A CLASS OF VECTOR COHERENT STATES DEFINED OVER MATRIX DOMAINS

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ABSTRACT. A general scheme is proposed for constructing vector coherent states, in analogy with the well-known canonical coherent states, and their deformed versions, when these latter are expressed as infinite series in powers of a complex variable \( z \). In the present scheme, the variable \( z \) is replaced by a matrix valued function over appropriate domains. As particular examples, we analyze the quaternionic extensions of the canonical coherent states and the Gilmore-Perelomov and Barut-Girardello coherent states arising from representations of \( SU(1,1) \).

1. INTRODUCTION

One way to define conventional coherent states, over complex domains, is by constructing linear superpositions \( | z \rangle \), parametrized by a single complex number \( z \), of vectors \( \{ \phi_m \}_{m=0}^{\infty} \), which form an orthonormal basis in an infinite dimensional, complex, separable Hilbert space \( \mathcal{H} \):

\[
| z \rangle = \mathcal{N}(|z|)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{z^m}{\rho(m)} \phi_m
\]

Here \( \{ \rho(m) \}_{m=0}^{\infty} \) is a sequence of non-zero, positive numbers, chosen so as to ensure the convergence of the sum in a non-empty open subset \( \mathcal{D} \), of the complex plane, and \( \mathcal{N}(|z|) \) is a normalization constant, ensuring the condition \( \langle z | z \rangle = 1 \). The coherent states \( | z \rangle \) are also required to satisfy a resolution of the identity condition:

\[
\int_{\mathcal{D}} | z \rangle \langle z | d\mu = I ,
\]

where \( d\mu \) is an appropriately chosen measure and \( I \) the identity operator on the Hilbert space \( \mathcal{H} \). These coherent states are known to have a large number of interesting properties, linking them to physical applications, orthogonal polynomials, generalized oscillator algebras, etc. [1], [6], [11].

In this paper we extend this definition to matrix domains, thereby generating families of vector coherent states. Vector coherent states are well-known mathematical objects, particularly when they are defined as orbits of vectors under the operators of unitary representations of groups (see, for example, [1], [3], [4], [15]). However, in the present paper we take a completely different route for constructing them, although in special cases the link to a group representation will also emerge.

Date: March 28, 2022.

Key words and phrases. coherent states, vector coherent states, quaternions.
2. Vector coherent states – the general set up

Let \( \mathcal{R} \) be a measure space, equipped with a measure \( d\mathcal{R} \) and \( \mathcal{K} \) a second measure space, equipped with a probability measure \( d\mathcal{K} \). For \( (r, k, \zeta) \in \mathcal{R} \times \mathcal{K} \times [0,2\pi) \), let

\[
\mathcal{Z} = A(r)e^{i\Theta(k)},
\]

where \( A(r), \Theta(k) \) are two (measurable) \( n \times n \) matrix-valued functions with the following properties (assumed to hold for almost all \( r \in \mathcal{R} \), with respect to the measure \( d\mathcal{R} \) and almost all \( k \in \mathcal{K} \), with respect to the measure \( d\mathcal{K} \)):

- \( \Theta(k) \) is hermitian, that is, \( \Theta(k) = \Theta(k)^\dagger \),
- \( \Theta(k)^2 = \mathbb{I}_n = n \times n \) unit matrix,
- \( [A(r), \Theta(k)] = A(r)\Theta(k) - \Theta(k)A(r) = 0 \),
- \( A(r)A(r)^\dagger = A(r)^\dagger A(r) \).

It is then straightforward to verify (e.g., by direct power series expansion) that,

\[
\mathcal{Z} = A(r)e^{i\Theta(k)} = A(r)[\cos \zeta + i\Theta(k)\sin \zeta].
\]

Let \( \mathcal{D} = \mathcal{R} \times \mathcal{K} \times [0,2\pi) \) and define the measure \( d\mu(r,k,\zeta) = d\mathcal{K}(k) \, d\mathcal{R}(r) \, d\zeta \), on it.

Let \( \chi^j, \quad j = 1,2,\ldots,n, \) be an orthonormal basis in \( \mathbb{C}^n \). Then, \( \{\chi^j \otimes \phi_m\} \), \( j = 1,2,\ldots,n, \quad m = 0,1,2,\ldots,\infty \), is an orthonormal basis in \( \mathcal{H} = \mathbb{C}^n \otimes \mathcal{H} \). For each \( \mathcal{Z} \) we define vector coherent states (VCS) as follows:

\[
|\mathcal{Z},j\rangle = \mathcal{N}(|\mathcal{Z}|)^{-\frac{1}{2}} \sum_{m=0}^\infty \frac{Z^m}{\sqrt{\rho(m)}} \chi^j \otimes \phi_m, \quad j = 1,2,\ldots,n.
\]

where once again, \( \mathcal{N}(|\mathcal{Z}|) \) is a normalization factor, which depends only on the positive part \( |\mathcal{Z}| = |\mathcal{Z}\mathcal{Z}^\dagger|^{\frac{1}{2}} \) of the matrix \( \mathcal{Z} \), and \( \{\rho(m)\}_{m=0}^\infty \) is a sequence of non-zero positive numbers, with \( \rho(0) = 1 \). These have to be chosen in a way such that the following two conditions are satisfied.

- **Normalization:** \( \sum_{j=0}^n \langle \mathcal{Z},j | \mathcal{Z},j \rangle = 1 \)
- **Resolution of the identity:** \( \sum_{j=1}^n \int_{\mathcal{D}} W(|\mathcal{Z}|) \ | \mathcal{Z},j \rangle \langle \mathcal{Z},j | \, d\mu = \mathbb{I}_n \otimes \mathbb{I} \),

where \( W(|\mathcal{Z}|) \) is an appropriately chosen positive weight function.

