Modeling the flyby anomalies with dark matter scattering

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We continue our exploration of whether the flyby anomalies can be explained by scattering of spacecraft nucleons from dark matter gravitationally bound to the earth. We formulate and analyze a simple model in which inelastic and elastic scatterers populate shells generated by the precession of circular orbits with normals tilted with respect to the earth’s axis. Good fits to the data published by Anderson et al. are obtained.

I. INTRODUCTION

In this paper we follow up our earlier investigation of the anomalous geocentric frame orbital energy changes that are observed during earth flybys of various spacecraft, as reported by Anderson et al. Some flybys show energy decreases, and others energy increases, with the largest anomalous velocity changes of order 1 part in $10^6$. While the possibility that these anomalies are artifacts of the orbital fitting method used in is still being actively explored, there is also a chance that they may represent new physics. In we explored the possibility that the flyby anomalies result from scattering of spacecraft nucleons from dark matter particles in orbit around the earth, with the observed velocity decreases arising from elastic scattering, and the observed velocity increases arising from exothermic inelastic scattering, which can impart an energy impulse to a spacecraft nucleon. Many constraints on this hypothesis were analyzed in , with the conclusion that the dark matter scenario is not currently ruled out, but requires dark matter to be non-self-annihilating, with the dark matter scattering cross section on nucleons much larger, and the dark matter mass much lighter, than usually assumed.

However, no attempt was made in to construct a model for the spatial and velocity distribution functions for dark matter populations in earth orbit, to see whether it can fit the flyby data reported in . Formulating such a model is the aim of the present paper. Our basic assumption is to consider two populations of dark matter particles, one of which scatters on nucleons elastically, and the other of which scatters inelastically, each with a shell-like distribution of orbits generated by the precession of a tilted circular orbit around the earth’s rotation axis. The formulas defining
this model are developed in Sec. II, with details of derivations in Appendices, and the results of numerical fits to the flyby data are given in Sec. III. We show that good fits to the data are possible, which leaves dark matter scattering as a viable candidate for explaining the flyby anomalies, pending further investigation of possible artifactual explanations\(^1\) of the flyby data, and further experiments aimed at directly detecting dark matter and determining its properties.

II. FORMULAS DEFINING THE MODEL

A. Velocity change formulas

We recall from \cite{1} formulas for the velocity change when a spacecraft nucleon of mass \(m_1 \approx 1\) GeV and initial velocity \(\vec{u}_1\) scatters from a primary dark matter particle of mass \(m_2\) and initial velocity \(\vec{u}_2\), into an outgoing nucleon of mass \(m_1\) and velocity \(\vec{v}_1\), and an outgoing secondary dark matter particle of mass \(m'_2 = m_2 - \Delta m\) and velocity \(\vec{v}_2\). The inelastic case corresponds to \(m'_2 \neq m_2\), while in the elastic case, \(m'_2 = m_2\) and \(\Delta m = 0\). Under the assumptions, (i) both initial particles are nonrelativistic, so that \(|\vec{u}_1|, |\vec{u}_2| \ll c\), (ii) the center of mass scattering amplitude \(f(\theta)\) depends only on the auxiliary polar angle \(\theta\) of scattering, and (iii) in the exothermic inelastic case, \(\Delta m/m_2\) and \(m'_2/m_2\) are both of order unity, a straightforward calculation gives the outgoing nucleon velocity change, averaged over scattering angles. In the elastic scattering case, with \(\Delta m = 0, m'_2 = m_2\), we have

\[
\langle \delta \vec{v}_1 \rangle = -2 \frac{m_2}{m_1 + m_2} (\vec{u}_1 - \vec{u}_2) \langle \sin^2(\theta/2) \rangle ,
\]

while in the inelastic case a good approximation is

\[
\langle \delta \vec{v}_1 \rangle \simeq \frac{\vec{u}_1 - \vec{u}_2}{|\vec{u}_1 - \vec{u}_2|} \left( \frac{2\Delta m}{m_1(m_1 + m'_2)} \right)^{1/2} c \langle \cos \theta \rangle ,
\]

with \(\langle ... \rangle\) denoting the angular average over the center of mass differential scattering cross section. Since \(\vec{u}_1\) and \(\vec{u}_2\) are typically of order 10 km s\(^{-1}\), the velocity change in the inelastic case is significantly larger than that in the elastic case.

\(^1\) A parameterized post-Newtonian analysis, given in an unpublished memo in the “Talks+Memos” section of the author’s home page, shows that deviations from Einstein gravity within the framework of metric theories of gravity obeying the equivalence principle cannot give residual accelerations large enough to explain the flyby anomalies.
B. Change in outgoing spacecraft velocity

Again as shown in [1], to get the force per unit spacecraft mass resulting from dark matter scatters, that is, the acceleration, one multiplies the velocity change in a single scatter \( \langle \delta \vec{v}_1 \rangle \) by the number of scatters per unit time. This latter is given by the flux \( |\vec{u}_1 - \vec{u}_2| \), times the scattering cross section \( \sigma \), times the dark matter spatial and velocity distribution \( \rho(\vec{x}, \vec{u}_2) \). Integrating out the dark matter velocity, one thus gets for the force acting at the point \( \vec{x}(t) \) on the spacecraft trajectory with velocity \( \vec{u}_1 = \frac{d\vec{x}(t)}{dt} \),

\[
\delta \vec{F} = \int d^3 u_2 \langle \delta \vec{v}_1 \rangle |\vec{u}_1 - \vec{u}_2| \sigma \rho(\vec{x}, \vec{u}_2) \cdot . \tag{3}
\]

Equating the work per unit spacecraft mass along a trajectory from \( t_i \) to \( t_f \) to the change in kinetic energy per unit mass (assuming that the initial and final times are in the asymptotic region where the potential energy can be neglected) we get

\[
\frac{1}{2}(\vec{v}_f^2 - \vec{v}_i^2) = \vec{v}_f \cdot \delta \vec{v}_f = \int_{t_i}^{t_f} dt (d\vec{x}/dt) \cdot \delta \vec{F} = \int_{t_i}^{t_f} dt \int d^3 u_2 (d\vec{x}/dt) \cdot \langle \delta \vec{v}_1 \rangle |\vec{u}_1 - \vec{u}_2| \sigma \rho(\vec{x}, \vec{u}_2) . \tag{4}
\]

C. Cross section and scattering-angle averaged kinematics

Let \( W \) be the center of mass scattering energy of the dark matter-spacecraft nucleon system. A simple calculation shows that to a good approximation we have

\[
\frac{W}{(m_1 + m_2)c^2} \simeq 1 + \frac{m_1 m_2}{2(m_1 + m_2)^2} \frac{(\vec{u}_1 - \vec{u}_2)^2}{c^2} , \tag{5}
\]

and so for \( m_2 \leq m_1 \) and for the nonrelativistic velocities \( \vec{u}_1, \vec{u}_2 \) of interest, the scattering is very close to threshold. Thus the cross section will be dominated by the lowest partial waves, which near threshold each have a characteristic power law dependence on the entrance channel momentum \( k = \frac{m_1 m_2}{m_1 + m_2} |\vec{u}_1 - \vec{u}_2| \) .

