.
$$

**Definition 2.8.** (Morrey-Banach spaces) [30] Let $X$ be a rearrangement-invariant Banach function space and $u : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. A measurable function $f$ belongs to the rearrangement-invariant Morrey spaces (Morrey-Banach spaces) $M^\lambda_X$ if it satisfies

$$
\|f\|_{M^\lambda_X} = \sup_{x_0 \in \mathbb{R}^n, r > 0} \frac{1}{u(r)} \|\chi_{B(x_0,r)}f\|_X < \infty
$$

**Remark 2.1.** Let $q = \infty$ and $\lambda \geq 0$, return to the Morrey-Banach spaces.

$$
\|f\|_{M^\lambda_X} = \sup_{x \in \mathbb{R}^n} \|f(\cdot + x)\|_{LM^\lambda_{X,\infty}} = \sup_{x \in \mathbb{R}^n} \sup_{r > 0} r^{-\lambda} \|f\chi_{B(x,r)}\|_X < \infty.
$$

**Definition 2.9.** (quasinorm derived from the ball quasi-Banach function space) For $f \in L^1_{1,\text{loc}}$, we consider the quasinorm where $X$ is the ball quasi-Banach function space.

$$
\|f\|_X = |B(0,r)| \sup_{\rho \geq r} |B(0,\rho)|^{-1} \|f\chi_{B(0,\rho)}\|_X
= r^n \sup_{\rho \geq r} \rho^{-n} \|f\chi_{B(0,\rho)}\|_X.
$$
Definition 2.10. (quasinorm derived from the local Morrey-type space associated with ball quasi-Banach function spaces) Let $\lambda \geq 0$, $0 < q \leq \infty$, and $f \in L^1_{\text{loc}}$, we consider quasinorm where $X$ is the ball quasi-Banach function space.

$$\|f\|_{\overline{LM}^\lambda_{X,q}} = \left(\int_0^\infty \left(r^{-\lambda}\|f\chi_{B(0,r)}\|_X\right)^q \frac{dr}{r}\right)^{\frac{1}{q}} < \infty.$$ 

Definition 2.11. [8] (the heat kernel) Let $t > 0$ and $f \in S'(\mathbb{R}^n)$. The heat kernel is defined by

$$e^{t\Delta}f(x) := \langle f, \frac{1}{\sqrt{(4\pi t)^n}}\exp\left(-\frac{|x - \cdot|^2}{4t}\right) \rangle_{x \in \mathbb{R}^n}.$$ 

Definition 2.12. (the Hardy local Morrey-type spaces associated with ball quasi-Banach function spaces) Let $\lambda \geq 0$, $0 < q \leq \infty$. Let $X$ be the ball quasi-Banach function space. The Hardy local Morrey-type spaces associated with ball quasi-Banach function spaces $HLM^\lambda_{X,q}(\mathbb{R}^n)$ collects all $f \in S'(\mathbb{R}^n)$ such that $\sup_{t > 0} |e^{t\Delta}f| \in LM^\lambda_{X,q}(\mathbb{R}^n)$.

$$\|f\|_{HLM^\lambda_{X,q}} := \left\|\sup_{t > 0} |e^{t\Delta}f|\right\|_{LM^\lambda_{X,q}} < \infty.$$ 

Proposition 2.1. For $\lambda \geq 0$, $0 < q, q_0, q_1 \leq \infty$, $f \in L^1_{\text{loc}}$ and $X$ is the ball quasi-Banach function space. If $q_0 < q_1$, then $LM^\lambda_{X,q_0} \subset LM^\lambda_{X,q_1}$ and $\overline{LM}^\lambda_{X,q_0} \subset \overline{LM}^\lambda_{X,q_1}$.

Proof. 1. First, suppose that $q_1 = \infty$, then

$$\|f\|_{LM^\lambda_{X,\infty}} = \sup_{r > 0} r^{-\lambda}\|f\chi_{B(0,r)}\|_X \leq (\lambda q_0)^{\frac{1}{q_0}} \sup_{r > 0} \left(\int_r^\infty t^{-\lambda q_0} dt\right)^{\frac{q_0}{q_1}} \|f\chi_{B(0,r)}\|_X \leq (\lambda q_0)^{\frac{1}{q_0}} \|f\|_{LM^\lambda_{X,q_0}}.$$ 

If $q_1 < \infty$, then it suffices to apply the interpolation inequality

$$\|f\|_{LM^\lambda_{X,q_1}} \leq \|f\|_{LM^\lambda_{X,\infty}}^{1 - \frac{q_0}{q_1}} \|f\|_{LM^\lambda_{X,q_0}}^{\frac{q_0}{q_1}} \leq \left((\lambda q_0)^{\frac{1}{q_0}} \|f\|_{LM^\lambda_{X,q_0}}\right)^{1 - \frac{q_0}{q_1}} \|f\|_{LM^\lambda_{X,q_0}} \leq (\lambda q_0)^{\frac{1}{q_0}} \|f\|_{LM^\lambda_{X,q_0}}.$$ 

2. First, suppose that $q_1 = \infty$, then

$$\|f\|_{\overline{LM}^\lambda_{X,\infty}} = \sup_{r > 0} r^{-\lambda}\sup_{\rho \geq r} r^{-n}\|f\chi_{B(0,r)}\|_X.$$
\[
= ((n - \lambda)q_0) \frac{1}{\gamma} V_n \sup_{r > 0} \left( \int_0^r \left( s^{(n-\lambda)q_0} \sup_{\rho \geq r} \rho^{-\lambda n} \|f\chi_{B(0,\rho)}\|_X \right) \frac{ds}{s} \right)^{\frac{1}{q_0}} \sup_{\rho \geq r} \rho^{-\lambda n} \|f\chi_{B(0,\rho)}\|_X
\]
\[
\leq ((n - \lambda)q_0) \frac{1}{\gamma} V_n \sup_{r > 0} \left( \int_0^r \left( s^{(n-\lambda)q_0} \sup_{\rho \geq r} \rho^{-\lambda n} \|f\chi_{B(0,\rho)}\|_X \right) \frac{ds}{s} \right)^{\frac{1}{q_0}} \frac{1}{\gamma} \sup_{\rho \geq r} \rho^{-\lambda n} \|f\chi_{B(0,\rho)}\|_X
\]
\[
= ((n - \lambda)q_0) \frac{1}{\gamma} \|f\|_{LM_{X,q_0}^\lambda}
\]

If \( q_1 < \infty \), it is proved in a similar way as above. □

**Remark 2.2.** [29] \( L_p(\mathbb{R}^n), L_{p'}(\mathbb{R}^n), L_{p(\cdot)}(\mathbb{R}^n) \) are proven to be ball quasi-Banach function spaces.

**Remark 2.3.** In the rest of the article, since the properties and applications of \( LM_{X,q_0}^\lambda(\mathbb{R}^n) \) will be discussed when \( X \) is \( L_p(\mathbb{R}^n), L_{p'}(\mathbb{R}^n), L_{p(\cdot)}(\mathbb{R}^n) \), all satisfied \( \sigma := \log_r \|\chi_{B(0,\rho)}\|_X \) and \( X \) is the ball quasi-Banach function space.

**Remark 2.4.** When \( X \) is \( L_p \), for \( LM_{X,q}^\lambda \) back to local Morrey-type spaces [6] \( LM_{p,q}^\lambda \). In particular, because of Remark 2.5, we also obtain the norm equivalence of \( LM_{p,q}^\lambda \) and homogeneous Herz spaces [45] \( \dot{K}_{q}^{\alpha,p}(\mathbb{R}^n) \).

When \( X \) is \( L_{p'} \), for \( LM_{X,q}^\lambda \) back to local mixed Morrey Spaces [23] \( LM_{p,q}^\lambda \). In particular, because of Remark 2.5, we also obtain the norm equivalence of \( LM_{p,q}^\lambda \) and homogeneous Mixed Herz spaces [46] \( \dot{K}_{q}^{\alpha,p}(\mathbb{R}^n) \).

When \( X \) is \( L_{p(\cdot)} \), for \( LM_{X,q}^\lambda \) obtains local Morrey-type spaces with variable exponents \( LM_{p(\cdot),q}^\lambda \). In particular, because of Remark 2.5, we also obtain the norm equivalence of \( LM_{p(\cdot),q}^\lambda \) and homogeneous variable exponents Herz spaces [47] \( \dot{K}_{q}^{\alpha,p}(\mathbb{R}^n) \) (\( \alpha(\cdot) = \alpha \)).

**Proposition 2.2.** Let \( 0 < q \leq \infty, 0 \leq \lambda < \sigma \). Then \( LM_{X,q}^\lambda(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n) \) in the sense of continuous embedding.

**Proof.** Denote by \( B_x \) the set of all open balls in \( \mathbb{R}^n \) which contain \( x \). The Hardy-Littlewood maximal operator \( M \) is bounded on \( LM_{X,q}^\lambda(\mathbb{R}^n) \) (As demonstrated in Section 3 of this paper). Therefore,

\[
\frac{\alpha}{|B(R)|} \int_{B(R)} |f(y)|dy \leq \|\chi_{B(1)} Mf\|_{LM_{X,q}^\lambda(\mathbb{R}^n)} \leq \|Mf\|_{LM_{X,q}^\lambda(\mathbb{R}^n)} \lesssim \|f\|_{LM_{X,q}^\lambda(\mathbb{R}^n)},
\]

where \( \alpha \equiv \|\chi_{B(1)}\|_{LM_{X,q}^\lambda(\mathbb{R}^n)} \). Then for all \( \kappa \in S(\mathbb{R}^n) \) and \( f \in LM_{X,q}^\lambda(\mathbb{R}^n) \)

\[
\int_{\mathbb{R}^n} |\kappa(x)f(x)|dx = \int_{B(1)} |\kappa(x)f(x)|dx + \sum_{j=1}^{\infty} \int_{B(j+1)\setminus B(j)} |\kappa(x)f(x)|dx
\]
\[
\leq \|\kappa\|_{L^\infty(B(1))} \|f\|_{L^1(B(1))} + \sum_{j=1}^{\infty} \int_{B(j+1)\setminus B(j)} |x|^{2n+1} |\kappa(x)f(x)|dx
\]
\[
\lesssim \|f\|_{LM_{X,q}^\lambda(\mathbb{R}^n)} \left( \sup_{x \in \mathbb{R}^n} (1 + |x|^{2n+1} |\kappa(x)|) \right)
\]

□

**Assumption 2.1.** If the Hardy-Littlewood maximal operator \( M \) is bounded from \( X(\mathbb{R}^n) \) to \( X(\mathbb{R}^n) \), then \( X \in \mathcal{M} \).
**Proposition 2.3.** The local Morrey space associated with ball quasi-Banach function spaces $LM_{X,q}$ is called a ball quasi-Banach function space.

