Beatwave Excitation of Plasma Waves Based on Relativistic Bi-Stability

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A nonlinear beatwave regime of plasma wave excitation is considered. Two beatwave drivers are considered: intensity-modulated laser pulse and density-modulated (microbunched) electron beam. It is shown that a long beatwave pulse can excite strong plasma waves in its wake even when the beatwave frequency is detuned from the electron plasma frequency. The wake is caused by the dynamic bi-stability of the nonlinear plasma wave if the beatwave amplitude exceeds the analytically calculated threshold. In the context of a microbunched beam driven plasma wakefield accelerator, this excitation regime can be applied to developing a femtosecond electron injector.

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Beatwave excitation of electron plasma waves continues attracting significant attention as a basic nonlinear plasma phenomenon, and as a viable approach to plasma-based particle acceleration. Beatwave excitation mechanism is realized when the driver intensity (laser or particle beam) is modulated with the temporal periodicity of the plasma wave. The linear one-dimensional theory of the beatwave-driven plasma wave generation is well understood, and its most important predictions are as follows. First, the effectiveness of plasma wave excitation is strongly dependent on the difference between the beat wave frequency, understood, and its most important predictions are as follows. First, the effectiveness of plasma wave excitation is strongly dependent on the difference between the beatwave frequency $\omega_B$ and plasma wave frequency $\omega_p = \sqrt{4\pi e^2 n_0 / m}$ (where $-e$ and $m$ are the electron charge and mass, and $n_0$ is the plasma density): the smaller is the frequency detuning $\Delta \omega \equiv \omega_B - \omega_p$, the larger is the resulting plasma wave inside the beatwave. Second, only if the beatwave pulse is short enough for its bandwidth to be comparable to $\Delta \omega$, an appreciable plasma wave is left in its wake.

In this Letter I demonstrate that these conclusions are no longer valid when the relativistic nonlinearity of a plasma wave is accounted for. In particular, a strong plasma wave can be excited in the wake of a relatively long beatwave pulse of duration $t_L \gg 1/\Delta \omega$ due to the nonlinear phenomenon of dynamic relativistic bi-stability (RB). Another manifestation of RB is that, at a certain critical strength of the beatwave driver, a weak driven plasma wave becomes unstable, and a much higher amplitude wave is excited. Linear estimates of the plasma wave amplitude fail when the beatwave amplitude exceeds this detuning-dependent critical strength. As the time-dependent beatwave strength increases and exceeds the critical value, significant pulsations of the plasma wave amplitude occur. These pulsations indicate that significant energy exchange takes place between the plasma wave and the driver. This effect can be exploited in a plasma wakefield accelerator driven by a microbunched electron beam: bunches in the head of the beam excite while those in the back deplete plasma waves, thereby gaining energy.

Relativistic bi-stability was originally described for a magnetized electron subjected to cyclotron heating. Applications of RB to electron cyclotron heating of fusion plasmas have been later suggested. Although the nonlinear nature of electron plasma waves has been noted before, the RB of plasma waves has not been explored, either as a basic phenomenon or in the context of plasma-based accelerators.

The one-dimensional relativistic dynamics of the cold plasma driven by a beatwave can be described using a Lagrangian displacement of the plasma element originally located at $z_0$: $z(t) = z_0 + \zeta(t, z_0)$. It is assumed that the beatwave generated by either a pair of frequency-detuned laser beams, or a modulated electron beam, is moving with the speed close to the speed of light, and, therefore, all beatwave quantities are functions of the co-moving coordinate $\tau' = \omega_p(t - z/c) \equiv \tau - \omega_p \zeta/c$. Introducing the normalized displacement $\zeta' = \omega_p \zeta/c$ and longitudinal relativistic momentum $\tilde{p} = \gamma d\zeta/d\tau$, where $\gamma = \sqrt{1 - \tilde{v}^2/c^2}$, equations of motion take on the form

$$\frac{d\tilde{z}}{d\tau} = \frac{\tilde{p}}{\sqrt{1 + \tilde{p}^2}}, \quad \frac{d\tilde{p}}{d\tau} = -\tilde{\zeta} + a(\tau') \cos \omega \tau' \quad \left(1\right)$$

