Maximum number of limit cycles for certain piecewise linear dynamical systems

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1 Introduction and statement of the main results

Non-smooth dynamical systems emerge in a natural way modeling many real processes and phenomena, for instance, recently piecewise linear differential equations appeared as idealized models of cell activity, see [10,37,38]. Due to that, in these last years, the mathematical community became very interested in understanding the dynamics of these kind of systems. In general, some of the main source of motivation to study non-smooth systems can be found in control theory [4], impact and friction mechanics [5,8,27], nonlinear oscillations [1,34], economics [19,24], and biology [3,26]. See for more details the book [11] and the references therein. In this paper, we are interested in discontinuous piecewise linear differential systems. The study of this particular class of non-smooth dynamical systems has started with Andronov and coworkers [1].

We start with a historical fact. Lum and Chua [33] conjectured that a continuous piecewise linear vector field in the plane with two zones separated by a straight line, which is the easiest example of this kind of system, has at most one limit cycle. This conjecture was proved by Freire et al. [13]. Even this relatively easy case demanded a hard work to show the existence of at most one limit cycle.

In this paper, we address the problem of Lum and Chua, for non-sliding limit cycles, extended to the class of discontinuous piecewise linear differential systems in the plane with two zones separated by a straight line. Here, a non-sliding limit cycle is a limit cycle that does
not contain any sliding segment in $\Sigma$. This problem is very related to the Hilbert’s 16th problem [23].

Limit cycles of discontinuous piecewise linear differential systems with two zones separated by a straight line have been studied recently by several authors, see among others [2, 7, 9, 17, 18, 20–22, 28, 30–32]. Nevertheless, the problem of Lum and Chua remains open for this class of differential equations. In this work, we give a partial solution for this problem. We note that in [12] the authors proved that if one of the two linear systems has its singular point on the discontinuity straight line, then the number of limit cycles of such a system is at most 4. Our results reduce this upper bound to 2, and, additionally, we prove that it is reached.

Our point of interest in the Lum and Chua problem is aligned with two directions which face serious technical difficulties. First, while solutions in each linear region are easy to find, the times of passage along the regions are not simple to achieve. It means that matching solutions across regions is a very difficult task. Second, to control all possible configurations one must generally consider a large number of parameters.

It was conjectured in [18] that a planar piecewise linear differential systems with two zones separated by a straight line have at most 2 non-sliding limit cycles. A negative answer for this conjecture was provided in [20] via a numerical example having 3 non-sliding limit cycles. Analytical proofs for the existence of these 3 limit cycles were given in [15, 31]. Finally, in [16] it was studied general conditions to obtain 3 non-sliding limit cycles in planar piecewise linear differential systems with two zones separated by a straight line. Recently, perturbative techniques (see [29, 30]) were used together with newly developed tools on Chebyshev systems (see [36]) to obtain 3 limit cycles in such systems when they are near to non-smooth centers.

When a general curve of discontinuity is considered instead of a straight line, there is no upper bound for the maximum number of non-sliding limit cycles that a system of this family can have. It is a consequence of a conjecture stated by Braga and Mello in [6] and then proved by Novaes and Ponce in [35].

In this paper, we deal with planar vector fields $Z$ expressed as $\dot{z} = F(z) + \text{sign}(x)G(z)$, where $z = (x, y) \in \mathbb{R}^2$, and $F$ and $G$ are linear vector fields in $\mathbb{R}^2$ or, equivalently,

$$\dot{z} = \begin{cases} X(z) & \text{if } x > 0, \\ Y(z) & \text{if } x < 0, \end{cases}$$

where $X(z) = F(z) + G(z)$ and $Y(z) = F(z) - G(z)$. The line $\Sigma = \{x = 0\}$ is called discontinuity set. Our main goal is to study the maximum number of non-sliding limit cycles that the discontinuous piecewise linear differential system (1) can have.

The systems $\dot{z} = X(z)$ and $\dot{z} = Y(z)$ are called lateral linear differential systems (or just lateral systems) and more specifically right system and left system, respectively.

A linear differential system is called degenerate if its determinant is zero; otherwise, it is called non-degenerate. From now on in this paper, we only consider non-degenerate linear differential systems.

System (1) can be classified according to the singularities of the lateral linear differential systems. A non-degenerate linear differential system can have the following singularities: saddle ($S$), node ($N$), focus ($F$), and center ($C$). Among the above classes of singularities, we shall also distinguish the following ones: a weak saddle, i.e., a saddle such that the sum of its eigenvalues is zero ($S^0$); a diagonalizable node with distinct eigenvalues ($N$); star node, i.e., a diagonalizable node with equal eigenvalues ($N^*$); and an improper node, i.e., a non-diagonalizable node ($iN$). We say that the discontinuous differential system (1) is an LR-system with $L, R \in \{S, S^0, N, N^*, iN, F, C\}$, when the left system has a singularity of type $L$ and the right system has a singularity of type $R$.

We define subclasses of LR-systems according to the position of the singularity of each lateral system. The right system can have a virtual singularity ($R_v$), i.e., a singularity $p = (p_x, p_y)$ with $p_x < 0$; a boundary singularity ($R_b$), i.e., a singularity $p = (p_x, p_y)$ with $p_x = 0$; or a real singularity ($R_r$), i.e., a singularity $p = (p_x, p_y)$ with $p_x > 0$. Accordingly, the left system can have a virtual singularity ($L_v$), i.e., a singularity $p = (p_x, p_y)$ with $p_x > 0$; a boundary singularity ($L_b$), i.e., a singularity $p = (p_x, p_y)$ with $p_x = 0$; or a real singularity ($L_r$), i.e., a singularity $p = (p_x, p_y)$ with $p_x < 0$.

We denote by $\mathcal{N}(L, R)$ the maximum number of non-sliding limit cycles that an LR-system can have. Clearly, $\mathcal{N}(L, R) = \mathcal{N}(R, L)$.

In this paper, we compute the exact value of $\mathcal{N}(L, R)$ always when one of the lateral systems is a saddle of kind $S_x, S_y, S^0$, a node of kind $N_r, N_b, N^*$, $iN_r, iN_b$, a focus of kind $F_b$, and a center $C$. Particularly, we obtain that $\mathcal{N}(L, R) \leq 2$ in all the above cases.
It is easy to see that if one of the lateral linear differential systems is of type $S_h$, $S_b$, $N_r$, $N_b$, $N^*$, or $iN_r$, or $iN_b$, then the first return map on the straight line $x = 0$ of system (1) is not defined. Consequently, system (1) does not admit non-sliding limit cycles in all these cases. So $\mathcal{N}(R, L) = 0$ for the systems having one of these kind of equilibria.

It remains to study the cases when one of the lateral system is $F_b$, $C$ or $S^0_b$. For these cases, we shall prove the following theorems.

**Theorem 1** All numbers $\mathcal{N}(F_b, F_b)$, $\mathcal{N}(F_b, F_r)$, $\mathcal{N}(F_b, iN_b)$, $\mathcal{N}(F_b, S_r)$ are equal to 2, and all numbers $\mathcal{N}(F_b, S_b)$ are equal to 1.

**Theorem 2** The equality $\mathcal{N}(S^0_b, S_r) = 2$ holds, all numbers $\mathcal{N}(C_b, F_b)$, $\mathcal{N}(C_b, b)$, $\mathcal{N}(C_b, iN_b)$, $\mathcal{N}(C_b, iN_r)$, $\mathcal{N}(C_b, S_r)$, $\mathcal{N}(C_b, S_b)$ are equal to 1, and all numbers $\mathcal{N}(S^0_b, C)$ and $\mathcal{N}(S^0_b, S_b)$ are equal to 0.

We shall see that the next result can be obtained as an immediately corollary of the proofs of Theorems 1 and 2.

**Corollary 3** The equality $\mathcal{N}(C_b, F_r) = 2$ holds, all numbers $\mathcal{N}(C_b, F_r)$, $\mathcal{N}(C_b, F_b)$, $\mathcal{N}(C_b, N_r)$, $\mathcal{N}(C_b, iN_r)$ and $\mathcal{N}(C_b, S_r)$ are equal to 1, and all numbers $\mathcal{N}(C_b, C)$ and $\mathcal{N}(C_b, S_b)$ are equal to 0.

The equalities of Corollary 3 can be extended for all linear centers.

**Theorem 4** The equality $\mathcal{N}(C, F_r) = 2$ holds, all numbers $\mathcal{N}(C, F_r)$, $\mathcal{N}(C, F_b)$, $\mathcal{N}(C, N_r)$, $\mathcal{N}(C, iN_r)$ and $\mathcal{N}(C, S_r)$ are equal to 1, and all numbers $\mathcal{N}(C, C)$ and $\mathcal{N}(C, S_b)$ are equal to 0.

Theorems 1, 2, and 4, and Corollary 3 are proved in Sect. 3.

Our results give sufficient conditions in order to guarantee that system (1) has at most 2, 1, or 0 limit cycles. We study the non-degenerate cases for which the expression of the time that a trajectory starting in $p \in \Sigma$ remains in the region $x > 0$ (or $x < 0$) is known. The remaining cases are those ones whose this associated time is not explicitly determined for both regions.

The systems studied in [15,16,20,29,31], possessing 3 limit cycles, have in one side a real focus, and in the other side either a real focus or a linear system with trace distinct from zero. Thus, they do not satisfy the conditions of our theorems.

## 2 Preliminary results

A linear change of variables in the plane preserving the vertical lines will be called a vertical lines-preserving linear change of variables.

