Abstract

These notes aim to take the reader from an elementary understanding of functional analysis and probability theory to a robust construction of the stochastic integral in Hilbert Spaces. We consider integrals driven at first by real valued martingales and later by Cylindrical Brownian Motion, introducing this concept and expanding into a basic set up for Stochastic Partial Differential Equations (SPDEs). The framework that we establish facilitates an exceedingly broad class of SPDEs and noise structures, in which we build upon standard SDE theory and rigorously deduce a conversion between the Stratonovich and Itô Forms. The study of Stratonovich equations is largely motivated by the stochastic variational principle of SALT [4] for fluid dynamics, and we discuss an application of the framework to a Navier-Stokes Equation with Stochastic Lie Transport as seen in [1]. Moreover we prove a fundamental existence and uniqueness result (which to the best of our knowledge, is not present in the literature) for SPDEs evolving in a finite dimensional Hilbert Space driven by Cylindrical Brownian Motion, assuming the analogy to the Lipschitz and linear growth conditions for the standard existence and uniqueness theory for SDEs.

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Introduction

Stochastic Differential Equations have rich applications in physics and finance, for example in Langevin Equations modelling the movement of a particle in space [5] or the Black-Scholes options pricing model for the dynamics of the price of a stock [6]. These are applications of classical Itô Calculus, where we construct the integral of a process taking values in Euclidean space. Whilst this theory is adequate in such applications, mathematical models for physical phenomena far exceed those for the position of a particle; it is natural to ask the question of how to stochastically perturb an equation modelling (for example) the velocity or temperature of a particle, as a function of space and time. The ways one might do this are vast [7, 8] and to this end we address the problem of how to define the stochastic integral

\[ \int_0^t \Psi_s dW_s \]  

for \( \Psi : \Omega \times [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^d \) and \( W \) a standard real valued Brownian Motion. The idea is to regard \( \Psi \) not as a pointwise defined function but rather an element of a function space, which is our motivating context for stochastic integration of Hilbert Space valued processes. The first section of these notes deals with a classical construction of the Itô stochastic integral, largely following the one dimensional case so well explicated in [9, 10]. We introduce the notion of a Cylindrical Brownian Motion which facilitates an infinite dimensional driving noise in our evolution equations, and the first section concludes by considering the integrals of Hilbert-Schmidt valued processes against a Cylindrical Brownian Motion.

The second section begins to address this stochastic integration in the context of SPDEs, where we establish an abstract framework to define notions of solutions for equations in Itô and Stratonovich form. We are keen to address Stratonovich equations for their physical significance, emphasised greatly in the seminal work [4] where the author deduces a new class of stochastic equations formulated with Stratonovich integration that serve as much improved fluid dynamics models. These models are highly technical, due to the nonlinearity persisting from the deterministic form and the differential operator appearing in the diffusion term. Such models are therefore good motivation for us to establish a very general framework for SPDEs, which aren’t covered by the standard monographs in the field. In this framework we make the conversion between Stratonovich and Itô equations rigorous, and this framework is returned to for a very strong existence and uniqueness result in [1].

1 Stochastic Calculus in Infinite Dimensions

1.1 Elementary Notation

Throughout these notes we work with a fixed filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\). In the following \( \mathcal{O} \) represents a subset of \( \mathbb{R}^N \) and we shall always consider Banach Spaces as measure spaces equipped with the Borel \( \sigma \)-algebra. We will on occasion use \( \lambda \) to represent the Lebesgue Measure.

Definition 1.1.1. Let \((\mathcal{X}, \mu)\) denote a general measure space, \((\mathcal{Y}, \|\cdot\|_Y)\) and \((\mathcal{Z}, \|\cdot\|_Z)\) be Banach Spaces, and \((\mathcal{U}, \langle \cdot, \cdot \rangle_\mathcal{U}), (\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H})\) be general Hilbert spaces. \( \mathcal{O} \) is equipped with Euclidean norm.

- \( L^p(\mathcal{X}; \mathcal{Y}) \) is the usual class of measurable \( p \)-integrable functions from \( \mathcal{X} \) into \( \mathcal{Y} \), \( 1 \leq p < \infty \),
which is a Banach space with norm
\[ \|\phi\|^p_{L^p(X;Y)} := \int_X \|\phi(x)\|^p_Y \mu(dx). \]
The space \( L^2(X;Y) \) is a Hilbert Space when \( Y \) itself is Hilbert, with the standard inner product
\[ \langle \phi, \psi \rangle_{L^2(X;Y)} = \int_X \langle \phi(x), \psi(x) \rangle_Y \mu(dx). \]

In the case \( X = \mathcal{O} \) and \( Y = \mathbb{R}^N \) note that
\[ \|\phi\|^2_{L^2(\mathcal{O};\mathbb{R}^N)} = \sum_{l=1}^N \|\phi^l\|^2_{L^2(\mathcal{O};\mathbb{R})} \]
for the component mappings \( \phi^l : \mathcal{O} \rightarrow \mathbb{R} \).

- \( L^\infty(X;Y) \) is the usual class of measurable functions from \( X \) into \( Y \) which are essentially bounded, which is a Banach Space when equipped with the norm
\[ \|\phi\|_{L^\infty(X;Y)} := \inf \{ C \geq 0 : \|\phi(x)\|_Y \leq C \text{ for } \mu\text{-a.e. } x \in X \}. \]

- \( L^\infty(\mathcal{O};\mathbb{R}^N) \) is the usual class of measurable functions from \( \mathcal{O} \) into \( \mathbb{R}^N \) such that \( \phi^l \in L^\infty(\mathcal{O};\mathbb{R}) \) for \( l = 1, \ldots, N \), which is a Banach Space when equipped with the norm
\[ \|\phi\|_{L^\infty} := \sup_{l \leq N} \|\phi^l\|_{L^\infty(\mathcal{O};\mathbb{R})}. \]

- \( C(X;Y) \) is the space of continuous functions from \( X \) into \( Y \).

- \( W^{m,p}(\mathcal{O};\mathbb{R}) \) for \( 1 \leq p < \infty \) is the sub-class of \( L^p(\mathcal{O};\mathbb{R}) \) which has all weak derivatives up to order \( m \in \mathbb{N} \) also of class \( L^p(\mathcal{O};\mathbb{R}) \). This is a Banach space with norm
\[ \|\phi\|^p_{W^{m,p}(\mathcal{O};\mathbb{R})} := \sum_{|\alpha| \leq m} \|D^\alpha \phi\|^p_{L^p(\mathcal{O};\mathbb{R})} \]
where \( D^\alpha \) is the corresponding weak derivative operator. In the case \( p = 2 \), \( W^{m,2}(U,\mathbb{R}) \) is a Hilbert Space with inner product
\[ \langle \phi, \psi \rangle_{W^{m,2}(\mathcal{O};\mathbb{R})} := \sum_{|\alpha| \leq m} \langle D^\alpha \phi, D^\alpha \psi \rangle_{L^2(\mathcal{O};\mathbb{R})}. \]

- \( W^{m,\infty}(\mathcal{O};\mathbb{R}) \) for \( m \in \mathbb{N} \) is the sub-class of \( L^\infty(\mathcal{O};\mathbb{R}) \) which has all weak derivatives up to order \( m \in \mathbb{N} \) also of class \( L^\infty(\mathcal{O};\mathbb{R}) \). This is a Banach space with norm
\[ \|\phi\|_{W^{m,\infty}(\mathcal{O};\mathbb{R})} := \sup_{|\alpha| \leq m} \|D^\alpha \phi\|_{L^\infty(\mathcal{O};\mathbb{R}^N)}. \]

- \( W^{m,p}(\mathcal{O};\mathbb{R}^N) \) for \( 1 \leq p < \infty \) is the sub-class of \( L^p(\mathcal{O};\mathbb{R}^N) \) which has all weak derivatives up to order \( m \in \mathbb{N} \) also of class \( L^p(\mathcal{O};\mathbb{R}^N) \). This is a Banach space with norm
\[ \|\phi\|^p_{W^{m,p}(\mathcal{O};\mathbb{R}^N)} := \sum_{l=1}^N \|\phi^l\|^p_{W^{m,p}(\mathcal{O};\mathbb{R})}. \]
In the case $p = 2$ the space $W^{m,2}(\mathcal{O}, \mathbb{R}^N)$ is Hilbertian with inner product

$$\langle \phi, \psi \rangle_{W^{m,2}(\mathcal{O}, \mathbb{R}^N)} := \sum_{l=1}^{N} \langle \phi^l, \psi^l \rangle_{W^{m,2}(\mathcal{O}, \mathbb{R})}.$$  

- $W^{m,\infty}(\mathcal{O}; \mathbb{R}^N)$ is the sub-class of $L^\infty(\mathcal{O}, \mathbb{R}^N)$ which has all weak derivatives up to order $m \in \mathbb{N}$ also of class $L^\infty(\mathcal{O}, \mathbb{R}^N)$. This is a Banach space with norm

$$\| \phi \|_{W^{m,\infty}(\mathcal{O}, \mathbb{R}^N)} := \sup_{l \leq N} \| \phi^l \|_{W^{m,\infty}(\mathcal{O}, \mathbb{R})}.$$  

- $\mathcal{L}(\mathcal{Y}; \mathcal{Z})$ is the space of bounded linear operators from $\mathcal{Y}$ to $\mathcal{Z}$. This is a Banach Space when equipped with the norm

$$\| F \|_{\mathcal{L}(\mathcal{Y}; \mathcal{Z})} = \sup_{\| y \|_{\mathcal{Y}} = 1} \| F y \|_{\mathcal{Z}}.$$  

It is the dual space $\mathcal{Y}^*$ when $\mathcal{Z} = \mathbb{R}$, with operator norm $\| \cdot \|_{\mathcal{Y}^*}$.

- $\mathcal{L}^2(\mathcal{U}; \mathcal{H})$ is the space of Hilbert-Schmidt operators from $\mathcal{U}$ to $\mathcal{H}$, defined as the elements $F \in \mathcal{L}(\mathcal{U}; \mathcal{H})$ such that for some basis $(e_i)$ of $\mathcal{U}$,

$$\sum_{i=1}^{\infty} \| F e_i \|^2_{\mathcal{H}} < \infty.$$  

This is a Hilbert space with inner product

$$\langle F, G \rangle_{\mathcal{L}^2(\mathcal{U}; \mathcal{H})} = \sum_{i=1}^{\infty} \langle F e_i, G e_i \rangle_{\mathcal{H}}$$

which is independent of the choice of basis.

For the reader’s convenience we also collect here notation that is introduced in the paper, referencing where it is defined if needed.

- $\lambda$ denotes the Lebesgue measure
- $1_A$ denotes the indicator function of the set $A$
- $I_T^f$ is defined in 1.2.3
- $\mathcal{T}_H$ is defined in 1.2.4
- $\mathcal{M}^2, \mathcal{M}^2_c$ are defined in 1.3.1
- $\mathcal{M}^2_c(\mathcal{H})$ is defined in 1.5.2
- $\mathcal{M}^2_c(\mathcal{H})$ are the corresponding semi-martingale spaces
- $\mathcal{T}_M^T, \mathcal{T}_M^H$ are defined in 1.3.2
- $\mathcal{T}_M^T, \mathcal{T}_M^H$ are defined in 1.3.3
- $[\cdot]$ is defined in 1.5.5
1.2 A Classical Construction for Hilbert Space Valued Processes

As alluded to, the construction precisely mirrors the standard one-dimensional Itô integral; as such we start from simple processes. \( \mathcal{H} \) will denote a general Hilbert space, with norm and inner product \( \| \cdot \|_{\mathcal{H}} \) and \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) respectively.

**Definition 1.2.1.** Let \( 0 = t_0 < \cdots < t_i < t_{i+1} < \cdots \) be a time index such that \( (t_i) \) approaches infinity. A simple \( \mathcal{H} \) valued process is one that for a.e. \( \omega \) takes the form

\[
\Psi_t(\omega) = a_0(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{i=0}^{\infty} a_i(\omega)\mathbb{1}_{(t_i,t_{i+1})}(t)
\]

where each \( a_i \in L^2(\Omega; \mathcal{H}) \) and is \( \mathcal{F}_{t_i} \) measurable, with respect to the Borel sigma algebra on \( \mathcal{H} \). The limit is taken in \( \mathcal{H} \).

**Definition 1.2.2.** The Itô integral of a simple \( \mathcal{H} \) valued process \( (\Psi_t) \), with respect to Brownian motion, is defined as

\[
\int_0^t \Psi_s dW_s := \sum_{i=0}^{\infty} a_i(W_{t_{i+1}} - W_{t_i})
\]

Note that in reality the above is a finite sum, so there is no danger in how we take the limit. Indeed it can alternatively be expressed as

\[
\sum_{i=0}^{k-1} a_i(W_{t_{i+1}} - W_{t_i}) + a_k(W_t - W_{t_k})
\]  

(2)

where \( k \) is such that \( t \in (t_k, t_{k+1}] \). Unsurprisingly now we define the integral for a more general class of integrands, using approximations by simple processes.

**Definition 1.2.3.** We use \( \mathcal{I}_T^\mathcal{H} \) to denote the class of \( \mathcal{H} \) valued processes \( (\Psi_t) \) which are \( (\mathcal{F}_t) \) progressively measurable and satisfy the square integrability condition

\[
\mathbb{E}\left( \int_0^T \| \Psi_s \|^2_{\mathcal{H}} ds \right) < \infty.
\]

(3)

In other words, \( \Psi \in L^2(\Omega \times [0,T]; \mathcal{H}) \) where the domain space \( \Omega \times [0,T] \) is a measure space equipped with the product measure \( \mathbb{P} \times \lambda \).

We have made the definition for progressively measurable processes, not previsible processes as will commonly be seen in the literature. Progressive measurability is a weaker condition than previsibility, but thankfully most reasonably behaved processes (adapted and left continuous for example) are both progressively measurable and previsible. We make the definition here for the
more general class of integrands in the cases where the integrator is continuous. Other authors may opt for previsible processes as these become necessary in retaining nice properties (e.g. martingality) when defining the stochastic integral with respect to discontinuous integrators, or even in making the definition itself.

**Definition 1.2.4.** The class of processes \( (\Psi_t) \) such that \( (\Psi_t) \in I^H_T \) for all \( T > 0 \), is denoted by \( I^H \).

\( I^H \) represents our class of integrands.

**Proposition 1.2.5.** For any \( (\Psi_t) = \Psi \in I^H_T \), there exists a sequence of simple processes \( (\Psi^n) \) which converge to \( \Psi \) in \( L^2(\Omega \times [0,T]; \mathcal{H}) \).

**Definition 1.2.6.** We can now define the Itô stochastic integral (1) for processes \( \Psi \in I^H \), by

\[
\int_0^t \Psi_s dW_s := \lim_{n \to \infty} \int_0^t \Psi^n_s dW_s
\]

where \( (\Psi^n) \) is the sequence of simple processes postulated in 1.2.5 which approach \( \Psi \) in \( L^2(\Omega \times [0,t]; \mathcal{H}) \), and the limit is taken in \( L^2(\Omega; \mathcal{H}) \).

The fact that this is the natural topology in which to take the limit of simple stochastic integrals falls from the Itô Isometry for simple processes.

**Proposition 1.2.7.** For a simple process \( \Psi^n \) and any time \( t > 0 \),

\[
\mathbb{E}\left( \left\| \int_0^t \Psi^n_s dW_s \right\|_{\mathcal{H}}^2 \right) = \mathbb{E}\left( \int_0^t \left\| \Psi^n_s \right\|_{\mathcal{H}}^2 ds \right).
\]

**Proof.** Let’s suppose that \( \Psi^n \) takes the form

\[
\Psi^n_t(\omega) = a^n_0(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{i=0}^{\infty} a^n_i(\omega)\mathbb{1}_{(\frac{t}{n}, \frac{t}{n}+\frac{1}{n})}(t)
\]

as outlined in 1.2.1. Then applying 1.2.2, we deconstruct the LHS of (5):

\[
\mathbb{E}\left( \left\| \int_0^t \Psi^n_s dW_s \right\|_{\mathcal{H}}^2 \right) = \mathbb{E}\left( \left\| \sum_{i=0}^{\infty} a^n_i(W_{t+1/n}^i - W_{t}^i) \right\|_{\mathcal{H}}^2 \right) = \mathbb{E}\left( \left\langle \sum_{i=0}^{\infty} a^n_i(W_{t+1/n}^i - W_{t}^i), \sum_{j=0}^{\infty} a^n_j(W_{t+1/n}^j - W_{t}^j) \right\rangle_{\mathcal{H}} \right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbb{E}\left( \langle a^n_i, a^n_j \rangle_{\mathcal{H}}(W_{t+1/n}^i - W_{t}^i)(W_{t+1/n}^j - W_{t}^j) \right).
\]

For \( i \neq j \), and without loss of generality \( i < j \), the random inner product is \( \mathcal{F}_t \) measurable as the continuity of the inner product preserves measurability, and therefore \( \langle a_i, a_j \rangle_{\mathcal{H}}(W_{t+1/n}^i - W_{t}^i) \) and \( (W_{t+1/n}^i - W_{t}^i) \) are independent from the independent increments of Brownian Motion. The terms thus vanish and we are left with

\[
\sum_{i=0}^{\infty} \mathbb{E}\left( \left\| a_i \right\|_{\mathcal{H}}^2(W_{t+1/n}^i - W_{t}^i)^2 \right)
\]
to which we note independence again and assert that this is just
\[ \sum_{i=0}^{\infty} \mathbb{E}(\|a_i\|_{\mathcal{H}}^2)(t_{i+1}^n \wedge t - t_i^n \wedge t) \]
which is precisely the integral
\[ \int_0^t \sum_{i=0}^{\infty} \mathbb{E}(\|a_i\|_{\mathcal{H}}^2) 1_{(t_i^n, t_{i+1}^n]}(s) ds. \]
Tonelli’s Theorem and absorbing the indicator into the norm then gives the result.

So, why is this useful in terms of the limit in (4)? First and foremost it ensures that the limit is uniquely defined; given that \( L^2(\Omega; \mathcal{H}) \) is complete we need only show that the sequence of stochastic integrals is Cauchy in this space. The Itô Isometry tells us that \( (\int_0^t \Psi^n_s dW_s) \) is Cauchy in \( L^2(\Omega; \mathcal{H}) \) if and only if \( (\Psi^n) \) is Cauchy in \( L^2(\Omega \times [0, t]; \mathcal{H}) \), which is of course true as by definition the \( (\Psi^n) \) are convergent (to \( \Psi \)) in this space. Furthermore the Isometry extends to the general integral defined in 1.2.6, as a trivial corollary of the discussion here.

**Corollary 1.2.7.1.** *The Itô Isometry (5) holds for all processes \( \Psi \in \mathcal{T}^\mathcal{H} \).*

Without direct appeal to the formal construction, we may also understand the integral (1) as a random element of the dual space \( \mathcal{H}^* \) and identify the functional with its counterpart in \( \mathcal{H} \) in the usual sense.

**Theorem 1.2.8.** *The Itô stochastic integral defined in 1.2.6 is the unique element of \( \mathcal{H} \) satisfying the duality relation*

\[ \langle \int_0^t \Psi_s dW_s, \phi \rangle_{\mathcal{H}} = \int_0^t \langle \Psi_s, \phi \rangle_{\mathcal{H}} dW_s \]  

*for each \( \phi \in \mathcal{H} \). The above are random inner products, defined by

\[ \langle \Psi_s, \phi \rangle_{\mathcal{H}}(\omega) := \langle \Psi_s(\omega), \phi \rangle_{\mathcal{H}} \]

*and similarly for the LHS. Therefore by the Riesz-Representation Theorem, it is consistent to define (1) as an \( \mathcal{H}^* \) valued random variable via the mapping

\[ \phi \mapsto \left( \int_0^t \langle \Psi_s, \phi \rangle_{\mathcal{H}} dW_s \right)(\omega). \]

**Proof.** Given that we have defined (1) as a limit of simple processes, it will come as no surprise that we must use this approach to prove the relation (7). We will demonstrate that this holds for simple processes \( \Psi^n \), and later that it is preserved in the \( L^2(\Omega \times [0, t]; \mathcal{H}) \) limit. Firstly though we ought to verify that the RHS of (7) makes sense, that is to say \( (\Psi, \phi)_{\mathcal{H}} \) is a valid (1-dimensional) integrand. Thus we must show the standard progressive measurability and square integrability conditions: for the former, note that the progressive measurability of \( \Psi \) is preserved under composition with the continuous mapping \( (\cdot, \phi)_{\mathcal{H}} \). The latter is straightforward, as

\[ \mathbb{E} \left( \int_0^T \langle \Psi_s, \phi \rangle_{\mathcal{H}}^2 ds \right) \leq \mathbb{E} \left( \int_0^T \|\Psi_s\|_{\mathcal{H}}^2 \|\phi\|_{\mathcal{H}}^2 ds \right) = \|\phi\|_{\mathcal{H}}^2 \mathbb{E} \left( \int_0^T \|\Psi_s\|_{\mathcal{H}}^2 ds \right) \]  

*Corollary 1.2.7.1.* The Itô Isometry (5) holds for all processes \( \Psi \in \mathcal{T}^\mathcal{H} \).
which is finite by (3).

Let’s suppose that $\Psi^n$ takes the form

$$\Psi^n_t(\omega) = a^n_0(\omega) \mathbb{1}_{\{0\}}(t) + \sum_{i=0}^{\infty} a^n_i(\omega) \mathbb{1}_{(t^n_i, t^n_{i+1}]}(t)$$

as outlined in 1.2.1. Then applying 1.2.2, we deconstruct the LHS of (7):

$$\langle \int_0^t \Psi^n_s dW_s, \phi \rangle_{\mathcal{H}} = \langle \sum_{i=0}^{\infty} a^n_i(W_{t^n_{i+1} \wedge t} - W_{t^n_i \wedge t}), \phi \rangle_{\mathcal{H}}$$

$$= \sum_{i=0}^{\infty} \langle a^n_i(W_{t^n_{i+1} \wedge t} - W_{t^n_i \wedge t}), \phi \rangle_{\mathcal{H}}$$

$$= \sum_{i=0}^{\infty} \langle a^n_i, \phi \rangle_{\mathcal{H}}(W_{t^n_{i+1} \wedge t} - W_{t^n_i \wedge t})$$

and proceed similarly for the RHS, observing that the integrand

$$\langle \Psi_s, \phi \rangle_{\mathcal{H}} = \langle a^n_0(\mathbb{1}_{\{0\}}(s)), \phi \rangle_{\mathcal{H}} + \sum_{i=0}^{\infty} a^n_i(\mathbb{1}_{(t^n_i, t^n_{i+1}]}(s)), \phi \rangle_{\mathcal{H}}$$

$$= \langle a^n_0(\mathbb{1}_{\{0\}}(s)), \phi \rangle_{\mathcal{H}} + \sum_{i=0}^{\infty} \langle a^n_i(\mathbb{1}_{(t^n_i, t^n_{i+1}]}(s)), \phi \rangle_{\mathcal{H}}$$

$$= \langle a^n_0, \phi \rangle_{\mathcal{H}}(\mathbb{1}_{\{0\}}(s)) + \sum_{i=0}^{\infty} \langle a^n_i, \phi \rangle_{\mathcal{H}}(\mathbb{1}_{(t^n_i, t^n_{i+1}]}(s))$$

is again simple (this is completely analogous to showing that $\langle \Psi, \phi \rangle_{\mathcal{H}}$ was a valid integrand) Applying the definition 1.2.2 to the RHS gives precisely the LHS as above. The result is proved for simple processes, so all that remains to show is preservation in the limit. We have of course

$$\langle \int_0^t \Psi^n_s dW_s, \phi \rangle_{\mathcal{H}} = \langle \lim_{n \to \infty} \int_0^t \Psi^n_s dW_s, \phi \rangle_{\mathcal{H}}$$

and a reminder that this limit is taken in $L^2(\Omega; \mathcal{H})$. We would like to take the limit outside of the inner product, in some appropriate topology, and use the result for simple functions: the steps would be

$$\langle \lim_{n \to \infty} \int_0^t \Psi^n_s dW_s, \phi \rangle_{\mathcal{H}} = \lim_{n \to \infty} \langle \int_0^t \Psi^n_s dW_s, \phi \rangle_{\mathcal{H}}$$

$$= \lim_{n \to \infty} \langle \Psi^n_s, \phi \rangle_{\mathcal{H}}dW_s$$

so it should be clear that the topology we want to take this limit in is that of $L^2(\Omega; \mathbb{R})$, as the last line would be precisely the RHS of (7) by definition if we can show that the simple real valued process $\langle \Psi^n, \phi \rangle_{\mathcal{H}}$ converges to $\langle \Psi, \phi \rangle_{\mathcal{H}}$ in $L^2(\Omega \times [0, t]; \mathbb{R})$. Thankfully it is straightforward to justify taking this limit outside of the inner product: if $(f^n)$ converges to $f$ in $L^2(\Omega; \mathcal{H})$ then

$$\mathbb{E} \left( \langle f_n, \phi \rangle_{\mathcal{H}} - \langle f, \phi \rangle_{\mathcal{H}} \right)^2 = \mathbb{E} \left( \langle f_n - f, \phi \rangle_{\mathcal{H}} \right)^2 \leq \mathbb{E} \left( \|f_n - f\|_{\mathcal{H}}^2 \right) = \|\phi\|_{\mathcal{H}}^2 \mathbb{E} \left( \|f_n - f\|_{\mathcal{H}}^2 \right) \rightarrow 0$$
so \( \langle f_n, \phi \rangle_{\mathcal{H}} \) converges to \( \langle f, \phi \rangle_{\mathcal{H}} \) in \( L^2(\Omega; \mathbb{R}) \), as required to justify the interchange. To show the convergence of \( \langle \Psi^n, \phi \rangle_{\mathcal{H}} \) to \( \langle \Psi, \phi \rangle_{\mathcal{H}} \) in \( L^2(\Omega \times [0,t]; \mathbb{R}) \) we apply the same trick:

\[
\| \langle \Psi^n, \phi \rangle_{\mathcal{H}} - \langle \Psi, \phi \rangle_{\mathcal{H}} \|_{L^2(\Omega \times [0,t]; \mathbb{R})} = \| \langle \Psi^n, \phi \rangle_{\mathcal{H}} \|_{L^2(\Omega \times [0,t]; \mathbb{R})} \\
= \mathbb{E} \left( \int_0^t \| \Psi^n_s - \Psi_s \|_{\mathcal{H}}^2 ds \right) \\
\leq \mathbb{E} \left( \int_0^t \| \Psi^n_s - \Psi_s \|^2_{\mathcal{H}} ds \right) \\
= \| \phi \|^2_{\mathcal{H}} \mathbb{E} \left( \int_0^t \| \Psi^n_s - \Psi_s \|^2_{\mathcal{H}} ds \right) \\
= \| \phi \|^2_{\mathcal{H}} \| \Psi^n \|_{L^2(\Omega \times [0,t]; \mathcal{H})} \\
\to 0
\]

where convergence to 0 is by definition of the approximating sequence \( \Psi^n \). We are done. \( \square \)

We provide two applications of this result below, both of which will be fundamental to our SPDE framework.

