Quadratic algebras: Three-mode bosonic realizations and applications

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Quadratic algebras of the type 
\([Q_0, Q_\pm] = \pm Q_\pm, [Q_+, Q_-] = aQ_0^2 + bQ_0 + c\)
are studied using three-mode bosonic realizations. Matrix representations and single variable differential operator realizations are obtained. Examples of physical relevance of such algebras are given.

1 Introduction

In recent times there has been a great deal of interest in non-linear deformations of Lie algebras because of their significant applications in several branches of physics. This is largely based on the realization that the physical operators relevant for defining the dynamical algebra of a system need not be closed under a linear (Lie) algebra, but might obey a nonlinear algebra. Such nonlinear algebras are, in general, characterized by commutation relations of the form

\[ [T_i, T_j] = C_{ij} (T_k), \]

where the functions \(C_{ij}\) of the generators \(\{T_k\}\) are constrained by the Jacobi identity

\[ [T_i, C_{jk}] + [T_j, C_{ki}] + [T_k, C_{ij}] = 0. \]

The functions \(C_{ij}\) can be infinite power series in \(\{T_k\}\) as is in the case of quantum algebras (with further Hopf algebraic restrictions) and \(q\)-oscillator algebras. When \(\{C_{ij}\}\) are polynomials of the generators one gets the so-called
polynomially nonlinear, or simply polynomial, algebras. A special case of interest is when the commutation relations \((1.1)\) take the form

\[
[T_i, T_j] = c^{k}_{ij} T_k, \quad [T_i, T_\alpha] = t^\beta_{i\alpha} T_\beta, \quad [T_\alpha, T_\beta] = f^\alpha\beta (T_k),
\]

(1.3)
containing a linear subalgebra. Simplest examples of such algebras occur when one gets

\[
[N_0, N_\pm] = \pm N_\pm, \quad [N_+, N_-] = f (N_0).
\]

(1.4)
In general, the Casimir operator of this algebra \((1.4)\) is seen to be given by

\[
\mathcal{C} = N_+ N_- + g (N_0 - 1) = N_- N_+ + g (N_0),
\]

(1.5)
where \(g (N_0)\) can be determined from the relation

\[
g (N_0) - g (N_0 - 1) = f (N_0).
\]

(1.6)

If \(f (N_0)\) is quadratic in \(N_0\) we have a quadratic algebra and if \(f (N_0)\) is cubic in \(N_0\) we have a cubic algebra. The nonlinear algebras, in particular the quadratic and cubic algebras, and their representations have been studied in connection with several problems in quantum mechanics, statistical physics, field theory, Yang-Mills type gauge theories, two-dimensional integrable systems, etc. \((1-16)\). As in the case of the Lie algebras, compact nonlinear algebras have finite dimensional unitary irreducible representations and noncompact algebras have infinite dimensional unitary irreducible representations. These nonlinear algebras have also been found to be very useful in quantum optics with the observation that quantum optical Hamiltonians describing multiphoton processes have symmetries described by polynomially deformed \(su(2)\) and \(su(1,1)\) algebras \((17)\). Coherent states of different kinds of nonlinear oscillator algebras have been presented by several authors \((18-21)\). Recently, a general unified approach for finding the coherent states of polynomial algebras, relevant for various multiphoton processes in quantum optics, has been presented by us \((24)\).

Algebras of the type \((1.4)\) have been studied in mathematics literature also and shown to have a surprisingly rich theory of representations \((24)\). Algebras of the type \((1.4)\) with commutator \([N_+, N_-]\) replaced by the anticommutator \(\{N_+, N_-\}\) leading to polynomial deformations of the superalgebra \(osp(1|2)\) have also been investigated \((26)\). As is well known, in the case
of classical Lie algebras bosonic realizations play a very useful role in the representation theory and applications to physical problems. The main purpose of this work is to study some aspects of quadratic algebras relating to three-mode bosonic realizations, and the associated matrix representations, differential operator realizations, coherent states, and physical applications. In Section 1 we briefly review the two-mode bosonic construction of \(su(2)\) and \(su(1, 1)\) algebras. Following closely our earlier work [24], in Sections 2 and 3 we construct the compact and noncompact quadratic algebras, respectively, using three bosonic modes and study the associated representations. In Section 4 we briefly discuss the coherent states of these algebras. Finally, in Section 5 we conclude with a few remarks on the physical and mathematical significance of these quadratic and higher order nonlinear algebras.

