Monodromies of projective structures on surface of finite-type

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Abstract
We characterize the monodromy of projective structures with Fuchsian-type singularities. Namely, any representation from the fundamental group of a Riemann surface of finite-type in $\text{PSL}_2(\mathbb{C})$ can be represented as the holonomy of branched projective structure with Fuchsian-type singularities over the cusps. We made a geometrical/topological study of all local conical projective structures whose Schwarzian derivative admits a simple pole at the cusp. Finally, we explore isomonodromic deformations of such projective structures and the problem of minimizing the branching order.

Keywords
General geometric structures on low-dimensional manifolds · Isomonodromic deformations for ordinary differential equations in the complex domain · Fuchsian groups and automorphic functions (aspects of compact Riemann surfaces and uniformization) · Singularities, monodromy and local behavior of solutions to ordinary differential equations in the complex domain, normal forms

Mathematics Subject Classification
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1 Introduction

This theory has its roots in the study of automorphic functions and differential equations by Klein [11, Part 1], Poincaré [16], Riemann [17], and others in the late nineteenth century (see Hejhal’s works [10], [9] for further historical discussion and references).

A complex projective structure on an oriented surface is a distinguished system of local coordinates modeled in $\mathbb{CP}^1$ in such a way that transition maps extend to homographies, i.e., lie in $\text{PSL}_2(\mathbb{C})$. Branched projective structures on closed orientable surfaces are given by atlases where local charts are finite branched coverings and transition maps lie in $\text{PSL}_2(\mathbb{C})$.

We know that if $S$ is a surface with a complex projective structure modeled in the projective space $\mathbb{CP}^1$, then there is a pair $(dev, \rho)$ (unique up to the action of automorphisms of $\mathbb{CP}^1$), where $dev : \tilde{S} \rightarrow \mathbb{CP}^1$, defined on the universal covering of $S$ is a projective immersion.
equivariant with respect to the homomorphism $\rho : \pi_1(S) \to Aut(\mathbb{C}P^1)$. We say that two projective structures are equivalent if the developing maps differ by homography.

We can associate a projective structure on $S$ to a triple $(\pi, \mathcal{F}, \sigma)$ due to Goldman’s thesis (\cite{7}). In detail, let $\rho : \pi_1(S) \to Aut(\mathbb{C}P^1)$ be a representation, there exists a natural bijection between equivalence classes of complex projective structures on $S$ with monodromy $\rho$ and sections $\sigma$ of the $\mathbb{C}P^1$-bundle $S \times_\rho \mathbb{C}P^1$, the suspension of $\rho$, that are transversal to the foliation $\mathcal{F}$. This foliation is obtained by quotient the horizontal foliation of $\tilde{S} \times \mathbb{C}P^1$ by the action of $\pi_1(S)$ in $\tilde{S}$ and the action of monodromy in $\mathbb{C}P^1$, i.e., $\gamma \cdot (x, z) = (\gamma \cdot x, \rho(\gamma) \cdot z)$, where $\gamma \in \pi_1(S)$, $x \in \tilde{S}$, $z \in \mathbb{C}P^1$.

A natural question about complex projective structures and their monodromy representations is to describe which representations can be realized as monodromy of a projective structure. In the case of closed surfaces of genus $g \geq 2$, Gallo-Kapovich-Marden \cite{6, 2000} showed that non-elementary representations are monodromy of projective structures with at most one branch point. Also in \cite{6, 2000}, they listed some open problems that we study here:

**Problem 1** Prove and/or explore the existence and non-uniqueness of complex projective structures with given monodromy in punctured surfaces.

**Problem 2** Make precise and optimize the connection between branching divisors and monodromy. Namely, compute the function $d : Hom(\pi_1(S), PSL_2(\mathbb{C})) \to \mathbb{Z}$, where $d(\rho)$ is the smallest integer for which there exists a branched complex projective structure with branching divisor of degree $d$ and monodromy $\rho$.

The problem of building complex projective structures on surfaces of finite-type had already been explored by Poincaré through his studies in solving linear differential equations to come to the Uniformization Theorem even though this was not his initial goal.

We define a singularity of *Fuchsian-type* as a point such that around it there is a map that, up to local holomorphic coordinate change, is given by $z^\alpha$, $\alpha \in \mathbb{C}^*$, or $log z + \frac{1}{zn}$, $n \in \mathbb{N}$. We define a singular projective structure of *Fuchsian-type* in $S$ as projective structures where a finite number of the singularities of this type are allowed. These singularities were considered by Fuchs in his studies about differential equations \cite{18}.

We know that every projective structure on a surface has a subjacent complex structure. If we consider a surface of finite-type $S^* := S \setminus P$, where $S$ is a compact Riemann surface and $P$ is a finite subset $\{p_1, \ldots, p_k\}$ of $S$, the complex structure extends in an unique way to $S$.

In order to obtain a result, analogous to the Gallo-Kapovich-Marden’s Theorem to Riemann surfaces of finite-type, we prove:

**Theorem 1.1** Let $S$ be a compact Riemann surface of any genus and $\{p_1, p_2, \ldots, p_k\} \subset S$ a finite subset with $S \setminus S^* = \{p_1, \ldots, p_k\}$. Given a representation $\rho : \pi_1(S^*) \to PSL_2(\mathbb{C})$ there exists a singular projective structure of the Fuchsian-type in $S$ with monodromy $\rho$.

In the proof, we use techniques from algebraic geometry precisely that ruled surfaces have sections and ideas developed by Loray and Marin \cite{12}. Differently from Gallo-Kapovich-Marden, we fix a complex structure before building the projective structure. They do not prescribe the complex structure in advance, rather it is determined as part of the solution. We do not control the local models, i.e., they cannot be optimal, it can exist a finite number of singularities with trivial local monodromy outside the cusps $S \setminus S^*$, i.e., branch points, and branching order nonzero at the cusps. It is necessary to introduce a branch point in the Theorem is however reminiscent of the need for "apparent singularities" in the theory about linear ordinary differential equations on Riemann surfaces introduced by Poincaré.
This result goes back to a work by Loray and Pereira [13] that restricted to projective surfaces, it is possible to build transversely projective foliations with prescribed monodromy. The approach is similar to ours, although they use other tools such as Deligne’s work on the Riemann-Hilbert problem used to build a meromorphic plane connection in a rank 2 vector bundle whose projectivization gives the $\mathbb{C}\mathbb{P}^1$-bundle and a meromorphic section generally transversal to the foliation by fiber bundle’s theory.

In Sect. 4, we explore the non-uniqueness of projective structures with given monodromy in surfaces of finite-type proposed in Problem 1. We prove that Theorem 1.1 is not rigid: we can deform isomonodromically the projective structure for the models $\log z + \frac{1}{\alpha}, n \geq 2$ and $\zeta^\alpha, \Re \alpha > 1$ in the singularities, for this, we use the inverse of moving branch points. When one of the singularities involved is $\zeta^\alpha, \Re \alpha > 1$, we generalize this surgery using the topological/geometrical description of projective charts around the singularities.

Finally, we discuss about representations that are monodromy of projective structures of Fuchsian-type that do not minimize branching order, we need to extend the notion of branching order seen previously in the Problem 2 to Fuchsian-type singularities. In fact, let $\rho : \pi_1(S^*) \to PSL_2(\mathbb{C})$ a representation, around each Fuchsian-type singularity $p$ of the projective structure $\sigma$ with monodromy $\rho$, we prove in Theorem 1.1 that the projective charts are defined by $\zeta^\alpha + n_p, 0 < \Re \alpha \leq 1$ or $\log z + \frac{1}{\alpha_p}$. We can define as $n_p \in \mathbb{Z}$ the branching order at $p$ and the sum $e(\sigma) = \sum_{p \in S} n_p$ as the branching order of the projective structure $\sigma$. We define $d(\rho) = \min \{e(\sigma) : \sigma$ is a projective structure of Fuchsian-type with monodromy $\rho\}$. The main result of the last section with advances in the answer to the Problem 2 is

**Theorem 1.2** There exist representations $\rho : \pi_1(S^*) \to PSL_2(\mathbb{C})$ such that $d(\rho)$ is odd.

For the case of closed surfaces, Gallo-Kapovich-Marden in [6] proved that $d(\rho) = 0$ for all liftable non-elementary representations $\rho$ and $d(\rho) = 1$ for all non-liftable non-elementary representations $\rho$. We observe that the representations covered by the Theorem 1.2, we will necessarily have $d(\rho) \geq 1$. Let $\rho : \pi_1(S^*) \to PSL_2(\mathbb{C})$ representation, we can rewrite the Problem 2: What is the minimum branching order of a projective structure of Fuchsian-type with monodromy $\rho$?

Independently, Gupta [8] has also shown a version of the Gallo-Kapovich-Marden’s Theorem with an analogous statement by using techniques from hyperbolic geometry in dimension 3. The biggest difference with our work is a restrictive hypothesis in monodromy representations that will not provide branch points outside the cusps. No recent results minimize the function $d$.

Calsamiglia, Deroin, and Francaviglia in [4] proved that two-branched projective structures on compact complex surfaces with the same quasi-Fuchsian holonomy and the same branching order are related by moving branch points. We believe that it would be interesting to do the same for the case of projective structures of Fuchsian-type. Another idea would use the surgery debubbling to reduce the branching order as in [4].

**2 Preliminaries**

**2.1 Projective structures**

On a surface $S$, a complex projective structure is defined by an atlas $\{U_i, f_i\}$ of homeomorphisms $f_i : U_i \to V_i, V_i \subset \mathbb{C}\mathbb{P}^1$, where the transition maps $f_i = \phi_{ij} \circ f_j$ are restrictions of Möbius transformations $\phi_{ij} \in PGL_2(\mathbb{C})$. 
We can define of alternative way: let \( \tilde{S} \to (S, z_0) \) be a universal covering of \( S \) based on \( z_0 \). A pair \( (\text{dev}, \rho) \) where \( \text{dev} : \tilde{S} \to \mathbb{CP}^1 \) is a local homeomorphism equivariant to respect the monodromy representation \( \rho : \pi_1(S, z_0) \to \text{PGL}_2(\mathbb{C}) \) defines a complex projective structure on \( S \). Two projective structures on \( S \) where the developing maps differ by homography are equivalent.

