Some observations on the Baireness of $C^k(X)$ for a locally compact space $X$

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Abstract

We prove some consistency results concerning the Moving Off Property for locally compact spaces, and thus the question of whether their function spaces are Baire.

1 Introduction

The Moving Off Property was introduced in [11] to characterize when $C^k(X)$ satisfies the Baire Category Theorem, for $q$-spaces $X$. Here we shall only be concerned with locally compact spaces (which are $q$), and so won’t define $q$. We shall assume all spaces are Hausdorff.

Definition. A moving off collection for a space $X$ is a collection $\mathcal{K}$ of non-empty compact sets such that for each compact $L$, there is a $K \in \mathcal{K}$ disjoint from $L$. A space satisfies the Moving Off Property (MOP) if each moving off collection includes an infinite subcollection with a discrete open expansion.

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Theorem 1. A locally compact space $X$ satisfies the MOP if and only if $C_k(X)$ is Baire, i.e., satisfies the Baire Category Theorem.

There is a less onerous equivalent of the MOP for locally compact spaces:

**Lemma 2.** Let $X$ be a locally compact space. Then $X$ has the MOP if and only if every moving off collection for $X$ includes an infinite discrete subcollection.

We give a proof for the benefit of readers who are not topologists.

**Proof.** Let $K$ be a moving off collection for $X$. By local compactness, each $K \in K$ can be fattened to an open set with compact closure. Let $K'$ be the collection of all compact closures of open sets around members of $K$. Then $K'$ is moving off. For let $C$ be a compact subset of $X$. There is a $K \in K$ disjoint from $C$. By regularity and local compactness, there is an open $U \supseteq K$ with compact closure $\overline{U}$ disjoint from $C$. Then $\overline{U} \in K'$. Since we have established that $K'$ is moving off, by hypothesis it includes an infinite discrete collection $\{\overline{U}_n\}_{n<\omega}$. But each $U_n$ included some $K_n \in K$. Then $\{K_n\}_{n<\omega}$ is discrete and has the discrete open expansion $\{U_n\}_{n<\omega}$.  

In [14], [15], and [24], assuming the existence of a supercompact cardinal, a model of set theory is constructed, which we shall refer to as a model of $PFA(S)[S]$. We refer the reader to those papers for a discussion of what $PFA(S)[S]$ is. In these papers various propositions concerning locally compact normal spaces are established in this model. We shall use:

**Lemma 3.** In this model, locally compact hereditarily normal spaces which do not include a perfect pre-image of $\omega_1$ are paracompact.

**Corollary 4.** In this model, locally compact, perfectly normal spaces are paracompact.

**Lemma 5.** In this model, locally compact normal spaces with Lindelöf number $\leq \aleph_1$ which do not include a perfect pre-image of $\omega_1$ are paracompact.

Let us also quote several useful results concerning the MOP.

**Lemma 6.** Countably compact spaces satisfying the MOP are compact.

**Lemma 7.** First countable spaces satisfying the MOP are locally compact.

**Lemma 8.** Locally compact, paracompact spaces satisfy the MOP.

A stronger result is in Lemma 24 below.
2 Locally compact, perfectly normal spaces and the MOP

Marion Scheepers asked us whether locally compact, perfectly normal spaces satisfy the MOP, and whether - if they do - they are paracompact. Here are the answers, modulo a supercompact cardinal.

Theorem 9. There is a model of PFA(S)[S] in which locally compact, perfectly normal spaces are paracompact and hence satisfy the MOP.

Theorem 10. There is a model in which there is a locally compact, perfectly normal space which does not satisfy the MOP.

Proofs. Theorem 9 follows from Corollary 4 plus Lemma 8. Theorem 10 follows from Lemma 6, since Ostaszewski’s space \([18]\), constructed from \(\diamond\), is locally compact, perfectly normal, countably compact, but not compact.

For the other question, obviously Corollary 4 answers it one way; for the other, we quote:

Lemma 11 \([16]\). MA(\(\omega_1\)) implies there is a locally compact perfectly normal space with the MOP which is not paracompact.

