Abstract. Let $U(G)$ be a maximal unipotent subgroup of one of classical groups $G = GL(V), O(V), Sp(V)$. Let $W$ be a direct sum of copies of $V$ and its dual $V^*$. For the natural action $U(G) : W$, we describe a minimal system of homogeneous generators for the algebra of $U(G)$-invariant regular functions on $W$. For $G = GL(V)$, we also describe the syzygies among these generators in some particular cases.

1 Main theorem.

Let $V$ be a finite-dimensional vector space over an algebraically closed field $k$ of characteristic zero. Let $H \subseteq GL(V)$ be an algebraic subgroup. For any $l \in \mathbb{N}$, we denote by $lV$ the direct sum of $l$ copies of $V$; similarly, we define $mV^*$ for any $m \in \mathbb{N}$. Consider the natural action of $H$ on $W = lV \oplus mV^*$ and assume that the algebra $k[W]^H$ of invariants is finitely generated for any $l, m$. Then First Fundamental Theorem of Invariant Theory of $H$ refers to a description of a minimal system of homogeneous generators of $k[W]^H$ for all $l, m$.

Such a description exists when $H$ is classical, i.e., $H$ is one of groups $GL(V), SL(V), O(V), SO(V), Sp(V)$ (see e.g. [PV, §9]).

Let now $G$ be one of the groups $GL(V), O(V), Sp(V)$; let $U(G)$ be a maximal unipotent subgroup of $G$. By [PV, Theorem 3.13], the algebra $k[W]^{U(G)}$ is finitely generated. Also the invariants of $U(G)$ are linear combinations of highest vectors of irreducible factors for $G$-module $k[W]$. So the $U(G)$-invariants are the $G$-covariants and the First Fundamental Theorem for covariants of $G$ means that for the invariants of $H = U(G)$.

Using a result (and some ideas) of [Ho], we prove in this paper First Fundamental Theorem for covariants of each of the above classical $G$.

Note that for $G = Sp(V), O(V)$, $V$ and $V^*$ are isomorphic as $G$-modules, hence, as $U(G)$-modules. Therefore we may assume $m = 0$ in these cases.
Elements of $\wedge^k V^* \subseteq \otimes^k V^* \subseteq k[V]$, $k \leq \dim V$ are said to be multilinear anisymmetric functions as well as their analogs in $k[V^*]$.  

**Theorem 1** The algebra $k[W]^{U(G)}$ is generated by the subalgebra $k[W]^G$ and multilinear antisymmetric invariants. Moreover, a set $\mathcal{M}$ described below is a minimal system of homogeneous generators of $k[W]^{U(G)}$. 

We describe a minimal system $\mathcal{M}$ of homogeneous generators of $k[W]^{U(G)}$ in a coordinate form. Set $n = \dim V$, choose a basis of $V$ and denote by $\overline{V}$ the corresponding $n \times t$-matrix of coordinates on $IV$. Similarly, denote by $\overline{V}^*$ the $m \times n$-matrix of coordinates on $mV^*$, in the dual basis of $V^*$. A minor of order $k$ of a matrix is said to be left, if it involves the first $k$ columns. Analogously, we call it lower, if it involves the last $k$ rows.  

A) Let $G = GL(V)$ and define $U(GL) = U(GL(V))$ to be the subgroup of the strictly upper triangular matrices, in the above basis. Then $\mathcal{M}$ is:  

- the matrix elements of the product $\overline{V}^*\overline{V}$,  
- the lower minor determinants of order $k$ of $\overline{V}$, $k = 1, \cdots, \min\{l, n\}$,  
- the left minor determinants of order $p$ of $\overline{V}^*$, $p = 1, \cdots, \min\{m, n\}$.  

Let $T(GL)$ be the diagonal matrices in the above basis. Then $T(GL)$ is a maximal torus of $G$ normalizing $U(GL)$. The pair $T(GL), U(GL)$ defines a system of simple roots of $T(GL)$. Here and in what follows, we use the enumeration of simple roots of simple groups as in [OV] and denote by $\varphi_1, \cdots, \varphi_n$ the fundamental weights. The torus $T(GL)$ acts on $k[W]^{U(GL)}$ and the elements of $\mathcal{M}$ are weight vectors of $T(GL)$. The set of their degrees and weights is (for $l, m \geq n$):  

$$(2,0), (1, \varphi_1), (2, \varphi_2), \cdots, (n, \varphi_n),$$  
$$(1, \varphi_{n-1} - \varphi_n), (2, \varphi_{n-2} - \varphi_n), \cdots, (n - 1, \varphi_1 - \varphi_n), (n, -\varphi_n).$$  

Furthermore, let $Q$ be a bilinear symmetric (antisymmetric) form having in the above basis a matrix with $\pm 1$ on the secondary diagonal and with zero entries outside it. Define $G = O(V)$ ($G = Sp(V)$) to be the stabilizer of this form. Then $U(G) = G \cap U(GL)$ is a maximal unipotent subgroup in $G$. Moreover, set $T(O) = T(GL) \cap SO(V)$, $T(Sp) = T(GL) \cap Sp(V)$. Then $T(G)$ is a maximal torus of $G$ of rank $r = \left\lceil \frac{n}{2} \right\rceil$. Denote by $\varphi_1, \cdots, \varphi_r$ the fundamental weights of $T(G)$ with respect to $U(G)$. For $x \in W = IV$, denote by $v_i$ the projections of $x$ on the $i$-th $V$-factor, $i = 1, \cdots, l$.  

