Directed Deza graphs

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Abstract

Deza graphs were introduced in 1999 by Erickson, Fernando, Haemers, Hardy and Hemmeter, as a generalization of strongly regular graphs. Further, Duval in 1988 introduced directed strongly regular graphs as directed graph version of strongly regular graphs. In this paper we introduce Deza digraphs as directed graph version of Deza graphs, as well as a generalization of directed strongly regular graphs. We establish some properties of Deza digraphs and give several constructions. Further, we introduce twin and Siamese twin Deza graphs and digraphs and construct several example. Also, we classify Deza digraphs with parameters \((n, k, b, a)\) having the property that \(b = t\), where \(t\) is the number of vertices adjacent to a vertex of the digraph.

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1 Introduction

Deza graphs were introduced in [4], as a generalization of strongly regular graphs. Further, Duval in [3] introduced directed strongly regular graphs as directed graph version of strongly regular graphs. For recent results on Deza graphs we refer the readers to [5, 6, 8]. In this paper we introduce Deza digraphs as directed graph version of Deza graphs, as well as a generalization of directed strongly regular graphs.

Let $D = (V, A)$ be a directed graph, where $V$ is the set of vertices and $A$ be the set of arcs. For $u, v \in V$ we will write $u \rightarrow v$ if there is an arc (directed edge) from $u$ to $v$, and we will say that $u$ dominates $v$ or that $v$ is dominated by $u$. We will also write $u \sim v$ if $u \rightarrow v$ and $v \rightarrow u$. In this case, we will count these as one undirected edge, and say that $u$ is adjacent to $v$. A digraph $D$ is called regular of degree $k$ if each vertex of $D$ dominates exactly $k$ vertices and is dominated by exactly $k$ vertices. The digraphs that we use will not have more than one arc from one vertex to another, and will not have any arcs from a vertex to itself.

A digraph $D$ on $n$ vertices is characterized by the $(n \times n)$ $(0, 1)$-matrix $M = [m_{i,j}]$, where $m_{ij} = 1$ if and only if $i \rightarrow j$ (or $i \sim j$), called the adjacency matrix of $D$. If the adjacency matrix $M$ of a digraph $D$ has the property that $M + M^t$ is a $(0, 1)$-matrix, the $D$ is called asymmetric. If $M$ satisfies the conditions

$$MJ_n = J_nM = kJ_n,$$

$$M^2 = tI_n + \lambda M + \mu(J_n - I_n - M).$$

where $I_n$ is the identity matrix of order $n$, and $J_n$ is the $n \times n$ matrix of all 1s, then $D$ is called a directed strongly regular graph (DSRG) with parameters $(n, k, \lambda, \mu)$.

The rest of the paper is organized as follows. In Section 2 we introduce the notion of directed Deza graphs and give some basic properties, and in Section 3 we give constructions of Deza digraphs. Further we introduce twin and Siamese twin Deza (reflexive) graphs and digraphs and construct several example. Also, we classify Deza digraphs with parameters $(n, k, b, a)$ having the property that $b = t$, where $t$ is the number of vertices adjacent to a vertex of the Deza digraph.
2 Deza digraphs

In this section we introduce the notion of directed Deza graphs and give some basic properties.

**Definition 2.1.** Let \( n, k, b, \) and \( a \) be integers such that \( 0 \leq a \leq b \leq k \leq n \) and \( 0 \leq t \leq k \).

A digraph \( D = (V, A) \) is a directed \((n, k, b, a)\)-Deza graph if

1. \( |V| = n \);

2. Every vertex has in-degree and out-degree \( k \), and is adjacent to \( t \) vertices.

3. Let \( u \) and \( v \) be distinct vertices. The number of vertices \( w \) such that \( u \to w \to v \) is \( a \) or \( b \).