3 Counterexamples

Although the question of whether locally compact normal spaces with the MOP are paracompact has not been answered in ZFC, there are a number of consistent counterexamples which repurpose spaces familiar to normal Moore space fans. a)-f) are not collectionwise Hausdorff, hence not paracompact. Each is normal in some model.

a) \([16]\) The Cantor tree on a set of reals of size \(\aleph_1\) is normal and has the MOP under MA(\(\omega_1\)).

Definition. A ladder system \(\{\lambda_\alpha\}_{\alpha \in S}\), where \(S\) is a subset of some ordinal \(\lambda\), is a set of sequences, where each \(\lambda_\alpha\) is strictly increasing, converges to \(\alpha\), and has range disjoint from \(S\). The corresponding ladder system space on \(S \cup \bigcup\{\text{range} \lambda_\alpha : \alpha \in S\}\) has the points in each range \(\lambda_\alpha\) isolated, while a basic open set about \(\alpha \in S\) is \(\{\alpha\} \cup \text{a tail of} \ \lambda_\alpha\).
b) A ladder system space on a stationary subset of \( \omega_1 \) has the MOP, and also is normal under MA\( \omega_1 \).

Note the first example is separable, while countable sets have countable closures in the second one.

c) There is also a version of b) consistent with CH, indeed with \( \Diamond \). See [21], [4], [14].

We shall show that the idea of the proof of the MOP for b) (and hence c)) can be used to establish the MOP for:

d) The tree topology on a a special Aronszajn tree. This is known to be non-collectionwise Hausdorff, and to be normal under MA\( \omega_1 \) [6].

as well as for the space of:

e) Devlin and Shelah [2] isolate some points of a special Aronszajn tree and manage to force normality while keeping CH.

Generalizing the proof in [11] that a ladder system space on a stationary subset of \( \omega_1 \) has the MOP, we obtain:

**Theorem 12.** Suppose \( X \) is locally compact, locally countable, countable sets have countable closures, and \( X = \bigcup_{\gamma<\omega_1} X\gamma \), where each \( X\gamma \) is countable, \( X\gamma \subseteq X_{\gamma+1} \), and for \( \gamma \) a limit, \( X\gamma = \bigcup_{\alpha<\gamma} X\alpha \). Further suppose that for \( \gamma \) a limit, for each \( x \) in the boundary of \( X\gamma \), there is a compact neighborhood \( N(x) \) such that for each \( \alpha < \gamma \), \( N(x) \cap X\alpha \) is compact. Then \( X \) has the MOP.

**Proof.** Since countable sets have countable closures, without loss of generality we may assume that \( X_\alpha \subseteq X_{\alpha+1} \). Since compact sets are countable,

\[
C = \{ \alpha : x \in X_\alpha \text{ implies } N(x) \subseteq X_\alpha \}
\]

is closed unbounded. Since \( X \) is first countable, each \( X_\alpha \) has a countable base \( B_\alpha \) of compact sets open in \( X_\alpha \). For \( \alpha \in C \), \( X_\alpha \) is open, so these sets are open in \( X \).

Let \( A \) be a moving off collection for \( X \). For any \( \alpha < \omega_1 \), there is a countable ordinal \( \delta(\alpha) \geq \alpha \) such that for \( B \in B_\alpha \), there is an \( A \in A \) such that \( A \subseteq X_\delta(\alpha) \) and \( A \) is disjoint from \( B \). Then

