Regular and irregular modulation of frequencies in limit cycle oscillator networks

M. TACHIKAWA(a)

ERATO Complex Systems Biology Project, JST - 3-8-1, Komaba, Meguro-ku, Tokyo 153-8902, Japan

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Abstract – Nonisochronicity (frequency change caused by amplitude deviation) is the essential trait that differentiates limit cycle oscillators from phase oscillators. We studied networks of nonisochronic limit cycle oscillators, and demonstrated that the frequencies assume distributed values in a sustainable manner. We observed two complex phenomena in the networks: stationary distributed frequencies at a regular interval (quantization of frequencies), and continuous irregular modulation of frequencies. In the analysis we reveal the mechanisms by which the frequencies are distributed, and show how the nonisochronicity produces these complex phenomena. We also illustrate that the topology of the networks determines the behavior of the system.

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Oscillator networks are widely accepted as a standard model for understanding dynamical systems with many degrees of freedom [1–7]. The synchronization phase transition for simple phase oscillator networks has been extensively investigated since the well-known research by Winfree [1] and Kuramoto [2]. In recent times this field has been developed by incorporating statistical knowledge about complex networks [8–11]. At the same time, the detailed dynamic behavior characteristics of limit cycle or chaotic oscillators has also attracted attention [3,4,12–15]. Various types of coherent motions have been classified, and concepts such as phase synchronization [4], clustering [3,14,15], and chaotic itinerancy [3] have been proposed. However, a large number of complex phenomena still remain unclassified, and an integrated understanding of them is still lacking compared with the understanding of simple phase oscillator systems.

This raises the question of what the essential differentiator is between limit cycle oscillators and simple phase oscillators. Among other differentiators, the nonisochronicity —frequency (phase velocity) change caused by amplitude deviation— must be emphasized. While any perturbation on a phase oscillator results in only a phase shift, in a limit cycle oscillator, the frequency too can be modulated by the perturbations. When such oscillators are assembled to form an interacting network, the interactions work as perturbations to modify their frequencies, and the resulting system may have distributed frequencies. Distributed frequencies are observed in many complex systems [3,16–19], and in fact, variable frequency effects have been noted in some of them [16–18]. Several theoretical studies have drawn attention to the variable frequency effect in complex systems [14,15,18,20].

In this paper, we highlight the nonisochronicity of limit cycle oscillators, and investigate how this property induces complex behavior in oscillator networks. In particular, we report that the frequencies of oscillators in networks are spontaneously distributed. We observe that there are two types of distributed frequency states depending on the network topologies: frequencies take quantized values, or show continuous irregular modulations. The analysis reveals the mechanism that generates these states. We also analyze how the topology of the networks determines the behaviors of the system.

Let us consider a perturbed limit cycle oscillator with a two-dimensional slow manifold that can be oriented (i.e., it is weakly stable in a direction transverse to the rotation direction). Applying the reduction method proposed by Kuramoto [2], it is reduced into the following two-variable oscillator:

\[
\dot{\Theta} = \frac{1}{\Gamma(r)} + \text{(perturbation)},
\]

\[
\dot{r} = \Gamma(r)
\]
where \( \Theta \) represents the phase variable, \( r \) represents the one-dimensional deviation of the orbit in the amplitude direction (amplitude deviation), and \( \Gamma(r) \) satisfies the conditions \( \Gamma(0) = 0 \) and \( \partial \Gamma(r)/\partial r < 0 \). If the limit cycle shows large nonisochronicity, defined isochrones in the reduction procedure have highly distorted geometries in the original phase space. We focus on such a case and introduce a new phase variable \( \theta = \Theta + f(r) \) to eliminate the distortion. Then we get

\[
\begin{align*}
\dot{\theta} &= 1 + \frac{\partial f(r)}{\partial r} \Gamma(r) + \text{(perturbation)}, \\
\dot{r} &= \Gamma(r)
\end{align*}
\]

where \( f(r) \) is chosen so that the isochrones of \( \theta \) have minimum distortion. The equations for the new coordinate show that the phase velocity is a function of \( r \), and thus perturbations on \( r \) result in changes of its frequency. In this paper, we mainly consider the oscillator, including the first order of \( r \) with \( \partial f(r)/\partial r \Gamma(r) = \omega r + O(r^2) \) and \( \Gamma(r) = -\gamma r + O(r^2) \).

