Research Article

Artificial Triangular Points by Lorentz Force in the Restricted Three-Body Problem

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The restricted three body problem was outlined. The acceleration due to planetary magnetic field, in terms of spacecraft’s orbital elements, was analysed. The conditions for calculating the liberation points including the mutual gravitational attraction and the effect of Lorentz acceleration were derived for the case of circular planer restricted three bodies. The stability of the solution for the artificial Lorentz triangular liberation points was studied. Finally, numerical investigation for the case of Sun-Jupiter system was calculated as case study. The results show the ability of changing the position of the triangular liberation points by an order from $10^{-7}$ to $10^{-6}$ for the dimensionless $x, y$ coordinates and distance $r$ from Jupiter. This is equivalent to about hundreds of Kilometers which is considerable.

1. Introduction

The history of the RTBP begins in 1772 by Euler and Lagrange, continues in 1836 by Hill [1], and is followed by Poincare [2], Levi-Civita [3], Birkhoff [4], and then by Szebehley [5]. Till now there are great names and important contributions to the RTBP.

Most perturbation techniques are applied to solve the equations of motion of the RTBP. Delva [6], used a Lie series to determine the orbits of massless bodies in the elliptic RTBP. Also Sandor et al. [7], applied the method of the short time Lyapunov indicators to the RTBP in order to study the structure of the phase space in some selected regions.

The idea of the effect of geomagnetic field on the rotation and motion of Earth’s artificial satellite was initiated by Fain and Greer [8]. Westerman, [9], used their idea to study the perturbation on circular orbit in the plane of the geomagnetic equator. Sehnal [10], concluded, at that time, no precise information available to evaluation of the effects of the Earth’s magnetic field on the orbit of a charged satellite.

The concept of generating AEP in the RTBP was initiated by Dusek [11]. The stability and location of the AEP were investigated since that time, from different aspects, by several authors [12–16]. On the other side, McInnes and others produced an extensive subsequently researches to find the equilibrium positions using solar sail propulsion such as [17–21]. Such work lead to find infinite equilibrium surfaces depending on the magnitude of the propulsive acceleration.

In the direction to finding the AEP using low-thrust systems Morimoto, et al. [22], proved the existence of infinite equilibrium surfaces depending on the magnitude of the propulsive acceleration.

However, only a subset of the potentially achievable AEPs turns out to be stable and, as thus, could not be exploited by a spacecraft without the use of a suitable control system. The topology of such subset of stable AEPs is strictly dependent on the propulsion system type employed by the spacecraft. In fact, as was recently pointed out by Bombardelli and Peláez [23], if the available propulsive acceleration is low, the stable AEPs are confined to a very restricted region around the classical Lagrange points.

In the last decade, several important articles worked on the effects of Lorentz force to control both orbit motion and
attitude of a spacecraft whether alone or combined with other forces, for example, in 2011 Pollock et al. [24] proposed analytical solutions for the relative motion of a Lorentz spacecraft, while in the year 2020, Mostafa et al. [25], studied the use of Lorentz force to control the perturbations due to solar radiation pressure, and Sun et al. [26], studied the coupled effect of Lorentz force with aerodynamics on controlling a spacecraft. Huang et al. [27], introduced a hybrid control system that contains both the Lorentz force and torque to control the orbit and attitude of a satellite.

In this article we study the effect of Lorentz force on the position and stability of the triangular liberation points in the restricted three body problem with an application given to the Sun-Jupiter system.

2. Formulation of the Problem

Consider the problem of three-bodies with the masses of the primaries $m_1, m_0$ and the mass of spacecraft (SC) $m$ in which $m$ is negligibly small compared to $m_0$ and $m_1$. Assume that the primaries revolve in circular orbits around their center of mass with mean motion $n$ and constant separation $L$. Then, $n^2 L^2 = k^2 M$ with $k$ is the Gaussian constant and $M = (m_1 + m_0)$.

