Optimal Strategies in a Production Inventory Control Model

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Abstract
We consider a production-inventory control model with finite capacity and two different production rates, assuming that the cumulative process of customer demand is given by a compound Poisson process. It is possible at any time to switch over from the different production rates but it is mandatory to switch-off when the inventory process reaches the storage maximum capacity. We consider holding, production, shortage penalty and switching costs. This model was introduced by Doshi, Van Der Duyn Schouten and Talman in 1978. In their paper they found a formula for the long-run average expected cost per unit time as a function of two critical levels, in this paper we consider expected discounted cumulative costs instead. We seek to minimize this discounted cost over all admissible switching strategies. We show that the optimal cost functions for the different production rates satisfy the corresponding Hamilton-Jacobi-Bellman system of equations in a viscosity sense and prove a verification theorem. The way in which the optimal cost functions solve the different variational inequalities gives the switching regions of the optimal strategy, hence it is stationary in the sense that depends only on the current production rate and inventory level. We define the notion of finite band strategies and derive, using scale functions, the formulas for the different costs of the band strategies with one or two bands. We also show that there are examples where the switching strategy with two critical levels is not optimal.

Keywords Inventory/Production model · Optimal switching strategies · Compound Poisson process · Scale functions · Hamilton-Jacobi-Bellman equations · Viscosity solutions

Mathematics Subject Classification 49L20 · 49L25 · 90B05 · 90B30
1 Introduction

The classical production-inventory model considers a single machine that produces a certain product. Finished products are stored and the storage capacity can be finite or infinite. Moreover, the classical model assumes a constant production rate, customer demands arriving according to a Poisson process and size demands distributed as i.i.d random variables. When the stock on hand is less than the demand then either the excess of the demand is lost or backlogged. In the first case the inventory level is always positive, while in the latter it can be negative. The costs associated with this model are holding cost and lost-sales cost. Higher production rates yield fewer lost-sale cost but higher holding cost and vice versa. Thus, there is a trade-off between holding and lost-sales costs. Therefore, researchers have looked for the optimal strategy to minimize the expected cost. One of the prominent strategy discussed in the literature is the two regime switching policy. Under this policy, the production rate switches from high to low rate when the inventory increases above a given level $y_1$; also, the production rate switches from low to high rate when the inventory becomes smaller than a given level $y_2$, where $y_2 < y_1$.

In the operations research literature, most articles have considered the average cost per time unit assuming that the system is at steady state. Gavish and Graves (1980) and Graves and Keilson (1981) studied the case where once the inventory level reaches a given threshold $y_1$, productions stops; and production resumes when the inventory level down-crosses another threshold $y_2$, where $y_2 < y_1$. In these two papers, customers arrive according to Poisson process and backlogging is permitted. In the first paper the demand is always for one item and the machine produces one item per time unit, and in the second one the demand is exponentially distributed. In both papers, the average cost per time unit is obtained. De Kok et al. (1984) and De Kok (1985, 1987) studied an infinite capacity production inventory system where demand occurs according to a compound Poisson process and unsatisfied demand is backlogged. They considered two production rates $\sigma_2 < \sigma_1$ where the production rate is switched to $\sigma_2$ once the inventory level is above $y_1$ and it is switched back to $\sigma_1$ when the inventory level down-crosses $y_2$. In the first paper, unsatisfied demand is backlogged and in the second one, unsatisfied demand is lost. Performance measures that are considered under some constrains on the switching and holding costs are: the average amount of stock-out per unit time, the fraction of demand to be met directly from stock on hand (in the backlog case) and the average amount of lost sales. Doshi et al. (1978) considered a finite capacity production inventory model with lost sales and similar production rate policy as in De Kok et al. (1984) and De Kok (1985, 1987). They obtained the steady-state distribution of the inventory level for this model and hence the average cost per time unit.

The optimization problem of minimizing the expected discounted cost control of production planning in a manufacturing systems with infinite capacity (and no costs for changing the production rate) has been studied, among others, by Fleming et al. (1987) and Sethi and Zhang (1994) in finite state Markov and diffusion demand processes using Dynamic Programming Theory.

In the last decade, there have been a growing interest on the production-inventory problem with expected discounted costs and compound Poisson demands. Shi et al. (2014) considered an infinite capacity production-inventory model with lost sales and constant production rate. They obtained the expected discounted cost with infinite horizon and the long-run time average and then the production rates which minimizes each of them. Shi (2016) and Chang et al.
(2019) considered the same model and obtained results on the expected discounted cost subject to a risk of stockout. The works mentioned above are mainly focused on the performance analysis; namely, they obtained the expected discounted cost for given cost rates, production rate and demand process, and then they found the optimal production rate that minimize this expected discounted cost. Barron et al. (2014) considered the model of Doshi et al. (1978) under the assumption of compound Poisson arrival process and phase-type demand and obtained the expected discounted cost. In Barron et al. (2016), Barron, Perry and Stadje addressed the case of a Markov additive arrival process with phase type demands. The papers Barron et al. (2014, 2016) are focused on performance analysis and do not contain optimization results.

In this paper we consider a production-inventory control model with finite capacity with two prespecified production rates introduced by Doshi et al. (1978) where the cumulative process of customer demand is given by a compound Poisson process. It is allowed to switch from one production rate to the other at any time but it is mandatory to switch-off when the inventory process reaches the storage maximum capacity. We consider holding, production, shortage penalty and switching costs. Our aim is to minimize the expected discounted cumulative costs up to infinity over all admissible switching strategies. Using the method of dynamic programming, we derive the Hamilton-Jacobi-Bellman system of equations associated to the problem and show that there exists an optimal switching strategy. We define the notion of finite band strategies and derive, using scale functions, the formulas for the different costs of the band strategies with one or two bands. We also show that there are examples where the switching strategy with two critical levels is not optimal. We show in the examples that the optimal costs functions could be non-differentiable, so it is essential to use the viscosity solution approach.

Continuous-time Markov decision processes (CTMDPs) are related optimization problems with applications in inventory control, queueing, etc. These problems have been studied for instance by Guo and Hernández-Lerma (2009), Dufour and Piunovskiy (2015), and Piunovskiy and Zhang (2014). In these works, the uncontrolled process have jumps but no drift, so the Hamilton-Jacobi-Bellman equation does not involve derivatives and as a consequence, the viscosity solution approach is not needed.

In the compound Poisson demand setting, the contribution of this article has two aspects. From the optimization point of view, unlike Shi et al. (2014), Shi (2016), and Chang et al. (2019), we use Dynamic Programming Theory to analyze the optimal dynamic policy which minimizes the expected discounted cost. From the performance analysis point of view, we add to the model in Shi et al. (2014) a constraint of finite capacity and two possible production rates which interchange according to a band strategy; these band strategies are a generalization of the switching strategies considered in Doshi et al. (1978). Moreover, the performance analysis in Barron et al. (2014) and Barron et al. (2016) is generalized to a general demand distribution.

Similar optimal starting-and-stopping problem has been studied extensively. In the diffusion setting and some special profit functions, Brekke and Oksendal (1994) apply a verification approach for solving the variational inequality associated with this impulse control problem. Pham and Ly Vath (2007), Hamadène and Jeanblanc (2007), and Bayraktar and Egami (2010) between others, studied various extensions of this model. Also in the diffusion setting, Pham et al. (2009) considered the case of multiple-regime switching. Azcue and Muler (2015) studied a mixed singular control/switching problem for multiple regimes in the compound Poisson setting.

This article is structured as follows. Section 2 describes the model setup and some basic results are derived in Section 3. In Section 4, we show that the optimal cost functions for
the different production rates satisfy the corresponding Hamilton-Jacobi-Bellman system of equations in a viscosity sense and prove both characterization and verification results. Moreover, we prove that there exists an optimal production-inventory strategy and that it has a band structure. In Section 5, we introduce the concept of finite band strategies depending on the number of connected components of the non-action regions; and in Section 6 we use the scale functions to find the formulas of the holding, shortage and switching cost functions for the band strategies with one or two connected components. Finally, in Section 7, we identify the optimal strategies and the corresponding cost functions for a number of concrete examples with exponentially distributed customer demands.

Some technical proofs are delegated to an Appendix.

2 Model

In this paper we address a production-inventory control model with finite storage capacity $b > 0$ and two production rates: $\sigma_1$ and $\sigma_2$ such that $0 < \sigma_2 < \sigma_1$; this model was introduced by Doshi et al. (1978). We say that the production is in phase $i = 1, 2$ when the production rate is $\sigma_i$, whenever the inventory level reaches level $b$, the production is stopped i.e. $\sigma_0 = 0$ at inventory level $b$. We say that the production is in phase $i = 0$ when the production is stopped.

We assume that the inter-arrival times between customer demands in phase $i$ are exponential with rate of arrival $\lambda_i$ and that the sizes of the demands are i.i.d positive random variables with distribution $F_i$ with finite mean, so in each phase the cumulative process of customer demands is a Compound Poisson process with rate $\lambda_i$. Moreover, let us assume that the compound Poisson processes of the cumulative customer demands in the three phases $i = 0, 1, 2$ are independent of each other. For a given phase $i$, let us call $N_i^j$ as the Poisson process with rate of arrival $\lambda_i$, $\tau_n^i$ as the arrival time of customer demand $n$ and $Y_n^i$ as the size of the customer demand.

We can describe this model in a rigorous way by defining its filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. We define first the probability space as a product

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega^0, \mathcal{F}^0, \mathbb{P}^0) \times (\Omega^1, \mathcal{F}^1, \mathbb{P}^1) \times (\Omega^2, \mathcal{F}^2, \mathbb{P}^2).$$

The sample space $\Omega^i = \{(\tau_n^i, Y_n^i)_{n \geq 1} : 0 \leq \tau_n^i < \tau_{n+1}^i, \lim_{n \to \infty} \tau_n^i = \infty \text{ and } Y_n^i > 0\}$; $\mathcal{F}^i$ is the $\sigma$-algebra generated by the sets

$$B^i_{j,t,y} = \{ (\tau_n^i, Y_n^i)_{n \in \mathbb{N}} \in \Omega^i : \tau_j^i \leq t \text{ and } Y_j^i \leq y \}$$

for $j \in \mathbb{N}$, $t \geq 0$ and $y > 0$; $\mathbb{P}^i$ is the unique probability measure which satisfies

$$\mathbb{P}^i \left( \bigcap_{j=1}^k B^i_{j,t,y} \right) \cap \{ \tau_{k+1}^i > t \} = \left( \frac{\lambda_i t^k}{k!} e^{-\lambda_i t} \right) \prod_{j=1}^k F_i(y_j).$$

Finally, we define the filtration $(\mathcal{F}_t)_{t \geq 0}$ where $\mathcal{F}_i$ is the $\sigma$-algebra generated by the random variables $\tau_n^i$ and $Y_n^i$ for $i = 0, 1, 2$ and for all $n \in \mathbb{N}$ such that $\tau_n^i \leq t$. Note that the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right continuous, that is $\mathcal{F}_{t+} = \mathcal{F}_t^c$. For more details on compound Poisson processes, see for instance Varadhan (2007).

We also assume that $I \leq 0$ is the minimum level below which the inventory is not allowed to decrease.
If the inventory drops below \( l \) the part of the demand below \( l \) is lost and production resumes at inventory level \( l \).

The following costs are considered:

- **Holding and production costs.** \( h_i : [l, b) \rightarrow [0, \infty) \) for \( i = 1, 2 \) correspond to the holding and production cost in phase \( i \) when the inventory level is \( x \in [l, b) \). We assume that it is bounded with finitely many discontinuities and Lipschitz between discontinuities with Lipschitz constant \( m_i \), \( h_0(b) \geq 0 \) corresponds to the holding cost at inventory level \( b \).

- **Shortage penalty costs.** \( p_i : [0, \infty) \rightarrow [0, \infty) \) corresponds to the penalty function cost when the amount \( y \) of the demand of a customer is lost and the production is in phase \( i = 0, 1, 2 \). We assume that it is non-negative and non-decreasing. Moreover,

\[
\int_0^\infty p_i(y) dF_i(y) < \infty. \tag{2.2}
\]

- **Switching costs.** \( K_{ij} \) corresponds to the fixed cost of switching from phase \( i \) to phase \( j \) where \( i, j = 0, 1, 2 \). Here we include the costs of switch on (\( K_{0i} \)) where \( i = 1, 2 \) and the costs of switch off (\( K_{0i} \)) where \( i = 1, 2 \) the production process when the inventory reaches level \( l \). We add the following conditions on the switching costs in order to penalize simultaneous changes of phases:

\[
K_{0i} \leq K_{ij} + K_{ji} \text{ for } \{i, j\} = \{1, 2\}, \quad K_{12} + K_{21} > 0. \tag{2.3}
\]

**Remark 2.1** We assume here that it is possible at any time to switch over from phase \( i \) to phase \( j \) where \( 1 \leq i, j \leq 2 \) but it is mandatory to switch off (namely to go to phase 0) when the inventory process reaches level \( b \). On top of that, the phase should be 0 or 2 (that corresponds to positive production rate) whenever the inventory process is in the interval \([l, b)\). Moreover, if a demand arrives and the inventory level before this arrival minus the demand of the customer is less than the backlog \( l \leq 0 \), this demand is covered up to \( l \) paying the corresponding penalty cost of the part of the demand that has been lost given by function \( p_i \) depending on the phase \( i = 0, 1, 2 \).

Our aim is to minimize the expected discounted cumulative costs over all possible production strategies. A production strategy can be defined as \( \pi = (T_k, J_k)_{k \geq 1} \) where \( T_k \) are the switching times from phase \( J_{k-1} \) to phase \( J_k \) and \( J_k \in \{0, 1, 2\} \). We call \( T_0 = 0 \) and \( J_0 \) as the initial phase. In addition, we assume that \( T_1 < T_2 < T_3 < \cdots \), and \( J_k \neq J_{k-1} \).

Given a initial inventory level \( x \), an initial phase \( J_0 = i \) and a production strategy \( \pi = (T_k, J_k)_{k \geq 1} \), the controlled process is defined recursively as \( X_{T_0}^\pi = x, T_0 = 0 \), and

\[
X_t^\pi = X_{T_k}^\pi + \sigma_{J_k}(t - T_k) - \sum_{n = \min \{j: \tau_{J_n}^{J_k} \geq T_k\}}^{N_{J_k}^J} \min \{Y_n^J, X_{\tau_{J_n}^{J_k} - l}^\pi\} \text{ for } t \in [T_k, T_{k+1}). \tag{2.4}
\]

Let us define the auxiliary inventory process,

\[
\bar{X}_t^\pi := X_t^\pi \text{ for } t \neq \tau_n \text{ and } \bar{X}_{\tau_n}^\pi = X_{\tau_n}^\pi - Y_n^J, \text{ for } t \in [T_k, T_{k+1})
\]

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so $X^*_e = l \lor X^*_n$, this corresponds to the controlled process before it eventually resumes at inventory level $l$.

