THIRD HOMOLOGY OF PERFECT CENTRAL EXTENSIONS

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Abstract. For a central perfect extension of groups \( A \rightarrowtail G \rightarrow Q \), first we study the natural image of \( H_3(A, \mathbb{Z}) \) in \( H_3(G, \mathbb{Z}) \). As a particular case, we show that if the extension is universal this image is 2-torsion. Moreover when the plus-construction of the classifying space of \( Q \) is an \( H \)-space, we also study the kernel of the surjective homomorphism \( H_3(G, \mathbb{Z}) \rightarrow H_3(Q, \mathbb{Z}) \).

Introduction

Homologies and cohomologies are important invariants that one can assign to a given group. Usually these (co)homology groups are too complicated to be computed explicitly. Therefore in many cases results allowing to compare the homologies of different groups become quite important.

In this article, we study such homomorphism for the third homology groups of a perfect central extension. A central extension

\[ A \rightarrowtail G \rightarrow Q \]

is called perfect if \( G \) is a perfect group, i.e. if \( G = [G, G] \). The aim of the current paper is to study the natural maps \( H_3(A, \mathbb{Z}) \rightarrow H_3(G, \mathbb{Z}) \) and \( H_3(G, \mathbb{Z}) \rightarrow H_3(Q, \mathbb{Z}) \) for such extensions.

The interest to this problem comes from two sources. First from algebraic and Hermitian \( K \)-theory, where various type of universal central extensions appear [13, §5], [14, Chap. 4], [9, §1.4, §5.5]. Second from algebraic topology and homology of groups that often one has to deal with different types of spectral sequences that usually are difficult to deal with [5, Chap. VII].

In Section 1, we give a quick overview of the (lower) homologies of a central extension and discuss a result of Eckmann and Hilton.

In Section 2 we study Whitehead’s quadratic functor which plays an important role in this article.

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In Section 3 we show that if $A$ is a central subgroup of a group $G$ such that $A \subseteq G'$, e.g. $G$ a perfect group, then the image of the natural map

\[(0.1)\quad H_3(A, \mathbb{Z}) \to H_3(G, \mathbb{Z})/\rho_*(A \otimes \mathbb{Z} H_2(G, \mathbb{Z}))\]

in 2-torsion, where $\rho : A \times G \to G$ is the usual product map. In particular, if $A \hookrightarrow G \twoheadrightarrow Q$ is a universal central extension, then the image of $H_3(A, \mathbb{Z})$ in $H_3(G, \mathbb{Z})$ is 2-torsion.

Section 4 has $K$-theoretic flavour. We show that if $A \hookrightarrow G \twoheadrightarrow Q$ is a perfect central extension such that $K(Q, 1)^+$, the plus-construction of the classifying space of $Q$, is an $H$-space, then we have the exact sequence

\[(0.2)\quad A/2 \to H_3(G, \mathbb{Z})/\rho_*(A \otimes \mathbb{Z} H_2(G, \mathbb{Z})) \to H_3(Q, \mathbb{Z}) \to 0.\]

Moreover we prove that with this extra condition, the map (0.1) is trivial. In particular, if the extension is universal, then the image of $H_3(A, \mathbb{Z})$ in $H_3(G, \mathbb{Z})$ is trivial.

In Section 5, we study the third Hermitian $K$-group of a ring. Let $R$ be a ring with involution and a central element $\epsilon$ such that $\epsilon^2 = 1$. Using the exact sequence (0.2) we show that we have the exact sequence

$$\epsilon K^h_2(R)/2 \to \epsilon K^h_3(R) \to H_3(\epsilon \text{EO}(R), \mathbb{Z}) \to 0,$$

where $\epsilon \text{EO}(R)$ is the elementary subgroup of the stable orthogonal group $\epsilon O(R)$ of the pair $(R, \epsilon)$.

Finally in Section 6 we prove a cohomological version of the exact sequence (0.2). In fact we show that if $A \hookrightarrow G \twoheadrightarrow Q$ is a perfect central extension such that $K(Q, 1)^+$ is an $H$-space, then we have the exact sequence

$$0 \to \text{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \to H^3(Q, \mathbb{Z}) \to H^3(G, \mathbb{Z}) \xrightarrow{\rho^*} (A \otimes \mathbb{Z} H_2(G, \mathbb{Z}))^*,\]

where for an abelian group $M$, $M^*$ is the dual group $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. In particular, if the extension is universal, then we have the exact sequence

$$0 \to \text{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \to H^3(Q, \mathbb{Z}) \to H^3(G, \mathbb{Z}) \to 0.$$

**Notations.** We denote the commutator subgroup of a group $G$, by $G'$ or $[G, G]$. If $A \to B$ is a homomorphism of abelian groups, by $B/A$ we mean $\text{coker}(A \to B)$ and by $\text{im}(A)$ we mean the image of $A$ in $B$.

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1. THE HOMOLOGY OF CENTRAL EXTENSIONS

Let $A \hookrightarrow G \rightarrow Q$ be a central extension. Standard classifying space theory gives a (homotopy theoretic) fibration \cite[Chap. 6]{6} of Eilenberg-MacLane spaces

$$K(A,1) \to K(G,1) \to K(Q,1)$$

(see \cite[5.1.28]{14}). From this we obtain the fibration

$$K(G,1) \to K(Q,1) \to K(A,2)$$

(see \cite[Lemma 3.4.2]{11}). By studying the Serre spectral sequences associated to the morphism of fibrations

$$\Omega K(A,2) \longrightarrow PK(A,2) \longrightarrow K(A,2)$$

\[6, \text{ Chap. 9}, \text{Section 6.4}, \] Eckmann and Hilton proved the following theorem.

**Theorem 1.1.** (\cite[Theorem 1.1]{7}) For any central extension $A \hookrightarrow G \rightarrow Q$, there is a natural map $\tau: H_4(K(A,2),\mathbb{Z}) \to A \otimes \mathbb{Z} H_1(G,\mathbb{Z})$ and a natural exact sequence

$$H_4(Q,\mathbb{Z}) \to H_4(K(A,2),\mathbb{Z}) \to H_3(G,\mathbb{Z})/\rho^* (A \otimes \mathbb{Z} H_2(G,\mathbb{Z}) \oplus \text{Tor}_1^\mathbb{Z} (A, \mathbb{Z})) \to H_3(Q,\mathbb{Z}) \to 0,$$

where $\rho: A \times G \to G$ is the product map $(a,g) \mapsto ag$.

**Corollary 1.2.** If $A \hookrightarrow G \rightarrow Q$ is a central perfect extension, then we have the exact sequence

$$H_4(Q,\mathbb{Z}) \to H_4(K(A,2),\mathbb{Z}) \to H_3(G,\mathbb{Z})/\rho^* (A \otimes \mathbb{Z} H_2(G,\mathbb{Z})) \to H_3(Q,\mathbb{Z}) \to 0.$$

The group $H_4(K(A,2),\mathbb{Z})$ plays a very important role in this article. It has interesting properties and has been studied extensively in \cite[Chap. 2]{19} and \cite[Section 13]{8}. It is closely connected to Whitehead’s quadratic functor, which is the topic of the next section.

2. WHITEHEAD’S QUADRATIC FUNCTOR

A function $\theta: A \to B$ of (additive) abelian groups is called a quadratic map if

1. for any $a \in A$, $\theta(a) = \theta(-a)$,
2. the function $A \times A \to B$ with $(a,b) \mapsto \theta(a+b) - \theta(a) - \theta(b)$ is bilinear.

For any abelian group $A$, there is a universal quadratic map

$$\gamma: A \to \Gamma(A)$$
such that for any quadratic map \( \theta : A \to B \), there is a unique group homomorphism \( \Theta : \Gamma(A) \to B \) such that
\[
\Theta \circ \gamma = \theta.
\]
It is easy to see that \( \Gamma \) is a functor from the category of abelian groups to itself.