Next, we determine the manner of interaction between oscillators. Pairs of interacting oscillators are represented by a symmetric interaction matrix \( k_{ij} = k_{ji} \in \{0, 1\} \), which identifies the bidirectional interactions. We suppose that the interaction makes oscillators with close \( \theta \) and \( r \) values to synchronize in phase. Such interacting functions satisfy \( \partial \theta/\partial \Delta \theta |_{\Delta r = 0} \neq 0 \), \( \partial r/\partial \Delta \theta |_{\Delta r = 0} = 0 \), and \( \partial^2 \Delta \theta^2 |_{\Delta r = 0} \neq 0 \), where \( \Delta \theta \) and \( \Delta r \) are the phase and amplitude deviation differences. Thus, the oscillator networks are given by

\[
\begin{align*}
\dot{\theta}_\alpha &= 1 + \omega r_\alpha + D_\theta \sum_\beta k_{\alpha \beta} \sin(2\pi(\theta_\beta - \theta_\alpha)), \\
\dot{r}_\alpha &= -\gamma r_\alpha + D_r \sum_\beta k_{\alpha \beta} \{1 - \cos(2\pi(\theta_\beta - \theta_\alpha))\}
\end{align*}
\]

where \( \alpha, \beta \) are oscillator indexes. In this study, we set \( D_\theta = D_r = 0.01, \gamma = 0.1 \). \( \omega \) is the parameter that controls the magnitude of the nonisochronicity. In all simulations, uniformly distributed random initial conditions in \( r_\alpha \in (0, 1], \theta_\alpha \in (0, 1] \) are used.

We now consider two examples of networks shown in fig. 1A and D, which display the two typical behaviors of this system.

Figure 1C shows the time evolution of \( r_\alpha \) in network A (fig. 1A) after the system has reached the stationary states. The case with \( \omega = 1, \varepsilon = 0.5 \) is shown. In the figure, \( r_\alpha \) are distributed and stay around the discrete stationary points that are arranged at regular intervals of 0.1. The fluctuations around the stationary points are small and seem to be periodic or quasi-periodic. In other words, the amplitude deviations are quantized. The variances in \( r_\alpha \) are plotted against \( \omega \) in fig. 1B. With \( \omega \leq 0.38 \), all oscillators achieve complete synchronization (\( r_\alpha = 0, \theta_\alpha = \theta_\beta \) for all \( \alpha, \beta \)), and no fluctuations appear. With \( \omega > 0.38 \), the amplitude deviations are distributed and the average values are independent of \( \omega \) (data not shown). In this larger \( \omega \) region, the variances of \( r_\alpha \) decay with \( 1/\omega^2 \).

Network D (fig. 1D) exhibits a different behavior as shown in fig. 1E and F. In the steady state, the quantization of amplitude deviations is partially broken and \( r_c, r_f, r_g \) and \( r_h \) show large fluctuations (fig. 1F). The widths of their fluctuations are greater than the differences between quantized values, and their fluctuations decay slowly with the increase in \( \omega \) (fig. 1E). Other oscillators (\( a, b, d \) and \( e \)) still stay around the stationary points. We emphasize that network D is constructed by changing only one interaction from network A: \( c \rightarrow d \) to \( c \rightarrow g \). Thus, these behaviors are quite sensitive to the topology of the networks.