The proof is simple, interested readers can prove it among themselves.

**Proposition 2.4.** Let $1 \leq q < \infty$ and $0 \leq \lambda < \sigma$ and $X$ is the ball quasi-Banach function space. Then for any $w \subset \mathbb{R}^n$

$$\|f\|_{LM_{X,q}} \sim \left( \sum_{j=-\infty}^{\infty} \left( 2^{-\lambda j} \|f\chi_{B(0,2^j)}\|_X \right)^q \right)^{1/q}$$

**Proof.** We start with the equality

$$\|f\|_{LM_{X,q}} = \left( \int_0^\infty \left( t^{-\lambda} \|f\chi_{B(0,2^j)}\|_X \right)^q \frac{dt}{t} \right)^{1/q} = \left( \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left( t^{-\lambda} \|f\chi_{B(0,2^j)}\|_X \right)^q \frac{dt}{t} \right)^{1/q}.$$  

On one hand

$$\left( \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left( t^{-\lambda} \|f\chi_{B(0,2^j)}\|_X \right)^q \frac{dt}{t} \right)^{1/q} \leq 2^\lambda (\ln 2)^{1/q} \left( \sum_{j=-\infty}^{\infty} \left( 2^{-\lambda j} \|f\chi_{B(0,2^j)}\|_X \right)^q \right)^{1/q}.$$  

On the other hand

$$\left( \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left( t^{-\lambda} \|f\chi_{B(0,2^j)}\|_X \right)^q \frac{dt}{t} \right)^{1/q} \geq 2^{-\lambda} (\ln 2)^{1/q} \left( \sum_{j=-\infty}^{\infty} \left( 2^{-\lambda j} \|f\chi_{B(0,2^j)}\|_X \right)^q \right)^{1/q}$$

and we obtain the required equivalence.

Let $B_{j} = \{x \in \mathbb{R}^n : |x| \leq 2^j\}$ and $A_{j} = B_{j} \setminus B_{j-1}$ for any $k \in \mathbb{Z}$. Denote $\chi_j = \chi_{A_j}$, where $\chi_E$ is the characteristic function of set $E$.

**Remark 2.5.** Let $1 < q \leq \infty$, $0 \leq \lambda < \sigma$ and for all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$. Then

$$\|f\|_{LM_{\lambda,q}} \sim \left( \sum_{j=-\infty}^{\infty} 2^{-\lambda j} \|f\chi_j\|_X \right)^{1/q}.$$  

**Proof.** It is clear from that

$$\|f\|_{LM_{\lambda,q}} \geq \left\{ \sum_{j=-\infty}^{\infty} \left( 2^{-\lambda j} \|f\chi_j\|_X \right)^q \right\}^{1/q}.$$
To prove the reverse estimate,

$$\|f\|_{L^{M^\lambda}_{p\theta}} \sim \left( \sum_{j=-\infty}^{\infty} \left( 2^{-\lambda j} \|f \chi_{B(0,2^j)} \|_X \right)^q \right)^{1/q}$$

$$= \left\{ \sum_{j=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} 2^{-\lambda j} \|f \chi_j \|_X \right)^q \right\}^{1/q}$$

$$= \left\{ \sum_{j=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} \chi_{(-\infty,j]}(k) 2^{-\lambda j} \|f \chi_j \|_X \right)^q \right\}^{1/q}$$

$$\leq \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{\infty} \chi_{(-\infty,j]}(k) 2^{-\lambda j} \|f \chi_j \|_X \right)^q \right\}^{1/q}$$

$$= \left\{ \sum_{k=-\infty}^{\infty} \left( \frac{1}{1-2^{-\lambda}} 2^{-\lambda k} \|f \chi_j \|_X \right)^q \right\}^{1/q}.$$

3 Boundedness of Hardy-littlewood Operators on the local Morrey space associated with ball quasi-Banach function spaces

The Hardy operator $H$ and its dual operators $H^*$, given by:

$$Hg(r) = \int_0^r g(t) dt, \quad H^*g(r) = \int_r^\infty g(t) dt.$$

Since the following article requires, we must need the following relationship. For $1 \leq q < \infty$ and a measurable function $v : (0, \infty) \to (0, \infty)$, whose norm is given by

$$\|f\|_{L^{q,v}(0,\infty)} := \|vf\|_{L^q(0,\infty)}.$$

Consider first the following "partial" maximum function.

$$\overline{M} f = \sup_{0 < t \leq r} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy;$$

$$\overline{M} f = \sup_{t > r} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

Lemma 3.1. Let $f \in L^1_{loc}(\mathbb{R}^n)$, then for $B(y,2r)$ in $\mathbb{R}^n$

$$\|M \left( f \chi_{B(x,2r)} \right) \chi_{B(0,t)} \|_X \geq r^n \overline{M} f(x).$$
Proof. If \( y \in B(x, r), B(x, \frac{t}{2}) \subset B(y, t) \cap \mathbb{C}B(x, 2r) \),

\[
M \left( f \chi_{B(x, 2r)} \right)(y) = \sup_{t > 0} \frac{1}{|B(y, t)|} \int_{B(y, t) \cap \mathbb{C}B(x, 2r)} |f(z)| dz \\
\quad \geq \sup_{t \geq 2r} \frac{1}{|B(x, 2t)|} \int_{B(x, 2t)} |f(y)| dy = Mf(x).
\]

\[
\|M \left( f \chi_{B(x, 2r)} \right) \chi_{B(0,t)}\|_X \geq r^\sigma Mf(x).
\]

\[\square\]

**Lemma 3.2.** Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), then

\[
\|Mf\chi_{B(0,t)}\|_X = \|M(f\chi_{B(x, 2r)})\chi_{B(0,t)}\|_X + r^\sigma Mf(x).
\]

**Proof.** It is obvious that for \( B(x, r) \),

\[
\|Mf\chi_{B(0,t)}\|_X \leq \|M(f\chi_{B(x, 2r)})\chi_{B(0,t)}\|_X + \|M(f\chi_{B(x, 2r)})\chi_{B(0,t)}\|_X.
\]

Let \( y \in B(x, r) \). If \( B(y, t) \cap \mathbb{C}B(x, 2r) \neq \emptyset \), then \( t > r \). Indeed \( z \in B(y, t) \cap \mathbb{C}B(x, 2r) \neq \emptyset \), then \( t > |z - y| \geq |z - x| - |x - y| > 2r - r < r \).

Another, \( B(y, t) \cap \mathbb{C}B(x, 2r) \subset B(x, 2t) \). Indeed , if \( z \in B(y, t) \cap \mathbb{C}B(x, 2r) \), then \( |z - x| \leq |z - y| + |y - x| < t + r < 2t \).

Hence

\[
M \left( f \chi_{B(x, 2r)} \right)(y) = \sup_{t > 0} \frac{1}{|B(y, t)|} \int_{B(y, t) \cap \mathbb{C}B(x, 2r)} |f(z)| dz \\
\quad \leq \sup_{t \geq r} \frac{1}{|B(x, 2t)|} \int_{B(x, 2t)} |f(y)| dy = Mf(x).
\]

\[
\|M \left( f \chi_{B(x, 2r)} \right) \chi_{B(0,t)}\|_X \leq r^\sigma Mf(x).
\]

On the one hand,

\[
\|M(f\chi_{B(x, 2r)})\chi_{B(0,t)}\|_X \leq \|Mf\chi_{B(0,t)}\|_X.
\]

On the other hand, if \( y \in B(x, r), \ z \in B(y, t) \cap \mathbb{C}B(x, 2r) \) and Lemma 3.1 then

\[
\|Mf\chi_{B(0,t)}\|_X \geq \|M(f\chi_{B(x, 2r)})\chi_{B(0,t)}\|_X \geq r^\sigma Mf(x).
\]

The proof is complete.

\[\square\]

**Lemma 3.3.** Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n), X \in \mathbb{M} \). Then for any \( B(x, r) \) in \( \mathbb{R}^n \),

\[
\|Mf\chi_{B(0,t)}\|_X \leq r^\sigma \int_r^\infty \|f\chi_{B(0,t)}\|_X dt \frac{dt}{t^{\sigma+1}}.
\]

**Proof.** By Lemma 3.2

\[
\|Mf\chi_{B(0,t)}\|_X \leq \|M(f\chi_{B(x, 2r)})\chi_{B(0,t)}\|_X + r^\sigma Mf(x) := I + II
\]
On the one hand,
\[
I \leq \|M(f_{\chi B(x,r)})\|_X \leq \|f_{\chi B(y,2r)}\|_X \leq \|f_{\chi B(y,2r)}\|_X \\
\leq r^\sigma \int_{2r}^\infty \|f_{\chi B(x,2r)}\|_X \frac{dt}{t^{\sigma+1}} \leq r^\sigma \int_r^\infty \|f_{\chi B(x,t)}\|_X \frac{dt}{t^{\sigma+1}}.
\]

On the other hand,
\[
II = r^\sigma \sup_{t>\tau} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy \\
\leq r^\sigma \int_r^\infty \|f_{\chi B(x,t)}\|_X \frac{dt}{t^{\sigma+1}}.
\]

Lemma 3.4. Let \(0 < q \leq \infty\), \(0 \leq \lambda < \sigma\), \(X \in \mathcal{M}\), for all \(f \in L^1_{\text{loc}}\), where
\[
g_X(t) = \|f_{\chi B(0,t-\frac{1}{\sigma})}\|_X,
\]
\[
v(r) = r^{\frac{\lambda}{\sigma} - 1 - \frac{1}{q}}.
\]

Then
\[
\|Mf\|_{L^\lambda_{X,q}} \lesssim \|H g_X\|_{L^q,v(0,\infty)}.
\]

