Exact diagonalization of the generalized 
supersymmetric \( t-J \) model with boundaries

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Abstract

We study the generalized supersymmetric \( t-J \) model with boundaries in three different gradings: FFB, BFF and FBF. Starting from the trigonometric \( R \)-matrix, and within the framework of the graded quantum inverse scattering method (QISM), we solve the eigenvalue problem for the supersymmetric \( t-J \) model. A detailed calculation is presented to obtain the eigenvalues and Bethe ansatz equations of the supersymmetric \( t-J \) model with boundaries in three different backgrounds.

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1 Introduction

One-dimensional strongly correlated electron models, such as the \( J \) model, have been attracting a great deal of interests in the context of high-\( T_c \) superconductivity. The Hamiltonian of the \( J \) model includes the nearest-neighbor hopping (\( t \)) and antiferromagnetic exchange (\( J \)) \cite{1,2}.

\[
H = \sum_{j=1}^{L} \left( t \mathbf{P} \cdot \mathbf{c}_{j+1}^\dagger \mathbf{c}_{j} + H.c. \right) + \frac{1}{2} \sum_{n} \mathbf{S}_n \cdot \mathbf{S}_{n+1} \tag{1}
\]

It is known that this model is supersymmetric and integrable for \( J = 2t \) \cite{3,4}. The supersymmetric \( t \) \( J \) model was also studied in Refs. \cite{5,6,7,8,9}, for a review, see Ref. \cite{10} and the references therein. Esterl and Korepin et al. showed that the one-dimensional Hamiltonian can be obtained from the transfer matrix of the two-dimensional supersymmetric exactly solvable lattice model \cite{11,12}. They used the graded QISM \cite{13,14} and obtained the eigenvalues and eigenvectors for the supersymmetric \( t \) \( J \) model with periodic boundary conditions in three different backgrounds, for related works, see for example \cite{15}. In this paper, we shall start from the trigonometric \( R \)-matrix which is a generalization of the \( R \)-matrix used in \cite{16}. The Hamiltonian is also a generalization of the supersymmetric \( t \) \( J \) model. We shall discuss the reflecting boundary condition cases. By using the graded QISM, we obtain the eigenvalues of the transfer matrix with boundaries in three different backgrounds.

The exactly solvable models are generally solved by imposing periodic boundary conditions. Recently, solvable models with reflecting (open) boundary conditions have been extensively studied (14-39). Besides the original Yang-Baxter equation \cite{17,18}, the reflection equations also play a key role in proving the commutativity of the transfer matrices under reflecting boundary conditions \cite{19,20}. The Hamiltonian includes non-trivial boundary terms which are determined by the boundary K-matrices.

In our previous paper \cite{21}, we used the algebraic Bethe ansatz method to solve the eigenvalue and eigenvector problems of the supersymmetric \( t \) \( J \) model with reflecting boundary conditions in the framework of the graded QISM (FFB grading). Here we shall extend the results in Ref. \cite{21}. We start from the trigonometric \( R \)-matrix proposed by Perk and Schultz \cite{22} and change the formulae to the graded case. Three kinds of grading are imposed, so there are three \( R \)-matrices for different grading. Solving the reflecting equation equation, we give general diagonal solutions. There are altogether four kinds of different boundary conditions for each choice of grading. Using the graded algebraic Bethe ansatz method in three possible grading FFB, BFF and FBF, we obtain the eigenvalues of the transfer matrix with general diagonal boundary matrices.

The graded method was proposed in \cite{23}, and it was applied for the reflection equation in \cite{24}, and later was applied to fermionic models \cite{25,26}. In this paper, we shall use the graded reflection equation to study the supersymmetric \( t \) \( J \) model. For the supersymmetric \( t \) \( J \) model, the spin of the electrons and the charge "hole" degrees of freedom play a very similar role forming a graded superalgebra with two fermions and one boson. The holes obey boson commutation relations, while the spinors are fermions, see Ref. \cite{27} and the references therein. The graded approach has the advantage of making a clear distinction between bosonic and fermionic degrees of freedom. So, it is interesting to study the supersymmetric \( t \) \( J \) model with reflecting boundary conditions by the graded algebraic Bethe ansatz method. In this paper, we give a detailed analysis for the Bethe ansatz in three different backgrounds. We should mention that the trigonometric \( R \)-matrix related to the supersymmetric \( t \) \( J \) model with reflecting boundary conditions...
was studied in [22,23] by using the usual reflection equation, the results have also been extended to more general cases [24,25]. And the thermodynamic limit of the Bethe ansatz was calculated in Ref. [26]. The nine-size corrections in the supersymmetric J model with boundary fields are presented in Ref. [27].

The integrable bulk Hamiltonian was derived previously by Karowski and Forster and by Gonzales-Ruiz [22,23]. Bariev also showed that it is integrable and studied physical properties for the Hermitian case [8].

As mentioned in Ref. [8], the formulae and the results for these different gradings are significantly different, so we shall write out in detail the graded algebraic Bethe ansatz for the generalized supersymmetric J model with four kinds of boundaries.

The paper is organized as follows: In section 2, we review the supersymmetric J model and its generalization. We start from the Perk-Shultz [13] model and change it to the graded case. In section 3, the general solutions of the reflection equation are presented. In section 4, in the FFB grading, we use the algebraic Bethe ansatz method to obtain the eigenvalues and eigenvectors of the transfer matrix with boundaries. In sections 5 and 6, we study the case of BFF grading and FBF grading. Section 7 includes a brief summary and some discussions.

## 2 Supersymmetric J Model and its Generalization

We first review the supersymmetric J model. For convenience, we adopt the notations in Ref. [8]. The Hamiltonian of the supersymmetric J model is given as:

\[
H = \sum_{j=1}^{N} (\mathbf{c}_j^c \mathbf{c}_j + \mathbf{c}_j \mathbf{c}^c_j) + J \sum_{j=1}^{N} \left[ c_j^z s_j^z + \frac{1}{2} (s_j^x s_{j+1}^x + s_j^y s_{j+1}^y) + \frac{1}{4} n_j n_{j+1} \right].
\]

This form is an equivalent expression of the Hamiltonian (1). The operators \( \mathbf{c}_j \) and \( \mathbf{c}_j^c \) mean the annihilation and creation operators of electron with spin \( c \) on a lattice site \( j \), and we assume the total number of lattice sites is \( N \), \( \mathbf{c}_j \) represent spin down and up, respectively. These operators are canonical Fermi operators satisfying anticommutation relations

\[
f \mathbf{c}_j^c \mathbf{c}_j g = \delta_{ij} \quad (3)
\]

We denote by \( n_j = \mathbf{c}_j^c \mathbf{c}_j \) the number operator for the electron on a site \( j \) with spin \( c \), and by \( n_j = \sum_i \mathbf{c}_i \mathbf{c}_j \) the number operator for the electron on a site \( j \). The Fock vacuum state \( \mathbf{0} \) satisfies \( \mathbf{c}_j \mathbf{0} = 0 \). There are altogether three possible electronic states at a given lattice site \( j \) due to excluding double occupancy:

\[
\mathbf{0} \quad \mathbf{n} \quad \mathbf{n}^* \quad \mathbf{j} \quad \mathbf{j}^* \quad \mathbf{j}^* \quad \mathbf{0} \quad (4)
\]

\( S_j^x, S_j^y, S_j^z \) are spin operators satisfying su(2) algebra and can be expressed as:

\[
S_j^x = c_{j+1}^c c_j c_j \quad ; \quad S_j^y = c_{j+1}^c c_j c_{j+1} \quad ; \quad S_j^z = \frac{1}{2} n_j n_{j+1} - n_j n_{j+1} \quad (5)
\]
It has been proved that for a special value \( J = 2t = 2 \), the Hamiltonian of the supersymmetric \( J \) model can be written as the a graded permutation operator \([3,7,9]\):

\[
H = \sum_{j=1}^{N} P_{j+1} \ 2N.t
\]

Here we have omitted a constant term. The total number operator \( N = \sum_{j=1}^{N} n_j \) commutes with the Hamiltonian and is dedicated to the chemical potential. We shall also omit the second term in the following. The graded permutation operator can be represented as

\[
P_{bd} = \text{ad bc} (1) =
\]

Here, different from the non-graded case, we have the Grassmann parities \( \alpha = 1; 0 \) representing fermion and boson, respectively. The Hamiltonian can also be represented by the generators of \( \mathfrak{u}(1|\mathfrak{p}) \),

\[
H = \sum_{j=1}^{N} \left( \begin{array}{c}
0 \\
0 \\
0 \\
0
\end{array} \right) = \sum_{j=1}^{N} (Q_{j}^{Y} + Q_{j}^{Y}) ; \quad 2S_{j}S_{j+1}^{Y} ; \quad S_{j}S_{j+1}^{Y} + S_{j+1}S_{j}^{Y} + 2T_{j}T_{j+1}^{Y}.
\]

The generators of the algebra \( \mathfrak{u}(1|\mathfrak{p}) \) are given by relation (5) and the following:

\[
Q_{j} = (1 - n_{j})c_{j} ; \quad Q_{j}^{Y} = (1 - n_{j})c_{j}^{Y} ; \quad T_{j} = \frac{1}{2} n_{j}.
\]

The fundamental representations of these operators take the following form:

\[
S_{j}^{Y} = \left( \begin{array}{ccc}
2 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) ; \quad T_{j} = \frac{1}{2} n_{j}.
\]

where \( e_{ij}^{k} \) is a \( 3 \times 3 \) matrix acting on the \( k \)-th space with elements \( (e_{ij}^{k})_{i,j} = 1 \).

The above Hamiltonian can be obtained from the logarithmic derivative at zero spectral parameter of the transfer matrix constructed by the rational \( R \)-matrix. In this paper, we shall study the trigonometric \( R \)-matrix. Let’s start from the \( R \)-matrix of the Perk-Schultz model \([4,3]\), the non-zero entries of the \( R \)-matrix are given by

\[
R_{a}^{b} = \sin (a + b) ; \quad R_{a}^{b} = q_{ab} \sin (a) ; \quad R_{a}^{b} = e^{i \text{sign}(a, b) \sin (a)} ; \quad a \neq b.
\]

where

\[
\text{sign}(a, b) = \begin{cases} 1 & \text{if } a > b \\ 0 & \text{if } a = b \\ 1 & \text{if } a < b \end{cases}
\]
As mentioned above, \( a \) is the Grassmann parity, \( a = 0 \) for boson and \( a = 1 \) for fermion. We denote \( q_{ab}q_{ca} = 1 \), in the following, and let \( q_{ab} = ( )^a b \). This \( R \)-matrix of the Perk-Schultz model satisfies the usual Yang-Baxter equation:

\[
R_{12}( )R_{13}( )R_{23}( ) = R_{23}( )R_{13}( )R_{12}( ) \tag{13}
\]

Introducing a diagonal matrix \( r^a_{bc} = ( )^a b c \), we change the original \( R \)-matrix to the following form

\[
R( ) = IR( ) \tag{14}
\]

Considering the non-zero elements of the \( R \)-matrix \( R_{ab} \), we have \( a + b + c + d = 0 \). One can show that the \( R \)-matrix satisfies the graded Yang-Baxter equation

\[
R( )b_1b_2R( )c_1c_2R( )d_1d_2( )b_2 = R( )c_1c_2R( )b_1b_2R( )d_1d_2( )d_2c_2 \tag{15}
\]

In the framework of the QISM, we can construct the L operator from the \( R \)-matrix as:

\[
L_{aq}( )R_{aq}( ) \tag{16}
\]

where \( a \) represents the auxiliary space and \( q \) represents the quantum space. Thus we have the (graded) Yang-Baxter relation

\[
R_{12}( )L_1( )L_2( ) = L_2( )L_1( )R_{12}( ) \tag{17}
\]

Here the tensor product is in the sense of super tensor product defined as

\[
\mathcal{P} G^b_{ac} = \mathcal{P}^a_b G^d_c ( )^a b c \tag{18}
\]

In the rest of this paper, all tensor products are in the super sense. However, there are two kinds of super tensor product, we shall point it out later.

