Renormalization and convergence in law for the derivative of intersection local time in $\mathbb{R}^2$

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Abstract
In this paper we will examine the derivative of intersection local time of Brownian motion and symmetric stable processes in $\mathbb{R}^2$. These processes do not exist when defined in the canonical way. The purpose of this paper is to exhibit the correct rate for renormalization of these processes.

1 Introduction

Let $B_t$ be a Brownian motion in $\mathbb{R}^d$, and let

\begin{equation}
\alpha_\varepsilon(T) = \int_0^T \int_0^t f_\varepsilon(B_s - B_u)dsdt
\end{equation}

where $f_\varepsilon$ denotes the Gaussian density function on $\mathbb{R}^d$ with variance $\varepsilon$. If $\alpha_\varepsilon(T)$ converges to a process as $\varepsilon \to 0$ we denote this process $\alpha_T$ and call it the intersection local time (henceforth abbreviated as ILT).

In one dimension the ILT does exist, as can be seen easily by using the occupation times formula (see, e.g. [8]). In dimension 2 it does not exist as defined above, as $\alpha_\varepsilon(T)$ blows up as $\varepsilon \to 0$ due to the set \{s = u\}. In [9], however, Varadhan showed that $\alpha_\varepsilon(T) - E[\alpha_\varepsilon(T)]$ does converge in law to a process, which is referred to as the renormalized intersection local time. This process was originally considered due to its relevance to quantum field theory (see [9]), but it has found several other uses, for example in Edwards’ work on polymers (see [2]) and Le Gall’s work on Wiener sausages (see [4], [5]).
In [10], Yor proved that in $d = 3$
\begin{equation}
\left\{ \frac{1}{\sqrt{\log(1/\varepsilon)}} \left( \alpha_{\varepsilon}(T) - E[\alpha_{\varepsilon}(T)] \right), T \geq 0 \right\}
\end{equation}
converges in law as $\varepsilon \rightarrow 0$ to the process $\{\frac{1}{\sqrt{2\pi}}B_T, T \geq 0\}$, where $B_T$ is a one dimensional Brownian motion. This theorem inspired a similar result from Rosen in [7] involving symmetric stable processes. Rosen considered the process
\begin{equation}
\alpha_{\varepsilon}(T) = \int_0^T \int_0^t f_{\varepsilon}(X_t - X_s)dsdt
\end{equation}
where now $X$ is a symmetric stable process of index $\beta$, and $f_t$ denotes the density of $X_t$. Rosen proved that if $4/3 < \beta \leq 2$, then $\alpha_{\varepsilon}(T) - E[\alpha_{\varepsilon}(T)]$ converges pathwise as $\varepsilon \rightarrow 0$ to a finite random variable. If $\beta = 4/3$, then
\begin{equation}
\left\{ \frac{1}{\sqrt{\log(1/\varepsilon)}} \left( \alpha_{\varepsilon}(T) - E[\alpha_{\varepsilon}(T)] \right), T \geq 0 \right\}
\end{equation}
converges in law as $\varepsilon \rightarrow 0$ to $\{k(\beta)B_T, T \geq 0\}$ where $k(\beta)$ is a constant which depends on $\beta$. Similarly, if $1 < \beta < 4/3$, then
\begin{equation}
\left\{ \varepsilon^{2/\beta - 3/2} \left( \alpha_{\varepsilon}(T) - E[\alpha_{\varepsilon}(T)] \right), T \geq 0 \right\}
\end{equation}
converges in law as $\varepsilon \rightarrow 0$ to $\{k(\beta)B_T, T \geq 0\}$ where $k(\beta)$ is a constant depending on $\beta$. When $\beta = 2$, $X$ is Brownian motion, and this gives a different proof of Varahdan’s renormalization. The method employed by Rosen in proving this also gives an alternate proof of Yor’s result in 3 dimensions.

In [8] Rosen introduced the notion of the derivative of the intersection local time of Brownian motion in $\mathbb{R}^1$. It is defined as
\begin{equation}
\alpha'(T) = \lim_{\varepsilon \rightarrow 0} \int_0^T \int_0^t f_{\varepsilon}'(B_t - B_s)dsdt
\end{equation}
provided the limit exists. Formally, we can write
\begin{equation}
\alpha'(T) = \int_0^T \int_0^t \delta'(B_t - B_s)dsdt
\end{equation}
Rosen was able to show that this integral converges as $\varepsilon \rightarrow 0$, and proved an occupation time formula, as well as some other facts about $\alpha'_T$. This paper
deals with the derivative of ILT in 2 dimensions. In two dimensions, we will use $f'_\varepsilon$ to denote $\frac{\delta}{\delta x} f_\varepsilon$. We let

$$\alpha'_\varepsilon(T) = \int_0^T \int_0^t f'_\varepsilon(B_t - B_s)dsdt$$

(1.8)

Our main result is that $\alpha'_\varepsilon(T)$ does not converge as $\varepsilon \rightarrow 0$. We will prove that the asymptotic behavior as $\varepsilon \rightarrow 0$ is very similar to that which occurs for the ILT in 3 dimensions as discovered by Yor. In particular, our main theorem is

**Theorem 1** \(\left\{ (\log(1/\varepsilon))^{-1} \alpha'_\varepsilon(T), T \geq 0 \right\} \) converges in law to \(\left\{ \frac{\sqrt{5}}{\pi \sqrt{2}} W_T, T \geq 0 \right\} \) as $\varepsilon \rightarrow 0$, where $W_T$ is a one-dimensional Brownian motion.

Remark: $f'_\varepsilon$ is an odd function, so $E[\alpha'_\varepsilon(T)] = 0$, which is why we need not subtract the expectation to obtain convergence, as was required in the theorems of Yor, Varadhan, and Rosen.

We also will prove an analogous theorem about symmetric stable processes. We let $X_t$ be a symmetric stable process of index $\beta$ with $1 < \beta < 2$, let $f_t$ be the density of $X_t$, and let $f'_t = \frac{\delta}{\delta x} f_t$. Again we will consider

$$\alpha'_\varepsilon(T) = \int_0^T \int_0^t f'_\varepsilon(X_t - X_s)dsdt$$

(1.9)

and we will prove the following:

**Theorem 2** \(\left\{ \varepsilon^{3/\beta - 3/2} \alpha'_\varepsilon(T), T \geq 0 \right\} \) converges in law to \(\left\{ c(\beta) W_T, T \geq 0 \right\} \) as $\varepsilon \rightarrow 0$ where $W_T$ is a one-dimensional Brownian motion and $c(\beta)$ is given by

$$\frac{1}{2\sqrt{2}\pi^2} \left( \int \int \frac{1}{p^3} \frac{1}{q^3} \frac{1}{(p+q)^3} e^{-(p^\beta + q^\beta)} p_1 q_1 dpdq + \int \int \frac{1}{p^{3\beta}} \frac{1}{(p+q)^3} e^{-(p^\beta + q^\beta)} p_1 q_1 dpdq \right)$$

(1.10)

Included in the proof is the definition of the second integral in the definition of $c(\beta)$. This integral does not converge absolutely for $\beta \geq 3/2$, so we must clarify what it means.

The paper is organized as follows. Section 2 gives the outline of the proof. Sections 3 and 4 prove the bulk of the required technical details. Section 5 wraps up the proof of Theorem \(\square\). Section 6 deals with with the symmetric stable case, and gives the proof of Theorem \(\square\).
2 Outline of the proof

The outline of the proof of Theorem 1 follows closely the proofs of Theorems 1, 2, and 3 given in [7], though the details, given in sections 3 and 4 below, are quite different. The reader may refer to that paper to see a slightly different presentation of the ideas of this section. We will show first that the moments of \( \alpha'(T)(\log(1/\varepsilon))^{-1} \) converge to the moments of a Brownian motion times \( \sqrt{\frac{5}{\pi \sqrt{128\sqrt{2}}} \log(1/\varepsilon)} \). Recall that

\[
(2.1) \quad f_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-\frac{|x|^2}{2\varepsilon}}
\]

We will express this in a form which is easier for us to use:

\[
(2.2) \quad f_\varepsilon(x) = \frac{1}{(2\pi)^2} \int e^{ipx-\varepsilon p^2} d^2 p
\]

This gives

\[
(2.3) \quad f'_\varepsilon(x) = \frac{i}{(2\pi)^2} \int p_1 e^{ipx-\varepsilon p^2} d^2 p
\]

We have

\[
E(\alpha'(T)^n) = \frac{i^n}{(2\pi)^{2n}} \int_{(\mathbb{R}^2)^n} \int_{D_T^n} e^{-\varepsilon \sum_j p_j^2} \prod_{j=1}^n p_j E[ \prod_{j=1}^n e^{ip_j(x_{t_j} - x_{t_j})}] \prod_{j=1}^n ds_j dt_j d^2 p_j
\]

(2.4)

This is obtained by combining \( n \) copies of the integral (1.8) which defines \( \alpha'(T) \), using the definition (2.2) of \( f_\varepsilon \). Now, if \( n \) is odd, then the integrand is an odd function of \( p \), and the expectation is therefore 0. Since all odd moments of Brownian motion are 0, we need only show that the even moments converge to the right values. The \( 2n \)-th moment of Brownian motion at time \( T \) is \( \frac{(2n)!}{2^{2n}n!} T^n \), so we will show that

\[
(2.5) \quad E[\alpha'_\varepsilon(T)^{2n}] = \frac{(2n)!}{2^{2n}n!} \left( \frac{\sqrt{5}}{\pi \sqrt{128\sqrt{2}}} (\log(1/\varepsilon)) \right)^{2n} T^n + o(\log(1/\varepsilon))^{2n}
\]

This expectation is given by (2.4) with \( n \) replaced by \( 2n \). The expectation on the right side of (2.4) will depend on the ordering of the \( s_j \)'s and \( t_j \)'s in
$D_{T}^{2n}$. By independence, the expectation will factor into the product of several expectations, each corresponding to a component of the set $\bigcup_{j} [s_j, t_j]$. Each of these resulting expectations are then integrated separately. Following [7], we will say that a component consisting of $m$ intervals $[s_j, t_j]$ is of order $m$. As in [7], it turns out that the dominant contribution comes from regions consisting of $n$ components of order 2. Suppose, for the time being, that we are integrating over only such regions. We will hold fixed the initial points of the components $r_1 < ... < r_n$. In this case we will show (Section 3) that each of the $n$ components contribute

\begin{equation}
(\frac{5}{\pi^2 128 \sqrt{2}} (\log(1/\varepsilon))^2 + o(\log(1/\varepsilon))^2)
\end{equation}

The contribution of each configuration with $n$ components of order 2 to (2.4) is therefore

\begin{equation}
\left(\frac{5}{\pi^2 128 \sqrt{2}} (\log(1/\varepsilon))^2 + o(\log(1/\varepsilon))^2\right)^n \int_{0 \leq r_1 \leq ... \leq r_n \leq T} dr_1...dr_n
\end{equation}

$$= \frac{T^n}{2^n n!} \left(\frac{\sqrt{5} \sqrt{2}}{\pi \sqrt{128 \sqrt{2}}} (\log(1/\varepsilon))^2 + o(\log(1/\varepsilon))^2\right)^n$$

$$= \frac{T^n}{2^n n!} \left(\frac{\sqrt{5}}{\pi^8 \sqrt{2}} (\log(1/\varepsilon))^2 + o(\log(1/\varepsilon))^2\right)^n$$

where we have used the identity

$$\int_{0 \leq r_1 \leq ... \leq r_n \leq T} dr_1...dr_n = T^n / n!$$

There are $(2n)!$ different ways to order the set $\{s_1,...,s_{2n}\}$, and each contributes (2.7), so the total is

\begin{equation}
\frac{(2n)!}{2^n n!} \left(\frac{\sqrt{5}}{\pi^8 \sqrt{2}} (\log(1/\varepsilon))^2\right)^n T^n + o(\log(1/\varepsilon))^2n
\end{equation}

which is, in fact, what we were aiming for. We must therefore show that the contribution from all regions which have other than $n$ components of order 2 is $o(\log(1/\varepsilon))^{2n}$. We will do this in Section 4 by showing that each component of order $m$ with $m \geq 3$ contributes $o(\log(1/\varepsilon))^m$. Note that any component of order 1, or indeed of any odd order, will in fact contribute 0,
since the integrand is an odd function. It will follow from all of this that the moments of \((\log(1/\varepsilon))^{-1} x'(T)\) converge to the right values. We will prove that the processes \((\log(1/\varepsilon))^{-1} x'(T)\) are tight in Section 5, so that there is a limiting process which has all of the same moments as \(\frac{\sqrt{5}}{\pi \sqrt{128\sqrt{2}}} W_t\). In addition, this process has independent increments, as will also be shown in Section 5. These facts, taken together, will prove Theorem 1.

3 Components of order 2

Here we will deal with the aforementioned components of order 2. We’ll begin by proving

**Proposition 1**

\[
E[x'(T)^2] = T \log(1/\varepsilon)^2 \left( \frac{10}{128\sqrt{2}\pi} + o(1) \right)
\]

To begin the proof, we write

\[
E[x'(T)^2] = -\frac{1}{(2\pi)^4} \int \int \int \int \frac{e^{-\varepsilon(p^2+q^2)} E[e^{ip(X_{t_1}-X_{s_1})+iq(X_{t_2}-X_{s_2})}]}{p_1q_1s_1d_1t_1d_1tdpdq}
\]

where \(D_T\) denotes the triangle \(\{(s_i, t_i) | 0 \leq s_i \leq t_i \leq T\}\). For simplicity, we’ll assume for the time being that \(T = 1\). To handle the expectation in the integrand, we must consider different orderings of the \(s_i\)'s and \(t_i\)'s. By symmetry, we may assume \(s_1 < s_2\). We’ll suppress the \(\frac{1}{(2\pi)^4}\) in front of the integral for the time being.