A straightforward computation, using the fact that

\[
\mathcal{Z}^m = A(r)^m e^{im\zeta \Theta(k)} = A(r)^m (\cos m\zeta + i\Theta(k) \sin m\zeta),
\]

shows that the normalization condition (2.8) implies the finiteness of the sum:

\[
\mathcal{N}(|\mathcal{Z}|) = \sum_{m=0}^\infty \frac{\text{Tr}[A(r)^{2m}]}{\rho(m)},
\]

\( |A(r)| = [A(r)A(r)^\dagger]^{\frac{1}{2}} \) denoting the positive part of the matrix \( A(r) \).

The resolution of the identity condition (2.9) imposes the following restriction on the weight function \( W(|\mathcal{Z}|) \) and the matrices \( A(r) \):

\[
\int_{\mathcal{R}} \frac{2\pi W(|\mathcal{Z}|) |A(r)|^{2m}}{\mathcal{N}(|\mathcal{Z}|)} \, d\mathcal{R} = \rho(m)\mathbb{I}_n,
\]
which can be interpreted as a sort of “matrix moment condition”.

To see this we note that,

\[
\int_{D} W(|Z|) \sum_{j=1}^{n} |Z,j⟩⟨Z,j| dμ
\]

\[
= \int_{D} W(|Z|) \sum_{j=1}^{n} N(|Z|)^{-1} |Z|^j \otimes φ_m⟩⟨Z|^j \otimes φ_l| dμ
\]

\[
= \sum_{m=0}^{∞} \sum_{l=0}^{∞} \int_{D} \frac{W(|Z|)}{N(|Z|)\sqrt{ρ(m)ρ(l)}} A(r)^m e^{imζ(Ω)} \left( \sum_{j=1}^{n} |χ_j⟩⟨χ_j| \right)
\]

\[
\times A(r)^l e^{-ilζ(Ω)} ⊗ |φ_m⟩⟨φ_l| dμ,
\]

Using

\[
\sum_{j=1}^{n} |χ_j⟩⟨χ_j| = I_n, \quad Ω(k)^\dagger = Ω(k) \quad \text{and}
\]

\[
\int_{0}^{2π} e^{im(−l)ζ(Ω)} dζ = \begin{cases} 0 & \text{if } l ≠ m, \\
2πI_n & \text{if } l = m,
\end{cases}
\]

we reduce the last line to

\[
\sum_{m=0}^{∞} \int_{R} \int_{K} \frac{2πW(|Z|)}{N(|Z|)\sqrt{ρ(m)}} A(r)^m A(r)^m ⊗ |φ_m⟩⟨φ_m| dRdK
\]

\[
= \sum_{m=0}^{∞} \int_{R} \int_{K} \frac{2πW(|Z|)}{N(|Z|)\sqrt{ρ(m)}} A(r)^m ⊗ |φ_m⟩⟨φ_m| dRdK.
\]

Since dK is a probability measure, using the fact that \(\sum_{m=0}^{∞} |φ_m⟩⟨φ_m| = I\) and imposing the condition (2.11), we immediately arrive at (2.9).

There is an associated matrix-valued reproducing kernel, \(K(Z^\dagger, Z')\), with matrix elements,

\[
(2.12) \quad K_{jℓ}(Z^\dagger, Z') = ⟨Z, j | Z', ℓ⟩ = \sum_{m=0}^{∞} \frac{1}{ρ(m)\sqrt{N(|Z|)N(|Z'|)}} \times ⟨e^{-imζ(Ω)} A(r')^m A(r)^m |χ| k⟩.
\]

In view of (2.10), this kernel satisfies the reproducing condition,

\[
(2.13) \quad \int_{D} K(Z^\dagger, Z'')K(Z''^\dagger, Z') dμ(k'', r'', ζ'') = K(Z^\dagger, Z').
\]

\section{Generalized annihilation, creation and number operators}

There are a number of operators, associated with the coherent states (1.1), which define the so called generalized oscillator algebras \[7\], \[8\], \[12\], \[13\]. Similar operators can also be constructed in the context of the VCS (2.7). In order to do that, let us first define \(x_m = \frac{ρ(m)}{ρ(m-1)}\), for \(m = 1, 2, 3, \ldots\). Thus we write
\[ \rho(m) = x_m x_{m-1} \cdots x_1 = x_m! \] and define \( x_0! = 1 \). The generalized annihilation or lowering operator, defined on the Hilbert space \( \mathcal{H} \), with respect to the basis \( \{ \phi_m \}_{m=0}^{\infty} \) is then written as

\[ a \phi_m = \sqrt{x_m} \phi_{m-1} \quad \text{with} \quad a \phi_0 = 0. \]

In the case where \( x_m = m \), we recover from this the standard annihilation operator for a harmonic oscillator. It is also easy to see that this operator acts on the coherent states \( |z\rangle \) in the expected manner:

\[ a |z\rangle = z |z\rangle. \]

Using \( a \), we construct the creation or raising operator \( a^\dagger \) and the number operator \( N' = a^\dagger a \):

\[ a^\dagger \phi_m = \sqrt{x_m+1} \phi_{m+1}, \quad N' \phi_m = x_m \phi_m. \]

These three operators generate a Lie algebra (under composition given by the commutator bracket). This is the so-called generalized oscillator algebra, which we denote by \( \mathfrak{A}_{osc} \). In general, the dimension of this algebra is not finite.