Proof. By Lemma 3.3,
\[
\|Mf\|_{L^\lambda_{X,q}} = \|r^{-\lambda} \|Mf_{\chi B(0,r)}\|_X\|_{L^q(0,\infty)} \\
\lesssim \|r^{-\lambda} r^\sigma \int_r^\infty \|f_{\chi B(0,t)}\|_X \frac{dt}{t^{\sigma+1}}\|_{L^q(0,\infty)} \\
\lesssim \|r^{-\lambda} r^\sigma \int_0^r \|f_{\chi B(0,t-\frac{1}{\sigma})}\|_X \frac{dt}{t^{\sigma+1}}\|_{L^q(0,\infty)} \\
\lesssim \|r^{\frac{\lambda}{\sigma} - 1 - \frac{1}{q}} H g_X(r)\|_{L^q(0,\infty)} \\
\sim \|H g_X\|_{L^q,v(0,\infty)}.
\]

Lemma 3.5. Let \(0 < q \leq \infty\), for all \(f \in L^1_{\text{loc}}\), where
\[
g_X(t) = \|f_{\chi B(0,t-\frac{1}{\sigma})}\|_X,
\]
\[
v(r) = r^{\frac{\lambda}{\sigma} - 1 - \frac{1}{q}}.
\]

Then
\[
\|g_X\|_{L^q,v(0,\infty)} \lesssim \|f\|_{L^\lambda_{X,q}},
\]
\[
\|g_X\|_{L^q,v(0,\infty)} = \|v(r)\|_{L^q,v(0,\infty)}.
\]
= \| r^{\frac{\lambda}{q} - \frac{1}{q}} f_{\chi_{B(0,r^{-\frac{1}{q}})}} \|_{L_q(0,\infty)}
\lesssim \| r^{-\lambda - \frac{1}{q}} f_{\chi_{B(0,r)}} \|_{L_q(0,\infty)}
= \| f \|_{LM_{X,q}^\lambda}.

**Lemma 3.6.** Let $0 < q \leq \infty$, $0 \leq \lambda < \sigma$, for all $f \in L^1_{loc}$,

$$g_X(t) = \| f_{\chi_{B(0,t^{-\frac{1}{q}})}} \|_X,$$

$$v_2(r) = r^{\frac{\lambda}{\sigma} - \frac{1}{q}}$$

$$v_1(r) = r^{\frac{\lambda}{\sigma} - \frac{1}{q}}.$$

Then

$$\| H g_X \|_{L_{q,v_2}(0,\infty)} \lesssim \| g_X \|_{L_{q,v_1}(0,\infty)}.$$

**Proof.**

$$\| H g_X \|_{L_{q,v_2}(0,\infty)} = \left( \int_0^\infty \left( \int_0^r \| f_{\chi_{B(0,t^{-\frac{1}{q}})}} \|_X \right) dt \right)^{\frac{1}{q}} dr
\leq \left( \int_0^\infty \left( \int_0^t \| f_{\chi_{B(0,t^{-\frac{1}{q}})}} \|_X \right) dt \right)^{\frac{1}{q}}
\leq \left( \frac{1}{q} + \frac{\lambda}{\sigma} \right)^{-1} \left( \int_0^\infty \left( \int_0^t \| f_{\chi_{B(0,t^{-\frac{1}{q}})}} \|_X \right) dt \right)^{\frac{1}{q}}
\lesssim \| g_X \|_{L_{q,v_1}(0,\infty)}.$$

\[\square\]

**Theorem 3.1.** Let $0 \leq \lambda < \sigma$, $0 < q \leq \infty$, $X \in M$. Then the operator $M$ is bounded from $LM_{X,q}^\lambda$ to $LM_{X,q}^\lambda$.

**Proof.** By Lemma 3.4, 3.5 and 3.6.

$$\| M f \|_{LM_{X,q}^\lambda} \lesssim \| H g_X \|_{L_{q,v_1}(0,\infty)} \lesssim \| g_X \|_{L_{q,v_1}(0,\infty)} \lesssim \| f \|_{LM_{X,q}^\lambda}.$$

\[\square\]

### 4 Interpolation theorem

**Lemma 4.1.** Let $0 < q \leq \infty$, $0 \leq \lambda < \sigma$. Then

$$\widetilde{LM}_{X,q}^\lambda = LM_{X,q}^\lambda$$

Moreover, for $q < \infty$,

$$\left( \frac{\lambda - \lambda q}{n} \right)^{\frac{1}{q}} \| f \|_{LM_{X,q}^\lambda} \leq \| f \|_{LM_{X,q}^\lambda} \leq \| f \|_{LM_{X,q}^\lambda}.$$

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For $q = \infty$, 
\[ \|f\|_{LM_{X,\infty}^{\lambda}} = \|f\|_{LM_{X,\infty}^{\lambda}}. \]

Proof. For $q < \infty$,
\[
\|f\|_{LM_{X,q}^{\lambda}}^q = \int_0^{\infty} \left( r^{-\lambda+n} \sup_{\rho \geq r} \rho^{-n} \|f_{B(0,\rho)}\|_X \right)^q \frac{dr}{r} 
= n \int_0^{\infty} \left( r^{-\lambda+n} \sup_{\rho \geq r} \int_{\rho}^{\infty} t^{-n} \frac{dt}{t} \|f_{B(0,\rho)}\|_X \right)^q \frac{dr}{r} 
\leq n \int_0^{\infty} r^{-\lambda+qn} \sup_{\rho \geq r} \left( \int_{\rho}^{\infty} t^{-qn} \|f_{B(0,t)}\|_X^q \frac{dt}{t} \right) \frac{dr}{r} 
= n \int_0^{\infty} t^{-qn} \|f_{B(0,t)}\|_X^q \left( \int_{0}^{t} r^{-(\lambda+n)} \frac{dr}{r} \right) \frac{dt}{t} 
= \frac{n}{nq - \lambda q} \|f\|_{LM_{X,q}^{\lambda}}^q.
\]

If $q = \infty$, then
\[
\|f\|_{LM_{X,\infty}^{\lambda}} = \sup_{r > 0} r^{-\lambda} \|f_{B(0,t)}\|_X 
= \sup_{r > 0} r^{-\lambda} r^n \sup_{r \leq t} t^{-n} \|f_{B(0,t)}\|_X 
= \sup_{t > 0} t^{-\lambda} \|f_{B(0,t)}\|_X 
= \|f\|_{LM_{X,\infty}^{\lambda}}.
\]

\[ \square \]

**Theorem 4.1.** Let $0 < q_0, q_1, q \leq \infty$, $0 \leq \lambda_0, \lambda_1, \lambda < \sigma$, $0 < \theta < 1$ and $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$. Then
\[
\left( LM_{X,q_0}^{\lambda_0}, LM_{X,q_1}^{\lambda_1} \right)_{\theta,q} = LM_{X,q}^{\lambda}
\]

Proof. 1. If $q < \infty$, 1.1 first, let us prove that
\[
\left( LM_{X,q_0}^{\lambda_0}, LM_{X,q_1}^{\lambda_1} \right)_{\theta,q} \subset LM_{X,q}^{\lambda}
\]

for $\lambda_0 < \lambda_1$. Let $f \in \left( LM_{X,q_0}^{\lambda_0}, LM_{X,q_1}^{\lambda_1} \right)_{\theta,q}$ and $f = \varphi + \psi$ with $\varphi \in LM_{X,q_0}^{\lambda_0}$ and $\psi \in LM_{X,q_1}^{\lambda_1}$. According to Proposition 2.1 and Lemma 4.1, then
\[
r^{-\lambda} \|f_{B(0,r)}\|_X \lesssim r^{-\lambda} \left( \|\varphi_{B(0,r)}\|_X + \|\psi_{B(0,r)}\|_X \right) 
= r^{\lambda_0 - \lambda} \left( r^{-\lambda_0} \|\varphi_{B(0,r)}\|_X + r^{\lambda_1 - \lambda_0} r^{-\lambda_1} \|\psi_{B(0,r)}\|_X \right) 
\lesssim r^{\lambda_0 - \lambda} \left( \sup_{s > 0} s^{-\lambda_0} \|\varphi_{B(0,r)}\|_X + r^{\lambda_1 - \lambda_0} \sup_{s > 0} s^{-\lambda_1} \|\psi_{B(0,r)}\|_X \right) 
= r^{\lambda_0 - \lambda} \left( \|\varphi\|_{LM_{X,\infty}^{\lambda_0}} + r^{\lambda_1 - \lambda_0} \|\psi\|_{LM_{X,\infty}^{\lambda_1}} \right) 
\]

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Since the representation $f = \varphi + \psi$ is arbitrary,
\[
\|f\|_{LM^{\lambda}_{X,q}} \lesssim r^{-\lambda} \left( \|\varphi\|_{LM^{\lambda}_{X,q,0}} + r^{\lambda_0 - \lambda} \|\psi\|_{LM^{\lambda}_{X,\infty}} \right)
\]
\[
\lesssim r^{-\lambda} \left( \|\varphi\|_{LM^{\lambda}_{X,\infty}} + \|\psi\|_{LM^{\lambda}_{X,q,0}} \right)
\]
\[
\lesssim r^{-\lambda} \left( \|\varphi\|_{LM^{\lambda}_{X,\infty}} + r^{\lambda_0 - \lambda} \|\psi\|_{LM^{\lambda}_{X,q,0}} \right)
\]
where \[41\]
\[
K(t, f) = \inf_{\varphi \in LM^{\lambda}_{X,q,0}, \psi \in LM^{\lambda}_{X,q,1}} \left( \|\varphi\|_{LM^{\lambda}_{X,q,0}} + t \|\psi\|_{LM^{\lambda}_{X,q,1}} \right), \quad t > 0.
\]
Hence,
\[
\|f\|_{LM^{\lambda}_{X,q}} \leq \|f\|_{LM^{\lambda}_{X,q}} = \left( \int_0^{\infty} \left( r^{-\lambda} \|f\chi_{B(0,r)}\|_{\tilde{X}} \right)^q \frac{dr}{r} \right)^{\frac{1}{q}} \lesssim \left( \int_0^{\infty} \left( r^{-\theta(\lambda_1 - \lambda_0)} K \left( r^{\lambda_1 - \lambda_0}, f \right) \right) \frac{dr}{r} \right)^{\frac{1}{q}}
\]
\[
\sim (\lambda_1 - \lambda_0)^{-\frac{1}{q}} \left( \int_0^{\infty} \left( t^{-\theta} K(t, f) \right) \frac{dt}{t} \right)^{\frac{1}{q}} \sim (\lambda_1 - \lambda_0)^{-\frac{1}{q}} \|f\|_{LM^{\lambda}_{X,q_0,0},LM^{\lambda}_{X,q_1}}.
\]

1.2. On the contrary, let us prove that
\[
LM^{\lambda}_{X,q} \subset \left( LM^{\lambda}_{X,q_0,0}, LM^{\lambda}_{X,q_1} \right)_{\theta,q}
\]
for $\lambda_0 < \lambda_1$. Set $\tilde{q}_0 = \min \{q_0, q\}$ and $\tilde{q}_1 = \min \{q_1, q\}$. It suffices to prove that
\[
LM^{\lambda}_{X,q} \subset \left( LM^{\lambda}_{X,q_0,0}, LM^{\lambda}_{X,q_1} \right)_{\theta,q},
\]
because
\[
\left( LM^{\lambda}_{X,q_0,0}, LM^{\lambda}_{X,q_1} \right)_{\theta,q} \subset \left( LM^{\lambda}_{X,q_0,0}, LM^{\lambda}_{X,q_1} \right)_{\theta,q}
\]
by Proposition 2.1. Therefore, without loss of generality we assume that $0 < q_0, q_1 \leq q \leq \infty$.

