On Bazilevič Functions and Umezawa’s Lemma

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Abstract. We consider some properties on $|z| = r < 1$ of analytic functions in the unit disk $|z| < 1$. Applying Umezawa’s lemma, On the theory of univalent functions, Tohoku Math J. 7(1955) 212–228, we prove some sufficient conditions for functions to be in the class of Bazilevič functions and some related results.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}_p$ denote the class of all functions analytic in the unit disk $D$ which have the form

\[ f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n}, \quad z \in D. \tag{1} \]

A function $f(z)$ meromorphic in a domain $D \subset \mathbb{C}$ is said to be $p$-valent in $D$ if for each $w$ the equation $f(z) = w$ has at most $p$ roots in $D$, where roots are counted in accordance with their multiplicity, and there is some $v$ such that the equation $f(z) = v$ has exactly $p$ roots in $D$. In [6] S. Ozaki proved that if $f(z)$ of the form (1) is analytic in a convex domain $D \subset \mathbb{C}$ and for some real $\alpha$ we have

\[ \Re\left\{\exp(i\alpha) f'(z) f(z)\right\} > 0, \quad z \in D, \]

then $f(z)$ is at most $p$-valent in $D$. Ozaki’s condition is a generalization of the well known Noshiro-Warschawski univalence condition, [4], [12]. In recent paper [10] there are some other conditions for a function to be $p$-valent in $D$. Further, a function $f \in \mathcal{A}_p, p = 1, 2, 3, \ldots$, is said to be $p$-valently starlike, if

\[ \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in D. \]

The class of all such functions is usually denoted by $S^*_p$. For $p = 1$ we receive the well known class of normalized starlike univalent functions. Recall that $f(z)$ of the form (1) is called the $p$-valently Bazilevič function of type $\beta$ if there exists a $p$-valently starlike function

\[ g(z) = z^p + \sum_{n=p+1} b_n z^n, \quad z \in \mathbb{D} \]

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such that
\[ \Re \left\{ \frac{z f'(z)}{f'(z) z^\beta(z)} \right\} > 0 \quad z \in \mathbb{D}, \]
where \( \beta > 0 \). Let \( \mathcal{P} \) denote the class of analytic functions \( q(z) \) in \( \mathbb{D} \) of the form
\[ q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n, \quad z \in \mathbb{D} \]
(2)
such that \( \Re \{ q(z) \} > 0 \), for \( z \in \mathbb{D} \). Functions in \( \mathcal{P} \) are sometimes called Carathéodory functions.

**Lemma 1.1.** [7, Lemma 2] see also [11, pp.224–225] Let us denote by \( D_z \) a simply connected closed domain including \( z = 0 \) inside and by \( C_z \) the boundary of \( D_z \). Let
\[ w = f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \]
(3)
be regular on \( D_z \) and \( f(z)/z^p \neq 0, f'(z) \neq 0 \) on \( D_z \). If \( f(z) \) is at least \((p+1)\)-valent then \( C_z \) has at least one arc \( C'_z \) such that
\[ \int_{C'_z} \frac{\partial}{\partial \theta} \left[ \arg \left\{ z f'(z) \right\} \right] d\theta \leq -\pi. \]
(4)
and
\[ \int_{C'_z} \frac{\partial}{\partial \theta} \arg \left\{ f(z) \right\} d\theta = 0, \quad z \in C'_z. \]
(5)
hold, and \( f(z_1) = f(z_2) \), where \( z_1 = re^{i\theta_1}, z_1 = re^{i\theta_2}, \theta_1 < \theta_2 \) are the initial and the end point of \( C'_z \) respectively.

**Lemma 1.2.** [11, p.224–225] Let \( f(z) \) be analytic in a simply connected domain \( D \) where boundary \( \Gamma_z \) consists of a regular curve and \( f'(z) \neq 0 \) on \( \Gamma_z \). Suppose that
\[ \int_{\Gamma_z} \frac{\partial}{\partial \theta} \left[ \arg \left\{ z f'(z) \right\} \right] d\theta = \int_{\Gamma_z} \Re \left( 1 + \frac{z f''(z)}{f'(z)} \right) d\theta = 2k\pi. \]
If we have for arbitrary \( p-k+1 \) arcs \( C_1, C_2, \ldots, C_{p-k+1} \) on the boundary \( \Gamma_z \) of \( D \) which doesn’t overlap one another
\[ \int_{C_1+C_2+\ldots+C_{p-k+1}} \frac{\partial}{\partial \theta} \left[ \arg \left\{ z f'(z) \right\} \right] d\theta > -\pi \]
(6)
or, if for arbitrary \( p-k+1 \) arcs \( C_1, C_2, \ldots, C_{p-k+1} \) on the boundary \( \Gamma_z \) of \( D \) which doesn’t overlap one another
\[ \int_{C_1+C_2+\ldots+C_{p-k+1}} \frac{\partial}{\partial \theta} \left[ \arg \left\{ z f'(z) \right\} \right] d\theta < (p+k+1)\pi, \]
(7)
then \( f(z) \) is at most \( p \)-valent in \( D \).

