FIXED POINTS AND HOMOLOGY OF SUPERELLIPTIC JACOBIANS

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Abstract. Let \( \eta : C_{f,N} \to \mathbb{P}^1 \) be a cyclic cover of \( \mathbb{P}^1 \) of degree \( N \) which is totally and tamely ramified for all the ramification points. We determine the group of fixed points of the cyclic group \( \mu_N \cong \mathbb{Z}/N\mathbb{Z} \) acting on the Jacobian \( J_N := \text{Jac}(C_{f,N}) \). For each prime \( \ell \) distinct from the characteristic of the base field, the Tate module \( T_\ell J_N \) is shown to be a free module over the ring \( \mathbb{Z}[T]/(\sum_{i=0}^{N-1} T^i) \). We also calculate the degree of the induced polarization on the new part \( J_{N_{\text{new}}} \) of the Jacobian.

1. Introduction

Throughout this paper, \( K \) is an algebraically closed field except when specified otherwise. The characteristic of \( K \) is denoted by \( \text{Char}(K) \). If \( A \) is an abelian variety over \( K \), we write \( A^\vee \) for the dual abelian variety of \( A \), and \( \text{End}(A) \) for the endomorphism ring of \( A \). The endomorphism algebra \( \text{End}_0(A) := \text{End}(A) \otimes \mathbb{Q} \) is a finite-dimensional semisimple algebra over \( \mathbb{Q} \). Given an abelian group \( G \) (or a commutative group scheme \( G \) over \( K \)), \( G[m] \) denotes the kernel of the homomorphism \( G \to \mathbb{Z}/m\mathbb{Z} \). We often identify a finite étale group scheme \( G/K \) with \( G(K) \).

The cardinality of a finite set \( S \) is denoted by \( |S| \). In particular, for any prime \( \ell \neq \text{Char}(K) \), one has \( |A[\ell]| = \ell^{2 \text{dim} A} \). The letters \( p \) and \( \ell \) always denote primes in \( \mathbb{N} \).

Fix an integer \( N > 1 \) coprime to \( \text{Char}(K) \). Then \( T^N - 1 \in K[T] \) is separable over \( K \). Let \( \xi_N \in K \) be a primitive \( N \)-th root of unity in \( K \), and \( \mu_N := \langle \xi_N \rangle \), the group of \( N \)-th root of unity in \( K \). Suppose that \( f(X) = \prod_{i=1}^{n} (X - \alpha_i)^{e_i} \in K[X] \) is a monic polynomial with

\[
\gcd(\deg(f), N) = 1, \quad \gcd(e_i, N) = 1, \quad \forall 1 \leq i \leq n.
\]

For example, if \( N \) is even, then all \( e_i \) must be odd, and hence \( n \) must be odd as well to ensure that \( \gcd(\deg(f), N) = 1 \).

Let \( C_{f,N} \) be the smooth projective curve defined by the affine equation \( Y^N = f(X) \), and \( J_N := \text{Jac}(C_{f,N}) \) be the Jacobian variety of \( C_{f,N} \). The map

\[
\eta : C_{f,N} \to \mathbb{P}^1, \quad (X, Y) \mapsto X
\]

realizes \( C_{f,N} \) as a cyclic cover of degree \( N \) of the projective line \( \mathbb{P}^1 \). There is a canonical isomorphism \( \rho_C : \mu_N \xrightarrow{\cong} \text{Aut}(\eta) \subseteq \text{Aut}_K(C_{f,N}) \), given by

\[
\rho_C(\xi) : C_{f,N} \to C_{f,N}, \quad (X, Y) \mapsto (X, \xi Y), \quad \forall \xi \in \mu_N.
\]

We denote \( \rho_C(\xi_N) \) by \( \delta_N \).

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The Albanese functoriality induces an action $\rho_J : \mu_N \to \text{Aut}(J_N)$ of $\mu_N$ on the Jacobian $J_N$. By an abuse of notation, we still write $\delta_N$ for the map $\rho_J(\xi_N) : J_N \to J_N$ induced from $\delta_N : C_f,N \to C_{f,N}$. For each $D \in \mathbb{N}$, let $\Phi_D(T) \in \mathbb{Z}[T]$ be the $D$-th cyclotomic polynomial, which is a monic irreducible polynomial of degree $\varphi(D)$. It will be shown in Subsection 5.3 (cf. also [21] Lemma 4.8) in the case $N = p'$ is a prime power and $\text{Char}(K) = 0$ that the minimal polynomial over $\mathbb{Z}$ of $\delta_N \in \text{End}(J_N)$ is

\begin{equation}
P_N(T) := \frac{T^N - 1}{T - 1} = \prod_{D|N,D > 1} \Phi_D(T) = \sum_{i=0}^{N-1} T^i.
\end{equation}

So there is an embedding

\begin{equation}
\iota : \mathbb{Z}[T]/(P_N(T)) \hookrightarrow \text{End}(J_N), \quad T \mapsto \delta_N.
\end{equation}

Hence for each prime $\ell \neq \text{Char}(K)$, the Tate-module $T_\ell J_N := \varprojlim_{i \geq 1} J_N[\ell^i]$ is naturally a $\mathbb{Z}_\ell[T]/(P_N(T))$-module.

**Theorem 1.1** (Main Theorem). For any prime $\ell \neq \text{Char}(K)$, $T_\ell J_N$ is a free $\mathbb{Z}_\ell[T]/(P_N(T))$-module of rank $n - 1$. In particular, if $K = \mathbb{C}$, the first homology group $H_1(C_{f,N}(\mathbb{C}), \mathbb{Z})$ is a projective $\mathbb{Z}[T]/(P_N(T))$-module of rank $n - 1$.

Let $\zeta_N \in \bar{\mathbb{Q}}$ be a primitive $N$-th root of unity in a fixed algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. For each $D \mid N$, we set $\zeta_N^D := \zeta_N^{N/D}$. The embedding $\iota$ induces an embedding

\begin{equation}
\iota : \mathbb{Q}[T]/(P_N(T)) \cong \prod_{D|N,D > 1} \mathbb{Q}(\zeta_N^D) \hookrightarrow \text{End}^0(J_N).
\end{equation}

**Corollary 1.2.** We have $\text{End}(J_N) \cap (\mathbb{Q}[T]/(P_N(T))) = \mathbb{Z}[T]/(P_N(T))$, where the intersection is taken within $\text{End}^0(J_N)$. In other words, the embedding $\iota : \mathbb{Z}[T]/(P_N(T)) \hookrightarrow \text{End}(J_N)$ is optimal.

**Theorem 1.3.** Given any $G(T) = \prod_{i=1}^t \Phi_{D_i}(T) \in \mathbb{Z}[T]$ such that $G(T) \mid P_N(T)$, the kernel of $G(\delta_N) : J_N \to J_N$ is an abelian subvariety of $J_N$ of dimension $(n - 1)\deg G(T)/2$. Moreover, $\ker G(\delta_N) = \sum_{i=1}^t \ker \Phi_{D_i}(\delta_N)$. Let $J_N^{\text{new}} = \ker \Phi_N(\delta_N)$, then $J_N^{\text{new}}$ is isomorphic to the dual $(J_N^{\text{new}})^\vee$. If $q = p'$ is a prime power with $p \neq 2$, then there exists a principal polarization $\lambda_q^{\text{new}} : J_q^{\text{new}} \to (J_q^{\text{new}})^\vee$.

**Remark 1.4.** In this remark, the ground field $K$ is not necessarily assumed to be algebraically closed. Let $\bar{K}$ be its algebraic closure. Since $J_N^{\text{new}} = \ker \Phi_N(\delta_N)$, there exists an embedding $\mathbb{Z}[\zeta_N] \hookrightarrow \text{End}(J_N^{\text{new}})$ given by $\zeta_N \mapsto \delta_N |_{J_N^{\text{new}}}$. Suppose that $\text{Char}(K) = 0$, $q := N = p'$ is a prime power, and $f(x)$ has multiple roots. In a series of papers [20, 21, 22, 23], Yuri G. Zarhin showed that $\text{End}_K(J_q^{\text{new}}) = \mathbb{Z}[\zeta_q]$ assuming that $\deg f(x) \geq 4$ and $f(x)$ is irreducible over $K$ with “large” Galois group (For example, $\text{Gal}(f)$ is either the full symmetric group $S_n$ or the alternating group $A_n$ when $\deg f(x) \geq 5$, or $\text{Gal}(f) = S_4$ when $\deg f(x) = 4$). When $K = \mathbb{C}$ and $\deg f(x) = 3$, the endomorphism algebra $\text{End}^0(J_q^{\text{new}})$ has been classified. In particular, if $p > 7$, then $\text{End}^0(J_q^{\text{new}})$ is either $\mathbb{Q}(\zeta_q)$, a quadratic field extension of $\mathbb{Q}((\zeta_q))$, or $\mathbb{Q}(\zeta_q) \oplus \mathbb{Q}((\zeta_q))$. The generic case was treated in [22] by Zarhin and the classification was given in [17] jointly by the second and third named authors. Now suppose $K \subseteq \mathbb{C}$, $\deg f(x) \geq 3$, and $\text{End}_K^0(J_q^{\text{new}}) = \mathbb{Q}((\zeta_q))$. With some further mild conditions on $q$, the special Mumford-Tate group of $J_q^{\text{new}}$ has been determined.
in another series of papers [16], [18], [19] jointly by Zarhin and the second named author.

The paper is organized as follows. Section 2 studies the kernel of endomorphisms of abelian varieties. The theorems above and their corollaries are proved in Section 3, where we study the superelliptic Jacobian $J_N$ and apply the results obtained in Section 2. Section 4 contains some arithmetic results that are used in the previous sections.

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2. Decomposition of Abelian varieties

Throughout this section, $F(T)$ and $G(T)$ denote polynomials in $\mathbb{Z}[T]$. Let $A$ be an abelian variety of positive dimension over $K$. The minimal polynomial of an endomorphism of $A$ is monic over $\mathbb{Z}$ [10, Theorem 19.4]. If $\phi \in \text{End}(A)$ has minimal polynomial $P(T) \in \mathbb{Z}[T]$, then there is an embedding $\mathbb{Z}[T]/(P(T)) \rightarrow \text{End}(A)$ with $T \mapsto \phi$. Given $F(T) | P(T)$, the kernel of $\beta := F(\phi)$ is a group scheme over $K$. In this section, we give a criterion to determine when $\ker \beta$ is an abelian subvariety of $A$. This question turns out to be closely related to the torsion subgroup of $A(K)$.

Suppose $R$ is a commutative ring. For any $F(T) \in \mathbb{Z}[T]$, we write $\overline{F}(T) := F(T) \otimes 1 \in \mathbb{Z}[T] \otimes_\mathbb{Z} R \simeq \mathbb{R}[T]$. For example, if $R = \mathbb{Z}/m\mathbb{Z}$, then $\overline{F}(T)$ is just $F(T)$ modulo $m$. For each $m \in \mathbb{Z}$ coprime to $\text{Char}(K)$, the $m$-torsion group $A[m] \subset A(K)$ is naturally a $\mathbb{Z}/m\mathbb{Z}/T]/(P(T))$-module.