**Example 2.2.** Let us define the matrices \( M_1 \) and \( M_2 \) as follows

\[
M_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0
\end{bmatrix}.
\]

The matrix \( M_1 \) is the adjacency matrix of a directed \((8,3,3,1)\)-Deza graph \( D_1 \). The digraph \( D_1 \) is asymmetric, so \( t = 0 \). The matrix \( M_2 \) is the adjacency matrix of a directed \((8,4,3,1)\)-Deza graph \( D_2 \) with \( t = 1 \).

Let \( M \) be the adjacency matrix of a directed graph \( D \) on \( n \) vertices. Then \( D \) is a directed \((n, k, b, a)\)-Deza graph if and only if \( M^2 = aX + bY + tI_n \) for some \((0,1)\)-matrices \( X \) and \( Y \) such that \( X + Y + I_n = J_n \). Note that \( G \) is a directed strongly regular graph if and only if \( X \) or \( Y \) is \( M \).

Suppose that we have a directed Deza graph with \( M \), \( X \), and \( Y \) as above. Then \( X \) and \( Y \) are adjacency matrices of digraphs. We will denote these graphs by \( D_X \) and \( D_Y \), and
call them the Deza children of $D$. Deza children are sometimes directed Deza graphs, and sometimes even directed strongly regular graphs. Deza children can be used to distinguish between nonisomorphic Deza digraphs with the same parameters.

In case $X$ and $Y$ are symmetric, we have a Deza graph, and that case is excluded here. This means that we require that $t < k$.

Let $D = (V, A)$ be a directed $(n, k, b, a)$-Deza graph. For a vertex $u$, let $|N_{uw}|$ be the number of vertices $w$ such that $u \rightarrow w \rightarrow v$. Further, for a vertex $u$ we define

$$
\alpha = |\{ v \in V : |N_{uv}| = a \}|, \quad \beta = |\{ v \in V : |N_{uv}| = b \}|.
$$

In Proposition 2.3 we give conditions on the parameters.

**Proposition 2.3.** Let $D$ be a directed $(n, k, b, a)$-Deza graph. The numbers $\alpha$ and $\beta$ do not depend on the vertex $u$ and

$$
\alpha = \begin{cases} 
\frac{b(n - 1) - k^2 + t}{b - a}, & a \neq b, \\
\frac{k^2 - t}{a}, & a = b.
\end{cases}
$$

$$
\beta = \begin{cases} 
\frac{a(n - 1) - k^2 + t}{a - b}, & a \neq b, \\
\frac{k^2 - t}{a}, & a = b.
\end{cases}
$$

**Proof.** Let $N$ be the number of triples $(u, w, v)$ such that $u \rightarrow w \rightarrow v$, and $u \neq v$. We have that $N = t(k - 1) + (k - t)k = k^2 - t$. If $a = b$, then $N = a\alpha = b\beta$. Otherwise, $N = a\alpha + b\beta$. Using extra condition, $\alpha + \beta = n - 1$ we have the expressions.

**Remark 2.4.** For $k = t$ we have a Deza graph for which conditions given in [4, Proposition 1.1] follow form the conditions given in Proposition 2.3.

If $a = b$ then a directed $(n, k, b, a)$-Deza graph is a directed strongly regular graph. Hence, we are interested in the case $a \neq b$. The following corollary is a direct consequence of Proposition 2.3.
Corollary 2.5. Let $D$ be a directed $(n, k, b, a)$-Deza graph. If $a \neq b$, then $b - a$ divides $b(n - 1) - k^2 + t$ and $a(n - 1) - k^2 + t$. If $a < b$ and $\alpha, \beta \neq 0$, then

$$a(n - 1) < k^2 - t < b(n - 1).$$

3 Constructions of Deza digraphs

In this section we give some constructions of directed Deza graphs. Also, we classify directed $(n, k, b, a)$-Deza graphs with $b = t$.

3.1 A construction from the lexicographical product

For two digraphs $D = (V_1, A_1)$ and $H = (V_2, A_2)$ the lexicographical product digraph (or the composition) $D[H]$ is the digraph with vertex set $V_1 \times V_2$. There is an arc from a vertex $(x_1, y_1)$ to a vertex $(x_2, y_2)$ in $D[H]$ if and only if either $x_1 \rightarrow x_2$ in $D$ or $x_1 = x_2$ and $y_1 \rightarrow y_2$ in $H$.