The functions
\[
\phi : A \to A/2, \quad \phi(a) = \overline{a}, \quad \text{and} \quad \psi : A \to A \otimes \mathbb{Z} A, \quad \psi(a) = a \otimes a,
\]
are quadratic maps. Thus we get the canonical homomorphisms
\[
\Phi : \Gamma(A) \to A/2, \quad \gamma(a) \mapsto \overline{a} \quad \text{and} \quad \Psi : \Gamma(A) \to A \otimes \mathbb{Z} A, \quad \gamma(a) \mapsto a \otimes a.
\]
Clearly \( \Phi \) is surjective. Moreover \( \text{coker}(\Psi) = A \wedge A \cong H_2(A, \mathbb{Z}) \) [5, Theorem 6.4(iii), Chap. V] and hence we have the exact sequence
\[(2.1) \quad \Gamma(A) \xrightarrow{\Psi} A \otimes \mathbb{Z} A \to H_2(A, \mathbb{Z}) \to 0.\]

Furthermore we have the bilinear pairing
\[
\Omega : A \otimes \mathbb{Z} A \to \Gamma(A), \quad a \otimes b \mapsto [a, b] := \gamma(a + b) - \gamma(a) - \gamma(b).
\]
It is easy to see that for any \( a, b, c \in A \),
\[
\begin{align*}
(1) \quad & [a, b] = [b, a], \\
(2) \quad & [a + b, c] = [a, c] + [b, c], \\
(3) \quad & \Phi([a, b]) = 0, \\
(4) \quad & \Psi([a, b]) = a \otimes b + b \otimes a.
\end{align*}
\]

Using (1) and (2), for any \( a, b, c \in A \), we obtain
\[
\begin{align*}
(1) \quad & \gamma(a) = \gamma(-a), \\
(2) \quad & \gamma(a + b + c) - \gamma(a + b) - \gamma(a + c) - \gamma(b + c) + \gamma(a) + \gamma(b) + \gamma(c) = 0.
\end{align*}
\]

Using these properties we can construct \( \Gamma(A) \). Let \( A \) be the free abelian group generated by the symbols \( w(a), a \in A \). Set
\[
\Gamma(A) := A/\mathcal{R},
\]
where \( \mathcal{R} \) is the subgroup generated by the elements
\[
- w(a) - w(-a) \quad \text{and} \quad - w(a + b + c) - w(a + b) - w(a + c) - w(b + c) + w(a) + w(b) + w(c),
\]
with \( a, b, c \in A \). Now
\[
\gamma : A \to \Gamma(A)
\]
is given by \( a \mapsto \overline{w(a)} \). It is easy to show that
\[
[a, a] = 2\gamma(a).
\]
Thus the composite
\[
\Gamma(A) \xrightarrow{\Psi} A \otimes \mathbb{Z} A \xrightarrow{\Omega} \Gamma(A)
\]
coincides with multiplication by 2. This implies that the kernel of $\Psi$ is 2-torsion. Moreover, one sees easily that the composite

$$A \otimes \mathbb{Z} A \xrightarrow{\Omega} \Gamma(A) \xrightarrow{\Psi} A \otimes \mathbb{Z} A$$

sends $a \otimes b$ to $a \otimes b + b \otimes a$. It is known that the sequence

$$(2.2) \quad A \otimes \mathbb{Z} A \xrightarrow{\Omega} \Gamma(A) \xrightarrow{\Phi} A/2 \to 0$$

is exact.

**Proposition 2.1.** For any abelian group $A$, $\Gamma(A) \simeq H_4(K(A,2),\mathbb{Z})$.

**Proof.** See [8, Theorem 21.1] □

Now a topological proof of the exact sequence (2.1) can be obtained by applying Theorem 1.1 to the central extension

$$A \xrightarrow{\sim} A \to \{1\}.$$

A topological proof of the exact sequence (2.2) may be obtained by studying the Serre spectral sequence associated to the path space fibration

$$\Omega K(A,2) \to PK(A,2) \to K(A,2).$$

We refer the reader to [7, pages 349–350] for analysis of the Serre spectral sequence associated to a fibration $F \to E \to B$ such that $F$ is 0-connected, $B$ is 1-connected and $H_3(B,\mathbb{Z}) = 0$. Observe that

$$\Omega K(A,n) = K(A,n+1)$$

and for $n \geq 3$

$$H_{n+2}(K(A,n),\mathbb{Z}) \simeq A/2$$

[20, Theorem 3.20, Chap. XII]. We should mention that $K(A,2)$ is an $H$-space [20, Theorem 7.11, Chap. V] and the pairing

$$A \otimes \mathbb{Z} A \to H_4(K(A,2),\mathbb{Z})$$

is induced by the $H$-space structure of $K(A,2)$ (see [7, page 349]).

### 3. Third homology of central subgroups

Let $A$ be a central subgroup of $G$ such that $A \subseteq G' = [G,G]$. The condition $A \subseteq G'$ is equivalent to the triviality of the homomorphism of homology groups $H_1(A,\mathbb{Z}) \to H_1(G,\mathbb{Z})$. Let $n$ be a positive integer. From the commutative diagram

$$\begin{array}{ccc}
A \times A & \xrightarrow{\mu} & A \\
\downarrow & & \downarrow \\
A \times G & \xrightarrow{\rho} & G,
\end{array}$$

(3.1)
where $\mu$ and $\rho$ are the usual product maps, we obtain the commutative diagram
\[
\begin{array}{ccc}
H_{n-1}(A, \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(A, \mathbb{Z}) & \longrightarrow & H_n(A, \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_{n-1}(A, \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(G, \mathbb{Z}) & \longrightarrow & H_n(G, \mathbb{Z}).
\end{array}
\]
The Pontryagin product (see [5, Chap. V, §5]) induces a natural map
\[
\bigwedge^n A \rightarrow H_n(A, \mathbb{Z})
\]
which always is injective [5, Theorem 6.4(i), Chap. V]. The above diagram shows that the composite
\[
\bigwedge^n A \rightarrow H_n(A, \mathbb{Z}) \rightarrow H_n(G, \mathbb{Z})
\]
is trivial. Since the natural map
\[
\bigwedge^n A \rightarrow H_n(A, \mathbb{Z})
\]
is an isomorphism [5, Theorem 6.4(ii), Chap. V], the group $H_n(A, \mathbb{Z})/\bigwedge^n A$ is torsion. This implies that the image of $H_n(A, \mathbb{Z})$ in $H_n(G, \mathbb{Z})$ is torsion.

Our first main result concerns the map $H_3(A, \mathbb{Z}) \rightarrow H_3(G, \mathbb{Z})$. For the study of this map we need the following well-known result.

**Proposition 3.1.** For any abelian group $A$ we have the exact sequence
\[
0 \rightarrow \bigwedge^3 A \rightarrow H_3(A, \mathbb{Z}) \rightarrow \text{Tor}_1^\mathbb{Z}(A, A)^{\Sigma_2} \rightarrow 0,
\]
where $\Sigma_2 = \{\text{id}, -\sigma\}$. The homomorphism on the right side of the exact sequence is obtained from the composition
\[
H_3(A, \mathbb{Z}) \xrightarrow{\Delta} H_3(A \times A, \mathbb{Z}) \rightarrow \text{Tor}_1^\mathbb{Z}(A, A),
\]
where $\Delta$ is the diagonal map $A \rightarrow A \times A$, $a \mapsto (a, a)$, and the action of $\sigma$ on $\text{Tor}_1^\mathbb{Z}(A, A)$ is induced by the involution $\iota : A \times A \rightarrow A \times A$, $(a, b) \mapsto (b, a)$.

**Proof.** See [16, Lemma 5.5] or [4, Section 6]. \qed

Here is our first main result.

**Theorem 3.2.** Let $A$ be a central subgroup of $G$ such that $A \subseteq G'$. Then the image of the natural map
\[
H_3(A, \mathbb{Z}) \rightarrow H_3(G, \mathbb{Z})/\rho_*(A \otimes_{\mathbb{Z}} H_2(G, \mathbb{Z}))
\]
is 2-torsion.
**Proof.** By Proposition 3.1 we have the exact sequence

$$0 \to \wedge_3^3 A \to H_3(A, \mathbb{Z}) \to \text{Tor}_1^\mathbb{Z}(A, A) \to 0.$$ 

From the diagram (3.1), we obtain the commutative diagram

$$\begin{array}{ccc}
\tilde{H}_3(A \times A, \mathbb{Z}) & \xrightarrow{\mu_*} & H_3(A, \mathbb{Z}) \\
\downarrow & & \downarrow \\
\tilde{H}_3(A \times G, \mathbb{Z}) & \xrightarrow{\rho_*} & H_3(G, \mathbb{Z}),
\end{array}$$

where

$$\tilde{H}_3(A \times A, \mathbb{Z}) := \text{ker}(H_3(A \times A, \mathbb{Z}) \xrightarrow{(p_1, p_2, \alpha)} H_3(A, \mathbb{Z}) \oplus H_3(A, \mathbb{Z})).$$

