RANDOMIZED LIMIT THEOREMS
FOR STATIONARY ERGODIC RANDOM PROCESSES AND FIELDS

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Abstract

We consider "randomized" statistics constructed by using a finite number of observations a random field at randomly chosen points. We generalize the invariance principle (the functional CLT), the Glivenko–Cantelli theorem, the theorem about convergence to the Brownian bridge and the Kolmogorov theorem about the limit distribution of the empirical distribution function, as well as an improved version of the CLT in A. Tempelman, Randomized multivariate central limit theorems for ergodic homogeneous random fields, Stochastic Processes and their Applications. 143 (2022), 89-105. The randomized approach, introduced in the mentioned work, allows to extend these theorems to all ergodic homogeneous random fields on $\mathbb{Z}^m$ and $\mathbb{R}^m$.

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1 Preliminaries

1.1 Short review of the article

The article is devoted to the extension of the main Probability limit theorems to ergodic stationary random processes and to ergodic homogeneous random fields. Our approach is based on consideration of “randomized” statistics, i.e. statistics constructed by using a finite number of observations of a random field at randomly chosen points, introduced in [62].

In subsection 1.4 and §2 we define finite randomizing sets in the “time” space and randomized statistics, calculated on restrictions of the fields to these sets.

In §3 the fulfillment of a ”randomized” form of the Lindeberg condition is proved for ergodic homogeneous random fields, possessing the second moment; it is essentially used in the sequel.

In §4 we consider two kinds of the CLT. Subsection 4.3 is devoted to ”randomized” versions of the classical CLTs which are valid for all ergodic homogeneous fields; in these theorems the condition $E[|X(0)|^2] < \infty$ replaces the condition: $E[|X(0)|^{2+\delta}] < \infty$ for some $\delta$, imposed by the second author in [62]. The ”randomized” functional central limit theorem (invariance principle) is considered in subsection 4.5.

In §5 two theorems related to the limiting behavior of the randomized empirical distribution functions (EDF) are considered: a randomized version of the Glivenko–Cantelli theorem for multivariate distributions of ergodic homogeneous random fields and a general theorem devoted the limit of the distributions of the EDF; as a corollary, the randomized version of the Kolmogorov convergence theorem for empirical distributions is derived (of course, these theorems are also valid for ergodic homogeneous random fields).

1.2 The ”time” set $T$

In this paper we study random fields defined on a set $T$, which is the $m$-dimensional Euclidean space $\mathbb{R}^m$ or the $m$-dimensional integer lattice $\mathbb{Z}^m$, $m \geq 1$ (when $m = 1$, the random fields turn into random processes or random sequences). We denote by $\mathcal{B}$ the Borel $\sigma$-field on $T$ (if $T = \mathbb{Z}^m$, then $\mathcal{B}$ coincides with the collection of all subsets of $\mathbb{Z}^m$, and each function $f$ on $\mathbb{Z}^m$ is $\mathcal{B}$-measurable); $\lambda$ is the Lebesgue measure on $\mathbb{R}^m$ and the counting measure on $\mathbb{Z}^m$ (in the latter case $\lambda(A)$ is the cardinality of $A \subset \mathbb{Z}^m$, and $\int_A f(t) \lambda(dt) = \sum_{t \in A} f(t)$ if $\lambda(A) < \infty$).
1.3 Random fields

We consider a \(d\)-dimensional random field \(X(t) = (X^1(t), ..., X^d(t)), t \in T\), over a probability space \((\Omega_X, \mathcal{F}_X, P_X)\).

Let us recall several definitions. The field \(X\) is (strict-sense) homogeneous if all finite dimensional distributions of \(X\) are shift-invariant:

\[
P_X\{\omega : X(t_1 + t, \omega) \in A_1, ..., X(t_k + t, \omega) \in A_k\} = P_X\{\omega : X(t_1, \omega) \in A_1, ..., X(t_k, \omega) \in A_k\}
\]

for all \(t, t_i \in T, \ i, k \in \mathbb{N}\) and for all sets \(A_i\), belonging to the Borel \(\sigma\)-field \(\mathcal{B}(\mathbb{R}^d)\).

A family of invertible transformations \(\gamma = \{\gamma_t, t \in T\}\) of \(\Omega_X\) is said to be a group if \(\gamma_0 \equiv \omega, \gamma_{s+t} = \gamma_s \gamma_t, \gamma^{-1}_t = \gamma_{-t}\), where \(s, t \in T, \omega \in \Omega_X\). The field \(X\) is generated by a group \(\gamma\), if \(X(t, \omega) = X(0, \gamma_t \omega), t \in T, \omega \in \Omega_X\); since \(X(0, \gamma_{s+t} \omega) = X(0, \gamma_s \gamma_t \omega) = X(s, \gamma_t \omega)\), this implies:

\[
X(s + t, \omega) = X(s, \gamma_t \omega), s, t \in T. \tag{1}
\]

A family \(\gamma\) is measure preserving, if the transformations \(\gamma_t\) are \(\mathcal{F}_X\)-measurable and \(P_X(\gamma_t \Lambda) = P_X(\Lambda), t \in T, \Lambda \in \mathcal{F}_X\). If the random field \(X\) is generated by a measure preserving group of transformations of \(\Omega_X\), then, by Property (1), it is homogeneous. Denote by \(\mathcal{I}_X\) the \(\sigma\)-field of all events \(\Lambda \in \mathcal{F}_X\), which are invariant mod \(P_X\) with respect to all transformations \(\gamma_t\), i.e. \(P_X(\Lambda \Delta \gamma_t \Lambda) = 0, t \in T\). The group \(\gamma\) is said to be metrically transitive if \(\mathcal{I}_X = \{\Lambda : \Lambda \in \mathcal{F}_X, P_X(\Lambda) = 0 \text{ or } 1\}\). The field is ergodic if it is generated by a metrically transitive measure preserving group \(\gamma\).

Without loss of generality, we assume that \((\Omega_X, \mathcal{F}_X, P_X)\) is the probability space of function type, i.e., \(\Omega_X\) is the space of \(\mathbb{R}^d\)-valued "sample functions" \(x(\cdot)\) on \(T\), \(\mathcal{F}_X\) is the \(\sigma\)-field generated by the sets \(\{x(\cdot) : x(t) \in A\}, t \in T, A \in \mathcal{B}(\mathbb{R}^d)\), and \(P_X\) is a probability measure on \(\mathcal{F}_X\). \(X\) is the coordinate random field: \(X(t, x(\cdot)) = x(t), \) where \(t \in T, x(\cdot) \in \Omega_X\) (sometimes, we write \(\omega\) instead of \(x(\cdot)\), when we refer to an element of \(\Omega_X\)).

Consider the invertible measurable "shift" transformations \(\gamma_t, t \in T\), of \(\Omega_X\), defined as follows: \(\gamma_t x(\cdot) = x(\cdot + t)\); it is clear that the family \(\{\gamma_t\}\) is a group; moreover, \(X\) is generated by this group:

\[
X(0, \gamma_t x(\cdot)) = X(0, x(\cdot + t)) = x(0 + t) = x(t) = X(t, \cdot).
\]

The field is homogeneous, if and only if the shift transformations preserve the measure \(P_X\).
Each component field \(X^l\) may be considered over the probability space \((\Omega'_X, \mathcal{F}_X', P'_X)\) where \(\Omega'_X\) is the set of scalar functions \(x^l(\cdot)\) on \(T\). \(\mathcal{F}_X'\) is the \(\sigma\)-field generated by the events \(\{x^l(\cdot) : x^l(t) \in D\}, t \in T, D \in \mathcal{B}(\mathbb{R})\). Let \(A^l\) be an event \(A^l \in \mathcal{F}_X\); the event \(\Lambda_{A^l} := \{x(\cdot) : x^l(\cdot) \in A^l\} \in \mathcal{F}_X\), and \(P'_X\) is the projection of the measure \(P_X\) onto \(\mathcal{F}_X\); for each \(A^l \in \mathcal{F}_X\), \(P'_X(A^l) = P_X(\Lambda_{A^l})\). \(X^l\) is the coordinate field: \(X^l(t, x^l(\cdot)) = x^l(t)\), where \(t \in T\), \(x^l(\cdot) \in \Omega'_X\).

In what follows we assume that the fields \(X^l\) are homogeneous, i.e., for each \(l\) all transformations \(\gamma^l\) preserve the measure \(P'_X\). It is also assumed that \(E_X[|X^l(0)|] < \infty (l = 1, \ldots, d)\) then we set \(\mu^l := E_X[X^l(0)]\). If \(E_X[|X^l(0)|^2] < \infty\), then the variances are denote by \(\sigma^l := Var_X[X^l(0)]\).

If \(T = \mathbb{R}^m\), we always assume that the random fields \(X^l(t) = X^l(t, x')\) are \(B \times \mathcal{F}_l\)-measurable; by the Fubini theorem, these assumptions imply that the sample functions \(x^l(\cdot)\) are Borel measurable with \(P'_X\)-probability 1 and for each finite Borel measure \(Q\) the integral \(\int_T x^l(t)Q(dt)\) exists with \(P'_X\)-probability 1.

Denote by \(\mathcal{I}_X^l\) the \(\sigma\)-field of all \(\gamma\)-invariant \(mod\ (P'_X)\) events in \(\mathcal{I}_X^l\) (we remind that \(X^l\) is said to be ergodic if \(\mathcal{I}_X^l\) is trivial). \(E_X[X^l(0)|\mathcal{I}_X^l]\), \(Var_X[X^l(0)|\mathcal{I}_X^l]\) are the conditional expectation and variance; in the sequel it is assumed that for each \(l\) \(Var_X[X^l(0)|\mathcal{I}_X^l] > 0\) with \(P'_X\)-probability 1.

In some cases it is assumed that the \(\mathbb{R}^d\)-valued random field \(X^l\) is homogeneous, that is the transformations \(\gamma^l\) preserve the measure \(P'_X\); then the transformations \(\gamma^l\) preserve the measures \(P'_X\), and the components \(X^l\) are also homogeneous; if the random field \(X\) is ergodic, then all its component fields \(X^l\) are ergodic, too.

### 1.4 Randomizing random vectors

Let \(\{q^l_n\}\) be sequences of probability measures on \(\mathcal{B}\), \((l = 1, \ldots, d), d \in \mathbb{N}\) (these sequences may coincide for some or even all \(l\)). For each natural \(n\) we consider \(d\) mutually independent \(k_n\)-dimensional random vectors \(\tau^l_n = (\tau^l_{n,1}, \ldots, \tau^l_{n,k_n}), l = 1, \ldots, d\), over a probability space \((\Omega_{\tau}, \mathcal{F}_{\tau}, P_{\tau})\) possessing the following properties:

a) the vectors \(\tau^l_n\) do not depend on the random field \(X\),

and

b) the components of each vector \(\tau^l\) are i.i.d. \(T\)-valued random vectors with the distribution \(q^l_n\).

(Of course, if \(T = \mathbb{Z}^m\) and if for some \(l\) the support of \(q^l_n\) is finite, then some points may appear in the sample \(\tau^l_{n,1}, \ldots, \tau^l_{n,k_n}\) several times). It is assumed that \(k_n \uparrow \infty\) as \(n \to \infty\) (the sequence \(\{k_n\}\) may also depend on \(l\); to simplify
the notation, we always drop the index \( l \) in \( k^l_n \). Since each scalar field
\[ X^l(t, \omega) \]
is a \( B \times \mathcal{F}_X \)-measurable function, \( X^{(i)}_{\tau_{n,i}}(\omega) \) are random variables over the probability space
\[ (\Omega_{X, \tau}, \mathcal{F}_{X, \tau}, P_{X, \tau}) := (\Omega_X \times \Omega_{\tau}, \mathcal{F}_X \times \mathcal{F}_{\tau}, P_X \times P_{\tau}) \]

### 1.5 Notation

In the sequel we also use the following notation:
- \( E_X, E_{\tau} \) and \( E_{X, \tau} \) denote the expectation with respect to the measure \( P_X, P_\tau \) and \( P_{X, \tau} \).
- \( Y_n \rightarrow Y \) \( P \)-a.s., \( Y_n \overset{P}{\rightarrow} Y \) mean convergence almost sure, respectively in probability.
- \( Y_n \overset{Q}{\Rightarrow} Y \) means convergence in distribution with respect to the measure \( Q \).
- \( a := b \) means that the quantity \( a \) is defined by the expression \( b \).
- \( \mathbb{R}, \mathbb{Z}, \mathbb{N} \) denote the sets of real numbers, of integers and of natural numbers.
- \( T = \mathbb{R}^m \) or \( \mathbb{Z}^m \) (\( m \in \mathbb{N} \)).
- \( 1_A \) or \( \text{Ind}(A) \) denotes the indicator of the set \( A \).

### 1.6 Terminology

All results are stated in the terms of homogeneous random fields (of course, they are valid for stationary random sequences and processes, too). If \( m = 1 \), the words "homogeneous random field" mean "stationary random process" or "stationary random sequence"; the words "ball", "cube" and "parallelepiped" mean "interval" and, if \( m = 2 \), these words mean "sphere" "square" and "parallelogram", respectively.

