K3 SURFACES WITH 9 CUSPS IN CHARACTERISTIC $p$

TOSHIYUKI KATSURA AND MATTHIAS SCHÜTT

ABSTRACT. We study K3 surfaces with 9 cusps, i.e. 9 disjoint $A_2$ configurations of smooth rational curves, over algebraically closed fields of characteristic $p \neq 3$. Much like in the complex situation studied by Barth, we prove that each such surface admits a triple covering by an abelian surface. Conversely, we determine which abelian surfaces with order three automorphisms give rise to K3 surfaces. We also investigate how K3 surfaces with 9 cusps hit the supersingular locus.

1. INTRODUCTION

In two papers from the 1990’s [2], [3], Barth studied complex K3 surfaces with 9 cusps, i.e. with 9 disjoint $A_2$ configurations of smooth rational curves. Barth’s arguments were of topological nature, using a triple cover by some suitable abelian surface. In this paper, we follow a more algebraic approach which lends itself to investigate the same problem over any algebraically closed field $k$ of characteristic $p \neq 3$ (which we fix throughout this paper). This enables us to detect several interesting phenomena; in particular, we also include the Zariski K3 surfaces in characteristics $p \equiv -1 \mod 3$ from [6]. Combined with explicit calculations for abelian surfaces (in positive characteristic) and the characteristic-free divisibility results for certain divisor classes from [18], we prove the following results:

**Theorem 1.1.** If $X$ is a K3 surface with 9 cusps, then $X$ admits a triple covering by an abelian surface with an automorphism of order 3.

**Theorem 1.2.** If $X$ is a supersingular K3 surface with 9 cusps, then

- either $X$ is the supersingular K3 surface of Artin invariant $\sigma = 1$,
- or $X$ has Artin invariant $\sigma = 2$ and $p \equiv -1 \mod 3$.

Both theorems are supported by ample examples, starting from suitable abelian surfaces with an automorphism of order 3. In fact, for an abelian surface to admit a triple K3 quotient is quite restrictive, both in the simple and non-simple case.

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Proposition 1.3. Let $A$ be an abelian surface such that $A$ is isogenous to $E_1 \times E_2$ with elliptic curves $E_i$ ($i = 1, 2$). Assume $E_1$ is not isogenous to $E_2$. Let $\sigma$ be an automorphism of $A$ of order 3. Then, $A/\langle \sigma \rangle$ is not birationally equivalent to a K3 surface.

In comparison, simple abelian surfaces are quite delicate to treat as we shall explore in Sections 6, 7. In the context of this paper, it turns out that ordinarity enters as an essential ingredient:

Theorem 1.4. Let $A$ be a simple ordinary abelian surface with an automorphism $\sigma$ of order 3. Assume that $\sigma$ is not a translation. Then, the quotient surface $A/\langle \sigma \rangle$ is birationally equivalent to a K3 surface.

Simple abelian surfaces in positive characteristic turn out to be quite hard to exhibit explicitly, especially in characteristic 2. Therefore we conclude the paper with an explicit one-dimensional family of K3 surfaces with 9 cusps valid in any characteristic $\neq 3, 5$ such that the generic covering abelian surface is simple (and we also provide an alternative family covering characteristic 5).

Remark 1.5. Many of our arguments also work in characteristic zero, but to ease the presentation we decided to restrict to the positive characteristic case.

2. Lattice theory for K3 surfaces with 9 cusps

Let $X$ be an algebraic K3 surface over an algebraically closed field $k$ of characteristic $p \neq 3$. Assume that $X$ contains 9 disjoint $A_2$ configurations of smooth rational curves. Then we have to determine the primitive closure of the resulting sublattice in $\text{NS}(X)$:

$$L := (A_2^9)' \subset \text{NS}(X).$$

From general lattice theory (see e.g. [13]), we know that $L$ is determined by some isotropic subgroup $H$ of the discriminant group $G = (A_2^9)/A_2^0$. Here the latter space is identified with the vector space $\mathbb{F}_9^9$, so the given problem can be analysed using coding theory. In [3], Barth achieves this by showing (topologically over $\mathbb{C}$) the following lemma:

Lemma 2.1. Any non-zero vector in $H$ has length 6 or 9.

Proof. The same holds true in arbitrary characteristic since isotropic vectors of length 3 would yield a vector in $L \setminus A_2^0$ of square $-2$, a contradiction to [18] (which is valid in any characteristic). \qed

In order to determine $L$, it will be instrumental to work out a suitable reference lattice $\Lambda$ into which $L$ embeds primitively. If $X$ has finite height,
then it is known by work of Deligne that $X$ lifts to characteristic zero with a full set of generators of $\text{NS}(X)$. Hence we can take $\Lambda$ to be the standard even unimodular lattice of rank $22$ and signature $(3, 19)$,

$$\Lambda = \Lambda_{K3} = U^3 + E^2_8$$

(to which $H^2(Y, \mathbb{Z})$ of any complex K3 surface $Y$ is isomorphic). On the other hand, if $X$ is supersingular, say of Artin invariant $\sigma$, then we may just take

$$\Lambda = \Lambda_{p, \sigma} = \text{NS}(X)$$

the unique even hyperbolic lattice of rank $22$ and discriminant group

$$A_{\Lambda} = \mathbb{F}_p^{2\sigma}.$$

What unifies both variants is that they have the same rank while being prime to $3$ in the sense that, by assumption, the discriminant is not divisible by $3$. In comparison, $L$ is also prime to $p$ since it has discriminant $-3^r$, where $r = 9 - 2 \cdot |H|$. More precisely, $L$ has discriminant group

$$A_L \cong \mathbb{F}_3^r.$$  

By construction, $L$ embeds primitively into $\Lambda$. Since $L$ and $\Lambda$ are relatively prime in the above terminology, the orthogonal complement $L^\perp$ admits a subgroup $H \subseteq A_{L^\perp}$ such that not only the discriminant groups are isomorphic,

$$H \cong A_L,$$

but also the discriminant forms agree up to sign:

$$q_L = -q_{L^\perp}|_H.$$  

In particular, $H$ and $A_L$ share the same length (i.e. minimum number of generators). Presently this is $r$ by (2.1), and on the other hand, the length is a priori bounded by the rank of $L^\perp$, i.e. $r \leq 4$.

