THE BARGMANN TRANSFORM ON A BROAD FAMILY OF BANACH SPACES, WITH APPLICATIONS TO TOEPLITZ AND PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. We investigate mapping properties for the Bargmann transform on modulation spaces whose weights and their reciprocals are allowed to grow faster than exponentials. We prove that this transform is isometric and bijective from modulation spaces to convenient Lebesgue spaces of analytic functions. We use this to prove that such modulation spaces fulfill most of the continuity properties which are well-known when the weights are moderated. Finally we use the results to establish continuity properties of Toeplitz and pseudo-differential operators in the context of these modulation spaces.

0. INTRODUCTION

In this paper we introduce and establish basic continuity properties for a broad family of (quasi-)Banach spaces of functions and distributions of Gelfand-Shilov types, in the framework of harmonic analysis. We establish close links between these spaces and (weighted) Lebesgue spaces $A_{\omega}^{p,q}$ of analytic functions related to Bargmann-Fock spaces. The family of spaces consists of modulation spaces, where each modulation space is obtained by imposing a weighted mixed norm estimate on the short-time Fourier transforms of the involved distributions. Important cases of such spaces are given by $M_{\omega}^{p,q}$, where the weighted mixed norm estimate is constituted by the $L_{\omega}^{p,q}$ norm.

Among the involved parameters $p$, $q$ and the weight $\omega$, it follows that $\omega$ is most important concerning imposing regularity, or relaxing growth, oscillations and singularity conditions on the involved distributions. In comparison to already established theories of such spaces (cf. [22,27,47] and the references therein) the conditions for the involved weight functions are significantly relaxed in the present paper. This leads to that our family of modulation spaces are significantly larger compared to the ”usual” families of such spaces. For example, for each fixed $s > 1/2$, the modulation space $M_{\omega}^{p,q}$ can be made ”arbitrary close” to the Gelfand-Shilov space $S_s$ or to $S'_s$, by choosing the weight $\omega$ in appropriate ways.

An essential part of our investigations concerns the establishment of the links between the modulation spaces and the $A_{\omega}^{p,q}$ spaces, by proving that the Bargmann transform is isometric and bijective between these spaces. One of the benefits is that any property valid for the $A_{\omega}^{p,q}$ spaces, carry over to the modulation spaces, and vise versa. For example, we prove that any modulation space is a Banach or quasi-Banach space, and that convenient density, duality and interpolation properties
hold for such spaces, because similar properties are valid for corresponding spaces of analytic functions.

Finally we use our results to extend the theory of pseudo-differential operators to involve more extreme symbols and target distributions comparing to earlier investigations.

We recall that the (classical) modulation space $M_{\omega}^{p,q}$, $p, q \in [1, \infty]$, as introduced and carefully investigated by Feichtinger and Gröchenig in [18–21, 27], consists of all tempered distributions whose short-time Fourier transforms (STFT) have finite mixed $L_{\omega}^{p,q}$ norm. Here the weight $\omega$ quantifies the degree of asymptotic decay and singularity of the distribution in $M_{\omega}^{p,q}$. In general it is assumed that $\omega$ should be moderate, which imposes several properties on $\omega$ and thereby on the modulation space $M_{\omega}^{p,q}$. (See Sections 1 and 2 for strict definitions.) For example, the moderate property implies that $\omega$ is not allowed to grow or decay faster than exponentials, that $M_{\omega}^{p,q}$ are invariant (but not norm invariant) under pullbacks of translations, and that several properties valid for weighted Lebesgue spaces (e.g. density, duality and interpolation properties) carry over to classical modulation spaces.

A major idea behind the design of these spaces was to find useful Banach spaces, which are defined in a way similar to Besov spaces, in the sense of replacing the dyadic decomposition on the Fourier transform side, characteristic to Besov spaces, with a uniform decomposition. From the construction of these spaces, it turns out that modulation spaces and Besov spaces in some sense are rather similar, and sharp embeddings between these spaces can be found in [54,55], which are improvements of certain embeddings in [26]. (See also [48, 60] for verification of the sharpness.)

During the last 15 years, several results have been proved which confirm the usefulness of the modulation spaces in time-frequency analysis, where they occur naturally. For example, in [19, 28, 30], it is shown that all modulation spaces admit reconstructible sequence space representations using Gabor frames.

Parallel to this development, modulation spaces have been incorporated into the calculus of pseudo-differential operators, which also involve Toeplitz operators. (See e.g. [28,31,32,35,36,41,54–58] and the references therein concerning symbol classes embedded in $S'$, and [15, 28, 42, 44, 49, 50] for results involving ultra-distributions.

Here and in what follows we use the usual notations for the usual function and distribution spaces, see e.g. [37].

The Bargmann transform can easily be reformulated in terms of the short-time Fourier transform, with a particular Gauss function as window function. By reformulating the Bargmann transform in such way, and using the fundamental role of the short-time Fourier transform in the definition of modulation spaces, it easily follows that the Bargmann transform is continuous and injective from $M_{\omega}^{p,q}$ to $A_{\omega}^{p,q}$. Furthermore, by choosing the window function as a particular Gaussian function in the $M_{\omega}^{p,q}$ norm, it follows that $\mathcal{V} : M_{\omega}^{p,q} \to A_{\omega}^{p,q}$ is isometric.

These facts and several other mapping properties for the Bargmann transform on (classical) modulation spaces were established in [20,22,27,33,47]. In fact, here
it is proved that the Bargmann transform from $M_{(\omega)}^{p,q}$ to $A_{(\omega)}^{p,q}$ is not only injective, but in fact bijective.

For the modulation space $M_{(\omega)}^{p,q}$, the weight function $\omega$ is important for imposing or relaxing conditions on the distributions $f$ in $M_{(\omega)}^{p,q}$. More precisely, the weight $\omega = \omega(x, \xi)$ depends on both the space (or time) variables $x$ as well as the momentum (or frequency) variables $\xi$. Roughly speaking, the weight function possesses (cf. [16, 18, 28, 31, 32]):

- $\omega$ tending rapidly to infinity as $x$ tends to infinity, imposes that $f$ tends rapidly to zero at infinity;
- $\omega$ tending rapidly to zero as $x$ tends to infinity, relaxes the growth conditions on $f$ at infinity;
- $\omega$ tending rapidly to infinity as $\xi$ tends to infinity, imposes high regularity for $f$;
- $\omega$ tending rapidly to zero as $\xi$ tends to infinity, relaxes the conditions on singularities of $f$;
- $\omega$ tending rapidly to infinity as both $x$ and $\xi$ tend to infinity, imposes stronger restrictions on oscillations for $f$ at infinity;
- $\omega$ tending rapidly to zero as both $x$ and $\xi$ tend to infinity, relax the restrictions on oscillations for $f$ at infinity.

The condition that $\omega$ should be moderate implies that

$$\omega + 1/\omega \leq v$$

for some $v = Ce^{c|\cdot|}$, where $c, C > 0$ are constants. In this case, $\omega$ is called a weight of exponential type. We remark that corresponding modulation spaces $M_{(\omega)}^{p,q}$ are subsets of appropriate spaces of Gelfand-Shilov distributions, and for certain choices of $\omega$ we may have that $M_{(\omega)}^{p,q}$ is contained in $\mathcal{S}$, or that $\mathcal{S}'$ is contained in $M_{(\omega)}^{p,q}$. A more restrictive case appears when (0.1) is true for some $v = Ce^{c|\cdot|^s}$, with $0 \leq s < 1$. In this case, $\omega$ is called a weight of subexponential type. If instead $v$ in (0.1) can be chosen as polynomial, then $\omega$ is said to be of polynomial type. In this case, $M_{(\omega)}^{p,q}$ contains $\mathcal{S}$, and is contained in $\mathcal{S}'$.

Several properties for the modulation spaces might be violated when passing from the subexponential type weights into exponential type weights. For example, if $\omega$ is of exponential type, then $M_{(\omega)}^{p,q}$ might be contained in the set of real analytic functions, which in particular implies that there are no non-trivial compactly supported elements in $M_{(\omega)}^{p,q}$. Consequently, there are no compactly supported Gabor atoms, implying the time-frequency machinery breaks in those parts were compactly supported Gabor atoms are needed.

In the present paper we go beyond these situations and relax the assumptions on $v$ even more. For example, we permit $v$ in (0.1) to be superexponential, i.e. $v = Ce^{c|\cdot|^\gamma}$, where $1 < \gamma < 2$. In this situation, almost no arguments in classical modulation space theory can be used, because the main results in that theory are
based on the fact that $\omega$ should be moderate. This condition is violated when $v$ in (0.1) has to be superexponential.

In Sections 1 and 2 we give the explicit conditions on the weight functions, and in Sections 3 and 4 we prove:

1. any extended weight class contains all weights in classical modulation space theory, including weights which are moderated by exponential type weights. Furthermore, any superexponential weight with $\gamma$ above less than 2 are included, as well as weights of the form $\omega = \langle \cdot \rangle^{\gamma}$ and $\omega = \Gamma(\langle \cdot \rangle + 1)$. Here $\langle x \rangle = (1 + |x|^2)^{1/2}$ and $\Gamma$ is the Gamma function.

2. $A_{(\omega)}^{p,q}$ and $M_{(\omega)}^{p,q}$ are Banach spaces and fulfill convenient density, duality and interpolation properties.

3. the Bargmann transform is isometric and bijective from $M_{(\omega)}^{p,q}$ to $A_{(\omega)}^{p,q}$.

In the last section we establish new forms of pseudo-differential calculi in the framework of these modulation spaces. This means that the spans of the spaces for operator symbols, target functions and image functions, are significantly larger comparing to earlier theories. Therefore, these spaces may be smaller as well as larger comparing the usual situations. The approach here is similar to [50, 57, 59], where similar results were obtained in background of classical modulation space theory. The results here are, to some extent, also related to the results in [11, 42–44, 46, 49, 50], when $v$ in (0.1) is bounded by a subexponential function.

We remark that in contrast to classical theory of pseudo-differential operators, (cf. e.g. [37]), there are no explicit regularity assumptions on the symbols. On the other hand, if $1 < \gamma_1 < \gamma < \gamma_2 < 2$ with $c > 0$, and the weight $\omega$ is given by

$$\omega(x, \xi) = e^{c(|x|^\gamma + |\xi|^\gamma)},$$  

then the corresponding modulation spaces are contained in the Gelfand-Shilov space $S_{1/\gamma_1}$, and contain $S_{1/\gamma_2}$. In particular, this means that the involved functions and their derivatives are extendable to entire analytic functions and fulfill estimates of the form

$$|f(x)| \leq Ce^{-c|x|^{\gamma_1}}, \quad \text{and} \quad |\hat{f}(\xi)| \leq Ce^{-c|\xi|^{\gamma_1}},$$

for some positive constants $c$ and $C$. It is therefore obvious that in this situation, the elements in these modulation spaces possess strong regularity properties.

On the other hand, if $c < 0$ in (0.2), then the corresponding modulation spaces contain the dual $S'_{1/\gamma_1}$ of $S_{1/\gamma_1}$, which in turn is significantly larger than e.g. $\mathcal{S}'$, the space of tempered distributions.

Finally, in Section 5 we apply the continuity results for modulation spaces to establish continuity properties for Toeplitz operators with symbols in weighted mixed norm space of Lebesgue types. (Cf. [5, 7, 10, 13, 14, 58] and the references therein for similar and related investigations.)
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1. Preliminaries

In this section we give some definitions and recall some basic facts. The proofs are in general omitted. In the first part we consider appropriate conditions on the involved weight functions. Thereafter we review some facts for Gelfand-Shilov spaces. Then we discuss basic properties of the short-time Fourier transform, which is thereafter used in the definition of modulation spaces, and obtaining basic properties for such spaces. The last part of the section is devoted to the Bargmann transform and appropriate Banach spaces of entire functions, which are appropriate in the background of the Bargmann transform.

1.1. Weight functions. We start by discussing general properties on the involved weight functions. A weight on $\mathbb{R}^d$ is a positive function $\omega$ on $\mathbb{R}^d$ such that $\omega \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, and for each compact set $K \subseteq \mathbb{R}^d$, there is a constant $c > 0$ such that

$$\omega(x) \geq c \quad \text{when} \quad x \in K.$$

A usual condition on $\omega$ is that it should be $v$-moderate for some positive function $v \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. This means that

$$\omega(x + y) \leq C \omega(x)v(y), \quad x, y \in \mathbb{R}^d, \quad (1.1)$$

for some constant $C$ which is independent of $x, y \in \mathbb{R}^d$. We note that (1.1) implies that $\omega$ fulfills the estimates

$$C^{-1}v(-x)^{-1} \leq \omega(x) \leq Cv(x).$$

We say that $v$ is submultiplicative when (1.1) holds with $\omega = v$. In the sequel, $v$ and $v_j$ for $j \geq 0$, always stand for submultiplicative weights if nothing else is stated.

The weight $\omega$ is called a weight of exponential type, if $v$ in (1.1) can be chosen as $v(x) = Ce^{c|x|}$ for some $c, C > 0$. If, more restrictive, $v$ can be chosen as a polynomial, then $\omega$ is called a weight of polynomial type. We let $\mathcal{P}(\mathbb{R}^d)$ and $\mathcal{P}_E(\mathbb{R}^d)$ be the sets of all weights on $\mathbb{R}^d$ of polynomial type and exponential type, respectively. Obviously, $\mathcal{P}(\mathbb{R}^d) \subseteq \mathcal{P}_E(\mathbb{R}^d)$.

A broader class of moderate weights comparing to $\mathcal{P}(\mathbb{R}^d)$ is obtained by replacing the polynomial assumption on $v$ by the so called GRS condition (Gelfand-Raikov-Shilov condition). That is, $v \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ is positive and satisfies

$$\lim_{n \to \infty} \frac{\log v(nx)}{n} = 0.$$
An important class of submultiplicative weights which fulfills the GRS condition is the so called subexponential weights, i.e. weights of the form
\[ \omega(x) = Ce^{c|x|}, \]  
(1.2)
when \( \omega = v \), and \( c, C \) and \( s \) are positive constants such that \( s < 1 \). On the other hand, if \( v \) is a weight of exponential type, then the GRS condition is violated. Furthermore, if \( \omega \) is \( v \)-moderate for some \( v \), then it is moderated by an exponential type weight. Consequently, the \( \mathcal{P}_E(\mathbb{R}^d) \) contains all weights on \( \mathbb{R}^d \) which are moderated by some functions, including those weights moderated by \( v \) which fulfills the GRS-conditions. We refer to [29] and the references therein for these facts.

In this paper we permit weights where the moderate condition (1.1) on \( \omega \) has been relaxed by appropriate local and global conditions. In most of the situations, the local condition is
\[ C^{-1}\omega(x) \leq \omega(x+y) \leq C\omega(x) \quad \text{when} \quad Rc \leq |x| \leq c/|y|, \quad R \geq 2, \]  
(1.3)
for some positive constants \( c \) and \( C \). However, in most of the situations, the condition (1.3) is relaxed into
\[ C^{-1}\omega(x)^2 \leq \omega(x+y)\omega(x-y) \leq C\omega(x)^2 \quad \text{when} \quad Rc \leq |x| \leq c/|y|, \quad R \geq 2, \]  
(1.3)^{'}
for some positive constants \( c \) and \( C \).

Important examples of weights satisfying (1.3) are those which satisfy (1.2), when \( C \) and \( s \) being positive such that \( s \leq 2 \), and \( c \in \mathbb{R} \). Especially we note that if \( \omega \) is given by (1.2) with \( 1 < s \leq 2 \), then \( \omega \) is not moderated by any weight \( v \), but satisfies (1.3) for some choices of \( c > 0 \) and \( C > 0 \). On the other hand, if \( v > 0 \) and satisfies (1.3), then Proposition 2.6 in Section 2 shows that
\[ C^{-1}e^{-c|x|^2} \leq \omega(x) \leq Ce^{c|x|^2}, \]  
(1.4)
holds for some positive constants \( c \) and \( C \).

**Definition 1.1.** Let \( \omega \) be a weight on \( \mathbb{R}^d \).

(1) \( \omega \) is called a weight of Gaussian type (weakly Gaussian type) on \( \mathbb{R}^d \), if (1.3) holds (if (1.3)^{'} holds) for some positive \( c \) and \( C \), and (1.4) holds for some positive \( c \) and \( C \). The set of Gaussian type and weakly Gaussian type weights on \( \mathbb{R}^d \) are denoted by \( \mathcal{P}_G(\mathbb{R}^d) \) and \( \mathcal{P}_Q(\mathbb{R}^d) \), respectively;

(2) \( \omega \) is called a weight of subgaussian type (weakly subgaussian type) on \( \mathbb{R}^d \), if (1.3) holds (if (1.3)^{'} holds) for some positive \( c \) and \( C \), and for every \( c > 0 \), there is a constant \( C > 0 \) such that (1.4) holds. The set of subgaussian type and weakly subgaussian type weights on \( \mathbb{R}^d \) are denoted by \( \mathcal{P}_G^0(\mathbb{R}^d) \) and \( \mathcal{P}_Q^0(\mathbb{R}^d) \), respectively.

We note that each one of the families of weight functions in Definition 1.1 are groups under the ordinary multiplications.

**Remark 1.2.** The family \( \mathcal{P}_Q^0 \) is larger than \( \mathcal{P}_G^0 \), but its definition is somewhat more complicated. An important reason for introducing this family is that we may prove that the general modulation spaces, introduced later on, can be made close
to Gelfand-Shilov spaces in the sense of Proposition 3.9 in Section 3. So far we are unable to prove any similar result when the family $P^0_Q$ is replaced by $P^0_G$.

On the other hand, for any weight in $P^0_Q$, one may find an equivalent smooth weight (cf. Proposition 2.6 in Section 2). So far we are unable to extend this property to all weights in $P^0_Q$.

We note that if $\omega \in P^0_Q(\mathbb{R}^d)$, then $\omega$ satisfies the following conditions:

(1) there are invertible $d \times d$-matrices $T_1, \ldots, T_N$ whose norms are at most one, i.e. $\|T_j\| \leq 1$, $j = 1, \ldots, N$, and such that

$$C^{-1}\omega(x)^N \leq \prod_{j=1}^N \omega(x + T_j y) \leq C\omega(x)^N$$

when $Rc \leq |x| \leq c/|y|$, $R \geq 2$, \((1.3)''\)

for some positive constants $c$ and $C$;

(2) for every $c > 0$, there is a constant $C > 0$ such that \((1.4)\) holds.

Hence if we modify the definition of $P^0_Q(\mathbb{R}^d)$ in such way that it should contain all weights $\omega$ satisfying (1) and (2), then we obtain a larger family of weights, comparing to Definition 1.1. By straight-forward computations it follows that all results in the paper are true after the definition of $P^0_Q$ in Definition 1.1 has been modified in this way.

A special situation appears for Proposition 3.9 in Section 3, where the symmetry condition in $y$ in \((1.3)'\) is essential for its proof. However, it follows that Proposition 3.9 is true, after \((1.3)'\) in the definition of $P^0_Q$ has been replaced by

$$C^{-1}\omega(x)^{2N} \leq \prod_{j=1}^N \omega(x + T_j y)\omega(x - T_j y) \leq C\omega(x)^{2N},$$

when $Rc \leq |x| \leq c/|y|$ and $R \geq 2$, where $T_j$ are invertible matrices with norm at most one.

In Section 2 we introduce other convenient subfamilies of $P_Q(\mathbb{R}^d)$.

**Example 1.3.** Let $c, s \in \mathbb{R}$, $C > 0$ and $t > 1/2$. Then

$$\sigma_s(x) \equiv \langle x \rangle^s = (1 + |x|^2)^{s/2}, \quad \omega_1(x) = e^{c|x|^{1/t}} \quad \text{and} \quad \omega_2(x) = e^{c|x|^2},$$

(1.5)

are weights of polynomial type, subgaussian type and Gaussian type, respectively.

**Definition 1.4.** Let $\Omega \subseteq P_Q(\mathbb{R}^d)$. Then $\Omega$ is called an *admissible family of weights*, if there is a rotation invariant function $0 < \omega_0(x) \in L^\infty_{\text{loc}}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ which decreases with $|x|$ and such that

$$\omega \cdot \omega_0 \in \Omega \quad \text{and} \quad \omega/\omega_0 \in \Omega \quad \text{when} \quad \omega \in \Omega.$$

We note that for some choice of $\omega_0$ in Definition 1.4 we have

$$\omega_0(x) \leq C(x)^{-d}$$

(1.6)

for some constant $C > 0$. 7
Example 1.5. Every family in Definition [11] are admissible. Moreover, if $\omega_0 \in \mathcal{P}_Q(\mathbb{R}^d)$ and $\Omega$ is a family of admissible weights, then

(1) $\{ \sigma_N; N \in \mathbb{Z} \}$ is admissible;

(2) $\omega_0 \cdot \Omega \equiv \{ \omega_0 \omega; \omega \in \Omega \}$ is admissible.

1.2. Gelfand-Shilov spaces. Next we recall the definition of Gelfand-Shilov spaces.

Let $0 < h, s \in \mathbb{R}$ be fixed. Then we let $\mathcal{S}_{s,h}(\mathbb{R}^d)$ be the set of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$
\| f \|_{\mathcal{S}_{s,h}} \equiv \sup_{\| \xi \| \leq h} \frac{|x^\beta \partial^{\alpha} f(x)|}{h^{\| \alpha \|+\| \beta \|} (\alpha! \beta!)^s}
$$

is finite. Here the supremum should be taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$. Obviously $\mathcal{S}_{s,h} \subseteq \mathcal{S}$ is a Banach space which increases with $h$ and $s$. Furthermore, if $s > 1/2$ or $s = 1/2$ and $h \geq 1$, then $\mathcal{S}_{s,h}$ contains all finite linear combinations of Hermite functions. Since such linear combinations are dense in $\mathcal{S}$, it follows that the dual $\mathcal{S}_{s,h}'(\mathbb{R}^d)$ of $\mathcal{S}_{s,h}(\mathbb{R}^d)$ is a Banach space which contains $\mathcal{S}'(\mathbb{R}^d)$.

The Gelfand-Shilov spaces $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$ are the inductive and projective limit respectively of $\mathcal{S}_{s,h}(\mathbb{R}^d)$. This implies that

$$
\mathcal{S}_s(\mathbb{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma_s(\mathbb{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}(\mathbb{R}^d),
$$

(1.7)

and that the topology for $\mathcal{S}_s(\mathbb{R}^d)$ is the strongest possible one such that each inclusion map from $\mathcal{S}_{s,h}(\mathbb{R}^d)$ to $\mathcal{S}_s(\mathbb{R}^d)$ is continuous. The space $\Sigma_s(\mathbb{R}^d)$ is a Fréchet space with semi norms $\| \cdot \|_{\mathcal{S}_{s,h}}$, $h > 0$.

We remark that the space $\mathcal{S}_s(\mathbb{R}^d)$ is zero when $s < 1/2$, and that $\Sigma_s(\mathbb{R}^d)$ is zero when $s \leq 1/2$. Furthermore, for each $\varepsilon > 0$ and $s \geq 1/2$ we have

$$
\Sigma_s(\mathbb{R}^d) \subseteq \Sigma_{s+\varepsilon}(\mathbb{R}^d).
$$

On the other hand, in [11] there is an alternative elegant definition of $\Sigma_{s_1}(\mathbb{R}^d)$ and $\mathcal{S}_{s_2}(\mathbb{R}^d)$ such that these spaces agrees with the definitions above when $s_1 > 1/2$ and $s_2 \geq 1/2$, but $\Sigma_{1/2}(\mathbb{R}^d)$ is non-trivial and contained in $\mathcal{S}_{1/2}(\mathbb{R}^d)$.

From now on we assume that $s > 1/2$ when considering $\Sigma_s(\mathbb{R}^d)$.

The Gelfand-Shilov distribution spaces $\mathcal{S}'_s(\mathbb{R}^d)$ and $\Sigma'_s(\mathbb{R}^d)$ are the projective and inductive limit respectively of $\mathcal{S}'_{s,h}(\mathbb{R}^d)$. This means that

$$
\mathcal{S}'_s(\mathbb{R}^d) = \bigcap_{h>0} \mathcal{S}'_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma'_s(\mathbb{R}^d) = \bigcup_{h>0} \mathcal{S}'_{s,h}(\mathbb{R}^d).
$$

(1.7)'

We remark that already in [21] it is proved that $\mathcal{S}'_s(\mathbb{R}^d)$ is the dual of $\mathcal{S}_s(\mathbb{R}^d)$, and if $s > 1/2$, then $\Sigma'_s(\mathbb{R}^d)$ is the dual of $\Sigma_s(\mathbb{R}^d)$ (also in topological sense).

The Gelfand-Shilov spaces are invariant under several basic transformations. For example they are invariant under translations, dilations, tensor products and under any Fourier transformation.

From now on we let $\mathcal{F}$ be the Fourier transform which takes the form

$$
(\mathcal{F} f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i(x, \xi)} \, dx.
$$
when \( f \in L^1(\mathbb{R}^d) \). Here \( \langle \cdot, \cdot \rangle \) denotes the usual scalar product on \( \mathbb{R}^d \). The map \( \mathcal{F} \) extends uniquely to homeomorphisms on \( \mathcal{S}'(\mathbb{R}^d), \mathcal{S}'_s(\mathbb{R}^d) \) and \( \Sigma_s(\mathbb{R}^d) \), and restricts to homeomorphisms on \( \mathcal{S}(\mathbb{R}^d), \mathcal{S}_s(\mathbb{R}^d) \) and \( \Sigma_s(\mathbb{R}^d) \), and to a unitary operator on \( L^2(\mathbb{R}^d) \).

The following lemma shows that elements in Gelfand-Shilov spaces can be characterized by estimates of the form

\[
|f(x)| \leq C e^{-\varepsilon|x|^{1/s}} \quad \text{and} \quad |\hat{f}(\xi)| \leq C e^{-\varepsilon|\xi|^{1/s}}. \tag{1.8}
\]

The proof is omitted, since the result can be found in e.g. [12, 24].

**Lemma 1.6.** Let \( f \in \mathcal{S}'_{1/2}(\mathbb{R}^d) \). Then the following is true:

1. if \( s \geq 1/2 \), then \( f \in \mathcal{S}_s(\mathbb{R}^d) \), if and only if there are constants \( \varepsilon > 0 \) and \( C > 0 \) such that (1.8) holds;
2. if \( s > 1/2 \), then \( f \in \Sigma_s(\mathbb{R}^d) \), if and only if for each \( \varepsilon > 0 \), there is a constant \( C' \) such that (1.8) holds.

Gelfand-Shilov spaces posses several other convenient properties. For example, they can easily be characterized by Hermite functions. We recall that the Hermite function \( h_\alpha \) with respect to the multi-index \( \alpha \in \mathbb{N}^d \) is defined by

\[
h_\alpha(x) = \pi^{-d/4}(-1)^{\alpha/2} \alpha!^{-1/2} e^{x^2/2} e^{-|x|^2/2} \text{e}^{-\xi^2/2}.
\]

The set \( (h_\alpha)_{\alpha \in \mathbb{N}^d} \) is an orthonormal basis for \( L^2(\mathbb{R}^d) \). In particular,

\[
f = \sum_\alpha c_\alpha h_\alpha, \quad c_\alpha = \langle f, h_\alpha \rangle_{L^2(\mathbb{R}^d)}, \tag{1.9}
\]

and

\[
\|f\|_{L^2} = \|\{c_\alpha\}_\alpha\|_{l^2} < \infty,
\]

when \( f \in L^2(\mathbb{R}^d) \). Here and in what follows, \( \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)} \) denotes any continuous extension of the \( L^2 \) form on \( \mathcal{S}_{1/2}(\mathbb{R}^d) \).

It is well-known that \( f \) here belongs to \( \mathcal{S}(\mathbb{R}^d) \), if and only if

\[
\|\{c_\alpha\}_\alpha\|_{l^2} < \infty \tag{1.10}
\]

for every \( t \geq 0 \). Furthermore, for every \( f \in \mathcal{S}'(\mathbb{R}^d) \), the expansion (1.3) still holds, where the sum converges in \( \mathcal{S}' \), and (1.10) holds for some choice of \( t \in \mathbb{R} \), depending on \( f \). The same conclusion holds after the \( L^2 \) norm has been replaced by any \( l^p \) norm with \( 1 \leq p \leq \infty \).

The following proposition, which can be found in e.g. [25], shows that similar conclusion for Gelfand-Shilov spaces hold, after the estimate (1.10) is replaced by

\[
\|\{c_\alpha e^{\alpha t^2}\}_\alpha\|_{l^p} < \infty. \tag{1.11}
\]

(Cf. formula (2.12) in [25].)

**Proposition 1.7.** Let \( p \in [1, \infty] \), \( f \in \mathcal{S}'_{1/2}(\mathbb{R}^d) \), \( s \geq 1/2 \) and let \( c_\alpha \) be as in (1.9). Then the following is true:

1. if \( f \in \mathcal{S}'_{1/2}(\mathbb{R}^d) \), if and only if (1.11) holds for every \( t < 0 \). Furthermore, (1.9) holds where the sum converges in \( \mathcal{S}'_{1/2} \);
(2) \( f \in \Sigma'_s(\mathbb{R}^d) \), if and only if (1.11) holds for some \( t < 0 \). Furthermore, (1.9) holds where the sum converges in \( \Sigma'_s \);

(3) \( f \in \mathcal{S}_s(\mathbb{R}^d) \), if and only if (1.11) holds for some \( t > 0 \). Furthermore, (1.9) holds where the sum converges in \( \mathcal{S}_s \);

(4) \( f \in \Sigma_s(\mathbb{R}^d) \), if and only if (1.11) holds for every \( t > 0 \). Furthermore, (1.9) holds where the sum converges in \( \Sigma_s \).

