Effective Action for Self-Interacting Scalar Field in 3-dimensional Ball

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Abstract

In this paper we have considered the renormalized one-loop effective action for massless self-interacting scalar field in the 3-dimensional ball. The scalar field satisfies Dirichlet boundary condition on the ball. Using heat kernel expansion method we calculate the divergent part of effective action, then by bag model renormalization procedure we obtain the renormalized one-loop effective action.
1 Introduction

The problem of calculating the determinant of a Laplacian-like operator $A$ on a manifold $M$ is very important in mathematics [1] and also in physics [2, 3, 4]. In the cases where $A$ has a discrete spectrum the determinant of $A$ is generally divergent. The zeta function regularization is an appropriate way for these calculations. The zeta function method is a particular useful tool for the determination of effective action, where one-loop effective action is given by $\frac{1}{2} \ln \det A$. Using the relation between zeta function and heat-kernel for operator $A$, one can find the zeta function. Heat kernel coefficients play an important role in many areas of theoretical physics. In quantum field theory, heat kernel coefficients define the one-loop counter terms and quantum anomalies, as well as the large mass expansion of the effective action.

In this paper we would like to consider a self-interacting massless scalar field which satisfies the Dirichlet boundary condition on a 3-dimensional ball and calculate the relevant effective action.

The study of a massless scalar field with quartic self-interaction is very important in different subject of physics, for example in the Winberg-Salam model of weak interaction, fermions masses generation, in solid state physics [5, 6], inflationary models [7], solitons [8, 9] and Casimir effect [10, 11].

The outline of the paper is as follows: In section 2 after a brief review of heat kernel expansion we calculate the heat kernel coefficients for operator $A = \Box + V''(\hat{\phi})$ in 3-dimensional ball. In section 3 we obtain the divergent part of effective action, and introduce the classical part of effective action, then using the bag model renormalization procedure [12, 13] the renormalized one-loop effective action can be obtained. Section 4 is devoted to conclusions.

2 Zeta function and Heat-Kernel coefficients

Our aim is to derive the effective action of a self-interacting massless scalar field on a 3-dimensional ball which is given by

$$B^3 = \{ x \in \mathbb{R}^3 ; |x| \leq R \},$$

with boundary $S^2$, two dimensional sphere. The classical action is given by [14]

$$S(\phi) = - \int_{B^3} \left[ -1/2 \phi \Box \phi + V(\phi) \right] d^3x,$$

where $\Box$, is the D’Alembert operator of the 3-dimensional ball $B^3$ and $V(\phi)$ is a potential of self interacting scalar field. The above action has a minimum at $\phi = \phi$ which satisfies the classical equation of motion

$$- \Box \phi + V'(\phi) = 0.$$  \hspace{1cm} (3)

Quantum fluctuations $\phi = \phi - \phi$ around the classical background $\phi$ satisfy the following equation

$$A\phi = ( - \Box + V''(\phi))\phi = 0.$$  \hspace{1cm} (4)
The effective action in the one-loop approximation is as follow

\[ \Gamma^{(1)} = \frac{1}{2} \ln \text{det} A/\mu^2, \]  

(5)

where \( \mu \) is an arbitrary parameter with the dimensions of a mass necessary from dimensional consideration. In the zeta function regularization method the one-loop effective action (5) is given by

\[ \Gamma^{(1)} = -\frac{1}{2} \zeta'_{\Lambda/\mu^2}(0), \]  

(6)

where \( \zeta_{\Lambda}(s) \) is the zeta function related to the operator \( A \) and is defined by

\[ \zeta_{\Lambda}(s) = \sum_j \lambda_j^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \text{tr} K(x, x, t), \]  

(7)

where \( K(x, x', t) \) satisfies the heat-kernel equation

\[ \left( \frac{\partial}{\partial t} + A \right) K(t, x, x') = 0, \]  

(8)

with the initial condition

\[ K(0, x, x') = g(x)^{-1/2} \delta(x, x'). \]  

(9)

Here we impose Dirichlet boundary condition

\[ K(t, x, x')|_{x \in \mathbb{S}^2} = 0. \]  

(10)