**Lemma 2.2.** $L$ is an overlattice of $A^0_9$ of index $27$, determined uniquely up to isometries by its discriminant form

$$q_L = -q_M$$

for $M = U(3) + A_2(-1)$.

**Proof.** The construction of $L$ follows exactly the lines of [3], just using the existence of some isomorphism (2.2) and Lemma 2.1. In particular, this shows that $L$ is unique up to isometries. In loc. cit. it was also proved that $L^\perp$ inside $\Lambda_{K3}$ is isometric to $M = U(3) + A_2(-1)$. Since the shape of $L$ does not depend on the characteristic, the statement on the discriminant forms is always valid. $\square$
Remark 2.3. The above argument also shows as in [3] that the subgroup $H$ of $G$ contains a vector of length 9. This will be quite useful in the proof of Theorem 1.1.

3. PROOF OF THEOREM 1.2

We are now ready to prove Theorem 1.2. To support it, we recall the following two well-known constructions of K3 surfaces with 9 cusps (cf. Katsura [5], for instance).

Example 3.1. Let $E$ be an elliptic curve defined by

$$y^2 + y = x^3,$$

and let $\sigma$ be an automorphism of $E$ given by

$$x \mapsto \omega x, \quad y \mapsto y$$

with $\omega$ a primitive cube root of unity. Then, $\sigma \times \sigma^2$ is an automorphism of the abelian surface $A = E \times \tilde{E}$ of order 3 and the quotient surface $A/\langle \sigma \times \sigma^2 \rangle$ is birationally equivalent to a K3 surface with 9 cusps. Note that in case $p \equiv 1 \pmod{3}$, $A$ is ordinary, and in case $p \equiv -1 \pmod{3}$, $A$ is supersingular (since the same holds for the elliptic curve $E$).

Example 3.2. Let $E$ be an elliptic curve, and we set $A = E \times E$. Let $\sigma$ be the automorphism of $A$ defined by

$$
\begin{pmatrix}
0 & \iota \\
\text{id} & \iota
\end{pmatrix}
$$

where $\iota$ is the inversion of $E$. Then $\sigma$ has order 3 and the quotient surface $A/\langle \sigma \rangle$ is birationally equivalent to a K3 surface with 9 cusps. Note that in case $E$ is ordinary, $A$ is also ordinary, and in case $E$ is supersingular, $A$ is also supersingular.

Accidentally, we treated the case of supersingular K3 surfaces of Artin invariant $\sigma = 2$ in characteristic $p \equiv -1 \pmod{3}$ in [6]. Namely we proved that all these K3 surfaces are Zariski (i.e. unirational, admitting an inseparable covering by $\mathbb{P}^2$ of degree $p$) by exploiting exactly the structures imposed by a configuration of 9 disjoint $\mathbb{A}^2$’s.

In order to prove Theorem 1.2 it remains to treat the cases of Artin invariants $\sigma > 2$ as well as $\sigma = 2$ in characteristic $p \equiv 1 \pmod{3}$.

3.1. Artin invariant $\sigma > 2$. In the previous section, we bounded the length of $A_L$ by considering the orthogonal complement $L^\perp$ inside the reference lattice $\Lambda$ (which was coprime to $L$). Here we can argue along similar lines for $\Lambda$ itself (cf. [6, Thm. 6.1]). Namely, for the same reason as above, the
discriminant group $A_\Lambda \cong \mathbb{F}_p^{2\sigma}$ has to be supported on $A_{L\perp}$. But again, $L^\perp$ has rank 4, so $\sigma \leq 2$ as claimed.

3.2. **Artin invariant $\sigma = 2$ in characteristic $p \equiv 1 \mod 3$.** For reasons to become clear in a moment, we omit the restriction on the characteristic for the time being. That is, we just assume that $\sigma = 2$, and for simplicity that $p > 3$ (because for computations with even lattices it is often easier to exclude $p = 2$). Suppose that $L$ admits a primitive embedding $L \hookrightarrow \Lambda = \Lambda_{p,2}$ and let $L^\perp$ denote the orthogonal complement as before. We have seen above that $A_\Lambda$ is supported on $A_{L\perp}$. Presently, this means that $A_{L\perp}$ has $p$-length 4, i.e. $L^\perp$ is $p$-divisible as an even lattice (since $p > 2$). We can thus scale $L^\perp$ by $1/p$ and obtain an even hyperbolic lattice

$$N = L^\perp \left( \frac{1}{p} \right),$$

of rank 4 and discriminant $-27$ (the same as the discriminant of $L$ up to sign). We claim that

$$N \cong U(3) + A_2.$$  

To see the claim, we impose a duality in the spirit of [7] to derive the even hyperbolic lattice $N^\vee(3)$ of same rank 4, but discriminant $-3$. The invariants are small enough to infer that $N^\vee(3) \cong U + A_2$. Now the claim follows by applying the duality again (since $A_2^\vee(3) \cong A_2$).

To conclude, we return to the subgroup

$$\mathbb{F}_3^3 \cong H \subset A_{L\perp} \cong \mathbb{F}_p^4 \times \mathbb{F}_3^3$$

from [2.2]. The discriminant form on $H$ can be read off from (3.1) as follows:

$$q_{L^\perp}|_H = p \cdot q_N = \begin{cases} q_N & \text{if } p \equiv 1 \mod 3, \\ -q_N & \text{if } p \equiv -1 \mod 3, \end{cases}$$

(3.2)  

(the quadratic forms taking values in $\mathbb{Q}/2\mathbb{Z}$). Recall that gluing $L$ to $L^\perp$ along $H$ requires exactly that

$$q_L = -q_{L^\perp}|_H$$

Independent of $p$, we already know that $q_L = q_N$ since $N = M(-1)$, see the proof of Lemma[2.2]. Note that this asserts the case $p \equiv -1 \mod 3$ in (3.2). Hence the other alternative, with $p \equiv 1 \mod 3$, can only persist (for some $p$) if $q_N = -q_N$. But this is absurd – for instance, it would imply that $N$ glues to itself to give an even unimodular lattice, but this would have signature $(2,6)$, contradiction. This completes the proof of Theorem[1.2]
4. Proof of Theorem 1.1