On the Hilbert space, \( \mathbb{C}^n \otimes \mathcal{H} \), of the VCS \( |Z, j\rangle \), we define the corresponding operators as,

\[ A = \mathbb{I}_n \otimes a \quad \text{annihilation operator} \]
\[ A^\dagger = \mathbb{I}_n \otimes a^\dagger \quad \text{creation operator} \]
\[ N = \mathbb{I}_n \otimes N' \quad \text{number operator}. \]

They act on the VCS as

\[ A |Z, j\rangle = Z |Z, j\rangle, \]
\[ A^\dagger |Z, j\rangle = \mathcal{N}(Z)^{-\frac{1}{4}} \sum_{m=0}^{\infty} \frac{x_m + 1}{x_m} Z^m \chi^j \otimes \phi_{m+1}, \]
\[ N |Z, j\rangle = \mathcal{N}(Z)^{-\frac{1}{4}} \sum_{m=1}^{\infty} \frac{x_m}{x_m!} Z^m \chi^j \otimes \phi_m, \]

and generate the Lie algebra \( \mathbb{I}_n \otimes \mathfrak{A}_{osc} \), which again is generally not finite dimensional.

Using the operators \( a \) and \( a^\dagger \), we may also define the (formally) self-adjoint operators,

\[ \hat{q} = \frac{a + a^\dagger}{\sqrt{2}} \quad \text{and} \quad \hat{p} = \frac{a - a^\dagger}{\sqrt{2}i}, \]

and the related operators

\[ Q = \frac{A + A^\dagger}{\sqrt{2}} = \mathbb{I}_n \otimes \hat{q} \quad \text{and} \quad P = \frac{A - A^\dagger}{\sqrt{2}i} = \mathbb{I}_n \otimes \hat{p}. \]

We shall need these operators later, when constructing minimal uncertainty states.

To end this section, let us note that, as a consequence of the resolution of the identity (2.9), there is a natural isometric embedding of the Hilbert space of the VCS into a space of vector valued functions on the domain \( \mathcal{D} \). Indeed, let \( \mathcal{H} = L^2(\mathcal{D}, d\mu) \). Then,

\[ W : \mathbb{C}^n \otimes \mathcal{H} \longrightarrow \mathbb{C}^n \otimes \mathcal{H} \quad \text{where,} \quad (W \Psi)(Z) = \langle Z, j | \Psi \rangle, \]

is easily seen to be an isometry.
4. Quaternionic canonical coherent states

As a first example of our general construction, we build in this section VCS using the complex representation of quaternions by $2 \times 2$ matrices. Using the basis matrices,

$$
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
$$

where $\sigma_1, \sigma_2$ and $\sigma_3$ are the usual Pauli matrices, a general quaternion is written as

$$q = x_0\sigma_0 + ix_1\sigma_1,$$

with $x_0 \in \mathbb{R}$, $\mathbf{r} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\mathbf{\sigma} = (\sigma_1, -\sigma_2, \sigma_3)$. Thus,

$$q = \begin{pmatrix} x_0 + ix_3 \\ -x_2 + ix_1 \\ x_2 + ix_1 \\ x_0 - ix_3 \end{pmatrix}.$$

It is convenient to introduce the polar coordinates:

$$x_0 = r \cos \theta, \quad x_1 = r \sin \theta \cos \phi \cos \psi, \quad x_2 = r \sin \theta \cos \phi \sin \psi, \quad x_3 = r \sin \theta \cos \phi,$$

where $r \in [0, \infty)$, $\theta, \phi \in [0, \pi]$ and $\psi \in [0, 2\pi)$. In terms of these,

$$q = A(r)e^{i\theta\sigma(\hat{n})}$$

where

$$A(r) = r\sigma_0, \quad \sigma(\hat{n}) = \begin{pmatrix} \cos \phi & \sin \phi e^{i\psi} \\ \sin \phi e^{-i\psi} & -\cos \phi \end{pmatrix} \quad \text{and} \quad \sigma(\hat{n})^2 = \sigma_0.$$

We denote the field of quaternions by $\mathbb{H}$.

The matrices $A(r)$ and $\sigma(\hat{n})$ satisfy the conditions \ref{2.2} and \ref{2.5}. Thus, with $\{ \phi_m \}_{m=0}^{\infty}$ an orthonormal basis of an abstract Hilbert space $\mathcal{H}$ and $\chi^1, \chi^2$ an orthonormal basis of $\mathbb{C}^2$, we can define the VCS,

$$\mathcal{N}(|q|) = \mathcal{N}(|q|)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{q^m}{\sqrt{x_m}} \chi^j \otimes \phi_m \in \mathbb{C}^2 \otimes \mathcal{H}, \quad j = 1, 2$$

where $\mathcal{N}(|q|)$ and $x_m!$ have to be chosen appropriately.

In order to determine the normalization constant $\mathcal{N}(|q|)$, and the resolution of the identity, first note that in order for the norm of the vector $|q, j\rangle$ to be finite, we must have,

$$\langle q, j \mid q, j \rangle = \mathcal{N}(|q|)^{-1} \sum_{m=0}^{\infty} \frac{r^{2m}}{x_m} < \infty.$$

Thus if $\lim_{m \to \infty} x_m = x$, we need to restrict $r$ to $0 \leq r < L = \sqrt{x}$ for the convergence of the above series. In this case, we define

$$\mathcal{D} = \{(r, \theta, \psi) \mid 0 \leq r < L, \ 0 \leq \phi \leq \pi, \ 0 \leq \theta, \psi < 2\pi\},$$

and note that

$$\mathcal{N}(|q|) = \mathcal{N}(r) = 2 \sum_{m=0}^{\infty} \frac{r^{2m}}{x_m}.$$

In the special case when $x_m = m$, $\mathcal{N}(|q|) = 2 \exp[r^2]$, and $\mathcal{D} = \mathbb{R}^+ \times [0, 2\pi] \times S^2$, where $S^2$ is the surface of the unit two-sphere and $(\phi, \psi)$ are the angular coordinates of a point on it. Note that $\mathcal{D}$ can also be identified with $T\mathcal{S}^2$, the tangent bundle.
of $S^2$. On $D$ we introduce the measure: $d\mu(r, \theta, \phi, \psi) = r \, dr \, d\theta \, d\Omega(\phi, \psi)$ with $d\Omega(\phi, \psi) = \frac{1}{4\pi} \sin \phi \, d\phi \, d\psi$.

