Abstract

In this work we study the problem of reconstructing shapes from simple nonasymptotic densities measured only along shape boundaries. The particular density we study is also known as the integral area invariant and corresponds to the area of a disk centered on the boundary that is also inside the shape. It is easy to show uniqueness when these densities are known for all radii in a neighborhood of $r = 0$, but much less straightforward when we assume we know it for (almost) only one $r > 0$. We present variations of uniqueness results for reconstruction of polygons and (a dense set of) smooth curves under certain regularity conditions.

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1 Introduction

This work discusses the integral area invariant introduced by Manay et al. [10], particular with regard to reconstructability of shapes. This topic has been considered previously by Fidler, et al. [7] [8] for the case of star-shaped regions. Recent results have shown local injectivity in the neighborhood of a circle [5] and for graphs in a neighborhood of constant functions [6].

The present work discusses does not assume a star-shaped condition, but does make use of a tangent-cone graph-like condition which is local to the integral area circle. We also present an interpretation of the integral area invariant as a nonasymptotic density. This is based on a poster presented by the authors. [9]

We discuss what the derivatives of these nonasymptotic densities represent (section 5) and show that all tangentially graph-like boundaries can be reconstructed given sufficient information about the nonasymptotic density and its derivatives (sections 6 and A). Under our tangent-cone graph-like condition we show all polygons (section 7) and a $C^1$-dense set of $C^2$ boundaries (section 8) are reconstructible.

This is a theoretical paper about a measure that is useful in applications: we do not pretend that the reconstruction techniques in our proofs are practically useful. In fact the reconstructions we use to show uniqueness would be seriously disturbed by the noise that any practical application would encounter. We do, however, comment on some possible approaches to reconstruction (section 9) using the ORTHOMADS direct search algorithm [2] to successfully reconstruct shapes which are not predicted by our theory.

The authors would like to thank David Caraballo for introducing us to integral area invariants as well as Simon Morgan and William Meyerson for initial discussions and work on related topics not in this paper.
2 Notation and preliminaries

Unless otherwise specified, we will be assuming throughout this paper that \( \Omega \subset \mathbb{R}^2 \) is a compact set with simple closed, piecewise continuously differentiable boundary \( \partial \Omega \) of length \( L \). Let \( \gamma : [0, L] \to \partial \Omega \) be a continuous arc-length parameterization of \( \partial \Omega \) (see Figure 1). We will adopt the convention that \( \gamma \) traverses \( \partial \Omega \) in a counterclockwise direction so always keeps the interior of \( \Omega \) on the left (there is no compelling reason for this particular choice, but adopting a consistent convention allows us to avoid some ambiguities later). Note that \( \gamma(0) = \gamma(L) \) and that \( \gamma \) restricted to \( [0, L] \) is a bijection. Denote by \( D(p, r) \) the closed disk and \( C(p, r) \) the circle of radius \( r \) centered at the point \( p \in \mathbb{R}^2 \).

In geometric measure theory, the \( m \)-dimensional density of a set \( A \subset \mathbb{R}^n \) at a point \( p \in \mathbb{R}^n \) is given by

\[
\Theta^m(A, p) = \lim_{r \to 0} \frac{\mathcal{H}^m(A \cap D(p, r))}{\alpha_m r^m}
\]

where \( \mathcal{H}^m \) is the \( m \)-dimensional Hausdorff measure and \( \alpha_m \) is the volume of
the unit ball in $\mathbb{R}^m$ [11]. In the current context, the 2-dimensional density of $\Omega$ at $\gamma(s)$ is simply

$$\Theta^2(\Omega, \gamma(s)) = \lim_{r \to 0} \frac{\text{Area}(\Omega \cap D(\gamma(s), r))}{\pi r^2}.$$  

While we can evaluate this for all $s \in [0, L)$, just knowing the density at every point along the boundary is certainly not sufficient to reconstruct the original shape (e.g., any two shapes with $C^1$ boundary yield a density of $\frac{1}{2}$ along the entire boundary and thus be indistinguishable).

One natural question to ask (and the focus of the present work) is whether failing to pass to the limit (i.e., using some fixed radius $r$) and collecting the values for all points along the boundary preserves enough information to reconstruct the original shape. That is, can a nonasymptotic density be used as a signature for shapes?

**Definition 1.** In the current context, the integral area invariant [10] is denoted by $g : [0, L) \times \mathbb{R}^+ \to \mathbb{R}^+$ and given by

$$g(s, r) = \int_{D(\gamma(s), r) \cap \Omega} dx = \text{Area}(D(\gamma(s), r) \cap \Omega).$$

**Remark 2.** Note the lack of the normalizing factor $\pi r^2$ in the definition of $g(s, r)$. Since we presume that $r$ is fixed and known for the situations we study, it’s trivial to convert data between the forms $g(s, r)$ and $\frac{g(s, r)}{\pi r^2}$; we choose to leave out the normalizing factor in the definition of $g(s, r)$ as it is the integral area invariant of Manay et al. [10] and this form proves useful when computing derivatives in section 5.

**Definition 3.** For a fixed radius $r$, we say that $\partial \Omega$ is graph-like (GL) at a point $p \in \partial \Omega$ (or graph-like on $D(p, r)$) if it is possible to impose a Cartesian coordinate system such that the set of points $\partial \Omega \cap D(p, r)$ is the graph of some function $f$ in this coordinate system. Without loss of generality, we adopt the convention that $p$ is the origin so that $f(0) = 0$. We define tangentially graph-like (TGL) in the same way but further require that $\partial \Omega$ be continuously differentiable and $f'(0) = 0$ (noting that $f$ is $C^1$ because $\partial \Omega$ is). This is illustrated in figure 2(a). Without loss of generality (and in keeping with our convention that $\gamma$ traverses $\partial \Omega$ counterclockwise), we assume that the interior of $\Omega$ is “up” in the circle (i.e., that $(0, \epsilon) \in \Omega$ for sufficiently small
\( \epsilon > 0 \). If \( \partial \Omega \) is (tangentially) graph-like on \( D(p,r) \) for all \( p \in \partial \Omega \), we say that \( \partial \Omega \) is (tangentially) graph-like for radius \( r \).

**Definition 4.** Given a piecewise \( C^1 \) function \( \gamma : \mathbb{R} \to \mathbb{R}^2 \), we define the tangent cone of \( \gamma \) at a point \( s \) (which we denote by \( T_\gamma(s) \)) in terms of the one-sided derivatives. In particular, we let \( T_\gamma(s) = \{ \alpha \Gamma^- + \beta \Gamma^+ | \alpha, \beta \geq 0, \alpha + \beta > 0 \} \) where \( \Gamma^- = \lim_{t \uparrow s} \gamma'(t) \) and \( \Gamma^+ = \lim_{t \downarrow s} \gamma'(t) \).

**Definition 5.** We extend the tangentially graph-like notion to boundaries that are piecewise \( C^1 \) by defining \( \partial \Omega \) to be tangent-cone graph-like (TCGL) at a point \( \gamma(s) \in \partial \Omega \) if it is graph-like at \( \gamma(s) \) for every orientation in the tangent cone of \( \partial \Omega \) at \( s \). More precisely, for every \( w \in T_\gamma(s) \) and every pair of distinct points \( u, v \in \partial \Omega \cap D(p,r) \), we have \( \langle w, u - v \rangle \neq 0 \). See figure 2(b).

**Remark 6.** It is clear that \( T_\gamma(s) \) in definition 4 is a convex cone. The tangent cone is dependent on the direction in which \( \gamma \) traverses \( \partial \Omega \) (which by convention was counterclockwise) since an arc-length traversal \( \hat{\gamma}(s,r) = \gamma(L - s, r) \) would have different tangent cones (namely, \( w \in T_\gamma(s) \) iff \( -w \in T_{\hat{\gamma}}(s) \)). However, these differences are irrelevant to the application of definition 5.

**Remark 7.** Note that the definitions of tangentially graph-like and tangent-cone graph-like coincide when \( \partial \Omega \) is \( C^1 \) so every tangentially graph-like boundary is tangent-cone graph-like.
Lemma 8. Let $r \in \mathbb{R}^+$ and $p \in \partial \Omega$. If $\partial \Omega$ is graph-like on $D(p, r)$, then $|\partial \Omega \cap C(p, r)| \geq 2$.

Proof. Suppose by way of contradiction that $|\partial \Omega \cap C(p, r)| < 2$. Since $\partial \Omega$ is a simple closed curve, we have $\partial \Omega \subseteq D(p, r)$. As $\partial \Omega$ is graph-like at $p$ with radius $r$, there exists some orientation for which $\partial \Omega \cap D(p, r) = \partial \Omega$ is the graph of a well-defined function. However, $\partial \Omega$ is a simple closed curve so is not the graph of a function in any orientation, yielding a contradiction.

