Some classes of Wiener–Hopf plus Hankel operators and the Coburn-Simonenko Theorem

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Abstract

Wiener–Hopf plus Hankel operators \( W(a) + H(b) : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+) \) with generating functions \( a \) and \( b \) from a subalgebra of \( L^\infty(\mathbb{R}) \) containing almost periodic functions and Fourier images of \( L^1(\mathbb{R}) \)-functions are studied. For \( a \) and \( b \) satisfying the so-called matching condition

\[
a(t)a(-t) = b(t)b(-t), \quad t \in \mathbb{R},
\]

we single out some classes of operators \( W(a) + H(b) \) which are subject to Coburn–Simonenko theorem.

1 Introduction

The classical Coburn–Simonenko Theorem states that for Toeplitz or Wiener–Hopf operator \( A \) with a scalar non-zero generating function, at least one of the numbers \( \dim \ker A \) or \( \dim \text{coker} A \) is equal to zero. Thus if it is known that the corresponding operator is Fredholm with index zero, the

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Coburn–Simonenko Theorem implies that this operator is invertible. Note that Fredholmness of such operators with generating functions from various classes is well understood. On the other hand, for Toeplitz plus Hankel operators $T(a) + H(b)$ with piecewise continuous generating functions $a$ and $b$ their Fredholm properties can be derived by a direct applications of results [11, Sections 4.95-4.102], [12, Sections 4.5 and 5.7], [13]. The case of quasi piecewise continuous generating functions has been studied in [15], whereas formulas for the index of the operators $T(a) + H(b)$ considered on various Banach and Hilbert spaces and with various assumptions about the generating functions $a$ and $b$ have been established in [8, 14]. It is also worth mentioning that lately a lot of effort has been spent to obtain information concerning the kernel and cokernel dimensions of Toeplitz plus Hankel or Wiener–Hopf plus Hankel operators. Here we are not going to discuss the history of these investigations in much details but just mention a few important developments. Thus in the works of T. Ehrhardt [9, 10] and T. Ehrhardt and E. Basor [11, 12, 13], Toeplitz plus Hankel operators have been studied in $H^p$-spaces on the unit circle $\mathbb{T}$ mainly under the assumption that the generating functions of these operators are piecewise continuous and satisfy an algebraic relation, and that the operators are Fredholm. Wiener–Hopf plus Hankel operators have received less attention in the literature and results are scarce (see, for example, [5] and references there). In addition, in most cases the conditions imposed on the generating functions are very restrictive and ensure that the problem can be handled in a more or less straightforward way.

Let us now describe the problem studied in the present paper. Consider the set $G$ of all functions of the form

$$a(t) = \sum_{j=-\infty}^{\infty} a_j e^{i\delta_j t} + \int_{-\infty}^{\infty} k(s)e^{its} \, ds, \quad -\infty < t < \infty, \quad (1.1)$$

where $\delta_j \in \mathbb{R}$ and

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty, \quad \int_{-\infty}^{\infty} |k(s)| \, ds < \infty.$$

The set $G$ actually forms a commutative unital Banach algebra under pointwise operations and the norm

$$||a|| := \sum_{j=-\infty}^{\infty} |a_j| < \infty + \int_{-\infty}^{\infty} |k(s)| \, ds.$$

This algebra $G$ contains both the algebra $AP_w$ of all almost periodic functions with absolutely convergent Fourier series and the algebra $L_0$ of all Fourier transforms of functions from $L^1(\mathbb{R})$. Moreover, the algebra $G$ is the direct sum of $AP_w$ and $L_0$, and $L_0$ is an ideal in $G$. A function $a \in G$ is invertible in $G$ if and only if it satisfies the condition

$$\inf_{t \in \mathbb{R}} |a(t)| > 0.$$
Moreover, if \( b \in \mathcal{AP}_w, k \in \mathcal{L}_0, \) and \( b+k \) is invertible in \( G \), then \( b \) is also invertible in \( \mathcal{AP}_w \) (see [11, Chapter VII]). Further, let us introduce the subalgebra \( G^+ \) (\( G^- \)) of the algebra \( G \), which consists of all functions \((1.1)\) such that all indices \( \delta_j \) are non-negative (non-positive) and the functions \( k \) vanishing on the negative (positive) semi-axis. It is clear that the functions from \( G^+ \) and \( G^- \) admit holomorphic extensions to the upper and to the lower half-plane, respectively, and the intersection of the sets \( G^+ \) and \( G^- \) contains constant functions only.

If \( b \in \mathcal{AP}_w, k \in \mathcal{L}_0, \) and the element \( b+k \) is invertible in \( G \), then the numbers
\[
\nu(a) := \lim_{l \to \infty} \frac{1}{2l} [\arg b(t)]^l_{-l},
\]
and
\[
n(a) := \frac{1}{2\pi} [\arg(1 + b^{-1}(t)k(t))]^\infty_{-\infty},
\]
are well-defined. In particular, the first limit exists because \( b \) is an almost periodic function.

Let \( \mathbb{R}^+ := (0, \infty) \) and let \( P \) be the projection operator from \( L^p(\mathbb{R}), 1 \leq p \leq \infty \) onto \( L^p(\mathbb{R}^+) \), that is
\[
P : f \mapsto f|_{\mathbb{R}^+}.
\]
Analogously, \( Q \) is the projection operator from \( L^p(\mathbb{R}) \) onto \( L^p(\mathbb{R}^-) \), \( \mathbb{R}^- := (-\infty, 0) \). In what follows we will identify the space \( L^p(\mathbb{R}^+) \) (\( L^p(\mathbb{R}^-) \)) with the subspace of \( L^p(\mathbb{R}) \) consisting of all functions vanishing on \( \mathbb{R}^- \) (\( \mathbb{R}^+ \)).

Each function \( a \in G, \)
\[
a(t) = \sum_{j = -\infty}^{\infty} a_j e^{i\delta_j t} + \int_{-\infty}^{\infty} k(s)e^{its} ds,
\]
generates two operators \( W^0(a) : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \) and \( W(a) : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+) \) defined by
\[
(W^0(a)f)(t) := \sum_{j = -\infty}^{\infty} a_j f(t - \delta_j) + \int_{-\infty}^{\infty} k(t-s)f(s) ds, \tag{1.2}
\]
\[
W(a)f := PW^0(a)f.
\]
Note that the convolution operator \((1.2)\) is shift invariant that is \( W^0(a)\tau_v = \tau_v W^0(a) \) for any \( v \in \mathbb{R} \), where \( \tau_v \) is the operator defined by \((\tau_v f)(t) := f(t-v)\). The operator \( W(a) \) is called integro-difference operator [11, Chapter VII]. It is shown in [4, Sections 9.4 and 9.21] that integro-difference operators are indeed Wiener–Hopf integral operators. If \( a \) does not vanish identically, then \( W(a) \) has a trivial kernel or a dense range in \( L^p(\mathbb{R}^+) \) at least for \( 1 < p < \infty \) and this is the Coburn–Simonenko Theorem for such class of operators (see [4, Section 9.5 (d)].

Now we can formulate the following result.
Theorem 1.1 (Gohberg/Feldman) If \( a \in G \), then the operator \( W(a) \) is one-sided invertible in \( L^p(\mathbb{R}^+) \) for \( 1 \leq p \leq \infty \) if and only if \( a \) is invertible in \( G \). Further, if \( a \in G \) is invertible in \( G \), then

(i) If \( \nu(a) > 0 \), then the operator \( W(a) \) is invertible from the left and \( \dim \ker W(a) = \infty \).

