On the sum of $k$-th largest distance eigenvalues of graphs

Huiqiu Lin *

Department of Mathematics, East China University of Science and Technology, Shanghai 200237, P.R. China

Abstract
For a connected graph $G$ with order $n$ and an integer $k \geq 1$, we denote by

$$S_k(D(G)) = \lambda_1(D(G)) + \cdots + \lambda_k(D(G))$$

the sum of $k$ largest distance eigenvalues of $G$. In this paper, we consider the sharp upper bound and lower bound of $S_k(D(G))$. We determine the sharp lower bounds of $S_k(D(G))$ when $G$ is connected graph and is a tree, respectively, and characterize both the extremal graphs. Moreover, we conjecture that the upper bound is attained when $G$ is a path of order $n$ and prove some partial result supporting the conjecture. To prove our result, we obtain a sharp upper bound of $\lambda_2(D(G))$ in terms of the order and the diameter of $G$, where $\lambda_2(D(G))$ is the second largest distance eigenvalue of $G$. As applications, we prove a general inequality involving $\lambda_2(D(G))$, the independence number of $G$, and the number of triangles in $G$. An immediate corollary is a conjecture of Fajtlowicz, which was confirmed in [10] by a different argument. We conclude this paper with some open problems for further study.

Keywords: Distance matrix, distance eigenvalue, eigenvalue sum, the second largest distance eigenvalue

Mathematics Subject Classification (2010): 05C50

1 Introduction
Throughout this paper, we consider simple, undirected and connected graphs. Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$. Let $N(v)$ denote the neighbor set of $v$ in $G$. Let $S \subset V(G)$. We use $G[S]$ to denote the subgraph of $G$ induced by $S$. The distance between vertices $v_i$ and $v_j$, denoted by $d_G(v_i, v_j)$, is the length of a shortest path from $v_i$ to $v_j$ in $G$. The diameter of a graph is the maximum distance between any two vertices of $G$.

*Supported by the National Natural Science Foundation of China (Nos. 11401211 and 11471121) and Fundamental Research Funds for the Central Universities (No. 222201714049). E-mail: huiqiu1in@126.com (H.Q. Lin)
Throughout this note, we use $M(G)$ to denote a real symmetric matrix respect to a connected graph $G$. We use $\lambda_1(M(G)) \geq \lambda_2(M(G)) \geq \cdots \geq \lambda_n(M(G))$ to denote all eigenvalues of $M(G)$ and denote by

$$S_k(M(G)) = \lambda_1(M(G)) + \cdots + \lambda_k(M(G)),$$

where $k \geq 1$ is an integer. The distance matrix of $G$, denoted by $D(G)$ (or simply by $D$), is the real symmetric matrix with $(i, j)$-entry being $d_G(v_i, v_j)$ (or $d_{ij}$). The distance eigenvalues (resp., distance spectrum) of $G$, are denoted by

$$\lambda_1(D(G)) \geq \lambda_2(D(G)) \geq \cdots \geq \lambda_n(D(G)).$$

Recently, the distance matrix of a graph has received increasing attention. Aouchiche and Hansen \[2\] and Lin, Das and Wu \[12\] proved some results on the relations between the distance eigenvalues and some graphic parameters. Lin \[11\] proved an upper bound on the least distance eigenvalue of a graph in terms of it order and diameter. By using some graph operations, Pokorný, Híc, Stevanović and Milošević \[16\] obtained many infinite families of distance integral graphs. Very recently, Lu, Huang and Huang \[13\] characterized all graphs with exactly two distance eigenvalues different from -1 and -3. Huang, Huang and Lu \[9\] characterized all graphs with exactly three distance eigenvalues different from -1 and -2. For more results on the distance matrix of graphs, we refer the reader to the survey \[3\].