Theorem 9. If $\partial \Omega$ is tangent-cone graph-like with radius $r \in \mathbb{R}^+$ at $p \in \partial \Omega$, then $|\partial \Omega \cap C(p, r)| = 2$ and $\partial \Omega$ crosses $C(p, r)$ transversely at these points. As a result, for every $q_1, q_2 \in \partial \Omega \cap D(p, r)$, there is a unique arc along $\partial \Omega$ between them in $D(p, r)$.

Proof. By Lemma 8, we have that $|\partial \Omega \cap C(p, r)| \geq 2$. Note that $\partial \Omega$ contains an interior point ($p$) and at least two boundary points of the disk $D(p, r)$ (since $|\partial \Omega \cap C(p, r)| \geq 2$). As $\partial \Omega$ is connected and simply closed, there must exist an arc of $\partial \Omega$ within the disk going from some point on $C(p, r)$ through $p$ to another point on $C(p, r)$.

Suppose $|\partial \Omega \cap C(p, r)| > 2$; that is, there are other points of intersection. Letting $q$ denote one of these, there are two cases to consider.

(a) $\partial \Omega$ does not cross $C(p, r)$ at $q$.

As $\partial \Omega$ is tangent-cone graph-like at $q$, then $\partial \Omega \cap C(q, r)$ is a graph in every orientation in the tangent cone of $\partial \Omega$ at $q$. In particular, note
that the tangent line to $C(p, r)$ at $q$ is in this cone. However, the line from $p$ to $q$ is normal to this line and thus $\partial \Omega \cap C(q, r)$ is not graph-like in this orientation, a contradiction. Therefore, this case cannot occur. This argument applies to all points in $\partial \Omega \cap C(p, r)$ so we immediately have the result that $\partial \Omega$ always crosses $C(p, r)$ transversely.

(b) $\partial \Omega$ crosses $C(p, r)$ at $q$.

There exists $q' \in \partial \Omega \cap C(p, r)$ such that there is a path along $\partial \Omega$ in $D(p, r)$ from $q$ to $q'$. That is, there exist $s_1, s_2 \in [0, L)$ (without loss of generality, $s_1 < s_2$) such that $\gamma(s_1) = q$, $\gamma(s_2) = q'$ and the image of $[s_1, s_2]$ under $\gamma$ is contained in $D(p, r)$ (but does not include $p$, since it is on another arc and $\partial \Omega$ is simple). Thus $\gamma$ enters $C(p, r)$ at $s_1$ and exits at $s_2$.

If we can find $s \in [s_1, s_2]$ and $w$ in the tangent cone of $\partial \Omega$ at $\gamma(s)$ satisfying $\langle w, p - \gamma(s) \rangle = 0$, we will contradict that $\partial \Omega$ is tangent-cone graph-like.

Define $v : [s_1, s_2] \rightarrow \mathbb{R}^2$ by

$$v(s) = \begin{cases} 
\lim_{t \downarrow s_1} \gamma'(s), & s = s_1, \\
\lim_{t \uparrow s} \gamma'(s), & s \in (s_1, s_2].
\end{cases}$$

Note that $v(s)$ is in the tangent cone of $\partial \Omega$ at $\gamma(s)$ so that $\partial \Omega \cap D(\gamma(s), r)$ is graph-like using the orientation given by $v(s)$.

Define $\phi(s) : [s_1, s_2] \rightarrow \mathbb{R}$ by $\phi(s) = \langle v(s), p - \gamma(s) \rangle$. Note that from $\gamma(s_1)$ both $v(s_1)$ and $p - \gamma(s_1)$ are directions pointing into the circle so $\phi(s_1) > 0$. Similarly, $v(s_2)$ points out and $p - \gamma(s_2)$ points in so that $\phi(s_2) < 0$.

Observe that $v$ (and therefore $\phi$) is piecewise continuous since $\gamma$ is piecewise $C^1$. By a piecewise continuous analogue of the intermediate value theorem, there exists $\bar{s} \in [s_1, s_2]$ such that

$$\lim_{t \rightarrow \bar{s}^-} \phi(t) \leq 0 \leq \lim_{t \rightarrow \bar{s}^+} \phi(t).$$

By continuity of the inner product and $\gamma$, we have

$$\lim_{t \rightarrow \bar{s}^-} \phi(t) = \langle \lim_{t \rightarrow \bar{s}^-} \gamma'(t), p - \gamma(\bar{s}) \rangle.$$
Similarly, \( \lim_{t \to \bar{s}^+} \phi(t) = \langle \lim_{t \to \bar{s}^+} \gamma'(t), p - \gamma(\bar{s}) \rangle \)

If \( \gamma \) is differentiable at \( \bar{s} \), then \( \phi(\bar{s}) = \lim_{t \to \bar{s}^+} \phi(t) = 0 \) and we have our contradiction. Otherwise, let \( w_1 = \lim_{t \to \bar{s}^-} \gamma'(t) \) and \( w_2 = \lim_{t \to \bar{s}^+} \gamma'(t) \).

As both \( w_1 \) and \( w_2 \) are in the convex tangent cone of \( \partial \Omega \) at \( \gamma(\bar{s}) \), any positive linear combination of them is as well. Letting \( \psi(\lambda) = \lambda w_1 + (1 - \lambda)w_2 \), we have

\[
\langle \psi(0), p - \gamma(\bar{s}) \rangle \leq 0 \leq \langle \psi(1), p - \gamma(\bar{s}) \rangle.
\]

Noting that \( \psi \) is continuous in \( \lambda \), we apply the intermediate value theorem to obtain \( \bar{\lambda} \in (0, 1) \) such that \( \langle \psi(\bar{\lambda}), p - \gamma(\bar{s}) \rangle = 0 \). Letting \( w = \psi(\bar{\lambda}) \), we obtain our contradiction.

Therefore, there are no other points of intersection and \( |\partial \Omega \cap C(p, r)| = 2 \). \( \square \)

**Lemma 10.** Suppose \( \partial \Omega \) is tangent-cone graph-like on \( D(\gamma(s), r) \) and \( s_1 < s_2 \) such that \( \gamma(s_1), \gamma(s_2) \in D(\gamma(s), r) \). Further suppose that \( w_1 \in T_\gamma(s_1) \), \( w_2 \in T_\gamma(s_2) \) and \( \alpha \in (0, 1) \), and let \( w' = \alpha w_1 + (1 - \alpha)w_2 \). There exists \( s' \in [s_1, s_2] \) such that either \( w' \) or \( -w' \) is in \( T_\gamma(s') \).

**Proof.** Let \( n \) be a unit vector in \( \mathbb{R}^2 \) with \( n \perp (\alpha w_1 + (1 - \alpha)w_2) \). We have \( \alpha \langle n, w_1 \rangle = -(1 - \alpha) \langle n, w_2 \rangle \). It suffices to consider only \( \langle n, w_1 \rangle \leq 0 \leq \langle n, w_2 \rangle \) as the argument is identical in the other case. Note that since \( 0 \leq \langle n, w_2 \rangle = c_1 \langle n, \lim_{t \to s_2} \gamma'(t) \rangle + c_2 \langle n, \lim_{t \to s_2} \gamma'(t) \rangle \) for nonnegative constants \( c_1, c_2 \) not both zero, at least one of the inner products on the right is nonnegative.

Using the notation of definition 4, we define \( M_2 = \arg\max_{\Gamma \in \{\Gamma^+, \Gamma^-\}} \langle n, \Gamma \rangle \) and have \( \langle n, M_2 \rangle \geq 0 \). We similarly define \( M_1 \) with respect to \( w_1 \) such that \( \langle n, M_1 \rangle \leq 0 \).

Define

\[
v(t) = \begin{cases} 
M_i, & t = s_i, i = 1, 2 \\
\lim_{t \uparrow t} \gamma'(s) & \text{otherwise}
\end{cases}
\]

and \( \phi(t) = \langle n, v(t) \rangle \). Since \( \phi(s_1) \leq 0 \leq \phi(s_2) \), the argument proceeds as in theorem 9 to yield \( \tilde{s} \in [s_1, s_2] \) and \( \tilde{w} \in T_\gamma(\tilde{s}) \) such that \( \langle n, \tilde{w} \rangle = 0 \). Thus \( \tilde{w} = k w' \) for some \( k \neq 0 \). In particular, \( w' = \frac{1}{k} \tilde{w} \) so either \( w' \in T_\gamma(\tilde{s}) \) or \(-w' \in T_\gamma(\tilde{s}) \) (depending on the sign of \( k \)). \( \square \)
Lemma 11. Suppose $\gamma : [a,b] \to \mathbb{R}^2$ is a simple, arc-length parameterized curve with piecewise continuous derivative defined on $(a,b)$ except possibly on finitely many points. Further suppose that the image of $\gamma$ has no cusps. Then there exists $c$ in $(a,b)$ such that either $\gamma(b) - \gamma(a)$ or $-(\gamma(b) - \gamma(a))$ is in $T_\gamma(c)$.