**Proposition 5** Let $M = (m_{ij})_{i,j}$ be a $2 \times 2$ matrix. If the linear differential system

$$(\dot{x}, \dot{y}) = M(x, y)^T \tag{2}$$

is a

(a) $S$-system then after a vertical lines-preserving linear change of variables and a time-rescaling system (2) becomes $$(\dot{x}, \dot{y}) = M_1(x, y)^T;$$

(b) $N$-system then after a vertical lines-preserving linear change of variables and a time-rescaling system (2) becomes $$(\dot{x}, \dot{y}) = M_2(x, y)^T;$$

(c) $F$-system ($C$-system) then after a vertical lines-preserving linear change of variables and a time-rescaling system (2) becomes $$(\dot{x}, \dot{y}) = M_3(x, y)^T$$

with $a \neq 0$ ($a = 0$);

(d) $iN$-system then after a vertical lines-preserving linear change of variables and a time-rescaling system (2) becomes $$(\dot{x}, \dot{y}) = M_4(x, y)^T,$$

where

$$M_1 = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \text{ with } |a| < 1;$$

$$M_2 = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \text{ with } |a| > 1;$$

$$M_3 = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \text{ with } a \in \mathbb{R}; \text{ and}$$

$$M_4 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \text{ with } \lambda = \pm 1.$$

**Proof of Proposition 5** Let $S = (s_{ij})_{i,j}$ be a $2 \times 2$ matrix. The change of variables $(u, v)^T = S(x, y)^T$ is a vertical lines-preserving linear change of variables if and only if $s_{12} = 0$ and $s_{11} = 1$. Indeed, $S(x, y) = (s_{11}x + s_{12}y, s_{21}x + s_{22}y)$ and $s_{11}x + s_{12}y = x$ for every $x \in \mathbb{R}$ if and only if $s_{11} = 1$ and $s_{12} = 0$. So in what follows we fix $s_{12} = 0$ and $s_{11} = 1$.

**Claim 1** The statement (a) holds.

Since we are assuming that we have a saddle at the origin and in the expression of its eigenvalues appears $\sqrt{4m_{12}m_{21} + (m_{11} - m_{22})^2}$, we must assume that $4m_{12}m_{21} + (m_{11} - m_{22})^2 > 0$. Taking
\[ s_{21} = \frac{m_{11} - m_{22}}{\sqrt{-4m_{12}m_{21} - (m_{11} - m_{22})^2}}, \quad \text{and} \]
\[ s_{22} = \frac{2m_{12}}{\sqrt{-4m_{12}m_{21} - (m_{11} - m_{22})^2}}, \]

it follows that

\[ SM^{-1} = \frac{1}{2} \left( \begin{array}{cc} m_{11} + m_{22} & \sqrt{-4m_{12}m_{21} - (m_{11} - m_{22})^2} \\ \sqrt{-4m_{12}m_{21} - (m_{11} - m_{22})^2} & m_{11} + m_{22} \end{array} \right). \]

Then, we can rescale the time by

\[ \tau = \frac{1}{2} \sqrt{4m_{12}m_{21} + (m_{11} - m_{22})^2} t. \]

Denoting \( a = (m_{11} + m_{22}) / \sqrt{4m_{12}m_{21} + (m_{11} - m_{22})^2} \), system (2) becomes

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \]

where now the prime denotes the derivative with respect to the new time variable \( \tau \). Computing the eigenvalues of the above system \( \{-i + a, i + a\} \), we conclude that \( |a| < 1 \), because this system is a saddle, i.e., the eigenvalues have different sign. Therefore, we have proved statement \( (a) \).

**Claim 2** The statement \( (b) \) holds.

The proof of statement \( (b) \) follows similarly to the proof of statement \( (a) \). Nevertheless, we conclude that \( |a| > 1 \), because in this case the system is a diagonalizable node, i.e., the eigenvalue have the same sign. Thus, we have proved statement \( (b) \).

**Claim 3** The statement \( (c) \) holds.

Taking

\[ s_{21} = \frac{m_{11} - m_{22}}{\sqrt{-4m_{12}m_{21} - (m_{11} - m_{22})^2}}, \quad \text{and} \]
\[ s_{22} = \frac{2m_{12}}{\sqrt{-4m_{12}m_{21} - (m_{11} - m_{22})^2}}, \]

it follows that

\[ SM^{-1} = \frac{1}{2} \left( \begin{array}{cc} m_{11} + m_{22} & \sqrt{-4m_{12}m_{21} - (m_{11} - m_{22})^2} \\ \sqrt{-4m_{12}m_{21} - (m_{11} - m_{22})^2} & m_{11} + m_{22} \end{array} \right). \]

From hypotheses, this system is a focus; thus, \(-4m_{12}m_{21} - (m_{11} - m_{22})^2 > 0\). So we can rescale the time by \( \tau = \frac{1}{2} \sqrt{-4m_{12}m_{21} - (m_{11} - m_{22})^2} t \). Denoting \( a = (m_{11} + m_{22}) / \sqrt{-4m_{12}m_{21} - (m_{11} - m_{22})^2} \), system (2) becomes

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \]

where now the prime denotes the derivative with respect to the new time variable \( \tau \). Computing the eigenvalues of the above system \( \{-i + a, i + a\} \), we conclude that \( a \neq 0 \) this system has a focus and a center when \( a = 0 \). Hence, statement \( (c) \) is proved.

**Claim 4** The statement \( (d) \) holds.

One of the entries \( m_{12} \) or \( m_{21} \) is distinct of zero. Indeed, suppose that \( m_{12} = 0 \), so \( \{m_{11}, m_{22}\} \) are the eigenvalues of the matrix \( M \). Since system (2) is a non-diagonalizable node, we have that \( m_{11} = m_{22} \) which implies that \( m_{21} \neq 0 \), in other way the matrix \( M \) would be diagonalizable. On the other hand, supposing that \( m_{21} = 0 \) we obtain \( m_{12} \neq 0 \). From here we assume, without loss of generality, that \( m_{12} \neq 0 \).

We also have that \( m_{11} + m_{22} \neq 0 \), we prove this by reduction to the absurd. Suppose that \( m_{11} + m_{22} = 0 \), then \( \pm \sqrt{m_{11}^2 + m_{12}m_{21}} \) are the eigenvalues of the matrix \( M \). Since system (2) is a non-diagonalizable node, we have that the matrix \( M \) has only one eigenvalue with multiplicity 2. This implies that the eigenvalues are zero, which is a contradiction with the fact that we are working with non-degenerate linear differential systems. In short, we have proved that \( m_{11} + m_{22} \neq 0 \).

From the expression of the eigenvalues, it is also easy to see that \( 4m_{12}m_{21} + (m_{11} - m_{22})^2 = 0 \).

Taking

\[ s_{21} = \frac{m_{11} - m_{22}}{2m_{12}} \quad \text{and} \quad s_{22} = \frac{2m_{12}}{m_{11} + m_{22}}, \]

it follows that

\[ SM^{-1} = \frac{1}{2} \begin{pmatrix} m_{11} + m_{22} & m_{11} + m_{22} \\ 0 & m_{11} + m_{22} \end{pmatrix}. \]
So we can rescale the time by \( \tau = \frac{1}{2|m_{11} + m_{22}|} t \) system, (2) becomes
\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
\lambda & \lambda \\
0 & \lambda
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix},
\]
where \( \lambda = \pm 1 \), and now the prime denotes the derivative with respect to the new time variable \( \tau \). This completes the proof of statement (c).

A limit cycle of our piecewise linear differential system (1) expends a time \( t_R \) in the region \( x > 0 \) and a time \( t_L \) in the region \( x < 0 \). As we shall see later on we know explicitly the time \( t_L \), and we do not know explicitly the time \( t_R \). The next lemma will help us to work with one of the intersection points of the limit cycle with the discontinuity straight line instead of the unknown time \( t_R \).

**Lemma 6** We consider the functions
\[
F(t) = e^{-at} \csc(t) - \cot(t), \quad G(t) = e^{-at} \csch(t) - \coth(t), \quad H(t) = \frac{e^{-t} - 1}{t}.
\]

The following statements hold.

(a) For every \( \alpha \in \mathbb{R} \), \( F(t) \) is a monotonic increasing function in the interval \((-\pi, \pi)\) such that \( F(t) < -\alpha \) for \( t \in (-\pi, 0) \), and \( F(t) > -\alpha \) for \( t \in (0, \pi) \).

(a') For every \( \alpha > 0 \) (resp. \( \alpha < 0 \)), \( F(t) \) is a monotonic increasing function in the interval \((\pi, 2\pi)\) (resp. \((-2\pi, -\pi)\)).

(b) For \( |\alpha| > 1 \), \( G(t) \) is a monotonic increasing function on \( \mathbb{R} \) such that \( G(t) > -\alpha \) for \( t > 0 \); and for \( |\alpha| < 1 \) \( G(t) \) is a monotonic decreasing function on \( \mathbb{R} \) such that \( G(t) < -\alpha \) for \( t > 0 \).

(c) \( H(t) \) is a monotonic increasing function on \( \mathbb{R} \) such that \( H(t) \leq -1 \) for \( t \leq 0 \).

**Proof** To prove statement (a), we compute
\[
F'(t) = \csc^2(t) \left(1 - e^{-at} \cos(t) + a \sin(t)\right) = \csc^2(t) p(t),
\]
where \( p(t) = 1 - e^{-at}(\cos(t) + a \sin(t)) \) and \( p'(t) = (1 + a^2)e^{-at} \sin(t) \). Clearly, \( p'(t) > 0 \) when \( 0 < t < \pi \), and \( p'(t) < 0 \) when \( -\pi < t < 0 \). So \( p(t) \) is a decreasing function in the interval \((-\pi, 0)\), and it is an increasing function in the interval \((0, \pi)\). Since \( p(0) = 0 \), we conclude that \( p(t) > 0 \) for \( t \in (-\pi, \pi) \setminus \{0\} \).