A Riemann surface \( S^* \) is of \textit{finite-type} if it is biholomorphic to \( S^* := S \setminus P \), where \( S \) is a compact Riemann surface and \( P \) is a finite subset \( \{p_1, \ldots, p_k\} \) of \( S \), we call \( p_i \) of cusps.

We define a \textit{singular projective structure} in \( S \) as a complex projective structure in \( S^* = S \setminus \{p_1, \ldots, p_k\} \), where \( \{p_1, \ldots, p_k\} \subset S \) and each \( p_i \) is called the singularity of the structure.

The restriction of a projective structure to an open subset \( U \subset S^* \) induces a projective structure in \( U \). We can consider the structures as a germ in the local ring and consider the equivalence of germs of projective structures around their singularities.

The monodromy of a singularity is the monodromy of the restriction of the projective structure to a disk around it.

**Example 2.1** In \( S^* = \mathbb{CP}^1 \setminus \{0, \infty\} \) with non-trivial monodromy, we can build projective charts as the branches of the multivalued map \( z^\alpha \) with \( \alpha \in \mathbb{C} \setminus \mathbb{Z} \) fixed and monodromy around the cusps conjugate to \( w \mapsto e^{2\pi i \alpha} w \), as well as, the branches of \( \log z + \frac{1}{z} \) also define a singular projective structure with monodromy \( w \mapsto w + 2\pi i \).

We define a singularity of \textit{Fuchsian-type} as a point such that around it there is a map that, up to local holomorphic coordinate change, is given by \( z^\alpha, \alpha \in \mathbb{C}^\ast \), or \( \log z + \frac{1}{z^n}, n \in \mathbb{N} \). We define a \textit{singular projective structure of Fuchsian-type} in \( S \) as projective structures where only singularities of this type are allowed.

We remark that singularities of Fuchsian-type with trivial monodromy have a simple topological description that comes from branched coverings. In particular, a branch point is a singularity of Fuchsian-type.

In an analytic approach, a projective structure is represented by a quadratic differential on a Riemann surface, which is extracted from the Schwarzian derivative. We want to explain the association between projective structures of Fuchsian-type and meromorphic quadratic differentials at most double pole at the cusps.

We say that a reduced linear differential equation with a \( h \) meromorphic coefficient

\[
\frac{d^2 u}{dz^2} + hu = 0 \tag{1}
\]

is Fuchsian on \( z = z_0 \) if \( h \) has a maximum of a double pole in \( z_0 \).

The chart \( w \) of the projective structure around \( z_0 \) is the quotient \( w = \frac{u_1}{u_2} \) of two independent solutions of the Eq. (1) around \( z_0 \), or better, as a solution of the Schwarzian equation

\[
S_z(w) := \left\{ \left( \frac{w''(z)}{w'(z)} \right)' - \frac{1}{2} \left( \frac{w''(z)}{w'(z)} \right)^2 \right\} = 2h, \tag{2}
\]

where \( h \) is the coefficient of the Eq. (1). Then, we will say that the projective structure around \( z_0 \) has a Fuchsian-type singularity in \( z_0 \). A meromorphic quadratic differential defined by the Schwarzian derivative of projective charts has the form

\[
\left\{ \frac{1 - \alpha^2}{2z^2} + \sum_{n \geq -1} b_n z^n \right\} dz^2, \tag{3}
\]
in local coordinates around each singularity of Fuchsian-type, $\alpha, b_n \in \mathbb{C}$. Conversely, the quotient of two linearly independent solutions of Schwarzian Eq. (2) defines a projective chart of Fuchsian-type.

Fuchs-Schwarz [18, Théorème IX.1.1.] solve the Schwarzian equation in the neighborhood of a double pole which the quotient of solutions are $y^\alpha, \alpha \in \mathbb{C}^*$ and $\frac{1}{y'} + \log y$ if $\alpha \in \mathbb{N}$ in local coordinate.

We can see the Schwarzian derivative explicitly of the projective structure of Fuchsian-type over the three-punctured sphere. In the next example, we fix the local models of the projective structure around the cusps, we are not prescribing the monodromy:

**Example 2.2** Let $\alpha_0, \alpha_1, \alpha_\infty \in \mathbb{C} \setminus \mathbb{Z}$, the meromorphic quadratic differential is
\begin{equation}
\left\{ \frac{1 - \alpha_0^2}{2z^2} + \frac{1 - \alpha_1^2}{2(z - 1)^2} - \frac{\alpha_0^2 + \alpha_1^2 - \alpha_\infty^2 - 1}{2z(z - 1)} \right\} \; dz^2
\end{equation}
defines the projective structure in $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ with charts projectively equivalent to $z \mapsto z^{\alpha_i}, i = 0, 1, \infty$, at the cusps.

If $\alpha_0, \alpha_1, \alpha_\infty \in \mathbb{Z}$, then the Laurent series expansion around $z_i \in \{0, 1, \infty\}$ of the Schwarzian derivative is
\begin{equation}
\left\{ \frac{1 - \alpha_i^2}{2(z - z_i)^2} + \sum_{n \geq -1} a_n^{(i)}(z - z_i)^n \right\} \; dz^2.
\end{equation}

This meromorphic quadratic differential is associated to a branched projective structure on $\mathbb{C}P^1$ with singularities on $z_i$ if and only if the coefficients $a_n^{(i)}$ satisfy the indicial equation $A_{\alpha_i}(a_{-1}^{(i)}, \ldots, a_{\infty}^{(i)}) = 0$ where $A_{\alpha_i}$ is a polynomial in $\alpha_i$ variables with coefficients in $\mathbb{C}$. If there is not a polynomial equation on the coefficients $a_n^{(i)}$ is satisfied, then the charts around the cusps $z_i$ is projectively equivalent to $z \mapsto \log z + \frac{1}{z}$ (see [18] and [4]).

Up to a M"{o}bius transformation, we can assume any three points in $\mathbb{C}P^1$ as 0, 1 and $\infty$. Thus, the projective structures with three singularities on $\mathbb{C}P^1$ are completely determined by their indexes. The uniqueness of quadratic differential induces the local monodromy up to conjugation of the (global) monodromy.

We want to define the complex projective structure in $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ starting with a group homomorphism. Let $[\gamma_0], [\gamma_1], [\gamma_\infty] \in \pi_1(\mathbb{C}P^1 \setminus \{0, 1, \infty\})$ be loops around each $i$, for $i = 0, 1, \infty$, with the same base point, satisfy $[\gamma_0] \cdot [\gamma_1] \cdot [\gamma_\infty] = Id$. The monodromy representation $\rho : \pi_1(\mathbb{C}P^1 \setminus \{0, 1, \infty\}) \to PSL_2(\mathbb{C})$ must satisfy
\begin{equation}
\rho([\gamma_0]) \cdot \rho([\gamma_1]) \cdot \rho([\gamma_\infty]) = Id.
\end{equation}

where the transformation $\rho([\gamma_i])$ is a local monodromy around $i = 0, 1, \infty$ and conjugate in $PSL_2(\mathbb{C})$, in the non-parabolic case, to $w \mapsto e^{2\pi i \alpha_i} w, \alpha_i \in \mathbb{C}$. At the cusps with parabolic monodromy is conjugate to $w \mapsto w + 2\pi i$. Therefore, the relation (5) is equivalent to $\alpha_0 + \alpha_1 + \alpha_\infty \in \mathbb{Z}$ and the representation $\rho$ induces a complex projective structure in $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$.

### 2.2 Compactification of $\mathbb{C}P^1$-bundles

Firstly, we recall some basic concepts needed for develop this work.
A foliation $\mathcal{F}$ on a compact connected complex surface $X$ is called Riccati foliation if there exists a $\mathbb{CP}^1$-bundle $\pi : X \rightarrow B$ (possibly with singular fibers) whose generic fiber is transverse to $\mathcal{F}$.

Let $C \subset X$ be a curve which is non-invariant by a holomorphic foliation $\mathcal{F}$. Given $p \in C$, let $f = 0$ a local reduced equation of $C$ and $\mathcal{F}$ represented by $v$ an holomorphic field in some neighborhood $U$ of $p$. The tangency order of $\mathcal{F}$ with respect to $C$ at $p$ is defined as

$$tang(\mathcal{F}, C, p) = \text{dim}_\mathbb{C} \frac{\mathcal{O}_p}{\langle f, v(f) \rangle}.$$ 

Moreover,

$$tang(\mathcal{F}, C) = \sum_{p \in \text{Sing}(\mathcal{F}) \cap C} tang(\mathcal{F}, C, p).$$

Differently from Gallo-Kapovich-Marden, we prescribe a complex structure on $S$ before building the projective structure. It can exist a finite number of singularities with trivial local monodromy outside the cusps $S \setminus S^*$, i.e., branch points. As in Goldman’s thesis in [7], we construct these projective structures through sections of the $\mathbb{CP}^1$-bundle obtained from the suspension of the given representation.

We denote by $S \times_\rho \mathbb{CP}^1$ the suspension of a representation $\rho : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$, the construction can be found in [5, Chapter 5].

Let $S^* = S \setminus \{p_1, \ldots, p_k\}$ be a Riemann surface of finite-type and $\rho : \pi_1(S^*) \rightarrow PSL_2(\mathbb{C})$ a representation. We can compactify the suspension $S^* \times_\rho \mathbb{CP}^1$ as a fiber bundle over $S$ provided with a Ricatti (possibly singular) foliation $\mathcal{F}$, where the fibers over the cusps are invariant by $\mathcal{F}$ and contains one or two singularities of foliation as can be found in Brunella [3].

