Bianchi Identities for Non-Geometric Fluxes
- From Quasi-Poisson Structures to Courant Algebroids -

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Abstract

Starting from a (non-associative) quasi-Poisson structure, the derivation of a Roytenberg-type algebra is presented. From the Jacobi identities of the latter, the most general form of Bianchi identities for fluxes \((H, f, Q, R)\) is then derived. It is also explained how this approach is related to the mathematical theory of quasi-Lie and Courant algebroids.
1 Introduction

One of the most distinctive features of string theory certainly is T-duality. Applying this transformation to configurations which are already well understood has led to new insight about the theory and to the discovery of new structures such as D-branes and mirror symmetry for Calabi-Yau three-folds. More recently, T-duality has been applied to closed-string backgrounds with non-vanishing three-form flux, resulting in configurations not known previously in the framework of supergravity. More concretely, starting from a background with $H$-flux and performing a T-duality transformation along a single direction of isometry gives a configuration with geometric flux $f$. After further T-dualities, as illustrated in [2, 3], backgrounds with so-called non-geometric fluxes $Q$ and $R$ are obtained. This chain of transformations can be summarized by

$$H_{abc} \xrightarrow{T_c} f_{ab} \xrightarrow{T_b} Q_a \xrightarrow{T_a} R^{abc},$$

(1)

where we would like to note that lower indices are form indices and upper ones are vector indices.

For the case of $Q$-flux, the underlying structure can be understood using the notion of T-folds [4, 5, 6]. However, the nature of backgrounds with $R$-flux is less clear, as the manifold is expected to not even be locally geometric. In fact, in [7, 8] it has been argued that these configurations have a non-associative structure. Further evidence for this observation was obtained by pursuing a conformal field theory (CFT) analysis [9, 10, 11], where T-duality is realized via a reflection of right-moving coordinates. The main results of this work can be summarized by the following commutator and Jacobiator of the coordinates (see also [12])

$$[x^i, x^j] = \oint_{C_a} Q_k \, dy^k, \quad [x^i, x^j, x^k] = R^{ijk}.$$

(2)

These expressions imply that, depending on the fluxes $Q$ and $R$, the coordinates can be both non-commutative and non-associative (NCA). As can be seen from the first expression, the commutator of two coordinates is related to a Wilson line and so only a string with a non-vanishing winding number can detect such a non-commutative structure. This was shown in the recent work [13]. On the other hand, as illustrated in [9], a non-associative structure indicated by a non-vanishing Jacobiator does not depend on the winding number of the closed string.

Recently, double-field theory (DFT) [14] has been established as a manifestly T-duality invariant formulation of the low-energy effective action of string theory. As such, it provides a powerful tool to analyze non-geometric fluxes (at least at tree-level). The new ingredient in DFT is a bi-vector field $\beta$ which can be seen as the “T-dual” of the Kalb-Ramond field $B$, and the fluxes $Q$ and $R$ are expressed in terms of $\beta^{ij}$ as $Q^{ij} = \partial_k \beta^{ij}$ and $R^{ijk} = \beta^{im} \partial_m \beta^{jk} + \text{cycl}[1]$. A bi-vector field also appears in [15] and [16], where the expressions for the fluxes $Q$ and $R$ can be found as well.

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the double-field theory action for a non-vanishing bi-vector field has been worked out explicitly. Furthermore, in [17] a Bianchi identity for the $R$-flux was found, which was discovered independently in [19] following a different approach based on the Schouten–Nijenhuis bracket. In the latter paper, two more Bianchi identities for the non-geometric fluxes were presented.

In this letter, our central objection is to generalize the above-mentioned Bianchi identities to the most general case in which all fluxes $(H, f, Q, R)$ are non-vanishing and non-constant. For constant fluxes a derivation was given in [3, 20, 21], but here we aim for the derivative corrections to these identities. Supporting space-time dependent structure "constants", the theory of Lie and Courant algebroids has been proposed in the literature (see e.g. [22, 15, 23]) as a suitable framework for describing these fluxes. After providing the relevant definitions, we show how the fluxes $(H, f, Q, R)$ fit into this scheme. In fact, they allow to define two quasi-Lie algebroids, which combine into a Courant algebroid. To verify the axioms of the latter, the generalized Bianchi identities turn out to play an essential role.