To obtain a resolution of the identity, we now have to find a density function $W(|q|) = W(r)$, such that

$$\int_D |q, j\rangle W(r) \langle q, j| \, d\mu = I_2 \otimes I.$$  

Since

$$\int_0^{2\pi} \int_0^{2\pi} \int_0^\pi e^{i(m-l)\theta} \sin \phi d\phi d\theta d\psi = \begin{cases} 2\pi I_2 & \text{if } m = l \\ 0 & \text{if } m \neq l \end{cases},$$

the moment condition (2.11) becomes

$$\int_0^\infty \frac{2\pi W(r)r^{2m+1}}{N(r)} \, dr = x_m! I_2.$$

Writing $W(r) = \frac{N(r)}{2\pi} \lambda(r)$, this is equivalent to solving the moment problem

$$\int_0^L \lambda(r)r^{2m+1} \, dr = x_m! ,$$

for determining the auxiliary density $\lambda(r)$. With this choice of $\lambda$ the resolution of the identity (4.5) will be satisfied. As an example, if $x_m! = m!$ we have $L = \infty$ and then $W(r) = \frac{2}{\pi}$. We shall call the corresponding VCS

$$|q, j\rangle = e^{-\frac{r^2}{2}} \sum_{m=0}^{\infty} \frac{q^m}{\sqrt{m!}} \chi^j \otimes \phi_m \in \mathbb{C}^2 \otimes \mathcal{F},$$

*quaternionic canonical coherent states*. These are the natural generalizations, to quaternions, of the well known canonical coherent states [1].

$$|z\rangle = e^{-\frac{z^2}{2}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} \phi_m \in \mathcal{F},$$

defined over $\mathbb{C}$. Treating the vectors $|q, 1\rangle$ and $|q, 2\rangle$ as elements of a basis, we shall define a general quaternionic VCS as a linear combination,

$$|q, \chi\rangle = \sum_{j=1}^{2} c_j |q, j\rangle, \text{ where } c_1, c_2 \in \mathbb{C}, \quad |c_1|^2 + |c_2|^2 = 1, \quad \chi = \sum_{j=1}^{2} c_j \chi^j.$$

### 5. Minimum Uncertainty and Analyticity Properties

It is well-known that the canonical coherent states [12] are also states of minimum uncertainty, in the sense that for any one of these states $|z\rangle$,

$$\langle \Delta \hat{q}\rangle_z \langle \Delta \hat{p}\rangle_z = \frac{1}{2}, \quad (\text{assuming } \hbar = 1),$$

where, for any operator $A$ on $\mathcal{F}$ and any vector $\phi \in \mathcal{F}$,

$$\langle \Delta A\rangle_{\phi} = \left[\langle \phi | A^2 \phi \rangle - \langle \phi | A\phi \rangle^2 \right]^{\frac{1}{2}}.$$
It is possible to construct quaternionic VCS with similar properties. To see this, first note that the matrix \( q \) can be diagonalized as,

\[
q = u(\theta, \phi) \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} u(\theta, \phi)^\dagger,
\]

where,

\[
u(\theta, \phi) = \begin{pmatrix} i e^{i \frac{\theta}{2}} \cos \frac{\phi}{2} & -e^{i \frac{\theta}{2}} \sin \frac{\phi}{2} \\ e^{-i \frac{\theta}{2}} \sin \frac{\phi}{2} & -i e^{-i \frac{\theta}{2}} \cos \frac{\phi}{2} \end{pmatrix} \quad \text{and} \quad z = re^{i \psi}.
\]

Let \( \chi^+(\theta, \phi) \) and \( \chi^-(\theta, \phi) \) be the two (normalized) eigenvectors, corresponding to the eigenvalues \( z \) and \( \bar{z} \), respectively. Define the two quaternionic VCS,

\[
|q, +\rangle = e^{-z^2} \sum_{m=0}^{\infty} q^m \sqrt{\frac{1}{m!}} \chi^+(\theta, \phi) \otimes \phi_m = e^{-\bar{z}^2} \sum_{m=0}^{\infty} \bar{q}^m \sqrt{\frac{1}{m!}} \chi^+(\theta, \phi) \otimes \phi_m ,
\]

\[
|q, -\rangle = e^{-\bar{z}^2} \sum_{m=0}^{\infty} \bar{q}^m \sqrt{\frac{1}{m!}} \chi^-(\theta, \phi) \otimes \phi_m = e^{-z^2} \sum_{m=0}^{\infty} q^m \sqrt{\frac{1}{m!}} \chi^-(\theta, \phi) \otimes \phi_m .
\]

The normalization of these states has been chosen to ensure that \( \langle q, \pm | q, \pm \rangle = 1 \).

