LINKING NUMBER AND WRITHE IN RANDOM LINEAR EMBEDDINGS OF GRAPHS

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Abstract. In order to model entanglements of polymers in a confined region, we consider the linking numbers and writhes of cycles in random linear embeddings of complete graphs in a cube. Our main results are that for a random linear embedding of $K_n$ in a cube, the mean sum of squared linking numbers and the mean sum of squared writhes are of the order of $\Theta(n(n!))$. We obtain a similar result for the mean sum of squared linking numbers in linear embeddings of graphs on $n$ vertices, such that for any pair of vertices, the probability that they are connected by an edge is $p$. We also obtain experimental results about the distribution of linking numbers for random linear embeddings of these graphs. Finally, we estimate the probability of specific linking configurations occurring in random linear embeddings of the graphs $K_6$ and $K_{3,3,1}$.

1. Introduction

Long polymers become tangled up as a result of being tightly packed in a confined region. For example, 46 human chromosomes are packed together inside the nucleus of a cell whose diameter can be as little as $10^{-5}$ times the length of a single chromosome. The entanglement that results affects the processes of replication and transcription of the DNA. For synthetic polymers, tangling is correlated with viscoelastic properties, and hence is important in the design and synthesis of new elastic materials. As these examples illustrate, understanding the tangling of polymers is useful for explaining and controlling molecular behaviour. However, since detailed visualizations of molecular entanglements are not yet technologically possible, their study has been approached through mathematical modeling rather than experimental observation.

Many authors have considered uniform random distributions of open and closed polygonal chains in a cube as a model for long molecular chains in a confined region (see for example [1], [2], [5], [4], [13], [14], [15]). Of particular note, Arsuaga et al [1] obtained a formula for the mean squared linking number of two uniform random $n$-gons in a cube, and showed that the probability of linking between a given simple closed curve in the cube and a uniform random $n$-gon grows at a rate of at least $1-O\left(\frac{1}{\sqrt{n}}\right)$. More recently,
Panagiotou et al [13] has shown that the mean squared linking number, the mean squared writhe, and the mean squared self-linking number of oriented uniform random open or closed chains with \( n \) vertices in a cube all grow at a rate of \( O(n^2) \).

Many of the results that have been obtained about the linking of two random uniform polygons in a cube are restricted to pairs of polygons of the same length. However, there is no biological reason for all polymers in a given region to have the same length. While no theoretical results have been proven thus far about linking between uniform random \( n \)- and \( m \)-gons in a confined region, Arsuaga et al [1] has observed from numerical simulations that the linking probability of a random linear \( n \)-gon and \( m \)-gon in a cube seems to be bounded below by

\[
1 - O\left(\frac{1}{\sqrt{nm}}\right).
\]

In order to obtain theoretical results about linking between random chains of different lengths as well as to measure entanglement in a more general way, we take a new approach. In particular, the above models use ordered sequences of \( n \) points chosen from a uniform random distribution of points in a cube to define one or two \( n \)-gons with linear edges. By contrast, we begin with an unordered set of \( n \) points chosen from a uniform random distribution of points in a cube. We then consider every possible pair of disjoint polygons obtained by adding line segments between some number of points in the set. By taking the sum of the squared linking numbers of all such pairs of polygons, we obtain a single number which represents the linking of all pairs of polygons with vertices in this set regardless of whether the polygons are of the same or distinct lengths. In addition, in order to measure the entanglement of individual polymers in a confined region, we consider the sum of the mean squared writhes over all polygons with vertices in our set.

In addition to modeling entanglement of polymers in confined regions, our results can be seen in the context of linear embeddings of graphs in \( \mathbb{R}^3 \) (that is, embeddings whose edges are realized by straight line segments). In particular, the set of polygons we are considering are the cycles in a linear embedding of the complete graph \( K_n \) in \( \mathbb{R}^3 \). Probably the most significant result in the study of such embeddings was the proof by Negami [11] that for every knot or link \( J \), there is an integer \( R(J) \) such that every linear embedding of the complete graph \( K_{R(J)} \) in \( \mathbb{R}^3 \) contains \( J \). In addition, several authors have obtained results characterizing what links can occur in linear embeddings of specific graphs. In particular, Hughes [7] and Huh and Jeon [8] gave combinatorial proofs that every linear embedding of \( K_6 \) contains either one or three Hopf links and no other links. More recently, Nikkuni [12] obtained the same result with a topological proof. Naimi and Pavelescu [9] proved that every linear embedding of \( K_9 \) contains a non-split link of three components, and showed in [10] that every linear embedding of \( K_{3,3,1} \) contains either 1, 2, 3, 4, or 5 non-trivial links.
Our main results concern the rate of growth of two measures of entanglement. Since rates of growth can be measured in several ways, for clarity we make the following definitions.

