Locally Scale-Invariant Gravity

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Abstract. We put forward the idea that in addition to diffeomorphism invariance of general relativity (GR) the gravitational interaction is invariant under arbitrary scale-deformations of the metric field. In addition, we assume that the scaling field has an internal symmetry. The global charges that are associated with this symmetry could potentially source the gravitational field. In the case that isotropic deformations are considered, the theory reduces to a Weyl-invariant (WI) version of GR. In the case that Minkowski spacetime is deformed the vierbein formalism is recovered, rendering GR a field theory on Minkowski spacetime. A few implications of a classical Weyl-invariant scalar-tensor (WIST) generalization of general relativity (GR) are considered. As an example, we recast the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) spacetime in the WIST form with static space and monotonically evolving masses.

1. Introduction

It is hard to conceive any physically realistic description of the Universe in the absence of either inertia or gravitation. Inertia is thought to be a universal phenomenon; the Higgs field that is responsible for the generation of particle masses in the electroweak sector is ubiquitous. Likewise, gravitation is everywhere. It is normally identified with the omnipresence of spacetime, the substratum of physical reality, at least down to spacetime singularities, where the very concepts of space and time are believed to break down.

Our current best theory of the (classical) gravitational interaction is general relativity (GR). The latter has successfully passed a wide range of tests in our solar system. GR is one of the simplest of all conceivable metric theories; it obeys the metricity condition, it is torsionless and it is described by a lagrangian linear in the Ricci scalar. GR provides a diffeomorphism-invariant field theoretical framework that incorporates the principle of relativity applied to non-inertial observers and is realized on curved spacetimes. The curvature in the presence of a given energy-momentum distribution is governed by $G$, the universal coupling constant of gravitation that converts energy-momentum density units to length units. This specific coupling between matter and geometry was adopted in GR so as to guarantee consistency with Newton law of gravitation in the weak field limit, assuming that the locally inferred $G$ is universal.
Naturally, GR is expected to depart from the Newtonian limit in the strong gravity regime, i.e. in physical systems which are dominated by the gravitational interaction while other interactions play a secondary role, e.g. in black holes or on the largest cosmological scales. However, in contrast to expectations Newtonian gravity does not adequately describe gravitational phenomena on galactic scales and larger, e.g. at the outskirts of galaxies and galaxy clusters, even though gravity is weak on these scales, i.e. the dimensionless gravitational potential is as low as $\Phi \sim 10^{-4}$ or $10^{-3}$. A straightforward remedy for this problem (and others in cosmology) involves introducing some form of non-relativistic (NR) invisible matter, i.e. ‘dark matter’ (DM). By far the leading candidate is some yet-to-be-determined beyond-the-standard-model (SM) particle. According to another, less popular, possibility primordial black holes may account for the missing non-luminous matter.

As of yet, no DM particle candidate has been found in spite of decades of intensive search. In addition, the status of beyond-the-SM theories of particle physics – our best hope for new DM particles – is unclear with persistent null results coming from the large hadron collider (LHC), although certain anomalies have been reported recently with mild indications for new physics. A recent high precision measurement of the Z boson mass at the Fermilab Tevatron collider, that significantly departs from SM expectations, calls the integrity of the SM into question. In any case, these are either inconclusive as of yet or need independent corroborations by other experiments, and even if eventually ratified considerable amount of effort is still needed before the existence of a DM particle that adequately accounts for DM phenomena on Hubble down to galactic scales is compellingly established.

Another possibility is that particulate DM simply does not exist, and that on galactic scales or larger gravitation is not strictly described by GR, but rather by a modified version thereof. Indeed, the validity of GR coupled to purely ordinary matter (described by the SM of particle physics) has been only compellingly established within our own solar system. One possible extension of GR, invokes a higher symmetry – Weyl invariance (WI), where $G$ is determined by a scalar field (while the other three fundamental interactions are assumed to have their standard forms, i.e. they are non-Weyl invariant). In GR this scalar field is replaced by default with a fixed $G$ throughout space and time. However, its local determination by Cavendish-type experiments does not necessarily imply its universality. In a WI version of GR, $G$ can vary in space and time, freely departing from its locally measured value, and simultaneously inducing scalar metric perturbations. From this perspective, there is a priori no preferred conformal frame, and whereas the choice $G = \text{const.}$ seems to work impressively well in our solar system it clearly fails on galactic scales and larger. It has been shown recently, that fractional variations of $G$ and active gravitational masses at the $O(10^{-3})$ or $O(10^{-4})$ level on galactic and galaxy cluster scales are sufficient for explaining away DM phenomena without particulate DM at the reasonable ‘cost’ of endowing GR with WI [1]. Another application of a WI version of GR is a possible resolution of the ‘Hubble tension’: the apparent statistically significant $\gtrsim 4\sigma$ tension between local and
cosmological inference of the Hubble constant, \( H_0 [2] \).

On the largest cosmological scales observations require some form of a non-clustering vacuum-like DE, which is puzzling in a few respects. First, it is \( \sim 120 \) orders of magnitude suppressed relative to naive expectations based on our current understanding of particle physics. This might be hinting towards a gravitational/geometric origin for DE rather than material one. Second, it is found empirically to be comparable in energy density terms to that of the NR matter at the present time although the two species greatly differ in their evolution history. In this sense, there seems to be a fine-tuning of the initial conditions that are associated with DE [3].

The present work focuses on the foundations of a certain generalization of GR that allows for a fully anisotropic locally scale-invariant generalization of GR by introducing a scale deformation (tensor) field \( \Phi_{\mu\nu} \) living on an arbitrary spacetime. In the special case that the latter is Minkowski \( \Phi_{\mu\nu} \) reduces to the well-known vierbein field. In the case of an isotropic scale-invariance on an arbitrary background the theory becomes a WI scalar-tensor (WIST) version of GR, hereafter referred to in abbreviation as ‘WIST’. In addition to local scale-invariance we endow \( \Phi_{\mu\nu} \) with an internal symmetry with which a global conserved charge is associated. The latter provides an additional source for spacetime curvature.

Although various variants of WIST have been contemplated over the past fifty years, e.g. [4-15], objections to this framework range between two extremes. On the one hand, it is often claimed that the WIST is ‘GR in a guise’, that extending the symmetry of GR to allow WI does not make it genuinely so. In particular it is argued that there is classically no preferred conformal frame, so WIST is equivalent to GR. On the other hand, it is sometimes implied that dressing GR with WI brings about a ghost scalar field that inflicts ‘disastrous instabilities’ on the theory, so much so as to disqualify the theory (while the same is not similarly claimed about the ‘equivalent’ GR). We address these claims in the present work, and as a side we explicitly show that, not surprisingly, the action of the Friedmann-Robertson-Walker (FRW) spacetime, a legitimate solution of the Einstein equations, can be presented in a WIST form. By doing so, we illustrate with a specific example, with well-known general-relativistic description of spacetime, that there are no ‘fatal instabilities’ associated with WIST that do not already exist in GR. Again, we stress that it is our assumption that WIST is entirely classical to the extent that GR is, and indeed the validity of GR has only been experimentally established on macroscopic down to \( \mu m \), i.e. mesoscopic, scales, and not smaller.

The paper is organized as follows. The general anisotropic scale-invariant version of GR is considered in section 2, followed by its special isotropic case, the WIST theory, in section 3. The FRW example is discussed in section 4, followed by endowing the theory with global symmetries in section 5 and a summary in section 6. In Appendix A the transformation from a purely gravitational to non-gravitational description of the weak field approximation is described, and in Appendix B the case of U(2) global symmetry is briefly discussed. Throughout, we adopt a mostly-positive signature for the spacetime metric \((-1, 1, 1, 1)\), with the speed of light \( c \) and reduced Planck constant \( \hbar \) set to unity.
2. Locally anisotropic scale-invariant theory of gravitation

GR is described by the diffeomorphism-invariant Einstein-Hilbert (EH) action

\[ I_{EH} = \int \left( \frac{1}{16\pi G} (R - 2\Lambda) + \mathcal{L}_M \right) \sqrt{-g} d^4x, \tag{1} \]

where \( R \) is the Ricci curvature scalar (obtained from the metric field \( g_{\mu\nu} \) and its first and second derivatives in the standard way) and \( \Lambda \) is the cosmological constant, respectively, \( \mathcal{L}_M \) is the lagrangian density of matter, and \( g \) is the metric determinant.