**Case 1: \(s_1 < s_2 < t_2 < t_1\)**

We rewrite the exponent in the expectation as \(ip(X_{t_1}-X_{t_2})+i(p+q)(X_{t_2}-X_{s_2})+ip(X_{s_2}-X_{s_1})\), and then factor the expectation using independence. As a result, the expectation becomes

\[
e^{-p^2(a+c)-(p+q)^2b}
\]

where \(a = t_1 - t_2\), \(b = t_2 - s_2\), and \(c = s_2 - s_1\). Upon making this linear transformation, the integral in question becomes
(3.4) $\int_0^T \left[ \int \int \int_{a+b+c \leq t_1} dadbdc \int \int e^{-p^2(a+c+\varepsilon)}e^{-(p+q)^2b}e^{-q^2\varepsilon}p_1q_1dpdq \right] dt_1$

**Case 2:** $s_1 < s_2 < t_1 < t_2$

We rewrite the exponent in the expectation as $iq(X_t - X_{t_1}) + i(p+q)(X_{t_1} - X_{s_2}) + ip(X_{s_2} - X_{s_1})$, and proceed as in Case 1. The integral in question here is then

(3.5) $\int_0^T \left[ \int \int \int_{a+b+c \leq t_1} dadbdc \int \int e^{-p^2(c+\varepsilon)}e^{-(p+q)^2b}e^{-q^2(a+\varepsilon)}p_1q_1dpdq \right] dt_1$

where now $a = t_2 - t_1$, $b = t_1 - s_2$, and $c = s_2 - s_1$.

**Case 3:** $s_1 < t_1 < s_2 < t_2$

Here the expectation factors, and since the integrand of (3.2) is then an odd function of $p$, the contribution to (3.2) of this case is 0.

Remark: Before we begin, let us clear up a technical point that will be necessary later. To do the computations, we will in fact integrate $da$, $db$, and $dc$ first. However, if we were to begin with $dp$ and $dq$, so that the integrals in cases 1 and 2 were of the form

(3.6) $\frac{-1}{(2\pi)^4} \int \int \int h(a, b, c, t)dadbdc dt$

then $h$ would be everywhere negative. Intuitively, this is because when $p_1$ and $q_1$ are the same sign, $|p + q|$ is larger than when they are opposite signs. In order to rigorously prove this, just note that the map $\phi((p_1, p_2), ((q_1, q_2)) = ((p_1, p_2), (-q_1, q_2))$ is a linear isometry which maps $U = \{p_1q_1 > 0\}$ bijectively onto $V = \{p_1q_1 < 0\}$, and the function $|p + q|$ is greater at $(p, q) \in U$ than at $\phi(p, q) \in V$. It will be pointed out later where we have used this fact.

We’ll attack case 1 first. In order to compute the required integral, we’ll first examine the following:

(3.7) $\int \int \int_{0 \leq a, b, c \leq 1} dadbdc \int \int e^{-p^2(a+c+\varepsilon)}e^{-(p+q)^2b}e^{-q^2\varepsilon}p_1q_1dpdq$

Upon integrating $da$, $db$, and $dc$, we obtain
\[
(3.8) \quad \int \int \frac{(1 - e^{-p^2})^2}{p^4} \frac{(1 - e^{-(p+q)^2}/\epsilon)}{(p+q)^2} e^{-\epsilon(p^2+q^2)} p_1 q_1 dp dq
= \int \int \frac{(1 - e^{-p^2/\epsilon})^2}{p^4} \frac{(1 - e^{-(p+q)^2}/\epsilon)}{(p+q)^2} e^{-(p^2+q^2)} p_1 q_1 dp dq
\]

Upon converting to polar coordinates, with \( p = r e^{i\theta}, \ q = s e^{i\phi}, \) we arrive at

\[
(3.9) \quad \int \int \int \frac{(1 - e^{-r^2/\epsilon})^2}{r^2} s^2 \cos(\phi) e^{-(r^2+s^2)} \\
\int_0^{2\pi} \left(1 - e^{-|re^{i(\theta+\phi)} + se^{i\phi}|^2/\epsilon}\right) \cos(\theta) d\theta d\phi dr ds
\]

We’ll isolate the \( d\theta \) integral. We may replace \( \theta \) with \( \theta + \phi \). Note that

\[
(3.10) \quad \frac{(1 - e^{-|re^{i(\theta+\phi)} + se^{i\phi}|^2/\epsilon}}{|re^{i(\theta+\phi)} + se^{i\phi}|^2} = \int_0^{1/\epsilon} e^{-|re^{i(\theta+\phi)} + se^{i\phi}|^2 x} dx
\]

so the \( d\theta \) integral is

\[
(3.11) \quad \int_0^{1/\epsilon} \int_0^{2\pi} e^{-|re^{i(\theta+\phi)} + se^{i\phi}|^2 x} \cos(\theta + \phi) d\theta dx
= \int_0^{1/\epsilon} \int_0^{2\pi} e^{-|re^{i\theta} + s|^{2 x}} [\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)] d\theta dx
\]

We expand this and pull the \( \cos(\phi) \) and \( \sin(\phi) \) terms outside of the \( d\theta \) integral. Integrating the \( \sin(\phi) \) term with the \( \cos(\phi) \) term already present outside the \( d\theta \) integral gives 0. Integrating the \( \cos(\phi) \) together with the other \( \cos(\phi) \) term already present gives \( \pi/2 \). We have therefore eliminated \( \phi \) from \( (3.9) \). The contribution of \( (3.10) \) to \( (3.9) \) is therefore equal to

\[
(3.12) \quad \frac{\pi}{2} \int_0^{1/\epsilon} e^{-(r^2+s^2)x} \left( \int_0^{2\pi} e^{-2rsx\cos(\theta)} \cos(\theta) d\theta \right) dx
\]

By \[3\] (p. 958, 8.431.5 with \( \nu = 1 \), replace \( \theta \) with \( \theta + \pi \)), what remains of the \( d\theta \) integral is equal to \(-2\pi I_1(2rsx)\), where \( I_1 \) denotes the modified Bessel function of the first kind. Thus, \( (3.9) \) is equal to
\[ (3.13) \quad -\pi^2 \int_0^\infty \int_0^\infty \int_0^{1/\varepsilon} \int_0^{1/\varepsilon} e^{-r^2/\varepsilon} s^2 e^{-(r^2+s^2)(x+1)} I_1(2rsx) dxdydrds \]
\[ = -\pi^2 \int_0^\infty \int_0^\infty \int_0^{1/\varepsilon} \int_0^{1/\varepsilon} e^{-r^2/\varepsilon} s^2 e^{-(r^2+s^2)(x+1)} I_1(2rsx) dydxdrds \]

Expand this into two integrals, and rewrite the first one as

\[ (3.14) \quad -\pi^2 \int_0^\infty \int_0^{1/\varepsilon} \int_0^{1/\varepsilon} e^{-s^2(x+1)} s^2 \int_0^\infty e^{-r^2(y+x+1)} I_1(2rsx) dxdyds \]

By \[3\] (p. 711, 6.618.4) the \(dr\) integral is equal to

\[ (3.15) \quad \frac{\sqrt{\pi}}{2\sqrt{x+y+1}} e^{(2sx)^2/(x+y+1)} I_{1/2}(8(x+y+1)) \]

This is \(\frac{(e^{x+y+1}-1)}{\sqrt{2sx}}\), since \(I_{1/2}(z) = \frac{1}{\sqrt{2\pi z}} (e^z - e^{-z})\) by \[3\] (p.967 8.467). Thus, \(3.13\) is

\[ (3.16) \quad -\pi^2 \int_0^{1/\varepsilon} \int_0^{1/\varepsilon} \int_0^\infty e^{-s^2(x+1)} s \left( \frac{e^{x+y+1}}{x} - 1 \right) dxdyds \]
\[ = -\pi^2 \int_0^{1/\varepsilon} \int_0^{1/\varepsilon} \frac{1}{x} \int_0^\infty s \left( e^{-s^2(x+1)} - e^{-s^2(x+1)} \right) dsdydx \]

Since

\[ (3.17) \quad \int_0^\infty se^{-bs^2} ds = (1/2)b^{-1} \]

we see that \(3.14\) is

\[ (3.18) \quad -\pi^2 \int_0^M \int_0^M \frac{1}{x} \left( x + 1 - \frac{x^2}{x+y+1} \right)^{-1} - (x + 1)^{-1} dxdy \]
\[ = -\pi^2 \int_0^M \int_0^M \frac{x}{(x+1)(xy+2x+y+1)} dxdy \]

where we have substituted \(M = 1/\varepsilon\) to simplify what follows. This last integral is explicitly computable, but it is easier to calculate the derivative and then apply L’Hospital’s rule. To do so, we’ll make use of the following lemma:
Lemma 1 If \( h(x, y, M) \) is bounded, continuous in \( x \) and \( y \) on \( \{ x, y \geq 0 \} \), differentiable in \( M \) with bounded derivative, then

\[
\frac{d}{dM} \int_0^M \int_0^M h(x, y, M) \, dx \, dy = \int_0^M h(M, y, M) \, dy
\]
\[
+ \int_0^M h(x, M, M) \, dx + \int_0^M \int_0^M \frac{d}{dM} h(x, y, M) \, dx \, dy
\]

In the case of (3.18) the integrand does not depend on \( M \), so the last term is 0. We include the last term because it will be used later. By the lemma, the derivative with respect to \( M \) of (3.18) is

\[
\frac{-\pi^2}{2\sqrt{2}} \left[ \frac{M}{M+1} \int_0^M \frac{dy}{(M+1)y+2M+1} 
+ \int_0^M \frac{x}{(x+1)((M+2)x+M+1)} \, dx \right]
\]
\[
= \frac{-\pi^2}{2\sqrt{2}} \left[ \frac{M}{M+1} \int_0^M \frac{dy}{(M+1)y+2M+1}
+ \int_0^M \frac{1}{(x+1) - \frac{M+1}{(M+2)x+M+1}} \, dx \right]
\]

Performing the integration gives

\[
\frac{-\pi^2}{2\sqrt{2}} \left[ \frac{M}{(M+1)^2} \log \left( \frac{(M+1)M+2M+1}{2M+1} \right) + \log(M+1)
\right.
\]
\[
- \frac{M+1}{(M+2)} \log \left( \frac{(M+2)M+M+1}{M+1} \right)
\]

The expression inside the brackets is asymptotic to \( \frac{2\log M}{M} \). This is an immediate consequence of the following easily verified facts:

1) \( \frac{M}{(M+1)^2} = \frac{1}{M} + O\left(\frac{1}{M^2}\right) \)
2) \( \log \left( \frac{(M+1)M+2M+1}{2M+1} \right) = \log M + O(1) \)
3) \( \log(M+1) = \log(M) + O\left(\frac{1}{M}\right) \)
4) \( \log \left( \frac{(M+2)M+M+1}{M+1} \right) = \log M + O\left(\frac{1}{M}\right) \)
5) \( \frac{(M+1)}{(M+2)} = 1 - \frac{1}{(M+1)} = 1 - \frac{1}{M} + O\left(\frac{1}{M^2}\right) \).
Thus, (3.20) is equal to $\log \frac{M}{M} \left(-\pi^2 \sqrt{2} + o(1)\right)$, and it follows from l'Hospital's rule that (3.18) is equal to $(\log M)^2 \left(-\pi^2 \frac{1}{2\sqrt{2}} + o(1)\right)$. Recall that we split (3.13) into two integrals. We must now deal with the second, namely

\[
\pi^2 \int_0^\infty \int_0^{1/\varepsilon} e^{-s^2(x+1)} s^2 \int_0^\infty e^{-r^2(y+x+1+1/\varepsilon)} I_1(2rsx) dr dx dy ds
\]

We can follow steps (3.14)-(3.18) exactly, with the only difference being that we have $y + M$ in place of $y$. We get

\[
\pi^2 \int_0^M \int_0^M \frac{1}{x} \left( (x + 1 - \frac{x^2}{x + y + 1 + M})^{-1} - (x + 1)^{-1} \right) dx dy
\]

\[
= -\pi^2 \int_0^M \int_0^M \frac{x}{(x + 1)(xy + 2x + y + 1 + M(x + 1))} dx dy
\]

where, again, $M = 1/\varepsilon$. We take the derivative as before, using Lemma 1, and this time the integrand depends on $M$:

\[
\pi^2 \int_0^M \int_0^M \frac{dy}{(M + 1)y + 2M + 1 + M(M + 1)}
\]

\[
\quad + \int_0^M \frac{x}{(x + 1)((2M + 2)x + 2M + 1)} dx
\]

\[
- \int_0^M \int_0^M \frac{x}{(xy + 2x + y + 1 + M(x + 1))^2} dx dy
\]