First, suppose that $q < \infty$.

1.2.1. Let $f \in LM^{\lambda}_{X,q}$ and $t > 0$. For $x \in \mathbb{R}^n$, set
\[
\varphi_t(x) = \begin{cases} 
  f(x) & \text{if } x \in B(0,t), \\
  0 & \text{if } x \notin B(0,t)
\end{cases} \quad \text{and} \quad \psi_t(x) = f - \varphi_t(x).
\]
Then, applying the change of variables $t^{\lambda_1 - \lambda_0} = s$,
\[
\|f\|_{LM^{\lambda}_{X,q_0,0},LM^{\lambda}_{X,q_1}} \leq \left( \int_0^{\infty} \left( s^{-\theta} K(s, f) \right) \frac{ds}{s} \right)^{\frac{1}{q}}
\]
\[
= (\lambda_1 - \lambda_0)^{\frac{1}{q}} \left( \int_0^{\infty} \left( t^{-(\lambda_1 - \lambda_0)\theta} K(t^{\lambda_1 - \lambda_0}, f) \right) \frac{dt}{t} \right)^{\frac{1}{q}}.
\]
where

\[
\bar{K}(t^{\lambda_1-\lambda_0}, f) = \inf_{f = \varphi + \psi, \varphi \in LM_{X,q_0}, \psi \in LM_{X,q_1}} \left( \|\varphi\|_{LM_{X,q_0}} + t^{\lambda_1-\lambda_0} \|\psi\|_{LM_{X,q_1}} \right)
\]

\[
\leq \|\varphi\|_{LM_{X,q_0}} + t^{\lambda_1-\lambda_0} \|\psi\|_{LM_{X,q_1}}.
\]

1.2.2. Let us estimate \(\|\psi_t\|_{LM_{X,q_1}}\). Since \(\|\psi_t(x)\| \leq |f(x)|\),

\[
\|\psi_t\|_{LM_{X,q_1}} = \left( \int_0^\infty \left( s^{-\lambda_1} \|\psi_t \chi_B(0,s)\|_{\tilde{L}_X} \frac{q_1}{s} \right) \right)^{-\frac{1}{q_1}}
\]

\[
\leq \left( \left( \int_0^t \left( s^{-\lambda_1} \|\psi_t \chi_B(0,s)\|_{\tilde{L}_X} \frac{q_1}{s} \right) \right)^{\frac{1}{q_1}} \right)^{q_1}
\]

\[
+ \left( \int_t^\infty \left( s^{-\lambda_1} \|\chi_B(0,s)\|_{\tilde{L}_X} \right) \frac{q_1}{s} \right)^{\frac{1}{q_1}}.
\]

Since (2)

\[
\|\psi_t \chi_B(0,r)\|_{\tilde{X}} = sup_{\rho \geq s} \rho^{-n} \|\chi_B(0,\rho)\|_{X} = \left( \frac{s}{t} \right)^n \|\chi_B(0,t)\|_{X} \quad 0 < s < t.
\]

Therefore,

\[
\left( \int_0^t \left( s^{-\lambda_1} \|\psi_t \chi_B(0,s)\|_{\tilde{X}} \right) \frac{q_1}{s} \right)^{\frac{1}{q_1}} \leq t^{-n} \|\chi_B(0,t)\|_{\tilde{X}} \left( \int_0^t \left( s^{-\lambda_1} \frac{q_1}{s} \right) \right)^{\frac{1}{q_1}}
\]

\[
= ((n - \lambda_1) q_1)^{-\frac{1}{q_1}} t^{-\lambda_1} \|\chi_B(0,t)\|_{\tilde{X}}
\]

\[
= C_1 t^{\lambda_0 - \lambda_1} - n \|\chi_B(0,t)\|_{\tilde{X}} \left( \int_0^t \left( s^{-\lambda_0} \sup_{\rho \geq t} \rho^n \|\chi_B(0,\rho)\|_{X} \right) \frac{q_0}{s} \right)^{\frac{1}{q_0}}
\]

\[
\leq C_1 t^{\lambda_0 - \lambda_1} \left( \int_0^t \left( s^{-\lambda_0} \|\chi_B(0,s)\|_{\tilde{X}} \right) \frac{q_0}{s} \right)^{\frac{1}{q_0}},
\]

where \(C_1 = ((n - \lambda_1) q_1)^{-1/q_1} ((n - \lambda_0) q_0)^{1/q_0}\).

Thus,

\[
\|\psi_t\|_{LM_{X,q_1}} \leq \left( t^{\lambda_0 - \lambda_1} \left( \int_0^t \left( s^{-\lambda_0} \|f\|_{\tilde{X}(B(0,s))} \right) \frac{q_0}{s} \right)^{\frac{1}{q_0}} \right)^{q_0} + \left( \int_0^\infty \left( s^{-\lambda_1} \|f\|_{\tilde{X}(B(0,s))} \right) \frac{q_1}{s} \right)^{\frac{1}{q_1}}
\]

\[
\leq C_2 (I_1(t) + I_2(t)),
\]

where \(C_2 = \max \{C_1, 1\}\).

1.2.3. Let us estimate \(\|\varphi_t\|_{LM_{X,q_0}}\). Since (2),

\[
\|\varphi_t\|_{LM_{X,q_0}} = \left( \int_0^\infty \left( s^{-\lambda_0} \|\varphi_t \chi_B(0,s)\|_{\tilde{X}} \right) \frac{q_0}{s} \right)^{\frac{1}{q_0}}
\]
\[
\varphi_t \mapsto \left( \int_0^t \left( s^{-\lambda_0} \|f \chi_B(0,s) \|_{X} \right)^{q_0} \frac{ds}{s} \right)^{\frac{1}{q_0}} + \left( \int_t^\infty \left( s^{-\lambda_0} \|\varphi_t \chi_B(0,s) \|_{\tilde{X}} \right)^{q_0} \frac{ds}{s} \right)^{\frac{1}{q_0}}.
\]
and
\[
\|\varphi_t \chi_B(0,s) \|_{\tilde{X}} = s^n \sup_{\rho \geq s} \rho^{-n} \|f \chi_B(0,\rho) \|_X = \|f \chi_B(0,t) \|_X \quad s > t.
\]
Therefore,
\[
\left( \int_t^\infty \left( s^{-\lambda_0} \|\varphi_t \chi_B(0,s) \|_{\tilde{X}} \right)^{q_0} \frac{ds}{s} \right)^{\frac{1}{q_0}} = (\lambda_0 q_0)^{-\frac{1}{q_0}} t^{-\lambda_0} \|f \chi_B(0,t) \|_X
\]
\[
= C_3 \left( \int_0^t \left( s^{-\lambda_0} t^{-n} \|f \chi_B(0,t) \|_X \right)^{q_0} \frac{ds}{s} \right)^{\frac{1}{q_0}}
\]
\[
\leq C_3 \left( \int_0^t \left( s^{-\lambda_0} s^n \sup_{\rho \geq s} \rho^{-n} \|f \chi_B(0,\rho) \|_X \right)^{q_0} \frac{ds}{s} \right)^{\frac{1}{q_0}}
\]
\[
= C_3 \left( \int_0^t \left( s^{-\lambda_0} \|f \chi_B(0,s) \|_{\tilde{X}} \right)^{q_0} \frac{ds}{s} \right)^{\frac{1}{q_0}} := I_3,
\]
where \( C_3 = \left( \frac{n - \lambda_0}{\lambda_0 p_0} \right)^{\frac{1}{q_0}} \).

Hence,
\[
\|\varphi_t \|_{L_{\tilde{X}}^{\lambda_0}} \leq C_4 I_3(t), \quad C_4 = 1 + C_3
\]
1.3. So,
\[
\tilde{K} \left( t^{\lambda_1 - \lambda_0}, f \right) \leq C_5 \left( I_3(t) + t^{\lambda_1 - \lambda_0} I_2(t) \right),
\]
where \( C_5 = C_2 + C_4 \), and
\[
\|f\|_{(L_{X}^{\lambda_0}, L_{X}^{\lambda_1})_{\theta,q}} \leq \left( \int_0^\infty \left( t^{-q(\lambda_1 - \lambda_0)} I_3(t) \right)^{\frac{q}{\theta}} dt \right)^{\frac{1}{q}} + \left( \int_0^\infty \left( t^{(1-q)(\lambda_1 - \lambda_0)} I_2(t) \right)^{\frac{q}{\theta}} dt \right)^{\frac{1}{q}}
\]
\[
\lesssim (C_5 (J_1 + J_2)).
\]
Note that
\[
J_1^{q_0} = \left( \int_0^\infty \left( t^{-\theta(\lambda_1 - \lambda_0)q_0} \int_0^t s^{-\lambda_0 q_0 - 1} \|f \chi_B(0,s) \|_{X}^{q_0} ds \right)^{\frac{q}{\theta}} dt \right)^{\frac{q}{\theta}}
\]
\[
J_2^{q_1} = \left( \int_0^\infty \left( t^{(1-\theta)(\lambda_1 - \lambda_0)q_1} \int_0^\infty s^{\lambda_1 q_1 - 1} \|f \chi_B(0,s) \|_{X}^{q_1} ds \right)^{\frac{q}{\theta}} dt \right)^{\frac{q}{\theta}}
\]
1.4. Applying the Hardy inequality in the form
\[
\left( \int_0^\infty \left( t^{-\alpha} \int_0^t g(s) ds \right)^{\sigma} dt \right)^{\frac{1}{\sigma}} \leq \left( \int_0^\infty \left( t^{1-\alpha} g(t) \right)^{\sigma} dt \right)^{\frac{1}{\sigma}},
\]
where \( \sigma \geq 1, \alpha > 0 \), and \( g \) is a nonnegative measurable function on \((0, \infty)\),

\[
J_1^{q_0} \leq (\theta (\lambda_1 - \lambda_0) q_0)^{-\frac{1}{q_0}} \| f \|_{L^\lambda_{X,q}}^{\lambda} \left( \int_0^\infty \left( t^{-\lambda} \| f \chi_{B(0,s)} \|_X \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}
\]

which implies, according to Lemma 4.1, that

\[
J_1 \leq (\theta (\lambda_1 - \lambda_0) q_0)^{-\frac{1}{q_0}} \left( \frac{np}{n - \lambda p} \right)^{\frac{1}{q}} \| f \|_{LM_{X,q}^\lambda}^{\lambda}
\]

Similar considerations yield

\[
J_2 \leq ((1 - \theta) (\lambda_1 - \lambda_0) q_1)^{-\frac{1}{q_1}} \left( \frac{np}{n - \lambda p} \right)^{\frac{1}{q}} \| f \|_{LM_{X,q}^\lambda}^{\lambda}
\]

Thus, the theorem is proved under the additional assumption \( s_1 \lambda_0 < \lambda_1 \) and \( q < \infty \).