The cosmological constant, \( \Lambda \), can be absorbed in \( \mathcal{L}_M \) as a species with vacuum-like properties. Diffeomorphism-invariance implies that equation (1) is invariant under the transformation \( g_{\mu\nu} \rightarrow g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} \) (where Greek indices stand for spacetime coordinates and repeated indices are summed over). This symmetry of equation (1) under change of coordinate basis relies on the commutativity of derivatives.

A natural generalization of diffeomorphism (that to the best of our knowledge has not been considered so far in the literature) could be the following replacement everywhere in equation (1)

\[ g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} \equiv \Phi^\alpha_{\mu} g_{\alpha\beta} \Phi^\beta_{\nu}, \]
\[ \mathcal{L}_M \rightarrow \bar{\mathcal{L}}_M / |\Phi|, \tag{2} \]

where \( \Phi^\alpha_{\mu}(x) \) is an arbitrary spacetime-dependent real scale-deformation tensor of rank 2, and \( |\Phi| \equiv \det(\Phi^\nu_{\mu}) \). This transformation represents a local anisotropic rescaling of the metric field and the lagrangian density (in which \( \Lambda \) is absorbed), thereby providing us with a generalization of the ‘isotropic’ local rescaling, known also as WI, that will be discussed below and in the next section. In this framework gravitation is described by both a tensor field \( \Phi^\mu_{\nu} \) and the spacetime metric \( g_{\mu\nu} \) on which \( \Phi^\mu_{\nu} \) lives. We assume that \( \Phi^\mu_{\nu} \) is all-pervading much like the Higgs field is universal.

Naturally, the local (in general anisotropic) rescaling of the various fields is by physical dimension, e.g. the dimensions of \( g_{\mu\nu} \), \( g^{\mu\nu} \) and \( \mathcal{L}_M \) are \( \text{length}^2 \), \( \text{length}^{-2} \) and \( \text{length}^{-4} \), respectively. The contravariant coordinates, \( x^\mu \), are dimensionless, and the covariant coordinates \( x_\mu \) are of dimension \( \text{length}^{-2} \). Specifically, the anisotropic local scale transformations for the metric and tensor fields, as well as the lagrangian density, are

\[ g_{\mu\nu} \rightarrow \Omega^\alpha_{\mu} g_{\alpha\beta} \Omega^\beta_{\nu}, \]
\[ \Phi^\alpha_{\mu} \rightarrow \Phi^\rho_{\mu} (\Omega^{-1})^\alpha_{\rho}, \]
\[ \Phi^\alpha_{\mu} \rightarrow (\Omega^{-1})^\alpha_{\rho} \Phi^\rho_{\mu}, \]
\[ \mathcal{L}_M \rightarrow \mathcal{L}_M / |\Omega(x)|, \tag{3} \]

where \( |\Omega(x)| \) is the determinant of an arbitrary spacetime-dependent matrix \( \Omega^\nu_{\mu} \), leaves \( \bar{g}_{\mu\nu} \) and \( \bar{\mathcal{L}}_M / |\Phi| \) that are defined in equation (2), invariant. Therefore, if we replace \( g_{\mu\nu} \) and \( \mathcal{L}_M \) with \( \bar{g}_{\mu\nu} \) and \( \bar{\mathcal{L}}_M \), respectively, everywhere in equation (1) then the latter is invariant under equation (3) in addition to being diffeomorphism-invariant.
In the special case that the scaling matrix is isotropic, $\Phi^\nu_{\mu} = \phi(x)\delta^\nu_{\mu}$, i.e. the metric scale deformation is isotropic, the proposed symmetry reduces to a WI version of GR, i.e. the aforementioned WIST theory, in which case the replacement $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu}\phi^2(x)$ and $\mathcal{L}_M \rightarrow \mathcal{L}_M/\phi^4$ is made in equation (1), where $\phi(x)$ is an arbitrary scalar function. This implies that the theory is invariant under the Weyl transformation

$$
g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu}\Omega^2(x)$$

$$
\phi \rightarrow \tilde{\phi} = \phi \Omega^{-1}(x)
$$

$$
\mathcal{L}_M \rightarrow \tilde{\mathcal{L}}_M = \mathcal{L}_M\Omega^{-4}(x)
$$

(4)

This isotropic case will be further explored in the next section.

In the special case $\eta_{\mu\nu} \rightarrow e_\alpha^\mu\eta_{\alpha\beta}e_\beta^\nu = g_{\mu\nu}(x)$, where $\eta_{\mu\nu}$ is the Minkowski metric, the tensor $\Phi^\nu_{\mu} = e_\nu^\mu$ is the vierbein field. Tensor-mode perturbations, i.e. gravitational waves, could therefore describe massless tensor perturbations in the $\Phi^\nu_{\mu}$ field, i.e. in ‘$G$’ and in active gravitational masses, rather than in spacetime itself. When spacetime is fixed to be Minkowski all the gravitational degrees of freedom that are normally represented by the various components of the spacetime metric are now accounted for by the tensor field, which in this case is merely the vierbein field, $e_\nu^\mu$. In this choice the metric field itself, that describes Minkowski spacetime, is non-dynamical, space and time are infinite and and all the singularities are in $\Phi^\nu_{\mu}$, its derivatives, and combinations thereof that replace, e.g. the standard spacetime curvature. This differs from the standard interpretation of GR where distortions and singularities are understood to be in spacetime itself, where e.g. the Big Bang singularity is normally understood as the beginning of time, and perhaps even of spacetime. To illustrate the description of gravitational phenomena within the WIST theory on Minkowski spacetime we examine the transformation of the geodesic equation for a point test particle in the ‘weak field’ approximation in Appendix A.

Setting $G = 3/(8\pi)$ and replacing $\mathcal{L}_M$ with $V$ (the latter contains no derivatives of the various fields), and $g_{\mu\nu}$ with $\tilde{g}_{\mu\nu}$ (see equation 2) everywhere in equation (1), we obtain

$$
\tilde{\mathcal{I}} = \int \left( \frac{1}{6}(\tilde{R} - 2\Lambda) + V \right) \sqrt{-\tilde{g}} d^4x.
$$

(5)

As will be detailed below we assume that $V$, the source of spacetime curvature, is a potential in $\Phi \equiv \sqrt{\Phi_\alpha^\beta\Phi^\beta_\alpha}$. In particular, we make the assumption that it is an analytic polynomial in $\Phi$. Consider the transformations described by equation (3). To first order, an infinitesimal transformation results in

$$
\tilde{g}_{\mu\nu} = \Omega_\alpha^\nu g_{\alpha\beta}\Omega^\beta_\mu \approx g_{\mu\nu} + \delta\Omega_\alpha^\nu g_{\alpha\nu} + g_{\mu\alpha}\delta\Omega^\alpha_\nu,
$$

(6)

where we used $\Omega_\mu^\nu \approx \delta_\mu^\nu + \delta\Omega_\mu^\nu$. Similarly,

$$
\tilde{\Phi}^\nu_{\mu} = (\Omega^{-1})^\alpha_\mu \Phi^\nu_\alpha \approx (\delta_\mu^\alpha - \delta\Omega_\mu^\alpha)\Phi^\nu_\alpha.
$$

(7)

from which it follows that

$$
\delta\Phi^\nu_{\mu} \approx -\delta\Omega_\mu^\alpha \Phi^\nu_\alpha.
$$

(8)
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Employing the relation between variations of inverse matrices, \( \delta g^{\mu\nu} = -g^{\mu\alpha} \delta g_{\alpha\beta} g^{\beta\nu} \) and using equation (6) we obtain

\[
\delta g^{\mu\nu} \approx -g^{\mu\alpha} \delta \Omega_{\alpha}^{\nu} - \delta \Omega_{\alpha}^{\mu} g^{\alpha\nu}.
\]  

(9)

In our convention that \( x^\mu \) are dimensionless (see discussion below equation 2), \( \int V \sqrt{-g} \) is dimensionless and so \( \delta (V \sqrt{-g}) = \frac{\delta V \sqrt{-g}}{\delta g} \delta g^{\mu\nu} + \frac{\delta V \sqrt{-g}}{\delta g_{\mu}^{\nu}} \delta g_{\mu}^{\nu} = 0 \) under the transformations Eqs.(8) and (9). Requiring invariance under arbitrary \( \delta \Omega_{\mu}^{\nu} \) we obtain

\[
S_{\mu}^{\nu} = \Phi_{\mu}^{\alpha} \delta V \frac{\delta}{\delta \Phi_{\nu}^{\alpha}} + \frac{\delta V}{\delta \Phi_{\nu}^{\alpha}} \Phi_{\mu}^{\alpha}
\]

(10)

where \( S_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta V \sqrt{-g}}{\delta g^{\mu\nu}} \). This provides a strong constraint on the potential if anisotropic local scale invariance, Eqs. (1)-(3), is to be a symmetry of the gravitational interaction. We will rederive the analog of this result in the isotropic scaling case and delve into the implications in sections 3 and 4.

