Simple heteroclinic cycles in $\mathbb{R}^4$

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Abstract
In generic dynamical systems heteroclinic cycles are invariant sets of codimension at least one, but they can be structurally stable in systems which are equivariant under the action of a symmetry group, due to the existence of flow-invariant subspaces. For dynamical systems in $\mathbb{R}^n$ the minimal dimension for which such robust heteroclinic cycles can exist is $n = 3$. In this case the list of admissible symmetry groups is short and well known. The situation is different and more interesting when $n = 4$. In this paper, we list all finite groups $\Gamma$ such that an open set of smooth $\Gamma$-equivariant dynamical systems in $\mathbb{R}^4$ possesses a simple heteroclinic cycle (a structurally stable heteroclinic cycle satisfying certain additional constraints). This work extends the results which were obtained by Sottocornola in the case when all equilibria in the heteroclinic cycle belong to the same $\Gamma$-orbit (in this case one speaks of homoclinic cycles).

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1. Introduction

Heteroclinic cycles are flow-invariant sets produced by dynamical systems, which have the property to carry recurrent dynamics with intermittent, cycling switching between equilibria (or more complicated bounded invariant sets but we shall restrict ourselves here to steady states). These objects are known to exist and in addition to be structurally stable within certain classes of $\Gamma$-equivariant systems, where $\Gamma$ is a finite or compact Lie group. Here we consider continuous dynamical systems

$$\dot{x} = f(x), \quad f : \mathbb{R}^n \to \mathbb{R}^n$$  \hspace{1cm} (1)
with the equivariance condition
\[ f(γx) = γf(x) \quad \text{for all } γ ∈ Γ ⊂ O(n). \]  
\[ Γ \text{ finite}. \]  

(2)

Let \( ξ_1, \ldots, ξ_m \), be a collection of (hyperbolic) saddle equilibria of the above system and set \( ξ_{m+1} = ξ_1 \). Let \( W^u(ξ_j) \), respectively \( W^s(ξ_j) \), be the unstable, respectively stable manifold of \( ξ_j \). Suppose that for each \( j = 1, \ldots, m \), \( W^u(ξ_j) \) intersects \( W^s(ξ_{j+1}) \), then the equilibria and their heteroclinic orbits form a heteroclinic cycle. Heteroclinic orbits between saddles are generically destroyed by small perturbations, hence such objects are unlikely to exist in generic systems. They can, however, be structurally stable, or robust, in a restricted class of equations, under the equivariance condition (2) for some group \( Γ \). Indeed this symmetry condition forces the existence of flow-invariant subspaces, which are formed by the points in \( \mathbb{R}^n \) fixed by isotropy subgroups of \( Γ \). We write \( \text{Fix}(Σ) \) for the set of points which are fixed by \( Σ \). This is a linear subspace of \( \mathbb{R}^n \), and moreover it is invariant by the flow of equation (1). Suppose now that there exists a collection of isotropy subgroups \( Σ_j \) such that \( ξ_j \) is a saddle and \( ξ_{j+1} \) is a sink in \( \text{Fix}(Σ_j) \), with the convention that \( ξ_{m+1} = ξ_1 \). Suppose, in addition, that a saddle–sink connection exists from \( ξ_j \) to \( ξ_{j+1} \) in \( \text{Fix}(Σ_j) \), then this connection is robust against (smooth) perturbations in the class of \( Γ \)-equivariant systems.

Many examples of robust heteroclinic cycles have been discovered and studied, especially in the context of hydrodynamical flows; see [2, 9] for an overview.

The question that we address in this paper is the following: for which groups \( Γ \) do there exist dynamical systems as above, which possess a structurally stable heteroclinic cycle? The answer to this question depends on \( n \) and we have to be more specific on this issue.

The case \( n = 3 \) is the simplest one in which robust heteroclinic cycles can occur and it can easily be handled. However, when \( n = 4 \) the situation is considerably more involved. Examples of four-dimensional heteroclinic cycles have been known and studied because they provide ‘non-trivial’ stability and bifurcation properties [6]. A classification of genuinely four-dimensional robust homoclinic cycles was achieved by Sottocornola in [20, 21]. A homoclinic cycle is a heteroclinic cycle in which all equilibria belong to the same \( Γ \)-orbit. Sottocornola listed all finite subgroups of \( O(4) \) for which robust homoclinic cycles exist. An outcome of his work is that one can find in \( \mathbb{R}^4 \) robust homoclinic cycles which connect \( 2k \) equilibria with \( k > 2 \) arbitrarily large.

Our aim is to extend these results to robust heteroclinic cycles in \( \mathbb{R}^4 \). The first classification of heteroclinic cycles was proposed in [10]. Assuming that all \( P_j \)'s are planes, the authors introduced the concept of ‘simple’ heteroclinic cycles, which were further divided into classes A, B and C. Although finite groups admitting cycles of types B and C can be easily found, the list of groups admitting type A was unknown. It is the aim of this paper to fill the gap. It was implicitly assumed in [10, 11] that simple heteroclinic cycles are such that each equilibrium in the cycle has generically only simple eigenvalues. We shall see in the next section that this is not always the case and we complete the definition of simple heteroclinic cycles accordingly.

As in [16, 20] our analysis exploits the quaternionic presentation of finite subgroups of \( SO(4) \). It does, however, not rely on the Galois theory as in [20] and it provides elementary proofs.

The paper is organized as follows: in section 2 we introduce basic notions about robust heteroclinic cycles and about the presentation of \( SO(4) \) and \( O(4) \) with quaternions. These are the basic material which will be used in the rest of the paper. In section 3 the main theorems are stated and their proof is given through a series of lemmas. The case \( Γ ⊂ SO(4) \) is considered first, then \( Γ ⊂ O(4) \). In theorem 2 the proofs that a subgroup \( Γ \) admits, or does not admit, simple heteroclinic cycles are presented only for selected \( Γ ⊂ SO(4) \). For other subgroups of \( SO(4) \) the proofs are similar, and therefore are omitted. They follow from appendices B–D.
in the extended arXiv version of this paper [18], which contain detailed information on the
geometry of finite subgroups of SO(4).

In section 4 we show several examples of heteroclinic cycles in $\mathbb{R}^4$ and in section 5 we
discuss the results together with some open questions.

Simple heteroclinic cycles, which are discussed in this paper, suppose the existence of
one-dimensional fixed-point subspaces for the action of the group in $\mathbb{R}^4$. In the appendix, we
list finite subgroups of O(4), which act irreducibly but do not possess such a subspace. This
provides an alternative and simple approach to a problem which was addressed by Lauterbach
and Matthews in [13].

2. Background and notation

2.1. Simple heteroclinic cycles in $\mathbb{R}^4$

In this section we make precise the framework in which we look for robust heteroclinic cycles.
Our notation will follow those of [10].

Let $\xi_1, \ldots, \xi_M$ be hyperbolic equilibria of the $\Gamma$-equivariant system (1)–(2) with stable
and unstable manifolds $W^s(\xi_j)$ and $W^u(\xi_j)$, respectively. Assuming $\xi_{M+1} = \xi_1$, we denote by
$\kappa_j, j = 1, \ldots, M$, the set of trajectories from $\xi_j$ to $\xi_{j+1}$:
\[
\kappa_j = W^u(\xi_j) \cap W^s(\xi_{j+1}) \neq \emptyset.
\]

Definition 1. (i) The union of equilibria \{\xi_1, \ldots, \xi_M\} and their connecting orbits
\{\kappa_1, \ldots, \kappa_M\} is called a heteroclinic cycle.
(ii) a homoclinic cycle is a heteroclinic cycle in which $\xi_j$ belong to the same group orbit.

We recall that the isotropy group of a point $x \in \mathbb{R}^n$ is the subgroup of $\Gamma$ satisfying
\[
\Sigma_x = \{\gamma \in \Gamma : \gamma x = x\}.
\]
The fixed-point subspace of a subgroup $\Sigma \subset \Gamma$ is the subspace
\[
\text{Fix} (\Sigma) = \{x \in \mathbb{R}^n : \sigma x = x \text{ for all } \sigma \in \Sigma\}.
\]
When $\dim \text{Fix} (\Sigma) = 1$ (respectively 2) the subspace is sometimes called an axis of symmetry
(respectively a plane of symmetry). We shall use either denominations. If a point $x$ has isotropy
$\Sigma$, then the point $\gamma x$ has isotropy $\gamma \Sigma \gamma^{-1}$. There is a bijection between the $\Gamma$-orbit of a point and the conjugacy class of its isotropy subgroup in $\Gamma$. Another useful property is that the
largest subgroup of $\Gamma$ which leaves the subspace $\text{Fix} (\Sigma)$ invariant is the normalizer
$N(\Sigma)$ of $\Sigma$.

The following definition gives sufficient conditions for a heteroclinic cycle to persist under
small enough $\Gamma$-equivariant perturbations.

Definition 2. [10] The heteroclinic cycle is structurally stable (or robust) if for any $j, 1 \leq j \leq M$, there exist $\Sigma_j \subset \Gamma$ and $P_j = \text{Fix}(\Sigma_j)$ such that
\begin{itemize}
  \item[(i)] $\xi_j$ is a sink in $P_j$;
  \item[(ii)] $\xi_{j-1}, \xi_j$ and $\kappa_j$ belong to $P_j$.
\end{itemize}

In case of a homoclinic cycle, it is enough to assume the existence of a transformation $\gamma \in \Gamma$
such that a saddle–sink connection exists from $\xi_1$ to $\xi_2 = \gamma \xi_1$ in a fixed-point subspace $P$.
Heteroclinic cycles in $\mathbb{R}^4$ have been classified by Sottocornola [20].

In what follows we use the notation $L_j = P_{j-1} \cap P_j = \text{Fix}(\Delta_j)$.

In [11], it was assumed that for all $j$, $\dim (P_j) = 2$ and the heteroclinic cycle intersects
each connected component of $L_j \setminus \{0\}$ in at most one point. They called simple any
robust heteroclinic cycle with these properties. Figure 1 sketches the sequence of inclusions
Figure 1. The graph structure of the isotropy types for a simple heteroclinic cycle. In parentheses: dimensions of the fixed-point subspaces.

between isotropy types corresponding to the groups $\Sigma_i$ and $\Delta_j$ when the heteroclinic cycle is simple.

This assumption imposes constraints on the eigenvalues and eigenvectors of the Jacobian matrix $J_j = df(\xi_j)$. Because $P_j$ are flow-invariant planes, $J_j$ has three eigenvectors which belong to respectively $L_j$, $P_j \ominus L_j$ and $P_j \ominus L_j$ where $X \ominus Y$ denotes a complementary subspace of $Y$ in $X$. We call radial the eigenvalue $r_j$ along the axis $L_j$, contracting the eigenvalue $-c_j$ with eigenspace $V_j = P_{j-1} \ominus L_j$ (with $c_j > 0$), expanding the eigenvalue $e_j$ with eigenspace $W_j = P_j \ominus L_j$ ($e_j > 0$), transverse the remaining eigenvalue and $T_j$ the corresponding eigenspace. Note that by construction, all eigenvalues of $J_j$ must be real.

We recall that the isotypic decomposition of a representation $T$ of a (finite) group $G$ in a vector space $V$ is the decomposition $V = V^{(1)} \oplus \cdots \oplus V^{(r)}$ where $r$ is the number of equivalence classes of irreducible representations of $G$ in $V$ and each $V^{(j)} = T|_{V_j}$ is the sum of the equivalent irreducible representations in the $j$th class. This decomposition is unique. The subspaces $V^{(j)}$ are mutually orthogonal (if $G$ acts orthogonally).

**Lemma 1.** Let a robust heteroclinic cycle in $\mathbb{R}^4$ be such that for all $j$: (i) $\text{dim } P_j = 2$, (ii) each connected component of $L_j \setminus \{0\}$ is intersected at most at one point by the heteroclinic cycle. Then the isotypic decomposition of the representation of $\Delta_j$ in $\mathbb{R}^4$ is of one of the following types:

1. $L_j \oplus \perp V_j \oplus \perp W_j \oplus \perp T_j$ (the symbol $\oplus \perp$ indicates the orthogonal direct sum).
2. $L_j \oplus \perp V_j \oplus \perp W_j$ where $W_j = W_j \oplus T_j$ has dimension 2.
3. $L_j \oplus \perp V_j \oplus \perp W_j$ where $V_j = V_j \oplus T_j$ has dimension 2.

In cases 2 and 3, $\Delta_j$ acts in $\widetilde{W}_j$ (respectively, $\widetilde{V}_j$) as a dihedral group $\mathbb{D}_m$ for some $m \geq 3$. It follows that in case 2, $e_j$ is double (and $e_j = t_j$) while in case 3, $-c_j$ is double (and $-c_j = t_j$).

**Proof.** $L_j$ is the axis on which $\Delta_j$ acts trivially, so it is a component of the isotypic decomposition. There cannot be a 3-dimensional component because from the existence of a heteroclinic cycle the eigenvalues of $J_j$ along $V_j$ and $W_j$ must be of opposite signs. Therefore, the remaining possibilities are that there are, in addition to $L_j$, three 1-dimensional components or one 1-dimensional and one 2-dimensional components. The action of $\Delta_j$ on a 1-dimensional component different from $L_j$ is isomorphic to $\mathbb{Z}_2$ (taking any non-zero vector to its opposite). The action on a 2-dimensional component allows a priori more possibilities: it can be isomorphic to the $k$-fold rotation group $C_k$ with $k \geq 3$, or to the dihedral group $\mathbb{D}_k$. The former case is excluded because this 2-dimensional space must contain at least one invariant axis (and therefore at least three of them by the $k$-fold rotations). Another way to prove this is that if the action were isomorphic to $C_m$ only, then in general the eigenvalues of $J_j$ along $V_j$...
these components would be complex. Hence there is a double eigenvalue, which can be either
\(-c_j = t_j\) or \(e_j = t_j\), the corresponding isotypic component being either \(V_j\) or \(W_j\).

\[QED\]

Cases 2 and 3 of the above lemma were not accounted for in [11]. For the sake of clarity we, therefore, introduce the following definition.

