Virtual Neighborhood Technique for Holomorphic Curve Moduli Spaces

An-Min Li and Li Sheng
Department of Mathematics, Sichuan University Chengdu, PRC

Abstract

In this paper we use the approach of Ruan ([34]) and Li-Ruan ([18]) to construct virtual neighborhoods and show that the Gromov-Witten invariants can be defined as an integral over top strata of virtual neighborhood. We prove that the invariants defined in this way satisfy all the Gromov-Witten axioms of Kontsevich and Manin.

1 Introduction

Ruan and Tian established the theory of Gromov-Witten invariants for semi-positive symplectic manifold about 90’s: Ruan [33] first introduced a new invariant to the symplectic manifold by counting $J$-holomorphic maps from $S^2$ to a fixed semi-positive symplectic manifold. Nowadays these invariants are called Gromov-Witten invariants. Later Ruan and Tian studied the higher genus case and proved the associativity for the quantum cup product and the WDVV equations in 1997.

In the effort to remove semi-positive condition, the technology went on a significant change. There had been several different approaches to define Gromov-Witten Invariants for general symplectic manifolds, such as Fukaya-Ono [10], Li-Tian [21], Liu-Tian [22], Ruan [34], Siebert [37] and etc.

Recently, there is a great deal of interest among symplectic geometric community to re-visit the latter approach with the purpose to clean up some of issues (see [4, 5], [6, 7], [11]-[13], [26]-[29], [38]). The main complication is that the moduli space has various lower strata. How to deal with these lower strata is one of main issues discussed recently. Our idea is that if we can show that the relevant differential form decays in certain rate near lower strata, the Gromov-Witten invariants can be defined as an integral over top strata of virtual neighborhood. Therefore, all the complication of lower strata of the of virtual neighborhood can be avoided entirely.

In this paper we use the approach of Ruan ([34]) and Li-Ruan ([18]). Let us describe the main idea.

1.1 Local regularizaton

We explain the construction of local regularization for the top strata $\mathcal{M}_{g,n}(A)$. For details and for lower strata please see the section §4 and §5. Consider the universal curve over the Teichmüller space (for detail see section §2.2)

$$\pi_T : \mathcal{Q} \to \mathcal{T}_{g,n}.$$  

We assume that $n > 2 - 2g$, and $(g, n) \neq (1, 1), (2, 0)$. For any $[b_0] = [(p_0, u)] \in \mathcal{M}_{g,n}(A)$ let $\gamma_0 \in \mathcal{T}_{g,n}$ such that $\pi_M(\gamma_0) = [p_0]$, where $\pi_M : \mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$ is the projection. We choose a local slice for $\mathcal{Q}$, which gives a local coordinate chart on $U \subset \mathcal{T}_{g,n}$ and a local trivialization on $\pi_T^{-1}(U)$:

$$\psi : U \to A, \quad \Psi : \pi_T^{-1}(U) \to A \times \Sigma,$$  

$^1$partially supported by a NSFC grant

$^2$anminliscu@126.com, lshengscu@gmail.com
with $\psi(\gamma_a) = a_\alpha$, where $U \subset T_{g,n}$ is an open set. We have a continuous family of Fredholm system
\[
\left( \tilde{B}(a), \tilde{E}(a), \tilde{\partial}_{j,a} \right)
\]
 parameterized by $a \in A$. Denote by $j_a$ the complex structure on $\Sigma$ associated with $a = (j, y)$ and put $j_a := j_o$. For any $v \in B(a)$ let $b = (a, v)$ and denote $\tilde{E}(a)_b := \tilde{E}_b$. Let $b_o = (a_o, u)$, denote by $G_{b_o}$ the isotropy group at $b_o$. We can choose a $G_{b_o}$-invariant finite dimensional subspace $\tilde{K}_{b_o} \subset \tilde{E}|_{b_o}$ such that every member of $\tilde{K}_{b_o}$ is in $C^\infty(\Sigma, u^*TM \otimes \wedge^{0,1} T^*\Sigma)$ and
\[
\tilde{K}_{b_o} + \text{image}D_{b_o} = \tilde{E}|_{b_o},
\]
where $D_{b_o} = D\tilde{\partial}_{j_o,a}$ is the vertical differential of $\tilde{\partial}_{j_o,a}$ at $u$.

The Weil-Petersson metric $g_{wp}$ on $T_{g,n}$ induces a $Diff^+(\Sigma)$-invariant distance $d_A(a_o, a)$ on $A$. Set
\[
\tilde{O}_{b_o}(\delta, \rho) := \{(a, v) \in A \times \tilde{B} \mid d_A(a_o, a) < \delta, \|h\|_{j, a, k, 2} < \rho\};
\]
\[
O_{[b_o]}(\delta, \rho) = \tilde{O}_{b_o}(\delta, \rho)/G_{b_o},
\]
where $h \in W_{j, k, 2}^2(\Sigma; u^*TM)$, $v = \exp_a(h)$. Note that both $d_A$ and $\|h\|_{j, k, 2}$ are $Diff^+(\Sigma)$-invariant, we may identified $O_{[b_o]}(\delta, \rho)$ with a neighborhood of $[b_o] \in \mathcal{M}_{g,n}(A)$. We can choose $\delta$, $\rho$ so small such that there is an isomorphism
\[
P_{b_o, b} : \tilde{E}_{b_o} \rightarrow \tilde{E}_b \quad \forall \ b \in \tilde{O}_{b_o}(\delta, \rho).
\]
Now we define a thickened Fredholm system $(\tilde{K}_{b_o}, \times \tilde{O}_{b_o}(\delta, \rho), \tilde{K}_{b_o} \times \tilde{E}|_{b_o}(\delta, \rho), \Sigma)$. Let $(\kappa, b) \in \tilde{K}_{b_o} \times \tilde{O}_{b_o}(\delta, \rho)$, define
\[
S(\kappa, b) = \tilde{\partial}_{j_o,a}u + P_{b_o, b}\kappa.
\]
We can choose $(\delta, \rho)$ small such that the linearized operator $DS(\kappa, b)$ is surjective for any $b \in \tilde{O}_{b_o}(\delta, \rho)$.

### 1.2 Global regularization and virtual neighborhoods

There exist finite points $[b_i] \in \overline{\mathcal{M}}_{g,n}(A)$, $1 \leq i \leq m$, such that

1. The collection $\{O_{[b_i]}(\delta_i/3, \rho_i/3) \mid 1 \leq i \leq m\}$ is an open cover of $\overline{\mathcal{M}}_{g,n}(A)$.

2. Suppose that $\tilde{O}_{b_i}(\delta_i, \rho_i) \cap \tilde{O}_{b_j}(\delta_j, \rho_j) \neq \phi$. For any $b \in \tilde{O}_{b_i}(\delta_i, \rho_i) \cap \tilde{O}_{b_j}(\delta_j, \rho_j)$, $G_b$ can be imbedded into both $G_{b_i}$ and $G_{b_j}$ as subgroups.

Set
\[
U = \bigcup_{i=1}^m O_{[b_i]}(\delta_i, \rho_i).
\]
There is a forget map
\[
forg : U \rightarrow \overline{\mathcal{M}}_{g,n}, \quad [(j, y, u)] \mapsto [(j, y)].
\]
In Section 3, we construct a finite rank orbi-bundle $\tilde{F}$ over $U$ such that, for every $i \leq m$, $\tilde{F}|_{b_i}$ contains a copy of group ring $\mathbb{R}[G_{b_i}]$. The construction imitates Siebert’s construction.

Then we construct a bundle map $i([\kappa, b]) : F \rightarrow E$ and define a global regularization to be the bundle map $S : F \rightarrow E$
\[
S([\kappa, b]) = [\tilde{\partial}_{j,a}u] + i([\kappa, b])
\]
such that $DS$ is surjective. Denote

$$U = S^{-1}(0)|_{U}.$$  

By restricting the bundle $F$ to $U$ we have a bundle $E$ of finite rank with a canonical section $\sigma$. We call $(U, E, \sigma)$ a virtual neighborhood for $\overline{M}_{g,n}(A)$. Denote by $U^T$ the top strata of $U$. In Section 7 we prove

Theorem 1.1. $U^T$ is a smooth oriented, effective orbifold of dimension $N = \text{rank}(F) + \text{ind} DS$.

1.3 Gromov-Witten invariants

Recall that we have a natural evaluation map

$$ev_i : U^T \to M \quad (\Sigma, j, y_i, (\kappa, u)) \mapsto u(y_i)$$

for $i \leq n$ defined by evaluating at marked points. We have another map

$$\mathcal{P} : U^T \to \mathcal{M}_{g,n} \quad (\Sigma, j, y, (\kappa, u)) \mapsto (\Sigma, j, y).$$

Choose a smooth metric $h$ on the bundle $E$. Using $h$ we construct a Thom form $\Theta$ supported in a small $\varepsilon$-ball of the 0-section of $E$. The Gromov-Witten invariants are defined as

$$\Psi_{A,g,n}(K; \alpha_1, \ldots, \alpha_n) = \int_{U^T} \mathcal{P}^*(K) \wedge \prod_{j=1}^{n} ev_j^*\alpha_j \wedge \sigma^*\Theta$$  \hspace{1cm} (4)$$

for $\alpha_i \in H^*(M, \mathbb{R})$ represented by differential form and $K$ represented by a good differential form defined on $\mathcal{M}_{g,n}$ in Mumford’s sense.

In [19] and [20] we proved the exponential decay of the derivatives of the gluing maps with respect to the gluing parameter near lower strata. Using these estimates we prove in [9] and [10]

Theorem 1.2. The integral (4) is convergent.

Theorem 1.3.  (1). $\Psi_{A,g,n}(K; \alpha_1, \ldots, \alpha_n)$ is well-defined, multi-linear and skew symmetry.

(2). $\Psi_{A,g,n}(K; \alpha_1, \ldots, \alpha_n)$ is independent of the choices of forms $K, \alpha_i$ representing the cohomology classes $[K], [\alpha_i]$ and is independent of the choice of $\Theta$.

(3). $\Psi_{A,g,n}(K; \alpha_1, \ldots, \alpha_n)$ is independent of the choices of the regularization.

(4). $\Psi_{A,g,n}(K; \alpha_1, \ldots, \alpha_n)$ is independent of $J$ and is a symplectic deformation invariant.

(5). When $M$ is semi-positive, $\Psi_{A,g,n}(K; \alpha_1, \ldots, \alpha_n)$ agrees with the definition of [36].

Theorem 1.4. Suppose that $(g, n) \neq (0, 3), (1, 1)$. Let $\pi : \overline{M}_{g,n} \to \overline{M}_{g,n-1}$ be the map by forgetting the last marked point.

(1) For any $\alpha_1, \cdots, \alpha_{n-1}$ in $H^*(M, \mathbb{R})$, we have

$$\Psi_{A,g,n}(K; \alpha_1, \ldots, \alpha_{n-1}, 1) = \Psi_{A,g,n-1}(\pi_* K; \alpha_1, \cdots, \alpha_{n-1}),$$

(2) Let $\alpha_n$ be in $H^2(Y, \mathbb{R})$, then

$$\Psi_{A,g,n}(\pi^* K; \alpha_1, \cdots, \alpha_{n-1}, 0) = \alpha_n(A)\Psi_{A,g,n-1}(K; \alpha_1, \cdots, \alpha_{n-1}).$$

3
Theorem 1.5. For any \( K_1 \times K_2 \in H^*(\overline{M}_{g_1, g_2, n_1, n_2}, \mathbb{R}) \), \( \alpha_1, \ldots, \alpha_n \in H^*(M, \mathbb{R}) \), represented by smooth forms, we have

\[
\Psi(A_{g,n})(\langle \theta \rangle; (K_1 \times K_2)); \{ \alpha_i \}) = e(K, \alpha) \sum_{A=A_1+A_2} \sum_{a,b} \Psi(A_{g_1,g_1+1})(K_1; \{ \alpha_i \}_{i \leq n_1}, \beta_a) \eta^{ab} \Psi(A_{g_2,g_2+1})(K_2; \beta_b, \{ \alpha_j \}_{j > n_1}),
\]

where \( e(K, \alpha) = (-1)^{\deg(K_2)} \sum_{i=1}^{\deg(\alpha_i)} \).

We conclude that the invariants defined in this way satisfy all the Gromov-Witten axioms of Kontsevich and Manin.

Acknowledgement: We would like to thank Yongbin Ruan, Huijun Fan, Jianxun Hu and Bohui Chen for many useful discussions.

2 Preliminary

First all, we recall some results on the Deligne-Mumford moduli space \( \overline{M}_{g,n} \) of stable curves, for detail see [39], [40], [41].

2.1 Metrics on \( \Sigma \)

Let \((\Sigma, j, y)\) be a smooth Riemann surface of genus \( g \) with \( n \) marked points. In this paper we assume that \( n > 2-2g \), and \((g, n) \neq (1, 1), (2, 0)\). It is well-known that there is a unique complete hyperboloc metric \( g_0 \) in \( \Sigma \setminus \{ y \} \) of constant curvature \(-1\) of finite volume, in the given conformal class \( j \) (see [40]). Let \( \mathbb{H} = \{ \zeta = \lambda + \sqrt{-1} \mu | \mu > 0 \} \) be the half upper plane with the Poincare metric

\[
g_0(\zeta) = \frac{1}{(Im(\zeta))^2} d\zeta d\bar{\zeta}.
\]

Let

\[
\mathbb{D} = \{ \zeta \in \mathbb{H} | \Im(\zeta) \geq 1 \}
\]

be a cylinder, and \( g_0 \) induces a metric on \( \mathbb{D} \), which is still denoted by \( g_0 \). Let \( z = e^{2\pi i \zeta} \), through which we identify \( \mathbb{D} \) with \( D(e^{-2\pi}) := \{ z | |z| < e^{-2\pi} \} \). An important result is that for any punctured point \( y_i \) there exists a neighborhood \( O_i \) of \( y_i \) in \( \Sigma \) such that

\[
(O_i \setminus \{ y_i \}, g_0) \cong (D(e^{-2\pi}) \setminus \{ 0 \}, g_0),
\]

moreover, all \( O_i \)’s are disjoint with each other. Then we can view \( D_{y_i}(e^{-2\pi}) \) as a neighborhood of \( y_i \) in \( \Sigma \) and \( z \) is a local complex coordinate on \( D_{y_i}(e^{-2\pi}) \) with \( z(y_i) = 0 \). For any \( c > 0 \) denote

\[
D(c) = \bigcup D_{y_i}(c), \quad \Sigma(c) = \Sigma \setminus D(c).
\]

Let \( g' = dzd\bar{z} \) be the standard Euclidean metric on each \( D_{y_i}(e^{-2\pi}) \). We fix a smooth cut-off function \( \chi(|z|) \) to glue \( g_0 \) and \( g' \), we get a smooth metric \( g \) in the given conformal class \( j \) on \( \Sigma \) such that

\[
g = \begin{cases} g_0 & \text{on } \Sigma \setminus D(e^{-2\pi}), \\ g' & \text{on } D(\frac{1}{2} e^{-2\pi}) \end{cases}.
\]
Let \( g^c = ds^2 + d\theta^2 \) be the cylinder metric on each \( D^*_g(e^{-2\pi}) \), where \( z = e^{s+2\pi\sqrt{-1}\theta} \). We also define another metric \( g^\circ \) on \( \Sigma \) as above by glue \( g_0 \) and \( g^c \), such that
\[
g^\circ = \begin{cases} 
g_0 & \text{on } \Sigma \setminus D(e^{-2\pi}), \\
g^c & \text{on } D(\frac{1}{2}e^{-2\pi}) .
\end{cases}
\]

The metric \( g \) (resp. \( g^\circ \)) can be generalized to marked nodal surfaces in a natural way. Let \( (\Sigma, j, y) \) be a marked nodal surfaces with \( n \) nodal points \( p = (p_1, \ldots, p_n) \). Let \( \sigma : \Sigma = \sum_{\nu=1}^r \Sigma_\nu \rightarrow \Sigma \) be the normalization. For every node \( p_i \) we have a pair \( \{a_i, b_i\} \). We view \( a_i, b_i \) as marked points on \( \Sigma \) and define the metric \( g_\nu \) (resp. \( g^\circ_\nu \)) for each \( \Sigma_\nu \). Then we define
\[
g := \bigoplus_{\nu=1}^r g_\nu, \quad g^\circ := \bigoplus_{\nu=1}^r g^\circ_\nu.
\]

### 2.2 Teichmüller space

Denote by \( \mathcal{J}(\Sigma) \subset \text{End}(T\Sigma) \) the manifold of all \( C^\infty \) complex structures on \( \Sigma \), let \( \mathcal{G} \) denote the manifold of \( C^\infty \) Riemannian metrics with constant scalar curvature \(-1\) on \( \Sigma \). Denote by \( \text{Diff}^+(\Sigma) \) the group of orientation preserving \( C^\infty \) diffeomorphisms of \( \Sigma \), by \( \text{Diff}^+_0(\Sigma) \) the identity component of \( \text{Diff}^+(\Sigma) \). \( \text{Diff}^+(\Sigma) \) acts on \( \mathcal{J}(\Sigma) \) and \( \mathcal{G} \) by
\[
(\phi^*J)_x := (d\phi_x)^{-1}J_{\phi(x)}d\phi_x, \quad (\phi^*g)(x)(w, v) := g(\phi(x))(d\phi(x)w, d\phi(x)v)
\]
for all \( \phi \in \text{Diff}^+(\Sigma), x \in \Sigma, w, v \in T_x\Sigma \). There is a bijective, \( \text{Diff}^+(\Sigma) \)-equivariant correspondence between \( \mathcal{J}(\Sigma) \) and \( \mathcal{G} \):
\[
\mathcal{J}(\Sigma) \cong \mathcal{G}.
\]
Put
\[
P := \mathcal{J}(\Sigma) \times (\Sigma^n \setminus \Delta),
\]
where \( \Delta \subset \Sigma^n \) denotes the fat diagonal. The orbit spaces are
\[
\mathcal{M}_{g,n} = (\mathcal{J}(\Sigma) \times (\Sigma^n \setminus \Delta)) / \text{Diff}^+(\Sigma), \quad T_{g,n} = (\mathcal{J}(\Sigma) \times (\Sigma^n \setminus \Delta)) / \text{Diff}^+_0(\Sigma).
\]
\( \mathcal{M}_{g,n} \) is called the Deligne-Mumford space, \( T_{g,n} \) is called the Teichmüller space. The mapping class group of \( \Sigma \) is
\[
\text{Mod}_{g,n} = \text{Diff}(\Sigma) / \text{Diff}_0(\Sigma).
\]
It is well-known that \( \text{Mod}_{g,n} \) acts properly discontinuously on \( T_{g,n} \) and
\[
\mathcal{M}_{g,n} = T_{g,n} / \text{Mod}_{g,n}
\]
is a complex orbifold of dimension \( N := 3g - 3 + n \). Let \( \pi_M : T_{g,n} \rightarrow \mathcal{M}_{g,n} \) be the projection.

Consider the principal fiber bundle
\[
\text{Diff}^+_0(\Sigma) \rightarrow P \rightarrow T_{g,n}
\]
and the associated fiber bundle
\[
\pi_T : \mathcal{Q} := P \times_{\text{Diff}^+_0(\Sigma)} \Sigma \rightarrow T_{g,n},
\]
which has fibers isomorphic to \( \Sigma \) and is equipped with \( n \) disjoint sections
\[
\mathcal{Y}_i := \{[j, y_1, \ldots, y_n, z] \in \mathcal{Q} : z = y_i\}, \quad i = 1, \ldots, n.
\]
It is commonly called the universal curve over \( T_{g,n} \). The following result is well-known (cf [32]):
Lemma 2.1. Suppose that \( n + 2g \geq 3 \). Then for any \( \gamma_o = [(j_o, y_o)] \in T_{g,n} \), and any \( (j_o, y_o) \in P \) with \( \pi_T(j_o, y_o) = \gamma_o \) there is an open neighborhood \( A \) of zero in \( C^{5g-3+n} \) and a local holomorphic slice \( t = (t_0, \cdots, t_n) : A \rightarrow P \) such that

\[
t_0(o) = j_o, \quad t_i(o) = y_{i o}, \quad i = 1, \ldots, n, \tag{5}
\]

and the map

\[
A \times Diff\_o(\Sigma) \rightarrow P : (a, \phi) \mapsto (\phi^* t_0(a), \phi^{-1}(t_1(a)), \cdots, \phi^{-1}(t_n(a))
\]

is a diffeomorphism onto a neighborhood of the orbit of \( (j_o, y_o) \).

From the local slice we have a local coordinate chart on \( U \) and a local trivialization on \( \pi_T^{-1}(U) \):

\[
\psi : U \rightarrow A, \quad \Psi : \pi_T^{-1}(U) \rightarrow A \times \Sigma, \tag{6}
\]

where \( U \subset T_{g,n} \) is a open set. We call \( (\psi, \Psi) \) in (6) a local coordinate system for \( Q \). Suppose that we have two local coordinate systems

\[
(\psi, \Psi) : (O, \pi_T^{-1}(O)) \rightarrow (A, A \times \Sigma), \tag{7}
\]

\[
(\psi', \Psi') : (O', \pi_T^{-1}(O')) \rightarrow (A', A' \times \Sigma). \tag{8}
\]

Suppose that \( O \cap O' \neq \emptyset \). Let \( W \) be a open set with \( W \subset O \cap O' \). Denote \( V = \psi(W) \) and \( V' = \psi'(W) \). Then (see [32])

**Lemma 2.2.** \( \psi' \circ \Psi^{-1}|_V : V \rightarrow V' \) and \( \Psi' \circ \Psi^{-1}|_V : V \times \Sigma \rightarrow V' \times \Sigma \) are holomorphic.

The diffeomorphism group \( Diff^+(\Sigma) \) acts on \( \Sigma^n \setminus \Delta \) by

\[
\varphi^*(j, y_1, \ldots, y_n) := (\varphi^* j, \varphi^{-1}(y_1), \ldots, \varphi^{-1}(y_n)). \tag{9}
\]

It is easy to see that \( g \) is \( Diff^+(\Sigma) \)-invariant.

Let \( \overline{M}_{g,n} \) be the Deligne-Mumford compactification space, \( g_{wp} \) be the Weil-Petersson metric on \( \overline{M}_{g,n} \). Denote by \( \overline{E}_{g,n} \) the groupoid whose objects are stable marked nodal Riemann surfaces of type \( (g, n) \) and whose morphisms are isomorphisms of marked nodal Riemann surfaces. J. Robbin, D. Salamon [32] used the universal marked nodal family to give an orbifold groupoid structure on \( \overline{E}_{g,n} \). Then \( \overline{M}_{g,n} \) has the structure of a complex orbifold, and \( M_{g,n} \) is an effective orbifold. It is possible that \( (g_i, n_i) = (1, 1) \) for some smooth component \( \Sigma_i \), in this case we consider the reduced effective orbifold structure.

**2.3 The moduli space of stable holomorphic maps**

Let \( (M, \omega, J) \) be a closed \( C^\infty \) symplectic manifold of dimension \( 2m \) with \( \omega \)-tame almost complex structure \( J \), where \( \omega \) is a symplectic form. Then there is a Riemannian metric

\[
G_J(v, w) := \langle v, w \rangle_J = \frac{1}{2} (\omega(v, Jw) + \omega(w, Jv)) \tag{10}
\]

for any \( v, w \in TM \). Following [25] we choose the complex linear connection

\[
\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J) Y
\]

induced by the Levi-Civita connection \( \nabla \) of the metric \( G_J \).
Let \((\Sigma, j, y)\) be a smooth Riemann surface of genus \(g\) with \(n\) marked points. Let \(u : \Sigma \rightarrow M\) be a smooth map. For any section \(h \in C^\infty(\Sigma; u^*TM)\) and section \(\eta \in C^\infty(\Sigma, u^*TM \otimes \wedge^0_1 T^*\Sigma)\) and given integer \(k > 4\) we define the norms
\[
\|h\|_{j,k,2} = \left( \int_{\Sigma} \sum_{i=0}^{k} |\nabla^i h|^2 dvol_{\Sigma} \right)^{1/2},
\]
\[
\|\eta\|_{j,k-1,2} = \left( \int_{\Sigma} \sum_{i=0}^{k-1} |\nabla^i \eta|^2 dvol_{\Sigma} \right)^{1/2}.
\]

Here all norms and covariant derivatives are taken with respect to the metric \(G_j\) on \(u^*TM\) and the metric \(g\) on \((\Sigma, j, y)\), \(dvol_{\Sigma}\) denotes the volume form with respect to \(g\). Denote by \(W^{k,2}(\Sigma; u^*TM)\) and \(W^{k-1,2}(\Sigma, u^*TM \otimes \wedge^0_1 T^*\Sigma)\) the complete spaces with respect to the norms (11) and (12) respectively. Denote
\[
\tilde{B} = \{ u \in W^{k,2}(\Sigma, M) | u_*([\Sigma]) = A \}.
\]

\(\tilde{B}\) is an infinite dimensional Banach manifold. For any \(h_1, h_2 \in W^{k,2}(\Sigma, u^*TM)\) we define
\[
\ll h_1, h_2 \gg = \int_{\Sigma} G_j(h_1, h_2) dvol_{\Sigma}.
\]

Then \(W^{k,2}(\Sigma; u^*TM)\) is a Hilbert space, \(\tilde{B}\) is a Hilbert manifold. The map \(u\) is called a \((j, J)\) holomorphic map if \(du \circ j = J \circ du\). Alternatively
\[
\partial_j J(u) := \frac{1}{2} (du + J(u)du \circ j) = 0.
\]

Let \(\tilde{E}\) be the infinite dimensional Banach bundle over \(\tilde{B}\) whose fiber at \(b = (j, y, u)\) is
\[
W^{k-1,2}(\Sigma, u^*TM \otimes \wedge^0_1 T^*\Sigma).
\]

The Cauchy-Riemann operator defines a Fredholm section \(\overline{\partial}_{j,J}, \tilde{B} \rightarrow \tilde{E}\).

The diffeomorphism group \(Diff^+(\Sigma)\) acts on \((\Sigma^n \setminus \Delta) \times \tilde{B}\) and \((\Sigma^n \setminus \Delta) \times \tilde{E}\) by
\[
\varphi^*(j, y_1, \ldots, y_n, u) := (\varphi^*j, \varphi^{-1}(y_1), \ldots, \varphi^{-1}(y_n), u \circ \varphi)
\]
\[
\varphi^* \kappa = \kappa \cdot d\varphi \quad \forall \kappa \in W^{k-1,2}(\Sigma, u^*TM \otimes \wedge^0_1 T^*\Sigma)
\]

for \(\varphi \in Diff^+(\Sigma)\). Put
\[
Aut(j, y, u) = \{ \varphi \in Diff^+(\Sigma)| \varphi^*(j, y_1, \ldots, y_n, u) = (j, y_1, \ldots, y_n, u) \}.
\]

We call it the automorphism group at \((j, y, u)\).

