A POISSON ANALOGUE OF NOETHER’S PROBLEM

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ABSTRACT. In this paper we show that the Poisson analogue of the Noether’s Problem has a positive solution for essentially all finite symplectic reflection groups — the analogue of complex reflection groups in the symplectic world. Our proofs are constructive, and generalize and refines previously known results. As an interesting consequence of the solution of this problem for complex reflection groups, we obtain the Poisson rationality of the Calogero-Moser spaces associated to any complex reflection group. The results of this paper can be thought as analogues of the Noncommutative Noether Problem (cf. [1], [19]) and the Gelfand-Kirillov Conjecture for rational Cherednik algebras (cf. [16]) in the ‘quasi-classical limit’. In the second half of the paper, an abstract framework to understand these results is introduced, and it is shown that every Coloumb branch of a $3d\mathcal{N} = 4$ gauge theory is Poisson rational as an application. We also obtain the Gelfand-Kirillov Conjecture for trigonometric Cherednik algebras and the Poisson rationality of their trigonometric Calogero-Moser spaces at the same time.

1. Introduction

We assume a base field $k$ with $char k = 0$. All rings in discussion are algebras over this field.

If $A$ is a Poisson domain and $S$ any multiplicatively closed subset of $A$, then the localization $A_{S}$ has natural structure of Poisson domain, uniquely extending the one from $A$:

$$\{as^{-1}, bt^{-1}\} = \{a, b\}s^{-1}t^{-1} - \{a, t\}bs^{-1}t^{-2} - \{s, b\}as^{-2}t^{-1} + \{s, t\}abs^{-2}t^{-2},$$

$a, b \in A, s, t \in S$.

In virtue of this simple fact, a question that has attracted a considerable amount of interest is the following: given two distinct Poisson domains $A, A'$, when are their field of fractions isomorphic as Poisson algebras, i.e., when they are Poisson birational? In case $X$ and $Y$ are two affine Poisson varieties, we say that they are Poisson birationally equivalent in case $O(X)$ is Poisson birational to $O(Y)$. One of the most significant Poisson domains in these questions is the following:

Definition 1.1. We consider the following Poisson algebra: $k[x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}]$, with Poisson brackets defined by $\{x_{i}, x_{j}\} = \{y_{i}, y_{j}\} = 0, \{x_{i}, y_{j}\} = \delta_{ij}, 1 \leq i, j \leq n$. Following [13], [21], we call it the Poisson-Weyl algebra, and denote it by $P_{n}$.

$P_{n}$ is, of course, the standard Poisson structure on the symmetric algebra of a non-degenerate symplectic vector space. We shall denote $K_{n}$ the Poisson field obtained from $P_{n}$ by localizing the set of all non-zero elements.

A version of the important Gelfand-Kirillov Conjecture ([20]), relating the field of fractions of envelopping algebras of finite dimensional algebraic Lie algebras with the field of fractions of the Weyl algebras, has been considered for the Poisson
algebra $S(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ with the Konstant-Kirillov bracket; namely, when its field of fractions is Poisson birational with the field of fractions of a Poisson-Weyl Algebra. It was shown to be true for nilpotent $\mathfrak{g}$ by Vergne ([36]), and later generalized to algebraic solvable Lie algebras by Tauvel and Yu ([34]), for algebraically closed fields. The question is also discussed in [27]. In a similar way, versions of the quantized Gelfand-Kirillov Conjecture ([7] I.2.11, II.10.4) for certain Poisson algebras have been considered by Goodearl and Launois ([21]), and Launois and Lecoutre ([28]).

In this paper, we are interested in a similar question, considered by Julie Baudry [3] and François Dumas [13], which is a Poisson analogue of the Noether’s Problem ([32]) about the rationality of invariants of the field of rational functions:

**Problem 1.** (Poisson Noether’s Problem) Let $G$ be a finite group of linear Poisson automorphisms of $P_n$, i.e., those Poisson automorphisms that fix the subspace $kx_1 \oplus \ldots \oplus kx_n \oplus ky_1 \oplus \ldots \oplus ky_n$ — and hence can be naturally seen as a subgroup of $SP_{2n}(k)$. When we have $K^G_n \simeq K_n$ as Poisson fields?

**Definition 1.2.** Whenever we have a Poisson isomorphism of a Poisson field with an certain $K_n$, we will call it Poisson rational. In case we have $X$ is an affine Poisson variety and $\operatorname{Frac}O(X)$ is Poisson rational, we also call $X$ Poisson rational.

