**On relative separability in hypergraphs of models of theories**

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**Abstract**

In the paper, notions of relative separability for hypergraphs of models of a theory are defined. Properties of these notions and applications to ordered theories are studied: characterizations of relative separability both in a general case and for almost $\omega$-categorical quite o-minimal theories are established.

**Keywords:** hypergraph of models, elementary theory, separability, relative separability.

Hypergraphs of models of a theory are related to derivative objects allowing to obtain an essential structural information both on theories themselves and related semantical objects including graph ones [1, 2, 3, 4, 5, 6, 7, 8].

In the present paper, notions of relative separability for hypergraphs of models of a theory are defined. Properties of these notions and applications to ordered theories are studied: characterizations of relative separability both in a general case and for almost $\omega$-categorical quite o-minimal theories are established.

**1 Preliminaries**

Recall that a *hypergraph* is a pair of sets $(X, Y)$, where $Y$ is some subset of the Boolean $P(X)$ of the set $X$.

Let $\mathcal{M}$ be some model of a complete theory $T$. Following [5], we denote by $H(\mathcal{M})$ a family of all subsets $N$ of the universe $M$ of $\mathcal{M}$ that are universes of elementary submodels $\mathcal{N}$ of the model $\mathcal{M}$: $H(\mathcal{M}) = \{ N \mid \mathcal{N} \preceq \mathcal{M} \}$. The pair $(M, H(\mathcal{M}))$ is called the hypergraph of elementary submodels of the model $\mathcal{M}$ and denoted by $\mathcal{H}(\mathcal{M})$.

For a cardinality $\lambda$ by $H_\lambda(\mathcal{M})$ and $\mathcal{H}_\lambda(\mathcal{M})$ are denoted restrictions for $H(\mathcal{M})$ and $\mathcal{H}(\mathcal{M})$ respectively on the class of elementary submodels $\mathcal{N}$ of models $\mathcal{M}$ such that $|N| < \lambda$.

By $\mathcal{H}_p(\mathcal{M})$ we denote the restriction of the hypergraph $\mathcal{H}_{\omega_1}(\mathcal{M})$ on the class of elementary submodels $\mathcal{N}$ of the model $\mathcal{M}$ that are prime over finite sets. Similarly by $H_p(\mathcal{M})$, is denoted the corresponding restriction for $H_{\omega_1}(\mathcal{M})$.

**Definition 1.1** [5, 9]. Let $(X, Y)$ be a hypergraph, $x_1, x_2$ be distinct elements of $X$. We say that the element $x_1$ is *separated* or *separable* from the element $x_2$, or $T_0$-separable if there is $y \in Y$ such that $x_1 \in y$ and $x_2 \notin y$. The elements $x_1$ and $x_2$ are called *separable*, $T_2$-separable, or Hausdorff separable if there are disjoint $y_1, y_2 \in Y$ such that $x_1 \in y_1$ and $x_2 \in y_2$.

**Theorem 1.2** [5]. Let $\mathcal{M}$ be an $\omega$-saturated model of a countable complete theory $T$, $a$ and $b$ be elements of $\mathcal{M}$. The following are equivalent:

1. the element $a$ is separable from the element $b$ in $\mathcal{H}(\mathcal{M})$;
(2) the element $a$ is separable from the element $b$ in $H_{\omega_1}(M)$;
(3) $b \notin \text{acl}(a)$.

**Theorem 1.3** [5]. Let $M$ be an $\omega$-saturated model of a countable complete theory $T$, $a$ and $b$ be elements of $M$. The following are equivalent:

1. the elements $a$ and $b$ are separable in $H(M)$;
2. the elements $a$ and $b$ are separable in $H_{\omega_1}(M)$;
3. $\text{acl}(a) \cap \text{acl}(b) = \emptyset$.

**Corollary 1.4** [5]. Let $M$ be an $\omega$-saturated model of a countable complete theory $T$, $a$ and $b$ be elements of $M$, and there exists the prime model over $a$. The following are equivalent:

1. the element $a$ is separable from the element $b$ in $H(M)$;
2. the element $a$ is separable from the element $b$ in $H_{\omega_1}(M)$;
3. $\text{acl}(a) \cap \text{acl}(b) = \emptyset$.

**Corollary 1.5** [5]. Let $M$ be an $\omega$-saturated model of a countable complete theory $T$, $a$ and $b$ be elements of $M$, and there exist the prime models over $a$ and $b$ respectively. The following are equivalent:

1. the elements $a$ and $b$ are separable in $H(M)$;
2. the elements $a$ and $b$ are separable in $H_{\omega_1}(M)$;
3. the elements $a$ and $b$ are separable in $H_p(M)$;
4. $\text{acl}(a) \cap \text{acl}(b) = \emptyset$.

**Definition 1.6** [5]. Let $(X,Y)$ be a hypergraph, $X_1,X_2$ be disjoint nonempty subsets of the set $X$. We say that the set $X_1$ is separated or separable from the set $X_2$, or $T_0$-separable if there is $y \in Y$ such that $X_1 \subseteq y$ and $X_2 \cap y = \emptyset$. The sets $X_1$ and $X_2$ are called separable, $T_2$-separable, or Hausdorff separable if there are disjunct $y_1,y_2 \in Y$ such that $X_1 \subseteq y_1$ and $X_2 \subseteq y_2$.

