Anomalous Anharmonicity in Doped Cuprate Superconductors

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Abstract

Strong nonlinear elastic effects arise due to the Lifshitz transition associated with a change in the Fermi surface topology. It is shown that an electronic contribution to the elastic characteristics of cuprates becomes important if the chemical potential is near but not too near the Van Hove critical energy \( \varepsilon_c \). In the case of a saddle point singularity in the energy spectrum an enhancement factor arises for the "electronic" anharmonic constants of the 3rd and 4th order, that is proportional to \( \varepsilon_F/(\varepsilon_F - \varepsilon_c) \) to the 1st and 2nd power, respectively. The behaviour of the Grueneisen coefficient in the vicinity of the Lifshitz transition is considered.

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High $T_c$ cuprate superconductors show unusual elastic properties in the normal as well as in the superconducting states. There is experimental evidence of a lattice softening in the normal state of $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ (LSCO) at $x = 0.14$ [1]. At the same time an unexpected lattice stiffening has been observed in the superconducting state. For $\text{YBa}_2\text{Cu}_3\text{O}_x$ (YBCO) the values of the Grueneisen parameter determined by ultrasonic techniques are spread over an unusually broad range [2]. The decrease of the Grueneisen parameter with $x$ observed in [3] was recently confirmed by the oxygen concentration dependence of Raman active phonons [3]. It is of interest to consider the unusual behaviour of the Grueneisen coefficient as a function of the oxygen content. As we will show below, a reason for such dependence may be a change of electronic structure with doping. In this work we demonstrate how the elastic properties depend on the peculiarities of the electronic structure of the doped cuprate superconductors. In particular we investigate the modification of the thermodynamic and elastic properties due to the change of the Fermi surface (FS) topology in the nonsuperconducting metallic state.

An essential aspect of the electronic band structure of the cuprate superconductors is the quasi two-dimensionality. Van Hove [4] has shown that, in the 2D case, inside each band the electron density of states has at least one infinite logarithmic singularity (at the critical energy $\varepsilon_c$). The influence of the Van Hove singularities on the properties of the high-$T_c$ superconductors has been widely studied [5]. The effect of a Lifshitz electronic topological transition (LETT) in the 2D electron system of cuprates has also attracted considerable attention [6], [7], as this presents the possibility for a unified approach to studying the anomalies in both normal and superconducting states. The LETT occurs when the Fermi energy $\varepsilon_F$ passes through $\varepsilon_c$, and a change in the FS topology takes place [8]. The LETT in the 2D case for conventional metals has been investigated in [7]. The angular resolved photo-emission experiments [8] have revealed that $\varepsilon_c$ is located close to $\varepsilon_F$ in the "optimally" doped cuprate superconductors. The possibility of the Lifshitz transition in the cuprates was suggested by Freeman [9], and it was considered recently in the high-$T_c$ superconductors by means of a microscopic $t - t' - J$ model, [10].
The metallic properties of cuprates are highly peculiar leading up to a deviation from the Fermi liquid (FL) behaviour of electrons \[11\]. In particular, a dependence of magnetic susceptibility and Hall’s constant on temperature, as well as the linear dependence of the resistivity on \(T\) for HTCS materials with the hole conductivity have been observed. Different aspects of these anomalies have been studied in connection with the characteristic features of the high \(T_c\) superconductors in normal phase, as it is customary to assume they define a behaviour of cuprates in superconducting state. There exists a pseudogap in the underdoped regime above \(T_c\), a rise of the localized magnetic moments and strong magnetic fluctuations over a wide doping range outside an area of antiferromagnetic phase. These effects can be considered as a manifestation of the strong electron correlations in cuprates. Therefore, a simple mean field treatment is not possible in this case.