A particularly important case of linear Poisson automorphisms arise in the following way (cf. [13]):

**Definition 1.3.** Let $G \subset GL_n(k)$ be a finite group of automorphisms acting an $n$ dimensional vector space $V$, and consider its diagonal action on $W = V^* \oplus V$, where $k[V^*] = k[x_1, \ldots, x_n]$ and $k[V] = k[y_1, \ldots, y_n]$. Then $G$ acts by Poisson automorphisms on $k[W] = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. In this case we call the action diagonally linear.

In [3], it was proved that Poisson Noether’s Problem has positive solution in case $k = \mathbb{C}$ and $G$ is a finite group of $SL_2(\mathbb{C})$ acting on $P_1$ and for diagonally linear action of the Weyl group $B_2$ on $P_2$. So, in particular, the Kleinian singularities are Poisson rational. In [13], for arbitrary $k$, a positive solution was found when $G$ acts diagonally linear on $P_n$ and arises from a representation that decomposes in a direct sum of one dimensional components — in particular, when $G$ is abelian and $k$ is algebraically closed. An important aspect of these results is an explicit exhibition of the isomorphism $K^G_n \simeq K_n$ as Poisson fields. We remark that in [3], a more general version of the Poisson Noether’s Problem was also considered, with results on ‘quasi-classical limits’ of the quantum torus ([3], [4]).

We remark that before the aforementioned works, Iain Gordon, in an unpublished manuscript [23], has already shown that, in fact, for any finite $\Gamma < SL_2(\mathbb{C})$, the Poisson Noether’s Problem has a positive solution for the natural action of the wreath product $\Gamma \wr S_n$ on $\mathbb{C}^{2n}$, using the work of [44] on symplectic resolutions of the quotient variety of this type. However, the argument does not allow immediately an explicit description of the Poisson rationality, and the idea of using symplectic resolutions of singularities cannot be applied in general ([6]).

In the first half of this paper, we generalize the work of Baudry and Gordon in finding an explicit description of the positive solution for the Poisson Noether’s Problem for (essentially) all non-exceptional indecomposable symplectic reflection groups (recalled in Section 2.1).

Our first main result is the following
Theorem 1.4. Let $k = \mathbb{C}$ and $G$ an irreducible complex reflection group or the symmetric group $S_n$ with a diagonally linear action on $P_n$. Then:

1. Poisson Noether’s Problem has a positive solution.
2. We can exhibit explicitly the isomorphism of Poisson fields $K_n^G \simeq K_n$.

With this result we show:

Corollary 1. Let $G$ be any complex reflection group. Then Poisson Noether’s Problem has a positive solution.

Using the result for the symmetric group and work of Baudry for finite subgroups of $SL_2(\mathbb{C})$, we can refine Gordon’s result and show:

Corollary 2. For $k = \mathbb{C}$ and $G = \Gamma \wr S_n$ a wreath product, we have an explicit Poisson isomorphisms $K_n^G \simeq K_n$.

Theorem 1.4 and Corollary 2 cover essentially all indecomposable symplectic reflection groups (cf. Section 2.1).

As a second main result, we show:

Theorem 1.5. The Calogero-Moser space associated to any finite complex reflection group is Poisson rational.

The second half of this paper builds on the fact that the cases of isomorphism between Poisson fields discussed above are the ‘quasi-classical limit’ of similar questions on the noncommutative Ore domains that quantize the relevant Poisson algebras. For instance, Poisson Noether’s Problem is the ‘quasi-classical limit’ of Noncommutative Noether’s Problem ([1]). In [19], Thm. 1.2, the author together with V. Futorny, have shown that, given an affine complex variety $X$ with the action of a finite group $G$ such that $X/G$ is birationally equivalent to an affine variety $Y$, then $\text{Frac} D(X)^G \simeq \text{Frac} D(Y)$. In Theorem 4.4 we prove a version of this result in the quasi-classical limit, which provides an abstract understanding of our approach in proving Theorem 1.4. Following these ideas, as applications we prove our third and fourth main results:

Theorem 1.6. Every Coulomb branch of a $3d N = g$ gauge theory, with symplectic representation of cotangent type, is Poisson rational.

