I. INTRODUCTION

Stochastic thermodynamics has proposed a general and unified scheme for addressing central issues in thermodynamics [1–5]. It includes not only an extension of concepts from equilibrium to nonequilibrium systems but also it deals with the existence of new definitions and bounds [6–9], general considerations about the efficiency of engines at finite time operations [1–3] and others as aspects. In all cases, the concept of entropy production [1, 2] plays a central role, being a quantity continuously produced in nonequilibrium steady states (NESS), whose main properties and features have been extensively studied in the last years, including its usage for typifying phase transitions [11, 12].

Basically, a NESS can be generated under two fundamental ways: from fixed thermodynamic forces [13, 14] or from time-periodic variation of external parameters [15, 16]. In this contribution, we address a different kind of periodic driving, suitable for the description of engineered reservoirs, at which a system interacts sequentially and repeatedly with different (and independent) set of interactions. Exact expressions for thermodynamic properties are derived and the entropy production presents a bilinear form, in which Onsager coefficients are obtained as function of period. Considerations about the efficiency are undertaken and a suited regime for the system operating as an efficient thermal machine is investigated.

The present study sheds light for fresh perspectives in nonequilibrium thermodynamics, including the possibility of experimental implementations of heat engines based on collisional models [21–23], with sequential reservoirs. Also, they provide us the extension and validation of recent bounds between currents and entropy production, the so-called thermodynamic uncertainty relations (TURs) [8, 9, 37–41], which has aroused a recent and great interest.

This paper is organized as follows: Secs. II and III present the model description and its exact thermodynamic properties. In Sec. IV we extend analysis for external forces and considerations about efficiency are performed in Sec. V. Conclusions and perspectives are drawn in Sec. VI.

II. MODEL AND FOKKER-PLANCK EQUATION

We are dealing with a Brownian particle with mass \( m \) sequentially placed in contact with \( N \) different thermal reservoirs. Each contact has a duration of \( \tau/N \) and occurs during the intervals \( \tau_{i-1} \leq t < \tau_i \), where \( \tau_i = i\tau/N \) for \( i = 1, \ldots, N \), in which the particle evolves in time according to the following Langevin equation

\[
\frac{dv_i}{dt} = -\kappa_i v_i + F_i(t) + B_i(t),
\]

where quantities \( v_i \), \( \kappa_i \), and \( F_i(t) \) denote the particle velocity, the viscous constant, and external force, respectively. From now on, we shall express them in terms of reduced quantities: \( \gamma_i = \kappa_i/m \) and \( f_i(t) = F_i(t)/m \). The stochastic force \( \zeta_i(t) = B_i(t)/m \) accounts for the interaction between particle and the \( i \)-th environment and...
satisfies the properties
\[ \langle \zeta_i(t) \rangle = 0, \] (2)
and
\[ \langle \zeta_i(t) \zeta_i(t') \rangle = 2\gamma_i T_i \delta(t - t'), \] (3)
respectively, where \( T_i \) is the bath temperature. Let \( P_i(v, t) \) be the velocity probability distribution at time \( t \), its time evolution is described by the Fokker-Planck (FP) equation \[ \partial P_i / \partial t = - \partial J_i / \partial v + f_i(t) \partial P_i / \partial v, \] (4)
where \( J_i \) is given by
\[ J_i = -\gamma_i v P_i - \gamma_i k_b T_i \partial P_i / m \partial v. \] (5)

It is worth mentioning that above equations are formally identical to description of the overdamped harmonic oscillator subject to the harmonic force \( f_h = -k x \) just by replacing \( x \to v \), \( k/\alpha \to \gamma_i \), \( 1/\alpha \to \gamma_i/m \).

From the FP equation and by performing appropriate partial integrations together boundary conditions in which both \( P_i(v, t) \) and \( J_i(v, t) \) vanish at extremities, the time variation of the energy system \( U_i = \langle E_i \rangle \) in contact with the \( i \)-th reservoir is given by
\[ dU_i / dt = -m \int v^2 \left[ \partial J_i / \partial v + f_i(t) \partial P_i / \partial v \right] dv. \] (6)
The right side of Eq. (6) can be rewritten as \( dU_i / dt = -(\dot{W}_i + \dot{Q}_i) \), where \( \dot{W}_i \) and \( \dot{Q}_i \) denote the work per unity of time and heat flux from the system to the environment (thermal bath) given by
\[ \dot{W}_i = -m(\langle v_i \rangle f_i(t)) \quad \text{and} \quad \dot{Q}_i = \gamma_i (m \langle v_i^2 \rangle - k_b T_i), \] (7)
respectively. In the absence of external forces \( \dot{W}_i = 0 \) and all heat flux comes from/goes to the thermal bath.

