Problems of the CASCADE Protocol and Renyi Entropy Reduction in Classical and Quantum Key Generation

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Abstract

It is shown that the interactive error correction protocol ‘CASCADE’ should be analyzed taking the correlation between passes and finite length of sequence into account. Furthermore we mention some problems in quantifying the reduction of Renyi entropy by information announced during the error correction process.

1 Introduction

The CASCADE protocol for error correction was studied by Brassard and Salvail \cite{Brassard1988} who gave asymptotic estimates of its success probability and the number of information bits leaked during protocol execution. In a practical implementation of CASCADE, with a fixed number of passes and a finite length sequence, we show that calculation of the success probability and number of information bits leaked are very involved problems which have not been properly addressed so far. Even the estimates in \cite{Brassard1988} are made under what may be called an ‘independent pass’ approximation because the correlation between operations taking place at different passes of the protocol has been neglected. On the implementation side, we study practically important parameters such as the communication complexity and the round-trip delay time of CASCADE.

For cryptographic purposes, in both classical and quantum protocols, one is interested in the Renyi information leakage from public discussion during the error correction process. This is because the standard privacy amplification theorem \cite{Maurer1993} requires the Renyi entropy before privacy amplification to be known. Such estimates of Renyi entropy reduction, especially for the case of identical individual attacks by Eve, have been studied by Cachin and Maurer \cite{Cachin1995}, who claimed to show that for linear error-correcting codes, the Renyi entropy reduction is roughly equal to the number of announced bits. We explain
why this claim is incorrect. However, we suggest that covering the announced bits with a pre-shared secret key would result in no Renyi entropy reduction and can be used as long as the net key generation rate remains positive. For CASCADE, for which the Renyi entropy reduction has not been studied, we suggest a similar procedure. In doing so, we find that, unlike the case of linear error-correcting codes, there is still an entropy reduction that is difficult to quantify.

2 CASCADE PROTOCOL

2.1 ‘Independent Pass’ and ‘Infinite-length Sequence’ Approximation

The CASCADE protocol proceeds in several passes in which block-parity comparison and binary-search-type error correction are executed. CASCADE uses correlation between the sequences of passes, so that it performs better as to the number of announced bits during the error correction process than the BBBSS protocol[1] which is also based on parity check and binary search, but does not use the correlation of sequences. Upper bounds on the number of announced bits were derived in [1]. The number of announced bits seems to depend heavily on error patterns and permutation functions used to produce sequences of all passes. Furthermore the process is nonlinear. Therefore, the expected value should be obtained by averaging the number of announced bits of all specific error patterns and permutation functions. However, the upper bound on the number of announced bits was derived by calculating expected amount of announced bits pass by pass. That is, the correlation of sequences was not considered. We performed computer simulation 1,000,000 times to analyze the performance of CASCADE. The number of announced bits during the error correction process obtained by the computer simulation is shown in Tab.1. The upper bound derived in [1] is also shown in the bracket. It is found that the upper bound is very close to the simulation brackets. In the case that the private channel error probability $\varepsilon$ is 0.05 and 0.1, the simulation results are larger than the upper bound. This indicates that a nonrigorous method was used to derive the upper bound.

In [1], the size of a block for each pass was designed so that the number of errors contained in a block of the initial pass reduces to less than half as passes proceed. However, the success probability was discussed in [1] on the basis of 100 empirical tests in which all errors were corrected at the end of the protocol. Analysis of the success probability is very complicated. Only a slight difference of error positions may determine whether the protocol succeeds or fails. We have not derived the probability that CASCADE protocol succeeds. In this situation, the results of a large number of computer simulations may help us to understand its properties. The protocol failure probability obtained by averaging the results of 1,000,000 computer simulation is shown in Tab.2. It is found that the protocol failure probability reduces as the sequence length becomes longer. Furthermore it is found that the failure probability decreases as the error probability of the private channel increases. According to our computer simulation, we found that the average number of errors remaining after pass 2 is almost the same in the range from 0.3 to 0.37. As the error probability of the private channel decreases, block sizes become longer and
the number of blocks decreases, so that the probability that two or more of the remaining errors move to the same block in the next pass becomes large. Then, the protocol failure probability increases. From this result, the protocol failure probability should be analysed for sequences with finite length and asymptotic limits are not reliable.

Table 1: **Number of Announced Bits during Error Correction**

| $n$  | $\varepsilon = 0.01$ | 0.05 | 0.1 | 0.15 |
|------|----------------------|------|-----|-----|
| 100  | 13.43(9.33)          | 35.37(33.14) | 59.3(57) | 77.56(82.4) |
| 200  | 20.64(18.66)         | 69.3(66.29) | 117.39(114) | 153.86(164.8) |
| 500  | 45.93(46.64)         | 170.12(165.71) | 288.67(285) | 384.76(412) |
| 1000 | 90.99(93.29)         | 339.01(331.43) | 576.53(570) | 768.19(824) |
| 2000 | 182.16(186.58)       | 677.25(662.86) | 1151.7(1140) | 1536.02(1648) |
| 5000 | 454.38(466.44)       | 1692.09(1657.14) | 2878.41(2850) | 3839.52(4120) |
| 10000| 906.18(932.88)       | 3382.11(3314.29) | 5753.93(5700) | 7678.68(8240) |

$\varepsilon$: error probability of private channel, $n$: sequence length.

The upper bound [1] is given in parentheses.