Theorem 1.7. Gelfand-Kirillov Conjecture holds for all trigonometric Cherednik algebras, and the trigonometric Calogero-Moser spaces are Poisson rational.

The structure of the paper is as follows. In Section 2 we discuss the relevant preliminaries on symplectic reflection groups, Cherednik algebras, Calogero-Moser spaces and Coulomb branches. In Section 3.1 we prove Theorem 1.4 for irreducible complex reflection groups, following the idea in [19] in the ‘quasi-classical limit’. The part (2) of this Theorem is Proposition 3.6. In Section 3.2, as applications, we prove Corollaries 1 and 2 and Theorem 1.5. In Section 4.1 we discuss the connection between birational equivalence and Poisson birational equivalence of cotangent bundles, which provides an abstract understanding of the proof of Theorem 1.4 — cf. Theorem 4.4. In Section 4.2, as applications, we prove Theorem 1.6 and Theorem 1.7 (in a slightly more general form).
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2. Preliminaries

Through the rest of the paper, the base field is the complex numbers.

2.1. Symplectic reflection groups and generalized Calogero-Moser spaces.

Let recall the important notion of symplectic reflection groups, which are analogues for vector spaces with a non-degenerate symplectic form of a complex reflection group.

Definition 2.1. Let $V$ be a complex vector space of dimension $2n$, with a non-degenerate skew-symmetric form $\omega$. Let $\Gamma$ be a finite subgroup of $\text{SP}_{2n}(\mathbb{C})$ generated by symplectic reflections: that is, elements $g \in \Gamma$ such that $1-g$ has rank two. Then $\Gamma$ is called a symplectic reflection group.

We call the data above a symplectic triple and denote it by $(V,\omega,\Gamma)$.

Their importance lies, among other things, in the study of symplectic singularities and symplectic reflection algebras; the later were introduced by Etingof and Ginzburg in [16], and the former by Beauville in [5]. It was shown in [35] that a quotient of a complex symplectic vector space $V$ by a finite subgroup $\Gamma$ of $\text{SP}_{2n}(\mathbb{C})$ has a symplectic resolution if and only if $\Gamma$ is a symplectic reflection group.

We now discuss our two main sources of symplectic reflection groups.

(1) Let $W$ be a finite complex reflection group acting on a vector space $h$ and in its dual $h^*$ in contragradient manner. Then if calling $V = h \oplus h^*$ we define a non-degenerate symplectic form:

$$\omega((y,f),(u,g)) = g(y) - f(u), y,u \in h, f,g \in h^*,$$

then we have that the diagonal action of $W$ turns it into a symplectic reflection group.

(2) The natural action of the wreath product $\Gamma \wr S_n$ on $\mathbb{C}^{2n}$, $\Gamma < S\text{L}_2(\mathbb{C})$ a finite group.

Like the case of complex reflection groups, we have a decomposition of a symplectic triple into indecomposable ones, i.e., triples $(V,\omega,\Gamma)$ such that $V$ cannot be expressed as a direct sum of two non-trivial $\Gamma$-stable subspaces $V_1,\ldots,V_2$ with $\omega(V_1,V_2) = 0$. They have been classified ([22]), and the bulk of the classification theorem says that, apart from certain exceptional cases, all indecomposable triples are of the form (1) or (2) above.

Now we remember the notion of Calogero-Moser spaces.

They were first considered by Kazhdan, Konstant and Sternberg ([26]) and studied by Wilson ([28]) and many others (cf. [14]), and generalized for any complex reflection group (the original case being $S_n$) in [16]. To introduce it we recall some basic notions of rational Cherednik algebras ([16]). They are a double step degeneration of the double affine Hecke algebras considered by Cherednik ([11]). The first step is the so called trigonometric Cherednik algebra, discussed below.
Let $W$ be a finite complex reflection group acting on a complex vector space $h$. Let $S$ be the set of reflections and, for each $s \in S$ take $\alpha_s \in h^*$ and $\alpha_s^\vee \in h$ such that $\alpha_s$ is an eigenvector of $\lambda_s$ (the non-trivial eigenvalue of $s$ in $h^*$); and $\alpha_s^\vee$ is an eigenvector of $\lambda_s^{-1}$ (the non-trivial eigenvalue of $s$ in $h$). Normalize them such that in the natural pairing $h^* \times h \to \mathbb{C}$ we have $(\alpha_s, \alpha_s^\vee) = 2$. Finally, let $c : S \to \mathbb{C}$ a conjugation invariant function.