(ii) If \( \nu(a) < 0 \), then the operator \( W(a) \) is invertible from the right and \( \dim \ker W(a) = \infty \).

(iii) If \( \nu(a) = 0 \), then the operator \( W(a) \) is invertible from the left (right) if \( n(a) > 0 \) \( (n(a) \leq 0) \) and

\[
\dim \ker W(a) = n(a) \quad (\dim \ker W(a) = -n(a)).
\]

Remark 1.2 Using the Coburn–Simonenko Theorem, one can show that if \( W(a) \) is normally solvable and \( a \neq 0 \), then \( a \) is invertible in \( G \), at least in the case where the operator \( W(a) \) acts on the space \( L^p(\mathbb{R}^+) \), \( p \in (1, \infty) \).

Let us introduce Hankel operators. For, consider first the operator \( J : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \) defined by \( J\varphi := \tilde{\varphi} \), where \( \tilde{\varphi}(t) := \varphi(-t) \). If \( a \in G \) and \( 1 \leq p \leq \infty \), then on the space \( L^p(\mathbb{R}^+) \) the Hankel operators \( H(a) \) and \( H(\bar{a}) \) are defined as follows

\[
H(a) : \varphi \mapsto PW^0(a)QJ\varphi,
\]

\[
H(\bar{a}) : \varphi \mapsto JQW^0(a)P\varphi.
\]

Note that \( JQW^0(a)P = PW^0(\bar{a})QJ \), and the last identity is the consequence of the following relations

\[ J^2 = I, \quad JQ = PJ, \quad JP = QJ, \quad JW^0(a)J = W^0(\bar{a}). \quad (1.3) \]

On the space \( L^p(\mathbb{R}) \), \( 1 \leq p \leq \infty \) we also consider the operators \( \mathcal{U} \) and \( \mathcal{U}^{-1} \) defined by

\[
(\mathcal{U}\varphi)(t) := \varphi(t) - 2 \int_{-\infty}^{t} e^{s-t}\varphi(s) \, ds, \quad -\infty < t < \infty,
\]

\[
(\mathcal{U}^{-1}\varphi)(t) := \varphi(t) - 2 \int_{t}^{\infty} e^{t-s}\varphi(s) \, ds, \quad -\infty < t < \infty.
\]

It is well-known \([11]\) that

\[ \mathcal{U} = W^0(\chi), \quad \mathcal{U}^{-1} = W^0(\chi^{-1}), \]

where \( \chi(t) := (t-i)/(t+i) \), \( \chi^{-1}(t) := (t+i)/(t-i) \), \( t \in \mathbb{R} \). Moreover, since \( W^0(\chi)W^0(\chi^{-1}) = W^0(\chi\chi^{-1}) \), we get \( \mathcal{U}\mathcal{U}^{-1} = \mathcal{U}^{-1}\mathcal{U} = I \).

One of the aims of this work is to establish Coburn–Simonenko Theorem for the operators \( W(a) + H(a\chi) \) and \( W(a) - H(a\chi^{-1}) \) for elements \( a \in G \) invertible in \( G \). Recall that the semi-Fredholmness of the operators \( W(b) + H(c) : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+) \), \( b, c \in G \) implies that the element \( b \) is invertible in
$G$ at least in the case where $1 < p < \infty$. Therefore, the above requirement of the invertibility of the element $a$ is not too restrictive.

Finally, let us also mention that if $a, b \in G$, then

$$W^0(ab) = W^0(a)W^0(b),$$

and if $a \in G^-, c \in G^+$, and $b \in G$, then

$$W(abc) = W(a)W(b)W(c).$$

Moreover, in the following we will make use of the identities

$$W(ab) = W(a)W(b) + H(a)H(\tilde{b}),$$
$$H(ab) = W(a)H(b) + H(a)W(\tilde{b}).$$

\[1.4\]

## 2 Kernels of Wiener–Hopf plus Hankel operators. General properties.

In this section we establish certain relations between the kernels of Wiener–Hopf plus Hankel operators and matrix Wiener-Hopf operators in the case where the generating functions $a, b \in G$. The corresponding results for Toeplitz plus Hankel operators $T(a) + H(b)$, $a, b \in L^\infty$ have been obtained recently [7]. Taking into account Theorem 1.1 we can always assume that $a$ is invertible in $G$. Along with the operator $W(a) + H(b)$ let us also consider Wiener–Hopf minus Hankel operator $W(a) - H(b)$ and Wiener–Hopf operator $W(V(a,b)))$ defined by the matrix

$$V(a,b) := \begin{pmatrix} a - b\tilde{a}^{-1} & d \\ -c & \tilde{a}^{-1} \end{pmatrix},$$

where $c := \tilde{b}a^{-1}$, $d := b\tilde{a}^{-1}$.

The following result shows connections between the solutions of homogeneous equations with Wiener–Hopf plus/minus Hankel operators and the solutions of the associated homogeneous equation with the matrix Wiener–Hopf operator $W(V(a,b)))$.

**Lemma 2.1** Assume that $a, b \in G$, $a$ is invertible in $G$, and the operators $W(a) \pm H(b)$ are considered on the space $L^p(\mathbb{R}^+)$, $1 \leq p \leq \infty$. Then

- If $(\varphi, \psi)^T \in \ker W(V(a,b)))$, then
  $$((\Phi, \Psi)^T = (\varphi - JQW^0(c)\varphi + JQW^0(\tilde{a}^{-1})\psi, \varphi + JQW^0(c)\varphi - JQW^0(\tilde{a}^{-1})\psi)^T \in \ker \text{diag}(W(a) + H(b), W(a) - H(b))$$

- If $(\Phi, \Psi)^T \in \ker \text{diag}(W(a) + H(b), W(a) - H(b))$, then
  $$((\Phi + \Psi, P(W^0(\tilde{b})(\Phi + \Psi) + W^0(\tilde{a})JP(\Phi - \Psi)))^T \in \ker W(V(a,b))).$$

\[2.2\]
Moreover, the operators

\[ E_1 : \ker W(V(a, b)) \to \ker \text{diag} \left( W(a) + H(b), W(a) - H(b) \right), \]
\[ E_2 : \ker \text{diag} \left( W(a) + H(b), W(a) - H(b) \right) \to \ker W(V(a, b)), \]

defined, respectively, by the relations (2.1) and (2.2) are mutually inverses to each other.

**Proof.** Consider the operators

\[
A := \begin{pmatrix} I & 0 \\ W^0(b) & W^0(a) \end{pmatrix} \begin{pmatrix} I & I \\ J & -J \end{pmatrix},
\]

\[
B_1 := 2 \begin{pmatrix} I & J \\ I & -J \end{pmatrix},
\]

\[
B_2 := \text{diag} (I, I) - \text{diag} (P, Q) \begin{pmatrix} W^0(a) & W^0(b) \\ W^0(b) & W^0(a) \end{pmatrix} \text{diag} (Q, P),
\]

\[
B_3 := \text{diag} (I, I) + \text{diag} (P, P) \begin{pmatrix} 0 & W^0(d) \\ -W^0(c) & W^0(d) \end{pmatrix} \text{diag} (Q, Q).
\]

It turns out that the operator \( \text{diag} \left( W(a) + H(b) + Q, W(a) - H(b) + Q \right) \) can be represented as the product of three matrix operators, viz.