Our moduli space \(\mathcal{M}_{g,n}(A)\) is the quotient space
\[
\mathcal{M}_{g,n}(A) = ((\Sigma^n \setminus \Delta) \times \overline{\partial}_{j,J}^{-1}(0))/Diff^+(\Sigma).
\]

In order to compactify the moduli space we need to consider the holomorphic maps from marked nodal Riemann surfaces. A configuration in \(M\) is a tuple \((\Sigma, j, y, \nu, u)\) where \((\Sigma, j, y, \nu)\) is a marked nodal Riemann surface of genus \(g\) with \(n\) distinct marked points, and \(u : \Sigma \rightarrow M\) is a smooth map satisfying the nodal conditions
\[
\{p, q\} \in \nu \implies u(p) = u(q).
\]
The configuration \((\Sigma, j, y, \nu, u)\) is called holomorphic if the restriction of \(u\) to each smooth component of \(\Sigma\) satisfies (14). The configuration \((\Sigma, j, y, \nu, u)\) is called stable, if the restriction on every smooth component is stable. Let \(\overline{\mathcal{M}}_{g,n}(A)\) be the Gromov compactification of \(\mathcal{M}_{g,n}(A)\). It is well-known that one can define a topology on \(\overline{\mathcal{M}}_{g,n}(A)\) such that \(\overline{\mathcal{M}}_{g,n}(A)\) is a compact Hausdorff space.

Denote by \(\overline{\mathcal{M}}_{g,n}\) the set of domains of each element in \(\mathcal{M}_{g,n}(A)\). Denote

\[
\overline{\mathcal{B}}_{g,n}(A) = \left\{ [(j, y, u)] \mid u \in W^k \mathcal{S}(\Sigma, M), u_{\ast}(\Sigma) = A, [(j, y)] \in \overline{\mathcal{M}}_{g,n} \right\},
\]

\[
\mathcal{B}_{g,n}(A) = \left\{ [(j, y, u)] \in \overline{\mathcal{B}}_{g,n}(A) \mid [(j, y)] \in \mathcal{M}_{g,n} \right\}.
\]

For any \([b_o] = [(p_o, u)] \in \mathcal{M}_{g,n}(A)\) with \(p_o \in \mathcal{M}_{g,n}\) let \(\gamma_o = [(j_o, y_o)] \in \mathcal{T}_{g,n}, (j_o, y_o) \in \mathcal{P}\) with \(\pi_{\mathcal{M}}(\gamma_o) = [p_o] \) and \(\pi_{\mathcal{T}}(j_o, y_o) = g_o\). Choose a local coordinate system \((\psi, \Psi)\) on \(U\) with \(\psi(\gamma_o) = a_o\) for \(Q\) as in (16), we have a local coordinate chart on \(U\) and a local trivialization on \(\pi_{\mathcal{T}}^{-1}(U)\):

\[
\psi : U \rightarrow A, \quad \Psi : \pi_{\mathcal{T}}^{-1}(U) \rightarrow A \times \Sigma,
\]

where \(U \subset T_{g,n}\) is an open set. We can view \(a = (j, y)\) as parameters, and the domain \(\Sigma\) is a fixed smooth surface. Denote by \(j_a\) the complex structure on \(\Sigma\) associated with \(a = (j, y)\) and put \(j_a := j_o\). The Weil-Petersson metric induces a \(Diff^+(\Sigma)\)-invariant distance \(d_{A}(a_o, a)\) on \(A\) such that \(d_{A}(a) := d_{A}(a_o, a)\) is a smooth function on \(A\). Denote by \(G_a\) the isotropy group at \(a\), that is

\[
G_a = \{ \phi \in Diff^+(\Sigma) \mid \phi^\ast(j, y) = (j, y) \}.
\]

Since \(\mathcal{M}_{g,n}\) is an effective orbifold, we can choose \(\delta\) small such that \(G_a\) can be imbedded into \(G_{a_o}\) as a subgroup for any \(a\) with \(d_{A}(a_o, a) < \delta\). Denote by \(im(G_a)\) the imbedding.

Let \(b_o = (a_o, u) = (j_o, y_o, u)\) be the expression of \([\gamma_o, u]\) in this local coordinates. Set

\[
\tilde{O}_{b_o}(\delta, \rho) := \{ (a, v) \in A \times \tilde{B} \mid d_{A}(a_o, a) < \delta, \| h \|_{j_o, k, 2} < \rho \},
\]

\[
O_{[b_o]}(\delta, \rho) = \tilde{O}_{b_o}(\delta, \rho) / G_{b_o},
\]

where \(v = \exp_a(h), G_{b_o}\) is the isotropy group at \(b_o\), that is

\[
G_{b_o} = \{ \phi \in Diff^+(\Sigma) \mid \phi^\ast(j, y, u) = (j, y, u) \}.
\]

Obviously, \(G_{b_o}\) is a subgroup of \(G_{a_o}\). Note that both \(d_{A}\) and \(\|h\|_{j_o, k, 2}\) are \(Diff^+(\Sigma)\)-invariant, we may identified \(O_{[b_o]}(\delta, \rho)\) with a neighborhood of \([b_o] \in \mathcal{M}_{g,n}(A)\) in \(\overline{\mathcal{B}}_{g,n}(A)\).

### 3 Gluing

Let \((\Sigma, j, y, q)\) be a marked nodal Riemann surface of genus \(g\) with \(n\) marked points \(y = (y_1, ..., y_n)\) and one nodal point \(q\). We write the marked nodal Riemann surface as

\[
(\Sigma = \Sigma_1 \wedge \Sigma_2, j = (j_1, j_2), y = (y_1, y_2), q = (p_1, p_2)),
\]

where \((\Sigma_i, j_i, y_i, q_i)\), \(i = 1, 2\), are smooth Riemann surfaces. We say that \(q_1, q_2\) are paired to form \(q\). Assume that \((\Sigma_i, j_i, y_i, q_i)\) is stable, i.e., \(n_i + 2g_i + 1 \geq 3, i = 1, 2\). We choose metric \(g_i\) on each \(\Sigma_i\) as in (2.1) Let \(z_i\) be the cusp coordinates around \(q_i, z_i(q_i) = 0, i = 1, 2\). Let

\[
z_1 = e^{-s_1 - 2\pi \sqrt{-1} t_1}, \quad z_2 = e^{s_2 + 2\pi \sqrt{-1} t_2}.
\]
Here all norms and covariant derivatives are taken with respect to the metric $g$. In terms of the holomorphic cusp cylindrical coordinates we write

$$\tilde{\Sigma}_1: = \Sigma \setminus \{q_1\} \cong \Sigma_1 \cup \{(0, \infty) \times S^1\}, \quad \tilde{\Sigma}_2: = \Sigma \setminus \{q_2\} \cong \Sigma_2 \cup \{(-\infty, 0) \times S^1\}.$$ 

Here $\Sigma_i \subset \Sigma$, $i = 1, 2$, are compact surfaces with boundary. Put $\tilde{\Sigma} = \Sigma \setminus \{q_1, q_2\} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$. We introduce the notations

$$\Sigma_i(R_0) = \Sigma_i \cup \{(s_i, t_i) | |s_i| \leq R_0\}, \quad \Sigma(R_0) = \Sigma_1(R_0) \cup \Sigma_2(R_0).$$

For any gluing parameter $(r, \tau)$ with $r \geq R_0$ and $\tau \in \Sigma^1$ we construct a surface $\Sigma_{(r)}$ with the gluing formulas:

$$s_1 = s_2 + 2r, \quad t_1 = t_2 + \tau.$$

(18) where we use $(r)$ to denote gluing parameters.

We will use the cusp cylinder coordinates to describe the construction of $u(r) : \Sigma_{(r)} \to M$. Write

$$u = (u_1, u_2), \quad u_i : \Sigma_i \to M \text{ with } u_1(q) = u_2(q).$$

We choose local normal coordinates $(x^1, \cdots, x^{2n})$ in a neighborhood $O(q)$ of $u(q)$ and choose $R_0$ so large that $u(\{|s_i| \geq \frac{r}{2}\})$ lie in $O(q)$ for any $r > R_0$. We glue the map $(u_1, u_2)$ to get a pregling maps $u_{(r)}$ as follows. Set

$$u_{(r)} = \begin{cases} 
  u_1 & \text{on } \Sigma_1 \cup \{(s_1, \theta_1) | 0 \leq s_1 \leq \frac{r}{2}, \theta_1 \in S^1\}
  
  u_1(q) = u_2(q) & \text{on } \{(s_1, \theta_1) | \frac{2r}{3} \leq s_1 \leq \frac{5r}{4}, \theta_1 \in S^1\}

  u_2 & \text{on } \Sigma_2 \cup \{(s_2, \theta_2) | 0 \leq s_2 \leq \frac{r}{2}, \theta_2 \in S^1\}
\end{cases}.$$ 

To define the map $u_{(r)}$ in the remaining part we fix a smooth cutoff function $\beta : \mathbb{R} \to [0, 1]$ such that

$$\beta(s) = \begin{cases} 
  1 & \text{if } s \geq 1 \\
  0 & \text{if } s \leq 0
\end{cases} \quad (19)$$

and $\sqrt{1 - \beta^2}$ is a smooth function, $0 \leq \beta'(s) \leq 4$ and $\beta^2\left(\frac{1}{2}\right) = \frac{1}{2}$. We define

$$u_{(r)} = u_1(q) + \left(\beta \left(3 - \frac{4s_1}{r}\right) (u_1(s_1, \theta_1) - u_1(q)) + \beta \left(\frac{4s_1}{r} - 5\right) (u_2(s_1 - 2r, \theta_1 - \tau) - u_2(q))\right).$$

We define weighted norms. Fix a positive function $W$ on $\Sigma$ which has order equal to $e^{\alpha|s|}$ on each end of $\Sigma_i$, where $\alpha$ is a small constant such that $0 < \alpha < 1$. We will write the weight function simply as $e^{\alpha|s|}$. For any $h \in C^\infty_c(\Sigma; u^*_iTM)$ and any section $\eta \in C^\infty_c(\Sigma_i; u^*_iTM \otimes \Lambda^0,1T^* \Sigma_i)$ we define the norms

$$\|h\|_{j, k, 2, \alpha} = \left(\int_{\Sigma} e^{2\alpha|s|} \sum_{i=0}^{k} |\nabla^i h|^2 dv_{\Sigma}\right)^{1/2}, \quad (20)$$

$$\|\eta\|_{j, k-1, 2, \alpha} = \left(\int_{\Sigma} e^{2\alpha|s|} \sum_{i=0}^{k-1} |\nabla^i \eta|^2 dv_{\Sigma}\right)^{1/2}. \quad (21)$$

Here all norms and covariant derivatives are taken with respect to the metric $G_j$ on $u^*_iTM$ and the metric $g^\alpha$ on $(\Sigma, j, y)$, $dv_{\Sigma}$ denotes the volume form with respect to $g^\alpha$. Denote by $W^{k, 2, \alpha}(\Sigma_i; u^*_iTM)$ and $W^{k-1, 2, \alpha}(\Sigma_i; u^*_iTM \otimes \Lambda^0,1T^* \Sigma_i)$ the complete spaces with respect to the norms respectively.
We choose $R_0$ so large that $u_i(\{|s| \geq \frac{r}{2}\})$ lie in $O_{u_i(q)}$ for any $r > R_0$. In this coordinate system we identify $T_xM$ with $T_{u_i(q)}M$ for all $x \in O_{u_i(q)}$. Any $h_0 \in T_{u_i(q)}M$ may be considered as a vector field in the coordinate neighborhood. We fix a smooth cutoff function $\varphi$:  

$$
\varphi(s) = \begin{cases} 
1, & \text{if } |s| \geq \frac{r}{2} \\
0, & \text{if } |s| \leq \frac{r}{2}
\end{cases}
$$

where $\frac{r}{2}$ is a large positive number. Put

$$
\hat{h}_0 = \varphi h_0.
$$

Then for $\frac{r}{2}$ large enough $\hat{h}_0$ is a section in $C^\infty(\Sigma_i; u_i^*TM)$ supported in the tube $\{(s, t)| |s| \geq \frac{r}{2}, t \in S^1\}$. Denote

$$
W_h^{k,2,\alpha}(\Sigma_i; u_i^*TM) = \left\{ h + \hat{h}_0 | h \in W_h^{k,2,\alpha}(\Sigma_i; u_i^*TM), h_0 \in T_{u_i(q)}M \right\}.
$$

We define weighted Sobolev norm on $W_h^{k,2,\alpha}$ by

$$
\|h + \hat{h}_0\|_{W_h^{j,k,2,\alpha}} = \sum_{i=1}^2 \|h\|_{j,k,2,\alpha} + |h_0|,
$$

where $|h_0| = [G_{\beta}(h_0, h_0^*)]^2$.

Denote

$$
\beta_{1;R}(s_1) = \beta \left( \frac{1}{2} + \frac{r - s_1}{R} \right), \quad \beta_{2;R}(s_2) = \sqrt{1 - \beta^2} \left( \frac{1}{2} - \frac{s_2 + r}{R} \right),
$$

where $\beta$ is the cut-off function defined in (19). For any $\eta \in C^\infty(\Sigma_i; u_i^*TM \otimes \Lambda_1^0 \Sigma(r))$, let

$$
\eta_i(p) = \begin{cases} 
\eta, & \text{if } p \in \Sigma_i \cup \{|s| \leq r - 1\} \\
\beta_{i;2}(s_i) \eta(s_i, t_i), & \text{if } p \in \{r - 1 \leq |s| \leq r + 1\} \\
0, & \text{otherwise}.
\end{cases}
$$

If no danger of confusion we will simply write $\eta_i = \beta_{i;2}\eta$. Then $\eta_i$ can be considered as a section over $\Sigma_i$. Define

$$
\|\eta\|_{r,k,2,\alpha} = \|\eta_1\|_{\Sigma_1,j_1,k-1,2,\alpha} + \|\eta_2\|_{\Sigma_2,j_2,k-1,2,\alpha}.
$$

We now define a norm $\| \cdot \|_{r,k,2,\alpha}$ on $C^\infty(\Sigma(r); u_i^*TM)$. For any section $h \in C^\infty(\Sigma(r); u_i^*TM)$ denote

$$
\hat{h}_0 = \int_{S^1} h(r, t) dt,
$$

$$
h_1(s_1, t_1) = (h - \hat{h}_0)(s_1, t_1) \beta_{1;2}(s_1), \quad h_2(s_2, t_2) = (h - \hat{h}_0)(s_2, t_2) \beta_{2;2}(s_2).
$$

We define

$$
\|h\|_{r,k,2,\alpha} = \|h_1\|_{\Sigma_1,j_1,k,2,\alpha} + \|h_2\|_{\Sigma_2,j_2,k,2,\alpha} + |h_0|.
$$

Denote the resulting completed spaces by $W_h^{k-1,2,\alpha}(\Sigma(r); u_i^*TM \otimes \Lambda_1^0 T\Sigma(r))$ and $W_h^{k,2,\alpha}(\Sigma(r); u_i^*TM)$ respectively.

The above construction can be generalized to the case with several nodes. Let $\Sigma \in \mathcal{M}_{g,n}$. Suppose that $\Sigma$ has $\epsilon$ nodal points $p = (p_1, \cdots, p_\epsilon)$, $n$ marked points $y_1, \cdots, y_n$ and $\epsilon$ smooth components. We can choose the plumbing coordinates $(s, t)$ following [40]. Set $\Sigma = \Sigma \setminus \{p_1, \cdots, p_\epsilon, y_1, \cdots, y_n\}$. Then
\( \tilde{\Sigma} \) is a Riemann surface with additional punctures \( a_j, b_j \) in the place of the \( j \)th node of \( \Sigma \), \( j = 1, \cdots, \epsilon \). For each node \( p_j, j = 1, \cdots, \epsilon \), there is a neighborhood isomorphic to 
\[
\{ (z_j, w_j) \in \mathbb{C}^2 \| |z_j| < 1, |w_j| < 1, z_j w_j = 0 \}.
\]

Denote by \( \Sigma_i \) the connected components of \( \tilde{\Sigma} \), \( i = 1, \cdots, \ell \). Suppose that \( \Sigma_i \) has \( n_i \) marked points, \( q_i \) punctures and has genus \( g_i \).

We can parameterize a neighborhood of \( \tilde{\Sigma} \) in the deformation space by Beltrami differentials. Let \( z_i \) (resp. \( w_i \)) be a local coordinate around \( a_i \) (resp. \( b_i \)). \( z_i(a_i) = 0, w_i(b_i) = 0, i = 1, \cdots, \epsilon \). Let \( U_j = \{ p \in \Sigma \| |z_j(p)| < 1 \} \) and \( V_j = \{ p \in \Sigma \| |w_j(p)| < 1 \} \) be disjoint neighborhoods of the punctures \( a_j \) and \( b_j, j = 1, \cdots, \epsilon \). We pick an open set \( U_o \subset \Sigma \) such that each component of \( \tilde{\Sigma} \) intersects \( U_o \) in a nonempty relatively compact set and the intersection \( U_o \cap (U_j \cup V_j) \) is empty for all \( j \). Denote \( N = \sum_{i=1}^\epsilon (3g_i - 3 + n_i + q_i) \). Choose Beltrami differentials \( \nu_j, j = 1, \cdots, N \) which are supported in \( U_o \) and form a basis of the deformation space at \( \Sigma \). Let \( s = (s_1, \cdots, s_N) \in \mathbb{C}^N, \nu = \sum_{i=1}^N s_i \nu_i \). Assume \( |s| \) small enough such that \( |\nu| < 1 \). The nodal surface \( \Sigma_{s,0} \) is obtained by solving the Beltrami equation \( \tilde{\partial} w = \nu(s) w \).

We recall the plumbing construction for \( \Sigma \) with a pair of punctures \( a_j, b_j \). Let \( z_{s,j}, w_{s,j} \) be cusp coordinates in \( U_j, V_j \) near \( a_j, b_j \) respectively, thus
\[
ds_{s,0}^2(z_j, w_j) = \frac{|dz_j|^2}{|z_j|^2 |\log |z_j|| dz_j|}, \quad ds_{s,0}^2(w_j) = \frac{|dw_j|^2}{|w_j|^2 |\log |w_j|| dw_j|},
\]
where \( ds_{s,0}^2 \) be the normalized hyperbolic metric on \( \Sigma_{s,0} \) of curvature \( -1 \). As \([9]\) denote \( F_{j,s} = z_j \circ z_{j,s}^{-1}, \quad G_{j,s} = w_j \circ w_{j,s}^{-1} \).

By the removable singularity theorem and setting \( \tilde{F}_{j,s} = F_{j,s}/F_{j,s}(0) \) and \( \tilde{G}_{j,s} = G_{j,s}/G_{j,s}(0) \), if necessary, we can assume that
\[
F_{j,s}(0) = 0, \quad F'_{j,s}(0) = 1, \quad G_{j,s}(0) = 0, \quad G'_{j,s}(0) = 1.
\]
Since \( U_o \) is disjoint from the \( U_j, V_j \), the \( F_{j,s}, G_{j,s} \) are also holomorphic onto their image. For any \( t = (t_1, \cdots, t_\epsilon) \) with \( 0 < |t_j| < 1 \), remove the discs \( |z_j| < |t_j| \) and \( |w_j| < |t_j| \) when \( |t_j| \) small, and identify \( z_j \) via the plumbing equation
\[
w_j = \frac{t_j}{z_j}.
\]
We can rewrite the equation as
\[
(F_{j,s} \circ z_{j,s}) \cdot (G_{j,s} \circ w_{j,s}) = t_j.
\]
Then we form a new Riemann surface \( \Sigma_{s,t} \). We call \( (t_1, \cdots, t_\epsilon) \) plumbing coordinate. We obtain a family of Riemann surfaces over \( \Delta_s \times \Delta_t \), whose fiber over \( (s, t) \) is the Riemann surface \( \Sigma_{s,t} \), where \( \Delta_s = (\Delta)^N \subset \mathbb{C}^N, \Delta_t = (\Delta)^\epsilon \subset \mathbb{C}^\epsilon \) are polydiscs.

In the coordinate system \( (s, t) \) the \( g_{op} \) metric induces a \( Diff(\Sigma) \)-invariant distance \( d_{s,t}((\cdot, \cdot), (\cdot, \cdot)) \) on \( \Delta_s \times \Delta_t \). Put
\[
O(\delta) = \{(s, t) \mid d_{s,t}((0, 0), (s, t)) < \delta \}.
\]
We can choose \( \delta \) small such that \( G_{(s,t)} \) can be imbedded into \( G_{(0,0)} \) as a subgroup for any \( (s, t) \in O(\delta) \). Denote by \( im(G_{(s,t)}) \) the imbedding.
Let $u_{s,0} : \Sigma_{s,0} \to M$ be a $W^{k,2,\alpha}$-map. We can construct $u_{s,t} : \Sigma_{s,t} \to M$. For any $h \in C^\infty_c(\Sigma_{s,0}; u_{s,0}^*TM \otimes \bigwedge^{0,1}_a T^* \Sigma_{s,0})$ and any section $\eta \in C^\infty_c(\Sigma_{s,0}, u_{s,0}^*TM \otimes \bigwedge^{0,1}_a T^* \Sigma_{s,0})$ we define the norms $\|h\|_{s,k,2,\alpha}$ and $\|\eta\|_{s,k-1,2,\alpha}$, and for any section $h \in C^\infty(\Sigma_{s,t}; u_{s,t}^*TM)$ and any $\eta \in C^\infty(\Sigma_{s,t}; u_{s,t}^*TM \otimes \bigwedge^{0,1}_a T^* \Sigma_{s,t})$, we define the norms $\|h\|_{s,t,k,2,\alpha}$ and $\|\eta\|_{s,t,k-1,2,\alpha}$ as for one node case.

Let $b_o = (\Sigma, 0, 0, u)$. Set
\[
\bar{O}_{b_o}(\delta_{b_o}, \rho_{b_o}) := \left\{ ((s, t), v_{s,t}) \mid d_{s,t}((0, 0), (s, t)) < \delta_{b_o}, \|h\|_{s,t,k,2,\alpha} < \rho_{b_o} \right\},
\]
\[
O_{b_o}(\delta_{b_o}, \rho_{b_o}) = \bar{O}_{b_o}(\delta_{b_o}, \rho_{b_o}) / G_{b_o},
\]
where $v_{s,t} = \exp_{u_{s,t}}(h)$.

4 Local regularization-Top strata

When the transversality fails we need to take the regularization.

4.1 Local regularization for the top strata

Let $[b_o] = [(p_o, u)] \in \mathcal{M}_{q,n}(A)$ and let $\gamma_o \in \mathcal{T}_{q,n}$ such that $\pi(\gamma_o) = [p_o]$, where $\pi : \mathcal{T}_{q,n} \to \mathcal{M}_{q,n}$ is the projection. We choose a local coordinate system $(\psi, \Psi)$ on $U$ with $\psi(\gamma_o) = a_o$ for $Q$. We view $a = (j, y)$ as a family of parameters defined to a fixed $\Sigma$. Denote
\[
\bar{B}(a) = \left\{ u \in W^{k,2}(\Sigma, M) \mid u_s([\Sigma]) = A \right\}.
\]
Let $\bar{E}(a)$ be the infinite dimensional Banach bundle over $\bar{B}(a)$ whose fiber at $v$ is
\[
W^{k-1,2}(\Sigma, v^*TM \otimes \bigwedge^{0,1}_{j_a} T^* \Sigma),
\]
where we denote by $j_a$ the complex structure on $\Sigma$ associated with $a = (j, y)$. We will denote $j_{a_o} := j_o$. We have a continuous family of Fredholm system
\[
\left( \bar{B}(a), \bar{E}(a), \partial_{j_o,j} \right)
\]
parameterized by $a \in A$ with $dA(a_o, a) < \delta$. For any $v \in \bar{B}(a)$ let $b = (a, v)$ and denote $\bar{E}(a)|_v := \bar{E}|_b$. Let $b_o = (a_o, u)$. Choose $\bar{K}_{b_o} \subset \bar{E}|_{b_o}$ to be a finite dimensional subspace such that every member of $\bar{K}_{b_o}$ is in $C^\infty(\Sigma, u^*TM \otimes \bigwedge^{0,1}_{j_a} T^* \Sigma)$ and
\[
\bar{K}_{b_o} + \text{image}D_{b_o} = \bar{E}|_{b_o},
\]
where $D_{b_o} = D\partial_{j_o,j}$ is the vertical differential of $\partial_{j_o,j}$ at $u$.

Let $G_{b_o}$ be the isotropy group at $b_o$. In case the isotropy group $G_{b_o}$ is non-trivial, we must construct a $G_{b_o}$-equivariant regularization. Note that $G_{b_o}$ acts on $W^{k-1,2}(\Sigma, u^*TM \otimes \bigwedge^{0,1}_{j_a} T^* \Sigma)$ in a natural way: for any $\kappa \in W^{k-1,2}(\Sigma, u^*TM \otimes \bigwedge^{0,1}_{j_a} T^* \Sigma)$ and any $g \in G_{b_o}$
\[
g \cdot \kappa = \kappa \circ dg \in W^{k-1,2}(\Sigma, u^*TM \otimes \bigwedge^{0,1}_{j_a} T^* \Sigma).
\]
Set
\[
\bar{K}_{b_o} = \bigoplus_{g \in G_{b_o}} g\bar{K}_{b_o}.
\]
Then $\bar{K}_{b_o}$ is $G_{b_o}$-invariant. To simplify notations we assume that $\bar{K}_{b_o}$ is already $G_{b_o}$-invariant.
Lemma 4.1. There are constants $\delta > 0$, $\rho > 0$ depending on $b_o$ such that there is an isomorphism

$$P_{b_o, b} : \tilde{\mathcal{E}}_{b_o} \to \tilde{\mathcal{E}}_b \quad \forall \ b \in \tilde{\mathcal{O}}_{b_o}(\delta, \rho).$$

Proof. Denote by $inj_M$ the injective radius of $(M, G)$. Given $h \in W^{k,2}(\Sigma, u^*TM)$ with $\|h\|_{L^\infty} < inj_M$, let

$$\Phi : u^*TM \to (\exp_u h)^*TM$$

denote the complex bundle isomorphism, given by parallel transport with respect to the connection $\nabla$, along the geodesics $s \to \exp_u (sh)$. We choose $\rho < inj_M$. For any $j \in \mathcal{J}(\Sigma)$ near $j_o$ we can write $j = (I + H)j_o(I + H)^{-1}$ where $H \in T_{j_o}\mathcal{J}(\Sigma)$. We define two maps

$$\Psi_{j_o, j} : u^*TM \otimes \wedge^j_{j_o} T^*\Sigma \to u^*TM \otimes \wedge^j_{j_o} T^*\Sigma$$

and

$$\Psi_{j_o, j} : u^*TM \otimes \wedge^j_{j_o} T^*\Sigma \to u^*TM \otimes \wedge^j_{j_o} T^*\Sigma$$

by

$$\Psi_{j_o, j}(\eta) = \frac{1}{2}(\eta - \eta \cdot j_o j_o), \quad \Psi_{j_o, j}(\varpi) = \frac{1}{2}(\varpi - \varpi \cdot j_o j_o).$$

Since $J\eta = -\eta j_o$ and $J\varpi = -\varpi j_o$, one can check that $J\Psi_{j_o, j}(\eta) = -\Psi_{j_o, j}(\eta) j_o$ and $J\Psi_{j_o, j}(\varpi) = -\Psi_{j_o, j}(\varpi) j_o$. Then $\Psi_{j_o, j}$ and $\Psi_{j_o, j}$ are well defined. The proof of the following claim can be found in [19], we omit the proof here.