By using proofs of theorems 1.2 and 1.3 the following generalizations of these theorems are established.

**Theorem 1.7** [5] Let $M$ be a $\lambda$-saturated model of a complete theory $T$, $\lambda \geq \max \{ |\Sigma(T)|, \omega \}$, $A$ and $B$ be nonempty sets in $M$ having the cardinalities $< \lambda$. The following are equivalent:

1. the set $A$ is separable from the set $B$ in $H(M)$;
2. the set $A$ is separable from the set $B$ in $H_{\lambda}(M)$;
3. $\text{acl}(A) \cap B = \emptyset$.

**Theorem 1.8** [5] Let $M$ be a $\lambda$-saturated model of a complete theory $T$, $\lambda \geq \max \{ |\Sigma(T)|, \omega \}$, $A$ and $B$ be nonempty sets in $M$ having the cardinalities $< \lambda$. The following are equivalent:

1. the sets $A$ and $B$ are separable in $H(M)$;
2. the sets $A$ and $B$ are separable in $H_{\lambda}(M)$;
3. $\text{acl}(A) \cap \text{acl}(B) = \emptyset$.

We obtain by analogy with corollaries 1.4 and 1.5
Corollary 1.9 [5]. Let $\mathcal{M}$ be an $\omega$-saturated model of a small theory $T$, $A$ and $B$ be finite nonempty sets in $\mathcal{M}$. The following are equivalent:

1. the set $A$ is separable from the set $B$ in $\mathcal{H}(\mathcal{M})$;
2. the set $A$ is separable from the set $B$ in $\mathcal{H}_{\omega_1}(\mathcal{M})$;
3. the set $A$ is separable from the set $B$ in $\mathcal{H}_p(\mathcal{M})$;
4. $\text{acl}(A) \cap B = \emptyset$.

Corollary 1.10 [5]. Let $\mathcal{M}$ be an $\omega$-saturated model of a small theory $T$, $A$ and $B$ be finite nonempty sets in $\mathcal{M}$. The following are equivalent:

1. the sets $A$ and $B$ are separable in $\mathcal{H}(\mathcal{M})$;
2. the sets $A$ and $B$ are separable in $\mathcal{H}_{\omega_1}(\mathcal{M})$;
3. the sets $A$ and $B$ are separable in $\mathcal{H}_p(\mathcal{M})$;
4. $\text{acl}(A) \cap \text{acl}(B) = \emptyset$.

The following proposition extends Theorem 1.8 with an additional criterion.

Proposition 1.11 Let $T$ be a theory, $\mathcal{M} \models T$, $\emptyset \neq A \subseteq M$, $\emptyset \neq B \subseteq M$, $\mathcal{M}$ be $|A \cup B|^+$-saturated. Then $A$ and $B$ are separable from each other in $\mathcal{H}(\mathcal{M})$ if and only if the following conditions hold:

1. $\text{acl}(A) \cap \text{acl}(B) = \emptyset$;
2. For any isolated type $p \in S_1(\emptyset)$, $p(\mathcal{M}) \setminus \text{acl}(A) \neq \emptyset$ and $p(\mathcal{M}) \setminus \text{acl}(B) \neq \emptyset$.

Proof of Proposition 1.11 If $A$ and $B$ are separable from each other in $\mathcal{H}(\mathcal{M})$ then by Theorem 1.8 we have $\text{acl}(A) \cap \text{acl}(B) = \emptyset$. If there is an isolated type $p \in S_1(\emptyset)$ such that $p(\mathcal{M}) \subseteq \text{acl}(A)$ then there is $\mathcal{M}_2 < \mathcal{M}$ with $B \subseteq M_2$ and $p(\mathcal{M}) \cap M_2 = \emptyset$, i.e. $p$ is not realized in $\mathcal{M}_2$. Similarly, $p(\mathcal{M}) \not\subseteq \text{acl}(A)$.

If the conditions (1), (2) hold then $A$ and $B$ are separable from each other in $\mathcal{H}(\mathcal{M})$ by Theorem 1.8. □

Recall that a subset $A$ of a linearly ordered structure $M$ is called convex if for any $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$. A weakly o-minimal structure is a linearly ordered structure $M = (M, =, <, \ldots)$ such that any definable (with parameters) subset of the structure $M$ is a union of finitely many convex sets in $M$.

In the following definitions $M$ is a weakly o-minimal structure, $A, B \subseteq M$, $M$ be $|A|^+$-saturated, $p, q \in S_1(A)$ be non-algebraic types.

Definition 1.12 [12] We say that $p$ is not weakly orthogonal to $q$ ($p \not\perp^w q$) if there exist an $A$-definable formula $H(x, y)$, $\alpha \in p(M)$ and $\beta_1, \beta_2 \in q(M)$ such that $\beta_1 \in H(M, \alpha)$ and $\beta_2 \not\in H(M, \alpha)$.

Definition 1.13 [13] We say that $p$ is not quite orthogonal to $q$ ($p \not\perp^q q$) if there exists an $A$-definable bijection $f : p(M) \to q(M)$. We say that a weakly o-minimal theory is quite o-minimal if the notions of weak and quite orthogonality of 1-types coincide.