Let $M_1$ be the adjacency matrix of $D$ and $M_2$ be the adjacency matrix of $H$. Then

$$M_1 \otimes J_{|V_2|} + I_{|V_1|} \otimes M_2$$

is the adjacency matrix of $D[H]$, where $\otimes$ denotes the Kronecker product of matrices and the vertices of $D[H]$ are ordered lexicographically.

Theorem 3.1. Let $D_1 = (V_1, A_1)$ be a DSRG with parameters $(n, k, \lambda, \mu)$ and $D_2 = (V_2, A_2)$ be a directed $(n', k', b, a)$-Deza graph. Then $D_1[D_2]$ is a $(k' + kn')$-regular digraph on $nn'$ vertices. It is a Deza digraph if and only if

$$|\{a + kn', b + kn', \mu n', \lambda n' + 2k'\}| \leq 2.$$  

Proof. The adjacency matrix $M_1$ of the digraph $D_1$ satisfies the conditions

$$M_1 J_n = J_n M_1 = k J_n,$$

$$M_1^2 = t I_n + \lambda M_1 + \mu(J_n - I_n - M_1).$$

Further, there are $(0,1)$-matrices $X$ and $Y$, $X + Y + I_{n'} = J_{n'}$, such that

$$M_2^2 = aX + bY + t' I_{n'},$$
where $M_2$ is the adjacency matrix of the Deza digraph $D_2$.

It is straightforward to see that $D_1[D_2]$ is $(k'+kn')$-regular. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be in $V_1 \times V_2$. Then

$$|N_{uv}| = \begin{cases} 
  t' + tn', & \text{if } u_1 = v_1 \text{ and } u_2 = v_2, \\
  a + tn', & \text{if } u_1 = v_1 \text{ and } |N_{u_2v_2}| = a \text{ in } D_2, \\
  b + tn', & \text{if } u_1 = v_1 \text{ and } |N_{u_2v_2}| = b \text{ in } D_2, \\
  \mu n', & \text{if } u_1 \neq v_1 \text{ and } u_1 \rightarrow v_1 \text{ in } D_1, \\
  \lambda n' + 2k', & \text{if } u_1 \neq v_1 \text{ and } u_1 \rightarrow v_1 \text{ in } D_1.
\end{cases}$$

$D_1[D_2]$ is a Deza digraph precisely when $|N_{uv}|$ takes on at most two values for $u \neq v$.  

**Example 3.2.** Let $D_1 = (V_1, A_1)$ be a DSRG with parameters $(n, k, \lambda, \lambda)$, with the adjacency matrix $M_1$ having the property

$$M_1^2 = tI_n + \lambda(J_n - I_n),$$

and $D_2 = (V_2, A_2)$ be the empty digraph on $n'$ vertices (i.e., $A_2 = \emptyset$). Then $D_1[D_2]$ is a Deza digraph with parameters $(nn', kn', tn', \lambda n')$ and the adjacency matrix $M$ with the property

$$M^2 = tn'I_{nn'} + tn'X + \lambda n'Y,$$

for (0,1)-matrices $X$ and $Y$, where $X + Y + I_{nn'} = J_{nn'}$.

In the next theorem we show that the digraphs from Example 3.2 are characterized by their parameters. The proof is similar to the proof of [4, Theorem 2.6.].

**Theorem 3.3.** Let $D$ be a directed $(n, k, b, a)$-Deza graph, where each vertex is adjacent to $t$ vertices. Then $b = t$ if and only if $D$ is isomorphic to $D_1[D_2]$, where $D_1$ is a DSRG with parameters $(n_1, k_1, b_1, a_1)$, and $D_2$ is an empty digraph of $n_2$ vertices. Moreover, the parameters of $D$ satisfy

$$n = n_1n_2, \quad k = k_1n_2, \quad b = t = t_1n_2, \quad a = \lambda n_2,$$

where each vertex of $D_1$ is adjacent to $t_1$ vertices.
Proof. If $D_1 = (V_1, A_1)$ is a DSRG with parameters $(n_1, k_1, b_1, a_1)$, $D_2 = (V_2, A_2)$ is an empty digraph ($A_2 = \emptyset$) of $n_2$ vertices and $D = D_1[D_2]$, then the statement follows from Theorem 3.1 and Example 3.2.