We emphasize that only gravitation is endowed with WI in the proposed framework. All other fundamental interactions, namely the strong and electroweak interactions are described by the SM of particle physics, with no modifications. Since any realistic physical system is governed by both gravitational and non-gravitational interactions, and since the fundamental interactions are minimally-coupled to gravitation, it then follows that this particular framework is clearly not merely a field re-definition of GR and the SM of particle physics. In any case, introducing the global charge that is generally associated with \( \Phi_{\mu\nu} \) due to internal symmetry as discussed in section 5 and in Appendix B renders the theory described here inequivalent to the SM even in purely gravitational systems such as late time cosmological scales.

Much like the nonvanishing and universality of VEV of the Higgs field guarantees that inertia phenomena are ubiquitous so it is the case with spacetime that never ceases to exist as is reflected by the fact that the metric field can always be so chosen to be Minkowski in the most general anisotropic locally scale-invariant gravitation. Since the SM of particle physics is minimally coupled to gravitation then the tensor field \( \Phi_{\mu\nu} \) locally couples to the SM fields instead of them being living on a curved spacetime. For example, \( \mathcal{I}_{kin} = \int g^{\mu\nu} (D_{\mu} H)^{\dagger} D_{\nu} H \sqrt{-g} d^{4}x \), the kinetic term of the Higgs field \( H \) (where \( D_{\mu} \) is the gauge-covariant derivative), transforms to \( \mathcal{I}_{kin} = \int (D_{\mu} H)^{\dagger} \bar{g}^{\mu\nu} D_{\nu} H \sqrt{-\bar{g}} d^{4}x \), which in the Minkowski background simplifies to \( \mathcal{I}_{kin} = \int (D_{\mu} H)^{\dagger} \bar{\Phi}_{\sigma}^{\mu} \bar{\Phi}_{\nu}^{\sigma} D_{\nu} H |\Phi| d^{4}x \). The tensor field \( \Phi^{\mu\nu} \) then directly couples to \( H \), (where \( \Phi_{\mu}^{\nu} \) is actually the vierbein field in the case of Minkowski background) thereby constituting a field theory on Minkowski spacetime (spacetime indices are raised and lowered by means of the Minkowski metric), as opposed to the standard interpretation according to which \( H \) interacts with the geometry of spacetime.
3. WIST theory

In this section we focus on the isotropic case, i.e. $\Phi^\nu_\mu = \delta^\nu_\mu \phi(x)$, where $\phi(x)$ is a scale deformation (scalar) field. Although it was assumed in equation (2) that $\Phi^\nu_\mu$ is real (in order for the new metric field, $\bar{g}_{\mu\nu}$, to be real) it is clear that at least in the special case considered in this section, i.e. $\Phi^\nu_\mu = \delta^\nu_\mu \phi(x)$, promoting $\phi(x)$ to a complex scalar field renders $\bar{g}_{\mu\nu}$ real. Since the scalar field is assumed to be complex, it has a U(1) internal symmetry. The more general case of an arbitrary internal symmetry is briefly discussed in the section 5. Gravitation is then fundamentally endowed with WI, i.e. an isotropic locally scale-invariance, within a WIST theory, and standard GR is recovered in a particular conformal frame, where $\phi = constant$.

Rather than using the more general results from the previous section we derive them directly in the isotropic case. The WIST theory readily follows from locally rescaling the spacetime metric $g_{\mu\nu} \rightarrow \phi \phi^* g_{\mu\nu}$ everywhere in equation (1). The resulting theory is then described by the following action

$$I_{WIST} = \int \left( \frac{1}{6} |\phi|^2 R - \phi^* \Box \phi + V(|\phi|, \{\Psi\}) \right) \sqrt{-g} d^4x$$

$$= \int \left( \frac{1}{6} |\phi|^2 R + \phi_\mu \phi^{*\mu} + V(|\phi|, \{\Psi\}) \right) \sqrt{-g} d^4x,$$

where $\phi_\mu \equiv \frac{\partial \phi}{\partial x^\mu}$, $V$ that replaces $L_M$ is now allowed to explicitly depend on $|\phi|$ but not on its derivatives, and the second equality follows from integration by parts of the kinetic term associated with the scalar field. All other fields (including $g_{\mu\nu}$) are collectively denoted by $\{\Psi\}$. We emphasize that in the new formulation, equation (11), gravitation is sourced by $V$ rather than by ‘matter’, $L_M$. In general, GR is consistent with observations within the solar system and so on these scales $V$ can be replaced by $L_M$, i.e. ‘ordinary’ matter. However, on larger scales, where both DM and DE are required by observations we favor the form equation (11) which does not imply the existence of DM particles. In addition, in equation (11) geometry and matter are not presented as two different entities. Rather, the existence of $\phi$ in both the source, $V$, and the kinetic term, reflects the viewpoint that they are actually intertwined. They are ‘unified’ in the sense that WI of equation (11), a symmetry that is not recognized by GR, depends on both the scalar and metric fields.

With $V$ being explicitly $\phi$-dependent, equation (11) is now a scalar-tensor theory of the Bergmann-Wagoner type [16], [17]. It was first obtained by Deser [4] and later in [10] with $L_M$ [referred to as $V$ in equation (11)] that does not necessarily depend on $\phi$. In [4] $\phi$ was assumed to be real. A similar procedure to the replacement $g_{\mu\nu} \rightarrow \phi \phi^* g_{\mu\nu}$ that transforms equation (1) to equation (10) was employed by Chamseddine and Mukhanov in their Mimetic Gravity [18] but in the latter case the scalar field was an irrotational velocity potential $\varphi$, i.e. $g_{\mu\nu} \rightarrow \varphi_\mu \varphi^{*\mu} g_{\mu\nu}$.

Formally, the kinetic term associated with the scalar field that appears in equation (11), $L_\phi \equiv \phi_\mu \phi^{*\mu}$ or $L_\phi \equiv -\phi^* \Box \phi$, can be viewed as a new source of the gravitational interaction. This term is completely ignored in GR where $\phi$ is set to a constant value,
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\(\phi_0 \equiv \sqrt{\frac{3}{8\pi G}}\). Invariance of equation (11) implies that \(V \to |\phi|^{-4}V\) under \(g_{\mu\nu} \to |\phi|^2 g_{\mu\nu}\).

This simple derivation also underscores the origin of the ‘ghost’ scalar field whose kinetic term, \(\mathcal{L}_\phi \equiv \phi_{\mu\nu} \phi^{\mu\nu}\), appears with the ‘wrong’ sign in equation (11); it simply stems from locally stretching/squeezing the spacetime metric, or equivalently stretching/squeezing the yardstick with which distances are measured, in our case it is the Planck length \(l_p \propto \sqrt{G}\) in vacuum, or \(\propto (G\rho)^{-1/2}\) where \(\rho\) is the energy density in non-vacuum configurations. This observation is crucial for evaluation of the significance of the ghost field. We see that \(\phi\) is tightly related to \(g_{\mu\nu}\) (merely a local stretch of the spacetime metric), and as \(g_{\mu\nu}\) is treated classically so should be \(\phi\) and \(V\); the ghostly nature of \(\phi\) derives solely from the specific form in which \(g_{\mu\nu}\) and its derivatives appear in the EH action. Below, we explicitly show (although it should be clearly evident already from the very local scaling procedure \(g_{\mu\nu} \to \phi\phi^{\star \mu\nu}\) that the spacetime dependence of \(\phi\) is arbitrary, and so whereas it is spacetime-dependent it is not dynamical in the sense that it is not determined by a dynamical differential equation which has an attractor solution, that, e.g. drives the kinetic term, \(\mathcal{L}_\phi \equiv \phi_{\mu\nu} \phi^{\mu\nu}\), to arbitrary negative values, as is the case with generic ghost fields. Actually, we will see in section 4 that exactly the opposite takes place in our expanding Universe when the action of the FRW spacetime is recast in the form of equation (11) instead of equation (1); the negative kinetic term evolves from negative infinity at the Big Bang towards vanishingly negative values at the remote future assuming a monotonic expansion history, compatible with the concordance cosmological model.

