Defensive Alliances in Graphs

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Abstract. A set $S$ of vertices of a graph is a defensive alliance if, for each element of $S$, the majority of its neighbours are in $S$. We study the parameterized complexity of the Defensive Alliance problem, where the aim is to find a minimum size defensive alliance. Our main results are the following: (1) The Defensive Alliance problem has been studied extensively during the last twenty years, but the question whether it is FPT when parameterized by feedback vertex set has still remained open. We prove that the problem is W[1]-hard parameterized by a wide range of fairly restrictive structural parameters such as the feedback vertex set number, treewidth, pathwidth, and treedepth of the input graph; (2) the problem parameterized by the vertex cover number of the input graph does not admit a polynomial compression unless coNP $\subseteq$ NP/poly, (3) it does not admit $2^{o(n)}$ algorithm under ETH, and (4) the Defensive Alliance problem on circle graphs is NP-complete.

Keywords: Defensive alliance · Parameterized Complexity · FPT · W[1]-hard · treedepth · feedback vertex set · ETH · circle graph

1 Introduction

In real life, an alliance is a collection of people, groups, or states such that the union is stronger than individual. The alliance can be either to achieve some common purpose, to protect against attack, or to assert collective will against others. This motivates the definitions of defensive and offensive alliances in graphs. The properties of alliances in graphs were first studied by Kristiansen, Hedetniemi, and Hedetniemi [19]. They introduced defensive, offensive and powerful alliances. An alliance is global if it is a dominating set. The alliance problems have been studied extensively during last fifteen years [10, 21, 23, 22, 25], and generalizations called $r$-alliances are also studied [23]. Throughout this article, $G = (V, E)$ denotes a finite, simple and undirected graph of order $|V| = n$.

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The subgraph induced by \( S \subseteq V(G) \) is denoted by \( G[S] \). For a vertex \( v \in V \), we use \( N_G(v) = \{ u : (u,v) \in E(G) \} \) to denote the (open) neighbourhood of vertex \( v \) in \( G \), and \( N_G[v] = N_G(v) \cup \{ v \} \) to denote the closed neighbourhood of \( v \). The degree \( d_G(v) \) of a vertex \( v \in V(G) \) is \( |N_G(v)| \). For a subset \( S \subseteq V \), we define its closed neighbourhood as \( N_G[S] = \bigcup_{v \in S} N_G[v] \) and its open neighbourhood as \( N_G(S) = N_G[S] \setminus S \). For a non-empty subset \( S \subseteq V \) and a vertex \( v \in V(G) \), \( N_S(v) \) denotes the set of neighbours of \( v \) in \( S \), that is, \( N_S(v) = \{ u \in S : (u,v) \in E(G) \} \). We use \( d_S(v) = |N_S(v)| \) to denote the degree of vertex \( v \) in \( G[S] \). The complement of the vertex set \( S \) in \( V \) is denoted by \( S^c \).

**Definition 1.** A non-empty set \( S \subseteq V \) is a defensive alliance in \( G = (V,E) \) if \( d_S(v) + 1 \geq d_{S^c}(v) \) for all \( v \in S \).

A vertex \( v \in S \) is said to be protected if \( d_S(v) + 1 \geq d_{S^c}(v) \). A set \( S \subseteq V \) is a defensive alliance if every vertex in \( S \) is protected. In this paper, we consider Defensive Alliance under structural parameters. We define the problem as follows:

**Defensive Alliance**

**Input:** An undirected graph \( G = (V,E) \) and an integer \( k \geq 1 \).

**Question:** Is there a defensive alliance \( S \subseteq V(G) \) such that \( |S| \leq k \)?

For standard notations and definitions in graph theory and parameterized complexity, we refer to West [26] and Cygan et al. [4], respectively. The graph parameters we explicitly use in this paper are feedback vertex set number, pathwidth, treewidth and treedepth.

**Definition 2.** For a graph \( G = (V,E) \), the parameter feedback vertex set is the cardinality of the smallest set \( S \subseteq V(G) \) such that the graph \( G - S \) is a forest and it is denoted by \( fvs(G) \).

We now review the concept of a tree decomposition, introduced by Robertson and Seymour in [21]. Treewidth is a measure of how “tree-like” the graph is.

**Definition 3.** [5] A tree decomposition of a graph \( G = (V,E) \) is a tree \( T \) together with a collection of subsets \( X_t \) (called bags) of \( V \) labeled by the vertices \( t \) of \( T \) such that \( \bigcup_{t \in T} X_t = V \) and (1) and (2) below hold:

1. For every edge \( (u,v) \in E(G) \), there is some \( t \) such that \( \{u,v\} \subseteq X_t \).
2. (Interpolation Property) If \( t \) is a vertex on the unique path in \( T \) from \( t_1 \) to \( t_2 \), then \( X_{t_1} \cap X_{t_2} \subseteq X_t \).

**Definition 4.** [5] The width of a tree decomposition is the maximum value of \( |X_t| - 1 \) taken over all the vertices \( t \) of the tree \( T \) of the decomposition. The treewidth \( tw(G) \) of a graph \( G \) is the minimum width among all possible tree decomposition of \( G \).
**Definition 5.** If the tree $T$ of a tree decomposition is a path, then we say that the tree decomposition is a *path decomposition*, and use *pathwidth* in place of *treewidth*.

A rooted forest is a disjoint union of rooted trees. Given a rooted forest $F$, its *transitive closure* is a graph $H$ in which $V(H)$ contains all the nodes of the rooted forest, and $E(H)$ contain an edge between two vertices only if those two vertices form an ancestor-descendant pair in the forest $F$.

**Definition 6.** The *treedepth* of a graph $G$ is the minimum height of a rooted forest $F$ whose transitive closure contains the graph $G$. It is denoted by $td(G)$.

### 1.1 Our Main Results

Our main results are as follows:

– the **Defensive Alliance** is $W[1]$-hard when parameterized by the vertex deletion set into trees of height at most two, even when restricted to bipartite graphs.

– the **Defensive Alliance** problem parameterized by the vertex cover number of the input graph does not admit a polynomial compression unless $\text{coNP} \subseteq \text{NP/poly}$.

– the **Defensive Alliance** problem does not admit $2^{o(n)}$ algorithm under ETH.

– the **Defensive Alliance** problem on circle graphs is NP-complete.

### 1.2 Known Results

The decision version for several types of alliances have been shown to be NP-complete. For an integer $r$, a nonempty set $S \subseteq V(G)$ is a *defensive $r$-alliance* if for each $v \in S$, $|N(v) \cap S| \geq |N(v) \setminus S| + r$. A set is a defensive alliance if it is a defensive $(-1)$-alliance. A defensive $r$-alliance $S$ is *global* if $S$ is a dominating set. The defensive $r$-alliance problem is NP-complete for any $r \geq 2$. The defensive alliance problem is NP-complete even when restricted to split, chordal and bipartite graph [14]. For an integer $r$, a nonempty set $S \subseteq V(G)$ is an *offensive $r$-alliance* if for each $v \in N(S)$, $|N(v) \cap S| \geq |N(v) \setminus S| + r$. An offensive 1-alliance is called an offensive alliance. An offensive $r$-alliance $S$ is *global* if $S$ is a dominating set. Fernau et al. showed that the offensive $r$-alliance and global offensive $r$-alliance problems are NP-complete for any fixed $r$ [8]. They also proved that for $r > 1$, $r$-offensive alliance is NP-hard, even when restricted to $r$-regular planar graphs. There are polynomial time algorithms for finding minimum alliances in trees [21,3]. A polynomial time algorithm for finding minimum defensive alliance in series parallel graph is presented in [13]. Fernau and Raible showed in [7] that the defensive, offensive and powerful alliance problems and their global variants are fixed parameter tractable when parameterized by solution size $k$. Kiyomi and Otachi showed in [17], the problems of finding smallest alliances of...
all kinds are fixed-parameter tractable when parameterized by the vertex cover number. The problems of finding smallest defensive and offensive alliances are also fixed-parameter tractable when parameterized by the neighbourhood diversity [11]. Enciso [6] proved that finding defensive and global defensive alliances is fixed parameter tractable when parameterized by domino treewidth. Bliem and Woltran [1] proved that deciding if a graph contains a defensive alliance of size at most \( k \) is \( \text{W}[1] \)-hard when parameterized by treewidth of the input graph. This puts it among the few problems that are FPT when parameterized by solution size but not when parameterized by treewidth (unless \( \text{FPT}=\text{W}[1] \)).

