Abstract: Inspired by recent work of Carlson, Friedlander and Pevtsova concerning modules for p-elementary abelian groups $E_r$ of rank $r$ over a field of characteristic $p > 0$, we introduce the notions of modules with constant $d$-radical rank and modules with constant $d$-socle rank for the generalized Kronecker algebra $K_r = \Gamma_r$ with $r \geq 2$ arrows and $1 \leq d \leq r - 1$. We study subcategories given by modules with the equal $d$-radical property and the equal $d$-socle property. Utilizing the simplification method due to Ringel, we prove that these subcategories in $\text{mod} \ K_r$ are of wild type. Then we use a natural functor $\mathcal{S}: \text{mod} \ K_r \to \text{mod} \ kE_r$ to transfer our results to $\text{mod} \ kE_r$.

Keywords: Kronecker algebra, Auslander–Reiten theory, constant socle rank, wild representation type

MSC 2010: 16G20, 16G60, 16G70

Introduction

Let $r \geq 2$. Let $k$ be an algebraically closed field of characteristic $p > 0$, and let $E_r$ be a $p$-elementary abelian group of rank $r$. It is well-known that the category of finite-dimensional $kE_r$-modules $\text{mod} \ kE_r$ is of wild type, whenever $p \geq 3$ or $p = 2$ and $r > 2$. Therefore, subclasses with more restrictive properties have been studied; in [5], the subclass of modules of constant rank $\text{CR}(E_r)$ and modules with even more restrictive properties, called equal images property and equal kernels property, were introduced. Let $\langle x_1, \ldots, x_r \rangle_k$ be a complement of $\text{Rad}^2(kE_r) = \text{Rad}(kE_r)$, and set $x_\alpha := \sum_{i=1}^r a_i x_i$ for $\alpha \in k^r$. We say that $M \in \text{mod} \ kE_r$ has constant Jordan type if the Jordan canonical form of the nilpotent operator $x_\alpha^{[M]}: M \to M$, $m \mapsto x_\alpha \cdot m$ is independent of $\alpha \in k^r \setminus \{0\}$. If the image (kernel) of $x_\alpha^{[M]}$ does not depend on $\alpha$, we say that $M$ has the equal images (kernels) property.

In [21], the author defined analogous categories $\text{CR}$, $\text{EIP}$ and $\text{EKP}$ in the context of the generalized Kronecker algebra $\mathcal{K}_r$, and in more generality for the generalized Beilinson algebra $\mathcal{B}(n, r)$. Using a natural functor $\mathcal{S}: \text{mod} \ \mathcal{K}_r \to \text{mod} \ kE_r$ with nice properties, she gave new insights into the categories of equal images and equal kernels modules for mod $kE_r$ of Loewy length $\leq 2$. A crucial step is the description of CR, EIP and EKP in homological terms, involving a family $\mathcal{P}^{r-1}$-family of regular “test”-modules.

Building on this approach, we show that the recently introduced modules [6] of constant socle rank and constant radical rank can be described in the same fashion. For $1 \leq d < r$, we introduce modules of constant $d$-radical rank $\text{CRR}_d$ and constant $d$-socle rank $\text{CSR}_d$ in mod $\mathcal{K}_r$. More restrictive — and also easier to handle — are modules with the equal $d$-radical property $\text{ERP}_d$ and the equal $d$-socle property $\text{ESP}_d$. For $d = 1$, we have $\text{ESP}_1 = \text{EKP}$, $\text{ERP}_1 = \text{EIP}$ and $\text{CSR}_1 = \text{CR} = \text{CRR}_1$. Studying these classes in the hereditary module category mod $\mathcal{K}_r$ allows us to use tools not available in mod $kE_r$. 

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As a first step, we establish a homological characterization of CSR$_d$, CRR$_d$, ESP$_d$ and ERP$_d$. We denote by Gr$_d$, the Grassmanian of $d$-dimensional subspaces of $k'$. In generalization of [21], we define a Gr$_d$-family of “test”-modules $(X_U)_{U \in \text{Gr}_d}$, and show that the modules in this family can be described in purely combinatorial terms by being indecomposable of dimension vector $(1, r - d)$. This allows us to construct many examples of modules of equal socle rank in mod$\mathcal{K}_s$ for $s \geq 3$ by considering pullbacks along natural embeddings $\mathcal{K}_r \rightarrow \mathcal{K}_s$.

Since $\mathcal{K}_r$ is a wild algebra for $r > 2$ and every regular component in the Auslander–Reiten quiver of $\mathcal{K}_r$ is of type $ZA_\infty$, it is desirable to find invariants that give more specific information about the regular components. It is shown in [21] that there are uniquely determined quasi-simple modules $M_\mathfrak{c}$ and $W_\mathfrak{c}$ in $\mathfrak{c}$ such that the cone $(M_\mathfrak{c} \rightarrow) \subseteq \mathfrak{c}$ consisting of all modules lying on an oriented path starting in $M_\mathfrak{c}$ satisfies $(M_\mathfrak{c} \rightarrow) = \text{EKP} \cap \mathfrak{c}$, and the cone $(\rightarrow W_\mathfrak{c})$ consisting of all modules lying on an oriented path ending in $W_\mathfrak{c}$ satisfies $(\rightarrow W_\mathfrak{c}) = \text{EIP} \cap \mathfrak{c}$. Using results on elementary modules, we generalize this statement for ESP$_d$ and ERP$_d$. Our main results may be summarized as follows.

**Theorem.** Let $r \geq 3$ and $\mathfrak{c}$ be a regular component of the Auslander–Reiten quiver of $\mathcal{K}_r$.

(a) For each $1 \leq i < r$, the category $\Delta_i := \text{ESP}_i \setminus \text{ESP}_{i-1}$ is wild, where $\text{ESP}_0 := \emptyset$.

(b) For each $1 \leq i < r$, there exists a unique quasi-simple module $M_i$ in $\mathfrak{c}$ such that $\text{ESP}_i \cap \mathfrak{c} = (M_i \rightarrow)$.

(c) There exists at most one number $1 < m(\mathfrak{c}) < r$ such that $\Delta_{m(\mathfrak{c})} \cap \mathfrak{c}$ is non-empty. If such a number exists, $\Delta_{m(\mathfrak{c})} \cap \mathfrak{c}$ is the ray starting in $M_{m(\mathfrak{c})}$.

If there is no such number as in (c), we set $m(\mathfrak{c}) = 1$. An immediate consequence of (b) and (c) is that, for $1 \leq i \leq j < r$, we have $M_i = M_j$ or $\tau M_i = M_j$. Moreover, statement (a) shows the existence of a lot of AR-components such that $\Delta_i \cap \mathfrak{c}$ is a ray, and for every such component, we have $m(\mathfrak{c}) = i$. With the dual result for ERP, we assign a number $1 \leq w(\mathfrak{c}) < r$ to each regular component $\mathfrak{c}$, giving us the possibility to distinguish $(r - 1)^2$ different types of regular components.

To prove statement (a), we exploit the fact that every regular module $M$ in mod$\mathcal{K}_r$ has self-extensions with $\dim_k \text{Ext}(M, M) \geq 2$, by applying the process of simplification. This method was introduced in [17] and produces extension closed subcategories, whose objects may be filtered by pairwise orthogonal bricks. For a $p$-elementary abelian group $E$, of rank $r$ over an algebraically closed field of characteristic $p > 0$, mod $kE$, is the only such subcategory. We therefore use the functor $\mathfrak{x} : \text{mod} \mathcal{K}_r \rightarrow \text{mod} kE$, whose essential image (the full subcategory of mod $kE$ formed by all modules isomorphic to modules of the form $\mathfrak{x}(M)$) consists of all modules of Loewy length $\leq 2$, to transfer our results to mod $kE$. We denote by ESP$_{2, d}(E_r)$ the category of modules in mod $kE_r$ of Loewy length $\leq 2$ with the equal $d$-socle property and arrive at the following result.

**Corollary.** Let $\text{char}(k) > 0$, $r \geq 3$ and $1 \leq d < r$. Then ESP$_{2, d}(E_r) \setminus \text{ESP}_{2, d-1}(E_r)$ has wild representation type, where ESP$_{2, 0}(E_r) := \emptyset$.
For $r = 2$, we consider the Beilinson algebra $B(3, 2)$. The fact that $B(3, 2)$ is a concealed algebra of type $Q = 1 \rightarrow 2 \Rightarrow 3$ allows us to apply the simplification process in mod $kQ$. We find a wild subcategory in the category of all modules in mod $B(3, 2)$ with the equal kernels property and conclude the following.

**Corollary.** Assume that char$(k) = p > 2$; then the full subcategory of modules with the equal kernels property in mod $kE_2$ and Loewy length $\leq 3$ has wild representation type.

In particular, we generalize results by Benson [3] and Bondarenko and Lytvynchuk [4] concerning the wildness of various subcategories of $kE_r$-modules. We also construct examples of regular components $\mathcal{C}$ such that each module in $\mathcal{C}$ has constant $d$-socle rank, but no module in $\mathcal{C}$ is $GL_d$-stable in the sense of [6].

### 1 Preliminaries

Throughout this article, let $k$ be an algebraically closed field and $r \geq 2$. If not stated otherwise, $k$ is of arbitrary characteristic. We denote by $Q = (Q_0, Q_1)$ a finite and connected quiver without oriented cycles. For an arrow $a: x \rightarrow y \in Q_1$, we define $s(a) = x$ and $t(a) = y$. We say that $a$ starts in $s(a)$ and ends in $t(a)$. A (finite-dimensional) representation $M = ((M_x)_{x \in Q_0}, (M(a))_{a \in Q_1})$ over $Q$ consists of vector spaces $M_x$ and linear maps $M(a): M_{s(a)} \rightarrow M_{t(a)}$ such that $\dim_k M := \sum_{x \in Q_0} \dim_k M_x$ is finite. A morphism $f: M \rightarrow N$ between representations is a collection of linear maps $(f_x)_{x \in Q_0}$ such that, for each arrow $a: x \rightarrow y$, there is a commutative diagram

$$
\begin{array}{ccc}
M_x & \xrightarrow{M(a)} & M_y \\
\downarrow f_x & & \downarrow f_y \\
N_x & \xrightarrow{N(a)} & N_y
\end{array}
$$

The category of finite-dimensional representations over $Q$ is denoted by $\text{rep}(Q)$, and $kQ$ is the path algebra of $Q$ with idempotents $e_x, x \in Q_0$. The $k$-algebra $kQ$ is a finite-dimensional, associative, basic and connected $k$-algebra. Let mod $kQ$ be the class of finite-dimensional $kQ$ left modules. Given $M \in \text{mod} kQ$, we set $M_x := e_x M$. The categories mod $kQ$ and $\text{rep}(Q)$ are equivalent (see for example [1, Theorem III 1.6]). We will therefore switch freely between representations of $Q$ and modules of $kQ$ if one of the approaches seems more convenient for us. We assume that the reader is familiar with Auslander–Reiten theory and basic results on wild hereditary algebras. For a well written survey on the subjects, we refer to [1, 13, 14].

**Definition.** Recall the definition of the *dimension function*

$$\dim: \text{mod} kQ \rightarrow \mathbb{Z}^{Q_0}, \quad M \mapsto (\dim_k M_x)_{x \in Q_0}.$$

If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence, then $\dim A + \dim C = \dim B$. The quiver $Q$ defines a (non-symmetric) bilinear form

$$\langle -,- \rangle: \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z},$$

given by $(x_i, (y_j)) \mapsto \sum_{i \in Q_0} x_i y_i - \sum_{a \in Q_1} x_{s(a)} y_{t(a)}$. For the case that $x, y$ are given by dimension vectors of quiver representations, there is another description of $\langle -,- \rangle$ known as the Euler–Ringel form [17]

$$\langle \dim M, \dim N \rangle = \dim_k \text{Hom}(M, N) - \dim_k \text{Ext}(M, N).$$

We denote by $q = q_Q: \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$ the corresponding quadratic form. A vector $d \in \mathbb{Z}^{Q_0}$ is called a real root if $q(d) = 1$ and an imaginary root if $q(d) \leq 0$.

Denote by $\Gamma$, the $r$-Kronecker quiver, which is given by two vertices $1, 2$ and arrows $y_1, \ldots, y_r$: $1 \rightarrow 2$.

We set $\mathcal{K}_r := k\Gamma$, and $P_1 := \mathcal{K}_r e_2$, $P_2 := \mathcal{K}_r e_1$. $P_1$ and $P_2$ are the indecomposable projective modules of mod $\mathcal{K}_r$, $\dim_k \text{Hom}(P_1, P_2) = r$ and $\dim_k \text{Hom}(P_2, P_1) = 0$. As Figure 2 suggests, we write

$$\dim M = (\dim_k M_1, \dim_k M_2).$$
For example, \( \dim P_1 = (0, 1) \) and \( \dim P_2 = (1, r) \). The Coxeter matrix \( \Phi \) and its inverse \( \Phi^{-1} \) are

\[
\Phi = \begin{pmatrix} r^2 - 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \Phi^{-1} = \begin{pmatrix} -1 & r \\ -r & 1 \end{pmatrix}.
\]

For \( M \) indecomposable, \( \dim \tau M = \Phi(\dim M) \) holds if \( M \) is not projective and \( \dim \tau^{-1} M = \Phi^{-1}(\dim M) \) if \( M \) is not injective. The quadratic form \( q \) is given by \( q(x, y) = x^2 + y^2 - ry \).

Figure 3 shows the notation we use for the components \( \mathcal{P}, \mathcal{J} \) in the Auslander–Reiten quiver of \( \mathcal{K}_{\tau} \) which contain the indecomposable projective modules \( P_1, P_2 \) and indecomposable injective modules \( I_1, I_2 \). The set of all other components is denoted by \( \mathcal{R} \).

