A REAL-VARIABLE CONSTRUCTION WITH APPLICATIONS TO BMO-TEICHMÜLLER THEORY

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Abstract. With the use of real-variable techniques, we construct a weight function \( \omega \) on the interval \([0, 2\pi)\) that is doubling and satisfies \( \log \omega \) is a BMO function, but which is not a Muckenhoupt weight \( (A_\infty) \). Applications to the BMO-Teichmüller space and the space of chord-arc curves are considered.

1. Introduction

Let \( \Gamma \) be a bounded Jordan curve in the extended complex plane \( \hat{\mathbb{C}} \). We can consider three objects associated to \( \Gamma \): the Riemann mapping \( \Phi \) from the unit disk \( \mathbb{D} \) onto the bounded component \( \Omega \) of \( \hat{\mathbb{C}} \setminus \Gamma \); the Riemann mapping \( \Psi \) from the exterior of the unit disk \( \mathbb{D}^* \) onto the unbounded component \( \Omega^* \) of \( \hat{\mathbb{C}} \setminus \Gamma \); the conformal welding corresponding to \( \Gamma \), \( h = \Psi^{-1} \circ \Phi \), which is a sense-preserving homeomorphism of the unit circle \( S \). Let \( S(\mathbb{D}) \) be the set of all mappings \( \log f'(z) \) where \( f \) is conformal (i.e. holomorphic and injective) in \( \mathbb{D} \). By the Koebe distortion theorem, \( S(\mathbb{D}) \) is a bounded subset of the Bloch space \( B(\mathbb{D}) \) which consists of holomorphic functions \( \varphi \) in \( \mathbb{D} \) with finite norm \( \| \varphi \|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2) |\varphi'(z)| \).

In particular \( \log \Phi' \in S(\mathbb{D}) \) and \( \log \Psi' \in S(\mathbb{D}^*) \), which is defined in a similar way.

A bounded Jordan curve \( \Gamma \) is called quasicircle if there exists a constant \( C > 0 \) such that

\[
\text{diam}(\gamma) \leq C|z - \zeta|
\]

for any \( z, \zeta \in \Gamma \), where \( \gamma \) is the smaller of the two subarcs of \( \Gamma \) joining \( z \) and \( \zeta \). The quasicircle \( \Gamma \) can be characterized from the viewpoint of the universal Teichmüller space in the following equivalent ways (see [1, 12, 13]):

(a) \( \log \Phi' \) belongs to the interior of \( S(\mathbb{D}) \) in Bloch space \( B(\mathbb{D}) \).
(b) \( \log \Psi' \) belongs to the interior of \( S(\mathbb{D}^*) \) in Bloch space \( B(\mathbb{D}^*) \).
(c) \( h \) is a quasisymmetric homeomorphism of \( S \), namely, there exists a constant \( C > 0 \) such that for any adjacent intervals \( I, I^* \subset S \) of length \( |I| = |I^*| \leq \pi \), we have

\[
C^{-1}|h(I)| \leq |h(I^*)| \leq C|h(I)|,
\]

where \( | \cdot | \) denotes the Lebesgue measure. The optimal value of such \( C \) is called the doubling constant for \( h \).

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It is well known that a quasisymmetric homeomorphism need not be absolutely continuous, and may be totally singular. If however it is absolutely continuous, we say $|h'|$ is a doubling weight.

There are analogs of the above statements in the setting of the BMO Teichmüller theory, introduced by Astala and Zinsmeister [3], and investigated in depth later by Fefferman, Kenig and Pipher [8], Bishop and Jones [4], Cui and Zinsmeister [6], Shen and Wei [16].

Let $\Gamma$ be a bounded quasicircle. Then the following three statements are equivalent:

1. $\log \Phi' \in \text{BMOA}(D)$, the space of analytic functions in $D$ of bounded mean oscillation.
2. $\log \Psi' \in \text{BMOA}(D^*)$, the space of analytic functions in $D^*$ of bounded mean oscillation.
3. $h$ is a strongly quasisymmetric homeomorphism of $S$, namely, for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|E| \leq \delta |I| \Rightarrow |h(E)| \leq \epsilon |h(I)|$$

whenever $I \subset S$ is an interval and $E \subset I$ a measurable subset: we say that $h$ is absolutely continuous and $|h'|$ is an $A_\infty$-weight (or Muckenhoupt weight).

The set of strongly quasisymmetric homeomorphisms of $S$ is a group; more precisely, it is the group of homeomorphisms $h$ such that $P_h : b \mapsto b \circ h$ is an isomorphism of the BMO space $\text{BMO}(S)$ (see [11]). Naturally an $A_\infty$-weight is doubling. Fefferman and Muckenhoupt [9] gave this a direct computation, and they also provided an example of a function that satisfies the doubling condition but not $A_\infty$.

Noting that $h = \Psi^{-1} \circ \Phi$, we conclude that

$$\log h' = \log \Phi' - \log \Psi' \circ h.$$

If one of the above three characterizations is true, then it holds that

4. $h$ is absolutely continuous and $\log h' \in \text{BMO}(S)$.

