On some $p(x)$ anisotropic elliptic equations in unbounded domain

A. Aberqi · B. Aharrouch · J. Bennouna

Abstract We study a class of nonlinear elliptic problems with Dirichlet conditions in the framework of the Sobolev anisotropic spaces with variable exponent, involving an anisotropic operator on an unbounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$).

We prove the existence of entropy solutions avoiding sign condition and coercivity on the lowers order terms.

Keywords Anisotropic sobolev spaces · Nonlinear elliptic problems · Entropy solution · Unbounded domain

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1 Introduction

Partial differential equations with non-standard growth have been widely studied in recent years, it thus finds applications in different fields of physics, image processing, filtration in porous media, optimal control and electro rheological fluids (smart fluids) which change their mechanical properties dramatically.

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when an external electric field is applied, for a model in the case of isothermal, homogeneous incompressible smart fluids see e.g. Rajagopal and Rusička (27), see also (20).

This paper concerns the following problem:

\[
\begin{aligned}
\{(P)\} & \quad \begin{cases}
- \text{div}(a(x,u,\nabla u)) + H(x,u,\nabla u) + |u|^{p_0(x)} - 2u = f \quad \text{in} \quad \Omega, \\
\quad u = 0 \quad \text{on} \quad \partial \Omega,
\end{cases}
\end{aligned}
\]

where \( a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) with \( a(x,s,\xi) = (a_1(x,s,\xi),...,a_N(x,s,\xi)) \), we assume that \( a_i(x,s,\xi) \) for \( i = 1,...,N \) and \( H(x,s,\xi) \) are Carathéodory functions satisfying assumptions (10–13) below.

The exponent \( p_0(x) \) is a measurable function defined on arbitrary domain \( \Omega \); which satisfies (8), the source \( f \) is merely integrable.

• Starting by the case where the domain \( \Omega \) is bounded in \( \mathbb{R}^N \), and consider the following problem:

\[
\begin{aligned}
- \sum_{i=1}^{i=N} \partial_{x_i}(a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}) - \sum_{i=1}^{i=N} \partial_{x_i}g_i(u) + |u|^{p_0(x)} - 2u = f(x),
\end{aligned}
\]

\( x \in \Omega \subset \mathbb{R}^N, \quad \Omega \) is a bounded domain.

The model (2) is very well understood, in the isotropic case, i.e. \( \overrightarrow{p} = (p_1,...,p_N) = p \equiv \text{cte} \), for an lucid, yet precise comprehensive papers see (6, 14, 24, 26).

For the anisotropic operator with polynomial growth i.e. \( \overrightarrow{p} = (p_1,p_2,...,p_N) \), \( p_i \in \mathbb{R} \) we mention the reference works of A. G. Korolev (22) and N. T. Chung et al. (12), for more works in the classical anisotropic spaces \( W^{1,\overrightarrow{p}}(\Omega) \) we refer the reader to (2, 11, 13, 15, 19).

• In the arbitrary domain \( \Omega \), there are many studies which establish the existence of solutions in an unbounded domain, in particular, Bendahmanne and Karlsen (4) proved the solvability and regularity of (2) where \( x \) lies in \( \mathbb{R}^N \), in the classical anisotropic spaces, (9) and (16) solved (2), in the framework of anisotropic spaces with variable exponents, without lower order terms \( g \) and perturbation \( |u|^{p_0(x)} - 2u \), they have shown the well-posedness without constraint on the growth.

Our paper continues the work in this direction. We will show the existence of entropy solutions of (2) with a general operator of type Leray-lions, and the presence of a lower order \( g(x,u,\nabla u) \) which do not satisfy the sign condition, and a perturbation \( |u|^{p_0(x)} - 2u \) in arbitrary domain of \( \mathbb{R}^N \).

Let us summarize the outline of this paper: In Section 2 we recall some basic notations and Sobolev inequality for anisotropic Sobolev spaces. Our main results are stated in Section 3, while the appendix in Section 4.
2 Framework Space

Let \( \Omega \) be an unbounded domain of \( \mathbb{R}^N \), \( N \geq 2 \), we denote

\[
C_+(\Omega) = \{ \text{measurable function } p(\cdot): \Omega \to \mathbb{R}, \text{ such that } 1 < p^- \leq p^+ < \infty \}
\]

where

\[
p^− = \text{essinf}\{p(x)/x \in \Omega\} \quad \text{and} \quad p^+ = \text{esssup}\{p(x)/x \in \Omega\}
\]

In this section we present the anisotropic variable exponent Sobolev space, used in the study of the elliptic problem (1).

Let \( p_0(x), p_1(x), ..., p_N(x) \) be \( N + 1 \) variable exponents in \( C_+(\Omega) \). We denote

\[
\overline{p}(\cdot) = \left(p_0(\cdot), p_1(\cdot), ..., p_N(\cdot)\right) \in \left(C_+(\Omega)\right)^{N+1}, \quad \underline{p} = \min\left(p_0^-, p_1^-, ..., p_N^-\right)
\]

and \( \overline{p} = \max\left(p_0^+, p_1^+, ..., p_N^+\right) \). Denoting by \( L^{\overline{p}(\cdot)}(\Omega) \) the product space \( \prod_{i=1}^N L^{p_i}(\Omega) \)
endowed with the product norm \( \| u \|_{\overline{p}(\cdot)} = \sum_{i=1}^N \| u \|_{p_i} \). We define the anisotropic variable exponent Sobolev space \( W^{1,\overline{p}(\cdot)}(\Omega) \) as follow:

\[
W^{1,\overline{p}(\cdot)}(\Omega) = \{ u \in L^{p_0(\cdot)}(\Omega), \quad \frac{\partial u}{\partial x_i} \in L^{p_i(\cdot)}(\Omega), \quad \text{for } i = 1, ...N \},
\]

where \( L^{p_i(\cdot)}(\Omega) \), for \( i = 0, ..., N \) are the Lebesgue spaces with variable exponent \( p_i(\cdot) \). We define also the space \( W^{1,\overline{p}(\cdot)}_0(\Omega) \) as the closure of \( C_0^\infty(\Omega) \) in

\[
W^{1,\overline{p}(\cdot)}(\Omega) \quad \text{with respect to the norm } \| u \|_{1,\overline{p}(\cdot)} = \sum_{i=0}^N \| \frac{\partial u}{\partial x_i} \|_{p_i(\cdot)}.
\]

The space \( \left(W^{1,\overline{p}(\cdot)}_0(\Omega), \| \cdot \|_{1,\overline{p}(\cdot)}\right) \) is a Banach reflexive space see [13].

Let \( p(x) = N \left( \sum_{i=1}^N \frac{1}{p_i(x)} \right)^{-1}, \quad p_+ = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) \leq N, \\ +\infty & \text{if } p(x) \geq N. \end{cases} \)

And \( p_\infty = \max(p_+(x), p_+(x)) \).

2.1 Basic Lemmas

**Lemma 1** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \) and \( p(\cdot) = (p_1(\cdot), p_2(\cdot), ..., p_N(\cdot)) \in \left(C^+(\bar{Q})\right)^N \). If \( q(\cdot) \in C^+(\bar{Q}) \) and

\[
q(x) < p_\infty(x) \quad \forall x \in \Omega,
\]

then the embedding \( W^{1,p(\cdot)}_0(\Omega) \subset L^{q(\cdot)}(\Omega) \) is continuous and compact.