For elastic scattering, the cross section is \( S \)-wave dominated, and tends to a \( k \)-independent constant \( \sigma_{el} \) near threshold, and the angular average \( 2\langle \sin^2(\theta/2) \rangle \) reduces to \( 1 - \langle \cos \theta \rangle = 1 \). Thus when Eq. [1] is substituted into Eqs. [3] and [4], we can effectively replace \( 2\langle \sin^2(\theta/2) \rangle \sigma \) by the \( k \)-independent constant \( \sigma_{el} \).
For exothermic inelastic scattering, the leading contribution to $\langle \cos \theta \rangle$ comes from the interference term between the $S$- and $P$-waves in the cross section, which scales as $k^{-2}k^{1/2}k^{3/2} \sim$ constant near threshold. Writing near threshold

$$\frac{d\sigma}{d\Omega} = \frac{A_{\text{inel}}}{4\pi}k^{-1} + \frac{B_{\text{inel}}}{4\pi} \cos \theta + ..., \quad (7)$$

we have

$$\sigma \simeq A_{\text{inel}}k^{-1},$$

$$\langle \cos \theta \rangle \simeq \frac{B_{\text{inel}}}{A_{\text{inel}}k^{-1}}.$$ \quad (8)

So when Eq. (2) for the inelastic exothermic case is substituted into Eqs. (3) and (4), we can effectively replace $\langle \cos \theta \rangle \sigma$ by the $k$-independent constant $B_{\text{inel}}$, remembering, however, that this is not the total cross section (which approaches $A_{\text{inel}}k^{-1}$ near threshold) but is proportional to the coefficient of the $S$-wave $P$-wave interference term in the differential cross section.

D. The dark matter distribution function $\rho(\vec{x}, \vec{u}_2)$

We now address the task of formulating a model for the distribution function $\rho(\vec{x}, \vec{u}_2)$ that describes dark matter postulated to be in orbit around the earth. The simplest model would be a disk composed of dark matter in circular orbits in earth’s equatorial plane, but attempts to fit the flyby anomaly data with such a model were unsuccessful, since for any reasonable disk inner radius, some of the flybys (such as NEAR) pass inside the disk. We thus proceed to the next simplest model, which is constructed from dark matter in a circular orbit, of radius $r$ and tilted at an angle $\psi$ ($0 \leq \psi \leq \pi$) with respect to earth’s equatorial plane. If the earth were exactly spherically symmetric, its gravitational field would be strictly monopole, and such a tilted orbit would be stable. But in fact the earth’s rotation produces an equatorial bulge, and so its mass distribution is only axially symmetric around its rotation axis, giving rise to quadrupole and higher moments in its gravitational field. As a result of these higher moments, the tilted orbit precesses around the earth’s rotation axis, in such a way that the angular momentum component $L_z$ along the earth’s axis is conserved. Over a long period of time, this precession will smear an initial cluster of tilted orbits into a uniform shell, obtained by averaging the tilted circle over the azimuthal angle that its normal makes with respect to the earth’s rotation axis.

To give this picture a mathematical description, let $x, y, z$ be a Cartesian axis system, with positive $z$ pointing to the earth’s North pole (so that the rotation sense of the earth is from
\(x \text{ to } y\). Let the normal \(\hat{n}\) to the tilted orbit have polar angle \(\psi\) and azimuthal angle \(\phi\) with respect to this system, so that \(\hat{n}(\psi, \phi) = (\sin \psi \cos \phi, \sin \psi \sin \phi, \cos \psi)\), and let the angle of rotation within the plane of the dark matter orbit be \(\theta\), with increasing \(\theta\) corresponding, at \(\psi = 0\), to the direction of earth’s rotation. Then a parametric description of the tilted circle is \(\vec{P}(r, \theta, \phi) \equiv (P_x(r, \theta, \phi), P_y(r, \theta, \phi), P_z(r, \theta, \phi))\), with

\[
\begin{align*}
P_x(r, \theta, \phi) &= r(\cos \theta \cos \psi \cos \phi - \sin \theta \sin \phi), \\
P_y(r, \theta, \phi) &= r(\cos \theta \cos \psi \sin \phi + \sin \theta \cos \phi), \\
P_z(r, \theta, \phi) &= -r \cos \theta \sin \psi, \\
|P_z(r, \theta, \phi)| &\leq r \sin \psi.
\end{align*}
\]

The corresponding velocity unit vector of a dark matter particle in the tilted circular orbit is 
\(\vec{U}(\theta, \phi) = (U_x(\theta, \phi), U_y(\theta, \phi), U_z(\theta, \phi)) = r^{-1}d\vec{P}/d\theta\), with

\[
\begin{align*}
U_x &= -\sin \theta \cos \psi \cos \phi - \cos \theta \sin \phi, \\
U_y &= -\sin \theta \cos \psi \sin \phi + \cos \theta \cos \phi, \\
U_z &= \sin \theta \sin \psi.
\end{align*}
\]

The velocity vector is obtained by multiplying the velocity unit vector by the velocity magnitude \((GM_\oplus/r)^{1/2}\) for a particle in a circular orbit of radius \(r\), with \(G\) the Newton gravitational constant and \(M_\oplus\) the earth mass.