As we have seen, the condition $A \subseteq G'$ implies that the composite

$$\wedge_3^3 A \to H_3(A, \mathbb{Z}) \to H_3(G, \mathbb{Z})$$

is trivial. From this fact together with the Künneth formula [5, Corollary 5.8, Chap. V] for $\tilde{H}_3(A \times A, \mathbb{Z})$ we obtain the commutative diagram

$$\begin{array}{ccc}
\text{Tor}_1^\mathbb{Z}(A, A) & \xrightarrow{\mu_*} & \text{Tor}_1^\mathbb{Z}(A, A) \oplus_2 \text{Tor}_1^\mathbb{Z}(A, A)
\end{array}$$

Note that

$$\text{im}(H_2(A, \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(G, \mathbb{Z}) \to H_3(G, \mathbb{Z})) \subseteq \text{im}(A \otimes_{\mathbb{Z}} H_2(G, \mathbb{Z}) \to H_3(G, \mathbb{Z}))$$

(see [15, Proposition 4.4, Chap. V]). Since the map

$$\text{Tor}_1^\mathbb{Z}(A, A) = \text{Tor}_1^\mathbb{Z}(A, H_1(A, \mathbb{Z})) \to \text{Tor}_1^\mathbb{Z}(A, H_1(G, \mathbb{Z}))$$

is trivial, we see that $\rho_* \circ \tilde{\alpha} \circ \bar{\alpha}^{-1}$ is trivial. This shows that $\text{inc}_* \circ \beta^{-1} \circ \bar{\mu}_*$ is trivial. Therefore the image of $H_3(A, \mathbb{Z})$ in $H_3(G, \mathbb{Z})$ is equal to the image of

$$\text{Tor}_1^\mathbb{Z}(A, A) \oplus_2 \text{Tor}_1^\mathbb{Z}(A, A).$$

By the above arguments, one sees that the homomorphism

$$\bar{\mu}_* : \text{Tor}_1^\mathbb{Z}(A, A) \to \text{Tor}_1^\mathbb{Z}(A, A) \oplus_2 \text{Tor}_1^\mathbb{Z}(A, A)$$

is induced by the composition $A \times A \xrightarrow{\mu} A \xrightarrow{\Delta} A \times A.$
If we apply Theorem 1.1 to the central extension \( A \hookrightarrow A \twoheadrightarrow 1 \), we obtain the exact sequence
\[
0 \to \ker(\Psi) \to H_4(K(A,2),\mathbb{Z}) \xrightarrow{\Psi} A \otimes_{\mathbb{Z}} A \to H_2(A,\mathbb{Z}) \to 0,
\]
where
\[
\ker(\Psi) \cong H_3(A,\mathbb{Z})/\mu_*(A \otimes_{\mathbb{Z}} H_2(A,\mathbb{Z})) \oplus \text{Tor}^\mathbb{Z}_1(A,\mathbb{Z}).
\]
Clearly \( \mu_*(A \otimes_{\mathbb{Z}} H_2(A,\mathbb{Z})) = \bigwedge^3 A \subseteq H_3(A,\mathbb{Z}) \). Therefore
\[
\ker(\Psi) \cong \text{Tor}^\mathbb{Z}_1(A,\mathbb{Z})/(\Delta_A \circ \mu)_*(\text{Tor}^\mathbb{Z}_1(A,\mathbb{Z})).
\]
But in the previous section we have seen that \( \ker(\Psi) \) is 2-torsion. This proves the claim.

\[\square\]

**Corollary 3.3.** If \( A \hookrightarrow G \twoheadrightarrow Q \) is a universal central extension, then the image of \( H_3(A,\mathbb{Z}) \) in \( H_3(G,\mathbb{Z}) \) is 2-torsion.

**Proof.** Since the extension is universal, the homology groups \( H_1(G,\mathbb{Z}) \) and \( H_2(G,\mathbb{Z}) \) are trivial [14, Corollary 4.1.18]. Thus the claim follows from the previous theorem. \( \square \)

**Remark 3.4.** (i) If \( A \) is a central subgroup of a group \( G \), then the same argument as in the proof of Theorem 3.2 shows that the image of the natural map
\[
H_3(A,\mathbb{Z}) \to H_3(G,\mathbb{Z})/\rho_*\bar{H}_3(A \times G,\mathbb{Z})
\]
is 2-torsion.

(ii) In Proposition 4.5 below, we show that if \( A \hookrightarrow G \twoheadrightarrow Q \) is a universal central extension such that \( K(Q,1)^+ \) is an \( H \)-space, then the natural image of \( H_3(A,\mathbb{Z}) \) in \( H_3(G,\mathbb{Z}) \) is trivial.

### 4. Third homology of central extensions over \( H \)-groups

For any sequence of abelian groups \( A_n, n \geq 2 \), Berrick and Miller constructed a perfect group \( Q \) such that \( H_n(Q,\mathbb{Z}) \cong A_n \) [3, Theorem 1].

Let \( A \) be an abelian group. By the result of Berrick and Miller, there is a perfect group \( Q \) such that \( H_2(Q,\mathbb{Z}) \cong A \) and \( H_4(Q,\mathbb{Z}) = 0 \). Now if \( A \hookrightarrow G \twoheadrightarrow Q \) is the universal central extension of \( Q \) [14, Theorem 4.1.3], then by Corollary 1.2 we have the exact sequence
\[
0 \to H_4(K(A,2),\mathbb{Z}) \to H_3(G,\mathbb{Z}) \to H_3(Q,\mathbb{Z}) \to 0.
\]
This example shows that in general for an universal central extension \( A \hookrightarrow G \twoheadrightarrow Q \), the kernel of \( H_3(G,\mathbb{Z}) \to H_3(Q,\mathbb{Z}) \) can be very complicated.

A group \( Q \) is called *quasi-perfect* if its commutator subgroup is perfect, i.e., \( [Q',Q'] = Q' \).

A quasi-perfect group \( Q \) is called an *\( H \)-group* if \( K(Q,1)^+ \), the plus-construction of \( K(Q,1) \) with respect to \( Q' \) (see [10, Section 1.1]), is an \( H \)-space. Note that for a group \( G \), \( K(G,1) \) is an \( H \)-space if and only if \( G \) is abelian.
Example 4.1. (a) A quasi-perfect group $Q$ with an internal “direct sum”, that is a homomorphism $Q \oplus Q \to Q$, is called a direct sum group if it satisfies in the following two conditions:

(i) for any $g_1, \ldots, g_k \in Q'$ and any $g \in Q$, there is $h \in Q'$ such that
$$gg_i g^{-1} = hg_i h^{-1} \text{ for } 1 \leq i \leq k,$$
(ii) for any $g_1, \ldots, g_n \in Q$, there are $c, d \in Q$ such that
$$c(g_i \oplus 1)c^{-1} = d(1 \oplus g_i)d^{-1} = g_i.$$

By a theorem of Wagoner direct sum groups are $H$-groups [18, Proposition 1.2].

For a ring $R$ with unit, the stable general linear group $GL(R)$, the stable orthogonal group $O(R)$ (see the next section) are direct sum groups [10, Example on page 323]. For more examples and some details see [10, Section 1.3].

(b) For any abelian group $A$, Berrick has constructed a perfect group $Q$ such that $K(Q, 1)^+$, is homotopy equivalence to $K(A, 2)$ [2, Corollary 1.4]. Since $K(A, 2)$ is an $H$-space [20, Theorem 7.11, Chap. V], $Q$ is an $H$-group.

The next theorem is our second main result.