**Definition 3.** Let a robust heteroclinic cycle in \(\mathbb{R}^4\) satisfy the conditions (i) and (ii) of lemma 1. The cycle is called *simple* if case 1 holds true for all \(j\), and pseudo-simple otherwise.

**Remark 1.** It can be easily shown that in \(\mathbb{R}^4\) the notions of simple heteroclinic cycle in [11] and in the above definition do coincide in the following cases: (a) the heteroclinic cycle is homoclinic; (b) the heteroclinic cycle is asymptotically stable (hence the stability analysis for simple heteroclinic cycles in [11] is correct).

Also note that for simple heteroclinic cycles, \(N(\Sigma_j)/\Sigma_j \cong \mathbb{D}_1\), where \(\mathbb{D}_1 \cong \mathbb{Z}_2\).

In this paper we consider simple heteroclinic cycles. Pseudo-simple heteroclinic cycles will be considered in a forthcoming work. We give in section 4.2 an example of a pseudo-simple heteroclinic cycle.

The property of being simple imposes strong geometrical constraints on the symmetries allowing for a robust heteroclinic cycle. Our aim in the following sections will be to exploit these constraints in order to determine all these symmetries. For this we still need some definitions and preliminary important properties.

**Lemma 2.** (See proof in [11]) Consider a simple heteroclinic cycle in \(\mathbb{R}^4\). For all \(j\), either \(\Delta_j \cong \mathbb{Z}_2^2\) and \(\Sigma_j \cong \mathbb{Z}_2\), or \(\Delta_j \cong \mathbb{Z}_3^2\) and \(\Sigma_j \cong \mathbb{Z}_2^2\). Moreover, the planes \(P_j = \text{Fix}(\Sigma_j)\) and \(P_{j+1}\) intersect orthogonally.

**Remark 2.** An order two element \(\sigma\) in SO(4) whose fixed point subspace is a plane \(P\) must act as \(-Id\) in the plane \(P\perp\) fully perpendicular to \(P\). Nevertheless, to distinguish it from other rotations fixing the points on \(P\), we call \(\sigma\) a *plane reflection*.

In the case \(\Sigma_j \cong \mathbb{Z}_2\) for all \(j\), the heteroclinic cycle does not intersect with any hyperplane of symmetry (a hyperplane which is the fixed-point subspace of some subgroup of \(\Gamma\)), while in the second case at least one such hyperplane exists. Indeed if \(\Sigma_j \cong \mathbb{Z}_2\) then \(P_j = \text{Fix}(\Sigma_j)\) cannot be included in a lower isotropy proper fixed-point subspace of \(\mathbb{R}^4\). Based on this property, Krupa and Melbourne [11] separated heteroclinic cycles in \(\mathbb{R}^4\) into three types.

**Definition 4.** A simple robust heteroclinic cycle is of type A if \(\Sigma_j \cong \mathbb{Z}_2\) for all \(j\). It is of type B if the heteroclinic cycle lies entirely in a fixed-point hyperplane. Otherwise it is of type C.

Krupa and Melbourne have determined in [11] the simple heteroclinic cycles of types B and C. We give this list in the following theorem, using their notation: \(B_m^\pm\) indicates a heteroclinic cycle of type B with \(m\) different types of equilibria (two equilibria have the same type if their isotropy groups are conjugate) and either \(-I \in \Gamma\) (sign \(-\)) or not (sign \(+\)). The same notation is used for heteroclinic cycles of type C. The coordinates \((x_1, x_2, x_3, x_4)\) are chosen to correspond to the isotypic decomposition of \(\Delta_1\) with the trivial component along the first coordinate. We only indicate the main features of the heteroclinic cycles, since the geometry is simple but cumbersome to describe.

**Theorem 1** (see [11]). There are four different types of simple heteroclinic cycles of type B and three types of simple heteroclinic cycles of type C.
1. \( B^*_1 \) with \( \Gamma = \mathbb{Z}_4^3 \) consisting of the reflections \((x_1, \pm x_2, \pm x_3, \pm x_4)\). There are three different hyperplanes and in each of them, a heteroclinic cycle with two equilibria, one on each connected component of \( L_1 \setminus \{0\} \).

2. \( B^*_1 \) with \( \Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2^3 \) where \( \mathbb{Z}_2^3 \) acts as above and \( \mathbb{Z}_2 \) is generated by \((-x_1, x_3, x_2, x_4)\). The structure of the heteroclinic cycle is the same as above but \( \xi_1 \) and \( \xi_2 \) are interchanged by \( \mathbb{Z}_2 \), hence the cycle is homoclinic.

3. \( B^*_1 \) with \( \Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2^3 \) generated by reflections through the four hyperplanes of coordinates. Similar heteroclinic cycles exist in each hyperplane. For example in the hyperplane \((x_1, x_2, x_3, 0)\) heteroclinic cycles connect equilibria lying on any three axes \( x_1, x_2, x_3 \) and the heteroclinic connections lie in the corresponding planes of coordinates.

4. \( B^*_1 \) with \( \Gamma = \mathbb{Z}_1 \times \mathbb{Z}_4^3 \) where \( \mathbb{Z}_3 \) is generated by the circular permutation of \( x_1, x_2, x_3 \). Same as above but with all three equilibria in the same \( \mathbb{Z}_3 \)-orbit, hence the cycle is homoclinic.

5. \( C^*_1 \) with \( \Gamma = \mathbb{Z}_4^4 \) acting as in case 3. These cycles connect equilibria lying on the four coordinate axes.

6. \( C^*_1 \) with \( \Gamma = \mathbb{Z}_4^4 \) with \( \mathbb{Z}_4 \) acting by circular permutation of the coordinates. Same as above but all equilibria in the same group orbit, hence the cycle is homoclinic.

7. \( C^*_1 \) with \( \Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2^2 \) and \( \mathbb{Z}_2 \) generated by the permutation \((x_1, x_2) \mapsto (x_3, x_4)\). Same as above but the four equilibria are pairwise of the same type.

### 2.2. Quaternionic presentation of the group SO(4)

In this section we recall some useful properties of quaternions [3, 4]. A real quaternion is a set of four real numbers, \( q = (q_1, q_2, q_3, q_4) \). Introducing the elements \( i = (0, 1, 0, 0), j = (0, 0, 1, 0) \) and \( k = (0, 0, 0, 1) \), any quaternion has the form \( q_1 + q_2 i + q_3 j + q_4 k \), where the first component is called the real part of the quaternion. Multiplication is defined by the rules \( i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j \), which implies

\[
qw = (q_1w_1 - q_2 w_2 - q_3 w_3 - q_4 w_4, q_1 w_2 + q_2 w_1 + q_3 w_4 - q_4 w_3, q_1 w_3 - q_2 w_4 + q_3 w_1 + q_4 w_2, q_1 w_4 + q_2 w_3 - q_3 w_2 + q_4 w_1).
\]

The conjugate of \( q \) is \( \bar{q} = q_1 - q_2 i - q_3 j - q_4 k \) and \( q \bar{q} = q_1^2 + q_2^2 + q_3^2 + q_4^2 = |q|^2 \) is the square of the norm of \( q \). Hence \( \bar{q} \) is also the inverse \( q^{-1} \) of a unit quaternion \( q \). We denote by \( \mathcal{Q} \) the multiplicative group of unit quaternions; obviously, its identity element is \((1, 0, 0, 0)\).

A unit quaternion can be represented as \( q = (\cos \theta, u \sin \theta) \), where \( u = (q_2, q_3, q_4) \in \mathbb{R}^3 \) is a unit vector. The three-dimensional subspace \( w = 0 \) in the four-dimensional vector space of all quaternions \( v = (w, x, y, z) \) can be identified with \( \mathbb{R}^3 \). The transformation \( v \mapsto qvq^{-1} \) is the rotation of angle \( 2\theta \) around \( u \) in \( \mathbb{R}^3 = \{(0, x, y, z)\} \), it is an element of SO(3). The respective homomorphism of \( \mathcal{Q} \) on SO(3) is 2-to-1 and its kernel is comprised of \((\pm 1, 0, 0, 0)\).

Therefore, any finite subgroup of \( \mathcal{Q} \) falls into one of the following cases, which are pre-images of the respective subgroups of SO(3):

\[
\begin{align*}
\mathbb{Z}_n &= \oplus_{r=0}^{n-1}(\cos 2r\pi/n, 0, 0, \sin 2r\pi/n), \\
\mathbb{D}_n &= \mathbb{Z}_{2n} \oplus \oplus_{r=0}^{n-1}(0, \cos r\pi/n, \sin r\pi/n, 0), \\
\mathbb{V} &= (\pm 1, 0, 0, 0), \\
\mathbb{T} &= \mathbb{V} \oplus \left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}\right), \\
\mathbb{O} &= \mathbb{T} \oplus \mathbb{Z}_{2}((\pm 1, \pm 1, 0, 0)), \\
\mathbb{I} &= \mathbb{T} \oplus \mathbb{Z}_{2}((\pm \tau, \pm 1, \pm \tau^{-1}, 0)),
\end{align*}
\]

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where $\tau = (\sqrt{5} + 1)/2$. Double parentheses denote all possible permutations of quantities within the parentheses and for $\prod$ only even permutations of $(\pm \tau, \pm 1, \pm \tau^{-1}, 0)$ are elements of the group. Any other finite subgroup of $Q$ is conjugate to one of these under an inner automorphism of $Q$.

The group of eight elements $\mathcal{V} = \{(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1)\}$ is classically known as the quaternion group.

The four numbers $(q_1, q_2, q_3, q_4)$ can be regarded as the Euclidean coordinates of a point in $\mathbb{R}^4$. For any pair of unit quaternions $(l, r)$, the transformation $q \mapsto lqr^{-1}$ is a rotation in $\mathbb{R}^4$, i.e. an element of the group $SO(4)$. The mapping $\Phi : Q \times Q \to SO(4)$ that relates the pair $(l, r)$ with the rotation $q \mapsto lqr^{-1}$ is a homomorphism onto, whose kernel consists of two elements, $(1, 1)$ and $(-1, -1)$; thus the homomorphism is two to one.

Therefore, a finite subgroup of $SO(4)$ is a subgroup of a product of two finite subgroups of $Q$. There is, however, an additional subtlety. Let $\Gamma$ be a finite subgroup of $SO(4)$, $\mathcal{G} = \Phi^{-1}(\Gamma)$ and $(l_j, r_j), 1 \leq j \leq J$, its elements. Denote by $L$ and $R$ the finite subgroups of $Q$ generated by $l_j$ and $r_j$, $1 \leq j \leq J$, respectively. To any element $l \in L$ there are several corresponding elements $r$, such that $(l, r) \in \mathcal{Q}$, and similarly for any $r \in R$. This establishes a correspondence between $L$ and $R$. Denote by $L_K$ and $R_K$ the subgroups of $L$ and $R$ corresponding to the unit elements in $R$ and $L$, respectively. The groups $L/L_K$ and $R/R_K$ are isomorphic [4] and characterize the group $\mathcal{G}$. Moreover, $\mathcal{G}_\tau = L_K \times R_K$ is normal in $\mathcal{G}$ and $L/L_K \cong \mathcal{G}/\mathcal{G}_\tau$. This relation allows us to compute the order of $\mathcal{G}$ and $\Gamma$ from the knowledge of $L, L_K$ and $R, R_K$.

**Notation.** Following [4] we write $(L \mid L_K; R \mid R_K)$ for the group $\Gamma$.

The isomorphism between $L/L_K$ and $R/R_K$ may not be unique, and different isomorphisms give rise to different subgroups of $SO(4)$. For instance, the correspondence

$$p^r \leftrightarrow p^{r^*},$$

where $r \in Z_{mr}/Z_{mr}, r^* \in Z_{mr}/Z_{mr}, p = (\cos 2\pi/mr, 0, 0, \sin 2\pi/mr)$ and $p^* = (\cos 2\pi/nr, 0, 0, \sin 2\pi/nr)$, for different $s < r/2$ prime to $r$, gives geometrically distinct subgroups of $SO(4)$, which are denoted by $(Z_{mr}/Z_{mr}; Z_{mr}/Z_{mr})$. The isomorphism extended to the one between $D_{mr}/Z_{mr}$ and $D_{mr}/Z_{mr}$ defines a group $(D_{mr}/Z_{mr}; D_{mr}/Z_{mr})$. The isomorphism between $\mathbb{Z}/\mathbb{Z}_1$ and $\mathbb{Z}/\mathbb{Z}_1$ can be the identity, or it can be $r = \pm 1$ for $r \in \mathbb{T}$ and $r = -1$ for $r \in \mathbb{T}_1$, where $\mathbb{T}_1$ is the coset of $\mathbb{T}$ in $\mathbb{O}$. The latter subgroup is denoted by $(\mathbb{O}/\mathbb{Z}_1; \mathbb{O}/\mathbb{Z}_1)'$. The complete list of finite subgroups of $SO(4)$ is given in table 1.

Here we are interested in subgroups $\Gamma$ of $SO(4)$ such that a $\Gamma$-equivariant system can possess a heteroclinic cycle. As will be shown in lemma 7, a pre-image $\Phi^{-1}\Gamma = (L \mid L_K; R \mid R_K)$ must satisfy $D_2 \subset L$ and $D_2 \subset R$. The subgroups of $SO(4)$ where both $L$ and $R$ contain $D_6$ ($n > 1$) are the groups 10–32 and 34–39 in the table.