In Figure 1, let the XY plane be the plane of motion of $m_0$ and $m_1$, with X axis is the line joining $m_0$ and $m_1$, Y axis is normal to X in their orbital plane while Z is perpendicular to the XY plane. Let $(X, Y, Z)$ be the coordinates of the SC in this coordinate system (the synodic coordinate system), then the equations of motion of SC in the framework of the circular restricted three body problem (CRTBP), using dimensionless units, become.

\[
\ddot{X} - 2n\dot{Y} = n^2 X - k^2 \left[ m_0 \left( \frac{X + a}{r_0^3} + m_1 \left( \frac{X - b}{r} \right) \right) \right],
\]

\[
\ddot{Y} + 2n\dot{X} = n^2 Y - k^2 \left[ m_0 \left( \frac{Y}{r_0} + m_1 \frac{Y}{r} \right) \right],
\]

\[
\dddot{Z} = -k^2 \left( m_0 \frac{Z}{r_0^3} + m_1 \frac{Z}{r} \right).
\]

If we transform the coordinate system to the center of the second primary $m_1$, with the coordinate $x, y,$ and $z$, the equations of motion will be:

\[
\ddot{x} - 2ny = n^2 (x + b) - k^2 \left[ m_0 \left( \frac{x + a + b}{r_0^3} + m_1 \frac{x}{r} \right) \right],
\]

\[
\ddot{y} + 2nx = n^2 y - k^2 \left[ m_0 \left( \frac{y}{r_0} + m_1 \frac{y}{r} \right) \right],
\]

\[
\dddot{z} = -k^2 \left( m_0 \frac{z}{r_0^3} + m_1 \frac{z}{r} \right).
\]

With,

\[
r_0 = \sqrt{(x + L)^2 + y^2 + z^2},
\]

\[
r = \sqrt{x^2 + y^2 + z^2},
\]

\[
x = \frac{1}{2L} (r_0^2 - r^2 - L^2),
\]

\[
y = \pm \frac{1}{2L} \sqrt{4L^2 r^2 - (r_0^2 - r^2 - L^2)^2}.
\]

Equations (2) and (3) give the base to calculate the liberation points (Lagrange points). To change the position of such points, additional acceleration, artificially, must be added to the equations. Let such acceleration be denoted by:

\[
a_L = \left[ a_{xL}, a_{yL}, a_{zL} \right]^T,
\]

with $a_L$ is adopted such as:

\[
a_{xL} = -\left[ n^2 (x + b) - k^2 \left( m_0 \left( \frac{x + a + b}{r_0^3} + m_1 \frac{x}{r} \right) \right) \right],
\]

\[
a_{yL} = -\left[ n^2 y - k^2 \left( m_0 \left( \frac{y}{r_0} + m_1 \frac{y}{r} \right) \right) \right],
\]

\[
a_{zL} = -\left[ k^2 \left( m_0 \frac{z}{r_0^3} + m_1 \frac{z}{r} \right) \right].
\]

3. Lorentz Acceleration

The Lorentz acceleration for charged SC with mass $m$ experienced by a charge $q$ (Coulombs) moving through a planet magnetic field $B$, in planet center coordinate system $\hat{i}, \hat{j}$ and $\hat{k}$, is given by Brett Jordan Streetman [28]:

\[
a_L = \frac{F}{m} = \frac{q}{m} \left( \mathbf{V} - \frac{q}{m} \mathbf{k} \times \mathbf{r} \right) \times \mathbf{B},
\]

where $\hat{i}$ and $\hat{j}$ are unit vectors in the equatorial plane of the planet while $\hat{k}$ in the direction of rotational axis. With $q/m$ is the charge-to-mass ratio of the SC. in Coulombs per
kilogram (C/kg). \( \mathbf{V} \) is the inertial spacecraft velocity while \( \mathbf{\theta} \times \mathbf{r} \) is the velocity of the magnetic field which is rotate with the spin speed \( \mathbf{\theta} \), of the planet and can written as \( \mathbf{\theta} = \mathbf{\theta} \mathbf{k} \).

The general vector model of a dipole magnetic field is [29]:

\[
\mathbf{B} = \frac{B_0}{r^3} [3(\mathbf{n} \cdot \mathbf{r})\mathbf{r} - \mathbf{n}],
\]

where \( \mathbf{n} \) a unit is vector along the north magnetic pole and \( B_0 \) is the strength of the field in Weber-meters. The unit vectors included in Eq. (12) are described in Figure 2.

For a tilted magnetic dipole, the angle \( \alpha \) between \( \mathbf{n} \) and \( \mathbf{k} \), in the Earth case is 11.7° and is independent of time.

