A note of “pointwise estimates of the SDFEM for convection–diffusion problems with characteristic layers”

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Abstract

We propose some useful estimates for the pointwise error estimates of the streamline diffusion finite element method (SDFEM) on Shishkin meshes, when SDFEM is applied for problems of characteristic layers.

1 Problem

We consider the singularly perturbed boundary value problem

\[ -\varepsilon \Delta u + bu_x + cu = f \quad \text{in } \Omega = (0,1)^2, \]
\[ u = 0 \quad \text{on } \partial \Omega \]

where \( b, c > 0 \) are constants and \( b \geq \beta \) on \( \Omega \) with a positive constant \( \beta \). It is assumed that \( f \) is sufficiently smooth. Here \( 0 < \varepsilon \ll 1 \) is a small perturbation parameter whose presence gives rise to an exponential layer of width \( O(\varepsilon) \) near the outflow boundary at \( x = 1 \) and to two characteristic (or parabolic) layers of width \( O(\sqrt{\varepsilon}) \) near the characteristic boundaries at \( y = 0 \) and \( y = 1 \).

2 The SDFEM on Shishkin meshes

In this Section we describe our mesh, our finite element method and the assumptions of our analysis.

2.1 The regularity result

As mentioned before the solution \( u \) of (1.1) possesses an exponential layer at \( x = 1 \) and two characteristic layers at \( y = 0 \) and \( y = 1 \). For our later analysis we shall suppose that \( u \) can be split into a regular solution component and various layer parts:

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Assumption 2.1. The solution $u$ of (1.1) can be decomposed as

$$u = S + E_1 + E_2 + E_{12},$$

where for all $x = (x, y) \in \bar{\Omega}$ and for $0 \leq i + j \leq 3$, the regular part satisfies

$$\left| \frac{\partial^{i+j} S}{\partial x^i \partial y^j}(x, y) \right| \leq C,$$

while for the layer terms and $0 \leq i + j \leq 3$, the following bounds hold true:

$$\left| \frac{\partial^{i+j} E_1}{\partial x^i \partial y^j}(x, y) \right| \leq C e^{-\beta (1-x)/\varepsilon},$$

$$\left| \frac{\partial^{i+j} E_2}{\partial x^i \partial y^j}(x, y) \right| \leq C e^{-j/2} (e^{-y/\sqrt{\varepsilon}} + e^{-(1-y)/\sqrt{\varepsilon}}),$$

and

$$\left| \frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j}(x, y) \right| \leq C e^{-(i+j)/2} e^{-\beta (1-x)/\varepsilon} (e^{-y/\sqrt{\varepsilon}} + e^{-(1-y)/\sqrt{\varepsilon}}).$$

For constant coefficients Kellogg and Stynes [4, 5] give sufficient compatibility conditions on the data that ensure the existence of (2.1a)–(2.1e).

2.2 Shishkin meshes

When discretizing (1.1), we use a piecewise uniform mesh — a so-called Shishkin mesh —with $N$ mesh intervals in both $x$- and $y$-direction which condenses in the layer regions. For this purpose we define the two mesh transition parameters

$$\lambda_x := \min \left\{ \frac{1}{2}, \rho \varepsilon \beta \ln N \right\} \quad \text{and} \quad \lambda_y := \min \left\{ \frac{1}{4}, \rho \sqrt{\varepsilon} \ln N \right\}.$$

In this paper, we define $\rho = 2.5$.

Assumption 2.2. We assume in our analysis that $\varepsilon \leq N^{-1}$. Furthermore we assume that $\lambda_x = \rho \varepsilon \beta^{-1} \ln N$ and $\lambda_y = \rho \sqrt{\varepsilon} \ln N$ as otherwise $N^{-1}$ is exponentially small compared with $\varepsilon$.

The domain $\Omega$ is dissected into four(six) parts as $\Omega = \Omega_s \cup \Omega_1 \cup \Omega_2 \cup \Omega_{12}$, where

$$\Omega_s := [0, 1 - \lambda_x] \times [\lambda_y, 1 - \lambda_y], \quad \Omega_2 := [0, 1 - \lambda_x] \times ([0, \lambda_y] \cup [1 - \lambda_y, 1]), \quad \Omega_1 := [1 - \lambda_x, 1] \times [\lambda_y, 1 - \lambda_y], \quad \Omega_{12} := [1 - \lambda_x, 1] \times ([0, \lambda_y] \cup [1 - \lambda_y, 1]).$$

Remark 2.1. The mesh transition parameters have been chosen such that the boundary layer function $E$ which can be any of $E_1, E_2$ and $E_{12}$ satisfies $|E| \leq C N^{-\rho}$ on $\Omega_s$. 

2
We introduce the set of mesh points \( \{(x_i, y_j) \in \Omega : i, j = 0, \cdots, N\} \) defined by
\[
x_i = \begin{cases} 
2i(1 - \lambda_x)/N, & \text{for } i = 0, \cdots, N/2, \\
1 - 2(N - i)\lambda_x/N, & \text{for } i = N/2 + 1, \cdots, N
\end{cases}
\]
and
\[
y_j = \begin{cases} 
3j\lambda_y/N, & \text{for } j = 0, \cdots, N/3, \\
(3j/N - 1) - 3(2j - N)\lambda_y/N, & \text{for } j = N/3 + 1, \cdots, 2N/3, \\
1 - 3(N - j)\lambda_y/N, & \text{for } j = 2N/3 + 1, \cdots, N.
\end{cases}
\]

By drawing lines through these mesh points parallel to the \( x \)-axis and \( y \)-axis the domain \( \Omega \) is partitioned into rectangles. This triangulation is denoted by \( \Omega^N \).

If \( D \) is a mesh subdomain of \( \Omega \), we write \( D^N \) for the triangulation of \( D \). The mesh sizes \( h_{x,\tau} = x_i - x_{i-1} \) and \( h_{y,\tau} = y_j - y_{j-1} \) satisfy
\[
\begin{align*}
h_{x,\tau} &= \begin{cases} 
H_x := \frac{1 - \lambda_x}{N/2}, & \text{for } i = 1, \cdots, N/2, \\
h_x := \frac{\lambda_x}{N/2}, & \text{for } i = N/2 + 1, \cdots, N,
\end{cases} \\
h_{y,\tau} &= \begin{cases} 
H_y := \frac{1 - 2\lambda_y}{N/3}, & \text{for } j = N/3 + 1, \cdots, 2N/3, \\
h_y := \frac{\lambda_y}{N/3}, & \text{otherwise}
\end{cases}
\end{align*}
\]

and
\[
N^{-1} \leq H_x, H_y \leq 3N^{-1}, \\
C_1 \varepsilon^{-1} N \ln N \leq h_x \leq C_2 \varepsilon^{-1} N \ln N, \\
C_1\sqrt{\varepsilon}N^{-1} \ln N \leq h_y \leq C_2\sqrt{\varepsilon}N^{-1} \ln N.
\]

The above properties are essential when inverse inequalities are applied in our later analysis.