Consider the algebra $H_{c,t}(W,h)$, $t \in \mathbb{C}$, with the following generators and relations: the quotient of $\mathbb{C}W \ltimes T(h \oplus h^*)$ by the relations:

$$[x, x'] = [y, y'] = 0; [y, x] = tx(y) - \sum_{s \in S} c_s(y, \alpha_s)(x, \alpha_s^\vee)s,$$

with $x, x' \in h^*, y, y' \in h$.

**Definition 2.2.** $H_{c,t}(W,h)$ is called the Rational Cherednik algebra.

In case $t \neq 0$, the center of this algebra is just $\mathbb{C}$; but in case $t = 0$, the center — which we will denote by $Z_c(W,h)$ — is quite big. In fact, the rational Cherednik algebra is a finite module over it (16).

We have

**Proposition 2.3.** (16) $Z_c(W,h)$ is a finitely generated algebra without zero divisors. It is also a Poisson algebra.

**Definition 2.4.** The Calogero-Moser space, denoted $\mathcal{M}_c(W,h)$, is $\text{Spec} Z_c(W,h)$

2.2. **Trigonometric Cherednik algebras.** As we discussed above, the trigonometric Cherednik algebra is also a degeneration of the double affine Hecke algebra. We will introduce them following the approach in [15]. It can be shown that it coincides with the usual definition in [16] (cf. [17], Example 7.10).

Let $X$ be an affine complex variety and $G$ a finite group of automorphisms of it. If $X^g$ is the set of fixed points of $X$ under the action of $g \in G$, the components $Y$ of $X^g$ of codimension 1 in $X$ are called reflection hypersurfaces.

We have a canonical surjection of $\mathcal{O}_X$-modules, for each such $Y$, $(\xi_Y : TX \to \mathcal{O}_X(Y)/\mathcal{O}_X$ (cf. [15], Section 2.4), where $\mathcal{O}_X(Y)$ is the sheaf of regular functions on $X \setminus Y$ with pole of order at most one in $Y$. Fix $t \in \mathbb{C}$ and $c$ a $G$-invariant complex valued function on the set $S = \{(g, Y)\}, g \in G, Y \subset X^g$ a reflection hypersurface. We have the following generalized version of the Dunkl-Opdam operators for such pairs $X$ and $G$:

$$D = tL_v + \sum_{(g, Y) \in S} \frac{2c(g, Y)}{1 - \lambda_{Y, g}} f_Y(x)(1 - g); D \in D(X)_v[c] \rtimes G,$$

where $D(X)_v$ is the ring of differential operators on $X$ with rational coefficients; $\lambda_{g,Y}$ is the eigenvalue of $g$ in the conormal bundle of $Y$; $L_v$ is the Lie derivative, $v$ a vector field on $X$; $f_Y \in \Gamma(X, \mathcal{O}_X(Y))$ an element in the coset of $\xi_Y(v)$ in the map considered above $(\xi)$. When $t = 0$, $tL_v$ does not vanish, but turns into the classical momentum (cf. [15], Definition 2.12).

**Definition 2.5.** $H_{t,c}(X, G)$ is the subalgebra of $D_c(X)$ generated by $\mathcal{O}(X), G$ and the Dunkl-Opdam operators. It is called the Cherednik algebra of $X$ and $G$. When $W$ is a Weyl group and $H$ the corresponding torus, we have the trigonometric Cherednik algebra, denoted by $\mathbb{H}_{t,c}(W)$. 

Lemma 3.1. gives a criterion of recognition of Poisson rationality.

Proof of Theorem 1.4. varieties of dimension \( r_k G \) locus. 

H (cf. [9]). In its equivariant Borel-Moore homology 

Theorem 2.9. M Coulomb branch is the scheme 

can introduced a convolution product, which turns out to be commutative, and the 

is a purely transcendental extension generated by algebraically independent elements 

Now we recall the following well known facts:
Theorem 3.2. Let $X$ be an affine smooth algebraic variety and $G$ a finite group that acts freely on it. Then the injection $\mathcal{O}(X)^G \to \mathcal{O}(X)$ induces, by restriction of differential operators, a filtration preserving isomorphism $\theta : D(X)^G \to D(X/G)$.

Proof. Theorem 3.7 (1) in [10].

