The double-slit quantum eraser experiments and Hardy’s paradox in the quantum linguistic interpretation

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Abstract

Recently we proposed the linguistic quantum interpretation (called quantum and classical measurement theory), which was characterized as a kind of metaphysical and linguistic turn of the Copenhagen interpretation. This turn from physics to language does not only extend quantum theory to classical systems but also yield the quantum mechanical worldview (i.e., quantum philosophy or quantum language). The purpose of this paper is to formulate the double slits experiment, the quantum eraser experiment, Wheeler’s delayed choice experiment, Hardy’s paradox and the three boxes paradox (the weak value associated with a weak measurement due to Aharonov, et al.) in the linguistic interpretation of quantum mechanics. Through these arguments, we assert that the linguistic interpretation is just the final version of so called Copenhagen interpretation. And therefore, the Copenhagen interpretation does not belong to physics (i.e., the realistic world view) but the linguistic world view.

Keywords: Copenhagen Interpretation, Many-worlds Interpretation, Operator Algebra, Quantum and Classical Measurement Theory, Schrödinger’s Cat, Wigner’s Friend, Double-Slit Experiment, Quantum eraser experiment, Wheeler’s delayed choice experiment, Hardy’s paradox and the three boxes paradox

1 Quantum language (Axioms and Interpretation)

1.1 The overview of quantum language

As mentioned in the above abstract, our purpose is to understand the double slits experiment, the quantum eraser experiment, Wheeler’s delayed choice experiment, Hardy’s paradox and the three boxes paradox in the linguistic interpretation of quantum mechanics, which is proposed in [10]-[22].

According to ref. [14], we shall mention the overview of quantum language (or, measurement theory, in short, MT). Quantum language is characterized as the linguistic turn of the Copenhagen interpretation of quantum mechanics. Quantum language (or, measurement theory ) has two simple rules (i.e. Axiom 1(concerning measurement) and Axiom 2(concerning causal relation)) and the linguistic interpretation (= how to use the Axioms 1 and 2). That is,

\[
\text{Quantum language} = \text{Axiom 1 (measurement)} + \text{Axiom 2 (causality)} + \text{Linguistic interpretation (how to use Axioms)}
\]

(cf. refs. [10]-[22]).
Measurement theory is, by an analogy of quantum mechanics (or, as a linguistic turn of quantum mechanics), constructed as the scientific theory formulated in a certain $C^*$-algebra $\mathcal{A}$ (i.e., a norm closed subalgebra in the operator algebra $B(H)$ composed of all bounded linear operators on a Hilbert space $H$, cf. [24, 26]). Let $\mathcal{N}$ be the weak$^\ast$ closure of $\mathcal{A}$, which is called a $W^*$-algebra. The structure $[\mathcal{A} \subseteq \mathcal{N} \subseteq B(H)]$ is called a fundamental structure of MT.

When $\mathcal{A} = \mathcal{C}(H)$, the $C^*$-algebra composed of all compact operators on a Hilbert space $H$, the MT is called quantum measurement theory (or, quantum system theory), which can be regarded as the linguistic aspect of quantum mechanics. Also, when $\mathcal{A}$ is commutative (that is, when $\mathcal{A}$ is characterized by $C_0(\Omega)$, the $C^*$-algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space $\Omega$ (cf. [26]), the MT is called classical measurement theory. Thus, we have the following classification:

1. Quantum MT (when non-commutative $\mathcal{A} = \mathcal{C}(H)$, $\mathcal{N} = B(H)$)
2. Classical MT (when commutative $\mathcal{A} = C_0(\Omega)$, $\mathcal{N} = L^\infty(\Omega, \nu) \subseteq B(L^2(\Omega, \nu))$)

Also, we assert that quantum language is located as follows:

![Figure 1: The history of the world-view](image)

**1.2 Observables**

Let $[\mathcal{A} \subseteq \mathcal{N} \subseteq B(H)]$ be the fundamental structure of measurement theory. Let $\mathcal{N}_s$ be the pre-dual Banach space of $\mathcal{N}$. That is, $\mathcal{N}_s = \{\rho \mid \rho$ is a weak$^\ast$ continuous linear functional on $\mathcal{N}\}$, and the norm $\|\rho\|_{\mathcal{N}_s}$ is defined by $\sup\{\|\rho(F)\| : F \in \mathcal{N}$ such that $\|F\|_{\mathcal{N}}(= \|F\|_{B(H)}) \leq 1\}$. The bi-linear functional $\rho(F)$ is also denoted by $\langle \rho, F \rangle_{\mathcal{N}_s}$ or in short $(\rho, F)$. Define the mixed state $\rho (\in \mathcal{N}_s)$ such that $\|\rho\|_{\mathcal{N}_s} = 1$ and $\rho(F) \geq 0$ for all $F \in \mathcal{N}$ satisfying $F \geq 0$. And put

$$\mathcal{S}^m(\mathcal{N}_s) = \{\rho \in \mathcal{N}_s \mid \rho$ is a mixed state\}.
According to the noted idea (cf. ref. [3]) in quantum mechanics, an observable $O \equiv (X, F, F)$ in the $W^*$-algebra $\mathcal{N}$ is defined as follows:

(B$_1$) [σ-field] $X$ is a set, $F(\subseteq 2^X$, the power set of $X$) is a σ-field of $X$, that is, “$\Xi_1, \Xi_2, \Xi_3, \ldots \in F$ ⇒ $\cup_{k=1}^{\infty} \Xi_k \in F$,” “$X \in F$” and “$\Xi \in F$ ⇒ $X \setminus \Xi \in F$.”

(B$_2$) [Countably additivity] $F$ is a mapping from $F$ to $\mathcal{N}$ satisfying: (a): for every $\Xi \in F$, $F(\Xi)$ is a non-negative element in $\mathcal{N}$ such that $0 \leq F(\Xi) \leq I$, (b): $F(\emptyset) = 0$, where 0 and $I$ is the 0-element and the identity in $\mathcal{N}$ respectively. (c): for any countable decomposition $\{\Xi_1, \Xi_2, \ldots\}$ of $\Xi \in F$ (i.e., $\Xi_k, \Xi \in F$ such that $\bigcup_{k=1}^{\infty} \Xi_k = \Xi, \Xi_i \cap \Xi_j = \emptyset (i \neq j)$), it holds that

$$\lim_{K \to \infty} \langle \rho, F\left(\bigcup_{k=1}^{K} \Xi_k\right)\rangle_{\mathcal{N}} = \langle \rho, F(\Xi)\rangle_{\mathcal{N}} \quad (\forall \rho \in \mathcal{S}^m(\mathcal{N}))$$

(2)

i.e., $\lim_{K \to \infty} F(\bigcup_{k=1}^{K} \Xi_k) = F(\Xi)$ in the sense of weak* convergence in $\mathcal{N}$.

Remark 1. In the above (b), it is usual to assume the condition: $F(X) = I$. In fact, through all this paper except Section 5, the condition: $F(X) = I$ is assumed. However, for the reason mentioned in Remark 9 later, we start from the above (b).

1.3 Quantum language (Axioms)

With any system $S$, a fundamental structure $[\mathcal{A} \subseteq \mathcal{N} \subseteq B(H)]$ can be associated in which the pure measurement theory (A$_1$) of that system can be formulated. A pure state of the system $S$ is represented by an element $\rho^p(\in \mathcal{S}^p(\mathcal{A}^*)=$"pure state class" (cf. ref. [14])) and an observable is represented by an observable $O = (X, F, F)$ in $\mathcal{N}$. Also, the measurement of the observable $O$ for the system $S$ with the pure state $\rho^p$ is denoted by $M_N(O, S_{[\rho^p]})$ (or more precisely, $M_N(O = (X, F, F), S_{[\rho^p]})$). An observer can obtain a measured value $x (\in X)$ by the measurement $M_N(O, S_{[\rho^p]})$.