**Proof** The previous embedding theorem for the space \( W^{1,\overline{p}(\cdot)}_0(\Omega) \) is proved in Theorem 2.5 of [18].
Lemma 2 Let the assumptions (10)–(12) be satisfied in $Q$, and for some fixed $k > 0$ there hold
\[
\begin{align*}
&u^j \rightharpoonup u \text{ in } L^p(\cdot)(Q), \quad j \to \infty, \\
&T_k(u^j) \rightharpoonup T_k(u) \text{ a.e. in } Q, \quad j \to \infty, \\
&\lim_{j \to \infty} \int_Q (a(x, T_k(u^j), \nabla u^j) - a(x, T_k(u), \nabla u)).(\nabla u^j - \nabla u) = 0.
\end{align*}
\]
Then along a subsequence,
\[
\begin{align*}
&\nabla u^j \rightharpoonup \nabla u \text{ a.e. in } Q, \quad j \to \infty, \\
&\nabla u^j \rightarrow \nabla u \text{ strongly } L^p(\cdot)(Q), \quad j \to \infty.
\end{align*}
\]

Proof The convergence (6) is established analogously as in the proof of Assertion 2. Apparently, the first statement of this kind is Lemma 3.3 from the work [23].

Lemma 3 Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain, $(u_n)_{n \in \mathbb{N}}$ and $u$ be functions from $L^{\overrightarrow{p}(\cdot)}(\Omega)$, such that $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^{\overrightarrow{p}(\cdot)}(\Omega)$ and $u_n \rightharpoonup u$ a.e. in $\Omega$. Then
\[
\begin{align*}
u_n \rightarrow u \text{ weakly in } L^{\overrightarrow{p}(\cdot)}(\Omega).
\end{align*}
\]
Theorem 1

The main result of the present work is the following theorem.

\[ a_i(x, s, \xi) \xi_i \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \]

(12)

with \( \tilde{a}_i : \mathbb{R}^+ \rightarrow \mathbb{R}^{++} \). Furthermore, we assume the nonlinear term \( H(x, s, \xi) \) is a Carathéodory function which satisfies only the growth condition:

\[ |H(x, s, \xi)| \leq \hat{h}(|s|) \sum_{i=1}^{N} |\xi_i|^{p'_i(x)} + h_0(x), \quad \text{with} \quad h_0 \in L^1(\Omega), \]

(13)

and \( \hat{h} : \mathbb{R}^+ \rightarrow \mathbb{R}^{++} \). The source data \( f \in L^1(\Omega) \). \( T_k, k > 0 \), denotes the truncation function at level \( k \) defined on \( \mathbb{R} \) by \( T_k(r) = \max(-k, \min(k, r)) \).

We set:

\[ T^1_0, p^1(\cdot)(\Omega) = \{ u : \Omega \rightarrow \mathbb{R}, \text{measurable}, T_k(u) \in W^{1,p^1(\cdot)}_0(\Omega), \quad \forall k > 0 \}. \]

**Definition 1** A measurable function \( u \) is said be an entropy solution for the problem \((\mathcal{P})\), if \( u \in T^1_0, p^1(\cdot)(\Omega) \), such that

1. \( H(x, u, \nabla u) \in L^1(\Omega) \) and \( |u|^{p_0(x)-2} u \in L^1(\Omega) \),

2. for all \( k > 0 \),

\[ \int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - \xi) \, dx + \int_{\Omega} \left( H(x, u, \nabla u) + |u|^{p_0(x)-2} u \right) T_k(u - \xi) \, dx \leq \int_{\Omega} f(x) T_k(u - \xi) \, dx, \quad \forall \xi \in C^1_0(\Omega). \]

The main result of the present work is the following theorem.

**Theorem 1** Let \( \Omega \subset \mathbb{R}^N \) be an unbounded domain, suppose that the assumptions \((\text{S}), (\text{T}), (\text{L})\) and \( f \in L^1(\Omega) \) hold true. Then there exists at least one entropy solution of the problem \((\mathcal{P})\).

**Remark 1** The Theorem above remains valid for \( \xi \in W^{1,p^1(\cdot)}_0(\Omega) \cap L^\infty(\Omega) \).

**Proof of theorem**

Let \( \Omega(n) = \{ x \in \Omega : |x| < n \} \), and \( f^n(x) = \frac{f(x)}{1 + \frac{|f(x)|}{n}} \chi_{\Omega(n)} \), we have

\[ f^n \rightarrow f \text{ in } L^1(\Omega), \quad n \rightarrow +\infty, \quad |f^n(x)| \leq |f|, \quad |f^n| \leq n \chi_{\Omega(n)} \]

(14)

\[ a^n(x, s, \xi) = \left( a^n_1(x, s, \xi), \ldots, a^n_N(x, s, \xi) \right) \]

where \( a^n_i(x, s, \xi) = a_i(x, T_n(s), \xi), \) for \( i = 1, \ldots, N; \)

\[ H^n(x, s, \xi) = T_n(H(x, s, \xi)) \chi_{\Omega(n)}; \quad |H^n(x, s, \xi)| \leq H(x, s, \xi) \chi_{\Omega(n)}. \]

(15)

Consider the following regularized equations:

\[ (\mathcal{P}_n) \left\{ \begin{array}{l}
\int_{\Omega} a(x, T_n(u^n), \nabla u^n) \nabla v \, dx + \int_{\Omega} \left( H(x, u^n, \nabla u^n) + |u^n|^{p_0(x)-2} u^n \right) v \, dx \\
= \int_{\Omega} f^n v \, dx \quad \text{for any} \quad v \in W^{1,p^1(\cdot)}_0(\Omega).
\end{array} \right. \]

(16)
The approximate problem \((P_n)\) admits a solution, this being based on the theorem of Lions [23, Chapter II, 2, Theorem 2.7] for pseudo-monotone operators. For the proof, see the Lemma 5 in the Appendix.

**Step 1:** A priori estimate of the sequence \(\{u^n\}\).

Let \(v = T_k(u^n) \exp(A(|u_n|))\) with \(A(s) = \int_0^s \frac{\hat{h}(r)}{\alpha} \, dr\). Since \(v \in W^{1,p}_0(\Omega)\), taking \(v\) as the test function in the problem \((P_m)\), we get

\[
\begin{align*}
\int_\Omega a^m(x, u^n, \nabla u^n) \nabla (T_k(u^n) \exp(A(|u_n|))) \, dx &+ \int_\Omega H^m(x, u^n, \nabla u^n) T_k(u^n) \exp(A(|u_n|)) \, dx \\
&+ \int_\Omega |u^n|^{p_0(x)} - 2u^n T_k(u^n) \exp(A(|u_n|)) \, dx \\
&\leq \int_\Omega f_m(x) T_k(u^n) \exp(A(|u_n|)) \, dx.
\end{align*}
\]

(17)