By assuming the system entropy \( S \) is given by \( S_i(t) = -k_b \int P_i(v, t) \ln[P_i(v, t)] dv \) and from the expression for \( J_i \), one finds that its time derivative is given by
\[ dS_i / dt = -k_b \int \left( \frac{J_i}{F_i} \right) \left( \partial P_i / \partial v \right) dv. \] (8)
As for the mean energy, above expression can be rewritten in the following form
\[ dS_i / dt = \frac{m}{\gamma_i T_i} \left( \int \frac{J_i^2}{F_i} dv + \gamma_i \int v J_i dv \right). \] (9)
Above expression can be interpreted according to the following form \( dS_i / dt = \Pi_i(t) - \Phi_i(t) \) \cite{16,42}, where the former term corresponds to the entropy production rate \( \Pi_i(t) \) and it is strictly positive (as expected). The second term is the the flux of entropy and can also be rewritten more conveniently as
\[ \Phi_i(t) = \frac{\dot{Q}_i}{T_i} = \gamma_i \left( \frac{m \langle v_i^2 \rangle}{T_i} - k_b \right). \] (10)

If external forces are null and the particle is placed in contact to a single reservoir, the probability distribution approaches for large times the Gibbs (equilibrium) distribution \( P_i^{eq}(v) = e^{-E/k_B T_i}/Z \), being \( E = mv^2/2 \) its kinetic energy and \( Z \) the partition function. In such case, \( \langle v_i^2 \rangle = k_b T_i/m \) and therefore \( \Pi_i^{eq} = \Phi_i^{eq} = 0 \) (as expected). Conversely, it will evolve to a nonequilibrium steady state (NESS) when placed in contact with sequential and distinct reservoirs, in which heat is dissipated and the entropy is produced and hence \( \Pi_{NESS} > 0 \).

### III. Exact Solution for Arbitrary Set of Sequential Reservoirs

From now on, quantities will be expressed in terms of the “reduced temperature” \( \Gamma_i = 2\gamma_i k_b T_i/m \) and \( k_B = 1 \). Since we are dealing with a linear force on the velocity, the NESS will also be characterized by a Gaussian probability distribution \( P_i(v, t) = e^{-E/k_B T_i}/\sqrt{2\pi k_B T_i} \) in which both mean \( \langle v_i(t) \rangle \) and the variance \( \langle v_i^2 \rangle \) will be in general time dependent. Their expressions can be calculated from Eqs. \( 4,11 \) and \( 15 \) and read
\[ \frac{d}{dt} \langle v_i \rangle = -\gamma_i \langle v_i \rangle + f_i(t), \] (11)
and
\[ \frac{d}{dt} \langle v_i^2 \rangle = -2\gamma_i \langle v_i^2 \rangle + \Gamma_i, \] (12)
respectively, where appropriate partial integrations were performed. Their solutions are given by the following expressions
\[ \langle v_i(t) \rangle = e^{-\gamma_i(t-\tau_i)}[v_{i-1}' + \int_{\tau_i-1}^{t} e^{\gamma_i(t'-\tau_i)} f_i(t') dt'], \] (13)
and
\[ \langle v_i^2(t) \rangle = A_{i-1} e^{-2\gamma_i(t-\tau_i)} + \frac{\Gamma_i}{2\gamma_i}, \] (14)
respectively, where quantities \( v_{i-1}' \equiv \langle v_i \rangle(\tau_i-1) \) and \( A_i \)'s are evaluated by taking into account the set of continuity relations for the averages and variances, \( \langle v_i \rangle(\tau_i) = \langle v_{i+1} \rangle(\tau_i) \) and \( b_i(\tau_i) = b_{i+1}(\tau_i) \) (for all \( i = 1, ..., N \)), respectively. Since the system returns to the initial state after a complete period, \( \langle v_i \rangle(0) = \langle v_N \rangle(\tau) \) and \( b_i(0) = b_N(\tau) \), all coefficients can be solely calculated in terms of model parameters, temperature reservoirs and
the period. Also, above conditions state that the probability at each point returns to the same value after every period.

For simplicity, from now on we shall assume the same viscous constant \( \gamma_l = \gamma \) for all \( i \)'s. In the absence of external forces, all \( e_l' \)'s vanish and the entropy production only depends on the coefficients \( A_i \)'s and \( \Gamma_i \)'s. Hence, the coefficient \( A_i \) becomes

\[
A_{i+1} = x A_i + \frac{1}{2\gamma} (\Gamma_i - \Gamma_{i+1}),
\]

where \( x = e^{-2\gamma t/N} \) and all of them can be found from a linear recurrence relation

\[
A_i = x^{-1} A_1 + \frac{1}{2\gamma} \sum_{l=2}^{i} x^{i-l} (\Gamma_{l-1} - \Gamma_l),
\]

for \( i = 2, \ldots, N \). As the particle returns to the initial configuration the after a complete period, \( A_N \) then reads

\[
A_N = x^{-1} A_1 + \frac{1}{2\gamma} (\Gamma_1 - \Gamma_N).
\]