In the work [14] the countable spectrum for quite o-minimal theories with non-maximal number of countable models has been described:

Theorem 1.14 Let $T$ be a quite o-minimal theory with non-maximal number of countable models. Then $T$ has exactly $3^k \cdot 6^s$ countable models, where $k$ and $s$ are natural numbers. Moreover, for any $k, s \in \omega$ there exists a quite o-minimal theory $T$ having exactly $3^k \cdot 6^s$ countable models.
Realizations of these theories with a finite number of countable models are natural generalizations of Ehrenfeucht examples obtained by expansions of dense linear orderings by a countable set of constants, and they are called theories of Ehrenfeucht type. Moreover, these realizations are representative examples for hypergraphs of prime models [11 5 5].

2 Relative separability in hypergraphs of models of theories

Observe that since by Theorem 1.8 and Corollary 1.10 separability of sets A and B in hypergraphs H(M) is possible only when acl(A) ∩ acl(B) = ∅, such a separability doesn’t hold when acl(∅) ≠ ∅. Thus, it is natural to consider the following notions of relative separability.

**Definition 2.1** Let (X, Y) be a hypergraph, x₁, x₂ be distinct elements of X, Z ⊆ X, x₂ ∉ Z. We say that the element x₁ is Z-separated or Z-separable from the element x₂, or (T₀, Z)-separable if there is y ∈ Y such that x₁ ∈ y ∪ Z and x₂ ∉ y. In this case the set y is called Z-separating x₁ from x₂. At the additional condition x₁ ∉ Z the elements x₁ and x₂ are called Z-separable, (T₂, Z)-separable, or Hausdorff Z-separable if there are y₁, y₂ ∈ Y such that (y₁ ∩ y₂) \ Z = ∅, x₁ ∈ y₁, and x₂ ∈ y₂.

Let X₁, X₂ be nonempty subsets of the set X, (X₁ ∩ X₂) \ Z = ∅, X₂ ⊆ Z. We say that the set X₁ is Z-separated or Z-separable from the set X₂, or (T₀, Z)-separable if there is y ∈ Y such that X₁ ⊆ y ∪ Z and (X₂ ∩ y) \ Z = ∅. At the additional condition X₁ ⊆ Z the sets X₁ and X₂ are called Z-separable, (T₂, Z)-separable, or Hausdorff Z-separable if there are y₁, y₂ ∈ Y such that (y₁ ∩ y₂) \ Z = ∅, X₁ ⊆ y₁ ∪ Z and X₂ ⊆ y₂ ∪ Z.

**Remark 2.2** 1. The notions of separability given in Section 1 correspond Z-separability for Z = ∅, X₁ ≠ ∅, X₂ ≠ ∅.
2. If X₂ ⊆ Z then the set X₂ also can be assumed Z-separable from X₁, although there is no reason to say on real separability of elements of the set X₂ from X₁.

For a tuple a and a set Z we denote by aZ the union of the set Z with the set of all elements containing in a.

The following theorem modifies Theorem 1.2 and it is a generalization of the theorem for acl(∅) = ∅.

**Theorem 2.3** Let M be an ω-saturated model of a countable complete theory T, Z be the algebraic closure of some finite set in M, a and b be elements of M, b ∉ Z. The following are equivalent:

1. the element a is Z-separable from the element b in H(M) by some set y from H(M) containing Z;
2. the element a is Z-separable from the element b in Hω₁(M) by some set y from Hω₁(M) containing Z;
3. b ∉ acl(aZ).

Proof. The implications (2) ⇒ (1) and (1) ⇒ (3) are obvious (clearly, if b ∈ acl(Z ∪ {a}) then b belongs to any model N ≺ M containing Z ∪ {a}).

To prove the implication (3) ⇒ (2) we need the following lemma.

**Lemma 2.4** Let a be a tuple, B be a finite set for which (acl(aZ) ∩ B) \ Z = ∅, and ϕ(x, a) be some consistent formula. Then there is an element c ∈ ϕ(M, a) such that (acl(acZ) ∩ B) \ Z = ∅.
Proof of Lemma 2.4. If $\varphi(M, \bar{a}) \cap Z \neq \emptyset$ then there is nothing to prove since as $c$ we can take an arbitrary element of $\varphi(M, \bar{a}) \cap Z$.

Suppose that $\varphi(M, \bar{a}) \cap Z = \emptyset$. By compactness and using consistent formulas $\varphi'(x, \bar{a})$ with the condition $\varphi'(x, \bar{a}) \vdash \varphi(x, \bar{a})$ instead of $\varphi(x, \bar{a})$, it is sufficiently to prove that for any $d \in B \setminus Z$ and the finite set of formulas $\psi_1(x, \bar{a}, y), \ldots, \psi_n(x, \bar{a}, y)$ with the condition

$$
\psi_i(x, \bar{a}, y) \vdash \varphi'(x, \bar{a}) \land \forall x \left( \varphi'(x, \bar{a}) \rightarrow \exists^{=k_i} y \psi_i(x, \bar{a}, y) \right)
$$

for some natural $k_i$, $i = 1, \ldots, n$, there is an element $c \in \varphi'(M, \bar{a})$ such that