The first term in the left hand side can be written as

\[
\begin{align*}
\int_\Omega a^m(x, u^n, \nabla u^n) \nabla (T_k(u^n) \exp(A(|u_n|))) \, dx &+ \int_\Omega a^m(x, u^n, \nabla u^n) \nabla T_k(u^n) \exp(A(|u_n|)) \, dx \\
&+ \int_\Omega a^m(x, u^n, \nabla u^n) \nabla |u^n| T_k(u^n) \frac{\hat{h}(|u^n|)}{\alpha} \exp(A(|u_n|)) \, dx \\
&\geq \int_\Omega a^m(x, u^n, \nabla u^n) \nabla T_k(u^n) \exp(A(|u_n|)) \, dx \\
&+ \sum_{i=1}^N \int_\Omega |\partial_x u^n|^{p_i(x)} \hat{h}(|u^n|) |T_k(u^n)| \exp(A(|u_n|)) \, dx,
\end{align*}
\]

(18)

for the second term in the left hand side, by increasing condition for \(H\), we have

\[
\begin{align*}
\int_\Omega H^m(x, u^n, \nabla u^n) T_k(u^n) \exp(A(|u_n|)) \, dx &\leq \sum_{i=1}^N \int_\Omega |\partial_x u^n|^{p_i(x)} \hat{h}(|u^n|) |T_k(u^n)| \exp(A(|u_n|)) \, dx \\
&+ \int_\Omega |h_0(x)| |T_k(u^n)| \exp(A(|u_n|)) \, dx.
\end{align*}
\]

(19)
By combining (17), (18), and (19), we obtain

\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_m(u^m), \nabla u^m) D^i T_k(u^m) \exp(A(|u_m|)) \, dx \\
+ \int_{\Omega} |u^m|^{p_0(x)-2} u^m T_k(u^m) \exp(A(|u_m|)) \, dx \\
\leq \int_{\Omega} |h_0(x)||T_k(u^m)| \exp(A(|u_m|)) \, dx \\
+ \int_{\Omega} f_m(x) T_k(u^m) \exp(A(|u_m|)) \, dx,
\]

which gives

\[
\sum_{i=1}^{N} \int_{\Omega} |D^i T_k(u^m)|^{p_i(x)} \, dx + \int_{\{u^m \leq k\}} |u^m|^{p_0(x)} \exp(A(|u_m|)) \, dx \\
+ k \int_{\{|u^m| > k\}} |u^m|^{p_0(x)-1} \exp(A(|u_m|)) \, dx \leq C_1 k.
\]

(20)

**Step 2**: Almost everywhere convergence of sequence \(\{u^m\}\)

**Lemma 4** \(\text{meas}\{x \in \Omega, \ |u^m| > k\} \text{ tends to zeros as } k \text{ to infinity.}\)

**Proof** By (21), we have

\[
\int_{\{|u^m| > k\}} |u^m|^{p_0(x)-1} \, dx \leq C_2,
\]

which gives \(\text{meas}\{x \in \Omega, \ |u^m| > k\} k^{p-1} \leq C_2\), for \(k > 1\),

then, \(\text{meas}\{x \in \Omega, \ |u^m| > k\} \rightarrow 0 \text{ as } k \rightarrow \infty.\)

Let \(g(k) = \sup_{m \in \mathbb{N}} \text{meas}\{x \in \Omega, \ |u^m| > k\} \rightarrow 0 \text{ as } k \rightarrow \infty.\) Since \(\Omega\) is an unbounded domain in \(\mathbb{R}^N\), we define \(\eta_R\) defined as

\[
\eta_R(r) = \begin{cases} 
1 & \text{if } r < R, \\
R + 1 - r & \text{if } R \leq r < R + 1, \\
0 & \text{if } r \geq R + 1.
\end{cases}
\]

For \(R, h > 0\), we have

\[
\sum_{i=1}^{N} \int_{\Omega} D^i(\eta_R(|x|)T_k(u^m)) \, dx \leq C_2 \sum_{i=1}^{N} \int_{\Omega} D^i(u^m \eta_R(|x|)) \, dx \\
+ C_3 \sum_{i=1}^{N} \int_{\Omega} T_k(u^m) D^i(\eta_R(|x|)) \, dx \leq C(h, R),
\]
which implies that the sequence \( \{ \eta_R(\Omega) T_h(u^m) \} \) is bounded in \( W_0^{1,p(\cdot)}(\Omega(R + 1)) \), and by embedding Theorem, and since \( \eta_R = 1 \) in \( \Omega(R) \) we have

\[
\eta_R T_h(u^m) \rightarrow v_h \text{ strongly in } L^{p(\cdot)}(\Omega((R + 1))), \\
T_h(u^m) \rightarrow v_h \text{ strongly in } L^{p(\cdot)}(\Omega((R))).
\]

By Egorov’s theorem, we can choose \( E_h \subset \Omega(R) \) such that \( \text{meas}(E_h) < \frac{1}{n} \) and \( T_h(u^m) \rightarrow v_h \) uniformly in \( \Omega(R) \) \( \setminus E_h \).

Let \( \Omega^h(R) = \{ x \in \Omega(R) \setminus E_h : |v_h(x)| \geq h - 1 \} \). Since \( T_h(u^m) \) converges uniformly to \( v_h \) in \( \Omega(R) \) \( \setminus E_h \), there exists \( m_0 \) such that for any \( m \geq m_0, |T_h(u^m)| \geq h - 2 \) on \( \Omega^h(R) \), i.e. \( |u^m| \geq h - 2 \), then by Lemma 4 we obtain

\[
\text{meas } \Omega^h(R) \leq \sup \text{meas}\{ x \in \Omega : |u^m| \geq h - 2 \} = g(h - 2) \xrightarrow{h \to 0} 0
\]

holds true. Then, by the diagonalisation argument with respect to \( R \in \mathbb{N} \), we establish the almost everywhere convergence of \( u^m \) to \( u \).

**Step 3** : Weak convergence of the gradient

By (21), we have

\[
\|T_h(u^m)\|_{W_0^{1,p(\cdot)}(\Omega)} \leq C(k).
\]

So we can extract a weakly convergent subsequence in \( W_0^{1,p(\cdot)}(\Omega) \), such that

\[
T_h(u^m) \rightharpoonup v_k \text{ weakly in } W_0^{1,p(\cdot)}(\Omega), \\
\nabla T_h(u^m) \rightharpoonup \nabla v_h \text{ weakly in } L^{p(\cdot)}(\Omega), \\
T_k(u^m) \rightarrow v_k \text{ strongly in } L^{p_0(\cdot)}(\Omega).
\]

Beside, \( T_h(u^m) \rightarrow T_k(u) \) a.e. in \( \Omega \), gives \( T_k(u^m) \rightarrow T_k(u) \) strongly in \( L^{p_0(\cdot)}(\Omega) \), and we conclude that \( \nabla T_k(u^m) \rightarrow \nabla T_k(u) \) weakly in \( L^{p(\cdot)}(\Omega) \).

**Step 4** : Strong convergence of the gradient

Show that \( \nabla T_k(u^m) \rightarrow \nabla T_k(u) \) in \( L^{p(\cdot)}_{\text{loc}}(\Omega) \).