## 2 Two-mode construction of \(su(2)\) and \(su(1, 1)\): A brief review

Let us briefly recall the study of \(su(2)\) and \(su(1, 1)\) in terms of two-mode bosonic realizations, to fix the framework and notations for our work. The \(su(2)\) algebra,

\[
[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0,
\]

(2.1)
is satisfied by the Jordan-Schwinger realization

\[
J_0 = \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2), \quad J_+ = a_2^\dagger a_1, \quad J_- = J_+^\dagger = a_1 a_2^\dagger.
\]

(2.2)

In this realization the Casimir operator becomes

\[
\mathcal{C} = J^2 = J_+J_- + J_0 (J_0 - 1) = \frac{1}{4} \left( a_1^\dagger a_1 + a_2^\dagger a_2 \right) \left( a_1^\dagger a_1 + a_2^\dagger a_2 + 2 \right). \tag{2.3}
\]

Consequently, the application of the realization (2.2) on a set of \(2j + 1\) two-mode Fock states \(|n_1, n_2\rangle\) with constant \(n_1 + n_2 = 2j\) leads to the \((2j + 1)\)-dimensional unitary irreducible representation for each \(j = 0, 1/2, 1, \ldots\). Thus, with \(\{ |j, m\rangle = |n_1 = j + m, n_2 = j - m\rangle \mid m = j, j - 1, \ldots, -j \} \) as the basis states, one gets the \(j\)-th unitary irreducible representation

\[
J_0 |j, m\rangle = m |j, m\rangle, \\
J_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle, \\
m = j, j - 1, \ldots, -j.
\]

(2.4)
associated with the constant value \( j(j + 1) \) for the Casimir operator \( J^2 \).

Let us now consider the single variable differential operator realization corresponding to the above matrix representation (2.4). With the Fock-Bargmann correspondence

\[
(a^\dagger, a) \rightarrow \left( z, \frac{d}{dz} \right), \quad |n\rangle \rightarrow \frac{z^n}{\sqrt{n!}},
\]

we can make the association

\[
|j, m\rangle \rightarrow \frac{z^{j+m} z^{-m}}{\sqrt{(j+m)!(j-m)!}} = \frac{z^{2j} (z_1/z_2)^{j+m}}{\sqrt{(j+m)!(j-m)!}},
\]

\[
m = -j, -j + 1, \ldots, j - 1, j.
\]

Since \( j \) is a constant for a given representation we can rewrite the above as a mapping to monomials

\[
|j, m\rangle \rightarrow \psi_{j,n}(z) = \frac{z^n}{\sqrt{n!(2j-n)!}}, \quad n = 0, 1, 2, \ldots, 2j.
\]

Then, it is obvious that the above set of \((2j+1)\) monomials (2.7) forms the basis carrying the finite dimensional representation (2.4) corresponding to the single variable realization

\[
J_0 = z \frac{d}{dz} - j, \quad J_+ = -z^2 \frac{d}{dz} + 2jz, \quad J_- = \frac{d}{dz}.
\]

In an analogous way, for \( su(1, 1) \) the two-mode bosonic realization

\[
K_0 = \frac{1}{2} (a_1^\dagger a_1 + a_2^\dagger a_2 + 1), \quad K_+ = a_1^\dagger a_2^\dagger, \quad K_- = K_+^\dagger = a_1 a_2,
\]

satisfies the algebra

\[
[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0.
\]

The Casimir operator is

\[
C = K^2 = K_+ K_- - K_0 (K_0 - 1) = \frac{1}{4} \left[ 1 - \left( a_1^\dagger a_1 - a_2^\dagger a_2 \right)^2 \right].
\]
Consequently, the application of the realization (2.9) on any infinite set of two-mode Fock states \( \{|k, n\rangle = |n, n + 2k - 1\rangle | n = 0, 1, 2, \ldots \} \) (or \( \{|k, n\rangle = |n + 2k - 1, n\rangle | n = 0, 1, 2, \ldots \} \)), with constant \(|n_1 - n_2| = 2k - 1\), leads to the infinite dimensional unitary irreducible representation, the so-called positive discrete representation \( D^+(k) \), for each \( k = 1/2, 1, 3/2, \ldots \).

\[
\begin{align*}
K_0 |k, n\rangle &= (k + n) |k, n\rangle, \\
K_+ |k, n\rangle &= \sqrt{(2k + n)(n + 1)} |k, n + 1\rangle, \\
K_- |k, n\rangle &= \sqrt{(2k + n - 1)n} |k, n - 1\rangle,
\end{align*}
\]

(2.12)

with the Casimir operator \( K^2 \) taking the value \( k(1 - k) \) in the representation.