In fact, the presence of the kinetic term in \(\mathcal{L}_\phi \equiv \phi_{\mu\nu} \phi^{\star \mu\nu}\) is expected once \(G\) is promoted to a field. More specifically, since the curvature scalar depends on derivatives of the metric field it transforms inhomogeneously under equation (12)

\[
R \to \Omega^{-2} \left( R - 6 \frac{\Box \Omega}{\Omega} \right),
\]

where \(\Box\) is the d’Alambertian. Invariance of equation (11) under equation (12) then requires the presence of the kinetic term \(\mathcal{L}_\phi \equiv \phi_{\mu\nu} \phi^{\mu\nu}\) that transforms inhomogeneously and guarantees the mutual cancellation of derivatives of \(\Omega\) (provided that the appropriate integration by parts has been carried out). The prefactor \(\frac{1}{6}\) in front of the curvature...
coupling term in equation (11) guarantees that the inhomogeneous term on the right hand side of equation (13) is compensated by a similar inhomogeneous term from the transformation of the kinetic term, $\phi^\mu \phi^\nu$. In other words, under the transformation equation (12) the combination of the first two terms in equation (11) transforms as, 
$$\frac{1}{6}|\phi|^2 R - \phi \Box \phi \rightarrow \Omega^{-4}\left(\frac{1}{6}|\phi|^2 R - \phi \Box \phi^*\right),$$
i.e. it has the appropriate well-defined conformal weight of $\text{length}^{-4}$, such that its product with the volume element in equation (11) is WI.

In the special case that $\phi$ is a real field and $V$ is independent of $\phi$ equation (11) can be cast in the form of a Brans-Dicke (BD) theory [19] 
$$I_{BD} = \int \left(\Phi R - \omega_{BD} \frac{\Phi}{\Phi} \Box \Phi + \mathcal{L}_M\right) \sqrt{-g} d^4 x$$
with $\Phi$ (not to be confused with the norm of $\Phi_{\mu\nu}$ that is defined just below equation 5) and $\omega_{BD}$ being the BD scalar field and dimensionless BD parameter, respectively, where $\Phi \equiv \phi^2/6$ and $\omega_{BD} \equiv -3/2$. Whereas a tight lower limit, $\omega_{BD} > 40000$, has been reported in [20], it should be stressed that it was obtained from satellite observations directly exploring the impact of varying $G$ within our own solar system, and not beyond. In addition, in the (generally non-WI) BD framework the source term $\mathcal{L}_M$ does not in general depend on $\phi$ while in WIST the potential $V$ does.

It is constructive at this point to further elucidate the issue of stability of Eq. (11) unlike what is sometimes claimed or implied. While it is true that the kinetic term appears in equation (11) with the ‘wrong’ sign relative to $V(|\phi|)$ and $\phi$ is formally a ‘ghost’ field it is merely a gauge artifact that can be seen when the latter is set to a constant in the unitary gauge [21]. The ‘fatal instability’ usually ascribed to quantum ghost fields stems from the fact that in their presence there is either no stable vacuum state, or non-conservation of probability takes place, e.g. [22]. However, GR is classical, and so is WIST. As was already mentioned above, in the special case of WIST, it is exactly WI that prevents classical instability from taking place because there is essentially no equation of motion for $\phi$ to dynamically drive it towards instability. This is not surprising since at least classically, two conformally-related theories are equivalent, e.g. [23], [24]. Actually, it has been argued that WI is a sham/fake symmetry of GR [25], [26], i.e. that equation (11) is equivalent to GR. On face value, the conclusion that the scalar field is a ‘spurion’ [25] implies that its formal ghostly nature is of no practical significance. As mentioned above, adding a global symmetry to $\Phi_{\mu\nu}$ renders both the anisotropic and isotropic (equation 11) locally scale invariant theories inequivalent to GR. We also re-iterate in this context that in the framework proposed here gravitation is WI, whereas the SM of particle physics is not, and since in general the other three interactions are non-negligible in typical physical systems, then the SM of particle physics coupled to WIST is not equivalent to the SM of particle physics coupled to GR. This was a key observation that was exploited in [1] in explaining away particulate DM in galaxies and clusters of galaxies. Had the SM of particle physics been WI as well then the proposed remedy for the DM problem would gauge away.

Ultimately, the ‘appropriate’ conformal frame to be used in the description of
a physical system should be determined by observations, much like the ‘appropriate’ coordinate system employed in the description of, e.g., a black hole in GR is determined by observations, e.g. [24]. For example, a spherically symmetric static black hole as seen from the perspective of a static observer would be most naturally described by a Schwarzschild metric, while a freely falling observer towards the same black hole will likely employ a different, time-dependent, metric in describing the same black hole. The diffeomorphism symmetry of GR allows observers to describe the system in the most symmetric fashion, which clearly depends on the (a priori arbitrary) observers states. In the same fashion, at the classical level, theory cannot pre-select the ‘appropriate’ units to be used in the description of a physical system. This freedom is clearly not a property of GR, but is certainly a desirable tenet of its WI generalization described by equation (11) or the more general, locally anisotropic scale-invariant, theory described in the previous section.

In the quantum regime the situation generally changes as quantization and Weyl transformations do not in general commute, e.g. [23], [24], and in fact ghosts are known to violate unitarity, e.g. [27]. However, quantization of GR has never been demonstrated to be feasible (at least this is the consensus), nor are there available measurements indicating that quantum gravitational effects are in play so as to necessitate its quantization. Actually, the standard theoretical arguments in favor of its quantization might be strong, yet inconclusive (e.g. [28] and references within), and at the very best GR could be considered an effective low energy limit of a more fundamental quantum theory of gravity, if it should indeed be ultimately quantized. Therefore, in the same fashion that the non-renormalizability of GR does not disqualify it from being the backbone of the standard cosmological model what might be a ghost ‘problem’ of equation (11) at the quantum level is not an issue for physical systems that are adequately described by GR, e.g. [29]. Therefore, the inequivalence between conformal frames at the quantum level need not concern us in the present work.

It has been claimed in [27] that fixing the scalar field [and thereby rendering equation (11) equivalent to GR] in order to go around the ‘ghost problem’ in practice ‘robb’s’ equation (11) from its claimed WI. Again, while this argument might be valid at the quantum level it definitely does not apply classically. Moreover, and as discussed below, the set of fields $\phi$ and $g_{\mu\nu}$ that appears in equation (11) is under-determined and $\phi$ could be chosen to be any arbitrary function, not necessarily of a fixed value, thereby still ‘manifesting’ WI, while at the same time avoiding driving the field configuration away from its equilibrium state. In summary, the ‘wrong’ sign of the scalar field in a WI theory is not an issue at the classical regime, which is the only domain at which equation (11), exactly as GR, can be trusted.

It is constructive at this point to explore a slightly more general case of a dynamical scalar field which is non-minimally coupled to spacetime curvature

$$I_{ST} = \int \left( \xi |\phi|^2 R + \phi^* \phi \partial^\mu + V(|\phi|) \right) \sqrt{-g} d^4 x,$$

where $\xi$ is an arbitrary dimensionless coupling parameter, thereby generalizing equation
(11), i.e., the theory is in general not WI, unless $\xi = \frac{1}{6}$.