The first term

\[
\frac{M}{M + 1} \int_0^M \frac{dy}{(M + 1)y + 2M + 1 + M(M + 1)}
\]

is bounded above by

\[
\int_0^M \frac{dy}{M^2} = \frac{1}{M}
\]

We may ignore it, as it is $o(\log M/M)$. The second term
\[
(3.27) \quad \int_0^M \frac{x}{(x + 1)((2M + 2)x + 2M + 1)} \, dx = \int_0^M \frac{1}{(x + 1)} - \frac{2M + 1}{((2M + 2)x + 2M + 1)} \, dx
\]

is the same as the second integral in (3.20), with \(2M\) replacing \(M\). We can follow steps (3.20) and (3.21), and use fact (3) above along with

6) \( \log \left( \frac{(2M + 2)M + 2M + 1}{M + 1} \right) = \log M + O(\frac{1}{M}) \)

7) \( \frac{(M + 1)}{(M + 2)} = 1 - \frac{1}{(2M + 2)} = 1 - \frac{1}{2M} + O(\frac{1}{M^2}) \)

to see that this term is asymptotic to \((1/2) \log(M)/M\). The third term

\[
(3.28) \quad \int_0^M \int_0^M \frac{x}{(xy + 2x + y + 1 + M(x + 1))^2} \, dx \, dy
\]

is bounded above by

\[
(3.29) \quad \int_0^M \int_0^M \frac{dx \, dy}{M^2(x + 1)^2} = O(1/M)
\]

and may also be ignored. Thus, (3.22) is \((\log(1/\varepsilon))^2(\frac{\pi^2}{8\sqrt{2}} + o(1))\). We have found the asymptotics for (3.22) and (3.14). They are

\[
(3.30) - \pi^2 \int_0^1 \int_0^{1/\varepsilon} e^{-s^2(x + 1)} s^2 \int_0^\infty e^{-r^2(y + x + 1)} I_1(2rsx) \, dr \, dx \, dy \, ds
\]

\[
= (\log(1/\varepsilon))^2(\frac{\pi^2}{2\sqrt{2}} + o(1))
\]

and

\[
(3.31) - \pi^2 \int_0^\infty \int_0^{1/\varepsilon} e^{-s^2(x + 1)} s^2 \int_0^\infty e^{-r^2(y + x + 1/\varepsilon)} I_1(2rsx) \, dr \, dx \, dy \, ds
\]

\[
= (\log(1/\varepsilon))^2(\frac{\pi^2}{8\sqrt{2}} + o(1))
\]

Combining these as in (3.13), we see that (3.27) is \((\log(1/\varepsilon))^2(\frac{3\pi^2}{8\sqrt{2}} + o(1))\)

Now let us consider
\begin{equation}
\int \int \int_{0<a,b,c \leq t} \, dabdc \int \int e^{-p^2(a+c+\varepsilon)} e^{-(p+q)^2b} e^{-q^2\varepsilon} p_1 q_1 \, dp dq
\end{equation}

By a simple scaling we now show that this is equal to \((\log(t/\varepsilon))^2 \left( \frac{-3\pi^2}{8\sqrt{2}} + o(1) \right)\).

The scaling is as follows:

\begin{equation}
= t^2 \int \int \int_{0<a,b,c \leq 1} \, dabdc \int \int e^{-p^2 t(a+c+\varepsilon/t)} e^{-(p+q)^2 t b} e^{-q^2 t(\varepsilon/t)} (\sqrt{tp_1}) (\sqrt{tq_1}) \, dp dq
\end{equation}

Now replace \((p, q)\) with \((p/\sqrt{t}, q/\sqrt{t})\). The \(t^2\) in front of the integral is canceled, and we are left with

\begin{equation}
= (\log(t/\varepsilon))^2 \left( \frac{-3\pi^2}{8\sqrt{2}} + o(1) \right)
\end{equation}

We must now examine the same integral, but over the region \(\{a+b+c < t_1\}\) rather than \(\{0 < a, b, c \leq t_1\}\). However, the remark following Case 3 shows that if \(U \subseteq V\), then

\begin{equation}
\bigg| \int_{a,b,c \in U} \bigg| \leq \bigg| \int_{a,b,c \in V} \bigg|
\end{equation}

(Recall that both are negative) We also note that

\begin{equation}
\frac{(\log(t/\varepsilon))^2}{(\log(1/\varepsilon))^2} \rightarrow 1
\end{equation}

We can then write

\begin{equation}
\bigg| \int_{a,b,c < (t_1/3)} \bigg| \leq \bigg| \int_{a+b+c < t_1} \bigg| \leq \bigg| \int_{a,b,c < t_1} \bigg|
\end{equation}

The first and last integrals are both \((\log(1/\varepsilon))^2 \left( \frac{-3\pi^2}{8\sqrt{2}} + o(1) \right)\), and it follows that the middle one is as well. Thus, the integrand in the \(dt_1\) integral \([3.3]\)
is \((\log(1/\varepsilon))^2(\frac{-3\pi^2}{8\sqrt{2}} + o(1))\), with the \(o(1)\) term uniformly bounded on \(\{0 < \delta < t_1 < T\}\). We will split up (3.4) as:

\[
(3.38) \int_0^T \int \int_{a+b+c \leq t_1} dadbdc \int \int e^{-p^2(a+c+\varepsilon)} e^{-(p+q)^2b} e^{-q^2\varepsilon} p_1 q_1 dp dq = \int_0^\delta h(t_1, \varepsilon) dt_1 + \int_\delta^T h(t_1, \varepsilon) dt_1
\]

where \(h\) denotes the result after doing the integrals in the other variables. We know that, for the second integral, \(h(t_1, \varepsilon)(\log(1/\varepsilon))^{-2} \to \frac{3\pi^2}{8\sqrt{2}}\) uniformly. Thus, the second integral is \((T - \delta)(\log(1/\varepsilon))^2(\frac{-3\pi^2}{8\sqrt{2}} + o(1))\). The first integral is bounded above in absolute value by (assuming \(\delta < 1\))

\[
(3.39) \int_0^\delta \int \int_{a+b+c \leq 1} dadbdc \int \int e^{-p^2(a+c+\varepsilon)} e^{-(p+q)^2b} e^{-q^2\varepsilon} p_1 q_1 dp dq = \delta(\log(1/\varepsilon))^2(\frac{3\pi^2}{8\sqrt{2}} + o(1))
\]

By letting \(\delta \to 0\), we may finally conclude that the contribution from Case 1 is

\[
(3.40) T(\log(1/\varepsilon))^2(\frac{-3\pi^2}{8\sqrt{2}} + o(1))
\]

Similar techniques will yield Case 2. We'll give only the outline here. Recall that we are evaluating

\[
(3.41) \int_0^T \int \int_{a+b+c \leq t_1} dadbdc \int \int e^{-p^2(c+\varepsilon)} e^{-(p+q)^2b} e^{-q^2(a+\varepsilon)} p_1 q_1 dp dq
\]

As before, we begin by changing the domain to \(\{0 \leq a, b, c \leq 1\}\) and integrating \(da, db, \text{ and } dc\). We arrive at

\[
(3.42) \int \int \frac{(1 - e^{-p^2})}{p^2} \frac{(1 - e^{-q^2})}{q^2} \frac{(1 - e^{-(p+q)^2})}{(p+q)^2} e^{-\varepsilon(p^2+q^2)} p_1 q_1 dp dq = \int \int \frac{(1 - e^{-p^2/\varepsilon})}{p^2} \frac{(1 - e^{-q^2/\varepsilon})}{q^2} \frac{(1 - e^{-(p+q)^2/\varepsilon})}{(p+q)^2} e^{-(p^2+q^2)} p_1 q_1 dp dq
\]
We convert to polar coordinates again:

\[
\begin{align*}
\int \int \int & \left(1 - e^{-r^2/\varepsilon}\right) \left(1 - e^{-s^2/\varepsilon}\right) \cos(\phi) e^{-(r^2+s^2)} \\
& \int_{0}^{2\pi} \frac{\left(1 - e^{-|re^{i\theta} + se^{i\phi}|^2/\varepsilon}\right)}{|re^{i\theta} + se^{i\phi}|^2} \cos(\theta) d\theta d\phi dr ds.
\end{align*}
\]

We follow the same steps for the \(d\theta\) integral as before (steps (3.9)-(3.13)) to arrive at

\[
\begin{align*}
& \int \int \int \left(1 - e^{-|re^{i\theta} + se^{i\phi}|^2/\varepsilon}\right) \left(1 - e^{-s^2/\varepsilon}\right) \left(1 - e^{-r^2/\varepsilon}\right) \\
& e^{-(r^2+s^2)(x+1)} I_1(2rsx) dx dr ds.
\end{align*}
\]

We’ll expand this into 4 integrals and do each separately. The first is

\[
\begin{align*}
& -\pi^2 \int_{0}^{\infty} \int_{0}^{1/\varepsilon} \int_{0}^{1/\varepsilon} e^{-s^2(x+1)} \left(\int_{0}^{\infty} e^{-t^2(x+1)} I_1(2rsx) dr\right) dx ds.
\end{align*}
\]

The \(dr\) integral, as in step (3.15), is equal to \(\frac{(e^{s^2x^2} - 1)}{\sqrt{2sx}}\), and we obtain:

\[
\begin{align*}
& \frac{-\pi^2}{\sqrt{2}} \int_{0}^{\infty} \int_{0}^{1/\varepsilon} e^{-s^2(x+1)} \left(e^{s^2x^2} - 1\right) dx ds.
\end{align*}
\]

We use the identity

\[
\begin{align*}
& s \int_{0}^{\infty} e^{st} dt = \frac{(e^{s^2x^2} - 1)}{s}
\end{align*}
\]

And the integral in question becomes

\[
\begin{align*}
& \frac{-\pi^2}{\sqrt{2}} \int_{0}^{\infty} \int_{0}^{1/\varepsilon} \frac{1}{x} \int_{0}^{x+1} \int_{0}^{\infty} e^{-s^2(x+1-t)} s ds dt dx
\end{align*}
\]

Note that \(t \leq \frac{x^2}{x+1} < x + 1\), which implies \((x+1-t) > 0\), so there’s no problem with convergence. Tackling this integral again reduces to basic calculus. Begin by substituting \(u = s^2(x+1-t)\) to get
\[
\frac{-\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \left( \frac{1}{x} \int_0^{x^2} \frac{1}{x+1-t} \int_0^\infty e^{-u} du dt dx \right)
\]

\[= \frac{-\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \frac{1}{x} \int_0^{x^2} \frac{1}{x+1-t} dt dx \]

\[= \frac{-\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \frac{1}{x} \log \left( \frac{x+1}{x+1-x^2/(x+1)} \right) dx \]

If \( M = 1/\varepsilon \), then, by the fundamental theorem of calculus, \( \frac{d}{dM} \) of the above is

\[
\frac{-\pi^2 \log \left( \frac{M+1}{M+1-M^2/(M+1)} \right)}{2\sqrt{2}}
\]

Since

\[
\log \left( \frac{M+1}{M+1-M^2/(M+1)} \right) = \log(M) + O(1)
\]

we see that (3.50) is \( \frac{\log M}{M} \left( -\pi^2 + o(1) \right) \), and thus our original integral (3.45) is \( (\log M)^2 \left( -\pi^2 + o(1) \right) \). Recall that (3.44) was divided into four integrals. The remaining three give a contribution of

\[
-\pi^2 \int_0^\infty \int_0^\infty \int_0^{1/\varepsilon} \left( -e^{-r^2/\varepsilon} - e^{-s^2/\varepsilon} + e^{-r^2/\varepsilon} e^{-s^2/\varepsilon} \right)
\]

\[e^{-(r^2+s^2)(x+1)} I_1(2rsx) dx dr ds \]

The integral corresponding to the first and second term will be identical, and so if we follow steps (3.44) through (3.49) we get

\[
\frac{-\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \frac{1}{x} \log \left( \frac{x + 1 + 1/\varepsilon}{x + 1 + 1/\varepsilon - x^2/(x + 1 + 1/\varepsilon)} \right) dx
\]

\[+ \frac{2\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \frac{1}{x} \log \left( \frac{x + 1 + 1/\varepsilon}{x + 1 + 1/\varepsilon - x^2/(x + 1 + 1/\varepsilon)} \right) dx
\]

\[= \frac{\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \frac{1}{x} \log \left( \frac{1}{x + 1 + 1/\varepsilon} \right) dx
\]

\[= -2 \frac{-\pi^2}{2\sqrt{2}} \int_0^{1/\varepsilon} \frac{1}{x} \log \left( \frac{x^2}{(x+1)(x+1+1/\varepsilon)} \right) dx \]

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Each of these integrals is $O(\log(1/\varepsilon))$. To see this, note that the absolute values of the integrands are bounded above by $g(x) = \frac{-1}{x} \log(1-x^2)$ on $\{0 < x < \frac{1}{2}\}$. $g(x)$ is bounded on this interval, so the integral over $\{0 < x < \frac{1}{2}\}$ is bounded. For $\{x > \frac{1}{2}\}$ the integrand is bounded by $\frac{-1}{x} \log(1-\frac{1}{2})$, and it follows from this that the integral is $O(\log(1/\varepsilon))$. This proves that (3.43) is $\log(1/\varepsilon)^2(\frac{-\pi^2}{4\sqrt{2}} + o(1))$. The remainder of the proof that the integral we began with in case 2, namely (3.44), is $\log(1/\varepsilon)^2(-\pi^2 \frac{5}{128\sqrt{2}\pi^2} + o(1))$ is identical to the steps (3.35) through (3.40).