Ringel has proven [18, Theorem 2.3] that every component in \( \mathcal{R} \) is of type \( Z\text{A}_{\infty} \) if \( r \geq 3 \) or a homogeneous tube \( Z\text{A}_{\infty}/(r) \) if \( r = 2 \). A module in such a component is called regular. An irreducible morphism in a regular component is injective if the corresponding arrow is uprising (see Figure 1 for \( r \geq 3 \) and surjective otherwise. A regular module \( M \) is called quasi-simple if the AR-sequence terminating in \( M \) has an indecomposable middle term. If \( M \) is quasi-simple in a regular component \( \mathcal{C} \), there is an infinite chain (a ray) of irreducible monomorphisms

\[
M = M[1] \rightarrow M[2] \rightarrow M[3] \rightarrow \cdots \rightarrow M[l] \rightarrow \cdots
\]

and an infinite chain (a coray) of irreducible epimorphisms

\[
\cdots \rightarrow (l)M \rightarrow \cdots \rightarrow (3)M \rightarrow (2)M \rightarrow (1)M = M,
\]

and for each regular module \( X \), there are unique quasi-simple modules \( N, M \) and \( l \in \mathbb{N} \) with \( (l)M = X = N[l] \). The number \( l \) is called the quasi-length of \( X \). We fix the orientation of each regular component in such a way that the quasi-simple modules form the bottom layer of the component (see Figure 1).

The indecomposable modules in \( \mathcal{P} \) are called preprojective modules and the modules in \( \mathcal{J} \) are called preinjective modules. Moreover, we call an arbitrary module preprojective (resp. preinjective, regular) if all its indecomposable direct summands are preprojective (resp. preinjective, regular). It is \( P \) in \( \mathcal{P} \) (in \( \mathcal{J} \)) if and only if there is \( l \in \mathbb{N}_0 \) with \( r^l P = P_i \) \((r^{-1}I = I_i)\) for \( i \in \{1, 2\} \). Recall that there are no homomorphisms from right to left [1, Corollary VIII.2.13]. To emphasize this result later on, we just write

\[
\text{Hom}(\mathcal{J}, \mathcal{P}) = 0 = \text{Hom}(\mathcal{J}, \mathcal{R}) = 0 = \text{Hom}(\mathcal{R}, \mathcal{P})
\]

Using the canonical equivalence ([1, Theorem III.1.6]) of categories \( \text{mod } \mathcal{K}_{\tau} \cong \text{rep}(\Gamma_{\tau}) \), we introduce the duality \( \delta : \text{mod } \mathcal{K}_{\tau} \rightarrow \text{mod } \mathcal{K}_{\tau} \) by setting \( (\delta M)_x := (M_{\psi(x)})^* \) and \( (\delta M(y)) := (M(y))^* \) for each \( M \in \text{rep}(\Gamma_{\tau}) \), where \( \psi : \{1, 2\} \rightarrow \{1, 2\} \) is the permutation of order 2. Note that \( \delta(P_i) = I_i \) for all \( i \in \mathbb{N} \). We state a simplified version of Kac’s theorem [12, Theorem 1.10] for the Kronecker algebra in combination with results on the quadratic form proven in [17, Lemma 2.3].
Theorem 1.1. Let \( r \geq 2 \) and \( d \in \mathbb{N}_0^2 \).

(a) If \( d = \dim M \) for some indecomposable module \( M \), then \( q(d) \leq 1 \).

(b) If \( q(d) = 1 \), then there exists a unique indecomposable module \( X \) with \( \dim X = d \). In this case, \( X \) is preprojective or preinjective, and \( X \) is preprojective if and only if \( \dim_k X_1 < \dim_k X_2 \).

(c) If \( q(d) \leq 0 \), then there exist infinitely many indecomposable modules \( Y \) with \( \dim Y = d \) and each \( Y \) is regular.

Since there is no pair \( (a, b) \in \mathbb{N}_0^2 \setminus \{(0, 0)\} \) satisfying \( a^2 + b^2 - rab = q(a, b) = 0 \) for \( r \geq 3 \), we conclude together with [1, Lemma VIII.2.7] the following.

Corollary 1.2. Let \( M \) be an indecomposable \( \mathcal{K}_r \)-module. Then \( \Ext(M, M) = 0 \) if and only if \( M \) is preprojective or preinjective. If \( r \geq 3 \) and \( M \) is regular, then \( \dim_k \Ext(M, M) \geq 2 \).

Let \( \text{mod}_{\rho} \mathcal{K}_r \) be the subcategory of all modules without non-zero projective direct summands and \( \text{mod}_{\tau} \mathcal{K}_r \) the subcategory of all modules without non-zero injective summands. Since \( \mathcal{K}_r \) is a hereditary algebra, the Auslander–Reiten translation \( \tau : \text{mod} \mathcal{K}_r \to \text{mod} \mathcal{K}_r \) induces an equivalence from \( \text{mod}_{\rho} \mathcal{K}_r \) to \( \text{mod}_{\tau} \mathcal{K}_r \). In particular, if \( M \) and \( N \) are indecomposable with \( M, N \) not projective, we get \( \Hom(M, N) \cong \Hom(\tau M, \tau N) \). The Auslander–Reiten formula [1, Theorem II.2.13] simplifies to the following.

Theorem 1.3 ([13, Theorem 2.3]). For \( X, Y \) in mod \( \mathcal{K}_r \), there a functorial isomorphisms

\[
\Ext(X, Y) \cong \Hom(Y, \tau X)^* \cong \Hom(\tau^{-1} Y, X)^*.
\]

2 Modules of constant radical and socle rank

2.1 Elementary modules of small dimension

Let \( r \geq 3 \). The homological characterization in [21] uses an algebraic family of modules of projective dimension 1 for the Beilinson algebra \( B(n, r) \) on \( n \) vertices. For \( n = 2 \), we have \( B(2, r) = \mathcal{K}_r \), and \( \text{mod} \mathcal{K}_r \) is a hereditary category. Hence every non-projective module is of projective dimension 1. In the following, we study the module family \( (X_a)_{a \in \mathbb{N}_0^2 \setminus \{0\}} \) for \( n = 2 \). We will see later on that each non-zero proper submodule of \( X_a \) is isomorphic to a finite number of copies of \( P_1 \), and \( X_a \) itself is regular. In particular, we do not find a short exact sequence \( 0 \to A \to X_a \to B \to 0 \) such that \( A \) and \( B \) are regular and non-zero. In the language of wild hereditary algebras, we therefore deal with elementary modules.

Definition ([16, Definition 1]). A non-zero regular module \( E \) is called elementary if there is no short exact sequence \( 0 \to A \to E \to B \to 0 \) with \( A \) and \( B \) regular non-zero. In particular, elementary modules are indecomposable and quasi-simple.

Elementary modules are analogues of quasi-simple modules in the tame hereditary case \( (r = 2) \). If \( X \) is a regular module, then \( X \) has a filtration \( 0 = X_0 \subset X_1 \subset \cdots \subset X_r = X \) such that \( X_i/X_{i-1} \) is elementary for all \( 1 \leq i \leq r \) and the elementary modules are the smallest class with that property. For basic results on elementary modules, used in this section, we refer to [16].

We are grateful to Otto Kern for pointing out the following helpful lemma.

Lemma 2.1.1. Let \( E \) be an elementary module and \( X, Y \) regular with non-zero morphisms \( f : X \to E \) and \( g : E \to Y \). Then \( g \circ f \neq 0 \). In particular, \( \End(E) = k \).

Proof. Since \( f \) is non-zero and \( \Hom(\mathcal{R}, \mathcal{R}) = 0 = \Hom(\mathcal{I}, \mathcal{R}) \), \( \text{im} f \) is a regular non-zero submodule of \( E \). Consequently, since \( E \) is elementary, \( \text{coker} f \) is preinjective by [16, Proposition 1.3]; hence \( g \) cannot factor through \( \text{coker} f \).

We use the theory on elementary modules to generalize [21, Corollary 2.7] in the hereditary case.

Proposition 2.1.2. Let \( \mathcal{E} \) be a non-empty family of elementary modules of bounded dimension, and put

\[
\mathcal{T}(\mathcal{E}) := \ker \Ext(\mathcal{E}, -) = \{ M \in \text{mod} \mathcal{K}_r \mid \Ext(E, M) = 0 \text{ for all } E \in \mathcal{E} \}.
\]
Then the following statements hold.

(1) \( \mathcal{T}(\mathcal{E}) \) is closed under extensions, images and \( \tau \).

(2) \( \mathcal{T}(\mathcal{E}) \) contains all preinjective modules.

(3) For each regular component \( \mathcal{C} \), the set \( \mathcal{T}(\mathcal{E}) \cap \mathcal{C} \) forms a non-empty cone in \( \mathcal{C} \), which consists of the predecessors of a uniquely determined quasi-simple module in \( \mathcal{C} \), i.e. there is \( W \in \mathcal{C} \) quasi-simple such that \( \mathcal{T}(\mathcal{E}) \cap \mathcal{C} = (- \rightarrow W) := \{ t^iW \mid i, l \in \mathbb{N}_0 \} \).

Proof. (1) Since \( \text{Ext}^2 = 0 \), the category is closed under images and extensions. Let \( M \in \mathcal{T}(\mathcal{E}) \); then the Auslander–Reiten formula yields \( 0 = \dim_k \text{Hom}(M, \tau M) \) for all \( E \in \mathcal{E} \). We first show that \( M \) is not preprojective. Assume to the contrary that \( P = M \) is preinjective. Let \( l \in \mathbb{N}_0 \) such that \( t^lP \) is projective; then \( t^lP = P_i \) for an \( i \in \{ 1, 2 \} \) and

\[
0 = \dim_k \text{Hom}(P, \tau E) = \dim_k \text{Hom}(t^lP, t^{l+1}E) = (\dim(t^{l+1}E))_{3-l}.
\]

This is a contradiction since every non-sincere \( \mathcal{X}_i \)-module is semi-simple. Hence \( M \) is regular or preinjective.

Now we show \( \tau M \in \mathcal{T}(\mathcal{E}) \). In view of the Auslander–Reiten formula, we get

\[
\dim_k \text{Ext}(E, \tau M) = \dim_k \text{Hom}(M, E).
\]

Let \( f : M \rightarrow E \) be a morphism, and assume that \( f \neq 0 \). Since \( 0 = \text{Hom}(\mathcal{T}, \mathcal{R}) \), the module \( M \) is not preinjective and therefore regular. As a regular module \( E \) has self-extensions (see Corollary 1.2), and therefore \( E \not\in \mathcal{T}(\mathcal{E}) \).

Hence \( 0 \neq \dim_k \text{Ext}(E, E) = \dim_k \text{Hom}(E, \tau E) \), and we find \( 0 \neq g \in \text{Hom}(E, \tau E) \). Lemma 2.1.1 provides a non-zero morphism

\[
M \xrightarrow{f} E \xrightarrow{g} \tau E.
\]

We conclude \( 0 \neq \dim_k \text{Hom}(M, \tau E) = \dim_k \text{Ext}(E, M) = 0 \), a contradiction. Hence

\[
0 = \dim_k \text{Hom}(M, \tau E) = \dim_k \text{Ext}(E, \tau M) \quad \text{for all } E \in \mathcal{E} \text{ and } \tau M \in \mathcal{T}(\mathcal{E}).
\]

(2) The injective modules \( I_1, I_2 \) are contained in \( \mathcal{T}(\mathcal{E}) \). Now apply (1).

(3) The existence of the cones can be shown as in [21, Theorem 3.3]. We sketch the proof. Let \( X \in \mathcal{C} \) be a quasi-simple module, and denote the upper bound by \( L \). By [13, Lemma 4.6, Proposition 10.5], we find \( n_0 \in \mathbb{N} \) such that \( \text{Ext}(Y, t^lX) = 0 \) for all \( l \geq n_0 \) and each regular representation \( Y \) with \( \dim_k Y \leq L \) and \( \text{Ext}(E, t^{-l}X) \neq 0 \) for some \( E \in \mathcal{E} \). In particular, we have \( t^{-l}X \in \mathcal{T}(\mathcal{E}) \) and \( t^{-l}X \not\in \mathcal{T}(\mathcal{E}) \). Now (1) shows that \( \mathcal{T}(\mathcal{E}) \cap \mathcal{C} = (- \rightarrow M) \) for the uniquely determined quasi-simple module \( M \in \mathcal{C} \) with \( M \in \mathcal{T}(\mathcal{E}) \) and \( t^{-l} \not\in \mathcal{T}(\mathcal{E}) \).

The next result follows by the Auslander–Reiten formula and duality since \( \delta(\mathcal{E}) \) is elementary if and only if \( E \) is elementary.

**Proposition 2.1.3.** Let \( \mathcal{E} \) be a family of elementary modules of bounded dimension, and put

\[
\mathcal{T}(\mathcal{E}) := \{ M \in \text{mod} \mathcal{X}_r \mid \text{Hom}(E, M) = 0 \text{ for all } E \in \mathcal{E} \}.
\]

Then the following statements hold.

(1) \( \mathcal{T}(\mathcal{E}) \) is closed under extensions, submodules and \( \tau^{-1} \).

(2) \( \mathcal{T}(\mathcal{E}) \) contains all preprojective modules.

(3) For each regular component \( \mathcal{C} \), the set \( \mathcal{T}(\mathcal{E}) \cap \mathcal{C} \) forms a non-empty cone in \( \mathcal{C} \), which consists of the successors of a uniquely determined quasi-simple module in \( \mathcal{C} \).

Note that \( \mathcal{T}(\mathcal{E}) \) is a torsion-free class of some torsion pair \( (\mathcal{T}, \mathcal{T}(\mathcal{E})) \) and \( \mathcal{T}(\mathcal{E}) \) is the torsion class of some torsion pair \( (\mathcal{T}(\mathcal{E}), \mathcal{T}) \) (see for example [1, Proposition VI.1.4]).

**Lemma 2.1.4.** Let \( M, N \) be indecomposable modules with \( \dim M = (c, 1), \dim N = (1, c), 1 \leq c < r \). Then the following statements hold.

(a) \( \tau^z M \) and \( \tau^z N \) are elementary for all \( z \in \mathbb{Z} \). Moreover, every proper factor of \( M \) is injective, and every proper submodule of \( N \) is projective.

(b) Every proper factor module of \( \tau^z M \) is preinjective, and every proper submodule of \( \tau^{-1} N \) is preprojective for \( l \in \mathbb{N}_0 \).
Proof. We will give the proofs for $M$. The statements for $N$ follow by duality.

(a) By [16, Lemma 1.1], $M$ is elementary if and only if all elements in its $r$-orbit are elementary. Since $(c, 1)$ is an imaginary root of the quadratic form, $M$ is regular. Now let $0 < X \subset M$ be a proper submodule with dimension vector \( \dim X = (a, b) \). Then $b = 1$ since $\text{Hom}(\mathcal{J}, X) = 0$. Hence $\dim M/X = (c - a, 0)$, and $M/X$ is injective.

