On the divisibility of characteristic classes of non-oriented surface bundles

Johannes Ebert¹, Oscar Randal-Williams *²

Mathematical Institute, 24-29 St Giles', Oxford, OX1 3LB, United Kingdom

Article history:
Received 6 June 2008
Received in revised form 7 July 2008
Accepted 14 July 2008

MSC:
57R20

Keywords:
Non-oriented manifold bundles
Mapping class groups
Characteristic classes

1. Introduction

The mapping class group \( \mathcal{N}_g \) of a non-orientable surface \( S_g \) of genus \( g \) (that is, the connected sum of \( g \) copies of \( \mathbb{R}P^2 \)) is defined to be

\[
\mathcal{N}_g := \pi_0(\text{Diff}(S_g)),
\]

the group of components of the diffeomorphism group of that surface. If \( g \geq 3 \), the components of \( \text{Diff}(S_g) \) are contractible [3], hence \( B\mathcal{N}_g \cong B\text{Diff}(S_g) \), and so the cohomology of \( B\mathcal{N}_g \) (or the group cohomology of \( \mathcal{N}_g \)) can be interpreted as the ring of characteristic classes for \( S_g \)-bundles.

Wahl [10] has proved a homological stability theorem for these groups, which says that in degrees \( * \leq (g - 3)/4 \) the cohomology groups \( H^*(\mathcal{N}_g) \) are independent of the genus \( g \). We call this range of degrees the stable range. Combining Wahl’s result with that of Galatius, Madsen, Tillmann and Weiss [7], the stable rational cohomology of these groups can be identified: there are certain integrally defined characteristic classes \( \zeta_i \) in degrees \( 4i \) (defined in Section 3) and the map

\[
\mathbb{Q}[\zeta_1, \zeta_2, \zeta_3, \ldots] \to H^*(\mathcal{N}_g; \mathbb{Q})
\]

is an isomorphism in the stable range. In [9] the second author calculates these stable groups with coefficients in a finite field, and tabulates some low-dimensional integral groups.

The classes \( \zeta_i \) are analogues of the even Miller–Morita–Mumford classes, for non-oriented surface bundles. Galatius, Madsen and Tillmann [6] have studied the divisibility of the Miller–Morita–Mumford classes \( \kappa_i \in H^*(\Gamma_{\infty}; \mathbb{Z}) \) in the free

* Corresponding author.

E-mail addresses: ebert@maths.ox.ac.uk (J. Ebert), randal-w@maths.ox.ac.uk (O. Randal-Williams).

1 Supported by a fellowship within the Postdoc-Programme of the German Academic Exchange Service (DAAD).

2 Supported by an EPSRC Studentship, DTA grant number EP/P502667/1.

0166-8641/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.topol.2008.07.004
Theorem A. The universal zeta classes, \( \zeta_n \in H^{4n}(\mathcal{N}_g; \mathbb{Z}) \), are not divisible in the stable range. Indeed, they are not divisible in the free quotient \( H^\text{free}(\mathcal{N}_g; \mathbb{Z}) \) of \( H^{4n}(\mathcal{N}_g; \mathbb{Z}) \) in this range.

This gives the trend that extra structure on the vertical tangent bundle, such as an orientation or a spin structure, gives extra divisibility of characteristic classes of surface bundles.

2. Lifting diffeomorphisms to orientation coverings

In this section, we will construct a natural homomorphism from the diffeomorphism group Diff(M) of a smooth d-manifold to the group Diff\(^+(M)\) of orientation-preserving diffeomorphisms of the orientation covering of M. This implies that any smooth fiber bundle \( p : E \to B \) admits a two-fold covering \( \pi : \tilde{E} \to E \), such that \( p \circ \pi : \tilde{E} \to B \) is an oriented smooth fiber bundle and that the restriction of \( \pi : \tilde{E} \to E \) to a fiber of \( p \) is the orientation covering.