In summary, we can say that the amplitude deviations of the oscillators in the networks are widely distributed when the nonisochronicity is large. The amplitude
Modulation of frequencies in oscillator networks

Fig. 2: (Colour on-line) Scatter plot distributions of averages and variances of amplitude deviations of all oscillators in two large networks with 100 and 1000 oscillators. Data on the 1000-oscillator network are plotted with red open circles, and data on the 100-oscillator network are plotted with blue filled diamonds. Panels A and B show results of linear oscillator networks (eqs. (3), (4)), with \( \omega = 1 \). Vertical dashed lines indicate \( \tau = mD_\omega / \gamma \) with \( m = 1, \ldots, 7 \). Networks in panel A are generated randomly with connection probabilities \( 2/(\text{number of oscillators}) \), and networks in panel B are generated randomly with connection probabilities \( 3/(\text{number of oscillators}) \). Panel C shows results of Stuart-Landau oscillator networks (eq. (5)) with connection probabilities \( 2/(\text{number of oscillators}) \), and \( \gamma = 0.1, D = 0.01, \omega = 10 \). The averages of amplitude deviations \( (1 - \bar{z}) \) are plotted, instead of averages of amplitudes \( (\bar{z}) \). Vertical dashed lines indicate \( |z| = \sqrt{1 - mD/\gamma} \) with \( m = 1, \ldots, 8 \). They correspond to the values of quantized amplitudes (discussed in the text).

behaviors are classified into two types: i) those that take a quantized value, and ii) those that display large fluctuations. All oscillators in some networks show type-i behavior (e.g., network A), while oscillators showing both behavior types coexist in other networks (e.g., network D). Although it generally depends on the topology of the network, larger networks more frequently show coexisting behaviors. When 1000 networks with 10 and 50 oscillators were generated, 532 networks with 10-oscillator systems showed coexisting behavior, as did 975 networks with 50-oscillator systems. Figure 2A and B shows scatter plot distributions of the averages and variances of amplitude deviations for two random networks that have 100 and 1000 oscillators. The variances are widely distributed. For less variant oscillators (variance < 0.001), the average values show the order at regular intervals as seen in fig. 1.

Note that, in this model, the complete synchronization state is always stable in any network. However, if the nonisochronicity is high, this state is not reached from most initial conditions. Instead, distributed frequency states are observed.

Although these results are gained with oscillators that depend linearly on \( r \), some nonlinear oscillators display similar behavior. Here we take networks of Stuart-Landau oscillators as an example,

\[
\dot{z}_\alpha = \{ \gamma + i - (\gamma - i\omega)|z_\alpha|^2 \}z_\alpha + D \sum_\beta (z_\beta - z_\alpha), \tag{5}
\]

where \( z_\alpha, z_\beta \) are complex variables. Figure 2C shows scatter plot distributions of the random networks of Stuart-Landau oscillators. The figure also displays the quantization in amplitude values, as shown in fig. 2A and B.

These distributed frequency states are basically sustained by a feedback mechanism: the difference in amplitude deviations leads to the difference in frequencies and the difference in frequencies maintains the difference in amplitude deviations. In the following, we illustrate this feedback mechanism and examine how these behaviors are generated depending on the network topologies.

We begin by investigating the mechanism of the amplitude quantization. For amplitude deviations to take stationary values, interaction terms should be approximately constant. There are two possible mechanisms; one is phase locking and the other is averaging. If the phase difference between two interacting oscillators takes a constant value (a phase-locked state), the interaction term also becomes constant. However, it is unlikely that the phase differences between all interacting oscillators take constant values except in complete synchronization. In distributed frequency states, limited pairs of interacting oscillators may be in phase-locked states. On the other hand, the averaging assumption for an interaction holds when the two oscillators have considerably different frequencies so that the time evolution of amplitude deviations cannot follow the alternating interaction term. Since the difference in amplitude deviations produces a difference in frequencies, the averaging assumption is expected to hold.