B) Let $n = 2r + 1, G = O(V)$. Then $\mathcal{M}$ is:  

- $Q(v_i, v_j), 1 \leq i \leq j \leq l$,  
- the lower minor determinants of order $k$ of $\overline{V}$, $k = 1, \cdots, \min\{l, n\}$,  

The set of degrees and weights of the above generators is (for $l \geq n$):  

$$(2,0), (1, \varphi_1), \cdots, (r - 1, \varphi_{r-1}), (r, 2\varphi_r), (r + 1, 2\varphi_r), \cdots, (n - 1, \varphi_1), (n, 0).$$
C) Let $G = \text{Sp}(V)$. Then $\mathcal{M}$ is:
\begin{itemize}
  \item $Q(v_i, v_j), 1 \leq i < j \leq l$,
  \item the lower minor determinants of order $k$ of $\nabla$, $k = 1, \cdots, \min\{l, r\}$.
\end{itemize}

The set of degrees and weights of the above generators is (for $l \geq r$):

$$(2, 0), (1, \varphi_1), (2, \varphi_2), \cdots, (r, \varphi_r).$$

Note that the lower minor determinants of order $k$ of $\nabla$ with $k > r$ are $U(\text{Sp})$-invariant, too. It is not hard to check that these can be expressed in the above generators.

D) Let $n = 2r, G = O(V)$. Then $\mathcal{M}$ is:
\begin{itemize}
  \item $Q(v_i, v_j), 1 \leq i \leq j \leq l$,
  \item the lower minor determinants of order $k$ of $\nabla$, $k = 1, \cdots, \min\{l, n\}$,
  \item for $l \geq r$, the minor determinants of order $r$, involving the $r$-th row and the last $r - 1$ rows of $\nabla$.
\end{itemize}

The set of degrees and weights of the above generators is (for $l \geq n$):

$$(2, 0), (1, \varphi_1), \cdots, (r - 2, \varphi_{r-2}), (r - 1, \varphi_{r-1} + \varphi_r), (r, 2\varphi_{r-1}), (r, 2\varphi_r),$$

$$(r + 1, \varphi_{r-1} + \varphi_r), \cdots, (n - 1, \varphi_1), (n, 0).$$

2 Proof of Theorem 1.

First we state a result of [Ho] that is a starting point of our proof. We keep the notation of loc.cit. but consider a slightly more general setting.

Let $W$ be a finite dimensional $k$-vector space. Denote by

$$\mathfrak{gr} = \mathfrak{gr}^{(2,0)} \oplus \mathfrak{gr}^{(1,1)} \oplus \mathfrak{gr}^{(0,2)} \subseteq \text{End}k[W]$$

the linear subspace of differential operators with the prescribed by the index degree and order. Namely, $\mathfrak{gr}^{(2,0)}$ are the homogeneous regular functions on $W$ of degree 2 acting on $k[W]$ by multiplication; $\mathfrak{gr}^{(0,2)}$ are the constant coefficients differential operators of order 2; $\mathfrak{gr}^{(1,1)}$ is nothing but the Lie algebra $\mathfrak{gl}(W)$.

Clearly, $\mathfrak{gr}$ is a Lie subalgebra in $\text{End}k[W]$, and moreover, $\mathfrak{gr}$ is isomorphic to $\mathfrak{sp}(W \oplus W^*)$, with respect to the natural symplectic form on $W \oplus W^*$.

Assume now that $G \subseteq GL(W)$ is a reductive subgroup. Then $G$ acts on $\mathfrak{gr}$; consider the invariants:

$$\Gamma' = \mathfrak{gr}^G, \Gamma'_{(2,0)} = \mathfrak{gr}_{(2,0)}^G, \Gamma'_{(1,1)} = \mathfrak{gr}_{(1,1)}^G, \Gamma'_{(0,2)} = \mathfrak{gr}_{(0,2)}^G.$$ Clearly, $\Gamma' = \Gamma'_{(2,0)} \oplus \Gamma'_{(1,1)} \oplus \Gamma'_{(0,2)}$ is also a Lie subalgebra in $\text{End}k[W]$.

Let $k[W] = \bigoplus_{k=1}^{\infty} I_k$ be the decomposition of $G$-module $k[W]$ into isotypic components. Let $I$ be one of $I_k$. Clearly, $I$ is stable under the action of $\Gamma'$. 

3
Theorem 2 (loc.cit. Theorem 8]) Assume that the algebra $k[W \oplus W^*]^G$ of invariants is generated by elements of degree 2. Then $I$ is an irreducible joint $(G,G')$-module.

By the First Fundamental Theorem for the classical groups, the assumption of Theorem 2 holds for the pairs $(G,W)$ from section 1. For these particular cases the above Theorem is (a part of) Theorem 8 of loc.cit.. However, one can see that the proof in loc.cit. works whenever the assumption of Theorem 2 holds.

Note that for the classical $(G,W)$ we have: $\Gamma'_{(1,1)} = \mathfrak{gl}_l \oplus \mathfrak{gl}_m$.

$\Gamma' \cong \mathfrak{gl}_{l+m}$, if $G = GL(V)$, $\Gamma' \cong \mathfrak{sp}_{2l}$, if $G = O(V)$, $\Gamma' \cong \mathfrak{so}_{2l}$, if $G = Sp(V)$.

We now show that Theorem 2 reduces Theorem 1 to a more simple statement. The below reasoning is an analog of that from the proof of Theorem 9 in loc.cit.

Clearly, $I$ is a homogeneous submodule of $k[W]$; denote by $I^{min}$ the subspace of the elements of $I$ of minimal degree. Let $A \subseteq k[W]^U$ be the subalgebra generated by $M$. Let $Z \subseteq k[W]$ be the $G$-submodule generated by $A$. Then Theorem 1 can be reformulated as follows: $Z = k[W]$. Assume that $X = Z \cap I^{min}$ is nonzero.