We denote by $D = \{z \in \mathbb{C} \mid |z| < 1\}$ the unit disk centered on the origin and $D^* = D \setminus \{0\}$. We denote by $(D^*, \mathcal{F}, \pi)$ a Riccati foliation $\mathcal{F}$ defined in $D^* \times \mathbb{CP}^1$ where $\pi$ a $\mathbb{CP}^1$-bundle transversal to the foliation $\mathcal{F}$.

Let $(D^*, \mathcal{F}, \pi)$ and $(D^*, \mathcal{F}', \pi')$ be Riccati foliations. We say that two representations $\rho_1 : \pi_1(D^*, z_1) \rightarrow Aut(\pi_1^{-1}(z_1))$ and $\rho_2 : \pi_1(D^*, z_2) \rightarrow Aut(\pi_1^{-1}(z_2))$ are analytically conjugated if there is a biholomorphism $h : \pi_1^{-1}(z_1) \rightarrow \pi_1^{-1}(z_2)$ such that, for every $[\gamma] \in \pi_1(D^*, z_1)$, we have $\rho_1([\gamma]) = h^{-1} \circ \rho_2([\gamma]) \circ h$.

**Proposition 2.3** Let $(D^*, \mathcal{F}, \pi)$ and $(D^*, \mathcal{F}', \pi')$ be Riccati foliations. There is a biholomorphism $\phi : D^* \times \mathbb{CP}^1 \rightarrow D^* \times \mathbb{CP}^1$ that takes leaves of $\mathcal{F}$ to leaves $\mathcal{F}'$ and such that $\pi$ and $\pi'$ are equivalent fiber bundles if and only if the representations of holonomy are analytically conjugated.

This is just a adapted statement of Theorem 2 [5] p. 98. A complete proof can be found in [15, Proposição 2.1] (see also note after Theorem 2 [5] p. 99).

**Lemma 2.4** Every suspension $S^* \times_\rho \mathbb{CP}^1$ admits a compactification $\pi : S^* \times_\rho \mathbb{CP}^1 \rightarrow S$, $\mathbb{CP}^1$-bundle over $S$, provided with a Riccati foliation $\mathcal{F}_\rho$ with invariant fibers over the cusps with non-trivial monodromy.

**Proof** There is a regular “horizontal” foliation on $S^* \times_\rho \mathbb{CP}^1$. Let $D_t$ be a disk (image of a disk of the complex plane by a chart of complex structure of $S$) on $S$ around of $p_t$, we have to the foliation over $D_t \setminus \{p_t\}$ is determined by $\rho(\partial D_t) \in PSL_2(\mathbb{C})$. We choose a biholomorphism that maps $p_t$ to 0 and $D_t$ to $D$.

We can choose on $D \times \mathbb{CP}^1$ a singular Riccati foliation with any prescribed monodromy, where in coordinates $(z, w) \in D \times \mathbb{CP}^1$ of fiber bundle trivialization around of invariant
fiber, the foliation will be generated by a meromorphic 1-form defined in $\mathbb{D} \times \mathbb{CP}^1$ rational in the variable $w$. According to the local monodromy, we consider the following models:

1. In the case of non-parabolic monodromy, conjugated to $w \mapsto e^{2\pi i \alpha_i} w$, the 1-form $\alpha_i w dz - zdw = 0$, $\alpha_i \in \mathbb{C}$, or
2. In the case of parabolic monodromy, conjugated to $w \mapsto w + 1$, the 1-form $dz - zdw = 0$, or
3. In the case of trivial monodromy, conjugated to the identity, the 1-form $m w dz - zdw = 0$, for some $m \in \mathbb{N}$.

Then, locally the monodromies are the same, thus by the Proposition 2.3 the foliations over $D_i \setminus \{p_i\}$ are biholomorphic, so we can glue and obtain a singular Riccati foliation in $S^* \times_\rho \mathbb{CP}^1$. We observe that over the cusps with non-trivial monodromy, the foliation has invariant fibers in $\{z = 0\}$ and one or two singularities in the fiber.

In the non-parabolic case, the singularities of foliation are $(0, 0)$ and $(0, \infty)$ with separatrix $\{w = 0\}$ and $\{w = \infty\}$, and in the parabolic case, the foliation has a saddle-node singularity in $(0, \infty)$ and a separatrix $\{w = \infty\}$.

\[\square\]

**Remark 2.5** In this compactification, the fibers over the cusps with non-trivial monodromy are always invariant by $F_\rho$. Over the cusps of trivial monodromy, only in the case $m = 0$ in the model 3 above, we would have a compactification defined in the neighborhood of these points by product foliation without invariant fibers and singularities.

Let $\rho : \pi_1(S^*) \to PSL_2(\mathbb{C})$ a representation. Every projective structure of Fuchsian-type in the Riemann surface $S$ with monodromy $\rho$ gives us a holomorphic section of $\mathbb{P}^1$-bundle $\pi : P \to S$ equipped with a Riccati foliation.

### 2.3 Flippings and existence of holomorphic sections

We will show that there exists a holomorphic section generically transversal to the foliation of the $\mathbb{CP}^1$-bundle obtained in the Lemma 2.4. This step is very important for describing the singularities of the projective structures obtained through holomorphic sections of $S^* \times_\rho \mathbb{CP}^1$.

Recall that a $\mathbb{CP}^1$-bundle, suspension of a representation $\rho : \pi_1(S^*) \to PSL_2(\mathbb{C})$, has an invariant holomorphic section if and only if $\rho$ has fixed points. Each fixed point determines an invariant holomorphic section transporting the fixed point through the holonomy of the foliation. Since we have at most two fixed points for non-trivial representations, we can affirm that there are at most two invariant sections.

We say that a representation $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ is elementary if the action of $Im(\rho)$ on $\mathbb{H}^3$ fixes one point or two in $\mathbb{H}^3 \cup \partial \mathbb{H}^3$, otherwise, we call it non-elementary. If the representation is non-elementary, then this $\mathbb{CP}^1$-bundle has not an invariant section.

We will show that if the monodromy of a Riccati foliation in a $\mathbb{CP}^1$-bundle over a Riemann surface $S$ is non-trivial, then the fiber bundle has at least three holomorphic sections to assure the existence of at least one non-invariant. We use a result by Tsen ([1] p. 140) states that any $\mathbb{CP}^1$-bundle over a compact Riemann surface has a holomorphic section.

Since the monodromy representation of a Riccati foliation gives a complete description of the foliation module birational isomorphisms, according to Brunella [3, Chapter 4].

When the monodromy representation is non-parabolic (including the trivial case) we choose the foliation around an invariant fiber $\alpha w dz - zdw = 0$, $\alpha \in \mathbb{C}$, by flipping (elementary transformation) of that fiber, i.e., related through a sequence of blowings up at the singular points and contractions of invariant fibers, one finds also that $\alpha$ can be changed to
\( \alpha + n, n \in \mathbb{Z} \). Similarly, when the monodromy is parabolic the foliations are \( dz - zdw = 0 \) or \((nw + zn)dz - zdw = 0, n \in \mathbb{N} \), and also related by “flipping” of the invariant fiber. Note that the flipping of the fiber does not change the local monodromy.

For showing that there exist infinitely many sections in \( \mathbb{C}P^1 \)-bundles, it follows immediately of next result:

**Theorem 2.6** ([12]) The composition of a finite number of flippings in a trivial bundle \( S \times \mathbb{C}P^1 \) gives a \( \mathbb{C}P^1 \)-bundle over a compact Riemann surface \( S \). Every \( \mathbb{C}P^1 \)-bundle over \( S \) can be obtained of this way.

If we can take the images of infinitely many sections of the trivial bundle over \( S \) by the flippings of the above theorem, then there is at least a non-invariant section among these.

### 2.4 Projective structures with prescribed monodromy

In order to prove the Theorem 1.1, we can calculate the projective charts by projecting the section along the leaves in a fiber transversal to the foliation. This construction was already known for the case without branch points, but we can extend in a similar way when there are branch points. It can be calculated through the image of the section \( \sigma(S) \) by local first integral \( h \) composing with the inverse of local first integral restricted to a transversal fiber \( F_1 \). Therefore, the local submersions that define the regular foliation restricted to the curve \( \sigma(S) \) define charts of branched projective structure on \( S \). The tangency points between \( \sigma(S) \) and the foliation produce the critical points of the charts.

A priori, we do not have this control at the cusps with singularities of Fuchsian-type. We can perform this, if the surface is finite-type, after the compactification of the suspension \( S^* \times \rho \mathbb{C}P^1 \), the foliation becomes a singular foliation. We can only use the construction above when the section does not intercept the singular points of the foliation, because in these points there exists no local submersion. However, we can extend the construction to the points that there is a closed meromorphic 1-form that defines the foliation locally. For example, the form \( \omega = \frac{dz}{z} + \lambda \frac{dw}{w} \) is a closed meromorphic and it has Liouvillian first integral \( h(z,w) = zw^\lambda \). Therefore, the foliation \( \mathcal{F}_\rho \) is defined by a closed meromorphic 1-form with a Liouvillian first integral around a singular point. Finally, the projective charts can be calculated in the same way above where the developing map is the local inverse of the holonomy germ \( f \) between the transversal fiber \( F_1 \) and the section.

Let \( \pi : P \to S \) be a \( \mathbb{P}^1 \)-bundle over \( S \) associated to the monodromy representation \( \rho : \pi_1(S^*) \to PSL_2(\mathbb{C}) \). We can equip this \( \mathbb{P}^1 \)-bundle with a Riccati foliation \( \mathcal{F}_\rho \) (see Brunella [3]) with the same monodromy obtained by compactification of the suspension of the representation \( \rho \).