Finally, \( F'(0) = (1 + a^2)/2 > 0 \) so \( F'(t) > 0 \) for every \( t \in (-\pi, \pi) \), which implies that \( F \) is monotonic increasing for \( t \in (-\pi, \pi) \). The proof of statement (a) follows by noting that \( \lim_{t \to 0} F(t) = -\alpha \). The proof of statement (a') is completely analogous to the proof of statement (a).

To prove statement (b), we compute
\[
G'(t) = \csch(t) \left(\csc(t) - e^{-at}(\alpha + \coth(t))\right) = -\frac{e^{-at}\csch(t)}{(e^t - 1)(e^t + 1)} \left(a - e^{2t} - 2ae^{t+at}\right)
\]
where \( q(t) = a - e^{2t} - 2ae^{t+at} \) and \( q'(t) = -2(1 + a)e^t(e^t - e^{at}) \). When \( |\alpha| > 1 \), \( q'(t) \geq 0 \) for \( t \geq 0 \); because \( e^t - e^{at} \geq 0 \) for \( t \leq 0 \). When \( q'(t) \geq 0 \) for \( t \leq 0 \). Hence, for \( |\alpha| < 1 \), we conclude that \( G \) is a monotonic decreasing function on \( \mathbb{R} \) such that \( G(t) < -\alpha \) for every \( t > 0 \). It concludes the proof of statement (b).

To prove statement (c), we compute
\[
H'(t) = \frac{e^{-t}}{t^2} \left(e^t - t - 1\right) = \frac{e^{-t}}{t^2} r(t),
\]
where \( r(t) = e^t - t - 1 \) and \( r'(t) = e^t - 1 \). Since \( r(0) = 0 \) and \( r'(0) \leq 0 \) for \( t \leq 0 \), we conclude that \( r(t) > 0 \) and consequently \( H'(t) > 0 \), for \( t \neq 0 \). It implies that \( H \) is an monotonic increasing function for \( t > 0 \). The proof of statement (c) follows by noting that \( \lim_{t \to 0} H(t) = -1 \).  

Some important tools we shall use to prove our main results lie in the theory of Chebyshev systems (for more details, see, for instance, the book of Karlin and Studden [25]). In the sequel, the concept of Chebyshev systems is introduced.

Consider an ordered set of smooth real functions \( \mathcal{F} = (f_0, f_1, \ldots, f_n) \) defined on an interval \( I \). The maximum number of zeros counting multiplicity admitted by any non-trivial linear combination of functions in \( \mathcal{F} \) is denoted as \( Z(\mathcal{F}) \).

**Definition 1** We say that \( \mathcal{F} \) is an Extended Chebyshev system or ET-system on \( I \) if and only if \( Z(\mathcal{F}) \leq n \).
We say that $\mathcal{F}$ is an Extended Complete Chebyshev system or an ECT-system on $I$ if and only if for any $k$, $0 \leq k \leq n$, $(f_0, f_1, \ldots, f_k)$ is an ET-system.

The next proposition relates the property of an ordered set of functions $(f_0, f_1, \ldots, f_k)$ being an ECT-system with the nonvanishing property of their Wronskians

$$W(f_0, f_1, \ldots, f_k)(t) = \begin{vmatrix} f_0(t) & f_1(t) & \cdots & f_k(t) \\ f'_0(t) & f'_1(t) & \cdots & f'_k(t) \\ \vdots & \vdots & \ddots & \vdots \\ f^{(k)}_0(t) & f^{(k)}_1(t) & \cdots & f^{(k)}_k(t) \end{vmatrix}.$$

**Proposition 7** ([25]) A ordered set of functions $\mathcal{F} = (f_0, f_1, \ldots, f_k)$ is an ECT-system on $I$ if and only if $W(f_0, f_1, \ldots, f_i)(t) \neq 0$ on $I$ for $0 \leq i \leq k$.

The next result has been recently proved by Novaes and Torregrosa in [36].

**Proposition 8** ([36]) Let $\mathcal{F} = (u_0, u_1, \ldots, u_n)$ be an ordered set of smooth functions on $[a, b]$. Assume that all the Wronskians are nonvanishing except $W_n(x)$ which have $\ell \geq 0$ zeros on $(a, b)$ and all these zeros are simple. Then, $Z(\mathcal{F}) = n$ when $\ell = 0$, and $n + 1 \leq Z(\mathcal{F}) \leq n + \ell$ when $\ell \neq 0$.

Now consider the functions

$$\xi_1(t) = 1,$$

$$\xi^1_2(t) = \cot(t) - e^{at} \csc(t), \quad \xi^2_2(t) = \coth(t) - e^{at} \csch(t), \quad \xi^3_2(t) = \frac{1 - e^t}{t},$$

$$\xi^4_2(t) = \frac{\xi^1_2(t) - \xi^2_2(t)}{2} = \csc(t) \sinh(at),$$

$$\xi^5_2(t) = \frac{\xi^2_2(t) - \xi^3_2(t)}{2} = \csch(t) \sinh(at),$$

$$\xi^6_2(t) = \frac{\xi^3_2(t) - \xi^4_2(t)}{2} = \frac{\sinh(t)}{t}.$$

We define the ordered sets of functions $\mathcal{F}^i = (\xi_1, \xi^1_2, \xi^2_2)$ and $\tilde{\mathcal{F}}^i = (\xi_i, \xi^i_2, \xi^2_2)$ for $i = 1, 2, 3$, and $\mathcal{F}^i = (\xi_1, \xi^2_2)$ for $i = 4, 5, 6$.

The next two technical lemmas together with Definition 1 and Propositions 7 and 8 will be used later on in the proofs of Theorems 1, 2, and 4 to establish sharp upper bounds for the maximum numbers of non-sliding limit cycles that system (1) can have.

**Lemma 9** The following statements hold.

(a) The set of functions $\mathcal{F}^1$ is an ECT-system on the intervals $(0, \pi)$ and $(-\pi, 0)$ for every $a \neq 0$.

(a') The set of functions $\tilde{\mathcal{F}}^1$ is an ECT-system on the interval $(\pi, 2\pi)$ (resp. $(-2\pi, -\pi)$) for every $a > 0$ (resp. $a < 0$).

(b) The set of functions $\mathcal{F}^2$ is an ECT-system on $\mathbb{R}^+$ for every $a \notin \{0, \pm 1\}$.

(c) The set of functions $\mathcal{F}^3$ is an ECT-system on $\mathbb{R}^+$.

(d) The set of functions $\mathcal{F}^4$ is an ECT-system on the intervals $(0, \pi)$ and $(-\pi, 0)$ for every $a \neq 0$.

(d') The set of functions $\tilde{\mathcal{F}}^4$ defined on the intervals $(\pi, 2\pi)$ (or $(-2\pi, -\pi)$) satisfies $Z(\tilde{\mathcal{F}}^4) = 2$ for every $a \neq 0$.

(e) The set of functions $\mathcal{F}^5$ is an ECT-system on $\mathbb{R}^+$ for every $a \notin \{0, \pm 1\}$.

(f) The set of functions $\mathcal{F}^6$ is an ECT-system on $\mathbb{R}^+$.

**Proof** To prove the statements (a)–(f), we compute the Wronskians $W_1(t) = W(\xi_1(t), W_2(t) = W(\xi_1, \xi_2(t)(t))$ for $i = 1, 2, \ldots, 6$, and $W_3(t) = W(\xi_1, \xi_2, \xi_3(t))$ for $i = 1, 2, 3$.

$$W_1(t) = 1,$$

$$W_2(t) = \csc(t) \left(e^{at} (\cot(t) - a) - \csc(t)\right),$$

$$W_3(t) = 2 \left(1 + a^2\right) \csc^2(t) (a - \csc(t) \sinh(at)),$$

$$W_4(t) = \csc(t) (e^{at} (\cot(t) - a) - \csc(t)),$$

$$W_5(t) = 2 \left(1 - e^{at}\right) \csch^2(t) (\csc(t) \sinh(at) - a),$$

$$W_6(t) = e^{at} (1 + t^3) - 1,$$

$$W_7(t) = \frac{2 (t - \sinh(t))}{t^3},$$

$$W_8(t) = \csc(t) (a \cosh(at) - \cot(t) \sinh(at)),$$

$$W_9(t) = \csc(t) (a \cosh(at) - \cot(t) \sinh(at)),$$

$$W_10(t) = \frac{t \cosh(t) - \sinh(t)}{t^2}.$$

From here, it is easy to see that for each $a \neq 0$ the Wronskians $W_1, W_3, W_2$ do not vanish at any point of the intervals $(0, \pi)$ and $(-\pi, 0)$; for each $a \notin \{0, \pm 1\}$ the Wronskians $W_2, W_3, W_2$ do not vanish at any point of $\mathbb{R}^+$, and the Wronskians $W_2, W_3, W_2$ do not vanish at any point of $\mathbb{R}^+$. So statements (a)–(f) are proved.
To see statement (a’), we compute the Wronskians
\[
\tilde{W}_2^1(t) = W(\xi_1, \xi_2^1(t)) = \csc(t) \left( e^{-at} (\cot(t) - a) - \csc(t) \right),
\]
\[
\tilde{W}_2^3(t) = W(\xi_1, \xi_2^3(t)) = -W_2^1(t).
\]
Again it is easy to see that for each \(a > 0\) (resp. \(a < 0\)) the Wronskian \(W_2^2(t)\) does not vanish at any point of the interval \((\pi, 2\pi)\) and \((-2\pi, -\pi)\).