Recall that an integrable function $u$ on $S$ is said to have bounded mean oscillation, i.e. $u \in \text{BMO}(S)$, if

$$\|u\|_* = \sup_{I \subset S} \frac{1}{|I|} \int_I |u(z) - u_I| dz < \infty,$$

where the supremum is taken over all bounded intervals $I$ on $S$ and $u_I$ denotes the integral mean of $u$ over $I$. This is regarded as a Banach space with norm $\| \cdot \|_*$ modulo constants since obviously constant functions have norm zero. An integrable function $u$ on $S$ is said to have vanishing mean oscillation, i.e. $u \in \text{VMO}(S)$, if $\|u\|_* < \infty$ and moreover

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_I |u(z) - u_I| dz = 0.$$

This is a closed subspace of $\text{BMO}(S)$, actually the closure of the space of all continuous functions on $S$ under the norm $\| \cdot \|_*$. If $\log h' \in \text{BMO}(S)$ with a small norm, or if $\log h' \in \text{VMO}(S)$, then it can be checked easily that $|h'|$ is an $A_\infty$-weight by the John-Nirenberg inequality (see [10])(see also [17, Proposition 5.4] for a proof).
In the present paper, in section 2 we will construct an example of a weight function with the use of real-variable techniques that shows (4) \( \not\Rightarrow \) (3) in the premise of \( \Gamma \) being a bounded quasicircle. Besides that, this construction implies more. More precisely, we will prove the followings in sections 3, 4, 5, respectively.

**Theorem 1.** There exists a sense-preserving homeomorphism \( h \) of \( S \) such that \( h \) is absolutely continuous, \( |h'| \) is a doubling weight, \( \log h' \in \text{BMO}(S) \), but \( |h'| \) is not an \( A_\infty \)-weight.

**Corollary 2.** There exist a sequence \( \{\Gamma_t\} \) \( (0 \leq t \leq 1) \) of quasicircles and a constant \( C > 0 \) such that \( \| \log h'_t \|_* \leq C \) for any \( 0 \leq t \leq 1 \), \( \log \Phi'_t \in \text{BMOA}(D) \) for any \( 0 \leq t < 1 \), but \( \| \log \Phi'_t \|_* \to \infty \) as \( t \to 1 \).

**Theorem 3.** For any \( \epsilon > 0 \) there exists a rectifiable quasicircle \( \hat{\Gamma} \) for which \( \log \hat{\Phi}' \in \text{BMOA}(D) \) with \( \| \log \hat{\Phi}' \|_B < \epsilon \) such that \( \hat{\Gamma} \) is not a chord-arc curve.

2. A “real-variable” construction

For \( 0 < \epsilon < 1 \), set

\[
P_n(t) = \prod_{j=0}^{n-1} (1 + \epsilon \cos(3^j t)), \quad t \in [0, 2\pi).
\]

This is a trigonometric polynomial, and it is known that \( P_n(t) \geq 0 \) and \( \int_0^{2\pi} P_n(t)dt = 2\pi \). Set \( \mu_n(x) = \int_0^x P_n(t)dt \) so that \( d\mu_n(x) = P_n(x)dx \). Here, \( P_n(x) \) and \( \mu_n(x) \) can also be defined for \( x \in \mathbb{R} \) by the periodic extension \( P_n(x + 2\pi) = P_n(x) \) and by the condition \( \mu_n(x + 2\pi) - \mu_n(x) = 2\pi \). The sequence \( \{\mu_n\} \) converges to a non-decreasing limit function \( \mu \), singular with respect to the Lebesgue measure on \([0, 2\pi)\) (see [19, Vol. I, Theorem 7.6]), which implies that the corresponding Lebesgue-Stieltjes measure sequence \( \{d\mu_n\} \) converges weakly to a measure \( d\mu \) of total mass \( 2\pi \). A direct proof of this fact was given in [18, P. 125]. We recall that this means convergence in the following sense:

\[
\int_0^{2\pi} \varphi(x)d\mu_n(x) \to \int_0^{2\pi} \varphi(x)d\mu(x), \quad n \to \infty
\]

for any \( \varphi \in C_b([0, 2\pi)) \), the set of continuous and bounded functions on \([0, 2\pi)\). The Riesz product

\[
P(t) = \prod_{j=0}^{\infty} (1 + \epsilon \cos(3^j t)), \quad t \in [0, 2\pi)
\]

may then be expanded as a well-defined trigonometric series, which is the Fourier series of the measure \( d\mu \) (also called the Fourier-Stieltjes series of the function \( \mu \) in the literature).