2.1. Let $F(T), G(T) \in \mathbb{Z}[T]$ be two monic polynomials with gcd($F(T), G(T)$) = 1. The resultant ring $R := \mathbb{Z}[T]/(F(T))$ is a free $\mathbb{Z}$-module of rank $\deg F(T)$. By an abuse of notation, we still write $G(T)$ for the canonical image of $G(T)$ in $R$. The resultant of $F(T)$ and $G(T)$ is defined to be (See [2, Section IV.6.6])

$$\text{Res}(F(T), G(T)) = N_{R/\mathbb{Z}}(G(T)) = \prod_{F(x) = 0, G(y) = 0} (x - y)$$

where $N_{R/\mathbb{Z}} : R \rightarrow \mathbb{Z}$ is the norm map, and $x, y \in \overline{\mathbb{Q}}$ are roots of $F(T)$ and $G(T)$ respectively. Since $F(T)$ and $G(T)$ are coprime, $\text{Res}(F(T), G(T)) \neq 0$. There exists $a(T), b(T) \in \mathbb{Z}[T]$ such that

$$a(T)F(T) + b(T)G(T) = \text{Res}(F(T), G(T)) \in \mathbb{Z}.$$  

Both $a(T)$ and $b(T)$ are uniquely determined if we further require that $\deg a(T) < \deg G(T)$ (or equivalently, $\deg b(T) < \deg F(T)$). The resultant may be calculated as the determinant of a matrix whose entries are coefficients of $F(T)$ and $G(T)$.

Let $\overline{F}(T) := F(T) \otimes 1 \in K[T] = \mathbb{Z}[T] \otimes_\mathbb{Z} K$, and define $\overline{G}(T)$ similarly. Then $\overline{F}(T)$ and $\overline{G}(T)$ share a common root in $K$ if and only if $\text{Res}(\overline{F}(T), \overline{G}(T))$ is divisible by $\text{Char}(K)$. We write $\text{Disc} F(T) \in \mathbb{Z}$ for the discriminant of $F(T)$ ([2] Section...
Then $\bar{F}(T)$ is separable if and only if $\text{Disc}(F(T))$ is coprime to $\text{Char}(K)$. Clearly,

$$\text{Res}(F(T), G(T)) = \text{Disc}(F(T)G(T)).$$

We refer to Subsection 3.1.1 for some further discussion of $\text{Res}(F(T), G(T))$.

**Lemma 2.2.** Let $P(T) = F(T)G(T) \in \mathbb{Z}[T]$ be the minimal polynomial of $\phi \in \text{End}(A)$, and $\beta := F(\phi), \gamma := G(\phi) \in \text{End}(A)$. Suppose that $\text{Res}(F(T), G(T))$ is coprime to $\text{Char}(K)$. Then both $\ker \beta$ and $\ker \gamma$ are reduced group schemes over $K$, and

$$\dim \ker \beta + \dim \ker \gamma = \dim A.$$

**Proof.** Let $\text{Lie}(\beta) : \text{Lie}(A) \to \text{Lie}(A)$ be the induced endomorphism of the Lie algebra of $A$. To show that $\ker \beta$ is reduced, it is enough to prove that

$$\dim \ker \beta = \text{dim}_K (\ker (\text{Lie}(\beta))) = \text{dim}_K (\ker (\text{Lie}(\beta))).$$

A priori, $\text{dim}_K (\ker (\text{Lie}(\beta))) \geq \text{dim}_K (\ker (\beta))$. Similarly for $\gamma$.

The subring $\mathbb{Z}[\phi] \subseteq \text{End}(A)$ generated by $\phi$ is isomorphic to $\mathbb{Z}[T]/(P(T))$. So $\text{Lie}(A)$ carries a natural $\mathbb{Z}[T]/(P(T)) \otimes \mathbb{Z}K$-module structure. Since $\text{Res}(F(T), G(T))$ is coprime to $\text{Char}(K)$, $F(T)$ and $G(T)$ share no common factors. By the Chinese Reminder Theorem,

$$(\mathbb{Z}[T]/(P(T))) \otimes \mathbb{Z}K = \mathbb{K}[T]/(\bar{F}(T)) \cong \mathbb{K}[T]/(\bar{F}(T)) \oplus \mathbb{K}[T]/(\bar{G}(T)).$$

Correspondingly, $\text{Lie}(A) = \text{Lie}(A)_F \oplus \text{Lie}(A)_G$. Here $\text{Lie}(A)_F = \ker(\text{Lie}(\beta))$, which is naturally equipped with a $\mathbb{K}[T]/(\bar{F}(T))$-module structure, and $\text{Lie}(A)_G = \ker(\text{Lie}(\gamma))$, which has a natural $\mathbb{K}[T]/(\bar{G}(T))$-module structure.

Necessarily, $F(T)$ and $G(T)$ are coprime over $\mathbb{Q}$. For simplicity, let $m := \text{Res}(F(T), G(T))$. We may choose $a(T), b(T) \in \mathbb{Z}[T]$ and such that (2.2) holds. Then $\ker \beta \cap \ker \gamma \subseteq A[m]$, a finite étale group scheme over $K$. Since $A(K)$ is divisible ([10, Application 6.2]),

$$A = \beta(A) + \gamma(A) \subseteq \ker \gamma + \ker \beta.$$

We have

$$\dim A = \dim \ker \beta + \dim \ker \gamma \leq \text{dim}_K (\ker (\text{Lie}(\beta))) + \text{dim}_K (\ker (\text{Lie}(\gamma))) = \text{dim}_K \text{Lie}(A) = \dim A.$$

It follows that $\dim \ker (\beta) = \dim_K \ker(\text{Lie}(\beta))$ and similarly for $\gamma$. \qed

**Corollary 2.3.** We keep the notation and assumptions of Lemma 2.2. Let $d := \dim \ker \beta$. For any prime $p \neq \text{Char}(K)$, $|\ker(\beta)[p]| \geq p^{2d}$, and $\ker \beta$ is connected if and only if the equality holds for any $p | \text{Res}(F(T), G(T))$.

**Proof.** Since $\ker \beta$ is reduced, its identity component $(\ker \beta)^\circ$ is an abelian subvariety of $A$, and $\ker \beta$ is an extension of $(\ker \beta)^\circ$ by a finite étale group scheme $\pi_0(\ker \beta)$ over $K$:

$$(2.3) \quad 0 \to (\ker \beta)^\circ \to \ker \beta \to \pi_0(\ker \beta) \to 0.$$

Because $(\ker \beta)^\circ(K)$ is divisible, it follows from the Snake Lemma [3 Exercise A.3.10] that there is an exact sequence

$$(2.4) \quad 0 \to (\ker \beta)^\circ[p] \to (\ker \beta)[p] \to \pi_0(\ker \beta)[p] \to 0$$

for any prime $p$. In particular, if $p \neq \text{Char}(K)$,

$$(2.5) \quad |(\ker \beta)[p]| = |(\ker \beta)^\circ[p]| \cdot |\pi_0(\ker \beta)[p]| \geq |(\ker \beta)^\circ[p]| = p^{2d}.$$
Recall that $\gamma(A) \subseteq (\ker \beta)^\circ$ and $\dim \gamma(A) = \dim A - \dim \ker \gamma = \dim (\ker \beta)^\circ$, so $(\ker \beta)^\circ = \gamma(A)$. Let $m := \operatorname{Res}(F(T), G(T))$, and $a(T), b(T) \in \mathbb{Z}[T]$ be polynomials such that (2.7) holds. For all $x \in \ker \beta$, we have

$$mx = a(\phi)x + b(\phi)\gamma x = b(\phi)\gamma x \in \gamma(A) = (\ker \beta)^\circ.$$ 

It follows that $\forall y \in \pi_0(\ker \beta)$, $my = 0$. Therefore, $\pi_0(\ker \beta)$ is trivial if and only if $\pi_0((\ker \beta)[p])$ is trivial for all $p \mid m$. By (2.4), this holds if and only if $|(\ker \beta)[p]| = p^{2d} = |(\ker \beta)^\circ[p]|$ for all $p \mid m$. \hfill \square

**Lemma 2.4.** Suppose that $\dim A = r \deg P(T)/2 \in \mathbb{N}$ for some $r \in \mathbb{N}$. Let $\ell$ be a prime distinct from $\operatorname{Char}(K)$. The following are equivalent:

1. $A[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})[T]/(\bar{P}(T))$-module of rank $r$.
2. $T_A \cong (\mathbb{Z}/\ell\mathbb{Z})[T]/(\bar{P}(T))$-module of rank $r$.

**Proof.** Since $A[\ell] \cong T_A \otimes_{\mathbb{Z}[\ell]} \mathbb{F}[\ell]$, clearly (2.4.ii) implies (2.4)i. To show that (2.4)i implies (2.4.ii), it is enough to show that $A[\ell]$ is a free $(\mathbb{Z}/\ell\mathbb{Z})[T]/(\bar{P}(T))$-module of rank $r$ for all $i \in \mathbb{N}$. Now fix $i$, and let $M := A[\ell^i]$ and $R := (\mathbb{Z}/\ell\mathbb{Z})[T]/(\bar{P}(T))$. The ideal $\mathfrak{a} = (\ell^i) \subseteq R$ is nilpotent. We have $M/\mathfrak{a}M = M/\ell^iM \cong A[\ell^i]$. By assumption, $M/\mathfrak{a}M$ is a free $(R/\mathfrak{a})$-module of rank $r$. It follows from Nakayama’s lemma [3 Corollary 4.8] that $M$ can be generated by $r$ elements. In other words, we have a surjective map $R^r \to M$. On the other hand,

$$|M| = |A[\ell^i]| = (\ell^i)^{2\dim A} = \ell^{ir} \deg P(T) = |R^r|.$$ 

Therefore, the map must be injective as well, and hence $M$ is free of rank $r$. \hfill \square

We refer to [3 Chapter 21] for the concept of Gorenstein rings.

**Lemma 2.5.** The Artinian ring $R = \mathbb{Z}[T]/(m, P(T))$ is Gorenstein for all positive integer $m > 1$.

**Proof.** This follows directly from [3 Corollary 21.19] since $\mathbb{Z}[T]$ is a regular ring and $m, P(T)$ form a regular sequence. \hfill \square

**Lemma 2.6.** Let $(R, m)$ be a local Artinian ring with residue field $k = R/m$, and $M$ a finitely generated $R$-module of length $l_R(M)$. The socle of $M$ is defined to be the submodule $M_0 := \{x \in M \mid yx = 0, \forall y \in m\}$, which is the sum of all simple submodules of $M$. Suppose $R$ is Gorenstein, then $l_R(M) = l_R(R) \dim_k(M_0)$, and the equality holds if and only if $M$ is a free $R$-module of rank $\dim_k M_0$.

**Proof.** For simplicity, we will write $l(M) := l_R(M)$ for the length of $M$ if the ring $R$ is clear from context. The socle of a local Gorenstein ring is simple (See [3 Proposition 21.5]), i.e., it has dimension 1 over the residue field. If $M$ is free, then $\dim_k M_0 = \text{rank}_R M = l(M)/l(R)$.

Let $\mathcal{M}$ be the category of all finitely generated $R$-modules. Since $R$ is Gorenstein, the functor $M \mapsto M^\vee := \operatorname{Hom}_R(M, R)$, $\forall M \in \mathcal{M}$ is a dualizing functor from $\mathcal{M}$ to itself. In other words,

- it is contravariant, $R$-linear and exact;
- $\forall M \in \mathcal{M}$, $(M^\vee)^\vee$ is canonically isomorphic to $M$.

In particular, the exactness implies that $l(M^\vee) = l(M)$, $\forall M \in \mathcal{M}$. We also note that $l(M_0) = \dim_k(M_0)$. For simplicity, let $r := \dim_k M_0$. 

\hfill 

By definition, $M_0$ is the maximal submodule of $M$ annihilated by $m$. Dualizing, we see that $M_0^\vee$ is the maximal quotient of $M^\vee$ annihilated by $m$. That is, $M_0^\vee \cong M^\vee/(mM^\vee)$. Therefore,
\[
\dim_k M^\vee/(mM^\vee) = \dim_k M_0^\vee = l(M_0^\vee) = l(M_0) = \dim_k(M_0) = r.
\]
By Nakayama’s lemma, $M^\vee$ can be generated by $r$ elements. In other words, we have an exact sequence
\[
0 \to \ker \theta \to R^r \xrightarrow{\theta} M^\vee \to 0.
\]
Therefore,
\[
(2.6) \quad l(M) = l(M^\vee) = l(R)r - l(\ker \theta) \leq l(R)r.
\]
If $l(M) = l(R)r$, then $l(\ker \theta) = 0$. Hence $\ker \theta = \{0\}$ and $M^\vee \cong R^r$. We conclude that $M$ is free as well since $M \cong (M^\vee)^\vee \cong \hom_R(R^r, R)$.