The asymptotic expansion of the trace of the heat kernel is given by \[15\]

\[ T_{\text{rf}} K(t, x, x) = (4\pi t)^{-3/2} \text{tr} \left( \int_{B^3} dx \sqrt{g} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} (f a_k) + \int_{S^2} d\theta \sqrt{\gamma} \sum_{k=0}^{\infty} t^{k+1/2} c_{k+1} (f) \right) \]  

(11)

where \( f(x) \) is an arbitrary smooth function on the ball and \( \gamma = \text{det} \gamma_{ij} \), in which \( \gamma_{ij} \) is the metric on the boundary. The \( a_k \) and \( c_{k+1} \) are the heat kernel coefficients. The heat kernel coefficients \( a_k \) are independent of the applied boundary conditions. The \( c_{k+1} \) coefficients depend on the boundary conditions imposed. The several first and simplest \( a_k \) coefficients are given by \[15\]

\[ a_0 = 1, \]  

(12)

\[ a_1 = Q - \frac{1}{6} R, \]  

(13)

\[ a_2 = (Q - R/6)^2 - \frac{1}{3} \Box Q - \frac{1}{90} R_{\mu\nu} R^{\mu\nu} + \frac{1}{90} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \frac{1}{15} \Box R + \frac{1}{6} \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu}, \]  

(14)

where \( Q \) is a potential term, in our problem \( Q \) is given by

\[ Q = V''(\hat{\phi}). \]  

(15)

As one can see the \( a_k \) coefficients are functions of geometric quantities, \( R_{\mu\nu\alpha\beta}, R_{\mu\nu} \) and \( R \) are respectively, Rieman, Ricci and scalar curvature tensor.

\[ \tilde{R}_{\mu\nu} = [\nabla_\mu, \nabla_\nu], \]  

(16)
where $\nabla_\mu$ is covariant derivative. Several first boundary coefficients in asymptotic expansion for Dirichlet boundary condition are as follow \[15\]

\[ c_{1/2} = -\frac{\sqrt{\pi}}{2}, \quad (17) \]

\[ c_1 = \frac{1}{3} K - \frac{1}{2} f^{(1)}, \quad (18) \]

\[ c_{3/2} = \frac{\sqrt{\pi}}{2} ((\frac{-1}{6} \hat{R} - \frac{1}{4} R^0_{\mu\mu} + \frac{3}{32} K^2 - \frac{1}{16} K_{ij} K^{ij} + Q) + \frac{5}{16} K f^{(1)} - \frac{1}{4} f^{(2)}), \quad (19) \]

where

\[ f^{(1)}(z) = 1/6 + \frac{z^2}{6} (2 + \frac{z^2}{2} - z (z^2 + 6)) h(z), \quad (20) \]

\[ f^{(2)}(z) = -1/6 + \frac{z^2}{6} (-4 + \frac{z^2}{2} - 4 z^3 h(z)), \quad (21) \]

\[ h(z) = \int_0^\infty \exp(-x^2 + 2zx) \quad (22) \]

Here $\hat{R}$ is the scalar curvature of the boundary, $K$ is trace of extrinsic curvature tensor on the $S^2$,

\[ K_{ij} = \nabla_i N_j, \quad (23) \]

where $N_j$ is outward unit normal vector. Now we rewrite the Eq.(11) as follow

\[ tr K(t) = (4\pi t)^{-3/2} \sum_{k=0,1,2, \ldots} (\int_{B^3} dva_k + \int_{S^2} dsc_k) \exp(-t V''(\hat{\phi})) t^k, \quad (24) \]

The heat kernel coefficients of the Laplace operator on the 3-dimensional ball with Dirichlet boundary condition have been calculated by Bordag et al \[4\]. In \[4\] the scalar field is free, but in our problem, we have a self-interacting scalar field with

\[ V(\hat{\phi}) = \frac{\lambda}{4!} \hat{\phi}^4, \quad (25) \]

therefore we must calculate the heat kernel coefficients for the following operator

\[ A = -\square + V''(\hat{\phi}) = -\square + \frac{\lambda}{2} \hat{\phi}^2. \quad (26) \]

Now we introduce the following heat kernel coefficients for interacting case

\[ B_k = (\int_{B^3} dva_k + \int_{S^2} dsc_k) (V''(\hat{\phi}))^{3/2 - k}. \quad (27) \]