Let \( M \) be a smooth d-manifold, \( d > 0 \), and let \( \Lambda^dTM \) be the highest exterior power of the tangent bundle, which is a real line bundle. The total space of the orientation covering of \( M \) can be defined as the quotient

\[
\tilde{M} := (\Lambda^dTM \setminus 0)/\mathbb{R}_{>0}.
\]

The canonical map \( \pi : \tilde{M} \to M \) is a two-sheeted covering. The space \( \tilde{M} \) is a smooth oriented manifold with a preferred orientation. To see this, recall that an orientation of a d-dimensional real vector space \( V \) is a component of \( \Lambda^dV \setminus 0 \), or in other words, one of the two points of \( (\Lambda^dV \setminus 0)/\mathbb{R}_{>0} \). Thus a point in \( x \in M \) is by definition an orientation of the tangent space \( T_xM \). The differential \( T_x\pi \) at \( x \in \tilde{M} \) is a linear isomorphism \( T_x\pi : T_xM \to T_{\pi(x)}M \) so the orientation of \( T_{\pi(x)}M \) given by \( x \) gives us a preferred orientation of \( T_xM \). Using local coordinates on \( M \), it is easy to see that these orientations of the tangent spaces \( T_x\tilde{M} \) fit together continuously and define an orientation of \( \tilde{M} \), the preferred orientation.

Moreover, this construction is natural: a diffeomorphism \( f : M \to N \) of smooth manifolds induces a diffeomorphism \( \tilde{f} : \tilde{M} \to \tilde{N} \) which covers \( f \). It is easy to see that \( \tilde{f} \) is orientation-preserving provided \( \tilde{M} \) and \( \tilde{N} \) are endowed with the preferred orientations. If \( g : N \to P \) is another diffeomorphism, then \( g \circ f = g \circ \tilde{f} \). Also, \( \text{id}_\tilde{M} = \text{id}_\tilde{N} \). Finally, we did not use that \( f \) is a diffeomorphism, but only that the differential of \( f \) was nonsingular. It follows that the assignments \( M \mapsto \tilde{M} \) and \( f \mapsto \tilde{f} \) define a functor \( \mathcal{L} \) from the category \( \mathcal{X}_d \) of smooth d-manifolds and local diffeomorphisms to the category \( \mathcal{X}_d^+ \) of oriented d-manifolds and orientation-preserving local diffeomorphisms. In particular, we define a group homomorphism \( \mathcal{L}_M : \text{Diff}(M) \to \text{Diff}^+(M) \).

For a manifold \( M \), we denote by \( \pi_M \) the covering map \( \tilde{M} \to M \) and by \( \iota_M : \tilde{M} \to \tilde{M} \) the unique nontrivial deck transformation. If \( f : M \to N \) is a (local) diffeomorphism, the following relations hold

\[
\pi_N \circ \tilde{f} = f \circ \pi_M; \quad \tilde{f} \circ \iota_M = \iota_N \circ \tilde{f}.
\]

The morphism spaces of the categories \( \mathcal{X}_d \) and \( \mathcal{X}_d^+ \) have a natural topology, the weak \( C^\infty \)-topology, with respect to which the composition maps are continuous. Thus \( \mathcal{X}_d \) and \( \mathcal{X}_d^+ \) are topological categories. Using local coordinates, it is easy to see that the functor \( \mathcal{L} \) is continuous. In particular, the homomorphism \( \mathcal{L}_M : \text{Diff}(M) \to \text{Diff}^+(M) \) is continuous.