(b) Let $l \geq 1$ and $0 \to \ker f \to \tau^l M \to X \to 0$ be exact with $X$ regular and non-zero. We have to show that $\tau^l M \cong X$. We assume that $\ker f \neq 0$. Since $\text{Hom}(\mathcal{J}, X) = 0$, we conclude that $\ker f$ has no indecomposable preinjective direct summand. Since $\tau^{-1}$ is right exact [1, Corollary VII.1.9], we get an exact sequence $\tau^{-1} \ker f \to M \to \tau^{-1} X \to 0$. We conclude with (a) that $\tau^{-1} X \cong M$; hence $X \cong \tau^{-1} M$. \qed

2.2 An algebraic family of test-modules

Let $r \geq 2$. Now we take a closer look at the modules $\langle X_a \rangle_{a \in k^r \setminus \{0\}}$. Let us start this section by recalling some definitions from [21] and the construction of the module family. We use a slightly different notation since we are only interested in the case $B(2, r) = \mathcal{K}_r$

For $a \in k^r$ and $M \in \mathcal{M}$, we define $x_a := a_1 y_1 + \cdots + a_r y_r$ and denote by $x_a^M : M \to M$ the linear operator associated to $x_a$.

Definition. For $a \in k^r \setminus \{0\}$, the map $\overline{a} : (e_j)_k = P_1 \to P_2$, $e_2 \mapsto a_1 y_1 + \cdots + a_r y_r = x_a$ defines an embedding of $\mathcal{K}_r$-modules and is just the left multiplication by $x_a$. We now set $X_a := \text{coker } \overline{a}$.

These modules are the “test”-modules introduced in [21]. In fact, $\text{im } \overline{a}$ is a 1-dimensional submodule of $P_2$ contained inside the radical $\text{rad}(P_2)$ of the local module $P_2$. From the definition, we get an exact sequence $0 \to P_1 \to P_2 \to X_a \to 0$ and $\dim X_a = (1, r - (0, 1))$. Since $P_2$ is local with semi-simple radical $\text{rad}(P_2) = P_1^r$, it now seems natural to study embeddings $P_1^d \to P_2$ for $1 \leq d < r$ and the corresponding cokernels. This motivates the next definition. We restrict ourselves to $d < r$ since otherwise the cokernel is the simple injective module.

Definition. Let and $1 \leq d < r$. For $T = (u_1, \ldots, u_d) \in (k^r)^d$, we define $\overline{T} : (P_1)^d \to P_2$ as the $\mathcal{K}_r$-linear map

$$
\overline{T}(x) = \sum_{i=1}^d \overline{u}_i \circ \pi_i(x),
$$

where $\pi_i : (P_1)^d \to P_1$ denotes the projection onto the $i$-th coordinate.

The map $\overline{T}$ is injective if and only if $T$ is linearly independent; then we have

$$
\dim \text{coker } \overline{T} = \dim P_2 - d \dim P_1 = (1, r - d),
$$

and $\text{coker } \overline{T}$ is indecomposable because $P_2$ is local. Moreover, $(1, r - d)$ is an imaginary root of $S$, and therefore coker $\overline{T}$ is regular indecomposable and by Lemma 2.1.4 elementary. We define $\langle T \rangle := \langle u_1, \ldots, u_d \rangle_k$.

Lemma 2.2.1. Let $T, S \in (k^r)^d$ such that $\dim_k \langle T \rangle = d = \dim_k \langle S \rangle$; then $\text{coker } \overline{T} \cong \text{coker } \overline{S}$ if and only if $\langle T \rangle = \langle S \rangle$.

Proof. If $\langle T \rangle = \langle S \rangle$, then the definition of $\overline{T}$ and $\overline{S}$ implies $\text{im } \overline{T} = \text{im } \overline{S}$. Hence coker $\overline{T} = P_2/\text{im } \overline{T} = \text{coker } \overline{S}$.

Now let $\langle S \rangle \neq \langle T \rangle$, and assume that $0 \neq \varphi : \text{coker } \overline{T} \to \text{coker } \overline{S}$ is $\mathcal{K}_r$-linear. Since coker $\overline{S}$ is local with radical $P_1^r$ and $\text{Hom}(\mathcal{R}, P) = 0$, the map $\varphi$ is surjective and therefore injective. Recall that $P_2$ has $\{e_1, y_1, \ldots, y_r\}$ as a basis. Let $x \in P_2$ such that $\varphi(e_1 + \text{im } \overline{T}) = x + \text{im } \overline{S}$. Since $\varphi$ is $\mathcal{K}_r$-linear, we get

$$
x + \text{im } \overline{S} = e_1 \varphi(e_1 + \text{im } \overline{T}) = e_1 x + \text{im } \overline{S}
$$

and hence $x - e_1 x \in \text{im } \overline{S}$. Write $x = \mu e_1 + \sum_{i=1}^r \mu_i y_i$; then

$$
x - \mu e_1 = \sum_{i=1}^r \mu_i y_i = x - e_1 x \in \text{im } \overline{S}
$$

and $x + \text{im } \overline{S} = \mu e_1 + \text{im } \overline{S}$. \qed
The assumption \( \langle S \rangle \neq \langle T \rangle \) yields \( y \in \text{im} \bar{S} \setminus \text{im} \bar{T} \subset \langle y_1, \ldots, y_i \rangle_k \). Then \( y + \text{im} \bar{T} \neq 0 \) and

\[
\varphi(y + \text{im} \bar{T}) = y\varphi(e_1 + \text{im} \bar{T}) = \mu y(e_1 + \text{im} \bar{S}) = \mu y + \text{im} \bar{S} = \text{im} \bar{S},
\]

a contradiction to the injectivity of \( \varphi \). Hence \( \text{Hom}(\text{coker} \bar{T}, \text{coker} \bar{S}) = 0 \) and \( \text{coker} \bar{T} \neq \text{coker} \bar{S} \). \( \square \)

**Definition.** Let \( r \geq 2 \) and \( U \in \text{Gr}_{d,r} \) with basis \( T = (u_1, \ldots, u_d) \). We define \( X_U := \text{coker} \bar{T} \).

**Remark.** \( X_U \) is well defined (up to isomorphism) with dimension vector \( \dim X_U = (1, r - d) \), and \( X_U \) is elementary for \( r \geq 3 \) and quasi-simple for \( r = 2 \).

For a module \( X \), we define add \( X \) as the category of summands of finite direct sums of \( X \), and \( Q^d \) denotes the set of isomorphism classes \([M] \) of indecomposable modules \( M \) with dimension vector \( (1, r - d) \) for \( 1 \leq d < r \).

**Proposition 2.2.2.** Let \( M \) be indecomposable.

(a) If \([M] \in Q^d \), then there exists \( U \in \text{Gr}_{d,r} \) with \( M \cong X_U \).

(b) The map \( \varphi : \text{Gr}_{d,r} \to Q^d \), \( U \mapsto [X_U] \) is bijective.

(c) Let \( 1 \leq c \leq d \) and \([M] \in Q^d \). There is \([N] \in Q^c \) and an epimorphism \( \pi : N \to M \).

**Proof.** (a) Let \( 0 \subseteq X \subseteq M \) be a submodule of \( M \). Then \( X \subseteq \text{rad}(M) = P_1^{r - d} \), and \( X \) is in add \( P_1 \). It is

\[
1 = \dim_k M = \dim_k \text{Hom}(X,e_1,M) = \dim_k \text{Hom}(P_2,M),
\]

so we find a non-zero map \( \pi : P_2 \to M \). Since every proper submodule of \( M \) is in add \( P_1 \) and \( \text{Hom}(P_2,P_1) = 0 \), the map \( \pi : P_2 \to M \) is surjective and yields an exact sequence \( 0 \to P_1^{r - d} \to P_2 \xrightarrow{\pi} M \to 0 \). For \( 1 \leq i \leq d \), there exist uniquely determined elements \( \beta_i, a_i, \ldots, a_i' \in k \) such that

\[
g_i(e_2) = \beta_i e_1 + a_i' y_1 + \cdots + a_i' y_r \in P_2 = \langle y_1, e_1 \mid 1 \leq i \leq r \rangle_k,
\]

where \( g_i : P_1 \to P_1^d \) denotes the embedding into the \( i \)-th coordinate. Since \( e_2 \) is an idempotent with \( e_2 y_j = y_j \) \( (1 \leq j \leq r) \) and \( e_2 e_1 = 0 \), we get

\[
a_i' y_1 + \cdots + a_i' y_r = e_2 (t \circ g_i (e_2)) = (t \circ g_i) (e_2 \cdot e_2) = (t \circ g_i)(e_2) = \beta_i e_1 + a_i' y_1 + \cdots + a_i' y_r.
\]

Hence \( \beta_i = 0 \). Now define \( a_i := (a_i', \ldots, a_i') \), \( T := (a_1, \ldots, a_d) \) and \( U := \langle T \rangle \). It is \( t = \overline{T} \), and by the injectivity of \( t \), we conclude that \( T \) is linearly independent, and therefore \( U \in \text{Gr}_{d,r} \). Now we conclude

\[
X_U = \text{coker} \bar{T} = \text{coker} t = M.
\]

(b) This is an immediate consequence of (a) and Lemma 2.2.1.

(c) By (a), we find \( U \) in \( \text{Gr}_{d,r} \) with basis \( T = (u_1, \ldots, u_d) \) such that \( X_U \cong M \). Let \( V \) be the subspace with basis \( S = (u_1, \ldots, u_c) \). Then \( X \subseteq \text{im} \bar{T} \), and we get an epimorphism \( \pi : X_U = P_2 / \text{im} \bar{S} \to P_2 / \text{im} \bar{T} = X_U, x + \bar{S} \to x + \bar{T} \) with \( \dim X_U = (1, r - c) \). \( \square \)

As a generalization of \( \chi^M_T : M \to M \), we introduce maps \( x_M^T : M \to M^d \) and \( y_M^T : M^d \to M \) for \( 1 \leq d < r \) and \( T \in (k^r)^d \). Note that \( x_M^T = y_M^T \) if and only if \( d = 1 \).

**Definition.** Let \( 1 \leq d < r \) and \( T = (a_1, \ldots, a_d) \in (k^r)^d \). We denote by \( x_M^T \) and \( y_M^T \) the operators

\[
x_M^T : M \to M^d, \quad m \mapsto (x_{a_1}^M(m), \ldots, x_{a_d}^M(m)),
\]

\[
y_M^T : M^d \to M, \quad (m_1, \ldots, m_d) \mapsto x_{a_1}^M(m_1) + \cdots + x_{a_d}^M(m_d).
\]

It is \( \text{im} x_M^T \subseteq M_2 \oplus \cdots \oplus M_2, M_2 \oplus \cdots \oplus M_2 \subseteq \ker y_M^T \) and \( (x_M^T)^* = y_M^T \) since, for \( f = (f_1, \ldots, f_d) \in (\delta M)^d \) and \( m \in M \), we have

\[
(x_M^T)^* (f)(m) = (x_{a_1}^M)^*(f_1, \ldots, f_d)(m) = \sum_{i=1}^d (f_i \circ x_{a_i}^M)(m)
\]

\[
= \sum_{i=1}^d f_i(x_{a_i}, m) = \sum_{i=1}^d (x_{a_i}, f_i)(m)
\]

\[
y_M^T (f_1, \ldots, f_d)(m) = y_M^T (f)(m).
\]
Lemma 2.2.3. Let $1 \leq d < r$ and $U \in \text{Gr}_{d,r}$. Every non-zero quotient $Q$ of $X_U$ is indecomposable. $Q$ is preinjective (injective) if $\dim Q = (1, 0)$ and regular otherwise.

Proof. Since $X_U$ is regular, we conclude with $\text{Hom}(R, P) = 0$ that every indecomposable non-zero quotient of $X_U$ is preinjective or regular. Let $Q$ be such a quotient with $\dim Q = (a, b)$ and $Q \neq X_U$. Since $X_U$ is local with radical $P_1$ and $\dim (1, r-d)$, it follows $(1, r-d) = (a, b) + (0, c)$ for some $c > 0$. Hence $a = 1$, and $Q$ is an injective module if $b = 0$. Otherwise, $Q$ is also indecomposable since $b > 0$ and $Q = A \oplus B$ with $A, B \neq 0$ imply w.l.o.g. $(\dim B) = 1$. Hence $B \in \text{add} P_1$, which is a contradiction to $\text{Hom}(R, P) = 0$. \hfill \Box

2.3 Modules for the generalized Kronecker algebra

In the following, we will give the definition of $\mathcal{K}_r$-modules ($r \geq 2$) with constant radical rank and constant socle rank.

Definition. Let $M$ be in $\text{mod} \mathcal{K}_r$ and $1 \leq d < r$.
(a) $M$ has constant $d$-radical rank if the dimension of
\[
\text{Rad}_U(M) := \sum_{u \in U} x^M_u(M) \subseteq M_2
\]
is independent of the choice of $U \in \text{Gr}_{d,r}$.
(b) $M$ has constant $d$-socle rank if the dimension of
\[
\text{Soc}_U(M) := \{m \in M \mid x^M_u(M) = 0 \text{ for all } u \in U\}
= \bigcap_{u \in U} \ker(x^M_u) = M_2
\]
is independent of the choice of $U \in \text{Gr}_{d,r}$.
(c) $M$ has the equal $d$-radical property if $\text{Rad}_U(M)$ is independent of the choice of $U \in \text{Gr}_{d,r}$.
(d) $M$ has the equal $d$-socle property if $\text{Soc}_U(M)$ is independent of the choice of $U \in \text{Gr}_{d,r}$.

Definition. Let $1 \leq d < r$. We define
(a) $\text{ESP}_d := \{M \in \text{mod} \mathcal{K}_r \mid M_2 = \text{Soc}_U(M) \text{ for all } U \in \text{Gr}_{d,r}\}$,
(b) $\text{ERP}_d := \{M \in \text{mod} \mathcal{K}_r \mid M_2 = \text{Rad}_U(M) \text{ for all } U \in \text{Gr}_{d,r}\}$,
(c) $\text{CSR}_d := \{M \in \text{mod} \mathcal{K}_r \mid \text{there exists } c \in \mathbb{N}_0 \text{ such that } \text{dim}_k \text{Soc}_U(M) = c \text{ for all } U \in \text{Gr}_{d,r}\}$,
(d) $\text{CRR}_d := \{M \in \text{mod} \mathcal{K}_r \mid \text{there exists } c \in \mathbb{N}_0 \text{ such that } \text{dim}_k \text{Rad}_U(M) = c \text{ for all } U \in \text{Gr}_{d,r}\}$.