Let us also define the controlled phase process

$$J^*_t := J_k \text{ for } t \in [T_k, T_{k+1}).$$

A production strategy $\pi = (T_k, J_k)_{k \geq 0}$ starting at phase $i$ and inventory level $x$ is admissible if it is $\mathcal{F}_T$-adapted, càdlàg and satisfies,

- $T_0 = 0$ and $J_0 = i$.
- If the current inventory level is less than $b$, then the phase should be either 1 or 2. More precisely, if $X^*_i < b$ then $J^*_t$ must be 1 or 2.
- If at time $t$, the phase process $J^*_t = i$ with $i = 1, 2$ and the current inventory level $X^*_t$ level reaches $b$, it is mandatory to switch off the production. Hence, this time $t$ should coincide with the next switching time $T_k$ for some $k$ and $J^*_t = J_k = 0$. Afterwards, $X^*_t = b$ for $t \in [T_k, T_{k+1})$, and $T_{k+1}$ would be the time of the arrival of the next customer demand and $J_{k+1}$ would be either 1 or 2.

If $\pi$ is admissible then the controlled process $X^*_t$ is $\mathcal{F}_T$-adapted and right continuous.

We denote as $\Pi_{i,b}$ the set of admissible strategies starting at phase $i$ and inventory level $x$. If the initial phase is $i \in \{1, 2\}$, given an initial inventory level $x \in [l, b)$, and $\pi = (T_k, J_k)_{k \geq 0} \in \Pi_{i,b}$, the associated cost function is given by,

$$V^*_i(x) = \mathbb{E}\left[\int_0^\infty e^{-q_i t} h_{j_i}(X^*_t) dt\right] + \mathbb{E}\left[\sum_{k=0}^{\infty} e^{-q_i T_{k+1}} K_{l_i, J_{k+1}} \right]$$

$$+ \mathbb{E}\left[\sum_{k=0}^{\infty} p^i_{T_k, T_{k+1}}(X^*_n) \right],$$

where $p^i_{T_k, T_{k+1}}(X^*_n)$ corresponds to the cumulative discounted shortage cost between switching times. More precisely,

$$p^i_{T_k, T_{k+1}}(X^*_n) = \sum_{n=\min\{j: e_j \geq x\}}^{N_i} e^{-q_i t_j} 1\{X^*_{t_j} - l \in Y_n\} p_i(Y_n - X^*_{t_j} + l)$$

We define the optimal cost functions for $i = 1, 2$ as

$$V_i(x) = \inf_{\pi \in \Pi_{i,b}} V^*_i(x)$$

for $x \in [l, b)$.

Given an initial inventory level $b$ and an admissible inventory strategy $\pi = (T_k, J_k)_{k \geq 0} \in \Pi_{b,0}$ the cost value of this strategy is given by

$$V^*_0(b) = \mathbb{E}\left[\int_0^\infty e^{-q_i t} h_{J}(X^*_t) dt\right] + \mathbb{E}\left[\sum_{k=0}^{\infty} e^{-q_i T_{k+1}} K_{l, J_{k+1}} \right]$$

$$+ \mathbb{E}\left[\sum_{k=0}^{\infty} p^i_{T_k, T_{k+1}}(X^*_n) \right].$$

In this case, the optimal value for inventory level $b$ is given by

$$V_0(b) = \inf_{\pi \in \Pi_{b,0}} V^*_0(b).$$
**Remark 2.2** The Markovian nature of the problem suggests that one could only consider Markov control policies; however in this paper, we prefer for technical reasons, to allow more general policies called admissible strategies, which could depend on the past in some more complex ways (for a discussion about it, see for instance Chapter III in Fleming and Soner (2006)). Nevertheless, we will see in Theorem 4.7, that there exists and optimal strategy and it is indeed a Markov control policy.

### 3 Basic Properties

In this section we study the existence and regularity of the optimal cost functions. Let us start proving that they are well defined.

**Proposition 3.1** \( V_0(b) \) is finite and the optimal cost functions \( V_i \) are bounded in \([l, b)\) for \( i = 1, 2 \).

**Proof** Take \( i \in \{1, 2\} \), \( x \in [l, b) \) and the admissible production strategy \( \pi = (T_k, J_k)_{k \geq 1} \in \Pi_{x, i} \) that only switch off from phase \( i \) to phase 0 when the current inventory level is \( b \) and remain in phase \( i \) otherwise. Let us call

\[
\tilde{h} = \max \left\{ \sup_{x \in [l, b]} h_1(x), \sup_{x \in [l, b]} h_2(x), h_0(b) \right\}.
\]

Then, we have

\[
\mathbb{E} \left[ \int_0^\infty e^{-qt} h_0(X_t^x) dt \right] \leq \frac{\tilde{h}}{q}. \tag{3.2}
\]

Moreover,

\[
\mathbb{E} \left[ \sum_{k=0}^\infty P^J_{T_k, T_{k+1}} (X_{T_k}^x) \right] \\
\leq \mathbb{E} \left[ \sum_{n=1}^\infty e^{-q_{T_k}} p_i(Y_n^i) \right] + \mathbb{E} \left[ \sum_{n=1}^\infty e^{-q_{T_k}} p_0(Y_n^0) \right] \\
= \mathbb{E} \left[ \sum_{n=1}^\infty e^{-q_{T_k}} \mathbb{E}[p_i(Y_n^i)] \right] + \mathbb{E} \left[ \sum_{n=1}^\infty e^{-q_{T_k}} \mathbb{E}[p_0(Y_n^0)] \right] \tag{3.3}
\]

and so it is finite from (2.2). Finally,

\[
\mathbb{E} \left[ \sum_{k=1}^\infty e^{-qT_k} K_{J_k, J_{k+1}} \right] \leq \mathbb{E} \left[ \sum_{k=1}^\infty 1 \left( J_k = r \right) 1 \left( J_{k+1} = r \right) e^{-qT_k} K_{0,0} \right] + \mathbb{E} \left[ \sum_{k=1}^\infty 1 \left( J_k = r \right) 1 \left( J_{k+1} = r \right) e^{-qT_k} K_{0,0} \right] \\
\leq K_{i,0} + (K_{i,0} + K_{0,0}) \mathbb{E} \left[ \sum_{k=1}^\infty e^{-qT_k} \right] \\
\leq K_{i,0} + (K_{i,0} + K_{0,0}) h_0/q. \tag{3.4}
\]

so from (3.2), (3.3) and (3.4), the function \( V_i \) is bounded in \([l, b)\). With a similar proof it can be shown that \( V_0 \) is finite and so we have the result.

**Proposition 3.2** The optimal cost functions \( V_i \) are Lipschitz for \( i = 1, 2 \) in \([l, b)\)
The proof is given in the Appendix.

4 Hamilton-Jacobi-Bellman Equations

From the definitions (2.6) and (2.7), we can obtain recursive equations relating the optimal cost $V_0(b)$ and the optimal cost functions $V_i$ for $i = 1, 2$; these recursive equations will be used to find the Hamilton-Jacobi-Bellman equations of the optimization problem.

It follows immediately from (2.7) that

$$V_0(b) = \mathbb{E} \left[ \int_0^{t^*_1} e^{-q_0t}h_0(b)ds + 1\{t^*_1 \geq T_1\} e^{-q_0T_1} \overline{V}(b - Y^0_1) \right]$$

$$+ \mathbb{E} \left[ \int_0^{t^*_1} e^{-q_0t} \left( p_0(Y^0_1 - b + l) + \overline{V}(l) \right) ds \right]$$

$$= \frac{1}{q + \lambda_0} h_0(b) + \frac{\lambda_0}{q + \lambda_0} \int_b^h \overline{V}(b - \alpha)dF_0(\alpha)$$

$$+ \frac{\lambda_0}{q + \lambda_0} \left( \int_b^h p_0(\alpha - b + l)dF_0(\alpha) + \overline{V}(l)(1 - F_0(b - l)) \right).$$

(4.1)

where

$$\overline{V}(x) = \min \{ K_{01} + V_1(x), K_{02} + V_2(x) \}.$$  (4.2)

For $x \in [l, b)$, let us define

$$t^*_x := \min \{ t : x + \sigma_i t = b \} = \frac{b - x}{\sigma_i}.$$  (4.3)

Take \{i, j\} = \{1, 2\} and consider any switching time $T_1 \geq 0$ from phase $i$ to phase $j$ and $0 < h < t^*_i$. Define $\tau = \tau_1 \wedge T_1 \wedge h$ and

$$R_i(x, T_1, h) = \mathbb{E} \left[ \int_0^{T_1} e^{-q_0t} h_i(x + \sigma_i, t)e^{-q_0h} \right] +$$

$$+ \mathbb{E} \left[ \int_{t^*_1}^{T_1} e^{-q_0t} h_i(x + \sigma_i, t) ds \right]$$

$$+ \mathbb{E} \left[ \int_{t^*_1}^{T_1} 1\{t^*_1 \leq T_1 \wedge h\} e^{-q_0t} \overline{V}(x + \sigma_i t^*_1 - Y^1_1) \right]$$

$$+ \mathbb{E} \left[ \int_{t^*_1}^{T_1} 1\{t^*_1 \leq T_1 \wedge h\} \left( p_i \left( Y^1_1 - (x + \sigma_i t^*_1 - l) \right) + V_j(l) \right) \right]$$

$$+ \mathbb{E} \left[ \int_{T_1}^{t^*_1} e^{-q_0t} h_i(x + \sigma_i, t) ds \right]$$

$$+ \mathbb{E} \left[ \int_{t^*_1}^{T_1} e^{-q_0t} \overline{V}(x + \sigma_i, t) ds \right]$$

$$+ \mathbb{E} \left[ \int_{T_1}^{t^*_1} e^{-q_0t} \overline{V}(x + \sigma_i, t) ds \right] + K_{ij} e^{-qT_1}.$$  (4.4)

We obtain the following recursive equations

$$V_i(x) = \inf_{T_i \geq 0} R_i(x, T_1, h).$$  (4.5)

Let us define the operators,

$$\mathcal{L}_i(V_i)(x) := \sigma_i V'_i(x) - (\lambda_i + q)V_i(x) + \lambda_i \int_{-\infty}^{x-l} V_i(x - \alpha)dF_i(\alpha) + \lambda_i \int_{x-l}^{\infty} p_i(\alpha - x + l)dF_i(\alpha)$$

$$+ \lambda_i V_i(l)(1 - F_i(x - l)) + h_i(x).$$

(4.4)

for $i = 1, 2$. Then, the Hamilton-Jacobi-Bellman equations for $V_i$, are

$$\min \{ \mathcal{L}_i(V_i)(x), V_j(x) + K_{ij} - V_j(x) \} = 0,$$  (4.5)

for $x \in [l, b], \{i, j\} = \{1, 2\}$. Also, defining
\[
\mathcal{L}_0(V_0)(b) := -(q + \lambda_0)V_0(b) + \lambda_0 \left( \int_0^{b-l} V(b - \alpha) dF_0(\alpha) + \int_{l-b}^0 p_0(\alpha - b + l) dF_0(\alpha) \right) \\
+ \lambda_0 V(l)(1 - F_0(b - l) + h_0(b),
\]

we obtain from (4.1), that
\[
\mathcal{L}_0(V_0)(b) = 0.
\]

In this kind of optimal control problems, the optimal value function could be not smooth enough to satisfy the HJB equation in a classical sense, therefore a weak formulation of solutions to these equations is necessary. The viscosity solution framework is well suited for this task and it is a standard tool for studying HJB equations, see for instance Fleming and Soner (2006). Crandall and Lions (1983) introduced the concept of viscosity solutions for first-order Hamilton-Jacobi equations. We show for instance, in the first example in Section 7, that the optimal value function is indeed a viscosity solution of the HJB equation, but not a solution in a classical sense, see Remark 7.1-(1).

**Definition 4.1** A function \( u_i : [l, b] \to \mathbb{R} \) is a viscosity subsolution of \((4.5)\) at \( x \in [l, b) \) for \( \{i, j\} = \{1, 2\} \) if it is Lipschitz and any continuously differentiable function \( \psi_i : [l, b) \to \mathbb{R} \) with \( \psi_i(x) = u_i(x) \) such that \( u_i - \psi_i \) reaches the minimum at \( x \) satisfies

\[
\min \{ \mathcal{L}_i(\psi_i)(x), V_j(x) + K_{ij} - u_i(x) \} \leq 0.
\]

A function \( \tilde{u}_i : [l, b] \to \mathbb{R} \) is a viscosity supersolution of \((4.5)\) at \( x \in [l, b] \) for \( \{i, j\} = \{1, 2\} \) if it is Lipschitz and any continuously differentiable function \( \varphi_i : [l, b] \to \mathbb{R} \) with \( \varphi_i(x) = \tilde{u}_i(x) \) and such that \( \tilde{u}_i - \varphi_i \) reaches the maximum at \( x \) satisfies

\[
\min \{ \mathcal{L}_i(\varphi_i)(x), V_j(x) + K_{ij} - \tilde{u}_i(x) \} \geq 0.
\]

The functions \( \psi_i \) and \( \varphi_i \) are called test-functions for subsolution and supersolution respectively. If a function \( u_i \) is both a subsolution and a supersolution at \( x \) it is called a viscosity solution of \((4.5)\) at \( x \).

Crandall and Lions (1983) introduced the concept of viscosity solutions for first-order Hamilton-Jacobi equations. It is the standard tool for studying HJB equations, see for instance Fleming and Soner (2006).

**Proposition 4.2** The optimal cost functions \( V_i \) satisfy \((4.5)\) in a viscosity sense, for \( x \in [l, b) \) and \( i = 1, 2 \).

The proof is similar to the one of Proposition 3.2 of Azcue and Muler (2015).

In the following proposition, we prove that the optimal cost functions are the largest viscosity supersolutions of their corresponding HJB equations with suitable boundary conditions.

**Proposition 4.3** Fix \( x \in [l, b) \) and \( j = 1, 2 \) or \( x = b \) and \( j = 0 \). Let \( \tilde{u}_1 \) and \( \tilde{u}_2 \) be non-negative viscosity supersolution of the corresponding HJB Eq. \((4.5)\) in \([l, b)\) and consider any admissible strategy \( \pi = (T_k, J_k)_{k \geq 0} \in \Pi_{xj} \). Defining

\[
\bar{u}(x) = \min \{ K_{01} + \tilde{u}_1(x), K_{02} + \tilde{u}_2(x) \}
\]
and since $\mathcal{L}_0(\bar{u}_0)(b) = 0$,

$$\bar{u}_0(b) = \frac{\lambda_0}{q + \lambda_0} \left( \int_0^{b-l} \bar{u}(b - \alpha)dF_0(\alpha) + \int_{b-l}^\infty (p_0(\alpha - b + l) + \bar{u}(l))dF_0(\alpha) \right) + \frac{h_0(b)}{q + \lambda_0}.$$ 

If we assume that

$$\bar{u}_1(b) \leq \bar{u}_0(b) + K_{10}, \bar{u}_2(b) \leq \bar{u}_0(b) + K_{20},$$

then $\bar{u}_j(x) \leq V^e_i(x)$ for $j = 1, 2$ and $\bar{u}_0(b) \leq V^e_0(b)$.