If we neglect the tilt angle of the magnetic field (i.e. \( \alpha = 0 \)), the unit vector \( \mathbf{n} \) will be coincidence with the unit vector \( \mathbf{k} \). Then, we can write the last equation in the form:

\[
\mathbf{B} = \frac{B_0}{r^3} [3 \cos \mathbf{bZ} \mathbf{r} - \mathbf{k}].
\]

Noting that from the spherical triangle \( ZbL, \cos \mathbf{b} \mathbf{Z} \) is related to the orbital elements, and equals \( \sin(\nu + \omega) \sin i \) and we will denote it “\( \mathbf{T} \)”. Substituting Eq. (13) into Eq. (8), (9) and (10) to describe Lorentz acceleration as:

\[
\begin{bmatrix}
\alpha_{lx} \\
\alpha_{ly} \\
\alpha_{lz}
\end{bmatrix}
= \frac{q B_0}{m r^3}
\begin{bmatrix}
\mathbf{x}^T \\
\mathbf{y} \\
\mathbf{z}
\end{bmatrix}
- \mathbf{\theta}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\times
\begin{bmatrix}
3 \mathbf{i} \\
0 \\
1
\end{bmatrix}
\]

Completing the required vector operations, we get:

\[
\alpha_{lx} = \frac{Q}{r^3} \left[ \frac{3 \mathbf{i}}{r} (\mathbf{y} \mathbf{z} - \mathbf{y} z - \mathbf{\theta} x \mathbf{z} + \mathbf{\theta} x z) + \mathbf{\theta} x - \mathbf{\theta} y \right],
\]

\[
\alpha_{ly} = \frac{Q}{r^3} \left[ \frac{3 \mathbf{i}}{r} (x \mathbf{z} - x \mathbf{z} - \mathbf{\theta} y \mathbf{z} + \mathbf{\theta} y x) + \mathbf{\theta} y - \mathbf{\theta} x \right],
\]

\[
\alpha_{lz} = \frac{3 \mathbf{i} Q}{r^3} [\mathbf{\theta} x - \mathbf{\theta} y + \mathbf{\theta} (x^2 + y^2)],
\]

where, \( Q = q B_0/m \) and \( \Gamma = \sin(\nu + \omega) \sin i \in \).

4. Controlling Equations

To control the position of liberation points using Lorentz force, we must add the Lorentz acceleration to the equations of motion and applying the stationary conditions.

If we neglect the inclination of the equatorial plane of the planet on its orbit, and choose the \( x \)-axis to be the line joining the Sun with planet, \( y \)-axis normal to it in the planet’s orbital plane and \( z \)-axis normal to the orbital plane, then

![Figure 2: Orbital motion in planet dipole magnetic field.](image)

Eqns. (3), (4) and (15) will be described in the same coordinate system.

Substituting Eqns. (15) into Eqns. (3), (4) the equations of motion are derived in the form:

\[
\ddot{x} - 2n \dot{y} = n^2 (x + b) - k^2 \left[ \frac{m_0 (x + L)}{r_0^3} + m_1 x \right] + \frac{Q}{r^3} \left[ \frac{3 \mathbf{i}}{r} (\mathbf{y} \mathbf{z} - \mathbf{y} z - \mathbf{\theta} x \mathbf{z} + \mathbf{\theta} x z) + \mathbf{\theta} x - \mathbf{\theta} y \right],
\]

\[
\ddot{y} + 2n \ddot{x} = n^2 y - k^2 \left[ \frac{m_0 y}{r_0^3} + m_1 y \right] + \frac{Q}{r^3} \left[ \frac{3 \mathbf{i}}{r} (x \mathbf{z} - x \mathbf{z} - \mathbf{\theta} y \mathbf{z} + \mathbf{\theta} y x) + \mathbf{\theta} y + \mathbf{\theta} x \right],
\]

\[
\ddot{z} = -k^2 \left[ \frac{m_0 z}{r_0^3} + m_1 z \right] + \frac{3 \mathbf{i} Q}{r^3} [\mathbf{\theta} y - \mathbf{\theta} x + \mathbf{\theta} (x^2 + y^2)].
\]

Equations (16), (17) and (18) are the governing equations for the circular restricted three body problem with the magnetic field of the primary \( m_1 \) considered, and the small mass is controlled through a charge \( q \).

5. Dimensionless Form of Controlling Equations

In what follows, we put equations (16), (17) and (18) in the usual dimensionless form of the restricted three body problem so that it is easy to compare the case of artificial points by the addition of the Lorentz force with the natural points when \( q \) is set equal zero.