Lemma 2.3.1. Let $M$ be indecomposable and not simple.
(a) $M$ has the equal $d$-socle property if and only if $M \in \text{ESP}_d$.
(b) $M$ has the equal $d$-radical property if and only if $M \in \text{ERP}_d$.

Proof. (a) Assume that $M$ is in $\text{ESP}_d$, and let $W := \text{Soc}_U(M)$ for $U \in \text{Gr}_{d,r}$. Denote by $e_1, \ldots, e_r \in k'$ the canonical basis vectors. We find a $k'$-complement of $\bigcap_{i=1}^r \ker(x^M_{e_i}) \cap M_1$ in $M_1$, say $K \subseteq M_1$. Then $M$ decomposes into submodules $M = (K + M_2) \oplus \bigcap_{i=1}^r \ker(x^M_{e_i}) \cap M_1$. Since $M$ is not simple, we have $0 \neq M_2$ and conclude $0 = \bigcap_{i=1}^r \ker(x^M_{e_i}) \cap M_1$, i.e. $\bigcap_{i=1}^r \ker(x^M_{e_i}) = M_2$ (see also [10, Lemma 5.1.1]). Denote by $S(d)$ the set of all subsets of $\{1, \ldots, r\}$ of cardinality $d$. Then
\[
\bigcap_{S \in S(d)} \bigcap_{j \in S} \ker(x^M_{e_j}) = \bigcap_{i=1}^r \ker(x^M_{e_i}) = M_2.
\]
Since $\langle e_j \mid j \in S \rangle \subseteq \text{Gr}_{d,r}$ and $M \in \text{ESP}_d$, we get $\bigcap_{S \in S(d)} \ker(x^M_{e_j}) = W$ and hence $M_2 = \bigcap_{S \in S(d)} W = W = \text{Soc}_U(M)$.

(b) Let $M \in \text{ERP}_d$, $U \in \text{Gr}_{d,r}$ and $W := \text{Rad}_U(M)$. Since $M$ is not simple, it is [10, Lemma 5.1.1] $\sum_{i=1}^r x^M_{e_i} = M_2$ and hence
\[
W = \sum_{S \in S(d)} \sum_{j \in S} x^M_{e_j}(M) = \sum_{i=1}^r x^M_{e_i}(M) = M_2.
\] \hfill \Box
Remark. For the benefit of the reader, we recall the definitions of the classes CR, EKP and EIP given in [21].

\[
\text{EKP} := \{ M \in \text{mod} \mathcal{K}_r | M_2 = \text{ker}(x^M_{\alpha}) \text{ for all } \alpha \in k' \setminus \{0\} \},
\]

\[
\text{EIP} := \{ M \in \text{mod} \mathcal{K}_r | M_2 = \text{im}(x^M_\alpha) \text{ for all } \alpha \in k' \setminus \{0\} \},
\]

\[
\text{CR} := \{ M \in \text{mod} \mathcal{K}_r | \text{there exists } c \in \mathbb{N}_0 \text{ such that } c = \dim_k \text{ker}(x^M_{\alpha}) \text{ for all } \alpha \in k' \setminus \{0\} \}.
\]

Note that \( \text{CR}_r = \text{CR} = \text{CSR}_1, \text{ERP}_1 = \text{EIP}, \text{ESP}_1 = \text{EKP} \), and for \( U \in \text{Gr}_{d,r} \) with basis \((u_1, \ldots, u_d)\), we have \( \text{Rad}_U(M) = \sum_{i=1}^{d} x^M_{\alpha_i} = \text{im} x^M_{\alpha_i} \) and \( \text{Soc}_U(M) = \bigcap_{i=1}^{d} \ker(x^M_{\alpha_i}) = \ker(x^M_{\alpha_1, \ldots, \alpha_d}) \). We restrict the definition to \( d < r \) since \( \text{Gr}_{r,r} = \{ k' \} \), and therefore every module in \( \text{mod} \mathcal{K}_r \) is of constant \( r \)-socle and \( r \)-radical rank.

Lemma 2.3.2. Let \( M \in \text{mod} \mathcal{K}_r \) and \( 1 \leq d < r \).

(a) \( M \in \text{CSR}_d \) if and only if \( \delta M \in \text{CRR}_d \).

(b) \( M \in \text{ESP}_d \) if and only if \( \delta M \in \text{ERP}_d \).

Proof. Note that \( \text{Rad}_U(\delta M) = \text{im}(x^\delta M) = \text{im}(x^M)^* \equiv (\text{im}(x^M)^* \) and hence

\[
M - \dim_k \text{Soc}_U(M) = \dim_k M - \dim_k \ker x^M_T = \dim_k \text{im} x^M_T
\]

\[
= \dim_k (\text{im} x^M_T)^* = \dim_k \text{Rad}_U(\delta M).
\]

Hence \( M \in \text{CSR}_d \) if and only if \( \delta M \in \text{CRR}_d \). Moreover, \( M \in \text{ESP}_d \) if and only if \( \delta M = M_2 \), and hence \( \dim_k \text{Rad}_U(\delta M) = \dim_k M_1 = \dim_k(\delta M)_2 \).

For the proof of the following proposition, we use the same methods as in [21, Theorem 2.5].

Proposition 2.3.3. Let \( 1 \leq d < r \in \mathbb{N} \). Then

\[
\text{ESP}_d = \{ M \in \text{mod} \mathcal{K}_r | \text{Hom}(X_U, M) = 0 \text{ for all } U \in \text{Gr}_{d,r} \},
\]

\[
\text{CSR}_d = \{ M \in \text{mod} \mathcal{K}_r | \text{there exists } c \in \mathbb{N}_0 \text{ such that } \dim_k \text{Hom}(X_U, M) = c \text{ for all } U \in \text{Gr}_{d,r} \}.
\]

Proof. Let \( U \in \text{Gr}_{d,r} \) with basis \( T = (\alpha_1, \ldots, \alpha_d) \). Consider the short exact sequence

\[
0 \to (P_1)^d \to P_2 \to X_U \to 0.
\]

Application of \( \text{Hom}(\cdot, M) \) yields

\[
0 \to \text{Hom}(X_U, M) \to \text{Hom}(P_2, M) \xrightarrow{\text{Hom}(T)} \text{Hom}(P_1^d, M) \to \text{Ext}(X_U, M) \to 0.
\]

Moreover, let

\[
f : \text{Hom}(P_2, M) \to M_1, \quad g : \text{Hom}(P_1^d, M) \to M_2
\]

be the natural isomorphisms, where \( t : P_1 \to P_1^d \) denotes the embedding into the \( i \)-th coordinate. Let \( \pi_M : M^d \to M^d_1 \) be the natural projection. The equality \( g \circ T = \pi_M \circ x^M_{T^1} \circ f \) holds since both maps are applied to a homomorphism. Hence \( \dim_k \ker(\pi_M) = \dim_k \ker(T) = \dim_k \text{Hom}(X_U, M) \).

Now let \( c \in \mathbb{N}_0 \). We conclude

\[
\dim_k \text{Hom}(X_U, M) = c \iff \dim_k \ker(\pi_M) = c \iff \dim_k \ker(x^M_{T^1}) = c
\]

\[
\dim_k \ker(x^M_{T^1}) = c + \dim_k M_2
\]

\[
\dim_k \text{Soc}_U(M) = c + \dim_k M_2.
\]

This finishes the proof for \( \text{CSR}_d \). Moreover, note that \( c = 0 \) together with Lemma 2.3.1 yields

\[
M \in \text{ESP}_d \iff \text{there exists } W \leq M \text{ such that } \text{Soc}_U(M) = W \text{ for all } U \in \text{Gr}_{d,r}
\]

\[
\iff \text{Soc}_U(M) = M_2 \text{ for all } U \in \text{Gr}_{d,r} \iff \dim_k \text{Soc}_U(M) = 0 + \dim_k M_2 \text{ for all } U \in \text{Gr}_{d,r}
\]

\[
\iff \text{Hom}(X_U, M) = 0 \text{ for all } U \in \text{Gr}_{d,r}.
\]

\[ \square \]
Since $\tau \circ \delta = \delta \circ \tau^{-1}$, the next result follows from the Auslander–Reiten formula and Lemma 2.3.2.

**Proposition 2.3.4.** Let $1 \leq d < r \in \mathbb{N}$. Then

\[
\text{ERP}_d = \{ M \in \text{mod} \mathcal{K}_r \mid \text{Ext}(\delta \tau X_U, M) = 0 \text{ for all } U \in \text{Gr}_d \},
\]

\[
\text{CRR}_d = \{ M \in \text{mod} \mathcal{K}_r \mid \text{there exists } c \in \mathbb{N}_0 \text{ such that } \dim_k \text{Ext}(\delta \tau X_U, M) = c \text{ for all } U \in \text{Gr}_d \}.
\]

**Remark.** For $d = 1$, we have $U = (\alpha)_k$ with $\alpha \in k^* \setminus \{0\}$, $X_U \cong X_{\alpha}$, and [21, Proposition 3.1] yields $\delta \tau X_U \cong X_U$. However, this identity holds if and only if $d = 1$. From Proposition 2.3.3 and Lemma 2.1.4, it follows immediately that, for $1 \leq d < r - 1$ and $V \in \text{Gr}_d$, the module $X_V$ is in $\text{CSR}_{d+1} \setminus \text{CSR}_d$. If not stated otherwise, we assume from now on that $r \geq 3$.

In view of Proposition 2.1.3, Lemma 2.1.4 and the definitions of ESP$_d$ and ERP$_d$, we immediately get the following proposition.

**Proposition 2.3.5.** Let $1 \leq d < r$ and $C$ a regular component of $\mathcal{K}_r$.

(a) ESP$_1 \subseteq$ ESP$_2 \subseteq \cdots \subseteq$ ESP$_{r-1}$ and ERP$_1 \subseteq$ ERP$_2 \subseteq \cdots \subseteq$ ERP$_{r-1}$.

(b) ESP$_d$ is closed under extensions, submodules and $\tau^{-1}$. Moreover, ESP$_d$ contains all preprojective modules, and ESP$_d \cap C$ forms a non-empty cone in $C$, i.e. there is a quasi-simple module $M_d \in C$ such that

\[
\text{ESP}_d \cap C = (M_d \rightarrow) := \{ \tau^{-1}M_d[i+1] \mid i, l \in \mathbb{N}_0 \}.
\]

(c) ERP$_d$ is closed under extensions, images and $\tau$. Moreover, ERP$_d$ contains all preinjective modules, and ERP$_d \cap C$ forms a non-empty cone in $C$, i.e. there is a quasi-simple module $W_d \in C$ such that

\[
\text{ERP}_d \cap C = (\leftarrow W_d) := \{ (i+1)\tau^lW_d \mid i, l \in \mathbb{N}_0 \}.
\]

**Definition.** For $1 \leq i < r$, we set $\Delta_i := \text{ESP}_i \setminus \text{ESP}_{i-1}$ and $\forall_i := \text{ERP}_i \setminus \text{ESP}_{i-1}$, where ESP$_0 = \emptyset = \text{ERP}_0$.

The next result suggests that, for each regular component $C$ and $1 < i < r$, only a small part of vertices in $C$ corresponds to modules in $\Delta_i$. Nonetheless, we will see in Section 4 that, for $1 \leq i < r$, the categories $\Delta_i$ and $\forall_i$ are of wild type.

**Proposition 2.3.6.** Let $C$ be a regular component and $M_1, W_i (1 \leq i \leq r)$ in $C$ the uniquely determined quasi-simple modules such that ESP$_1 \cap C = (M_1 \rightarrow)$ and ERP$_1 \cap C = (\leftarrow W_i)$. Then

(a) There exists at most one number $1 < m(\mathcal{C}) < r$ such that $\Delta_{m(\mathcal{C})} \cap C$ is non-empty. If such a number exists, then $\Delta_{m(\mathcal{C})} \cap C = \{ M_{m(\mathcal{C})}[l] \mid l \geq 1 \}$.

(b) There exists at most one number $1 < w(\mathcal{C}) < r$ such that $\forall_{w(\mathcal{C})} \cap C$ is non-empty. If such a number exists, then $\forall_{w(\mathcal{C})} \cap C = \{ (l)W_{w(\mathcal{C})} \mid l \geq 1 \}$.

**Proof.** (a) By Proposition 2.3.5, there are $n_1, \ldots, n_{r-1} \in \mathbb{N}_0$ such that

\[
0 = n_1 \leq n_2 \leq \cdots \leq n_{r-1} \quad \text{and} \quad M_i = \tau^{n_i}M_1 \quad \text{for all } i \in \{ 1, \ldots, r-1 \}.
\]

We will show that either $0 = n_1 = \cdots = n_{r-1}$ or that there exists a uniquely determined $1 < i < r$ such that $n_i > n_{i-1}$.

Let $M$ be in $C$, and assume $M \notin \text{ESP}_1$. In the following, we show that $\tau M \notin \text{ESP}_{r-1}$. There exists $\alpha \in k^* \setminus \{0\}$ with $\text{Hom}(X_U, M) \neq 0$ for $U = (\alpha)_k$. Hence we find a non-zero map $f : \tau X_U \rightarrow \tau M$. Consider an exact sequence $0 \rightarrow P_{1}^{2} \rightarrow X_U \rightarrow N \rightarrow 0$. Then $\dim N = (1, 1)$, and by Lemma 2.2.3, $N$ is indecomposable. By Proposition 2.2.2, there exists $V \in \text{Gr}_{r-1}$, with $X_V \cong \delta N$. Since $\delta X_U = \tau X_U$ (see [21, Proposition 3.1]), we get a non-zero morphism $g : X_V \rightarrow \tau X_U$ and by Lemma 2.1.1 a non-zero morphism

\[
X_V \xrightarrow{g} X_U \xrightarrow{f} \tau M.
\]

Therefore, $\tau M \notin \text{ESP}_{r-1}$ by Proposition 2.3.3.

