Static and dynamic overdetermined problems in elasticity, plasticity and post-limit deformation

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Abstract. Some overdetermined problems, formulated for the Laplace equation in a circle (for arbitrary class of functions, not necessarily analytical) and a half-plane, the heat equation, the one-dimensional wave equation, the elasticity equations for planar deformation, the plasticity and deformations theory problem for a plane with a circular hole, dynamic elasticity theory problems for a half-plane and a half-space with simultaneously known on one its boundary the Dirichlet condition, and the Neumann condition are investigated. Analytical and numerical solutions are constructed, stress-strain states, thermal and other states are restored, internal structure of the body, concentrated sources are determined.

1. Introduction
There are various statements of boundary-value problems: Dirichlet, Neumann, Robin [1-3]. The paper deals with boundary value problems in Cauchy’s formulation, when both the Dirichlet and the Neumann conditions are described on the known contour. Some examples of analytical solutions are presented.

2. Laplace equation for the half-plane
Let the Laplace equation for the half-plane have the form:
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (-\infty < x < \infty, \ y \leq 0), \] (1)
with boundary conditions:
\[ u \big|_{y=0} = 2g_1(x), \quad \frac{\partial u}{\partial y} \big|_{y=0} = 2g_2'(x). \] (2)
General solution \( u = f(z) + \overline{f(z)} \) [4], where \( f(z) = g_1(z) - ig_2(z), \ z = x + iy, \ g_1, \ g_2 \) – given boundary functions.

3. Elasticity theory plane problem for a half-plane
Boundary conditions at \( y = 0 \) take the following form
\[ \sigma_y = f_1(x), \ \tau_{xy} = f_2(x), \ u_z = f_3(x), \ u_y = f_4(x), \] (3)
where \( f_1, f_2, f_3, f_4 \) – arbitrary boundary functions. For this special problem, there is a solution in Kolosov-Muskhelishvili potentials:

\[
\varphi(z) = \frac{1}{1 + N} \int f_1(z) dz + \frac{2\mu}{1 + N} [f_3(z) + if_4(z)] + C_1,
\]

\[
\psi(z) = \frac{N}{1 + N} \int f_1(z) dz - \frac{z}{1 + N} [f_3(z) - if_4(z)] - \frac{2\mu}{1 + N} [f_3(z) + if_4(z)] + C_2,
\]

(4)

where \( N \) is the Muskhelishvili parameter. A similar solution has other specified problems.

4. One-dimensional case of the wave equation

Consider the classic formulation of the one-dimensional wave equation. In the beginning, we consider a simplified situation. We have a homogeneous semi-infinite rod immersed into the soil. For it there is an equation describing the motion of its elements:

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2},
\]

(5)

where \( a = \sqrt{\rho/E} \) (\( \rho \) – density, \( E \) – Young's modulus), \( u \) – particle displacement.

There is a classic statement of the problem, when along the axis of the rod at some fixed time \( t_0 \) function of displacement and velocity function are assumed to be given (derivative of displacement function with reference to time \( t \)).

Let the initial conditions be given in the form:

\[
u \bigg|_{t=t_0} = \alpha(x), \quad \frac{\partial u}{\partial t} \bigg|_{t=t_0} = \alpha \cdot \beta'(x),
\]

(6)

where \( \beta'(x) \) – derivative of the function \( \beta(x) \) with reference to \( x \) (function \( \beta(x) \) is considered to be known).

Solution of (5) has the form

\[
u = 1/2[\alpha(x - at) - \beta(x - at)] - 1/2[\alpha(x + at) + \beta(x + at)].
\]

(7)

The characteristic triangle \( ABC \) (Figure 1) appears in a view of two progressive waves: the one going up is the first group of characteristic lines (6), another wave going down is the second. It is clear that on the boundary of the rod \( CD \) the information only from the first characteristic item will come as a difference

\[
1/2[\alpha(-at) - \beta(-at)] \text{ (at } x = 0 \).
\]

Another group of characteristic lines: \( 1/2[\alpha(x + at) + \beta(x + at)] \) will go down to infinity.

To solve the problem in the triangle \( CBD \) it is necessary to put on a \( CD \) boundary problem (Dirichlet; Neumann, Robin).