As in the \( su(2) \) case, we can make the association

\[
|k, n\rangle \rightarrow \frac{z^n z_1^{n+2k-1}}{\sqrt{n!(n + 2k - 1)!}} = \frac{(z_1 z_2)^n z_2^{2k-1}}{\sqrt{n!(n + 2k - 1)!}}.
\]

(2.13)

Then, with \( k \) being constant in a given representation, it is obvious that the infinite set of monomials

\[
\psi_{k,n}(z) = \frac{z^n}{\sqrt{n!(n + 2k - 1)!}}, \quad n = 0, 1, 2, \ldots \,
\]

(2.14)

forms the basis carrying the representation (2.12) corresponding to the single variable realization

\[
K_0 = z \frac{d}{dz} + k, \quad K_+ = z, \quad K_- = z \frac{d^2}{dz^2} + 2k \frac{d}{dz}.
\]

(2.15)

### 3 Three-mode construction of compact quadratic algebras

A quadratic algebra is defined, in general, by the commutation relations

\[
[Q_0, Q_\pm] = \pm Q_\pm, \quad [Q_+, Q_-] = aQ_0^2 + bQ_0 + c,
\]

(3.1)
where \((a, b, c)\) are constants, or commute among themselves and with \((Q_0, Q_+, Q_-)\) so that they take constant values in any irreducible representation. When \(a = c = 0\) and \(b = \pm 2\) one gets, respectively, \(su(2)\) and \(su(1, 1)\) as special cases. Let us now construct a class of compact quadratic algebras using three bosonic modes. To this end, let

\[
Q_0 = \frac{1}{4} \left( a_1^+ a_1 + a_2^+ a_2 - 2a_3^+ a_3 + 1 \right) = \frac{1}{2} \left( K_0 - a_3^+ a_3 \right),
\]

\[
Q_+ = a_1^+ a_2^+ a_3 = K_+ a_3, \quad Q_- = Q_+^* = a_1 a_2^+ a_3^* = K_- a_3^*,
\]

\[
\mathcal{K} = \frac{1}{4} \left[ 1 - \left( a_1^+ a_1 - a_2^+ a_2 \right)^2 \right] = \mathcal{K}^2,
\]

\[
\mathcal{L} = \frac{1}{4} \left( a_1^+ a_1 + a_2^+ a_2 + 2a_3^+ a_3 + 1 \right) = \frac{1}{2} \left( K_0 + a_3^+ a_3 \right),
\]

(3.2)

where \((K_0, K_+, K_-)\) generate \(su(1, 1)\) with \(K^2\) as the Casimir operator. Then, it is found that

\[
[\mathcal{K}, \mathcal{L}] = 0, \quad [\mathcal{K}, Q_{0,\pm}] = 0, \quad [\mathcal{L}, Q_{0,\pm}] = 0, \quad [Q_0, Q_+] = \pm Q_+, \quad [Q_+, Q_-] = 3Q_0^2 + (2\mathcal{L} - 1)Q_0 + (\mathcal{K} - \mathcal{L}(\mathcal{L} + 1)) \quad .
\]

(3.3)

Thus, \((Q_0, Q_+, Q_-)\) generate a quadratic algebra with \(\mathcal{K}\) and \(\mathcal{L}\) taking constant values in any irreducible representation. The Casimir operator of this algebra is given by

\[
\mathcal{C} = Q_+ Q_- + Q_0^2 + (\mathcal{L} - 2)Q_0^2 + \left( \mathcal{K} - \mathcal{L}^2 - 2\mathcal{L} + 1 \right) Q_0,
\]

(3.4)

apart from an additional constant function of \(\mathcal{K}\) and \(\mathcal{L}\), following the recipe contained in (1.4)-(1.6).