Assuming that $|\Delta \omega| \ll \omega_p$ (near-resonance excitation), transverse momentum of the plasma has been neglected and the relativistic $\gamma$-factor simplified to $\gamma = \sqrt{1 + \tilde{v}^2}$. The first term in the force equation is the restoring force of the ions, and the second term signifies the beatwave with the frequency $\omega_B \equiv \omega \omega_p$. The nonlinear in $\zeta$ modification of the beatwave in the rhs of Eqs. (1) is neglected in what follows. For a pair of linearly polarized laser pulses with electric field amplitudes $E_1$ and $E_2$ and the corresponding frequencies $\omega_1$ and $\omega_2 = \omega_1 - \omega_B$ the normalized beatwave amplitude $a = (e/mc^2)E_1 E_2 / 2\omega_1 \omega_2$. For a driving electron bunch with the density profile $n_0 = n_0 + \delta n_0 \sin \omega \tau$ it can be shown that $a = \delta n_0 / n_0$. Although arbitrary profiles of $a(\tau)$ are allowed, it is assumed that $|da/d\tau| \ll |a|$. The total energy density of the plasma wave $U_p/n_0 mc^2 = \sqrt{1 + \tilde{p}^2} + \tilde{\zeta}^2/2$ is changed via the interaction with the
beatwave. The effect of the plasma wave on the beatwave is neglected for the moment and addressed towards the end of the Letter.

Although Eqs. (1) can be solved numerically at this point, further simplification is made by assuming \( \rho = u \cos(\omega \tau + \phi) \), where \( u \) and \( \phi \) are slowly varying functions of \( \tau \). In the weakly relativistic approximation \( \rho^2 \ll 1 \) obtain:

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\begin{align*}
\frac{du}{d\tau} &= \frac{a}{2} \cos \phi \\
\frac{d\phi}{d\tau} &= -\frac{a}{2} \sin \phi - \frac{u}{2\omega}(\omega^2 - 1 + 3u^2/8). 
\end{align*}
\]

Depending on the beatwave frequency \( \omega \) and the amplitude \( a \), equilibrium solutions \( du/d\tau = 0 \) (steady amplitude) and \( d\phi/d\tau = 0 \) (phase-locking to the beatwave) of Eqs. (2,3) can have one or three real roots. For any \( \omega \) there is a stable equilibrium point: \( \phi_0 = -\pi/2 \) and \( u_0 > 0 \) found as the root of the third-order polynomial equation \( \mathcal{P}(u_0) = u_0(\omega^2 - 1 + 3/8u_0^2) = \omega a \). For the most interesting \( \omega < 1 \) regime additional solutions \( \phi_0 = \pi/2 \) and \( u_0 > 0 \), where \( u_0 \) is the positive root of \( \mathcal{P}(u_0) = -\omega a \), may be found, depending on the beatwave amplitude. Specifically, there are no additional positive roots for \( a > a_{\text{crit}} \), where \( a_{\text{crit}} = 4\sqrt{2}(1 - \omega^2)^{3/2}/9\omega \), and two positive roots \( u_{1,2} \) for \( a < a_{\text{crit}} \) (one of them unstable). Stable equilibrium amplitudes \( u_0 \) with \( \phi_0 = \pi/2 \) (Branch 1) and \( \phi_0 = -\pi/2 \) (Branch 3), as well as the unstable one (Branch 2) are plotted in Fig. 1 as a function of the beatwave strength \( a \) for \( \omega = 0.95 \) (\( a_{\text{crit}} = 0.02 \)). Equilibrium bi-stability corresponding to Branches 1 and 3 is universal for any nonlinear pendulum, including a weakly damped one. Equilibrium solutions are meaningful only if the plasma wave is phase-locked to the beatwave: \( d\phi/d\tau \approx 0 \). As shown below, this is not the case when the peak beatwave amplitude exceeds \( a_{\text{crit}} \). Nonetheless, a dynamic RB described below occurs even in the absence of phase-locking.