2 Hardness Results of Defensive Alliance

In this section we show that the Defensive Alliance problem is \( \text{W}[1] \)-hard when parameterized by the size of a vertex deletion set into trees of height at most 2, even when restricted to bipartite graphs, via a reduction from the Multidimensional Relaxed Subset Sum (MRSS) problem.

**Multidimensional Subset Sum (MSS)**

**Input:** An integer \( k \), a set \( S = \{s_1, \ldots, s_n\} \) of vectors with \( s_i \in \mathbb{N}^k \) for every \( i \) with \( 1 \leq i \leq n \) and a target vector \( t \in \mathbb{N}^k \).

**Parameter:** \( k \)

**Question:** Is there a subset \( S' \subseteq S \) such that \( \sum_{s \in S'} s = t \)?

We consider a variant of MSS that we require in our proofs. In the Multidimensional Relaxed Subset Sum (MRSS) problem, an additional integer \( k' \) is given (which will be part of the parameter) and we ask whether there is a subset \( S' \subseteq S \) with \( |S'| \leq k' \) such that \( \sum_{s \in S'} s \geq t \). This variant can be formalized as follows:

**Multidimensional Relaxed Subset Sum (MRSS)**

**Input:** An integer \( k \), a set \( S = \{s_1, \ldots, s_n\} \) of vectors with \( s_i \in \mathbb{N}^k \) for every \( i \) with \( 1 \leq i \leq n \), a target vector \( t \in \mathbb{N}^k \) and an integer \( k' \).

**Parameter:** \( k + k' \)

**Question:** Is there a subset \( S' \subseteq S \) with \( |S'| \leq k' \) such that \( \sum_{s \in S'} s \geq t \)?

It is known that MRSS is \( \text{W}[1] \)-hard when parameterized by the combined parameter \( k + k' \), even if all integers in the input are given in unary [12]. We now show that the Defensive Alliance problem is \( \text{W}[1] \)-hard parameterized by the size of a vertex deletion set into trees of height at most 2, via a reduction from MRSS. We now prove the following theorem:

**Theorem 1.** The Defensive Alliance problem is \( \text{W}[1] \)-hard when parameterized by the size of a vertex deletion set into trees of height at most 2, even when restricted to bipartite graphs.
Proof. Let $I = (k, k', S, t)$ be an instance of MRSS. From this we construct an instance $I' = (G, r)$ of DEFENSIVE ALLIANCE the following way. See Figure 1 for an illustration. Let $s = (s(1), s(2), \ldots, s(k)) \in S$ and let $\text{max}(s)$ denote the value of the largest coordinate of $s$. We set $N = \sum_{s \in S} (2\text{max}(s) + 2)$. First we introduce a set of $k$ new vertices $U = \{u_1, u_2, \ldots, u_k\}$. For every $u_i \in U$, we introduce a set $V^\Box_{u_i} = \sum_{s \in S} s(i) + 2N - 2(\sum_{s \in S} s(i) - t(i))$ square vertices and make $u_i$ adjacent to every vertex of $V^\Box_{u_i}$. We also introduce a set $F = \{f_1, f_2, f_3\}$ of three new vertices and make each $f \in F$ adjacent to a set $V^\Box_{f}$ of $2N$ square vertices. We consider the vertices of $U$ and $F$ in the following order: $u_1, u_2, \ldots, u_k, f_1, f_2, f_3, u_1$. Let

$$P = \{(u_1, u_2), \ldots, (u_{k-1}, u_k), (u_k, f_1), (f_1, f_2), (f_2, f_3), (f_3, u_1)\}$$

be the set of pairs of consecutive vertices. For each pair $(x, y) \in P$, we add a set $H_{xy} = \{h_{xy}^1, \ldots, h_{xy}^N\}$ of $N$ new vertices and add a special vertex $h_{xy}^0$ which is
adjacent to all the vertices of $H_{xy}$; $h^0_{xy}$ is also adjacent to a set $V^0_{xy}$ of $N$ new square vertices. Finally, we introduce a set $H^\square = \{h^1, h^2, h^3\}$ of three new square vertices and for each pair $(x, y) \in P$, make vertices of $H_{xy}$ adjacent to square vertices of $H^\square$.

Recall that a star $S_k$ is the complete bipartite graph $K_{1, k}$: a tree with one internal node and $k$ leaves. For each vector $s \in S$, we introduce two stars $S^1_{\max(s) + 1}$ and $S^2_{\max(s) + 1}$. The first star has internal node $x_s$ and $\max(s) + 1$ leaves $A_s = \{a_1^s, \ldots, a_{\max(s) + 1}^s\}$; the second star has internal node $y_s$ and $\max(s) + 1$ leaves

$$B_s = \{b_1^s, \ldots, b_{\max(s) + 1}^s\}.$$  

For each $i \in \{1, 2, \ldots, k\}$ and for each $s \in S$, we make $u_i$ adjacent to exactly $s(i)$ many vertices of $A_s$ in an arbitrary manner. We make the three vertices $f_1, f_2, f_3$ from the set $F$ adjacent to all vertices of $\bigcup A_s \cup B_s$. Finally, we add a set $V_\Delta = \{v_1, \ldots, v_k\}$ of $k + 5$ square vertices. We make each vertex in $A_s$ adjacent to exactly $|N_U(a^s)| + 5$ many vertices of $V_\Delta$ arbitrarily. We define the set of square vertices as $V^\square = V^\square \cup H^\square \cup \bigcup_{(x, y) \in P} V^\square_{xy} \cup V^\square_u \cup V^\square_f$. and we also define $V^\bigtriangleup = \bigcup_{(x, y) \in P} H_{xy} \cup \{h^0_{xy}\}$. We set $r = (k + 3)N + 2k + 6 + \sum_{i=1}^n (\max(s_i) + 1) + k'$. For every vertex $x \in V^\square$, we introduce a set $V_x = \{x_1, x_2, \ldots, x_{2r + 2}\}$ of $2r + 2$ many vertices adjacent to $x$. We also add two vertices $t$ and $t'$. The vertex $t$ is adjacent to all the vertices in $\bigcup \{V_x \mid x \in \bigcup_{(x, y) \in P} V^\square_{x,y} \cup V^\square_u \cup V^\square_f\}$. Similarly, the vertex $t'$ is adjacent to all the vertices in $\bigcup \{V_x \mid x \in V^\square \cup H^\square\}$. This completes the construction of graph $G$. The strategy is to force all the square vertices outside the solution and all the vertices in $V^\bigtriangleup$ inside the solution. Observe that if we remove the set $U \cup F \cup H^\square \cup V^\square_a \cup \{h^0_{xy} \mid (x, y) \in P\} \cup \{t, t'\}$ of $3k + 16$ vertices from $G$ then we are left with only star graphs. It is easy to see that $G$ is a bipartite graph with bipartition