Combining our work in cases 1 and 2, and reinserting the constant $\frac{-1}{(2\pi)^4}$ which was suppressed throughout, we see that

\[
E[\alpha'_x(T)^2] = T \log(1/\varepsilon)^2 \left(\frac{5}{128\sqrt{2}\pi^2} + o(1)\right)
\]

We assumed at the outset that $s_1 < s_2$, so this must be multiplied by 2 to obtain the correct answer. \hfill \square

**Corollary 1.** The contribution to (2.4) of any component of order 2 is $\log(1/\varepsilon)^2(\frac{5}{128\sqrt{2}\pi^2} + o(1))$. That is, if $r_j$ is the left endpoint of a component of order 2, and $r_{j+1}$ is the maximal right endpoint (see (2.7)), then the integral over this region is $\log(1/\varepsilon)^2(\frac{5}{128\sqrt{2}\pi^2} + o(1))(r_{j+1} - r_j)$.

**Proof:** Suppose that the component of order 2 is composed of $[s_k, t_k]$ and $[s_k', t_k']$. Then the integral in question is

\[
\frac{-1}{(2\pi)^4} \int_{s_k, t_k, s_k', t_k' \in [r_j, r_{j+1}]} e^{-\varepsilon(p^2+q^2)} E[e^{ip(X_{t_k}-X_{s_k})+iq(X_{t_k'}-X_{s_k'})}] p_1 q_1 d^2s d^2t dp dq
\]

We can write $X_{t_k} - X_{s_k}$ as $(X_{t_k} - X_{r_j}) - (X_{s_k} - X_{r_j})$, and likewise for $X_{t_k'} - X_{s_k'}$. Since $X_{r_{j+1}} - X_{r_j}$ is itself a Brownian motion we see that (3.55) is equal to (3.2) with $D_T$ replaced by $D_{r_{j+1} - r_j}$. \hfill \square

### 4 Components of order $n \geq 3$

We now turn to components of order $n$, where $n \geq 3$. We’ll show
Proposition 2. The contribution to (2.4) of any component of order \( n \geq 3 \) is \( o(\log(1/\varepsilon)^n) \).

This entire section is devoted to the proof of this proposition. Suppose that a component of order \( n \) is can be formed by a specific arrangement of intervals \([s_1, t_1], ..., [s_n, t_n]\) corresponding to the variables \( p_1, ..., p_n \). Let \( \{r_1, ..., r_{2n}\} \) be a relabeling of the \( s_i \)'s and \( t_i \)'s so that \( 0 := r_0 \leq r_1 \leq r_2 \leq ... \leq r_{2n} \). We split up the expectation in the integrand by independence and change coordinates as we did in the order 2 case. The contribution of this arrangement of intervals is then given by the integral

\[
\int e^{-\varepsilon \sum p_i^2} \prod (p_i) \left( \int \sum_{c_j < T} e^{-u_j c_j} \prod dc_j \right) \prod dp_i
\]

where \( c_j = r_j - r_{j-1} \) and, as before, \( (p_i)_1 \) denotes the x-coordinate of \( p_i \). Each \( u_j \) is a linear combination of \( p_i \)'s, and is ordered in the natural way. That is, \( u_1 = p_1, u_2 = p_1 + p_2, u_3 = p_2 \) or \( p_1 \) or \( p_1 + p_2 + p_3 \), etc. For each \( j \), either \( u_j - u_{j-1} = p_i \) or \( u_j - u_{j-1} = -p_i \) for some \( i \). In the first case we’ll refer to \( u_j \) as increasing (abbreviated as \( u_j \uparrow \)) and in the second case we’ll say \( u_j \) is decreasing (abbreviated as \( u_j \downarrow \)). To simplify some of the notation that follows, let \( u_0 = u_{2n} = 0 \). We will suppose first that our component contains no isolated intervals. That is, there does not exist an interval \([s_i, t_i]\) such that \( t_k, s_k \notin (s_i, t_i) \) for all \( k \). In this case, we may obtain a sufficient bound for (4.1) by replacing the integrand with the absolute value of the integrand. We may then also replace the region \( \sum c_j < T \) with \( 0 < c_j < T \) for all \( j \) and use the simple fact that

\[
\int_0^T e^{-u_j c_j} dc_j \leq \frac{k}{(1 + |u_j|)^2}
\]

for a constant \( k \) to reduce our problem to bounding

\[
\int \frac{e^{-\varepsilon \sum p_i^2} \prod |p_i|}{\prod_{j=1}^{2n-1} (1 + |u_j|)^2} \prod dp_i
\]

We will eventually need the following lemmas:

Lemma 2

\[
\int_{R^2} \frac{e^{-\varepsilon u^2}}{(1 + |u|)^2} d^2 u = O(\log(1/\varepsilon))
\]
Proof: Replace $u$ with $u/\sqrt{\varepsilon}$ and the integral becomes

$$\int_{\mathbb{R}^2} \frac{e^{-u^2}}{(\sqrt{\varepsilon} + u)^2} d^2u$$

(4.5)

The integral over $|u| > 1$ is clearly $O(1)$, and for $|u| < 1$ it suffices to bound

$$\int_{|u|<1} \frac{1}{(\sqrt{\varepsilon} + u)^2} d^2u$$

(4.6)

Converting to polar coordinates this is

$$2\pi \int_0^1 \frac{r}{(\sqrt{\varepsilon} + r)^2} dr \leq 2\pi \int_0^1 \frac{1}{\sqrt{\varepsilon} + r} dr = O(\log(1/\varepsilon))$$

(4.7)

$\square$

**Lemma 3** There is a constant $c < \infty$ such that, independently of $\varepsilon, a$ and $n \geq 1$, we have

$$\int_{\mathbb{R}^2} \frac{e^{-\varepsilon u^2}}{(1+|u|)^2(1+|a+u|)^n} d^2u < c$$

(4.8)

**Proof:** We can ignore the numerator. By Hölder’s inequality the integral is bounded by

$$\left( \int \frac{1}{(1+|u|)^3} d^2u \right)^{2/3} \left( \int \frac{1}{(1+|u+a|)^3n} d^2u \right)^{1/3} < c < \infty$$

(4.9)

$\square$

These lemmas motivate the intuition behind our approach, which we first describe informally. The lemmas essentially say that a square in the denominator gives a log, whereas a cube or higher gives a constant. We will write (4.3) in such a way that we can cancel the $\prod |p_i|$ in the numerator with powers in the denominator. We will use the Cauchy-Schwarz inequality to cut down on the number of different terms in the numerator, and we will change variables by a linear transformation. After all of this we will obtain a product of a collection of integrals of the form in Lemma 2 with at least one integral in the form of Lemma 3. Each one of the type in Lemma 2 contributes a $\log(1/\varepsilon)$, whereas the Lemma 3 type doesn’t. When we multiply everything
out, the power of $\log(1/\varepsilon)$ will be less than $n$. (As a side note, Lemma 2 also indicates why we are initially not considering the case of the isolated intervals. In that case there is some $p_i$ which is only present as a term in one $u_j$, so that if we were to put the absolute value inside the integral as we are doing here, we would have only a square of $p_i$ in the denominator with a $|p_i|$ present in the numerator. This would essentially give us

$$\int \frac{e^{-sp_i^2}}{1 + |p_i|} d^2p_i$$

(4.10)

And this is only $O(\sqrt{1/\varepsilon})$, which is not good enough.)

To make good on this approach, we will need a way to make sure that, after we cancel the terms in the numerator, we have enough terms left in the denominator to obtain adequate convergence. In terms of the sheer number of powers in the denominator there is no problem. Lemma 2 suggests we need more than $2n$ powers on the bottom, but there are $n$ powers on top versus $4n - 2$ on the bottom, for a total of $3n - 2$ on the bottom. This is enough since $n \geq 3$. The tricky part is making sure that we have a proper assortment on the bottom. The details are rather involved, so we first will prove several technical lemmas. To state the first lemma, we need another bit of terminology: We will say that $p_i$ is $t$-free if there is no $t_k$ contained in $(s_i, t_i)$. Note that if $p_i$ is $t$-free and $[s_i, t_i]$ is not an isolated interval, then at least one $s_k$ is contained in $(s_i, t_i)$. We have the following lemma, which was proved in [8].

**Lemma 4** The span of the decreasing $u_j$’s is equal to the span of the set of all $p_i$’s which are not $t$-free. Furthermore, suppose that for each $t$-free $p_i$ we chose $u(p_i)$ to be any one of the increasing $u_j$’s which contains $p_i$ as a term. Then, if we let $D = \{\text{set of decreasing } u_j \text{'s}\} \cup \{\text{set of all } u(p_i) \text{'s}\}$, $D$ spans the entire set \{p_1, ..., p_n\}

**Proof:** To begin with, suppose that $p_i$ is $t$-free. Then $p_i$ only appears as a term in increasing $u_j$’s, and so is not in the span of the decreasing $u_j$’s. Conversely, suppose we have a configuration of intervals such that the set of decreasing $u_j$’s does not span the set of all $p_i$’s which are not $t$-free. Let $p_{i_o}$ be the non-$t$-free $p$ with largest $s$ value which is not in the span of the decreasing $u_j$’s. That is, if $s_i > s_{i_o}$ and $p_i$ is not $t$-free, then $p_i$ is in the span of decreasing $u_j$’s. Now, let $u_{j_o}$ be the $u$ with largest $j$ value which contains $p_{i_o}$,
i.e. such that \( u_{j_0} - u_{j_0+1} = p_{i_0} \). Then \( u_{j_0+1} \) is decreasing, and we will obtain a contradiction if we can show that \( u_{j_0} \) is in the span of decreasing \( u_j \)'s. If \( u_{j_0} \) is decreasing there is nothing to prove, so suppose \( u_{j_0} \) is increasing. Let \( v > 0 \) be chosen as small as possible so that \( u_{j_0-v} \) is decreasing. The fact that \( p_{i_0} \) is not t-free implies that \( p_{i_0} \) appears as a term in \( u_{j_0-v} \). Furthermore, we can write \( p_{i_0} = u_{j_0-v} + (u_{j_0-v+1} - u_{j_0-v}) + \ldots + (u_{j_0} - u_{j_0-1}) - u_{j_0+1} \). Now, \( u_{j_0-v} \) and \( u_{j_0+1} \) are decreasing, and the terms \((u_{j_0-v+1} - u_{j_0-v}), \ldots, (u_{j_0} - u_{j_0-1})\) are all equal to \( p_i \)'s which have the properties that (i) they are not t-free, because they appear as a term in \( u_{j_0+1} \), and (ii) they have larger \( s \) values than \( p_{i_0} \). We conclude that they are in the span of the decreasing \( u_j \)'s, which means that \( p_{i_0} \) is as well. This is a contradiction, and establishes the first part of the lemma. To prove the second part, just note that \( u(p_i) \) contains \( p_i \) as a term as well as several other \( p_k \)'s which cannot be t-free. These \( p_k \)'s are in the span of \( D \) then, and thus \( p_i \) is as well.

In order to state the next lemma, we must consider (4.3) again. Let \( u_{j_1}, \ldots, u_{j_n} \) be the increasing \( u \)'s in order. That is, \( u_{j_i} - u_{j_{i-1}} = p_i \). We see that (4.3) is bounded by

\[
(4.11) \quad \int \frac{e^{-\varepsilon \sum p_i^2} \prod(|u_{j_1}| + |u_{j_1-1}|)}{\prod_{j=1}^{2n-1}(1 + |u_j|^2)} \prod dp_i
\]

Expand the numerator completely, and break this integral into the sum of many integrals, each of which we do individually. Each of these integrals has a product of \(|u|\)'s in the numerator, but no \( u \) can appear more than twice. This allows us to cancel all of the \( u \)'s in the numerator with \( u \)'s in the denominator (Note: The word ”canceling”, in this context, means replacing \( \frac{|u|}{1+|u|} \) with 1). We arrive at the following integral:

\[
(4.12) \quad \int \frac{e^{-\varepsilon \sum p_i^2}}{\prod_{j=1}^{2n-1}(1 + |u_j|)^{m_j}} \prod dp_i
\]

where \( m_j = 0, 1, \) or \( 2 \), depending on what power of \( u_j \) appeared in the numerator. The following lemma relates \( m_j \) with the properties of \( u_j \) in the configuration of intervals.

**Lemma 5**

1. If \( u_j \downarrow \) then \( m_j, m_{j-1} \geq 1 \).
2. If \( u_j \downarrow \) and \( m_j = 1 \), then \( u_{j+1} \uparrow \) and \( m_{j+1} \geq 1 \).
3. If \( u_j, u_{j+1} \downarrow \) then \( m_j = 2 \).
Proof: Each term in the numerator is of the form \(|u_j| + |u_{j-1}|\) where \(u_j \uparrow\). We see that we can only have \(m_j = 0\) if \(u_j\) appears in two terms in the numerator, and this can only happen if both \(u_j\) and \(u_{j+1}\) are increasing. This proves (1). If \(u_j \downarrow\) then \(u_j\) appears at most once in the numerator in the term \(|u_{j+1}| + |u_j|\) where \(u_{j+1}\) must be increasing. Furthermore, \(u_{j+1}\) can appear in at most one other term, and so if \(m_j = 1\) then \(m_{j+1} \geq 1\). This proves (2). As for (3), if \(u_j, u_{j+1} \downarrow\) then \(u_j\) does not appear in the numerator at all, so \(m_j = 2\).