We want to study the relation between projective structures with Fuchsian singularities and flipping of a fiber. In the next proposition, we prove that, up to birational morphism, the choice of the local model in the compactification is related to the section of the \( \mathbb{C}P^1 \)-bundle intersects the singularity of the foliation or not. Let us consider the \( \mathbb{C}P^1 \)-bundle restrict to a disk around a cusp as \( \pi \big|_{\pi^{-1}(\mathbb{D})} : \pi^{-1}(\mathbb{D}) \to \mathbb{D} \) and the Riccati foliation \( \mathcal{F}_\rho \) restrict to \( \mathbb{D} \times \mathbb{C}P^1 \).

**Proposition 2.7** Given a non-trivial holomorphic section \( \sigma \) of \( \pi \) and non-invariant by \( \mathcal{F}_\rho \) that intersects a singularity of \( \mathcal{F}_\rho \). Then, after a flipping the projective chart is preserved, and the tangency order of the foliation with the section is \( \text{tang}(\mathcal{F}_\rho, \sigma(\mathbb{D})) - 1 \).

**Proof** There are two cases: parabolic and non-parabolic monodromy.
1. Non-parabolic Monodromy

We suppose that locally \( F_\rho \) is given by \( \alpha w dz - zdw = 0, \alpha \in \mathbb{C} \). In coordinates, we suppose \( \sigma(z) = (z, \sigma_1(z)) \) where \( \sigma_1 \) is a holomorphic germ, up to coordinate changing, the section intercepts the singularity \((0,0)\), then we can rewrite as \( \sigma_1(z) = z^n \phi(z), n \geq 1 \) and \( \phi(0) \neq 0 \). Let \( h(z, w) = z^n w^{1-n} \) be a multi-valued first integral of the foliation in a neighborhood \( U \) of \((0, \sigma_1(0))\).

Let us study the projection of \( \sigma \cap U \) along the leaves in the transversal fiber to a foliation \( F_1 = \pi^{-1}(z_1), z_1 \in \mathbb{D}^* \) for calculating the projective chart. Since \( \sigma_1(0) = 0, \) it follows that \( f(z) = z_1^\alpha \cdot \sigma_1(z) \), up to an automorphism of \( \mathbb{C}P^1 \), we have \( f(z) = (z \phi(z))^{n-\alpha} = (k(z))^{n-\alpha} \) where the holomorphic function \( \phi \) is the only solution in the neighborhood of \( 0 \) of the equation \( \phi(z)^{n-\alpha} = \phi(z), \phi(0) \neq 0 \), and \( k \) is an invertible germ. Thus, \( f(k^{-1}(z)) = z^{n-\alpha} \).

After one blowing up \((z = z \text{ and } w = zy)\) and one contraction of the fiber \( \{z = 0\}\), we obtain a section \( \beta(z) = (z^n \phi(z)) \), thus the tangency order is \( \text{tang}(F_\rho, \sigma(\mathbb{D})) = 1 \).

The foliation becomes \((\alpha - 1) y dz - z dy = 0 \) with first integral \( z^{\alpha-1} y^{-1} \) different from the initial. We obtain using the same arguments above the projective chart equals \( z^{n-\alpha} \).

2. Parabolic Monodromy

We suppose that \( F_\rho \) is given by \( dz - zdw = 0 \) to compactify who the first integral is \( h(z, w) = w - \log z \).

Let \( h(z, w) = \log z - w \) be the multi-valued first integral of the foliation in a neighborhood \( U \) of \((0, \sigma_1(0))\). Using a similar argument to calculate the projective chart as above, we obtain \( f(z) = \log z^{1-n} \).

We affirm that there exists diffeomorphism germ \( w \) such that \( \log \left( w(z) e^{\frac{1}{w(z)}} \right) = \log(z^{n-\sigma_1(z)}) \) where \( n \) satisfies \( \sigma_1(z) = \frac{1}{z^n} \cdot \sigma_2(z), \sigma_2(0) \neq 0 \).

In fact, if we put \( w(z) = zh(z), \) then \( \log h(z) + \frac{1}{z^nh^n} = -\frac{\sigma_2(z)}{z^n}. \) We take \( F(z, \xi) = z^n \log z + \frac{1}{z} + \sigma_2(z), \) where \( F(z, h(z)) \equiv 0 \). We have \( \frac{\partial F}{\partial \xi}(0, h(0)) = -\frac{n}{h(0)^{n+1}} \neq 0 \) for \( \frac{1}{h(0)^n} = -\sigma_2(0) \neq 0 \), thus \( h(0) \neq 0 \).

By Implicit Function Theorem, there exists \( h(z) \) holomorphic germ in the neighborhood of \( 0, \) thus \( w(z) \) is an invertible germ \( (w'(0) = h(0) \neq 0) \). Thus, \( f(w^{-1}(z)) = \log z + \frac{1}{z} \).

Blow up on \((0, \infty)\) and after the contraction of the invariant fiber, we put \( w = \frac{1}{u}, \) we obtain \( dz + \frac{1}{nu} du = 0 \), and after this coordinate changing, blow up on \((0, 0)\) \( (u = z t e z = z) \), thus we obtain \( \frac{1}{z} \left( (z + \frac{1}{z}) dz + \frac{z}{\sqrt{t}} \right) = 0 \). Changing the coordinates one more time \((v = \frac{1}{z}) \) and contracting the fiber \( \{z = 0\}\), we obtain \((v + z)dz - zdv = 0 \). After this flipping, the first integral in this new model is \( h(z, v) = \frac{v}{z} + \log z \).

We suppose that the section \( \sigma \) contains the singularity \((0, \infty)\), then we can write it as \( \sigma(z) = (z, \sigma_1(z)) = (z, z^{-n} \sigma_2(z)), \sigma_2(0) \neq 0 \), that goes to \( \beta(z) = (z, z^{-n+1} \sigma_2(z)) \) after the flipping above, where \( n = \text{tang}(F_\rho, \sigma(\mathbb{D})) \). Up to composition with a Möbius transformation, the projective chart can be calculated through the projection of the section along the leaves in a transversal fiber to the foliation will be \( \frac{\sigma_2(z)}{z^n} + \log z \).

We show that there is a diffeomorphism germ \( \psi \) such that
\[
\log \psi(z) + \frac{1}{\psi(z)^n} = \frac{\sigma_2(z)}{z^n} + \log z,
\]
through of Implicit Function Theorem, thus, \( d\psi(\psi^{-1}(z)) = \log z + \frac{1}{z} \).

Therefore, any flipping changes the section, the foliation, and therefore the first integral, but preserving the projective structure.

\[\square\]
The flipping given by the composition of $n$ blowings up and contractions sends the section $\sigma$ in the section $\beta(z) = (z, \phi(z))$, $\phi$ biholomorphism germ, $\phi(z) \neq 0$, i.e., a transversal section to the foliation around the invariant fiber. This is the only flipping that happens this, by Proposition 2.7. Thus, we can state

**Corollary 2.8** There is one only model, up to flipping, where the section does not intersect the singularities of foliation.

Therefore, we can choose a local model to compactify the $\mathbb{CP}^1$-bundle over $S^*$ such that the section is transversal to the foliation around invariant fibers, i.e, it does not intersect the foliation’s singularities.

### 3 Proof of the existence theorem

**Proof of the Theorem 1.1** By discussion in the previous section, the $\mathbb{CP}^1$-bundle $\pi : S^* \times_{\rho} \mathbb{CP}^1 \to S$ has at least one non-trivial, and non-invariant holomorphic section $\sigma$.

We separate the proof in two cases: regular points and cusps.

1st case: Regular points.

At regular points of the surface, we obtain, up to appropriate coordinates changing, complex projective charts or branched coverings.

In fact, at regular point $p = (z_0, w_0) \in S^* \times \mathbb{CP}^1$ of a non-trivial and non-invariant section $\sigma$ we have to $\mathcal{F}_\rho \cap \pi$. We study two cases: $\sigma$ is transversal to $\mathcal{F}_\rho$ at $p$ or not.

By introducing coordinates $(z, w) \in S^* \times \mathbb{CP}^1$ centered on $p$, in a neighborhood of $p$ to foliation $\mathcal{F}_\rho$ is regular, we can think as "horizontal" foliation $\frac{\partial}{\partial z}$.

Putting $\sigma(z) := (z, \sigma(z))$, let $U$ be a neighborhood of $(0, \sigma_1(0))$ in $\mathbb{D} \times \mathbb{CP}^1$, $h(z, w) = z$ the holomorphic first integral of $\mathcal{F}_\rho$ in $U$ and $F_1 = \pi^{-1}(z_1)$ a fiber near to 0. The restriction of $h$ to $F_1 \cap U$ is a diffeomorphism and $\left( h\big|_{F_1 \cap U} \right)^{-1} \circ h(\sigma(z)) = (z_1, f(z))$, then $f(z) = \sigma_1(z)$. If the foliation is transversal to the section in $(0, \sigma_1(0))$, then $f$ is holomorphic and we obtain that projective chart around $0$ is a homeomorphism. Otherwise, $\sigma_1(z) = 0$. We can rewrite as $\sigma_1(z) = z^n \phi(z)$, $\phi(z) \neq 0$, or better, $\sigma_1(z) = (\phi(z))^n = (k(z))^n$, where $tang(\mathcal{F}_\rho, \sigma(S)) = n - 1$. Since $k'(0) \neq 0$, we conclude that $k$ is an invertible holomorphic germ.

Thus, $f(k^{-1}(z)) = z^n$, i.e., ramified covering with $n$ sheets.