1.3. The short-time Fourier transform. Let \( \phi \in \mathscr{S}(\mathbb{R}^d) \setminus 0 \) be fixed. For every \( f \in \mathscr{S}'(\mathbb{R}^d) \), the short-time Fourier transform \( V_\phi f \) is the distribution on \( \mathbb{R}^{2d} \) defined by the formula

\[
( V_\phi f ) (x, \xi) = \mathcal{F} ( f \phi(\cdot - x))(\xi). \tag{1.12}
\]

We note that the right-hand side defines an element in \( \mathscr{S}'(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d}) \), and that if \( f \in L^q_\omega \) for some \( \omega \in \mathscr{P}(\mathbb{R}^d) \), then \( V_\phi f \) takes the form

\[
V_\phi f (x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) \phi(y-x) e^{-i(y,\xi)} \, dy. \tag{1.12'}
\]

In order to extend the definition of the short-time Fourier transform we reformulate (1.12) in terms of partial Fourier transforms and tensor products. More presiely, we let \( \mathcal{F}_2 F \) be the partial Fourier transform of \( F(x,y) \in \mathcal{S}'(\mathbb{R}^{2d}) \) with respect to the \( y \)-variable, and we let \( U \) be the map which takes \( F(x,y) \) into \( F(y,y-x) \). Then it follows that

\[
V_\phi f = (\mathcal{F}_2 \circ U)(f \otimes \phi) \tag{1.13}
\]

when \( f \in \mathcal{S}'(\mathbb{R}^d) \) and \( \phi \in \mathcal{S}(\mathbb{R}^d) \).

We remark that tensor products of elements in Gelfand-Shilov spaces are defined in similar ways as for tensor products for distributions (cf. Chapter V in \[37\]). Let \( f, g \in \mathcal{S}'_{1/2}(\mathbb{R}^d) \) and let \( s \geq 1/2 \). Then it follows that \( f \otimes g \in \mathcal{S}'_s(\mathbb{R}^{2d}) \), if and only if \( f, g \in \mathcal{S}'_s(\mathbb{R}^d) \). Similar fact holds for any other choice of Gelfand-Shilov spaces of functions or distributions.

The following result is essentially a restatement of ?? in \[15\] and concerns the map

\[
(f, \phi) \mapsto V_\phi f, \tag{1.14}
\]

and follows immediately from (1.13), and the facts that tensor products, \( \mathcal{F}_2 \) and \( U \) are continuous on Gelfand-Shilov spaces. (See also \[15\][34] for general properties of the short-time Fourier transform in background of Gelfand-Shilov spaces.)

**Proposition 1.8.** Let \( s \geq 1/2 \) and let \( f, \phi \in \mathcal{S}'_{1/2}(\mathbb{R}^d) \setminus 0 \). Then the map (1.14) from \( \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^{2d}) \) is uniquely extendable to a continuous map from \( \mathcal{S}'_{1/2}(\mathbb{R}^d) \times \mathcal{S}'_{1/2}(\mathbb{R}^d) \) to \( \mathcal{S}'_{1/2}(\mathbb{R}^{2d}) \). Furthermore, the following is true:

1. The map (1.14) restricts to a continuous map from \( \mathcal{S}_s(\mathbb{R}^d) \times \mathcal{S}_s(\mathbb{R}^d) \) to \( \mathcal{S}_s(\mathbb{R}^{2d}) \). Moreover, \( V_\phi f \in \mathcal{S}_s(\mathbb{R}^{2d}) \), if and only if \( f, \phi \in \mathcal{S}_s(\mathbb{R}^d) \);

2. The map (1.14) restricts to a continuous map from \( \mathcal{S}'_s(\mathbb{R}^d) \times \mathcal{S}'_s(\mathbb{R}^d) \) to \( \mathcal{S}'_s(\mathbb{R}^{2d}) \). Moreover, \( V_\phi f \in \mathcal{S}'_s(\mathbb{R}^{2d}) \), if and only if \( f, \phi \in \mathcal{S}'_s(\mathbb{R}^d) \).
Similar facts hold after the spaces $\mathcal{S}_s$ and $\mathcal{S}_s'$ have been replaced by $\Sigma_s$ and $\Sigma_s'$ respectively.

We also have the following proposition.

**Proposition 1.9.** Let $s \geq 1/2$, $\phi \in \mathcal{S}_s(\mathbb{R}^d) \setminus \{0\}$ be even, and let $f \in \mathcal{S}'_{1/2}(\mathbb{R}^d)$. Then the following is true:

1. $f \in \mathcal{S}_s(\mathbb{R}^d)$, if and only if for some $\varepsilon > 0$ and some constant $C_\varepsilon$ it holds
   \[ |V_\phi f(x, \xi)| \leq C_\varepsilon e^{-\varepsilon(|x|^{1/s}+|\xi|^{1/s})}; \tag{1.15} \]

2. if $f \in \mathcal{S}'_s(\mathbb{R}^d)$, then there are constants $\varepsilon > 0$ and $C_\varepsilon > 0$ such that
   \[ |V_\phi f(x, \xi)| \leq C_\varepsilon e^{\varepsilon(|x|^{1/s}+|\xi|^{1/s})}; \tag{1.16} \]

3. if for every $\varepsilon > 0$, there is a constant $C_\varepsilon$ such that (1.16) holds, then $f \in \mathcal{S}'_s(\mathbb{R}^d)$.

Proposition 1.9 can be found in [15] and to some extent also in [34]. Since the arguments in the proof are important later on, we present here an explicit proof, based on reformulation of the statements in terms of Wigner distributions.

First let $f, g \in L^2(\mathbb{R}^d)$. Then the *Wigner distribution* of $f$ and $g$ is defined by the formula

\[
W_{f,g}(x, \xi) = (2\pi)^{-d/2} \int f(x - y/2)g(x + y/2)e^{i\langle y, \xi \rangle} \, dy.
\]

We note that the Wigner distribution is closely connected to the short-time Fourier transform, since

\[
V_\phi f(x, \xi) = 2^{-d}e^{i(x,\xi)/2}W_{f,\hat{\phi}}(-x/2, \xi/2),
\]

which follows by straightforward computations. Here $\hat{f}(x) = f(-x)$. From this relation it follows that most of the properties which involve short-time Fourier transforms also hold after replacing the short-time Fourier transforms by Wigner distributions. For example, Propositions 1.8 and 1.9 remain the same after such replacements.

**Proof.** (1) If $f \in \mathcal{S}_s(\mathbb{R}^d)$, then it follows from Lemma 1.6 and Proposition 1.8 that (1.15) holds for some constants $\varepsilon > 0$ and $C_\varepsilon > 0$.

Now assume instead that (1.15) holds for some constants $\varepsilon > 0$ and $C_\varepsilon > 0$. Then (1.15) still holds after $V_\phi f$ has been replaced by $W_{f,\hat{\phi}} = W_{f,\phi}$, provided the constants $\varepsilon$ and $C_\varepsilon$ have been replaced by other suitable ones, if necessary. Since

\[
|\mathcal{F}(W_{f,\phi})(\xi, x)| = |V_\phi f(-x, \xi)|,
\]

by Parseval’s formula, it follows that (1.15) holds for both $W_{f,\phi}$ and $\mathcal{F}(W_{f,\phi})$. Hence, $f \in \mathcal{S}_s(\mathbb{R}^d)$ by Lemma 1.6. This proves (1).

The assertion (2) follows by straight-forward computations, using the fact that

\[
V_\phi f(x, \xi) = \langle f, \phi(\cdot - x)e^{-i(\cdot, \xi)} \rangle
\]

in combination with Lemma 1.6.
On the other hand, if for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that \((1.16)\) holds and $\varphi$ is a finite sum of Hermite functions, then $V_\phi \varphi \in S_\nu(\mathbb{R}^d)$, and
\[
(f, \varphi)_{L^2(\mathbb{R}^d)} = c(V_\phi f, V_\phi \varphi)_{L^2(\mathbb{R}^{2d})}
\]
is well-defined. Here $c = \|\phi\|_{L^2}^2 > 0$. Now, by \((1.16)\), \((1)\) and the fact that finite sums of Hermite functions are dense in $S_\nu(\mathbb{R}^d)$, it follows that the right-hand side of \((1.17)\) defines a continuous linear form on $S_\nu(\mathbb{R}^d)$ with respect to $\varphi$. Hence, $f \in S_\nu'(\mathbb{R}^d)$, which gives \((3)\), and the proof is complete. $\square$

**Remark 1.10.** There is obviously a gap between the necessary and sufficiency conditions in \((2)\) and \((3)\) of Proposition \[1.9\]. In general it seems to be difficult to find convenient equivalent conditions for the short-time Fourier transform of $f \in S_{1/2}^\prime$ in order for $f$ should belong to $S_s^\prime$, for some $s > 1/2$.

On the other hand, for each $s \geq 1/2$ and $\phi \in S_s(\mathbb{R}^d) \setminus 0$, let $\Upsilon_{s,\phi}(\mathbb{R}^d)$ be the space which consists of all $f \in S_s(\mathbb{R}^d)$ such that for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that \((1.15)\) holds. Then $\Upsilon_{s,\phi}(\mathbb{R}^d)$ is still a ”large space” in the sense that it contains every $S_s^\prime(\mathbb{R}^d)$ for $t > s$.

For future references we set $\Upsilon = \Upsilon_{s,\phi}$ when $s = 1/2$ and $\phi(x) = \pi^{-d/4} e^{-|x|^2/2}$.

### 1.4. Modulation spaces

We shall now discuss modulation spaces and recall some basic properties. In what follows we let $\mathcal{B}$ be a mixed norm space on $\mathbb{R}^d$. This means that for some $p_1, \ldots, p_n \in [1, \infty]$ and vector spaces

\[
V_1, \ldots, V_n \subseteq \mathbb{R}^d \text{ such that } V_1 \oplus \cdots \oplus V_n = \mathbb{R}^d,
\]
then $\mathcal{B} = \mathcal{B}_n$, where $\mathcal{B}_j, j = 1, \ldots, n$ is inductively defined by
\[
\mathcal{B}_j = \begin{cases}
L^{p_j}(V_j), & j = 1 \\
L^{p_j}(V_j; \mathcal{B}_{j-1}), & j = 2, \ldots, n.
\end{cases}
\]

The minimal and maximal exponents $\min(p_1, \ldots, p_n)$ and $\max(p_1, \ldots, p_n)$ are denoted by $\nu_1(\mathcal{B})$ and $\nu_2(\mathcal{B})$ respectively, and the norm of $\mathcal{B}$ is given by $\|f\|_{\mathcal{B}} \equiv \|F_{n-1}\|_{L^{p_n}(V_n)}$, where $F_0 = f$ and
\[
F_j(x_{j+1}, \ldots, x_n) = \|F_{j-1}(\cdot, x_{j+1}, \ldots, x_n)\|_{L^{p_j}(V_j)}, \quad j = 1, \ldots, n - 1.
\]

In several situations the notation $L^p(V)$ is used instead of $\mathcal{B}$, where
\[
V = (V_1, \ldots, V_n) \quad \text{and} \quad p = (p_1, \ldots, p_n).
\]

We set $\mathcal{B}' = L^{p'}(V)$, where $p' = (p'_1, \ldots, p'_2)$ and $p'_j \in [1, \infty]$ is the conjugate exponent of $p_j$, $j = 1, \ldots, n$. That is, $p_j$ and $p'_j$ should satisfy $1/p_j + 1/p'_j = 1$. If $\nu_2(\mathcal{B}) < \infty$, then the dual of $\mathcal{B}$ with respect to $(\cdot, \cdot)_{L^2}$ is given by $\mathcal{B}'$.

In some situations we relax the conditions on $p_1, \ldots, p_n$ in such way that they should belong to $(0, \infty]$ instead of $[1, \infty]$. Still we set
\[
\|f\|_{L^{p_j}(V_j)} \equiv \begin{cases}
\left(\int_{V_j} |f(x_j)|^{p_j} \, dx_j\right)^{1/p_j}, & \text{when } 0 < p_j < \infty \\
\text{ess sup}_{x_j \in V_j} |f(x_j)|, & \text{when } p_j = \infty,
\end{cases}
\]

12
where $f$ is measurable on $V_j$. (Cf. [3].) We note that $\| \cdot \|_{L^p_j(V_j)}$ is a quasi-norm, but not a norm, when $p_j < 1$. Furthermore, in this case, $L^p_j(V_j)$ is a quasi-Banach space, with topology defined by this quasi-norm.

Now, if $p_1, \ldots, p_n \in (0, \infty]$, and $V = (V_1, \ldots, V_n)$ is the same as above, then $L^p(V)$ is called mixed quasi-norm space on $\mathbb{R}^d$, and is defined as $\mathcal{B}_n$ in (1.19).

Example 1.11. Let $p, q \in [1, \infty]$, and $L^{p,q}(\mathbb{R}^d)$ and its twisted space $L^{p,q}_{tw}(\mathbb{R}^d)$ be the Banach spaces, which consist of all $f \in L^{1}_{loc}(\mathbb{R}^d)$ such that

$$\|F\|_{L^{p,q}} \equiv \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x, \xi)|^p \, dx \right)^{q/p} \, d\xi \right)^{1/q} < \infty,$$

and

$$\|F\|_{L^{p,q}_{tw}} \equiv \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x, \xi)|^{q/p} \, dx \right)^{1/p} \, d\xi \right)^{1/q} < \infty,$$

respectively (with obvious modifications when $p = \infty$ or $q = \infty$). Then it follows that both $L^{p,q}(\mathbb{R}^d)$ and $L^{p,q}_{tw}(\mathbb{R}^d)$ are mixed norm spaces.

If instead $p, q \in (0, \infty]$, then $L^{p,q}$ and $L^{p,q}_{tw}$ are defined in analogous ways, where the condition $F \in L^{1}_{loc}(\mathbb{R}^d)$ has to be replaced by $F \in L^{r}_{loc}(\mathbb{R}^d)$ with $r = \min(p, q)$. In this case, one obtains mixed quasi-norm spaces.

The definition of modulation spaces is given in the following.

Definition 1.12. Let $\mathcal{B}$ be a mixed quasi-norm space on $\mathbb{R}^d$, $\omega \in \mathcal{P}_0^0(\mathbb{R}^d)$, and let $\phi = \pi^{-d/4}e^{-|x|^2/2}$. Then the modulation space $M(\omega, \mathcal{B})$ consists of all $f \in \mathcal{S}_{1/2}^1(\mathbb{R}^d)$ such that

$$\|f\|_{M(\omega, \mathcal{B})} \equiv \|V_{\phi}f\|_{\mathcal{B}} < \infty. \quad (1.21)$$

If $\omega = 1$, then the notation $M(\mathcal{B})$ is used instead of $M(\omega, \mathcal{B})$.

Since the cases $\mathcal{B} = L^{p,q}(\mathbb{R}^d)$ and $\mathcal{B} = L^{p,q}_{tw}(\mathbb{R}^d)$ are especially important to us we set $M^{p,q}(\mathbb{R}^d) = M(\omega, L^{p,q}(\mathbb{R}^d))$ and $W^{p,q}(\omega)(\mathbb{R}^d) = M(\omega, L^{p,q}_{tw}(\mathbb{R}^d))$. We recall that if $\omega \in \mathcal{B}(\mathbb{R}^d)$, then the former space is a "classical modulation space", and the latter space is related to certain types of classical Wiener amalgam spaces. For convenience we set $M^p(\omega) = M^{p,p}(\omega) = W^{p,p}(\omega)$. Furthermore, we set $M^p_{\omega} = M^{p,q}(\omega)$ and $M^p_{\sigma} = M^p_{\omega}(\sigma)$, where $\sigma$ is given by (1.39), and if $\omega = 1$, then we use the notations $M(\mathcal{B}), M^{p,q}, W^{p,q}$ and $M^p$, instead of $M(\omega, \mathcal{B}), M^{p,q}(\omega), W^{p,q}(\omega)$ and $M^p(\omega)$, respectively. Here we note that

$$\sigma_s(x) = (x)^s = (1 + |x|^2 + |\xi|^2)^{s/2} \quad \text{and} \quad \sigma_s(x, \xi) = (x, \xi)^s = (1 + |x|^2 + |\xi|^2)^{s/2}.$$

For exponential type weights we have the following proposition. We omit the proof, since the result can be found in [18, 21, 24, 28, 56]. Here and in what follows we write $p_1 \leq p_2$, when

$$p_1 = (p_{1,1}, \ldots, p_{1,n}) \in (0, \infty]^n \quad \text{and} \quad p_2 = (p_{2,1}, \ldots, p_{2,n}) \in (0, \infty]^n \quad (1.22)$$

satisfy $p_{1,j} \leq p_{2,j}$ for every $j = 1, \ldots, n$. 

13
Proposition 1.13. Let $p, q, p_j, q_j \in [1, \infty]$, $\omega, \omega_j, v \in \mathcal{P}_E(\mathbb{R}^{2d})$ for $j = 1, 2$ be such that $v$ is submultiplicative and even, $\omega$ is $v$-moderate, and let $\mathcal{B}$ be a mixed normed space on $\mathbb{R}^{2d}$. Then the following is true:

1. if $\phi \in M_1^v(\mathbb{R}^d) \setminus \{0\}$, then $f \in M(\omega, \mathcal{B})$ if and only if (1.21) holds, i.e. $M(\omega, \mathcal{B})$ is independent of the choice of $\phi$. Moreover, $M(\omega, \mathcal{B})$ is a Banach space under the norm in (1.21), and different choices of $\phi$ give rise to equivalent norms;

2. if (1.18), (1.20) and (1.22) hold with $p_1 \leq p_2$, and $\omega_2 \leq C\omega_1$ for some constant $C$, then

$$
\Sigma_1(\mathbb{R}^d) \subseteq M(\omega_1, L^{p_1}(V)) \subseteq M(\omega_2, L^{p_2}(V)) \subseteq \Sigma_1'(\mathbb{R}^d);
$$

3. the sesquilinear form $(\cdot, \cdot)_{L^2}$ on $\Sigma_1(\mathbb{R}^d)$ extends to a continuous map from $M^{p,q}_\omega(\mathbb{R}^d) \times M^{p,q}_\omega(\mathbb{R}^d)$ to $\mathbb{C}$. This extension is unique, except when $p = q' \in \{1, \infty\}$. On the other hand, if $\|a\| = \sup |(a, b)_{L^2}|$, where the supremum is taken over all $b \in M^{p,q}_\omega(\mathbb{R}^d)$ such that $\|b\|_{M^{p,q}_\omega(\mathbb{R}^d)} \leq 1$, then $\| \cdot \|$ and $\| \cdot \|_{M^{p,q}_\omega(\mathbb{R}^d)}$ are equivalent norms;

4. if $p, q < \infty$, then $\Sigma_1(\mathbb{R}^d)$ is dense in $M^{p,q}_\omega(\mathbb{R}^d)$, and the dual space of $M^{p,q}_\omega(\mathbb{R}^d)$ can be identified with $M^{p,q}_\omega(\mathbb{R}^d)$, through the form $(\cdot, \cdot)_{L^2}$. Moreover, $\Sigma_1(\mathbb{R}^d)$ is weakly dense in $M^{\infty}_\omega(\mathbb{R}^d)$.

1.5. The Bargmann transform. We shall now consider the Bargmann transform which is defined by the formula

$$
(\mathfrak{B}f)(z) = \pi^{-d/4} \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2} \langle z, z \rangle + |y|^2 \right) f(y) dy,
$$

when $f \in L^2(\mathbb{R}^d)$. We note that if $f \in L^2(\mathbb{R}^d)$, then the Bargmann transform $\mathfrak{B}f$ of $f$ is the entire function on $\mathbb{C}^d$, given by

$$
(\mathfrak{B}f)(z) = \int \mathfrak{A}_d(z, y) f(y) dy,
$$
or

$$
(\mathfrak{B}f)(z) = \langle f, \mathfrak{A}_d(z, \cdot) \rangle,
$$

where the Bargmann kernel $\mathfrak{A}_d$ is given by

$$
\mathfrak{A}_d(z, y) = \pi^{-d/4} \exp \left( -\frac{1}{2} \langle z, z \rangle + |y|^2 \right) + 2^{1/2} \langle z, y \rangle.
$$

Here

$$
\langle z, w \rangle = \sum_{j=1}^d z_j w_j, \quad \text{when } z = (z_1, \ldots, z_d) \in \mathbb{C}^d \quad \text{and} \quad w = (w_1, \ldots, w_d) \in \mathbb{C}^d,
$$

and otherwise $(\cdot, \cdot)$ denotes the duality between test function spaces and their corresponding duals. We note that the right-hand side in (1.23) makes sense when
$f \in \mathcal{S}_{1/2}(\mathbb{R}^d)$ and defines an element in $A(\mathbb{C}^d)$, since $y \mapsto \mathfrak{M}(z, y)$ can be interpreted as an element in $\mathcal{S}_{1/2}(\mathbb{R}^d)$ with values in $A(\mathbb{C}^d)$. Here and in what follows, $A(\mathbb{C}^d)$ denotes the set of all entire functions on $\mathbb{C}^d$.

It was proved by Bargmann that $f \mapsto \mathfrak{M}f$ is a bijective and isometric map from $L^2(\mathbb{R}^d)$ to the Hilbert space $A^2(\mathbb{C}^d)$, the set of entire functions $F$ on $\mathbb{C}^d$ which fulfills

$$
\|F\|_{A^2} \equiv \left( \int_{\mathbb{C}^d} |F(z)|^2 d\mu(z) \right)^{1/2} < \infty.
$$

(1.24)

Here $d\mu(z) = \pi^{-d} e^{-|z|^2} d\lambda(z)$, where $d\lambda(z)$ is the Lebesgue measure on $\mathbb{C}^d$, and the scalar product on $A^2(\mathbb{C}^d)$ is given by

$$(F, G)_{A^2} \equiv \int_{\mathbb{C}^d} F(z)\overline{G(z)} d\mu(z), \quad F, G \in A^2(\mathbb{C}^d).$$

(1.25)

Furthermore, Bargmann proved that there is a convenient reproducing kernel on $A^2(\mathbb{C}^d)$, given by the formula

$$F(z) = \int_{\mathbb{C}^d} e^{(z,w)} F(w) d\mu(w), \quad F \in A^2(\mathbb{C}^d),$$

(1.26)

where $(z, w)$ is the scalar product of $z \in \mathbb{C}^d$ and $w \in \mathbb{C}^d$ (cf. [38]). Note that this reproducing kernel is unique in view of (38).

From now on we assume that $\phi$ in (1.12), (1.12)′ and (1.21) is given by

$$\phi(x) = \pi^{-d/4} e^{-|x|^2/2},$$

(1.27)

if nothing else is stated. Then it follows that the Bargmann transform can be expressed in terms of the short-time Fourier transform $f \mapsto V_\phi f$. More precisely, let $S$ be the dilation operator given by

$$(SF)(x, \xi) = F(2^{-1/2}x, -2^{-1/2}\xi),$$

(1.28)

when $F \in L^1_{\text{loc}}(\mathbb{R}^{2d})$. Then it follows by straight-forward computations that

$$(\mathfrak{M}f)(z) = (\mathfrak{M}f)(x + i\xi) = (2\pi)^{d/2} e^{(|x|^2 + |\xi|^2)/2} e^{-i(x, \xi)} V_\phi f(2^{1/2}x, -2^{1/2}\xi) - (2\pi)^{d/2} e^{(|x|^2 + |\xi|^2)/2} e^{-i(x, \xi)} (S^{-1}(V_\phi f))(x, \xi),$$

(1.29)

or equivalently,

$$V_\phi f(x, \xi) = (2\pi)^{-d/2} e^{-(|x|^2 + |\xi|^2)/4} e^{-i(x, \xi)/2} (\mathfrak{M}f)(2^{-1/2}x, -2^{-1/2}\xi) - (2\pi)^{-d/2} e^{-i(x, \xi)/2} S(e^{-1/2}\mathfrak{M}f))(x, \xi).$$

(1.30)

For future references we observe that (1.29) and (1.30) can be formulated into

$$\mathfrak{M} = U_{\mathfrak{M}} \circ V_\phi, \quad \text{and} \quad U_{\mathfrak{M}}^{-1} \circ \mathfrak{M} = V_\phi,$$

where $U_{\mathfrak{M}}$ is the linear, continuous and bijective operator on $\mathcal{D}'(\mathbb{R}^{2d}) \simeq \mathcal{D}'(\mathbb{C}^d)$, given by

$$U_{\mathfrak{M}} F(x, \xi) = (2\pi)^{d/2} e^{(|x|^2 + |\xi|^2)/2} e^{-i(x, \xi)} F(2^{1/2}x, -2^{1/2}\xi).$$

(1.31)
Definition 1.14. Let $\omega_1 \in \mathcal{P}_Q^0(\mathbb{R}^{2d})$, $\omega_2 \in \mathcal{P}_Q(\mathbb{R}^{2d})$, $\mathcal{B}$ be a mixed quasi-norm space on $\mathbb{R}^{2d} = \mathbb{C}^d$, and let $r > 0$ be such that $r \leq \nu_1(\mathcal{B})$.

1. The space $B(\omega_2, \mathcal{B})$ is the modified weighted $\mathcal{B}$-space which consists of all $F \in L^r_{\text{loc}}(\mathbb{R}^{2d}) = L^r_{\text{loc}}(\mathbb{C}^d)$ such that

$\|F\|_{B(\omega_2, \mathcal{B})} \equiv \|(U_0^{-1}F)\omega_2\|_\mathcal{B} < \infty.$

Here $U_0$ is given by (1.31);

2. The space $A(\omega_2, \mathcal{B})$ consists of all $F \in A(\mathbb{C}^d) \cap B(\omega_2, \mathcal{B})$ with topology inherited from $B(\omega_2, \mathcal{B})$;

3. The space $A_0(\omega_1, \mathcal{B})$ is given by

$A_0(\omega_1, \mathcal{B}) \equiv \{ (\mathfrak{W} f) : f \in M(\omega_1, \mathcal{B}) \},$

and is equipped with the quasi-norm $\|F\|_{A_0(\omega_1, \mathcal{B})} \equiv \|f\|_{M(\omega_1, \mathcal{B})}$, when $F = \mathfrak{W} f$.

We note that the spaces in Definition 1.14 are normed spaces when $\nu_1(\mathcal{B}) \geq 1$.

For convenience we set $\|F\|_{B(\omega, \mathcal{B})} = \infty$, when $F \not\in B(\omega, \mathcal{B})$ is measurable, and $\|F\|_{A(\omega, \mathcal{B})} = \infty$, when $F \in A(\mathbb{C}^d) \setminus B(\omega, \mathcal{B})$. We also set

$B^{p,q}_{\omega} = B^{p,q}_{(\omega)}(\mathbb{C}^d) = B(\omega, \mathcal{B})$, $A^{p,q}_{\omega} = A^{p,q}_{(\omega)}(\mathbb{C}^d) = A(\omega, \mathcal{B})$

when $\mathcal{B} = L^{p,q}(\mathbb{C}^d)$, $A^p_{\omega} = A^p_{(\omega)}$, and if $\omega = 1$, then we use the notations $B^{p,q}$, $A^{p,q}$, $B^p$ and $A^p$ instead of $B^{p,q}_{\omega}$, $A^{p,q}_{\omega}$, $B^p_{\omega}$ and $A^p_{\omega}$, respectively.

For future references we note that the $B^{p}_{\omega}$ quasi-norm is given by

$\|F\|_{B^p_{\omega}} = 2^{d/p}(2\pi)^{-d/2} \left(\int_{\mathbb{C}^d} |e^{-|z|^2/2}F(z)\omega(2^{1/2}z)|^p \ d\lambda(z)\right)^{1/p}$

$= 2^{d/p}(2\pi)^{-d/2} \left(\int_{\mathbb{R}^{2d}} |e^{-|x|^2+|\xi|^2/2}F(x+i\xi)\omega(2^{1/2}x,-2^{1/2}\xi)|^p \ dx d\xi\right)^{1/p}$

(1.32)

(with obvious modifications when $p = \infty$). Especially it follows that the norm and scalar product in $B^2_{\omega}(\mathbb{C}^d)$ take the forms

$\|F\|_{B^2_{\omega}} = \left(\int_{\mathbb{C}^d} |F(z)\omega(2^{1/2}z)|^2 \ d\mu(z)\right)^{1/2}$

$= \left(\int_{\mathbb{C}^d} |F(z)\overline{G(z)}\omega(2^{1/2}z)|^2 \ d\mu(z)\right)^{1/2}$

$\quad F, G \in B^2_{\omega}(\mathbb{C}^d)$

(cf. (1.24) and (1.25)).

The following result shows that the norm in $A_0(\omega, \mathcal{B})$ is well-defined.

**Proposition 1.15.** Let $\omega \in \mathcal{P}_Q^0(\mathbb{R}^{2d})$, $\mathcal{B}$ be a mixed norm space on $\mathbb{R}^{2d}$ and let $\phi$ be as in (1.27). Then $A_0(\omega, \mathcal{B}) \subseteq A(\omega, \mathcal{B})$, and the map $\mathfrak{W}$ is an isometric injection from $M(\omega, \mathcal{B})$ to $A(\omega, \mathcal{B})$. 

Proof. The result is an immediate consequence of \(1.29\), \(1.30\) and Definition 1.14. \(\square\)

In the case \(\omega = 1\) and \(B = L^2\), it follows from [3] that Proposition 1.15 holds, and the inclusion is replaced by equality. That is, we have \(A_0^2 = A^2\) which is called the Bargmann-Fock space, or just the Fock space. In Section 4 we improve the latter property and show that for any choice of \(\omega \in \mathcal{P}_Q^0\) and every mixed quasi-norm space \(B\), we have \(A_0(\omega, B) = A(\omega, B)\).

2. Weight functions

In this section we establish results on weight functions which are needed. In the first part we investigate weights belonging to \(\mathcal{P}_E\). Here we are especially focused on finding properties which are needed to show that \(\mathcal{P}_E\) is contained in convenient subfamilies of \(\mathcal{P}_Q\), which are introduced in the second part of the section.

2.1. Moderate weights. For a moderate weight \(\omega\) there are convenient ways to find smooth weights \(\omega_0\) which are equivalent in the sense that for some constant \(C > 0\) we have

\[ C^{-1} \omega_0 \leq \omega \leq C \omega_0. \tag{2.1} \]

In fact, we have the following result, which extends Lemma 1.2 (4) in [55]. Here the weight \(\omega \in \mathcal{P}_E(\mathbb{R}^d)\) is called elliptic if \(\omega \in C^\infty(\mathbb{R}^d)\), and for each multi-index \(\alpha\), we have

\[ (\partial^\alpha \omega_0)/\omega_0 \in L^\infty(\mathbb{R}^d). \tag{2.2} \]

**Lemma 2.1.** Let \(\omega \in \mathcal{P}_E(\mathbb{R}^d)\). Then it exists an elliptic weight \(\omega_0 \in \mathcal{P}_E(\mathbb{R}^d)\) such that (2.1) holds.