For the mesh elements we shall use some notations: \( \tau_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \) for a specific element, \( \tau \) for a generic mesh rectangle (see Fig.1) and
\[
x_\tau = (x_{i-1} + x_i)/2, \quad y_\tau = (y_{j-1} + y_j)/2 \quad \text{if} \quad \tau = [x_{i-1}, x_i] \times [y_{j-1}, y_j].
\]

### 2.3 The streamline diffusion finite element method

The weak formulation of the problem (1.1) is: Find \( u \in H^1_0(\Omega) \) such that
\[
\varepsilon(\nabla u, \nabla v) + (bu_x + cu, v) = (f, v), \quad \forall v \in H^1_0(\Omega).
\]

Note that the variational formulation (2.2) has a unique solution by means of the Lax-Milgram Lemma.
On the above Shishkin mesh we define a finite element space
\[ V^N := \{ v^N \in C(\bar{\Omega}) : v^N|_{\partial \Omega} = 0 \text{ and } v^N|_{\tau} \text{ is bilinear}, \forall \tau \in \Omega \}. \]
Then we can state the standard Galerkin discretisation of (2.2) is: Find \( U \in V^N \) such that
\[ \varepsilon(\nabla U, \nabla v^N) + (bU_x + cU, v^N) = (f, v^N), \forall v^N \in V^N. \]

The SDFEM adds weighted residuals to the standard Galerkin finite element method: Find \( U \in V^N \) such that
\[ B(U, v^N) = (f, v^N + \delta b v^N_x), \forall v^N \in V^N, \]
where
\[ B(U, v^N) := \varepsilon(\nabla U, \nabla v^N) + (bU_x + cU, v^N) + (bU_x + cU, \delta b v^N_x). \]
The term \((-\varepsilon \Delta U, \delta b v^N_x)\) is neglected in our case. \( \delta = \delta(x) \) is a user-chosen parameter (see [3, 9]). In this paper, we set
\[ \delta(x) := \begin{cases} C^* N^{-1}, & \text{if } x \in \Omega_1 \cup \Omega_2, \\ 0, & \text{otherwise} \end{cases} \]
where \( C^* \) is chosen so that \( 0 < \delta(x) \leq 1/c \) for \( x \in \Omega_1 \cup \Omega_2 \) (see [9], §III 3.2.1).
Finally, we define a special energy norm associated with \( B(\cdot, \cdot) \):
\[ |||U|||^2 := ((\varepsilon + b^2 \delta)U_x, U_x) + \varepsilon(U_y, U_y) + \varepsilon(U, U). \]

For any subdomain \( D \) of \( \Omega \), let \( B_D(\cdot, \cdot), (\cdot, \cdot)_D \) and \( ||| \cdot |||_D \) mean that the integrations in (2.5) and (2.6) are restricted to \( D \). We denote by \( \| \cdot \|_D \) the \( L^2 \) norm in \( L^2(D) \), i.e.,
\[ \|v\|_D^2 = (v, v)_D \text{ for all } v \in L^2(D). \]
If \( D = \Omega \) then we drop \( \Omega \) from the notation.
3 Interpolation error estimates

We start our analysis by quoting some previous results. In the following analysis, we shall frequently use the bilinear interpolation $g^I$ of a given function $g$.

**Lemma 3.1.** Let $\tau \in \Omega^N$ and $p \in [1, \infty]$. Assume that $g$ lies in $C^3(\tau)$. Then

$$
\| (g - g^I)_x \|_{L^p(\tau)} \leq C \left( h_{x,\tau}^2 \| g_{xxx} \|_{L^p(\tau)} + h_{x,\tau} h_{y,\tau} \| g_{xxy} \|_{L^p(\tau)} + h_{y,\tau}^2 \| g_{yyy} \|_{L^p(\tau)} \right) + C h_{x,\tau} \| g_{xx} \|_{L^p(\tau)}
$$

$$
\| (g - g^I)_y \|_{L^p(\tau)} \leq C \left( h_{y,\tau}^2 \| g_{xyy} \|_{L^p(\tau)} + h_{x,\tau} h_{y,\tau} \| g_{xxy} \|_{L^p(\tau)} + h_{y,\tau}^2 \| g_{yyy} \|_{L^p(\tau)} \right) + C h_{y,\tau} \| g_{yy} \|_{L^p(\tau)}
$$

**Proof.** See [1, Theorem 4].

**Lemma 3.2.** Assume that $u$ satisfies Assumption 2.1. Then on our Shishkin mesh

$$
\| u - u^I \|_{L^\infty(\tau)} \leq \begin{cases} 
CN^{-2}, & \text{if } \tau \in \Omega_s, \\
CN^{-2} \ln^2 N, & \text{otherwise}.
\end{cases}
$$

**Proof.** See [2, Theorem 3].

**Lemma 3.3.** For any function $g \in C^3(\tau)$ and any $w \in V^N$, we have the identities

$$
\int_{\tau} (g - g^I)_x w_x \, dx \, dy = \int_{\tau} g_{xxy} J_* (y) \left( w_x - \frac{2}{3} (y - y_\tau) w_{xy} \right) \, dx \, dy,
$$

$$
\int_{\tau} (g - g^I)_y w_y \, dx \, dy = \int_{\tau} g_{xyy} F_* (x) \left( w_y - \frac{2}{3} (x - x_\tau) w_{xy} \right) \, dx \, dy
$$

and

$$
\int_{\tau} (g - g^I)_x w \, dx \, dy = \int_{\tau} R(g, w) \, dx \, dy + \frac{h_{x,\tau}^2}{12} \left( \int_{l_2} - \int_{l_4} \right) g_{xx} w \, dy
$$

where

$$
F_* (x) = \frac{1}{2} \left( (x - x_\tau)^2 - h_{x,\tau}^2 / 4 \right) \quad \text{and} \quad J_* (y) = \frac{1}{2} \left( (y - y_\tau)^2 - h_{y,\tau}^2 / 4 \right)
$$

and

$$
R(g, w) = \frac{1}{3} F_* (x) (x - x_\tau) g_{xxx} w_x - \frac{h_{x,\tau}^2}{12} g_{xxx} w + J_* (y) g_{xxy} \left[ w - (x - x_\tau) w_x - \frac{2}{3} (y - y_\tau) w_y + \frac{2}{3} (x - x_\tau) (y - y_\tau) w_{xy} \right].
$$

**Proof.** See [6, 7] or [12, the Appendix] for details.
Lemma 3.4. Let $U$ be the solution of (2.4) on our Shishkin mesh. Then

$$
\| u^I - U \| \leq C(N^{-2} \ln^2 N + \varepsilon^{1/4} N^{-1} \ln^{1/2} N)
$$

Proof. See [2, Theorem 5].

Throughout the remaining analysis we shall make frequent use of the following inverse estimates. Let $\chi$ be a polynomial on the mesh rectangle $\tau$. Then

$$
\| \chi_x \|_{L^p(\tau)} \leq Ch_x^{-1} \| \chi \|_{L^p(\tau)},
$$

(3.2)

$$
\| \chi_y \|_{L^p(\tau)} \leq Ch_y^{-1} \| \chi \|_{L^p(\tau)},
$$

(3.3)

$$
\int_{y_{j-1}}^{y_j} |\chi(x_i, y)| \, dy \leq Ch_x^{-1} \| \chi \|_{L^1(\tau_{ij})},
$$

(3.4)

where the function $\phi_{ij}$–(3.5c) and the first inequality of (3.5d) is similar.