Proof. Let $n$ be a unit vector with $\langle \gamma(b) - \gamma(a), n \rangle = 0$. Consider $\psi(t) = \gamma(t)$ and note that $\psi'(t) = \langle \gamma'(t), n \rangle$ is defined wherever $\gamma(t)$ is differentiable. We have $\int_a^b \psi'(t) = \psi(b) - \psi(a) = \langle \gamma(b) - \gamma(a), n \rangle = 0$. Thus either $\psi'(t) = 0$ everywhere it is defined or it takes on both positive and negative values. In particular, there exists a point $c \in (a,b)$ such that either $\psi'(c) = 0$ or $\lim_{t \downarrow c} \psi'(t) \leq 0 \leq \lim_{t \uparrow c} \psi'(t)$.

If $\psi'(c) = 0$, then we have $\langle \gamma'(c), n \rangle = 0$ so that $\gamma'(c) = k(\phi(b) - \phi(a))$ for some $k \neq 0$. As $\gamma'(c) \in T_\gamma(c)$, we have $\frac{k}{|k|}(\phi(b) - \phi(a)) \in T_\gamma(c)$ which gives us our conclusion.

If $\lim_{t \downarrow c} \psi'(t) \leq 0 \leq \lim_{t \uparrow c} \psi'(t)$, there exists $\alpha \in (0,1)$ such that $0 = \alpha \lim_{t \downarrow c} \psi'(t) + (1 - \alpha) \lim_{t \uparrow c} \psi'(t)$. Note that $\lim_{t \downarrow c} \psi'(t) = \langle w_1, n \rangle$ and $\lim_{t \uparrow c} \psi'(t) = \langle w_2, n \rangle$ for some $w_1, w_2 \in T_\gamma(c)$ and let $w' = \alpha w_1 + (1 - \alpha) w_2$.

By the convexity of $T_\gamma(c)$, we have $w' \in T_\gamma(c)$ with $\langle w', n \rangle = 0$ which follows as in the previous case. \hfill \Box

Lemma 12. If $\partial \Omega$ is tangent-cone graph-like for some radius $r$, then $\partial \Omega$ has no cusps.

Proof. Suppose $\partial \Omega$ has a cusp at $\gamma(s)$. Then, using the terminology of definition 4 and that $\gamma$ is arc length parameterized, we have $\Gamma^+ = -\Gamma^-$. We let $w = 0$ and note that $w = \Gamma^+ + \Gamma^- \in T_\gamma(s)$. Letting $u, v \in \partial \Omega \cap D(\gamma(s), r)$ with $u \neq v$, we have $\langle w, u - v \rangle = 0$, contradicting that $\partial \Omega$ is tangent-cone graph-like. Therefore $\partial \Omega$ has no cusps. \hfill \Box

Lemma 13. Suppose $\partial \Omega$ is tangent-cone graph-like with radius $r$ and points $p_1, p_2 \in \partial \Omega$ with $d(p_1, p_2) < r$. Then one of the arcs (call it $P$) along $\partial \Omega$ between $p_1$ and $p_2$ is such that for any two points $q_1, q_2 \in P$, we have $d(q_1, q_2) < r$. 

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Proof. Note that \( p_2 \in D(p_1, r) \) so that there is an arc along \( \partial \Omega \) from \( p_1 \) to \( p_2 \) which is fully contained in the interior of \( D(p_1, r) \) by theorem 9. We will call this arc \( P \).

For all \( x \) on \( P \), let \( P_x \) denote the subpath of \( P \) from \( p_1 \) to \( x \) (so \( P = P_{p_2} \)). We claim that \( P_x \) is contained in \( D(x, r) \) for all \( x \) on \( P \) (thus \( P \) is contained in \( D(p_2, r) \)). Indeed, if this were not the case, then there must be some \( \hat{x} \) on \( P \) such that \( P_{\hat{x}} \) is contained in \( D(\hat{x}, r) \) but \( C(\hat{x}, r) \cap P_{\hat{x}} \) is nonempty (i.e., we can move the disk along \( P \) until some part of the subpath hits the boundary). That is, the subpath \( P_{\hat{x}} \) has a tangency with the disk \( D(\hat{x}, r) \) which is impossible because of theorem 9.

Let \( q_1 \in P \) and note that since \( P_x \) is contained in \( D(x, r) \) for all \( x \) on \( P \), we have that \( P \) is contained in \( D(q_1, r) \). Therefore, \( d(q_1, q_2) < r \) for all \( q_1, q_2 \in P \) as desired.

**Lemma 14.** If \( q_1 = \gamma(s_1), q_2 = \gamma(s_2) \in P \) where \( P \) is as in the previous lemma, then the arc length between \( q_1 \) and \( q_2 \) along \( P \) is at most \( \sqrt{2}d(q_1, q_2) \).

Proof. Since \( \Omega \) is tangentially graph-like, for any \( w_1 \in T_{\gamma}(s_1), w_2 \in T_{\gamma}(s_2) \), the angle between \( w_1 \) and \( w_2 \) is at most \( \frac{\pi}{2} \). Since this is true for all \( q \in P \), there is a point \( q' = \gamma(s') \in P \) and \( w' \in T_{\gamma}(s') \) such that the angle between \( w' \) and tangent vectors for any other point \( q \in P \) is at most \( \frac{\pi}{4} \).

This means that \( P \) the graph of a Lipschitz function \( g \) of rank 1 in the orientation defined by \( w' \). This does not necessarily imply that \( D(q', r) \cap \partial \Omega \), \( D(p_1, r) \cap \partial \Omega \) or \( D(p_2, r) \cap \partial \Omega \) is the graph of a Lipschitz function; we explore a Lipschitz condition for the disks in section 4. Let \( x_1, x_2 \in [-r, r] \) with \( p_1 = (x_1, g(x_1)), p_2 = (x_2, g(x_2)) \). Then the arclength from \( p_1 \) to \( p_2 \) is given by

\[
\int_{x_1}^{x_2} \sqrt{1 + g'(x)^2} \, dx \leq \int_{x_1}^{x_2} \sqrt{2} \, dx = \sqrt{2}(x_2 - x_1) \leq \sqrt{2}d(p_1, p_2). \]

Suppose \( \partial \Omega \) is tangent-cone graph-like with radius \( r \) and we have some \( s \in [0, L] \) such that \( \partial \Omega \) is tangentially graph-like at \( \gamma(s) \) with radius \( r \). Since \( \partial \Omega \) is TGL at \( \gamma(s) \), it has two points of intersection with \( C(\gamma(s), r) \) by theorem
9. In the orientation forced by the TGL condition, one of these points of intersection must be on the right side of the circle and one must be on the left side.

With reference to figure 4 we define $s^+(s)$ and $s^-(s) \in [0, L)$ so that $\gamma(s^+(s))$ is the point of intersection on the right and $\gamma(s^-(s))$ is the point of intersection on the left. The notation is motivated by the fact that $0 < s^-(s) < s < s^+(s) < L$ in general due to our convention that $\gamma$ traverses $\partial \Omega$ counterclockwise. The only case where this is not true is when $\gamma(L) = \gamma(0)$ is in the disk but even then it will hold for a suitably shifted $\hat{\gamma}$ that starts at some point outside the current disk.

The quantities $\theta_1(s)$ and $\theta_2(s)$ are the angles that the rays from the origin to the right and left points of intersection respectively make with the positive $x$ axis. We can assume $\theta_1(s) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\theta_2(s) \in (\frac{\pi}{2}, \frac{3\pi}{2})$.

We define $\nu_1(s)$ as the angle between the vector $\gamma(s^+(s)) - \gamma(s)$ and the vector $\lim_{t \uparrow s^+(s)} \gamma'(t)$, the one-sided tangent to $\partial \Omega$ at the point of intersection on the right. That is, we are measuring the angle between the outward normal to the disk at the point of intersection and the actual direction $\gamma$ is going as it exits the disk. We define $\nu_2(s)$ similarly. We have $\nu_1, \nu_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ due to

Figure 4: notation
the fact that all circle crossings are transverse by theorem 9.

When the proper $s$ to use is implied by context, we will often simply write $s^+$, $s^-$, $\theta_1$, $\theta_2$, $\nu_1$ and $\nu_2$.