Suppose that $D$ is a directed $(n, k, b, a)$-Deza graph and $b = t$. Let us consider the equivalence relation $R$ on the set of vertices of $D$ given by

$$uRv \iff |N_{uv}| = b.$$ 

Proposition 2.3 implies that each equivalence class has the same size, namely $\beta + 1$. Let $D_2$ be the empty digraph on $n_2 = \beta + 1$ vertices. Let us define the digraph $D_1$. The vertices of $D_1$ are the equivalence classes of the relation $R$, where $C_1 \rightarrow C_2$ in $D_1$ if and only if there is an arc in $D$ which joins a vertex from $C_1$ to a vertex from $C_2$.

Let us show that $D$ is isomorphic to $D_1[D_2]$. Let $V_2 = \{u_1, \ldots, u_{n_2}\}$, and let the equivalence class $C_i$ be $\{v_{i1}, v_{i2}, \ldots, v_{in_2}\}$. The mapping $f$ given by $f(C_i, u_j) = v_{ij}$ is a digraph isomorphism from $D_1[D_2]$ to $D$. \hfill \Box

### 3.2 A construction from association schemes

We assume that the reader is familiar with the basic facts of theory of association schemes. For background reading in theory of association schemes we refer the reader to [1].

Let $X$ be a finite set of size $n$ and $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ be relations defined on the set $X$. Let $\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$ be the set of $(0,1)$-adjacency matrices such that $[A_i]_{xy} = 1$ if $(x, y) \in R_i$. Then a pair $(X, \mathcal{R})$ is called an association scheme with $d$ classes if

1. $A_0 = I$,
2. $\sum_{i=0}^{d} A_i = J$,
3. $A_i^T \in \mathcal{A}$, for all $i \in \{0, 1, \ldots, d\}$,
4. $A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k$.

We view $A_1, \ldots, A_d$ as adjacency matrices of directed graphs $D_1, \ldots, D_d$, with common vertex set. An association scheme is symmetric if each matrix in it is symmetric.
Theorem 3.4. Let \((X, \mathcal{R})\) be an association scheme, and \(F \subset \{1, 2, \ldots, d\}\). Let \(D\) be the directed graph with adjacency matrix \(\sum_{f \in F} A_f\). Then \(D\) is a directed Deza graph if and only if
\[
\sum_{f, g \in F} p_{fg}^k
\]
takes on at most two values, when \(k \in \{1, \ldots, d\}\).

Proof. Each \(A_i\) can be regarded as an adjacency matrix of a regular digraph (see [7]). Since \(\sum_{i=0}^d A_i = J\), any sum is an adjacency matrix of a regular digraph, i.e. every vertex has constant in-degree and out-degree. Moreover, every vertex of \(D\) is adjacent with
\[
t = \sum_{f \in F} p_{ff}^0
\]
vertices. Let \(u, v \in V(D)\), and \(d(u, v) = k\). Then
\[
|N_{uv}| = \sum_{f, g \in F} p_{fg}^k.
\]
When these numbers take on at most two values \(D\) is a directed Deza graph, which completes the proof. □

3.3 A construction from Hadamard matrices

We say that a \((0, 1)\)-matrix \(X\) is skew if \(X + X^t\) is a \((0, 1)\)-matrix. A Hadamard matrix \(H\) of order \(n\) is called skew if \(H + H^t = 2I_n\).