**Definition 1.1.** Let $f(n)$ be a function of the naturals. 
- $f(n)$ is said to be of the order of $O(g(n))$ if there exists a constant $C > 0$ such that for sufficiently large $n$,
  $$f(n) \leq Cg(n).$$
- $f(n)$ is said to be of the order of $\theta(g(n))$, if there exist constants $c, C > 0$ such that for sufficiently large $n$,
  $$cg(n) \leq f(n) \leq Cg(n).$$

Section 2 is devoted to the proofs of the following two theorems about entanglement of random linear embeddings of complete graphs inside a cube $C^3 = [0,1]^3$. That is, embeddings of complete graphs whose vertices are given by a random uniform distribution of $n$ points in the cube and whose edges are realized by straight line segments.

**Theorem 2.4.** Let $n \geq 6$, and let $K_n$ be a random linear embedding of the complete graph on $n$ vertices in the cube $C^3$. Then the mean sum of squared linking numbers for $K_n$ is of the order of $\theta(n(n!))$.

**Theorem 2.6.** Let $n \geq 3$, and let $K_n$ be a random linear embedding of the complete graph on $n$ vertices in the cube $C^3$. Then the mean sum of squared writhe for $K_n$ is of the order of $\theta(n(n!))$.

The ideas of the proofs of these results are as follows. The first theorem is proved using Lemma 2.3 which shows that the expected linking number of random cycles of length $k$ and length $l$ is on the order of $\theta(kl)$. The complete graph $K_n$ contains on the order of $\theta((n-1)!)$ links, both of whose cycles are of length $\frac{n}{2}$. Heuristically, the linking number from these cycles dominate the sum of squared linking numbers, resulting in a mean sum of squared linking numbers on the order of $\theta(n(n!))$. To prove the second theorem, we observe that there are many more cycles of length $n$ than of any other length, and the number of such cycles is on the order of $\theta((n-1)!)$. We then use the result of Panagiotou et al [13] that the mean squared writhe of an $n$-cycle is on the order of $\theta(n^2)$ to obtain the desired result.

In Section 3, we consider a set of $n$ points chosen from a uniform random distribution of points in a cube, and then assign a probability that a given pair of vertices is joined with an edge. In this way, we obtain a subgraph of $K_n$ with a given probability. In particular, we use the following definition, originally due to Gilbert [6].

**Definition 1.2.** Let $n \in \mathbb{N}$ and $p \in (0,1)$. Then a $(n,p)$-graph is a graph on $n$ vertices, such that for any pair of vertices, the probability that they are connected by an edge is $p$. 
We obtain several results on knotting and linking in random linear embeddings of \((n, p)\)-graphs, including the following.

**Theorem 3.1.** For any knot \(J\), the probability that a random linear embedding of an \((n, p)\)-graph contains a cycle isotopic to \(J\) goes to 1 as \(n \to \infty\).

**Theorem 3.3.** For any \(n \geq 6\) and \(p \in (0, 1)\), the mean sum of squared linking numbers of a random linear embedding of an \((n, p)\)-graph is of the order of \(\theta(p^n n(n!))\).

In Section 4, we apply the inequalities that we obtain in Section 2 to random linear embeddings of the graphs \(K_6\) and \(K_{3,3,1}\) in order to estimate the probability of specific linking configurations occurring.

Finally, in Section 5, we sample random linear graph embeddings and make some observations about the distribution of linking numbers.

## 2. Random linear embeddings of \(K_n\)

**Definition 2.1.** A random linear embedding of a graph \(G\), is an embedding of \(G\) in the unit cube \(C^3 = [0, 1]^3\) such that the vertices of \(G\) are embedded with a uniform distribution, and every edge \((v_i, v_j)\) of \(G\) is realized by a straight line segment between \(v_i\) and \(v_j\).

Arsuaga et al [1] prove the following lemma.