At the same time the appearance of various anomalies in the thermal and electronic properties of the free electron system due to the Lifshitz transition may be responsible for the anomalies in the interacting electron system. In this work we wish to focus our attention on the anomalies of the thermodynamic and elastic characteristics of cuprates which come from the peculiarities of the 2D electronic structure and layered anisotropy. So, we suppose that the FL approach is a suitable starting point as applied to normal metallic state of cuprates at some distance from the ”optimal” doping. Below we discuss the influence of the electron interaction and a renormalization of the electron spectrum in cuprates.

In the conventional metals a topological transition leads to strong nonlinear effects \[12\]. One can expect that elastic anomalies due to the LETT arise in the cuprate materials also. For simplicity, our investigation is performed for the 2D case. As well, we restrict our consideration to the case of a simple saddle point in band structure of the YBCO and LSCO compounds.

To derive the behaviour of the elastic constants in the metallic normal-state of doped cuprates it is necessary to determine how a basic parameter of the LETT, the difference \(z = \mu - \varepsilon_c\), depends on the strain tensor \(u_{ij} (i, j = 1, 2)\) of the 2D crystal. The renormalization of the chemical potential \(\mu\) (at \(T = 0\)) due to deformation has been given in \[12\] and it
can be easily generalized to the 2D case. Then, we obtain for the deformed crystal

\[ z = z_0 + \Lambda_{ij} u_{ij}, \]  

(1)

where \( z_0 = (\varepsilon_F - \varepsilon_c)|_{u_{ik}=0} \), \( \Lambda_{ij} = \lambda_{ij} - \lambda_{ji} \), \( \lambda_{ij}(\vec{k}) = \lambda_{ji}(\vec{k}) \) is the tensor of the deformation potential at \( u_{ij} = 0 \), \( \vec{k} \) is an electron wave vector, \( \lambda_{ij} \) denotes its average over the FS. The Lifshitz transition point is defined by the condition \( z = 0 \). We suppose that the term \( z_0 \) can be varied by doping so that it becomes sufficiently small. In addition, we consider an elastic field of deformation in the form of a sound wave. Then the transition parameter \( z \) in cuprates consists of a static and a dynamic part \[13\].

The electron density of states (DOS) in the 2D system is given by the expression

\[ \nu(\varepsilon) = 2S \frac{d}{d\varepsilon} \int \theta(\varepsilon(\vec{p}) - \varepsilon) \frac{d^2\vec{p}}{(2\pi\hbar)^2}, \]  

(2)

where \( S \) is the area occupied by the electrons, \( \theta(x) = 1 \) for \( x < 0 \), and 0 else is a step-function. In the case of a simple saddle point, near the Van Hove critical energy \( \varepsilon_c \), in the 2D band, the electron dispersion relation may be written as

\[ \varepsilon(\vec{p}) = \varepsilon_c + \frac{p_x^2}{2m_1^*} - \frac{p_y^2}{2m_2^*}, \]  

(3)

where \( p_x, p_y \) are the projections of \( \vec{p} \) and \( m_1^*, m_2^* > 0 \) are the effective masses. Substituting Eq. (3) in Eq. (2), it may be easily seen that a logarithmic singularity arises in \( \nu(\varepsilon) \). We decompose the electron density of states into a regular part \( \nu_0(\varepsilon) \) and a singular part \( \delta\nu(\varepsilon) \): \( \nu(\varepsilon) = \nu_0(\varepsilon) + \delta\nu(\varepsilon) \). The singular part results from the integration over the Fermi surface in the break-off region

\[ \delta\nu(\varepsilon) = -\beta \ln(|\varepsilon - \varepsilon_c|/\varepsilon_c), \]  

(4)

where \( \beta = S\pi^{-2}\hbar^{-2}(|m_1^*m_2^*|)^{1/2} \). For the sake of definiteness, we will consider the case when an equienergy surface neck appears with increasing energy, then the sign of a singular part of the electron density of states is positive, \( \delta\nu(\varepsilon) > 0 \). It is easy to verify that in the case of the neck disruption, as the energy grows, a singular part \( \delta\nu(\varepsilon) \) has the opposite sign relatively
to Eq. (4), \(\delta \nu(\varepsilon) < 0\).