The row-to-row monodromy matrix \( T \) is defined as the matrix product over the \( N \) operators on all sites of the lattice,

\[
T_a( ) = L_{aN}( )L_{aN-1}( )a\mathcal{U}( ) \tag{19}
\]

where \( a \) still represents the auxiliary space, and the tensor product is in the graded sense. Explicitly we write

\[
f[T( )]^{ab}_{1N} = L_{N}( )^{c_a}_{a}L_{N-1}( )^{c_a}_{a} \cdots L_{1}( )^{c_a}_{a} \mathcal{P}_{j_1}^{b_1}( j_{N+1}^{1}) \mathcal{P}_{j_1}^{b_1}( j_{N+1}^{1}) \cdots \mathcal{P}_{j_1}^{b_1}( j_{N+1}^{1}) \tag{20}
\]

By repeatedly using the Yang-Baxter relation (17), one can prove easily that the \( m \) monodromy matrix also satisfies the Yang-Baxter relation

\[
R( )T_1( )T_2( ) = T_2( )T_1( )R( ) \tag{21}
\]
For periodic boundary condition, the transfer matrix $\text{peri}(\cdots)$ of this model is defined as the supertrace of the monodromy matrix in the auxiliary space. In general case, the supertrace is defined as

$$X_{\text{peri}} = \text{str} T(\cdots) = (1)^{T(\cdots)}_{\text{aa}}; \quad (22)$$

As a consequence of the Yang-Baxter relation (21) and the unitarity property of the $R$-matrix, we can prove that the transfer matrix commutes with each other for different spectral parameters.

$$\{ \text{peri}(\cdots); \text{peri}(\cdots) \} = 0 \quad (23)$$

Generally in this sense, we mean the model is integrable. Expanding the transfer matrix in the powers of $\omega$, we can find conserved quantities, the first non-trivial conserved quantity is the Hamiltonian.

For the rational $R$-matrix, it has been proved that the Hamiltonian obtained by taking the first logarithmic derivative at the zero spectral parameter, $H = \frac{\text{dlog}(\cdots)}{\text{d}}_{j=0} = \sum_{k=1}^{N} P_{k,k+1}$, is equivalent to the Hamiltonian of the supersymmetric $J$-model [3].

Here we shall study the trigonometric case. Noting $R_{ij}(0) = \sin(i)P_{ij}$, the Hamiltonian can be defined as:

$$H = \sin(i) \frac{\text{dln}(\cdots)}{\text{d}}_{j=0} = \sum_{j} X_{jj+1}; \quad (24)$$

with $H_{jj+1} = P_{jj+1} L_{jj+1} \theta (0)$.

As an example, we choose Fermionic, Fermionic and Bosonic (FFB) grading that means $1 = 2 = 1; 3 = 0$. Explicitly, we can write the $R$-matrix as:

$$R(\cdots) = \begin{cases} a(\cdots) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(\cdots) & 0 & c(\cdots) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b(\cdots) & 0 & 0 & 0 & c(\cdots) & 0 & 0 \\ 0 & c_{a}(\cdots) & 0 & b(\cdots) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a(\cdots) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(\cdots) & 0 & c(\cdots) & 0 \\ 0 & 0 & c_{a}(\cdots) & 0 & 0 & 0 & b(\cdots) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{a}(\cdots) & 0 & b(\cdots) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w(\cdots) & 0 \end{cases} \quad (25)$$

where

$$a(\cdots) = \sin(\cdots); \quad w(\cdots) = \sin(\cdots); \quad b(\cdots) = \sin(\cdots); \quad c(\cdots) = e^{i\sin(\cdots)}; \quad c(\cdots) = e^{i\sin(\cdots)}; \quad \quad (26)$$

The rational limit of this $R$-matrix is completely the same as the one used by Essler and Korepin in Ref.[3]. In the framework of the QISM, we define the $L$ operator as

$$L_{n}(\cdots) = \begin{cases} 0 & b(\cdots) & a(\cdots) \theta^{n}_{11} & c(\cdots) \theta^{n}_{21} & c(\cdots) \theta^{n}_{31} \\ c_{a}(\cdots) \theta^{n}_{12} & b(\cdots) & a(\cdots) \theta^{n}_{22} & c(\cdots) \theta^{n}_{32} & A \\ c_{a}(\cdots) \theta^{n}_{13} & c_{a}(\cdots) \theta^{n}_{23} & b(\cdots) & b(\cdots) & w(\cdots) \theta^{n}_{33} \end{cases} \quad (27)$$

Here $e^{n}_{ab}$ acts on the $n$-th quantum space.
We denote explicitly the row-to-row monodromy matrix as

\[ T(0) = \begin{pmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{pmatrix} \]

If we choose the FFB grading, the transfer matrix is then given as

\[ (\text{peri}) = A_{11} + A_{22} + D \]

Thus we can write

\[ L(0) \begin{pmatrix} 1 \\ (1 \cos(\theta))e_{11} \\ \sin(\theta)e_{11} \\ \sin(\theta)e_{21} \\ \sin(\theta)e_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \sin(\theta)e_{12} \\ \sin(\theta)e_{22} \\ \sin(\theta)e_{32} \end{pmatrix} \]

With the help of the fundamental representation of algebra \( u(1|\mathcal{F}) \), we have

\[ H_{j+j+1} = \sum_{j'=1}^{X} \left[ Q_{j'} \{ \begin{array}{c} 1_n \ 1_n \\ \mathcal{y}_{j+1} \ \mathcal{y}_{j+1} \end{array} \} + Q_{j'} \{ \begin{array}{c} 1_n \ 1_n \\ \mathcal{y}_{j+1} \ \mathcal{y}_{j+1} \end{array} \} \right] + \sum_{j'=1}^{X} \left[ S_j S_{j+1} + S_j S_{j+1} + \cos(\theta) \right] \]

As mentioned in introduction, this Hamiltonian was previously obtained by Karowski and Feser and by Gonzales-Ruiz. Explicitly, using the fermionic representation (5) and (9), we can write the Hamiltonian of the generalized supersymmetric J model as following:

\[ H = \sum_{j=1}^{X} \left[ c_{j+1} \{ \begin{array}{c} 1_n \ 1_n \\ \mathcal{y}_{j+1} \ \mathcal{y}_{j+1} \end{array} \} + c_{j+1} \{ \begin{array}{c} 1_n \ 1_n \\ \mathcal{y}_{j+1} \ \mathcal{y}_{j+1} \end{array} \} \right] + \sum_{j=1}^{X} \left[ \frac{1}{2} S_j S_{j+1} + S_j S_{j+1} + \cos(\theta) \frac{1}{4} n_j n_{j+1} \right] \]

\[ + \sin(\theta) \left[ S_j n_{j+1} + S_{j+1} n_j \right] \]

Here periodic boundary condition is assumed. We remark that this Hamiltonian is in general not Hermitian.

In this paper, we shall study the reflecting boundary conditions, which may cause non-trivial boundary terms in the Hamiltonian.

### 3 Integrable reflecting boundary conditions and the solutions of reflection equation

In this paper, we consider the reflecting boundary condition case. In the end of 80's, Sklyanin proposed a systematic approach to handle the exactly solvable models with reflecting (open) boundary conditions, which includes a so-called reflection equation proposed by Cherednik.
For the graded case, the above form of the reflection equation remains the same. We only need to change the usual tensor product to the graded tensor product [13]. We write it explicitly as

$$R((b_1 b_2)_{a_1 a_2} K((c_1 c_2)_{b_1 b_2} R((d_1 d_2)_{c_1 c_2}((e_1 e_2)_{a_1 a_2}))) = K((a_1 a_2) R((b_1 b_2) K((c_1 c_2) R((d_1 d_2) ((e_1 e_2)_{a_1 a_2}))))$$

We concentrate the discussion to the diagonal solutions of the reflection equation. Suppose $K((a)_{b} = a b k_a()$). Inserting this relation into the reflection equation, we end there are only one non-trivial relation to be solved:

$$R((b_1 b_2)_{a_1 a_2} R((c_1 c_2)_{a_1 a_2} k((a)_1)_{a_1} + R((d_1 d_2)_{a_1 a_2} R((e_1 e_2)_{a_1 a_2} k((a)_2)_{a_2}) = R((b_1 b_2) R((c_1 c_2)_{a_1 a_2} k((a)_1)_{a_1} + R((d_1 d_2)_{a_1 a_2} R((e_1 e_2)_{a_1 a_2} k((a)_2)_{a_2}$$

Suppose $a_2 > a_1$, and substitute the exact form of the elements of the matrix into the above relation. We end a general diagonal solution:

$$\frac{k((a)_1)}{k((a)_2)} = \frac{\sin(+e^{2i})}{\sin(\theta)} e^{2i};$$

where $\theta$ is an arbitrary parameter. In a special limit we can see the identity is also a solution of the reflection equation. For the cases (FFB, BFF and FBF grading) we study in this paper, there are two types of solutions to the reflection equation

$$K_1() = \begin{pmatrix} 0 & \sin(+e^{2i}) & A \\ \sin(\theta) & 1 & \sin(\theta) \\ 0 & \sin(+e^{2i}) & A \end{pmatrix}$$

$$K_{12}() = \begin{pmatrix} 0 & \sin(\theta) & A \\ \sin(\theta) & 1 & \sin(\theta) \\ 0 & \sin(\theta) & A \end{pmatrix}$$

Instead of the monodromy matrix $T(\theta)$ for periodic boundary conditions, we consider the double-row monodromy matrix

$$T(\theta) = T(\theta) K(\theta) T^{-1}(\theta)$$

for the reflecting boundary conditions. Using the Yang-Baxter relation, and considering the boundary $K$-matrix which satisfies the reflection equation, one can prove that the double-row monodromy matrix $T(\theta)$ also satisfies the reflection equation

$$R((b_1 b_2)_{a_1 a_2} T((c_1 c_2)_{b_1 b_2} R((d_1 d_2)_{c_1 c_2} T((e_1 e_2)_{a_1 a_2}))) = T((b_1 b_2) R((c_1 c_2)_{a_1 a_2} T((d_1 d_2)_{c_1 c_2} R((e_1 e_2)_{a_1 a_2}))))$$

Next, we shall study the properties of the $R$-matrix. We define the super-transposition as

$$(A^{st})_{ij} = A_{jk}(1^{j+1})$$

As an example, we take the FFB grading, that means $1 = 2 = 1; 3 = 0$. We can rewrite the above relation explicitly as

$$\begin{pmatrix} 0 & A_{11} & A_{12} & B_1 & 1 \\ 0 & A_{11} & A_{21} & C_1 & 1 \\ \theta A_{21} & A_{22} & B_2 & A & 0 \\ \theta A_{21} & A_{22} & B_2 & C_2 & A \\ C_1 & C_2 & D & B_1 & B_2 & D \end{pmatrix}$$

\[ (41) \]
We also define the inverse of the super-transposition as $f A^{ST} g^{ST} = A$.