### 2 Randomization using uniform distributions on subsets \( T^l_n \) of \( T \)

Let \( X(t) = (X^1(t), ..., X^d(t)) \), \( t \in T \), be a homogeneous random field on \( T \), \( \{T^l_n\} \) be sequences of bounded Borel sets of positive measure in \( T \) \((l = 1, ..., d), \ d \in \mathbb{N} \) (these sequences may coincide for some or even all \( l \)). It is supposed that \( \lambda(T^l_n) \rightarrow \infty \) as \( n \rightarrow \infty \).
We consider the $T_n$-valued random vectors $\tau_{n,1}, ..., \tau_{n,k_n}, \ n \in \mathbb{N}$, introduced in Subsect. 1.4.

**Lemma 1.** If $f$ is a measurable function on $\mathbb{R}$ such that $E_X[|f(X(0))|] < \infty$, then $E_{X,\tau}[|f(X(\tau_{n,i}))|] < \infty$ and with $P_X$-probability 1 $E_{\tau}[|f(X(\tau_{n,i}))|] < \infty$.

**Proof.** It follows immediately from the Fubini - Tonelli Theorem. \qed

Assume that for each $l \in \{1, ..., d\}$ and for each natural number $n$ the random vectors $\tau_{n,i}, \ i = 1, ..., k_n$, introduced in Subsect. 1.4, are uniformly distributed (with respect to $\lambda$) on the set $T_{n}^l$. In this section we use the following estimators of $\mu^l$ and $(\sigma^l)^2$:

$$M_n^l := \frac{1}{\lambda(T_{n}^l)} \int_{T_{n}^l} X^l(t) \lambda(dt),$$

$$V_n^l := \frac{1}{\lambda(T_{n}^l)} \int_{T_{n}^l} (X^l(t) - M_n^l)^2 \lambda(dt) = \frac{1}{\lambda(T_{n}^l)} \int_{T_{n}^l} (X^l(t))^2 \lambda(dt) - (M_n^l)^2.$$

If $\omega \in \Omega_X$ is fixed, the random variables $X^l(\tau_{n,i}, \omega)$ form $d$ triangular arrays and for each $n$ they are independent and identically distributed. For each measurable function $f$ on $\mathbb{R}$, such that $f(X^l(t))$ is integrable on $T_{n}^l$,

$$E_{\tau}[f(X^l(\tau_{n,i}))] = \frac{1}{\lambda(T_{n}^l)} \int_{T_{n}^l} f(X^l(t)) \lambda(dt), \ i = 1, ..., k_n, \ l = 1, ..., d.$$ 

In particular,

$$E_{\tau}[X^l(\tau_{n,i})] = M_n^l; \ Var_{\tau}[X^l(\tau_{n,i})] = V_n^l.$$

The above relation implies:

$$E_{\tau}\left(\sum_{i=1}^{k_n} f(X^l(\tau_{n,i}))\right) = k_n \frac{1}{\lambda(T_{n}^l)} \int_{T_{n}^l} f(X^l(t)) \lambda(dt) \ (l = 1, ..., d). \ (2)$$

**Definition 1.** We say that a sequence of Borel sets $\{T_n\}$ is pointwise averaging if the following Pointwise Ergodic Theorem (PET) holds with this sequence:

Let $f(\cdot)$ be a measurable function on $\mathbb{R}$. If $X$ is a scalar homogeneous random field on $T$ over $(\Omega_X, \mathcal{F}_X, P_X)$ with $E_X[|f(X(0))|] < \infty$ and $\mathcal{I}_X$ is the $\sigma$-field of shift-invariant mod $(P_X)$ events in $\mathcal{F}_X$, then with $P_X$-probability 1

$$\lim_{n \to \infty} \frac{1}{\lambda(T_n)} \int_{T_n} f(X(t)) \lambda(dt) = E_X[f(X(0)|\mathcal{I}_X].$$
Example 1. In $\mathbb{R}^m$ each increasing sequence of bounded convex sets $T_n \in \mathbb{R}^m$, containing balls $B_n$ of radii $r(B_n) \to \infty$, is pointwise averaging (see Corollary 3.3 in Ch. 6 in \cite{64}). In particular, any sequence of concentric balls $B_n$ with $r(B_n) \to \infty$ and any increasing sequence of parallelepipeds $T_n \subset \mathbb{R}^m$ with infinitely growing edges is pointwise averaging (in particular, the sequence of cubes $[0,n]^m$ has this property). The intersections of the mentioned sets with $\mathbb{Z}^m$ form pointwise averaging sequences in $\mathbb{Z}^m$ (see Subsect. 5.2 in \cite{61}).

Example 2. Let $A$ be a compact set in $\mathbb{R}^m$, with $\lambda(A) > 0$ and star-shaped with respect to 0. If $s_n \uparrow \infty$, the sequence of homothetic sets $\{s_n A\}$ is pointwise averaging (see Example 2.9 in Ch. 5 in \cite{64} or Subsection 5.4.1 in \cite{61}).

Remark 1. Along with the pointwise averaging sequences of sets, mean averaging sequences may be considered, i.e. sequences $T_n$ for which The Mean Ergodic Theorem (MET) for each homogeneous random field with $E[(X(0))^2] < \infty$:

$$E_X \left[ \frac{1}{\lambda(T_n)} \int_{T_n} X(t) \lambda(dt) - E_X[X(0)|I_X] \right]^2 \to 0.$$ 

Under our condition $E[(X(0))^2] < \infty$, each pointwise averaging sequence is mean averaging (see Corollary 2 in Subsect 9.4 in \cite{45}). Consider the Hilbert subspace $H_X$ of $L^2(\Omega_X, \mathcal{F}_X, P_X)$, spanned by the random variables $X(t)$, $t \in T$; it is invariant with respect to the shift transformations: $W(\gamma \omega) \in H_X$ if $W(\omega) \in H_X$; denote by $I$ the subspace of all random variables invariant with respect to all $\gamma$. The conditional expectation $E_X[X(0)|I_X] = \tilde{E}_X[X(0)|I]$, the orthogonal projection of $X(0)$ onto $I$. This random variable equals $E[X(0)]$, if $X$ is "wide-sense ergodic", i.e., if $I = \mathbb{R}$; of course this condition is much weaker than ergodicity, which assures that the subspace of all $\gamma$-invariant random variables in $L^2(\Omega_X, \mathcal{F}_X, P_X)$ coincides with $\mathbb{R}$. The class of the mean averaging sequences is rather wide; in particular, the monotonicity condition in Example 1 is redundant. The MET is essentially used in the sequel (see also Remark 8 where the speed of convergence in the MET is discussed).

### 3 Randomized Lindeberg condition

Let us remind that the main probability space has the following form $(\Omega_X, \mathcal{F}_X, P_X) = (\Omega_X \times \Omega, \mathcal{F}_X \times \mathcal{F}, P_X \times P)$.
Proposition 1. Let $X$ be a homogeneous random field, $E[(X(0))^2] < \infty$, and let $\{T_n\}$ be a pointwise averaging sequence of sets. Then for $P_X$-almost all $\omega \in \Omega_X$ the Lindeberg condition is fulfilled:

for each $\varepsilon$ as $n \to \infty$

$$L_n(\omega) := \frac{\sum_{i=1}^{k_n} E_r[(X(\tau_{n,i},\omega) - M_n(\omega))^2 1_{B_n}]}{k_n V_n(\omega)} \to 0,$$

where $B_n = \{|X(\tau_{n,i},\omega) - M_n(\omega)| > \varepsilon k_n^\frac{1}{2} (V_n)^\frac{1}{2}\}$.

Proof. For a fixed "good" $\omega$, we will apply the Lindeberg theorem (see, e.g., Theorem 27.2 in [3]) to the random variables over the space $(\Omega, F, P)$.

By relation (2), for each $\omega \in \Omega_X$ and for each $\varepsilon > 0$, the Lindeberg fraction for the random variables $X(\tau_{n,i}(\omega), \omega), i = 1, \ldots, k_n$, is given by

$$L_n(\omega) = \frac{1}{V_n(\omega)} \frac{1}{\lambda(T_n)} \int_{T_n} (X(t, \omega) - M_n(\omega))^2 1_{C_n(t)} \lambda(dt),$$

where $C_n(t) = \{|X(t,\omega) - M_n(\omega)| > \varepsilon k_n^\frac{1}{2} (V_n)^\frac{1}{2}\}$.

We have to prove that with $P_X$-probability 1 $L_n(\omega) \to 0$. Since the sequence $\{T_n\}$ is pointwise averaging, by the PET, with $P_X$-probability 1

$$M_n(\omega) \to z(\omega),$$

$$V_n(\omega) = \frac{1}{\lambda(T_n)} \int_{T_n} (X(t, \omega) - M_n(\omega))^2 \lambda(dt) \to v(\omega),$$

where

$$z(\omega) = E_X[X(0, \omega) | I_X](\omega),$$

$$v(\omega) = Var_X[(X(0), \omega)^2 | I_X](\omega) > 0.$$

It is clear that

$$L_n(\omega) \leq \Sigma_n^{(1)} + \Sigma_n^{(2)},$$

where

$$\Sigma_n^{(1)} = \frac{1}{V_n(\omega)} \frac{2}{\lambda(T_n)} \int_{T_n} (X(t, \omega) - z(\omega))^2 1_{C_n(t)} \lambda(dt),$$

$$\Sigma_n^{(2)} = \frac{1}{V_n(\omega)} \frac{2}{\lambda(T_n)} \int_{T_n} (z(\omega) - M_n)^2 1_{C_n(t)} \lambda(dt).$$
Due to (3) with $P_X$-probability 1
\[ \Sigma_n^{(2)} \leq (z(\omega) - M_n(\omega))^2 \frac{2}{V_n(\omega)} \rightarrow 0. \] (7)

Consider $\Sigma_n^{(1)}$. Denote by $\Lambda \in \mathcal{F}_X$ the set of $P_X$-measure 1 in $\Omega$ where the limits (3) and (4) exist, and fix some $\omega \in \Lambda$. It is clear that $k_n V_n(\omega) \rightarrow \infty$ as $n \rightarrow \infty$. For each $C > 0$, let us also fix some $m_C(\omega) \in \mathbb{N}$ such that, if $n > m_C(\omega)$, then $|M_n(\omega) - z(\omega)| < \frac{1}{2} \varepsilon C$ and $(k_n V_n(\omega))^\frac{1}{2} > C > 0$.

As for $n > m_C(\omega)$
\[ \{|X(t,\omega) - M_n| > \varepsilon k_n^{\frac{1}{2}} V_n^{\frac{1}{2}}\} \subset \{|X(t,\omega) - z(\omega)| > \frac{\varepsilon C}{2}\} \cup \{|z(\omega) - M_n| > \frac{\varepsilon C}{2}\}, \]
we have:
\[ \Sigma_n^{(1)} \leq \delta_n^{(1)} + \delta_n^{(2)}, \] (8)
where
\[ \delta_n^{(1)} = \frac{1}{V_n(\omega)} \frac{2}{\lambda(T_n)} \int_{T_n} (X(t,\omega) - z(\omega))^2 1_{\{|X(t,\omega) - z(\omega)| > \frac{\varepsilon C}{2}\}} \lambda(dt), \]
\[ \delta_n^{(2)} = \frac{1}{V_n(\omega)} \frac{2}{\lambda(T_n)} \int_{T_n} (X(t,\omega) - z(\omega))^2 1_{\{|z(\omega) - M_n| > \frac{\varepsilon C}{2}\}} \lambda(dt). \]

By the PET, with $P_X$-probability 1
\[ \lim_n \delta_n^{(1)} = \frac{2}{v(\omega)} E \left[ (X(0,\omega) - z(\omega))^2 1_{\{|X(0,\omega) - z(\omega)| > \frac{\varepsilon C}{2}\}} \right], \] (9)
and due to (3)
\[ \lim_n \delta_n^{(1)} = \frac{2}{v(\omega)} \lim_n \frac{1}{\lambda(T_n)} \int_{T_n} (X(t,\omega) - z(\omega))^2 \lambda(dt) \cdot \lim_n 1_{\{|z(\omega) - M_n| > \frac{\varepsilon C}{2}\}} = 0. \] (10)

It follows from (7)-(10) that
\[ \limsup_n L_n(\omega) \leq 2E \left[ (X(0,\omega) - z(\omega))^2 1_{\{|X(0,\omega) - z(\omega)| > \frac{\varepsilon C}{2}\}} \right]. \]

Letting $C$ tend to infinity, we finally obtain with $P_X$-probability 1
\[ \lim_n L_n(\omega) = 0. \]
4 Generalizations of the Central Limit Theorem

4.1 On the CLT

The Central Limit Theorem (CLT) is one of the remarkable statements of Probability. Originally proved for sequences of independent random variables, in many works this theorem was generalized to stationary random processes and homogeneous random fields under some additional assumptions: Markov, strong mixing and/or rather strong moment conditions, etc. (see, e.g., [4, 7, 8, 10, 11, 12, 15, 19, 21, 31, 32, 34, 43, 46, 47, 50, 51, 53, 63-68]).

However, the CLT may fail even when the sequence $X_1, X_2, \ldots$ is stationary, quite strongly mixing, orthogonal and with $\lim \frac{1}{n} \text{Var}(\sum_{i=1}^{n} X_i) = \sigma^2 > 0$ [28]; two other interesting counterexamples are provided in [8]. Earlier Chung [13] and Davydov [16, 17] proved that the CLT may fail for strictly stationary irreducible aperiodic Markov chains with a countable set of states (these and other counterexamples can be found in vol. 3, Chapters 30 and 31 in the book [8]). Ibragimov and Linnik [33] have constructed a strongly mixing stationary sequence of random variables $X_1, X_2, \ldots$ with finite variances such that the self-normalized sums $(\text{Var}[\sum_{i=1}^{n} X_i]^{-\frac{1}{2}} \sum_{i=1}^{n} X_i)$ converge in distribution to a non-normal random variable (see Ch. 19, §5 therein). A rather common case when the CLT fails is when $\lim \frac{1}{n} \text{Var}[\sum_{i=1}^{n} X_i] = 0$ (see, e.g., [20], p.153).