The condition that \(\mathcal{K}\) and \(\mathcal{L}\) take constant values in an irreducible representation fixes the basis to be the set of three-mode Fock states

\[
|k, l, n\rangle = |n, n + 2k - 1, 2l - k - n\rangle, \quad n = 0, 1, 2, \ldots, (2l - k),
\]

(3.5)

with

\[
2l - k = 0, 1, 2, \ldots, \quad k = 1/2, 1, 3/2, \ldots,
\]

(3.6)

and

\[
\mathcal{K} |k, l, n\rangle = k(1 - k) |k, l, n\rangle, \quad \mathcal{L} |k, l, n\rangle = l |k, l, n\rangle.
\]

(3.7)
The basis states (3.5) carry the $(2l - k + 1)$-dimensional unitary irreducible representation of the quadratic algebra (3.3) which can be labeled by the values of the pair $(k, l)$. Explicitly, the $(k, l)$-th representation is:

\[ Q_0 |k, l, n\rangle = (k - l + n) |k, l, n\rangle, \]
\[ Q_+ |k, l, n\rangle = \sqrt{(n + 1)(n + 2k)(2l - n - k)} |k, l, n + 1\rangle, \]
\[ Q_- |k, l, n\rangle = \sqrt{n(n + 2k - 1)(2l - n - k + 1)} |k, l, n - 1\rangle, \]
\[ K |k, l, n\rangle = k(1 - k) |k, l, n\rangle, \quad L |k, l, n\rangle = l |k, l, n\rangle, \quad n = 0, 1, 2, \ldots, (2l - k). \]

The Casimir operator has the value $(l^2 + (l + 1)[k(1 - k) - 1] + 1)$ in this representation.

It should be noted that for $k > 1/2$ the same representation is obtained by the choice of basis states as $|k, l, n\rangle = |n + 2k - 1, n, 2l - k - n\rangle$, with $n = 0, 1, 2, \ldots, (2l - k)$. It should also be noted that, unlike in the case of the $su(2)$ algebra, the quadratic algebra (3.3) has infinitely many inequivalent unitary irreducible representations of the same dimension. For example, for each value of $k = 1/2, 1, 3/2, \ldots$, there is a 2-dimensional representation of the algebra (3.3) given by

\[ Q_0 = \frac{1}{2} \begin{pmatrix} k - 1 & 0 \\ 0 & k + 1 \end{pmatrix}, \quad Q_+ = \begin{pmatrix} 0 & 0 \\ \sqrt{2k} & 0 \end{pmatrix}, \quad Q_- = \begin{pmatrix} 0 & \sqrt{2k} \\ 0 & 0 \end{pmatrix}, \]
\[ K = k(1 - k), \quad L = \frac{1}{2}(k + 1), \quad C = \frac{1}{8} \left(-3k^3 - 5k^2 + 11k - 3\right), \]

as can be verified directly.

As before, let us make the association

\[ |k, l, n\rangle \longrightarrow \frac{z_2^{2k-1}z_3^{2l-k}(z_1z_2/z_3)^n}{\sqrt{n!(n + 2k - 1)!(2l - k - n)!}}. \]

Since $k$ and $l$ are constants for a given representation we can take

\[ \phi_{k,l,n}(z) = \frac{z^n}{\sqrt{n!(n + 2k - 1)!(2l - k - n)!}}, \quad n = 0, 1, 2, \ldots, (2l - k), \]

(3.11)
as the set of basis functions for the single variable realization

\[ Q_0 = z \frac{d}{dz} + k - l, \quad Q_+ = -z^2 \frac{d}{dz} + (2l - k)z, \quad Q_- = z \frac{d^2}{dz^2} + 2k \frac{d}{dz}, \quad (3.12) \]

leading to the representation (3.8).

4 Three-mode construction of noncompact quadratic algebras

Let us choose

\[
\begin{align*}
Q_0 &= \frac{1}{4} \left( a_1^\dagger a_1 + a_2^\dagger a_2 + 2a_3^\dagger a_3 + 1 \right) = \frac{1}{2} \left( K_0 + a_3^\dagger a_3 \right), \\
Q_+ &= a_1^\dagger a_2^\dagger a_3^\dagger = K_+ a_3^\dagger, \\
Q_- &= Q_+^\dagger = a_1 a_2 a_3 = K_- a_3, \\
K &= \frac{1}{4} \left[ 1 - \left( a_1^\dagger a_1 - a_2^\dagger a_2 \right)^2 \right] = K^2, \\
L &= \frac{1}{4} \left( a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3 + 1 \right) = \frac{1}{2} \left( K_0 - a_3^\dagger a_3 \right), \quad (4.1)
\end{align*}
\]

