Quasideterminants and Casimir elements for the general linear Lie superalgebra

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Abstract

We apply the techniques of quasideterminants to construct new families of Casimir elements for the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ whose images under the Harish-Chandra isomorphism are respectively the elementary, complete and power sums supersymmetric functions.

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1 Introduction

Let \( A \) be a square matrix over a ring. Its quasideterminants are certain rational expressions in the entries of \( A \). The theory of quasideterminants originates from the papers by Gelfand and Retakh \([2, 3]\) and since then a number of applications of the theory has been found; see \([4]\) for an overview. In particular, the techniques of quasideterminants is fundamental in the theory of noncommutative symmetric functions developed by Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon \([1]\). The symmetric functions associated with a matrix whose entries are elements of a noncommutative ring is one of the interesting specializations of the general theory. When applied to the matrix \( E \) formed by the generators of the general linear Lie algebra \( \mathfrak{gl}(n) \) the theory produces a new family of Casimir elements for \( \mathfrak{gl}(n) \) as well as a distinguished set of generators of the Gelfand–Tsetlin subalgebra of \( \mathfrak{U}(\mathfrak{gl}(n)) \); see \([1\, Section\, 7.4]\). These results were extended to the orthogonal and symplectic Lie algebras in \([8]\) with the use of the twisted Yangians and quantum determinants; see also a review paper \([10]\).

In this paper we use the techniques of quasideterminants to get new families of Casimir elements for the general linear Lie superalgebra \( \mathfrak{gl}(m|n) \) and calculate their images with respect to the Harish-Chandra isomorphism. They can be regarded as super-analogs of those constructed in \([1\, Section\, 7.4]\). Three families of Casimir elements are given explicitly in terms of some oriented graphs associated with \( \mathfrak{gl}(m|n) \). The Harish-Chandra images turn out to be respectively the elementary, complete and power sums supersymmetric functions.

The starting point for our construction is a result of Nazarov \([12]\). He produced a formal series \( B(t) \) called quantum Berezinian with coefficients in the center of the universal enveloping algebra \( \mathfrak{U}(\mathfrak{gl}(m|n)) \). Our first result is a quasideterminant factorization of \( B(t) \) (Theorem 3.1). We then use it to get graph presentations for the Casimir elements (Theorem 4.1).

Some other families of Casimir elements for \( \mathfrak{gl}(m|n) \) were constructed e.g. in \([9]\). This work is a super-version of the earlier constructions of \([13, 14]\) for \( \mathfrak{gl}(n) \) and it provides a linear basis of the center of \( \mathfrak{U}(\mathfrak{gl}(m|n)) \) formed by the so-called quantum immanants.

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2 Preliminaries

Let \( x = (x_1, \ldots, x_m) \) and \( y = (y_1, \ldots, y_n) \) be two families of variables. A polynomial \( P \) in \( x \) and \( y \) is called supersymmetric if \( P \) is symmetric separately in \( x \) and \( y \) and satisfies the following cancellation property: the result of setting \( x_m = -y_n = z \) in \( P \) is independent of \( z \). We denote by \( \Lambda(m|n) \) the algebra of supersymmetric polynomials in \( x \) and \( y \). The algebra \( \Lambda(m|n) \) is generated by the polynomials

\[
p_k = x_1^k + \cdots + x_m^k + (-1)^{k-1}(y_1^k + \cdots + y_n^k), \quad k \geq 1,
\]

called the power sums supersymmetric functions. Two other families of generators of \( \Lambda(m|n) \) are comprised by the elementary and complete supersymmetric functions defined respectively by the formulas

\[
e_k = \sum_{p+q=k} \sum_{i_1 \leq \cdots \leq i_p} \sum_{j_1 \leq \cdots \leq j_q} x_{i_1} \cdots x_{i_p} y_{j_1} \cdots y_{j_q};
\]
\[
h_k = \sum_{p+q=k} \sum_{i_1 \leq \cdots \leq i_p} \sum_{j_1 \leq \cdots \leq j_q} x_{i_1} \cdots x_{i_p} y_{j_1} \cdots y_{j_q};
\]

see [15], [16].