From the nature of the operators \( Q \) and \( P \), defined in (3.10), it is then clear that these states also have minimum uncertainty:

\[
\langle \Delta Q \rangle \pm \langle \Delta P \rangle = \frac{1}{2}.
\]

Next let us look a little more closely at the nature of the isometry (3.11), for the quaternionic VCS. Recall that in this case, \( D = \mathbb{R}^+ \times (0, 2\pi] \times S^2 \simeq TS^2 \). Once again, let \( \mathfrak{H} = L^2(D, d\mu) \). We are interested in the isometry

\[
\mathcal{W} : \mathbb{C}^2 \otimes \mathfrak{H} \rightarrow \mathbb{C}^2 \otimes \bar{\mathfrak{H}} \quad \text{with} \quad \langle \mathcal{W} \Psi | q, j \rangle = \langle \Psi | q, j \rangle.
\]

A general vector \( \Psi \in \mathbb{C}^2 \otimes \mathfrak{H} \) has the form, \( \Psi = \sum_{j=1}^{2} \chi^j \psi_j \), with \( \psi_j \in \mathfrak{H} \). We write \( F = \mathcal{W} \Psi \) and introduce the functions,

\[
f_j(z) = \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} \langle \phi_m | \psi_j \rangle_{\mathfrak{H}} , \quad j = 1, 2 , \quad f(z) = \sum_{j=1}^{2} \chi^j f_j(z) ,
\]

\[
f_j(\bar{z}) = \sum_{m=0}^{\infty} \frac{\bar{z}^m}{\sqrt{m!}} \langle \phi_m | \psi_j \rangle_{\mathfrak{H}} , \quad j = 1, 2 , \quad f(\bar{z}) = \sum_{j=1}^{2} \chi^j f_j(\bar{z}) .
\]

A straightforward computation then shows that the image of the isometry (5.6) consists of vector valued functions of the type,

\[
F(z, \bar{z}, \theta, \phi) = \frac{1}{\sqrt{2}} e^{-|z|^2} \left[ P^+(\theta, \phi) f(z) + P^-(\theta, \phi) f(\bar{z}) \right] .
\]

where \( P^\pm(\theta, \phi) \) are eigenprojectors corresponding to the eigenvectors \( \chi^\pm(\theta, \phi) \), respectively. Thus, for fixed \( (\theta, \phi) \), each component function \( F^j(z, \bar{z}, \theta, \phi) \) is a linear combination of two holomorphic functions \( f_1(z), f_2(z) \) and their antiholomorphic counterparts.
6. Relation to the Weyl-Heisenberg group

The canonical coherent states (4.8) can be expressed (see, for example, [1]) in the form,

\begin{equation}
\ket{z} = e^{za \dagger - z a} \phi_0 = e^{i(p\hat{q} - q\hat{p})} \phi_0, \quad \text{where} \quad z = \frac{q - ip}{\sqrt{2}} .
\end{equation}

We now show that the quaternionic canonical coherent states (4.7) also have the analogous representation:

\begin{equation}
\ket{q, j} = \frac{1}{\sqrt{2}} e^{q \otimes a \dagger - q \dagger \otimes a} \chi^j \otimes \phi_0
\end{equation}

To see this, note that,

\[ [q \dagger \otimes a, q \otimes a] = r^2 \mathbb{1}_2 \otimes I . \]

Next, since for two operators \( A \) and \( B \), the commutator of which commutes with both \( A \) and \( B \), the Baker-Campbell-Hausdorff identity,

\[ e^{A+B} = e^{-\frac{1}{2} [A,B]} e^A e^B , \]

holds, we may write

\[ e^{q \otimes a \dagger - q \dagger \otimes a} = e^{-\frac{1}{2} [q \otimes a \dagger, q \dagger \otimes a]} e^{q \otimes a \dagger - q \dagger \otimes a} . \]

Since \( a^m \phi_0 = 0 \) for all \( m \geq 1 \), we have

\[ e^{-q \otimes a} \chi^j \otimes \phi_0 = \chi^j \otimes \phi_0 \]

and

\[ e^{q \otimes a \dagger} \chi^j \otimes \phi_0 = \sum_{m=0}^{\infty} \frac{(q \otimes a)^m}{m!} \chi^j \otimes \phi_0 = \sum_{m=0}^{\infty} \frac{q^m \chi^j \otimes a^m \phi_0}{m!} \]

\[ = \sum_{m=0}^{\infty} \frac{q^m}{m!} \chi^j \otimes \phi_m . \]

Thus,

\[ \frac{1}{\sqrt{2}} e^{q \otimes a \dagger - q \dagger \otimes a} \chi^j \otimes \phi_0 = \frac{1}{\sqrt{2}} e^{\frac{z^2}{2}} \sum_{m=0}^{\infty} \frac{q^m}{\sqrt{m!}} \chi^j \otimes \phi_m = \ket{q, j} . \]

To develop a group theoretical interpretation for the quaternionic canonical coherent states, we go back to the canonical coherent states as written out in (6.1).

The operators \( \hat{q}, \hat{p} \) and \( I \) generate an irreducible representation of \( g_{WH} \), the Lie algebra of the Weyl-Heisenberg group \( G_{WH} \), on the Hilbert space \( H \). A unitary irreducible representation of \( G_{WH} \) on \( H \) is given by the operators \( U(\vartheta, q, p) = e^{i(\vartheta \hat{q} - p\hat{q} - q\hat{p})} \). Thus, \( \ket{z} = U(0, q, p) \phi_0 \). Turning now to the quaternionic canonical coherent states, as expressed in (6.2), we find using (5.2),

\[ q \otimes a \dagger - q \dagger \otimes a = u(\vartheta, \phi) \begin{pmatrix} za \dagger - a \vartheta & 0 \\ 0 & za \dagger - a \vartheta \end{pmatrix} u(\vartheta, \phi) \dagger . \]