where \((K_0, K_+, K_-)\) generate \(su(1,1)\) with \(K^2\) as the Casimir operator. Now, the algebra becomes

\[
\begin{align*}
[K, L] &= 0, \quad [K, Q_{0,\pm}] = 0, \quad [L, Q_{0,\pm}] = 0, \\
[Q_0, Q_{\pm}] &= \pm Q_{\pm}, \quad [Q_+, Q_-] = -3Q_0^2 - (2L + 1)Q_0 - (K - L(L - 1)). \quad (4.2)
\end{align*}
\]

The Casimir operator of this algebra is given by

\[
C = Q_+ Q_- - Q_0^3 + (L - 1)Q_0^2 - (K - L^2)Q_0 \quad (4.3)
\]

apart from additional constant functions of \(K\) and \(L\).

The constancy of \(K\) and \(L\) in an irreducible representation leads to the choice of basis consisting of the infinite three-mode Fock states

\[
|k, l, n\rangle = |n, n + 2k - 1, n + k - 2l\rangle, \quad n = 0, 1, 2, \ldots, \quad (4.4)
\]

with

\[
k - 2l = 0, 1, 2, \ldots, \quad k = 1/2, 1, 3/2, \ldots, \quad (4.5)
\]
The set of basis states (4.4) carry the infinite dimensional unitary irreducible representation of the quadratic algebra (4.2) given by

\[
\begin{align*}
Q_0 |k,l,n\rangle &= (k - l + n) |k,l,n\rangle , \\
Q_+ |k,l,n\rangle &= \sqrt{(n + 1)(n + 2k)(n + k - 2l + 1)} |k,l,n + 1\rangle , \\
Q_- |k,l,n\rangle &= \sqrt{n(n + 2k - 1)(n + k - 2l)} |k,l,n - 1\rangle , \\
\mathcal{K} |k,l,n\rangle &= k(1 - k) |k,l,n\rangle , \\
\mathcal{L} |k,l,n\rangle &= l |k,l,n\rangle ,
\end{align*}
\]

(4.7)

The Casimir operator has the value \( l(l - k^2) \) in this \((k,l)\)-th representation. As in the compact case, to obtain this representation one can also use an alternative set of basis states \(|k,l,n\rangle = |n, n + 2k - 1, n + k - 2l\rangle\), with \( n = 0, 1, 2, \ldots \).

From the association

\[
|k,l,n\rangle \rightarrow \frac{(z_1 z_2 z_3)^n z_2^{2k-1} z_3^{k-2l}}{\sqrt{n!(n + 2k - 1)!(n + k - 2l)!}}
\]

(4.8)

it is clear that we can take

\[
\psi_{k,l,n}(z) = \frac{z^n}{\sqrt{n!(n + 2k - 1)!(n + k - 2l)!}}
\]

(4.9)

as the set of basis functions for a single variable realization of the algebra (4.2). The corresponding realization is

\[
\begin{align*}
Q_0 &= z \frac{d}{dz} + k - l , \\
Q_+ &= z , \\
Q_- &= z^2 \frac{d^3}{dz^3} + (3k - 2l + 2)z \frac{d^2}{dz^2} + (2k^2 - 4kl + 2k) \frac{d}{dz}.
\end{align*}
\]

(4.10)

5 Coherent states of the quadratic algebras

As is well known, coherent states form an overcomplete set of nonorthogonal states, generally labeled by a continuous index, say \( \alpha \), providing a resolution
of identity $\int |\alpha\rangle \langle \alpha| \, d\alpha = I$ and hence a useful set of basis states. Coherent states associated with the unitary irreducible representations of the dynamical algebra of a physical system are very useful in certain studies of the system.