**Proposition 1.2.9.** The Itô Isometry holds for a multi-dimensional driving Brownian motion, in the sense that if \( (\Psi^i)_{i=1}^n \) are a collection of processes in \( \mathcal{I}^\mathcal{H} \), and \( (W^i)_{i=1}^n \) are independent Brownian Motions, then

\[
\mathbb{E} \left( \left\| \sum_{i=1}^n \int_0^t \Psi^i_s dW^i_s \right\|_{\mathcal{H}}^2 \right) = \sum_{i=1}^n \mathbb{E} \left( \int_0^t \| \Psi^i_s \|_{\mathcal{H}}^2 ds \right)
\]

**Proof.** We look to simplify the left hand side of the required equality, swiftly applying Parseval’s identity for a basis \( \{e_k\} \) of \( \mathcal{H} \):

\[
\mathbb{E} \left( \left\| \sum_{i=1}^n \int_0^t \Psi^i_s dW^i_s \right\|_{\mathcal{H}}^2 \right) = \mathbb{E} \sum_{k=1}^\infty \left( \sum_{i=1}^n \int_0^t \Psi^i_s dW^i_s, e_k \right)^2_{\mathcal{H}} \\
= \mathbb{E} \sum_{k=1}^\infty \left( \sum_{i=1}^n \int_0^t \Psi^i_s, e_k \right)^2_{\mathcal{H}}
\]

having used linearity of the inner product to pull out the sum, and indeed 1.2.8. We can now regard the infinite sum as an integral with respect to the counting measure and apply Tonelli’s Theorem, obtaining

\[
\sum_{k=1}^\infty \mathbb{E} \left( \sum_{i=1}^n \int_0^t \Psi^i_s, e_k \right)^2_{\mathcal{H}} = \sum_{k=1}^\infty \mathbb{E} \sum_{i=1}^n \sum_{j=1}^n \left( \int_0^t \Psi^i_s, e_k \right)_{\mathcal{H}} \left( \int_0^t \Psi^j_s, e_k \right)_{\mathcal{H}} \\
= \sum_{k=1}^\infty \sum_{i=1}^n \mathbb{E} \left( \int_0^t \Psi^i_s, e_k \right)_{\mathcal{H}} \left( \int_0^t \Psi^j_s, e_k \right)_{\mathcal{H}}
\]

For the cross terms \( i \neq j \) we make use of the independence of the Brownian Motions and hence the respective stochastic integrals, and furthermore the standard property that the Itô integral has zero expectation to nullify these terms. Our expression reduces to

\[
\sum_{k=1}^\infty \sum_{i=1}^n \mathbb{E} \left( \int_0^t \Psi^i_s, e_k \right)_{\mathcal{H}} \left( \int_0^t \Psi^j_s, e_k \right)_{\mathcal{H}}
\]
to which we can apply the Itô Isometry \textup{1.2.7.1} for the Hilbert Space $\mathbb{R}$ (which is of course the standard Isometry) giving us
\[
\sum_{k=1}^{\infty} \sum_{i=1}^{n} \mathbb{E} \int_{0}^{t} \langle \psi_{i}^{k}, e_{k} \rangle_{\mathcal{H}}^{2} ds
\]
from which we apply Tonelli twice more to take the infinite sum all the way through:
\[
\sum_{i=1}^{n} \mathbb{E} \int_{0}^{t} \sum_{k=1}^{\infty} \langle \psi_{i}^{k}, e_{k} \rangle_{\mathcal{H}}^{2} ds.
\]
A final application of Parseval’s identity gives us precisely what we want.

Whilst we chose to prove \textup{1.2.7} and subsequently \textup{1.2.7.1} from first principles in the Hilbert Space setting, the method of proof here touches upon a fundamental aspect of this theory: with a good understanding of the standard $\mathbb{R}$ valued setting, we can apply \textup{1.2.8} to straightforwardly deduce key properties here. Indeed if we accepted the Itô Isometry in $\mathbb{R}$, we could have just proved \textup{1.2.7.1} in the simple vein of \textup{1.2.9}.

\textbf{Theorem 1.2.10.} Suppose that $\mathcal{H}_{1}, \mathcal{H}_{2}$ are Hilbert spaces such that $\Psi \in \mathcal{I}^{\mathcal{H}_{1}}$ and $T \in \mathcal{L}(\mathcal{H}_{1}; \mathcal{H}_{2})$. Then the process $TX$ defined by
\[
T \Psi_{s}(\omega) = T(\Psi_{s}(\omega))
\]
belongs to $\mathcal{I}^{\mathcal{H}_{2}}$ and is such that
\[
T\left( \int_{0}^{t} \Psi_{s} dW_{s} \right) = \int_{0}^{t} T \Psi_{s} dW_{s}. \tag{9}
\]

\textit{Proof.} We shall prove first that $T \Psi \in \mathcal{I}^{\mathcal{H}_{2}}$. The progressive measurability is preserved under the continuity of $T$, and for $C$ the (square of the) boundedness constant associated to $T$ we have that at any time $t$,
\[
\mathbb{E}\left( \int_{0}^{t} \| T \Psi_{s} \|_{\mathcal{H}_{2}}^{2} ds \right) \leq C \mathbb{E}\left( \int_{0}^{t} \| \Psi_{s} \|_{\mathcal{H}_{1}}^{2} ds \right) < \infty
\]
as $\Psi \in \mathcal{I}^{\mathcal{H}_{1}}$, showing that $T \Psi \in \mathcal{I}^{\mathcal{H}_{2}}$. To commute $T$ with the integral we shall use the characterisation \textup{1.2.8}, having now established that the right hand side of (9) is well defined in $\mathcal{H}_{2}$. Observe that for any $\phi \in \mathcal{H}_{2}$,
\[
\langle T\left( \int_{0}^{t} \Psi_{s} dW_{s} \right), \phi \rangle_{\mathcal{H}_{2}} = \langle \int_{0}^{t} \Psi_{s} dW_{s}, T^{*} \phi \rangle_{\mathcal{H}_{1}}
\]
\[
= \int_{0}^{t} \langle \Psi_{s}, T^{*} \phi \rangle_{\mathcal{H}_{1}} dW_{s}
\]
\[
= \int_{0}^{t} \langle T \Psi_{s}, \phi \rangle_{\mathcal{H}_{2}} dW_{s}
\]
\[
= \langle \int_{0}^{t} T \Psi_{s} dW_{s}, \phi \rangle_{\mathcal{H}_{2}}
\]
applying \textup{1.2.8} twice. As this equality holds for arbitrary $\phi \in \mathcal{H}_{2}$ then we have proven (9), which is of course an identity almost everywhere in $\mathcal{H}_{2}$.

\qed
1.3 Martingale and Local Martingale Integrators

As expected, we can extend the definition to integrators beyond Brownian motion, in the same manner as the standard Itô integral. We begin the extension to continuous square integrable martingales, and then to continuous local martingales.

**Definition 1.3.1.** We shall denote the class of real valued martingales $(M_t) = M$ (with respect to the filtration $(\mathcal{F}_t)$) such that $M_t \in L^2(\Omega;\mathbb{R})$ for every $t \geq 0$ by $\mathcal{M}^2$. The subclass of such martingales with continuous paths will be represented by $\mathcal{M}^2_c$.

**Definition 1.3.2.** For any $M \in \mathcal{M}^2_c$, define $I_T^M$ to be the class of $\mathcal{H}$ valued processes $(\Psi_t)$ which are $(\mathcal{F}_t)$ progressively measurable on $[0, T] \times \Omega$ and satisfy the square integrability condition

$$
\mathbb{E}\left(\int_0^T \|\Psi_s\|^2_{\mathcal{H}} d[M]_s\right) < \infty
$$

where $([M]_s)$ is the quadratic variation process associated with $(M_t)$. We similarly define $I_H^M$ to be the class of processes $(\Psi_t)$ such that $(\Psi_t) \in I_T^M$ for all $T > 0$.

We do not put explicitly the space $\mathcal{H}$ into the time restricted notation for simplicity; once constructed, we will rarely need this notation, and when needed the space will be mentioned separately. Constructing the integral

$$
\int_0^t \Psi_s dM_s
$$

for $\Psi \in I_H^M$ now falls from what we have already done for (1). We use simple processes $\Psi^n$ as in 1.2.1 to approximate $\Psi$, in the sense that

$$
\lim_{n \to 0} \mathbb{E}\left(\int_0^T \|\Psi_s - \Psi^n_s\|^2_{\mathcal{H}} d[M]_s\right) = 0.
$$

Simply replacing $W$ by $M$ in 1.2.2 and 1.2.6 completes the construction. Let’s now move on to the more delicate matter of integration with respect to a local martingale. This begins again with notation for our set of integrands.

**Definition 1.3.3.** For a continuous local martingale $\tilde{M}$, define $I_T^{\tilde{M}}$ to be the class of progressively measurable processes $\Psi$ such that

$$
\int_0^T \|\Psi_s\|^2_{\tilde{H}} d[\tilde{M}]_s < \infty \quad \text{a.e.}
$$

Also define $I_H^{\tilde{M}}$ to be those $\Psi$ in $I_T^{\tilde{M}}$ for every $T$.

Suppose that $\tilde{M}$ is localised by the stopping times $(T_n)$. Without loss of generality this sequence of stopping times can be chosen such that the stopped processes $\tilde{M}^{T_n}$ defined by

$$
\tilde{M}^{T_n}_t := \tilde{M}_{t \wedge T_n}
$$

are bounded; if $(T'_n)$ are localising stopping times, then we can simply set

$$
T_n = T_n' \wedge \inf\{0 \leq t < \infty : |\tilde{M}_t| \geq n\}
$$

so that for each $n$, $\tilde{M}^{T_n}$ is a bounded continuous martingale and hence in $\mathcal{M}^2$. Note of course that the new stopping times $(T_n)$ are still non-decreasing and approach infinity a.s. by the pathwise
continuity of \( \widetilde{M} \). Continuing in this theme, for a process \( \Psi \in \mathcal{H}_{\widetilde{M}} \) let’s define some more non-decreasing random times \((R_n)\) by

\[
R_n := n \land \inf\{0 \leq t < \infty : \int_0^t \|\Psi_s\|_H^2 d[\widetilde{M}]_s \geq n\}
\]

(11)

taking the convention that the infimum of the empty set is infinite. The \((R_n)\) are stopping times as they are simply first hitting times of the continuous and \((\mathcal{F}_t)\)-adapted random variable

\[
\int_0^t \|\Psi_s\|_H^2 d[\widetilde{M}]_s.
\]

Again these times tend to infinity a.s. by condition (10). Now define \(\tau_n\) by

\[
\tau_n = R_n \land T_n
\]

and the truncated processes \(\Psi^n\) as

\[
\Psi^n_t := \Psi_t \mathbb{1}_{t \leq \tau_n}.
\]

We use the fact that for \(m \leq n\), and \(t \leq \tau_m\), we have

\[
\Psi_t \mathbb{1}_{t \leq \tau_m} = \Psi_t \mathbb{1}_{t \leq \tau_m}
\]

and also that

\[
\widetilde{M}^\tau_n = \widetilde{M}^\tau_m
\]

so we can make the consistent definition that

\[
\left( \int_0^t \Psi_s d\widetilde{M}_s \right)(\omega) := \left( \int_0^t \Psi^n_s d\widetilde{M}^\tau_n \right)(\omega)
\]

(12)

at almost every \(\omega\) for any \(n\) such that \(t \leq \tau_n(\omega)\), noting that such an \(n\) exists (for almost every \(\omega\)). There is subtlety in this, as \(n\) itself is a random variable (it is dependent on \(\omega\)) and the logic in which we are proceeding is vital. To be clear, we are not considering the \(n\) of (12) as a random variable; the random selection of \(n\) occurs prior. That is to say we understand (12) as a definition of the left hand side pointwise a.e. by fixing such an \(\omega\), then fixing our \(n\) as outlined, and then considering the right hand side as a process which can be evaluated at any \(\omega' \in \Omega\), but is such that if \(\omega' \neq \omega\) then it will not necessarily be true that \(t \leq \tau_n(\omega')\). We simply evaluate this process at \(\omega\) to make the definition. Of course to do this we require that at this choice of \(n\), \(\Psi^n \in \mathcal{H}_{\widetilde{M}^\tau_n}\); the process \((\mathbb{1}_{t \leq \tau_n})\) is progressively measurable, as it is both left continuous and adapted (adaptedness becomes clear when for each fixed \(t\), we write the random variable \(\mathbb{1}_{t \leq \tau_n} \text{ as } 1 - \mathbb{1}_{t > \tau_n}\)). The square integrability in 1.3.2 comes from the fact that the random variable

\[
\int_0^t \|\Psi^n_s\|_H^2 d[\widetilde{M}^\tau_n]_s
\]

is bounded by \(n\) a.e. (owing to (11)), hence the expectation satisfies the same bound. It is critical again here that the \(n\) in (12) is not allowed to be random, as we would have instead a bound

\[
\left( \int_0^t \|\Psi^n_s\|_H^2 d[\widetilde{M}^\tau_n]_s \right)(\omega) \leq n(\omega)
\]

so we cannot deduce a finite expectation as required because the bound is not uniform in \(\omega\).

Of course where \(\widetilde{M}\) is itself a genuine martingale, this procedure defines the stochastic integral for processes with only the regularity (10). In this case we do not have to stop the integrator, just truncate the integrand.
Definition 1.3.4. In the special case where the continuous local martingale is given by the genuine martingale of Brownian Motion, we denote $\mathcal{I}^H_W$ by simply $\mathcal{I}^H$. This class of processes differs to $\mathcal{I}^H$ because we only assume a bound almost everywhere, not in expectation.

Being able to work with this larger class of processes allows us to extend the analysis of the integral, such as 1.2.8 which is done in the following.

**Proposition 1.3.5.** Let $Ψ ∈ \bar{\mathcal{I}}^H$ and $φ ∈ L^∞(Ω; H)$ be $\mathcal{F}_0$—measurable. Then $⟨Ψ, φ⟩ ∈ \bar{\mathcal{I}}^H$ and for every $t > 0$ we have that

$$\mathbb{P} − a.s.. \text{ The above are random inner products defined by }$$

$$⟨Ψ_s, φ⟩_H(ω) := ⟨Ψ_s(ω), φ(ω)⟩_H$$

and similarly for the left hand side.

**Proof.** We should first justify that $⟨Ψ, φ⟩_H ∈ \bar{\mathcal{I}}^R$. The progressive measurability follows as for every $T > 0$ the mapping

$$⟨Ψ, φ⟩_H : t × Ω × \tilde{ω} → ⟨Ψ_t(ω), φ(\tilde{ω})⟩_H$$

is $\mathcal{B}([0, T]) × \mathcal{F}_T × \mathcal{F}_0$ measurable, so in particular it is $\mathcal{B}([0, T]) × \mathcal{F}_T × \mathcal{F}_T$ measurable and as such

$$⟨Ψ_s, φ⟩_H : t × Ω → ⟨Ψ_t(ω), φ(ω)⟩_H$$

is $\mathcal{B}([0, T]) × \mathcal{F}_T$ measurable as required. Note that we have used the progressive measurability requirement on $Ψ$. We also appreciate that for $\mathbb{P} − a.e. \, ω,$

$$\int_0^T ⟨Ψ_t(ω), φ(ω)⟩^2_H dτ ≤ ∥φ(ω)∥^2_H \int_0^T ∥Ψ_t(ω)∥^2_H dτ < ∞$$

again by assumption on $Ψ ∈ \bar{\mathcal{I}}^H$. Thus $⟨Ψ, φ⟩_H ∈ \bar{\mathcal{I}}^R$. To compute the integrals we introduce the stopping times

$$τ_j := j ∧ \inf \left\{ 0 ≤ t < ∞ : (1 + ∥φ∥^2_H) \int_0^t ∥Ψ_t∥^2_H dτ ≥ j \right\}$$

such that for every $j ∈ \mathbb{N}$, $Ψ_1\cdot 1_{≤τ_j} ∈ \mathcal{I}^H$, $⟨Ψ_1\cdot 1_{≤τ_j}, φ⟩_H ∈ \mathcal{I}^R$. For $ω$ fixed and $t$ fixed as in (13) then the choose $n$ sufficiently large so that $τ_j(ω) ≥ t$. The integrals are then defined at this $ω$ by

$$⟨\int_0^t Ψ_1\cdot 1_{≤τ_j} dW_r, φ⟩_H = \int_0^t ⟨Ψ_r\cdot 1_{≤τ_j}, φ⟩_H dW_r$$

so we in fact show that (14) holds $\mathbb{P} − a.e.$ for every $j$. We now fix arbitrary $j ∈ \mathbb{N}$. Our plan is as follows: we consider a sequence of simple processes $(Φ^n)$ which approximate $Ψ_1\cdot 1_{≤τ_j}$ in $L^2(Ω × [0, t]; H)$ as postulated in 1.2.5. We then claim that $(⟨Φ^n, φ⟩_H)$ is a sequence of $\mathbb{R}$ valued simple processes which converge to $⟨Ψ_1\cdot 1_{≤τ_j}, φ⟩_H$ in $L^2(Ω × [0, t]; \mathbb{R})$. Following this we shall prove (13) for this simple case and show the identity holds in the limit.
We first show that for each \( n \in \mathbb{N} \), \( \langle \Phi^n, \phi \rangle_H \) is a simple process. Let \( \Phi^n \) have the representation (1.2.1). Then

\[
\langle \Phi^n, \phi \rangle_H = \left( a^n_0 \mathbb{1}_{\{0\}} + \sum_{i=0}^{\infty} a^n_i \mathbb{1}_{[t^n_i, t^n_{i+1}]} \right) \phi
\]

so this would satisfy the requirements of an \( \mathbb{R} \) valued simple process if for each \( i \in \mathbb{N} \), \( \langle a^n_i, \phi \rangle_H \in L^2(\Omega; \mathbb{R}) \) and is \( \mathcal{F}_{t_i} \)-measurable. For the square integrability constraint, observe that

\[
\mathbb{E} \left( \|a^n_i \|^2_H \right) \leq \mathbb{E} \left( \|a^n_i \|^2 \right) \leq \|\phi\|^2_{L^\infty(\Omega; H)} \mathbb{E} \left( \|a^n_i \|^2_H \right) < \infty
\]

by the assumptions of \( a_i \in L^2(\Omega; H) \) and \( \phi \in L^\infty(\Omega; H) \). The \( \mathcal{F}_{t_i} \) measurability follows in the same way as the progressive measurability of \( \langle \Psi, \phi \rangle_H \). Indeed the required \( L^2(\Omega \times [0, t]; \mathbb{R}) \) convergence follows similarly as

\[
\left\| \langle \Psi \mathbb{1}_{\leq \tau_j}, \phi \rangle_H - \langle \Phi^n, \phi \rangle_H \right\|_{L^2(\Omega \times [0, t]; \mathbb{R})} = \left\| \langle \Psi \mathbb{1}_{\leq \tau_j} - \Phi^n, \phi \rangle_H \right\|_{L^2(\Omega \times [0, t]; \mathbb{R})}
\]

and by assumption

\[
\left\| \langle \Psi \mathbb{1}_{\leq \tau_j} - \Phi^n, \phi \rangle_H \right\|_{L^2(\Omega \times [0, t]; \mathbb{R})} \rightarrow 0
\]

as \( n \rightarrow \infty \), so the convergence is proved. To show the identity (13) in the case of the simple process \( \Phi^n \), observe that

\[
\langle \int_0^t \Phi^n_r dW_r, \phi \rangle_H = \left\langle \sum_{i=0}^{\infty} a^n_i \left( W^n_{t^n_{i+1} \wedge t} - W^n_{t^n_i \wedge t} \right), \phi \right\rangle_H
\]

\[
= \sum_{i=0}^{\infty} \langle a^n_i, \phi \rangle_H \left( W^n_{t^n_{i+1} \wedge t} - W^n_{t^n_i \wedge t} \right)
\]

as required. In order to conclude the argument, by definition of the integral we have that

\[
\int_0^t \Psi_r \mathbb{1}_{t \leq \tau_j} dW_r = \lim_{n \rightarrow \infty} \int_0^t \Phi^n_r dW_r
\]

\[
\int_0^t \langle \Psi, \mathbb{1}_{t \leq \tau_j}, \phi \rangle_H dW_r = \lim_{n \rightarrow \infty} \int_0^t \langle \Phi^n, \phi \rangle_H dW_r
\]

where the first limit is taken in \( L^2(\Omega; \mathcal{H}) \) and the second one in \( L^2(\Omega; \mathbb{R}) \). For each we can thus extract a \( \mathbb{P} - a.s. \) convergent subsequence in the appropriate space, so by taking successive subsequences we can find one common subsequence indexed by \( (n_k) \) such that the above limits hold.
\[ \left\langle \int_0^t \Psi_r \mathbf{1}_{r \leq \tau_j} dW_r, \phi \right\rangle_{\mathcal{H}} = \lim_{n_k \to \infty} \left\langle \int_0^t \Phi_{r}^{n_k} dW_r, \phi \right\rangle_{\mathcal{H}} \]

\[ = \lim_{n_k \to \infty} \left\langle \int_0^t \Phi_{r}^{n_k} dW_r, \phi \right\rangle_{\mathcal{H}} \]

\[ = \lim_{n_k \to \infty} \int_0^t \langle \Phi_{r}^{n_k}, \phi \rangle_{\mathcal{H}} dW_r \]

\[ = \int_0^t \langle \Psi_r \mathbf{1}_{r \leq \tau_j}, \phi \rangle_{\mathcal{H}} dW_r \]

so (14) is justified and the proof is complete.

In fact, the result can be extended to unbounded \( \phi \). To do this we shall prove a Stochastic Dominated Convergence Theorem.

**Lemma 1.3.6.** Let \((\Psi^n)\) be a sequence in \(\tilde{\mathcal{H}}\) such that there exists processes \(\Psi : \Omega \times [0, \infty) \to \mathcal{H}\) and \(\Phi \in \tilde{\mathcal{H}}\) with the properties that for every \(T > 0\), \(\mathbb{P} \times \lambda - \text{a.e.} \ (\omega, t) \in \Omega \times [0, T]\):

1. \(\|\Psi^n_t(\omega)\|_{\mathcal{H}} \leq \|\Phi_t(\omega)\|_{\mathcal{H}}\) for all \(n \in \mathbb{N}\);

2. \((\Psi^n_t(\omega))\) is convergent to \(\Psi_t(\omega)\) in \(\mathcal{H}\).

Then \(\Psi \in \tilde{\mathcal{H}}\) and for every \(t > 0\), there exists a subsequence indexed by \((n_k)\) such that

\[ \lim_{n_k \to \infty} \int_0^t \Psi_{r}^{n_k} dW_r = \int_0^t \Psi_{r} dW_r \quad (15) \]

\(\mathbb{P} - \text{a.s.}\)

**Proof.** Immediately we note that \(\Psi\) inherits the progressive measurability from \(\Psi^n\) from the almost everywhere limit in the product space \(\Omega \times [0, T]\) when equipped with product sigma algebra \(\mathcal{F}_T \times \mathcal{B}([0, T])\). Similarly we must have that for \(\mathbb{P} \times \lambda - \text{a.e.} \ (\omega, t)\), \(\|\Psi_t(\omega)\|_{\mathcal{H}} \leq \|\Phi_t(\omega)\|_{\mathcal{H}}\) so \(\Psi\) must too satisfy the integrability constraints and hence belongs to \(\tilde{\mathcal{H}}\). We look to find a common sequence of localising times for the stochastic integrals, and then demonstrate (15) by the showing the identity holds true when stopped at each localising time. To this end we introduce the stopping times

\[ \tau_j := j \wedge \inf \left\{ 0 \leq t < \infty : \int_0^t \|\Phi_r\|^2_{\mathcal{H}} dr \geq j \right\} \]

which from item 1 serve as a sequence of localising times for every \(\Psi^n\), and too for \(\Psi\). Thus for any fixed \(t > 0\) and \(j \in \mathbb{N}\) we wish to show that

\[ \lim_{n_k \to \infty} \int_0^t \Psi_{r}^{n_k} \mathbf{1}_{r \leq \tau_j} dW_r = \int_0^t \Psi_{r} \mathbf{1}_{r \leq \tau_j} dW_r \quad (16) \]

for a subsequence \((n_k)\) \(\mathbb{P} - \text{a.s.}\), or equivalently that

\[ \lim_{n_k \to \infty} \int_0^t (\Psi_{r}^{n_k} - \Psi_{r}) \mathbf{1}_{r \leq \tau_j} dW_r = 0. \]
We first assess the convergence in \(L^2(\Omega; \mathcal{H})\), applying 1.2.7.1 for each fixed \(n\) to see that
\[
\mathbb{E} \left\| \int_0^t (\Psi^n_r - \Psi_r) 1_{r \leq \tau_j} dW_r \right\|_{\mathcal{H}}^2 = \mathbb{E} \left( \int_0^t \left\| (\Psi^n_r - \Psi_r) 1_{r \leq \tau_j} \right\|_{\mathcal{H}}^2 dr \right).
\]
Observing that for \(\mathbb{P} \times \lambda - a.e. (\omega, t)\),
\[
\left\| (\Psi^n_r(\omega) - \Psi_r(\omega)) 1_{r \leq \tau_j}(\omega) \right\|_{\mathcal{H}}^2 \leq \left( \left\| \Psi^n_r(\omega) 1_{r \leq \tau_j(\omega)} \right\|_{\mathcal{H}} + \left\| \Psi_r(\omega) 1_{r \leq \tau_j(\omega)} \right\|_{\mathcal{H}} \right)^2 \leq 4 \left\| \Psi_r(\omega) 1_{r \leq \tau_j(\omega)} \right\|_{\mathcal{H}}^2
\]
Then with dominating function \(4 \| \Phi \cdot \mathbf{1}_{1 \leq \tau_j} \|^2_{\mathcal{H}}\) we can apply the standard Dominated Convergence Theorem for the integral over the product space (we face no problems with the order and configuration of integration from Tonelli’s Theorem given the progressive measurability) to deduce that
\[
\lim_{n \to \infty} \mathbb{E} \left( \int_0^t \left\| (\Psi^n_r - \Psi_r) 1_{r \leq \tau_j} \right\|_{\mathcal{H}}^2 dr \right) = 0
\]
and therefore
\[
\lim_{n \to \infty} \mathbb{E} \left\| \int_0^t (\Psi^n_r - \Psi_r) 1_{r \leq \tau_j} dW_r \right\|_{\mathcal{H}}^2 = 0.
\]
Thus we have demonstrated the convergence (16) but for the whole sequence in \(L^2(\Omega; \mathcal{H})\), from which we can deduce a \(\mathbb{P} - a.s.\) convergent subsequence and the result is proved.

\[\square\]

**Proposition 1.3.7.** Let \(\Psi \in \tilde{I}^\mathcal{H}\) and \(\phi : \Omega \to \mathcal{H}\) be \(\mathcal{F}_0\)-measurable. Then \(\langle \Psi, \phi \rangle \in \tilde{I}^\mathcal{H}\) and for every \(t > 0\) we have that
\[
\left\langle \int_0^t \Psi_s dW_r, \phi \right\rangle_{\mathcal{H}} = \int_0^t \langle \Psi_r, \phi \rangle_{\mathcal{H}} dW_r
\]
\((17)\)
\(\mathbb{P} - a.s.\).