We shall denote by \( E_{ij}, i, j = 1, \ldots, m + n \) the standard basis of the Lie superalgebra \( \mathfrak{gl}(m|n) \). The \( \mathbb{Z}_2 \)-grading on \( \mathfrak{gl}(m|n) \) is defined by \( E_{ij} \mapsto \bar{i} + \bar{j} \), where \( \bar{i} \) is an element of \( \mathbb{Z}_2 \) which equals 0 or 1 depending on whether \( i \leq m \) or \( i > m \). The commutation relations in this basis are given by

\[
[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj} (-1)^{(i+j)(k+l)}.
\]

(2.3)

Given a \( m+n \)-tuple \( (\lambda|\mu) = (\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_n) \in \mathbb{C}^{m+n} \) we consider a highest weight \( \mathfrak{gl}(m|n) \)-module \( L(\lambda|\mu) \) with the highest weight \( (\lambda|\mu) \). That is, \( L(\lambda|\mu) \) is generated by a nonzero vector \( \xi \) such that

\[
E_{ii} \xi = \lambda_i \xi \quad \text{for} \quad i = 1, \ldots, m,
\]
\[
E_{m+j,m+j} \xi = \mu_j \xi \quad \text{for} \quad j = 1, \ldots, n,
\]
\[
E_{ij} \xi = 0 \quad \text{for} \quad 1 \leq i < j \leq m + n.
\]

(2.4)

Any element \( z \) of the center \( Z(\mathfrak{gl}(m|n)) \) of the universal enveloping algebra \( U(\mathfrak{gl}(m|n)) \) acts in \( L(\lambda|\mu) \) as a scalar \( \chi(z) \). For a fixed \( z \) the scalar \( \chi(z) \) is a polynomial in \( \lambda_i \) and \( \mu_i \) which is supersymmetric in the shifted variables defined by

\[
x_i = \lambda_i - i + 1 \quad \text{for} \quad i = 1, \ldots, m,
\]
\[
y_j = \mu_j + m - j \quad \text{for} \quad j = 1, \ldots, n.
\]

(2.5)

Furthermore, the map \( z \mapsto \chi(z) \) defines an algebra isomorphism

\[
\chi : Z(\mathfrak{gl}(m|n)) \to \Lambda(m|n),
\]

(2.6)

which is called the Harish-Chandra isomorphism; see [5], [15], [16].


\section{Decomposition of the Quantum Berezinian}

Introduce the super-matrix $\hat{E}$ of size $(m + n) \times (m + n)$ whose ij-th entry is $\hat{E}_{ij} = (-1)^\bar{\nu} E_{ij}$. By the quantum Berezinian we mean the formal series $B(t)$ defined by

$$B(t) = \sum_{\sigma \in S_m} \text{sgn} \, \sigma \left( 1 + t \hat{E} \right)_{\sigma(1),1} \cdots \left( 1 + t (\hat{E} - m + 1) \right)_{\sigma(m),m}$$

$$\times \sum_{\tau \in S_n} \text{sgn} \, \tau \left( 1 + t (\hat{E} - m + 1) \right)_{m+1,m+\tau(1)}^{-1} \cdots \left( 1 + t (\hat{E} - m + n) \right)_{m+n,m+n}^{-1}.$$

(3.1)

The quantum Berezinian was constructed by Nazarov \cite{12}. He also proved that all its coefficients are central in the universal enveloping algebra $U(\mathfrak{gl}(m|n))$. The image of $B(t)$ under the Harish-Chandra isomorphism is given by

$$\chi(B(t)) = \frac{(1 + tx_1) \cdots (1 + tx_m)}{(1 - ty_1) \cdots (1 - ty_n)},$$

(3.2)

cf. \cite{9}. Our first result is a decomposition of $B(t)$ into a product of quasideterminants. If $X$ is a square matrix over a ring with 1 such that there exists the inverse matrix $X^{-1}$ and its ji-th entry $(X^{-1})_{ji}$ is an invertible element of the ring, then the ij-th quasideterminant of $X$ is defined by the formula

$$|X|_{ij} = \left( (X^{-1})_{ji} \right)^{-1},$$

see \cite{2, 3} for other equivalent definitions of the quasideterminants and their properties.