$$V_1 = H^\square \cup V^\square_a \cup U \cup F \cup \{t'\} \bigcup_{x \in V^\square \setminus (H^\square \cup V^\square_a)} V_x \bigcup_{(x, y) \in P} \{h^0_{xy}\} \bigcup_{s \in S} \{x, y_s\}$$

and

$$V_2 = \{t\} \bigcup_{(x, y) \in P} (V^\square_{xy} \cup H_{xy}) \bigcup_{u \in U} V^\square_u \bigcup_{f \in F} V^\square_f \bigcup_{x \in H^\square \cup V^\square_a} V_x \bigcup_{s \in S} (A_s \cup B_s).$$

We claim that $I = (k, k', S, t)$ is a positive instance of MRSS if and only if $I' = (G, r)$ is a positive instance of DEFENSIVE ALLIANCE. Let $S' \subseteq S$ be such that $|S'| \leq k'$ and $\sum_{s \in S'} s \geq t$. We claim that the set

$$R = U \cup F \bigcup_{(x, y) \in P} H_{xy} \cup \{h^0_{xy}\} \bigcup_{s \in S'} A_s \cup \{x_s\} \bigcup_{s \in S \setminus S'} B_s$$

is a defensive alliance in $G$ such that $|R| \leq r$. Let $x$ be an arbitrary element of $R$.  

Case 1: If \( x = u_i \in U \), then

\[
d_R(u_i) = \sum_{s \in S'} s(i) + 2N
\]

and

\[
d_{R'}(u_i) = \sum_{s \in S', S} s(i) + |V_{u_i}| = \sum_{s \in S} s(i) + \sum_{s \in S} s(i) + 2N - 2(\sum_{s \in S} s(i) - t(i))
\]

\[
d_{R'}(u_i) = 2N + \sum_{s \in S, S'} s(i) + \sum_{s \in S} s(i) - 2(\sum_{s \in S} s(i) - t(i))
\]

\[
= 2N - \left( \sum_{s \in S} s(i) - \sum_{s \in S'} s(i) \right) + 2t(i)
\]

\[
= 2N - \sum_{s \in S'} s(i) + 2t(i)
\]

\[
= 2N + \sum_{s \in S'} s(i) + 2\left(t(i) - \sum_{s \in S'} s(i)\right)
\]

\[
\leq 2N + \sum_{s \in S'} s(i)
\]

\[
= d_R(u_i)
\]

Therefore, we have \( d_R(u_i) + 1 \geq d_{R'}(u_i) \), and hence \( u_i \) is protected.

Case 2: If \( x = a^* \in A_s \), then \( d_R(a^*) = |N_U(a^*)| + |\{f_1, f_2, f_3, x_s\}| = |N_U(a^*)| + 4 \) and \( d_{R'}(a^*) = |N_U(a^*)| + 5 \). Therefore, we get \( d_R(a^*) + 1 \geq d_{R'}(a^*) \).

Case 3: If \( x = f_1 \in F \), then \( N_R(f_1) = \bigcup_{s \in S'} A_s \bigcup_{s \in S'} B_s \cup H_{u_2 f_1} \cup H_{f_1 f_2} \) and

\[
N_{R'}(a) = \bigcup_{s \in S'} B_s \bigcup_{s \in S'} A_s \cup V_{f_1} \cap A_s = |A_s| = |B_s| \cup H_{u_2 f_1} \cup H_{f_1 f_2} = N \text{ and } |V_{f_1}| = 2N + 1, \text{ we have } d_R(f_1) + 1 \geq d_{R'}(f_1). \]

We can similarly check that \( \{f_2, f_2\} \) are also protected.

For the rest of the vertices in \( R \), it is easy to see that \( d_R(x) + 1 \geq d_{R'}(x) \).

Therefore, \( I' = (G, r) \) is a yes instance.

For the reverse direction, suppose that \( G \) has a defensive alliance \( R \) of size at most \( r \). It is easy to see that \( (V_{\bigcup} \{t, t'\}) \cap R = \emptyset \) as any defensive alliance of size at most \( r \) cannot contain vertices of degree greater than \( 2r \). This also shows that \( \bigcup_{x \in V'} V_x \cap R = \emptyset \). Now we show that \( V_\Delta = U \cup F \bigcup_{(x, y) \in P} H_{xy} \cup \{h^0_{xy}\} \subseteq R \).

We claim that if \( V_\Delta \cap R \neq \emptyset \) then \( V_\Delta \subseteq R \).

Case 1: Suppose \( R \) contains \( u_1 \) from \( U \). We observe that if some \( h_{u_1 u_2} \in H_{u_1 u_2} \) is not in \( R \) then \( h^0_{u_1 u_2} \) is also not in \( R \). This implies that no vertex from
Therefore we proved that if $u_1 \implies R$ implies that $H_{u_1 u_2}$ is in $R$. In this case $N_{R-}(u_1) \geq 3N + 2t(i) - \sum_{s \in S} s(i)$ and $N_R(u_1) \leq N + \sum_{i=1}^{n} (\max(s_i) + 2)$. This implies that $u_1$ is not protected in $R$ which is a contradiction as $u_1 \in R$. This implies that $H_{u_1 u_2} \cup \{h_{u_1 u_2}^0\} \subseteq R$. Applying the same argument for $h_{f_3 u_1} \in H_{f_3 u_1}$, we see that $H_{f_3 u_1} \cup \{h_{f_3 u_1}^0\} \subseteq R$. Clearly, this shows that $f_3$ and $u_2$ are in $R$ for protection of vertices in $H_{u_1 u_2} \cup H_{f_3 u_1}$. Applying the same argument for $u_2$ and $f_3$, we get $H_{u_2 u_3} \cup \{h_{u_2 u_3}^0\} \subseteq R$ and $H_{f_3 u_1} \cup \{h_{f_3 u_1}^0\} \subseteq R$, respectively. Repeatedly applying the above argument, we get $V_\Delta \subseteq R$. Observe that this argument can be easily extended to all the vertices of $U \cup F$. Therefore, we see that if $(U \cup F) \cap R \neq \emptyset$, then $V_\Delta \subseteq R$.

**Case 2:** Suppose $R$ contains $h_{xy}$ from $h_{xy}$ for some $(x, y) \in P$. Clearly, this implies $R$ contains both $x$ and $y$ from $U \cup F$. Using Case 1, we get $V_\Delta \subseteq R$.

**Case 3:** Suppose $R$ contains $h_{xy}^0$ for some $(x, y) \in P$. Clearly, this implies that $h_{xy} \subseteq R$. Using Case 2, we get $V_\Delta \subseteq R$.

Therefore we proved that if $V_\Delta \cap R \neq \emptyset$ then $V_\Delta \subseteq R$. Next we claim that if $R$ is non-empty then $R$ contains the set $V_\Delta$. Since $R$ is non-empty, we see that $R$ must contain a vertex from graph $G$. We consider the following cases:

**Case 1:** Suppose $R$ contains $a^*$ from $A_s$ for some $s \in S$. Then we know that $d_R(a^*) \geq |N_U(a^*)| + 5$. We see that $a^*$ is protected if and only if $F \cap R \neq \emptyset$. This implies that $V_\Delta \cap R \neq \emptyset$ which implies $V_\Delta \subseteq R$.