The Axiom 1 presented below is a kind of mathematical generalization of Born’s probabilistic interpretation of quantum mechanics.

Axiom 1 [Pure Measurement]. The probability that a measured value $x (\in X)$ obtained by the measurement $M_N(O \equiv (X, F, F), S_{[\rho^p]})$ belongs to a set $\Xi (\in F)$ is given by $\rho^p(F(\Xi))$, if $F(\Xi)$ is essentially continuous at $\rho^p$ (cf. ref. [14]).

Next, we explain Axiom 2 in (A$_1$). Let $(T, \leq)$ be a tree, i.e., a partial ordered set such that “$t_1 \leq t_2$ and $t_2 \leq t_3$” implies “$t_1 \leq t_2$ or $t_2 \leq t_1$”. Assume that there exists an element $t_0 \in T$, called the root of $T$, such that $t_0 \leq t$ (\forall $t \in T$) holds. Put $T^2_\leq = \{(t_1, t_2) \in T^2 | t_1 \leq t_2\}$. The family $\{\Phi_{t_1, t_2} : N_{t_2} \to N_{t_1}\}_{(t_1, t_2) \in T^2_\leq}$ is called a causal relation (due to the Heisenberg picture), if it satisfies the following conditions (C$_1$) and (C$_2$).

(C$_1$) With each $t \in T$, a fundamental structure $[\mathcal{A}_t \subseteq \mathcal{N}_t \subseteq B(H_t)]$ is associated.

(C$_2$) For every $(t_1, t_2) \in T^2_\leq$, a continuous Markov operator $\Phi^{t_1, t_2} : N_{t_2}(\text{with the weak* topology}) \to N_{t_1}$ (with the weak* topology) is defined (i.e., $\Phi^{t_1, t_2} \geq 0$, $\Phi^{t_1, t_2}(I_{N_{t_2}}) = I_{N_{t_1}}$). And it satisfies that $\Phi^{t_1, t_2}\Phi^{t_2, t_3} = \Phi^{t_1, t_3}$ holds for any $(t_1, t_2), (t_2, t_3) \in T^2_\leq$. 
The family of pre-dual operators \( \{ \Phi^{t_1,t_2}_s : \mathfrak{S}^m((\mathcal{N}_{t_1})_s) \rightarrow \mathfrak{S}^m((\mathcal{N}_{t_2})_s) \}_{(t_1,t_2) \in \mathbb{Z}^2_T} \) is called a pre-dual causal relation (due to the Schrödinger picture). If we can regard that \( \Phi^{t_1,t_2}_s(\mathfrak{S}^p((\mathcal{A}_{t_1})^*)) \subseteq \mathfrak{S}^p((\mathcal{A}_{t_2})^*) \), the causal relation \( \{ \Phi_{t_1,t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1} \}_{(t_1,t_2) \in \mathbb{Z}^2_T} \) is said to be deterministic.

Now Axiom 2 in the measurement theory (1) is presented as follows:

**Axiom 2 [Causality].** The causality is represented by a causal relation \( \{ \Phi^{t_1,t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1} \}_{(t_1,t_2) \in \mathbb{Z}^2_T} \).

### 1.4 Linguistic Interpretation

Next, we have to study the linguistic interpretation (i.e., the manual of how to use the above axioms, ) as follows.

That is, we present the following interpretation (D) \([= (D_1), (D_2)]\), which is characterized as a kind of linguistic turn of so-called Copenhagen interpretation (cf. refs. \[14, 15\]). That is,

(D1) Consider the dualism composed of “observer” and “system (=measuring object)”. And therefore, “observer” and “system” must be absolutely separated.

(D2) Only one measurement is permitted. And thus, the state after a measurement is meaningless since it can not be measured any longer. Therefore, the wave collapse is prohibited. Also, the causality should be assumed only in the side of system, however, a state never moves. Thus, the Heisenberg picture should be adopted. And thus, the Schrödinger picture is rather makeshift. Thus, the problem “when and where a measurement is performed?” is nonsense.

and so on. For example, the axioms seem the rule of how to move the piece of a chess game. On the other hand, the linguistic interpretation resembles the standard tactics of chess game. In this sense, we cannot completely say all about the linguistic interpretation.

The following argument is a consequence of the above (D2). For each \( k = 1, 2, \ldots, K \), consider a measurement \( M_{\mathcal{N}}(O_k \equiv (X_k, \mathcal{F}_k, F_k, S_{[p]}) \). However, since the (D2) says that only one measurement is permitted, the measurements \( \{ M_{\mathcal{N}}(O_k, S_{[p]}) \}_{k=1}^K \) should be reconsidered in what follows. Under the commutativity condition such that

\[
F_i(\Xi_i) F_j(\Xi_j) = F_j(\Xi_j) F_i(\Xi_i) \quad \forall \Xi_i \in \mathcal{F}_i, \forall \Xi_j \in \mathcal{F}_j, \ i \neq j,
\]

we can define the product observable (or, simultaneous observable) \( \times^K_{k=1} O_k = (\times^K_{k=1} X_k, \times^K_{k=1} \mathcal{F}_k, \times^K_{k=1} F_k) \) in \( \mathcal{N} \) such that

\[
(\times^K_{k=1} F_k)(\times^K_{k=1} \Xi_k) = F_1(\Xi_1) F_2(\Xi_2) \cdots F_K(\Xi_K) \quad \forall \Xi_k \in \mathcal{F}_k, \forall k = 1, \ldots, K.
\]

Here, \( \times^K_{k=1} \mathcal{F}_k \) is the smallest \( \sigma \)-field including the family \( \{ \times^K_{k=1} \Xi_k : \Xi_k \in \mathcal{F}_k \ k = 1, 2, \ldots, K \} \). Then, the above \( \{ M_{\mathcal{N}}(O_k, S_{[p]}) \}_{k=1}^K \) is, under the commutativity condition (3), represented by the simultaneous measurement \( M_{\mathcal{N}}(\times^K_{k=1} O_k, S_{[p]}) \).

Consider a finite tree \( (T \equiv \{ t_0, t_1, \ldots, t_n \}, \leq) \) with the root \( t_0 \). This is also characterized by the map \( \pi : T \setminus \{ t_0 \} \rightarrow T \) such that \( \pi(t) = \max \{ s \in T \mid s < t \} \). Let \( \{ \Phi^{t,t'}_s : \mathcal{N}_{t'} \rightarrow \mathcal{N}_{t} \}_{(t,t') \in \mathbb{Z}^2_T} \) be a causal relation, which is also represented by \( \{ \Phi_{\pi(t),t} : \mathcal{N}_{t} \rightarrow \mathcal{N}_{\pi(t)} \}_{t \in T \setminus \{ t_0 \}} \). Let an observable \( O_t \equiv (X_t, \mathcal{F}_t, F_t) \) in the \( \mathcal{N}_t \) be given for each \( t \in T \). Note that \( \Phi^{t,t}_s O_t (\equiv (X_t, \mathcal{F}_t, \Phi^{t,t}_s F_t)) \) is an observable in the \( \mathcal{N}_{\pi(t)} \). For the case that a tree \( T \) is not finite, see \[11\].
Remark 2 [Particle or wave]. The argument about the ”particle vs. wave” is meaningless in quantum language. As seen in the following table, this argument is traditional:

| Theories \ P or W | Particle(=symbol) | Wave(=mathematical representation) |
|-------------------|-------------------|-------------------------------------|
| Aristotles        | hyle              | eidos                               |
| Newton mechanics  | point mass        | state (=position, momentum)         |
| Statistics        | population        | parameter                            |
| Quantum mechanics | particle          | state (≈ wave function)              |
| Quantum language  | system (=measuring object) | state |

In the above table, Newtonian mechanics (i.e., mass point ↔ state) may be easiest to understand. Thus, ”particle” and ”wave” are not confrontation concepts. In this sense, the ”wave or particle” is meaningless. In the linguistic interpretation of quantum mechanics, this should be usually understood as the problem ”interference or no interference”.