Let \(D\) be a regular asymmetric digraph of degree \(k\) on \(v\) vertices. \(D\) is called a divisible design digraph (DDD for short) with parameters \((v, k, \lambda_1, \lambda_2, m, n)\) if the vertex set can be partitioned into \(m\) classes of size \(n\), such that for any two distinct vertices \(x\) and \(y\) from the same class, the number of vertices \(z\) that dominates or being dominated by both \(x\) and \(y\) is equal to \(\lambda_1\), and for any two distinct vertices \(x\) and \(y\) from different classes, the number of vertices \(z\) that dominates or being dominated by both \(x\) and \(y\) is equal to \(\lambda_2\).

An incidence structure with \(v\) points and the constant block size \(k\) is a (group) divisible design with parameters \((v, k, \lambda_1, \lambda_2, m, n)\) whenever the point set can be partitioned into \(m\) classes of size \(n\), such that two points from the same class are incident with exactly \(\lambda_1\)
common blocks, and two points from different classes are incident with exactly \( \lambda_2 \) common blocks. A divisible design \( D \) is said to be symmetric (or to have the dual property) if the dual of \( D \) is a divisible design with the same parameters as \( D \). If \( D \) is a divisible design digraph with parameters \( (v, k, \lambda_1, \lambda_2, m, n) \) then its adjacency matrix is the incidence matrix of a symmetric divisible design \( (v, k, \lambda_1, \lambda_2, m, n) \).

The following construction of Deza digraphs follows from the construction of divisible design digraphs given in [2, Theorem 10.]. By \( O_n \) we denote the \( n \times n \) zero-matrix.

**Theorem 3.5.** Let \( H \) be a skew Hadamard matrix of order \( 4u \) with diagonal entries equal to 1. Then there exists an asymmetric Deza digraph with parameters \((8u, 4u - 1, 4u - 1, 2u - 1)\).

**Proof.** Replace each diagonal entry of \( H \) by \( O_2 \), each entry value 1 of \( H \) by \( I_2 \), and each entry value \(-1\) by \( J_2 - I_2 \). By [2, Theorem 10.], the obtained matrix is the adjacency matrix of a DDD with parameters \((8u, 4u - 1, 0, 2u - 1, 4u, 2)\). By the definition of a DDD, the obtained digraph is asymmetric. By the construction, each row (and column) of the Hadamard matrix correspond to a class of size two of vertices of the DDD. Let \( r_i \) be the row of the adjacency matrix of the DDD corresponding to the vertex \( v_i \), and \( c_i \) be the column of the adjacency matrix corresponding to \( v_i \). Since the digraph is asymmetric, the dot product \( r_i \cdot c_i = 0 \). If the vertex \( v_j \) belongs to the same class as \( v_i \), then \( r_i \cdot c_j = 4u - 1 \), i.e. \( r_i = c_j \). Since the adjacency matrix \( M \) of \( D \) is the incidence matrix of a symmetric divisible design and there is a column of \( M \) equal to \( r_i \), the statement of the theorem follows.

**Example 3.6.** The matrix \( M_1 \) from Example 2.2 can be obtained by Theorem 3.5 using the Hadamard matrix

\[
H = \begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}.
\]

### 3.4 Constructions of twin and Siamese twin Deza graphs and digraphs

A digraph \( D \) on \( v \) vertices is called doubly regular with parameters \((v, k, \lambda)\) if it is regular of degree \( k \) and, for any distinct vertices \( x \) and \( y \), the number of vertices \( z \) that dominate
both $x$ and $y$ is equal to $\lambda$ and the number of vertices $z$ that are dominated by both $x$ and $y$ is equal to $\lambda$. A doubly regular asymmetric digraph (DRAD) with parameters $(v, k, \lambda)$ is a directed $(v, k, \lambda, \lambda)$-Deza graph with $t = 0$.

A $(0, \pm 1)$-matrix $T$ is called a twin Deza graph (digraph), if $T = K - L$, where $K, L$ are non-zero $(0, 1)$-matrices and both $K$ and $L$ are adjacency matrices of Deza graphs (digraphs) with the same parameters.