Since the system $M$ of generators of $A$ is symmetric with respect to permutations of isomorphic $G$-factors of $W$, $A$ is $GL_l \times GL_m$-stable, i.e., $\Gamma'_{(1,1)}$-stable. Hence, $Z$ and $X$ are stable with respect to both $G$ and $\Gamma'_{(1,1)}$.

Let $R, R_{(2,0)}$ etc. be the subalgebras in $\text{End} k[W]$ generated by $\Gamma', \Gamma'_{(2,0)}$ etc. Consider $R$ as a representation of the universal enveloping algebra of $\Gamma'$. Using the PBW theorem, we obtain

(1) \[ R = R_{(2,0)} R_{(1,1)} R_{(0,2)}. \]

Differentiating a polynomial, we decrease its degree; hence, $\Gamma'_{(0,2)} I^{min} = 0$. Therefore $R_{(0,2)} X = X$. Moreover, since $X$ is $\Gamma'_{(1,1)}$-stable, we have by (1): $RX = R_{(2,0)} X = k[W]^G X$. On the other hand, $RX$ is a non-zero joint $(G,\Gamma')$-submodule of $I$. By Theorem 2 $I = RX = k[W]^G X \subseteq Z$.

Thus to prove Theorem 1 we need to check for any isotypic component $I$:

(2) \[ A \cap I^{min} \neq \{0\}. \]

Note that it is sufficient to prove Theorem 1 with $l, m \geq n$, in the case $G = GL(V)$, and with $l \geq n, m = 0$, in the case $G = O(V), Sp(V)$.

Denote by $G^0$ the connected component of the unity of $G$; $GL(V)$ and $Sp(V)$ are connected, but for $G = O(V)$, $G^0 = SO(V)$. Recall that the irreducible finite dimensional $G^0$-modules are in one-to-one correspondence with their highest weights with respect to $U(G)$ and $T(G)$. Denote by $P$ the set of highest weights of irreducible factors for $G^0$-module $k[W]$. For any graded algebra $B$ and $t \in N$, for any
we denote by $B_t$ the subspace of the elements of degree $t$. For any $\chi \in P$ we set:

- $R(\chi)$ is the irreducible representation of $G^0$ with highest weight $\chi$
- $I_\chi$ is the $R(\chi)$-isotypic component of $G^0$-module $k[W]$
- $m(\chi) = \min\{t | k[W]_{t} \cap I_\chi \neq 0\}$
- $n(\chi) = \min\{t | A_t \cap I_\chi \neq 0\}$

By definition, $n(\chi) \geq m(\chi)$. For $G = GL(V), Sp(V)$ the condition (3) is equivalent to $n(\chi) = m(\chi)$ for any $\chi \in P$.

**Lemma 1**  
For any $\chi \in P, c \in \mathbb{N}$ we have: $n(c\chi) = cn(\chi)$.

Denote by $t$ the Lie algebra of $T(G)$. Let $C \subseteq t^*$ be the Weyl chamber corresponding to $U(G)$. Consider the set

$$\Delta = \{ \frac{\chi^*}{t} | I(\chi) \cap k[W]_t \neq 0 \} \subseteq C,$$

where $\chi^*$ denotes the highest weight of the $G^0$-module dual to that with highest weight $\chi$. By [Br87], if $k$ is the field of complex numbers, then $\Delta$ is the set of rational points in the momentum polytope for the action of the maximal compact subgroup $K \subseteq G^0$ on the projective space $P(W)$. Further, we set:

$$\tilde{\Delta} = \{ \frac{\chi^*}{t} | I(\chi) \cap Z_t \neq 0 \} \subseteq \Delta.$$

Let now $\Phi \subseteq t^*$ be the convex hull over the rational numbers of the weights for the action $T(G) : W$.

**Lemma 2**  
$\tilde{\Delta} \supseteq \Phi \cap C$.

By definition, we have: $\Delta \subseteq \Phi \cap C$. Therefore $\Delta = \tilde{\Delta} = \Phi \cap C$.

Suppose that $k[W]^{U(G)}$ contains an element of degree $t$ and weight $\chi$. Then by definition, $\frac{\chi^*}{t} \in \Delta$. Hence, the equality $\Delta = \tilde{\Delta}$ implies that for some $c \in \mathbb{N}$ there exists an element of $A$ of degree $ct$ and weight $c\chi$. Thus $ct \geq n(c\chi) = cn(\chi)$ and $t \geq n(\chi)$. In other words, $m(\chi) \geq n(\chi)$, hence $m(\chi) = n(\chi)$. This completes (modulo Lemmas 1 and 2) the proof of Theorem for $G = GL(V), Sp(V)$.

Let $G$ be $O(V)$; to prove Theorem, we apply induction on $n = \dim V$.

For $n = 2$, $U(O)$ is trivial and one can see $A = k[W]$.

For $n = 3$, $(SO_3, k^3) \cong (SL_2, S^2k^2)$. Since the stabilizer of a point on the dense orbit for the action $SL_2 : k^2$ is a maximal unipotent subgroup in $SL_2$, we obtain an isomorphism:

$$k[k^2 + lk^3]^{SL_2} \cong k[W]^{U(O)}.$$

\[\text{For } k = \mathbb{C}, \text{ one can directly prove for the moment polytope } \Delta \otimes \mathbb{R} = (\Phi \otimes \mathbb{R}) \cap C.\]
Lemma 3 There exists an isomorphism $k[k^2 + l k^3]^{SL_2} \cong k[(l + 1)k^3]^{SO_3}/(d)$, where $d = Q(v_{l+1}, v_{l+1})$.

Proof: Consider the morphism 

$$\varphi : k^2 + l k^3 \to (l + 1)k^3, \varphi(e, Q_1, \cdots, Q_l) = (Q_1, \cdots, Q_l, e^2).$$

Clearly, $\varphi$ is $SL_2$-equivariant; moreover, $\varphi$ is the quotient map with respect to the center of $SL_2$. Furthermore, the image of $\varphi$ is the zero level of $d$. This completes the proof.