**Corollary 2.7.** Suppose that $\dim A = r \deg P(T)/2$. Let $\ell$ be a prime distinct from $\operatorname{Char}(k)$, and $\tilde{P}(T) = \prod_{l=1}^{s} h_i(T)^{t_i}$ be the factorization of $\tilde{P}(T) := P(T) \otimes 1 \in \mathbb{F}_l[T] = \mathbb{Z}/\mathbb{Z} \otimes \mathbb{F}_l$ into irreducible factors over $\mathbb{F}_l$. For each $1 \leq i \leq s$, let $W_{\ell,i} := \{x \in A[\ell] \mid h_i(\phi)x = 0\}$\footnote{A priori, $H(\phi)$ only make sense if $H(T) \in \mathbb{Z}[T]$. We may choose $H_i(\ell) \in \mathbb{Z}[T]$ such that its reduction mod $\ell$ is $h_i(T)$. Then for any $x \in A[\ell]$, the element $H_i(\phi)x$ does not depends on the choice of $H_i(T)$. By an abuse of notation, we will denote this element by $h_i(\phi)x$.} Then $A[\ell]$ is a free $\mathbb{F}_l[T]/(\tilde{P}(T))$-module of rank $r$ if and only if $\dim_{\mathbb{F}_l} W_{\ell,i} = r \deg h_i(T)$.

**Proof.** By the Chinese Reminder Theorem, we may decompose the Artinian ring $\mathbb{F}_l[T]/(\tilde{P}(T))$ into a direct sum of local Artinian rings:
\[
R := \mathbb{F}_l[T]/(\tilde{P}(T)) \cong \bigoplus_{i=1}^{s} \mathbb{F}_l[T]/(h_i(T)^{t_i}).
\]
Clearly, $m_i = (h_i(T))$ is the unique maximal ideal in $R_i := \mathbb{F}_l[T]/(h_i(T)^{t_i})$ and its residue field is $k_i := R_i/m_i \cong \mathbb{F}_l$ with $d_i := \deg h_i(T)$. Correspondingly, we have a direct sum decomposition
\[
M := A[\ell] = \bigoplus_{i=1}^{s} M_i,
\]
where each $M_i$ is an $R_i$-module. By definition, $W_{\ell,i}$ is the socle of $M_i$.

If $M$ is a free $R$-module of rank $r$, then each $M_i$ is a free $R_i$-module of rank $r$. By Lemma 2.6, $\dim_{\bigoplus_{i=1}^{s} \mathbb{F}_l[T]} W_{\ell,i} = [k_i : \mathbb{F}_l] \dim_{\mathbb{F}_l} W_{\ell,i} = d_i r$ for all $1 \leq i \leq s$.

Now suppose that $\dim_{\mathbb{F}_l} W_{\ell,i} = r$ for all $i$. Then by Lemma 2.6, $l_{R_i}(M_i) \leq rl_{R_i}(R_i)$. Hence $M_i \leq [R_i]^r = \ell^{r[d_i t_i]}$. On the other hand,
\[
\ell^{r \deg P(T)} = |A[\ell]| = \prod_{i=1}^{s} |M_i| \leq \ell^{r \sum_{i=1}^{s} d_i t_i} = \ell^{r \deg P(T)}.
\]
So we must have equality at all places. In particular, $l_{R_i}(M_i) = rl_{R_i}(R_i)$. By Lemma 2.7 again, $M_i$ is a free $R_i$-module of rank $r$ for all $1 \leq i \leq s$. Hence $M$ is a free $R$-module of rank $r$. \qed

**Lemma 2.8.** Let $\iota : \mathcal{O} \hookrightarrow \operatorname{End}(A)$ be an embedding of the ring of integers $\mathcal{O} \subset L$ of a number field $L$ into $\operatorname{End}(A)$. Then for any prime $\ell$ distinct from $\operatorname{Char}(K)$, $T_{\ell}A$ is a free $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$-module of rank $2 \dim A/[L : \mathbb{Q}]$. 
Proof. This is a well-known fact. By [10] Theorem 4, p. 180, we have \( \text{Tr}(\ell(a)); V_1(A) \) \( \in \mathbb{Q} \) for all \( a \in L \), where \( V_1(A) = T_1(A) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \). It follows that \( V_1(A) \) is a free \( L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \)-module. Since \( \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \) is a product of complete discrete valuation rings, the freeness of \( T_1(A) \) follows. \( \square \)

**Theorem 2.9.** Suppose that \( \dim A = r \deg P(T)/2 \), and \( \text{Disc}(P(T)) \) is coprime to \( \text{Char}(K) \). Consider the following statements.

\( (2.9a') \) \( \text{A}[\ell] \) is a free \( \mathbb{F}_\ell[T]/(P(T)) \)-module of rank \( r \) for any prime \( \ell \) | \( \text{Disc}(P(T)) \).

\( (2.9b) \) \( \ker G(\phi) \) is an abelian subvariety of \( A \) of dimension \( r \deg G(T)/2 \) for all \( G(T) | P(T) \).

\( (2.9c) \) \( \ker F(\phi) \) is an abelian subvariety of \( A \) of dimension \( r \deg F(T)/2 \) for all irreducible \( F(T) | P(T) \).

Then \( (2.9a') \Rightarrow (2.9b) \Rightarrow (2.9c) \). If \( (2.9b) \) holds, \( \ker G(\phi) = \sum \ker F(\phi) \), where the sum is over all irreducible factors \( F(T) \) of \( G(T) \). We further assume that \( \mathbb{Z}[T]/(F(T)) \) is a normal integral domain for all irreducible factors \( F(T) | P(T) \). Then \( (2.9a') \Rightarrow (2.9b) \Rightarrow (2.9c) \), where \( (2.9a') \) is the following variant of \( (2.9a) \):

\( (2.9a') \) \( A[\ell] \) is a free \( \mathbb{F}_\ell[T]/(P(T)) \)-module of rank \( r \) for any prime \( \ell \neq \text{Char}(K) \).

Proof. Clearly \( (2.9a') \Rightarrow (2.9b) \). We prove that \( (2.9a') \Rightarrow (2.9b) \). Let \( G'(T) = P(T)/G(T) \), then \( \text{Res}(G(T), G'(T)) \mid \text{Disc}(P(T)) \) by Subsection 2.1. In particular, \( \text{Res}(G(T), G'(T)) \) is coprime to \( \text{Char}(K) \). Let \( \beta := G(\phi) \), \( \gamma := G'(\phi) \in \text{End}(A) \). Clearly, \( (\ker \beta)[p] = \{ x \in A[p] \mid \beta x = 0 \} \). If \( A[p] \) is a free \( \mathbb{F}_p[T]/(P(T)) \)-module of rank \( r \), then \( |(\ker \beta)[p]| = p^{r \deg G'(T)} \). By Lemma 2.3 \( \dim \ker \beta \leq r \deg G(T)/2 \). Similarly, \( \dim \ker \gamma \leq r \deg G'(T)/2 \). However, by Lemma 2.2 \( \dim \ker \beta + \dim \ker \gamma = \dim A \). So we have

\[
\dim A = r \deg P(T)/2 = r \deg G(T)/2 + r \deg G'(T)/2 \\
\geq \dim \ker \beta + \dim \ker \gamma = \dim A.
\]

Therefore, \( \dim \ker \beta = r \deg G(T)/2 \). We conclude that \( \ker \beta \) is connected by Lemma 2.3 again. This proves that \( (2.9a') \Rightarrow (2.9b) \).

If \( G_1(T) \) and \( G_2(T) \) are coprime divisors of \( P(T) \), then \( (\ker G_1(\phi)) \cap (\ker G_2(\phi)) \) is a finite étale group scheme over \( K \), so

\[
\dim(\ker G_1(\phi) + \ker G_2(\phi)) = \dim \ker G_1(\phi) + \dim \ker G_2(\phi).
\]

Suppose that \( G(T) = \prod_{i=1}^t F_i(T) \) with each \( F_i(T) \) irreducible and pairwise distinct. By induction, \( \dim \ker G(\phi) = \dim(\sum_{i=1}^t \ker F_i(\phi)) \). Clearly

\[
\ker G(\phi) \supseteq \sum_{i=1}^t \ker F_i(\phi).
\]

Suppose that \( (2.9b) \) holds. Then both sides of \( (2.9b) \) are abelian varieties of the same dimension. So they must be the same.

Suppose that \( \mathbb{Z}[T]/(F(T)) \) is integrally closed for all irreducible factors \( F(T) \) of \( P(T) \). To show the statements are equivalent, it is enough to prove \( (2.9c) \Rightarrow (2.9a') \). Suppose that \( \bar{P}(T) = \prod_{i=1}^s h_i(T)^{t_i} \) is the prime factorization of \( \bar{P}(T) \) over \( \mathbb{F}_\ell \). Let \( W_{\ell,i} = \{ x \in A[\ell] \mid h_i(\phi)x = 0 \} \). By Corollary 2.7 \( A[\ell] \) is a free \( \mathbb{F}_\ell[T]/(P(T)) \)-module of rank \( r \) if we can prove that \( \dim_{\mathbb{F}_\ell} W_{\ell,i} = r \deg h_i(T) \) for all \( 1 \leq i \leq s \).
For each fixed $h_i(T)$, there exists an irreducible factor $F(T)$ of $P(T)$ such that $h_i(T) | F(T)$. Therefore,

$$W_{i,i} = \{ x \in A[\ell] \mid h_i(\phi)x = 0 \} = \{ x \in \ker F(\phi)[\ell] \mid h_i(\phi)x = 0 \}.$$ 

Suppose that $\ker F(\phi)$ is an abelian subvariety of $A$ of dimension $r \deg(F(T))/2$. There is an embedding $\mathbb{Z}[T]/(F(T)) \hookrightarrow \text{End}(\ker F(\phi))$ given by $T \mapsto \phi |_{\ker F(\phi)}$. Since $\mathbb{Z}[T]/(F(T))$ is a normal integral domain, it follows from Lemma 2.4 and Lemma 2.8 that $(\ker F(\phi))[\ell]$ is a free $\mathbb{F}_\ell[T]/(F(T))$-module of rank $r$. By Corollary 2.7, the $\mathbb{F}_\ell$-vector space $\{ x \in \ker F(\phi)[\ell] \mid h_i(\phi)x = 0 \}$ has dimension $r \deg h_i(T)$. We obtain the desired result. \hfill \Box

3. Superelliptic Jacobians

In this section, we prove the theorems and their corollaries stated in the introduction. Certain simple arithmetic results are postponed to Section 4. We keep the notations and assumptions of Section 1. Recall that $K$ is an algebraically closed field, $C_f,N$ is the smooth projective curve over $K$ defined by $Y^N = f(X) = \prod_{i=1}^n (X - \alpha_i)^{s_i}$ with $f(x)$ satisfying the conditions in (1.1), and $J_N := \text{Jac}(C_f,N)$ is the Jacobian of $C_f,N$. There is a natural action of $\rho_J : \mu_N \to \text{Aut}(J_N)$ of $\mu_N \subset K^\times$ on $J_N$, and $\rho_J(\mu_N) = (\delta_N)$.