\[
\begin{pmatrix} W(a) + H(b) + Q & 0 \\ 0 & W(a) - H(b) + Q \end{pmatrix} = B(W(V(a, b)) + \text{diag} (Q, Q))A,
\]

where \( B := B_1B_2B_3 \). The operator \( A : L^p(\mathbb{R}) \times L^p(\mathbb{R}) \to L^p(\mathbb{R}) \times L^p(\mathbb{R}) \) is invertible because \( a \) is invertible in \( G \), and it is well-known that all the operators \( B_1, B_2, B_3 \) are invertible as well. Therefore, relations (2.3) and (2.4) imply that for any \( (\varphi, \psi)^T \in \ker W(V(a, b)) \), the element \( A^{-1}((\varphi, \psi)^T) \) belongs to the set

\[
\ker \text{diag} \left( W(a) + H(b) + Q, W(a) - H(b) + Q \right) = \ker \text{diag} \left( W(a) + H(b), W(a) - H(b) \right).
\]

Hence

\[
\text{diag} (P, P)A^{-1}((\varphi, \psi)^T) = A^{-1}((\varphi, \psi)^T).
\]

Computing the left-hand side of the last equation, one obtains the relation (2.1). Analogously, if \( (\Phi, \Psi)^T \in \ker \text{diag} \left( W(a) + H(b), W(a) - H(b) \right) \), then \( A((\Phi, \Psi)^T) \in \ker W(V(a, b)) \) and

\[
\text{diag} (P, P)A((\Phi, \Psi)^T) = A((\Phi, \Psi)^T),
\]

so the representation (2.2) follows.

Now let \( (\varphi, \psi) \) and \( (\Phi, \Psi) \) be as above. Then

\[
\text{diag} (P, P)A \text{diag} (P, P)A^{-1}((\varphi, \psi)^T) = AA^{-1}((\varphi, \psi)^T),
\]

and

\[
\text{diag} (P, P)A^{-1} \text{diag} (P, P)A((\Phi, \Psi)^T) = A^{-1}A((\Phi, \Psi)^T),
\]
which completes the proof.

From now on we will always assume that the generating functions $a$ and $b$ satisfy the condition
\[ \tilde{a} a = \tilde{b} b. \] \hfill (2.5)

Analogously to [6], relation (2.5) is called matching condition, and if $a$ and $b$ satisfy (2.5), then the duo $(a, b)$ is called matching pair. For each matching pair $(a, b)$ one can assign another matching pair $(c, d)$ with $c := \tilde{b} a^{-1}$ and $d := b \tilde{a}^{-1}$. Such a pair $(c, d)$ is called the subordinated pair for $(a, b)$, and it is easily seen that the functions which constitutes a subordinated pair have a specific property, namely $c \tilde{c} = 1 = d \tilde{d}$. Throughout this paper any function $g \in G$ satisfying the condition
\[ \tilde{g} g = 1, \]
is called matching function. In passing note that the matching functions $c$ and $d$ can also be expressed in the form
\[ c = a b^{-1}, \quad d = \tilde{b}^{-1} a. \]

Besides, if $(c, d)$ is the subordinated pair for a matching pair $(a, b)$, then $(\tilde{a}, \tilde{c})$ is the subordinated pair for the matching pair $(\tilde{a}, \tilde{b})$ which defines the adjoint operator
\[ (W(a) + H(b))^* = W(\tilde{a}) + H(\tilde{b}), \] \hfill (2.6)

for the operator $W(a) + H(b)$. Further, a matching pair $(a, b)$ is called Fredholm, if the Wiener–Hopf operators $W(c)$ and $W(d)$ are Fredholm.

If $(a, b)$ is a matching pair, then the corresponding matrix–function $V(a, b)$ takes the form
\[ V(a, b) = \begin{pmatrix} 0 & d \\ -c & \tilde{a}^{-1} \end{pmatrix}, \]
where $(c, d)$ is the subordinated pair for the pair $(a, b)$. Moreover, similarly to the corresponding representation of the matrix Toeplitz operator $T(V(a, b))$ from [6], the operator $W(V(a, b))$ can be represented as the product of three matrix Wiener–Hopf operators
\[ W(V(a, b)) = \begin{pmatrix} 0 & W(d) \\ -W(c) & W(\tilde{a}^{-1}) \end{pmatrix} \]
\[ = \begin{pmatrix} -W(d) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & W(\tilde{a}^{-1}) \end{pmatrix} \begin{pmatrix} -W(c) & 0 \\ 0 & I \end{pmatrix}, \] \hfill (2.7)

where the operator
\[ D := \begin{pmatrix} 0 & -I \\ I & W(\tilde{a}^{-1}) \end{pmatrix} \]
in the right–hand side of (2.7) is invertible and
\[ D^{-1} = \begin{pmatrix} W(\tilde{a}^{-1}) & I \\ -I & 0 \end{pmatrix}. \]
Note that a useful representation for the kernel of the block Toeplitz operator $T(V(a, b))$ defined by a matching pair $(a, b)$, has been derived recently. Following the proof of [7, Proposition 3.3], one can also obtain a similar result for the block Wiener-Hopf operator $W(V(a, b))$.

**Proposition 2.2** Let $(a, b) \in G \times G$ be a matching pair such that the operator $W(c)$, $c = ab^{-1}$, is invertible from the right. Then

$$\ker W(V(a, b)) = \Omega(c) + \hat{\Omega}(d)$$

where

$$\Omega(c) := \{(\varphi, 0)^T : \varphi \in \ker W(c)\},$$

$$\hat{\Omega}(d) := \{(W_r^{-1}(c)W(\overline{a}^{-1})s, s)^T : s \in \ker W(d)\},$$

and $W_r^{-1}(c)$ is one of the right inverses for the operator $W(c)$.

Thus the inclusion $\varphi \in \ker W(c)$ implies that $(\varphi, 0)^T \in \ker W(V(a, b))$ and by Lemma 2.1 one obtains

$$\varphi - JQW^0(c)P\varphi \in \ker(W(a) + H(b)), \quad \varphi + JQW^0(c)P\varphi \in \ker(W(a) - H(b)).$$

It is even more remarkable that the functions $\varphi - JQW^0(c)P\varphi$ and $\varphi + JQW^0(c)P\varphi$ belong to the kernel of the operator $W(c)$ as well.

**Proposition 2.3** If $g \in G$ is a matching function, i.e. $g\tilde{g} = 1$, then

(i) If $f \in \ker W(g)$, then $JQW^0(g)Pf \in \ker W(g)$.

(ii) If $f \in \ker W(g)$, then $(JQW^0(g)P)^2f = f$.

**Proof.** If $g\tilde{g} = 1$ and $f \in \ker W(g)$, then

\[ W(g)(JQW^0(g)Pf) = PW^0(g)PJQW^0(g)Pf = JQW^0(\tilde{g})QW^0(g)Pf = JQW^0(\tilde{g})W^0(g)Pf - JQW^0(\tilde{g})PW^0(g)Pf = 0, \]

and assertion (i) follows. On the other hand, for any $f \in \ker W(g)$ one has

\[ (JQW^0(g)P)^2f = JQW^0(g)PJQW^0(g)Pf = PW^0(\tilde{g})QW^0(g)Pf = PW^0(\tilde{g})W^0(g)Pf - PW^0(\tilde{g})PW^0(g)Pf = f, \]

which completes the proof.