A well-known source of counterexamples for the classical CLT for strict sense stationary random sequences is the class of coboundaries with respect to measure preserving transformations. Let $\gamma : \omega \mapsto \gamma \omega$ be an ergodic measure preserving transformation of a probability space $(\Omega, \mathcal{F}, P)$; let $\alpha \geq 1$; then for each $f \in L^\alpha(\Omega, \mathcal{F}, P), f \neq 0$ a.s., the random sequence $X_i^{(f)} = f(\gamma^i \omega), i = 1, 2, \ldots,$ is an ergodic strict sense stationary sequence and $E[|X_i^{(f)}|^\alpha] < \infty$ (see, e.g., [41], §1.4). Let $g \in L^\alpha(\Omega, \mathcal{F}, P)$ and let $f$ be the coboundary of $g$: $f(\omega) = g(\gamma \omega) - g(\omega)$. It is clear that $E(X_i^{(f)}) = 0$, $i = 1, 2, \ldots$. We have also: $\sum_{i=1}^{n} X_i^{(f)} = \sum_{i=1}^{n} f(\gamma^i \omega) = (g^{n+1} \omega) - g(\gamma \omega)$, hence $E[\sum_{i=1}^{n} X_i^{(f)}]^\alpha \leq 2\alpha ||g||_{L^\alpha}^\alpha$, and $E[|n^{-\beta} \sum_{i=1}^{n} X_i^{(f)}|^\alpha] \to 0$ if $\beta > 0$ (when $\alpha > 1$, the converse is true: if $E[f] = 0$ and $\sup_n E[|\sum_{i=1}^{n} X_i^{(f)}|^\alpha] < \infty$, it is a coboundary of some $g \in L^\alpha$; see Lemma 5 in [9]); the set of coboundaries is dense in the space $L^\alpha(\Omega, \mathcal{F}, P) \ominus \mathbb{R}$, by the ergodic decomposition. It is evident that, if $f$ is a coboundary of a function $g \in L^\alpha$, then $n^{-\beta} \sum_{i=1}^{n} X_i^{(f)} \to 0$ in probability for any $\beta > 0$, and, if $g$ is
bounded, then this is true in the sense of a.s. convergence.

There is an extensive literature related to limit theorems for “long memory” (or “long-range dependent”) stationary sequences, where the slowly decreasing dependence between receding terms is characterized by the rate of decrease of the correlation function or by properties of the spectrum near 0) - see [1], [26], [56] and the references therein.

4.2 Main results of this section

In the previous work by the second author [62], CLTs for homogeneous random fields on $\mathbb{R}^m$ and $\mathbb{Z}^m$ ($m \geq 1$), satisfying the condition: $E[|X(0)|^{2+\delta}] < \infty$ have been proved. In the present article the authors prove that these theorems are valid under the weaker condition $E[(X(0))^2] < \infty$.

In this paper we present randomized versions of the CLT, which are valid for each ergodic homogeneous measurable random field $X(\cdot)$ on $\mathbb{R}^m$ or $\mathbb{Z}^m$ ($m \geq 1$), (in particular, for each ergodic stationary random process and each ergodic stationary random sequence) with a finite second moment (in some versions ergodicity may be omitted). Specifically, observations of the random field at randomly chosen points are used.

If the field $X$ is multivariate and each of its components satisfies the above conditions, another feature is obtained by the randomization: in Theorems 129 and Corollaries 1, 2 the components of the limit normal vector are independent whatever are the components of $X$. When $E[|X(0)|^2] < \infty$, these theorems are valid also in all cases, mentioned in Subsection 1.1, when the conventional CLT fails.

For example, consider the stationary random sequences $\{X_i^{(f)}\}$, which have been introduced in Subsection 111 (each sequence is specified by some coboundary $f \in L^2(\Omega, \mathcal{F}, P)$); we have: $E[|X_i^{(f)}|^2] < \infty$; as mentioned above, the set of coboundaries is dense in $L^2(\Omega, \mathcal{F}, P) \ominus \mathbb{R}$, and the conventional CLT fails for $\{X_i^{(f)}\}$, if $f$ a coboundary: the limit is degenerate ($\delta_0$) and not normal. However, the randomized CLTs are valid for all sequences $\{X_i^{(f)}\}$ with $E[|X_i^{(f)}|^2] < \infty$.

Randomization allowed us to assume that the field is only homogeneous (in some statements it is assumed that the field is also ergodic) and its second moment is finite.

The main tools, used in the proofs, are the Lindeberg CLT and the Pointwise and Mean Ergodic Theorems.

To illustrate our results, we state a simple corollary of Theorem 3.

Let $X(t), t \in \mathbb{R}$, be an ergodic stationary measurable random process and
\[ E[(X(0))^2] < \infty; \text{ denote } \sigma^2 = \text{Var}[X(0)], M_n = \frac{1}{n} \int_0^n X(t) dt. \text{ Let } \tau_{n,i} (n = 1, 2, ..., \text{ and } i = 1, ..., n) \text{ be random variables, independent of } X, \text{ and, for each } n, \text{ independent of each other and uniformly distributed on } [0, n]. \text{ Then, if } n \to \infty, \]

\[ \frac{\sum_{i=1}^n (X(\tau_{n,i}) - M_n)}{\sigma \sqrt{n}} \to N(0, 1). \]

A slightly more complex corollary with \( \mu = E[X(0)] \) instead of \( M_n \) can be derived from Theorem 4; as usual, if \( \sigma \) is known, this statement may be used for consistent statistical inference on the expectation \( \mu \).

4.3 Randomized Central Limit Theorem

Assumptions 1. In Theorems 1–5, all \( X_l, l = 1, ..., d, \) are homogeneous random fields and, for all \( l, \)

\[ E_X[|X^l(0)|^2] < \infty. \]  

(11)

If \( X^l \) is not ergodic, we assume \( \text{Var}_X[X^l(0)|T^l_X] > 0 \) with \( P^l_X \)-probability 1, \( l = 1, ..., d. \)

We remark that \( k_n \in \mathbb{N}, k_n \uparrow \infty. \)

For each \( l, \) \( \{T^l_n\} \) is a pointwise averaging sequence of sets.

Remark 2. We will prove that, together with the conditions mentioned above, in the non-ergodic case, the condition \( \text{Var}_X[X^l(0)|T^l_X] > 0 \) with \( P^l_X \)-a.s., \( l = 1, ..., d, \) is sufficient for the relation (13) to hold with \( P_X \)-probability 1. And, for each fixed \( l, \) it is also necessary for this statement to hold for the component \( X^l. \) Indeed, if the variance of the conditional distribution \( \text{Var}_X[X^l(0)|T^l_X] = 0, \) then \( X^l(0) = E_X[X^l(0)|T^l_X] P^l_X \)-a.s., hence \( X^l(0) \) is \( T^l_X \)-measurable and \( X^l(t, \omega^l) = X^l(0, \omega^l) = X^l(0, \omega^l) \ P^l_X \)-a.s., so, for each \( t \in T, \) \( X^l(t) = X^l(0) \ P^l_X \)-a.s. It is clear, that this dependence is too strong for any version of the CLT to hold for \( X^l. \) In this case, it becomes senseless; note that \( M_n^l = X^l(0), V_n^l = 0; \) therefore, the expression

\[ \frac{\sum_{i=1}^{k_n} (X^l(\tau^l_{n,i}) - M_n^l)}{k_n^\frac{3}{2} (V_n^l)^\frac{1}{2}} = \frac{k_n^\frac{3}{2} (V_n^l)^\frac{1}{2}}{k_n^\frac{3}{2} (V_n^l)^\frac{1}{2}} \]

(see (13) ) is of type \( 0 \); conditions (17) and (28) for \( X^l \) do not hold, and the expressions (3), (4) for \( X^l \) are of type \( 0 \).
Let us remind that the probability space
\[(\Omega_{X,\tau}, \mathcal{F}_{X,\tau}, P_{X,\tau}) = (\Omega_X \times \Omega_{\tau}, \mathcal{F}_X \times \mathcal{F}_\tau, P_X \times P_\tau).\]

**Lemma 2.** 1. Let \(A_n(z)\) be a sequence of events in \(\Omega_{X,\tau}\) depending on \(z \in \mathbb{R}^d\); for each \(\omega \in \Omega_X\) denote: \(A_n(\omega, z) = \{t \in \Omega_{\tau} : (\omega, t) \in A_n(z)\}\), the \(\omega\)-section of \(A(z)\); if for each \(z \in \mathbb{R}^d\) the limit
\[
\lim_{n \to \infty} P_\tau(A_n(\omega, z)) = \psi(z)
\]
evenly exists for \(P_X\)-almost all \(\omega\), then for each \(z \in \mathbb{R}^d\)
\[
\lim_{n \to \infty} P_{X,\tau}(A_n(z)) = \psi(z). \tag{12}
\]
2. Let \(\{f_n(X, \tau) = (f^1_n(X, \tau), ..., f^d_n(X, \tau)), n \in \mathbb{N}\}\) be a sequence of random vectors over \(\Omega_{X,\tau}\) and let \(Z = (Z^1, ..., Z^d)\) be a random vector over the same probability space. If \(f_n(X, \tau) \to Z\) in \(P_\tau\)-distribution with \(P_X\)-probability 1, then \(f_n(X, \tau) \to Z\) in \(P_{X,\tau}\)-distribution.

**Proof.** 1. By the Fubini theorem, \(A_n(\omega, z) \subset \mathcal{F}_\tau\) for \(P_X\)-almost all \(\omega \in \Omega_X\). By virtue of the Fubini theorem and the Lebesgue Dominated Convergence theorem, we have for each \(z \in \mathbb{R}^d\):
\[
\lim_{n \to \infty} P_{X,\tau}(A_n(z)) = \lim_{n \to \infty} E_\tau[1_{A_n(z)}] = E_\tau[\lim_{n \to \infty} 1_{A_n(z)}] = E_\tau P_\tau(A_n(\omega, z)) = E_\tau \psi(z) = \psi(z).
\]
2. Apply Statement 1 with
\[
A_n(z) := \{(\omega, t) \in \Omega_{X,\tau} : f^1_n(\omega, t) \leq z^1, ..., f^d_n(\omega, t) \leq z^d\},
\]
\[
\psi(z) = P_X(Z^1 \leq z^1, ..., Z^d \leq z^d).
\]
\[\square\]

We denote: \(Z = (Z^1, ..., Z^d)\) where \(Z^1, ..., Z^d\) are independent standard normal random variables.

**Theorem 1.** Under Assumptions 1, with \(P_X\)-probability 1
\[
\left(\frac{\sum_{i=1}^{k_n} (X^1_{n,i} - M^1_n)}{k^\frac{1}{2} \sqrt{V^1_n}}, ..., \frac{\sum_{i=1}^{k_n} (X^d_{n,i} - M^d_n)}{k^\frac{1}{2} \sqrt{V^d_n}}\right) \overset{P_\tau}{\Rightarrow} Z; \tag{13}
\]
if all $X^l$ are ergodic, then with $P_X$-probability 1

$$\left( \frac{\sum_{i=1}^{k_n} (X^1(\tau_{n,i}^1) - M_n^1)}{k_n^l \sigma^1}, \ldots, \frac{\sum_{i=1}^{k_n} (X^d(\tau_{n,i}^d) - M_n^d)}{k_n^d \sigma^d} \right) \xrightarrow{P_x} Z. \quad (14)$$

Proof. We may apply the Lindeberg theorem and obtain: with $P_X$-probability 1 for each $l$

$$\frac{\sum_{i=1}^{k_n} (X^l(\tau_{n,i}^l) - M_n^l)}{k_n^l (V_n^l)^{1/2}} \xrightarrow{P_x} Z^l. \quad (15)$$

If $X$ is ergodic, then $(V_n^l)^{1/2} \to \sigma^l$ with $P_X$-probability 1, and, by the Slutsky theorem ([42], Theorem 23.3), with $P_X$-probability 1

$$\frac{\sum_{i=1}^{k_n} (X^l(\tau_{n,i}^l) - M_n^l)}{k_n^l \sigma^l} \xrightarrow{P_x} Z^l. \quad (16)$$

Since for each $n$ the $d$ vectors $(\tau_{n,i}^l, i = 1, \ldots, k_n), \ l = 1, \ldots, d,$ are mutually independent, using (15) and (16) we obtain the convergence (13), respectively (14), of vectors.