Lemma 2.1 follows by similar arguments as in the proof of Lemma 1.2 (4) in [55]. In order to be self-contained we here present a proof.

**Proof.** Let \(\psi \in C^\infty(\mathbb{R}^d)\) be a positive function which is bounded by Gauss functions together with all its derivatives, and let \(\omega_0 = \psi * \omega\). Also let \(v \in \mathcal{P}_E(\mathbb{R}^d)\) be chosen such that \(\omega\) is \(v\)-moderate. Then \(\omega_0\) is smooth, and

\[ \omega_0(x) = \int \omega(x - y)\psi(y) \, dy \leq C_1 \omega(x), \]

with

\[ C_1 = \int v(-y)\psi(y) \, dy < \infty. \]

Furthermore, if

\[ C_2 = \int v(y)^{-1}\psi(y) \, dy < \infty, \]

then

\[ C_2 \omega(x) = \int \omega(x - y + y)\psi'(y) v(y) \, dy \leq C \int \omega(x - y)\psi(y) \, dy = C \omega_0(x), \]

for some constant \(C\). This proves that \(\omega_0 \in \mathcal{P}_E \cap C^\infty\), and that (2.1) is fulfilled.
By differentiating $\omega_0$, the first part of the proof gives that for some Gaussian $\psi_1$, and some constants $C_1$ and $C_2$ we have

$$|\partial^\omega \omega_0| = |(\partial^\omega \psi) * \omega| \leq |\partial^\omega \psi| * \omega \leq \psi_1 * \omega \leq C_1 \omega \leq C_2 \omega_0.$$ 

Hence, \(2.2\) is fulfilled for $\omega = \omega_0$. The proof is complete.

We also need some properties concerning the minimal weight which moderates a specific weight $\omega \in P_E(R^d)$. For any such weight, the fact that $\omega$ is $v$-moderate for some function $v$, implies that

$$v_0(x) \equiv \sup_{y_0 \in R^d} \frac{\omega(x + y_0)}{\omega(y_0)}$$

is well-defined. Furthermore, by straight-forward computations it follows that

$$\omega(x + y) \leq \omega(x)v(y) \quad \text{and} \quad v(x + y) \leq v(x)v(y), \quad x, y \in R^d,$$

holds for $v = v_0$. The following result shows that $v_0$ is minimal among elements which moderates $\omega$, and satisfies $2.4$. Furthermore, here we establish differential properties of $v_0$ in terms of the functionals

$$(J_1 \omega)(x, y) \equiv \inf_{y_0 \in R^d} \frac{(\partial_y \omega)(x + y_0)}{\omega(y_0)} = \inf_{y_0 \in R^d} \frac{(|\nabla \omega)(x + y_0), y)}{\omega(y_0)},$$

$$(J_2 \omega)(x, y) \equiv \sup_{y_0 \in R^d} \frac{(\partial_y \omega)(x + y_0)}{\omega(y_0)} = \sup_{y_0 \in R^d} \frac{(|\nabla \omega)(x + y_0), y)}{\omega(y_0)},$$

when in additional $\omega$ is elliptic. We note that $J_1 \omega$ and $J_2 \omega$ satisfy

$$|(J_k \omega)(x, y)| \leq \sup_{y_0 \in R^d} \frac{|(|\nabla \omega)(x + y_0)|}{\omega(y_0)} |y| \leq Cv_0(x)|y|, \quad k = 1, 2,$$

for such $\omega$ and some constant $C$, depending on $\omega$ only.

**Lemma 2.2.** Let $\omega \in P_E(R^d)$, $0 < v \in L^\infty_{loc}(R^d)$ be such that $\omega(x + y) \leq \omega(x)v(y)$ when $x, y \in R^d$, and let $v_0$ be as in $2.3$. Then $v_0 \leq v$, and $2.4$ holds.

Moreover, if in addition $\omega$ is elliptic, then the following is true:

1. $(J_1 \omega)(x, y)$ and $(J_2 \omega)(x, y)$ are continuous functions which are positively homogeneous in the $y$-variable of order one;

2. If $x, y \in R^d$, then

$$\inf_{0 \leq t \leq 1} (J_1 \omega)(x + ty, y) \leq v_0(x + y) - v_0(x) \leq \sup_{0 \leq t \leq 1} (J_2 \omega)(x + ty, y).$$

In particular, $v_0$ is locally Lipschitz continuous;

3. If $x, y \in R^d$, then

$$(J_1 \omega)(x, y) \leq \liminf_{h \to 0} \frac{v_0(x + hy) - v_0(x)}{h} \leq \limsup_{h \to 0} \frac{v_0(x + hy) - v_0(x)}{h} \leq (J_2 \omega)(x, y).$$
Proof. The first part and (1) are simple consequences of the definitions. The details are left for the reader. It is also obvious that (3) follows from (1) and (2) if we replace \( y \) in (2) by \( hy \), and let \( h \) turns to zero.

It remains to prove (2), and then we may assume that \( y \neq 0 \) and \( x \) are fixed. By (2.3) it follows that for every \( \varepsilon > 0 \), there is an element \( y_0 \in \mathbb{R}^d \) such that

\[
\omega(x + y) \leq \frac{\omega(x + y + y_0)}{\omega(y_0)} + \varepsilon.
\]

By Taylor’s formula we get for some \( \theta \in [0, 1] \),

\[
v_0(x + y) \leq \frac{\omega(x + y_0)}{\omega(y_0)} + \frac{\langle (\nabla \omega)(x + \theta y + y_0), y \rangle}{\omega(y_0)} + \varepsilon
\]

\[
\leq v_0(x) + \omega(x + y, y) + \varepsilon \leq v_0(x) + \sup_{0 \leq \varepsilon \leq 1} (J_2 \omega)(x + \varepsilon y, y) + \varepsilon.
\]

Since \( \varepsilon \) was arbitrary chosen, the last inequality in (2) follows.

In the same way, let \( \varepsilon > 0 \) be arbitrary and choose \( y_1 \in \mathbb{R}^d \) such that

\[
v_0(x) \leq \frac{\omega(x + y_1)}{\omega(y_1)} + \varepsilon.
\]

By Taylor’s formula we get for some \( \theta \in [0, 1] \),

\[
v_0(x + y) \geq \frac{\omega(x + y + y_1)}{\omega(y_1)} = \frac{\omega(x + y_1)}{\omega(y_1)} + \frac{\langle (\nabla \omega)(x + \theta y + y_1), y \rangle}{\omega(y_1)} + \varepsilon
\]

\[
\geq v_0(x) + (J_1 \omega)(x + \theta y, y) - \varepsilon \geq v_0(x) + \inf_{0 \leq \varepsilon \leq 1} (J_1 \omega)(x + \varepsilon y, y) - \varepsilon,
\]

and the first inequality in (2) follows. The proof is complete. □

2.2. Subfamilies of Gaussian type weights. Next we discuss further appropriate conditions for subfamilies to \( \mathcal{P}_Q^0(\mathbb{R}^d) \) and show that these subfamilies contain both \( \mathcal{P}_Q(\mathbb{R}^d) \) as well as all weights of the form \( \omega(x) = Ce^{c|x|^\gamma} \), when \( C > 0, c \in \mathbb{R} \) and \( 0 \leq \gamma < 2 \).

In the following definition we list most of the needed properties. The definitions involve global conditions of the form

\[
\frac{\omega(2^{1/2} x)e^{-\varepsilon(1-\lambda^2)|x|^2/2}}{\omega(2^{1/2} \lambda x)} \leq C_{\varepsilon}, \quad 1 - \theta \varepsilon < \lambda < 1, \quad x \in \mathbb{R}^d,
\]

and

\[
\lim_{\lambda \to 1^-} \left( \sup_{x_2 \in V^\perp} \frac{\omega_0(2^{1/2} x)e^{-\varepsilon(1-\lambda^2)|x|^2/2}}{\omega_0(2^{1/2} \lambda x)} \right) \leq 1,
\]

\[
x_1 \in V, \ x_2 \in V^\perp, \ x = x_1 + x_2 \in \mathbb{R}^d,
\]

for some vector space \( V \). Here the dilation factor \( 2^{1/2} \), is needed because of the relation between the Bargmann transform and the short-time Fourier transform in (1.29), and the quasi-norms in Definition 1.14.
**Definition 2.3.** Let $V \subseteq \mathbb{R}^d$ be a vector space.

1. The weight $\omega \in \mathcal{P}_Q^0(\mathbb{R}^d)$ is called *dilated suitable* with respect to $\varepsilon \in (0, 1]$, if there are constants $C_\varepsilon > 0$ and $\theta_\varepsilon \in (0, 1)$ such that (2.5) holds. If both $\omega$ and $1/\omega$ are dilated suitable with respect to every $\varepsilon \in (0, 1]$, then $\omega$ is called *strongly dilated suitable*;

2. The weight $\omega \in \mathcal{P}_Q^0(\mathbb{R}^d)$ is called *narrowly dilated suitable* with respect to $\varepsilon \in (0, 1]$ and $V$, if it is dilated suitable with respect to $\varepsilon$, and (2.6) holds for every $x_1 \in V$ and some equivalent continuous weight $\omega_0$ to $\omega$. If $\omega$ and $1/\omega$ are narrowly dilated suitable with respect to $V$ and every $\varepsilon \in (0, 1]$, then $\omega$ is called *strongly narrowly dilated suitable*.

The set of strongly dilated and strongly narrowly dilated suitable weights with respect to $V$ are denoted by $\mathcal{P}_D^0(\mathbb{R}^d)$ and $\mathcal{P}_{D,V}^0(\mathbb{R}^d)$ respectively. Furthermore we set $\mathcal{P}_D^0(\mathbb{R}^d) = \mathcal{P}_{D,V}^0(\mathbb{R}^d)$, when $V = \{0\}$. We note that $\mathcal{P}_{D,V}^0(\mathbb{R}^d)$ is increasing with respect to $V$, and that $\mathcal{P}_{D,V}^0(\mathbb{R}^d) \subseteq \mathcal{P}_D^0(\mathbb{R}^d)$.

If $\omega$ is $v$-moderate for some $v$, then $\omega$ is moderated by some function which grow exponentially. It follows easily that $\omega$ satisfies (2.5) in this case. Hence, any weight in $\mathcal{P}_E$ is dilated suitable.

In the following proposition we stress the latter property and prove that we in fact have that $\mathcal{P}_E \subseteq \mathcal{P}_D^0$.

**Proposition 2.4.** Let $V \subseteq \mathbb{R}^d$ be a vector space. Then the following inclusions are true:

\[ \mathcal{P}(\mathbb{R}^d) \subseteq \mathcal{P}_E(\mathbb{R}^d) \subseteq \mathcal{P}_D^0(\mathbb{R}^d) \cap \mathcal{P}_G^0(\mathbb{R}^d), \quad \mathcal{P}_D^0(\mathbb{R}^d) \subseteq \mathcal{P}_{D,V}^0(\mathbb{R}^d) \subseteq \mathcal{P}_D(\mathbb{R}^d), \]

\[ \mathcal{P}_G^0(\mathbb{R}^d) \subseteq \mathcal{P}_Q^0(\mathbb{R}^d) \cap \mathcal{P}_G^0(\mathbb{R}^d) \subseteq \mathcal{P}_Q^0(\mathbb{R}^d) \cup \mathcal{P}_G^0(\mathbb{R}^d) \subseteq \mathcal{P}_Q(\mathbb{R}^d) \]

Furthermore, if $C > 0$, $0 \leq \gamma < 2$, $t \in \mathbb{R}$, and $\omega(x) = C e^{t|x|^\gamma}$, $x \in \mathbb{R}^d$, then $\omega \in \mathcal{P}_D^0(\mathbb{R}^d) \cap \mathcal{P}_G^0(\mathbb{R}^d)$.

**Proof.** All the inclusions, except $\mathcal{P}_E \subseteq \mathcal{P}_D^0$ are immediate consequences of the definitions.

Therefore, let $\omega \in \mathcal{P}_E(\mathbb{R}^d)$, and let $\nu_0$ be the same as in (2.3). By Lemma 2.1 we may assume that $\omega$ is elliptic.

We shall prove that (2.5) and (2.6) holds for $\omega$ and $1/\omega$, for every choice of $\varepsilon \in (0, 1]$, and since $\mathcal{P}_E$ is a group under multiplications, it suffices to prove these relations for $\omega$. Since the left-hand side of (2.5) is equal to 1 as $x = 0$, the result follows if we prove

\[ \frac{\omega(x)e^{-\varepsilon(1-\lambda^2)|x|^2/2}}{\omega(\lambda x)} \leq 1 + C_0 \sqrt{1 - \lambda}, \quad 0 < \lambda < 1, \quad (2.7) \]

for some constant $C_0$ which depends on $\varepsilon \in (0, 1)$.

Let $\varepsilon \in (0, 1)$ be fixed. We have $\nu_0(x) \leq C e^{C|x|}$ for some $C > 1$, and we choose $R > 0$ such that

\[ C e^{R C - \varepsilon R^2/2} < 1. \]
Since $0 < \lambda < 1$ we have $\lambda^2 < \lambda$. Hence Lemma 2.2 gives

$$\frac{\omega(x)e^{-\varepsilon(1-\lambda^2)}|x|^2/2}{\omega(\lambda x)} \leq v_0((1-\lambda)x)e^{-\varepsilon(1-\lambda)|x|^2/2}. \quad (2.8)$$

We shall estimate the right-hand side, and start by considering the case when $|x| \leq R/\sqrt{1-\lambda}$. Since $v_0(0) = 1$ and $(1-\lambda)x$ stays bounded, the right-hand side of (2.8) can be estimated by

$$v_0((1-\lambda)x)e^{-\varepsilon(1-\lambda)|x|^2/2} \leq |v_0((1-\lambda)x) - v_0(0)|e^{-\varepsilon(1-\lambda)|x|^2/2} + e^{-\varepsilon(1-\lambda)|x|^2/2}.$$

$$\leq C_1(1-\lambda)|x|e^{-\varepsilon(1-\lambda)|x|^2/2} + 1,$$

where the last inequality follows from the fact that $v_0$ is locally Lipschitz continuous, in view of Lemma 2.2. Since

$$(1-\lambda)|x|e^{-\varepsilon(1-\lambda)|x|^2/2} \leq R\sqrt{1-\lambda}$$

as $|x| \leq R/\sqrt{1-\lambda}$, we obtain

$$v_0((1-\lambda)x)e^{-\varepsilon(1-\lambda)|x|^2/2} \leq 1 + C_1 R\sqrt{1-\lambda}, \quad |x| \leq R/\sqrt{1-\lambda},$$

and (2.7) follows in this case.

Next we consider the case $R/\sqrt{1-\lambda} \leq |x| \leq R/(\varepsilon(1-\lambda))$. Then we have

$$v_0((1-\lambda)x) \leq Ce^{C(1-\lambda)|x|} \leq Ce^{RC/\varepsilon}$$

and

$$e^{-(1-\lambda)|x|^2/2} \leq e^{-R^2/2},$$

which gives

$$v_0((1-\lambda)x)e^{-\varepsilon(1-\lambda)|x|^2/2} \leq Ce^{RC/\varepsilon-R^2/2} = C^{1-1/\varepsilon}\left(Ce^{RC-\varepsilon R^2/2}\right)^{1/\varepsilon} < 1, \quad (2.9)$$

where the last inequality follows from our choice of $R$, together with the fact that $C > 1$. This proves (2.7) in this case.

Finally, we consider the case $R/(\varepsilon(1-\lambda)) \leq |x|$. From the assumptions on $R$, it follows that $R > 2C/\varepsilon$, which implies that

$$C - |x|/2 < C - R/(2\varepsilon) < C - \varepsilon R/2 < 0. \quad (2.10)$$

Then

$$v_0((1-\lambda)x)e^{-(1-\lambda)|x|^2/2} \leq Ce^{C(1-\lambda)|x|}e^{-(1-\lambda)|x|^2/2}$$

$$\leq Ce^{(1-\lambda)|x|(C-|x|/2)} \leq Ce^{R(C-\varepsilon R^2/2)/\varepsilon} < 1,$$

where the last inequalities follows from (2.9) and (2.10). This gives (2.7) also for $R/(\varepsilon(1-\lambda)) \leq |x|$, and the proof is complete.

In most of our investigations, the pairs of weights and mixed norm spaces fulfill the conditions in the following definition.
Definition 2.5. Let $\mathcal{B} = L^p(V)$ be a mixed norm space on $\mathbb{R}^d$ such that (1.18) and (1.20) hold for some $p \in [1, \infty]^n$, and let $\omega \in \mathcal{P}_Q(\mathbb{R}^d)$. Then the pair $(\mathcal{B}, \omega)$ is called feasible (strongly feasible) on $\mathbb{R}^d$ if one of the following conditions hold:

1. $\nu_1(\mathcal{B}) > 1$ and $\omega$ is dilated suitable with respect to $\varepsilon = 1$ (\omega is strongly dilated suitable);

2. $\nu_2(\mathcal{B}) < \infty$, and $\omega$ is dilated suitable with respect to $\varepsilon = 1$ (\omega is strongly dilated suitable);

3. $p_1 = \infty, 1 < p_2, \ldots, p_{n-1} < \infty, p_n = 1$, and $\omega$ is narrowly dilated suitable with respect to $\varepsilon = 1$ and $V = \{0\}$ (\omega is strongly narrowly dilated suitable with respect to $V = \{0\}$).

In some situations it is convenient to separate the case (3) in Definition 2.5 from the other ones. Therefore we say that the pair $(\mathcal{B}, \omega)$ is narrowly feasible (strongly narrowly feasible) if it is feasible and satisfies (3) in Definition 2.5.

We note that if $\mathcal{B}$ fulfills (1) or (2) in Definition 2.5 and $\omega \in \mathcal{P}_E(\mathbb{R}^d)$, then the pair $(\mathcal{B}, \omega)$ is feasible. If instead $\mathcal{B}$ fulfills (3), then Proposition 2.4 shows that the pair $(\mathcal{B}, \omega)$ is narrowly feasible.

The following result related to Lemma 2.1 shows that for any weight in $\mathcal{P}_G(\mathbb{R}^d)$, it is always possible to find a smooth equivalent weight.

Proposition 2.6. Let $\omega$ be a weight on $\mathbb{R}^d$ such that (1.3) holds. Then $\omega \in \mathcal{P}_G(\mathbb{R}^d)$. Furthermore, there is a weight $\omega_0 \in \mathcal{P}_G(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ such that the following is true:

1. for every multi-index $\alpha$, there is a constant $C_\alpha$ such that $|\partial^\alpha \omega_0(x)| \leq C_\alpha |\omega_0(x)|^{|\alpha|};$

2. (2.1) holds for some constant $C$;

3. if in addition $\omega$ is rotation invariant, then $\omega_0$ is rotation invariant.

In the proof and later on we let $B_r(a)$ denote an open ball in $\mathbb{R}^d$ or $\mathbb{C}^d$ with radius $r \geq 0$ and center at $a$ in $\mathbb{R}^d$ or $\mathbb{C}^d$, respectively.

Proof. Let $B = B_r(0)$, where $c$ be the same as in (1.3), and let $0 \leq \psi \in C_0^\infty(B)$ be rotation invariant such that $\int \psi \, dx = 1$. Also let $\psi_0 \in C_0^\infty(3B)$ be rotation invariant such that $0 \leq \psi_0 \leq 1$ and $\psi_0(x) = 1$ when $x \in 2B$. Then it follows by straight-forward computations that (1)–(3) are fulfilled when

$$\omega_0(x) \equiv \psi_0(x) + (1 - \psi_0(x))|x|^d \int \psi(|x|(x - y))\omega(y) \, dy.$$ 

It remains to prove that $\omega_0, \omega \in \mathcal{P}_G(\mathbb{R}^d)$, and then it suffices to prove that $\omega_0$ satisfies (1.4). Let $1 \leq t \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$ be fixed such that $|x_0| = 1$, and let

$$C_1 = \inf_{|x|=1} \omega_0(x) \quad \text{and} \quad C_2 = \sup_{|x|=1} \omega_0(x).$$

Then (1) implies that

$$\psi(t) = \psi_{x_0}(t) \equiv \omega_0(tx_0), \quad t \geq 1,$$
satisfies
\[ \psi'(t) - C t \psi(t) \leq 0 \quad \text{and} \quad \psi'(t) + C t \psi(t) \geq 0, \quad 0 < C_1 \leq \psi(1) \leq C_2, \]
for some constant \( C > 0 \) which is independent of \( x_0 \). This implies that
\[ C_1 e^{-C(t^2-1)/2} \leq \psi(t) \leq C_2 e^{C(t^2-1)/2}, \quad t \geq 1. \]
Since \( C_1 \) and \( C_2 \) are independent of the choice of \( x_0 \) on the unit sphere, (1.3) follows from the latter inequalities. Hence \( \omega_0 \in \mathcal{P}_G \), and the proof is complete. \( \square \)

We finish this section by proving the following result on existence rotation invariant weights in our families of weights.

**Proposition 2.7.** Let \( \mathcal{P} \) be equal to
\[ \mathcal{P}(\mathbb{R}^d), \quad \mathcal{P}_E(\mathbb{R}^d), \quad \mathcal{P}_D(\mathbb{R}^d), \quad \mathcal{P}_G(\mathbb{R}^d), \quad \mathcal{P}_E(\mathbb{R}^d), \quad \mathcal{P}_D(\mathbb{R}^d), \quad \mathcal{P}_G(\mathbb{R}^d) \text{ or } \mathcal{P}_Q(\mathbb{R}^d). \]
If \( \omega \in \mathcal{P} \), then there are rotation invariant weights \( \omega_1, \omega_2 \in \mathcal{P} \) such that
\[ C^{-1} \omega_1 \leq \omega \leq C \omega_2. \]

For the case \( \mathcal{P} = \mathcal{P}_Q \) we need the following lemma.

**Lemma 2.8.** Let \( f \in L_{\infty}^\text{loc}(\mathbb{R}^d) \) be such that for each \( \varepsilon > 0 \), there is a constant \( C_\varepsilon > 0 \) such that
\[ |f(x)| \leq C_\varepsilon e^{\varepsilon |x|^2}. \]
Then there is a rotation invariant \( \omega \in \mathcal{P}_Q^0(\mathbb{R}^d) \) such that \( |f| \leq \omega \).

**Proof.** Let
\[ g \equiv |f| + e \quad \text{and} \quad h_0(x) \equiv \frac{\log g(x)}{|x|^2}. \quad (2.11) \]
Then
\[ \lim_{|x| \to \infty} h(x) = 0, \quad (2.12) \]
for \( h = h_0 \) due to the assumptions.

Now set
\[ h_1(x) = \int \varphi(x-y) \left[ \sup_{|y| \geq |x|^{-1}} h_0(y) \right] dy, \quad |x| \geq 2, \]
where \( 0 \leq \varphi \in C_0^\infty(\mathbb{R}^d) \) is rotation invariant, supported in the unit ball, and satisfies \( \|\varphi\|_{L^1} = 1 \). Then it follows that \( h_1 \) is rotation invariant, smooth and larger than \( h_0 \) in \( \Omega_2 \), where \( \Omega_r \equiv \mathbb{R}^d \setminus B_r(0) \). Furthermore, (2.12) holds for \( h = h_1 \), and since \( h_0 \) is bounded in \( \Omega_1 \), it follows that \( h_1^{(\alpha)} \) is bounded in \( \Omega_2 \) for every multi-index \( \alpha \).

Hence, if \( |y| \leq 1/|x| \) with \( x \in \Omega_3 \), Taylor’s formula gives
\[ |h_1(x+y) + h_1(x-y) - 2h_1(x)| \leq C |y|^2 \leq C/|x|^2. \]
By again using the fact that $h_1$ is bounded in $\Omega_2$, it follows that

$$\omega(x) \equiv \begin{cases} 
\sup_{|y| \leq 3} |g(y)| & \text{when } |x| \leq 3, \\
\epsilon h_1(x|x|^2) & \text{when } |x| \geq 3,
\end{cases}$$

is rotation invariant, larger than $g$ and fulfills \((1.3)'\). This together with \((2.12)\) shows that $\omega \in \mathcal{P}_Q^0(\mathbb{R}^d)$, and the result follows.

**Proof of Proposition 2.7.** If $\mathcal{P} = \mathcal{P}_Q^0$, then the result is an immediate consequence of Lemma 2.8 and the fact that $\mathcal{P}_Q^0$ is a group under multiplications. By straightforward computations it also follows that $\omega$ in Lemma 2.8 is dilated suitable or strongly dilated suitable when this is true for $f$. This proves the statement for $\mathcal{P} = \mathcal{P}_D(\mathbb{R}^d)$ and for $\mathcal{P} = \mathcal{P}_E^0(\mathbb{R}^d)$.

For $\mathcal{P} = \mathcal{P}$ we may choose $\omega_j = C_j \langle \cdot \rangle_{N_j}^N$ for appropriate constants $C_j$ and $N_j$, and if $\mathcal{P} = \mathcal{P}_E$ we may choose $\omega_j = C_j e^{\epsilon_j |x|}$ for appropriate constants $0 < s_j \leq 1$, $c_j$ and $C_j$. If instead $\mathcal{P} = \mathcal{P}_G$ or $\mathcal{P} = \mathcal{P}_Q$, then similar is true after the condition on $s_j$ is replaced by $s_j = 2$.

It remains to consider the case $\mathcal{P} = \mathcal{P}_G^0$. Therefore, assume that $\omega \in \mathcal{P}_G^0$, and set

$$\omega_2(x) = \sup_{|z| = |x|} \omega(z) \quad \text{and} \quad \omega_1(x) = \inf_{|z| = |x|} \omega(z).$$

Now choose $c$ and $C$ such that \((1.3)\) holds, and let $x, y$ be such that $|x| \geq 2c$ and $|y| \leq c/|x|$. Then for each $\epsilon > 0$, there exists $z \in \mathbb{R}^d$ such that $|z| = |x|$ and

$$\omega_2(x) \leq \omega(z) + \epsilon.$$

By \((1.3)\) we get

$$\omega_2(x) \leq \omega(z) + \epsilon \leq C \inf_{|y_0| \leq c/|x|} \omega(z + y_0) + \epsilon \leq C \inf_{|y_0| \leq c/|x|} \omega_2(x + y_0) + \epsilon \leq C \omega_2(x + y) + \epsilon.$$

This proves that $\omega_2(x) \leq C \omega_2(x + y)$. In the same way it follows that $\omega_2(x + y) \leq C \omega_2(x)$. Since $\omega_2$ fulfills similar types of estimates as $\omega$, it follows that $\omega_2$ is subgaussian. In the same way it follows that $\omega_1$ is subgaussian, and the result follows for $\mathcal{P} = \mathcal{P}_G^0$. \(\square\)

### 2.3. Examples.

Next we give some examples on weights in $\mathcal{P}_D^0(\mathbb{R}^d)$. First we note that any weight of the form $\sigma_\epsilon$ and $\omega_1$ in Example 1.3 belongs to $\mathcal{P}(\mathbb{R}^d)$ in view of Proposition 2.4. In order to give other examples it is convenient to consider corresponding logarithmic conditions on those weights. We note that if $\omega$ is a weight on $\mathbb{R}^d$ and $\varphi(x) = \log \omega(x)$, then \((1.4)\) is equivalent to

$$|\varphi(x)| \leq C + \epsilon |x|^2, \quad (2.13)$$

for some positive constants $C$ and $\epsilon$. The conditions \((1.3)\) and \((1.3)'\) are the same as

$$|\varphi(x + y) - \varphi(x)| \leq C, \quad |x| \geq 2c, \quad |y| \leq c/|x|, \quad (2.14)$$
\[ |\varphi(x + y) + \varphi(x - y) - 2\varphi(x)| \leq C, \quad |x| \geq 2c, \quad |y| \leq c/|x|, \quad (2.14) \]

respectively, for some positive constants \( c \) and \( C \). Finally, \( \omega \) and \( 1/\omega \) are dilated suitable with respect to \( \varepsilon \in (0, 1] \), if and only if there are constants \( C_\varepsilon > 0 \) and \( \theta_\varepsilon \in (0, 1) \) such that

\[ |\varphi(\lambda x) - \varphi(x)| \leq C_\varepsilon + \varepsilon(1 - \lambda^2)|x|^2, \quad 1 - \theta_\varepsilon < \lambda < 1, \quad (2.15) \]

and \( \omega \) and \( 1/\omega \) are narrowly dilated suitable with respect to \( \varepsilon \in (0, 1] \) and \( V = \{0\} \)

when

\[ \limsup_{\lambda \to 1} \left( |\varphi(\lambda x) - \varphi(x)| - \varepsilon(1 - \lambda^2)|x|^2 \right) = 0. \quad (2.16) \]

In particular, the following lemma is an immediate consequence of the definitions.

**Lemma 2.9.** Let \( \omega \) be a weight on \( \mathbb{R}^d \), and let \( \varphi(x) = \log \omega(x) \). Then the following is true:

1. \( \omega \in \mathcal{P}_0^0(\mathbb{R}^d) \), if and only if \( (2.14) \) holds for some positive constants \( c \) and \( C \), and for every \( \varepsilon > 0 \), there is a positive constant \( C \) such that \( (2.13) \) holds:

2. \( \omega \in \mathcal{P}_0^0(\mathbb{R}^d) \), if and only if \( (2.14) \) holds for some positive constants \( c \) and \( C \), and for every \( \varepsilon > 0 \), there is a positive constant \( C \) such that \( (2.13) \) holds:

3. \( \omega \in \mathcal{P}_0^0(\mathbb{R}^d) \), if and only if \( (2.14) \) holds for some positive constants \( c \) and \( C \), and for every \( \varepsilon > 0 \), there are positive constants \( C, C_\varepsilon \) and \( \theta_\varepsilon \in (0, 1) \) such that \( (2.13), (2.15) \) and \( (2.16) \) hold.