Table 2: **Protocol Failure Probability of CASCADE**

| $\varepsilon$ | $n=100$ | 200 | 500 | 1000 | 2000 | 5000 | 10000 |
|---------------|---------|-----|-----|------|------|------|-------|
| 0.01          | 0.023603| 0.056623| 0.078354| 0.044117| 0.011357| 0.002677| 0.001005|
| 0.05          | 0.072621| 0.038288| 0.005492| 0.001657| 0.000355| 0.000075| 0.000066|
| 0.1           | 0.032161| 0.009446| 0.001129| 0.000349| 0.000119| 0.000009| 0.000011|
| 0.15          | 0.016245| 0.006394| 0.000415| 0.000158| 0.000039| 0 | 0.00001 |

$\varepsilon$: error probability of private channel, $n$: sequence length.

2.2 Communication Complexity

For interactive error correction like CASCADE, a round-trip is necessary for comparing block and sub-block parities. From a practical viewpoint, time for the round-trips should be considered in evaluating the net key generation rate. The number of round-trips is shown in Tab.3 for CASCADE and BBBSS protocol which does not use the correlation between sequences[4]. It is found in Tab.3 that the number of the round-trip increases as error probability of the private channel becomes large and the sequence length becomes longer, and it is almost proportionally to the sequence length in CASCADE, while that of BBBSS protocol is almost constant. The difference comes from the fact that each error has to be corrected one by one after pass two in CASCADE, while errors can be corrected simultaneously in each pass in BBBSS protocol. It is found that as sifted key generation rate over a private channel increases, error correction process would dominate the net key generation rate, especially for CASCADE.
Table 3: Number of Round Trips

| \( n \) | CASCADE          | BBBS |       |       |       |       |
|--------|------------------|------|-------|-------|-------|-------|
|        | \( \varepsilon \) | 0.01 | 0.05  | 0.1   | 0.14  | 0.01  | 0.1   | 0.1   | 0.15  |
| 500    | 26.12            | 58.66| 81.95 | 105.21| 26.08 | 36.04 | 38.77 | 38.57 |
| 1000   | 42.70            | 107.41| 154.89| 202.34| 33.26 | 44.16 | 47.60 | 47.56 |
| 2000   | 74.86            | 205.09| 300.89| 396.46| 42.47 | 54.17 | 57.87 | 57.17 |
| 5000   | 170.21           | 498.91| 739.75| 979.01| 56.20 | 68.58 | 72.29 | 71.17 |
| 10000  | 329.23           | 987.53| 1470.49| 1949.51| 67.42 | 80.14 | 82.73 | 82.27 |

3 REDUCTION OF RENYI ENTROPY BY ERROR CORRECTION

The amount of information leaked during the error correction process would reduce Eve’s Renyi entropy about the legitimate users’ sequence. The precise estimation of the amount of this reduction is very important for evaluating the key generation rate. It was shown in Th. 9 of [3] that in the case where the raw key is generated by many independent repetitions of a random experiment, each bit leaked during the error correction process reduces Eve’s Renyi entropy by only about one. The properties of the so-called \( \varepsilon \)-strongly typical sequences defined below are used to prove this result.

Definition 1 (\( \varepsilon \)-strongly typical set[3]) Let \( X \) be a random variable distributed according to \( P_X \) over some finite set \( \mathcal{X} \) where it is assumed that \( P_X(x) > 0 \) for all \( x \in \mathcal{X} \). Let \( x^n = [\{x_1, \ldots, x_n\}] \) be a sequence of \( n \) digits of \( \mathcal{X} \) and define \( N_a(x^n) \) to be the number of occurrences of the symbol \( a \in \mathcal{X} \) in the sequence \( x^n \). A sequence \( x^n \in \mathcal{X}^n \) is called \( \varepsilon \)-strongly typical if and only if \((1 - \varepsilon)P_X(a) \leq \frac{N_a(x^n)}{n} \leq (1 + \varepsilon)P_X(a)\) for all \( a \in \mathcal{X} \).

It was claimed that the occurrence probability of each sequence in the \( \varepsilon \)-strongly typical set is asymptotically identical, and this property was used to prove Th. 9 in [3]. This claim is wrong, however – the ratio of the maximum occurrence probability of a sequence in the \( \varepsilon \)-strongly typical set to the minimum one increases as the sequence length becomes longer[5]. For the binary case, assuming that \( P_X(0) = p(\leq 1/2) \), the ratio is \((1 - p)^{2np\varepsilon} \). As an example, let us consider the case that \( n=100,000 \), \( P_X(0)=0.1 \), \( \varepsilon=0.01 \). The minimum occurrence probability \( P_X^{\min}(x^n) = 2.52 \times 10^{-14214} \), the maximum \( P_X^{\max}(x^n) = 1.78 \times 10^{-14923} \), then the ratio \( P_X^{\max}(x^n)/P_X^{\min}(x^n) = 7.06 \times 10^{190} \). In this case, the probability \( Pr [X^n \in S^n] \) that the sequence is in a \( \varepsilon \)-strongly typical set \( S^n(\varepsilon) \) is 0.711. Therefore, Th. 9 does not apply to this case. The amount of information additional to that predicted by Th. 9 can be obtained by Th. 6 of [3]. We find the upper bound on the reduction of Eve’s Renyi entropy is 2485 bits more than that predicted by Th. 9.

Next, let us consider the case that the legitimate users cover the announced bits by a pre-shared secret key. The covered announced bits never reduces Eve’s Renyi entropy for error correction by non-interactive linear code. This is because the pre-shared secret key and the announced bits are mutually independent. Then, the reduction of the size of the
final key by the error correction process is just as the same as the number of announced bits. In the case of interactive error correction, however, it is not apparent whether the announced bits reduce Eve’s Renyi entropy or not, because interactive error correction reveals possible information on the positions of all errors, which could help Eve. Thus, it should be verified that this problem is addressed in, e.g., [6], where the security of CASCADE using the above method of covering the parity information has been studied.

References

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