By equating Eqs. (16) and (17) for \( i = N \), all coefficients \( A_i \)'s can be finally calculated and are given by

\[
A_1 = \frac{1}{2\gamma} \frac{x^N}{1-x^N} \sum_{l=1}^{N} x^{-l} (\Gamma_{l-1} - \Gamma_{l+1}),
\]

and

\[
A_i = \frac{1}{2\gamma} \frac{x^{i-1}}{1-x^N} \left[ \sum_{l=1}^{i-1} x^{-l} (\Gamma_{l-1} - \Gamma_{l+1}) + \sum_{l=i}^{N} x^{N-l} (\Gamma_{l-1} - \Gamma_{l+1}) \right],
\]

for \( i = 1 \) and \( i > 1 \), respectively. As we are focusing on the steady-state time-periodic regime, thermodynamic quantities can be averaged over one period \( \tau \). The mean entropy production then \( \Pi \) reads

\[
\Pi = \frac{1}{\tau} \sum_{i=1}^{N} \int_{\tau_{i-1}}^{\tau_i} \Phi_i(t) \, dt = \frac{1-e^{-2\gamma \tau/N}}{2\gamma \tau} \sum_{i=1}^{N} A_i \Gamma_i.
\]

From Eqs. (18) and (19), it follows that

\[
\sum_{i=1}^{N} \frac{A_i}{\Gamma_i} = \frac{x^N}{1-x^N} \sum_{i=1}^{N} x^{-l} \left( \frac{\Gamma_{i+1} - \Gamma_{i+2}}{\Gamma_{i}} \right),
\]

and we arrive at an expression for \( \Pi \) solely dependent on the model parameters

\[
\Pi = \frac{N}{2\gamma \tau} \left( \frac{1-x}{x} \right) + \frac{1}{2\gamma \tau} x^{N-1} (1-x)^2 \sum_{i=1}^{N} x^{-l} \frac{\Gamma_{i+1}}{\Gamma_{i}}.
\]

In order to show that \( \Pi \geq 0 \), we resort to the inequality \( \sum_{i=1}^{N} \Gamma_{i+1}/\Gamma_i \geq N \sqrt{\prod_{i=1}^{N} \Gamma_{i+1}/\Gamma_i} \) for showing that \( \sum_{i=1}^{N} \Gamma_{i+1}/\Gamma_i \geq N \), and hence Eq. (22) fulfills the condition

\[
\Pi \geq -\frac{N}{2\gamma \tau} \left( \frac{1-x}{x} \right) + \frac{N}{2\gamma \tau} \left( \frac{1-x}{x} \right) = 0,
\]

in consistency with the second law of thermodynamics.

As an concrete example, we derive explicit results for the two sequential reservoirs case. From Eqs. (13) and (14), coefficients \( A_1 \) and \( A_2 \) reduce to the following expressions

\[
A_1 = \frac{\Gamma_2 - \Gamma_1}{2\gamma} \left( \frac{1 - e^{-\gamma t}}{1 - e^{-2\gamma t}} \right) = \frac{\Gamma_2 - \Gamma_1}{2\gamma} \left( \frac{1}{1 + e^{\gamma t}} \right),
\]

where \( A_2 = -A_1 \) and hence

\[
\Phi_1(t) = \gamma \left( \frac{\Gamma_2 - \Gamma_1}{\Gamma_2} \right) \left( \frac{1}{1 + e^{\gamma t}} \right) e^{-2\gamma t},
\]

for \( 0 \leq t < \tau/2 \) and

\[
\Phi_2(t) = \gamma \left( \frac{\Gamma_1 - \Gamma_2}{\Gamma_2} \right) \left( \frac{1}{1 + e^{\gamma t}} \right) e^{-2\gamma (t-\frac{\tau}{2})},
\]

\( \tau/2 \leq t < \tau \), respectively average mean entropy production reads

\[
\Pi = \left[ \frac{\Gamma_1 \Gamma_2}{2\tau} \tanh \left( \frac{\gamma \tau}{2} \right) \right] \left( \frac{1}{\Gamma_1} - \frac{1}{\Gamma_2} \right)^2.
\]

Note that \( \Pi \geq 0 \) and it vanishes when \( \Gamma_1 = \Gamma_2 \). In the limit of slow \( (\tau \gg 1) \) and fast \( (\tau \ll 1) \) oscillations, \( \Pi \) approaches to the following asymptotic expressions

\[
\Pi \approx \frac{\Gamma_1 \Gamma_2}{2\tau} \left( \frac{1}{\Gamma_1} - \frac{1}{\Gamma_2} \right)^2 \quad \text{and} \quad \frac{\Gamma_1 \Gamma_2}{4} \left( \frac{1}{\Gamma_1} - \frac{1}{\Gamma_2} \right)^2,
\]

respectively and such a latter expression is independent on the period.

Eq. (27) can be conveniently written down as a flux-times-force expression, where the thermodynamic force attempts to the difference of temperatures of reservoirs. Given that the viscous coefficient is the same for all switchings, the thermodynamic force can be more conveniently expressed in terms of difference of \( \Gamma_i \)'s. More specifically, we have that \( \Pi = J_{TT} f_T \), where \( f_T = (1/\Gamma_2 - 1/\Gamma_1) \) and \( J_T \) can also be rewritten as \( J_T = L_{TT} f_T \), where \( L_{TT} \) is the Onsager coefficient given by

\[
L_{TT} = \frac{\Gamma_1 \Gamma_2}{2\tau} \tanh \left( \frac{\gamma \tau}{2} \right).
\]

Note that \( L_{TT} \geq 0 \) (as expected).