The superscript $\dagger$ is employed to denote subgroups of $SO(4)$ where the isomorphism between the quotient groups $L/L_K$ and $R/R_K \cong L/L_K$ is not the identity. The group $\mathbb{O}^\dagger$, isomorphic to $\mathbb{O}$, involves the elements $((\pm \tau^*, \pm 1, \pm (\tau^*)^{-1}, 0))$, where $\tau^* = (-\sqrt{5} + 1)/2$. Groups 1–32 contain the central rotation $-I$, and groups 33–39 do not.

A reflection in $\mathbb{R}^4$ can be expressed in the quaternionic presentation as $q \mapsto aq b$, where $a$ and $b$ are a pair of unit quaternions. We write this reflection as $(a; b)^\tau$. The transformations $q \mapsto aqa$ and $q \mapsto -aqa$ are, respectively, the reflections about the axis $a$ and through the hyperplane orthogonal to the vector $a$. Therefore, if $a \perp b$ are two orthogonal unit quaternions, the rotation of angle $\pi$ about the plane $(a, b)$ is $q \mapsto -abq(ba)$. We call this transformation the plane reflection about $(a, b)$ (see remark 2).
A group $\Gamma^* \subset \text{O}(4)$, $\Gamma^* \not\subset \text{SO}(4)$, can be decomposed as

$$\Gamma^* = \Gamma \oplus \sigma \Gamma,$$

where $\Gamma \subset \text{SO}(4)$ and $\sigma = (a; b)^* \not\in \text{SO}(4)$.

If $\Gamma^*$ is finite, then in the quaternion form of $\Gamma$, $\Phi^{-1}\Gamma = (L \mid L_K; R \mid R_K)$, the groups $L$ and $R$ are isomorphic, and so are $L_K$ and $R_K$ [4]. The elements $a$ and $b$ belong to a subgroup $H$ of $Q$ in which $G = L = R$ and $G_K = L_K = R_K$ are invariant subgroups. Moreover, $a$ and $b$ are in the same coset of $G$ in $H$. If $\phi$ denotes the isomorphism between $L = L/K$ and $R = R/K$, $a$ and $b$ are the isomorphisms from $R'$ to $L'$ defined by $\alpha : R' \to aR'a^{-1}$ and $\beta : R' \to aR'a^{-1}$ then $\phi\alpha\phi^*\beta = 1$. The list of finite subgroups of $O(4)$, which was derived from these arguments, can be found in [4].

3. Classification of simple heteroclinic cycles in $\mathbb{R}^4$

In this section we state and prove classification of simple heteroclinic cycles in $\mathbb{R}^4$. More precisely, we list all finite subgroups $\Gamma \subset O(4)$ such that $\Gamma$-equivariant systems exist, which possess a simple heteroclinic cycle. Note, that the subgroups $\Gamma$ giving rise to cycles of types B and C were found in [11] and are listed in theorem 1 (see the previous section).

The proof of our main theorems is given in section 3.3 and will proceed from a series of lemmas which are stated in section 3.2.

3.1. Statement of the main results

We begin with a definition.

**Definition 5.** We say that a subgroup $\Gamma$ of $O(n)$ admits robust heteroclinic cycles if there exists an open subset of the set of smooth $\Gamma$-equivariant vector fields in $\mathbb{R}^n$, such that vector fields in this subset possess a (robust) heteroclinic cycle.

The following theorems exhibit all finite subgroups of $O(4)$, which admit robust simple heteroclinic cycles. In theorem 2 we list those groups which are included in $SO(4)$ and in theorem 3 we list the groups which contain elements not in $SO(4)$. We use the notation introduced in 2.2.
Theorem 2. A group $\Gamma \subset SO(4)$ admits simple homoclinic cycles, if and only if it is one of the following:

| $\langle D_{2k}, D_{2k}, D_{2k}, D_{2k} \rangle$ | $\langle D_{2k}, Z_{4k}, D, T \rangle, \ K \neq 3k$ |
| $\langle D_{2k}, Z_{4k}, D_{2k}, Z_{4k} \rangle$, $K_1$, $K_2$, $r$, $s$ satisfy (19) | $\langle D_{2k}, D_{2k}, Z_{4k} \rangle$, $K_1$, $K_2$, $r$ odd, $\langle D_{2k}, D_{2k}, T \rangle$ |
| $\langle D_{2k}, Z_{4k}, D_{2k}, Z_{4k} \rangle$, $K_1$, $K_2$, $r$, $s$ satisfy (22) | $\langle D_{2k}, Z_{4k}, D_{2k}, Z_{4k} \rangle$, $K_1$, $K_2$, $r$, $s$ satisfy (23) |
| $\langle D_{2k}, D_{2k}, T \rangle$ | $K_1$, $K_2$, $r$, $s$ satisfy (23) |
| $\langle D_{2k}, D_{2k}, D_{2k}, D_{2k} \rangle \ K$ odd | $\langle D_{2k}, D_{2k}, Z_{4k}, D_{2k} \rangle$, $K_1$, $K_2$, $r$ odd, $\langle D_{2k}, D_{2k}, T \rangle$ |
| $\langle D_{2k}, D_{2k}, D_{2k}, D_{2k} \rangle$ | $\langle D_{2k}, D_{2k}, D_{2k}, D_{2k} \rangle$ |

Remark 3. A subgroup $\Gamma \subset O(n)$ was called in [20, 21] a minimal admissible group if

- $\Gamma$ admits simple homoclinic cycles;
- any proper subgroup of $\Gamma$ does not admit homoclinic cycles.

Minimal admissible groups, subgroups of $O(4)$, were found in [20, 21]. In the quaternion form, subgroups of $SO(4)$ are ($D_4 \mid D_4 \mid D_4 \mid Z_4$) (with $\alpha = \pi/2$) and ($D_4 \mid D_4 \mid T \mid T$) (with $\alpha = \pi/4$). Any group admitting simple homoclinic cycles (see the table in appendix D in [18]), except for ($T \mid Z_2$, $T \mid Z_2$), has at least one of these groups as a subgroup. A homoclinic cycle which can exist in a ($T \mid Z_2$, $T \mid Z_2$)-equivariant system belongs to a three-dimensional hyperplane, such cycles were not considered in [ibid].

Theorem 3. A group $\Gamma^+ \subset O(4)$,

$\Gamma^+ = \Gamma \oplus \sigma \Gamma^+$, where $\Gamma \subset SO(4)$ and $\sigma \notin SO(4)$,

admits simple homoclinic cycles, if and only if $\Gamma$ and $\sigma$ are one of the following:

| $\Gamma$ | $\sigma$ |
|-----------------|------------------|
| $\langle D_4 \mid Z_2, D_4 \mid Z_2 \rangle$ | $((0, 1, 0, 1), (0, 1, 0, 1))^*/2$ |
| $\langle D_4 \mid Z_1, D_4 \mid Z_1 \rangle$ | $((0, 1, 0, 1), (0, 1, 0, 1))^*/2$ |
| $\langle D_2 \mid Z_2, D_2 \mid Z_2 \rangle$ | $((0, 1, 0, 0), (0, 1, 0, 0))^*$ |
| $\langle T \mid Z_2, T \mid Z_2 \rangle$ | $((0, 1, 0, 0), (0, 1, 0, 0))^*$ |
| $\langle D_2 \mid Z_1, D_2 \mid Z_1 \rangle$ | $((1, 0, 0, 0), (1, 0, 0, 0))^*$ |
| $\langle D_{2k} \mid D_{2k}, D_{2k}, D_{2k} \rangle \ (\cos \theta, 0, 0, \sin \theta), (1, 0, 0, 0))^*, \ \theta = \pi/(2K)$ |

Remark 4. The groups listed in theorem 1, which admit heteroclinic cycles of types B or C, are not subgroups of $SO(4)$ and therefore decompose as $\Gamma^* = \Gamma \oplus \sigma \Gamma, \Gamma \subset SO(4)$. In quaternion formulation the groups $\Gamma$ are the following:

- ($D_2 \mid Z_1, D_2 \mid Z_1$) (for $B^+_2$); ($D_4 \mid Z_1, D_4 \mid Z_1$) (for $B^+_4$); ($D_2 \mid Z_2, D_2 \mid Z_2$) (for $B^+_3$ and $C^+_3$); ($T \mid Z_2, T \mid Z_2$) (for $B^-_1$); ($D_2 \mid D_2, D_2 \mid D_2$) (for $C^-_3$); ($D_4 \mid D_2, D_4 \mid D_2$) (for $C^-_1$).
Remark 5. There exists only one group $\Gamma^* \subset O(4)$, $\Gamma^* \not\subset SO(4)$, admitting homoclinic cycles which are not of type B or C [20, 21]. In the quaternion form its rotation subgroup is $(D_{2K} \mid D_K \mid D_K)$.

These theorems are proven in section 3.3, but we need first several lemmas which are provided in the next section.

3.2. Lemmas

According to definition 3, if $X$ is a simple heteroclinic cycle then $\dim P_j = 2$ and the plane $P_j$ intersects with $P_{j+1}$ orthogonally for any $j$. Denote by $P_j^\bot$ the orthogonal complement to $P_j$ in $\mathbb{R}^4$. We assume that the bases $(h_1, h_2)$ in $P_j$ and $(h_3, h_4)$ in $P_j^\bot$ constitute a positively oriented basis $(h_1, h_2, h_3, h_4)$ in $\mathbb{R}^4$. The plane $P_j^\bot$ intersects orthogonally with $P_{j-1}$ and $P_{j+1}$.

Definition 6. Denote by $\alpha_j$ the oriented angle between $L_j$ and $L_{j+1}$; by $\beta_j$ the oriented angle between intersections of $P_j^\bot$ with $P_{j-1}$ and $P_{j+1}$. The angles $\alpha_j$ and $\beta_j$, $1 \leq j \leq M$, are called the structure angles of the heteroclinic cycle $X$.

Remark 6. The structure angles can be alternatively defined as the angles between: (i) the expanding eigenvector of $df (\xi_j)$ and the contracting eigenvector of $df (\xi_{j+1})$ (the angle $\alpha_j$); (ii) the contracting eigenvector of $df (\xi_j)$ and the expanding eigenvector of $df (\xi_{j+1})$ (the angle $\beta_j$).

Remark 7. The definition of structure angles can be generalized to simple heteroclinic cycles in $\mathbb{R}^n$ by introducing subspaces $U_j = \mathbb{R}^4$ such that $P_s \subset U_j$ for $s = j - 1$, $j$ and $j + 1$.

Lemma 3. (See proof in [16]) Let $N_1$ and $N_2$ be two planes in $\mathbb{R}^4$ and $p_j$, $j = 1, 2$, be the elements of $SO(4)$ which act on $N_j$ as identity, and on $N_j^\bot$ as $-I$, and $\Phi^{-1} p_j = (l_j; r_j)$, where $\Phi$ is the homomorphism defined in the previous section. Denote by $(l_1 l_2)_1$ and $(r_1 r_2)_1$ the first components of the respective quaternion products. The planes $N_1$ and $N_2$ intersect if and only if $(l_1 l_2)_1 = (r_1 r_2)_1 = \cos \alpha$ and $\alpha$ is the angle between the planes.

In order to ensure the existence of a heteroclinic cycle in terms of definition 1, it is enough to find $m \leq M$ and $\gamma \in \Gamma$ such that a minimal sequence of robust heteroclinic connections $\xi_1 \to \cdots \to \xi_{m+1}$ exists with $\xi_{m+1} = \gamma \xi_1$ (minimal in the sense that no other equilibrium inside this sequence belongs to the $\Gamma$-orbit of $\xi_1$). It follows that $\gamma^k = 1$ where $k$ is a divisor of $M$.

Definition 7. The sequence $\xi_1 \to \cdots \to \xi_m$ and the element $\gamma$ define a building block of the heteroclinic cycle

Lemma 4. Let $\xi_1 \to \cdots \to \xi_m$, $m \geq 2$, be a building block of a simple heteroclinic cycle in $\mathbb{R}^n$ and $\alpha_j = \pi/k_j$ be its structure angles according to definition 6. Then

(a) One of the following takes place:

(i) all $k_j$ are even and $\Delta_i$ and $\Delta_j$ are not conjugate for any $1 \leq i, j \leq m, i \neq j$;

(ii) $m = 2$, $k_1$ and $k_2$ are odd and $\Delta_1$, $\Delta_2$ are conjugate. The case $k_j = 1$ corresponds to having only one axis $L_j$ in $P_j$.

(b) The groups $\Sigma_i$ and $\Sigma_j$ are not conjugate for any $1 \leq i, j \leq m, i \neq j$. 

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Proof. We start from proving that either all \( k_j \) are odd, or all \( k_j \) are even. Suppose that this is not true and there exists \( j \) such that \( k_{j-1} \) is odd and \( k_j \) is even. Denote the two connected components of \( L_j \setminus \{0\} \) by \( L'_j \) and \( L''_j \) and assume that \( \xi_j \in L'_j \). Since \( k_{j-1} \) is odd, \( \Delta_{j-1} \) and \( \Delta_j \) are conjugate by some \( \kappa \in N(\Sigma_{j-1})/\Sigma_{j-1} \cong \mathbb{D}_{k_{j-1}} \). The symmetry \( \kappa \) satisfies \( \kappa L_{j-1} = L_j \) and \( \kappa \xi_{j-1} \in L''_j \). Since \( k_j \) is even, there exists \( \sigma \in N(\Sigma_j)/\Sigma_j \), such that \( \sigma L''_j = L'_j \). The image of \( \xi_{j-1} \) under \( \sigma \kappa \) satisfies \( \sigma \kappa \xi_{j-1} \in L'_j \), which contradicts the condition \( m \geq 2 \). Hence, either all \( k_j \) are even or all \( k_j \) are odd.