Consider plasma response to a Gaussian beatwave pulse \( a(\tau) = a_0 \exp(-\tau^2/\tau_L^2) \), where \( \tau_L \gg 1/(1 - \omega) \) is the normalized pulse duration. For \( a_0 < a_{\text{crit}} \) the plasma response is as follows: amplitude \( u \) adiabatically follows \( a(\tau) \) by staying on the Branch 1 and following the equilibrium trajectory schematically shown by arrows in Fig. 1. The adiabaticity condition is \( \Omega_B \tau_L \gg 1 \), where \( \Omega_B \) is the bounce frequency around the equilibrium point \( u_0 \) such that \( \mathcal{P}(u_0) = -\omega a(\tau) \). Linearizing Eqs. (2,3) around \( \phi = \pi/2 \) and \( u = u_0 \) yields \( \Omega_B^2 = a(u_{\text{crit}}^2 - u_0^2)/4\omega u_0 \), where \( u_{\text{crit}} = 2\sqrt{2}(1 - \omega^2)^{3/2}/3 \) is the critical plasma wave amplitude corresponding to the merging point between Branches 1 and 2 in Fig. 1. For \( a_0 < a_{\text{crit}} \) plasma oscillation is indeed phase-locked to the beatwave at \( \phi_0 \approx \pi/2 \) during the ramp-up and most of the ramp-down of the laser pulse (although phase-locking is lost when the pulse amplitude becomes very small on the down-ramp). As the result, plasma wave amplitude returns to a very small value in the wake of the beatwave, as shown by a dot-dashed line in Fig. 2. The longer is the beatwave pulse duration \( \tau_L \), the smaller is the wake because its non-vanishing amplitude is due to the adiabaticity violation for finite \( \tau_L \).

Situation changes for \( a_0 > a_{\text{crit}} \) as \( a(\tau) \) approaches \( a_{\text{crit}} \); the adiabatic condition is violated (noted in the context of electron cyclotron heating), and phase-locking at \( \phi_0 = \pi/2 \) is no longer possible. Thus, the transfer to Branch 3 schematically shown by a vertical arrow in Fig. 1 becomes feasible, and the plasma wave amplitude can
dramatically increase. In the presence of a finite plasma wave damping this indeed happens: the subsequent decrease of the beatwave amplitude results in phase-locking at $\varphi_0 = -\pi/2$, with $u$ following along the Branch 3. Without damping, there is no mechanism for the plasma wave to reach the equilibrium amplitude given by the upper Branch 3. As shown below, a conservation law prohibits the jump between Branches 1 and 3.

Nevertheless, even without damping, a significant plasma wave is left behind the finite-duration beatwave pulse (Fig. 2 solid line). The previously unaccessible finite-amplitude solution has been reached due to the effect of the dynamic RB which is best understood through the conservation of the effective Hamiltonian of the driven plasma wave. The effective Hamiltonian

$$H = \frac{1}{2} au \sin \phi + \frac{(\omega^2 - 1)u^2}{4\omega} + \frac{3u^4}{64\omega}$$

(4)

can be used to express Eqs. 2,3 in the form of $\dot{u} = (1/u)dH/d\phi, \dot{\phi} = -(1/u)dH/du$. For a slowly changing beatwave amplitude $a(\tau)$ the Hamiltonian is almost conserved: $dH/d\tau = 0.5u \sin \phi \dot{a} / \dot{\tau} \approx 0$. This constitutes the conservation law preventing the jump between Branches 1 and 3. For the initially quiescent plasma $a = 0$ and $u = 0$ before the arrival of the beatwave. Therefore, $H \approx 0$ after its passage, as confirmed by numerical simulations of various pulse durations and amplitudes. Remarkably, in addition to the trivial quiescent plasma solution $u = 0$, there is a second $u_\infty = 4\sqrt{(1-\omega^2)/3}$ solution satisfying $H(u_\infty) = 0$. Thus, a plasma wave with $H = 0$ is dynamically bi-stable: after the passage of the beatwave it can be either quiescent, or have the finite amplitude $u_\infty$. It is conjectured that, by using a beatwave pulse with $a_0 > a_{\text{crit}}$, the latter solution can be accessed, thereby leaving a wake of a substantial plasma wave with amplitude $u_\infty$.