We now turn our attention to (4.3). Suppose that we can form sets \(A = \{a_1, ..., a_r\}\) and \(B = \{b_1, ..., b_s\}\) with the following properties:

i) Each of \(a_1, ..., a_r\) and \(b_1, ..., b_s\) are equal to some \(u_j\).

ii) \(A\) and \(B\) each span \(\{p_1, ..., p_n\}\).

iii) If \(a_i = u_j\) or \(b_i = u_j\), then \(m_j \geq 1\).

iv) If \(a_i = b_k = u_j\), then \(m_j = 2\).

Note that if we can find such sets we can, simply by deleting elements if necessary, find two sets \(A_o\) and \(B_o\) which satisfy the above properties and which each have \(n\) elements. So in the calculations which follow we’ll assume that \(n = s = r\), even though when we eventually construct \(A\) and \(B\) they may have more than \(n\) elements. Given two such sets, we could bound (4.12) by

\[
\int \frac{e^{-\varepsilon \sum \rho_i^2}}{K(a_1, ..., a_n) \prod_{j=1}^n (1 + |a_j|)(1 + |b_j|)} \prod dp_i
\]

where \(K\) is of the form \((1 + |u_j|)\) for some \(j\). Recall that we have \(3n - 2\) powers of \(u\)'s in the denominator, so there will always be at least one term left over after choosing our sets \(A\) and \(B\). This term will contain a linear combination of \(p_i\)'s, but since \(A\) spans \(\{p_1, ..., p_n\}\) we may write it as a linear combination of \(a_j\)'s. It is irrelevant what the linear combination present in \(K\) actually is, except that it must be nonzero, of course. Now, we can apply the Cauchy-Schwarz inequality to bound (4.13) by

\[
\left( \int \frac{e^{-\varepsilon \sum \rho_i^2}}{K(a_1, ..., a_n)^2 \prod_{j=1}^n (1 + |a_j|)^2} \prod dp_i \right)^{1/2}
\]

\[
\times \left( \int \frac{e^{-\varepsilon \sum \rho_i^2}}{\prod_{j=1}^n (1 + |b_j|)^2} \prod dp_i \right)^{1/2}
\]
There is a constant $c > 0$ so that $\sum p_i^2 > c \sum a_i^2$ and $\sum p_i^2 > c \sum b_i^2$; this is because the functions $\sum a_i^2$ and $\sum b_i^2$ are homogeneous of degree 2 in the $p_i$'s and bounded on $\sum p_i^2 = 1$. Thus, (4.14) is bounded by

\begin{equation}
\frac{e^{-\varepsilon c} \sum a_j^2}{K(a_1, \ldots, a_n)^2 \prod_{j=1}^n (1 + |a_j|)^2} \prod dp_i \right)^{1/2}
\end{equation}

\times \left( \int \frac{e^{-\varepsilon c} \sum b_j^2}{\prod_{j=1}^n (1 + |b_j|)^2} \prod dp_i \right)^{1/2}

Now, we apply a linear change of coordinates to these integrals so that we are integrating with respect to $a_j$ and $b_j$ instead of $p_i$. Relabel if necessary so that $a_1$ is one of the $a$'s which appears as a term in $K(a_1, \ldots, a_n)$. We see that the first integral in (4.15) is bounded by a constant times

\begin{equation}
\int \left( \int \frac{e^{-\varepsilon c a_1^2}}{K(a_1, \ldots, a_n)^2 (1 + |a_1|)^2} \prod_{j=2}^n \frac{e^{-\varepsilon c a_j^2}}{(1 + |a_j|)^2} \prod da_j \right) \prod_{j=2}^n \frac{e^{-\varepsilon c a_j^2}}{(1 + |a_j|)^2} da_j
\end{equation}

By Lemma 3 the inner integral is $O(1)$ and by Lemma 2 the others are all $O(\log(1/\varepsilon))$. Lemma 2 also shows that the second integral in (4.15) is $O(\log^n(1/\varepsilon))$. We see that (4.15) is $O((\log(1/\varepsilon))^{n-1/2})$, and this shows that (4.1) is $o((\log(1/\varepsilon))^n)$, which is what we set out to prove.

All that remains, then, is to show that we can always find sets $A$ and $B$ of this form. For this, we'll use Lemmas 4 and 5. Lemma 4 gives us a good first initial candidate for $A$ and $B$. We can let $A$ be equal to the set of (distinct) decreasing $u_j$'s together with elements $u(p_i)$ for each t-free $p_i$. (Recall that all decreasing $u_j$'s have $m_j \geq 1$). The possible problem with this is that every increasing $u_i$, and in particular each possibility for $u(p_i)$, appears at least once in the numerator of (4.11), so that we need to make sure that we really can appropriately choose the $u(p_i)$'s. Nevertheless, as will be shown below, this works for $A$. $B$ cannot be chosen the same way, however. This is because if $u_j$ is decreasing but $u_{j+1}$ is increasing, then $u_j$ appears exactly once in the numerator of (4.11) and we may have $m_j = 1$, so that $u_j$ cannot be in both $A$ and $B$. $B$ will have to be formed in a different manner.

In order to eventually create the set $B$, we’ll begin by creating an increasing collection of sets $B_n$ by considering decreasing $u_j$’s for increasing values of $j$. Start with the smallest $j$ such that $u_j$ is decreasing. If $m_j = 2$, then let $B_1 = \{u_j\}$. If $m_j = 1$, then we know from Lemma 5 that $u_{j+1}$ is increasing.
We can then choose \( d > 0 \) such that \( u_j = u_{j+1} - (u_{j+d} - u_{j+d+1}) \); \( d \) is simply chosen to be the largest value such that \( u_{j+d} \) contains \( p_i = u_{j+1} - u_j \) as a term. Let \( B_1 = \{u_{j+1}, u_{j+d}, u_{j+d+1}\} \). We will essentially repeat this for each decreasing \( u_j \). Suppose that the set \( B_n \) has already been formed. Let \( j \) be as small as possible so that \( u_j \notin \text{span}\{B_n\} \) and \( u_j \) is decreasing. If \( m_j = 2 \), then let \( B_{n+1} = B_n \cup \{u_j\} \). If \( m_j = 1 \), then let \( B_{n+1} = B_n \cup \{u_{j+1}, u_{j+d}, u_{j+d+1}\} \setminus \{u_j\} \), where again \( d \) is such that \( u_{j+d+1} - u_{j+d} = u_{j+1} - u_j \). The reason for subtracting the set \( \{u_j\} \) is that it may already be in the set \( B_n \), having been of the form \( u_{j'+d'} \) or \( u_{j'+d'+1} \) for an earlier \( j' \). Repeat this process through all of the decreasing \( u_j \)'s. The final set obtained, say \( B_N \), will span the set of decreasing \( u_j \)'s. To see this, suppose to the contrary, and let \( u_{j_0} \) be the \( u_j \) with largest \( j \) value which is not in the span of \( B_N \). Clearly then \( m_{j_0} = 1 \), which means that an earlier \( B_n \) must have contained \( u_{j_0+1}, u_{j_0+d_0}, \) and \( u_{j_0+d_0+1} \). Any of these elements which are increasing must be present in \( B_N \), and any decreasing ones have larger \( j \) values than \( u_{j_0} \), which means they are in the span of \( B_N \). Thus, \( u_{j_0} = u_{j_0+1} - (u_{j_0+d_0} - u_{j_0+d_0+1}) \) is also in the span of \( B_N \), a contradiction. The set \( B_N \) also satisfies property (iii) above. This will be shown using the following lemma:

**Lemma 6** 1. If \( u_i \) is in \( B_N \) then \( u_i \) is either decreasing or else neighbors on a decreasing interval (i.e. at least one of \( u_i-1 \) and \( u_{i+1} \) is decreasing).

2. If \( u_i, u_{i+1} \) are both increasing and \( u_i \in B_N \) then \( u_{i-1} \downarrow \), \( m_{i-1} = 1 \), and \( m_i \geq 1 \).

**Proof:** If \( u_i \) is increasing and in \( BN \) then \( u_i \) must be of the form \( u_{j+1}, u_{j+d}, \) or \( u_{j+d+1} \) for some \( j \) where \( j \) and \( d \) are as in the construction of the set \( B' \) above, i.e. \( u_j \) is decreasing, \( m_j = 1 \), and \( u_j = u_{j+1} - (u_{j+d} - u_{j+d+1}) \). It is always true that \( u_{j+d+1} \) is decreasing, so this cannot be \( u_i \). (1) is proved by noting that \( u_{j+1}, u_{j+d} \) are neighbors to the decreasing intervals \( u_j, u_{j+d+1} \) respectively. If, in addition, the situation in (2) arises then \( u_i \) cannot be of the form \( u_{j+d} \) since in that case \( u_{j+d+1} \) would be decreasing. Thus, \( u_i \) is of the form \( u_{j+1} \). In order for \( u_{j+1} \) to be included in \( B' \) it was necessary that \( u_j \downarrow \) and \( m_j = 1 \). By part 2 in Lemma 3 \( m_i \geq 1 \). \( \Box \)

If \( u_j \in B_N \) then either \( u_j \downarrow \), in which case \( m_j \geq 1 \) by part 1 of Lemma 3 or \( u_j \uparrow \) in which case \( m_j \geq 1 \) by part 1 of Lemma 3 or part 2 of Lemma 6 depending on whether \( u_{j+1} \downarrow \) or \( \uparrow \). Thus, \( B_N \) satisfies (iii) as claimed. Let \( B' = B_N \) and \( A' \) be the set of all decreasing \( u_j \)'s. We know from the discussion above
that $B'$ satisfies (i) and (iii). $A'$ clearly satisfies (i), and satisfies (iii) by part 1 of Lemma 5. $A'$ and $B'$ together satisfy (iv) because of the way that $B'$ was constructed, and both span the set of all decreasing $u_j$'s. We need now only extend them to sets $A$ and $B$ which span all of $\{p_1, ..., p_n\}$. $A'$ and $B'$ already span the set of all non-t-free $p_i$'s, by the first part of Lemma 4. In light of the second part of Lemma 4 all that remains is to show that, for any t-free $p_i$, we can always choose $u_1(p_i), u_2(p_i)$ which contain $p_i$ as a term, and which we may include in $A$ and $B$ respectively without violating rules (iii) and (iv).

Suppose $p_i$ is t-free, and $k$ is chosen as large as possible so that $s_i < s_{i+1} < ... < s_{i+k} < t_1$ (Note: $k$ here is not the same as $d$ above; they differ by 1). Let $u_j - u_{j-1} = p_i$. The term $p_ip_{i+1}...p_{i+k}$ in (4.1) becomes $|u_j| + |u_{j-1}|...(|u_{j+k}| + |u_{j+k-1}|)$ in (4.11), with $u_j, ..., u_{j+k}$ not appearing anywhere else in the numerator. If $k > 1$ we can just note that, upon expanding this expression, the sum of the powers of $u_{j+k}$ and $u_{j+k-1}$ in the numerator is at most two. This means that $m_{j+k} + m_{j+k-1}$ must be at least 2, and we can choose $u_1(p_i), u_2(p_i)$ as some combination of $u_{j+k}$ and $u_{j+k-1}$. It is possible that $u_{j+k}$ is already in $B'$, and so we must make sure that if $u_1(p_i) = u_{j+k} \neq u_2(p_i)$ that we interchange $u_1(p_i)$ and $u_2(p_i)$, so that $u_{j+k}$ is not in both $A$ and $B$, so as to not violate (iv). Note that if $k > 1$ then $u_{j+k-1} \notin B'$ by Lemma 6 since $u_{j+k-1} \uparrow$ and neighbors only on increasing intervals. In the case that $k = 1$ we still have $m_j + m_{j+1} \geq 2$, but now it is possible that both $u_j$ and $u_{j+1}$ are in $B'$, since both neighbor upon intervals which may be decreasing. However, if this is the case then, since $u_j, u_{j+1} \uparrow$, we have by Lemma 6 $u_{j-1} \downarrow$ and $m_{j-1} = 1$. Recall that we have the term $|(u_j| + |u_{j-1}|)(|u_{j+1}| + |u_j|)$ in the numerator, with $u_{j-1}, u_j, u_{j+1}$ appearing nowhere else in the numerator. The sum of the powers of $u_{j-1}, u_j,$ and $u_{j+1}$ is two, and thus $m_{j-1} + m_j + m_{j+1} = 4$. Since $m_{j-1} = 1$, one of $m_j$ and $m_{j+1}$ is 2. We can then let $u_1(p_i) = u_2(p_i) = u_j$ or $u_{j+1}$, depending on whether $m_j$ or $m_{j+1}$ is 2. This handles the case $k = 1$. (If $k = 0$ then we would have an isolated interval, and this argument doesn’t work. This is the only place where we used the fact that we had no isolated intervals.) Doing this for each t-free $p_i$ we create the sets $A$ and $B$, which are guaranteed by Lemma 4 to satisfy the property (ii). $A$ and $B$ also satisfy properties (i), (ii), and (iv) by construction, so we have completed the proof in the case where no isolated intervals are present.