(a.i) Let all \( k_j \) be even and assume that \( \Delta_i \) and \( \Delta_j, i \neq j \), are conjugate by some \( \sigma \in \Sigma \), which implies \( \sigma L_i = L_j \). Denote by \( L'_j \) the connected component of \( L_j \setminus \{0\} \) such that \( \xi_j \in L'_j \) and \( L''_j = L_j \setminus \{0\} \setminus L'_j \). Since \( k_j \) is even, there exists \( \kappa \in \Sigma \) such that \( \kappa L'_j = L''_j \). Hence, either \( \sigma \kappa \xi_i \in L'_j \) or \( \sigma \kappa \xi_i \in L''_j \). Therefore, the assumption that \( \Delta_i \) and \( \Delta_j \) are conjugate contradicts definition 7.

(a.ii) If all \( k_j \) are odd, then there exist \( k_1 \) and \( k_2 \) such that \( k_1 L'_1 = L''_1 \) and \( k_2 L''_2 = L'_2 \) (as above, \( \xi_j \in L'_j \) for \( j = 1, 2, 3 \)). Therefore, \( k_2 k_1 \xi_1 \in L'_3 \), which implies \( m \leq 2 \).

(b) If all \( k_j \) are even, then conjugacy of \( \Sigma_i \) and \( \Sigma_j \) implies that \( \Delta_i \) is conjugate to \( \Delta_j \) or to \( \Delta_{j+1} \), which is not possible due to (a.i).

If all \( k_j \) are odd and \( m = 2 \), then the conjugacy of \( \Sigma_1 \) and \( \Sigma_2 \) and the definition of the building block imply existence of \( \sigma \in \Gamma \), such that \( \sigma \Sigma_1 \sigma^{-1} = \Sigma_2 \) and \( \sigma \xi_1 = \xi_2 \). Hence, the symmetry \( \sigma \) maps the connection \( \xi_1 \rightarrow \xi_2 \subset \Sigma_1 \) to the one \( \xi_1 \rightarrow \xi_3 \subset \Sigma_2 \), \( \xi_3 = \gamma \xi_1 \gamma^{-1} \), while the connection \( \xi_2 \rightarrow \xi_3 \) is needed to complete the heteroclinic cycle.

QED

In the next lemma we list the conditions for a finite subgroup of \( O(4) \) to admit (see definition 5) simple heteroclinic cycles. This lemma generalizes to heteroclinic cycles a theorem which was stated for homoclinic cycles in [1] (theorem 4.1).

**Lemma 5.** A finite subgroup \( \Gamma \) of \( O(4) \) admits simple heteroclinic cycles in \( \mathbb{R}^n \) (see definition 5) if and only if there exist two sequences of isotropy subgroups \( \Sigma_j, \Delta_j, j = 1, \ldots, m \), and an element \( \gamma \) in \( \Gamma \) satisfying the following conditions:

1. **C1.** Denote \( P_j = \text{Fix}(\Sigma_j) \) and \( L_j = \text{Fix}(\Delta_j) \). Then \( \dim P_j = 2 \) and \( \dim L_j = 1 \) for all \( j \).

2. **C2.** For \( i \neq j \), \( \Sigma_i \) and \( \Sigma_j \) are not conjugate.

3. **C3.** For \( j = 2, \ldots, m \), \( L_j = P_{j-1} \cap P_j \) and \( L_1 = \gamma^{-1} P_m \gamma \cap P_1 \). We set \( \Delta_{m+1} = \gamma \Delta_1 \gamma^{-1} \).

4. **C4.** \( N(\Sigma_j)/\Sigma_j \cong \mathbb{D}_{k_j} \), the dihedral group of order \( 2k_j \). Either all \( k_j \) are even and the groups \( \Delta_i, \Delta_j \) are not conjugate, or all \( k_j \) are odd and \( m \leq 2 \). Moreover, any isotropy subgroup which contains \( \Sigma_j \) is conjugate to either \( \Delta_j \) or \( \Delta_{j+1} \) (for any \( j = 1, \ldots, m \)).

5. **C5.** For all \( j \), the subspaces \( L_j, P_{j-1} \oplus L_j \) and \( P_j \oplus L_j \) are one-dimensional isotypic components in the isotypic decomposition of \( \Delta_j \) in \( \mathbb{R}^n \).

**Proof.** We prove sufficiency. Necessity follows from the definition of a simple heteroclinic cycle and the fact that if an invariant axis exists in \( P_j \), which is not an axis of symmetry of \( \mathbb{D}_{k_j} \), then its orthogonal complement in \( P_j \) cannot be an isotypic component for the action of \( \Delta_j \) (hence a heteroclinic cycle involving a connection in \( P_j \) with that axis cannot be simple).

Hypothesis C3 results in the following property of the invariant subspaces: for \( j = 2, \ldots, m, L_j = P_{j-1} \cap P_j \), and \( \gamma L_1 = P_m \cap \gamma P_1 \) (which also means that \( L_1 = \gamma^{-1} P_m \cap P_1 \)). Condition C4 takes care of the case when the heteroclinic cycle connects equilibria which have the same isotropy type. In this case, the building block reduces to two equilibria.
Now let $X_1$ be the set of $\Gamma$-equivariant smooth vector fields in $\mathbb{R}^n$ which have a hyperbolic equilibrium $\xi_j$ with isotropy $\Delta_j$ for all $j = 1, \ldots, m$, and such that the linearization at $\xi_j$ has a negative eigenvalue along $L_j = \text{Fix}(\Delta_j)$, a positive eigenvalue in $P_j = \text{Fix}(\Sigma_j)$ (in the direction orthogonal to $L_j$) and a negative eigenvalue in $P_{j-1} = \text{Fix}(\Sigma_{j-1})$ (in the direction orthogonal to $L_j$). Condition C3 implies that $\gamma L_1 \subset P_m$ and we assume that $\gamma \xi_1$ has a negative eigenvalue in $P_m$ in the direction orthogonal to $\gamma L_1$. This set is non-empty and open in the space of $\Gamma$-equivariant, smooth vector fields in $V = \mathbb{R}^n$.

Let $X_2$ be the set of vector fields in $X_1$ such that for all $j$, a heteroclinic orbit connecting $\xi_j$ to $\xi_{j+1}$ exists in $P_j$ (we set $\xi_{m+1} = \gamma \xi_1$). Since these trajectories realize saddle–sink connections in invariant subspaces $P_j$, the set $X_2$ is open. We need to show it is not empty.

Let $V = \mathbb{R}^n$. We need to recall first some properties of the orbit space $V/\Gamma$ of a finite group action, see [8, 2] for details. The orbit space can be realized as the image of the map $\Pi : V \to \mathbb{R}^p$, which to any point $x$ associates $(\theta_1(x), \ldots, \theta_p(x))$ where $\theta_1, \ldots, \theta_p$ are a (minimal) generating family of the ring of $\Gamma$-invariant polynomials in $V$. The set $\Pi(V)$ is a stratified semi-algebraic set. Each stratum is an algebraic manifold, image under $\Pi$ of the set of points in $V$ which have the same orbit type (that is, points which have conjugate isotropy subgroups). Despite the fact that $V/\Gamma$ is not a manifold we can give a meaning to a ‘smooth’ vector field in $V/\Gamma$ by saying that it is the restriction to $\Pi(V)$ of a smooth vector field in $\mathbb{R}^p$, which in addition is tangent to each stratum in $\Pi(V)$. The projection of a smooth $\Gamma$-equivariant vector field in $V$ under $\Pi$ is a smooth vector field in $V/\Gamma$. Conversely, any smooth vector field in $V/\Gamma$ lifts to a smooth $\Gamma$-equivariant vector field in $V$ [19]. Another important property of the orbit space is that given a point $x \in V$ with isotropy $\Sigma$, there exists a neighbourhood of $x$ in which $V/\Gamma$ is isomorphic to a neighbourhood of $0$ in $V/\Sigma$.

Now let $\tilde{\xi}_j$ be the image in $V/\Gamma$ of the equilibria $\xi_j$ for a vector field $f$ in $X_1$ and let $\tilde{f}$ be the image of $f$ in $\mathbb{R}^p$. We call $\tilde{L}_j$ the Jacobian matrix of $\tilde{f}$ at $\tilde{\xi}_j$. We also write $S_j$ the stratum corresponding to the orbit type of the subgroup $\Sigma_j$. Note that $\dim(S_j) = 2$. It follows from the properties of the orbit space studied in [8] that the unstable manifold of $\tilde{\xi}_j$ intersects $S_j$ along a one-dimensional curve $\tilde{w}_j$ while the stable manifold of $\tilde{\xi}_{j+1}$ contains a neighbourhood of this point in $S_j$. Due to second part of the condition C4 one can build a smooth path in $S_j$ which joins $\tilde{\xi}_j$ and $\tilde{\xi}_{j+1}$ and coincides with $w_j$ in the vicinity of $\tilde{\xi}_j$. The union of these paths for $j = 1$ to $p$ is a closed path $C$. Taking a tubular neighbourhood of $C$ in $\mathbb{R}^p$ we can build a smooth vector field $\tilde{f}$ which vanishes outside this neighbourhood, coincides with $\tilde{L}_j$ in a neighbourhood of $\tilde{\xi}_j$, is tangent to the strata in $\Pi(V)$ and such that the unstable manifold at $\tilde{\xi}_j$ intersects the stable manifold at $\tilde{\xi}_{j+1}$ in $S_j$ (see [1] for details). This vector field lifts to a $\Gamma$-equivariant vector field in $V$, which belongs to $X_2$.

Finally, the assumption C5 ensures that the heteroclinic cycles are simple. QED

**Lemma 6.** Let $P_1$ and $P_2$ be two-dimensional planes in $\mathbb{R}^n$, $\dim(P_1 \cap P_2) = 1$, $\rho \in O(n)$ is a plane reflection about $P_1$ and $\sigma \in O(n)$ maps $P_1$ into $P_2$. Suppose that $\rho$ and $\sigma$ are elements of a finite subgroup $\Delta \subset O(n)$. Then $\Delta \supset \mathbb{D}_m$, where $m \geq 3$.

**Proof.** Let $e_1$ denote a vector in $P_1$, which is orthogonal to $P_1 \cap P_2$. According to the statement of the lemma, $\sigma^l e_1 = e_1$ for a finite $l$. The subspace of $\mathbb{R}^n$ spanned by $e_1, \sigma e_1, \ldots, \sigma^{l-1} e_1$ has at least one $\sigma$- and $\rho$-invariant plane, which cannot be decomposed as a sum of two one-dimensional invariant subspaces. The action of group generated by $\rho$ and $\sigma$ on this plane is isomorphic to a dihedral group $\mathbb{D}_k$ for a $k > 2$. QED
Lemma 7. Let $X$ be a simple heteroclinic cycle in a $\Gamma$-equivariant system (1)–(2) in $\mathbb{R}^4$ and $\alpha_j$ and $\beta_j$, $j = 1, \ldots, m$, be its structure angles. Denote by $s_j$ the plane reflection through $P_j$. Then

(i) $\alpha_j = \pi/K_j$, $K_j \in \mathbb{Z}$;
(ii) $\beta_j = M_j \alpha_j/2$, $M_j \in \mathbb{Z}$;
(iii) if $P_j$ intersects with a plane $P_0$ such that $s_0 \in \Gamma$, then $P_j \perp P_0$ and the intersection $L_0 = P_j \cap P_0$ satisfies either $L_0 = \sigma L_0$ or $L_0 = \sigma L_{j+1}$ for some $\sigma \in \Gamma$. The angle between $L_j$ and $L_0$ is $k \alpha_j$ with an integer $k$;
(iv) the left and right subgroups $L$ and $R$ in the expression $\Gamma = \langle L \mid L \rangle$ satisfy $\mathbb{D}_2 \subset L$ and $\mathbb{D}_2 \subset R$.

Proof. As is noted in section 2, lemma 2, either $\Sigma_j \cong \mathbb{Z}_2$ or $\Sigma_j \cong (\mathbb{Z}_2)^2$. In both cases $s_j$ is an element of the group.

(i) To prove that $\alpha_j = \pi/K_j$, it is enough to remark that $N(\Sigma_j)/\Sigma_j \cong \mathbb{D}_{2K_j}$ (dihedral group of order $2K_j$) for some integer $K_j > 1$. Then, since $s_j s_j^{-1}$ is a rotation acting in $P_j$, it has to be in $D_{2K_j}$, so the only possibility is that $2\alpha_j = 2\pi/K_j$.

(ii) $\alpha_j = \pi/K_j$ implies that $(s_{j+1}s_{j-1})^{K_j} = \xi_j$, therefore $(s_{j+1}s_{j-1})^{K_j} \in \Sigma_j$. This transformation acts on $P_j$ as a rotation by $2\beta_j K_j$. Since $\Sigma_j \cong \mathbb{Z}_2$ or $\Sigma_j \cong (\mathbb{Z}_2)^2$, $2\beta_j K_j = k\pi$, which implies $\beta_j = k\alpha_j/2$.

(iii) Note that $L_0$ is one of the axes of symmetries, otherwise for some $\rho \in \mathbb{D}_{K_j}$, the axis $\rho L_0$ intersects with $K_j$. Since $L_0$ is an axis of symmetry, $L_0 = \sigma L_0$ or $L_0 = \sigma L_{j+1}$, and the definition of simple cycles implies that the intersection is orthogonal.