3.1. The assumptions in (1.1) guarantee that there is exactly one point in $C_f,N(K)$ corresponding to the point $(\alpha_i,0)$ on the affine curve $Y^N = f(X)$, and moreover, there is a unique point (denoted by $\infty$) in $C_f,N(K)$ that lies above the point at infinity on $\mathbb{P}^1(K)$ for the map $\eta : C_f,N \to \mathbb{P}^1$. Clearly, $\mu_N$ fixes the following set of points

$$\mathfrak{S}(C_f,N) := \{ Q_1 = (\alpha_1,0), \ldots, Q_n = (\alpha_n,0), \infty \} \subset C_f,N(K),$$

and it acts freely outside $\mathfrak{S}(C_f,N)$. Therefore $\eta : C_f,N \to \mathbb{P}^1$ is totally ramified at each point of $\mathfrak{S}(C_f,N)$ with ramification index $N$, and unramified everywhere else. All the ramifications are tame since the characteristic of $K$ does not divide $N$. By the Hurwitz formula [5 Corollary IV.2.4], the genus of $C_f,N$ is (cf. [7] for the case $K = \mathbb{C}$)

$$g(C_f,N) = \frac{(N - 1)(n - 1)}{2}.$$ 

3.2. A natural question is to describe the group of all fixed points of $\mu_N$ on $J_N$. Let us denote it by

$$\mathfrak{S}_N := (J_N)_{\mu_N} = \{ x \in J_N(K) \mid \delta_N x = x \}.$$ 

It contains an obvious subgroup consisting of the linear equivalence classes of divisors of degree zero supported on $\mathfrak{S}(C_f,N)$:

$$\mathfrak{S}_N := \left\{ [D] \in \text{Pic}^0(C_f,N) = J_N(K) \mid D = \sum a_i Q_i, \deg D = b + \sum a_i = 0 \right\}.$$ 

We will describe the group structure of $\mathfrak{S}_N$. Given a rational function $g \in K(C_f,N)$ on $C_f,N$, let $\text{Div}(g)$ be its divisor. Then

$$\text{Div}(Y) = \sum_{i=1}^n c_i Q_i - \deg(f)\infty,$$

$$\text{Div}(X - \alpha_i) = NQ_i - N\infty.$$
Since \( \gcd(\deg(f), N) = 1 \), we may find \( a, b \in \mathbb{Z} \) such that \( a \deg(f) + bN = 1 \). Then
\[
\text{Div}(Y^a(X - \alpha_i)^b) = a \sum_{j=1}^{n} e_j Q_j + bNQ_i - (a \deg(f) + bN)\infty = a \sum_{j=1}^{n} e_j Q_j + bNQ_i - \infty.
\]
Therefore, any divisor of degree zero supported on \( \mathcal{Y}(C_{f,N}) \) is linear equivalent to one supported on the set
\[
\mathcal{R} := \{ Q_1, \ldots, Q_n \}.
\]
By \cite{17} Lemma 4.1, a divisor of degree zero of the form \( D = \sum_{i=1}^{n} a_i Q_i \) is linear equivalent to zero if and only if there exists \( c \in \mathbb{Z} \) such that \( a_i \equiv ce_i (\mod N) \) for all \( 1 \leq i \leq n \). For this \( c \in \mathbb{Z} \), we have \( 0 = \sum_{i=1}^{n} a_i \equiv c \sum_{i=1}^{n} e_i (\mod N) \). Since \( \deg f(x) = \sum_{i=1}^{n} e_i \) is coprime to \( N \), \( c \equiv 0 (\mod N) \). In other words, \( D \) is linear equivalent to zero if and only if \( a_i \equiv 0 (\mod N) \).

Let \( M_\mathcal{R} \) be the free \((\mathbb{Z}/N\mathbb{Z})\) module of rank \( n \) generated by elements of \( \mathcal{R} \), and
\[
M^0_\mathcal{R} := \left\{ m \in M_\mathcal{R} \mid m = \sum_{i=1}^{n} a_i Q_i, \ a_i \in \mathbb{Z}/N\mathbb{Z}, \sum_{i=1}^{n} a_i = 0 \right\},
\]
\[
\mathcal{E}_0 := \sum_{i=1}^{n} e_i Q_i \in M_\mathcal{R}.
\]
Then \( M_\mathcal{R} = M^0_\mathcal{R} \oplus (\mathbb{Z}/N\mathbb{Z})\mathcal{E}_0 \) and \( M^0_\mathcal{R} \cong (\mathbb{Z}/N\mathbb{Z})^{n-1} \). We have a canonical isomorphism
\[
(3.3) \quad \mathcal{G}_N \cong M^0_\mathcal{R} \cong M_\mathcal{R}/((\mathbb{Z}/N\mathbb{Z})\mathcal{E}_0).
\]
Our first goal in this section is to show that \( \mathcal{G}_N = \mathcal{G}_N \).

### 3.3.
We refer to \cite{14} Section VI.2] for the notation \( a_Q \) below. It is the character of the Artin representation of \( \mu_N \) at \( Q \in C_{f,N}(\mathbb{K}) \) which encodes the ramification information at each point \( Q \) for the map \( \eta : C_{f,N} \to \mathbb{P}^1 = C_{f,N}/\mu_N \). If \( \eta \) is unramified at \( Q \), then \( a_Q(\xi) = 0 \) for all \( \xi \in \mu_N \). If \( \eta \) is totally ramified at \( Q \), \( a_Q \) may be defined in the following way (Combining \cite{14} Lemma III.6.3 and \cite{14} Section IV.1). Let \( \pi_Q \in K(C_{f,N})^\times \) be a local parameter at \( Q \), and \( \text{val}_Q : K(C_{f,N})^\times \to \mathbb{Z} \) be the valuation of \( K(C_{f,N}) \) associated to \( Q \). Then \( \forall \xi \in \mu_N \),
\[
a_Q(\xi) = -\text{val}_Q(\rho_C(\xi)\pi_Q - \pi_Q) \quad \text{if } \xi \neq 1, \quad a_Q(1) = -\sum_{\xi \in \mu_N, \xi \neq 1} a_Q(\xi).
\]
For all \( Q' \in \mathbb{P}^1(\mathbb{K}) \), \( a_{Q'} \) is defined to be \( \sum_{Q \to Q'} a_Q \).

Let us fixed a prime \( \ell \neq \text{Char } K \). Since \( \eta \) is totally and tamely ramified at each point \( Q \in \mathcal{Y}(C_{f,N}) \), we have
\[
a_{\eta(Q)} = a_Q = r_{\mu_N} - 1_{\mu_N} = t_{\mu_N},
\]
where \( r_{\mu_N}, 1_{\mu_N}, t_{\mu_N} : \mu_N \to \mathbb{Q}_\ell \) are the characters of the regular representation, the 1-dimensional trivial representation, and the augmentation representation of \( \mu_N \) respectively. More precisely, \( 1_{\mu_N}(\xi) = 1, \forall \xi \in \mu_N \); \( r_{\mu_N}(1) = N \), and \( r_{\mu_N}(\xi) = 0 \) for all \( \xi \neq 1 \). Hence
\[
\forall \xi \in \mu_N, \quad t_{\mu_N}(\xi) = \begin{cases} N - 1 & \text{if } \xi = 1, \\ -1 & \text{if } \xi \neq 1. \end{cases}
\]
Let $h_1 : \mu_N \to \mathbb{Q}_\ell$ be the character of the representation of $\mu_N$ defined by $T_\ell J_N$. By \cite[Section VI.4]{14}, we have

$$h_1 = \sum_{\xi \in \mathfrak{F}(C_{1,N})} a_{\eta(\xi)} + 2 \cdot 1_{\mu_N} - E(\mathbb{P}^1) \cdot \iota_{\mu_N},$$

$$= (n - 1)(\iota_{\mu_N} - 1_{\mu_N}) = (n - 1)u_{\mu_N}. $$

Here $E(\mathbb{P}^1) = 2$ is the Euler characteristic of $\mathbb{P}^1$.

3.4. Since $\rho_\ell(\mu_N)$ is generated by $\delta_N$, $\mathfrak{f}_N = (J_N)^{\mu_N} = \ker(1 - \delta_N)$, so

$$|\mathfrak{f}_N| = |\ker(1 - \delta_N)| = \deg(1 - \delta_N).$$

By \cite[Theorem 19.4]{10}, $\deg(1 - \delta_N) = \deg(T_\ell(1 - \delta_N))$. We may choose the prime $\ell$ such that $\ell \equiv 1 \pmod{N}$. Then $\ell$ splits completely in $\mathbb{Z}[\zeta_N]$, and $u_{\mu_N} = \sum_{\chi \neq 1} \chi$, where the sum is over all nontrivial characters $\chi : \mu_N \to \mathbb{Q}_\ell^\times$. It follows from Subsection 3.3 that the characteristic polynomial of $T_\ell(\delta_N)$ is

$$\det(T - T_\ell(\delta_N)) = \left(\prod_{i=1}^{N-1} (T - (\zeta_N)^i)\right)^{n-1} = P_N(T)^{n-1},$$

where $P_N(T)$ is given by \cite[1.2]{12}, and the minimal polynomial of $T_\ell(\delta_N)$ is $P_N(T)$. Since the natural map $\text{End}(J_N) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \text{End}_{\mathbb{Z}_\ell}(T_\ell J_N)$ is an embedding \cite[Theorem 19.3]{10}, the minimal polynomial of $\delta_N$ is equal to $P_N(T)$. We have

$$\deg(1 - \delta_N) = P_N(1)^{n-1} = N^{n-1}.$$ 

Recall that $\mathfrak{f}_N$ contains the subgroup $\mathfrak{G}_N \simeq (\mathbb{Z}/N\mathbb{Z})^{n-1}$ by \cite[3.3]{13}. They must coincide by comparing the cardinality. We have proven the following theorem:

\textbf{Theorem 3.5.} We have $\mathfrak{f}_N = \mathfrak{G}_N \simeq (\mathbb{Z}/N\mathbb{Z})^{n-1}$.

For the case $N = p$ is a prime, Theorem 3.5 was already contained in \cite[Section 6]{12} and \cite[Proposition 3.2]{13}.

3.6. Since $P_N(T) = \prod_{D \vert N, D > 1} \Phi_D(T)$, by the Chinese Remainder Theorem,

\begin{equation}
\mathbb{Q}[T]/(P_N(T)) \cong \prod_{D \vert N, D > 1} \mathbb{Q}[T]/(\Phi_D(T)) \cong \prod_{D \vert N, D > 1} \mathbb{Q}[\zeta_D].
\end{equation}

On the other hand, it is important to note that if $N$ is not prime, the embedding

\begin{equation}
\mathbb{Z}[T]/(P_N(T)) \hookrightarrow \prod_{D \vert N, D > 1} \mathbb{Z}[T]/(\Phi_D(T)).
\end{equation}

is not an isomorphism. For example, if $N = p^r$ for some $r > 1$, then

\begin{equation}
(\mathbb{Z}[T]/(P_N(T))) \otimes_{\mathbb{Z}} \mathbb{F}_p \cong \mathbb{F}_p[T]/(P_N(T)) = \mathbb{F}_p[T]/((T - 1)^{N-1}).
\end{equation}

The right hand side is a local ring, and therefore not a direct product of proper subrings. We leave it to the reader to prove that 3.5 is not an isomorphism for arbitrary $N$ not a prime. However, from an explicit construction (cf. \cite[Lemma 5.2]{14} or \cite[Subsection 2.6]{17}), one may show that the idempotents in $\mathbb{Q}[T]/(P_N(T))$ lie in $\frac{1}{N} \mathbb{Z}[T]/(P_N(T))$. Therefore,

$$\frac{1}{N} \mathbb{Z}[T]/(P_N(T)) \supset \prod_{D \vert N, D > 1} \mathbb{Z}[T]/(\Phi_D(T)).$$
and the cokernel of (3.5) are \(N\)-torsions. There is an isomorphism

\[(3.6) \quad \mathbb{Z}[1/N, T]/(P_N(T)) \cong \prod_{D \mid N, D > 1} \mathbb{Z}[1/N, T]/(\Phi_D(T)).\]

We leave it to the reader to show that

\[(3.7) \quad \text{Disc}(P_N(T)) = (-1)^{(N-1)(N-2)/2} N^{N-2}.\]

**Proposition 3.7.** For any prime \(\ell \not | (N \text{ Char}(K))\), the Tate module \(T_\ell J_N\) is a free \(\mathbb{Z}_\ell[T]/(P_N(T))\)-module of rank \(n - 1\).