There is another formulation of the problem. We assume that on the boundary \( x = 0 \) (Figure 1) the very function of the displacement and its derivative with reference to coordinate \( x \) (deformation) is set. In other words, the function and its normal derivative to the surface are defined on the boundary \( x = 0 \) (Cauchy problem on coordinate!). It is assumed that these functions are independent from each other, and at the same time depend on the variable \( t \).
The feature of this approach, unlike the previous one, is that these functions can be measured on the boundary $x = 0$ at any time $t$. Knowing their distribution on the boundary $x = 0$, our purpose is to find their distribution along the axis of the rod at any time $t$.

**Figure 1.** Characteristics of the first problem.

Let us consider the solution of the following problem. The following conditions are set on the boundary $x = 0$:

$$u|_{x=0} = \alpha(t), \quad \frac{\partial u}{\partial t}|_{x=0} = \frac{\beta'(t)}{\alpha}. \quad (8)$$

D’Alembert solution in our case is

$$u = \frac{1}{2} \left[ \alpha \left( t - \frac{x}{a} \right) + \alpha \left( t + \frac{x}{a} \right) \right] + \frac{1}{2} \left[ \beta \left( t + \frac{x}{a} \right) - \beta \left( t - \frac{x}{a} \right) \right]. \quad (9)$$

**Figure 2.** Decision in characteristics.
There are also two waves – the one that comes to the boundary $x=0$, and the other, that leaves the boundaries (Figure 2).

5. The problem of elasticity (plane strain case)

To simplify the situation, we consider not the spatial pattern but the plane strain case. In the coordinate system $xOy$ we have a boundary with the equation $y=0$ where the normal stress $\sigma_y$, tangent stress $\tau_{xy}$, normal displacement $u_y$, tangent displacement $u_x$ are given. All these quantities are considered to be the functions of the coordinate $x$ and time $t$:

$$\sigma_y = \sigma_y(x,t), \ \tau_{xy} = \tau_{xy}(x,t), \ u_y = u_y(x,t), \ u_x = u_x(x,t). \ (10)$$

The procedure of determination of tension, deformations and displacements on the boundary $y=0$ and below it is the following: we find the derivative $\partial u_x / \partial x$ thereby on boundary $y=0$, thus the deformation $\varepsilon_x$ is found; then, according to Hooke's law in case of plane strain on this boundary the tension $\sigma_x$ from the relation $\varepsilon_x = \frac{1-\nu^2}{E} \sigma_x + \frac{\nu(1+\nu)}{E} \sigma_y$ is found. And in this case

$$\sigma_x = \frac{\nu \sigma_y}{1-\nu^2} + \frac{E \sigma_y}{1-\nu^2}. \ (11)$$

According to the known stresses $\sigma_y$ and $\sigma_x$ from Hooke's law we find deformation $\varepsilon_y$:

$$\varepsilon_y = -\frac{\nu(1+\nu)}{E} \sigma_x + \frac{1-\nu^2}{E} \sigma_y. \ (12)$$

To find a derivative $\partial u_y / \partial y$ we consider one more relation of Hooke's law

$$\frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) = \frac{\tau_{xy}}{2\mu} \ (13)$$

from which

$$\frac{\partial u_x}{\partial y} = \frac{2 \tau_{xy}(1+\nu)}{E} - \frac{\partial u_y}{\partial x}. \ (14)$$

Next, vector components $\omega_x = \frac{1}{2} \left( \frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y} \right)$ are found. Further, consider the layer $y=-h$. For this purpose we use the balance equations. From which we find

$$\tau_{xy}(x,-h,t) = \tau_{xy}(x,0,t) - h \cdot \left( \frac{u_x^0 - 2u_x^0 + u_x^0}{\Delta x^2} - \frac{\sigma_y(x+\Delta x) - \sigma_y(x)}{\Delta x} \right), \ (15)$$

$$\sigma_x(x,-h,t) = \sigma_x(x,0,t) - h \cdot \left( \frac{u_y^0 - 2u_y^0 + u_y^0}{\Delta y^2} - \frac{\tau_{xy}(x+\Delta x) - \tau_{xy}(x)}{\Delta x} \right). \ (16)$$