Finally, statement (d’) follows by showing that the Wronskian \(W_2^4(t)\) has exactly one zero in each one of the intervals \((\pi, 2\pi)\) and \((-2\pi, -\pi)\). Indeed,
\[
W_2^4(t) = \csc(t) \cosh(at)(a - \cot(t) \tanh(at)) = \csc(t) \csch(t) Pa(t).
\]
Since \(\csc(t) \cosh(at)\) is nonvanishing for every \(a \in \mathbb{R}\), it is sufficient to study the zeros of \(Pa(t)\) in order to study the zeros of \(W_2^2(t)\). For \(a > 0\)
\[
\lim_{t \to 2\pi} P_a(t) = -\lim_{t \to \pi} P_a(t) = \infty \quad \text{and} \quad \lim_{t \to -2\pi} P_a(t) = -\lim_{t \to -\pi} P_a(t) = \infty,
\]
and for \(a < 0\)
\[
\lim_{t \to \pi} P_a(t) = -\lim_{t \to 2\pi} P_a(t) = \infty \quad \text{and} \quad \lim_{t \to -\pi} P_a(t) = -\lim_{t \to -2\pi} P_a(t) = \infty.
\]
So, for \(a \neq 0\), there exist \(\tilde{t}_a \in (\pi, 2\pi)\) and \(t_a \in (2\pi, -\pi)\) such that \(P_a(\tilde{t}_a) = P_a(t_a) = 0\). Indeed, function \(P_a(t)\) is continuous on the intervals \((\pi, 2\pi)\) and \((-2\pi, -\pi)\). Computing \(P_a'(t) = \csc^2(t) \tanh(at) - a \cot(t) \csch^2(at)\), we see that \(P_a'(t) \neq 0\) for every \(a \neq 0\) and \(t \in (\pi, 2\pi) \cup (-2\pi, -\pi)\), which implies that \(P_a(t)\) has at most one zero in each one of these intervals. This proof ends by applying Proposition 8 for \(n = \ell = 1\). \(\square\)

Lemma 9 was stated assuming \(a \neq 0\). For \(a = 0\), we define the sets of functions \(G^i = \{\xi_1, \xi_2^i\}\) for \(i = 1, 2\) and we prove the next lemma.

**Lemma 10** Then, following statements hold.

(a) The set of functions \(G^1\) is an ECT-system on the intervals \((0, \pi), (-\pi, 0), (\pi, 2\pi), \) and \((-2\pi, -\pi)\).

(b) The set of functions \(G^2\) is an ECT-system on \(\mathbb{R}^+\).

**Proof** Assuming \(a = 0\) and proceeding analogously to the proof of Lemma 9, we compute the Wronskians.
\[
W_1(t) = 1,
\]
\[
W_2^1(t) = \csc(t) \cot(t) - \csc^2(t),
\]
\[
W_2^3(t) = \csc(t) \coth(t) - \csch^2(t).
\]
From here, it is easy to see that the Wronskian \(W_2^2\) does not vanish at any point of the interval \((0, \pi), (-\pi, 0), (\pi, 2\pi), \) and \((-2\pi, -\pi)\) and that the Wronskian \(W_2^3\) does not vanish at any point of \(\mathbb{R}^+\). \(\square\)

### 3 Proof of Theorems 1, 2 and 4, and Corollary 3

The proofs of Theorem 1 and Corollary 3 will be immediate consequences of Propositions 11–16; the proof of Theorem 2 will be an immediate consequence of Propositions 16–21; and the proof of Theorem 4 will be an immediate consequence of Propositions 22–25 and Corollary 3.

We note that some of the partial results contained in this section could be obtained using different approaches. Particularly, the results in [14] may lead to the Propositions 11, 12, and 13. For sake of completeness, we shall prove all propositions using the same technique.

Using Proposition 5, the matrix which defines the right system \(X\) of (1) is transformed into one of the matrices of the statements (a)–(d), namely \(A = (a_{ij})_{i,j}\). Of course, the transformation is applied to the whole system (1), so the matrix which defines the left system \(Y\) is also transformed into a (general) matrix \(B = (b_{ij})_{i,j}\). Then, system (1), after this transformation, reads
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} \\
&a_{21} & a_{22}
\end{pmatrix} \begin{pmatrix} x + u_1 \\
y + u_2 \end{pmatrix} \quad \text{if} \quad x > 0,
\]
\[
\begin{pmatrix}
\dot{y} \\
\dot{x}
\end{pmatrix} = \begin{pmatrix}
b_{11} & b_{12} \\
&b_{21} & b_{22}
\end{pmatrix} \begin{pmatrix} x + v_1 \\
y + v_2 \end{pmatrix} \quad \text{if} \quad x < 0. \quad (4)
\]

The solution of (4) can be easily computed, because it is a piecewise linear differential system. So let \(\varphi^+(t, x, y) = (\varphi_1^+(t, x, y), \varphi_2^+(t, x, y))\) be the solution of (4) for \(x > 0\) such that \(\varphi^+(0, x, y) = (x, y)\). Similarly, let \(\varphi^-(t, x, y) = (\varphi_1^-(t, x, y), \varphi_2^-(t, x, y))\) be the solution of (4) for \(x < 0\) such that \(\varphi^-(0, x, y) = (x, y)\).

In what follows, let \(t^+(y) > 0\) be the smallest positive time such that \(\varphi_1^+(t^+(y), 0, y) = 0\), and let \(t_+(y) < 0\) be the biggest negative time such that \(\varphi_1^+(t_+(y), 0, y) = 0\). Analogously, let \(t^-(y) < 0\) be the biggest negative time such that \(\varphi_1^+(t^-(y), 0, y) = 0\), and \(t_-(y) > 0\) be the smallest positive time such that \(\varphi_1^+(t_-(y), 0, y) = 0\). Observe that the functions \(t^+(y)\),
Down the line, the zeros of the function \( f(y) = \varphi_2^+(t^+(y), 0, y) - \varphi_2^-(t^-(y), 0, y), \) (5) on the domain \( J^*. \)

Equivalently, if \( t_+(y) < 0 \) and \( t_-(y) > 0 \) are defined, then there exists a limit cycle passing through the point \( (0, y) \) with \( y \in J_0 = \text{Dom}(t_+) \cap \text{Dom}(t_-) \) if and only if \( \varphi_2^+(t^+(y), 0, y) = \varphi_2^-(t^-(y), 0, y). \) Thus, in this case, we must study the zeros \( y^* \) of the function

\[
f(y) = \varphi_2^+(t^+(y), 0, y) - \varphi_2^-(t^-(y), 0, y),
\]

(6) on the domain \( J_0. \)

Since the vectors fields \( X \) and \( Y \) are linear, then a limit cycle passing through a point \((x_0, y_0)\) must contain points of kind \((0, y^*)\) and \((0, y_s)\) such that \( y^* \in J^* \) and \( y_s \in J_0. \) Therefore, detecting all the zeros of (5) or (6) we must detect all non-sliding limit cycles of (4).

Let \( X = (X_1, X_2) \) and \( Y = (Y_1, Y_2). \) We say that a point \((0, y)\) is an

(a) **invisible fold point for the right system** when

\[
X_1(0, y) = 0 \quad \text{and} \quad \frac{\partial X_1}{\partial y}(0, y)X_2(0, y) < 0;
\]

(c) **invisible fold point for the left system** when

\[
Y_1(0, y) = 0 \quad \text{and} \quad \frac{\partial Y_1}{\partial y}(0, y)Y_2(0, y) > 0.
\]

An affine (linear) change of variables in the plane preserving the straight line \( x = 0 \) will be called in what follows a \( \Sigma^- \)-preserving affine (linear) change of variables, and a \( \Sigma^- \)-preserving affine (linear) change of variables which also preserves the semiplane \( x > 0 \) will be called in what follows a \( \Sigma^+ \)-preserving affine (linear) change of variables. Clearly, a \( \Sigma^+ \)-preserving affine (linear) change of variables also preserves the semiplane \( x < 0. \)

The case when the left system has a focus or a center on \( \Sigma \) will be studied in Sect. 3.1, the case when the left system has a weak saddle will be studied in Sect. 3.2, and the case when the left system has a virtual or real center will be studied in Sect. 3.3.
compute the zeros of the function \( g_1(t) = f(y^+(t)) \).

Since

\[
f(y) = v_2 + e^{\frac{(b_{11} + b_{22})\pi}{\Gamma}} (v_2 + y) + e^{\alpha t^+(y)} \left( y \cos(t^+(y)) - u_1 \sin(t^+(y)) \right),
\]

taking \( \delta = e^{-\frac{(b_{11} + b_{22})\pi}{\Gamma}} \) we obtain

\[
g_1(t) = v_2(1 + \delta) + u_1 \left( \cot(t) - e^{\alpha t} \csc(t) \right) - \delta u_1 \left( \cot(t) - e^{-\alpha t} \csc(t) \right)
= k_1 \xi_1 + k_2 \xi_2 + k_3 \xi_3^4,
\]

for \( t \in I \subset (0, \pi) \). Clearly \( k_1 = v_2(1 + \delta) \), \( k_2 = u_1 \), \( k_3 = -\delta u_1 \), and \( I = t^+((Y_M, \infty)) \). Note that \( I = (0, \pi) \) provided \( v_2 \geq au_1 \).

Applying Lemma 9(a), we obtain the inequality \( \mathcal{N}(F_b, F_r) \leq 2 \). The equality is not a direct consequence of Lemma 9(a) and Proposition 8 because the parameters \( k_1 \) and \( k_2 \) are not free to be chosen among the real numbers. However, choosing \( a = -1, u_1 = 8, v_2 = -40/9, \) and \( b_i, j \) for \( i, j = 1, 2 \) such that \( \delta = 1/8 \) we obtain \( k_1 = -5, k_2 = 8, \) and \( k_3 = -1 \). We claim that for these choice of parameters the function (7) has exactly 2 simple zeros in \((0, \pi)\).