The equations are divided by \((L \cdot n^2)\) which is clearly the dimensions of acceleration to have it in dimensionless form, we further define the quantities:

\[
t' = n \cdot t \text{ dimensionless time, and } x' = x/L, y' = y/L, z' = z/L \text{ dimensionless distances. Also let } r' = r/L, r_0' = r_0/L, \text{ and finally define } \bar{w} = \theta/n \text{ dimensionless angular velocity.}
\]
\[
\begin{align*}
\frac{d^2x^*}{dt^2} - 2\frac{dy^*}{dt} &= \left( x^* + \frac{bL}{n^2} \right) - k^2 \left[ m_0(x^* + 1) + m_1x^* \right] + \frac{Q}{nL^3r^3} \left[ 3x^* \left( -y^* \right) - \frac{y^*}{n} \right] + \frac{9}{n} \left( -x^* - dy^* \right), \\
\frac{d^2y^*}{dt^2} + 2\frac{dx^*}{dt} &= y^* - \frac{k^2}{n^2} \left[ m_0y^* + m_1y^* \right] + \frac{Q}{nL^3r^3} \left[ 3y^* \left( -x^* \right) - \frac{x^*}{n} \right] + \frac{9}{n} \left( -y^* + dx^* \right), \\
\frac{d^2z^*}{dt^2} &= -\frac{k^2}{n^2} \left[ m_0z^* + m_1z^* \right] + \frac{3lQ}{nL^3r^4} \left[ dx^* - \frac{dy^*}{dt} x^* + \frac{9}{n} \left( x^2 + y^2 \right) \right].
\end{align*}
\]

We note that the quantity \( Q/nL^3 \) is dimensionless since the units of \( Q = qB_0/m \) is \( \text{Cm}^{-1} \) \((\text{MC}^{-1} T^{-1} \text{L}^3) \) which is after simplifying \( T^{-1} \text{L} \) where we use \( C \) for unit of electric charge, \( M \) for unit of mass, \( T \) for time and \( L \) for length. So, we also define the dimensionless quantity \( Q^* = Q/nL^3 \). Next, we substitute \( k^2 = n^2L^2/(m_0 + m_1) \), then introduce the well-known parameter in the three-body problem, \( \mu = m_1/(m_0 + m_1) = a/L \). The equations take the simpler dimensionless form,

\[
\begin{align*}
\dot{x}^* - 2\dot{y}^* &= (x^* + 1 - \mu) - \left[ (1 - \mu) \frac{(x^* + 1)}{r^3} + \mu \frac{x^*}{r^3} \right] + \frac{Q^*}{r^3} \left[ 3x^* - \frac{y^*}{n} \right] + \frac{Q^*}{r^3} \left[ 3y^* - \frac{x^*}{n} \right] + \frac{3lQ^*}{r^3} \left[ x^* - y^* + \frac{x^2 + y^2}{n} \right],
\end{align*}
\]

Finally, we omit the stars for simplicity of writing, keeping in mind that all involved quantities are now dimensionless,

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= (x + 1 - \mu) - \left[ (1 - \mu) \frac{(x + 1)}{r^3} + \mu \frac{x}{r^3} \right] + \frac{Q}{r^3} \left[ 3x - yz - \bar{w}xz \right] + \bar{w}x - \dot{y},
\end{align*}
\]

Equations (21), (22) and (23) are the dimensionless controlling equations of the motion of a small body with an electric charge, under the effect of two big bodies one of them has a significant magnetic field. The equations for planar problem are:

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= (x + 1 - \mu) - \left[ (1 - \mu) \frac{(x + 1)}{r_0^3} + \mu \frac{x}{r_0^3} \right] + \frac{Q}{r_0^3} \left[ \bar{w}x - \dot{y} \right],
\end{align*}
\]

\[
\begin{align*}
\ddot{y} + 2\dot{x} &= y - \left[ (1 - \mu) \frac{y}{r_0^3} + \mu \frac{y}{r_0^3} \right] + \frac{Q}{r_0^3} \left[ xz - \bar{w}yz \right] + \bar{w}y + \dot{x},
\end{align*}
\]

6. Artificial Equilibrium Points

The liberation points are found by letting \( \ddot{x} = \dot{x} = \ddot{y} = \dot{y} = 0 \). We also take \( L \) as the unit of distance = 1, and \( 1/n \) the unit of time, hence \( n = 1 \), thus Eqns (24) and (25) will give,

\[
\begin{align*}
(1 - \mu + x)r^3 - (1 - \mu)(1 + x) \frac{r^3}{r_0^3} - \mu x + Q\bar{w}x = 0, \\
yr^3 - (1 - \mu)\frac{r^3}{r_0^3} - \mu y + Q\bar{w}y = 0.
\end{align*}
\]
Equations (26) and (27) represent the controlling equations depending on the position of the space craft’s liberation points and the charge (per unit mass) required to generate Lorentz acceleration.