Here, \( \arg df(z) \) means the argument of the tangent to the curve \( f(re^{i\theta}), 0 \leq \theta \leq 2\pi \) or \( \arg \{iz f'(z)\} \). Applying Umezawa’s Lemma 1.2, we can have the following contraposition of it.
Theorem 1.3. Let $f(z)$ be of the form (1) be analytic in $D$ and $f'(z) \neq 0$ in $D$ and let for arbitrary $r$, $0 < r < 1$, $f(z)$ satisfies
\[
\int_{|z|=r} \frac{\partial}{\partial \theta} \arg(zf'(z)) \, d\theta = 2\pi n.
\]
Then, if $f(z)$ is at least $(p+1)$-valent in $D$, then there exists an arc $\Gamma$ on the circle $|z| = r$, $0 < r < 1$, for which
\[
\int_{\Gamma} \frac{\partial}{\partial \theta} \arg(zf'(z)) \, d\theta \leq -\pi
\]
or
\[
\int_{\Gamma} \frac{\partial}{\partial \theta} \arg(zf'(z)) \, d\theta \geq (2p + 1)\pi.
\]

2. Results and Discussion

Applying Theorem 1.3 gives the following theorem.

Theorem 2.1. Let
\[
f(z) = z^p + \sum_{n=p+1} a_n z^n, \quad z \in D
\]
be analytic in $D$. Assume that there exists a $p$-valently starlike function
\[
g(z) = z^p + \sum_{n=p+1} b_n z^n, \quad z \in D
\]
such that
\[
\Re \left\{ \frac{zf'(z)}{f'(z)g'(z)} \right\} > 0, \quad z \in D,
\]
where $\beta > 0$. Then $f(z)$ is $p$-valent in $D$.

Proof. From the hypothesis (12), we have
\[
\int_{|z|=r} \frac{\partial}{\partial \theta} \arg \left\{ \frac{zf'(z)}{f'(z)g'(z)} \right\} \, d\theta
= \int_{|z|=r} \left( \frac{\partial}{\partial \theta} \arg \left\{ \frac{zf'(z)}{f'(z)g'(z)} \right\} \right) d\theta - \frac{\partial}{\partial \theta} \arg \left\{ g'(z) \right\}
= \int_{|z|=r} \left( \frac{\partial}{\partial \theta} \arg \left\{ zf'(z) \right\} - (1-\beta) \frac{\partial}{\partial \theta} \arg \left\{ f(z) \right\} - \beta \frac{\partial}{\partial \theta} \arg \left\{ g(z) \right\} \right) d\theta
\geq -\pi.
\]
It is trivial that $f(z)$ is at least $p$-valent in $D$ because
\[
f(z) = z^p + \sum_{n=p+1} a_n z^n
\]
is at least $p$-valent in at the neighborhood of the origin. Then if $f(z)$ is not $p$-valent in $D$ or $f(z)$ is at least $(p+1)$-valent in $D$, then by Lemma 1.1, there exists an arc on the circle $|z| = r$, $0 < r < 1$, for which we have the following picture Fig. 1. which is a part of the image of $w = f(z)$, $|z| = r$. 
\[ \Gamma = \{ f(z) : f(re^{i\theta}), 0 \leq \theta_1 \leq \theta \leq \theta_2, \ z_j = re^{i\theta_j}, \ j = 1, 2, \ f(z_1) = f(z_2) \ \text{and} \ \frac{\partial}{\partial \theta} \arg \{zf'(z)\} \bigg|_{z_2} = \frac{\partial}{\partial \theta} \arg \{zf'(z)\} \bigg|_{z_1} - \pi \}. \]