(iv) We choose a basis in $\mathbb{R}^4$ such that $\xi_2 = (0, a, 0, 0)$ and invariant planes containing the trajectories that join $\xi_2$ with $\xi_1$ and $\xi_3$ are

\[ P_1 = \langle e_1, e_2 \rangle, \quad P_2 = \langle e_2, e_3 \rangle. \]  

Denote by $(l_j; r_j)$ a pre-image of $s_j$ under the homomorphism $\Phi$. We have

\[ \Phi^{-1}s_1 = (l_1; r_1) = ((0, 1, 0, 0); (0, 1, 0, 0)), \quad \Phi^{-1}s_2 = (l_2; r_2) = ((0, 0, 0, 1); (0, 0, 0, -1)). \]  

The group generated by $l_1$ and $l_2$ is $\mathbb{D}_2$, and so is the one generated by $r_1$ and $r_2$. \( \text{QED} \)

Lemma 8. Suppose that a finite group $\Gamma^* \subset O(4)$, $\Gamma^* \not\subset SO(4)$, admits simple heteroclinic cycles. Then the group $\Gamma = \Gamma^* \cap SO(4)$ admits simple heteroclinic cycles.

Proof. Let $\Sigma_j^*, \Delta_j^* \subset \Gamma^*$, $j = 1, \ldots, m^*$, and $\gamma^* \in \Sigma_j^*$ be the sequences of isotropy subgroups, and the symmetry, satisfying C1–C5 in the statement of lemma 5. Define the subgroups $\Sigma_j, \Delta_j \subset \Gamma$ as follows:

- If $\Sigma_j^* \cong \mathbb{Z}_2$ (this is satisfied or not satisfied simultaneously for all $j$), then $\Sigma_j^* \subset SO(4)$ and we set $\Sigma_j = \Sigma_j^*$ and $\Delta_j = \Delta_j^*$, $j = 1, m^*$.
- If $\Sigma_j^* \cong (\mathbb{Z}_2)^2$, then there exists a plane reflection $\sigma_j \in \Sigma_j^*, \sigma_j \in SO(4)$. We set $\Sigma_j = \langle \sigma_j \rangle$ and $\Delta_j = \langle \sigma_{j-1}, \sigma_j \rangle$, $j = 1, m^*$.
- If $\gamma^* \in SO(4)$, then $\gamma = \gamma^*$ and $m = m^*$.
- If $\gamma^* \not\in SO(4)$, then $\gamma = (\gamma^*)^2$, $m = 2m^*$, $\Sigma_{j+m^*} = \gamma^* \Sigma_j (\gamma^*)^{-1}$ and $\Delta_{j+m^*} = \gamma^* \Delta_j (\gamma^*)^{-1}$, $j = 1, m^*$.

Evidently, $\Sigma_j, \Delta_j \subset \Gamma$, $j = 1, \ldots, m$, and $\gamma \in \Sigma_j$, satisfy C1–C5. Hence, if the group $\Gamma^*$ admits simple heteroclinic cycles, then so does $\Gamma$. \( \text{QED} \)
Lemma 9. Let \( r, s, k_1, k_2, n_1, n_2 \) and \( n_3 \) be integers satisfying the relation
\[
\frac{n_1}{k_1} + \frac{n_3}{k_1} = \frac{n_2}{k_2} + \frac{s n_3}{r k_2} = v.
\] (9)

(i) If \( k_1 \) and \( k_2 \) are co-prime; \( r \) and \( k_2 - sk_1 \) are co-prime
\[
\text{then } v \in \mathbb{Z}.
\]
(ii) If \( v \notin \mathbb{Z} \) then at least one of the conditions in (10) is not satisfied.

Proof. First, we notice that if \( k_1 = m K_1 \) and \( k_2 = m K_2 \) with \( m > 1 \), then \( n_1 = K_1, n_2 = K_2 \) and \( n_3 = 0 \) is a solution to (9) with \( v \notin \mathbb{Z} \). Now we suppose \( k_1 \wedge k_2 = 1 \). Assume, that there exists a solution to (9) such that \( v \notin \mathbb{Z} \). Since \( k_1 \) and \( k_2 \) are co-prime, for this solution \( n_3 \neq r K_3 \). Re-write (9) as
\[
n_1 k_2 - n_2 k_1 = n_3 \frac{s k_1 - k_2}{r}.
\]
If \( sk_1 - k_2 \) and \( r \) are co-prime, then the above equation does not have solutions with \( n_3 \neq r K_3 \); and if they are not co-prime, then it does. QED

3.3. Proof of theorems 2 and 3

3.3.1. Proof of theorem 2. According to lemma 7(iv), if a \( \Gamma \)-equivariant system possesses a heteroclinic cycle then the left and the right groups of \( \Gamma = (L \mid L_K; R \mid R_K) \) satisfy \( \mathbb{D}_2 \subset L \) and \( \mathbb{D}_2 \subset R \). Such subgroups of SO(4) are the groups 10–32 and 34–39 listed in table 1.

By definition of simple heteroclinic cycles and thanks to lemma 7(iii), an admissible group \( \Gamma \subset SO(4) \) involves at least two plane reflections \( s_1 \) and \( s_2 \), such that
\[
\text{I dim } P_j \cap P_0 = 1, \text{ where } P_j = \text{Fix } (s_j), \ j = 1, 2; \text{ II if } P_j \text{ or } P_0 \text{ intersects with a plane } P_0 = \text{Fix } (s_0), \text{ where } s_0 \in \Gamma, \text{ then } P_j \perp P_0; \text{ III if } L' = \text{Fix } (\Delta') \text{ for some } \Delta' \subset \Gamma \text{ satisfies dim } L' = 1 \text{ and } L' \subset P_j, \ j = 1 \text{ or } 2, \text{ then } \Delta' \cong (\mathbb{Z}_2)^2.
\]

To study whether \( \Gamma \subset SO(4) \) admits simple heteroclinic cycles, we proceed in three steps.

In step [i] we identify all plane reflections, which are elements of groups 10–32 and 34–39 in table 1. A plane reflection \( g = (l; r) \in \Gamma = (L \mid L_K; R \mid R_K) \) satisfies
\[
\text{I } \ell^2 = (-1, 0, 0, 0) \text{ and } r^2 = (-1, 0, 0, 0).
\]
Using (4) and the correspondence between \( L \) and \( R \) discussed in section 2.2, we obtain all such pairs \((l, r)\). In particular, we identify subgroups of SO(4) which do not possess plane reflections satisfying I and II.

In step [ii] we determine the conjugacy classes of subgroups of \( \Gamma \), isomorphic to \( \mathbb{Z}_2 \), which are generated by a plane reflection, and conjugacy classes of \( \Delta' \cong (\mathbb{Z}_2)^2 \), such that \( \Delta' \) is generated by two plane reflections and \( \dim \text{Fix } (\Delta') = 1 \). Using lemma 3 we then identify those \( \Gamma \)'s, which do not have plane reflections satisfying I–III.

Finally, in step [iii], we identify those sequences \( \Sigma_j \) and \( \Delta_j \) which satisfy C1–C5 and calculate structure angles \( \alpha_j \) and \( \beta_j \). If \( L = (\cos \omega, v \sin \omega) \) and \( r = (\cos \omega', v' \sin \omega') \), then the transformation \( q \rightarrow l q r^{-1} \) is a rotation of angles \( \omega \pm \omega' \) in a pair of absolutely perpendicular planes. Let \( E_j \cong \mathbb{Z}_2 \) be represented as \( E_j = \{ e, \sigma_j \} \). If \( \alpha_j \) and \( \beta_j \) are the structure angles of heteroclinic cycles according to definition 6, then the product \( \sigma_{j+1} \sigma_{j-1} \) acts as rotation by angles \( 2\alpha_j \) in \( P_j \) and \( 2\beta_j \) in \( P_j^L \), which allows us to calculate the angles \( \alpha_j \) and \( \beta_j \) from \( \sigma_{j+1} \).
and $\sigma_{j-1}$. The angle $\alpha_j$ can also be found as $\alpha_j = \pm \pi/k_j$, where $D_{k_j} = N(\Sigma_j)/\Sigma_j$. To find the structure angles we first determine $\alpha_j$. Then we represent in the quaternionic form $\sigma_j \sigma_{j-1} = (\cos \omega, v \sin \omega); (\cos \omega', v' \sin \omega')$ and note that $2\alpha_j = \omega \pm \omega'$ and $2\beta_j = \omega \mp \omega'$, which allows us to find $\beta_j$.

Below we show that the groups

$$(D_{2K_i} \mid D_{2K_k}; D_{2K_j} \mid D_{2K_k}) \text{ and } (T \mid Z_2; T \mid Z_2)$$

admit simple heteroclinic cycles, while the groups

$$(T \mid T; T \mid T), (O \mid O; O \mid O) \text{ and } (O \mid Z_2; O \mid Z_2)$$

do not. We derive the conditions (the relations between $n, k, r$ and $s$ for the first group, the restrictions on $K$ for the second) for the groups

$$(D_{nr} \mid Z_{2n}; D_{kr} \mid Z_{2k}); \text{ and } (D_{2K} \mid D_K; O \mid T)$$

to admit simple heteroclinic cycles. For other groups the proofs are similar and we omit them.

The proofs follow from appendices B–D in the extended arXiv variant of this paper [18], where

- In appendix B we list all plane reflections, which are elements of groups 10–32 and 34–39. Subgroups of SO(4), which do not possess plane reflections satisfying I and II, can be found from this list.
- In appendix C for all $\Gamma$’s satisfying I and II, we list conjugacy classes of subgroups of $\Gamma$, isomorphic to $Z_2$, which are generated by a plane reflection, and conjugacy classes of $\Delta' \cong \langle Z_2 \rangle^2$, $\dim \text{Fix}(\Delta') = 1$.
- In appendix D we list sequences $\Sigma_j$ and $\Delta_j$ which satisfy C1–C5 and structure angles $\alpha_j$ and $\beta_j$.

The group $(D_{2K_i} \mid D_{2K_k}; D_{2K_j} \mid D_{2K_k})$.

[i] The group $D_n$ (see (4)) is comprised of the elements

$$\rho_0(t) = (\cos t\pi/n, 0, 0, \sin t\pi/n), \quad \rho_0(t) = (0, \cos t\pi/n, \sin t\pi/n, 0), \quad 0 \leq t < 2n.$$ (12)

The pairs $(l; r) \in (D_{2K_i} \mid D_{2K_k}; D_{2K_j} \mid D_{2K_k})$ satisfy $l \in D_{2K_i}, r \in D_{2K_j}, \text{ where all possible combinations are elements of the group}$. Hence, in view of (11), the plane reflections are

$$\begin{align*}
\kappa_1(\pm) &= ((0, 0, 0, 1); (0, 0, 0, \pm 1)), \\
\kappa_2(n_1) &= ((0, \cos(n_1\theta_1), \sin(n_1\theta_1), 0); (0, 0, 0, 1)), \\
\kappa_3(n_2) &= ((0, 0, 0, 1); (0, \cos(n_2\theta_2), \sin(n_2\theta_2), 0)), \\
\kappa_4(n_1, n_2) &= ((0, \cos(n_1\theta_1), \sin(n_1\theta_1), 0); (0, \cos(n_2\theta_2), \sin(n_2\theta_2), 0)),
\end{align*}$$ (13)

where $\theta_1 = \pi/(2K_1), \theta_2 = \pi/(2K_2), 0 \leq n_1 < 4K_1$ and $0 \leq n_2 < 4K_2$. Lemma 3 implies that for any $n_1$ and $n_2$ the plane reflections $s_1 = \kappa_2(n_1)$ and $s_2 = \kappa_3(n_2)$ satisfy I and II.

[ii] In the group $D_n$, the elements $(0, \cos(t\pi/n), \sin(t\pi/n), 0)$ split into two conjugacy classes, corresponding to odd and even $t$. The elements $(0, 0, 0, 1)$ and $(0, 0, 0, -1)$ are conjugate. Therefore, the group has nine conjugacy classes of isotropy subgroups $\Sigma \cong Z_2$:

$$\begin{align*}
\{e, \kappa_1(\pm)\}, \\
\{e, \kappa_2(n_1)\} &\text{ : } n_1 \text{ even or odd}, \\
\{e, \kappa_3(n_2)\} &\text{ : } n_2 \text{ even or odd}, \\
\{e, \kappa_4(n_1, n_2)\} &\text{ : } n_1 \text{ even or odd, } n_2 \text{ even or odd}.
\end{align*}$$ (14)
We set:

\[ \Delta = \{e, \kappa_2(n_1), \kappa_3(n_2), \kappa_4(n_1 - K_1, n_2 + K_2)\}. \quad (15) \]

Similarly, we do not discuss the largest possible heteroclinic network admitted by other groups.

and not to find the largest heteroclinic network which can possibly exist in a \( \Gamma_1 \)

of \( \kappa \) if \( K \).

The condition \( 1 \) and \( 1 \) are co-prime. The goal of the study is to prove existence of heteroclinic cycles, and to find the largest heteroclinic network which can possibly exist in a \( \Gamma \)-equivariant system. Since the isotropy subgroups (16) satisfy C1–C5, the additional axes are not discussed. Similarly, we do not discuss the largest possible heteroclinic networks admitted by other groups.

The group \( (\mathbb{D}_{4K}; \mathbb{Z}_{2K}) \).