With these lattice theoretic preparations, it is not hard to give a proof of Theorem 1.1. Starting from a K3 surface $X$ containing 9 disjoint $A_2$ configurations of smooth rational curves, we not only have the sublattice $L$ inside $\text{NS}(X)$ from Lemma 2.2, but we are also equipped with a vector $v$ of length 9 inside the discriminant group $G$ of $A_2^9$ which in fact is integral, i.e. belongs to $L$ (see Remark 2.3). Explicitly, $v$ may be represented as

$$v = \frac{1}{3} \sum_{i=1}^{9} (C_i + 2C'_i)$$

where the $C_i, C'_i$ are the smooth rational curves supporting the nine $A_2$ configurations (up to exchanging the two curves). Following classical theory (e.g. [11]), this divisor determines a triple covering of $X$ which we can use to our advantage. Indeed, we can proceed exactly as in [6, §5], so we just give the rough outline of the construction:

1. blow up the intersection points $C_i \cap C'_i$ to get $\tilde{X}$;
2. switch to the smooth triple covering $\tilde{A}$;
3. minimalize to $A$ by first blowing down the strict transforms of the $C_i, C'_i$ and then those of the exceptional curves in $\tilde{X}$;
4. check using the classification of algebraic surfaces that $A$ is an abelian surface.

For each step, the arguments from [6] go through – regardless of the characteristic and of the question whether $X$ is supersingular or not. Automatically, the triple covering endows $A$ with an automorphism $\sigma$ of order 3 such that $X$ can be recovered as minimal desingularization of the quotient $A/\langle \sigma \rangle$, and this completes the proof. (The following diagram of maps is only reproduced for the convenience of the reader.)

\[
\begin{array}{ccc}
\tilde{A} & \rightarrow & \hat{A} \\
\downarrow & & \downarrow \\
\tilde{X} & \rightarrow & X \\
& & \rightarrow \ A/\langle \sigma \rangle
\end{array}
\]

5. Non-simple Abelian Surfaces with Automorphism of Order 3

This section provides a proof of Proposition 1.3 so we let $A$ be an abelian surface such that $A$ is isogenous to $E_1 \times E_2$ with elliptic curves $E_i$ ($i = 1, 2$). We assume that $E_1$ is not isogenous to $E_2$ and that $A$ admits an automorphism $\sigma$ of order 3.

If $\sigma$ is fixed-point free, then it is clear that the quotient surface is either abelian or hyperelliptic. It remains to consider the case where $\sigma$ has a fixed
point on $A$, say $P$. By our assumption, we have an exact sequence

$$0 \to E \to A \xrightarrow{f} A/E \to 0$$

with an elliptic curve $E$. Let $F$ be a fiber of $f$ such that $F$ passes through the fixed point $P$. Then we have $\sigma(F) \cap F \ni P$. If $\sigma(F) \neq F$, then $\sigma(F)$ would be a multi-section of $f$ and we would have an isogeny from $\sigma(F)$ to $A/E$. Therefore, $A$ would be isogenous to $F \times A/E$ with $F$ isogenous to $A/E$, a contradiction to our assumption. It follows that $F = \sigma(F)$. We put $f(F) = Q$. Then, we have $3F = f^*(3Q)$ and, by Riemann–Roch, $3Q$ is a very ample divisor on $A/E$. Therefore, the linear system $|3F|$ gives the morphism $f$. Since $F$ is invariant under the action of $\sigma$, $\sigma$ acts on the vector space $L(3F)$. Therefore, $\sigma$ induces an action on $A/E$ fixing $Q$.

Suppose first that $\sigma$ does not act as the identity on $F$ nor on $A/E$. Thus both elliptic curves admit an automorphism of order 3, but this curve is unique up to isomorphism (j-invariant zero), so $F \cong A/E$, contradiction.

Suppose that $\sigma$ acts as identity on $A/E$. Then we have a morphism $A/\langle \sigma \rangle \to A/E$. Therefore, we have the dimension $q(A/\langle \sigma \rangle) \geq 1$ of the Albanese variety of $A/\langle \sigma \rangle$. In particular, $A/\langle \sigma \rangle$ cannot be birationally equivalent to a K3 surface.

Suppose that $\sigma$ acts as identity on $F$. Since $F$ is non-singular, there exist local coordinates $x, y$ such that $\sigma(x) = x$ and $\sigma(y) = \omega y$ with $\omega$, a cube root of unity (it may be 1). Applying the same argument to any fixed point of $\sigma$, we see that the quotient surface $A/\langle \sigma \rangle$ is non-singular. Therefore, we have $H^0(A/\langle \sigma \rangle, \Omega^1_{A/\langle \sigma \rangle}) \cong H^0(A, \Omega^1_A)^{\langle \sigma \rangle}$. Since we have a natural isomorphism $m_P/m_P^2 \cong H^0(A, \Omega^1_A)$ and $\dim(m_P/m_P^2)^{\langle \sigma \rangle} \geq 1$, we obtain

$$\dim H^0(A/\langle \sigma \rangle, \Omega^1_{A/\langle \sigma \rangle}) \geq 1,$$

and again $A/\langle \sigma \rangle$ cannot be a K3 surface. This concludes the proof of Proposition 1.3.

**Corollary 5.1.** Let $A$ be an abelian surface with $p$-rank 1. Then, there exists no automorphism $\sigma$ of order 3 on $A$ such that $A/\langle \sigma \rangle$ is birationally equivalent to a K3 surface.

**Proof.** If $A$ is simple, we will show this corollary in the next section. If $A$ is non-simple, $A$ is isogenous to a product $E_1 \times E_2$ of two elliptic curves $E_i$ ($i = 1, 2$). Since the $p$-rank of $A$ is 1, one of $E_i$’s is ordinary and the other is supersingular. Therefore, $E_1$ is not isogenous to $E_2$. Hence, the result follows from Proposition 1.3.

6. **Endomorphism algebras of simple abelian surfaces**

As before, we let $k$ be an algebraically closed field of characteristic $p > 0$. We summerize here some results by Mumford on the endomorphism
algebras of simple abelian surfaces ([12] Section 21, Theorem 2) over \( k \).