$$
\models \bigwedge_{i=1}^n \neg \psi_i(c, \bar{a}, d).
$$

Assume the contrary: for any $c \in \varphi'(M, \bar{a})$ there is $i$ such that $\models \psi_i(c, \bar{a}, d)$. Then the formula $\chi(x, \bar{a}, y) \equiv \bigvee_{i=1}^n \psi_i(x, \bar{a}, y)$ satisfies the following condition: for any $c \in \varphi'(M, \bar{a})$, $\models \chi(c, \bar{a}, d)$ and $\chi(c, \bar{a}, y)$ has finitely many, no more than $m = k_1 + \ldots + k_n$ solutions. Consequently, the formula

$$
\theta(\bar{a}, y) \equiv \exists x(\chi(x, \bar{a}, y) \land \forall z((\varphi'(z, \bar{a}) \rightarrow (\chi(x, \bar{a}, y) \land \chi(z, \bar{a}, y))))
$$

satisfies $d$ and has no more than $m$ solutions. This fact contradicts the condition $d \notin acl(\bar{a}Z)$.

\[\Box\]

Assuming that $b \notin acl(aZ)$, we construct by induction a countable model $N \preceq M$ such that $acl(aZ) \subset N$, $b \notin N$, and $N = \bigcup_{n \in \omega} A_n$ for a chain of some sets $A_n$.

In the initial step we consider the set $A_0 = acl(aZ)$ and renumber all consistent formulas of the form $\varphi(x, \bar{a})$, $\bar{a} \in A_0$: $\Phi_0 = \{\varphi_0, m(x, a_m) \mid m \in \omega\}$. According this numeration construct at most a countable set $A_1 = \bigcup_{m \in \omega \cup \{1\}} A_{1,m} \supset A_0$ with the condition $b \notin acl(A_1)$.

Let $A_{1,-1} = A_0$ and $A_{1,0} = \varphi_0, n(x, a_n)$. If the set $A_{1,m+1}$ had already been already defined and $\varphi_0, n(m, a_n) \cap A_{1,m+1} \neq \emptyset$ then we put $A_{1,m} = A_{1,m+1}$; if $\varphi_0, n(m, a_n) \cap A_{1,m-1} = \emptyset$ choose by Lemma 2.4 an element $c_m \in \varphi_0, n(M, a_n)$ such that $b \notin acl(c_m A_{1,m-1})$ and put $A_{1,0} = acl(c_m A_{1,m-1})$.

If at most a countable set $A_n$ had been already constructed, renumber all consistent formulas of the form $\varphi(x, \bar{a})$, $\bar{a} \in A_n$: $\Phi_n = \{\varphi_n, m(x, a_m) \mid m \in \omega\}$. According this numeration construct at most a countable set $A_{n+1} = \bigcup_{m \in \omega \cup \{1\}} A_{n+1,m} \supset A_n$ with the condition $b \notin acl(A_{n+1})$. Let $A_{n+1,0} = A_n$. If the set $A_{n+1,m-1}$ had already been already defined and $\varphi_n, m(m, a_n) \cap A_{n+1,m-1} \neq \emptyset$ then put $A_{n+1,m} = A_{n+1,m-1}$; if $\varphi_n, m(M, a_n) \cap A_{n+1,m-1} = \emptyset$, choose by Lemma 2.4 an element $c_m \in \varphi_n, m(M, a_n)$ such that $b \notin acl(c_m A_{n+1,m-1})$ and put $A_{n+1,0} = acl(c_m A_{n+1,m-1})$.

By constructing the set $\bigcup_{n \in \omega} A_n$ forms a required universe $N$ of a countable model $N \preceq M$ such that $acl(Z \cup \{a\}) \subset N$ and $b \notin N$.

Applying Lemma 2.4 we obtain the following lemma.

**Lemma 2.5** Let $M$ be an $\omega$-saturated model of a complete theory $T$, $\bar{a}, \bar{b} \in M$, $Z$ be the algebraic closure of some finite set in $M$. If $(acl(\bar{a}Z) \cap acl(\bar{b}Z)) \setminus Z = \emptyset$ and $\varphi(x, \bar{b})$ is a consistent formula, $\bar{a}' \in \bar{a}Z$, then there is $c \in \varphi(M, \bar{a}')$ such that $(acl(cZ) \cap acl(\bar{b}Z)) \setminus Z = \emptyset$. 

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Theorem 2.6 Let $\mathcal{M}$ be an $\omega$-saturated model of a countable complete theory $T$, $Z$ be the algebraic closure of some finite set in $\mathcal{M}$, $a$ and $b$ be elements of $\mathcal{M}$, $a, b \notin Z$. The following are equivalent:

1. the elements $a$ and $b$ are $Z$-separable in $H(\mathcal{M})$ by some sets $y$ and $z$ from $H(\mathcal{M})$ containing $Z$;
2. the elements $a$ and $b$ are $Z$-separable in $H_{\omega_1}(\mathcal{M})$ by some sets $y$ and $z$ from $H_{\omega_1}(\mathcal{M})$ containing $Z$;
3. $(\text{acl}(aZ) \cap \text{acl}(bZ)) \setminus Z = \emptyset$.