**Claim 1.** For any \( p > 1 \), it holds that

\[
\lim_{n \to \infty} \|P_n\|_p = \infty.
\]
Proof. Assuming that there exists a positive constant \( C \) and a subsequence \( \{n_k\} \) such that
\[
\|P_{n_k}\|_p \leq C
\]
for any \( k \), then, by the Banach-Alaoglu theorem, the sequence \( \{P_{n_k}\} \) has a weak-star convergent subsequence \( \{P_{n'_k}\} \), converging to some \( \hat{P} \in L^p([0, 2\pi]) \), namely,
\[
\int_0^{2\pi} \varphi(x)P_{n'_k}(x)dx \to \int_0^{2\pi} \varphi(x)\hat{P}(x)dx, \quad k \to \infty
\]  
(2)
for any \( \varphi \in L^q([0, 2\pi]) \). In particular, taking \( \varphi \in C_b([0, 2\pi]) \), we conclude by (1) and (2) that \( d\mu(x) = \hat{P}(x)dx \), which contradicts the fact that \( d\mu \) is singular with respect to the Lebesgue measure on \([0, 2\pi]) \). □

Set \( p_n = 1 + 1/n \) so that \( p_n \to 1 \) as \( n \to \infty \). By Claim 1, for any \( n \geq 1 \) there exists an integer \( N_n \) such that
\[
\|P_{N_n}\|_{p_n} \geq 4^n.
\]  
(3)
Set
\[
\tilde{f}(x) = \sum_{n \geq 1} 2^{-n} \frac{P_{N_n}}{\|P_{N_n}\|_{L\log L}}, \quad x \in [0, 2\pi).
\]
Here, \( \|g\|_{L\log L} = \int_I |g(x)| \log(e + |g(x)|)dx \) for any integrable function \( g \) on the bounded interval \( I \), and we say that \( g \in L \log L(I) \) if \( \|g\|_{L\log L} < \infty \). Set
\[
\omega(x) = M\tilde{f}(x), \quad x \in [0, 2\pi).
\]
Here, \( M\tilde{f} \) is the Hardy-Littlewood maximal function of \( \tilde{f} \). For any \( t \in [0, 1] \), set \( h_t(e^{ix}) = e^{igt(x)} \) by
\[
g_t(x) = \int_0^x \omega^t(s)ds, \quad x \in [0, 2\pi).
\]
Then, \( |h'_t(e^{ix})| = \omega^t(x) \) and
\[
\log h'_t(e^{ix}) = \log g'_t(x) + i(g_t(x) - x)
\]
whose imaginary part is clearly a continuous function, and in particular a BMO function on the interval \([0, 2\pi]) \). Let \( \Gamma_t \) be the bounded Jordan curve whose conformal welding is \( h_t \), and \( \Phi_t, \Psi_t \) the Riemann mappings associated to the two components \( \Omega_t \) and \( \Omega'_t \) of \( \mathbb{C} \setminus \Gamma_t \), respectively. We will denote \( h_1 \) by \( h \) for simplicity in the following.

3. Proof of Theorem 1

In this section, we prove the function \( h \) above is a desired one for Theorem 1. Before that, We need to show that \( \tilde{f} \notin L^p([0, 2\pi]) \) for any \( p > 1 \) (see Claim 2), but \( \tilde{f} \in L \log L([0, 2\pi]) \) (see Claim 3). For this purpose we first recall some well-known facts.

Let \( f \) be a measurable function on a measure space \((X, \nu)\). The distribution function
\[
m(t) = \nu(\{x \in X : |f(x)| > t\})
\]
defined for \( t > 0 \) is a decreasing function of \( t \), and it determines the \( L^p \) norms of \( f \). If \( f \in L^\infty \) then \( \| f \|_\infty = \sup\{ t : m(t) > 0 \} \), and if \( f \in L^p \) (\( 0 < p < \infty \)) then the Chebychev inequality says

\[
m(t) \leq \frac{1}{t^p} \int_{|f| \geq t} |f|^p \, d\nu,
\]

(4)

and in particular \( m(t) \leq \| f \|_p^p / t^p \).

**Lemma 4.** Let \( \psi : [0, \infty) \to [0, \infty) \) be an increasing differentiable function such that \( \psi(0) = 0 \). If \( f(x) \) is a non-negative measurable function in a measure space \( (X, \nu) \), then

\[
\int_X \psi(f(x)) \, d\nu = \int_0^\infty \psi'(x) m(t) \, dt.
\]

(5)

**Proof.** We may assume \( f \) vanishes except on a set of \( \sigma \)-finite measure because otherwise both sides of (5) are infinite. Then the Fubini theorem shows that both sides of (5) equal the product measure of the ordinate set \( \{ (x, t) : 0 < t < \psi(f(x)) \} \). That is,

\[
\int_X \psi(f(x)) \, d\nu = \int_X \int_0^{f(x)} \psi'(t) \, dt \, d\nu = \int_0^\infty \psi'(t) \nu(\{ f > t \}) \, dt
\]

and

\[
= \int_0^\infty \psi'(t) m(t) \, dt.
\]

\( \square \)

**Proposition 5.** Let \( f \) be a non-negative measurable function in a measure space \( (X, \nu) \) with norm \( \| f \|_1 = 1 \). If \( \| f \|_p \geq 2 \) for some \( p > 1 \), then there is a positive constant \( C \) depending only on \( p \) such that

\[
\int_X f(x) \log(e + f(x)) \, d\nu \leq \frac{C}{(p-1)^2} \log \| f \|_p.
\]

**Proof.** By Lemma 4 and taking \( \psi(t) = t \log(e + t) \) we have

\[
\int_X f(x) \log(e + f(x)) \, d\nu = \int_0^\infty \psi'(t) m(t) \, dt < 2 \int_0^\infty \log(e + t) m(t) \, dt.
\]