For the $R$-matrix with all three different grading, FFB, BFF and FBF, we can prove directly that the $R$-matrix satisfy the following unitarity and cross-unitarity relations:

\begin{align}
R_{12} ( )R_{21} ( ) &= ( ) \text{id} ; \quad ( ) = \sin ( + ) \sin ( - ) ; \\
R_{12}^{ST} ( )M_1 R_{21}^{ST} ( )M_1^{-1} &= ( ) \text{id} ; \quad ( ) = \sin ( + ) \sin ( - ) ;
\end{align}

Here the matrix $M$ is diagonal and is determined by the $R$-matrix. For three different gradings, the form of $M$ are different. We have: $M = \text{diag}(e^{2i}; 1; 1)$ for FFB grading, $M = \text{diag}(1; 1; e^{2i})$ for BFF grading and $M = 1$ for FBF grading.

In order to construct the commuting transform matrix with boundaries, besides the re-fection equation, we need the dual re-fection equation. Generally, the dual re-fection equation which depends on the unitarity and cross-unitarity relations of the $R$-matrix takes different forms for different models. For the models considered in this paper, the cross-unitarity relation remains the same for three different backgrounds. We can write the dual re-fection equation in the following form:

\begin{equation}
R_{12} ( )K_1^+ ( )M_1^{-1} R_{21} ( )K_2^+ ( )M_2^{-1} = K_2^+ ( )M_2^{-1} R_{12} ( )K_1^+ ( )M_1^{-1} R_{21} ( ) ;
\end{equation}

One finds that there is an isomorphism between the re-fection equation (33) and the dual re-fection equation (44):

\begin{equation}
K^+ ( ) = K^+ ( ) = M K ( + =2) ;
\end{equation}

Here we mean: given a solution of the re-fection equation (33), we can find a solution of the dual re-fection equation (44). Note, however, that in the sense of the commuting transform matrix, the re-fection equation and the dual re-fection equation are independent of each other.

The transform matrix with boundaries is defined as:

\begin{equation}
t ( ) = \text{str} K^+ ( ) T ( ) ;
\end{equation}

The commutativity of $t ( )$ can be proved by using unitarity and cross-unitarity relations, re-fection equation and the dual re-fection equation. The detailed proof of the commuting transfer matrix with boundaries for super (graded) case can be found, for instance, in Ref. [27, 23, 42, 43] etc.

We also define the Hamiltonian by a relation

\begin{equation}
H = \frac{1}{2} \sin ( ) \frac{d \text{ln} t ( )}{d} \bigg|_{j=0} \frac{K^1}{\prod_{j=1} \sin ( )} \left[ L^0 \left( 0 \right) + \frac{1}{2} \sin ( ) \frac{K^0}{\sin ( )} \right] + \frac{\text{str} K^+_a (0) P_{N_a} L^0}{\text{str} K^+_a (0)} ;
\end{equation}

We still take the FFB grading as an example, and thus $M = \text{diag}(e^{2i}; 1; 1)$. We have two types of the solutions to the dual re-fection equation

\begin{align}
K_1^+ ( ) &= \theta \left[ \sin ( + ) e^{i2} + \right] \sin ( + ) e^{i\zeta} \quad \zeta = A ; \\
K_1^+ ( ) &= \theta \left[ \sin ( + ) e^{i2} + \right] \sin ( + ) e^{i\zeta} \quad A ;
\end{align}
\begin{align*}
K_{11}^{+} &= 0 \sin(\theta \, +) e^{i(2 \theta +)} \\
&\quad \sin(\theta \, +) A \\
&\quad \sin(\theta \, +) \quad (48)
\end{align*}

where \( \theta \) is also an arbitrary boundary parameter. Since the reaction equation and the dual reaction equation are independent of each other, there are altogether four different types of boundary matrices: \( fK_1; K_1^{+} g, fK_1; K_1^{*} g, K_1; K_1^{+} g, K_1; K_1^{*} g. \)

The Hamiltonian of the generalized supersymmetric \( J \) model with boundaries is written as

\begin{equation}
H = \sum_{j=1}^{N-1} \left[ \sum_{j} \left( X_{ij} \left( 1 \quad n_{j+1} \right) c_{j+1,1} \left( 1 \quad n_{j+1} \right) + c_{j+1,1}^{*} \left( 1 \quad n_{j+1} \right) c_{j+1,1} \left( 1 \quad n_{j} \right) \right) \\
+ \frac{2}{j} \left( S_{j} S_{j+1} + S_{j} S_{j+1}^{*} \right) \cos(\theta) S_{j} S_{j+1} \right] + \sin(\theta) \left( S_{j} S_{j+1}^{*} \right) 2 \cos(\theta) n_{j} + e^{i} n_{j} + e^{i} n_{j} + H_{1} + H_{N} \right] \\
\end{equation}

where \( H_{1} \) and \( H_{N} \) are determined by the reacting matrices. Explicitly, they are

\begin{align*}
H_{1}^{I} &= \frac{\sin(\theta)}{2 \sin(\theta \, +)} e^{i \theta} n_{1} \quad H_{1}^{II} = \frac{\sin(\theta)}{2 \sin(\theta \, +)} e^{i \theta} S_{1}^{+} \\
H_{N}^{I} &= \frac{\sin(\theta)}{2 \sin(\theta \, +)} e^{i \theta} n_{N} \quad H_{N}^{II} = \frac{\sin(\theta)}{2 \sin(\theta \, +)} e^{i \theta} S_{N}^{+} \quad (50)
\end{align*}

We remark that there are four types of boundary terms in the Hamiltonian.

The solution of the graded reaction equation is identical to that of the non-graded reaction equation, because we focus our attention on the diagonal solutions of the reaction equation, and the two cases for graded and non-graded is completely the same. The solution of the dual reaction equation for FFB case is similar to the non-graded case in Ref.23 except a minus in the last diagonal elements. And the boundary terms appeared in the Hamiltonian (49,50) are similar to the previous results23 (the anisotropic parameter should be redone as ).

4 Algebraic Bethe ansatz method for FFB grading

In this section, the FFB grading is assumed. We shall use the nested algebraic Bethe ansatz method to obtain the eigenvalues of the transfer matrix with boundaries described above.

4.1 Commutation relations necessary for the algebraic Bethe ansatz method

We write solution of the dual reaction equation \( K^{+} \) and the double-row monodromy matrix \( T \) respectively in the following form:

\begin{align*}
K^{+} &= \text{diag} \left( k_{1}^{+} ; k_{2}^{+} ; k_{3}^{+} \right) \\
&\quad A_{11} \quad A_{12} \quad B_{1} \\
T &= \begin{pmatrix}
A_{21} \\
A_{22} \\
C_{1} \\
C_{2} \\
D
\end{pmatrix}
\end{align*}

(51)
Instead of $A_{ab}$, we shall use $A'_{ab}$ in the algebraic Bethe ansatz method so that there will exist only one type wanted term in the commutation relation. The transformation takes the form

$$A_{ab} = A'_{ab} + \frac{e^{2i\sin (\theta)}}{\sin (2\theta)}D_{ab};$$

(53)

So, the transformation with boundaries can be rewritten as

$$t() = k_1^+ (A_{11}) + k_2^+ (A_{22}) + k_3^+ (D) = k_1^+ (A'_{11}) + k_2^+ (A'_{22}) + U_3^+ (D);$$

(54)

where

$$U_3^+ = k_3^+ \frac{e^{2i\sin (\theta)}}{\sin (2\theta)}k_1^+ + k_2^+);$$

(55)

For type I, II solutions of the dual reflection equation $K^+$, we have

$$U_3^+ = \frac{\sin (\theta)\sin (\theta + \theta)}{\sin (\theta + \theta)}; \quad \text{for } K^+_1;$$

(56)

$$U_3^+ = \frac{\sin (\theta)\sin (\theta + \theta e^{i\theta})}{\sin (\theta + \theta)}; \quad \text{for } K^+_2;$$

(57)

As mentioned above, the double-row monodromy matrix also satisfies the graded reflection equation (39). Setting the indices in that relation to be special values, we can find the commutation relations which are necessary for the algebraic Bethe ansatz method. The detailed calculation is tedious and complicated, so we do not present it here. The result is

$$C_{d_i} (C_{d_i}) = \frac{r_{12}(\theta)_{d_i d_i}}{\sin (\theta + \theta)}C_{c_i} (C_{c_i});$$

(58)

$$D (C_d) = \frac{\sin (\theta)\sin (\theta \theta)}{\sin (\theta \theta)}C_d (D) + \frac{\sin (\theta)\sin (\theta)e^{i\theta}}{\sin (\theta \theta)}C_d (D) = \frac{\sin (\theta)\sin (\theta \theta)}{\sin (\theta \theta)}C_d (D) + \frac{\sin (\theta)\sin (\theta \theta)}{\sin (\theta \theta)}C_d (D) + \frac{\sin (\theta)\sin (\theta \theta)}{\sin (\theta \theta)}C_d (D);$$

(59)

$$K_{a_1 d_i} (C_{d_i}) = \frac{r_{12}(\theta)_{a_1 d_i}}{\sin (\theta + \theta)}C_{c_i} (K_{c_i d_i}) + \frac{\sin (\theta)\sin (\theta \theta)}{\sin (\theta \theta)}r_{12}(\theta)_{a_1 d_i}C_{c_i} (K_{c_i d_i}) + \frac{\sin (\theta)\sin (\theta \theta)}{\sin (\theta \theta)}r_{12}(\theta)_{a_1 d_i}C_{c_i} (K_{c_i d_i});$$

(60)

Here the indices take values 1, 2, and the r-m matrix is defined as

$$r_{12}(\theta) = \frac{b_{12}(\theta)_{a_1 b_1}}{\sin (\theta + \theta)}C_{c_1} (A_{c_1 d_1}) + \frac{\sin (\theta)\sin (\theta \theta)}{\sin (\theta \theta)}r_{12}(\theta)_{a_1 b_1}C_{c_1} (A_{c_1 d_1}) + \frac{\sin (\theta)\sin (\theta \theta)}{\sin (\theta \theta)}r_{12}(\theta)_{a_1 b_1}C_{c_1} (A_{c_1 d_1});$$

(61)

In fact, the elements of the r-m matrix are equal to those of the original R-m matrix when its indices just take values 1, 2.
4.2 Vacuum State

According to the definition of the double-row monodromy matrix, we write it explicitly as

\[
T \rightarrow T = \begin{pmatrix}
0 & A_{11} & A_{12} & B_1 \\
0 & A_{21} & A_{22} & B_2 & A \\
C_1 & C_2 & D &
\end{pmatrix}
\]

where we have used the unitarity relation of the R-matrix and a whole factor is omitted.