2 Non-geometric fluxes and quasi-Poisson structures

As it is well-known, at the massless level the gravity sector of string theory contains the metric $g_{ij}$, the anti-symmetric Kalb-Ramond field $B_{ij}$ and the dilaton $\phi$. Starting then from a geometric background with non-vanishing three-form flux $H = dB$ and applying T-dualities, one obtains backgrounds which are no longer geometric. For these non-geometric configurations, the degrees of freedom are more conveniently described by a dual metric $\tilde{g}_{ij}$, an anti-symmetric bi-vector field $\beta^{ij}$ and a dilaton $\tilde{\phi}$. Defining $\tilde{\mathcal{E}}_{ij} = g_{ij} + B_{ij}$ and $\tilde{\mathcal{E}}_{ij} = \tilde{g}^{ij} + \beta^{ij}$, the relation between the two sets of fields is given by $\tilde{\mathcal{E}} = \mathcal{E}^{-1}$, which in components reads [24, 16] (see also [22, 15] for a derivation from a world-sheet point of view)

$$
\tilde{g}_{ij} = g_{ij} - B_{im}g^{mn}B_{nj}, \quad \beta^{ij} = -g^{im}B_{mn}g^{nj},
$$

and $\tilde{\phi}$ is defined via $\sqrt{-\tilde{g}} \exp(-2\tilde{\phi}) = \sqrt{-g} \exp(-2\phi)$. In double-field theory, which is an explicitly $O(D,D)$-invariant framework consisting of a space with doubled coordinates $(x^i, \tilde{x}^i)$, the relation (3) is just a particular $T$-duality transformation [25, 12].

We therefore see that an anti-symmetric bi-vector field $\beta \in \Gamma(\Lambda^2 TM)$ plays an important role for non-geometric fluxes. In coordinates, it can be written as $\beta = \frac{1}{2} \beta^{ij} e_i \wedge e_j$, where $e_i = \partial_i$ denotes a basis vector in $TM$. Such a bi-vector induces two new structures: a quasi-Poisson structure and an anchor map, which we will discuss in the present section.
Quasi-Poisson structure and NCA geometry

Given an anti-symmetric bi-vector field $\beta$, one can define a quasi-Poisson structure as follows

$$\{f, g\} = \beta^{ij} (\partial_i f) (\partial_j g),$$

(4)

where $f, g \in C^\infty(M)$. In general, this bracket does not satisfy the Jacobi identity but one finds

$$\{\{f, g\}, h\} + \text{cycl.} = R^{ijk}(\partial_i f)(\partial_j g)(\partial_k h).$$

(5)

Here, $R^{ijk}$ is given by

$$2R^{ijk} = 3\beta^{im}\partial_m\beta^{jk}$$

which takes the form of the non-geometric $R$-flux mentioned earlier. For vanishing $R \in \Gamma(\Lambda^3 TM)$, one obtains a Poisson structure.

Next, we consider the second non-geometric flux $Q$ which can be expressed as $Q^{ij} = \partial_k \beta^{ij}$, and which in general is not to be considered a tensor. To make contact with the CFT results (2), we write the quasi-Poisson structure (4) for two coordinates $x^i$ and $x^j$ as

$$\{x^i, x^j\} = \int_{C_x} Q^{ij} dy^k.$$ 

(6)

That means, a non-trivial Wilson line of the $Q$-flux can detect a non-trivial Poisson bracket between two coordinates.

Let us mention that both (5) and (6) can be considered as the classical limits of the quantum non-commutativity and non-associativity investigated from a conformal field theory point of view in [9,10,11,13]. For instance, in [11] it was found that a string moving in a background with constant $R$-flux gives rise to a non-trivial three-product of the form

$$f \triangle g \triangle h(x) = f g h + R^{ijk}(\partial_i f)(\partial_j g)(\partial_k h) + \mathcal{O}(R^2).$$

(7)

The quantum version of (6) was considered in [10,13]. Therefore, in analogy to the Moyal-Weyl product, (7) points towards the existence of a deformation quantization of the classical quasi-Poisson structure. However, this certainly very interesting question is beyond the scope of this letter.

Anchor map

In addition to the quasi-Poisson structure, the bi-vector $\beta$ also induces a natural map $\beta^\sharp : T^*M \to TM$ from the co-tangent to the tangent space of a manifold $M$. It is defined by the relation

$$\beta^\sharp(\eta)(\xi) = \beta(\eta, \xi)$$

for all $\xi \in T^*M$.

\[\text{Here and in the following, underlined indices are anti-symmetrized and anti-symmetrization is defined as } A_{[a_1, a_2, \ldots, a_n]} = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign}(\sigma) A_{a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}}. \]

\[\text{This map is closely related to the } \beta\text{-transform in [24].} \]
and, as we will discuss later, such a map is called an anchor. In components, equation (8) reads as follows: denoting by \( \{ e^i \} \in T^* M \) the basis dual to \( \{ e_i \} \), that is \( e^i (e_j) = \delta^i_j \), one finds
\[
e_i^j = \beta^i (e^j) = \beta^{ij} \partial_j .
\]
Thus, there are two kinds of derivative operators: \( e_i \) and \( e^i \). We also observe that the derivative \( \tilde{D}^i = \tilde{\partial}^i - \beta^{ij} \partial_j \) introduced in [17], in the case of vanishing winding derivatives \( \tilde{\partial}^i \), is related to \( \beta \) as \( \tilde{D}^i = -e_i^i \).