The free energy \(F\) can be also decomposed into a regular and a singular part: \(F = F_0 + \delta F\). \(\delta F\) due to electronic transition may be represented in terms of \(z = \mu - \varepsilon_c\), where \(\mu\) is the chemical potential, \([7]\):

\[
\delta F = \beta \frac{z^2}{2} \ln \frac{|z|}{\varepsilon_c} \left[ 1 + \frac{\pi^2 T^2}{3} \right],
\]

(5)

here we suppose that \(\mu \gg z \gg T\) (temperature is in energy units). \(F_0\) is assumed to be an analytical function of \(z\). We may neglect the temperature dependence of the elastic constants in the considered temperature range.

The standard equations of the theory of elasticity are

\[
\rho \ddot{u}_i = \partial \sigma_{ij} / \partial x_j, \quad \sigma_{ij} = \sigma_{ij}^{\text{latt}} + \sigma'_{ij},
\]

(6)

where \(\rho\) is a mass density, \(u_i\) is a displacement vector. The lattice term \(\sigma_{ij}^{\text{latt}}\) holds the linear contribution \(\lambda_{ijklm} u_{ij}\), where \(\lambda_{ijklm}\) is a tensor of elastic moduli, and the conventional anharmonic terms of expansion. The electronic term \(\sigma'_{ij} \sim \partial \delta F / \partial u_{ij}\) contains a logarithmic singularity,

\[
\sigma'_{ij} = \beta_0 z \Lambda_{ij} \ln \frac{|z|}{\varepsilon_c} + \frac{\beta_0}{2} z \Lambda_{ij},
\]

(7)

The constant \(\beta_0 = \beta/V\) is of the order \(\sim \nu_0(\varepsilon_F)/\pi^2 v_0\), where \(\nu_0(\varepsilon_F)\) is the regular part of density on the FS, \(v_0\) is the volume of the unit cell of the crystal.

There are two different limiting cases from Eq. (1): (i) \(|z_0| = |\varepsilon_F - \varepsilon_c|u_{ij}=0\) is comparable with \(|\Lambda_{ij}u_{ij}|\) in magnitude, or (ii) far exceeds it. We consider the latter case, when the sound wave propagates along the metallic phase of the cuprates, and the values \(u_{ij}\) are such that

\[
|z_0| \gg |\Lambda_{ij}u_{ij}|.
\]

(8)

Then, using the expansion of \(\delta F\), Eq. (5), with respect to the small parameter \(|\Lambda_{ij}u_{ij}|/|z_0| \ll 1\), (here the singular terms at \(|z_0|/\varepsilon_c \ll 1\) are retained)

\[
\delta F \simeq \frac{\beta}{2} (\Lambda_{ij}u_{ij})^2 \ln \frac{|z_0|}{\varepsilon_c} \pm \frac{\beta}{6} (\Lambda_{pq}u_{pq})^3 \pm \frac{\beta}{24} (\Lambda_{lm}u_{lm})^4 z_0^2,
\]

(9)
the electronic stress tensor, Eq. (7), may be expanded up to the second order terms as:

\[
\sigma'_{ij} = \beta_0 \Lambda_{ij} \Lambda_{lm} u_{lm} \ln \frac{|z_0|}{\varepsilon_c} \pm \frac{\beta_0}{2} \Lambda_{ij} (\Lambda_{lm} u_{lm})^2.
\]  

(10)

Here the upper sign corresponds to the region \(z_0 > 0\) where the Fermi surface neck exists, and a lower sign refers to region \(z_0 < 0\) where the neck has been disrupted. The quadratic term in Eq. (10) has the same form as the lattice contribution due to the cubic anharmonicity. Using Eq. (9) and the estimate \(|\Lambda_{ij}| \sim \Lambda \delta_{ij}\), where \(\Lambda > 0\) is of the order of the deformation potential constant, the ”electronic” anharmonic constants of 3rd and 4th order, \(\Gamma_3, \Gamma_4\), can be written as

\[
\Gamma_3 \cong \pm \frac{\beta_0}{6} \Lambda^2 |\eta|, \quad \Gamma_4 \cong -\frac{\beta_0}{24} \Lambda^2 \eta^2,
\]