\[
C' = \{ \alpha \in C : \beta < \alpha \text{ implies } \delta(\beta) < \alpha \}
\]
is closed unbounded. Take a strictly increasing sequence \( \{ \gamma_n \} \) in \( C' \) and let \( \gamma = \sup \gamma_n \). Let \( \{ B_{m,k} : m < \omega \} \) enumerate the basic compact open sets of \( X_{\gamma} \). Note \( \bigcup \{ B_{m,k} : m, k < \omega \} \) is a basis for \( X_{\gamma} \). Let \( \{ x_{\gamma,i} : i < \omega \} \) enumerate \( X_{\gamma} - X_{\gamma} \). For each \( j < \omega \), there is an \( A_j \in A \) with \( A_j \subseteq X_{\gamma} \) and \( A_j \cap \left( \bigcup_{k < j} A_k \cup \bigcup_{m,k < j} B_{m,k} \cup \bigcup_{n,i < j} (N(x_{\gamma,i}) \cap X_{\gamma_n}) \right) = \emptyset \). Then \( \{ A_j \}_{j<\omega} \) is locally finite in \( X_{\gamma} \), since each \( B_{m,k} \) eventually misses the \( A_j \)'s. The \( x_{\gamma,i} \)'s are then the only possible limits of the \( A_j \)'s. But \( N(x_{\gamma,i}) \) is disjoint from \( A_j \) for \( j > i \). Thus the \( A_j \)'s are locally finite in \( X \). Since the \( A_j \)'s are also closed disjoint, in fact the collection is discrete. \( \square \)

Note that by Lemma 3 Theorem 12 does not offer a roadmap for constructing a locally compact normal space with the MOP which is not paracompact.

We note, for future reference, that:

**Corollary 13.** A countable topological sum of spaces satisfying the hypotheses of Theorem 12 also has the MOP.

**Proof.** The sum also satisfies these hypotheses. \( \square \)

**Theorem 14.** Suppose \( X \) is locally compact, locally countable, \( |X| \geq 2^{\aleph_0} \), and every closed subspace of size \( 2^{\aleph_0} \) has the MOP. Then \( X \) has the MOP.

**Corollary 15.** CH implies if \( X \) is locally compact, locally countable, and closed subspaces of size \( \aleph_1 \) have the MOP, then so does \( X \).

**Corollary 16.** CH implies if \( X \) is locally compact, locally countable, and closed subspaces of size \( \aleph_1 \) are paracompact, then \( X \) has the MOP.

**Corollary 17.** CH implies if \( X \) is locally compact, locally countable, countable subsets have countable closures, and each closed \( Y \subseteq X \) of size \( \aleph_1 \) satisfies the conditions for \( X \) in Theorem 12, then \( X \) has the MOP.

The first and third corollaries are immediate. The second is because local compactness is closed-hereditary, and locally compact, paracompact spaces have the MOP.

**Proof of Theorem 14.** Let \( M \) be a countably closed elementary submodel of size \( 2^{\aleph_0} \) containing the space \( X \) and a moving off collection \( A \) for it. By first countability, \( X \cap M \) is a closed subspace of \( X \), so it will suffice to find
a discrete collection \( \{A_n\}_{n<\omega} \) included in \( \mathcal{A} \), with each \( A_n \subseteq X \cap M \), and \( \{A_n\}_{n<\omega} \) discrete in \( X \cap M \). It suffices to show \( \mathcal{A} \cap M \) is moving off for \( X \cap M \). Let \( F \) be a compact subspace of \( X \cap M \). Since compact sets are countable and \( M \) is countably closed, \( F \in M \). Then, since \( M \models \mathcal{A} \) is moving off, \( M \models (\exists A \in \mathcal{A})(F \cap A = \emptyset) \). But then there is an \( A \in \mathcal{A} \cap M \) such that \( F \cap A = \emptyset \).

Our previous counterexamples were not \( \aleph_1 \)-collectionwise Hausdorff; now we can get one that satisfies that property:

f) A consistent-with-CH example of a locally compact, normal, \( \aleph_1 \)-collectionwise Hausdorff space with the MOP which is not paracompact.

A ladder system space \( X \) on a non-reflecting stationary set \( E \) of \( \omega \)-cofinal ordinals in \( \omega_2 \) is easily seen to be \( \aleph_1 \)-collectionwise Hausdorff, because initial segments of \( E \) are non-stationary. In fact, subspaces of size \( \leq \aleph_1 \) are paracompact, and hence such small closed ones have the MOP. \( X \) is not paracompact because it is not \( \aleph_2 \)-collectionwise Hausdorff. Shelah [20] forced to make \( X \) normal, consistent with CH.

\[ \square \]

g) A Souslin tree with the usual tree topology is collectionwise normal [7]. It has countable extent but is not Lindelöf, so is not paracompact. By Theorem 12 it has the MOP.