Assume that the speeds of the phase differences between all coupled oscillators are sufficiently fast for the interaction functions in eq. (4) to be approximately replaced by the average values \( \cos(2\pi(\theta_\beta - \theta_\alpha)) \approx 0 \). Then, eq. (4) is solved and the approximate amplitude deviation takes

\[
r_\alpha \approx \frac{N_\alpha D_\omega}{\gamma}, \tag{6}
\]

where \( N_\alpha \) is the number of interactions for the oscillator \( \alpha \). Note that the approximate amplitude deviations

1000 networks for each size were generated randomly with connection probabilities \( 2/(\text{number of oscillators}) \). \( \omega = 1 \) were used to test the behaviors.
are proportional to $N_\alpha$, and hence they take quantized values. With the parameters used in fig. 1, $D_r/\gamma = 0.1$ which agrees with the observed interval in the figure. Applying the averaging assumption to the Stuart-Landau oscillators (eq. (5)), we get $|z| \approx \sqrt{1 - N_\alpha D/\gamma}$, which also explains the values of the quantized amplitudes (fig. 2C).

Next, we choose two interacting oscillators in the networks, and examine under which conditions the averaging assumption holds. Replacing amplitude deviations in eq. (3) with approximate values, we get an approximate equation for the phase difference between two interacting oscillators from eq. (3)

$$\Delta \theta_{\alpha\beta} = (N_\alpha - N_\beta) \frac{D_r \omega}{\gamma} - D_r \sin(2\pi \Delta \theta_{\alpha\beta}),$$  

(7)

where the interaction effects from other oscillators are assumed to be replaced by average values. If $N_\alpha \neq N_\beta$ and $D_r \omega/(\gamma D_\theta) \gg 1$ hold, the interaction has little effect and the averaging assumption between the phases of these two oscillators also holds. Then a uniform oscillation with frequency $F_{\alpha\beta} = (N_\alpha - N_\beta) D_\omega/\gamma$ is an approximate solution of eq. (7). By putting the solution in eq. (4), we get the equation for the amplitude deviation

$$\dot{r}_\alpha = -\gamma r_\alpha + (N_\alpha - 1) D_r + D_r \{1 - \cos (2 \pi \Delta \theta_{\alpha\beta})\},$$  

(8)

where the effects from other interactions are again replaced by average values. The solution is

$$r_\alpha = \frac{N_\alpha D_r}{\gamma} + \frac{D_r}{\sqrt{(2 \pi F_{\alpha\beta})^2 + \gamma^2}} \sin(2 \pi F_{\alpha\beta} t + \theta_0),$$  

(9)

where $\theta_0$ is the initial phase. This equation indicates that the average of $r_\alpha$ (the first term) agrees with eq. (6). The second term denotes that the variance of the amplitude deviation decreases with the increase in $\omega/\gamma$. This is also consistent with the decrease of fluctuation in fig. 1B, since $1/\sqrt{\text{variance}} \approx F_{\alpha\beta} \propto \omega$. Therefore, if this estimate is applicable to all pairs of interacting oscillators in a network, the averaging assumptions are fulfilled and the quantized state of amplitude deviations is expected for the network.

When $N_\alpha = N_\beta$ holds, the averaging assumption does not hold, and the phase-synchronous state (a special case of phase-locked states)

$$\Delta \theta_{\alpha\beta} = 0$$  

(10)

becomes the attractor for eq. (7). Substituting it into eq. (8), we get another solution

$$r_\alpha = r_\beta = \frac{(N_\alpha - 1) D_r}{\gamma},$$  

(11)

which differs from the eq. (6). Since the amplitude deviations of both oscillators still take the same value, the phase synchronous state is also the attractor at the new amplitude deviation value. Besides, if $N_\alpha - 1 \neq N_\gamma$ holds for all oscillators ($\gamma$) that interact with oscillator $\alpha$ or $\beta$, this local synchrony does not change the averaged behaviors of other oscillators and is sustained.