A $(0, \pm 1)$-matrix $S$ is called a Siamese twin Deza graph (digraph) sharing the entries of $N$, if $S = N + K - L$, where $N, K, L$ are non-zero $(0, 1)$-matrices and both $N + K$ and $N + L$ are adjacency matrices of Deza graphs (digraphs) with the same parameters.

Let $G$ be a graph on $v$ vertices. A reflexive graph $RG$ is obtained from $G$ by including a loop at every vertex. If $A$ is the adjacency matrix of the graph $G$, then $A + I_v$ is the adjacency matrix of $RG$. In the following theorem we give constructions of twin Deza graphs and Siamese twin Deza reflexive graphs.

**Theorem 3.7.** Let there exists a Hadamard matrix of order $n$. Then there exist a twin Deza graph with parameters $((2n - 1)n, (n - 1)n, \frac{n(n-2)}{2}, \frac{n(n-1)}{2})$, and a Siamese twin Deza reflexive graph with parameters $((2n - 1)n, n^2, \frac{n^2}{2}, \frac{n^2(n+1)}{2})$.

**Proof.** Let $H$ be a normalized Hadamard matrix of order $n$. Let $r_i$ be the $i$-th row of the matrix $H$ and let $C_i = r_i^T r_i$, $i = 1, 2, 3, \ldots, n$. Further, let $C$ be the circulant matrix of order $2n - 1$ with the first row $(1, 2, 3, \ldots, n, n, n - 1, \ldots, 2)$. Note that every pair of rows of this matrix have exactly one common entry in the same column. Now replace $i$ with $C_i$ in $C$.

We get a $(2n - 1)n \times (2n - 1)n$ matrix $K$, which is a $(1, -1)$-matrix such that $K^2$ has all its off-diagonal entries $n$ or $-n$. By changing the diagonal blocks of size $n \times n$ in $K$ to the zero matrices, and split the remaining matrix into $A - B$, where $A$ and $B$ are $(0, 1)$-matrices, we obtain the adjacency matrices $A$ and $B$ of two graphs. The rows and the columns of $A$ and $B$ are divided into $2n - 1$ groups of size $n$, according to the circulant matrix of order $2n - 1$. From [9, Lemma 3.] it follows that for two rows $r_i$ and $r_j$, $i \neq j$, of $A$ (or $B$) belonging to the same group $r_i \cdot r_j = \frac{n(n-2)}{2}$. The properties of the matrices $C_i$, $i = 1, 2, 3, \ldots, n$, given in the proof of [9, Theorem 1.] lead us to conclusion that for two rows $r_i$ and $r_j$ from different groups the dot product $r_i \cdot r_j$ is equal to $\frac{n(n-2)}{2}$ or $\frac{n(n-1)}{2}$. Hence, $A$ and $B$ are adjacency.
matrices of Deza graphs with parameters \(((2n - 1)n, (n - 1)n, \frac{n(n-2)}{2}, \frac{n(n-1)}{2})\) and \(K\) is the adjacency matrix of a twin Deza graph.

If we now add the matrix \(I_{2n-1} \otimes C_1\) to each of \(A\) or \(B\), we get two Deza reflexive graphs with parameters \(((2n - 1)n, n^2, \frac{n^2}{2}, \frac{n(n+1)}{2})\) sharing the cliques of size \(n\).

**Example 3.8.** Let us take the matrix

\[
H = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & - & - \\
1 & - & 1 & - \\
1 & - & - & 1
\end{bmatrix},
\]

which is a a normalized Hadamard matrix of order 4. Then \(r_1 = (1, 1, 1, 1), r_2 = (1, 1, -, -), r_3 = (1, -, 1, -), r_4 = (1, -, -, 1),\) and

\[
C_1 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
1 & 1 & - & - \\
1 & 1 & - & - \\
- & - & 1 & 1 \\
- & - & 1 & 1
\end{bmatrix},
\]

\[
C_3 = \begin{bmatrix}
1 & - & 1 & - \\
- & 1 & - & 1 \\
1 & - & 1 & - \\
- & 1 & - & 1
\end{bmatrix}, \quad C_4 = \begin{bmatrix}
1 & - & - & 1 \\
- & 1 & 1 & - \\
- & 1 & 1 & - \\
1 & - & - & 1
\end{bmatrix}.
\]

Let \(C\) be the circulant matrix of order 7 with the first row \((1, 2, 3, 4, 4, 3, 2)\). By applying Theorem 3.7 one gets a twin Deza graph with parameters \((28, 12, 4, 6)\), and a Siamese twin Deza reflexive digraph with parameters \((28, 16, 8, 10)\).