**Theorem 4.2.** Let $A \rightarrow G \rightarrow Q$ be a perfect central extension. If $Q$ is an $H$-group, then we have the exact sequence

$$A/2 \to H_3(G, \mathbb{Z})/\rho_*(A \otimes_{\mathbb{Z}} H_2(G, \mathbb{Z})) \to H_3(Q, \mathbb{Z}) \to 0.$$ 

**Proof.** From the central extension and the fact that $Q$ is perfect we have the fibration

$$K(A, 1) \to K(G, 1)^+ \to K(Q, 1)^+$$

[21, Proposition 1], [1, Theorem 6.4]. From this we obtain the fibration

(4.1) $$K(G, 1)^+ \to K(Q, 1)^+ \to K(A, 2)$$

[11, Lemma 3.4.2]. It is known that $K(A, 2)$ is an $H$-space. Moreover the map $K(Q, 1)^+ \to K(A, 2)$ is an $H$-map [22, Proposirion 2.3.1]. Since the plus-construction does not change the homology [10, Theorem 1.1.1], from the Serre spectral sequence of the fibration (4.1) we obtain (see [7, pages 347–349]) the exact sequence

$$H_4(Q, \mathbb{Z}) \to H_4(K(A, 2), \mathbb{Z}) \to H_3(G, \mathbb{Z})/\rho_*(A \otimes_{\mathbb{Z}} H_2(G, \mathbb{Z})) \to H_3(Q, \mathbb{Z}) \to 0.$$ 

From the commutative diagram, up to homotopy, of $H$-spaces and $H$-maps

$$\begin{array}{ccc}
K(Q, 1)^+ \times K(Q, 1)^+ & \rightarrow & K(Q, 1)^+ \\
\downarrow & & \downarrow \\
K(A, 2) \times K(A, 2) & \rightarrow & K(A, 2)
\end{array}$$
we obtain the commutative diagram
\[
\begin{array}{ccc}
H_2(Q, \mathbb{Z}) \otimes \mathbb{Z} H_2(Q, \mathbb{Z}) & \longrightarrow & H_4(Q, \mathbb{Z}) \\
\downarrow & & \downarrow \\
A \otimes \mathbb{Z} A & \longrightarrow & H_4(K(A, 2), \mathbb{Z}).
\end{array}
\]

Since $G$ is perfect, $H_2(Q, \mathbb{Z}) \rightarrow A$ is surjective. This gives us the surjective map
\[H_4(K(A, 2), \mathbb{Z})/\text{im}(A \otimes \mathbb{Z} A) \twoheadrightarrow H_4(Q, \mathbb{Z})/\text{im}(H_4(Q, \mathbb{Z})).\]

From this, Proposition 2.1 and the exact sequence (2.2) we obtain the desired exact sequence. \hfill \square

**Corollary 4.3.** Let $A \rightarrow G \rightarrow Q$ be an universal central extension. If $Q$ is an $H$-group, then we have the exact sequence
\[A/2 \rightarrow H_3(G, \mathbb{Z}) \rightarrow H_3(Q, \mathbb{Z}) \rightarrow 0.\]

For a perfect central extension $A \rightarrow G \rightarrow Q$, Theorem 3.2 implies that the image of $H_3(A, \mathbb{Z})$ in $H_3(G, \mathbb{Z})/\rho_\ast(A \otimes \mathbb{Z} H_2(G, \mathbb{Z}))$ is 2-torsion. In the next proposition we show that this image is trivial provided that $Q$ is an $H$-group. We need the following well-known fact.

**Lemma 4.4.** For any abelian group $A$, we have the short exact sequence
\[0 \rightarrow A/2 \xrightarrow{\bar{\psi}} (A \otimes \mathbb{Z} A)_\sigma \xrightarrow{\beta} \wedge_2 \mathbb{Z} A \rightarrow 0,
\]
where $(A \otimes \mathbb{Z} A)_\sigma := (A \otimes \mathbb{Z} A)/\langle a \otimes b + b \otimes a | a, b \in A \rangle$, $\bar{\psi}(\pi) = a \otimes a$ and $\beta(a \otimes b) = a \wedge b$.

**Proof.** This exact sequence is well-known and has appeared in [12, page 70] without a proof. But its proof is rather classic.

First note that $A \simeq \varinjlim_{i \in I} A_i$, where $\{A_i : i \in I\}$ is the direct system of all finitely generated subgroups of $A$. Now it is easy to see that
\[A \otimes \mathbb{Z} A \simeq \varinjlim_{i \in I} (A_i \otimes \mathbb{Z} A_i), \quad A/2 \simeq \varinjlim_{i \in I} (A_i/2), \quad \wedge_2 \mathbb{Z} A \simeq \varinjlim_{i \in I} \wedge_2 \mathbb{Z} A_i
\]
(see [5, 6.3, Chap. V] for the third isomorphism). Furthermore, observe that
\[(A \otimes \mathbb{Z} A)_\sigma = H_0(\Sigma_2, A \otimes \mathbb{Z} A) = \text{Tor}^\mathbb{Z}[\Sigma_2](\mathbb{Z}, A \otimes \mathbb{Z} A),
\]
where $\Sigma_2 = \{1, \sigma\}$ and the action of $\sigma$ on $A \otimes \mathbb{Z} A$ is given by $\sigma(a \otimes b) = b \otimes a$. Since the direct limite (with direct system index) is an exact functor and commutes with the Tor-functor, we may assume that $A$ is finitely generated. If $A = A_1 \oplus A_2$, it is easy to see that the claim holds for $A$ if and only if it holds for $A_1$ and $A_2$. Thus we can reduce the problem to cyclic groups. Since for any cyclic group $A$, $\wedge_2 \mathbb{Z} A = 0$ and $(A \otimes \mathbb{Z} A)_\sigma \simeq A/2$, the claim follows easily. \hfill \square
Proposition 4.5. Let $A \rightarrow G \rightarrow Q$ be a perfect central extension. If $Q$ is an $H$-group, then the natural map

$$H_3(A, \mathbb{Z}) \rightarrow H_3(G, \mathbb{Z})/\rho_*(A \otimes \mathbb{Z} H_2(G, \mathbb{Z}))$$

is trivial. In particular, if the extension is universal, then the natural map $H_3(A, \mathbb{Z}) \rightarrow H_3(G, \mathbb{Z})$ is trivial.

Proof. If we apply Theorem 1.1 to the morphism of extensions

$$\begin{array}{ccc}
A & \rightarrow & A \\
\downarrow & & \downarrow \\
A & \rightarrow & G \\
\downarrow & & \downarrow \\
A & \rightarrow & Q,
\end{array}$$

we obtain the commutative diagram with exact rows

$$\begin{array}{ccc}
\ker(\Psi) & \cong & \overline{H_3(A, \mathbb{Z})} \\
\downarrow & & \downarrow \\
H_4(K(A, 2), \mathbb{Z}) & \rightarrow & H_3(G, \mathbb{Z})/\rho_*(A \otimes \mathbb{Z} H_2(G, \mathbb{Z})) \rightarrow H_3(Q, \mathbb{Z}) \rightarrow 0,
\end{array}$$

where $\overline{H_3(A, \mathbb{Z})}$ is a quotient of $H_3(A, \mathbb{Z})$ and the map

$$\Psi : \Gamma(A) = H_4(K(A, 2), \mathbb{Z}) \rightarrow A \otimes \mathbb{Z} A$$

is discussed in Section 2. Since $\Gamma(A)/\Omega(A) \cong A/2$ (see the exact sequence (2.2)), from Theorem 4.2 and the above diagram we obtain the commutative diagram

$$\begin{array}{ccc}
\ker(\Psi) & \cong & \overline{H_3(A, \mathbb{Z})} \\
\downarrow & & \downarrow \\
A/2 & \rightarrow & H_3(G, \mathbb{Z})/\rho_*(A \otimes \mathbb{Z} H_2(G, \mathbb{Z})).
\end{array}$$

We have seen that the composite

$$A \otimes \mathbb{Z} A \xrightarrow{\Omega} \Gamma(A) \xrightarrow{\Psi} A \otimes \mathbb{Z} A,$$

takes $a \otimes b$ to $a \otimes b + b \otimes a$. Thus from the commutative diagram

$$\begin{array}{ccc}
0 & \rightarrow & \ker(\Omega) \\
\downarrow & & \downarrow \Omega \\
0 & \rightarrow & A \otimes \mathbb{Z} A \xrightarrow{\Omega} \text{im}(\Omega) \rightarrow 0 \\
\downarrow & & \downarrow \Psi \\
\ker(\Psi) & \rightarrow & \Gamma(A) \xrightarrow{\Psi} A \otimes \mathbb{Z} A
\end{array}$$

and the exact sequence (2.2) we obtain the exact sequence

$$\ker(\Psi) \rightarrow A/2 \xrightarrow{\mathcal{Q}} (A \otimes \mathbb{Z} A)_\sigma \rightarrow H_2(A, \mathbb{Z}) \rightarrow 0.$$

By Lemma 4.4 the sequence

$$0 \rightarrow A/2 \xrightarrow{\mathcal{Q}} (A \otimes \mathbb{Z} A)_\sigma \rightarrow H_2(A, \mathbb{Z}) \rightarrow 0.$$
is exact. Therefore the map \( \ker(\Psi) \rightarrow A/2 \) is trivial. Now it follows from the diagram (4.2) that the map \( \overline{H_3(A,\mathbb{Z})} \rightarrow H_3(G,\mathbb{Z})/\rho_*(A \otimes_{\mathbb{Z}} H_2(G,\mathbb{Z})) \) is trivial. 

\[ \square \]

**Example 4.6.** Let \( A \twoheadrightarrow G \twoheadrightarrow Q \) be a perfect central extension and let \( Q \) be an \( H \)-group. Here we would like to calculate the homomorphism

\[ A/2 \rightarrow H_3(G,\mathbb{Z})/\rho_*(A \otimes_{\mathbb{Z}} H_2(G,\mathbb{Z})) \]

from Theorem 4.2.