**Proof.** A justification that \(\langle \Psi_r, \phi \rangle_{\mathcal{H}} \in \tilde{I}^\mathbb{R}\) is precisely as in 1.3.5. To apply this result, we rewrite \(\phi\) in a trivial way as
\[
\phi := \sum_{k=1}^{\infty} \phi 1_{k \leq \|\phi\|_{\mathcal{H}} < k+1}
\]
where the limit is taken \(\mathbb{P} - a.s.\) in \(\mathcal{H}\) (similarly to (2) this is just a finite sum at each fixed \(\omega\), or more precisely just a single element of the sum). Introducing the notation
\[
\phi^n := \sum_{k=1}^{n} \phi 1_{k \leq \|\phi\|_{\mathcal{H}} < k+1}
\]
then clearly \(\phi^n \in L^\infty(\Omega; \mathcal{H})\) and is still \(\mathcal{F}_0\)-measurable, so we can apply 1.3.5 to see that
\[
\left\langle \int_0^t \Psi_s dW_r, \phi^n \right\rangle_{\mathcal{H}} = \int_0^t \langle \Psi_r, \phi^n \rangle_{\mathcal{H}} dW_r.
\]
We can take the \(\mathbb{P} - a.s.\) limit in \(\mathcal{H}\) outside of the inner product on the left hand side, so it is sufficient to show that
\[
\lim_{n \to \infty} \int_0^t \langle \Psi_r, \phi^n \rangle_{\mathcal{H}} dW_r = \int_0^t \langle \Psi_r, \phi \rangle_{\mathcal{H}} dW_r
\]
\((18)\)
or at least that this is true for a subsequence. This is an immediate application of 1.3.6, with dominating function simply the limit \(\langle \Psi, \phi \rangle_{\mathcal{H}}\).

\[\square\]
The same is true for multiplication by real valued random variables, where the proof is identical. We state the result here.

**Proposition 1.3.8.** Let $\Psi \in \mathcal{I}^H$ and $\eta : \Omega \rightarrow \mathbb{R}$ be $\mathcal{F}_0-$measurable. Then $\eta \Psi \in \mathcal{I}^H$ and for every $t > 0$ we have that

$$\eta \int_0^t \Psi_r dW_r = \int_0^t \eta \Psi_r dW_r$$

$\mathbb{P} - a.s.$.

### 1.4 Cylindrical Processes

Having now addressed the question of how to integrate a Hilbert Space valued process with respect to a finite dimensional driving martingale, we look to extend this theory to the case of an infinite dimensional driving martingale. The aforementioned construction then arises from the one dimensional projections of the driving process. Some schools prefer to take this driving martingale explicitly as a process indexed by an additional spatial variable, typically leading to the Brownian sheet [7]. We prefer to take the same logic already applied to the integrand, considering the driving process to take values in some more general topological vector space. This leads to the notion of Cylindrical Brownian Motion, which we explore in this subsection.

We will denote by $Y$ a real-valued zero-mean Gaussian process with correlation function $R(t, s)$, $\mathcal{H}$ a Hilbert space and $Q$ a bounded positive self-adjoint operator on $\mathcal{H}$.

**Definition 1.4.1.** A $Q$-Cylindrical $Y$ process over $\mathcal{H}$ is a process $X^Q$ taking values in the space of functions from $\mathcal{H}$ to $\mathbb{R}$, that is

$$X^Q : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^\mathcal{H}$$

such that for each $h \in \mathcal{H}$, $X^Q_h : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ is a process of zero mean Gaussian random variables, and for every $g, h \in \mathcal{H}$ and all times $t, s$,

$$\mathbb{E}(X^Q_g(t)X^Q_h(s)) = \langle Qg, h \rangle_{\mathcal{H}}R(t, s).$$

**Definition 1.4.2.** A $Q$-Cylindrical Brownian Motion is defined as above where the $Y$ process is a Brownian Motion. A Cylindrical Brownian Motion is a $Q$-Cylindrical Brownian Motion such that $Q$ is the identity operator.

To make this abstract definition workable, we look to how such processes can be represented. This motivates the notion of a regular process.

**Definition 1.4.3.** A $Q$-Cylindrical $Y$ process $X^Q$ is said to be regular if there exists a square integrable $\mathcal{H}$ valued process $\tilde{X}^Q$ (that is, $\tilde{X}^Q_t \in L^2(\Omega; \mathcal{H})$ for all $t \geq 0$) such that for every $h \in \mathcal{H}$, $X^Q_h$ has the same distribution as the process $\langle \tilde{X}^Q, h \rangle_{\mathcal{H}}$ defined by

$$\langle \tilde{X}^Q, h \rangle_{\mathcal{H}}(t, \omega) := \langle \tilde{X}^Q(t, \omega), h \rangle_{\mathcal{H}}.$$

We will not hesitate to identify the functional valued process $X^Q$ with the Hilbert space one $\tilde{X}^Q$. In the next theorem however, we keep the distinction for clarity:

**Theorem 1.4.4.** A $Q$-Cylindrical $Y$ process $X^Q$ is regular if and only if $Q$ is trace-class. In this case $X^Q$ admits the regular representation

$$\tilde{X}^Q(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i Y^i_t$$  \hspace{1cm} (19)
where the \((e_i)\) are an orthonormal basis of \(H\) consisting of eigenvectors of the self-adjoint trace-class (hence compact) operator \(Q\), \((\lambda_i)\) are the corresponding eigenvalues, and \((Y^i)\) are independent copies of the process \(Y\). The limit is taken in \(L^2(\Omega; H)\).

**Proof.** We consider the two directions:

\(\Leftarrow\) : A sensible place to start would be to verify that for trace-class \(Q\), (19) does indeed define an element of \(L^2(\Omega; H)\). We will rely on completeness of the space and show that the sequence of partial sums is Cauchy. Observe that

\[
\mathbb{E}(\|n \sum_{i=m}^{n} \sqrt{\lambda_i} e_i Y_t^i\|_H^2) = \mathbb{E}(\|n \sum_{i=m}^{n} |\lambda_i| |Y_t^i|^2\|_H^2) = \mathbb{E}(|Y_t|^2) \sum_{i=m}^{n} \lambda_i.
\]

All we then require to conclude the Cauchy property is that \(\sum_{i=1}^{\infty} \lambda_i < \infty\) which it is given that \(Q\) is trace class. To conclude that (19) is a regular representation of \(X^Q\), it is sufficient to show that the one dimensional processes \(\langle \tilde{X}^Q, h \rangle_H\) satisfy the conditions postulated of the \(X^Q_h\) in 1.4.1 as these characterise the distribution.

First of all for each \(h \in H\) and time \(t\), we must verify that the random variable

\[
\omega \mapsto \left\langle \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i Y_t^i(\omega), h \right\rangle_H
\]

is zero mean Gaussian. Note that

\[
\left\langle \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\lambda_i} e_i Y_t^i, h \right\rangle_H = \lim_{n \to \infty} \left\langle \sum_{i=1}^{n} \sqrt{\lambda_i} e_i Y_t^i, h \right\rangle_H
\]

where the second limit is in \(L^2(\Omega; \mathbb{R})\) just as we did in 1.2.8. The random variable (20) is thus an \(L^2\) limit of zero mean Gaussian random variables, so is itself zero mean Gaussian (convergence in \(L^2\) implies that in distribution so we have Gaussianity, and \(L^2\) convergence implies \(L^1\) from which we readily deduce the zero mean property). It remains to show that for each \(g, h \in H\) and \(s, t > 0\) that

\[
\mathbb{E}\left(\left\langle \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i Y_t^i, g \right\rangle_H \left\langle \sum_{j=1}^{\infty} \sqrt{\lambda_j} e_j Y_s^j, h \right\rangle_H\right) = \langle Qg, h \rangle_H R(t, s).
\]

We take the limit through the first inner product on the LHS as above, that is

\[
\mathbb{E}\left(\left( \lim_{n \to \infty} \left\langle \sum_{i=1}^{n} \sqrt{\lambda_i} e_i Y_t^i, g \right\rangle_H \right) \left\langle \sum_{j=1}^{\infty} \sqrt{\lambda_j} e_j Y_s^j, h \right\rangle_H\right)
\]

and argue that for a sequence of functions \((f_n)\) convergent to \(f\) in \(L^2(\Omega; \mathbb{R})\), and \(a \in L^2(\Omega; \mathbb{R})\), that

\[
\lim_{L^2} (f_n)(a) = \lim_{L^1} (f_n a).
\]

The right side is well defined as the limit of a Cauchy sequence:

\[
\mathbb{E}(|f_n a - f_m a|) = \mathbb{E}(|(f_n - f_m)a|) \leq \mathbb{E}(|f_n - f_m|^2) \mathbb{E}(|a|^2)
\]
and a similar calculation shows that this element of $L^1(\Omega; \mathbb{R})$ is the left side. Applying to (21) and pulling the $L^1$ limit through the expectation produces

$$\lim_{n \to \infty} \mathbb{E}\left( \left< \sum_{i=1}^{n} \sqrt{\lambda_i} e_i Y_i^j, g \right>_\mathcal{H} \left< \sum_{j=1}^{\infty} \sqrt{\lambda_j} e_j Y_s^j, h \right>_\mathcal{H} \right)$$

and playing the same game, this is

$$\lim_{n \to \infty} \lim_{m \to \infty} \mathbb{E}\left( \left< \sum_{i=1}^{n} \sqrt{\lambda_i} e_i Y_i^j, g \right>_\mathcal{H} \left< \sum_{j=1}^{m} \sqrt{\lambda_j} e_j Y_s^j, h \right>_\mathcal{H} \right)$$

and further

$$\lim_{n \to \infty} \lim_{m \to \infty} \mathbb{E}\left( \sum_{i=1}^{n} \sum_{j=1}^{m} \sqrt{\lambda_i} e_i, g \left< \sqrt{\lambda_j} e_j, h \right>_\mathcal{H} Y_i^j Y_s^j \right).$$

Independence of the copies $(Y^i)$ and definition of the correlation function gives that this is equal to

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left< \sqrt{\lambda_i} e_i, g \right>_\mathcal{H} \left< \sqrt{\lambda_i} e_i, h \right>_\mathcal{H} R(t, s) = \lim_{n \to \infty} \sum_{i=1}^{n} \left< \lambda_i e_i, g \right>_\mathcal{H} \left< e_i, h \right>_\mathcal{H} R(t, s)$$

which is just

$$\left< \sum_{i=1}^{\infty} \lambda_i e_i, g \right>_\mathcal{H} \left< \sum_{i=1}^{\infty} \lambda_i e_i, h \right>_\mathcal{H} R(t, s) = \langle Q g, h \rangle_\mathcal{H} R(t, s)$$

as required.

$\implies$: For the reverse direction, assume that $X^Q$ is regular with corresponding process $\bar{X}^Q$. We want an expression in terms of the trace of $Q$, so we exploit the definition 1.4.1:

$$\sum_{i=1}^{\infty} \mathbb{E}(\|X^Q_{e_i}(t)\|^2) = \sum_{i=1}^{\infty} \langle Q e_i, e_i \rangle_\mathcal{H} R(t, t)$$

and use an alternative expression from the assumed regular representation:

$$\sum_{i=1}^{\infty} \mathbb{E}(\|X^Q_{e_i}(t)\|^2) = \sum_{i=1}^{\infty} \mathbb{E}(\|\bar{X}^Q(t), e_i \|^2_\mathcal{H}) = \mathbb{E}\left( \sum_{i=1}^{\infty} \left< \bar{X}^Q(t), e_i \right>_\mathcal{H}^2 \right) = \mathbb{E}(\|\bar{X}^Q(t)\|^2_\mathcal{H}) < \infty$$

where the infinite sum is pulled inside the expectation from the monotone convergence theorem, and finiteness is by assumption on $\bar{X}^Q \in L^2(\Omega; \mathcal{H})$. Hence $Q$ is trace class.

$\square$

A standard Cylindrical Brownian Motion, that is where $Q$ is the identity, is thus not regular. However it would be convenient to have such a representation for Cylindrical Brownian Motion (denote this $W(t)$); we would like this to be something along the lines of

$$W(t) = \sum_{i=1}^{\infty} e_i W^i_t$$
where the \((e_i)\) form an orthonormal basis of \( \mathcal{H} \), and \((W^i)\) are standard independent Brownian Motions. We can in fact explicitly construct a larger Hilbert space \( \mathcal{H}' \) such that the inclusion mapping \( J : \mathcal{H} \hookrightarrow \mathcal{H}' \) is Hilbert-Schmidt. The composition \( Q := JJ^* \) is then trace-class on \( \mathcal{H}' \), and indeed \( \mathcal{W}_t \) is a \( Q \)-Cylindrical Brownian Motion on \( \mathcal{H}' \).

To be precise, that is \( J(e_i) = \sqrt{\lambda_i} \eta_i \) for each \( i \), where the \((\eta_i)\) form an orthonormal basis (of \( \mathcal{H}' \)) of eigenfunctions of \( JJ^* \) with eigenvalues \( \lambda_i \), and for any \( h \in \mathcal{H} \),

\[
\mathcal{W}_h(t) = \sum_{i=1}^{\infty} J(e_i)W^i_t, J(h))_{\mathcal{H}'}.
\]

## 1.5 Cylindrical and Hilbert Space Valued Martingales

As we look to make cylindrical processes the driving force in our stochastic integral, it will come as no surprise that we introduce martingality in this setting.

**Definition 1.5.1.** A \( Q \)-Cylindrical \( Y \) process over \( \mathcal{H} \), \( X^Q \), is said to be a martingale if for every \( h \in \mathcal{H} \), the real valued process \( X^Q_h \) is a martingale.

Whilst the choice is natural, there seems no way to extend the definition to include two key aspects of our martingale integrators for a one dimensional driving process, namely continuity and square integrability. As such we consider the analogy in Hilbert spaces, covering at least the cylindrical processes which have a regular representation, with an understanding that this would most likely be the starting point for developing the integration theory with respect to an infinite dimensional martingale.

**Definition 1.5.2.** A process \( (M_t) \) taking values in a Hilbert space \( \mathcal{H} \) is said to be a martingale if for every \( h \in \mathcal{H} \), the inner product \( \langle M_t, h \rangle_\mathcal{H} \) is a martingale. The martingale is said to be continuous if for a.e. \( \omega \) and every \( T > 0 \), \( M(\omega) : [0,T] \rightarrow \mathcal{H} \) is continuous. The martingale is said to be square integrable if for every \( t \geq 0 \), \( \mathbb{E}(\|M_t\|_\mathcal{H}^2) < \infty \). The class of continuous square integrable martingales will be denoted \( \mathcal{M}^2_c(\mathcal{H}) \).

**Proposition 1.5.3.** For \( \Psi \in \mathcal{I}_\mathcal{H} \), the Itô stochastic integral

\[
\int_0^t \Psi_s d\mathcal{W}_s
\]

belongs to \( \mathcal{M}^2_c(\mathcal{H}) \).

**Proof.** The martingality follows immediately from 1.2.8 and the standard one dimensional result. Similarly the square integrability was proved in 1.2.7.1. For the continuity we simply defer to the finite dimensional construction as this works in the same manner; see [9] for details.

This result extends to the case of a general martingale integrator completely synonymous with the finite dimensional setting. Local martingality is then defined as we would expect, and we have the following result.

**Proposition 1.5.4.** For a continuous local martingale \( \widetilde{M} \) and \( \Psi \in \mathcal{I}_{\widetilde{M}}^\mathcal{H} \), the Itô stochastic integral

\[
\int_0^t \Psi_s d\widetilde{M}_s
\]

is itself a continuous local martingale.
Proof. We claim that the localising stopping times are given simply by the \((\tau_n)\) used in the definition of the integral. We have already seen that these tend to infinity almost surely, so it just remains to show that any fixed \(n \in \mathbb{N}\) the stopped process is a continuous martingale. So we fix such \(n\) and look at how the stopped process
\[
\int_0^{t \wedge \tau_n} \Psi_s d\tilde{M}_s
\]
is actually defined. Well, at each fixed \(\omega\) we need to choose an \(m\) such that \(t \wedge \tau_n(\omega) \leq \tau_m(\omega)\), so we can simply choose \(m\) to be \(n\). Indeed this can be done uniformly across all \(\omega\), so the stopped process is genuinely the integral
\[
\int_0^t \Psi_s^n d\tilde{M}_s^{\tau_n}
\]
for this fixed \(n\), which is a continuous martingale by the standard result.

The next ingredient would be a definition of quadratic variation, which we look to do via a Doob-Meyer decomposition. That is, can we show that \(\|M_t\|_H^2\) defines a sub-martingale? Well we have integrability by definition, and
\[
\|M_t\|_H^2 = \sum_{i=1}^{\infty} \langle M_t, e_i \rangle_H^2
\]
where the limit is defined \(a.e.\). Again by definition the above projections are martingales, and so the squares are submartingales. The process \(\|M\|_H^2\) is adapted as each \(\|M_t\|_H^2\) is the \(a.e.\) limit of \(\mathcal{F}_t\) measurable random variables (on the complete measure space), and it is a submartingale as we can apply the monotone convergence theorem to take the limit through the expectation for the defining submartingale property. As such, we have the following:

**Definition 1.5.5.** For \(M \in \mathcal{M}_c^2(\mathcal{H})\), the quadratic variation process \([M]\) associated to \(M\) is defined to be the unique continuous adapted non-decreasing process starting from zero specified in the Doob-Meyer decomposition such that
\[
\|M\|_H^2 - [M]
\]
is a real valued martingale.

**Proposition 1.5.6.** Suppose that \(\Psi \in \mathcal{I}^H\), then
\[
\left[ \int_0^t \Psi_r dW_r \right]_t = \int_0^t \|\Psi_r\|_H^2 dr. \tag{22}
\]

Proof. The fact that the process \((22)\) is continuous, adapted and starting from zero is clear. It simply remains to show the required martingality. To this end observe that at each time \(t\),
\[
\left\| \int_0^t \Psi_r dW_r \right\|_H^2 - \int_0^t \|\Psi_r\|_H^2 dr = \sum_{i=1}^{\infty} \left( \int_0^t \langle \Psi_r dW_r, e_i \rangle_H \right)^2 - \int_0^t \sum_{i=1}^{\infty} \langle \Psi_r, e_i \rangle_H^2 dr
\]
\[
= \sum_{i=1}^{\infty} \left( \int_0^t \langle \Psi_r, e_i \rangle_H dW_r \right)^2 - \sum_{i=1}^{\infty} \int_0^t \langle \Psi_r, e_i \rangle_H^2 dr
\]
\[
= \sum_{i=1}^{\infty} \left[ \left( \int_0^t \langle \Psi_r, e_i \rangle_H dW_r \right)^2 - \int_0^t \langle \Psi_r, e_i \rangle_H^2 dr \right]
\]
having applied 1.2.8 to the first term and the Monotone Convergence Theorem to the second term, where the infinite sum is a limit taken $\mathbb{P} - a.s.$ From the standard one dimensional theory, for each $n$

$$\sum_{i=1}^{n} \left( \int_{0}^{t} \langle \mathbf{W}_{r}, e_{i} \rangle_{H} dW_{r} \right)^{2} - \int_{0}^{t} \langle \mathbf{W}_{r}, e_{i} \rangle_{H}^{2} dr$$

is a real valued martingale so to conclude the proof we only need to justify that the $\mathbb{P} - a.e.$ limit also holds in $L^{1}(\Omega; \mathbb{R})$ as convergence in this space preserves martingality. This is a straightforward applications of the monotone convergence theorem (each term is not only integrable but in fact zero from the standard Itô Isometry), and we are done.

\[\square\]

We also look to reconcile this definition with the one often stated in the real valued case, as a limit in probability over any time partition with mesh approaching zero.

**Proposition 1.5.7.** Let $\mathbf{W} \in \mathcal{T}^{t}_{H}$ and consider any sequence of partitions

$$I_{i} := \left\{ 0 = t_{0}^{i} < t_{1}^{i} < \cdots < t_{k_{i}}^{i} = T \right\}$$

with $\max_{j} |t_{j}^{i} - t_{j-1}^{i}| \to 0$ as $l \to \infty$. Then for all $t \in [0, T]$,

$$\lim_{l \to \infty} \mathbb{E} \left( \sum_{t_{j+1}^{i} \leq t} \left\| \int_{t_{j}^{i}}^{t_{j+1}^{i}} \mathbf{W}_{r} dW_{r} \right\|_{H}^{2} - \int_{0}^{t} \left\| \mathbf{W}_{r} \right\|_{H}^{2} dr \right) = 0.$$

**Proof.** We once again look to prove this result by considering the finite dimensional projections on which the result is known to be true, before showing that it is preserved in the limit. Identically to the proof of 1.5.6 we have that

$$\mathbb{E} \left( \sum_{t_{j+1}^{i} \leq t} \left\| \int_{t_{j}^{i}}^{t_{j+1}^{i}} \mathbf{W}_{r} dW_{r} \right\|_{H}^{2} - \int_{0}^{t} \left\| \mathbf{W}_{r} \right\|_{H}^{2} dr \right)$$

$$= \mathbb{E} \left( \sum_{t_{j+1}^{i} \leq t} \sum_{i=1}^{\infty} \left( \int_{t_{j}^{i}}^{t_{j+1}^{i}} \langle \mathbf{W}_{r}, e_{i} \rangle_{H} dW_{r} \right)^{2} - \int_{0}^{t} \langle \mathbf{W}_{r}, e_{i} \rangle_{H}^{2} dr \right)$$

$$\leq \mathbb{E} \left( \sum_{i=1}^{\infty} \sum_{t_{j+1}^{i} \leq t} \left( \int_{t_{j}^{i}}^{t_{j+1}^{i}} \langle \mathbf{W}_{r}, e_{i} \rangle_{H} dW_{r} \right)^{2} - \int_{0}^{t} \langle \mathbf{W}_{r}, e_{i} \rangle_{H}^{2} dr \right)$$

$$= \sum_{i=1}^{\infty} \mathbb{E} \left( \sum_{t_{j+1}^{i} \leq t} \left( \int_{t_{j}^{i}}^{t_{j+1}^{i}} \langle \mathbf{W}_{r}, e_{i} \rangle_{H} dW_{r} \right)^{2} - \int_{0}^{t} \langle \mathbf{W}_{r}, e_{i} \rangle_{H}^{2} dr \right)$$

where on the last line we have applied the Monotone Convergence Theorem. From the standard theory we know that for each $i \in \mathbb{N}$,

$$\lim_{l \to \infty} \mathbb{E} \left( \sum_{t_{j+1}^{i} \leq t} \left( \int_{t_{j}^{i}}^{t_{j+1}^{i}} \langle \mathbf{W}_{r}, e_{i} \rangle_{H} dW_{r} \right)^{2} - \int_{0}^{t} \langle \mathbf{W}_{r}, e_{i} \rangle_{H}^{2} dr \right) = 0$$
so we would be done if we can justify the interchange of infinite sum and limit in \( l \). We look to apply the Dominated Convergence Theorem, noting that for each fixed \( i, l \in \mathbb{N} \),

\[
\mathbb{E} \left( \left\| \sum_{t_{j+1}^l \leq t} \left( \int_{t_j^l}^{t} \langle \Psi_r, e_i \rangle_{\mathcal{H}} dW_r \right)^2 - \int_0^t \langle \Psi_r, e_i \rangle_{\mathcal{H}}^2 dr \right\| \right) \\
\leq \sum_{t_{j+1}^l \leq t} \mathbb{E} \left( \left( \int_{t_j^l}^{t} \langle \Psi_r, e_i \rangle_{\mathcal{H}} dW_r \right)^2 \right) + \mathbb{E} \left( \int_0^t \langle \Psi_r, e_i \rangle_{\mathcal{H}}^2 dr \right) \\
= \sum_{t_{j+1}^l \leq t} \mathbb{E} \left( \int_{t_j^l}^{t} \langle \Psi_r, e_i \rangle_{\mathcal{H}}^2 dr \right) + \mathbb{E} \left( \int_0^t \langle \Psi_r, e_i \rangle_{\mathcal{H}}^2 dr \right) \\
= 2 \mathbb{E} \left( \int_0^t \langle \Psi_r, e_i \rangle_{\mathcal{H}}^2 dr \right)
\]

which is a bound uniform in \( l \) and summable in \( i \). This is our dominating function which justifies the application of the Dominated Convergence Theorem, concluding the proof.

**Proposition 1.5.8.** Suppose that \((M^n) = ((M^n_t))\) is a sequence of martingales in \( \mathcal{M}_c^2(\mathcal{H}) \) which at every time \( t \) converges to a continuous \( M = (M_t) \) in \( L^2(\Omega; \mathcal{H}) \). Then at any time \( t \), \( [M^n]_t \) converges to \([M]_t \) in \( L^1(\Omega; \mathbb{R}) \).

**Proof.** It should first be noted that \( M \in \mathcal{M}_c^2(\mathcal{H}) \) as the \( L^2(\Omega; \mathcal{H}) \) convergence preserves square integrability and implies \( L^1(\Omega; \mathcal{H}) \) convergence which implies martingality. From here simply observe that for each \( n \), by definition

\[
\|M^n_t\|_{\mathcal{H}}^2 - [M^n]_t
\]

is a real valued martingale, so the \( L^1(\Omega; \mathbb{R}) \) limit

\[
\lim_{n \to \infty} (\|M^n_t\|_{\mathcal{H}}^2 - [M^n]_t)
\]

of this is again a real valued martingale. But this is just

\[
\lim_{n \to \infty} \|M^n_t\|_{\mathcal{H}}^2 - \lim_{n \to \infty} [M^n]_t = \|M_t\|_{\mathcal{H}}^2 - \lim_{n \to \infty} [M^n]_t
\]

by definition of the limit, and therefore the \( L^1(\Omega; \mathbb{R}) \) limit of \([M^n]_t \) is an adapted process such that \( \|M_t\|_{\mathcal{H}}^2 - \lim_{n \to \infty} [M^n]_t \) is a real valued martingale, which is also clearly non-decreasing by its construction as the limit of non-decreasing processes. So it must be the unique such process, and we are done.

**Definition 1.5.9.** The cross-variation process between two continuous square integrable martingales, one \( \Psi \in \mathcal{M}_c^2(\mathcal{H}) \) and another \( Y \in \mathcal{M}_c^2 \), is the unique continuous and adapted finite variation \( \mathcal{H} \) valued process \( \langle [\Psi, Y]_t \rangle \) which starts from zero such that

\[
\Psi Y - [\Psi, Y]
\]

is an \( \mathcal{H} \) valued martingale.
Proposition 1.5.10. Suppose that \((\Psi_n) = ((\Psi_{nt})_t)_t\) is a sequence of martingales in \(M^2_c(\mathcal{H})\) which at every time \(t\) converges to a continuous \(\Psi = (\Psi_t)_t\) in \(L^2(\Omega; \mathcal{H})\), and that \(Y \in M^2_c\). Then at any time \(t\), \([\Psi^n, Y]_t\) converges to \([\Psi, Y]_t\) in \(L^1(\Omega; \mathcal{H})\).