Let us now discuss smooth fiber bundles. Let \( p : E \to B \) be a smooth fiber bundle with fiber a d-dimensional smooth manifold \( M \) and structural group \( \text{Diff}(M) \) (with the weak \( C^\infty \)-topology). Consider the associated \( \text{Diff}(M) \)-principal bundle \( Q \to B \), which has the property that \( Q \times_{\text{Diff}(M)} M \cong E \). Via the homomorphism \( \mathcal{L}_M \), the manifold \( M \) has a \( \text{Diff}(M) \)-action by orientation-preserving diffeomorphisms. Hence the fiber bundle

\[
q : \tilde{E} := Q \times_{\text{Diff}(M)} \tilde{M} \to B
\]

is an oriented smooth fiber bundle with fiber \( \tilde{M} \). Because of (2.2), there is a two-fold covering \( \pi_E : \tilde{E} \to E \), such that \( q = p \circ \pi_E \). Furthermore, there is a fiber-preserving and orientation-reversing involution \( \iota_E \) on \( \tilde{E} \). We call \( \tilde{E} \) the fiberwise orientation cover of \( E \). We summarize the results of this section.

Theorem 2.1. The fiberwise orientation covering \( \pi_E : \tilde{E} \to E \) of a smooth fiber bundle \( p : E \to B \) is a two-sheeted covering whose restriction to any fiber \( E_b \) of \( p \) is the orientation covering of \( E_b \). The composition \( q = p \circ \pi_E \) is an oriented fiber bundle. Furthermore, \( \tilde{E} \) and \( \pi_E \) are uniquely determined by these properties (up to orientation-preserving isomorphism).

We conclude with a simple remark. All the constructions in this section make sense when the manifold \( M \) (or the fiber bundle \( E \)) is orientable. If this is the case, then \( \tilde{M} \) is the disjoint sum of two copies of \( M \). The choice of an orientation of \( M \) singles out a component of \( M \).
3. Characteristic classes of surface bundles

In this section, we give a brief review of the theory of characteristic classes of surface bundles, both oriented and non-oriented. First we discuss the oriented case. Let \( \pi : E \to B \) be an oriented surface bundle and let \( T_v E \) be the vertical tangent bundle. It is an oriented 2-dimensional real vector bundle on \( E \) and thus it has an Euler class \( e(T_v E) \in H^2(B; \mathbb{Z}) \). We can consider \( T_v E \) also as a complex line bundle (there is a unique structure on it, which is unique up to isomorphism) and the Euler class agrees with the first Chern class. The Miller–Morita–Mumford classes are defined to agree with the usual cohomology of the space \( X \). Thus we can form the map \( \text{trf}_p^* \circ p^* : H^*(B; \mathbb{Z}) \to H^*(B; \mathbb{Z}) \), and for all \( x \in H^*(B; \mathbb{Z}) \) we have

\[
\text{trf}_p^* \circ p^*(x) = \chi(F) \cdot x, \tag{3.1}
\]

where \( \chi(F) \) denotes the Euler number of the fiber [1, Theorem 5.5]. Furthermore, if \( q : \tilde{E} \to E \) is another smooth fiber bundle with compact fibers, then \( p \circ q \) is also such a fiber bundle. In this situation the composition of the transfers is homotopic to the transfer of the composition (see [2, Eq. (2.3), p. 137]):

\[
\text{trf}_{pq} \simeq \text{trf}_q \circ \text{trf}_p. \tag{3.2}
\]

A diffeomorphism \( f : M \to N \) of manifolds can be considered as a fiber bundle whose fiber is a point. By (3.1),

\[
\text{trf}_f^* \circ f^* = \text{id}_{H^*(N; \mathbb{Z})}, \quad f^* \circ \text{trf}_f^* = \text{id}_{H^*(M; \mathbb{Z})}. \tag{3.3}
\]

In fact, \( \text{trf}_f \) and \( \Sigma^\infty(f^{-1}) \) are homotopic, but we do not need this fact. The transfer of an oriented fiber bundle \( p : E \to B \) is closely related to the umkehr map. For all \( x \in H^*(E; \mathbb{Z}) \), one has (see [1, Theorem 4.3])

\[
\text{trf}_p^*(x) = p_!(x \cup e(T_v E)). \tag{3.4}
\]