(11)

where \(|\eta| = \Lambda/|z_0| \sim \varepsilon_F/|\varepsilon_F - \varepsilon_c| \gg 1\); \(\Lambda > \varepsilon_F\) as we shall discuss below. The case considered here could be formally interpreted as an increase of the anharmonic constants \(\Gamma_3, \Gamma_4\) with an enhancement factor \(\eta\), to the corresponding power. Thus the electronic contribution to the anharmonicity increases as the LETT point is approached.

In conventional metals \(\Lambda\) is usually of the same order as \(\varepsilon_F\) [12]. Since a value of \(\varepsilon_F\) in cuprates is well below, at first sight it would seem that the influence of electrons on elastic properties of these materials is modest. But in the metallic state of cuprates, owing to a relatively low density of carriers and a poor screening (compared to the standard metals), the Coulomb energy of interaction is large, and can cause the value of \(\Lambda\) to increase. The quasi-two dimensionality of the electronic structure in cuprate superconductors also provides an explanation for \(\Lambda\) increasing. For the quantitative estimates of \(\Lambda\) we used a familiar procedure from the method of deformation potential [14]. Hence we take into account that the electrons of conductivity in cuprates are basically localized in the \(CuO_2\) planes. Using an effective density of electronic states per atom of \(Cu\), \(\nu_0 (\varepsilon_F) \cong 4.8\ (eV \cdot \text{at.Cu})^{-1}\) in the case of \(YBCO\) and \(\nu_0 (\varepsilon_F) \cong 4.14\ (eV \cdot \text{at.Cu})^{-1}\) for \(LSCO\), [15], we obtain \(\Lambda \cong 1.25\) eV and \(\Lambda \cong 0.82\) eV, respectively, which are larger than the accepted values of \(\varepsilon_F\) in these systems.
The relation between the energy distance \( z_0 \) and the doping parameter \( x \) is of great importance \[16\]. For simplicity, we treat the dependence of \( z_0 \) on the doping \( x \) as an empirical fact. The dopant dependence of \( z_0 \) has been estimated from experimental data for the chemical potential shift in doped \( La_{2-x}Sr_xCuO_4 \), \[17\], and \( YBa_2Cu_3O_x \), \[18\]. From the above-mentioned estimates we have obtained the following values of the "electronic" anharmonic constants in \( LSCO \) for \( |z_0| \sim 100K: |\Gamma_3| \approx 1.33 \text{GPa}, \Gamma_4 \approx -30 \text{GPa}; \) in \( YBCO \) for \( |z_0| \sim 250K: |\Gamma_3| \approx 14 \text{GPa}, \Gamma_4 \approx -189 \text{GPa}. \) The dependence of "electronic" anharmonic elastic constants of 3rd and 4th order for compounds \( LSCO \) and \( YBCO \) on doping are shown in Fig.1 (b), (d). A dependence of \( \Gamma_3, \Gamma_4 \) on doping, Eq. \[11\], and the asymmetry of \( \Gamma_3 \) relatively to the point \( z_0 = 0 \) offer a reason for a distinction between a metallic phase of cuprates and the conventional metals in the kinetic characteristics and transport properties. Also, the anharmonic dynamics is highly sensitive to the ion mass \[15\], so that one can expect a modification of isotope shift with doping.