3 Two-arc property

Definition 15. We say that $\Omega$ has the two-arc property for a given radius $r$ if for every point $p \in \partial \Omega$, we have that $D(p, r)$ divides $\partial \Omega$ into two connected arcs: $\partial \Omega \cap D(p, r)$ and $\partial \Omega \setminus D(p, r)$. Equivalently, $\Omega$ has the two-arc property if the circle $C(p, r)$ is divided into two connected arcs by $\partial \Omega$ for every $p \in \partial \Omega$.

Lemma 16. If $\Omega$ is tangent-cone graph-like for some radius $r$, then it has the two-arc property.

Proof. This is a trivial consequence of Theorem 9.

Corollary 17. If $\Omega$ is tangentially graph-like for some radius $r$, then it has the two-arc property for radius $r$.

Remark 18. While the assumption of the two-arc property for disks of radius $r = \hat{r}$ does not imply the two-arc property for all $r < \hat{r}$ (see Figure 5), it is the case that TGL for $r = \hat{r}$ does imply that $\gamma$ is TGL for all $0 < r < \hat{r}$. That $\gamma$ is TGL for all $0 < r < \hat{r}$ follows easily from the definition of TGL and the fact that $D(p, r) \subseteq D(p, \hat{r})$. 

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Figure 5: The two-arc property for $r = \hat{r}$ does not imply it holds for all $r < \hat{r}$
4 TCGL polygonal approximations

If $\Omega$ is tangent-cone graph-like with radius $r$, it can sometimes be nice to know that there is an approximating polygon to $\Omega$ which is also tangent-cone graph-like. The following lemmas explore this idea.

**Lemma 19.** If $\partial \Omega$ is TCGL with radius $r$ then for each $\epsilon \in (0, r)$, then there is a polygonal approximation to $\partial \Omega$ that is TCGL with radius $r - \epsilon$ and such that every point on $\partial \Omega$ is within distance $\frac{\epsilon}{6}$ of the polygon.

*Proof.* First, choose a finite number of points along the boundary such that the arc length along $\gamma$ between any two neighboring points is no more than $\frac{\epsilon}{3}$. These will be the vertices of our polygon and we have the $\frac{\epsilon}{6}$ bound as an immediate result. Similarly to $\gamma$, we let $\phi$ be an arclength parameterization of this polygon so that they both encounter their common points in the same order.

Consider a point $p = \phi(t)$ on a side of the polygon (i.e., not a vertex) and its neighboring vertices $v_1 = \phi(t_1) = \gamma(s_1)$ and $v_2 = \phi(t_2) = \gamma(s_2)$ (chosen with $t_1 < t < t_2$ and $s_1 < s_2$). By lemma 11, there exists $s \in (s_1, s_2)$ such that $v_2 - v_1 \in T_{\gamma}(s)$. Note that this is the only member of $T_{\phi}(t)$ up to positive scalar multiplication.

Combining the arcs along $\gamma$ and $\phi$ between $v_1$ and $v_2$, we obtain a closed curve with total length at most $\frac{2}{3} \epsilon$, so that the distance between any two points on the curve is at most $\frac{\epsilon}{3}$. That is, for any $s' \in [s_1, s_2]$ and $t' \in [t_1, t_2]$, we have $d(\gamma(s'), \phi(t')) \leq \frac{\epsilon}{3}$.

Let $x \in D(\phi(t), r - \epsilon)$. Then $d(x, \gamma(s)) \leq d(x, \phi(t)) + d(\phi(t), \gamma(s)) \leq r - \frac{2\epsilon}{3}$ so that $D(\phi(t), r - \epsilon)$ is contained in $D(\gamma(s), r - \frac{2\epsilon}{3})$.

Let $a, b$ be distinct points on the polygon in $D(\phi(t), r - \epsilon)$ and consider the line connecting them. This line also intersects $a', b'$ on $\gamma$ such that we have $a' \neq b'$, $d(a, a') \leq \frac{\epsilon}{3}$ and $d(b, b') \leq \frac{\epsilon}{3}$ so that $a', b' \in \partial \Omega \cap D(\gamma(s), r)$. As $a - b = c(a' - b')$ for some scalar $c > 0$, we have

$$\langle v_2 - v_1, a - b \rangle = c\langle v_2 - v_1, a' - b' \rangle \neq 0$$
since $\gamma$ is TCGL at $\gamma(s)$ with radius $r$ and $v_2 - v_1 \in T_\gamma(s)$. Thus $\phi$ is TCGL at $p$ with radius $r - \epsilon$.

The case where $p = \phi(t)$ is a vertex is similar but we must consider an arbitrary vector $w \in T_\phi(t)$ in the inner product. We wish to show that for every $w \in T_\phi(t)$, there is a $s'$ such that either $w$ or $-w \in T_\gamma(s')$ and $d(p, \gamma(s')) \leq \frac{\epsilon}{3}$, after which the proof follows as in the first case with $w$ (or $-w$) in place of $v_2 - v_1$. We let $\gamma(s) = \phi(t) = p$ and let $v_1 = \phi(t_1) = \gamma(s_1)$ and $v_2 = \phi(t_2) = \gamma(s_2)$ be the neighboring vertices (so $t_1 < t < t_2$ and $s_1 < s < s_2$).

As above, there exist $s'_1, s'_2$ such that $s_1 \leq s'_1 \leq s \leq s'_2 \leq s_2$, $\gamma(s) - \gamma(s_1) \in T_\gamma(s'_1)$ and $\gamma(s_2) - \gamma(s) \in T_\gamma(s'_2)$. Note that $T_\phi(t)$ is exactly the set of positive linear combinations of these vectors. By lemma 10, for every $w \in T_\phi(t)$, there is a $s' \in [s'_1, s'_2]$ such that $w \in T_\gamma(s')$. As $d(p, \gamma(s')) < \frac{\epsilon}{3}$, the proof is complete.

**Definition 20.** We say that $\Omega$ is tangentially graph-like and Lipschitz (TGLL) with radius $r$ if $\Omega$ is tangentially graph-like with radius $r$ and there is some constant $0 < K < \infty$ such that for every $p \in \partial \Omega$, the arc $D(p, r) \cap \partial \Omega$ is the graph of a Lipschitz function (in the same orientation used by the tangentially graph-like definition) and that the Lipschitz constant is at most $K$.

**Remark 21.** Note that tangentially graph-like does not imply tangentially graph-like and Lipschitz: taking $\gamma$ to be a square with side length 5 whose corners are replaced by quarter circles of radius 1 and then considering disks of radius $\sqrt{2}$ centered on $\gamma$ yields one example.

Because $\gamma$ is arclength parameterized by $s$, $||\gamma'(s)|| = 1$ for all $s$. Since $\gamma$ is assumed $C^1$ on its compact domain $[0, L]$, $\gamma'$ is uniformly continuous: for any $\epsilon > 0$, there is a $\delta_\epsilon$ such that if $|s_2 - s_1| < \delta_\epsilon$ then $||\gamma'(s_2) - \gamma'(s_1)|| < \epsilon$.

We shall the fact that $\gamma$ always crosses $\partial D$ transversely to prove that $\gamma$ is in fact TGLL on slightly bigger disks of radius $r + \delta$ as long as one takes a somewhat bigger Lipschitz constant $\hat{K}$. It is then an immediate result of lemma 19 that we can find an approximating polygon that is TCGL with radius $r$.
Lemma 22. If $\gamma$ is TGLL with radius $r$, then it is TGLL with radius $r + \delta$ for some $\delta > 0$ and there is a approximating polygon $P_\gamma$ which is TCGL with radius $r$.

Proof. **Step 1:** Show that the quantities $\nu_1$ and $\nu_2$ are continuous as a function of $s \in [0, L]$ (see Fig. 4)

Define $R^2(s, t) \equiv ||\gamma(s) - \gamma(t)||^2$. Taking the derivative, we get

$$DR = \left[ \left\langle \frac{\gamma(s) - \gamma(t)}{R(s, t)}, \gamma'(s) \right\rangle, \left\langle \frac{\gamma(t) - \gamma(s)}{R(s, t)}, \gamma'(t) \right\rangle \right].$$

Because $\nu_1$ and $\nu_2$ are both less than $\pi/2$ and $\gamma$ is graph-like in the disk, we have that both elements of this derivative are nowhere zero. By the implicit function theorem, we get that $s^-(s)$ and $s^+(s)$ are continuous functions of $s$. From this it follows that $\nu_1$ and $\nu_2$ are continuous on $[0, L]$.

**Step 2:** From the previous step and the compactness of $[0, L]$ we get that $\nu_1(s)$ and $\nu_2(s)$ are both bounded by $M_\nu < \pi/2$. We define $\epsilon_\nu \equiv \pi/2 - M_\nu > 0$. 