Thus,

\begin{equation}
\begin{aligned}
e^{q \otimes a \dagger - q \dagger \otimes a} &= u(\vartheta, \phi) \begin{pmatrix} 0 & U(0, q, p) \\ U(0, q, -p) & 0 \end{pmatrix} u(\vartheta, \phi) \dagger.
\end{aligned}
\end{equation}

Writing

\begin{equation}
\tilde{U}(\vartheta, q) = \tilde{U}(\vartheta, q, p, \theta, \phi) := u(\vartheta, \phi) \begin{pmatrix} 0 & U(\vartheta, q, p) \\ U(\vartheta, q, -p) & 0 \end{pmatrix} u(\vartheta, \phi) \dagger ,
\end{equation}

we obtain

\[ e^{q \otimes a \dagger - q \dagger \otimes a} = \tilde{U}(\vartheta, q) . \]
we observe that for fixed \((\theta, \phi)\) these operators realize a unitary (reducible) representation of \(G_{\mathbb{W}-H}\) on \(\mathbb{C}^2 \otimes \mathcal{H}\). In terms of these operators,

\[
|q,j \rangle = \frac{1}{\sqrt{2}} \tilde{U}(q) \chi^j \otimes \phi_0 = \frac{1}{\sqrt{2}} \tilde{U}(q,p,\theta,\phi) \chi^j \otimes \phi_0,
\]

in complete analogy with the case of the canonical coherent states.

7. Quaternionic VCS from \(SU(1,1)\) representations

As a second example of the construction of VCS using quaternions, we shall obtain, in this section, analogues of the Gilmore-Perelomov \([10], [14]\) and Barut-Girardello \([5]\) coherent states. Both these families of states arise from the discrete series representations of \(SU(1,1)\). Writing \(D_1 = \{z \in \mathbb{C} \mid |z| < 1\}\), the Gilmore-Perelomov coherent states, labelled by points of \(D_1\), are defined to be,

\[
|z; \text{G-P} \rangle = (1 - r^2)^\kappa \sum_{m=0}^{\infty} \left[ \frac{(2\kappa)_m}{m!} \right]^{1/2} z^m \phi_m \in \mathcal{H}, \quad r = |z|, \quad \kappa = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots,
\]

where we have used the Pochhammer symbol,

\[
(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)(a+2)\ldots(a+m-1),
\]

and as before, the \(\phi_m\) constitute an orthonormal basis of the Hilbert space \(\mathcal{H}\). The index \(\kappa\) labels the unitary irreducible representation of \(SU(1,1)\), to which the above coherent states are associated. This representation is carried by the Hilbert space \(\mathcal{H}_{\text{hol}}(D_1)\), which is the subspace of all holomorphic functions in \(L^2(D_1, 2\kappa - 1) d\mu\), where

\[
d\mu(z, \bar{z}) = \frac{(1 - r^2)^{2\kappa - 2}}{\pi} r dr d\theta, \quad z = re^{i\theta}.
\]

An element \(g \in SU(1,1)\) is a complex \(2 \times 2\) matrix,

\[
g = \begin{pmatrix} \alpha & \beta \\ \beta & \bar{\alpha} \end{pmatrix}, \quad \det g = |\alpha|^2 - |\beta|^2 = 1,
\]

and the unitary irreducible representation \(U^\kappa\), labelled by \(\kappa\), acts on vectors \(f \in \mathcal{H}_{\text{hol}}(D_1)\) in the manner

\[
(U^\kappa(g)f)(z) = (\alpha - \beta \bar{z})^{-2\kappa} f \left( \frac{\alpha z - \beta}{\alpha - \beta \bar{z}} \right).
\]

The monomials

\[
u_m(z) = \left[ \frac{(2\kappa)_m}{m!} \right]^{1/2} z^m,
\]

form an orthonormal basis in \(\mathcal{H}_{\text{hol}}(D_1)\). Moreover, identifying the abstract Hilbert space \(\mathcal{H}\) with \(\mathcal{H}_{\text{hol}}(D_1)\) and \(\phi_m\) with \(u_m\), it can be shown \([4]\) that the coherent states \([8,1]\) can also be written in the form

\[
|z; \text{c.r.} \rangle = U^\kappa(Z) \phi_0, \quad \text{where,} \quad Z = \frac{1}{\sqrt{1 - r^2}} \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} \in SU(1,1).
\]

Observe, that in the notation introduced in \([14]\), in this case we have,

\[
\mathcal{N}(|z|) = (1 - r^2)^{-2\kappa}, \quad \rho(m) = \left[ \frac{(2\kappa)_m}{m!} \right]^{-1}.
\]
Thus, $x_m = \frac{m}{2\kappa + m - 1}$ and since $\lim_{m \to \infty} x_m = 1$, this determines the radius of convergence of the infinite series in (7.1) and hence the appearance of the unit disc.

The coherent states (7.1) satisfy the resolution of the identity,

$$\frac{2\kappa - 1}{\pi} \int_{D_1} |z; G-P\rangle \langle z; G-P| \frac{r \, dr \, d\theta}{(1 - r^2)^2} = I.$$  

(7.3)

The representation of the Lie algebra of $SU(1, 1)$ on $\mathcal{H}_{hol}(D_1)$ is generated by the three operators $K_+, K_-$ and $K_3$, which satisfy the commutation relations

$$[K_3, K_{\pm}] = \pm K_{\pm}, \quad [K_-, K_+] = 2K_3.$$  

(7.4)

They act on the vectors $\phi_m$ in the manner,

$$K_+ \phi_m = \sqrt{m(2\kappa + m - 1)} \phi_{m-1}, \quad K_+ = K_+\dagger, \quad K_3 \phi_m = (\kappa + m) \phi_m.$$  