Proof. As you may have guessed the proof is largely similar to that of 1.5.8: on this occasion for each \(n\) the martingale in question is the \(\mathcal{H}\) valued one \(\Psi_n Y - [\Psi^n, Y]\).

We use again that the \(L^1(\Omega; \mathcal{H})\) limit preserves martingality, that by Hölder’s inequality the \(L^1(\Omega; \mathcal{H})\) limit of \(\Psi^n_t Y_t\) is \(\Psi_t Y_t\), that this limit also retains the finite variation and adaptedness properties (and too starting from zero, via the continuity), and the uniqueness of the cross-variation process to conclude the result.

If the given processes were only (continuous) local martingales, then we can make a slightly modified version of the definition. Assuming without loss of generality that \(\Psi\) and \(Y\) are locally square integrable (see the discussion after 1.3.3), localised by stopping times \((R_n)\) and \((T_n)\) respectively, then for a new sequence of stopping times defined by

\[\tau_n = R_n \wedge T_n\]

the stopped processes \(\Psi^{\tau_n}\) and \(Y^{\tau_n}\) are genuine square integrable martingales (in their respective spaces), so the cross variation \([\Psi^{\tau_n}, Y^{\tau_n}]\) can be defined. The canonical localisation procedure is evident once more, as the \((\tau_n)\) go to infinity almost surely, the consistency conditions that for \(m \leq n\) and \(t \leq \tau_m\) we have

\[\Psi_t^{\tau_m} = \Psi_t^{\tau_n}\quad\text{and}\quad Y_t^{\tau_m} = Y_t^{\tau_n}\]

allow us once more to define the process at almost every \(\omega\) and any \(t\) by

\[\left(\left(\Psi, Y\right)_t\right)(\omega) := \left(\left[\Psi^{\tau_n}, Y^{\tau_n}\right]_t\right)(\omega)\]

(24)

for any \(n\) such that \(t \leq \tau_n(\omega)\), independently of this choice of \(n\). Then the process

\[\Psi Y - [\Psi, Y]\]

is itself a local martingale, localised by the stopping times \((\tau_n)\). The argument justifying this is identical to 1.5.4, also containing the straightforward justification that

\[[\Psi, Y]^{\tau_n} = [\Psi^{\tau_n}, Y^{\tau_n}]\].

In the traditional way, these notions can all be extended to semi-martingales (that is, a martingale plus a finite variation process). The notion of finite variation in a Hilbert Space is completely analogous to the real valued setting. The quadratic and cross variation of such semi-martingales is then simply the quadratic/cross variation of the corresponding martingale parts. To this end we introduce the notation \(\bar{M}^2_c\) and \(\bar{M}^2_c(\mathcal{H})\) to be the corresponding spaces of square integrable continuous semi-martingales, and similarly \(\bar{M}_c, \bar{M}_c(\mathcal{H})\) to be the spaces continuous semi-martingales.
1.6 Integration Driven by Cylindrical Brownian Motion

For our analysis now we will need to make reference to two distinct Hilbert Spaces; one over which \( \mathcal{W} \) is a Cylindrical Brownian Motion, and the other in which our integrand maps to. Henceforth we introduce \( \mathcal{U} \) as the Hilbert Space over which \( \mathcal{W} \) is a Cylindrical Brownian Motion.

**Definition 1.6.1.** Denote by \( \mathcal{I}^H_T(\mathcal{W}) \) the class of progressively measurable operator valued processes \( B \) belonging to the set \( L^2(\Omega \times [0,T]; L^2(\mathcal{U}; \mathcal{H})) \). Measurability here is of course defined with respect to the Borel Sigma algebra on \( L^2(\mathcal{U}; \mathcal{H}) \).

Note that we make no explicit reference to \( \mathcal{U} \), the space on which \( \mathcal{W} \) is a cylindrical Brownian Motion. This is because, in practice, the space \( \mathcal{U} \) will be arbitrarily chosen; this shall be discussed later.

**Definition 1.6.2.** The class of processes \( B \) such that \( B \in \mathcal{I}^H_T(\mathcal{W}) \) for all \( T \) will be denoted by \( \mathcal{I}^H(\mathcal{W}) \).

Recall that if \( \mathcal{W} \) is a Cylindrical Brownian motion over \( \mathcal{U} \), it can be formally represented by

\[
\mathcal{W}(t) = \sum_{i=1}^{\infty} e_i \mathcal{W}^i_t
\]  

where the \( (e_i) \) form an orthonormal basis of \( \mathcal{U} \) and the \( (\mathcal{W}^i) \) are standard independent one-dimensional Brownian motions.

**Definition 1.6.3.** For \( B \in \mathcal{I}^H(\mathcal{W}) \) we define the Itô stochastic integral

\[
\int_0^t B(s) d\mathcal{W}_s
\]

as the \( \mathcal{H} \) valued random variable

\[
\sum_{i=1}^{\infty} \int_0^t B_{e_i}(s) d\mathcal{W}^i_s
\]

where each integral is defined as in 1.2.6 and the infinite sum is taken in \( L^2(\Omega; \mathcal{H}) \).

The immediate response to this definition is to prove that (27) is well defined; that is the integrals are well defined, as is the limit. Firstly for each \( i, B_{e_i} \) is trivially in \( L^2(\Omega \times [0,T]; \mathcal{H}) \) as this norm is bounded by the \( L^2(\Omega \times [0,T]; L^2(\mathcal{U}; \mathcal{H})) \) norm of \( B \). The progressive measurability is inherited from that of \( B \).

In order to show that the limit of partial sums is well defined, we proceed similarly to the method applied for (19) and argue that the sequence of partial sums is Cauchy. Observe that

\[
\left\| \sum_{i=m}^{n} \int_0^t B_{e_i}(s) d\mathcal{W}^i_s \right\|_{L^2(\Omega; \mathcal{H})}^2 = \mathbb{E} \left\| \sum_{i=m}^{n} \int_0^t B_{e_i}(s) d\mathcal{W}^i_s \right\|_{\mathcal{H}}^2 = \sum_{i=m}^{n} \mathbb{E} \int_0^t \| B_{e_i}(s) \|_{\mathcal{H}}^2 ds
\]

having applied the Itô Isometry 1.2.9 to the above. But by assumption that \( B \in L^2(\Omega \times [0,t]; L^2(\mathcal{U}; \mathcal{H})) \) we know

\[
\mathbb{E} \int_0^t \sum_{i=1}^{\infty} \| B_{e_i}(s) \|_{\mathcal{H}}^2 ds < \infty
\]
and thus, by Tonelli’s theorem regarding the infinite sum as an integral with respect to the counting measure,

\[ \sum_{i=1}^{\infty} \mathbb{E} \int_{0}^{t} \| B_{e_{i}}(s) \|_{H}^{2} ds < \infty \]

demonstrating the required Cauchy property. Of course the \( L^{2}(\Omega; \mathcal{H}) \) norm of the limit is the limit of the \( L^{2}(\Omega; \mathcal{H}) \) norms, so we have justified the following.

**Proposition 1.6.4.** For \( B \in \mathcal{T}^{\mathcal{H}}(\mathcal{W}) \), we have

\[ \mathbb{E} \left( \left\| \int_{0}^{t} B(s) dW_{s} \right\|_{\mathcal{H}}^{2} \right) = \mathbb{E} \left( \int_{0}^{t} \| B(s) \|_{L^{2}(\mathcal{H})}^{2} ds \right). \]

It is worth noting that whilst we impose the condition

\[ \mathbb{E} \int_{0}^{t} \sum_{i=1}^{\infty} \| B_{e_{i}}(s) \|_{H}^{2} ds < \infty \]

one may instead require the looser condition

\[ \int_{0}^{t} \sum_{i=1}^{\infty} \| B_{e_{i}}(s) \|_{H}^{2} ds < \infty \quad a.e. \tag{28} \]

or equivalently that \( B : \Omega \to L^{2}([0, t]; \mathcal{L}^{2}(\mathcal{H})) \) for almost every \( \omega \). Our formulation follows the classical construction as laid out in 1.2, ensuring that the integral is a genuine square integrable martingale. We can just as straightforwardly follow the arguments from 1.3.3, which are laid out here.

**Definition 1.6.5.** Denote by \( \mathcal{T}^{\mathcal{H}}_{T}(\mathcal{W}) \) the class of progressively measurable operator valued processes \( B \) such that \( B(\omega) \) belongs to the set \( L^{2}([0, T]; \mathcal{L}^{2}(\mathcal{H})) \) for almost every \( \omega \).

**Definition 1.6.6.** The class of processes \( B \) such that \( B \in \mathcal{T}^{\mathcal{H}}_{T}(\mathcal{W}) \) for all \( T \) will be denoted by \( \mathcal{T}^{\mathcal{H}}(\mathcal{W}) \).

Using the template after 1.3.3, for a process \( B \in \mathcal{T}^{\mathcal{H}}(\mathcal{W}) \) let’s introduce

\[ \tau_{n} := n \land \inf\{ 0 \leq t < \infty : \int_{0}^{t} \| B(s) \|_{\mathcal{L}^{2}(\mathcal{H})}^{2} ds \geq n \} \]

taking the convention that the infimum of the empty set is infinite. The \( \{ \tau_{n} \} \) are stopping times as they are simply first hitting times of the continuous and \( (\mathcal{F}_{t}) \)-adapted random variable

\[ \int_{0}^{t} \| B(s) \|_{\mathcal{L}^{2}(\mathcal{H})}^{2} ds. \]

These times tend to infinity \( a.s. \) by condition (28). Now define the truncated processes \( B^{n} \) as

\[ B^{n}(t) := B(t) 1_{t \leq \tau_{n}} \]

and using the fact that for \( m \leq n \), and \( t \leq \tau_{m} \), we have

\[ B(t) 1_{t \leq \tau_{n}} = B(t) 1_{t \leq \tau_{m}} \]

we can make the consistent definition
Definition 1.6.7. In the setting described, we define
\[
\left( \int_0^t B(s) dW_s \right)(\omega) := \left( \int_0^t B^n(s) dW_s \right)(\omega)
\] (29)
at almost every \(\omega\) for any \(n\) such that \(t \leq \tau_n(\omega)\), noting that such an \(n\) exists (for almost every \(\omega\)) from the assumed condition (28).

The justification of this definition is identical to that discussed in 1.3, such that for this fixed \(n\) we have \(B^n \in \mathcal{I}^H (\mathcal{W})\) and subsequently the complete definition
\[
\left( \int_0^t B(s) dW_s \right)(\omega) := \left( \sum_{i=1}^{\infty} \int_0^t B_{e_i}(s) \mathbb{1}_{s \leq \tau_n} dW_s^i \right)(\omega)
\] (30)
with the limit again in \(L^2(\Omega; \mathcal{H})\), which is of course a limit averaging over all \(\Omega\) which is taken for the fixed \(n\) chosen with respect to the specific \(\omega\) in which we are evaluating the limit. We will have no quarrels in writing (29) as a formal expression
\[
\sum_{i=1}^{\infty} \int_0^t B_{e_i}(s) dW_s^i
\] (31)
motivated by the fact that as \(B_{e_i} \in \mathcal{I}^H (1.3.4)\) then by definition
\[
\int_0^t B_{e_i}(s) dW_s^i = \int_0^t B_{e_i}(s) \mathbb{1}_{s \leq \tau_n} dW_s^i
\]
at this same choice of \(\omega\), for the same fixed \(n\). We say the expression is only formal though, as the infinite sum in (31) is not the \(L^2(\Omega; \mathcal{H})\) limit of the partial sums of the local martingales as presented. We understand (31) only by (30), that is by choosing the \(\omega\) at which we evaluate (31), then fixing our \(n\) associated to this \(\omega\), before then taking the limit in \(L^2(\Omega; \mathcal{H})\) of the genuine square integrable martingales (given by stopping the local martingales at \(\tau_n\)) which is finally then evaluated at \(\omega\).

Proposition 1.6.8. For \(B \in \mathcal{I}^H (\mathcal{W})\), the Itô stochastic integral
\[
\int_0^t B(s) dW_s
\]
belongs to \(\mathcal{M}^2(\mathcal{H})\).

Proof. This follows immediately from 1.5.3, the \(L^2(\Omega; \mathcal{H})\) convergence and the fact that the space \(\mathcal{M}^2(\mathcal{H})\) is closed in this topology.

Corollary 1.6.8.1. For \(B \in \mathcal{I}^H (\mathcal{W})\), the Itô stochastic integral
\[
\int_0^t B(s) dW_s
\]
is a continuous local martingale.

Proof. This follows identically to 1.5.4 using 1.6.8.
Due to this martingale property, we have a Burkholder-Davis-Gundy type inequality for the stochastic integral.

**Theorem 1.6.9.** There exists a constant $c$ such that for any $B \in \mathcal{I}^H(W)$ and $t \geq 0$, we have that

$$
\mathbb{E} \sup_{r \in [0,t]} \left\| \int_0^t B(s)dW_s \right\|_H \leq c \mathbb{E} \left( \int_0^t \|B(s)\|_{L^2(\Omega;H)}^2 ds \right)^{1/2}.
$$

**Remark 1.** There is no a priori assumption on the finiteness of the expectation.

For a proof and discussion of this result please see [7] Theorem 4.36. The martingality of the integral also allows us to consider the quadratic variation as defined in 1.5.5.

**Proposition 1.6.10.** For $B \in \mathcal{I}^H(W)$, we have that

$$
\left[ \int_0^t B_r dW_r \right]_t = \int_0^t \|B_r\|_{L^2(\Omega;H)}^2 dr. \tag{32}
$$

**Proof.** At each time $t$, the integral

$$
\int_0^t B_r dW_r
$$

is defined to be the $L^2(\Omega;H)$ limit of the sequence

$$
\sum_{i=1}^n \int_0^t B_r(e_i) dW_r^i.
$$

We look to infer the quadratic variation of this sequence of processes using 1.5.6 and apply 1.5.8. We claim that

$$
\left[ \sum_{i=1}^n \int_0^t B_r(e_i) dW_r^i \right]_t = \int_0^t \sum_{i=1}^n \|B_r(e_i)\|_{L^2(\Omega;H)}^2 dr
$$

which is to say

$$
\left\| \sum_{i=1}^n \int_0^t B_r(e_i) dW_r^i \right\|^2_H = \int_0^t \sum_{i=1}^n \|B_r(e_i)\|^2_H dr \tag{33}
$$
is a martingale. As seen in 1.5.6, for an orthonormal basis $(a_k)$ of $\mathcal{H}$,

$$\left\| \sum_{i=1}^{n} \int_0^t B_r(e_i) dW^i_r \right\|_{\mathcal{H}}^2 - \int_0^t \sum_{i=1}^{n} \left\| B_r(e_i) \right\|_{\mathcal{H}}^2 \, dr$$

$$= \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n} \int_0^t B_r(e_i) dW^i_r, a_k \right)_{\mathcal{H}}^2 - \int_0^t \sum_{i=1}^{n} \left\| B_r(e_i) \right\|_{\mathcal{H}}^2 \, dr$$

$$= \sum_{k=1}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \int_0^t B_r(e_i) dW^i_r, a_k \right)_{\mathcal{H}} \left( \int_0^t B_r(e_j) dW^j_r, a_k \right)_{\mathcal{H}} - \int_0^t \sum_{i=1}^{n} \left\| B_r(e_i) \right\|_{\mathcal{H}}^2 \, dr$$

$$= \left( \sum_{k=1}^{\infty} \sum_{i=1}^{n} \left( \int_0^t B_r(e_i) dW^i_r \right)^2_{\mathcal{H}} - \int_0^t \left\| B_r(e_i) \right\|_{\mathcal{H}}^2 \, dr \right)$$

$$+ \left( \sum_{k=1}^{\infty} \sum_{i \neq j} \left( \int_0^t B_r(e_i) dW^i_r, a_k \right)_{\mathcal{H}} \left( \int_0^t B_r(e_j) dW^j_r, a_k \right)_{\mathcal{H}} \right).$$

Inspecting the last equality, by 1.5.6 we have that

$$\sum_{i=1}^{n} \left( \int_0^t B_r(e_i) dW^i_r \right)^2_{\mathcal{H}} - \int_0^t \left\| B_r(e_i) \right\|_{\mathcal{H}}^2 \, dr$$

is a finite sum of martingales, so we would prove that the process defined in (33) also belongs to this class if we show that the same is true of

$$\sum_{k=1}^{\infty} \sum_{i \neq j} \left( \int_0^t B_r(e_i) dW^i_r, a_k \right)_{\mathcal{H}} \left( \int_0^t B_r(e_j) dW^j_r, a_k \right)_{\mathcal{H}}. \quad (34)$$

We consider the above for each fixed $k$, rewriting it as

$$\sum_{i \neq j} \left( \int_0^t \langle B_r(e_j), a_k \rangle_{\mathcal{H}} dW^j_r \right) \left( \int_0^t \langle B_r(e_i), a_k \rangle_{\mathcal{H}} dW^i_r \right). \quad (35)$$
Adaptedness of this process is clear, and to show integrability observe that

\[
\mathbb{E}\left| \sum_{i \neq j} \left( \int_0^t \langle B_r(e_j), a_k \rangle_{\mathcal{H}} dW_r^j \right) \left( \int_0^t \langle B_r(e_i), a_k \rangle_{\mathcal{H}} dW_r^i \right) \right|
\]

\[
\leq \sum_{i \neq j} \mathbb{E} \left| \left( \int_0^t \langle B_r(e_i), a_k \rangle_{\mathcal{H}} dW_r^i \right) \left( \int_0^t \langle B_r(e_i), a_k \rangle_{\mathcal{H}} dW_r^i \right) \right|
\]

\[
\leq \sum_{i \neq j} \left( \mathbb{E} \int_0^t \langle B_r(e_i), a_k \rangle_{\mathcal{H}}^2 dW_r^i \right)^{1/2} \left( \mathbb{E} \int_0^t \langle B_r(e_i), a_k \rangle_{\mathcal{H}}^2 dW_r^i \right)^{1/2}
\]

\[
= \sum_{i \neq j} \left( \mathbb{E} \int_0^t \langle B_r(e_i), a_k \rangle_{\mathcal{H}}^2 dr \right)^{1/2} \left( \mathbb{E} \int_0^t \langle B_r(e_i), a_k \rangle_{\mathcal{H}}^2 dr \right)^{1/2}
\]

\[
\leq n^2 \mathbb{E} \int_0^t \sum_{i=1}^n \| B_r(e_i) \|_{\mathcal{H}}^2 dr
\]

\[
\leq n^2 \mathbb{E} \int_0^t \sum_{i=1}^n \| B_r(e_i) \|_{\mathcal{H}}^2 dr < \infty.
\]

As for the martingale property, for any times \( s < t \),

\[
\mathbb{E} \left( \sum_{i \neq j} \left( \int_s^t \langle B_r(e_j), a_k \rangle_{\mathcal{H}} dW_r^j \right) \left( \int_s^t \langle B_r(e_i), a_k \rangle_{\mathcal{H}} dW_r^i \right) \middle| \mathcal{F}_s \right)
\]

\[
= \mathbb{E} \left( \sum_{i \neq j} \left( \int_s^t \langle B_r(e_j), a_k \rangle_{\mathcal{H}} dW_r^j \right) \left( \int_s^t \langle B_r(e_i), a_k \rangle_{\mathcal{H}} dW_r^i \right) \right)
\]

\[
= 0 \quad (36)
\]

where passage from the first line to the second is through the independent increments property of the Brownian Motions, and the second to the third is from the independence of the Brownian Motions. So the process defined in (35) is shown to be a martingale, where we wish to show that this property remains true in the limit of the infinite sum for (34). Convergence of the infinite sum is defined \( \mathbb{P} - a.s. \), and it is sufficient to show that the convergence also holds in \( L^1(\Omega; \mathbb{R}) \). For this we show that the sequence is Cauchy in \( L^1(\Omega; \mathbb{R}) \), taking the difference of the \( l^{th} \) and \( m^{th} \) terms to
see that
\[
\mathbb{E} \left| \sum_{k=m+1}^{l} \sum_{i \neq j} \left( \int_{0}^{t} \langle B_r(e_j), a_k \rangle \mathcal{H} dW_r^j \right) \left( \int_{0}^{t} \langle B_r(e_i), a_k \rangle \mathcal{H} dW_r^i \right) \right| \\
\leq \sum_{k=m+1}^{l} \mathbb{E} \left| \sum_{i \neq j} \left( \int_{0}^{t} \langle B_r(e_j), a_k \rangle \mathcal{H} dW_r^j \right) \left( \int_{0}^{t} \langle B_r(e_i), a_k \rangle \mathcal{H} dW_r^i \right) \right| \\
\leq n^2 \sum_{k=m+1}^{l} \mathbb{E} \int_{0}^{t} \sum_{i=1}^{n} \langle B_r(e_i), a_k \rangle^2 \mathcal{H} dr \\
\leq n^2 \sum_{k=m+1}^{l} \mathbb{E} \int_{0}^{t} \sum_{i=1}^{n} \langle B_r(e_i), a_k \rangle^2 \mathcal{H} dr
\]

which is a monotone decreasing sequence to zero in \( m \), hence the Cauchy property is shown so there exists a limit in \( L^1(\Omega; \mathbb{R}) \) which must agree with the \( \mathbb{P} \) - \( a.s. \) limit (we can take a \( \mathbb{P} \) - \( a.s. \) convergent subsequence from the \( L^1(\Omega; \mathbb{R}) \) convergence) and the martingale property of the process defined in (34) and hence (33) is shown. We thus apply 1.5.8 and deduce that \( \left[ \int_{0}^{t} B_r dW_r \right]_t \) is the \( L^1(\Omega; \mathbb{R}) \) limit of the sequence
\[
\int_{0}^{t} \sum_{i=1}^{n} \| B_r(e_i) \|_{\mathcal{H}}^2 dr
\]
in \( n \). Similarly to the analysis just conducted we can show that this sequence is Cauchy in \( L^1(\Omega; \mathbb{R}) \) and agrees with the \( \mathbb{P} \) - \( a.s. \) limit, which is of course
\[
\int_{0}^{t} \| B_r \|_{\mathcal{H}}^2 \mathcal{H} dr
\]
taking the infinite sum through the integral with either Tonelli’s Theorem (identifying the infinite sum as a integral with respect to the counting measure) or the monotone convergence theorem. The proof is concluded.

We also have the analogous result to 1.5.7.

**Proposition 1.6.11.** Let \( B \in \mathcal{I}_T(\mathcal{W}) \) and consider any sequence of partitions
\[
I_l := \left\{ 0 = t^l_0 < t^l_1 < \cdots < t^l_{k_l} = T \right\}
\]
with \( \max \lvert t^l_{j+1} - t^l_{j} \rvert \to 0 \) as \( l \to \infty \). Then for all \( t \in [0, T] \),
\[
\lim_{l \to \infty} \mathbb{E} \left( \left( \sum_{j=1}^{k_l} \left\| \int_{t^l_j}^{t^l_{j+1}} B_r dW_r \right\|_{\mathcal{H}}^2 - \int_{0}^{t} \| B_r \|_{\mathcal{H}}^2 \mathcal{H} dr \right) \right) = 0. \tag{37}
\]

**Proof.** Following the method used in 1.5.7 we again would like to reduce this to a familiar case and extrapolate the result to the limit. Let \( (a_k) \) be an orthonormal basis of \( \mathcal{H} \), then identically to 1.5.7
we have that
\[
\mathbb{E} \left( \sum_{t_{j+1}^l \leq t} \left\| \int_{t_j}^{t_{j+1}} B_r dW_r \right\|^2_{\mathcal{H}} - \int_0^t \left\| B_r \right\|^2_{L^2(\mathcal{H}; \mathcal{L})} dr \right)
\]
\[
= \mathbb{E} \left( \sum_{t_{j+1}^l \leq t} \left\| \int_{t_j}^{t_{j+1}} B_r dW_r \right\|^2_{\mathcal{H}} - \int_0^t \sum_{i=1}^{\infty} \left\| B_r(e_i) \right\|^2_{\mathcal{H}} dr \right)
\]
\[
= \mathbb{E} \left( \sum_{t_{j+1}^l \leq t} \sum_{i=1}^{\infty} \left( \int_{t_j}^{t_{j+1}} (B_r, a_k)_{\mathcal{H}} dW_r \right)^2 - \int_0^t \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (B_r(e_i), a_k)_{\mathcal{H}}^2 dr \right)
\]
\[
= \mathbb{E} \left( \sum_{t_{j+1}^l \leq t} \sum_{i=1}^{\infty} \left( \int_{t_j}^{t_{j+1}} (B_r, a_k)_{\mathcal{H}} dW_r \right)^2 - \int_0^t \sum_{k=1}^{\infty} \left\| (B_r(e_i), a_k)_{\mathcal{H}} \right\|^2_{L^2(\mathcal{H}; \mathcal{L})} dr \right)
\]
\[
\leq \sum_{k=1}^{\infty} \mathbb{E} \left( \sum_{t_{j+1}^l \leq t} \left( \int_{t_j}^{t_{j+1}} (B_r, a_k)_{\mathcal{H}} dW_r \right)^2 - \int_0^t \left\| (B_r(e_i), a_k)_{\mathcal{H}} \right\|^2_{L^2(\mathcal{H}; \mathcal{L})} dr \right)
\]

having applied 1.6.12.1, used Tonelli’s Theorem to justify the interchange in order of summation, and applied the Monotone Convergence Theorem to take the infinite sum in \( k \) through the time integral and expectation. From 1.6.12.1 and 1.6.8 then \( \int_0^t (B_r, a_k)_{\mathcal{H}} dW_r \) belongs to \( \mathcal{M}_c^2 \), with quadratic variation \( \int_0^t \left\| (B_r(e_i), a_k)_{\mathcal{H}} \right\|^2_{L^2(\mathcal{H}; \mathcal{L})} dr \) coming from 1.6.10. So by the theory of martingales in \( \mathbb{R} \), we have that for each fixed \( k \in \mathbb{N} \),

\[
\lim_{l \to \infty} \mathbb{E} \left( \sum_{t_{j+1}^l \leq t} \left( \int_{t_j}^{t_{j+1}} (B_r, a_k)_{\mathcal{H}} dW_r \right)^2 - \int_0^t \left\| (B_r(e_i), a_k)_{\mathcal{H}} \right\|^2_{L^2(\mathcal{H}; \mathcal{L})} dr \right) = 0
\]

so it is sufficient to justify the interchange of limit in \( l \) and summation in \( k \). This follows identically to the justification in 1.5.7, appealing this time to the Itô Isometry 1.6.4. \( \square \)

It is also critical for our analysis that we have the corresponding result of 1.2.10.