By definition, the reference state in the n-th quantum space and the vacuum |0\rangle as:

\[
|0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

By use of the definition of the row-to-row monodromy matrix (19) and its inverse (63), we have

\[
A_{ab} |\psi\rangle = a b \sin N |\psi\rangle; \quad \text{D} (|\psi\rangle) = \sin N (|\psi\rangle + |\psi\rangle);
\]

\[
B_a (|\psi\rangle) = 0; \quad C_a (|\psi\rangle) = 6 |\psi\rangle.
\]

So, with the help of T's definition relation (62), we can show that

\[
\text{D} (|\psi\rangle) = k_3 \sin^{2N} (|\psi\rangle + |\psi\rangle);
\]

\[
\text{K}_{\text{ab}} (|\psi\rangle) = 0; \quad a b \in \mathbb{N}
\]

To obtain the actions of operator K_{ab} on the vacuum state, we use the following relation obtained from the Yang-Baxter relation

\[
[T^{-1} (|\psi\rangle)]_{a_1 a_2} R (|\psi\rangle)_{a_1 b_2} T (|\psi\rangle)_{b_1 c_2} T^{-1} (|\psi\rangle)_{c_1 b_2} = T (|\psi\rangle)_{a_1 b_2} R (|\psi\rangle)_{a_1 b_2} T^{-1} (|\psi\rangle)_{c_1 b_2} T^{-1} (|\psi\rangle)_{c_1 b_2}:
\]

(68)
Actually, we have already used \( \pm \) to obtain the results \( A_{ab} (t) \mathcal{D} = 0; \ a \neq b \).

Then, we have

\[
A_{11} (t) \mathcal{D} = k_1 (t) \left( \frac{\sin \theta}{\sin \left( \theta + \frac{\pi}{2} \right)} \right) \mathcal{D} = W_1 (t) \mathcal{D} > 0.
\]

(69)

For case I, II rectifying K-matrix, we have a same \( W_1 \) which takes the form:

For \( K_1 \) and \( K_11 \):

\[
W_1 (t) = e^{2it} \left( \frac{\sin \theta \sin \left( \theta + \frac{\pi}{2} \right)}{\sin \left( \theta + \frac{\pi}{2} \right)} \right).
\]

(70)

Similarly, we have

\[
A_{22} (t) \mathcal{D} = k_2 (t) \left( \frac{\sin \theta}{\sin \left( \theta + \frac{\pi}{2} \right)} \right) \mathcal{D} = W_2 (t) \mathcal{D} > 0.
\]

(71)

For case I, II rectifying K-matrix, \( W_2 \) take the following form:

For \( K_1 \):

\[
W_2 (t) = e^{2it} \left( \frac{\sin \theta \sin \left( \theta + \frac{\pi}{2} \right)}{\sin \left( \theta + \frac{\pi}{2} \right)} \right).
\]

(72)

For \( K_{11} \):

\[
W_2 (t) = e^{it} \left( \frac{\sin \theta \sin \left( \theta + \frac{\pi}{2} \right)}{\sin \left( \theta + \frac{\pi}{2} \right)} \right).
\]

(73)

4.3 Bethe ansatz

We construct a set of the eigenvectors of the transfer matrix with rectifying boundary conditions as

\[
C_{d_1} (t_1)C_{d_2} (t_2) \mathcal{D} > F^{d_1} \mathcal{D} > F^{d_2} \mathcal{D} > d_e \mathcal{D}.
\]

(74)

Here \( F_d \) is a function of the spectral parameters \( \lambda \). Acting the transfer matrix on this eigenvectors, we find the eigenvalues \( \lambda \) of the transfer matrix \( t \) and a set of Bethe ansatz equations. This technique is standard for the algebraic Bethe ansatz method. Acting \( D \) on the eigenvector defined above, use next the commutation relation (59), consider the value of \( D \) acting on the vacuum state (67). Then we have

\[
D (t)C_{d_1} (t_1)C_{d_2} (t_2) \mathcal{D} > F^{d_1} \mathcal{D} > F^{d_2} \mathcal{D} > d_e \mathcal{D} > F^{d_1} \mathcal{D} > d_e \mathcal{D} = \frac{1}{\sin \left( \theta + \frac{\pi}{2} \right)} \epsilon_1 (1 + \cdots + \epsilon_1 C_{d_1} (t_1)C_{d_2} (t_2) \mathcal{D}) = \frac{1}{\sin \left( \theta + \frac{\pi}{2} \right)} \epsilon_1 \mathcal{D}.
\]

(75)

where \( u \) means the unwanted term \( s \).

We act \( A_{ab} (t) \) on the assumed eigenvector (74). Using repeatedly the commutation relations (60), we have

\[
A_{ab} (t_1)C_{d_1} (t_1)C_{d_2} (t_2) \mathcal{D} = F^{d_1} \mathcal{D} > F^{d_2} \mathcal{D} > d_e \mathcal{D} = \frac{1}{\sin \left( \theta + \frac{\pi}{2} \right)} \epsilon_1 \mathcal{D}.
\]

(76)
Summarizing relations (67, 69, 71), we obtain
\[
A_{a, b, n}(\mathcal{D}) = a_n b_n W_{a, n}(\mathcal{D}) \sin^{2L}(\mathcal{D}) > 0 \tag{77}
\]
We can rewrite the transfer matrix as
\[
t(\mathcal{D}) = k_1^t(\mathcal{A}^{11}(\mathcal{D})) + k_2^t(\mathcal{A}^{22}(\mathcal{D}) + U_1^t(\mathcal{D}) > 0 \tag{78}
\]
Thus the eigenvalue of the transfer matrix with re-acting boundary condition is written as
\[
t(\mathcal{D}) = U_1^t(\mathcal{D}) \sin^{2N}(\mathcal{D}) > 0 \tag{79}
\]
where \(t^{(1)}(\mathcal{D})\) is the so-called nested transfer matrix, and with the help of the relation (76), it can be defined as
\[
t^{(1)}(\mathcal{D}) = d''_{n, d} \times_{s} \times_{a, c} = \begin{cases}
kn_{n, d}(\mathcal{D}) \sin^{2N}(\mathcal{D}) > 0 \\
\sin^{2N}(\mathcal{D}) > 0
\end{cases} \tag{79}
\]
We find that this nested transfer matrix can be defined as a transfer matrix with re-acting boundary conditions corresponding to the anisotropic case
\[
t^{(1)}(\mathcal{D}) = \sin^{2N}(\mathcal{D}) > 0 \tag{81}
\]
with the grading \(k = 2 = 1\). Here, we denote \(\mathcal{D} = +\mathbf{i}, \mathcal{E} = +\mathbf{j} \mapsto +\mathbf{k}, \mathcal{E}^* = +\mathbf{k} \mapsto -\mathbf{j}, \mathcal{D} = -\mathbf{i}, \mathcal{E} = -\mathbf{j} \mapsto -\mathbf{k}, \mathcal{E}^* = -\mathbf{k} \mapsto +\mathbf{j}\), and the same notation will be used, for instance, \(\mathcal{E}^* \mapsto +\mathbf{j}, \mathcal{E} = -\mathbf{j} \mapsto -\mathbf{k}, \mathcal{E}^* = -\mathbf{k} \mapsto +\mathbf{j}\). Explicitly we have
\[
K^{(1)}(\mathcal{D}) = \sin^{2N}(\mathcal{D}) > 0 \tag{82}
\]
and
\[
K^{(1)}(\mathcal{D}) = \sin^{2N}(\mathcal{D}) > 0 \tag{83}
\]
Corresponding to \(K^{(1)}_i(\mathcal{D})\) and \(K^{(1)}_j(\mathcal{D})\), respectively. We also have
\[
K^{(1)}_i(\mathcal{D}) = e^{i(2\mathbf{i})} \sin(2\mathbf{i}) \sin(2\mathbf{j}) > 0 \tag{84}
\]
\[
K^{(1)}_j(\mathcal{D}) = e^{i(2\mathbf{j})} \sin(2\mathbf{j}) \sin(2\mathbf{i}) > 0 \tag{85}
\]
corresponding to \( K_1 \) and \( K_{11} \). The row-to-row monodromy matrix \( T^{(1)}(\tau; \gamma) \) (corresponding to the periodic boundary condition) is defined as

\[
T^{(1)}_{abc} (\tau; \gamma) = 2 \sin(z_1) \sin(z_2) \sin(z_3) \sin(z_4)
\]

The L-operator takes the form

\[
L^{(1)}_{k} = \begin{pmatrix} b(\tau) & a(\tau) \end{pmatrix} e^{i \frac{\pi}{2}} \begin{pmatrix} c(\tau) e^{i \frac{\pi}{2}} \end{pmatrix} \begin{pmatrix} b(\tau) \end{pmatrix} a(\tau) e^{i \frac{\pi}{2}}
\]

And we also have

\[
T^{(1)}_{k} (\tau; \gamma) = 2 \sin(z_1) \sin(z_2) \sin(z_3) \sin(z_4)
\]

where we have used the unitarity relation of the \( r \)-matrix \( r_{12}(\tau) r_{21}(\tau) = \sin(\theta) \sin(\theta') \) and.

In this section, we show that a problem to nd the eigenvalue of the original transfer matrix \( T(\tau) \) reduces to a problem to nd the eigenvalue of the nested transfer matrix \( T^{(1)}(\tau) \). In relation (79), one can see that besides the wanted term which gives the eigenvalue, we also have the unwanted terms which must be cancelled so that the assumed eigenvector is indeed the eigenvector of the transfer matrix. With the help of the symmetry property (58) of the assumed eigenvector (74), we nd that, if \( r_{1; n} \) satisfies the following Bethe ansatz equations, the unwanted term will vanish.

\[
U_{j}^{(1)}(\tau) k_{3}(\tau) \sin^{2N}(j + \theta) \sin(j + \theta) \sin(j - \theta) = \sin^{2N}(j) (j ; 1) (j ; 2)
\]

where we have used the notation \( (\.\) to denote the eigenvalue of the nested transfer matrix \( T^{(1)}(\tau) \).

Thus what we should do next is to nd the eigenvalue of the nested transfer matrix \( T^{(1)}(\tau) \).

4.4 The nested algebraic Bethe ansatz method

We expect that the eigenvalue of the nested transfer matrix can be solved similarly as that of the original transfer matrix. So, we should first prove that the above de ned nested transfer matrix indeed constitutes a commuting family. Note that the grading is \( g_1 = g_2 = 1 \). Actually, because all grading is Fermionic, the graded method is simply the same as the usual one.

We note that the \( r \)-matrix satisfies the unitarity and and cross-unitarity relations:

\[
r_{12}(\tau) r_{21}(\tau) = \sin(\theta) \sin(\theta') \text{id};
\]

\[
r^{(1)}_{12}(\tau) r^{(1)}_{21}(\tau) = \sin(\theta) \sin(\theta') \text{id};
\]

The matrix \( M^{(1)} \) is a diagonal matrix, \( M^{(1)} = \text{diag}(e^{2\theta j}; 1) \).