For the noncompact algebra (4.2) the Barut-Girardello type coherent states associated with the $(k, l)$-th irreducible representation can be defined as the eigenstates of $Q_-:

\begin{align*}
Q_- |k, l, \alpha\rangle &= \alpha |k, l, \alpha\rangle , \quad |k, l, \alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha) |k, l, n\rangle ,
\end{align*}

(5.1)

where the complex $\alpha$ labels these states. Solving for $|k, l, \alpha\rangle$ using the representation (4.7) we get easily

\begin{align*}
|k, l, \alpha\rangle &= \left[ \frac{(2k)! (k - 2l + 1)!}{\left( \binom{2k}{k} - 2k - 2l + 1 \right)} \right]^\frac{1}{2} \times \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n! (n + 2k - 1)! (n + k - 2l)!}} |k, l, n\rangle .
\end{align*}

(5.2)

In terms of the single variable realization the coherent state equation (5.1) can be written as

\begin{align*}
\left[ z^2 \frac{d^3}{dz^3} + (3k - 2l + 2)z \frac{d^2}{dz^2} + \left( 2k^2 - 4kl + 2k \right) \frac{d}{dz} \right] \Psi_{k,l}(\alpha, z) &= \alpha \Psi_{k,l}(\alpha, z) ,
\end{align*}

(5.3)

which is the equation for

\begin{align*}
\Psi_{k,l}(\alpha, z) = \binom{2k}{k} - 2k - 2l + 1 \, \alpha z .
\end{align*}

(5.4)

The resolution of the identity is given by:

\begin{align*}
\int d\sigma(\alpha, \alpha^*; k, l) |\alpha; k, l\rangle \langle \alpha; k, l| = \hat{1}.
\end{align*}

(5.5)

With a polar decomposition ansatz $\alpha = re^{i\theta}$ we get,

\begin{align*}
d\sigma(\alpha^*, \alpha; k, l) = N_{i,k}(r^2)(\sigma(r^2))d\theta dr dr ,
\end{align*}

(5.6)
where

\[ N_{l,k}(r^2) = \binom{F_2(-; 2k, k - 2l + 1; r^2)}{0} \] (5.7)

The integral (5.5) reduces to the following condition on \( \sigma(r^2) \):

\[ \frac{1}{2} \int_0^\infty d(r^2) \sigma(r^2) (r^2)^{(n+1)-1} = \frac{1}{2\pi} \Gamma(n + 1) \frac{\Gamma(2k + n)\Gamma(k - 2l + 1 + n)}{\Gamma(2k)\Gamma(k - 2l + 1)}. \] (5.8)

\( \sigma(r^2) \) is found by an inverse Mellin transform to be:

\[ \sigma(r^2) = \frac{1}{\pi \Gamma(2k)(\Gamma(k - 2l + 1)\Gamma(2k))} C_{03}^{30}(r^2|0, k-2l+1, 2k), \] (5.9)

where \( G \) is the Meijers G function \([21]\).

Another useful set of coherent states for the quadratic algebras will be the analogues of the well known Perelemov-type coherent states. A technique for generalizing the Perelemov-type construction of coherent states for the nonlinear algebras, using a mapping of the given compact or noncompact algebra to \( su(2) \) or \( su(1, 1) \) respectively, has been presented in detail in our earlier work \([24]\). For completeness we give the expressions for these states. For the non compact case the \textit{Perelemov type} states are:

\[ |\beta\rangle = N \sum_n (\beta)^n \sqrt{\frac{\Gamma(2k + n)\Gamma(k - 2l + 1 + n)}{\Gamma(n + 1)\Gamma(2k)\Gamma(k - 2l + 1)}} |l, k, n>, \] (5.10)

where the normalization coefficient \( N \) is given by

\[ N = (\binom{F_0(2k, k - 2l + 1; (|\beta|^2))}{-\frac{1}{2}}. \] (5.11)

The resolution of the identity in this case reduces to finding \( \sigma(r^2) \) such that,

\[ \int_0^\infty d(r^2) \sigma(r^2) (r^2)^{(n+1)-1} = \frac{1}{\pi} \Gamma(n + 1) \frac{\Gamma(2k)\Gamma(k - 2l + 1)}{\Gamma(2k + n)\Gamma(k - 2l + 1 + n)}. \] (5.12)

and the resultant \( \sigma(r^2) \) is given by

\[ \sigma(r^2) = \frac{1}{\pi} \Gamma(k - 2l + 1)\Gamma(2k)G_{2,1}^{1,0}(r^2|0, k-2l+1, 2k-1), \] (5.13)

where \( G \) is the Meijers G function.
The corresponding states for the compact case are given by:

\[ |\alpha, k, l> = N \sum_{n=0}^{2l-k} (\alpha)^n \sqrt{\frac{(2l-k)!(2k-1+n)!}{n!(2l-k-n)!(2k-1)!}} |l, k, n >. \]  