(6)

Let now \( T > 0 \) to be determined later. We divide the right integral of the above inequality by \( T \) into two parts and then estimate them respectively:

\[
\int_0^\infty \log(e + t) m(t) \, dt = \int_0^T \log(e + t) m(t) \, dt + \int_T^\infty \log(e + t) m(t) \, dt.
\]

By Lemma 4 and \( \| f \|_1 = 1 \) we have

\[
\int_0^T \log(e + t) m(t) \, dt \leq \log(e + T) \int_0^T m(t) \, dt
\]

\[
\leq \log(e + T) \int_X f(x) \, d\nu \leq \log(e + T).
\]
By the Chebychev inequality we have
\[
\int_T^\infty \log(e + t)m(t)dt \leq \|f\|_p^p \int_T^\infty \frac{\log(e + t)}{t^p} dt < \frac{\|f\|_p^p}{(p - 1)^2 T^{p-1}} ((p - 1) \log(e + T) + 1).
\]

Then, by choosing \( T = \|f\|_p^\frac{p}{p-1} \) and substituting what have obtained into (6), it follows from \( \|f\|_p \geq 2 \) that
\[
\int_X f(x) \log(e + f(x))d\nu \leq C(p - 1)^2 \log \|f\|_p.
\]

\[\boxed{}\]

**Claim 2.** For any \( p > 1 \), \( \tilde{f} \) is not in \( L^p([0, 2\pi]) \).

**Proof.** It follows from Proposition [5] and (3) that
\[
\|\tilde{f}\|_{p_n} > \frac{\|P_{N_n}\|_{p_n}}{2^n \|P_{N_n}\|_{L_{\log L}}} \geq C^{-1}(p_n - 1)^2 \frac{\|P_{N_n}\|_{p_n}}{2^n \log(\|P_{N_n}\|_{p_n})} > \left( \frac{3}{2} \right)^n
\]
as \( n \) is sufficiently large.

Assuming \( \tilde{f} \in L^p([0, 2\pi]) \) for some \( p > 1 \) with \( \|\tilde{f}\|_p \) being a constant \( C_1 \). Then, \( \tilde{f} \in L^q([0, 2\pi]) \) for any \( 1 < q < p \) and \( \|\tilde{f}\|_q \leq C_1 \). However, taking some integer \( n \) so that \( \left( \frac{3}{2} \right)^n > C_1 \) and \( p_n < p \), we then have \( \|\tilde{f}\|_{p_n} > C_1 \). This leads to a contradiction. \( \boxed{} \)

**Claim 3.** It holds that
\[
\tilde{f}(x) \log(e + \tilde{f}(x)) \leq \sum_{n \geq 1} 2^{-n} \frac{P_{N_n}(x) \log(e + P_{N_n}(x))}{\|P_{N_n}\|_{L_{\log L}}},
\]
and moreover \( \|\tilde{f}\|_{L_{\log L}} \leq 1 \).

**Proof.** Since \( \psi(t) = t \log(e + t) \) is convex on \( \mathbb{R}^+ \), we have
\[
\tilde{f}(x) \log(e + \tilde{f}(x)) = \psi \circ \tilde{f}(x) = \psi \left( \frac{\sum_{n \geq 1} 2^{-n} P_{N_n}(x)}{\sum_{n \geq 1} 2^{-n} \|P_{N_n}\|_{L_{\log L}}} \right)
\leq \sum_{n \geq 1} 2^{-n} \frac{1}{\|P_{N_n}\|_{L_{\log L}}} P_{N_n}(x) \log \left( e + \left( \frac{P_{N_n}(x)}{\|P_{N_n}\|_{L_{\log L}}} \right) \right).
\]
Then,
\[
\tilde{f}(x) \log(e + \tilde{f}(x)) \leq \sum_{n \geq 1} 2^{-n} \frac{P_{N_n}(x) \log(e + P_{N_n}(x))}{\|P_{N_n}\|_{L_{\log L}}}
\]
and thus \( \|\tilde{f}\|_{L_{\log L}} \leq 1 \). \( \boxed{} \)
The Hardy-Littlewood maximal function $M_\nu$ of the signed measure $\nu$ is defined as

$$M_\nu(x) = \sup_{x \in I} \frac{1}{|I|} |\nu|(I),$$

where the supremum is taken over all bounded intervals. In particular, for the signed measure of the form $d\nu(x) = g(x)dx$, $M_\nu$ is usually denoted by $Mg$ and called the Hardy-Littlewood maximal function of the function $g$ in the literature. This is a quantitation of the Lebesgue theorem which says that if $g(x)$ is locally integrable on $\mathbb{R}$ then

$$\lim_{h,k \to 0^+} \frac{1}{h+k} \int_{x-h}^{x+k} g(t)dt = g(x)$$

for almost every $x \in \mathbb{R}$. If $g \in L^p(\mathbb{R})$ for $p \in [1, \infty]$, then $Mg(x)$ is finite almost everywhere. Moreover, $Mg \in L^p(\mathbb{R})$ if $g \in L^p(\mathbb{R})$ for $p \in (1, \infty)$, while $Mg$ is weak $L^1$ if $g \in L^1(\mathbb{R})$ (see [10] Page 23). On the other hand, Stein [15] and Zygmund [19] proved that if $g$ is supported on a finite interval $I$ then $Mg \in L^1(I)$ if and only if $g \in L \log L(I)$.