The proof is given in the Appendix.

From Propositions 4.2 and 4.3, we obtain the following verification result.

**Theorem 4.4** Consider two families of admissible strategies $\{\pi_{x,i} \in \Pi_{x,i} : x \in [l, b]\}$ for $i = 1, 2$. Assume that the functions $w_i(x) := V^{x,i}(x)$ for $i = 1, 2$ are viscosity supersolutions of the corresponding HJB Eq. (4.5) for $x \in (l, b)$ and that satisfy the boundary conditions

$$w_1(b) \leq w_0(b) + K_{10}, w_2(b) \leq w_0(b) + K_{20},$$

where

$$w_0(b) = \frac{\lambda_0}{q + \lambda_0} \left( \int_0^{b-l} \bar{w}(b - \alpha)dF_0(\alpha) + \int_{b-l}^\infty (p_0(\alpha - b + l) + \bar{w}(l))dF_0(\alpha) \right) + \frac{h_0(b)}{q + \lambda_0} \text{ and}$$

$$\bar{w}(x) = \min\{K_{01} + w_1(x), K_{02} + w_2(x)\}.$$

Then, $w_0(b) = V_0(b)$ and $w_i = V_i$ for $i = 1, 2$.

In the remainder of the section, we show that there exists an optimal production-inventory strategy and it is stationary in the sense that depends only on the phase and the inventory level.

**Definition 4.5** Given two disjoint closed sets $A_{12}$ and $A_{21}$ in $[l, b)$ and a closed set $C_1$ in $[l, b)$ with $A_{21} \subset C_1$ and $A_{12} \subset [l, b) - C_1$, we define the production-inventory band strategy associated to the sets $(A_{12}, A_{21}, C_1)$ as follows:

1. If the current phase is $i = 1$ and the current inventory level is $x \in A_{12}$, change immediately to phase 2, if the current inventory level $x \in [l, b) - A_{12}$ stay in phase 1.
2. If the current phase is $i = 2$ and the current inventory level is $x \in A_{21}$, change immediately to phase 1, if the current inventory level $x \in [l, b) - A_{21}$ stay in phase 2.
3. If the current phase is $i = 0$ with current inventory level $b$, then in the event of an arrival of the next customer demand of size $Y$, switch on the production to phase 1 if $\max\{b - Y, l\} \in C_1$ and switch on the production to phase 2 if $\max\{b - Y, l\} \in [l, b) - C_1$.
4. If the inventory level reaches $b$, it is mandatory to switch to phase 0.

The sets $A_{ij}$ are called the switching zone from the phase $i$ to phase $j$, and the sets $C_1$ and $C_2 = [l, b) - C_1$ are called the selection zones for phases 1 and 2 respectively. Also, the set $[l, b) - (A_{12} \cup A_{21})$ is called the non-action zone. Note that the sets $A_{ij}$ could be empty (so, there is no switching from phase $i$ to phase $j$), and also the set $C_1$ could be empty or $[l, b)$.
Remark 4.6 Given the sets $A = (A_{12}, A_{21}, C_1)$, an initial inventory level $x$ and an initial phase $i$, we define and admissible strategy $\pi_{x,i} = (T_k, J_k)_{k \geq 0} \in \Pi_{x,i}$ where $J_0 = i$ and $T_k$ is the $k$-th switching (from phase $J_{k-1}$ to $J_k$) given by (1), (2), (3) and (4). Note that the switching times $T_k$ are the times in which the controlled inventory process in $[l, b]$ exit the sets $[l, b) - A_{12}$, $[l, b) - A_{21}$ or $\{b\}$. Let us denote the cost function of this admissible strategies as

$$W_i^A(x) = V_i^{\pi_{x,i}}(x) \text{ for } i = 1, 2 \text{ and } x \in [l, b);$$

and

$$W_0^A(b) = V_i^{\pi_{x,i}}(b).$$

We can characterize the triple $(W_0^A(b), W_1^A, W_2^A)$ as the unique fixed point of a contraction operator: Let $C[l, b)$ be the set of all the functions $W : [l, b) \to \mathbb{R}$ continuous and bounded and let consider the Banach space

$$\mathcal{B} = \mathbb{R} \times C[l, b) \times C[l, b)$$

with norm

$$\| (f_0, f_1, f_2) \| = \max\{ |f_0|, \sup_{x \in [l, b)} |f_1(x)|, \sup_{x \in [l, b)} |f_2(x)| \}. $$

We define, the operator $T^A : \mathcal{B} \to \mathcal{B}$ as

$$T^A(f_0, f_1, f_2) = (T_0^A(f_0, f_1, f_2), T_1^A(f_0, f_1, f_2), T_2^A(f_0, f_1, f_2)). \quad (4.8)$$

We define $T_0^A$ as

$$T_0^A(f_0, f_1, f_2) = \mathbb{E} \left[ \int_0^{\tau_1} e^{-q_s} h_0(b) ds \right] + \mathbb{E} \left[ \sum_{k=1}^{\infty} \mathbb{I}_{\{\tau_k \leq \tau_1\}} e^{-q_{\tau_k}} \left( \tau_k - Y_{\tau_k} \right) \right] + \mathbb{E} \left[ \int_0^{\tau_1} e^{-q_s} \left( p_0(Y_0 - b + l) + f(l) \right) \right],$$

where

$$\tau_1 := (f_1(x) + K_{01})1_{\{x \in C_1\}} + (f_2(x) + K_{02})1_{\{x \notin C_1\}}.$$ 

Here $(\tau_1, Y_1)$ is the time and size of the first customer demand. Let us define now $T_i^A$. Take the admissible strategy $\pi_{x,i}^A = (T_i, J_i)_{k \geq 0} \in \Pi_{x,i}$ as defined in Definition 4.5 and consider the associated controlled inventory process $X_i$ and the process $J_i$ defined in (2.5). Let us denote by $d$ the phase corresponding to the first demand, more precisely $d = J_{\tau_n}$ where

$$n = \max \left\{ m : T_m \leq \tau_n \right\} .$$

Hence,

$$T_i^A(f_0, f_1, f_2)(x) = \mathbb{E} \left[ \int_0^{\tau_d} e^{-q_s} h_{d}(X_s) ds \right] + \mathbb{E} \left[ \sum_{k=1}^{\infty} \mathbb{I}_{\{\tau_k \leq \tau_d\}} e^{-q_{\tau_k}} K_{d-1, \tau_k} \right] + \mathbb{E} \left[ \int_0^{\tau_d} e^{-q_s} \left( f_d(X_{\tau_d} - Y_{\tau_d}) \right) \right] + \mathbb{E} \left[ \int_0^{\tau_d} e^{-q_s} \left( p_d(l - X_{\tau_d} + Y_{\tau_d} + f_d(l) \right) \right].$$
for $x \in [l, b)$ and $i \in \{1, 2\}$. Note that

$$
\left| T^A_i(f_0, f_1, f_2) - T^A_i(g_0, g_1, g_2) \right| \leq \mathbb{E}\left( e^{-q t_i} \right) \left\| (f_0, f_1, f_2) - (g_0, g_1, g_2) \right\| \\
\leq \frac{\overline{\lambda}}{q + \overline{\lambda}} \left\| (f_0, f_1, f_2) - (g_0, g_1, g_2) \right\|
$$

where $\overline{\lambda} = \max_{i=0,1,2} \lambda_i$ and so $T^A : \mathcal{B} \to \mathcal{B}$ is a contraction operator with a unique fixed point.

Finally, by the definition of the production-inventory strategy associated to the sets $\mathcal{A}(A_{12}, A_{21}, C_1)$, it follows immediately that the triple $(W_0^A, W_1^A, W_2^A)$ is a fixed point of the operator $T^A$.

In the following theorem we prove that there exists an optimal strategy and that it comes from a production-inventory band strategy as defined in Definition 4.5.

**Theorem 4.7** The optimal strategy of problem (2.6) and (2.7), is the production-inventory strategy associated to the sets $A^* = (A^*_{12}, A^*_{21}, C^*_1)$ where

$$
A^*_{12} = \{ x \in [l, b) : V_1(x) + K_{12} - V_2(x) = 0 \}, \\
A^*_{21} = \{ x \in [l, b) : V_2(x) + K_{21} - V_1(x) = 0 \}, \\
C^*_1 = \{ x \in [l, b) : K_{01} + V_1(x) \leq K_{02} + V_2(x) \}.
$$

**Proof** By Remark 4.6, it is enough to prove that the triple $(V_0(b), V_1, V_2)$ is a fixed point of the operator $T^A$ for the sets $A^* = (A^*_{12}, A^*_{21}, C^*_1)$. By definition of the sets $A^*_j$ and $C^*_1$, we obtain immediately that $T^A_0(V_0(b), V_1, V_2) = V_0(b)$. Let us prove now that $T^A_i(V_0(b), V_1, V_2)(x) = V_i(x)$ for $x \in [l, b)$ and $i = 1, 2$. As before, let us denote by $d$ the phase corresponding to the first demand. Since $\mathcal{L}_0(V_0)(b) = 0$; and for $\{i,j\} = \{1,2\}$ the functions $t \to V_i(X_t)$ are absolutely continuous, $\mathcal{L}_i(V_i) = 0$ a.e. in $[l, b) - A^*_j$ and $V_j(x) + K_{ij} - V_i(x) = 0$ in $A^*_j$, we can prove, with arguments similar to the proof of Proposition 4.3, and using the martingales introduced in (8.10) that

Calling for simplicity $X_t$ as the controlled process $X_t^{e^{qT_i}}$, we can write,

$$
T^A_i(V_0, V_1, V_2)(x) - V_i(x) = \mathbb{E}\left[ \int_0^{t_i} e^{-q s} h_{\mathcal{B}}(X_s) ds \right] + \\
+ \mathbb{E}\left[ \sum_{k=1}^{\infty} \mathbb{I}_{t_i < \tau_k} e^{-q \tau_k} K_{h \tau_k - \tau_k} \right] \\
+ \mathbb{E}\left[ \mathbb{I}_{\{ x_{t_i} < \tau_{i+1} \}} e^{-q \tau_i} (V_{i+1}(X_{t_i} - Y_{t_i}^i)) \right] \\
+ \mathbb{E}\left[ \mathbb{I}_{\{ x_{t_i} < \tau_{i+1} \}} e^{-q \tau_i} (p_{a} (l - X_{t_i}^i + Y_{t_i}^i) + V_{a}(l)) \right] - V_i(x).
$$

Hence, $W_0^A(b) = V_0(b)$, $W_1^A = V_1$ and $W_2^A = V_2$. 

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5 Finite Band Strategies

We define the finite band strategies as the production-inventory band strategies in which the non-action set \([l, b) - (A_{12} \cup A_{21})\) has a finite number of connected components. We say that a production-inventory band strategy is of type \(n\) if the non-action set has \(n\) connected components.

Doshi et al. (1978) studied the production-inventory band strategies with switching zones \(A_{12} = [y_1, b)\) and \(A_{21} = [l, y_2]\) and selection zones \(C_1 = [l, y_2]\) and \(C_2 = (y_2, b)\) for \(l \leq y_2 < y_1 < b\); we denote these strategies as two-levels strategies of type one.

Assuming that the optimal strategy is a finite band strategy, we look for it in the following way:

**First step.** We find the best two-levels strategy of type one, that is we construct the cost functions \(W_0^A(b), W_1^A, W_2^A\) for \(A \in \{[y_1, b), [l, y_2], (y_2, b)\}\); then we minimize the \(W_0^A(b)\) among the two levels \(0 < y_2 < y_1 < b\). We check whether the associated cost functions \(W_0^A(b), W_1^A, W_2^A\) of this strategy satisfy the conditions of Theorem 4.4, if they do this is the optimal strategy; if this is not the case, we go to the second step.

**Second step.** We consider the three-levels strategies of type one. Here, the switching zones are of the form \(A_{12} = [y_1, b)\) and \(A_{21} = [l, y_2]\) and the selection zones are of the form \(C_1 = [l, y_3]\) and \(C_2 = (y_3, b)\) for \(l \leq y_2 \leq y_3 < y_1 < b\); the non-action zone is \((y_2, y_1)\). Then we minimize \(W_0^A(b)\) among the three variables \(y_2, y_3, y_1\). As before, we check whether the associated cost functions \(W_0^A(b), W_1^A, W_2^A\) of this strategy satisfy the conditions of Theorem 4.4, if they do this is the optimal strategy; if this is not the case, we go to the third step. Note that the two-levels strategies of type one are the band strategies of type one in which \(y_2 = y_3\).

**Third step.** We consider the band strategies of type two. Here, the switching zones are of the form \(A_{12} = [y_1, y_4]\) and \(A_{21} = [l, y_2]\) and the selection zones are of the form \(C_1 = [l, y_3]\) and \(C_2 = (y_3, b)\) for \(l \leq y_2 \leq y_3 < y_1 < y_4 < b\); in these band strategies, the non-action zone is \((y_2, y_4)\). Now, we minimize \(W_0^A(b)\) among the four variables \(y_2, y_3, y_1, y_4\). Again, we check whether the associated cost functions \(W_0^A(b), W_1^A, W_2^A\) of this strategy satisfy the conditions of Theorem 4.4, if they do this is the optimal strategy; if this is not the case, we consider band strategies where the non-action zone has more connected components. And so on.

In the next section, we describe how to find the cost functions of band strategies with one and two connected components using scale functions. We also show how to find the decomposition into the different types of costs: holding, production, switching and shortage costs.

In Section 7, we show examples where the optimal strategy is a two-levels strategy of type one (Fig. 1), is a three-levels strategy of type one (Fig. 6) and is a band strategy of type two (Fig. 11).

6 The Value Functions of Band Strategies

In this section we derive the cost functions for the band strategies of type one and two. Throughout this section we assume that \(l = 0\). We further assume that the holding cost per time unit in phase \(i\) when the inventory level is \(x\) is \(h_i(x) = a_i + c_i x\) for \(i = 1, 2\), where \(a_i, c_i \geq 0\) are given. To obtain the value function we apply the fluctuation theory for Lévy processes as described in Chapter 8 in Kyprianou (2014) and Avram et al. (2019).
6.1 Preliminaries

For $i = 1, 2$ let

$$X_{i,t} := x + \sigma_i t - \sum_{n=1}^{N_i} Y_{i,n}$$

be the uncontrolled process at phase $i$ with initial inventory level $x$. The processes $X_i$ are spectrally negative bounded variation Lévy processes. Let us define

$$\varphi_i(\theta) = \log \mathbb{E} \left[ e^{\theta (X_{i,1}-t)} \right] = \sigma_i \theta - \lambda_i + \lambda_i \mathcal{L}_Y(\theta),$$

where $\mathcal{L}_Y(\theta) := \mathbb{E}[e^{-\theta Y_1}]$. Let us also define the exit times $\tau_{i,a}^- = \inf \{ t : X_{i,t} < a \}$ and $\tau_{i,d}^+ = \inf \{ t : X_{i,t} = d \}$.