Since the limiting distributions in (13) and (14) with $P_X$-probability 1 are the same, we immediately deduce from the previous theorem the nonconditional weak convergence:

**Theorem 2.** Under Assumptions 1,

$$\left( \frac{\sum_{i=1}^{k_n} (X^1(\tau_{n,i}^1) - M_n^1)}{k_n^1 (V_n^1)^{1/2}}, \ldots, \frac{\sum_{i=1}^{k_n} (X^d(\tau_{n,i}^d) - M_n^d)}{k_n^d (V_n^d)^{1/2}} \right) \xrightarrow{P_{X,\tau}} Z;$$

if all $X^l$ are ergodic, then with $P_X$-probability 1

$$\left( \frac{\sum_{i=1}^{k_n} (X^1(\tau_{n,i}^1) - M_n^1)}{k_n^1 \sigma^1}, \ldots, \frac{\sum_{i=1}^{k_n} (X^d(\tau_{n,i}^d) - M_n^d)}{k_n^d \sigma^d} \right) \xrightarrow{P_{X,\tau}} Z.$$

The following Theorems 3 and 4 can be deduced subsequently from Theorems 12 literally as Th. 3 and Th. 4 in [62].
Theorem 3. In addition to the above Assumptions 1, assume that
\[
\frac{1}{k^n} (M^l_n - \mu^l) \to 0 \text{ with } P_X\text{-probability 1, } l = 1, \ldots, d. \tag{17}
\]
Then with \(P_X\)-probability 1
\[
\left( \frac{\sum_{i=1}^{k_n} (X^1(\tau^1_{n,i}) - \mu^1)}{\frac{k_n^2}{\sigma^1} V_n^1} , \ldots, \frac{\sum_{i=1}^{k_n} (X^d(\tau^d_{n,i}) - \mu^d)}{\frac{k_n^2}{\sigma^d} V_n^d} \right) \xrightarrow{P_X} Z;
\]
if all \(X^l\) are ergodic, then with \(P_X\)-probability 1
\[
\left( \frac{\sum_{i=1}^{k_n} (X^1(\tau^1_{n,i}) - \mu^1)}{\frac{k_n^2}{\sigma^1} \sigma^1} , \ldots, \frac{\sum_{i=1}^{k_n} (X^d(\tau^d_{n,i}) - \mu^d)}{\frac{k_n^2}{\sigma^d} \sigma^d} \right) \xrightarrow{P_X} Z.
\]

Theorem 4. Let the conditions stated in Assumptions 1 be fulfilled and
\[
\frac{1}{k_n} (M^l_n - \mu^l) \xrightarrow{P_X} 0. \tag{18}
\]
Then
\[
\left( \frac{\sum_{i=1}^{k_n} (X^1(\tau^1_{n,i}) - \mu^1)}{\frac{k_n^2}{\sigma^1} V_n^1} , \ldots, \frac{\sum_{i=1}^{k_n} (X^d(\tau^d_{n,i}) - \mu^d)}{\frac{k_n^2}{\sigma^d} V_n^d} \right) \xrightarrow{P_X} Z;
\]
if all \(X^l\) are ergodic, then
\[
\left( \frac{\sum_{i=1}^{k_n} (X^1(\tau^1_{n,i}) - \mu^1)}{\frac{k_n^2}{\sigma^1} \sigma^1} , \ldots, \frac{\sum_{i=1}^{k_n} (X^d(\tau^d_{n,i}) - \mu^d)}{\frac{k_n^2}{\sigma^d} \sigma^d} \right) \xrightarrow{P_X} Z.
\]

Condition (18) puts some restrictions to the growth of the number \(k_n\) of observations of the values of the field \(X\). It is often of the form \(k_n = o(\lambda(T_n))\) (see Remark 3). On the other hand, the accuracy of the approximation of the normal distribution by the CLT is higher when \(k_n\) is large. An obvious way to increase \(k_n\) without violation of this condition is to increase \(\lambda(T_n)\) proportionally. Another way to increase the number of observations without breaking this condition puts more work for the statistician; it is based on a special case of Theorem 4 when, instead of \(X\), an auxiliary \(\mathbb{R}^{w^d}\)-valued random field \(Y\) is considered, in which each component \(X^l\) participates \(w\) times.

\(^1\)See §A for remarks on this condition and condition (18).
Corollary 1. Let the conditions of Theorem 4 be fulfilled. Consider the family of independent random variables
\( \{\tau_{n,i}^{l,u}, n \in \mathbb{N}, 1 \leq i \leq k_n, 1 \leq l \leq d, 1 \leq u \leq w\} \) which, for each \( n, l, u, i, \) are distributed uniformly on \( T_n^l \). Then
\[
\sum_{i=1}^{k_n}(X_l^{\tau_{n,i}^{l,u}} - \mu_l) \xrightarrow{P} Z_l^{l,u}, \quad l = 1, \ldots, d, \quad u = 1, \ldots, w.
\]
where all random variables \( Z_l^{l,u}, \quad l = 1, \ldots, d, \quad u = 1, \ldots, w, \) are standard normal and independent.

If the field \( X \) is ergodic, the "empirical" standard deviations \( V_n^l \) can be replaced by the "theoretical" standard deviations \( \sigma^l \).

Proof. Apply Theorem 4 to the random fields \( Y_l^{l,u}(t) := X_l(t), \quad t \in T, \quad l = 1, \ldots, d, \quad u = 1, \ldots, w. \)

We consider a generalization of this corollary to various parameters of the marginal or multidimensional distributions of the field \( X \); for simplicity we assume that \( X \) is \( \mathbb{R} \)-valued. Let \( \theta^l, l = 1, \ldots, d, \) be parameters of the distribution of a vector \( (X(t_1), \ldots, X(t_k)) \). We assume that there exists a \( \mathcal{B}(\mathbb{R}^k) \)-measurable functions \( f^l(x_1, \ldots, x_k) \) such that
\[
E_X[|f^l(X(t_1), \ldots, X(t_k))|^2] < \infty,
\]
and \( \theta^l = E_X[f^l(X(t_1), \ldots, X(t_k))] \). Denote:
\[
\sigma_f^l := \left[ Var_X[f^l(X(t_1), \ldots, X(t_k))] \right]^\frac{1}{2},
\]
\[
M_{f,n}^l := \frac{1}{\lambda(T_n)} \int_{T_n} f^l(X(t_1 + t), \ldots, X(t_k + t)) \lambda(dt),
\]
\[
V_{f,n}^l := \frac{1}{\lambda(T_n)} \int_{T_n} (f^l(X(t_1 + t), \ldots, X(t_k + t)) - M_{f,n}^l)^2 \lambda(dt).
\]
For each \( l \) we consider the random field
\[
Y_f^l(t) := f^l(X(t_1 + t), \ldots, X(t_k + t)), \quad t \in T,
\]
over \( (\Omega_X, \mathcal{F}_X, P_X) \). Note that \( Y_f^l(0) := f^l(X(t_1), \ldots, X(t_k + t)) \), so the condition \( E_X[Y_f^l(0)]^2 < \infty \) is fulfilled; the finite-dimensional distributions
of the field $Y^l_f$ coincide with finite-dimensional distributions of $X^l$, and $Y^l_f$ is generated by the shift transformations: by Property (1),

$$Y^l_f(0, \gamma_t \omega) = f^l(X(t_1, \gamma_t \omega), \ldots, X(t_k, \gamma_t \omega)) = f^l(X(t_1 + t), \omega, \ldots, X(t_k + t), \omega) = Y^l_f(t, \omega);$$

therefore, the field $Y^l_f$ is homogeneous, and, if the field $X^l$ is ergodic, the field $Y^l_f$, is ergodic, too.

For example, if $\theta$ is a mixed moment, i.e.

$$\theta = E[(X(t_1))^v_1 \ldots (X(t_k))^v_k], \ (v_1, \ldots, v_k \in \mathbb{N}),$$

we have: $f^l(x_1, \ldots, x_k) = (x(t_1))^v_1 \ldots (x(t_k))^v_k$,

$$Y^l_f(t) = (X(t_1 + t))^v_1 \ldots (X(t_k + t))^v_k.$$

Application of Corollary 1 to the fields $Y^l_f(t)$ brings us to the following statement.

**Corollary 2.** Let $X$ be a homogeneous scalar random field. Under the conditions of Corollary 1 (with $Y^l_f, M^l_n, V^l_n, \theta^l$ instead of $X^l, M^l_n, V^l_n, \mu^l$, respectively), for $l = 1, \ldots, d, \ u = 1, \ldots, w$,

$$\sum_{i=1}^{k_n} (f^l(X(t_1 + \tau_{n,i}^l), \ldots, X(t_k + \tau_{n,i}^l)) - \theta^l) \ p_{X, \tau} \Rightarrow \mathcal{N}(\sigma^l_f).$$

where all random variables $Z^l_{i,u}$ are standard normal and independent.

If $X$ is ergodic, the "empirical" standard deviations $(V^l_n)^{\frac{1}{2}}$ can be replaced by the "theoretical" ones $\sigma^l_f$.

Now let $\theta := F(t_1, \ldots, t_k; x^1, \ldots, x^k) := P(X(t_1) \leq x^1, \ldots, X(t_k) \leq x^k)$, the $k$-dimensional distribution function of the random field $X$ with fixed "time" values $t_1, \ldots, t_k \in \mathbb{R}^m$ and fixed space values $x^1, \ldots, x^k \in \mathbb{R}$. Let $Y(t) = Ind\{X(t_1 + t) \leq x^1, \ldots, X(t_k + t) \leq x^k\}$. Note that

$$E_X[Y(0)] = F(t_1, \ldots, t_k; x^1, \ldots, x^l).$$

Denote:

$$M_n := \frac{1}{\lambda(T_n)} \int_{T_n} Y(t) \lambda(dt), \ V_n := \frac{1}{\lambda(T_n)} \int_{T_n} (Y(t) - M_n)^2 \lambda(dt),$$

$$t := \{t_1, \ldots, t_k\}, \ x := (x_1, \ldots, x_k).$$
Corollary 3. Let $X$ be an ergodic homogeneous scalar random field. Let 
$\sqrt{k_n}(M_n - F(t,x)) \rightarrow 0$ in $P_X$-probability. Consider the family of independent random variables \(\{\tau_{n,i}^u, n \in \mathbb{N}, 1 \leq i \leq k_n, 1 \leq u \leq w\}\) which, for each $n, u, i$, are distributed uniformly on $T_n$. Then
$$
\sum_{i=1}^{k_n} \left( \text{Ind}\{X(t_1 + \tau_{n,i}^u) \leq x^1, ..., X(t_k + \tau_{n,i}^u) \leq x^k\} - F(t;x) \right) \overset{P_X}{\Rightarrow} Z^u,
$$
where all random variables $Z^u, u = 1, ..., w$, are standard normal and independent.

The sequence $V_n$ may be replaced by its limit $F(t;x) - (F(t;x))^2$.

Now we present a randomized version of the CLT when the limit normal distribution is not standard and its covariance matrix coincides with the marginal covariance matrix of the field $X$. This statement is a generalization of the classical CLT for i.i.d. random vectors (see Theorem 29.5 in [3]) and can be readily deduced from our Theorem 2 and 3 (with $d = 1$) by using the Cramér - Wold theorem (compare with the proof of the mentioned Theorem 29.5).

Theorem 5. Let the conditions of Theorem 1 be fulfilled and, moreover, let the field $X$ be strict sense stationary and ergodic. Let the pointwise averaging sequence $\{T_n\}$ and the randomizing sequence $(\tau_{n,i})$ be the same for all components $X^l$. Denote by $\Sigma$ the (nonsingular) covariance matrix of the vector $X(0)$ and let $V$ be a Gaussian vector with mean $0$ and covariance matrix $\Sigma$. Then
$$
\left( k_n^{-\frac{1}{2}} \sum_{i=1}^{k_n} (X^1(\tau_{n,i}) - M_n^1), ..., k_n^{-\frac{1}{2}} \sum_{i=1}^{k_n} (X^d(\tau_{n,i}) - M_n^d) \right) \overset{P_X}{\Rightarrow} V.
$$
If the condition (18) takes place, then also
$$
\left( k_n^{-\frac{1}{2}} \sum_{i=1}^{k_n} (X^1(\tau_{n,i}) - \mu^1), ..., k_n^{-\frac{1}{2}} \sum_{i=1}^{k_n} (X^d(\tau_{n,i}) - \mu^d) \right) \overset{P_X}{\Rightarrow} V.
$$

Our theorems and Examples 1 and 2 imply the following statement.

Corollary 4. 1. Let $T = \mathbb{R}^m$. Then Theorems 1-3 hold if $\{T_n^1\}$ is
- an increasing sequence of bounded convex sets, containing balls of radii $r_n \rightarrow \infty$, or
- a sequence of homothetic sets $\{s_n A\}$ considered in Example 2.

2. The same is true if $T = \mathbb{Z}^m$ and $T_n^1$ are restrictions onto $\mathbb{Z}^m$ of convex sets in $\mathbb{R}^m$ considered above.
4.4 Rate of convergence

We remind that \( k_n \in \mathbb{N}, \ k_n \uparrow \infty, \ \{T_n\} \) is a pointwise averaging sequence of sets and

\[
V_n := \frac{1}{\lambda(T_n)} \int_{T_n} (X(t) - M_n)^2 \lambda(dt).
\]

Let \( F_n \) be the distribution function of the normalized sum

\[
Z_n = \sum_{i=1}^{k_n} \frac{(X(\tau_{n,i}) - \mu)}{k_n^{\frac{1}{2}}V_n^{\frac{1}{2}}}
\]

and \( \Phi \) be the distribution function of the standard Gaussian law. We are interested in the estimation of

\[
\Delta_n := \sup_{x \in \mathbb{R}} \Delta_n(x),
\]

where

\[
\Delta_n(x) = |F_n(x) - \Phi(x)|.
\]

**Theorem 6.** Suppose that \( \{X(t), \ t \in T\}, \ X(t) \in \mathbb{R}, \) is a stationary ergodic field such that for some \( \delta, \ 0 < \delta \leq 1, \ E|X(t)|^{2+\delta} < \infty. \)

There exist a constant \( C \) (depending only of \( \delta \)) such that for all \( \varepsilon > 0 \) and every \( n \)

\[
\Delta_n \leq P\{V_n < \varepsilon\} + C k_n^{-\frac{\delta}{2}} \varepsilon^{-\frac{2+\delta}{2}} E|X(0)|^{2+\delta}.
\]  \hspace{1cm} (19)

**Remark 3.** Below, in Lemma 3, we present several sufficient conditions for estimating the probability \( P\{V_n < \varepsilon\} \). Their proof is easily deduced from Theorem 4.4 in [60].