The Grueneisen constant is defined as the ratio of the full coefficient of thermal expansion \( \alpha_V \) to the full specific heat of the crystal at constant volume \( C_V \)

\[
\gamma = \frac{V \alpha_V B_T}{C_V}, \quad \text{or} \quad \gamma = \frac{V}{T} \left( \frac{\partial^2 F}{\partial T \partial V} \right) / \frac{\partial^2 F}{\partial T^2}, \quad (12)
\]

where \( B_T \) is the isothermal bulk modulus and \( F \) is the free energy. The spectrum of oscillations in cuprates as anisotropic layered crystal structure is characterized by more that one Debye temperature. In particular, a temperature range, where the interaction between the layers is not essential, is \[19\],

\[
\left( \frac{E_\perp}{E_\parallel} \right)^{1/2} \Theta \ll T \ll \Theta, \quad E_\perp \ll E_\parallel, \quad (13)
\]

here \( \Theta = \pi \hbar s_\parallel / a \) is a characteristic temperature, corresponding to a velocity \( s_\parallel \) of propagation of sound wave in the \( ab \)-plane of layer \( CuO_2 \), \( s_\parallel = (E_\parallel / \rho)^{1/2} \); \( a \) is a cell parameter, \( E_\parallel \) and \( E_\perp \) are the Young’s moduli in the plane \( ab \) (\( \lambda_{xxxx} \)) and along the \( c \)-axis (\( \lambda_{zzzz} \)) accordingly. Let us estimate the temperature range defined by Eq. \[13\]. In the case of \( YBa_2Cu_3O_7 \), using \( E_\parallel = 2.30 \times 10^{11} \text{Pa}, \ E_\perp = 1.60 \times 10^{11} \text{Pa} \) \[20\], and a mass density
\[ \rho = 6.4 \text{g/cm}^3 \text{, we obtain } \Theta = 356 \text{K and } (E_\perp/E_\parallel)^{1/2} \Theta = 296 \text{K. For LSCO respectively, using } E_\parallel = 2.48 \times 10^{11} \text{Pa, } E_\perp = 2.05 \times 10^{11} \text{Pa} \text{, } \rho = 6.88 \text{g/cm}^3 \text{, we find } \Theta = 361 \text{K and } (E_\perp/E_\parallel)^{1/2} \Theta = 328 \text{K. Notice that the temperature boundaries in Eq. (13) in turn themselves depend on the doping.} \]

A singular part of the lattice Grueneisen parameter \( \delta \gamma^{ph} \) can be obtained by estimating the corrections to the squared phonon frequency averaged over the Brillouin zone, resulting from the topological transition. Our estimates have shown that the latter are relatively small. Much more sensitivity to the doping in the region of the LETT was found for the electronic Grueneisen coefficient, \( \gamma^e \). To calculate \( \gamma^e \), we shall present, as previously, the free energy as a sum \( F = F_0 + \delta F \), then \( \gamma^e = \gamma^e_0 + \delta \gamma^e \).

The regular parts of the electronic specific heat \( C_V \) and the thermal coefficient of pressure are

\[
C_V = \frac{\pi^2}{3} \nu_0(\varepsilon_F)T; \quad \left( \frac{\partial P}{\partial T} \right)_V = \frac{\pi^2}{3} \frac{\partial \nu_0(\varepsilon_F)}{\partial V} T. \tag{14}
\]

According to Eq. (13), the singular part of the electronic specific heat follows as

\[
\delta C_V = -T \frac{\partial^2 \delta F}{\partial T^2} \approx -\frac{\pi^2 \beta_0}{3} T \ln |\eta| \varepsilon_c \Lambda. \tag{15}
\]

The singular part of the thermal coefficient of pressure is

\[
\frac{1}{T} \delta \left( \frac{\partial P}{\partial T} \right)_V = -\frac{1}{T} \frac{\partial^2 \delta F}{\partial T \partial V} \approx -\frac{\pi^2 \beta_0}{3} \eta. \tag{16}
\]

