Topological chiral magnonic edge mode in a magnonic crystal

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Topological phases have been explored in various fields in physics such as spintronics, photonics, liquid helium, correlated electron system and cold-atomic system. This leads to the recent foundation of emerging materials such as topological band insulators, topological photonic crystals and topological superconductors/superfluid. In this paper, we propose a topological magnonic crystal which provides protected chiral edge modes for magnetostatic spin waves. Based on a linearized Landau-Lifshitz equation, we show that a magnonic crystal with the dipolar interaction acquires spin-wave volume-mode band with non-zero Chern integer. We argue that such magnonic systems are accompanied by the same integer numbers of chiral spin-wave edge modes within a band gap for the volume-mode bands. In these edge modes, the spin wave propagates in a unidirectional manner without being scattered backward, which implements novel fault-tolerant spintronic devices.

I. INTRODUCTION

Topological phases in condensed matters have been attracting much attention because of their fascinating physical properties. Discoveries of topological band insulators open up emerging research paradigm on spin-orbit interaction physics. Relativistic spin-orbit interaction in the topological band insulator endows its Bloch electron bands with non-trivial global phase structures, which lead to novel surface metallic states. Superconductor analogues of topological insulators have exotic edge modes, which are composed only of Majorana fermion. Some aspects of these bound states are experimentally confirmed, fostering much prospect of the realization of quantum computers. Photons analogue of quantum Hall phase with chiral edge modes are proposed theoretically and are subsequently designed in actual photonics crystals. Unidirectional propagations of electromagnetic wave along these edge modes were experimentally observed, which provides these metamaterials with unique photonic functionality.

In this paper, we theoretically propose a spin-wave analogue of topological phases, which has topologically-protected chiral edge mode for the spin wave propagation. Spin wave is a collective propagation of precessional motions of magnetic moments in magnets. Depending on its wavelength, spin waves are classified into two categories. One is exchange spin-wave with the shorter wavelength, whose motion is driven by the quantum-mechanical exchange interactions (‘exchange-dominated’ region). The other is magnetostatic spin wave with the longer wavelength whose propagation is caused by the long-range dipolar interaction (‘dipolar’ region). Magnonics research investigates how these spin waves propagate in the sub-micrometer length scale and sub-nanosecond time scale. Like in other solid-state technologies such as photonics and plasmonics, the main application direction is to explore ability of the spin wave to carry and process information. Especially, the propagation of spin waves in periodically modulated magnetic materials dubbed as magnonic crystals are of one of its central concern. Owing to the periodic structuring, the spin-wave volume-mode spectrum in magnonic crystal acquires allowed frequency bands of spin wave states and forbidden-frequency bands (band gap).

In the next section, we introduce a topological Chern number in these volume-mode bands with a band gap. In sec. III, we consider a two-dimensional magnonic crystal (MC), where the Chern number for the lowest spin-wave volume mode takes non-zero integer values in the dipolar region. In sec. IV, we argue that the nonzero Chern integer for the lowest volume-mode band results in the same integer numbers of topological chiral edge modes (surface modes; Fig. 0), whose dispersions run across the band gap between the lowest volume-mode band and the second lowest volume-mode band. The existence of the topological chiral edge mode in the MC is further justified by a micromagnetic simulation in sec. V. The relevant length scale for the magnonic crystal turns out to be from sub-μm to sub-mm, well within the range of nanosize fabrication. Unidirectional propagations of spin waves along the edge modes are experimentally measurable especially in yttrium iron garnet (YIG), where the coherence length of magnons is on the order of centimeters. Based on these observations, we argue in sec. VI that the topological chiral edge modes can be easily channelized, twisted, split and manipulated, which enables to construct novel magnonic devices such as spin-wave logic gate and spin-wave current splitter.

II. CHERN NUMBER IN BOSON SYSTEMS

A. bosonic BdG Hamiltonian

To introduce topological Chern number for spin-wave volume-mode band, we first consider a quadratic form of
general boson Hamiltonian:
\[
\hat{H} = \frac{1}{2} \sum_k \begin{bmatrix} \beta_k^1 & \beta_{-k}^1 \\ \beta_k^2 & \beta_{-k}^2 \end{bmatrix} \cdot H_k \cdot \begin{bmatrix} \beta_k^1 \\ \beta_{-k}^1 \end{bmatrix},
\]
where \( \beta_k^j = [\beta_{1,k}^j, \cdots, \beta_{N,k}^j] \) denote spin-wave (boson) creation operators. Describing volume-type modes, the operators are already Fourier-transformed in a two-dimensional space with the periodic boundary conditions and the wavevector \( \mathbf{k} \equiv (k_x, k_y) \). \( N \) is a number of internal degrees of freedom considered within a unit cell. A \( 2N \times 2N \) Hermitian matrix \( (H_k) \) stands for a bosonic Bogoliubov-de Gennes (BdG) Hamiltonian, whose explicit form will be derived from a linearized Landau-Lifshitz equation later. With the magnetic dipolar interaction, the Hamiltonian thus derived acquires not only \( N \) by \( N \) normal parts (particle-hole channel), \( a_k \) and \( a_{-k}^\dagger \), but also \( N \) by \( N \) anomalous parts (particle-particle channel), \( b_k \) and \( b_{-k}^\dagger \):
\[
H_k = \begin{bmatrix} a_k & b_k \\ b_{-k}^\dagger & a_{-k}^\dagger \end{bmatrix}.
\]

Such a bosonic BdG Hamiltonian is diagonalized in terms of a para-unitary matrix \( T_k \) instead of a unitary matrix \( U_k \):
\[
T_k^\dagger H_k T_k = \begin{bmatrix} E_k & E_{-k} \\
E_{-k} & -E_k \end{bmatrix},
\]
with \( \gamma_k = [\gamma_{1,k}^1, \cdots, \gamma_{N,k}^1] \). \( E_k \) is a diagonal matrix, whose diagonal element gives a dispersion relation for respective volume-mode band. The orthogonality and completeness of a new basis (\( \gamma \)-field) are derived as
\[
T_k \sigma_3 T_k^\dagger = \sigma_3, \ T_k \sigma_3 T_k^\dagger = \sigma_3
\]
respectively, where a diagonal matrix \( \sigma_3 \) takes \( \pm 1 \) in the particle/hole space, i.e. \( [\sigma_3]_{jm} = \delta_{jm} \sigma_j \) with \( \sigma_j = \pm 1 \) for \( j = 1, \cdots, N \) and \( \sigma_j = -1 \) for \( j = N + 1, \cdots, 2N \). This additional structure comes from the fact that the magnon obeys the boson statistics. Each column vector encoded in the paraunitary matrix \( T_k \) stands for (periodic part of) Bloch wavefunction for the respective volume-mode band.