From Eq. (27), we get:

\[ y = 0, \]  
\[ \text{Or } r^3 - (1 - \mu) \frac{r^3}{r_0^3} - \mu + Q\tilde{\omega} = 0, \]  
\[ y = 0, \]  

is the solution of collinear liberation points while \( r^3 - (1 - \mu) \left( \frac{r^3}{r_0^3} \right) - \mu + Q\tilde{\omega} = 0 \). This leads to the equilateral triangle solution.

In the equilateral triangle solution, solving for \( r \), we get:

\[ r = r_0 \sqrt{\frac{\mu - Q\tilde{\omega}}{\mu + \mu - 1}}. \]  
\[ \text{(29)} \]

In other side, solving Eq. (28) for \( Q \) we get:

\[ Q = \frac{1}{\tilde{\omega}} \left[ \mu - r^3 + (1 - \mu) \left( \frac{r}{r_0} \right)^3 \right]. \]  
\[ \text{(30)} \]

We can express another useful relation since \( Q \) must satisfy both Eqns. (26) and (27), then:

\[ (1 - \mu + x) \frac{r^3}{x} - (1 - \mu) (1 + x) \frac{r^3}{x r_0} = \mu \]
\[ = r^3 - (1 - \mu) \frac{r^3}{r_0^3} - \mu \implies r_0 = 1. \]  
\[ \text{(31)} \]

Simplifying yields,

\[ (1 - \mu) \frac{r^3}{x} \left[ 1 - \frac{1}{r_0^3} \right] = 0 \implies r_0 = 1. \]  
\[ \text{(32)} \]

Which leads to conclude that the artificial liberation points due to Lorentz force generated from the small primary exist on a circular arc with center at the big primary with radius the unit distance, with angles \( \pm \left( \frac{\pi}{3} \right) \pm \epsilon \) where \( \epsilon \) is the small deviation from the original point of equilateral triangular case, and the positive sign is for the small deviation from the original point of equilateral triangular case, and the positive sign is for \( L_4 \) while the negative sign is for \( L_5 \), as shown in Figure 3.

The charge per unit mass \( q \), on the SC can be expressed from Eq. (30), as:

\[ \frac{\theta}{B_0\tilde{\omega}} \left[ \mu - r^3 + (1 - \mu) \left( \frac{r}{r_0} \right)^3 \right]. \]  
\[ \text{(33)} \]

Applying the constraint (32), with \( r_0 = 1 \), leads to:

\[ \frac{\mu L^3}{B_0\tilde{\omega}} (1 - r^3), \]
\[ r = \sqrt[3]{1 - \frac{Q\tilde{\omega}}{\mu}}. \]  
\[ \text{(34)} \]

When the body is not charged \((Q = 0)\), we get the classical case of the Lagrangian triangular points \( L_4 \) and \( L_5 \) with \( r = 1 \).

7. Triangular Liberation Points Coordinates

We proved in the previous section that the liberation points exist. In this section we will calculate their position as a function of the charge, of the SC, per unit mass.

Using Eqns. (5) and (6), with \( r_0 = 1, L = 1, a = \mu, b = 1 - \mu \),

\[ x = \frac{-r^2}{2} = \frac{1}{2} \left( 1 - \frac{Q\tilde{\omega}}{\mu} \right)^{2/3}, \]  
\[ \text{(35)} \]

\[ y = \pm \frac{1}{2} \sqrt{4r^2 - r^4} = \pm \frac{1}{2} \sqrt{4 \left( 1 - \frac{Q\tilde{\omega}}{\mu} \right)^{2/3} - \left( 1 - \frac{Q\tilde{\omega}}{\mu} \right)^{4/3}}. \]  
\[ \text{(36)} \]

Equations (35) and (36) give the coordinates of the artificial triangular points in terms of the parameter \( \mu \), the magnetic field strength \( B_0 \) of the primary at \( m_1 \), the dimensionless angular velocity \( \tilde{\omega} \) of the primary \( m_1 \), and the charge per unit mass \( q \).