Now assume that $n_i \neq n_j$ for some $i$ and $j$. Then, in particular, $M_1 \neq M_{r-1}$. Hence $n_{r-1} \geq 1$. By definition, we have $M := \tau M_1 \notin \text{ESP}_1$, and the above considerations yield $\tau (\tau M_1) = \tau M \notin \text{ESP}_{r-1}$. Therefore,
In [17, Theorem 1.2], the author shows that \( 1 \leq n_{r-1} < 2 \) since \( \text{ESP}_{r-1} \cap C \) is closed under \( r^{-1} \). Therefore, \( n_{r-1} = 1 \) and \( M_{r-1} = \tau M_1 \). We conclude that there is a uniquely determined \( 1 < i < r \) such that \( n_i > n_{i-1} \), and in this case, \( n_i = n_{i-1} + 1 \). Now we set \( m(C) := i \).

(b) This follows by duality.

We state two more results that follow from Proposition 2.3.3 and will be needed later on. The first one is a generalization of [21, Lemma 3.5] and follows with the same arguments.

**Lemma 2.3.7.** Let \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) be an almost split sequence such that two modules of the sequence are of constant \( d \)-socle rank. Then the third module also has constant \( d \)-socle rank.

**Definition.** Let \( r \geq 2 \) and \( 1 \leq d < r \), and let \( X_{d,r} := \{X_U \mid U \in \text{Gr}_{d,r}\} \). Let \( X_{d,r}^+ \) be the right orthogonal category \( X_{d,r}^+ = \{M \in \text{mod} X_r \mid \text{Hom}(X_U, M) = 0 \text{ for all } U \in \text{Gr}_{d,r}\} \), and let \( X_{d,r}^- \) be the left orthogonal category \( X_{d,r}^- := \{M \in \text{mod} X_r \mid \text{Hom}(M, X_U) = 0 \text{ for all } U \in \text{Gr}_{d,r}\} \). Then we set \( X_{d,r} := X_{d,r}^+ \cap X_{d,r}^- \).

Note that every module in \( X_{d,r} \) is regular by Proposition 2.3.5.

**Lemma 2.3.8.** Let \( r \geq 3 \) and \( 1 \leq d < r \), and let \( M \) be quasi-simple regular in a component \( C \) such that \( M \in X_{d,r} \). Then every module in \( C \) has constant \( d \)-socle rank.

**Proof.** Let \( V \in \text{Gr}_{d,r} \). We have shown in Proposition 2.3.5 that the set

\[
\{N \mid \text{Hom}(X_V, N) = 0\} \cap C \quad \text{(resp. } \{N \mid \text{Ext}(X_V, N) = 0\} \cap C)\]

is closed under \( r^{-1} \) (resp. \( r \)). Since \( 0 = \dim_k \text{Hom}(M, X_V) = \dim_k \text{Ext}(X_V, \tau M) \), we have \( \text{Ext}(X_V, \tau^q M) = 0 \) for \( l \geq 1 \). The Euler–Ringel form yields

\[
0 = \dim_k \text{Ext}(X_V, \tau^q M) = \langle \dim X_V, \dim \tau^q M \rangle + \dim \text{Hom}(X_V, \tau^q M).
\]

Since \( \langle \dim X_V, \dim \tau^q M \rangle = \langle (1, r - d), \dim M \rangle \) is independent of \( V \), \( \tau^q M \) has constant \( d \)-socle rank. On the other hand, \( \text{Hom}(X_V, M) = 0 \) implies that \( \tau^{-q} M \) has constant \( d \)-socle rank for all \( q \geq 0 \). It follows that each quasi-simple module in \( C \) has constant \( d \)-socle rank. Now apply Lemma 2.3.7. \( \square \)

### 3 Process of simplification and applications

#### 3.1 Representation type

Denote by \( \Lambda := kQ \) the path algebra of a connected, wild quiver \( Q \). We use the notation introduced in [14]. Recall that a module \( M \) is called brick if \( \text{End}(M) = k \), and two modules \( M, N \) are called orthogonal if we have \( \text{Hom}(M, N) = 0 = \text{Hom}(N, M) \).

**Definition.** Let \( \mathcal{X} \) be a non-empty class of pairwise orthogonal bricks in \( \text{mod} \, \Lambda \). The full subcategory \( \mathcal{E}(\mathcal{X}) \) is by definition the class of all modules \( Y \) in \( \text{mod} \, \Lambda \) with an \( \mathcal{X} \)-filtration, that is, a chain

\[
0 = Y_0 \subset Y_1 \subset \cdots \subset Y_{n-1} \subset Y_n = Y
\]

with \( Y_i/Y_{i-1} \in \mathcal{X} \) for all \( 1 \leq i \leq n \).

In [17, Theorem 1.2], the author shows that \( \mathcal{E}(\mathcal{X}) \) is an exact abelian subcategory of \( \text{mod} \, \Lambda \), closed under extensions, and \( \mathcal{X} \) is the class of all simple modules in \( \mathcal{E}(\mathcal{X}) \). In particular, a module \( M \) in \( \mathcal{E}(\mathcal{X}) \) is indecomposable if and only if it is indecomposable in \( \text{mod} \, kQ \).

**Proposition 3.1.1.** Let \( r \geq 3 \), and let \( \mathcal{X} \subset \text{mod} \, \mathcal{X}_r \) be a non-empty class of pairwise orthogonal bricks with self-extensions (and therefore regular).

(a) Every module in \( \mathcal{E}(\mathcal{X}) \) is regular.

(b) Every regular component \( C \) contains at most one module of \( \mathcal{E}(\mathcal{X}) \).

(c) Every indecomposable module \( N \in \mathcal{E}(\mathcal{X}) \) is quasi-simple in \( \text{mod} \, \mathcal{X}_r \).

(d) \( \mathcal{E}(\mathcal{X}) \) is a wild subcategory of \( \text{mod} \, \mathcal{X}_r \).
Hence, Lemma 3.2.1 follows by [15, Proposition 1.4] and the fact that every regular brick in mod $\mathcal{K}$ is quasi-simple [13, Proposition 9.2]. Let $M \in \mathcal{X}$. Then we have $t := \dim_0 \text{Ext}(M, M) \geq 2$ by Corollary 1.2. Due to [11, Section 7] and [15, Remark 1.4], the category $E((M)) \subseteq E(\mathcal{X})$ is equivalent to the category of finite-dimensional modules over the power-series ring $k\langle X_1, \ldots, X_t \rangle$ in non-commuting variables $X_1, \ldots, X_t$. Since $t \geq 2$, the category $E((M)) \subseteq E(\mathcal{X})$ is wild, and also $E(\mathcal{X})$.

We will use the above result to prove the existence of numerous components such that all of its vertices correspond to modules of constant d-rank. By duality, all results also follow for constant radical rank. As a by-product, we verify the wildness of $EKP = ESP_1$ and $EIP = ERP_1$. Using the functor $\bar{\gamma} : \text{mod } \mathcal{X}_r \to \text{mod } kE_r$, we show the wildness of the corresponding full subcategories in $\text{mod}_2 kE_r$ of $E_r$-modules of Loewy length $\leq 2$.

### 3.2 Passage between $\mathcal{K}_r$ and $\mathcal{K}_s$

Let $2 \leq r < s \in \mathbb{N}$. Denote by $\inf^r_2 : \text{mod } \mathcal{K}_r \to \text{mod } \mathcal{K}_s$ the functor that assigns to a $\mathcal{K}_r$-module $M$ the module $\inf^r_2(M)$ with the same underlying vector space so that the action of $e_1, e_2, y_1, \ldots, y_r$ on $\inf^r_2(M)$ unchanged and all other arrows act trivially on $\inf^r_2(M)$. Moreover, let $i : \mathcal{K}_r \to \mathcal{K}_s$ be the natural $k$-algebra monomorphism given by $i(e_i) = e_i$ for $i \in \{1, 2\}$ and $i(y_i) = y_i$. Then each $\mathcal{K}_s$-module $N$ becomes a $\mathcal{K}_r$-module $N^*$ via pullback along $i$. Denote the corresponding functor by $\text{res}^r_2 : \text{mod } \mathcal{K}_s \to \text{mod } \mathcal{K}_r$. In the following, $r, s$ will be fixed, so we suppress the index and write just $\inf$ and $\text{res}$.

**Lemma 3.2.1.** Let $2 \leq r < s \in \mathbb{N}$. The functor $\inf : \text{mod } \mathcal{K}_r \to \text{mod } \mathcal{K}_s$ is fully faithful and exact. The essential image of $\inf$ is a subcategory of $\text{mod } \mathcal{K}_s$ closed under factors and submodules. Moreover, $\inf(M)$ is indecomposable if and only if $M$ is indecomposable in $\text{mod } \mathcal{K}_r$.

**Proof.** Clearly, $\inf$ is fully faithful and exact. Now let $M \in \text{mod } \mathcal{K}_r$, and let $U \subseteq \inf(M)$ be a submodule. Then $y_i$ ($i > r$) acts trivially on $U$, and hence the pullback $\text{res}(U) = U^*$ is a $\mathcal{K}_r$-module with $\inf(U^*) = \inf \circ \text{res}(U) = U$. Now let $V \in \text{mod } \mathcal{K}_s$, and let $f \in \text{Hom}_{\mathcal{K}_s}(\inf(M), V)$ be an epimorphism. Let $v \in V$ and $m \in M$ such that $f(m) = v$. It follows $y_i(v) = y_i(f(m)) = f(y_i(m)) = 0$ for all $i > r$. This shows that $f(V^*) = \inf \circ \text{res}(V) = V$.

Since $\inf$ is fully faithful, we have $\text{End}_{\mathcal{K}_s}(\inf(M)) \cong \text{End}_{\mathcal{K}_r}(M)$. Hence $\text{End}_{\mathcal{K}_s}(\inf(M))$ is local if and only if $\text{End}_{\mathcal{K}_r}(M)$ is local.

Statement (a) of the following lemma is stated in [7, Proposition 3.1] without proof.

**Lemma 3.2.2.** Let $2 \leq r < s$, and let $M$ be an indecomposable $\mathcal{K}_r$-module that is not simple. The following statements hold.

(a) $\inf(M)$ is regular and quasi-simple.
(b) $\inf(M) \in \text{CSR}_m$ for all $m \in \{1, \ldots, s-r\}$.

**Proof.** (a) Write $\dim M = (a, b) \in \mathbb{N}_0 \times \mathbb{N}_0$. Since $M$ is not simple, $ab \neq 0$ and $q(\dim M) = a^2 + b^2 - rab \leq 1$. It follows

$$q(\dim \inf(M)) = a^2 + b^2 - sab = a^2 + b^2 - rab - (s-r)ab \leq 1 - (s-r)ab < 1.$$ 

Hence $q(\dim \inf(M)) \leq 0$, and $\inf(M)$ is regular.

Assume that $\inf(M)$ is not quasi-simple; then $\inf(M) = U[i]$ for $U$ quasi-simple with $i \geq 2$. By Lemma 3.2.1, we have $U[-1] = \inf(A)$ and $\tau^{-1}U[i-1] = \inf(B)$ for some $A, B$ indecomposable in $\text{mod } \mathcal{K}_r$. Fix an irreducible monomorphism $f : \inf(A) \to \inf(M)$. Since $\inf$ is full, we find $g : A \to M$ with $\inf(g) = f$. The faithfulness of $\inf$ implies that $g$ is an irreducible monomorphism $g : A \to M$. By the same token, there exists an irreducible epimorphism $M \to B$. As all irreducible morphisms in $\mathcal{P}$ are injective and all irreducible morphisms in $\mathcal{J}$ are surjective, $M$ is located in a $\mathbb{Z}\mathcal{A}_\infty$ component. It follows that $\tau B = A$ in $\text{mod } \mathcal{X}_r$. Let $\dim B = (c, d)$; then the Coxeter matrices for $\mathcal{K}_r$ and $\mathcal{K}_s$ yield

$$((r^2-1)c-rd, rc-d) = \dim \tau B = \dim A = \dim \inf(A) = \dim \tau \inf(B) = ((s^2-1)c-sd, sc-d).$$

This is a contradiction since $s \neq r$. 

(b) Denote by \( \{e_1, \ldots, e_d\} \) the canonical basis of \( k^d \). Let \( 1 \leq m \leq s - r \), and set \( U := \langle e_{r+1}, \ldots, e_{r+m} \rangle_k \). Then \( \text{Soc}(M) = \bigcap_{i=1}^m \ker(\gamma_i M) = M \). Let \( j \in \{1, \ldots, r\} \) such that \( y_j \) acts non-trivially on \( M \). Let \( V \in \text{Gr}_{m,s} \) such that \( e_j \in V \). Then \( \text{Soc}(\text{im}(M)) \neq M \), and \( M \) does not have constant \( m \)-socle rank.

**Proposition 3.2.3.** Let \( 2 \leq r < s \in \mathbb{N} \) and \( 1 \leq d < r \), and let \( M \) be an indecomposable and non-simple \( \mathcal{X}_r \)-module. Then the following statements hold.

(a) If \( M \in \mathcal{X}_{d,r} \), then \( \text{im}(M) \in \mathcal{X}_{d+s-r,s} \).

(b) If \( M \in \mathcal{X}_{d,r}^\perp \), then \( \text{im}(M) \in \mathcal{X}_{d+s-r,s}^\perp \).

(c) If \( M \in \mathcal{X}_{d,r} \), then \( \text{im}(M) \) is contained in a regular component \( \mathcal{C} \) with \( \mathcal{C} \subseteq \text{CSR}_{d+s-r} \).

**Proof.** By definition, it is \( 1 \leq d + s - r < s \). Now fix \( V \in \text{Gr}_{d+s-r,s} \), and note that

\[
\dim X_V = (1, s - (d + s - r)) = (1, r - d),
\]

which is the dimension vector of every \( \mathcal{X}_r \)-module \( X_U \) for \( U \in \text{Gr}_{d,r} \).

(a) Assume that \( \text{Hom}(\text{im}(M), X_U) \neq 0 \), and let \( 0 \neq f : \text{im}(M) \to X_U \). By Lemmata 3.2.1 and 2.1.4, the \( \mathcal{X}_s \)-module \( \text{im}(M) \) is regular and every proper submodule of \( X_U \) is preprojective. Hence \( f \) is surjective onto \( X_V \). Again, Lemma 3.2.1 yields \( Z \in \text{mod} \mathcal{X}_r \) indecomposable with \( \dim Z = (1, r - d) = \dim X_V \) such that \( X_V = \text{im}(Z) \). By Proposition 2.2.2, there exists \( U \in \text{Gr}_{d,r} \) with \( Z = X_U \). Since \( \text{im} \) is fully faithful, it follows \( 0 = \text{Hom}(M, X_U) = \text{Hom}(\text{im}(M), X(U)) = \text{Hom}(\text{im}(M), X_V) \neq 0 \), a contradiction.