Provided that a Hermite matrix \( H_k \) is unitarily equivalent to a positive-definite diagonal matrix, a para-unitary matrix \( T_k \) which diagonalizes \( H_k \) can be obtained by a method based on the Cholesky decomposition. In the method, we first decompose \( H_k \) into a product between an upper triangle matrix \( K_k \) and its Hermite conjugate; \( H_k = K_k^\dagger K_k \). The unitarily positive definiteness of \( H_k \) always allows this decomposition and also guarantees the existence of \( K_k^{-1} \). We next introduce a unitary matrix \( U_k \) which diagonalizes a Hermite matrix \( W_k \equiv K_k \sigma_3 K_k^\dagger \):
\[
U_k^\dagger W_k U_k = \begin{bmatrix} E_k & \cdot \\
\cdot & -E_k \end{bmatrix}.
\]
Owing to the Sylvester’s law of inertia, both \( E_k \) and \( -E_k \) can be made positive-definite \( N \) by \( N \) diagonal matrices. One can see a posteriori that these two diagonal matrices are nothing but those in the right hand side of Eq. \( (6) \). Namely, the following paraunitary matrix satisfies Eq. \( (3) \):
\[
T_k = K_k^{-1} U_k \begin{bmatrix} E_k^\dagger & \cdot \\
\cdot & E_{-k}^\dagger \end{bmatrix}.
\]
and it diagonalizes the Hamiltonian as;
\[
H_k T_k = \sigma_3 T_k \begin{bmatrix} E_k & \cdot \\
\cdot & -E_{-k} \end{bmatrix}.
\]

The upper \( N \) by \( N \) diagonal matrix in the right hand side, \( E_k \), is positive definite, so that we will referred to them as (dispersions for) ‘particle bands’, while the lower \( N \) by \( N \) diagonal matrix, \( -E_{-k} \), is negative definite, whose diagonal elements are thus referred to as (dispersion for) ‘hole bands’. Due to the trivial redundancy, \( \sigma_1 H_k \sigma_1 = H_{-k} \) with \( [\sigma_1]_{jm} = \delta_{|j-m|,N} \), either one of these two \( N \) by \( N \) diagonal matrices gives the full information of the dispersions for the volume-mode bands.

B. Chern integers in bosonic BdG systems

To introduce the Chern number for the \( j \)-th volume-mode band, let us first define a projection operator \( P_j \) in the \( 2N \) dimensional vector space, which filters out those bands other than the \( j \)-th volume-mode band at each momentum point \( k \):
\[
P_j \equiv T_k \Gamma_j \sigma_3 T_k^\dagger \sigma_3.
\]
Here \( \Gamma_j \) is a diagonal matrix taking \( +1 \) for the \( j \)-th diagonal component and zero otherwise. Eq. \( (6) \) suggests
that the operator obeys \( \sum_j P_j = 1 \) and \( P_j P_m = \delta_{jm} P_j \). In terms of the projection operator, the Chern number for the \( j \)-th band is given as follows:

\[
C_j = \frac{i e_{j\mu\nu}}{2\pi} \int_{BZ} dk \text{Tr} [(1 - P_j) (\partial_{\nu\mu} P_j) (\partial_{\mu\nu} P_j)],
\]

(7)

where the integral is over the first Brillouin zone (BZ). To see this, we follow the same argument as in the quantum Hall case and introduce different field strength (Berry’s curvature) \( B_j \) and gauge connection (gauge field) \( (A_{j,x}, A_{j,y}) \) for each volume-band mode:

\[
B_j(k) = \partial_{k_x} A_{j,y}(k) - \partial_{k_y} A_{j,x}(k),
\]

(8)

\[
A_{j,\nu}(k) = i \text{Tr} [\Gamma_j \sigma_3 T_k^j \sigma_3 (\partial_{k_\nu} T_k)],
\]

(9)

with \( j = 1, \cdots, 2N \). The Chern number for a volume-mode band reduces to an integral of the respective Berry’s curvature over the BZ,

\[
C_j = \frac{1}{2\pi} \int_{BZ} d^2k B_j(k).
\]

(10)

Such a surface integral is zero, provided that the respective gauge field can be defined uniquely and smoothly over the first BZ. When \( T_k \) has a zero somewhere on the BZ, for any \( m \) \((m = 1, \cdots, 2N)\), however, the gauge field for the \( j \)-th band cannot be chosen uniquely over the BZ. In this case, it is necessary to decompose the BZ into two overlapped regions \((H_1 \cup H_2 = \text{BZ})\) and \( H_1 \cap H_2 = \partial H_1 = -\partial H_2 \equiv S \), so that \( T_k \) does not have any zero within one region \((H_1)\), while \( T_k \) has no zero inside the other \((H_2)\). In the former region, we then take the gauge, \([T_k]^{(1)}_{m,j}\), such that \([T_k]^{(1)}_{1,j}\) is always real positive, while take another gauge in the other, making \([T_k]^{(2)}_{2,j}\) to be always real positive. Provided that the \( j \)-th band considered is isolated from the others \((E_{k,j} \neq E_{k,m\neq j} \) for any \( k \)), these two gauge choices are related to each other by a \( U(1) \) transformation,

\[
T_k^{(2)} \Gamma_j = T_k^{(1)} \Gamma_j e^{i\theta_k},
\]

(11)

on \( k \in H_1 \cap H_2 \). Now that the gauge of \( T_k^{(1)} \Gamma_j \) and the gauge of \( T_k^{(2)} \Gamma_j \) are uniquely defined in \( H_1 \) and \( H_2 \) respectively,

\[
A_{j,\nu}^{(m)}(k) = i \text{Tr} [\Gamma_j \sigma_3 T_k^{(m)} \sigma_3 (\partial_{k_\nu} T_k^{(m)})]
\]

(12)

\((m = 1, 2)\) are smooth functions in each of these two regions respectively. The Stokes theorem is applied separately, so that Eq. (11) is calculated as,

\[
C_j = \frac{1}{2\pi} \int_S dk \cdot (A_j^{(1)} - A_j^{(2)}) = \frac{1}{2\pi} \int_S dk \cdot \nabla_k \theta_k,
\]

(13)

with \( \nabla_k \equiv (\partial_{k_x}, \partial_{k_y}) \). Two regions share a boundary \((S)\), which forms a closed loop, \( \theta_k \) in Eq. (11) has a \( 2\pi n \) phase winding along the loop. This leads to \( C_j = n \) \((n \in \mathbb{Z})\).