It is now straightforward to show that the differential operators \( \{ e_i, e^i \} \in T M \) satisfy the following commutation relations (see [21] for closely related expressions)
\[
\begin{align*}
[e_i, e_j] &= 0 , \\
[e_i, e^j] &= Q^{jk} e_k , \\
[e^i, e^j] &= R^{ijk} e_k + Q^{kij} e^k ,
\end{align*}
\]
where \( R^{ijk} = 3 \beta^{[am} \partial_m \beta^{jk]} \) and \( Q^{kij} = \partial_k \beta^{ij} \) as defined above. The Jacobi identities for this Lie bracket imply Bianchi identities for the non-geometric \( Q \)- and \( R \)-fluxes. For vanishing \( H \)-flux and vanishing geometric flux \( f \), they read
\[
\begin{align*}
0 &= 3 \beta^{[am} \partial_m Q_{d]bc} - \partial_d R^{abc} + 3 Q_d [a[m Q_{]bc} , \\
0 &= 2 \beta^{[am} \partial_m R^{bcd] - 3 R^{[alm} Q_{m]cd} .
\end{align*}
\]
These relations were derived in [19] using a different approach based on the Schouten–Nijenhuis bracket.\(^4\) The second identity also appeared in [17] in the context of double-field theory. In the next section, we generalize the algebra (10) and the Bianchi identities (11) to the situation when all fluxes \( (H, f, Q, R) \) are non-vanishing.

### 3 Bianchi identities for all fluxes

In order to implement the geometric flux \( f \) into our discussion, we have to introduce a vielbein basis. The notation we employ is the following. The vielbeins are denoted by \( \{ e_a^i \} \), and we define in the usual way
\[
e_a = e_a^i \partial_i .
\]
The \( \{ e_a^i \} \) are required to be orthonormal with respect to the metric \( g_{ij} \) on \( M \), that is \( e_a^i e_b^j g_{ij} = \delta^{ab} \), and the inverse of \( e_a^i \) is denoted by \( e^a_i \). Furthermore, in general the fields \( e_a \) do not commute but have a non-vanishing Lie bracket
\[
[e_a, e_b] = f_{ab}^c e_c ,
\]
\(^4\)Note that in [19] we used a convention for \( R \) which differs by a factor of two.
where \( f_{abc} \) is called geometric flux and is given by
\[
f_{abc} = e^j_i (e_a^i \partial_i e_b^j - e_b^i \partial_i e_a^j).
\]
The basis dual to \( \{ e_a \} \) will be denoted by \( \{ e^a \} \) and is constructed as \( e^a = e^a_i e_i \).
The map \( \beta \) induced by the quasi-Poisson structure \( \beta \) acting on \( e^a \) then reads
\[
e^a_i = \beta^i (e^a) = e^a_i \beta^j \partial_j = \beta^{ab} e_b,
\]
where we employed \( \beta^{ab} = e^a_i e^b_j \beta^i j \). Recall that for the case of a vanishing torsion tensor we have the relation
\[
d e^a = -\frac{1}{2} f_{abc} e^b \wedge e^c,
\]
and the connection coefficients \( \Gamma^c_{ab} \) in the non-coordinate basis satisfy
\[
f_{abc} = \Gamma^c_{ab} - \Gamma^c_{ba} \Gamma \text{ satisfies. The corresponding covariant derivative is denoted by } \nabla_a.
\]

**Pre-Roytenberg algebra**

After having introduced our notation for the vielbeins, let us now evaluate two Lie brackets. In particular, we compute
\[
\begin{align*}
[e_a, e^b] & = Q^{bc} e^c - f_{ac} e^b, \\
[e^a, e^b] & = R^{abc} e_c + Q^{ab} e^c.
\end{align*}
\]
The terms appearing on the right-hand side are given by
\[
\begin{align*}
Q^{bc} & = \partial_a \beta^{bc} + f_{am}^b \beta^{mc} - f_{am}^c \beta^{mb}, \\
R^{abc} & = 3 \left( \beta^{[am} \partial_m \beta^{bc]} + f_{mn}^a \beta^{bm} \beta^{nc} \right) = 3 \beta^{[am} \nabla_m \beta^{bc]},
\end{align*}
\]
and correspond to the non-geometric \( Q \)- and \( R \)-fluxes in the presence of non-vanishing geometric flux \( f \). However, the remaining flux \( H \in \Gamma(\Lambda^3 T^*M) \) does not yet appear in the Lie algebra defined by the commutators \( [13] \) and \( [16] \). Let us therefore perform the following redefinitions
\[
\begin{align*}
H_{abc} & = H_{abc}, \\
F_{ab}^c & = f_{ab}^c - H_{abm} \beta^{mc}, \\
Q_a^{bc} & = Q_a^{bc} + H_{amm} \beta^{mb} \beta^{nc}, \\
R^{abc} & = R^{abc} - H_{mnp} \beta^{ma} \beta^{mb} \beta^{pc},
\end{align*}
\]
where \( H_{abc} \) denotes the components of the usual \( H \)-field in the non-coordinate basis. Recalling its definition \( H = dB \) in terms of the two-form gauge field \( B \), we can infer
\[
H_{abc} = H_{abc} = 3 \nabla_{[a} B_{bc]}.
\]

Employing \( [18] \), the commutators \( [13] \) and \( [16] \) then take the form (see also \[24\])
\[
\begin{align*}
[e_a, e_b] & = F_{ab}^c e^c + H_{abc} e^c, \\
[e_a, e^b] & = Q_a^{bc} e^c - F_{ab}^c e^c, \\
[e_a^b, e^b] & = R^{abc} e_c + Q_c^{ab} e^c.
\end{align*}
\]
These relations are very similar to the Roytenberg bracket \[26, 27\] for a particular basis (see \[13\]), with the only difference that the latter is not given by a Lie bracket on \( TM \) but rather by a so-called Courant bracket on \( TM \oplus T^*M \). Therefore, we call \[20\] a pre-Roytenberg algebra, and we will clarify the relation between these two structures in the next section.