In order to prove the commutativity of the nested transfer matrices, we need the reflection equation and the dual reflection equation, which take the following form:

\[
r_{12}(\tau) K^{(1)}_{1}(\tau) r_{21}(\tau + \eta) K^{(1)}_{2}(\tau) = K^{(1)}_{2}(\tau) r_{12}(\tau + \eta) K^{(1)}_{1}(\tau) r_{21}(\tau);
\]
By a direct calculation, we can prove that the above defined rectifying matrices $K_1^{(1)}$ and $K_2^{(1)}$ satisfy the rectifying equation, and also $K_1^{(1)^+}$, $K_2^{(1)^+}$ satisfy the dual rectifying equation.

We know that the following graded Yang-Baxter relation is satisfied:

$$r(\ )L_1^{(1)}(\ )L_2^{(1)}(\ ) = L_2^{(1)}(\ )L_1^{(1)}(\ )r(\ )$$  \hspace{1cm} (94)

Therefore, we also have the Yang-Baxter relation for the row-to-row monodromy matrix

$$r(\ )T_2^{(1)}(\ ;f\ ,g)T_2^{(1)}(\ ;f\ ,g) = T_2^{(1)}(\ ;f\ ,g)T_2^{(1)}(\ ;f\ ,g)r(\ )$$  \hspace{1cm} (95)

Since we already know $K^{(1)}$ satisfy the rectifying equation (92), we can show that the nested double-row monodromy matrix

$$T^{(1)}(\ ;f\ ,g)T^{(1)}(\ ;f\ ,g)K^{(1)}(\ )T^{(1)}(\ ;f\ ,g)$$

also satisfy the the rectifying equation

$$r_{22}(\ )T_2^{(1)}(\ ;f\ ,g)r_{21}(\ ) + T_2^{(1)}(\ ;f\ ,g)r_{21}(\ ) = T_2^{(1)}(\ ;f\ ,g)r_{21}(\ ) + T_2^{(1)}(\ ;f\ ,g)r_{21}(\ )$$  \hspace{1cm} (97)

Parallel to the procedures presented above, with the help of unitarity, cross-unitarity relations, and rectifying equation, dual rectifying equation, one can prove the de nested transfer matrix indeed constitutes a commuting family.

Now, let us use again the algebraic Bethe ansatz method to obtain the eigenvalue $(t^{(1)}(\ ))$ of the nested transfer matrix $t^{(1)}(\ )$. We write the nested double-row monodromy matrix as

$$T^{(1)}(\ ;f\ ,g) = \begin{pmatrix} A^{(1)}(\ ) & B^{(1)}(\ ) \\ C^{(1)}(\ ) & D^{(1)}(\ ) \end{pmatrix}$$

$$= T^{(1)}(\ ;f\ ,g)K^{(1)}(\ )T^{(1)}(\ ;f\ ,g)$$

$$= \begin{pmatrix} A^{(1)}(\ ) & B^{(1)}(\ ) \\ C^{(1)}(\ ) & D^{(1)}(\ ) \end{pmatrix}$$

$$= \begin{pmatrix} A^{(1)}(\ ) & B^{(1)}(\ ) \\ C^{(1)}(\ ) & D^{(1)}(\ ) \end{pmatrix} k^{(1)}(\ )$$

$$= \begin{pmatrix} A^{(1)}(\ ) & B^{(1)}(\ ) \\ C^{(1)}(\ ) & D^{(1)}(\ ) \end{pmatrix} k^{(1)}(\ )$$

$$= \begin{pmatrix} A^{(1)}(\ ) & B^{(1)}(\ ) \\ C^{(1)}(\ ) & D^{(1)}(\ ) \end{pmatrix} k^{(1)}(\ )$$

$$= \begin{pmatrix} A^{(1)}(\ ) & B^{(1)}(\ ) \\ C^{(1)}(\ ) & D^{(1)}(\ ) \end{pmatrix} k^{(1)}(\ )$$

$$= \begin{pmatrix} A^{(1)}(\ ) & B^{(1)}(\ ) \\ C^{(1)}(\ ) & D^{(1)}(\ ) \end{pmatrix} k^{(1)}(\ )$$

$$= \begin{pmatrix} A^{(1)}(\ ) & B^{(1)}(\ ) \\ C^{(1)}(\ ) & D^{(1)}(\ ) \end{pmatrix} k^{(1)}(\ )$$

For convenience, we introduce again a transformation

$$A^{(1)}(\ ) = K^{(1)}(\ ) \cdot \frac{\sin(\ ) e^{2i}}{\sin(2 \ )} D^{(1)}(\ );$$  \hspace{1cm} (99)

Because the nested double-row monodromy matrix satisfies the rectifying equation (97), we can find the following commutation relations:

$$D^{(1)}(\ )C^{(1)}(\ ) = \frac{\sin(\ ) e^{2i}}{\sin(2 \ )} C^{(1)}(\ )D^{(1)}(\ )$$

$$= \frac{\sin(\ ) e^{2i}}{\sin(2 \ )} C^{(1)}(\ )D^{(1)}(\ ) + \frac{\sin(\ ) e^{2i}}{\sin(2 \ )} C^{(1)}(\ )K^{(1)}(\ );$$  \hspace{1cm} (100)
As the reference states for the nesting, we choose
\[ \mathcal{D} > (1) = 0 \quad ; \quad \mathcal{D} > (1) = \sum_{k=1}^{n} \mathcal{D} > (1) ; \quad (103) \]

With the help of the definition (86, 88), we know the actions of the nested monodromy matrix and the inverse of the monodromy matrix on the reference state

\[
T^{(1)} (g) \mathcal{D} > (1) = A^{(1)} (g) \mathcal{D} > (1) ; \\
= \begin{cases} 
A^{(1)} (g) & B^{(1)} (g) \\
C^{(1)} (g) & D^{(1)} (g)
\end{cases} \mathcal{D} > (1) ; \\
= Q_{n} \sin \left( + \frac{i}{1} \right) \mathcal{D} > (1) ; \\
= Q_{n} \sin \left( - \frac{i}{1} \right) \mathcal{D} > (1) ; \\
(104) \]

\[
T^{-1} (g) \mathcal{D} > (1) = A^{(1)} (g) \mathcal{D} > (1) ; \\
= \begin{cases} 
A^{(1)} (g) & B^{(1)} (g) \\
C^{(1)} (g) & D^{(1)} (g)
\end{cases} \mathcal{D} > (1) ; \\
= Q_{n} \sin \left( - \frac{i}{1} \right) \mathcal{D} > (1) ; \\
= Q_{n} \sin \left( + \frac{i}{1} \right) \mathcal{D} > (1) ; \\
(105) \]

Repeating almost the same calculation in the former sections, we obtain the results of the nested double-row monodromy matrix acting on the nested vacuum state \( \mathcal{D} > (1) \),

\[ B^{(1)} (g) \mathcal{D} > (1) = 0 ; \quad C^{(1)} (g) \mathcal{D} > (1) = 0 ; \quad (106) \]

\[ D^{(1)} (g) \mathcal{D} > (1) = U_{2}(g) \sum_{i=1}^{n} \sin \left( + \frac{i}{1} \right) \sin \left( - \frac{i}{1} \right) \mathcal{D} > (1) ; \quad (107) \]

Here we use the notation \( U_{2} = k_{2}^{(1)} \),

\[ U_{2}(g) = e^{i \frac{2}{2} \sin \left( - \frac{g}{2} \right) \sin \left( + \frac{g}{2} \right)} \quad (108) \]

for \( K \) case.

\[ U_{2}(g) = e^{i \frac{2}{2} \sin \left( - \frac{g}{2} \right) \sin \left( + \frac{g}{2} \right)} \quad (109) \]
for $K_{II}$ case.

Using the Yang-Baxter relation, we also have

$$A^{(1)}(\gamma) \mathcal{D}^{(1)} = k_1^{(1)}(\gamma) A^{(1)}(\gamma) A^{(1)}(\gamma) \mathcal{D}^{(1)} + k_2^{(1)}(\gamma) \frac{b(\gamma)}{a(\gamma)} A^{(1)}(\gamma) A^{(1)}(\gamma) \mathcal{D}^{(1)}$$

$$= \left[ k_1^{(1)}(\gamma) + k_2^{(1)}(\gamma) \frac{\sin(\gamma) e^{2i}}{\sin(\gamma)} \right] \mathcal{D}^{(1)} \prod_{i=1}^{\infty} \left[ \sin(\gamma) + \sin(-\gamma) \right] \mathcal{D}^{(1)}$$

With the help of the transformation (99), we find

$$K^{(1)}(\gamma) \mathcal{D}^{(1)} = U_1(\gamma) \left[ \prod_{i=1}^{\infty} \sin(\gamma) + \sin(-\gamma) \right] \mathcal{D}^{(1)}$$

where we denote

$$U_1(\gamma) = k_1^{(1)}(\gamma) + k_2^{(1)}(\gamma) \frac{\sin(\gamma) e^{2i}}{\sin(\gamma)}$$

Here $U_1$ takes the following form explicitly: For $K_{I}$, $U_1(\gamma) = e^{2i} \sin(\gamma)$; For $K_{II}$, $U_1(\gamma) = e^{i\gamma} \sin(\gamma)$.

The nested transformation takes the form

$$U^{(1)}(\gamma) = U_1(\gamma) A(\gamma) \mathcal{D}(\gamma)$$

$$= \left[ U_1^{(1)}(\gamma) A(\gamma) \right] D(\gamma)$$

where we denote $U_1^{(1)} = k_1^{(1)}$,

$$U_2^{(1)}(\gamma) = k_2^{(1)} \frac{\sin(\gamma) e^{2i}}{\sin(\gamma)} k_1^{(1)}$$

that means:

For $K_{I}$ case:

$$U_1^+(\gamma) = \sin(\gamma e^{i\gamma})$$

$$U_2^+(\gamma) = \frac{\sin(\gamma e^{i\gamma})}{\sin(\gamma e^{i\gamma})} e^{i\gamma}$$

For $K_{II}$ case:

$$U_1^+(\gamma) = \sin(\gamma e^{i\gamma}) e^{2i\gamma}$$

$$U_2^+(\gamma) = \frac{\sin(\gamma e^{i\gamma}) \sin(\gamma e^{i\gamma})}{\sin(\gamma e^{i\gamma})}$$
Following the standard algebraic Bethe ansatz method, we assume that the eigenvector of the nested transfer matrix is constructed as $C (-1) \vec{C} (-1) \vec{P} > (1)$. Acting the nested transfer matrix on this eigenvector, using repeatedly the commutation relations (100,101), we have the eigenvalue

$$
(1) (') = \prod_{i=1}^{n} \frac{Y}{2} \sin \left( \alpha + z_i - z_i \right) \sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right) \frac{Y}{2} \frac{\sin \left( \bar{z}_i - \bar{z}_i \right) \sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right)}{\sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right)}
$$

where $z_i^{(1)}$, $\bar{z}_i^{(1)}$ should satisfy the following Bethe ansatz equations

$$
\prod_{i=1}^{n} \frac{Y}{2} \sin \left( \alpha + z_i - z_i \right) \sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right) \frac{Y}{2} \frac{\sin \left( \bar{z}_i - \bar{z}_i \right) \sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right)}{\sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right)} = \prod_{i=1}^{n} \frac{Y}{2} \sin \left( \alpha + z_i - z_i \right) \sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right) \frac{Y}{2} \frac{\sin \left( \bar{z}_i - \bar{z}_i \right) \sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right)}{\sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right)}
$$