Fig. 3 depicts the average entropy production \( \Pi \) versus \( \tau \) for distinct values of \( \Gamma_2 \) and \( \Gamma_1 = 1, \gamma = 1 \). Note that it is monotonically increasing with \( f_T \) and reproduces above asymptotic limits.
for the first or second half of each period, respectively.

The average work and heat per time are given by

$$\overline{W} = \overline{W}_1 + \overline{W}_2$$
$$\overline{Q} = \overline{Q}_1 + \overline{Q}_2$$

respectively and straightforwardly evaluated from Eq. (10), whose \(\overline{W}_1\) and \(\overline{Q}_1\) read

$$\overline{W}_1 = -\frac{m f_1}{\tau} \int_0^{\tau/2} \langle v_1 \rangle \, dt =$$
$$= \frac{m f_1}{\gamma^2 \tau} (f_1 - f_2) \tanh \left( \frac{\gamma \tau}{4} \right) - \frac{m f_2^2}{2 \gamma},$$

and

$$\overline{Q}_1 = \frac{m}{4 \gamma \tau} (\Gamma_2 - \Gamma_1) \tan \left( \frac{\gamma \tau}{2} \right) + \frac{m}{2 \gamma^2 \tau} (f_1 + f_2)^2 \times$$
$$\times \tanh \left( \frac{\gamma \tau}{4} \right) + \frac{2 m f_2^2}{\gamma^2} \left[ \frac{\gamma \tau}{4} - \tan \left( \frac{\gamma \tau}{4} \right) \right],$$

respectively. Analogous expressions are obtained for \(\overline{W}_2\) and \(\overline{Q}_2\) just by exchanging \(1 \leftrightarrow 2\). Note that \(\overline{Q}_1 + \overline{Q}_2 + \overline{W}_1 + \overline{W}_2 = 0\), in consistency with the first law of thermodynamics.

In the same way as before, the steady entropy production per period \(\Pi\) can be evaluated from Eq. (10) (by taking \(k_b = 1\)) and reads

$$\Pi = \frac{2 \gamma}{m} \left( \overline{Q}_1 \Gamma_1 + \overline{Q}_2 \Gamma_2 \right),$$

and we arrive at the following expression

$$\Pi = \frac{1}{2 \tau} \left( \frac{\Gamma_2 - \Gamma_1}{\Gamma_1 \Gamma_2} \right)^2 \tan \left( \frac{\gamma \tau}{2} \right) + \frac{1}{\gamma \tau} \left( \frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} \right) \times$$
$$\times \tan \left( \frac{\gamma \tau}{4} \right) (f_1 + f_2)^2 + \left( \frac{f_1^2}{\Gamma_1} + \frac{f_2^2}{\Gamma_2} \right) \left[ 1 - \frac{4}{\gamma \tau} \tan \left( \frac{\gamma \tau}{4} \right) \right].$$

Since \(\gamma \tau \geq 0\) and \(1 - \tan(x)/x \geq 0\), it follows that \(\Pi \geq 0\). Note that \(\Pi\) reduces to Eq. (27) as \(f_1 = f_2 = 0\).

1. Bilinear form and Onsager coefficients

The shape of Eq. (30) is similar to the linear irreversible thermodynamics [18, 19, 40], in which the entropy production is written down as a sum of flux-times-force expression. This similarity provides to reinterpret Eq. (30) in the following form

$$\Pi = J_T f_T + J_1 f_1 + J_2 f_2,$$

where forces \(f_T = (1/\Gamma_1 - 1/\Gamma_2)\) and \(f_1, f_2\) have associated fluxes \(J_T, J_1, J_2\) given by

$$J_T = L_T f_T,$$
$$J_1 = L_{11} f_1 + L_{12} f_2,$$
$$J_2 = L_{21} f_1 + L_{22} f_2,$$

for the first or second half of each period, respectively.

The average work and heat per time are given by

$$\overline{W} = \overline{W}_1 + \overline{W}_2$$
$$\overline{Q} = \overline{Q}_1 + \overline{Q}_2$$

respectively and straightforwardly evaluated from Eq. (10), whose \(W_1\) and \(Q_1\) read

$$W_1 = -\frac{m f_1}{\tau} \int_0^{\tau/2} \langle v_1 \rangle \, dt =$$
$$= \frac{m f_1}{\gamma^2 \tau} (f_1 - f_2) \tanh \left( \frac{\gamma \tau}{4} \right) - \frac{m f_2^2}{2 \gamma},$$

and

$$Q_1 = \frac{m}{4 \gamma \tau} (\Gamma_2 - \Gamma_1) \tan \left( \frac{\gamma \tau}{2} \right) + \frac{m}{2 \gamma^2 \tau} (f_1 + f_2)^2 \times$$
$$\times \tanh \left( \frac{\gamma \tau}{4} \right) + \frac{2 m f_2^2}{\gamma^2} \left[ \frac{\gamma \tau}{4} - \tan \left( \frac{\gamma \tau}{4} \right) \right],$$

respectively. Analogous expressions are obtained for \(W_2\) and \(Q_2\) just by exchanging \(1 \leftrightarrow 2\). Note that \(Q_1 + Q_2 + W_1 + W_2 = 0\), in consistency with the first law of thermodynamics.