2. If \( q = \infty \), the proof follows the same lines in which the integrals should be replaced by the corresponding upper bounds.

If \( \lambda_1 < \lambda_0 \), then one should take into account that

\[
\| f \|_{(LM_{X,q_0}^\lambda,LM_{X,q_1}^\lambda)} \leq \left( \int_0^\infty \left( t^{-\theta} \inf_{\varphi \in LM_{X,q_0}, \psi \in LM_{X,q_1}} \left( \| \varphi \|_{LM_{X,q_0}^\lambda} + t \| \psi \|_{LM_{X,q_1}^\lambda} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \| f \|_{LM_{X,q}^\lambda}
\]

According to what has been proved above, this expression is equivalent (with constants depending only on the parameters \( q_0,q_1, X, \lambda_0, \lambda_1, \) and \( q \)) to the quasinorm \( \| f \|_{LM_{X,q}^\mu} \), where

\[
\mu = (1 - (1 - \theta)) \lambda_1 + (1 - \theta) \lambda_0 = \lambda,
\]

to the quasinorm \( \| f \|_{LM_{X,q}^\lambda} \).

\[
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\]

5 Vector valued maximal inequalities

Assumption 5.1. Let \( 1 < u \leq \infty \), for every sequence \( \{ f_j \}_{j=1}^\infty \subset L_{1}^{loc}(\mathbb{R}^n), X \in M \). If

\[
\left\| \left( \sum_{j=1}^\infty |M f_j|^u \right) \right\|_X \lesssim \left\| \left( \sum_{j=1}^\infty |f_j|^u \right) \right\|_X.
\]

Especially \( u = \infty \),

\[
\left\| M \left[ \sup_{j \in \mathbb{N}} |f_j| \right] \right\|_X \lesssim \left\| \sup_{j \in \mathbb{N}} |f_j| \right\|_X.
\]

Then \( X \in M' \).
Theorem 5.1. Let $1 < q \leq \infty$, $0 \leq \lambda < \sigma$, $1 < v < \infty$, $X \in \mathbb{M}^r$. Then

$$\left\| \left( \sum_{j=1}^{\infty} (Mf_j)^v \right)^{\frac{1}{v}} \right\|_{LM^\lambda_{X,q}} \leq \left\| \left( \sum_{j=1}^{\infty} |f_j|^v \right)^{\frac{1}{v}} \right\|_{LM^\lambda_{X,q}}. \quad (3)$$

In particular,

$$\left\| M \left[ \sup_{j \in \mathbb{N}} |f_j| \right] \right\|_{LM^\lambda_{X,q}} \leq \left\| \sup_{j \in \mathbb{N}} |f_j| \right\|_{LM^\lambda_{X,q}}. \quad (4)$$

Proof. In Section 3, we prove that the Hardy-Littlewood maximal operator the operator $M$ is bounded from $LM^\lambda_{X,q}$ to $LM^\lambda_{X,q}$. Refer to Section 3 to obtain (3), we only suppose $\theta < \infty$; the case $\theta = \infty$ can be dealt similarly.

$$\left\| \chi_{B(0,r)} \left( \sum_{j=1}^{\infty} |Mf_j|^v \right)^{\frac{1}{v}} \right\|_X \leq r^\sigma \int_{2r}^{\infty} t^{-\sigma-1} \left\| \chi_{B(0,t)} \left( \sum_{j=1}^{\infty} |f_j|^v \right)^{\frac{1}{v}} \right\|_X dt.$$

Referring to the method of Section 3, with the help of the boundedness of $H^\sigma$, we obtain the desired result. \qed

6 Predual spaces of the local Morrey space associated with ball quasi-Banach function spaces

Let $1 < q \leq \infty$. If $supp(A) \subset B(R)$ and $\|A\|_X \leq R^{\lambda - \frac{1}{q}}$, which we call the function $A$ is a $(X, R)$-block. The local block space $LH^\lambda_{X',q'}(\mathbb{R}^n)$ is the set of all measurable functions $g$. There exists a decomposition

$$g(x) = \sum_{j=-\infty}^{\infty} \lambda_j A_j(x).$$

Where each of $A$ is a $(X, 2^j)$-block and $\{\lambda_j\}_{j=-\infty}^{\infty} \in l^{q'}$ and the convergence of almost all $x \in (\mathbb{R}^n)$, the norm of $g$ is given by:

$$\|g\|_{LH^\lambda_{X',q'}} := \inf \left( \sum_{j=-\infty}^{\infty} |\lambda_j|^{q'} \right)^{\frac{1}{q'}},$$

where $\{\lambda_j\}_{j=-\infty}^{\infty}$ covers all the above admissible expressions.

Theorem 6.1. Let $1 < q \leq \infty$ and $0 \leq \lambda < \sigma$, then $LM^\lambda_{X,q}(\mathbb{R}^n)$ is the dual of $LH^\lambda_{X',q'}(\mathbb{R}^n)$ in the following sense:

(i) Let $f \in LM^\lambda_{X,q}(\mathbb{R}^n)$, for any $g \in LH^\lambda_{X',q'}(\mathbb{R}^n)$, then $fg \in L^1(\mathbb{R}^n)$, the mapping

$$g \in LH^\lambda_{X',q'}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} f(x)g(x)dx \in \mathbb{C}$$

may define a continuous linear functional $L_f$ on $LH^\lambda_{X',q'}$.
(ii) Conversely, any continuous linear functional \( L \) on \( LH^\lambda_{X',q'}(\mathbb{R}^n) \) can be realized as \( L = L_f(\mathbb{R}^n) \) with a certain \( f \in LM^\lambda_{X,q}(\mathbb{R}^n) \).

Furthermore, for all \( f \in LM^\lambda_{X,q}(\mathbb{R}^n) \) the operator norm of \( L_f \) is equivalent to \( \| f \|_{LM^\lambda_{X,q}} \), scilicet there exists a constant \( C > 0 \) such that

\[
C^{-1}\| f \|_{LM^\lambda_{X,q}} \leq \| L_f \|_{LH^\lambda_{X',q'}} \leq C\| f \|_{LM^\lambda_{X,q}}.
\]

**Proof.** (1) Let \( g \) be such that

\[
g := \sum_{j=-\infty}^{\infty} \lambda_j A_j,
\]

where each \( A_j \) is a \( (X',2^j) \)-block and \( \{ \lambda_j \}_{j=1}^{\infty} \in l^q' \) satisfies

\[
\left( \sum_{j=-\infty}^{\infty} |\lambda_j|^{q'} \right)^{1/q'} \leq 2 \| g \|_{LH^\lambda_{X',q'}}.
\]

Then

\[
\|fg\|_{L_1} \leq \sum_{j=-\infty}^{\infty} |\lambda_j| \int_{B(0,2^j)} |f(x)A_j(x)| \, dx
\]

\[
\leq \sum_{j=-\infty}^{\infty} |\lambda_j| \| f \|_{X'} \| A_j \|_{X'} \| g \|_{LH^\lambda_{X',q'}}
\]

\[
\leq \left( \sum_{j=-\infty}^{\infty} |\lambda_j|^{q'} \right)^{1/q'} \left( \sum_{j=-\infty}^{\infty} \left( 2^{-j\lambda - j/q} \| f \|_{X'} \right)^q \right)^{1/q}
\]

\[
\lesssim \| f \|_{LM^\lambda_{X,q}} \| g \|_{LH^\lambda_{X',q'}}.
\]

(2) Let \( L \) be a bounded linear functional on \( LH_{p,q',w}(\mathbb{R}^n) \), since the mapping

\[
g \in X'(\mathbb{R}^n) \mapsto L(g\chi_{B(0,2^j)}) \in \mathbb{C}
\]

is a bounded linear functional, we see that \( L \) is realized by an \( L^1_{loc}(\mathbb{R}^n) \)-function \( f \) satisfy

\[
L(g\chi_{B(0,2^j)}) = \int_{B(0,2^j)} g(x)f(x) \, dx
\]

for all \( g \in X'(\mathbb{R}^n) \) and \( j \in \mathbb{Z} \). We have to check \( f \in LM^\lambda_{X,q}(\mathbb{R}^n) \), or equivalently (Proposition 2.4),

\[
\left( \sum_{j=-\infty}^{\infty} \left( 2^{-j\lambda - j/q} \| f \|_{X} \right)^q \right)^{1/q}
\]

To this end, choose a nonnegative \( \ell^{q'} \)-sequence \( \{ \rho_j \}_{j=-\infty}^{\infty} \) arbitrarily so that \( \rho_j = 0 \) with \( |j| \gg 1 \).
and estimate
\[ \sum_{j=-\infty}^{\infty} 2^{-j\lambda - \frac{j}{q}} \rho_j \| f \chi_{B(0,2^j)} \|_X. \]