We already know the exact form of $(1)$, so we can change the former Bethe ansatz equation presented in relation (89) as follows:

$$
1 = \frac{U_2^+ (-1) U_1 (-1) \sin (2) \frac{Y}{2} \sin (2) + n \sin (2) + }{U_3^+ (-1) U_3 (-1) \sin (2) + n \sin (2) + } \frac{Y}{2} \frac{\sin \left( \bar{z}_i - \bar{z}_i \right) \sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right)}{\sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right)}
$$

The eigenvalue of the transfer matrix $t(\cdot)$ with respect to boundary condition (46) is obtained as

$$
(1) (') = \prod_{i=1}^{n} \frac{Y}{2} \sin \left( \alpha + z_i - z_i \right) \sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right) \frac{Y}{2} \frac{\sin \left( \bar{z}_i - \bar{z}_i \right) \sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right)}{\sin \left( \bar{\alpha} + \bar{z}_i - \bar{z}_i \right)}
$$

Here for convenience, we give a summary of the values $U$ and $U^+$. Case I:

$$
U_1^+ (') = \sin \left( \alpha + \bar{\alpha} \right) e^{2i\nu} ; \\
U_2^+ (') = \frac{\sin \left( \bar{\alpha} + \bar{\alpha} \right) e^{2i\nu}}{\sin \left( \alpha + \bar{\alpha} \right)} ; \\
U_3^+ (') = \frac{\sin \left( \bar{\alpha} + \bar{\alpha} \right) e^{2i\nu}}{\sin \left( \alpha + \bar{\alpha} \right)}
$$

Case II:

$$
U_1^+ (') = \sin \left( \alpha + \bar{\alpha} \right) e^{2i\nu} ;
$$
\[
U_2(\gamma) = \frac{\sin(\gamma + \gamma) \sin(2\gamma)}{\sin(2\gamma)}; \\
U_3(\gamma) = \frac{\sin(2\gamma) \sin(\gamma + \gamma)}{\sin(2\gamma + \gamma)} e^i .
\]

\[\text{(L23)}\]

\text{Case I:}

\[
U_1(\gamma) = \sin(\gamma + \gamma) e^{2i\gamma}; \\
U_2(\gamma) = e^{i\gamma^2} \frac{\sin(2\gamma) \sin(\gamma + \gamma)}{\sin(2\gamma)}; \\
U_3(\gamma) = \sin(\gamma) .
\]

\[\text{(L24)}\]

\text{Case II:}

\[
U_1(\gamma) = \sin(\gamma + \gamma) e^{i\gamma^2} ; \\
U_2(\gamma) = e^{i\gamma^2} \frac{\sin(2\gamma) \sin(\gamma + \gamma)}{\sin(2\gamma)}; \\
U_3(\gamma) = \sin(\gamma) .
\]

\[\text{(L25)}\]

Since \(U\) and \(U^+\) are independent of each other, there are four combinations for \(fU; U^+g\) such as \(fI; Ig, fII; Ig, fII; Ig\) and \(fII; Ig\).

In a special limit, the solution of the reflection equation becomes identity, our result should be reduced to the results obtained by Foerster and Karowski [22]. And in the rational limit, the results are equivalent to the previous results [22, 57].

5 \text{ Algebraic Bethe ansatz for BFF grading}

5.1 The rst-level Bethe ansatz

For the case of BFF grading, the calculations proceed parallel to the case of FFB. However, for the nested algebraic Bethe ansatz method, the low-level r-matrix is BF grading which is significantly different from the FF grading r-matrix. Actually, as we observed in the last section, the graded method is the same as the usual method for FF grading r-matrix. We shall study the supersymmetric t-J model in the BFF grading.

The r-matrix is now

\[
R(\gamma) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & b() & 0 & c() & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b() & 0 & 0 & 0 & c() & 0 & 0 \\
0 & c() & 0 & b() & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a() & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b() & 0 & c() & 0 \\
0 & 0 & c() & 0 & 0 & 0 & b() & 0 & 0 \\
0 & 0 & 0 & 0 & c() & 0 & b() & 0 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a()
\end{pmatrix}.
\]

\[\text{(L26)}\]
The diagonal solutions of the dual reflection equation are
\[
K^+ ( ) = \begin{bmatrix} 0 & \sin ( + ) \text{e}^{i2} \\ \sin ( + ) \text{e}^{i2} & 1 \end{bmatrix} A ;
\]
\[
K^- ( ) = \begin{bmatrix} 0 & \sin ( + ) \text{e}^{2i} \\ \sin ( + ) \text{e}^{2i} & 1 \end{bmatrix} A ;
\]
where \(+\) is an arbitrary boundary parameter.

We still denote solution of the dual reflection equation \(K^+\) and the double-row monodromy matrix \(T\) respectively in the following form:
\[
K^+ ( ) = \text{diag}:k_1^+ ( ):\text{k}_2^+ ( ):\text{k}_1^+ ( ); \quad T ( ) = \begin{bmatrix} A_{11} ( ) & A_{12} ( ) & B_1 ( ) \\ A_{21} ( ) & A_{22} ( ) & B_2 ( ) & A ; \\ C_1 ( ) & C_2 ( ) & D ( ) \end{bmatrix}.
\]

In order to obtain the commutation relations, we need the following transformation:
\[
A ( )_{ab} = \hat{A} ( )_{ab} \frac{\text{e}^{2i} \sin ( )}{\sin (2)} D ( ) ;
\]
Because the double-row monodromy matrix satisfies the reflection equation, we obtain the following commutation relations after some tedious calculations:
\[
C_{d_1 ( )} C_{d_2 ( )} = \frac{\text{r}_{12} ( )_{d_1 d_2 c_1 c_2}}{\sin ( + ) \sin ( + )} C_{d_1 ( )} D ( ) ;
\]
\[
D ( ) C_{d_1} ( ) = \frac{\sin ( + ) \sin ( + )}{\sin ( + ) \sin ( + )} C_{d_1 ( )} D ( ) ;
\]
\[
\hat{A}^a_{a_1 d_1 ( )} C_{d_2 ( )} = ( 1 ) a_1^+ a_2^+ a_1 a_2 \frac{\text{r}_{12} ( + ) a_1 a_2}{\sin ( + ) \sin ( + )} C_{d_2 ( )} \hat{A}_{a_1 b_1} ( ) ;
\]

Here the indices take values 1,2, and the Grassmann parities are BF, \( 1 = 0 ; 2 = 1 \). The r-matrix is defined as
\[
r_{12} ( ) = \begin{bmatrix} 0 & \sin ( ) & 0 & 0 \\ 0 & \sin ( ) \text{e}^i & 0 & 0 \\ 0 & 0 & \sin ( ) \text{e}^i & 0 \\ 0 & 0 & 0 & \sin ( ) \end{bmatrix}.
\]
The elements of the r-matrix are equal to those of the original R-matrix when its indices just take values 1,2, and the Grassmann parities also remain the same as before if we just take values 1,2. This r-matrix has the su(1,1) symmetry.

Let the elements of the double-row monodromy matrix act on the vacuum state $|\mathcal{D}\rangle$: 

\[
\begin{align*}
D(\cdot)|\mathcal{D}\rangle &= U_3(\cdot)\sin^{2N}(\cdot)|\mathcal{D}\rangle; \\
A_{aa}(\cdot)|\mathcal{D}\rangle &= W_a(\cdot)\sin^{2N}(\cdot)|\mathcal{D}\rangle; \\
K_{ab}(\cdot)|\mathcal{D}\rangle &= 0; \quad a \neq b \\
B_a(\cdot)|\mathcal{D}\rangle &= 0; \\
C_a(\cdot)|\mathcal{D}\rangle &= 0. 
\end{align*}
\]

Here we have defined

\[
U_3(\cdot) = k_3(\cdot); \quad W_a(\cdot) = k_a(\cdot) + \frac{\sin(\cdot)\sin 2i}{\sin 2}\frac{e^{2i}}{k_3(\cdot)}; \quad (135)
\]

Substituting the exact forms of the reflecting type I and type II K-matrices into the above relation, we have

\[
\begin{align*}
W_1^{(1)}(\cdot) &= W_1^{(2)}(\cdot) = \frac{\sin(\cdot)\sin(\cdot + \frac{i}{2})}{\sin 2}e^{i\frac{2i}{2}}; \\
W_2^{(1)}(\cdot) &= W_2^{(2)}(\cdot) = \frac{\sin(\cdot)\sin(\cdot + \frac{i}{2})}{\sin 2}e^{i\frac{2i}{2}}; \\
W_3^{(1)}(\cdot) &= W_3^{(2)}(\cdot) = \frac{\sin(\cdot)\sin(\cdot + \frac{i}{2})}{\sin 2}e^{i\frac{2i}{2}}; \quad (137)
\end{align*}
\]

The transform matrix with boundaries for BFF grading is written as

\[
t(\cdot) = k_1^{+}(\cdot)A_{11}(\cdot) \quad k_2^{+}(\cdot)A_{22}(\cdot) \quad k_3^{+}(\cdot)D(\cdot) \\
= (1)e^{k_a(\cdot)}K_{aa}(\cdot) + U_3(\cdot)D(\cdot) \\
\]

where $U_3^{+}$ is defined by

\[
U_3^{+}(\cdot) = k_3^{+}(\cdot) + \frac{e^{2i}}{\sin 2}\frac{\sin(\cdot)}{k_3^{+}(\cdot)}k_2^{+}(\cdot) \\
\]

For type I, II solutions of the dual reflection equations $K^+$, we have

\[
U_3^{+}(\cdot) = k_3^{+}(\cdot) = \sin(\cdot + \frac{i}{2})e^{2i}; \quad \text{for } K_1^{+}; \\
U_3^{+}(\cdot) = \sin(\cdot + \frac{i}{2})e^{i}; \quad \text{for } K_{11}; \\
\]