The following notations are used:

- $W_{i}^{(q)}(x)$ – the scale function associated with $X_i$. This function is defined by its Laplace transform

$$\int_0^\infty e^{-\theta x} W_{i}^{(q)}(x) dx = \frac{1}{q - \varphi_i(\theta)}.$$

$$Z_{i}^{(q)}(x, \theta) = e^{\theta x} \left( 1 + (q - \varphi_i(\theta)) \int_0^x e^{-\theta y} W_{i}^{(q)}(y) dy \right).$$

Denote $Z_{i}^{(q)}(x) = Z_{i}^{(q)}(x, 0) = 1 + q \int_0^x W_{i}^{(q)}(y) dy$.

- $W_{i}^{(q)}(x)$
- $\overline{W}_{i}^{(q)}(x)$
- $\overline{Z}_{i}^{(q)}(x)$

The following results are applied
\[ \mathbb{E}_A\left[ e^{-q\tau^+_i} 1_{\tau^+_i < \tau^-_i} \right] = \frac{W_i^q(x - a)}{W_i^q(d - a)}, \]  
\( (6.1) \)

\[ \mathbb{E}_A\left[ e^{-q\tau^+_i + \theta X_i \tau^-_i} 1_{\tau^-_i < \tau^+_i} \right] = Z_i^q(x, \theta) - \frac{W_i^q(x)}{W_i^q(d)} Z_i^q(d, \theta). \]  
\( (6.2) \)

\[ \mathbb{E}_A\left[ e^{-q\tau^-_i} 1_{\tau^-_i < \tau^+_i} \right] = Z_i^q(x) - \frac{W_i^q(x)}{W_i^q(d)} Z_i^q(d). \]  
\( (6.3) \)

1. For \( 0 < y < d \), let us define the \( q \)-potential measure of \( X_i \) as

\[ U_i^q(a, d, x, dy) = \int_0^\infty e^{-qt} \mathbb{P}_x \left[ X_{i,t} \in dy, \tau^-_i \wedge \tau^+_i > t \right] dt. \]  
\( (6.4) \)

2. By Theorem 8.7 in Kyprianou (2014), \( U_i^q(a, d, x, dy) = U_i^q(x, y)dy \), where

\[ U_i^q(a, d, x, y) = \frac{W_i^q(x - a)}{W_i^q(d - a)} - W_i^q(x - y). \]  
\( (6.5) \)

Throughout, we denote by \( \mathbb{E}^i \) the expectation according to the probability law \( \mathbb{P}^i \) induced by the process \( X_i \) for \( i = 1, 2 \).

### 6.2 Cost Functions for Strategies of Type One

As defined in the previous sections the switching zone are \( A_{12} = [y_1, b] \) and \( A_{21} = [0, y_2] \) and the selection zones are \( C_1 = [0, y_3] \) and \( C_2 = (y_3, b) \) for \( 0 \leq y_2 \leq y_3 < y_1 < b \); the non-action zone is \( (y_2, y_1) \). The value function is obtained in three steps, first we obtain the expected discounted holding cost, then the expected discounted shortage cost and finally the expected discounted switching cost.

#### 6.2.1 Expected Discounted Holding Cost

Here, we compute the formulas for

1. \( H_i^q(x) \)-- the expected discounted holding cost starting at \( x \) at phase \( i, i = 1, 2 \).
2. \( H_0^q(b) \)-- the expected discounted holding cost starting at \( b \).
3. \( H_1^q(x, y_1) \)-- the expected discounted holding cost until reaching \( y_1 \) starting at \( x \) at phase 1, \( 0 \leq x < y_1 \).
4. \( H_2^q(x, y_2, b) \)-- the expected discounted holding cost until reaching \( b \) or down-crossing \( y_2 \) starting at \( x \) at phase 2, \( y_2 < x < b \).
Let us define \( X_{1,t} = \inf \{ s \leq t, X_{1,s} \} \), \( L_t = -(X_{1,t} \wedge 0) \). Let \( R_t = X_{1,t} + L_t \). Let \( \kappa_+^* = \inf \{ t : R_t \geq y_1 \} \) be the first time that \( R \) reaches \( y_1 \). Notice that when the inventory is less than \( y_1 \) and the phase is 1, the inventory evolves as \( R \). By Theorem 8.1 (ii) in Kyprianou (2014),

\[
\mathbb{E}_x^1[e^{-q\kappa_+^*}] = \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}.
\]  

(6.6)

**Remark 6.1** The main tool to evaluate the expected discounted holding cost is the Kella–Whitt martingale, Kella and Whitt (1992): let \( X \) be a spectrally negative Lévy process with Laplace exponent \( \varphi(a) = \log \mathbb{E}[e^{a(X_t-X_0)}] \), \( Y_t \) an adapted process with bounded expected variation on finite intervals and \( V_t = X_t + Y_t \). Let \( \Delta Y_s = Y_s - Y_{s^-} \) and \( Y^c \) the continuous part of \( Y \), i.e. \( Y^c = Y_t - \sum_{0 \leq s \leq t} \Delta Y_s \). Then:

\[
M_t = \varphi(\alpha) \int_0^t e^{\alpha Y_s} ds + e^{\alpha Y_0} - e^{\alpha Y_t} + \alpha \int_0^t e^{\alpha Y_s} dY^c_s + \sum_{0 \leq s \leq t} e^{\alpha Y_s} (1 - e^{-\alpha \Delta Y_s})
\]  

is a zero mean martingale.

From the strong Markov property at \( y_1 \) and (6.6), it follows that for \( 0 < x < y_1 \),

\[
\mathcal{H}_1^{(q)}(x) = H_1^{(q)}(x, y_1) + \mathbb{E}_x^1[e^{-q\kappa_+^*}] \mathcal{H}_2^{(q)}(y_1) = H_1^{(q)}(x, y_1) + \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \mathcal{H}_2^{(q)}(y_1). 
\]  

(6.8)

Similarly, for \( y_2 < x < b \),

\[
\mathcal{H}_2^{(q)}(x) = H_2^{(q)}(x, y_2, b) + \mathbb{E}_x \left[ e^{-q \tau_{x,y_2}^+} 1_{\tau_{x,y_2}^+ < \tau_{x,b}^+} \mathcal{H}_1^{(q)}(X_{2,\tau_{x,y_2}^+}) \right] + \mathbb{E}_x \left[ e^{-q \tau_{x,b}^+} 1_{\tau_{x,b}^+ < \tau_{x,y_2}^+} \mathcal{H}_0^{(q)}(b) \right]. 
\]  

(6.9)

Denote by \( \text{Exp}(\lambda_0) \) an exponentially distributed random variable with rate \( \lambda_0 \),

\[
\mathcal{H}_0^{(q)}(b) = h_0(b) \mathbb{E} \left[ \int_0^{\text{Exp}(\lambda_0)} e^{-q t} dt \right] + \mathbb{E} \left[ e^{-q \text{Exp}(\lambda_0)} \left( \int_0^{b-y_3} \mathcal{H}_2^{(q)}(b-z) dF_0(z) + \int_{b-y_3}^{\infty} \mathcal{H}_1^{(q)}(b-z) dF_0(z) \right) \right].
\]  

(6.10)

**Remark 6.2** For \( x < 0 \), we define \( H_1^{(q)}(x, y_1) = H_1^{(q)}(0, y_1) \) and \( \mathcal{H}_1^{(q)}(x) = \mathcal{H}_1^{(q)}(0) \).
Next, we obtain $H_2^{(q)}(x, y_2, b)$.

$$H_2^{(q)}(x, y_2, b) = \mathbb{E}_x^2 \left[ \int_0^{\tau_{2,b}^+ \wedge \tau_{2,2}^+} e^{-qt} (a_2 + c_2 X_{2,t}) dt \right]$$

$$= \frac{a_2}{q} \left( 1 - \mathbb{E}_x^2 \left[ e^{-q(\tau_{2,b}^+ \wedge \tau_{2,2}^+)} \right] \right) + c_2 \mathbb{E}_x^2 \left[ \int_0^{\tau_{2,b}^+ \wedge \tau_{2,2}^+} e^{-qt} X_{2,t} dt \right].$$

Let

$$h_{2,1}(x, y_2, b) = \frac{1}{q} \left( 1 - \mathbb{E}_x^2 \left[ e^{-q(\tau_{2,b}^+ \wedge \tau_{2,2}^+)} \right] \right).$$

Applying (6.3) and (6.1) yields

$$h_{2,1}(x, y_2, b) = \frac{1}{q} \left( 1 - \mathbb{E}_x^2 \left[ e^{-q\tau_{2,2}^+} 1_{\tau_{2,2}^+ < \tau_{2,b}^+} \right] - \mathbb{E}_x^2 \left[ e^{-q(\tau_{2,b}^+ \wedge \tau_{2,2}^+)} \right] \right)$$

$$= \frac{1}{q} \left( 1 - Z_2^{(q)}(x - y_2) + \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} Z_2^{(q)}(b - y_2) - \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} \right).$$

In order to obtain $\mathbb{E}_x^2 \left[ \int_0^{\tau_{2,b}^+ \wedge \tau_{2,2}^+} e^{-qt} X_{2,t} dt \right]$, we apply Kella-Whitt martingale (6.7) for $X_{2,t}$, where $\varphi_2(\alpha) = \log \mathbb{E} [e^{\alpha (X_{2,t} - X_{1,t})}]$, and $Y_t = -qt/\alpha$, so

$$\mathbb{E}_x^2 \left[ (\varphi_2(\alpha) - q) \int_0^{\tau_{2,b}^+ \wedge \tau_{2,2}^+} e^{aX_{2,t} - qt} dt + e^{\alpha x} - e^{aX_{2,b}^+ \wedge \tau_{2,2}^+} - q \right] = 0.$$

Taking derivative of (6.14) with respect to $\alpha$ at $\alpha = 0$, we obtain

$$\varphi_2'(0) \mathbb{E}_x^2 \left[ \int_0^{\tau_{2,b}^+ \wedge \tau_{2,2}^+} e^{-qt} dt \right] + x - \frac{\partial}{\partial \alpha} \mathbb{E}_x^2 \left[ e^{aX_{2,\tau_{2,b}^+} \wedge \tau_{2,2}^+} - q \tau_{2,2}^+ \wedge \tau_{2,b}^+ \right]_{\alpha=0} = q \mathbb{E}_x^2 \left[ \int_0^{\tau_{2,b}^+ \wedge \tau_{2,2}^+} X_{2,t} e^{-qt} dt \right].$$

By (6.2), (6.14) and (6.15), we get

$$\mathbb{E}_x^2 \left[ e^{aX_{2,\tau_{2,2}^+} \wedge \tau_{2,b}^+} - q \tau_{2,2}^+ \wedge \tau_{2,b}^+ \right]$$

$$= \mathbb{E}_x^2 \left[ e^{a(b - y_2) - q \tau_{2,b}^+} 1_{\tau_{2,b}^+ < \tau_{2,2}^+} \right] + \mathbb{E}_x^2 \left[ e^{aX_{2,\tau_{2,2}^+} - q \tau_{2,2}^+} 1_{\tau_{2,2}^+ < \tau_{2,b}^+} \right]$$

$$= e^{ab} \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} + e^{ay_2} \left( Z_2^{(q)}(x - y_2, \alpha) - \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} Z_2^{(q)}(b - y_2, \alpha) \right).$$

Taking derivative of (6.16) with respect to $\alpha$ at $\alpha = 0$, as in (53) of Avram et al. (2019),
\[
\begin{align*}
    h_{2,2}(x,y_2,b) &= \frac{\partial}{\partial \alpha} \mathbb{E}_x^1\left[e^{\alpha X_1} e^{\alpha \tau_{1,Y_1} - \eta(\tau_{1,Y_1} - \tau_{1,Y_2})}\right]\bigg|_{\alpha = 0} = b \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} \\
    &+ y_2 \left(Z_2^{(q)}(x-y_2) - \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} Z_2^{(q)}(b-y_2)\right) \\
    &+ \frac{Z_2^{(q)}(x-y_2) - \phi'_2(0) W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} - \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} \left(Z_2^{(q)}(b-y_2) - \phi'_2(0) W_2^{(q)}(b-y_2)\right).
\end{align*}
\]

Combining (6.11), (6.13) and (6.17), we have

\[
H_2^{(q)}(x,y_2,b) = \left(a_2 + \frac{c_2 \phi'_2(0)}{q}\right) h_{2,1}(x,y_2,b) + \frac{c_2}{q} (x-h_{2,2}(x,y_2,b)).
\]

Next, we obtain \(H_1^{(q)}(x,y_1)\) for \(0 \leq x < y_1\) - the expected discounted holding cost starting at inventory level \(x\) at phase 1 until reaching \(y_1\):

\[
H_1^{(q)}(x,y_1) = a_1 \mathbb{E}_x^1\left[\int_0^{\kappa_{1,Y_1}} e^{-qs} ds\right] + c_1 \mathbb{E}_x^1\left[\int_0^{\kappa_{1,Y_1}} e^{-qs} L_s\right].
\]

By (6.6), the first term on the right-hand side of (6.19) is

\[
\mathbb{E}_x^1\left[\int_0^{\kappa_{1,Y_1}} e^{-qs} ds\right] = \frac{1}{q} \left(1 - \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}\right).
\]

To obtain the second term on the right-hand side of (6.19), we apply the Kella-Whitt martingale (6.7) for the process \(X_1\) with \(\varphi_1(\alpha) = \log \mathbb{E}[e^{\alpha(X_{1,1}-X_{1,0})}], Y_s = L_s - qs/\alpha\) and \(V_s = X_{1,s} + L_s - (q/\alpha)s = R_s - (q/\alpha)s\). Then

\[
\mathbb{E}_x^1\left[(\varphi_1(\alpha) - q) \int_0^{\kappa_{1,Y_1}} e^{\alpha R_s - qs} ds + e^{\alpha R_0} - e^{\alpha \tau_{1,Y_1} - q\kappa_{1,Y_1}}\right]
\]

\[+ \alpha \int_0^{\kappa_{1,Y_1}} e^{\alpha R_s - qs} dL_s + \sum_{0 \leq s \leq \kappa_{1,Y_1}} e^{\alpha R_s - qs} (1 - e^{-a\Delta L_s}) = 0.
\]

Note that \(R(\kappa_{1,Y_1}) = y_1\) and that \(dL_s \neq 0\) or \(\Delta L_s \neq 0\) implies that \(R_s = 0\). Thus (6.21) reduces to

\[
\mathbb{E}_x^1\left[(\varphi_1(\alpha) - q) \int_0^{\kappa_{1,Y_1}} e^{\alpha R_s - qs} ds + e^{\alpha \tau_{1,Y_1} - q\kappa_{1,Y_1}}\right]
\]

\[+ \alpha \int_0^{\kappa_{1,Y_1}} e^{-qs} dL_s + \sum_{0 \leq s \leq \kappa_{1,Y_1}} e^{-qs} (1 - e^{-a\Delta L_s}) = 0.
\]
Taking derivative of (6.22) with respect to \( x \) at \( \alpha = 0 \), and applying (6.6) yields:

\[
\varphi'(0) \int_0^{\kappa_1} e^{-qs} \, ds + x - y_1 \frac{Z^{(q)}_1(x)}{Z^{(q)}_1(y_1)} = q \mathbb{E}_x \left[ \int_0^{\kappa_1^+} R_x e^{-qs} \, ds \right].
\]  

(6.24)

Using (6.6) and (6.24) we obtain after some algebra that

\[
\mathbb{E}_x \left[ \int_0^{\kappa_1^+} R_x e^{-qs} \, ds \right] = \frac{Z^{(q)}_1(x)}{Z^{(q)}_1(y_1)} \overline{W}_1(y_1) - \overline{W}_1(x).
\]  

(6.25)

By (6.19), the expected discounted “fixed” part of the holding cost until reaching \( y_1 \) is given by

\[
a_1 \int_0^{\kappa_1^+} e^{-qs} \, dt = a_1 \left( 1 - \frac{Z^{(q)}_1(x)}{Z^{(q)}_1(y_1)} \right).
\]

Applying (6.19), (6.20) and (6.25), we get

\[
H_1^{(q)}(x, y_1) = a_1 \left( 1 - \frac{Z^{(q)}_1(x)}{Z^{(q)}_1(y_1)} \right) + c_1 \left( \frac{Z^{(q)}_1(x)}{Z^{(q)}_1(y_1)} \overline{W}_1(y_1) - \overline{W}_1(x) \right).
\]  

(6.26)

In order to obtain \( \mathcal{H}_2^{(q)}(x) \) –the expected discounted holding cost starting at inventory level \( x \) at phase 2-, we first obtain \( \mathbb{E}_x^2 \left[ e^{-qs_{12}} 1_{\tau_{22}^{(q)} < \tau_0} H_2^{(q)}(X_{2, \tau_{22}^{(q)}}) \right] \).