Theorem 3.3. Let $X$ be an affine smooth algebraic variety, and $G$ any finite group of automorphisms of it. It acts in a natural way by Poisson automorphisms on $\mathcal{O}(T^*X)$. Then, with the filtration by order of differential operators, $D(X)$ and $D(X)^G$ are filtered quantizations of $\mathcal{O}(T^*X)$ and $(\mathcal{O}(T^*X))^G$, respectively. In both cases the Poisson bracket has degree $-1$.

Proof. From this follow the following general result, which is of independent interest — a quasi-classical analogue of Theorem 3.2.

Theorem 3.4. Let $X$ be an affine smooth algebraic variety, and $G$ a finite group that acts freely on it. Then we have an isomorphism of Poisson varieties $\phi : T^*(X)/G \to T^*(X/G)$.

Proof. Immediate consequence of Theorems 3.2 and 3.3.

With this preparation, we can begin our proof of Theorem 1.4. The idea of the proof is similar to the one of Noncommutative Noether’s Problem for pseudo-reflection groups ([19], Section 5.2), in the ‘quasi-classical limit’.

Let $G$ be an irreducible complex reflection group or $S_n$. Consider the invariants $\mathbb{C}[y_1, \ldots, y_n]^G \simeq \mathbb{C}[e_1, \ldots, e_n]$ (by Chevalley-Shephard-Todd Theorem). We introduce the $n \times n$ matrix $M$ with entries $ij$ being $\partial_{y_i} e_j$, $1 \leq i, j \leq n$ and consider its determinant $J(M)$. Define $\sigma(M) = J(M)^G$ in case $G$ is an irreducible complex reflection group, and $\sigma(M) = J(M)^2$ in case it is the symmetric group. $\sigma(M)$ is non-null and $G$-invariant polynomial on the $y_1, \ldots, y_n$ ([25], 20-2, Props. A, B and 21-1, Props. A, B). Write also, in any case, $\sigma(M)'$ as the expression of $\sigma(M)$ as a polynomial in $\mathbb{C}[e_1, \ldots, e_n]$.

We are interested now in the solutions of the systems

\[(*) ML_i = Y_i,
\]

where $Y_i$, $i = 1, \ldots, n$, is the column vector with 1 in the $i$th position and 0 in all others. Let, for each $i$,

\[L_i = \begin{pmatrix} l_{i1} \\ \vdots \\ l_{in} \end{pmatrix}
\]

be a solution of the linear system $(*)$. By the Kramer’s rule, $l_{ij} \in \mathbb{C}[y_1, \ldots, y_n]_{J(M)} = \mathbb{C}[y_1, \ldots, y_n]_{\sigma(M)}$, $1 \leq i, j \leq n$.

In any case, consider the algebra $P_{\sigma(M)}$ with its natural Poisson structure, as well its invariant subalgebra $P_{\sigma(M)}^G$. We are going to interpret them as regular functions on a certain contangent bundle and its invariants. Introduce $h = \text{Spec} \mathbb{C}[y_1, \ldots, y_n]$; $h_v = \text{Spec} \mathbb{C}[y_1, \ldots, y_n]_{\sigma(M)}$. $P_{\sigma(M)}$ is $\mathbb{C}[h_v \times h^*] \simeq \mathcal{O}(T^* h_v)$.

$G$ is a finite group of automorphisms that acts freely on $h_v$, since localizing by $\sigma(M)$ remove the reflecting hyperplanes ([25], 20-2, Props. A, B and 21-1, Props.
A, B). The induced action of $G$ on $\mathcal{O}(T^* (h_r))$ is exactly the $G$ action on $P_{n\sigma(M)}$. By Theorem 3.4 we have an isomorphism of Poisson varieties

\[(\tilde{\theta}) : T^* (h_r)/G \to T^* (h_r)/G) = T^* (h_r),\]

$h_r/G = h_{r'}$, where $h_{r'} = \text{Spec} \mathbb{C} [e_1, \ldots, e_{n\sigma(M)}]$. In the level of rings of regular functions, we have by $\theta$ that $\mathcal{O}(T^* (h_r))^{G} \simeq P_{n\sigma(M)}^{G}$ is isomorphic as a Poisson algebra to a localization of $P_n$ — namely $P_{n\sigma(M)}^{G} \simeq \mathcal{O}(T^* (h_r))$. So, in view of Lemma 3.1, Theorem 1.4 (1) holds.