8. Stability of Artificial Lorentz Triangular Liberation Points

Recalling the equations of motion in planar case:

\[ \ddot{x} - 2\dot{y} = 1 - \mu + \left[ (1 - \mu) \frac{(x + 1) r}{r_0^3} + \mu \frac{x}{r^3} \right] + \frac{Q}{r^3} \left[ \tilde{\omega} y - \dot{y} \right], \]
\[ \ddot{y} + 2\dot{x} = y - \left[ (1 - \mu) \frac{y}{r_0^3} + \mu \frac{y}{r^3} \right] + \frac{Q}{r^3} \left[ \tilde{\omega} x - \dot{x} \right]. \]  
\[ \text{(37)} \]

These equations can be written as:

\[ \ddot{x} - 2\dot{y} = \frac{\tilde{\omega}}{\dot{y}} - \frac{\mu}{r^3} \dot{y}, \]
\[ \ddot{y} + 2\dot{x} = \frac{\tilde{\omega}}{\dot{y}} + \frac{\mu}{r^3} \dot{x}, \]  
\[ \text{(38)} \]
where,
\[
\Omega = \Omega - \frac{Q\omega}{r},
\]
\[
\Omega = \frac{1}{r^2} \left( \left( x + 1 - \mu \right)^2 + y^2 \right) + \left( 1 - \mu \right) \frac{r_0 + \mu}{r}
\]
(39)

Then,
\[
\ddot{x} - 2\dot{y} + \frac{Q}{r^2} \dot{y} = \frac{\partial \Omega}{\partial x},
\]
\[
\ddot{y} + 2\dot{x} - \frac{Q}{r^2} \dot{x} = \frac{\partial \Omega}{\partial y}
\]
(40)

Now, we linearize the equations about the equilibrium points \((x_0, y_0)\) by letting,
\[
x = x_0 + \varepsilon \xi,
\]
\[
y = y_0 + \varepsilon \eta,
\]
(41)

where, \(\varepsilon \ll 1\).

Then, we expand the equations keeping only first order terms. This gives,
\[
\ddot{\xi} + (c - 2)\dot{\eta} = a_{11}\xi + a_{12}\eta,
\]
\[
\ddot{\eta} + (c - 2)\dot{\xi} = a_{12}\xi + a_{22}\eta,
\]
(42)

where,
\[
a_{11} = \frac{\partial^2 \Omega}{\partial x^2}(x_0, y_0),
\]
\[
a_{12} = \frac{\partial^2 \Omega}{\partial x \partial y}(x_0, y_0),
\]
\[
a_{22} = \frac{\partial^2 \Omega}{\partial y^2}(x_0, y_0),
\]
\[
c = \frac{Q}{r^3}(x_0, y_0)
\]
(43)

The characteristic equation for the system (40) is then,
\[
\lambda^4 + \lambda^2 \left( 4c - 4c^2 - a_{11} + a_{22} \right) + a_{11}a_{22} - a_{12}^2 = 0
\]
(44)

This can be written as:
\[
\lambda^4 + F\lambda^2 + H = 0,
\]
(45)

where,
\[
F = \left( 4 - 4c + c^2 - a_{11} - a_{22} \right),
\]
\[
H = a_{11}a_{22} - a_{12}^2.
\]
(46)

The condition for \(\lambda\) to be imaginary is that \(\lambda^2\) is a real negative value. This is guaranteed by the satisfaction of the conditions:

\[
F^2 - 4H > 0,
\]
(47)

\[
F > 0,
\]
(48)

\[
H > 0.
\]
(49)

Equations (47), (48) and (49) are the necessary conditions for the artificial liberation points to be stable.
9. Numerical Investigation for the Case of the Sun-Jupiter System