Our main motivations of this note come from two aspects. The first one is a paper of Mohar \[15\], in which he proved that $S_k(A(G)) \leq \frac{1}{2}(\sqrt{k} + 1)n$ for any graph of order $n$ and an integer $k \geq 1$. His theorem is originally motivated by a result of Gernert which states that $S_2(A(G)) \leq n$ for any regular graphs $G$ and the upper bound is best possible. Another motivations are the Grone-Merris conjecture \[7\] \[8\] and Brouwer’s conjecture \[5\]. For any graph $G$ on $n$ vertices with degree sequence $d_1 \geq \cdots \geq d_n$, its conjugate degree sequence is defined as the sequence $d'_1 \geq d'_2 \geq \cdots \geq d'_n$ where $d'_k := |\{v_i : d_i \geq k\}|$. The Grone-Merris conjecture, which is proved by Bai \[4\], states that and for any $k \in \{1, \ldots, n\}$, $S_k(L(G)) \leq \sum_{i=1}^{k} d'_i$. Brouwer’s conjecture says that $S_k(L(G)) \leq m + \binom{k}{2}$ holds for any simple graph $G$ of order $n$ and size $m$ and any $1 \leq k \leq n$. These topics have received much attentions and Brouwer’s conjecture is widely open now. However, till to now, it seems to have no study on upper and lower bounds of $S_k(D(G))$.

In this paper, we try to bound it in terms of some parameters of graphs. We will study the following general problem which seems interesting and non-trivial.

**Problem 1.** For a connected graph $G$ and an integer $k \geq 2$, to give tight upper and lower bounds on $S_k(G)$ and to characterize the extremal graphs corresponding to them, respectively.

We first give a lower bound of $S_k(D(G))$.

**Theorem 1.** Let $k \geq 2$ be an integer and $n$ be sufficiently large with respect to $k$. Let $G$ be a connected graph of order $n$.

(i) Then $S_k(D(G)) \geq n - k$ where the equality holds if and only if $G \cong K_n$.

(ii) If $G$ is a tree, then $S_k(D(G)) \geq 2n - 2k$ where the equality holds if and only if $G \cong K_{1,n-1}$.
We then consider the upper bound of \( S_k(D(G)) \). The following problem is our original motivation.

**Problem 2.** Let \( G \not\cong P_n \) be a connected graph with order \( n \). For an integer \( k \geq 2 \) and sufficiently large \( n \) with respect to \( k \), does there hold \( S_k(D(G)) < S_k(D(P_n)) \)?

Very interesting for us, in order to prove some results supporting this problem, we need to obtain a sharp upper bound on \( \lambda_2(D) \), which may be of its own interest.

**Theorem 2.** Let \( G \) be a connected graph of order \( n \) with diameter \( d \). Then \( \lambda_2(D(G)) \leq \frac{n(d-1)}{2} - d \), where the equality holds if and only if \( G \cong K_n \) or \( G \cong K_{\frac{n+1}{2}, \frac{n-1}{2}} \).

As an application of Theorem 2, we can prove the following result. In particular, we can reprove a conjecture by Fajtlowicz [6], which was confirmed in [10].

**Theorem 3.** Let \( G \) be a connected graph of order \( n > s^3 + s^2 - 2s + 1 \), where \( s \geq 2 \). Suppose that the independence number \( \alpha(G) \leq s \). Then there hold:

(i) \( \lambda_2(D(G)) < 3s^3 \cdot \frac{\binom{n}{m}}{m} \).

(ii) (Lin [10] Theorem 1.2) If \( t = 2 \), then \( \lambda_2(D(G)) < t(G) \), where \( t(G) \) denotes the number of triangles in \( G \).

If there is some information on the diameter of a connected graph, we can prove the following result supporting Problem 2 affirmatively.

**Proposition 1.** Let \( G \) be a connected graph with order \( n \) and diameter \( d \). For an integer \( k \geq 2 \) and sufficiently large \( n \) with respect to \( k \), if \( d < \frac{2n}{3(k+2)} \) then \( S_k(D(G)) < S_k(D(P_n)) \).

## 2 Preliminaries

In this section, we will list some preliminaries and prove some lemmas. Our one main tool is Cauchy Interlacing Theorem.

**Theorem 4** (Cauchy Interlacing Theorem). Let \( A \) be a Hermitian matrix with order \( n \) and let \( B \) be a principal submatrix of \( A \) with order \( m \). If \( \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A) \) are the eigenvalues of \( A \) and \( \mu_1(B) \geq \mu_2(B) \geq \cdots \geq \mu_m(B) \) are the eigenvalues of \( B \), then \( \lambda_{n-m+i}(A) \leq \mu_i(B) \leq \lambda_i(A) \) for \( i = 1, \ldots, m \).

Another tool is the famous Ramsey Theorem, which has already turned out to be powerful for problems in spectral graph theory. For example, see [19] due to Zhang and Cao.