Let us set
\[ g_j(x) := \begin{cases} P(x) & (\text{satisfy } \|P\|_{X'} = 1), \text{ if } \| f \chi_{B(0,2^j)} \|_X > 0 \\ 0, & \text{otherwise.} \end{cases} \]
Then each \( g_j \) is a \((X', R)\)-block
\[ g := \sum_{j=-\infty}^{\infty} \rho_j g_j \in LH^{\lambda}_{X',q'}(\mathbb{R}^n) \]
and satisfies
\[ \int_{\mathbb{R}^n} |f(x)g(x)| \, dx = \sum_{j=-\infty}^{\infty} w(2^j) \rho_j \| f \chi_{B(0,2^j)} \|_X. \]
Therefore, by letting \( h(x) := \text{sgn}(f(x))g(x) \) for \( x \in \mathbb{R}^n \), since \( \text{supp}(h) \subset B(0,2^j) \) for some large \( j \),
\[ \int_{\mathbb{R}^n} |f(x)g(x)| \, dx = L(h) \text{ thanks to } (5) \text{ and the fact that } \rho_j = 0 \text{ if } |j| \gg 1. \]
Thus
\[ \sum_{j=-\infty}^{\infty} 2^{-j\lambda - \frac{j}{q}} \rho_j \| f \chi_{B(0,2^j)} \|_X = \int_{\mathbb{R}^n} |f(x)g(x)| \, dx \]
\[ = L(h) \leq \|L\|_{LH^{\lambda}_{X',q'} \to C} \|h\|_{LH^{\lambda}_{X',q'}} \]
\[ \leq \|L\|_{LH^{\lambda}_{X',q'} \to C} \left( \sum_{j=-\infty}^{\infty} |\rho_j|^q \right)^{\frac{1}{q}}. \]

\[ \square \]

7 Characterization of the Hardy local Morrey-type spaces associated with ball quasi-Banach function spaces in terms of the grand maximal operators and the heat kernel

We characterize the space \( LM^{\lambda}_{X,q}(\mathbb{R}^n) \) in terms of the heat kernel. Let us show that \( LM^{\lambda}_{X,q}(\mathbb{R}^n) \) and \( HLM^{\lambda}_{X,q}(\mathbb{R}^n) \) are isomorphic.

**Theorem 7.1.** Let \( 1 < q \leq \infty, 0 \leq \lambda < \sigma \), and \( X \in M' \).

(i) If \( f \in LM^{\lambda}_{X,q}(\mathbb{R}^n) \), then \( f \in HLM^{\lambda}_{X,q}(\mathbb{R}^n) \).

(ii) If \( f \in HLM^{\lambda}_{X,q}(\mathbb{R}^n) \), then \( f \) is represented by a measurable function \( g \in LM^{\lambda}_{X,q}(\mathbb{R}^n) \).

If \( f \in LM^{\lambda}_{X,q}(\mathbb{R}^n) \), then
\[ \|f\|_{LM^{\lambda}_{X,q}} \leq \|f\|_{HLM^{\lambda}_{X,q}} \leq C\|f\|_{LM^{\lambda}_{X,q}} \tag{6} \]

**Proof.** (1) Proposition 2.2 has proved that \( LM^{\lambda}_{X,q}(\mathbb{R}^n) \leftrightarrow S'(\mathbb{R}^n) \). Also \[ \sup_{t>0} |e^{t\Delta} f| \leq Mf. \]
In Section 3, the $LM^λ_{X,q}(\mathbb{R}^n)$-boundedness of the Hardy-Littlewood maximal operator, we see that $f \in HLM^λ_{X,q}(\mathbb{R}^n)$ and that the right inequality in (6) follows.

(2) Due to Theorem 6.1, the dual of $LH^λ_{X',q'}(\mathbb{R}^n)$ is isomorphic to $LM^λ_{X,q}(\mathbb{R}^n)$. Let $L : h \in LM^λ_{X,q}(\mathbb{R}^n) \mapsto L_h \in \left( LH^λ_{X',q'}(\mathbb{R}^n) \right)^\ast$ be an isomorphism in Theorem 6.1. By the Banach-Alaoglu theorem, there exists a positive decreasing sequence $\{t_j\}_{j=1}^\infty \subset (0,1)$ such that $Le^{t_j}f$ is convergent to $G = L_g \in \left( LH^λ_{X',q'}(\mathbb{R}^n) \right)^\ast$ for some $g \in LM^λ_{X,q}(\mathbb{R}^n)$ in the weak-$*$ sense. Observe that

$$
\|g\|_{LM^λ_{X,q}} \sim \|L_g\|_{\left( LH^λ_{X',q'} \right)^\ast} \\
\leq \liminf_{j \to \infty} \|Le^{t_j}\|_{\left( LH^λ_{X',q'} \right)^\ast} \\
\sim \liminf_{j \to \infty} \|e^{t_j}\|_{LM^λ_{X,q}} \leq \|f\|_{HLM^λ_{X,q}}.
$$

Meanwhile, since $f \in S'(\mathbb{R}^n)$, $e^{t_j}f$ is convergent to $f \in S'(\mathbb{R}^n)$. Thus, we conclude $S'(\mathbb{R}^n) \ni f = g \in LM^λ_{X,q}(\mathbb{R}^n)$. The left inequality in (6) follows since the spaces $LM^λ_{X,q}(\mathbb{R}^n)$ is isomorphic to the dual of $LH^λ_{X',q'}(\mathbb{R}^n)$. Thus, from Lebesgue’s differentiation theorem,

$$
\|f\|_{LM^λ_{X,q}} \leq \sup_{t>0} \|e^{t}\|_{LM^λ_{X,q}} = \|f\|_{HLM^λ_{X,q}}.
$$

□

In terms of the grand maximal operator in Definition 2.3, can rephrase Theorem 7.1 as follows:

**Theorem 7.2.** Let $1 < q \leq \infty$, $0 \leq \lambda < \sigma$, and $X \in \mathcal{M}'$.

(i) If $f \in LM^λ_{X,q}(\mathbb{R}^n)$, then $Mf \in LM^λ_{X,q}(\mathbb{R}^n)$.

(ii) Let $f \in S'(\mathbb{R}^n)$, if $Mf \in LM^λ_{X,q}(\mathbb{R}^n)$, then $f$ is represented by a measurable function $g \in LM^λ_{X,q}(\mathbb{R}^n)$.

If $f \in LM^λ_{X,q}(\mathbb{R}^n)$, then $C^{-1}\|f\|_{LM^λ_{X,q}} \leq \|Mf\|_{LM^λ_{X,q}} \leq C\|f\|_{LM^λ_{X,q}}$.

**Proof.** The implication $(i) \implies (ii)$ immediately follows from the pointwise inequality $Mf(x) \lesssim Mf(x)$. The converse implication $(ii) \implies (i)$ follows from the pointwise estimate $|e^{t}\Delta f(x)| \lesssim Mf(x)$. Indeed, from this pointwise estimate, we conclude $\sup_{t>0} |e^{t}\Delta f| \in LM^λ_{X,q}(\mathbb{R}^n)$. Thus, we are in the position of applying Theorem 7.1 to receive $f \in LM^λ_{X,q}(\mathbb{R}^n)$. □

**Remark 7.1.** [43] $HLM^λ_{X,q}$ returne to the norm of Hardy-type spaces for ball quasi-Banach function spaces $H_{LM^λ_{X,q}}$.

$$
\|f\|_{H_{LM^λ_{X,q}}}(\mathbb{R}^n) \sim \sup_{t \in (0,\infty)} \|e^{t}\Delta f\|_{LM^λ_{X,q}}
$$
8 Atomic decomposition of the local Morrey space associated with ball quasi-Banach function spaces

Theorem 8.1. Let $1 < q \leq \infty$, $0 \leq \lambda < \sigma$, $X \in \mathcal{M}$ and

$$\sigma_1 < \sigma - \lambda \quad (7)$$

And $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^{\infty} \subset X_1(\mathbb{R}^n)$, $\{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty)$ satisfying

$$\|a_j\|_{X_1} \leq \|\chi_{Q_j}\|_{X_1} = r_{\sigma_1}, \text{ supp } (a_j) \subset Q_j, \sum_{j=1}^{\infty} \|\lambda_j \chi_{Q_j}\|_{LM_{\lambda_{X},q}} < \infty.$$ Then the series $f := \sum_{j=1}^{\infty} \lambda_j a_j$ converges in $L_{loc}^1(\mathbb{R}^n)$ and in the Schwartz space $S'(\mathbb{R}^n)$ of tempered distributions and satisfies the estimate

$$\|f\|_{LM_{\lambda_{X},q}} \lesssim \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{LM_{\lambda_{X},q}} \quad (8)$$

Lemma 8.1. Let $1 < q \leq \infty$, $0 \leq \lambda < \sigma$, each $A_j$ be a $(X', 2^j)$-block and $\{\rho_j\}_{j=-\infty}^{\infty} \in \ell^q$. Suppose $\sigma$ and $\sigma_1$ satisfies (7). Then

$$h := \sum_{j=-\infty}^{\infty} \rho_j M \left[\|A_j \chi_{B(0,r)}\|_{X'_1}\right] \in LH_{X',q'}(\mathbb{R}^n), \quad \|h\|_{LH_{X',q'}} \leq C \left(\sum_{j=-\infty}^{\infty} |\rho_j|^q\right)^{1/q'}.$$ Proof. By the $X'_1$-boundedness of the Hardy-Littlewood maximal operator and $q' < \infty$,

$$\sum_{j=-\infty}^{\infty} \rho_j \chi_{B(0,2^{j+1})} M \left[\|A_j \chi_{B(0,r)}\|_{X'_1}\right] \in LH_{X',q'}(\mathbb{R}^n)$$

and

$$\left\|\sum_{j=-\infty}^{\infty} \rho_j \chi_{B(0,2^{j+1})} M \left[\|A_j \chi_{B(0,r)}\|_{X'_1}\right]\right\|_{LH_{X',q'}} \lesssim \left(\sum_{j=-\infty}^{\infty} |\rho_j|^q\right)^{1/q'}.$$ Meanwhile, combining $\|A_j\|_{X'_1} \leq (2^j)^{\sigma - \sigma_1} \|A_j\|_{X'} \leq (2^j)^{\sigma - \sigma_1 - \lambda - \frac{1}{q}}$ and (7), therefore,

$$\sum_{j=-\infty}^{\infty} \rho_j \chi_{B(2^{j+1})} M \left[\|A_j \chi_{B(0,r)}\|_{X'_1}\right]$$

$$= \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} \rho_j \chi_{B(2^{j+k+1}) \setminus B(2^{j+k+1})} M \left[\|A_j \chi_{B(0,r)}\|_{X'_1}\right]$$

$$\leq C \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} \rho_j (2^{(j+k+1)} \sigma_1) \|A_j \chi_{B(2^{j+k+2})} \setminus B(2^{j+k+1}) \chi_{B(2^{j+k+2})} \|_{X'_1}.$$
\[
C \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} \rho_j \frac{2^{j\sigma_1-j\sigma_2}}{2^{(j+k)\sigma_1}} \| A_j \chi_{B(0,2^k)} \|_{X^1_x} \chi_{B(2^{j+k+2}) \setminus B(2^{j+k+1})} \\
\leq C \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} \rho_j \frac{2^{j\sigma_1-j\sigma_2} 2^{j-k}}{2^{(j+k)\sigma_1}} \chi_{B(2^{j+k+2}) \setminus B(2^{j+k+1})} \lesssim \sum_{j=-\infty}^{\infty} \rho_j.
\]

Next, to prove Theorem 8.1.