**Theorem 1.6.12.** Suppose that \( \mathcal{H}_1, \mathcal{H}_2 \) are Hilbert spaces such that \( B \in \mathcal{F}^{\mathcal{H}_1}(\mathcal{W}) \) and \( T \in \mathcal{L}(\mathcal{H}_1; \mathcal{H}_2) \). Then the process \( TB \) defined by

\[
TB_{e_i}(s, \omega) = T(B_{e_i}(s, \omega))
\]

belongs to \( \mathcal{F}^{\mathcal{H}_2}(\mathcal{W}) \) and is such that

\[
T \left( \int_0^t B(s) dW_s \right) = \int_0^t TB(s) dW_s.
\]

In addition, the two integrals are defined pointwise a.e. with respect to the same stopping times.
Proof. We shall prove first that $TB \in \overline{\mathcal{T}}^{H_2}(\mathcal{W})$. The progressive measurability is preserved under the continuity of $T$, and for $C$ the (square of the) boundedness constant associated to $T$ we have

$$
\int_0^t \sum_{i=1}^{\infty} \|TB_{e_i}(s)\|_{H_2}^2 \, ds \leq C \int_0^t \sum_{i=1}^{\infty} \|B_{e_i}(s)\|_{H_1}^2 \, ds < \infty
$$

holding a.e. as $B \in \overline{\mathcal{T}}^{H_1}(\mathcal{W})$. In addition for any stopping time $\tau_n$ as in 1.6.7,

$$
\mathbb{E}\left( \int_0^t \sum_{i=1}^{\infty} \|TB_{e_i}(s)\|_{H_2}^2 \, ds \right) = \mathbb{E}\left( \int_0^t \sum_{i=1}^{\infty} \|T(B_{e_i}(s)1_{s \leq \tau_n})\|_{H_2}^2 \, ds \right)
$$

$$
\leq C \mathbb{E}\left( \int_0^t \sum_{i=1}^{\infty} \|B_{e_i}(s)1_{s \leq \tau_n}\|_{H_1}^2 \, ds \right) < \infty
$$

so the new stochastic integral

$$
\int_0^t TB(s) \, dW_s
$$

can be constructed using the same sequence of stopping times. We will freely use linearity of $T$ to commute it with the indicator function, not showing this explicitly with brackets. To carry $T$ through the integral however let’s avoid danger and write out explicitly what we want to show, which is for almost any choice of $\omega$ with the associated $\tau_n$ and fixed $n$ as in 1.6.7, then

$$
T\left( \left( \int_0^t B(s) \, dW_s \right)(\omega) \right) := T\left( \left( \sum_{i=1}^{\infty} \int_0^t B^n_{e_i}(s) \, dW_s^i \right)(\omega) \right) = \left( \sum_{i=1}^{\infty} \int_0^t TB^n_{e_i}(s) \, dW_s^i \right)(\omega)
$$

where the left hand side limit is taken in $L^2(\Omega; H_1)$ and the right side in $L^2(\Omega; H_2)$. From the $L^2(\Omega; H_1)$ limit there exists a subsequence convergent almost everywhere in $H_1$ (we assume w.l.o.g that $\omega$ belongs to this set of full probability). Working with this subsequence, we can pass the a.e. limit through the continuous $T$ such that it is now the a.e. limit in $H_2$. Linearity of $T$ allows us to hit each term in the sum individually, so we have now that

$$
T\left( \left( \int_0^t B(s) \, dW_s \right)(\omega) \right) = \left( \lim_{n_k \to \infty} \sum_{i=1}^{n_k} T\left( \int_0^t B^n_{e_i}(s) \, dW_s^i \right) \right)(\omega)
$$

(38)

for the limit a.e. of the subsequence indexed by $(n_k)$. Applying 1.2.10, we can commute $T$ with the integral on the right side of (38). However we have justified already that the limit over the whole sequence in $L^2(\Omega; H_2)$ exists, agreeing with the $L^2(\Omega; H_2)$ limit of the subsequence, which in turn agrees with the a.e. limit. Thus by definition of the stochastic integral, (38) is simply the equality

$$
T\left( \left( \int_0^t B(s) \, dW_s \right)(\omega) \right) = \left( \int_0^t TB(s) \, dW_s \right)(\omega)
$$

completing the proof.

□

This is stated in the more general form for a process only in $\overline{\mathcal{T}}^{H_1}(\mathcal{W})$, but from the proof the following is clear.
Corollary 1.6.12.1. Suppose that \( H_1, H_2 \) are Hilbert spaces such that \( B \in \mathcal{D}(H_1; \mathcal{W}) \) and \( T \in \mathcal{L}(H_1; H_2) \). Then the process \( TB \) defined by

\[
TB_{e_i}(s, \omega) = T(B_{e_i}(s, \omega))
\]

belongs to \( \mathcal{D}(H_2; \mathcal{W}) \) and is such that

\[
T \left( \int_0^t B(s)dW_s \right) = \int_0^t TB(s)dW_s.
\]

Proposition 1.6.13. Let \( B \in \tilde{\mathcal{D}}(\mathcal{W}) \) and \( \phi : \Omega \rightarrow \mathcal{H} \) be \( \mathcal{F}_0 \)-measurable. Then for every \( t > 0 \) we have that

\[
\left\langle \int_0^t B_r dW_r, \phi \right\rangle_{\mathcal{H}} = \int_0^t \langle B_r, \phi \rangle_{\mathcal{H}} dW_r
\]

\( \mathbb{P} - \text{a.s.} \).

Proof. Firstly we make clear that \( \langle B, \phi \rangle_{\mathcal{H}} \) is understood as a process defined by the mapping

\[
ediag{i} \times \omega \times t \rightarrow \langle B_{\langle ed i, \omega \rangle}, \phi(\omega) \rangle_{\mathcal{H}}.
\]

The fact that \( \langle B, \phi \rangle_{\mathcal{H}} \in \tilde{\mathcal{D}}(\mathcal{W}) \) is completely analogous to 1.3.5, where the progressive measurability follows as the mapping

\[
\langle B, \phi \rangle_{\mathcal{H}} : t \times \omega \times \bar{\omega} \rightarrow \langle B_t(\omega), \phi(\bar{\omega}) \rangle_{\mathcal{H}}
\]

is \( B([0, T]) \times \mathcal{F}_T \times \mathcal{F}_0 \) measurable as a mapping into \( \mathcal{L}^2(\Omega; \mathbb{H}) \). Similarly we have that

\[
\|\langle B_t(\omega), \phi(\omega) \rangle_{\mathcal{H}}\|_{\mathcal{L}^2(\Omega; \mathbb{H})} \leq \|\phi(\omega)\|_{\mathcal{H}}\|B_t(\omega)\|_{\mathcal{L}^2(\Omega; \mathbb{H})}
\]

which is sufficient to justify that \( \langle B, \phi \rangle_{\mathcal{H}} \in \tilde{\mathcal{D}}(\mathcal{W}) \). The extension of 1.3.7 to this result is then identical to the extension of 1.2.10 to 1.6.12, so we conclude the proof here.

\( \square \)

Proposition 1.6.14. Let \( B \in \tilde{\mathcal{D}}(\mathcal{W}) \) and \( \eta : \Omega \rightarrow \mathbb{R} \) be \( \mathcal{F}_0 \)-measurable. Then \( \eta B \in \tilde{\mathcal{D}}(\mathcal{W}) \) and for every \( t > 0 \) we have that

\[
\eta \int_0^t B_r dW_r = \int_0^t \eta B_r dW_r
\]

\( \mathbb{P} - \text{a.s.} \).

Proof. The proof follows identically to 1.6.13 in analogy with 1.3.8. We make explicit that \( \eta B \) is defined by the mapping

\[
ediag{i} \times \omega \times t \rightarrow \eta(\omega)B_t(e_i, \omega).
\]

\( \square \)

Remark 2. Although we have not explicitly addressed the construction of the integral over a time interval \([s, t]\) where \( s > 0 \), this can be done without any extra difficulty just as in the standard real valued case. If we were to just consider the integral over \([s, t]\) in 1.6.13, then the results extends to any \( \mathcal{F}_s \)-measurable \( \phi, \eta \) in 1.6.13, 1.6.14. To show this we of course revisit 1.3.5, appreciating that the \( \mathcal{F}_s \)-measurability does not disturb the measurability requirements of the simple process.

We also extend the Stochastic Dominated Convergence Theorem to this setting.
Lemma 1.6.15. Let \((B^n)\) be a sequence in \(\bar{I}^H(W)\) such that there exists processes \(B : \Omega \times [0, \infty) \rightarrow \mathcal{L}^2(\mathcal{H})\) and \(Q \in \bar{I}^H(W)\) with the properties that for every \(T > 0\), \(\mathbb{P} \times \lambda - \text{a.e.} (\omega, t) \in \Omega \times [0, T] :\)

1. \(\|B^n_t(\omega)\|_{\mathcal{L}^2(\mathcal{H})} \leq \|Q_t(\omega)\|_{\mathcal{L}^2(\mathcal{H})}\) for all \(n \in \mathbb{N}\);

2. \((B^n_t(\omega))\) is convergent to \(B_t(\omega)\) in \(\mathcal{L}^2(\mathcal{H})\).

Then \(B \in \bar{I}^H(W)\) and for every \(t > 0\), there exists a subsequence indexed by \((n_k)\) such that

\[
\lim_{n_k \to \infty} \int_0^t B_{n_k}^r \, dW_r = \int_0^t B_r \, dW_r \quad (40)
\]

\(\mathbb{P} - \text{a.s.}\).

Proof. The proof is mechanically identical to that of 1.3.6, simply replacing \(\mathcal{H}\) by \(\mathcal{L}^2(\mathcal{H})\) when dealing with the integrands and using the appropriate Itô Isometry 1.6.4. \(\square\)
2 A Framework for SPDEs

In this section we consider an abstract framework to define an SPDE, which will be general enough to include the highly involved SALT derived fluid dynamics models as discussed in the introduction.

2.1 The Stratonovich Integral

We look at first to define the Stratonovich Integral of a martingale with respect to a one dimensional martingale, before then doing with respect to a Cylindrical Brownian Motion. We make the definition here only for integrands which are martingales in the Hilbert Space, though we note that this can be extended to a wider class of processes by considering the cross-variation defined as a limit in probability over nested partitions in the traditional way.

Definition 2.1.1. For \( M \in \mathcal{M}^2_c \) and \( \Psi \in \mathcal{I}^H_M \cap \mathcal{M}^2_c(\mathcal{H}) \), the Stratonovich stochastic integral is defined as

\[
\int_0^t \Psi_s \circ dM_s := \int_0^t \Psi_s dM_s + \frac{1}{2} [\Psi, M]_t.
\]

Definition 2.1.2. For \( B \in \mathcal{I}^{H}(W) \) such that \( B_{e_i} \in \mathcal{M}_c(H) \) for every \( e_i \) and the limit

\[
\sum_{i=1}^{\infty} [B_{e_i}, W^i]_t
\]

is well defined in \( L^2(\Omega; \mathcal{H}) \), the Stratonovich stochastic integral is defined as

\[
\int_0^t B(s) \circ dW_s := \sum_{i=1}^{\infty} \left( \int_0^t B_{e_i}(s) dW^i_s + \frac{1}{2} [B_{e_i}, W^i]_t \right)
\]

where the limit is taken in \( L^2(\Omega; \mathcal{H}) \). The class of such processes will be denoted \( \mathcal{I}^{H}(W) \).

It will be necessary to extend this definition to processes \( B \in \mathcal{I}^{H}(W) \), but we encounter more technical issues in trying to construct a sequence of stopping times such that the stopped process belong to \( \mathcal{I}^{H}(W) \). We find it simplest to give the definition below.

Definition 2.1.3. If there exists a sequence of stopping times \( (\tau_n) \) which are a.s. monotone increasing and convergent to infinity such that for every \( n: B^n \in \mathcal{I}^H(W) \), \( B^n_{e_i} \in \mathcal{M}_c(H) \) for every \( e_i \) and the limit

\[
\sum_{i=1}^{\infty} [B^n_{e_i}, W^i]_t
\]

is well defined in \( L^2(\Omega; \mathcal{H}) \), the Stratonovich stochastic integral is defined at a fixed \( \omega \) for any \( \tau^n(\omega) \geq t \) as

\[
\left( \int_0^t B(s) \circ dW_s \right)(\omega) := \left( \sum_{i=1}^{\infty} \left( \int_0^t B^n_{e_i}(s) dW^i_s + \frac{1}{2} [B^n_{e_i}, W^i]_t \right) \right)(\omega)
\]

where the limit is taken in \( L^2(\Omega; \mathcal{H}) \).

This definition is of course completely analogous to the localisation procedure used in the previous constructions, except we postulate in the first instance the existence of the localising sequence \( (\tau_n) \).
2.2 Strong Solutions in the Abstract Framework

We work with a quartet of embedded Hilbert Spaces

$$V \hookrightarrow H \hookrightarrow U \hookrightarrow X$$

where the embedding is assumed as a continuous linear injection. We introduce at first the Itô SPDE

$$\Psi_t = \Psi_0 + \int_0^t Q\Psi_s ds + \int_0^t G\Psi_s dW_s$$ (41)

where $W$ continues to be a Cylindrical Brownian Motion over $\mathcal{U}$ relative to our fixed filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with representation (25). We impose now some conditions on the operators relative to these spaces. To do so we define the general operator $\tilde{K} : H \to \mathbb{R}$ by

$$\tilde{K}(\phi) := c \left( 1 + \|\phi\|_U^p + \|\phi\|^q \right)$$

for any constants $c, p, q$ independent of $\phi$.

**Assumption 2.2.1.** $Q : V \to U$ is measurable and for any $\phi \in V$,

$$\|Q\phi\|_U \leq \tilde{K}(\phi)[1 + \|\phi\|^2_V].$$

**Assumption 2.2.2.** $G$ is understood as a measurable operator

$$G : V \to \mathcal{L}^2(\mathcal{U}; H)$$

$$G : H \to \mathcal{L}^2(\mathcal{U}; U)$$

$$G : U \to \mathcal{L}^2(\mathcal{U}; X)$$

defined over $\mathcal{U}$ by its action on the basis vectors

$$G(\cdot, e_i) := G_i(\cdot).$$

Each $G_i$ is linear and there exists constants $c_i$ such that for all $\phi \in V$, $\psi \in H$, $\eta \in U$:

$$\|G_i\phi\|_H \leq c_i \|\phi\|_V$$

$$\|G_i\psi\|_U \leq c_i \|\psi\|_H$$

$$\|G_i\eta\|_X \leq c_i \|\eta\|_U$$

$$\sum_{i=1}^{\infty} c_i^2 < \infty.$$ 

It is worth clarifying how $G$ is defined over $\mathcal{U}$: fix a $\phi \in V$ and consider $G(\phi, \cdot) : \mathcal{U} \to H$ (the arguments here apply for the larger spaces as well). Any $\alpha \in \mathcal{U}$ has the representation

$$\sum_{i=1}^{\infty} \langle \alpha, e_i \rangle_{\mathcal{U}} e_i$$

where

$$\sum_{i=1}^{\infty} \langle \alpha, e_i \rangle_{\mathcal{U}}^2 < \infty.$$
Then
\[ G(\phi, \cdot) : \alpha \mapsto \sum_{i=1}^{\infty} \langle \alpha, e_i \rangle_u G_i \phi \]
is well defined as an element of \( H \). This is justified by showing that the sequence of partial sums is Cauchy in \( H \): note that from Cauchy-Schwartz,
\[
\left\| \sum_{i=m}^{n} \langle \alpha, e_i \rangle_u G_i \phi \right\|_H^2 \leq \left( \sum_{i=m}^{n} \langle \alpha, e_i \rangle_u^2 \right) \left( \sum_{i=m}^{n} \| G_i \phi \|_H^2 \right) \leq \left( \sum_{i=m}^{\infty} \langle \alpha, e_i \rangle_u^2 \right) \left( \sum_{i=m}^{\infty} c_i^2 \| \phi \|_V^2 \right)
\]
which approaches zero as \( m \to \infty \) as the sums are finite. We introduce now the first notion of our strong solutions and frame this definition for a specified initial condition, which is an \( \mathcal{F}_0 \)-measurable mapping \( \Psi_0 : \Omega \to H \). Measurability here is again defined with respect to the Borel Sigma Algebra on \( H \).

**Definition 2.2.3.** A pair \((\Psi, \tau)\) where \( \tau \) is a \( \mathbb{P} \)-a.s. positive stopping time and \( \Psi \) is a process such that for \( \mathbb{P} \)-a.e. \( \omega \), \( \Psi(\omega) \in C([0, T]; H) \) and \( \Psi(\omega) 1_{\leq \tau(\omega)} \in L^2([0, T]; V) \) for all \( T > 0 \) with \( \Psi 1_{\leq \tau} \) progressively measurable in \( V \), is said to be a local strong solution of the equation (41) if the identity
\[
\Psi_t = \Psi_0 + \int_0^{\tau \wedge \tau} Q_t \Psi_s ds + \int_0^{\tau \wedge \tau} G_t \Psi_s d\mathcal{W}_s
\]
holds \( \mathbb{P} \)-a.s. in \( U \) for all \( t \geq 0 \).

Let's take a few moments to process this definition and ensure that the integrals make sense with the given regularity of the solution and the operators \( Q, G \). As an initial aside note that if \((\Psi, \tau)\) is a \( V \)-valued local strong solution of the equation (41), then \( \Psi_t = \Psi_t_{\tau} \). Moreover the progressive measurability condition on \( \Psi 1_{\leq \tau} \) may look a little suspect as \( \Psi_0 \) itself may only belong to \( H \) and not \( V \) making it impossible for \( \Psi 1_{\leq \tau} \) to be even adapted in \( V \). We are mildly abusing notation here; what we really ask is that there exists a process \( \Phi \) which is progressively measurable in \( V \) and such that \( \Phi_t = \Psi_t 1_{\leq \tau} \) almost surely over the product space \( \Omega \times [0, \infty) \) with product measure \( \mathbb{P} \times \lambda \).

The time integral in (42) is well defined in \( U \) as a Bochner Integral: first of all \( \Psi(\omega) 1_{\leq \tau(\omega)} \in L^2([0, T]; V) \) for \( \mathbb{P} \)-a.e. \( \omega \) so \( \Psi(\omega) 1_{\leq \tau(\omega)} : [0, t] \to V \) is measurable hence \( Q(\Psi(\omega) 1_{\leq \tau(\omega)}) : [0, t] \to U \) is measurable from 2.2.1. Moreover the mapping \( Q(\Psi(\omega) 1_{\leq \tau(\omega)}) 1_{\leq \tau(\omega)} : [0, t] \to U \) is again measurable and we have that
\[
\int_0^{\tau \wedge \tau(\omega)} Q(\Psi_s(\omega)) ds = \int_0^t Q(\Psi_s(\omega) 1_{s \leq \tau(\omega)}) 1_{s \leq \tau(\omega)} ds
\]
so the required measurability in order to define the integral is satisfied. In this vein we have that
\[
\int_0^{\tau \wedge \tau(\omega)} \| Q(\Psi_s(\omega)) \|_U ds \leq \int_0^{\tau \wedge \tau(\omega)} \tilde{K}(\Psi_s(\omega)) \left[ 1 + \| \Psi_s(\omega) \|_V^2 \right] ds
\]
\[
\leq \sup_{s \in [0, \tau \wedge \tau(\omega)]} \left[ \tilde{K}(\Psi_s(\omega)) \right] \int_0^{\tau \wedge \tau(\omega)} 1 + \| \Psi_s(\omega) \|_V^2 ds
\]
\[
< \infty
\]
employing 2.2.1 again and using the regularity specified by the local solution to deduce finiteness, which justifies that the integral is well defined.
As for the stochastic integral in (42), this is well defined in the sense of 1.6.7 in \( H \) (which then embeds into \( U \)), though it is done so formally via the process \( \Phi \) stipulated in the above remark regarding progressive measurability. We understand again that 
\[
\int_{0}^{t \wedge \tau_0} G \Psi_s dW_s = \int_{0}^{t \wedge \tau_0} G(\Psi_s 1_{s \leq \tau}) 1_{s \leq \tau} dW_s
\]
should the integral be well defined, and so we actually define the integral by
\[
\int_{0}^{t \wedge \tau_0} G \Psi_s dW_s := \int_{0}^{t} G\Phi_s 1_{s \leq \tau} dW_s.
\]
The progressive measurability of \( G\Phi 1_{s \leq \tau} : [0, t] \times \Omega \to L^2(\mathcal{U}; H) \) is immediate from the measurability assumption of 2.2.2, the same arguments above for the time integral and the progressive measurability of \( 1_{s \leq \tau} \) coming from the definition of the stopping time. Similarly we have that
\[
\int_{0}^{t \wedge \tau} \sum_{i=1}^{\infty} \|G_i(\Phi_s(\omega))\|^2_H ds = \int_{0}^{t \wedge \tau} \sum_{i=1}^{\infty} \|G_i(\Psi_s 1_{s \leq \tau})\|^2_H ds < \infty
\]
\( \mathbb{P} - a.s. \), using again 2.2.2, validating that this term is well defined as a local martingale in \( H \) and thus in \( U \) from the continuous embedding. We make no claims that this is genuinely a square integrable martingale.

We treat the local solution here to deal with the additional technicalities which arise from accounting for the stopping time. Global solutions are similarly defined however.

**Definition 2.2.4.** A process \( \Psi \) such that for \( \mathbb{P} - a.e. \ \omega \), \( \Psi_\omega(\omega) \in C([0, T]; H) \cap L^2([0, T]; V) \) for all \( T > 0 \) with \( \Psi \) progressively measurable in \( V \), is said to be a strong solution of the equation (41) if the identity (41) holds \( \mathbb{P} - a.s. \) in \( U \) for all \( t \geq 0 \).

### 2.3 Stratonovich SPDEs in the Abstract Framework

We work now with the same initial condition \( \Psi_0 \) and operators \( Q, G \) but instead pose the question of how to understand the Stratonovich SPDE
\[
\Psi_t = \Psi_0 + \int_0^t Q\Psi_s ds + \int_0^t G\Psi_s \circ dW_s. \quad (43)
\]

We are slightly hesitant to define our strong solution for the equation (43) as understanding the Stratonovich integral is delicate given the necessary semi-martingale structure. If we can show that \( \Psi \) satisfies an evolution equation of an Itô stochastic integral plus a time integral then we can deduce the required semi-martingality and identify the martingale part, but we would need this assumption on semi-martingality a priori in order to define the Stratonovich integral in the sense of 2.1.3 in order to show the desired representation. To this end, we identify a Stratonovich SPDE with an Itô one in the sense given here.

**Theorem 2.3.1.** If \( \Psi \) is a strong solution of the equation
\[
\Psi_t = \Psi_0 + \int_0^t \left( Q\Psi_s + \frac{1}{2} \sum_{i=1}^{\infty} G_i^2\Psi_s \right) ds + \int_0^t G\Psi_s dW_s \quad (44)
\]
then \( \Psi \) satisfies the identity (43) \( \mathbb{P} - a.s. \) in \( X \) for all \( t \geq 0 \).
There is a little to unpack here before going on to the proof of this result. The first is how we understand the infinite sum of (44) and subsequently the SPDE. The operator
\[ \sum_{i=1}^{\infty} G_i^2 : V \to U \]
is defined as the pointwise limit of the partial sums, which is well defined as for any fixed \( \phi \in V \),
\[
\left\| \sum_{i=m}^{n} G_i^2 \phi \right\|_U \leq \sum_{i=m}^{n} \| G_i^2 \phi \|_U \leq \sum_{i=m}^{n} c_i \| G_i \phi \|_H \leq \sum_{i=m}^{n} c_i^2 \| \phi \|_V
\]
which as seen before approaches zero as \( m \to \infty \). To understand the strong solution as defined in 2.2.4 we need to show that the new operator \( Q + \frac{1}{2} \sum_{i=1}^{\infty} G_i^2 \) satisfies the assumptions postulated in 2.2.1, but this is clear as \( G_i : V \to H \) and \( G_i : H \to U \) are bounded linear hence continuous so measurable, thus too is \( G_i^2 : V \to U \) and therefore the partial sums and the pointwise limit are as well. Moreover
\[
\left\| \sum_{i=1}^{\infty} G_i^2 \phi \right\|_U \leq \sum_{i=1}^{\infty} c_i^2 \| \phi \|_V
\]
as seen in (45) so the boundedness is also satisfied so we can understand the SPDE (44) in the same manner as (41). It also remains to be checked that the Stratonovich integral of (43) is well defined for \( \Psi \) a strong solution of (44). We show this in the sense of 2.1.3 for \( H = X \), the space in which the identity is satisfied. We show that the sequence of stopping times
\[ \tau_n(\omega) := n \wedge \inf \left\{ s \geq 0 : \int_0^s \| \Psi_r \|_V^2 \, dr \geq n \right\} \]
fits the requirements of 2.1.3, noting immediately that the sequence is a.s. monotone increasing and convergent to infinity as the process
\[ s \mapsto \int_0^s \| \Psi_r \|_V^2 \, dr \]
is continuous. Moreover the process \( \mathcal{G} \Psi^n := \mathcal{G}(\Psi \mathbb{1}_{\leq \tau_n}) = \mathcal{G} \Psi \mathbb{1}_{\leq \tau_n} \) (using linearity of \( \mathcal{G} \)) is progressively measurable in \( H \) (we remark again that this is really \( \mathcal{G} \Phi^n \) as stipulated but we identify the two) and satisfies the bound
\[ \mathbb{E} \int_0^t \sum_{i=1}^{\infty} \| G_i \Psi^n_r \|_H^2 \, dr \leq \sum_{i=1}^{\infty} c_i^2 \int_0^t \| \Psi^n_r \|_V^2 \, dr \leq \sum_{i=1}^{\infty} c_i^2 n < \infty \]
freely applying Tonelli’s Theorem between the expectation, integral and sum. Thus \( \mathcal{G} \Psi^n \in \mathcal{I}^H(\mathcal{W}) \) so the integral can be constructed in \( H \) and then embedded into \( X \), but we note that the embedding \( J : H \to X \) is a continuous linear operator and so from 1.6.12.1 then \( J(\mathcal{G} \Psi^n) \in \mathcal{I}^X(\mathcal{W}) \) and in particular
\[ J \left( \int_0^t \mathcal{G} \Psi^n_s \, d\mathcal{W}_s \right) = \int_0^t J(\mathcal{G} \Psi^n_s) \, d\mathcal{W}_s \]
so there is no ambiguity in how we understand the integral as an element of \( X \). Indeed we simply make the identification \( \mathcal{G} \Psi^n \) with \( J(\mathcal{G} \Psi^n) \) and will make no explicit reference to the embeddings.
in our analysis henceforth. As for showing that $\mathcal{G}_t \Psi^n \in \bar{\mathcal{M}}_c(X)$, we look at the evolution equation satisfied by $\Psi^n$ which is

$$\Psi^n_t = \Psi^n_0 + \int_0^t \left( Q + \frac{1}{2} \sum_{i=1}^{\infty} G^2_i \right) (\Psi^n_s) \mathbb{1}_{s \leq \tau_n} ds + \int_0^t G_i \Psi^n_s dW_s$$

a.s. in $U$, therefore from 1.6.12.1 we have that

$$\mathcal{G}_t \Psi^n_t = \mathcal{G}_t \Psi^n_0 + \int_0^t \left( \mathcal{G}_i \left( Q + \frac{1}{2} \sum_{i=1}^{\infty} G^2_i \right) \right) (\Psi^n_s) \mathbb{1}_{s \leq \tau_n} ds + \int_0^t \mathcal{G}_i \Psi^n_s dW_s$$