**Case 2:** Suppose $R$ contains $x_s$ for some $s \in S$. We know that $x_s$ has at least two neighbours in $A_s$ as $\max(s) + 1 \geq 2$. This implies that $x_s$ is protected if and only if at least one vertex $a_s \in A_s$ is in $R$. Now, Case 1 implies that $V_\Delta \subseteq R$.

**Case 3:** Suppose $R$ contains $b^* \in B_s$ for some $s \in S$. Then we know that $N(b^*) \subseteq \{y_s\} \cup F$. Clearly, the protection of $b^*$ requires at least one vertex from $F$. This implies that $F \cap R \neq \emptyset$. Therefore, we have $V_\Delta \cap R \neq \emptyset$ and hence $V_\Delta \subseteq R$.

**Case 4:** Suppose $R$ contains $y_s$ for some $s \in S$. We know that $y_s$ has at least two neighbours in $B_s$ as $\max(s) + 1 \geq 2$. This implies that $y_s$ is protected if and only if at least one vertex $b_s \in B_s$ is in $R$. Now, Case 3 implies that $V_\Delta \subseteq R$.

This shows if $R$ is non-empty then $V_\Delta \subseteq R$. We know $V_\Delta$ contains exactly $(k+3)N + 2k + 6$ many vertices; thus besides the vertices of $V_\Delta$, there are at most $\sum_{i=1}^{n} (\max(s_i) + 1) + k'$ vertices in $R$. Since $f_1 \in R$ and $d_{V_\Delta}(f_1) = d_{V_\Delta}(f_1) = 2N$, it must have at least $\sum_{i=1}^{n} (\max(s_i) + 1)$ many neighbours in $R$ from the set $\bigcup A_s \cup B_s$. We also observe that if a vertex $a^*$ from the set $A_s$ is in the solution
then \( x_s \) also lie in the solution for the protection of \( a_s \). This shows that at most \( k' \) many sets of the form \( A_s \) contribute to the solution as otherwise the size of solution exceeds \( r \). Therefore, any arbitrary defensive alliance \( R \) of size at most \( r \) can be transformed to another defensive alliance \( R' \) of size at most \( r \) as follows:

\[
R' = V_\Delta \bigcup_{x_s \in R} A_s \cup \{x_s\} \bigcup_{x_s \in V(G) \setminus R} B_s.
\]

We define a subset \( S' = \{ s \in S \mid x_s \in R' \} \). Clearly, \( |S'| \leq k' \). We claim that \( \sum_{s \in S'} s(i) \geq t(i) \) for all \( 1 \leq i \leq k \). Assume for the sake of contradiction that \( \sum_{s \in S'} s(i) < t(i) \) for some \( i \in \{1, 2, \ldots, k\} \). Note that

\[
d_R'(u_i) = \sum_{s \in S'} s(i) + 2N
\]

and

\[
d_R'^c(u_i) = \sum_{s \in S \setminus S'} s(i) + |V_{u_i}| = \sum_{s \in S \setminus S'} s(i) + \sum_{s \in S} s(i) + 2N - 2(\sum_{s \in S} s(i) - t(i))
\]

Then, we have

\[
d_R'^c(u_i) = 2N + \sum_{s \in S \setminus S'} s(i) + \sum_{s \in S} s(i) - 2(\sum_{s \in S} s(i) - t(i)) = 2N - \left( \sum_{s \in S} s(i) - \sum_{s \in S \setminus S'} s(i) \right) + 2t(i) = 2N - \sum_{s \in S'} s(i) + 2t(i) = 2N + \sum_{s \in S'} s(i) + 2\left( t(i) - \sum_{s \in S'} s(i) \right) > 2N + \sum_{s \in S'} s(i) = d_{R'}(u_i)
\]

We also know that \( u_i \in R' \), which is a contradiction to the fact that \( R' \) is a defensive alliance. This shows that \( I = (k, k', S, t) \) is a yes instance.

Clearly trees of height at most two are trivially acyclic. Moreover, it is easy to verify that such trees have pathwidth [18] and treedepth [20] at most two, which implies:

**Theorem 2.** The Defender Alliance problem is \( W[1] \)-hard when parameterized by any of the following parameters:

- the feedback vertex set number,
- the treewidth and clique width of the input graph,
- the pathwidth and treedepth of the input graph,

even when restricted to bipartite graphs.
3 No Polynomial Kernel Parameterized by Vertex Cover Number

A set $C \subseteq V$ is a vertex cover of $G = (V, E)$ if each edge $e \in E$ has at least one endpoint in $X$. The minimum size of a vertex cover in $G$ is the **vertex cover number** of $G$, denoted by $vc(G)$. Parameterized by vertex cover number $vc$, the **Defensive Alliance** problem is FPT [17] and in this section we prove the following kernelization hardness of the **Defensive Alliance** problem.

**Theorem 3.** The **Defensive Alliance** problem parameterized by the vertex cover number of the input graph does not admit a polynomial compression unless coNP $\subseteq$ NP/poly.

To prove Theorem 3, we give a polynomial parameter transformation (PPT) from the well-known **Red Blue Dominating Set** problem (RBDS) to **Defensive Alliance** parameterized by vertex cover number. Recall that in RBDS we are given a bipartite graph $G = (T \cup S, E)$ and an integer $k$, and we are asked whether there exists a vertex set $X \subseteq S$ of size at most $k$ such that every vertex in $T$ has at least one neighbour in $X$. We also refer to the vertices of $T$ as **terminals** and to the vertices of $S$ as **sources** or **nonterminals**. The following theorem is known:

**Theorem 4.** [9] RBDS parameterized by $|T|$ does not admit a polynomial compression unless coNP $\subseteq$ NP/poly.

### 3.1 Proof of Theorem 3

By Theorem 4, RBDS parameterized by $|T|$ does not admit a polynomial compression unless coNP $\subseteq$ NP/poly. To prove Theorem 3, we give a PPT from RBDS parameterized by $|T|$ to **Defensive Alliance** parameterized by the vertex cover number. Given an instance $(G = (T \cup S, E), k)$ of RBDS, we construct an instance $(G', k')$ of **Defensive Alliance** as follows. Take three distinct copies $T_0, T_1, T_2$ of $T$, and let $t_i$ be the copy of $t \in T$ in $T_i$. Similarly, take two distinct copies $S_0, S_1$ of $S$, and let $s_i$ be the copy of $s \in S$ in $S_i$. Now for every vertex in $t \in T_1 \cup T_2$, we introduce a set $V_t = \{t_1, \ldots, t_4\}$ of vertices adjacent to $t$ where the number $\ell$ is defined later in the proof. Moreover, create three vertices $a$, $b$ and $c$. The vertices $a$, $b$ and $c$ are adjacent to all the vertices in $\bigcup_{t \in T_1 \cup T_2} V_t$.