### C. Sum rule for Chern integer

When all volume-mode bands in a system are physically stable, the sum of the Chern integer over all particle bands and that over all hole bands are zero respectively;

\[
\sum_{j=1}^N C_j = \sum_{j=N+1}^{2N} C_j = 0.
\]

(14)

To see this, let us linearly interpolate a \( 2N \) by \( 2N \) spin-wave Hamiltonian and the \( 2N \) by \( 2N \) unit matrix;

\[
H_{k,\lambda} = (1 - \lambda)H_k + \lambda 1
\]

(15)

We assume that \( H_k \) is paraunitarily equivalent to a diagonal matrix whose elements are all positive for any wavevector \( k \); all volume-mode bands obtained from original spin-wave Hamiltonian \((\lambda = 0)\) are physically stable. Thanks to the Sylvester’s law of inertia, such an Hermite matrix is unitarily equivalent to a diagonal matrix whose elements are all positive definite. Clearly, so is any \( H_{k,\lambda} \) during \( 0 \leq \lambda \leq 1 \). Given the unitarily positive definiteness, we can then introduce from Eq. (1) a paraunitary matrix \( T_k,\lambda \) which transforms \( H_{k,\lambda} \) into a diagonal form as;

\[
H_{k,\lambda} T_{k,\lambda} = \sigma_3 T_k \begin{bmatrix} E_{k,\lambda} & 0 \\ 0 & -E_{-k,\lambda} \end{bmatrix}
\]

with positive-definite \( N \) by \( N \) diagonal matrices \( E_{k,\lambda} \) and \( E_{-k,\lambda} \). With \( T_{k,\lambda} \), the Chern integer can be explicitly defined as a function of \( \lambda \) for \( 0 \leq \lambda \leq 1 \), \( C_j(\lambda) \).

The sum of Chern integer over a group of bands does not change, unless some band in the group forms a band touching (frequency degeneracy) with bands outside the group. The positive definiteness of \( E_{k,\lambda} \) and \( E_{-k,\lambda} \) means that particle bands obtained from \( H_{k,\lambda} \) are always in positive frequency regime, while hole bands are in negative frequency regime: during the interpolation \((0 \leq \lambda \leq 1)\), they are always disconnected from each other in frequency. Thus, the sum of the Chern integer over all particle bands does not change during the interpolation,

\[
\sum_{j=1}^N C_j(\lambda = 0) = \sum_{j=1}^N C_j(\lambda = 1).
\]

Since a paraunitary matrix at \( \lambda = 1 \) is trivial, \( T_{k,\lambda = 1} = 1 \), the right hand side reduces to zero and so is the case with the original spin-wave Hamiltonian \((\lambda = 0)\). In summary, Eq. (14) is derived only from the para-unitarily positive definiteness of \( 2N \) by \( 2N \) Hermite matrix \( H_k \). As a corollary of eq. (15), one can argue that any topological chiral edge modes obtained from proper spin-wave approximations appear only at a finite frequency region (see the sec. IV for the argument).

In the following, we show that a two-dimensional bi-component magnonic crystal (MC) with the dipolar interaction supports spin-wave bands with non-zero Chern integers.
III. MAGNONIC CRYSTAL, DIPOLAR INTERACTION AND CHERN INTEGER FOR VOLUME-MODE BANDS

A. plane-wave theory for magnonic crystals

The MC considered is a ferromagnetic system with its magnetization and exchange interaction modulated periodically in the 2-dimensional \((x-y)\) direction. For simplicity, we assume that the system is translationally symmetric along the \(z\)-direction, whereas the subsequent results are expected to be similar when a thickness of the system in the \(z\)-direction becomes finite (but large). The system is composed of two kinds of ferromagnets; iron and YIG. The unit cell of the MC is an \(a_x \times a_y\) rectangle, inside which iron is embedded into circular regions, while the remaining region is filled with YIG (Fig. I). A uniform magnetic field \(H_0\) is applied along the \(z\) direction, such that the static ferromagnetic moment \(M_s\) in both regions is fully polarized in the longitudinal direction. Propagation of the transverse moments \((m_x, m_y)\) is described by a linearized Landau-Lifshitz equation \(^{27–30}\):

\[
\frac{1}{|\gamma|\mu_0} \frac{dm_{\pm}}{dt} = 2iM_s(\nabla \cdot Q\nabla)m_{\pm} \mp 2im_s(\nabla \cdot Q\nabla)M_s \\
\mp iH_0m_{\pm} \pm ih_{\pm}M_s \tag{16}
\]

with \(\nabla \equiv (\partial_x, \partial_y)\), \(m_{\pm} = m_x \pm im_y\) and \(h_{\pm} = h_x \pm ih_y\); \((h_x, h_y)\) stands for transverse component of long-ranged magnetic dipolar field \(h\), which is related to the ferromagnetic moment \(m \equiv (m_x, m_y, M_s)\) via the Maxwell equation, i.e. \(\nabla \times h = c^{-1}\partial_t e_z\) and \(\nabla \cdot (h + m) = 0\). The former two terms in the right hand side of Eq. \((16)\) come from short-ranged exchange interaction, where \(Q\) denotes a square of the exchange interaction length. \(M_s\) and \(Q\) take values of iron inside the circular region and those of YIG otherwise. A filling fraction of the circular region with respect to the total area of the unit cell is represented by \(f\). We further employ magneto-static approximation, replacing the Maxwell equations by \(\nabla \times h = 0\) and \(\nabla \cdot (h + m) = 0\):

\[h_\nu = -\partial_\nu \Psi, \quad \Delta \Psi = \partial_x m_x + \partial_y m_y, \tag{17}\]

with \(\nu = x, y\). This in combination with Eq. \((10)\) gives a closed equation of motion (EOM) for the transverse moment.

In order to obtain band dispersions and Chern integers for volume-mode bands, we need to reduce the EOM into a generalized eigenvalue problem with a BdG Hamiltonian \((H_k)\) defined in the form of Eq. \((10)\). To this end, we first normalize the transverse moment, to introduce a Holstein-Primakov (HP) field as \(^{39}\):

\[
\beta(r) = \frac{m_{\pm}(r)}{\sqrt{2M_s(r)}}, \quad \beta^\dagger(r) = \frac{m_{-}(r)}{\sqrt{2M_s(r)}}. \tag{18}\]