**Proof.** Since \(\ell \not | N\), \(\mathbb{Z}_\ell[T]/(P_N(T))\) is a product of discrete valuation rings by (3.6). Because \(T_\ell(J_N)\) is \(\mathbb{Z}_\ell\)-torsion free, it is enough to prove that \(V_\ell(J_N) := T_\ell(J_N) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell\) is a free \(\mathbb{Q}_\ell[T]/(P_N(T))\)-module of rank \(n - 1\). This follows directly from Subsection 3.8, noting that the representation space of the augmentation representation \(u_{\mu_N}\) over \(\mathbb{Q}_\ell\) is isomorphic to \(\mathbb{Q}_\ell[T]/(P_N(T))\). \(\square\)

**3.8.** For each integer \(D \mid N\) and \(D > 1\), there exists a finite morphism

\[\eta_D : C_{f,N} \rightarrow C_{f,D}, \quad (X, Y) \mapsto (X, Y^{N/D}).\]

It induces two maps between the Jacobians

\[\eta_D : J_N \rightarrow J_D, \quad \eta_D^* : J_D \rightarrow J_N,\]

by Albanese functoriality and Picard functoriality respectively. We describe the action of these maps on closed points. Recall that \(J_D(K) = \text{Div}^0(C_{f,D})/\sim\), the group of divisors of degree zero modulo linear equivalence. Given a divisor \(D \in \text{Div}(C_{f,D})\) on \(C_{f,D}\), we write \([D]\) for the linear equivalence class of \(D\). The abelian group \(J_D(K)\) is generated by the set of elements \(\{[Q - \infty_D]\}_{Q \in C_{f,D}(K)}\), where \(\infty_D\) is the unique point at infinity on \(C_{f,D}\). Let \(\nu_D : C_{f,D} \rightarrow J_D\) be the closed immersion defined by \(Q \mapsto [Q - \infty_D]\) (cf. [8] Section 2). Then by definition, \(\eta_D : J_N \rightarrow J_D\) is the unique homomorphism such that the following commutative diagram holds (cf. [8] Proposition 6.1)

\[
\begin{array}{ccc}
C_{f,N} & \xrightarrow{\eta_D} & C_{f,D} \\
\downarrow{\nu_N} & & \downarrow{\nu_D} \\
J_N & \xrightarrow{\eta_D^*} & J_D
\end{array}
\]

It follows that

\[(3.8) \quad \eta_D([Q - \infty_N]) = [\eta_D(Q) - \eta_D(\infty_N)] = [\eta_D(Q) - \infty_D] \quad \forall Q \in C_{f,N}(K).\]

On the other hand, let \(M = N/D\), then \(\eta_D : C_{f,N} \rightarrow C_{f,D}\) realizes \(C_{f,D}\) as a quotient of \(C_{f,N}\) by the group \(\mu_M := \underbrace{\langle \xi_N \rangle} \subseteq \mu_N\). For each \(Q \in C_{f,N}(K)\), we write \(e_Q\) for the ramification index of \(\eta_D\) at \(Q\). Then \(\forall Q' \in C_{f,D}(K)\),

\[(3.9) \quad \eta_D^*([Q' - \infty_D]) = \left[ \sum_{Q \rightarrow Q'} e_Q Q - M \infty_N \right] = \left[ \sum_{\xi \in \mu_M} \rho_f(\xi) Q_0 - M \infty_N \right],\]

where \(Q_0 \in C_{f,N}(K)\) is a fixed point in \(\eta_D^{-1}(Q')\).

Combining (3.8) and (3.9), we see that

\[\eta_D \circ \eta_D^* = \text{deg}(\eta_D) \cdot \text{Id}_{J_D} = M \cdot \text{Id}_{J_D}.\]
Moreover, the composition of \( J_N \xrightarrow{\eta_D} J_D \xrightarrow{\eta_N} J_N \) is given by
\[
[D] \mapsto \sum_{\xi \in \mu_M} \rho_J(\xi)[D] = Q_{N,D}(\delta_N)([D]), \quad \forall [D] \in J_N(K),
\]
where
\[
(3.10) \quad Q_{N,D}(T) := \frac{T^N - 1}{T^D - 1} = \sum_{i=0}^{D-1} T^{iD} = \prod_{D'|N\cdot D^0} \Phi_{D'}(T) \in \mathbb{Z}[T].
\]

Clearly \( \delta_N = \delta_N^{-1} \in \text{End}(J_N) \). Let \( \xi_D := \xi_N^{N/D} \in \mu_D \subseteq K^\times \) and define \( \delta_D : J_D \to J_D \) similarly to \( \delta_N \). Then \( \delta_D \eta_D = \eta_D \delta_N \) and \( \eta_D^2 \delta_D = \delta_N \eta_D^2 \). Both \( \eta_D \) and \( \eta_D^2 \) are \( \mu_N \)-equivariant if we let \( \mu_N \) act on \( J_D \) via the map \( \mu_N \to \mu_D \) that sends \( \xi_N \) to \( \xi_D \).

3.9. Let \( \lambda_N : J_N \to J_N' \) be the canonical principal polarization of \( J_N \). Under the canonical identification \( J_N = (J_N')^\vee \), \( \lambda_N' = \lambda_N \). It induces a Rosati involution on \( \text{End}(J_N) \) defined by \( \phi \mapsto \phi' := \lambda_N' \circ \phi \circ \lambda_N \), \( \forall \phi \in \text{End}(J_N) \). The polarization \( \lambda_N \) is \( \rho_J(\mu_N) \)-invariant. For any \( \ell \notin \text{Char}(K) \), let \( E^{\lambda_N}(T) : T_c J_N \times T_c J_N \to \mathbb{Z}_l(1) := \lim_{\epsilon \to 0} \mathbb{Z}_l[\mu_N] \), be the nondegenerate Riemann form (See [10, Section 20]) defined by \( \lambda_N \). Then
\[
E^{\lambda_N}(T_c(\delta_N)x, T_c(\delta_N)y) = E^{\lambda_N}(x, y), \quad \forall x, y \in T_c J_N.
\]
In particular, \( \delta_N' = (\delta_N)^{-1} \in (\delta_N)^{-1} \), and \( \mathbb{Z}[\delta_N] \subseteq \text{End}(J_N) \) is invariant under the Rosati involution. For any \( D \mid N \) with \( D > 1 \), we have (cf. [1] Proposition 11.11.6) or [11, Section 17.5] in the case \( K = \mathbb{C} \), and [9, Proposition A.6] in general
\[
(3.11) \quad \eta_D = \lambda_D^{-1} \circ (\eta_D^2)^\vee \circ \lambda_N.
\]

We refer to [17, Subsection 2.11] for the following proposition.

**Proposition 3.10.** The map \( \eta^*_M : J_M \to J_N \) is a closed immersion for all \( M \mid N \) and \( M > 1 \).

Since \( P_M(\delta_M)J_M = \{0\} \), one has \( P_M(\delta_N)(\eta^*_M J_M) = \eta^*_M P_M(\delta_M)J_M = \{0\} \). We prove that the image \( \eta^*_M J_M \) is in fact uniquely characterized as a subvariety of \( J_N \) by this property.

**Proposition 3.11.** For each integer \( M \mid N \) and \( M > 1 \), the kernel of \( P_M(\delta_N) : J_N \to J_M \) is \( \eta^*_M J_M \).

**Proof.** Let \( \beta_M := P_M(\delta_N) = \sum_{i=0}^{M-1} (\delta_N)^i \in \text{End}(J_N) \). Since \( P_N(T) \) is separable in \( K[T] \), \( \ker \beta_M \) is reduced by Lemma 2.2. As remarked, \( \eta^*_M J_M \subseteq \ker \beta_M \).

For any divisor \( D \in \text{Div}(C_{f,N}) \), we write \( \mathcal{L}(D) \) for the invertible sheaf on \( C_{f,N} \) associated to \( D \) [3, Section II.6, p144],
\[
L(D) := \text{H}^0(C_{f,N}, \mathcal{L}(D)) = \{ g \in K(C_{f,N}) \mid \text{Div}(g) + D \geq 0 \},
\]
and
\[
l(D) := \dim_K L(D) \mid \text{Div}(g) \in J_N(K) \text{ is nonzero, then } l(D) = 0. \text{ By the Riemann-Roch theorem } [3, \text{Theorem IV.1.3}], l(D + t\infty) = t + 1 - g(C_{f,N}) > 0 \text{ if } t \in N \text{ is large enough. Since } l(D + (t + 1)\infty) - l(D + t\infty) \leq 1 \text{ for all } t \geq 0, \text{ there exists a smallest } t \text{ such that } l(D + t\infty) = 1. \text{ In other words, for this } t \text{ there exists a unique effective divisor } D_0 > 0 \text{ such that } D + t\infty \sim D_0. \text{ Clearly, } D_0 \text{ depends only on } [D]. \text{ The coefficient } b_\infty \text{ in } D_0 = \sum_{P \in C_{f,N}(K)} b_P P \text{ is necessarily zero by the minimality of } t. \)
Any point $x = [D] \in \ker \beta_M$ is fixed by $(\delta_N)^M$. Choose $t$ and $D_n$ for $[D]$ as above. By the uniqueness of $D_n$, we must have $(\delta_N)^m D = D = 0$ (Equality of divisors). Let $D'_0 = D'_0 + D'_0$, with $D'_0 = \sum_{i=1}^\ell b_i Q_i$, and the support of $D'_0$ disjoint from $\delta(C_{f,N}) = \{Q_1, \ldots, Q_n, \infty\}$. We write $t_1 = \deg D'_0$ and $t_2 = \deg D'_0$, then $t_1 + t_2 = t$ and $D \sim (D'_0 - t_1 \infty) + (D'_0 - t_2 \infty)$. Clearly $D''_0$ is fixed by $(\delta_N)^M$. In other words, if $P \in \supp D''_0$, then $P'' \in \supp D''_0$ for all $P'' \in \eta_{M}^{-1}(\eta_{M}(P)) = \{P, (\delta_N)^M P, \ldots, (\delta_N)^N - M P\}$. Therefore, $y := [D'_0 - t_2 \infty] \in \eta_{M}^2(J_M)(K)$.

Now we have $z := [D'_0 - t_1 \infty] = [D] - [D'_0 - t_2 \infty] = \ker \beta_M$. By construction $\delta_N z = z$. So $\beta_M z = \sum_{i=0}^{M-1} (\delta_N)^i z = Mz$, and hence $z \in \mathfrak{F}_N[M]$. We claim that $\mathfrak{F}_N[M] = \eta_{M}(\mathfrak{F}_M)$. Indeed, by Theorem 3.3, $\mathfrak{F}_N \simeq (\mathbb{Z}/M\mathbb{Z})^{-1}$, so $\mathfrak{F}_N[M] \simeq (\mathbb{Z}/M\mathbb{Z})^{-1}$. On the other hand, $\eta_{M} : J_M \to J_N$ is a closed immersion by Proposition 3.11 so

$$(\mathbb{Z}/M\mathbb{Z})^{-1} \simeq \eta_{M}^2(\mathfrak{F}_M) \subseteq \mathfrak{F}_N[M] \simeq (\mathbb{Z}/M\mathbb{Z})^{-1}.$$  

It follows that $z \in \eta_{M}^2(\mathfrak{F}_M) \subseteq \eta_{M}^2(J_M)(K)$. So $x = y + z \in \eta_{M}^2(J_M)(K)$, and $\ker \beta_M = \eta_{M}(J_M)$.