Let \( A \) be a simple abelian surface and let \( \text{End}(E) \) be the endomorphism ring of \( A \). We denote by \( A_n \) the (reduced) \( n \)-torsion group of \( A \). We set \( D = \text{End}^0(E) = \text{End}(E) \otimes \mathbb{Z} \mathbb{Q} \). Then, \( D \) is a central simple division algebra. We denote by \( K \) the center of \( D \) and by \( K_0 \) the subfield of \( K \) which is fixed by the Rosati involution. We put \( [D : K] = d^2 \), \( [K : \mathbb{Q}] = e \) and \( [K_0 : \mathbb{Q}] = e_0 \). We also put \( S = \{ x \in D \mid x' = x \} \). It is known that \( \dim_{\mathbb{Q}} S \) is equal to the Picard number \( \rho(A) \) of \( A \). We put \( \eta = \frac{\dim_{\mathbb{Q}} S}{\dim_{\mathbb{Q}} D} \). Then, Mumford gave the following table for the possible numerical invariants of \( D \).

| Type | \( e \) | \( d \) | \( \eta \) | \text{char } p > 0 |
|------|------|------|------|------------------|
| I    | \( e_0 \) | 1    | 1    | \( e \mid 2 \)   |
| II   | \( e_0 \) | 2    | \( \frac{3}{2} \) | \( 2e \mid 2 \)  |
| III  | \( e_0 \) | 2    | \( \frac{1}{2} \) | \( e \mid 2 \)    |
| IV   | \( 2e_0 \) | \( d \) | \( \frac{1}{2} \) | \( e_0 \eta d \mid 2 \) |

Using this list, we get the following detailed list.

| Type  | \( e \) | \( d \) | \( \eta \) | \( \dim_{\mathbb{Q}} D \) | \( \rho(A) \) |
|-------|------|------|------|--------------------|--------|
| (I-i) | 1    | 1    | 1    | 1                 | 1      |
| (I-ii)| 2    | 1    | 1    | 2                 | 2      |
| (II)  | 1    | 1    | \( \frac{3}{4} \) | 4      | 3      |
| (III-i)| 1   | 1    | \( \frac{1}{4} \) | 4      | 1      |
| (III-ii)| 2   | 2    | \( \frac{1}{4} \) | 8      | 2      |
| (IV-i)| 2    | 1    | \( \frac{1}{2} \) | 2      | 1      |
| (IV-ii)| 2  | 1    | \( \frac{1}{2} \) | 8      | 4      |
| (IV-iii)| 4  | 2    | \( \frac{1}{7} \) | 4      | 2      |

We will show that the cases (III-ii), (IV-ii) and (IV-iii) cannot occur for a simple abelian surface \( A \). We denote the \( p \)-adic Tate module of \( A \) by \( T_p(A) \). First we show the following lemma (see also [12], Section 19, Theorem 3).

**Lemma 6.1.** Let \( A \) be a simple abelian surface. Then, the natural homomorphism

\[
\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \text{End}_{\mathbb{Z}_p}(T_p(A))
\]

is injective.

**Proof.** In dimension 2, \( A \) is supersingular if and only if the \( p \)-rank of \( A \) is 0. Since \( A \) is simple, \( A \) is not supersingular (cf. Oort [15]). Therefore, the rank of \( T_p(A) \) is either 1 or 2. Since \( A \) is simple, the kernel of a non-zero endomorphism \( f \) is a finite group scheme. Therefore, for a large positive integer \( m \), the induced homomorphism \( f : A_{p^m} \longrightarrow A_{p^m} \) is not the zero-map.
Therefore, the natural homomorphism $\text{End}(A) \otimes \mathbb{Z}_p \rightarrow \text{End}_{\mathbb{Z}_p}(T_p(A))$ is injective.

**Lemma 6.2.** In the list above, the cases (III-ii), (IV-ii) and (IV-iii) cannot occur.

**Proof.** In cases (III-ii) and (IV-ii), we have $\dim_{\mathbb{Q}_p} D \otimes_{\mathbb{Q}} \mathbb{Q}_p = 8$. On the other hand, $\dim_{\mathbb{Q}_p} \text{End}_{\mathbb{Q}_p}(T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is equal to 1 or 4, according to the $p$-rank of $A = 1$ or 2, which is impossible by Lemma 6.1. In case (IV-iii), since we have $\dim_{\mathbb{Q}} D = 4$, the $p$-rank of $A$ should be 2 and $\dim_{\mathbb{Q}_p} \text{End}_{\mathbb{Q}_p}(T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = 4$, and we have $D \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \text{End}_{\mathbb{Q}_p}(T_p(A)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. However, $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is commutative and $\text{End}_{\mathbb{Q}_p}(T_p(A)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is non-commutative, a contradiction.

By the list above, we have the following corollary.

**Corollary 6.3.** For simple abelian surfaces, we have $\rho(A) \leq 3$.

Note how this fits together with the classic result of Shioda–Mitani [20] that a complex abelian surface $A$ with $\rho(A) = 4$ is isomorphic to a product of elliptic curves.

**Proposition 6.4.** Let $A$ be a simple abelian surface with an automorphism $\sigma$ of order 3. Then, the structure of the endomorphism algebra $\text{End}^0(A)$ of $A$ is one of the following.

(i) A division algebra over $\mathbb{Q}$ which contains $\mathbb{Q}(\sigma)$.
(ii) $\text{End}^0(A) = \mathbb{Q}(\sigma)$ with $K_0 = \mathbb{Q}$ and $K = \mathbb{Q}(\sigma)$.

**Proof.** This follows from the above classification of division algebras.

7. **SIMPLE ABELIAN SURFACES WITH AUTOMORPHISM OF ORDER 3**

We shall now start working towards the proof of Theorem 1.4. First comes the ordinarily condition imposed by automorphisms of order 3:

**Proposition 7.1.** Let $A$ be a simple abelian surface with an automorphism $\sigma$ of order 3. Assume that $p \neq 3$ and that $\sigma$ is not a translation. Then, $A$ is an ordinary abelian surface.