Integrating the position and velocity distribution for a tilted circular orbit over the angles \(\theta, \phi\) gives the distribution for the corresponding shell, and integrating over the shell parameters \(r, \psi\) with a general weighting function \(w(r, \psi)\) gives as the model for the dark matter distribution function

\[
\rho(\vec{x}, \vec{u}_2) = \int dr \int d\psi w(r, \psi) \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \delta^3(\vec{x} - \vec{P}(r, \theta, \phi)) \delta^3(\vec{u}_2 - (GM_\oplus/r)^{1/2}\vec{U}(\theta, \phi)) ,
\]

with the corresponding total number of particles in the shell given by

\[
N \equiv \int d^3x \int d^3u_2 \rho(\vec{x}, \vec{u}_2) = 4\pi^2 \int dr \int d\psi w(r, \psi) .
\]

Referring to Eq. (11), we have to evaluate an integral over the distribution function of the form

\[
I = \int dt \int d^3u_2 F(\vec{x}(t), d\vec{x}(t)/dt, \vec{u}_2) \rho(\vec{x}(t), \vec{u}_2) ,
\]

with \(F(\vec{x}(t), d\vec{x}(t)/dt, \vec{u}_2)\) given by

\[
F(\vec{x}(t), d\vec{x}(t)/dt, \vec{u}_2) = (d\vec{x}(t)/dt) \cdot (\delta \vec{v}_1)|_{\vec{u}_1 = d\vec{x}(t)/dt} |d\vec{x}(t)/dt - \vec{u}_2| \sigma .
\]
On substituting Eq. (11) and noting that the coordinate delta function constrains \( r = |\vec{P}(r, \theta, \phi)| = |\vec{x}(t)| \equiv r(t) \), we obtain

\[
I = \int dt \int d\psi w(r(t), \psi) \int dr \times \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi F(\vec{x}(t), d\vec{x}(t)/dt, (GM_\oplus/r(t))^{1/2} \vec{U}(\theta, \phi)) \delta^3(\vec{x}(t) - \vec{P}(r, \theta, \phi)) .
\]

(15)

As shown in Appendix A, by making changes of variable one can carry out the integrations over \( r, \phi \) and \( \theta \) in Eq. (15), leaving an integral in which \( \theta \) and \( z \) have been replaced, by virtue of the delta function constraints, by \( \theta(\vec{x}(t)) \) and \( z(t) \equiv z(\vec{x}(t)) \),

\[
I = \int dt \int d\psi w(r(t), \psi) \sum_{\pm} F(\vec{x}(t), d\vec{x}(t)/dt, (GM_\oplus/r(t))^{1/2} \vec{U}_\pm(\theta(\vec{x}(t)), \phi(\vec{x}(t))))

\times \frac{1}{r(t) \sqrt{r(t)^2 \sin^2 \psi - z(t)^2}} .
\]

(16)

Note that by virtue of Eq. (9), the integration domain extends only over \( |z(t)| \leq r(t) \sin \psi \), and hence the argument of the square root is nonnegative. In Eq. (16) the sum over \( \pm \) is over the two roots \( \theta(\vec{x}(t)) \) of the equation \( \cos \theta(\vec{x}(t)) = -z(t)/(r(t) \sin \psi) \), which differ in the sign of \( \sin \theta \),

\[
\sin \theta(\vec{x}(t)) = \pm \sqrt{1 - z(t)^2/(r(t)^2 \sin^2 \psi)} ,
\]

(17)

while the values of \( \phi(\vec{x}(t)) \) corresponding to these two roots \( \theta(\vec{x}(t)) \) are obtained by equating \( \vec{x}(t) \) to \( \vec{P} \) and then solving Eq. (9) for \( \cos \phi \) and \( \sin \phi \). The two roots correspond to the fact that a circular orbit with tilt angle \( \psi \) consists of two semicircular segments, with opposite directions of the velocity component normal to the equatorial plane. Thus the intersection of the spacecraft trajectory \( \vec{x}(t) \) with the dark matter shell generated by azimuthal rotation of such a tilted circular orbit will intersect two segments of circular orbits, one up-going and one down-going relative to the equatorial plane.

It will be useful for what follows to express the unit velocities \( \vec{U}_\pm(\theta(\vec{x}(t)), \phi(\vec{x}(t))) \) in terms of their components on unit vectors \( \hat{n}_\parallel(t) = (\hat{z} \times \hat{x}(t))/|\hat{z} \times \hat{x}(t)| \) and \( \hat{n}_\perp(t) = \hat{x}(t) \times \hat{n}_\parallel(t) \), normal to \( \hat{x}(t) = \vec{x}(t)/r \), that are respectively parallel (in the sense of earth rotation) and perpendicular to the earth equatorial plane. A simple calculation given in Appendix B shows that \( \vec{U}_\pm \) are given on this basis by

\[
\vec{U}_\pm(\theta(\vec{x}(t)), \phi(\vec{x}(t))) = C(t)\hat{n}_\parallel \pm D(t)\hat{n}_\perp ,
\]

(18)
with the coefficients $C(t)$ and $D(t)$ given by

\[
C(t) = \frac{r(t) \cos \psi}{\sqrt{r(t)^2 - z(t)^2}}, \quad D(t) = \frac{\sqrt{r(t)^2 \sin^2 \psi - z(t)^2}}{\sqrt{r(t)^2 - z(t)^2}},
\]

which obey $C(t)^2 + D(t)^2 = 1$. Explicit expressions for $\hat{n}_\parallel(t)$ and $\hat{n}_\perp(t)$ in the flyby plane basis are given in the next subsection, which together with Eqs. (18) and (19) give the formulas for the unit velocities $\vec{U}_\pm(\theta(\vec{x}(t)), \phi(\vec{x}(t)))$ on the flyby plane basis needed in the numerical computations.

E. Flyby orbital plane kinematics

The Anderson et al. paper [2] gives the flyby orbit parameters in terms of coordinates on the celestial sphere, but it will be more convenient for our purposes to carry out all flyby orbit calculations in the flyby orbital plane. Let $x_o, y_o, z_o$ be a Cartesian axis system, with $z_o$ normal to the flyby orbital plane. The flyby orbit can then be written in parametric form as

\[
\begin{align*}
    x_o(t) &= r(t) \cos \theta_o(t) , \\
    y_o(t) &= r(t) \sin \theta_o(t) , \\
    r(t) &= \frac{p}{1 + e \cos \theta_o(t)} , \\
    R_f &= \frac{p}{1 + e} , \\
    \frac{dx_o(t)}{dt} &= \frac{-V_f \sin \theta_o(t)}{1 + e} = \frac{-y_o(t)}{1 + e \cos \theta_o(t)} \frac{d\theta_o(t)}{dt} , \\
    \frac{dy_o(t)}{dt} &= \frac{V_f (e + \cos \theta_o(t))}{1 + e} = \frac{er(t) + x_o(t)}{1 + e \cos \theta_o(t)} \frac{d\theta_o(t)}{dt} , \\
    \frac{d\theta_o(t)}{dt} &= \frac{R_f V_f}{r(t)^2} .
\end{align*}
\]