To see the claim, it is sufficient to prove the existence of two distinct zeros in \((0, \pi)\). Indeed, once proved their existence, Lemma 9(a) and Proposition 8 imply, directly, that they are simple and that the function \( g_1 \) has no more zeros in \((0, \pi)\). This argumentation will be recurrently used, with less details, throughout the proofs in this section. Accordingly, we compute \( g_1(1/2) \approx 1.13 > 0 \), \( g_1(3/2) \approx -1.80 < 0 \), and \( g_1(5/2) \approx 4.89 > 0 \). Thus, from continuity, there exist at least two zeros in the interval \((1/2, 5/2) \subset (0, \pi)\), which leads to the claim. We may also estimate \( t_1 \approx 0.770 \) and \( t_2 \approx 2.203 \). Hence, for \( y^+(t_1) \approx 16.572 > Y_M = 8 \) and \( y^+(t_2) \approx 95.667 > 8 \) there exist two limit cycles of system (4) passing, respectively, through the points \((0, y^+(t_1))\) and \((0, y^+(t_2))\).

Now, the right system is a center if and only if \( a = 0 \), in this case \( \xi_2^2(t) = \xi_3^4(t) = \cot(t) - \csc(t) \), so the function (7) becomes

\[
g_1(t) = k_1 \xi_1 + \tilde{k}_2 \xi_2^4,
\]

where \( \tilde{k}_2 = k_2 + k_3 \). Since now \( k_1 \) and \( \tilde{k}_2 \) can be chosen freely, we conclude that \( \mathcal{N}(F_b, C_v) = 1 \) we obtain, from Lemma 10(a), that \( \mathcal{N}(F_b, C_v) \leq 1 \).

The left system has a center if and only if \( b_{22} = -b_{11} \) and \( b_{11}^2 + b_{12} b_{21} < 0 \). In this case, \( \delta = 1, k_1 = 2v_2, k_3 = -k_2 = -u_1 \), so the function (7) becomes

\[
g_4(t) = k_1 \xi_1 - 2k_2 \xi_2^4.
\]

Multiplying \( g_4 \) by a parameter, if needed, we see that \( k_1 \) and \( k_2 \) can be chosen freely. Hence, applying Lemma 9(d) we conclude that \( \mathcal{N}(C_b, F_v) = 1 \).

Finally, the lateral systems are centers if and only if \( a = 0, b_{22} = -b_{11} \) and \( b_{11}^2 + b_{12} b_{21} < 0 \). In this case, the function (7) becomes \( g_1(t) = k_1 \). So if \( k_1 \neq 0 \), that is \( v_2 \neq 0 \), then there is no solutions for the equation \( g_1(t) = 0; \) if \( k_1 = 0 \), that is \( v_2 = 0 \), then \( g_1 = 0 \), which implies that all the equations of system (4) passing through \((0, y)\) for \( y > Y_M \) are periodic solutions, in other words there are no limit cycles. Hence, we conclude that \( \mathcal{N}(C_b, C_v) = 0 \).

\[\square\]

**Proposition 12** *The equalities \( \mathcal{N}(F_b, F_r) = \mathcal{N}(C_b, F_r) = 2, \mathcal{N}(F_b, C_r) = 1 \) and \( \mathcal{N}(C_b, C_r) = 0 \) hold.*

**Proof** From Proposition 5(c) and by a \( \Sigma^+\)-preserving translation, we can assume that \( a_{11} = a_{22} = a \) with \( a \in \mathbb{R}, a_{12} = -a_{21} = 1, u_2 = 0, \) and \( u_1 < 0 \) because the right system has a focus which is real for system (4).

In the case that \( a < 0 \) it is easy to see that the focus \((-u_1, 0)\) is an attractor singularity and that the point \((0, -au_1) \in \Sigma\) is a visible fold point for the right system. So the function \( t_+\rangle(y) < 0 \) is defined for
every $y < - au_1$. Moreover, its image is the interval $(- \tau, - \pi)$, where $\tau = - t_1(- au_1)$ so $\pi < \tau < 2 \pi$. Indeed, given $y < - au_1$ consider the line $\ell(y)$ passing through the focus point $(- u_1, 0)$ and $(0, y)$. The trajectory of the left system starting at $(0, y)$ returns to the line $\ell(y)$ at $t = - \pi$, so it must return to $\Sigma$ for $-2 \pi < - \tau < t < - \pi$. Thus, $t_+(y) \in (- \pi, - \pi)$ for every $y < - au_1$ (see Fig. 2 left).

In the other case, $a > 0$ the focus $(- u_1, 0)$ is a repulsive singularity. Considering now the function $t_+(y) > 0$ defined for every $y > - au_1$, the same analysis can be done (see Fig. 2 right).

From now on in this proof we assume, without loss of generality, that $a < 0$.

We know that $\phi_1^+(t_+(y), 0, y) = 0$ for every $y < - au_1$, that is

$$- u_1 + e^{a t_+(y)}(u_1 \cos(t_+(y)) + y \sin(t_+(y))) = 0.$$  

Hence, taking $y_+(t) = u_1 F(t)$ for $t \in (- \tau, - \pi)$ we have that $y_+(t_+(y)) = y$ for every $y < - au_1$.

Now, we claim that $t_+(y_+(t)) = t$ for every $t \in (- \tau, - \pi)$, Indeed, for $t_0 \in (- \tau, - \pi)$, let $y_0 = y_+(t_0)$.

From Lemma 6(a’) $y_+(t)$ is decreasing on the interval $(- \tau, - \pi) \subset (-2 \pi, - \pi)$, and since $y_+(\tau) = - au_1$ it follows that $y_0 < - au_1$. So from the above comments, we obtain that $y_0 = y_+(t_+(y_0))$. Thus, $y_+(t_0) = y_+(t_+(y_0))$. Again from Lemma 6(a’) $y_+(t) = u_1 F(t)$ is injective on the interval $(\tau, - \pi) \subset (-2 \pi, - \pi)$, so $t_0 = t_+(y_0)$. Hence, $t_0 = t_+(y_0) = t_+(y_+(t_0))$. Since $t_0$ was arbitrarily chosen in $(- \tau, - \pi)$, we conclude that $t_+(y_+(t)) = t$ for every $t \in (- \tau, - \pi)$. Therefore, the function $t_+: (- \infty, - au_1) \to (- \tau, - \pi)$ is invertible with inverse equal to $y_+: (- \tau, - \pi) \to (- \infty, - au_1)$.

Let $Y_m = \min(- au_1, - v_2)$, so computing the zeros of the function (6) for $y < Y_m$ is also equivalent to compute the zeros of the function (7) now for $t \in I \subset (- \tau, - \pi)$, where $I = t_+ ((- \infty, Y_m))$. Note that $I = (- \tau, - \pi)$ provided $v_2 \leq au_1$.

Applying Lemma 9(a’), we conclude that $\mathcal{N}(F_b, F_r) \leq 2$. Now choosing $a = -3/4, u_1 = -1/10, v_2 = -3/22, \text{ and } b_{i,j}$ for $i, j = 1, 2$ such that $\delta = 10$ we obtain $k_1 = -3/2, k_2 = -1/10, \text{ and } k_3 = 1$ which implies, analogously to the proof of Proposition 11, that (7) has exactly 2 simple zeros, at first, in $(- 2 \pi, - \pi)$, namely $t_1 \approx -3.733$ and $t_2 \approx -3.250$. We wish to conclude that $t_1, t_2 \in (- \tau, - \pi)$ or, equivalently, $y^+(t_1), y^+(t_2) < Y_m$. Indeed, $y^+(t_1) \approx -0.160 < Y_m = -0.075$ and $y^+(t_2) \approx -0.999 < -0.075$. So there exist two limit cycles of system (4) passing, respectively, through the points $(0, y_+(t_1))$ and $(0, y_+(t_2))$.

The equalities $\mathcal{N}(F_b, C_r) = 1$ and $\mathcal{N}(C_b, C_r) = 0$ follow in a similar way to the proof of Proposition 11.

The inequality $\mathcal{N}(C_b, C_r) \leq 2$ also follows in a similar way to the proof of Proposition 11 but now applying Lemma 9(a’) to the function $g_4(t) = 2v_2 \xi_1 - 2u_1 \xi_2^4$ for $t \in (- \tau, - \pi)$. To see the equality, we take $a = u_1 = -1/10, \text{ and } v_2 = -1/20$. It implies that $g_4$ has exactly 2 simple zeros, at first, in $(- 2 \pi, - \pi)$, namely $t_1 \approx -4.176$ and $t_2 \approx -4.796$. Again, $y^+(t_1) \approx -0.136 < Y_m = -0.01$ and $y^+(t_2) \approx -0.545 < -0.01$. So there exist two limit cycles of system (4) passing, respectively, through the points $(0, y_+(t_1))$ and $(0, y_+(t_2))$. 

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and \((0, y_+(t_2))\). It concludes the proof of this proposition.

**Proposition 13** The equalities \(N(F_b, F_b) = N(F_b, C_b) = 1\) and \(N(C_b, C_b) = 0\) hold.

**Proof** Here \(u_1 = 0\), because the right system has its focus on the line \(\Sigma\). From Proposition 5(c) and by a \(\Sigma^+\)-preserving translation, we can assume that \(a_{11} = a_{22} = a\) with \(a \in \mathbb{R}, a_{12} = -a_{21} = 1\), and \(u_2 = 0\).

The function \(t^+(y) > 0\) is defined for every \(y > 0\), because the point \((0, 0)\) is a focus for the right system. Moreover, we compute \(t^+(y) = \pi\).

Let \(Y_M = \max\{0, -v_2\}\), so computing the zeros of the function (5) for \(y > Y_M\) is equivalent to compute the zeros of the linear function

\[f_1(y) = k_1 + k_2 y, \tag{8}\]

for \(y > Y_M\), where \(k_1 = v_2(1 + \delta)\) and \(k_2 = (\delta - e^{a\pi})\). Hence, \(N(F_b, F_b) \leq 1\). Nevertheless, we can choose coefficients such that \(\tilde{y} = k_1 + k_2 y > Y_M\) is the unique zero of (8).