**Proof.** If the $p$-rank of $A$ is 0, then in case of dimension 2 $A$ is a supersingular abelian surface as we have used above. Therefore, $A$ is not simple (cf. Oort [15]). Assume the $p$-rank of $A$ is equal to 1. Then, $T_p(A)$ has rank 1 over $\mathbb{Z}_p$ and so $\text{End}(T_p(A)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is 1-dimensional over $\mathbb{Q}_p$, which contradicts Proposition 6.4 and Lemma 6.1. Hence, the $p$-rank of $A$ is 2, that is, $A$ is ordinary as claimed.

We use the Harder-Narashimhan theorem frequently.
Theorem 7.2 (Harder-Narashimhan [4], Proposition 3.2.1). Let $X$ be a nonsingular projective variety on which a finite group $G$ acts. Let $\ell$ be a prime number which is prime to both $p$ and the order of $G$. Then, the étale cohomology $H^i(X/G, \mathbb{Q}_\ell)$ is isomorphic to the subspace $H^i(X, \mathbb{Q}_\ell)^G$ of $G$-invariants in $H^i(X, \mathbb{Q}_\ell)$:

$$H^i(X/G, \mathbb{Q}_\ell) \cong H^i(X, \mathbb{Q}_\ell)^G.$$ 

Applied to quotients of projective surfaces, we obtain the following:

Lemma 7.3. Let $X$ be a nonsingular projective surface on which a finite group $G$ acts. Let $\ell$ be a prime number which is prime to both $p$ and the order of $G$. Moreover, assume $G$ has only isolated fixed points, and let $\phi : Y \to X/G$ be a minimal resolution of $X/G$. Then, we have an isomorphism

$$\phi^*: H^1_c(X/G, \mathbb{Q}_\ell) \cong H^1_c(Y, \mathbb{Q}_\ell)$$

and an injective homomorphism

$$\phi^*: H^2_c(X/G, \mathbb{Q}_\ell) \to H^2_c(Y, \mathbb{Q}_\ell).$$

Proof. Let $W$ be the set of singular points of $X/G$, and $E$ be the exceptional divisor of $\phi$ on $Y$. Then, we have an isomorphism

$$\phi|_{Y \setminus E} : Y \setminus E \to X/G \setminus W.$$ 

Therefore, we have an isomorphism $H^i_c(A/G \setminus W, \mathbb{Q}_\ell) \cong H^i_c(Y \setminus E, \mathbb{Q}_\ell)$. There is a commutative diagram of long exact sequences of étale cohomology groups with compact support whose coefficients are in $\mathbb{Q}_\ell$ (cf. Milne [10]):

$$\begin{array}{cccccccc}
\to & H^{-1}_c(W, \mathbb{Q}_\ell) & \to & H^1_c(A/G \setminus W, \mathbb{Q}_\ell) & \to & H^1_c(A/G, \mathbb{Q}_\ell) & \to & H^1_c(W, \mathbb{Q}_\ell) & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\to & H^{-1}_c(E, \mathbb{Q}_\ell) & \to & H^1_c(Y \setminus E, \mathbb{Q}_\ell) & \to & H^1_c(Y, \mathbb{Q}_\ell) & \to & H^1_c(E, \mathbb{Q}_\ell) & \to
\end{array}$$

The singularities of $A/G$ are rational by [16, p. 149] (which assumes characteristic zero, but the trace argument works in characteristic $p$ as long as the order of $G$ is prime to $p$). Hence $E$ consists of trees of $\mathbb{P}^1$’s. Therefore, we have $H^1_c(E, \mathbb{Q}_\ell) = 0$. We also have

$$H^1_c(W, \mathbb{Q}_\ell) = H^2_c(W, \mathbb{Q}_\ell) = 0,$$

$$H^i_c(A/G, \mathbb{Q}_\ell) \cong H^i(A/G, \mathbb{Q}_\ell) \quad (i = 1, 2),$$

$$H^i_c(Y, \mathbb{Q}_\ell) \cong H^i(Y, \mathbb{Q}_\ell) \quad (i = 1, 2).$$

The results follow from these facts.

We will also need the following helpful property.

Lemma 7.4. Let $A$ be an abelian surface, and $C$ be a nonsingular complete curve of genus $g \geq 2$. Then, there exists no non-trivial rational map from $A$ to $C$. 

Proof. Suppose there exists a non-trivial rational map \( f : A \rightarrow C \). Then, by composition, there exists a homomorphism from \( A \) to the Jacobian variety \( J(C) \) of \( C \). Since the homomorphism factors through \( C \), it is absurd. \( \square \)

**Lemma 7.5.** Let \( A \) be a simple abelian surface with an automorphism \( \sigma \) of order 3. Assume that \( p \neq 3 \) and that \( \sigma \) is not a translation. Then, \( \sigma \) has at least one fixed point and the fixed locus consists of finitely many points.

**Proof.** If \( \sigma \) is fixed-point-free, then the quotient surface \( A/\langle \sigma \rangle \) is either an abelian surface or a hyperelliptic surface. If it is an Abelian surface, \( \sigma \) must be a translation, which contradicts our assumption. If it is a hyperelliptic surface, then the Albanese variety \( \text{Alb}(A/\langle \sigma \rangle) \) is an elliptic curve and we have a surjective morphism from \( A \) to \( \text{Alb}(A/\langle \sigma \rangle) \), which contradicts our assumption that \( A \) is simple. Now, we may choose a fixed point of \( \sigma \) as the zero point of \( A \). Then, \( \sigma \) is a homomorphism. Since \( A \) is simple, the kernel of the homomorphism \( \sigma - \text{id}_A \) is finite. Therefore, the fixed locus of \( \sigma \) is a finite set. \( \square \)

We denote by \( \omega \) a primitive cube root of unity.

**Lemma 7.6.** Let \( A \) be a simple abelian surface with an automorphism \( \sigma \) of order 3. Assume that \( p \neq 3 \) and that \( \sigma \) is not a translation. Then, the eigenvalues of \( \sigma \) on the étale cohomology group \( H^1(A, \mathbb{Q}_\ell) \) are given by \( \omega, \omega, \omega^2 \) and \( \omega^2 \).