**Lemma 2.2 ([1]).** Let \(l_1, l_2, l'_1, l'_2\) denote edges in a random linear embedding of a graph with an orientation assigned to each edge. Let \(\epsilon_i\) denote the signed crossing of \(l_i\) and \(l'_i\), and let \(E[\epsilon_1\epsilon_2]\) denote the expected value of \(\epsilon_1\epsilon_2\).

1. If the endpoints of \(l_1, l_2, l'_1, l'_2\) are distinct, then \(E[\epsilon_1\epsilon_2] = 0\).
2. If \(l_1 = l_2\), the endpoints of \(l'_1\) and \(l'_2\) are distinct, and both \(l'_1, l'_2\) are disjoint from \(l_1 = l_2\), then \(E[\epsilon_1\epsilon_2] = 0\).
3. Define variables as follows:
   - Let \(2s\) denote the probability that \(l_1\) and \(l'_1\) cross when \(l_1\) and \(l'_1\) are disjoint.
   - Let \(u = E[\epsilon_1\epsilon_2]\) when \(l_1 = l_2, l'_1\) and \(l'_2\) share exactly one endpoint, and \(l'_1 \cup l'_2\) is disjoint from \(l_1 = l_2\).
   - Let \(v = E[\epsilon_1\epsilon_2]\) when \(l_1\) and \(l_2\) share exactly one endpoint, \(l'_1\) and \(l'_2\) share exactly one endpoint, and \(l_1 \cup l_2\) and \(l'_1 \cup l'_2\) are disjoint.

Then, \(q = s + 2(u + v) > 0\).

Arsuaga et al [1] use the above lemma to prove that the mean squared linking number of two uniform random polygons of length \(n\) is \(\frac{1}{2}n^2q\) where \(q\) is defined in Case (3) of the lemma. We now apply the above lemma in a similar way to obtain a formula for the mean squared linking number of two uniform random polygons where the number of vertices in the two polygons may differ.
**Lemma 2.3.** Let \( n, m \geq 3 \), and let the graph \( G \) be the disjoint union of an \( n \)-cycle \( L \) and an \( m \)-cycle \( L' \). Then the mean squared linking number of a random linear embedding of \( G \) in the cube \( C^3 \) is \( \frac{1}{2}nmq \), where \( q \) is defined in Case (3) of Lemma 2.2.

**Proof.** Let the edges of \( L \) be \( l_1, l_2, \ldots, l_n \) and the edges of \( L' \) be \( l'_1, l'_2, \ldots, l'_m \), both cyclically ordered and oriented. Let \( \epsilon_{ij} \) denote the signed crossing of \( l_i \) and \( l'_j \). Then, the linking number of \( L \) and \( L' \) is given by

\[
lk(L, L') = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \epsilon_{ij}.
\]

Hence the expected value of the mean squared linking number is given by:

\[
E \left[ \left( \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \epsilon_{ij} \right)^2 \right] = \frac{1}{4} \left[ E \left( \sum_{i=1}^{n} \sum_{j=1}^{m} \epsilon_{ij} \right)^2 \right]
\]

\[
= \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{m} E[\epsilon_{ij}^2] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} (E[\epsilon_{ij}\epsilon_{i(j-1)}] + E[\epsilon_{ij}\epsilon_{i(j+1)}])
\]

\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} (E[\epsilon_{ij}\epsilon_{(i+1)(j+1)}] + E[\epsilon_{ij}\epsilon_{(i-1)(j+1)}]).
\]

Note that those cross terms which we know are 0 by Cases (1) and (2) of Lemma 2.2 have been omitted from the above expansion.

Let \( 2s \) denote the probability that a pair of edges \( l_i \) and \( l'_j \) cross. Then the first term in the above expansion is

\[
\frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{m} E[\epsilon_{ij}^2] = \frac{1}{2} nm s.
\]

Let \( u \) denote the expected value of the product of signed crossings of an edge \( l_i \) of \( L \) with consecutive edges \( l'_j \) and \( l'_{j+1} \) of \( L' \). Then the second term in the above expansion is given by

\[
\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} (E[\epsilon_{ij}\epsilon_{i(j-1)}] + E[\epsilon_{ij}\epsilon_{i(j+1)}]) = nm u.
\]

Let \( v \) denote the expected value of the product of signed crossings of consecutive edges \( l_i \) and \( l_{i\pm1} \) of \( L \) with consecutive edges \( l'_j \) and \( l'_{j\pm1} \) respectively of \( L' \). Then the third term in the above expansion is given by

\[
\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} (E[\epsilon_{ij}\epsilon_{(i+1)(j+1)}] + E[\epsilon_{ij}\epsilon_{(i-1)(j+1)}]) = nv.
\]
Finally, let \( q = s + 2(u + v) \). Then we have

\[
E \left[ \left( \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \epsilon_{ij} \right)^2 \right] = \frac{1}{2} nms + nmu + nmv = \frac{1}{2} nm(s + 2(u + v)) = \frac{1}{2} nmq.
\]

\( \square \)

Using the above lemma, we now prove the following theorem.