**Example 2.10.** Let \( \omega(x) = \langle x \rangle^{t(x)} \), \( x \in \mathbb{R}^d \), for some choice of \( t \in \mathbb{R} \). Then it follows by straight-forward computations that \( \varphi(x) = \log \varphi(x) = \langle x \rangle \log \langle x \rangle \) satisfies \( (2.13), (2.16) \). Hence \( \omega \in \mathcal{P}_0^0(\mathbb{R}^d) \cap \mathcal{P}_G^0(\mathbb{R}^d) \).

**Example 2.11.** Let \( \omega(x) = \Gamma(\langle x \rangle + 1 + r) \) and \( \varphi(x) = \log \omega(x) \), where \( \Gamma \) is the gamma function, and \( r > -2 \) is real. Then we have

\[ \omega(x) = (2\pi(\langle x \rangle + r))^{1/2} \left( \frac{\langle x \rangle + r}{e} \right)^{(\langle x \rangle + r)} \left( 1 + o(\langle x \rangle^{-1}) \right), \]

by Stirling’s formula. This is gives

\[ \varphi(x) = \frac{1}{2} \log(2\pi) + \frac{1 + 2r}{2} \log \langle x \rangle + \langle x \rangle(\log \langle x \rangle - 1) + \psi(x), \quad (2.17) \]

where \( \psi(x) \) is continuous and satisfies

\[ \lim_{|x| \to \infty} \langle x \rangle \psi(x) = 0. \quad (2.18) \]

By straight-forward computations it follows that the first three terms in \( (2.17) \) satisfy the conditions \( (2.13) - (2.16) \). Furthermore, the condition \( (2.18) \) together with the proof of Proposition \( 2.4 \) show that also \( \psi(x) \) satisfies \( (2.13) - (2.16) \). Consequently, \( \omega \in \mathcal{P}_0^0(\mathbb{R}^d) \cap \mathcal{P}_G^0(\mathbb{R}^d). \)

The following result shows that there are weights in \( (\mathcal{P}_G^0 \cap \mathcal{P}_0^0) \setminus \mathcal{P} \) which fulfills \((0.1)\) for some polynomial \( v \).
Proposition 2.12. Let \( \omega(x, \xi) = (1 + |x|^{r} |\xi|^{s})^{s} \) for some \( r > 0 \) and \( s \in \mathbb{R} \setminus 0 \). Then \( \omega \) belongs to \( (P_{D}^{0}(\mathbb{R}^{2d}) \cap P_{G}^{0}(\mathbb{R}^{2d})) \setminus P(\mathbb{R}^{2d}) \) and fulfills (0.1) for some polynomial \( v \).

Proof. Since \( P_{D}^{0}, P_{G}^{0} \) and \( P \) are groups under multiplications, and invariant under mappings \( \omega \mapsto \omega^{t} \) for \( t \neq 0 \), and that \( \omega \) is equivalent to

\[
\omega_{0}(x, \xi) = (1 + |x|^{1/2} |\xi|^{1/2})^{2rs},
\]

we may assume that \( r = 1/2 \) and \( s = 1 \).

It is obvious that \( \omega \) fulfills (0.1) for some polynomial \( v \), and by straight-forward computations it also follows that (1.3) is fulfilled. Furthermore, by choosing \( x, y, \xi, \eta \in \mathbb{R}^{d} \) in such way that \( |\eta| = 1/|y| \) and \( |x| = |\xi| = 1 \), it follows that

\[
\sup_{y, \eta} \frac{\omega(x + y, \xi + \eta)}{\omega(y, \eta)} \geq \sup_{|\eta| = 1/|y|} \frac{1 + |x + y|^{1/2} |\xi + \eta|^{1/2}}{2} = \infty.
\]

This implies that \( \omega \notin P \).

It remains to prove that \( \omega \in P_{D}^{0} \), which follows if we prove that for every \( \varepsilon > 0 \), then (2.6) holds for \( V = \{0\} \), after \( \mathbb{R}^{d} \) and \( x \) have been replaced by \( \mathbb{R}^{2d} \) and \( (x, \xi) \), respectively.

Here it is obvious that (2.6) is true with \( \omega = \omega_{0} \), since

\[
\frac{\omega(\lambda x, \lambda \xi)e^{-\varepsilon(1-\lambda^{2})(|x|^{2}+|\xi|^{2})/2}}{\omega(x, \xi)} \leq 1
\]

with equality when \( x = \xi = 0 \). Let

\[
h(t_{1}, t_{2}) = \frac{(1 + \sqrt{t_{1}t_{2}})e^{-\varepsilon(1-\lambda^{2})(t_{1}^{2}+t_{2}^{2})/2}}{1 + \lambda \sqrt{t_{1}t_{2}}}, \quad t_{1}, t_{2} \geq 0.
\]

Since

\[
h(|x|, |\xi|) = \frac{\omega(x, \xi)e^{-\varepsilon(1-\lambda^{2})(|x|^{2}+|\xi|^{2})/2}}{\omega(\lambda x, \lambda \xi)} \geq \frac{\omega(x, \xi)e^{-\varepsilon(1-\lambda^{2})(|x|^{2}+|\xi|^{2})/2}}{\omega(\lambda x, \lambda \xi)},
\]

and \( h(0, 0) = 1 \), the result follows if we prove

\[
\lim_{\lambda \to 1^{-}} \left( \sup_{t_{1}, t_{2} \geq 0} h(t_{1}, t_{2}) \right) = 1.
\]

Now, by straight-forward computations it follows that

\[
\sup_{t_{1}, t_{2} \geq 0} h(t_{1}, t_{2}) = h(t_{0}, t_{0}), \quad \text{where} \quad t_{0} = -\frac{1 + \lambda}{2\lambda} + \sqrt{\left(\frac{1 + \lambda}{2\lambda}\right)^{2} + \frac{1 - \varepsilon}{\varepsilon\lambda}},
\]

and it is straight-forward to control that \( h(t_{0}, t_{0}) \to 1 \) as \( \lambda \to 1^{-} \). The proof is complete. \( \square \)
3. Harmonic estimates and mapping properties for the Bargmann transform on modulation spaces

In the first part of the section we establish certain invariance properties of spaces of harmonic or analytic functions. Thereafter we apply these properties to prove that $A_0(\omega, B) = A(\omega, B)$, for appropriate weights $\omega$. In the end of the section we use these results to prove general properties of $A(\omega, B)$ and $M(\omega, B)$, for example that they are Banach spaces when $B$ is a mixed norm space.

3.1. Analytic and harmonic estimates. Let $\Omega$ be an admissible family of weights on $\mathbb{R}^d$, and let $B$ be a mixed quasi-norm space on $\mathbb{R}^d$. (Cf. Definition 1.4.) Then we prove that subsets of $E_1(\Omega, B) \equiv \{ f \in L^r_{\text{loc}}(\mathbb{R}^d); \ f \omega \in B \text{ for every } \omega \in \Omega \}$ and $E_2(\Omega, B) \equiv \{ f \in L^r_{\text{loc}}(\mathbb{R}^d); \ f \omega \in B \text{ for some } \omega \in \Omega \}, \ r = \nu_1(B)$, of analytic or harmonic functions are independent of the choice of $B$. Recall here that $\nu_1(B)$ is the smallest involved Lebesgue exponent in $B$, and belongs to $(0, \infty]$. Also recall that $B$ is a mixed norm space, if and only if $\nu_1(B) \geq 1$. Some restrictions are needed when considering subsets of harmonic functions.

It is easy to prove one direction. In fact, by the definitions and Hölder’s inequality we get

$$E_j(\Omega, L^\infty(\mathbb{R}^d)) \subseteq E_j(\Omega, B) \subseteq E_j(\Omega, L^r(\mathbb{R}^d)), \quad j = 1, 2$$

where $j = 1, 2$ and $r = \min(1, \nu_1(B))$ (3.1)

(with continuous inclusions).

In order to establish opposite inclusions to (3.1), for corresponding subsets of analytic or harmonic functions, we will use techniques based on harmonic estimates for such functions.

We start with the following result. Here and in what follows we let $\mathcal{H}(\mathbb{R}^d)$ be the set of harmonic functions on $\mathbb{R}^d$.

**Proposition 3.1.** Let $\Omega \subseteq \mathcal{P}_G(\mathbb{R}^d)$ be an admissible family of weights on $\mathbb{R}^d$, and let $B_1$ and $B_2$ be mixed norm space on $\mathbb{R}^d$. Then

$$E_j(\Omega, B_1) \cap \mathcal{H}(\mathbb{R}^d) = E_j(\Omega, B_2) \cap \mathcal{H}(\mathbb{R}^d), \quad j = 1, 2.$$  (3.2)

**Proof.** It suffices to prove $E_j(\Omega, L^1) \cap \mathcal{H} \subseteq E_j(\Omega, L^\infty) \cap \mathcal{H}$, in view of (3.1).

Assume that $f \in E_2(\Omega, L^1) \cap \mathcal{H}$. By (1.6) and the assumptions we have

$$\|f\|_{L^1(\omega)} < \infty \quad \text{and} \quad \omega_1 \leq C(\cdot)^{-d}\omega,$$

for some $\omega, \omega_1 \in \Omega$ and some constant $C > 0$. Let $c$ and $C$ be the same as in (1.3). Then the result follows if we prove

$$\|f \chi\|_{L^\infty(\omega_1)} < \infty,$$  (3.4)
when \( \chi \) is the characteristic function of \( \{ x \in \mathbb{R}^d ; |x| \geq 2c \} \).

Since \( f \in \mathcal{H}(\mathbb{R}^d) \), the mean-value property for harmonic functions gives

\[
f(x) = c_d^{-1}(|x|/c)^d \int_{|y| \leq c/|x|} f(x + y) \, dy,
\]

where \( c_d \) is the volume of the \( d \)-dimensional unit ball. If \( |x| \geq 2c \), then \( (3.5) \) holds.

Furthermore, for every fixed \( \omega \in \Omega \), there are \( \omega_1, \omega_2 \in \Omega \) and constant \( C > 0 \) such that

\[C^{-1}\|F\|_{A(\omega_1, \mathcal{B}_2)} \leq \|F\|_{A(\omega, \mathcal{B}_1)} \leq C\|F\|_{A(\omega_2, \mathcal{B}_2)}, \quad F \in A(\mathbb{C}^d).\]  

For the proof we need the following lemma, concerning mean-value properties for analytic functions.

**Lemma 3.3.** Let \( \nu \) be a positive Borel measure on \( \mathbb{C}^d \) which is rotation invariant under each coordinate \( z_1, \ldots, z_d \in \mathbb{C} \), \( T_1, \ldots, T_n \) be (complex) \( d \times d \)-matrices, and let \( F_1, \ldots, F_n \in A(\mathbb{C}^d) \). Also let \( r > 0 \), and let \( \Omega \subseteq \mathbb{C}^d \) be compact and convex, which is rotation invariant under each coordinate \( z_1, \ldots, z_d \in \mathbb{C} \). Then

\[
\prod_{j=1}^n F_j(z) = \frac{1}{\nu(\Omega)} \int_{\Omega} \prod_{j=1}^n F_j(z + T_jw) \, d\nu(w)
\]

and

\[
\prod_{j=1}^n |F_j(z)|^r \leq \frac{1}{\nu(\Omega)} \int_{\Omega} \prod_{j=1}^n |F_j(z + T_jw)|^r \, d\nu(w).
\]

Next we discuss similar questions for spaces of analytic functions, i.e. we present sufficient conditions in order for the identity

\[E_j(\Omega, \mathcal{B}_1) \cap A(\mathbb{C}^d) = E_j(\Omega, \mathcal{B}_2) \cap A(\mathbb{C}^d), \quad j = 1, 2.\]  

should hold.

**Theorem 3.2.** Let \( \Omega \) be an admissible family of weights on \( \mathbb{C}^d \approx \mathbb{R}^{2d} \), and let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) be mixed quasi-norm spaces on \( \mathbb{C}^d \). Then (3.5) holds.

Furthermore, for every fixed \( \omega \in \Omega \), there are \( \omega_1, \omega_2 \in \Omega \) and constant \( C > 0 \) such that

\[
C^{-1}\|F\|_{A(\omega_1, \mathcal{B}_2)} \leq \|F\|_{A(\omega, \mathcal{B}_1)} \leq C\|F\|_{A(\omega_2, \mathcal{B}_2)}, \quad F \in A(\mathbb{C}^d).
\]
Proof. Let $G(z, w) = \prod_{j=1}^{n} F_j(z + T_j w)$. Then $w \mapsto G(z, w)$ is analytic. By the mean-value property for harmonic functions we get

$$G(z, 0) = \frac{1}{\nu(\Omega)} \int_{\Omega} G(z, w) \, d\nu(w),$$

which is the same as (3.7).

From the same analyticity property it follows that the map $w \mapsto |G(z, w)|^r$ is subharmonic, in view of [38, Corollary 2.1.15]. Hence, by [38, Theorem 2.1.4] we get

$$|G(z, 0)|^r \leq \frac{1}{\nu(\Omega)} \int_{\Omega} |G(z, w)|^r \, d\nu(w),$$

which is the same as (3.8), and the proof is complete. \qed

Proof of Theorem 7.2. We only prove (3.5) for $j = 2$, leaving the small modifications of the case $j = 1$ for the reader. By (3.1) it suffices to prove

$$E_2(\Omega, L^r) \cap A(C^d) \subseteq E_{\infty}(\Omega, L^r) \cap A(C^d), \quad r = \nu_1(\mathcal{B}).$$

Therefore, assume that $F \in E(\Omega, L^r) \cap A(C^d)$. We shall mainly follow the ideas in Proposition 3.2. By (1.6) and the assumptions we have

$$\|F\|_{L^r_\omega} < \infty \quad \text{and} \quad \omega_1 \leq C(\cdot)^{-2d}\omega,$$

for some $\omega, \omega_1 \in \Omega$ and some constant $C > 0$. Let $c$ and $C$ be the same as in (1.3)’. Then the result follows if we prove that (3.4) holds when $\chi$ is the characteristic function of $\{z \in C^d; |z| \geq 2c\}$.

By Lemma 3.3 and Cauchy-Schwartz inequality we get

$$|F(z)\omega_1(z)|^r \leq c_2 \|\chi\|_{L^r_\omega} \int_{|w| \leq c/|z|} |F(z + w)F(z - w)|^{r/2} \, d\lambda(w) \leq C_1 \int_{|w| \leq c/|z|} |F(z + w)F(z - w)|^{r/2} \, d\lambda(w) \leq C_2 \int_{\mathbb{C}^d} |F(z + w)\omega(z + w)|^{r/2}|F(z - w)\omega(z - w)|^{r/2} \, d\lambda(w) \leq C_2 \left( \int_{\mathbb{C}^d} |F(z + w)\omega(z + w)|^r \lambda(w) \right)^{1/2} \cdot \left( \int_{\mathbb{C}^d} |F(z - w)\omega(z - w)|^r \lambda(w) \right)^{1/2} \leq C_2 \|F\|_{L^r_\omega} < \infty,$$

for some constants $C_1$ and $C_2$. Here recall that $d\lambda(z)$ is the Lebesgue measure on $\mathbb{C}^d$. This proves (3.5) for $j = 2$, and (3.6). By similar arguments, (3.5) follows for $j = 1$. The details are left for the reader, and the proof is complete. \qed
3.2. Mapping properties for the Bargmann transform on modulation spaces. Next we prove that $A_0(\omega, \mathcal{B})$ is equal to $A(\omega, \mathcal{B})$ for every choice of $\omega$ in $\mathcal{B}_Q$ and mixed quasi-norm space $\mathcal{B}$.

**Theorem 3.4.** Let $\mathcal{B}$ be a mixed quasi-norm space on $\mathbb{R}^{2d} \simeq \mathbb{C}^d$ and let $\omega \in \mathcal{B}_Q^0(\mathbb{C}^d)$. Then $A_0(\omega, \mathcal{B}) = A(\omega, \mathcal{B})$, and the map $f \mapsto \mathfrak{M}f$ from $M(\omega, \mathcal{B})$ to $A(\omega, \mathcal{B})$ is isometric and bijective.

We need some preparations for the proof, and start to show that $M(\omega, \mathcal{B})$ is a Banach space when $\nu_1(\mathcal{B}) \geq 1$, which is a consequence of the following result. Here, for each $\nu \in \mathcal{S}_{1/2}(\mathbb{R}^d) \setminus \{0\}$, $0 < \omega \in L^\infty_0(\mathbb{R}^{2d})$ and mixed norm space $\mathcal{B}$ on $\mathbb{R}^{2d}$, we let $M_\omega(\mathcal{B}, \mathcal{B})$ be the set of all $f \in (\mathcal{S}_{1/2})'(\mathbb{R}^d)$ such that

$$\|f\|_{M_\omega(\mathcal{B}, \mathcal{B})} \equiv \|V_\psi f \cdot \omega\|_{\mathcal{B}} < \infty.$$ 

**Proposition 3.5.** Let $\psi \in \mathcal{S}_{1/2}(\mathbb{R}^d) \setminus \{0\}$ and $\omega \in \mathcal{L}_0^\infty(\mathbb{R}^{2d})$ be such that for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\omega \geq C_\varepsilon^{-1} e^{-| \cdot |^2}. \tag{3.9}$$

Also let $\mathcal{B}$ be a mixed norm space on $\mathbb{R}^{2d}$. Then $M_\psi(\omega, \mathcal{B})$ is a Banach space.

**Proof.** We may assume that $\|\psi\|_{L^2} = 1$, and start by proving that $M_\psi(\omega, \mathcal{B})$ is continuously embedded in $(\mathcal{S}_{1/2})'(\mathbb{R}^d)$, for every choice of $\varepsilon > 0$. We have $\psi \in \mathcal{S}_{1/2, e_0}$, for some $e_0 > 0$.

Therefore let $\varepsilon > 0$ be arbitrary. By a straight-forward combination of Lemma 1.6 and Proposition 1.8 and their proofs it follows that for some $\delta > 0$ we have

$$\|V_\psi f\|_{\mathcal{S}_{1/2, \delta}} \leq C_\varepsilon \|\varphi\|_{\mathcal{S}_{1/2, \varepsilon}} \|\psi\|_{\mathcal{S}_{1/2, \varepsilon_0}}, \tag{3.10}$$

for some constant $C_\varepsilon > 0$, which only depends on $\varepsilon$. (Cf. [15].)

Now we recall that

$$(f, \varphi)_{L^2(\mathbb{R}^d)} = (V_\psi f, V_\psi \varphi)_{L^2(\mathbb{R}^{2d})}, \quad \varphi \in \mathcal{S}_{1/2, \varepsilon}(\mathbb{R}^d),$$

for any $f \in M_\psi(\omega, \mathcal{B})$. Hence, (3.9), (3.10) and H"older’s inequality give

$$|(f, \varphi)_{L^2}| = |(V_\psi f, V_\psi \varphi)_{L^2}| \leq \|f\|_{M_\psi(\omega, \mathcal{B})} \|V_\psi \varphi / \omega\|_{\mathcal{B}} \leq C_1 \|f\|_{M_\psi(\omega, \mathcal{B})} \|V_\psi \varphi / \omega\|_{\mathcal{B}} \leq C_2 \|M_\psi(\omega, \mathcal{B})\| \|V_\psi \varphi / \omega\|_{L^\infty} \|\langle \cdot \rangle\|_{\mathcal{B}}^{-2d-1} \leq C_3 \|f\|_{M_\psi(\omega, \mathcal{B})} \|V_\psi \varphi\|_{\mathcal{S}_{1/2, \delta}} \leq C_4 \|f\|_{M_\psi(\omega, \mathcal{B})} \|\varphi\|_{\mathcal{S}_{1/2, \delta}},$$

for some constants $C_1, \ldots, C_4 > 0$. Consequently, $M_\psi(\omega, \mathcal{B})$ is continuously embedded in $(\mathcal{S}_{1/2})'(\mathbb{R}^d)$ and in $(\mathcal{S}_{1/2})'(\mathbb{R}^d)$ for every $\varepsilon > 0$.

Now let $\{f_j\}_{j=1}^\infty$ be a Cauchy sequence in $M_\psi(\omega, \mathcal{B})$. Since $(\mathcal{S}_{1/2, \varepsilon})'(\mathbb{R}^d)$ decreases when $\varepsilon$ decreases, (1.7) in combination with the previous embedding properties show that there is an element $f \in (\mathcal{S}_{1/2})'(\mathbb{R}^d)$ (which is independent of $\varepsilon$) such that $f_j \to f$ in $(\mathcal{S}_{1/2, \varepsilon})'$ as $j \to \infty$. 

30
Since $V_\psi f_j \to V_\psi f$ pointwise, it now follows that $f \in M_\psi(\omega, \mathcal{B})$ by Fatou’s lemma. This shows that $M_\psi(\omega, \mathcal{B})$ is a Banach space, and the proof is complete.

Next we consider the case when $\mathcal{B} = L^2$ and $\omega$ is rotation invariant in each coordinate. In this case we have the following.

**Lemma 3.6.** Let $\omega \in \mathcal{B}_Q^0(\mathbb{R}^{2d})$ be rotation invariant. If

$$c_\alpha = \|z_\alpha\|_{A^2_\omega}^{-1},$$

then $\{c_\alpha z_\alpha\}_{\alpha \in \mathbb{N}^d}$ is an orthonormal basis for $A^2_\omega(\mathbb{C}^d)$.

In the proof of Lemma 3.6 and in several other situations later on, we encounter the integral

$$\int_{\Delta_d} e^{i(n, \theta)} d\theta = \begin{cases} (2\pi)^d, & n = 0, \\ 0, & n \neq 0, \end{cases} \text{ when } \Delta_d = [0, 2\pi]^d \text{ and } n \in \mathbb{Z}^d. \quad (3.11)$$

**Proof.** First we prove that the scalar product $(z_\alpha, z_\beta)_{A^2_\omega}$ is zero when $\alpha \neq \beta$. By polar coordinates we have

$$z = (r_1 e^{i\theta_1}, \ldots, r_d e^{i\theta_d}),$$

where

$$r = (r_1, \ldots, r_d) \in [0, \infty)^d, \quad \text{and} \quad \theta = (\theta_1, \ldots, \theta_d) \in \Delta_d.$$ 

Furthermore it follows from the assumptions that $\omega(z) = \omega_0(2^{-1/2}r)$, for some positive function $\omega_0$ on $[0, \infty)^d$. Hence (3.32) and (3.11) give

$$(z_\alpha, z_\beta)_{A^2_\omega} = \pi^{-d} I_{\omega_0}(\alpha + \beta) \cdot \int_{\Delta_d} e^{i(\alpha-\beta, \theta)} d\theta = 2^d I_{\omega_0}(2\alpha) \delta_{\alpha,\beta},$$

where

$$I_{\omega_0}(\alpha) = \int_{[0, \infty)^d} \omega_0(r) r^\alpha e^{-|r|^2} r_1 \cdots r_d \, dr > 0, \quad \alpha \in \mathbb{N}^d,$$

and $\delta_{\alpha,\beta}$ is the Kronecker’s delta function. In particular, $(z_\alpha, z_\beta)_{A^2_\omega} = 0$, if and only if $\alpha \neq \beta$, and we have proved that $\{c_\alpha z_\alpha\}$ is an orthonormal system for $A^2_\omega$.

It remains to prove that the set of linear combinations of $c_\alpha z_\alpha$ spans $A^2_\omega$. By Hahn-Banach’s theorem, it suffices to prove that if $F \in A^2_\omega$, and

$$(F, z_\alpha)_{A^2_\omega} = 0 \quad \text{for all } \alpha \in \mathbb{N}^d, \quad (3.12)$$

implies that $F = 0$.

Therefore assume that (3.12) holds. Since $F$ is entire, it follows that its Taylor series expansion

$$F(z) = \sum_\beta a_\beta z_\beta, \quad a_\beta = \frac{F^{(\beta)}(0)}{\beta!} \quad (3.13)$$
is locally uniformly convergent, and that
\[ \sum_{\beta} |a_{\beta}r^\beta| < \infty \] holds. Hence (3.11) gives
\[ 0 = (F, z^\alpha)_{A^2(\omega)} = \int_{\mathbb{C}^d} \left( \sum_{\beta} a_{\beta}z^\beta \right) z^\alpha \omega(2^{1/2}z) \, d\mu(z) \]
\[ = \pi^{-d} \int_{[0,\infty)^d} \left( \int_{\Delta^d} \left( \sum_{\beta} a_{\beta}r^{\alpha+\beta} e^{i(\beta-\alpha,\theta)} \right) \, d\theta \right) \omega_0(r) e^{-|r|^2} r_1 \cdots r_d \, dr \]
\[ = \pi^{-d} \int_{[0,\infty)^d} \left( \sum_{\beta} a_{\beta}r^{\alpha+\beta} \left( \int_{\Delta^d} e^{i(\beta-\alpha,\theta)} \, d\theta \right) \right) \omega_0(r) e^{-|r|^2} r_1 \cdots r_d \, dr \]
\[ = 2^d I_{(\omega_0)}(2\alpha)a_\alpha. \]

Since \( I_{(\omega_0)}(2\alpha) > 0 \), we get \( a_\alpha = 0 \) for every \( \alpha \). Consequently, \( F \) is identically zero, and the proof is complete. \( \square \)

We may now prove Theorem 3.4 in the important special case that \( \mathcal{B} = L^2 \) and \( \omega \) is rotation invariant in every coordinate.

**Proposition 3.7.** If \( \omega \in P^0_Q(\mathbb{C}^d) \) is rotation invariant in each coordinate, then \( A_0(\omega, L^2) = A(\omega, L^2) \).

**Proof.** We use the same notations as in the proof of Lemma 3.6. The image under the Bargmann transform \( \mathfrak{B} \) of the hermite function \( h_\alpha \) is \( z^\alpha/(\alpha!)^{1/2} \). Since \( M^2(\omega) \) is a Banach space in view of Proposition 3.3 and \( \mathfrak{B} \) is isometric and injective from \( M^2(\omega) \) to \( A^2(\omega) \), it follows from Lemma 3.6 that
\[ \{(\alpha!)^{1/2} c_\alpha h_\alpha \}_{\alpha \in \mathbb{N}^d} \]
is an orthonormal basis of \( M^2(\omega) \), and that \( \mathfrak{B} \) is bijective from \( M^2(\omega) \) to \( A^2(\omega) \). \( \square \)

**Remark 3.8.** Lemma 3.6 and Proposition 3.7 give equalities between weighted \( L^p \)-norms of the Taylor coefficients and weighted \( L^p \) norm of corresponding entire functions, when the involved weights are rotation invariant and \( p = 2 \). In general it is difficult to find such equalities between coefficients and functions in other situations when the weights are not rotation invariant, or \( p \) is not equal to 2. We refer to [9] for positive results in this directions.

**Proof of Theorem 3.4.** By Proposition 1.15 it follows that the map \( f \mapsto \mathfrak{B} f \) is an isometric injective map from \( M(\omega, \mathcal{B}) \) to \( A(\omega, \mathcal{B}) \). We have to show that this map is surjective.

Therefore assume that \( F \in A(\omega, \mathcal{B}) \). Since \( M(\omega_1, \mathcal{B}) \subseteq M(\omega_2, \mathcal{B}) \), as \( \omega_2 \leq C\omega_1 \), it follows from Theorem 3.2 and Proposition 3.7 that there is an element

\[ 32 \]
\[ f \in M^2_{(\omega_1)}(\mathbb{R}^d) \subseteq S'_1(\mathbb{R}^d) \] such that \( F = \mathcal{U} f \), for some \( \omega_1 \). We have
\[
\| f \|_{M_{(\omega_1)}} = \| \mathcal{U} f \|_{A_{(\omega_1)}} = \| F \|_{A_{(\omega_1)}} < \infty.
\]

Hence, \( f \in M(\omega, \mathcal{B}) \), and the result follows. The proof is complete. \( \square \)

As a consequence of Theorems \ref{thm:3.2} (2) and \ref{thm:3.4} we may identify the space \( \tilde{\Upsilon}(\mathbb{R}^d) \) in Remark \ref{rem:1.10} with unions of modulation spaces. More precisely we have the following.

**Proposition 3.9.** Let \( \mathcal{B} \) be a mixed quasi-norm space on \( \mathbb{R}^{2d} \). Then
\[
\bigcup M(\omega, \mathcal{B}) = \Upsilon(\mathbb{R}^d),
\]
where the union is taken over all rotation invariant \( \omega \in \mathcal{P}_0^0(\mathbb{R}^{2d}) \).

Furthermore, if \( 0 < \gamma < 2, 0 < \varepsilon < \min(\gamma, 2 - \gamma) \) and \( \omega(x, \xi) = C e^{c(\|x\|^{2\gamma} + \|\xi\|^{2\gamma})} \), for some constants \( C > 0 \) and \( c \in \mathbb{R} \) which are independent of \( x, \xi \in \mathbb{R}^d \), then the following is true:

1. if \( c > 0 \), then \( S_{1/(\gamma+\varepsilon)}(\mathbb{R}^d) \subseteq M(\omega, \mathcal{B}) \subseteq S_{1/(\gamma-\varepsilon)}(\mathbb{R}^d) \);
2. if \( c < 0 \), then \( S'_{1/(\gamma-\varepsilon)}(\mathbb{R}^d) \subseteq M(\omega, \mathcal{B}) \subseteq S'_{1/(\gamma+\varepsilon)}(\mathbb{R}^d) \).

**Proof.** By Theorems \ref{thm:3.2} (2) and \ref{thm:3.4} we may assume that \( \mathcal{B} = L^\infty \). By the assumptions on \( \omega \) we get \( M^\infty_{(\omega)} \subseteq \Upsilon \).

Let \( f \in \Upsilon(\mathbb{R}^d) \). Then it follows from Proposition \ref{prop:1.9} Lemma \ref{lem:2.8} and the fact that \( \mathcal{P}_0^0 \) is a group under multiplications that
\[
|V_\phi f| \leq 1/\omega
\]
for some \( \omega \in \mathcal{P}_0^0(\mathbb{R}^{2d}) \), where \( \phi(x) = \pi^{-d/4} e^{-|x|^2/2} \). This is the same as \( f \in M^\infty_{(\omega)} \), and the first part follows.