**Theorem 5** (Ramsey [17]). Given any positive integers \( k \) and \( l \), there exists a smallest integer \( R(k, l) \) such that every graph on \( R(k, l) \) vertices contains either a clique of \( k \) vertices or an independent set of \( l \) vertices.

The third one is a theorem due to Merris, which helps us to obtain bounds of distance eigenvalues.

**Theorem 6** (Merris [14]). Let \( G \) be a tree of order \( n \). Let \( \lambda_1(D(G)) \geq \cdots \geq \lambda_n(D(G)) \) be the eigenvalues of \( D(G) \) and let \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq 0 \) be the eigenvalues of \( L(G) \). Then

\[
0 > -\frac{2}{\mu_1} > \lambda_2(D(G)) \geq -\frac{2}{\mu_2} \geq \cdots \geq \lambda_{n-1}(D(G)) \geq -\frac{2}{\mu_{n-1}} \geq \lambda_n(D(G)).
\]
Lemma 1. Let \( G \) be a graph of order \( n \). If \( \Delta(G) \leq l \) and \( \text{diam}(G) \leq d \), then
\[
n \leq 1 + l + l(l-1) + l(l-1)^2 + \cdots + l(l-1)^{d-1}.
\]

Now we shall give the proof of Lemma 2 whose proof relies on Theorems 1 and 3. Part of technique is inspired by Zhang and Cao [19].

Lemma 2. Let \( G \) be a connected graph of order \( n \). For any integer \( k \geq 2 \), if \( n \) is sufficiently large with respect to \( k \) then \( \lambda_k(D(G)) \geq -2 \).

Proof. We divide the proof into two cases.

Case 1. \( \Delta(G) \geq R(k-1, k-1) \)

Let \( v \in V(G) \) with \( d_G(v) = \Delta(G) \). By Theorem 5, \( G' := G[N(v)] \) either contains a clique \( A \) of size \( k-1 \) or an independent set \( B \) of size \( k-1 \).

If the first case occurs, then \( H = G[A \cup \{v\}] \cong K_k \). By Theorem 4 and the inequality \( k \geq 2 \), \( \lambda_k(D(G)) \geq \lambda_k(D(H)) = -1 \).

If the second case occurs, then \( H = G[B \cup \{v\}] \cong K_{1,k-1} \). By Theorem 4 and the inequality \( k \geq 2 \), \( \lambda_k(D(G)) \geq \lambda_k(D(H)) = -2 \).

Case 2. \( \Delta(G) < R(k-1, k-1) \)

Take \( l = R(k-1, k-1) - 1 \) and \( d \geq 2k \). By Lemma 1 if \( n \geq 1 + l + l(l-1) + l(l-1)^2 + \cdots + l(l-1)^{d-1} \), then \( \text{diam}(G) \geq d + 1 \). Thus, the distance matrix \( D' \) of \( P_d \) is a principle submatrix of \( D \). Note that \( \mu_i(P_d) = 2 + 2 \cos \frac{\pi}{d} \) for \( i = 1, \ldots, d \). Thus, by Theorems 4 and 6 we have \( \lambda_k(D(G)) \geq \lambda_k(D(P_d)) \geq \frac{n^2}{\mu_k(P_d)} \geq -1 \) for \( 2k \leq d \). The proof is complete.

The following three theorems are used in the proof of Proposition 1.

Theorem 7 (Merris [14]). Let \( T \) be a tree with diameter \( d \). Then \( \lambda_{\frac{d}{2}}(D(T)) > -1 \).

Theorem 8 (Zhou and Ilić [20]). Let \( G \) be a connected graph on \( n \) vertices with diameter \( d \), minimum degree \( \delta_1 \) and the second minimum degree \( \delta_2 \). Then
\[
\lambda_1(D(G)) \leq \sqrt{[dn - \frac{d(d-1)}{2} - 1 - \delta_1(d-1)][dn - \frac{d(d-1)}{2} - 1 - \delta_2(d-1)]},
\]
where the equality holds if and only if \( G \) is a regular graph with \( d \leq 2 \).

Theorem 9 (Ruzieh and Powers [18] Corollary 2.2). The distance spectral radius of the path \( P_n \) is \( \lambda_1(D(P_n)) = \frac{n^2}{2a} - \frac{2n^2 - 2 + a^2}{6a} + O\left(\frac{1}{n^2}\right) \), where \( a \) is the root of \( \tanh a = 1 \). (\( a \approx 1.199679 \).)