Proof. As in the proof of Theorem 2.3 it is sufficiently to prove the implication (3) $\Rightarrow$ (2). Assuming $(\text{acl}(aZ) \cap \text{acl}(bZ)) \setminus Z = \emptyset$, we construct by induction countable models $N_a, N_b \preceq \mathcal{M}$ such that $\text{acl}(aZ) \subseteq N_a$, $\text{acl}(bZ) \subseteq N_a \cap N_b \setminus Z = \emptyset$, $N_a = \bigcup_{n \in \omega} A_n$ for a chain of some sets $A_n$ and $N_b = \bigcup_{n \in \omega} B_n$ for a chain of some sets $B_n$.

In the initial step we consider the sets $A_0 = \text{acl}(aZ)$, $B_0 = \text{acl}(bZ)$ and enumerate all consistent formulas of the form $\varphi(x, \bar{a})$, $\bar{a} \in A_0$: $\Phi_0 = \{\varphi_{0,m}(x, \bar{a}_m) \mid m \in \omega\}$. According to this enumeration we construct at most countable set $A_1 = \bigcup_{m \in \omega \cup \{-1\}} A_{1,m} \supseteq A_0$ with the condition $(\text{acl}(A_1 \cap B_0)) \setminus Z = \emptyset$. Let $A_{1,-1} \models A_0$. If the set $A_{1,m-1}$ had been already defined and $\varphi_{0,m}(\mathcal{M}, \bar{a}_m) \cap A_{1,m-1} \neq \emptyset$, then put $A_{1,m} \models A_{1,m-1}$; if $\varphi_{0,m}(\mathcal{M}, \bar{a}_m) \cap A_{1,m-1} = \emptyset$ then by Lemma 2.5 choose an element $c_m \in \varphi_{m}(\mathcal{M}, \bar{a}_m)$ such that $(\text{acl}(c_m A_{1,m-1}) \cap \text{acl}(B_0)) \setminus Z = \emptyset$ and put $A_{1,m} = \text{acl}(c_m A_{1,m-1})$.

If the set $A_1$ had been already defined, extend symmetrically the set $B_0$ to an algebraically closed set $B_1$ such that $B_1 \supseteq Z$, all consistent formulas $\varphi(x, \bar{b})$, $\bar{b} \in B_0$, are realized in $B_1$ $(\text{acl}(A_1) \cap \text{acl}(B_1)) \setminus Z = \emptyset$.

If at most countable sets $A_n$ and $B_n$ had been already constructed, renumber all consistent formulas of the form $\varphi(x, \bar{a})$, $\bar{a} \in A_n$: $\Phi_n = \{\varphi_{n,m}(x, \bar{a}_m) \mid m \in \omega\}$. According to this enumeration construct at most a countable set $A_{n+1} = \bigcup_{m \in \omega \cup \{-1\}} A_{n+1,m} \supseteq A_n$ with the condition $(\text{acl}(A_{n+1}) \cap \text{acl}(B_1)) \setminus Z = \emptyset$. Let $A_{n+1,-1} \models A_n$. If the set $A_{n+1,m-1}$ had been already defined and $\varphi_{0,m}(\mathcal{M}, \bar{a}_m) \cap A_{n+1,m-1} \neq \emptyset$, then put $A_{n+1,m} \models A_{n+1,m-1}$; if $\varphi_{0,m}(\mathcal{M}, \bar{a}_m) \cap A_{n+1,m-1} = \emptyset$ then by Lemma 2.5 choose an element $c_m \in \varphi_{n,m}(\mathcal{M}, \bar{a}_m)$ such that $(\text{acl}(c_m A_{n+1,m-1}) \cap \text{acl}(B_1)) \setminus Z = \emptyset$, and put $A_{n+1,m} = A_{n+1,m-1} \cup \{c_m\}$.

If we have the set $A_{n+1}$ then extend symmetrically the set $B_n$ to at most a countable set $B_{n+1}$ such that all consistent formulas $\varphi(x, \bar{b})$, $\bar{b} \in B_n$, are realized in $B_{n+1}$ $(\text{acl}(A_{n+1}) \cap \text{acl}(B_{n+1})) \setminus Z = \emptyset$.

By constructing the sets $\bigcup_{n \in \omega} A_n$ and $\bigcup_{n \in \omega} B_n$ form required universes $N_a$ and $N_b$ respectively of $Z$-separable countable models $N_a, N_b \preceq \mathcal{M}$ such that $a \in N_a$ and $b \in N_b$.

Combining proofs of Claims 1.4.1.10 and Theorems 2.3, 2.6, we obtain the following assertions.

Corollary 2.7 Let $\mathcal{M}$ be an $\omega$-saturated model of a small theory $T$, $Z$ be the algebraic closure of some finite set in $\mathcal{M}$, $a$ and $b$ be elements of $\mathcal{M}$, $a, b \notin Z$. The following are equivalent:

1. the element $a$ is $Z$-separable from the element $b$ in $H(\mathcal{M})$ by some set $y$ from $H(\mathcal{M})$ containing $Z$;
2. the element $a$ is $Z$-separable from the element $b$ in $H_{\omega_1}(\mathcal{M})$ by some set $y$ from $H_{\omega_1}(\mathcal{M})$ containing $Z$;
(3) the element \(a\) is \(Z\)-separable from the element \(b\) in \(H_p(M)\) by some set \(y\) from \(H_p(M)\) containing \(Z\);
(4) \(b \notin \text{acl}(aZ)\).