\textbf{Theorem 3.1.} We have the following decomposition of $B(t)$ in the algebra of formal series with coefficients in $U(\mathfrak{gl}(m|n))$

$$B(t) = |1 + t \hat{E}^{(1)}|_{11} \cdots |1 + t (\hat{E}^{m} - m + 1)|_{mm}$$

$$\times |1 + t (\hat{E}^{(m+1)} - m + 1)|_{m+1,m+1}^{-1} \cdots |1 + t (\hat{E}^{(m+n)} - m + n)|_{m+n,m+n}^{-1},$$

(3.3)

where $\hat{E}^{(k)}$ denotes the submatrix of $\hat{E}$ corresponding to the first $k$ rows and columns. Moreover, the factors in the decomposition are pairwise permutable.

\textbf{Proof.} We employ a quasideterminant decomposition of the quantum determinant for the Yangian $Y(\mathfrak{gl}(r))$. The latter is the associative algebra with the generators $t_{ij}^{(1)}, t_{ij}^{(2)} \ldots$ where $1 \leq i, j \leq r$ and the following defining relations

$$[t_{ij}(u), t_{kl}(v)] = \frac{1}{u - v} \left( t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u) \right),$$

(3.4)
where
\[ t_{ij}(u) = \delta_{ij} + t^{(1)}_{ij} u^{-1} + t^{(2)}_{ij} u^{-2} + \cdots \in Y(\mathfrak{gl}(n))[u^{-1}]. \] (3.5)

Consider the quantum determinant of the matrix \( T(u) = [t_{ij}(u)] \) defined by the following equivalent formulas
\[
q\text{det} T(u) = \sum_{\sigma \in S_r} \text{sgn} \sigma \cdot t_{\sigma(1),1}(u) \cdots t_{\sigma(r),r}(u-r+1)
= \sum_{\sigma \in S_r} \text{sgn} \sigma \cdot t_{1,\sigma(1)}(u-r+1) \cdots t_{r,\sigma(r)}(u).
\] (3.6)

It is well-known that the coefficients of this series are algebraically independent generators of the center of the algebra \( Y(\mathfrak{gl}(r)) \); see e.g. [11] for a proof. For \( 1 \leq k \leq n \) denote by \( T^{(k)}(u) \) the submatrix of \( T(u) \) corresponding the first \( k \) rows and columns. We have the following quasideterminant decomposition of \( q\text{det} T(u) \) in the algebra \( Y(\mathfrak{gl}(m))[u^{-1}] \)
\[
q\text{det} T(u) = |T^{(1)}(u)|_{(u-n+1)} \cdots |T^{(m)}(u-m+1)|_{m,m},
\] (3.7)
where the factors are pairwise permutable; see [8] and also [2], [6] for analogous decompositions in the case of noncommutative determinants of different types. Now we apply the algebra homomorphism \( Y(\mathfrak{gl}(m)) \rightarrow U(\mathfrak{gl}(m|n)) \) given by
\[ T(u) \mapsto 1 + \hat{E}(m) u^{-1} \] (3.8)
to (3.7), set \( u = t^{-1} \) and multiply both sides by \( (1-t) \cdots (1-(m-1)t) \). This will represent the first determinant factor in (3.1) as a product of quasideterminants which comprise the first \( m \) factors in (3.3); cf. [8].