Remark 3 [Reality]. Since quantum language is a kind of metaphysics, we are not concerned with the reality such as discussed in [4] and [2]. Also, since space and time are independent in quantum language (cf. [15]), we can not expect it to yield a good physical theory (i.e., (5) in Figure 1).

Remark 4 [The Schrödinger’s cat]. Axiom 2 allows us to deal with more than the deterministic causal relation, for example, the Brownian motion and the quantum decoherence, etc. Therefore, we can easily describe the Schrödinger’s cat by quantum language. Thus, this is not a paradox in quantum language. However, quantum language (due to dualism composed of “observer” and “system”) does not have a power to describe Wigner’s friend as well as Descartes’ proposition ”I think, therefore I am” (cf. [16]).

Remark 5 [The commutative condition (3)].(i): If the commutative condition (3) does not hold, \( \times_{k=1}^{K} O_k = (\times_{k=1}^{K} X_k, \otimes_{k=1}^{K} F_k, \times_{k=1}^{K} F_k) \) is not an observable, but the \( \mathcal{N} \)-valued measure space.

(ii): Assume that there exists an observable \( \hat{O} = (\times_{k=1}^{K} X_k, \otimes_{k=1}^{K} F_k, \hat{F}) \) such that

\[
\hat{F}(X_1 \times X_2 \times \cdots \times X_{j-1} \times \Xi_j \times X_{j+1} \times \cdots \times X_K) = F_j(\Xi_j) \quad (\Xi_j \in \mathcal{F}_j, \ j = 1, 2, \ldots, K)
\]

Then, there is a reason to call the \( \hat{O} \) a simultaneous observable.

(iii): Also, it may be worth while investigating the concept such that \( \hat{O} = (\times_{k=1}^{K} X_k, \otimes_{k=1}^{K} F_k, \hat{F}) \) is an simultaneous observable concerning \( \rho \).

2 The double-slit experiment

Although Feynmann’s enthusiasm is transmitted in the explanation of the double-slit experiment in [6], we do not think that his explanation is sufficient. That is because the double-slit experiment and so on should be explained after the answer to ”What kind of measurement is taken?".
Figure 2(1). Potential $V_1(x, y) = \infty$ on the thick line, = 0 (elsewhere)

Figure 2(2). Potential $V_2(x, y) = \infty$ on the thick line, = 0 (elsewhere)

That is,

$$V_2 = V_1 + \text{"the line segment } \overline{ab}\text{"}$$

Consider a tree $T, \leq$ with the two branches such that

$$T = \{0\} \cup T_1 \cup T_2$$

where

$$T_1 = \{(1, s) \mid s > 0\}, \quad T_2 = \{(2, s) \mid s > 0\}$$

$$0 \leq (i, s_i) \quad (i = 1, 2, \quad 0 < s_i)$$

$$(i, s_i) \leq (i, s'_i) \quad (i = 1, 2, \quad s_i \leq s'_i)$$
For each \( t \in T \), define the fundamental structure

\[ [C(H_t) \subseteq B(H_t) \subseteq B(H_t)] \]

where \( H_t = L^2(\mathbb{R}^2) \) (\( \forall t \in T \)). Let \( u_0 \in H_0 = L^2(\mathbb{R}^2) \) be a initial wave-function such that (\( k_0 > 0 \), small \( \sigma > 0 \)):

\[ u_0(x, y) \approx \psi_x(x, 0)\psi_y(y, 0) = \frac{1}{\sqrt{\pi/2\sigma}} \exp \left( i k_0 x - \frac{x^2}{2\sigma^2} \right) \cdot \frac{1}{\sqrt{\pi/2\sigma}} \exp \left( - \frac{y^2}{2\sigma^2} \right) \]

where the average momentum \((p_0^0, p_0^0)\) is calculated by

\[ (p_0^0, p_0^0) = \left( \int_{\mathbb{R}} \overline{\psi}_x(x, 0) \cdot \frac{h\partial\psi_x(x, 0)}{i\partial x} dx, \int_{\mathbb{R}} \overline{\psi}_y(y, 0) \cdot \frac{h\partial\psi_y(y, 0)}{i\partial y} dy \right) = (hk_0, 0) \]

That is, we assume that the initial state of the particle \( P \) (in Figures 2(1) and 2(2)) is equal to \(|u_0\rangle\langle u_0|\).

As mentioned in the above, consider two branches \( T_1 \) and \( T_2 \). Thus, concerning \( T_1 \), we have the following Schrödinger equation:

\[ ih \frac{\partial}{\partial t} \psi_t(x, y) = \mathcal{H}_1 \psi_t(x, y), \quad \mathcal{H}_1 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + V_1(x, y) \]

Also, concerning \( T_2 \), we have the following Schrödinger equation:

\[ ih \frac{\partial}{\partial t} \psi_t(x, y) = \mathcal{H}_2 \psi_t(x, y), \quad \mathcal{H}_2 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + V_2(x, y) \]

Let \( s_1, s_2 \) be sufficiently large positive numbers. Put \( t_1 = (1, s_1) \in T_1, t_2 = (2, s_2) \in T_2 \). Define the subtree \( T'(\subseteq T) \) such that \( T' = \{0, t_1, t_2\} \) and \( 0 < t_1, 0 < t_2 \). Thus, we have the causal relation: \( \{\Phi_i^{0, t_i} : B(H_{s_i}) \to B(H_0)\}_{i=1, 2} \) where

\[ \Phi_1^{0, t_1} F = e^{\frac{2\pi i}{\hbar} \int_{H_0} F_1} e^{-\frac{2\pi i}{\hbar} \int_{H_1}} \quad (\forall F_1 \in B(H_1) = B(L^2(\mathbb{R}^2))) \]

\[ \Phi_2^{0, t_2} F = e^{\frac{2\pi i}{\hbar} \int_{H_0} F_2} e^{-\frac{2\pi i}{\hbar} \int_{H_2}} \quad (\forall F_2 \in B(H_2) = B(L^2(\mathbb{R}^2))) \]

Put \( Z = \{0, \pm 1, \pm 2, \cdots\} \). Let \( \delta \) be a sufficiently small positive number. For each \( n \in Z \), define the region \( D_n(\subseteq \mathbb{R}^2) \) such that

\[ D_0 = \{(x, y) \in \mathbb{R}^2 \mid x < b\} \]

\[ D_n = \begin{cases} 
\{ (x, y) \in \mathbb{R}^2 \mid b \leq x, \delta(n - 1) < y \leq \delta n \} & (n = 1, 2, \cdots) \\
\{ (x, y) \in \mathbb{R}^2 \mid b \leq x, \delta n < y \leq \delta(n + 1) \} & (n = -1, -2, \cdots) 
\end{cases} \]

Define the observable \((Z, 2^Z, F)\) in \( B(L^2(\mathbb{R}^2)) \) such that

\[ [F(\{n\})](x, y) = \chi_{D_n}(x, y) \quad (\forall n \in Z, \forall (x, y) \in \mathbb{R}^2) \]

where \( \chi_{D_n}(x, y) = 1 \) (\( (x, y) \in D_n \)), 0 (elsewhere).