Proof of Theorem 1. Recall that

$$\omega(x) = M\tilde{f}(x), \quad x \in [0, 2\pi).$$

Then, we have that $\omega$ is finite almost everywhere, $\omega \in L^1([0, 2\pi))$ by Claim 3 and $\omega \notin L^p([0, 2\pi))$ for any $p > 1$ by Claim 2 since it holds that $\omega(x) \geq \tilde{f}(x)$ from the Lebesgue theorem, and thus $\omega$ is not an $A_\infty$-weight. Indeed, if $\omega$ were an $A_\infty$-weight, then the reverse Hölder inequality holds for $\omega$ (see [10]), namely, there are $\delta > 0$ and $C > 0$ such that

$$\left( \frac{1}{|I|} \int_I \omega(x)^{1+\delta} dx \right)^{1/(1+\delta)} \leq C \int_I \omega(x)dx$$

for any interval $I \subset [0, 2\pi)$. This contradicts that $\omega \notin L^p([0, 2\pi))$ for any $p > 1$.

Furthermore, we note that the claim $\log \omega \in \text{BMO}([0, 2\pi))$ follows from a result by Coifman and Rochberg [5]: assuming $\nu$ is a locally finite signed Borel measure on $\mathbb{R}$ for which the maximal function $M_\nu(x)$ is finite almost everywhere we have $\log M_\nu \in \text{BMO}(\mathbb{R})$. Thus, by taking $\nu(x) = \tilde{f}(x)dx$ we get the claim $\log \omega = \log Mg \in \text{BMO}([0, 2\pi))$.

It remains to show that $\omega$ is a doubling weight on $[0, 2\pi)$. It is sufficient to check the doubling condition $\int_I \omega(x)dx \approx \int_I, \omega(x)dx$ holds when $I$ and $I^*$ are two adjacent intervals of length $\frac{2\pi}{3\pi}$.

We cut the sum giving $\tilde{f}$ into two parts, each term in the first part with the subscript $N_k$ such that $N_k \leq n - 1$. Then we can write

$$\tilde{f}(x) = g_n(x) + f_n(x)h_n(x).$$

Here, we split the second part into the product of $f_n$ and $h_n$, $f_n$ with the subscript $N_k$’s such that $N_k \leq n - 1$, and $h_n$ being $\frac{2\pi}{3\pi}$ periodic. Then,

$$\exp\left( -\frac{\pi\epsilon}{1 - \epsilon} \right) f_n(x) \leq f_n(x + \frac{2\pi}{3\nu}) \leq \exp\left( \frac{\pi\epsilon}{1 - \epsilon} \right) f_n(x)$$
and
\[
\exp \left( -\frac{\pi \epsilon}{1 - \epsilon} \right) g_n(x) \leq g_n(x + \frac{2\pi}{3^n}) \leq \exp \left( \frac{\pi \epsilon}{1 - \epsilon} \right) g_n(x).
\]

It follows that
\[
\exp \left( -\frac{\pi \epsilon}{1 - \epsilon} \right) \tilde{f}(x) \leq \tilde{f}(x + \frac{2\pi}{3^n}) \leq \exp \left( \frac{\pi \epsilon}{1 - \epsilon} \right) \tilde{f}(x). \tag{7}
\]

Indeed, for any \( k \leq n - 1 \), we write
\[
P_k(x) = \prod_{j=1}^{k} (1 + \epsilon \cos(3^j x)) = \prod_{j=1}^{k} \varphi_j(x),
\]
so that
\[
\log P_k(x) = \sum_{j=1}^{k} \log \varphi_j(x).
\]

By using the finite increment theorem we get
\[
| \log P_k(x + \frac{2\pi}{3^n}) - \log P_k(x) | \leq \sum_{j=1}^{k} | \log \varphi_j(x + \frac{2\pi}{3^n}) - \log \varphi_j(x) |
\]
\[
\leq \sum_{j=1}^{k} \| \frac{d}{dx} \log \varphi_j(x) \|_\infty \frac{2\pi}{3^n}
\]
\[
\leq \frac{2\pi \epsilon}{1 - \epsilon} \sum_{j=1}^{k} 3^{j-n} \leq \frac{\pi \epsilon}{1 - \epsilon},
\]

which implies
\[
\exp \left( -\frac{\pi \epsilon}{1 - \epsilon} \right) P_k(x) \leq P_k(x + \frac{2\pi}{3^n}) \leq \exp \left( \frac{\pi \epsilon}{1 - \epsilon} \right) P_k(x).
\]

For any \( x \in [0, 2\pi) \) and any interval \( I = [a, b] \) with \( a \leq x \leq b \), set \( J = [a + \frac{2\pi}{3^n}, b + \frac{2\pi}{3^n}] \) so that \( x + \frac{2\pi}{3^n} \in J \). This gives
\[
\int_I \tilde{f}(t) dt = \int_J \tilde{f}(t + \frac{2\pi}{3^n}) dt.
\]