Consider now the operator $P(g) := JQW^0(g)P|_{\ker W(g)}$. Proposition 2.3 implies that $P(g) : \ker W(g) \rightarrow \ker W(g)$ and $P^2(g) = I$. Thus on the space $\ker W(g)$ the operators $P^-(g) := (1/2)(I - P(g))$ and $P^+(g) := (1/2)(I + P(g))$ are complimentary projections, so they generate a decomposition of $\ker W(g)$. Moreover, inclusions (2.8) lead to the following result.
Corollary 2.4 Let \((c, d)\) be the subordinated pair for a matching pair \((a, b)\) \(\in G \times G\). Then
\[
\ker W(c) = \text{im } P^-(c) + \text{im } P^+(c),
\]
and
\[
\begin{align*}
\text{im } P^-(c) &\subset \ker(W(a) + H(b)), \\
\text{im } P^+(c) &\subset \ker(W(a) - H(b)),
\end{align*}
\]
hold.

Relations \([2.9]\) show the influence of the operator \(W(c)\) on the kernels of the operators \(W(a) + H(b)\) and \(W(a) - H(b)\). Let us now clarify the role of another operator—viz. the operator \(W(d)\), in the structure of the kernels of the operators \(W(a) + H(b)\) and \(W(a) - H(b)\). Assume additionally that the operator \(W(c)\) is invertible from the right. If \(s \in \ker W(d)\), then the element \((W_r^{-1}(c)W(a^{-1})s, s)^T \in \ker W(V(a, b))\). By Lemma 2.1, the element
\[
\varphi^\pm(s) := W_r^{-1}(c)W(a^{-1})s \mp JQW^0(c)PW_r^{-1}(c)W(a^{-1})s \pm JQW^0(\tilde{a}^{-1})s
\]
belongs to the null space \(\ker(W(a) \pm H(b))\) of the corresponding operator \(W(a) \pm H(b)\).

Lemma 2.5 The mapping \(s \mapsto \varphi^\pm(s)\) is a one-to-one function from the space \(\text{im } P^\pm(d)\) to the space \(\ker(W(a) \pm H(b))\).

Proof. Assuming that \(s \in \ker W(d)\), one can show that the operator \((1/2)(PW^0(\tilde{b})P + PW^0(\tilde{a})JP)\) sends \(\varphi^+(s)\) into \(P^+(d)s\) and the operator \((1/2)(PW^0(\tilde{b})P - PW^0(\tilde{a})JP)\) sends \(\varphi^-(s)\) into \(P^-(d)s\). The proof of these facts is based on the relations \([1.3]\) and runs similarly to the proof of [7, Lemma 3.6].

Proposition 2.6 Let \((c, d)\) be the subordinated pair for a matching pair \((a, b)\) \(\in G \times G\). If the operator \(W(c)\) is right-invertible, then
\[
\begin{align*}
\ker(W(a) + H(b)) &= \varphi^+(\text{im } P^+(d)) + \text{im } P^-(c), \\
\ker(W(a) - H(b)) &= \varphi^-(\text{im } P^-(d)) + \text{im } P^+(c).
\end{align*}
\]

Proof. As was mentioned in the proof of Lemma 2.5, for \(s \in \ker W(d)\) one has
\[
\begin{align*}
(1/2)(PW^0(\tilde{b})P + PW^0(\tilde{a})JP)\varphi^+(s) &= P^+(d)s, \\
(1/2)(PW^0(\tilde{b})P - PW^0(\tilde{a})JP)\varphi^-(s) &= P^-(d)s.
\end{align*}
\]
On the other hand, if \(s \in \ker W(c)\), then
\[
(1/2)(\varphi^+(s), \varphi^-(s))^T = (\text{im } P^-(c)s, \text{im } P^+(c)s)^T \in \ker \text{diag } (W(a) + H(b), W(a) - H(b)),
\]
and it is easily seen that
\[
\begin{align*}
(PW^0(\tilde{b})P + PW^0(\tilde{a})JP)(s - JQcPs) &= 0, \\
(PW^0(\tilde{b})P - PW^0(\tilde{a})JP)(s + JQcPs) &= 0.
\end{align*}
\]
Using these relations one can observe that if \( s \in \text{im} \, P^+(d) \), then
\[
E_2((1/2)(\varphi^+(s), 0)^T) = (1/2)(\varphi^+(s), s)^T,
\]
and if \( s \in \text{im} \, P^-(c) \), then
\[
E_2((1/2)(\varphi^+(s), 0)^T) = (1/2)(\varphi^+(s), 0)^T = (P^-(c)s, 0)^T.
\]
Thus
\[
E_2(\varphi^+(\text{im} \, P^+(d))) \cap E_2(\varphi^+(\text{im} \, P^-(c))) = \{0\}.
\]
But \( E_2 \) is an isomorphism, hence
\[
\varphi^+(\text{im} \, P^+(d)) \cap \text{im} \, P^-(c) = \{0\},
\]
and it is clear that \( \varphi^+(\text{im} \, P^+(d)) \) and \( \text{im} \, P^-(c) \) are closed subspaces of \( \ker(W(a) + H(b)) \). Moreover, the direct sum of \( \varphi^+(\text{im} \, P^+(d)) \) and \( \text{im} \, P^-(c) \) is a closed subspace. In order to show that \( Y := \varphi^+(\text{im} \, P^+(d)) \oplus \text{im} \, P^-(c) \) is \( \ker(W(a) + H(b)) \) we have to show that for \( s \in \text{P}^-(d) \), the element \( \varphi^+(s) \) belongs to \( Y \). Thus assume that \( s \in \text{P}^-(d) \) and consider the element
\[
E_2((\varphi^+(s), 0)^T) = (\varphi^+(s), 0)^T \in \ker W(V(a, b)).
\]
By Proposition 2.2 we have \( \varphi^+(s) \in \ker W(c) \). Moreover, \( \varphi^+(s) \in \text{P}^-(c) \) since otherwise \( E_1((\varphi^+(s), 0)^T) \notin \ker(W(a) + H(b)) \), which is a contradiction.

The related result for \( \ker(W(a) - H(b)) \) can be proved analogously. \( \blacksquare \)

**Remark 2.7** If \( \dim \ker W(d) < \infty \) and \( \dim \ker W(c) < \infty \), the proof of Proposition 2.7 follows from simple dimension considerations. The proof above covers the general case \( \dim \ker W(d) \leq \infty \) and \( \dim \ker W(c) \leq \infty \).

**Remark 2.8** If \( (a, b) \in G \times G \) is a Fredholm matching pair, i.e. if \( W(c), W(d) \) are Fredholm operators, then \( W(a) \pm H(b) \) are Fredholm operators and
\[
\text{ind } (W(a) + H(b)) + \text{ind } (W(a) - H(b)) = \text{ind } W(c) + \text{ind } W(d). \tag{2.11}
\]
We conjecture that if one of the operators \( W(a) + H(b) \) or \( W(a) - H(b) \) is Fredholm, then so is the other and the relation (2.11) holds.

### 3 Kernels of Wiener–Hopf plus Hankel operators. Specification.

In this section we study the kernels of Wiener–Hopf plus Hankel operators \( W(a) + H(b) \) in the case where the generating function \( a, b \in G \) satisfy the matching condition (2.5) and \( W(c), W(d) \) are mainly Fredholm operators such that
\[
0 \leq |\text{ind } W(c)|, |\text{ind } W(d)| \leq 1.
\]
Recall that \( a \) is supposed to be invertible in \( G \). In view of Theorem 1.1 on has \( \nu(c) = \nu(d) = 0 \) and \( 0 \leq |n(c)|, |n(d)| \leq 1 \). In order to formulate our first theorem, we need the following lemma.
Lemma 3.1 If \( a \in G \), then

(i) If the function \( \psi \) is defined by

\[
\psi(t) := \begin{cases} 
  e^{-t} & \text{if } t > 0, \\
  0 & \text{if } t < 0,
\end{cases}
\]

then \( W^0(\chi^{-1})\psi = \tilde{\psi} \).