**Bianchi identities**

Using the commutation relations \[20\], we can now deduce the Bianchi identities for the various fluxes from the Jacobi identities of the pre-Roytenberg algebra. First, we have the usual Bianchi identity for the \( H \)-flux which in the presence of geometric flux can be written as

\[
I : 0 = \nabla_{[a} \mathcal{H}_{bcd]} = \partial_{[a} \mathcal{H}_{bcd]} - \frac{3}{2} F_{[ab}^\ m \mathcal{H}_{mcd]} .
\]

Next, since the Lie bracket satisfies the Jacobi identity, we can evaluate the following four equations

\[
II : 0 = [[e_a, e_b], e_c] + \text{cycl.}, \quad III : 0 = [[e_a, e_b], e_c^\sharp] + \text{cycl.}, \\
IV : 0 = [[e_a, e_b^\sharp], e_c^\sharp] + \text{cycl.}, \quad V : 0 = [[e_a^\sharp, e_b^\sharp], e_c^\sharp] + \text{cycl.},
\]

which lead to the four Bianchi identities

\[
II : 0 = \left( \partial_{[a} F_{b]d}^\ d + F_{[ab}^\ m F_{cd]}^\ m + \mathcal{H}_{[ab|m} Q_{cd]}^\ m \right) \\
+ \left( \partial_{[a} \mathcal{H}_{b]m} - 2 F_{[ab}^\ m \mathcal{H}_{mcd]} \right) \beta^{md},
\]

\[
III : 0 = \left( \beta^{cm} \partial_m F_{ab}^\ d + 2 \partial_{[a} \mathcal{Q}_{b]cd} - \mathcal{H}_{mab} R_{mcd} + F_{[ab}^\ m Q_{mcd]} + 4 Q_{[a}^m F_{m[b]}^\ d \right) \\
+ \left( \beta^{cm} \partial_m \mathcal{H}_{abn} - 2 \partial_{[a} F_{b]n}^\ c - 3 \mathcal{H}_{m[a} Q_{b]m}^{mc} + 3 F_{[ab}^\ m F_{mcd]} \right) \beta^{nd},
\]

\[
IV : 0 = \left( - \partial_a R_{bcd} - 2 \beta^{cm} \partial_m Q_{a}^b^d + 3 Q_{a}^m [^b_m Q_{m}^{cd} - 3 F_{am}^b R^{cd]} \right) \\
+ \left( 2 \beta^{cm} \partial_m F_{an}^b + \partial_a Q_{n}^{bc} + Q_{m}^{bc} F_{an}^m \right) \\
+ \left( R^{bcm} \mathcal{H}_{man} - 4 Q_{[a}^m F_{m[b]}^\ d \right) \beta^{nd},
\]

\[
V : 0 = \left( \beta^{cm} \partial_m R_{ab}^b_d - 2 R_{[abm} Q_{m}^{cd]} \right) \\
+ \left( \beta^{cm} \partial_m Q_{m}^{ab] + R_{[abm} F_{mn]}^\ d + Q_{m}^{ab} Q_{n}^{m} \right) \beta^{nd}.
\]
Remarks

We close this section with some remarks.

• Note that for vanishing fluxes $H$ and $f$, the equations IV and V reduce to the Bianchi identities (11) in the geometric basis. Furthermore, equation III reduces to $\partial_a Q_{\alpha \beta \gamma \delta} = 0$.

• Equation II is the Bianchi identity for the usual Riemann curvature tensor, while in the case of vanishing geometric flux equation IV is the Bianchi identity of the second curvature tensor $\tilde{R}^{i}_{jkl}$ defined in [17, 12].