Proof of the main theorem by induction. If $N = \ell$ is a prime, then $\mathbb{Z}[T]/(P_{\ell}(T)) = \mathbb{Z}[T]/(\Phi_{\ell}(T))$ is isomorphic to the ring of integers $\mathbb{Z}[\zeta_{\ell}]$ in the cyclotomic field $\mathbb{Q}(\zeta_{\ell})$. The theorem follows from Lemma 2.8.

Suppose that the theorem holds for all $J_D$ with $D \mid N$ and $D \neq N$. The case $\ell \nmid N \text{ Char } (K)$ is already treated in Proposition 3.7. Now fix a prime $\ell \mid N$. By Lemma 2.4, it is enough to prove that $J_N[\ell]$ is a free $\mathbb{F}_{\ell}[T]/(P_N(T))$-module of rank $n - 1$. Let $N = q M$ with $q = \ell^r$ for some $r \in \mathbb{N}$ and $\gcd(q, M) = 1$. In $\mathbb{F}_{\ell}[T]$, $P_N(T) = (T - 1)^{n-1}(\ell \in \mathbb{Z})$ factors as

$$T^n - 1 = \frac{(T - 1)^{q}-1}{T - 1} = \prod_{i=1}^{\ell} h_i(T)^q,$$

where each $h_i(T)$ is a monic irreducible factor of $P_M(T)$. Because $\gcd(M, \ell) = 1$, $P_M(T)$ is separable over $\mathbb{F}_{\ell}$, so all $h_i(T)$ in (3.12) are distinct. By Theorem 3.3,

$$W_{\ell,0} = \{x \in J_N[\ell] \mid (\delta_N - 1)x = 0\} \cong \mathfrak{F}_N[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{-1}.$$  

On the other hand, by Proposition 3.11

$$W_{\ell,i} = \{x \in J_N[\ell] \mid h_i(\delta_N)x = 0\} \subseteq \eta_{M}^2(J_M[\ell]).$$

By the induction hypothesis, $J_M[\ell]$ is a free $\mathbb{F}_{\ell}[T]/(\overline{P}_M(T))$-module of rank $n - 1$. So $\dim_{\mathbb{F}_{\ell}} W_{\ell,i} = (n - 1) \deg h_i(T)$ by Corollary 2.7. Applying the same Corollary again, we see that $J_N[\ell]$ is a free $\mathbb{F}_{\ell}[T]/(P_N(T))$-module of rank $n - 1$. Therefore, $T_{\ell} J_N$ is a free $\mathbb{Z}_\ell[T]/(P_N(T))$-module of rank $n - 1$ by Lemma 2.4.

Proof of Corollary 3.2. Recall that we have an embedding of $\mathbb{Q}$-algebras

$$\iota : E := \mathbb{Q}[T]/(P_N(T)) = \mathbb{Z}[T]/(P_N(T)) \otimes \mathbb{Z} \mathbb{Q} \hookrightarrow \text{End}^{\mathbb{Q}}(J_N),$$

and we want to show that $E \cap \text{End}(J_N) = \mathbb{Z}[T]/(P_N(T))$, where the intersection is taken within $\text{End}^{\mathbb{Q}}(J_N)$. Since $\tilde{R}_\ell := \mathbb{Z}_\ell[T]/(P_N(T))$ is the maximal order in $E_\ell := E \otimes \mathbb{Q}_\ell$ for any prime $\ell \nmid N$, it is enough to prove that for all $p \mid N$,

$$R_p := \mathbb{Z}_p[T]/(P_N(T)) = E_p \cap (\text{End}(J_N) \otimes \mathbb{Z}_p) \text{ inside } \text{End}^{\mathbb{Q}}(J_N) \otimes \mathbb{Q}_p.$$
By [10] Theorem 19.3], \( E_p \cap (\text{End}(J_N) \otimes \mathbb{Z}_p) \subseteq \text{End}_{R_p}(T_p(J_N)) \). So it reduces to prove that

\[
R_p = E_p \cap \text{End}_{R_p}(T_p(J_N)) \quad \text{inside} \quad \text{End}_{E_p}(V_p(J_N)) = \text{End}_{R_p}(T_p(J_N)) \otimes \mathbb{Z}_p \mathbb{Q}_p.
\]

Now by Theorem 1.1, End \( \text{Mat} \) \( \simeq \) Mat \( -1 \) \( \text{(} R_p \text{)} \) and hence End \( E_p \) \( \simeq \) Mat \( -1 \) \( (E_p ) \). The embedding \( \iota \otimes \mathbb{Q}_p \) identifies \( E_p \) with the scalar matrices \( E_p \cdot \text{Id} \). Clearly \( E_p \cdot \text{Id} \cap \text{Mat} \( -1 \) \( (R_p ) = R_p \cdot \text{Id} \).

We assume that \( K \) is not algebraically closed exclusively for the following theorem. Let \( K \) be a fixed algebraic closure of \( K \).

**Theorem 3.12.** Let \( K \) be a field of characteristic zero, and \( f(X) \in K[X] \) be a polynomial with no multiple roots and \( n = \deg f \geq 5 \). Suppose \( p \) is a prime that does not divide \( n \). Let \( r > 1 \) be a positive integer and \( q = p^r \). Assume also that either \( n = q + 1 \) or \( q \) does no divides \( n - 1 \). If \( p = 2 \) then we assume additionally that \( n = kq + c \) with nonnegative integers \( k \) and \( c < q/2 \). Suppose \( \text{Gal}(f) \) contains a doubly transitive simple non-abelian subgroup \( \mathcal{G} \). Then \( \text{End}_K(\text{Jac}(f,q)) \simeq \mathbb{Z}/(P_q(T)) \).

**Proof.** By [23] Corollary 5.4, \( \text{End}_K(\text{Jac}(f,q)) = \mathbb{Q}[\delta_q] \simeq \mathbb{Z}[T]/(P_q(T)) \) under the above assumptions. Now the theorem follows directly from Corollary 1.2.

The rest of this section is devoted to the proof of Theorem 1.3.

**3.13.** For each \( D \mid N \) and \( D > 1 \), let \( \gamma_D := \Phi_D(\delta_N) \in \text{End}(J_N) \). By Theorem 2.9 \( \ker \gamma_D \) is an abelian subvariety of \( J_N \) of dimension \( \varphi(D)(n - 1)/2 \). We give a more geometric description of these subvarieties. Suppose that \( D_1 \mid D_2 \mid N \) with \( D_1 > 1 \), then the map \( \eta_{D_1} : C_{D_1} \rightarrow C_{D_2} \) factors as \( C_{D_1} \rightarrow C_{D_2} \rightarrow C_{D_1} \). By functoriality, \( \eta_{D_1}^*: J_{D_1} \rightarrow J_N \) factors as \( J_{D_1} \rightarrow J_{D_2} \rightarrow J_N \). In particular, \( \eta_{D_1}^* \) factors as \( J_{D_1} \rightarrow J_{D_2} \rightarrow J_N \). Following [3] Section 5], we define

\[
J_{N,D}^{\text{old}} := \sum_{D \mid N, 1 < D < N} \eta_{D}^* J_D = \sum_{p \mid N} \eta_{N/p}^* J_{N/p},
\]

The orthogonal complement of \( J_{N,D}^{\text{old}} \) with respect to the canonical principal polarization \( \lambda_N \) is called the new part of the Jacobian and denoted by \( J_{N,D}^{\text{new}} \). We write \( \epsilon_N: J_{N}^{\text{new}} \rightarrow J_N \) for the canonical embedding. If \( N = p \) is a prime, \( J_{N,D}^{\text{new}} \) is defined to be \( J_p = \text{Jac}(C_{f,p}) \).

Let \( G(T) = P_N(T)/\Phi_N(T) \in \mathbb{Z}[T] \). By Theorem 2.9 and Proposition 3.11

\[
J_N^{\text{old}} = \sum_{D \mid N, 1 < D < N} \ker \gamma_D.
\]

Therefore, \( J_N^{\text{old}} = \ker G(\delta_N) \). In particular,

\[
\dim J_N^{\text{old}} = (n - 1) \deg G(T)/2 = (n - 1)(N - 1 - \varphi(N))/2,
\]

and \( \dim J_N^{\text{new}} = \dim J_N - \dim J_N^{\text{old}} = \varphi(N)(n - 1)/2 \) (cf. also [2] Corollary 5.4). The map \( \prod_{p \mid N} \eta_{N/p}^* \rightarrow J_N^{\text{old}} \) factors as

\[
\prod_{p \mid N} J_{N/p}^{\text{old}} \rightarrow J_N^{\text{old}} \rightarrow J_N.
\]
The orthogonal complement of $J^\text{old}_N$ with respect to $\lambda_N$ is defined to be the identity component (with the reduced subscheme structure) of the map (See [10, Theorem 19.1])

\begin{equation}
J_N \xrightarrow{\lambda_N} J^\vee_N \xrightarrow{J^{\text{old}}_N} (J^\text{old}_N)^\vee.
\end{equation}

Now compose the map in \ref{3.13} with

\begin{equation}
(J^\text{old}_N)^\vee \xrightarrow{\pi^\vee} \prod_{p | N} J^\vee_{(N/p)} \xrightarrow{\prod_{p | N} \lambda^{-1}_{(N/p)}} \prod_{p | N} J_{(N/p)} \xrightarrow{\prod_{p | N} \eta_{(N/p)}} \prod_{p | N} J_N
\end{equation}

and then apply \ref{3.11}, we obtain the map

\begin{equation}
\prod_{p | N} Q_{N,N/p}(\delta_N) : J_N \to \prod_{p | N} J_N,
\end{equation}

where $Q_{N,N/p}(T) \in \mathbb{Z}[T]$ is defined in \ref{3.10}. By Lemma \ref{3.3}, the ideal in $\mathbb{Z}[T]$ generated by $Q_{N,N/p}(T)$ for all $p | N$ is $(\Phi_N(T))$. So the kernel of \ref{3.14} coincides with $\ker \gamma_N$. On the other hand, $\ker(j^\vee \circ \lambda_N)$ is contained in the kernel of \ref{3.15}.

Comparing dimensions, we obtain that

\begin{equation}
J^\text{new}_N = \ker(j^\vee \circ \lambda_N) = \ker \gamma_N = \ker \Phi_N(\delta_N).
\end{equation}

As a side result, $\pi^\vee$ must be a closed immersion since otherwise the kernel of $j^\vee \circ \lambda_N$ will be properly contained in that of \ref{3.15}. There is an exact sequence of abelian varieties

\begin{equation}
0 \to (J^\text{old}_N)^\vee \xrightarrow{\pi^\vee \circ \lambda_N} J_N \xrightarrow{\phi} J^\text{new}_N \to 0.
\end{equation}

For $1 < D < N$ and $D \nmid N$, we have $\Phi_D(T) \mid P_D(T)$, so $\ker \gamma_D \subseteq \ker(P_D(\delta_N)) = \eta_D^p \phi_D \phi_D(\delta_N)$. Recall that we have $\delta_N \eta_D^p = \eta_D^p \phi_D \phi_D(\delta_N)$ by Subsection 3.8. Therefore,

\begin{equation}
\ker \gamma_D = \ker \Phi_D(\delta_N) = \eta_D^p(\ker(J_D \phi_D^p(\delta_D) \phi_D \phi_D(\delta_D))) = \eta_D^p(J_D^\text{new}).
\end{equation}

\textbf{3.14.} Since $\eta_D^p : J_D \to J_N$ is a closed immersion for each $D \mid N$ and $D > 1$, we may and will regard $J_D$ and $J^\text{new}_D$ as subvarieties of $J_N$ via $\eta_D^p$. Let $\omega(N)$ be the number of distinct prime factors of $N$. If $\omega(N) = 1$, then $q := N = p^r$ is a prime power. In this case $J_q^\text{old} = J_q / p$. By \cite[Subsection 2.11]{17}, $J^\text{new}_q \cap J_q = J_q / p$. It follows that

\begin{equation}
\forall 1 \leq i \leq r - 1, \quad J^\text{new}_q \cap J_{p^i} = J_{p^i} / p, \quad J^\text{new}_q \cap J_{p^r} = J_{p^r} / p.
\end{equation}