(46)
a.s. in $X$ ($\mathcal{G}_i : U \to X$ is bounded and linear). Of course the time integral is of finite variation in $X$ and from 1.6.8 we have the result. The last thing to prove here is that the infinite sum

$$\sum_{i=1}^{\infty} [\mathcal{G}_i \Psi^n, W^i]_t$$

converges in $L^2(\Omega; X)$. From the identity (46) and the definition of the cross-variation for the semi-martingale,

$$[\mathcal{G}_i \Psi^n, W^i]_t = \left[ \int_0^t \mathcal{G}_i \mathcal{G}_s \Psi^n_s dW_s, W^i \right]_t$$

and from the definition of the integral and 1.5.10,

$$\left[ \int_0^t \mathcal{G}_i \mathcal{G}_s \Psi^n_s dW_s, W^i \right]_t = \left[ \sum_{j=1}^{\infty} \int_0^t \mathcal{G}_i \mathcal{G}_j \Psi^n_s dW^j_s, W^i \right]_t = \lim_{n \to \infty} \left[ \sum_{j=1}^{n} \int_0^t \mathcal{G}_i \mathcal{G}_j \Psi^n_s dW^j_s, W^i \right]_t$$

where the limit is taken in $L^1(\Omega, X)$. Whilst the techniques are not proven here in the Hilbert Space setting, it follows in the same way from the real valued scenario that the cross-variation of independent processes is null and that

$$\left[ \int_0^t \mathcal{G}_i^2 \Psi^n_s dW^i_s, W^i \right]_t = \int_0^t \mathcal{G}_i^2 \Psi^n_s ds.$$

So from the independence of the Brownian Motions (and hence the stochastic integral integrated against $W^j$ is independent from $W^i$ for $i \neq j$), we have that

$$[\mathcal{G}_i \Psi^n, W^i]_t = \int_0^t \mathcal{G}_i^2 \Psi^n_s ds.$$

The convergence of the infinite sum in $L^2(\Omega, X)$ will follow from our standard Cauchy argument, though we have to work a little harder here. The Cauchy argument requires showing that

$$\mathbb{E} \left\| \sum_{i=m}^{k} \int_0^t \mathcal{G}_i^2 \Psi^n_s ds \right\|_{X}^2 \to 0$$

as $m, k \to \infty$ to which end we note that

$$\mathbb{E} \left\| \sum_{i=m}^{k} \int_0^t \mathcal{G}_i^2 \Psi^n_s ds \right\|_{X}^2 \leq \mathbb{E} \left( \sum_{i=m}^{k} \int_0^t \left\| \mathcal{G}_i^2 \Psi^n_s \right\|_{X} ds \right)^2$$
\[
\mathbb{E}\left(\sum_{i=1}^{\infty} \int_0^t \|G^2_i \Psi_s^n\|_X \, ds\right)^2 \leq \mathbb{E}\left(\sum_{i=1}^{\infty} c_i^2 \int_0^t \|\Psi_s^n\|_H \, ds\right)^2 \\
\leq \mathbb{E}\left(\sum_{i=1}^{\infty} c_i^2 \int_0^t c \|\Psi_s^n\|_V \, ds\right)^2 \\
\leq \left(\sum_{i=1}^{\infty} c_i^2 \right)^2 \mathbb{E}\left[ t \int_0^t \|\Psi_s^n\|_V^2 \, ds\right] \\
\leq \left(\sum_{i=1}^{\infty} c_i^2 \right)^2 tn
\]

where \(c\) is the constant from the embedding of \(V \hookrightarrow H\). Thus defining

\[Y_k := \mathbb{E}\left(\sum_{i=1}^{k} \int_0^t \|G^2_i \Psi_s^n\|_X \, ds\right)^2\]

then \((Y_k)\) is a real valued sequence monotone increasing and bounded above, hence it is convergent so Cauchy. But

\[\mathbb{E} \left\|\sum_{i=m}^{k} \int_0^t G^2_i \Psi_s^n \, ds \right\|_X^2 \leq \mathbb{E}\left(\sum_{i=m}^{k} \int_0^t \|G^2_i \Psi_s^n\|_X \, ds\right)^2 \leq Y_k - Y_{m-1}\]

where we’ve just said that this sequence \((Y_k)\) is Cauchy, proving the required convergence.

**Proof of 2.3.1:** We must show that

\[\int_0^t G[\Psi_s] \circ d\mathcal{W}_s = \int_0^t G[\Psi_s] \circ d\mathcal{W}_s + \frac{1}{2} \int_0^t \sum_{i=1}^{\infty} G^2_i \Psi_s \, ds\]

a.s. in \(X\) for all \(t \geq 0\). So working with fixed arbitrary \(t\) and \(\omega\), we choose any \(n\) such that \(\tau_n(\omega) \geq t\) and have that at this \(\omega\),

\[\int_0^t G[\Psi_s] \circ d\mathcal{W}_s = \sum_{i=1}^{\infty} \int_0^t G_i \Psi_s^n \, d\mathcal{W}_s + \frac{1}{2} \sum_{i=1}^{\infty} [G_i \Psi_s^n, W^i]_t\]

\[= \int_0^t G[\Psi_s] \circ d\mathcal{W}_s + \frac{1}{2} \sum_{i=1}^{\infty} [G_i \Psi_s^n, W^i]_t\]

where this limit is taken in \(L^2(\Omega; X)\). Of course we have just shown that

\[\sum_{i=1}^{\infty} [G_i \Psi_s^n, W^i]_t = \sum_{i=1}^{\infty} \int_0^t G_i^2 \Psi_s^n \, ds\]

is well defined in this topology, but we must show that it is equal to

\[\int_0^t \sum_{i=1}^{\infty} G_i^2 \Psi_s \, ds\]
evaluated at this fixed $\omega$, for the pointwise limit in $U$ as it was defined. Firstly note that
\[
\int_0^t \sum_{i=1}^{\infty} G_i^2 \Psi_s(\omega) ds = \int_0^t \sum_{i=1}^{\infty} G_i^2 \Psi_s^n(\omega) ds
\]
and by an application of the dominated convergence theorem with dominating function
\[
\sum_{i=1}^{\infty} c_i^2 \|\Psi^n\|_V
\]
we can rewrite
\[
\int_0^t \sum_{i=1}^{\infty} G_i^2 \Psi_s^n ds = \sum_{i=1}^{\infty} \int_0^t G_i^2 \Psi_s^n ds
\]
as a limit $a.s.$ in $U$, and thus $a.s.$ in $X$. However the convergence in $L^2(\Omega; X)$ implies that of a subsequence $a.s.$ in $X$, which clearly agrees with the limit of the whole sequence $a.s.$ in $X$ thus giving the result.

This theorem has been stated for the strong solution 2.2.4, though we note that all arguments follow in the corresponding local case by incorporating the stopping time as done in the justification that the integrals in 2.2.3 are well defined. The result is stated below.

**Corollary 2.3.1.1.** If $(\Psi, \tau)$ is a local strong solution of the equation (44) then $\Psi$ satisfies the identity

\[
\Psi_t = \Psi_0 + \int_0^{t\land \tau} Q\Psi_s ds + \int_0^{t\land \tau} G\Psi_s \circ dW_s
\]
$\mathbb{P}$ – a.s. in $X$ for all $t \geq 0$.

To conclude this discussion we should emphasise why we the identity (43) holds only in $X$ and not in $U$, the space in which (44) is satisfied. The evolution equation (46) allowed us to identify the semi-martingale structure in $X$ as is this where the integrals are constructed. We can, however, construct the stochastic integral in (46) in $U$, which allows us to conclude that the time integral is itself an element of $U$ (as $\mathcal{G}_t\Psi^n_t, \mathcal{G}_t\Psi^n_0$ are as well). Unfortunately we cannot say that this is actually an integral in $U$ (just an integral in $X$ which is in turn an element of $U$) so poignantly we cannot say that this is of finite variation in $U$, which would be necessary when considering the cross-variation $[\mathcal{G}_t\Psi^n, W^r]$ in $U$.

### 2.4 Time-Dependent Operators

We did not facilitate time dependence in the operators $Q, G$ as solely for the fact that if $G$ was time dependent then the conversion from Stratonovich to Itô Form would be much more troublesome. There is no real additional difficulty in establishing a framework for the Itô Form for time-dependent operators, so we briefly do so now. There is no longer a need for the space $X$ so we work with the triple

\[
V \hookrightarrow H \hookrightarrow U
\]

and now the SPDE

\[
\Psi_t = \Psi_0 + \int_0^t Q(s, \Psi_s) ds + \int_0^t G(s, \Psi_s) dW_s .
\]

We require the assumptions now as:
Assumption 2.4.1. For any $T > 0$, the operators $Q : [0, T] \times V \to U$ and $G : [0, T] \times V \to \mathcal{L}^2(\mathcal{U}; H)$ are measurable.

Assumption 2.4.2. There exists a $C : [0, \infty) \to \mathbb{R}$ bounded on $[0, T]$ for every $T$, and constants $c_i$ such that for every $\phi \in V$ and $t \in [0, \infty)$,

$$
\|Q(t, \phi)\|_U \leq C_t \tilde{K}(\phi) [1 + \|\phi\|^2_V]
$$

$$
\|G_i(t, \phi)\|^2_H \leq C_t c_i (1 + \|\phi\|^2_V)
$$

$$
\sum_{i=1}^\infty c_i < \infty
$$

Definitions of solutions in this framework, and a justification that the integrals are well-defined, then follows largely in the same way as for (41) so we omit the details here. What is slightly more delicate is the progressive measurability of the process $G(\cdot, \Phi \cdot) \mathds{1}_{\leq \tau}$ in $\mathcal{L}^2(\mathcal{U}; H)$. From the measurability of $G$ and the progressive measurability of $\Phi$ we have that for any fixed $t$, the mapping $G(\cdot, \Phi \cdot) : [0, t] \times [0, t] \times \Omega \to \mathcal{L}^2(\mathcal{U}; H)$ defined by

$$(s, r, \omega) \mapsto G(s, \Phi_r(\omega))$$

is $\mathcal{B}([0, t]) \times \mathcal{B}([0, t]) \times \mathcal{F}_t$ measurable, and hence the mapping

$$(s, \omega) \mapsto G(s, \Phi_s(\omega))$$

is $\mathcal{B}([0, t]) \times \mathcal{F}_t$ measurable as is the indicator up to the stopping time, so the product retains the required measurability. For completeness we define the notion of a strong solution here.

Definition 2.4.3. A process $\Psi$ such that for $\mathbb{P} - a.e.$ $\omega$, $\Psi \cdot (\omega) \in C([0, T]; H) \cap L^2([0, T]; V)$ for all $T > 0$ with $\Psi$ progressively measurable in $V$, is said to be a strong solution of the equation (47) if the identity (47) holds $\mathbb{P} - a.s.$ in $U$ for all $t \geq 0$.

2.5 An Energy Equality

Having established this framework we introduce techniques to facilitate our analysis in it. The Itô Formula is well regarded as one of the most useful tools in stochastic analysis, and we formulate the infinite dimensional version here through first looking at a specific energy equality. We shall introduce a new setting in which our established solution framework falls, with the understanding that we would like to apply this to solutions whilst also using the results to deduce the existence of solutions when they are not a priori known. The ideas of this subsection are just a mild extension of [13] Theorem 4.2.5.

To prove this energy equality we shall rely on looking at partitions in time over which some nice properties are satisfied, before taking the limit as the increments go to zero. Towards this we recall the following lemma from [13] Lemma 4.2.6.

Lemma 2.5.1. Let $X_1 \hookrightarrow X_2$ be two Banach Spaces with continuous embedding and suppose that for some $T > 0$ and stopping time $\tau$, $\Phi : \Omega \times [0, T] \to X_2$ is such that for $\mathbb{P} - a.e.$ $\omega$, $\Phi_\cdot(\omega) \in C([0, T]; X_2)$ and $\Phi \mathds{1}_{\leq \tau} \in L^2(\Omega \times [0, T]; X_1)$. Then for any $A \subset [0, T]$ with $\lambda(A) = 0$ there exists a sequence of partitions $(I_i)$ such that
1. $I_l \cap A = \emptyset$;
2. $I_l \subset I_{l+1}$;
3. $I_l := \left\{ 0 = t^l_0 < t^l_1 < \cdots < t^l_k = T \right\}$, $\max_j |t^l_j - t^l_{j-1}| \to 0$ as $l \to \infty$;
4. For $\mathbb{P} - \text{a.e.} \, \omega$ and every $t^l_j$ with $1 \leq j \leq k-1$, $\Phi_{t^l_j} (\omega) 1_{t^l_j \leq \tau (\omega)} \in \mathcal{X}_1$;
5. The processes $\tilde{\Phi}^l, \tilde{\Phi}^l$ defined at each $t \in [0, T]$ and $\omega \in \Omega$ by

$$
\tilde{\Phi}^l (\omega) := \sum_{j=2}^{k_l} 1_{[t^l_{j-1}, t^l_j)} (t) \Phi_{t^l_{j-1}} (\omega) 1_{t^l_{j-1} \leq \tau (\omega)}, \quad \tilde{\Phi}^l := \sum_{j=1}^{k_l-1} 1_{[t^l_{j-1}, t^l_j]} (t) \Phi_{t^l_j} (\omega) 1_{t^l_j \leq \tau (\omega)}
$$

belong to $L^2 (\Omega \times [0, T]; \mathcal{X}_1)$ and both converge to $\Phi 1_{\leq \tau}$ in this space.

Before moving on we take a moment to dissect this result. In the statement of [13] there is no continuity assumption on $\Phi$ and indeed this is superfluous to requirement, however we want to make explicit that is genuinely the process $\Phi$ taken in item 5 and not some other representative of an equivalence class. The assumptions are of course reminiscent of 2.2.3 and just as was stressed there the progressively measurable process in $V$ was not necessarily the continuous process itself but just a $\mathbb{P} \times \lambda - \text{a.s.}$ equivalent representation, we are reminded again that elements of $L^2 (\Omega \times [0, T]; \mathcal{X}_1)$ are only an equivalence class of $\mathbb{P} \times \lambda - \text{a.s.}$ equal functions so the fact that we can fix the representation $\Phi$ in $\tilde{\Phi}^l, \tilde{\Phi}^l$ is significant.

We now fix a framework in which we conduct the analysis of this subsection. We work with a triple of embedded Hilbert Spaces

$$
V \hookrightarrow H \hookrightarrow U
$$

where the embeddings are continuous, $V$ is assumed dense in $H$, and there exists a continuous bilinear form $\langle \cdot, \cdot \rangle_{U \times V} : U \times V \to \mathbb{R}$ such that for every $\phi \in H, \psi \in V$,

$$
\langle \phi, \psi \rangle_{U \times V} = \langle \phi, \psi \rangle_H.
$$

We suppose that for some $T > 0$ and stopping time $\tau$:

1. $\Psi_0 \in L^2 (\Omega; H)$ is $\mathcal{F}_0-$measurable;
2. $\eta \in L^2 (\Omega; L^2 ([0, T]; U))$;
3. $B \in \mathcal{I}^H (\mathcal{W})$;
4. $\Psi.1_{\leq \tau} \in L^2 (\Omega; L^2 ([0, T]; V))$ and is progressively measurable in $V$;
5. The identity

$$
\Psi_t = \Psi_0 + \int_0^{t \wedge \tau} \eta_s ds + \int_0^{t \wedge \tau} B_s dW_s \quad (48)
$$

holds $\mathbb{P} - \text{a.s.}$ in $U$ for all $t \in [0, T]$.

**Remark 3.** Once more, it is clear from assumption 5 that for $\mathbb{P} - \text{a.e.} \, \omega$, $\Psi (\omega) \in C ([0, T]; U)$ but the progressively measurable process assumed in item 4 is only equivalent to this $\Psi.1_{\leq \tau} \mathbb{P} \times \lambda - \text{a.s.}$. It is a slight abuse of notation commonplace in the literature and we shall follow suit without excessive clarification at each use.
With this structure in place, we first look to deduce some improved regularity on $\Psi$. For this we fix an application of 2.5.1 relative to the assumptions laid out above. We take $T$ and $\tau$ as in the assumptions, $X_1 = V$, $X_2 = U$, $\Phi = \Psi$. From item 4 and the continuous embedding of $V$ into $H$ we have that for $\lambda - a.e. \ t \in [0, T]$, $E (\| \Psi_t \|^2_H) < \infty$ and we take $A$ to be the $\lambda$–zero set on which this does not hold. Then $\Psi^i_t, \Psi^i, I^i$ are defined as in 2.5.1 and we define

$$I := \bigcup_{t \in \mathbb{N}} I^i.$$

**Lemma 2.5.2.** We have that

$$E \left( \sup_{t \in [0,T]} \| \Psi_t \|^2_H \right) < \infty.$$

**Proof.** For $\mathbb{P} - a.e. \ \omega$ and every $s < t$ with $t \in I \cap [0, \tau(\omega)]$ and $s \in I \cup \{0\}$, observe that

$$\left\| \int_s^t B_r dW_r \right\|^2_H - \left\| \Psi_t - \Psi_s - \int_s^t B_r dW_r \right\|^2_H + 2 \left\langle \Psi_t, \int_s^t B_r dW_r \right\rangle_H = \left\| \int_s^t B_r dW_r \right\|^2_H - \left\| \Psi_t - \Psi_s \right\|^2_H - \left\| \Psi_t - \Psi_s \right\|^2_H + 2 \left\langle \Psi_t, \Psi_s \right\rangle_H$$

$$= 2 \left\langle \Psi_t, \int_s^t B_r dW_r \right\rangle_H - \left\| \Psi_t - \Psi_s \right\|^2_H$$

$$= 2 \left\langle \Psi_s, \Psi_t - \Psi_s - \int_s^t \eta_r dr \right\rangle_H - \left\| \Psi_t \right\|^2_H + \left\| \Psi_s \right\|^2_H + 2 \left\langle \Psi_t, \Psi_s \right\rangle_H$$

$$= \left\| \Psi_t \right\|^2_H - \left\| \Psi_s \right\|^2_H - 2 \left\langle \Psi_t, \int_s^t \eta_r dr \right\rangle_H$$

which we rewrite as the equality

$$\left\| \Psi_t \right\|^2_H - \left\| \Psi_s \right\|^2_H = 2 \int_s^t \langle \eta_r, \Psi_t \rangle_{U \times V} dr + 2 \left\langle \Psi_s, \int_s^t B_r dW_r \right\rangle_H + \left\| \int_s^t B_r dW_r \right\|^2_H - \left\| \Psi_t - \Psi_s - \int_s^t B_r dW_r \right\|^2_H.$$


Using this equality, for any \( l \in \mathbb{N} \) and \( t = t^l_i \in I_l \cap (0, \tau(\omega)]/\{T\}, \)

\[
\| \Psi_t \|_H^2 - \| \Psi_0 \|_H^2 = \sum_{j=0}^{i-1} \left( \| \Psi_{t_j} \|_H^2 - \| \Psi_{t_j} \|_H^2 \right)
\]

\[
= \sum_{j=0}^{i-1} \left( 2 \int_{t_j}^{t_j+1} \left\langle \eta_r, \Psi_{t_j} \right\rangle_{U \times V} dr + 2 \left\langle \Psi_{t_j}, \int_{t_j}^{t_j+1} B_r dW_r \right\rangle_H + \left\| \int_{t_j}^{t_j+1} B_r dW_r \right\|_H^2 - \left\| \Psi_{t_j+1} - \Psi_{t_j} - \int_{t_j}^{t_j+1} B_r dW_r \right\|_H^2 \right)
\]

\[
= 2 \sum_{j=0}^{i-1} \left( \int_{t_j}^{t_j+1} \left\langle \eta_r, \Psi_{t_j} \right\rangle_{U \times V} dr + \int_{t_j}^{t_j+1} \left\langle B_r, \Psi_{t_j} \right\rangle_H dW_r \right) + 2 \int_{t_j}^{t_j+1} \left\| B_r dW_r \right\|_H^2 - \left\| \Psi_{t_j+1} - \Psi_{t_j} - \int_{t_j}^{t_j+1} B_r dW_r \right\|_H^2 \right)
\]

\[
= \sum_{j=0}^{i-1} \left( \int_{t_j}^{t_j+1} \left\langle \eta_r, \tilde{\Psi}_{t_j} \right\rangle_{U \times V} dr + \int_{t_j}^{t_j+1} \left\langle B_r, \tilde{\Psi}_{t_j} \right\rangle_H dW_r + \int_{t_j}^{t_j+1} \left\langle B_r, \Psi_{t_j} \right\rangle_H dW_r + \int_{t_j}^{t_j+1} \left\langle B_r, \Psi_{t_j} \right\rangle_H dW_r \right) + \sum_{j=0}^{i-1} \left( \int_{t_j}^{t_j+1} \left\| B_r dW_r \right\|_H^2 - \left\| \Psi_{t_j+1} - \Psi_{t_j} - \int_{t_j}^{t_j+1} B_r dW_r \right\|_H^2 \right)
\]

where we have applied 1.6.13 and the associated remark thereafter. In particular we have that

\[
\| \Psi_t \|_H^2 \leq \| \Psi_0 \|_H^2 + 2 \int_0^t \left\langle \eta_r, \tilde{\Psi}_r \right\rangle_{U \times V} dr + 2 \int_0^t \left\langle B_r, \tilde{\Psi}_r \right\rangle_H dW_r + 2 \int_0^t \left\langle B_r, \Psi_0 \right\rangle_H dW_r + \sum_{j=0}^{i-1} \left( \int_{t_j}^{t_j+1} \left\| B_r dW_r \right\|_H^2 - \left\| \Psi_{t_j+1} - \Psi_{t_j} - \int_{t_j}^{t_j+1} B_r dW_r \right\|_H^2 \right).
\]

Our goal is to show that

\[
\mathbb{E} \left( \sup_{t \in I \cap (0, \tau]/\{T\}} \| \Psi_t \|_H^2 \right) \leq c
\]
for some constant $c$ independent of $l$. To this end, observe that

$$
\mathbb{E} \left( \sup_{t \in I_l \cap (0,\tau) \cap \{T\}} \| \Psi_t \|^2_H \right) \leq \mathbb{E} \left( \| \Psi_0 \|^2_H \right)
+ 2\mathbb{E} \left( \int_0^{T \wedge \tau} \left| \langle \eta_r, \hat{\Psi}_r^l \rangle_{U \times V} \right| \, dr \right) + 2\mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^{t \wedge \tau} \langle B_r, \hat{\Psi}_r^l \rangle_H \, dW_r \right| \right)
+ 2\mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^{t \wedge \tau} \langle B_r, \Psi_0 \rangle_H \, dW_r \right| \right)
+ \mathbb{E} \left[ \sup_{t \in I_l \cap (0,\tau) \cap \{T\}} \sum_{j=0}^{i-1} \left( \| \int_{t_j}^{t_{j+1}} B_r \, dW_r \|^2_H \right) \right].
$$

We shall treat each term individually. Firstly we have that

$$
2\mathbb{E} \left( \int_0^{T \wedge \tau} \left| \langle \eta_r, \hat{\Psi}_r^l \rangle_{U \times V} \right| \, dr \right) \leq \mathbb{E} \left( \int_0^{T \wedge \tau} \| \eta_r \|^2_U + \| \hat{\Psi}_r^l \|^2_V \, dr \right)
\leq \mathbb{E} \left( \int_0^{T \wedge \tau} \| \eta_r \|^2 \, dr \right) + \max_{m \leq L} \mathbb{E} \left( \int_0^{T \wedge \tau} \| \hat{\Psi}_r^m \|^2_V \, dr \right) + 1
$$

where $L$ is taken sufficiently large so that for all $m \geq L,$

$$
\mathbb{E} \left( \int_0^{T \wedge \tau} \left( \| \hat{\Psi}_r^m \|^2_V - \| \hat{\Psi}_r^l \|^2_V \right) \, dr \right) \leq \frac{1}{2}.
$$

For the first stochastic integral we apply 1.6.9, seeing that

$$
2\mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^{t \wedge \tau} \langle B_r, \hat{\Psi}_r^l \rangle_H \, dW_r \right| \right) \leq c\mathbb{E} \left( \int_0^{T \wedge \tau} \left( \| B_r \|^2_{L^2(U;H)} \right) \left( \| \hat{\Psi}_r^l \|^2_H \right) \, dr \right)^{\frac{1}{2}}
\leq c\mathbb{E} \left( \int_0^{T \wedge \tau} \| B_r \|^2_{L^2(U;H)} \left( \| \hat{\Psi}_r^l \|^2_H \right) \, dr \right)^{\frac{1}{2}}
\leq c\mathbb{E} \left( \sup_{r \in [0,T \wedge \tau]} \left( \| \hat{\Psi}_r^l \|^2_H \right) \int_0^{T \wedge \tau} \| B_r \|^2_{L^2(U;H)} \, dr \right)^{\frac{1}{2}}
= c\mathbb{E} \left( \sup_{r \in [0,T \wedge \tau]} \left( \| \hat{\Psi}_r^l \|^2_H \right) \right)^{\frac{1}{2}} \left( \int_0^{T \wedge \tau} \| B_r \|^2_{L^2(U;H)} \, dr \right)^{\frac{1}{2}}
= \frac{1}{2} \mathbb{E} \left( \sup_{r \in [0,T \wedge \tau]} \left( \| \hat{\Psi}_r^l \|^2_H \right) \right) + c\mathbb{E} \left( \int_0^{T \wedge \tau} \| B_r \|^2_{L^2(U;H)} \, dr \right)
= \frac{1}{2} \mathbb{E} \left( \sup_{t \in I_l \cap (0,\tau) \cap \{T\}} \| \Psi_t \|^2_H \right) + c\mathbb{E} \left( \int_0^{T \wedge \tau} \| B_r \|^2_{L^2(U;H)} \, dr \right)
$$

where $c$ here is a generic constant changing from line to line, coming initially from the constant in
which is independent of \( l \). Putting this together we now see that

\[
\frac{1}{2} \mathbb{E} \left( \sup_{t \in I_l \cap (0,T) \setminus \{ T \}} \| \Psi_t \|_H^2 \right) \leq \mathbb{E} \left( \| \Psi_0 \|_H^2 \right) + \mathbb{E} \left( \int_0^{T \wedge \tau} \| \eta_r \|_U^2 dr \right) + \max_{m \leq L} \mathbb{E} \left( \int_0^{T \wedge \tau} \| \tilde{\Psi}_m \|_U^2 dr \right) + 1 + c \mathbb{E} \left( \int_0^{T \wedge \tau} \| B_r \|_{\mathcal{L}(U;H)}^2 dr \right)
\]

\[
+ 2 \mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^t \langle B_r, \Psi_0 \rangle_H dW_r \right| \right)
\]

\[
+ 2 \mathbb{E} \left[ \sup_{t \in I_l \cap (0,T) \setminus \{ T \}} \sum_{j=0}^{i-1} \left( \left\| \int_{t_j}^{t_{j+1}} B_r dW_r \right\|_H^2 \right) \right]
\]