In terms of the HP field, the Landau-Lifshitz equation is properly symmetrized as,

\[
\frac{d\beta^\dagger}{dt} = 4i\alpha(\nabla \cdot Q \nabla)\beta - 4i\beta(\nabla \cdot Q \nabla)\alpha^2 - iH_0\beta - i\alpha \partial_\nu \Psi, \tag{19}\]

\[
\Delta \Psi = \partial_\nu^+ (\alpha \beta^\dagger) + \partial_\nu^-(\alpha \beta), \quad \partial_\nu^\pm = \partial_x \pm i\partial_y, \tag{20}\]

with \(\alpha(r) \equiv \sqrt{M_s(r)/2}\), \(|\gamma|\mu_0\) was omitted in Eq. \((19)\) for clarity. The static magnetization and exchange interaction are spatially modulated with the lattice periodicity;

\[
\alpha(r) = \sum_G \alpha(G) e^{iG \cdot r}, \quad Q(r) = \sum_G Q(G) e^{iG \cdot r},
\]

with the reciprocal vectors \(G\), \(\alpha^*(G) = \alpha(-G)\) and \(Q^*(G) = Q(-G)\). It follows from the Bloch theorem that \(m(r)\) and \(\Psi(r)\) take a form;

\[
\beta(r) = \sum_k \sum_G \beta_k(G) e^{i(k+G) \cdot r},
\]

\[
\beta^\dagger(r) = \sum_k \sum_G \beta_k^\dagger(G) e^{-i(k+G) \cdot r},
\]

and

\[
\Psi(r) = \sum_k \sum_G \Psi_k(G) e^{i(k+G) \cdot r},
\]

where the \(k\)-summation is taken over the first BZ.

In terms of these Fourier modes, an equivalent generalized eigenvalue problem with a quadratic Hamiltonian for the HP field is derived as

\[
i\frac{d}{dt} \begin{bmatrix} \beta_k \\ \beta_k^\dagger \end{bmatrix} = \begin{bmatrix} \beta_k \\ \beta_k^\dagger \end{bmatrix}, \quad \hat{\mathcal{H}} = \sigma_3 H_k \begin{bmatrix} \beta_k \\ \beta_k^\dagger \end{bmatrix}, \tag{21}\]

with

\[
\mathcal{H} = \sum_{k \nu > 0} \begin{bmatrix} \beta_k^\dagger & \beta_{-k} \end{bmatrix} H_k \begin{bmatrix} \beta_k \\ \beta_{-k} \end{bmatrix},
\]

the reciprocal vectors \(G\), \(\alpha^*(G) = \alpha(-G)\) and \(Q^*(G) = Q(-G)\). It follows from the Bloch theorem that \(m(r)\) and \(\Psi(r)\) take a form;

\[
\beta(r) = \sum_k \sum_G \beta_k(G) e^{i(k+G) \cdot r},
\]

\[
\beta^\dagger(r) = \sum_k \sum_G \beta_k^\dagger(G) e^{-i(k+G) \cdot r},
\]
\[
B_k[G,G'] \equiv 4 \sum_{G_1,G_2} \alpha(G - G_1)Q(G_1 - G_2)\alpha(G_2 - G') (k + G_1) \cdot (k + G_2)
- 4 \sum_{G_1,G_2} Q(G_1)\alpha(G_2)\alpha(G - G' - G_1 - G_2) (G - G') \cdot (G - G' - G_1).
\]

\[\alpha(r) \text{ has a discontinuity at the boundary between these two regions, the last term in Eq. (25), } Q(\partial_r \alpha)\partial_r \alpha, \text{ diverges, since it contains the second derivative with respect to a spatial coordinate along the radial direction. Physically, such an infrared divergence is removed by a smooth variation of the saturation magnetization at the boundary. For simplicity, we interpolate } \alpha(r) \text{ as a linear function of the radial coordinate measured from the center of the circular region;}
\]

\[\alpha(r) = \begin{cases} 
\alpha_1 & (|r| < R_0) \\
\alpha_1 - \frac{|r| - R_0}{R_1 - R_0} (\alpha_1 - \alpha_2) & (R_0 < |r| < R_1) \\
\alpha_2 & (R_1 < |r|) 
\end{cases}
\]

A discontinuity in \(Q(r)\) is also removed by the same linear interpolation. Actual numerics are carried out with \((r_0, r_1) \equiv (R_0/\lambda, R_1/\lambda) = (0.10, 0.125)\), where \(\lambda\) denotes the linear dimension of the unit cell size, \(\lambda \equiv \sqrt{a_x a_y}\). As for the material parameter, we used \((M_{s,1}, M_{s,2}) = (1.8, 0.19)\) [A/µm] and \((\sqrt{Q_1}, \sqrt{Q_2}) = (33, 130)\) [A] with \(\alpha_j \equiv \sqrt{M_{s,j}}\), while parameters of iron (Fe), cobalt (Co), and YIG are \((M_s [A/µm], \sqrt{Q} [A]) = (1.7, 21), (1.4, 29), \text{ and } (0.14, 184) \text{ respectively.}

Under \(T_k\), the EOM is para-unitarily equivalent to

\[i \frac{d}{dt} \begin{bmatrix} \gamma_k \\ \gamma_{-k} \end{bmatrix} = \begin{bmatrix} E_k \\ -E_{-k} \end{bmatrix} \begin{bmatrix} \gamma_k \\ \gamma_{-k} \end{bmatrix}.
\]

\([E_k]_j\) gives a dispersion relation for the \(j\)-th volume-mode band \((j = 1, \cdots)\), while the Chern integer is calculated from \(T_k\) via Eqs. (\text{[3910]}) In its numerical evaluation, we employed an algorithm based on a ‘manifestly gauge-invariant’ description of the Chern integer.\[\]
B. role of dipolar interaction

When a system considered is either time-reversal symmetric; $H_{-k} = H_k$, or mirror-symmetric with a mirror plane perpendicular to the $xy$ plane, e.g. $H_{(k_x, k_y)} = H_{(-k_x, -k_y)}$, the Berry’s curvature satisfies $-B_j(k) = B_j(-k)$ or $-B_j(k_x, k_y) = B_j(k_x, -k_y)$, respectively, which reduces the Chern integer to zero. In the present situation, however, the magnetic dipolar field brings about complex-valued phase factors in the anomalous part, $I_k \neq I_k^\star$, which removes from Eq. (22) both the time-reversal symmetry and the mirror symmetries. Without periodic modulation of the saturation magnetization, $\alpha = \alpha 1$, these phase factors can be erased by a proper gauge transformation, $\beta_k^\dagger \rightarrow \beta_k^\dagger \sqrt{T_k}$ and $\beta_{-k} \rightarrow \beta_{-k} \sqrt{T_k}$, so that both symmetries are recovered. In the presence of the periodic modulations, $\alpha \neq \alpha 1$, however, these two symmetries are generally absent in the 2-d MCs and the bosonic Chern integer can take a non-zero integer value.