**Lemma 3.** Let \( \{T_n\} \) be a sequence of Borel sets in \( T \) with the following property: each \( T_n \) is contained in a ball \( B(b_n, Cn) \) and \( \lambda(T_n) \geq cn^m \) \( (b_n \in T, \ 0 < C < \infty, \ 0 < c < C) \). Let \( \varepsilon < \sigma^2 \). We assume that \( \mu_4 = E[X^4] < \infty. \)

Suppose the covariance functions:

\[
R_X(t) := E[X(t)X(0)] - \mu^2 \quad \text{and} \quad R_{X^2}(t) := E[(X(t)X(0))^2] - \mu_2^2
\]

satisfy the conditions: for some \( \beta > 0 \)

\[
R_X(t) = O(|t|^{-\beta}), \quad R_{X^2}(t) = O(|t|^{-\frac{\beta}{2}}), \quad t \to \infty.
\]

**Case 1.** If \( \beta < m \), then \( P\{V_n < \varepsilon\} \leq (\sigma^2 - \varepsilon)^{-2} O(n^{-\frac{\beta}{2}}). \)

**Case 2.** If \( \beta = m \), then \( P\{V_n < \varepsilon\} \leq (\sigma^2 - \varepsilon)^{-2} O(n^{-\frac{m}{2}} (\log n)^{\frac{1}{2}}). \)

**Case 3.** If \( \beta > m \), then \( P\{V_n < \varepsilon\} \leq (\sigma^2 - \varepsilon)^{-2} O(n^{-\frac{m}{2}}). \)
If we take \( \varepsilon = \sigma^2/2 \), we get from (19) the following corollary.

**Corollary 5.** Under the conditions of Theorem 4 and Lemma 3 we have:

In Case 1: \( \Delta_n = O(\max\{n^{-\beta/2}, k_{n^{-\beta}}\}) \).

In Case 2: \( \Delta_n = O(\max\{n^{-\beta/2} (\log n)^{\beta/2}, k_{n^{-\beta}}\}) \).

In Case 3: \( \Delta_n = O(\max\{n^{-\beta/2}, k_{n^{-\beta}}\}) \).

Now let’s move on to the proof of the theorem 6.

**Proof.** Since for fixed path of the process \( X \) the random variables \( \{X(\tau_{n,i})\} \), \( i = 1, \ldots, n \), are independent and identically distributed, then denoting by \( F_{\tau_n} \) the function of conditional distribution \( Z_n \), we can apply the estimation ([54], Ch.5, 3, Th. 6):

There exists an absolute constant \( C \) such that

\[
\Delta_{\tau_n}(x) := |F_{\tau_n}(x) - \Phi(x)| \leq \frac{C}{V_n^{(2+\delta)/2} k_{\delta/2}} E_T \{|X(\tau_{n,1}) - M_n|^{2+\delta}\}.
\]

Therefore

\[
\Delta_n(x) = |F_n(x) - \Phi(x)| = |E(F_{\tau_n}(x) - \Phi(x))| \leq E|\Delta_{\tau_n}(x)| := I_1 + I_2,
\]

where

\[
I_1 = \int_{\{V_n < \varepsilon\}} |\Delta_{\tau_n}(x)| dP, \quad I_2 = \int_{\{V_n \geq \varepsilon\}} |\Delta_{\tau_n}(x)| dP.
\]

Evidently

\[
I_1 \leq P\{V_n < \varepsilon\}.
\]

Consider \( I_2 \). As

\[
E_T \{|X(\tau_{n,1}) - M_n|^{2+\delta}\} = \frac{1}{\lambda(T_n)} \int_{T_n} |X(t) - M_n|^{2+\delta} \lambda(dt) \leq 2^{1+\delta} \left\{ \frac{1}{\lambda(T_n)} \int_{T_n} |X(t)|^{2+\delta} \lambda(dt) + |M_n|^{2+\delta} \right\},
\]

we have

\[
I_2 \leq \frac{C}{k_{\delta/2}} \varepsilon^{-\frac{2+\delta}{2}} 2^{1+\delta} \left[E|X(0)|^{2+\delta} + E|M_n|^{2+\delta}\right].
\]

Since \( E|M_n|^{2+\delta} \leq E|X(0)|^{2+\delta} \), we get the estimation

\[
I_2 \leq C \frac{2^{2+\delta}}{k_{\delta/2}} \varepsilon^{-\frac{2+\delta}{2}} E|X(0)|^{2+\delta},
\]

which gives finally (19). \( \square \)
4.5 Invariance principle

The next natural step is to state the functional central limit theorem (invariance principle).

Let $X(t, \omega), t \in \mathbb{R}^m$, be a bi-measurable random field over a probability space $(\Omega_X, \mathcal{F}_X, P_X)$. Let $\{T_n\}$ be a sequence of increasing convex sets in $\mathbb{R}^m$ containing balls with radii $r_n \to \infty$. Let, for each $n \in \mathbb{N}$, $\tau_{n,i}, i = 1, ..., k_n$, be random variables over a probability space $(\Omega_{\tau}, \mathcal{F}_{\tau}, P_{\tau})$, each being uniformly distributed on $T_n$ and independent of each other and of $X$. Let $(\Omega_X, \mathcal{F}_X, \tau, P_X, \tau) := (\Omega_X \times \Omega_{\tau}, \mathcal{F}_X \times \mathcal{F}_{\tau}, P_X \times P_{\tau})$.

Using random variables $X(\tau_{n,i})$, we construct in usual way the continuous piecewise linear random process $Z_n = \{Z_n(t), t \in [0, 1]\}$. The process $Z_n$ has the vertices at the points

$$\left(\frac{j}{k_n}, \frac{S_j}{k_n^{1/2}V_n^{1/2}}\right), \quad j = 0, 1, \ldots, k_n,$$

where

$$S_j = \sum_{r=1}^{j} (X(\tau_{n,r}) - M_n), \quad V_n = Var_{\tau}(X(\tau_{n,r})).$$

Denote by $P_n$ the distribution of the process $Z_n$ in the space $C[0, 1]$ and by $W$ the distribution of standard Wiener process.

**Theorem 7.** Under mentioned conditions:

a) For $P_X$-almost all $\omega$

$$Z_n(\omega, \cdot) \xrightarrow{P} W. \quad (22)$$

b) The convergence

$$Z_n \xrightarrow{P_{X,\tau}} W. \quad (23)$$

also takes place.

**Remark 4.** The importance of weak convergence in (22, 23) comes from the fact that by continuous mapping theorem (see e.g., [3]) many corollaries emerge immediately, namely for each $W$-a.e. continuous mapping $f : \mathbb{C} \to \mathbb{R}$ we have the convergence $f(Z_n) \xrightarrow{W} f(W)$. Typical examples of such functionals are $f(x) = \sup_{t \in [0, 1]} x(t)$, $f(x) = \sup_{t \in [0, 1]} |x(t)|$, $f(x) = \int_{[0,1]} h(t)dt$, and so on...
Proof. As for triangular array \( \{ X(\tau_{n,i}) \} \) the Lindeberg condition is fulfilled, by Prokhorov’s theorem (see [55], Th.3.1.), we get (22).

The relation (23) follows from (22) since the limiting measure does not depend of \( \omega \).

Let us consider now the random broken lines \( \tilde{Z}_n \) with a different normalization. More exactly, let

\[
\tilde{Z}_n \left( \frac{j}{k_n} \right) = \frac{S_j}{k_n^{1/2} \sigma^{1/2}},
\]

where \( \sigma = \text{Var}\{X(0)\} \).

**Theorem 8.** Suppose that the field \( X(t) \) is ergodic. Then:

a) For \( P_X \)-almost all \( \omega \)

\[
\tilde{Z}_n(\omega, \cdot) \xrightarrow{P} W.
\] (24)

b) The convergence

\[
\tilde{Z}_n \xrightarrow{P_X, \tau} W.
\] (25)

also takes place.

**Proof.** The proof follows from the relation \( \tilde{Z}_n(t) = (\frac{\hat{V}_n}{\sigma})^{1/2} Z_n(t) \) and the fact that due to the ergodic theorem \( \frac{\hat{V}_n}{\sigma} \to 1 \) a.s.

With a little additional restriction, we can replace \( M_n \) by \( \mu \) which is the mean value of our field.

Define the process \( \tilde{Z}_n \) as a broken line with vertices at the points

\[
\left( \frac{j}{k_n}, \frac{S_j}{k_n^{1/2} \sigma^{1/2}} \right),
\]

where now \( S_j = \sum_{r=1}^j (X(\tau_{n,r} - \mu)) \).

**Theorem 9.** Suppose that the field \( X(t) \) is ergodic. Additionally suppose that a.s.\(^2\)

\[
k_n^{1/2} (M_n - \mu) \to 0.
\]

Then:

\(^2\)See §A for remarks on this condition.
a) For $P_X$-almost all $\omega$

\[ \tilde{Z}_n(\omega, \cdot) \Rightarrow W. \quad (26) \]

b) The convergence

\[ \tilde{Z}_n \overset{P_{X,\tau}}{\Rightarrow} W. \quad (27) \]

also takes place.

Proof. It is sufficient to remark that a.s.

\[ \sup_{t \in [0,1]} |\tilde{Z}_n(t) - \tilde{Z}_n(t)| \leq \sigma^{-1/2}(k_n^{1/2}(M_n - \mu)) \to 0. \]

\[ \square \]

5 Empirical distributions

5.1 Randomized versions of the Glivenko-Cantelli theorem

In this section we generalize the Glivenko-Cantelli theorem to multidimensional distributions of stationary random processes on $\mathbb{R}$ and of homogeneous random fields on $\mathbb{R}^m$ (to simplify the presentation we assume ergodicity; this assumption can be readily dropped by the consideration of the conditional expectations as in Lemma 4 below). Generalized versions of the Glivenko-Cantelli theorem for marginal distributions of stationary random sequences were proved in [58, 64]. We consider a randomized version of such theorem for ergodic homogeneous random fields on $\mathbb{R}^m$.

Let $X(t, \omega) = (X_1(t, \omega), ..., X_l(t, \omega))$ be an $l$-dimensional bi-measurable random field on $\mathbb{R}^m$ over a probability space $(\Omega_X, \mathcal{F}_X, P_X)$.

We consider the CDF of this random field:

\[ F(x_1, ..., x_s) := P(X_1(0) \leq x_1, ..., X_l(0) \leq x_l). \]

Let $\{T_n\}$ be a sequence of increasing convex sets in $\mathbb{R}^m$ containing balls with radii $r_n \to \infty$ and let, for each $n \in \mathbb{N}$, $\eta_{n,i}$, $i = 1, ..., k_n$, be random variables over a probability space $(\Omega_\eta, \mathcal{F}_\eta, P_\eta)$, each being uniformly distributed on $T_n$ and independent of each other and of $X$. Denote:

\[ (\Omega_{X,\eta}, \mathcal{F}_{X,\eta}, P_{X,\eta}) := (\Omega_X \times \Omega_\eta, \mathcal{F}_X \times \mathcal{F}_\eta, P_X \times P_\eta) \]

Lemma 4. If $Y(t, \omega)$, $t \in \mathbb{R}^m$, is an homogeneous random field over the probability space $(\Omega_X, \mathcal{F}_X, P_X)$ and for some $\delta > 0$

\[ E_X[|Y(0)|^{2+\delta}] < \infty, \quad (28) \]
then with $P_{X, \eta}$-probability 1

$$
\lim_{n \to \infty} \frac{1}{k_n} \sum_{i=1}^{k_n} Y(\eta_{n,i}, \omega) = E_X[Y(0)|I_X]
$$

(if the random field $X$ is ergodic, $E[Y(0)|I_X] = E_X[Y(0)]$).

Proof. Denote $M_n[Y] := \frac{1}{\lambda(T_n)} \int_{T_n} Y(t) \lambda(dt)$. Since with $P_X$-probability 1 for each $n \in \mathbb{N}$ the random variables $Y(\eta_{n,i}, \omega)$, $i = 1, \ldots, k_n$ are independent, condition (28) implies:

$$
\lim_{n \to \infty} \frac{1}{k_n} \sum_{i=1}^{k_n} Y(\eta_{n,i}, \omega) - M_n[Y] = 0,
$$

by virtue of Corollary 1 (with $p=1$) and Remark 3 in [30]. The Fubini theorem implies that (29) is valid with $P_{X, \eta}$-probability 1. By the PET, with $P_X$-probability 1

$$
\lim_{n \to \infty} M_n[Y] = E[Y(0)|I_Y].
$$

(30)

It remains to note that

$$
\frac{1}{k_n} \sum_{i=1}^{k_n} [Y(\eta_{n,i}, \omega) - E[Y(0)|I_Y]] = 
\left( \frac{1}{k_n} \sum_{i=1}^{k_n} Y(\eta_{n,i}, \omega) - M_n[Y] \right) + (M_n - E[Y(0)|I_Y]).
$$

\[ \square \]

Denote:

$$
F_n(x^1, \ldots, x^l; \omega, \eta) := \frac{1}{k_n} \sum_{i=1}^{k_n} \text{Ind}[X_1(\eta_{n,i}, \omega) \leq x^1, \ldots, X_l(\eta_{n,i}, \omega) \leq x^l].
$$

To simplify the notation we denote: $x = (x^1, \ldots, x^l) (\in \mathbb{R}^l)$.