Using Lemma 3 and the well-known description of $k[(l + 1)k^3]^{SO_3}$, one easily deduces the Theorem for $n = 3$.

The step of induction. Assume that Theorem is proven for $n - 2$. We apply now the Theorem of local structure of Brion-Luna-Vust ([BLV]) to get a local version of the assertion of Theorem.

Denote by $x_i^j = \nabla^j_i$ the $i$-th coordinate of $v_j$. Set $f = x_1^1 \in k[W]^U$, $W_f = \{x \in W | f(x) \neq 0\}$. Define a mapping:

$$\psi_f : W_f \to \sigma(V)^*, \psi(x)(\xi) = \frac{(\xi f)(x)}{f(x)}.$$

Denote by $P_f$ the stabilizer in $SO(V)$ of the line $(f)$. Clearly, $P_f$ is a parabolic subgroup in $SO(V)$ containing $U(O)$ and $\psi_f$ is $P_f$-equivariant.

Furthermore, we denote by $e_l^j$ the $i$-th element of the above basis in the $j$-th copy of $V$, $x = e_1^1, \Sigma = \psi_f^{-1}(\psi_f(x))$. Denote by $L$ the stabilizer of $\psi_f(x)$ in $P_f$. By [BLV], $L$ is a Levi subgroup of $P_f$ and the natural morphism  

$$P_f \ast L \Sigma \to W_f, (p, \sigma) \to p\sigma$$

is a $P_f$-equivariant isomorphism. Therefore we have:

$$k[W]^U_f \cong k[W]^U_f \cong k[P_f \ast L \Sigma]^U,$$

Also, $P_f = U(O)L$. Hence,

$$k[P_f \ast L \Sigma]^U \cong k[\Sigma]^{U(O) \cap L} = k[\Sigma]^{U(L)},$$

where $U(L)$ is a maximal unipotent subgroup in $L$. Calculating, we have:

$$(L, \Sigma) \cong (SO_2 \times SO_{n-2}, \langle e_1^1, e_n^1 \rangle_f \times (l-1)V).$$

In other words,

$$k[W]^U_{x_n^1} \cong k[x_1^1, x_n^1, x_1^2, x_n^2, \ldots, x_1^l, x_n^l]_{x_1^1} \otimes k[(l - 1)k^{n-2}]^{U(O_{n-2})}.$$  

The induction hypothesis yields the generators of $k[(l - 1)k^{n-2}]^{U(O_{n-2})}$. Restricting the elements of $M$ to $\Sigma$, one can easily deduce:

$$k[W]^U_{x_n^1} = A_{x_n^1}.$$
We return to our consideration of the isotypic components of \( O(V) : k[W] \). Consider an irreducible representation \( \rho \) of \( O(V) \) and its restriction \( \rho' \) to \( SO(V) \). Here two cases occur:

- either \( \rho' \) is also irreducible, \( \rho' = R(\chi) \) for some \( \chi \in P \)
- or else \( n = 2r \), \( \rho' = R(\chi) + R(\tau(\chi)) \), where \( \tau \) is the automorphism of the system of simple roots of \( O(V) \) interchanging the \( r-1 \)-th and the \( r \)-th roots.

The latter case is more simple: elements of minimal degree in the \( \rho \)-isotypic component are the elements of minimal degree in both \( I(\chi) \) and \( I(\tau(\chi)) \) (clearly, \( n(\chi) = n(\tau(\chi)) \) and \( m(\chi) = m(\tau(\chi)) \)). Hence, the above equality \( n(\chi) = m(\chi) \) implies the assertion for such an isotypic component.

Now consider the former case. Here for any \( \rho' = R(\chi) \) there exist two possibilities for \( \rho \): \( R(\chi) \) and \( R(\chi) = R(\chi) \otimes \text{det} \), where \( \text{det} \) is the unique nontrivial character of \( O(V) \). Moreover, we define explicitly \( R(\chi) \) and \( R(\chi) \) as follows. Let \( \theta \in O(V) \setminus SO(V) \) be an element normalizing \( T(O) \) as follows.

For \( n \) odd, \( \theta = \text{Id} \). For \( n \) even, \( \theta \) is the operator interchanging the \( r \)-th and the \( r+1 \)-th elements of the above basis and acting trivially on the other basis elements. Note that in both cases \( \theta(\chi) = \chi \) for any \( \chi \), if \( n \) is odd and for all \( \chi \) such that \( \tau(\chi) = \chi \), if \( n \) is even. Now we define \( R(\chi) \) by the condition:

\[
R(\chi)(\theta)(u_\chi) = \pm u_\chi
\]

for the highest vector \( u_\chi \) of \( T(O) \) and \( U(O) \) in \( R(\chi) \). For instance, if \( n \) is even, \( k \leq r-2 \), minor determinants of order \( k \) of \( \chi \) generate \( R(\varphi_{k+}) \) and minor determinants of order \( n-k \) generate \( R(\varphi_{k-}) \). Moreover, multiplying two highest vectors of \( k[W] \), we add their weights and multiply as usual their \( \pm \) subscripts. Thus we control the structure of the \( O(V) \)-module \( Z \).