2nd case: Cusps

At the points $\{p_1, \ldots, p_k\}$ we will obtain, up to appropriate coordinates changing, singular projective charts $z \mapsto z^\alpha, \alpha \in \mathbb{C}^*$ when the monodromy around the point is non-parabolic. Otherwise, we will obtain $z \mapsto \log z + \frac{1}{n}$, $n \in \mathbb{N}$.

The foliations of Lemma 2.4 have Liouvillian first integrals. We shall separate in parabolic, non-parabolic, and trivial cases.

In coordinates $(z, w) \in \mathbb{D} \times \mathbb{CP}^1$, we can consider the section $\sigma : \mathbb{D} \to \mathbb{D} \times \mathbb{CP}^1$ given by $\sigma(z) = (z, \sigma_1(z))$. We can assume that $\sigma$ do not intersect singularities of the foliation, i.e., $\sigma_1(0) \neq 0, \infty$, by Corollary 2.8. We have three cases:

(i) Non-parabolic Monodromy

The foliation is induced by $\omega = awdz - zdw$. We know that $F = \pi^{-1}(0)$ is a invariant fiber by the foliation whose monodromy is given by $w \mapsto e^{2\pi i \alpha} w$.

Let $h(z, w) = z^\alpha w^{-1}$ be a multi-valued first integral defined at $\mathbb{D} \times \mathbb{CP}^1$ of the foliation in a neighborhood $U$ of $(0, \sigma_1(0))$. We remark that the (multi-valued) graphs of $w = \ldots$
Let \( \sigma \cap U \) along the leaves in transversal fiber to a foliation \( F_1 = \pi^{-1}(z_1), z_1 \in \mathbb{D}^* \).

In fact, \( (h \mid_{F_1})^{-1} \circ (h(\sigma(z))) = (h \mid_{F_1})^{-1} \circ \left( \frac{z^\alpha}{\sigma_1(z)} \right) = \left( z_1, z_1^\alpha \cdot \frac{\sigma_1(z)}{z^\alpha} \right) \). Thus, \( f(z) = z_1^\alpha \cdot \frac{\sigma_1(z)}{z^\alpha} \). Since \( \sigma_1(z) \) is a holomorphic germ with \( \sigma_1(0) \neq 0 \), up to an automorphism of \( \mathbb{CP}^1 \), \( f \) is \( \frac{\sigma_1(z)}{z^\alpha} = (\phi(z))^{-\alpha} = (k(z))^{-\alpha} \). We remark that the equation \( \phi(z)^{-\alpha} = \sigma_1(z), \sigma_1(0) \neq 0 \), admits only one solution \( \phi(z) = e^{-\frac{1}{\alpha} \log(\sigma_1(z))} \) holomorphic in the neighborhood of 0. We have that \( k \) a invertible germ, since \( k' \neq 0 = \phi(0) \neq 0 \). Thus, \( f(k^{-1}(\frac{1}{\alpha})) = z^\alpha \).

(ii) Parabolic Monodromy

We have to \( f(z) = \log z_1 - \log z + \sigma_1(z) \), as above, with coordinates appropriate changing, we have \( dev(w^{-1}(z)) = \log z \).

(iii) Trivial Monodromy

The foliation is induced by \( \omega = mwdz - zdw \) where \( m \in \mathbb{N} \). Analogously, we obtain \( f(z) = z^{\alpha - m} \), when \( \sigma_1(0) \neq \infty \).

Remark 3.1 The case of trivial monodromy around a cusp, the charts are as in the case of the regular points where the section is not transversal to \( F_\rho \). In fact, if in Lemma 2.4 we choose the model \( dw = 0 \) instead of \( mwdz - zdw = 0 \), for some \( m \in \mathbb{Z} \), then the foliation (also the first integral) would extend holomorphically at the cusps with trivial monodromy.

Let \( \rho : \pi_1(S^*) \to PSL_2(\mathbb{C}) \) a representation. We obtain a dictionary between a triple \((\pi, F_\rho, \sigma)\), where \( \pi : P \to S \) is a \( \mathbb{P}^1 \)-bundle equipped with a Riccati foliation \( F_\rho \) and \( \sigma \) is a holomorphic section of \( \pi \) generically transversal to \( F_\rho \), and a singular projective structure of Fuchsian-type in the Riemann surface \( S \) with monodromy \( \rho \).

4 Isomonodromic deformations

In this section, we study the geometry and topology of the local structures around the cusps to construct continuous deformations of projective structures on surfaces of finite-type that preserve the monodromy, i.e., the problem about the non-uniqueness of projective structures with Fuchsian-type singularities with prescribed monodromy. We prove the inverse of surgery of moving branch points when one of the singularities is \( z^\alpha, \Re \alpha > 1 \).

4.1 Geometry and topology of Fuchsian-type singularities

We would like to answer the problem of Gallo-Kapovich-Marden about non-uniqueness of projective structure on surfaces of finite-type, but singularities of Fuchsian-type have complicated behaviors. We need to study this behavior to support surgeries like moving branch points.

We denote by \( \mathbb{D}^* \) a deleted open neighborhood of a cusp, \( p : T \to \mathbb{D}^* \) universal covering of \( \mathbb{D}^* \) where \( T = \{ x \in \mathbb{C} \mid \Re x < 0 \} \) and \( p(x) = e^x \), here \( \Re x \) represents the real part of \( x \). Let \( f \) be a multi-valued map, we denote by \( \tilde{f} \) a lifting of \( f \) to universal covering.

Definition 4.1 The degree of a multi-valued map \( f : \mathbb{D}^* \to \mathbb{D}^* \) is the maximum number of preimage of each \( z \in \mathbb{D}^* \) by \( \tilde{f} \) restricted to a fundamental domain.
We remark that if \( f \) is a multi-valued map of degree 1, then \( \tilde{f} \) is injective on each fundamental domain.

**Example 4.2** The multi-valued map \( z^3 \) defines a projective chart around a cusp with monodromy \( w \mapsto -w \) and developing map \( \text{dev}: T \to \mathbb{C}P^1 \) given by \( \text{dev}(x) = e^{3x} \) defined in \( T \). The points of \( \{ z \in \mathbb{D}^\ast | \Im z < 0 \} \), where \( \Im x \) represents the imaginary part of \( x \), has a preimage in the fundamental domain \( T_0 = \{ x \in T | 0 < \Im x < 2\pi \} \), while in \( \{ z \in \mathbb{D}^\ast | \Im z > 0 \} \) has two preimages in \( T_0 \), therefore this map has degree 2.

We consider a local non-parabolic monodromy conjugate to \( w \mapsto e^{2\pi i \alpha} w \), \( \Re \alpha \neq 0 \). The projective structure defined around one of the fixed points of this monodromy, which we can assume to be the origin, the projective structure can be thought as a sector of \( \mathbb{C}P^1 \setminus \{0, \infty\} \) centered on 0 with angle \( 2\pi \Re \alpha \) and length sides 1 and \( e^{-2\pi \Im \alpha} \) identified by the local monodromy.

Geometrically, two points \( u + iv, u' + iv' \in T \) have the same image by \( \text{dev} \) if and only if \( (u', v') = (u, v) - \beta \mathbb{Z} \), where \( \beta = \frac{2\pi i}{\alpha} \).

We put \( \alpha = a + ib \). The semi-plane \( au - bv < 0 \) can be decomposed in biholomorphic strips to the disk minus the radius \( [0, 1) \) by \( \text{dev} \). This decomposition is given by equidistant parallel lines to \( au + bu = 0 \) with distance \( \frac{2\pi}{|a|} \).

We change the universal covering of \( \mathbb{D}^\ast \) such that the new fundamental domain is given by a strip whose boundary consists of lines \( bu + av = 0 \) and \( bu + av = 2\pi|a| \). Therefore, the maximum number of preimages of \( z \in \mathbb{D}^\ast \) for \( \text{dev} \) restricted to fundamental domain is \( \lceil \Re \alpha \rceil \), i.e., \( z^\alpha \).

In the case \( \Re \alpha = 0 \), the fundamental domain \( T_0 \) covers an annulus \( A = \{ z \in \mathbb{C} | e^{-2\pi b} < |z| < 1 \} \) by \( \text{dev}(x) = e^{ibx} \). The action of \( \text{dev} \) is defined by the translation \( w \mapsto w - \frac{2\pi}{b} \) where the semi-plane \( v > 0 \) will be decomposed by the equidistant lines parallel to \( u = 0 \) with distance \( \frac{2\pi}{|b|} \), see Fig. 2. In this case, the projective structure can be seen as the annulus \( A \) with the boundary lines identified by the transformation \( w \mapsto e^{-2\pi b}w \). Observe this is topologically a torus.

We prove that two actions in \( T \) classify projective structures of type \( z^\alpha, \Re \alpha > 0 \), thus, classify every singular projective structures of type \( z^\alpha, \Re \alpha \neq 0 \).
Fig. 2 Decomposition of $dev(x) = e^{\alpha x}$, $\Re \alpha = 0$

**Proposition 4.3** The projective structure defined by the branches of $z^\alpha$, $\Re \alpha > 0$, in $\mathbb{D}^*$ is represented by a pair of vectors $(2\pi i, \frac{2\pi i}{\alpha})$, where $x \mapsto x + 2\pi i$ and $x \mapsto x + \frac{2\pi i}{\alpha}$ are in $\pi_1(\mathbb{D}^*)$ acting in $T$. Conversely, this pair defines the singular projective structure defined by the branches of $z^\alpha$ in $\mathbb{D}^*$.

**Proof** By the discussion above, the first assumption follows. Conversely, given the pair $(2\pi i, \frac{2\pi i}{\alpha})$, we establish that $2\pi i$ is the vector of $\pi_1(\mathbb{D}^*)$ action in $T$ and $\frac{2\pi i}{\alpha}$ is the vector of equivalence action of $dev$ by the monodromy representation.