**Proof.** Since \( \sigma^3 = \text{id}_A \) and \( \sigma - \text{id}_A \) is an isogeny, we see \( \sigma^2 + \sigma + \text{id}_A = 0 \). Therefore, the minimal polynomial of \( \sigma \) is \( x^2 + x + 1 \) (cf. Mumford [12], Section 19, Theorem 4). Therefore, the possibilities of the eigenvalues of \( \sigma \) on \( H^1(A, \mathbb{Q}_\ell) \) are the following.

Case (i) \( 1, 1, 1, 1 \).

Case (ii) \( 1, 1, \omega, \omega^2 \).

Case (iii) \( \omega, \omega, \omega^2, \omega^2 \).

Suppose Case (i). Then, since \( H^1(A, \mathbb{Q}_\ell) \cong \wedge^1 H^1(A, \mathbb{Q}_\ell) \), we see that all the eigenvalues of \( \sigma \) on \( H^*(A, \mathbb{Q}_\ell) \) are 1. Hence, the alternating sum of traces of \( \sigma \) on \( H^*(A, \mathbb{Q}_\ell) \) is equal to 0. Hence, by the Lefschetz trace formula, \( \sigma \) is fixed-point-free on \( A \), which contradicts Lemma 7.5. Therefore, Case (i) is excluded.

Now, we denote by \( Y \rightarrow A/\langle \sigma \rangle \) a resolution of singularities of \( A/\langle \sigma \rangle \). Then, by Lemma 7.3, we have an isomorphism \( H^1(A/\langle \sigma \rangle, \mathbb{Q}_\ell) \cong H^1(Y, \mathbb{Q}_\ell) \), and we have \( \dim H^1(Y, \mathbb{Q}_\ell) = \dim H^1(A, \mathbb{Q}_\ell)(\sigma) \).

Suppose Case (ii). Then we have \( H^1(Y, \mathbb{Q}_\ell) = \dim H^1(A, \mathbb{Q}_\ell)(\sigma) = 2 \). Therefore, the dimension \( q(Y) \) of the Albanese variety of \( Y \) is equal to 1. Therefore, we have a surjective homomorphism from \( A \) to the Albanese variety (an elliptic curve), which contradicts the assumption that \( A \) is simple.
Hence, we conclude that Case (iii) holds.

**Corollary 7.7.** Let $A$ be a simple abelian surface with an automorphism $\sigma$ of order 3. Assume that $p \neq 3$ and that $\sigma$ is not a translation. Then, the number of fixed points of $\sigma$ is equal to 9.

**Proof.** Since $A$ is simple, the fixed loci of $\sigma$ are isolated. Since $H^2(A, \mathbb{Q}_\ell) \cong \wedge^2 H^1(A, \mathbb{Q}_\ell)$, the eigenvalues of $\sigma$ on $H^2(A, \mathbb{Q}_\ell)$ are given by

$$1, 1, 1, \omega, \omega^2,$$

and on $H^1(A, \mathbb{Q}_\ell)$ they are the same as on $H^3(A, \mathbb{Q}_\ell)$. By the Lefschetz trace formula, we see that the number of fixed points is equal to 9.

We are now ready to prove Theorem 1.4. Let $Y \to A/\langle \sigma \rangle$ be a resolution of singularities of $A/\langle \sigma \rangle$. Since we have a separable dominating rational map from $A$ to $Y$, we see

$$0 = \kappa(A) \geq \kappa(Y).$$

By the Enriques–Kodaira classification (extended to positive characteristic by Bombieri–Mumford), $Y$ is a K3 surface, an Abelian surface, a hyperelliptic surface, an Enriques surface or a ruled surface. If $Y$ is an Abelian surface, the rational map from $A$ to $Y$ is a homomorphism. Therefore, $\sigma$ must coincide with a translation, which contradicts our assumption. If $Y$ is a ruled surface with $q(Y) \geq 2$. Then, we have a rational map from $A$ to $Y$. Therefore, we have a rational map from $A$ to the base curve of $Y$, which is a curve of genus $\geq 2$. A contradiction to Lemma 7.4. If $Y$ is either hyperelliptic or ruled with $q(Y) = 1$, then we have a homomorphism from $A$ to an elliptic curve – which contradicts that $A$ is simple. If $Y$ is either rational or Enriques, then we have an inclusion

$$H^2(A, \mathbb{Q}_\ell)^{(\sigma)} \cong H^i(A/\langle \sigma \rangle, \mathbb{Q}_\ell) \hookrightarrow H^2(Y, \mathbb{Q}_\ell).$$

Since $Y$ is supersingular in the sense of Shioda, that is, $H^2(Y, \mathbb{Q}_\ell)$ is generated by algebraic cycles, we see that $H^2(A, \mathbb{Q}_\ell)^{(\sigma)}$ is generated by algebraic cycles. Since $\dim H^2(A, \mathbb{Q}_\ell)^{(\sigma)} = 4$ by (7.1), we see the Picard number $\rho(A) \geq 4$, which contradicts Corollary 6.3. Hence, $A$ is a K3 surface. This completes the proof of Theorem 1.4.

Summarizing these results, we have the following corollary.

**Corollary 7.8.** Let $A$ be a simple ordinary abelian surface with an automorphism $\sigma$ of order 3. Assume $p \neq 3$ and $\sigma$ is not a translation. Then, $A/\langle \sigma \rangle$ has just 9 $A_2$-rational double points as singular points, and the minimal resolution is a K3 surface with $\rho = 19$. 
8. Explicit Quotients of Simple Abelian Surfaces

Exhibiting explicit simple abelian surfaces turns out to be a non-trivial problem in positive characteristic – especially in characteristic two. For this reason, we decided to include families of K3 surfaces with nine cusps in any characteristic $p \neq 3$ such that the covering abelian surfaces are generically simple.