We can find an explicit isomorphism of Poisson algebras using the results in [18] and [19], where we obtained an explicit isomorphism $D(h_r)^{G} \simeq D(h_{r'})$, and the fact that this isomorphism is a quantization of $\mathcal{O}(T^* (h_r))^{G} \simeq \mathcal{O}(T^* (h_{r'}))$ - cf. Theorem 3.3. Namely, using the standard generators $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ for the $n$-th Weyl algebra, and noticing that $D(h_{r'})$ is just its localization by $\Delta'$, we have

**Proposition 3.5.** When $G$ is either the symmetric group or an irreducible complex reflection group, we have that the following map defines an isomorphism between $D(h_r)^{G}$ and $D(h_{r'})$:

\[e_1 \mapsto x_1; i = 1, \ldots, n.\]

**Proof.** By [18] Lemma 1 and [19] Proposition 5.7. \(\square\)

Hence, if we set for $i = 1, \ldots, n$ $y'_i = e_i, x'_i = l_1 x_1 + \ldots + l_n x_n$, then we have an explicit description of the isomorphism $\theta$ above (\(\tilde{\theta}\)):

**Proposition 3.6.**

\[\theta : P_{n\sigma(M)}^{G} \to P_{n\sigma(M')}^{G},\]

\[y'_i \mapsto y_i, x'_i \mapsto x_i, i = 1, \ldots, n.\]

The extension of this map to the field of fractions gives us an explicit isomorphism $\theta : K_n^{G} \to K_n$ (cf. lemma 3.3), and so Theorem 1.4 (2) is proved.

**Remark 3.7.** Localization is necessary. It is always false that $P_{n}^{G}$ is isomorphic to $P_{n}$ as a Poisson algebra, for any finite group acting linearly diagonally ([2], Thm. 4).

**Remark 3.8.** If we were only interested in showing that $P_{n}^{G}$ is rational (instead of Poisson rational), the proof would follow immediately from the Chevalley-Shephard-Todd Theorem and [30], Remark 3.

### 3.2. Applications of Poisson Noether’s Problem.

We recall the following well known result (cf. [19], Prop. 5.10):

**Proposition 3.9.** Let $W$ be a finite complex reflection group acting on a vector space $V$. Then $W = W_1 \times \ldots \times W_m$, $V = V_1 \oplus \ldots \oplus V_m \oplus V'$, where each $W_i$ acts as an irreducible complex reflection group on $V_i$, and trivially on $V_j, j \neq i; \text{ and } V'$ is fixed by the whole $W$.

Let us now prove Corollary 1.

Let $W$ be an arbitrary finite complex reflection group acting diagonally linear on some $P_n$. By the Proposition above $P_n^W \simeq P_{n_1}^{W_1} \otimes \ldots \otimes P_{n_m}^{W_m} \otimes P_k, n_1 + \ldots + n_m + k = n$ ($P_k$ corresponds to the part fixed by the whole $W$). An application of Theorem 1.4 gives our result.
The proof of Corollary 2 is similar. Let $\Gamma < SL_2(\mathbb{C})$. Let $u, v \in K^T$ be two elements such that the map $K^T \to K_1$, $u \mapsto x, v \mapsto y$, is an isomorphism of Poisson fields (cf. [3]). In the wreath product case $G = \Gamma \wr S_n$ action in $P_n$, consider the elements $u_i, v_i$, $i = 1, \ldots, n$ — a copy of each $u, v$ in the $n$-factors of the action. Then, repeating the procedure (with $u_i, v_i$ instead of $x, y$, respectively) in the proof of Theorem 1.4 (2), cf. Proposition 3.3 we obtain our result.

Example 3.10. Here we have an example showing how combining Baudry’s results and Theorem 1.4 (2) we can explicitly exhibit the isomorphisms. Let $D_n$ be the binary dihedral group of order $4n$ acting on $\mathbb{C}(x_1, y_1)$ with $\{x_1, y_1\} = 1$. Then $\mathbb{C}(x_1, y_1)^{D_n} \simeq \mathbb{C}(u_1, v_1)$, with:

$$u_1 = 1/8n((x_1^{-1}y_1)^{-n} - (x_1^{-1}y_1)^n)((x_1^{-1}y_1)^n - 1)^2 x_1 y_1; v_1 = ((x_1^{-1}y_1)^n + 1)/(x_1^{-1}y_1)^n - 1)^2,$$

and $\{u_1, v_1\} = 1$ ([3]).