Using Eqs. (12-14), (17-19) and Eq.(27) one can find

\[ B_0 = \int_{B^3} dva_0 V''^3/2 = \int_{B^3} dv (\frac{\lambda}{2} \hat{\phi}^2)^{3/2}, \quad (28) \]

\[ B_{1/2} = \int_{S^2} dsc_{1/2} V'' = \frac{\sqrt{\pi}}{2} \int_{S^2} (\frac{\lambda}{2} \hat{\phi}^2) ds, \quad (29) \]
\[ B_1 = (\int_{B^3} dva + \int_{S^2} dsc_1) V^{m1/2} = \int_{B^3} (\frac{\lambda}{2} \phi^2)^{3/2} dv + \frac{2}{3R} \int_{S^2} (\frac{\lambda}{2} \phi^2)^{1/2} ds \] (30)

\[ B_{3/2} = \int_{S^2} dsc_{3/2} = -\frac{\pi^{3/2}}{6} + \frac{\sqrt{\pi}}{2} \int_{s^2} \phi^2 ds, \] (31)

\[ B_2 = (\int_{B^3} dva_2 + \int_{S^2} dsc_2) V^m - V^{1/2} = \int_{B^3} \left[ \frac{\lambda^2}{4} \phi^4 \right] - \frac{\lambda}{6} \Box \phi^2 (\lambda \phi^2)^{-1/2} dv + \int_{S^2} \frac{4}{315R^3} (\lambda \phi^2)^{-1/2} ds, \] (32)

Here \( R \) is radius of 3-ball. Now using Eq.(24), the zeta function, Eq.(7) has the form

\[ \zeta_A(s) = \frac{1}{(4\pi)^{3/2} \Gamma(s)} \sum_{k=0,1/2,1,...} \Gamma(s + k - 3/2) (\int_{B^3} a_k dv + \int_{S^2} c_k ds) V^{m3/2 - k} \] (33)

**3 The renormalize one-loop effective action**

Now we can find the one-loop effective action Eq.(6) for a space with odd dimension. The unrenormalized one-loop effective action is given by [14],

\[ \Gamma^{(1)} = \frac{-1}{(4\pi)^{3/2}} \sum_{k=0,1/2,1,...} \Gamma(k - 3/2) (\int_{B^3} a_k dv + \int_{S^2} c_k ds) V^{m3/2 - k} \] (34)

Using Eq.(27) we have

\[ \Gamma^{(1)} = \frac{-1}{(4\pi)^{3/2}} [\Gamma(-3/2) \int_{B^3} (\lambda \phi^2)^{3/2} dv - \Gamma(-1) \frac{\sqrt{\pi}}{2} \int_{s^2} (\lambda \phi^2)^{1/2} ds + \Gamma(1/2) \lambda \phi^2] + \frac{2}{3R} \int_{s^2} (\lambda \phi^2)^{1/2} ds \] (35)

The first five coefficients of heat kernel expansion contribute to divergences of the one-loop effective action. Calling these first five terms \( \Gamma^{(1)}_{\text{div}} \), we have

\[ \Gamma^{(1)}_{\text{div}} = \frac{-1}{(4\pi)^{3/2}} \Gamma(-3/2) \int_{B^3} (\lambda \phi^2)^{3/2} dv - \Gamma(-1) \frac{\sqrt{\pi}}{2} \int_{s^2} (\lambda \phi^2)^{1/2} ds \]

\[ + \Gamma(-1/2) \left[ \int_{B^3} (\frac{\lambda}{2} \phi^2)^{3/2} dv + \frac{2}{3R} \int_{s^2} (\frac{\lambda}{2} \phi^2)^{1/2} ds \right] + \Gamma(0) \left[ \frac{-\pi^{3/2}}{6} + \frac{\sqrt{\pi}}{2} \int_{s^2} \phi^2 ds \right] \]

\[ + \Gamma(1/2) \left[ \int_{B^3} \left( \frac{\lambda}{4} \phi^4 + \frac{\lambda}{6} \Box \phi^2 \right) (\lambda \phi^2)^{-1/2} dv + \int_{s^2} \frac{4}{315R^3} (\lambda \phi^2)^{-1/2} ds \right]. \] (36)