A similar proof of the MOP works for any other \( \omega_1 \)-tree with the tree topology, but normal ones that are not paracompact will not be found in ZFC - see [7]. Gruenhage [10] proved earlier that any Aronszajn tree has the MOP.

4 More results in a model of PFA(\( S \))[\( S \)]

There are some easy observations about the MOP in the model of PFA(\( S \))[\( S \)] we have mentioned earlier.

**Theorem 18.** In the model of Lemma 3, Theorem 9, etc., locally compact, hereditarily normal, countably tight spaces with the MOP are paracompact.

**Proof.** In a countably tight space, countably compact subspaces are closed 3. Closed subspaces of a space satisfying the MOP also satisfy it. Perfect pre-images of \( \omega_1 \) are countably compact but not compact. Now apply Lemmas 3 and 6. \[ \square \]
Corollary 19. In this model, first countable hereditarily normal spaces satisfying the MOP are paracompact.

Proof. By Theorem 18 and Lemma 7

Theorem 20. In this model, locally compact, normal, countably tight spaces with Lindelöf number ≤ ℵ₁ satisfying the MOP are paracompact.

Proof. They do not include a perfect pre-image of ω₁, so we can apply Lemma 5

Corollary 21. In this model, first countable, normal spaces with Lindelöf number ≤ ℵ₁ satisfying the MOP are paracompact.

Proof. Apply Lemma 7 and Theorem 20

Corollary 22. In this model, locally compact, normal, countably tight spaces satisfying the MOP (in particular, first countable normal spaces satisfying the MOP) are paracompact, provided countable sets have Lindelöf closures.

Proof. In [24] it is shown that in this model,

Lemma 23. In this model, locally compact normal spaces not including a perfect pre-image of ω₁ are paracompact, provided countable sets have Lindelöf closures.

5 Baire powers of function spaces

Definition. A space is weakly α-favorable [1] if Nonempty has a winning strategy in the Banach-Mazur game. In that game, players take turns picking an open set included in their opponent’s pick. The first player, Empty, wins if, after ω plays, the intersection of the open sets is empty; otherwise the second player, Nonempty, wins.

Lemma 24 [16]. A locally compact X is paracompact if and only if Cₖ(X) is weakly α-favorable.

Galvin and Scheepers [9] note that White [25] showed that all box powers of weakly α-favorable spaces are Baire, and then prove:
Theorem 25. If it is consistent there is a proper class of measurable cardinals, then it is consistent that if all box powers of a space are Baire, then the space is weakly \( \alpha \)-favorable.

They then ask whether there are any consistent counterexamples. Let us consider the particular case of \( C_k(X) \) for \( X \) locally compact. Their result then entails:

Corollary 26. If it is consistent there is a proper class of measurable cardinals, then it is consistent that if all box powers of \( C_k(X) \) are Baire, where \( X \) is locally compact, then \( X \) is paracompact.

Scheepers pointed out to me that Oxtoby [19] proved that any product of Baire spaces with a countable base is Baire, but that a Bernstein set of reals is Baire but not weakly \( \alpha \)-favorable, so in the Theorem, ordinary powers are not enough.

In fact, they are not even sufficient for the Corollary. Example b) is a counterexample:

Theorem 27. Suppose \( X \) satisfies the hypotheses of Theorem 12. Then arbitrary powers of \( C_k(X) \) are Baire.

Proof. Fleissner and Kunen [8] prove

Lemma 28. Let \( \kappa \geq \omega \). If \( X^\omega \) is Baire, then \( X^\kappa \) is Baire.

McCoy and Ntantu [17] prove

Lemma 29. Let \( \bigoplus_{\alpha<\lambda} X_\alpha \) be the topological sum of copies of \( X \). Then \( C_k \left( \bigoplus_{\alpha<\lambda} X_\alpha \right) \) is homeomorphic to \( (C_k(X))^\lambda \).

Thus, by Corollary [13] our assertion that b) is a counterexample is verified.