• For constant fluxes, the five Bianchi identities above reduce to the system of Bianchi identities derived in [3, 20, 21] which, in our notation, read

\begin{align}
0 &= H_{k[a} \mathcal{F}^{k]cd} , \\
0 &= H_{k[a} \mathcal{Q}_{\alpha \beta}^{\gamma j} - \mathcal{F}_{k[a}^{j} \mathcal{F}_{\beta]k} , \\
0 &= H_{kab} \mathcal{R}^{kcd} + \mathcal{F}_{ab}^{k} \mathcal{Q}_{\alpha \beta}^{cd} - 4 \mathcal{F}_{k[a}^{[e} \mathcal{Q}_{\beta]}^{d]k} , \\
0 &= \mathcal{F}^{[a}_{k[bi} \mathcal{R}^{bc]k} - \mathcal{Q}_{i}^{[a} \mathcal{Q}^{bc]}_{k} , \\
0 &= \mathcal{Q}_{k}^{[ab} \mathcal{R}^{cd]k} .
\end{align}

4 Lie and Courant algebroids

In the previous two sections, we have considered a framework based on the bi-vector field $\beta$ to compute Bianchi identities. However, it has been noted in the literature that the underlying mathematical structures for non-geometric fluxes are Lie and Courant algebroids, which generalize Lie algebras such that structure “constants” become space-time dependent.

In this section, we first briefly review the relevant notions for algebroids and then show how our previous analysis fits into this scheme. For mathematically more rigorous and complete definitions we would like to refer the reader to the existing literature, in particular to [26, 27, 28].

Gerstenhaber algebra and Schouten–Nijenhuis bracket

Let us start by introducing the Gerstenhaber algebra which is a graded, associative, super-commutative algebra $G^* = \bigoplus_k G^k$ with respect to a product $\wedge$, together with a graded Lie bracket $[\cdot, \cdot]_G$ such that the following Leibniz rule holds

\begin{equation}
[a, b \wedge c]_G = [a, b]_G \wedge c + (-1)^{(k-1)l} b \wedge [a, c]_G ,
\end{equation}

where $a \in G^k, b \in G^l$ and $c \in G^*$. Super-commutativity means

\begin{equation}
[a, b]_G = -(-1)^{(k-1)(l-1)} [b, a]_G .
\end{equation}
In addition, for a Gerstenhaber algebra the graded Jacobi identity is satisfied
\[[a, [b, c]]_G = [[a, b], c]_G + (-1)^{(k-1)(l-1)} [b, [a, c]]_G .\] (27)

An example of a Gerstenhaber algebra is the Schouten–Nijenhuis bracket \([\cdot, \cdot]_{SN}\), which for functions \(f, g \in C^\infty(M)\) and vector fields \(X, Y \in \Gamma(TM)\) it is defined by
\[[f, g]_{SN} = 0 , \quad [X, f]_{SN} = X(f) , \quad [X, Y]_{SN} = [X, Y]_L ,\] (28)
with \([\cdot, \cdot]_L\) being the Lie bracket. The Schouten–Nijenhuis bracket is uniquely extended to arbitrary alternating multi-vector fields in \(\Gamma(\Lambda^*T^*M)\) with usual exterior product by demanding (25) and (26). Similarly, any exterior algebra of a Lie algebra is a Gerstenhaber algebra.

**Lie algebroids**

Next, we turn to Lie algebroids. A vector bundle \(E\) over a manifold \(M\) is called a **Lie algebroid**, if it is equipped with a Lie bracket \([\cdot, \cdot]_E\) and a bundle homomorphism \(\rho : E \to TM\), called an **anchor**, such that the following Leibniz rule holds
\[[s_1, f s_2]_E = f [s_1, s_2]_E + (\rho(s_1)f) s_2 ,\] (29)
where \(s_i\) are sections of \(E\) and \(f \in C^\infty(M)\). The Lie algebroid \((E, [\cdot, \cdot]_E, \rho)\) then has the following important properties. First, the space of sections \(\Gamma(\Lambda^*E)\) is a Gerstenhaber algebra with the bracket determined by
\[[f, g]_G = 0 , \quad [f, s]_G = -\rho(s) f , \quad [s_1, s_2]_G = [s_1, s_2]_E ,\] (30)
as well as by (25) and (26). Second, \(\Gamma(\Lambda^*E^*)\) is a graded differential algebra, and the differential with respect to the multiplication \(\wedge\) is given by
\[(d_E \omega)(s_0, \ldots, s_k) = \sum_{i=0}^{k} (-1)^i \rho(s_i) (\omega(s_0, \ldots, \hat{s}_i, \ldots, s_k)) \]
\[+ \sum_{i<j} (-1)^{i+j} \omega([s_i, s_j]_E, s_0, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_k) ,\] (31)
where \(\omega \in \Gamma(\Lambda^kE^*)\) and \(\{s_i\} \in \Gamma(E)\). Let us mention that these two properties are equivalent to the definition of a Lie algebroid given in [29].

There are two standard examples for Lie algebroids which are also of importance for our analysis. We discuss them in turn.