The magnetic field of Jupiter has the values $B_0 = 1.5812 \times 10^{20}$ Tesla meter$^{-2}$. The parameter $\mu = 0.000954$, the mean motion $n = 0.0006042$ hour$^{-1}$, the sun-Jupiter distance $L = 779 \times 10^{6}$ km, and the angular velocity $\dot{\omega} = 0.63301$ hour$^{-1}$ [30]. The dimensionless quantity $Q$ will thus have the value $\eta B_0 / n L^2 = 5.80366 \times 10^{-3} \eta$. Also, in this case $\dot{\omega} = \theta / n \approx 1047.829$. If the distance from the original triangular liberation points to the small primary (Jupiter in this case) be $r = 1$, the $x$ and $y$ coordinates $X_{4,5} = -0.5$ and $Y_{4,5} = \pm (\sqrt{3}/2)$. These values will differ by a small amount due to the existence of the Lorentz force. We plot the deviations of $r$, $X_{4,5}$, and $Y_{4,5}$ denoted by $\delta r$, $\delta X_4$ and $\delta Y_4$ respectively against the charge per unit mass $\eta$ in Figure 4–Figure 6, where we note that $X_4 = X_5$ and $Y_4 = -Y_5$, so we plot only the graphs for $L_4$. The results show a variation of an order from $10^{-2}$ to $10^{-6}$ for the dimensionless $x$, $y$ coordinates and distance $r$ from Jupiter. This is equivalent to about hundreds of kilometers which is considerable. The range of $\eta$ is from $-0.1$ to $0.1$ C/kg.

Concerning the stability, the graphs of $F$, $H$, and $F^2 - 4H$ are plotted in the same range of the specific charge $-0.1$ to $0.1$ to check conditions (47), (48), and (49). The graphs show values greater than zero in the whole range which guarantees the stability of the artificial triangular points as the case of the natural points. The results are given in Figure 7–Figure 9.

Finally, the Eigen values of equation (44) for different values of $\eta$ are given in Table 1.

10. Conclusion

The artificial triangular liberation points $L_{4,5}$ in the circular restricted three body problem in the case when the small primary has a significant magnetic field are studied. In this work the control for the motion is taken the charge per unit mass of the charged space craft under study. The origin of the coordinates was taken at the small primary which has the center of the magnetic field. A numerical application is taken for the case of Sun-Jupiter system. The choice of this system is because of the small eccentricity of Jupiter’s orbit around the sun and the weak effects of other perturbing bodies on the motion, and most important for this study is because of the strong magnetic field of Jupiter.

The results show the ability of varying the triangular liberation points by an order from $10^{-7}$ to $10^{-6}$ for the dimensionless $x$, $y$ coordinates and distance $r$ from Jupiter. This is equivalent to about hundreds of kilometers which is considerable. The artificial points lie on a circular arc with center at the big primary with radius the unit distance, with angles $\pm (\pi/3 \pm \epsilon)$ where $\epsilon$ is the small deviation from the
original point of equilateral triangular case, and the positive
sign is for $L_4$ while the negative sign is for $L_5$. It should be
mentioned that the results are made for a range of charge per
unit mass up to order 0.1 C/kg, where the existing values till
now for the specific charge are 0.001–0.01 C/kg. However, it is
predicted to overcome the technical problems of this issue to
reach values greater than 0.1 C/kg in the future [31] (Peck et al.).
The results also show the stability of the Lorentz artificial
$L_{4,5}$ in the range chosen for the specific charge, as the
original points for Sun-Jupiter system.

With the improvement of the studies with artificial
liberation points, future studies may generalize the work
either to stronger kinds of control or to work with the 5 body
problem or even the n-body problem [32, 33].

Abbreviations

| SC: Space craft | TBP: Three body problem |
|----------------|------------------------|
| RTBP: Restricted three body problem |
| CRTBP: Circular restricted three body problem |
| AEP: Artificial equilibrium points |
| LA: Lorentz acceleration |
| LF: Lorentz force |
| $(X, Y, Z)$: The center of mass coordinates system (synodic coordinate system) |
| $(x, y, z)$: The planet centered coordinates system |
| $x^*, y^*, z^*$ and $r^*$: Dimensionless distances |
| $\tilde{n}$: Unit vector along the north magnetic pole of the planet |
| $k$: Unit in the direction of rotational axis of the planet |
| $a_L$: Lorentz acceleration vector |
| $\bar{q}$: Charge-to-mass ratio of the SC. in coulombs per kilogram (C/kg) |
| $B$: Planet magnetic field vector |
| $B_0$: The strength of the field in weber-meters |
| $m_0, m_1$: The masses of the primaries |
| $m$: Mass of space craft (SC) |
| $V$: The rotational velocity vector of the planet |
| $\delta$: The small deviation from the original point |
| $N$: Mean motion of the primaries around their center of mass |
| $\bar{\omega}$: Dimensionless angular velocity of the smaller primary. |

Data Availability

Data available on request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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