The assertions (1) and (2) follow by similar arguments in combination of Proposition \ref{prop:1.9} and are left for the reader. The proof is complete. \( \square \)

### 4. Basic properties for spaces of analytic functions and modulation spaces

In this section we establish basic properties for the spaces \( A(\omega, \mathcal{B}) \) and \( M(\omega, \mathcal{B}) \) when \( \omega \) is an appropriate weight and \( \mathcal{B} \) is a mixed quasi-norm space. In view of Theorem \ref{thm:3.4} any property of \( A(\omega, \mathcal{B}) \) carry over to \( M(\omega, \mathcal{B}) \), and vice versa. We start by proving that these spaces are quasi-Banach spaces. Then we prove that if \( \nu_2(\mathcal{B}) < \infty \), then \( P(C^d) \) is dense in \( A(\omega, \mathcal{B}) \), and that the dual of \( A(\omega, \mathcal{B}) \) can be identified with \( A(1/\omega, \mathcal{B}^\prime) \) through a unique extension of the \( A^2 \) form on \( P(C^d) \). A straight-forward consequence of the latter results is that \( P(C^d) \) is dense in \( A(\omega, \mathcal{B}) \) with respect to the weak*-topology, when \( \nu_1(\mathcal{B}) > 1 \). (Recall Subsection \ref{sect:1.4} for the definitions of \( \nu_1(\mathcal{B}) \) and \( \nu_2(\mathcal{B}) \).) Thereafter we introduce the concept of narrow convergence to get convenient density properties for certain \( \mathcal{B} \) with \( \nu_1(\mathcal{B}) = 1 \) and \( \nu_2(\mathcal{B}) = \infty \). Finally we formulate corresponding results for modulation spaces.
A cornerstone of these investigations concerns the projection operator

\[ (\Pi_A F)(z) = \int F(w) e^{(z,w)} \, d\mu(w), \]  

(4.1)

related to the reproducing formula (1.26). Here recall that \( d\mu(z) = \pi^{-d} e^{-|z|^2} \, d\lambda(z) \), where \( d\lambda(z) \) is the Lebesgue measure on \( \mathbb{C}^d \). The minimal assumption on \( F \) is that it should be locally integrable on \( \mathbb{C}^d \) and satisfy

\[ \|F \cdot e^{N\cdot|\cdot|^2}\|_{L^p} < \infty \quad \text{for every} \quad N \geq 0, \]  

(4.2)

where \( p \in (0, \infty] \) is fixed. We note that (4.2) is fulfilled for \( p = 1 \) if \( F \in L^1_{\text{loc}}(\mathbb{C}^d) \) and satisfies

\[ \int |F(z)| e^{-\gamma|z|^2} \, d\lambda(z) < \infty \quad \text{for some} \quad \gamma < 1. \]  

(4.3)

**Lemma 4.1.** Let \( \gamma, \delta > 0 \) and let \( F \in L^1_{\text{loc}}(\mathbb{C}^d) \). Then the following is true:

1. if \( p \in [1, \infty] \) and \( F \) satisfies (1.2), then \( \Pi_A F \) is an entire function on \( \mathbb{C}^d \);
2. if \( p \in (0, \infty] \) and \( F \) satisfies (1.2) and in addition is entire, then \( \Pi_A F = F \);
3. if \( 4\delta(1 - \gamma) \geq 1 \), then for some constant \( C > 0 \) it holds

\[ \|\Pi_A F \cdot e^{-\delta|\cdot|^2}\|_{L^1} \leq C \|F e^{-\gamma|\cdot|^2}\|_{L^1} \]  

when \( F \in L^1_{\text{loc}}(\mathbb{C}^d) \) and satisfies (4.3);
4. if \( \gamma < 3/4 \) and (4.3) is fulfilled, then

\[ (F,G)_{B^2} = (\Pi_A F, G)_{B^2} = (\Pi_A F, G)_{B^2}, \]  

(4.4)

for every polynomial \( G \) (which is analytic) on \( \mathbb{C}^d \).

**Proof.** By Hölder’s inequality we may assume that \( p = 1 \) when proving (1). Let \( E(z, w) = F(w) e^{(z,w)-|w|^2} \). The condition (1.2) implies that if \( z \in K \), where \( K \subseteq \mathbb{C}^d \) is compact, then for each multi-index \( \alpha \) we have that \( \partial_z^\alpha E(z, w) \) is uniformly bounded in \( L^1 \) with respect to \( w \). The assertion (1) is now a consequence of the fact that \( z \mapsto E(z, w) \) is entire.

Since \( \{ e^{t|z|^2} \in L^\infty(\mathbb{C}^d); t \in \mathbb{R} \} \) is an admissible family of weights, we may assume that \( p = 1 \) in view of Theorem 3.2 when proving (2). The assertion then follows from the same arguments as for the proof of Lemma A.2 in [47]. In order to be self-contained we give here a proof.

For every multi-index \( \alpha \) we have

\[ (\partial_z^\alpha E)(z, w) = e^{(z,w)-|w|^2} \, \bar{w}^\alpha F(w) \]  

and

\[ (\partial_{\bar{z}}^\alpha \partial_z E)(z, w) = 0. \]

Hence, by the assumptions it follows that the map \( w \mapsto (\partial_z^\alpha E)(z, w) \) belongs to \( L^1(\mathbb{C}^d) \) for every \( z \in \mathbb{C}^d \), and that \( F_0 \equiv \Pi_A F \in (1.1) \) is analytic with derivatives

\[ \left( \partial^\alpha F_0 \right)(z) \equiv \int e^{(z,w)} \, \bar{w}^\alpha F(w) \, d\mu(w). \]  

(4.5)
In particular we have

\[(\partial^\alpha F_0)(0) \equiv \int_{C^d} \overline{w}^\alpha F(w) \, d\mu(w). \tag{4.3}'\]

We have to prove that \(F_0 = F\). Since both \(F\) and \(F_0\) are entire functions it suffices to prove

\[\partial^\alpha F_0(0) = \partial^\alpha F(0),\]

for every multi-index \(\alpha\).

If \(w = (r_1 e^{i\theta_1}, \ldots, r_d e^{i\theta_d})\), where \(r = (r_1, \ldots, r_d) \in [0, \infty)^d\) and \(\theta = (\theta_1, \ldots, \theta_d) \in \Delta_d = [0, 2\pi]^d\), then \((3.13)\) and \((4.5)\)' give

\[\partial^\alpha F_0(0) = \int_{C^d} \overline{w}^\alpha F(w) \, d\mu(w) = \pi^{-d} \int_0^\infty \left( \int_{\Delta_d} r^\alpha e^{-i(\theta, \alpha)} F(r_1 e^{i\theta_1}, \ldots, r_d e^{i\theta_d}) r_1 \cdots r_d e^{-|r|^2} \, d\theta \right) \, dr = \pi^{-d} \int_0^\infty r^\alpha r_1 \cdots r_d e^{-|r|^2} J_\alpha(r) \, dr, \tag{4.6}\]

where

\[J_\alpha(r) = \int_{\Delta_d} e^{-i(\theta, \alpha)} F(r_1 e^{i\theta_1}, \ldots, r_d e^{i\theta_d}) \, d\theta = \int_{\Delta_d} \left( \sum_\beta a_\beta r^\beta e^{i(\theta, \beta - \alpha)} \right) \, d\theta\]

By \((3.14)\) it follows that we may interchange the order of summation and integration. This gives

\[J_\alpha(r) = \sum_\beta a_\beta r^\beta \int_{\Delta_d} e^{i(\theta, \beta - \alpha)} \, d\theta = (2\pi)^d a_\alpha r^\alpha, \tag{4.7}\]

in view of \((3.11)\).

By inserting \((4.7)\) into \((4.6)\) and taking \(u_j = r_j^2\) as new variables of integration, \((3.13)\) gives

\[\partial^\alpha F_0(0) = 2^d a_\alpha \int_0^\infty r_1^{2\alpha_1} \cdots r_d^{2\alpha_d} e^{-|r|^2} \, dr = a_\alpha \int_0^\infty u^{\alpha_1 + \cdots + \alpha_d} e^{-u/2} \, du = a_\alpha \alpha! = \partial^\alpha F(0). \tag{4.8}\]

This proves (2).

The assertion (3) follows from the inequality

\[|F(w)e^{(z,w)-|w|^2}e^{-\delta|z|^2}| \leq |F(w)|e^{-\gamma|w|^2}e^{-(1-\gamma)|w-(1-\gamma)^{-1}z/2|^2},\]

and (4) is obtained by choosing \(\delta > 1\) in the latter estimate, giving that

\[(z, w) \mapsto F(w)G(z)e^{(z,w)-|w|^2}e^{-|z|^2}\]
belongs to \( L^1(C^d \times C^d) \) when \( G \) is a polynomial. The relation (4.4) is now an immediate consequence of the reproducing formula (1.26) applied on \( G \), and Fubini’s theorem.

Remark 4.2. We note that if \( F \in A(C^d) \) and satisfies (4.2), then

\[
z^\alpha F(z) = \int w^\alpha F(w)e^{(z,w)} \, d\mu(w)
\]

and

\[
\partial^\alpha F(z) = \int \overline{w}^\alpha F(w)e^{(z,w)} \, d\mu(w),
\]

giving that

\[
\partial^\alpha F(0) = (F, z^\alpha)_{A^2} \tag{4.9}
\]

(see also [3]).

In fact, the first formula follows by replacing \( F \) by \( z^\alpha F \) in the reproducing formula and using (2) in Lemma 4.1. For the second formula we note that the condition (4.2) and reproducing formula give

\[
\partial^\alpha F(z) = \partial^\alpha \left( \int F(w)e^{(z,w)} \, d\mu(w) \right)
\]

\[
= \int \partial^\alpha_z (F(w)e^{(z,w)}) \, d\mu(w) = \int \overline{w}^\alpha F(w)e^{(z,w)} \, d\mu(w),
\]

and the result follows.

Remark 4.3. It follows from the the proof, and especially (4.8), of the previous lemma that if \( F_1, F_2 \in A(C^d) \), (4.2) holds for \( F_j \) and \( p = 1, j = 1, 2 \), and that

\[
(F_1, G)_{A^2} = (F_2, G)_{A^2}, \quad G \in P(C^d),
\]

then \( F_1 = F_2 \).

Next we prove that \( A(\omega, \mathcal{B}) \) and \( M(\omega, \mathcal{B}) \) are Banach spaces when \( \nu_1(\mathcal{B}) \geq 1 \).

Theorem 4.4. Let \( \omega_1 \in \mathcal{R}_Q^0(C^d) \), \( \omega_2 \in \mathcal{R}_Q(C^d) \) and \( \mathcal{B} \) be a mixed quasi-norm space on \( C^d \). Then the following is true:

1. \( M(\omega_1, \mathcal{B}) \) and \( A(\omega_2, \mathcal{B}) \) are quasi-Banach spaces;
2. if in addition \( \nu_1(\mathcal{B}) \geq 1 \), then \( M(\omega_1, \mathcal{B}) \) and \( A(\omega_2, \mathcal{B}) \) are Banach spaces.

Proof. By Theorem 3.4 it suffices to prove the result for \( A(\omega_2, \mathcal{B}) \). Since the statement is invariant under dilations, we may assume that (1.4) holds for \( \omega = \omega_2 \) and \( c = 1/8 \). Furthermore, since it is obvious that \( \| \cdot \|_{\mathcal{B}(\omega_2, \mathcal{B})} \) is a norm when \( \nu_1(\mathcal{B}) \geq 1 \), it suffices to prove (1).

By Theorem 3.2 it follows that \( A(\omega_2, \mathcal{B}) \) is continuously embedded in \( A(\omega_0, L^1) \), for some choice of \( \omega_0 \in \mathcal{R}_Q \). Furthermore, it follows from the proof of Theorem 3.2 that we may choose \( \omega_0 \) such that it satisfies (1.4) with \( c = 1/6 \). Consequently, any \( F \) in \( A(\omega_0, L^1) \) fulfills (1.2) with \( p = 1 \).
Now let \((F_j)_{j=1}^{\infty}\) be a Cauchy sequence in \(A(\omega, \mathcal{B})\). Since both \(B(\omega_2, \mathcal{B})\) and \(B(\omega_0, L^1)\) are quasi-Banach spaces, it exists an element \(F \in B(\omega_2, \mathcal{B}) \cap B(\omega_0, L^1)\) such that \(F_j \to F\) in \(B(\omega_2, \mathcal{B})\) and \(B(\omega_0, L^1)\) as \(j \to \infty\).

We have to prove that \(F \in A(C^d)\). By the assumptions and Lemma 4.1, it follows that \(F_0 = \Pi_A F\) in (4.1) defines an analytic function, and that

\[
F_j(z) = \int F_j(w) e^{\langle z, w \rangle\mu} \, dw
\]

for every \(j\). Furthermore, for each compact set \(K \subseteq C^d\) there is a constant \(C > 0\) such that

\[
\sup_{K} |F_j(z) - F_0(z)| \leq \pi^{-d} \int |F_j(w) - F(w)| e^{-|w|^2 + C|w|} \, d\lambda(w) = \pi^{-d} \int \left| (F_j(w) - F(w)) e^{-|w|^2/2 \omega_0(2^{1/2}w)} \right| \cdot (e^{-|w|^2/2 + C|w|} \omega_0(2^{1/2}w))^{-1} \, d\lambda(w) \leq C_{\omega_0} \int |F_j(w) - F(w)| e^{-|w|^2/2 \omega_0(2^{1/2}w)} \, d\lambda(w) = C \|F_j - F\|_{B(\omega_0, L^1)},
\]

where

\[
C_{\omega_0} = \operatorname{ess} \sup_{w \in C^d} \left( e^{-|w|^2/2 + C|w|} \omega_0(2^{1/2}w) \right) < \infty,
\]

and \(C\) is a constant. Since the right-hand side of (4.10) turns to zero as \(j \to \infty\), it follows that \(F_j \to F_0\) locally uniformly as \(j \to \infty\). This proves that \(F = F_0\), which is analytic, and the result follows.

4.1. Density and duality properties. Next we prove that if \(\mathcal{B}\) is a mixed norm space with \(\nu_2(\mathcal{B}) < \infty\), and that the weight \(\omega \in \mathcal{P}_Q^0(C^d)\) in addition should be dilated suitable, then the set \(P(C^d)\) of polynomials on \(C^d\) is dense in \(A(\omega, \mathcal{B})\). Furthermore, in this situation we also prove that the dual of \(A(\omega, \mathcal{B})\) can be identified with \(A(1/\omega, \mathcal{B})\), through a unique extention of the \(A^2\) form on \(P(C^d)\).

An important part of these considerations concerns possibilities to approximate elements \(F\) in \(A(\omega, \mathcal{B})\) with their dilations \(F(\lambda \cdot)\) for \(0 < \lambda < 1\). We note that the latter functions belong to

\[
A_P(C^d) \equiv \{ F \in A(C^d) ; F \cdot e^{-(1-\varepsilon)|z|^2/2} \in \mathcal{B} \text{ for some } \varepsilon > 0 \},
\]

and that

\[
P(C^d) \subseteq A_P(C^d) \subseteq A(\omega, \mathcal{B}), \quad \text{when } \omega \in \mathcal{P}_Q^0(C^d).
\]

This is a straight-forward consequence of Theorem 3.2 and the definitions.

**Proposition 4.5.** The set \(A_P(C^d)\) in (4.11) is independent of the mixed quasi-norm space \(\mathcal{B}\) on \(C^d\). Furthermore, if \(\mathcal{B}\) is a mixed norm space on \(C^d\) and \(\omega \in \mathcal{P}_Q^0(C^d)\), then \(P(C^d)\) is dense in \(A_P(C^d)\) with respect to the topology in \(A(\omega, \mathcal{B})\).

**Proof.** The first part follows immediately from Theorem 3.2 and the observation that

\[
\Omega = \{ e^{-(1-\varepsilon)|z|^2/2} ; 0 < \varepsilon < 1, \ z \in C^d \}
\]
is an admissible family of weights on $\mathbf{C}^d$.

The first part then shows that we may assume that $\mathcal{B} = L^1$ and $\omega = e^{\varepsilon_0 |\cdot|^2}$ for some small $\varepsilon_0 > 0$ which depends on $\lambda > 0$, when proving the second part. The result is then an immediate consequence of [17, Prop. 3.2]. The proof is complete.

\[ \square \]

Remark 4.6. By similar arguments, using Proposition 3.1 instead of Theorem 3.2 it follows that

\[ \{ f \in \mathcal{H}(\mathbf{R}^d) ; f \cdot e^{-(1-\varepsilon)|x|^2/2} \in \mathcal{B} \text{ for some } \varepsilon > 0 \} \]

is independent of the mixed norm space $\mathcal{B}$ on $\mathbf{R}^d$. Here recall that $\mathcal{H}(\mathbf{R}^d)$ is the set of harmonic functions on $\mathbf{R}^d$.

Our result on duality is the following.

Theorem 4.7. Let $\omega \in \mathcal{P}_Q^0(\mathbf{C}^d)$ be dilated suitable and $\mathcal{B}$ be a mixed norm space on $\mathbf{C}^d$ such that $\nu_2(\mathcal{B}) < \infty$. Then the following is true:

1. the $A^2$ form on $P(\mathbf{C}^d)$ extends uniquely to a continuous sesqui-linear form on $A(\omega, \mathcal{B}) \times A(1/\omega, \mathcal{B}')$;

2. the dual of $A(\omega, \mathcal{B})$ can be identified by $A(1/\omega, \mathcal{B}')$ through the extension of the $A^2$ form on $P(\mathbf{C}^d)$.

Here we recall that $\mathcal{B}' = L^p(V)$, when $\mathcal{B} = L^p(V)$, and $p \in [1, \infty]^n$ and $V$ are given by \[.]\[20\].

We also have the following result on density, which is strongly connected to the proof of Theorem 4.7.

Theorem 4.8. Let $\omega \in \mathcal{P}_Q^0(\mathbf{C}^d)$ be dilated suitable, $\mathcal{B}$ be a mixed quasi-norm space on $\mathbf{C}^d$ such that $\nu_2(\mathcal{B}) < \infty$, and let $F \in A(\omega, \mathcal{B})$ and $G \in A(\omega, \mathcal{B}')$. Then the following is true:

1. $P(\mathbf{C}^d)$ is dense in $A(\omega, \mathcal{B})$. If in addition $\nu_1(\mathcal{B}) \geq 1$, then $P(\mathbf{C}^d)$ is dense in $A(\omega, \mathcal{B}')$ with respect to the weak$^*$-topology;

2. if $0 < \lambda < 1$, then $F(\lambda \cdot) \rightarrow F$ in $A(\omega, \mathcal{B})$. If in addition $\nu_1(\mathcal{B}) \geq 1$, then $G(\lambda \cdot) \rightarrow G$ with respect to the weak$^*$-topology in $A(\omega, \mathcal{B}')$.

We start by proving the first parts of (1) and (2) in Theorem 4.8. Thereafter we prove Theorem 4.7 and finally we prove the last parts of (1) and (2) in Theorem 4.8.

Some preparations for the proofs are needed. We start by recalling the following generalization of Lebesgue’s theorem.

Lemma 4.9. Let $p \in (0, \infty)$, $d\mu$ be a positive measure, and let $f_j, f, g_j, g \in L^p(d\mu)$ be such that $f_j \rightarrow f$ and $g_j \rightarrow g$ pointwise a.e. as $j \rightarrow \infty$, $|f_j| \leq g_j$, $|f| \leq g$ and $\|g_j\|_{L^p(d\mu)} \rightarrow \|g\|_{L^p(d\mu)}$ as $j \rightarrow \infty$. Then $f_j \rightarrow f$ in $L^p(d\mu)$ as $j \rightarrow \infty$.

Proof. Let $q = \max(1, p)$. Then the result follows by applying Fatou’s lemma on $2^q(g^p + g_j^p) - |f - f_j|^p$. The details are left for the reader. \[ \square \]
The first part of Theorem 4.8 (2) is an immediate consequence of the following lemma.

**Lemma 4.10.** Let \( 0 < \omega_0 \in L^\infty_{\text{loc}}(\mathbb{R}^d) \) be such that
\[
\omega_0(x) \leq C\omega_0(\lambda x), \quad x \in \mathbb{R}^d, \quad 1 - \theta < \lambda < 1,
\]
for some constants \( \theta \in (0, 1) \) and \( C > 0 \), and let and \( \mathcal{B} \) be a mixed quasi-norm space on \( \mathbb{R}^d \) with \( \nu_2(\mathcal{B}) < \infty \). Also let \( f \in L^r_{\text{loc}}(\mathbb{R}^d) \) be such that \( f \cdot \omega_0 \in \mathcal{B} \), where \( r = \nu_1(\mathcal{B}) \). Then \( f(\lambda \cdot)\omega_0 \in \mathcal{B} \) when \( 1 - \theta < \lambda < 1 \), and
\[
\lim_{\lambda \to 1^-} \|(f - f(\lambda \cdot))\omega_0\|_{\mathcal{B}} = 0. \tag{4.12}
\]

**Proof.** We only consider the case when \( \mathcal{B} = L^p(\mathbb{R}^d) \), \( p \in (0, \infty) \). The straightforward modifications to the general case are left for the reader.

By the assumptions we have
\[
|f(\lambda x)\omega_0(x)| \leq C|f(\lambda x)\omega_0(\lambda x)|,
\]
and it follows that \( \|f(\lambda \cdot)\omega_0\|_{L^p} \leq C\lambda^{-d/p}\|f\omega_0\|_{L^p} < \infty \), by a simple change of variables in the integral. Hence \( \{f(\lambda \cdot)\omega_0\}_{1 - \theta < \lambda < 1} \) is a bounded set in \( L^p \).

By straightforward approximations, it follows that we may assume that \( f \in C_0(\mathbb{R}^d) \) and \( \omega_0 \in C(\mathbb{R}^d) \) when proving (4.12). Then the result follows if we choose \( g_\lambda(x) = |f(\lambda x)| \), \( g(x) = |f(x)| \) and \( d\mu(x) = \omega_0(x) \, dx \) in Lemma 4.9. The proof is complete. \( \square \)

**Proof of the first parts of Theorem 4.8.** By Lemma 4.10 it suffices to prove that if \( F \in A(\omega, \mathcal{B}) \) and \( \lambda < 1 \) with \( 1 - \lambda \) small enough, then \( \|F(\lambda \cdot) - G\|_{A(\omega, \mathcal{B})} \) can be made arbitrarily small as \( G \in P(\mathcal{C}^d) \).

Let \( s \in \mathbb{R} \) be chosen such that \( \lambda < s < 1 \). Then it follows from the assumptions that
\[
C^{-1}\omega(z)e^{-|z|^2/2} \leq e^{-s|z|^2/2} \leq C\omega(\lambda z)e^{-\lambda^2|z|^2/2},
\]
for some constant \( C \). In particular, we have
\[
\|F(\lambda \cdot) - G\|_{A(\omega, \mathcal{B})} \leq C\|(F(\lambda \cdot) - G)e^{-s|\cdot|^2/2}\|_{\mathcal{B}} \tag{4.13}
\]
and
\[
\|F(\lambda \cdot) e^{-s|\cdot|^2/2}\|_{\mathcal{B}} < \infty, \tag{4.14}
\]
for some constant \( C > 0 \).

By (4.13) it follows that \( F(\lambda \cdot) \) belongs to the set
\[
\mathcal{A}_{p,t}(\mathcal{C}^d) \equiv \{ F \in A(\mathcal{C}^d) ; \|F \cdot e^{-|\cdot|^2/2}\|_{\mathcal{B}} < \infty \},
\]
when \( t = s \). Since the same is true for any choice of \( t \in (\lambda, s) \), \( \mathcal{A}_{p,t} \) increases with \( t \), and that \( \{ e^{-t|\cdot|^2/2} ; \lambda < t < s \} \) is an admissible family of weights, it follows from Theorem 3.2 that (4.13) and (4.14) hold after \( s \) has been replaced by an appropriate \( t \in (\lambda, s) \), and \( p = (1, \ldots, 1) \). Since \( P(\mathcal{C}^d) \) is dense in \( \mathcal{A}_{p,t}(\mathcal{C}^d) \) with \( p = (1, \ldots, 1) \), in view of [47, Proposition 3.2], it follows that the right-hand side of (4.13) can be made arbitrarily small, and the assertion follows. This completes the proof of the first parts of Theorem 4.8 (1) and (2). \( \square \)
Next we turn to the proof of Theorem 4.7. First we note that if \( A(\omega, \mathcal{B}) \) is the same as in Theorem 4.7 and \( l \in (A(\omega, \mathcal{B}'))' \), then

\[
l(F) = \int F(z)\overline{G(z)} \, d\mu(z), \quad F \in A(\omega, \mathcal{B}),
\]

(4.15)

for some \( G \in B(1/\omega, \mathcal{B}) \). (Note that the mixed norm space \( \mathcal{B}' \) is the dual to \( \mathcal{B} \) when \( \nu_2(\mathcal{B}) < \infty \) and the \( L^2 \) form is used.) In fact, it follows from the definitions that there is a constant \( C \geq 0 \) such that

\[
|l(F)| \leq C\|F\|_{B(\omega, \mathcal{B})}
\]

(4.16)

when \( F \in A(\omega, \mathcal{B}) \). By Hahn-Banach’s theorem it follows that \( l \) is extendable to a linear continuous form on \( B(\omega, \mathcal{B}) \) and that (4.16) still holds for \( F \in B(\omega, \mathcal{B}) \).

The formula (4.15) now follows from well-known results in measure theory.

From now on we let \( I_G \) be the continuous linear form on \( A(\omega, \mathcal{B}) \), defined by (4.15) when \( G \in B(1/\omega, \mathcal{B}') \). Then it follows from the previous investigations that the map \( G \mapsto I_G \) is surjective from \( B(1/\omega, \mathcal{B}') \) to \( (A(\omega, \mathcal{B}'))' \).

In what follows we link the kernel of the latter map with the kernel

\[
N(\omega, \mathcal{B}) \equiv \{ F \in B(\omega, \mathcal{B}) ; \Pi_A F = 0 \}
\]

of the projection operator. Here we note that every \( F \in B(\omega, \mathcal{B}) \) for \( \omega \in \mathcal{P}^0_Q(\mathbb{C}^d) \) and mixed norm space \( \mathcal{B} \), fulfill the required properties in Lemma 4.1.

**Lemma 4.11.** Let \( \mathcal{B} \) be a mixed norm space on \( \mathbb{C}^d \) and assume that \( \omega \in \mathcal{P}^0_Q(\mathbb{C}^d) \) is dilated suitable. Then the following is true:

1. \( N(\omega, \mathcal{B}) \) is a closed subspace of \( B(\omega, \mathcal{B}) \);
2. if \( \nu_2(\mathcal{B}) < \infty \), then the kernel of the map \( G \mapsto I_G \), from \( B(1/\omega, \mathcal{B}') \) to \( (A(\omega, \mathcal{B}'))' \) is equal to \( N(1/\omega, \mathcal{B}') \).

**Proof.** Assume that \( F_j \in N(\omega, \mathcal{B}), j \geq 1 \), converges to \( F \in B(\omega, \mathcal{B}) \) in \( B(\omega, \mathcal{B}) \), as \( j \to \infty \). If \( K \subseteq \mathbb{C}^d \) is compact, then for some constant \( C_K \), depending on \( K \), Hölder’s inequality gives

\[
\sup_{z \in K} |\Pi_A F| = \sup_{z \in K} |\Pi_A (F - F_j)| \leq C_K \|F - F_j\|_{B(\omega, \mathcal{B})} \to 0
\]

as \( j \to \infty \). This proves (1).

(2) By the first part of Theorem 4.8 (1) it suffices to prove that \( I_G(F) = 0 \) for every polynomial \( F \) on \( \mathbb{C}^d \), if and only if \( G \in N(1/\omega, \mathcal{B}') \). By Lemma 4.1 (4) it follows that

\[
I_G(F) = I_{\Pi_A G}(F) = (F, G)_{A \omega}, \quad F \in P(\mathbb{C}^d),
\]

(4.17)

and \( G \in B(1/\omega, \mathcal{B}') \). This proves that \( I_G = 0 \) when \( G \in N(1/\omega, \mathcal{B}') \).

On the other hand, if \( I_G(F) = 0 \) for every polynomial \( F \), then (4.15) and (4.17) show that the entire function \( \Pi_A G \) satisfies

\[
(\partial^\alpha \Pi_A G)(0) = 0,
\]

for every \( \alpha \). This implies that \( \Pi_A G = 0 \), i.e. \( G \in N(1/\omega, \mathcal{B}') \), and the proof is complete. \( \Box \)
Remark 4.12. Let $\omega \in \mathcal{P}_Q^0(\mathbb{C}^d)$, $\mathcal{B}$ be a mixed norm space on $\mathbb{C}^d$, and let $A_*(\omega, \mathcal{B})$ be the completion of $P(\mathbb{C}^d)$ under the norm $\| \cdot \|_{A(\omega, \mathcal{B})}$. Then the same arguments as in the proof of Lemma 4.11 give that the kernel of the map $G \mapsto l_G$, from $B(1/\omega, \mathcal{B}')$ to $(A_*(\omega, \mathcal{B}))'$ is equal to $N(1/\omega, \mathcal{B}')$.

We note that this generalizes Lemma 4.11 (2), since if in addition $\nu_2(\mathcal{B}) < \infty$, then $A_*(\omega, \mathcal{B}) = A(\omega, \mathcal{B})$ in view of the first part of Theorem 4.8 (1).

As a consequence of Lemma 4.11 it follows that if $\nu_2(\mathcal{B}) < \infty$, then the surjective and continuous map $G \mapsto l_G$ from $B(1/\omega, \mathcal{B}')$ to $(A(\omega, \mathcal{B}))'$ induces a homeomorphism from the quotient space

$$C(1/\omega, \mathcal{B}') \equiv B(1/\omega, \mathcal{B}')/N(1/\omega, \mathcal{B}')$$

to $(A(\omega, \mathcal{B})){'}$. Here note that Lemma 4.11 implies that $(1/\omega, \mathcal{B}')$ is a Banach space under the usual quotient topology.