This situation is quite analogous to what the relativistic spin-orbit interaction does in ferromagnetic metals. Moreover, contrary to the spin-orbit interaction, a strength of the dipolar interaction is an experimentally tunable parameter in MCs. When characteristic length scale of MC (linear dimension of the unit cell size $\lambda \equiv \sqrt{a_x a_y}$) becomes larger than the typical exchange length $\sqrt{\mu_a}$, the dipolar interaction is expected to prevail over the exchange interaction.

C. Chern integer of volume-mode bands and role of band touchings

In fact, we found that the Chern integer of the lowest magnonic band, $C_1$, is always quantized to be 2 for the longer $\lambda$, while the integer reduces to zero for the shorter $\lambda$ (Fig. 2b). The respective quantization is protected by a finite direct band gap between the lowest band and the second lowest band (Fig. 2b). In the intermediate regime of $\lambda$, these two bands get closer to each other. With a four-fold rotational symmetry ($r = a_y/a_x = 1$), the gap closes at the two $X$-points at a critical value of $\lambda$ ($\sim 0.28\mu$m), where the two bands form gapless Dirac spectra (Fig. 2). Without the four-fold symmetry ($r \neq 1$), the band touching at one of the two $X$ points and that of the other occur at different values of $\lambda$. These band-touchings are denoted as $P_1(\pi, 0, \lambda_{c,1})$ and $P_2(0, \pi, \lambda_{c,2})$ in a 3-dimensional parameter space subtended by two wavevectors $k_x$, $k_y$ and the unit cell size $\lambda$ (Fig. 3).

The band touchings endow the lowest volume-mode band with non-zero Chern integers in the longer $\lambda$ region. In generalized eigenvalue problems as well as usual eigenvalue problems, a band-touching point in the 3-d parameter space plays role of a dual magnetic monopole (charge). The corresponding dual magnetic field is generalized from Eq. (8) as a rotation of three component gauge field $A_j = (A_{j,x}, A_{j,y}, A_{j}\lambda)$;

$$B_j = \nabla \times A_j$$

with $\nabla \equiv (\partial_{k_x}, \partial_{k_y}, \partial_\lambda)$. Here the third component of the gauge field $A_{j,\lambda}$ is introduced in the same way as in Eq. (28);

$$A_{j,\lambda} \equiv i\text{Tr}[\Gamma_j \sigma_3 T_{k}^\dagger \sigma_3 (\partial_\lambda T_k)]$$

$j$ specifies either one of the two magnonic bands which form the band-touching. At the band-touching point, the dual magnetic field for the respective bands has a dual magnetic charge, whose strength is quantized to be $2\pi$ times integer (see Appendix). A numerical evaluation tells that the dual magnetic charges for the lowest band.
at the band touching points at P_1 and P_2 are both +2π (Fig. 3):

\[ \nabla \cdot B_1 = 2\pi \delta(\lambda - \lambda_{c,1})\delta(k_x - \pi)\delta(k_y) + 2\pi \delta(\lambda - \lambda_{c,2})\delta(k_x)\delta(k_y - \pi), \]

where \( \lambda_{c,1} < \lambda_{c,2} \) for \( a_y > a_x \) (\( r > 1 \)) and \( \lambda_{c,1} > \lambda_{c,2} \) for \( a_y < a_x \) (\( r < 1 \)).

Because the Chern integer can be regarded as the total dual magnetic flux penetrating through the constant λ plane (see Eq. (10)), the Gauss theorem suggests that, when λ goes across either λ = \( \lambda_{c,1} \) plane or λ = \( \lambda_{c,2} \) plane, the Chern integer for the lowest magnonic band always changes by unit, e.g.

\[ C_1|\lambda > \lambda_{c,1} > C_1|\lambda_{c,2} > \lambda = 1. \]

Without the four-fold rotational symmetry (\( r \neq 1 \)), the two critical values of λ bound three phases for the lowest magnonic band, the phase with \( C_1 = 0 \) (phase IV), that with \( C_1 = 1 \) (phase III or I) and that with \( C_1 = 2 \) (phase II). In the presence of the four-fold rotational symmetry (\( r = 1 \)), two band touchings occur at the same critical value of λ, where \( C_1 \) increases by two on increasing λ. This leads to a phase diagram shown in Fig. 2, which describes the Chern integer of the lowest magnonic band as a function of the unit cell size λ and the aspect ratio r.

A dual magnetic charge is a quantized topological object, so that, upon any small change of parameters, it cannot disappear by itself. Instead, it only moves around in the 3-d parameter space. As a result, the global structure of the phase diagram depicted in Fig. 2 widely holds true for other combinations of material parameters. For \( r = 1 \) and \( f = \pi \times 10^{-2} \), we found \( \lambda_e = 0.370\text{μm} \) for iron (circular region) and YIG (host), and \( \lambda_e = 0.372\text{μm} \) for cobalt (circular region) and YIG (host). When varying the filling fraction for iron (circular region) and YIG (host) with \( r = 1 \), we found \( \lambda_e = 0.274\text{μm} \) for \( f = 4\pi \times 10^{-2} \), and \( \lambda_e = 0.348\text{μm} \) for \( f = 9\pi \times 10^{-2} \).

IV. CHIRAL SPIN-WAVE EDGE MODE IN MC

The chiral phases with non-zero Chern integers have chiral spin-wave edge modes, which are localized at a boundary with the phase with zero Chern integer (phase IV) or the vacuum. The edge modes have chiral dispersions which go across the band gap between the lowest and the second lowest band.

As an illustrative example, we consider a boundary (y axis) between the MC in phase III (\( C_1 = 1 \)) and MC in the phase IV (\( C_1 = 0 \)). The existence of a chiral spin-wave edge mode at the boundary is shown from a following 2×2 Dirac Hamiltonian derived near their phase boundary \( \lambda = \lambda_{c,1} \) (see appendix),

\[ \mathcal{H}_{\text{eff}} = \omega_0 \tau_0 + \kappa(x) \tau_3 - i \alpha \partial_x \tau_1 - i b \partial_y \tau_2. \] (30)

\( \tau_j \) denotes the Pauli matrices subtended by the two-fold degenerate eigenstates at \( P_j \). \text{a} and \text{b} are positive material parameters. The difference of the Chern integers (\( C_1 \)) for the two phases is represented as a change of sign of a Dirac mass term \( \kappa(x) \): \( \kappa(x) > 0 \) for \( x > 0 \) (phase III) and \( \kappa(x) < 0 \) for \( x < 0 \) (phase IV) (see Fig. 3); \( \lim_{x \to \pm \infty} \kappa(x) = \pm \kappa_{\infty} \). The Hamiltonian has a following eigenstate \( \psi_k(r) \propto e^{i\mu y} e^{-f/2} \kappa(x) dx [1, i]^t \), which is localized at the boundary (\( x = 0 \)). In terms of the surface wavevector \( k \) along the y axis, the corresponding eigen-frequency is given by \( E = \omega_0 + bk \). This connects the lowest magnonic band lying at \( E \leq \omega_0 - \kappa_{\infty} \) and the second lowest band at \( E \geq \omega_0 + \kappa_{\infty} \) (see Fig. 3b,c). The mode is chiral, since the group velocity is always positive, \( v_k \equiv \partial_k E = b \). Similarly, we can easily show that the phase with \( C_1 = 1 \) at \( r > 1 \) (Phase III) or \( r < 1 \) (Phase I) has a chiral edge mode at its boundary with vacuum, whose dispersion crosses the direct band gap at the \( (\pi, 0) \) or \( (0, \pi) \)-point respectively, while the phase with \( C_1 = 2 \) (Phase II) has both at its boundary with vacuum. A number of chiral modes localized at the interface between two MCs with different Chern integers is equal to the difference of the two Chern integers.