Claim. $\Psi_{j_o, j}$ is an isomorphism when $|H|$ small enough.

$\Psi_{j_o, j}$ induces an isomorphism

$$\Psi_{j_o, j} : W^{k-1,2}(\Sigma, u^*TM \otimes \wedge^j_{j_o} T^*\Sigma) \to W^{k-1,2}(\Sigma, u^*TM \otimes \wedge^j_{j_o} T^*\Sigma)$$

in a natural way. Let $P_{b_o, b} = \Phi \circ \Psi_{j_o, j}$. We can choose $\delta, \rho$ such that there is an isomorphism

$$P_{b_o, b} : \tilde{\mathcal{E}}_{b_o} \to \tilde{\mathcal{E}}_b \quad \forall \ b \in \tilde{\mathcal{O}}_{b_o}(\delta, \rho).$$

Now we define a thickened Fredholm system $(\tilde{K}_{b_o} \times \tilde{\mathcal{O}}_{b_o}(\delta, \rho), \tilde{K}_{b_o} \times \tilde{\mathcal{E}}_{b_o}(\delta, \rho), S)$. Let $(\kappa, b) \in \tilde{K}_{b_o} \times \tilde{\mathcal{O}}_{b_o}(\delta, \rho), b = (a, v) \in \tilde{\mathcal{O}}_{b_o}(\delta, \rho)$. Define

$$S(\kappa, b) = \tilde{\delta}_{j_o, j} u + P_{b_o, b} \kappa. \quad (25)$$

We can choose $(\delta, \rho)$ small such that the linearized operator $DS(\kappa, b)$ is surjective for any $b \in \tilde{\mathcal{O}}_{b_o}(\delta, \rho)$.

4.2 The norm $\|h\|_{j_o, k, 2}^2$ and the isotropy group on the top strata

If we fix the complex structure $j_o$, then $W^{k,2}(\Sigma; u^*TM)$ is a Hilbert space. It is well-known that $\|h\|_{j_o, k, 2}^2$ is a smooth function ( see [43]). Now the $\|h\|_{j_o, k, 2}^2$ is a family of norms, so the following lemma is important.

Lemma 4.2. For any $[b_o] = [(p_o, u)] \in \mathcal{M}_{g,n}(A)$ and any local coordinates $(\psi, \Psi)$ on $U$ with $\psi : U \to A \ni a_o$ the norm $\|h\|_{j_o, k, 2}^2$ is a smooth function in $\tilde{\mathcal{O}}_{b_o}(\delta, \rho)$. 

13
Lemma 4.3. There exist two constants the following hold.

Proof. We first prove that, in the coordinate system \((\psi, \Psi)\) on \(U\), \(\|h\|_{j_a,k,2}^2\) is smooth. Since the slice we choose is smooth, the complex structure \(j_a\) smoothly depends on \(a\), so the associated hyperbolic metric \(g\) smoothly depends on \(a\). It follows that \(\|h\|_{j_a,k,2}^2\) is smooth in \((\psi, \Psi)\). Now we choose another coordinate system \((\psi', \Psi')\), then

\[
a = \psi \circ (\psi')^{-1}(a'), \quad u = u' \circ \vartheta_a,
\]

where \(\vartheta_a = \Psi' \circ \Psi^{-1}|_a \in Diff^+(\Sigma)\) is a family of diffeomorphisms. On the other hand, in terms of the two coordinate system we have

\[
\|h\|_{j_a,k,2}^2 = \|h\|_{j_{a'},k,2}^2.
\]

Then

\[
\frac{\partial}{\partial a} \|h\|_{j_{a'},k,2}^2 = \frac{\partial}{\partial a'} \|h\|_{j_a,k,2}^2 = \frac{\partial}{\partial a} \|h\|_{j_a,k,2}^2 \frac{\partial a}{\partial a'}.
\]

By Lemma \[\ref{2.2}\] \(\psi' \circ \psi^{-1}\) is smooth. So \(\|h\|_{j_{a'},k,2}^2 \in C^1\). By the same argument we conclude that \(\|h\|_{j_{a'},k,2}^2\) is smooth. \(\square\)

\[\text{Lemma 4.3. There exist two constants } \delta_0, \rho_0 > 0 \text{ depend only on } b_0 \text{ such that for any } \delta < \delta_0, \rho < \rho_0 \text{ the following hold.}\]

(1) For any \(p \in \tilde{O}_{b_0}(\delta, \rho)\), let \(G_p\) be the isotropy group at \(p\), then \(im(G_p)\) is a subgroup of \(G_{b_0}\).

(2) Let \(p \in \tilde{O}_{b_0}(\delta, \rho)\) be an arbitrary point with isotropy group \(G_p\), then there is a \(G_p\)-invariant neighborhood \(O(p) \subset \tilde{O}_{b_0}(\delta, \rho)\) such that for any \(q \in O(p)\), \(im(G_q)\) is a subgroup of \(G_p\), where \(G_p, G_q\) denotes the isotropy groups at \(p\) and \(q\) respectively.

Proof. We only prove (2), the proof of (1) is similar. If (2) is not true, there exists a point \(p = (a, \hat{u}) \in \tilde{O}_{b_0}(\delta, \rho)\) and a sequence \(b_i = (a_i, u_i) \in \tilde{O}_p(\delta_i, \rho_i)\) such that

(a) \(\delta_i \to 0, \rho_i \to 0,\)

(b) \(im(G_{b_i})\) is not a subgroup of \(G_p\).

By (a) we have

\[
\lim_{i \to \infty} a_i = a, \quad \lim_{i \to \infty} \|h_i\|_{k,2} = 0,
\]

where \(u_i = \exp_a(h_i)\). Since \(\mathcal{M}_{g,n}\) is an orbifold, by \[\ref{26}\] we have \(im(G_{a_i})\) is a subgroup of \(G_a\). Obviously, \(G_{b_i}\) is a subgroup of \(G_{a_i}\). Then \(im(G_{b_i})\) is a subgroup of \(G_a\). Since \(G_a\) is a finite group, by choosing subsequence we may assume that \(im(G_{b_i})\) is a fixed subgroup of \(G_a\) independent of \(i\), denoted by \(im(G_{b_i})\). By (b), \(im(G_{b_i})\) is not a subgroup of \(G_p\). On the other hand, by \[\ref{26}\] \(G_{b_i} \cdot u_i\) converges to \(G_b \cdot \hat{u}\) and \(u_i\) converges to \(\hat{u}\) in \(W^{k,2}\). Hence for any \(g \in G_b\), we have

\[
\|g \cdot \hat{u} - \hat{u}\|_{k,2} = 0.
\]

It follows that \(im(G_{b_i})\) is a subgroup of \(G_p\). We get a contradiction. \(\square\)

Remark 4.4. It is easy to see from the proof that the Lemma \[\ref{2.3}\] also hold for lower stratum.
5 Local regularization-Lower strata

5.1 Local regularization for lower stratum: without bubble tree

Let $\Sigma, \Sigma, \Sigma_i$ be as in \cite{6,8}. We choose local plumbing coordinates $(s, t)$ and construct $\Sigma_{s,t} \to \Delta_s \times \Delta_t$. Consider the family of Banach manifold

$$\tilde{\mathcal{B}}(s, t) = \{ u \in W^{k,2,\alpha}(\Sigma_{s,t}, M) \mid u_*([\Sigma]) = A \}.$$  

Let $\tilde{E}(s, t)$ be the infinite dimensional Banach bundle over $\tilde{\mathcal{B}}(s, t)$ whose fiber at $b = (s, t, u)$ is $W^{k-1,2,\alpha}(\Sigma_{s,t}, u^* TM \otimes \wedge_{j,s}^1 T^* \Sigma_{s,t})$. We have a continuous family of Fredholm system

\[
\left( \tilde{\mathcal{B}}(s, t), \tilde{E}(s, t), \tilde{\partial}_{j,s,t} \right)
\]

parameterized by $(s, t) \in \Delta_s \times \Delta_t$. Let $b_o = (0, 0, u), b = (s, t, v)$. We use the same method as in subsection 4.1 to choose $K_{b_o} = \bigoplus_{k=1}^{t} K_{b_{o,k}} \subset \tilde{E}|_{b_o} = \bigoplus_{k=1}^{t} \tilde{E}_{b_{o,k}}$ to be a finite dimensional subspace such that

1. Every member of $\tilde{K}_{b_o}$ is in $C^\infty(\Sigma_{s,0}, u_1^* TM \otimes \wedge_{j,s}^1 T^* \Sigma_{s,0})$ and supports in the compact subset $\Sigma_{0,0}(R_0)$ of $\Sigma_{0,0}$.

2. $\tilde{K}_{b_o} + \text{image} D_{b_o} = \tilde{E}|_{b_o}, \forall i = 1, 2, \ldots, t$.

3. $\tilde{K}_{b_o}$ is $G_{b_o}$-invariant.

where we denote by $j_{oi}$ the complex structure on $\Sigma_i$ associated with $(0, 0)$, and

$$W(R_0) := \bigcup_{i=1}^{n} \left( \{ |z_i| < e^{-R_0} \} \cup \{ |w_i| < e^{-R_0} \} \right) \cup D(e^{-R_0}), \quad \Sigma_{s,0}(R_0) = \Sigma_{s,t} \setminus W(R_0). \quad (27)$$

for a constant $R_0 > 1$. We identify each $\Sigma_{s,t}(R_0)$ with $\Sigma_{0,0}(R_0) := \Sigma(R_0)$ for $|s|/|t|$ small. Denote by $j_{s,t}$ the family of complex structure on $\Sigma(R_0)$. Denote $j_0 := j_{0,0}$. Then when $|H|$ small

$$\Psi_{j_0,j_{s,t}} : W^{k-1,2,\alpha}(\Sigma(R_0), u_1^* TM \otimes \wedge_{j,s}^1 T^* \Sigma(R_0)) \to W^{k-1,2,\alpha}(\Sigma(R_0), u_1^* TM \otimes \wedge_{j,s}^1 T^* \Sigma(R_0))$$

is an isomorphism. Let $P_{b_o,b} = \Phi \circ \Psi_{j_0,j_{s,t}}$. We fix a smooth cutoff function $\beta_{R_0} : \mathbb{R} \to [0, 1]$ such that

$$\beta_{R_0}(s) = \begin{cases} 0 & \text{if } |s| \geq R_0 \\ 1 & \text{if } |s| \leq R_0 - 1. \end{cases} \quad (28)$$

Lemma 5.1. Let $\tilde{E}(s, t)$ be the infinite dimensional Banach bundle over $\tilde{\mathcal{B}}(s, t)$ whose fiber at $b = (s, t, u)$ is

$$\tilde{E}(s,t,u) := \{ \beta_{R_0}(s) \eta \mid \eta \in \tilde{E}(s,t,u) \}.$$  

Then there are constants $\delta > 0, \rho > 0$ depending on $b_o$ such that there is an isomorphism

$$P_{b_o,b} : \tilde{E}_{b_o} \to \tilde{E}_b \quad \forall b \in \tilde{O}_{b_o}(\delta, \rho).$$

The proof is the same as in Lemma 4.1.

Now we define a thickened Fredholm system $(\tilde{K}_{b_o} \times \tilde{O}_{b_o}(\delta, \rho), \tilde{K}_{b_o} \times \tilde{E}|_{\tilde{O}_{b_o}(\delta, \rho)}, S)$. Let $(\kappa, b) \in \tilde{K}_{b_o} \times \tilde{O}_{b_o}(\delta, \rho), b = (a, v) \in \tilde{O}_{b_o}(\delta, \rho)$. Define

$$S(\kappa, b) = \tilde{\partial}_{j,s} u + P_{b_o,b} \kappa. \quad (29)$$

We can choose $(\delta, \rho)$ small such that the linearized operator $DS_{(\kappa,b)}$ is surjective for any $b \in \tilde{O}_{b_o}(\delta, \rho)$. 

15
5.2 Local regularization for lower stratum : with bubble tree

A-G-F procedure. We introduce the A-G-F procedure.

Consider a strata $\mathcal{M}^Γ$ of $\overline{\mathcal{M}}_{g,n}(A)$. Let $b_o = [(\Sigma, j, y, u)] \in \mathcal{M}^Γ$. Then $(\Sigma, j, y)$ is a marked nodal Riemann surface. Suppose that $\Sigma$ has a principal part $\Sigma^P$ and some bubble tree $\Sigma^B$ attaching to $\Sigma^P$ at $q$. Let $u = (u_1, u_2)$ where $u_1 : \Sigma^P \to M$ and $u_2 : S^2 \to M$ are $J$-holomorphic maps. We consider the simple case $\Sigma^B = (S^2, q)$ with $[u_2(S^2)] \neq 0$, the general cases are similar. Denote $b_{oo} := (S^2, q, u_2)$,

$$\tilde{O}_{b_{oo}}(\rho_o) = \{ v \in W^{k,2,\alpha}(S^2, q, u_2^*TM) \parallel h \parallel_{k,2,\alpha} \leq \rho_o, \text{ where } v = \exp_{u_2}(h) \} .$$

$$O_{b_{oo}}(\rho_o) = \tilde{O}_{b_{oo}}(\rho_o)/G_{b_{oo}}$$

where $G_{b_{oo}} = \{ \phi \in Diff(S^2) \mid \phi^{-1}(q) = q, \ u_2 \circ \phi = u_2 \}$ is the isotropy group at $b_{oo}$.

We can choose a local smooth codimension-two submanifold $Y$ such that $u_2(S^2)$ and $Y$ transversally intersects, and $u_2^{-1}(Y) = x = (x_1, ..., x_\ell)$ (see [33] and [31]). We add these intersection points as marked points to $S^2$ such that $S^2$ is stable. Denote the Riemann surface by $(S^2, q, x)$. We may choose $\rho_o$ such that for any $(S^2, q, v) \in O_{b_{oo}}(\rho_o)$, $v(S^2)$ and $Y$ transversally intersects, and $v^{-1}(Y)$ has $\ell$ points. Denote

$$\tilde{O}_{b_{oo}}(1 + \ell, \rho_o) = \{ (S^2, q, x, v) \mid v(x) \in Y, \ v \in \tilde{O}_{b_{oo}}(\rho_o) \} .$$

Note that the additional marked points are unordered, so we consider the space

$$\tilde{O}_{b_{oo}}(1 \mid \ell, \rho_o) = \tilde{O}_{b_{oo}}(1 + \ell, \rho_o)/Sy(\ell)$$

where $Sy(\ell)$ denotes the symmetric group of order $\ell$. Denote $\tilde{b}_{oo} := (S^2, q \mid x, u_2)$, where the points after “$|$” are unordered. Denote

$$G_{\tilde{b}_{oo}} = \{ \phi \in Diff(S^2) \mid \phi^{-1}(q) = q, \ u_2 \circ \phi = u_2, \ \phi^{-1}\{x_1, ..., x_\ell\} = \{x_1, ..., x_\ell\} \} .$$

For any $\phi \in G_{\tilde{b}_{oo}}$, since $u_2 \circ \phi = u_2$, we have $\phi^{-1}\{x_1, ..., x_\ell\} = \{x_1, ..., x_\ell\}$. Then the following lemma holds.

**Lemma 5.2.** $G_{\tilde{b}_{oo}} = G_{b_{oo}}$.

Let $\tilde{b}_{oo} := (S^2, q, x, u_2)$ be a representative of $\tilde{b}_{oo} := (S^2, q \mid x, u_2)$, where $x = (x_1, ..., x_\ell)$ is an ordered set. We choose cusp coordinates $z$ on $\Sigma^P$ and $w$ on $S^2$ near $q$. We can construct a metric $g$ on $(S^2, q, x)$ as in section 2.1 such that $g^o$ is the standard cylinder metric near marked points and nodal points. Put $\Sigma_1 = \Sigma^P, \Sigma_2 = S^2, b_o = (b_{o1}, b_{o2})$. Let $G_{b_{oi}}$ be the isotropy group at $b_{oi}$. Denote $b_o = (b_{o1}, b_{o2})$, where $b_{o1}$ is a lift of $b_{o1}$ to the uniformization system, and $b_{o2} := b_{oo}$. Note that the cusp coordinates $z$ and $w$ are unique modulo rotations near nodal point $q$ and the metric $g$ on $\Sigma^P$ is $G_{b_{oi}}$-invariant and $g$ on $(S^2, q, x)$ is $G_{b_{oo}}$-invariant. In the coordinates $z, w$ for any $\phi_i \in G_{b_{oi}}$,

$$\phi_1(z) = e^{-\sqrt{-1}\gamma_1 z}, \quad \phi_2(w) = e^{-\sqrt{-1}\gamma_2 w}.$$

By the finiteness of $G_{b_{oi}}$, we have $\gamma_i = \frac{2j_i\pi}{l_i}$ where $j_i < l_i, j_i, l_i \in \mathbb{Z}, i = 1, 2$.

We choose

$$\tilde{K}_{b_o} = \bigoplus_{i=1}^{2} \tilde{K}_{b_{oi}} \subset \tilde{E}_{|b_o} = \bigoplus_{i=1}^{2} \tilde{E}_{b_{oi}}$$

16
representives of\( \Sigma \) be a finite dimensional subspace satisfying (1), (2) and (3) in §5.1.

Then we glue \( \tilde{b}_{o1} \) and \( \tilde{b}_{o2} \) at \( q \) with gluing parameters \( (r^*, \tau^*) \) in the coordinates \( z, w \) to get representives of \( \tilde{p}^* := (\Sigma_x, y \mid x) \) and pregluing map \( \tilde{u}_{(r^*)} \). Let \( \tilde{b}^*_o := (\tilde{p}^*, \tilde{u}_{(r^*)}) \), denote by \( G_{\tilde{b}^*_o} \) the isotropy group at \( \tilde{b}^*_o \). Now we forget \( Y \) and the additional marked points \( x \). We get a element \( \Sigma^* := \Sigma_x \), which is a point \( p^* = (\Sigma_x, y) \in \overline{\mathcal{M}}_{g,n} \). Let \( b^*_o = (p^*, u_{(r^*)}) \), denote by \( G_{p^*} \) and \( G_{b^*_o} \) the isotropy groups at \( p^* \) and \( b^*_o \) respectively. The following lemma is obvious.

**Lemma 5.3.** \( G_{b^*_o} = G_{b^*_o} \).

We call this procedure a A-G-F procedure (Adding marked points-Gluing-Forgetting \( Y \) and marked points). This procedure can be extended to bubble tree and bubble chain in an obvious way.

We use the same method as in §5.1 to construct the local regularization.

## 6 Global regularization

### 6.1 A finite rank orbi-bundle over \( \overline{\mathcal{M}}_{g,n}(A) \)

By the compactness of \( \overline{\mathcal{M}}_{g,n}(A) \) there exist finite points \( [b_i] \in \overline{\mathcal{M}}_{g,n}(A) \), \( 1 \leq i \leq m \), such that

1. The collection \( \{O_{[b_i]}(\delta_i/3, \rho_i/3) \mid 1 \leq i \leq m\} \) is an open cover of \( \overline{\mathcal{M}}_{g,n}(A) \).

2. Suppose that \( O_{b_i}(\delta_i, \rho_i) \cap O_{b_j}(\delta_j, \rho_j) \neq \emptyset \). For any \( b \in O_{b_i}(\delta_i, \rho_i) \cap O_{b_j}(\delta_j, \rho_j) \), \( G_b \) can be imbedded into both \( G_{b_i} \) and \( G_{b_j} \) as subgroups.

**Remark 6.1.** We may choose \( [b_i], 1 \leq i \leq m \), such that if \( [b_i] \) lies in the top strata for some \( i \), then \( O_{[b_i]}(\delta_i, \rho_i) \) lies in the top strata.

Set 

\[
\mathcal{U} = \bigcup_{i=1}^{m} O_{[b_i]}(\delta_i/2, \rho_i/2).
\]

There is a forget map

\[
\text{forg} : \mathcal{U} \to \overline{\mathcal{M}}_{g,n}, \quad [(j, y, u)] \mapsto [(j, y)].
\]

We construct a finite rank orbi-bundle \( \mathbf{F} \) over \( \mathcal{U} \). The construction imitates Siebert’s construction. First of all, we can slightly deform \( \omega \) to get a rational class \( [\omega^*] \). By taking multiple, we can assume that \( [\omega^*] \) is an integral class on \( \Sigma \). Therefore, it is the Chern class of a complex line bundle \( L \) over \( \Sigma \). Let \( i \) be the complex structure on \( L \). We choose a Hermitian metric \( G^L \) and the associate unitary connection \( \nabla^L \) on \( L \).

Let \( (\Sigma, j, y) \) be a marked nodal Riemann surface of genus \( g \) with \( n \) marked points. Let \( u : \Sigma \to M \) be a \( W^{k,2} \) map. We have complex line bundle \( u^*L \) over \( \Sigma \) with complex structure \( u^*i \). The unitary connection \( u^*\nabla^L \) splits into \( u^*\nabla^L := u^*\nabla^{L,(1,0)} \oplus u^*\nabla^{L,(0,1)} \). Denote

\[
D^L := u^*\nabla^{L,(0,1)} : W^{k,2}(\Sigma, u^*L) \to W^{k-1,2}(\Sigma, u^*L \otimes \Lambda^{(0,1)}T^*\Sigma).
\]

\( D^L \) takes \( s \in W^{k,2}(\Sigma, u^*L) \) to the \( \mathbb{C} \)-antilinear part of \( \nabla^L \), where \( s \) is a section of \( L \). One can check that

\[
D^L(f\xi) = \partial\Sigma f \otimes \xi + f \cdot D^L\xi.
\]

17
$D^L$ determines a holomorphic structure on $u^*L$, for which $D^L$ is an associated Cauchy-Riemann operator (see [15, 16]). Then $u^*L$ is a holomorphic line bundle.

Let $\lambda_{(\Sigma', \beta)}$ be the dualizing sheaf of meromorphic 1-form with at worst simple pole at the nodal points and for each nodal point $p$, say $\Sigma_1$ and $\Sigma_2$ intersects at $p$,

$$\operatorname{Res}_p(\lambda_{(\Sigma_1, \beta_1)}) + \operatorname{Res}_p(\lambda_{(\Sigma_2, \beta_2)}) = 0.$$  

Let $\Pi : \overline{\mathcal{C}}_g \to \mathcal{M}_g$ be the universal curve. Let $\lambda$ be the relative dualizing sheaf over $\overline{\mathcal{C}}_g$, the restriction of $\lambda$ to $(\Sigma, \beta)$ is $\lambda_{(\Sigma, \beta)}$.

Set $\Lambda_{(\Sigma, \beta)} := \lambda_{(\Sigma, \beta)} (\sum_{i=1}^n y_i)$. Let $(\psi, \Psi) : (O, \pi\pi^{-1}(O)) \to (A, A \times \Sigma)$ be a local coordinate systems, where $O \subset T_{g,n}$ is an open set. $\lambda$ induces a line bundle over $A \times \Sigma$, denoted by $\lambda$. Then $\Lambda \mid_b := \mathcal{P}^* \lambda \otimes u^* L$ is a holomorphic line bundle over $\Sigma$, where $\mathcal{P}$ denote the forgetful map. We have a Cauchy-Riemann operator $\partial_b$. Then $H^0(\Sigma, \Lambda \mid_b)$ is the $\ker \partial_b$. Here the $\partial$-operator depends on the complex structure $j$ on $\Sigma$ and the bundle $u^* L$, so we denote it by $\partial_b$.

If $\Sigma'$ is not a ghost component, there exist a constant $\tilde{h}_\alpha > 0$ such that

$$\int_{u(\Sigma')} \omega^* > \tilde{h}_\alpha.$$  

Therefore, $c_1(u^* L)(\Sigma') > 0$. For ghost component $\Sigma'$, $\lambda_{\Sigma'} (\sum_{i=1}^n y_i)$ is positive. So for any $b = (a, v) \in \tilde{O}_b(\delta, \rho)$ by taking the higher power of $L \mid_b$, if necessary, we can assume that $L \mid_b$ is very ample. Hence, $H^1(\Sigma, \Lambda \mid_b) = 0$. Therefore, $H^0(\Sigma, \Lambda \mid_b)$ is of constant rank ( independent of $b \in \tilde{O}_\beta(\delta, \rho)$). We have a finite rank bundle $\tilde{F}$ over $\tilde{O}_b(\delta, \rho)$, whose fiber at $b = (j, y, v) \in \tilde{O}_b(\delta, \rho)$ is $H^0(\Sigma, \Lambda \mid_b)$. The finite group $G_b$ acts on the bundle on $\tilde{F} \mid_b$ in a natural way.

**Lemma 6.2.** For any $\varphi \in Di f f^+(\Sigma)$ denote

$$b' = (j', y', u') = \varphi \cdot (j, y, u) = (\varphi^* j, \varphi^{-1} y, \varphi^* u).$$  

Then the following hold

(a). $L^\mid_{b'} = \varphi^* L \mid_b$,

(b). $D^L \mid_{b'} (\varphi^* \xi) = \varphi^* (D^L \mid_b (\xi)).$

It follows from (b) above that if we choose another coordinate system $A'$ and another local model $\tilde{O}_{b'}(\delta', \rho')/G_{b'}$, we have

$$H^0(\Sigma, \tilde{L} \mid_{b'}) = H^0(\Sigma, \tilde{L'} \mid_{b'}).$$  

But the coordinate transformation is continuous. So we get a continuous bundle $F \to U$. Moreover, by (1) and (2) we conclude that $F$ has a “orbi-vector bundle” structure over $U$.