Finally, we prove an easy but useful fact to conclude this section.

Lemma 3. Let \( G = G[V_1, V_2] \) be a connected bipartite graph with \( |V_1| = r \) and \( |V_2| = n - r \) and \( e(G) = m \). Then
\[
\lambda_1(D(G)) \geq \frac{2(n^2 + (r-1)n - r^2 - 2m)}{n}.
\]

Proof. Note that
\[
W(G) \geq m + 2 \left( \binom{r}{2} + \binom{n-r}{2} \right) + 3(r(n-r) - m)
= r(r-1) + (n-r)(n-r-1) + 3r(n-r) - 2m
= n^2 + (r-1)n - r^2 - 2m.
\]
Then the result follows from that \( \lambda_1(D(G)) \geq \frac{2W}{n} \). \( \square \)
3 Proofs

Proof of Theorem 1. (i) By a simple calculation, we have $S_k(D(K_n)) = n - k$ for $k \geq 1$. Let $G \not\cong K_n$ be a connected graph with order $n$. Then

$$\lambda_1(D(G)) \geq \frac{2W(G)}{n} \geq \frac{2[m + 2\binom{n}{2} - m]}{n} = 2n - 1 - \frac{2m}{n}.$$  

If $m \leq \frac{n(n-k)}{2}$, then $\lambda_1(D(G)) \geq n + k - 2$. Note that $\lambda_2(D(G)) \geq -1$. By Lemma 2 if $n$ is sufficiently large with respect to $k$, then $\lambda_k(D(G)) \geq -2$. So we obtain

$$S_k(D(G)) = \lambda_1(D(G)) + \lambda_2(D(G)) + \cdots + \lambda_k(D(G)) \geq n + k - 2 - 2(k - 2) = n - k + 1$$

$$= n - k > n - k$$

$$= S_k(D(K_n)).$$

If $m > \frac{n(n-k)}{2}$, then $m > (1 - \frac{1}{2})n^2$ when $n \geq k^2$ (recall that $n$ is sufficiently large with respect to $k$). By Turán’s theorem, $G$ contains a $K_k$. Thus, we have

$$\lambda_i(D(G)) \geq \lambda_i(D(K_k)) = -1 \text{ for } i = 2, \ldots, k.$$  

Since $G \not\cong K_n$, we obtain $\lambda_1(D(G)) > n - 1$. It follows that $S_k(D(G)) > n - 1 - (k - 1) = n - k = S_k(D(K_n))$. If $G = K_n$, it is easy to find that $S_k(D(G)) = n - k$. This completes the proof.

(ii) Let $T \not\cong K_{1,n-1}$. Similar to the proof of Lemma 2 we have either $\Delta(T) \geq k - 1$ or $\text{diam}(T) \geq 2k + 1$, where $k \geq 2$. It follows that either $K_{1,k-1} \subset G$ or $P_{2k+2} \subset G$. Then by Theorem 4 and Lemma 2, either $\lambda_k(D(T)) \geq -2$ or $\lambda_k(D(T)) > -1$. Thus, we have $\lambda_3(D(T)) + \cdots + \lambda_k(D(T)) \geq -2(k - 2)$. Note that $\text{diam}(T) \geq 3$. Then there exists a bipartite partition of $T = T(V_1, V_2)$ such that $|V_1| = r$ and $|V_2| = n - r$ with $2 \leq r \leq \frac{n}{2}$. Then by Lemma 3, we have

$$\lambda_1(D(T)) \geq \frac{2n^2 + 2(r - 1)n - 2r^2 - 4m}{n}$$

$$= \frac{2n^2 + 2(r - 1)n - 2r^2 - 4(n - 1)}{n}$$

$$\geq \frac{2n^2 + 2(2-1)n - 2 \times 2^2 - 4(n - 1)}{n}$$

$$= 2n - 2 - \frac{4}{n}$$

$$> 2n - 3.$$  

Combing with $\lambda_2(D(T)) \geq -1$, we have

$$S_k(T) > 2n - 3 - 1 - 2(k - 2) = 2n - 2k = S_k(K_{1,n-1}).$$

If $T = K_{1,n-1}$, then $S_k(T) = 2n - 2k$. This completes the proof.