Let \( j > k > 0 \), and

\[
h_j(s) = \begin{cases} 
1 & \text{ if } |s| \leq j, \\
1 - |s - j| & \text{ if } j \leq |s| \leq j + 1, \\
0 & \text{ if } r > j + 1.
\end{cases}
\]
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Taking $v = \exp(A(|u^m|))(T_k(u^m) - T_k(u))h_j(u^m)\eta_R(|x|)$ as test function in the approximate problem $(P_m)$, we have

$$
\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_m(u^m), \nabla u^m) D^i(\exp(A(|u^m|))(T_k(u^m) - T_k(u))h_j(u^m)\eta_R(|x|)) \, dx
$$

$$
+ \int_{\Omega} H^m(x, u^m, \nabla u^m) \exp(A(|u^m|))(T_k(u^m) - T_k(u))h_j(u^m)\eta_R(|x|) \, dx
$$

$$
+ \int_{\Omega} |u^m|^{p_0(x)-2} u^m \exp(A(|u^m|))(T_k(u^m) - T_k(u))h_j(u^m)\eta_R(|x|) \, dx
$$

$$
\leq \int_{\Omega} f_m(x) \exp(A(|u^m|))(T_k(u^m) - T_k(u))h_j(u^m)\eta_R(|x|) \, dx.
$$

Denoting $J_1, J_2, J_3,$ and $J_4$ by

$$
J_1 = \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_m(u^m), \nabla u^m) D^i(\exp(A(|u^m|))(T_k(u^m) - T_k(u))h_j(u^m)\eta_R(|x|)) \, dx,
$$

$$
J_2 = \int_{\Omega} H^m(x, u^m, \nabla u^m) \exp(A(|u^m|))(T_k(u^m) - T_k(u))h_j(u^m)\eta_R(|x|) \, dx,
$$

$$
J_3 = \int_{\Omega} |u^m|^{p_0(x)-2} u^m \exp(A(|u^m|))(T_k(u^m) - T_k(u))h_j(u^m)\eta_R(|x|) \, dx,
$$

$$
J_4 = \int_{\Omega} f_m(x) \exp(A(|u^m|))(T_k(u^m) - T_k(u))h_j(u^m)\eta_R(|x|) \, dx.
$$

$J_1$ can be rewritten as

$$
J_1 = \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_m(u^m), \nabla u^m) \times \frac{\hat{h}(u^m)}{\alpha} \times D^i u^m \text{sign}(u^m) \exp(A(|u^m|))(T_k(u^m) - T_k(u))h_j(u^m)\eta_R(|x|) \, dx
$$

$$
+ \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_m(u^m), \nabla u^m) \exp(A(|u^m|))(D^i T_k(u^m) - D^i T_k(u))h_j(u^m)\eta_R(|x|) \, dx
$$

$$
+ \sum_{i=1}^{N} \int_{|u^m| \leq j+1} a_i(x, T_m(u^m), \nabla u^m) D^i u^m \exp(A(|u^m|))(T_k(u^m) - T_k(u))h_j(u^m)\eta_R(|x|) \, dx
$$

$$
+ \sum_{i=1}^{N} \int_{|u^m| \leq j} a_i(x, T_m(u^m), \nabla u^m) D^i u^m \exp(A(|u^m|))(T_k(u^m) - T_k(u))h_j(u^m)D^i \eta_R(|x|) \, dx
$$

$$
= J_1^1 + J_1^2 + J_1^3 + J_1^4.
$$
Since $h_j \geq 0$ and $\eta_R(|x|) \geq 0$ and $u^m(T_k(u^m) - T_k(u)) \geq 0$, then

$$J_1^3 \geq \sum_{i=1}^{N} \int_{\Omega} \hat{h}(|u^m|)|D^i u^m| p_i(x) \exp(A(|u^m|))|T_k(u^m) - T_k(u)|\eta_R(|x|)dx.$$  

$$J_2^3 \geq \sum_{i=1}^{N} \int_{|u^m| \leq k} a_i(x, T_m(u^m), \nabla u^m) \exp(A(|u^m|))|D^i T_k(u^m) - D^i T_k(u)|\eta_R(|x|)dx$$

$$- \sum_{i=1}^{N} \int_{k < |u^m| \leq j+1} |a_i(x, T_m(u^m), \nabla u^m)| \exp(A(|u^m|))|D^i T_k(u)|\eta_R(|x|)dx.$$  

(24)

$$J_3^3 \geq \sum_{i=1}^{N} \int_{|u^m| \leq j+1} a_i(x, u^m, \nabla u^m)|D^i u^m|T_k(u^m) - T_k(u)|\eta_R(|x|) dx.$$  

$$J_4 \leq \sum_{i=1}^{N} \int_{|u^m| \leq k} \hat{h}(|u^m|)|D^i u^m| p_i(x) \exp(A(|u^m|))|T_k(u^m) - T_k(u)|\eta_R(|x|) dx$$

$$- \sum_{i=1}^{N} \int_{\Omega} |h_0(x)||T_k(u^m) - T_k(u)|\eta_R(|x|) \exp(A(|u^m|)) dx.$$  

(25)

By (23), (24), (25), $J_3$, and $J_4$, we have

$$\sum_{i=1}^{N} \int_{|u^m| \leq k} a_i(x, T_k(u^m), \nabla T_k(u^m)) \exp(A(|u^m|))|D^i T_k(u^m) - D^i T_k(u)|\eta_R(|x|)dx$$

$$+ \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_m(u^m), \nabla u^m) \exp(A(|u^m|))|T_k(u^m) - T_k(u)|\eta_R(|x|) dx$$

$$+ \delta \sum_{i=1}^{N} \int_{|u^m| \leq k} |T_k(u^m)| p_i(x)|2 T_k(u^m) \exp(A(|u^m|))|T_k(u^m) - T_k(u)|\eta_R(|x|)dx$$

$$\leq \int_{\Omega} (|h_0(x)| + |f^m|) \exp(A(|u^m|))|T_k(u^m) - T_k(u)|\eta_R(|x|) dx$$

$$+ \sum_{i=1}^{N} \int_{k < |u^m| \leq j+1} a_i(x, T_{j+1}(u^m), \nabla T_{j+1}(u^m)) \exp(A(|u^m|))|D^i T_k(u)|\eta_R(|x|)dx$$

$$+ \sum_{i=1}^{N} \int_{j < |u^m| \leq j+1} a_i(x, T_m(u^m), \nabla u^m) \times$$

$$D^i u^m \exp(A(|u^m|))|T_k(u^m) - T_k(u)|\eta_R(|x|) dx.$$  

(26)

The first term in the right hand side goes to zeros as $m$ tends to infinity, since $T_k(u^m) \to T_k(u)$ weakly in $L^\infty(\Omega)$. Since $(a_i(x, T_{j+1}(u^m), \nabla T_{j+1}(u^m))m)$ is bounded in $L^p(\Omega_R)$, there exists $\xi \in L^p(\Omega_R)$, such that $|a_i(x, T_{j+1}(u^m), \nabla T_{j+1}(u^m) | \to \xi$, in $L^p(\Omega_R)$, the second term of the left hand side tends to zeros.

The third term in the left hand side go to zeros.
Indeed, by taking $v = \exp(A(|u^m|))T_i(u^m - T_j(u^m))\eta_R(|x|)$ as test function in the approximate problem $(P_m)$, we obtain

$$\sum_{i=1}^{N} \int_{|u^m| \leq j+1} a_i(x, u^m, \nabla u^m) D^i u^m \eta_R(|x|) \, dx = 0,$$

and since $T_k(u^m) \to T_k(u)$ weak * in $L^\infty(\Omega)$, we conclude the result. Since $T_k(u^m) \to T_k(u)$ strongly in $L_{loc}^{p,i}(\Omega)$, the second term of the right hand side increased by a quantity that tends to 0 as $m$ tends to zero.