Lemma 3.5. Let Assumption 2.1 hold true. Then there exists a constant $C$ such that the following interpolation error estimates hold true

$$
\| E - E^I \|_{L^\infty(\Omega_1)} \leq CN^{-p},
$$

(3.5a)

$$
\| E_1 - E_1^I \|_{L^\infty(\Omega_2)} \leq CN^{-p},
$$

(3.5b)

$$
\| (E_1 - E_2)^I \|_{L^\infty(\Omega_2)} \leq CN^{-p},
$$

(3.5c)

$$
\| (E_1 - E_1^I)_y \|_{L^\infty(\Omega_2)} \leq CN^{-p},
$$

(3.5d)

$$
\| (E_2 - E_2^I)_x \|_{L^\infty(\Omega_1)} \leq CN^{-p},
$$

(3.5e)

$$
\| \nabla (u - u^I) \|_{L^1(\Omega_1)} \leq CN^{-1},
$$

(3.5f)

where the function $E$ can be any one of $E_1$, $E_2$ or $E_{12}$. 

Proof. At first, we will prove the second inequality of (3.5d). The proof of (3.5a)–(3.5c) and the first inequality of (3.5d) is similar.

Each bilinear basis function $\phi^{ij}(x, y)$ satisfies $\phi^{ij}(x, y) = \phi_i(x)\phi_j(y)$ where $\phi_i$ and $\phi_j$ are piecewise linear basis functions. If $x \in [x_{i-1}, x_i],

$$
\phi_{i-1}(x) = \frac{x_i - x}{h_{x, \tau}}, \quad \phi_i(x) = \frac{x - x_{i-1}}{h_{x, \tau}}.
$$

The functions $\phi^{j-1}(y)$ and $\phi^j(y)$ are defined similarly in $[y_{j-1}, y_j]$. We define $z(x, y) := E_{12}(x, y)$ and $w(x, y) := e^{-\beta(1 - x)/\varepsilon}(e^{-y/\sqrt{\varepsilon}} + e^{-(1-y)/\sqrt{\varepsilon}}).

A direct calculation and (2.1e) give

$$
\| (E_{12})_x \|_{L^1(\Omega_2)} \leq C \varepsilon^{-1} \| w(x, y) \|_{L^1(\Omega_2)} \leq C \varepsilon^{1/2} N^{-p}.
$$

(3.6)

For $(E_{12}^I)_x$, we have

$$
(E_{12}^I)_x|_{\tau_{ij}} := (z^I)_x|_{\tau_{ij}} = \frac{z_{ij-1} - z_{ij}}{h_{x, \tau}} \phi^{j-1}(y) + \frac{z_{ij} - z_{ij-1}}{h_{x, \tau}} \phi^j(y)
$$

(3.7)

$$
= h_{x, \tau}^{-1} \left( \phi^{j-1}(y) \int_{x_{i-1}}^{x_i} z_x(x, y_{j-1}) \, dx + \phi^j(y) \int_{x_{i-1}}^{x_i} z_x(x, y_j) \, dx \right)
$$

$$
= h_{x, \tau}^{-1} \left( \int_{y_{j-1}}^{y_j} z_x(x, y_{j-1}) \, dx + \phi^j(y) \int_{x_{i-1}}^{x_i} z_{xy}(x, t) \, dt \right).
$$
Similarly, we have

\begin{equation}
(z^I)_x|_{\tau_{ij}} = h_{x, \tau}^{-1} \left( \int_{x_{i-1}}^{x_i} z_x(x, y_j) dx - \varphi^{j-1}(y) \int_{\tau_{ij}} z_y dx dt \right).
\end{equation}

By direct calculations, we can obtain

\begin{equation}
w(x, y) \geq w(x, y_j) > 0 \quad \text{if } [y_{j-1}, y_j] \subset [0, \lambda_y],
\end{equation}

\begin{equation}
w(x, y) \geq w(x, y_j - 1) > 0 \quad \text{if } [y_{j-1}, y_j] \subset [1 - \lambda_y, 1]
\end{equation}

for \( y \in [y_{j-1}, y_j] \). Then, for any \( \tau_{ij} \in \Omega_y \) and \( (x, y) \in \tau_{ij} \), from (2.1e) we have

\begin{equation}
|z_{xy}(x, y)| \leq C \varepsilon^{-3/2} w(x, y)
\end{equation}

and

\begin{equation}
|z_x(x, y_j)| \leq C \varepsilon^{-1} w(x, y) \quad \text{if } [y_{j-1}, y_j] \subset [0, \lambda_y]
\end{equation}

\begin{equation}
|z_x(x, y_j - 1)| \leq C \varepsilon^{-1} w(x, y) \quad \text{if } [y_{j-1}, y_j] \subset [1 - \lambda_y, 1]
\end{equation}

where we have used (3.9). Combining (3.7), (3.10), (3.12) or (3.8), (3.10), (3.11) and considering \( 0 \leq \phi^{j-1}(y), \phi^j(y) \leq 1 \), we obtain

\[
\| (E_{12}^I)_x \|_{L^1(\tau_{ij})} = \int_{\tau_{ij}} \| (z^I)_x \| dx dy \\
\leq C \int_{\tau_{ij}} h_{x, \tau}^{-1} \left( \int_{x_{i-1}}^{x_i} \varepsilon^{-1} w(x, y) dx + \int_{\tau_{ij}} \varepsilon^{-3/2} w(x, t) dx dt \right) dx dy \\
\leq Ch_{x, \tau}^{-1} (h_{x, \tau} \varepsilon^{-1} + h_{x, \tau} h_{y, \tau} \varepsilon^{-3/2}) \| w \|_{L^1(\tau_{ij})}.
\]

Then, we have

\[
\| (E_{12}^I)_x \|_{L^1(\Omega_\varepsilon)} = \sum_{\tau_{ij} \in \Omega_y} \| (E_{12}^I)_x \|_{L^1(\tau_{ij})} \\
\leq C (\varepsilon^{-1} + \varepsilon^{1/2} N^{-1} \ln N \cdot \varepsilon^{-3/2}) \| w(x, y) \|_{L^1(\Omega_\varepsilon)} \\
\leq C \varepsilon^{-1/2} N^{-\rho}.
\]

From (2.1a), we have

\[
u - u^I = (S - S^I) + (E_1 - E_1^I) + (E_2 - E_2^I) + (E_{12} - E_{12}^I).
\]

For the proof of the first inequality of (3.5a), from the standard interpolation theory and (2.1b), we have

\begin{equation}
\| \nabla (S - S^I) \|_{L^1(\Omega_\varepsilon)} \leq CN^{-1} \sum_{i+j=2} \left\| \frac{\partial^{i+j} S}{\partial x^i \partial y^j} \right\|_{L^1(\Omega_\varepsilon)} \leq CN^{-1}.
\end{equation}

(2.1c) and the inverse estimates (3.9) give

\begin{equation}
\| \nabla (E_1 - E_1^I) \|_{L^1(\Omega_\varepsilon)} \leq \| \nabla E_1 \|_{L^1(\Omega_\varepsilon)} + \| \nabla E_1^I \|_{L^1(\Omega_\varepsilon)} \\
\leq \| \nabla E_1 \|_{L^1(\Omega_\varepsilon)} + CN \| E_1^I \|_{L^1(\Omega_\varepsilon)} \leq CN^{1-\rho}
\end{equation}
where we have used \(|E_1(x, y)| \leq CN^{-\rho}\) for \((x, y) \in \Omega_s\). Similarly, we have

\[(3.15) \quad \|\nabla (E_2 - E_2^I)\|_{L^1(\Omega_\ast)} + \|\nabla (E_{12} - E_{12}^I)\|_{L^1(\Omega_\ast)} \leq CN^{1-\rho}.\]

Combining (3.13), (3.14), (3.15), we are done.