The field equations derived from variation of equation (14) with respect to the metric and scalar field are, respectively

\[ 2\xi |\phi|^2 G_{\mu\nu} = S_{M,\mu\nu} + \Theta_{\mu\nu}, \]  

(15)

and

\[ \xi \phi R - \Box \phi + \frac{\partial V}{\partial \phi} = 0, \]  

(16)

where

\[ 3\Theta_{\mu\nu} \equiv 6\xi \phi^* \phi_{;\mu;\nu} + (6\xi - 3) \phi^*_\mu \phi^*_\nu \]

\[ - 6\xi g_{\mu\nu} \left[ \phi^* \Box \phi - \left( \frac{1}{4\xi} - 1 \right) \phi^*_\alpha \phi^*_\alpha \right] + c.c. \]  

(17)

Here and throughout $S_{M,\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} V)}{\delta g^{\mu\nu}}$ is the analog of the energy-momentum tensor. Equation (15) is a generalization of Einstein equations with $2\xi |\phi|^2$ replacing $1/(16\pi G)$, and $\Theta_{\mu\nu}$ is an effective contribution to the energy-momentum tensor essentially due to gradients of $G$ and active gravitational masses, the latter appear in the source term, $L_M$. Multiplying equation (16) by $\phi^*$, adding the result to its complex conjugate and to the trace of equation (15) results in

\[ \phi \frac{\partial V}{\partial \phi} + \phi^* \frac{\partial V}{\partial \phi^*} = S_M + (1 - 6\xi)(\phi^* \Box \phi + \phi^*_\alpha \phi^*_\alpha + c.c.). \]  

(18)

Together, equation (15) and equation (18) provide a system of dynamical equations for the ten metric components and the complex scalar field. In the BD form ($\phi$ is real and $L_M$, that is here replaced by $V$, is independent of $\phi$) equation (18) reduces to the well-known form $(3 + 2\omega_{BD}) \Box \Phi = T_M$. It follows from equation (18) that for any $\xi \neq \frac{1}{6}$ (equivalently $\omega_{BD} \neq -3/2$) the scalar field is dynamically determined by the matter distribution via a differential equation, which thereby embodies the Machianity of BD theory of gravity.

In contrast, in the special case $\xi = \frac{1}{6}$ (equivalently $\omega_{BD} = -3/2$ [4-15]) Equation (14) is identical to (11), and equation (18) reduces to a constraint that only determines the dependence of $V$ on the scalar field (as in the more general equation 10). In this special case, $\phi$ is indeed non-dynamical; it can be set to any desired function, irrespective of the matter distribution, and Machianity is lost. Although $\phi$ still appears in equation (15) we now have only ten differential equations for ten metric components and a scalar field – this under-determined system simply reflects the freedom to locally rescale the fields in the special case $\xi = \frac{1}{6}$, merely a manifestation of WI, equation (12), [4-15]. The latter implies invariance of equation (11) under local-rescaling of fields such that $V \rightarrow |\phi|^{-4} V$. This requires in particular that, e.g.,

\[ \psi \rightarrow \Omega^{-\frac{2}{5}} \psi \]

\[ A_\mu \rightarrow A_\mu, \]  

(19)
where $\psi$ and $A_\mu$ are Dirac and vector fields, respectively, in addition to the transformation of the scalar and metric fields described in equation (12) – each field is rescaled by its mass/length dimension. In the special case $\xi = \frac{1}{6}$, equation (18) could be derived solely from the requirement that $V \sqrt{-g}$ is WI, as was done in the previous section in the more general case, in equation (10).

Any mass terms that appear in $V$ are by definition active gravitational masses, which need not be equivalent to inertial or passive gravitational masses. While the three types of mass are not necessarily equivalent [30], the notion of a passive gravitational mass in any theory of gravity that satisfies the equivalence principle is vacuous. If the ratio of the latter two mass types is a universal constant then the equivalence principle is satisfied – an assumption that we indeed make in the present work. Again, we emphasize that the other three fundamental interactions are described by the lagrangian of the SM of particle physics, $L_{SM}$, where all masses are inertial, whether generated via the Higgs mechanism in the electroweak sector or via the explicitly broken chiral symmetry in QCD.

Since $\phi(x) = \Omega^{-1}(x)\phi_0$, where $\phi_0$ is the fixed GR value, then for any given such a choice $\Theta_{\mu\nu}$ can be calculated according to equation (17) and used in equation (15) to solve for the corresponding $g_{\mu\nu}$. In practice, however, it is easier to start from any known solution of the Einstein equations, $\phi_0$ and $g_{\mu\nu}$, and transform it to $\Omega^{-1}(x)\phi_0$ and $\Omega^2(x)g_{\mu\nu}$ with arbitrary $\Omega(x)$ by virtue of WI. We emphasize, once again, that this latter procedure is only possible assuming (as is virtually always assumed) that there are no global charges associated with the field as is discussed in section 5.

In the theory described by equation (11), the scalar field $\phi$ determines not only $G$ but also active gravitational masses. Only pure radiation $S_{rad} = 0$ is consistent with the theory described by equation (14) with $\xi = \frac{1}{6}$ unless $V$ explicitly depends on $|\phi|$. In the case of perfect fluid $S = V$ on shell [31], and with EOS $w_M$ the trace is $S = -\rho_M(1 - 3w_M)$. It then follows from equation (18) that $V \propto |\phi|^{-3w_M}$ in case that $\xi = \frac{1}{6}$, i.e it is linear and quartic in $|\phi|$ in cases of NR- and vacuum-like terms, respectively, and is independent of $|\phi|$ in case of pure radiation. Linearity of $V$ in $|\phi|$ in the case of vanishing EOS, $w_M = 0$, suggests that active gravitational masses are regulated by $|\phi|$. Not only that the same $\phi$ determines both Planck mass and active gravitational masses is a necessary condition for the consistency of non-radiation sources with this WI theory, it is also a conceptually natural “conclusion” as the concept of active gravitational mass is meaningless unless it couples to G, and in this sense it seems natural that both quantities are determined by the same scalar field. This clearly does not have necessarily to be the case in general (as in e.g, BD theory), but it is a nice merit of the WI model described by equation (11).

We emphasize that in the original BD proposal [19] the matter lagrangian, $L_M$, does not explicitly depend on the scalar field. Notably, Brans & Dicke required $\omega_{BD} > 0$ to guarantee the positivity of the Hamiltonian in their original proposal [19], a fact that was emphasized and reinterpreted in [32]. The instability of BD theories with $\omega_{BD} < 0$ was further emphasized in [33], but as we argued above, the non-positive kinetic term
of the scalar field is not an issue in the special case $\omega_{BD} = -3/2$ and $V = V(|\phi|)$ due to WI. The observational lower limit $\omega_{BD} \geq 40000$ reported in [20] implies that the BD field $\Phi$ is very nearly a constant, essentially reducing the theory to GR. But, again, this conclusion applies to the BD theory, where $\mathcal{L}_M$ is independent of the scalar field, and in addition (perhaps not less important and relevant to astrophysics and cosmology) this tight limit has been obtained in our solar system and clearly does not readily apply on, e.g., galactic scales.

4. The FRW spacetime In WIST representation

In the present section we cast the FRW action in the form of equation (11) in a particular conformal frame where spacetime is static. The conventional initial curvature singularity at the Big Bang is replaced in this alternative description by the vanishing of $\phi$, i.e. of Planck and active gravitational masses, at $t = 0$. The redshifting Universe is then a reflection not of space expansion, but rather of temporal evolution of masses, i.e. the redshifting Universe is a manifestation of shrinking Planck length and Compton wavelengths. The latter is a result of the fact that the time coordinate is now the conformal time, $\eta$, rather than the cosmic time, $t$, as will be discussed below.

The FRW action

$$\mathcal{I}_{FRW} = \int \left[ -a'^2 + K a^2 - \Lambda a^4/3 + a^4 \mathcal{L}_M(a) \right] \sqrt{-\gamma} d^4x,$$

is obtained from equation (1) in units where $3/(8\pi G) \equiv 1$ and assuming that the metric is $g_{\mu\nu} = a^2 \gamma_{\mu\nu}$, where the two metrics $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ are conformally related, and $\gamma_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1)$ is the metric in the comoving frame, $\gamma \equiv \det(\gamma_{\mu\nu})$, and after the term proportional to the corresponding curvature scalar $a^4 R = 6a^4(a''/a + K)/a^2$ is integrated by parts. Here, $K$ is the spatial curvature parameter, $a(\eta)$ is the purely time-dependent scale factor, a prime denotes derivatives with respect to $\eta$, where $d\eta \equiv dt/a$ and $d^4x = d\eta dr d\theta d\phi$.