Both $\tilde{K}_b$ and $\tilde{F} \mid_b$ are representation spaces of $G_b$. Hence they can be decomposed as sum of irreducible representations. There is a result in algebra saying that the irreducible factors of group ring contain all the irreducible representations of finite group. Hence, it is enough to find a copy of group ring in $\tilde{F}(b) \mid_b$. This is done by algebraic geometry. We can assume that $L$ induces an embedding of $\Sigma$ into $\mathbb{C}P^{N_i}$ for some $N_i$. Furthermore, since $L$ is invariant under $G_{b_1}$, $G_{b_2}$ also acts effectively naturally on $\mathbb{C}P^{N_i}$. Pick any point $x_0 \in im(\Sigma) \subset \mathbb{C}P^{N_i}$ such that $\sigma_k(x_0)$ are mutually different for any $\sigma_k \in G_{b_i}$. Then, we can find a homogeneous polynomial $f$ of some degree, say $k_i$, such that $f(x_0) \neq 0$, $f(\sigma_k(x_0)) = 0$ for $\sigma_k \neq I_{d_i}$. Note that $f \in H^0(\mathbb{C}(\Sigma))$. By pull back over $\Sigma$, $f$ induces a section $v \in H^0(\Sigma, L^k)$. We replace $L$ by $L^k$ and redefine $F_i \mid_b = H^0(\Sigma, L^k \mid_b)$. Then $G_{b_2} \cdot v$ generates a group ring, denoted by $\ll G_{b_2} \cdot v \gg$. It is obvious that $\ll G_{b_2} \cdot v \gg$ is isomorphic to $\mathbb{R}[G_{b_2}]$, so $F_i \mid_b$ contains a copy of group ring. We denote the obtained bundle by $F(k_i)$.  

18
Lemma 6.3. We have a continuous “orbi-vector bundle” $F(k_i) \to U$ such that $F(k_i) \mid_{b_i}$ contains a copy of group ring $\mathbb{R}[G_{b_i}]$.

In [20] we proved

Lemma 6.4. For the top strata, in the local coordinate system $A$ the bundle $\mathcal{F}$ is smooth. Furthermore, for any base $\{e_\alpha\}$ of the fiber at $b_0$ we can get a smooth frame fields $\{e_\alpha(a, h)\}$ for the bundle $\mathcal{F}$ over $\mathcal{O}_{b_0}(\delta_0, \rho_0)$.

Remark 6.5. Let $G_{b_0}$ be the isotropy group at $b_0$. $D^L$ is $G_{b_0}$-equivariant and $G_{b_0}$ acts on $\ker D^L\mid_{b_0}$. We may choose a $G_{b_0}$-equivariant right inverse $Q^{L \mid_{b_0}}$. So we have a $G_{b_0}$-equivariant version of Lemma 6.3. In particular, for any base $\{e_\alpha\}$ of the fiber at $b_0$ we can get a smooth $G_{b_0}$-equivariant frame fields $\{e_\alpha(a, h)\}$ for the bundle $\mathcal{F}$ over $\mathcal{O}_{b_0}(\delta_0, \rho_0)$.

Put $F = \bigoplus_{i=1}^m F(k_i)$.

6.2 Gluing the finite rank bundle $\mathcal{F}$

We recall some results in [20]. Let $(U, z)$ be a local coordinates on $\Sigma$ around a nodal point (or a marked point) $q$ with $z(q) = 0$. Let $b = (s, u) \in \mathcal{O}_{b_0}(\delta_0, \rho_0)$ and $e$ be a local holomorphic section of $u^*L\mid_U$ with $\|e\|_{C^2}(q) \neq 0$ for $q \in U$. Then for any $\phi \in \mathcal{F}\mid_b$ we can write

$$\phi\mid_U = f \left( \frac{dz}{z} \otimes e \right)^k, \quad \text{where } f \in \mathcal{O}(U). \quad (30)$$

In terms of the holomorphic cylindrical coordinates $(s, t)$ defined by $z = e^{-s+2\pi \sqrt{-1}t}$ we can re-written (30) as

$$\phi(s, t)\mid_U = f(s, t) \left( (ds + 2\pi \sqrt{-1}dt) \otimes e \right)^k,$$

where $f(z) \in \mathcal{O}(U)$. It is easy to see that $|f(s, t) - f(-\infty, t)|$ uniformly exponentially converges to 0 with respect to $t \in S^1$ as $|s| \to \infty$.

For any $\zeta \in C^\infty_c(\Sigma, \widetilde{L}\mid_b)$ and any section $\eta \in C^\infty_c(\Sigma, \widetilde{L}\mid_b \otimes \wedge^1 1^*\Sigma)$ we define weighted norms $\|\zeta\|_{j,k,2,\alpha}$ and $\|\eta\|_{j,k-1,2,\alpha}$. Denote by $W^{k,2,\alpha}(\Sigma, \widetilde{L}\mid_b)$ and $W^{k-1,2,\alpha}(\Sigma, \widetilde{L}\mid_b \otimes \wedge^1 1^*\Sigma)$ the complete spaces with respect to the norms respectively. We also define the space $W^{k,2,\alpha}(\Sigma, \widetilde{L}\mid_b)$.

Let $(\Sigma, j, y)$ be a marked nodal Riemann surface of genus $g$ with $n$ marked points. Suppose that $\Sigma$ has $e$ nodal points $p = (p_1, \cdots, p_e)$ and $e$ smooth components. We fix a local coordinate system $s \in A$ for the strata of $\overline{M}_{g,n}$, where $A = A_1 \times A_2 \times \cdots \times A_e$. Let $b_0 = (s, u)$ where $u : \Sigma \to M$ be $(j, J)$-holomorphic map. For each node $p_i$ we can glue $\Sigma$ and $u$ at $p_i$ with gluing parameters $(r) = ((r_1, \tau_1), \cdots, (r_e, \tau_e))$ to get $\Sigma_{(r)}$ and $u_{(r)}$, then we glue $\mathcal{F}\mid_b$ to get $\mathcal{F}\mid_{b_{(r)}}$. Denote $|r| = \min_{i=1}^e |r_i|$.

Lemma 6.6. $D^L\mid_{b_{(r)}}$ is surjective for $|r|$ large enough. Moreover, there is a $G_{b_{(r)}}$-equivariant right inverse $Q^{L \mid_{b_{(r)}}}$ such that

$$\|Q^{L \mid_{b_{(r)}}}\| \leq C \quad (31)$$

for some constant $C > 0$ independent of $(r)$. 

19
Lemma 6.7. (1) \( I^L_{(r)} : \ker D^L|_{b_0} \to \ker D^L|_{b(r)} \) is a \( \frac{|G_{b_0}|}{|G_{b(r)}|} \) multiple covering map for \( r_i, 1 \leq i \leq \alpha \), large enough, and
\[
\| I^L_{(r)} \| \leq C,
\]
for some constant \( C > 0 \) independent of \( (r) \).

(2) \( I^L_{(r)} \) induces an isomorphism \( I^L_{(r)} : \ker D^L|_{b_0} \to \ker D^L|_{b(r)} \).

For fixed \( (r) \) we consider the family of maps:
\[
F_{(r)} : A \times W^{k,2,\alpha}(\Sigma(r), u^*TM) \times \mathcal{W}^{k,2,\alpha}(\Sigma(r), \tilde{L}|_{b(r)}) \to W^{k-1,2,\alpha}(\Sigma(r), \wedge^0 T\Sigma(r) \otimes \tilde{L}|_{b(r)})
\]
defined by
\[
F_{(r)}(s, h, \xi) = P^L_{b(r)} \circ D^L_b \circ (P^L_{b(r)})^{-1}\xi,
\]
where \( b = ((r), s, v_r) \) and \( v_r = \exp_{a(r)} h \). By implicit function theorem we have

Lemma 6.8. There exist \( \delta > 0, \rho > 0 \) and a small neighborhood \( \tilde{O}_{(r)} \) of \( 0 \in \ker D^L|_{b(r)} \) and a unique smooth map
\[
f^L_{(r)} : \tilde{O}_{b(r)}(\delta, \rho) \times \tilde{O}_{(r)} \to W^{k-1,2,\alpha}(\Sigma(r), \wedge^0 T\Sigma(r) \otimes \tilde{L}|_{b(r)})
\]
such that for any \( (b, \zeta) \in \tilde{O}_{b(r)}(\delta, \rho) \times \tilde{O}_{(r)} \)
\[
D^L_b \circ (P^L_{b(r)})^{-1}\left( \zeta + Q^L_{b(r)} \circ f^L_{s,h,(r)}(\zeta) \right) = 0.
\]

Together with \( I^L_{(r)} \) we have gluing map
\[
Glu^L_{(r)} : F|_{[b_0]} \to F|_{[b]} \quad \text{for any} \ [b] \in O_{[b(r)]}(\delta, \rho)
\]
defined by
\[
Glu^L_{(r)}([\zeta]) := \left( (P^L_{b(r)})^{-1}\left( I^L_{(r)}\zeta + Q^L_{b(r)} \circ f^L_{s,h,(r)}I^L_{(r)}\zeta \right) \right), \quad \forall [\zeta] \in F|_{[b_0]}.
\]

Given a frame \( e_\alpha(z) \) on \( \tilde{F}|_{b_0}, 1 \leq \alpha \leq \text{rank } \tilde{F} \), as Remark 6.5 we have a \( G_{b_0} \)-equivariant frame field
\[
e_\alpha((r), s, h)(z) = (P^L_{b,b(r)})^{-1}\left( I^L_{(r)}e_\alpha + Q^L_{b(r)} \circ f^L_{s,h,(r)}I^L_{(r)}e_\alpha \right)(z)
\]
over \( D^*_{R_0}(0) \times \tilde{O}_{b_0}(\delta_0, \rho_0) \), where \( z \) is the coordinate on \( \Sigma \), and
\[
D^*_{R_0}(0) := \bigoplus_{i=1}^\infty \{ (r, \tau) \mid R_0 < r < \infty, \tau \in S^1 \}.
\]
For any fixed \( (r) \), \( e_\alpha \) is smooth with respect to \( s, h \) over \( \tilde{O}_{b_0}(\delta_0, \rho_0) \).

To discuss the smoothness with respect to \( (r), s, h \) we need to fix a Riemann surface \( \Sigma_{(R_0)} \). Let \( \alpha_{(r)} : [0, 2r_i] \to [0, 2R_0] \) be a smooth function satisfying
\[
\alpha_{(r)}(s) = \begin{cases} 
\frac{R_0}{2} + \frac{R_0}{2r_i - R_0}(s - R_0/2) & \text{if } s \in [0, \frac{R_0}{2} - 1] \\
\frac{R_0}{2} + \frac{R_0}{2r_i + 2R_0}(s - R_0/2) & \text{if } s \in [\frac{R_0}{2r_i - R_0}, \frac{R_0}{2} + 1, 2r_i]
\end{cases}
\]
Set \( \alpha_{(r)} : [-2r, 0] \rightarrow [-2R_0, 0] \) by \( \alpha_{(r)}(s) = -\alpha_{(r)}(-s) \). Let \((s_1^i, t_1^i)\) and \((s_2^i, t_2^i)\) be cusp cylinder coordinates around \( p_i \), thus \( z_i = e^{-s_1^i-2\pi\sqrt{-1}t_1^i} \) and \( u_i = e^{s_2^i+2\pi\sqrt{-1}t_2^i} \). Denote

\[
W_i(R) = \{|s_1^i| > R\} \cup \{|s_2^i| > R\}.
\]

Obviously, \( W(R) = \bigcup_{i=1}^c W_i(R) \). We can define a map \( \varphi(r) : \Sigma(r) \rightarrow \Sigma(R) \) as follows:

\[
\varphi(r) = \begin{cases} p, & p \in \Sigma(R_0/4), \\ (\alpha_{(r)}(s_i), t_i), & (s_i^i, t_i) \in W_i(R_0/4), \ i = 1, \ldots, c. \end{cases}
\]

Then we obtain a family of Riemann surfaces \( \left( \Sigma(R_0), (\varphi^{-1}_r)^*f_r, (\varphi^{-1}_r)(y) \right) \). Denote \( u_0^r := u(r) \circ \varphi_r^{-1} \).

In [20] we have proved the following lemma.

**Lemma 6.9.** There exists positive constants \( d, R \) such that for any \( h \in W^{k,2,\alpha}(\Sigma(R_0), (u(R_0))^*TM) \), \( \zeta \in \ker D^L|_{b_0} \) with

\[
\|\zeta\|_{W^{k,2,\alpha}} \leq d, \quad \|h - \hat{h}(r)\| < d, \quad |r| \geq R,
\]

\( (\varphi_r^{-1})(\text{Gl}_{\mu_b(r,h)}(e)) \) is smooth with respect to \((s_i, t_i), h\) for any \( e \in \ker D^L|_{b_0} \), where \( h' = \exp_{u_i'(r)} \circ \exp_{u_i(R_0)}(h) \circ \varphi(r) \). In particular \( \text{Gl}_{\mu_b(r,h)}(e) |_{\Sigma(R_0)} \) is smooth.

### 6.3 Global regularization and virtual neighborhoods

We are going to construct a bundle map \( i : F \rightarrow \mathcal{E} \). We first define a bundle map \( i : F|_{k_i} \rightarrow \mathcal{E} \). Consider two different cases:

**Case 1.** \([b_i]\) lies in the top strata \( \mathcal{M}_{g,n}(A) \). Denote \( b_0 = b_i \). Choose a local coordinate system \((\psi, \Psi)\) for \( Q \) and a local model \( \tilde{O}_{b_0}(\delta_{b_0}, \rho_{b_0})/G_{b_0} \) around \([b_0]\). We have an isomorphism

\[
P_{b_0,b} = \Phi \circ \Psi_{j_0,3a} : \tilde{E}_{b_0} \rightarrow \tilde{E}_b, \quad \forall \ b \in \tilde{O}_{b_0}(\delta_{b_0}, \rho_{b_0}). \tag{34}
\]

To simplify notations we denote \( \tilde{F}(k_i) = \tilde{H} \), \( P_{b_0,b} = P \) in this section.

Choosing a base \( \{e_{a}\} \) of the fiber \( \tilde{H}|_{b_0} \), by Lemma 6.4 we can get a smooth frame fields \( \{e_{a}\} \) for the bundle \( \tilde{H} \) over \( \tilde{O}_{b_0}(\delta_{b_0}, \rho_{b_0}) \), which induces another isomorphism

\[
Q : \tilde{H}|_{b_0} \rightarrow \tilde{H}|_{b}, \quad \forall \ b \in \tilde{O}_{b_0}(\delta_{b_0}, \rho_{b_0}) \tag{35}
\]

\[
\sum c_{a}e_{a}|_{b_0} \rightarrow \sum c_{a}e_{a}|_{b}. \tag{36}
\]

Let \( \rho_{\tilde{K}_{b_0}} : G_{b_0} \rightarrow GL(\tilde{K}_{b_0}) \) be the natural linear representation, and let \( \rho_{R} : G_{b_0} \rightarrow GL(\mathbb{R}[G_{b_0}]) \) be the standard representation. Both \( \tilde{K}_{b_0} \) and \( \tilde{H}|_{b_0} \) can be decomposed as sum of irreducible representations. Without loss of generality we assume that \( \rho_{\tilde{K}_{b_0}} \) is an irreducible representation. Let \( \eta_1, \ldots, \eta_l \) be a base of \( \tilde{K}_{b_0} \), let \( \tilde{H}|_{b_0} = \bigoplus_{i=1}^l E_i \) be the decomposition of irreducible representations such that \( E_i \) has base \( e_1, \ldots, e_i \). Define map \( em(\eta_i) = e_i, i = 1, \ldots, l \). Thus we have map \( p : \tilde{H}|_{b_0} \rightarrow \tilde{K}_{b_0} \) with \( p \cdot em = id \).

Let \( \mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\} \) and \( f_{\delta_{o},\rho_{o}} : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be a smooth cut-off function such that

\[
f_{\delta_{o},\rho_{o}}(x, y) = \begin{cases} 1 & \text{on } \{(x, y)| 0 \leq x \leq \delta_{o}/3, 0 \leq y \leq \rho_{o}/3\}, \\ 0 & \text{on } \{(x, y)| x \geq 2\delta_{o}/3 \cup \{(x, y)| y \geq 2\rho_{o}/3\}. \end{cases}
\]
We define a cut-off function $\alpha_{b_o} : \vec{O}_b_o(\delta_{b_o}, \rho_{b_o}) \to [0, 1]$ by

$$\alpha_{b_o}(b) = f_{\delta_{b_o}, \rho_{b_o}}(d_A^2(a_o, a), \|h\|^2_{j_{a,k,2}}).$$  \hfill (37)

For any $\kappa \in \vec{H} \mid_b$ with $b \in \vec{O}_b_o(\delta_{b_o}, \rho_{b_o})$, in terms of the local coordinate system $(\psi, \Psi)$, we define

$$i(\kappa, b)_{b_o} = \begin{cases} \alpha_{b_o}(b) P \circ p \circ Q^{-1}(\kappa) & \text{if } \|h\|_{j_{a,k,2}} < \rho_{b_o}, \text{ and } d_A^2(a_o, a) < \delta_{b_o} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6.10. In the local coordinates $(\psi, \Psi)$ on $U$ and in $\vec{O}_b_o(\delta_{b_o}, \rho_{b_o})$ the bundle map $i(\kappa, b)_{b_o} : \vec{F}(k_i) \to \vec{E}$ is smooth with respect to $(\kappa, a, h)$.

Proof. By Lemma 4.2, we immediately obtain that the cut-off function $\alpha_{b_o}(b)$ is a smooth function. Note that, in the local coordinates $(\psi, \Psi)$, $P$, $p$ and $Q^{-1}$ are smooth. We conclude that $i(\kappa, b)_{b_o}$ is a smooth function of $(\kappa, a, h)$.

We can transfer the definition to other local coordinate system $(\psi', \Psi')$ and local model $\vec{O}_b_o(\delta'_{b_o}, \rho'_{b_o})$. Suppose that in the coordinate system $(\psi, \Psi)$

$$b_o = (a_o, u_o), \quad b = (a, v), \quad v = \exp_{u_o} h,$$

and in the coordinate system $(\psi', \Psi')$

$$b'_o = (a'_o, u'_o), \quad b' = (a', v'), \quad v' = \exp_{u'_o} h', \quad \text{where } [b] = [b'].$$

We have

$$(\psi' \circ \psi^{-1}, \Psi' \circ \Psi^{-1}) \cdot (a, v) = (a', v'), \quad a' = \psi' \circ \psi^{-1}(a), \quad v' = v \circ (\Psi' \circ \Psi^{-1})_{|a}.$$

$$(\psi' \circ \psi^{-1}, \Psi' \circ \Psi^{-1}) \cdot (a_o, u_o) = (a'_o, u'_o), \quad a'_o = \psi' \circ \psi^{-1}(a), \quad u'_o = u_o \circ (\Psi' \circ \Psi^{-1})_{|a_o}.$$

$(\psi' \circ \psi^{-1}, \Psi' \circ \Psi^{-1})$ send $e_{a_o}$ to $e'_{a_o}$. Then $(\Psi' \circ \Psi^{-1})_{|a}$ induces an isomorphism $\varphi_a : \vec{H} \mid (a,v) \to \vec{H}' \mid (a',v')$. In $(\psi', \Psi')$ we have isomorphism

$$Q' : \vec{H}' \mid (a'_o, u'_o) \to \vec{H} \mid (a_o, v), \quad \forall b \in \vec{O}_b_o(\delta'_{b_o}, \rho'_{b_o}).$$

We have chosen a finite dimensional subspace $\vec{K}(a,v) \subset \vec{E} \mid (a,v)$ in $(\psi, \Psi)$. Denote $\vartheta_a = (\Psi' \circ \Psi^{-1})_{|a}$. Define $\vec{K}'(a',v') = \{ \kappa \circ d\vartheta_a^{-1} \mid \forall \kappa \in \vec{K}(a,v) \}$. Then $(\Psi' \circ \Psi^{-1})_{|a}$ induces a map

$$\phi_a : \vec{K}(a,v) \to \vec{K}'(a',v'), \quad \phi_a(\kappa) = \kappa \circ d\vartheta_a^{-1}, \quad \forall \kappa \in \vec{K}(a,v).$$  \hfill (38)

Denote $\kappa' = \phi_a(\kappa)$. Define

$$P' : \vec{E}(a'_o, u'_o) \to \vec{E}'(a', v'), \quad \text{by } P' = \phi_a \circ P \circ \phi_a^{-1},$$

and

$$p' : \vec{H}' \mid (a'_o, u'_o) \to \vec{K}'(a'_o, u'_o), \quad \text{by } p' = \phi_a \circ p \circ \varphi_a^{-1}.$$  

$(\Psi' \circ \Psi^{-1})_{|a}$ also induces a map

$$\lambda_a : G(a_o, u_o) \to G(a'_o, u'_o) \quad g \longrightarrow g' = d\vartheta_a \circ g \circ (d\vartheta_a)^{-1}.$$
It is easy to check that $\rho_{\tilde{K}(a_u, u_o)} : G(a_u, u_o) \to GL(\tilde{K}(a_u, u_o))$ and $\rho_{\tilde{K}'(a'_u, u'_o)} : G(a'_u, u'_o) \to GL(\tilde{K}'(a'_u, u'_o))$ are equivariant. Let

$$\eta_i = \phi_a(\eta_i), \ e'_i = \varphi_a(e_i), \ em'(\eta_i) = e'_i, \ i = 1, 2, \ldots, l.$$ 

Then $em'(\tilde{K}'(a'_u, u'_o)) = \text{span}\{e'_1, \ldots, e'_l\} \subset \tilde{H}' |_{(a'_u, u'_o)}$. In the coordinate system $(\psi', \Psi')$ we define

$$i(\kappa', b')_{b_o} = \begin{cases} \alpha_{b_o}(b') P' \circ p' \circ (Q')^{-1}(\kappa') & \text{if } ||h||_{j_a, k_2} < \rho_{b_o}, \text{ and } d^2_{\tilde{A}}(a'_o, a') < \delta_{b_o} \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$i(\kappa', b')_{b_o} = \phi_a \circ i(\kappa, b)_{b_o} \circ \varphi_a^{-1}. \quad (39)$$

If we choose three local coordinate systems $(\psi, \Psi), (\psi', \Psi')$ and $(\psi'', \Psi'')$, since

$$(\Psi \circ (\Psi'')^{-1}) \circ (\Psi'' \circ (\Psi')^{-1}) \circ (\Psi' \circ (\Psi')^{-1}) = Id,$$

one can easily check that

$$\phi''_{a'} \circ \phi_{a'} \circ \phi_a = Id, \quad \varphi''_{a'} \circ \varphi_{a'} \circ \varphi_a = Id. \quad (40)$$

It follows from (39) and (40) that the bundle map $i : F(k_i) \to E$ is well defined. Obviously, $i([\kappa_i, b]) = [i(\kappa_i, b)]$.

**Remark 6.11.** Let $(\psi', \Psi')$ be a local coordinate system in $O(b'_o)(\delta'_{b'_o}, \rho'_{b'_o}) \subset O(b_o)(\delta_{b_o}, \rho_{b_o})$ such that $[b_o] \notin O(b'_o)(\delta'_{b'_o}, \rho'_{b'_o})$. The restriction of $[i(\kappa, b)_{b_o}]$ to $O(b'_o)(\delta'_{b'_o}, \rho'_{b'_o})$ is an element in $E |_{O(b'_o)(\delta'_{b'_o}, \rho'_{b'_o})}$. We can transfer it to $(\psi', \Psi')$ by (38).

**Case 2.** $[b_i]$ lies in lies in a lower strata. We choose $(s, t)$ coordinates. Put $t_i = e^{-2\pi \tau}t_i$, sometimes we use $(s, (r))$ coordinates, where $(r) = ((r_1, \tau_1), \ldots, (r_t, \tau_t))$. Denote $b_o = b_i = (0, 0, u)$. $F(k_i) = H(s, t), F(k_i) |_{b_o} = H(0, 0)$. We choose $|s|, |t|$ small enough. In terms of $(s, t)$ we have an isomorphism

$$P : \tilde{E}_{b_o} \to \tilde{E}_b, \quad \forall \ b \in \tilde{O}_{b_o}(\delta_o, \rho_o).$$

Denote $H = \{\zeta | \Sigma(R_o) \mid \zeta \in \tilde{H}\}$. Choosing a base $\{e_a\}$ of the fiber $H |_{b_o}$, by (33) we can get a frame fields $\{e_a((r), a, h) \mid \Sigma(R_o)\}$ for the bundle $\tilde{H}$ over $\tilde{O}_{b_o}(\delta_o, \rho_o)$. We have another isomorphism in the $(s, t)$ coordinates

$$Q : H(0, 0) \to \tilde{H}(s, t), \quad \forall \ b \in \tilde{O}_{b_o}(\delta_o, \rho_o).$$

Denote $O(\delta_o) = \{p \in \tilde{M}_{A,n} | d^2_{\text{map}}(0, p) < \delta_o\}$. Since $\tilde{M}_{A,n}$ has a natural effective orbifold structure, we can choose a smooth cut-off function in orbit sense $\beta_{b_o} : O(\delta_o) \to [0, 1]$ such that

$$\beta_{b_o} |_{O(\delta_o/3)} = 1, \quad \beta_{b_o} |_{O(\delta_o), O(2\delta_o/3)} = 0.$$

We define a cut-off function $\alpha_{b_o} : \tilde{O}_{b_o}(\delta_o, \rho_o) \to [0, 1]$ by

$$\alpha_{b_o}(b) = f_{\delta_o, \rho_o}(\beta_{b_o}(s, t), ||\beta_{R_o}(h)||^2_{j_a, k_2}), \quad (41)$$

where $\beta_{R_o}$ is the function in (28). Using $\alpha_{b_o}(b)$ defined in (41), we can define the bundle map $i : F(k_i) \to E$ by

$$i(\kappa, b)_{b_o} = \begin{cases} \alpha_{b_o}(b) P \circ p \circ Q^{-1}(\kappa) & \text{if } ||h||_{j_a, k_2} < \rho_{b_o}, \text{ and } \beta_{b_o}(s, t) < \delta_{b_o} \\ 0 & \text{otherwise.} \end{cases}$$

23
For any fixed \((r), i(\kappa, b)_{b_0}\) and \(Q\) are smooth with respect to \((s, h)\) in the coordinates \((s, (r))\). In order to study the smoothness with respect to \((r)\) we note that \(i(\kappa, b)_{b_0}\) is supported in \(\Sigma(R_0)\). For any \(v = \exp_{u(r)} h\), we let

\[
h^v_0 = \left((h - \tilde{h}_0)(s_1, t_1), \beta_1; \ldots, (h - \tilde{h}_0)(s_2, t_2), \beta_2; \ldots\right),
\]

where

\[
h_0 = \int_{s^1} h(r, t) dt.
\]

Denote \(v^0 = \exp_{u} h^0\). We can view \(\tilde{E}|_v\) to be \(\tilde{E}|_{v^0}\). Then we view \(P\) to be a family of operators in \(E\) over \(W^{k,2}(\Sigma; u^*TM)\), where \(E \rightarrow W^{k,2}(\Sigma; u^*TM)\) is independent of \((r)\). Consider the map

\[
i(\kappa, b)_{b_0} \circ Q : \tilde{H}(0, 0) \times A \times D^*_{R_0}(0) \times W^{k,2}(\Sigma; u^*TM) \rightarrow E
\]

\[
i(\kappa, b)_{b_0} \circ Q(\kappa, s, (r), h) = \alpha_{b_0}(s, (r), v) P \circ p(\kappa).
\]

Lemma 6.12. In the local coordinates \((s, (r))\), the bundle map \(i(\kappa, b)_{b_0} \circ Q\) is smooth with respect to \((\kappa, s, (r), h)\) in \(\tilde{O}_{b_0}(\delta_0, \rho_0)\).