Now let’s consider the action of the group $G = \mathbb{D}_n \wr S_3$ on $\mathbb{C}(x_1, y_1, x_2, y_2, x_3, y_3) = K_3$. We have an isomorphism $\psi : K^{D_n}_3 \to \mathbb{C}(U_1, V_1, U_2, V_2, U_3, V_3)$. $J(M)$ is $(v_1 - v_2)(v_2 - v_3)(v_3 - v_2)$. Now we define $V_1 = v_1 + v_2 + v_3; V_2 = v_1 v_2 + v_2 v_3 + v_3 v_1; V_3 = v_1 v_2 v_3$; and

$$U_1 = \frac{v_1^2(v_2 - v_3)}{J} u_1 + \frac{v_2^2(v_3 - v_1)}{J} u_2 + \frac{v_3^2(v_1 - v_2)}{J} u_3;$$

$$U_2 = \frac{v_1(v_3 - v_2)}{J} u_1 + \frac{v_2(v_3 - v_1)}{J} u_2 + \frac{v_3(v_2 - v_1)}{J} u_3;$$

$$U_3 = \frac{(v_2 - v_3)}{J} u_1 + \frac{(v_3 - v_1)}{J} u_2 + \frac{(v_1 - v_2)}{J} u_3.$$

We have $\{V_i, V_j\} = \{U_i, U_j\} = 0, \{U_i, V_j\} = \delta_{ij}, i, j = 1, 2, 3$, and the explicit isomorphism $\psi : K^{D_n}_3 \to \mathbb{C}(U_1, V_1, U_2, V_2, U_3, V_3)$ is $u_i \mapsto U_i, v_i \mapsto V_i, i = 1, 2, 3$ (cf. [19], Ex. 5.12).

Finally, considering Calogero-Moser spaces, we need the following result from [10].

Proposition 3.11. $M_c(W, h)$ is birationally equivalent to $h \oplus h^* / W$ as a Poisson variety.

Proof. [10], Proposition 17.7*

Combining this result with Theorem 1.4 we immediately obtain Theorem 1.5.

We finish this section with a Conjecture:

Conjecture 1. For every finite symplectic reflection group $\Gamma$, Poisson Noether’s Problem for the Poisson-Weyl algebra holds.

4. Birational Equivalence and Poisson Birational Equivalence

In this section we show that birational equivalence of quotient varieties implies Poisson birational equivalence of the quotient for the induced action of the group on the cotangent bundles.

Proposition 4.1. Let $X, Y$ be two affine smooth varieties, birationally equivalent. Then $O(T^*X)$ and $O(T^*Y)$ have isomorphic Poisson field of fractions.
Proof. Since \( X \) and \( Y \) are birationally equivalent, there exists \( U \subset X, V \subset Y \) open subsets that are isomorphic as varieties. We can suppose both of them affine: \( U = \text{Spec} A, V = \text{Spec} B \), \( \mathcal{O}(X) \subset A, \mathcal{O}(Y) \subset B \). Then we have \( T^*U \cong T^*V \) as Poisson varieties. Since we have \( \mathcal{O}(T^*X) \subset \mathcal{O}(T^*U); \mathcal{O}(T^*Y) \subset \mathcal{O}(T^*V) \) as Poisson subalgebras with the same field of fractions, we are done. \( \square \)

In case of smooth curves, we have a converse:

Proposition 4.2. Let \( X, Y \) be two affine smooth curves, such that \( \mathcal{O}(T^*X) \) and \( \mathcal{O}(T^*Y) \) have isomorphic Poisson field of fractions. Then they are birationally equivalent.

Proof. Let \( Z \) be an affine smooth variety of dimension \( n \). Since the sheaf of Kahler differentials is locally free, the cotangent bundle is locally trivial — namely, there is a dense affine open set \( U \subset Z \) such that \( T^*(U) = \mathcal{O}(U)[\xi_1, \ldots, \xi_n] \); and \( \text{Frac}\mathcal{O}(Z) = \text{Frac}\mathcal{O}(U) \). So, if \( X \) and \( Y \) are two curves whose cotangent bundles are birationally equivalent, then \( \text{Frac}\mathcal{O}(X)(\xi) \cong \text{Frac}\mathcal{O}(Y)(\xi') \). So the two curves are stably birational equivalent (cf. [14]), and hence birationally equivalent ([24], V, Exercise 2.1). \( \square \)

It seems an interesting question to understand whether the converse of Proposition 4.1 holds for varieties of higher dimension.