**Theorem 2.4.** Let \( n \geq 6 \), and let \( K_n \) be a random linear embedding of the complete graph on \( n \) vertices in the cube \( C^3 \). Then the mean sum of squared linking numbers for \( K_n \) is of the order of \( \Theta(n(n!)) \).

**Proof.** Let \( k, l \geq 3 \) such that \( k + l \leq n \). If \( k \neq l \), then the number of disjoint pairs of cycles in \( K_n \) such that one cycle has \( k \) vertices and the other cycle has \( l \) vertices is given by

\[
\binom{n}{k} \binom{n-k}{l} \frac{(k-1)! (l-1)!}{2} \frac{2}{2}.
\]

If \( k = l \), then this number is given by

\[
\frac{1}{2} \binom{n}{k} \binom{n-k}{l} \frac{(k-1)! (l-1)!}{2} \frac{2}{2}.
\]

By Lemma 2.3, we know that the mean squared linking number of a \( k \)-cycle and an \( l \)-cycle in \( K_n \) is \( \frac{1}{2} klq \), where \( q \) is defined in Case (3) of Lemma 2.2. Thus, we obtain the mean sum of squared linking numbers over all disjoint pairs of cycles in \( K_n \) as

\[
\frac{q}{4} \sum_{k=3}^{n-3} \sum_{l=3}^{n-k} kl \binom{n}{k} \binom{n-k}{l} \frac{(k-1)! (l-1)!}{2} \frac{2}{2} = \frac{q}{4} \sum_{k=3}^{n-3} \sum_{l=3}^{n-k} \binom{n}{k} \binom{n-k}{l} \frac{k! l!}{2} \frac{2}{2} = \frac{q}{16} \sum_{k=3}^{n-3} \sum_{l=3}^{n-k} \binom{n}{n-k-l}.
\]

Observe that the double sum

\[
\sum_{k=3}^{n-3} \sum_{l=3}^{n-k} \frac{n!}{(n-k-l)!}
\]

counts the number of ways to obtain disjoint subsets of \( k \geq 3 \) and \( l \geq 3 \) ordered points from the set of \( n \) points. This same quantity can alternatively
be counted by choosing an ordered list of \( i = k + l \) points out of \( n \), then picking a number \( 3 \leq j \leq i - 3 \), so that the first \( j \) points are in the subset of \( k \) points, and the rest are in the subset of \( l \) points. Hence, we have the equality:

\[
\sum_{k=3}^{n-3} \sum_{i=3}^{n-k} \frac{n!}{(n-k-l)!} = \sum_{i=6}^{n} \frac{n!}{(n-i)!} (i-5).
\]

If we only consider the \( i = n \) term in the sum, we obtain the following lower bound for the mean sum of squared linking numbers.

\[
\frac{q}{16} \sum_{i=6}^{n} \frac{n!}{(n-i)!} (i-5) \geq \frac{q}{16} \frac{n!}{16!} (n-5) = \frac{q}{16} (n-5) n!.
\]

For sufficiently large \( n \), we have \( n-5 > \frac{n}{2} \). Thus we have the lower bound

\[
\frac{q}{16} \sum_{i=6}^{n} \frac{n!}{(n-i)!} (i-5) \geq \frac{q}{32} (n) n!.
\]

For an upper bound, we find that,

\[
\frac{q}{16} \sum_{i=6}^{n} \frac{n!}{(n-i)!} (i-5) \leq \frac{q}{16} n! \left( \sum_{i=6}^{n} \frac{n}{(n-i)!} \right)
= \frac{q}{16} n (n!) \left( \sum_{i=6}^{n} \frac{1}{(n-i)!} \right)
= \frac{q}{16} n (n!) \sum_{m=1}^{\infty} \frac{1}{m!}
= \frac{q}{16} n (n!) e.
\]

Putting these inequalities together, we see that the mean sum of squared linking numbers is of the order of \( \theta(n(n!)) \). \( \square \)