(b) Assume that \( \text{Hom}(X_V, \text{im}(M)) \neq 0 \), and let \( f : X_V \to \text{im}(M) \) be non-zero. Since \( \text{im}(M) \) is regular indecomposable, the module \( \text{im} \subseteq \text{im}(M) \) is not injective, and Lemma 2.2.3 yields that \( \text{im} \) is indecomposable and regular. As \( \text{im} \) is a submodule of \( \text{im}(M) \), there exists an indecomposable module \( Z \in \text{mod} \mathcal{X}_r \) with \( \text{im}(Z) \supseteq \text{im}(\text{im}(M)) \). Since \( \text{im}(M) \) is not composite and \( \text{im}(M) \) is not regular, we have \( \dim \text{im}(M) = (1, r - c) \) for \( 1 \leq r - c \leq r - d \). Hence \( Z = X_U \) for \( U \in \text{Gr}_{d,r} \), and by Proposition 2.2.2(d), there exists \( W \in \text{Gr}_{d,r} \) and an epimorphism \( \pi : X_W \to X_U \). We conclude with \( 0 \neq \text{Hom}(\text{im} f, \text{im}(M)) = \text{Hom}(\text{im}(X_W), \text{im}(M)) = \text{Hom}(X_W, M) \) and the surjectivity of \( \pi : X_W \to X_U \) that \( \text{Hom}(X_W, M) \neq 0 \), a contradiction to the assumption.

(c) By Lemma 3.2.2, the module \( \text{im}(M) \) is quasi-simple in a regular component and satisfies the conditions of Lemma 2.3.8 for \( q := d + s - r \) by (a) and (b).

**Examples.** The following two examples will be helpful later on.

1. Let \( r = 3 \). Ringel has shown that the representation \( F = (k^2, k^2, F(y_1), F(y_2), F(y_3)) \) with the linear maps \( F(y_1) = 1d_{k^2}, F(y_2)(a, b) = (b, 0) \) and \( F(y_3)(a, b) = (0, a) \) is elementary. Let \( E \) be the corresponding \( \mathcal{X}_3 \)-module. Then \( \dim E = (2, 2) \), and it is easy to see that every indecomposable submodule of \( E \) has dimension vector \( (0, 1) \) or \( (1, 2) \). In particular, \( \text{Hom}(W, E) = 0 \) for each indecomposable module with dimension vector \( \dim W = (1, 1) \). Assume now that \( f : E \to W \) is non-zero; then \( f \) is surjective since every proper submodule of \( W \) is projective. Since \( E \) is elementary, \( f \) is a preprojective module with dimension vector \( (1, 1) \), a contradiction. Hence \( E \in \mathcal{X}_{3,3} \).

2. Recall that \( \text{ESP}_1 = \text{EKP} \) and \( \text{ERP}_1 = \text{EIP} \). Given a regular component \( \mathcal{C} \), there are unique quasi-simple modules \( M_\mathcal{C} \) and \( W_\mathcal{C} \) in \( \mathcal{C} \) such that \( \text{EIP} \cap \mathcal{C} = (\to W_\mathcal{C}) \) and \( \text{EKP} \cap \mathcal{C} = (M_\mathcal{C} \to) \). The width \( \nu(\mathcal{C}) \in \mathbb{Z} \) is defined as the unique integer satisfying \( \tau^{\nu(\mathcal{C})} M_\mathcal{C} = W_\mathcal{C} \). In fact, it is shown that \( \nu(\mathcal{C}) \in \mathbb{N}_0 \), and an example of a regular component \( \mathcal{C} \) with \( \nu(\mathcal{C}) = 0 \) and \( \text{End}(M_\mathcal{C}) = k \) is given. Since \( X_U \cong \delta X_U \) for \( U \in \text{Gr}_{1,r} \) (see [21, Theorem 3.1]), we conclude for an arbitrary regular component \( \mathcal{C} \) that

\[
\nu(\mathcal{C}) = 0 \iff \tau M_\mathcal{C} = W_\mathcal{C},
\]

\[
M_\mathcal{C} \in \text{EKP} \text{ and } \tau M_\mathcal{C} \in \text{EIP},
\]

\[
\iff \text{Hom}(X_U, M_\mathcal{C}) = 0 = \text{Ext}(X_U, \tau M_\mathcal{C}) \text{ for all } U \in \text{Gr}_{1,r},
\]

\[
\iff \text{Hom}(X_U, M_\mathcal{C}) = 0 = \text{Hom}(M_\mathcal{C}, X_U) \text{ for all } U \in \text{Gr}_{1,r},
\]

\[
\iff M_\mathcal{C} \in \mathcal{X}_{1,1},
\]

**Lemma 3.2.4.** Let \( s \geq 3 \) and \( 2 \leq \delta < s \). Then there exists a regular module \( E_\delta \) with the following properties.

(a) \( E_\delta \) is a (quasi-simple) brick in mod \( \mathcal{X}_s \).

(b) \( E_\delta \in \mathcal{X}_{d,s} \).

(c) There exist \( V, W \in \text{Gr}_{1,s} \) with \( \text{Hom}(X_V, E_\delta) = 0 \neq \text{Hom}(X_W, E_\delta) \).
Proof. We start by considering $s = 3$ and $d = 2$. Pick the elementary module $E_d := E$ from the preceding example. $E$ is a brick, and $E \in \overline{X}_{d,s}$. Set $a := (1, 0, 0), \beta := (0, 1, 0) \in k^3$ and $V := \langle a \rangle, W := \langle \beta \rangle$. By the definition of $E$, we have

$$\dim_k \ker x_\alpha^E = 2 \neq 3 = \dim_k \ker x_\beta^E,$$

and therefore

$$\dim_k \Hom(X_V, E_d) = 0 \neq 1 = \dim_k \Hom(X_W, E_d).$$

Now let $s > 3$. If $d = s - 1$, consider $E_d := \inf_2^d(E)$. In view of Proposition 3.2.3, we have $E_d \in \overline{X}_{2s-3,s} = \overline{X}_{d,s}$.

Moreover, $\inf(E)$ is a brick in mod $\mathcal{K}_s$ and for the canonical basis vectors $e_1, e_2 \in k^3$ and $V = \langle e_1 \rangle, W := \langle e_2 \rangle$ we get as before

$$\dim_k \Hom(X_V, \inf(E)) = 0 \neq 1 = \Hom(X_W, \inf(E)).$$

Now let $1 < d < s - 1$. Set $r := 1 + s - d \geq 3$, consider a regular component for $\mathcal{K}_r$ with $\mathcal{W}(\mathcal{E}) = 0$ such that $M_\mathcal{E}$ is a brick and set $M := M_\mathcal{E}$. Then $M \in \overline{X}_{1,r}$, and Proposition 3.2.3 yields $E_d := \inf(M) \in \overline{Y}_{1s-(1+s-d),s} \subseteq \overline{X}_{d,s}$.

Since $M$ is a brick, $\inf(M)$ is a brick in mod $\mathcal{K}_s$. Recall that $\Hom(X_V, M) = 0$ for all $U \in \text{Gr}_{d,r}$ implies that, viewing $M$ as a representation, the linear map $M(y_1) : M_1 \rightarrow M_2$ corresponding to $y_1$ is injective. Since the map is not affected by $\inf$, $\inf(M)(y_1) : M_1 \rightarrow M_2$ is also injective. Therefore, we conclude for the first basis vector $e_1 \in k^3$ and $V = \langle e_1 \rangle$ that $0 = \Hom(X_V, \inf(M))$. By Lemma 3.2.2, we find $W \in \text{Gr}_{1,s}$ with $0 \neq \Hom(X_W, \inf(M))$. □

3.3 Numerous components lying in CSR$_d$

In this section, we use the simplification method to construct a family of regular components such that every vertex in such a regular component corresponds to a module in CSR$_d$. By the next result, it follows that $\mathcal{X} \subseteq \overline{X}_{d,r}$ implies $\mathcal{E}(\mathcal{X}) \subseteq \overline{X}_{d,r}$.

Lemma 3.3.1 ([13, Lemma 1.9]). Let $X, Y$ be modules with $\Hom(X, Y)$ non-zero. If $X$ and $Y$ have filtrations

$$X = X_0 \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} = 0, \quad Y = Y_0 \supset Y_1 \supset \cdots \supset Y_s \supset Y_{s+1} = 0,$$

then there are $i, j$ with $\Hom(X_i/X_{i+1}, Y_j/Y_{j+1}) \neq 0$.

For a regular module $M \in \mathcal{K}_r$, denote by $\mathcal{E}_M$ the regular component that contains $M$.

Proposition 3.3.2. Let $1 \leq d < r$, and let $\mathcal{X}$ be a family of pairwise orthogonal bricks in $\overline{X}_{d,r}$. Then

$$\varphi : \text{ind} \mathcal{E}(\mathcal{X}) \rightarrow \mathcal{X}, \quad M \mapsto \mathcal{E}_M$$

is an injective map such that, for each component $\mathcal{E}$ in im $\varphi$, we have $\mathcal{E} \subseteq \text{CSR}_d$. Here $\text{ind} \mathcal{E}(\mathcal{X})$ denotes the category of a chosen set of representatives of non-isomorphic indecomposable objects of mod $\mathcal{X}_r$ in $\mathcal{E}$.

Proof. Since each module in $\overline{X}_{d,r}$ is regular, Proposition 3.1.1 implies that every module $N \in \text{ind} \mathcal{E}(\mathcal{X})$ is contained in a regular component $\mathcal{E}_N$ and is quasi-simple. By Lemma 3.3.1, the module $N$ satisfies $\Hom(X_U, N) = 0 = \Hom(N, X_U)$ for all $U \in \text{Gr}_{d,r}$. But now Lemma 2.3.8 implies that every module in $\mathcal{E}_M$ has constant $d$-socle rank. The injectivity of $\varphi$ follows immediately from Proposition 3.1.1. □

Corollary 3.3.3. There exists an infinite set $\Omega$ of regular components such that, for all $\mathcal{E} \in \Omega$,

(a) $\mathcal{W}(\mathcal{E}) = 0$, in particular, every module in $\mathcal{E}$ has constant rank,

(b) $\mathcal{E}$ does not contain any bricks.

Proof. Let $\mathcal{E}$ be a regular component that contains a brick and $\mathcal{W}(\mathcal{E}) = 0$ (such a component exists by the example above). Let $M := M_\mathcal{E}$; then $M \in \overline{X}_{1,r}$. Apply Proposition 3.3.2 with $\mathcal{X} = \{M\}$, and set $\Omega := \text{im} \varphi \setminus \mathcal{E}_M$. Let $N \in \mathcal{E}(\mathcal{X}) \setminus \{M\}$ be indecomposable. $N$ is quasi-simple in $\mathcal{E}_N$ and has a $\{M\}$-filtration $0 = N_0 \subset \cdots \subset N_l = N$ with $l \geq 2$ and $N_1 = M = N_l/N_{l-1}$. Hence $N \rightarrow N_i/N_{i-1} \rightarrow N_1 \rightarrow N$ is a non-zero homomorphism that is not injective. Therefore, $N$ is not a brick. This finishes the proof since every regular brick in mod $\mathcal{K}_r$ is quasi-simple [13, Proposition 9.2] and $\End(r^tN) \cong \End(N) \neq k$ for all $l \in \mathbb{Z}$. □
Now we apply our results on the simplification method to modules $E_d$ constructed in Lemma 3.2.4.

**Definition** ([6, Proposition 3.6]). Denote with $\text{GL}_r$ the group of invertible $r \times r$-matrices which acts on $\bigoplus_{i=1}^{r} k y_i$ via $g \cdot y_i = \sum_{j=1}^{r} g_{ij} y_j$ for $1 \leq j \leq r$, $g \in \text{GL}_r$. For $g \in \text{GL}_r$, let $\varphi_g : \mathcal{X}_r \to \mathcal{X}_r$ be the algebra homomorphism with $\varphi_g(e_1) = e_1$, $\varphi_g(e_2) = e_2$ and $\varphi_g(y_i) = g y_i$, $1 \leq i \leq r$. For a $\mathcal{X}_r$-module $M$, denote the pullback of $M$ along $\varphi_g$ by $M^{[g]}$. The module $M$ is called $\text{GL}_r$-stable if $M^{[g]} \cong M$ for all $g \in \text{GL}_r$.

**Theorem 3.3.4.** Let $2 \leq d < r$; then there exists a wild full subcategory $\mathcal{E} \subseteq \text{mod} \mathcal{X}_r$ and an injection

$$\varphi_d : \text{ind} \mathcal{E} \to \mathcal{R}, \quad M \mapsto \mathcal{C}_M,$$

such that, for each component $\mathcal{C}$ in $\text{im} \varphi_d$, we have $\mathcal{C} \subseteq \text{CSR}_d$ and no module in $\mathcal{C}$ is $\text{GL}_r$-stable.

**Proof.** Fix $2 \leq d < r$, and let $E_d$ be as in Lemma 3.2.4 with $V, W \in \text{Gr}_1, r$ and $\text{Hom}(X_V, E_d) = 0 \neq \text{Hom}(X_U, E_d)$. Set $\mathcal{X} := \{E_d\}$, and let $M \in \mathcal{E}(\mathcal{X})$. By Proposition 3.3.2, we get an injective map

$$\varphi_d : \text{ind} \mathcal{E}(\mathcal{X}) \to \mathcal{R}, \quad M \mapsto \mathcal{C}_M$$

such that each component $\mathcal{C}$ in $\text{im} \varphi_d$ satisfies $\mathcal{C} \subseteq \text{CSR}_d$.