The extension \( A \twoheadrightarrow G \twoheadrightarrow Q \) is an epimorphic image of the universal extension of \( Q \), say \( A_1 \twoheadrightarrow G_1 \twoheadrightarrow Q \), which is unique up to isomorphism. By Theorem 4.2, this gives us the commutative diagram

\[ \begin{array}{ccc}
A_1/2 & \longrightarrow & H_3(G_1,\mathbb{Z}) \\
\downarrow & & \downarrow \\
A/2 & \longrightarrow & H_3(G,\mathbb{Z})/\rho_*(A \otimes_{\mathbb{Z}} H_2(G,\mathbb{Z})).
\end{array} \]

Thus we may assume that our extension is universal.

Thus let \( A \twoheadrightarrow G \twoheadrightarrow Q \) be an universal central extension such that \( Q \) is an \( H \)-group. From Corollary 1.2 we have the exact sequence

\[ H_4(Q,\mathbb{Z}) \rightarrow \Gamma(A) \rightarrow H_3(G,\mathbb{Z}) \rightarrow H_3(Q,\mathbb{Z}) \rightarrow 0. \]

By studying the Lyndon/Hochschild-Serre spectral sequences of the extension [5, §, Chap. VII], we obtain the exact sequence

\[ H_4(Q,\mathbb{Z}) \rightarrow B \rightarrow H_3(G,\mathbb{Z})/H_3(A,\mathbb{Z}) \rightarrow H_3(Q,\mathbb{Z}) \rightarrow 0, \]

where

\[ B = \ker(A \otimes_{\mathbb{Z}} A \rightarrow H_2(A,\mathbb{Z})) = \langle a \otimes a : a \in A \rangle. \]

By Proposition 4.5, \( H_3(G,\mathbb{Z}) = H_3(G,\mathbb{Z})/H_3(A,\mathbb{Z}). \) Now it is easy to see that the following diagram is commutative:

\[ \begin{array}{ccc}
H_4(Q,\mathbb{Z}) & \longrightarrow & \Gamma(A) \\
\downarrow \quad \rotatebox{90}{$\cong$} & \downarrow \quad \rotatebox{90}{$\cong$} & \downarrow \quad \rotatebox{90}{$\cong$} \\
H_4(Q,\mathbb{Z}) & \longrightarrow & B \\
\downarrow \Psi & \longrightarrow & \downarrow \Psi \\
H_3(G,\mathbb{Z}) & \longrightarrow & H_3(G,\mathbb{Z}) \\
\downarrow \quad \rotatebox{90}{$\cong$} & \downarrow \quad \rotatebox{90}{$\cong$} \\
H_3(Q,\mathbb{Z}) & \longrightarrow & H_3(Q,\mathbb{Z}) \\
\downarrow \quad \rotatebox{90}{$\cong$} & \downarrow \quad \rotatebox{90}{$\cong$} \\
H_3(Q,\mathbb{Z}) & \longrightarrow & 0.
\end{array} \]

Since the map \( A/2 \rightarrow H_3(G,\mathbb{Z}) \) factors through \( \Gamma(A)/H_4(Q,\mathbb{Z}) \), it is also factors through the group \( B/H_4(Q,\mathbb{Z}) \). In fact it factors throughout \( B/\langle a \otimes b + b \otimes a | a, b \in A \rangle \). Thus it is enough to calculate the map

\[ B/\langle a \otimes b + b \otimes a | a, b \in A \rangle \rightarrow \longrightarrow H_3(G,\mathbb{Z}). \]
Let $Q = F/S$ be a free presentations of $Q$. By a theorem of Hopf $H_2(Q, \mathbb{Z}) \simeq (S \cap [F, F])/[S, F]$ \cite[Theorem 5.3, Chap. II]{Huntington}. This isomorphism can be given by the following explicit formula

$$
\Lambda : (S \cap [F, F])/[S, F] \overset{\approx}{\longrightarrow} H_2(Q, \mathbb{Z}) = H_2(B_\bullet(Q)_Q),
$$

$$
\left( \prod_{i=1}^{g} [a_i, b_i] \right) \in [S, F] \longmapsto \sum_{i=1}^{g} \left( [s_i^{-1} | \hat{a}_i] + [\hat{s}_i | \hat{b}_i] - [\hat{s}_i | \hat{b}_i] \right),
$$

where $s_i = [a_1, b_1] \cdots [a_i, b_i]$ and for $x \in F$ we set $\bar{x} = xS \in F/S = Q$ \cite[Exercise 4, §5, Chap. II]{Huntington}. Note that $\bar{s}_g = 1$. Here $B_\bullet(Q) \to \mathbb{Z}$ is the bar resolution of $Q$.

Let $G = F/R$, $Q = F/S$ and $A = S/F$ be free presentations of $G$, $Q$ and $A$, respectively. Since $A$ is central we have $[S, F] \subseteq R$ and thus the following diagram

$$
\begin{array}{ccc}
H_2(Q, \mathbb{Z}) & \overset{\approx}{\longrightarrow} & A = S/R \\
\Lambda \downarrow & & \downarrow (S \cap [F, F])/[S, F]. \\
(S \cap [F, F])/[S, F] & \longrightarrow & (S \cap [F, F])/[S, F].
\end{array}
$$

commutes, where $(S \cap [F, F])/[S, F] \to S/R = A$ is given by $s[S, F] \mapsto sR$. For any $a \in F$, we denote $aR \in G = F/R$ by $\hat{a}$ and for any $s \in S \cap [F, F]$, we denote $s[S, F]$ by $\bar{s}$.

The Lyndon-Hochschild-Serre spectral sequence

$$
\mathcal{E}_{p,q}^2 = H_p(Q, H_q(A, \mathbb{Z})) \Rightarrow H_{p+q}(G, \mathbb{Z})
$$

gives us a filtration of $H_3(G, \mathbb{Z})$

$$
0 = F_{-1}H_3 \subseteq F_0H_3 \subseteq F_1H_3 \subseteq F_2H_3 \subseteq F_3H_3 = H_3(G, \mathbb{Z}),
$$

such that $\mathcal{E}_{i,3-i}^\infty = F_iH_3/F_{i-1}H_3$. Now by an easy analysis of the above spectral sequence one sees that $F_0H_3 = F_1H_3 = 0$ and the map $\eta$ is induced by the composite

$$
(4.3) \quad B \to E_{2,1}^3 \simeq E_{\infty,1}^\infty \simeq F_2H_3 \subseteq H_3(G, \mathbb{Z}).
$$

If $s_g = \prod_{i=1}^{g} [a_i, b_i] \in S \cap [F, F]$, then we need to compute

$$
\eta(\Lambda(\bar{s}_g) \otimes \bar{s}_g) \in H_3(G, \mathbb{Z})
$$

under the composition (4.3). By direct calculation, which we delete the details here, this element maps to the following element of $H_3(G, \mathbb{Z})$:

$$
\Lambda(s_g) := [\bar{s}_g | \hat{s}_i | \hat{s}_g] + \\
\sum_{i=1}^{g} \left( [\hat{a}_i | \hat{s}_i^{-1} | \hat{s}_g] - [\hat{a}_i^{-1} | \hat{s}_i^{-1} | \hat{s}_g] + [\hat{a}_i^{-1} | \hat{s}_i^{-1} | \hat{s}_g] [\hat{a}_i | \hat{s}_i^{-1} | \hat{s}_g] + [\hat{a}_i^{-1} | \hat{s}_i^{-1} | \hat{s}_g] [\hat{a}_i^{-1} | \hat{s}_i^{-1} | \hat{s}_g] + [\hat{a}_i | \hat{s}_i^{-1} | \hat{s}_g] [\hat{a}_i^{-1} | \hat{s}_i^{-1} | \hat{s}_g] + [\hat{a}_i^{-1} | \hat{s}_i^{-1} | \hat{s}_g] [\hat{a}_i^{-1} | \hat{s}_i^{-1} | \hat{s}_g] \right).
$$
5. The Hermitian $K$-theory

Let $Q$ be a perfect group. Then $K(Q,1)^+$ is a 1-connected CW-complex [14, Theorem 5.2.2] and by the theorem of Hurewicz

$$\pi_2(K(Q,1)^+) \to H_2(Q,\mathbb{Z})$$

is an isomorphism and

$$\pi_3(K(Q,1)^+) \to H_3(Q,\mathbb{Z})$$

is surjective.