This pair is associated to the projective structure of the branches of $z^\alpha$, if we show that there is a biholomorphism $\tilde{\phi} : T \to T$ such that $\tilde{\phi}(t + 2\pi i) = \tilde{\phi}(t) + 2\pi i$ and $dev = e^{\alpha x} \circ \tilde{\phi}$ but, the result follows when we put $\tilde{\phi} = id$. \(\square\)

We recall that translation structure on a surface is defined as an atlas such that the coordinate changes are translations. The branches of $\log z$ and $\log z + 1/z$ define different translation structures in $\mathbb{D}^*$, for example. We will show that these structures and their pull-backs by covering maps of degree $\geq 2$ provide us with a list of translation structures in $\mathbb{D}^*$ modulo projective equivalence.

**Proposition 4.4** The translation structures in $\mathbb{D}^*$ induced by the pull-back of $\log z$ by the map $z \mapsto z^n$, $n \in \mathbb{N}$, $n \geq 2$, are projectively equivalent to those induced by $\log z$.

The translation structure defined by $\log z$ can be seen as an infinite cylinder with one end. We know that a branch of $\log z$ is injective in its domain, the same happens with $\log z + 1/z$. We use some ideas from Sect. 2.2 of [2]. He studied local models of poles of meromorphic forms that induce translation structures on compact Riemann surfaces.

We consider $U_R = \{z \in \mathbb{C} \mid |z| > R\}$. Let $V_R$ be the Riemann surface obtained after removing from $U_R$ the $\pi$-neighborhood of the real half-line $\mathbb{R}^-$, and identifying the lines $-i\pi + \mathbb{R}^-$ and $i\pi + \mathbb{R}^-$ by the translation $z \mapsto z + 2\pi i$.

We choose the usual determination of $\log z$ in $\mathbb{C} \setminus \mathbb{R}^-$ restricted to $U_R'$, we obtain the map $z \mapsto z + \log z$ from $U_R' \setminus \mathbb{R}^-$ to $\mathbb{C}$. 

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Proposition 4.5 The map \( z + \log z \) extends to a injective holomorphic map \( f : U_{R'} \to V_R \), if \( R' \) is large enough.

**Proof** See [2, Section 2.2] or [15, Proposição 4.3] for more details. \( \square \)

We conclude that \( \log z + \frac{1}{z} \) is also injective when we restrict to a deleted neighborhood of the origin. We affirm that \( f \) is surjective in a neighborhood of infinity. In fact, we have

Proposition 4.6 The map \( f : U_{R'} \to V_R \) is surjective in a neighborhood of infinity, i.e., for \( Z \in V_R \) with large enough modulus, there exists \( z \in U_{R'} \) such that \( f(z) = Z \).

**Proof** See [2, Section 2.2] or [15, Proposição 4.4] for more details. \( \square \)

Thus, we conclude by Proposition 4.5 that the chart \( \log z + \frac{1}{z} \), defined in the neighborhood of origin, is topologically \( V_R \).

Proposition 4.7 The projective structure on \( D^{*} \) given by the pull-back of \( \log z + \frac{1}{z} \) by \( z \mapsto z^n \) is projectively equivalent to \( \log z + \frac{1}{z^n} \).

Therefore, we can think the chart defined by the branches of \( \log z + \frac{1}{z^n} \), in terms of projective structure, is a suitable rotating and rescaling covering of \( V_R \) of order \( n \).

4.2 Generalization of surgery

Moving branch points is a surgery that consists of deformation of branched local projective charts; it can be thought as a configuration analogous to Schiffer’s variations in Riemann surface’s theory (see [14]). These movements were introduced by Tan in [19] for projective structures with simple branch points and then generalized in [4] for higher-order branch points. Schiffer variations, in particular the moving branch points, produce deformations of the projective structure without changing the monodromy but, in general, do not preserve the underlying complex structure.

Let \( S \) be a closed Riemann surface with a singular projective structure of Fuchsian-type with developing map \( \text{dev} \). Let \( p \) be a singularity of Fuchsian-type.

**Definition 4.8** We define a pair of twins embedding in \( S \) as a pair of embedded curves \( \gamma = \{\gamma_1, \gamma_2\} \) starting from \( p \) such that there is a determination of developing map around \( \gamma_1 \cup \gamma_2 \) which maps \( \gamma_1, \gamma_2 \) into a simple curve \( \hat{\gamma} \subset \mathbb{C}p^1 \).

According to the study of degree developing map of projective structure in the previous section, we conclude that there are twin curves starting from singularities of type \( z^{\alpha} \) where \( \Re \alpha > 1 \). Since \( \log z \) and \( \log z + \frac{1}{z} \) are injective in \( D^{*} \), thus they do not have twins. As a consequence of Proposition 4.7, for \( n \geq 2 \), \( \log z + \frac{1}{z^n} \) is a branched covering of order \( n \) of \( \log z + \frac{1}{z} \). The preimage of a segment starting from origin of \( D \) has \( n \) copies in \( D^{*} \) by \( \log z + \frac{1}{z^n} \) are the candidate twins in this model.

Let us describe the moving branch points. Let \( p \) be a branch point of \( S \), we take twin curves \( \gamma_1, \gamma_2 \) starting from \( p \) with endpoints \( q_1 \) and \( q_2 \). We denote by \( \alpha \) and \( \beta \) angles in \( p \), and \( \theta_i \) the angles in \( q_i, i = 1, 2 \), where \( \theta_i = 2\pi \), if \( q_i \) is a regular point. A new branched projective structure in \( S \) will be obtained by cutting \( S \) along \( \gamma_1 \cup \gamma_2 \) and pasting the copies according to the identifications made in Fig. 3.
After this process, we obtain two new twin curves $\gamma'_1$ and $\gamma'_2$ starting from a point $q$ with total angle $\theta_1 + \theta_2$ and the endpoints $p_1$ and $p_2$ of the new twins have angles $\beta$ and $\alpha$, respectively. Note that the image of $\gamma'_1$ and $\gamma'_2$ by the developing map is the same that $\gamma_1$ and $\gamma_2$, i.e., the surgery does not change the image by develop, then we have a pair of twins embedded into the new structure, and we will return to the initial structure if we move the points along of that pair.

Note that the angles $\alpha$, $\theta_i$ and $\beta$ are multiples of $2\pi$ and if $p$ is a single branch point, then $\alpha = \beta = 2\pi$. This process describes locally a continuous deformation in the space of classes of branched projective structures over $S$ with monodromy representation fixed. Besides collapsing branch points, the process can be used to change the position of branch points and separating a higher-order branch point into several branch points of lower order.

Calsamiglia, Deroin, and Francaviglia proved that two-branched projective structures in compact surfaces with the same quasi-Fuchsian holonomy and the same degree of branching are related by a movement of branch points, so it is possible to use this surgery to show the non-uniqueness of projective structures with the same monodromy representation.

A cone-angle $\theta$ is produced from a sector of angle $\theta$ by identification of their boundaries by an isometry. Then the singularities conical singularities, i.e., with cone-angle still have the same notion of angle that we have in the case of branch points. Thus the surgery to move branch points works in the same way.

In [20], Troyanov characterized orientable compact surfaces with conical singularities. The invariants that represent the opening of the cone are real numbers, and he obtained a classification of these surfaces. More precisely, given $p_1, \ldots, p_k \in S$ and $\theta_1, \ldots, \theta_k > 0$, if $\chi(S) + \sum_{i=1}^k (2\pi - \theta_i) < 0$ (respectively, $= 0$ or $= 1$), then there exists a hyperbolic metric (respectively, Euclidean or spherical) in $S \setminus \{ p_1, \ldots, p_k \}$ with a conical angle $\theta_i$ in $p_i$.

Now, we will prove the inverse surgery for the case $z^\alpha$, $\Re \alpha > 1$, i.e., when the degree of multi-valued function $z^\alpha$ is at least 2. Given two twin curves starting of $z^\alpha$, we remove an angle $2\pi$, that we see in a fundamental domain, it would be to remove one of the biholomorphic strips of the disk minus a radius through of $dev(x) = e^{\alpha x}$. Finally, we glue in the perpendicular way to the boundary of strip.

**Proposition 4.9** Let $\gamma_1$, $\gamma_2$ be a pair of twins starting from a singularity $p$ of type $z^\alpha$, with $\Re \alpha > 1$, forming a sector with angle $2\pi$ and end-points $q_1$ and $q_2$ are regular points. The inverse surgery of the movement that removes the angle $2\pi$ in $p$ results in a simple branch point where start two twin curves whose end-points are a singularity of type $z^{\alpha - 1}$ and a regular point.
Proof After the cut and paste process, we identify two regular points $q_1$ and $q_2$, making a simple branch point $q$. We need to show that removing an angle $2\pi$ of $z^\alpha$, this results in a singularity of type $z^{\alpha-1}$.

We use the decomposition of the universal covering of $\mathbb{D}^*$ as in Fig. 1 and we note the degree of $dev$ is $[\Re\alpha]$.

We remove a biholomorphic strip from the disk minus the radius by $dev$ and all its copies via the action of fundamental group $\pi_1(\mathbb{D}^*)$. We define a relation in the lines $av + bu = 2\pi l$, $l \in \mathbb{Z}$, given by $u + iv \sim u + iv + j\beta$, $j \in \mathbb{Z}$. This identifies the boundaries of strips in the direction of vector $\beta = \frac{2\pi i}{\alpha}$.

The initial $dev$ is given by $D(x) = e^{\alpha x}$. Since the family of lines are twins of the projective structure in $\mathbb{D}^*$, we have that $D(0) = D(\beta)$ and it follows from the equivalence of $D$ by the monodromy representation $\rho : \pi_1(\mathbb{D}^*) \to PSL_2(\mathbb{C})$ given by $\rho([\gamma]) = e^{2\pi i\alpha}w$ that $D(2\pi i) = D(0 + 2\pi i) = D(\beta) \cdot e^{2\pi i\alpha}$, we have used the action $x \mapsto x + 2\pi i$ of fundamental group in $T$.