Generalizing the arguments so far into the linearly interpolated Hamiltonian defined in eq. (15), we can argue that, in general, a number of those chiral spin-wave edge modes whose dispersions run across a gap between the \( m \)-th and the \( (m + 1) \)-th particle bands, \( N_m \), is equal to the sum of the Chern integer over those particle bands...
FIG. 4: Chiral spin-wave edge modes. a, geometry of the system. bc, wavevector-frequency dispersions for Dirac Hamiltonian with the Pöschl-Teller potential $\kappa(x) = \kappa_\infty \tanh(x/d)$ for $\kappa_\infty d = 0.9a$ (b) and $\kappa_\infty d = 2.9a$ (c). The volume-mode magnonic bands have a band gap, inside which an edge mode has a chiral dispersion. In c there are some nonchiral edge states, whereas in b there is not. Nonetheless, the number of chiral edge modes is one, which is determined solely from the difference between the Chern integers for the two phases.

Below the gap:

$$N_m \equiv \sum_{j=1}^{m} C_j.$$  

(31)

Here clockwise (counterclockwise) chiral edge modes contribute by $+1$ ($-1$) to the number of chiral edge modes, $N_m$ (see Fig. 4a). Namely, the sum rule, eq. (14), suggests that the right hand side of eq. (31) counts the total number of the band touchings (including the sign of the respective dual magnetic charges) which happen between the $m$-th particle band and $(m+1)$-th particle band during the interpolation, $\lambda = 1 \rightarrow \lambda = 0$. On the one end, each band touching (dual magnetic monopole) is accompanied by the emergence of a chiral edge mode between these two bands, whose sense of rotation is either clockwise or counterclockwise, depending on the sign of the respective dual magnetic charge. Since there are no chiral edge modes in the trivial limit ($\lambda = 1$), we can safely conclude eq. (31) at $\lambda = 0$. As a corollary of eq. (31), one can also see that any topological chiral edge modes obtained from legitimate spin-wave approximation appear only at a finite frequency region; never be a gapless mode.

Chiral edge modes proposed in this paper share similar physical properties with well-known Damon-Eshbach (DE) surface mode. The topological modes as well as DE surface mode are propagating in a chiral way along boundaries (surfaces) of the systems, where the propagation directions are parallel (or anti-parallel) to vector products between the polarized ferromagnetic moment and the normal vectors associated with surfaces. Experimental techniques for measuring DE surface mode can be also utilized for detecting the proposed topological chiral edge modes. Possible experiments include Brillouin light scattering (BLS) measurements, time-resolved Kerr microscopy, infrared thermography and scanning local magnetic fields in terms of a wire loop or antenna.

The topological edge mode always has a chiral dispersion within a band gap for volume-mode bands. When a radio frequency (rf) of applied microwave is chosen inside the band gap, spin waves are excited only along these chiral edge modes, while other volume modes remain intact. One can test this situation by changing the position of an input antenna from the boundaries to an interior far from the boundaries. Being protected by the topological Chern integers defined for volume modes, the proposed chiral edge modes are expected to be robust against various perturbations introduced near the boundaries. The robustness can be also tested by changing boundary shape or introducing boundary roughness and obstacle.

Contrary to the DE mode, eq. (31) dictates that the number and the sense of rotation of the topological chiral edge modes are determined by the Chern integer for volume modes below the band gap. In fact, the chiral edge mode in the proposed magnonic crystal rotates along the boundary in the clockwise manner with an up-headed magnetic field (Fig. 4), while the DE surface mode with the same geometry rotates in the counterclockwise way with the up-headed field. Moreover, the Chern integer for a volume-mode band itself can be changed by closing the band gap, as was shown in the previous section. Thus, using band gap manipulation, one can even control the chiral direction or the number of the mode, which enable intriguing spintronic device such as a spin-current splitter (see also sec VI). To our best knowledge, the DE mode in a uniform thin film does not have such properties.

V. MICROMAGNETIC SIMULATION FOR CHIRAL TOPOLOGICAL EDGE MODES

A. simulation procedure and material parameters

To justify the existence of topological chiral spin-wave edge modes in the present MC model, we have numeri-
where $\overline{\sigma}$ is set to the Gilbert damping coefficient of YIG; $\overline{\sigma} = 6.7 \times 10^{-12}$. The magnetic field $\mathbf{H}_{\text{eff}}$ includes a short-ranged exchange field, long-ranged dipolar field $\mathbf{h}$, the static longitudinal external field $H_0 = 0.1T$ and a small temporally alternating transverse field ($H_1 e_x \cos \omega t$) with $H_1 = 1.0$ Oe. A simulated magnonic crystal (MC) is as large as either $28 \mu m \times 28 \mu m$ or $7 \mu m \times 7 \mu m$ in the $x$-$y$ plane, which is composed of $80 \times 80$ or $20 \times 20$ MC unit cells respectively; each unit cell size is $350 \, \text{nm} \times 350 \, \text{nm}$ within the plane. A MC unit cell consists of YIG region and Fe region, where the size of the Fe region is $140 \, \text{nm} \times 140 \, \text{nm}$ ($f = 0.16$). The MC studied in the previous sections is translationally symmetric along the $z$-direction. To mimic this situation in micromagnetic simulation, we take the thickness along the $z$-direction $L_z$ to be sufficiently large ($L_z = 1 \, \text{mm}$).