The identity (3.4) implies that

\[
\kappa_n(E) = \text{trf}_p^*(e(T_v E)^n) \tag{3.5}
\]

for the Miller–Morita–Mumford classes of an oriented surface bundle \( p : E \to B \). Because of the identity \( p_1(L) = e(L)^2 \) for the Pontrjagin class of a 2-dimensional oriented real vector bundle \( L \), we see that

\[
\kappa_{2n}(E) = \text{trf}_p^*(p_1(T_v E)^n). \tag{3.6}
\]

This can be generalized to the non-oriented case. Wahl defines [10, p. 3]

\[
\zeta_i(E) := \text{trf}_p^*(p_1(T_v E)^i) \in H^{4i}(B; \mathbb{Z}). \tag{3.7}
\]

for a non-oriented surface bundle \( p : E \to B \), where \( p_1(T_v E) \in H^4(E; \mathbb{Z}) \) is the first Pontrjagin class of the vertical tangent bundle.

The spaces \( B \text{ Diff}^+(F_g) \) and \( B \text{ Diff}(S_g) \) carry universal oriented and non-orientable surface bundles of a fixed genus, so the above constructions define classes \( \kappa_i \in H^{2i}(B \text{ Diff}^+(F_g); \mathbb{Z}) \) and \( \zeta_i \in H^{4i}(B \text{ Diff}(S_g); \mathbb{Z}) \) which we call the universal classes, and omit the universal bundle from the notation.

Now we can state and prove the main result of this section.

**Theorem 3.1.** Let \( p : E \to B \) be a non-oriented surface bundle with compact fibers and let \( c : \tilde{E} \to E \) be its fiberwise orientation covering. Denote \( q := p \circ c : \tilde{E} \to B \). Then the following relations hold for all \( n \geq 0 \):

1. \( \kappa_{2n}(\tilde{E}) = 2 \cdot \zeta_n(E) \).
2. \( 2 \cdot \kappa_{2n+1}(\tilde{E}) = 0 \).
Proof. For the identity (1), we compute
\[
\kappa_{2n}(\tilde{E}) = \text{trf}_p^c(p_1(T_\nu \tilde{E})^n)
\]
\[
= \text{trf}_p^c(\text{trf}_c^i(c^*(p_1(T_\nu \tilde{E})^n)))
\]
\[
= \text{trf}_p^c(2 \cdot p_1(T_\nu E)^n)
\]
\[
= 2 \cdot \kappa_n(E).
\]
The first equality is (3.6). Because \(c: \tilde{E} \to E\) is a smooth covering in every fiber, \(c^*(T_\nu E) \cong T_\nu(\tilde{E})\), whence \(p_1(T_\nu \tilde{E}) = c^*(p_1(T_\nu E))\). Together with (3.2), this fact implies the second equality. Because \(c\) is a double covering, the Euler number of its fiber is 2. Thus \(\text{trf}_p^c \circ c^* = 2\), which gives the third equality. The fourth equality is the definition.

For the proof of identity (2), we use the orientation-reversing involution \(i\) on \(\tilde{E}\). By (3.3), \(\text{trf}_p^c = (i^*)^{-1} = i^c\). Because \(c \circ i = c\), it follows that \(\text{trf}_p^c = \text{trf}_p^c \circ \text{trf}_c^i = \text{trf}_c^i \circ i^c\). Because \(i\) is an orientation-reversing fiberwise diffeomorphism, it induces an orientation-reversing vector bundle isomorphism \(d_i: T_\nu \tilde{E} \to i^* T_\nu \tilde{E}\). Thus \(e(T_\nu \tilde{E}) = -i^c e(T_\nu \tilde{E})\). Thus
\[
\kappa_{2n+1}(\tilde{E}) = \text{trf}_p^c(\text{trf}_c^i(e(T_\nu \tilde{E})^{2n+1}))
\]
\[
= \text{trf}_p^c(\text{trf}_c^i(e(T_\nu \tilde{E})^{2n+1}))
\]
\[
= (-1)^{2n+1} \text{trf}_p^c(\text{trf}_c^i(e(T_\nu \tilde{E})^{2n+1}))
\]
\[
= -\kappa_{2n+1}(\tilde{E}). \quad \square
\]