Define \( m(\chi), n(\chi) \) as above. Then the condition (3) is equivalent to the equality \( m(\chi_{+}) = n(\chi_{+}) \) for any \( \chi \in P \) (\( \tau \)-invariant for \( n \) even). For any \( \chi = \sum_{i=1}^{q} k_i \psi_i, k_q > 0 \), set \( t = r-1 \), if \( q = r, n = 2r \) and \( t = q \) otherwise. Then we have:

\[
\text{(4)} \quad \min\{n(\chi_{+}), n(\chi_{-})\} = n(\chi), |n(\chi_{+}) - n(\chi_{-})| = n - 2t.
\]

Let \( g \) be a highest vector of \( k[W] \) generating \( R(\chi_{-}) \). Then by (3), for some even \( j \) we have \( (x_n^2)^j g \in A \). Since \( (x_n^2)^j g \) generates \( R(\chi + j \varphi_1)_{-} \), we have:
\[
\text{deg } g + j \geq n((\chi + j \varphi_1)_{-}). \quad \text{Clearly, } n(\chi + j \varphi_1) = n(\chi) + j \quad \text{see formulae (3), (4) below). Hence, (3) yields } n((\chi + j \varphi_1)_{-}) = n(\chi_{-}) + j. \quad \text{Thus we have } \text{deg } g \geq n(\chi_{-}) \text{ implying } m(\chi_{-}) = n(\chi_{-}). \text{ The same is true for } \chi_{+}. \text{ This completes the proof of Theorem for } G = O(V). \Box
\]

Thus we reduced Theorem 1 to Lemma 1 and Lemma 2. Both are properties of degrees and weights of the given generators, and we consider case by case.

3 Proof of Lemmas 1 and 2.

*Proof of Lemma 1.*
Recall that $n(\chi)$ is the minimum of degree of the monomials in the elements of $\mathcal{M}$ having weight $\chi$. Clearly, we should not involve the $G$-invariants in a monomial of minimal degree. Then for $G = \text{Sp}(V), \text{O}(V)$ we have no much choice for such a monomial and we can write down formulae for $n(\chi)$ as follows.

Let $\chi = k_1 \varphi_1 + \cdots + k_r \varphi_r$.

For $G = \text{Sp}(V)$, we have: $n(\chi) = k_1 + 2k_2 + \cdots + rk_r$.

For $G = \text{O}(V), n = 2r + 1$, $k_r$ is even for $\chi \in P$, and we have:

\[
(5) \quad n(\chi) = k_1 + 2k_2 + \cdots + (r - 1)k_{r-1} + r \frac{k_r}{2}.
\]

For $G = \text{O}(V), n = 2r$, $k_{r-1} + k_r$ is even for $\chi \in P$, and we have:

\[
(6) \quad n(\chi) = k_1 + 2k_2 + \cdots + (r - 2)k_{r-2} + r \frac{k_{r-1} + k_r}{2} - \min(k_r, k_{r-1}).
\]

These formulae yield the assertion of Lemma.

Consider the case $G = \text{GL}(V)$. The elements of $\mathcal{M}$ with non-zero weights have the following weights endowed with degrees:

\[
\alpha_i = \varphi_i, \deg \alpha_i = i, i = 1, \cdots, n,
\]

\[
\beta_j = \varphi_j - \varphi_n, \deg \beta_j = n - j, j = 1, \cdots, n - 1, \beta_n = -\varphi_n, \deg \beta_n = n.
\]

For $\chi = k_1 \varphi_1 + \cdots + k_n \varphi_n$ consider the presentations of $\chi$ as linear combinations of the above weights with positive integer coefficients. Define the degree of such a combination as the sum of degrees of the summands. We claim that there is a unique presentation of minimal degree.

For any $j = 1, \cdots, n-1$, all the presentations of $\chi$ contain $k_j$ summands $\alpha_j$ or $\beta_j$. Set $r = \left\lfloor \frac{n}{2} \right\rfloor$. The linear combination

\[
\chi' = k_1 \alpha_1 + \cdots + k_n \alpha_n + k_{r+1} \beta_{r+1} + \cdots + k_{n-1} \beta_{n-1}
\]

has the minimal degree among the linear combinations equal to $\chi$ modulo $\langle \varphi_n \rangle$. If $\chi' = \chi$, then this presentation of $\chi$ has the minimal degree and no presentation of the same degree exists. Otherwise, we can:

(a) replace some $\alpha_i$ by $\beta_i$, (b) add $\beta_n$,

(c) replace some $\beta_j$ by $\alpha_j$, (d) add $\alpha_n$.

The steps (a),(b) decrease the $n$-th coordinate by 1, the steps (c),(d) increase it by 1. The increasing of the degree is: $n$ for (b),(d), $n - 2i$ for (a), $2j - n$ for (c). If $\chi' = \chi + t \varphi_n$, then to obtain the minimal presentation, we apply $t$ times (a) and (b), if $t > 0$, and we apply $-t$ times (c) and (d), if $t < 0$. Clearly, there exists a unique sequence of steps giving $\chi$ with the minimal possible degree. Therefore the presentation of $\chi$ with the minimal degree is unique. Moreover, from its construction follows that the presentation of $c\chi$ with the minimal degree is just the sum of $c$ minimal presentations for $\chi$. This completes the proof. $\square$
Proof of Lemma 2.

Consider the case $G = GL(V)$. Let $\varepsilon_1, \ldots, \varepsilon_n$ be the weights of $T$ acting on $V$, a basis of the character lattice of $T$. Let $\chi_1, \ldots, \chi_n$ be the dual basis. The fundamental weights are: $\varphi_i = \varepsilon_1 + \cdots + \varepsilon_i, i = 1, \ldots, n$. Furthermore, $C$ is given by the inequalities $\chi_1 \geq \chi_2 \cdots \geq \chi_n$, $\Phi = \text{conv}(\pm \varepsilon_1, \ldots, \pm \varepsilon_n)$, and $\widetilde{\Delta}$ is the convex hull of 

$$
\frac{\varepsilon_1 + \varepsilon_2}{2}, \ldots, \frac{\varepsilon_1 + \cdots + \varepsilon_n}{n}, -\frac{\varepsilon_n - \varepsilon_{n-1}}{2}, \ldots, -\frac{\varepsilon_n - \cdots - \varepsilon_1}{n}.
$$