**Proof.** To prove (8), we resort to the duality obtained in Theorem 6.1.

\[
\|f\|_{L^M_{X,q}} = \sup \left\{ \int_{\mathbb{R}^n} f(x)g(x)dx : \|g\|_{L^H_{X',q'}} = 1 \right\}.
\]

We can assume that \(\{\lambda_j\}_{j=1}^{\infty}\) is finitely supported thanks to the monotone convergence theorem. Let us assume in addition that the \(a_j\) are non-negative without loss of generality.

\[
g := \sum_{k=-\infty}^{\infty} \rho_k A_k, \quad G := \sum_{k=-\infty}^{\infty} |\rho_k| M \left[ \|A_j \chi_{B(0,r)}\|_{X^1_x} \right],
\]

where each \(A_k\) is a \((p',2^j)\)-block, Lemma 8.1 and

\[
\sum_{k=-\infty}^{\infty} |\rho_k| q' \leq 1.
\]

Then

\[
\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \leq \sum_{(j,k) \in \mathbb{N} \times \mathbb{Z}} \lambda_j |\rho_k| \int_{B(2^k) \cap Q_j} a_j(x) |A_k(x)| dx \\
\leq \sum_{(j,k) \in \mathbb{N} \times \mathbb{Z}} \lambda_j |\rho_k| \|a_j \chi_{B(0,r)}\|_{X^1_x} \|A_j \chi_{B(0,r)}\|_{X^1_x} \\
\lesssim \sum_{(j,k) \in \mathbb{N} \times \mathbb{Z}} \lambda_j |\rho_k| \int_{Q_j} M \left[ \|A_j \chi_{B(y,r)}\|_{X^1_x} \right] dx < \infty.
\]

With the aid of Proposition 2.2, we extend into Theorem 8.2.

**Theorem 8.2.** Satisfying the conditions of theorem 8.1 but where \(\{a_j\}_{j=1}^{\infty} \subset L^\infty(\mathbb{R}^n)\) such that \(f := \sum_{j=1}^{\infty} \lambda_j a_j\) converges in \(S'(\mathbb{R}^n) \cap L^1_{\text{loc}}(\mathbb{R}^n)\), that

\[
|a_j| \leq \chi_{Q_j}, \quad \int_{\mathbb{R}^n} x^\alpha a_j(x)dx = 0,
\] (9)
for all multi-indices \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) with \(|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n < \infty \) and, that for all \( v > 0 \)

\[
\left\| \left( \sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{1/v} \right\|_{L^\infty_{X,q}} \leq C_v \| f \|_{L^\infty_{X,q}}.
\]  \( (10) \)

Here the constant \( C_v > 0 \) is independent of \( f \).

**Lemma 8.2.** [8] Let \( \varphi \in \mathcal{S}(\mathbb{R}^n) \). With the same notation as Lemma 2.1, then

\[
|\langle b_j, \varphi \rangle| \leq C_\varphi \left\{ \sum_{l=0}^{\infty} \left( \frac{1}{2^ln} \| \mathcal{M} f \cdot \chi_{\mathcal{O}_j} \|_{L^1(B(2^l))} \right)^\mu \right\}^{1/\mu},
\]  \( (11) \)

where \( \mu := \frac{n + d + 1}{n} \) and the constant \( C_\varphi \) in (11) depends on \( \varphi \) but not on \( j \) or \( k \).

**Lemma 8.3.** Let \( 1 < q \leq \infty, 0 \leq \lambda < \sigma, X \in M', f \in L^\infty_{X,q} (\mathbb{R}^n) \) and \( \sigma, \sigma_1 \) satisfies (7). Then in the notation of Lemma 2.1, in the topology of \( \mathcal{S}'(\mathbb{R}^n) \), \( g_j \to 0 \) as \( j \to -\infty \) and \( b_j \to 0 \) as \( j \to \infty \). In particular

\[
f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j).
\]

**Proof.** Observe that

\[
\frac{1}{2^ln} \| \mathcal{M} f \cdot \chi_{\mathcal{O}_j} \|_{L^1(B(2^l))} \lesssim \frac{1}{2^ln} \| \mathcal{M} f \|_{L^1(B(2^l))} \lesssim \frac{1}{2^\sigma} \| \mathcal{M} f \|_{L^1(B(2^l))} \lesssim \frac{1}{2^{\sigma}} \| f \|_{L^1_{\mathcal{X},q}} \lesssim \frac{1}{2^{\sigma}} \| f \|_{L^\infty_{\mathcal{X},q}}.
\]

Note that (7) and \( \mu := \frac{n + d + 1}{n} \).

\[
\sum_{l=1}^{\infty} \left( \frac{1}{2^\sigma 2^{l\lambda - \frac{l}{q}}} \right)^\mu < \infty
\]

Consequently, we may use the Lebesgue convergence theorem to conclude that \( b_j \to 0 \) as \( j \to \infty \). Hence, it follows that \( f = \lim_{j \to \infty} g_j \) in \( \mathcal{S}'(\mathbb{R}^n) \). Consequently, it follows from Lemma 2.1 that

\[
f = \lim_{j \to \infty} g_j = \lim_{j,k \to \infty} \sum_{l=-k}^{j} (g_{l+1} - g_l) \text{ in } \mathcal{S}'(\mathbb{R}^n).
\]

Next, to prove Theorem 8.2.

**Proof.** For each \( j \in \mathbb{Z} \), consider the level set

\[
\mathcal{O}_j := \{ x \in \mathbb{R}^n : \mathcal{M} f(x) > 2^j \}
\]
Then it follows immediately from the definition that

$$\mathcal{O}_{j+1} \subset \mathcal{O}_j.$$  

Apply Lemma 2.1, then $f$ can be decomposed as

$$f = g_j + b_j, \quad b_j = \sum_k b_{j,k}, \quad b_{j,k} = (f - c_{j,k}) \eta_{j,k}$$

where each $b_{j,k}$ is supported in a cube $Q_{j,k}$ as described in Lemma 2.1.

$$f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j)$$

with the series converging in the sense of distributions from Lemma 6.4.

$$f = \sum_{j,k} A_{j,k}, \quad g_{j+1} - g_j = \sum_k A_{j,k} \quad (j \in \mathbb{Z})$$

in the sense of distributions, where each $A_{j,k}$, supported in $Q_{j,k}$, satisfies the pointwise estimate $|A_{j,k}(x)| \leq C_0 2^j$ for some universal constant $C_0$ and the moment condition $\int_{\mathbb{R}^n} A_{j,k}(x) q(x) dx = 0$ for every $q(x) \in \mathcal{P}_d (\mathbb{R}^n)$. With these observations in mind, write

$$a_{j,k} := \frac{A_{j,k}}{C_0 2^j}, \quad \kappa_{j,k} := C_0 2^j.$$

Then we shall obtain that each $a_{j,k}$ satisfies

$$|a_{j,k}| \leq \chi_{Q_{j,k}}, \quad \int_{\mathbb{R}^n} x^\alpha a_{j,k}(x) dx = 0$$

and that $f = \sum_{j,k} \kappa_{j,k} a_{j,k}$ in the topology of $\mathcal{HLM}^\lambda_{X,q} (\mathbb{R}^n)$. Rearrange $\{a_{j,k}\}$ to obtain $\{a_j\}$. Do the same rearrangement for $\{\lambda_{j,k}\}$. To establish (10), write

$$\beta := \left\| \left( \sum_{j=-\infty}^{\infty} |\lambda_j \chi_{Q_j}|^v \right)^{1/v} \right\|_{\mathcal{HLM}^\lambda_{X,q}}.$$

Since

$$\{(\kappa_{j,k};Q_{j,k})\}_{j,k} = \{ (\lambda_j;Q_j) \}_{j},$$

we have

$$\beta = \left\| \left( \sum_{j=-\infty}^{\infty} \sum_{k \in K_j} |\kappa_{j,k} \chi_{Q_{j,k}}|^v \right)^{1/v} \right\|_{\mathcal{HLM}^\lambda_{X,q}}.$$
By using the definition of $\kappa_j$, we then have \[
\beta = C_0 \left\| \left( \sum_{j=-\infty}^{\infty} \sum_{k \in K_j} |2^j \chi_{Q_j,k}| \right)^v \right\|_{LM^\lambda_{X,q}} = C_0 \left\| \left( \sum_{j=-\infty}^{\infty} 2^j \sum_{k \in K_j} \chi_{Q_j,k} \right)^v \right\|_{LM^\lambda_{X,q}}.
\]

Observe that (1), together with the bounded overlapping property, yields \[
\chi_{\mathbb{O}_j}(x) \leq \sum_{k \in K_j} \chi_{Q_j,k}(x) \leq \sum_{k \in K_j} \chi_{200Q_j,k}(x) \lesssim \chi_{\mathbb{O}_j}(x) \quad (x \in \mathbb{R}^n).
\]

Thus, \[
\beta \lesssim \left\| \left( \sum_{j=-\infty}^{\infty} (2^j \chi_{\mathbb{O}_j})^v \right)^{1/v} \right\|_{LM^\lambda_{X,q}}.
\]

Recalling that $\mathbb{O}_j \supset \mathbb{O}_{j+1}$ for each $j \in \mathbb{Z}$, \[
\sum_{j=-\infty}^{\infty} (2^j \chi_{\mathbb{O}_j}(x))^v \sim \left( \sum_{j=-\infty}^{\infty} 2^j \chi_{\mathbb{O}_j}(x) \right)^v \sim \left( \sum_{j=-\infty}^{\infty} 2^j \chi_{\mathbb{O}_j \setminus \mathbb{O}_{j+1}}(x) \right)^v \quad (x \in \mathbb{R}^n).
\]

Then, \[
\beta \lesssim \left\| \sum_{j=-\infty}^{\infty} 2^j \chi_{\mathbb{O}_j \setminus \mathbb{O}_{j+1}} \right\|_{LM^\lambda_{X,q}}.
\]

It follows by the definition of $\mathbb{O}_j$ that $2^j < Mf(x)$ for all $x \in \mathbb{O}_j$. Hence, \[
\beta \lesssim \left\| \sum_{j=-\infty}^{\infty} \chi_{\mathbb{O}_j \setminus \mathbb{O}_{j+1}} Mf \right\|_{LM^\lambda_{X,q}} \lesssim \|Mf\|_{LM^\lambda_{X,q}},
\]

So we receive the proof of Theorem 8.2.