Then, we have
\[ \int_{\Gamma} \frac{\partial}{\partial \theta} \arg \{zf'(z)\} \, d\theta = -\pi. \] (14)

From (13), we must have
\[ \int_{\Gamma} \frac{\partial}{\partial \theta} \arg \{zf'(z)\} \, d\theta = (1 - \beta) \int_{\Gamma} \frac{\partial}{\partial \theta} \arg \{f(z)\} \, d\theta - \beta \int_{\Gamma} \frac{\partial}{\partial \theta} \arg \{g(z)\} \, d\theta \]
\[ > -\pi \]
because \( f(z_1) = f(z_2) \). Therefore, we have
\[ \int_{\Gamma} \frac{\partial}{\partial \theta} \arg \{zf'(z)\} \, d\theta > \int_{\Gamma} \frac{\partial}{\partial \theta} \arg \{g(z)\} \, d\theta - \pi > -\pi \] (16)
because \( 0 < \beta \) and \( g(z) \) is \( p \)-valently starlike in \( \mathbb{D} \). This contradicts (14) and it completes the proof of Theorem 2.1. \( \square \)

**Corollary 2.2.** If \( f(z) \in \mathcal{A}_p \) and there exist \( g(z) \in \mathcal{S}_p \), \( q(z) \in \mathcal{P} \) and a positive integer \( k \geq 2 \) such that
\[ f^k(z) = kp \int_0^z \frac{g^k(t)q(t)}{t} \, dt \quad z \in \mathbb{D}, \] (17)
then \( f(z) \) is \( p \)-valent in \( \mathbb{D} \).
Proof. Equality (17) may be written in the form
\[ zf^{-1}(z)f'(z) = pq(z) q(z) \]
or
\[ \frac{zf'(z)}{f^{-1}(z)g'(z)} = pq(z), \]
and \( \Re[pq(z)] > 0 \) in \( D \). This gives (12) hence \( f(z) \) is \( p \)-valent in \( D \).

For \( k = 2 \), Corollary 2.2 becomes the following corollary.

**Corollary 2.3.** Let \( f(z) \in A_p \) and there exist \( q(z) \in S_p^* \) and \( q(z) \in P \) such that
\[ zf(z)f'(z) = p \frac{z}{f(z)} q(z) \quad z \in D, \]
Then \( f(z) \) is \( p \)-valent in \( D \).

**Theorem 2.4.** Let
\[ f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \]
be analytic in \( D \). Assume that there exists a \( p \)-valently starlike function
\[ g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \]
such that
\[ \frac{zf'(z)}{p f^{-1}(z)g'(z)} < \left( \frac{1+z}{1-z} \right)^2. \]

Then
\[ \int_0^{2\pi} \left| \Re \left\{ \frac{zf'(z)}{p f^{-1}(z)g'(z)} \right\} \right| d\theta \leq 2\pi, \quad |z| < \sqrt{2} - 1. \]

Proof. If \( \Phi(z) < \Phi_0(z) \), then [1]
\[ \int_0^{2\pi} \left| \Re \{ \Phi(z) \} \right| d\theta \leq \int_0^{2\pi} \left| \Re \{ \Phi_0(z) \} \right| d\theta \quad \text{for} \quad 0 < \rho < 1. \]

From (18) and from (19), for all \( z = \rho e^{i\theta} \), \( \rho \in (0, 1) \), we have
\[ \int_0^{2\pi} \left| \Re \left\{ \frac{zf'(z)}{p f^{-1}(z)g'(z)} \right\} \right| d\theta \leq \int_0^{2\pi} \left| \Re \left\{ \left( \frac{1+z}{1-z} \right)^2 \right\} \right| d\theta. \]

If \( 0 < r \leq \sqrt{2} - 1 \), then \( (1-r^2)^2 - 4r^2 \sin^2 \theta \geq 0 \) and we have
\[ \int_0^{2\pi} \left| \Re \left\{ \frac{1+z}{1-z} \right\} \right| d\theta = 2 \int_0^{r} \left| \frac{(1-r^2)^2 - 4r^2 \sin^2 \theta}{(1+r^2 - 2r \cos \theta)^2} \right| d\theta \]
\[ = 2 \left. \frac{4r \sin \theta}{r^2 - 2r \cos \theta + 1} + \theta \right|_0^\pi \]
\[ = 2\pi. \]

where \( z = \rho e^{i\theta} \).
Theorem 2.5. Assume that \( f(z) \in \mathcal{A}_p, g(z) \in \mathcal{A}_p \). If there are positive integer \( m, n \in \{1, \ldots, p\} \) such that

\[
\left| \arg \left( \frac{zf^{(m)}(z)}{f^{(m-1)}(z)} \right) \right| < \frac{\gamma \pi}{2}, \quad z \in \mathbb{D},
\]  