For a function \( g \) satisfying the conditions of Theorem 2 in Loeffen (2018), let us define

\[
\Omega^2(g(x)) = \mathbb{E}_x^2 \left[ e^{-qs_{12}} g(X_{2, \tau_{22}^{(q)}}) 1_{\tau_{22}^{(q)} < \tau_{21}^{(q)}} \right].
\]  

(6.27)

Then, by the aforementioned Theorem 2,
\[ \Omega^2(g(x)) = g(x) - \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} g(b) \]
\[ + \int_{y_2}^{b} (G_2 - q)g(z) \left( \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} W_2^{(q)}(b - z) - W_2^{(q)}(x - z) \right) dz, \]  
(6.28)

where for \( i = 1, 2 \), \( G_i \) is the infinitesimal generator of \( X_i \), i.e.

\[ G_i g(x) = \sigma_i g'(x) + \lambda_i \int_0^{\infty} (g(x) - g(y))dF_i(y). \]

It can be shown that

\[ G_2 g(x) = \sigma_2 g'(x) - \lambda_2 g' * \overline{F}_2(x), \]  
(6.29)

where \( \overline{F}_2(x) = 1 - F_2(x) \) and \( * \) is the convolution operator.

**Remark 6.3** It is well known that

\[ (G_1 - q)Z_1^{(q)}(x) = 0 \]  
(6.30)

and as a result

\[ (G_1 - q)\overline{W}_1^{(q)}(x) = x. \]  
(6.31)

therefore, when \( F_1 = F_2 \)

\[ (G_2 - q)\overline{Z}_1^{(q)}(x) = (\sigma_2 - \sigma_1)qW_1^{(q)}(x) \]

and

\[ (G_2 - q)\overline{W}_1^{(q)}(x) = x + (\sigma_2 - \sigma_1)\overline{W}_1^{(q)}(x). \]

Substituting Eq. (6.8) in (6.26) using (6.3), (6.26), (6.9) yield

\[ \mathcal{H}_2^{(q)}(x) = H_2(x, y_2, b) \]
\[ + \frac{a_1}{q} \left( \frac{Z_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} - \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} Z_2^{(q)}(b - y_2) - \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)} \right) \]
\[ + c_1 \left( \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)} \overline{W}_1^{(q)}(y_1) - \Omega^2(\overline{W}_1^{(q)}(x)) \right) \]
\[ + \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)} \mathcal{H}_2^{(q)}(y_1) + \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} \mathcal{H}_0^{(q)}(b). \]  
(6.32)

Let
\[ A(x) := H_2^{(q)}(x, y_2, b) \]
\[ + \frac{a_1}{q} \left( Z_2^{(q)}(x - y_2) - \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} Z_2^{(q)}(b - y_2) - \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)} \right) \]  
\[ + c_1 \left( \Omega^2(Z_1^{(q)}(x)) \frac{\Omega^2(Z_1^{(q)}(y_1))}{W_1^{(q)}(y_1)} - \Omega^2(W_1^{(q)}(x)) \right), \tag{6.33} \]

then

\[ \mathcal{H}_2^{(q)}(x) = A(x) + \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)} \mathcal{H}_2^{(q)}(y_1) + \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} \mathcal{H}_0^{(q)}(b). \tag{6.34} \]

Substituting \( x = y_1 \) in (6.34) and solving for \( \mathcal{H}_2^{(q)}(y_1) \) yield:

\[ \mathcal{H}_2^{(q)}(y_1) = \frac{A(y_1) + \frac{W_2^{(q)}(y_1 - y_2)}{W_2^{(q)}(b - y_2)} \mathcal{H}_0^{(q)}(b)}{1 - \frac{\Omega^2(Z_1^{(q)}(y_1))}{Z_1^{(q)}(y_1)}}. \tag{6.35} \]

Equations (6.32)-(6.35) yield that

\[ \mathcal{H}_2^{(q)}(x) = \alpha_2(x) \mathcal{H}_0^{(q)}(b) + \beta_2(x), \tag{6.36} \]

where

\[ \alpha_2(x) = \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} + \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1) - \Omega^2(Z_1^{(q)}(y_1))} \frac{W_2^{(q)}(y_1 - y_2)}{W_2^{(q)}(b - y_2)} \tag{6.37} \]

\[ \beta_2(x) = A(x) + \frac{\Omega^2(Z_1^{(q)}(y_1)) A(y_1)}{Z_1^{(q)}(y_1) - \Omega^2(Z_1^{(q)}(y_1))}. \tag{6.38} \]

Substituting (6.36) in (6.8), we get

\[ \mathcal{H}_1^{(q)}(x) = H_1^{(q)}(x, y_1) + \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \alpha_2(y_1) \mathcal{H}_0^{(q)}(b) + \beta_2(y_1) \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)}, \tag{6.39} \]

\[ = \alpha_1(x) \mathcal{H}_0^{(q)}(b) + \beta_1(x), \]

where

\[ \alpha_1(x) = \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \alpha_2(y_1), \tag{6.40} \]

\[ \beta_1(x) = H_1^{(q)}(x, y_1) + \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \beta_2(y_1). \]
In order to obtain $\mathcal{H}^{(q)}_i$ for $i = 1, 2$, we substitute (6.40) and (6.36) in (6.10) and get the following linear equation for $\mathcal{H}^{(q)}_0(b)$.

$$\mathcal{H}^{(q)}_0(b) = \frac{h_0(b)}{q + \lambda_0} + \frac{\lambda_0}{\lambda_0 + q} \left( \int_0^{b-y_3} \mathcal{H}^{(q)}_2(b-z) dF_0(z) + \int_{b-y_3}^{\infty} \mathcal{H}^{(q)}_1(b-z) dF_0(z) \right)$$

$$= \frac{h_0(b)}{q + \lambda_0} + \frac{\lambda_0}{\lambda_0 + q} \left( \int_0^{b-y_3} \beta_2(b-z) dF_0(z) + \int_{b-y_3}^{\infty} \beta_1(b-z) dF_0(z) \right)$$

$$+ \mathcal{H}^{(q)}_0(b) \left( \int_0^{b-y_3} \alpha_2(b-z) dF_0(z) + \int_{b-y_3}^{\infty} \alpha_1(b-z) dF_0(z) \right).$$

(6.41)

We obtain $\mathcal{H}^{(q)}_0(b)$ solving the linear Eq. (6.41); from this, we get $\mathcal{H}^{(q)}_i$ for $i = 1, 2$.

### 6.2.2 Expected Discounted Shortage Cost

Here, we derive formulas for the expected discounted shortage cost $S^{(q)}_i(x)$ starting at inventory level $x$ at phase $i$ for $i = 1, 2$ together with the expected discounted shortage cost $S^{(q)}_0(b)$ starting at inventory level $b$.

Let us define $S^{(q)}_1(x, y_1)$ as the expected discounted shortage cost starting at inventory level $x \in [0, y_1)$ until the inventory level reaches $y_1$. By (6.6), we can write

$$S^{(q)}_1(x) = S^{(q)}_1(x, y_1) + \mathbb{E}^{1}_{x}[e^{-S^{(q)}_1(y_1)}]S^{(q)}_2(y_1) = S^{(q)}_1(x, y_1) + \frac{Z^{(q)}_1(x)}{Z^{(q)}_1(y_1)}S^{(q)}_2(y_1).$$

(6.42)

Equations (6.5) and (6.1) yield

$$S^{(q)}_2(x) = \int_{y_2}^{b} u_2^{(q)}(y_2, b, x, z) \lambda_2 \left( \int_{v=z}^{\infty} (p_2(v-z) + S^{(q)}_1(0)) dF_2(v) \right) dz$$

$$+ \int_{y_2}^{b} u_2^{(q)}(y_2, b, x, z) \lambda_2 \left( \int_{v=z-y_2}^{\infty} S^{(q)}_1(z-v) dF_2(v) \right) dz$$

$$+ \frac{W_2^{(q)}(x-y_2)}{W_2^{(q)}(b-y_2)} S^{(q)}_0(b).$$

(6.43)

And also,

$$S^{(q)}_0(b) = \frac{\lambda_0}{\lambda_0 + q} \left( \int_{b}^{\infty} (p_0(z-b) + S^{(q)}_1(0)) dF_0(z) \right)$$

$$+ \int_{0}^{b-y_3} S^{(q)}_2(b-z) dF_0(z) + \int_{b-y_3}^{b} S^{(q)}_1(b-z) dF_0(z).$$

(6.44)

First we consider $S^{(q)}_1(x, y_1)$ which corresponds to the expected discounted shortage cost starting at inventory level $x$ at phase $1$, $0 < x < y_1$ until the process reaches $y_1$. The definition of $u_1^{(q)}$ given in (6.5) yields

$$S^{(q)}_1(x, y_1) = \int_{0}^{y_1} u_1^{(q)}(0, y_1, x, z) \lambda_1 \left( \int_{v=z}^{\infty} p_1(v-z) dF_1(v) \right) dz$$

(6.45)
where (6.45) describes the expected discounted shortage cost occurring before the inventory level reaches $y_1$, and (6.46) describes the expected discounted shortage costs occurring after the first downcrossing level 0. Applying Eq. (6.3) yields:

$$
S_{1}^{(q)}(x, y_1) = \int_{0}^{y_1} u_1^{(q)}(0, y_1, x, z)\lambda_1 \left( \int_{v=z}^{\infty} p_1(v-z)dF_1(v) \right)dz
$$

$$
Z_{1}^{(q)}(x) = \frac{w_{1}^{(q)}(y_1)}{w_{1}^{(q)}(y_1)} \int_{0}^{y_1} u_1^{(q)}(0, y_1, 0, z)\lambda_1 \left( \int_{v=z}^{\infty} p_1(v-z)dF_1(v) \right)dz.
$$

(6.47)

Next, we introduce linear equations to obtain $S_{i}^{(q)}(x)$, $i = 0, 1, 2$. Let us define

$$
\mu(x) := \int_{y_2}^{b} u_2^{(q)}(y_2, b, x, z)\lambda_2 \left( \int_{v=z}^{\infty} p_2(v-z)dF_2(z) \right)dz
$$

$$
+ \lambda_2 S_{1}^{(q)}(0, y_1) \int_{y_2}^{b} u_2^{(q)}(y_2, b, x, z)\bar{F}_2(z)dz
$$

$$
+ \int_{y_2}^{b} u_2^{(q)}(y_2, b, x, z)\lambda_2 \left( \int_{v=-y_2}^{-z} S_{1}^{(q)}(z-v, y_1)dF_2(v) \right)dz,
$$

(6.48)

where $\bar{F}(z) = 1 - F(z)$. Let us also define

$$
\gamma(x) := \lambda_2 \frac{1}{Z_{1}^{(q)}(y_1)} \int_{y_2}^{b} u_2^{(q)}(y_2, b, x, z)\bar{F}_2(z)dz
$$

$$
+ \lambda_2 \int_{y_2}^{b} u_2^{(q)}(y_2, b, x, z) \left( \int_{v=-y_2}^{-z} \frac{Z_{1}^{(q)}(z-v)}{Z_{1}^{(q)}(y_1)}dF_2(z) \right)dz.
$$

(6.49)

Substituting (6.42) in (6.43), we have that $S_{2}^{(q)}(x)$ can be written as follows:

$$
S_{2}^{(q)}(x) = \mu(x) + \gamma(x)S_{2}^{(q)}(y_1) + \frac{w_{2}^{(q)}(x-y_2)}{w_{2}^{(q)}(b-y_2)} S_{0}^{(q)}(b).
$$

(6.50)

Solving (6.50) for $x = y_1$ yield

$$
S_{2}^{(q)}(y_1) = \frac{\mu(y_1) + \frac{w_{2}^{(q)}(y_1-y_2)}{w_{2}^{(q)}(b-y_2)} S_{0}^{(q)}(b)}{1 - \gamma(y_1)}.
$$

(6.51)
\[ \mu_2(x) := \mu(x) + \gamma(x) \frac{\mu(y_1)}{1 - \gamma(y_1)}, \]

\[ \gamma_2(x) := \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} + \frac{W_2^{(q)}(y_1 - y_2)}{W_2^{(q)}(b - y_2)} \frac{\gamma(x)}{1 - \gamma(y_1)}, \]

and

\[ \mu_1(x) := S_1^{(q)}(x, y_1) + \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \mu_2(y_1), \]

\[ \gamma_1(x) := \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \gamma_2(y_1). \]

Thus, Eqs. (6.43), (6.42), (6.50) and (6.51) yield:

\[ S_i^{(q)}(x) = \mu_i(x) + S_0^{(q)}(b) \gamma_i(x) \] for \( i = 1, 2. \) (6.53)