**Theorem 3.9.** Let there exists a Hadamard matrix of order \(n\). Then there exist a Siamese twin Deza reflexive digraph with parameters \(((2n - 1)n, n^2, \frac{n^2}{2}, \frac{n(n+1)}{2})\).

**Proof.** Let \(H\) be a normalized Hadamard matrix of order \(n\) and let the matrices \(C_i, i = 1, 2, \ldots, n\) be defined as in the proof of Theorem 3.7. Further, let \(D\) be the circulant matrix with the first row \((1, 2, 3, \ldots, n, -n, -n+1, \ldots, -2)\). Now replace \(i\) with \(C_i\) in \(C\) if \(i > 0,\)
and replace $i$ with $-C_{[i]}$ if $i < 0$. We get a $(2n - 1)n \times (2n - 1)n (1, -1)$-matrix $K'$ such that $K'K''$ has all its off-diagonal entries $n$ or $-n$. By changing the block diagonals of $K'$ to the zero matrices, we get a matrix which splits as $A - B$, where $A$ and $B$ are $(0, 1)$-matrices. If we now add the matrix $I_{2n - 1} \otimes C_1$ to each of $A$ or $B$, we get adjacency matrices of two Deza reflexive digraphs with parameters $((2n - 1)n, \frac{n(n-1)}{2}, 2n - 1, n)$.

**Remark 3.10.** The matrices $A$ and $B$ from the proof of Theorem 3.9 are the adjacency matrices of divisible design digraphs with parameters $((2n - 1)n, (n - 1)n, \frac{n(n-2)}{2}, \frac{n(n-1)}{2}, 2n - 1, n)$.

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**References**

[1] E. Bannai, T. Ito, Algebraic combinatorics I: Association schemes, Benjamin-Cummings Lecture Note Series 58, Benjamin/Cummings, London, 1984.

[2] D. Crnković, H. Kharaghani, Divisible design digraphs, in: Algebraic Design Theory and Hadamard Matrices, (C. J. Colbourn, Ed.), Springer Proc. Math. Stat., Vol. 133, Springer, New York, 2015, 43–60.

[3] A. Duval, A directed graph version of strongly regular graphs, J. Combin. Theory Ser. A 47 (1988), 71–100.

[4] M. Erickson, S. Fernando, W. H. Haemers, D. Hardy, J. Hemmeter, Deza graphs: A generalization of strongly regular graphs, J. Comb. Designs. 7 (1999), 395–405.

[5] S. Goryainov, W. H. Haemers, V. Kabanov, L. Shalaginov, Deza graphs with parameters $(n, k, k - 1, a)$ and $\beta = 1$, J. Combin. Des. 27 (2019), 188–202.
[6] S. Goryainov, D. Panasenko, On vertex connectivity of Deza graphs with parameters of the complements to Seidel graphs, European J. Combin. 80 (2019), 143–150.

[7] A. Hanaki, I. Miyamoto, Classification of association schemes of small order, Discrete Math. 264 (2003) 75–80.

[8] V. Kabanov, N. V. Maslova, L. V. Shalaginov, On strictly Deza graphs with parameters \((n, k, k - 1, a)\), European J. Combin. 80 (2019), 194–202.

[9] H. Kharaghani, On the Siamese twin designs, in: Finite fields and applications, Proceedings of The Fifth International Conference on Finite Fields and Applications \(F_{q^5}\), held at the University of Augsburg, Germany, August 2–6, 1999 (D. Jungnickel, H. Niederreiter, Eds.), Springer, Berlin, 2001, 303–312.