Using the standard algebraic Bethe ansatz method, acting the above defined transform matrix on the ansatz of eigenvector $C_{d_i}(1)C_{d_j}(2)_{d_i\notin C_{n}}|\mathcal{D}\rangle$, we have

\[
t(\cdot)C_{d_i}(1)C_{d_j}(2)_{d_i\notin C_{n}}|\mathcal{D}\rangle |\mathcal{D}\rangle > F^{d_i}_{d_j} \\
= U_3^{+}(\cdot)U_3(\cdot)\sin^{2N}(\cdot)\sin(\cdot + \frac{i}{2})(\sin(\cdot + \frac{i}{2})e^{-i})_{i=1}^{\nu}C_{d_i}(1)_{d_i\notin C_{n}}|\mathcal{D}\rangle + \sin^{2N}(\cdot)_{i=1}^{\nu}C_{d_i}(1)_{d_i\notin C_{n}}|\mathcal{D}\rangle > t^{(1)}_{d_i\notin C_{n}} |\mathcal{D}\rangle + \sin^{2N}(\cdot)_{i=1}^{\nu}C_{d_i}(1)_{d_i\notin C_{n}}|\mathcal{D}\rangle + \sin^{2N}(\cdot)_{i=1}^{\nu}C_{d_i}(1)_{d_i\notin C_{n}}|\mathcal{D}\rangle > t^{(1)}_{d_i\notin C_{n}} |\mathcal{D}\rangle \\
\]

\[
\]

\[
(142)
\]

\[
22
\]
where the nested transfer matrix \( t^{(1)} \) is defined as

\[
t^{(1)}(\mathbf{C}) = \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \\
1 & \cdots & 1 \\
\end{pmatrix} \in \mathbb{C}^{n \times n}
\]

Here we have used \( a_{b} = c_{d} \) for a non-zero element of the \( r \times m \) matrix \( r_{ab} \). We also know that for non-zero \( r_{ab} \), we have \( a + c = b + d \). Considering \( a + c = 0 \), we can write

\[
a_i + b_i = a_i + 2a_{i+1} + \cdots + b_{i+1} + \cdots + b_n,
\]

which

\[
X_i = (a_i + b_i)(l + e_i) = X_i X_1 X_2 \cdots X_{i-1} X_i
\]

Thus this nested transfer matrix can still be interpreted as a transfer matrix with respecting boundary conditions corresponding to the anisotropic case

\[
t^{(1)}(\mathbf{C}) = \text{str} K^{(1)+} T^{(1)}(\gamma; f^{-1} g) K^{(1)} T^{(1)}(\gamma; f^{-1} g)
\]

with the grading \( BF_1 = 0; 2 = 1 \), where we denote \( x = \gamma_x = \cdots \gamma_x \). According to the definition, we have nested re-acting matries:

\[
K^{(1)}(\gamma) = \begin{pmatrix} W_1 (\gamma; \gamma_t) \\ 1 \\
\end{pmatrix}
\]

\[
K^{(1)+}(\gamma) = \begin{pmatrix} k_+ (\gamma; \gamma_t) \\ 1 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix} \sin (\gamma \gamma_t) \sin (\gamma \gamma_t) \cos (\gamma \gamma_t) \\ \sin (\gamma \gamma_t) \sin (\gamma \gamma_t) \cos (\gamma \gamma_t) \\
\end{pmatrix} \text{diag} : \sin (\gamma \gamma_t) \sin (\gamma \gamma_t) \cos (\gamma \gamma_t)
\]

and

\[
= \begin{pmatrix} \sin (\gamma \gamma_t) \sin (\gamma \gamma_t) \cos (\gamma \gamma_t) \\ \sin (\gamma \gamma_t) \sin (\gamma \gamma_t) \cos (\gamma \gamma_t) \\
\end{pmatrix} \text{diag} : \sin (\gamma \gamma_t) \sin (\gamma \gamma_t) \cos (\gamma \gamma_t)
\]

The row-to-row monodromy matrix \( T^{(1)}(\gamma; f^{-1} g) \) and \( T^{(1)+}(\gamma; f^{-1} g) \) are defined respectively as

\[
T^{(1)}(\gamma; f^{-1} g) = \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \\
1 & \cdots & 1 \\
\end{pmatrix} \in \mathbb{C}^{n \times n}
\]

\[
= \begin{pmatrix} \sin (\gamma \gamma_t) \sin (\gamma \gamma_t) \cos (\gamma \gamma_t) \\ \sin (\gamma \gamma_t) \sin (\gamma \gamma_t) \cos (\gamma \gamma_t) \\
\end{pmatrix} \text{diag} : \sin (\gamma \gamma_t) \sin (\gamma \gamma_t) \cos (\gamma \gamma_t)
\]

\[
= \begin{pmatrix} \sin (\gamma \gamma_t) \sin (\gamma \gamma_t) \cos (\gamma \gamma_t) \\ \sin (\gamma \gamma_t) \sin (\gamma \gamma_t) \cos (\gamma \gamma_t) \\
\end{pmatrix} \text{diag} : \sin (\gamma \gamma_t) \sin (\gamma \gamma_t) \cos (\gamma \gamma_t)
\]
\[
T^{(1)}(\tau F^{-1} g) = r_{21}(\tau^{-1})^{b_0}_{a_0} L^{(1)}_{r_2} L^{(1)}_{r_1} \left(\begin{array}{cc}
1 & \frac{1}{n} \sum_{k=1}^{n-1} (s_k + e_k) \\
\frac{1}{n} \sum_{k=1}^{n-1} (s_k + e_k) & 1
\end{array}\right) \left(\begin{array}{c}
\frac{1}{n} \sum_{k=1}^{n-1} (s_k + e_k) \\
\frac{1}{n} \sum_{k=1}^{n-1} (s_k + e_k)
\end{array}\right) \left(\begin{array}{c}
\frac{1}{n} \sum_{k=1}^{n-1} (s_k + e_k) \\
\frac{1}{n} \sum_{k=1}^{n-1} (s_k + e_k)
\end{array}\right)
\]

(150)

where we have used the unitarity relation of the \(r\)-matrix. The
L-operator is obtained from the \(r\)-matrix and takes the form

\[
L^{(1)}_{k} (\tau) = \left(\begin{array}{cc}
\mathbf{b}(\tau) & \mathbf{c}(\tau)
\end{array}\right) e_{k}^{(1)} \left(\begin{array}{c}
\mathbf{e}_{k}^{(1)}
\end{array}\right) = \left(\begin{array}{c}
\mathbf{b}(\tau) \mathbf{a}(\tau)
\end{array}\right) e_{k}^{(2)}
\]

(151)

We end that the super tensor product in the above de ned monodromy m
atrix is different from the original definition. Nevertheless, as in the periodic boundary condition case, we can de ne another
graded tensor product as follows [1]:

\[
F^{ac} = F^{ac} (1)^{s_{a} + s_{b}} (1^{+} e_{c})
\]

(152)

Effectively the graded tensor product switches even and odd Grassmann
parities. The graded tensor product in the above monodromy m
atries follows the new de ned rule.

The L-operator sat is the following Yang-Baxter relation

\[
r(\tau) = \left(\begin{array}{cc}
b_{1} b_{2} L^{(1)} & C_{1} L^{(1)}
\end{array}\right) \left(\begin{array}{cc}
b_{2} b_{1} L^{(1)} & C_{2} L^{(1)}
\end{array}\right) \left(\begin{array}{c}
1^{s_{a} + s_{b}} (1^{+} e_{c})
\end{array}\right) = L^{(1)} (\tau) b_{2} b_{1} L^{(1)} (\tau) b_{1} b_{2} r(\tau) \left(\begin{array}{cc}
b_{1} b_{2} L^{(1)} & C_{1} L^{(1)}
\end{array}\right) \left(\begin{array}{cc}
b_{2} b_{1} L^{(1)} & C_{2} L^{(1)}
\end{array}\right) \left(\begin{array}{c}
1^{s_{a} + s_{b}} (1^{+} e_{c})
\end{array}\right)
\]

(153)

Multiplying both sides of this Yang-Baxter relation by \((1)^{s_{a} + s_{b}}\), we obtain

\[
f(\tau) b_{2} b_{1} L^{(1)} (\tau) b_{1} b_{2} L^{(1)} (\tau) c_{2} c_{1} (1)^{s_{a} + s_{b}} (1^{+} e_{c}) = L^{(1)} (\tau) b_{2} b_{1} L^{(1)} (\tau) b_{1} b_{2} r(\tau) \left(\begin{array}{cc}
b_{1} b_{2} L^{(1)} & C_{1} L^{(1)}
\end{array}\right) \left(\begin{array}{cc}
b_{2} b_{1} L^{(1)} & C_{2} L^{(1)}
\end{array}\right) \left(\begin{array}{c}
1^{s_{a} + s_{b}} (1^{+} e_{c})
\end{array}\right)
\]

(154)

This is just the graded Yang-Baxter relation in the new \(\tau\) de ned graded tensor product. And we have another \(r\)-matrix

\[
f(\tau) b_{2} b_{1} L^{(1)} (\tau) b_{1} b_{2} L^{(1)} (\tau) c_{2} c_{1} (1)^{s_{a} + s_{b}} (1^{+} e_{c}) = L^{(1)} (\tau) b_{2} b_{1} L^{(1)} (\tau) b_{1} b_{2}
\]

(155)

For the row-to-row monodromy m
atrix, we also have

\[
f(1) = (1)^{s_{a} + s_{b}} b_{2} b_{1} L^{(1)} (\tau) b_{1} b_{2} L^{(1)} (\tau) c_{2} c_{1} (1)^{s_{a} + s_{b}} (1^{+} e_{c})
\]

(156)

In order to prove that the nested monodromy m
atrix is indeed the transfer m
atrix with re ecting boundary conditions, we need to prove that it constitutes a commuting family. As discussed in the last
sections, we should prove that \(K^{(1)}\) and \(K^{(1)^{+}}\) satisfy someth ing like re ection equations. One can prove that \(K^{(1)}\) and \(K^{(1)^{+}}\) satisfy the following graded re ection equations in the new \(\tau\) de ned graded sense.

\[
f(\tau) = K^{(1)} (\tau) b_{1} b_{2} L^{(1)} (\tau) b_{2} b_{1} L^{(1)} (\tau) c_{2} c_{1} (1)^{s_{a} + s_{b}} (1^{+} e_{c})
\]

(157)
We have the following commutation relations:

\[
\mathcal{O}(1) (\mathcal{O}(2) K^{(1)} (\mathcal{O}(1))^+) = \mathcal{O}(1) (\mathcal{O}(2) K^{(1)} (\mathcal{O}(1))^+),
\]

Thus, the nested transfer matrix is proved to constitute a commuting family. We can still use the graded algebraic Bethe ansatz method to find its eigenvalue and eigenvector.