We also make $a$ and $b$ adjacent to every vertex in $T_0$, and make $a$ adjacent to every vertex in $S_1$. If $(t, s) \in E(G)$ then we add the edges $(t_0, s_0), (t_0, s_1), (t_1, s_1)$ and $(t_2, s_0)$ in $E(G')$. Finally, we add a vertex $x^*$ which is adjacent to every vertex in $S_1$ and also adjacent to exactly $|S|$ many arbitrary vertices from $V_t$ for some $t \in T_1 \cup T_2$. We observe that $C = T_0 \cup T_1 \cup T_2 \cup \{a, b, c, x^*\}$ is a vertex cover of $G'$. Therefore the vertex cover size of $G'$ is bounded by $3|T| + 4$. We set $k' = |T| + |S| + k + 1$ and $l = 4k'$. See Figure 2 for an illustration. We now claim that $G$ is a yes-instance of RBDS if and only if $G'$ is a yes-instance of **Defensive Alliance**.
Suppose there exists a vertex set $X \subseteq S$ of size at most $k$ in $G$ such that every vertex in $T$ has at least one neighbour in $X$. We claim that the set $R = S_1 \cup \{s_0 \in S_0 \mid s \in X\} \cup T_0 \cup \{x^*\}$ is a defensive alliance in graph $G'$. Let $x$ be an arbitrary element of $R$. We prove that $x$ is protected in $R$.

Case 1: Suppose $x \in S_1$. Note that $N_R(x) = N_{T_0}(x) \cup \{x^*\}$. Thus, including itself, it has $d_G(x) + 2$ defenders in $G'$. The attackers of $x$ consist of elements of $N_{T_1}(x)$ and element $a$. Hence $x$ has $d_G(x) + 1$ attackers. This shows that $x$ has at least as many defenders as attackers; hence $x$ is protected.

Case 2: Suppose $x \in \{s_0 \in S_0 \mid s \in X\}$. Note that $N_R(x) = N_{T_0}(x)$. Thus, including itself, it has $d_G(x) + 1$ defenders in $G'$. The attackers of $x$ consist of elements of $N_{T_2}(x)$. Hence $x$ has $d_G(x)$ attackers in $G'$. This shows that $x$ is protected.

Case 3: Suppose $x \in T_0$. Clearly, including itself, $x$ has $2d_G(x) + 3$ neighbours in $G'$. Thus it requires at least $d_G(x) + 2$ many defenders in $G'$. Note that, including itself, $x$ has $d_G(x) + 1$ neighbours in $S_1 \subseteq R$. Therefore, it requires at least one neighbour from the set $\{s_0 \in S_0 \mid s \in X\}$ inside the solution and this is true.
because $G$ is a yes instance.

**Case 4:** Suppose $x = x^*$. It has the same number of defenders and attackers in $G'$, this shows that $x$ is protected.

Conversely, suppose there exists a defensive alliance $R$ of size at most $k'$ in $G'$. We observe that no vertex from the set $Q = T_1 \cup T_2 \cup \{a, b, c\} \bigcup_{t \in T_1 \cup T_2} V_t$ can be part of $R$ as otherwise its size will exceed $k'$. Since $R$ is non-empty, it must contain a vertex from one of the sets $\{x^*\}, S_1, T_0$ or $S_0$.

**Case 1:** Suppose $x^* \in R$. Since $x^*$ has $|S|$ many neighbours in $Q$, it implies that all the neighbours of $x^*$ in $S_1$ must be inside the solution for protection of $x^*$. This implies that $S_1 \subseteq R$. Let $v$ be an arbitrary vertex in $S_1$. Note that $v$ has $d_G(v)$ neighbours in $T_0$, and it has $d_G(v) + 1$ neighbours in $Q$. For protection of $v$ all the neighbours of $v$ in $T_0$ must be part of the solution. This implies that $T_0 \subseteq R$ as all the vertices in $S_1$ must be protected. Note that till now we have added $|S| + |T| + 1$ many vertices in the solution. Therefore, we can add at most $k$ vertices to the solution from the set $S_0$ as otherwise the solution size will exceed $k'$. Suppose we add a set $X \subseteq S_0$ of size at most $k$ to the solution. Consider the protection of vertices in $T_0$. If $v$ is a vertex of $T_0$, then it has $d_G(v)$ neighbours in $S_0$ and similarly $d_G(v)$ neighbours in $S_1$. Excluding itself, $v$ has $2d_G(v) + 2$ neighbours in $G'$. Thus it requires at least $d_G(v) + 1$ many neighbours inside the solution. We know that $d_G(v)$ neighbours are inside the solution due to the fact that $S_1 \subseteq R$. Therefore, it requires at least one neighbour from $S_0$ inside the solution. Since there exists a set $X \subseteq S$ of size at most $k$ such that all the vertices in $T_0$ are protected, it shows that all vertices in $T_0$ have at least one neighbour in $X$. This proves that $G$ is a yes instance.

**Case 2:** Suppose $R$ contains a vertex $v$ from the set $S_1$. In this case, the protection of $v$ requires $x^*$ to be inside the solution and then the same argument as in Case 1 will lead to the proof.

**Case 3:** Suppose $R$ contains a vertex $v$ from the set $T_0$. Excluding itself, $v$ has $2d_G(v) + 2$ neighbours in $G'$. Thus it requires at least $d_G(v) + 1$ many neighbours from $S_0 \cup S_1$ inside the solution. This implies that at least one neighbour from the set $S_1$ must be inside the solution. Now the same argument as in Case 2 will lead to the proof.

**Case 4:** Suppose $R$ contains a vertex $v$ from the set $S_0$. Clearly, it has $d_G(v)$ neighbours in $T_2 \subseteq Q$ and $d_G(v)$ neighbours in $T_0$. Since the vertices in the set $Q$ cannot be part of the solution, the protection of $v$ will imply that all the neighbours of $v$ in $T_0$ are part of the solution. In other words, there exists a vertex in $T_0$ which is inside the solution. Now the same argument as in Case 3 will lead to the proof.
This proves that $G$ is a yes-instance.

4 Defensive Alliance has no Subexponential Algorithm

In this section, we prove lower bound based on ETH for the time needed to solve the Defensive Alliance problem. In order to prove that a too fast algorithm for Defensive Alliance contradicts ETH, we give a reduction from Vertex Cover in graphs of maximum degree 3 and argue that a too fast algorithm for Defensive Alliance would solve Vertex Cover in graphs of maximum degree 3 in time $2^{o(n)}$. Johnson and Szegedy [15] proved that, assuming ETH, there is no algorithm with running time $2^{o(n)}$ to compute a minimum vertex cover in graphs of maximum degree 3.

Theorem 5. Unless ETH fails, Defensive Alliance does not admit a $2^{o(n)}$ algorithm where $n$ is the number of vertices of the input graph.

Proof. We give a linear reduction from Vertex Cover in graphs of maximum degree 3 to Defensive Alliance, that is, a polynomial-time algorithm that takes an instance $(G, k)$ of Vertex Cover, where $G$ has $n$ vertices and $m = O(n)$ edges, and outputs an equivalent instance of Defensive Alliance whose size is bounded by $O(n)$. We construct an equivalent instance $(G', k')$ of Defensive Alliance the following way. See Figure 3.