In the same way as before, the steady entropy production per period \(\Pi\) can be evaluated from Eq. (10) (by taking \(k_b = 1\)) and reads

$$\Pi = \frac{2 \gamma}{m} \left( \frac{Q_1}{\Gamma_1} + \frac{Q_2}{\Gamma_2} \right),$$

and we arrive at the following expression

$$\Pi = \frac{1}{2 \tau} \left( \frac{\Gamma_2 - \Gamma_1}{\Gamma_1 \Gamma_2} \right)^2 \tan \left( \frac{\gamma \tau}{2} \right) + \frac{1}{\gamma \tau} \left( \frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} \right) \times$$
$$\times \tan \left( \frac{\gamma \tau}{4} \right) (f_1 + f_2)^2 + \left( \frac{f_1^2}{\Gamma_1} + \frac{f_2^2}{\Gamma_2} \right) \left[ 1 - \frac{4}{\gamma \tau} \tan \left( \frac{\gamma \tau}{4} \right) \right].$$

Since \(\gamma \tau \geq 0\) and \(1 - \tan(x)/x \geq 0\), it follows that \(\Pi \geq 0\). Note that \(\Pi\) reduces to Eq. (27) as \(f_1 = f_2 = 0\).

1. Bilinear form and Onsager coefficients

The shape of Eq. (30) is similar to the linear irreversible thermodynamics [18, 19, 40], in which the entropy production is written down as a sum of flux-times-force expression. This similarity provides to reinterpret Eq. (30) in the following form

$$\Pi = J_T f_T + J_1 f_1 + J_2 f_2,$$

where forces \(f_T = (1/\Gamma_1 - 1/\Gamma_2)\) and \(f_1, f_2\) have associated fluxes \(J_T, J_1, J_2\) given by

$$J_T = L_T f_T,$$
$$J_1 = L_{11} f_1 + L_{12} f_2,$$
$$J_2 = L_{21} f_1 + L_{22} f_2,$$
respectively, where \( L_{11}, L_{12}, L_{21}, \) and \( L_{22} \) denote their Onsager coefficients given by

\[
L_{11} = \frac{1}{\Gamma_1} \left[ 1 - \frac{3}{\gamma \tau} \tanh \left( \frac{2 \gamma \tau}{4} \right) \right] + \frac{1}{\gamma \tau \Gamma_2} \tanh \left( \frac{2 \gamma \tau}{4} \right),
\]

and

\[
L_{12} = L_{21} = \frac{1}{\gamma \tau} \left( \frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} \right) \tanh \left( \frac{2 \gamma \tau}{4} \right),
\]

respectively. Coefficients \( L_{22} \) and \( L_{21} \) have the same shape of \( L_{11} \) and \( L_{12} \) by replacing \( 1 \leftrightarrow 2 \), respectively. Besides, \( L_{11} \) and \( L_{22} \geq 0 \) (as should be) and they satisfy the inequality \( 4L_{11}L_{22} - (L_{12} + L_{21})^2 \geq 0 \), in consistency with the positivity of the entropy production.

### B. Time dependent external forces

By repeating the previous calculations for linear external forces the mean velocities \( \langle v_i(t) \rangle \)'s are given by

\[
\langle v_i(t) \rangle = \frac{1}{\gamma \tau} \left\{ \lambda_1 \left( \gamma t - 1 \right) + e^{-\gamma t} \left[ \lambda_1 + \left( \lambda_2 e^{-\gamma t} - \lambda_1 \right) \alpha(\gamma, \tau) \right] \right\},
\]

\[
\langle v_2(t) \rangle = \frac{1}{\gamma \tau} \left\{ - \lambda_2 \left[ \gamma \left( t - \frac{\tau}{2} \right) - 1 \right] + e^{-\gamma (t-\frac{\tau}{2})} \left[ \left( \lambda_1 + \lambda_2 \right) e^{-\gamma \left( t - \frac{\tau}{2} \right)} - \lambda_2 \right] \alpha(\gamma, \tau) - \lambda_2 \right\},
\]

where

\[
\alpha(\gamma, \tau) = \frac{2 - e^{-\gamma \left( \gamma \tau - 2 \right)}}{2(e^{\gamma \tau} - 1)},
\]

respectively. Although more complex than the previous case, the mean work and heat per time are evaluated analogously from expressions for \( \langle v_i(t) \rangle \)'s and \( b_i(t) \)'s, whose values averaged over a cycle read

\[
\overline{W} = -\overline{Q} = -\overline{A} \left\{ e^{\gamma \tau} \varphi_+(\gamma, \tau, \xi) + 12 e^{\frac{\gamma \tau}{2}} \left( \gamma^2 \tau^2 \xi - 4 \right) - \varphi_-(\gamma, \tau, \xi) \right\},
\]

where parameters \( \overline{A}, \xi \) and \( \varphi_\pm(\gamma, \tau, \xi) \) read

\[
\overline{A} = \frac{m(\lambda_1 + \lambda_2)^2}{24 \gamma^2 \tau (e^{\gamma \tau} - 1)}, \quad \xi = \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2},
\]

and

\[
\varphi_\pm(\gamma, \tau, \xi) = \gamma^2 \tau^2 (2 \xi - 1) (3 \pm \gamma \tau) + 24 (1 \pm \gamma \tau \xi),
\]

respectively.