The scale parameter $p$, the eccentricity $e$, the velocity at closest approach to earth $V_f$, the radius at closest approach $R_f$, and the velocity at infinity $V_\infty$ are given in Table I for each of the six flybys discussed in [2], together with the polar angle $I$ and azimuthal angle $\alpha$ of the earth’s north pole with respect to the $x_o, y_o, z_o$ coordinate system. The quantities $V_f$ and $V_\infty$ are given directly
in [2], while \( R_f, p, \) and \( e \) can be calculated from them using the formulas

\[
R_f = \frac{2GM_{\oplus}}{V_f^2 - V_\infty^2},
\]

\[
e = 1 + \frac{2V_\infty^2}{V_f^2 - V_\infty^2},
\]

\[
p = \frac{4GM_{\oplus}}{V_\infty^2} \left[ \left( \frac{V_\infty^2}{V_f^2 - V_\infty^2} \right)^2 + \frac{V_\infty^2}{V_f^2 - V_\infty^2} \right].
\]

(21)

The earth axis polar angle \( I \) is also directly given in [2], while the azimuthal angle \( \alpha \) can be calculated from the formula

\[
\cos \alpha = \frac{\sin \phi'}{\sin I},
\]

with \( \phi' \) the geocentric latitude at closest approach (which is called \( \phi \) in [2]; with the orbit parametrization of Eq. (20), \( \phi' \) is the latitude of the positive \( x_o \) axis). This formula does not determine the quadrant in which \( \alpha \) lies, but this can be fixed from the additional orbital parameters given in [2] (with some corrections supplied to me by J.K. Campbell [4]). Enough orbit parameters are given in [2] to provide several redundancies that serve as cross-checks on these calculations.

| TABLE I: Flyby orbital parameters |
|-----------------------------------|
| GLL-I | GLL-II | NEAR | Cassini | Rosetta | Messenger |
|-------|--------|------|---------|---------|-----------|
| \( V_f \) (km/s) | 13.740 | 14.080 | 12.739 | 19.026 | 10.517 | 10.389 |
| \( R_f \) (km) | 7,334 | 6,674 | 6,911 | 7,544 | 8,332 | 8,715 |
| \( V_\infty \) (km/s) | 8.949 | 8.877 | 6.851 | 16.010 | 3.863 | 4.056 |
| \( e \) | 2.474 | 2.320 | 1.814 | 5.851 | 1.312 | 1.360 |
| \( p \) (km) | 25,480 | 22,160 | 19,450 | 51,690 | 19,260 | 20,570 |
| \( I \) (deg) | 142.9 | 138.7 | 108.0 | 25.4 | 144.9 | 133.1 |
| \( \alpha \) (deg) | -45.1 | -147.4 | -55.1 | -158.4 | -53.1 | 0.0 |
To carry out the computation of the flyby velocity change in the flyby plane basis \( x_o, y_o, z_o \) we will need the components of \( \hat{n}_\parallel \) and \( \hat{n}_\perp \) on this basis. In Eq. (11) we gave their components on the earth centered basis \( x, y, z \); these can be rotated to the flyby plane basis, but it is simpler to calculate them directly by going back to the defining cross product relations, using the components of \( \vec{x}(t) \) and of the earth axis \( \hat{z} \) on the flyby plane basis,

\[
\vec{x}(t) = (x_o(t), y_o(t), 0) ,
\hat{z} = (\sin I \cos \alpha, \sin I \sin \alpha, \cos I) .
\]

From these we find

\[
\hat{n}_\parallel(t) = \hat{z} \times \vec{x}(t) = \langle \frac{1}{\sqrt{r(t)^2 - z(t)^2}} \rangle \left( -y_o(t) \cos I, x_o(t) \cos I, (y_o(t) \cos \alpha - x_o(t) \sin \alpha) \sin I \right) ,
\]

\[
\hat{n}_\perp(t) = \vec{x}(t) \times \hat{n}_\parallel(t) = \langle \frac{1}{r(t) \sqrt{r(t)^2 - z(t)^2}} \rangle \left( (y_o(t) \cos \alpha - x_o(t) \sin \alpha) \sin I, -x_o(t)(y_o(t) \cos \alpha - x_o(t) \sin \alpha) \sin I, r(t)^2 \cos I \right) ,
\]

with

\[
r(t) = |\vec{x}(t)| = \sqrt{x_o(t)^2 + y_o(t)^2} ,
\]

\[
z(t) = \vec{x}(t) \cdot \hat{z} = (x_o(t) \cos \alpha + y_o(t) \sin \alpha) \sin I .
\]

Substituting Eq. (20) for \( x_o(t) \) and \( y_o(t) \) into Eq. (25) we have

\[
z(t) = r(t) \sin I \cos \left( \theta_o(t) - \alpha \right) ,
\]

which allows one to rewrite the Jacobian factor appearing in Eq. (16) as

\[
\frac{1}{r(t) \sqrt{r(t)^2 \sin^2 \psi - z(t)^2}} = \frac{1}{r(t)^2 \sin \psi \sqrt{1 - \left( \frac{\sin I}{\sin \psi} \right)^2 \cos^2 \left( \theta_o(t) - \alpha \right)}} .
\]

when the argument of the square root is nonnegative.

F. Simplified model used for numerical work

The model as defined above involves a general weighting function \( w(r, \psi) \), but for an initial survey we make the simplifying assumption of only a single tilt angle \( \psi_i, \psi_e \) for the inelastic and
elastic scatterers, respectively, and Gaussian distributions in \( r \) with different centers and widths for each. Thus we take for the inelastic scatterers

\[
w_i(r, \psi) = K_i e^{-(r-R_i)^2/D_i^2} \delta(\psi - \psi_i),
\]

(28)

and for the elastic scatterers

\[
w_e(r, \psi) = K_e e^{-(r-R_e)^2/D_e^2} \delta(\psi - \psi_e).
\]

(29)

With this choice, the integral of Eq. (12) becomes

\[
N_\ell = 4\pi^{5/2} K_\ell D_\ell, \quad \ell = i, e.
\]

(30)

It is now convenient to combine the constants \( K_{i,e} \) with the mass-dependent constants appearing in Eqs. (1) and (2) of Sec. IIA, and the constants \( \sigma_{el} \) and \( B_{inel} \) introduced in Sec. IIC, giving new parameters \( \rho_i, \rho_e \) characterizing the effective density times cross section for the inelastic and elastic scatterer distributions,

\[
\rho_i \equiv \frac{m_2}{m_1 + m_2} \sigma_{el} K_e,
\]

\[
\rho_e \equiv \left( \frac{2\Delta m m'_2}{m_1(m_1 + m'_2)} \right)^{1/2} B_{inel} K_i.
\]