Using Eqs. (12), (14)–(16) and detaching a standard electronic Grueneisen coefficient \( \gamma^e_0 = \frac{\nu_0}{\nu_0} \frac{\partial \ln \nu_0}{\partial \ln V} = \partial \ln \nu_0/\partial \ln V \), we obtain the electronic contribution to the Grueneisen parameter near the point of the LETT as

\[
\gamma^e \approx \gamma^e_0 / c(\eta) + \delta \gamma^e, \quad \delta \gamma^e = -|\eta| / \pi^2 c(\eta), \tag{17}
\]

where \( c(\eta) = (1 + \frac{1}{\pi^2} \ln \frac{|\eta| \varepsilon_c}{\Lambda})(1 + C^ph_V/C^e_V) \). The dependence of the singular electronic correction \( \delta \gamma^e \) due to the proximity to the LETT on doping accordingly to Eq. (17) is plotted in Fig.1 (a), (c). Under certain conditions its variations may lead to negative values of the
Grueneisen coefficient. In particular, important point is the relation between the phonon and electronic specific heats $C_{V}^{ph}/C_{V}^{e}$ which enters in Eq. (17). At relatively high temperatures ($T > T_{c}$) the phonon contribution to the specific heat $C_{V}^{ph}$ is, as a rule, greater than the electronic one, $C_{V}^{e}$. But in the temperature range Eq. (13), where $C_{V}^{ph} \sim T/\Theta$ due to the layered anisotropy [19], $C_{V}^{ph}$ is comparable with $C_{V}^{e}$. For the metallic normal state of $LSCO$ another low temperature interval where $C_{V}^{e} \gg C_{V}^{ph}$ is available: $T \ll \left( E_{\perp}/E_{\parallel} \right)^{3/4} \Theta(\Theta/\varepsilon_{F})^{1/2}$. This case is the only one plotted in Fig.1 (a).

Let us take up the assumptions underlying the results obtained above. We assumed $\varepsilon_{F} \gg |z| \gg T$. The restriction $\varepsilon_{F} \gg |z|$ is obvious from setting up a problem, since the ratio $|z|/\varepsilon_{F}$ is a dimensionless parameter that defines the distance to the transition point. From Eq. (8) it follows that $|u_{ij}| \ll |z_{0}|/|\Lambda_{ij}| \sim |z_{0}|/\Lambda \ll 1$. Thus, the present treatment is appropriate to ultrasound waves (with finite amplitudes) as well as to long-wavelength phonons. In this connection of special interest is a problem of interplay between the ”electronic” anharmonicity in plane $ab$ due to the proximity to the Lifshitz transition and a ”tilting” mode in $LSCO$, or, in case of $YBCO$, a pyramidal apex oxygen motion along the $c$-axis.

For temperatures in considered $T \ll \Theta_{D}, z_{0}$ a boundary of superconducting state gives a restriction from below, $T > T_{c}$. At $T \neq 0$ the singularities caused by the LETT are smeared out. The width of the temperature smearing is $\Delta z \sim T$. So, the obtained results are valid at temperatures such that the smearing is small: $T/|z_{0}| \sim \Delta z/|z_{0}| \ll 1$. The inequality $T/|z_{0}| \ll 1$ coincides with a condition for a degeneracy of the electron gas in the normal metallic state of cuprates. The Fermi–liquid approach in this case may be considered as a close approximation if $\lambda \ll (\Theta_{D}/T)^{2}$ [22], where $\lambda$ is an electron–phonon coupling constant. The simple estimates of $\lambda$ for $YBCO$ and $LSCO$ show that above mentioned requirement is fulfilled for these compounds at $\lambda \simeq 1.3 \div 4.2$. While a behaviour of $\lambda$ depends on the details of anharmonic interactions [23]. Taking into account the ”electronic” contribution to the anharmonic constants, Eq. (14), one can find $\lambda$ as a function of temperature [24] and doping, to refine the regions of the Fermi–liquid theory feasibility.
Up to now we considered a free electron system in the presence of the simple saddle point in the electron spectrum. Now we discuss how the fact that the electrons in the cuprates are a strongly correlated system will influence the results. Recently in [25] it was shown that in 2D interacting system the density of states deviates from the bare DOS. Particularly, for the doping range around the critical point corresponding to the LETT, it is characterised by a shifted logarithmic singularity and two jumps on the right in energy of the logarithmic divergence point. Ignoring these jumps, one can conclude that the obtained results are qualitatively not changed for a renormalized spectrum in the presence of interaction. The quantitative changes are appreciably defined by the magnitude of the deformation potential. At the same time the renormalization of the deformation potential in the system of strong correlated electrons remains to be solved and demands further investigation. In this case, it is of special interest to extend the deformation potential theorem for the electron phonon coupling due to Bardin to the electron-spin fluctuation coupling as given by Schrieffer [26].