Substituting (6.53) in (6.44), we get the following linear equation for \( S_0(b), \)

\[ S_0^{(q)}(b) = \frac{\lambda_0}{\lambda_0 + q} \left( \int_b^{\infty} (p_0(z - b) + \mu_1(0) + \gamma_1(0)S_0^{(q)}(b))dF_0(z) \right. \]

\[ + \int_0^{b-y_3} (\mu_2(b - z) + S_0^{(q)}(b)\gamma_2(b - z))dF_0(z) + \left. \int_{b-y_3}^b (\mu_1(b - z) + \gamma_1(b - z)S_0^{(q)}(b))dF_0(z) \right), \]

and so

\[ S_0^{(q)}(b) = \frac{\lambda_0}{\lambda_0 + q} \left( \gamma_1(0)\bar{F}_0(b) + \int_0^{b-y_3} \gamma_2(b - z)dF_0(z) + \int_{b-y_3}^b \gamma_1(b - z)dF_0(z) \right) \]

\[ \int_0^{b-y_3} \gamma_2(b - z)dF_0(z) + \int_{b-y_3}^b \gamma_1(b - z)dF_0(z) \right). \] (6.54)

Finally, from (6.53), we get the formulas for \( S_i^{(q)}(x) \) for \( i = 1, 2. \)

### 6.2.3 Expected Discounted Switching Cost

Here, we compute the formulas for the expected discounted switching cost. Let \( K_i^{(q)}(x), \)

\( i = 1, 2 \) be the expected discounted switching cost starting at inventory level \( x \) and phase \( i \) for \( i = 1, 2 \) and let \( i = 1, 2 \) be the expected discounted switching cost starting at \( b. \) Assume that initially the inventory level \( x \) is at phase 1, then the first switching from phase 1 to phase 2 occurs at \( \kappa_1^{+}. \) By (6.6),

\[ K_1^{(q)}(x) = \mathbb{E}_1[e^{-\alpha x_1}] \left( K_{12} + K_2^{(q)}(y_1) \right) = \frac{Z_1^{(q)}(x)}{Z_1^{(q)}(y_1)} \left( K_{12} + K_2^{(q)}(y_1) \right). \] (6.55)

If initially the inventory level is \( x \) at phase 2, then the first switching from phase 2 to phase 1 occurs when the inventory level downcrosses \( y_2 \) before reaching \( b. \) If the inventory reaches \( b \)

before downcrossing \( y_2, \) there is a switching from phase 2 to phase 0. By (6.1),
\begin{align}
K_2^{(q)}(x) &= \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} \left( K_{20} + K_0^{(q)}(b) \right) + \mathbb{E}_x \left[ e^{-\eta \tau_{x,y_2}^+} \left( 1_{\tau_{x,y_2}^+ < \tau_y^+} (K_{21} + K_1^{(q)}(X_2,\tau_{x,y_2}^+)) \right) \right]. 
\end{align} 

Substituting (6.55) in (6.56) and using (6.3) and (6.27) yield that
\begin{align}
K_2^{(q)}(x) &= \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} \left( K_{20} + K_0^{(q)}(b) \right) + \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)} \left( K_{12} + K_2^{(q)}(y_1) \right) \\
&+ K_{21} \left( Z_2^{(q)}(x - y_2) - \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} Z_2^{(q)}(b - y_2) \right).
\end{align} 

Substituting \( x \) by \( y_1 \) in (6.57) and solving for \( K_2^{(q)}(y_1) \) yields:
\begin{align}
K_2^{(q)}(y_1) &= \frac{1}{1 - \frac{\Omega^2(Z_2^{(q)}(y_1))}{Z_2^{(q)}(y_1)}} \left( K_{20} \frac{W_2^{(q)}(y_1 - y_2)}{W_2^{(q)}(b - y_2)} + K_{12} \frac{\Omega^2(Z_1^{(q)}(y_1))}{Z_1^{(q)}(y_1)} \right) \\
&+ K_{21} \left( Z_2^{(q)}(y_1 - y_2) - \frac{W_2^{(q)}(y_1 - y_2)}{W_2^{(q)}(b - y_2)} Z_2^{(q)}(b - y_2) \right) + \frac{W_2^{(q)}(y_1 - y_2)}{W_2^{(q)}(b - y_2)} K_0^{(q)}(b). 
\end{align} 

Thus
\begin{align}
K_2^{(q)}(x) &= \omega_2(x) + \delta_2(x) K_0^{(q)}(b), 
\end{align} 

where,
\begin{align}
\omega_2(x) &= K_{20} \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} + K_{21} \left( Z_2^{(q)}(x - y_2) - \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} Z_2^{(q)}(b - y_2) \right) \\
&+ \frac{\Omega^2(Z_1^{(q)}(x))}{Z_1^{(q)}(y_1)} \left( K_{12} \frac{W_2^{(q)}(y_1 - y_2)}{W_2^{(q)}(b - y_2)} + K_{21} \left( Z_2^{(q)}(y_1 - y_2) - \frac{W_2^{(q)}(y_1 - y_2)}{W_2^{(q)}(b - y_2)} Z_2^{(q)}(b - y_2) \right) \right) \\
&+ \frac{1 - \frac{\Omega^2(Z_2^{(q)}(y_1))}{Z_2^{(q)}(y_1)}}{Z_2^{(q)}(y_1)} 
\end{align} 

and
\begin{align}
\delta_2(x) &= \frac{W_2^{(q)}(x - y_2)}{W_2^{(q)}(b - y_2)} + \frac{\Omega^2(Z_2^{(q)}(x)) W_2^{(q)}(y_1 - y_2)}{Z_2^{(q)}(y_1) W_2^{(q)}(b - y_2)} + \frac{1 - \frac{\Omega^2(Z_2^{(q)}(y_1))}{Z_2^{(q)}(y_1)}}{Z_2^{(q)}(y_1)}.
\end{align} 

By (6.55),
\[ K^{(q)}_{1}(x) = \omega_{1}(x) + \delta_{1}(x)K^{(q)}_{0}(b), \quad (6.62) \]

where

\[ \omega_{1}(x) := \frac{Z^{(q)}_{1}(x)}{Z^{(q)}_{1}(y_{1})}(K_{12} + \omega_{2}(y_{1})) \]

and

\[ \delta_{1}(x) := \frac{Z^{(q)}_{1}(x)}{Z^{(q)}_{1}(y_{1})}\delta_{2}(y_{1}). \quad (6.63) \]

Moreover, \( K^{(q)}_{0}(b) \) satisfies the following linear equation:

\[
\begin{align*}
K^{(q)}_{0}(b) &= \frac{\lambda_{0}}{q + \lambda_{0}} \left( \int_{0}^{b-y_{3}} (K_{02} + K^{(q)}_{2}(b - z))dF_{0}(z) \\
&+ \int_{b-y_{3}}^{b} (K_{01} + K^{(q)}_{1}(b - z))dF_{0}(z) + \int_{b}^{\infty} (K_{01} + K^{(q)}_{1}(0))dF_{0}(z) \right) \\
&= \frac{\lambda_{0}}{q + \lambda_{0}} \left( \int_{0}^{b-y_{3}} (K_{02} + \omega_{2}(b - z) + \delta_{2}(b - z)K^{(q)}_{0}(b))dF_{0}(z) \\
&+ \int_{b-y_{3}}^{b} (K_{01} + \omega_{1}(b - z) + \delta_{1}(b - z)K^{(q)}_{0}(b))dF_{0}(z) \\
&+ \int_{b}^{\infty} (K_{01} + \omega_{1}(0) + \delta_{1}(0)K^{(q)}_{0}(b))dF_{0}(z) \right),
\end{align*}
\]

thus

\[
K^{(q)}_{0}(b) = \frac{\lambda_{0}}{q + \lambda_{0}} \left( \int_{0}^{b-y_{3}} (K_{02} + \omega_{2}(b - z))dF(z) + \int_{b-y_{3}}^{b} (K_{01} + \omega_{1}(b - z))dF_{0}(z) + (K_{01} + \omega_{1}(0))F(b) \right) \\
1 \frac{\lambda_{0}}{q + \lambda_{0}} \left( \int_{0}^{b-y_{3}} \delta_{2}(b - z)dF_{0}(z) + \int_{b-y_{3}}^{b} \delta_{1}(b - z)dF_{0}(z) + \delta_{1}(0)F_{0}(b) \right). 
\]

\[ (6.65) \]

### 6.2.4 Total Discounted Cost

As a summary, we have that the total discounted cost starting at inventory level \( x \) and phase \( i \) is:

| Inventory level | Phase | Expected discounted cost |
|-----------------|-------|--------------------------|
| \( 0 \leq x < y_{1} \) | 1 | \( \mathcal{H}^{(q)}_{1}(x) + \mathcal{S}^{(q)}_{1}(x) + K^{(q)}_{1}(x) \) |
| \( 0 \leq x \leq y_{2} \) | 2 | \( \mathcal{H}^{(q)}_{1}(x) + \mathcal{S}^{(q)}_{1}(x) + K^{(q)}_{1}(x) + K_{21} \) |
| \( y_{2} < x < b \) | 2 | \( \mathcal{H}^{(q)}_{2}(x) + \mathcal{S}^{(q)}_{2}(x) + K^{(q)}_{2}(x) \) |
| \( y_{1} \leq x < b \) | 1 | \( \mathcal{H}^{(q)}_{2}(x) + \mathcal{S}^{(q)}_{2}(x) + K^{(q)}_{2}(x) + K_{12} \) |
| \( x = b \) | 0 | \( \mathcal{H}^{(q)}_{0}(b) + \mathcal{S}^{(q)}_{0}(b) + K^{(q)}_{0}(b) \) |
6.3 Cost Functions for Strategies of Type Two

Here the switching zone from 1 to 2 is \( A_{12} = [y_1, y_4] \), the switching zone from 2 to 1 is \( A_{21} = [0, y_2] \), the selection zones are \( C_1 = [0, y_3] \) and \( C_2 = (y_3, b) \) and the non-action zone \( (y_2, y_1) \cup (y_4, b) \) for \( 0 \leq y_2 \leq y_3 < y_1 < y_4 < b \).

The analysis of the value function in this case is very similar to the analysis of the strategy of type one. The only difference is in the case when initially the inventory level is \( x \in (y_4, b) \) at phase 1. Thus we consider only this case.

Let us start with \( H^{(q)}_1(x, y_4) \) –the expected discounted holding cost in the case \( y_4 < x < b \). Consider the expected discounted holding until reaching \( b \) or down-crossing \( y_4 \).

\[
H^{(q)}_1(x, y_4, b) = \mathbb{E}_x \left[ \int_0^{\tau^+_{1,b} \lor \tau^+_{1,2}} e^{-qs}(a_1 + c_1 X_{1,s}) ds \right]. \tag{6.66}
\]

Similarly to Eqs. (6.11)-(6.18), we have

\[
H^{(q)}_1(x, y_4, b) = \left( a_1 + \frac{c_1 \varphi_1'(0)}{q} \right) h_{1,1}(x, y_4, b) + \frac{c_1}{q} (x - h_{1,2}(x, y_4, b)), \tag{6.67}
\]

where

\[
h_{1,1}(x, y_4, b) = \frac{1}{q} \left( 1 - \mathbb{E}_x \left[ e^{-q \tau_{1,y_4}} 1_{\tau_{1,y_4} < \tau^+_{1,2}} \right] \right) - \mathbb{E}_x \left[ e^{-q \tau^+_{1,b} 1_{\tau^+_{1,b} < \tau^+_{1,2}}} \right]
\]

\[
= \frac{1}{q} \left( 1 - Z^{(q)}_1(x - y_4) + \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)} Z^{(q)}_1(b - y_4) - \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)} \right) \tag{6.68}
\]

and

\[
h_{1,2}(x, y_4, b) = \frac{\partial}{\partial a} \mathbb{E}_x \left[ e^{a(X_{1,\tau^+_{1,y_4}} \lor \tau^+_{1,2}) - q(\tau_{1,y_4} \lor \tau^+_{1,2})} \right] \bigg|_{a = 0} = b \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)}
\]

\[
+ y_4 \left( Z^{(q)}_1(x - y_4) - \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)} Z^{(q)}_1(b - y_4) \right) \tag{6.69}
\]

\[
+ \bar{Z}^{(q)}_1(x - y_4) - \varphi_1'(0) \bar{W}^{(q)}_1(x - y_4)
\]

\[
- \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)} \left( Z^{(q)}_1(b - y_4) - \varphi_1'(0) \bar{W}^{(q)}_1(b - y_4) \right).
\]

Once the inventory level reaches \( b \), the expected discounted holding cost is \( H^{(q)}_0(b) \). In the case that the inventory level down-crosses \( y_4 \) before reaching \( b \) there are two scenarios:

1. If \( X_{1,\tau^+_{1,y_4}} \) lies in \([y_1, y_4]\), then the expected discounted holding cost is \( H^{(q)}_2\left( X_{1,\tau^+_{1,y_4}} \right) \).
2. If the inventory level immediately after the jump is \( X_{1,\tau^+_{1,y_4}} \) lies in \((\infty, y_1)\), then the expected discounted holding cost is \( H^{(q)}_3\left( X_{1,\tau^+_{1,y_4}} \right) \). Thus,
\[
\tilde{F}_1^{(q)}(x) = H_1^{(q)}(x, y_4, b) + \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)} \cdot \mathcal{H}_0^{(q)}(b)
\]
\[
+ \lambda_1 \int_{y_4}^b u_1^{(q)}(y_4, b, x, y) \left( \int_{\gamma - y_1}^{y - y_1} \mathcal{H}_2^{(q)}(y - z) dF_1(z) \right) dy
\]
\[
+ \mathcal{K}_1^{(q)}(x) = \mathcal{K}_1^{(q)}(x) = \mathcal{K}_1^{(q)}(x) = \mathcal{K}_1^{(q)}(x) = \mathcal{K}_1^{(q)}(x)
\]
\[
\tilde{S}_1^{(q)}(x) = \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)} \tilde{S}_0^{(q)}(b)
\]
\[
+ \lambda_1 \int_{y_4}^b u_1^{(q)}(y_4, b, x, y) \int_{\gamma - y_1}^{y - y_1} \tilde{S}_2^{(q)}(y - z) dF_1(z) dy
\]
\[
+ \lambda_1 \int_{y_4}^b u_1^{(q)}(y_4, b, x, y) \int_{\gamma - y_1}^{y - y_1} \tilde{S}_1^{(q)}(y - z) dF_1(z) dy
\]
\[
+ \lambda_1 \int_{y_4}^b u_1^{(q)}(y_4, b, x, y) \int_{\gamma - y_1}^{y - y_1} p_1(z - y) dF_1(z) dy
\]
\[
+ \mathcal{S}_1^{(q)}(0) \lambda_1 \int_{y_4}^b u_1^{(q)}(y_4, b, x, y) \tilde{F}_1(y) dy.
\]

Consider now \( \tilde{S}_1^{(q)}(x) \) – the expected discounted shortage cost starting at \( x \in (y_4, b) \) at phase 1. If the process reaches \( b \) before down-crossing \( y_4 \), then the expected discounted shortage cost is \( \tilde{S}_2^{(q)}(x) \) in the case that \( y_1 \leq X_{1, r_{1, y_4}} \leq y_4 \); \( \tilde{S}_1^{(q)}(x) \) in the case that \( 0 \leq X_{1, r_{1, y_4}} < y_1 \), and is \( p(-X_{1, r_{1, y_4}} + \tilde{S}_1^{(q)}(0) \) in the case that \( X_{1, r_{1, y_4}} < 0 \). Applying (6.1) and (6.5) yields:

\[
\tilde{S}_1^{(q)}(x) = \frac{W_1^{(q)}(x - y_4)}{W_1^{(q)}(b - y_4)} \tilde{S}_0^{(q)}(b)
\]
\[
+ \lambda_1 \int_{y_4}^b u_1^{(q)}(y_4, b, x, y) \int_{\gamma - y_1}^{y - y_1} \tilde{S}_2^{(q)}(y - z) dF_1(z) dy
\]
\[
+ \lambda_1 \int_{y_4}^b u_1^{(q)}(y_4, b, x, y) \int_{\gamma - y_1}^{y - y_1} \tilde{S}_1^{(q)}(y - z) dF_1(z) dy
\]
\[
+ \lambda_1 \int_{y_4}^b u_1^{(q)}(y_4, b, x, y) \int_{\gamma - y_1}^{y - y_1} p_1(z - y) dF_1(z) dy
\]
\[
+ \mathcal{S}_1^{(q)}(0) \lambda_1 \int_{y_4}^b u_1^{(q)}(y_4, b, x, y) \tilde{F}_1(y) dy.
\]
7 Examples

In this section, we find the optimal strategies for three different situations. In the first one, the optimal strategy is a two-levels strategy of type one, in the second is a three-levels strategy of type one and in the third is of type two.