Finally (26), rewrite as

$$\sum_{i=1}^{N} \int_{|u^m| \leq k} a_i(x, T_k(u^m), \nabla T_k(u^m)) \exp(A(|u^m|)) |D^i T_k(u^m) - D^i T_k(u)| \eta_R(|x|) \, dx$$

$$+ \delta \sum_{i=1}^{N} \int_{|u^m| \leq k} |T_k(u_m)|^{p(x) - 2} T_k(u_m) \exp(A(|u^m|)) (T_k(u^m) - T_k(u)) \eta_R(|x|) \, dx$$

$$\leq \varepsilon(j, m).$$

(27)

then,

$$\sum_{i=1}^{N} \int_{\Omega} (a_i(x, T_k(u^m), \nabla T_k(u^m)) - a_i(x, T_k(u^m), \nabla T_k(u))) \times$$

$$\exp(A(|u^m|)) (D^i T_k(u^m) - D^i T_k(u)) \eta_R(|x|) \, dx$$

$$\leq - \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u^m), \nabla T_k(u)) \exp(A(|u^m|)) |D^i T_k(u^m) - D^i T_k(u)| \eta_R(|x|) \, dx$$

$$- \sum_{i=1}^{N} \int_{|u^m| \leq k} a_i(x, T_k(u^m), \nabla T_k(u^m)) \exp(A(|u^m|)) D^i T_k(u) \eta_R(|x|) \, dx$$

$$+ \varepsilon(j, m).$$

(28)

In view of Lebesgue dominated convergence theorem, we have $T_k(u^m) \to T_k(u)$ strongly in $L_{loc}^{p,i}(\Omega)$ and $D^i T_k(u^m) \to D^i T_k(u)$ weakly in $L^{p(i)}(\Omega)$, then the terms on the right hand side of (28) go to zeros as $m$ and $j$ tend to infinity, which gives

$$\sum_{i=1}^{N} \int_{\Omega \cap R} \left( a_i(x, T_k(u^m), \nabla T_k(u^m)) - a_i(x, T_k(u^m), \nabla T_k(u)) \right) \times$$

$$\left( D^i T_k(u^m) - D^i T_k(u) \right) \, dx \to 0.$$

(29)

Thanks to Lemma 2, we conclude that

$$\nabla T_k(u^m) \to \nabla T_k(u) \text{ a.e. in } \Omega(R).$$

(30)
As before, applying Egorov’s theorem, we can find a set $E_k \subset \Omega(R)$, such that \( \text{meas} E_k < 1/k \) and \( T_k(u^m) \rightarrow T_k(u) \) uniformly in \( \Omega(R) \setminus E_k \).

Recall that \( \text{meas} \Omega_k(R) > \text{meas} \Omega(R) - 1/k - g(k - 2) \), with

\[
\Omega_k(R) = \{ x \in \Omega(R) \setminus E_k : |u(x)| < k - 1 \},
\]

which gives \( |T_k(u^m)| < k \), on \( \Omega_k(R) \), for any \( m \geq m_0 \), then

\[
\nabla u^m \rightarrow \nabla u \quad \text{a.e. in} \quad \Omega_k(R).
\]

Thus by the diagonalisation argument with respect to \( R \), we obtain

\[
\nabla u^m \rightarrow \nabla u \quad \text{a.e. in} \quad \Omega, \quad \text{and} \quad \nabla T_k(u^m) \rightarrow \nabla T_k(u) \quad \text{a.e. in} \quad \Omega. \tag{31}
\]

Since \( H^m(x, u^m, \nabla u^m) \rightarrow H(x, u, \nabla u) \) a.e. in \( \Omega \), by Fatou’s Lemma,

\[
H(x, u, \nabla u) \in L^1(\Omega).
\]

**Step 5**: Equi integrability of \( |u^m|^{p_0(x)} - 2u^m \) and \( H^m(x, u^m, \nabla u^m) \)

In this section we will prove that \( H^m(x, u^m, \nabla u^m) \rightarrow H(x, u, \nabla u) \) and \( |u^m|^{p_0(x)} - 2u^m \rightarrow |u|^{p_0(x)} - 2u \) strongly in \( L^1_{\text{loc}}(\Omega) \).

We have

\[
H^m(x, u^m, \nabla u^m) \rightarrow H(x, u, \nabla u) \quad \text{a.e. in} \quad \Omega
\]

and

\[
|u^m|^{p_0(x)} - 2u^m \rightarrow |u|^{p_0(x)} - 2u \quad \text{a.e. in} \quad \Omega.
\]

In view of Vitali’s Theorem, it’s sufficient to prove that \( H^m(x, u^m, \nabla u^m) \) and \( |u^m|^{p_0(x)} - 2u^m \) are uniformly equi-integrable.

Let \( v = \exp(2A(|u^m|))T_1(u^m) - T_h(u^m) \), remark that \( v \in W^{1, \text{loc}}(\Omega) \cap L^\infty(\Omega) \), so taking \( v \) as test function in the approximate problem \( (P_m) \), we have

\[
\sum_{i=1}^N \int_\Omega a_i(x, T_m(u^m), \nabla u^m) \frac{D^i}{\exp(2A(|u^m|))} T_1(u^m - T_h(u^m)) \, dx \\
+ \int_\Omega H^m(x, u^m, \nabla u^m) \frac{\exp(2A(|u^m|)) T_1(u^m - T_h(u^m))}{} \, dx \\
+ \int_\Omega |u^m|^{p_0(x)} - 2u^m \frac{\exp(2A(|u^m|)) T_1(u^m - T_h(u^m))}{} \, dx \\
\leq \int_\Omega f_m(x) \exp(2A(|u^m|)) T_1(u^m - T_h(u^m)) \, dx.
\tag{32}
\]
By coercivity of \( a_i \) and increasing conditions of \( H^m \), we obtain

\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_m(u^m), \nabla u^m) D^i u^m \frac{\hat{h}(|u^m|)}{\alpha} \exp(A(|u^m|)) |T_1(u^m - T_h(u^m))| \, dx \\
+ \int_{k \leq |u^m| \leq k+1} a_i(x, u^m, \nabla u^m) \exp(2A(|u^m|)) D^i u^m \, dx \\
+ \delta \int_{\Omega} |u^m|^{p(x)-2} u^m \exp(2A(|u^m|)) T_1(u^m - T_h(u^m)) \, dx \\
\leq C_1 \int_{\Omega} (|f(x)| + |c(x)|) |T_1(u^m - T_h(u^m))| \, dx,
\]

it follows that

\[
\sum_{i=1}^{N} \int_{|u^m| > h+1} \hat{h}(|u^m|) |D^i u^m|^{p_i(x)} \, dx + \delta \int_{|u^m| > h+1} |u^m|^{p(x)-1} \, dx \\
\leq C_1 \int_{|u^m| > h} (|f(x)| + |c(x)|) \, dx.
\]

Thus, for \( \varepsilon > 0 \), there exists \( h(\varepsilon) > 0 \) such that \( \forall h > h(\varepsilon) \)

\[
\sum_{i=1}^{N} \int_{|u^m| > h+1} \hat{h}(|u^m|) |D^i u^m|^{p(x)} \, dx + \delta \int_{|u^m| > h+1} |u^m|^{p(x)-1} \, dx \leq \varepsilon/2. \tag{35}
\]

Let \( Q \) be an arbitrary bounded subset for \( \Omega \).