(5.14)

For the purposes of calculating the measure for the resolution of id entity we define \( \gamma = \frac{1}{\alpha} \). The coherent state \( |\gamma, k, l> \) becomes

\[ |\gamma, k, l> = N\gamma^{k-2l} \sum_{n=0}^{2l-k} (\gamma)^n \sqrt{\frac{(2l-k)!(k+2l-1-n)!}{n!(2l-k-n)!(2k-1)!}} |l, k, n >, \]  

(5.15)

with the normalization coefficient \( N \) given by:

\[ N = \frac{\Gamma(k+2l)}{\Gamma(\gamma^2)^{2l-k}} \Phi(k-2l, 1-2l-k; (\gamma^2))^{-\frac{1}{2}}, \]  

(5.16)

where \( \Phi(a, b, x) \) is the Confluent Hypergeometric function \( (1F_1) \).

The resolution of the identity for the coherent states \( |\gamma, k, l> \) is given by

\[ \int \mu(\gamma, \gamma; k, l) |\gamma; k, l> \langle \gamma; k, l| = 1_{l,k}, \]  

(5.17)

where \( 1_{l,k} \) is the projection operator on the subspace \( \mathcal{H}_{2l-k}: \)

\[ 1_{l,k} = \sum_{n=0}^{2l-k} |l, k, n> \langle l, k, n|. \]  

(5.18)

Again defining \( \gamma = re^{i\theta} \) we have,

\[ \sum_{n=0}^{2l-k} \frac{\Gamma(2l+k-n)}{n!(2l-k-n)!} \left[ \int_0^\infty (r^2)^n M(r^2; k, l) d(r^2) \right] |l, k, n> \langle l, k, n| = 1_{l,k}, \]  

(5.19)

where we have defined

\[ M(r^2; k, l) \equiv \frac{\pi(2l-k)!}{\Gamma(2l+k)} \frac{\sigma(r^2; k, l)}{(\Phi(k-2l, 1-2l-k; r^2))}. \]  

(5.20)

By using the integral [23],

\[ \int_0^\infty r^{b-1}\Phi(a; c; -r) dr = \frac{\Gamma(b)\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}, \]  

(5.21)
we obtain

\[ M(r^2; k, l) = \frac{\Gamma(2l - k + 2)}{\Gamma(2l + k + 1)} \Phi(2l - k + 2; 2l + k + 1; -r^2). \]

This gives us the final expression for the integration measure:

\[ d\mu(\gamma, \gamma^*; k, l) = \frac{1}{2\pi} \frac{(2l - k + 1)}{(2l + k + 1)} \Phi(k - 2l; 1 - 2l - k; r^2) \Phi(2l - k + 2; k + 2l + 1; -r^2) d(r^2) d\theta. \]  (5.22)

The resolution of the identity is important because it allows the use of the coherent states as a basis in the state space. All three sets of coherent states can also be easily shown to be overcomplete [24].

6 Concluding remarks: Physical applications

As already mentioned in the introduction the nonlinear algebras are finding several applications in physical problems. Here we shall like to make some observations with reference to the bosonic constructions of the quadratic algebras we have presented.

Let us consider the Hamiltonian

\[ H = a_1^\dagger a_1 + a_2^\dagger a_2 + 2a_3^\dagger a_3 \]  (6.1)

which describes, with units \( \hbar = 1 \) and \( \omega = 1 \), and apart from the additional zero-point level constant, a three-dimensional anisotropic quantum harmonic oscillator with the frequency in the third direction twice that in the perpendicular plane. From the construction of the compact quadratic algebra given above we recognize that

\[ H = 4\mathcal{L} - 1, \]  (6.2)

where \((\mathcal{L}, \mathcal{K}, Q_0, Q_\pm)\) generate the quadratic algebra (3.3). Thus, \((\mathcal{K}, Q_0, Q_\pm)\) are constants of motion for the system (6.1) and the quadratic algebra (3.3) is its dynamical algebra. Since \(\mathcal{L}\) has the spectrum

\[ \mathcal{L} = l = n/4, \quad n = 1, 2, 3, \ldots, \]  (6.3)

it is clear that the Hamiltonian (6.1) has the spectrum

\[ H = N, \quad N = 0, 1, 2, \ldots. \]  (6.4)
Each level can be labeled by the eigenvalues of the complete set of commuting operators \((H, K, Q_0)\).