**Theorem 10.** Let the random field $X(t, \omega)$, $t \in \mathbb{R}^m$, be ergodic and homogeneous. With $P_{X, \eta}$-probability 1

$$
\lim_{n \to \infty} \sup_{x \in \mathbb{R}^l} |F_n(x; \omega, \eta) - F(x)| = 0.
$$

(31)
Proof. Since the random field $X$ is bi-measurable, the function $(\omega, \eta) \mapsto F_n(x; \omega, \eta)$ is $\mathcal{F}_{X,\eta}$-measurable. Let $\mathbb{Q}$ be the set of rational numbers. It is clear that the function $(\omega, \eta) \mapsto \sup_{x \in \mathbb{Q}^l} |F_n(x; \omega, \eta) - F(x)|$ is $\mathcal{F}_{X,\eta}$-measurable, and, since $\mathbb{Q}^l$ is dense in $\mathbb{R}^l$,

$$\sup_{x \in \mathbb{R}^l} |F_n(x; \omega, \eta) - F(x)| = \sup_{x \in \mathbb{Q}^l} |F_n(x; \omega, \eta) - F(x)|,$$

the function $\sup_{x \in \mathbb{R}^l} |F_n(x; \omega, \eta) - F(x)|$ is also $\mathcal{F}_{X,\eta}$-measurable.

For each $k$, $0 \leq k \leq M - 1$, we denote:

$$L_{k,M} = \{x \in \mathbb{R}^l : \frac{k}{M} \leq F(x) < \frac{k+1}{M}\}.$$ 

Of course, if $F$ is continuous, then each $y \in \mathbb{R}^l$ has at least one preimage with respect to the function $F$ and, for each $k$, $F(L_{k,M}) = [\frac{k}{M}, \frac{k+1}{M})$; otherwise, some points in $[0, 1]$ do not have preimages, and, therefore, there are $k$ such that $[\frac{k}{M}, \frac{k+1}{M}) \setminus F(L_{k,M}) \neq \emptyset$, and even such that $L_{k,M} = \emptyset$. If $\frac{k}{M}$ possesses at least one preimage with respect to $F(x)$, we denote by $x_{k,M}$ one of these preimages; in general, if $L_{k,M} \neq \emptyset$, consider the set $D_{k,M} := F(L_{k,M})$, i.e., the set of all points $y$ in $[\frac{k}{M}, \frac{k+1}{M})$ possessing preimages.

Let $0 \leq k \leq M - 1$ and $L_{k,M} \neq \emptyset$.

Denote:

$$G(k+1, M) := P\{(X_1(0), ..., X_l(0)) \in F^{-1}([0, \frac{k+1}{M}))\},$$

$$H(k, M) := 1 - P\{(X_1(0), ..., X_l(0)) \in F^{-1}([\frac{k}{M}, 1])\},$$

and

$$G_n(k+1, M; \omega, \eta) := \frac{1}{k_n} \sum_{i=1}^{k_n} \text{Ind}[(X_1(\eta_n,i, \omega), ..., X_l(\eta_n,i, \omega)) \in F^{-1}([0, \frac{k+1}{M})],$$

$$H_n(k, M; \omega, \eta) := 1 - \frac{1}{k_n} \sum_{i=1}^{k_n} \text{Ind}[(X_1(\eta_n,i, \omega), ..., X_l(\eta_n,i, \omega)) \in F^{-1}([\frac{k}{M}, 1])].$$

Note: a) if $l = 1$, then $G(k, M) = F(x_{k+1,M-0}); H(k, M) = F(x_{k+1,M+0});$

b) if the function $F$ is continuous at the point $x_{k+1,M}$, then $G(k+1, M) = F(x_{k+1,M}); H(k, M) = F(x_{k,M}).$
c) if $x \in F(L_{k,M})$, then $H(k,M) \leq F(x) \leq G(k+1,M)$;
$H_n(k,M;\omega,\eta) \leq F_n(x;\omega\eta) \leq G_n(k+1,M;\omega,\eta)$;

d) $0 \leq G(k+1,M) - H(k,M) \leq \frac{1}{M}$.

By Lemma 4, for each $k$ with $P_{X,\eta}$-probability 1

$$|H_n(k,M;\omega,\eta) - H(k,M)| \to 0$$  \hspace{1cm} (32)

and

$$|G_n(k+1,M;\omega,\eta) - G(k+1,M)| \to 0.$$  \hspace{1cm} (33)

If $x \in L_{k,M}$,

$$F_n(x;\omega,\eta) - F(x) \geq H_n(k,M;\omega,\eta) - H(k,M) - \frac{1}{M}.$$  

Therefore, if $0 \leq k \leq [M]$, $L_{k,M} \neq \emptyset$ and $x \in L_{k,M}$,

$$|F_n(x;\omega,\eta) - F(x)| \leq \max\{|G_n(k+1,M;\omega,\eta) - G(k+1,M)|, |H_n(k,M;\omega,\eta) - H(k,M)|\} + \frac{1}{M}.$$  

It is clear that $\cup_{0 \leq k \leq M-1} L_{k,M} = \mathbb{R}^l$; therefore,

$$\sup_{x \in \mathbb{R}^l} |F_n(x;\omega,\eta) - F(x)| \leq \max\{|G_n(k+1,M;\omega,\eta) - G(k+1,M)|, |H_n(k,M;\omega,\eta) - H(k,M)|\} + \frac{1}{M}.$$  

and, by relations (32) and (33), with $P_{X,\eta}$-probability 1

$$0 \leq \limsup_{n \to \infty} \sup_{x \in \mathbb{R}^l} |F_n(x;\omega,\eta) - F(x)| \leq \limsup_{n \to \infty} \max_{k:L_{k,M} \neq \emptyset, 0 \leq k \leq M-1} \sup_{x \in L_{k,M}} |F_n(x;\omega,\eta) - F(x)| \leq \frac{1}{M}.$$  

Since $M$ is chosen arbitrarily, with $P_{X,\eta}$-probability 1

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}^l} |F_n(x;\omega,\eta) - F(x)| = 0.$$  

\[\square\]
Let $X(t, \omega)$, $t \in \mathbb{R}^m$, be a random field over a probability space $(\Omega_X, \mathcal{F}_X, P_X)$. We fix a set of "time" points $t = \{t_1, ..., t_l \in \mathbb{R}^m\}$ where $l \in \mathbb{N}$, and study the multidimensional CDF

$$F(t; x^1, ..., x^l) := P(X(t_1) \leq x^1, ..., X(t_l) \leq x^l).$$

Denote:

$$F_n(t; x^1, ..., x^l; \omega, \eta) := \frac{1}{k_n} \sum_{i=1}^{k_n} \text{Ind}[X(t_1 + \eta_n,i, \omega) \leq x^1, ..., X(t_l + \eta_n,i, \omega) \leq x^l];$$

It is the $l$-dimensional randomized empirical distribution function, based on observations of $X$ on randomly chosen points in $\bigcup_{j=1}^{l}(t_j + T_n)$.

We apply the latter theorem to the ergodic homogeneous random field $X(t, \omega) = (X(t_1 + t), ..., X(t_l + t))$ and come to the following statement.

**Theorem 11.** Let the random field $X(t, \omega)$, $t \in \mathbb{R}^m$, be homogeneous and ergodic. With $P_{X, \eta}$-probability 1

$$\lim_{n \to \infty} \sup_{(x^1, ..., x^l) \in \mathbb{R}^l} |F_n(t; x^1, ..., x^l; \omega, \eta) - F(t; x^1, ..., x^l; \omega, \eta)| = 0. \quad (34)$$

Let $\eta^u = \{\eta^u_{n,i}\}, u = 1, ..., r$ be $r$ independent arrays of randomizing random variables.

$$F_n(t; x^1, ..., x^l; \omega, \eta^u) := \frac{1}{k_n} \sum_{i=1}^{k_n} \text{Ind}[\{(\omega, \eta) : X(t_1 + \eta^u_{n,i}, \omega) \leq x^1, ..., X(t_l + \eta^u_{n,i}, \omega) \leq x^l\}];$$

$$F_n^r(t; x^1, ..., x^l; \omega, \eta) := \frac{1}{r} \sum_{u=1}^{r} F_n(t; x^1, ..., x^l; \omega, \eta^u).$$

**Corollary 6.** Let the random field $X(t, \omega)$, $t \in \mathbb{R}^m$, be ergodic and homogeneous. With $P_{X, \eta}$-probability 1

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}^l} |F_n^r(t; x^1, ..., x^l; \omega, \eta) - F(x)| = 0. \quad (35)$$
This follows from Theorem 11 and the relation:
\[
\sup_{(x^1,\ldots,x^l)\in\mathbb{R}^l} |F_n^r(t; x^1, \ldots, x^l; \omega, \eta) - F(x)| \leq \frac{1}{r} \sum_{u=1}^r \sup_{(x^1,\ldots,x^l)\in\mathbb{R}^l} |F_n(t; x^1, \ldots, x^l; \omega, \eta^u) - F(x)|.
\]
If \( r \geq 2 \), each estimator \( F_n^r(t; x; \omega, \eta) \) of the CDF involves more observations and, therefore, it is more precise than the estimator \( F_n(t; x; \omega, \eta) \).

5.2 Convergence of the distributions of empirical processes

We suppose in this section that \( \{X(t, \omega), t \in \mathbb{R}^m\} \), is a bi-measurable stationary random field with values in \( \mathbb{R}^l \). Moreover we suppose that a.s. \( X(t) \in (0,1)^l \), as the general case can be reduced to this by the standard transformation \( \beta^i = \frac{\arctan x^i}{\pi} + \frac{1}{2}, i = 1, \ldots, l \).

Let
\[
G_n(x) := \frac{1}{k_n^{1/2}}(F_n(x) - F(x)), \quad x \in [0,1]^l, \tag{36}
\]
where \( F_n \) is the empirical distribution function for the series of r.v. \( \{X(\tau_{n,i}), i = 1, 2, \ldots, k_n\} \),

\[
F_n(x) = \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{1}_{[0,x]}(X(\tau_{n,i})),
\]
where \( [0,x] = \prod_{i=1}^l [0,x_i] \) and \( F \) is the distribution function of \( X(0) \). Below we suppose that \( F \) is continuous.

We will use also empirical processes with different centering:

\[
L_n(x) = \frac{1}{n(T_n)} \int_{T_n} \mathbb{1}_{[0,x]}(X(s)) \lambda(ds),
\]
where \( M_n(x) = \frac{1}{n(T_n)} \int_{T_n} \mathbb{1}_{[0,x]}(X(s)) \lambda(ds) \).

Let \( B([0,1]^l) \) be the set of bounded, real-valued, and measurable functions defined on the \( l \)-dimensional cube \([0,1]^l\). Let \( C([0,1]^l) \) be the set of all continuous functions. Furthermore, let \( \mathcal{D} \) be a subset of \( B([0,1]^l) \) such that, first, it contains \( C([0,1]^l) \) and, second, the supremum norm of every function in \( \mathcal{D} \) is determined by the supremum of the function over a countable subset of \([0,1]^l\). Furthermore, assume that there is a metric \( \rho \) that makes \( \mathcal{D} \) a complete separable space whose topology is weaker than the topology of uniform convergence, and such that each time when \( \rho(x_n, x) \to 0 \) and \( x \) is continuous it follows that \( x_n \) converges to \( x \) uniformly. Examples of the
space $\mathbb{D}$ with various metrics/topologies with mentioned properties can be found in a number of works (cf., e.g., [57], [23] and references therein). For comments on weak convergence in non-separable spaces we refer to Section 4 of [18].

It is clear that we can consider $G_n, L_n$ as random elements of $\mathbb{D}$.

We use the notation $\tilde{W}$ for the continuous Gaussian centered process with correlation function $K(x, y) = F(x \wedge y) - F(x)F(y)$, $x, y \in [0, 1]^l$, where $x \wedge y = (x_1 \wedge y_1, \ldots, x_l \wedge y_l)$.

Our aim is to state the weak convergence of $G_n$ and $L_n$ to $\tilde{W}$.

**Theorem 12.** If the field $\{X_t\}$ is ergodic, then

$$L_n \Rightarrow \tilde{W}. \tag{37}$$

If additionally

$$\sqrt{k_n} \sup_{x \in [0, 1]^l} |M_n(x) - F(x)| \xrightarrow{P} 0, \tag{38}$$

then

$$G_n \Rightarrow \tilde{W}. \tag{39}$$

**Remark 5.** Condition (38) imposes a restriction on the rate of growth of the sequence $k_n$. In the case, when the a.s. convergence is considered in this condition, the admissible growth of the integers $k_n$ has been studied in [24] for iid random sequences $X(1), X(2), \ldots$; in [67], stationary sequences $X$ are studied, and it is established, how the admissible rate of increase of $k_n$ is specified by the rate of mixing of $X$ and by rate of growth of its metric entropy. Condition (38) encourages to study the admissible rate of the growth of $k_n$ when convergence in probability is considered (we suspect it will be higher).