□

Proof of Theorem 2. If $d = 1$, then $G \cong K_n$ and $\lambda_2(D(G)) = -1$, hence the result holds. In the following, set $\lambda_2(D) = \lambda_2(D(G))$. Assume that $d \geq 2$. Let $X$
be an eigenvector of $D(G)$ corresponding to $\lambda_2(D)$. We use $x_v$ to denote the entry of $X$ corresponding to the vertex $v \in V(G)$. Define $S^+ = \{v \in V(G) : x_v > 0\}$ and

\[ S^- = \{v \in V(G) : x_v < 0\}. \]

For $v \in S^+$, we have

\[ \lambda_2(D)x_v = \sum_{u \in S^+ \setminus \{v\}} d(u,v)x_u + \sum_{w \in S^-} d(w,v)x_w \leq d \sum_{u \in S^+} x_u + \sum_{w \in S^-} x_w, \]

that is, $(\lambda_2(D) + d)x_v \leq d \sum_{w \in S^+} x_v + \sum_{w \in S^-} x_w$. So

\[ (\lambda_2(D) + d) \sum_{v \in S^+} x_v \leq |S^+|d \sum_{u \in S^+} x_u + |S^-| \sum_{w \in S^-} x_w, \]

that is,

\[ (\lambda_2(D) + d - d|S^+|) \sum_{v \in S^+} x_v \leq |S^+| \sum_{w \in S^-} x_w. \tag{1} \]

For $v \in S^-$, we have

\[ \lambda_2(D)x_v = \sum_{u \in S^+} d(u,v)x_u + \sum_{w \in S^+ \setminus \{v\}} d(w,v)x_w \geq \sum_{u \in S^-} x_u + d \sum_{w \in S^- \setminus \{v\}} x_w. \]

Similarly,

\[ (\lambda_2(D) + d - d|S^-|) \sum_{v \in S^-} x_v \geq |S^-| \sum_{w \in S^+} x_w. \tag{2} \]

Combining Eqs. (1) and (2), we obtain

\[ (\lambda_2(D) + d - d|S^+|)(\lambda_2(D) + d - d|S^-|) \sum_{v \in S^+} x_v \sum_{u \in S^-} x_u \leq |S^+| |S^-| \sum_{v \in S^+} x_v \sum_{u \in S^-} x_u, \]

that is,

\[ (\lambda_2(D) + d - d|S^+|)(\lambda_2(D) + d - d|S^-|) \geq |S^+| |S^-|. \]

Let

\[ f(y) = (y + d - d|S^+|)(y + d - d|S^-|) - |S^+| |S^-|. \]

Clearly, the roots of $f(y) = 0$ are

\[ y_1 = \frac{d(|S^+| + |S^-| - 2) + \sqrt{d^2(|S^+| + |S^-|)^2 - 4(d^2 - 1)|S^+||S^-|}}{2}, \]

and

\[ y_2 = \frac{d(|S^+| + |S^-| - 2) - \sqrt{d^2(|S^+| + |S^-|)^2 - 4(d^2 - 1)|S^+||S^-|}}{2}, \]

respectively. From Eqs. (1) and (2), we can see that $\lambda_2(D) \leq d|S^+| - d$ and $\lambda_2(D) \leq d|S^-| - d$. Hence, $y_1 > \frac{d(|S^+| + |S^-| - 2)}{2} \geq \min\{d|S^+| - d, d|S^-| - d\} \geq \lambda_2(D)$. Since $f(\lambda_2(D)) = (\lambda_2(D) - y_1)(\lambda_2(D) - y_2) \geq 0$, we have

\[ \lambda_2(D) \leq y_2 = \frac{d(|S^+| + |S^-| - 2) - \sqrt{d^2(|S^+| + |S^-|)^2 - 4(d^2 - 1)|S^+||S^-|}}{2} \leq \frac{d(|S^+| + |S^-| - 2) - (|S^+| + |S^-|)}{2} \leq \frac{d - 1}{2} n - d, \]

6
where the second inequality holds since \( 4|S^+||S^-| \leq (|S^+| + |S^-|)^2 \) and the last inequality holds since \( |S^+| + |S^-| \leq n \).