For the estimate of the second inequality of (3.5e), on the one hand, we apply Lemma 3.1 to \(\nabla (S - S^I), \nabla (E_1 - E_1^I), \nabla (E_2 - E_2^I)\) and \((E_{12} - E_{12}^I)\), and on the other hand, apply the similar analytic techniques as for the second inequality of (3.5d) to \((E_{12} - E_{12}^I)\).

4 The discrete Green’s function

In this section, we will introduce the discrete Green’s function and derive some estimates of it.

Let \(x^* = (x^*, y^*)\) be a mesh node in \(\Omega\). The discrete Green’s function \(G \in V^N\) associated with \(x^*\) is defined by

\[(4.1) \quad B(v^N, G) = v^N(x^*), \quad \forall v^N \in V^N.\]

We introduce

\[(4.2) \quad \sigma_x = kN^{-1} \ln N, \quad \sigma_y = kN^{-1/2}.\]

The constants \(k > 0\), sufficiently large and independent of \(N\) and \(\varepsilon\), are chosen according to the derivation of Theorem 4.1.

**Theorem 4.1.** For \(x^* \in \Omega_s \cup \Omega_1\) we have

\[\|G\|^2 \leq 8\|G\|^2_{\omega} \leq CN \ln N.\]

**Proof.** See [11, Theorem 4.1] \(\square\)

For pointwise bounds on \(G\) and its first-order derivatives, we define a sub-domain of \(\Omega\) as

\[\Omega_0 := \{x \in \Omega : x - x^* \leq K\sigma_x \ln N \text{ and } |y - y^*| \leq K\sigma_y \ln N\}.\]

The constant \(K > 0\) will be chosen later.

We extend \(\Omega_0\) to the smallest mesh domain \(\Omega'_0 = \Omega'_0(x^*), i.e.,\)

\[\Omega'_0 = \cup \{\tau \in \Omega^N : \text{meas}(\Omega_0 \cap \tau) \neq 0\}.\]

Note that \(\text{meas}(\Omega'_0) \leq C\sigma_y \ln N.\)

**Theorem 4.2.** Assume that \(\sigma_x = kN^{-1} \ln N\) and \(\sigma_y = kN^{-1/2}\), where \(k > 0\) is sufficiently large and independent of \(\varepsilon\) and \(N\). Let \(x^* \in \Omega_s \cup \Omega_1\), then for each
nonnegative integer $\nu$, there exists a positive constant $C = C(\nu)$ and $K = K(\nu)$ such that

\[
\|G\|_{W^{1,\infty}(\Omega \setminus \Omega_0')} \leq C N^{-\nu},
\]

\[
\varepsilon \|G\|_{W^{1,\infty}(\Omega \setminus \Omega_0')} + \|G\|_{L^\infty(\Omega \setminus \Omega_0')} \leq C N^{-\nu},
\]

\[
\varepsilon \|G\|_{W^{1,\infty}(\Omega_1 \setminus \Omega_0')} + \|G\|_{L^\infty(\Omega_1 \setminus \Omega_0')} \leq C \varepsilon^{-1/4} N^{-\nu}
\]

and

\[
\varepsilon^{1/4} \|G_x\|_{L^\infty(\Omega_2 \setminus \Omega_0')} + \varepsilon^{3/4} \|G_y\|_{L^\infty(\Omega_2 \setminus \Omega_0')} \leq C \varepsilon^{-1/4} N^{-\nu}.
\]

Proof. The proof is similar to the one of [8, Theorem 4.2]. \qed

5 Maximum-Norm error estimates

In this section we shall derive bounds on $|(u - U)(x)|$ for $x$ lying in the various subregions of $\Omega$. Taking $v^N = U - u^I$ in (4.1) yields

\[
(U - u)(x^*) = (U - u^I)(x^*) = B(U - u^I, G)
\]

where $x^*$ is a mesh node. Let $e$ denote the interpolation error, i.e.,

\[
e(x) := (u - u^I)(x).
\]

Then from (2.2), (2.4) and (2.5) we have

\[
(U - u)(x^*) = -\varepsilon(\Delta u, \delta b G_x) + B(e, G).
\]

The various terms on the right-hand side are bounded separately.

For the following analysis, we derive some useful local estimates.

Lemma 5.1. If Assumption 2.1 hold true, then there exists a constant $C$ such that

\[
\|E - E^I\|_{L^1(\Omega_1 \cap \Omega_0')} \leq C N^{-\rho \sigma_y \ln N},
\]

\[
\|(E_1 + E_{12}) - (E_1^I + E_{12}^I)\|_{L^1(\Omega_1 \cap \Omega_0')} \leq C N^{-\rho \sigma_y \ln N},
\]

\[
\|\Delta (E_1 + E_{12})\|_{L^1(\Omega_1 \cap \Omega_0')} \leq C \varepsilon^{-1} N^{-\rho \sigma_y \ln N}
\]

where $\sigma_y$ as in (4.2) and the function $E$ can be any one of $E_1$, $E_2$ or $E_{12}$.

Proof. The proof of (5.2a) and (5.2b) is similar to [10, Lemma 4.1]. Inequality (5.2c) can be deduced directly by Assumption 2.1 and the definition of $\Omega_0'$. \qed

Theorem 5.1. Assume that $\sigma_x = kN^{-1} \ln N$, $\sigma_y = kN^{-1/2}$ and $\varepsilon \leq N^{-1}$. Then, for $x^* \in \Omega_2 \cup \Omega_1$, we have

\[
|B(e, G)| \leq C (N^{-9/4} + \varepsilon^{1/4} N^{-2}) (\ln^3 N) \cdot \|G\|.
\]
Proof. For the following analysis, we define
\[ \tilde{\Omega} := ((\Omega_\ast \cup \Omega_1) \cap \Omega_0') \cup (\Omega_2 \cup \Omega_{12}) \]
and modify the bilinear form by mean of integration by parts and the decomposition (2.1a) as follows:
\[
B(e, G) = ((\varepsilon + b^2 \delta)e_x, G_x) + \varepsilon (e_y, G_y) + (b(S - S^I), G)
+ (b(E_2 - E^I_2), G)_{\Omega_2 \cup \Omega_{12}} + (b(E_2 - E^I_2), Gn_x)_{\partial(\Omega_2 \cup \Omega_{12})}
- (b(E - E^I), G_x)_{\Omega_2 \cup \Omega_1} - (b(E_1 - E^I_1), G_x)_{\Omega_2 \cup \Omega_{12}}
- (b(E_12 - E^I_12), G_x)_{\Omega_2 \cup \Omega_{12}} + (ce, \delta bG_x) + (ce, G)
\]
where \( E = E_1 + E_2 + E_{12}, \) \( n_x \) is the \( x \)-axis coordinate of the outward normal vector of \( \partial(\Omega_\ast \cup \Omega_1) \). Note that \( (b(E_2 - E^I_2), Gn_x)_{\partial(\Omega_2 \cup \Omega_{12})} = 0 \) because \( G = 0 \) on \( \partial \Omega \).