Another way to model entanglement is to consider the tangling of individual cycles rather than the linking between cycles. In particular, given a fixed oriented \( k \)-cycle \( J_k \) in \( \mathbb{R}^3 \), we define the directional writhe \( \text{Wr}_\xi(J_k) \) projected in a direction perpendicular to a unit vector \( \xi \in S^2 \) as the algebraic sum of the signed crossings of \( J_k \). In order to avoid issues of sign, it is preferable to work instead with the directional squared writhe, which is defined as \( \text{Wr}_\xi^2(J_k) = (\text{Wr}_\xi(J_k))^2 \). Now if we average the directional squared
writhe $\text{Wr}_\xi(J_k)$ over all possible direction vectors $\xi \in S^2$, we obtain the \textit{mean squared writhe} denoted by $\text{Wr}^2(J_k)$. More formally, we define

$$\text{Wr}^2(J_k) = \frac{1}{4\pi} \int_{S^2} \text{Wr}_\xi(J_k) d\xi$$

Panagiotou et al \cite{13} prove that the mean squared writhe of a random linear embedding of a $k$-cycle is of the order of $O(k^2)$. Rather than focusing on a single cycle, we are interested in obtaining a single value representing the complexity of the entanglement of all cycles $C$ in a random linear embedding of $K_n$. Thus we define the \textit{mean sum of squared writhe} of $K_n$ as the expected value

$$E[\sum_{C \subseteq K_n} \text{Wr}^2(C)]$$

over all random linear embeddings of $K_n$.

We will make use of the following lemma from \cite{13} which is similar to Lemma \ref{lem:2.2}

\textbf{Lemma 2.5} (\cite{13}). Let $l_1, l_2, l'_1, l'_2$ denote edges in a random linear embedding of a graph with an orientation assigned to each edge, let $\epsilon_i$ denote the signed crossing of $l_i$ and $l'_i$, and let $E[\epsilon_1 \epsilon_2]$ denote the expected value of $\epsilon_1 \epsilon_2$. Also, let $s, u,$ and $v$ be defined in Case (3) of Lemma \ref{lem:2.2} and let $w = E[\epsilon_1 \epsilon_2]$ when $l_1, l_2, l'_1, l'_2$ are consecutive edges. Then $q' = 3s + 2(2u + v + w) > 0$.

\textbf{Theorem 2.6.} Let $n \geq 3,$ and let $K_n$ be a random linear embedding of the complete graph on $n$ vertices in the cube $C^3$. Then the mean sum of squared writhe for $K_n$ is of the order of $\Theta(n(n!))$.

\textit{Proof.} For some $k \leq n,$ let $J_k$ be a $k$-cycle in $K_n$. It follows from Panagiotou et al \cite{13} that the mean squared writhe satisfies

$$\text{Wr}^2(J_k) = qk^2 - (6q - q')k$$

where $q$ is defined in Case (3) of Lemma \ref{lem:2.2}. Also, by Lemma \ref{lem:2.5} we know that $q' > 0$. Thus we have

$$\text{Wr}^2(J_k) > qk^2 - 6qk.$$ 

Hence, we have the following lower bound for the mean sum of squared writhe.

$$E[\sum_{J \subseteq K_n} \text{Wr}^2(J)] \geq \sum_{k=3}^{n} (qk^2 - 6qk) \frac{n!}{(n-k)!(2k)} = \frac{qn!}{2} \sum_{k=3}^{n} \frac{k - 6}{(n-k)!}$$
Taking only the term when $k = n$, we see that
\[
\frac{qn!}{2} \sum_{k=3}^{n} \frac{k-6}{(n-k)!} \geq \left(\frac{qn!}{2}\right)(n-6).
\]

For sufficiently large $n$, we have $n - 6 > \frac{n}{2}$. Thus we obtain the lower bound
\[
E[\sum_{J \subseteq K_n} \text{Wr}^2(J)] \geq \left(\frac{n}{4}\right)(n)!.
\]