(ii) On each space \( L^p(\mathbb{R}^+) \), \( 1 \leq p \leq \infty \) the operator \( W(\chi^{-1}) \) has a one-dimensional kernel generated by the function \( \psi_0(t) = e^{-t}, t > 0 \).

Proof. Assertion (i) can be obtained by using the relation

\[
W^0(\chi^{-1})g(t) = g(t) - 2 \int_{t}^{\infty} e^{t-s} g(s) \, ds, \quad -\infty < t < \infty,
\]

which is valid for all \( g \in L^p(\mathbb{R}) \), \( 1 \leq p \leq \infty \), \([11]\).

Assertion (ii) is well-known. It can be proved by the differentiation of the identity

\[
\varphi(t) = 2 \int_{t}^{\infty} e^{t-s} \varphi(s) \, ds.
\]

Moreover, one has

\[
W(\chi^{-1})g(t) = g(t) - 2 \int_{t}^{\infty} e^{t-s} g(s) \, ds, \quad 0 < t < \infty,
\]

(see \([11]\)). Note that assertion (ii) also follows from assertion (i). \( \blacksquare \)

Now we can derive the following version of Coburn–Simonenko theorem.

Theorem 3.2 Let \( a \in G \) be invertible and let \( A \) denote any of the four operators \( W(a) + H(a\chi) \), \( W(a) - H(a\chi^{-1}) \), \( W(a) \pm H(a) \). Then at least one of the spaces \( \ker A \) or \( \operatorname{coker} A \) is trivial.

Proof. Part 1: Let us start with the operator \( W(a) + H(a\chi) \). The function \( \chi \) satisfies the relation \( \tilde{\chi} = \chi^{-1} \), so the duo \( (a, a\chi) \) is a matching pair with the subordinated pair \( (c, d) \) with \( c = \chi^{-1} \) and \( d = a\tilde{a}^{-1}\chi \). Moreover, the operator \( W(\chi^{-1}) \) is invertible from the right and one of its right inverses is the operator \( W(\chi) \). Thus the theory of Section 2 applies. As it was pointed out earlier, the kernel of this operator is

\[
\ker W(\chi^{-1}) = \{ c\psi_0 : c \in \mathbb{C} \},
\]

where \( \psi_0(t) = e^{-t}, t > 0 \). In order to apply Proposition 2.6 we have to identify, in particular, the projections \( P^\pm(\chi^{-1}) \) acting on the space \( \ker W(\chi^{-1}) \). But \( P^+(\chi^{-1}) \) and \( P^-(\chi^{-1}) \) are complimentary projections on the one-dimensional space \( \ker W(\chi^{-1}) \). Therefore, one of these projections is just the identity operator whereas the other one is the zero operator. Consider next the expression \( JQW^0(\chi^{-1})P\psi_0 \). By Lemma 3.1(ii) one has

\[
JQW^0(\chi^{-1})P\psi_0 = JQW^0(\chi^{-1})\psi = JQ\tilde{\psi},
\]

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so that $JQW^0(\chi^{-1})P\psi_0 = \psi_0$ and $P^+(\chi^{-1}) = I$ on $\ker W(\chi^{-1})$.

According to Proposition 2.6, the kernels of the operators $W(a) + H(a\chi)$ and $W(a) - H(a\chi)$ can be represented in the form

$$\ker(W(a) + H(a\chi)) = \varphi^+(\im P^+(d)),$$
$$\ker(W(a) - H(a\chi)) = \varphi^-(\im P^-(d)) + \{c\psi_0 : c \in \mathbb{C}\}. \quad (3.1)$$

If $\dim \ker W(d) > 0$, then $\coker(W(a) \pm H(a\chi)) = \{0\}$. Indeed, relation (2.7) and the familiar Coburn–Simonenko Theorem for the operator $W(d)$ show that $\coker W(V(a, a\chi)) = \{0\}$. Taking into account the representation (2.4), one obtains that the cokernel of each of the operators $W(a) + H(a\chi)$ and $W(a) - H(a\chi)$ contains the zero element only.

Let us now assume that $\ker W(d) = \{0\}$. Then the first relation (3.1) implies that

$$\ker(W(a) + H(a\chi)) = 0.$$

Thus, the operator $W(a) + H(a\chi)$ is subject to Coburn–Simonenko Theorem.

**Part 2:** Consider the operator $W(a) - H(a\chi^{-1})$ and note that $W(c) = W(\chi)$ is not right-invertible, so that Proposition 2.6 cannot be directly used in this situation. Nevertheless, the case at hand can be reduced to the operators studied. Thus the operators $W(a) \pm H(a\chi^{-1})$ can be represented in the form

$$W(a) \pm H(a\chi^{-1}) = (W(a\chi^{-1}) \pm H(a\chi^{-1}))(W(\chi)). \quad (3.2)$$

The proof of (3.2) follows from (1.4) and relation $H(\chi)W(\chi) = 0$. Setting $\alpha := a\chi^{-1}$, we get

$$W(a) \pm H(a\chi^{-1}) = (W(\alpha) \pm H(a\chi))(W(\chi)). \quad (3.3)$$

The operators of the form $W(\alpha) \pm H(a\chi)$ in the right-hand side of (3.2) have been just studied, and we already know that the function $\psi_0$ belongs to the kernels of both operators $W(\alpha) - H(a\chi)$ and $W(\chi)^{-1}$. Since $W(\chi^{-1}W(\chi) = I$ it follows that $\psi_0 \notin \im W(\chi)$. Consider now the projection $Q_0 := W(\chi)W(\chi)^{-1}$ which projects the space $L^p(\mathbb{R}^+)$, $1 \leq p \leq \infty$ onto $\im W(\chi)$ parallel to $\ker W(\chi^{-1})$.

Assume first that $\ker W(d) = \{0\}$ and note that for the matching pairs $(a, a\chi^{-1})$ and $(\alpha, a\chi)$, the corresponding subordinated pairs $(c, d)$ have the same element $d$, namely, $d = a\alpha^{-1}\chi$. Then (3.3) shows that $\ker(W(a) - H(a\chi^{-1})) = \{0\}$. Further, if $\dim \ker W(d) > 0$, then the space $\ker(W(\alpha) - H(a\chi))$ decomposes as follows

$$\ker(W(\alpha) - H(a\chi)) = \ker W(\chi^{-1}) \oplus Q_0(\ker(W(\alpha) - H(a\chi))).$$

However, as was already shown, the operator $W(\alpha) - H(a\chi)$ is right-invertible and

$$\ker W(\chi^{-1}) \subset \ker W(\alpha) - H(a\chi).$$

Therefore, relation (3.2) implies that the operator $W(\alpha) - H(a\chi^{-1})$ maps $L^p(\mathbb{R}^+)$ onto $L^p(\mathbb{R}^+)$, so it is subject to Coburn–Simonenko Theorem.

**Part 3:** It remains to consider the operators $W(a) \pm H(a)$. For these operators the element $c$ in the corresponding subordinated pair is either 1 or −1, and our claim follows immediately from the Coburn–Simonenko Theorem for scalar Wiener–Hopf operators and from the relations (2.7) and (2.4).
Remark 3.3 The proof of Theorem 3.2 shows that this theorem remains true for more general generating functions, for instance, in the case where \( a \) and \( b \) belong to the algebras \( G_p, 1 \leq p \leq \infty \) studied in Chapter VII.