Let $A \to G \to Q$ be a central extension with $Q$ perfect. Then

$$K(A,1) \to K(G,1)^+ \to K(Q,1)^+$$

is a fibration (see [1, Corollary 8.4] or [21, Proposition 1]), where the plus-construction $K(G,1)^+$ is taken with respect to the maximal perfect subgroup of $G$. From this we obtain the exact sequence

$$\cdots \to \pi_3(K(A,1)) \to \pi_3(K(G,1)^+) \to \pi_3(K(Q,1)^+) \to \pi_2(K(A,1)) \to \cdots$$

Since $\pi_n(K(A,1)) = 0$ for $n \neq 1$, this implies that for $n \geq 3$,

$$\pi_n(K(G,1)^+) \cong \pi_n(K(Q,1)^+).$$

**Lemma 5.1.** Let $A \to G \to Q$ be the universal central extension of the perfect group $Q$. Then $\pi_3(K(Q,1)^+) \cong H_3(G,\mathbb{Z})$ and the Hurewicz map $\pi_3(K(Q,1)^+) \to H_3(Q,\mathbb{Z})$ coincides with the natural map $H_3(G,\mathbb{Z}) \to H_3(Q,\mathbb{Z})$.

**Proof.** We proved in above that $\pi_3(K(G,1)^+) \cong \pi_3(K(Q,1)^+)$. Since the extension is universal, $H_1(G,\mathbb{Z}) = H_2(G,\mathbb{Z}) = 0$. Now Hurewicz’s theorem implies that

$$\pi_1(K(G,1)^+) = \pi_2(K(G,1)^+) = 0.$$ 

Thus $K(G,1)^+$ is 2-connected and again by Hurewicz’s theorem

$$\pi_3(K(G,1)^+) \cong H_3(G,\mathbb{Z}).$$

This implies that $\pi_3(K(Q,1)^+) \cong H_3(G,\mathbb{Z})$. The other claim follows from the commutative diagram

$$\begin{array}{ccc}
\pi_3(K(G,1)^+) & \cong & H_3(G,\mathbb{Z}) \\
\downarrow & & \downarrow \\
\pi_3(K(Q,1)^+) & \longrightarrow & H_3(Q,\mathbb{Z}).
\end{array}$$

\[ \Box \]

**Proposition 5.2.** If $Q$ is an $H$-group, then we have the exact sequence

$$\pi_2(K(Q,1)^+)/2 \to \pi_3(K(Q,1)^+) \to H_3(Q,\mathbb{Z}) \to 0.$$

**Proof.** Let $A \to G \to Q$ be the universal central extension of $Q$. Since $\pi_2(K(Q,1)^+) \cong H_3(Q,\mathbb{Z}) \cong A$, the claim follows immediately from Proposition 5.1 and Corollary 4.3. \[ \Box \]
Let $R$ be an associative ring with unit. Let there be an involution on $R$, that is an automorphism of the additive group of $R$, $R \to R$ with $r \mapsto \bar{r}$, such that $\bar{\bar{r}} = r$ and $\bar{r} \bar{s} = \bar{s} \bar{r}$.

Let $\epsilon$ be an element in the center of $R$ such that $\epsilon^2 = 1$. Set

$$R_\epsilon := \{ r - \epsilon \bar{r} : r \in R \} \quad \text{and} \quad R^\epsilon := \{ r \in R : \epsilon \bar{r} = -r \}$$

and observe that $R_\epsilon \subseteq R^\epsilon$. Let

$$M_n(R)_\epsilon := \{ A - \epsilon \bar{A}^T : A \in M_n(R) \},$$

where $(a_{ij}) = (\bar{a}_{ij})$. Let $e_{i,j}(r)$ be the $2n \times 2n$-matrix with $r \in R$ in the $(i,j)$-place and zero elsewhere.

We define the *unitary and orthogonal group* of the pair $(R, \epsilon)$ as follow

$$\epsilon U_{2n}(R) := \{ A \in GL_{2n}(R) : \bar{A}^T F_n A = F_n \},$$

$$\epsilon O_{2n}(R) := \{ A \in GL_{2n}(R) : \bar{A}^T Q_n A - Q_n \in M_n(R)_\epsilon \},$$

respectively, where

$$F_n = \sum_{i=1}^{n} (e_{2i-1,2i}(1) + e_{2i,2i-1}(\epsilon)), \quad Q_n := \sum_{i=1}^{n} e_{2i-1,2i}(1).$$

We have always

$$\epsilon O_{2n}(R) \subseteq \epsilon U_{2n}(R).$$

If $R_\epsilon = R^\epsilon$, then $\epsilon U_{2n}(R) = \epsilon O_{2n}(R)$. Observe that if there is an element $s$ in the center of $R$ such that $s + \bar{s} \in R^\times$, in particular if $2 \in R^\times$, then $R_\epsilon = R^\epsilon$.

**Example 5.3.** (i) Let $\epsilon = -1$ and let the involution be the identity map $id_R$. Then

$$\epsilon U_{2n}(R) = Sp_{2n}(R)$$

is the usual symplectic group. Note that $R$ is commutative in this case.

(ii) Let $\epsilon = 1$ and let the involution be the identity map $id_R$. Then

$$\epsilon O_{2n}(R) = O_{2n}(R)$$

is the usual orthogonal group. As in the symplectic case, $R$ is necessarily commutative.

(iii) Let $\epsilon = -1$ and let the involution is not the identity map $id_R$. Then

$$\epsilon U_{2n}(R) = U_{2n}(R)$$

is the classical unitary group corresponding to the involution.
Let $\alpha$ be the permutation of the set of natural numbers given by $\alpha(2i) = 2i - 1$ and $\alpha(2i - 1) = 2i$. For $1 \leq i, j \leq 2n$, $i \neq j$, and every $r \in R$ define

$$E_{i,j}(r) = \begin{cases} 
I_{2n} + e_{i,j}(r) & \text{if } i = 2k - 1, j = \alpha(i), r = -\tau \\
I_{2n} + e_{i,j}(r) & \text{if } i = 2k, j = \alpha(i), r = -\epsilon \\
I_{2n} + e_{i,j}(r) + e_{\alpha(j),\alpha(i)}(-\tau) & \text{if } i + j = 2k, i \neq j \\
I_{2n} + e_{i,j}(r) + e_{\alpha(j),\alpha(i)}(-\epsilon^{-1}\tau) & \text{if } i \neq \alpha(j), i = 2k - 1, j = 2l \\
I_{2n} + e_{i,j}(r) + e_{\alpha(j),\alpha(i)}(\epsilon\tau) & \text{if } i \neq \alpha(j), i = 2k, j = 2l - 1
\end{cases}$$

where $I_{2n}$ is the identity element of $GL_{2n}(R)$. It is easy to see that $E_{i,j}(r) \in \epsilon U_{2n}(R)$.

Let $\epsilon EU_{2n}(R)$ be the subgroup of $\epsilon U_{2n}(R)$ generated by the matrices $E_{i,j}(r), r \in R$, and $\epsilon EO_{2n}(R)$ be the group of $\epsilon O_{2n}(R)$ generated by the matrices $E_{i,j}(r), r \in R$, which are in $\epsilon U_{2n}(R)$. We call $\epsilon EU_{2n}(R)$ the elementary unitary group and $\epsilon EO_{2n}(R)$ the elementary orthogonal group.