(31)

Thus in Eq. (15) we effectively replace (see Eq. (14))

\[
\int d\psi w(r(t), \psi) F(\bar{x}(t), d\bar{x}(t)/dt, \bar{u}_2)
\]

(32)

by

\[
\sum_{\ell=i,e} \left\{ |d\bar{x}(t)/dt - \bar{u}_2|(|d\bar{x}(t)/dt|/|d\bar{x}(t)/dt - \bar{u}_2|) \cdot \bar{V}_\ell \rho_\ell e^{-(r(t)-R_\ell)^2/D_\ell^2} \right\}_{\psi = \psi_\ell},
\]

(33)

with \( \bar{V}_\ell \) given by

\[
\bar{V}_i = c (d\bar{x}(t)/dt - \bar{u}_2) /|d\bar{x}(t)/dt - \bar{u}_2|, \quad \bar{V}_e = - (d\bar{x}(t)/dt - \bar{u}_2),
\]

(34)

and with \( \bar{u}_2 \) evaluated as \( \bar{U}_\pm \) of Eqs. (16) and (18). The simplified model thus defined has eight parameters, four parameters \( \psi_i, \rho_i, R_i, D_i \) characterizing the inelastic scatterers, and four
parameters $\psi_e$, $\rho_e$, $R_e$, $D_e$ characterizing the elastic scatterers. Finally, we note that by combining Eqs. (30) and (31), and approximating
\[
\frac{m_2}{m_1 + m_2} \sim \left( \frac{2\Delta m}{m_1(m_1 + m_2)} \right)^{1/2} \sim \frac{m_2}{m_1},
\]
we find the following estimates for the total mass in the dark matter shells,
\[
M_e \equiv m_2 N_e = 4\pi^{5/2} \rho_e D_e m_1 / \sigma_{el},
\]
\[
M_i \equiv m_2 N_i = 4\pi^{5/2} \rho_i D_i m_1 / B_{inel}.
\]

III. NUMERICAL RESULTS AND DISCUSSION

Let us turn now to numerical fitting of the eight parameter model to the flyby anomalies reported in [2]. In carrying out the needed integrals over flyby orbits, we replaced the integration over $t$ by an integration over orbit angle $\theta_o$, using the expression for $d\theta_o/dt$ given in Eq. (20). To utilize integration mesh points efficiently, the integrations were restricted to the parts of the orbits where the Gaussian factors $e^{-(r-R_\ell)^2/D_\ell^2}$ were larger than $e^{-9} = 0.00012$, that is, to the parts of the orbits where $|r - R_\ell| \leq 3D_\ell$.

In attempting to search for good fits with coarse meshes, we found that the infinite jump in the Jacobian factor of Eq. (27) at the dark matter shell edges led to the search program settling on false minima reflecting truncation errors, which were unstable with respect to small changes in the integration mesh or fitting parameters. To avoid this problem, we replaced the original Jacobian by a smoothed Jacobian, as follows. Abbreviating $W \equiv z(t)^2 / (r(t)^2 \sin^2 \psi)$, and using $\Theta$ to denote the usual step function, the original Jacobian contains the function with an infinite jump at $W = 1$,
\[
f(W) = \frac{\Theta(1 - W)}{\sqrt{1 - W}}.
\]
We replaced this by the following function, which is continuous and has a continuous first derivative,
\[
f_\epsilon(W) = \begin{cases} \frac{1}{\sqrt{1 - W}} & \text{for } W \leq 1 - \epsilon, \\ e^{-P_\epsilon(W)} & \text{for } W \geq 1 - \epsilon, \end{cases}
\]
\[
P_\epsilon(W) = -\frac{1}{2\epsilon} (W - 1 + \epsilon) + \frac{1}{\epsilon^2} (W - 1 + \epsilon)^2.
\]
For our initial searches we took $\epsilon = 10^{-2}$.

Our numerical searches were carried out by minimizing a least squares likelihood function $\chi^2$, defined as

$$\chi^2 = \sum_{k=1}^{6} \frac{(\delta v_{k;\text{th}} - \delta v_{k;A})^2}{\sigma^2_{k;A}} ,$$  

(39)

where $k$ indexes the six flybys reported by Anderson et al. [2], the $\delta v_{k;\text{th}}$ are the theoretical values of the velocity discrepancies computed from our model, the $\delta v_{k;A}$ are the observed values for these discrepancies reported in [2], and the $\sigma_{k;A}$ are the corresponding estimated errors in these discrepancies given in [2]. Since the quoted $\sigma_{k;A}$ values contain both systematic and statistical components, a least squares likelihood function is not a true statistical chi square function, but having a quadratic form is very convenient for the following reason. Because the theoretical values $\delta v_{k;\text{th}}$ are linear in the dark matter density times cross section parameters $\rho_{i,e}$,

$$\delta v_{k;\text{th}} = \rho_i \delta v_{k;i} + \rho_e \delta v_{k,e} ,$$  

(40)

with $\delta v_{k;i,e}$ the respective contributions from the inelastic and elastic scatterers computed with $\rho_{i,e} = 1$, the likelihood function is a positive semi-definite quadratic form in these two parameters. Hence for fixed values of the other six parameters $\psi_{i,e}, R_{i,e}, D_{i,e}$, the minimization of $\chi^2$ with respect to the parameters $\rho_{i,e}$ can be accomplished algebraically by solving a pair of linear equations in the two variables $\rho_{i,e}$, with the result

$$\rho_i = \frac{C_{ee} G_i - C_{ii} G_e}{C_{ii} C_{ee} - C_{ie} C_{ei}} ,$$

$$\rho_e = \frac{C_{ii} G_e - C_{ie} G_i}{C_{ii} C_{ee} - C_{ie} C_{ei}} ,$$  

(41)

with coefficients given by

$$C_{\ell m} = \sum_{k=1}^{6} \frac{\delta v_{k;\ell} \delta v_{k;m}}{\sigma^2_{k;A}} , \quad \ell, m = i, e ,$$

$$G_{\ell} = \sum_{k=1}^{6} \frac{\delta v_{k;A} \delta v_{k;\ell}}{\sigma^2_{k;A}} , \quad \ell = i, e .$$  

(42)