Note that $D(\beta + w) = D(\beta) \cdot e^{2\pi i\alpha}$, where $w = 2\pi i - \beta$. We affirm that $D$ is equivalent for the monodromy representation $\rho$ and the new action of the fundamental group is given by $x \mapsto x + w$. We need to show that, $\forall x \in T$, $D(x + w) = D(x) \cdot e^{2\pi i\alpha}$. In fact, $D(x + w) = D(x + 2\pi i - \beta) = D(x - \beta) \cdot e^{2\pi i\alpha} = D(x) \cdot e^{2\pi i\alpha}$, since $D(x - \beta) = D(x)$.

We will obtain a new domain $T'$ of the covering map and a new developing map $D_1$ equivariant with respect to $\rho$ with the new action of the fundamental group and the images will coincide with the initial developing map at the respective paste points.

The domain $T'$ is simply connected, its quotient by the action of $x \mapsto x + w$ is homeomorphic to $\mathbb{D}^*$ and therefore can be taken as a universal cover of $\mathbb{D}^*$.

Using the classification obtained in Proposition 4.3, we have to after the surgery, they are given by $(w, \beta) = (2\pi i \left(\frac{\alpha - 1}{\alpha}\right), \frac{2\pi i}{\alpha})$ and the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L(t) = \frac{\alpha}{\alpha - 1} t$ takes $(w, \beta)$ to $(2\pi i, \beta')$ where $\beta' = \frac{2\pi i}{\alpha - 1}$. Therefore the projective structure obtained is biholomorphically equivalent to $z^{\alpha-1}$ in $\mathbb{D}^*$ with new developing map is $D_1 \circ L^{-1}$.

We conclude the models $\log z + \frac{1}{z^n}, n \geq 2$ and $z^\alpha, \Re\alpha > 1$ can have twins and candidates to be deformed isomonodromically. Otherwise, the other models $\log z, \log z + \frac{1}{z}$ and $z^\alpha, \Re\alpha \leq 1$ are rigid, i. e., those that do not have twins (they do not have branching order nonzero).
5 Branching order

In this section, we explore the Problem 2 proposed by Gallo-Kapovich-Marden about minimizing angles in projective structures. Let \( \rho : \pi_1(S^*) \to PSL_2(\mathbb{C}) \) be a representation, we want to answer this question: What is the minimum branching order of a projective structure of Fuchsian-type with monodromy this representation?

We find an obstruction for prescribing local models at the singularities with branching order zero on a compact Riemann surface.

5.1 Algebro-geometric interpretations of projective structures of Fuchsian-type

There are exactly two oriented topologically \( S^2 \)-bundle over the closed Riemann surface \( S \) and they are distinguished by the 2nd Stiefel-Whitney class \( w_2(P) \) of the bundle \( \pi : P \to S \), \( \sigma^2 \equiv w_2(P)(\text{mod} \ 2) \), where \( \sigma \) is section of \( \pi \). Then, the parity of the self-intersection \( \sigma^2 \) depends only on the bundle: \( \sigma^2 \) is even if the bundle is diffeomorphic to the trivial bundle and it is odd, otherwise.

**Proposition 5.1** Let \( \sigma \) and \( \sigma' \) be two holomorphic sections of holomorphic \( \mathbb{CP}^1 \)-bundles on a compact Riemann surface that have the same 2nd class of Stiefel-Whitney. We have to

\[
\sigma^2 \equiv \sigma'^2(\text{mod} \ 2).
\]

In particular, holomorphic sections of the same \( \mathbb{CP}^1 \)-bundle have self-intersection with the same parity.

Let \( \pi : P \to S \) be a \( \mathbb{CP}^1 \)-bundle with a Riccati foliation, after a flipping of an invariant fiber we get another \( \mathbb{CP}^1 \)-bundle \( \pi' : P' \to S \) also equipped with an equivalent birationally Riccati foliation equivalent to \( \pi \). In fact, flipping changes the topological class of the bundle:

**Proposition 5.2** The 2nd classes Stiefel-Whitney \( w_2(P) \) and \( w_2(P') \) have different parities.

**Proof** Let \( \sigma \) be an holomorphic section of fiber bundle \( \pi : P \to S \). If we consider a blow-up at a point outside the section, after a flipping we have the new section \( \tilde{\sigma} \) de \( \pi' : P' \to S \) with \( \tilde{\sigma}^2 = \sigma^2 + 1 \), if we consider blow-up at a point in the section, after a flipping \( \tilde{\sigma} \) has self-intersection \( \sigma^2 - 1 \).

In general, the intersection numbers of holomorphic sections are either all even, or all odd: \( \sigma^2 \mod 2 \) is the topological invariant of the bundle. Thus, at the same compactification, two holomorphic sections have the same parity of tangency order with the foliation.

Let \( \pi : P \to S \) be a \( \mathbb{CP}^1 \)-bundle over \( S \) associated to the monodromy representation \( \rho : \pi_1(S^*) \to PSL_2(\mathbb{C}) \) of a projective structure of Fuchsian-type in \( S \). This \( \mathbb{CP}^1 \)-bundle is equipped with a Riccati foliation \( F_\rho \) (see Brunella [3][Section 4.1]) with the same monodromy \( \rho \). The developing map of the projective structure defined in \( S \) defines a non-trivial holomorphic section \( \sigma \) of \( \pi \) non-invariant by \( F_\rho \). We obtain a formula that relates topological invariants of the surface with the tangency order of Riccati foliation with the holomorphic section (see [3, p. 22]) of the suspension and its self-intersection.

**Proposition 5.3** Under the conditions above, the self-intersection of \( \sigma(S) \) in \( P \) is

\[
\sigma(S) \cdot \sigma(S) = \text{tang}(F_\rho, \sigma(S)) + \chi(S) - k_0,
\]

where \( k_0 \) represents the number of fibers invariant by foliation \( F_\rho \).
Proof It’s a consequence of Brunella’s formula [3]: the cotangent bundle of a Riccati foliation is
\[ T^*_\mathcal{F}_\rho = \pi^*(K_S) \otimes \mathcal{O}_P \left( \sum_{j=1}^{n} k_j F_j \right), \]
where \( K_S \) is the canonical bundle and \( F_1, \ldots, F_n \) are the \( \mathcal{F}_\rho \)-invariant fibres of multiplicity \( k_1, \ldots, k_n \). Since after the compactification, there are \( k_0 \) \( \mathcal{F}_\rho \)-invariant fibers with multiplicity 1, we have \( T_{\mathcal{F}_\rho} \cdot \sigma = 2 - 2g - k_0 \) with the formula \( T_{\mathcal{F}_\rho} \cdot \sigma = \sigma \cdot \sigma - \text{tang}(\mathcal{F}_\rho, \sigma) \) (see [3, Proposition 2.2]), the result follows. \( \square \)

Remark 5.4 We can prove this Proposition with the same arguments of Proposition 11.2.2 of [6], for a complete proof see [15, Teorema 5.3].

Corollary 5.5 Let \( \sigma \) and \( \sigma' \) be two non-trivial holomorphic sections of \( \mathbb{CP}^1 \)-bundle over \( S \) associated to monodromy representation \( \rho : \pi_1(S^*) \to PSL_2(\mathbb{C}) \). Then,
\[ \text{tang}(\mathcal{F}_\rho, \sigma(S)) \equiv \text{tang}(\mathcal{F}_\rho, \sigma'(S)) \mod 2, \]
where \( \mathcal{F}_\rho \) is Riccati foliation of \( \mathbb{CP}^1 \)-bundle compactified.

Proof It follows immediately from the propositions 5.1 and 5.3. \( \square \)

5.2 Minimum branching order

Let \( \rho : \pi_1(S^*) \to PSL_2(\mathbb{C}) \) be a representation. Minimizing the branching order of a projective structure of Fuchsian-type on \( S \) with monodromy \( \rho \) is equivalent to minimizing the index \( \text{tang}(\mathcal{F}_\rho, \sigma(S)) \) of a Riccati foliation and a section \( \sigma \) of the \( \mathbb{CP}^1 \)-bundle with a specific compactification that defines a projective structure.

We recall that a branched projective structure induces a complex structure and thus angles on \( S \). Unbranched points are called regular points and the total angle around them is \( 2\pi \). The cone-angle around a point \( p \) whose branching order is \( n_p \geq 2 \) is \( 2\pi n_p \). The branching divisor of \( \sigma \) is the divisor \( \sum_{p \in S} (n_p - 1)p \). Its degree \( \sum_{p \in S} (n_p - 1) \) is called the total branching order of \( \sigma \).

We extend the notion of branching order to singular points of Fuchsian-type. In fact, it follow of Theorem 1.1 that around each singularity \( p \) of the projective structure \( \sigma \) with given monodromy \( \rho \) the projective charts are defined by \( z^{\alpha+n_p}, 0 < \Re \alpha \leq 1 \) or \( \log z + \frac{1}{z^{n_p}} \). We define \( n_p \in \mathbb{Z} \) as the branching order at each singularity \( p \) and \( e(\sigma) = \sum_{p \in S} n_p \) as the branching order of projective structure \( \sigma \). We also define \( d(\rho) = \min\{e(\sigma) : \sigma \text{ is a projective structure of Fuchsian-type with monodromy } \rho \} \).

Gallo-Kapovich-Marden proved that \( d(\rho) = 0 \) for all liftable non-elementary representations \( \rho \) and \( d(\rho) = 1 \) for all non-liftable non-elementary representations \( \rho \).