Thus $K_+ \phi_0 = 0$ and

$$\phi_m = \frac{1}{\sqrt{m! (2\kappa)_m}} K_+^m \phi_0.$$  

Furthermore, it can be shown [9] that, 

$$|z; G-P\rangle = e^{wK_+} \text{hol} \phi_0, \quad w \in \mathbb{C},$$  

(7.6)

where $z$ and $w$ are related by

$$z = \frac{w \tanh(|w|)}{|w|}.$$  

(7.7)

Equation (7.6) should be compared to (6.1). Note however, that unlike in that case, the operators $K_+$ and $K_-$ appearing in (7.6) are not the creation and annihilation operators naturally associated with the expansion in (7.1) (see (3.1)). Indeed, in the present case the operator $a$ has the form:

$$a_{G-P} |z; G-P\rangle = z |z; G-P\rangle, \quad a_{G-P} \phi_m = \sqrt{\frac{m}{2\kappa + m - 1}} \phi_{m-1}.$$  

(7.8)

On the other hand, it is possible to define [5] a second set of coherent states $|w; b-G\rangle$ for this same representation of $SU(1, 1)$, using $K_-$ as the generalized annihilation operator:

$$K_- |w; b-G\rangle := a_{b-G} |w; b-G\rangle = w |w; b-G\rangle, \quad w \in \mathbb{C}.$$  

(7.9)

These states, known as the Barut-Girardello coherent states, are defined for all $w \in \mathbb{C}$ and they are of the form:

$$|w; b-G\rangle = \frac{|w|^{2\kappa - 1}}{\sqrt{I_{2\kappa - 1}(|w|)}} \sum_{m=0}^{\infty} \frac{w^m}{\sqrt{m! (2\kappa + m - 1)!}} \phi_m,$$  

(7.10)

where $I_{\nu}(x)$ is the order $\nu$ modified Bessel function of the first kind. These coherent states satisfy the resolution of the identity,

$$\frac{2}{\pi} \int_{\mathbb{C}} |w; b-G\rangle \langle w; b-G| K_{2\kappa-1}(2q) I_{2\kappa-1}(2q) \, q \, dq \, d\theta, \quad w = q e^{i\theta},$$  

(7.11)

where again, $K_{\nu}(x)$ is the order $\nu$ modified Bessel function of the second kind.

It is now straightforward to write down quaternionic VCS which extend (7.1):

$$|q, j; G-P\rangle = \frac{(1 - r^2)^{\kappa}}{\sqrt{2}} \sum_{m=0}^{\infty} \left[ \frac{(2\kappa)_m}{m!} \right]^\frac{j}{2} q^m \chi^j \otimes \phi_m, \quad r = |q| = |qq^+|^{\frac{1}{2}},$$  

(7.12)
where \( q \) is a quaternionic variable with domain \( D_1 \times S^2 \), and a similar set of VCS extending (7.11):

\[
| w, j; B-G \rangle = r_2^{2\kappa - 1} \sum_{m=0}^{\infty} \frac{w^m}{m!(2\kappa + m - 1)!} \chi^j \otimes \phi_m, \quad r = |w|,
\]

the quaternionic variable \( w \) being defined over the domain \( TS^2 \).

In the case of the vectors (7.12), it is also possible, using (7.2), to give a representation theoretic interpretation along the lines of (6.2) – (6.5). Indeed, by virtue of (7.1), (7.2) and the decomposition (5.2) of the quaternion \( q \), we can immediately rewrite (7.12) as

\[
| q, j; G-P \rangle = \frac{1}{\sqrt{2}} U^\kappa(q) \chi^j \otimes \phi_0.
\]

Writing

\[
\tilde{U}^\kappa(q) = u(\theta, \phi) \begin{pmatrix} U^\kappa(Z) & 0 \\ 0 & U^\kappa(Z) \end{pmatrix} u(\theta, \phi)^\dagger \chi^j \otimes \phi_0,
\]

this immediately yields,

\[
| q, j; c-v \rangle = \frac{1}{\sqrt{2}} \tilde{U}^\kappa(q) \chi^j \otimes \phi_0,
\]

which is the analogue of (6.5). Moreover, since by (7.10),

\[
\tilde{U}^\kappa(q) \chi^j \otimes \phi_0 = u(\theta, \phi) \begin{pmatrix} e^{wK^+ - \overline{w}K^-} & 0 \\ 0 & e^{\overline{w}K^+ - wK^-} \end{pmatrix} u(\theta, \phi)^\dagger \chi^j \otimes \phi_0
\]

with \( z \) and \( w \) being related by (7.7), we can now immediately transform this to

\[
| q, j; c-v \rangle = \frac{1}{\sqrt{2}} e^{w \otimes K^+ - \overline{w} \otimes K^-} \chi^j \otimes \phi_0,
\]

where now the quaternionic variables \( q \) and \( w \) are related by

\[
q = \frac{w \tanh(|w|)}{|w|}.
\]

Note that while \( 0 \leq |q| < 1 \), for the transformed variable \( w \) we have, \( 0 \leq |w| < \infty \).