- **We first consider** \(\mathcal{A} = (TM, [\cdot, \cdot]_L, \rho = \text{id})\), where the anchor is the identity map and the bracket is given by the usual Lie bracket \([X, Y]_L\) of vector fields. The Gerstenhaber bracket on \(\Gamma(\Lambda^*TM)\) is given by the Schouten–Nijenhuis bracket (28) and the differential on \(\Gamma(\Lambda^*T^*M)\) is the usual de Rham differential. Note that for a vielbein basis \(\{e_a\}\) with geometric flux \(f\) we obtain (13).
For the second example, we let \((M, \beta)\) be a Poisson manifold with Poisson tensor \(\beta = 1/2 \beta_{ij} e_i \wedge e_j\). In view of (5), this means \(\mathcal{R} = 1/2 [\beta, \beta]_{SN} = 0\). The Lie algebroid is then given by \(\mathcal{A}^* = (T^* M, [\cdot, \cdot]_K; \rho = \beta^\sharp)\), where the anchor is defined as in (8). The bracket on \(T^* M\) is the Koszul bracket defined on one-forms as

\[
[\xi, \nu]_K = \mathcal{L}_{\beta^\sharp(\xi)} \nu - \iota_{\beta^\sharp(\nu)} d\xi ,
\]

where the Lie derivative reads \(\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X\). The associated Gerstenhaber bracket is called the Koszul–Schouten bracket. The corresponding differential on \(T M\) is given in terms of the Schouten–Nijenhuis bracket

\[
d_\beta = [\beta, \cdot]_{SN} .
\]

Note that for the basis \(\{e^a\}\) we obtain \([e^a, e^b]_K = Q^{ab}_c e^c\), with the non-geometric flux \(Q\) as in (17).

These two Lie algebroids can be combined into a Lie bi-algebroid \((\mathcal{A}, \mathcal{A}^*)\) where, in this particular case, the definition of the latter requires the de Rham differential to be a derivation of the Koszul–Schouten bracket. That means

\[
d[e^a, e^b]_K = [de^a, e^b]_K + [e^a, de^b]_K ,
\]

which can be brought into the form

\[
0 = \beta^{am} \left( \partial_p f_{qm}^{ \; [b} + f_{pq}^{ \; n} f_{mn}^{ \; b} \right) = \beta^{[am} R^{b]}_{[pq]m} ,
\]

where \(R^{ab}_{cd}\) is the curvature tensor of the connection \(\Gamma_a^{bc}\). Note that due to the Bianchi identity for the curvature tensor, the relation (35) is automatically satisfied. Thus, the above two examples indeed combine into a Lie bi-algebroid.

Let us remark that the framework of Lie (bi-)algebroids provides a natural way of implementing non-constant \(f\)- and \(Q\)-fluxes, which is not possible in the realm of Lie algebras.

**Quasi-Lie algebroids**

In order to also describe \(H\)- and \(R\)-fluxes, we have to generalize the above structure to include twists (for a review see for instance [30]). For the two Lie algebroids of interest, twists can be realized as follows.

- We first consider \(\mathcal{A}_H = (T M, [\cdot, \cdot]_L^H; \text{id}_{TM}; H)\) with an \(H\)-twisted Lie bracket

\[
[X, Y]_L^H = [X, Y]_L - \beta^\sharp \left( \iota_Y \iota_X H \right) .
\]

The bracket (36) does not satisfy the Jacobi identity and is therefore called a quasi-Lie algebroid. However, the Jacobi identity is satisfied upon setting
\( H = 0 \); thus \( H \) measures the defect of \( \mathcal{A}_H \) to be a Lie algebroid. In particular, let us evaluate (36) for two fields \( e_a \) in a vielbein basis \( \{e_a\} \) with geometric flux \( f \). We obtain
\[
[e_a, e_b]^H_L = f_{ab}^p e_p - H_{abm} \beta^{mp} e_p = F_{ab}^c e_c .
\] (37)

- For the second example, we consider \( \mathcal{A}_H^* = (T^*M, [\cdot, \cdot]^H_K, \beta^*; \mathcal{R}) \) with the \( H \)-twisted Koszul bracket
\[
[\xi, \nu]^H_K = [\xi, \nu]^K + \iota_{\beta^*(\nu)} \iota_{\beta^*(\xi)} H .
\] (38)

Again, this bracket does not satisfy the Jacobi identity but defines a quasi-Lie algebroid, and the defect of \( \mathcal{A}_H^* \) is measured by \( \mathcal{R} \in \Gamma(\Lambda^3TM) \) given by
\[
\mathcal{R}^{abc} = \frac{1}{2} [\beta, \beta]^{abc}_{SN} + \beta^{am} \beta^{bn} \beta^{ck} H_{mnk} .
\] (39)

That is, the Jacobi identity is satisfied iff \( \mathcal{R} = 0 \), which also guarantees \( \beta^* \) to be an algebra homomorphism. This condition is called the quasi-Poisson condition [31]. Evaluating the twisted Koszul bracket (38) for the dual basis \( \{e_a^*\} \), we obtain
\[
[e^a, e^b]^K = \partial_p \beta^{ab} e^p + 2 f_{pm} \beta^m \beta^p H_{mp} e^p = Q_{abc} e^c .
\] (40)

To summarize, we see that the fluxes \( \mathcal{H} \) and \( \mathcal{R} \) introduced in (18) have a direct interpretation as the defects to the Lie algebroid properties. Furthermore, also the fluxes \( \mathcal{F} \) and \( \mathcal{Q} \) appear naturally via the brackets of two basis fields.