7.1 First Example: A Two-Levels Strategy of Type One is Optimal

In this example we consider two equal manufacturing units, in phase 1 both units are producing together, in phase 2 only one manufacturing unit is working and in phase 0 none of the units are working. We assume that the cost of shutdown each unit is equal to 1, the cost of restarting each unit is equal to 2 and the rate of production of each unit is equal to 3/2. The cost rate of production of each unit is 1/100, and there is also a fix cost rate (independent of the production) equal to 1/1000. Moreover, the holding cost rate is equal to 3/2. The cost rate of production of each unit is 1/100, and the storage capacity is b = 10. So we have the following parameters: σ₁ = 3, σ₂ = 3/2, K₁₂ = 1, K₂₁ = 2, K₂₀ = 2, K₀₂ = 4, K₀₁ = 4, h₂(x) = (21 + x)/1000, h₁(x) = (41 + x)/1000, h₀(b) = (1 + b)/1000. We assume that the rate of arrival of the customer demands are A₀ = λ₁ = λ₂ = 2, and the demands are distributed as Exp(1.5), the discount rate is q = 0.1 and the shortage cost when an amount y of a customer is lost is given by p₀(y) = p₁(y) = p₂(y) = (80 + 40y)/100 (here we are taking l = 0).

We find that the best two-levels strategy of type one is given by the sets A₁₂ = [y₁, 10), A₂₁ = [0, y₂], C₁ = [0, y₂] and C₂ = (y₂, 10) with y₂ = 1.526 and y₁ = 5.077. We check that the value functions of this strategy are viscosity solutions of the Eq. (4.5) and satisfy the conditions of Theorem 4.4; so the best two-levels strategy of type one is the optimal one. We show this optimal strategy en Fig. 1.

In Fig. 2, we show the discounted total cost V₁(x) (dotted), V₂(x) (dashed) and V₀(b) (solid point) of the optimal strategy; in Fig. 3, the discounted holding cost H₁(q)(x) (dotted), H₂(q)(x) (dashed) and H₀(q)(b) (solid point) of the optimal strategy; in Fig. 4, the discounted shortage cost S₁(q)(x) (dotted), S₂(q)(x) (dashed) and S₀(q)(b) (solid point) of the optimal strategy; and finally in Fig. 5, the discounted switching cost K₁(q)(x) (dotted), K₂(q)(x) (dashed) and K₀(q)(b) (solid point) of the optimal strategy.

Remark 7.1 (1) V₂ − V₁ is equal to K₁₁ in A₁₁ and equal to −K₁₂ in A₁₂; also V₂(b) − V₀(b) = K₂₀ and V₁(b) − V₀(b) = K₁₂ + K₁₀. V₂ is not differentiable at y₂, and so it is necessary to use the notion of viscosity solution.

(2) H₁(q)(x) = H₂(q)(x) in the switching zones A₁₂ ∪ A₁₁, H₂(q)(x) is not continuous at the boundary y₂ between the switching zone A₁₁ and the non-action zone (y₂, y₁). The jump of H₂(q)(x) at y₂ is downward because, for an initial inventory level x in the non-action zone, Xₙ > X₁₁ for t > 0 while these processes remain in the non-action zone. Also note that H₂(q)(b) = H₁(q)(b) = H₀(q)(b).

(3) S₂(q)(x) = S₁(q)(x) in the switching zones A₁₂ ∪ A₁₁. As in the previous case and for similar reasons, S₂(q) has a discontinuity at y₂, but in this case the jump is upward. Also note that S₂(q)(b) = S₁(q)(b) = S₀(q)(b).

(4) K₂(q) − K₁(q) is equal to K₁₁ in A₁₁ and equal to −K₁₂ in A₁₂; also K₂(q)(b) = K₂₀ and K₁(q)(b) = K₁₂ + K₁₀. K₂(q)(b) has a downward jump at y₂ because this point is the boundary between the switching zone A₁₁ and the non-action zone (y₂, y₁).
7.2 Second Example: A Three-Levels Strategy of Type One is Optimal

In this example, we consider that the demands in all the phases are distributed as $\text{Exp}(1)$, the parameters are $q = 0.1$, $\lambda_0 = \lambda_1 = \lambda_2 = 2$, $l = 0$, $b = 20$, the rates of production are $\sigma_1 = 2.5$, $\sigma_2 = 2.2$, and the costs are given by $K_{12} = K_{21} = 0.05$, $K_{20} = 1/200$, $K_{10} = 11/2000$, $K_{01} = K_{02} = 0$, $h_2(x) = (20 + x)/1000$, $h_1(x) = (30 + x)/1000$, $h_0(b) = (2 + 10b)/10000$, $p_0(y) = p_1(y) = p_2(y) = (80 + 40y)/100$.

In this case, the value functions of the best two-levels strategy of type one do not satisfy the condition of Theorem 4.4, so we look for the best three-levels strategies of type one, which is given by the sets $A_{12} = [y_1, 20)$, $A_{21} = [0, y_2)$, $C_1 = [0, y_3]$ and $C_2 = (y_3, 20)$ for $y_2 = 6.213$, $y_3 = 9.805$ and $y_1 = 17.294$. The value functions of this strategy are viscosity solutions of the Eq. (4.5) and satisfy the conditions of Theorem 4.4, so this is the optimal strategy. We show this optimal strategy in Fig. 6.

In Fig. 7, we show the discounted total cost $V_1(x)$ (dotted), $V_2(x)$ (dashed) and $V_0(b)$ (solid point) of the optimal strategy; in Fig. 8, the discounted holding cost $\mathcal{H}_1^{(q)}(x)$ (dotted), $\mathcal{H}_2^{(q)}(x)$ (dashed) and $\mathcal{H}_0^{(q)}(b)$ (solid point) of the optimal strategy; in Fig. 9, the discounted shortage cost $\mathcal{S}_1^{(q)}(x)$ (dotted), $\mathcal{S}_2^{(q)}(x)$ (dashed) and $\mathcal{S}_0^{(q)}(b)$ (solid point) of the optimal strategy; and finally in Fig. 10, the discounted switching cost $\mathcal{K}_1^{(q)}(x)$ (dotted), $\mathcal{K}_2^{(q)}(x)$ (dashed) and $\mathcal{K}_0^{(q)}(b)$ (solid point) of the optimal strategy.

The observations of Remark 7.1 hold for this example.

7.3 Third Example: Strategy of Type Two is Optimal

In this last example, we consider that the demands are distributed as $\text{Exp}(1)$ in all the phases, the parameters are $q = 0.1$, $\lambda_i = 2$ for $i = 0, 1, 2$, $l = 0$, $b = 10$, the rates of...
production are $\sigma_1 = 3.5$, $\sigma_2 = 2.5$, and the costs are given by $K_{12} = K_{21} = 0.05$, $K_{20} = 0$, $K_{10} = 1/100$, $K_{01} = K_{02} = 0$, $h_1(x) = h_2(x) = (1 + 12x)/100$, $h_0(b) = (1 + 10b)/100$, $p_i(y) = 2 + 1.1y$ for $i = 0, 1, 2$.

Fig. 3  First example, holding cost

Fig. 4  First example, shortage cost
In this case, the value functions of the best strategy of type one do not satisfy the condition of Theorem 4.4, so we look for the best band strategies of type two, which is given by the sets \( A_{12} = [y_1, y_4] \), \( A_{21} = [0, y_2] \), \( C_1 = [0, y_3] \) and \( C_2 = (y_3, 10) \) for \( y_2 = 2.468 \), \( y_3 = 3.114 \), \( y_1 = 4.610 \), \( y_4 = 7.660 \). The value functions of this strategy are viscosity solutions of the Eq. (4.5) and satisfy the conditions of Theorem 4.4, so this is the optimal strategy. We show this optimal strategy in Fig. 11.

In Fig. 12, we show the discounted total cost \( V_1(x) \) (dotted), \( V_2(x) \) (dashed) and \( V_0(b) \) (solid point) of the optimal strategy; in Fig. 13, the discounted holding cost \( H_1(q)(x) \) (dotted), \( H_2(q)(x) \) (dashed) and \( H_0(q)(b) \) (solid point) of the optimal strategy; in Fig. 14, the discounted...
shortage cost $S_1^{(q)}(x)$ (dotted), $S_2^{(q)}(x)$ (dashed) and $S_0^{(q)}(b)$ (solid point) of the optimal strategy; and finally in Fig. 15, the discounted switching cost $K_1^{(q)}(x)$ (dotted), $K_2^{(q)}(x)$ (dashed) and $K_0^{(q)}(b)$ (solid point) of the optimal strategy.

Fig. 7 Second example, total cost

Fig. 8 Second example, holding cost
The observations (1), (2) and (3) of Remark 7.1 also hold for this example. In this case $V_1$ is not differentiable at $y_4$. Also note, that $K^{(q)}_2 - K^{(q)}_1$ is equal to $K_{21}$ in $A_{21}$ and equal...
to $-K_{12}$ in $A_{12}$; also $K_2^{(q)}(b^-) - K_0^{(q)}(b) = K_{20}$ and $K_1^{(q)}(b^-) - K_0^{(q)}(b) = K_{10}$ because $(y_4, b)$ is the second component of the non-action zone. As in the previous examples, $K_2^{(q)}$ has a downward jump at $y_2$ and, in this case, $K_1^{(q)}$ has a downward jump at $y_4$ because this point is the boundary between the switching zone $A_{12} = [y_1, y_4]$ and the non-action zone $(y_4, b)$.

Fig. 11 Optimal strategy in third example

Fig. 12 Third example, total cost
Fig. 13  Third example, holding cost

Fig. 14  Third example, shortage cost
Appendix

Proof of Proposition 3.2

We call $S_i$ the positive upper bounds of the functions $V_i$ for $i = 1, 2$. Given initial inventory level $x \in [l, b)$ and initial phase $i = 1, 2$, take $\delta \in (0, b - x]$ and consider an admissible strategy $\pi_{x+\delta} \in \Pi_{x+\delta, i}$ such that $V_i^{\pi_{x+\delta}}(x + \delta) \leq V_i(x + \delta) + \epsilon$, where $0 < \epsilon < \delta$. Let us now define the admissible strategy $\pi_x \in \Pi_{x, i}$ as follows: stay in phase $i$ until the controlled inventory level $X_i^{\pi_x}$ reaches $x + \delta$ and then follow $\pi_{x+\delta} \in \Pi_{x+\delta, i}$. Then, from (3.1) and Proposition 3.1, we get

$$V_i(x) \leq V_i^{\pi_x}(x) \leq \int_0^{\tau_{i+1}} e^{-q_j t} f_i(x + \sigma_j t) dt + \mathbb{P}[\tau_{i+1} > \frac{\delta}{\sigma_i}] e^{-q_i \frac{\delta}{\sigma_i}} V_i^{\pi_{x+\delta}}(x + \delta) + \mathbb{P}[\tau_{i+1} \leq \frac{\delta}{\sigma_i}] S_i$$

$$\leq \tilde{h} \frac{\delta}{\sigma_i} + e^{-\left(\frac{\lambda_i}{\sigma_i} + q_i\right) \frac{\delta}{\sigma_i}} \left(V_i(x + \delta) + \epsilon\right) + (1 - e^{-\frac{\lambda_i}{\sigma_i} \frac{\delta}{\sigma_i}}) S_i.$$

Hence, we have

$$V_i(x) - V_i(x + \delta) \leq \tilde{h} \frac{\delta}{\sigma_i} + e^{-\left(\frac{\lambda_i}{\sigma_i} + q_i\right) \frac{\delta}{\sigma_i}} \left(V_i(x + \delta) + \epsilon\right) - V_i(x + \delta) + (1 - e^{-\frac{\lambda_i}{\sigma_i} \frac{\delta}{\sigma_i}}) S_i$$

$$\leq \tilde{h} \frac{\delta}{\sigma_i} + \epsilon + \frac{\lambda_i}{\sigma_i} \frac{\delta}{\sigma_i} S_i.$$

So, taking
we obtain

\[ V_i(x) - V_i(x + \delta) \leq m^1_i \delta. \quad (8.1) \]

Let us prove now that there exists \( m^2_i > 0 \) such that,

\[ V_i(x + \delta) - V_i(x) \leq m^2_i \delta. \quad (8.2) \]

We start showing that there exists \( m \) such that,

\[ V_i(y) - V_i(l) \leq m \delta \quad (8.3) \]

for all \( y \in [l, l + \delta] \). Given \( \epsilon > 0 \) and an initial inventory level \( l \), consider the strategy \( \pi_i \in \Pi_{l,i} \) for \( i = 1, 2 \) such that \( V_i^{\pi_i}(l) \leq V_i(l) + \epsilon \) and call \( X_i^{\pi_i} \) the associated process with initial inventory level \( l \). Take also a strategy \( \pi_j \in \Pi_{b,0} \) such that \( V_0^{\pi_j}(b) \leq V_0(b) + \epsilon \).

Let us define the admissible strategy \( \pi_y \in \Pi_{y,i} \) for initial inventory level \( y \in [l, l + \delta] \) as:

- For \( 0 \leq t \leq T \), follow \( \pi_i \) (and so the associated controlled processes \( X_i^{\pi_i} = X_i^{\pi_i} + (y - l) \) for \( t < T \), where \( T := \min\{ t : X_i^{\pi_i} = b \) or \( X_i - (y - l) = X_i < l \} \).