Interestingly, there is yet another family of coherent states, again related to the \( SU(1, 1) \) group, which can be constructed using the two number operators,

\[
N_{G-P} = a_{G-P}^\dagger a_{G-P} \quad \text{and} \quad N_{B-G} = a_{B-G}^\dagger a_{B-G}.
\]

Indeed, from (7.6) and (7.8),

\[
N_{G-P} \phi_m = \frac{m}{2\kappa + m - 1} \phi_m \quad \text{and} \quad N_{B-G} \phi_m = m(2\kappa + m - 1)\phi_m.
\]

Thus we define a third number operator \( N_{\text{INT}} \), essentially as one which interpolates between these two:

\[
N_{G-P} N_{\text{INT}} = N_{B-G} \quad \Rightarrow \quad N_{\text{INT}} \phi_m = (2\kappa + m - 1)^2 \phi_m,
\]

and the related annihilation operator,

\[
a_{\text{INT}} \phi_m = (2\kappa + m - 1)\phi_{m-1}
\]

The corresponding coherent states, defined for all \( w \in \mathbb{C} \), are

\[
| w; \text{INT} \rangle = N(r)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{w^m}{(2\kappa + m - 1)!} \phi_m, \quad r = |w|,
\]
where the normalization constant is given by
\[ N(r) = \frac{1 F_2(1; 2\kappa, 2\kappa; r^2)}{[\Gamma(2\kappa)]^2}, \]
in terms of the hypergeometric function
\[ 1 F_2(a; b, c; x) = \sum_{m=0}^{\infty} \frac{(a)_m}{(b)_m (c)_m} \frac{x^m}{m!} . \]
The moment problem for determining the resolution of the identity is now
\[ \pi \int_0^\infty r^m \lambda(r) \, dr = [(2\kappa + m - 1)!]^2 . \]
This can be explicitly solved to yield,
\[ \lambda(r) = \frac{2}{\pi} r^{2\kappa - 1} K_0(2\sqrt{r}) , \]
where once again, \( K_0 \) is the order-0 modified Bessel function of the second kind. Finally, one obtains
\[ \int_C |w; \text{INT}\rangle \langle w; \text{INT}| \, d\mu_{\text{INT}}(w, \overline{w}) = I , \]
with
\[ d\mu_{\text{INT}}(w, \overline{w}) = \frac{2r^{4\kappa - 1}}{\pi [\Gamma(2\kappa)]^2} K_0(2r) \frac{1 F_2(1; 2\kappa, 2\kappa; r^2)}{[\Gamma(2\kappa)]^2} \, d\theta \, dr . \]
The corresponding quaternionic VCS are then
\[ |w, j; \text{INT}\rangle = \frac{1}{\sqrt{2}} \frac{\Gamma(2\kappa)}{[1 F_2(1; 2\kappa, 2\kappa; r^2)]^{1/4}} \sum_{m=0}^{\infty} \frac{w^m}{(2\kappa + m - 1)!} \chi^j \otimes \phi_m , \]
where \( r = |w| \) and \( w \in TS^2 \).

The fact that the coherent states (7.20) are indeed related to the \( SU(1, 1) \) group is brought out more clearly by the following observation: computing the commutator \([a_{\text{INT}}, a^{\dagger}_{\text{INT}}]\) we find,
\[ [a_{\text{INT}}, a^{\dagger}_{\text{INT}}] \phi_m = [2(2\kappa + m) - 1] \phi_m \]
Let us define a new “number operator” \( \tilde{N}_{\text{INT}} \) by the action
\[ \tilde{N}_{\text{INT}}\phi_m = (2\kappa + m - \frac{1}{2}) \phi_m , \]
on the basis vectors \( \phi_m \). Then we easily establish the commutation relations,
\[ [a_{\text{INT}}, a^{\dagger}_{\text{INT}}] = 2 \tilde{N}_{\text{INT}} , \quad [\tilde{N}_{\text{INT}}, a^{\dagger}_{\text{INT}}] = a^{\dagger}_{\text{INT}} , \quad [\tilde{N}_{\text{INT}}, a_{\text{INT}}] = -a_{\text{INT}} . \]
Comparing with (7.24), we find that the three operators \( a_{\text{INT}}, a^{\dagger}_{\text{INT}} \) and \( \tilde{N}_{\text{INT}} \) satisfy exactly the same commutation relation as the three generators, \( K^- , K^+ \) and \( K_3 \) of \( su(1, 1) \), the Lie algebra of \( SU(1, 1) \). Thus, they also realize a representation of this algebra on \( \mathfrak{H} \). The two number operators \( \tilde{N}_{\text{INT}} \) and \( N_{\text{INT}} \) are related as,
\[ N_{\text{INT}} = \tilde{N}_{\text{INT}}^2 - \tilde{N}_{\text{INT}} + \frac{1}{4} = [\tilde{N}_{\text{INT}} - \frac{1}{2}]^2 . \]
A similar situation was seen to arise in the case of temporally stable coherent states related to the infinite well and Pöschl-Teller potentials, where the Lie algebra \( su(1, 1) \) appeared as a dynamical algebra. It ought to be pointed out, however, that
the representation of $\mathfrak{su}(1, 1)$, generated by the operators $K_+, K_3$ in (7.4)-(7.5), is different from the one generated by the operators $a_{\text{INT}}^+, a_{\text{INT}}$ and $\tilde{N}_{\text{INT}}$. Indeed, computing the Casimir operators in the two cases, we find that \( \frac{1}{2}(K_+K_- + K_-K_+ - K_3^2) = \kappa(1 - \kappa) \) while, \( \frac{1}{2}(a_{\text{INT}}a_{\text{INT}}^+ + a_{\text{INT}}^+a_{\text{INT}}) - \tilde{N}_{\text{INT}}^2 = \frac{1}{4} \).

8. Conclusion

As amply evident from the above discussion, the method just elaborated for constructing vector coherent states is generic. One could in this manner associate families of VCS to almost any hypergeometric function. More interestingly, the method enables one to associate VCS to certain Clifford algebras and to reducible representations from the principal series of locally compact groups. Some of these results will be presented in forthcoming publications.

Acknowledgements

We would like to thank M. Bertola for interesting discussions. This work was partly supported by research grants from the NSERC (Canada) and the FCAR (Québec).

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