(20)

for some \( \gamma \in (0, 1) \),

\[
\left| \arg \left( \frac{zg^{(n)}(z)}{g^{(n-1)}(z)} \right) \right| < \frac{\pi}{2}, \quad z \in \mathbb{D},
\]  

(21)

and

\[
\left| \arg \left( \frac{f^{(n)}(z)}{g^{(n)}(z)} \right) \right| < \frac{(1 - \gamma)\pi}{2\beta}, \quad z \in \mathbb{D},
\]  

(22)

for some \( \beta > 1 - \gamma \), then

\[
\Re \left( \frac{zf'}{f^{(1-\beta)}g^{(\beta)}(z)} \right) > 0, \quad z \in \mathbb{D}.
\]  

(23)

This means that \( f(z) \) is a \( p \)-valently Bazilević function of type \( \beta \).

Proof. Let

\[
q(z) = \left\{ \frac{zf^{(m-1)}(z)}{(p - m + 2)f^{(m-2)}(z)} \right\}, \quad q(0) = 1.
\]

If there exists a point \( z_0, |z_0| < 1 \), such that

\[
|\arg \{q(z)\}| < \frac{\pi \gamma}{2}
\]  

(24)

for \( |z| < |z_0| \) and

\[
|\arg \{q(z_0)\}| = \frac{\pi \gamma}{2}
\]  

(25)

for some \( \gamma \in (0, 1) \), then from [5], we have

\[
\frac{z_0q'(z_0)}{q(z_0)} = \frac{2ik \arg \{q(z_0)\}}{\pi},
\]  

(26)

for some \( k \geq (a + a^{-1})/2 \geq 1 \), where \( |q(z_0)|^{1/\gamma} = \pm ia \), and \( a > 0 \). If we consider (25) for the case \( \arg \{q(z_0)\} = \pi \gamma/2 \), then from (26) we have

\[
\left| \arg \left( \frac{zf^{(m)}(z)}{f^{(m-1)}(z)} \right) \right| = \left| \arg \left( (p - m + 2)q(z_0) - 1 + \frac{z_0q'(z_0)}{q(z_0)} \right) \right|  
\]

\[
= \left| \arg \left( (p - m + 2)q(z_0) - 1 + ik\gamma \right) \right|  
\]

\[
\geq \arg \left( (p - m + 2)q(z_0) \right) = \pi \gamma/2.
\]

This contradicts (20), so supposition (25) is false and (24) holds true in whole unit disc \( \mathbb{D} \). The same argumentation shows that (24) holds true if we consider (25) for the case \( \arg \{q(z_0)\} = -\pi \gamma/2 \). Applying this method again and again we obtain that (20) implies the same inequality for all smaller numbers than \( m \) namely

\[
\left| \arg \left( \frac{zf^{(m)}(z)}{f^{(m-1)}(z)} \right) \right| < \frac{\gamma \pi}{2} \quad \forall k \in \{1, \ldots, m\} :  
\]

\[
\left| \arg \left( \frac{zf^{(k)}(z)}{f^{(k-1)}(z)} \right) \right| < \frac{\gamma \pi}{2}.
\]
Also, in the same way, from (21) we have
\[
\left| \arg \left\{ \frac{zg^{(n)}(z)}{g^{(n-1)}(z)} \right\} \right| < \frac{\pi}{2} \Rightarrow \forall k \in \{1, \ldots, n\} : \left| \arg \left\{ \frac{zg^{(k)}(z)}{g^{(k-1)}(z)} \right\} \right| < \frac{\pi}{2}.
\]

Furthermore, it is known, [3, p.200], that if \( q \) is convex univalent in \( \mathbb{D} \) and \( F(z), G(z) \) are analytic in \( \mathbb{D} \), \( G(0) = F(0) \) and
\[
\Re \left\{ \frac{zG'(z)}{G(z)} \right\} > 0, \quad (z \in \mathbb{D}),
\]
then we have
\[
\frac{F'(z)}{G'(z)} < q(z) \Rightarrow \frac{F(z)}{G(z)} < q(z), \quad (z \in \mathbb{D}).
\]