### V. EFFICIENCY

Distinct works have tackled the conditions in which periodically driven systems can operate as thermal machines [43-45]. The conversion of a given type of energy into another one requires the existence of a generic force \( X_1 \) operating against its flux \( J_1X_1 \leq 0 \) counterbalancing with driving forces \( X_2 \) and \( X_T \) in which \( J_2X_2 + J_TX_T \geq 0 \). A measure of efficiency \( \eta \) is given by

\[
\eta = \frac{-J_1X_1}{J_2X_2 + J_TX_T} = \frac{-L_{11}X_1^2 + L_{12}X_1X_2}{L_{21}X_1^2 + L_{22}X_2^2 + L_{TT}X_T^2},
\]

where in such case \( X_T = f_j \) and we have taken into account Eq. (37) for relating fluxes and Onsager coefficients. Taking into account that the best machine aims at maximizing the efficiency and minimizing the dissipation \( \overline{P} \) for a given power output \( \overline{P} = -\Gamma_1 J_1 X_1 \), it is important to analyze the role of three load forces, \( X_{1mP} \), \( X_{1mE} \) and \( X_{1mS} \), in which the power output and efficiency are maximum and the dissipation is minimum, respectively [47]. Their values can be obtained straightforwardly from expressions for \( \overline{P} \) and Eq. (41), respectively. Due to the present symmetric relation between Onsager coefficients \( L_{12} = L_{21} \) (in both cases), they acquire simpler forms and read

\[
X_{1mE} = \frac{1}{L_{11}L_{12}X_2} \left[ -L_{11}(L_{22}X_2^2 + L_{TT}X_T^2) + A(X_2, X_T) \right],
\]

with \( A(X_2, X_T) \) being given by

\[
A(X_2, X_T) = \sqrt{L_{11}(L_{22}X_2^2 + L_{TT}X_T^2)} \times
\]

\[
\sqrt{[L_{11}(L_{22}X_2^2 + L_{TT}X_T^2) - L_{12}^2X_2^2]},
\]

respectively.
and $X_{1mS} = -L_{12}X_2/L_{11} = 2X_{1mP}$, respectively, where $X_i = f_i$ and $\lambda_i$ for the constant and linear drivings, respectively. The efficiencies at minimum dissipation, maximum power and its maximum value become $\eta_{mS} = 0$, $\eta_{mP} \approx \frac{L_{12}^2X_2^2}{2(2L_{22}L_{11} - L_{12}^2)X_2^2 + 4LTTL_{11}X_2^2}$, (48) and

$$\eta_{mE} = \frac{1}{L_{12}X_2^2}[2L_{11}(L_{22}X_2^2 + L_{TT}X_2^3T) - L_{12}X_2^2 - 2A(X_2, X_T)],$$

(49) respectively, and finally their associated power outputs read $P_{mS} = 0$, $P_{mP} = \Gamma_1L_{12}X_2^2/4L_{11}$ and

$$P_{mE} = \frac{\Gamma_1}{L_{11}L_{12}X_2^2} \times \left[ L_{11}(L_{22}X_2^2 + L_{TT}X_2^3T) - A(X_2, X_T) - L_{12}X_2^2 \right] \times \left[ L_{11}(L_{22}X_2^2 + L_{TT}X_2^3T) - A(X_2, X_T) \right],$$

(50) respectively. We pause to make a few comments: First, above expressions extend the findings from Ref. [47] for a couple of driving forces. Second, both efficiency and power vanish when $X_1 = X_{1mS}$ and $X_1 = 0$ and are strictly positive between those limits. Hence the physical regime in which the system can operate as an engine is bounded by the lowest entropy production $\Pi_{mS} = L_{TT}X_2^2 + (L_{22} - L_{12}/L_{11})X_2^2$ and the value $\Pi = L_{TT}X_2^2 + L_{22}X_2^2$. Third, despite the long expressions for Eqs. (49) and (50), powers $P_{mP}$, $P_{mE}$ and efficiencies $\eta_{mP}$, $\eta_{mE}$ are linked through a couple of simple expressions (in similarity with Refs. [46, 47]):

$$\eta_{mP} = \frac{\eta_{mE}}{1 + \eta_{mE}} \quad \text{and} \quad \frac{P_{mE}}{P_{mP}} = 1 - \eta_{mE}^2.$$  

(51)

and they imply that $0 \leq \eta_{mP} < \eta_{mE}$ (with $0 \leq \eta_{mE} \leq 1$ and $0 \leq \eta_{mP} \leq 1/2$) and $0 \leq P_{mE} \leq P_{mP}$. Fourth and last, the achievement of most efficient machine $\eta_{mE} = 1$ implies that the system has to be operated at null power $P_{mE} = 0$ and hence the projection of a machine operating for finite $P_{mP}/P_{mE}$ will imply at a loss of its efficiency.