Remark 3.2. An implication of this theorem is that for an oriented surface bundle \(E' \to B\), the characteristic classes \(2 \cdot \kappa_{2n+1}(E')\) are obstructions to \(E'\) admitting an orientation-reversing, fixed-point free, fiberwise involution. Furthermore, for bundles which do admit such an involution, it gives an interpretation of \(\frac{1}{2} \kappa_{2n}(E)\) as the zeta classes of the associated quotient bundle of non-orientable surfaces.

4. An example

In this section, we consider the example of a genus zero surface bundle. Let \(\gamma_3 \to BSO(3)\) be the universal 3-dimensional oriented Riemannian real vector bundle and let \(S(\gamma_3) \to BSO(3)\) be its unit sphere bundle. It is known that this is the universal smooth oriented bundle with fiber \(S^2\), but we do not need this fact. In [4, Proposition 5.2.4] the first author has computed that \(\kappa_{2n}(S(\gamma_3)) = 2p_n^3\). The bundle \(S(\gamma_3)\) admits an orientation-reversing, fixed-point free involution on its fibers, namely the antipodal map \(-\text{id}\). The quotient is \(P(\gamma_3)\), the \(\mathbb{RP}^2\)-bundle associated to \(\gamma_3\). By Theorem 3.1, we have
\[
2\zeta_n(P(\gamma_3)) = 2p_n^3. \quad \text{(4.1)}
\]
It is well known that the free quotient \(H^*_{\text{free}}(BSO(3); \mathbb{Z})\) is the polynomial algebra \(\mathbb{Z}[p_1]\). In particular, the powers \(p_n^3\) are not divisible in the free quotient of \(H^*(BSO(3); \mathbb{Z})\). We have shown:

Proposition 4.1. The class \(\zeta_n(P(\gamma_3))\) is not divisible in \(H^*_{\text{free}}(BSO(3); \mathbb{Z})\).

5. A review of the stable homotopy theory of surfaces and proof of Theorem A

In this section, we give a brief introduction to the modern homotopy theory of surface bundles developed by Galatius, Madsen, Tillmann and Weiss. A good survey can be found in [6, Sections 2 and 3], and full proofs can be found in [7]. Let us first discuss the oriented case. Consider the universal complex line bundle \(L \to BSO(2)\). There does not exist a vector bundle \(V\) such that \(V \oplus L\) is trivial, but we can define an additive inverse \(L^\perp\) of \(L\) as a stable vector bundle. The Madsen–Tillmann spectrum \(MTSO(2)\) is defined to be the Thom spectrum of \(L^\perp\). For any oriented surface bundle \(E \to B\), there exists a natural map
\[
\alpha_E: B \to \Omega_0^\infty MTSO(2)
\]
into the unit component of the infinite loop space of the Madsen–Tillmann spectrum. In particular, it can be defined for the universal oriented surface bundle with fibers a surface \(F_g\) of genus \(g\), to obtain a universal map
\[
\alpha_g: B \text{Diff}^+(F_g) \to \Omega_0^\infty MTSO(2).
\]
For each \(n > 0\), there is a cohomology class \(y_n \in H^{2n}(\Omega_0^\infty MTSO(2); \mathbb{Z})\) such that for any surface bundle as above
\[
\alpha^*_E(y_n) = \kappa_n(E).
\]
The rational cohomology of $\Omega^\infty_0 \text{MTSO}(2)$ is isomorphic to the polynomial ring $\mathbb{Q}[y_1, y_2, \ldots]$. The main result of [7], originally due to Madsen and Weiss [8], implies that the map $\alpha_g$ induces an isomorphism on homology groups in the stable range, that is,

$$H_k(\alpha_g): H_k(B \text{Diff}^+(F_g); \mathbb{Z}) \to H_k(\Omega^\infty_0 \text{MTSO}(2); \mathbb{Z})$$

is an isomorphism as long as $g \geq 2k + 2$.