Now consider the second factor in (3.1). We shall use the subscript \((k)\) of a matrix to indicate its submatrix obtained by removing the first \( k-1 \) rows and columns. Here we need another version of the decomposition (3.7) given by
\[
q\text{det} T(u) = |T_{(1)}(u-n+1)|_{11} \cdots |T_{(m)}(u)|_{m,m}.
\] (3.9)
Apply another homomorphism \( Y(\mathfrak{gl}(n)) \rightarrow U(\mathfrak{gl}(m|n)) \) defined by
\[ T(u) \mapsto [(1 + \hat{E} u^{-1})^{-1}]_{(m+1)}, \] (3.10)
(see [12]) to both sides of (3.9) with \( q\text{det} T(u) \) expanded by the second formula in (3.6). Now observe that by the Inversion Theorem for quasiminors [2, 3], we have for any \( k \in \{1, \ldots, n\} \)
\[
\left[ (1 + \hat{E} (u-n+k)^{-1})^{-1} \right]_{m+k,m+k} = [1 + \hat{E}^{(m+k)} (u-n+k)^{-1}]_{m+k,m+k}^{-1}.
\] (3.11)
To complete the argument, it remains to set \( u = t^{-1} + n - m \) and divide both sides of the relation by the product \( (1 + t(1 - m)) \cdots (1 + t(n - m)) \).

Finally, note that the product of the first \( m + n - 1 \) factors in (3.3) coincides with the quantum Berezinian for the subalgebra \( gl(m|n-1) \) of \( gl(m|n) \). Therefore the last factor in (3.3) is permutable with the elements of \( gl(m|n-1) \) by the centrality of the quantum Berezinian. The proof is completed by an obvious induction.

\[ \square \]

4 Casimir elements

Let \( A = (A_{ij}) \) be a square matrix of size \( l \times l \) with entries from an arbitrary ring and let \( t \) be a formal variable. Fix an integer \( i \) between 1 and \( l \). Following [1, Definition 7.19] introduce the noncommutative symmetric functions associated with the matrix \( A \) and the index \( i \) as follows. The elementary symmetric functions \( \Lambda_k^{(i)} \), the complete symmetric functions \( S_k^{(i)} \), the power sums symmetric functions of the first kind \( \Psi_k^{(i)} \) and the power sums symmetric functions of the second kind \( \Phi_k^{(i)} \) are defined by the formulas

\[
1 + \sum_{k=1}^{\infty} \Lambda_k^{(i)} t^k = |1 + tA|_{ii},
\]
\[
1 + \sum_{k=1}^{\infty} S_k^{(i)} t^k = |1 - tA|_{ii}^{-1},
\]
\[
\sum_{k=1}^{\infty} \Psi_k^{(i)} t^{k-1} = |1 - tA|_{ii} \frac{d}{dt} |1 - tA|_{ii}^{-1},
\]
\[
\sum_{k=1}^{\infty} \Phi_k^{(i)} t^{k-1} = -\frac{d}{dt} \log (|1 - tA|_{ii}).
\]

These functions are polynomials in the entries of the matrix \( A \) and can be interpreted in terms of graphs in the following way. Let us consider the complete oriented graph \( \mathcal{A} \) with \( l \) vertices \( \{1, 2, \ldots, l\} \), the arrow from \( i \) to \( j \) being labelled by \( A_{ij} \). Then every path in the graph going from \( i \) to \( j \) defines a monomial of the form \( A_{ir_1}A_{r_1r_2}\cdots A_{r_{k-1}j} \). A simple path is a path such that \( r_s \neq i, j \) for every \( s \). Then by [1, Proposition 7.20], \((-1)^{k-1} \Lambda_k^{(i)} \) is the sum of all monomials labelling simple paths in \( \mathcal{A} \) of length \( k \) going from \( i \) to \( i \); \( S_k^{(i)} \) is the sum of all monomials labelling paths in \( \mathcal{A} \) of length \( k \) going from \( i \) to \( i \); \( \Psi_k^{(i)} \) is the sum of all monomials labelling paths in \( \mathcal{A} \) of length \( k \) going from \( i \) to \( i \), where the coefficient of each monomial is the length of the first return to \( i \); \( \Phi_k^{(i)} \) is the sum of all monomials labelling paths in \( \mathcal{A} \) of length \( k \) going from \( i \) to \( i \), where the coefficient of each monomial is the ratio of \( k \) to the number of returns to \( i \).
For any \(i = 1, \ldots, m\) consider the matrix \(\hat{E}(i) - i + 1\) and the noncommutative symmetric functions associated with this matrix and the index \(i\). We keep the above notation for these functions. Similarly, for any \(j = 1, \ldots, n\) consider the matrix \(-\hat{E}(m+j) + m - j\) and the noncommutative symmetric functions associated with this matrix and the index \(m + j\). Again, we denote the functions by the same symbols and distinguish them by the upper index \(m + j\).