Here, derivatives $\partial \sigma_x / \partial x$ and $\partial \tau_{xy} / \partial x$ are also calculated at $t=t_0$. To find $u_x$ and $u_y$ on a layer $y=-h$ we use formulas for calculation $\varepsilon_x$, $\varepsilon_y$. We act in the same way for all other times. That is on the layer $y=-h$ all components of stresses and strains, components of a displacement vector are calculated for times $t=t_0 + \tau, \ t=t_0 + 2\tau, \ t=t_0 + 3\tau, \ldots, \ t=t_0 + nh \tau$.

Then we go down to the layer $y=-2h$. Here the equilibrium equations are used again. On the layer $y=-h$ we have displacements $u_x$ at the time points $t=t_0 + \tau, t=t_0 + 2\tau, t=t_0 + 3\tau$ etc. On this layer it is required to make the second derivative from the displacement $u_x$. This can be done at the time

$$t=t_0 + \tau : \left[ \frac{\partial^2 u_x}{\partial t^2} \right]_{t=t_0+\tau} = \frac{u_x^2 - 2u_x^1 + u_x^0}{\tau^2}. \ (17)$$

Therefore, for a layer $y=-h$ all stresses and strains referred by the time $t=t_0 + n\tau$ are considered. As the stress $\sigma_x$ at $y=-h$ is known, the derivative $\partial \sigma_x / \partial x$ at
\( t = t_0 + \tau \nu \) is known as well. From the balance equation the derivative \( \partial \tau_{xy} / \partial x \) is found again and as \( \tau_{xy} \) is known at \( y = -h \), we find \( \tau_{xy} \) on a layer \( y = -2h \) for time \( t = t_0 + \tau \). In the same way all quantities \( (\sigma, \sigma, \tau_{xy}, \varepsilon_x, \varepsilon_y, \varepsilon_{xy}) \), including \( u_x, u_y, u_t \) on this layer, are found. All these calculations are repeated for all other times. Further going down to \( y = -3h \) layer, the second derivatives from \( u_x, u_y \) with reference to time \( t \) in a time point \( t = t_0 + 2\tau \) are also determined. On a layer \( y = -2h \)

the derivative \( \left. \frac{\partial^2 u}{\partial t^2} \right|_{t=t_0+2\tau} = \frac{u_x^1 - 2u_x^2 + u_x^1}{\tau^2} \) is formed.

Process thus repeats. For verification of the proposed scheme the test is constructed.

6. Test problem

Suppose that at the origin of some coordinate system \( xOy \) we have a source of cylindrical disturbance. Environment deformation law around the source is described by equation:

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} = \frac{1}{\rho} \frac{\partial^2 u}{\partial t^2},
\]

(11)

where \( a^2 = \frac{(1 - \nu)E}{\rho(1 + \nu)(1 - 2\nu)} \), \( \rho \) – density, \( E \) – Young's modulus, \( \nu \) – Poisson's ratio.

The solution of (11) is found by separation of variables method.

For the analysis we take a displacement function \( g \) as:

\[
u = C_1 J_1(\lambda r) \sin(a\lambda t),
\]

(12)

where \( J_1(\lambda r) \) – Bessel function of the 1st kind. Then

\[
\varepsilon_r = \frac{\exp}{r} = \lambda C_1 \sin(a\lambda t) \frac{J_1(\lambda r)}{\lambda^2},
\]

\[
\varepsilon_\varphi = \frac{\partial u}{\partial r} = \lambda \left[ J_0(\lambda r) - \frac{J_1(\lambda r)}{\lambda r} \right] C_1 \sin(a\lambda t),
\]

\[
\sigma_r = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ (1 - \nu)\lambda J_0(\lambda r) - (1 + \nu)\lambda \frac{J_1(\lambda r)}{\lambda r} \right] C_1 \sin(a\lambda t),
\]

\[
\sigma_\varphi = \frac{E}{(1 + \nu)(1 - 2\nu)} \left[ \nu \lambda J_0(\lambda r) - (1 - \nu)\lambda \frac{J_1(\lambda r)}{\lambda r} \right] C_1 \sin(a\lambda t).
\]