**The associated Courant algebroid**

Our aim is to identify a framework in which all the fluxes in (18) appear. So far, we have described \( (\mathcal{H}, \mathcal{F}) \) on \( TM \) as well as \( (\mathcal{R}, \mathcal{Q}) \) on \( T^*M \). Hence, we are naturally lead to seek for a suitable structure on \( TM \oplus T^*M \) respecting both \( \mathcal{A}_H \) and \( \mathcal{A}_H^* \). Following [32, 26, 27], we therefore consider \( TM \oplus T^*M \) equipped with

- a bi-linear form for \( (X + \xi) \in \Gamma(TM \oplus T^*M) \), where \( X \in \Gamma(TM) \) and \( \xi \in \Gamma(T^*M) \), which reads
\[
\langle X + \xi, Y + \nu \rangle = \xi(Y) + \nu(X) .
\] (41)

- a skew-symmetric bracket \( [\cdot, \cdot] \) on \( \Gamma(TM \oplus T^*M) \) composed of
\[
\begin{align*}
[X, Y] &= [X, Y]^H_L + \iota_Y \iota_X H , \\
[X, \xi] &= [\iota_X H]^+ + \xi - [\iota_X, d_H^\beta]^+ X + \frac{1}{2} (d_H - d_H^\beta) \langle X, \xi \rangle , \\
[\xi, X] &= [\iota_\xi H]^+ X - [\iota_\xi, d_H^\beta]^+ X + \frac{1}{2} (d_H - d_H^\beta) \langle \xi, X \rangle , \\
[\xi, \nu] &= [\xi, \nu]^H_K + \iota_\nu \iota_\xi \mathcal{R} ,
\end{align*}
\] (42)
with $d^H$ the $H$-twisted de Rham differential given by (31) via (36), $d^H_\beta$ the $H$-twisted Poisson differential associated to (38), and $[\cdot, \cdot]_+$ the anticommutator.

- an algebra homomorphism (anchor) given by $\alpha(X + \xi) = X + \beta^\sharp(\xi)$.

This additional structure makes $TM \oplus T^*M$ into a Courant algebroid. The required axioms for the latter are the following [32]:

1. The anchor $\alpha$ satisfies $\alpha([s_1, s_2]) = [\alpha(s_1), \alpha(s_2)]$ for sections $s_1, s_2 \in \Gamma(TM \oplus T^*M)$.

2. The Courant bracket $[\cdot, \cdot]$ satisfies the modified Leibniz rule

$$[s_1, f s_2] = f [s_1, s_2] + (\alpha(s_1)f) s_2 - \frac{1}{2} \langle s_1, s_2 \rangle_+ D f ,$$

where $D = d^H + d^H_\beta$ and $f \in C^\infty(M)$.

3. The anchor satisfies $\alpha \circ D = 0$.

4. For $s_1, s_2, s_3 \in \Gamma(TM \oplus T^*M)$ the following relation holds

$$\alpha(s_1) \langle s_2, s_3 \rangle_+ =$$

$$\langle [s_1, s_2] + \frac{1}{2} D\langle s_1, s_2 \rangle_+, s_3 \rangle_+ + \langle s_2, [s_1, s_3] \rangle_+ + \frac{1}{2} D\langle s_1, s_3 \rangle_+ \rangle_+ .$$

5. The Jacobiator $\text{Jac}(s_1, s_2, s_3) = [[s_1, s_2], s_3] + \text{cycl.}$ satisfies

$$\text{Jac}(s_1, s_2, s_3) = D T(s_1, s_2, s_3)$$

where $T = \frac{1}{6} \langle [s_1, s_2], s_3 \rangle_+ + \text{cycl.}$.

The first four properties are checked directly, while the last one will become apparent in the following.

To make contact with our results in section 3, we evaluate the Courant bracket $[[\cdot, \cdot]]$ on basis sections $\{e_a, e^b\} \in TM \oplus T^*M$. We obtain

$$[[e_a, e_b]] = \mathcal{F}_{ab}^c e_c + \mathcal{H}_{abc} e^c ,$$

$$[[e_a, e^b]] = Q_a^{bc} e_c - \mathcal{F}_{ac}^b e^c ,$$

$$[[e^a, e^b]] = Q_{c}^{ab} e^c + \mathcal{R}^{abc} e_c .$$

\[5\] Strictly speaking, this is true provided $(A_H, A_H^*)$ is a proto bi-algebroid [20]. However, to avoid technical details we will argue directly that the above structure gives a Courant algebroid. Note that an analogous construction can be made for the untwisted Lie bi-algebroid $(A, A^*)$ leading to the untwisted version of this Courant algebroid.
which will be denoted the Roytenberg algebra \[26, 15\]. Applying the anchor map to these relations gives the pre-Roytenberg algebra \[20\] found in the previous section. Furthermore, evaluating the Jacobiators we find

\[
\text{Jac}(e_a, e_b, e_c) = -3 \left( \partial_a F_{[ab]}{}^d + F_{[ab}{}^m F_{cd]}{}^m + H_{[abm} Q_{cd]}{}^{md} \right) e_d
\]