From here, the equalities \(N(F_b, C_b) = 1\) and \(N(C_b, C_b) = 0\) follow similarly to the proof of Proposition 11. It concludes the proof of this proposition. \(\square\)

**Proposition 14** The equalities \(N(F_b, N_v) = 2\) and \(N(C_b, N_v) = 1\) hold.

**Proof** From Proposition 5(b) and by a \(\Sigma^+\)-preserving translation, we can assume that \(a_{11} = a_{22} = a\) with \(|a| > 1, a_{12} = a_{21} = 1, u_2 = 0\), and \(u_1 > 0\), because the right system is a diagonalizable node, which is virtual for system (4).

It is easy to see that the point \((0, -a u_1) \in \Sigma\) is an invisible fold point for the right system.

In the case \(a < 1\), the node \((-u_1, 0)\) is an attractor singularity. The stable manifold and the strong stable manifold of the node intersect \(\Sigma\) at the points \((0, y^s)\) and \((0, y^{ss})\), respectively, where \(y^s = u_1 < -a u_1\) and \(y^{ss} = u_1 < -u_1\). So the function \(t^+(y) > 0\) is defined for every \(y > -au_1\) (see Fig. 3 left).

In the other case \(a > 1\), the node \((-u_1, 0)\) is an repulsive singularity. The stable manifold and the strong stable manifold of the node intersect \(\Sigma\) at the points \((0, y^s)\) and \((0, y^{ss})\), respectively, where \(y^s = -u_1 > -a u_1\) and \(y^{ss} = u_1 > -u_1\). So the function \(t^+(y) < 0\) is defined for every \(y < -au_1\) (see Fig. 3 right).

From now on in this proof we assume, without loss of generality, that \(a < -1\).

We know that \(\varphi^+_1(t^+(y), 0, y) = 0\) for every \(y > -au_1\), that is

\[-u_1 + e^{at^+(y)} (u_1 \cosh(t^+(y)) + y \sinh(t^+(y))) = 0.\]

Hence, taking \(y^+(t) = u_1 G(t)\) for \(t \in \mathbb{R^+}\) we have that \(y^+ (t^+(y)) = y\) for every \(y > -au_1\).
The image of the function \( t^+ \) is \( \mathbb{R}^+ \). Indeed, computing implicitly the derivative in the variable \( y \) of the identity \( y^+ (t^+ (y)) = y \) we obtain
\[
\frac{dt^+ (y)}{dy} = P \left( t^+ (y) \right),
\]
where
\[
P(\theta) = \frac{\sinh(\theta)}{u_1 (\cosh(\theta) - e^{-a\theta} (a + \coth(\theta)))}.
\]
It is easy to see that \( P(\theta) > 0 \) for every \( \theta > 0 \). So any solution \( \theta(y) \) of the differential equation \( \dot{\theta} = F(\theta) \) starting at \( \theta = \bar{\theta} > 0 \) and \( y = \tilde{y} \), i.e., \( \theta(\tilde{y}) = \bar{\theta} \), keeps itself positive for every \( y > \tilde{y} \); moreover, this solution will be strictly increasing. Hence, we conclude that \( t^+ (y) > 0 \) is a positive strictly increasing function for \( y > -a u_1 \).

We claim that \( t^+ (y^+(t)) = t \) for every \( t > 0 \). Indeed, for \( t_0 > 0 \), let \( y_0 = y^+(t_0) \). From Lemma 6(b) \( y_0 > -a u_1 \), so from the above comments we obtain that \( y_0 = y^+ (t^+ (y_0)) \). Thus, \( y^+(t_0) = y^+ (t^+ (y_0)) \).

Again from Lemma 6(b) \( y^+(t) = u_1 G(t) \) is injective on \( \mathbb{R}^+ \), so \( t_0 = t^+ (y_0) \). Hence, \( t_0 = t^+ (y_0) = t^+ (y^+(t_0)) \). Since \( t_0 > 0 \) was arbitrarily chosen, we conclude that \( t^+ (y^+(t)) = t \) for every \( t > 0 \). Therefore, the function \( t^+ : (-a u_1, \infty) \to \mathbb{R}^+ \) is invertible with inverse equal to \( y^+ : \mathbb{R}^+ \to (-a u_1, \infty) \).

Computing the zeros of the function (5) for \( y > Y_M = \max \{-a u_1, -v_2\} \) is equivalent to compute the zeros of the function
\[
g_2(t) = f (y^+(t)) = k_1 \xi_1 + k_2 \xi_2^2 + k_3 \xi_3^2 \tag{9}
\]
for \( t \in I \subset \mathbb{R}^+ \), where \( k_1 = v_2 (1 + \delta), k_2 = u_1, k_3 = -\delta u_1, \delta = e^{-b_1 (1+2 y_t^2)}, \) and here \( I = t^+ ((Y_M, \infty)) \).

Note that \( I \in \mathbb{R}^+ \) provided \( v_2 \geq a u_1 \).

Applying Lemma 9(b), we conclude that \( N(F_b, N_v) \leq 2 \). Now choosing \( a = -3/2, u_1 = 75, v_2 = -375/4, \) and \( b_{1,j} \) for \( i, j = 1, 2 \) such that \( \delta = 1/15 \) we obtain \( k_1 = -100, k_2 = 75, \) and \( k_3 = -5 \) which implies, analogously to the proof of Proposition 11, that (9) has 2 zeros in \( \mathbb{R}^+ \), namely \( t_1 \approx 0.704 \) and \( t_2 \approx 2.069 \). Hence, for \( y^+ (t_1) \approx 158.781 > Y_M = 112.5 \) and \( y^+ (t_2) \approx 351.490 > 112.5 \) there exist two limit cycles of system (4) passing, respectively, through the points \((0, y^+ (t_1))\) and \((0, y^+ (t_2))\).

From here, the equality \( N(F_b, N_v) = 1 \) follows similarly to the proof of Proposition 11 but now applying Lemma 9(e) to the function \( g_2(t) = k_1 \xi_1 - 2k_2 \xi_2^5 \) it completes the proof of this proposition.

**Proposition 15** The equalities \( \mathcal{N}(F_b, i N_v) = 2 \) and \( \mathcal{N}(C_b, i N_v) = 1 \) hold.

**Proof** From Proposition 5(d) and by a \( \Sigma^+ \)-preserving translation, we can assume that \( a_{11} = a_{12} = a_{22} = \lambda \) with \( \lambda = \pm 1, a_{21} = 0, u_2 = 0, \) and \( u_1 > 0 \), because the right system is a non-diagonalizable node, which is virtual for system (4).

It is easy to see that for \( \lambda = \pm 1 \) the point \((0, -u_1) \in \Sigma \) is a invisible fold point for the right system and that the invariant manifold of the node intersects \( \Sigma \) at the origin \((0, 0) \) (see Fig. 4). In order to fix the clockwise orientation of the flow of system (4), we assume that \( \lambda = 1 \); otherwise, the first return map would not be defined and there would not exist limit cycles. In this case, the function \( t^+(y) < 0 \) is defined for every \( y < -u_1 \).

We know that \( \varphi_1^+(t^+(y), 0, y) = 0 \) for every \( y < -u_1 \), that is
\[
-u_1 + e^{t^+(y)} (u_1 + y t) = 0.
\]

Hence, taking \( y^+(t) = u_1 H(t) \) for \( t \in \mathbb{R}^- \) we have that \( y^+ (t^+(y)) = y \) for every \( y < -u_1 \).

The image of the function \( t^+ \) is \( \mathbb{R}^- \). Indeed, computing implicitly the derivative in the variable \( y \) of the identity \( y^+ (t^+(y)) = y \) we obtain
\[
\frac{dt^+(y)}{dy} = Q (t^+(y)), \text{ where } Q(\theta) = \frac{e^{\theta^2}}{u_1 (e^{\theta^2} - \theta - 1)}.
\]

So the function \( t^+ \) is the solution \( \theta(y) \) of the above differential equation such that \( \theta(-u_1) = 0 \). It is easy to see that \( Q(\theta) > 0 \); moreover, by continuity we have that \( Q(0) = 2 \). So it follows that the solution \( \theta(y) \) is strictly increasing. Hence, we conclude that \( t^+(y) < 0 \) is strictly increasing function such that \( t^+ (-u_1) = 0 \), which implies that \( t^+(y) < 0 \) for \( y < -u_1 \).

Now, we claim that \( t^+ (y^+(t)) = t \) for every \( t < 0 \). Indeed, for \( t_0 < 0 \), let \( y_0 = y^+(t_0) \). From Lemma 6(c)
Proof From Proposition 5(a), we can assume that \(a_{11} = a_{22} = a\) with \(|a| < 1\), \(a_{12} = a_{21} = 1\), \(u_2 = 0\), and \(u_1 < 0\), because the right system is a saddle, which is real for system (4).

It is easy to see that the point \((0, -au_1) \in \Sigma\) is an invisible fold point for the right system and that the stable and unstable invariant manifolds of the saddle intersect \(\Sigma\) at the points \((0, y^s)\) and \((0, y^u)\), respectively, where \(y^s = -u_1\) and \(y_u = u_1\). So the function \(t^+(y)\) is defined for every \(-au_1 < y < -u_1\).

We know that \(\varphi^+_c(t^+(y), 0, y) = 0\) for every \(-au_1 < y < -u_1\), that is \(-u_1 + e^{au^+(y)}(u_1 \cosh(t^+(y)) + y \sinh(t^+(y))) = 0\).

Hence, taking \(y^+(t) = u_1 G(t)\) for \(t \in \mathbb{R}^+\) we have that \(y^+(t^+(y)) = y\) for every \(-au_1 < y < -u_1\).