Since null geodesics are blind to conformal metric transformations we expect light to still be redshifted in the WIST description with the spatially-static metric $\gamma_{\mu\nu}$ now replacing the standard expanding space metric $g_{\mu\nu}$. This is explained by the fact that $-d\eta^2 = -\frac{dr^2}{a^2}$, and so the the lapse function, $\gamma_{00}$, is $a^{-2}$ if the metric $\gamma_{\mu\nu}$ is presented in cosmic time, $t$, coordinates, rather than conformal $\eta$. Incoming photon wavelengths are not stretched by space expansion in this description (space is static) but rather by the temporaly-evolving gravitational potential, $\gamma_{00}(t)$, which is $\propto a^{-2}(\eta)$, which as we see below is $\propto \phi^{-2}(\eta)$. In other words, cosmological redshift is (indirectly) due to the growing (decreasing) Planck mass (length). A more direct way to explain redshift is by the fact that in the comoving frame, described by $\gamma_{\mu\nu}$, inertial masses $M_0$ are transformed to $M = a(\eta) M_0$, i.e. the Rydberg constant monotonically increases with time.

We thus see that the ratio between either the Planck or active gravitational mass to the inertial mass is time-independent on cosmological scales where the distribution of
matter is assumed to be homogeneous and isotropic and Eq. (20) applies. However, we stress that this is not the case in general spacetimes and matter distributions since the inertial mass is independent of the scalar field. It is only due to the specific symmetries of the cosmological model making it amenable to a particular combination of Weyl (e.g. $g_{\mu\nu} \rightarrow \gamma_{\mu\nu} = g_{\mu\nu}/a^2$) and coordinate (i.e. $dt \rightarrow d\eta = dt/a$) transformations that, e.g. the ratio of inertial-to-Planck mass is fixed at the background level. However, in the presence of small perturbations of the scalar field the Planck and active gravitational mass are perturbed while inertial masses are not. This difference is critical for certain applications. For example, it guarantees that CDM perturbations can be ascribed in the present framework to scalar field perturbations (essentially to $\lesssim 10^{-3} - 10^{-4}$ fractional variations of $G$) on galactic and galaxy cluster scales, [1], [34]. Otherwise they could be gauged away and DM phenomena cannot be explained this way.

It can be readily verified that the Euler-Lagrange equation for the scale factor $a(\eta)$ that extremizes the action $I_{FRW}$ is indeed the Friedmann equation $H^2 + K = a^2 \rho_M + \Lambda a^2 + \text{const.}/a^2$, where $H \equiv a'/a$ is the conformal Hubble function, and it is assumed that $\mathcal{L}_M(a)$ is a power-law in $a$, much like $\rho_M$ is a power-law in standard cosmology. Here, $\mathcal{L}_M(a)$ accounts for all sources (e.g. dust, radiation, etc.) other than spatial curvature and cosmological constant (which are characterized by effective equations of state -1/3 and -1, respectively). Completing the derivation requires that the integration constant is related to the energy density of radiation $\rho_r = \text{const.}/a^4$. Since $\rho_M > 0$ it is clear that the kinetic and potential terms in equation (20) have the ‘wrong’ relative sign (unless $K > 0$ or $\Lambda < 0$), yet the GR-based FRW model is not considered to be ‘plagued with ghosts’, or ‘disastrous instabilities’ at the classical level. On the contrary, FRW is the backbone of the remarkably successful standard cosmological model.

Re-defining the scale factor $\phi \equiv a$, the FRW action is reformulated as a WIST theory with a non-positive kinetic term, defined on a static background

$$I_{FRW} = \int \left( \frac{1}{6} R \phi^2 - \phi'^2 - \lambda \phi^4 + \tilde{\mathcal{L}}_M(\phi) \right) \sqrt{-\gamma} d^4x. \quad (21)$$

Here, $\tilde{\mathcal{L}}_M = a^4 \mathcal{L}_M$, $R = 6K$ and $\lambda \equiv \Lambda/3$. The single degree of freedom of standard FRW spacetime, $a(\eta)$, is here replaced by the scalar field $\phi(\eta)$. Clearly, renaming the scale factor, $a \rightarrow \phi$, in going from the general relativistic equation (20) to the WIST form, equation (21), does not introduce any new dynamics, in particular no unacceptable new instabilities that are not already present in the FRW spacetime arise due to this field redefinition. The relative ‘wrong’ sign between the kinetic and potential term is simply manifested in that the integration of the Friedmann equation results in $a(\eta)$ [or equivalently $\phi(\eta)$] that evolves monotonically with $\eta$ rather than having oscillatory behavior.

As equation (20) and equation (21) are equivalent any perturbation is either $a(\eta)$ of equation (20) or $\phi(\eta)$ of equation (21) will have precisely the same dynamics. Exactly as in the standard cosmological model $\delta a$ is never considered as a dynamical degree of freedom, so should be the case with $\delta \phi$. This is so since $g_{\mu\nu} = a^2 \gamma_{\mu\nu}$, and therefore any
perturbation in $a$ can be absorbed in $\delta \gamma_{\mu
u}$, a perturbation of $\gamma_{\mu
u}$. Considering scalar metric perturbations only—these are the Newtonian potentials. Therefore, invoking WI implies that any perturbation in $G$ induces scalar metric perturbations \cite{2}, and vice versa any excessive gravitational potential could be explained by corresponding perturbations in $G$ \cite{1}.

As we just saw, local isotropic scaling has sufficient freedom to allow for reformulation of the (background) FRW model on a static background. However, this is not generally the case with Bianchi Universes. The latter are homogeneous but not isotropic. As such, they may have up to three different expansion rates along the three principal axes for which a corresponding number of functional degrees of freedom are required. For that purpose, the more general anisotropic local rescaling freedom described in section 2 is required. In addition, at the perturbation level, vector and tensor perturbation modes cannot be accounted for by a single function $\phi$, and for that purpose the more general theory described in section 2 is required as well, as was already mentioned above.