**Corollary 2.8** Let \(M\) be an \(\omega\)-saturated model of a small theory \(T\), \(Z\) be the algebraic closure of some finite set in \(M\), \(a\) and \(b\) be elements of \(M\), \(a, b \notin Z\). The following are equivalent:

1. the elements \(a\) and \(b\) are \(Z\)-separable in \(H(M)\) by some sets \(y\) and \(z\) from \(H(M)\) containing \(Z\);
2. the elements \(a\) and \(b\) are \(Z\)-separable in \(H_{\omega_1}(M)\) by some sets \(y\) and \(z\) from \(H_{\omega_1}(M)\) containing \(Z\);
3. the elements \(a\) and \(b\) are separable in \(H_p(M)\) by some sets \(y\) and \(z\) from \(H_p(M)\) containing \(Z\);
4. \((\text{acl}(aZ) \cap \text{acl}(bZ)) \setminus Z = \emptyset\).

**Theorem 2.9** Let \(M\) be a \(\lambda\)-saturated model of a complete theory \(T\), \(\lambda \geq \max\{\vert \Sigma(T)\vert, \omega\}\), \(A\) and \(B\) be nonempty sets in \(M\) having cardinalities \(< \lambda\), \(Z\) be the algebraic closure of some set of cardinality \(< \lambda\) in \(M\). The following are equivalent:

1. the set \(A\) is \(Z\)-separable from the set \(B\) in \(H(M)\) by some set \(y\) from \(H(M)\) containing \(Z\);
2. the set \(A\) is \(Z\)-separable from the set \(B\) in \(H_\lambda(M)\) by some set \(y\) from \(H_\lambda(M)\) containing \(Z\);
3. \((\text{acl}(A \cup Z) \cap B) \setminus Z = \emptyset\).

**Theorem 2.10** Let \(M\) be a \(\lambda\)-saturated model of a complete theory \(T\), \(\lambda \geq \max\{\vert \Sigma(T)\vert, \omega\}\), \(A\) and \(B\) be nonempty sets in \(M\) having cardinalities \(< \lambda\), \(Z\) be the algebraic closure of some set of cardinality \(< \lambda\) in \(M\). The following are equivalent:

1. the sets \(A\) and \(B\) are \(Z\)-separable in \(H(M)\) by some sets \(y\) and \(z\) from \(H(M)\) containing \(Z\);
2. the sets \(A\) and \(B\) are \(Z\)-separable in \(H_\lambda(M)\) by some sets \(y\) and \(z\) from \(H_\lambda(M)\) containing \(Z\);
3. \((\text{acl}(A \cup Z) \cap \text{acl}(B \cup Z)) \setminus Z = \emptyset\).

**Corollary 2.11** Let \(M\) be an \(\omega\)-saturated model of a small theory \(T\), \(A\) and \(B\) be finite nonempty sets in \(M\), \(Z\) be the algebraic closure of some finite set in \(M\). The following are equivalent:

1. the set \(A\) is \(Z\)-separable from the set \(B\) in \(H(M)\) by some set \(y\) from \(H(M)\) containing \(Z\);
2. the set \(A\) is \(Z\)-separable from the set \(B\) in \(H_{\omega_1}(M)\) by some set \(y\) from \(H_{\omega_1}(M)\) containing \(Z\);
3. the set \(A\) is \(Z\)-separable from the set \(B\) in \(H_p(M)\) by some set \(y\) from \(H_p(M)\) containing \(Z\);
4. \((\text{acl}(A \cup Z) \cap B) \setminus Z = \emptyset\).

**Corollary 2.12** Let \(M\) be an \(\omega\)-saturated model of a small theory \(T\), \(A\) and \(B\) be finite nonempty sets in \(M\), \(Z\) be the algebraic closure of some finite set in \(M\). The following are equivalent:

1. the sets \(A\) and \(B\) are \(Z\)-separable in \(H(M)\) by some sets \(y\) and \(z\) from \(H(M)\) containing \(Z\);
(2) the sets $A$ and $B$ are $Z$-separable in $H_{\omega_1}(\mathcal{M})$ by some sets $y$ and $z$ from $H_{\omega_1}(\mathcal{M})$ containing $Z$;
(3) the sets $A$ and $B$ are $Z$-separable in $H_p(\mathcal{M})$ by some sets $y$ and $z$ from $H_p(\mathcal{M})$ containing $Z$;
(4) $(\text{acl}(A \cup Z) \cap \text{acl}(B \cup Z)) \setminus Z = \emptyset$.

3 On separability in hypergraphs of models of ordered theories

Definition 3.1 Let $p_1(x_1), \ldots, p_n(x_n) \in S_1(T)$. A type $q(x_1, \ldots, x_n) \in S(T)$ is called $(p_1, \ldots, p_n)$-type if $q(x_1, \ldots, x_n) \supseteq \bigcup_{i=1}^n p_i(x_i)$. The set of all $(p_1, \ldots, p_n)$-types of a theory $T$ is denoted by $S_{p_1,\ldots,p_n}(T)$. A countable theory $T$ is called almost $\omega$-categorical if for any types $p_1(x_1), \ldots, p_n(x_n) \in S(T)$ there exist only finitely many types $q(x_1, \ldots, x_n) \in S_{p_1,\ldots,p_n}(T)$.