Hence, we can consider two observables \( O_{t_1} = (Z, 2^Z, F) \) in \( B(H_{t_1})(= B(L^2(\mathbb{R}^2))) \) and \( O_{t_2} = (Z, 2^Z, F) \) in \( B(H_{t_2})(= B(L^2(\mathbb{R}^2))) \).

Since \( \Phi_1^{0, t_1} O_{t_1} = (Z, 2^Z, \Phi_1^{0, t_1} F) \) is the observable in \( B(H_0) \), we have the measurement

\[ M_{B(H_0)}(\Phi_1^{0, t_1} O_{t_1}, S_{[\rho_0]}) \]

We consider that this is just the description of the standard double-slit experiment. The following is well known:
Fig. 2(2) says that

Also, since $\Phi_1 \neq 0$, we have the sequential causal observable

where the interference term (i.e., the third term) appears.

Further, note that we have the sequential causal observable

so that the particle $P$ passed through the hole $A$.

However, it should be noted that

Remark 6 Although, strictly speaking, we have to say that the statement "the particle $P$ passed through the hole $A" can not be described in terms of quantum language, it should be allowed to say the statement $(E_2)$. Also, concerning the statement $(E_3)$, note that

but the observables $O_{t_1}$ and $O_{t_2}$ are in different worlds (i.e., different branches), except while $\Phi_1^{0,t_1} = \Phi_2^{0,t_2}$. We consider that, the double-slit experiment can not be completely explained without branches In this sense, our argument may be similar to Everett’s (cf. [5]). Also, for our other understanding of the double-slit experiment, see [8] and [9].

3 The quantum eraser experiment

3.1 Usual situation

Let $H$ be a Hilbert space. And let $O = (X, F, F)$ be an observable in $B(H)$. Let $u_1$ and $u_2$ ($\in H$) be orthonormal elements, i.e., $\|u_1\|_H = \|u_2\|_H = 1$ and $\langle u_1, u_2 \rangle = 0$. Put

$u = \alpha_1 u_1 + \alpha_2 u_2$

where $\alpha_i \in \mathbb{C}$ such that $|\alpha_1|^2 + |\alpha_2|^2 = 1$.

Consider the measurement:

$M_{B(H)}(O, S_{[u]}(u))$. (7)

Then, the probability that a measured value $x(\in X)$ belongs to $\Xi(\in F)$ is given by

$\langle u, F(\Xi)u \rangle$

$= \langle \alpha_1 u_1 + \alpha_2 u_2, F(\Xi)(\alpha_1 u_1 + \alpha_2 u_2) \rangle$

$= |\alpha_1|^2 \langle u_1, F(\Xi)u_1 \rangle + |\alpha_2|^2 \langle u_2, F(\Xi)u_2 \rangle + \overline{\alpha_1}\alpha_2 \langle u_1, F(\Xi)u_2 \rangle + \alpha_1 \overline{\alpha_2} \langle u_2, F(\Xi)u_1 \rangle$

$= |\alpha_1|^2 \langle u_1, F(\Xi)u_1 \rangle + |\alpha_2|^2 \langle u_2, F(\Xi)u_2 \rangle + 2\text{Real part} \langle \overline{\alpha_1}\alpha_2 \langle u_1, F(\Xi)u_2 \rangle \rangle$

where the interference term (i.e., the third term) appears.
3.2 Tensor Hilbert space

Let $\mathbb{C}^2$ be the two dimensional Hilbert space, i.e., $\mathbb{C}^2 = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mid z_1, z_2 \in \mathbb{C} \right\}$. And put

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Here, define the observable $O_x = (\{-1, 1\}, 2^{(-1, 1)}, F_x)$ in $B(\mathbb{C}^2)$ such that

$$F_x(\{1\}) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad F_x(\{-1\}) = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

Here, note that

$$F_x(\{1\})e_1 = \frac{1}{2}(e_1 + e_2), \quad F_x(\{1\})e_2 = \frac{1}{2}(e_1 + e_2)$$

$$F_x(\{-1\})e_1 = \frac{1}{2}(e_1 - e_2), \quad F_x(\{-1\})e_2 = \frac{1}{2}(-e_1 + e_2)$$

Also, define the existence observable $O_E = (\{1\}, 2^{(1)}, F_E)$ in $B(\mathbb{C}^2)$ such that

$$F_E(\{1\}) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Further, define $\psi \in \mathbb{C}^2 \otimes H$ (the tensor Hilbert space of $\mathbb{C}^2$ and $H$) such that

$$\psi = \alpha_1 e_1 \otimes u_1 + \alpha_2 e_2 \otimes u_2$$

where $\alpha_i \in \mathbb{C}$ such that $|\alpha_1|^2 + |\alpha_2|^2 = 1$.

3.3 No interference

Consider the measurement:

$$M_{B(\mathbb{C}^2 \otimes H)}(O_E \otimes O, S_{[\psi]}(\psi)) \quad (8)$$

Then, we see

$(F_1)$ the probability that a measured value $(1, x) \in \{1\} \times \Xi$ belongs to $\{1\} \times \Xi$ is given by

$$\langle \psi, (I \otimes F(\Xi))\psi \rangle$$

$$= \langle \alpha_1 e_1 \otimes u_1 + \alpha_2 e_2 \otimes u_2, (I \otimes F(\Xi))(\alpha_1 e_1 \otimes u_1 + \alpha_2 e_2 \otimes u_2) \rangle$$

$$= \langle \alpha_1 e_1 \otimes u_1 + \alpha_2 e_2 \otimes u_2, \alpha_1 e_1 \otimes F(\Xi)u_1 + \alpha_2 e_2 \otimes F(\Xi)u_2 \rangle$$

$$= |\alpha_1|^2 \langle u_1, F(\Xi)u_1 \rangle + |\alpha_2|^2 \langle u_2, F(\Xi)u_2 \rangle$$

where the interference term disappears.
3.4 Interference

Consider the measurement:

\[ M_{B(C^2\otimes H)}(O_x \otimes O, S_{||\psi||}(\psi)) \] (9)

Then, we see:

\[ (F_2) \text{ the probability that a measured value } (1, x)(\in \{-1, 1\} \times X) \text{ belongs to } \{1\} \times \Xi \text{ is given by} \]

\[
\langle \psi, (F_x(\{1\}) \otimes F(\Xi))\psi \rangle \\
= (\alpha_1 e_1 \otimes u_1 + \alpha_2 e_2 \otimes u_2, (F_x(\{1\}) \otimes F(\Xi)))(\alpha_1 e_1 \otimes u_1 + \alpha_2 e_2 \otimes u_2) \\
= \frac{1}{2}(\alpha_1 e_1 \otimes u_1 + \alpha_2 e_2 \otimes u_2, \alpha_1(e_1 + e_2) \otimes F(\Xi)u_1 + \alpha_2(e_1 + e_2) \otimes F(\Xi)u_2) \\
= \frac{1}{2}\left(|\alpha_1|^2\langle u_1, F(\Xi)u_1 \rangle + |\alpha_2|^2\langle u_2, F(\Xi)u_2 \rangle + \overline{\alpha}_1 \alpha_2 \langle u_1, F(\Xi)u_1 \rangle + \alpha_1 \overline{\alpha}_2 \langle u_2, F(\Xi)u_1 \rangle \right) \\
= \frac{1}{2}\left(|\alpha_1|^2\langle u_1, F(\Xi)u_1 \rangle + |\alpha_2|^2\langle u_2, F(\Xi)u_2 \rangle + 2[\text{Real part}](\overline{\alpha}_1 \alpha_2 \langle u_1, F(\Xi)u_2 \rangle) \right)
\]

where the interference term (i.e., the third term) appears.