Proof. \(\alpha_{b_0}(s, (r), v)\) is smooth with respect to \((s, (r), h)\). For any \(l \in Z^+\), denote \(b_k = (s, \exp_{u}(h + \sum_{i=1}^{l} t_i h_i))\) and

\[
T^l(h; h_1, \ldots, h_l) = \nabla_{h_1} \cdots \nabla_{h_1} (P_{b_0,b_k})|_{t=0}
\]

By the same method as in the proof of Lemma 3.1 of [20] we can show that \(T^l(h; \cdots)\) is a bounded linear operator. The proof is complete. \(\square\)

By Case 1, Case 2 we have defined \(i([\kappa_i, b])_t\) for all \(i = 1, \ldots, m\). Set

\[
i([\kappa, b]) = \sum_{i=1}^{m} i([\kappa_i, b])_t \text{ for any } \kappa = (\kappa_1, \ldots, \kappa_m) \in F|_b.
\]

Then \(i : F \rightarrow E\) is a bundle map. We define a global regularization to be the bundle map \(S : F \rightarrow E\)

\[
S([\kappa, b]) = [\tilde{\partial}_{\mu_j} v] + i([\kappa, b])
\]

It is obvious that \(DS\) is surjective. Denote \(p : F \rightarrow \mathcal{U}\) by the projection of the bundle. Set

\[
\mathcal{U} = S^{-1}(0)|_{p^{-1}(\mathcal{U})}.
\]

By restricting the bundle \(p^*F\) to \(\mathcal{U}\) we have a bundle \(p : E \rightarrow \mathcal{U}\) of finite rank with a canonical section \(\sigma\) defined by

\[
\sigma([\kappa, b]) = ([([\kappa, b], \kappa)]) \text{, } \forall ([\kappa, b]) \in \mathcal{U}.
\]

We call

\[(\mathcal{U}, E, \sigma),\]

a virtual neighborhood for \(\mathcal{M}_{g,n}(A)\).

7 Smoothness of the top strata

Proof of Theorem 1.1

The proof is divided into two steps, the subsections §7.1 and §7.2.
7.1 Smoothness

Let \([\kappa_o, b_o] \in U_T\). To simplify notations we consider the following case, for the general case the argument are the same. We assume that

\[ [b_o] \in \mathcal{O}_{[b_1]}(2\delta_1/3, 2\rho_1/3) \bigcap \mathcal{O}_{[b_2]}(2\delta_2/3, 2\rho_2/3) \]

and

\[ [b_o] \notin \overline{\mathcal{O}}_{[b_i]}(2\delta_i/3, 2\rho_i/3) \quad \forall i = 3, \ldots, m. \]

We choose a local coordinate system \((\psi, \Psi)\) for \(Q\) and local model \(\tilde{O}_{b_o}(\delta_o, \rho_o)/G_{b_o}\) around \(b_o\). Let \(b_o = (a_o, u)\), and let \(\tilde{U}^T\) be the local expression of \(U^T\) in terms of \((\psi, \Psi)\). We choose \((\delta_o, \rho_o)\) so small that

\[ \mathcal{O}_{[b_i]}(\delta_o, \rho_o) \notin \mathcal{O}_{[b_i]}(2\delta_i/3, 2\rho_i/3) \quad \forall i = 3, \ldots, m. \]

Then we only need to consider the bundles \(F(k_1)\) and \(F(k_2)\). We consider two different cases.

**Case 1.** Both \([b_1]\) and \([b_2]\) lie in the top strata. By Remark 6.1 we may assume that both \(\mathcal{O}_{[b_1]}(2\delta_1/3, 2\rho_1/3)\) and \(\mathcal{O}_{[b_2]}(2\delta_2/3, 2\rho_2/3)\) lie in the top strata. Let

\[ b_1 = (a_1, u_1) \text{ in } (\psi_1, \Psi_1), \quad b_2 = (a_2, u_2) \text{ in } (\psi_2, \Psi_2). \]

In terms of the coordinate system \((\psi, \Psi)\), let \(b = (a, v) \in \tilde{O}_{b_o}(\delta_o, \rho_o)\). Suppose that, in the coordinate system \((\psi_1, \Psi_1)\),

\[ [b'] = [b], \quad b' = (a', v'), \quad v' = \exp_{a_1} h_1, \]

and in the coordinate system \((\psi_2, \Psi_2)\),

\[ [b'''] = [b], \quad b''' = (a'', v''), \quad v''' = \exp_{a_2} h_2. \]

The bundle maps are given respectively by

\[ i(\kappa_1, b')_{b_1} = \alpha_{b_1}(b) P_1 \circ p_1 \circ Q^{-1}_1(\kappa_1) : (\tilde{H}_1) \bigg|_{b'} \rightarrow \tilde{K}_1 \bigg|_{b'} \quad \text{in } (\psi_1, \Psi_1), \]

\[ i(\kappa_2, b''')_{b_2} = \alpha_{b_2}(b) P_2 \circ p_2 \circ Q^{-1}_2(\kappa_2) : (\tilde{H}_2) \bigg|_{b'''} \rightarrow \tilde{K}_2 \bigg|_{b'''} \quad \text{in } (\psi_2, \Psi_2), \]

where \(P_1 = P_{b_1,b'}\) in \((\psi_1, \Psi_1)\), \(P_2 = P_{b_2,b'''}\) in \((\psi_2, \Psi_2)\). By Lemma 6.10, \(i(\kappa_1, b)_{b_1}\) in \((\psi_1, \Psi_1)\) (resp. \(i(\kappa_2, b)_{b_2}\) in \((\psi_2, \Psi_2)\)) is smooth with respect to \(\kappa_1, b\) (resp. \(\kappa_2, b\)).

We transfer from both the local coordinate systems \((\psi_1, \Psi_1)\) and \((\psi_2, \Psi_2)\) to the coordinates \((\psi, \Psi)\). We have

\[ (\psi \circ \psi^{-1}_1, \Psi \circ \Psi^{-1}_1) \cdot (a', v') = (a, v), \quad a = \psi \circ \psi^{-1}_1(a'), \quad v = v' \circ (\Psi \circ \Psi^{-1}_1) \big|_{a'}, \]

\[ (\psi \circ \psi^{-1}_2, \Psi \circ \Psi^{-1}_2) \cdot (a'', v'') = (a, v), \quad a = \psi \circ \psi^{-1}_2(a''), \quad v = v'' \circ (\Psi \circ \Psi^{-1}_2) \big|_{a''}. \]

The \((\psi \circ \psi^{-1}_i, \Psi \circ \Psi^{-1}_i), i = 1, 2\), induces maps

\[ \phi_{a'}^1 : \tilde{K}_1 \rightarrow \tilde{K}_1^\circ, \quad \phi_{a''}^2 : \tilde{K}_2 \rightarrow \tilde{K}_2^\circ \]

\[ \varphi_{a'}^1 : \tilde{H}_1 \rightarrow \tilde{H}_1^\circ, \quad \varphi_{a''}^2 : \tilde{H}_2 \rightarrow \tilde{H}_2^\circ. \]

Put

\[ \tilde{H}^\circ = (\tilde{H}^\circ_1) |_{b_o} \oplus (\tilde{H}^\circ_2) |_{b_o}, \quad \kappa = (\kappa_1, \kappa_2) \in \tilde{H}^\circ \quad (Q^1_1\kappa_1, Q^2_2\kappa_2) := Q^\circ \kappa. \]

Here \(\tilde{H}^\circ, \tilde{K}^\circ\) and \(Q^\circ\) denote the spaces and operator in \((\psi, \Psi)\). By Remark 6.11 the bundle map in \((\psi, \Psi)\) becomes

\[ i(\kappa, b) = i(\kappa_1, b')_{b_1} \circ d\varphi_{a'}^1 + i(\kappa_2, b'')_{b_2} \circ d\varphi_{a''}^2, \]
Consider the map

\[ F_{(\kappa_o, b_o)} : \mathbf{A} \times \tilde{H}^\circ \times W^{k,2}(\Sigma; u^*TM) \rightarrow W^{k-1,2}(u^*TM \otimes \wedge^{0,1}) \]

\[ F_{(\kappa_o, b_o)}(a, \kappa, h) = P_{b_o b_o} \left( \partial_{j_a, J} v + i(Q^o \kappa, b) \right), \]

where \( b = (a, v) \), \( v = \exp_a(h) \) for some \( h \in W^{k,2}(\Sigma, u^*TM) \). For any \( (a, \kappa, h) \in F_{(\kappa_o, b_o)}(0) \) we have

\[ \partial_{j_a, J} v + i(Q^o \kappa, b) = 0, \]

where \( b = (a, v) \). For any fixed \( a \), it follows from the standard elliptic estimates and the smoothness of \( i \) that \( v \in C^\infty(\Sigma, M) \). Then by Lemma 6.10 and the smoothness of the frame field \( e_a \) we conclude that \( i|_a \) and \( Q^o |_a \) are smooth with respect to \( (a, \kappa, h) \). It is easy to see that \( F_{(\kappa_o, b_o)}(a, \kappa, h) \) is smooth with respect to \( (a, \kappa, h) \). Then we use the implicit theorem with parameter \( a \) to conclude that \( v \) is smooth with respect to \( (a, \kappa, h) \). It follows that \( \tilde{U}^T \cap p^*\bar{O}_{b_o}(\delta_o, \rho_o) \) is smooth, where \( p : \tilde{U}^T \rightarrow \tilde{B} \) is the projection.

**Case 2.** \([b_o]\) lies in the top strata, \([b_1]\) lies in a lower strata. Without loss of generality we assume that \( b_1 = (\Sigma, j, y, u) \), where \( \Sigma \) has one node \( q \), \( s_o \in \mathbf{A} = \mathbf{A}_1 \times \mathbf{A}_2 \). We glue \( \Sigma \) at \( q \) with gluing parameter \( (r) \). We have bundle maps \( i(\kappa_1, b_1)_b = \alpha_1(b)P_1 \circ p_1 \circ Q^{-1}_1(\kappa_1) \) and \( i(\kappa_2, b'_2)_b = \alpha_2(b')P_2 \circ p_2 \circ Q^{-1}_2(\kappa_2) \). Then we transfer to the coordinates \((\psi, \Psi)\), and choose \((s, t)\)-coordinates. We use Lemma 6.12 and the same method as in **Case 1** to prove that \( v \) is smooth with respect to \((s, \kappa, h)\). Then we use Lemma 6.9 to prove that \( Q_o^1 \) is smooth with respect to \((s, \kappa, h)\). Then we can prove the smoothness of \( \tilde{U}^T \cap p^*\bar{O}_{b_o}(\delta_o, \rho_o) \).

The proof of the orientation of \( \tilde{U}^T \) is standard, we omit here.

### 7.2 The orbifold structure

We introduce a notation. For any \((\kappa_o, b_o) \in U\) we choose a local coordinate system \((\psi, \Psi)\) on \( U \ni a_o \) and local model \( \bar{O}_{b_o}(\delta_o, \rho_o)/G_{b_o} \). Set

\[ \tilde{U}_{\kappa_o, b_o}(\varepsilon, \delta_o, \rho_o) = \left\{ (\kappa, b) \in \tilde{U} \mid |\kappa - \kappa_o|_h < \varepsilon, b \in \bar{O}_{b_o}(\delta_o, \rho_o) \right\}, \]

\[ U_{\kappa_o, b_o}(\varepsilon, \delta_o, \rho_o) = \tilde{U}_{\kappa_o, b_o}(\varepsilon, \delta_o, \rho_o)/G_{\kappa_o, b_o}, \]

where \( G_{\kappa_o, b_o} \) is the isotropy group at \((\kappa_o, b_o)\). For any \((\kappa, b) \in \tilde{U}_{\kappa_o, b_o}(\varepsilon, \delta_o, \rho_o)\) denote by \( G_{\kappa, b} \) the isotropy group at \((\kappa, b)\). Any element \( \varphi \in G_{\kappa, b} \) satisfies \( \varphi^*(\kappa, b) = (\kappa, b) \). It follows that \( G_{\kappa, b} \) is a subgroup of \( G_a \).

**Lemma 7.1.** Let \([(\kappa_o, b_o)] \in U^T\). Suppose that \( \tilde{U}_{\kappa_o, b_o}(\varepsilon, \delta_o, \rho_o) \subset U^T\). The following hold

1. For any \( p \in \tilde{U}_{\kappa_o, b_o}(\varepsilon, \delta_o, \rho_o) \) let \( G_p \) be the isotropy group at \( p \), then \( \text{im}(G_p) \) is a subgroup of \( G_{\kappa_o, b_o} \).

2. Let \( p \in \tilde{U}_{\kappa_o, b_o}(\varepsilon, \delta_o, \rho_o) \) be an arbitrary point with isotropy group \( G_{p_o} \), then there is a \( G_p \)-invariant neighborhood \( O(p) \subset \tilde{U}_{\kappa_o, b_o}(\varepsilon, \delta_o, \rho_o) \) such that for any \( q \in O(p) \), \( \text{im}(G_q) \) is a subgroup of \( G_p \), where \( G_p \) denotes the isotropy groups at \( p \) and \( q \) respectively.
**Proof:** We only prove (1), the proof of (2) is similar. Denote $b_o = (a_o, u)$. If the lemma not true, we can find a sequence $(\kappa_i, b_i) = (\kappa_i, a_i, u_i) \in \tilde{U}_{n_o} (\varepsilon, \delta_o, \rho_o)$ such that

1. $\delta_i \to 0$, $\rho_i \to 0$, $\kappa_i \to \kappa_o$, 
2. $\text{im}(G_{\kappa_i, b_i})$ is not a subgroup of $G_{\kappa_o, b_o}$.

It is obvious that $G_{\kappa_o, b_o}$ is a subgroup of $G_{\kappa_i, b_i}$, $G_{\kappa_i, b_i}$ is a subgroup of $G_{\kappa_i}$ and $G_{\kappa_i}$ can be imbedded into $G_{\kappa_a}$ as a subgroup for $i$ large enough. So we can view $\text{im}(G_{\kappa_i, b_i})$ as a subgroup of $G_{\kappa_o}$. By choosing a subsequence we may assume that $\text{im}(G_{\kappa_i, b_i})$ converges to a subgroup $G_{\kappa_o, b}$ of $G_{\kappa_o}$ and $\text{im}(G_{\kappa_i, b_i}) \cdot u_i$ converges to $\text{im}(G_{\kappa_o, b}) \cdot u$ and $U_i$ converges to $u$ in $W^{k,2}$. By Sobolev imbedding theorem and elliptic estimates we have $\text{im}(G_{\kappa_i, b_i}) \cdot (\kappa_i, u_i)$ converges to $\text{im}(G_{\kappa_o, b}) \cdot (\kappa_o, u)$, $(\kappa_i, u_i)$ converges to $(\kappa_o, u)$ in $C^\infty$ for any $\ell > 1$. It follows that $\text{im}(G_{\kappa_o, b}) \subset G_{\kappa_o, b_o}$. Since there are only finite many subgroups of $G_{\kappa_o}$, for $i$ large enough we have $\text{im}(G_{\kappa_i, b_i}) = G_{\kappa_o, b}$. So $G_{\kappa_i, b_i}$ can be imbedded into $G_{\kappa_o, b_o}$ as a subgroup for $i$ large enough. We get a contradiction.

As corollary of Lemma 7.1 we conclude that $U^T$ is an orbifold. Since $(g, n) \neq (1, 1), (2, 0)$, $U^T$ has the structure of an effective orbifold.

Combination of the subsections 7.1, 7.2 give us the proof of Theorem 1.1.

### 7.3 A metric on E

In this section we construct a metric on $E|_{U}$. By the compactness of $U_{2\varepsilon}$ we may find finite many points $(\kappa_1, b_1), \ldots, (\kappa_n, b_n) \in U_{\varepsilon}$ such that

- $\{U_{[(\kappa_i, b_i)]}(\varepsilon, \delta_{\kappa_i}, \rho_{\kappa_i}), 1 \leq a \leq n\}$ is a covering of $U_{2\varepsilon}$.
- For any $a \in \{1, \ldots, n\}$ there is $i_a \in \{1, \ldots, m\}$ such that $p(U_{[(\kappa_i, b_i)]}(\varepsilon_a, \delta_a, \rho_a)) \subset O_{b_{i_a}}(\delta_{i_a}, \rho_{i_a})$,

where $O_{b_{i_a}}(\delta_{i_a}, \rho_{i_a})$ is as in subsection 6.1.

- $\tilde{U}_{(\kappa_n, b_n)}(\varepsilon, \delta_n, \rho_n) \subset \tilde{U}^T$ for all $1 \leq a \leq n$.

Let $\{e_{\alpha}^a\}_{1 \leq \alpha \leq \ell}$ be a local smooth frame field of $F$ over $O_{b_{i_a}}(\delta_{i_a}, \rho_{i_a})$ as in section 6.3. Let $p : U \to \mathcal{U}$ denote the projection. Denote $e_{\alpha}^a = p^* e_{\alpha}^a|_{U_{[(\kappa_n, b_n)]}(\varepsilon_a, \delta_{\kappa_n}, \rho_{\kappa_n})}$. Then we have a smooth frame field $\{e_{\alpha}^a\}_{1 \leq \alpha \leq \ell}$ of $F$ over $\tilde{U}_{[(\kappa_n, b_n)]}(\varepsilon_a, \delta_n, \rho_n)$, where $r$ denotes the rank of $E$. We define a local metric $h_a$ on $E|_{U_{[(\kappa_n, b_n)]}(\varepsilon_a, \delta_n, \rho_n)}$ by $h_a(e_{\alpha}^a, e_{\beta}^a) = \delta_{\alpha\beta}$.

Now we choose smooth cutoff functions $\Gamma'$ as follows. Let $(\kappa_o, b_o)$ be one of $(\kappa_1, b_1), \ldots, (\kappa_n, b_n)$. We consider two cases:

1. $(\kappa_o, b_o)$ lies in $\tilde{U}^T$. We define a cut-off function $\alpha_{b_o} : \tilde{O}_{b_o}(\delta_{b_o}, \rho_{b_o}) \to [0, 1]$ by (37) and let $\Gamma'_o = p^* \alpha_{b_o}(b)$.

2. $(\kappa_o, b_o)$ lies in a lower strata. We define a cut-off function $\alpha_{b_o} : \tilde{O}_{b_o}(\delta_o, \rho_o) \to [0, 1]$ by (41) and let $\Gamma'_o = p^* \alpha_{b_o}(b)$.

Thus we have $\Gamma'_a$ for every $1 \leq a \leq n$. Set $G_a = \frac{\Gamma'_a}{\sum_{l=1}^n \Gamma'_l}$. 

27
Then $\sum \Gamma_a = 1$ and $\Gamma_a$ is smooth on $U_\epsilon^T$ in orbifold sense. We define a metric $h$ on $E$ over $U_\epsilon$ by

$$h = \sum_{a=1}^n \Gamma_a h_a.$$  

We define a connection on $E$ as follows. Let $\{e_a^\alpha\}_{1\leq a\leq r}$ be a local smooth frame field of $E$ over $U_{[(\kappa_a, b_a)]}(\epsilon, \delta_a, \rho_a)$ as above. Consider the Gram-Schmidt process with respect to the metric $h$ and denote by $\hat{e}_1^a, \ldots, \hat{e}_r^a$ the Gram-Schmidt orthonormalization of $\{e_a^\alpha\}$. We define a local connection $\nabla^a$ by

$$\nabla^a \hat{e}_\alpha^a = 0, \quad \alpha = 1, \cdots, r.$$  

For any section $e \in E|_{U_\epsilon}$, we define

$$\nabla e = \sum \Gamma_a \nabla^a (e|_{U_{[(\kappa_a, b_a)]}(\epsilon, \delta_a, \rho_a)}).$$  

It is easy to see that $\nabla$ is a compatible connection of the metric $h$. Denote

$$\nabla \hat{e}_\alpha^a = \sum_{\beta} \omega^a_{\alpha \beta} \hat{e}_\beta^a, \quad \nabla^2 \hat{e}_\alpha^a = \sum \Omega^a_{\alpha \beta} \hat{e}_\beta^a.$$  

For any $U_{[(\kappa_a, b_a)]}(\epsilon, \delta_a, \rho_a) \cap U_{[(\kappa_c, b_c)]}(\epsilon, \delta_c, \rho_c) \neq \emptyset$, let $(\hat{a}_c^{ac})_{1\leq \alpha, \beta\leq r}$ be functions such that

$$\hat{e}_\alpha^a = \sum_{\beta=1}^r \hat{a}_c^{ac} \hat{e}_\beta^c, \quad \alpha = 1, \cdots, r.$$  

It is easy to see that

$$\omega^a_{\alpha \beta} = \sum_{c} \sum_{\beta=1}^r \alpha_{bc} d\hat{a}_c^{ac} \hat{e}_\gamma^c.$$  

We get a metric $h$ and a connection $\nabla$ in $E$ over $U_\epsilon$.  

8 Gluing estimates

8.1 Gluing maps

Let $\Sigma$ be a marked nodal Riemann surfaces. Suppose that $\Sigma$ has nodes $p_1, \cdots, p_e$ and marked points $y_1, \cdots, y_n$. We choose local coordinate system $A$. Let $u : \Sigma \to M$ be perturbed $J$-holomorphic map. We glue $\Sigma$ and $u$ at each node with gluing parameters $(r)$ to get $\Sigma(r)$ and the pregluing map $u(r) : \Sigma(r) \to M$. Set

$$t_i = e^{-2r_i - 2\pi i}, \quad |r| = \min\{r_1, \ldots, r_e\}, \quad b(r) := (0, (r), u(r)).$$

The following lemma is proved in [19].

**Lemma 8.1.** For $|r| > R_0$ there is an isomorphism

$$I_{(r)} : \ker DS_{(\kappa, b_\alpha)} \longrightarrow \ker DS_{(\kappa, b_{(r)})}.$$  

Using Theorem 5.3 in [19] and the implicit function theorem with parameters we immediately obtain
Lemma 8.2. There are constant $\varepsilon > 0$, $R_0 > 0$ and a neighborhood $O_1 \subset A$ of $s_o$ and a neighborhood $O$ of 0 in $\ker DS_{(\kappa_o, b_o)}$ such that

$$glu : O_1 \times (\mathbb{D}_c^*(0))^s \times O \to glu(O_1 \times (\mathbb{D}_c^*)^s \times O) \subset \tilde{U}^T$$

is an orientation preserving local diffeomorphisms, where

$$\mathbb{D}_c^*(0) := \{ t \mid 0 < |t| < c \}, \quad c = e^{-2R_0}.$$ 

Denote

$$Glu_{s,(r)} = I_r + Q_{b(r)} \circ f_{s,(r)} \circ I_r.$$ 

8.2 Equivariant gluing

We consider the case with one node, the construction in this section can be generalized to the case with several nodes. Let $(\kappa_o, b_o) \in \tilde{U}$ where

$$\kappa_o = (\kappa_o^1, \kappa_o^2), \quad b_o = (a_{oi}, u_i), \quad a_{oi} = (\Sigma_i, j_{oi}, y_i, q), \quad i = 1, 2,$$

$\Sigma_1$ and $\Sigma_2$ are smooth Riemann surfaces joining at $q$. Assume that $(\Sigma_i, y_i, q)$ is stable. Let $G_{(\kappa_o, b_o)} = (G_{(\kappa_o^1, b_o^1)}, G_{(\kappa_o^2, b_o^2)})$ be the isotropy group at $(\kappa_o, b_o)$, thus,

$$G_{(\kappa_o, b_o)} = \{ \phi = (\phi_1, \phi_2) \mid \phi_i \in Diff^+(\Sigma_i), \phi_i^* (j_{oi}, y_i, q, \kappa_{oi}, u_i) = (j_{oi}, y_i, q, \kappa_{oi}, u_i), i = 1, 2 \}.$$ 

Obviously, $G_{(\kappa_o, b_o)}$ is a subgroup of $G_{\kappa_o}$. The following lemma can be easily to check.

Lemma 8.3. For any $(\kappa, b) \in \tilde{U}_{(\kappa_o, b_o)}(\varepsilon, \delta, \rho)$, $\varphi \in Diff^+(\Sigma)$ denote

$$\kappa' = \varphi^* \kappa, \quad b' = (j', y', u') = \varphi \cdot (j, y, u) = (\varphi^* j, \varphi^{-1} y, \varphi^* u).$$

Then

$$\tilde{\partial}_{j', y'} = \varphi^* (\tilde{\partial}_{j, y}), \quad i(\kappa', b') = \varphi^* (i(\kappa, b)).$$

It is easy to check that the operator $DS_{(\kappa_o, b_o)}$ is $G_{(\kappa_o, b_o)}$-equivariant. Then we may choose a $G_{(\kappa_o, b_o)}$-equivariant right inverse $Q_{(\kappa_o, b_o)}$. In fact, let $Q_{(\kappa_o, b_o)}$ be a right inverse of $DS_{(\kappa_o, b_o)}$, we define

$$Q_{(\kappa_o, b_o)}(\eta) = \frac{1}{|G_{(\kappa_o, b_o)}|} \sum_{\varphi \in G_{(\kappa_o, b_o)}} \varphi^{-1} \cdot \hat{Q}_{(\kappa_o, b_o)}(\varphi \cdot \eta).$$

Then, for any $\varphi' \in G_{(\kappa_o, b_o)}$, we have

$$Q_{(\kappa_o, b_o)}(\varphi' \cdot \eta) = \frac{1}{|G_{(\kappa_o, b_o)}|} \sum_{\varphi \in G_{(\kappa_o, b_o)}} \varphi^{-1} \cdot \hat{Q}_{(\kappa_o, b_o)}(\varphi \cdot \varphi' \cdot \eta) =$$

$$= \frac{1}{|G_{(\kappa_o, b_o)}|} \sum_{\varphi \in G_{(\kappa_o, b_o)}} \varphi' \cdot (\varphi')^{-1} \varphi^{-1} \cdot \hat{Q}_{(\kappa_o, b_o)}(\varphi \cdot \varphi' \cdot \eta) = \varphi' \cdot Q_{(\kappa_o, b_o)}(\eta)$$

By the $G_{(\kappa_o, b_o)}$-equivariance of $DS_{(\kappa_o, b_o)}$, $G_{(\kappa_o, b_o)}$ acts on $\ker DS_{(\kappa_o, b_o)}$ in a natural way.