1. We introduce the vertex sets $X$ and $Y$ into $G'$, where $X = V(G) = \{v_1, \ldots, v_n\}$ and $Y = E(G) = \{e_1, e_2, \ldots, e_m\}$, the edge set of $G$. We make $v_i$ adjacent to $e_j$ if and only if $v_i$ is an endpoint of $e_j$.
2. For every $1 \leq i \leq m$, we introduce a cycle $C_i$ of length 4. For every $1 \leq i \leq m - 1$, make every vertex of $C_i$ adjacent to $e_i$ and $e_{i+1}$; and make every vertex of $C_m$ adjacent to $e_m$ and $e_1$.
3. We add a set $F = \{f_1, f_2, \ldots, f_8\}$ of 8 new vertices into $G'$. Set $k' = 5m + k$. For every vertex $f \in F$ we introduce a set $V_f$ of $4k'$ new vertices into $G'$ and make them adjacent to $f$. We make every vertex of $\{f_1, f_2, f_3, f_4, f_5\}$ adjacent to every vertex of $C_i$ for $i = 1, 2, \ldots, m$. We also make every vertex of $F$ adjacent to every vertex of $Y$.
4. Finally, we introduce a vertex $a$ and make it adjacent to every vertex of $X \cup \bigcup_{f \in F} V_f$.

We now argue equivalence of the instances. Suppose there exists a vertex cover $S$ of size at most $k$ in $G$. We show that $D = S \cup Y \bigcup_{i=1}^{m} V(C_i)$ is a defensive alliance of size at most $k'$ in $G'$. It is easy to verify that all the vertices in $D$ are protected.

To prove the reverse direction of the equivalence, suppose now that $D$ is a defensive alliance of size at most $k'$ in $G'$. Observe that no vertex from
Fig. 3. The reduction from Vertex Cover to Defensive Alliance.

\[ Q = \{a\} \cup F \cup \bigcup_{f \in F} V_f \] can be part of \( D \) as otherwise the size of \( D \) will exceed \( k' \). To prove this theorem we need the following simple claim:

**Claim.** Every defensive alliance \( D \) of \( G' \) contains the set \( Y \bigcup_{i=1}^{m} V(C_i) \).

**Proof.** Since the defensive alliance is non-empty, it must contain a vertex from \( X \bigcup Y \bigcup_{i=1}^{m} V(C_i) \).

**Case 1:** Suppose \( D \) contains \( e_i \) from \( Y \). We observe that \( \text{deg}_{G'}(e_i) = 18 \). Note that eight neighbours of \( e_i \) in \( F \) cannot be part of the solution as they belong to the forbidden set \( Q \). This implies that we need to add at least one vertex from the set \( V(C_{i-1}) \bigcup V(C_i) \) to the solution for the protection of \( e_i \). Without loss of generality, suppose we include one vertex from \( V(C_i) \) in the solution. Inclusion of one vertex from the set \( V(C_i) \) in the solution forces \( V(C_i) \subseteq D \). This in turn forces \( e_{i+1} \) in the solution. Repeatedly applying the above argument, we see that \( Y \bigcup_{i=1}^{m} V(C_i) \subseteq D \).

**Case 2:** Suppose \( D \) contains an arbitrary vertex from the set \( X \bigcup_{i=1}^{m} V(C_i) \). Then the protection of that vertex forces at least one vertex from \( Y \) in the solution. Using the argument in Case 1, it implies that \( Y \bigcup_{i=1}^{m} V(C_i) \subseteq D \). \( \Box \)
We observe that for every vertex $e \in Y$, we have included eight out of its 18 neighbours in the solution. For the protection of $e$, we need to include at least one more of its neighbours from $X$ in the solution. As we have already added $5m$ vertices in the solution, we can add a set $S \subseteq X$ of at most $k$ vertices in $D$ such that every vertex $e \in Y$ has at least one neighbour in $S$. If such a set $S$ exists then it forms a vertex cover of $G$. This shows that $I$ is a yes instance.

5 Defensive Alliance on Circle Graphs

A circle graph is the intersection graph of a set of chords of a circle. That is, it is an undirected graph whose vertices can be associated with chords of a circle such that two vertices are adjacent if and only if the corresponding chords cross each other. Here, we prove that the Defensive Alliance problem is NP-complete even when restricted to circle graphs, via a reduction from Dominating Set. It is known that the Dominating Set problem on circle graphs is NP-hard [16].

Theorem 6. The Defensive Alliance problem on circle graphs is NP-complete.

On the way towards this result, we provide hardness result for a variant of the Defensive Alliance problem which we require in the proof of Theorem 6. The problem Defensive Alliance$^F$ generalizes Defensive Alliance where some vertices are forced to be outside the solution; these vertices are called forbidden vertices. This variant can be formalized as follows:

**Defensive Alliance$^F$**

**Input:** An undirected graph $G = (V, E)$, a positive integer $r$ and a set $V_\square \subseteq V(G)$ of forbidden vertices.

**Question:** Is there a defensive alliance $S \subseteq V$ such that $1 \leq |S| \leq r$, and $S \cap V_\square = \emptyset$?

**Lemma 1.** The Defensive Alliance$^F$ problem on circle graphs is NP-complete.

**Proof.** It is easy to see that the problem is in NP. To show that the problem is NP-hard we give a polynomial reduction from Dominating Set on circle graphs. Let $(G, k)$ be an instance of Dominating Set, where $G$ is a circle graph. Suppose we are also given the circle representation $C$ of $G$. Without loss of generality, we can assume that none of the endpoints of chords overlap with each other. We create a graph $G'$ and output the instance $(G', V_\square, k')$. See Figure 6. The steps given below describe the construction of $G'$:

- **Step 1:** Take two distinct copies $G_1$ and $G_2$ of $G$ and let $v_i$ be the copy of $v \in V(G)$ in graph $G_i$. For each $v \in V$, make $v_1$ adjacent to every vertex of $N_{G_2}(v_2) \cup \{v_2\}$ and similarly make $v_2$ adjacent to every vertex of $N_{G_1}(v_1) \cup \{v_1\}$. Note that this operation can be easily incorporated in the circle representation by replacing the chord corresponds to $v$ with two crossing cords correspond to $v_1$ and $v_2$ as shown in Figure 6.
- **Step 2:** For every $v \in V$, create two sets of vertices $X^v = \{x^v_1, \ldots, x^v_{2n+1}\}$ and $Y^v = \{y^v_1, \ldots, y^v_{2n+1}\}$ and make $v_1, v_2$ adjacent to every vertex of $X^v \cup Y^v$. This can be easily incorporated in circle representation by introducing $2n+1$ parallel chords for the vertices $x^v_1, \ldots, x^v_{2n+1}$ which cross the chords for $v_1, v_2$. Similarly, introduce $2n+1$ parallel chords for the vertices $y^v_1, \ldots, y^v_{2n+1}$ which cross the chords for $v_1, v_2$, as shown in Figure 5.