Combined with (7), it implies
\[
\exp \left( -\frac{\pi \epsilon}{1 - \epsilon} \right) \frac{1}{|I|} \int_I \tilde{f}(t) dt \leq \frac{1}{|J|} \int_J \tilde{f}(t) dt \leq \exp \left( \frac{\pi \epsilon}{1 - \epsilon} \right) \frac{1}{|I|} \int_I \tilde{f}(t) dt,
\]

and then by taking the supremum over the interval \( I \) containing \( x \) we pass (7) to the maximal function \( \omega(x) \), namely,
\[
\exp \left( -\frac{\pi \epsilon}{1 - \epsilon} \right) \omega(x) \leq \omega(x + \frac{2\pi}{3^n}) \leq \exp \left( \frac{\pi \epsilon}{1 - \epsilon} \right) \omega(x). \tag{8}
\]

For any two adjacent intervals \( I \) and \( I^* \) of length \( \frac{2\pi}{3^n} \), we have
\[
\int_{I^*} \omega(x) dx = \int_I \omega(x + \frac{2\pi}{3^n}) dx.
\]
By combining this with (8) we conclude that
\[ \exp \left( -\frac{\pi \epsilon}{1 - \epsilon} \right) \int_I \omega(x) dx \leq \int_{I^*} \omega(x) dx \leq \exp \left( \frac{\pi \epsilon}{1 - \epsilon} \right) \int_I \omega(x) dx. \] 
This completes the proof of the doubling condition. □

Remark. We see from the above arguments that
\[ \exp \left( -\frac{\pi \epsilon t}{1 - \epsilon} \right) \int_I \omega^t(x) dx \leq \int_{I^*} \omega^t(x) dx \leq \exp \left( \frac{\pi \epsilon t}{1 - \epsilon} \right) \int_I \omega^t(x) dx, \]
which implies \( \omega^t \) is also a doubling weight for any \( 0 \leq t \leq 1 \). Since \( \epsilon \in (0, 1) \) may be arbitrarily small, we may moreover assume that the doubling constant of the weight \( \omega^t \) is as close to 1 as we like.

4. Proof of Corollary 2

A locally integrable function \( \omega \geq 0 \) on the real line \( \mathbb{R} \) is called an \( A_p \)-weight for \( 1 < p < \infty \) if
\[ \sup_I \left( \frac{1}{|I|} \int_I \omega(x) dx \right) \left( \frac{1}{|I|} \int_I \left( \frac{1}{\omega(x)} \right)^{\frac{1}{p-1}} dx \right)^{p-1} < \infty, \]
where the supremum is taken over all bounded intervals. It is known that \( A_p \subset A_q \) if \( p < q \) and \( A_\infty = \bigcup_{p>1} A_p \). A locally integrable function \( \omega \geq 0 \) on the real line \( \mathbb{R} \) is called an \( A_1 \)-weight, if there is a constant \( C > 0 \) such that for all bounded intervals \( I \)
\[ \omega_I \leq C \inf_I \omega, \]
or equivalently, there is a constant \( C > 0 \) such that
\[ M\omega(x) \leq C \omega(x) \]
almost everywhere on \( \mathbb{R} \). If \( \omega(x) \) satisfies \( A_1 \), then \( \omega(x) \) satisfies \( A_p \) for any \( p > 1 \).

The following result by Coifman and Rochberg [5] establishes the relationship between \( A_1 \)-weights and Hardy-Littlewood maximal functions.

Proposition 6. If \( \nu \) is a locally finite signed Borel measure with \( M\nu(x) < \infty \) almost everywhere, then \( (M\nu)^t \) is an \( A_1 \)-weight for any \( 0 \leq t < 1 \).

As was observed in [5], this construction yields essentially all the elements of \( A_1 \)-weights and in fact essentially all of \( A_1 \)-weights are obtained using only signed measures of the form \( g(x)dx \).

Proof of Corollary 2. We come back to our constructions of the function \( \tilde{f} \) and the weight function \( \omega = M\tilde{f} \) in section 2. Set the measure \( \nu(x) = \tilde{f}(x)dx \) so that \( M\nu = \omega \). We conclude by Proposition 6 that \( |h'_t| = \omega^t \) is an \( A_1 \)-weight, and thus \( A_\infty \)-weight for any \( 0 \leq t < 1 \). On the other hand, it follows from Theorem 1 that \( |h'_t| = \omega \) is not an \( A_\infty \)-weight. Moreover, \( |h'_t| = \omega^t \) is a doubling weight for any \( 0 \leq t \leq 1 \).
It is observed that
\[ \| \log |h'|_* \| \leq t \log |h'|_* \leq \| \log |h'|_* \| \leq C, \]
where \( C > 0 \) is some constant. By the equivalences of (1), (2) and (3) in section 1, we have \( \log \Phi_t' \in \text{BMO}(\mathbb{D}) \) and \( \log \Psi_t' \in \text{BMO}(\mathbb{D}^*) \) for any \( 0 \leq t < 1 \), but \( \log \Phi_t' \notin \text{BMO}(\mathbb{D}) \) and \( \log \Psi_t' \notin \text{BMO}(\mathbb{D}^*). \)