In order to get an upper bound, first observe that for any $k$-cycle $J_k$, by Panagiotou et al.\cite{13} we have
\[
\text{Wr}^2(J_k) = qk^2 - (6q - q')k
\]
Thus, taken over all cycles $J$ in $K_n$, we have the expected value
\[
E[\sum_{J \subseteq K_n} \text{Wr}^2(J)] = \sum_{k=3}^{n} (qk^2 - (6q - q')k) \frac{n!}{(n-k)!(2k)}.
\]
Now, by Lemma 2.2 $q > 0$. Thus we obtain the following upper bound.
\[
\sum_{k=3}^{n} (qk^2 - (6q - q')k) \frac{n!}{(n-k)!(2k)} \leq \sum_{k=3}^{n} (qk + q') \frac{n!}{(n-k)!(2)}
\]
\[
= \frac{n!}{2} \sum_{k=3}^{n} \frac{qk + q'}{(n-k)!}
\]
\[
\leq (qn + q'n) \frac{n!}{2} \sum_{j=1}^{\infty} \frac{1}{j!}
\]
\[
\leq \left(\frac{q + q'}{2}\right) (n)n!e.
\]

Putting these inequalities together we see, that the mean sum of squared writhe of a random linear embedding of $K_n$ is of order of $\theta(n(n!)).$

We remark that the calculations above can be modified to show that the total number of links (resp. cycles) in a random linear embedding of $K_n$ is of the order of $\theta(n^4)$, so that the mean average linking number (resp. writhe), where we average over all two component links (resp. cycles) in $K_n$, is of the order of $\theta(n^2)$. This agrees with the results of \cite{11} and \cite{13}, and shows that links (resp. cycles) of length $\theta(n)$ dominate the mean sum of linking number (resp. writhe) of the embedded graph.
3. Random linear embeddings of \((n, p)\)-graphs

**Theorem 3.1.** For any knot \(J\), the probability that a random linear embedding of an \((n, p)\)-graph contains a cycle isotopic to \(J\) goes to 1 as \(n \to \infty\).

**Proof.** By Negami [11], there is an integer \(R(J)\) such that every linear embedding of the complete graph \(K_{R(J)}\) in \(\mathbb{R}^3\) contains \(J\). Thus, given a set of vertices \(\{v_1, v_2, \ldots, v_{R(J)}\}\) in general position in the cube \(C^3\), the linear embedding of \(K_{R(J)}\) defined by these vertices necessarily contains the knot \(J\) as a cycle. Furthermore, since no cycle in \(K_{R(J)}\) has length more than \(R(J)\), a random linear embedding of an \((R(J), p)\)-graph has probability at least \(p^{R(J)}\) of containing the knot \(J\).

For any \(n \geq R(J)\), by partitioning the vertices into sets of \(R(J)\) vertices, we see that the probability that a random linear embedding of an \((n, p)\)-graph contains \(J\) is at least \(1 - (1 - p^{R(J)})^{\lfloor n/R(J) \rfloor}\). This value goes to 1 as \(n \to \infty\). \(\square\)

Similar results can be obtained for other intrinsic properties of spatial embeddings of graphs. For example,

**Theorem 3.2.** For an \((n, p)\)-graph \(G\),

1. The probability that a random linear embedding of \(G\) contains a non-trivial link of two components is at least \(1 - (1 - p^6)^{\lfloor n/6 \rfloor}\). In particular, it goes to 1 as \(n \to \infty\).
2. The probability that a random linear embedding of \(G\) contains a non-split link of three components is at least \(1 - (1 - p^9)^{\lfloor n/9 \rfloor}\). In particular, it goes to 1 as \(n \to \infty\).

**Proof.** The proof is similar to that of Theorem 3.1. For part (1), we apply the result of Conway and Gordon [3] that every embedding of \(K_6\) contains a non-trivial link; and for part (2), we apply the result of Naimi and Pavelescu [9] that every linear embedding of \(K_9\) contains a non-split link of three components. \(\square\)

Suppose that \(n \geq 6, p \in (0, 1)\), and \(G\) is a random linear embedding of an \((n, p)\)-graph. Then for any \(k, l \geq 3\) such that \(k + l \leq n\), the probability that \(G\) contains a pair of disjoint cycles where one is a \(k\)-cycle and the other is an \(l\)-cycle is \(p^{k+l}\). We can now modify the proof of Theorem 2.4 to obtain the following.

**Theorem 3.3.** For any \(n \geq 6\) and \(p \in (0, 1)\), the mean sum of squared linking numbers of a random linear embedding of an \((n, p)\)-graph is of the order of \(\theta(p^n n(n!))\).