The preceding two lemmas prove that:

Theorem 30. If countable sums of copies of a locally compact \( X \) have the MOP, then arbitrary sums of copies of \( X \) have the MOP.

Surprisingly, there is a consistent example of locally compact spaces \( X \) and \( Y \), each having the MOP, but \( X \oplus Y \) does not have the MOP [16].

We do have one necessity theorem for large cardinals, but do not know whether the hypothesis is vacuous:
**Theorem 31.** Suppose that whenever all usual powers of $C_k(X)$ are Baire, for locally compact, $\mathfrak{N}_1$-collectionwise Hausdorff $X$, then $X$ is paracompact. Then it is consistent that there is a strong cardinal.

**Proof.** Take a non-reflecting stationary set $E$ of $\omega$-cofinal ordinals in $\lambda^+$, for some $\lambda \geq \mathfrak{c}$. It is known (attributed to R. Jensen) that if it’s consistent no such set exists, then it’s consistent there is a strong cardinal - see [12]. Form a ladder system space $X$ on $E$. $X$ is not paracompact, but initial segments of it are. Consider an arbitrary sum $\bigoplus_{\alpha \in S} X_\alpha$ of copies of $X$. I claim that $\bigoplus_{\alpha \in S} X_\alpha$ has the MOP, whence $C_k(X)\mid S$ is Baire. By Corollary 15 it suffices to show closed subspaces of $\bigoplus_{\alpha \in S} X_\alpha$ of size $\leq 2^{\mathfrak{c}}$ have the MOP. But they are all paracompact, so they do. But $X$ is not paracompact. □

A strong cardinal (see [13] for the definition) has arbitrarily large measurable cardinals below it. Thus, in $V_\kappa$, where $\kappa$ is the least strong cardinal, there is a proper class of measurable cardinals, but no strong cardinal in an inner model. Collapsing these cardinals as in [9] yields a model in which there is a ladder system space $X$ on a non-reflecting stationary set as above. Some box power of $C_k(X)$ is then not Baire.

Large cardinals can be used to destroy non-reflecting stationary sets; this translates into results about small subspaces being paracompact implying the whole space is paracompact. For example:

**Theorem 32** [23]. Martin’s Maximum implies that if a first countable space is either generalized ordered or monotonically normal and closed subspaces of size $\mathfrak{N}_1$ are paracompact, then the space is paracompact.

For more results of this sort, see [23].

In [5], Fleissner raises the question of whether, if the box product of a collection of Baire spaces is Baire, its Tychonoff product is Baire. Also see [8]. The converse is not true [8]. Note that for box powers, in the model of Galvin and Scheepers, this is true, since Tychonoff products of weakly $\alpha$-favorable spaces are Baire [25]. Fleissner also asks whether the box product of Baire spaces with a countable base is Baire [5]. In this model, this is not true - consider the box powers of a Bernstein set.

One might be tempted, in view of the countable nature of the Baire Category Theorem and of weak $\alpha$-favorability, to conjecture that the Baireness of countable box powers would consistently be sufficient to imply weak $\alpha$-favorability, at least for spaces with a countable base. This is not true. L. Zsilinszky [26] proved:
Theorem 33. The countable box power of a Baire space with a countable base is Baire.

But again, a Bernstein set is not weakly $\alpha$-favorable.

6 Problems

The most interesting open question in this area is raised in [11]:

Problem 1. Is there in ZFC a locally compact, normal space with the MOP which is not paracompact (equivalently, $C_k(X)$ is not weakly $\alpha$-favorable)?

None of the examples we have mentioned exist in the model of $PFA(S)[S]$ we have been using, since in that model there are no Souslin trees, and normal first countable spaces are collectionwise Hausdorff [14].

Problem 2 (5). Are large cardinals necessary for Theorem 25?

Problem 3. Can one prove in ZFC that some box power of $C_k$ of a ladder system space on a (non-reflecting?) stationary set of $\omega$-cofinal ordinals is not Baire?

Problem 4 (5). Can one prove in ZFC that if a box product of a collection of Baire spaces is Baire, then its Tychonoff product is Baire?

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