Lemma 4.3. Let \( X \) be an affine smooth variety, \( 0 \neq f \in \mathcal{O}(X), X_f = \text{Spec} \mathcal{O}(X)_f \). \( \mathcal{O}(T^*X)_f \cong \mathcal{O}(T^*X_f) \).

Proof. Let \( \Theta_X \) be the Lie algebra of vector fields on \( X \). It is an \( \mathcal{O}(X) \)-module. It is well known that \( \Theta_X \) is isomorphic to \( \mathcal{O}(X)_f \otimes_{\mathcal{O}(X)} \Theta_{X_f} \) as an \( \mathcal{O}(X)_f \)-module ([24], 15.1.24). Since, for an affine smooth variety \( Y \), \( \mathcal{O}(T^*Y) = \text{Sym}_{\mathcal{O}(Y)}\Theta_Y \), we are done. \( \square \)

Theorem 4.4. Let \( G \) be a finite group acting on an affine smooth variety \( X \), such that \( X/G \) is birationally equivalent to an affine smooth variety \( Y \). Then \( \mathcal{O}(T^*X)_G \) and \( \mathcal{O}(T^*Y) \) have isomorphic Poisson field of fractions.

Proof. There exists a principal open subset of \( X, X_f \), where the action of \( G \) is free (cf. lemma 4.4 [19]). By Lemma 4.3 we have \( (\mathcal{O}(T^*X)_f)_G = (\mathcal{O}(T^*X)_f)^G \cong \mathcal{O}(T^*X_f)_G \). By Theorem 5.3 this is isomorphic to \( \mathcal{O}(T^*X_f/G) \). By Proposition 4.4 the Poisson field of fractions of \( \mathcal{O}(T^*X_f/G) \) is the same as the one of \( \mathcal{O}(T^*Y) \).

First we consider Coulomb branches.

Theorem 4.5. Let \( H \) be a maximal torus of \( G \). \( \mathcal{M}_C(G, M) \) is birationally equivalent to \( T^*(H^*/W) \) as a Poisson variety.

Proof. [9], Corol. 5.21. \( \square \)
This already implies that the birational equivalence class of the Coulom branch depends only on $G$. The quotient of $H^\vee$ by $W$, since the Weyl group acts by automorphisms of algebraic groups on $H^\vee$, is again an affine connected algebraic group; a connected linear algebraic group. Connected linear algebraic groups over the complex numbers are all rational, as is well known. Hence $H^\vee/W$ is a rational variety. Hence, by Proposition 4.1 we have:

For every Coulomb branch, if $A = O(M_C(G,M))$, then the Poisson field of fractions of $A$ is isomorphic to $K_n = C(x_1, \ldots, x_s, y_1, \ldots, y_s)$, where $s = rk H^\vee$. In other words, the Coulomb branch is Poisson rational.

This proves Theorem 1.6.

Now to the proof of Theorem 1.7. In fact, we are going to show a slightly more general result, about the Cherednik algebra of $X$ an affine variety and $G$ a finite group acting on it (cf. Definition 2.5).

**Theorem 4.6.** \cite{15} $U_{t,c}(X,G)$, when $t \neq 0$, is an Ore domain whose quotient ring of fractions is isomorphic to $D(X)^G$. $\mathcal{M}(X,G)$ is Poisson birationally equivalent to $(T^*X)/G$.

**Proof.** \cite{15}, Proposition 2.15 and Theorem 2.29. \qed

**Theorem 4.7.** In the notation of the above theorem, if $X/G$ is a rational variety, then $(t \neq 0) \text{Frac} U_{t,c}(X,G)$ is isomorphic to quotient ring of fractions of $A_n(C)$, the Weyl algebra, $n = \dim X$. $O(T^*X)^G$ has Poisson field of fractions isomorphic to $K_n$ — so $\mathcal{M}(X,G)$ is Poisson rational.

**Proof.** The first claim follows from \cite{19}, Thm 1.2. The second one follows from Theorem 4.6. \qed

Theorem 1.7 follows immediately, noting that given a Weyl group $W$ with maximal torus $H$, $H/W$ is a rational variety (cf. proof of Theorem 1.6).

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