Now for the isolated intervals case. Recall that the integral which gives us the contribution from this configuration is
\[ (4.17) \int e^{-\varepsilon \sum p_i^2} \prod (p_i) \prod_{j=1}^{2n-1} \left( \int_{\sum t_j < T} \prod_j e^{-u_j^2 t_j} \prod_j dt_j \right) \prod dp_i \]

As mentioned before, here we cannot replace the integrand with its absolute value, for in that case each isolated interval would contribute a \( \sqrt{1/\varepsilon} \) to the integral. Cancelation occurs in the integral, however, since the integrand is positive in some regions and negative in others. It turns out that it is enough to integrate each of the variables corresponding to isolated intervals first, and then to bring the absolute value inside the integral. After we have "removed" the initial set of isolated intervals in this fashion, we will have created a new configuration of intervals, which may again contain isolated intervals. We can remove these isolated intervals by a different method than was used for the first set. This brings us to a new configuration, which may again have isolated intervals, which we again remove, etc. After a finite number of steps we either have removed all intervals or we have arrived at an arrangement with no isolated intervals. In the second case we are reduced to the case we have already done, and the first is handled easily in a slightly different way.

Let us bring in some definitions in order to make this rigorous. Let our initial configuration of intervals be denoted \( K_0 \), and let \( K_m \) be the configuration of intervals obtained upon removing the isolated intervals from \( K_{m-1} \). We will say \( p_i \in K_m \) to mean that the interval \((s_i, t_i)\) appears in the configuration \( K_m \), and we will define the \textit{order} of \( K_m \) to be the number of \( p_i \)'s in \( K_m \). Let \( u_{(m,1)}, \ldots, u_{(m,n_m)} \) be the linear combinations of \( p_i \)'s which appear in the configuration \( K_m \), ordered from left to right. Let us define \( I_m \) to be the set of all \( j \) values corresponding to isolated intervals in \( K_m \); that is, \( I_m = \{ j : u_{(m,j)} = p_i \text{ where } (s_i, t_i) \text{ contains no } s_k \text{ or } t_k \text{ in } K_m \} \). A \( \hat{p} \) will refer to the \( p \) associated to an isolated interval. That is, if \( j \in I_m \) and \( p_i \) is the \( p \) which appears only in \( u_{(m,j)} \), label \( p_i \) as \( \hat{p}_{m,j} \). We can bound \( (4.17) \) by

\[ \int \prod e^{-\varepsilon p_i^2} |p_i| \left( \int \prod_{j \notin I_0} e^{-u_{(0,j)}^2 t_j} \prod_{j \in I_0} \int e^{-\varepsilon \hat{p}_{(0,j)}^2 (\hat{p}_{(0,j)})} (\hat{p}_{(0,j)})_1 e^{-u_{(0,j)}^2 t_j} dt_j d\hat{p}_{(0,j)} \right) \prod dp_i \]

\[ (4.18) \]

The first and last products over all \( i \) such that \( p_i \neq \hat{p}_{(0,j)} \) for all \( j \). We will get a good bound on the \( dt_j d\hat{p}_j \) integrals. Note that we have suppressed the region of integration in \( t_j \), since it may be quite complicated. We do know that the upper limit of integration is bounded above by \( T \), and this allows us to get a sufficient bound, as the following lemma shows.
Lemma 7 For any $a$ with $0 < a < T$ and any $k \in \mathbb{R}^2$, we have

\[
| \int_0^a e^{-\varepsilon p^2} p_1 e^{-(p+k)^2 t} dt dp | = |k| O(\log(1/\varepsilon))
\]

independently of $a$.

Proof:

\[
\int_0^a e^{-\varepsilon p^2} p_1 e^{-(p+k)^2 t} dt dp
\]

\[
= \int_0^a \left( \int e^{-\varepsilon p_1^2} p_1 e^{-(p_1+k_1)^2 t} dp_1 \int e^{-\varepsilon p_2^2} e^{-(p_2+k_2)^2 t} dp_2 \right) dt
\]

Now, for $p, k \in \mathbb{R}^2$, $(p+k)^2 = p^2 + k^2 + 2p \cdot k$, so this is

\[
\int_0^a e^{-k^2 t} \left( \int e^{-\varepsilon p_1^2} p_1 e^{-(p_1+k_1)^2 t} dp_1 \int e^{-\varepsilon p_2^2} e^{-(p_2+k_2)^2 t} dp_2 \right) dt
\]

\[
= \int_0^a e^{-k^2 t} \frac{k_1^2}{\varepsilon + t} \left( \int e^{-(\varepsilon+t)p_1^2} (p_1 - \frac{k_1 t}{\varepsilon + t}) dp_1 \int e^{-(\varepsilon+t)p_2^2} dp_2 \right) dt
\]

We now split the $p_1$ integral into two pieces, and we see that the first one,

\[
\int e^{-(\varepsilon+t)p_1^2} p_1 dp_1
\]

is 0 by symmetry (this is what will give us the extra convergence). We use the fact that, for $d = 1, 2$ we have

\[
\int e^{-(\varepsilon+t)p_d^2} dp_d = \frac{c}{\sqrt{\varepsilon + t}}
\]

for some constant $c$. We’ll also replace $\frac{1}{\varepsilon + t}$ and $e^{-k^2 t} \frac{k_1^2}{\varepsilon + t}$ by the trivial bound of 1. This shows us that we can bound (4.21) by

\[
c^2 |k_1| \int_0^a \frac{1}{\varepsilon + t} dt
\]

Since $a < T$, this is $|k| O(\log(1/\varepsilon))$, independently of $a$. \qed
We integrate the $\hat{p}_{0,j}$'s first, and by the previous lemma each one gives $|u_{0,j-1}|O(\log(1/\varepsilon))$ ($|u_{0,j-1}|$ is the $u_j$ which appears immediately before and after the isolated interval corresponding to $\hat{p}_{0,j}$). Equation (4.18) is thus

\begin{equation}
O(\log(1/\varepsilon))|I_0| \sum_{p_i \neq \hat{p}_{0,j}} e^{-\varepsilon p_i^2 |p_i|} \prod_{(j+1) \in I_0} |u_{(0,j)}| \prod_{p_i \neq \hat{p}_{0,j}} \prod_{j \neq I_0} dp_i
\end{equation}

Since the integrand is now positive we can extend the region of integration for the $t_i$'s to be $0 < t_i < T$ and use (4.2) to bound (4.25) by

\begin{equation}
O(\log(1/\varepsilon))|I_0| \sum_{p_i \neq \hat{p}_{0,j}} e^{-\varepsilon p_i^2 |p_i|} \prod_{(j+1) \in I_0} |u_{(0,j)}| \prod_{j \neq I_0} \prod_{p_i \neq \hat{p}_{0,j}} dp_i
\end{equation}

Suppose that $u_{m,j}$ is an isolated interval in $K_m$. Then $u_{m,j-1} = u_{m,j+1}$. We will say in this case that $u_{m,j-1}$ contains $u_{m,j}$. If $u_{m,j} = u_{m,j+1}$, where $m > m'$, and $u_{m',j+1}$ is isolated in $K_m'$, we will also say that $u_{m,j}$ contains $u_{m',j+1}$. We will let $l_{m,j}$ denote the total number of isolated intervals which the interval $u_{m,j}$ contained in all $K_m'$'s, where $m' < m$. Each $u_{1,j}$ which contained one or more isolated intervals in $K_0$ will appear to a power $l_{1,j}$ in the numerator of (4.26) as a result of Lemma 7, but the term $(1 + u_{2,j}^2)$ will also appear an extra $l_{1,j}$ times in the denominator. We see that (4.26) is

\begin{equation}
O(\log(1/\varepsilon))|I_0| \sum_{p_i \neq \hat{p}_{0,j}} e^{-\varepsilon p_i^2 |p_i|} \prod_{(j+1) \in I_0} |u_{(0,j)}| \prod_{j \neq I_0} \prod_{p_i \neq \hat{p}_{0,j}} \prod_{1 \leq j \leq n_1} \prod_{p_i \in K_1} dp_i
\end{equation}

We must have some idea how the integral (4.26) can be bounded as we remove successive stages of isolated intervals, and Lemma 8 below gives us that. The following lemma prepares us to prove Lemma 9.

**Lemma 8**

\begin{equation}
\int e^{-\varepsilon p^2 |p|} \prod_{p_i \in K_1} dp_i \prod_{1 \leq j \leq n_1} |u_{(1,j)}|^{l_{1,j}} \prod_{p_i \in K_1} \frac{1}{(1 + u_{2,j}^2)} dp_i
\end{equation}

We must have some idea how the integral (4.28) can be bounded as we remove successive stages of isolated intervals, and Lemma 8 below gives us that. The following lemma prepares us to prove Lemma 9.

**Lemma 8**

\begin{equation}
\int e^{-\varepsilon p^2 |p|} \prod_{p_i \in K_1} dp_i \prod_{1 \leq j \leq n_1} |u_{(1,j)}|^{l_{1,j}} \prod_{p_i \in K_1} \frac{1}{(1 + u_{2,j}^2)} dp_i
\end{equation}

if $m = 3$, and is $(1 + |k|)O(1)$ if $m > 3$. 28
Proof: \((4.28)\) is bounded by

\[
\int e^{-\varepsilon(p-k)^2} \frac{1}{(|p| + |k|)(1 + |p|)^m} dp
\]

Divide this into two integrals. The one with \(|k|\) in the numerator is bounded by

\[
|k| \int \frac{1}{(1 + |p|)^m} dp = |k|O(1)
\]

The other is bounded by

\[
c \int e^{-\varepsilon(p-k)^2} \frac{1}{1 + |p|^{m-1}} dp
\]

Again if \(m > 3\) this is \(O(1)\). If \(m = 3\), divide the region into \(||p| > 2|k|\)| and \(||p| < 2|k|\). On \(||p| > 2|k|\) we can bound the integral by

\[
\int e^{\varepsilon p^2/2} \frac{1}{1 + p^2} = O(\log(1/\varepsilon))
\]

by Lemma 2. On \(||p| < 2|k|\) we can bound it by

\[
\int_{|p|<2|k|} \frac{1}{1 + p^2} dp \leq \log(|k| + 1) \leq |k|
\]

These bounds combine to prove the lemma.

\[\Box\]

Lemma 9 Suppose that \(K_m\) contains isolated intervals. Then \((4.17)\) is

\[
O(\log(1/\varepsilon)) \prod_{i=0}^{m} \int e^{-\frac{1}{2} \sum_{p_i \in K_{m+1}} p_i^2} \prod_{p_i \in K_{m+1}} |p_i|
\]

\[
\prod_{1 \leq j \leq n_{m+1}} (1 + \frac{1}{u_{(m+1,j)}})^{t_{m+1,j}} \frac{1}{(1 + \frac{2 + 2l_{m+1,j}}{u_{(m+1,j)}})} \prod_{p_i \in K_{m+1}} dp_i
\]

Proof: By induction. We know that it is true for \(m = 0\)(see (1.27)). Assume that it is true for \(m - 1\), so \((4.17)\) is
We will integrate the variables in $K_m$ corresponding to isolated intervals. We can rewrite the integral in (4.35) as

$$\int e^{-\varepsilon \sum_{p_i \in K_m} p_i^2} \prod_{p_i \in K_m} |p_i| \prod_{1 \leq j \leq n_m} \left(1 + |u_{(m,j)}|\right)^{l_{m,j}} \prod_{j \notin I_m} (1 + u_{(m,j)}^{2+2l_{m,j}}) \prod_{p_i \in K_{m,j}} dp_i$$

(4.36)

It is simply to verify that

$$\frac{(1 + |u_{(m,j)}|)^{l_{m,j}}}{(1 + u_{(m,j)}^{2+2l_{m,j}})} \leq K \frac{1}{(1 + u_{(m,j)}^{2+2l_{m,j}})}$$

(4.37)

For some constant $K$ depending on $l_{m+1,j}$. Each $\hat{p}_{m,j}$ integral is

$$O(\log(1/\varepsilon))$$

by Lemma 8. Plugging this into (4.36) and relabeling the $u$’s with index $m + 1$ instead of $m$ gives (4.34).

To complete the proof of Proposition 2 let us consider several cases. Recall that order refers to how many intervals $[s_i, t_i]$ make up a configuration.

**Case 1:** There is a $K_m$ of order greater than or equal to 3 which contains no isolated intervals.

In this case our integral in (4.34) is almost the same as what would have been obtained if we had started with the configuration $K_m$. The only difference is the presence of the extra powers $l_{m,j}$, which in fact cause greater convergence. Thus, by what we did earlier in this section, the remaining integral is $o(\log(1/\varepsilon))|K_m|$. Since

$$|I_0| + ... + |I_{m-1}| + |K_m| = n$$

(4.39)
we see that (4.17) is \( o(\log(1/\varepsilon)) \), which is what we set out to prove.

**Case 2:** There is a \( K_m \) of order 2 with no isolated intervals.

As before we get \( O(\log(1/\varepsilon)) \) times an integral nearly identical to what we would have had if starting with \( K_m \). Again there will be extra factors which aid convergence. The integral in question can be bounded by one of the following integrals:

\[
\int \int \frac{1}{(1 + |p|)^2(1 + |q|)^2(1 + |p + q|)^3} e^{-\varepsilon(p^2 + q^2)} |p||q| dpdq
\]

\[
\int \int \frac{1}{(1 + |p|)^2(1 + |q|)^3(1 + |p + q|)^2} e^{-\varepsilon(p^2 + q^2)} |p||q| dpdq
\]

\[
\int \int \frac{1}{(1 + |p|)^3(1 + |q|)^2(1 + |p + q|)^2} e^{-\varepsilon(p^2 + q^2)} |p||q| dpdq
\]

And therefore the following lemma completes the proof in this case.