In the proof of Proposition 14, we have seen that the function \(t^+: (-\infty, \infty) \to \mathbb{R}^+\) is invertible with inverse equal to \(y^+: \mathbb{R}^+ \to (-\infty, -u_1)\). So its restriction to \(-au_1 < y < -u_1\) is also invertible with inverse defined in \(t^+(-au_1, -u_1)\).

Computing the zeros of the function (5) for max \((-au_1, -v_2) = Y_M < y < -u_1\) is equivalent to computing the zeros of the function (9) for \(t \in I \subset \mathbb{R}^+\), where \(k_1 = v_2(1 + \delta), k_2 = u_1, k_3 = -\delta u_1, \delta = e^{-\frac{b_{11} + b_{22}}{2}}, \) and here \(I = y^-((\infty, Y_M))\).

Note that \(I = \mathbb{R}^+\) provided \(v_2 \geq u_1\).

Applying Lemma 9(c), we conclude that \(N(F_b, iN_v) \leq 2\). Now choosing \(u_1 = 149, v_2 = 298/3\), and \(b_{i,j}\) for \(i, j = 1, 2\) such that \(\delta = 1/149\), we obtain \(k_1 = 100, k_2 = 149, k_3 = -1\), which implies, analogously to the proof of Proposition 11, that (10) has 2 zeros in \(\mathbb{R}^+, \) namely \(t_1 \approx -6.146\) and \(t_2 \approx -0.897\).

Hence, for \(y^+(t_1) \approx -11, 295, 600 < Y_m = -149\) and \(y^+(t_2) \approx -241.197 < -149\) there exist two limit cycles of system (4) passing, respectively through the points \((0, y^+(t_1))\) and \((0, y^+(t_2))\).

From here, the equality \(N(C_b, iN_v) = 1\) follows similarly to the proof of Proposition 11 but now applying Lemma 9(f) to the function \(g_3(t) = \frac{k_1}{\xi_1} - \frac{2k_2}{\xi_2}\).

It concludes the proof of proposition.

**Proposition 16** The equalities \(\mathcal{N}(F_b, S_v) = 2, \mathcal{N}(F_b, S_v^0) = \mathcal{N}(C_b, S_v) = 1\) and \(\mathcal{N}(C_b, S_v^0) = 0\) hold.

**Proof** From Proposition 5(a) and by a \(\Sigma^+\)-preserving translation, we can assume that \(a_{11} = a_{22} = a\) with \(|a| < 1\), \(a_{12} = a_{21} = 1\), \(u_2 = 0\), and \(u_1 < 0\), because the right system is a saddle, which is real for system (4).

3.2 Left system has a weak saddle

In this case, \(b_{22} = -b_{11}, b_{11}^2 + b_{12}b_{21} > 0\) and \(v_1 > 0\) and the point \((-v_1, -v_2)\) is a singularity of saddle type.

Let \(\Gamma = \sqrt{b_{11}^2 + b_{12}b_{21}}\), let \(y^u\) be the \(y\)-coordinate of the intersection between the unstable manifold with \(\Sigma\), and let \(y^s\) be the \(y\)-coordinate of the intersection between the stable manifold with \(\Sigma\). We compute

\[
y^u = -v_2 + \frac{v_1 (\Gamma - b_{11})}{b_{12}}\quad\text{and}\quad y^s = -v_2 - \frac{v_1 (\Gamma + b_{11})}{b_{12}}.
\]

In order to fix the clockwise orientation of the flow of system (4), we assume that \(y^s < y^u\), which is equivalent to assume that \(b_{12} > 0\).
The left system has an invisible fold point \((0, \bar{y})\) given by
\[
\bar{y} = -v_2 - \frac{b_{11}v_1}{b_{12}}.
\]
For \(y^s < y < y^u\), we define
\[
t^s(y) = \frac{1}{\Gamma} \log \left( \frac{v_1 (\Gamma - b_{11}) - b_{12} (v_2 + y)}{v_1 (\Gamma + b_{11}) + b_{12} (v_2 + y)} \right).
\]
So \(t^- = t^s \leq 0\) for \(\bar{y} < y < y^u\) and \(t^-(y) = t^s(y) > 0\) for \(y^s < y < \bar{y}\).

**Proposition 17** The equalities \(N(S_r^0, F_v) = 1\) and \(N(S_v^0, C_v) = 0\) hold.

**Proof** From Proposition 5(c), we can assume that \(a_{11} = a_{22} = a\) with \(a \in \mathbb{R}\), \(a_{12} = -a_{21} = 1\), and by a \(\Sigma^+\)-preserving translation we can take \(u_2 = 0\). Moreover, \(u_1 > 0\) because the right system has a focus which is virtual for system (4).

From the proof of Proposition 11, we know that the function \(t^+ : (-a u_1, \infty) \to (0, \pi)\), such that \(\varphi^+(t^+(y), 0, y) = 0\) for \(y > -a u_1\), is invertible with inverse \(y^+ : (0, \pi) \to (-a u_1, \infty)\) given by \(y^+(t) = u_1 F(t)\).

Let \(Y_M = \max \{-a u_1, \bar{y}\}\), so computing the zeros of the function (5) for \(Y_M < y < y^u\) is equivalent to compute the zeros of the function
\[
g_4(t) = f(y^+(t)) = k_1 \xi_1 + k_2 \xi_2^4 \tag{11}
\]
for \(t \in I \subset (0, \pi)\), where \(k_1 = 2(b_{11}v_1 + b_{12}v_2)/b_{12}\) and \(k_2 = -2u_1\), and here \(I = t^+ ((Y_M, y^u))\). Multiplying the function \(g_4\) by a parameter, if necessary, we see that \(k_1\) and \(k_2\) can be chosen freely. So applying Lemma 9(d), we conclude that \(N(S_r^0, F_v) = 1\).

The right system has a center if and only if \(a = 0\). In this case, \(\xi_2^4 = 0\) and the function (11) becomes \(g_4(t) = k_1\). So if \(k_1 \neq 0\), that is \(b_{11}v_1 \neq -b_{12}v_2\), then there are no solutions for the equation \(g_4(t) = 0\); and if \(k_1 = 0\), that is \(b_{11}v_1 = -b_{12}v_2\), then \(g_4 = 0\), that is system (4) is a center. Hence, we conclude that \(N(S_v^0, C_v) = 0\). \(\square\)

**Proposition 18** The equalities \(N(S_r^0, F_r) = 2\) and \(N(S_v^0, C_r) = 0\) hold.

**Proof** From Proposition 5(c), we can assume that \(a_{11} = a_{22} = a\) with \(a \in \mathbb{R}\), \(a_{12} = -a_{21} = 1\), and by a \(\Sigma^+\)-preserving translation we can take \(u_2 = 0\). Moreover, \(u_1 < 0\) because the right system has a focus, which is real for system (4).

From the proof of Proposition 12, we know that the function \(t^+ : (-\infty, -au_1) \to (-\pi, -\pi)\) is invertible with inverse \(y^+ : (-\pi, -\pi) \to (-\infty, -au_1)\) given by \(y^+(t) = u_1 F(t)\). Here, as we have done in the proof of Proposition 12, we are assuming, without loss of generality, that \(a < 0\).

Let \(Y_M = \min \{-au_1, \bar{y}\}\), so computing the zeros of the function (6) for \(y^s < y < y^u\) is equivalent to compute the zeros of the function (11) now for \(t \in I \subset (-\pi, -\pi)\), where \(I = t^+ ((y^s, Y_M))\).

Applying Lemma 9(d), we conclude that \(N(S_r^0, F_v) \leq 2\). Now choosing \(b_{11} = b_{12} = b_{21} = v_1 = 1\), \(a = -1/10\), \(u_1 = -1/20\), \(u_2 = -21/20\), we obtain \(b_{11}^2 + b_{12}b_{21} = 2 > 0\), and \(k_2 = -k_1 = 1/10\) It implies, analogously to the proof of Proposition 11, that (9) has 2 zeros in \((-\pi, -\pi)\), namely \(t_1 \approx -3.508\) and \(t_2 \approx -5.646\). Hence, for \(y^+(t_1) \approx -0.048 \in (y^s, Y_M) \approx (-1.364, -0.005)\) and \(y^+(t_2) \approx -0.05 \in (-1.364, -0.005)\) there exist two limit cycles of system (4) passing, respectively, through the points \((0, y^+(t_1))\) and \((0, y^+(t_2))\).

The equality \(N(S_v^0, C_v) = 0\) follows similarly to the proof of Proposition 17. It concludes the proof of this proposition. \(\square\)

**Proposition 19** The equality \(N(S_r^0, N_v) = 1\) holds.

**Proof** From Proposition 5(b) and by a \(\Sigma^+\)-preserving translation, we can assume that \(a_{11} = a_{22} = a\) with \(|a| > 1\), \(a_{12} = a_{21} = 1\), \(u_2 = 0\), and \(u_1 > 0\), because the right system has a diagonalizable node, which is virtual for system (4).

Following the proof of Proposition 14, the function \(t^+ : (-au_1, \infty) \to \mathbb{R}^+\) is invertible with inverse \(y^+ : \mathbb{R}^+ \to (-au_1, \infty)\) given by \(y^+(t) = u_1 G(t)\). Here, as we have done in the proof of Proposition 13 we are assuming, without loss of generality, that \(a < 1\).

Let \(Y_M = \max \{-au_1, \bar{y}\}\), so computing the zeros of the function (5) for \(Y_M < y < y^u\) is equivalent to compute the zeros of the function
\[
g_5(t) = f(y^+(t)) = k_1 \xi_1 + k_2 \xi_2^5 \tag{12}
\]
for \(t \in I \subset \mathbb{R}^+\), where \(k_1 = 2(b_{11}v_1 + b_{12}v_2)/b_{12}\), \(k_2 = -2u_1\), and \(I = y^+ ((Y_M, y^u))\). Multiplying the function \(g_5(t)\) by a parameter, if necessary, we see that the parameters \(k_1\) and \(k_2\) can be chosen freely. So applying Lemma 9(e), we conclude that \(N(S_v^0, N_v) = 1\). \(\square\)
Proposition 20 The equality $\mathcal{N}(S_r^0, iN_e) = 1$ holds.