It remains to show that \( \| \log \Phi_t' \| \rightarrow \infty \) as \( t \rightarrow 1 \). We suppose that there exists a subsequence \( \{t_n\} \) converging to 1 such that \( \| \log \Phi_{t_n}' \| \) is bounded and we argue toward a contradiction. Since \( \text{BMO}(\mathbb{D}) = H^2 \cap \text{BMO}(\mathbb{S}) \) is the dual of the classical space \( H^1 \), the sequence \( \{\log \Phi_{t_n}'\} \) has a weak-star convergent subsequence \( \{\log \Phi_{t_{n_k}}'\} \) converging to some function \( \varphi \in \text{BMOA} \) in the following sense:
\[ \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \log \Phi_{t_{n_k}}'(e^{i\theta}) d\theta \rightarrow \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \varphi(e^{i\theta}) d\theta \]
as \( k \rightarrow \infty \) for any \( f \in H^1 \). In particular, by taking \( f \equiv 1 \) we get \( \log \Phi_{t_{n_k}}' \rightarrow \varphi \) almost everywhere on \( \mathbb{S} \), and then by taking the Poisson integral we have \( \log \Phi_{t_{n_k}}' \rightarrow \varphi \) almost everywhere on \( \mathbb{D} \). On the other hand, this subsequence \( \{\log \Phi_{t_{n_k}}'\} \) converges in the universal Teichmüller space in \( \mathbb{D} \) to \( \log \Phi' \), namely, \( \| \log \Phi_{t_{n_k}}' - \log \Phi' \|_B \rightarrow 0 \). Thus, we conclude that \( \log \Phi_t' = \varphi \in \text{BMOA}(\mathbb{D}) \). This leads to a contradiction. \( \square \)

**Question.** Let \( \mathcal{C} = \{\log |h'|, |h'| \in A_\infty(\mathbb{S})\} \). It is an open convex subset of the real Banach space \( \text{BMOR}(\mathbb{S}) \), the space of all real-valued BMO functions on \( \mathbb{S} \). A paraphrase of our results (Theorem 1 and Corollary 2) is that there exists a quasisymmetric homeomorphism \( h \) of \( \mathbb{S} \) which is absolutely continuous with \( \log |h'| \in \text{BMOR}(\mathbb{S}) \), and moreover \( \log |h'| \in \mathcal{C} \setminus \mathcal{C} \), the boundary of \( \mathcal{C} \) for the BMO topology. We thus address the question: Does there exist a quasisymmetric homeomorphism of \( \mathbb{S} \) which is absolutely continuous with \( \log |h'| \in \text{BMOR}(\mathbb{S}) \) such that \( \log |h'| \notin \mathcal{C} \)?

5. **Proof of Theorem 3**

**Proof of Theorem 3.** Recall that \( |h'(e^{i\theta})| = \omega(\theta) \). We use \( \omega(\theta) \) to denote \( |h'(e^{i\theta})| \) for simplicity. Here, \( \omega \) is the weight function constructed in section 2. Since \( \log \omega \in \text{BMO}(\mathbb{S}) \), it is in particular integrable on the unit circle \( \mathbb{S} \). If \( z = re^{i\phi} \), then the Poisson integral of \( \log \omega \),
\[ u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\varphi - \theta) \log \omega(\theta) d\theta = P_r(\log \omega)(\varphi), \]
is harmonic on \( \mathbb{D} \). We let
\[ \log \Phi' = u(z) + iv(z), \]
where \( v(z) \) is the harmonic conjugate function of \( u(z) \), normalized so that \( v(0) = 0 \). By \( \log \omega \in \text{BMO}(\mathbb{S}) \) again, we have that \( v(z) \) has nontangential limit almost everywhere on \( \mathbb{S} \) which we denote by \( b(\theta) \), \( b \in \text{BMO}(\mathbb{S}) \), and thus \( \log \Phi' \in \text{BMOA}(\mathbb{D}) \). By the univalence criterion of Ahlfors-Weill \([2]\), \( \Phi \) is a conformal map onto a quasidisk and we
call \( \hat{\Gamma} \) its boundary whenever the doubling constant of \( \omega \) is sufficiently small. By the Jensen inequality we have

\[
|\hat{\Phi}'(z)| = \exp(u(z)) = \exp\left( \frac{1}{2\pi} \int_0^{2\pi} P_r(\varphi - \theta) \log \omega(\theta) d\theta \right) \leq \frac{1}{2\pi} \int_0^{2\pi} P_r(\varphi - \theta) \omega(\theta) d\theta = P_r * \omega(\varphi).
\]

Since \( \omega \in L^1 \) on \( S \), we conclude that \( P_r * \omega \in L^1 \) on \( S \) for any \( 0 \leq r < 1 \) and the sequence \( \| P_r * \omega \|_1 \) increases to \( \| \omega \|_1 \) as \( r \to 1 \). Then,

\[
\| \hat{\Phi}' \|_{H^1} = \sup_r \frac{1}{2\pi} \int_0^{2\pi} |\hat{\Phi}'(re^{i\varphi})| d\varphi < \infty,
\]

which implies that \( \hat{\Phi}' \in H^1(\mathbb{D}) \), actually \( \hat{\Phi}' \) is an outer function, and thus \( \hat{\Gamma} \) is rectifiable. Moreover, since the boundary function \( \omega \) of \( |\hat{\Phi}'| \) is not an \( A_\infty \)-weight, \( \hat{\Gamma} \) is not a chord-arc curve ([14, Theorem 7.11][18]).