Proof of Theorem 4.7. Let $C_w(1/\omega, \mathcal{B}')$ be equal to $C(1/\omega, \mathcal{B}')$ equipped by the induced weak*-topology on $(A(\omega, \mathcal{B})){'}$. For each $G \in B(1/\omega, \mathcal{B}')$ we write $G^* = G$ mod $N(1/\omega, \mathcal{B}')$ for its image in $C(1/\omega, \mathcal{B}')$ under the quotient map. Then it follows that $C_w(1/\omega, \mathcal{B}')$ is a local convex topological vector space, and that the separating vector space of linear functionals on $C_w(1/\omega, \mathcal{B}')$ is equal to

$${\{ \Lambda ; \Lambda(G^*) = l_G(F) \text{ for some } F \in A(\omega, \mathcal{B}) \}}.$$ 

Note here that the equality $\Lambda(G^*) = l_G(F)$ makes sense since Lemma 4.11 gives $l_{G_1}(F) = l_{G_2}(F)$ when $G_1, G_2 \in B(1/\omega, \mathcal{B})$ are two different representatives of $G$ mod $N(1/\omega, \mathcal{B}')$ and $F \in A(\omega, \mathcal{B})$.

Since $\Pi_A(F) = F$ when $F \in A(1/\omega, \mathcal{B}')$, it follows that the map $G \mapsto G$ mod $N(1/\omega, \mathcal{B}')$ from $A(1/\omega, \mathcal{B}')$ to $C(1/\omega, \mathcal{B}')$ is continuous and injective. Let $C_0(1/\omega, \mathcal{B}')$ be the image of this map. The result follows if we prove that $C_0(1/\omega, \mathcal{B}') = C(1/\omega, \mathcal{B}')$.

By Hahn-Banach’s theorem it suffices to prove that if $\Lambda$ is a linear and continuous functional on $C_w(1/\omega, \mathcal{B}')$ which is zero on $C_0(1/\omega, \mathcal{B}')$, then $\Lambda$ is identically zero. Since the dual of $C_w(1/\omega, \mathcal{B}')$ is equal to $A(\omega, \mathcal{B})$ when using the $A^2$ form, we have

$$\Lambda(G^*) = \Lambda_F(G^*) \equiv (G, F)_A^2,$$

for some $F \in A(\omega, \mathcal{B})$.

Now $\Lambda_F(G^*) = 0$ when $G \in A(1/\omega, \mathcal{B}')$. In particular $\Lambda_F(\varepsilon^\alpha) = (\varepsilon^\alpha, F)_A^2 = 0$ for every multi-index $\alpha$. Since $\partial^\alpha F(0) = (F, \varepsilon^\alpha)_A^2$ in view of Remark 4.2, it follows that the entire function $F$ is zero together with all its derivatives at origin. This implies that $F$ is identically zero, and hence, $\Lambda$ is zero. This proves the result. \hfill \Box

Remark 4.13. By Theorem 4.7 and its proof it follows that if $\omega \in \mathcal{P}_Q^0(\mathbb{C}^d)$ is dilated suitable and $\mathcal{B}$ is a mixed norm space such that $\nu_2(\mathcal{B}) < \infty$, then the mappings

$$G \mapsto l_G \quad \text{and} \quad G \mapsto G^*$$

from $A(1/\omega, \mathcal{B}')$ to $(A(\omega, \mathcal{B})){'}$ and from $A(1/\omega, \mathcal{B}')$ to $C(1/\omega, \mathcal{B}')$ respectively, are bijective and continuous. Hence these mappings are homeomorphisms by the
open mapping theorem. In particular, the norms in respective space are equivalent, giving that for some \( C > 0 \) it holds
\[
C^{-1} \| G \|_{(A_\omega, \mathcal{B})'} \leq \| G^* \|_{C(1/\omega, \mathcal{B}')'} \leq \| G \|_{A(1/\omega, \mathcal{B})'} \leq C \| G \|_{(A_\omega, \mathcal{B})'},
\]
when \( G \in A(1/\omega, \mathcal{B})').

The end of the proof of Theorem 4.8. We start to prove the second part of (2). Let \( F \in A(1/\omega, \mathcal{B}) \) be fixed, and choose the vector spaces \( V_j \subseteq \mathbb{C}^d \) and \( p \in [1, \infty) \) such that \( \mathcal{B} = L^p(V) \) when \( V = (V_1, \ldots, V_n) \). Then it follows by straight-forward computations that
\[
\|(G - G(\lambda \cdot), F)_{A_2}\|
\leq C\left| \pi^{-d} \int (G(z)e^{-|z|^2/2})(F(z)e^{-|z|^2/2} - \lambda^{-2d}F(z/\lambda)e^{-|2-\lambda^2|z|^2/(2\lambda^2)})d\lambda(z) \right|
\leq C\|S(F \cdot e^{-1\cdot|z|^2/2} - \lambda^{-2d}F(\cdot / \lambda)e^{-|2-\lambda^2|\cdot}|^2/(2\lambda^2))/\omega\|_{L^p(V)},
\]
where \( C = C_0 \|G\|_{A_\omega(\mathcal{B})} < \infty \), for some constant \( C_0 > 0 \), and \( S \) is the operator in (4.28). By taking \( 2^{-1/2}\pi/\lambda \) as new variables of integration we get
\[
\|(G - G(\lambda \cdot), F)_{A_2}\|
\leq C\|S(F \cdot e^{-1\cdot|z|^2/2} - \lambda^{-2d}F(\cdot / \lambda)e^{-|1-\lambda^2/2|\cdot}|^2)/(2\lambda^2)\omega(S^{-1}(\lambda \cdot))/\omega\|_{L^p(V)},
\]
for some constants \( c \) and \( C_1 \).

Now we set
\[
\Phi_\lambda(z) = (F(\lambda z)e^{-|\lambda z|^2/2} - \lambda^{-2d}F(z)e^{-|1-\lambda^2/2|z|^2}/\omega(2^{1/2}\lambda^2)z)
\]
for the last integrand. By (2.5), we get \( |\Phi_\lambda| \leq \Psi_\lambda \), where
\[
\Psi_\lambda(z) = |F(\lambda z)|e^{-|\lambda z|^2/2}/\omega(2^{1/2}\lambda^2) + C\lambda^{-2d}|F(z)|e^{-|z|^2/2}/\omega(2^{1/2}\lambda^2),
\]
provided the constant \( C \) is chosen sufficiently large. Since \( \Phi_\lambda \to 0 \) pointwise, and
\[
\Psi_\lambda(z) \to (C + 1)F(z)e^{-|z|^2/2}/\omega(2^{1/2}\lambda^2)
\]
in \( L^p(V) \) as \( \lambda \to 1 \)— in view of Lemma 4.10 it follows from Lemma 4.9 that \( \Phi_\lambda \to 0 \) in \( L^p(V) \) as \( \lambda \to 1 \). This implies that the right-hand side of (4.18) turns to zero as \( \lambda \to 1 \), and the second part of (2) follows.

It remains to prove the second part of (1). Let \( G \in A(\omega, \mathcal{B})' \) and \( F \in A(1/\omega, \mathcal{B}) \). By the second part of (2) it suffices to prove that for some \( 0 < \lambda < 1 \) and some polynomials \( G_j \) we have
\[
\|(G(\lambda^2 \cdot) - G_j, F)_{A_2}\| \to 0 \quad \text{as} \quad j \to \infty.
\]

Therefore, let \( G_j \) be a sequence of polynomials on \( \mathbb{C}^d \). By Proposition 4.5 we get \( G(\lambda \cdot)e^{-1\cdot|z|^2/2} \in L^1(\mathbb{C}^d) \), and
\[
\|(G(\lambda^2 \cdot) - G_j, F)_{A_2}\| \leq \|G(\lambda^2 \cdot) - G_j\|_{A(\omega, \mathcal{B})'}\|F\|_{A(1/\omega, \mathcal{B})'}
\leq C \|\|G(\lambda \cdot) - G_j(\cdot / \lambda)\|e^{-1\cdot|z|^2/2}_{L^1}\|F\|_{A(1/\omega, \mathcal{B})'},
\]
(4.20)
for some constant $C$. In the last inequality we have applied Theorem 3.2 with $\mathcal{B}_1 = L^1$ and $\mathcal{B}_2 = \mathcal{B}$ on the estimates

$$|G(\lambda w) - G_j(w/\lambda)|e^{-|w|^2/(2\lambda^2)}\omega(2^{1/2}w/\lambda) \leq C|G(\lambda w) - G_j(w/\lambda)|e^{-|w|^2/2},$$

for some constant $C$.

Consequently, if $\omega_0 = 1$ and $G_j$ are chosen such that $G_j(\cdot/\lambda) \to G(\cdot/\lambda)$ in $A(\omega_0, L^1)$ as $j \to \infty$, then (4.19) follows from (4.20). The proof is complete. \qed

4.2. Narrow convergence. Next we introduce the narrow convergence for $A(\omega, \mathcal{B})$. We note that Theorem 4.8 give no explicit possibilities to approximate elements in $A(\omega, \mathcal{B})$ with polynomials when $\nu_1(\mathcal{B}) = 1$ and $\nu_2(\mathcal{B}) = \infty$. In this context, the narrow convergence makes such approximations possible in some of these situations. The assumptions on the involved weight functions and $\mathcal{B}$ is that the pair $(\mathcal{B}, \omega)$ should be narrowly feasible (cf. Definition 2.5).

In order to define the narrow convergence we introduce the functional

$$J_{F,\omega}(\zeta) \equiv \sup_{\zeta_1 \in V_1} \left(|S(F(z)e^{-|z|^2/2})\omega(z)| \right), \quad \zeta = (\zeta_1, \zeta_2) \in V_1 \oplus V_1^\perp \simeq \mathbb{C}^d,$$

when $F \in A(\omega, \mathcal{B})$. Here we recall that $S$ is the dilatation operator, given by (1.28).

**Definition 4.14.** Let $(\mathcal{B}, \omega)$ be a narrowly feasible space weight pair on $\mathbb{C}^d$, let $p$ and $V$ be the same as in Definition 2.5 and let $q = (p_2, \ldots, p_n)$ and $U = (V_2, \ldots, V_n)$. Also let $F, F_j \in A(\omega, \mathcal{B})$, $j \geq 1$. Then $F_j$ is said to converge to $F$ narrowly as $j \to \infty$, if the following conditions are fulfilled:

1. $S(F_j e^{-|\cdot|^2/2})\omega \to S(F e^{-|\cdot|^2/2})\omega$ in $S'_{1/2}(\mathbb{C}^d)$ as $j \to \infty$;

2. $J_{F_j,\omega,p} \to J_{F,\omega,p}$ in $L^q(U)$ as $j \to \infty$.

The following result gives motivations for introducing the narrow convergence.

**Theorem 4.15.** Let $(\mathcal{B}, \omega)$ be a narrowly feasible pair on $\mathbb{C}^d$. Then the following is true:

1. $P(\mathbb{C}^d)$ is dense in $A(\omega, \mathcal{B})$ with respect to the narrow convergence;

2. if $F \in A(\omega, \mathcal{B})$ and $0 < \lambda < 1$, then $F(\lambda \cdot) \to F$ narrowly as $\lambda \to 1$.

**Proof.** We start by proving (2). We may assume that $\omega = \omega_0$, where $\omega_0$ is the same as in Definition 4.14. Then

$$S(F(\lambda \cdot)e^{-|\cdot|^2/2})\omega \to S(F e^{-|\cdot|^2/2})\omega$$

pointwise and in $S'_{1/2}(\mathbb{R}^d)$ as $\lambda \to 1$.

Furthermore, if $z = (\zeta_1, \zeta_2) \in V_1 \oplus V_1^\perp = \mathbb{C}^d$, then

$$J_{F(\lambda \cdot),\omega,p}(\zeta) \leq C_\lambda J_{F,\omega,p}(\lambda \zeta),$$

where

$$C_\lambda = \sup_{z \in \mathbb{C}^d} \frac{\omega(z) e^{-|\lambda^2| z^2/4}}{\omega(\lambda z)}.$$
Furthermore, by Hölder’s inequality we get

$$\|J_{F(\lambda \cdot), \omega, p} - J_{F_j, \omega, p}\|_{L^p(U)} \leq \|F(\lambda \cdot) - F_j\|_{A(\omega, \mathcal{B})},$$

in view of the definition of $S$. Since $C_\lambda \to 1$ as $\lambda \to 1^-$, and $J_{F, \omega, p}(\lambda \cdot) \to J_{F, \omega, p}$ in $L^q(U)$ as $\lambda \to 1^-$, in view of Lemma 4.10 it follows from Lemma 4.9 that $J_{F(\lambda \cdot), \omega, p} \to J_{F, \omega, p}$ in $L^q(U)$ as $\lambda \to 1^-$. This proves (2).

It remains to prove (1). Let $F \in A(\omega, \mathcal{B})$ and $0 < \lambda < 1$. By Cantor’s diagonal principle it suffices to prove that there is a sequence $F_j$ of polynomials which converges to $F(\lambda \cdot)$ narrowly as $j \to \infty$. Since

$$\|J_{F(\lambda \cdot), \omega, p} - J_{F_j, \omega, p}\|_{L^p(U)} \leq \|F(\lambda \cdot) - F_j\|_{A(\omega, \mathcal{B})},$$

it suffices to prove that $F_j \to F(\lambda \cdot)$ in $A(\omega, \mathcal{B})$. However, this fact is an immediate consequence of (2.5), Proposition 4.5 and (2). The proof is complete.

**Proposition 4.16.** Let $(\mathcal{B}, \omega)$ be a narrowly feasible pair on $\mathcal{C}^d$, $F_j, F \in A(\omega, \mathcal{B})$, $j = 1, 2, \ldots$, be such that $F_j \to F$ narrowly as $j \to \infty$, and let $G \in B(1/\omega, \mathcal{B}')$. Then

$$(F_j, G)_{B^2} \to (F, G)_{B^2} \quad \text{as} \quad j \to \infty.$$ 

**Proof.** We may assume that $\omega = \omega_0$, where $\omega_0$ is the same as in Definition 4.14 and we let $p, q, U$ and $V$ be the same as in Definition 4.14. It follows from the assumptions that

$$\lim_{j \to \infty} F_j(z)\overline{G(z)} e^{-|z|^2} = F(z)\overline{G(z)} e^{-|z|^2}, \quad (4.21)$$

and that

$$|S(F_j \overline{Ge^{-|\cdot|^2}})(z)| \leq J_{F_j, \omega_0, p}(\zeta_2)|S(\overline{Ge^{-|\cdot|^2}})(z)|/\omega(z),$$

$$|S(F \overline{Ge^{-|\cdot|^2}})(z)| \leq J_{F, \omega_0, p}(\zeta_2)|S(\overline{Ge^{-|\cdot|^2}})(z)|/\omega(z). \quad (4.22)$$

Furthermore, by Hölder’s inequality we get

$$J_{F_j, \omega_0, p}(\zeta_2)|S(\overline{Ge^{-|\cdot|^2}})(z)|/\omega(z) \to J_{F, \omega_0, p}(\zeta_2)|S(\overline{Ge^{-|\cdot|^2}})(z)|/\omega(z)$$

in $L^1(\mathcal{C}^d)$ as $j \to \infty$. A combination of (4.21), (4.22) and Lemma 4.3 now implies that $S(F_j \overline{Ge^{-|\cdot|^2}}) \to S(F \overline{Ge^{-|\cdot|^2}})$ in $L^1(\mathcal{C}^d)$ as $j \to \infty$, which implies

$$(F_j, G)_{B^2} = \int F_j(z)\overline{G(z)} \, d\mu(z) \to \int F(z)\overline{G(z)} \, d\mu(z) = (F, G)_{B^2} \quad \text{as} \quad j \to \infty.$$ 

The proof is complete.

### 4.3. General properties for modulation spaces

We finish the section by using Theorem 3.4 in order to carry over basic results on $A(\omega, \mathcal{B})$ spaces into corresponding result for modulation spaces.

The following result is an immediate consequences of Theorems 3.4, 4.7 and 4.8. Here and in what follows we let $\mathcal{H}_0(\mathcal{R}^d)$ be the vector space which consists of all finite linear combinations of Hermite functions.

**Theorem 4.17.** Let $\omega \in \mathcal{P}_0(\mathcal{C}^d)$ be dilated suitable and $\mathcal{B}$ be a mixed norm space on $\mathcal{R}^{2d}$ such that $\nu_2(\mathcal{B}) < \infty$. Then the following is true:

1. the $L^2$ form on $S_{1/2}(\mathcal{R}^d)$ extends uniquely to a continuous sesqui-linear form on $M(\omega, \mathcal{B}) \times M(1/\omega, \mathcal{B}');$
(2) the dual of $M(\omega, \mathcal{B})$ can be identified by $M(1/\omega, \mathcal{B}')$ through the extension of the $L^2$ form on $S_{1/2}(R^d)$;

(3) $\mathcal{J}_0(R^d)$ is dense in $M(\omega, \mathcal{B})$, and dense in $M(\omega, \mathcal{B}')$ with respect to the weak$^*$-topology.

The definition of narrow convergence for elements in certain modulation spaces is the following. Here the functional which corresponds to $J_{f,\omega}$ is

$$H_{f,\omega}(\zeta_2) \equiv \sup_{\zeta_1 \in \Omega_1} (|V_\phi f(x, \xi)| \omega(x, \xi)), \quad z = (\zeta_1, \zeta_2) \in \Omega_1 \oplus \Omega_{1}^\perp = R^{2d},$$

Definition 4.18. Let $(\mathcal{B}, \omega)$ be a narrowly feasible pair on $R^{2d}$, let $p$ and $V$ be the same as in Definition 2.3 and let $q = (p_2, \ldots, p_n)$ and $U = (V_2, \ldots, V_n)$. Also let $f, f_j \in M(\omega, \mathcal{B}), j \geq 1$. Then $f_j$ is said to converge to $f$ narrowly as $j \to \infty$, if the following conditions are fulfilled:

1. $f_j \to f$ in $S'_{1/2}(R^d)$ as $j \to \infty$;
2. $H_{f_j,\omega,p} \to H_{f,\omega,p}$ in $L^q(U)$ as $j \to \infty$.

By (1.30) it follows that $H_{f,\omega} = (2\pi)^{-d/2} J_{f,\omega}$. Consequently, $f_j \to f$ narrowly in $M(\omega, \mathcal{B})$ as $j \to \infty$, if and only if $\mathfrak{M} f_j \to \mathfrak{M} f$ narrowly in $A(\omega, \mathcal{B})$ as $j \to \infty$.

The following result is therefore an immediate consequence of Theorems 3.4, 4.15 and Proposition 4.16.

Theorem 4.19. Let $(\mathcal{B}, \omega)$ be a narrowly feasible pair on $R^{2d}$. Also let $f_j, f \in M(\omega, \mathcal{B}), j = 1, 2, \ldots,$ be such that $f_j \to f$ narrowly as $j \to \infty$, and let $g \in M(1/\omega, \mathcal{B}')$. Then the following is true:

1. $\mathcal{J}_0(R^d)$ is dense in $M(\omega, \mathcal{B})$ with respect to the narrow convergence;
2. $(f_j, g)_{L^2} \to (f, g)_{L^2}$ as $j \to \infty$.

In Section 6 we shall use these results to form a pseudo-differential calculus involving symbols and target distributions in background of the modulation space theory presented here.

5. Some consequences

In this section we show some consequences of the results in the previous sections. We start by establishing continuity properties of $\Pi_A$ on appropriate $B(\omega, \mathcal{B})$ spaces. From these results it follows that $A(\omega, \mathcal{B})$ is a retract of $B(\omega, \mathcal{B})$ under $\Pi_A$. Thereafter we use this property for establishing interpolation properties for $A(\omega, \mathcal{B})$, and continuity properties of Toeplitz operators.

5.1. Continuity properties of $\Pi_A$ on $B(\omega, \mathcal{B})$. We start with the following result related to Lemma 4.11 (3).

Theorem 5.1. Let $\mathcal{B}$ be a mixed norm space on $R^{2d}$ and assume that $\omega \in \mathcal{P}_0(R^{2d})$ is dilated suitable. Then the map $\Pi_A$ is continuous from $B(\omega, \mathcal{B})$ to $A(\omega, \mathcal{B})$. In particular, $A(\omega, \mathcal{B})$ is a retract of $B(\omega, \mathcal{B})$ under $\Pi_A$. 

45
Proof. By Remark 4.12 it follows that the mappings

$$F \mapsto L_F$$ and $$F \mod N(\omega, \mathcal{B})$$

from $$A(\omega, \mathcal{B})$$ and $$B(\omega, \mathcal{B})/N(\omega, \mathcal{B})$$ respectively to

$$\{ l_F; F \in B(\omega, \mathcal{B}) \} \subseteq (A_*(1/\omega, \mathcal{B}'))'$$

are well-defined, continuous and bijective. By the open mapping theorem it follows that the inverse of the map $$F \mapsto F \mod N(\omega, \mathcal{B})$$ is continuous and bijective, and that

$$\|F_0\|_{A(\omega, \mathcal{B})} \leq C \inf_{G \in N(\omega, \mathcal{B})} \|F + G\|_{B(\omega, \mathcal{B})}, \quad F - F_0 \in N(\omega, \mathcal{B}), \quad F_0 \in A(\omega, \mathcal{B}),$$

for some constant $$C$$.

Now let $$F \in B(\omega, \mathcal{B})$$, set $$F_1 = \Pi_A F$$, and choose $$F_0 \in A(\omega, \mathcal{B})$$ such that $$l_{F_0} = l_{F_1}$$. Then $$l_{F_0} = l_{F_1}$$ on $$P(C^d)$$, and Lemma 4.1 and Remark 4.2 shows that $$\partial^\alpha F_0(0) = \partial^\alpha F_1(0)$$ for every multi-index $$\alpha$$. Consequently, $$F_0(z) = F_1(z)$$, since both $$F_0$$ and $$F_1$$ are entire. Furthermore, $$F - F_0 \in N(\omega, \mathcal{B})$$, since the restriction of $$\Pi_A$$ on $$A(\omega, \mathcal{B})$$ is the identity map.

By (5.1) we now get

$$\|\Pi_A F\|_{A(\omega, \mathcal{B})} = \|F_0\|_{A(\omega, \mathcal{B})} \leq C \inf_{G \in N(\omega, \mathcal{B})} \|F + G\|_{B(\omega, \mathcal{B})} \leq C \|F\|_{B(\omega, \mathcal{B})},$$

and the assertion follows. □

We recall that (real and complex) interpolation properties carry over under retracts. These properties also include interpolation techniques with more than two spaces involved. (Cf. [1], [2], [8].) In particular, the following result is an immediate consequence of Theorem 5.1. Here recall that $$(B_1, B_2)_\theta$$ denotes the complex interpolation space with respect to $$\theta \in [0, 1]$$, when $$(B_1, B_2)$$ is a compatible pair.

**Proposition 5.2.** Any interpolation property valid for the $$B(\omega, \mathcal{B})$$ spaces also holds for the $$A(\omega, \mathcal{B})$$ spaces, when the weights $$\omega$$ are dilated suitable and belong to $$(\mathcal{B}_Q^0(C^d))$$, and $$\mathcal{B}$$ are mixed norm spaces on $$C^d$$. In particular, if $$0 \leq \theta \leq 1$$, $$(\omega, \mathcal{B}_1)$$ and $$(\omega, \mathcal{B}_2)$$ are feasible pairs on $$C^d$$, and that $$\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)_\theta$$, for $$\theta \in [0, 1]$$, then

$$(B(\omega, \mathcal{B}_1), B(\omega, \mathcal{B}_2))_\theta = B(\omega, \mathcal{B}) \quad \text{and} \quad (A(\omega, \mathcal{B}_1), A(\omega, \mathcal{B}_2))_\theta = A(\omega, \mathcal{B}).$$

Next we discuss further density properties of $$A(\omega, \mathcal{B})$$. We recall that $$A_*(\omega, \mathcal{B})$$ denotes the completion of $$P(C^d)$$ under the norm $$\| \cdot \|_{B(\omega, \mathcal{B})}$$, when $$\omega \in (\mathcal{B}_Q^0(C^d))$$ and $$\mathcal{B}$$ is a mixed norm space on $$C^d$$. We also let $$B_*(\omega, \mathcal{B})$$ be the completion of $$C_0^\infty(C^d)$$ under the norm $$\| \cdot \|_{B(\omega, \mathcal{B})}$$. The following result links the $$A_*(\omega, \mathcal{B})$$ with $$B_*(\omega, \mathcal{B})$$.  

**Proposition 5.3.** Let $$\omega \in (\mathcal{B}_Q^0(C^d))$$ be dilated suitable, and let $$\mathcal{B}$$ be a mixed norm space on $$C^d$$. Then

$$A(C^d) \cap B_*(\omega, \mathcal{B}) = A(\omega, \mathcal{B}) \cap B_*(\omega, \mathcal{B}) = A_*(\omega, \mathcal{B}).$$
First we prove the first inclusion in (5.2). Since $A_+(\omega, \mathcal{B}) \subseteq A(\omega, \mathcal{B})$, it suffices to prove that $A_+(\omega, \mathcal{B}) \subseteq B_+(\omega, \mathcal{B})$. Let $F \in A_+(\omega, \mathcal{B})$. Then there is a sequence $F_j \in P(\mathbb{C}^d)$ such that $\|F - F_j\|_{A(\omega, \mathcal{B})} \to 0$ as $j \to \infty$. Hence (5.2) follows if we prove that for each $G \in P(\mathbb{C}^d)$, there are elements $G_j \in C_0^\infty(\mathbb{C}^d)$ such that $\|G - G_j\|_{B(\omega, \mathcal{B})} \to 0$ as $j \to \infty$.

Let $\varphi_j \in C_0^\infty(\mathbb{C}^d)$ be chosen such that $0 \leq \varphi_j \leq 1$ and $\varphi_j(z) = 1$ when $|z| \leq j$, and let $G_j = \varphi_jG$. By Hölder’s inequality, there is a constant $C$ such that
\[ \|G - G_j\|_{B(\omega, \mathcal{B})} \leq C \sup_{z \in \mathbb{C}^d} (|G(z) - G_j(z)|e^{-|z|^2/4}) \leq C \left( \sup_{|z| \geq j} |G(z)|e^{-|z|^2/4} \right) \to 0 \]
as $j \to \infty$. This gives the first inclusion in (5.2).

In order to prove the second inclusion we instead assume that $F \in A(\mathbb{C}^d) \cap B_+(\omega, \mathcal{B})$, and we let $F_j \in C_0^\infty(\mathbb{C}^d)$ be a sequence such that
\[ \|F - F_j\|_{B(\omega, \mathcal{B})} \to 0 \quad \text{as} \quad j \to \infty. \] (5.3)
By straightforward computations it follows that $|\Pi_A F_j(z)| \leq C_1 e^{C_2|z|}$, for some constants $C_1 > 0$, which implies that $\Pi_A F_j \in A_P(\mathbb{C}^d) \subseteq A_+(\omega, \mathcal{B})$. Hence Theorem 5.1 and (5.3) give
\[ \|F - \Pi_A F_j\|_{B(\omega, \mathcal{B})} = \|\Pi_A (F - F_j)\|_{B(\omega, \mathcal{B})} \leq C \|F - F_j\|_{B(\omega, \mathcal{B})} \to 0 \quad \text{as} \quad j \to \infty, \]
for some constant $C$. Since any element in $A_P(\mathbb{C}^d)$ can be approximated by elements in $P(\mathbb{C}^d)$ with respect to the norm $A(\omega, \mathcal{B})$, in view of Proposition 4.5 the result follows. The proof is complete.

We may now extend (4.3). In fact, we have the following.

**Proposition 5.4.** Let $(\mathcal{B}, \omega)$ be a feasible pair on $\mathbb{C}^d$, and let $F \in A(\omega, \mathcal{B})$ and $G \in B(1/\omega, \mathcal{B}')$. Then (4.3) holds.

**Proof.** First assume that $\nu_2(\mathcal{B}) < \infty$. Then $A(\omega, \mathcal{B}) = A_+(\omega, \mathcal{B})$. Let $F_j \in P(\mathbb{C}^d)$ be such that $F_j \to F$ in $A(\omega, \mathcal{B})$ as $j \to \infty$. Since $\Pi_A F = F$ and $\Pi_A F_j = F_j$, it follows from Lemma 4.1 and Theorem 5.1 that
\[ (\Pi_A F, G)_{B_2} = (F, G)_{B_2} = \lim_{j \to \infty} (F_j, G)_{B_2} = \lim_{j \to \infty} (F_j, \Pi_A G)_{B_2} = (F, \Pi_A G)_{B_2}, \] (5.4)
and the result follows in this case.

Next we consider the case when $\nu_1(\mathcal{B}) > 1$. Then $B(1/\omega, \mathcal{B}') = B_+(1/\omega, \mathcal{B}')$, and the result follows by similar arguments after approximating $G$ with elements in $C_0^\infty$ and using the fact that
\[ |(\Pi_A G)(z)| \leq C e^{C|z|} \quad \text{for some} \quad C > 0, \]
when $G \in C_0^\infty$ which is needed when applying Lemma 4.1 (cf. (4.3)).

Finally, if $(\mathcal{B}, \omega)$ is narrowly feasible, then we choose a sequence $F_j \in P(\mathbb{C}^d)$ which converges to $F$ narrowly as $j \to \infty$. By Lemma 4.1, Proposition 4.16 and Theorem 5.1 it follows that (5.4) holds also in this case. The proof is complete. \qed
The next result concerns convenient equivalent norms on $A(\omega, \mathcal{B})$.

**Proposition 5.5.** Let $(\mathcal{B}, \omega)$ be a feasible pair on $C^d$, and let
\[ ||F|| \equiv \sup |(F, G)|_{A^2}, \quad F \in A(\omega, \mathcal{B}), \]
where the supremum is taken over all $G \in A(1/\omega, \mathcal{B}^\prime)$ such that $||G||_{A(1/\omega, \mathcal{B}^\prime)} \leq 1$. Then $|| \cdot ||$ is a norm on $A(\omega, \mathcal{B})$ which is equivalent to $|| \cdot ||_{A(\omega, \mathcal{B}^\prime)}$.

**Proof.** By Hölder’s inequality we get $||F|| \leq ||F||_{A(\omega, \mathcal{B})}$. We have to prove that
\[ ||F||_{A(\omega, \mathcal{B})} \leq C||F||, \quad F \in A(\omega, \mathcal{B}), \] (5.5)
for some constant $C$ which is independent of $F \in A(\omega, \mathcal{B})$.