The discussion of \( B(e, G) \) will be separated into three parts. In (a), we will analyze \( ((\varepsilon + b^2 \delta)e_x, G_x) \) and \( \varepsilon (e_y, G_y) \). In (b), \( (b(S - S^I), G) \) and \( (b(E_2 - E^I_2), G)_{\Omega_2 \cup \Omega_{12}} \) will be discussed. In (c), we will analyze the residual terms of \( B(e, G) \).

(a) In this part, based on the boundary layer behavior of \( u \), we discuss \( u - u^I \) by mean of the following decomposition
\[
u - u^I = (S - S^I) + (E_1 - E^I_1) + (E_2 - E^I_2) + (E_{12} - E^I_{12}).
\]
According to Lemma 3.3, we have
\[
((S - S^I)_{x}, G_x) = \int_\tau S_{xyy}J_r(y) \left( G_x - \frac{2}{3}(y - y_r)G_{xy} \right) dx dy.
\]
Then
\[
\left\| (S - S^I)_{x}, G_x \right\| \leq CN^{1/2} \| S_{xyy} \|_r \| G_x \|_r
\]
where we have used the inverse inequalities (3.2). Thus,
\[
(5.3) \quad \left\| ((\varepsilon + b^2 \delta)(S - S^I)_{x}, G_x) \right\|_r
\leq CN^{-2} \| (\varepsilon + b^2 \delta)^{1/2} S_{xyy} \|_{\tilde{\Omega}} \| (\varepsilon + b^2 \delta)^{1/2} G_x \|_{\Omega}
\leq CN^{-5/2}(\sigma_y \ln N)^{1/2} \| G \| \leq CN^{-1/4}(\ln^{1/2} N) \| G \|
\]
where we have used \( \text{meas}(\tilde{\Omega}) \leq C\sigma_y \ln N + C\varepsilon^{1/2} \ln N \leq C\sigma_y \ln N.\)
Similarly, we have

\begin{align}
(5.4) & \quad \|(\varepsilon + \delta^2)(E_1 - E_1^l)_x, G_x)_{\Omega_1 \cap \Omega_0'}\| \leq CN^{-\frac{9}{4}}(\ln^{1/2} N)|||G|||,
(5.5) & \quad \|(\varepsilon + \delta^2)(E_1 - E_1^l)_x, G_x)_{\Omega_1 \cup \Omega_1^c}\| \leq C\varepsilon^{3/4}N^{-3}(\ln^{5/2} N)|||G|||,
(5.6) & \quad \|(\varepsilon + \delta^2)(E_2 - E_2^l)_x, G_x)_{\Omega_1 \cup \Omega_1^c}\| \leq C\varepsilon^{1/4}N^{-5/2}(\ln^2 N)|||G|||,
(5.7) & \quad \|(\varepsilon + \delta^2)(E_12 - E_12^l)_x, G_x)_{\Omega_1^c}\| \leq C\varepsilon^{1/4}N^{-2}(\ln^2 N)|||G|||,
(5.8) & \quad \varepsilon \left[\|(S - S^l)_y, G_y)_{\Omega}\| \leq C\varepsilon^{1/2}N^{-9/4}(\ln^{1/2} N)|||G|||,
(5.9) & \quad \varepsilon \left[\|(E_2 - E_2^l)_y, G_y)_{\Omega_1 \cap \Omega_0'}\| \leq C\varepsilon^{1/4}N^{-2}|||G|||,
(5.10) & \quad \varepsilon \left[\|(E - E^l)_y, G_y)_{\Omega_1 \cap \Omega_0'}\| \leq C\varepsilon^{3/4}N^{-2}(\ln^2 N)|||G|||,
(5.11) & \quad \varepsilon \left[\|(E - E^l)_y, G_y)_{\Omega_1^c}\| \leq C\varepsilon^{3/4}N^{-2}(\ln^2 N)|||G|||,
(5.12) & \quad \varepsilon \left[\|(E_2 - E_2^l)_y, G_y)_{\Omega_1^c}\| \leq C\varepsilon^{1/4}N^{-2}|||G|||.
\end{align}

Furthermore, Lemma 5.1 and the inverse inequality (3.4) imply

\begin{align}
(5.13) & \quad \|(\varepsilon + \delta^2)(E - E^l)_x, G_x)_{\Omega_1 \cap \Omega_0'}\| \\
& \leq CN^{-1}\|\|E - E^l\|_x, L^1(\Omega_1 \cup \Omega_0') \cdot \|G_x\|_{L^\infty(\Omega_1 \cap \Omega_0')} \\
& \leq CN^{-1}N^{-\rho}\sigma_y(\ln N) \cdot N\|G_x\|_{\Omega_1 \cap \Omega_0'} \\
& \leq CN^{-\rho}(\ln N)|||G|||.
\end{align}

Similar argument shows

\begin{align}
(5.14) & \quad \varepsilon \left[\|(E_1 - E_1^l)_y, G_y)_{\Omega_1 \cap \Omega_0'}\| \leq C\varepsilon^{1/2}N^{1/2-\rho}(\ln N)|||G|||,
(5.15) & \quad \varepsilon \left[\|(E_12 - E_12^l)_y, G_y)_{\Omega_1 \cap \Omega_0'}\| \leq C\varepsilon^{1/2}N^{1/2-\rho}(\ln N)|||G|||,
(5.16) & \quad \|(\varepsilon + \delta^2)(E_12 - E_12^l)_x, G_x)\|_{\Omega_1^c}\| \leq C\varepsilon^{1/4}N^{1/2-\rho}(\ln^{-1/2} N)|||G|||.
\end{align}

where we have used (5.5d) in the last inequality.

Lemma 3.5 and Hölder inequalities give

\begin{align}
(5.17) & \quad \|(\varepsilon + \delta^2)(E_2 - E_2^l)_x, G_x)\|_{\Omega_1 \cap \Omega_0'}\| \\
& \leq \varepsilon \|(E_2 - E_2^l)_x\|_{L^\infty(\Omega_1 \cap \Omega_0')}\|G_x\|_{L^1(\Omega_1 \cap \Omega_0')} \\
& \leq C\varepsilon N^{-\rho}(\varepsilon\sigma_y\ln N)^{1/2}\|G_x\|_{\Omega_1 \cap \Omega_0'} \\
& \leq C\varepsilon N^{-(1/4+\rho)}(\ln N)|||G|||.
\end{align}
Similarly, we have
\begin{align}
(5.18) \quad & \left| (\varepsilon + b^2 \delta)(E_{12} - E'_{12})_x, G_x)_{\Omega_1 \cap \Omega_6} \right| \leq C N^{-1/4(\rho)} (\ln N) \|G\|, \\
(5.19) \quad & \varepsilon \left| (E_1 - E'_1)_y, G_y)_{\Omega_2} \right| \leq C \varepsilon^{3/4} N^{-\rho} (\ln^{1/2} N) \|G\|, \\
(5.20) \quad & \varepsilon \left| ((E_{12} - E'_{12})_y, G_y)_{\Omega_2} \right| \leq C \varepsilon^{1/4} N^{-\rho} (\ln^{1/2} N) \|G\|.
\end{align}