\[
-3 \left( \partial_a \mathcal{H}_{[ab]}{}^d - 2 F_{[ab}{}^m \mathcal{H}_{cd]m} \right) e^d + \frac{3}{2} D \mathcal{H}_{abc},
\]

\[
\text{Jac}(e_a, e_b, e^c) = - \left( \partial_m \mathcal{F}_{ab}{}^d + 2 \partial_{[a} Q_{b]}{}^{cd} - H_{mab} \mathcal{R}^{med} - F_{ab}{}^m Q_m{}^{cd} \right.
\]

\[
+ 4 Q_{[ab}{}^m F_{mcd]} \right) e_d - \left( \partial_m \mathcal{H}_{abcd} - 2 \partial_{[a} F_{b]d}{}^c \right.
\]

\[
- 3 \mathcal{H}_{m[ab} Q_{d]}{}^{mc} + 3 F_{ab}{}^m F_{mcd]} \right) e^d + \frac{3}{2} D F_{abc},
\]

\[
\text{Jac}(e_a, e^b, e^c) = \left( - \partial_a \mathcal{R}^{bcd} - 2 \partial_m \mathcal{Q}_{a}{}^{b}{}^{cd} + 3 Q_{a}{}^{[lm} \mathcal{Q}_{m}{}^{cd]} \right.
\]

\[
- 3 \mathcal{F}_{am}{}^{[b} \mathcal{R}{}^{cdm]} \right) e_d + \left( 2 \partial_m \mathcal{F}_{ad}{}^{b} - \partial_a \mathcal{Q}_{d}{}^{bc} + \mathcal{Q}_{m}{}^{bc} \mathcal{F}_{ad}{}^{m} \right.
\]

\[
+ \mathcal{R}{}^{bcm} \mathcal{H}_{mad} + 4 Q_{[a}{}^{[lm} \mathcal{F}_{mcd]} \right) e^d + \frac{3}{2} D \mathcal{Q}_{a}{}^{bc},
\]

\[
\text{Jac}(e^a, e^b, e^c) = -3 \left( \partial_m \mathcal{R}_{ab}{}^{cd} - 2 \mathcal{R}_{[adm} \mathcal{Q}_{c]{}^{e}{}^{m} \right) e_d - 3 \left( \partial_m \mathcal{Q}_{a}{}^{ab} \right.
\]

\[
+ \mathcal{R}^{[admn} \mathcal{F}_{md]}{}^{c} + \mathcal{Q}_{m}{}^{ab} \mathcal{Q}_{d}{}^{c} \right) e^d + \frac{3}{2} D \mathcal{R}_{abc}.
\]

Let us note that the parenthesis multiplying \( e_d \) and \( e^d \) contain the same terms appearing in the four Bianchi identities \[23\]. In fact, \[23\] can be obtained by applying the anchor \( \alpha \) to the Jacobiators above since \( \alpha \circ D = 0 \). Employing then \[21\] and \[23\], we can simplify the Jacobiators considerably and bring them into the form

\[
\text{Jac}(e_a, e_b, e_c) = D T(e_a, e_b, e_c) = \frac{1}{2} D \mathcal{H}_{abc},
\]

\[
\text{Jac}(e_a, e_b, e^c) = D T(e_a, e_b, e^c) = \frac{1}{2} D \mathcal{F}_{ab}{}^c,
\]

\[
\text{Jac}(e_a, e^b, e^c) = D T(e_a, e^b, e^c) = \frac{1}{2} D \mathcal{Q}_{a}{}^{bc},
\]

\[
\text{Jac}(e^a, e^b, e^c) = D T(e^a, e^b, e^c) = \frac{1}{2} D \mathcal{R}_{abc}.
\]

These expressions are the expected defects for a Courant algebroid mentioned in \[45\]. We therefore have verified that the underlying structure to describe the fluxes \( (\mathcal{H}, \mathcal{F}, \mathcal{Q}, \mathcal{R}) \) is indeed given by a Courant algebroid.
5 Conclusions

We conclude this letter with a brief summary. Starting from a bi-vector field $\beta$, which can be considered as the T-dual of the Kalb-Ramond two-form $B$, we have followed a quite straightforward and logical path leading to an intricate structure for non-geometric fluxes. The approach we followed is summarized in the diagram below.

More concretely, we have achieved a systematic identification of the mathematical structure to describe $(H, F, Q, R)$ fluxes in a combined way. It is that of a Courant algebroid arising from twisting the standard Lie algebroid of the tangent and co-tangent bundle by $H$. Note that the Bianchi identities \cite{23} are embedded crucially in this picture.

In view of future work, we recall again that an effective action for the fields $(\tilde{g}^{ij}, \beta^{ij}, \tilde{\phi})$ has been derived in \cite{12} using double-field theory. This action is formulated on an ordinary commutative space-time, however, the existence of the quasi-Poisson structure and the evidence from CFT computations suggests that there might exist an alternative description in terms of an NCA geometry. We hope to come back to this question in the future.

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