- If \( X_i^{\pi_i} = b \), follow \( \pi_i \) for \( t \geq T \).
- If \( X_i \leq l \) (and so \( X_i^{\pi_i} = X_i^{\pi_i} = l \), follow \( \pi_i \) for \( t \geq T \).
- If \( l \leq X_i^{\pi_i} < y \) (and so \( X_i^{\pi_i} = l \) and \( X_i^{\pi_i} = X_i^{\pi_i} \), also follow the strategy \( \pi_i \) for \( t \geq T \).

Given any stopping time \( \tau \), let us define \( \tilde{V}_i^{\pi_i}(y, \tau) \) as the expected discounted cost of the strategy before \( \tau \) and \( \tilde{V}_i^{\pi_i}(y, \tau) \) as the expected discounted cost of the strategy after \( \tau \). Thus,

\[ V_i(y) - V_i(l) - \epsilon \leq \tilde{V}_i^{\pi_i}(y) - V_i^{\pi_i}(l) \leq \mathbb{E} \left[ \int_0^T e^{-\delta t} \left( h_{\gamma_i} (X_i^{\pi_i} + (y - l)) - h_{\gamma_i}(X_i^{\pi_i}) \right) dt \right] + \]

\[ + \mathbb{E} \left[ \left\{ X_i^{\pi_i} = b \right\} \left( e^{-\delta T} \left( K_{\gamma_i,0} + V_0(b) + \epsilon \right) - \tilde{V}_i^{\pi_i}(l, T) \right) \right] + \]

\[ + \mathbb{E} \left[ \left\{ X_i^{\pi_i} < l \right\} e^{-\delta T} \left( V_{X_i^{\pi_i}}(l + \epsilon + p_{X_i^{\pi_i}}(l - X_i^{\pi_i})) - (V_{X_i^{\pi_i}}(l) + p_{X_i^{\pi_i}}(l - X_i^{\pi_i} + y - l)) \right) \right] \]

\[ + \mathbb{E} \left[ \left\{ i \leq X_i^{\pi_i} < y \right\} e^{-\delta T} \left( V_{X_i^{\pi_i}}(X_i^{\pi_i}) - V_{X_i^{\pi_i}}(l) + 2\epsilon \right) \right]. \]

Let \( n_D \) be the sum of the numbers of discontinuities of \( h_1 \) and \( h_2 \). Note that between two customer demands, the inventory level \( X_i^{\pi_i} \) goes through at most \( n_D \) points of discontinuities of \( h_{\gamma_i} \). Hence, calling \( \tau_0 = 0 \) and \( \lambda = \max_{i=0,1,2} \lambda_i \), we have
\[
\mathbb{E} \left[ \int_0^T e^{-qt} \left( h_{\mathcal{D}}(X^T_i + (y - l)) - h_{\mathcal{D}}(X^T_i) \right) dt \right] \leq \left( \frac{m_\mathcal{D}}{q} + n_\mathcal{D} \frac{\delta}{\sigma_2} \right) \left( 1 + \frac{\lambda}{q} \right) \delta. \tag{8.4}
\]

Let us call \( \tilde{T} := \inf \{ t : X^T_i = b \} \) and \( \tau \) the time of the first customer demand after \( T \); we have that \( \mathbb{P}[\tilde{T} > \tau] \leq 1 - e^{-\frac{\lambda \tau}{\sigma_2}} \) and so

\[
\begin{align*}
\mathbb{E} \left[ 1 \{ X^T_i = b \} \left( e^{-qT} (K_{\mathcal{D},0} + V_0(b)) - \widetilde{\nu}^{x_i}_i(l, T) \right) \right] & \leq \left( 1 - e^{-\frac{\lambda \tau}{\sigma_2}} \right) (V_0(b) + (K_{1,0} \lor K_{2,0})) \\
& + \mathbb{E} \left[ 1 \{ X^T_i = b, \tilde{T} < \tau \} e^{-qT} (K_{\mathcal{D},0} + V_0(b)) - \widetilde{\nu}^{x_i}_i(l, T) \right] \\
& \leq \frac{\lambda \tau}{\sigma_2} (V_0(b) + \max \{ K_{1,0}, K_{2,0} \}) \\
& + \mathbb{E} \left[ 1 \{ X^T_i = b, \tilde{T} < \tau \} e^{-qT} (K_{\mathcal{D},0} + V_0(b)) - \widetilde{\nu}^{x_i}_i(l, T) \right]. \tag{8.5}
\end{align*}
\]

Let \( \Delta \) be the length of time after \( T \) in which the process \( X^T_i \) reaches \( b \) in the event of no arrivals of demands. In this case, we have

\[
X^T_{t+\Delta} = b - (y - l) + \int_T^{t+\Delta} e^{-qs} \sigma_{\mathcal{D}} ds = b
\]

and so \( \frac{\delta}{\sigma_2} \leq \Delta \leq \frac{\delta}{\sigma_2} \). Hence, from (2.3),

\[
\begin{align*}
\mathbb{E} \left[ 1 \{ X^T_i = b \} 1 \{ \tilde{T} < \tau \} \widetilde{\nu}^{x_i}_i(l, T) \right] & \geq \mathbb{P} \left[ \text{no demands in } t \in [T, T + \frac{\delta}{\sigma_2}] \right] \mathbb{E} [e^{-q(t+\Delta)} (K_{\mathcal{D},0} + V_0(b))] \\
& \geq e^{-q(t+\Delta)} \frac{\delta}{\sigma_2} \mathbb{E} [e^{-qT} (K_{\mathcal{D},0} + V_0(b))].
\end{align*}
\]

Therefore,

\[
\begin{align*}
\mathbb{E} \left[ 1 \{ X^T_i = b \} 1 \{ \tilde{T} < \tau \} e^{-qT} (K_{\mathcal{D},0} + V_0(b)) - \widetilde{\nu}^{x_i}_i(l, T) \right] & \leq \left( 1 - e^{-q(t+\Delta)} \frac{\delta}{\sigma_2} \right) ((K_{1,0} \lor K_{2,0}) + V_0(b)) \\
& \leq \frac{q(t+\Delta)}{\sigma_2} ((K_{1,0} \lor K_{2,0}) + V_0(b)) \delta. \tag{8.6}
\end{align*}
\]

Since the penalty functions \( p_i \) are non-decreasing, we also have,

\[
\mathbb{E} \left[ 1 \{ X^T_i < y \} e^{-qT} \left( p_{\mathcal{D},0} (l - X^T_i) - p_{\mathcal{D},0} (l - X^T_i + y - l) \right) \right] \leq 0. \tag{8.7}
\]

Finally, since the event \( l \leq X^T_i < y \) coincides with the arrival of a customer demand,
$$\mathbb{E} \left[ 1 \{ t X_i^x < \} e^{-qT} \left( V_{J_{T_i}}^x (X_{T_i}^x) - V_{J_{T_i}} (l) \right) \right] = \mathbb{E} \left[ 1 \{ t X_i^x < \} 1 \{ t = t_{n+1} \text{ for some } n \} e^{-qT} \left( V_{J_{T_i}}^x (X_{T_i}^x) - V_{J_{T_i}} (l) \right) \right] \leq \mathbb{E} e^{-qT} \max_{z \in [l, l+\delta]} \left( V_{J_{T_i}}^x (z) - V_{J_{T_i}} (l) \right) \leq \frac{q}{q + \lambda} \max_{z \in [l, l+\delta]} \left( V_{J_{T_i}}^x (z) - V_{J_{T_i}} (l) \right) \leq \bar{m} \delta. \tag{8.8}$$

Hence, from (8.5), (8.6), (8.7) and (8.8), there exists $\bar{m}$ large enough such that

$$\frac{q}{q + \lambda} \max_{z \in [l, l+\delta]} \left( V_{J_{T_i}}^x (z) - V_{J_{T_i}} (l) \right) \leq \bar{m} \delta.$$ 

So, we obtain (8.3) with $m = \bar{m}(q + \lambda)/q$. The argument to show (8.2) is analogous.

**Proof of Proposition 4.3**

Consider $\pi \in \Pi_{i,j}$. Let us extend $\bar{u}_i$ and $\bar{u}_j$ as $\bar{u}_i(x) = \bar{u}_i(l)$ and $\bar{u}_j(x) = \bar{u}_j(l)$ for $x < l$. Consider the controlled risk process $X_i^x$ starting at $x$ and the function $J_i$ defined in (2.5). Since $\bar{u}_i$ is Lipschitz for $i = 1, 2$, we obtain that the function $t \to e^{-qt} \bar{u}_i(X_i^x)$ is absolutely continuous in between the stopping times $(0) \cup \{ \tau_n : n \geq 1 \} \cup \{ T_k : k \geq 1 \}$. So, taking

$$m_i := \max \{ k : T_k \leq t \},$$

we have

$$\bar{u}_i(X_i^x)e^{-qT} - \bar{u}_i(x) = \sum_{k=0}^{m_i-1} \left( \bar{u}_i(X_{T_{k+1}}^x)e^{-qT_{k+1}} - \bar{u}_i(X_{T_k}^x)e^{-qT_k} \right) + \left( \bar{u}_i(X_i^x)e^{-qT} - \bar{u}(X_i^x) e^{-qT} \right). \tag{8.9}$$

Let us define

$$M^i(z_0, t_0, t) = \bar{u}_i(Z_t^i) e^{-qT} - \bar{u}_i(z_0) e^{-qT_0} + \sum_{n=N_0}^{N_i} e^{-qT_n - Y_n} \nu_n \left( Z_{t_n}^i - Y_{t_n}^i \right) + \int_{t_0}^{t} e^{-qs} \left( \pi_j \nu_j(Z_s^j) - (q + \lambda_j) \nu_j(Z_s^j) + \lambda_j \int_{Z_s^{-1}}^Z \nu_j(Z_s^j - \alpha) dF_t(\alpha) \right) ds,$$

with

$$Z_t^i = z_0 + \sigma_i(t - t_0) - \sum_{n=N_0}^{N_i} \min \{ Y_n^i, Z_t^{i_n} - l \} \text{ for } t \geq t_0 \geq 0,$$

it can be seen that $M^i(z_0, t_0, t)$ is a martingale with zero expectation for $t \geq t_0$.

Consider first the case $J_k = i$ and $J_{k+1} = j$ with $i = 1, 2$, $j = 0, 1, 2$ and $i \neq j$. Since $\bar{u}_i$ is absolutely continuous, the function $t \to \bar{u}_i(X_i^x)e^{-qt}$ is also absolutely continuous, between customer demands. Using an extension of the Dynkin’s Formula, we obtain
and so, since \( \tilde{u} \) is a supersolution of (4.5), we get that

\[
\mathbb{E} \left[ \tilde{u}(X_{T_{k+1}}^\pi) e^{-qT_{k+1}} - \tilde{u}(X_{T_k}^\pi) e^{-qT_k} \middle| F_{T_k} \right] \geq - \mathbb{E} \left[ K_0 e^{-qT_{k+1}} + \int_{T_k}^{T_{k+1}} e^{-qT_h} P_l(l - X_{T_k}^\pi + Y_n^i) 1_{\{X_{T_k}^\pi - Y_n^i < 0\}} \right] \mathcal{F}_{T_k}.
\]

In the case \( J_k = 0 \) we have \( J_{T_{k+1}} \neq 0 \), then \( X_{T_k} = b \) in \( \{T_k, T_{k+1}\} \), \( T_{k+1} = t_n^0 \) for some \( n \) and so, analogously to the previous case,

\[
\mathbb{E} \left[ \tilde{u}(X_{T_{k+1}}^\pi) e^{-qT_{k+1}} - \tilde{u}_0(X_{T_k}^\pi) e^{-qT_k} \middle| F_{T_k} \right] = e^{-qT_k} \left( \tilde{u}(b - Y_n^0) 1_{\{b - Y_n^0 \geq 0\}} + \tilde{u}_{J_{T_{k+1}}} (l) 1_{\{b - Y_n^0 < 0\}} e^{-q(T_{k+1} - T_k)} \right) \]
\[
\quad - e^{-qT_k} \left( K_0 e^{-qT_{k+1}} + \int_{T_k}^{T_{k+1}} e^{-qT_h} P_l(l - b + Y_n^0) 1_{\{b - Y_n^0 < 0\}} \right) \mathcal{F}_{T_k}.
\]

and, since \( T_{k+1} - T_k \) is distributed as \( \exp(\lambda_0) \), we obtain that

\[
0 = \mathbb{E} \left[ e^{-qT_k} \left( \tilde{u}(b - Y_n^0) 1_{\{b - Y_n^0 \geq 0\}} + \tilde{u}_{J_{T_{k+1}}} (l) 1_{\{b - Y_n^0 < 0\}} \right) \right] \mathcal{F}_{T_k}
\]

\[
+ \mathbb{E} \left[ e^{-qT_{k+1}} P_l(l - b + Y_n^0) 1_{\{b - Y_n^0 < 0\}} \right] \mathcal{F}_{T_k}
\]

and so

\[
\mathbb{E} \left[ \tilde{u}(X_{T_{k+1}}^\pi) e^{-qT_{k+1}} - \tilde{u}_0(X_{T_k}^\pi) e^{-qT_k} \middle| F_{T_k} \right] = - \mathbb{E} \left[ K_0 e^{-qT_{k+1}} + \int_{T_k}^{T_{k+1}} e^{-qT_h} P_l(l - b + Y_n^0) 1_{\{b - Y_n^0 < 0\}} \right] \mathcal{F}_{T_k}.
\]

Analogously, we can prove that
Taking the expected value in (8.9), we obtain
\[
\mathbb{E}\left[\overline{u}_{J_m} (X^\pi_t) e^{-qt} - \overline{u}_{j_m} (X^\pi_{T_m}) e^{-qT_m} \mid \mathcal{F}_{T_m}\right]
\geq -\mathbb{E}\left[\int_{T_m}^t e^{-q\tau} h_m (X^\pi_s) \, ds + \sum_{n=T_m}^{N_m} e^{-q\tau_m} p_{J_m} (l - X^\pi_{\tau_m} + Y^f_{J_m}) 1_{X^\pi_{\tau_m} - Y^f_{J_m} - l < 0} \mid \mathcal{F}_{T_m}\right].
\]

Taking the expected value in (8.9), we obtain
\[
\mathbb{E}\left[\overline{u}_{J} (X^\pi_t) e^{-qt} - \overline{u}_j (x)\right] = \mathbb{E}\left[\sum_{k=0}^{m_j-1} \mathbb{E}\left[\left(\overline{u}_{J_{k+1}} (X^\pi_{T_{k+1}}) e^{-qT_{k+1}} - \overline{u}_{J_k} (X^\pi_{T_k}) e^{-qT_k}\right) \mid \mathcal{F}_{T_k}\right]\right]
\geq -V^\pi_j (x)
\]

Taking the limit with \(t\) going to infinity, and using that \(X^\pi_t \in [l, b]\) we obtain that
\[
\overline{u}_j (x) \leq V^\pi_j (x)
\]
for \(j = 1, 2\).

Considering instead the controlled risk process \(X^\pi_t\) starting at \(b\), we obtain with a similar proof that
\[
\overline{u}_0 (b) \leq V^\pi_0 (b).
\]

Declarations

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