5.2 Algebraic Bethe ansatz method for BF six vertex model with boundaries and the nal results for BFF case

Denote the double-row monodromy matrix as

\[
T^{(1)} (i g) = \begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & D^{(1)} \end{pmatrix}.
\]

For convenience, we need the following transformation

\[
T^{(1)} (i g) = \begin{pmatrix} A^{(1)} & B^{(1)} \\ C^{(1)} & D^{(1)} \end{pmatrix} = \begin{pmatrix} \sin (\theta) e^{2i} \\ \sin (\theta) \end{pmatrix}
\]

Because the nested double-row monodromy matrix satisfy the reflection equation

\[
\mathcal{O}(1) (\mathcal{O}(2) K^{(1)} (\mathcal{O}(1))^+) = \mathcal{O}(1) (\mathcal{O}(2) K^{(1)} (\mathcal{O}(1))^+),
\]

we have the following commuting relations:

\[
D^{(1)} (C^{(1)}) = \frac{\sin (\theta) \sin (\theta) \sin (\theta) \sin (\theta)}{\sin (\theta) \sin (\theta) \sin (\theta) \sin (\theta)} C^{(1)} (D^{(1)}),
\]

\[
A^{(1)} (C^{(1)}) = \frac{\sin (\theta) \sin (\theta) \sin (\theta) \sin (\theta)}{\sin (\theta) \sin (\theta) \sin (\theta) \sin (\theta)} C^{(1)} (A^{(1)}),
\]

\[
C^{(1)} (C^{(1)}) = \frac{\sin (\theta) \sin (\theta) \sin (\theta) \sin (\theta)}{\sin (\theta) \sin (\theta) \sin (\theta) \sin (\theta)} C^{(1)} (C^{(1)}).
\]

For the local vacuum state \( \mathcal{O}^{(1)} = \mathcal{O}^{(1)} \), we have

\[
B^{(1)} (\mathcal{O}^{(1)}) = 0;
\]

\[
C^{(1)} (\mathcal{O}^{(1)}) = 0;
\]

\[
A^{(1)} (\mathcal{O}^{(1)}) = U_1 \left[ \sin (\theta) \sin (\theta) \right] \mathcal{O}^{(1)};
\]

\[
D^{(1)} (\mathcal{O}^{(1)}) = U_2 \left[ \sin (\theta) \sin (\theta) \right] \mathcal{O}^{(1)}.
\]
Acting the transfer matrix $t^{(1)}(\gamma) = U_1^+(\gamma) A^{(1)}(\gamma) U_2^+(\gamma) B^{(1)}(\gamma)$ on the ansatz of the eigenvector $C (-_1^{(1)}) C (-_2^{(1)}) \quad \Omega \{ \gamma \} \geq \Omega$, we find the eigenvalue of the nested transfer matrix as follows:

$$
\begin{align*}
(1) ^{\ (1)}(\gamma) &= U_1^+ (\gamma) U_1 (\gamma) \prod_{i=1}^{\gamma} \left[ \sin (\gamma + -_i) \sin (\gamma + -_i) \right] \prod_{i=1}^\gamma \frac{\sin (-_1^{(1)} + -_i) \sin (-_1^{(1)} + -_i)}{\sin (-_1^{(1)} + -_1) \sin (-_1^{(1)} + -_1)} \\
&= U_2^+ (\gamma) U_2 (\gamma) \left[ \sin (\gamma + -_2) \sin (\gamma + -_2) \right] \prod_{i=1}^\gamma \frac{\sin (-_1^{(1)} + -_i) \sin (-_1^{(1)} + -_i)}{\sin (-_1^{(1)} + -_1) \sin (-_1^{(1)} + -_1)}; \quad (166)
\end{align*}
$$

where $-_1^{(1)}; \quad -_m^{(1)}$ should satisfy the following Bethe ansatz equations:

$$
\begin{align*}
&\frac{U_1^+ (-_2^{(1)}) U_1 (-_1^{(1)}) \prod_{i=1}^{\gamma} \sin (-_1^{(1)} + -_i) \sin (-_1^{(1)} + -_1)}{U_2^+ (-_2^{(1)}) U_2 (-_1^{(1)}) \prod_{i=1}^{\gamma} \sin (-_1^{(1)} + -_i) \sin (-_1^{(1)} + -_1)} = 1; \quad j = 1; \quad m: \quad (167)
\end{align*}
$$

The eigenvalue of the transfer matrix $t(\gamma)$ with reflecting boundary condition is finally obtained as

$$
\begin{align*}
(1) ^{\ (1)}(\gamma) &= U_3^+ (-_1^{(1)}) U_3 (\gamma) \sin 2n (\ \gamma) \prod_{i=1}^{\gamma} \frac{\sin (\gamma + -_1) \sin (\gamma + -_1)}{\sin (\gamma + -_1) \sin (\gamma + -_1)} \\
&= \sin (2_1^{(1)} + -_1) \prod_{i=1}^{\gamma} \frac{\sin (j + -_1) \sin (j + -_1)}{\sin (j + -_1) \sin (j + -_1)}; \quad (169)
\end{align*}
$$

and $1; \quad m; \quad$ should satisfy the Bethe ansatz equations:

$$
\begin{align*}
&\frac{\sin (2_1^{(1)} + -_1) \prod_{i=1}^{\gamma} \sin (j + -_1) \sin (j + -_1)}{\sin (2_1^{(1)} + -_1) \prod_{i=1}^{\gamma} \sin (j + -_1) \sin (j + -_1)} = \frac{\sin (2_1^{(1)} + -_1) \prod_{i=1}^{\gamma} \sin (j + -_1) \sin (j + -_1)}{\sin (2_1^{(1)} + -_1) \prod_{i=1}^{\gamma} \sin (j + -_1) \sin (j + -_1)}; \quad (169)
\end{align*}
$$

Finally, we give a summary of $U$ and $U^+$ for BFF grading.

Case I:

$$
\begin{align*}
U_1^+ (\gamma) &= \sin (-_1^{(1)} + -_1^{(1)}) e^{2i\gamma}; \\
U_2^+ (\gamma) &= \frac{\sin (2^\gamma) \sin (-_1^{(1)} + -_1^{(1)}) e^{2i\gamma}}{\sin (2^\gamma)}; \\
U_3^+ (\gamma) &= \sin (-_1^{(1)} + 2\gamma) e^{2i\gamma};
\end{align*}
$$

Case II:

$$
\begin{align*}
U_1^+ (\gamma) &= \sin (-_1^{(1)} + -_1^{(1)}) e^{2i\gamma}; \\
U_2^+ (\gamma) &= \frac{\sin (-_1^{(1)} + -_1^{(1)}) \sin (2^\gamma)}{\sin (2^\gamma)}; \\
U_3^+ (\gamma) &= \sin (-_1^{(1)} + 2\gamma) e^{i\gamma};
\end{align*}
$$

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Case I:

\[ U_1(\gamma) = \frac{\sin(2\gamma^+) \sin(\gamma^-)}{\sin(2\gamma^-)} e^{i(2\gamma^-)}; \]
\[ U_2(\gamma) = \frac{\sin(2\gamma^+) \sin(\gamma^-)}{\sin(2\gamma^-)} e^{i(2\gamma^-)}; \]
\[ U_3(\gamma) = \sin(\gamma^-); \]  \hspace{1cm} (172)

Case II:

\[ U_1(\gamma) = \frac{\sin(2\gamma^+) \sin(\gamma^-)}{\sin(2\gamma^-)} e^{i(2\gamma^-)}; \]
\[ U_2(\gamma) = \frac{\sin(2\gamma^+) \sin(\gamma^-)}{\sin(2\gamma^-)} e^{i}; \]
\[ U_3(\gamma) = \sin(\gamma^-); \]  \hspace{1cm} (173)

As before \( U \) and \( U^* \) are independent of each other, so there are four combinations for \( fU; U^*g \) such as \( fI; Ig, fi; IIg, fII; Ig \) and \( fIII; IIg \).

6 Results for FBF grading

The last possible grading is FBF, \( 1 = 3 = 1; 2 = 0 \). We can analyze it in the same way as the BFF grading. Here we just present the eigenvalue, the corresponding Bethe ansatz equation and the boundary factors. The eigenvalue of the transfer matrix with reflecting boundary condition is

\[
\rho = U_3^*(\gamma) U_3(\gamma) \sin^{2N} \left( \sum_{i=1}^{N} \sin(\gamma^- + i) \sin(\gamma^+ - i) \right) \sin(\gamma^- + i) \sin(\gamma^+ - i) + \sin^{2N} \left( \sum_{i=1}^{N} \sin(\gamma^- + i) \sin(\gamma^+ - i) \right) \tag{174}\]

where \( \gamma_1^{(i)} \); \( \gamma_1^{(i)} \) - should satisfy the following Bethe ansatz equations

\[
U_1^{(r)}(\gamma_1^{(i)}) U_1(\gamma_1^{(i)}) \sin(\gamma_1^{(i)} + i + 1) = 1; \quad j = 1, \ldots, m; \tag{175}
\]

and \( \gamma_1; \gamma_1; \) should satisfy

\[
1 = \frac{\sin^{2N} \left( \gamma_1^{(i)} \right) \sin^{2N} \left( \gamma_1^{(i)} \right) \sin^{2N} \left( \gamma_1^{(i)} \right) \sin^{2N} \left( \gamma_1^{(i)} \right) \sin^{2N} \left( \gamma_1^{(i)} \right)}{\sin^{2N} \left( \gamma_1^{(i)} \right) \sin^{2N} \left( \gamma_1^{(i)} \right) \sin(\gamma_1^{(i)} + i + 1) \sin(\gamma_1^{(i)} + i + 1)}; \quad j = 1, \ldots, n; \tag{176}
\]
The boundary factors are as follows:

Case I:

\[ U_1^+(\gamma) = \sin(-\gamma) e^{2i\gamma}; \]
\[ U_2^+(\gamma) = \frac{\sin(2\gamma) \sin(-\gamma)}{\sin(2\gamma+\gamma)} e^{i(2\gamma)}; \]
\[ U_3^+() = \sin(\gamma+\gamma); \] (177)

Case II:

\[ U_1^+(\gamma) = \sin(-\gamma) e^{2i\gamma}; \]
\[ U_2^+(\gamma) = \frac{\sin(-\gamma+\gamma) \sin(2\gamma)}{\sin(2\gamma+\gamma)}; \]
\[ U_3^+() = \sin(\gamma+\gamma) e^{i}; \] (178)

Case I:

\[ U_1(\gamma) = \sin\gamma e^{2i\gamma}; \]
\[ U_2(\gamma) = \frac{\sin(2\gamma+\gamma) \sin(-\gamma) e^{i(2\gamma+\gamma)}}{\sin(2\gamma)}; \]
\[ U_3() = \sin(); \] (179)

Case II:

\[ U_1(\gamma) = \sin\gamma e^{i(2\gamma+\gamma)}; \]
\[ U_2(\gamma) = \frac{\sin(2\gamma+\gamma) \sin(-\gamma) e^{i}}{\sin(2\gamma)}; \]
\[ U_3() = \sin(); \] (180)

7 Summary and discussions

We have studied the generalized supersymmetric \( J \) model with boundaries in the framework of the graded quantum inverse scattering method. The trigonometric R-matrix of the Perk-Shultz model is changed to the graded one. Solving the reflection equation and the dual reflection equation, we obtain two types of solutions each for three different backgrounds FFB, BFF and FBF. The transfer matrix is constructed from the R-matrix and the reflecting K-matrix. The Hamiltonian is the supersymmetric \( J \) model with boundary terms. Using the graded algebraic Bethe ansatz method, we obtain the eigenvalues of the transfer matrix for three possible gradings. The corresponding Bethe ansatz equations are obtained.

Comparing our results with the previous results in [23], we find that the form of Bethe ansatz equations for BFF case in section 5 are similar to the results obtained in [23].

It is important to investigate the thermodynamic limit of the results obtained in this paper. There, we may calculate some physical quantities such as free energy, surface free energy and interfacial tension.
etc.. It is also important to extend the supersymmetric \( J \) model to a more general supersymmetric case.

Recently, the boundary impurity problems have attracted considerable interests \([46, 47, 48, 49, 50, 51]\). Studying the boundary impurities by using three different grading is interesting and will be left for a future study.

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