If \( \lambda_2(D) = \frac{d-1}{2}n - d \), then \( |S^+| + |S^-| = n \), \( |S^+| = |S^-| \) and the equalities in Eqs. (1) and (2) hold. This implies that for any vertices \( v, w \in S^- \), \( d(w, v) = d \geq 2 \), and hence \( v, w \) are nonadjacent; for any vertex \( v \in S^- \) and \( u \in S^+ \), \( d(u, v) = 1 \), which implies that \( u \) and \( v \) are adjacent. Thus, \( G[S^-] \) and similarly, \( G[S^+] \) are independent sets and each vertex in \( S^+ \) is adjacent to each vertex in \( S^- \), which implies that \( G \cong K_{\frac{n}{2}, \frac{n}{2}} \).

Conversely, it is routine to check that \( \lambda_2(D(K_{\frac{n}{2}, \frac{n}{2}})) = \frac{n}{2} - 2 \), completing the proof. \( \square \)

Let \( G \) be a graph and \( v \in V(G) \). We denote by \( t(G, u) \) the number of triangles in \( G \) containing the vertex \( u \).

**Proof of Theorem 3**

(i) Since \( \alpha(G) \leq s \), we have \( \text{diam}(G) \leq 2s - 1 \). By Theorem 2

\[
\lambda_2(D) \leq (s - 1)n - (2s - 1). \quad (3)
\]

Recall that a corollary of Turán’s inequality says that for any graph of order \( n \) and size \( m \), \( \alpha(G) \geq \frac{m - \sqrt{m}}{1 + d} \), where \( d = \frac{2m}{n} \) (see Alon and Spencer [1, pp.95]). Thus, we obtain \( s \geq \alpha(G) \geq \frac{n^2}{n + 2m} \), that is,

\[
m \geq \frac{n^2 - sn}{2s}. \quad (4)
\]

For each vertex \( u \in V(G) \), \( t(G, u) = e(G[N(u)]) \). Note that \( \alpha(G[N(u)]) \leq s \). Then (1) becomes

\[
t(G, u) \geq \frac{d^2(u) - sd(u)}{2s}. \quad (5)
\]

Summing over all vertices for (5), we have

\[
3t(G) = \sum_{v \in V(G)} t(G; v) \geq \sum_{u \in V(G)} \frac{d^2(u) - sd(u)}{2s} = \frac{\sum_{v \in V(G)} d^2(u)}{2s} - m. \quad (6)
\]

By AG-mean inequality,

\[
\frac{\sum_{v \in V(G)} d^2(u)}{2s} - m \geq \frac{(\sum_{v \in V(G)} d(u))^2}{2sn} - m = m\left(\frac{2m}{sn} - 1\right) \geq m\left(\frac{n - s}{s^2} - 1\right). \quad (7)
\]

By (6) and (7), we obtain

\[
3s^3 \cdot \frac{t(G)}{m} \geq s(n - s) - s^3 > (s - 1)n - (2s - 1). \quad (8)
\]

By (3) and (8), the proof is completed.

(ii) Setting \( s = 2 \) in Theorem 3(i), we have \( \lambda_2(D(G)) < \frac{2n}{m} \cdot t(G) \) for \( n > 9 \). By Turán’s theorem, when \( \alpha(G) \leq 2 \) and \( n \geq 11 \), we get

\[
m \geq \frac{n^2 - 2n}{4} = \frac{n}{2} \left(\frac{n}{2} - 1\right) \geq \frac{11}{2} \cdot \frac{9}{2} = 24.75 > 24.
\]
The proof is completed. □

Proof of Proposition (i) By Theorem 9 and Lemma 2 we have $S_k(P_n) \geq \frac{n^2}{3} - 2(k - 1)$. From Theorem 8 we have $\lambda_1(D) < dn - \frac{d(d-1)}{2} - 1$. By Theorems 2 and 8 we have

$$S_k(G) \leq dn - \frac{d(d-1)}{2} - 1 + k\left(\frac{n(d-1)}{2} - d\right)$$

$$= (k\frac{d-1}{2} + d)n - \frac{d(d-1)}{2} - 1 - dk$$

$$< \frac{n^2}{3} - 2(k - 1) \text{ (since } d < \frac{2n}{3(k+2)}\text{)}$$

$$\leq S_k(P_n).$$

This completes the proof. □

4 Concluding remarks

It is known that $S_k(K_r,n-r) = 2n - 2k$ for $k \geq 2$. Theorem 1 (ii) shows that $S_k(D) \geq 2n - 2k = S_k(K_{1,n-1})$ if $G$ is a tree, so we may have the following more general problem.