For $\chi \in \langle \varepsilon_1, \ldots, \varepsilon_n \rangle_\mathbb{Q}$, set $\alpha_i = \chi_i(\xi)$. First assume

\[
\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0, \alpha_1 + \cdots + \alpha_n \leq 1. \tag{7}
\]

Then we can rewrite:

$$\xi = (\alpha_1 - \alpha_2)\varphi_1 + (\alpha_2 - \alpha_3)\varphi_2 + \cdots + (\alpha_n - \alpha_n - 1)\varphi_{n-1} + \alpha_n \varphi_n.$$ 

So $\xi$ is a linear combination of $\frac{\varphi_i}{t}, i = 1, \ldots, n$ with non-negative coefficients. Now we sum the coefficients:

$$(\alpha_1 - \alpha_2) + 2(\alpha_2 - \alpha_3) + \cdots + (n-1)(\alpha_{n-1} - \alpha_n) + n\alpha_n = \alpha_1 + \cdots + \alpha_n \leq 1.$$ 

Therefore we get:

$$\xi \in \text{conv}(0, \varepsilon_1, \frac{\varepsilon_1 + \varepsilon_2}{2}, \ldots, \frac{\varepsilon_1 + \cdots + \varepsilon_n}{n}) \subseteq \widetilde{\Delta}.$$

Analogously, assuming

\[
0 \geq \alpha_1 \geq \cdots \geq \alpha_n, \alpha_1 + \cdots + \alpha_n \geq -1, \tag{8}
\]

we obtain

$$\xi \in \text{conv}(0, -\varepsilon_n, \frac{-\varepsilon_n - \varepsilon_{n-1}}{2}, \ldots, \frac{-\varepsilon_n - \cdots - \varepsilon_1}{n}) \subseteq \widetilde{\Delta}.$$ 

Now assume $\xi \in \Phi \cap C$. Then $\xi \in \Phi$ implies $|\alpha_1| + \cdots + |\alpha_n| \leq 1$. If all the $\alpha_i$ are of the same sign, then either (7) or (8) holds and we are done. Otherwise for some $q < n$ we have

$$\alpha_1 \geq \cdots \geq \alpha_q \geq 0 \geq \alpha_{q+1} \geq \cdots \geq \alpha_n.$$ 

Then set:

$$t = \sum_{i=1}^{q} \alpha_i \leq 1, \xi_+ = \frac{\sum_{i=1}^{q} \alpha_i \varepsilon_i}{t}, \xi_- = \frac{\sum_{j=q+1}^{n} \alpha_j \varepsilon_j}{1-t}.$$ 

Clearly, (7) holds for $\xi_+$ and (8) holds for $\xi_-$. Hence, $\xi_+, \xi_- \in \widetilde{\Delta}$, and $\xi = t\xi_+ + (1-t)\xi_- \in [\xi_+, \xi_-] \subseteq \Delta$. 9
For $G = Sp(V), O(V)$, we let $\varepsilon_1, \cdots, \varepsilon_r$ to be basic characters of $T(G)$ and keep the notation of $\chi_i$-s. Then the fundamental weights are (see e.g. [OV]):

for $G = Sp(V)$, $\varphi_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $i = 1, \cdots, r$,
for $G = O(V)$, $n = 2r + 1$, $\varphi_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $i = 1, \cdots, r - 1$,
$\varphi_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r)$,
for $G = O(V)$, $n = 2r$, $\varphi_1 = \varepsilon_1 + \cdots + \varepsilon_i$ for $i = 1, \cdots, r - 2$,
$\varphi_{r-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r)$, $\varphi_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r)$.

For the cases $G = Sp(V)$, $n = 2r$ or $G = SO(V)$, $n = 2r + 1$, we have: $C$ is given by the inequalities $\chi_1 \geq \chi_2 \cdots \geq \chi_m \geq 0$,

$$\Phi = \text{conv}(\pm \varepsilon_1, \cdots, \pm \varepsilon_r), \quad \Delta = \text{conv}(0, \varepsilon_1, \frac{\varepsilon_1 + \varepsilon_2}{2}, \cdots, \frac{\varepsilon_1 + \cdots + \varepsilon_r}{r}).$$

Therefore for $\xi \in C \cap \Phi$ the assumption ( [F] ) holds, hence $\xi \in \Delta$.

For the case $G = O(V), n = 2r$, we have: $\Phi = \text{conv}(\pm \varepsilon_1, \cdots, \pm \varepsilon_r),

$$\Delta = \text{conv}(0, \varepsilon_1, \frac{\varepsilon_1 + \varepsilon_2}{2}, \cdots, \frac{\varepsilon_1 + \cdots + \varepsilon_r}{r}, \frac{\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r}{r}).$$

If $\xi \in C$, then we can write:

$$\xi = \alpha_1 \varepsilon_1 + \alpha_2 \frac{\varepsilon_1 + \varepsilon_2}{2} + \cdots + \alpha_r \frac{\varepsilon_1 + \cdots + \varepsilon_r}{r} + \beta \frac{\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r}{r},$$

where $\alpha_1, \cdots, \alpha_r, \beta \geq 0$, $\alpha_r \beta = 0$. Assume $\xi \in \Phi$. If $\alpha_r = 0$, then, taking into account the inequality $\chi_1(\xi) + \cdots + \chi_{r-1}(\xi) - \chi_r(\xi) \leq 1$, we obtain $\alpha_1 + \cdots + \alpha_{r-1} + \beta \leq 1$. Therefore $\xi \in \Delta$. Similarly, we consider the case $\beta = 0$. This completes the proof of Lemma [F] \(\square\)

4 Syzygies.

Since we found the generators of $k[W]^{U(G)}$, a natural question is to describe their syzygies. This is a subject of the Second Fundamental Theorem of Invariant Theory for the linear group $(U(G), V)$. In this section we present some results for $G = GL(V)$. Of course, syzygies that we present are also syzygies for the orthogonal and symplectic cases, if the involved generators are.