Let $\varepsilon > 0$ be fixed, and let $\Omega$ be the set of all $G \in B(1/\omega, \mathcal{B}^\prime)$ such that $||G||_{B(1/\omega, \mathcal{B}^\prime)} \leq 1$. Then the converse of Hölder’s inequality and Proposition 5.4 give
\[ ||F||_{A(\omega, \mathcal{B})} = \sup_{G \in \Omega} |(F, G)|_{B^2} \leq |(F, G_0)|_{B^2} + \varepsilon = |(F, \Pi A G_0)|_{B^2} + \varepsilon, \]
for some choice of $G_0 \in \Omega$. Since $\Pi A G_0 \in A(1/\omega, \mathcal{B}^\prime)$ and $||\Pi A G_0||_{A(1/\omega, \mathcal{B}^\prime)} \leq C||G_0||_{B(1/\omega, \mathcal{B}^\prime)}$ for some constant $C$, by Theorem 5.1 we obtain
\[ ||F||_{A(\omega, \mathcal{B})} \leq |(F, G_0)|_{B^2} + \varepsilon = |(F, \Pi A G_0)|_{B^2} + \varepsilon \leq C \sup |(F, G)|_{B^2} + \varepsilon, \]
where the supremum is taken over all $G \in \Omega \cap A(1/\omega, \mathcal{B}^\prime)$. Since $\varepsilon > 0$ was arbitrary chosen, (5.5) follows. The proof is complete. □

5.2. Consequences for modulation spaces and Toeplitz operators. Next we use Theorem 3.4 to carry over the previous results for $A(\omega, \mathcal{B})$ spaces into modulation spaces. First we need to investigate how $\Pi A$ is linked to the composition $V_\phi \circ V_\phi^*$, where $V_\phi^*$ is the (Hilbert-) adjoint of $V_\phi$. Here we let $V_\phi^* F$ be the unique element in $S'(1/2; \mathbb{R}^d)$ which satisfies
\[ (V_\phi^* F, g)_{L^2(\mathbb{R}^d)} = (F, V_\phi g)_{L^2(\mathbb{R}^d)}, \quad g \in \mathcal{S}_0(\mathbb{R}^d), \]
when $F \in S'(1/2; \mathbb{R}^d)$. We also let $\Pi_M$ be the continuous operator on $S'(1/2; \mathbb{R}^d)$ given by $\Pi_M = V_\phi \circ V_\phi^*$, and note that $\Pi_M$ is a projection from $S'(1/2; \mathbb{R}^d)$ onto $V_\phi(S'(1/2; \mathbb{R}^d))$.

**Lemma 5.6.** Let $\omega \in \mathcal{B}_Q^0(C^d)$, and let $\mathcal{B}$ be a mixed norm space on $\mathbb{R}^{2d}$. Then
\[ \Pi_A = U_{Q_1} \circ \Pi_M \circ U_{Q_1}^{-1}, \]
on $B(\omega, \mathcal{B})$, where $U_{Q_1}$ is given by (1.31).

**Proof.** Let $F \in B(\omega, \mathcal{B})$, $F_0 = U_{Q_1}^{-1} F$, $g \in \mathcal{S}_0(\mathbb{R}^d)$, $G_0 = V_\phi g$ and $G = U_{Q_1} G_0$. Then $\Pi_M G_0 = G_0$, and
\[ (F, G)_{B^2} = (U_{Q_1} F_0, U_{Q_1} G_0)_{B^2} = (F_0, G_0)_{L^2} = (F_0, V_\phi g)_{L^2} \]
\[ = (V_\phi^* F_0, g)_{L^2} = (\Pi_M F_0, V_\phi g)_{L^2} = (\Pi_M F_0, G_0)_{L^2} \]
\[ = (U_{Q_1} (\Pi_M F_0), G)_{B^2} = ((U_{Q_1} \circ \Pi_M \circ U_{Q_1}^{-1}) F, G)_{B^2}, \]
48
where $U_{\Omega}(\Pi MF_0) = \mathfrak{V}(V_\phi^*F_0)$ is analytic and satisfies (4.2). Furthermore, $(F, G)_{B^2} = (\Pi AF, G)_{B^2}$ in view of Lemma 4.4 (4). A combination of these equalities now gives

$$(U_{\Omega} \circ \Pi M \circ U_{\Omega}^{-1})F, G = (\Pi AF, G)_{B^2}.$$ 

Since also $\Pi \mathcal{A}F$ is analytic it follows from Remark 4.3 that

$$(U_{\Omega} \circ \Pi M \circ U_{\Omega}^{-1})F = \Pi \mathcal{A}F,$$

and the result follows. \hfill \Box

The following result is now an immediate consequence of Theorems 3.4 and 5.1, 5.4 and 5.5, and Lemma 5.6. Here and in what follows we let $\mathcal{B}$ and the result follows.

**Theorem 5.7.** Let $\mathcal{B}$ be a mixed norm space on $C^d$ and assume that $\omega \in \mathcal{P}_Q(R^d)$ is dilated suitable. Then the following is true:

1. $\Pi M$ is continuous from $\mathcal{B}(\omega)$ to $V_\phi(M(\omega, \mathcal{B}))$;
2. $V_\phi$ is continuous from $\mathcal{B}(\omega)$ to $M(\omega, \mathcal{B})$.

In particular, $V_\phi(M(\omega, \mathcal{B}))$ is a retract of $B_{\mathcal{B}}(\omega)$ under $\Pi M$.

The next results are immediate consequences of Theorem 3.4, Propositions 5.2, 5.4 and 5.5, and Lemma 5.6.

**Proposition 5.8.** Any interpolation property valid for the $B(\omega, \mathcal{B})$ spaces also hold for the $M(\omega, \mathcal{B})$ spaces, when the weights $\omega$ are dilated suitable and belong to $\mathcal{P}^0_Q(R^{2d})$, and $\mathcal{B}$ are mixed norm space on $R^{2d}$. In particular, if $0 \leq \theta \leq 1$, $(\omega, \mathcal{B}_1)$ and $(\omega, \mathcal{B}_2)$ are feasible pairs on $R^{2d}$, and that $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)_{[\theta]}$, for $\theta \in [0, 1]$, then

$$(M(\omega, \mathcal{B}_1), M(\omega, \mathcal{B}_2))_{[\theta]} = M(\omega, \mathcal{B}).$$

**Proposition 5.9.** Let $(\mathcal{B}, \omega)$ be a feasible pair on $R^{2d}$, and let $f \in M(\omega, \mathcal{B})$ and $G \in \mathcal{B}'(1/\omega)$. Then

$$(V_\phi f, G)_{L^2(R^{2d})} = (f, V_\phi^*G)_{L^2(R^{2d})}.$$

**Proposition 5.10.** Let $(\mathcal{B}, \omega)$ be a feasible pair on $R^{2d}$, and let

$$\|f\| \equiv \sup \|f, G\|_{L^2}, \quad f \in M(\omega, \mathcal{B}),$$

where the supremum is taken over all $g \in M(1/\omega, \mathcal{B}')$ such that $\|g\|_{M(1/\omega, \mathcal{B}')} \leq 1$. Then $\| \cdot \|$ is a norm on $M(\omega, \mathcal{B})$ which is equivalent to $\| \cdot \|_{M(\omega, \mathcal{B})}$.

Next we consider Toeplitz operators and Berezin-Toeplitz operators. Let $a \in S_{1/2}(R^{2d})$ be fixed. Then the **Toeplitz operator** $T_p(a)$ is the linear and continuous operator on $S_{1/2}(R^{2d})$, defined by the formula

$$(T_p(a)f, g)_{L^2(R^{2d})} = (a V_\phi f, V_\phi g)_{L^2(R^{2d})}. \quad (5.6)$$

There are several characterizations and several ways to extend the definition of Toeplitz and Berezin-Toeplitz operators (see e.g. [5, 7, 13, 14, 17, 23, 31, 32, 47, 54, 58] and the references therein). For example, the definition of $T_p(a)$ is uniquely extendable to every $a \in S_{1/2}')))
Furthermore, it follows from (5.6) that \( T_\omega(a) \) is uniquely extendable to a continuous operator from \( \Upsilon(\mathbb{R}^d) \) to \( S_1^{1/2}(\mathbb{R}^d) \).

Toeplitz operators arise in pseudo-differential calculus in [23,39], in the theory of quantization (Berezin quantization) in [7], and in signal processing in [17] (under the name of time-frequency localization operators or STFT multipliers). There are also strong connection between such operators and Berezin-Toeplitz operators which we shall consider now.

Let \( \mathcal{B}_1, \mathcal{B}_2 \) be mixed norm spaces on \( \mathbb{R}^{2d} \), \( \omega_1, \omega_2 \in \mathcal{P}_Q^0(\mathbb{R}^{2d}) \simeq \mathcal{P}_Q^0(\mathbb{C}^d) \), \( a \in \mathcal{B}_1(\omega_1) \), and \( S \) be as in (1.28). Then the Berezin-Toeplitz operator \( T_\omega(a) \) is the operator from \( A(\omega_2, \mathcal{B}_2) \) to \( A(\mathbb{C}^d) \), given by the formula

\[
T_\omega(a) F = \Pi_A((S^{-1}a)F), \quad F \in A(\omega_2, \mathcal{B}_2).
\]

It follows from (5.6) that if \( a \in S_1^{1/2}(\mathbb{R}^{2d}) \) and \( f \in S_1^{1/2}(\mathbb{R}^d) \), then

\[
(V_\phi \circ T_\omega(a)) f = \Pi_M(a \cdot F_0), \quad \text{where} \quad F_0 = V_\phi f.
\]

Hence

\[
T_\omega(a) \circ \mathcal{U} = \mathcal{U} \circ T_\omega(a), \quad \text{for appropriate} \ a \in S_1^{1/2}(\mathbb{R}^{2d}), \text{by Lemma 5.6.}
\]

We have now the following result. Here we recall that if \( p_j = (p_{j,1}, \ldots, p_{j,n}) \in [1, \infty]^n \), \( j = 0, 1, 2 \), then \( 1/p_1 + 1/p_2 = 1/p_0 \) means that \( 1/p_{1,k} + 1/p_{2,k} = 1/p_{0,k} \) for every \( k = 1, \ldots, n \).

**Proposition 5.11.** Let \( \mathcal{B}_j = L^{p_j}(V) \), \( j = 0, 1, 2 \), be mixed norm space such that \( 1/p_1 + 1/p_2 = 1/p_0 \), and let \( \omega_j \in \mathcal{P}_Q^0(\mathbb{R}^{2d}) \simeq \mathcal{P}_Q^0(\mathbb{C}^d) \) for \( j = 0, 1, 2 \). Also let \( a \in \mathcal{B}_1(\omega_1) \). Then the following is true:

1. the Toeplitz operator \( T_\omega(a) \) is continuous from \( M(\omega_2, \mathcal{B}_2) \) to \( M(\omega_0, \mathcal{B}_0) \);
2. the Berezin-Toeplitz operator \( T_\omega(a) \) is continuous from \( A(\omega_2, \mathcal{B}_2) \) to \( A(\omega_0, \mathcal{B}_0) \).

**Proof.** The assention (2) follows immediately from the definitions and Hölder’s inequality, and (1) is then an immediate consequence of (2) and (5.9). The proof is complete. \( \square \)

### 6. Pseudo-differential operators

In this section we state results on pseudo-differential operators in background of the modulation space theory from the previous sections. The proofs are in general omitted, since the results follows by the same arguments as in [57,59] in combination with the results in previous sections.

#### 6.1. General properties of pseudo-differential operators

We start with the definition of pseudo-differential operators. Let \( t \in \mathbb{R} \) be fixed and let \( a \in S_1^{1/2}(\mathbb{R}^{2d}) \). Then the pseudo-differential operator \( \text{Op}_t(a) \) with symbol \( a \) is the linear and continuous operator on \( S_1^{1/2}(\mathbb{R}^d) \), defined by the formula

\[
(\text{Op}_t(a)f)(x) = (2\pi)^{-d} \int \int a((1 - t)x + ty, \xi) f(y) e^{i(x-y, \xi)} dyd\xi.
\]
The definition of $\text{Op}_t(a)$ extends to each $a \in S^\prime_{1/2}(\mathbb{R}^{2d})$, and then $\text{Op}_t(a)$ is continuous from $S_{1/2}(\mathbb{R}^d)$ to $S^\prime_{1/2}(\mathbb{R}^d)$. (Cf. e.g. [15], and to some extent in [37].) More precisely, for any $a \in S^\prime_{1/2}(\mathbb{R}^{2d})$, the operator $\text{Op}_t(a)$ is defined as the linear and continuous operator from $S_{1/2}(\mathbb{R}^d)$ to $S^\prime_{1/2}(\mathbb{R}^d)$ with distribution kernel given by

$$K_{a,t}(x,y) = (\mathcal{F}_2^{-1}a)((1-t)x + ty, x-y).$$  \hspace{1cm} (6.2)

Here $\mathcal{F}_2 F$ is the partial Fourier transform of $F(x, y) \in S^\prime_{1/2}(\mathbb{R}^{2d})$ with respect to the $y$ variable. This definition makes sense, since the mappings $\mathcal{F}_2$ and $F(x, y) \mapsto F((1-t)x + ty, y-x)$ are homeomorphisms on $S^\prime_{1/2}(\mathbb{R}^{2d})$.

On the other hand, let $T$ be an arbitrary linear and continuous operator from $S(\mathbb{R}^d)$ to $S^\prime(\mathbb{R}^d)$. Then it follows from Theorem 2.2 in [10] that for some $K = K_T \in S^\prime_{1/2}(\mathbb{R}^{2d})$ we have

$$(T f, g)_{L^2(\mathbb{R}^d)} = (K, g \otimes \overline{f})_{L^2(\mathbb{R}^{2d})},$$

for every $f, g \in S(\mathbb{R}^d)$. Now by letting $a$ be defined by (6.2) after replacing $K_{a,t}$ with $K$ it follows that $T = \text{Op}_t(a)$. Consequently, the map $a \mapsto \text{Op}_t(a)$ is bijective from $S^\prime_{1/2}(\mathbb{R}^{2d})$ to $\mathcal{L}(S(\mathbb{R}^d), S^\prime(\mathbb{R}^d))$.

If $t = 1/2$, then $\text{Op}_t(a)$ is equal to the Weyl quantization $\text{Op}^w(a)$ of $a$. If instead $t = 0$, then the standard (Kohn-Nirenberg) representation $a(x, D)$ is obtained.

In particular, if $a \in S^\prime_{1/2}(\mathbb{R}^{2d})$ and $s, t \in \mathbb{R}$, then there is a unique $b \in S^\prime_{1/2}(\mathbb{R}^{2d})$ such that $\text{Op}_s(a) = \text{Op}_t(b)$. By straight-forward applications of Fourier’s inversion formula, it follows that

$$\text{Op}_s(a) = \text{Op}_t(b) \iff b(x, \xi) = e^{i(t-s)(D_x, D_\xi)}a(x, \xi).$$  \hspace{1cm} (6.3)

(Cf. Section 18.5 in [37].) Note here that the right-hand side makes sense, because $e^{i(t-s)(D_x, D_\xi)}$ on the Fourier transform side is a multiplication by the function $e^{i(t-s)(x, \xi)}$, which is a continuous operation on $S^\prime_{1/2}(\mathbb{R}^{2d})$, in view of the definitions.

Let $t \in \mathbb{R}$ and $a \in S^\prime_{1/2}(\mathbb{R}^{2d})$ be fixed. Then $a$ is called a rank-one element with respect to $t$, if the corresponding pseudo-differential operator is of rank-one, i.e.

$$\text{Op}_t(a)f = (f, f_2)_{L^2(\mathbb{R}^d)} f_1,$$  \hspace{1cm} (6.4)

for some $f_1, f_2 \in S^\prime_{1/2}(\mathbb{R}^d)$. Here $f \in S_{1/2}(\mathbb{R}^d)$. By straight-forward computations it follows that (6.4) is fulfilled, if and only if $a = (2\pi)^{d/2} W^t_{f_1, f_2}$, where the $W^t_{f_1, f_2}$ $t$-Wigner distribution, defined by the formula

$$W^t_{f_1, f_2}(x, \xi) \equiv \mathcal{F}(f_1(x + t \cdot) f_2(x - (1-t) \cdot))(\xi),$$

which takes the form

$$W^t_{f_1, f_2}(x, \xi) = (2\pi)^{-d/2} \int f_1(x + ty) f_2(x - (1-t)y) e^{-i(y, \xi)} dy,$$

when $f_1, f_2 \in S_{1/2}(\mathbb{R}^d)$. By combining these facts with (6.3), it follows that

$$W^t_{f_1, f_2} = e^{i(t-s)(D_x, D_\xi)} W^s_{f_1, f_2}.$$
for each $f_1, f_2 \in S'_1(\mathbb{R}^d)$ and $s, t \in \mathbb{R}$. Since the Weyl case is important to us, we set $W_{f_1,f_2} = W_{f_1,f_2}^{t/2}$ when $t = 1/2$. Then $W_{f_1,f_2}$ is the usual (cross-) Wigner distribution of $f_1$ and $f_2$.

Next we discuss the Weyl product, twisted convolution and related objects. Let $a, b \in S'_1(\mathbb{R}^{2d})$ be appropriate. Then the Weyl product $a \# b$ between $a$ and $b$ is the function or distribution which fulfills $\text{Op}_t(a \# b) = \text{Op}_t^w(a) \circ \text{Op}_t^w(b)$, provided the right-hand side makes sense. More generally, if $t \in \mathbb{R}$, then the product $\#_t$ is defined by the formula

$$\text{Op}_t(a \#_t b) = \text{Op}_t(a) \circ \text{Op}_t(b),$$

provided the right-hand side makes sense as a continuous operator from $S_{1/2}(\mathbb{R}^d)$ to $S'_{1/2}(\mathbb{R}^d)$.

The Weyl product can also, in a convenient way, be expressed in terms of the symplectic Fourier transform and twisted convolution. More precisely, the symplectic Fourier transform for $a \in S_{1/2}(\mathbb{R}^{2d})$ is defined by the formula

$$(\mathcal{F}_\sigma a)(X) = \pi^{-d} \int a(Y)e^{2i\sigma(X,Y)}dY.$$ 

Here $\sigma$ is the symplectic form, which is defined by

$$\sigma(X,Y) = \langle y, \xi \rangle - \langle x, \eta \rangle, \quad X = (x, \xi) \in \mathbb{R}^{2d}, \quad Y = (y, \eta) \in \mathbb{R}^{2d},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on $\mathbb{R}^d$, as before.

It follows that $\mathcal{F}_\sigma$ is continuous on $S_{1/2}(\mathbb{R}^{2d})$, and extends as usual to a homeomorphism on $S'_{1/2}(\mathbb{R}^{2d})$, and to a unitary map on $L^2(\mathbb{R}^{2d})$. Furthermore, $\mathcal{F}_\sigma^2$ is the identity operator.

Let $a, b \in S_{1/2}(\mathbb{R}^{2d})$. Then the twisted convolution of $a$ and $b$ is defined by the formula

$$(a \ast_\sigma b)(X) = (2/\pi)^{d/2} \int a(X - Y)b(Y)e^{2i\sigma(X,Y)}dY.$$ 

The definition of $\ast_\sigma$ extends in different ways. For example, it extends to a continuous multiplication on $L^p_v(\mathbb{R}^{2d})$ when $p \in [1, 2]$ when $v \in \mathcal{P}_E(\mathbb{R}^{2d})$ is sub-multiplicative (cf. [52]), and to a continuous map from $S'_{1/2}(\mathbb{R}^{2d}) \times S_{1/2}(\mathbb{R}^{2d})$ to $S'_{1/2}(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d})$. If $a, b \in S'_{1/2}(\mathbb{R}^{2d})$, then $a \# b$ makes sense if and only if $a \ast_\sigma \hat{b}$ makes sense, and then

$$a \# b = (2\pi)^{-d/2}a \ast_\sigma (\mathcal{F}_\sigma b).$$

We also remark that for the twisted convolution we have

$$\mathcal{F}_\sigma(a \ast_\sigma b) = (\mathcal{F}_\sigma a) \ast_\sigma b = \hat{a} \ast_\sigma (\mathcal{F}_\sigma b),$$

where $\hat{a}(X) = a(-X)$ (cf. [51][53]). A combination of (6.6) and (6.7) gives

$$\mathcal{F}_\sigma(a \# b) = (2\pi)^{-d/2}(\mathcal{F}_\sigma a) \ast_\sigma (\mathcal{F}_\sigma b).$$
For admissible $a, b, c \in S'_{1/2}(\mathbb{R}^d)$, it follows by straight-forward computations that
\[(a_1 \ast_x a_2, b) = (a_1, b \ast_x \tilde{a}_2) = (a_2, \tilde{a}_1 \ast_x b), \quad (a_1 \ast_x a_2) \ast_x b = a_1 \ast_x (a_2 \ast_x b)
\]
\[(a_1 \# a_2, b) = (a_1, b \# \tilde{a}_2) = (a_2, \tilde{a}_1 \# b), \quad (a_1 \# a_2) \# b = a_1 \# (a_2 \# b).
\]

6.2. **Pseudo-differential operators and modulation spaces.** Next we consider questions on Weyl quantizations of pseudo-differential operators in the context of modulation space theory. It is then convenient to use the symplectic Fourier transform and the symplectic short-time Fourier transform, instead of corresponding "ordinary" transformations. Here the symplectic short-time Fourier transform of $a \in S'_{1/2}(\mathbb{R}^d)$ with respect to the window function $\Psi \in S'_{1/2}(\mathbb{R}^{2d})$ is defined by
\[\mathcal{V}_\Psi a(X, Y) = \mathcal{F}_\sigma(a \Psi(\.) - X)(Y), \quad X, Y \in \mathbb{R}^d.
\]

Let $(\mathcal{B}, \omega)$ be an admissible pair on $\mathbb{R}^{2d}$. Then $\mathcal{M}(\omega, \mathcal{B})$ denotes the modulation spaces of Gelfand-Shilov distributions on $\mathbb{R}^{2d}$, where the symplectic short-time Fourier transform is used instead of the usual short-time Fourier transform and the window function $\Psi(X)$ here above is equal to
\[\Phi(X) = (2/\pi)^{d/2}e^{-|X|^2},
\]
in the definitions of the norms. In a way similar as for the usual modulation spaces, we set
\[\mathcal{M}^{p, q}_{\omega}(\mathbb{R}^{2d}) = \mathcal{M}(\omega, L^p(\mathbb{R}^{2d})),
\]
when $p, q \in [1, \infty]$. It follows that any property valid for the modulation spaces in previous sections carry over to spaces of the form $\mathcal{M}(\omega, \mathcal{B})$.

The choice of window function is motivated by the simple form that the following results attain. For the proof we refer to the results in previous sections in combination with the proof of Proposition 4.1 in [57].

**Proposition 6.1.** Let $p_j, q_j, p, q \in [1, \infty]$ such that $p \leq p_j, q_j \leq q$, for $j = 1, 2$, and that
\[1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 = 1/p + 1/q.
\]
Also let $\omega_1, \omega_2 \in \mathcal{P}_Q^0(\mathbb{R}^d)$ and $\omega \in \mathcal{P}_Q^0(\mathbb{R}^{2d})$ be such that $(L^{p_j, q_j}(\mathbb{R}^{2d}), \omega_j)$, $j = 1, 2$, and $(L^{p, q}(\mathbb{R}^{2d}), \omega)$ are admissible pairs and satisfy
\[\omega(X, Y) \leq C\omega_1(X - Y)\omega_2(X + Y).
\]
Then the map $(f_1, f_2) \mapsto W_{f_1, f_2}$ from $S'_{1/2}(\mathbb{R}^d) \times S'_{1/2}(\mathbb{R}^d)$ to $S'_{1/2}(\mathbb{R}^{2d})$ restricts to a continuous mapping from $M^{p_j, q_j}_{\omega_1}(\mathbb{R}^d) \times M^{p_2, q_2}_{\omega_2}(\mathbb{R}^d)$ to $M^{p, q}_{\omega}(\mathbb{R}^{2d})$, and
\[\|W_{f_1, f_2}\|_{M^{p, q}_{\omega}} \leq C\|f_1\|_{M^{p_j, q_j}_{\omega_1}} \|f_2\|_{M^{p_2, q_2}_{\omega_2}},
\]
where the constant $C$ is independent of $f_j \in M^{p_j, q_j}_{\omega_j}(\mathbb{R}^{2d})$, $j = 1, 2$.

We now arrive on the following continuity result for pseudo-differential operators. Again we omit the proof, since the result is a straight-forward consequence of Proposition 4.12 in [57] and its proof in combinations with the results on modulation spaces in previous sections.
Theorem 6.2. Let \( p, q, p_j, q_j \in [1, \infty] \) for \( j = 1, 2 \), be such that
\[
\frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{q_2} - \frac{1}{q_1} = \frac{1}{p} + \frac{1}{q} - 1, \quad q \leq p_2, q_2 \leq p. \tag{6.8}
\]
Also let \( \omega \in \mathcal{S}_q^0(\mathbb{R}^d) \) and \( \omega_1, \omega_2 \in \mathcal{S}_q^0(\mathbb{R}^d) \) be such that \((L^{p_j,q_j}(\mathbb{R}^d), \omega_j)\) and \((L^{p, q}(\mathbb{R}^d), \omega)\) are admissible pairs and satisfy
\[
\frac{\omega_2(X - Y)}{\omega_1(X + Y)} \leq C\omega(X, Y), \tag{6.9}
\]
for some constant \( C > 0 \). If \( a \in \mathcal{M}^{p, q}_{\omega}(\mathbb{R}^d) \), then \( \text{Op}^w(a) \) from \( S_{1/2}(\mathbb{R}^d) \) extends uniquely to a continuous map from \( M^{p_1, q_1}_{\omega_1}(\mathbb{R}^d) \) to \( M^{p_2, q_2}_{\omega_2}(\mathbb{R}^d) \).

Moreover, if in addition \( a \) belongs to the closure of \( \mathcal{S}^0(\mathbb{R}^d) \) under the norm \( \| \cdot \|_{\mathcal{M}^{p, q}_{\omega}} \), then \( \text{Op}^w(a) : M^{p_1, q_1}_{\omega_1}(\mathbb{R}^d) \to M^{p_2, q_2}_{\omega_2}(\mathbb{R}^d) \) is compact.

Next we consider algebraic properties of modulation spaces under twisted convolution and Weyl product. These investigations are based on the following lemma together with the observations \( \Phi \# \Phi = \pi^{-d}\Phi \) and \( \Phi \ast \sigma \Phi = 2^d\Phi \) (cf. [53]). We refer to [52, 59] for its proof.

Lemma 6.3. Assume that \( a_1 \in \Upsilon(\mathbb{R}^d), a_2 \in \mathcal{S}_{1/2}(\mathbb{R}^d), \Phi(X) = (2/\pi)^{d/2}e^{-|X|^2} \) and \( X, Y \in \mathbb{R}^d \). Then the following is true:

1. The map
   \[
   Z \mapsto e^{2i\sigma(Z,Y)}(\mathcal{V}_\Phi a_1)(X - Y + Z, Z) (\mathcal{V}_\Phi a_2)(X + Z, Y - Z)
   \]
   belongs to \( L^1(\mathbb{R}^d) \), and
   \[
   \mathcal{V}_\Phi(a_1 \# a_2)(X, Y) = \int e^{2i\sigma(Z,Y)}(\mathcal{V}_\Phi a_1)(X - Y + Z, Z) (\mathcal{V}_\Phi a_2)(X + Z, Y - Z) dZ;
   \]

2. The map
   \[
   Z \mapsto e^{2i\sigma(X - Z, Y - Z)}(\mathcal{V}_\Phi a_1)(X - Y + Z, Z) (\mathcal{V}_\Phi a_2)(Y - Z, X + Z)
   \]
   belongs to \( L^1(\mathbb{R}^d) \), and
   \[
   \mathcal{V}_\Phi(a_1 \ast_\sigma a_2)(X, Y) = \int e^{2i\sigma(X - Z, Y - Z)}(\mathcal{V}_\Phi a_1)(X - Y + Z, Z) (\mathcal{V}_\Phi a_2)(Y - Z, X + Z) dZ.
   \]

\(\text{54}\)
Theorem 6.5. Let
\[ \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_0} = 1 - \left( \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_0} \right), \] (6.10)
\[ 0 \leq \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_j} \leq \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_j}, \quad j = 0, 1, 2, \] (6.11)
and
\[ 0 \leq \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_j} \leq \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_j}, \quad j = 0, 1, 2. \] (6.12)
Furthermore, the involved weight functions should satisfy
\[ \omega_0(X,Y) \leq C \omega_1(X-Y + Z,Z) \omega_2(X+Z,Y-Z), \quad X,Y,Z \in \mathbb{R}^{2d}, \] (6.13)
or
\[ \omega_0(X,Y) \leq C \omega_1(X-Y + Z,Z) \omega_2(Y-Z,X+Z), \quad X,Y,Z \in \mathbb{R}^{2d}. \] (6.14)

**Theorem 6.4.** Let \( \omega_j \in \mathcal{P}_Q^0(\mathbb{R}^{4d}) \) and \( p_j, q_j \in [1, \infty) \) be such that \( (L^{p_j,q_j}(\mathbb{R}^{4d}), \omega_j) \) are admissible pairs for \( j = 0, 1, 2 \), and (6.10), (6.11) and (6.13) hold. Then the map \( (a_1, a_2) \mapsto a_1 \# a_2 \) on \( \mathcal{S}_0(\mathbb{R}^{2d}) \) extends uniquely to a continuous map from \( \mathcal{M}^{p_1,q_1}(\mathbb{R}^{2d}) \times \mathcal{M}^{p_2,q_2}(\mathbb{R}^{2d}) \) to \( \mathcal{M}^{p_0,q_0}(\mathbb{R}^{2d}) \), and
\[ \|a_1 \# a_2\|_{\mathcal{M}^{p_0,q_0}} \leq C\|a_1\|_{\mathcal{M}^{p_1,q_1}}\|a_2\|_{\mathcal{M}^{p_2,q_2}}, \]
where the constant \( C \) is independent of \( a_j \in \mathcal{M}^{p_j,q_j}(\mathbb{R}^{2d}), j = 1, 2. \)

**Theorem 6.5.** Let \( \omega_j \in \mathcal{P}_Q^0(\mathbb{R}^{4d}) \) and \( p_j, q_j \in [1, \infty) \) be such that \( (L^{p_j,q_j}(\mathbb{R}^{4d}), \omega_j) \) are admissible pairs for \( j = 0, 1, 2 \), and (6.10), (6.12) and (6.14) hold. Then the map \( (a_1, a_2) \mapsto a_1 \ast a_2 \) on \( \mathcal{S}_0(\mathbb{R}^{2d}) \) extends uniquely to a continuous map from \( \mathcal{W}^{p_1,q_1}(\mathbb{R}^{2d}) \times \mathcal{W}^{p_2,q_2}(\mathbb{R}^{2d}) \) to \( \mathcal{W}^{p_0,q_0}(\mathbb{R}^{2d}) \), and
\[ \|a_1 \ast a_2\|_{\mathcal{W}^{p_0,q_0}} \leq C\|a_1\|_{\mathcal{W}^{p_1,q_1}}\|a_2\|_{\mathcal{W}^{p_2,q_2}}, \]
where the constant \( C \) is independent of \( a_j \in \mathcal{W}^{p_j,q_j}(\mathbb{R}^{2d}), j = 1, 2. \)

**Remark 6.6.** We note that \( \omega_j, j = 0, 1, 2 \), fulfills all the required properties in Theorem 6.4 if
\[ \omega_0(X,Y) = \frac{\nu_3(X-Y)}{\nu_1(X+Y)}, \quad \omega_1(X,Y) = \frac{\nu_2(X-Y)}{\nu_1(X+Y)}, \] (6.15)
for some appropriate \( \nu_1, \nu_2, \nu_3 \in \mathcal{P}_Q^0(\mathbb{R}^{2d}) \). Note here that such types of conditions appears for \( \omega \) in Theorem 6.2
6.3. Examples on calculi of pseudo-differential operators. Next we give some examples on symbol classes and continuity properties for corresponding pseudo-differential operators. We are especially focused on symbol classes of the form $\mathcal{M}^{\infty,1}_{(\omega)}(\mathbb{R}^{2d})$, because they are to some extend linked to certain classical symbol classes in the pseudo-differential calculus. We have for example that

$$S^{(\omega)}(\mathbb{R}^{2d}) = \bigcap_{N \geq 0} \mathcal{M}^{\infty,1}_{(1/\omega_N)}(\mathbb{R}^{2d}), \quad (6.16)$$

where $\omega \in \mathcal{P}(\mathbb{R}^{2d})$ and $\omega_N(X,Y) = \omega(X)(Y)^{-N}$. Here $S^{(\omega)}(\mathbb{R}^{2d})$ is the symbol class which consists of all $a \in C^\infty(\mathbb{R}^{2d})$ such that $(\partial^a a)/\omega \in L^\infty(\mathbb{R}^{2d})$ for every multi-index $a$. (Cf. [36, Rem. 2.18].)