**Proof.** We note that mean sum of squared linking numbers for a random linear embedding of an \((n, p)\)-graph is

\[
\frac{q}{16} \sum_{i=6}^{n} p^i \frac{n!}{(n - i)!} (i - 5).
\]
Taking the term $i = n$ gives the lower bound:
\[
\frac{q}{16} p^n (n!) (n - 5) \geq \frac{q}{32} p^n n(n!).
\]
For the upper bound, we re-index over $k = n - i$, so that the sum becomes
\[
n! \sum_{i=6}^{n} \frac{p^i (i - 5)}{(n - i)!} = n! \sum_{k=0}^{n-6} \frac{p^{n-k}(n - k - 5)}{k!}
\]
\[
\leq n(n!) \sum_{k=0}^{n-6} \frac{p^{-k}}{k!}.
\]
\[
\leq n(n!) \sum_{k=0}^{\infty} \frac{p^{-k}}{k!} = n(n!) p^n e^{1/p}.
\]
\[\square\]

4. Random linear embeddings of $K_6$ and $K_{3,3,1}$

In this section, we apply the formulas in the proof of Theorem 2.4 to the graphs $K_6$ and $K_{3,3,1}$ in order to find bounds on the probability of specific types of linking occurring.

It follows from the formula
\[
\frac{q}{16} \sum_{k=3}^{n-3} \sum_{l=3}^{n-k} \frac{n!}{(n-k-l)!}
\]
for the mean sum of squared linking numbers for $K_n$ in the proof of Theorem 2.4 that the mean sum of squared linking numbers for $K_6$ is given by
\[
\frac{q}{16} \frac{6!}{16! 0!} = 45q.
\]

It has been shown independently by Hughes [7], Huh and Jeon [8], and Nikkuni [12] that every linear embedding of $K_6$ contains either exactly one or three Hopf links, and all other links in the embedding are trivial. Hence, for a given linear embedding of $K_6$, the sum of squared linking numbers is either 1 or 3.

This means that $45q = p_1 + 3p_3$, where $p_1$ is the probability that a random linear embedding of $K_6$ has exactly one Hopf link, and $p_3 = 1 - p_1$ is the probability that a random linear embedding of $K_6$ has exactly three Hopf links. This implies that
\[
p_1 = \frac{3 - 45q}{2}.
\]
We estimated the value of $q$ to be $0.033867 \pm 0.000013$ (see the numerical computation described in Appendix A). This value is consistent with the value of $q = 0.0338 \pm 0.024$ obtained by [11], which is verified in [13]. Thus we find that $p_1 = 0.7380 \pm 0.0003$.

For the complete tripartite graph $K_{3,3,1}$, Naimi and Pavelescu [10] show that every linear embedding contains either $1, 2, 3, 4,$ or $5$ non-trivial links. Furthermore, they show that if the number of non-trivial links is odd, all such links are Hopf links; whereas if the number of non-trivial links is even, then one link is a $(2,4)$-torus link and the rest are Hopf links. Since a Hopf link has linking number $\pm 1$ and a $(2,4)$-torus link has linking number $\pm 2$, it follows that the sum of squared linking numbers for any linear embedding of $K_{3,3,1}$ is either $1, 3, 5$ or $7$.

Now every pair of disjoint cycles in $K_{3,3,1}$ consists of one 3-cycle and one 4-cycle. By Lemma 2.3, the mean squared linking number of a random linear embedding of a disjoint union of a 3-cycle and a 4-cycle is $q(3)(4)$. Since there are nine pairs of disjoint cycles in $K_{3,3,1}$, it follows that the expected value of the sum of squared linking numbers of a linear embedding of $K_{3,3,1}$ is

$$\frac{q}{2}(9)(3)(4) = 54q.$$  

For each $k$, we let $p_k$ be the probability that there are $k$ non-trivial links in the embedding. Then, this expected value is equal to

$$1p_1 + 5p_2 + 3p_3 + 7p_4 + 5p_5 \geq p_1 + 3(1 - p_1).$$

Hence, it follows that the probability that there is precisely one non-trivial link in a random linear embedding of $K_{3,3,1}$ is given by

$$p_1 \geq \frac{3 - 54q}{2} = 0.5856 \pm 0.0004.$$  

5. Experimental data

In this section, we describe some experimental results we obtained for links in random linear embeddings of graphs.

The data was generated using a Python program, taking coordinates of the $n$ vertices to be uniformly distributed in $(0,1)$. An edge between two vertices is taken with probability $p$, and then the number of links with each linking number are tallied. To give a more accurate picture of the distribution of linking numbers and the average sum of linking numbers, we took multiple samples for each $(n, p)$.