For the stochastic integral involving the initial condition we can treat this identically to generate the bound

\[
2 \mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^{T \wedge \tau} \langle B_r, \Psi_0 \rangle_H dW_r \right| \right) \leq \mathbb{E} \left( \| \Psi_0 \|_H^2 \right) + c \mathbb{E} \left( \int_0^{T \wedge \tau} \| B_r \|_{\mathcal{L}(U;H)}^2 dr \right).
\]

As for the final term, it is clear that the supremum over all partitions can be bounded by taking the partition for \( t = t_{k_l} = T \). Thus

\[
\mathbb{E} \left[ \sup_{t \in I_l \cap (0,T) \setminus \{ T \}} \sum_{j=0}^{i-1} \left( \left\| \int_{t_j}^{t_{j+1}} B_r dW_r \right\|_H^2 \right) \right] \leq \mathbb{E} \left[ \sum_{j=0}^{k_l-1} \left( \left\| \int_{t_j}^{t_{j+1}} B_r dW_r \right\|_H^2 \right) \right]
\]

\[
\leq \sum_{j=0}^{k_l-1} \mathbb{E} \left( \left\| \int_{t_j}^{t_{j+1}} B_r dW_r \right\|_H^2 \right)
\]

\[
= \sum_{j=0}^{k_l-1} \mathbb{E} \left( \left\| \int_{t_j}^{t_{j+1}} B_r \right\|_{\mathcal{L}(U;H)}^2 dr \right)
\]

\[
= \mathbb{E} \left( \int_0^T \| B_r \|_{\mathcal{L}(U;H)}^2 dr \right)
\]

having applied 1.6.4. In total then we have that

\[
\mathbb{E} \left( \sup_{t \in I_l \cap (0,T) \setminus \{ T \}} \| \Psi_t \|_H^2 \right) \leq 4 \mathbb{E} \left( \| \Psi_0 \|_H^2 \right) + 2 \mathbb{E} \left( \int_0^{T \wedge \tau} \| \eta_r \|_U^2 dr \right) + 2 \max_{m \leq L} \mathbb{E} \left( \int_0^{T \wedge \tau} \| \tilde{\Psi}_m \|_U^2 dr \right) + 1 + c \mathbb{E} \left( \int_0^{T \wedge \tau} \| B_r \|_{\mathcal{L}(U;H)}^2 dr \right)
\]

which is a finite bound independent of \( l \). As \( I_l \subset I_{l+1} \) then the sequence

\[
\sup_{t \in I_l \cap (0,T) \setminus \{ T \}} \| \Psi_t \|_H^2
\]

is \( \mathbb{P} \) – a.s. monotone increasing in \( l \), so we can apply the Monotone Convergence Theorem to see that

\[
\mathbb{E} \left( \sup_{t \in I \cap (0,T) \setminus \{ T \}} \| \Psi_t \|_H^2 \right) = \lim_{l \to \infty} \mathbb{E} \left( \sup_{t \in I_{l} \cap (0,T) \setminus \{ T \}} \| \Psi_t \|_H^2 \right) < \infty.
\]
Thus for $\mathbb{P} - a.e. \omega$, we have that
\[
\sup_{t \in I \cap (0, \tau(\omega))/\{T\}} \|\Psi_t(\omega)\|_H^2 = \tilde{c} < \infty
\]
and $\Psi(\omega) \in C([0, T]; U)$. We fix such an $\omega$ and any $t \in [0, \tau(\omega)]$. As the mesh of the partitions go to zero then there is a sequence of times $(t_n)$ in $I \cap (0, \tau(\omega))/\{T\}$ such that $t_n \to t$. The sequence $(\Psi_{t_n}(\omega))$ is uniformly bounded in $H$ so admits a weakly convergent subsequence in this space, to a limit which we call $\psi$. From the continuous embedding of $H$ into $U$ this weak convergence also holds in $U$, but from the continuity of $\Psi(\omega)$ in $U$ we have that $(\Psi_{t_n}(\omega))$ converges strongly and therefore weakly to $\Psi_t(\omega)$ in $U$. By the uniqueness of limits in the weak topology, we conclude that $\Psi_t(\omega) = \psi$ and thus belongs to $H$. Moreover the weak limit preserves the boundedness in $H$, so $\|\Psi_t(\omega)\|_H \leq \tilde{c}$. Therefore
\[
\sup_{t \in I \cap (0, \tau(\omega))/\{T\}} \|\Psi_t(\omega)\|_H^2 = \sup_{t \in [0, \tau(\omega)]} \|\Psi_t(\omega)\|_H^2
\]
for our fixed $\omega$ in a full measure set, and thus $\mathbb{P} - a.s.$ We also know that $\Psi = \Psi_{\lambda, \tau}$ from the identity (48), so this equality extends to
\[
\sup_{t \in I \cap (0, \tau(\omega))/\{T\}} \|\Psi_t(\omega)\|_H^2 = \sup_{t \in [0, T]} \|\Psi_t(\omega)\|_H^2.
\]
Combining this with (49) concludes the proof. 

Having now justified that for $\mathbb{P} - a.e. \omega \Psi_t(\omega) \in H$, we move on to prove weak continuity in this space.

**Lemma 2.5.3.** For $\mathbb{P} - a.e. \omega$, $\Psi_t(\omega)$ is weakly continuous in $H$.

**Proof.** We fix $t \in [0, T]$ but now take any sequence of times $(t_n)$ such that $t_n \to t$. For $\mathbb{P} - a.e. \omega$ and any given $\phi \in H$ we must justify that
\[
\lim_{n \to \infty} \langle \Psi_{t_n}(\omega) - \Psi_t(\omega), \phi \rangle_H = 0.
\]
(50)
For any $\varepsilon > 0$, from the density of $V$ in $H$ there exists $\phi^k \in V$ such that
\[
\|\phi - \phi^k\|_H < \frac{\varepsilon}{4\|\Psi(\omega)\|_{L^\infty([0, T]; H)}}
\]
and then from the continuity in $U$ there exists an $N \in \mathbb{N}$ sufficiently large such that for all $n \geq N,
\[
\|\Psi_{t_n}(\omega) - \Psi_t(\omega)\|_U < \frac{\varepsilon}{2\|\phi^k\|_U}.
\]
Putting this together,
\[
\|\Psi_{t_n}(\omega) - \Psi_{t_n}(\omega)\|_H \leq \|\Psi_{t_n}(\omega) - \Psi_{t_n}(\omega), \phi - \phi^k\|_H + \|\Psi_{t_n}(\omega) - \Psi_{t_n}(\omega), \phi^k\|_H
\]
\[
\leq 2\|\Psi(\omega)\|_{L^\infty([0, T]; H)}\|\phi - \phi^k\|_H + \|\Psi_{t_n}(\omega) - \Psi_{t_n}(\omega), \phi^k\|_U \|\phi^k\|_U
\]
\[
< \varepsilon
\]
as required.

\[\square\]
Proposition 2.5.4. The equality
\[ \| \Psi_t \|_{H}^2 = \| \Psi_0 \|_{H}^2 + \int_0^{t \wedge \tau} \left( 2\langle \eta_s, \Psi_s \rangle_{U \times V} + \| B_s \|_{L^2(U; H)}^2 \right) ds + 2 \int_0^{t \wedge \tau} \langle B_s, \Psi_s \rangle_{H} dW_s \] (51)
holds for any \( t \in [0, T] \), \( \mathbb{P} \)-a.s. in \( \mathbb{R} \). Moreover for \( \mathbb{P} \)-a.e. \( \omega \), \( \Psi(\omega) \in C([0, T]; H) \).

Proof. The proof now directly follows from claim c onwards in the proof of Theorem 4.2.5 of [13]. \( \square \)

We wish to relax the integrability constraints over \( \Omega \), in accordance with 2.2.3. In the same setting of the Hilbert Spaces \( V, H, U \), we impose the new assumptions for some \( T > 0 \) and stopping time \( \tau \):

1. \( \Psi_0 : \Omega \rightarrow H \) is \( \mathcal{F}_0 \)-measurable;
2. For \( \mathbb{P} \)-a.e. \( \omega \), \( \eta(\omega) \in L^2([0, T]; U) \);
3. \( B \in \mathcal{I}^H(W) \);
4. For \( \mathbb{P} \)-a.e. \( \omega \), \( \Psi(\omega) \mathbb{1}_{\leq \tau(\omega)} \in L^2([0, T]; V) \) and \( \mathbb{1}_{\leq \tau} \) is progressively measurable in \( V \);
5. The identity
\[ \Psi_t = \Psi_0 + \int_0^{t \wedge \tau} \eta_s ds + \int_0^{t \wedge \tau} B_s dW_s \] (52)
holds \( \mathbb{P} \)-a.s. in \( U \) for all \( t \in [0, T] \).

We restate 2.5.4 for the new setting.

Proposition 2.5.5. The equality
\[ \| \Psi_t \|_{H}^2 = \| \Psi_0 \|_{H}^2 + \int_0^{t \wedge \tau} \left( 2\langle \eta_s, \Psi_s \rangle_{U \times V} + \| B_s \|_{L^2(U; H)}^2 \right) ds + 2 \int_0^{t \wedge \tau} \langle B_s, \Psi_s \rangle_{H} dW_s \] (53)
holds for any \( t \in [0, T] \), \( \mathbb{P} \)-a.s. in \( \mathbb{R} \). Moreover for \( \mathbb{P} \)-a.e. \( \omega \), \( \Psi(\omega) \in C([0, T]; H) \).

Proof. The idea is simply to apply 2.5.4 for some truncated versions of the processes. We consider the stopping times
\[ \tau_n^1 := n \wedge \inf \{ 0 \leq t < \infty : \int_0^t \| \eta_s \|_{U}^2 ds \geq n \} \]
\[ \tau_n^2 := n \wedge \inf \{ 0 \leq t < \infty : \int_0^t \| B_s \|_{L^2(U; H)}^2 ds \geq n \} \]
\[ \tau_n^3 := n \wedge \inf \{ 0 \leq t < \infty : \int_0^t \| \Psi_s \mathbb{1}_{\leq \tau} \|_{V}^2 ds \geq n \} \]
\[ \tau_n := \tau_n^1 \wedge \tau_n^2 \wedge \tau_n^3 \]

Then for every \( n \), we have that
\[ \Psi_0 \mathbb{1}_{\| \Psi_0 \| \leq n}, \quad \eta_s \mathbb{1}_{\leq \tau_n} \mathbb{1}_{\| \Psi_0 \| \leq n}, \quad B_s \mathbb{1}_{\leq \tau_n} \mathbb{1}_{\| \Psi_0 \| \leq n}, \quad \Psi_s \mathbb{1}_{\leq \tau_n} \mathbb{1}_{\| \Psi_0 \| \leq n} \]
satisfy the previous assumptions of 1, 2, 3, 4. Moreover from (52) we have that for any \( t \in [0, T] \)
\[
\Psi_{t \land \tau_n} = \Psi_0 + \int_0^{t \land \tau_n} \eta_s ds + \int_0^{t \land \tau_n} B_s dW_s
\]
\[
= \Psi_0 + \int_0^{t \land \tau_n} \eta_s 1_{s \leq \tau_n} ds + \int_0^{t \land \tau_n} B_s 1_{s \leq \tau_n} dW_s
\]
\[\mathbb{P} - \text{a.s.} \] and moreover
\[
\Psi_{t \land \tau_n} 1_{\|\Psi_0\|_H \leq n} = \Psi_0 1_{\|\Psi_0\|_H \leq n} + 1_{\|\Psi_0\|_H \leq n} \int_0^{t \land \tau_n} \eta_s 1_{s \leq \tau_n} ds + 1_{\|\Psi_0\|_H \leq n} \int_0^{t \land \tau_n} B_s 1_{s \leq \tau_n} dW_s
\]
having applied 1.6.14. Therefore we can apply 2.5.4 to see that the equality
\[
\left\| \Psi_{t \land \tau_n} 1_{\|\Psi_0\|_H \leq n} \right\|^2_H = \left\| \Psi_0 1_{\|\Psi_0\|_H \leq n} \right\|^2_H + \int_0^{t \land \tau_n} 2\langle \eta_s 1_{s \leq \tau_n}, \Psi_{s \land \tau_n} 1_{\|\Psi_0\|_H \leq n} \rangle_{U \times V} ds
\]
\[
+ \int_0^{t \land \tau_n} \| B_s 1_{s \leq \tau_n} \|_{\|\Psi_0\|_H \leq n}^2 dW_s
\]
\[
+ 2 \int_0^{t \land \tau_n} \langle B_s 1_{s \leq \tau_n}, \Psi_{s \land \tau_n} 1_{\|\Psi_0\|_H \leq n} \rangle_{H} dW_s
\]
holds for all \( t \in [0, T] \) \( \mathbb{P} - \text{a.s.} \). We rewrite this as
\[
1_{\|\Psi_0\|_H \leq n} \left\| \Psi_{t \land \tau_n} \right\|^2_H = 1_{\|\Psi_0\|_H \leq n} \left\| \Psi_0 \right\|^2_H + 1_{\|\Psi_0\|_H \leq n} \int_0^{t \land \tau_n} 2\langle \eta_s, \Psi_s \rangle_{U \times V} ds
\]
\[
+ 1_{\|\Psi_0\|_H \leq n} \int_0^{t \land \tau_n} \| B_s \|_{\mathcal{L}^2(U; H)}^2 ds
\]
\[
+ 1_{\|\Psi_0\|_H \leq n} 2 \int_0^{t \land \tau_n} \langle B_s, \Psi_s \rangle_{H} dW_s.
\]
Therefore for any \( t \in [0, T] \), \( \mathbb{P} - \text{a.e.} \) \( \omega \) with \( n \) sufficiently large so that \( \left\| \Psi_0(\omega) \right\|_H \leq n \) and \( t \leq \tau_n(\omega) \), the identity (52) holds. We can always find such a large enough \( n \), which completes the justification of this identity. The continuity then follows identically as we have again from 2.5.4 that for every \( n \) and \( \mathbb{P} - \text{a.e.} \) \( \omega \), \( \Psi_{t \land \tau_n}(\omega) \in C([0, T]; H) \), and we are done. \( \square \)

2.6 The General Itô Formula

The key consideration is the notion of derivatives, for which we use the standard functional analytic choice of the Fréchet Derivative. For a Fréchet differentiable function between Banach Spaces \( \mathcal{K} : \mathcal{Y} \to \mathcal{Z} \), recall that its derivative is defined as a mapping \( \mathcal{K}' : \mathcal{Y} \to \mathcal{L}(\mathcal{Y}; \mathcal{Z}) \) and subsequently its second derivative as \( \mathcal{K}'' : \mathcal{Y} \to \mathcal{L}(\mathcal{Y}; \mathcal{L}(\mathcal{Y}; \mathcal{Z})) \). In the case where \( \mathcal{Y} = \mathcal{H} \) is a Hilbert Space and \( \mathcal{Z} = \mathbb{R} \), then
\[
\mathcal{K}' : \mathcal{H} \to \mathcal{H}^*
\]
\[
\mathcal{K}'' : \mathcal{H} \to \mathcal{L}(\mathcal{H}; \mathcal{H}^*)
\]
so it is commonplace to apply the Riesz Representation and make the identification
\[
\mathcal{K}' : \mathcal{H} \to \mathcal{H}
\]
\[
\mathcal{K}'' : \mathcal{H} \to \mathcal{L}(\mathcal{H}).
\] (54)
(55)
We will be considering functions $F : [0, \infty) \times U \to \mathbb{R}$ with partial derivatives $F_t, F_x$ and $F_{xx}$ where the latter two are understood in the sense of (54) and (55).

**Theorem 2.6.1.** Suppose $(\Psi, \tau)$ is a local strong solution of (47), and $F : [0, \infty) \times U \to \mathbb{R}$ is such that $F_t, F_x$ and $F_{xx}$ are uniformly continuous on bounded subsets of $[0, T] \times U$. Then $F(\cdot, \Psi)$ satisfies the identity

$$F(t, \Psi_t) = F(0, \Psi_0) + \int_0^{t \wedge \tau} \left( F_s(s, \Psi_s) + \langle F_x(s, \Psi_s), Q(s, \Psi_s) \rangle_U \right) ds + \frac{1}{2} \int_0^{t \wedge \tau} \text{Tr} \left( F_{xx}(s, \Psi_s)(G(s, \Psi_s))(G(s, \Psi_s))^* \right) ds + \int_0^{t \wedge \tau} \langle F_x(s, \Psi_s), G(s, \Psi_s) \rangle_U dW_s$$

a.e. in $\mathbb{R}$ for every $t \in [0, T]$, where the trace term is simply a composition of operators and the stochastic integral is understood component wise similarly to 1.6.12.

One particularly important application of this Itô Formula is the so called energy equality, through the function $F : h \mapsto \|h\|^2_U$. The result is as follows:

**Proposition 2.6.2.** Suppose $(\Psi, \tau)$ is a local strong solution of (47). Then $\Psi$ satisfies the energy equality

$$\|\Psi_t\|^2_U = \|\Psi_0\|^2_U + \int_0^{t \wedge \tau} \left( 2\langle \Psi_s, Q(s, \Psi_s) \rangle_U + \|G(s, \Psi_s)\|_{L_2(U, U)}^2 \right) ds + 2 \int_0^{t \wedge \tau} \langle \Psi_s, G(s, \Psi_s) \rangle_U dW_s$$

a.e. in $\mathbb{R}$ for all $t \geq 0$.

We do not prove these results here, but instead refer to [7] Theorem 4.18 and related discussions therein. Also shown there is the Itô Formula for processes defined by a more general evolution equation in a Hilbert Space $\mathcal{H}$,

$$\Phi_t = \Phi_0 + \int_0^t \phi_s ds + \int_0^t B(s) dW_s \quad (56)$$

for any given $\Phi_0 : \Omega \to \mathcal{H}$ $\mathcal{F}_0-$measurable, $\phi : \Omega \times [0, t] \to \mathcal{H}$ progressively measurable and a.s. Bochner Integrable and $B \in \mathcal{L}^2(\mathcal{W})$. Then we have the corresponding result to 2.6.1.

**Theorem 2.6.3.** Suppose that $F$ is as in 2.6.1 for $\mathcal{H}$ replacing $U$ and $\Phi$ is defined by (56). Then $F(\cdot, \Phi)$ satisfies the identity

$$F(t, \Phi_t) = F(0, \Phi_0) + \int_0^{t \wedge \tau} \left( F_s(s, \Phi_s) + \langle F_x(s, \Phi_s), \Phi_s \rangle_{\mathcal{H}} \right) ds + \frac{1}{2} \int_0^{t \wedge \tau} \text{Tr} \left( F_{xx}(s, \Phi_s)(B(s))(B(s))^* \right) ds + \int_0^{t \wedge \tau} \langle F_x(s, \Phi_s), B(s) \rangle_{\mathcal{H}} dW_s$$

a.e. in $\mathbb{R}$ for every $t \in [0, T]$. 
2.7 The Case of Constant Multiplicative Noise

Many techniques in proving existence and uniqueness of an SPDE in this framework rely on simplifying the equation to one where we can apply the standard theory, and then constructing solutions in the original framework via some appropriate limit of solutions to the simplified equations. As such we shall briefly considered a special type of equation in this framework which reduces the driving noise from something infinite dimensional to one dimensional. For this we work again with an arbitrary Hilbert Space \( \mathcal{H} \).

**Proposition 2.7.1.** Suppose that \( \Psi \in \mathbb{I}^H \) and that the operator \( G : \mathfrak{A} \times \mathcal{H} \to \mathcal{H} \) is such that

\[ G_i : \phi \mapsto \lambda_i \phi \]

for each \( i \) with \( \lambda_i \in \mathbb{R} \), and

\[ \sum_{i=1}^{\infty} \lambda_i^2 < \infty. \]  

(57)

Then \( G \Psi \in \mathbb{I}^H(\mathcal{W}) \) and

\[ \int_0^t G \Psi_s d\mathcal{W}_s = \left( \sum_{i=1}^{\infty} \lambda_i^2 \right)^{1/2} \int_0^t \Psi_s d\mathcal{W}_s \]

(58)

where \( \mathcal{W} \) is a standard one dimensional Brownian Motion.

In order to prove the above, we use an intermediary lemma.

**Lemma 2.7.2.** In the setting of 2.7.1 the infinite sum

\[ M_s := \sum_{i=1}^{\infty} \lambda_i W^i_s \]  

(59)

is convergent in \( L^2(\Omega; \mathbb{R}) \) at every \( s \), and the limiting martingale has the representation

\[ M = \left( \sum_{i=1}^{\infty} \lambda_i^2 \right)^{1/2} W \]

(60)

where \( W \) is a standard Brownian Motion.

**Proof of 2.7.2.** Firstly let’s verify that the convergence in (59) does indeed hold, which is immediate from observing that

\[ \sum_{i=1}^{\infty} \lambda_i W^i_s = \sum_{i=1}^{\infty} \int_0^s \lambda_i d\mathcal{W}^i_r \]

which is simply the stochastic integral

\[ \int_0^s P(r) d\mathcal{W}_r \]

for the process \( P \in \mathbb{I}^R(\mathcal{W}) \) defined by \( P_i(r) = \lambda_i \). So \( M \) is a continuous genuine martingale, which we show is of the form (60) through Levy’s Characterisation of Brownian Motion. Indeed the quadratic variation process \( [M] \) is given precisely as in 1.5.8, where our approximating sequence of martingales

\[ M^n = \sum_{i=1}^{n} \lambda_i W^i_s \]
have quadratic variation

\[ [M^n]_s = \sum_{i=1}^{n} \lambda_i^2 s \]

which of course converges in \( L^1(\Omega; \mathbb{R}) \) to the infinite sum, from which we conclude

\[ [M]_s = \sum_{i=1}^{\infty} \lambda_i^2 s. \]

Therefore

\[ \left[ \frac{M}{(\sum_{i=1}^{\infty} \lambda_i^2)^{1/2}} \right]_s = s \]

and we immediately deduce the representation (60) from Levy’s Characterisation.

\[ \Box \]

Proof of 2.7.1. As \( \mathcal{G}: \mathcal{H} \to \mathcal{L}^2(\mathcal{U}; \mathcal{H}) \) is linear and bounded from

\[ \|\mathcal{G}\phi\|_{\mathcal{L}^2(\mathcal{U}; \mathcal{H})}^2 = \sum_{i=1}^{\infty} \|\lambda_i\phi\|_{\mathcal{H}}^2 = \sum_{i=1}^{\infty} \lambda_i^2 \|\phi\|_{\mathcal{H}}^2 \]

then it is continuous as a mapping between these spaces so preserves the progressive measurability and evidently the required boundedness to deduce that \( \mathcal{G}\Psi \in \tilde{\mathcal{H}}(\mathcal{W}) \). To show the identity (58) let’s rewrite

\[ \int_0^t \mathcal{G}\Psi_s dW_s = \lim_{n \to \infty} \sum_{i=1}^{n} \int_0^t \lambda_i\Psi_s dW^i_s = \lim_{n \to \infty} \int_0^t \Psi_s dM^n_s \]

with notation \( M^n \) as in 2.7.2 and understanding once more that this limit is taken in \( L^2(\Omega; \mathcal{H}) \) for the stopped integrals. The localising stopping times \( (\tau_m) \) defined by

\[ \tau_m = m \land \inf\{0 \leq t < \infty: \int_0^t \sum_{i=1}^{\infty} \lambda_i^2 \|\Psi_s\|_{\mathcal{H}}^2 ds \geq m\} \]

are precisely as in 1.6.7 and (11) for the left and right side of (58) respectively. It is therefore sufficient to show that for any \( m \),

\[ \lim_{n \to \infty} \int_0^t \Psi_s 1_{s \leq \tau_m} dM^n_s = \int_0^t \Psi_s 1_{s \leq \tau_m} dM_s \]

having simply inserted the representation (60) into our required identity. In other words we want that

\[ \mathbb{E}\left[ \left\| \int_0^t \Psi_s 1_{s \leq \tau_m} dM_s - \int_0^t \Psi_s 1_{s \leq \tau_m} dM^n_s \right\|^2_{\mathcal{H}} \right] \to 0 \]

as \( n \to \infty \) which is equivalent to the statement

\[ \mathbb{E}\left[ \left\| \int_0^t \Psi_s 1_{s \leq \tau_m} (M - M^n)_s \right\|^2_{\mathcal{H}} \right] \to 0. \]
The same arguments of 2.7.2 afford us that the martingale $M - M^n$ which is given at each time $s$ by

$$(M - M_n)_s = \sum_{i=n+1}^{\infty} \lambda_i W^i_s$$

has the representation

$$M - M^n = \left( \sum_{i=n+1}^{\infty} \lambda_i^2 \right)^{1/2} V$$

for a standard Brownian Motion $V$. So we have that

$$\mathbb{E} \left| \int_0^t \Psi_s 1_{s \leq \tau_m} d(M - M^n)_s \right|^2_{\mathcal{H}} = \mathbb{E} \left| \left( \sum_{i=n+1}^{\infty} \lambda_i^2 \right)^{1/2} \int_0^t \Psi_s 1_{s \leq \tau_m} dV_s \right|^2_{\mathcal{H}}$$

$$= \left( \sum_{i=n+1}^{\infty} \lambda_i^2 \right) \mathbb{E} \left| \int_0^t \Psi_s 1_{s \leq \tau_m} dV_s \right|^2_{\mathcal{H}}$$

$$= \left( \sum_{i=n+1}^{\infty} \lambda_i^2 \right) \int_0^t \| \Psi_s 1_{s \leq \tau_m} \|^2_{\mathcal{H}} ds$$

having used the Itô Isometry 1.2.7.1. By definition of the stopping time the integral is bounded uniformly in $\omega$ (a.e.) hence has finite expectation, so we conclude that this approaches zero in the limit by finiteness of the sum.

\[ \square \]

2.8 An Existence and Uniqueness Result in Finite Dimensions

Another such way that we look to simplify the equations in this framework to apply the standard theory is through taking finite dimensional approximations (which also motivated the previous result!). The scheme of application for this is referred to as a Galerkin Scheme, used traditionally in the analysis for highly non-trivial PDEs such as the Euler and Navier-Stokes Equations (see [11, 12]) and indeed is used in the abstract stochastic solution method [1].