- **Step 3:** For each $x^v \in X^v$, create two 3-vertex cliques $C^1_{x^v}$ and $C^2_{x^v}$, and make $x^v$ adjacent to every vertex of $C^1_{x^v}$ and $C^2_{x^v}$. For $1 \leq i \leq 2n$, make every vertex of $C^1_{x^v_i}$ adjacent to every vertex of $C^1_{x^v_{i+1}}$. Similarly, make every vertex of $C^2_{x^v_i}$ adjacent to every vertex of $C^2_{x^v_{i+1}}$ for $1 \leq i \leq 2n$. For each $y^v \in Y^v$, create two 3-vertex cliques $C^1_{y^v}$ and $C^2_{y^v}$, and make $y^v$ adjacent to every vertex of $C^1_{y^v}$ and $C^2_{y^v}$. Make every vertex of $C^1_{y^v_i}$ adjacent to every vertex of $C^1_{y^v_{i+1}}$ for $1 \leq i \leq 2n$. Similarly, make every vertex of $C^2_{y^v_i}$ adjacent to every vertex of $C^2_{y^v_{i+1}}$ for $1 \leq i \leq 2n$. We start at an arbitrary vertex on the circle representation of $C$ of $G$ and then traverse the circle in counter clockwise direction. We record the sequence in which the chords are visited.
For example, in Figure 4(i), if we start at the red vertex on the circle, then the sequence in which the chords are visited, is a, b, c, a, b, c. Note that every vertex appears twice in the sequence as every chord is visited twice while traversing the circle. Thus we get a sequence $S$ of length $2n$ where $n$ is the number of chords. We use the sequence to connect newly added cliques. For every consecutive pair $(u, v)$ in the sequence $S$, make every vertex of $C^2_{2n+1}$ adjacent to every vertex of $C^{1}_{2n+1}$, when both $u$ and $v$ appear for the first time in the sequence $S$; make every vertex of $C^2_{y_{2n+1}}$ adjacent to every vertex of $C^{1}_{y_{2n+1}}$ when both $u$ and $v$ appear for the second time; and make every vertex of $C^2_{y_{2n+1}}$ adjacent to every vertex of $C^{1}_{y_{2n+1}}$, when $u$ appears for the first time and $v$ appear for the second time. These adjacency are shown in green color in Figure 6 and 7.

Step 4: For every vertex $u$ in cliques, add $d$ forbidden vertices where $d$ is the degree of $u$ until now in $G'$. For every vertex $u \in X' \cup Y'$, add six forbidden vertices and make them adjacent with $u$. For every vertex $u \in V(G_1) \cup V(G_2)$, add $4n + 3$ forbidden vertices and make them adjacent to $u$. This completes the construction of $G'$. We set $k' = 7n(4n+2) + n + k$ and $V_\square$ be the set of all one degree forbidden vertices.

We observed that the constructed graph $G'$ is indeed a circle graph, and the construction can be performed in time polynomial in $n$. We now claim that $G$ admits a dominating set of size at most $k$ if and only if $G'$ admits a defensive alliance $D$ of size at most $k'$ such that $D \cap V_\square = \emptyset$. Assume first that $G$ admits a dominating set $S$ of size at most $k$. Consider

$$D = \left\{ v_1 : v \in S \right\} \bigcup_{v \in V(G)} X' \cup Y' \bigcup_{v \in V(G)} (C^1_{x_i'}) \cup V(C^2_{x_i'}) \cup V(C^1_{y_i'}) \cup V(C^2_{y_i'}).$$

Clearly, $|D| \leq 7n(4n+2) + n + k$ and $D \cap V_\square = \emptyset$, so it suffices to prove that $D$ is a defensive alliance in $G'$. We observe that every vertex in $V(G)$ has equal neighbours inside and outside the solution. Every vertex $v \in \bigcup_{v \in V(G)} X' \cup Y'$ is protected as it has at least 7 neighbours inside the solution and at most 7 neighbours outside the solution. Each $v \in \left\{ v_1 : v \in S \right\} \bigcup V(G_2)$ has at least $d + 1 + 4n + 2$ neighbours inside the solution and at most $d + 1 + 4n + 2$ neighbours outside the solution where $d = d_G(x)$. This shows that $D$ is a defensive alliance of size at most $k'$ in $G'$.

Conversely, suppose that $G'$ admits a defensive alliance $D$ of size at most $k'$ such that $D \cap V_\square = \emptyset$. We define

$$V_\triangle = \bigcup_{v \in V(G)} X' \cup Y' \bigcup_{v \in V(G)} (C^1_{x_i'}) \cup V(C^2_{x_i'}) \cup V(C^1_{y_i'}) \cup V(C^2_{y_i'}).$$
Fig. 6. The reduction of an instance $G$ of the DOMINATING SET problem on circle graphs to an instance $G'$ of the DEFENSIVE ALLIANCE problem in Theorem 6. Here $2n + 1 = 7$. One degree forbidden vertices introduced in Step 4 are not shown here.
We first show that \( V_\triangle \subseteq D \). Since \( D \) is non-empty, it should contain a vertex from either \( V_\triangle \) or \( V(G_1) \cup V(G_2) \). We consider the following cases:

**Case 1:** Suppose \( D \) contains a vertex from \( \bigcup_{i=1}^{2n+1} V(C^1_{x_i}) \cup V(C^2_{x_i}) \cup V(C^1_{y_i}) \cup V(C^2_{y_i}) \). Without loss of generality, we may assume that \( D \) contains a vertex \( u \) from \( V(C^1_{x^v}) \). It is easy to see that \( u \) is protected if and only if all its non-forbidden neighbours are inside \( D \) because the number of forbidden neighbours of \( u \) is equal to the number of non-forbidden neighbours. This implies that \( x^v \in D \).

It is easy to note that either \( C^1_{x^v} \subseteq D \) or \( C^1_{x^v} \cap D = \emptyset \). Since \( d_G(x^v) = 15 \) and \( x^v \) has 6 forbidden neighbours, the above observation implies that \( C^1_{x^v} \cup C^2_{x^v} \subseteq D \).

This implies that \( \bigcup_{i=1}^{2n+1} C^1_{x_i} \cup C^2_{x_i} \subseteq D \). This in turn implies that \( X^v \subseteq D \). Note that \( C^2_{x_{2n+1}} \) is adjacent to \( C^1_{x_{2n+1}} \) (resp. \( C^1_{y_{2n+1}} \)) for some \( w \in G \) such that \( u, w \) are consecutive elements in the sequence \( S \) and \( w \) appears for the first (resp. second) time in the sequence. Therefore \( \bigcup_{i=1}^{2n+1} C^1_{x^w} \cup C^2_{x^w} \subseteq D \) and also \( X^w \subseteq D \) if \( w \) appears for the first time in the sequence; whereas \( \bigcup_{i=1}^{2n+1} C^1_{y^w} \cup C^2_{y^w} \subseteq D \) and also \( Y^w \subseteq D \) if \( w \) appears for the second time in the sequence. Repeatedly applying the above argument, we get \( V_\triangle \subseteq D \).

**Case 2:** Suppose \( D \) contains a vertex from \( \bigcup_{v \in V(G)} X^v \cup Y^v \). Without loss of generality, we may assume that \( D \) contains \( x^v \) from \( X^v \). We observe that the protection of \( x^v \) clearly requires at least one vertex from the set \( C^1_{x^v} \cup C^2_{x^v} \). Now, Case 1 implies that \( V_\triangle \subseteq D \).
Case 3: Suppose $D$ contains a vertex $v$ from $V(G_1) \cup V(G_2)$. The protection of $v$ requires at least one vertex from the set $X^v \cup Y^v$. Now, Case 2 implies that $V_\Delta \subseteq D$.

Observe that $|V_\Delta| = 7n(4n + 2)$. Therefore $|D \cap (V(G_1) \cup V(G_2))| \leq n + k$. For each $v \in V(G)$, the protection of every vertex in $X^v \cup Y^v$ requires either $v_1$ or $v_2$ inside the solution. Since, $v_1$ and $v_2$ are twins, we can assume that $V(G_2) \subseteq D$. Let $v_2 \in V(G_2)$. We see that $v_2$ has $d + 1 + 4n + 2$ neighbours (including itself) inside the solution. The vertex $v_2$ has $4n + 3$ forbidden neighbours.