Also, we see:

\[ (F_3) \text{ the probability that a measured value } (-1, x)(\in \{-1, 1\} \times X) \text{ belongs to } \{-1\} \times \Xi \text{ is given by} \]

\[
\langle \psi, (F_x(\{-1\}) \otimes F(\Xi))\psi \rangle \\
= (\alpha_1 e_1 \otimes u_1 + \alpha_2 e_2 \otimes u_2, (F_x(\{-1\}) \otimes F(\Xi)))(\alpha_1 e_1 \otimes u_1 + \alpha_2 e_2 \otimes u_2) \\
= \frac{1}{2}(\alpha_1 e_1 \otimes u_1 + \alpha_2 e_2 \otimes u_2, \alpha_1(e_1 - e_2) \otimes F(\Xi)u_1 + \alpha_2(-e_1 + e_2) \otimes F(\Xi)u_2) \\
= \frac{1}{2}\left(|\alpha_1|^2\langle u_1, F(\Xi)u_1 \rangle + |\alpha_2|^2\langle u_2, F(\Xi)u_2 \rangle - \overline{\alpha}_1 \alpha_2 \langle u_1, F(\Xi)u_1 \rangle - \alpha_1 \overline{\alpha}_2 \langle u_2, F(\Xi)u_1 \rangle \right) \\
= \frac{1}{2}\left(|\alpha_1|^2\langle u_1, F(\Xi)u_1 \rangle + |\alpha_2|^2\langle u_2, F(\Xi)u_2 \rangle - 2[\text{Real part}](\overline{\alpha}_1 \alpha_2 \langle u_1, F(\Xi)u_2 \rangle) \right)
\]

where the interference term (i.e., the third term) appears.

Remark 7 Note that

\[
\begin{array}{c}
(F_1) \\
\text{no interference}
\end{array} = \begin{array}{c}
(F_2) + (F_3) \\
\text{interferences are canceled}
\end{array}
\]

This was experimentally examined in [27].

4 Wheeler’s delayed choice experiment

Let \( H \) be a two dimensional Hilbert space, i.e., \( H = \mathbb{C}^2 \). Let \( f_1, f_2 \in H \) such that

\[
f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

Put

\[
u = \frac{f_1 + f_2}{\sqrt{2}}
\]
Thus, we have the state $\rho = |u\rangle\langle u| \in \mathbb{S}^p(B(\mathbb{C}^2))$.

Let $U \in B(\mathbb{C}^2)$ be an unitary operator such that

$$U = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix}$$

and let $\Phi : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ be the homomorphism such that

$$\Phi(F) = U^* FU \quad (\forall F \in B(\mathbb{C}^2))$$

Consider two observable $O_f = (\{1, 2\}, 2^{(1,2)}, F)$ and $O_g = (\{1, 2\}, 2^{(1,2)}, G)$ in $B(\mathbb{C}^2)$ such that

$$F(\{1\}) = |f_1\rangle\langle f_1|, \quad F(\{2\}) = |f_2\rangle\langle f_2|$$

and

$$G(\{1\}) = |g_1\rangle\langle g_1|, \quad G(\{2\}) = |g_2\rangle\langle g_2|$$

where

$$g_1 = \frac{f_1 + f_2}{\sqrt{2}}, \quad g_2 = \frac{f_1 - f_2}{\sqrt{2}}$$

![Figure 3(1). $[D_1 + D_2]$=Observable$O_f$](image)

Firstly, consider the measurement:

$$M_{B(\mathbb{C}^2)}(\Phi O_f, S[\rho])$$

Then, we see:

$$(G_1)$$ the probability that \[
\begin{bmatrix} \text{a measured value 1} \\ \text{a measured value 2} \end{bmatrix}
\] is obtained by $M_{B(\mathbb{C}^2)}(\Phi O_f, S[\rho])$ is given by

$$\begin{bmatrix} \text{tr}(\rho \cdot \Phi F(\{1\})) \\ \text{tr}(\rho \cdot \Phi F(\{2\})) \end{bmatrix} = \begin{bmatrix} |\langle U u, F(\{1\}) U u \rangle |^2 \\ |\langle U u, F(\{2\}) U u \rangle |^2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$
Next, consider the measurement:

\[
M_{B(C^2)}(\Phi^2O_g, S_{[\rho]})
\] (11)

Figure 3(2). \([D_1 + D_2] = \text{Observable}_{O_g}\)

Then, we see:

\[
(G_2) \quad \text{the probability that} \quad \begin{bmatrix} \text{a measured value 1} \\ \text{a measured value 2} \end{bmatrix} \quad \text{is obtained by} \quad M_{B(C^2)}(\Phi^2O_g, S_{[\rho]}) \quad \text{is given by}
\]

\[
\begin{bmatrix}
tr(\rho \cdot \Phi^2G(\{1\})) \\
tr(\rho \cdot \Phi^2G(\{2\}))
\end{bmatrix} = \begin{bmatrix}
\langle u, \Phi^2G(\{1\})u \rangle \\
\langle u, \Phi^2G(\{2\})u \rangle
\end{bmatrix} = \begin{bmatrix}
|\langle u, UUg_1 \rangle|^2 \\
|\langle u, UUg_2 \rangle|^2
\end{bmatrix} = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

Also, consider the following Figure 3(3). This is clearly the same as the situation of Figure 3(1). Therefore, this is characterized by the same measurement \(M_{B(C^2)}(\Phi O_f, S_{[\rho]})\).

Figure 3(3). \([D_2 + D_1] = \text{Observable}_{O_f}\)

Remark 8. When the half mirror 2 is set in Figure 3(1) (i.e., when the observable \(O_f\) changes to the \(O_g\)), we see that the measurement \(M_{B(C^2)}(\Phi O_f, S_{[\rho]})\) changes to \(M_{B(C^2)}(\Phi^2O_g, S_{[\rho]})\). Thus, we think
that Wheeler’s delayed choice experiment (cf. [28]) is not surprising in the linguistic interpretation of quantum mechanics. That is because the problem is not "wave or particle" but "interference or no interference". On the other hand, the statement \(C_1\) concerning \(M_{B(\mathbb{C}^2)}(\Phi_O f, S_{[\hat{\rho}]} )\) is surprising, since it implies the non-locality. This surprising fact is essentially the same as the de Broglie’s paradox (in \(B(L^2(\mathbb{R}^3))\)).

5 Hardy’s paradox

Let \(H\) be a two dimensional Hilbert space, i.e., \(H = \mathbb{C}^2\). Let \(f_1, f_2, g_1, g_2 \in H\) such that

\[
\begin{align*}
  f_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & g_1 &= \frac{f_1 + f_2}{\sqrt{2}}, & g_2 &= \frac{f_1 - f_2}{\sqrt{2}} \\
  f_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & g_2 &= \frac{f_1 + f_2}{\sqrt{2}}.
\end{align*}
\]

Put

\[ u = \frac{f_1 + f_2}{\sqrt{2}} ( = g_1 ). \]

Now, consider the tensor Hilbert space \(H \otimes H = \mathbb{C}^2 \otimes \mathbb{C}^2\). Thus, put

\[ \hat{u} = u \otimes u, \quad \hat{\rho} = |u \otimes u \rangle \langle u \otimes u |. \]

Define the projection \(P : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2\) such that

\[
P(\alpha_{11} f_1 \otimes f_1 + \alpha_{12} f_1 \otimes f_2 + \alpha_{21} f_2 \otimes f_1 + \alpha_{22} f_2 \otimes f_2 ) = \alpha_{12} f_1 \otimes f_2 + \alpha_{11} f_2 \otimes f_1 + \alpha_{21} f_2 \otimes f_2
\]

and thus, define the \(\Psi : B(\mathbb{C}^2 \otimes \mathbb{C}^2) \to B(\mathbb{C}^2 \otimes \mathbb{C}^2)\) by

\[
\hat{\Psi}(\hat{A}) = P \hat{A} P \quad (\hat{A} \in B(\mathbb{C}^2 \otimes \mathbb{C}^2))
\]