This has the effect of reducing the parameter space that must be searched numerically from an eight parameter space to a six parameter space, which results in a substantial saving of computational effort.
Our search procedure was then as follows. Using a very coarse 10 point integration mesh for the model calculation of the flyby velocity changes, and with \( \epsilon = 10^{-2} \), we surveyed the six parameter space in 31 steps of \( \pi/32 \) for the tilt angles \( \psi_{i,e} \), going from \( \pi/64 \) to \( \pi - \pi/64 \), in 20 steps of 2,500 km for the Gaussian centers \( R_{i,e} \), going from 15,000 km to 62,500 km, and in 5 steps of 1000 km for the Gaussian widths \( D_{i,e} \), going from 1,000 km to 5,000 km. For each of the 9,610,000 steps in this survey, the values of \( \rho_{i,e} \) were then optimized by using Eqs. (41) and (42), and the resulting data for \( \chi^2 \) values less than 25 which also had positive \( \rho_e \) were written to a storage file. This left 18 potential starts for fits. For about a half dozen of these, we used the corresponding sets of parameter values as starting points for a six parameter minimization search using the CERN program Minuit, with successively 200 and then 2000 point integration meshes for the model calculation of the flyby velocity changes, and using double precision arithmetic throughout (as recommended in the Minuit documentation). Finally, using the optimized parameter values obtained this way, we tested for stability of the \( \chi^2 \) values and resulting fits with respect to program modifications, such as refinement of the integration mesh. The parameter space survey took several hours on our pentium processor laptop, the Minuit minimizations took typically minutes (or less) each, and the stability checks took of the order of seconds.

This procedure showed that for \( \epsilon = 10^{-2} \) excellent fits could be obtained with a wide range of values of the radius \( R_i \) of the inelastic dark matter scatterer shell. Using the parameters for these good fits as a starting point, we then did a series of 5 parameter fits, each for a different fixed value of \( R_i \). Also using the good fits as starting points, we did a similar series of 5 parameter fits, versus fixed \( R_i \), this time with \( \epsilon = 10^{-16} \) corresponding to no smoothing of the Jacobian discontinuity (up to the accuracy of double precision truncation errors), but using an adaptive integration program to adequately sample points on the trajectories where the Jacobian becomes large. These searches (as well as a 6 parameter fit in the \( \epsilon = 10^{-16} \) case) show that the model with no smoothing has a distinct \( \chi^2 \) minimum at \( R_i = 34,520 \) km. Results in both \( \epsilon \) cases are given in Tables II – IV. We caution that the \( \epsilon = 10^{-2} \) cases do not exactly obey the constraints between dark matter position and velocity required by orbital dynamics, so it is not clear at this point whether the wide range of \( R_i \) values and nearly exact fits obtained in this case are a reflection of just the smoothing, which will be present in a more realistic dark matter orbit model, or are an artifact associated with relaxing the orbital constraints.

From the products \( \rho_i D_i \) and \( \rho_e D_e \) for each fit, one can use Eq. (36) to estimate the total mass in the dark matter shells, in terms of the elastic and inelastic scattering parameters \( \sigma_{el} \) and \( B_{inel} \). Alternatively, given the upper bound [5] on the mass of dark matter in orbit around the earth
between the LAGEOS satellite orbit and the moon’s orbit, of \(4 \times 10^{-9} M_\oplus \sim 1.4 \times 10^{42} \text{GeV}/c^2\), one can turn these relations into lower bounds on \(\sigma_{\text{el}}\) and \(B_{\text{inel}}\). For example, from the values \(\rho_i D_i = 0.00304 \text{km}^2\) and \(\rho_e D_e = 19.2 \text{km}^2\) for fit 2d, one finds the bounds

\[
\begin{align*}
\sigma_{\text{el}} & \geq 9.4 \times 10^{-31} \text{cm}^2, \\
B_{\text{inel}} & \geq 1.5 \times 10^{-34} \text{cm}^2,
\end{align*}
\]  

(43)

which are consistent with the cross section range arrived at from various constraints in [1]. The spatial constraints found in [1], which require that the dark matter should be localized well away from the earth and the moon, are also obeyed.

In Table V we give the results of fitting the data with \(R_i\) constrained to the value 34,520 km found in fit 2d (repeated in the first line of this table), versus increasing values of the Gaussian width \(D_i\). These results, together with those for fits 2e–g in Tables II and IV, show that the range of widths \(D_i\) for good fits extends up to around 10,000 km. The fact that \(D_i\) is not well-determined is also seen in the calculation leading to Table VII, where in fit 4a we give the result of repeating fit 2d with a refined (4000 point) integration mesh. The parameter values for fit 4a agree to within 1% with those of fit 2d, except for \(D_i\), which in fit 4a is 2030 km, and \(10^6 \times \rho_i\), which in fit 4a is 1.49 km, with the product \(\rho_i D_i\) matching that of fit 2d to within 1%.

In Table VI, we show the results of basing the fit solely on a shell of inelastic scatterers, without a second shell of elastic scatterers. As seen, with this restriction it is not possible to get good fits, even when various combinations of the flyby data are excluded from the fits. For example, as shown on the last line of Table VI, the four parameter model with only inelastic scatterers cannot give a good fit to just the two flyby data points from NEAR and Messenger. In Table VII, following up on a suggestion by V. Toth [6], we give the results of fitting the full model, with both elastic and inelastic scatterers, to the flyby data, with one flyby at a time omitted from the fit. These results show that the predicted value for the anomaly of each omitted flyby is in qualitative accord with the experimental value.