The reader can also observe that, in fact, we have proved a bit more than Theorem 3.2 states. A more detailed result can be formulated as follows.

Corollary 3.4 Let \( a \in G \). Then

(i) If \( \dim \ker W(d) = 0 \), then

\[
\ker(W(a) + H(a\chi)) = \{0\}, \quad \ker(W(a) - H(a\chi)) = \{c\psi_0 : c \in \mathbb{C}\},
\]

and if \( \dim \ker W(d) > 0 \), then

\[
\coker(W(a) \pm H(a\chi)) = \{0\}.
\]

(ii) If \( \dim \ker W(d) = 0 \), then

\[
\ker(W(a) \pm H(a\chi^{-1})) = \{0\},
\]

and if \( \dim \ker W(d) > 0 \), then

\[
\coker(W(a) - H(a\chi^{-1})) = \{0\}.
\]

Let us emphasize that the description of the projections \( P^\pm(\chi^{-1}) \) did play an important role in our considerations. In a general case one has to study the projections \( P^\pm(g) \) for the functions \( g \) satisfying the relation \( g\tilde{g} = 1 \). Because of the space restriction, we are not going to pursue this matter here. Nevertheless, let us consider the case where \( \nu(g) = 0 \) and \( n(g) = -1 \), which is one of the simplest generalization of the situation \( g = \chi^{-1} \). In order to handle this case we need a result from Chapter VII.

Proposition 3.5 Each invertible function \( g \in G \) admits the factorization of the form

\[
g(t) = g_-(t)e^{\nu t}(t-i)^n g_+ (t), \quad -\infty < t < \infty, \quad (3.4)
\]

where \( g_{\pm}^\pm \in G^\pm, g_{\pm}^\pm \in G^-, \nu = \nu(g) \) and \( n = n(g) \). Moreover, under the agreement \( g_-(0) = 1 \), the factorization factors \( g_+ \) and \( g_- \) are uniquely defined.

Note that the proof of Theorem 3.1 is based on Proposition 3.5.

Definition 3.6 Suppose that \( g \in G \) satisfies the condition \( g\tilde{g} = 1 \) and set

\[
\xi(g) = (-1)^n g(0), \quad n = n(g).
\]
Theorem 3.7 If \( g \in G \) and \( g\overline{g} = 1 \), then \( \xi(g) = \pm 1 \) and the factorization (3.4) takes the form

\[
g(t) = (\xi(g) g_{+1}^{-1}(t)) e^{i\nu t} \left( \frac{t - i}{t + i} \right)^n \ g_{+}(t)
\]

with \( g_{+1}^{-1}(t) \in G^- \) and \( g_{-}(t) = \xi(g) g_{+1}^{-1}(t) \).

Proof. Using the condition \( g^{-1} = \overline{g} \), we get from (3.4) that

\[
g_{+1}^{-1}(t)e^{-i\nu t} \left( \frac{t - i}{t + i} \right)^n g_{-1}(t) = \overline{g}_{-}(t)e^{-i\nu t} \left( \frac{t - i}{t + i} \right)^n \overline{g}_{+}(t),
\]

where \( \nu = \nu(g) \), \( n = n(g) \). Note that \( \overline{g}_{+1}^{-1} \in G^+ \), \( \overline{g}_{+1}^{-1} \in G^- \), as easy computations show. Therefore,

\[
g_{+1}^{-1} \overline{g}_{-1}^{-1} = g_{-} \overline{g}_{+},
\]

and \( g_{+1}^{-1} \overline{g}_{-1}^{-1} \in G^+ \), \( g_{-} \overline{g}_{+} \in G^- \). It follows that there is a constant \( \xi \in \mathbb{C} \) such that

\[
g_{+1}^{-1} \overline{g}_{-1}^{-1} = \xi = g_{-} \overline{g}_{+},
\]

and

\[
g_{-} = \xi \overline{g}_{+}^{-1}
\]

For the function \( g_0 = g_{+}g_{-} \) we have \( g_0 \overline{g}_0 = 1 \). Therefore,

\[
1 = g_0 \overline{g}_0 = (\xi g_{+} \overline{g}_{+}^{-1})(\xi \overline{g}_{+} g_{+}^{-1}) = \xi^2.
\]

For \( t = 0 \), which is one of the fixed points of the operator \( J \), the equation \( g_0 = \xi g_{+} \overline{g}_{+}^{-1} \) implies \( g_0(0) = \xi \), and \( g_0(0) = g(0)(-1)^n \) (see (3.5)). Thus we obtain that \( \xi = g(0)(-1)^n \) which completes the proof.

Now we again use the notation

\[
\chi^{\pm 1}(t) = \left( \frac{t - i}{t + i} \right)^{\pm 1}, \quad t \in \mathbb{R}.
\]

Theorem 3.8 Let \( g \in G \), \( g\overline{g} = 1 \), \( \nu(g) = 0 \) and \( n(g) = -1 \). Then

\[
\text{im} \ P^\pm(g) = \left\{ c \left( \frac{1 \pm \xi(g)}{2} \right) W(g_{+}^{-1})\psi_0 : c \in \mathbb{C} \right\}.
\]

Proof. It is easily seen that

\[
\ker W(g) = \left\{ cW(g_{+}^{-1})\psi_0 : c \in \mathbb{C} \right\}.
\]

According to the definition of projections \( P^\pm(g) \) we have to compute the expression

\[
JQW^0(g)PW(g_{+}^{-1})\psi_0.
\]

We have

\[
JQW^0(g)PW(g_{+}^{-1}) = JQW^0(g)W(g_{-})W^0(\chi_{-1})W^0(g_{+})W^0(g_{+}^{-1})P
\]

\[
= JQW^0(g_{-})W^0(\chi_{-1})P.
\]
Recall that by Lemma 3.4, \( W^0(\chi^{-1})P\psi_0 = W^0(\chi^{-1})\psi = \tilde{\psi} \), and using Theorem 3.7 we get

\[
JQW^0(g)PW(g_+^{-1})\psi_0 = JQW^0(g_-)\tilde{\psi} = W^0(\tilde{g}_-)\psi
= P\xi(g)W^0(g_+^{-1})P\psi_0 = \xi(g)W^0(g_+^{-1})\psi_0,
\]

and we are done.

The next result is a generalization of Theorem 3.2.

**Theorem 3.9** Let \( a,b \in G \) constitute a matching pair, \( a \) be invertible in \( G \) and let \( (c,d) \) be the subordinated pair for \( (a,b) \). If \( A \) denotes one of the following operators

(i) \( W(a) \pm H(b) \) with \( \nu(c) = 0 \), \( n(c) = 1 \) and \( \xi(c) = \pm 1 \);

(ii) \( W(a) \pm H(b) \) with \( \nu(c) = 0 \), \( n(c) = -1 \) and \( \xi(c) = \pm 1 \);

(iii) \( W(a) \pm H(b) \) with \( \nu(c) = 0 \) and \( n(c) = 0 \)

considered on the space \( L^p(\mathbb{R}^+) \), then at least one of the spaces \( \ker A \) or \( \text{coker} A \) is trivial.