Now assume that $\omega(N) \geq 2$. Let $\mathcal{O} := \mathbb{Z}[\zeta_N]$ be the ring of integers in the cyclotomic field $\mathbb{Q}(\zeta_N)$. By \cite[Proposition 2.8]{15}, $1 - \zeta_N$ is a unit in $\mathcal{O}$. For each $D \mid N$ and $1 < D < N$, there is a natural action of $\mathbb{Z}[T] / (\Phi_N(T), P_D(T))$ on $J^\text{new}_N \cap J_D$. We have

\begin{equation}
\mathbb{Z}[T] / (\Phi_N(T), P_D(T)) \cong \mathcal{O} / (P_D(\zeta_N)) = \mathcal{O} / ((\zeta_N)^D - 1) \cong \mathbb{Z}[T] / (\Phi_N(T), T^D - 1)
\end{equation}

since $P_D(\zeta_N) = ((\zeta_N)^D - 1) / (\zeta_N - 1)$. Clearly, $(\zeta_N)^D$ is a primitive $(N/D)$-th root of unity, so if $\omega(N/D) > 1$, then $(\zeta_N)^D - 1$ is again a unit in $\mathcal{O}$. It follows that

\begin{equation}
J^\text{new}_N \cap J_D = \{0\} = J^\text{new}_N \cap J^\text{new}_D \quad \text{if} \quad \omega(N/D) > 1.
\end{equation}

Suppose that $N = Mp^r$ with $r \geq t > 0$ and $\gcd(p, M) = 1$. In particular, $p \nmid \text{Char}(K)$. Then $p\mathcal{O}$ is divisible by $(1 - \zeta_N)^D$, so $J^\text{new}_N \cap J_D \subseteq J_D / p$. It follows that $J^\text{new}_N \cap J_D$ is naturally a $\mathbb{F}_p[T] / (\Phi_N(T), T^D - 1)$-module. By Lemma \ref{4.2}

\begin{equation}
\Phi_N(T) = \Phi_M(T)^c(p^t). \quad \text{On the other hand,} \quad T^D - 1 = (T^M - 1)^{p^r - t} \quad \text{in} \quad \mathbb{F}_p[T].
\end{equation}
\[
gcd(p, M) = 1, \ T^M - 1 \text{ is separable over } \mathbb{F}_p. \quad \text{Because } \varphi(p^r) = p^{r-1}(p-1) \geq p^{r-t}, \ \gcd(\Phi_M(T)^{p^r}, (T^M - 1)^{p^{r-t}}) = \Phi_M(T)^{p^{r-t}}. \quad \text{We have}
\]

\[
\begin{align*}
\mathbb{Z}[T]/(\Phi_N(T), P_D(T)) & \cong \mathbb{F}_p[T]/(\Phi_N(T), T^D - 1) \cong \mathbb{F}_p[T]/(\Phi_M(T)^{p^{r-t}}).
\end{align*}
\]

Therefore, if \( N = p^rD = p^rM \) with \( r \geq t > 0 \) and \( p \nmid M \), then
\[
(3.18) \quad J^\mathrm{new}_N \cap J_D = \{ x \in J_D[p] \mid \Phi_M(\delta_N)^{p^{r-t}} = 0 \}.
\]

By Theorem 1.11 and Lemma 2.4 \( J_D[p] \) is a free \( \mathbb{F}_p[T]/(\bar{P}_D(T)) \)-module of rank \( n - 1 \). It follows that \( J^\mathrm{new}_N \cap J_D \) is a free \( \mathbb{F}_p[T]/(\Phi_M(T)) \)-module of rank \( n - 1 \). In particular, it has dimension \( (n-1)\varphi(M)p^{r-t} \) over \( \mathbb{F}_p \). Furthermore,
\[
J^\mathrm{new}_N \cap J^\mathrm{new}_D \subseteq J^\mathrm{new}_N \cap J_D \cap J^\mathrm{new}_D \subseteq J_D[p] \cap J^\mathrm{new}_D = J^\mathrm{new}_D[p].
\]

Since \( \bar{P}_D(T) = \Phi_M(T)^{\varphi(p^{r-t})} \) divides \( \Phi_N(T) \) in \( \mathbb{F}_p[T] \), one has
\[
J_D[p] = (\ker \Phi_D(\delta_N))[p] \subseteq (\ker \Phi_N(\delta_N))[p] = J^\mathrm{new}_N[p].
\]

Therefore,
\[
(3.19) \quad J^\mathrm{new}_N \cap J^\mathrm{new}_D = J^\mathrm{new}_D[p].
\]

If \( 1 < D_1 < D_2 \) are divisors of \( N \) with \( D_1 \nmid D_2 \), then \( J^\mathrm{new}_N \cap J^\mathrm{new}_D = \{0\} \) since \( (\Phi_{D_1}(T), \Phi_{D_2}(T)) = \mathbb{Z}[T] \) by Lemma 1.13.

3.15. Recall that \( G(T) = P_N(T)/\Phi_N(T) = \prod_{D|N, 1 < D < N} \Phi_D(T) \in \mathbb{Z}[T] \). Consider the map \( G(\delta_N) : J_N \to J_N \). By Subsection 3.13 \( \ker G(\delta_N) = J^\mathrm{old}_N \). By the proof of Corollary 2.3 \( G(\delta_N)J_N = (\ker \Phi_N(\delta_N))^\circ = \ker \Phi_N(\delta_N) = J^\mathrm{new}_N \). Therefore, \( G(\delta_N) \) factors as \( J_N \to J^\mathrm{new}_N \xrightarrow{\lambda_N} J_N \). By an abuse of notation, we will denote the map \( J_N \to J^\mathrm{new}_N \) thus obtained by \( G(\delta_N) \) as well. There is an exact sequence
\[
(3.20) \quad 0 \leftarrow J^\mathrm{new}_N \xleftarrow{G(\delta_N)} J_N \xrightarrow{\lambda_N} J^\mathrm{old}_N \leftarrow 0.
\]

On the other hand, taking the dual exact sequence (11 Exercise 10.1, p131, or Proposition 2.42 in the complex abelian variety case) of (3.17) and identifying \( J^\nu_N \) with \( J^\nu_N \) via \( \lambda_N \), we obtain another exact sequence
\[
(3.21) \quad 0 \leftarrow (J^\mathrm{new}_N)^\vee \xleftarrow{\epsilon_N^\vee \circ \lambda_N} J_N \xrightarrow{\kappa_N} J^\mathrm{old}_N \leftarrow 0.
\]

So there exists an isomorphism by comparing (3.20) and (3.21):
\[
(3.22) \quad \kappa_N : J^\mathrm{new}_N \cong (J^\mathrm{new}_N)^\vee.
\]

Moreover, \( G(\delta_N) = \kappa_N^{-1} \circ \epsilon_N^\vee \circ \lambda_N \).

The induced polarization on \( J^\mathrm{new}_N \) from \( \lambda_N : J_N \to J^\nu_N \) is defined to be the composition of maps
\[
\lambda^\mathrm{new}_N : J^\mathrm{new}_N \xrightarrow{\epsilon_N} J_N \xrightarrow{\lambda_N} J^\nu_N \xrightarrow{\epsilon_N^\nu} (J^\nu_N)^\vee.
\]

It follows that \( \kappa_N^{-1} \circ \lambda^\mathrm{new}_N = G(\delta_N)|_{J^\mathrm{new}_N} \), the restriction of \( G(\delta_N) \) on \( J^\mathrm{new}_N \). Since \( \kappa_N \) is an isomorphism, we have
\[
(3.23) \quad \ker \lambda^\mathrm{new}_N = \ker (G(\delta_N)|_{J^\mathrm{new}_N}).
\]
3.16. Again let \( \mathcal{O} := \mathbb{Z}[[q]] \cong \mathbb{Z}[T]/(\Phi_N(T)) \) be the ring of integers in the cyclotomic field \( \mathbb{Q}((\zeta_N)) \). There is an embedding \( \mathcal{O} \hookrightarrow \text{End}(J_N^{\text{new}}) \) by \( \zeta_N \mapsto \delta_N \mid_{J_N^{\text{new}}} \), and we will identify \( \mathcal{O} \) with its image. Then

\[
G(\delta_N) \mid_{J_N^{\text{new}}} = G(\zeta_N) = \prod_{0 < i < N \atop \gcd(i,N) > 1} (\zeta_N - \zeta_i^{-1}) = (\zeta_N)^{N-1-\varphi(N)} \prod_{0 < i < N \atop \gcd(i,N) > 1} (1 - \zeta_i^{-1}).
\]

If \( \omega(N) = 1 \), i.e., \( q := N = p^r \) is a prime power, then \( G(T) = \rho_q/p(T) \), and \( \zeta_i^{-1} \) is a primitive \( q \)-th root of unity for all \( i \) with \( \gcd(i,q) > 1 \). Therefore,

\[
G(\zeta_q) = u_q \cdot (1 - \zeta_q)^{p^{r-1} - 1} \quad \text{for some} \quad u_q \in \mathcal{O}^\times.
\]

Now suppose that \( N = p_1^{s_1} \cdots p_t^{s_t} \) with \( t > 1 \). We would like to find the set

\[ \{ i \in \mathbb{Z}/N\mathbb{Z} \mid i \neq 0, \gcd(i,N) > 1, 1 - \zeta_i^{-1} \not\in \mathcal{O}^\times \}. \]

First note that \( 1 - \zeta_i^{-1} \not\in \mathcal{O}^\times \) if and only if \( \zeta_i^{(t-1)p_i^s} = 1 \) for some \( 1 \leq s \leq t \). Since it is also required that \( \gcd(i,N) > 1 \), necessarily \( p_s \mid i \). We are reduced to solve the following equation:

\[
\begin{cases} 
  i = 1 \pmod{N/p_s^r}, \\
  i = 0 \pmod{p_s}.
\end{cases}
\]

By the Chinese Remainder Theorem, (3.25) has a unique solution in \( \mathbb{Z}/(N/p_s^{r-1})\mathbb{Z} \). Lifting it to \( \mathbb{Z}/N\mathbb{Z} \), we obtain \( p_s^r \cdot (1 - \zeta_i)^{p^{r-1} - 1} \) solutions of (3.25) in \( \mathbb{Z}/N\mathbb{Z} \). Let \( q_s := p_s^r \), and \( \zeta_i := (\zeta_q)^{N/q_s} \). Then \( \zeta_i^{-1} \) is a primitive \( q_s \)-th root of unity for all solutions of (3.25). It follows that if \( N = \prod_{s=1}^t q_s = \prod_{s=1}^t p_s^r \) with \( t > 1 \), then

\[
G(\zeta_N) = u_N \prod_{s=1}^t (1 - \zeta_i)^{p_s^{r-1} - 1} \quad \text{for some} \quad u_N \in \mathcal{O}^\times.
\]

In summary, let

\[
c_N := \begin{cases} 
  (1 - \zeta_q)^{p^r - 1} & \text{if } N = q = p^r, \\
  \prod_{s=1}^t (1 - \zeta_i)^{p_s^{r-1} - 1} & \text{if } N = \prod_{s=1}^t q_s = \prod_{s=1}^t p_s^r \text{ and } t > 1.
\end{cases}
\]

Then

\[
\ker \lambda_N^{\text{new}} = J_N^{\text{new}}[c_N] := \ker(J_N^{\text{new}} \xrightarrow{c_N} J_N^{\text{new}}).
\]

We leave it as an exercise to show that

\[
N_{\mathcal{O}/\mathbb{Z}}(c_N) = \begin{cases} 
  p^{(p^r - 1)} & \text{if } N = q = p^r, \\
  \left( \prod_{p \mid N} p^{\varphi(N)} \right)^{\varphi(N)} & \text{if } \omega(N) > 1.
\end{cases}
\]

Since \( T_\ell J_N^{\text{new}} \) is a free \( \mathcal{O} \otimes \mathbb{Z}_\ell \)-module of rank \( n - 1 \) for all \( \ell \neq \text{Char}(K) \),

\[
\deg \lambda_N^{\text{new}} = |\ker(\lambda_N^{\text{new}})| = N_{\mathcal{O}/\mathbb{Z}}(c_N)^n - 1.
\]

**Theorem 3.17.** Suppose that \( q = p^r \) is a prime power with \( p \neq 2 \). There exists a principal polarization \( \lambda_q^{\text{new}} : J_q^{\text{new}} \to (J_q^{\text{new}})^{\vee} \).