Set $U = U(GL)$ and denote by $W_U$ the spectrum of $k[|W|]^{U}$. Moreover, denote by $\pi_{U,W}$ the quotient map $\pi_{U,W} : W \to W_U$ corresponding to the inclusion $k[|W|]^{U} \subseteq k[|W|]$.

For any $p, l \in \mathbb{N}, 1 \leq p \leq l$, set $L = k^l \oplus \wedge^2 k^l \oplus \cdots \oplus \wedge^p k^l$. Let $F_{p,l}$ denote the set of all $(q_1, q_2, \cdots, q_p)$ in $L$ such that for $i = 2, \cdots, p$, the $i$-vector $q_i$ is decomposable, and $Ann(q_{i-1}) \subseteq Ann(q_i)$, where $Ann(q) = \{x \in V | q \wedge x = 0\}$.

The subset $F_{p,l}$ is not closed in $L$. In fact, assume $(q_1, \cdots, q_p) \in F_{p,l}$ is such that $q_2 \neq 0$. Then for any $t \in k^*$ the collection $(tq_1, q_2, \cdots, q_p)$ also belongs to $F_{p,l}$. But the limit $(0, q_2, \cdots, q_p)$ of such collections does not belong to $F_{p,l}$. Denote by $\overline{F}_{p,l}$ the Zariski closure of $F_{p,l}$.
Note that the subset $\mathcal{F}_{p,l}$ is stable under the natural action of the group $GL_l$ on $L$. Therefore $\mathcal{F}_{p,l}$ is acted upon by $GL_l$.

**Theorem 3** For $W = lV$, set $p = \min\{l, n\}, q = n - p + 1$. Consider the rows $u_1, \cdots, u_n$ of the matrix $V$ as the coordinates of some vectors in $k^l$. Then the map $W \to \mathcal{F}_{p,l} \subseteq L$ taking a tuple of vectors to the element with coordinates

$$(u_n, u_{n-1} \wedge u_n, \cdots, u_q \wedge u_{q+1} \wedge \cdots \wedge u_n)$$

is the $GL_L$-equivariant quotient map $\pi_{U,W}$ and its image is $\mathcal{F}_{p,l}$.

**Proof.** We only need to prove that the Plücker coordinates of the antisymmetric forms $u_q \wedge \cdots \wedge u_n, \cdots, u_{n-1} \wedge u_n, u_n$ generate $k[W]^U$. But these are just the lower minor determinants of $V$ and Theorem 2 implies Theorem 3.

A different proof of both Theorems for this case is as follows. Let the maximal unipotent subgroup $U' \subseteq GL_l$ consist of all the strictly upper triangular matrices, in the chosen basis of $k^l$. It is well known (see e.g. [Kr, 3.7]) that $k[W]_{U \times U'}$ is generated by the left lower minor determinants of $V$. Therefore the algebra $A$ generated by all the lower minor determinants contains $k[W]_{U \times U'}$. In other words, $A'' = (k[W]^U)^{U''}$. Since $A$ is $GL_l$-stable, we obtain $A = k[W]^U$. □

Thus the syzygies of the set of lower minor determinants of $V$ are the generators of the ideal in $k[L]$ vanishing on $\mathcal{F}_{p,l}$. These are the Plücker relations saying that each $q_i$ is decomposable, and the incidence relations saying $\text{Ann}(q_i) \subseteq \text{Ann}(q_j)$ for any $1 \leq i < j \leq p$.

The syzygies can be written down explicitly. For instance, if $i + j \leq p$, then we construct a $(i + j) \times l$ matrix of the last $i$ rows and the last $j$ rows of $V$. Clearly, any minor determinant of order $i + j$ of such a matrix is zero. This is a bilinear syzygy among the lower minor determinants of order $i$ and $j$.

There is also a $GL_l$-equivariant description of the ideal of syzygies, in the form of [Br85]. For $1 \leq i \leq j \leq p$, let $M_{i,j}$ be the $GL_l$-stable complementary subspace to the highest vector irreducible factor of $((\wedge^i k^l)^* \otimes (\wedge^j k^l)^*) \subseteq k[L]$, if $i < j$, or of $S^2((\wedge^i k^l)^*) \subseteq k[L]$, if $j = i$. Let $J$ be the ideal generated by $M_{i,j}$, for all $1 \leq i \leq j \leq p$.

**Lemma 4** The ideal in $k[L]$ vanishing on $\mathcal{F}_{p,l}$ is $J$.

**Proof.** Clearly, we have $\mathcal{F}_{p,l} = GL_l(U'')$. Then by the Theorem of [Br85] p.382, the set of zeros of $J$ is $\mathcal{F}_{p,l}$. Moreover, by the same theorem, $J$ is radical. This completes the proof. □

**Corollary 1** All the syzygies are of degree 2.

Clearly, for arbitrary $l$ and $m$, similar Plücker and incidence relations hold for the left minor determinants of $V^\tau$. 

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Theorem 4 Suppose that \( l > 0, m > 0 \) and set \( W = lV + mV^\ast \). Then the ideal of syzygies for the generators of \( k[W]^U \) is generated by the Plücker and the incidence relations for the lower minor determinants of \( V \) and for the left minor determinants of \( V^\ast \) if and only if \( l + m \leq n \).