In actual simulation, each MC unit cell is further discretized into a bunch of smaller elements, each of which is taken in this study as large as $70 \, \text{nm} \times 70 \, \text{nm} \times L_z$ and each element is assigned with one ferromagnetic spin, which is uniformly distributed over the element. Namely, every MC unit cell has 25 spins consisting of 4 Fe spins and 21 YIG spins. The short-ranged exchange stiffness between Fe elements is taken to be $A_{\text{Fe}} = 2.1 \times 10^{-11} \, \text{J/m}$, between YIG elements $A_{\text{YIG}} = 0.437 \times 10^{-11} \, \text{J/m}$, and between Fe and YIG elements $A_{\text{Fe-YIG}} = 1.0 \times 10^{-11} \, \text{J/m}$. These set material parameters to be those in the dipolar regime with $C_1 = 2$ ($H_0 = 0.1T$, $\lambda = 0.35 \mu m$, $f = 1$ and $f = 0.16$).

The super-elongated cell size ($70 \, \text{nm} \times 70 \, \text{nm} \times L_z$ with $L_z = 1 \, \text{mm}$) used in this simulation clearly ignores those spin-wave modes which have nodes along the $z$-direction. In the presence of very large thickness along the $z$-direction (e.g. $28 \mu m \times 28 \mu m \times 1 \, \text{mm}$), however, it is likely that the strong magnetic shape anisotropy pushes up such spin-wave modes into higher frequency regimes; spin-wave excitations in lower-frequency regime, which we focus on in the present study, are mainly dominated by those modes having no node along the $z$-direction. Even for spin-wave modes without nodes along the $z$-direction, their wavefunctions should be certainly modified near the boundaries along the direction. When the thickness along the direction is much larger than the linear dimension within the other two directions ($xy$-plane), however, the modifications of the wavefunctions are also expected to be small.

The time evolution is determined by Eq. (32), which is numerically integrated with a time interval of 1 ps by use of the 4th order Runge-Kutta method. The demagnetization field $\mathbf{h}$ is calculated by the convolution of a kernel which describes the dipole-dipole interaction. With the Gilbert damping term, the system eventually reaches a certain steady state, in which only the spin-wave modes around the external frequency $\omega$ are excited permanently. In order to deduce spatial distribution of spin-wave modes at a given frequency, we have studied steady states ($\omega t \gg 1$) with changing the external frequency.

To compare simulation results with dispersions for the volume-mode bands obtained from the plane-wave theory, we also take a Fourier-transformation of transverse moments over space and time in a steady state. Specifically, we perform a discrete Fourier transformation of $m_+ \equiv m_x + im_y$ with respect to space and time coordinate as:

$$m_+(k_x, k_y, \omega) \equiv \sum_{X, Y} m_+(X, Y, j \Delta T) e^{ik_x X + ik_y Y - i\omega j \Delta T}.$$

A contour-plot of (absolute value of) the left hand side as a function of $k_x$, $k_y$ and $\omega$ is expected to give dispersion relations for spin-wave volume-mode bands.

**B. result**

Throughout the micromagnetic simulation, we found that a lower frequency region can be roughly classified.
where frequency regimes correspond to the lowest volume-mode the transverse moments (Fig. 5) indicates that these two spin-wave excitations in steady states are always dis-

FIG. 6: Snapshots of spatial distribution of spin-wave excitations in steady states. In this calculation, the system size is taken to be 28μm × 28μm. A normalized transverse component of the ferromagnetic spin, $m_x \equiv n_x/n_{\text{max}}$, are plotted, where $n_{\text{max}}$ denotes the maximum value of $\sqrt{n_x^n + n_y^n}$ with $(n_x, n_y, n_z) \equiv m/M_s$. The static magnetic field is taken along the +z direction. a, snapshot of $m_x$ at $t = 100$ ns with the external frequency $\omega_0 < \omega < \omega_1$ ($\omega = 25.6$ GHz). b, snapshot at $t = 100$ ns with $\omega_1 < \omega < (\omega = 27.8$ GHz). c, snapshot at $t = 100$ ns with $\omega_1 < \omega < \omega_2$ ($\omega = 26.8$ GHz). d,e,f, Spin-wave edge modes at $\omega = 26.8$ GHz are propagating in a chiral way (d, $t = 100$ ns, e, $t = 100.02$ ns, f, $t = 100.04$ ns)

into three characteristic regimes: two volume-mode frequency regimes, (i) $\omega_0 < \omega < \omega_1$ and (ii) $\omega_2 < \omega$, and a band-gap regime for volume modes, (ii) $\omega_1 < \omega < \omega_2$. Below $\omega_0$, the system remains intact against the small alternating transverse field, indicating no spin-wave modes for $\omega < \omega_0$. Within the volume-mode frequency regimes (i) $\omega_0 < \omega < \omega_1$ (Fig. 3) and (iii) $\omega_2 < \omega$ (Fig. 5)), spin-wave excitations in steady states are always distributed over the entire system. A comparison between Fig. 2b, and the contour-plot of the Fourier-transform of the transverse moments (Fig. 5) indicates that these two frequency regimes correspond to the lowest volume-mode band and higher volume-mode bands respectively.

On the one hand, a steady state in the intermediate frequency regime, $\omega_1 < \omega < \omega_2$, has almost no weight for volume modes. Instead, spin-wave excitations in the intermediate regime are localized only around the boundaries of the system (see Fig. 3), indicating the existence of spin-wave edge modes. Moreover, these edge modes propagate in a chiral way (see Fig. 2i,e,f), whose direction is consistent with a chiral direction determined from the Chern integer in the dipolar regime found in Fig. 2. $C_1 = +2 > 0$. These observations indicate that the spin-wave edge modes found in the intermediate frequency regime is nothing but the topological chiral spin-wave edge modes described in the previous section. In fact, the three frequency regimes are qualitatively consistent with the plane-wave-theory calculation (Fig. 2b); $(\omega_0, \omega_1, \omega_2) = (25.5$ GHz, $26.0$ GHz, $27.5$ GHz) for $28\mu$m × $28\mu$m × $1$ mm, and (26.2 GHz, 27.1 GHz, 30.0 GHz) for $7\mu$m × $7\mu$m × $1$ mm. Our micromagnetic simulation with a shorter sample thickness ($L_z = 200\mu$m) also justifies the existence of topological chiral modes.

VI. DISCUSSION

By calculating a newly-introduced bosonic Chern integer for spin wave bands, we argue that a two-dimensional normally-magnetized magnonic crystal acquires chiral edge modes for magnetostatic wave in the dipolar regime. Each mode is localized at the boundary of the system, carrying magnetic energies in a unidirectional way. Thanks to the topological protection, spin-wave propagations along these edge modes are robust against imperfections of the lattice periodicity and boundary roughness; they are free from any types of elastic backward scatterings with moderate strength. This robustness makes it possible to implement novel fault-tolerant magnonic device such as spin-wave current splitter and a magnonic Fabry-Perot interferometer as discussed below.