We also remark that $(6.16)$ can be used to carry over properties valid for modulation spaces into $S^{(\omega)}$ spaces. For example, in [36, Rem. 2.18] it is proved that Theorem 6.4 and $(6.16)$ imply $S^{(\omega)}_1 \# S^{(\omega)}_1 \subseteq S^{(\omega_1 \omega_2)}_1$ when $\omega_1, \omega_2 \in \mathcal{P}$. (See [37, Sec. 18.5] for an alternative proof of the latter fact.)

As a consequence of Theorems 6.2 and 6.4, and Remark 6.6 we have the following.

**Proposition 6.7.** Let $p, q \in [1, \infty]$, $\nu_1, \nu_2, \nu_3 \in \mathcal{P}^0_D(\mathbb{R}^{2d})$ and $\omega_0, \omega_1, \omega_2 \in \mathcal{P}^0_D(\mathbb{R}^{2d})$ be such that $(6.15)$ is fulfilled. Then the following is true:

1. if $a_j \in \mathcal{M}^{\infty,1}_{(\nu_j)}(\mathbb{R}^{2d})$, then the mappings

$$\text{Op}^w(a_1) : M^{p,q}_{(\nu_1)}(\mathbb{R}^{d}) \to M^{p,q}_{(\nu_2)}(\mathbb{R}^{d}), \quad \text{Op}^w(a_2) : M^{p,q}_{(\nu_2)}(\mathbb{R}^{d}) \to M^{p,q}_{(\nu_3)}(\mathbb{R}^{d})$$

are continuous;

2. the map $(a_1, a_2) \mapsto a_1 \# a_2$ is continuous from $\mathcal{M}^{\infty,1}_{(\omega_1)}(\mathbb{R}^{2d}) \times \mathcal{M}^{\infty,1}_{(\omega_2)}(\mathbb{R}^{2d})$ to $\mathcal{M}^{\infty,1}_{(\omega_0)}(\mathbb{R}^{2d})$.

**Corollary 6.8.** Let $p, q \in [1, \infty]$, $\nu \in \mathcal{P}^0_D(\mathbb{R}^{2d})$ and

$$\omega(X,Y) = \frac{\nu(X-Y)}{\nu(X+Y)}. \quad (6.17)$$

Then the following is true:

1. if $a \in \mathcal{M}^{\infty,1}_{(\omega)}(\mathbb{R}^{2d})$, then Op$^w(a)$ is continuous on $M^{p,q}_{(\nu)}(\mathbb{R}^{d})$;

2. $(\mathcal{M}^{\infty,1}_{(\omega)}(\mathbb{R}^{2d}), \#)$ is an algebra.

**Proof.** The result follows by letting $\nu_1 = \nu_2 = \nu_3 = \nu$ and $\omega_0 = \omega_1 = \omega_2 = \omega$ in Proposition 6.7. \qed

**Example 6.9.** Let $\nu(X) = e^{c|X|^\gamma}$, for $0 \leq \gamma < 2$ and some constant $c \in \mathbb{R}$. In this case, $\omega$ in $(6.17)$ is given by

$$\omega(X,Y) = e^{c(|X-Y|^\gamma - |X+Y|^\gamma)}.$$

In the case $\gamma \leq 1$ one may use the inequality $\omega(X,Y) \leq e^{2c|X|^\gamma}$ to conclude that Op$^w(a)$ is continuous on $M^{p,q}_{(\nu)}(\mathbb{R}^{d})$, when $a \in \mathcal{M}^{\infty,1}_{(\omega_0)}$ and $\omega_0(X,Y) = e^{2c|X|^\gamma}$.
More generally, let \( \nu_j(X) = e^{c_j|X|^\gamma} \), for \( 0 \leq \gamma < 2 \) and some constants \( c_j \in \mathbb{R} \), \( j = 1, 2 \). In this case, \( \omega_1 \) in (6.15) is given by
\[
\omega_1(X, Y) = e^{c_2|X-Y|^\gamma - c_1|X+Y|^\gamma}.
\]

In the case \( \gamma \geq 1 \), \( c_1 = 2^{\gamma-1} \) and \( c_2 = 1 \) we have
\[
|X - Y|^\gamma - 2^{\gamma-1}|X + Y| \leq (|X + Y| + 2|Y|)^\gamma - 2^{\gamma-1}|X + Y| \leq 2^{2\gamma-1}|Y|^\gamma,
\]
and \( \omega(X, Y) \leq e^{2^{\gamma-1}|Y|^\gamma} \). Hence, if
\[
\omega_0(X, Y) = e^{2^{\gamma-1}|Y|^\gamma}, \quad \nu_1(X) = e^{2^{\gamma-1}|X|^\gamma}, \quad \nu_2(X) = e^{1|X|^\gamma},
\]
and \( a \in \mathcal{M}_{(\omega_0)}^{\infty,1} \), then \( \text{Op}^w(a) \) is continuous from \( M_{(\nu_1)}^{p,q} \) to \( M_{(\nu_2)}^{p,q} \).

**Example 6.10.** Let \( \nu_j(X) = \langle X \rangle^{c_j(X)} \) or \( \nu_j(X) = \Gamma(X)^{c_j} \) for some constant \( c_j \in \mathbb{R} \), \( j = 1, 2 \). In this case, \( \omega_1 \) in (6.15) is given by
\[
\omega_1(X, Y) = \langle X - Y \rangle^{c_2(X-Y)}\langle X + Y \rangle^{-c_1(X+Y)}
\]
or
\[
\omega_1(X, Y) = (\Gamma((X - Y)))^{c_2} (\Gamma((X + Y)))^{-c_1}.
\]

Hence, if \( a \in \mathcal{M}_{(\omega_1)}^{\infty,1} \), then \( \text{Op}^w(a) \) is continuous from \( M_{(\nu_1)}^{p,q} \) to \( M_{(\nu_2)}^{p,q} \).

We note that different situations appear depending on the sign on \( c_1 \) and \( c_2 \):

1. if \( c_1 > 0 \) and \( c_2 > 0 \), then the weights \( \nu_j(X) \), \( j = 1, 2 \), turn rapidly to infinity at infinity. This implies that the target space \( M_{(\nu_1)}^{p,q} \) as well as the image space \( M_{(\nu_2)}^{p,q} \) are small, in the sense that their elements turns rapidly to zero at infinity, fulfill hard restrictions on oscillations at infinity, and are extendable to entire functions on \( \mathbb{C}^d \).

   The corresponding weight \( \omega_1 \) turns rapidly to zero as \( X = Y \) and \( |X| \to \infty \), while \( \omega_1 \) turns rapidly to infinity as \( X = -Y \) and \( |X| \to \infty \).

2. if \( c_1 < 0 \) and \( c_2 < 0 \) (i.e. the adjoint situation comparing to (1)), then the target space \( M_{(\nu_1)}^{p,q} \) as well as the image space \( M_{(\nu_2)}^{p,q} \) are large, in the sense that their elements are allowed to turns rapidly to infinity at infinity, with small restrictions on oscillations and singularities at infinity.

   The corresponding weight \( \omega_1 \) turns rapidly to zero as \( X = -Y \) and \( |X| \to \infty \), while \( \omega_1 \) turns rapidly to infinity as \( X = Y \) and \( |X| \to \infty \);

3. if \( c_1 < 0 \) and \( c_2 > 0 \), then the target space \( M_{(\nu_1)}^{p,q} \) is large and the image space \( M_{(\nu_2)}^{p,q} \) is small.

   The corresponding weight \( \omega_1 \) turns rapidly to infinity at infinity. Hence, the symbols in \( \mathcal{M}_{(\omega_1)}^{\infty,1} \) turn rapidly to zero at infinity, fulfill hard restrictions on oscillations at infinity, and are extendable to entire functions on \( \mathbb{C}^{2d} \);

4. if \( c_1 > 0 \) and \( c_2 < 0 \), then the target space \( M_{(\nu_1)}^{p,q} \) is small and the image space \( M_{(\nu_2)}^{p,q} \) is large.

57
The corresponding weight \( \omega_1 \) turns rapidly to zero at infinity. Hence, the symbols in \( M^{\infty}_{(\omega_1)} \) are allowed to turn rapidly to infinity at infinity, with small restrictions on oscillations and singularities at infinity.

6.4. The case of moderate weights. It follows from the general results in previous sections that almost all results on pseudo-differential operators in \([56,57]\) can be extended to include weights in the class \( \mathcal{P}_E \). In what follows we state these extensions, and leave most of the verifications for the reader.

We start with the following general form of Feichtinger-Gröchenig’s kernel theorem. The proof is the same as in \([57, \text{Prop. 4.7}]\), where Theorem 2.2 in \([40]\) should be applied instead of the classical Schwartz kernel theorem.

**Proposition 6.11.** Let \( d = d_1 + d_2, \omega_j \in \mathcal{P}_E(\mathbb{R}^{2d_j}) \) for \( j = 1,2 \) and let \( \omega \in \mathcal{P}_E(\mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2}) \) be such that

\[
\omega(x,y,\xi,\eta) = \omega_2(x,\xi)/\omega_1(y,-\eta).
\]

Also let \( T \) be a linear and continuous map from \( S_{1/2}(\mathbb{R}^{d_1}) \to S'_{1/2}(\mathbb{R}^{d_2}) \). Then \( T \) extends to a continuous mapping from \( M^1_{(\omega_1)}(\mathbb{R}^{d_1}) \) to \( M^\infty_{(\omega_2)}(\mathbb{R}^{d_2}) \), if and only if it exists an element \( K \in M^\infty_{(\omega)}(\mathbb{R}^{d_2}) \) such that

\[
(T\phi)(x) = \langle K(x,\cdot),\phi \rangle.
\]

For the proof of the following result we refer to \([57, \text{Prop. 4.8}]\) and its proof.

**Proposition 6.12.** Let \( t \in \mathbb{R}, a \in S'_{1/2}(\mathbb{R}^{2d}) \), and let \( K \in S'_{1/2}(\mathbb{R}^{2d}) \) be the distribution kernel for the pseudo-differential operator \( \mathcal{O}_p(a) \). Also let \( p \in [1, \infty] \), and \( \omega,\omega_0 \in \mathcal{P}_E(\mathbb{R}^{2d_1} \oplus \mathbb{R}^{2d_2}) \) be such that

\[
\omega(x,\xi,\eta,y) = \omega_0(x-ty,x+(1-t)y,-\xi+(1-t)\eta,\xi+t\eta).
\]

Then \( a \in M^p_{(\omega)}(\mathbb{R}^{2d}) \) if and only if \( K \in M^p_{(\omega_0)}(\mathbb{R}^{2d}) \). Moreover, if \( \phi \in S_{1/2}(\mathbb{R}^{2d}) \) and

\[
\psi(x,y) = \int \phi((1-t)x+ty,\xi)e^{j(x-y,\xi)}d\xi,
\]

then \( \|a\|_{M^p_{(\omega)}} = \|K\|_{M^p_{(\omega_0)}} \).

The next result shows that pseudo-differential operators with symbols in modulation are to some extent invariant under the choice of \( t \) in (6.1). We refer to \([57, \text{Prop. 1.7}]\) for the proof. Here we let \( S_\Phi \) be the linear and continuous map on \( S_{1/2}(\mathbb{R}^d) \) and on \( S'_{1/2}(\mathbb{R}^d) \), defined by the formula

\[
f \mapsto S_\Phi f \equiv (e^{j\phi} \otimes \delta_{V_2}) \ast f, \tag{6.18}
\]

where \( \delta_{V_2} \) is the delta function on the vector space \( V_2 \subseteq \mathbb{R}^d \) and \( \Phi \) is a real-valued and non-degenerate quadratic form on \( V_1 = V_2^\perp \).

**Proposition 6.13.** Let \( \phi \in S_{1/2}(\mathbb{R}^d), \omega \in \mathcal{P}_E(\mathbb{R}^{2d}), p, q \in [1, \infty], V_1, V_2 \subseteq \mathbb{R}^d \) be vector spaces such that \( V_2 = V_1^\perp \). Also let \( \Phi \) be a real-valued and non-degenerate
quadratic form on $V_1$, and let $A_\phi/2$ be the corresponding matrix. If $\xi = (\xi_1, \xi_2)$ where $\xi_j \in V_j$ for $j = 1, 2$, then
\[ \|S_\phi f\|_{M_p^{\omega,\phi}} = \|f\|_{M_p^\omega}, \quad \text{where} \quad f \in \mathcal{S}'_{1/2}(\mathbb{R}^d), \]
\[ \omega_\phi(x, \xi) = \omega(x - A_\phi^{-1}\xi, \xi) \quad \text{and} \quad \psi = S_\phi \phi. \]

In particular, the following are true:

1. the map \((6.18)\) on $\mathcal{S}'_{1/2}(\mathbb{R}^d)$ restricts to a homeomorphism from $M_p^{\omega}(\mathbb{R}^d)$ to $M_p^{\omega,\phi}(\mathbb{R}^d)$;

2. if $t \in \mathbb{R}$, $\omega_0 \in \mathcal{P}_E(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$, and
\[ \omega_1(x, \xi, \eta, y) = \omega_0(x - ty, \xi - t\eta, y, \eta), \]
then the map $e^{it(D_x, D_\xi)}$ on $\mathcal{S}'_{1/2}(\mathbb{R}^{2d})$ restricts to a homeomorphism from $M_p^{\omega}(\mathbb{R}^{2d})$ to $M_p^{\omega,\phi}(\mathbb{R}^{2d})$.

By combining Propositions \[6.11, 6.12\] we get the following result (cf. \[57\] Thm. 4.6).

**Theorem 6.14.** Let $t \in \mathbb{R}$ and $p, q, p_j, q_j \in [1, \infty]$ for $j = 1, 2$, be such that \((6.8)\) holds. Also let $\omega \in \mathcal{P}_E(\mathbb{R}^{2d})$ and $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d})$ be such that
\[ \frac{\omega_2(x - ty, \xi + (1 - t)\eta)}{\omega_1(x + (1 - t)y, \xi - t\eta)} \leq C\omega(x, \xi, \eta, y), \quad (6.19) \]
for some constant $C > 0$. If $a \in M_p^{\omega}(\mathbb{R}^{2d})$, then $\text{Op}_t(a)$ from $\mathcal{S}_{1/2}(\mathbb{R}^{2d})$ to $\mathcal{S}'_{1/2}(\mathbb{R}^{2d})$ extends uniquely to a continuous map from $M_p^{\omega_1}(\mathbb{R}^{2d})$ to $M_p^{\omega_2}(\mathbb{R}^{2d})$.

Moreover, if in addition $a$ belongs to the closure of $\mathcal{I}_0(\mathbb{R}^{2d})$ under the norm $\| \cdot \|_{M_p^{\omega_1}}$, then
\[ \text{Op}_t(a) : M_p^{\omega_1}(\mathbb{R}^{2d}) \to M_p^{\omega_2}(\mathbb{R}^{2d}) \]
is compact.

**Theorem 6.15.** Let $t \in \mathbb{R}$, $a \in \mathcal{S}_{1/2}(\mathbb{R}^{2d})$, $\omega \in \mathcal{P}_E(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$, and $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d})$ such that \((6.9)\) holds. Then the following is true:

1. the operator $\text{Op}_t(a)$ from $\mathcal{S}_{1/2}(\mathbb{R}^{2d})$ to $\mathcal{S}'_{1/2}(\mathbb{R}^{2d})$ extends to a continuous mapping from $M^1_\omega(\mathbb{R}^{2d})$ to $M^\infty_\omega(\mathbb{R}^{2d})$, if and only if $a \in M^\infty_\omega(\mathbb{R}^{2d})$;

2. the map $a \mapsto \text{Op}_t(a)$ from $M^\infty_\omega(\mathbb{R}^{2d})$ to the set of linear and continuous operators from $M^1_\omega(\mathbb{R}^{2d})$ to $M^\infty_\omega(\mathbb{R}^{2d})$.

Finally we consider Schatten-von Neumann properties. Let $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbb{R}^{2d})$. Then the set $s_{t,p}(\omega_1, \omega_2)$ consists of all $a \in \mathcal{S}_{1/2}(\mathbb{R}^{2d})$ such that $\text{Op}_t(a)$ belongs to $\mathcal{I}_p(\omega_1, \omega_2)$, the set of Schatten-von Neumann operator of order $p \in [1, \infty]$ from $M^2_\omega(\mathbb{R}^{2d})$ to $M^2_\omega(\mathbb{R}^{2d})$. Note that
\[ \mathcal{I}_1(\omega_1, \omega_2), \quad \mathcal{I}_2(\omega_1, \omega_2) \quad \text{and} \quad \mathcal{I}_\infty(\omega_1, \omega_2), \]
are the sets of trace-class, Hilbert-Schmidt and continuous operators respectively, from $M^2_{(ω)}(R^d)$ to $M^2_{(ω)}(R^d)$. The space $s_{t,p}(ω_1, ω_2)$ is equipped by the norm

$$\|a\|_{s_{t,p}(ω_1, ω_2)} \equiv \|\text{Op}_t(a)\|_{J(ω_1, ω_2)}.$$  

By Theorem 6.15 it follows that the map $a \mapsto \text{Op}_t(a)$ from $s_{t,p}(ω_1, ω_2)$ to $J(ω_1, ω_2)$ is continuous and bijective.

It is easy to obtain a complete characterization of symbols to Hilbert-Schmidt operators. In fact, we have the following result. We refer to [57, Prop4.11] for the proof.

**Proposition 6.16.** Let $a ∈ S'_{1/2}(R^{2d})$, $ω_1, ω_2 ∈ \mathcal{P}_E(R^{2d})$ and that $ω ∈ \mathcal{P}_E(R^{2d} ⊕ R^{2d})$ be such that equality is attained in (6.9) for $t = 1/2$ and some constant $C$. Then $\text{Op}^w(a) ∈ J_2(ω_1, ω_2)$, if and only if $a ∈ M^2_{(ω)}(R^{2d})$. Moreover, for some constant $C > 0$ it holds

$$C^{-1}∥a∥_{M^2_{(ω)}} ≤ ∥a∥_{A^2(ω_1, ω_2)} ≤ C∥a∥_{M^2_{(ω)}},$$

for every $a ∈ S'_{1/2}(R^{2d})$.

We have now the following result.

**Theorem 6.17.** Let $t ∈ R$ and $p, q, p_j, q_j ∈ [1, ∞]$ for $j = 1, 2$, satisfy

$$p_1 ≤ p ≤ p_2, \quad q_1 ≤ \min(p, p') \quad \text{and} \quad q_2 ≥ \max(p, p').$$

Also let $ω ∈ \mathcal{P}_E(R^{2d} ⊕ R^{2d})$ and $ω_1, ω_2 ∈ \mathcal{P}_E(R^{2d})$ be such that equality is attained in (6.19), for some constant $C$. Then

$$M^{p_1,q_1}_{(ω)}(R^{2d}) ⊆ s_{t,p}(ω_1, ω_2) ⊆ M^{p_2,q_2}_{(ω)}(R^{2d})$$

(6.20)

Moreover, for some constant $C > 0$ it holds

$$C^{-1}∥a∥_{M^{p_2,q_2}_{(ω)}} ≤ ∥a∥_{s_{t,p}(ω_1, ω_2)} ≤ C∥a∥_{M^{p_1,q_1}_{(ω)}}$$

for every $a ∈ S'_{1/2}(R^{2d})$.

**Proof.** By Proposition 1.13 and Theorem 6.15 it follows that $s_{t,∞} ⊆ M^∞_{(ω)}$, and by Theorem 6.14 we get $M^∞_{(ω)} ⊆ s_{t,∞}$. By duality we obtain $M^1_{(ω)} ⊆ s_{t,1}$ and $s_{t,1} ⊆ M^1_{(ω)}$. Furthermore, if $p_1 = p_2 = q_1 = q_2 = 2$, then (6.20) follows from and Propositions 6.13 and 6.16. The result now follows for general $p$ by interpolating these cases. The proof is complete.

6.5. A pseudo-differential calculus in the Bargmann-Fock setting. In this section we show some possibilities to establish a pseudo-differential calculus on Banach spaces of analytic functions, in the frame-work of the theory of the Bargmann transform. The definition of the calculus is in some sense similar to the usual pseudo-differential calculus, defined in Section 1 (cf. (6.11)). We show that usual partial differential operators have convenient forms, and remark that the usual calculus in Section 1 to some extent, can be considered as a part of this pseudo-differential calculus on analytic functions.
Before the definition of the calculus on analytic functions, we consider properties of compositions of the Bargmann transform with the Fourier transform, translations or modulations. It is then convenient to introduce some notations.

The **Fourier-Bargmann** transform \( \mathcal{F}_{a,t} \) of any function or distribution \( F \) on \( \mathbb{C}^d \) of order \( t \in \mathbb{R} \) is given by

\[
(\mathcal{F}_{a,t} F)(z) = F(e^{-it\pi/2}z).
\]

We also set \( \mathcal{F}_0 = \mathcal{F}_{a,1} \), and call this map the Fourier-Bargmann transform. We note that \( \mathcal{F}_{a,1}^{-1} = \mathcal{F}_{a,-1} \), and that \( (\mathcal{F}_{a,1}^{-1} F)(z) = F(iz) \). The following lemma shows that the latter formula is strongly related to Fourier’s inversion formula.

**Lemma 6.18.** Let \( \omega \in \mathcal{P}_Q(\mathbb{C}^d) \), \( \mathcal{B} \) be a mixed norm space on \( \mathbb{C}^d \), and set \( \omega_i(z) = \omega(e^{-it\pi/2}z) = (\mathcal{F}_{a,1}\omega)(z) \). Then the following is true:

1. \( \mathcal{F}_{a,t} \) restricts to continuous bijective mappings from \( B(\omega, \mathcal{B}) \) to \( B(\omega_i, \mathcal{B}) \), and \( A(\omega, \mathcal{B}) \) to \( A(\omega_i, \mathcal{B}) \);
2. \( \mathcal{B} \circ \mathcal{F} \) is equal to \( \mathcal{F} \circ \mathcal{B} \) as mappings from \( M(\omega, \mathcal{B}) \) to \( A(\omega_i, \mathcal{B}) \);
3. if \( f \in M(\omega, \mathcal{B}) \), then

\[
(\mathcal{B}(f \cdot - x/\sqrt{2}))(z) = e^{(z,x) - |x|^2/4} (\mathcal{B}f)(z - x),
\]

\[
(\mathcal{B}(e^{i\sqrt{2}(\cdot - \xi)} f))(z) = (\mathcal{B}f)(z + i\xi).
\]

We note that (2) and (3) in Lemma 6.18 in some special cases were proved already in \([3, 22, 27, 30]\).

**Proof.** The assertions (1) and (3) follows immediately from the definitions, and (2) follows by a straight-forward application of Fourier’s inversion formula. The details are left for the reader. \( \square \)

By Lemma 6.18 and the investigations in Section 11, it follows that \( e^{i(x,\xi)} \), \( \mathcal{F} \) and \( dx \) in (6.1) concerning the usual pseudo-differential calculus correspond to \( e^{i(z,w)} \), \( \mathcal{F}_0 \) and \( d\mu(z) \) respectively. The following definition of our complex version of pseudo-differential operators, is based on these observations.

**Definition 6.19.** Let \( t \in \mathbb{R} \) and let \( a \in (S_{1/2}')((\mathbb{C}^d \oplus \mathbb{C}^d) \) be such that

1. \( (a(z,\cdot), e^{-|\cdot|^2-N(z)+\langle w,\cdot\rangle}) \in \mathcal{F}'((\mathbb{C}^d \oplus \mathbb{C}^d) \) for every \( N \geq 0 \);
2. if \( p \in P(\mathbb{C}^d) \), then \( z \mapsto (a(z, i \cdot), e^{-|\cdot|^2+N(z,i \cdot)p}) \) is entire.

Then the (complex) pseudo-differential operator \( Op_{a,t}(a) \) with respect to the symbol \( a \) is the linear operator from \( P(\mathbb{C}^d) \) to \( A(\mathbb{C}^d) \), given by

\[
(\text{Op}_{a,t}(a) F)(z) = \int \int a((1-t)z + tw_1, w_2) F(w_1) e^{i(z,w_2) - (w_2,w_1)} \, d\mu(w_1) d\mu(w_2)
\]

\[
= \int \int a((1-t)z + tw_1, iw_2) F(w_1) e^{i(z,w_2) + (w_2,w_1)} \, d\mu(w_1) d\mu(w_2), \quad (6.21)
\]
when \( F \in P(\mathbb{C}^d) \).

We note that the reproducing kernel in combination with the fact that \( w_1 \mapsto a((1-t)z + tw_1, w_2)F(w_1) \) in (6.21) is analytic and satisfying appropriate conditions give

\[
(\text{Op}_{\mathfrak{V}, t}(a)F)(z) = \int a((1-t)z + tw, w)F(w)e^{i(z,w)}d\mu(w).
\]

If \((w_1, w_2) \mapsto a((1-t)z + tw_1, w_2)F(w_1)e^{i((z,w_2)-(w_2,w_1))}\) in (6.21) is not an integrable function, then \(\text{Op}_{\mathfrak{V}, t}(a)\) is defined as the operator with kernel

\[
(z, w) \mapsto \pi^{-d}\Pi_A(a((1-t)z + tw, i\cdot)e^{i((z,\cdot)-(\cdot,w))}e^{-|w|^2}.
\]

For convenience we also set \(\text{Op}_{\mathfrak{V}} = \text{Op}_{\mathfrak{V}, 0}\).

The following proposition gives motivations for considering operators of the form \(\text{Op}_{\mathfrak{V}, t}(a)\).

**Proposition 6.20.** Let \( N \geq 0 \) be an integer, \( a_\beta \in A(\mathbb{C}^d) \) for every \( \beta \in \mathbb{N}^d \) such that \(|\beta| \leq N\), and let

\[
a(z, w) = \sum_{|\beta| \leq N} a_\beta(z)\overline{w}^\beta.
\]

Then

\[
(\text{Op}_{\mathfrak{V}}(a)F)(z) = \sum_{|\beta| \leq N} a_\beta(z)(D\beta F)(z), \quad F \in P(\mathbb{C}^d).
\]

**Proof.** The result follows by straightforward computations, using Remark 6.21. \(\square\)

**Remark 6.21.** We may use Lemma 6.18 and the mapping properties for the Bargmann transform to reformulate certain pseudo-differential operators of the form \(\text{Op}_t(a)\) into pseudo-differential operators given by Definition 6.19. The details are left for the reader.

**Remark 6.22.** If \(a(z, w) = (S^{-1}b)(w/i)\), then it follows by the definitions that \(\text{Op}_{\mathfrak{V}, t}(a) = T_{\mathfrak{V}}(b)\). Hence the set of Berezin-Toeplitz operators can be considered as a subclass of the Bargmann pseudo-differential operators.

**Remark 6.23.** Let \(a\) fulfills the conditions in Definition 6.19 and assume in addition that \(w \mapsto a(z, w)\) is analytic. Then it follows by the reproducing formula that

\[
(\text{Op}_{\mathfrak{V}, t}(a)F)(z) = a(z, z)F(z),
\]

when \(F \in P(\mathbb{C}^d)\).
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64
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