We have

$$\sqrt{k_n} \sup_{x \in [0, 1]^l} |M_n(x) - F(x)| = \sqrt{\frac{k_n}{\lambda(T_n)}} \sup_{x \in [0, 1]^l} |\xi_n(x)|,$$

where

$$\xi_n(x) = \lambda(T_n)^{-1/2} \int_{T_n} [1_{[0, x]}(X(s)) - F(x)]\lambda(ds).$$

We see that $\xi_n$ is empirical process associated with initial process $X$, hence, in good cases it has a continuous limit $\xi$. Then, as the functional $h \rightarrow$
sup_{x \in [0,1]^l} h(x) is a.e. continuous with respect to the distribution of \( \xi \), sup_{x \in [0,1]^l} |\xi_n(x)| is bounded in probability, and the condition \( k_n = o(\lambda(T_n)) \) will be sufficient for (38).

We start the proof of the Theorem 12 with the following Lemma.

Let \( Y_{n,j}, j = 1, \ldots, k_n \) be array of i.i.d. \( l \)-dimensional random vectors in each row having a common distribution function \( J_n \). It is supposed that \( Y_{n,j} \in [0,1]^l \) a.s.

Consider the empirical process

\[
V_n(x) = k_n^{1/2}(H_n(x) - J_n(x)), \quad x \in [0,1]^l,
\]

where \( H_n(x) = \frac{1}{k_n} \sum_{i=1}^{k_n} 1_{[0,x]}(Y_{n,i}). \)

**Lemma 5.** Suppose additionally that \( J_n \) converges uniformly to some continuous distribution function \( J \).

Then

\[
V_n \Rightarrow W_J. \tag{40}
\]

**Remark 6.** The case when \( J_n = J \) for all \( n \) is well known, see, for example, [2], [57] or [22]. But this result for triangular arrays we could not find in the literature and therefore we present here detailed proof. It is based on the approach proposed in [18].

**Proof.** At the beginning we consider the case when all one-dimensional marginal distributions of \( Y_{n,j} \) are uniform on \([0,1] \).

Following [18], we need to carry out three steps: establish the convergence of finite-dimensional distributions, check the moment condition of Theorem 1 of [18] and, finally, estimate the modulus of continuity of the function \( J_n \).

**Step 1.** The convergence of finite-dimensional distributions follows in the standard way from Lindeberg condition and Cramer-Wold device.

**Step 2.** Verification of the first condition of Th.1 from [18].

We shall find it technically more convenient to work with

\[
\|x\| := \max_{1 \leq i \leq l} |x_i|.
\]

We prove that for some \( C > 0 \) and for all \( x, y \in [0,1]^l \)

\[
E|V_n(x) - V_n(y)|^{2l+2} \leq C\|x - y\|^{l+1} \quad \text{whenever} \quad \|x - y\| \geq \frac{1}{k_n}. \tag{41}
\]

Indeed, \( V_n(x) - V_n(y) \) is a normalized sum of i.i.d. centered random variables

\[
\xi_i := 1_{[0,x]}(Y_{n,i}) - 1_{[0,y]}(Y_{n,i}) - (J_n(x) - J_n(y)).
\]
Since the symmetric difference of parallelepipeds \([0, x], [0, y]\) can be partitioned into a disjoint union of at most \(2^l\) parallelepipeds of the form \([a, b]\), it suffices to estimate the moments for the individual elements of this partition. Let

\[
\alpha_i = 1_{[a, b]}(Y_{n,i}), \quad p = P\{Y_{n,i} \in [a, b]\}.
\]

As every parallelepiped \([a, b]\) from our partition has at least one edge whose length is \(|x_i - y_i|\) for some \(i\), then \(p \leq |J_n(a) - J_n(b)| \leq \|x - y\|\).

Using inequality of Th.19, \[54\], ch.III, we have

\[
E \left| \sum_{1}^{m} (\alpha_i - p) \right|^{2l+2} \leq C(mE\alpha_1^{2l+2} + (mE\alpha_1^2)^{l+1}) \leq C(mp + m^{l+1}p^{l+1}), \quad (42)
\]

since \(E\alpha_1^{2l+2} = p(1-p)[(1-p)^{2l+1} + p^{2l+1}] \leq p\), and \(E\alpha_1^2 = p(1-p) \leq p\).

Returning to \(E|V_n(x) - V_n(y)|^{2l+2}\), we get for \(m = k_n\) and some \(C > 0\) the estimation

\[
E|V_n(x) - V_n(y)|^{2l+2} \leq C \left( \frac{p}{k_n} + p^{l+1} \right).
\]

The right-hand side does not exceed \(C\|x - y\|^{l+1}\) since \(p \leq \|x - y\|\) and \(\frac{1}{k_n} \leq \|x - y\|\).

**Step 3.** We need to estimate \(\sqrt{k_n}|J_n(x) - J_n(y)|\) when \(x, y\) are two adjacent points of the lattice \(\Gamma_n = \{j/k_n \mid j \in [0, k_n]\}\). Due to the condition that marginal distributions of \(Y_{n,i}\) are uniform we get immediately

\[
\sqrt{k_n} \sup_{x, y \in \Gamma_n, \|x - y\| = 1/k_n} |J_n(x) - J_n(y)| \leq l/\sqrt{k_n} \to 0
\]

when \(n \to \infty\), which completes the proof in this case.

Now we consider the case when limiting function \(J\) is continuous and coordinatewise strictly increasing. We modify slightly the arguments from the second part of the proof of the Th.16.4. \[3\].

For simplicity we use the notation \(Y_n\) for \(Y_{n,1}\) and \(Y\) for a random vector having distribution function \(J\). Let \(J_n^{(k)}, J^{(k)}\) be marginal distribution functions for respectively \(Y_n\) and \(Y\):

\[
J_n^{(k)}(x_k) = P\{Y_n^{(k)} \leq x_k\}, \quad J^{(k)}(x_k) = P\{Y^{(k)} \leq x_k\}, \quad x_k \in [0, 1].
\]

Let \(\psi_n, \psi : [0, 1]^l \to [0, 1]^l\) be define by

\[
\psi_n(x) = (J_n^{(l)}(x_1), \ldots, J_n^{(l)}(x_l)), \quad \psi(x) = (J^{(l)}(x_1), \ldots, J^{(l)}(x_l)).
\]

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Now let \( Z_{n,i}^{(k)} = J_n^{(k)}(Y_{n,i}^{(k)}) \), \( k = 1, \ldots, l \), \( i = 1, \ldots, k_n \). Then vectors \( Z_{n,i} = (Z_{n,i}^{(1)}, \ldots, Z_{n,i}^{(l)}) \) are i.i.d. and have \([0,1]\)-uniformly distributed coordinates.

Let \( Z_n = \psi_n(Y_n) \), \( Z = \psi(Y) \). If \( x_n \to x \), then \( \psi_n(x_n) \to \psi(x) \) and by Th.5.5 we get the convergence \( Z_n \Rightarrow \psi(Y) \) which gives the convergence of distribution functions \( F_{Z_n}(x) \to F_Z(x) \), and this convergence is uniform as \( F_Z \) is continuous. Hence we can apply to the triangular array \( \{Z_{n,i}\} \) our previous result which says that the empirical processes \( U_n \),

\[
U_n(x) = \sqrt{k_n}(A_n(x) - F_{Z_n}(x)), A_n(x) = \frac{1}{k_n} \sum_{i=1}^{k_n} 1_{[0,x]}(Z_{n,i}), \tag{43}
\]

converge weakly to \( \tilde{W}_{F_Z} \).

Now define two mappings \( \varphi_n, \varphi \) inverse to \( \psi_n, \psi \):

\[
\varphi_n(y) = (\ldots, \varphi_n^{(k)}(y_k), \ldots), \quad \varphi(y) = (\ldots, \varphi^{(k)}(y_k), \ldots),
\]

\[
\varphi_n^{(k)}(s) = \inf \{t \mid s \leq J_n^{(k)}(t)\}, \quad \varphi^{(k)}(s) = \inf \{t \mid s \leq J^{(k)}(t)\}.
\]

It is clear that the vectors \( (Y_{n,1}, \ldots, Y_{n,k_n}) \) and \( (\varphi_n(Z_{n,1}), \ldots, \varphi_n(Z_{n,k_n})) \) will have the same distribution.

Define \( f_n, f : \mathbb{D} \to \mathbb{D} \) respectively by

\[
(f_n y)(x) = y(J_n(x)); \quad (f y)(x) = y(J(x)), \quad y \in \mathbb{D}, \ x \in [0,1]^l.
\]

If \( y_n \) converges to \( y \) in \( \mathbb{D} \) and \( y \in \mathbb{C}[0,1]^l \), then the convergence is uniform and \( f_n(y_n) \) will converge to \( f(y) \) uniformly, due to the uniform convergence of \( J_n \) to \( J \). Hence, by Th.5.5,

\[
f_n(U_n) \Rightarrow f(\tilde{W}_{F_Z}),
\]

which gives the result since we have the equalities in distribution

\[
f_n(U_n) \overset{d}{=} V_n \quad \text{and} \quad f(\tilde{W}_{F_Z}) \overset{d}{=} \tilde{W}_J.
\]

To complete the proof of the lemma, it suffices bring into consideration a family \( \{L_\theta\} \) of triangular arrays, for which distribution functions \( J_\theta \) are continuous, strictly increasing, and uniformly converge to the distribution function \( J_{\theta_0} := J \) when \( \theta \to \theta_0 \). Let \( V_{n,\theta} \) be the empiric process associated with \( L_\theta \) and \( V_{n,\theta_0} = V_n \). Then we have the following properties:

1. For each \( \theta \), \( V_{n,\theta} \Rightarrow \tilde{W}_{J_\theta} \), as \( n \to \infty \) (by previous consideration).
2. For each \( n \), \( V_{n,\theta} \Rightarrow V_{n,\theta_0} \), as \( \theta \to \theta_0 \) (due to the convergence \( J_\theta \to J_{\theta_0} \)).
3. $\hat{W}_{J_0} \Rightarrow \hat{W}_{J_{\theta_0}}$, as $\theta \to \theta_0$ (due to the convergence of covariance functions.)

From these three properties it follows immediately the convergence $V_n = V_{n, \theta_0} \Rightarrow \hat{W}_{J_{\theta_0}} = \hat{W}_J$ which finishes the proof.

**Proof of Th.12.** Remark that by Glivenko-Cantelli theorem

$$M_n(x) = \frac{1}{\lambda(T_n)} \int_{T_n} 1_{[0,x]}(X(s))\lambda(ds).$$

a. s. converges uniformly to $F$. Therefore, applying previous Lemma 5 to the vectors $X(\tau_{n,i})$ which are i.i.d. conditionally given field $X$, we get the convergence :

$$L_n \stackrel{P}{\Rightarrow} W_F,$$

from which the convergence (37) follows in the usual way.

The convergence (39) follows immediately from (37) due to the condition (38):

$$\sup_{x \in [0,1]^l} |L_n(x) - G_n(x)| = \sqrt{k_n} \sup_{x \in [0,1]^l} |M_n(x) - F(x)| \to 0.$$

Let $l = 1$. Applying once more the continuous mapping theorem [3], we deduce from (37) an analog of the famous Kolmogorov’s result.

**Corollary 7.** Let $l = 1$, $X$ be an ergodic homogeneous random field and condition (38) be fulfilled. Then

$$P\{\sup_y |k_n^{1/2}(F_n(y) - F(y))| \leq x\} \to \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2x^2}, \ x > 0.$$

**Remark 7.** Theorem 12 allows one to analyze the asymptotic behavior of empirical processes associated with finite-dimensional distributions of the original field.

Let $X(t, \omega), t \in \mathbb{R}^m, X(t) \in \mathbb{R}^l$, be a random field over a probability space $(\Omega_X, \mathcal{F}_X, P_X).$ We fixe the points $t_1, ..., t_l$ and consider the distribution function

$$F(x) := P\{X(t_1) \leq x^1, ..., X(t_l) \leq x^l\}, \ x = (x^1, ..., x^l),$$

of the vector $(X(t_1), ..., X(t_l))$ and associated with it the $l$-dimensional randomized empirical distribution function $F_n,$

$$F_n(x) := \frac{1}{k_n} \sum_{i=1}^{k_n} \text{Ind}[X(t_1 + \tau_{n,i}) \leq x^1, ..., X(t_l + \tau_{n,i}) \leq x^l];$$
here, as before, \( \{\tau_{n,i}\} \) is a randomizing triangular array. Let

\[
M_n(x) = E_r(F_n(x)) = \frac{1}{\lambda(T_n)} \int_{T_n} \mathbb{1}_{[0,x]}(X(t_1 + s), ..., X(t_l + s)) \lambda(ds).
\]

Consider the empirical processes:

\[
L_n(x) = k_n^{1/2}(F_n(x) - M_n(x)), \quad x \in [0,1]^{l},
\]

\[
G_n(x) = k_n^{1/2}(F_n(x) - F(x)), \quad x \in [0,1]^{l}.
\]

We apply the Theorem 12 to the ergodic homogeneous random field \( X(t,\omega) = (X(t_1 + t), ..., X(t_l + t)) \) and come to the following statement.