The natural embeddings

$$\epsilon O_{2n}(R) \rightarrow \epsilon O_{2(n+1)}(R), \quad \epsilon U_{2n}(R) \rightarrow \epsilon U_{2(n+1)}(R),$$

given by $A \mapsto \left( \begin{array}{cc} A & 0 \\ 0 & I_2 \end{array} \right)$, embeds $\epsilon EO_{2n}(R)$ and $\epsilon EU_{2n}(R)$ naturally in $\epsilon EO_{2(n+1)}(R)$ and $\epsilon EU_{2(n+1)}(R)$, respectively. Define the stable unitary and orthogonal groups and their stable elementary subgroups as follow:

$$\epsilon O(R) := \bigcup_{n \geq 1} \epsilon O_{2n}(R), \quad \epsilon EO(R) := \bigcup_{n \geq 1} \epsilon EO_{2n}(R),$$

$$\epsilon U(R) := \bigcup_{n \geq 1} \epsilon U_{2n}(R), \quad \epsilon EU(R) := \bigcup_{n \geq 1} \epsilon EU_{2n}(R).$$

It is a classical result that $\epsilon EO(R)$ (resp. $\epsilon EU(R)$) is perfect and is normal in $\epsilon O(R)$ (resp. $\epsilon U(R)$). More precisely,

$$\epsilon EO(R) = [\epsilon O(R), \epsilon EO(R)] = [\epsilon O(R), \epsilon O(R)],$$

$$\epsilon EU(R) = [\epsilon EU(R), \epsilon EU(R)] = [\epsilon U(R), \epsilon U(R)],$$

(see [17, Theorems 1.4 and 1.4]). For $n \geq 1$, the Hermitian and Unitary $K$-groups of the pair $(R, \epsilon)$ are defined as follow

$$\epsilon K^h_n(R) := \pi_n(K(\epsilon O(R), 1)^+),$$
$$\epsilon K^n_n(R) := \pi_n(K(\epsilon U(R), 1)^+),$$

where the plus-constructions are taken with respect to the perfect subgroups $\epsilon EO(R)$ and $\epsilon EU(R)$, respectively. Since $\epsilon O(R)$ and $\epsilon U(R)$ are quasi-perfect, for $n \geq 2$ we have

$$\epsilon K^h_n(R) \simeq \pi_n(K(\epsilon EO(R), 1)^+),$$
$$\epsilon K^n_n(R) \simeq \pi_n(K(\epsilon EU(R), 1)^+).$$

One can show that the homomorphism

$$(A, B) \mapsto A \oplus B,$$
where

\[(A \oplus B)_{ij} := \begin{cases} A_{kl} & \text{if } i = 2k - 1, j = 2l - 1 \\ B_{kl} & \text{if } i = 2k, j = 2l \\ 0 & \text{otherwise} \end{cases}\]

defined on \(\epsilon O(R)\) and \(\epsilon U(R)\) satisfies in conditions (i) and (ii) of Example 4.1(a) ([10, page 324, Example (d)]). This implies that \(\epsilon O(R)\) and \(\epsilon U(R)\) are \(H\)-groups. Now by Proposition 5.2 we obtain the following result.

**Theorem 5.4.** Let \(R\) be a ring with an involution and a central element \(\epsilon\) such that \(\epsilon^2 = 1\). Then we have the exact sequences

\[\epsilon K^3_2(R)/2 \rightarrow \epsilon K^3_3(R) \rightarrow H_3(\epsilon EO(R), \mathbb{Z}) \rightarrow 0,\]
\[\epsilon K^4_2(R)/2 \rightarrow \epsilon K^4_3(R) \rightarrow H_3(\epsilon EU(R), \mathbb{Z}) \rightarrow 0.\]

With a similar argument we can prove a similar result for Algebraic \(K\)-groups of a ring: For any ring \(R\) with unit we have the exact sequence

\[K^2_2(R)/2 \rightarrow K^2_3(R) \rightarrow H_3(E(R), \mathbb{Z}) \rightarrow 0,\]

where \(E(R)\) is the elementary subgroup of the stable linear group \(GL(R)\). This exact sequence already was proved, with a different method, by Suslin in [16, Corollary 5.2]. We should mention that Suslin also studied the nature of the map \(K^2_2(R)/2 \rightarrow K^2_3(R)\).

6. The third cohomology of central perfect extensions

Let \(A \rightarrow G \rightarrow Q\) be a perfect central extension. From the fibration

\[K(G, 1)^+ \rightarrow K(Q, 1)^+ \rightarrow K(A, 2),\]

(see the proof of Theorem 4.2) we obtain the Serre spectral sequence

\[E_2^{p,q} = H^p(K(A, 2), H^q(G, \mathbb{Z})) \Rightarrow H^{p+q}(Q, \mathbb{Z}),\]

(see [6, Section 9.5]). By the universal coefficients theorem for the cohomology of groups [5, Exercise 3, Section 1, Chap. III] and spaces [6, Corollary 2.35] we have

\[H^1(G, \mathbb{Z}) = 0,\]
\[H^2(G, \mathbb{Z}) \simeq \text{Hom}(H_2(G, \mathbb{Z}), \mathbb{Z}),\]
\[H^1(K(A, 2), \mathbb{Z}) = 0,\]
\[H^2(K(A, 2), \mathbb{Z}) \simeq \text{Hom}(A, \mathbb{Z}),\]
\[H^3(K(A, 2), \mathbb{Z}) \simeq \text{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z})\]

and

\[H^4(K(A, 2), \mathbb{Z}) \simeq \text{Hom}(\Gamma(A), \mathbb{Z}).\]

For any \(n \geq 0\), the spectral sequence gives us a filtration of \(H^n(Q, \mathbb{Z})\) as follow

\[0 = F^{n+1}H^n \subseteq F^nH^n \subseteq \cdots \subseteq F^1H^n \subseteq F^0H^n = H^n(Q, \mathbb{Z})\]

such that

\[E_\infty^{p,n-p} \simeq F^pH^n/F^{p+1}H^n.\]
By the universal coefficients theorems for groups and spaces and the above calculations, we obtain

\[ E_2^{p,1} = 0, \quad E_2^{0,q} \cong H^q(G, \mathbb{Z}), \quad p, q \geq 0, \]

and

\[ E_2^{2,0} \cong \text{Hom}(A, \mathbb{Z}), \quad E_2^{3,0} \cong \text{Ext}_H^1(A, \mathbb{Z}), \quad E_2^{1,2} = 0, \]

\[ E_2^{2,2} \cong \text{Hom}(A, H^2(G, \mathbb{Z})), \quad E_2^{4,0} \cong \Gamma(A)^*, \]

where for an abelian group \( M, M^* \) is the dual group \( \text{Hom}_\mathbb{Z}(M, \mathbb{Z}) \). Now by a direct analysis of the filtrations of \( H^2(Q, \mathbb{Z}) \) and \( H^3(Q, \mathbb{Z}) \) we obtain the exact sequence

\[ (6.1) \quad 0 \to \text{Hom}(A, \mathbb{Z}) \to H^2(Q, \mathbb{Z}) \to H^2(G, \mathbb{Z}) \to \text{Ext}_H^1(A, \mathbb{Z}) \to H^3(Q, \mathbb{Z}) \]

\[ \to \ker(H^3(G, \mathbb{Z}) \to \text{Hom}(A, H^2(G, \mathbb{Z}))) \to \Gamma(A)^*. \]

By Theorem 1.1, we have the exact sequence

\[ 0 \to H_2(G, \mathbb{Z}) \to H_2(Q, \mathbb{Z}) \to A \to 0. \]

This together with the isomorphisms \( H^2(G, \mathbb{Z}) \cong \text{Hom}(H_2(G, \mathbb{Z}), \mathbb{Z}) \) and \( H^3(Q, \mathbb{Z}) \cong \text{Hom}(H_2(Q, \mathbb{Z}), \mathbb{Z}) \) imply that the natural map \( H^2(Q, \mathbb{Z}) \to H^2(G, \mathbb{Z}) \) is surjective. Thus from the exact sequence (6.1) and the isomorphism

\[ \text{Hom}(A, H^2(G, \mathbb{Z})) \cong \text{Hom}(A, \text{Hom}(H_2(G, \mathbb{Z}), \mathbb{Z})) \]

\[ \cong (A \otimes \mathbb{Z} H_2(G, \mathbb{Z}))^*, \]

we obtain the exact sequence

\[ 0 \to \text{Ext}_H^1(A, \mathbb{Z}) \to H^3(Q, \mathbb{Z}) \to \ker(H^3(G, \mathbb{Z}) \to (A \otimes \mathbb{Z} H_2(G, \mathbb{Z}))^*) \to \Gamma(A)^*. \]

This proves the first part of the following proposition.