**Theorem 2.8.1.** Fix a finite-dimensional Hilbert Space $\mathcal{H}$. Suppose the following:

1: For any $T > 0$, the operators $\mathcal{A} : [0, T] \times \mathcal{H} \rightarrow \mathcal{H}$ and $\mathcal{G} : [0, T] \times \mathcal{H} \rightarrow \mathcal{L}^2(\Omega; \mathcal{H})$ are measurable;

2: There exists a $C : [0, \infty) \rightarrow \mathbb{R}$ bounded on $[0, T]$ for every $T$, and constants $c_i$ such that for every $\phi, \psi \in \mathcal{H}$ and $t \in [0, \infty)$,

$$\| \mathcal{A}(t, \phi) \|^2_{\mathcal{H}} \leq C_t \left[ 1 + \| \phi \|^2_{\mathcal{H}} \right]$$

$$\| \mathcal{G}(t, \phi) \|^2_{\mathcal{H}} \leq C_t c_i \left[ 1 + \| \phi \|^2_{\mathcal{H}} \right]$$

$$\sum_{i=1}^{\infty} c_i < \infty$$

$$\| \mathcal{A}(t, \phi) - \mathcal{A}(t, \psi) \|^2_{\mathcal{H}} + \sum_{i=1}^{\infty} \| \mathcal{G}(t, \phi) - \mathcal{G}(t, \psi) \|^2_{\mathcal{H}} \leq C_t \| \phi - \psi \|^2_{\mathcal{H}}$$
Then there exists a process \( \Phi : [0, \infty) \times \Omega \to \mathcal{H} \) such that for \( \mathbb{P} \)-a.e. \( \omega \), \( \Phi(\omega) \in C([0, T]; \mathcal{H}) \) for every \( T > 0 \), \( \Phi \) is progressively measurable in \( \mathcal{H} \) and the identity

\[
\Phi_t = \Phi_0 + \int_0^t \mathcal{A}(s, \Phi_s)ds + \int_0^t \mathcal{B}(s, \Phi_s)dW_s
\]

holds a.s. in \( \mathcal{H} \) for every \( t \geq 0 \).

We remark that the operators satisfy the assumptions of 2.4.1, 2.4.2 for the spaces \( V = H = U := \mathcal{H} \) and that the conclusion of this theorem is the existence of a strong solution of (61) in the sense of 2.4.3.

Proof. With the finite-dimensionality of \( \mathcal{H} \) in place, we first restrict ourselves to finitely many Brownian Motions in our stochastic integral to make things classical. Let’s define the operator \( \mathcal{G}^k : [0, T] \times \mathcal{H} \times \mathcal{U} \to \mathcal{H} \) on the basis vectors \((e_i)\) of \( \mathcal{U}\) by \( \mathcal{G}^k(\cdot, \cdot, e_i) = \mathcal{A}(\cdot, \cdot, e_i)\) for \( i \leq k\), and zero otherwise. We consider at first the equation

\[
\Phi^k_t = \Phi^k_0 + \int_0^t \mathcal{A}(s, \Phi^k_s)ds + \int_0^t \mathcal{G}^k(s, \Phi^k_s)dW_s
\]

or equivalently,

\[
\Phi^k_t = \Phi^k_0 + \int_0^t \mathcal{A}(s, \Phi^k_s)ds + \sum_{i=1}^k \int_0^t \mathcal{G}_i^k(s, \Phi^k_s)dW^i_s
\]

for \( \Phi^k_0 := \Phi_0 \). The existence and uniqueness of solutions to this finite-dimensional system is then classical (for solutions defined as in the theorem). Consider now solutions \( \Phi^j \), \( \Phi^k \) for \( j < k \) arbitrary, which therefore satisfy the difference equation

\[
\Phi^k_r - \Phi^j_r = \int_0^r \mathcal{A}(s, \Phi^k_s) - \mathcal{A}(s, \Phi^j_s)ds + \int_0^r \mathcal{G}^k(s, \Phi^k_s) - \mathcal{G}^j(s, \Phi^j_s)dW_s
\]

for any \( r \in [0, \infty) \). By applying the Itô Formula for the energy identity 2.6.2 but in the general case of 2.6.3, we see further that the identity

\[
\left\| \Phi^k_r - \Phi^j_r \right\|^2 = 2 \int_0^r \langle \mathcal{A}(s, \Phi^k_s) - \mathcal{A}(s, \Phi^j_s), \Phi^k_s - \Phi^j_s \rangle_{\mathcal{H}} ds
\]

\[
+ \int_0^r \left\| \mathcal{G}^k(s, \Phi^k_s) - \mathcal{G}^j(s, \Phi^j_s) \right\|^2_{\mathcal{L}^2(\mathcal{U}; \mathcal{H})} ds + 2 \int_0^r \langle \mathcal{G}^k(s, \Phi^k_s) - \mathcal{G}^j(s, \Phi^j_s), \Phi^k_s - \Phi^j_s \rangle_{\mathcal{H}} dW_s
\]

holds a.s.. We use Cauchy-Schwartz to move to an inequality, and rewrite the quadratic variation term to give us the bound

\[
\left\| \Phi^k_r - \Phi^j_r \right\|^2 \leq \int_0^r \left( 2\left\| \mathcal{A}(s, \Phi^k_s) - \mathcal{A}(s, \Phi^j_s) \right\|^2_{\mathcal{H}} + \sum_{i=1}^j \left\| \mathcal{G}_i(s, \Phi^k_s) - \mathcal{G}_i(s, \Phi^j_s) \right\|^2_{\mathcal{H}} + \sum_{i=1}^k \left\| \mathcal{G}_i(s, \Phi^k_s) - \mathcal{G}_i(s, \Phi^j_s) \right\|^2_{\mathcal{H}} \right) ds
\]

\[
+ \int_0^r \sum_{i=j+1}^k \left\| \mathcal{G}_i(s, \Phi^k_s) \right\|^2_{\mathcal{H}} ds + 2 \int_0^r \langle \mathcal{G}^k(s, \Phi^k_s) - \mathcal{G}^j(s, \Phi^j_s), \Phi^k_s - \Phi^j_s \rangle_{\mathcal{H}} dW_s.
\]
In one step now we bound the stochastic integral by its absolute value, take the supremum over all such $r$ up to any arbitrary time $t \in [0, \infty)$ and employ the Lipschitz assumption to see that

$$
\sup_{r \in [0,t]} \| \Phi_r^k - \Phi_r^j \|_H^2 \leq c \int_0^t \| \Phi_s^k - \Phi_s^j \|_H^2 ds + \int_0^t \sum_{i=j+1}^k \| \mathcal{G}_i(s, \Phi_s^k) \|_H^2 ds + 2 \sup_{r \in [0,t]} \left| \int_0^r \langle \mathcal{G}^k(s, \Phi_s^k) - \mathcal{G}^j(s, \Phi_s^j), \Phi_s^k - \Phi_s^j \rangle_H dW_s \right|
$$

for a generic constant $c$, allowed to depend on $t$. We want to take the expectation here but have to be slightly careful in ensuring that the expectation is finite; to this end we consider the stopping times

$$
\tau_R := R \land \inf \{ s \geq 0 : \max \{ \| \Phi_s^k \|_H, \| \Phi_s^j \|_H \} \geq R \}
$$

and the process defined for any fixed $R$ by

$$
\tilde{\Phi}_s^k := \Phi_s^k 1_{s \leq \tau_R}, \quad \tilde{\Phi}_s^j := \Phi_s^j 1_{s \leq \tau_R}.
$$

From the continuity of the processes $\tilde{\Phi}_s^k, \tilde{\Phi}_s^j$ then $(\tau_R)$ is an $a.s.$ monotone increasing sequence convergent to infinity and $\max \{ \| \tilde{\Phi}_s^k \|_H, \| \tilde{\Phi}_s^j \|_H \} \leq R$ for any $s \geq 0$. It is trivial that these processes satisfy the same inequality

$$
\sup_{r \in [0,t]} \| \tilde{\Phi}_r^k - \tilde{\Phi}_r^j \|_H^2 \leq c \int_0^t \| \tilde{\Phi}_s^k - \tilde{\Phi}_s^j \|_H^2 ds + \int_0^t \sum_{i=j+1}^k \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \|_H^2 ds + 2 \sup_{r \in [0,t]} \left| \int_0^r \langle \mathcal{G}^k(s, \tilde{\Phi}_s^k) - \mathcal{G}^j(s, \tilde{\Phi}_s^j), \tilde{\Phi}_s^k - \tilde{\Phi}_s^j \rangle_H dW_s \right|
$$

and justify that the expectation of all terms involved is finite (indeed for the stochastic integral, using the Lipschitz assumption and the boundedness of $\tilde{\Phi}_r^k - \tilde{\Phi}_r^j$ then $\langle \mathcal{G}^k(\cdot, \tilde{\Phi}_s^k) - \mathcal{G}^j(\cdot, \tilde{\Phi}_s^j), \tilde{\Phi}_s^k - \tilde{\Phi}_s^j \rangle \in L^2(W)$ so by 1.6.9 the expectation of this term is finite). We do now take the expectation and apply 1.6.9 to give us the bound

$$
\mathbb{E} \sup_{r \in [0,t]} \| \tilde{\Phi}_r^k - \tilde{\Phi}_r^j \|_H^2 \leq c \mathbb{E} \int_0^t \| \tilde{\Phi}_s^k - \tilde{\Phi}_s^j \|_H^2 ds + \mathbb{E} \int_0^t \sum_{i=j+1}^k \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \|_H^2 ds + c \mathbb{E} \left( \int_0^t \sum_{i=1}^{\infty} \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) - \mathcal{G}_i(s, \tilde{\Phi}_s^j) \|_H^2 ds \right)^{\frac{1}{2}}
$$

which we promptly reduce to

$$
\mathbb{E} \sup_{r \in [0,t]} \| \tilde{\Phi}_r^k - \tilde{\Phi}_r^j \|_H^2 \leq c \mathbb{E} \int_0^t \| \tilde{\Phi}_s^k - \tilde{\Phi}_s^j \|_H^2 ds + \mathbb{E} \int_0^t \sum_{i=j+1}^k \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \|_H^2 ds
$$

$$
+ c \mathbb{E} \left( \int_0^t \left[ \sum_{i=1}^j \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) - \mathcal{G}_i(s, \tilde{\Phi}_s^j) \|_H^2 + \sum_{i=j+1}^k \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \|_H^2 \right] \| \tilde{\Phi}_s^k - \tilde{\Phi}_s^j \|_H^2 ds \right)^{\frac{1}{2}}.
$$
Employing our Lipschitz assumption once more, followed by an application of Young’s Inequality, we have that

\[ c \left( \int_0^t \left[ \sum_{i=1}^j \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) - \mathcal{G}_i(s, \tilde{\Phi}_s^j) \|^2 + \sum_{i=j+1}^k \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \|^2 \right] \| \tilde{\Phi}_s^k - \tilde{\Phi}_s^j \|^2 ds \right)^{\frac{1}{2}} \]

\[ \leq c \left( \int_0^t \left[ \| \tilde{\Phi}_s^k - \tilde{\Phi}_s^j \|^2 + \sum_{i=j+1}^k \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \|^2 \right] \| \tilde{\Phi}_s^k - \tilde{\Phi}_s^j \|^2 ds \right)^{\frac{1}{2}} \]

\[ \leq c \left( \sup_{r \in [0,t]} \| \mathcal{G}_r \|^2 \right)^{\frac{1}{2}} \left( \int_0^t \| \tilde{\Phi}_s^k - \tilde{\Phi}_s^j \|^2 + \sum_{i=j+1}^k \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \|^2 \right) \]

\[ \leq \frac{1}{2} \sup_{r \in [0,t]} \| \tilde{\Phi}_r \|^2 + c = \frac{1}{2} \int_0^t \left[ \| \tilde{\Phi}_s^k - \tilde{\Phi}_s^j \|^2 + \sum_{i=j+1}^k \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \|^2 \right] ds \]

and furthermore

\[ \mathbb{E} \sup_{r \in [0,t]} \| \tilde{\Phi}_r \|^2 \leq c \mathbb{E} \int_0^t \| \tilde{\Phi}_s^k - \tilde{\Phi}_s^j \|^2 ds + c \mathbb{E} \int_0^t \sum_{i=j+1}^k \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \|^2 ds. \]

It is then a standard application of the Grönwall Inequality that

\[ \mathbb{E} \sup_{r \in [0,t]} \| \tilde{\Phi}_r \|^2 \leq c \mathbb{E} \int_0^t \sum_{i=j+1}^k \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \|^2 ds \] (62)

where the \( c \) incorporates \( e^{ct} \). Observe also that, through very similar arguments just using the linear growth property instead of the Lipschitz one, we have that

\[ \mathbb{E} \sup_{r \in [0,t]} \| \tilde{\Phi}_r \|^2 \leq \mathbb{E} \| \tilde{\Phi}_0 \|^2 + \mathbb{E} \int_0^t \left( 2 \| \mathcal{G}_r(s, \tilde{\Phi}_s^k) \|^2 + \sum_{i=j+1}^k \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \|^2 \right) ds \]

\[ + 2 \mathbb{E} \sup_{r \in [0,t]} \left| \sum_{i=1}^k \int_0^r \langle \mathcal{G}_i(s, \tilde{\Phi}_s^k), \tilde{\Phi}_s^k \rangle_{\mathcal{H}} dW_s \right| \]

\[ \leq \mathbb{E} \| \tilde{\Phi}_0 \|^2 + c \mathbb{E} \int_0^t \left( 1 + \| \tilde{\Phi}_s^k \|^2 \right) \| \tilde{\Phi}_s^k \|^2 + 1 + \| \tilde{\Phi}_s^k \|^2 ds \]

\[ + c \mathbb{E} \left( \int_0^t \left( 1 + \| \tilde{\Phi}_s^k \|^2 \right) \| \tilde{\Phi}_s^k \|^2 ds \right)^{\frac{1}{2}} \]

to which we use that \( \| \tilde{\Phi}_s^k \|^2 \leq 1 + \| \tilde{\Phi}_s^k \|^2 \) to see that

\[ \mathbb{E} \sup_{r \in [0,t]} \| \tilde{\Phi}_r \|^2 \leq \mathbb{E} \| \tilde{\Phi}_0 \|^2 + c \mathbb{E} \int_0^t \left( 1 + \| \tilde{\Phi}_s^k \|^2 \right) \| \tilde{\Phi}_s^k \|^2 ds + c \mathbb{E} \left( \int_0^t \left( 1 + \| \tilde{\Phi}_s^k \|^2 \right) \| \tilde{\Phi}_s^k \|^2 ds \right)^{\frac{1}{2}} \]
and further
\[ \mathbb{E} \sup_{r \in [0,t]} \| \tilde{\Phi}_r^k \|_{\mathcal{H}}^2 \leq c \left[ \mathbb{E} \| \tilde{\Phi}_0 \|_{\mathcal{H}}^2 + 1 \right] + c \mathbb{E} \int_0^t \| \tilde{\Phi}_s^k \|_{\mathcal{H}}^2 ds \]

as above, integrating the 1 and adding it as a constant. Thus we have
\[ \mathbb{E} \sup_{r \in [0,t]} \| \tilde{\Phi}_r^k \|_{\mathcal{H}}^2 \leq c \left[ \mathbb{E} \| \tilde{\Phi}_0 \|_{\mathcal{H}}^2 + 1 \right] = c \left[ \mathbb{E} \| \Phi_0 \|_{\mathcal{H}}^2 + 1 \right]. \]

which is a bound uniform in \( k \) and independent of \( \tau_R \). Moreover, for each fixed \( k \) we appreciate that the sequence of random variables
\[ \sup_{r \in [0,t]} \| \tilde{\Phi}_r^k \|_{\mathcal{H}} \]

is monotone increasing (indexed by \( R \)) and convergent to \( \sup_{r \in [0,t]} \| \Phi_r^k \|_{\mathcal{H}} \) a.s. Thus we may apply the Monotone Convergence Theorem to this sequence of random variables to see that
\[ \mathbb{E} \sup_{r \in [0,t]} \| \tilde{\Phi}_r^k \|_{\mathcal{H}}^2 = \lim_{R \to \infty} \mathbb{E} \sup_{r \in [0,t]} \| \tilde{\Phi}_r^k \|_{\mathcal{H}}^2 \leq c \left[ \mathbb{E} \| \Phi_0 \|_{\mathcal{H}}^2 + 1 \right]. \]

With this bound established we can revert back to (62), combining with the boundedness of the \( \mathcal{G}_i \) to deduce that
\[ \mathbb{E} \int_0^t \sum_{i=j+1}^k \| \mathcal{G}_i(s, \Phi_s^k) \|_{\mathcal{H}}^2 ds < \infty \]

and clearly
\[ \mathbb{E} \int_0^t \sum_{i=j+1}^k \| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \|_{\mathcal{H}}^2 ds \leq \mathbb{E} \int_0^t \sum_{i=j+1}^k \| \mathcal{G}_i(s, \Phi_s^k) \|_{\mathcal{H}}^2 ds \]

so we can update (62) with the bound
\[ \mathbb{E} \sup_{r \in [0,t]} \| \tilde{\Phi}_r^k - \tilde{\Phi}_r^j \|_{\mathcal{H}}^2 \leq c \mathbb{E} \int_0^t \sum_{i=j+1}^k \| \mathcal{G}_i(s, \Phi_s^k) \|_{\mathcal{H}}^2 ds \]

to which we apply the same monotone convergence argument to deduce that
\[ \mathbb{E} \sup_{r \in [0,t]} \| \Phi_r^k - \Phi_r^j \|_{\mathcal{H}}^2 \leq c \mathbb{E} \int_0^t \sum_{i=j+1}^k \| \mathcal{G}_i(s, \Phi_s^k) \|_{\mathcal{H}}^2 ds. \] (63)
Moreover

\[ \sup_{k>j} \mathbb{E} \int_0^t \sum_{i=j+1}^k \| \mathcal{G}_i(s, \mathbf{F}_s^k) \|^2_{\mathcal{H}} ds \leq t \sup_{k>j} \mathbb{E} \sup_{r \in [0,t]} \sum_{i=j+1}^k \| \mathcal{G}_i(r, \mathbf{F}_s^k) \|^2_{\mathcal{H}} ds \]

\[ \leq t \sup_{k>j} \mathbb{E} \sup_{r \in [0,t]} \sum_{i=j+1}^k C_t c_i (1 + \| \Phi_s^k \|^2_{\mathcal{H}}) ds \]

\[ := c \sup_{k>j} \mathbb{E} \sup_{r \in [0,t]} \sum_{i=j+1}^k c_i (1 + \| \Phi_s^k \|^2_{\mathcal{H}}) ds \]

\[ = c \sup_{k>j} \mathbb{E} \sup_{r \in [0,t]} (1 + \| \Phi_s^k \|^2_{\mathcal{H}}) ds \]

\[ \leq c \sum_{i=j+1}^{\infty} c_i \sup_{k>j} \mathbb{E} (1 + \| \Phi_s^k \|^2_{\mathcal{H}}) ds \]

which is a sequence in \( j \) monotone decreasing to zero. Thus in view of (63),

\[ \lim_{j \to \infty} \sup_{k>j} \mathbb{E} \sup_{r \in [0,t]} (1 + \| \Phi_s^k \|^2_{\mathcal{H}}) = 0 \]

so the sequence (\( \Phi^k \)) is Cauchy in \( L^2(\Omega; C([0,t]; \mathcal{H})) \) and as such we can deduce the existence of a \( \Phi \) such that \( \Phi^k \to \Phi \) in this space (and hence, in \( L^2(\Omega; L^2([0,t]; \mathcal{H})) \)) for every \( t \in [0, \infty) \), and thus \( \Phi \) is also the \( \mathbb{P} - a.s. \) limit of a subsequence of the \( (\Phi^k) \) in \( C([0,t]; \mathcal{H}) \). This limit process inherits the progressive measurability (indeed it is trivially adapted and has continuous paths in \( \mathcal{H} \)). It simply remains to show that \( \Phi \) satisfies the identity (61), so we first consider the \( \mathbb{P} - a.s. \) convergent subsequence \( (\Phi^k) \). Looking at the stochastic integral, we have that

\[ \mathbb{E} \left\| \int_0^t \mathcal{G}(s, \Phi_s) dW_s - \int_0^t \mathcal{G}^{k_i}(s, \Phi_s^{k_i}) dW_s \right\|^2_{\mathcal{H}} = \mathbb{E} \left\| \int_0^t \mathcal{G}(s, \Phi_s) - \mathcal{G}^{k_i}(s, \Phi_s^{k_i}) dW_s \right\|^2_{\mathcal{H}} \]

\[ = \lim_{N \to \infty} \mathbb{E} \left\| \sum_{i=1}^N \int_0^t \mathcal{G}_i(s, \Phi_s) - \mathcal{G}^{k_i}_i(s, \Phi_s^{k_i}) dW_s \right\|^2_{\mathcal{H}} \]

\[ = \lim_{N \to \infty} \mathbb{E} \int_0^t \sum_{i=1}^N \left\| \mathcal{G}_i(s, \Phi_s) - \mathcal{G}^{k_i}_i(s, \Phi_s^{k_i}) \right\|^2_{\mathcal{H}} ds \]

\[ \leq \mathbb{E} \int_0^t \sum_{i=1}^{\infty} \left\| \mathcal{G}_i(s, \Phi_s) - \mathcal{G}^{k_i}_i(s, \Phi_s^{k_i}) \right\|^2_{\mathcal{H}} ds \]

\[ \leq \sum_{i=k_i+1}^{\infty} c_i \int_0^t \left\| \Phi_s - \Phi_s^{k_i} \right\|^2_{\mathcal{H}} ds \]

\[ \leq c \int_0^t \left\| \Phi_s - \Phi_s^{k_i} \right\|^2_{\mathcal{H}} ds \]
so from the known $L^2(\Omega; L^2([0,t]; \mathcal{H}))$ convergence, we have that

$$
\lim_{k_i \to \infty} \int_0^t \mathcal{G}_{k_i}(s, \Phi_{s}^{k_i}) d\mathcal{W}_s = \int_0^t \mathcal{G}(s, \Phi_s) d\mathcal{W}_s
$$

with the limit in $L^2(\Omega; \mathcal{H})$. We can thus extract a further subsequence which we denote $(\Phi_{s}^{k_m})$ such that this limit holds $\mathbb{P}_- a.s.$ in $\mathcal{H}$, and is of course still such that $(\Phi_{s}^{k_m}) \to \Phi$ $\mathbb{P}_- a.s.$ in $L^2([0,t]; \mathcal{H})$ (again a trivial consequence of the $C([0,t]; \mathcal{H})$ convergence). Therefore for $\mathbb{P}_- a.e. \omega$,

$$
\left\| \int_0^t \mathcal{G}(s, \Phi_s(\omega)) ds - \int_0^t \mathcal{G}(s, \Phi_{s}^{k_m}(\omega)) ds \right\|^2_{\mathcal{H}} \leq \int_0^t \left\| \mathcal{G}(s, \Phi_s(\omega)) - \mathcal{G}(s, \Phi_{s}^{k_m}(\omega)) \right\|^2_{\mathcal{H}} ds
$$

and so

$$
\lim_{k_m \to \infty} \int_0^t \mathcal{G}(s, \Phi_{s}^{k_m}(\omega)) ds = \int_0^t \mathcal{G}(s, \Phi_s(\omega)) ds
$$

with the limit in $\mathcal{H}$. Thus by taking the $\mathbb{P}_- a.s.$ limit in $\mathcal{H}$ of the identity satisfied by $\Phi_{s}^{k_m}$, we reach (61) as required.

\[\square\]

**Theorem 2.8.2.** Suppose $\Psi$ is another strong solution of (61). Then for all $t \in [0, \infty)$,

$$
\mathbb{P} \left\{ \{ \omega \in \Omega : \Phi_t(\omega) = \Psi_t(\omega) \} \right\} = 1.
$$

**Proof.** The method of proof here is entirely contained in that for the existence just seen. Indeed we look at the energy equality satisfied by the difference of the solutions, which is

$$
\left\| \Phi_r - \Psi_r \right\|^2_{\mathcal{H}} = 2 \int_0^r \left\langle \mathcal{G}(s, \Phi_s) - \mathcal{G}(s, \Psi_s), \Phi_s - \Psi_s \right\rangle_{\mathcal{H}} ds + \int_0^r \left\| \mathcal{G}(s, \Phi_s) - \mathcal{G}(s, \Psi_s) \right\|^2_{L^2(\Omega, \mathcal{H})} ds + 2 \int_0^r \left\langle \mathcal{G}(s, \Phi_s) - \mathcal{G}(s, \Psi_s), \Phi_s - \Psi_s \right\rangle_{\mathcal{H}} d\mathcal{W}_s.
$$

Following along the proof, we introduce

$$
\tau_R := R \wedge \inf \{ s \geq 0 : \max \{ \| \Phi_s \|_{\mathcal{H}}, \| \Psi_s \|_{\mathcal{H}} \} \geq R \}
$$

and the process defined for any fixed $R$ by

$$
\tilde{\Phi}_s := \Phi_s 1_{s \leq \tau_R}, \quad \tilde{\Psi}_s := \Psi_s 1_{s \leq \tau_R}.
$$

In this case we have the inequality

$$
\sup_{r \in [0,t]} \left\| \tilde{\Phi}_r - \tilde{\Psi}_r \right\|^2_{\mathcal{H}} \leq c \int_0^t \left\| \tilde{\Phi}_s - \tilde{\Psi}_s \right\|^2_{\mathcal{H}} ds + \sup_{r \in [0,t]} \left| \int_0^r \left\langle \mathcal{G}(s, \tilde{\Phi}_s) - \mathcal{G}(s, \tilde{\Psi}_s), \tilde{\Phi}_s - \tilde{\Psi}_s \right\rangle_{\mathcal{H}} d\mathcal{W}_s \right|
$$

so following all of the same steps, simply now without the $\sum_{i=j+1}^k \left\| \mathcal{G}_i(s, \tilde{\Phi}_s^k) \right\|^2_{\mathcal{H}}$ term, we deduce again that

$$
\mathbb{E} \sup_{r \in [0,t]} \left\| \tilde{\Phi}_r - \tilde{\Psi}_r \right\|^2_{\mathcal{H}} \leq 0
$$

in analogy with (62). By the same monotone convergence argument, we have that

$$
\mathbb{E} \sup_{r \in [0,t]} \left\| \Phi_r - \Psi_r \right\|^2_{\mathcal{H}} = 0
$$

which gives the result. 

\[\square\]
2.9 Applications

We have alluded quite heavily to the application of this framework for SALT [4] derived SPDEs, which is done for now in [1] and to be expanded upon in [2,3]. In [1] we establish an abstract solution method in the context of the Hilbert Spaces

\[ \mathbf{V} \hookrightarrow \mathbf{H} \hookrightarrow \mathbf{U} \hookrightarrow \mathbf{X} \]

posing additional assumptions on the operators to the ones given in 2.4.1, 2.4.2. In particular this is applied to the Stratonovich SPDE

\[
\begin{aligned}
    u_t - u_0 + \int_0^t \mathcal{L} u_s \, ds - \int_0^t \Delta u_s \, ds + \int_0^t B u_s \circ dW_s + \nabla \rho_t = 0
\end{aligned}
\]  

where \( B \) here is a differential (transport) operator and \( \mathcal{L} \) is the usual fluids non-linear term. Clearly this is a highly non-trivial SPDE, but we show that this fits our framework for (43) to swiftly convert into the Itô Form for which we then apply the general existence and uniqueness arguments in the paper. The notion of \( \mathbf{V} \)-valued solution in [1] is thus what corresponds precisely and rigorously to a solution of the Stratonovich form as laid out in this text. Indeed the general existence argument also used the result 2.8.1.

We hope that this very general framework can facilitate a much easier analysis of many SPDEs in the future.
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