5.1 Concerning the tensor observable \(O_g \otimes O_g\)

Define the observable \(\hat{O}_{gg} = \{(1, 2) \times \{1, 2\}, 2^{\{1,2\} \times \{1,2\}}, \hat{H}_{gg}\}\) in \(B(\mathbb{C}^2 \otimes \mathbb{C}^2)\) by the tensor observable \(O_g \otimes O_g\), that is,

\[
\begin{align*}
  \hat{H}_{gg}(\{(1, 1)\}) &= |g_1 \otimes g_1 \rangle \langle g_1 \otimes g_1 |, & \hat{H}_{gg}(\{(1, 2)\}) &= |g_1 \otimes g_2 \rangle \langle g_1 \otimes g_2 |, \\
  \hat{H}_{gg}(\{(2, 1)\}) &= |g_2 \otimes g_1 \rangle \langle g_2 \otimes g_1 |, & \hat{H}_{gg}(\{(2, 2)\}) &= |g_2 \otimes g_2 \rangle \langle g_2 \otimes g_2 |.
\end{align*}
\]

Consider the measurement:

\[
M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(\hat{\Psi} \hat{O}_{gg}, S_{[\hat{\rho}]})(12)
\]

Then, the probability that a measured value \((2, 2)\) is obtained by \(M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(\hat{\Psi} \hat{O}, S_{[\hat{\rho}]} )\) is given by

\[
\begin{align*}
  &\langle u \otimes u, P \hat{H}_{gg}(\{(2, 2)\})P(u \otimes u) \rangle \\
  = &\frac{1}{16} |\langle f_1 - f_2 \rangle \otimes (f_1 - f_2) , \langle f_1 \otimes f_2 + f_2 \otimes f_1 + f_2 \otimes f_2 \rangle |^2 \\
  = &\frac{1}{16} |\langle f_1 \otimes f_1 - f_1 \otimes f_2 - f_2 \otimes f_1 + f_2 \otimes f_2 , \langle f_1 \otimes f_2 + f_2 \otimes f_1 + f_2 \otimes f_2 \rangle |^2 \\
  = &\frac{1}{16} = \frac{1}{16}
\end{align*}
\]
Also, the probability that a measured value $(1, 1)$ is obtained by $\mathbf{M}_{B(C^2 \otimes C^2)}(\hat{\Psi} \hat{O}_{gg}, S_{[\bar{\rho}]} )$ is given by
\[
\langle u \otimes u, P\hat{H}_{gg}((1, 1))P(u \otimes u) \rangle = \frac{|\langle (f_1 + f_2) \otimes (f_1 + f_2), f_1 \otimes f_2 + f_2 \otimes f_1 + f_2 \otimes f_2 \rangle|^2}{16} = \frac{9}{16}
\]
Further, the probability that a measured value $(2, 1)$ is obtained by $\mathbf{M}_{B(C^2 \otimes C^2)}(\hat{\Psi} \hat{O}_{gg}, S_{[\bar{\rho}]} )$ is given by
\[
\langle u \otimes u, P\hat{H}_{gg}((2, 1))P(u \otimes u) \rangle = \frac{|\langle (f_1 - f_2) \otimes (f_1 - f_2), f_1 \otimes f_2 + f_2 \otimes f_1 + f_2 \otimes f_2 \rangle|^2}{16} = \frac{1}{16}
\]
Similarly,
\[
\langle u \otimes u, P\hat{H}_{gg}((2, 1))P(u \otimes u) \rangle = \frac{1}{16}
\]

**Remark 9.** Recalling Remark 1, note that
\[
\frac{1}{16} + \frac{9}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{4} < 1
\]
Thus, the probability that no measured value is obtained by the measurement $\mathbf{M}_{B(C^2 \otimes C^2)}(\hat{\Psi} \hat{O}, S_{[\bar{\rho}]} )$ is equal to $\frac{1}{4}$.

### 5.2 Concerning the tensor observable $O_g \otimes O_f$

Define the observable $\hat{O}_{gf} = \{(1, 2) \times \{1, 2\}, 2^{\{1,2\} \times \{1,2\}}, \hat{H}_{gf} \}$ in $B(C^2 \otimes C^2)$ by the tensor observable $O_g \otimes O_f$, that is,
\[
\hat{H}_{gf}((1, 1)) = |g_1 \otimes f_1\rangle \langle g_1 \otimes f_1|, \quad \hat{H}_{gf}((1, 2)) = |g_1 \otimes f_2\rangle \langle g_1 \otimes f_2|,
\]
\[
\hat{H}_{gf}((2, 1)) = |g_2 \otimes f_1\rangle \langle g_2 \otimes f_1|, \quad \hat{H}_{gf}((2, 2)) = |g_2 \otimes f_2\rangle \langle g_2 \otimes f_2|
\]
Consider the measurement:
\[
\mathbf{M}_{B(C^2 \otimes C^2)}(\hat{\Psi} \hat{O}_{gf}, S_{[\bar{\rho}]} )
\]
Then, the probability that a measured value $(2, 2)$ is obtained by $\mathbf{M}_{B(C^2 \otimes C^2)}(\hat{\Psi} \hat{O}_{gf}, S_{[\bar{\rho}]} )$ is given by
\[
\langle u \otimes u, P\hat{H}_{gf}((2, 2))P(u \otimes u) \rangle = \frac{|\langle (f_1 - f_2) \otimes f_2, f_1 \otimes f_2 + f_2 \otimes f_1 + f_2 \otimes f_2 \rangle|^2}{8} = 0
\]
Also, the probability that a measured value $(1, 1)$ is obtained by $\mathbf{M}_{B(C^2 \otimes C^2)}(\hat{\Psi} \hat{O}_{gf}, S_{[\bar{\rho}]} )$ is given by
\[
\langle u \otimes u, P\hat{H}_{gf}((1, 1))P(u \otimes u) \rangle = \frac{|\langle (f_1 + f_2) \otimes f_1, f_1 \otimes f_2 + f_2 \otimes f_1 + f_2 \otimes f_2 \rangle|^2}{8} = \frac{1}{8}
\]
Further, the probability that a measured value (1, 2) is obtained by \( M_{B(H)}(O_1, S_{[\rho]}) \) is given by

\[
\langle u \otimes u, \hat{P} \hat{H}_{gf} \{ \{1, 2\} \} \rangle P(u \otimes u) = \frac{|\langle f_1 + f_2 \otimes f_2, f_1 \otimes f_2 + f_2 \otimes f_1 + f_2 \otimes f_2 \rangle|^2}{16} = \frac{4}{8}
\]

Similarly,

\[
\langle u \otimes u, \hat{P} \hat{H}_{gf} \{ \{2, 1\} \} \rangle P(u \otimes u) = \frac{|\langle f_1 - f_2 \otimes f_2, f_1 \otimes f_2 + f_2 \otimes f_1 + f_2 \otimes f_2 \rangle|^2}{8} = \frac{1}{8}
\]

**Remark 10.** It is usual to consider that "Which way pass problem" is nonsense. However, for the other aspect of this problem, see Remarks 11 and 12 later.