To explain the approach, we recall from [3] that complex tori $A$ with an automorphism $\sigma$ of order 3 come in a two-dimensional analytic family such that generically

$$\text{NS}(A) = A_2, \quad T_A = M_0 = U + A_2(-1).$$

Algebraic subfamilies are obtained by enhancing the Néron–Severi lattice by a positive vector $H$ from $M_0$; the generic Néron–Severi lattice is thus promoted to the primitive closure $N$ of $\mathbb{Z}H + A_2$ inside $H^2(A, \mathbb{Z}) \cong U^3$. The very general member of the resulting one-dimensional family is simple if and only if $N$ does not represent zero non-trivially. In [3], an abstract example with $H^2 = 12$ is worked out; in contrast we will work out an explicit example with $H^2 = 10$, though admittedly, it is fully explicit only on the K3 side (which can be used to recover $A$ as explained in Section 4). To this end, take $H \in U \subset M_0$ with $H^2 = 10$ and postulate that $H \in \text{NS}(A)$. Then this determines a one-dimensional family of abelian surfaces $A$ with an automorphism $\sigma$ of order 3 such that generically

$$\text{NS}(A) = \mathbb{Z}H + A_2 \quad \text{and} \quad T_A = \mathbb{Z}(-10) + A_2(-1).$$

Consider the family of K3 surfaces $X$ which arise as minimal resolutions of the quotients $A/\langle \sigma \rangle$. Then these always have $L \subset \text{NS}(X)$, and following [3], the sublattice $M_0$ pushes down to $M = \mathbb{U}(3) + A_2(-1)$ (the lattice from Lemma 2.2). The algebraic enhancement means that $H$ induces a positive vector $v$ of square $v^2 = 30$ in $\text{NS}(X)$, such that generically

$$(8.1) \quad T_X = (v^\perp \subset M_0) = \mathbb{Z}(-30) + A_2(-1), \quad \text{NS}(X) \supset \mathbb{Z}v + L$$

where the last inclusion has index 3 for discriminant reasons.

**Lemma 8.1.** Generically, one has

$$(8.2) \quad \text{NS}(X) = U + 2E_6 + A_4 + A_1.$$ 

**Proof.** By [13] suffices to verify that the discriminant forms of Néron–Severi lattice and transcendental lattice generically agree up to sign; i.e. for $T_X$ from (8.1) and $\text{NS}(X)$ as in (8.2), we have $q_{\text{NS}} = -q_T$ which is readily verified. \qed

The above representation of $\text{NS}(X)$ is very convenient because it implies by standard arguments (see [19], for instance) that $X$ admits an elliptic
fibration such that generically there is only a single section (so most of NS is captured by fibre components). One can use this as a starting point to work out the following family of elliptic K3 surfaces with 9 cusps, given by in affine Weierstrass form with parameter $\lambda$:

$$y^2 + (\lambda + 1)txy = x^3 + t(3t^2 - t(\lambda^2 - 4\lambda + 1) + 3\lambda^2)x^2 + 3t^2(t - 1)^2(t + \lambda^3)(t + \lambda)x + t^3(t - 1)^4(t + \lambda^3)^2$$

**Proposition 8.2.** In any characteristic $\neq 3, 5$, the family $X$ has generically $\rho(X) = 19$ and $\text{NS}(X) = U + 2E_6 + A_4 + A_1$.

Before coming to the proof of the proposition, we note that we can recover the family of covering abelian surfaces from $X$ by the geometric argument from Section 4. In particular, Proposition 8.2 implies the following:

**Corollary 8.3.** The covering abelian surfaces are generically simple in any characteristic $\neq 3, 5$.

**Proof of Proposition 8.2.** We first prove that the family $X$ is non-isotrivial. To this end, we use that the discriminant $\Delta$ of the above elliptic fibration obviously varies with $\lambda$ – and so does the j-invariant. Hence, if the family were isotrivial, i.e. almost all members isomorphic to a single K3 surface $X_0$, then $X_0$ would admit infinitely many non-isomorphic elliptic fibrations. Over fields of characteristic $\neq 2$, this is ruled out by work of Sterk [21] and Lieblich–Maulik [8].

In characteristic 2, it suffices by [9] to exhibit two non-isomorphic smooth specializations within $X$. For this purpose, we endow special members of the family with a suitable section as follows. We start by arguing in characteristic zero with a root $\alpha$ of $\alpha^3 - 2\alpha^2 - 3\alpha + 9$. Let $L = \mathbb{Q}(\alpha)$. Then the special member $X$ of the family $X$ at $\lambda = \alpha$ over $L$ admits a section of height $29/30$ with $x$-coordinate $-t(t - 1)^2(t + \alpha^3)/\alpha$. It follows that $X$ is a singular K3 surface of discriminant $-87$. Note that $X$ has smooth reduction $X_2$ over $\mathbb{F}_8$. Arguing as in [17] proof of Claim 10.3, one finds that $X_2$ is ordinary (i.e. $\rho(X_2) = 20$ with $\text{NS}(X_2)$ of the same discriminant $-87$).

To compare with another member of the family, we work exclusively in characteristic 2 (to limit the complexity). Let $\beta \in \mathbb{F}_{256}$ be a root of $\beta^8 + \beta^5 + \beta^4 + 1$ and consider the member $X'$ of the family $X$ at $\lambda = \beta$. One finds ($\beta$ by requiring) that $X'$ admits a section of height $61/30$; its $x$-coordinate is $t(t + 1)(t + \beta^3)(t^2 + t + 1)/(t + \beta^6 + \beta^5 + \beta^4 + \beta^2)$. As before, this implies $\rho(X') = 20$ and $\det\text{NS}(X') = -183$. In particular, $X_2 \not\cong X'$, so the family $X$ is non-isotrivial in characteristic 2 as claimed.

We proceed by proving the statement about the generic Picard number – which clearly satisfies $\rho \geq 19$. Since we have a one-dimensional family, the only alternative to $\rho = 19$ is $\rho = 22$, and only in positive characteristic
(because K3 surfaces with \( \rho = 20 \) do not move in a family (just like over \( \mathbb{C} \)), and \( \rho = 21 \) is impossible, see [1]). So let us assume that \( \rho = 22 \) and \( \text{char}(k) = p > 0 \). By [14], there is a unique supersingular K3 surface of Artin invariant \( \sigma = 1 \), so we would require \( \sigma \geq 2 \). For \( p \neq 2 \) we can argue along the same lines as in Section 3: since \( N_0 = U + 2E_6 + A_4 + A_1 \) embeds primitively into \( \text{NS} = \Lambda = \Lambda_{p,\sigma} \), but the discriminants \( d(N_0) = 90 \) and \( d(\Lambda_{p,\sigma}) = -p^{2\sigma} \) are relatively prime, the discriminant group of \( A_\Lambda \) would be fully supported on \( A_{N_0} \). The length of this group is bounded by the rank of \( N_0 \), i.e. \( 2\sigma \leq 3 \), contradiction. To complete the argument, we appeal to the non-supersingular members \( X_2 \) or \( X' \) of the family \( X \) in characteristic 2 which we have already used above to prove the non-isotriviality of the family \( X \).