**Lemma 10** Each of the integrals in (4.40) is \( o(\log(1/\varepsilon))^2 \)

**Proof:** This is fairly straightforward to calculate using Lemmas 2 and 8. For example, by the Cauchy-Schwarz inequality and symmetry we can bound the first integral by

\[
(4.41) \quad k \int \int \frac{1}{(1 + |p|)^4(1 + |p + q|)^3} e^{-\varepsilon(p^2 + q^2)} |p||q| dpdq
\]

The \( dq \) integral is \((1 + |p|)O(1) + O(\log(1/\varepsilon))\) by Lemma 8 and thus (4.41) is

\[
(4.42) kO(1) \int \frac{1}{(1 + |p|)^2} e^{-\varepsilon p^2} dp + O(\log(1/\varepsilon)) \int \frac{1}{(1 + |p|)^3} e^{-\varepsilon p^2} dp
\]

which is \( O(\log(1/\varepsilon)) \) by Lemma 2 and the fact that \( \frac{1}{(1 + |p|)^3} \in L^1 \).

The second and third integrals are identical with \( p \) and \( q \) interchanged, so we need only do one, let us say the second one. This is bounded by

\[
(4.43) \quad k \int \int \frac{1}{(1 + |p|)(1 + |q|)^2(1 + |p + q|)^2} e^{-\varepsilon(p^2 + q^2)} dpdq
\]

By Cauchy-Schwarz, this is bounded by
This first integral is $O((\log(1/\epsilon))^2)$ by Lemma [2], and the second one is $O(\log(1/\epsilon))$, using Lemma [2] in conjunction with the fact that

\[ \left( \int \int \frac{1}{(1+|p|)^2(1+|q|)^2} e^{-\epsilon(p^2+q^2)} dp dq \right)^{1/2} \left( \int \int \frac{1}{(1+|q|)^2(1+|p+q|)^4} e^{-\epsilon(p^2+q^2)} dp dq \right)^{1/2} \tag{4.44} \]

As a simple alternate proof, one can recall our proof for the case with no isolated intervals where we constructed the sets $A$ and $B$. Here it is simple to verify in each case that we can form two sets with the same properties. The lemma is then proved by the reasoning in steps (4.13) through (4.16) above.

Case 3: There is a $K_m$ consisting of just one interval.

Here we must examine in closer detail the proof of Lemma [M]. First of all, if there was ever an isolated interval in some $K_{m'}$ which contained two or more isolated intervals in $K_{m'-1}$, then the variable corresponding to that interval, say $u_{(m',j)}$, would have had $l_{m',j} \geq 2$. In that case, by Lemma [N] the contribution to (4.36) of the $\hat{p}_{m',j}$ integral is $O(1)(1 + |u_{(m',j)} - \hat{p}_{m',j}|)$.

We see that we can replace the term $O((\log(1/\epsilon))^{|I_0|+\ldots+|I_{m-1}|} + \log(1/\epsilon))^{|I_0|+\ldots+|I_{m-1}|}_0$ in (4.34) with $O((\log(1/\epsilon))^{|I_0|+\ldots+|I_{m-1}|})$, which will finish the proof. Thus we need only consider the case where $s_1 < s_2 < \ldots < s_n < t_n < \ldots < t_2 < t_1$. In this case, consider what happens as we remove the first three intervals (recall that we are assuming that there are at least three intervals). After removing $(s_n, t_n)$ and then $(s_{n-1}, t_{n-1})$ we have

\[ (1 + |u_{(2,n-2)}| + O(\log(1/\epsilon))) \prod_{1 \leq j \leq n_2} \frac{1}{(1 + u_{(2,j)})} \prod_{1 \leq i \leq n-2} dp_i \]

Note that $u_{2,n-2} = p_1 + \ldots + p_{n-2}$. We can expand this into two integrals, namely

\[ O(\log(1/\epsilon)) \int e^{-\epsilon} \sum_{1 \leq i \leq n-2} p_i^2 \prod_{1 \leq i \leq n-2} |p_i| \]

\[(4.46)\]
\[(4.47) \quad O(\log(1/\varepsilon)) \int e^{-\varepsilon \sum_{1 \leq i \leq n-2} p_i^2} \prod_{1 \leq i \leq n-2} |p_i| (1 + |u_{(2,n-2)}|) \prod_{1 \leq j \leq n_2} \frac{1}{1 + u_{(2,j)}^2} \prod_{1 \leq i \leq n-2} dp_i \]

and

\[(4.48) \quad O(\log(1/\varepsilon))^2 \int e^{-\varepsilon \sum_{1 \leq i \leq n-2} p_i^2} \prod_{1 \leq i \leq n-2} |p_i| \prod_{1 \leq j \leq n_2} \frac{1}{1 + u_{(2,j)}^2} \prod_{1 \leq i \leq n-2} dp_i \]

The integral in \[(4.47)\] is \(O((\log(1/\varepsilon))^{n-2})\) by the same technique as was used to prove Lemma 9. Thus, \[(4.47)\] is \(O((\log(1/\varepsilon))^{n-1})\). As for \[(4.48)\], when we remove the next interval, \((s_{n-2}, t_{n-2})\), we have no powers of \(|u_{2,n-2}|\) in the numerator, and by Lemma 8 we do not pick up an \(O(\log(1/\varepsilon))\) term. Thus, \[(4.48)\] is \(o((\log(1/\varepsilon))^n)\) as well. This completes the proof of Proposition 2.

5 Completing the proof

All that remains is to prove that the processes \(\alpha'(T)\) are tight and that the limit process has independent increments. Both are essentially corollaries of the following lemma:

Lemma 11 If \(b \leq c\), then \((\log(1/\varepsilon))^{-1}\alpha'(\mathcal{I}[a, b] \times [c, d]) \longrightarrow 0\) in \(L^n\), for any \(n \geq 3\).

**Proof:** To compute \(E[\alpha'(\mathcal{I}[a, b] \times [c, d])]\), we multiply the integrals together as before (see \[(2.4)\]). Now, however, we have \(s_i \leq b \leq c \leq t_i\) for all \(i\), and it follows from this that the only configurations of intervals that can appear here are ones containing just one component of order \(n\). We have shown that these components contribute \(o(\log(1/\varepsilon))\) to the the expectation, and this is enough to prove the lemma.

Now that we have this lemma, we can show that the processes \(\alpha'(T)((\log(1/\varepsilon))^{-1}\) are tight. We will show that
(5.1) \[ E[(\log(1/\varepsilon))^{-1}(\alpha'_\varepsilon(T) - \alpha'_\varepsilon(S))]^{2n} \leq k(T - S)^n \]

where \( k \) depends on \( n \geq 2 \) but can be chosen independently of \( \varepsilon, S, \) and \( T, \) provided they are sufficiently small. This will prove tightness by, for example, Theorem 12.3 in \([1]\). We can rewrite the left side of (5.1) as

(5.2) \[ E[(\log(1/\varepsilon))^{-1}(\alpha'_\varepsilon([0, S] \times [S, T]) + \alpha'_\varepsilon(D_T \cap \{s, t \geq S\}))]^{2n} \]

We know by the lemma that \( (\log(1/\varepsilon))^{-1}\alpha'_\varepsilon([0, S] \times [S, T]) \rightarrow 0 \) in \( L^2 \), so that (5.2) is bounded by

(5.3) \[ kE[(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(D_T \cap \{s, t \geq S\})]^{2n} \]

Suppressing the \( \log \) for the time being, this is given by

(5.4) \[ \frac{(-1)^n}{(2\pi)^{4n}} \int_{D_T^2} e^{-\varepsilon \sum_j p_j^2} \prod_{j=1}^n p_{j,1} E\left[ \prod_{j=1}^n e^{ip_j(X_{t_j} - X_{s_j})} \right] \prod_{j=1}^n ds_j dt_j d^2p_j \]

If we rewrite \( X_{t_j} - X_{s_j} \) as \( (X_{t_j} - X_S) - (X_{s_j} - X_S) \), and let \( \beta_t = X_{S+t} - X_S \) be a new Brownian motion this is

(5.5) \[ \frac{(-1)^n}{(2\pi)^{4n}} \int_{D_2^{2n}} e^{-\varepsilon \sum_j p_j^2} \prod_{j=1}^n p_{j,1} E\left[ \prod_{j=1}^n e^{ip_j(\beta_{t_j} - \beta_{s_j})} \right] \prod_{j=1}^n ds_j dt_j d^2p_j \]

which is equal to (reinserting the \( \log \))

(5.6) \[ kE[(\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(T - S)]^{2n} \]

And this is \( O(1)|T - S|^n \), as we showed earlier. This establishes tightness. We can write

(5.7) \[ (\log(1/\varepsilon))^{-1}(\alpha'_\varepsilon(T) - \alpha'_\varepsilon(S)) = (\log(1/\varepsilon))^{-1}(\alpha'_\varepsilon([0, S] \times [S, T]) + \alpha'_\varepsilon(D_T \cap \{s, t \geq S\})) \]

Since \( (\log(1/\varepsilon))^{-1}\alpha'_\varepsilon([0, S] \times [S, T]) \rightarrow 0 \) and \( \alpha'_\varepsilon(D_T \cap \{s, t \geq S\}) \in \bigcup \{\sigma(X_t - X_s) : S \leq s, t \leq T\} \), we see that \( (\log(1/\varepsilon))^{-1}\alpha'_\varepsilon(T) \) has asymptotically independent increments. This shows that the limit process, \( W_T \), has independent increments, and completes the proof of Theorem 1.
6 Symmetric stable processes

We will now prove Theorem 2. The proof of this theorem is, naturally, very similar to the proof in the Brownian motion case, so we will in many cases just refer to steps undertaken in the previous proof. In particular, the general outline (Sections 2 and 5) is identical in both cases; the only difference lies in some of the calculations.

The main difficulty is in showing that the integrals corresponding to components of order two converge. \(X_t\) is a symmetric stable process of index \(\beta\) where \(1 < \beta < 2\). The density of \(X_t\) is given by

\[
 f_\varepsilon(x) = \frac{1}{(2\pi)^2} \int e^{ipx-\varepsilon p^\beta} d^2p \tag{6.1}
\]

Thus,

\[
 f'_\varepsilon(x) = \frac{i}{(2\pi)^2} \int p_1 e^{ipx-\varepsilon p^\beta} d^2p \tag{6.2}
\]

Proceeding as in section 3, the first integral is

\[
 \int \int \frac{(1 - e^{-p^\beta}) (1 - e^{-q^\beta}) (1 - e^{-(p+q)^\beta})}{p^\beta q^\beta (p+q)^\beta} e^{-\varepsilon(p^\beta+q^\beta)} p_1 q_1 dpdq \tag{6.3}
\]

\[
 = \varepsilon^{3-6/\beta} \int \int \frac{(1 - e^{-p^\beta/e}) (1 - e^{-q^\beta/e}) (1 - e^{-(p+q)^\beta/e})}{p^\beta q^\beta (p+q)^\beta} e^{-(p^\beta+q^\beta)} p_1 q_1 dpdq
\]

In order to prove that this integral converges as \(\varepsilon \rightarrow 0\), it is enough to show that

\[
 \int \int \frac{1}{p^{\beta-1} q^{\beta-1}} \frac{1}{(p+q)^\beta} e^{-(p^\beta+q^\beta)} dpdq \tag{6.4}
\]

converges, and then to apply the dominated convergence theorem. We need only consider the integral over \(\{|p|, |q| < 1\}\), for in order to evaluate the integral over, say, \(A = \{|p| > 1\}\) we may divide \(A\) into the disjoint union of \(B = \{|q| < 1/2\} \cap A\), \(C = \{|p+q| < 1/2\} \cap A\), and \(D = A - (B \cup C)\). The integrals over \(B\) and \(C\) are both bounded because \(\beta < 2\) i.e. we have only integrable singularities. And the integral over \(D\) is bounded by a constant times

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\[(6.5) \quad \int \int e^{-(p^2 + q^2)} dp dq < \infty \]

So we must consider the integral

\[(6.6) \quad \int \int \frac{1}{p^{\beta-1}} \frac{1}{q^{\beta-1}} \frac{1}{(p + q)^\beta} dp dq \]

We manipulate the integral as follows:

\[(6.7) \quad \int_{|p|<1} \frac{1}{p^{\beta-2}} \int_{|q|<1} \frac{1}{q^{\beta-1}} \frac{1}{(p + q)^\beta} dp dq \]

The argument of \(p\) (thought of as a complex number) is irrelevant, so we may replace \(p/|p|\) by 1, and substitute \(q' = q/|p|\) to get

\[(6.8) \quad \int_{|p|<1} \frac{1}{p^{\beta-4}} \left( \int_{|q|<1/|p|} \frac{1}{q^{\beta-1}} \frac{1}{(1 + q)^\beta} dq \right) dp \]

If \(\beta > 3/2\) then the \(dq\) integral is bounded independently of \(|p|\) (since then \(\frac{1}{q^{\beta-1}} \frac{1}{(1 + q)^\beta} \in L^1\)), so that \((6.8)\) is bounded by a constant times

\[(6.9) \quad \int_{|p|<1} \frac{1}{p^{\beta-4}} dp \]

which is finite, as \(\beta < 2\). If \(\beta < 3/2\) (resp. \(\beta = 3/2\)), then the \(dq\) integral in \((6.8)\) is \(O(|p|)^{2\beta-3}\) (resp. \(O(|\log |p||)|)), so that \((6.6)\) is bounded by a constant times

\[(6.10) \quad \int_{|p|<1} \frac{1}{p^{\beta-1}} dp \]

when \(\beta < 3/2\) and

\[(6.11) \quad \int_{|p|<1} \frac{|\log(|p||)|}{p^{1/2}} dp \]

when \(\beta = 3/2\). These integrals are both finite.