In view of Theorem 4.2 with \( v = 2 \) and Lemma 5.5, we see
\begin{align}
(5.21) \quad & \left| (\varepsilon + b^2 \delta)(u - u')_x, G_x)_{(\Omega_1 \cup \Omega_5) \setminus \Omega_6} + \varepsilon ((u - u')_y, G_y)_{(\Omega_1 \cup \Omega_5) \setminus \Omega_6} \right| \\
& \leq C \|\nabla (u - u')\|_{L^1(\Omega_1 \cup \Omega_5 \setminus \Omega_6)} (N^{-1} \|\nabla G\|_{L^\infty(\Omega_1 \setminus \Omega_6)} + N \|\nabla G\|_{L^\infty(\Omega_1 \setminus \Omega_6)}) \\
& \leq CN^{-2}.
\end{align}

(b) We see from Lemma 3.3 that
\[
((S - S^t)_x, G) = \sum_{\tau \in \Omega} \int_{\tau} R(S, G) \, dx \, dy + \sum_{\tau \in \Omega} \frac{h_x^2}{12} \left( \int_{t_2} - \int_{t_1} \right) S_{xx} G \, dy
\]
where \( R(\cdot, \cdot) \) as in Lemma 3.3. Based on our Shishkin mesh and the properties of the discrete Green function \( G \), we decompose the first term as follows:
\[
\sum_{\tau \in \Omega} \int_{\tau} R(S, G) \, dx \, dy = \left( \sum_{\tau \in \Omega} + \sum_{\tau \in (\Omega_1 \cup \Omega_5) \setminus \Omega_6} \right) \int_{\tau} R_1(S, G) + R_2(S, G) \, dx \, dy
\]
where
\[
R_1(S, G) = \frac{1}{3} F_\tau(x)(x - x_\tau) S_{xxx} G_x - \frac{h_x^2}{12} S_{xxx} G
\]
and
\[
R_2(S, G) = J_{\tau}(y) S_{xyy}(G - (x - x_\tau) G_x - \frac{2}{3} (y - y_\tau) G_y + \frac{2}{3} (x - x_\tau)(y - y_\tau) G_{xy}).
\]

Firstly, from Assumption 2.1 and the definition of \( \hat{\Omega} \), we have
\begin{align}
(5.22) \quad & \sum_{\tau \in \Omega} \int_{\tau} \frac{1}{3} |F_\tau(x)(x - x_\tau) S_{xxx} G_x| \, dx \, dy \\
& \leq C \sum_{\tau \in \Omega} h_{x, \tau} \|S_{xxx}\|_{L^\infty(\tau)} \|G_x\|_{L^1(\tau)} \\
& \leq CH_x^3 \|G_x\|_{L^1(\Omega_1 \cap \Omega_5 \cup \Omega_2)} + C h_x^3 \|G_x\|_{L^1(\Omega_1 \cap \Omega_6 \cup \Omega_{12})} \\
& \leq CN^{-1/4} (\ln^{1/2} N) \|G\|
\end{align}
and
\begin{align}
(5.23) \quad & \sum_{\tau \in \Omega} \int_{\tau} \frac{h_x^2}{12} |S_{xxx} G| \, dx \, dy \leq CN^{-9/4} (\ln^{1/2} N) \|G\|.
\end{align}
Applying inverse inequalities (3.2) to the last part of $R_2(S, G)$, we obtain

\begin{align}
(5.24) \quad & \sum_{\tau \in \Omega} \int_{\Omega} |R_2(S, G)| dxdy \leq C \sum_{\tau \in \Omega} h_{y, \tau}^2 \|S_{xyy}\|_\tau \|G\|_\tau \\
& \quad \leq CN^{-2} \cdot (\sigma_y \ln N)^{1/2} \cdot \|G\| \\
& \quad \leq CN^{-9/4} (\ln^{1/2} N) \|G\|. 
\end{align}

From Theorem 4.2 with $\nu = 1$, we have

\begin{align}
(5.25) \quad & \sum_{\tau \in (\Omega \cup \Omega_1) \setminus \Omega_0^1} \int_{\tau} |R_1(S, G)| dxdy \\
& \quad \leq C \sum_{\tau \in (\Omega \cup \Omega_1) \setminus \Omega_0^1} (h_{x, \tau}^3 \|S_{xxx}\|_{L^1(\tau)} \|G_x\|_{L^\infty(\tau)} + h_{x, \tau}^2 \|S_{xxx}\|_{L^1(\tau)} \|G\|_{L^\infty(\tau)}) \\
& \quad \leq CH_2^3 \|G_x\|_{L^\infty(\Omega \cup \Omega_1)} + Ch_2^2 \|G_x\|_{L^\infty(\Omega_1 \setminus \Omega_0^1)} + CN^{-2} \|G\|_{L^\infty(\Omega_1 \cup \Omega_1 \setminus \Omega_0^1)} \\
& \quad \leq CN^{-2}
\end{align}

and

\begin{align}
(5.26) \quad & \sum_{\tau \in (\Omega \cup \Omega_1) \setminus \Omega_0^1} \int_{\tau} |R_2(S, G)| \leq C \sum_{\tau \in (\Omega \cup \Omega_1) \setminus \Omega_0^1} H_2^2 \|S_{xyy}\|_{L^1(\tau)} \|G\|_{L^\infty(\tau)} \\
& \quad \leq CN^{-2}
\end{align}

where we have used inverse inequalities (3.2).

Secondly, we set $L := \{(1 - \lambda_x, y) : 0 \leq y \leq 1\}$. Then we have

\begin{align*}
& \left| \sum_{\tau \in \Omega} \frac{h_{x, \tau}^2}{12} \left( \int_{l^2} - \int_{l^4} \right) S_{xx} G dy \right| = \frac{1}{12} \left| \sum_{l \in L} (H_2^2 - h_x^2) \int_l S_{xx} G dy \right| \\
& \quad \leq CH_2^2 \sum_{l \in L} \int_l |S_{xx} G| dy \leq CH_2^2 \left( \sum_{l \in L \cap \Omega_0^1} \int_l |S_{xx} G| dy + \sum_{l \in L \setminus \Omega_0^1} \int_l |S_{xx} G| dy \right) \\
& \quad = CH_2^2 (I + II).
\end{align*}

The estimate of $I$ is straightforward:

\begin{align*}
I &= \sum_{l \in L \cap \Omega_0^1} \int_l |S_{xx} G| (1 - \lambda_x, y) dy \leq \int_{y_0^1} \left( \int_{1 - \lambda_x}^{y_0^1} (S_{xxx} G + S_{zzz} G_x) dx \right) dy \\
& \quad \leq \|S_{xxx}\|_{L^\infty(D_L)} \|G\|_{L^1(D_L)} + \|S_{zzz}\|_{L^\infty(D_L)} \|G_x\|_{L^1(D_L)} \\
& \quad \leq C (\varepsilon \sigma_y \ln^2 N)^{1/2} (\|G\|_{D_L} + \|G_x\|_{D_L}) \\
& \quad \leq CN^{-1/4} \ln N \cdot (\varepsilon^{1/2} \|G\|_{D_L} + \varepsilon^{1/2} \|G_x\|_{D_L}) \\
& \quad \leq CN^{-1/4} (\ln N) \|G\|.
\end{align*}
where \( \{1 - \lambda_z\} \times [y_1', y_2'] = L \cap \Omega'_0 \) and \( D_L := [1 - \lambda_x, 1] \times [y_1', y_2'] \).