Theorem 3.2 Let $T$ be an almost $\omega$-categorical quite $\omega$-minimal theory, $\mathcal{M}$ be an $\omega$-saturated model of the theory $T$, $Z$ be the algebraic closure of some finite set in $\mathcal{M}$, $a, b \in M \setminus Z$. Then the following conditions are equivalent:

1. $a$ is $Z$-separable from $b$ in $\mathcal{H}(\mathcal{M})$ by some set $y$ from $H(\mathcal{M})$ containing $Z$;
2. $b$ is $Z$-separable from $a$ in $\mathcal{H}(\mathcal{M})$ by some set $y$ from $H(\mathcal{M})$ containing $Z$;
3. the elements $a$ and $b$ are $Z$-separable in $H(\mathcal{M})$ by some sets $y$ and $z$ from $H(\mathcal{M})$ containing $Z$;
4. $a \notin \text{dcl}(\{bZ\})$;
5. $b \notin \text{dcl}(\{aZ\})$;
6. $(\text{dcl}(aZ) \cap \text{dcl}(bZ)) \setminus Z = \emptyset$.

Proof of Theorem 3.2 By Proposition 3.9 Exchange Principle for algebraic closure holds. By linear ordering of the model $\mathcal{M}$ $\text{dcl}(A) = \text{acl}(A)$ for any $A \subseteq M$. Then by proofs of Theorems 2.3 and 2.6 we have an equivalence of the conditions (1)–(6). □

Remark 3.3 1. Theorem 3.2 remains true for an arbitrary theory satisfying both Exchange Principle for algebraic closure and the condition $\text{dcl}(A) = \text{acl}(A)$ for any $A \subseteq M$.
2. If Exchange Principle for algebraic closure holds and the condition $\text{dcl}(A) = \text{acl}(A)$ for any $A \subseteq M$ doesn’t hold, Theorem 3.2 remains true if we replace $\text{dcl}$ by $\text{acl}$.
3. If the condition $\text{dcl}(A) = \text{acl}(A)$ for any $A \subseteq M$ holds and Exchange Principle for algebraic closure doesn’t hold, Theorem 3.2 splits into three independent statements (1) $\iff$ (5), (2) $\iff$ (4), (3) $\iff$ (6).

Theorem 3.2 immediately implies the following

Corollary 3.4 Let $T$ be an almost $\omega$-categorical quite $\omega$-minimal theory, $\mathcal{M}$ be an $\omega$-saturated model of the theory $T$, $a, b \in M \setminus \text{dcl}(\emptyset)$. Then the following conditions are equivalent:

1. $a$ is separable from $b$ in $\mathcal{H}(\mathcal{M})$;
2. $b$ is separable from $a$ in $\mathcal{H}(\mathcal{M})$;
3. $a \notin \text{dcl}(\{b\})$;
4. $b \notin \text{dcl}(\{a\})$. 

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Example 3.5 [11] Let $\mathcal{M} = \langle M; <, P_1^1, P_2^1, f^1 \rangle$ be a linearly ordered structure such that $M$ is the disjoint union of interpretations of unary predicates $P_1$ and $P_2$ so that $P_1(M) < P_2(M)$. We identify an interpretation of $P_2$ with the set of rational numbers $\mathbb{Q}$, ordered as usual, and $P_1$ with $\mathbb{Q} \times \mathbb{Q}$, ordered lexicographically. The symbol $f$ is interpreted by a partial unary function with $Dom(f) = P_1(M)$ and $Range(f) = P_2(M)$ and is defined by the equality $f((n, m)) = n$ for all $(n, m) \in \mathbb{Q} \times \mathbb{Q}$.

It is known that $\mathcal{M}$ is a countably categorical weakly o-minimal structure, and $Th(\mathcal{M})$ is not quite o-minimal. Take arbitrary $a \in P_1(M)$, $b \in P_2(M)$ such that $f(a) = b$. Then we obtain that $a$ is separable from $b$ in $\mathcal{H}(\mathcal{M})$, but $b$ is not separable from $a$ in $\mathcal{H}(\mathcal{M})$.

Proposition 3.6 Let $T$ be an almost $\omega$-categorical quite o-minimal theory, $\mathcal{M} \models T$, $A = \{a_1, \ldots, a_{n_1}\}$, $B = \{b_1, \ldots, b_{n_2}\} \subseteq M$ for some positive $n_1, n_2 < \omega$. Then the following conditions are equivalent:

1. $A$ and $B$ are separable from each other in $\mathcal{H}(\mathcal{M})$;
2. $dcl(A) \cap dcl(B) = \emptyset$.
3. $dcl\{a_i\} \cap dcl\{b_j\} = \emptyset$ for any $1 \leq i \leq n_1$, $1 \leq j \leq n_2$.