**Proof.** If \( q = p \), then \( J_p^{\text{new}} = J_p \) is the Jacobian of \( C_{p,p} \), which is automatically principally polarized. So assume \( r \geq 2 \). By Subsection 3.3 the Rosati involution on \( \mathbb{Q}(\zeta_q) \subseteq \text{End}_0(J_q^{\text{new}}) \) induced by the polarization \( \lambda_N^{\text{new}} : J_N^{\text{new}} \to (J_N^{\text{new}})^{\vee} \) is the

\[
\lambda_q^{\text{new}}(w) = \frac{1}{\zeta_q} \sum_{i=0}^{q-1} \zeta_i w_i.
\]
complex conjugation $x \mapsto \bar{x}$. Since $p$ is odd, $(p^r - 1)/2 \in \mathbb{N}$. We have $c_{\bar{q}} = \tau_q^2$ with $\tau_q := (1 - \zeta_q)/(p^r - 1)/2 \in \mathbb{Z}[\zeta_q]$. Because $\bar{\tau}_q$ differs from $\tau_q$ by a unit, \[
abla_{\bar{\lambda}}_{\bar{q}} \lambda_{\bar{q}} \text{ new} \xrightarrow{\tau_q \bar{\tau}_q} J_{q_{\text{new}}}^{\text{new}} = \nabla_{\lambda} \text{ new} \text{ new} = \ker(J_{q_{\text{new}}} \rightarrow J_{q_{\text{new}}}). \]
It follows that $\lambda_{q_{\text{new}}} : J_{q_{\text{new}}} \rightarrow (J_{q_{\text{new}}})^\vee$ factors as
\[
\lambda_{q_{\text{new}}} = \tilde{\lambda}_{q_{\text{new}}} \circ (\tau_q \bar{\tau}_q) \]
for some isomorphism $\tilde{\lambda}_{q_{\text{new}}} : J_{q_{\text{new}}} \cong (J_{q_{\text{new}}})^\vee$. Under the isomorphism
\[
\text{Hom}^0(J_{q_{\text{new}}}, (J_{q_{\text{new}}})^\vee) \cong \text{End}^0(J_{q_{\text{new}}}), \quad \phi \mapsto (\lambda_{q_{\text{new}}})^{-1} \circ \phi,
\]
$\tilde{\lambda}_{q_{\text{new}}}$ is identified with $1/(\tau_q \bar{\tau}_q)$. Because $1/(\tau_q \bar{\tau}_q)$ is fixed by the Rosati involution and totally positive, $\tilde{\lambda}_{q_{\text{new}}}$ is induced from an ample line bundle $\mathcal{L}$ on $J_{q_{\text{new}}}$ by [10] Application 21.III]. In other words, $\tilde{\lambda}_{q_{\text{new}}}$ is a polarization, which is necessarily principal since $\tilde{\lambda}_{q_{\text{new}}}$ is an isomorphism.

Proof of Theorem 1.3. The first part of the theorem is proved by combining Theorem 1.1 and Theorem 2.9, noting that $Z(T)/\langle \Phi_D(T) \rangle \cong Z[\zeta_D]$ is integrally closed for any $D \in \mathbb{N}$. The fact that $J_{q_{\text{new}}}$ is isomorphic to $(J_{q_{\text{new}}})^\vee$ is proven in Subsection 3.1. The existence of a principal polarization $\tilde{\lambda}_{q_{\text{new}}} : J_{q_{\text{new}}} \rightarrow (J_{q_{\text{new}}})^\vee$ when $q = p^r$ and $p \neq 2$ is shown in Theorem 3.1.

4. Arithmetic Results

In this section, we prove the arithmetic results that are referred to previously.

4.1. We retain the notations of Subsection 2.1. In particular, $F(T), G(T) \in Z[T]$ are monic polynomials with $\gcd(F(T), G(T)) = 1$. Since $F(T)$ and $G(T)$ are co-prime, $Z[T]/\langle F(T), G(T) \rangle$ is a finite ring, and its cardinality is given by the absolute value of $\text{Res}(F(T), G(T))$ by (2.1). It follows that
\[
\text{Res}(F(T), G(T)) \in (F(T), G(T)) \cap Z,
\]
where the intersection is taken within $Z[T]$. Suppose that the following is the invariant factor decomposition [2] Theorem VII.4.2] of $Z[T]/\langle F(T), G(T) \rangle$ as a $Z$-module:
\[
Z[T]/\langle F(T), G(T) \rangle \cong Z/d_1Z \oplus \cdots \oplus Z/d_rZ,
\]
where $d_i | d_{i+1}$ for all $1 \leq i \leq r - 1$ and $d_i > 0$ for all $i$. We claim that
\[
(F(T), G(T)) \cap Z = d_rZ.
\]
Indeed, $(F(T), G(T)) \cap Z$ may be characterized as the annihilator of $Z[T]/\langle F(T), G(T) \rangle$ as a $Z$-module. This coincides with $d_rZ$ by (4.1). Clearly,
\[
|\text{Res}(F(T), G(T))| = |Z[T]/\langle F(T), G(T) \rangle| = d_1 \cdots d_r.
\]
It follows that
\[
d_r | \text{Res}(F(T), G(T)), \quad \text{and} \quad p | d_r \Leftrightarrow p | \text{Res}(F(T), G(T)).
\]
Recall that $\Phi_M(T) \in Z[T]$ denotes the $M$-th cyclotomic polynomial, and $\Phi_M(T) \in F_p[T]$ is its reduction modulo $p$. The number of distinct prime factors of $M$ is denoted by $\omega(M)$. 

Lemma 4.2. Suppose that $\gcd(q, D) = 1$ and $q = p^r$, then $\Phi_{qD}(T) = (\Phi_{D}(T))^{\varphi(q)}$ in $\mathbb{F}_p[T]$.

Proof. Let $\mu : \mathbb{N} \to \{0, \pm 1\}$ be the M"obius $\mu$-function ([6] Section 2.2)). More explicitly, $\mu(1) = 1$, and for each $m > 1$, $\mu(m) = (-1)^{\omega(m)}$ if $m$ is square free, and $\mu(m) = 0$ otherwise. We have

$$\Phi_N(T) = \prod_{D \mid N, D > 0} (T^D - 1)^{\mu(N/D)}.$$ 

Therefore,

$$\Phi_{qD}(T) = \prod_{m \mid qD} (T^m - 1)^{\mu(qD/m)} = \prod_{m_1 \mid D \ m_2 \mid q} (T^{m_1 m_2} - 1)^{\mu(qD/(m_1 m_2))}$$

$$= \prod_{m_1 \mid D} \prod_{m_2 \mid q} (T^{m_1} - 1)^{m_2 \mu(D/m_1) \mu(q/m_2)}$$

$$= \left( \prod_{m_1 \mid D} (T^{m_1} - 1)^{\mu(D/m_1)} \right)^{\varphi(q)}$$

since $\sum_{m_2 \mid q} m_2 \mu(q/m_2) = \varphi(q)$. □

Lemma 4.3. For each positive integer $D \mid N$, let $Q_{N,D}(T) = (T^N - 1)/(T^D - 1) \in \mathbb{Z}[T]$ be the polynomial in [3,10]. The ideal in $\mathbb{Z}[T]$ generated by $Q_{N,N/p}(T)$ for all primes $p \mid N$ is the principal ideal $(\Phi_N(T))$.

Proof. If $N = p^r$ is a prime power, then $Q_{N,N/p}(T) = \Phi_{p^r}(T)$. Now suppose that $N = \prod_{i=1}^l p_i^{r_i}$ with $t > 1$. For each $p \mid N$, let

$$F_p(T) := \frac{T^N - 1}{(T^{N/p} - 1) \Phi_N(T)} \in \mathbb{Z}[T].$$

Since $\mathbb{Z}[T]$ is a unique factorization domain, it is enough to show that the ideal $I$ generated by all $F_p(T)$ with $p \mid N$ is the unit ideal in $\mathbb{Z}[T]$. Suppose otherwise, then $I$ is contained in a maximal ideal $\mathfrak{m}$ of $\mathbb{Z}[T]$. By Subsection 4.1 and (3.7), $\mathfrak{m} \cap \mathbb{Z} = p\mathbb{Z}$ for some $p \mid N$. Let $\bar{I}$ be the canonical image of $I$ in the quotient ring $F_p[T]/(p)$. It is contained in the maximal ideal $\bar{\mathfrak{m}} \subset F_p[T]$. Suppose that $N = qD$ with $q = p^r$ and $p \nmid D$. Let $\ell$ be a prime divisor of $D$.

$$\bar{F}_p(T) = \frac{T^N - 1}{(T^{N/p} - 1) \Phi_N(T)} = \left( \frac{T^D - 1}{\Phi_D(T)} \right)^{\varphi(q)} = \left( \prod_{M \mid D, M \neq D} \bar{\Phi}_M(T) \right)^{\varphi(q)},$$

$$\bar{F}_\ell(T) = \frac{T^N - 1}{(T^{N/\ell} - 1) \Phi_N(T)} = \frac{(T^D - 1)^q}{(T^{D/\ell} - 1)^q \Phi_D(T)^{\varphi(q)}} = \bar{\Phi}_D(T)^{q/p} \prod_{d \nmid D, \ d \mid \ell} \bar{\Phi}_d(T)^{q}. $$

Because $\gcd(p, D) = 1$, the polynomial $T^D - 1 = \prod_{M \mid D} \bar{\Phi}_M(T)$ is separable in $\mathbb{F}_p[T]$. In particular, for any two distinct divisors $M_1, M_1$ of $D$, $\gcd(\bar{\Phi}_{M_1}(T), \bar{\Phi}_{M_1}(T)) = 1$. Clearly, $\bar{F}_p(T)$ is not divisible by $\bar{\Phi}_D(T)$. For any $M \mid D$ and $M \neq D$, we take $\ell$ to be a prime divisor of $D/M$, then $\gcd(\bar{F}_\ell(T), \bar{\Phi}_M(T)) = 1$. It follows that $\gcd(\bar{F}_{p_1}(T), \cdots, \bar{F}_{p_t}(T)) = 1$ and hence $\bar{I} = \mathbb{F}_p[T]$, which leads to a contradiction. □
Lemma 4.4. Suppose $D_1 < D_2$ are two positive integers with $D_1 \nmid D_2$. Then

$$(\Phi_{D_1}(T), \Phi_{D_2}(T)) = \mathbb{Z}[T].$$

Proof. Let $d = \gcd(D_1, D_2) < D_1$. Clearly, $(\Phi_{D_1}(T), \Phi_{D_2}(T)) \supseteq (Q_{D_1,d}(T), Q_{D_2,d}(T))$. So it is enough to prove that $(Q_{D_1,d}(T), Q_{D_2,d}(T)) = \mathbb{Z}[T]$. By the Euclidean algorithm, there exist $a(T), b(T) \in \mathbb{Z}[T]$ such that

$$a(T)(T^{D_1} - 1) + b(T)(T^{D_2} - 1) = T^d - 1.$$ 

The lemma follows by diving both sides by $T^d - 1$. 

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