The results in Tables II – VII show that the dark matter scattering model, with inelastic and elastic scatterers, can account for the flyby anomaly data. One could argue that the fits are too good, and are indicative of “over-fitting”, since there are 8 parameters in the model (9 if one includes \(\epsilon\) in the smoothed case), and only 6 data points. On the other hand, it was not a priori obvious that such a simple model should be able to account for data from a complicated physical process with a three-dimensional geometry, and the results shown in Table VII support
the view that the success of the model is not attributable to over-fitting of the data. Further steps in this investigation would be: (1) incorporation of further flybys into the fits, when the flyby parameters in Table I and the corresponding velocity discrepancy and error values are available, or alternatively, using fit 2d (or 4a) to predict the velocity discrepancy for future flybys, given their orbital parameters; (2) incorporating constraints on residual drag coming from fitting satellite drag measurements to conventional drag sources; (3) as suggested to me by V. Toth [6], incorporating the time development of the velocity anomaly near perigee when such data becomes available from improved tracking of future flybys; (4) as suggested to me by J. Rosner [7], investigating possible constraints arising from the effect of the quadrupole moment of the dark matter shells on the precession of high-lying satellite orbits; (5) extending the model to include a general form of the weighting function $w(r, \psi)$; and (6) extending the model to include shells generated by precessing elliptical, as opposed to circular orbits, and shells generated by a precessing Schwarzschild disk [8]. The extensions (5) and (6), which can incorporate consistent smoothing of the Jacobian, will require computing resources well beyond those used here to analyze the 8 parameter model. It will also be necessary to address the question of mechanisms for producing dark matter shells. According to A. Peter [9], the accumulation cascade suggested in [1] is not viable as a mechanism. Another scenario, suggested by Dr. Peter’s comments and the structure of the model formulated here, would involve the gravitational capture by the earth of a dense (up to $\sim 10^{15}$ times galactic halo mean density, that is $\sim 10^{-9}$ times mean ordinary matter density) condensed ball of dark matter into an orbit tilted with respect to earth’s rotation axis; breakup of this by tidal forces could then lead to population of a shell of the type we have assumed.\(^2\) If the flyby anomalies are ultimately confirmed, detailed study of such a mechanism would be warranted.

**IV. ACKNOWLEDGEMENTS**

This work was supported by the Department of Energy under grant no DE-FG02-90ER40542, and parts of this work were done during the author’s stay at the Aspen Center for Physics. I wish to thank Scott Tremaine for helpful conversations about orbital dynamics, James Campbell for sending me corrections to some of the data published in [2], Michele Papucci for suggesting that I use the CERN minimization program Minuit, and Prentice Bisbal for assistance in downloading it to my computer. I also wish to thank Angelo Bassi, Annika Peter, Jonathan Rosner, and Viktor

\(^2\) The constraints derived in [1] on the sun-bound dark matter density are not relevant for this scenario for producing dark matter shells.
APPENDIX A: CHANGES OF VARIABLE TO INTEGRATE OUT THE SPATIAL DELTA FUNCTION

To eliminate the spatial delta function in Eq. (15), we note that rewriting $\vec{P}$ in terms of spherical coordinates,

$$
\vec{P} = r(\sin \omega \cos \beta, \sin \omega \sin \beta, \cos \omega),
$$

(A1)

the delta function $\delta^3(\vec{x} - \vec{P})$ becomes

$$
\delta^3(\vec{x} - \vec{P}) = r^{-2}\sin \omega|^{-1}\delta(|\vec{x}|-r)\delta(\omega(\vec{x})-\omega)\delta(\beta(\vec{x})-\beta) .
$$

(A2)

Equating Eq. (A1) with Eq. (9), we see that

$$
\beta = \phi + \Psi(\theta, \psi),
$$

$$
\cos \Psi(\theta, \psi) = \frac{\cos \theta \cos \psi}{\sqrt{1 - \cos^2 \theta \sin^2 \psi}}, \quad \sin \Psi(\theta, \psi) = \frac{\cos \theta}{\sqrt{1 - \cos^2 \theta \sin^2 \psi}},
$$

(A3)

and

$$
\cos \omega = -\cos \theta \sin \psi .
$$

(A4)

Using Eq. (A3), on substituting Eq. (A2) with $\vec{x} = \vec{x}(t)$ into Eq. (15), we can immediately eliminate the $r$ and $\phi$ integrations, leaving

$$
I = \int dt \int d\psi w(r(t), \psi)r(t)^{-2}\int_0^{2\pi} d\theta|\sin \omega|^{-1}
$$

$$
\times F(\vec{x}(t), d\vec{x}(t)/dt, (GM_{\odot}/r(t))^{1/2}U(\theta, \phi(\vec{x}(t))))\delta(\omega(\vec{x})-\omega) .
$$

(A5)

To carry out the $\theta$ integration, we differentiate Eq. (A1), giving

$$
\frac{d\theta}{\sin \omega} = \frac{-d\omega}{\sin \psi \sin \theta}
$$

$$
= \frac{-d\omega}{\sqrt{\sin^2 \psi(1 - \cos^2 \theta)}} = \frac{-d\omega}{\sqrt{\sin^2 \psi - \cos^2 \omega}}
$$

$$
= \frac{-rd\omega}{\sqrt{r^2 \sin^2 \psi - z^2}} ,
$$

(A6)
with \( z = r \cos \omega \). Substituting this into Eq. (A5), we can carry out the \( \theta \) integral, leaving an the integral given in Eq. (16) of the text.

**APPENDIX B: CALCULATION OF THE COEFFICIENTS \( C(t) \) AND \( D(t) \)**

From the defining cross product relations, we see that on the geocentric basis system with \( z \) aligned along the earth rotation axis, we have

\[
\mathbf{n}_\parallel(t) = \frac{\mathbf{z} \times \mathbf{x}(t)}{\left| \mathbf{z} \times \mathbf{x}(t) \right|} = \frac{1}{\sqrt{r(t)^2 - z(t)^2}} (-y(t), x(t), 0) ,
\]

\[
\mathbf{n}_\perp(t) = \mathbf{x}(t) \times \mathbf{n}_\parallel(t) = \frac{1}{r(t)} \frac{1}{\sqrt{r(t)^2 - z(t)^2}} (-x(t)z(t), -y(t)z(t), r(t)^2 - z(t)^2) .
\]

\[\text{(B1)}\]

To express the unit velocity of Eq. (10) on this basis, at the intersections where \( \mathbf{x}(t) = \mathbf{P}(r, \theta, \phi) \), we rewrite Eq. (9) as

\[
x(t)/r(t) = (\cos \theta \cos \psi \cos \phi - \sin \theta \sin \phi) ,
\]

\[
y(t)/r(t) = (\cos \theta \cos \psi \sin \phi + \sin \theta \cos \phi) ,
\]

\[
z(t)/r(t) = - \cos \theta \sin \psi .
\]

\[\text{(B2)}\]

The third of these equations determines \( \cos \theta \) and \( \sin \theta \) in terms of \( \mathbf{x}(t) \),

\[
\cos \theta = - \frac{z(t)}{(r(t) \sin \psi)} ,
\]

\[
\sin \theta = \pm \sqrt{1 - z(t)^2/(r(t)^2 \sin^2 \psi)} ,
\]

\[\text{(B3)}\]

while solving the first two gives \( \sin \phi \) and \( \cos \phi \) in terms of \( x(t) \) and \( y(t) \),

\[
\sin \phi = \frac{gy(t) - hx(t)}{r(g^2 + h^2)} ,
\]

\[
\cos \phi = \frac{gx(t) + hy(t)}{r(g^2 + h^2)} ,
\]

\[\text{(B4)}\]

with \( g = \cos \theta \cos \psi, \ h = \sin \theta \), which obey

\[
g^2 + h^2 = 1 - \cos^2 \theta \sin^2 \psi = 1 - z(t)^2/r(t)^2 .
\]

\[\text{(B5)}\]
Substituting Eqs. (B3) and (B4) into (10) gives the velocity components at the intersections expressed in terms of $\bar{x}(t)$, and comparing with Eq. (B1) then identifies the coefficients $C(t)$ and $D(t)$ appearing in the decomposition of $\bar{U}_\pm$ on the intrinsically defined basis $\hat{n}_\parallel$ and $\hat{n}_\perp$.