Similar results hold in the non-oriented case, and are detailed in [10, Section 6]. The Madsen–Tillmann spectrum is replaced by $\text{MTO}(2)$, which is the Thom spectrum of the stable inverse of the universal 2-dimensional real vector bundle over $BO(2)$. There is an analogue of the map $\alpha_E$ for any non-oriented surface bundle $E \to B$, and there are classes $x_n \in H^{4n}(\Omega^\infty_0 \text{MTO}(2); \mathbb{Z})$ for $n > 0$, such that $\alpha_E^*(x_n) = \zeta_n(E)$. The rational cohomology ring of $\Omega^\infty_0 \text{MTO}(2)$ is isomorphic to the polynomial ring $\mathbb{Q}[x_1, x_2, \ldots]$, in complete analogy to the oriented case.

The analogue of the Madsen–Weiss theorem is also true in the non-oriented case, by [10] and [7]. More precisely

$$H_k(\alpha_g; \mathbb{Z}): H_k(B \text{Diff}(S_g); \mathbb{Z}) \to H_k(\Omega^\infty_0 \text{MTO}(2); \mathbb{Z})$$

(5.2)

is an isomorphism as long as $4k + 3 \leq g$, and similarly in cohomology.

**Proof of Theorem A.** This is now straightforward. We assume that the universal class $\zeta_n$ is divisible in the stable range. Under the isomorphism (5.2), $\zeta_n$ corresponds to the class $x_n \in H^{4n}(\Omega^\infty_0 \text{MTO}(2); \mathbb{Z})$, which must also be divisible. We have seen in Proposition 4.1 that the image of $x_n \in H^{4n}(\Omega^\infty_0 \text{MTO}(2); \mathbb{Z})$ under the map $\alpha_{\Omega^\infty_0 \text{MTO}(2)}: BSO(3) \to \Omega^\infty_0 \text{MTO}(2)$ is $p_n^i$ in the free quotient and so not divisible. This is a contradiction. \hfill $\square$

**References**

[1] J.C. Becker, D.H. Gottlieb, The transfer map and fiber bundles, Topology 14 (1975) 1–12.
[2] G. Brumfiel, I. Madsen, Evaluation of the transfer and the universal surgery classes, Invent. Math. 32 (1976) 133–169.
[3] C.J. Earle, J. Eells, A fibre bundle description of Teichmüller theory, J. Differential Geom. 3 (1969) 19–43.
[4] J. Ebert, Characteristic classes of spin surface bundles: Applications of the Madsen–Weiss theory, PhD thesis, Bonner Math. Schriften 381 (2006).
[5] J. Ebert, Divisibility of Miller–Morita–Mumford classes of spin surface bundles, Quart. J. Math. 59 (2008) 207–212.
[6] S. Galatius, I. Madsen, U. Tillmann, Divisibility of the stable Miller–Morita–Mumford classes, J. Amer. Math. Soc. 19 (2006) 759–779.
[7] S. Galatius, I. Madsen, U. Tillmann, M. Weiss, The homotopy type of the cobordism category, Acta Math., in press, electronic preprint arXiv:math/0605249.
[8] I. Madsen, M. Weiss, The stable moduli space of Riemann surfaces: Mumford’s conjecture, Ann. of Math. (2) 165 (2007) 843–941.
[9] O. Randal-Williams, The homology of the stable non-orientable mapping class group, electronic preprint arXiv:0803.3825.
[10] N. Wahl, Homological stability for the mapping class groups of non-orientable surfaces, Invent. Math. 171 (2008) 389–424.