At this stage we recall that \( \Gamma^{(1)} \) is only one part of the total action. There is also a classical part. We can try to absorb \( \Gamma^{(1)}_{\text{div}} \) into the classical action. The classical action is

\[ \Gamma_{\text{class}} = PV + \sigma S + FR + K + \frac{h}{R}, \] (37)
where $V = \frac{4\pi R^3}{3}$ and $S = 4\pi R^2$ are, respectively, volume of $B^3$ and surface of $S^2$, and $P$ is pressure, $\sigma$ is surface tension and $F, K, h$ do not have special names. In order to obtain a well defined result for the total one-loop effective action, we have to renormalize the parameters of classical action according to below:

\begin{align*}
P & \rightarrow P + \frac{1}{(4\pi)^{3/2}} \Gamma(-3/2) \int_{B^3} \left( \frac{\lambda}{2} \phi^2 \right)^{3/2} dv. \tag{38} \\
\sigma & \rightarrow \sigma - \frac{1}{(4\pi)^{3/2}} \frac{1}{4\pi R^2} \Gamma(-1) \frac{\sqrt{\pi}}{2} \int_{S^2} \left( \frac{\lambda}{2} \phi^2 \right) ds. \tag{39} \\
F & \rightarrow F + \frac{1}{(4\pi)^{3/2}} \frac{\sqrt{\pi}}{2} \int_{B^3} \left( \frac{\lambda}{2} \phi^2 \right)^{3/2} dv + \frac{2}{3R} \int_{S^2} \left( \frac{\lambda}{2} \phi^2 \right)^{1/2} ds \tag{40} \\
K & \rightarrow K + \frac{1}{(4\pi)^{3/2}} \left[ \frac{-\pi^{3/2}}{6} + \frac{\sqrt{\pi}}{2} \int_{S^2} \left( \frac{\lambda}{2} \phi^2 \right) ds \right]. \tag{41} \\
h & \rightarrow h + \frac{1}{(4\pi)^{3/2}} R \left[ \int_{B^3} \left( \frac{\lambda^2}{4} \phi^4 - \frac{\lambda}{6} \phi^2 \right)^{1/2} dv + \int_{S^2} \frac{4}{315R^3} \left( \frac{\lambda}{2} \phi^2 \right)^{-1/2} ds \right]. \tag{42}
\end{align*}

Hence, the effect of the self-interacting scalar quantum field is to change, or renormalize the parameter of classical part of system. Using Eq.(37), the total action becomes

$$\Gamma^{\text{tot}} = \Gamma^{(1)} + \Gamma^{\text{class}},$$

where the parameter of $\Gamma^{\text{class}}$ are given by Eqs.(38-42). Once the terms named $\Gamma^{(1)}_{\text{div}}$ are removed from $\Gamma^{(1)}$, the remainder is finite and will be called the renormalized one-loop effective action

$$\Gamma^{(1)}_{\text{ren}} = \Gamma^{(1)} - \Gamma^{(1)}_{\text{div}} = \frac{-1}{(4\pi)^{3/2}} \sum_{k=5/2,3,\ldots} \Gamma(k - 3/2) \left( \int_{B^3} a_k dv + \int_{S^2} c_k ds \right) V^{n^{3/2-k}} \tag{44}$$

4 conclusion

In this paper we have considered the renormalized one-loop effective action for massless self-interacting scalar field in 3-dimensional ball. We assume the scalar field satisfies Dirichlet boundary condition on the ball. Unlike to the main part of previous studies on the scalar Casimir effect and one-loop effective action here we adopt the boundary condition problem with interacting quantum field. To obtain the divergent part of one-loop effective action we calculate heat kernel coefficients for operator $A = -\Box + V'(\hat{\phi})$. Previous result of heat kernel coefficients for Laplace operator on the 3-dimensional ball with Dirichlet boundary condition have been deformed for interacting case in our problem. The new result are given by Eqs.(28-32). The first five coefficients of heat kernel expansion contribute to divergences of the one-loop effective action. The renormalization procedure which is necessary to apply in this situation is similar to that of the bag model. At this stage we introduce the classical system and try to absorb divergent part into this classical action, therefore we renormalize the parameters of classical action. Extraction divergent part allows us to have the renormalize one-loop effective action.
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