Then, for any measurable set \( E \subset Q \), we have

\[
\sum_{i=1}^{N} \int_{E} \hat{h}(|u^m|) |D^i u^m|^{p_i(x)} \, dx + \delta \int_{E} |u^m|^{p(x)-1} \, dx \\
\leq \sum_{i=1}^{N} \int_{E} \hat{h}(|T_{h+1} u^m|) |D^i T_{h+1} u^m|^{p_i(x)} \, dx + \delta \int_{E} |T_{h+1}(u^m)|^{p(x)-1} \, dx \tag{36}
\]

\[
+ \sum_{i=1}^{N} \int_{|u^m| > h+1} \hat{h}(|u^m|) |D^i u^m|^{p_i(x)} \, dx + \delta \int_{|u^m| > h+1} |u^m|^{p(x)-1} \, dx.
\]

We conclude that for all \( E \subset Q \) with \( \text{meas}(E) < \beta(\varepsilon) \), and \( T_h(u^m) \rightarrow T_h(u) \)
in \( W_0^{1, p(1)}(\Omega_R) \),

\[
\sum_{i=1}^{N} \int_{E} \hat{h}(|T_{h+1}(u^m)|) |D^i T_{h+1} u^m|^{p_i(x)} \, dx + \delta \int_{E} |T_{h+1}(u^m)|^{p(x)-1} \, dx \leq \varepsilon/2. \tag{37}
\]

Finally, combining the last formulas, \( \forall E \subset Q \) such that \( \text{meas}(E) \leq \beta(\varepsilon) \) we obtain

\[
\sum_{i=1}^{N} \int_{E} \hat{h}(|u^m|) |D^i u^m|^{p_i(x)} \, dx + \delta \int_{E} |u^m|^{p(x)-1} \, dx \leq \varepsilon, \tag{38}
\]
which gives the results.

**Step 5**: Passage to the limit
Let \( v = \psi_l T_k(u^m - \varphi) \), \( \psi_l \in \mathcal{D}(\Omega) \) such that
\[
\psi_l(x) = \begin{cases} 
1 & \text{if } x \in \Omega(l) \\
0 & \text{if } x \in \Omega \setminus \Omega(l+1),
\end{cases}
\]
and taking \( \varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega) \) as test function in the approximate problem, we obtain
\[
\sum_{i=1}^N \int_{\Omega(l+1)} a_i(x, T_m(u^m), \nabla u^m) \psi_l D^iT_k(u^m - \varphi) \, dx \\
+ \sum_{i=1}^N \int_{\Omega(l+1) \setminus \Omega(l)} a_i(x, T_m(u^m), \nabla u^m) D^i \psi_l T_k(u^m - \varphi) \, dx \\
+ \int_{\Omega(l+1)} H^m(x, u^m, \nabla u^m) \psi_l T_k(u^m - \varphi) \, dx \\
+ \int_{\Omega(l+1)} |u^m|^p_0(x) u^m \psi_l T_k(u^m - \varphi) \, dx \\
= \int_{\Omega(l+1)} f_m \psi_l T_k(u^m - \varphi) \, dx.
\] (39)

Let \( M = k + \| \xi \|_\infty \). If \( |u^m| \geq M \), then \( |u^m - \xi| \geq |u^m| - \| \xi \|_\infty \geq k \). Therefore \( \{ |u^m - \xi| < k \} \subseteq \{ |u^m| < M \} \), and hence
\[
I_i^m = \int_{\Omega} a_i(x, T_m(u^m), \nabla u^m) \psi_l D^iT_k(u^m - \varphi) \, dx \\
= \int_{\Omega} a_i(x, T_M(u^m), \nabla T_M(u^m)) \psi_l D^iT_k(u^m - \varphi) \, dx \\
= \int_{\Omega} a_i(x, T_M(u^m), \nabla T_M(u^m)) \psi_l (D^iT_M(u^m) - D^i\varphi) \chi_{\{|u^m - \xi| < k\}} \, dx, \quad m \geq M.
\]

Let \( u^m = u^m - \varphi, w = u - \varphi \). We have
\[
D^iT_k(u^m) - D^iT_k(w) = (D^i w^m - D^i w) \chi_{\{|w^m| < \kappa\}} + D^i w (\chi_{\{|u^m| < \kappa\}} - \chi_{\{|w| < \kappa\}}) \to 0,
\] (40)
a.e. in \( \Omega, m \to \infty \). Using Young inequality and the assumptions (10), (12), we deduce for any \( \varepsilon \in (0, 1) \) that
\[
a_i(x, T_M(u^m), \nabla T_M(u^m)) (D^iT_M(u^m) - D^i\varphi) \chi_{\{|u^m - \xi| < \kappa\}} \geq -c_1(|D^i\xi|_1^p(x)).
\]

Since \( -c_1(|D^i\xi|_1^p(x)) \in L^1(\Omega) \) by Fatou’s lemma we have
\[
\lim_{m \to \infty} \inf_{\Omega} I_i^m \geq \int_{\Omega} a_i(x, T_M(u), \nabla u) \psi_l D^iT_k(u - \varphi) \, dx \\
= \int_{\Omega} a_i(x, u, \nabla u) \psi_l D^iT_k(u - \varphi) \, dx.
\] (41)
\[ \psi_l T_k (u^m - \varphi) \to \psi_l T_k (u^m - \varphi) \text{ weakly * in } L^\infty (\Omega), \ m \to \infty. \] (42)

By equi-integrability of \( H^m \) and \( |u^m|_{p_0(x)} - 2u^m \), passing to the limit on \( m \) in (39) we obtain

\[
\begin{align*}
\sum_{i=1}^{N} \int_{\Omega(l+1)} a_i (x, u, \nabla u) \psi_l D^i T_k (u - \varphi) \, dx \\
+ \sum_{i=1}^{N} \int_{\Omega(l+1), \Omega(l)} a_i (x, u, \nabla u) D^i \psi_l T_k (u - \varphi) \, dx \\
+ \int_{\Omega(l+1)} H(x, u, \nabla u) \psi_l T_k (u - \varphi) \, dx \\
+ \int_{\Omega(l+1)} |u|_{p_0(x)} - 2u \psi_l T_k (u - \varphi) \, dx \\
\leq \int_{\Omega(l+1)} f \psi_l T_k (u - \varphi) \, dx.
\end{align*}
\] (43)

Now passing to the limit to infinity in \( l \) we obtain the existence of entropy solution for the problem.

4 Appendix

Lemma 5 Let \( \Omega \subset \mathbb{R}^N \) be an unbounded domain, suppose that the assumptions (8), (10)–(13), there exists at least one weak solution of the problem \((P_n)\).