The comparison of the anharmonic constants of 3rd and 4th order, $\Gamma_3$, $\Gamma_4$, Eq.(11), with the experimental values is not possible because of lack of data. However there are indications that the results obtained with the use of $\Gamma_3$, $\Gamma_4$ are consistent with other experimental data. Fig.1 (c) shows that a change of the singular correction, $\delta\gamma^e$, of the electronic Grueneisen coefficient in $YBa_2Cu_3O_x$, as function of oxygen content $x$ (solid curve) is in qualitative agreement with a variation of a mean acoustic Grueneisen parameter with $x$ taken from experimental data [2]. A closer comparison would require further experimental data with monocrystal samples in the proximity to the ”optimal” doping. In the case of $La_{2-x}Sr_xCuO_4$, at $x=0.14$ the experimentally observed anomaly (negative value) of the in-plane $ab$ thermal expansion coefficient $\alpha_{ab}$ [27] gives one more evidence of agreement between our results and experiment. When $\alpha_{ab}$ is expressed in terms of the Grueneisen parameter

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1 What means that we have restricted ourselves to the region of the overdoped regime.

2 A role of the jumps in the underdoped regime for elastic properties we consider elsewhere.
using Eqs. (12), (17), we obtain $\alpha | < 0$ in the required temperature range.

In summary, we have presented the dependence of the elastic properties on the singularities of the 2D electronic energy spectrum in the metallic normal state of the doped cuprate superconductors, as applied to $LSCO$ and $YBCO$ compounds. It has been shown that the proximity to the point of a topological modification of the Fermi surface gives rise to the unusual "electronic" anharmonicity, which essentially depends on the doping and may exceed the usual lattice anharmonicity. The correction to the electronic Grueneisen coefficient due to this anomalous anharmonicity was found to be negative, what is in line with the experimental data for $YBCO$ and $LSCO$. In superconducting state, "electronic" anharmonicity can be of importance in the context of an enhancement of the electron-phonon coupling to explain high $T_c$ in cuprates.

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FIGURES

(a) \( \delta \gamma' \) for La\(_{2-x}\)Sr\(_x\)CuO\(_4\) composition \( x \) vs. oxygen content \( x \).

(b) \( \delta \gamma'' \) for La\(_{2-x}\)Sr\(_x\)CuO\(_4\) composition \( x \) vs. oxygen content \( x \).

(c) \( \delta \gamma' \) for YBa\(_2\)Cu\(_3\)O\(_x\) oxygen content \( x \) vs. oxygen content \( x \).

(d) \( \delta \gamma'' \) for YBa\(_2\)Cu\(_3\)O\(_x\) oxygen content \( x \) vs. oxygen content \( x \).
FIG. 1. Dependence of singular correction $\delta \gamma^e$ on doping $x$ for LSCO (a) and for YBCO (c) near the Lifshitz transition point. In (b), (d) the similar dependence of elastic constants $\Gamma_3, \Gamma_4$ is shown. Dashed lines separate the regions with $z_0 > 0$ (left), $z_0 < 0$ (right). In (c) the open circles are the experimental points for a change of mean acoustic Grueneisen parameter with $x$ [2]. A dotted line is intended as a guide to the eye.