Summarizing these arguments, the averages of amplitude deviations are expected to be given by

$$\bar{r}_\alpha = \frac{N_\alpha' D_r}{\gamma},$$  

(12)

where $N_\alpha'$ denotes the number of effective interactions after eliminating the interaction of synchronizing pairs.

Applying this to network A, the quantization observed in fig. 1C is explained. In network A, d-e and g-h pairs have the same numbers of interactions ($N_d = N_e = 2$, $N_g = N_h = 3$), and are expected to synchronize. In fact, the phase difference between oscillators $d$ and $e$ was always less than 0.013 and oscillators $g$ and $h$ had exactly the same phase in the simulations.

Next, we illustrate the continuous irregular motion of amplitude deviations observed in network D. In short, the motion is caused by the indeterminacy of a consistent set of synchronizing pairs. This leads to continuous changes in the synchronizing pairs, and irregular motion arises. Here we describe these processes in detail using network D as an example.

Following these rules, oscillators $f$ and $g$ in network D synchronize ($N_f = N_g = 4$), and their amplitude deviations converge to $3D_r/\gamma$. In addition, oscillators $c$ and $h$, which interact with $g$, have the same frequency as oscillator $g$, since $N_c = N_h = N_g - 1 = 3$. Thus, $g$ and either of the two oscillators start to synchronize, which brings about another change in the amplitude deviations of the synchronizing pair ($c-g$ or $g-h$). Their amplitude deviations converge to $2D_r/\gamma$. However, this second process introduces a difference in amplitude deviations between oscillators $f$ and $g$, i.e. a difference in frequencies, and hence their synchrony is disrupted. This effect also destroys the second synchrony ($c-g$ or $g-h$). Thus, the system returns to the original no-synchrony state. In this way, the synchronizing pairs in network D are never settled and the amplitude deviations of oscillators $c$, $f$, $g$, and $h$ continue to change spontaneously.

Lastly, we apply this analysis to a simple class of networks: the circular arrangements of oscillators with nearest-neighbor couplings. In this network, the number of effective connections can take $N'_\alpha = 0, 1, 2$ for all oscillators. If an oscillator takes $N'_\alpha = 0$, it is synchronous with both neighbors, and the neighbors also take $N'_\alpha = 0$. This situation can be globally stabilized only when the complete synchronization occurs. If an oscillator takes $N'_\alpha = 1$, one neighbor is the synchronizing partner and the other neighbor takes $N'_\alpha \neq 1$. If a oscillator takes $N'_\alpha = 2$, both sides take $N'_\alpha \neq 2$. To stabilize these states, unsynchronized oscillators ($N'_\alpha = 2$) must have neighbors that are synchronous with the other side of the oscillators, and the neighbors of synchronizing pairs of oscillators ($N'_\alpha = 1$) must be unsynchronized oscillators. This is realized only when unsynchronized oscillators and synchronizing pairs of oscillators are arranged alternately. This is the spatially
periodic solution with period three, and it is realized only when the number of oscillators is a multiple of three.

These are verified by simulations. The networks with 6, 9, 12, 15, 18 and 21 oscillators converge to the periodic solutions, where the averages and the variances $\langle V_0(r) \rangle$ of the amplitude deviations take $(\sigma_3, \sigma_9) = (0.99 \times 10^{-1}, 1.28 \times 10^{-5})$, $(\sigma_{3n+1}, \sigma_{3n+4}) = (1.97 \times 10^{-1}, 4.79 \times 10^{-5})$. The relaxing time approaching the periodic solutions increases with the size of the network as shown in fig. 3. If the number of oscillators is not a multiple of three, the system never settles to the periodic solutions, and is expected finally to converge to the complete synchronization state. This was observed in small networks with 4, 5, 7, 8, 10, or 11 oscillators, and the relaxing time approaching the complete synchronization also increases with network size (fig. 3). At the transient states before these relaxations occur, all oscillators display irregular modulations in the same way, because of the symmetry of the system, and no partial quantization is observed. The duration in the transient states can be long as seen in fig. 3. In particular, we have not observed the relaxation to the complete synchronization state in networks with 13, 14, 16, 17 or 20 oscillators within the simulation time frame ($t \approx 10^6$). Note that, the rings of oscillators have twisted states $\langle \theta_n - \theta_{n+1} = \pm 1/M, \ r_n = D_r \cos(2\pi/M)/\gamma, \ M \in \mathbb{Z}, M < \text{number of oscillators}/2 \rangle$ as attractors. However, they were not observed in any of the simulations performed in fig. 3.