Having shown that generically \( \rho = 19 \), it remains to prove that the Néron–Severi lattice generically assumes the given shape, i.e. \( \text{NS}(X) \cong N_0 \). By inspection of the discriminant \( d(N_0) = 90 \), \( \text{NS}(X) \) would otherwise have to be an index 3 overlattice \( N_0 \). But then one verifies that the discriminant group \( A_{N_0} \) does not contain any non-zero isotropic elements, so there is no integral overlattice at all. □

8.1. Comments on characteristic 5. In characteristic 5, the full family \( X \) turns out to be supersingular (quite remarkably, without the singular fibers degenerating). For instance, the generic fibre, base changed to \( k(\sqrt{\lambda}) \), admits a section of height \( 5/6 \) with \( x \)-coordinate \(-t(t + \lambda^3)(t + 1/\lambda)\).

In order to work out an analogue of Corollary 8.3 in characteristic 5, one can apply the same procedures as above to an initial positive vector \( H \in U \subset M_0 \) with \( H^2 = 4 \). Along similar lines, this leads to the following (non-isotrivial) family of K3 surfaces over \( \mathbb{Q} \):

\[
Y : \quad y^2 + t^2(t - 1)^2y = \mu(x^3 - 3t^3(t - 1)^2x - 2t^4(t - 1)^3).
\]

One shows as before that generically \( \rho(Y) = 19 \) and \( \text{NS}(Y) = U + 2E_6 + D_5 \) outside characteristics 2, 3, so one obtains simple abelian surfaces as in Corollary 8.3.

Specifically for characteristic 5, one can work, for instance, with the special member \( Y' \) at \( \mu = 8/17 \) admitting a \( \mathbb{Q}(\sqrt{17}) \)-rational section of height \( 17/12 \) with \( x \)-coordinate \( t^2(t - 1)(t - 19)/18 \). The reduction to characteristic 5 is seen to be ordinary.

In comparison, in characteristic 2, the full family \( Y \) turns out to be supersingular.

REFERENCES

[1] M. Artin, Supersingular K3 surfaces, Ann. Sci. Ecole Norm. Sup. 4 (1974), 543–568.
[2] W. Barth, K3 Surfaces with Nine Cusps, Geom. Dedic. 72 (1998), 171–178.
[3] W. Barth, On the Classification of K3 Surfaces with Nine Cusps, Complex Analysis and Algebraic Geometry - A Volume in Memory of Michael Schneider (Eds: T. Peternell and F.–O. Schreyer), 42–59 (2000).

[4] G. Harder and M. S. Narashimhan, On the cohomology group of moduli spaces of vector bundles on curves, Math. Ann. 212 (1975), 215–248.

[5] T. Katsura, Generalized Kummer surfaces and their unirationality in characteristic $p$, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 34 (1987), 1–41.

[6] T. Katsura and M. Schütt, Zariski K3 surfaces, preprint (2017), arXiv: 1710.08661.

[7] Kondō, S., Shimada, I.: On certain duality of Néron-Severi lattices of supersingular K3 surfaces and its application to generic supersingular K3 surfaces, Algebr. Geom. 1 (2014), 311–333.

[8] M. Lieblich and D. Maulik, A note on the cone conjecture for K3 surfaces in positive characteristic, to appear in Math. Res. Letters, arXiv: 1102.3377v5.

[9] T. Matsusaka, D. Mumford, Two fundamental theorems on deformations of polarized varieties, Amer. J. Math. 86 (1964), 668–684.

[10] J. S. Milne, Étale Cohomology, Princeton Univ. Press, Princeton, New Jersey, 1980.

[11] R. Miranda, Triple covers in algebraic geometry, Amer. J. Math. 107 (1985), 1123–1158.

[12] D. Mumford, Abelian Varieties, Oxford Univ. Press, London/New York, 1970.

[13] Nikulin, V. V.: Integral symmetric bilinear forms and some of their applications, Math. USSR Izv. 14, No. 1 (1980), 103–167.

[14] A. Ogus, Supersingular K3 crystals, Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. II, Astérisque 64, Paris: Société Mathématique de France, pp. 3–86.

[15] F. Oort, Which abelian varieties are products of elliptic curves?, Math. Ann. 214 (1975), 35–47.

[16] H. Pinkham, Singularités rationnelles de surfaces, Lecture Notes in Math. 777, Springer-Verlag, Berlin, Heidelberg (1980), 147–178.

[17] M. Schütt, $\mathbb{Q}_l$-cohomology projective planes and Enriques surfaces in characteristic two, preprint (2017), arXiv: 1703.10441v2.

[18] M. Schütt, Divisibilities among nodal curves, Math. Res. Letters 25 (2018), 1359–1368.

[19] M. Schütt and T. Shioda, Elliptic surfaces, Algebraic geometry in East Asia - Seoul 2008, Advanced Studies in Pure Math. 60 (2010), 51-160.

[20] T. Shioda and N. Mitani, Singular abelian surfaces and binary quadratic forms, Classification of algebraic varieties and compact complex manifolds, Lect. Notes in Math. 412 (1974), 259–287.

[21] H. Sterk, Finiteness results for algebraic K3 surfaces, Mathematische Zeitschrift 189 (1985), 507–513.

[22] Tate, J.: Algorithm for determining the type of a singular fibre in an elliptic pencil, in: Modular functions of one variable IV (Antwerpen 1972), Lect. Notes in Math. 476 (1975), 33–52.
Faculty of Science and Engineering, Hosei University, Koganei-shi, Tokyo 184-8584, Japan
E-mail address: toshiyuki.katsura.tk@hosei.ac.jp

Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany, and
Riemann Center for Geometry and Physics, Leibniz Universität Hannover, Appelstrasse 2, 30167 Hannover, Germany
E-mail address: schuett@math.uni-hannover.de