Problem 3. Let $G$ be a connected bipartite graph of order $n$. For an integer $k \geq 2$ and sufficiently large $n$ with respect to $k$, does there always hold $S_k(G) \geq 2n - 2k$, where the equality holds if and only if $G \cong K_{r,n-r}$ for $1 \leq r \leq n - 1$?

By Theorem 2 we have $\lambda_2(D) \leq \frac{n(d-1)}{2} - d$. If $d = 2$, then $\lambda_2(D) \leq \frac{n}{2} - 2$. It seems that the upper bound holds for every connected graph of order $n$, so we have the following problem.

Problem 4. Let $G$ be a connected graph with second largest distance eigenvalue $\lambda_2(D)$. Then $\lambda_2(D) \leq \frac{n}{2} - 2$ and the equality holds if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Acknowledgement

The author would like to thank Bo Ning for sharing the proof of theorem 8.

References

[1] N. Alon, J. H. Spencer, The probabilistic method. the third edition. Wiley-Interscience, New York.

[2] M. Aouchiche, P. Hansen, Proximity, remoteness and distance eigenvalues of a graph, Discrete Appl. Math., 213 (2016), 17–25.

[3] M. Aouchiche, P. Hansen, Distance spectra of graphs: A survey, Linear Algebra Appl., 458 (2014), 301–386.

[4] H. Bai, The Grone-Merris conjecture, Trans. Amer. Math. Soc., 363 (2011), no. 8, 4463–4474.
[5] A. E. Brouwer, W. H. Haemers, Spectra of graphs, Universitext. Springer, New York, 2012. xiv+250 pp.

[6] S. Fajtlowicz, Written on the wall: conjectures derived on the basis of the program Galatea Gabriella Graffiti, Technical report, University of Houston, 1998.

[7] R. Grone, R. Merris, Coalescence, majorization, edge valuations and the Laplacian spectra of graphs, *Linear Multilinear Algebra*, 27 No.2 (1990), 139–146.

[8] R. Grone, R. Merris, The Laplacian spectrum of a graph II, *SIAM J. Discrete Math.*, 7 (1994), 221–229.

[9] X. Huang, Q. Huang, L. Lu, Graphs with at Most Three Distance Eigenvalues Different from -1 and -2, *Graphs Combin.*, 34 (2018), 395–414.

[10] H. Lin, Proof of a conjecture involving the second largest $D$-eigenvalue and the number of triangles, *Linear Algebra Appl.*, 472 (2015), 48–53.

[11] H. Lin, On the least distance eigenvalue and its applications on the distance spread, *Discrete Math.*, 338 (2015), 868–874.

[12] H. Lin, K. Ch. Das, B. Wu, Remoteness and distance eigenvalues of a graph, *Discrete Appl. Math.*, 215 (2016), 218–224.

[13] L. Lu, Q. Huang, X. Huang, The graphs with exactly two distance eigenvalues different from -1 and -3, *J. Algebraic Combin.*, 45 (2017), 629–647.

[14] R. Merris, The distance spectrum of a tree, *J. Graph Theory*, 14 (1990), 365–369.

[15] B. Mohar, On the sum of $k$ largest eigenvalues of graphs and symmetric matrices, *J. Combin. Theory, Ser. B*, 99 (2009), 306–313.

[16] M. Pokorný, P. Híc, D. Stevanović, M. Milošević, On distance integral graphs, *Discrete Math.*, 338 (2015), 1784–1792.

[17] F. P. Ramsey, On a problem of formal logic, *Proc. Lond. Math. Soc.*, (1930), 264–286.

[18] S. Ruzieh, D. Powers, The distance spectrum of the path $P_n$ and the first distance eigenvector of connected graphs, *Linear Multilinear Algebra*, 28 (1990), 75–81.

[19] F. Zhang, Z. Chen, Ramsey numbers, graph eigenvalues, and a conjecture of Cao and Yuan, *Linear Algebra Appl.*, 458 (2014), 526–533.

[20] B. Zhou, A. Ilić, On distance spectral radius and distance energy of graphs, *MATCH Commun. Math. Comput. Chem.*, 64 (2010), 261–280.