Proof From Proposition 5(b) and by a $\Sigma^+$-preserving translation, we can assume that $a_{11} = a_{12} = a_{22} = \lambda$ with $\lambda = \pm 1$, $a_{21} = 0$, $u_2 = 0$, and $u_1 > 0$, because the right system has a non-diagonalizable node, which is virtual for system (4).

Following the proof of Proposition 15, the function $t^+ : (-u_1, \infty) \to \mathbb{R}^+$ is invertible with inverse $y^+ : \mathbb{R}^+ \to (-u_1, \infty)$ given by $y^+ (t) = u_1 H(t)$. Here, as we have done in the proof of Proposition 15 we are assuming, without loss of generality, that $\lambda = 1$.

Let $Y_M = \max \{-u_1, \tilde{y}\}$, so computing the zeros of the function (5) for $Y_M < y < y^u$ is equivalent to compute the zeros of the function
\[ g_6 (t) = f (y^+ (t)) = k_1 \xi_1 + k_2 \xi_2^6 \]
for $t \in I \subset \mathbb{R}^+$, where $k_1 = 2(b_{11} v_1 + b_{12} v_2)/b_{12}$, $k_2 = -2u_1$, and $I = y^+ ((Y_M, y^u))$. Multiplying the function $g_6 (t)$ by a parameter, if necessary, we see that $k_1$ and $k_2$ can be chosen freely. So applying Lemma 9(f), we conclude that $\mathcal{N}(S_r^0, iN_e) = 1$. □

Proposition 21 The equalities $\mathcal{N}(S_r^0, S_r) = 1$ and $\mathcal{N}(S_r^0, S_d^0) = 0$ hold.

Proof From Proposition 5(d) and by a $\Sigma^+$-preserving translation, we can assume that $a_{11} = a_{22} = a$ with $|a| < 1$, $a_{12} = a_{21} = 1$, $u_2 = 0$, and $u_1 < 0$, because the right system has a saddle, which is real for system (4).

Following the proof of Proposition 16, the function $t^+ : (-u_1, \infty) \to \mathbb{R}^+$ is invertible with inverse $y^+ : I \to (-u_1, u_1)$ given by $y^+ (t) = u_1 G(t)$, where $I = t^+ (0, \infty)$. Let $Y_M = \max \{-a_1, \tilde{y}\}$ and $Y_m = \min \{u_1, y^u\}$, so computing the zeros of the function (5) for $Y_M < y < Y_m$ is equivalent to compute the zeros of the function (12) for $t \in I \subset \mathbb{R}^+$, where $k_1 = 2(b_{11} v_1 + b_{12} v_2)/b_{12}$ and $k_2 = -2u_1$. Multiplying the function (12) by a parameter, if necessary, we see that $k_1$ and $k_2$ can be chosen freely. So applying Lemma 9(e) we conclude that $\mathcal{N}(S_r^0, S_r) = 1$.

The right system has a saddle with trace equal to 0 if and only if $a = 0$. In this case, $\xi_1^5 = 0$ and the function (12) becomes $g_5 (t) = k_1$. So if $k_1 \neq 0$, that is $b_{11} v_1 \neq 0$, then there are no solutions for the equation $g_5 (t) = 0$. If $k_1 = 0$, that is $b_{11} v_1 = 0$, then $g_5 = 0$, which implies that all the solutions of system (4) passing through $(0, y)$ for $Y_M < y < Y_m$ are periodic solutions; in other words, there are no limit cycles. Hence, we conclude that $\mathcal{N}(S_r^0, S_d^0) = 0$. □

3.3 Left system has a virtual or real center

In this case, $v_1 \neq 0$, $b_{22} = -b_{11}$, $v_{21} + b_{12} b_{21} < 0$ and the point $(-v_1, -v_2)$ is a singularity of center type.

The left system has a fold point $(0, \tilde{y})$ given by
\[ \tilde{y} = -v_2 - \frac{b_{11} v_1}{b_{12}}. \]
which is visible if $v_1 > 0$, and invisible if $v_1 < 0$.

In order to fix the clockwise orientation of the flow of system (4), we assume that $Y_1 (-v_1, 1 - v_2) = b_{12} > 0$.

Let $\Gamma = 2\sqrt{-b_{11}^2 - b_{12} b_{21}}$. We define
\[ t^*(y) = \frac{4}{\Gamma} \arctan \left( \frac{2(b_{11} v_1 + b_{12} v_2 + y)}{v_1 \Gamma} \right). \]
If $v_1 < 0$, then $t^-(y) = t^*(y)$ for $y > \tilde{y}$ and $t^-(y) = t^*(y)$ for $y < \tilde{y}$. If $v_1 > 0$, then $t^-(y) = t^*(y) - 4\pi / \Gamma$ for $y > \tilde{y}$ and $t^-(y) = t^*(y) + 4\pi / \Gamma$ for $y < \tilde{y}$.

Proposition 22 The equalities $\mathcal{N}(C, F_v) = 1$, $\mathcal{N}(C, F_r) = 2$ and $\mathcal{N}(C, C_v) = \mathcal{N}(C, C_r) = 0$ hold.

Proof In Corollary 3, these equalities have already been proved when the left system has a center in $\Sigma$. So we can take $v_1 \neq 0$.

To obtain $\mathcal{N}(C, F_v) = 1$, we follow the proof of Proposition 11, and then, we compute the solutions of the function (5) for $y > Y_M = \max \{\tilde{y}, -a_1 u_1\}$. To obtain $\mathcal{N}(C, F_r) = 2$, we follow the proof of Proposition 12 and then we compute the solutions of the function (6) for $y < Y_m = \min \{\tilde{y}, -a_1 u_1\}$. In both cases, the equations to be solved are equivalent to $k_1 + k_2 \xi_1^5 (t) = 0$, for $t \in (0, \pi)$ and $t \in (-\pi, -\pi)$, respectively. Here, $k_1 = (b_{11} v_1 + b_{12} v_2)/b_{12}$ and $k_2 = -u_1$. So applying statements (d) and (d') of Lemma 9, we conclude that $\mathcal{N}(C, F_v) = 1$ and $\mathcal{N}(C, F_r) \leq 2$, respectively. Moreover, since $\mathcal{N}(C_b, F_v) = 2$, we actually have the equality $\mathcal{N}(C, F_r) = 2$. The equality $\mathcal{N}(C, C_v) = \mathcal{N}(C, C_r) = 0$ follows similarly to the proof of Proposition 17. It concludes the proof of this proposition. □

Proposition 23 The equalities $\mathcal{N}(C, F_v) = 1$ and $\mathcal{N}(C, C_b) = 0$ hold.


**Proof** In Corollary 3, these equalities have already been proved when the left system has a center in $\Sigma$. So we can take $v_{1} \neq 0$.

To obtain $N(C, F_{\sigma}) = 1$, we follow the proof of Proposition 13, and then, we compute the solutions of the function (5) for $y > Y_{M} = \max\{\bar{\gamma}, -a u_{1}\}$, which is equivalent to compute the zeros of the linear equation $k_{1} + k_{2}y = 0$. Here, $k_{1} = 2(b_{11}v_{1} + b_{12}v_{2})/b_{12}$ and $k_{2} = (1 - e^{i\alpha})$. The equalities $N(C, F_{\sigma}) = 1$ and $N(C, S_{b_{0}}) = 0$ follow similarly to the proof of Proposition 11. It concludes the proof of this proposition.

□

**Proposition 24** The equalities $N(C, N_{e}) = N(C, S_{r}) = 1$ and $N(C, S^{0}_{b}) = 0$ hold.

**Proof** In Corollary 3, these equalities have already been proved when the left system has a center in $\Sigma$. So we can take $v_{1} \neq 0$.

To prove the equality $N(C, N_{e}) = 1$, we follow the proof of Proposition 14, and then, we compute the solutions of the function (5) for $y > Y_{M} = \max\{\bar{\gamma}, -a u_{1}\}$. To prove the equality $N(C, S_{r}) = 1$, we follow the proof of Proposition 16, and then, we compute the solutions of the function (5) for $Y_{M} < y < u_{1}$. In both cases, the equations to be solved are equivalent to $k_{1} + k_{2}x_{2}^{5} = 0$, where $k_{1} = 2(b_{11}v_{1} + b_{12}v_{2})/b_{12}$ and $k_{2} = -2u_{1}$. From here, the proofs of the equalities $N(C, N_{e}) = 1$ and $N(C, S_{r}) = 1$ follow similarly to the proofs of the Propositions 19 and 21, respectively. The equality $N(C, S^{0}_{b}) = 0$ follows similarly to the proof of Proposition 21. It concludes the proof of this proposition.

□

**Proposition 25** The equality $N(C, iN_{e}) = 1$ holds.

**Proof** In Corollary 3, this equality has already been proved when the left system has a center in $\Sigma$. So we can take $v_{1} \neq 0$.

Following the proof of Proposition 15, we compute the solutions of the function (6) for $y < Y_{M} = \{\bar{\gamma}, -u_{1}\}$, which is equivalent to compute the zeros of the following equation $k_{1} + k_{2}x_{2}^{6} = 0$, where $k_{1} = (b_{11}v_{1} + b_{12}v_{2})/b_{12}$ and $k_{2} = -u_{1}$. So analogously to the proof of Proposition 20, we conclude that $N(C, iN_{e}) = 1$. It concludes the proof of this proposition.

□

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