Figure 6: TGLL implies TCGL: Step one
0. Fix a \( t \in [0, L] \). Define \( \hat{\rho}(s) \) by \( \hat{\rho}^2(s) = R^2(s, t) = ||\gamma(s) - \gamma(t)||^2 \). Then 
\[
\hat{\rho}(s) = \frac{\langle \gamma(s) - \gamma(t), \gamma'(s) \rangle}{\rho} = \langle n_t(s), \gamma'(s) \rangle
\]
where \( n_t(s) = \frac{\gamma(s) - \gamma(t)}{||\gamma(s) - \gamma(t)||} = \frac{\gamma(s) - \gamma(t)}{\hat{\rho}} \),
the external normal to \( \partial D(\gamma(t), \rho) \) at \( \gamma(s) \). On any interval in \( s \) where \( \hat{\rho}(s) > 0 \) we have that \( \hat{\rho}(s) \) is one to one and strictly increasing. Define
\( s^* \equiv s^+(t) \) and \( s_* \equiv s^-(t) \). We showed above that \( \hat{\rho}(s^*) = \langle n_t(s^*), \gamma'(s^*) \rangle \geq \cos(M_\nu) > 0 \).

For \( \langle n_t(s), \gamma'(s) \rangle = 0 \), \( n_t(s) \) and \( \gamma' \) will have to have turned together by at least \( \pi/2 - M_\nu \) radians. And until they have turned this far, \( \langle n_t(s), \gamma'(s) \rangle > 0 \).
But \( \dot{n}_t(s) \leq \frac{1}{\rho} \leq \frac{1}{r_{\text{min}}} \) for some \( r_{\text{min}} > 0 \). (Choosing \( r_{\text{min}} = \frac{r}{2} \) works.) And \( \gamma' \) is uniformly continuous on \([0, L]\). Therefore, there is a \( \delta_s \) such that on \([s^*, s^* + \delta_s]\), \( n_t(s) \) and \( \gamma' \) both turn by less than \( \epsilon_\nu/3 \). Therefore, for \( s \in [s^*, s^* + \delta_s] \), we have that \( \langle n_t(s), \gamma'(s) \rangle > \cos(\pi/2 - \epsilon_\nu/3) \) and \( \gamma([s^*, s^* + \delta_s]) \) intersects \( C = \partial D(\gamma(t), \rho) \) once for each \( \rho \in [r, r + \delta_r] \), where \( \delta_r \equiv \delta_s \cos(\pi/2 - \epsilon_\nu/3) \).

A completely analogous argument works to show that \( \gamma([s_* - \delta_s, s_*]) \) intersects \( C = \partial D(\gamma(t), \rho) \) once for each \( \rho \in [r, r + \delta_r] \).

Define \( d(t) \) to be the distance from \( D(\gamma(t), r) \) to \( \gamma \setminus \gamma([s_* - \delta_s, s^* + \delta_s]) \).
Since \( \gamma \) is TGL, \( d(t) \) is greater than zero for all \( t \) and is continuous in \( t \). Therefore, there is a smallest distance \( \delta_d \) such that \( d(t) \geq \delta_d \) for all \( t \). Define \( \delta_{\gamma_0} = \min(\delta_d/2, \delta_r/2) \).

Therefore, \( \partial D(\gamma(t), \rho) \) intersects \( \gamma \) exactly twice for \( \rho \in [r, r + \delta_{\gamma_0}] \) for any \( t \in [0, L] \).

A similar argument shows that \( \partial D(\gamma(t), \rho) \) intersects \( \gamma \) exactly twice for \( \rho \in [r - \delta_{\gamma_1}, r] \) for any \( t \in [0, L] \). Defining \( \delta_{\gamma_1} \equiv \min(\delta_{\gamma_1}, \delta_{\gamma_0}) \) we get that \( \partial D(\gamma(t), \rho) \) intersects \( \gamma \) exactly twice for \( \rho \in [r - \delta_{\gamma}, r + \delta_{\gamma}] \), with the additional fact that \( \langle n_t(s), \gamma'(s) \rangle > \cos(\pi/2 - \epsilon_\nu/3) \) at all those intersections.

**Step 3:** TGLL implies that there is a constant \( K < \infty \) such that \( \gamma \cap D(\gamma(t), r) \) is the graph of a function whose x-axis direction is parallel to \( \gamma'(t) \) and this function is Lipschitz with Lipschitz constant \( K \).

Since \( \gamma' \) is uniformly continuous, there will be a \( \delta_1 \) such that if \( |u - v| < \delta_1 \), then \( \angle(\gamma'(u), \gamma'(v)) < \arctan 2K - \arctan K \). Define \( \delta_{K,s} = \min(\delta_s, \delta_1) \).
Define $\delta_{K,r} = \min(\delta\gamma, \delta_{K,s} \cos(\pi/2 - \epsilon/3))$. Then $\gamma \cap D(\gamma(t), r + \delta_{K,r})$ is the graph of a Lipschitz function with Lipschitz constant at most $2K$ when $\gamma'(t)$ is used as the $x$-axis direction. That is, for all $t$, $\gamma$ is TGLL with Lipschitz constant $2K$ for disks of radius $r + \delta_{K,r}$. The result follows by lemma 19.

5 Derivatives of $g(s,r)$

**Lemma 23.** If $\gamma$ is tangent-cone graph-like with radius $r$ and $0 \leq s_1 \leq s_2 < L$ with $d(\gamma(s_1), \gamma(s_2)) = \delta < r$, then the image of $[s_1, s_2]$ together with the straight line from $\gamma(s_1)$ to $\gamma(s_2)$ enclose a region with $O(\delta^2)$ area.

**Proof.** By Lemma 14, we have that the image of $[s_1, s_2]$ under $\gamma$ has arc length $s_2 - s_1 \leq \sqrt{2}\delta$. Therefore, the region of interest has perimeter at most $(\sqrt{2}+1)\delta$ so by the isoperimetric inequality has area at most $\frac{(\sqrt{2}+1)^2}{4\pi}\delta^2$ from which the conclusion follows.

![Figure 7: Deriving $\frac{\partial g}{\partial r}$ as the arclength of the circular segment.](image)

**Lemma 24.** Using the notation of figure 4, we have $\frac{\partial}{\partial r} g(s,r) = (\theta_2 - \theta_1)r$. That is, the derivative exists and equals the length of the curve $C(\gamma(s), r) \cap \Omega$. 
Proof. We have (see figure 7)

$$\frac{\partial}{\partial r} g(s, r) = \lim_{\Delta r \to 0} \frac{\text{Area}(\Omega \cap D(\gamma(s), r + \Delta r)) - \text{Area}(\Omega \cap D(\gamma(s), r))}{\Delta r}.$$ 

This difference of areas can be modeled by the difference in the circular sectors of $D(\gamma(s), r + \Delta r)$ and $D(\gamma(s), r)$ with angle $\theta_1 - \theta_2$. The actual area depends on the image of $\gamma$ outside of $D(\gamma(s), r)$, but this correction will be a subset of the circular segment of $D(\gamma(s), r + \Delta r)$ which is tangent to $D(\gamma(s), r)$ at the point $\gamma$ exits. This has area $O(\Delta r^2)$ by lemma 23.

Thus we have

$$\frac{\partial}{\partial r} g(s, r) = \lim_{\Delta r \to 0} \frac{(\theta_1 - \theta_2) r \Delta r + \frac{1}{2} (\theta_1 - \theta_2) \Delta r^2 + O(\Delta r^2)}{\Delta r} = (\theta_1 - \theta_2) r. \quad \square$$

Figure 8: Deriving $\frac{\partial g}{\partial s}$ as the difference in heights of the entry and exit points

Lemma 25. Using the notation of figures 4 and 8, we have $\frac{\partial}{\partial r} g(s, r) = h_2 - h_1 = r \sin(\theta_2) - r \sin(\theta_1)$.

Proof. We have

$$\frac{\partial}{\partial s} g(s, r) = \lim_{\Delta s \to 0} \frac{\text{Area}(\Omega \cap D(\gamma(s + \Delta s), r)) - \text{Area}(\Omega \cap D(\gamma(s), r))}{\Delta s}.$$
The situation is illustrated in figure 8 where we can see that the area being added as we go from \( s \) to \( s + \Delta s \) is the shaded region on the right with height \( r - h_1 \) and, considering first order terms only, uniform width \( \Delta s \) so has area \( (r - h_1)\Delta s \). Similarly, we are subtracting the area \( (r - h_2)\Delta s \) on the left. Therefore, we have

\[
\frac{\partial}{\partial s} g(s, r) = \lim_{\Delta s \to 0} \frac{(r - h_1)\Delta s - (r - h_2)\Delta s}{\Delta s} = h_2 - h_1.
\]

6 Reconstructing shapes from T-like data

![Diagram](image)

Figure 9: T-like data: we restrict the domain of \( g(s, r) \) to a fixed radius \( \hat{r} \) plus any vertical segment from \( r = 0 \) to \( r = \hat{r} \).