**Proposition 6.1.** Let \( A \to G \to Q \) be a perfect central extension. Then we have the exact sequence

\[ 0 \to \text{Ext}_H^1(A, \mathbb{Z}) \to H^3(Q, \mathbb{Z}) \to \ker(H^3(G, \mathbb{Z}) \to (A \otimes \mathbb{Z} H_2(G, \mathbb{Z}))^*) \to \Gamma(A)^*. \]

In particular, if \( Q \) is an \( H \)-group, then we have the exact sequence

\[ 0 \to \text{Ext}_H^1(A, \mathbb{Z}) \to H^3(Q, \mathbb{Z}) \to H^3(G, \mathbb{Z}) \to (A \otimes \mathbb{Z} H_2(G, \mathbb{Z}))^*. \]

**Proof.** We already proved the first part. So let \( Q \) be an \( H \)-group. First assume that the extension is universal. Then the above extension finds the following form

\[ 0 \to \text{Ext}_H^1(A, \mathbb{Z}) \to H^3(Q, \mathbb{Z}) \to H^3(G, \mathbb{Z}) \to \Gamma(A)^*. \]

From the proof of Theorem 4.2 we see that the map \( \Gamma(A) \to H_3(G, \mathbb{Z}) \) factors through \( A/2 = \Gamma(A)/\Omega(A) \). Thus

\[ H^3(G, \mathbb{Z}) \cong H_3(G, \mathbb{Z})^* \to \Gamma(A)^* \]

factors through \( (A/2)^* = 0 \), which implies that it is trivial.
In general the extension \( A \to G \to Q \) is an epimorphic image of a universal central extension of \( Q \), say \( A_1 \to G_1 \to Q \). Thus \( A_1 \cong H_2(Q, \mathbb{Z}) \) and we have a morphism of extensions

\[
\begin{array}{cccccc}
A_1 & \to & G_1 & \to & Q \\
\downarrow & & \downarrow & & \downarrow \\
A & \to & G & \to & Q.
\end{array}
\]

This gives us the commutative diagram of exact sequences

\[
\begin{array}{cccccccc}
0 & \to & \text{Ext}^1_Z(A, \mathbb{Z}) & \to & H^3(Q, \mathbb{Z}) & \to & \tilde{H}^3(G, \mathbb{Z}) & \to & \Gamma(A)^* \\
& & \downarrow & & \downarrow = & & \downarrow & & \downarrow \\
0 & \to & \text{Ext}^1_Z(A_1, \mathbb{Z}) & \to & H^3(Q, \mathbb{Z}) & \to & H^3(G_1, \mathbb{Z}) & \to & \Gamma(A_1)^*.
\end{array}
\]

where

\[
\tilde{H}^3(G, \mathbb{Z}) := \ker \left( H^3(G, \mathbb{Z}) \oto{\rho^*} (A \otimes \mathbb{Z} H_2(G, \mathbb{Z}))^* \right).
\]

(Note that since \( A_1 \to A \) is surjective, \( \Gamma(A_1) \to \Gamma(A) \) is surjective too. This implies that the map \( \Gamma(A)^* \to \Gamma(A_1)^* \) is injective.) Now from the above diagram we see that the map \( \tilde{H}^3(G, \mathbb{Z}) \to \Gamma(A)^* \) is trivial. This proves our claim. \( \square \)

Note that the second part of the above proposition is the cohomology analogue of Theorem 4.2.

**Corollary 6.2.** Let \( A \to G \to Q \) be an universal central extension. Then we have the exact sequence

\[
0 \to \text{Ext}^1_Z(A, \mathbb{Z}) \to H^3(Q, \mathbb{Z}) \to H^3(G, \mathbb{Z}) \to \Gamma(A)^*.
\]

In particular, if \( Q \) is an \( H \)-group, then we have the exact sequence

\[
0 \to \text{Ext}^1_Z(A, \mathbb{Z}) \to H^3(Q, \mathbb{Z}) \to H^3(G, \mathbb{Z}) \to 0.
\]

**Remark 6.3.** Let \( A \) be a central subgroup of \( G \) and let \( A \subseteq G' \). Let \( i : A \to G \) be the usual inclusion map. We have seen at the beginning of Section 3, that \( i_* : H_2(A, \mathbb{Z}) \to H_2(G, \mathbb{Z}) \) is trivial and the image of \( i_* : H_3(A, \mathbb{Z}) \to H_3(G, \mathbb{Z}) \) is torsion. Thus

\[
i^* : H_3(G, \mathbb{Z})^* \to H_3(A, \mathbb{Z})^*
\]

is trivial (because it factors through \( \text{Hom}(\text{im}(i_*), \mathbb{Z}) = 0 \)). Now it follows from the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \text{Ext}^1_Z(H_2(G, \mathbb{Z}), \mathbb{Z}) & \to & H^3(G, \mathbb{Z}) & \to & H^3(G, \mathbb{Z})^* & \to & 0 \\
& & \downarrow 0 & & \downarrow & & \downarrow 0 \\
0 & \to & \text{Ext}^1_Z(H_2(A, \mathbb{Z}), \mathbb{Z}) & \to & H^3(A, \mathbb{Z}) & \to & H^3(A, \mathbb{Z})^* & \to & 0,
\end{array}
\]
that
\[ \text{im}(H^3(G, \mathbb{Z}) \to H^3(A, \mathbb{Z})) \subseteq \text{Ext}^1_{\mathbb{Z}}(\Lambda^2_\mathbb{Z} A, \mathbb{Z}). \]

References

[1] Berrick, A. J. An Approach to Algebraic K-Theory. Research Notes in Math. No. 56, Pitman, London, 1982 9, 14
[2] Berrick, A. J. Two functors from abelian groups to perfect groups. J. Pure Appl. Algebra 44 (1987), no. 1-3, 35–43 9
[3] Berrick, A. J., Miller, C. F., III. Strongly torsion generated groups. Math. Proc. Cambridge Philos. Soc. 111 (1992), no. 2, 219–229 8
[4] L. Breen, L. On the functorial homology of abelian groups. Journal of Pure and Applied Algebra 142 (1999) 199–237. 6
[5] Brown, K. S. Cohomology of Groups. Graduate Texts in Mathematics, 87. Springer-Verlag, New York, 1994 1, 4, 6, 7, 10, 12, 13, 17
[6] Davis, J. F., Kirk, P. Lecture Notes in Algebraic Topology. Graduate Studies in Mathematics, 35. Providence, RI: American Mathematical Society, 2001 3, 17
[7] Eckmann, B., Hilton, P. J. On central group extensions and homology. Commentarii Mathematici Helvetici 46 (1971), 345–355. 3, 5, 9
[8] Eilenberg, S., MacLane, S. On the groups \( H(\mathbb{II}, n) \). II: Methods of computation. Ann. of Math. 70 (1954), no. 1, 49–139 3, 5
[9] Hahn, A. J., O’Meara, O. T. The classical groups and K-theory. Grundlehren der Mathematischen Wissenschaften 291, 1989 1
[10] Loday, J. L. K-théorie algébrique et représentations de groupes. Ann. Sci. École Norm. Sup. (4) 9 (1976), no. 3, 309–377 8, 9, 17
[11] May, J. P., Ponto, K. More concise algebraic topology: Localization, completion, and model categories. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012 3, 9
[12] Mikhailov, R. Polynomial Functors and Homotopy Theory. In: Franjou V., Touzé A. (eds) Lectures on Functor Homology. Progress in Mathematics, vol 311. Birkhäuser, 2015 10
[13] Milnor, J. Introduction to Algebraic K-Theory. Annals of Mathematics Studies, No. 72. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971 1
[14] Rosenberg, J. Algebraic K-theory and its application. Graduate text in Mathematics. Springer 147, 1996 1, 3, 8, 14
[15] Stammbach, U. Homology in group theory. Lecture Notes in Mathematics, Vol. 359. Springer-Verlag, Berlin-New York, 1973 7
[16] Suslin, A. A. \( K_3 \) of a field and the Bloch group. Proc. Steklov Inst. Math. 183 (1991), no. 4, 217–239 6, 17
[17] Vaserstein, L. N. Stabilization of unitary and orthogonal groups over a ring with involution. Math. USSR Sbornik 10 (1970), no. 3, 307–326 16
[18] Wagoner, J. Delooping classifying spaces in algebraic K-theory. Topology 11 (1972), 349–370 9
[19] Whitehead, J. H. C. A certain exact sequence. Ann. Math. 52 (1950), 51–110 3
[20] Whitehead, G. W. Elements of homotopy theory. Springer-Verlag. 1978 5, 9
[21] Wojtkowiak, Z. Central extension and coverings, Publ. Sec. Mat. Univ. Autónoma Barcelona 29 no. 2-3, (1985), 145–153 9, 14
[22] Zabrodsky, A. Hopf spaces, Moth-Holland Publishing Company, 1976 9