Moreover, $\mathcal{E}(\mathcal{X})$ is a wild full subcategory of $\text{mod} \mathcal{X}_r$ by Proposition 3.1.1. Let $M \in \mathcal{E}(\mathcal{X})$ be indecomposable. Then $M$ has a filtration $0 = Y_0 \subseteq Y_1 \cdots \subseteq Y_m$ with $Y_i/Y_{i-1} = E_d$ for all $1 \leq i \leq m$. By Lemma 3.3.1, we have $0 = \text{Hom}(X_U, M)$, and since $E_d = Y_1 \subseteq M$, we conclude $0 \neq \text{Hom}(X_U, M)$. This proves that $M$ does not have constant 1-socle rank. Therefore, $\mathcal{C}_M$ contains a module that is not of constant 1-socle rank. By [6, Proposition 3.6], the module $M$ is not $\text{GL}_r$-stable. Assume that $\mathcal{C}_M$ contains an $\text{GL}_r$-stable module $N$. Since $g \in G$ acts as an auto-equivalence on $\text{mod} \mathcal{X}_r$ (see also [9, Lemma 2.2]), we conclude that $g$ sends the Auslander–Reiten sequence $0 \to X \to E \to N \to 0$ to the Auslander–Reiten sequence $0 \to X^g \to E^g \to N \to 0$. Hence $X^g = X$ and $E^g = E$ for all $g \in \text{GL}_r$, and therefore $X$ and $E$ are $\text{GL}_r$-stable. If $E$ is not indecomposable, we write $E = E_1 \oplus E_2$ with $E_1, E_2$ indecomposable such that the quasi-lengths $q_l(E_1)$, $q_l(E_2)$ satisfy $q_l(E_1) = q_l(E_2) - 2$. We get $\text{dim}_k E_2 > \text{dim}_k E_1$ and therefore $(E_1)^g \equiv E_2$ and $(E_1)^g \equiv E_1$. Hence every direct summand in the Auslander–Reiten sequence is $\text{GL}_r$-stable. Now one can easily conclude that every module in $\mathcal{C}_M$ is $\text{GL}_r$-stable, a contradiction since $M$ is not $\text{GL}_r$-stable.

\[ \square \]

### 3.4 Components lying almost completely in CSR\(_d\)

The following definition and two lemmata are a generalization of [22, Definition 4.7, Proposition 4.13] and [21, Proposition 3.7]. We sketch the proof of Lemma 3.4.1.

**Definition.** Let $M$ be an indecomposable $\mathcal{X}_r$-module, $1 \leq d < r$ and $U \in \text{Gr}_d, r$. $M$ is called $U$-trivial if

$$\text{dim}_k \text{Hom}(X_U, M) \neq \text{dim}_k M_1.$$ 

Note that the sequence $0 \to P_1^{-d} \to P_2 \to X_U \to 0$ and left-exactness of $\text{Hom}(\cdot, M)$ imply that

$$\text{dim}_k \text{Hom}(X_U, M) \leq \text{dim}_k M_1.$$ 

**Lemma 3.4.1.** Let $M$ be a regular $U$-trivial module. If $M$ is not elementary, then

$$\text{Ext}(X_U, \tau M) = 0 = \text{Hom}(X_U, \tau^{-1} M)$$

for all $V \in \text{Gr}_d, r$.

**Proof.** Assume that $\text{Ext}(X_U, \tau M) \neq 0$; then we find an epimorphism $f : M \to X_U$ and an exact sequence $0 \to \text{ker} f \to M \to X_U \to 0$. Note that $\text{dim}_k \text{Hom}(X_U, \text{ker} f) \leq \text{dim}_k (\text{ker} f)_1 < \text{dim}_k M_1 = \text{Hom}(X_U, M)$. We apply $\text{Hom}(X_U, -)$ and conclude that $f : \text{Hom}(X_U, M) \to \text{Hom}(X_U, X_U), g \mapsto f \circ g$ is non-zero. In particular, we have $0 \neq \text{Hom}(X_U, X_U)$ and therefore $U = V$. Let $h \in \text{Hom}(X_U, M)$ such that $f \circ h \neq 0$. Since $X_U$ is a brick, we conclude that $f$ is an isomorphism and $M \cong X_U$ is elementary.

Assume that $\tau^{-1} M \notin \text{ESP}_d \subseteq \text{ESP}_1$. Consider $\langle \alpha \rangle_k = W \in \text{Gr}_1, r$ together with an epimorphism $p : X_W \to X_U$ (see Proposition 2.2.2). We conclude with $\text{dim}_k M_1 \geq \text{dim}_k \text{Hom}(X_W, M) \geq \text{dim}_k \text{Hom}(X_U, M) = \text{dim}_k M_1$ that
Let $M$ be regular quasi-simple in a regular component $\mathcal{C}$ such that
\[
\text{Ext}(X_U, \tau M) = 0 = \text{Hom}(X_U, \tau^{-1} M) \quad \text{for all } U \in \text{Gr}_{d, r}.
\]
If $M$ does not have constant $d$-socle rank, then a module $X$ in $\mathcal{C}$ has constant $d$-socle rank if and only if $X$ is in $(\tau M) \cup (\tau^{-1} M)$.

**Corollary 3.4.3.** Let $3 \leq r < s$ and $1 \leq d < r$, and let $b := d + s - r$ and $1 \leq l \leq s - r$. Let $M$ be an indecomposable $\mathcal{K}_r$-module in $\mathcal{X}_{d, r}$ that is not elementary. Denote by $\mathcal{C}$ the regular component that contains $\text{inf}(M)$.

(a) Every module in $\mathcal{C}$ has constant $b$-socle rank.

(b) $N \in \mathcal{C}$ has constant $l$-socle rank if and only if $N \in (\tau \text{inf}(M)) \cup (\tau^{-1} \text{inf}(M) \to)$.

**Proof.** (a) is an immediate consequence of Proposition 3.2.3.

(b) Consider the indecomposable projective module $P_2 = \mathcal{K}_r e_1$ in mod $\mathcal{K}_r$. We get
\[
\text{Hom}(\text{inf}(P_2), \text{inf}(M)) \cong \text{Hom}(P_2, M) = M_1 = \text{inf}(M)_1.
\]
Since
\[
\dim \text{inf}(P_2) = (1, r) = (s - (s - r)),
\]
we find $W \in \text{Gr}_{s-r, s}$ with $\text{inf}(P_2) = X_W$. Now let $1 \leq l \leq s - r$. By Proposition 2.2.2, there is $U \in \text{Gr}_{l, s}$ and an epimorphism $\pi : X_U \to X_W$. Let $\{f_1, \ldots, f_q\}$ be a basis of $\text{Hom}(\text{inf}(P_2), \text{inf}(M))$. Since $\pi$ is surjective, the set $\{f_1 \pi, \ldots, f_q \pi\} \subseteq \text{Hom}(X_U, \text{inf}(M))$ is linearly independent. Hence
\[
q \leq \dim_k \text{Hom}(X_U, \text{inf}(M)) \leq \dim_k \text{inf}(M)_1 = q
\]
holds, and $\text{inf}(M)$ is $U$-trivial.

Since $M$ is not elementary, $\text{inf}(M)$ is not elementary, and therefore Lemma 3.4.1 yields that
\[
\text{Ext}(X_W, \tau \text{inf}(M)) = 0 = \text{Hom}(X_W, \tau^{-1} \text{inf}(M)) \quad \text{for all } W \in \text{Gr}_{l, s}.
\]
By Lemma 3.2.2, the module $\text{inf}(M)$ does not have the constant $l$-socle rank for $1 \leq l \leq s - r$. Note that $M$ is regular, and therefore $\text{inf}(M)$ is a quasi-simple module. Now apply Lemma 3.4.2. □

**Example.** Let $r \geq 3$, and let $\mathcal{C}$ be a regular component with $\forall(\mathcal{C}) = 0$ such that $M_0$ is not a brick (see Corollary 3.3.3) and in particular not elementary. Then $M_0 \in \mathcal{X}_{1, r}$, and we can apply Corollary 3.4.3. Figure 4 shows the regular component $\mathcal{D}$ of $\mathcal{K}_s$ containing $\text{inf}(M_0)$. Every module in $\mathcal{D}$ has constant $b := 1 + s - r$ socle rank. But for $1 \leq q \leq s - r$, a module in this component has constant $q$-socle rank if and only if it lies in the shaded region.

![Figure 4: Regular component containing $\text{inf}(M_0)$](image-url)
4 Wild representation type

4.1 Wildness of strata

As another application of the simplification method and the inflation functor \( \text{inf}_s : \text{mod} \mathcal{K}_r \rightarrow \text{mod} \mathcal{K}_s \), we get the following result.

**Theorem 4.1.1.** Let \( s \geq 3 \) and \( 1 \leq d \leq s - 1 \). Then \( \Delta_d = \text{ESP}_d \setminus \text{ESP}_{d-1} \subseteq \text{mod} \mathcal{K}_s \) is a wild subcategory, where \( \text{ESP}_0 := \emptyset \).

**Proof.** For \( d = 1 \), consider a regular component \( \mathcal{C} \) for \( \mathcal{K}_s \) that contains a brick \( F \). By Proposition 2.3.5, we find a module \( E \) in the \( r \)-orbit of \( F \) that is in \( \text{ESP}_1 \) and set \( X := \{E\} \). Then \( E \) is brick since \( \text{Hom}(E,E) \cong \text{Hom}(F,F) = k \) and \( \dim_k \text{Ext}(E,E) \geq 2 \) by Corollary 1.2. Therefore, \( \mathcal{E}(X) \) is wild category (see Proposition 3.1.1). As \( \text{ESP}_1 \) is closed under extensions, it follows \( \mathcal{E}(X) \subseteq \text{ESP}_1 \). Note that this case does only require the application of Proposition 3.1.1.

Now let \( d > 1 \) and \( r := s - d + 1 \geq 2 \). Consider the projective indecomposable \( \mathcal{K}_r \)-module \( P := P_2 \) with \( \dim P = (1, r) \). By Lemma 3.2.2, \( \text{inf}(P) \) is a regular quasi-simple module in \( \text{mod} \mathcal{K}_s \) with

\[
\dim_k \text{Ext}(\text{inf}(P), \text{inf}(P)) \geq 2.
\]

Since \( P \) is in \( \text{ESP}_1 \), we have \( 0 = \text{Hom}(X_U, P) \) for all \( U \in \text{Gr}_{1,r} \). Hence Proposition 3.2.3 implies

\[
0 = \text{Hom}(X_U, \text{inf}(P)) \quad \text{for all } U \in \text{Gr}_{1+s-r} \text{, and } 0 \neq \text{End}(\text{inf}(P)) = \text{Hom}(X_U, \text{inf}(P)).
\]

That means \( \text{inf}(P) \notin \text{ESP}_{d-1} \). Since \( \text{ESP}_{d-1} \) is closed under submodules, we have \( \mathcal{E}(X) \cap \text{ESP}_{d-1} = \emptyset \). Hence \( \mathcal{E}(X) \subseteq \text{ESP}_d \setminus \text{ESP}_{d-1} \).

**Remarks.** Let us collect the following observations.

(i) Note that all indecomposable modules in the wild category \( \mathcal{E}(X) \) are quasi-simple in \( \text{mod} \mathcal{K}_s \) and \( \mathcal{E}(X) \subseteq \text{ESP}_d \setminus \text{ESP}_{d-1} \).

(ii) For \( 1 \leq d \leq r \), we define \( \text{EK}_d := \{M \in \text{mod} \mathcal{K}_r \mid \text{Hom}(\delta T X_U, M) = 0 \text{ for all } U \in \text{Gr}_{d,r}\} \). One can show that \( M \in \text{EK}_d \) if and only if \( y_T^M M_1^{d} \rightarrow M \) is injective for all linearly independent tuples \( T \) in \( (k^d)^d \). From the definitions, we get a chain of proper inclusions \( \text{ESP}_{r-1} \supset \text{ESP}_{r-2} \supset \cdots \supset \text{ESP}_1 = \text{EK}_1 \supset \text{EK}_2 \supset \cdots \supset \text{EK}_{r-1} \). By adapting the preceding proof, it follows that \( \text{EK}_{r-1} \) is wild. Moreover, it can be shown that, for each regular component \( \mathcal{C} \), the set \( \text{EK}_1 \setminus \text{EK}_{r-1} \cap \mathcal{C} \) is empty or forms a ray.

We will use the following result later on to prove the wildness of the subcategory in \( \text{mod} k E_2 \) consisting of modules of Loewy length 3 and the equal kernels property. We denote by \( B(3, 2) \) the Beilinson algebra with 3 vertices and 2 arrows.

**Proposition 4.1.2.** Let \( \text{EK}(3, 2) \) be the full subcategory of modules in \( \text{mod} B(3, 2) \) with the equal kernels property (see [21, Definition 2.1, Theorem 2.5]). The category \( \text{EK}(3, 2) \) is of wild representation type.

**Proof.** Consider the path algebra \( A \) of the extended Kronecker quiver \( Q = 1 \rightarrow 2 \rightarrow 3 \). Since the underlying graph of \( Q \) is not a Dynkin or Euclidean diagram, the algebra \( A \) is of wild representation type by [8]. It is known that there exists a preprojective tilting module \( T \) in \( \text{mod} A \) with \( \text{End}(T) = B(3, 2) \); see for example [20] or [23, Section 4]. We sketch the construction. The start of the preprojective component of \( A \) is illustrated in Figure 5, and the direct summands of \( T \) are marked with a dot.