We can see the sum \( e(\sigma) \) as a tangency order of a Riccati foliation with sections of fiber bundles from compactification of suspension of a representation \( \rho \). We fix a complex structure on \( S \), it follows from the proof of the Existence Theorem that \( n_p \) are tangency orders of foliation with the section:
\[ e(\sigma) = \text{tang}(\mathcal{F}_{\rho}^{\text{min}}, \sigma(S)), \]
where \( \mathcal{F}_{\rho}^{\text{min}} \) is the foliation provided the compactification which the local models are:
\begin{itemize}
  \item $\alpha w dz - zdw = 0, \alpha \in \mathbb{C}^*$ and $0 \leq \Re \alpha < 1$, at the cusps with non-parabolic monodromy;
  \item $zdw - dz = 0$ at the cusps with parabolic monodromy;
  \item $dw = 0$ at the cusps with trivial monodromy.
\end{itemize}

In that compactification, $e(\sigma) = \tan(F^\text{min}_\rho, \sigma) = 0$ if and only if the section $\sigma$ is transversal to $F^\text{min}_\rho$. For this reason, we will call it by minimum compactification.

First, it follows from the formula $\sigma^2 \equiv w_2(P) (mod 2)$ and Proposition 5.3 that $e(\sigma)$ has the same parity that $w_2(P) + k_0$:

**Lemma 5.6** Let $\rho : \pi_1(S^*) \to PSL_2(\mathbb{C})$ be a monodromy representation of a projective structure of Fuchsian-type $\sigma$ on $S$, we have:

$$w_2(P) + k_0 \equiv e(\sigma) \mod 2$$

where $w_2(P)$ is 2nd Stiefel-Whitney class of the minimum compactification $F^\text{min}_\rho$ and $k_0$ represents the number of points with non-trivial local monodromy.

Since minimum compactification only depends of the monodromy, and its 2nd Stiefel-Whitney class too, we conclude that the parity of $e(\sigma)$ only depends of the monodromy.

If $\rho$ is the monodromy of a projective structure with $e(\sigma) = 0$, we have to 2nd Stiefel-Whitney class of minimum compactification has the same parity that the number of invariant fibers by the foliation.

Let $\rho : \pi_1(S^*) \to PSL_2(\mathbb{C})$ be a representation of fundamental group of surface of finite-type $S^* = S \setminus \{p_1, \ldots, p_k\}$, where $S$ is a closed surface of genus $g \geq 1$, we consider a presentation of $\pi_1(S^*)$:

$$\langle a_i, b_i, c_j, i = 1, \ldots, g, j = 1, \ldots, k \mid \prod_{i=1}^g [a_i, b_i] \prod_{j=1}^k c_j = Id \rangle,$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ is the commutator of $a_i$ and $b_i$, in this presentation we can define the representation $\rho$ as $\rho(a_i) = A_i$, $\rho(b_i) = B_i$ and $\rho(c_j) = C_j$, where $A_i$, $B_i$ and $C_j$ are elements of $PSL_2(\mathbb{C})$ that satisfy

$$\prod_{i=1}^g [A_i, B_i] \prod_{j=1}^k C_j = Id.$$

For each generator $a_i$, $b_i$ and $c_j$, $\rho$ can lift in two ways,

$$\pm \tilde{A}_i, \pm \tilde{B}_i, \pm \tilde{C}_j \in SL_2(\mathbb{C}),$$

whose progetivizations give the Möbius transformations of $A_i$, $B_i$ and $C_j$, respectively, we choose a sign for each element and the product

$$\prod_{i=1}^g [\tilde{A}_i, \tilde{B}_i] \prod_{j=1}^k \tilde{C}_j$$  \hfill (6)

can be $\pm Id$. For the choices where the product gives $Id$ the representation lifts to $SL_2(\mathbb{C})$, if it gives $-Id$, the representation does not lift to $SL_2(\mathbb{C})$.

In the case of genus 0, the presentation of $\pi_1(S^*)$ there are only $c_j, j = 1, \ldots, k$, satisfying

$$\prod_{j=1}^k c_j = Id.$$
**Remark 5.7** Every representation \( \rho : \pi_1(S^\ast) \to PSL_2(\mathbb{C}) \) lifts to \( SL_2(\mathbb{C}) \), because the group \( \pi_1(S^\ast) \) is free. The question here is the representation lifts if we prescribe the local models (e.g., minimal angles) at the cusps or not. Namely, if we have a minimal angle at a point, then we choose a lift.

**Proposition 5.8** The parity of the 2nd Stiefel-Whitney class of the minimum compactification changes according to the representation lifts to \( SL_2(\mathbb{C}) \) or not.

**Proof** Let \( \pi : P \to S \) be a \( \mathbb{CP}^1 \)-bundle with minimum compactification, after a flipping of an invariant fiber we get another \( \mathbb{CP}^1 \)-bundle \( \pi' : P' \to S \) with a birationally equivalent Riccati foliation. It follows from Proposition 5.2. This flipping changes the topological class of bundles, and the 2nd Stiefel-Whitney classes \( w_2(P) \) and \( w_2(P') \) have distinct parities.

If \( \rho \) lifts to \( SL_2(\mathbb{C}) \), we have to each generator of \( \pi_1(S^\ast) \) lifts to a matrix in \( SL_2(\mathbb{C}) \), where the product is \( Id \). The matrices related to the local monodromy representations come from linear differential equations with simple poles used to projectivize and thus obtain the local Riccati model. When we do one flipping the sign of that matrix change, changing the compactification and therefore the product of all matrices is \(-Id\). The representation \( \rho \) does not lift in this compactification.

We will obtain two families that changing parity when doing a flipping: in one of the families, the parity is even and odd in the others. \( \square \)

We obtain a version analogous to Theorem 3.10 of Goldman’s thesis [7]:

**Proposition 5.9** At the minimum compactification, the representation \( \rho : \pi_1(S^\ast) \to PSL_2(\mathbb{C}) \) lifts to \( SL_2(\mathbb{C}) \) if and only if \( w_2(P) \) is even.

**Proof** Build a path in the character variety

\[
Hom(\pi_1(S^\ast), PSL_2(\mathbb{C}))/PSL_2(\mathbb{C})
\]

between one representation \( \rho \) which lifts and the trivial representation, that also lifts, preserving the lifting relation expressed in the Eq. (6) equal to \( Id \).

Using the continuity of 2nd class Stiefel-Whitney, we can conclude that along this path the 2nd classe is constant, and therefore equals to zero. \( \square \)

**Corollary 5.10** At the minimum compactification, the representation \( \rho : \pi_1(S^\ast) \to PSL_2(\mathbb{C}) \) lifts to \( SL_2(\mathbb{C}) \) if and only if \( e(\sigma) \equiv k_0 \mod 2 \).

**Proof** If the representation \( \rho \) lifts, it follows from the Proposition 5.9 that \( w_2(P) \) is even. By Lemma 5.6, we have to \( e(\sigma) \equiv k_0 \mod 2 \). Analogously, \( e(\sigma) \) has not the same parity that \( k_0 \) when the representation \( \rho \) does not lift. \( \square \)

**Theorem 5.11** Let \( \rho : \pi_1(S^\ast) \to PSL_2(\mathbb{C}) \) be a representation. A projective structure of Fuchsian-type with monodromy \( \rho \) has an odd branching order if and only if at the minimum compactification:

1. \( w_2(P) \) is even and the number of cusps with non-trivial local monodromy is odd; or
2. \( w_2(P) \) is odd and the number of cusps with non-trivial local monodromy is even.

where \( w_2(P) \) is the 2nd Stiefel-Whitney class of the bundle \( \pi : P \to S \).

**Proof** Suppose \( \text{tang}(\mathcal{F}_p^{\min}, \sigma(S)) \) is odd. If \( w_2(P) \) is even, then \( \sigma^2 \equiv w_2(P) \equiv 0 \mod 2 \). Therefore it follows from the Proposition 5.3 that \( \text{tang}(\mathcal{F}_p^{\min}, \sigma(S)) \equiv k_0 \mod 2 \), then \( k_0 \) is odd. Analogously, if \( w_2(P) \) is odd, then \( k_0 \) is even.

The other implication follows immediately from Lemma 5.6 above. \( \square \)
The Theorem 5.11 states what are the representations that are not realized as monodromy of projective structures of Fuchsian-type with minimal branching order. For these cases of Theorem 5.11, we will necessarily have \( d(\rho) \geq 1 \). For the representations covered by the Theorem 5.11, do we have \( d(\rho) = 1 \)? In other cases, do we have \( d(\rho) = 0 \)?

Example 5.12 Returning to the Example 2.2 of the three-punctured sphere \( \mathbb{C}P^1 \setminus \{0, 1, \infty\} \), let \( \rho : \pi_1(\mathbb{C}P^1 \setminus \{0, 1, \infty\}) \to PSL_2(\mathbb{C}) \) be the representation given by \( \rho(\gamma_j)w = e^{2\pi i \alpha_j}w \), for \( j = 0, 1, \infty \). At the minimum compactification, as a consequence of the Theorem 1.1 we have the local models of the projective structure \( e^{\alpha_j} \) around the cusps, \( \Re \alpha_j < 1 \), observe that there are all three cusps with non-trivial local monodromy. Fixed this models at the cusps, we can lift each local monodromy to \( SL_2(\mathbb{C}) \):

\[
e^{2\pi i \alpha_j}w \mapsto \begin{pmatrix} e^{\pi i \alpha_j} & 0 \\ 0 & e^{-\pi i \alpha_j} \end{pmatrix}
\]

Thus, we conclude that \( w_2(P) \) is even, as well as the parity of self-intersection \( \sigma^2 \) of the section defined by the developing map. It follows by the formula of Proposition 5.3 that it is necessary introduce one branch point of type \( \epsilon^2 \), thus the \( d(\rho) = 1 \) is odd.

Theorem 1.2 follows as consequence of Theorem 5.11 and the example of existence of such representations above.

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