We first investigated the mean sum of squared linking number for $p = 1$, $p = 0.5$, and $p = 0.25$, comparing experimental data with the expected number from the formula given in the proof of Theorem 3.3. For $p = 1$, we
Figure 1. Experimental vs. expected mean sum of squared linking number for $p = 1$. Experimental data used 1000 samples for $6 \leq n \leq 11$, and 100 samples for $n = 12$.

Figure 2. Experimental vs. expected mean sum of squared linking number for $p = 0.5$. Experimental data used 1000 samples for $6 \leq n \leq 14$, and 100 samples for $n = 15$. 
Figure 3. Experimental vs. expected mean sum of squared linking number for $p = 0.25$. Experimental data used 200000 samples for $6 \leq n \leq 7$, 50000 samples for $8 \leq n \leq 12$, 5000 samples for $13 \leq n \leq 15$, 500 samples for $n = 16$, and 100 samples for $n = 17$.

The experimental data follows the expected super-factorial growth. The deviation for the mean sum of squared linking number is within approximately 10% of the expected value (and most data points are within 5%), and the discrepancy for large numbers of vertices and $p < 1$ is due to the small number of samples taken due to computational constraints. In addition, from the $n = 6$, $p = 1$ case, we can determine that of the 1000 random linear embeddings sampled, 729 had exactly one Hopf link, giving a 99% confidence interval for the probability that a random linear embedding of $K_6$ has one Hopf link of $0.729 \pm 0.036$, which agrees with the theoretical computation from Section 4 and numerical value of $q$.

In addition, we computed the mean average squared linking number and the mean average absolute linking number, where the average is taken over all links in the graph, and then the mean is taken over all samples of a given size. The experimental data for the mean average squared linking number follows a quadratic growth, as expected. From the samples we computed, it appeared that the average absolute linking number was also quadratic,
which differs from the linear growth rate for absolute linking number of random polygons studied in [1] and [13] (see Figures 4, 5, and 6). However, because of the small number of data points that we could compute, this is inconclusive.

**Figure 4.** Mean average squared and absolute linking number for $p = 1$.

**Figure 5.** Mean average squared and absolute linking number for $p = 0.5$. 
We also investigated the distributions of links with a given linking number in random \((n,p)\) graphs (see Figures 7, 8, and 9). Omitted from the figures are links with linking number greater than 2, which were detected in fewer than 1% of the links in the samples that we generated. We expect that for a fixed linking number \(k\), the proportion of links with that linking number will increase, peak, then decrease as \(n \to \infty\). However, due to computational
Figure 8. Proportion of links in random linear embeddings of \((n, p)\) graphs with linking number 0, 1, and 2, when \(p = 0.5\).

Figure 9. Proportion of links in random linear embeddings of \((n, p)\) graphs with linking number 0, 1, and 2, when \(p = 0.25\).

constraints, we were not able to compute samples out to large enough \(n\) to see this behavior, even with linking number 1.
Appendix A. Computing $q$

In order to obtain better estimates of the linking probabilities for $K_6$ and $K_{3,3,1},$ we numerically estimated the value of $q.$ Consider two triangles described by the consistently oriented edges $l_1, l_2, l_3$ and $l'_1, l'_2, l'_3,$ and let $\epsilon_{ij}$ denote the signed crossing number between $l_i$ and $l'_j.$ Then it follows from the proof of Lemma 2.2 in [1] that

$$E \left( \left( \sum_{i,j=1}^{3} \epsilon_{ij} \right)^2 \right) = 18q.$$  

But the quantity $\sum_{i,j=1}^{3} \epsilon_{ij}$ is precisely twice the linking number of the two triangles, which for a linear embedding is either 0 or $\pm 1.$ Hence, $\frac{18q}{4}$ is the probability that a random linear embedding of two disjoint triangles is linked.

We wrote a Python program to generate random linear embeddings of two triangles by taking six random points in $\mathbb{C}^3,$ described as three (pseudo)random coordinates in $(0, 1),$ and computing the associated linking number. Out of the one billion samples generated, 152,402,780 were linked, giving a 99% confidence interval for $q$ of $0.033867 \pm 0.000013.$

All code is available at [http://math.berkeley.edu/~kozai/random_graphs/](http://math.berkeley.edu/~kozai/random_graphs/).

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