This conjecture is verified by numerically integrating Eqs. 2,3 for two different detunings (resonant, with $\omega = 1$, and non-resonant, with $\omega = 0.95$) and two beatwave amplitudes (sub-threshold, with $a_0 = 0.018$, and above-threshold, with $a_0 = 0.023$). In all cases the Gaussian pulse duration was chosen $\tau_L = 150$. In physical units, for the plasma density of $n_0 = 10^{19}\text{cm}^{-3}$ the corresponding pulse duration is $t_L \equiv \tau_L/\omega_p \approx 750$ fs. Simulation results are shown in Fig. 2 where the solid line corresponds to the most interesting of the three cases: $\omega = 0.95$ and $a_0 = 0.023$. The plasma wave amplitude of $u \approx 0.75$ in the wake of the laser pulse is in a good agreement with $u_\infty = 0.72$. This wake owes its existence to the dynamic RB: upon interacting with the above-threshold laser beatwave, plasma wave is transferred from the quiescent state of $u = 0$ to the excited state of $u = u_\infty$. The sub-threshold excitation (dot-dashed line) with the same detuning fails to transfer the plasma into the excited state, yielding a negligible wake that is an order of magnitude smaller than in the above-threshold regime.

Linear theory also fails to describe the strong wake in this example because the detuning and the pulse duration are chosen such that the linear prediction $u_{\text{lin}} = a_0/(1-\omega^2) \times \exp[-\tau_L^2(\omega - 1)^2/4] \approx 0$ is negligibly small. Resonant excitation (dashed line) also yields a much smaller wave. Moreover, the resonantly and the sub-threshold excited plasma waves would have been even smaller had the adiabatic assumption been fully satisfied. Indeed, it is numerically confirmed that the wake amplitudes for the resonant and the sub-threshold excitations rapidly decline for longer pulses,

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FIG. 2: Excitation of a plasma wave by a Gaussian beatwave pulse (dotted line), $a(\tau) = a_0 \exp[-\tau^2/\tau_L^2], \tau_L = 150$. Solid line: $\omega = 0.95$, above-threshold excitation with $a_0 = 0.023 > a_{\text{crit}} = 0.02$; dashed line: resonant excitation with $\omega = 1$ and $a_0 = 0.023$; dot-dashed line: $\omega = 0.95$, below-threshold excitation with $a_0 = 0.018 < a_{\text{crit}}$. 

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FIG. 3: (a) Excitation of a plasma wave by a pair of identical Gaussian beatwave pulses (dot-dashed line) separated by the
delay times $\tau_d = 920$ (solid line: wake depleted by the second pulse) and $\tau_d = 980$ (dashed line: wake unperturbed by the
second pulse). Pulse parameters: same as in Fig. 2: $a_0 = 0.023$, $\omega = 0.95$, and $\tau_L = 150$. (b) Sequence of phase lockings and
phase releases for $\tau_d = 920$.

FIG. 4: Dependence of the residual plasma wave amplitude $u$ in the wake of a pair of identical beatwave pulses on the time
delay between the pulses $t_d$. Zero corresponds to $t_d = 900/\omega_p$. Pulse parameters are the same as in Fig. 3.

whereas the amplitude of the non-resonant above-threshold excitation is insensitive to the beatwave pulse length $\tau_L$.