In summary, we have investigated networks of limit cycle oscillators whose frequencies change widely along with amplitude deviation changes, and we have reported novel complex behaviors: the quantization and the continuous large fluctuation of amplitude deviations. In the analysis, we explained a mechanism that determines the behaviors of amplitude deviations from the local structure of the network. The behavior of amplitude deviation depends primarily on the number of connections the oscillator has, but also on the states of interacting oscillators which are determined by the numbers of their connections. Therefore, the global structure of the network topology affects the behavior of each oscillator.

Here we summarize the parameter relationships assumed in the model derivation and the analysis, 

$$D_r \gamma \ll D_r \ll \gamma \ll 1.$$  \hspace{1cm} (13)

The left relationship is required for the time scale separation between oscillators with different quantized values, the middle relationships is required to ensure that $\gamma$ is small (since our model includes only the first-order term of $\gamma$), and the right relationship is required for the interaction between oscillators to be a function of the phase differences. Although these conditions limit the parameter ranges where the reported phenomena are expected, they do not deny the possibility of qualitatively similar phenomena in other parameter regions. In fact, we reported similar behavior in networks of Stuart-Landau oscillators.

Nonisochronicity is key for a variety of phenomena. In quantized states, the differences in amplitude deviations originate from the differences in frequencies, and this assures the decoupling of oscillator interactions by averaging. Thus, the differences in amplitude deviations are sustained. The same effect enables the globally coupled oscillators to form stable multi-cluster states [14,15]. In contrast, in fluctuating states, amplitude deviation changes result in switching between decoupled and influential interactions. This emergence and annihilation of effective interactions sustainably drives the fluctuation of amplitude deviations.

If $\gamma$ and $D_r$ are sufficiently large in our model, $r$ can be removed and the model will correspond to a phase oscillator model, where the nonisochronicity is represented by an asymmetry in the phase coupling function [2]. However, in such a case, the fluctuations in phase evolution speeds become quite large. In our model, small $\gamma$ and $D_r$ bring slow change in $r$ (i.e. slow change in frequency), which provides the clear time scale separation between oscillators. Although the averaged frequencies of oscillators in the phase oscillator model may be distributed as observed in [21], the detailed dynamics becomes more
chaotic, and the clear quantization seen in fig. 1C and F, will not be observed.

In this analysis, we have also shown that the topology of networks determines the behaviors of oscillator networks, and we have revealed how a small difference in topology brings about qualitative behavioral change. This demonstrates the importance of exploring the detailed topology of oscillator networks. Knowledge obtained from group structures of networks [22] or graph coloring problems [23] may be useful for understanding the behaviors of the system.

This change in effective frequencies (time scales) and the associated change in effective interactions are both important properties of complex systems. Such behavior is frequently observed in biological systems as an adaptation of time scales. For example, nerve systems are thought to code information by changing the firing frequencies of neurons as well as their phase relationships [16]. The complex dynamics of both properties are expected to work as unified systems [19]. These relationships are also observed in the adaptive behaviors of single-cell organisms that contain a variety of time scales [17,18]. Our study extracts the mechanism underlying the change in time scales from a general limit cycle system, and reveals the origins of these complex phenomena.

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