Our purpose here aims at not only extending relevant concepts about efficiency for Brownian particles in contact with sequential reservoirs, but also to show that a desired compromise between maximum power and maximum efficiency can be achieved by adjusting conveniently the model parameters (such as the period and the driving). From expressions for Onsager coefficients, aforementioned quantities are evaluated, as depicted in Figs. 2 and 3 for distinct periods $\tau$ and temperature differences $\Delta \Gamma$’s for constant and linear drivings, respectively.

In both cases, quantities follow theoretical predictions and exhibit similar portraits, in which efficiencies and power outputs present maximum values at $f_{1mE}(\lambda_{1mE})$ and $f_{1mP}(\lambda_{1mP})$, respectively. The loss of efficiency from the maximum $\eta_{mE}$ as $f_1(\lambda_1)$ goes up (down) is signed by the increase of dissipation (as expected) until vanishing when $\Pi = \Pi \ast$. For the constant driving, absolute values of forces and efficiencies increase as the period $\tau$ (see e.g. panels (a)) and/or temperature differences (see e.g. panels (b)) are lowered. In such a case, $\Gamma_1 \approx \Gamma_2 = \Gamma$, $\Delta \Gamma = \Gamma_1 - \Gamma_2 << 1$ and the thermodynamic force $f_T$ approaches to $f_T \approx \Delta \Gamma^2$. Onsager coefficients become simpler in the limit of fast switchings, $\tau \to 0$ and $L_{11}, L_{22}, L_{12}$ approach to ($\Gamma_1 + \Gamma_2)/(4\Gamma_1 \Gamma_2)$. Some remarkable quantities then approach to the asymptotic values $f_{1mS} \to -f_2 = 2f_{1mP}$ and

$$\eta_{mP} \to \frac{f_2^2(\Gamma_1 + \Gamma_2)}{2[f_2^2(\Gamma_1 + \Gamma_2) + 2\Delta \Gamma^2]},$$

(52) respectively. For $\Gamma_1 \approx \Gamma_2$, $\eta_{mP} \to 1/2$, $\eta_{mE} \to 1$ and $P_{mP}$ reads $P_{mP} = f_2^2/8$ and thereby the limit of an ideal machine is achieved for low periods and equal temperatures. Similar features are verified for the linear driving, including increasing efficiencies as both $\tau$ and $\Delta \Gamma$ decreases. However, they are marked by a reentrant behavior for $\tau << 1$ and $\Delta \Gamma \neq 0$ (see e.g. Figs. 4 and 5). It moves for lower $\tau$’s as $\Delta \Gamma$ goes down and the limit of ideal machine, $\eta_{mP} \to 1/2$ and $\eta_{mE} \to 1$, is also recovered when both $\tau \to 0$ for $\Delta \Gamma \to 0$.

Other differences between protocols are appraised in Figs. 4 and 5. For finite difference of temperatures, the constant driving is always more efficient than the linear one and their power outputs are also superior. The maximum efficiency curves (linear drivings) are also reentrant, whose maxima values increase and deviate for lower $\tau$’s as $\Delta \Gamma$ decreases.

We close this section by remarking that although short periods indicates a general route for optimizing the efficiency of thermal machines in contact to sequential reservoirs, the present description provides to properly tune the period and forces in order to obtain the desirable compromise between maximum efficiency and power.

VI. CONCLUSIONS

The thermodynamics of a Brownian particle periodically placed in contact with sequential thermal reservoirs is introduced. We have obtained explicit (exact) expressions for relevant quantities, such as heat, work and entropy production. Generalization for an arbitrary number of sequential reservoirs and the influence of external forces were considered. Considerations about the efficiency were undertaken, in which Brownian machines can be properly operated ensuring the reliable compromise between efficiency and power for small switching periods.
FIG. 2: Panels (a) and (b) depict the efficiency $\eta$ versus $f_1$ for distinct periods $\tau$ (for $\Delta \Gamma = 0.5$) and $\Delta \Gamma$'s (for $\tau = 1$), respectively. In both cases, $\Gamma_1 = 2$ and $f_2 = 1$. Symbols •, “stars” and “squares” denote the $f_{1mE}$, $f_{1mP}$ and $f_{1mS}$ respectively. Panels (c) and (d) show the corresponding power $\mathcal{P}$, whereas (e) and (f) the average entropy production rate $\Pi$. Dashed lines show the values of $f_1$ the system can not be operated as a thermal machine.