The second configuration of intervals gives rise to the following:
\[ \int \int \frac{(1 - e^{-p^\beta/\varepsilon})^2 (1 - e^{-(p+q)^\beta/\varepsilon})}{p^{2\beta}} e^{-(p^\beta+q^\beta)} p_1 q_1 dp dq \]  

This integral is more difficult as for some \( \beta \) the integrand is not in \( L^1 \) were we to remove the terms involving \( \varepsilon \) (there is a non-integrable singularity at \( p = 0 \) when \( \beta \geq 3/2 \)). We will first show that (6.12) is bounded independently of \( \varepsilon \). We isolate the \( dq \) integral:

\[ \int \frac{(1 - e^{-(p+q)^\beta/\varepsilon})}{(p + q)^\beta} e^{-q^\beta} q_1 dq \]

We will show that this is \(|p|O(1)\) (the \( O \) here refers to \( \varepsilon \)). Because we will refer to this result later, we isolate it as a lemma (which we state in slightly greater generality).

**Lemma 12** For any \( a \) with \( 0 < a < T \) and any \( p \in \mathbb{R}^2 \), we have

\[ \left| \int \frac{1 - e^{-(p+q)^\beta a/\varepsilon}}{(p + q)^\beta} e^{q^\beta} q_1 dq \right| = |p|O(1) \]

independently of \( a \).

**Proof:** We can drop the \( e^{-(p+q)^\beta a/\varepsilon} \) term. (6.14) is bounded by

\[ \left| \int \frac{1}{q^\beta} e^{-(q-p)^\beta} (q_1 - p_1) dq \right| \]

Expand the \((q_1 - p_1)\) term. The second term is bounded by

\[ |p| \int \frac{1}{q^\beta} e^{-(q-p)^\beta} dq \]

The integrand is bounded by the function

\[ \frac{1}{q^\beta} 1_{\{|q|<1\}} + e^{-(q-p)^\beta} 1_{\{|q|\geq1\}} \]

which is bounded in \( L^1 \) independently of \( p \). Thus, (6.16) is \(|p|O(1)\). To bound the first term we subtract

\[ \int \frac{1}{q^\beta} e^{-q^\beta} q_1 dq \]
which is 0 by symmetry. This gives us

\[
(6.19) \quad \left| \int \frac{1}{q^\beta} (e^{-(q-p)^\beta} - e^{-q^\beta}) dq \right| \\
\leq \int \frac{1}{q^\beta} |e^{-(q-p)^\beta} - e^{-q^\beta}| dq
\]

We split this up into the integral over the region \{ |q| < 2|p| \} and \{ |q| \geq 2|p| \}. The integral over the first region is bounded by

\[
(6.20) \quad k \int_{|q| < 2|p|} |q|^{1-\beta} dq \\
= k \int_0^{2|p|} r^{2-\beta} dr \\
= k|p|^{3-\beta}
\]

Here \( k \) is a constant which may change from line to line. This is \((|p| + |p|^2)O(1)\). On the region \{ |q| \geq 2|p| \} suppose first that \( e^{-(q-p)^\beta} \geq e^{-q^\beta} \).

Then

\[
(6.21) \quad |e^{-(q-p)^\beta} - e^{-q^\beta}| \\
\leq e^{-(|q|-|p|)^\beta} - e^{-q^\beta} \\
= \beta \int_{|q|-|p|}^{|q|} x^{\beta-1} e^{-x^\beta} dx \\
\leq k|p||q|^{\beta-1} e^{-(|q|-|p|)^\beta}
\]

The last inequality is the length of the interval being integrated over multiplied by a term which bounds the integrand. Plugging this into \((6.19)\) gives a bound of

\[
(6.22) \quad k|p| \int_{|q| \geq 2|p|} e^{-(|q|-|p|)^\beta} dq \leq k|p| \int_{|q| \geq 2|p|} e^{-(|q|/2)^\beta} dq = O(|p|)
\]

In the case \( e^{-(q-p)^\beta} < e^{-q^\beta} \) we have

\[
(6.23) \quad |e^{-(q-p)^\beta} - e^{-q^\beta}|
\]

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\[ \leq e^{-|q|^{\beta}} - e^{-(|q|+|p|)^{\beta}} \]
\[ = \beta \int_{|q|}^{(|q|+|p|)} x^{\beta-1} e^{-x^\beta} \, dx \]
\[ \leq k|p|(|q|+|p|)^{\beta-1} e^{-|q|^{\beta}} \]

Since \(|q| \geq 2|p|\) this is \(k|p||q|^{\beta-1}e^{-|q|^{\beta}}\). Thus, the contribution to (6.19) of this region is bounded by

\[ k|p| \int e^{-|q|^{\beta}} \, dq = |p|O(1) \]  
(6.24)

This shows that

\[ \int \frac{1}{q^\beta} e^{-(q-p)^{\beta}} q \, dq = (|p| + |p|^2)O(1) \]  
(6.25)

It is also \(O(1)\), however, since the integrand is bounded by

\[ \frac{1}{q^{\beta-1}1_{\{|q|<1\}}} + e^{-(q-p)^{\beta}} \]  
(6.26)

which is bounded in \(L^1\) independently of \(p\). So (6.25) is \(|p|O(1)\) for \(p\) small, and \(O(1)\) for \(p\) large. We conclude that (6.25) is \(|p|O(1)\) for all \(p\).  

This lemma allows us to see that (6.12) is bounded by

\[ k \int \frac{(1 - e^{-\frac{p^\beta}{\varepsilon}})^2}{p^{2\beta}} e^{-p^\beta} |p|^2 \, dp \]  
(6.27)

The extra powers of \(p\) in the numerator are enough to convert our singularity at 0 into an integrable one, and it follows that (6.27) is bounded by

\[ k \int \frac{1}{p^{2\beta}} e^{-p^\beta} |p|^2 \, dp < \infty \]  
(6.28)

We have showed that (6.12) is bounded independently of \(\varepsilon\). This alone does not show that (6.12) converges. However, convergence is proved using the same ideas, as follows. Let the value of (6.12) be denoted by \(A(\varepsilon)\). We will show that, for any \(\delta > 0\), there is an \(\varepsilon' > 0\) such that if \(0 < \varepsilon, \varepsilon_2 < \varepsilon'\) then \(|A(\varepsilon_1) - A(\varepsilon_2)| < \delta\). This will prove convergence. We will assume below that \(0 < \varepsilon_1 < \varepsilon_2 < \varepsilon'\). We have
\[
\int \int \left( \frac{(1 - e^{-p^\beta/\varepsilon_1})^2 (1 - e^{-(p+q)^\beta/\varepsilon_1})}{p^{2\beta}} - \frac{(1 - e^{-p^\beta/\varepsilon_2})^2 (1 - e^{-(p+q)^\beta/\varepsilon_2})}{p^{2\beta} (p + q)^\beta} \right) e^{-(p^\beta+q^\beta)p_1q_1} dp dq
\]

We will rewrite the difference

\[
(6.30) \quad (1 - e^{-p^\beta/\varepsilon_1})^2 (1 - e^{-(p+q)^\beta/\varepsilon_1}) - (1 - e^{-p^\beta/\varepsilon_2})^2 (1 - e^{-(p+q)^\beta/\varepsilon_2})
\]
as

\[
(6.31) \quad (1 - e^{-p^\beta/\varepsilon_1})^2 [(1 - e^{-(p+q)^\beta/\varepsilon_1}) - (1 - e^{-(p+q)^\beta/\varepsilon_2})] + [(1 - e^{-p^\beta/\varepsilon_1})^2 - (1 - e^{-p^\beta/\varepsilon_2})^2] (1 - e^{-(p+q)^\beta/\varepsilon_2})
\]

and handle each term in this sum separately. The first one gives rise to the integral

\[
(6.32) \quad \int \int \frac{(1 - e^{-p^\beta/\varepsilon_1})^2}{p^{2\beta}} \frac{e^{-(p+q)^\beta/\varepsilon_2} - e^{-(p+q)^\beta/\varepsilon_1}}{(p + q)^\beta} e^{-(p^\beta+q^\beta)p_1q_1} dp dq
\]

As in step (6.15) the \( dp \) integral is

\[
(6.33) \quad \int e^{-q^\beta/\varepsilon_2} \frac{1 - e^{-q^\beta/\varepsilon_3}}{q^\beta} e^{-(q-p)^\beta} (q_1 - p_1) dq
\]

where \( \varepsilon_3 = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 - \varepsilon_1} > 0 \) This is in turn bounded by

\[
(6.34) \quad \int e^{-q^\beta/\varepsilon'} \frac{1 - e^{-q^\beta/\varepsilon_3}}{q^\beta} e^{-(q-p)^\beta} (q_1 - p_1) dq
\]

We may now follow steps (6.15) through (6.28), and it is straightforward to verify in each case that the extra \( e^{-q^\beta/\varepsilon'} \) term allows us to replace the \( O(1) \) by \( o(1) \) (the \( o \) now refers to \( \varepsilon' \)). This implies that (6.32) can be made arbitrarily small by choosing \( \varepsilon' \) sufficiently small. As for the second integral
We can rewrite \((1 - e^{-\beta/\varepsilon_1})^2 - (1 - e^{-\beta/\varepsilon_2})^2\) as
\[
(1 - e^{-\beta/\varepsilon_1} + (1 - e^{-\beta/\varepsilon_2}))e^{-\beta/\varepsilon_2}(1 - e^{-\beta/\varepsilon_3})
\]
and we see that we can bound (6.35) by
\[
k \int e^{-\beta/\varepsilon'} |p| e^{-\beta} \sum_{i} \prod_{j} |p_i| e^{-u_j \beta} \prod_{j} \prod_{i} dp_i
\]
We have shown above that the \(dq\) integral is \((|p| + |p|^2)O(1)\), and that this implies that the entire integral converges. Furthermore, as \(\varepsilon' \to 0\), the dominated convergence theorem implies that the value of the integral approaches zero. Again we see that if we choose \(\varepsilon'\) sufficiently small we can make (6.35) arbitrarily small. This shows that if \(\varepsilon_n\) is a sequence converging to zero then \(A(\varepsilon_n)\) converges. Thus, \(\lim_{\varepsilon \to 0} A(\varepsilon)\) exists, and we define
\[
\int \int \frac{1}{p^{2\beta}} \frac{1}{(p + q)\beta} e^{-(p^2 + q^2)} p_1 q_1 dp dq
\]
to be this limit. This completes the calculation for components of order 2.

For a component of order \(n \geq 3\) we have the following integral:
\[
\int e^{-\varepsilon \sum p_i^\beta} \prod (p_i)_1 \left( \int \sum_{c_j < T} \prod_j e^{-u_j \beta} c_j \prod_{j} dc_j \right) \prod_{i} dp_i
\]
We must show that this is \(o(e^{-(3n/\beta - 3n/2)})\). This would be a bit of a chore were it not that we have done almost all of the work already in the Brownian motion case. For instance, suppose we have a configuration with no isolated intervals. Then (6.39) can be bounded by (see (6.34))
\[
\int \frac{e^{-\varepsilon \sum p_i^\beta} \prod |p_i|}{\prod_{j=1}^{2n-1} (1 + |u_j|)^\beta} \prod dp_i
\]
We are done if we can bound this integral effectively. We know from earlier work that if \(\beta\) were replaced by 2 in this integral then it would be
\( O(\log(1/\varepsilon))^n \), which is certainly good enough. We can bound as follows using Holder’s inequality:

\[
(6.41) \quad \int \frac{e^{-\sum \beta p_i} \prod |p_i| \prod dp_i}{\prod_{j=1}^{2n-1} (\varepsilon + |u_j|)^\beta} \leq \left( \int \frac{e^{-\sum \beta p_i} \prod |p_i| \prod dp_i}{\prod_{j=1}^{2n-1} (\varepsilon + |u_j|)^\beta} \right)^{\beta/2} \left( \int \frac{e^{-\sum \beta p_i} \prod |p_i| \prod dp_i}{\prod_{j=1}^{2n-1} (\varepsilon + |u_j|)^\beta} \right)^{(2-\beta)/2}
\]

A quick examination of the proofs of Lemmas 2, 3, 8, and 10 will show that the conclusions of these lemmas remain valid if any \( e^{-p^2} \)’s in the hypotheses are replaced by \( e^{-p^\beta} \). We can conclude that (6.41) is \( O(\log(1/\varepsilon))^n \), and this component is therefore sufficiently bounded. We do the same thing in the isolated interval case, with Lemma 7 replaced by Lemma 12. This completes the proof of Theorem 2.

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