Proof of Proposition 3.6 (1) $\Rightarrow$ (2) Let $A$ be separable from $B$ in $\mathcal{H}(\mathcal{M})$. This means that there is $\mathcal{M}_1 < \mathcal{M}$ such that $A \subseteq M_1$ and $B \cap M_1 = \emptyset$. Then we have: $dcl(A) \subseteq M_1$, whence we obtain that $dcl(A) \cap B = \emptyset$. Similarly, by the condition of separability of $B$ from $A$ in $\mathcal{H}(\mathcal{M})$ it can be established that $dcl(B) \cap A = \emptyset$.

Assume the contrary: $dcl(A) \cap dcl(B) \neq \emptyset$. Consequently, there is $c \in M$ such that $c \in dcl(A)$ and $c \in dcl(B)$. But then by binarity of $Th(\mathcal{M})$ there exist $a \in A$ and $b \in B$ such that $c \in dcl\{a\}$ and $c \in dcl\{b\}$. By holding Exchange Principle for algebraic closure we obtain that $b \in dcl\{a\}$. The last contradicts the condition $dcl(A) \cap B = \emptyset$. (2) $\Rightarrow$ (1) In this case we assert that $M_1 := M \setminus dcl(A)$ and $M_2 := M \setminus dcl(B)$ are universes of elementary submodels of the model $\mathcal{M}$.

(2) $\Leftrightarrow$ (3) By binarity of $Th(\mathcal{M})$. 

Proposition 3.7 Let $T$ be an almost $\omega$-categorical quite o-minimal theory, $\mathcal{M} \models T$, $Z = dcl(\emptyset)$, $A = \{a_1, \ldots, a_{n_1}\}$, $B = \{b_1, \ldots, b_{n_2}\} \subseteq M$ for some positive $n_1, n_2 < \omega$ so that $A \cap Z = B \cap Z = \emptyset$. Then the following conditions are equivalent:

1. $A$ and $B$ are $Z$-separable in $\mathcal{H}(\mathcal{M})$;
2. $dcl(A) \cap dcl(B) = Z$.
3. $dcl\{a_i\} \cap dcl\{b_j\} = Z$ for any $1 \leq i \leq n_1$, $1 \leq j \leq n_2$.

Proof of Proposition 3.7 (1) $\Rightarrow$ (2) Let $A$ and $B$ be $Z$-separable in $\mathcal{H}(\mathcal{M})$. Then there exist $\mathcal{M}_1, \mathcal{M}_2 < \mathcal{M}$ such that $(\mathcal{M}_1 \cap \mathcal{M}_2) \setminus Z = \emptyset$, $A \subseteq M_1$ and $B \subseteq M_2$. Consequently, $dcl(A) \cap dcl(B) \subseteq \mathcal{M}_1 \cap \mathcal{M}_2$. Then $[dcl(A) \cap dcl(B)] \setminus Z = \emptyset$, whence $dcl(A) \cap dcl(B) = Z$.

(2) $\Rightarrow$ (1) In this case we assert that $M_1 := [M \setminus dcl(A)] \cup Z$ and $M_2 := [M \setminus dcl(B)] \cup Z$ are universes of elementary submodels of the model $\mathcal{M}$.

Arguments for Propositions 1.11 and 3.6 imply the following

Proposition 3.8 Let $T$ be an almost $\omega$-categorical quite o-minimal theory, $\mathcal{M} \models T$, $\emptyset \neq A, B \subseteq M$, $\mathcal{M}$ be $|A \cup B|^+$-saturated. Then $A$ and $B$ are separable from each other in $\mathcal{H}(\mathcal{M})$ if and only if the following conditions hold:

1. $dcl\{a\} \cap dcl\{b\} = \emptyset$ for any $a \in A$ and $b \in B$;
2. For any isolated type $p \in S_1(\emptyset)$, $p(\mathcal{M}) \setminus dcl(A) \neq \emptyset$ and $p(\mathcal{M}) \setminus dcl(B) \neq \emptyset$. 

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Corollary 3.9 Let $T$ be an almost $\omega$-categorical quite o-minimal theory, $M \models T$, $Z = \text{dcl}(\emptyset)$, $A, B$ be non-empty subsets of $M$ such that $A \cap Z = B \cap Z = \emptyset$, $M$ be $|A \cup B|^{+}$-saturated. Then $A$ and $B$ are $Z$-separable in $H(M)$ if and only if the following conditions hold:

1. $\text{dcl}(\{a\}) \cap \text{dcl}(\{b\}) = Z$ for any $a \in A$ and $b \in B$;
2. For any isolated type $p \in S_1(\emptyset)$, $p(M) \setminus \text{dcl}(A) \neq \emptyset$ and $p(M) \setminus \text{dcl}(B) \neq \emptyset$.

Arguments for Propositions 1.11 and 3.6 as well as Theorem 2.10 imply the following

Proposition 3.10 Let $T$ be an almost $\omega$-categorical quite o-minimal theory, $M \models T$ be $\lambda$-saturated, $\lambda \geq \max\{|\Sigma(T)|, \omega\}$, $A$ and $B$ be nonempty sets in $M$ having cardinalities $< \lambda$, $Z$ be the algebraic closure of some set of cardinality $< \lambda$ in $M$. Then the following are equivalent:

1. $A$ and $B$ are $Z$-separable in $H(M)$;
2. $(\text{dcl}(aZ) \cap \text{dcl}(bZ)) \setminus Z = \emptyset$ for any $a \in A$ and $b \in B$.

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