The chiral spin-wave edge modes studied in this paper can be easily twisted or split by changes of the size ($L$) and shape ($r$) of the unit cell, which we demonstrate in Fig. 7a,b. In Fig. 7a, the MC in the phase II ($r = 1$) is connected with the other MC in the phase III ($r > 1$), whose Chern integer for the lowest band differ by unit. A boundary between these two MC systems supports the chiral spin-wave edge mode which runs across the direct band gap at (0, $\pi$) point. This means that the two chiral edge modes propagating along the boundary of the the MC in the phase II ($r = 1$) are spatially divided into two, where one mode goes along the boundary of the MC in the phase III ($r > 1$), while the other goes along the boundary between these two MCs. This configuration realizes a spin-wave current splitter, an alternative to those proposed in other geometries.

The Fabry-Perot interferometer is made up of a coupled of chiral spin-wave edge modes encompassing a single topological MC (see Fig. 7c). Parts of the MC are
spatially constricted by the hole inside, so as to play a role of the point contact between these edge modes. A unidirectional spin-wave propagation is induced in a chiral mode via either an antenna attached to the boundary (‘input’ in Fig. 7c) or a microwave-spin-wave transducer put near the boundary. The wave is divided into two chiral edge modes at a point contact (‘PC1’ in Fig. 7c). Two chiral propagations merge into a single chiral propagation at the other point contact (‘PC2’). Depending on a phase difference between these two, the superposed wave exhibits either a destructive or a constructive interference, which is detected as an electric signal from the other antenna (‘output’). Note that local application of magnetic fields (‘PS1’ or ‘PS2’ in Fig. 7c) change the velocities of the two chiral edge modes locally. These modifications result in phase shifts of the two chiral waves, which thus changes the interference pattern observed in the output signal. With the use of these local magnetic fields as an external control, the interferometer can serve as a solid-state based magnonic logic gate. A combination with a recently-proposed resonant microwave-to-spin-wave transducer would also expand further prospects of spintronic applications of these spin-wave devices.

Though chiral edge modes are robust against static perturbations, magnetic energies excited in the edge mode decay into either phonon states or other magnon states in the volume modes via inelastic scatterings. The associated decay time or coherence length depends on specific materials, and spin wave propagation along the chiral edge mode survives only within this coherence length. Due to the absence of conduction electrons, however, spin waves in magnetic insulators have long coherence lengths: The coherence length in YIG in the magnetostatic regime becomes on the order of centimeter. In such magnetic insulators, the characteristic spin-wave propagations depicted in Fig. 7a,b,c are experimentally realizable especially in sizable MC systems. Experimental measurement of these propagations in the space- and time-resolved manner is by itself remarkable, which surely leads to the development of innovative spintronic device in future.

After submission of the present manuscript into the preprint server, we found another submission which also theoretically explored the realization of similar topological chiral magnonic edge mode in localized spin system. Their edge modes come from the short-ranged Dzyaloshinskii-Moriya exchange interaction instead of the (more classical) magnetic dipole-dipole interaction studied here.

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Appendix A: Magnetic Monopole and Dirac Hamiltonian

When two bosonic (magnonic) bands form a band-touching point in the 3-dimensional \( p \)-parameter space with \( p \equiv (k_x, k_y, \lambda) \) and \( \sigma \equiv (-k_x, -k_y, \lambda) \). The diagonal matrix \( \sigma_3 \) takes \(+1\) for the particle space while \(-1\) in the hole space. \( H_p \) is a quadratic form of \( p \) takes +1 for the particle space while \(-1\) in the hole space. \( H_p \) is a quadratic form of boson Hamiltonian introduced in sec. I. \( E_p \) is a diagonal matrix, whose elements give dispersions for bosonic (magnonic) bands and are physically all positive definite.

We decompose this into the zero-th order part and the perturbation part;

\[
H_p = H_0 + V_p.
\]

with \( H_0 \equiv H_p=p_c \) and \( V_p \equiv H_p - H_0 \). Suppose that \( H_0 \) has two-fold degenerate eigenstates \( t_j \) \((j = 1, 2)\) with its eigen-frequency \( \omega_0(>0) \);

\[
H_0 t_j = \sigma_3 t_j \omega_0.
\]

where the states are normalized as \( t_j^\dagger \sigma_3 t_m = \delta_{jm} \). On introducing the perturbation \( V_p \), the degeneracy is split into two frequency levels. The eigenstate for the respective eigen-frequency is determined on the zero-th order of \( p - p_c \) as;

\[
T_p = T_0 U_p + \mathcal{O}(|p - p_c|),
\]

where \( T_0 \) diagonalizes \( H_0 \) with \( T_0^\dagger \sigma_3 T_0 = T_0 \sigma_3 T_0^\dagger = \sigma_3 \) and a unitary matrix \( U_p \) diagonalizes a 2 by 2 Hamiltonian \( V_{\text{eff}} \) formed by the two-fold degenerate eigenstates;

\[
V_{\text{eff}} = \begin{pmatrix} t_1^\dagger V_p t_1 & t_1^\dagger V_p t_2 \\ t_2^\dagger V_p t_1 & t_2^\dagger V_p t_2 \end{pmatrix}.
\]

By substituting Eq. (A2) into Eqs. (28), one can easily see that, near \( p = p_c \), the dual magnetic field is given only by the unitary matrix; in the leading order of \( p - p_c \), it is given as

\[
B_j = \nabla \times A_j + \mathcal{O}(|p - p_c|^{-1}), \quad A_j = i \text{Tr}[\Gamma_j U_p^\dagger \nabla U_p],
\]

with \( \nabla \equiv (\partial_{k_x}, \partial_{k_y}, \partial_{\lambda}) \). Now that Eq. (A3) reduces to a usual 2 by 2 Dirac-type Hamiltonian, we can show the quantization of the dual magnetic charge at the band-touching point exactly in the same way as in the 2 by 2 Dirac fermion system.\(^{37-38}\) Thereby, the sign and the strength of the magnetic charge is determined only by the 2 by 2 effective Dirac Hamiltonian. With a proper gauge transformation and scale transformation, the effective Hamiltonian at the band-touching points \( P_j \) \((j = 1, 2)\) takes a form;

\[
H_{\text{eff}} = \omega_0 \tau_j + (\lambda - \lambda_{x,j}) \tau_3 + ap_x \tau_1 + bp_y \tau_2,
\]

with \( a > 0, b > 0 \), \((p_x, p_y) \equiv (k_x - \pi, k_y) \) for \( j = 1 \) and \((p_x, p_y) \equiv (k_x, k_y - \pi) \) for \( j = 2 \). Eq. (30) is derived by the replacement of \( p_{\mu} \to -i \partial_{\mu} \).

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