### 6 The three boxes paradox

Let \( H \) be the three dimensional Hilbert space, i.e., \( H = \mathbb{C}^3 \). Let \( f_1, f_2, f_3 \in H \) such that

\[
f_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

Put

\[
u = \frac{f_1 + f_2 + f_3}{\sqrt{3}}, \quad g_1 = \frac{f_1 + f_2 - f_3}{\sqrt{3}}.
\]

And, put

\[
\rho = |u\rangle \langle u|
\]

Further, consider two observables \( O_1 = (\{1, 2\}, 2^{1, 2}, G) \) and \( O_2 = (\{1, 2, 3\}, 2^{1, 2, 3}, F) \) in \( B(H) \) such that

\[
G(\{1\}) = |g_1\rangle \langle g_1| = \frac{1}{3} f_1 + f_2 - f_3 \langle f_1 + f_2 - f_3 | \equiv P_1, \quad G(\{2\}) = I - |g_1\rangle \langle g_1| \equiv P_2
\]

and

\[
F(\{1\}) = |f_1\rangle \langle f_1|, \quad F(\{3\}) = |f_2\rangle \langle f_2|, \quad F(\{3\}) = |f_3\rangle \langle f_3|,
\]

And consider the measurements

\[
M_{B(H)}(O_1, S_{[\rho]}) \quad \text{and} \quad M_{B(H)}(O_2, S_{[\rho]}) \quad (14)
\]

Clearly, the probability that a measured value 1 obtained by \( M_{B(H)}(O_1, S_{[\rho]}) \) is given by

\[
\langle u, G(\{1\}) u \rangle = \langle u, |g_1\rangle \langle g_1| u \rangle = \frac{1}{9} |\langle f_1 + f_2 + f_3, f_1 + f_2 - f_3 \rangle|^2 = \frac{1}{9}
\]

and, the probability that a \[
\begin{bmatrix}
\text{measured value 1} \\
\text{measured value 2} \\
\text{measured value 3}
\end{bmatrix}
\]

obtained by \( M_{B(H)}(O_2, S_{[\rho]}) \) is given by

\[
\begin{bmatrix}
\langle u, |f_1\rangle \langle f_1| u \rangle = |\langle f_1, u \rangle|^2 = \frac{1}{3} \\
\langle u, |f_2\rangle \langle f_2| u \rangle = |\langle f_2, u \rangle|^2 = \frac{1}{3} \\
\langle u, |f_3\rangle \langle f_3| u \rangle = |\langle f_3, u \rangle|^2 = \frac{1}{3}
\end{bmatrix}
\]
Since $O_1$ and $O_2$ do not commute, the simultaneous observable $O_1 \times O_2$ does not exist. However, putting $X = \{1, 2\}$, $Y = \{1, 2, 3\}$, $O_1 \times O_2$ may be formally written by

$$O_1 \times O_2 = (X \times Y, 2^X \times Y, G \times F)$$

where

$$G(\{1\})F(\{1\}) = (\langle g_1, f_1 \rangle | g_1 \langle f_1 | = \frac{1}{3} | f_1 + f_2 - f_3 \rangle \langle f_1 |$$

$$G(\{1\})F(\{2\}) = (\langle g_1, f_2 \rangle | g_1 \langle f_2 | = \frac{1}{3} | f_1 + f_2 - f_3 \rangle \langle f_2 |$$

$$G(\{1\})F(\{3\}) = (\langle g_1, f_3 \rangle | g_1 \langle f_3 | = \frac{1}{3} | f_1 + f_2 - f_3 \rangle \langle f_3 |$$

$$\ldots$$

However, we try to consider the "measurement"

$$" M_{B(H)}(O_1 \times O_2, S_{[\rho]}".$$ (15)

And further, we can calculate as follows.

(H) under the condition that the measured value $(1, y)$ is obtained by "$M_{B(H)}(O_1 \times O_2 \times O_2, S_{[\rho]}")$,

the probability that

$$\begin{bmatrix} y = 1 \\ y = 2 \\ y = 3 \end{bmatrix}$$

is formally given by

$$\begin{bmatrix}
\langle u, G(\{1\})F(\{1\})u | \\
\langle u, G(\{1\})F(\{2\})u | \\
\langle u, G(\{1\})F(\{3\})u |
\end{bmatrix}
= \begin{bmatrix}
\langle u, (\langle g_1, f_1 \rangle | g_1 \langle f_1 | | u | \\
\langle u, (\langle g_1, f_2 \rangle | g_1 \langle f_2 | | u | \\
\langle u, (\langle g_1, f_3 \rangle | g_1 \langle f_3 | | u |
\end{bmatrix}
= \begin{bmatrix}
\langle g_1 | f_1 \rangle | f_1 | | u |
\langle g_1 | f_2 \rangle | f_2 | | u |
\langle g_1 | f_3 \rangle | f_3 | | u |
\end{bmatrix}
= \begin{bmatrix}
1 \\
1 \\
-1
\end{bmatrix}$$

which shows the strange fact (i.e., "minus probability").

**Remark 11.** Since $O_1$ and $O_2$ do not commute, $O_1 \times O_2$ is not an observable, but $B(H)$-valued measure space (cf. Remark 5). Thus, it is usual to consider that the above (H) is meaningless. However, if some will find the idea such that the (H) becomes meaningful, then the idea should be added to the linguistic interpretation mentioned in Section 1.4. For example, the idea in ref. [1] (the weak value associated with a weak measurement) may be somewhat hopeful, but we can not assure it.

**Remark 12** (Continued from Hardy's paradox in Section 5). Define the observable $\widehat{O}_{ff} = \{(1, 2) \times \{1, 2\}, 2^{\{1,2\}} \times \{1,2\}, \widehat{H}_{ff}\}$ in $B(\mathbb{C}^2 \otimes \mathbb{C}^2)$ by the tensor observable $O_f \otimes O_f$, that is,

$$\widehat{H}_{ff}(\{(1, 1)\}) = |f_1 \otimes f_1 \rangle \langle f_1 \otimes f_1 |, \quad \widehat{H}_{ff}(\{(1, 2)\}) = |f_1 \otimes f_2 \rangle \langle f_1 \otimes f_2 |,$$

$$\widehat{H}_{ff}(\{(2, 1)\}) = |f_2 \otimes f_1 \rangle \langle f_2 \otimes f_1 |, \quad \widehat{H}_{ff}(\{(2, 2)\}) = |f_2 \otimes f_2 \rangle \langle f_2 \otimes f_2 |$$

In spite of the non-commutativity of $\widehat{\Psi}O_{gg}$ and $\widehat{O}_{ff}$, consider the "measurement":

$$M_{B(\mathbb{C}^2 \otimes \mathbb{C}^2)}(\widehat{\Psi}O_{gg} \times \widehat{O}_{ff}, S_{[\rho]})$$ (16)

And we can calculate as follows.
(I) under the condition that the measured value \( (2, 2), (y_1, y_2) \) is obtained by 

\[ M_{B(C^2 \otimes C^2)}(\hat{\Psi} \hat{O}_g) \times (\hat{O}_f, S_\rho) \]

the probability (or precisely, weak value) that \( (y_1, y_2) = (1, 1) \) is formally given by

\[
\begin{bmatrix}
(y_1, y_2) = (1, 1) \\
(y_1, y_2) = (1, 2) \\
(y_1, y_2) = (2, 1) \\
(y_1, y_2) = (2, 2)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\langle \omega \otimes u, P \hat{H}_f ((2, 2)) \rangle P \hat{H}_f ((1, 1)) (\omega \otimes u)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\langle (f_1 + f_2) \otimes (f_1 + f_2), P[g_2 \otimes g_2] (g_2 \otimes g_2) | f_1 \otimes f_1 \rangle | f_1 \otimes f_1, (f_1 + f_2) \otimes (f_1 + f_2) \rangle \\
\langle (f_1 + f_2) \otimes (f_1 + f_2), P[g_2 \otimes g_2] (g_2 \otimes g_2) | f_1 \otimes f_2 \rangle | f_1 \otimes f_2, (f_1 + f_2) \otimes (f_1 + f_2) \rangle \\
\langle (f_1 + f_2) \otimes (f_1 + f_2), P[g_2 \otimes g_2] (g_2 \otimes g_2) | f_2 \otimes f_2 \rangle | f_2 \otimes f_2, (f_1 + f_2) \otimes (f_1 + f_2) \rangle \end{bmatrix}
\]

\[
= \begin{bmatrix}
0 \\
-1 \times (-1) \\
(-1) \times (-1)
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}
\]

This (I) and the idea in ref. [1] are superficially similar, but completely different in essence. However, if the latter says something good, we can expect that the (I) is somewhat meaningful. For completeness, note that quantum language is not physics but language. Therefore, we say that

(J) if this statement (I) can be used effectively, then the concept: "weak value" should be accepted in the linguistic interpretation, however, if this is not more than "even not wrong", we will not be concerned with "the weak value".