We choose cusp cylinder coordinates $(s_i, t_i)$ on $\Sigma_i$ near $q$. Choosing the gluing parameter $(r)$ we construct $\Sigma(r)$ and $u(r)$ as in [6.8]. Since the cut-off function $\beta(s)$ depends only on $s$, $G_{b_o}$ acts on $\tilde{\Theta}_{b_o}(\delta, \rho)$. Since the $\| \cdot \|_{k, 2, \alpha, r}$ is $G_{b_o}$ invariant, it induces a $G_{(\kappa_o, b_o)}$ action on $\tilde{U}_{(\kappa_o, b_o)}(\varepsilon, \delta, \rho)$.
Set \( a(r) = (\Sigma(r), f_o, y) \) and \( b(r) = (a(r), u(r)) \). Denote by \( G_{b(r)} \) (resp. \( G_{(\kappa, b(r))} \)) the isotropy group at \( b(r) \) (resp. \( (\kappa, b(r)) \)). It is easy to see that \( G_{b(r)} \) is a subgroup of \( G_{b_o} \). It follows that \( G_{(\kappa, b(r))} \) is a subgroup of \( G_{(\kappa, b_o)} \). Then \( G_{(\kappa, b(r))} \) can be seen as rotation in the gluing part. The gluing map is the \( \frac{|G_{(\kappa, b_o)}|}{|G_{(\kappa, b(r))}|} \)-multiple covering map. Since \( \beta_1, \tau \) is independent of \( \tau, Q_{(\kappa, b(r))} \) is \( G_{(\kappa, b(r))} \)-equivariant. By the definition of \( Q_{(\kappa, b(r))} \) and the \( G_{(\kappa, b(r))} \)-equivariance of \( DS_{(\kappa, b(r))} \), we can conclude that \( Q_{(\kappa, b(r))} \) is \( G_{(\kappa, b(r))} \)-equivariant. It follows from the definition of \( I(r) \) that \( I(r) \) is \( G_{(\kappa, b(r))} \)-equivariant. By the uniqueness of the implicit function \( f \), we conclude that \( f \) is \( G_{(\kappa, b(r))} \)-equivariant. Since

\[
Glu(r) = I(r) + Q_{(\kappa, b(r))} \circ f \circ I(r).
\]

\( Glu(r) \) is \( G_{(\kappa, b(r))} \)-equivariant. Denote

\[ \ker DS_{(\kappa, b_o)} = \ker DS_{(\kappa, b_o)}/G_{(\kappa, b_o)}, \quad \ker DS_{(\kappa, b(r))} = \ker DS_{(\kappa, b(r))}/G_{(\kappa, b(r))}. \]

Then we have

**Lemma 8.4.** (1) \( I(r) : \ker DS_{(\kappa, b_o)} \rightarrow \ker DS_{(\kappa, b(r))} \) is a \( \frac{|G_{(\kappa, b_o)}|}{|G_{(\kappa, b(r))}|} \)-multiple covering map.

(2) \( I(r) \) induces an isomorphism \( I(r) : \ker DS_{(\kappa, b_o)} \rightarrow \ker DS_{(\kappa, b(r))} \).

### 8.3 Exponential decay of gluing maps

The following theorem is proved in [19].

**Theorem 8.5.** Let \( l \in \mathbb{Z}^+ \) be a fixed integer. There exists positive constants \( C_l, d, R_0 \) such that for any \( (\kappa, \xi) \in \ker DS_{(\kappa, b_o)} \) with \( \| (\kappa, \xi) \| < d \) and for any \( X_i \in \{ \partial \cdot \partial_{e_i}, \partial_{e_i} \}), i = 1, \cdots, e, \) restricting to the compact set \( \Sigma(R_0) \), the following estimate hold

\[
\| X_i (Glu_{s, (\kappa, \xi)}) \|_{C^l(\Sigma(R_0))} \leq C_l e^{-(\varepsilon - 5\alpha) \frac{r}{4}},
\]

\[
\| X_i X_j (Glu_{s, (\kappa, \xi)}) \|_{C^l(\Sigma(R_0))} \leq C_l e^{-(\varepsilon - 5\alpha) \frac{r_i + r_j}{4}},
\]

\[ 1 \leq i \neq j \leq e, \text{for any } s \in \bigotimes_{i=1}^e O_i \text{ when } |r| \text{ big enough.} \]

### 8.4 Estimates of exponential decay of the line bundle

The following theorem is proved in [20].

**Theorem 8.6.** Let \( l \in \mathbb{Z}^+ \) be a fixed integer. Let \( u : \Sigma \rightarrow M \) be a \((j, J)\)-holomorphic map. Let \( \varepsilon \in (0, 1) \) be a fixed constant. For any \( 0 < \alpha < \frac{1}{100e} \) there exists positive constants \( C_l, d, R \) such that for any \( \zeta \in \ker D_{\lambda b_o}^i, (\kappa, \xi) \in \ker DS_{(\kappa, b_o)} \) with

\[
\| \zeta \| _{W_{4\lambda (2, \alpha)}} \leq d, \quad \| (\kappa, \xi) \| < d, \quad |r| \geq R,
\]

restricting to the compact set \( \Sigma(R_0) \), the following estimate hold.

\[
\| X_i (Glu_{s, (\kappa, \xi)}) \|_{C^l(\Sigma(R_0))} \leq C_l e^{-(\varepsilon - 5\alpha) \frac{r_i}{4}}, \quad \cdots \quad (45)
\]

\[
\| X_i X_j (Glu_{s, (\kappa, \xi)}) \|_{C^l(\Sigma(R_0))} \leq C_l e^{-(\varepsilon - 5\alpha) \frac{r_i + r_j}{4}}, \quad \cdots \quad (46)
\]

for any \( X_i \in \{ \partial \cdot \partial_{e_i}, \partial_{e_i} \}, i = 1, \cdots, e, s \in \bigotimes_{i=1}^e O_i \) and any \( 1 \leq i \neq j \leq e \), where \( h_{(r)} = \Pi_2(Glu_{s, (\kappa, \xi)}) \) and \( \Pi_2 : \mathcal{F}_{b(r)} \times T_{u(r)} \mathcal{B} \rightarrow T_{u(r)} \mathcal{B} \) denotes the projection.
8.5 Estimates of Thom forms

We estimate the derivatives of the metric $h$ near the boundary of $F|_{U_r}$. Let $(\kappa, b_0)$ be one of \{$(\kappa, b_0)$, $a = n_t + 1, \ldots, n$\} and $b_0 = (a_0, u)$. Fix a basis $\{e_1, \cdots, e_d\}$ of $Ker\ DS_{(\kappa, b_0)}$ and let $\tilde{z} = (\tilde{z}_1, \cdots, \tilde{z}_d)$ be the corresponding coordinates. Set $t_i = e^{-2r_i-2\pi r_i}, 1 \leq i \leq e$. Denote

$$L(s, r, \tilde{z}) := I_r(\sum_{i=1}^d \tilde{z}_i e_i) + Q_{(\kappa, b_0)} \circ f_{s_0}(r) \circ I_r(\sum_{i=1}^d \tilde{z}_i e_i),$$

where $b_{r_i} = (0, (r), u_i)$. Then $(s, r, \tilde{z})$ is a local coordinates of $U_{(\kappa, b_0)}(e, \delta, \rho)$. We say that $f(s, r, \tilde{z})$ satisfies (r)-exponential decay if

$$\left| \frac{\partial f}{\partial r_i} \right| + \left| \frac{\partial f}{\partial \tilde{z}_j} \right| \leq C e^{-\delta r_i}, \forall 1 \leq i \leq e$$

and

$$\left| \frac{\partial f}{\partial \tilde{z}_j} \right| + \left| \frac{\partial f}{\partial \tilde{z}_\alpha} \right| \leq C, \forall 1 \leq j \leq \tau, 1 \leq \alpha \leq d.$$ (47)

Let

$$\Pi_1 : \tilde{F}_{b(e)} \times T_{u(e)} \tilde{B} \rightarrow \tilde{F}_{b(e)}, \quad \Pi_2 : \tilde{F}_{b(e)} \times T_{u(e)} \tilde{B} \rightarrow T_{u(e)} \tilde{B}$$

be the projection. By Theorem 8.5 the implicit function theorem and (41), we conclude that $\Gamma_c$ satisfies (r)-exponential decay, where $\Gamma_a$ is the cutoff function defined in section 7.3.

For any $U_{(\kappa, b_a)}(e, \delta, \rho_a) \cap U_{(\kappa, b)}(e, \delta, \rho) \neq \emptyset$, let $a_{\alpha\beta}^{ac}, \alpha, \beta = 1, \cdots, r$ be functions such that $e_a = \sum_{\alpha=1}^r a_{\alpha\beta}^{ac} e^c, \alpha = 1, \cdots, r$. By the implicit function theorem, Theorem 8.6 we have, for any $p \in \Sigma(R_0), e_a(p), e^c(p)$ satisfies (r)-exponential decay. Since $a_{\alpha\beta}^{ac}$ is a function of $(s, (r), \tilde{z})$, we have

$$d(e_a(p)) = \sum_{\beta=1}^r e^c(p) \cdot da_{\alpha\beta}^{ac} + \sum_{\beta=1}^r a_{\alpha\beta}^{ac} \cdot d(e^c(p)), \forall p \in \Sigma(R_0),$$ (49)

Recall that $e_a = (\tilde{f}_L^{\tilde{r}}, Q_{(s, h(e), (r))}^{\tilde{L}_{(r)}}) (e_a_{(\kappa, b_0)}).$ Using the implicit function theorem we get

$$\|Q_{(s, h(e), (r))}^{\tilde{L}_{(r)}}(e_a_{(\kappa, b_0)})\|_{k_0, a, c, r} \leq 2C \|D_{b} \circ (P_{b(e)})^{-1}(\tilde{f}_L^{\tilde{r}}(e_a_{(\kappa, b_0)}))\|.$$

Choosing $\delta_a$ and $\rho_a$ small enough, by the exponential estimates of $e_a_{b_0}$ we have

$$\|e_a_{(\kappa, b_0)}\|_{\Sigma(R_0)} \geq 1 \|e_a||_{k_0, a, c}.$$ (50)

So $\max_{\Sigma(R_0)} |e_a|$ has uniform lower bound. Then we obtain the (r)-exponential decay of $a_{\alpha\beta}^{ac}$. Denote $h_{\alpha\beta}^{ac} = (e_a, e^\gamma)_{h_i}$. By the definition of $h$ and the (r)-exponential decay of $\Gamma_a$, $a_{\alpha\beta}^{ac}$ we conclude that $h_{\alpha\beta}^{ac}$ satisfies the (r)-exponential decay. By the GramSchmidt orthonormalization and the similar argument above we obtain the (r)-exponential decay of $a_{\alpha\beta}^{ac}$.

Let $\Delta_r$ be the open disk in $\mathbb{C}$ with radius $r$, let $\Delta^r_s = \Delta_r \setminus \{0\}$ and $\Delta^r = \Delta \setminus \{0\}$. Set $N = 3g - 3 + n$. For each point $p \in \partial M_{g,n}$ we can find a coordinate chart $(U, s_1, \cdots, s_{N-\epsilon}, t_1, \cdots, t_e)$ around $p$ in $\overline{M}_{g,n}$ such that $U \cong \Delta^N$ and $V = U \cap M_{g,n} \cong \Delta^{N-\epsilon} \times (\Delta^*)^\epsilon$. We assume that $U \cap \overline{M}_{g,n}$ is defined by the equation $t_1 \cdots t_e = 0$. Let $\{U_a\}$ be a local chart of $\overline{M}_{g,n}$. On each chart $V_a$ of $\overline{M}_{g,n}$ we can define a local Poincare metric:

$$g_{loc}^a = \sum_{i=1}^m \frac{|dt_i|^2}{|t_i|^2 \log |t_i|^2} + \sum_{a=1}^{N-m} |ds_a|^2.$$

31
We let \( U_\alpha(r) \cong \Delta^N_r \) for \( 0 < r < 1 \) and let \( V_\alpha(r) = U_\alpha(r) \cap \mathcal{M}_{g,n} \).

Let \( s, (r), 3 \) be the local coordinates of \( U_{(s, b_0)}^T \). In the coordinates \( (s, (r), 3) \) the local Poincaré metric \( g_{loc} \) can be written as

\[
g_{loc} = \sum_{i=1}^\epsilon \frac{4(d^2r_i + d^2\tau_i)}{r_i^2} + \sum_{i=1}^{3g-3+n-\epsilon} |ds_i|^2 + \sum_{i=1}^d d\delta_i^2.
\]

(51)

**Lemma 8.7.** There exists a constant \( C > 0 \) such that

\[
|\omega_{\alpha,\beta}(X_1)|^2 \leq g_{loc}(X_1, X_1), \quad |\Omega_{\alpha,\beta}(X_1, X_2)|^2 \leq \Pi_{i=1}^g g_{loc}(X_i, X_i)
\]

for any \( X_i \in T U_T^T, i = 1, 2, 3 \).

**Proof.** The first inequality follows from (44) and \( (r) \)-exponential decay of \( \hat{a}_{\alpha,\beta} \). By \( \Omega_{\alpha,\beta} = d\omega_{\alpha,\beta} + \sum_{\gamma} \omega_{\alpha,\gamma} \wedge \omega_{\gamma,\beta} \) and \( (r) \)-exponential decay of \( \hat{a}_{\alpha,\beta} \) and \( \Gamma_a \), we can get the second inequality. The last inequality follows from the Bianchi identity. \( \square \)

Let \( p^* E \) be the pull-back of the bundle \( E \) to a bundle over \( U \), where \( p : E \to U \) is the projection. Then the bundle \( p^* E \) has a metric \( p^* h \) with compatible connection \( p^* \nabla \). To simply notation we write these as \( h \) and \( \nabla \). Let \( \hat{\sigma} \) be the tautological section of \( p^* E \). Then the elements \( |\hat{\sigma}|_h^2 \in \mathcal{A}\left( E, \wedge^0(p^* E) \right) \), and the covariant derivative \( \nabla \sigma \in \mathcal{A}_1(E, \wedge^1(p^* E)) \). The curvature \( p^* \Omega \) of the connection \( \nabla \) on \( E \) can also seen as an element of \( \mathcal{A}_2(E, \wedge^2(p^* E)) \). By \( \mathcal{A}\left( E, \wedge^1(p^* E) \right) \), the Mathai-Quillen type Thom form can be written as

\[
\Theta_{MQ} = c(r) \int_B e^{-\frac{\sigma^2}{2\epsilon} - \nabla \sigma - p^* \Omega} \in \mathcal{A}'(E)
\]

(52)

where \( c(r) \) is a constant depending only \( r \), \( \int_B \) denotes the Berezin integral on \( \wedge^*(p^* E) \). Here \( \Theta_E \) is Gaussian shaped Thom form. Let \( B_\epsilon(0) \) denote the open \( \epsilon \)-ball in \( \mathbb{R}^2r \) and consider the map \( \rho_\epsilon : B_\epsilon(0) \to \mathbb{R}^2r \) defined by \( \rho_\epsilon(v) = \frac{v}{\sqrt{\sum_{x=-\epsilon}^{\epsilon} x^2}} \). If we extend \( \rho_\epsilon^* \Theta_{MQ} \) by setting it equal to zero outside \( B_\epsilon(0) \), still denoted by \( \Theta_E := \rho_\epsilon^* \Theta_{MQ} \), we obtain a form \( \Theta_E \) of compact support.

Finally, we have the following estimate for \( \sigma^* \Theta_E \).

**Lemma 8.8.** There exists a constant \( C > 0 \) such that

\[
|\sigma^* \Theta_E(X_1, \cdots, X_\epsilon)|^2 \leq C \Pi_{i=1}^g g_{loc}(X_i, X_i)
\]

for any \( X_i \in T U_T^T, i = 1, 2, 3 \).

**Proof.** One can easily check that

\[
\sigma^* \Theta_E = \sigma^* p^* \Theta_{MQ} = c(r) \int_B e^{-\frac{\sigma^2}{(\epsilon - |\sigma|^2)^2} - \nabla \sigma - p^* \Omega} \in \mathcal{A}'(M).
\]

Denote \( \sigma = \sum_{\alpha} \sigma_\alpha e_\alpha^a \). For any \( p \in \Sigma(R_0) \), by \( d\sigma(p) = \sum_{\alpha} \sigma_\alpha c_\alpha^a(p) \sigma_\alpha^a + \sum_{\alpha} \sigma_\alpha c_\alpha^a(p) d\sigma_\alpha^a \), as above we obtain the \( (r) \) exponential decay \( \sigma_\alpha^a \). Since \( \nabla \sigma = \sum_{\alpha} d\sigma_\alpha c_\alpha^a + \sum_{\alpha, \beta} \sigma_\alpha \omega_{\alpha,\beta} c_\alpha^a, \Omega = \sum \Omega_{\alpha,\beta} c_\alpha^a \wedge c_\beta^a \) and

\[
\int_B c_1^a \wedge \cdots \wedge c_\epsilon^a = 1,
\]

the lemma follows from Lemma 8.7 and a direct calculation. \( \square \)
9 Gromov-Witten invariants

9.1 The convergence of the integrals

Denote

\[ U_\varepsilon = \{ [(\kappa, b)] \in U \mid |\kappa|_h \leq \epsilon \}, \quad U_T^\varepsilon = \{ [(\kappa, b)] \in U^T \mid |\kappa|_h \leq \epsilon \}. \]

We choose open covering

\[ \{ U_{[(\kappa, b)],}(\varepsilon, \delta, \rho) \}, \quad 1 \leq a \leq n \}\]

of \( U_{2\varepsilon} \) and a family cutoff functions \( \{ \Gamma_a, \quad 1 \leq a \leq n \} \) as in [7,3]. Let \( \Theta_{E} \) be the Thom form of \( E \) supported in a small \( \varepsilon \)-ball of the 0-section of \( E \). To simply notation we denote \( \Theta_{E} \) by \( \Theta \).

Remark 9.1. \( \{ \Gamma_a \} \) is not a partition of unity in the classical sense, since it is not smooth on lower stratum, and it is not compactly supported. But it is smooth on \( U^T \) and \( \Gamma_a \sigma^* \Theta \) is compactly supported. This is enough to define Gromov-Witten invariants.

Denote

\[ V_{[(\kappa, b)],}(\varepsilon, \delta, \rho) := U_{[(\kappa, b)],}(\varepsilon, \delta, \rho) \cap U^T, \]
\[ \tilde{V}_{\kappa, b}(\varepsilon, \delta, \rho) := \tilde{U}_{\kappa, b}(\varepsilon, \delta, \rho) \cap \tilde{U}^T. \]

Sometimes we write the above two sets by \( V_a \) and \( \tilde{V}_a \) to simplify notations. Let \( p : \tilde{V}_a \rightarrow V_a \), let \( \Gamma_a, \tilde{K} \) and \( \Theta \) be the lift of \( \Gamma_a, K \) and \( \Theta \) to \( \tilde{V}_a \). We write the Gromov-Witten invariants as

\[ \Psi_{A,g,n}(K; \alpha_1, \ldots, \alpha_n) = \sum_{a=1}^{n} (I)_a \]

where

\[ (I)_a := \int \limits_{V_a} \Gamma_a \cdot \mathcal{P}^s(K) \wedge \prod_{j} ev_j^* \alpha_j \wedge \sigma^* \Theta. \]

Proof of Theorem 1.2. Note that the integration region \( U_a \) for \( 1 \leq a \leq n \) are compact set in \( U^T \) and the integrand in (54) are smooth we conclude that \( \sum_{a=1}^{n} (I)_a \) is bounded. So we only need to prove the convergence of \( (I)_a \) for \( a = n_t + 1, \ldots, n \). Denote

\[ (J)_a = \int \limits_{\tilde{V}_a} \tilde{\Gamma}_a \cdot \mathcal{P}^s(\tilde{K}) \wedge \prod_{j} \tilde{ev}_j^* \alpha_j \wedge \tilde{\sigma}^* \tilde{\Theta}. \]

It suffices to prove the convergence of \( (J)_a \).

Let \( (\kappa, b_0) \) be one of \( \{ (\kappa, b), a = n_t + 1, \ldots, n \} \) and \( b_0 = (a_0, u) \). We choose coordinates \( (s, t, \tilde{z}) \). To simplify notation we denote

\[ dV = \bigwedge_i (dr_i \wedge dr_t) \wedge \left( \bigwedge_j \left( \frac{\sqrt{-1}}{2} ds_j \wedge ds_{j} \right) \right) \wedge d\tilde{z}_1 \wedge \cdots \wedge d\tilde{z}_d \]

and

\[ \delta_i = glu(t)_* \left( \frac{\partial}{\partial r_i} \right), \quad \eta_i = glu(t)_* \left( \frac{\partial}{\partial r_t} \right), \quad 1 \leq i \leq \epsilon \]
\[ \delta_\alpha = glu(t)_* \left( \frac{\partial}{\partial \alpha_{\alpha - \epsilon}} \right), \quad \eta_\alpha = glu(t)_* \left( \frac{\partial}{\partial \alpha_{\alpha - \epsilon}} \right), \quad \epsilon + 1 \leq \alpha \leq 3g - 3 + n \]
\[ \delta_i = glu(t)_* \left( \frac{\partial}{\partial \tilde{z}_i} \right), \quad 1 \leq i \leq d. \]
We will denote by \( (E_1, E_2, \ldots, E_{6g-6+2n+d}) \) the frame
\[
(\delta_1, \ldots, \delta_n, \eta_1, \ldots, \eta_n, \delta_{n+1}, \ldots, \delta_{5g-3+n}, \eta_{n+1}, \ldots, \eta_{5g-3+n}, \ell_1, \ldots, \ell_d).
\]
Then, for \( a = n_t + 1, \ldots, n \),
\[
(J)_a = \int_{V_a} \widetilde{\Gamma}_a \cdot \left( \mathcal{P}^* \widetilde{K} \land \prod_{i} \tilde{e}_i^* \alpha_i \land \tilde{\sigma}^* \tilde{\Theta}(E_1, E_2, \ldots, E_{6g-6+2n+d}) \right) \, dV.
\]

1. Estimates for \( \mathcal{P}^* \widetilde{K} \)

We can choose \( s, t, \tilde{z}_1, \ldots, \tilde{z}_d \) as the local coordinates of \( U^T \). In this coordinates \( \mathcal{P} : U^T \to \mathcal{M}_{g,n} \) can be written as
\[
\mathcal{P}(s, t, \tilde{z}_1, \ldots, \tilde{z}_d) = (s, t).
\]
Notch that \( \mathcal{P} \cdot E_i = E_i \) for \( i \leq 6g - 6 + 2n \), \( \mathcal{P} \cdot E_i = 0 \) for \( i \geq 6g - 6 + 2n + 1 \). We assume that for any \( 1 \leq j \leq \deg(K) \), \( E_{ij} \in \{ E_1, \ldots, E_{6g-6+2n} \} \). Since \( K \) has Poincare growth we have
\[
|\mathcal{P}^* \widetilde{K}(E_{i_1}, \ldots, E_{i_{\deg(K)}})| = |\widetilde{K}(E_{i_1}, \ldots, E_{i_{\deg(K)}})| \leq C \left[ \Pi_{j=1}^{\deg(K)} g_{tloc}(E_{ij}, E_{ij}) \right]^{\frac{1}{2}}. \tag{55}
\]

2. Estimates for \( \prod_i \tilde{e}_i^* \alpha_i \)

For any \( p \in M \) and \( \xi \in T_pM \) we denote \( D \exp_p(\xi) : T_pM \to T_{\exp_p \xi}M \), then
\[
D \exp_p(\xi) := \frac{d}{dt} \exp_p(\xi + t \xi) \big|_{t=0}. \tag{56}
\]

Obviously, \( D \exp_p(\xi) \) is an isomorphism when \( |\xi| \) small enough. By a direct calculation we have, for any \( X \in \{ \frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_l}, \frac{\partial}{\partial r_1}, \ldots, \frac{\partial}{\partial r_l}, \frac{\partial}{\partial \tilde{z}_1}, \ldots, \frac{\partial}{\partial \tilde{z}_d} \}, 1 \leq i \leq 3g - 3 + n - \epsilon, 1 \leq l \leq \epsilon, 1 \leq j \leq d \},
\]
\[
|\tilde{e}_i^* g_lu(t) X| = |\Pi_{2,\alpha}(X(glu(s,t,\tilde{z})))(y_i)| = |D \exp_a(\Pi_{2,\alpha}(\mathcal{L})(\Pi_{2,\alpha}(\mathcal{L}))(y_i))| \tag{57}
\]

By Theorem 8.3 and (57) we have
\[
\|\tilde{e}_i^* E_i\| G_j + \|\tilde{e}_i^* E_{i+1}\| G_j \leq C e^{-\delta r_i}, \quad \|\tilde{e}_i^* E_j\| G_j \leq C,
\]
\[
[g_{tloc}(E_i, E_i)]^{\frac{1}{2}} = \frac{2}{r_i}, \quad [g_{tloc}(E_j, E_j)]^{\frac{1}{2}} = 1
\]
for \( 1 \leq i \leq \epsilon, 2\epsilon + 1 \leq j \leq 6g - 6 + 2n + d \). It follows that
\[
|\Pi \tilde{e}_i^* \alpha_i(E_{i_1}, \ldots, E_{i_\epsilon})| \leq C \prod_{i=1}^{\epsilon} [g_{tloc}(E_{ij}, E_{ij})]^{\frac{1}{2}}, \tag{58}
\]
where \( \{ E_{i_1}, \ldots, E_{i_\epsilon} \} \subset \{ E_1, E_2, \ldots, E_{6g-6+2n+d} \} \).

3. Estimates for the Thom form By Lemma 8.8 we have
\[
|\tilde{\sigma}^* \tilde{\Theta}(E_{i_1}, \ldots, E_{i_\epsilon})| \leq C \prod_{i=1}^{\epsilon} [g_{tloc}(E_{ij}, E_{ij})]^{\frac{1}{2}}, \tag{59}
\]
where \( \{ E_{i_1}, \ldots, E_{i_\epsilon} \} \subset \{ E_1, E_2, \ldots, E_{6g-6+2n+d} \} \).

It follows from (55), (58) and (59) that
\[
\mathcal{P}^* \widetilde{K} \land \prod_i \tilde{e}_i^* \alpha_i \land \tilde{\sigma}^* \tilde{\Theta}(E_1, \ldots, E_{6g-6+2n+d}) \leq C \prod_{i=1}^{6g-6+2n+d} [g_{tloc}(E_i, E_i)]^{\frac{1}{2}} \leq \frac{C}{\Pi_{i=1}^{r_i^2}}.
\]
Hence the integral \( (I)_a \) is convergence.
10 Properties of Gromov-Witten invariants

10.1 Some common properties

Denote by $U^T_n$ (resp. $U^T_{n-1}$) the top stratum of virtual neighborhood of $\overline{\mathcal{M}}_{g,n}(A)$ (resp. $\overline{\mathcal{M}}_{g,n-1}(A)$). We begin with the smoothness of the forgetful maps and the evaluation maps.

**Theorem 10.1.** Restricting to $U^T$, the following hold:

(a) the forgetful map $\chi$ is smooth,

(b) suppose that $(g, n) \neq (0, 3), (1, 1)$, the map $\pi : U^T_n \to U^T_{n-1}$ is smooth,

(c) the evaluation map $ev_i$ is smooth.