**Question.** For any \( 0 \leq t < 1 \), if we replace \( \omega \) with \( \omega^t \), set

\[
\hat{\Phi}_t(z) = \hat{\Phi}_t(0) + \int_0^z (\hat{\Phi}')^t(\zeta) d\zeta
\]

and denote the curve \( \partial \hat{\Phi}_t(\mathbb{D}) \) by \( \hat{\Gamma}_t \). Then, by the same arguments as the above we have that \( \hat{\Gamma}_t \) is a chord-arc curve since \( \omega^t \) is an \( A_\infty \)-weight. According to these observations, the conformal map \( \hat{\Phi} \) lies in the closure of chord-arc domain maps in the sense that \( \hat{\Phi}_t \) is a conformal map onto a chord-arc domain for any \( t \in [0, 1) \). Must every map satisfying the conclusions of Theorem 3 be such?

**Remark.** Let \( z(s) \) denote the arc-length parametrization of the chord-arc curve. Then, the set of all \( \arg z'(s) \) forms an open subset of real-valued BMO functions (see [7]).

We see from the above Question that the rectifiable curve \( \hat{\Gamma} \) is on the boundary of the closure of the space of chord-arc curves. Set

\[
z(s) = z(0) + \int_0^s e^{i\beta(x)} dx
\]

is an arc-length parametrization of \( \hat{\Gamma} \). Noting that the curve \( \hat{\Gamma} \) has a parametrization \( \gamma \) such that \( \gamma'(t) = \omega(t)e^{ib(t)} \), we conclude that

\[
z'(s) = e^{i\beta(s)} = e^{ib\alpha(s)}
\]

where \( \alpha(s) \) is the inverse of the function

\[
s(t) = \int_0^t |\gamma'(x)| dx = \int_0^t \omega(x) dx.
\]
Recall that \( b \in \text{BMO}(\mathbb{S}) \) and \( \omega \) is a positive \( L^1 \) function, but not an \( A_{\infty} \) weight. Thus, we cannot conclude that \( \beta \) is a BMO function, nor can we conclude that \( z'(s) \) is not of this form!

**References**

[1] L.V. Ahlfors, Conformal Invariants: Topics in Geometric Function Theory, AMS Chelsea Publishing, 2010.

[2] L.V. Ahlfors and G. Weill, A uniqueness theorem for Beltrami equation, Proc. Amer. Math. Soc. 13 (1962), 975–978.

[3] K. Astala and M. Zinsmeister, Teichmüller spaces and BMOA, Math. Ann. 289 (1991), 613–625.

[4] C. Bishop and P. Jones, Harmonic measure, \( L^2 \) estimates and the Schwarzian derivative, J. Anal. Math. 62 (1994), 77-113.

[5] R.R. Coifman and R. Rochberg, Another characterization of BMO, Proc. Amer. Math. Soc. 79 (1980), 249-254.

[6] G. Cui and M. Zinsmeister, BMO-Teichmüller spaces, Illinois J. Math. 48 (2004), 1223-1233.

[7] G. David, Thèse de troisième cycle, Université de Paris XI, Orsay, France.

[8] R.A. Fefferman, C.E. Kenig and J. Pipher, The theory of weights and the Dirichlet problems for elliptic equations, Ann. of Math. 134 (1991), 65–124.

[9] C. Fefferman and B. Muckenhoupt, Two nonequivalent conditions for weight functions, Proc. Amer. Math. Soc. 45 (1974), 99-104.

[10] J.B. Garnett, Bounded Analytic Functions, Academic Press, New York, 1981.

[11] P.W. Jones, Homeomorphisms of the line which preserve BMO, Ark. Mat. 21 (1983), 229-231.

[12] O. Lehto, Univalent Functions and Teichmüller Spaces, Graduate Texts in Math. 109, Springer, 1987.

[13] S. Nag, The Complex Analytic Theory of Teichmüller Spaces, Wiley-Interscience, 1988.

[14] C. Pommerenke, Boundary Behaviour of Conformal Maps, Springer, 1992.

[15] E.M. Stein, Note on the class \( L \log L \), studia Math. 32 (1969), 305-310.

[16] Y. Shen and H. Wei, Universal Teichmüller space and BMO, Adv. Math. 234 (2013), 129-148.

[17] H. Wei and K. Matsuzaki, Strongly symmetric homeomorphisms on the real line with uniform continuity, preprint.

[18] M. Zinsmeister, Domaines de Lavrentiev, Publi. Math. Orsay, 1985.

[19] A. Zygmund, Trigonometric Series, Vols. I and II, 2nd ed., Cambridge University Press, London, 1959.