In this section, we show that a curve for which nonasymptotic densities are known along a T shaped set can be reconstructed.

**Lemma 26.** Assume that \( \gamma \) is TGL for all \( r = \hat{r} \). Then if we know \( g(s, r) \), \( g_s(s, r) = \frac{dg(s, r)}{ds} \), and \( g_r(s, r) = \frac{dg(s, r)}{dr} \) for \((s, r) \in ([0, L] \times \{\hat{r}\}) \cup (\{\hat{s}\} \times [0, \hat{r}])\), we can reconstruct \( \gamma(s) \in \mathbb{R}^2 \) for all \( s \in [0, L] \). (See figure 9.)
Proof: As was shown in section 5, $g_r$ gives us the length of the arc $\partial D(s, \hat{r}) \cap \Omega$ and $g_s$ tells us precisely what position this arc is along $\partial D(s, \hat{r})$ with respect to the direction $\gamma'(s)$. The assumption of TGL for $r = \hat{r}$ implies TGL for $0 < r < \hat{r}$ (see remark 18) and this implies that $\gamma$ has the 2 arc property and transverse intersections with $\partial D(s, r)$ for all disks corresponding to $(s, r) \in ([0, L] \times \{\hat{r}\}) \cup (\{\hat{s}\} \times [0, \hat{r}])$. Since we care only about reconstructing a curve $\gamma$ isometric to the original curve, we choose $\gamma(\hat{s}, \hat{r}) = (0, 0) \in \mathbb{R}^2$ and $\gamma'(\hat{s}, \hat{r}) = (1, 0)$. Taken together, $g_s(\hat{s}, r)$ and $g_r(\hat{s}, r)$ locate both points in $\partial D(\hat{s}, r) \cap \gamma$ for all $r \in [0, \hat{r}]$. This yields $\gamma \cap D(\gamma(\hat{s}, \hat{r}))$. Now, simply increase $s$, sliding the center of a disk of radius $\hat{r}$ along $\gamma \cap D(\gamma(\hat{s}, \hat{r}))$, using $g_r(s, \hat{r})$ to find the element of $\gamma \cap D(\gamma(s, \hat{r}))$ outside $D(\gamma(\hat{s}, \hat{r}))$, using the fact that the other element of $\gamma \cap D(\gamma(s, \hat{r}))$ is inside $D(\gamma(\hat{s}, \hat{r}))$ and known. This process can be continued until the entire curve is traced out in $\mathbb{R}^2$.

7 TCGL polygon is reconstructible from $g_r$ and $g_s$ without tail

Theorem 27. For a tangent-cone graph-like polygon $\Omega$, knowing $g(s, r)$, $g_r(s, r)$ and $g_s(s, r)$ for all $s \in [0, L)$ and a particular $r$ for which $\partial \Omega$ is tangent-cone graph-like is sufficient to completely determine $\Omega$ up to translation and rotation; that is, we can recover the side lengths and angles of $\Omega$.

Proof. For a given $s$ and $r$ where $g_r$ and $g_s$ exist, we can use them to obtain $r(\theta_2 - \theta_1)$ as the length of the circular arc between the entry and exit points by Lemma 24 and $r(\sin \theta_2 - \sin \theta_1)$ as the difference in heights of the entry and exit points by Lemma 25.

We wish to recover $\theta_1$ and $\theta_2$ from these quantities. Note that if $(\theta_1, \theta_2) = (\phi_1, \phi_2)$ is one possible solution, then so is $(\theta_1, \theta_2) = (2\pi - \phi_2, 2\pi - \phi_1)$ so solutions always come in pairs.

We can imagine placing a circular arc with angle $\frac{\theta_2 - \theta_1}{r}$ on our circle and sliding it around until the endpoints have the appropriate height difference, yielding
our $\theta_1$ and $\theta_2$. Note that since $\Omega$ is tangent-cone graph-like, one endpoint must be on the left side of the circle and the other must be on the right and we cannot slide either endpoint to or beyond the vertical line through the center of the circle.

Therefore, as we slide the right endpoint down, the left endpoint slides up so that the height difference as a function of the slide is strictly monotonic. Therefore, the slide that gives us $\theta_1$ and $\theta_2$ is unique for a given starting arc placement. However, there are two starting arc placements: the first calls the angle for the right endpoint $\theta_1$ and the left endpoint $\theta_2$ (so the interior of $\Omega$ is “up” in the circle) and the second swaps these (so the interior of $\Omega$ is “down”). Since we have adopted the convention that $\partial\Omega$ is traversed in a counterclockwise direction (so the interior of $\Omega$ is up in the circles) we therefore pick the first option; this gives us a unique solution for $\theta_1$ and $\theta_2$.

This procedure works whenever $g_r$ and $g_s$ exist which is certainly true whenever the density disk does not touch a vertex of $\Omega$ either at its center or on its boundary because if we avoid these cases, then there is only one graph-like orientation to deal with and $\partial\Omega$ is $C^\infty$ for all the points that enter into the computation. In fact, with a moment’s thought we can make a stronger statement than this: $g_r$ always exists and $g_s$ exists as long as the center of the density disk is not a vertex of the polygon.

We can identify the $s$ values at which $g_s(s, r)$ doesn’t exist to obtain the arc length positions of the vertices (and therefore obtain side lengths). For a given $s$ corresponding to a vertex, we can find $g_r$ and the one-sided derivatives $g_{s-}$ and $g_{s+}$. These correspond to the graph-like orientations required by the polygon sides adjacent to the current vertex.

Referring to Figure 10, the one-sided derivatives along with the argument at the beginning of the proof yield the angles $\theta_1$, $\theta_2$, $\phi_1$, and $\phi_2$. Thus we can calculate $\psi = \theta_1 - \phi_1$ which means that the polygon vertex at $s$ has angle $\pi - \psi$.

Doing this for all $s$ corresponding to vertices, we can determine all of the angles of the polygon. With the side lengths identified earlier, this completely determines the polygon $\Omega$ up to translation and rotation. □
8 Simple closed curves are generically reconstructible using fixed radius data

We will assume $\gamma$ is TGL for the radius $\hat{r}$. We will also assume that we know the first, second and third derivatives of $g(s, r)$ for $r = \hat{r}$. Under these assumptions, $\gamma$ is generically reconstructible. By generic we mean the admittedly weak condition of density – reconstructible curves are $C^1$ dense in the space of $C^2$ simple closed curves.

**Theorem 28.** Define $\mathbb{G} \equiv \{ \gamma | \gamma$ is a $C^2$ simple closed curve and TGL for $r = \hat{r} \}$. Suppose that for $r = \hat{r}$, for all $s \in [0, L]$, and for each $\gamma \in \mathbb{G}$ we know the first, second and third order partial derivatives of $g_\gamma(s, r)$. Then the set of reconstructible $\gamma \in \mathbb{G}$ is $C^1$ dense in $\mathbb{G}$.

**Proof:** From the preceding sections we have that $\frac{\partial g(s, r)}{\partial r} = r(\theta_2 - \theta_1)$ and $\frac{\partial g(s, r)}{\partial s} = r(\sin(\theta_2) - \sin(\theta_1))$. Because $\gamma$ is TGL, we can solve for $\theta_1$ and $\theta_2$ from these two derivatives.

**Claim 1.** $\frac{\partial^2 g(s, r)}{\partial r^2} = \theta_2 - \theta_1 + r(\frac{\partial \theta_2}{\partial r} - \frac{\partial \theta_1}{\partial r})$ and $\frac{\partial^2 g(s, r)}{\partial r \partial s} = \sin(\theta_2) - \sin(\theta_1) + r(\cos(\theta_2) \frac{\partial \theta_2}{\partial r} - \cos(\theta_1) \frac{\partial \theta_1}{\partial r})$
Proof of Claim: Simply differentiate the expressions we already have for \( \frac{\partial g(s,r)}{\partial r} \) and \( \frac{\partial g(s,r)}{\partial s} \). □

We wish to express this in terms of \( \nu_1 \) and \( \nu_2 \). Note that if we expand the circle radius by \( \Delta r \), the right exit point \( s_+(s) \) moves approximately (i.e., considering first-order terms only) a distance of \( k \equiv \Delta r \sec(\nu_1) \) (so \( \frac{\partial k}{\partial r} = \sec \nu_1 \), a fact we will use later to compute curvature). Therefore,

\[
\frac{\partial \theta_1}{\partial r} = \lim_{\Delta r \to 0} \frac{\arctan \left( \frac{r \sin \theta_1 + k \sin(\theta_1 + \nu_1)}{r \cos \theta_1 + k \cos(\theta_1 + \nu_1)} \right) - \theta_1}{\Delta r}.
\]

Straightforward techniques yield \( \frac{\partial \theta_1}{\partial r} = \frac{\tan \nu_1}{r} \) and a similar calculation shows that \( \frac{\partial \nu_2}{\partial r} = \frac{\tan \nu_2}{r} \).