One can check that \( T = P(1) \oplus \tau^{-2} P(1) \oplus \tau^{-1} P(3) \) is a tilting module. Since preprojective components are standard [19, Proposition 2.4.11], one can show that \( \text{End}(T) \) is given by the quiver in Figure 6, bound by the relation \( \alpha_2 \alpha_1 + \beta_2 \beta_1 \). Moreover, it follows from the description as a quiver with relations that \( \text{End}(T) = B(3, 2) \).
Since $A$ is hereditary, it follows that the algebra $B(3, 2)$ is a concealed algebra [1, Definition 4.6]. By [2, Theorem XVIII.5.1], the functor $\text{Hom}(T, -): \mod A \to \mod B(3, 2)$ induces an equivalence $G$ between the regular categories add $\mathcal{R}(A)$ and add $\mathcal{R}(B(3, 2))$, and we have an isomorphism between the two Grothendieck groups $f: K_0(A) \to K_0(B(3, 2))$ with $\dim G(M) = \dim \text{Hom}(T, M) = f(\dim M)$ for all $M \in \mod A$. Now we make use of a homological characterization of the class $\text{EKP}(3, 2)$ given in [21, Theorem 2.5]: for each $\alpha \in k^2 \setminus \{0\}$, there exist certain indecomposable $B(3, 2)$-modules $X^0_{\alpha}, X^1_{\alpha}$ such that

$$\text{EKP}(3, 2) = \{ M \in \mod B(3, 2) \mid \text{Hom}(X^0_{\alpha}, X^1_{\alpha}, M) = 0 \text{ for all } \alpha \in k^2 \setminus \{0\}\}. $$

The modules $X^0_{\alpha}, X^1_{\alpha}$ arise as cokernels of embeddings similar to the embeddings studied in Section 2.2. We do not need the exact definition of $X^i_{\alpha}$. Let us show that each $X^1_{\alpha}$ is regular. Clearly, $0 \neq \text{Hom}_B(X^1_{\alpha}, Z)$ for $Z \in \{X^0_{\alpha}, X^1_{\alpha}\}$, so $Z$ is not in $\text{EKP}(3, 2)$. Moreover, the equality (see [22, Proposition 3.14])

$$\tau_{B(3, 2)} X^1_{\alpha} = DX^3_{\alpha} \cong DX^0_{\alpha}$$

holds ($D$ denotes a certain duality on $\mod B(3, 2)$). Since $X^0_{\alpha}$ is not in $\text{EKP}(3, 2)$, we conclude that $\tau_{B(3, 2)} X^1_{\alpha}$ is not in $\text{EKP}(3, 2)$. Since $\text{EIP}(3, 2)$ is closed under $\tau_{B(3, 2)}$, we conclude that $X^1_{\alpha}$ is not in $\text{EIP}(3, 2)$. The assumption $X^0_{\alpha}$ is in $\text{EIP}(3, 2)$ yields that $\tau_{B(3, 2)} X^1_{\alpha}$ is in $\text{EIP}(3, 2)$. Since $\text{EKP}(3, 2)$ is closed under $\tau_{B(3, 2)}$, we get that $X^1_{\alpha}$ is in $\text{EKP}(3, 2)$, a contradiction. Therefore, $X^0_{\alpha}, X^1_{\alpha}$ are not in $\text{EIP}(3, 2) \cup \text{EKP}(3, 2)$ and by [21, Corollary 2.7] regular, so $X^1_{\alpha}$ is a regular module as well.

Moreover, $\dim X^1_{\alpha}$ is independent of $\alpha$. Hence we find for each $\beta \in k^2 \setminus \{0\}$ a regular indecomposable module $U_{\beta}$ in $\mod A$ with $G(U_{\beta}) = X^1_{\beta}$ and $\dim U_{\beta} = \dim U_{\alpha}$ for all $\alpha \in k^2 \setminus \{0\}$. Now let $M$ be in $\mod A$ a regular brick with $\dim M \geq 2$ (see [15, Proposition 5.1]). By the dual version of [13, Lemma 4.6], we find $l \in \mathbb{N}$ with $\text{Hom}(U_{\alpha}, M^l) = 0$ for all $\alpha \in k^2 \setminus \{0\}$. Set $N := \tau^{-l}M$ and $\mathcal{E} := \{N\}$. $N$ is a regular brick with $\dim N \geq 2$, and therefore $\mathcal{E}$ is a wild category in add $\mathcal{R}(A)$ (see [15, Proposition 1.4]). By Lemma 3.3.1, we have $\text{Hom}(U_{\alpha}, L) = 0$ for all $\alpha \in k^2 \setminus \{0\}$ and $l \in \mathcal{E}(\mathcal{X})$ and all $\alpha \in k^2 \setminus \{0\}$. Hence $0 = \text{Hom}(U_{\alpha}, L) = \text{Hom}(G(U_{\alpha}), G(L)) = \text{Hom}(X^1_{\alpha}, L)$ for all $\alpha \in k^2 \setminus \{0\}$. This shows that the essential image of $\mathcal{E}$ under $G$ is a wild subcategory contained in $\text{EKP}(3, 2)$.

\[\square\]

4.2 The module category of $E_r$

Throughout this section, we assume that $\text{char}(k) = p > 0$ and $r \geq 2$. Moreover, let $E_r$ be a $p$-elementary abelian group of rank $r$ with generating set $\{g_1, \ldots, g_r\}$. For $x_i := g_i - 1$, we get an isomorphism

$$kE_r \cong k[X_1, \ldots, X_r] / (X^p_1, \ldots, X^p_r)$$

of $k$-algebras by sending $X_i$ to $x_i$ for all $i$. We recall the definition of the functor $\mathfrak{S}: \mod \mathcal{X} \to \mod kE_r$ introduced in [21]. Given a module $M$, $\mathfrak{S}(M)$ is by definition the vector space $M$ and

$$x_i \cdot m := y_i \cdot m = y_i \cdot m_1 + y_i \cdot m_2 = y_i \cdot m_1,$$
where $m_i = e_i \cdot m$. Moreover, $\overline{\alpha}$ is the identity map on morphisms, that is, $\overline{\alpha}(f) : \overline{\alpha}(M) \to \overline{\alpha}(N)$, $\overline{\alpha}(f)(m) = f(m)$ for all $f : M \to N$.

**Definition** ([6, Definition 2.1]). Let $V := \langle x_1, \ldots, x_r \rangle_k \leq \ker(kE_r)$. For $U$ in $Gr_{d,V}$ with basis $u_1, \ldots, u_d$ and a $kE_r$-module $M$, we set

$$\text{Rad}_U(M) := \sum_{u \in U} u \cdot M = \sum_{i=1}^d u_i \cdot M,$$

$$\text{Soc}_U(M) := \{m \in M \mid u \cdot m = 0 \text{ for all } u \in U\} = \bigcap_{i=1}^d \{m \in M \mid u_i \cdot m = 0\}.$$

**Definition** ([6, Definition 3.1]). Let $M \in \text{mod } kE_r$ and $1 \leq d < r$.

(a) $M$ has the **constant d-Rad rank** ($d$-Soc rank), respectively if the dimension of $\text{Rad}_U(M)$ ($\text{Soc}_U(M)$, respectively) is independent of the choice of $U \in Gr_{d,V}$.

(b) $M$ has the **equal d-Rad property** ($d$-Soc property, respectively) if $\text{Rad}_U(M)$ ($\text{Soc}_U(M)$, respectively) is independent of the choice of $U \in Gr_{d,V}$.

**Proposition 4.2.1.** Let $M$ be a non-simple indecomposable $\mathcal{K}_r$-module, and let $1 \leq d < r$.

(a) $M$ is in CSR$_d$ if and only if $\overline{\alpha}(M)$ has constant $d$-Soc rank.

(b) $M$ is in ESR$_d$ if and only if $\overline{\alpha}(M)$ has the equal $d$-Soc property.

**Proof.** We fix $V := \langle x_1, \ldots, x_r \rangle_k \leq \ker(kE_r)$. In the following, we denote for $u \in \ker(kE_r)$, with the induced linear map on $M$. Let $U \in Gr_{d,V}$ with basis $\{u_1, \ldots, u_d\}$, write $u \cdot m = \sum_{i=1}^r a_i x_i$, for all $1 \leq j \leq d$, and set $a_i = (a_i^1, \ldots, a_i^r)$. Then $T := (a_1, \ldots, a_d)$ is linearly independent, and

$$\ker(l(u_j)) = \ker\left(\sum_{i=1}^r a_i^j l(x_i)\right) = \ker\left(\sum_{i=1}^r a_i^j x_i\right) = \ker(x_i^M_{a_i^j}).$$

It follows

$$\text{Soc}_U(\overline{\alpha}(M)) = \bigcap_{i=1}^d \ker(l(u_i)) = \bigcap_{i=1}^d \ker(x_i^M_{a_i^j}) = \text{Soc}_T(M).$$

Hence $M \in CSR_d$ implies that $\overline{\alpha}(M)$ has constant $d$-Soc rank.

Now assume that $T := (a_1, \ldots, a_d)$ is linearly independent, and set $u_i := \sum_{j=1}^r a_i^j x_i$. Then

$$U := \langle u_1, \ldots, u_d \rangle \in Gr_{d,V} \text{ and } \text{Soc}_T(M) = \text{Soc}_U(\overline{\alpha}(M)).$$

We have shown that $M$ is in CSR$_d$ if and only if $\overline{\alpha}(M)$ has constant $d$-Soc rank. The other equivalence follows in the same fashion.

For $1 < d < r$, we denote by $\text{ESP}_{2,d}(E_r)$ the category of modules in mod $kE_r$ of Loewy length $\leq 2$ with the equal $d$-Soc property. As an application of Section 4.1, we get a generalization of [3, Theorem 5.6.12] and [4, Theorem 1].

**Corollary 4.2.2.** Let $\text{char}(k) > 0, r \geq 3$ and $1 \leq d \leq r - 1$. Then $\text{ESP}_{2,d}(E_r) \setminus \text{ESP}_{2,d-1}(E_r)$ has wild representation type.

**Proof.** Let $1 \leq c < r$. By [21, Proposition 2.3] and Proposition 4.2.1, a restriction of $\overline{\alpha}$ to ESP$_c$ induces a faithful exact functor

$$\overline{\alpha}_c : \text{ESP}_c \to \text{mod } kE_r$$

that reflects isomorphisms and with essential image $\text{ESP}_{2,c}(E_r)$. Let $\mathcal{E} \subseteq \text{ESP}_d \setminus \text{ESP}_{d-1}$ be a wild subcategory. Since $\overline{\alpha}_{d-1}$ and $\overline{\alpha}_d$ reflect isomorphisms, we have $\overline{\alpha}(E) \in \text{ESP}_{2,d}(E_r) \setminus \text{ESP}_{2,d-1}(E_r)$ for all $E \in \mathcal{E}$. Hence the essential image of $\overline{\alpha}$ under $\overline{\alpha}$ is a wild category.

**Corollary 4.2.3.** Assume that $\text{char}(k) = p > 2$; then the full subcategory of modules with the equal kernels property in mod $kE_r$ and Loewy length $\leq 3$ is of wild representation type.

**Proof.** By [21, Proposition 2.3] $(n = 3 \leq p, r = 2)$, the functor $\overline{\alpha}_{\text{EKP}(3,2)} : \text{mod } B(3,2) \to \text{mod } kE_2$ is a representation embedding with essential image in EKP($E_2$).
Acknowledgment: The results of this article are part of my doctoral thesis, which I have written at the University of Kiel. I would like to thank my advisor Rolf Farnsteiner for fruitful discussions, his continuous support and helpful comments on an earlier version of this paper. I also would like to thank the whole research team for the very pleasant working atmosphere and the encouragement throughout my studies. Furthermore, I thank Otto Kerner for answering my questions on hereditary algebras and giving helpful comments, and Claus Michael Ringel for sharing his insights on elementary modules for the Kronecker algebra. I would like to thank the anonymous referee for the detailed comments.

Funding: Partly supported by the DFG priority program SPP 1388 “Darstellungstheorie”.

References

[1] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras. I*, London Math. Soc. Stud. Texts 72, Cambridge University Press, Cambridge, 2006.
[2] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras. III*, London Math. Soc. Stud. Texts 72, Cambridge University Press, Cambridge, 2007.
[3] D. J. Benson, *Representations of Elementary Abelian p-groups and Vector Bundles*, Cambridge Tracts in Math. 208, Cambridge University, Cambridge, 2017.
[4] V. M. Bondarenko and I. V. Lytvynchuk, The representation type of elementary abelian p-groups with respect to the modules of constant Jordan type, *Algebra Discrete Math.* 14 (2012), no. 1, 29–36.
[5] J. F. Carlson, E. M. Friedlander and J. Pevtsova, Modules of constant Jordan type, *J. Reine Angew. Math.* 614 (2008), 191–234.
[6] J. F. Carlson, E. M. Friedlander and J. Pevtsova, Representations of elementary abelian p-groups and bundles on Grassmannians, *Adv. Math.* 229 (2012), no. 5, 2985–3051.
[7] B. Chen, Dimension vectors in regular components over wild Kronecker quivers, *Bull. Sci. Math.* 137 (2013), 730–745.
[8] P. Donovan and M. R. Freislich, *The Representation Theory of Finite Graphs and Associated Algebras*, Carleton Math. Lecture Notes 5, Carleton University, Ottawa, 1973.
[9] R. Farnsteiner, Categories of modules given by varieties of p-nilpotent operators, preprint (2011), https://arxiv.org/abs/1110.2706.
[10] R. Farnsteiner, Nilpotent operators, categories of modules, and auslander-reiten theory, Lectures notes (2012), http://www.math.uni-kiel.de/algebra/de/farnsteiner/material/Shanghai-2012-Lectures.pdf.
[11] P. Gabriel, Indecomposable representations. II, in: *Symposia Mathematica Vol. XI* (Rome 1971), Academic Press, London (1973), 81–104.
[12] V. G. Kac, Root systems, representations of quivers and invariant theory, in: *Invariant Theory* (Montecatini 1982), Lecture Notes in Math. 996, Springer, Berlin (1983), 74–108.
[13] O. Kerner, Representations of wild quivers, in: *Representation Theory of Algebras and Related Topics* (Mexico City 1994), CMS Conf. Proc. 19, American Mathematical Society, Providence (1996), 65–107.
[14] O. Kerner, More representations of wild quivers, in: *Expository Lectures on Representation Theory*, Contemp. Math. 607, American Mathematical Society, Providence (2014), 35–55.
[15] O. Kerner and F. Lukas, Regular modules over wild hereditary algebras, in: *Representations of Finite-dimensional Algebras* (Tsukuba 1990), CMS Conf. Proc. 11, American Mathematical Society, Providence (1991), 191–208.
[16] O. Kerner and F. Lukas, Elementary modules, *Math. Z.* 223 (1996), no. 3, 421–434.
[17] C. M. Ringel, Representations of K-species and bimodules, *J. Algebra* 41 (1976), no. 2, 269–302.
[18] C. M. Ringel, Finite dimensional hereditary algebras of wild representation type, *Math. Z.* 161 (1978), no. 3, 235–255.
[19] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, Berlin, 1984.
[20] L. Unger, The concealed algebras of the minimal wild, hereditary algebras, *Bayreuth. Math. Schr.* (1990), no. 31, 145–154.
[21] J. Worch, Categories of modules for elementary abelian p-groups and generalized Bellinson algebras, *J. Lond. Math. Soc. (2) 88* (2013), no. 3, 649–668.
[22] J. Worch, *Module categories and Auslander-Reiten theory for generalized Bellinson algebras*, PhD thesis, Christian-Albrechts-Universität zu Kiel, 2013.
[23] J. Worch, AR-components for generalized Bellinson algebras, *Proc. Amer. Math. Soc.* 143 (2015), no. 10, 4271–4281.