From Theorem 12 with \( \nu = 1 \), we have

\[
\Pi = \sum_{t \in L \cap (\Omega_2 \cap \Omega'_0)} \int |S_{xx} G| \, dy + \sum_{t \in L \cap (\Omega_2 \cap \Omega'_0)} \int |S_{xx} G| \, dy \\
\leq C \|G\|_{L^\infty(\Omega_2 \cap \Omega'_0)} + C \varepsilon^{1/2} (\ln N) \|G\|_{L^\infty(\Omega_2 \cap \Omega'_0)} \\
\leq CN^{-1}.
\]

Considering the estimates for I and II, we obtain

\[(5.27) \quad \left| \sum_{t \in \Omega} \frac{h^2_{x,t}}{12} \left( \int_{L_2} - \int_{L_4} \right) S_{xx} G \, dy \right| \leq CN^{-9/4} (\ln N) \|G\|.
\]

The estimates for \(( (E_2 - E_2')_x, G)_{\Omega_2 \cap \Omega_12} \) are the same as \(( (S - S')_x, G) \).

Thus we have

\[(5.28) \quad |b(E_2 - E_2', G)_{\Omega_2 \cap \Omega_12}| \leq C \varepsilon^{1/4} N^{-2} (\ln^2 N) \|G\|.
\]

(c) From Lemma 3.2, we have

\[(5.29) \quad \left| b(E - E', G)_{\Omega_2 \cap \Omega_12} \right| \leq C \|E - E'\|_{L^\infty(\Omega_2 \cap \Omega'_0)} \|G\|_{L^1(\Omega_2 \cap \Omega'_0)} \\
\leq CN^{-\rho} \cdot (\sigma_y \ln N)^{1/2} \|G\|_{L^1(\Omega_2 \cap \Omega'_0)} \\
\leq CN^{1/4 - \rho} (\ln^{1/2} N) \|G\|.
\]

Similarly, we have

\[(5.30) \quad \left| b(E_1 - E_1', G)_{\Omega_2} \right| \leq C \varepsilon^{1/4} N^{1/2 - \rho} (\ln^{1/2} N) \|G\|,
\]

\[(5.31) \quad \left| b(E_{12} - E_{12}', G)_{\Omega_2} \right| \leq C \varepsilon^{1/4} N^{1/2 - \rho} (\ln^{1/2} N) \|G\|.
\]

In view of Lemma 3.2, we obtain

\[(5.32) \quad \left| b(E - E', G)_{\Omega_2 \cap \Omega'_0} \right| \leq C \|E - E'\|_{L^\infty(\Omega_2 \cap \Omega'_0)} \|G\|_{L^1(\Omega_2 \cap \Omega'_0)} \\
\leq CN^{-2} (\ln^2 N) \cdot (\varepsilon \sigma_y \ln^2 N)^{1/2} \|G\|_{L^1(\Omega_2 \cap \Omega'_0)} \\
\leq CN^{-9/4} (\ln^3 N) \|G\|.
\]

Similar argument shows

\[(5.33) \quad \left| b(E_1 - E_1', G)_{\Omega_{12}} \right| \leq C \varepsilon^{1/4} N^{-2} (\ln^3 N) \cdot \|G\|,
\]

\[(5.34) \quad \left| b(E_{12} - E_{12}', G)_{\Omega_{12}} \right| \leq C \varepsilon^{1/4} N^{-2} (\ln^3 N) \cdot \|G\|,
\]

\[(5.35) \quad \left| c(u - u', \delta G) \right| \leq CN^{-5/2} (\ln^2 N) \|G\|,
\]

\[(5.36) \quad \left| c(u - u', G) \right| \leq C(N^{-9/4} \ln^1 N + \varepsilon^{1/4} N^{-2} \ln^{5/2} N) \|G\|.
\]
Theorem 4.2 with $\nu = 1$ and Lemma 3.2 yield

\[ b(E - E^I, G_x)_{\Omega_1 \setminus \Omega_0'} \leq C\| E - E^I \|_{L^\infty(\Omega_1 \setminus \Omega_0')} \| G_x \|_{L^\infty(\Omega_1 \setminus \Omega_0')} \leq CN^{-2}\ln^2 N(\varepsilon \ln N) \| G_x \|_{L^\infty(\Omega_1 \setminus \Omega_0')} \leq CN^{-2} \]

and

\[ |(u - u^I, G)_{(\Omega_1 \cup \Omega_2) \setminus \Omega_0'}| \leq CN^{-2}, \]

\[ |(u^I, G)_{(\Omega_1 \cup \Omega_2) \setminus \Omega_0'}| \leq CN^{-2}. \]

Collecting (5.3)–(5.39), we are done. \qed

**Theorem 5.2.** Assume that $u$ satisfies Assumption 2.1 and $\varepsilon \leq N^{-1}$. Let $\sigma_x = kN^{-1} \ln N$ and $\sigma_y = kN^{-1/2}$. Then for $x^* \in \Omega_1 \cup \Omega_2$

1. if $\Omega_0' \subset \Omega_1 \cup \Omega_2$, we have

\[ |(\varepsilon \Delta u, \delta bG_x)| \leq C(N^{-\rho} + \varepsilon N^{-5/4})(\ln N)\| G \|. \]

2. if $\Omega_0' \not\subset \Omega_1 \cup \Omega_2$, we have

\[ |(\varepsilon \Delta u, \delta bG_x)| \leq C(N^{-\rho} + \varepsilon^{1/4} \delta_y)(\ln N)\| G \|. \]

**Proof.** We set $E = E_1 + E_2 + E_{12}$ and define $\Gamma_{s,x} := \Omega_1 \cap \Omega_2$ and $\Gamma_{y,xy} := \Omega_2 \cap \Omega_{12}$. At the beginning, integration by parts and the definition of $\delta$ yield

\[ (\Delta S, \delta bG_x) = (\Delta S, \delta bG_x)_{\Omega_1 \cup \Omega_2} = (\Delta S, \delta_S bG)_{\Gamma_{s,x}} + (\Delta S, \delta_y bG)_{\Gamma_{y,xy}} - ((\Delta S)_x, \delta bG) \]

and

\[ (\Delta E_2, \delta bG_x) = (\Delta E_2, \delta_S bG)_{\Gamma_{s,x}} + (\Delta E_2, \delta_y bG)_{\Gamma_{y,xy}} - ((\Delta E_2)_x, \delta bG). \]

Thus,

\[ (\varepsilon \Delta u, \delta bG_x) = \varepsilon \delta_s (\Delta(S + E_2), bG)_{\Gamma_{s,x}} + \varepsilon \delta_y (\Delta(S + E_2), bG)_{\Gamma_{y,xy}} - \varepsilon((\Delta(S + E_2))_x, \delta bG) + \varepsilon((\Delta(E_1 + E_{12}), \delta bG_x). \]