(13)

Moving to Cartesian coordinates \( x, y \) we consider the trace of the solution (12), (13) on the boundary \( y = H \). In dimensionless quantities it has the form:

\[
\tilde{\sigma}_y = \frac{1}{2(1 - \nu)} \left[ J_0(\sqrt{\tilde{x}^2 + 1}) - (1 - 2\nu)J_0(\sqrt{\tilde{x}^2 + 1}) - 2(1 - 2\nu) J_1(\sqrt{\tilde{x}^2 + 1}) \frac{\tilde{x}^2 - 1}{\sqrt{\tilde{x}^2 + 1}} \right] C_1 \sin(\tilde{t}),
\]

\[
\tilde{\tau}_{xy} = \frac{(1 - 2\nu)}{2(1 - \nu)} \left[ (1 - 2\nu)J_0(\sqrt{\tilde{x}^2 + 1}) - 2(1 - 2\nu) J_1(\sqrt{\tilde{x}^2 + 1}) \frac{\tilde{x}}{\sqrt{\tilde{x}^2 + 1}} \right] C_1 \sin(\tilde{t}),
\]

\[
\tilde{\mu}_x = \tilde{C}_1 J_1(\sqrt{\tilde{x}^2 + 1}) \frac{\tilde{x}}{\sqrt{\tilde{x}^2 + 1}} \sin(\tilde{t}),
\]

\[
\tilde{\mu}_y = \tilde{C}_1 J_1(\sqrt{\tilde{x}^2 + 1}) \frac{1}{\sqrt{\tilde{x}^2 + 1}} \sin(\tilde{t}),
\]

(14)

(15)
Task: let on the boundary $y = H$ traces of the test solution be set. Figure 3 shows the dependencies of various quantities $(\sigma_y, \tau_{xy}, u_x, u_y)$ on $x$. The numerical and analytical solutions coincide.

7. Creating solutions and examples

It is visible that pictures are identical. The solid line in Figure 3 shows dependencies for the point with coordinates on $y = H$, the dashed line for $H - 2$, the dotted one for $H - 4$, the dotted-dashed line for $H - 6$.

Analyzing the curves it is possible to make the conclusion: on the ordinate where there is a source of perturbation firstly the tangent stresses turns into zero, secondly, vertical displacements $u_y$ are accepted as constants. In the reviewed example, $u_y$ is equal to zero. As for abscissa of perturbation source, at approximation to the perturbation source along the straight line $y = y_0$, stress $\sigma_y$ increases up to the maximum value with reduction of $x$ to zero (the point zero can be taken for the perturbation source).

![Figure 3. Analytical and numerical solution: a) $\sigma_y(x)$, b) $\tau_{xy}(x)$, c) $u_x(x)$, d) $u_y(x)$.](image)

abscissa). These features can serve as a criterion for defining the location of perturbation source.

Let us consider another example. Let the half-plane be free ($\sigma_y = 0, \tau_{xy} = 0$) and, besides, on its boundary distribution of displacements is set as following:

$$u_x = \frac{Ax \sin(\omega t)}{x^2 + b^2}, \quad u_y = \frac{AH \sin(\omega t)}{x^2 + b^2},$$

where $t \geq 0$. Using the proposed scheme of the solution, we obtain required distributions; they are considered in the dimensionless variables $(A = 1, H = 1, b = 1, \omega = 1)$. In Figure 4 curves of these values are shown for dimensionless time $t = \pi/2$. 


Figure 4. Numerical solution of the task of the offered scheme:  a) $\sigma_y(x)$, b) $\tau_{xy}(x)$, c) $u_x(x)$, d) $u_y(x)$.

8. Conclusions
The solutions of some basic boundary value problems with initial conditions are developed. Finite-difference algorithms for Cauchy problems numerical solutions are constructed. Its verification is carried out using the available analytical solutions. The problem of distinguishing features in numerical solutions with application of solutions for lumped sources in an infinite medium is considered. The problem of boundary functions smoothing (constructed from its discrete values on the boundary) is solved with correlation analysis usage to ensure correctness of assigned problems solutions (stable dependence on the input data).

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