**Proof.** The proof mimics that of Theorem 3.2 with minor modifications. First, we note that the case \( \xi(c) = -1 \) can be reduced to the case \( \xi(c) = 1 \) via rearrangements \( W(a) + H(b) = W(a) - H(-b) \) and \( W(a) - H(b) = W(a) + H(-b) \). Therefore, we only have to consider the situation \( \xi(c) = 1 \) in the cases (i) and (ii). Further, one has to use Theorem 3.8 instead of the description of the projections \( P^\pm(\chi^{\pm 1}) \). Consider the operator \( W(a) - H(b) \) in the case where \( \nu(c) = 0 \) and \( n(c) = 1 \). Representing the operator \( W(a) \pm H(b) \) in the form

\[
W(a) \pm H(b) = (W(a\chi^{-1}) \pm H(b\chi))W(\chi),
\]

we observe that \( (a\chi^{-1}, b\chi) \) is a matching pair with the subordinated pair \( (c, d) \) and \( \text{ind} W(c, d) = -1 \), \( \text{im} P^+(c, d) = \ker W(c, d) = \{cW(c\chi^{-1})\psi_0 : c \in \mathbb{C}\} \). Let us also note that \( \ker W(c\chi^{-2}) = \ker W(c\chi^{-1}) \) and

\[
W(c\chi^{-1}) W(c\chi^{-1}) = I.
\]

Hence,

\[
\ker W(c\chi^{-2}) \cap \text{im} W(c\chi^{-1}) = \{0\}.
\]

Since obviously \( \text{im} W(c\chi^{-1}) = \text{im} W(\chi) \), we obtain

\[
\ker W(c\chi^{-1}) \cap \text{im} W(\chi) = \{0\}.
\]

Now one can proceed similarly to Part 2 in the proof of Theorem 3.2.

**Corollary 3.10** Assume that \( a,b \in G \) constitute a matching pair with the subordinated pair \( (c,d) \) such that \( \xi(c) = 1 \). Then
(i) If \( \dim \ker W(d) = 0 \), and \( \text{ind} W(c) = 1 \), then
\[
\ker(W(a) + H(b)) = \{0\},
\]
\[
\ker(W(a) - H(b)) = \{cW(c^{-1})\psi_0 : c \in \mathbb{C}\},
\]
and if \( \dim \ker W(d) > 0 \), then
\[
coker (W(a) \pm H(b)) = \{0\}.
\]

(ii) If \( \dim \ker W(d) = 0 \), and \( \text{ind} W(c) = -1 \), then
\[
\ker(W(a) \pm H(b)) = \{0\},
\]
and if \( \dim \ker W(d) > 0 \), then
\[
coker (W(a) - H(b)) = \{0\}.
\]

An interesting and important subclass of the operators considered in this paper comprises the identity plus Hankel operators. Let us specify the above results in this situation

**Corollary 3.11** If \( b \in G \) is a matching function, then \((1, b)\) is a matching pair with the subordinated pair \((\tilde{b}, b)\), and if \( A \) denote any of the operators

(i) \( I - H(b) \) with \( \nu(\tilde{b}) = 0 \), \( n(\tilde{b}) = -1 \) and \( \xi(\tilde{b}) = 1 \);
(ii) \( I + H(b) \) with \( \nu(\tilde{b}) = 0 \), \( n(\tilde{b}) = 1 \) and \( \xi(\tilde{b}) = 1 \);
(iii) \( I \pm H(b) \) with \( \nu(\tilde{b}) = 0 \) and \( n(\tilde{b}) = 0 \),
considered on the space \( L^p(\mathbb{R}^+) \), then \( \ker A \) or \( \text{coker} A \) is trivial.

Now we revisit Theorem 3.2 and consider the operators \( W(a) \pm H(\chi) \) and \( W(a) \pm H(\chi^{-1}) \) under additional assumptions.

1. Suppose that \( \nu(a) = n(a) = 0 \) and \( b = \alpha \). The subordinated pair \((c, d)\) is given by the elements \( c = \chi^{-1} \) and \( d = \alpha \chi^{-1} \). Thus
\[
\text{ind} W(c) = 1, \quad \text{ind} W(d) = -1, \quad \xi(c) = \xi(d) = 1.
\]

According to (2.11) we have
\[
\text{ind} (W(a) + H(\alpha \chi)) + \text{ind} (W(a) - H(\alpha \chi)) = 0. \tag{3.6}
\]

Further, by Corollary 3.4(i) we also have
\[
\ker(W(a) + H(\alpha \chi)) = 0, \quad \ker(W(a) - H(\alpha \chi)) = \{c\psi_0 : c \in \mathbb{C}\}.
\]

In order to describe the cokernels of the above operators we make use of the adjoint operators. If \( p \in [1, \infty) \), then according to (2.6) the adjoint operators have the form \( W(\bar{\alpha}) \pm H(\alpha \chi) \), and the duo \((\bar{\alpha}, \alpha \chi)\) is a matching pair with the subordinated pair \( (\bar{d}, \bar{c}) \), so that \( \text{ind} W(\bar{d}) = 1, \text{ind} W(\bar{c}) = -1 \) and \( \xi(\bar{d}) = 1 \).
By Corollary 3.10 (ii), \( \ker(W(\bar{a}) + H(\bar{a}\chi)) = \{0\} \), which finally proves that the operator \( W(a) + H(a\chi) \) is invertible. Note that this result is also true for the space \( L^\infty(\mathbb{R}^+) \). Indeed, the operator \( W(\bar{a}) + H(\bar{a}\chi) \) acts on the space \( L^1(\mathbb{R}^+) \) and the above considerations show that \( \dim \ker(W(\bar{a}) + H(\bar{a}\chi)) = 0 \). The adjoint of this operator acts on the space \( L^\infty(\mathbb{R}^+) \) and is equal to the operator \( W(a) + H(a\chi) \), the kernel of which is trivial. Therefore, the operator \( W(\bar{a}) + H(\bar{a}\chi) \) is invertible on the space \( L^1(\mathbb{R}^+) \). Consequently, its adjoint \( W(a) + H(a\chi) \) is invertible on \( L^\infty(\mathbb{R}^+) \). Then the relation (3.6) immediately implies that \( \text{ind} (W(a) - H(a\chi)) = 0 \). Note that the operator \( W(a) - H(a\chi) \) provides an example of operators where both spaces \( \ker(W(a) - H(a\chi)) \) and \( \text{coker} (W(a) - H(a\chi)) \) are non-trivial.

2\(^0\). Suppose that \( \nu(a) = 0 \), \( n(a) = -1 \) and \( b = a\chi \). For the subordinated pair \((c, d)\) we have \( c = \chi^{-1} \) and \( d = a\bar{a}^{-1}\chi \) so that

\[
\text{ind} W(c) = 1, \quad \text{ind} W(d) = 1, \quad \xi(d) = 1.
\]

Since \( \text{ind} W(d) = 1 \), Corollary 3.4 (a) indicates that

\[
\text{coker} (W(a) \pm H(a\chi)) = \{0\}.
\]

besides, \( \dim \ker(W(a) \pm H(a\chi)) = 1 \) by Proposition 2.6.