5. Global Symmetry

As reviewed in section 3 one of the often-made claims about WIST is that the procedure of ‘conformalizing’ GR does not really result in modified physics. Specifically, it is argued, the conformal frame can always be so chosen so as to recover GR from equation (11) by setting $\phi$ to a constant. Irrespective of the validity of this claim in light of the discussion towards the end of section 2, it is clear that if a global symmetry is associated with the tensor field $\Phi^\alpha_{\mu\nu}$ then the presence of a global conserved charge definitely modifies the theory. Specifically, the field $\Phi^\alpha_{\mu\nu}$ itself can be generalized to carry the indices of an internal (non-coordinate) symmetry, i.e. $\Phi^\alpha_{\mu\nu} \to \Phi^\alpha_{\mu\nu} = (\Phi^K_I)^{\alpha}_{\mu\nu}$ where boldface letters indicate that the tensor field carries internal indices (capital Latin letters) of a (typically unitary) group. The metric on (internal) group space is denoted $G_{IJ}$, such that e.g., $\Phi^\alpha_{\mu\nu} \to \Phi^\alpha_{\mu\nu} = (\Phi^K_I)^{\alpha}_{\mu\nu}$ where boldface letters indicate that the tensor field carries internal indices (capital Latin letters) of a (typically unitary) group. The metric on (internal) group space is denoted $G_{IJ}$, such that e.g., $\Phi^\alpha_{\mu\nu} \to \Phi^\alpha_{\mu\nu} = (\Phi^K_I)^{\alpha}_{\mu\nu}$ where boldface letters indicate that the tensor field carries internal indices (capital Latin letters) of a (typically unitary) group. The metric on (internal) group space is denoted $G_{IJ}$, such that e.g., $\Phi^\alpha_{\mu\nu} \to \Phi^\alpha_{\mu\nu} = (\Phi^K_I)^{\alpha}_{\mu\nu}$ where boldface letters indicate that the tensor field carries internal indices (capital Latin letters) of a (typically unitary) group. The metric on (internal) group space is denoted $G_{IJ}$, such that e.g., $\Phi^\alpha_{\mu\nu} \to \Phi^\alpha_{\mu\nu} = (\Phi^K_I)^{\alpha}_{\mu\nu}$ where boldface letters indicate that the tensor field carries internal indices (capital Latin letters) of a (typically unitary) group. The metric on (internal) group space is denoted $G_{IJ}$, such that e.g., $\Phi^\alpha_{\mu\nu} \to \Phi^\alpha_{\mu\nu} = (\Phi^K_I)^{\alpha}_{\mu\nu}$ where boldface letters indicate that the tensor field carries internal indices (capital Latin letters) of a (typically unitary) group. The metric on (internal) group space is denoted $G_{IJ}$, such that e.g., $\Phi^\alpha_{\mu\nu} \to \Phi^\alpha_{\mu\nu} = (\Phi^K_I)^{\alpha}_{\mu\nu}$ where boldface letters indicate that the tensor field carries internal indices (capital Latin letters) of a (typically unitary) group. The metric on (internal) group space is denoted $G_{IJ}$, such that e.g., $\Phi^\alpha_{\mu\nu} \to \Phi^\alpha_{\mu\nu} = (\Phi^K_I)^{\alpha}_{\mu\nu}$ where boldface letters indicate that the tensor field carries internal indices (capital Latin letters) of a (typically unitary) group. The metric on (internal) group space is denoted $G_{IJ}$, such that e.g., $\Phi^\alpha_{\mu\nu} \to \Phi^\alpha_{\mu\nu} = (\Phi^K_I)^{\alpha}_{\mu\nu}$ where boldface letters indicate that the tensor field carries internal indices (capital Latin letters) of a (typically unitary) group. The metric on (internal) group space is denoted $G_{IJ}$, such that e.g., $\Phi^\alpha_{\mu\nu} \to \Phi^\alpha_{\mu\nu} = (\Phi^K_I)^{\alpha}_{\mu\nu}$ where boldface letters indicate that the...
not to any other fields. There is a conserved global charge that is associated with this
global symmetry that physically alters the field equations of gravitation. The presence
of this charge modifies the kinetic term associated with the field, as is demonstrated in
e.g., the U(2) case in Appendix B. An example of this within a WIST-based bouncing
cosmological model is described in [34].

6. Summary

Symmetry plays a central role in our current understanding of the fundamental
interactions. The SM of particle physics is based on a local SU(3)×SU(2)×U(1) gauge
symmetry, and GR is diffeomorphism-invariant. In the present work, the idea that
GR accommodates a broader symmetry, anisotropic local scale invariance (of which
WI is just a special case), is explored first. The underlying idea is that not only that
the gravitational interaction (and by induction also all the other interactions which take
place in the spacetime arena) does not depend on the reference frame of the observer (as
is manifested by diffeomorphism invariance of, e.g., the Einstein and geodesic equations)
it also does not depend on (a priori arbitrary) choices of units, i.e. yardsticks, e.g. the
Planck length scale, $l_P$. The latter is the natural yardstick in vacuum. In the non-
vacuum case the natural meter stick would be the dynamical gravitational time scale,
$(G \rho_M)^{-1/2} \propto l_P^{-1} \rho_M^{-1/2}$ where $G$ is Newton constant and $\rho_M$ is the energy density of
matter. The standard units choice adopted in GR is that both $G$ and particle masses
(active gravitational masses included) are fixed constants.

Newton constant, $G$, has been measured in our solar system at a reasonable
precision, and is purportedly also deduced (at the few percent precision) outside of
our solar system from Big Bang nucleosynthesis (BBN) via observational inference of
light element abundances, etc. However, all these estimates of $G$ are based on the
premise that all other universal constants are truly constants, as is clearly manifested
in the Brans-Dicke program. In addition, Newtonian gravity fails to account for a
few observed phenomena on galactic scales and beyond in spite of the fact that the
gravitational pull is weak in these systems and Newtonian gravity is expected, at least
naively, to provide a reasonably accurate description. The standard solution to this
problem involves an assumption about the existence of a non-luminous DM substance,
presumably exotic electrically neutral particles. Another way, albeit less popular, to
tackle the problem has involved modifying the Newtonian dynamics on sufficiently large
scales. Both remedies assume that $G$ is a fixed universal constant across space and
time; in the absence of compelling direct evidence for the universality of $G$ this has to
be assumed.

However, positing WI as an additional symmetry of (what is usually considered GR)
opens up new possibilities. Promoting $G$ to a tensor (section 2) or scalar (section 3)
scale-deformation field necessitates that active gravitational masses are regulated by the
same fields as well. Clearly, it is assumed here that (anisotropic or isotropic) local scale
invariance applies only within the domain of validity of GR. This is of course not the case
on microscopic scales. While GR is assumed here to be locally scale-invariant, the SM of particle physics is assumed to be as is, i.e. non-scale-invariant, with no modifications. The effective energy and pressure contained in spatio-temporal variations of the scalar field, i.e. $G$ and active gravitational masses, then provide additional (non-particulate) source for spacetime curvature, thereby accounting for, e.g. ‘DM phenomena’ with no recourse to DM. In the present work we explored a few key aspects of the underlying locally scale-invariant generalizations of GR. In addition to local scale invariance, we equip the ‘scaling’ fields ($\phi$ or $\Phi_{\mu\nu}$) with internal symmetry and generally non-vanishing global charge.

It should be emphasized that whereas the case isotropic local rescaling has long been explored, its generalization, i.e., the anisotropic case that is discussed in section 2, has not been considered in the literature in full generality (except for the special case of Minkowski background where it is known as the ‘vierbein formalism’) to the best of our knowledge.

While GR is modified in this framework, the SM is not. We stress that unlike in other frameworks that ‘conformalize’ all the four fundamental interactions and so any ‘new physics’ can be entirely gauged away by means of an appropriate Weyl transformation (in the absence of new global charges), this cannot be done in the framework adopted in the present work. It should be also emphasized that even if all the four fundamental interactions are ‘conformalized’ the presence of internal symmetry and global charge do modify the field equations, i.e. the dynamics, and consequently give rise to ‘new physics’.

Whereas GR is recovered from the locally anisotropic scale-invariant or WIST theory in a given conformal frame (in case that $G$ is set to a constant), the additional functional degrees of freedom (obtained by promoting $m_p$ to a scalar $\phi$ or tensor $\Phi_{\mu\nu}$ field) provides ample freedom that allows doing away with clustered DM as mentioned above. A notable property of $\phi$ is that it is formally a ‘ghost’ field, i.e. its kinetic and potential terms have the ‘wrong’ relative sign. However, this property by itself does not disqualify the theory because the problem with ghost-afflicted theories is that the field configuration dynamically runs away from its ground state, or equilibrium, by ever lowering the kinetic energy while increasing the potential term. However, as was explicitly shown in section 3, and as is equally clear from the underlying WI, the spacetime-dependent $\phi$ in the case of WIST, or $\Phi_{\mu\nu}$ in the anisotropic case, is non-dynamical; it is simply not required to obey any specific Euler-Lagrange equation. Consequently, physical systems described by locally scale-invariant theories of gravity are not doomed to runaway from stable field configurations. We also illustrated this conclusion with the redshifting Universe in section 4. The FRW spacetime can be recast in the canonical WIST form on a static background. Cosmological redshift is then explained not by space expansion but rather by contraction of our fundamental yardsticks, e.g. the Rydberg ‘constant’, which are (directly and indirectly) regulated by $\phi$. Whereas $\phi$ can be locally rescaled along with a corresponding rescaling of $g_{\mu\nu}$, adopting a specific conformal frame in which the metric is static implies that $\phi$ satisfies
exactly the same Friedmann equation which is normally satisfied by $a$, the scale factor that describes space expansion in the standard cosmological model. The conformal Hubble function $H = a'/a$ is now replaced by $\phi'/\phi$. In the WIST version of the FRW action the negative kinetic term has been actually increasing ($\phi'/\phi$ monotonically decreases) asymptotically towards zero over the entire cosmic history between the ‘Big Bang’ and the present time. This is a counter-example to often-raised concerns about instabilities associated with WIST. It is true that on the face of it we could have found ourselves in a contracting Universe with an ever decreasing negative kinetic term of either $a$ or $\phi$ (in the GR and WIST formulations respectively) culminating in a ‘big crunch’, but even in this case the WIST dynamics is not ‘worse’ than that of GR. More general cosmological models, e.g. Bianchi-types models, that are described by non-isotropic expansion rates are in general described by three scale factors. This family of models can be easily described by the *anisotropic* locally scale-invariant theory.