[ii] The condition \( \mathbb{D}_2 \subset \mathbb{D}_{4K}; \mathbb{D}_{2K} \), implies that \( \Gamma \) is either

\( (\mathbb{D}_{4K}; \mathbb{Z}_{2K}) \), with odd \( r \), or \( (\mathbb{D}_{4K}; \mathbb{Z}_{2K}) \). Because of (11) and (12), the reflections in the group \( (\mathbb{D}_{4K}; \mathbb{Z}_{2K}) \) are

\[ \kappa_1(\pm) = ((0, 0, 0, 1); (0, 0, 0, \pm 1)), \]

\[ \kappa_2(n_1, n_2, n_3) = ((0, \cos(n_1 \theta_1 + n_3 \theta_1^*), \sin(n_1 \theta_1 + n_3 \theta_1^*), 0); (0, \cos(n_1 \theta_2 + n_3 \theta_2^*), \cos(n_1 \theta_3 + n_3 \theta_3^*), 0)), \quad (17) \]

where \( 0 \leq n_1 < 4K \), \( \theta_j = \pi/(2K) \) and \( \theta_j^* = \theta_j/r, j = 1, 2 \). Denote by \( P(n_1, n_2, n_3) \) the fixed-point subspace of \( \kappa_2(n_1, n_2, n_3) \). Lemma 3 implies that planes \( P(n_1, n_2, n_3) \) and \( P(n'_1, n'_2, n'_3) \) intersect if

\[ \cos((n_1 - n'_1) \theta_1 + (n_3 - n'_3) \theta_1^*) = \cos((n_2 - n'_2) \theta_2 + (n_3 - n'_3) \theta_2^*). \quad (18) \]

By lemma 9, if

\[ K_1 \text{ and } K_2 \text{ are co-prime, } r \text{ and } K_2 - sK_1 \text{ are co-prime,} \quad (19) \]

then the only solutions to this equation are \( (n_1 - n'_1) \theta_1 + (n_3 - n'_3) \theta_1^* = M_1 \pi/2, (n_2 - n'_2) \theta_2 + (n_3 - n'_3) \theta_2^* = M_2 \pi/2 \), i.e. any intersection is orthogonal. If (19) is not
satisfied, then there exist solutions to (18) with \( \cos((n_1 - n_1')\theta_1 + (n_3 - n_3')\theta_1^o) \neq 0, \pm 1 \), hence the intersection is non-orthogonal. If (19) holds true, then \( s_1 = \kappa_2(0, 0, 0) \) and \( s_2 = \kappa_2(K_1, K_2, 0) \) satisfy I and II.

The elements of the group \( \mathbb{D}_{2K_J', \mathbb{Z}_{2K_J}} \), which are plane reflections are different for odd and even \( K_1 \) and \( K_2 \) (note that the case when both are even was considered above). If \( K_1 \) is even and \( K_2 \) is odd, then plane reflections are given by

\[
\kappa_1(n_1, n_2, n_3) = ((0, \cos(2n_1\theta_1 + n_3\theta_1^o)), \sin(2n_1\theta_1 + n_3\theta_1^o), 0);
\]

\[
(0, \cos(2n_2\theta_2 + n_3\theta_2^o), \cos(2n_2\theta_2 + n_3\theta_2^o), 0),
\]

\[
\kappa_2(n_1, n_2, n_3) = ((0, \cos((2n_1 + 1)\theta_1 + n_3\theta_1^o)), \sin((2n_1 + 1)\theta_1 + n_3\theta_1^o), 0);
\]

\[
(0, \cos((2n_2 + 1)\theta_2 + n_3\theta_2^o), \cos((2n_2 + 1)\theta_2 + n_3\theta_2^o), 0),
\]

where \( 0 \leq n_j < 2K_j, j = 1, 2 \). It can be easily shown that whenever \( \kappa_1(n_1, n_2, n_3) \) and \( \kappa_i(n'_1, n'_2, n'_3) \), \( i = 1, 2, \) intersect, the intersection is non-orthogonal.

The group \( \Gamma = (\mathbb{D}_{2K_J', \mathbb{Z}_{2K_J}} \cup \mathbb{D}_{2K_J, \mathbb{Z}_{2K_J}}) \), where \( K_1 \) and \( K_2 \) odd, involves plane reflections

\[
\kappa_1(\pm) = ((0, 0, 0, 1); (0, 0, 0, 1)),
\]

\[
\kappa_2(n_1, n_2, n_3) = ((0, \cos(2n_1\theta_1 + n_3\theta_1^o)), \sin(2n_1\theta_1 + n_3\theta_1^o), 0);
\]

\[
(0, \cos(2n_2\theta_2 + n_3\theta_2^o), \cos(2n_2\theta_2 + n_3\theta_2^o), 0),
\]

\[
\kappa_3(n_1, n_2, n_3) = ((0, \cos((2n_1 + 1)\theta_1 + n_3\theta_1^o)), \sin((2n_1 + 1)\theta_1 + n_3\theta_1^o), 0);
\]

\[
(0, \cos((2n_2 + 1)\theta_2 + n_3\theta_2^o), \cos((2n_2 + 1)\theta_2 + n_3\theta_2^o), 0).
\]

Lemma 9 implies that whenever

\[
K_1 \text{ and } K_2 \text{ are co-prime, } r \text{ and } (K_2 \pm sK_1)/2 \text{ are co-prime},
\]

a plane fixed by \( \kappa_1(n_1, n_2, n_3), j = 1 \text{ or } 2 \), intersect only orthogonally with another plane fixed by a plane reflection. Hence, we set \( s_1 = \kappa_2(0, 0, 0) \) and \( s_2 = \kappa_2((K_1 - 1)/2, (K_2 - 1)/2, 0) \).

For the group \( \Gamma = (\mathbb{D}_{2K_J', \mathbb{Z}_{2K_J}} \cup \mathbb{D}_{2K_J, \mathbb{Z}_{2K_J}}) \), where \( K_1 \) and \( K_2 \) odd, by lemma 9 the planes fixed by elements of the group intersect only orthogonally if and only if \( K_1 \) and \( K_2 \) are co-prime, \( r \) and \( (K_2 \pm sK_1)/2 \) are co-prime,

\[
r \text{ and } (K_2 \pm sK_1)/4 \text{ are co-prime},
\]

where plus or minus are taken so that the ratios are integer.

[iii] In the group \( \mathbb{D}_{2n} \) (see (12)) the elements \( \rho_{2n}(n) = (0, 0, 0, 1) \) and \( \rho_{2n}(3n) = (0, 0, 0, -1) \) are conjugate by \( \sigma_{2n}(t) \). The group \( (\mathbb{D}_{2n}, \mathbb{Z}_{2K}) \), involves \( \sigma \)'s only in pairs \((\sigma_{2n}(t_1); \sigma_{2n}(t_2))\), therefore \( \kappa_1(+) \) and \( \kappa_1(-) \) are not conjugate in this group. The splitting of \( \kappa_2 \) and \( \kappa_1 \) into conjugacy classes depends on whether \( K_1 \), \( K_2 \) and \( r \) are even or odd. Here we consider only the case of \( (\mathbb{D}_{2K_J', \mathbb{Z}_{4K_J}} \cup \mathbb{D}_{2K_J, \mathbb{Z}_{4K_J}}) \), where \( K_1 \), \( K_2 \), \( r \) and \( s \) satisfy (19), \( K_1 \), \( K_2 \), \( r \) and \( s \) are odd. The cases of other parities are similar and we do not present them. Under this assumption, the reflections \( \kappa_2(n_1, n_2, n_3) \) in (17) split into four conjugacy classes, a class is categorized by whether the sums \( n_1 + n_3 \) and \( n_2 + n_3 \) are even or odd. By arguments presented in part [i], if (19) is satisfied, then the reflections \( s_1 \) and \( s_2 \) satisfy III.

We set

\[
\Sigma_1 = \{ e, \kappa_2(0, 0, 0) \}, \Sigma_2 = \{ e, \kappa_2(K_1, K_2, 0) \},
\]

\[
\Delta_1 = \{ e, \kappa_1(-), \kappa_3(0, 0, 0), \kappa_2(K_1, 3K_2, 0) \},
\]

\[
\Delta_2 = \{ e, \kappa_2(+) \}, \kappa_2(0, 0, 0), \kappa_2(K_1, K_2, 0) \}.
\]
and $\gamma = ((1, 0, 0, 0); (0, 0, 0, 1))$. Since $\gamma \kappa_2^{-1}(n_1, n_2, n_3)\gamma^{-1} = \kappa_2(n_1 + 2K_1, n_2 + 2K_2, n_3)$, the sequences $\Sigma_j, \Delta_j, j = 1, \ldots, 2$, satisfy conditions C1–C5 of lemma 5. We have $N(\Sigma_1)/\Sigma_1 = D_2$ and $\sigma_1\sigma_2 = \gamma\sigma_2\gamma^{-1}\sigma_2 = ((-1, 0, 0, 0); (1, 0, 0, 0))$, therefore $\alpha_1 = \pi/2$ and $\beta_1 = \pi/2$. Similarly, $\alpha_2 = \pi/2$ and $\beta_2 = \pi/2$.

The group $(\mathbb{T} | \mathbb{T}; \mathbb{T} | \mathbb{T})$.

[i] The pairs $(l; r) \in (\mathbb{T} | \mathbb{T}; \mathbb{T} | \mathbb{T})$ satisfy $l \in \mathbb{T}, r \in \mathbb{T}$, where all possible combinations are elements of the group. Hence, the plane reflections are

$$k(\pm, r, s) = \pm(\rho \cdot u; \rho' \cdot u),$$

where $u = (0, 0, 0, 1)$ and the permutation $\rho$ acts as $\rho(a, b, c, d) = (a, c, d, b)$. By lemma 3, whenever two $\kappa$’s intersect, the intersection is orthogonal. Taking $s_1 = k(+, 0, 0)$ and $s_2 = k(+, 1, 1)$, we get plane reflections satisfying I and II.

[ii] The quaternions $(0, 0, 1, 0)$ and $(0, 1, 0, 0)$ are conjugate by $(a, b, b, a)$, the quaternions $(0, 0, 1, 0)$ and $(0, 0, -1, 0)$ are conjugate by $(0, a, b, 0)$, hence all plane reflections in the group are conjugate. The plane fixed by $k(\pm, r, s)$ intersects with the ones fixed by $k(\pm, r + j, s + k), j, k = 0, 1, 2$. Hence, the group $(\mathbb{T} | \mathbb{T}; \mathbb{T} | \mathbb{T})$ does not satisfy III.

The group $(\mathbb{T} | \mathbb{Z}_2; \mathbb{T} | \mathbb{Z}_2)$.

[i] Elements of the group are the pairs $(l; r)$ such that $l \in \mathbb{T}$ and $r = \pm l$. Therefore, the group involves plane reflections

$$k(\pm, r) = \pm(\rho \cdot u; \rho' \cdot u).$$

The planes $\text{Fix}(k(\pm, r))$ and $\text{Fix}(k(\pm, s))$ intersect whenever $r \neq s$ and the intersection is orthogonal. The plane reflections $s_1 = k(-, 0, 0)$ and $s_2 = k(-, 1)$ satisfy I and II.

[ii] The plane reflections split into two conjugacy classes: $k(+, r)$ and $k(-, r)$. There are two isotropy types of one-dimensional subspaces, their symmetry groups involve the following plane reflections:

$$(a): k(+, r), k(+, r + 1), k(+, r + 2),$$

$$(b): k(+, r), k(-, r + 1), k(-, r + 2).$$

In the former case the three plane reflections are cyclically conjugate by a symmetry $((1/2, 1/2, 1/2, 1/2); (1/2, 1/2, 1/2, 1/2))$ in $(\mathbb{T} | \mathbb{Z}_2; \mathbb{T} | \mathbb{Z}_2)$, hence the isotropy subgroup of this line is $\mathbb{D}_3$. In the latter case the isotropy subgroup is $(\mathbb{Z}_2)^2$.

[iii] Setting

$$\Sigma_1 = \{e, k(-, 0)\}, \ \Delta_1 = \{e, k(+, 2), k(-, 0), k(-, 1)\}$$

and $\gamma = ((1/2, 1/2, 1/2, 1/2); (1/2, 1/2, 1/2, 1/2))$, we get the sequences $(m = 1)$ satisfying C1–C5. The structure angles are $\alpha_1 = \pi/2$ and $\beta_1 = \pi$.  

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The group \((\mathbb{O} | \mathbb{O}; \mathbb{O} | \mathbb{O})\).

[i] The pairs \((l; r)\) \(\in (\mathbb{O} | \mathbb{O}; \mathbb{O} | \mathbb{O})\) are any combinations of \(l \in \mathbb{O}\) and \(r \in \mathbb{O}\). Hence, the plane reflections are

\[
\kappa_1(\pm, r, s) = \pm(\rho^r u, \rho^r u), \quad \kappa_2(\pm, r, s, \pm) = \pm(\rho^r u, \rho^r v_{\pm}),
\]

\[
\kappa_3(\pm, r, s, \pm) = (\rho^r v_{\pm}, \rho^r u), \quad \kappa_4(\pm, r, s, \pm) = (\rho^r v_{\pm}, \rho^r v_{\pm}),
\]

where \(u = (0, 0, 0, 1), \quad v_{\pm} = (0, 1, \pm 1, 0)/\sqrt{2}\) and the permutation \(\rho\) acts as \(\rho(a, b, c, d) = (a, c, d, b)\). Planes fixed by \(\kappa_1\) and \(\kappa_4\) intersect non-orthogonally and so do the ones fixed by \(\kappa_2\) and \(\kappa_3\). Therefore, the group does not have plane reflections satisfying I and II.

The group \((\mathbb{O} | \mathbb{Z}_1; \mathbb{O} | \mathbb{Z}_1)\).

[ii] The group is comprised of the pairs \((l; r)\), such that \(l \in \mathbb{O}\) and \(l = r\). The plane reflections are

\[
\kappa_1(r) = (\rho^r u, \rho^r u) \quad \text{and} \quad \kappa_2(r, \pm) = (\rho^r v_{\pm}, \rho^r v_{\pm}).
\]

Since the planes fixed by \(\kappa_1\) and \(\kappa_2\) intersect non-orthogonally, the group does not admit heteroclinic cycles.