Proof Let \( \Omega \subset \mathbb{R}^N \) be an unbounded domain, for all \( \forall u, v \in W^{1, \overline{p}(\cdot)}_0 (\Omega) \), we denote by \( A^n \) the operator defined from \( W^{1, \overline{p}(\cdot)}_0 (\Omega) \) into it's dual by:

\[ \langle A^n (u), v \rangle = \int_{\Omega} a(x, T_n(u), \nabla u) \nabla v \, dx + \int_{\Omega} (H^n(x, u, \nabla u) + |u|_{p_0(x)} - 2u) v \, dx \]

1) We show that \( A^n \) is bounded in \( W^{1, \overline{p}(\cdot)}_0 (\Omega) \):

Using (10) and the fact that \( \|u\|_{\overline{p}(\cdot)} \leq \left( \int_{\Omega} |u|_{p(\cdot)} + 1 \right)^{\frac{1}{p(\cdot)}} \) we obtain the estimate

\[
\|a(x, T_n(u), \nabla u)\|_{p(\cdot)} \to \gamma_{p(\cdot)} = \sum_{i=1}^{N} \|a_i (x, T_n(u), \nabla u)\|_{p(\cdot)},
\]

\[
\leq \sum_{i=1}^{N} \left( \int_{\Omega} |a_i (x, T_n(u), \nabla u)|^{p(x)} \, dx + 1 \right)^{\frac{1}{p}}, \quad (44)
\]

\[
\leq \sum_{i=1}^{N} \left( \hat{a}_i (n) \int_{\Omega} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} + 1 \right)^{\frac{1}{p}},
\]

\[
\leq C_1 (n, \| \nabla u \|_{\overline{p}(\cdot)}).
\]
Likewise, (43) yields
\[ \|H^n(x, u, \nabla u)\|_{p_0^-(\cdot)} \leq C_2(n) \quad \text{and} \quad \|u|^{p_0(\cdot)-2}u\|_{p_0^-(\cdot)} \leq C_3. \] (45)

The above estimates (43), (45) yield, for any \( v \in W_0^{1,p(\cdot)}(\Omega) \):
\[ (A^n(u), v) \leq C_1\|\nabla v\|_{\overline{p}(\cdot)} + 2\left( C_2(n) + C_3 + \|f^n\|_{\overline{p}(\cdot)} \right)(\|v\|_{\overline{p}(\cdot)} + \|\nabla v\|_{p_0(\cdot)}) \]
which gives the bounded of the operator \( A^n \).

2)- \( A^n \) is coercive:
In view of Hölder’s type inequality, we have for all \( u \in W_0^{1,p(\cdot)}(\Omega) \).
\[
\left| \int_\Omega H^n(x, u, \nabla u)u \, dx \right| \\
\leq \left( \frac{1}{p_0^-} + \frac{1}{p_0^+(\cdot)} \right) \left( \int_\Omega |H^n(x, u, \nabla u)|^{p_0^-(\cdot)} \, dx + 1 \right)^{1/(p_0^-)} \|u\|_{p_0^-} \]
\[
\leq 2\left( n^{(p_0^-)^+} |\Omega(n)| + 1 \right)^{1/(p_0^-)} \|u\|_{1,\overline{p}(\cdot)} \\
\leq C_n \|u\|_{1,\overline{p}(\cdot)}.
\] (46)

Indeed, by means of (42), (43), (45) and generalized Young inequality, we deduce that
\[
\langle A^n(u), v \rangle \\
= \int_\Omega a(x, T_n(u), \nabla u) \nabla u \, dx + \int_\Omega \left( H^n(x, u, \nabla u) + |u|^{p_0(x)-2}u + f^n \right)u \, dx \\
\geq \alpha \sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_0(x)} \, dx - C_n\|u\|_{1,\overline{p}(\cdot)} + (1 - \epsilon) \int_\Omega |u|^{p_0(x)} \, dx \\
- C_n \int_\Omega |f^n|^{p_0(x)} \, dx \\
\geq \min(\alpha, (1 - \epsilon)) \left( \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_0^-} + \|u\|_{p_0^-} \right) - C_n\|u\|_{1,\overline{p}(\cdot)} - C'.
\] (47)

In view of (43) we have
\[
\frac{\langle A^n(u), v \rangle}{\|u\|_{1,\overline{p}(\cdot)}} \geq \min(\alpha, (1 - \epsilon)) \left( \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_0^-} + \|u\|_{p_0^-} \right)^{\frac{1}{p_0^-}} - C_n - \frac{C'}{\|u\|_{1,\overline{p}(\cdot)}}
\]

Thus, \( \frac{\langle A^n(u), v \rangle}{\|u\|_{1,\overline{p}(\cdot)}} \to +\infty \) when \( \|u\|_{1,\overline{p}(\cdot)} \to +\infty \).

3)- We show that \( A^n \) is pseudo-monotone operator:
Let us now prove that if
\[
u^j \to u \quad \text{in} \quad W_0^{1,p(\cdot)}(\Omega),
\] (48)
\[ A^m(u^j) \rightharpoonup w \quad \text{in} \quad W_0^{1,p'}(\Omega), \quad \langle A^m(u^j), u^j \rangle \leq \langle w, u \rangle, \]

(49)

then

\[ A^m(u) = w \]

(50)

\[ \lim_{j \to \infty} \langle A^m(u^j), u^j \rangle = \langle A^m(u), u \rangle. \]

(51)

The convergence (48) yields the estimate

\[ \|u^j\|_{W_0^{1,p'}(\Omega)} \leq C_1, \quad j \in \mathbb{N}. \]

(52)

Let us show the following convergence along a subsequence:

\[ u^j \rightharpoonup u \quad \text{a.e. in} \quad \Omega, \quad j \to \infty. \]

(53)

For \( R, h > 0 \), we have

\[
\sum_{i=1}^{N} \int_{\Omega} D^i(\eta_R(|x|)T_h(u^m)) \, dx \leq C_2 \sum_{i=1}^{N} \int_{|u_m|^2 \leq h} |D^i u^m|^p(x) \, dx \\
+ C_3 \sum_{i=1}^{N} \int_{\Omega} |D^i(\eta_R(|x|)^p(x) \, dx \leq C(h, R),
\]

which implies that the sequence \( \{\eta_R(|x|)T_h(u^m)\} \) is bounded in \( W_0^{1,p'}(\Omega(R+1)) \), and by embedding theorem, and since \( \eta_R = 1 \) in \( \Omega(R) \) we have

\[
\eta_R T_h(u^m) \rightharpoonup v_h \quad \text{in} \quad L^{p'}(\Omega((R+1))), \\
T_h(u^m) \rightharpoonup v_h \quad \text{in} \quad L^{p'}(\Omega((R))).
\]

(54)

By Egorov’s Theorem, we can choose \( E_h \subset \Omega(R) \) such that \( \text{meas}(E_h) < 1/h \) and \( T_h(u^m) \rightharpoonup v_h \) uniformly in \( \Omega(R) \setminus E_h \).

Let \( \Omega^h(R) = \{x \in \Omega(R) \setminus E_h : |v_h(x)| \geq h - 1\} \). Since \( T_h(u^m) \) converges uniformly to \( v_h \) in \( \Omega(R) \setminus E_h \), there exists \( m_0 \) such that for any \( m \geq m_0, |T_h(u^m)| \geq h - 2 \) on \( \Omega^h(R) \), i.e. \( |u^m| \geq h - 2 \), then by Lemma 3 we obtain

\[
\text{meas } \Omega^h(R) \leq \sup_m \text{meas}\{x \in \Omega : |u^m| \geq h - 2\} = g(h-2) \to 0
\]

as \( h \) tends to zeros.