Therefore, rewriting the second derivatives of \( g(s,r) \) in terms of \( \nu_1 \) and \( \nu_2 \) we get:

\[
\frac{\partial^2 g(s,r)}{\partial r^2} = \theta_2 - \theta_1 + \tan(\nu_2) - \tan(\nu_1)
\]

\[
\frac{\partial^2 g(s,r)}{\partial r \partial s} = \sin(\theta_2) - \sin(\theta_1) + \cos(\theta_2) \tan(\nu_2) - \cos(\theta_1) \tan(\nu_1)
\]

Using these 2 derivatives, together with the previous two, we can solve for \( \nu_1 = \arctan \left( r \frac{\partial \theta_1}{\partial r} \right) \) and \( \nu_2 = \arctan \left( r \frac{\partial \theta_1}{\partial r} \right) \) whenever \( \cos(\theta_1) \neq \cos(\theta_2) \). Since we are assuming that the curve is a simple closed curve, \( \cos(\theta_1) \neq \cos(\theta_2) \) is always true.

Claim 2. Knowing \( \frac{\partial^3 g(s,r)}{\partial r^3} \) and \( \frac{\partial^3 g(s,r)}{\partial r^2 \partial s} \) gives us \( \kappa(s^+(s)) \) and \( \kappa(s^-(s)) \), the curvatures of \( \gamma \) at \( s^+(s) \) and \( s^-(s) \).

Proof of Claim: Computing, we get

\[
\frac{\partial^3 g(s,r)}{\partial r^3} = \frac{\partial \theta_2}{\partial r} - \frac{\partial \theta_1}{\partial r} + \sec^2(\nu_2) \frac{\partial \nu_2}{\partial r} - \sec^2(\nu_1) \frac{\partial \nu_1}{\partial r}
\]

\[
\frac{\partial^3 g(s,r)}{\partial r^2 \partial s} = \cos(\theta_2) \frac{\partial \theta_2}{\partial r} - \cos(\theta_1) \frac{\partial \theta_1}{\partial r} - \sin(\theta_2) \frac{\partial \theta_2}{\partial r} \tan(\nu_2) + \sin(\theta_1) \frac{\partial \theta_1}{\partial r} \tan(\nu_1) + \cos(\theta_2) \sec^2(\nu_2) \frac{\partial \nu_2}{\partial r} - \cos(\theta_1) \sec^2(\nu_1) \frac{\partial \nu_1}{\partial r}.
\]
\[ D \cap \Omega \]

\[ \gamma(s) \]

\[ D = D(\gamma(s), r) \]

\[ C = \partial D \]

\[ \theta_1(s) + \nu_1(s) = \pi - \theta_2(s) - \nu_2(s) \]

\[ \theta_1(s) + \nu_1(s) = \pi - \theta_2(s^+(s)) - \nu_2(s^+(s)) \]

Since \( \nu'_2 \equiv \frac{\partial \nu_2}{\partial r} \) and \( \nu'_1 \equiv \frac{\partial \nu_1}{\partial r} \) are the only unknowns, we end up having to invert

\[
\begin{bmatrix}
1 & -1 \\
\cos(\theta_2) & \cos(\theta_1)
\end{bmatrix}
\]

again and this is always nonsingular, giving us \( \nu'_1 \) and \( \nu'_2 \) as a function of \( s \), the coordinate of the center of the disk.

Relative to the horizontal, the angle of the curve at \( s^+(s) \) is \( \theta_1 + \nu_1 \) so the rate of change in angle as we expand the circle is \( \frac{\partial \theta_1}{\partial r} + \nu'_1 \). Recalling that rate of movement of this exit point as we expand the circle is given by \( \frac{\partial k}{\partial r} = \sec \nu_1 \), we have that the curvature is given by

\[
\kappa(s^+(s)) = \frac{\partial k}{\partial r}(\frac{\partial \theta_1}{\partial r} + \nu'_1) = \sec \nu_1(\frac{\partial \nu_1}{\partial r} + \nu'_1).
\]

Similarly, \( \kappa(s^-(s)) = \sec(\nu_2)(\frac{\partial \nu_2}{\partial r} + \nu'_2) \). □

**Claim 3.** Generically, we can deduce \( s^+(s) \) from knowledge of \( \nu_1(s) \), \( \nu_2(s) \), \( \theta_1(s) \) and \( \theta_2(s) \).

**Proof:** We outline of the proof without some of the explicit constructions that follow without much trouble from the outline. We have that \( \theta_1(s^-(s)) + \nu_1(s^-(s)) = \pi - \theta_2(s) - \nu_2(s) \) and \( \theta_1(s) + \nu_1(s) = \pi - \theta_2(s^+(s)) - \nu_2(s^+(s)) \).

All four of these quantities (the left- and right-hand sides of each of the 2
equations) are the turning angles between the tangent to the curve at the center of the disk and the tangent to the curve at a point \( r \) away from the center of the disk.

Now we use this correspondence between the \( \theta + \nu \) curves to solve for \( s^- (s) \) and \( s^+ (s) \). But these curves can differ by a homeomorphism of the domain. Thus, we can only find the correspondence if there is a distinguished point on those curves as well as no places where the values attained are constant. The turning angle curves having isolated critical points and a unique maximum or minimum, is enough.

To get isolated extrema, start by approximating the curve \( \gamma \) with another one, \( \hat{\gamma} \) that agrees in \( C^1 \) at a large but finite number of points \( \{ s_i \} \) (i.e. agrees in tangent direction as well as position), and has isolated critical points in the derivative of the tangent direction. Now perturb \( \hat{\gamma} \) to one that is \( C^1 \) close (but not \( C^2 \) close) by using oscillations about the curve so that the 2nd and 3rd derivatives are never simultaneously below the bounds on the 2nd and 3rd derivatives of the curve we started with. We do this in a way that alternates around the curve. See Figure 12. In a bit more detail, suppose that

\[
\max \left\{ \frac{d^2 \hat{\gamma}}{ds^2}, \frac{d^3 \hat{\gamma}}{ds^3} \right\} < L_1.
\]

Choose a starting point on the curve; \( s = 0 \) works. Now begin perturbing \( \hat{\gamma} \) at the point \( s_\hat{r} \) in the positive \( s \) direction such that

\[
|\hat{\gamma}(s_\hat{r}) - \hat{\gamma}(0)| = \hat{r}.
\]

We name the newly perturbed curve \( \hat{\hat{\gamma}} \) and we keep

\[
L_1 < \max \left\{ \frac{d^2 \hat{\hat{\gamma}}}{ds^2}, \frac{d^3 \hat{\hat{\gamma}}}{ds^3} \right\} < L_2.
\]

We continue perturbing until we have reached \( s_{2\hat{r}} \) defined by

\[
|\hat{\gamma}(s_2\hat{r}) - \hat{\gamma}(s_\hat{r})| = \hat{r}.
\]

We begin perturbing again when we reach \( s_{3\hat{r}} \). Continue in this fashion around \( \hat{\gamma} \). The last piece, shown in green in the figure will require a perturbation that is distinct in size due to the fact that it will interact with the perturbation that starts at \( s_\hat{r} \). On this last piece, we enforce

\[
L_2 < \max \left\{ \frac{d^2 \hat{\hat{\gamma}}}{ds^2}, \frac{d^3 \hat{\hat{\gamma}}}{ds^3} \right\} < L_3.
\]

All these perturbations can be chosen with isolated singularities in derivatives, thus giving us \( \theta + \nu \) curves that are monotonic between isolated singularities. (In fact, we might as well choose all perturbations to be piecewise polynomial perturbations. This immediately gives us the isolated singularities and monotonicity that we want.)

Finally, if there is not a distinct maximum, we can choose one of the maxima and add a small twist to the curve at that point. See Figure 13. The idea is that a small twist, applied to leading edge of the tangents we are comparing to get the turning angle, will increase the angle most at the center of the
twist. If this corresponds to a nonunique global maximum, we end up with a unique global maximum.

\[ L_1 < \max\{d^2\gamma/ds^2, d^3\gamma/ds^3\} < L_2 \]
\[ L_2 < \max\{d^2\gamma/ds^2, d^3\gamma/ds^3\} < L_3 \]

Figure 12: In this schematic figure, we illustrate the alternating perturbation around the curve, keeping the curve \( C^1 \) close to and messing with the second and third derivatives to eliminate any critical points other than isolated maxima and minima. Here the perturbation is of course greatly exaggerated.