**Theorem 13.** Let the random field \( X(t,\omega), \ t \in R^m, \) be homogeneous and ergodic. With \( P_{X,r} \)-probability 1

\[
L_n \Rightarrow \hat{W}_F. \tag{44}
\]

If additionally

\[
\sqrt{k_n} \sup_{x \in [0,1]^{l}} |M_n(x) - F(x)| \overset{P}{\longrightarrow} 0, \tag{45}
\]

then

\[
G_n \Rightarrow \hat{W}_F. \tag{46}
\]

**APPENDIX**

**A Comments on conditions (17) and (18)**

In what follows we put \( d = 1 \) for simplicity.

**Remark 8.** The proofs of Theorems 1 and 2 suggest that, in these theorems, it is reasonable to use sequences \( \{k_n\} \) growing to \( \infty \) rather fast, i.e., to chose the ”size of randomization” as large as possible. But, in Theorems 3 and 4 conditions (17) and (18) put some restrictions to this growth. These conditions are related to the speed of convergence in the Pointwise and Mean Ergodic Theorems, respectively. In the following notes we provide an idea of these conditions. We start with well known results related to condition (17) in the case \( T = \mathbb{Z}, \ T_n = \{1, ..., n\} \).
1. There are no universal sequences \( k_n \) satisfying condition (17): for each non-decreasing sequence \( k_n \to \infty \) there is a bounded ergodic stationary sequence \( \{X_n\} \) such that \( \{k_n\} \) does not satisfy this condition [41].

2. The sequence \( k_n = n^2 \) does not satisfy condition (17) [35].

3. For each ergodic stationary random field there is a non-decreasing sequence of integers \( k_n \to \infty \) that satisfies condition (17) [39].

4. Note 1 shows that the rate of convergence in the PET depends on the properties of the field \( X \); about this relation see [35, 38, 39] and references therein.

5. If \( \sup_n (n^\gamma Var_X(M_n)) < \infty \), where \( \gamma > 0 \), then (17) holds with \( k_n = n^\alpha \), where \( \alpha < \frac{\gamma}{2} \) ([14], Proposition 1). See Notes 6 and 9 - 11 below for examples.

Now we turn to condition (18).

6. By the Chebyshev inequality, condition (18) holds if \( k_n Var_X(M_n) \to 0 \). If the field \( X \) is wide-sense ergodic, then, by the Mean Ergodic Theorem, \( Var_X(M_n) \to 0 \) (see Remark 1); therefore, in this case (18) is always fulfilled if \( k_n \) grows sufficiently slow:

\[
k_n = o((Var_X(M_n))^{-1}). \tag{47}
\]

7. If \( \{X(k)\} \) is a mixing stationary random sequence, \( E_X[(X(0))^2] < \infty \) and \( T_n^l = \{1, ..., n\} \), then the sequence \( k_n = n^2 \) does not satisfy condition (47).

8. Let \( T = \mathbb{R}^m \) or \( \mathbb{Z}^m \), \( m \geq 1 \). Assume that the CLT holds in its classical form: \( (\lambda(T_n))^{1/2} \sigma^{-1}(M_n - \mu) \overset{D}{\to} Z \), where \( Z \) is the standard normal random variable. We have: \( k_n^{1/2}(M_n - \mu) = (k_n \sigma^{-2})^{1/2}(\lambda(T_n))^{1/2} \sigma^{-1}(M_n - \mu) \); therefore \( k_n^{1/2}(M_n - \mu) \overset{D}{\to} 0 \), if and only if \( k_n = o((\lambda(T_n))) \). A rather often case when the classical CLT fails is when \( (\lambda(T_n))^{1/2}(M_n - \mu) \overset{D}{\Rightarrow} 0 \), hence condition (18) is fulfilled with \( k_n = O((\lambda(T_n))) \).

9. Let \( \{\xi_k\} \) be an ergodic stationary Markov chain with the probability state space \( (\mathcal{X}, \mathcal{A}, m) \) (\( m \) the initial probability measure). Let \( Q(x, A) \) be the transition function, and let \( F \) be the (dense) set in \( L^2(\mathcal{X}, \mathcal{A}, m) \) consisting of all functions \( f : f(x) = g(x) - \int_{\mathcal{X}} g(y)Q(x, dy), g \in L^2(\mathcal{X}, \mathcal{A}, m) \). Then, for all \( f \in F \) the following alternative holds: the stationary random sequence \( X_k = f(\xi_k) \) either satisfies the CLT in the classical form or \( Var[M_n] = O(n^{-1}) \); this alternative holds also for \( f \) in a larger class of functions, which contains \( F \) (see [49]). According to the previous note, in both cases each sequence \( k_n = o(n) \) satisfies condition (18).
10. Let \( m \geq 1 \), \( X(t), t \in \mathbb{Z}^m \) be an wide-sense stationary random se-
quence on \( T \), satisfying Condition (A), \( E[X(0)] = \mu \). Let \( R(t) \) be its covari-
ance function.

Condition (18) holds if
- \( k_n = o(n^a) \) and \( |R(t)| = O(|t|^{-a}), \ 0 < a < m \), or
- \( k_n = o\left(\frac{m}{\log n}\right) \) and \( |R(t)| = O(|t|^{-m}) \), or
- \( k_n = o\left(\frac{n^a}{\log n}\right) \) and \( |R(t)| = O(|t|^{-a}) \), \( a > m \).

11. Let \( T = \mathbb{R}^m \) or \( \mathbb{Z}^m, m \geq 1 \), and let \( T_n = (0, n]^m, n \in \mathbb{N} \); if \( X \) is wide-
sense stationary and possesses a spectral density \( \psi \), which is continuous at
0, then condition (17) holds if \( k_n = o(n^m) \). This follows from the relation

\[
n^m \text{Var}_X \left( \int_{T_n} X(t) \lambda(dt) \right) = (2\pi)^m \psi(0) + o(1)
\]

(in the cases \( T = \mathbb{Z} \) and \( T = \mathbb{R} \) these are, respectively, Theorems 18.2.1 and
18.3.2 in [33]; if \( T = \mathbb{Z}^m, m \geq 2 \), this is Theorem 3 in [36]; the case \( T = \mathbb{R}^m \),
m \geq 2, can be treated similarly).

12. In the case \( m = 1 \) various restrictions on the spectrum and the cor-
relation function implying a given rate of convergence in the Mean Ergodic
Theorem, hence on restrictions on the choice of \( k_n \) in (18) are discussed in
[35, 37, 38, 40].

B Randomization using non-uniform
distributions

Let \( X(t) = (X^1(t), \ldots, X^d(t)), t \in T, \) be a \( \mathcal{B} \times \mathcal{F} \)-measurable random field
with homogeneous components \( (d \in \mathbb{N}) \) and \( E[|X(0)|^2] < \infty \) for some \( \delta > 0 \).
For each \( l = 1, \ldots, d, n \in \mathbb{N} \), consider a probability Borel measure \( \{q^l_n\} \) on \( T^l_n \),
which is the distribution of the \( T^l_n \)-valued i.i.d. randomizing random vectors
\( \tau^l_{n,i}, i = 1, \ldots, k_n \), introduced in Subsection 1.5. By the Fubini theorem, for
each measurable function \( f \) on \( \mathbb{R} \) such that \( E_P[f(X(0))] < \infty \) we have:

\[
\int_{T^l_n} |f(X(t))| q^l_n(dt) < \infty \text{ a.s., and}
\]

\[
E_r[f(X(\tau_{n,i}))] = \int_{T^l_n} f(X(t))q^l_n(dt), \ i = 1, \ldots, k_n.
\]
In particular,

\[ M_n^l := E_{\tau}[X(\tau_n^l, i)] = \int_{T_n^l} X(t)q_n^l(dt); \]

\[ V_n^l := Var_{\tau}[X(\tau_n^l, i)] = \int_{T_n^l} (X(t) - M_n^l)^2 q_n^l(dt). \]

Hence

\[ E_{\tau}\left(\sum_{i=1}^{k_n} f(X(\tau_n, i))\right) = k_n \int_{T_n^d} f(X(t))q_n^l(dt). \]

**Definition 2.** We say that the sequence of probability Borel measures \( \{q_n\} \) is pointwise averaging if for each homogeneous random field \( X \) the following PET with the “weights” \( q_n \) holds: if \( E_X[|f(X(0))|] < \infty \) then

\[ \lim_{n \to \infty} \int_{T_n^l} f(X(t))q_n(dt) = E_X[f(X(0))|I_X] \text{ with probability } 1; \]

in the case, when the measures \( q_n \) possess densities \( \varphi_n \) with respect to \( \lambda \), we say that the sequence \( \{\varphi_n\} \) is pointwise averaging.

We provide a simple lemma that helps to construct pointwise averaging sequences in \( \mathbb{R}^m \).

**Lemma 6.** Let \( \{\varphi_n\} \) be a sequence of densities on \( \mathbb{R}^m \) with compact supports \( S_n \) in \( T \); if there are positive numbers \( a_n \) such that the sets \( T_n^l := \{(x, y) : x \in S_n, 0 \leq y \leq a_n\varphi_n(x)\} \) form a pointwise averaging sequence in \( \mathbb{R}^{m+1} \), then \( \{\varphi_n\} \) is pointwise averaging.

**Proof.** Apply Definition 1 to the random field \( Y(t, s) \equiv X(t), t \in \mathbb{R}^m, s \in \mathbb{R}, \)
(note that \( \lambda(T_n^l) = a_n \), hence \( \frac{1}{\lambda(T_n^l)} \int_{T_n^l} Y(t, s)\lambda(dt, ds) = \int_{S_n} X(t)\varphi_n(t)dt; \)
here \( \lambda \) is the Lebesgue measure on \( \mathbb{R}^{m+1} \)).

The following two examples are implied by Lemma 6 and Examples 1 and 2 (see also Corollaries 4.2 and 4.3 in Ch. 6 in [64]).

**Example 3.** Let \( \{\varphi_n\} \) be a sequence of densities on \( \mathbb{R}^m \) which are concave on their supports \( S_n \) (the sets \( S_n \) are compact and convex). If the sequence \( \{S_n\} \) is increasing and the sets \( S_n \) contain balls of radii \( r_n \to \infty \), then the sequence \( \{\varphi_n\} \) is pointwise averaging; see §5.3 in [61].

**Example 4.** Let \( \{c_n\} \) be a sequence of positive numbers tending to \( \infty \). If \( \varphi \) is a bounded density on \( \mathbb{R}^m \) with a compact support \( S \) containing 0, then the sequence of rescaled densities \( \varphi_n(x) = c_n^{-m}\varphi(c_n^{-1}x) \) is pointwise averaging.
Remark 9. When \( \varphi(t) > 0 \) for all \( t \in \mathbb{R}^m \), some more conditions on \( \varphi \) are needed, but the class of "good" rescaled densities is still rather wide (see Proposition 5.3 in [61]); for example, if \( \varphi \) is the density of a nondegenerate symmetric normal distribution, then the sequence of rescaled densities is pointwise averaging.

For more examples, related to averages by convolutions of probability measures, see the next section.

It is easy to see that Theorems 2, 3 hold for all pointwise averaging sequences \( \{q_n^t\} \) (of course, in the proofs \( \int_{T_n^t} f(X(t))q^t_n(dt) \) has to be considered instead of \( \frac{1}{\lambda(T_n^t)} \int_{T_n^t} f(X(t))\lambda(dt) \)).

C Randomized CLT on general groups

In the above study the simplest groups \( \mathbb{R}^m \) and \( \mathbb{Z}^m \) were considered. But the results are valid for all groups \( T \) on which the PET holds with some sequence of sets \( \{T_n^t\} \) or "weights" \( \{q_n^t\} \). Then the proof of the generalized versions of Theorems 2 and 3 may be repeated almost word for word. There is a rather rich literature related to PETs on groups; see [44], [52], [64], [61] and the bibliographical survey therein; the PET with exponential rates of convergence on semisimple Lie groups is presented in [48]. In locally compact topological groups it is natural to consider pointwise averaging sequences of sets or densities with respect to the Haar measure. If the group is not locally compact, there is no Haar measure on it, and one has to consider pointwise averaging with sequences of probability Borel measures \( \{q_n\} \) as "weights" (see §B). If \( T \) is a second countable topological group and the smallest closed group containing the support of a probability Borel measure \( p \) is \( T \), then the sequence \( q_n := \frac{1}{n} \sum_{k=1}^n p^{*k} \) is pointwise averaging on \( T \) (see Corollary 5.3 in Ch. 3, Proposition 1.1 in Ch. 5 and Theorem 6.1 in Chapter 6 in [64]). In all these cases the analogs of our CLTs for homogeneous random fields with \( E[(X(0))^2] < \infty \) are valid.

If, in addition, \( p \) is symmetric (i.e. \( p(A^{-1}) = p(A) \) for each Borel set \( A \) in \( T \)) we may put \( q_n := p^{*n} \) (see Note 6.4 in Chapter 6 in [64] and the references therein); the PET with these weights is valid for a homogeneous field \( X \), if \( E_X[|X(0)|^{1+\delta}] < \infty \) for some \( \delta > 0 \); note that condition

\[
E_X[|X(0)|^{2+\delta}] < \infty \text{ for some } \delta > 0
\]
guarantees that the PET holds also for the random field \( (X(t))^2 \); this, in its turn, implies the analog of Lemma 1 and, hence, the analogs of the rest.
results. Therefore, under this condition, the CLTs hold with the "weights" $p^*n$, too.

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