Now we set \( \Omega_h(R) = \{x \in \Omega(R) \setminus E_h : |v_h(x)| < h - 1\} \), and remark that \( \Omega(R) = \Omega^h(R) \cup \Omega_h(R) \cup E_h \) by combining the last results we have

\[
\text{meas } \Omega_h(R) > \text{meas } \Omega(R) - 1/h - g(h-2). \]

The uniform convergence of \( T_h(u^m) \) implies that, there exists \( m_0 \in \mathbb{N} \) such that for \( m \geq m_0, |T_h(u^m)| < h \) on \( \Omega_h(R) \), which gives \( u^m \rightharpoonup v_h \) on \( \Omega_h(R) \), by classical argument we can prove that \( v_h \) not depend on \( h \), and the convergence

\[ u^m \rightharpoonup u \quad \text{a.e. in} \quad \Omega(R), \quad m \to \infty, \]

(55)
by (55), we get
\[ u^m \to u \text{ in } L^{p_1}((\Omega(R))), \tag{57} \]
holds true. Then, by the diagonalisation argument with respect to \( R \in \mathbb{N} \), we obtain the result (64).

From (53), (47) we have the estimate
\[ \|a(x,T_m(u^j),\nabla u^j)\|_{p_1'} \leq C_2(m), \quad j \in \mathbb{N}. \tag{58} \]

Therefore, there exist functions \( \tilde{a}^m \in L^{p_1'}(\Omega) \) such that
\[ a(x,T_m(u^j),\nabla u^j) \to \tilde{a}^m \text{ in } L^{p_1'}(\Omega), \quad j \to \infty. \tag{59} \]

The estimate (15) implies the existence of function \( \tilde{b}^m \in L^{p_0'}(\Omega) \) such that
\[ H^m(x,u^j,\nabla u^j) \to \tilde{H}^m \text{ in } L^{p_0'}(\Omega), \quad j \to \infty. \tag{60} \]

Then, from the inequality (13) and the convergence (57) we have
\[ \lim_{j \to \infty} \|u^j\|_{p_0}^2 \leq \|u^0\|_{p_0}^2 \tag{61} \]

Thus, in view of the convergence (54), we obtain from Lemma 4.3 that
\[ \|u_j\|_{p_0}^2 - 2u^j \to \|u^0\|_{p_0}^2 \text{ in } L^{p_1'}(\Omega), \quad j \to \infty. \tag{62} \]

By (49), and (50) - (62), for any \( v \in W_0^{1,p_1}(\Omega) \) we deduce that
\[ \langle w, v \rangle = \lim_{j \to \infty} \langle A^m(u^j), v \rangle = \lim_{j \to \infty} \langle a(x,T_m(u^j),\nabla u^j),\nabla v \rangle + \lim_{j \to \infty} \langle \tilde{a}^m,\nabla v \rangle + \langle \tilde{H}^m + |u|_{p_0}^2 - 2u - f_m, v \rangle. \tag{63} \]

Evidently, the following equality is satisfied:
\[ \langle A^m(u^j), u^j \rangle = \langle a(x,T_m(u^j),\nabla u^j),\nabla u^j \rangle + \langle (H^m(x,u^j,\nabla u^j) + |u|_{p_0}^2 - 2u - f_m), u^j \rangle. \tag{64} \]

(50) and (63) give
\[ \lim_{j \to \infty} \langle A^m(u^j), u^j \rangle \leq \langle \tilde{a}^m,\nabla u \rangle + \langle \tilde{H}^m + |u|_{p_0}^2 - 2u - f_m, u \rangle \tag{65} \]

It follows from the convergence (58) that
\[ \lim_{j \to \infty} \langle f^m, u^j \rangle = \langle f^m, u \rangle. \tag{66} \]

Then, from the inequality (14) and the convergence (57) we have
\[ \lim_{j \to \infty} \langle H^m(x,u^j,\nabla u^j)(u^j - u) \rangle \leq m \lim_{j \to \infty} \int_{\Omega(m)} |u^j - u| \, dx \]
\[ \leq C(m) \lim_{j \to \infty} \|u^j - u\|_{p_0,\Omega(m)} = 0. \]
Using this fact and the convergence (69), we conclude that
\[
\lim_{j \to \infty} \langle H^m(x, u^j, \nabla u^j)u^j \rangle = \langle \tilde{H}^m u \rangle,
\]
which gives
\[
\limsup_{j \to \infty} \langle a(x, T_m(u^j), \nabla u^j, \nabla u^j + |u^j|^{p_0(x)-2}u^j \rangle \leq \langle \tilde{a}^m \cdot \nabla u + |u|^{p_0(x)} \rangle. \tag{68}
\]
On the other hand, thanks to the assumption (10), we have
\[
\langle (a(x, T_m(u^j), \nabla u^j) - a(x, T_m(u^j), \nabla u)) \cdot \nabla (u^j - u) \rangle + \langle (|u^j|^{p_0(x)-2}u^j - |u|^{p_0(x)-2}u)(u^j - u) \rangle \geq 0.
\tag{69}
\]
Then
\[
\langle a(x, T_m(u^j), \nabla u^j) \rangle \cdot \nabla u^j + |u^j|^{p_0(x)} \rangle \geq \langle a(x, T_m(u^j), \nabla u) \rangle \cdot \nabla (u^j - u) \rangle + \langle (|u^j|^{p_0(x)-2}u^j) + (|u|^{p_0(x)-2}u)(u^j - u) \rangle.
\tag{70}
\]
Using (68) and (10) we obtain that
\[
a(x, T_m(u^j), \nabla u) \to a(x, T_m(u^j), \nabla u) \text{ strongly in } L^{p_1} (\Omega), j \to \infty. \tag{71}
\]
In view of the convergence (56), we deduce that
\[
\lim_{j \to \infty} \langle a(x, T_m(u^j), \nabla u^j) \cdot \nabla u^j + |u^j|^{p_0(x)-2}u^j \rangle \geq \langle \tilde{a}^m \cdot \nabla u + |u|^{p_0(x)} \rangle. \tag{72}
\]
Combining (68) and (72) we have
\[
\lim_{j \to \infty} \langle a(x, T_m(u^j), \nabla u^j) \cdot \nabla u^j + |u^j|^{p_0(x)-2}u^j \rangle = \langle \tilde{a}^m \cdot \nabla u + |u|^{p_0(x)} \rangle. \tag{73}
\]
and
\[
\lim_{j \to \infty} \langle A^m(u^j), u^j \rangle = \langle w, u \rangle, \tag{74}
\]
by combining (48), (59), (60), (71) and (73) we get
\[
\langle (a(x, T_m(u^j), \nabla u^j) - a(x, T_m(u^j), \nabla u)) \cdot \nabla (u^j - u) \rangle + \langle (|u^j|^{p_0(x)-2}u^j - |u|^{p_0(x)-2}u)(u^j - u) \rangle = 0. \tag{75}
\]
By Lemma 3 we have
\[
\nabla u^j \to \nabla u \text{ in } L^{p_1} (\Omega), j \to \infty, \tag{76}
\]
\[
\nabla u^j \to \nabla u \text{ a.e. in } \Omega, j \to \infty.
\]
Then Lemma 3
\[
\tilde{a}^m = a(x, T_m(u), \nabla u), \quad \tilde{H}^m = H^m(x, u, \nabla u).
\]
By (74) we conclude the result, and the existence of solutions for the approximate problem is proved.
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