**Theorem 4.1.** The algebra \(Z(\mathfrak{gl}(m|n))\) is generated by each of the families

\[
\Lambda_k = \sum_{i_1 + \cdots + i_{m+n} = k} \Lambda_{i_1}^{(1)} \cdots \Lambda_{i_m}^{(m)} S_{i_{m+1}}^{(m+1)} \cdots S_{i_{m+n}}^{(m+n)},
\]

\[
S_k = \sum_{i_1 + \cdots + i_{m+n} = k} S_{i_1}^{(1)} \cdots S_{i_m}^{(m)} \Lambda_{i_{m+1}}^{(m+1)} \cdots \Lambda_{i_{m+n}}^{(m+n)},
\]

\[
\Psi_k = \sum_{i=1}^m \Psi_i^{(i)} + (-1)^{k-1} \sum_{j=1}^n \Psi_j^{(m+j)},
\]

\[
\Phi_k = \sum_{i=1}^m \Phi_i^{(i)} + (-1)^{k-1} \sum_{j=1}^n \Phi_j^{(m+j)},
\]

where \(k = 1, 2, \ldots\). Moreover, \(\Psi_k = \Phi_k\) for any \(k\), and the Harish-Chandra images of these generators are respectively the elementary, complete and power sums supersymmetric functions,

\[
\chi(\Lambda_k) = e_k, \quad \chi(S_k) = h_k, \quad \chi(\Psi_k) = p_k.
\]

**Proof.** Introduce the generating functions for the supersymmetric polynomials (2.1) and (2.2) by

\[
p(t) = \sum_{k=1}^{\infty} p_k t^{k-1},
\]

\[
e(t) = 1 + \sum_{k=1}^{\infty} e_k t^k,
\]

\[
h(t) = 1 + \sum_{k=1}^{\infty} h_k t^k.
\]

These functions are related by

\[
h(t) = e(-t)^{-1}, \quad p(t) = -\frac{d}{dt} \log e(-t) = e(-t) \frac{d}{dt} e(-t)^{-1},
\]

see e.g. [7]. On the other hand, by Theorem 3.1 we have

\[
1 + \sum_{k=1}^{\infty} \Lambda_k t^k = B(t)
\]
which proves that the elements $\Lambda_k$ are central in $U(\mathfrak{gl}(m|n))$. Moreover, $\chi(B(t)) = e(t)$ due to (3.2) and so $\chi(\Lambda_k) = e_k$. The proof is completed by applying (4.5) and taking into account the fact that the factors in the decomposition (3.3) are mutually permutable; cf. the argument for the case of $\mathfrak{gl}(n)$ [11 Section 7.4]. 

**Example 4.2.** We have

$$
\Psi_1 = \sum_{i=1}^{m} (E_{ii} - i + 1) + \sum_{j=1}^{n} (E_{m+j,m+j} + m - j),
$$

$$
\Psi_2 = \sum_{i=1}^{m} \left( (E_{ii} - i + 1)^2 + 2 \sum_{k=1}^{i-1} E_{ik}E_{ki} \right)
$$

$$
- \sum_{j=1}^{n} \left( (E_{m+j,m+j} + m - j)^2 - 2 \sum_{l=1}^{m+j-1} (-1)^l E_{m+j,l}E_{l,m+j} \right).
$$

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