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| Fit     | $10^6 \times \rho_i$ (km) | $10^2 \times \rho_e$ (km) | $\psi_i$ (rad) | $\psi_e$ (rad) | $R_i$ (km) | $D_i$ (km) | $R_e$ (km) | $D_e$ (km) |
|---------|-----------------------------|-----------------------------|----------------|----------------|------------|-----------|------------|-----------|
| 1a      | 0.304                       | 0.268                       | 1.926          | 0.3939         | 30000      | 6278      | 28620      | 6303      |
| 1b      | 1.55                        | 0.245                       | 1.261          | 0.3945         | 40000      | 2185      | 27985      | 5890      |
| 1c      | 0.411                       | 0.261                       | 1.374          | 0.3952         | 50000      | 13540     | 28450      | 6299      |
| 1d      | 0.351                       | 0.253                       | 1.381          | 0.3946         | 60000      | 20193     | 28340      | 6334      |
| 1e      | 0.343                       | 0.248                       | 1.394          | 0.3942         | 70000      | 25780     | 28240      | 6367      |
| 2a      | 0.537                       | 0.323                       | 1.767          | 0.3902         | 25000      | 3030      | 29370      | 6678      |
| 2b      | 0.827                       | 0.316                       | 1.626          | 0.3902         | 30000      | 3030      | 29370      | 6678      |
| 2c      | 0.965                       | 0.309                       | 1.515          | 0.3902         | 32500      | 3030      | 29370      | 6678      |
| 2d      | 1.000                       | 0.288                       | 1.372          | 0.3902         | 34520      | 3030      | 29370      | 6678      |
| 2e      | 0.655                       | 0.288                       | 1.369          | 0.3902         | 35000      | 4663      | 29370      | 6678      |
| 2f      | 0.348                       | 0.286                       | 1.364          | 0.3902         | 37500      | 9223      | 29370      | 6678      |
| 2g      | 0.290                       | 0.286                       | 1.361          | 0.3902         | 40000      | 11681     | 29370      | 6678      |
TABLE V: Flyby anomaly fits with $R_i = 34520$ and indicated values of $D_i$

| $\delta v_A$ (mm/s) | $\chi^2$ | GLL-I | GLL-II | NEAR | Cassini | Rosetta | Messenger |
|---------------------|---------|-------|--------|------|---------|---------|-----------|
| $\delta v_{th}$ $D_i = 3030$ | 0.51 | 3.90 | -4.8 | 13.46 | -2 | 1.80 | 0.02 |
| $\delta v_{th}$ $D_i = 6060$ | 0.68 | 3.94 | -4.8 | 13.46 | -2.8 | 1.80 | 0.02 |
| $\delta v_{th}$ $D_i = 9090$ | 1.3 | 3.97 | -5.1 | 13.46 | -3.0 | 1.80 | 0.02 |
| $\delta v_{th}$ $D_i = 12120$ | 4.2 | 3.88 | -4.7 | 13.46 | -4.0 | 1.80 | 0.02 |

TABLE VI: Flyby anomaly attempted fits with only inelastic scatterers

| $\delta v_A$ (mm/s) | $\chi^2$ | GLL-I | GLL-II | NEAR | Cassini | Rosetta | Messenger |
|---------------------|---------|-------|--------|------|---------|---------|-----------|
| $\delta v_{th}$ fit 3a | $0.63 \times 10^5$ | 1.87 | 1.8 | 13.0 | 1.5 | 2.9 | 2.4 |
| $\delta v_{th}$ fit 3b | $0.63 \times 10^5$ | 1.87 | - | 13.0 | - | 2.9 | 2.4 |
| $\delta v_{th}$ fit 3c | $0.16 \times 10^4$ | 1.93 | - | 13.4 | - | 3.0 | - |
| $\delta v_{th}$ fit 3d | $0.61 \times 10^5$ | - | - | 13.0 | - | - | 2.5 |

Fit attempts with only inelastic scattering; entries labeled – were excluded from the corresponding fit. To two decimal places, all fits in this table correspond to the parameter values $\rho_i = 0.14$, $\psi_i = 1.13$, $R_i = 40000$, and $D_i = 2000$.

TABLE VII: Flyby anomaly fits to five of the six flybys

| $\delta v_A$ (mm/s) | $\chi^2$ | GLL-I | GLL-II | NEAR | Cassini | Rosetta | Messenger |
|---------------------|---------|-------|--------|------|---------|---------|-----------|
| $\delta v_{th}$ fit 4a | 0.49 | 3.90 | -4.6 | 13.46 | -2 | 1.80 | 0.02 |
| $\delta v_{th}$ fit 4b | 0.45 | 3.71 | -4.4 | 13.46 | -2.6 | 1.80 | 0.02 |
| $\delta v_{th}$ fit 4c | 0.49 | 3.91 | -4.6 | 13.46 | -2.7 | 1.80 | 0.02 |
| $\delta v_{th}$ fit 4d | 0.40 | 3.93 | -4.4 | 16.03 | -2.6 | 1.80 | 0.02 |
| $\delta v_{th}$ fit 4e | $0.63 \times 10^{-3}$ | 3.92 | -4.6 | 13.46 | -2.7 | 1.80 | 0.02 |
| $\delta v_{th}$ fit 4f | $0.40 \times 10^{-1}$ | 3.93 | -4.5 | 13.46 | -2.2 | 1.62 | 0.02 |
| $\delta v_{th}$ fit 4g | 0.19 | 3.94 | -4.3 | 13.46 | -2.3 | 1.80 | 0.12 |

Fit 4a is a fit with all six flybys included. Fits 4b–4g are fits with one flyby at a time excluded; the predicted value for the flyby omitted in each fit is in boldface. These fits use a factor of 2 finer mesh than fit 2d.