It is interesting to compute the degeneracy of the \(N\)-th level using the representation theory of the algebra (3.3). For the \(N\)-th level the value of \(L\) is \(l = (N + 1)/4\). Calculating the corresponding values of \(k\) for which finite dimensional representations are possible we find that the dimensions of the associated irreducible representations are \((1, 2, \ldots, 2m + 1)\) if \(N = 4m\) or \(4m + 1\), and \((1, 2, \ldots, 2m + 2)\) if \(N = 4m + 2\) or \(4m + 3\). The degeneracy of the level is the sum of the dimension of the \(k = 1/2\) representation and twice the dimensions of \(k > 1/2\) representations. One has to count the dimensions of \(k > 1/2\) representations twice in the sum since there are two possible choices for the bases leading to the same representation in these cases as already noted. Now, the four cases, \(N = 4m, 4m + 1, 4m + 2, 4m + 3\), are to be considered separately. The result is: the degeneracies of the levels, \(N = 4m, 4m + 1, 4m + 2, 4m + 3\), respectively, are \((2m + 1)^2\), \((2m + 1)(2m + 2)\), \(4(m + 1)^2\) and \(2(m + 1)(2m + 3)\). In other words, the number of compositions of the integer \(N\) (partitions with ordering taken into account) in the prescribed pattern \(n_1 + n_2 + 2n_3\), with the interchange of \(n_1\) and \(n_2\) taken into account, is \((2m + 1)^2\), \((2m + 1)(2m + 2)\), \(4(m + 1)^2\) and \(2(m + 1)(2m + 3)\), if \(N = 4m, 4m + 1, 4m + 2,\) and \(4m + 3\), respectively.

It is to be noted that in the above example the sum of all the dimensions of the irreducible representations associated with the given \(l = (N + 1)/4\) gives the number of partitions of \(N\) in the pattern \(n_1 + n_2 + 2n_3\), disregarding the interchange of \(n_1\) and \(n_2\). This leads to the result that the number of such partitions is \((m + 1)(2m + 1)\) for \(N = 4m\) or \(4m + 1\) and \((m + 1)(2m + 3)\) for \(N = 4m + 2\) or \(4m + 3\). Thus, it is interesting to observe this connection between the quadratic algebra and the theory of partitions.

An another interesting possibility is suggested by the structure of the compact quadratic algebra (3.3). Let us define

\[
N = Q_0, \quad A = \frac{1}{\sqrt{\mathcal{L}(\mathcal{L} + 1) - \mathcal{K}}} Q_-, \quad A^\dagger = \frac{1}{\sqrt{\mathcal{L}(\mathcal{L} + 1) - \mathcal{K}}} Q_+.
\]  

The the algebra (3.3) becomes

\[
[N, A] = -A, \quad [N, A^\dagger] = A^\dagger,
\]  

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\[ [A, A^\dagger] = 1 - \frac{3}{\mathcal{L}(\mathcal{L} + 1) - \mathcal{K}} N^2 - \frac{2\mathcal{L} - 1}{\mathcal{L}(\mathcal{L} + 1) - \mathcal{K}} N. \quad (6.6) \]

We may consider this as the defining algebra of a quadratic oscillator, corresponding to a special case of the general class of deformed oscillators (27–37):

\[ [N, A] = -A, \quad [N, A^\dagger] = A^\dagger, \quad [A, A^\dagger] = F(N). \quad (6.7) \]

The quadratic oscillator (6.6) belongs to the class of generalized deformed parafermions [12]. It should be interesting to study the physics of assemblies of quadratic oscillators. In fact, the canonical fermion, with

\[ N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \] (6.8)

is a quadratic oscillator! Observe that

\[ [N, f] = -f, \quad [N, f^\dagger] = f^\dagger, \quad [f, f^\dagger] = 1 - \frac{1}{2} N - \frac{3}{2} N^2. \quad (6.9) \]

To conclude, we find that the study of polynomial algebras should lead to interesting connections with the theory of partitions and new physical systems in which the polynomial oscillators replace the role of the canonical oscillators, besides having the other known applications.

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