If we put
\[
F(z) = f^{(n-1)}(z), \quad G(z) = g^{(n-1)}(z), \quad q(z) = \left( \frac{1 + z}{1 - z} \right)^\alpha, \quad \alpha = \frac{(1 - \gamma)\pi}{2\beta},
\]
then by (25) and (27), we have
\[
\left| \arg \left\{ \frac{f^{(n)}(z)}{g^{(n)}(z)} \right\} \right| < \frac{(1 - \gamma)\pi}{2\beta} \Rightarrow \left| \arg \left\{ \frac{f^{(n-1)}(z)}{g^{(n-1)}(z)} \right\} \right| < \frac{(1 - \gamma)\pi}{2\beta}, \quad (z \in \mathbb{D}).
\]

Applying this method again and again we obtain that
\[
\left| \arg \left\{ \frac{f^{(n)}(z)}{g^{(n)}(z)} \right\} \right| < \frac{(1 - \gamma)\pi}{2\beta} \Rightarrow \forall k \in \{0, \ldots, n\} : \left| \arg \left\{ \frac{f^{(k-1)}(z)}{g^{(k-1)}(z)} \right\} \right| < \frac{(1 - \gamma)\pi}{2\beta}, \quad (z \in \mathbb{D}).
\]

Note that (27) is an improvement of the earlier Pommerenke’s result [8, Lemma 1, p.180]: If \( f(z) \) is analytic and \( g(z) \) is convex in \( \mathbb{D} \), then
\[
\left| \arg \frac{f'(z)}{g'(z)} \right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D} \Rightarrow \left| \arg \frac{f(z)}{g(z)} \right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},
\]
where \( 0 < \alpha \leq 1 \).

From the above considerations, we can see that inequality (20) holds true for \( m = 1 \), inequality (21) holds true for \( n = 1 \) and inequality (22) holds true for \( n = 0 \). Therefore, we have
\[
\left| \arg \left\{ \frac{zf'(z)}{f^{(1-\beta)}(z)g^{\beta}(z)} \right\} \right| = \left| \arg \left\{ \frac{zf'(z)}{f(z)} \left[ \frac{f(z)}{g(z)} \right]^\beta \right\} \right| = \left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} + \beta \arg \left\{ \frac{f(z)}{g(z)} \right\} \right| \leq \frac{\gamma\pi}{2} + \beta \frac{(1 - \gamma)\pi}{2\beta} = \frac{\pi}{2}
\]
This is (23).
References

[1] G. Arzahichev, L. A. Aksent’ev, The subordination principle in sufficient conditions for univalence, Dokl. Akad. Nauk SSSR, 211(1)(1973), (Soviet Math. Dokl. 14(4)(1973)) 934–939.
[2] A. W. Goodman, Univalent Functions, Vols. I and II, Mariner Publishing Co.: Tampa, Florida (1983).
[3] S. S. Miller, P. T. Mocanu, Differential Subordinations, Theory and Applications, Series of Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York / Basel 2000.
[4] K. Noshiro, On the theory of schlicht functions, J. Fac. Sci. Hokkaido Univ. Jap. 1(2)(1934-35) 129–135.
[5] M. Nunokawa, On the order of strongly starlikeness of strongly convex functions, Proc. Japan Acad. Ser. A 69(7)(1993) 234–237.
[6] S. Ozaki, On the theory of multivalent functions II, Sci. Rep. Tokyo Bunrika Daigaku Sect. A (1941) 45–87.
[7] S. Ozawa, On some criteria for p-valence, J. Math. Soc. Japan 13(1961) 431–441.
[8] N. Pommerenke, On close to-convex functions, Trans. Amer. Math. Soc. 114(1)(1965) 176–186.
[9] K. Sakaguchi, A note on p-valent functions J. Math. Soc. Japan 14(3)(1962) 312–321.
[10] J. Sokół, M. Nunokawa, N. E. Cho, H. Tang, On Some Applications of Noshiro-Warschawski’s Theorem, Filomat, 31(1)(2017) 107–112.
[11] T. Umezawa, On the theory of univalent functions, Tohoku Math J. 7(1955) 212–228.
[12] S. Warschawski, On the higher derivatives at the boundary in conformal mapping, Trans. Amer. Math. Soc. 38(1935) 310–340.