3\(^0\). Suppose that \( \nu(a) = n(a) = 0 \) and \( b = a\chi^{-1} \). Since \( c = \chi \), the operator \( W(c) \) is not invertible from the right. Write

\[
W(a) \pm H(a\chi^{-1}) = (W(a\chi^{-1}) \pm H(a\chi^{-1}\chi))W(\chi), \tag{3.7}
\]

and set \( \alpha := a\chi^{-1} \). The operators \( W(\alpha) \pm H(\alpha\chi) \) are considered in 2\(^0\), so we have

\[
\dim \ker(W(\alpha) \pm H(\alpha\chi)) = 1
\]

\[
\dim \text{coker} (W(\alpha) \pm H(\alpha\chi)) = 0.
\]

According to the Part 2 in the proof of Theorem 3.2 one has \( \ker(W(\alpha) - H(\alpha\chi)) = \{0\} \). This and relation \( \dim \text{coker} (W(\alpha) - H(\alpha\chi)) = 0 \) show the invertibility of the operator \( W(\alpha) - H(\alpha\chi) \). Due to Proposition 2.6 (see also 4.1) we know that the kernel of the operator \( W(\alpha) + H(\alpha\chi) \) is spanned on the element

\[
\kappa = W(\chi)W(\bar{\alpha}^{-1})W(d_+^{-1})\psi_0
- \alpha W(\chi^{-1})PW(\bar{\alpha}^{-1})PW(\bar{\alpha}^{-1})W(d_+^{-1})\psi_0
+ \alpha W(\chi^{-1})PW(d_+^{-1})\psi_0., \tag{3.8}
\]

where we used the fact that \( W(\chi) \) is a right inverse for the operator \( W(\chi^{-1}) \) and \( d_+^{-1} \) arises from the factorization (3.5) of the function \( d = a\bar{a}^{-1}\chi^{-1} \). Note that the first item in (3.5) belongs to the set \text{im} W(\chi), whereas the second one is equal to zero. Thus, the operator \( W(a) + H(a\chi^{-1}) \) is invertible if and only if \( H(\alpha^{-1})W(d_+^{-1})\psi_0 \notin \text{im} W(\chi) \). However, if this condition is not satisfied, \( W(a) + H(a\chi^{-1}) \) gives an example of a Wiener–Hopf plus Hankel operator with one-dimensional kernel and cokernel.
4°. Suppose that \( \nu(a) = 0 \), \( n(a) = 1 \) and \( b = a\chi^{-1} \). Let us use the representation (3.7) and set \( \alpha = a\chi^{-1} \). It follows from part 1° that \( W(\alpha) + H(\alpha\chi) \) is invertible whereas the operator \( W(\alpha) - H(\alpha\chi) \) has one-dimensional kernel and cokernel. Since

\[
\ker(W(\alpha) - H(\alpha\chi)) = \{c\psi_0 : c \in \mathbb{C}\} \cap \text{im} W(\chi) = \{0\},
\]

we conclude that the operator \( W(\alpha) - H(\alpha\chi^{-1}) \) has trivial kernel and a cokernel of dimension 1. Of course, the same conclusion is valid for the operator \( W(\alpha) + H(\alpha\chi) \).

It is worth noting that similar consideration with natural amendments can be conducted in the contest of Theorem 3.9. Let us restrict ourself to the operators \( I + H(b) \) with the generating function \( b \) satisfying the condition \( \bar{b}b = 1 \). Then \( (1, b) \) is a matching pair with the subordinated pair \( \tilde{b}, b \).

5°. Suppose that \( \nu(b) = n(b) = 0 \). Then the operators \( W(b) \) and \( W(\tilde{b}) \) are invertible and relations (2.4), (2.7) already show that \( I + H(b) \) and \( I - H(b) \) are invertible operators.

Assume next that \( \nu(b) = 0 \) but \( n(b) = 1 \) and \( \xi(\tilde{b}) = 1 \). Then \( \text{ind} W(\tilde{b}) = 1 \) and \( \text{ind} W(b) = -1 \). By Corollary 3.10(i), one has

\[
\ker(I + H(b)) = \{0\}, \quad \ker(I - H(b)) = \{cW(b_+\psi_0) : c \in \mathbb{C}\}.
\]

Similarly to Part 1° one shows that the operator \( I + H(b) \) is invertible and \( \text{ind}(I - H(b)) = 0 \).

Finally, let us assume that \( \nu(b) = 0 \), \( n(b) = -1 \) and \( \xi(\tilde{b}) = 1 \). Since \( \text{ind} W(\tilde{b}) = -1 \), we will use the relation

\[
I \pm H(b) = (W(\chi^{-1}) \pm H(b\chi))W(\chi).
\]

It is clear that \( (\chi^{-1}, b\chi) \) is a matching pair with the subordinated pair \( \tilde{b}\chi^{-2}, b \) and \( \text{ind} W(\tilde{b}\chi^{-2}) = \text{ind} W(b) = 1 \). Analogously to Part 2° we obtain that

\[
\text{coker}(W(\chi^{-1}) \pm H(b\chi)) = \{0\}.
\]

Moreover, by Proposition 2.6, \( \dim \ker(W(\chi^{-1}) \pm H(b\chi)) = 1 \) and since \( \ker(W(\chi^{-1}) - H(b\chi)) = \{cW(b_+\psi_0) : c \in \mathbb{C}\} \cap \text{im} W(\chi) = \{0\} \), the operator \( I - H(b) \) is invertible. If \( \ker W(\chi^{-1}) + H(b\chi) \cap \text{im} W(\chi) = \{0\} \), then \( I + H(b) \) is invertible. Otherwise, \( \text{ind}(I + H(b)) = 0 \) but this operator is not invertible.

References

[1] E. L. Basor, T. Ehrhardt, Factorization theory for a class of Toeplitz + Hankel operators, J. Operator Theory 51(2) (2004), 411–433.

[2] E. L. Basor, T. Ehrhardt, Factorization of a class of Toeplitz + Hankel operators and the \( A_p \)-condition, J. Operator Theory 55(2) (2006), 269–283.
[3] E. L. Basor, T. Ehrhardt, *Fredholm and invertibility theory for a special class of Toeplitz + Hankel operators*, J. Spectral Theory 3(3) (2013), 171–214.

[4] A. Böttcher, B. Silbermann, *Analysis of Toeplitz operators*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006.

[5] L. P. Castro, A. S. Silva, *Defect numbers of singular integral operators with Carleman shift and almost periodic coefficients*, J. Math. Anal. Appl. 387(1) (2012), 66–76.

[6] V. D. Didenko, B. Silbermann, *Some results on the invertibility of Toeplitz plus Hankel operators*, arXiv:1306.2837, 2013.

[7] V. D. Didenko, B. Silbermann, *Structure of kernels and cokernels of Toeplitz plus Hankel operators*, arXiv:1309.7574, 2013.

[8] V. D. Didenko, B. Silbermann, *Index calculation for Toeplitz plus Hankel operators with piecewise quasi-continuous generating functions*, Bull. London Math. Soc. 45(3) (2013), 633–650.

[9] T. Ehrhardt, *Factorization theory for Toeplitz+Hankel operators and singular integral operators with flip*, Habilitation Thesis, Technische Universität Chemnitz, 2004.

[10] T. Ehrhardt, *Invertibility theory for Toeplitz plus Hankel operators and singular integral operators with flip*, J. Funct. Anal. 208(1) (2004), 64–106.

[11] I. C. Gohberg, I. A. Feldman, *Convolution equations and projection methods for their solution*, American Mathematical Society, Providence, R.I., 1974.

[12] S. Roch, P. A. Santos, B. Silbermann, *Non-commutative Gelfand theories. A tool-kit for operator theorists and numerical analysts*, Universitext, Springer-Verlag London Ltd., London, 2011.

[13] S. Roch, B. Silbermann, *Algebras of convolution operators and their image in the Calkin algebra*, Report MATH, vol. 90, Akademie der Wissenschaften der DDR Karl-Weierstrass-Institut für Mathematik, Berlin, 1990.

[14] S. Roch, B. Silbermann, *A handy formula for the Fredholm index of Toeplitz plus Hankel operators*, Indag. Math. 23 (2012), no. 4, 663–689.

[15] B. Silbermann, *The C*-algebra generated by Toeplitz and Hankel operators with piecewise quasicontinuous symbols*, Integral Equations Operator Theory 10(5) (1987), 730–738.