Proof: To prove the ”if” part, it is sufficient to consider the case \( l + m = n \). Recall that by Theorem 1, the generators of \( k[W]^U \) are the lower minor determinants of \( V \), the left minor determinants of \( V^\ast \), and the elements of the matrix \( C = V^\ast V \). Let \( \sum_\alpha a_\alpha c^\alpha = 0 \) be a relation among the generators, where \( c^\alpha \) is a monomial in the \( C^\alpha \)-s, \( a_\alpha \) is a polynomial in the minor determinants. The assertion of the Theorem amounts to prove that \( a_\alpha \) belongs to the ideal of syzygies, for any \( \alpha \). This will be proven if we check for generic fibers \( F = \pi_{U,V}(\xi), \xi \in \mathcal{F}_{l,m} \) and \( F^\ast = \pi_{U,mV^\ast}(\eta), \eta \in \mathcal{F}_{m,m} \) that the restrictions of the matrix elements of \( C \) to \( F \times F^\ast \) are algebraically independent. Fix a tuple of vectors in a generic fiber \( F \) such that \( V \) has the form

\[
\begin{pmatrix}
 & l & \\
* & * & * \\
* & * & * \\
 a_1 & 0 & 0 \\
* & \ddots & 0 \\
* & * & a_l \\
\end{pmatrix}
\]

with \( a_1a_2\cdots a_l \neq 0 \) and fix generic elements of the first \( m \) columns of \( V^\ast \). Then, varying the \( lm \) elements in the last \( l = n - m \) columns of \( V^\ast \), we do not change the minor determinants and we can obtain any \( m \times l \) matrix as \( C \). Thus the ”if” part is proven.

The ”only if” part. Take \( l, m \) such that \( 1 \leq l, m \leq n, l + m > n \) and set \( s = l + m - n, r = n - l + 1 \). Denote by \( a_i^j, b_i^j, c_i^j \) the element in the \( i \)-th row and the \( j \)-th column of the matrix \( V^\ast, V, \) and \( C \), respectively. Denote by \( \varepsilon^{s-a} \) and \( \varepsilon_{a-b} \) the determinant tensors. In this notation, \( \varepsilon_{i_{1}\cdots i_{m}}^{s_{1}\cdots s_{m}} a_{i_{1}}^{1} \cdots a_{i_{m}}^{m} \) is the left minor determinant of order \( m \) of \( V^\ast \) and \( \varepsilon_{j_{1}\cdots j_{n}}^{s_{1}\cdots s_{n}} b_{j_{1}}^{1} \cdots b_{j_{n}}^{n} \) is the lower minor determinant of order \( l \) of \( V \). We claim that the following relation holds:

\[
\varepsilon_{i_{1}\cdots i_{m}}^{s_{1}\cdots s_{m}} a_{i_{1}}^{1} \cdots a_{i_{m}}^{m} \varepsilon_{j_{1}\cdots j_{n}}^{s_{1}\cdots s_{n}} b_{j_{1}}^{1} \cdots b_{j_{n}}^{n} = \frac{1}{s!} \varepsilon_{i_{1}\cdots i_{m}}^{s_{1}\cdots s_{m}} a_{i_{1}}^{1} \cdots a_{i_{r-1}}^{r-1} \varepsilon_{j_{1}\cdots j_{n}}^{s_{1}\cdots s_{n}} b_{j_{m+1}}^{j_{m+1}} \cdots b_{j_{n}}^{j_{n}} c_{i_{r}}^{j_{r}} \cdots c_{i_{m}}^{j_{m}}.
\]

To prove this formula, we rewrite the right hand side, using \( c_i^j = a_i^k b_k^j \):

\[
\frac{1}{s!} \varepsilon_{i_{1}\cdots i_{m}}^{s_{1}\cdots s_{m}} a_{i_{1}}^{1} \cdots a_{i_{r-1}}^{r-1} a_{i_{r}}^{k_{r}} \cdots a_{i_{m}}^{k_{m}} \varepsilon_{j_{1}\cdots j_{n}}^{s_{1}\cdots s_{n}} b_{j_{1}}^{1} \cdots b_{j_{m+1}}^{j_{m+1}} \cdots b_{j_{n}}^{j_{n}}.
\]

\[\text{This relation with } m = n \text{ was indicated to us by E. B. Vinberg.}\]}
Let $S(k_1, \ldots, k_s)$ denote the sum of terms in formula (10) with fixed $k_1, \ldots, k_s$. Clearly, if $\{k_1, \ldots, k_s\} \neq \{r, r + 1, \ldots, m\}$, then $S(k_1, \ldots, k_s) = 0$. Moreover, if $\{k_1, \ldots, k_s\} = \{r, \ldots, m\}$, then $S(k_1, \ldots, k_s)$ equals the left hand side of (9).

Therefore, the relation (9) holds. Clearly, the right hand side is a polynomial in the left minor determinants of order $m - s$ of $\nabla^*$, the lower minor determinants of order $l - s$ of $\nabla$, and the matrix elements of $C$. It is not hard to check that this relation among the generators of $k[W]^U$ can not be obtained from relations of smaller degrees. 

**Remark.** Theorems 3, 4 yield an independent proof of Theorem 1 for the case $l + m \leq n$. Indeed, we prove in Theorem 4 that, in the case $l + m \leq n$, the syzygies among the elements of the set $\mathcal{M}$ are generated by those for $lV$ and those for $mV^*$. We did not use Theorem 1 for this. Hence, by Theorem 3 (that we also prove independently of Theorem 1), Spec$A \cong (lV)_U \times (mV^*)_U$. Since for an action of an algebraic group $H$ on a normal affine variety $X$, the algebra $k[X]^H$ is integrally closed, Spec$A$ is normal. Furthermore, as we did it for $O(V)$, one can prove $k[W]^U = A_f$ for all linear $U$-invariants $f$. Then any $g \in k[W]^U$ gives rise to a rational function on Spec$A$, regular outside the intersection of the divisors of these linear $U$-invariants. Since Spec$A$ is normal, we get $g \in A$.

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