Appendix A: The Weak gravitational field limit as an inertial force

The classical trajectories followed by particles in GR are described by geodesics
\[ \ddot{x}^\mu + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 0, \quad (23) \]
where $\Gamma^\mu_{\rho\sigma}$ is the Christoffel symbol that is calculated from the metric field and its derivatives in the usual way, and $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$, with $\lambda$ an affine parameter. Equation (23) is subject to the constraint $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = c$ where $c$ equals -1 or 0 in case of massive or massless particles, respectively. In the presence of additional non-gravitational forces, $F^\mu$, equation (23) generalizes to
\[ \ddot{x}^\mu + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = \frac{F^\mu}{m}, \quad (24) \]
in the case of a massive particle of mass $m$. Now, the Christoffel symbol transforms inhomogeneously under Weyl transformations (equation 12)
\[ \Gamma^\mu_{\rho\sigma} \rightarrow \tilde{\Gamma}^\mu_{\rho\sigma} = \Gamma^\mu_{\rho\sigma} + \Omega^{-1}(\delta^\mu_\rho \Omega_\sigma + \delta^\mu_\sigma \Omega_\rho - g_{\rho\sigma} g^{\mu\delta} \Omega_\delta). \quad (25) \]
Therefore, in the case $F^\mu = 0$, equation (23) is rewritten in the new conformal frame as
\[ \ddot{x}^\mu + \tilde{\Gamma}^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = \frac{2\Omega_\sigma}{\Omega} \dot{x}^\sigma \dot{x}^\mu - c g^{\mu\nu} \Omega_\nu / \Omega, \quad (26) \]
or equivalently
\[ \ddot{x}^\mu + \tilde{\Gamma}^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = -\frac{2\phi_\sigma}{\phi} \dot{x}^\sigma \dot{x}^\mu + c g^{\mu\nu} \phi_\nu / \phi. \quad (27) \]
We see that what is considered as a pure gravitational force in one conformal frame is interpreted as not pure gravitational in other frames. A striking example is the case where spacetime is described by Minkowski metric in the new frame, and in cartesian coordinates then $\tilde{\Gamma}^\mu_{\rho\sigma}$ vanishes. In this particular case equation (27) describes a purely non-gravitational interaction. The particles (massive or massless) follow exactly the same geodesics (that describe e.g. gravitational clustering of massive particles or light
bending of massless photons) in the two conformal frames, but while in the first frame it is due to spacetime curvature in the other frame it is due to an inertial ‘force’, \[ F \equiv -2\dot{x}^{\mu}\dot{x}^{\nu}\phi_{\nu}/\phi + \epsilon \eta^{\mu\nu}\phi_{\nu}/\phi, \] where \( \eta^{\mu\nu} \) is Minkowski metric.

For example, the line element that describes a weakly perturbed Minkowski spacetime in Cartesian coordinates and in case that stress is negligible, and neglecting vector and tensor modes, is \[ ds^{2} = -(1 + 2\phi)dt^{2} + (1 - 2\phi)dx^{i}dx^{j}. \] Here \( \phi(x) \) is the gravitational potential. Applying a Weyl transformation \( \Omega = 1 + \phi \) the metric in the new conformal frame becomes purely Minkowski, and \( \tilde{\Gamma}_{\rho\sigma}^{\mu} \) vanishes in Cartesian coordinates. However, the r.h.s of equation (27) does not vanish now. The entire kinematics of test particles is now due to the fact that \( \phi_{0} \rightarrow \tilde{\phi} \approx \phi_{0}[1 - \phi(x)] \). In other words, light bending and gravitational clustering are not due to spacetime curvature (which is flat in this case) but rather due to the spacetime dependence of \( G \) and active gravitational masses. In the presence of vector and/or tensor modes the single functional degree of freedom, \( \phi \), added by the WIST theory is insufficient for entirely flattening the spacetime geometry and the more general locally scale invariant theory that is described in section 2 is required.

The tensor field \( \Phi_{\alpha\beta} \) can be used reformulate GR on a Minkowski spacetime, in which case it reduces to the vierbein field. The generalization of equation (27) in that case is straightforward.

Appendix B: The Case of U(2) Global Symmetry

As an example for possibilities opened up by introducing internal symmetries associated with the gravitational interaction, we consider the case that the scalar field \( \phi \) is a U(2) multiplet. Elements of this group are represented by

\[
\phi = \rho \begin{pmatrix}
\cos(\theta)e^{i\varphi} & \sin(\theta)e^{ix} \\
-\sin(\theta)e^{-ix} & \cos(\theta)e^{-i\varphi}
\end{pmatrix},
\]

where \( \rho \) is the modulus of the field and \( \theta, \varphi \) and \( \chi \) are three phases. The line element on this group manifold is \( ds_{\phi}^{2} = G_{IJ}d\phi^{I}d\phi^{J} \), where contracted capital Latin indices are summed over. Since this is a unitary group then, \( ds_{\phi}^{2} = \frac{1}{2}Tr(d\phi^{I}d\phi) = -\frac{1}{2}Tr(\phi^{-1}d\phi)^{2} = d\rho^{2} + \rho^{2}[\cos^{2}(\theta)d\varphi^{2} + \sin^{2}(\theta)d\chi^{2} + d\theta^{2}] \). The lagrangian of the kinetic term, \( L_{kin} = G_{IJ}\phi^{I}\phi^{J} \) (which is a straightforward generalization of the kinetic term in Equation 11) is therefore \( L_{kin} = -\rho^{2} - \rho^{2}[\cos^{2}(\theta)d\varphi^{2} + \sin^{2}(\theta)d\chi^{2} + d\theta^{2}] \). The Euler-Lagrange equations associated with \( \chi \) and \( \varphi \) that are obtained from variations of \( L_{kin} \) can each be integrated once to give

\[
\dot{\chi} = \frac{c_{\chi}}{\rho^{2}\sin^{2}\theta}, \quad \dot{\varphi} = \frac{c_{\varphi}}{\rho^{2}\cos^{2}\theta},
\]

where \( c_{\chi} \) and \( c_{\varphi} \) are integration constants. Employing these in the field equation for \( \theta \) and integrating once we obtain

\[
\rho^{4}\dot{\theta}^{2} = -\frac{c_{\chi}^{2}}{\sin^{2}\theta} - \frac{c_{\varphi}^{2}}{\cos^{2}\theta} + c_{\theta}^{2},
\]
where \( c_\theta \) is another integration constant. Employing all these in the kinetic term

\[
\phi_\mu \phi^\mu = -\dot{\rho}^2 - \rho^2 [\cos^2(\theta) \dot{\phi}^2 + \sin^2(\theta) \dot{\chi}^2 + \dot{\theta}^2]
\]

we obtain \(-\dot{\rho}^2 - \frac{c_\theta^2}{\rho}\), and both integration constants \( c_\chi \) and \( c_\phi \) vanish from our final expression. All this amounts to replacing \( \phi \) with \( \rho \) everywhere in equation (11) in addition to introducing a negative contribution to the potential, \(-\frac{c_\theta^2}{\rho}\).

In the cosmological context this new term competes with the radiation term at sufficiently small \( \rho \). This term, obtained due to the \( U(2) \) symmetry of the scalar field (a symmetry which is not recognized by GR) rather than having to recourse to exotic forms of matter with negative energy density, is responsible to a bounce, where the cosmological expansion rate momentarily halts. As expected, this is exactly the result obtained in [34] in case of \( U(1) \) symmetry. The constant \( c_\theta \) is simply a global charge associated with the \( U(1) \) subgroup, which is the relevant symmetry in a ‘central’ potential, i.e. a potential which is independent on the phase, and depends only on \( \rho \). In cosmology the term \( \propto c_\theta \) serves as an effective stiff matter with a negative energy density, thereby replacing the initial Big Bang singularity with a Big Bounce [34].

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