The group \(\Gamma = (\mathbb{D}_{2K} | \mathbb{D}_K; \mathbb{O} | \mathbb{T})\), \(K\) even.

[iii] The group \((\mathbb{D}_{2K} | \mathbb{D}_K; \mathbb{O} | \mathbb{T})\) is comprised of the pairs \((l; r)\), where either \(l \in \mathbb{D}_K\) and \(r \in \mathbb{T}\), or \(l \in \mathbb{D}_{2K} \setminus \mathbb{D}_K\) and \(r \in \mathbb{O} \setminus \mathbb{T}\).

Therefore, for even \(K\) the group has the following plane reflections:

\[
\kappa_1(\pm, r) = ((0, 0, 0, \pm 1); \rho^r u), \quad \kappa_2(n, r) = ((0, \cos(2n\theta), \sin(2n\theta), 0); \rho^r u), \quad \kappa_3(n, r, \pm) = ((0, \cos((2n + 1)\theta), \sin((2n + 1)\theta), 0); \rho^r v_{\pm}),
\]

where \(\theta = \pi/(2K)\) and \(0 \leq n \leq 2K\). By lemma 3, if \(K = 2(2K + 1)\) then the planes fixed by \(\kappa_2\) and \(\kappa_3\) intersect non-orthogonally. Otherwise, plane reflections \(s_1 = \kappa_2(0, 0)\) and \(s_2 = \kappa_2(2K/2, 1)\) satisfy I and II.

The group has three conjugacy classes of isotropy subgroups satisfying \(\dim \text{Fix} (\Sigma) = 2\), they are

\[
[e, \kappa_1(\pm, r)], \quad [e, \kappa_2(n, r)], \quad [e, \kappa_3(n, r, \pm)].
\]

For \(K \neq 2(2K + 1)\) it has two isotropy types of symmetry axes, one of which has the isotropy subgroup

\[
[e, \kappa_1(\pm, r), \kappa_2(n + 1, r, + 1), \kappa_2(n + K/2, r, + 2)],
\]

isomorphic to \((\mathbb{Z}_2)^2\). (The other axis has isotropy subgroup generated by two \(\kappa_3\), it can be isomorphic to \((\mathbb{Z}_2)^2\), or it can be not, depending on \(K\).) The planes fixed by \(\kappa_2\) contain only symmetry axes with the group (29). Therefore, III holds true.

[iii] The isotropy subgroups

\[
\Sigma_1 = \{e, \kappa_2(0, 0)\}, \quad \Delta_1 = \{e, \kappa_1(+, 1), \kappa_2(0, 0), \kappa_2(K/2, 2)\},
\]

and the symmetry \(\gamma = ((1, 0, 0, 1)/\sqrt{2}; (1, 1, 1, 1)/2)\) satisfy conditions C1–C5 with \(m = 1\). The structure angles of this homoclinic cycle are \(\alpha_1 = \pi/4\) and \(\alpha_2 = \pi/4\)

\[QED\]
3.3.2. Proof of theorem 3. Recall, that a group $\Gamma^* \in O(4)$, $\Gamma^* \not\subseteq SO(4)$, can be decomposed as

$$\Gamma^* = \Gamma \oplus \sigma \Gamma,$$

where in the quaternion form $\Phi^{-1} \Gamma = (L \mid L_K \mid R \mid R_K)$, the groups $L$ and $R$ are isomorphic, and so are $L_K$ and $R_K$. A reflection $\sigma : q \rightarrow aqb$ is written as $\sigma = (a, b)^\ast$. By lemma 8, if the group $\Gamma^*$ admits simple heteroclinic cycles, then so does $\Gamma$.

Admissible subgroups of $\Gamma \subset SO(4)$ are listed in theorem 2, the ones which have isomorphic left and right groups are:

$$\begin{align*}
(D_{2K} \mid D_{2K} \mid D_{2K}), & (D_{2r} \mid Z_4; D_{2r} \mid Z_4), (D_{2r} \mid Z_2; D_{2r} \mid Z_2), \\
(D_{2K} \mid D_{2K} \mid D_{2K}), & (T \mid Z_2; T \mid Z_2), (D_{2r} \mid Z_1; D_{2r} \mid Z_1).
\end{align*}$$

(30)

A reflection $\sigma \not\subseteq SO(4)$ has $\pm 1$ for two of its eigenvalues, the other two being $e^{i\pi / 2}$. Hence, we obtain the first five groups listed in the statement of theorem 3.

First, we consider $\omega = k \pi$. If $\omega = 0$, then $\sigma$ is a reflection about a three-dimensional hyperplane orthogonal to a vector $e$, leaving unchanged all vectors in the hyperplane and reversing all orthogonal. If $\omega = \pi$, then $\sigma$ is an axial reflection about an axis along a vector $e'$. Any plane $P_0$ fixed by a subgroup $\Sigma_0 \subset \Gamma$ is mapped by $\sigma$ to a plane (perhaps, the same), fixed by $\Sigma_0 \subset \Gamma$. If $P_j$ is one of the planes involved in a simple heteroclinic cycle, then the orthogonal complement to $e$, or to $e'$, which we denote by $V$, is either orthogonal to $P_j$, or $P_j \subset V$. Since this holds true for all $1 \leq j \leq m$, the planes $P_j$ are coordinate planes in an appropriate basis, structure angles are multiples of $\pi / 2$ and $e$ (or $e'$) is a basis vector. The groups in (30) that have structure angles multiples of $\pi / 2$ are

$$(D_4 \mid Z_2; D_4 \mid Z_2), (D_4 \mid Z_4; D_4 \mid Z_4), (D_2 \mid Z_2; D_2 \mid Z_2), (T \mid Z_2; T \mid Z_2), (D_2 \mid Z_1; D_2 \mid Z_1).$$

For the first two group the direction of $L_1$ can be taken as $(0, 1, 0, 1)/\sqrt{2}$, for the next two groups as $(0, 1, 0, 0)$ and for the last as $(1, 0, 0, 0)$. Hence, we obtain the first five groups listed in the statement of theorem 3.

Second, we consider $\omega \neq 0, \pi$. The symmetry $\sigma$ maps any $P_j$ into another plane, which does not belong to the group orbit of $P_j$ in $\Gamma$, because otherwise the isotropy subgroup of $L_j \subset P_j$ has elements of order more than two. For $(D_{2K} \mid D_{2K}) \oplus (D_{2K} \mid D_{2K})$ the only possibility is $\sigma : P_j \rightarrow P_{j+2}$, and therefore $L_j \rightarrow L_{j+3}$ and $L_2 \rightarrow L_4$. Hence $\sigma^4$ maps $P_j \rightarrow P_{j+4}$ for any $j$. For this group, there exists a heteroclinic cycle with four equilibria, implying that $\sigma^4$ is an identity, which is possible only if $\sigma$ is an axial reflection, or a reflection about a three-dimensional hyperplane. Therefore, there is no heteroclinic group in $O(4)$, which has $(D_{2K} \mid D_{2K} \mid D_{2K} \mid D_{2K})$ as a reflection subgroup with $\omega \neq 0, \pi$. For $(T \mid Z_2; T \mid Z_2)$ such a $\sigma$ does not exist, because the group has only one group orbit of fixed planes.

For other groups in (30) the heteroclinic cycle (see appendix D in [18]) involves two group orbits of planes, hence $\sigma : P_j \rightarrow P_{j+1}$. Since for all groups, except for $(D_{2K} \mid D_{2K} \mid D_{2K} \mid D_{2K})$, $\alpha_2$ and $\beta_2$ are multiples of $\pi / 2$, they do not give rise to subgroups of $O(4)$, different from already obtained. For $(D_{2K} \mid D_{2K} \mid D_{2K} \mid D_{2K})$ the condition $\sigma : P_j \rightarrow P_{j+1}$ determines $\sigma$, up to multiplication by some $\gamma \in \Gamma$.

QED

4. Examples

In this section we provide some examples of simple heteroclinic cycles of type A in $\mathbb{R}^4$. We will also give an example of a pseudo-simple heteroclinic cycle.
4.1. Simple heteroclinic cycles of type A

4.1.1. The simplest case. Consider the following transformations in $\mathbb{R}^4$:

\[ \kappa_1 : (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, -x_3, -x_4) \]
\[ \kappa_2 : (x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, -x_4) \]
\[ \kappa_3 : (x_1, x_2, x_3, x_4) \mapsto (-x_1, -x_2, x_3, x_4). \]

They generate a group $\Gamma_0$ which is isomorphic to $\mathbb{Z}_2^3$; note however the difference with the case $B_2^0$ in theorem 1. There is no invariant hyperplane; however, each $\kappa_j$ has a planar fixed-point subspace and there are overall six such invariant planes. Moreover, each plane contains two axes of symmetry, which are the coordinate axes. In the list of theorem 2, this group is $(\mathbb{D}_2 \times \mathbb{Z}_2) \rtimes (\mathbb{D}_2 \rtimes \mathbb{Z}_2)$, with $m = n = 1$ and $r = 2)$. In terms of quaternionic presentation, we have

\[ \kappa_1 = [i, i], \kappa_2 = [k, -k], \kappa_3 = [i, -i], \]

where $i, j, k$ are the usual quaternion basis ‘imaginary’ elements.

Remark that $-I \in \Gamma_0$ acts non-trivially in $\mathbb{R}^4$. Simple robust heteroclinic cycles can easily be built from the knowledge of the general equivariant smooth vector fields. Indeed, one can easily check the following lemma (using Schwarz theorem on the structure of equivariant vector fields under smooth compact group actions):

**Lemma 10.** Every smooth, $\Gamma_0$ equivariant differential system has the following form:

\[ \dot{x}_1 = a_1(x_1^2, x_2^2, x_3^2, x_4^2, \theta) x_1 + b_1(x_1^2, x_2^2, x_3^2, x_4^2, \theta) x_2 x_3 x_4, \]
\[ \dot{x}_2 = a_2(x_1, x_2, x_3, x_4, \theta) x_2 + b_2(x_1, x_2, x_3, x_4, \theta) x_1 x_3 x_4, \]
\[ \dot{x}_3 = a_3(x_1^2, x_2^2, x_3^2, x_4^2, \theta) x_3 + b_3(x_1^2, x_2^2, x_3^2, x_4^2, \theta) x_1 x_2 x_4, \]
\[ \dot{x}_4 = a_4(x_1^2, x_2^2, x_3^2, x_4^2, \theta) x_4 + b_4(x_1^2, x_2^2, x_3^2, x_4^2, \theta) x_1 x_2 x_3, \]

where $\theta = x_1 x_2 x_3 x_4$ and $a_j, b_j$ are smooth functions.

It is then an elementary computation to check that the conditions of existence of a robust heteroclinic cycle connecting equilibria on the symmetry axes are generically fulfilled.

4.1.2. A non-trivial example. This example was studied first in the context of pattern formation on the hyperbolic plane [5]. Let $\Gamma_1$ be the group generated by the following $4 \times 4$ matrices:

\[ \kappa = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \rho = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}, \quad \sigma = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}. \]

(31)

This group has 96 elements. The generators can be identified with the following elements in the quaternionic presentation:

\[ \kappa = [i, j], \quad \rho = \frac{\sqrt{2}}{2} [1 - k, i], \quad \sigma = \frac{\sqrt{2}}{2} [j + k, i]. \]

In the nomenclature of theorem 2, $\Gamma_1 = (\mathbb{D} \rtimes \mathbb{T}; \mathbb{D}_2 \rtimes \mathbb{Z}_2)$.

The following groups are four-elements subgroups of $\Gamma_1$. They are isomorphic but belong to different conjugacy classes:

\[ \bar{C}_{2k} = \langle \sigma, \kappa \rangle \quad \text{and} \quad \bar{C}_{2k}^{-1} = \langle \rho^2 \sigma \rho^{-2}, \kappa \rangle. \]
The action of $\Gamma_1$ admits the following lattice of isotropy types [5], where $\kappa' = \rho \kappa$ is not conjugated to $\kappa$.

\[
\begin{array}{c}
\mathcal{C}_{2\kappa} \\
\langle \kappa \rangle \\
(1)
\end{array}
\begin{array}{c}
\mathcal{C}'_{2\kappa} \\
\langle \sigma \rangle \\
(2)
\end{array}
\begin{array}{c}
\mathcal{C}_{2\kappa}' \\
\langle \kappa' \rangle
\end{array}
\]

The numbers in parentheses are the dimensions of the corresponding fixed-point subspaces. Moreover the planes $\text{Fix}(\sigma)$ and $\text{Fix}(\kappa')$ contain one copy of each type of symmetry axes, while $\text{Fix}(\kappa)$ contains two copies of each.

The general form of $\Gamma_1$ equivariant vector fields is complicated but the polynomial form up to degree 5 has been computed in [5] and it was shown that a codimension 1 bifurcation from the trivial equilibria leads to robust heteroclinic cycles. These cycles are simple (as is clear from the isotropy subgroups). Also observe that there are in fact two types of cycles, hence a heteroclinic network. Their asymptotic stability depends upon terms of order 7.