The only unsettled neighbours of $v_2$ are in $V(G_1)$ and $v_2$ has $d + 1$ neighbours in $V(G_1)$. For protection of each $v_2 \in V(G_2)$, we require at least one neighbour from $V(G_1)$ inside the solution. We can add at most $k$ vertices from $V(G_1)$ to the solution as we have already added $7n(4n + 2) + n$ vertices. Clearly, $S = V(G_1) \cap D$ is a dominating set of size at most $k$.

5.1 Proof of Theorem 6

It is easy to see that the problem is in NP. To show that the problem is NP-hard we give a polynomial reduction from Defensive Alliance. Let $(G, k, V_{□})$ be an instance of Defensive Alliance, where $G$ is a circle graph. We construct an instance $(G', k')$ of Defensive Alliance the following way. For every $x \in V_{□}$, create a vertex $x'$ and a set of $2k'$ vertices $V_{x}^x$. Make both $x$ and $x'$ adjacent to every vertex in $V_{x}^x$. This completes the construction of $G'$. Set $k' = k$.

![Fig. 8. The circle representation to get rid of forbidden vertices when $k' = 2$.](image)

We observe in Figure 8 that the constructed graph $G'$ is indeed a circle graph, and the construction can be performed in time polynomial in $n$. We now claim that $G$ admits a defensive alliance $D$ of size at most $k$ such that $D \cap V_{□} = \emptyset$ if and only if $G'$ admits a defensive alliance $D'$ of size at most $k'$. Assume first that $D$ is a defensive alliance of size at most $k$ in $G$ such that $D \cap V_{□} = \emptyset$. Consider $D' = D$. Clearly, $D'$ is a defensive alliance of size at most $k'$ in $G'$. Conversely, suppose that $G'$ admits a defensive alliance $D'$ of size at most $k'$. Observe that
\[ D' \cap \bigcup_{x \in V_{D}} V_{x} \cup \{x, x'\} = \emptyset. \] As \( x \) and \( x' \) are of degree \( 2k' \), they cannot be part of a defensive alliance of size at most \( k' \). As \( x \) and \( x' \) are outside \( D' \), the vertices in \( V_{x} \) cannot be in \( D' \). Consider \( D = D' \). Clearly, \( D \) is a defensive alliance of size at most \( k \) in \( G \) such that \( D \cap V_{D} = \emptyset \).

6 Conclusions

In this work we proved that the Defensive Alliance problem is \( \text{W}[1] \)-hard parameterized by a wide range of fairly restrictive structural parameters such as the feedback vertex set number, pathwidth, treewidth, treedepth, and clique width of the input graph, even when restricted to bipartite graph. We also proved that the problem parameterized by the vertex cover number of the input graph does not admit a polynomial compression unless \( \text{coNP} \subseteq \text{NP/poly} \); it cannot be solved in time \( 2^{o(n)} \), unless ETH fails, and the Defensive Alliance problem on circle graphs is NP-complete. By the construction of our proofs in Section 2, it is clear that hardness also holds for problem variants that ask for defensive alliances exactly of a given size. In the future it may be interesting to study if our ideas can be useful for different kinds of alliances from the literature such as offensive and powerful alliances. The parameterized complexity of offensive and defensive alliance problems remain unsettled when parameterized by other important structural graph parameters like twin cover and modular-width.

References

1. B. Bliem and S. Woltran. Defensive alliances in graphs of bounded treewidth. *Discrete Applied Mathematics*, 251:334 – 339, 2018.
2. C.-W. Chang, M.-L. Chia, C.-J. Hsu, D. Kuo, L.-L. Lai, and F.-H. Wang. Global defensive alliances of trees and cartesian product of paths and cycles. *Discrete Applied Mathematics*, 160(4):479 – 487, 2012.
3. M. Chellali and T. W. Haynes. Global alliances and independence in trees. *Discuss. Math. Graph Theory*, 27(1):19–27, 2007.
4. M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015.
5. R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Springer, 2012.
6. R. Enciso. *Alliances in graphs: Parameterized algorithms and on partitioning series-parallel graphs*. PhD thesis, USA, 2009.
7. H. Fernau and D. Raible. Alliances in graphs: a complexity-theoretic study. In *Proceeding Volume II of the 33rd International Conference on Current Trends in Theory and Practice of Computer Science*, 2007.
8. H. Fernau, J. A. Rodriguez, and J. M. Sigarreta. Offensive r-alliances in graphs. *Discrete Applied Mathematics*, 157(1):177 – 182, 2009.
9. F. V. Fomin, D. Lokshtanov, S. Saurabh, and M. Zehavi. *Kernelization: Theory of Parameterized Preprocessing*. Cambridge University Press, 2019.
10. G. Fricke, L. Lawson, T. Haynes, M. Hedetniemi, and S. Hedetniemi. A note on defensive alliances in graphs. *Bulletin of the Institute of Combinatorics and its Applications*, 38:37–41, 2003.
11. A. Gaikwad, S. Maity, and S. K. Tripathi. Parameterized complexity of defensive and offensive alliances in graphs. In D. Goswami and T. A. Hoang, editors, *Distributed Computing and Internet Technology*, pages 175–187, Cham, 2021. Springer International Publishing.

12. R. Ganian, F. Klute, and S. Ordyniak. On structural parameterizations of the bounded-degree vertex deletion problem. *Algorithmica*, 2020.

13. L. H. Jamieson. *Algorithms and Complexity for Alliances and Weighted Alliances of Various Types*. PhD thesis, USA, 2007.

14. L. H. Jamieson, S. T. Hedetniemi, and A. A. McRae. The algorithmic complexity of alliances in graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 68:137–150, 2009.

15. D. S. Johnson and M. Szegedy. What are the least tractable instances of max independent set? In *Proceedings of the Tenth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA ’99, page 927–928, USA, 1999. Society for Industrial and Applied Mathematics.

16. J. Keil. The complexity of domination problems in circle graphs. *Discrete Applied Mathematics*, 42(1):51–63, 1993.

17. M. Kiyomi and Y. Otachi. Alliances in graphs of bounded clique-width. *Discrete Applied Mathematics*, 223:91 – 97, 2017.

18. T. Kloks. *Treewidth, Computations and Approximations*, volume 842 of *Lecture Notes in Computer Science*. Springer, 1994.

19. P. Kristiansen, M. Hedetniemi, and S. Hedetniemi. Alliances in graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 48:157–177, 2004.

20. J. Nesetril and P. O. de Mendez. *Sparsity: Graphs, Structures, and Algorithms*. Springer Publishing Company, Incorporated, 2014.

21. N. Robertson and P. Seymour. Graph minors. iii. planar tree-width. *Journal of Combinatorial Theory, Series B*, 36(1):49 – 64, 1984.

22. J. Rodríguez-Velázquez and J. Sigarreta. Global offensive alliances in graphs. Electronic *Notes in Discrete Mathematics*, 25:157 – 164, 2006.

23. J. Sigarreta, S. Bermudo, and H. Fernau. On the complement graph and defensive k-alliances. *Discrete Applied Mathematics*, 157(8):1687 – 1695, 2009.

24. J. Sigarreta and J. Rodríguez. On defensive alliances and line graphs. *Applied Mathematics Letters*, 19(12):1345 – 1350, 2006.

25. J. Sigarreta and J. Rodríguez. On the global offensive alliance number of a graph. *Discrete Applied Mathematics*, 157(2):219 – 226, 2009.

26. D. B. West. *Introduction to Graph Theory*. Prentice Hall, 2000.