Figure 13: A twist perturbation. Notice that if the twist is applied precisely at a global max of the turning angle (as measured by the tangent here and the one lagging it in \( s \)), we will increase the turning angle there and will end up with a unique global maximum.

Now the correspondence scheme works. That is, we know that the global maximums must match, and because the turning angle curves are monotonic between isolated critical points, we can find the homeomorphisms in \( s \) that move the turning angle curves into correspondence. □

Taken together, the last two claims give us the curvature as a function of arclength. This determines \( \gamma \) up to translations and rotations. □
9 Numerical experiments

In this section we consider a numerical curve reconstruction for the situation in which \( g(s, r) \) is known for a given radius \( r \) but no derivative information is available. This reconstruction is more strict than the scenarios of sections 6–8. Our motivation is to explore whether any \( \gamma \) can be uniquely and practically reconstructed with this limited information.

We consider \( \gamma_a(\bar{s}) \in \mathcal{P}^N \), the set of simple polygons of \( N \) ordered vertices \( \{(x_1, y_1), \ldots, (x_N, y_N)\} \) parameterized by the set \( \{\bar{s}_k\}_{k=1}^N \) with \( \bar{s}_k = k/N \) as

\[
\begin{align*}
x_k &= \sum_{j=0}^{m-1} a_{1,j} \cos(2\pi j \bar{s}_k/N) + a_{2,j} \sin(2\pi j \bar{s}_k/N), \\
y_k &= \sum_{j=0}^{m-1} a_{3,j} \cos(2\pi j \bar{s}_k/N) + a_{4,j} \sin(2\pi j \bar{s}_k/N),
\end{align*}
\]

for some coefficients \( a_{i,j} \in \mathbb{R} \). In this way, the polygon \( \gamma \) is a discrete approximation of a \( C^\infty \) curve. The sides of \( \gamma_a(\bar{s}) \) are not necessarily of equal length.

We take the vector signature \( g_a(\bar{s}, r) \in \mathbb{R}^N \) to be the discrete area densities of \( \gamma_a(\bar{s}) \) computed at each vertex. Given such a signature for fixed radius \( r \) and fixed partition \( \bar{s} \), we seek \( a^* \) satisfying

\[
a^* := \arg \min_{b \in \mathbb{R}^{4m}} \| g_b(\bar{s}, r) - g_a(\bar{s}, r) \|_2^2 \\
\text{s.t. } \gamma_b \in \mathcal{P}^N
\]

Equation (2) represents a nonlinearly constrained optimization problem with continuous nonsmooth objective. The constraint ensures that polygons are simple though any optimal reconstruction \( \gamma_{a^*} \) is not expected to lie on the feasible region boundary except in cases of noisy signatures. This approach to reconstructing curves seeks a polygon that matches a given discrete signature, rather than an analytic sequential point construction procedure.

We use the direct search OrthoMads algorithm \cite{Mads} to solve this problem. MADS class algorithms do not require objective derivative information \cite{Mads, Mads2}.
and converge to second order stationary points under reasonable conditions on nonsmooth functions [1]. We implement our constraint using the extreme barrier method [4] in which the objective value is set to infinity whenever constraints are not satisfied. We utilize the standard implementation with partial polling and minimal spanning sets of $4m + 1$ directions.

Figure 14: Shamrock reconstruction: comparing the original curve with those found for $m = 12$ and $m = 18$. Curves for $m \geq 20$ are visually indistinguishable from the original curve. The shape signatures are given at the bottom.

We performed a series of numerical tests using the synthetic shamrock curve shown in black in the upper portion of Figure 14. This curve is given as a polygon in $\mathcal{P}^{256}$ with discretization coefficients $a \in \mathbb{R}^{4 \times 20}$ ($m = 20$). A se-
quence of reconstructions was performed with all integer values $8 \leq m \leq 20$. The $m = 8$ reconstruction begins with initial coefficients, $a_{i,j}$, which determine a regular 256-gon with approximately the same interior area as the shamrock (as determined by the signature $g_{a}(\bar{s}, r)$). Subsequent reconstructions begin with initial coefficients optimal to the previous relatively coarse reconstruction. Curve reconstructions for $m = 12$ (blue) and $m = 18$ (red) are compared to the shamrock in the upper portion of Figure 14. Reconstructions for $m \geq 20$ are visually indistinguishable from the actual curve and are not shown. Corresponding area density signatures are shown in the lower portion of Figure 14. A representative disk of radius $r$ is shown in green along with corresponding location in the signature; note that the shamrock is not tangent-cone graph-like with this radius.

In comparing and interpreting the shamrock curves it is important to note that the scale of the curves is determined entirely by the fit parameters $a_{i,j}$. On the other hand, as the density signature is independent of curve rotation, the rotation is eyeball adjusted for easy visual comparison. Also note that the two-arc property does not hold for this example. The accuracies of both the curve reconstruction and area density signature fit suggest that somewhat general simple polygons are reconstructible from $g(s, r)$ for fixed $r$ and no derivative information.

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A Appendix: Easy reconstructability

For completeness, we include a short proof of the fact that knowing $g(s, r)$ for all $s$ and $r$ very easily gives us reconstructability. This follows from the
fact that knowing the asymptotic behavior of $g(s, r)$ as $r \to 0$ for any $s$ gives us $\kappa(s)$. That in turn implies that knowing $g(s, r)$ in any neighborhood of the set $(s, r) \in [0, L] \times \{r = 0\}$ also gives us $\kappa(s)$ and therefore the curve.

Figure 15: Using the osculating circle as a surrogate for $\partial \Omega$ in the (a) positive and (b) negative curvature cases.

**Theorem 29.** Suppose $\partial \Omega$ is $C^2$ and there exists $\epsilon > 0$ such that we know $g(s, r)$ for all $(s, r) \in [0, L) \times (0, \epsilon)$. This information is enough to determine the curvature of every point on $\partial \Omega$. In particular, if $\gamma : [0, L) \to \partial \Omega$ is a counterclockwise arclength parameterization of $\partial \Omega$, then $\kappa(\gamma(s)) = -3\pi \lim_{r \to 0} \frac{\partial g(s, r)}{\partial r} \frac{\pi r^2}{\pi r^2}$.

**Proof.** Fix $s \in [0, L)$. If the curvature of $\gamma$ at $s$ is positive, we consider what happens if we replace $\Omega$ with the disk whose boundary is the osculating circle of $\partial \Omega$ at $\gamma(s)$ (call its radius $R$). We have the following expression for the new normalized nonasymptotic density (see Figure 15(a)):

$$
g(s, r) = \int_{-p}^p \sqrt{r^2 - x^2} - (R - \sqrt{R^2 - x^2}) \, dx.
$$

where $x = p$ is the positive solution to $\sqrt{r^2 - x^2} = R - \sqrt{R^2 - x^2}$. Differentiating with respect to $r$ and taking the limit as $r$ goes to 0 gives us $-\frac{1}{3\pi R}$.

That is, for the case where $\Omega$ is locally a disk, the curvature at $\gamma(s)$ is given by $-3\pi \lim_{r \to 0} \frac{\partial g(s, r)}{\partial r} \frac{\pi r^2}{\pi r^2}$.
If the curvature of $\partial \Omega$ at $\gamma(s)$ is negative, we can set up a similar surrogate (see figure 15(b)) and again obtain that $\kappa(\gamma(s)) = -3\pi \lim_{r \to 0} \frac{\partial}{\partial r} \frac{g(s,r)}{\pi r^2}$.

Lastly, this calculation gives the right result in the curvature 0 case when $\partial \Omega$ is locally a straight line (so $\frac{g(s,r)}{\pi r^2} = \frac{1}{\pi r^2} \int_r^r \sqrt{r^2 - x^2} \, dx = \frac{1}{2}$ for sufficiently small $r$ and $-3\pi \lim_{r \to 0} \frac{\partial}{\partial r} \frac{g(s,r)}{\pi r^2} = 0$).

For the case where $\partial \Omega$ is not locally a circle or straight line, the corrections to the integrals are of order $O(x^3)$ as $r$ goes to 0 and have no impact on the final answer so the curvature at $\gamma(s)$ is always given by $-3\pi \lim_{r \to 0} \frac{\partial}{\partial r} \frac{g(s,r)}{\pi r^2}$.

The available data (the values $g(s,r)$ for all $s \in [0,L)$ and all $r \in (0,\epsilon)$) are sufficient to compute the relevant derivative and limit so we can use this process to determine the curvature of every point on the $C^2$ curve $\partial \Omega$. □

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