**Proof.** (a) and (b). Restricting to $U^T$ and in the fixed coordinate system $(\psi, \Psi)$ for $Q$, we may choose $(a, y_1, \ldots, y_n, s)$ as local coordinates of $U$ around $b_0$ as in subsection [6.3]. In this coordinates the maps $P$ and $\pi$ are given by $P(a, y_1, \ldots, y_n, s) = (a, y_1, \ldots, y_n)$, $\pi(a, y_1, \ldots, y_n, s) = (a, y_1, \ldots, y_{n-1}, s)$ respectively. It is obvious that $P$ and $\pi$ are smooth. Note that for (a), since $n + 2g \geq 3$, after forgetting the map the domain is still smooth. For (b), since $(g, n) \neq (0, 3), (1, 1)$ and we restrict to the top strata, forgetting the last marked point the domain is still stable. (c). Since $U^T$ is smooth, the evaluation map $ev_i$ is smooth. $\square$

**Proof of Theorem 1.3**

(1) follows from the definition, we omit it.

Proof of (2)

We only prove for $\alpha_i$, the proofs for $K$ and $\Theta$ are the same. If $\alpha_i' \in [\alpha_i]$ is another closed form, there is a form $\psi_i$ such that $\alpha_i' - \alpha_i = d\psi_i$.

For any $n_i + 1 \leq a \leq n$, let $U^T_{a,R} \subset V_a$ be an open set, defined in terms of the coordinates system $(s, (r), t)$ by $U^T_{a,R} := \{(s, (r), t) \mid r_i \leq R, i = 1, \ldots, n \}$.

Put $U^T_{e,R} = (\cup_{n=1}^{m-1} V_n) \bigcup (\cup_{n=n_e+1}^{n_e} U^T_{a,R})$.

Then we smoothen it at corners, and denote the resulted neighborhood by $U^T_{e,R}$.

By the Stokes theorem and Theorem 8.5, Theorem 8.6 we have

$$\int_{U^T_{e,R}} P^* K \wedge \prod_i ev_i^* \alpha_i \wedge \sigma^* \Theta - \int_{U^T_{e,R}} P^* K \wedge \prod_{j \neq i} ev_j^* \alpha_j \wedge ev_i^* \alpha_i' \wedge \sigma^* \Theta = \int_{\partial U^T_{e,R}} i^* (P^* K \wedge \prod_{j \neq i} ev_j^* \alpha_j \wedge ev_i^* \partial_i \wedge \sigma^* \Theta) \to 0, \quad \text{as } d \to 0,$$

where $i : \partial U^T_{e,R} \to U^T$ is the inclusion map. We explain $\int_{\partial U^T_{e,R}} i^* (\cdot) \to 0$: The area $\text{Area}$ of the hypersurface $\partial U^T_{e,R}$ satisfies $|\text{Area}| \leq C' R$ for some constant $C' > 0$. By the estimates in [9.1] we have

$$\left| \int_{\partial U^T_{e,R} \cap V_n} i^* (\cdot) \right| \leq \frac{C'' R}{\prod_{i=1}^n R^2} \to 0, \quad \text{as } R \to \infty.$$

Then (2) follows.
Proof of (3)

Suppose that \((\mathcal{H}', i'(\kappa', b'))\) is another choice and \((U', E', \sigma')\) is the virtual neighborhood constructed by \((\mathcal{H}', i'(\kappa', b'))\). Let \(\Theta'\) be the Thom form of \(\mathcal{H}'\) supported in a neighborhood of zero section. Let

\[
S(t)(\kappa, b) = \partial_{j,v} + (1 - t)i((\kappa, b)) + ti'(\kappa', b') : \mathcal{H} \oplus \mathcal{H}' \times [0, 1] \to \mathcal{E}.
\]

Let \((U(t), E \oplus E', \sigma(t))\) be the virtual neighborhood cobordism constructed by \(S(t)\). Using the same method as in (2), by Stokes theorem we have

\[
\int_{U^0_t} \mathcal{P} \wedge \prod_{i} ev^*(0) \alpha_i \wedge \sigma^*_0(\Theta \land \Theta') - \int_{U^1_t} \mathcal{P} \wedge \prod_{i} ev^*(1) \alpha_i \wedge \sigma^*_1(\Theta \land \Theta') = \int_{U^1_t} d \left( \mathcal{P} \wedge \prod_{i} ev^*(t) \alpha_i \wedge \sigma^*_t(\Theta \land \Theta') \right) = 0,
\]

where \(ev^*(t) : U_t \to M\) is the evaluation map. On the other hand, \(\sigma_0 = \sigma \times I_d, \pi : U^0_t \to U^T\) is a bundle with fibre \(E'\). It follows that

\[
\int_{U^0_t} \mathcal{P} \wedge \prod_{i} ev^*(0) \alpha_i \wedge \sigma^*_0(\Theta \land \Theta') = \int_{U^T} \mathcal{P} \wedge \prod_{i} ev^*_t \alpha_i \wedge \sigma^*(\Theta).
\]

By the same way we have

\[
\int_{U^1_t} \mathcal{P} \wedge \prod_{i} ev^*(1) \alpha_i \wedge \sigma^*_1(\Theta \land \Theta') = \int_{U^T} \mathcal{P} \wedge \prod_{i} ev^*_t \alpha_i \wedge \sigma^*(\Theta')
\]

where \(ev^*_t : U^T \to M\). Then (3) follows.

Proof of (4)

Let \(J'(\text{resp. } \omega')\) be another smooth almost complex structure (resp. symplectic form). Suppose that \(F'\) is another choice of finite rank bundle. Let \(\omega_t\) be a family of symplectic structures and \(J_t\) be a family of almost complex structures such that \(J_t\) is tamed with \(\omega_t\) and

\[
J_0 = J', \quad J_1 = J, \quad \omega_0 = \omega', \quad \omega_1 = \omega.
\]

Let \(D_t\) be the linearized operator \(\tilde{\partial}_{J,J_t}\). We cut the interval \([0, 1]\) into \([t_i, t_{i+1}]\), \(0 \leq i \leq l\) with \(t_0 = 0, t_{l+1} = 1\), and construct a finite rank bundle \(K_t = \bigoplus_i F_{t_i}\) with \(F_0 = F', F_1 = F\). We choose a smooth family bundle map \(i_t, t \in [0, 1]\) such that for any \(t \in [0, 1]\), \(D_t + di_t\) is surjective and

\[
i_0 = i', \quad i_1 = i.
\]

Let \(S(t)(\kappa, b) = \tilde{\partial}_{J,J_t} + i_t((\kappa, b))\) and \((U(t), \oplus E_i, \sigma(t))\) be the virtual neighborhood cobordism constructed by \(S(t)\). Then by the same argument of (3) we can prove (4).

Proof of (5)

When \(M\) is semi-positive, we can use the same method of Ruan ([34]) to complete the proof. 

Proof of Theorem [L.4]
We have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{M}_{g,n}(A) & \rightarrow & \mathcal{M}_{g,n} \\
\downarrow \pi & & \downarrow \pi \\
\mathcal{M}_{g,n-1}(A) & \rightarrow & \mathcal{M}_{g,n-1}
\end{array}
\]

We construct virtual manifold \( U_{n-1} \) for \( \mathcal{M}_{g,n-1}(A) \). By pulling back of \( \pi \), this is also as virtual manifold for \( \mathcal{M}_{g,n}(A) \). That is,
\[
E_n = \pi^* E_{n-1}, \quad U_n = \pi^* U_{n-1}, \quad S_n = S_{n-1} \circ \pi.
\]
Furthermore, \( \pi_* \mathcal{P}^*_{g,n} (K) = \mathcal{P}^*_{g,n-1}(\pi_*(K)) \). So
\[
\Psi_{(A,g,n)}(K; \alpha_1, \ldots, \alpha_{n-1}, 1) = \int_{U_{n-1}} \mathcal{P}^*_{g,n}(K) \wedge \prod_{i=1}^{n-1} ev_i^* \alpha_i \wedge 1 \wedge \Theta.
\]

On the other hand, for \( \alpha_n \in H^2(M, \mathbb{R}) \), one can check that \( \pi_*(ev_n^*(\alpha_n)) = \alpha_n(A) \). Therefore,
\[
\Psi_{(A,g,n)}(\pi_*(K); \alpha_1, \ldots, \alpha_{n-1}, \alpha_n) = \int_{U_n} \mathcal{P}^*_{g,n}(\pi_*(K)) \wedge \prod_{i=1}^{n-1} ev_i^* \alpha_i \wedge ev_n^* \alpha_n \wedge \Theta = \alpha_n(A) \Psi_{(A,g,n-1)}(K; \alpha_1, \ldots, \alpha_{n-1}).
\]

\[\square\]

### 10.2 Axioms for Gromov-Witten invariants

In [17] Kontsevich and Manin listed the following axioms for Gromov-Witten invariants.

**Effectivity Axiom.** If \( \omega(A) < 0 \) then \( \Psi_{(A,g,n)} = 0 \).

**Symmetry Axiom.** The symmetric group \( S_n \) acts naturally on marked points. This axiom asserts that \( \Psi_{(A,g,n)} \) is \( S_n \)-equivariant. This means that
\[
\Psi_{(A,g,n)}(K; \alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n) = (-1)^{deg_{ev_i} deg_{c_{i+1}} \psi_{(A,g,n)}}(K; \alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_n).
\]

**Grading Axiom.** If \( \Psi_{(A,g,n)}(K; \alpha_1, \ldots, \alpha_n) \neq 0 \) then
\[
\sum_{i=1}^{n} \deg \alpha_i + \deg K = 2(1-g)(m-3) + 2c_1(A) + 2n.
\]

**Fundamental Class Axiom.** For any \( \alpha_1, \ldots, \alpha_{n-1} \) in \( H^*(M, \mathbb{R}) \),
\[
\Psi_{A,g,n}(K; \alpha_1, \ldots, \alpha_{n-1}, 1) = \Psi_{A,g,n-1}(\pi_*(K); \alpha_1, \ldots, \alpha_{n-1}),
\]

**Divisor Axiom.** If \( (A, n) \neq (0, 3) \) and \( \deg \alpha_n = 2 \) then
\[
\Psi_{A,g,n}(\pi_*(K); \alpha_1, \ldots, \alpha_{n-1}, \alpha_n) = \alpha_n(A) \Psi_{A,g,n-1}(K; \alpha_1, \ldots, \alpha_{n-1}).
\]

**Zero Axiom.** If \( A = 0 \) then \( \Psi_{(A,g,n)}(K; \alpha_1, \ldots, \alpha_n) = 0 \) whenever \( \deg K > 0 \), and
\[
\Psi_{(A,g,n)}(PD([pt]); \alpha_1, \ldots, \alpha_n) = \int_M \wedge_{i=1}^{n} \alpha_i.
\]

**Deformation Axiom.** \( \Psi_{A,g,n}(K; \alpha_1, \ldots, \alpha_n) \) is independent of \( J \) and is a symplectic deformation invariant.

**Splitting Axiom.** cf. Theorem [11.8]

The Effectivity Axiom, the Symmetry Axiom, the Grading Axiom and the Zero Axiom are easy to prove. The Deformation Axiom is proved in Theorem [1.3] The Fundamental Class Axiom and the Divisor Axiom are proved in Theorem [1.4] In [11] we state and prove the Splitting axiom.
11 Splitting axiom

Assume $g = g_1 + g_2$ and $n = n_1 + n_2$ with $2g_1 + n_1 + 1 \geq 3$, $(g, n_i) \neq (1, 0), n_i > 0, i = 1, 2$. Fix a partition of the index set $\{1, \cdots, n\} = S_1 \cup S_2$, such that $n_i = |S_i|$ for $i = 1, 2$. We denote $\overline{M}_{g_1, g_2, n_1, n_2}$ the moduli space which identifies the last marked point of a stable curve in $\overline{M}_{g_1, n_1 + 1}$ with the first marked point of a stable curve in $\overline{M}_{g_2, n_2 + 1}$. Denote by $q$ the last marked point of of a stable curve in $\overline{M}_{g_1, n_1 + 1}$. The remaining indices have the unique ordering such that the relative order is preserved, the first $n_1$ points in $\overline{M}_{g_1, n_1 + 1}$ are mapped to the points indexed by $S_1$, and the last $n_2$ points in $\overline{M}_{g_2, n_2 + 1}$ are mapped to the points indexed by $S_2$. Let 

$$\theta : \overline{M}_{g_1, g_2, n_1, n_2} \to \overline{M}_{g, n}$$

be the map. Clearly, $\text{im}(\theta)$ is a submanifold of $\overline{M}_{g, n}$.

Let $C$ be the set of all decomposition of $A = A_1 + A_2$, $A_i \in H_2(M, \mathbb{Z})$. Given $C = (A_1, A_2) \in C$, let $\overline{M}_C(g_1, g_2, n_1, n_2)$ be the moduli space of all stable configuration $(\Sigma, y, \nu, j, u)$ with $(\Sigma, y, \nu, j) \in \overline{M}_{g_1, g_2, n_1, n_2}$, $u_i(\Sigma_i) = A_i$. Set

$$\overline{M}_A(g_1, g_2, n_1, n_2) = \bigcup_{C \in C} \overline{M}_C(g_1, g_2, n_1, n_2).$$

Denote by $\overline{M}_{g, n}(A, \theta)$ the moduli space of all stable configuration $(\Sigma, y, \nu, j, u)$ with homology class $A$ and $(\Sigma, y, \nu, j) \in \theta(\overline{M}_{g_1, g_2, n_1, n_2})$. Obviously, $\overline{M}_{g, n}(A, \theta) \subset \overline{M}_{g, n}(A)$. The map $\theta$ induces an isomorphism between $\overline{M}_A(g_1, g_2, n_1, n_2)$ and $\overline{M}_{g, n}(A, \theta)$. We identify $\overline{M}_A(g_1, g_2, n_1, n_2)$ with $\overline{M}_{g, n}(A, \theta)$ if no danger of confusion. Denote by $\overline{M}_A(g_1, g_2, n_1, n_2)$ the top strata, the element of which have one node $q$.

11.1 Constructing virtual neighborhoods

Let $C = (A_1, A_2)$. Consider $\overline{M}_C(g_1, g_2, n_1, n_2)$. Let

$$[b_o] = [(b_{o1}, b_{o2})] \in \overline{M}_C(g_1, g_2, n_1, n_2)$$

be a point. We view $[b_o]$ as a point in $\overline{M}_{g, n}(A)$ and choose a local coordinate system and a local orbifold model $\tilde{O}_{b_o}(\delta_0, \rho_o)/G_{b_o}$. Denote by $\tilde{O}_{b_i}(\delta_i, \rho_i)/G_{b_i}$ its restriction to $\overline{M}_C(g_1, g_2, n_1, n_2)$.

Let $b_o = (\Sigma, j, y, u)$. We write $(\Sigma, j, y, u) = (\Sigma_1, j_1, y_1, u_1) \cup (\Sigma_2, j_2, y_2, u_2)$, $u = (u_1, u_2)$, where $u_1 : \Sigma_1 \to M$, $u_2 : \Sigma_2 \to M$ with $u_1(q) = u_2(q)$. The following lemma is obtained by the same method as before.

Lemma 11.1. There exist finite points $[b_i] \in \overline{M}_C(g_1, g_2, n_1, n_2), 1 \leq i \leq m_c$, such that

1. The collection $\{O_{[b_i]}(\delta_i/3, \rho_i/3) \mid 1 \leq i \leq m_c\}$ is an open cover of $\overline{M}_C(g_1, g_2, n_1, n_2)$.

2. Suppose that $\tilde{O}_{b_i}(\delta_i, \rho_i) \cap \tilde{O}_{b_j}(\delta_j, \rho_j) \neq \emptyset$. For any $b \in \tilde{O}_{b_i}(\delta_i, \rho_i) \cap \tilde{O}_{b_j}(\delta_j, \rho_j)$, $G_b$ can be imbedded into both $G_{b_i}$ and $G_{b_j}$ as subgroups.

By Lemma 6.3 we have a continuous orbi-bundle $F(k_i) \to U$ such that $F(k_i) \mid [b_i]$ contains a copy of group ring $\mathbb{R}[G_{b_i}]$. Set

$$U^c = \bigcup_{i=1}^{m_c} O_{[b_i]}(\delta_i, \rho_i).$$
For each $C \in C$ we do this and put
\[
\mathcal{U}_\theta = \bigcup_{C \in \mathcal{C}} \mathcal{U}^c, \quad F' = \bigoplus_{C \in \mathcal{C}} \bigoplus_{i=1}^{m_c} F(k_i).
\]
Now we choose finite many points and local obifold models
\[
[b_\alpha] \in \mathcal{M}_{g,n}(A), \quad \tilde{O}_{b_\alpha} (\delta_\alpha, \rho_\alpha) / G_{b_\alpha}, \quad 1 \leq \alpha \leq m_o
\]
such that the collection
\[
\bigcup_{c \in \mathcal{C}} \{ O_{[b_i]}(\delta_i, \rho_i), \ 1 \leq i \leq m_c \} \bigcup_{c \in \mathcal{C}} \{ O_{[b_\alpha]}(\delta_\alpha, \rho_\alpha), \ 1 \leq \alpha \leq m_o \}
\]
is an open cover of $\overline{\mathcal{M}}_{g,n}(A)$. We have a continuous “orbi-bundle” $F(k_\alpha) \to \mathcal{U}$ such that $F(k_\alpha)|_{b_\alpha}$ contains a copy of group ring $\mathbb{R}[G_{b_\alpha}]$ for any $1 \leq \alpha \leq m_o$. Put
\[
F = \bigoplus_{\alpha=1}^{m_o} F(k_\alpha) \bigoplus F'.
\]
Define a bundle map $i : F \to \mathcal{E}$ as in \cite{6.3} We define a global regularization for $\overline{\mathcal{M}}_{g,n}(A)$ to be the bundle map $S : F \to \mathcal{E}$ by
\[
S([\kappa, b]) = [\bar{\partial}_{T,J,\nu}] + [i(\kappa, b)].
\]
Denote
\[
U = S^{-1}(0)|_{\mathcal{U}}.
\]
There is a bundle of finite rank $E$ over $U$ with a canonical section $\sigma$. We have a virtual neighborhood for $\overline{\mathcal{M}}_{g,n}(A)$:
\[
(U, E, \sigma).
\]
The map $\theta$ induce a bundle $\pi : \theta^*F \to \mathcal{U}^c$ and a bundle map $\theta^*S : \theta^*F \to \mathcal{E}$. Then restricting on $\mathcal{U}^c$ we have a virtual neighborhood for $\overline{\mathcal{M}}_C(g_1, g_2, n_1, n_2)$:
\[
(U_e, E_e, \sigma_e).
\]
One can check that
\[
U_e = \theta^*U|_{\mathcal{U}^c}, \quad E_e = \theta^*E|_{U_e}.
\]
Restricting on $\mathcal{U}_\theta$ we have a virtual neighborhood for $\overline{\mathcal{M}}_A(g_1, g_2, n_1, n_2)$:
\[
(U_\theta, E_\theta, \sigma_\theta).
\]
Then
\[
U_\theta = \bigcup_{C \in \mathcal{C}} U_e, \quad E_\theta|_{U_e} = E_e, \quad \sigma_\theta|_{U_e} = \sigma_e.
\]
Denote by $U_c^T$ (resp. $U_\theta^T$, $U^T$) the top strata of $U_e$ (resp. $U_\theta$, $U$). The element of $U_c^T$ has the form
\[
((\Sigma_1, \kappa_1, j_1, y_1, u_1), (\Sigma_2, \kappa_2, j_2, y_2, u_2)),
\]
where $u_1 : \Sigma_1 \to M$, $u_2 : \Sigma_2 \to M$ with $u_1(q) = u_2(q)$. Set
\[
U_{c,e} = \{(\kappa, b) \in U_e | |\kappa|_h \leq \varepsilon\}, \quad U_{\theta,e} = \{(\kappa, b) \in U_{\theta} | |\kappa|_h \leq \varepsilon\}.
\]
Let $(U_{ie}, E_{ie}, \sigma_{ie}), i = 1, 2$ be the virtual neighborhood of $\overline{\mathcal{M}}_A(g_i, n_i + 1), i = 1, 2$, where $A = A_1 + A_2$, $C = (A_1, A_2)$. Then
\[
U_e = \{(b_1, b_2) \in U_{1e} \times U_{2e} | e^{1e}_{n_1+1}(b_1) = e^{2e}_{n_2+1}(b_2)\}, \quad E_e = E_{1e} \times E_{2e}|_{U_e}, \quad \sigma_e = (\sigma_{1e}, \sigma_{2e})|_{U_e}.
\]
By the same method as in Theorem [1.1] we have

39
Theorem 11.2. $\mathbf{U}_c^T, \mathbf{U}_{1c}^T, \mathbf{U}_{2c}^T$, and $\mathbf{U}_b^T$ are smooth oriented effective orbifolds.

Note that $\{\Gamma_a\}$ are smooth on $\mathbf{U}_c^T, \mathbf{U}_{1c}^T$ and $\mathbf{U}_{2c}^T$, and $\Gamma_a \Theta$ is compactly supported. For any $[K] = [K_1 \times K_2] \in H^*(\mathcal{M}_{g_1,g_2,n_1,n_2}, \mathbb{R})$, let $[K] \in H^*(\mathcal{M}_{g_1,g_2,n_1,n_2}, \mathbb{R})$. We take a Thom form $\Theta$ supported in a small $\varepsilon$-ball of the 0-section of $\mathcal{E}$. Let $\Theta_c = \theta^c \Theta |_{\mathbf{U}_c}$. Then we define the GW-invariants $\Psi_{(A,g_1,g_2,n_1,n_2)}(K_1 \times K_2; \{\alpha_i\})$ as

$$
\Psi_{(A,g_1,g_2,n_1,n_2)}(K_1 \times K_2; \{\alpha_i\}) = \sum_{C \in \mathcal{C}} \int_{\mathbf{U}_{T,c}} \mathcal{P}(K) \wedge \prod_{j} ev_j'^* \alpha_j \wedge \sigma^* \Theta_c, \quad (61)
$$

where $\Theta_c$ is the Thom form of $p : \mathbf{E}_c \to \mathbf{U}_c$, $ev_j'$ denote the evaluation map $ev_j' : \mathbf{U}_{c,\varepsilon} \to M$ at $j$-th marked point. Using the same method as in Theorem 1.2 we can prove that the integrals are convergence. We can also define $\Psi_{(A,g_1,n_1+1)}(K_i; \{\alpha_i\})$ for $\mathcal{M}_{A_i}(g_i, n_i + 1)$, $i = 1, 2$.

Next we define the invariant $\Psi_{(A,g,n)}(\theta_i(K_1 \times K_2); \{\alpha_i\})$. First we define the following transfer map

**Definition 11.3.** Suppose that $X, Y$ are two topological space such that Poincare duality holds over $\mathbb{R}$. Let $f : X \to Y$. Then, the transfer map

$$
f_1 : H^*(X, \mathbb{R}) \to H^*(Y, \mathbb{R})
$$

is defined by $f_1(K) = PD(f_*(PD(K)))$.

We can identify a tubular neighborhood $\mathcal{O}$ of $im(\theta)$ with a neighborhood of zero section of the normal bundle $\mathcal{N}$ of $im(\theta)$ in $\mathcal{M}_{g,n}$. Let $im(\theta)^*$ be the Thom form of the bundle

$$
\pi : \mathcal{N} \to im(\theta),
$$

which can be chosen to be supported in the tubular neighborhood $\mathcal{O}$ of $im(\theta)$ and $im(\theta)^*$ can be seen as the Poincare dual of $im(\theta)$. For any $[K] = [K_1 \times K_2] \in H^*(\mathcal{M}_{g_1,g_2,n_1,n_2}, \mathbb{R})$, choose $K_i \in H^*(\mathcal{M}_{g_1,n_1+1}, \mathbb{R})$. Then $K = (K_1, K_2) \in \mathcal{M}_{g_1,g_2,n_1,n_2}$. Let $K_{\mathcal{M}}$ be the Poincare dual of $\theta_3(PD(K))$ in $\theta_3(\mathcal{M}_{g_1,g_2,n_1,n_2})$. Through $\pi$ we can pull $K_{\mathcal{M}}$ back to the total space of the normal bundle $\mathcal{N}$, denoted by $\pi^* K_{\mathcal{M}}$. Then, $\pi^* K_{\mathcal{M}}$ is defined over a tubular neighborhood of $im(\theta)$. Since $im(\theta)^*$ is supported in the tubular neighborhood, $im(\theta)^* \wedge \pi^* K_{\mathcal{M}}$ is a closed differential form defined over $\mathcal{M}_{g,n}$. One can check that

$$
\theta_i[K] = [im(\theta)^* \wedge \pi^* K_{\mathcal{M}}], \quad \pi_\ast[\theta_i(K)] = \theta_i[K]. \quad (62)
$$

Then we have

$$
\Psi_{(A,g,n)}(\theta_i(K_1 \times K_2); \{\alpha_i\}) = \int_{\mathbf{U}_c^T} \mathcal{P}(im(\theta)^* \wedge \pi^* K_{\mathcal{M}}) \wedge \prod_{j} ev_j'^* \alpha_j \wedge \sigma^* \Theta. \quad (63)
$$

where $\Theta$ is the Thom form of $p : \mathbf{E} \to \mathbf{U}_c$, $ev_j$ denote the evaluation map $ev_j : \mathbf{U}_c^T \to M$ at $j$-th marked point. As in the proof of Theorem 1.2 we can prove the convergence of $\Psi_{(A,g,n)}(\theta_i(K_1 \times K_2); \{\alpha_i\})$. By the same argument of Theorem 1.3 we can prove that

**Lemma 11.4.** $\Psi_{(A,g,n)}(\theta_i(K); \{\alpha_i\})$ is independent of the choice of $im(\theta)^*$. 
11.2 A gluing formula

In this subsection our main purpose is to prove the following theorem:

**Theorem 11.5.** For any $K_1 \times K_2 \in H^*(\mathcal{M}, \mathbb{R})$, $\alpha_1, \cdots, \alpha_n \in H^*(M, \mathbb{R})$, represented by smooth forms, we have

$$
\Psi_{(\alpha, \theta, n)}(\theta_1(\alpha_1 \times \alpha_2); \{\alpha_i\}) = \Psi_{(\alpha, \theta, n)}(K_1 \times K_2; \{\alpha_i\}).
$$

**Remark 11.6.** If $\mathbb{U}_\epsilon$ is smooth, by using (62) we can prove Theorem 11.5 directly. But what we know only the smoothness of the top strata $U^T$ and $U^T$ for $\mathbb{U}_\epsilon$. So we use our estimates in Section 8 to prove this theorem.