As a final comment, we mention the several new perspectives to be addressed. First, it might be very interesting to extend such study for other external forces protocols (e.g. sinusoidal time dependent ones) as well as for time asymmetric switchings, in order to compare their efficiencies, mainly with the linear driving case. Finally, it would be very remarkable to verify the validity of recent proposed uncertainties relations (TURs) for Fokker-Planck equations [39, 41], in such class of systems.

VII. ACKNOWLEDGMENT

We acknowledge Karel Proesmans and Mário J. de Oliveira for careful readings of the manuscript and useful suggestions. C. E. F acknowledges the financial support from FAPESP under grant 2018/02405-1.

[1] I. Prigogine, Introduction to thermodynamics of irreversible processes (Interscience New York, 1965).

[2] S. De Groot and P. Mazur, “North-holland,” (1962).
FIG. 3: Panels (a) and (b) depict the efficiency $\eta$ versus $\lambda_1$ for distinct periods $\tau$ (for $\Delta \Gamma = 0.5$) and $\Delta \Gamma$'s (for $\tau = 1$), respectively. In both cases, $\Gamma_1 = 2$ and $\lambda_2 = 1$. Symbols $\bullet$, “stars” and “squares” denote the $\lambda_{1mE}$, $\lambda_{1mP}$ and $\lambda_{1mS}$ respectively. Panels (c) and (d) show the corresponding power $P$, whereas (e) and (f) the average entropy production rate $\Pi$. Dashed lines show the values of $\lambda_1$ the system can not be operated as a thermal machine.

[3] T. Tomé and M. J. De Oliveira, *Stochastic dynamics and irreversibility* (Springer, 2015).
[4] U. Seifert, Reports on progress in physics 75, 126001 (2012).
[5] C. Van den Broeck and M. Esposito, Physica A: Statistical Mechanics and its Applications 418, 6 (2015).
[6] C. Jarzynski, Physical Review Letters 78, 2690 (1997).
[7] O.-P. Saira, Y. Yoon, T. Tanttu, M. Möttönen, D. Averin, and J. P. Pekola, Physical review letters 109, 180601 (2012).
[8] K. Proesmans and C. Van den Broeck, EPL (Europhysics Letters) 119, 20001 (2017).
[9] A. C. Barato and U. Seifert, Physical review letters 114, 158101 (2015).
[10] J. Schnakenberg, Reviews of Modern physics 48, 571 (1976).
[11] C. F. Noa, P. E. Harunari, M. de Oliveira, and C. Fiore, Physical Review E 100, 012104 (2019).
[12] T. Herpich, J. Thingna, and M. Esposito, Physical Review X 8, 031056 (2018).
[13] T. Herpich and M. Esposito, Physical Review E 99, 022135 (2019).
[14] B. O. Goes, C. E. Fiore, and G. T. Landi, Physical Review Research 2, 013136 (2020).
[15] H. Ge, M. Qian, and H. Qian, Physics Reports 510, 87 (2012).
[16] T. Tomé and M. J. de Oliveira, Physical review E 91, 042140 (2015).
[17] K. Brandner, K. Saito, and U. Seifert,
FIG. 4: For $\Gamma = 2$, $f = 1$ and distinct $\Delta \Gamma$'s, the comparison between maximum efficiency (panel (a)) and efficiency at maximum power (panel (b)) for constant drivings. Insets: The corresponding power outputs $P$'s versus $\tau$.

FIG. 5: For $\Gamma = 2$, $\lambda = 1$ and distinct $\Delta \Gamma$'s, the comparison between maximum efficiency (panel (a)) and efficiency at maximum power (panel (b)) for linear drivings. Insets: The corresponding power outputs $P$'s versus $\tau$.
[35] Y. Izumida and K. Okuda, The European Physical Journal B 77, 499 (2010).

[36] D. Abreu and U. Seifert, EPL (Europhysics Letters) 94, 10001 (2011).

[37] A. C. Barato and U. Seifert, The Journal of Physical Chemistry B 119, 6555 (2015).

[38] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, Physical review letters 116, 120601 (2016).

[39] Y. Hasegawa and T. Van Vu, Physical Review E 99, 062126 (2019).

[40] T. Van Vu and Y. Hasegawa, Physical Review E 100, 012134 (2019).

[41] T. Van Vu and Y. Hasegawa, Phys. Rev. Research 2, 013060 (2020).

[42] T. Tomé and M. J. de Oliveira, Physical Review E 82, 021120 (2010).

[43] K. Proesmans and C. Van den Broeck, Physical review letters 115, 090601 (2015).

[44] M. Bauer, K. Brandner, and U. Seifert, Physical Review E 93, 042112 (2016).

[45] A. Rosas, C. Van den Broeck, and K. Lindenberg, Physical Review E 94, 052129 (2016).

[46] K. Proesmans and C. Van den Broeck, Chaos: An Interdisciplinary Journal of Nonlinear Science 27, 104601 (2017).

[47] K. Proesmans, B. Cleuren, and C. Van den Broeck, Physical review letters 116, 220601 (2016).

[48] A. Rosas, C. Van den Broeck, and K. Lindenberg, Physical Review E 96, 052135 (2017).