To demonstrate the bi-stable nature of the relativistic plasma wake, excitation by a pair of identical beatwave pulses
is considered. By varying the time delay $\tau_d \equiv \omega_p t_d$ between the pulses, plasma wave can be either returned to the
original quiescent state $u = 0$ (Fig. 3(a), solid line, delay time $\tau_d = 920$), or brought into the excited state $u = u_\infty$
(Fig. 3(a), dashed line, delay time $\tau_d = 980$). Depending on the time delay $\tau_d$, there are, essentially, only two outcomes
for plasma wave amplitude: $u \approx 0$ or $u = u_\infty$. This result is remarkably nonlinear: the linear theory predicts that
the wake behind two pulses depends on their separation in a sinusoidal way: $u(t = \infty) = 2u(t_1) \cos^2[\pi \tau_d (\omega - 1)]$,
where $\tau_L \ll t_1 \ll \tau_d$ is the instance well after the end of the first and before the beginning of the second pulse. The
dependence of $u(t = \infty)$ on the delay time plotted in Fig. 4 illustrates the effect for the identical Gaussian pulses with
$\tau_L = 150$, $a_0 = 0.023$, and $\omega = 0.95$.

Dynamical RB described in this Letter is different from the standard equilibrium bi-stability of a weakly-damped
nonlinear oscillator \cite{6, 14} in that the former does not require phase-locking, only the conservation of the effective
Hamiltonian $H$. As Fig. 3(b) indicates, phase locking at $\phi_0 = \pi/2$ exists only during the switch-on half of the beatwave,
or deceleration gradient of the drive electron bunch $E_\text{red}(\text{blue})$ shifted if $\frac{du}{d\tau} > 0$ (per unit of the propagation length) can be found as beatwave (For concreteness, I concentrate on the above-threshold case plotted in Fig. 2 (solid line). The leading portion of the beatwave either lose or gain energy. In the weakly relativistic case, the plasma energy density frequency into the Stokes component. Assuming equal amplitude lasers, \cite{Wilks1990}. In the context of the laser beatwave the red-shifting corresponds to the scattering of the photons from the higher energy of the plasma wave on the driver has been neglected. Of course, the energy of the plasma wave is supplied by the beatwave. Since the plasma wave energy changes non-monotonically, different portions of the beatwave either lose or gain energy. In the weakly relativistic case, the plasma energy density $\approx n_0 mc^2 u^2/2$. For concreteness, I concentrate on the above-threshold case plotted in Fig. 2 (solid line). The leading portion of the beatwave $(-\infty < \tau < 64)$ contributes energy to the beatwave and is, therefore, depleted. If the beatwave is produced by a laser pulse, this depletion can be described in the language of photon deceleration, or red-shifting \cite{Wilks1990}. In the context of the laser beatwave the red-shifting corresponds to the scattering of the photons from the higher frequency into the Stokes component. Assuming equal amplitude lasers, $E_3 = E_2$, the rate of the frequency shifting (per unit of the propagation length) can be found as $-d\omega/dz = (\omega_p^2/4\epsilon_0\omega_1 a) \times d(u^2)/d\tau$. Therefore, the laser pulse is red (blue) shifted if $du/d\tau > 0$ ($du/d\tau < 0$).

If the beatwave is produced by a microbunched electron beam, the sign of $du/d\tau$ can be related to the acceleration or deceleration gradient of the drive electron bunch $E_z$ through

$$\frac{E_z(\tau)}{E_{\text{WB}}} = \frac{\delta n_b}{n_{60}} \left( \frac{1}{2a(\tau)} \frac{du^2}{d\tau} \right),$$

(5)

where $E_{\text{WB}} = mc\omega_p/e$ is the non-relativistic wavebreaking electric field. Again, the sign of $du/d\tau$ determines whether the driving bunch is accelerated or decelerated. For a microbunched electron driver consisting of femtosecond bunches with duration $\delta t \ll 1/\omega_p$, \cite{Esarey1996} produced by an inverse free-electron laser $\delta n_b \sim n_{60}$. It is estimated that in the plasma wave decay region of the driving bunch ($64 < \tau < 112$) the beam is decelerated at a rate of $E_z \approx 30$ GeV/m for $n_0 = 10^{13}$ cm$^{-3}$. Therefore, the marriage of the microbunched plasma wakefield accelerator and the dynamic relativistic bi-stability concepts yields a new advanced acceleration technique which takes advantage of the temporal drive beam structure to produce high energy femtosecond electron beams.

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