9 The Hardy operator on the local Morrey space associated with ball quasi-Banach function spaces

Theorem 9.1. Suppose $1 < q \leq \infty$, $0 \leq \lambda < \sigma$, $X \in \mathcal{M}^\prime$. Then $\|Hf\|_{LM^\lambda_{X,q}} \lesssim \|f\|_{LM^\lambda_{X,q}}$.

Proof. Let $f = \sum_{j=1}^{\infty} \lambda_j a_j$, $\mu$ stands for the Haar measure of $\text{SO}(n)$ [9]. \[
Sf(x) := \int_{\text{SO}(n)} f(Ax) d\mu(A)
\]

Note that \[
S : LM^\lambda_{X,q}(\mathbb{R}) \rightarrow LM^\lambda_{X,q}(\mathbb{R})
\]
is a bounded linear operator. Since

\[
Hf(x) \sim \frac{1}{|x|^n} \int_{B(|x|)} f(y)dy \\
= \int_{SO(n)} \frac{1}{|Ax|^n} \int_{B(|Ax|)} f(y)dyd\mu(A) \\
= \int_{SO(n)} \frac{1}{|x|^n} \int_{B(|x|)} f(Ay)dyd\mu(A) = HSf(x),
\]

therefore

\[
Hf = HSf = \sum_{j=1}^{\infty} \lambda_j HSa_j.
\]

since \(a_j\) is compactly supported \(|HSa_j| \lesssim S\chi_{Q_j}\), and Theorem 8.2,

\[
\|Hf\|_{LM^{\lambda}_{X,q}} \leq \left\| \sum_{j=1}^{\infty} \lambda_j HSa_j \right\|_{LM^{\lambda}_{X,q}} \lesssim \left\| \sum_{j=1}^{\infty} \lambda_j S\chi_{Q_j} \right\|_{LM^{\lambda}_{X,q}} \\
\lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{LM^{\lambda}_{X,q}} \lesssim \|f\|_{LM^{\lambda}_{X,q}}.
\]

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References

[1] C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math, 1938.
[2] D. R. Adams, A note on Riesz potentials, Duke Math, 1975.

[3] N. Samko, Weighted Hardy and singular operators in Morrey spaces. Journal of Mathematical Analysis and Applications, 2009.

[4] T. Iida, Y. Sawano, H. Tanaka, Atomic decomposition for Morrey spaces. Zeitschrift für Analysis und ihre Anwendungen, 2014.

[5] V.I. Burenkov, H. V. Guliyev, Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces, Studia Math, 2004.

[6] V. I. Burenkov, E. D. Nursultanov, Description of interpolation spaces for local Morrey-type spaces. Proceedings of the Steklov Institute of Mathematics, 2010.

[7] V. I. Burenkov, P. Jain, T. V. Tararykova, On boundedness of the Hardy operator in Morrey-type spaces. Eurasian Mathematical Journal, 2011.

[8] T. Batbold, Y. Sawano, Decompositions for local Morrey spaces, Eurasian Math, 2014.

[9] V. S. Guliyev, S. G. Hasanov, Y. Sawano, Decompositions of local Morrey-type spaces. Positivity, 2017.

[10] V. I. Burenkov, H. V. Guliyev, V. S. Guliyev, Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces, Comput. Appl. Math, 2007.

[11] V. I. Burenkov, H. V. Guliyev, V. S. Guliyev, On boundedness of the fractional maximal operator from complementary Morrey-type spaces to Morrey-type spaces, Contemp. Math, 2007.

[12] V. I. Burenkov, V. S. Guliyev, Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces, Potential Anal, 2009.

[13] V. I. Burenkov, A. Gogatishvili, V. S. Guliyev, R. Ch. Mustafayev, Boundedness of the fractional maximal operator in local Morrey-type spaces. Complex Var. Elliptic Equ, 2010.

[14] V. I. Burenkov, V. S. Guliyev, A. Serbetci, T. V. Tararykova, Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces, Eurasian Math, 2010.

[15] V. I. Burenkov, A. Gogatishvili, V. S. Guliyev, R. Ch. Mustafayev, Boundedness of the fractional maximal operator in local Morrey-type spaces, Potential Anal, 67-87 (2011)

[16] W. Orlicz, Über konjugierte exponentenfolgen, Studia Mathematica, 1931.

[17] V. Kokilashvili, A. Meskhi, Maximal and Potential Operators in Variable Morrey Spaces Defined on Nondoubling Quasimetric Measure Spaces[J]. Bulletin of the Georgian National Academy of Sciences, 2008.

[18] T. L. Yee, K. L. Cheung, K. P. Ho, C. K. Suen, Local sharp maximal functions, geometrical maximal functions and rough maximal functions on local Morrey spaces with variable exponents. Math. Inequal. Appl, 2020.
[19] K. P. Ho, Calderón operator on local Morrey spaces with variable exponents, Mathematics, 2021.

[20] K. P. Ho, Singular integral operators and sublinear operators on Hardy local Morrey spaces with variable exponents, Bulletin des Sciences Mathématiques, 2021.

[21] A. Benedek, R. Panzone, The space $L^{\vec{p}}$, with mixed norm, Duke Mathematical Journal, 1961.

[22] T. Nogayama, Mixed Morrey spaces, Positivity ,2019.

[23] H. Zhang, J. Zhou, The Boundedness of Fractional Integral Operators in Local and Global Mixed Morrey-type Spaces, Positivity, 2021.

[24] M. W. Shi, J. Zhou, The Hardy-Littlewood maximal operator in Local and Global mixed Morrey-type Spaces, Submitted.

[25] M. W. Shi, J. Zhou, Decompositions of Local mixed Morrey-type spaces and Application, Submitted.

[26] M. Wei, Boundedness criterion for some integral operators on generalized mixed Morrey spaces and generalized mixed Hardy–Morrey spaces, Banach Journal of Mathematical Analysis, 2022.

[27] Y. Sawano, K. P. Ho, D. Yang, S. Yang, Hardy spaces for ball quasi-Banach function spaces, Dissertationes mathematicae, 2017.

[28] Y. Zhang, D. Yang, W. Yuan, S. Wang, Weak Hardy-type spaces associated with ball quasi-Banach function spaces I: Decompositions with applications to boundedness of Calderón-Zygmund operators, Science China Mathematics, 2021.

[29] S. Wang, D. Yang, W. Yuan, Y. Zhang Weak Hardy-type spaces associated with ball quasi-Banach function spaces II: Littlewood–Paley characterizations and real interpolation, The Journal of Geometric Analysis, 2021.

[30] K. P. Ho. Approximation in vanishing rearrangement-invariant Morrey spaces and applications, Revista de la Real Academia de Ciencias Exactas, 2019.

[31] K. P. Ho, Weak type estimates of singular integral operators on Morrey–Banach spaces, Integral Equations and Operator Theory, 2019.

[32] Y. Zhang, L. Huang, D. Yang, W. Yuan New ball Campanato-type function spaces and their applications, The Journal of Geometric Analysis, 2022.

[33] K. P. Ho, Boundedness of operators and inequalities on Morrey–Banach spaces, Publications of the Research Institute for Mathematical Sciences, 2022.

[34] S. Wang, J. Zhou, Another proof of the boundedness of Calderón–Zygmund singular integrals on generalized Orlicz spaces, Bulletin des Sciences Mathématiques, 2022.

[35] L. Huang, J. Liu, D. Yang, W. Yuan, Atomic and Littlewood–Paley characterizations of anisotropic mixed-norm Hardy spaces and their applications, The Journal of Geometric Analysis, 2019.
[36] F. Wang, D. Yang, S. Yang, Applications of Hardy spaces associated with ball quasi-Banach function spaces, Results in Mathematics, 2020.

[37] D. C. Chang, S. Wang, D. Yang, Y. Zhang, Littlewood-Paley characterizations of Hardy-type spaces associated with ball quasi-Banach function spaces, Complex Analysis and Operator Theory, 2020.

[38] X. Yan, D. Yang, W. Yuan, Intrinsic square function characterizations of Hardy spaces associated with ball quasi-Banach function spaces, Frontiers of Mathematics in China, 2020.

[39] K. P. Ho, Fourier-type transforms on rearrangement-invariant quasi-Banach function spaces, Glasgow Mathematical Journal, 2019.

[40] C. Bennett, R. Sharpley, Interpolation of Operators, Pure Appl. Math, 1988.

[41] J. E. Gilbert, Interpolation between weighted $L^p$-spaces[J]. Arkiv för Matematik, 1972.

[42] E. M. Stein, Harmonic Analysis, real-variable methods, orthogonality and oscillatory integrals, Princeton University Press, Princeton 1993.

[43] Y. Sawano, K. P. Ho, D. Yang, S. Yang, Hardy spaces for ball quasi-Banach function spaces, Dissertationes mathematicae, 2017.

[44] A. Calderón, Lebesgue spaces of differentiable functions.” Proc. Sympos, Pure Math, 1961.

[45] S.Z. Lu, D. Yang, H. Guoen, Herz type spaces and their applications, Beijing: Science Press, 2008.

[46] M. Q. Wei, A characterization of $CMO^q$ via the commutator of Hardy-type operators on mixed Herz spaces, Applicable Analysis, 2021.

[47] A. Almeida, D. Drihem, Maximal, potential and singular type operators on Herz spaces with variable exponents, Journal of Mathematical Analysis and Applications, 2012.

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