The terms on the right-hand side are analyzed separately.
From Assumption 2.1 we have

$$
\varepsilon \delta_x \left| (\Delta S, bG)_{\Omega, x} \right| \leq C \varepsilon N^{-1} \| \Delta S \|_{L^\infty(\Omega, x)} \| G \|_{L^1(\Omega, x)}
$$

$$
\leq C \varepsilon N^{-1} \left( \int_{\Gamma, x} |G(1 - \lambda_x, y)| dy + \| G \|_{L^1(\Gamma, x)} \right)
$$

$$
\leq C \varepsilon N^{-1} \left( \int_{\Gamma, x} \int_{1-\lambda_x}^{1} |G_x(x, y)| dy + \| G \|_{L^\infty(\Omega, \lambda_\delta)} \right)
$$

$$
\leq C \varepsilon N^{-1} (\varepsilon \sigma_y \ln^2 N)^{1/2} \| G_x \|_{\Omega_1} + C \varepsilon N^{-1} \| G \|_{L^\infty(\Omega, \lambda_\delta)}
$$

$$
\leq C \varepsilon N^{-5/4} (\ln N) \| G \| + C \varepsilon N^{-2}
$$

where we have used Theorem 4.2 with \( \nu = 1 \). Similarly, we have

$$
\varepsilon \delta_x \left| (\Delta E_2, bG)_{\Gamma, x} \right| \leq C N^{-1} \left( (\ln N) \right) |G|,
$$

$$
\varepsilon \delta_y \left| (\Delta (S + E_2), bG)_{\Gamma, x} \right| \leq C \varepsilon \delta_y (\ln N) \| G \|.
$$

Considering (2.1b) and \( \text{meas}(\Omega \cap \Omega' \delta) \leq C \sigma_x \ln N \), we obtain

$$
\varepsilon \delta_x \left| ((\Delta S)_x, \delta bG)_{\Omega \cap \Omega'_\delta} \right| \leq C \varepsilon N^{-1} \| (\Delta S)_x \|_{L^\infty(\Theta \cap \Omega'_\delta)} \| G \|_{L^1(\Theta \cap \Omega'_\delta)}
$$

$$
\leq C \varepsilon N^{-1} (\sigma_y \ln N)^{1/2} \| G \|_{\Omega \cap \Omega'_\delta}
$$

$$
\leq C \varepsilon N^{-5/4} (\ln N) \| G \|.
$$

Assumption 2.4 and the inverse inequality (5.3) yield

$$
|\varepsilon |((\Delta E_2)_x, \delta G)_{\Theta \cap \Omega'_\delta} | \leq C \varepsilon \delta_x \| (\Delta E_2)_x \|_{L^1(\Theta \cap \Omega'_\delta)} \| G \|_{L^\infty(\Theta \cap \Omega'_\delta)}
$$

$$
\leq C \varepsilon \delta_x \cdot \varepsilon^{-1/2} N^{-\rho} \cdot (H_x H_y)^{-1/2} \| G \|_{\Theta \cap \Omega'_\delta}
$$

$$
\leq \varepsilon^{1/2} N^{-\rho} \cdot \| G \|.
$$

In view of (2.1d), we get

$$
|\varepsilon \delta_y ((\Delta E_2)_x, G)_{\Theta \cap \Omega'_\delta} | \leq C \varepsilon \delta_y \| (\Delta E_2)_x \|_{\Omega_2} \| G \|_{\Omega_2}
$$

$$
\leq C \varepsilon^{1/4} \delta_y \cdot \| G \|.
$$

Lemma 5.1 Assumption 2.4 and the inverse inequality (5.3) yield

$$
|\varepsilon \left| (\Delta (E_1 + E_1), \delta bG_x)_{\Omega_1, \cap \Omega'_\delta} \right| \leq C \varepsilon \delta_x \| (\Delta (E_1 + E_1))_{\Omega_1} \|_{L^1(\Theta \cap \Omega'_\delta)} \| G_x \|_{L^\infty(\Theta \cap \Omega'_\delta)}
$$

$$
\leq C \varepsilon \delta_x \cdot \varepsilon^{-1/2} N^{-\rho} \sigma_y \ln N \cdot (H_x H_y)^{-1/2} \| G_x \|_{\Theta \cap \Omega'_\delta}
$$

$$
\leq C N^{-\rho} (\ln N) \| G \|.$$
and
\[
\| (\Delta (E_1 + E_{12}), \delta b G_x)_{\Omega_2 \cap \Omega_0'} \| \\
\leq C \varepsilon \delta y \| \Delta (E_1 + E_{12}) \|_{L^1(\Omega_2 \cap \Omega_0')} \| \delta b G_x \|_{L^\infty(\Omega_2 \cap \Omega_0')} \\
\leq C \varepsilon \delta y \cdot \varepsilon^{-1/2} N^{-\rho} (\ln N) \cdot (H_x h_y)^{-1/2} \| \Delta (E_1 + E_{12}) \|_{L^\infty(\Omega_2 \cap \Omega_0')} \\
\leq C \varepsilon^{1/4} N^{1/2-\rho} (\ln^{1/2} N) \| G \|.
\]

Considering (2.1b)—(2.1e) and Theorem 4.2 with \( \nu = 1 \), we obtain
\[
\| (\Delta S)_x, \delta b G \|_{\Omega \cap \Omega_0'} \leq C \varepsilon \delta \| (\Delta S)_x \|_{L^1(\Omega \cap \Omega_0')} \| G \|_{L^\infty(\Omega \cap \Omega_0')} \\
\leq CN^{-2}.
\]

Similar argument shows
\[
(5.48) \quad \| (\Delta E_2)_x, \delta b G \|_{\Omega \cap \Omega_0'} \leq C \varepsilon N^{-2}
\]

We deduce the last inequality by analyzing separately for \( \Omega_s \setminus \Omega_0' \) and \( \Omega_2 \setminus \Omega_0' \).

If \( \Omega_0' \subset \Omega_s \cup \Omega_1 \), we can get \( \Omega_2 \cap \Omega_0' = \emptyset \). Then
\[
(5.51) \quad \| \varepsilon (\Delta E_2)_x, \delta b G \|_{\Omega_2 \cap \Omega_0'} = 0
\]
\[
(5.52) \quad \| \varepsilon (\Delta (E_1 + E_{12}), \delta b G_x)_{\Omega_2 \cap \Omega_0'} \| = 0.
\]

In this case, the modification of (5.42) is that
\[
(5.53) \quad \varepsilon \delta_y (|\Delta S| + |\Delta E_2|, |\delta b G|)_y \leq C \varepsilon \delta_y \| (\Delta S + E_2) \|_{L^1(\gamma_{y,x})} \| G \|_{L^\infty(\gamma_{y,x})} \\
\leq C \varepsilon^{1/2} \delta_y \| G \|_{L^\infty(\Omega_2 \setminus \Omega_0')} \\
\leq CN^{-2}
\]
where we have used Theorem 4.2 with \( \nu = 1 \). Collecting (5.40), (5.41), (5.43), (5.44), (5.46) and (5.48)–(5.53), we have
\[
| (\varepsilon \Delta u, \delta b G_x) | \leq C (N^{-\rho} + \varepsilon N^{-5/4} (\ln N)) \| G \|.
\]

If \( \Omega_0' \not\subset \Omega_s \cup \Omega_1 \), using (5.40)–(5.51) leads to
\[
| (\varepsilon \Delta u, \delta b G_x) | \leq C (N^{-\rho} + \varepsilon^{1/4} \delta_y) (\ln N) \| G \|.
\]
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