7 Conclusions

In this paper, we discussed the double slits experiment, the quantum eraser experiment, Wheeler’s delayed choice experiment, Hardy’s paradox and the three boxes paradox in the linguistic interpretation of quantum mechanics.

Quantum language says that everything should be described in terms of Axioms 1 and 2. Therefore, we always have to describe "measurement" explicitly. In fact, in this paper, any measurement was explicitly described such as the formula \([15]-14\), \([15]-16\) \}. Particularly, in Section 2, we say that the double-slit experiment can not be understood without the concept of "branch". And, in Section 4, we note that Wheeler’s delayed choice experiment is not surprising, since it should be regarded as the problem such as "interference or no interference”.

Through these arguments, we assert that the linguistic interpretation is just the final version of so called Copenhagen interpretation. And therefore, the Copenhagen interpretation does not belong to physics but the linguistic world view (cf. Figure 1).

We hope that our proposal will be discussed and examined from various view-points.
References

[1] Y. Aharonov, D. Z. Albert and L. Vaidman, How the Result of a Measurement of a Component of a spin-1/2 Particle Can Turn Out to be 100, Phys Rev. Lett., 60, 1351-1354, 1988
DOI: http://dx.doi.org/10.1103/PhysRevLett.60.1351

[2] J. S. Bell, “On the Einstein-Podolosky-Rosen Paradox,” Physics, Vol. 1, 1966, 195-200.

[3] E. B. Davies, Quantum Theory of Open Systems, Academic Press, 1976

[4] A. Einstein, B. Podolosky and N. Rosen, “Can Quantum-mechanical Description of Physical Reality be Considered Complete?” Physical Review, Vol. 47, No. 10, 1935, pp. 777-780.
doi: 10.1103/PhysRev.47.777

[5] H. Everett ‘Relative state’ Formulation of Quantum mechanics, Review of Modern Physics 29 1957

[6] R. P. Feynman The Feynman lectures on Physics; Quantum mechanics Addison-Wesley Publishing Company, 1965

[7] L. Hardy, Quantum mechanics, local realistic theories, and Lorentz-invariant realistic theories, Physical Review Letters 68 (20): 2981-2984 1992
DOI: http://dx.doi.org/10.1103/PhysRevLett.68.2981

[8] S. Ishikawa, Uncertainty relation in simultaneous measurements for arbitrary observables, Rep. Math. Phys., 9, 257-273, 1991
doi: 10.1016/0034-4877(91)90046-P

[9] S. Ishikawa, T. Arai, T. Kawai, Numerical Analysis of Trajectories of a Quantum Particle in Two-slit Experiment, International Journal of Theoretical Physics, Vol. 33, No. 6, 1265-1274, 1994
doi: 10.1007/BF00670793

[10] S. Ishikawa, A Quantum Mechanical Approach to Fuzzy Theory, Fuzzy Sets and Systems, Vol. 90, No. 3, 277-306, 1997
doi: 10.1016/S0165-0114(96)00114-5

[11] S. Ishikawa, T. Arai, T. Takamura, A dynamical system theoretical approach to Newtonian mechanics, Far east journal of dynamical systems 1, 1-34 (1999)
( http://www.pphmj.com/abstract/191.htm )

[12] S. Ishikawa, Statistics in measurements, Fuzzy sets and systems, Vol. 116, No. 2, 141-154, 2000
doi:10.1016/S0165-0114(98)00280-2

[13] S. Ishikawa, Mathematical Foundations of Measurement Theory, Keio University Press Inc. 335pages, 2006.
(http://www.keio-up.co.jp/kup/mfmt/ )

[14] S. Ishikawa, A New Interpretation of Quantum Mechanics, Journal of quantum information science, Vol. 1, No. 2, 35-42, 2011
doi: 10.4236/jqis.2011.12005
( http://www.scirp.org/journal/PaperInformation.aspx?paperID=7610 )

[15] S. Ishikawa, The linguistic interpretation of quantum mechanics, arXiv:1204.3892v1[physics.hist-ph], (2012),
(http://arxiv.org/abs/1204.3892 )

[16] S. Ishikawa, Quantum Mechanics and the Philosophy of Language: Reconsideration of traditional philosophies, Journal of quantum information science, Vol. 2, No. 1, 2-9, 2012
doi: 10.4236/jqis.2012.21002
( http://www.scirp.org/journal/PaperInformation.aspx?paperID=18194 )

[17] S. Ishikawa, A Measurement Theoretical Foundation of Statistics, Journal of Applied Mathematics, Vol. 3, No. 3, 283-292, 2012
doi: 10.4236/am.2012.33044
( http://www.scirp.org/journal/PaperInformation.aspx?paperID=18109 )
[18] S. Ishikawa, *Measurement theory in the philosophy of science*, arXiv:1209.3483 [physics.hist-ph] 2012. (http://arxiv.org/abs/1209.3483)

[19] S. Ishikawa, *A quantum linguistic characterization of the reverse relation between confidence interval and hypothesis testing*, arXiv:1401.2709v2 [math.ST] 2014. (http://arxiv-web3.library.cornell.edu/abs/1401.2709v2)

[20] S. Ishikawa, *Heisenberg uncertainty principle and quantum Zeno effects in the linguistic interpretation of quantum mechanics*, arXiv:1308.5469 [quant-ph] 2014. (http://arxiv.org/abs/1308.5469)

[21] S. Ishikawa, *Regression analysis in quantum language*, arXiv:1403.0060 [math.ST] 2014. (http://arxiv.org/abs/1403.0060)

[22] S. Ishikawa, *Kalman filter in quantum language*, arXiv:1404.2664 [math.ST] [math.ST] 2014. (http://arxiv.org/abs/1404.2664)

[23] A. Kolmogorov, *Foundations of probability (translation)*, Chelsea Publishing Co. 1950

[24] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Springer Verlag, Berlin, 1932

[25] T. Ravon and L. Vaidman, “The three-box paradox revisited,” J. Phys. A: Math. Theor. 40 (2007) 2873-2882 (http://iopscience.iop.org/1751-8121/40/11/021/)

[26] S. Sakai, *$C^*$-algebras and $W^*$-algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete (Band 60), Springer-Verlag, (1971)

[27] S. P. Walborn, et al. “Double-Slit Quantum Eraser,” Phys.Rev.A 65, (3), 2002 DOI: http://dx.doi.org/10.1103/PhysRevA.65.033818

[28] J. A. Wheeler, "The 'Past' and the 'Delayed-Choice Double-Slit Experiment','" pp 9-48, in A.R. Marlow, editor, *Mathematical Foundations of Quantum Theory*, Academic Press (1978)