4.2. A pseudo-simple heteroclinic cycle

Here we show that pseudo-simple cycles exist. An example is the (unique) four-dimensional irreducible representation of the group $GL(2, 3)$ ($2 \times 2$ invertible matrices over the field $\mathbb{Z}_3$). This group is generated by the elements $\rho$ (order 8) and $\sigma$ (order 2) below:

\[
\rho = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}
\]

The group has eight conjugacy classes and exactly one 4-dimensional irreducible representation. Writing $\epsilon = \sigma \rho^{-1}$, the conjugacy classes and character table of this representation is given below (see [12]):

| Representative | $Id$ | $\rho$ | $\rho^2$ | $-Id$ | $\rho^5$ | $\sigma$ | $\epsilon$ | $-\epsilon$ |
|----------------|------|-------|---------|-------|---------|--------|-----------|----------|
| Order          | 1    | 8     | 4       | 2     | 8       | 2      | 3         | 6        |
| # elements     | 1    | 6     | 6       | 1     | 6       | 12     | 8         | 8        |
| Character      | 4    | 0     | 0       | $-4$  | 0       | 0      | 1         | $-1$     |

From this table and using the trace formula for the computation of the dimension of fixed-point subspaces [2] one finds that there are exactly two submaximal isotropy types: their group representatives are $\Sigma_1 = \langle \sigma \rangle$ and $\Sigma_2 = \langle \epsilon \rangle$. Their fixed-point subspaces have dimension 2. Moreover, each of these planes contains exactly one copy of each of the two types of symmetry axes, the isotropy of which are isomorphic to the dihedral group $D_3$ but are not conjugate in $GL(2, 3)$. From this and using either the same proof as in lemma 5 or by explicit computation of an equivariant vector field, one can show the existence of robust heteroclinic cycles between equilibria on the symmetry axes. Clearly, these equilibria have isotropies which fall into cases 2 or 3 of lemma 1: $\Sigma_2 \cong \mathbb{Z}_3$ and $\Delta_2 \cong D_3$, which implies that the heteroclinic cycles are pseudo-simple.

In quaternionic form the group is $(O_3 \mid \mathbb{Z}_2; O \mid V)$. We do not pursue further in this example, which is one of a list of pseudo-simple cycles in $\mathbb{R}^4$ yet to be established.
5. Discussion

We have found a complete list of finite subgroup of O(4) admitting simple heteroclinic cycles, thus complementing the classification initiated by [11] (cycles of types B and C) and [20, 21] (homoclinic cycles). This led us to define pseudo-simple heteroclinic cycles, a case which had not been envisaged before. An example of a pseudo-simple cycle is given; however, their classification is yet to be completed.

This work was based on the quaternionic presentation of finite subgroups of SO(4). Note that, such an approach can be applied to other questions in equivariant bifurcation theory in \( \mathbb{R}^4 \). The appendix provides an example where a problem treated in [13] gets a shorter solution. The reconstruction of the matrix group actions, invariant planes and axes and equivariant systems with heteroclinic cycles, can be performed from the formulae in section 2.2 and from tables in appendices C–D in [18].

The subgroups of O(4) which do not admit simple heteroclinic cycles can admit pseudo-simple heteroclinic cycle, as is shown in section 4.2. A pseudo-simple cycle has at least one equilibria \( \xi_j \) where the expanding eigenvector belongs to the two-dimensional isotypic component in the decomposition of \( \Delta_j \). This implies that \( L_j \) is the intersection of several symmetric copies of \( P_j \), which gives rise to a new kind of potentially complex nearby dynamics. Subgroups, admitting pseudo-simple heteroclinic cycles, can be found and the cycles can be identified using the same technique as in the present paper. In fact, the subgroups of O(4) typically admit not just heteroclinic cycles, but more complex heteroclinic networks. (This should be clear from the tables in appendices C–D in [18]). Identification of such networks can also be achieved by the same approach.

According to [10, 11, 17], any simple heteroclinic cycle can be asymptotically stable, provided that eigenvalues of \( df(\xi_j) \) satisfy some inequalities stated ibid. If a cycle is not asymptotically stable, it can be stable in a weaker sense and attract a positive measure set of initial conditions, as discussed in [11, 14, 15]. The local extension of the basin of attraction can be described in terms of stability indices, which were introduced in [17]. However, this issue is beyond the scope of this paper.

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Appendix. Subgroups of O(4) that do not have one-dimensional fixed-point subspaces

Here we give a list of subgroups of O(4) which act irreducibly and do not possess axes of symmetry. The proof of the main theorem is based on a series of lemmas given below.

**Lemma 11.** Consider \( g \in SO(4) \), \( \Phi^{-1}g = ((\cos \alpha, \sin \alpha v); (\cos \beta, \sin \beta w)) \). Then \( \dim \text{Fix } < g >= 2 \) if and only if \( \cos \alpha = \cos \beta \).

**Lemma 12.** Consider \( g, s \in SO(4) \), where \( \Phi^{-1}g = ((\cos \alpha, \sin \alpha v); (\cos \alpha, \sin \alpha w)) \) and \( \Phi^{-1}s = ((0, v); (0, w)) \). Then \( \text{Fix } < g >= \text{Fix } < s > \).
Lemma 13. The action of $\Gamma = \Phi(Z_2 \mid Z_2 \mid Z_2)$ on $\mathbb{R}^4$ is reducible.

The proofs follow from the properties of quaternions and we do not present them.

Lemma 14. If a group $\Gamma \subset SO(4)$ has a one-dimensional fixed-point subspace then
\[ \Phi^{-1}(\Gamma) = (L \mid L_K; R \mid R_K) \] satisfies
\[ L \supset \mathbb{D}_s \text{ and } R \supset \mathbb{D}_t \text{ for some } s \geq 2. \] (32)

Proof. Any one-dimensional fixed-point subspace $L$ of $\Gamma \subset SO(4)$ is an intersection of two isotropy planes, $P_1$ and $P_2$. Denote by $s_j$ elements of $SO(4)$ such that $P_s = \text{Fix } s_j$. The group $\langle s_1, s_2 \rangle$ acting on $\mathbb{R}^3 = \mathbb{R}^4 \oplus L$ does not have fixed-point subspaces, therefore $s_1, s_2 \neq \mathbb{Z}_k$ for any $k$. Hence, $s_1, s_2 \supset \mathbb{D}_t$ for some $t \geq 2$, which implies (32). QED

Lemma 15. Suppose that a group $\Gamma \subset SO(4)$ satisfies
(i) $\Gamma$ is not a subgroup of $(Z_n \mid Z_n \mid Z_n \mid Z_n)$ for any $n$ and $k$;
(ii) $\Gamma$ does not have one-dimensional fixed-point subspaces.

Then the group $\Gamma$ acts on $\mathbb{R}^4$ irreducibly.

Proof. There exists a group $(Z_{rN} \mid Z_{rM} \mid Z_{rM}) \subset \Gamma$ where at least one of $rN \geq 3$ or $rM \geq 3$ is satisfied. The elements of $(Z_{rN} \mid Z_{rM} \mid Z_{rM})$ act as rotations in two absolutely perpendicular planes, $V_1$ and $V_2$. The condition (i) implies existence of $g \in \Gamma$, such that $g \neq (Z_{rN} \mid Z_{rM} \mid Z_{rM})$. If the action of $\Gamma$ is reducible, then both $V_1$ and $V_2$ are $g$-invariant and $g$ acts on both $V_1$ and $V_2$ as a reflection. The group, generated by any $q \in (Z_{rN} \mid Z_{rM} \mid Z_{rM})$, $q \neq e$, and $g$, contains $(\mathbb{D}_s \mid \mathbb{Z}_1 \mid \mathbb{D}_t \mid \mathbb{Z}_1)$ with some $s \geq 2$. According to lemma 14, such a group has an axis of symmetry, which contradicts (ii). Therefore, the group $\Gamma$ acts on $\mathbb{R}^4$ irreducibly. QED

Theorem 4. The following subgroups of $SO(4)$ act on $\mathbb{R}^4$ irreducibly and do not have one-dimensional fixed-point subspaces:

\[
\begin{array}{|c|c|c|}
\hline
(Z_{2K_1} \mid Z_{2K_2}; \mathbb{D}_{K_2} \mid \mathbb{D}_{K_2}) & (\mathbb{D}_{K_1}; \mathbb{D}_{K_2} \mid \mathbb{D}_{K_2}) & K_1, K_2 \text{ co-prime} \\
(Z_{2K_1} \mid Z_{2K_2}; \mathbb{D}_{K_2} \mid \mathbb{Z}_{K_2}) & (\mathbb{D}_{K_1}; \mathbb{Z}_{K_2} \mid \mathbb{Z}_{K_2}) & K_1, K_2 \text{ co-prime} \\
(Z_{2K_1} \mid Z_{2K_2}; \mathbb{D}_{K_2} \mid \mathbb{Z}_{K_2}) & (\mathbb{D}_{K_1}; \mathbb{D}_{K_2} \mid \mathbb{Z}_{K_2}) & K_1 \text{ odd, } K_1, K_2 \text{ co-prime} \\
(Z_{2K_1} \mid Z_{2K_2}; \mathbb{Z}_{K_2}; \mathbb{T} \mid \mathbb{T}) & (\mathbb{D}_{K_1}; \mathbb{Z}_{K_2} \mid \mathbb{T} \mid \mathbb{T}) & K_1 \neq 2k \\
(Z_{2K_1} \mid Z_{2K_2}; \mathbb{T} \mid \mathbb{V}) & (\mathbb{D}_{K_1}; \mathbb{Z}_{K_2} \mid \mathbb{T} \mid \mathbb{V}) & K_1 \neq 2k, 3k \\
(Z_{2K_1} \mid Z_{2K_2}; \mathbb{O} \mid \mathbb{O}) & (\mathbb{D}_{K_1}; \mathbb{Z}_{K_2} \mid \mathbb{O} \mid \mathbb{O}) & K_1 \neq 2k, 3k \\
(Z_{2K_1} \mid Z_{2K_2}; \mathbb{O} \mid \mathbb{T}) & (\mathbb{D}_{K_1}; \mathbb{Z}_{K_2} \mid \mathbb{O} \mid \mathbb{T}) & K_1 \neq 2k, 5k \\
(Z_{2K_1} \mid Z_{2K_2}; \mathbb{I} \mid \mathbb{I}) & (\mathbb{D}_{K_1}; \mathbb{Z}_{K_2} \mid \mathbb{I} \mid \mathbb{I}) & K_1 \neq 2k, 5k \\
(Z_{2K_1} \mid Z_{2K_2}; \mathbb{Z}_{K_2} \mid \mathbb{Z}_{K_2}) & (\mathbb{D}_{K_1}; \mathbb{Z}_{K_2} \mid \mathbb{Z}_{K_2}) & K_1, K_2 \text{ odd, co-prime} \\
\hline
\end{array}
\]

The proof follows from the list of finite subgroups of $SO(4)$ (see table 1), lemmas 14 and 15 and is not presented.
Remark 8. Note that the groups
\[
\left(\mathbb{Z}_{2K_1} | \mathbb{Z}_{2K_2}; \mathbb{D}_{K_1} | \mathbb{D}_{K_2}\right) \text{ with } K_1 \text{ odd, } K_1, K_2 \text{ co-prime;}
\]
\[
\left(\mathbb{Z}_{2K_1} | \mathbb{Z}_{2K_2}; \mathbb{T} | \mathbb{T}\right) \text{ with } K_1 \neq 2k, 3k;
\]
\[
\left(\mathbb{Z}_{2K_1} | \mathbb{Z}_{2K_2}; \mathbb{O} | \mathbb{O}\right) \text{ with } K_1 \neq 2k, 3k \text{ and } \left(\mathbb{Z}_{2K_1} | \mathbb{Z}_{2K_2}; \mathbb{I} | \mathbb{I}\right) \text{ with } K_1 \neq 2k, 3k, 5k
\]
do not have non-trivial fixed-point subspaces at all.

Lemma 16. Suppose that a finite group \( \Gamma^* \subset O(4) \), \( \Gamma^* \not\subset SO(4) \), acts irreducibly in \( \mathbb{R}^4 \). Then \( \Gamma^* \) possesses at least one axis of symmetry.

Proof. Recall that \( \Gamma^* \) can be decomposed as
\[
\Gamma^* = \Gamma \oplus \sigma \Gamma, \text{ where } \Gamma \subset SO(4) \text{ and } \sigma \not\subset SO(4).
\]
In the quaternion form \( \Phi^{-1}\Gamma = (G | G_K; G | G_K) \). If \( G \neq \mathbb{Z}_n \), then the existence of a one-dimensional fixed-point subspace follows from lemma 14.

Suppose that \( G = \mathbb{Z}_n \). Recall that \( \sigma \), a reflection in \( \mathbb{R}^4 \), has \( \pm 1 \) for two of its eigenvalues, the other two being of the form \( e^{i\theta} \). If \( \omega \neq k\pi \) then the reflection \( \sigma \) has a one-dimensional fixed-point subspace. If \( \omega = k\pi \), then it has a three-dimensional fixed-point subspace, \( Q \). As in the proof of lemma 15, denote by \( V_1 \) and \( V_2 \) two invariant subspaces of the group \( \Gamma^* \). Since \( \sigma \gamma \sigma \in \Gamma \) for any of \( \gamma \in \Gamma \), one of these subspaces belongs to \( Q \), therefore then the action of \( \Gamma^* \) on \( \mathbb{R}^4 \) is reducible.

QED

Remark 9. Lauterbach and Matthews [13] found three subgroups of \( SO(4) \) which act irreducibly and do not have one-dimensional fixed-point subspaces. The subgroups are denoted by \( G_j(m) \), where \( j = 1, 2, 3 \) and \( m \geq 3 \) is an odd integer. In our notation, \( G_1(m) \) is \( (\mathbb{D}_4 | \mathbb{D}_2; \mathbb{D}_m | \mathbb{Z}_{2m}) \) and \( G_3(m) \) is \( (\mathbb{D}_m | \mathbb{D}_m; \mathbb{D}_2 | \mathbb{D}_2) \).

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