As in [9,1] we choose finite many points $P = \{(\kappa, b) \in \mathbb{U}_\theta, a = 1, \cdots, n\}$ with

$$
[(\kappa, b)] \in U_\theta, a = 1, \cdots, n_t,
$$

such that \{$(\mathbb{U}_\theta, (\kappa, b)) \in U_\theta \{/U_\theta, a = n_t + 1, \cdots, n\}$ is an open cover of $\mathbb{U}_\theta$ and $\mathbb{U}_\theta(\kappa, b)] \subset U_\theta^T$ for all $1 \leq a \leq n_t$. Choose $\kappa, \delta, \rho, t$ small such that

$$
glu(\mathbb{U}_\theta, (\kappa, b)) \cap U_\theta^T \times \mathbb{D}_\kappa^*(0) \rightarrow glu((\mathbb{U}_\theta, (\kappa, b)) \cap U_\theta^T \times \mathbb{D}_\kappa^*(0))
$$

is an orientation preserving diffeomorphism in orbifold sense. To simplify notations we denote

$$
V_{\theta, a} = \mathbb{U}_\theta, (\kappa, b) \cap U_\theta^T, \text{ } W_{\theta, a} = glu(V_{\theta, a} \times \mathbb{D}_\kappa(0)),
$$

For any $d$ denote

$$
U_{d, \theta} = \bigcup_a glu(V_{\theta, a} \times \mathbb{D}_\kappa^*(0)).
$$

We fix a number $a$, let $d$ be a small constant with $0 < d < \epsilon/3$. The map $\theta$ induces a embedding $\mathbb{U}_\theta \rightarrow \mathbb{U}$. We choose $im(\theta) = supp (\mathbb{U}_\theta \cap \theta^*(\mathbb{D}_\kappa^*(0) \cap U_\theta\cap U_\theta) \subset \mathbb{U}_d^T, \theta. \text{ (64)}$

We can choose a partition of unit \{$(\kappa, b) \in \mathbb{U}_\theta \cap \theta^*(\mathbb{D}_\kappa^*(0) \cap U_\theta\cap U_\theta) \subset \mathbb{U}_d^T, \theta. \text{ (64)}$

Using Theorem 8.5 and by a direct calculation we have

$$
\left| \frac{\partial \hat{\Gamma}_a}{\partial r} \right| \leq C e^{-\epsilon_1 r}. \text{ (65)}
$$

where we used the smoothness of cut-off function.

We can choose finite many points $\{(\kappa, b) \in \mathbb{U}_\theta \cap \theta^*(\mathbb{D}_\kappa^*(0) \cap U_\theta\cap U_\theta) \subset \mathbb{U}_d^T, \theta. \text{ (64)}$

Using Theorem 8.5 and by a direct calculation we have

$$
\left| \frac{\partial \hat{\Gamma}_a}{\partial r} \right| \leq C e^{-\epsilon_1 r}. \text{ (65)}
$$

where we used the smoothness of cut-off function.

We can choose finite many points $\{(\kappa, b) \in \mathbb{U}_\theta \cap \theta^*(\mathbb{D}_\kappa^*(0) \cap U_\theta\cap U_\theta) \subset \mathbb{U}_d^T, \theta. \text{ (64)}$

Using Theorem 8.5 and by a direct calculation we have
(2) \( \mathbb{U}_{[(n_\theta', b_\theta')]}(\varepsilon_{a'}, \delta_{a'}, \rho_{a'}) \cap U_{d}^{T, \theta} = \emptyset \) as \( d \) small enough for any \( n_\theta + 1 \leq a' \leq n \).

As in section 7.3 we can construct finite many cut-off functions \( \hat{\Gamma}_{a'} \) supported in \( \mathbb{U}_{[(r_{a'}, b_{a'})]}(\varepsilon_{a'}, \delta_{a'}, \rho_{a'}) \), \( n_\theta + 1 \leq a' \leq n \), satisfying
\[
\sum_{a' = n_\theta + 1}^{n} \hat{\Gamma}_{a'}|_{U_{\epsilon}^\tau \setminus U_{\epsilon/3}^\tau} > 0.
\]

By (65) we have \( \sum_{a' = 1}^{n} \hat{\Gamma}_{a'}|_{U_{\epsilon}^\tau} > 0 \) as \( \epsilon \) small enough. Then \( \{ \hat{\Gamma}_{a}, 1 \leq a \leq n \} \) induces a partition of unity \( \{ \Gamma_{a'}, 1 \leq a \leq n \} \) of \( U_{\epsilon} \) defined by
\[
\Gamma_{a'} = \frac{\hat{\Gamma}_{a}}{\sum_{1 \leq a \leq n} \hat{\Gamma}_{a}}.
\]

By (2) it is easy to see that \( \{ \Gamma_{a'}, 1 \leq a \leq n_\theta \} \) is a partition of unity of \( U_{d}^{T, \theta} \) and in \( U_{d}^{T, \theta} \)
\[
\Gamma_{a'} = \frac{\hat{\Gamma}_{a}}{\sum_{1 \leq a \leq n_\theta} \hat{\Gamma}_{a}}. \tag{66}
\]

We use \( \mathcal{P} \) to denote both \( \mathcal{P} : U_{c, c} \rightarrow \overline{\mathcal{M}}_{g_1, g_2, n_1, n_2} \) and \( \mathcal{P} : U_{c} \rightarrow \overline{\mathcal{M}}_{g, n} \). We have
\[
supp(\Gamma_{a'} \cdot \mathcal{P}^* im(\theta)^* \wedge \sigma^* \Theta) \subset W_{a'}, \ \forall a \leq n_\theta. \tag{67}
\]

**Proof of Theorem 11.5.** Denote
\[
(A) = \sum_{C \in C} \int_{U_{c, c}^\epsilon} \mathcal{P}^*(K) \wedge \prod_j ev_j^{*} \alpha_j \wedge \sigma_c^* \Theta_c - \sum_{C \in C} \int_{U_{c, c}^\epsilon} \mathcal{P}^*(\theta^* K_{M}) \wedge \prod_j ev_j^{*} \alpha_j \wedge \sigma_c^* \Theta,
\]
\[
(B) = \int_{U_{c}^\tau} \mathcal{F}_r - \sum_{C \in C} \int_{\theta(U_{c}^\tau)} \mathcal{P}^*(K_{M}) \wedge \prod_j ev_j^{*} \alpha_j \wedge \sigma^* \Theta.
\]
where
\[
\mathcal{F}_r = \mathcal{P}^* (im(\theta)^* \wedge \pi^* K_{M}) \wedge \prod_j ev_j^{*} \alpha_j \wedge \sigma^* \Theta.
\]

Note that \( ev_j \cdot \theta = ev'_j, \ \theta \cdot \sigma_c = \sigma \cdot \theta, \ \mathcal{P} \cdot \theta = \theta \cdot \mathcal{P} \). We have
\[
\int_{\theta(U_{c}^\tau)} \mathcal{P}^*(K_{M}) \wedge \prod_j ev_j^{*} \alpha_j \wedge \sigma^* \Theta = \int_{U_{c}^\tau} \mathcal{P}^*(\theta^* K_{M}) \wedge \prod_j ev_j^{*} \alpha_j \wedge \sigma_c^* \Theta.
\]

By (61) and (63), we only need to prove that \( (A) - (B) = 0 \). Since \( K \) and \( \theta^* K_{M} \) are in the same cohomology, we have \( (A) = 0 \), so it suffices to prove \( (B) = 0 \).

For \( 1 \leq a \leq n_\epsilon \), we choose \( s \) as a local coordinates of \( U_{\theta, \epsilon} \). Let \( t_o = e^{-2r - 2\sqrt{-1} \tau} \) be the gluing parameter at node \( \epsilon \). Then \( (t_o, s) \) is a local coordinates of \( \pi^* U_{\theta, \epsilon} \). On the other hand, since the bundle \( \mathcal{N} \) has a Riemannian structure, we can choose a smooth orthonormal frame field. This defines a coordinate \( \eta \) over fiber. Denote \( \eta = e^{-2\tau - 2\sqrt{-1} \tau} \) and \( s = \pi^* s \). Then \( (\eta, s) \) is also a local coordinates of \( O_d \cap \pi^* U_{\theta, \epsilon} \). Denote the Jacobi matrix by \( (a_{ij}) = \frac{\partial(t_o, s)}{\partial(\eta, \dot{\eta})} \). Since \( \mathcal{M}_{g, n}^{ed} \) is a smooth orbifold, \( (a_{ij}) \) and the inverse matrix \( (a_{ij}^{-1}) \) are uniform bounded in the coordinates. Then \( (t_o, s, \dot{\eta}) \) and \( (\eta, s, \dot{\eta}) \) are the local coordinates of \( W_{a} \), where \( \dot{j} = (\dot{j}_1, \cdots, \dot{j}_d) \). We have the coordinates tranformation
\[
s = s(\eta, \dot{s}), \ t_o = t_o(\eta, \dot{s}), \ \dot{j}_j = \dot{j}_j.
\]
Denote by \((b_{ij}) = \frac{\partial (\eta, \delta)}{\partial (\gamma, \sigma)}\). It follows from the bound of \((a_{ij})\) and \((a_{ij}^{-1})\) that \((b_{ij})\) and the inverse matrix \((b_{ij}^{-1})\) are uniform bounded in the coordinates.

In each \(W_a\), the map \(\pi : \mathcal{M}_{g,n} \to \mathcal{M}_{g_1,n_1}^{g_2,n_2} \) induce a map \(\pi : W_a \to V_{\theta,a}\) defined by
\[
(\eta, \bar{s}, \bar{z}) \mapsto (s, z),
\]
Then
\[
(B_a) = \int_{W_a} \pi^* \Gamma_a \mathbb{H}_r - \int_{\theta(V_{\theta,a})} \Gamma_a \mathcal{P}^*(K_{\mathcal{M}}) \wedge \prod_j ev_j^* \alpha_j \wedge \sigma^* \Theta.
\]
Then \((B) = \sum(B_a) + \sum \int_{W_a} (\Gamma_a - \pi^* \Gamma_a) \mathbb{H}_r\). By (64), (66), (65) and the bound of matrix \((b_{ij}), (b_{ij}^{-1})\) we obtain
\[
\left| \int_{W_a} (\Gamma_a - \pi^* \Gamma_a) \mathbb{H}_r \right| \leq Cd^1.
\]
We only need estimate \((B_a)\).

We recall the expression of \(im(\theta)^*\), the detail can be found in [3]. Let \(\Gamma\) be an increasing function of the radius \(|\eta|\) such that
\[
\int_{\mathbb{R}^+} d\Gamma = 1, \quad \Gamma(0) = -1, \quad |\Gamma| \leq 1, \quad d\Gamma \text{ is a compact support form.} \quad (68)
\]
Let \(\phi_{ac} : U_a \cap U_c \to S^1\) the transformation function of \(\mathcal{N}\). Then
\[
im(\theta)^* = d\Gamma \wedge \psi_N - \Gamma \pi^* (e(\mathcal{N})), \quad e(\mathcal{N}) = \frac{\sqrt{-1}}{2\pi} \sum_{e} d(\Gamma_a d \log \phi_{ac}). \quad (69)
\]
Denote \(i : \theta(U_c) \to U\) is the inclusion map. Then \((B_a)\) can be re-written as
\[
\int_{W_a} \pi^* \Gamma_a \mathcal{P}^* \gamma^*(\theta)^* \wedge \left( \pi^* K_{\mathcal{M}} \wedge \prod_j ev_j^* \alpha_j \wedge \sigma^* \Theta - \pi^* \gamma^* \left( \pi^* K_{\mathcal{M}} \wedge \prod_j ev_j^* \alpha_j \wedge \sigma^* \Theta \right) \right).
\]
As in the proof of Theorem [12] denote by \((E_1, \ldots, E_{6g-6+2n+d})\) (resp. \((\bar{E}_1, \ldots, \bar{E}_{6g-6+2n+d})\)) the induced vector field by \((t_0, \bar{s}, \bar{z})\) (resp. \((\eta, \bar{s}, \bar{z})\)). Set
\[
dV_{\theta} = \bigwedge_j \left( \frac{\sqrt{-1}}{2} d\bar{s}_j \wedge d\bar{\theta}_j \right) \wedge d\bar{\theta}_1 \wedge \cdots \wedge d\bar{\theta}_d.
\]
Then \(\pi^* K_{\mathcal{M}} \wedge \prod_j ev_j^* \alpha_j \wedge \sigma^* \Theta\) can be written as
\[
f_1(\eta, \bar{s}, \bar{z}) d\bar{V} + f_2(\eta, \bar{s}, \bar{z}) \wedge d\bar{r} \wedge \bar{\tau},
\]
where \(f_1 = \pi^* K_{\mathcal{M}} \wedge \prod_j ev_j^* \alpha_j \wedge \sigma^* \Theta(\bar{E}_3, \cdots, \bar{E}_{6g-6+2n+d})\). Since
\[
\pi^* \gamma^* f_1(\eta, \bar{s}, \bar{z}) = f_1(0, \bar{s}, \bar{z}), \quad \pi^* \gamma^* (f_2(\eta, \bar{s}, \bar{z}) \wedge d\bar{r} \wedge \bar{\tau}) = 0,
\]
by (69) we have
\[
(I)_a = \int_{W_a} \pi^* \Gamma_a \left[ (f_1(\eta, \bar{s}, \bar{z}) - f_1(0, \bar{s}, \bar{z})) d\bar{V} \wedge d\Gamma \wedge \psi_N \right] \quad (70)
\]
\[
(II)_a = \int_{W_a} \pi^* \Gamma_a \Gamma f_2(\eta, \bar{s}, \bar{z}) \mathcal{P}^* \gamma^*(e(\mathcal{N})) \wedge d\bar{r} \wedge \bar{\tau} \quad (71)
\]
43
As in the proof of Theorem 1.2 using (68), Theorem 8.5 and Lemma 8.8 we can prove that

\[ |(I)_{\alpha}| \to 0, \quad \text{as } d \to 0. \]

Similarly, by Theorem 8.5, Lemma 8.7 and smoothness of \( b_{ij} \) we have

\[ |(f_1(\eta, s, \hat{t}) - f_1(0, \hat{s}, \hat{t}))| \leq C(e^{-\sigma} + \hat{\delta}) \to 0, \quad \text{as } d \to 0. \]

Using (68) and (69) we get

\[ |(I)_{\alpha}| \to 0, \quad \text{as } d \to 0. \]

Then for \( 1 \leq a \leq n_t \)

\[ |(B_{\alpha})| \to 0, \quad \text{as } d \to 0. \]  (72)

For \( \alpha > n_t \), we choose \((t_0, s, t, \hat{t})\) and \((\eta, \hat{s}, \hat{t}, \hat{\eta})\) as local coordinates of \( \mathcal{W}_a \), where \((\hat{s}, \hat{t}) = (\pi^*s, \pi^*t)\).

By the similar argument above we have (72) also holds.

By Lemma 11.4 \( \Psi_{(A,g,n)}(\theta_i(K); \{\alpha_i\}) \) is independent of the choice of \( im(\theta)^* \). Hence

\[ |\Psi_{(A,g,n)}(\theta_i(K); \{\alpha_i\}) - \Psi_{(A,g,1,2,n_1,n_2)}(K; \{\alpha_i\})| = |(A) - (B)| \to 0, \quad \text{as } d \to 0. \]

Then the lemma is proved. □

11.3 Splitting axiom

Denote by \( \Delta \subset M \times M \) the diagonal. Let \( \pi : N \to \Delta \) be the normal bundle in \( M \times M \), and let \( \Phi \) be a Thom form on \( N \). There is a natural map

\[ ev_{n_1+1,n_2+1}^c : U_{1c}^T \times U_{2c}^T \to M \times M \]

defined by

\[ ev_{n_1+1,n_2+1}^c(b_1, b_2) = (ev_{n_1+1}^c(b_1), ev_{n_2+1}^c(b_2)). \]

Then \((ev_{n_1+1,n_2+1}^c)^* N \) is a vector bundle on \( U_{1c}^T \). Set \( \Phi_c = (ev_{n_1+1,n_2+1}^c)^* \Phi \). Then for any differential form \( \alpha \in U_{1c}^T \) with exponential decay on \( \partial U_{1c}^T \) we have

\[ \int_{U_{1c}^T} \alpha = \int_{U_{1c}^T \times U_{2c}^T} \Phi_c \wedge \pi^* \alpha. \]

Choose a homogeneous basis \( \{\beta_b\}_{1 \leq b \leq L} \) of \( H^*(M, \mathbb{R}) \). Let \( (\eta_{ab}) \) be its intersection matrix. Note that \( \eta_{ab} = \delta_a \cdot \beta_b = 0 \) if the dimensions of \( \delta_a \) and \( \beta_b \) are not complementary to each other. Put \( (\eta^{ab}) \) to be the inverse of \( (\eta_{ab}) \). Then, the Poincare dual of \( \Delta \) is

\[ \Delta^* = \sum_{a,b} \eta^{ab} \beta_a \otimes \beta_b. \]

There is a smooth form \( \sigma \in C^\infty(M \times M) \) such that

\[ \Phi - \Delta^* = d\sigma. \]

Then

\[ \int_{U_{1c}^T} \alpha = \int_{U_{1c}^T \times U_{2c}^T} (ev_{n_1+1,n_2+1}^c)^*(\Delta^* + d\sigma) \wedge \alpha = \int_{U_{1c}^T \times U_{2c}^T} (ev_{n_1+1,n_2+1}^c)^* \Delta^* \wedge \alpha. \]  (73)

In the last equality we used the Stokes theorem and the same argument as in the proof of (2) in Theorem 1.3.
Lemma 11.7. Let $K_1 \times K_2 \in H^*(\overline{M}_{g_1,g_2,n_1,n_2}, \mathbb{R})$, $\alpha_1, \cdots, \alpha_n \in H^*(M, \mathbb{R})$ be represented by smooth forms. Then

$$
\Psi_{(A_1,A_2,g_1,g_2,n_1,n_2)}(K_1 \times K_2; \{\alpha_i\}) = \epsilon(K, \alpha) \sum_{a,b} \eta^{ab} \Psi_{(A_1,g_1,n_1+1)}(K_1; \{\alpha_i\}_{i \leq n_1}, \beta_a) \Psi_{(A_2,g_2,n_2+1)}(K_2; \{\alpha_j\}_{j > n_1}, \beta_b),
$$

where $\epsilon(K, \alpha) = (-1)^{\deg(K_2) \sum_{i=1}^{n_1}(\deg(\alpha_i))}$.

Proof. Let $K = (K_1, K_2)$ be the smooth form as in subsection [11.1]. Let $\Theta_{ic}$ be the Thom form of $E_{ic}$ supported in a neighborhood of the zero section. Then $\Theta_{ic} = \Theta_{1c} \land \Theta_{2c}$ is the Thom form of $E_c$. Using (73), a direct calculation gives us

$$
\Psi_{(A_1,A_2,g_1,g_2,n_1,n_2)}(K_1 \times K_2; \{\alpha_i\}) = \epsilon(K, \alpha) \sum_{a,b} \eta^{ab} \Psi_{(A_1,g_1,n_1+1)}(K_1; \{\alpha_i\}_{i \leq n_1}, \beta_a) \Psi_{(A_2,g_2,n_2+1)}(K_2; \{\alpha_j\}_{j > n_1}, \beta_b).
$$

The lemma is proved.

Combination of Lemmas [11.7] and [11.5] gives us

Theorem 11.8. For any $K_1 \times K_2 \in H^*(\overline{M}_{g_1,g_2,n_1,n_2}, \mathbb{R})$, $\alpha_1, \cdots, \alpha_n \in H^*(M, \mathbb{R})$, represented by smooth forms, we have

$$
\Psi_{(A,g,n)}((\theta);(K_1 \times K_2)); \{\alpha_i\}) = \epsilon(K, \alpha) \sum_{A_1+A_2=A} \sum_{a,b} \Psi_{(A_1,g_1,n_1+1)}(K_1; \{\alpha_i\}_{i \leq n_1}, \beta_a) \eta^{ab} \Psi_{(A_2,g_2,n_2+1)}(K_2; \{\alpha_j\}_{j > n_1}, \beta_b),
$$

where $\epsilon(K, \alpha) = (-1)^{\deg(K_2) \sum_{i=1}^{n_1}(\deg(\alpha_i))}$.

References

[1] K. Behrend, Gromov-Witten invariants in algebraic geometry. Invent. Math. 127(3), 601-617, 1997.

[2] N. Berline, E. Getzler, M. Vergne, Heat kernels and Dirac operators. Corrected reprint of the 1992 original. Grundlehren Text Editions. Springer-Verlag, Berlin, 2004.

[3] R. Bott and L.W. Tu. Differential forms in algebraic topology. Springer-Verlag. 1982

[4] R. Castellano, Smoothness of Kuranishi atlases on Gromov-Witten moduli spaces, arXiv:1511.04350

[5] R. Castellano, Genus zero Gromov-Witten axioms via Kuranishi atlases, arXiv:1601.04048

[6] B. Chen, A.-M. Li and B. Wang, Virtual neighborhood technique for pseudo-holomorphic spheres, arXiv:1306.3276.
[7] B. Chen, A.-M. Li and B. Wang, Gluing principle for orbifold stratified spaces, Geometry and topology of manifolds, 15-57, Springer Proc. Math. Stat., 154, Springer, [Tokyo], 2016.

[8] K. Cieliebak, K. Mohnke, Symplectic hypersurfaces and transversality in Gromov-Witten theory, J. Symplectic Geom. 3 (2005), no.4, 589-654.

[9] G. Daskalopoulos, C. Mese, $C^1$ estimates for the Weil-Petersson metric. Trans. Amer. Math. Soc. 369 (2017), no. 4, 2917-2950.

[10] K. Fukaya and K. Ono, Arnold conjecture and Gromov-Witten invariants, Topology 38, 1999, 933-1048.

[11] K. Fukaya, Y. Oh, H. Ohta, K. Ono, Technical details on Kuranishi structure and virtual fundamental chain, arXiv:1209.4410.

[12] K. Fukaya, Y. Oh, H. Ohta, K. Ono, Kuranishi structure, Pseudo-holomorphic curve, and Virtual fundamental chain: Part 1, arXiv:1503.07631.

[13] K. Fukaya, Y. Oh, H. Ohta, K. Ono, Kuranishi structure, Pseudo-holomorphic curve, and Virtual fundamental chain: Part 2, arXiv:1704.01848.

[14] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, Invent. math., 82 (1985), 307-347.

[15] H. Hofer, V. Lizan, J.-C. Sikorav: On genericity for holomorphic curves in four-dimensional almost-complex manifolds, J. Geom. Anal. (1997) 7, 149-159.

[16] S. Ivashkovich, V. Shevchishin: Pseudo-holomorphic curves and envelopes of meromorphy of two-spheres in $\mathbb{C}P^2$, arXiv:math/9804014.

[17] M. Kontsevich, Yu. Manin, Gromov-Witten classes, quantum cohomology, and enumerative geometry, Comm.Math.Phys. 164 (1994) 525-562. M. Gromov, Pseudo holomorphic curves in symplectic manifolds, Invent. math., 82 (1985), 307-347.

[18] A.-M. Li and Y. Ruan, Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds, Invent. Math. 145, 151-218(2001).

[19] A.-M. Li and Li, Sheng, The Exponential Decay of Gluing Maps and Gromov-Witten Invariants, arXiv:1506.06333.

[20] A.-M. Li, Li, Sheng , A Finite Rank Bundle over $J$-Holomorphic map Moduli Spaces arXiv:1711.04228.

[21] J. Li and G. Tian, Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. J. Amer. Math. Soc., 11(1), 119-174,1998.

[22] G. Liu and G. Tian, Floer homology and Arnold conjecture. J. Differential Geom. 49 (1998), no. 1, 1-74.

[23] H. Masur. The Extension of the Weil-Petersson Metric to the Boundary of Teichmüller Space. Duke Math. Jour. 43(1976), no.3, 623-635.

[24] D. McDuff, Notes on Kuranishi Atlases, arXiv:1411.4306.

[25] D. McDuff and D. Salamon, $J$-holomorphic curves and symplectic topology. American Mathematical Society Colloquium Publications, 52. American Mathematical Society, Providence, RI, 2004.
[26] D. McDuff, K. Wehrheim, Kuranishi atlases with trivial isotropy - the 2013 state of affairs, arXiv:1208.1340.

[27] D. McDuff, K. Wehrheim, Smooth Kuranishi atlases with isotropy, Geom. Topol. 21 (2017), no. 5, 2725-2809.

[28] D. McDuff, K. Wehrheim, The fundamental class of smooth Kuranishi atlases with trivial isotropy, J. Topol. Anal. 10 (2018), no. 1, 71-243.

[29] D. McDuff, K. Wehrheim, The topology of Kuranishi atlases, Proc. Lond. Math. Soc. (3) 115 (2017), no. 2, 221-292.

[30] D. Mumford, Hirzebruch’s proportionality theorem in the noncompact case, Invent. Math. 42 (1977), 239-272.

[31] J. Pardon, An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves, Geometry & Topology 20 (2016) 779-1034.

[32] J. Robbin, D. Salamon, A construction of the Deligne-Mumford orbifold. J. Eur. Math. Soc. (JEMS) 8 (2006), no. 4, 611-699.

[33] Y. Ruan, Topological Sigma model and Donaldson type invariants in Gromov theory, Math. Duke J. 83(1996), 461-500.

[34] Y. Ruan, Virtual neighborhoods and pseudo-holomorphic curves, Turkish Jour. of Math. 1(1999), 161-231.

[35] Y. Ruan, G. Tian, A mathematical theory of quantum cohomology. J. Differential Geom. 42 (1995), no. 2, 259-367.

[36] Y. Ruan, G. Tian, Higher genus symplectic invariants and sigma models coupled with gravity. Invent. Math. 130 (1997), no. 3, 455-516.

[37] B. Siebert, Gromov-Witten invariants for general symplectic manifolds, arXiv:9608005

[38] M. Tehrani, Kenji Fukaya, Gromov-Witten theory via Kuranishi structures, arXiv:1701.07821

[39] A.J. Tromba, Teichmüller theory in Riemannian geometry, Lecture notes prepared by Jochen Denzler. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1992.

[40] S. Wolpert, Cusps and the family hyperbolic metric, Duke Jou. Math., vol. 138, no. 3, 423-443, 2007.

[41] S. Wolpert, The hyperbolic metric and the geometry of the universal curve, J. Diff. Geom. 31(1990),417-472

[42] S. Wolpert. On families of holomorphic differentials on degenerating annuli. Quasiconformal mappings, Riemann surfaces, and Teichmüller spaces, 363-370, Contemp. Math., 575, Amer. Math. Soc., Providence, RI, 2012.

[43] Marian Fabian, Vicente Montesinos and Vaclav Zizler, Smoothness in Banach spaces. Selected problems, R. Acad. Cien. Serie A. Mat.VOL. 100 (1-2), 2006, pp. 101-125

[44] W. Zhang, Lectures on Chern-Weil theory and Witten deformations. (English summary) Nankai Tracts in Mathematics, 4. World Scientific Publishing Co., Inc., River Edge, NJ, 2001.