On the maximal reduction of games

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Abstract

We study the conditions under which the iterated elimination of strictly dominated strategies is order independent and we identify a class of discontinuous games for which order does not matter. In this way, we answer the open problem raised by M. Dufwenberg and M. Stegeman (2002) and generalize their main results. We also establish new theorems concerning the existence and uniqueness of the maximal game reduction when the pure strategies are dominated by mixed strategies.

Keywords: Game theory, strict dominance, iterated elimination, order independence, maximal reduction.

1. Introduction

The question raised by Pearce (1984), concerning the rationalizable strategic behaviour of the players in noncooperative strategic situations was followed by a great amount of literature. It seemed to attract the interest of researchers from Game theory. The first step of research in this area was made by Bernheim (1984) and Pearce (1984), who defined the rationalizable strategies of a strategic game by using iterative processes of elimination of dominated strategies that were considered 'undesirable'.

This procedure led to the issue of order independence, which was studied by many authors. They searched for classes of games and defined dominance relations under which the result of the iterative process of removal of dominated strategies does not depend on the order of removal.
Gilboa, Kalai and Zemel (1990) provided conditions (including strict dominance) which guarantee the uniqueness of the reduced games. Marx and Swinkels (1997) defined nice weak dominance, proved that under this order relation, order does not matter. The main result of Dufwenberg and Stegeman (2002) concerns a class of games for which a unique and nonempty maximal reduction exists. The properties satisfied by games for which the iterated elimination for strictly dominated strategies (IESDS) preserves the set of Nash equilibria are the compactness of the strategy spaces and the continuity of payoff functions. The authors also proved that if, in addition, the payoff functions are upper semicontinuous in own strategies, then the order does not matter. Chen, Long and Luo (2007) provided a new definition of IESDS that proved to be suitable for all types of games and also order-independent. Apt’s approach (2007) uses operators on complete lattice and their transfinite iterations. The monotonicity of the operators assures the order independence of iterated eliminations. Apt’s paper (2007) provides an analysis of different ways of iterated eliminations of strategies. The notions of dominance and rationalizability are involved by other two strategy elimination procedures studied by Apt (2005). In order to study the problem of order independence for rationalizability, the author considers three reduction relations on games and belief structures.

In this paper, we identify a class of discontinuous games for which order independence holds, generalizing the main results of Dufwenberg and Stegeman (2002). The payoff functions are transfer weakly upper continuous in the sense of Tian and Zhou (1995). These authors defined the transfer upper continuity and proved generalizations of Weierstrass and of the maximum theorem. We also establish results for game reductions in which the pure strategies are dominated by mixed strategies. We use some notions of measurability and especially some results of Robson (1990).

The paper is organized in the following way: Section 2 contains preliminaries and notations. Generalizations of Dufwenberg-Stegeman Lemma are presented in Section 3. The mixed strategy case is treated in Section 4. The concluding remarks follow at the end.

2. Preliminaries

Dufwenberg and Stegeman (2002) concluded that it remained an open problem to identify classes of games for which order independence holds, outside of the compact and continuous class.
We are searching to solve this problem. In order to reach this aim, we first introduce the notions of games, parings, dominance and game reduction, following that, in the next subsection, we discuss the transfer upper continuity, a concept due to Tian and Zhou, which characterizes the payoff functions of a class of games which generalizes than that one of Dufwenberg and Stegeman. In section 3, we will prove that, in this case the iterated elimination of strictly dominated strategies (IESDS) also produces a unique maximal reduction.

2.1. Games, Parings, Dominance and Reduction

In the paper called "Equilibrium points in n-person games" (1950), Nash describes without formalizing, the concepts of the n-person game and the equilibrium of the attached game. He defines the n-person game, where each player has a finite number of strategies and each n-tuple of strategies corresponding to a given set of players wins. Any n-tuple of strategies can be regarded as a point in the product space of sets of players’ strategies. A point of equilibrium is an n-tuple of strategies such that every player’s strategy brings the maximum payout for that player, against n-1 strategies of the other ones.

We give the formal definition of an n-person game below.

**Definition 1.** The normal form of an n-person game is $G = (I, (G_i)_{i \in I}, (r_i)_{i \in I})$, where, for each $i \in I = \{1, 2, ..., n\}$, $G_i$ is a non-empty set (the set of individual strategies of player $i$) and $r_i$ is the preference relation on $\prod_{i \in I} G_i$ of player $i$.

The individual preferences $r_i$ are often represented by utility functions, i.e. for each $i \in \{1, 2, ..., n\}$ there exists a real valued function $u_i : \prod_{i \in I} G_i \to \mathbb{R}$ (called the utility function of $i$), such that $x r_i y \iff u_i(x) \geq u_i(y), \forall x, y \in \prod_{i \in I} G_i$.

Then the normal form of n-person game is $(I, (G_i)_{i \in I}, (u_i)_{i \in I})$.

**Notation.** Denote $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ and $G_{-i} = \prod_{j \in I \setminus \{i\}} G_j$.

**Definition 2.** The Nash equilibrium for the game $(I, (G_i)_{i \in I}, (u_i)_{i \in I})$ is a point $x^* \in \prod_{i \in I} G_i$ which satisfies for each $i \in \{1, 2, ..., n\} : u_i(x^*) \geq u_i(x_{-i}^*, x_i)$ for each $x_i \in G_i$.

Further we will assume that for each $i \in I$, the set $G_i$ is a Hausdorff topological space and $\prod_{i \in I} G_i$ is endowed with the product topology.
Definition 3. The game $G$ is called

i) compact if $G_i$ is compact for each $i \in I$;

ii) own-uppersemicontinuous if $u_i(\cdot, s_{-i})$ is upper semicontinuous for each $i \in I$ and for each $s_{-i} \in G_{-i}$;

iii) continuous if $u_i$ is continuous for each $i \in I$.

Definition 4. (Dufwenberg and Stegeman, 2002). A paring of $G$ is a triple $H = (I, (H_i)_{i \in I}, (u'_i)_{i \in I})$, where $H_i \subseteq G_i$ and $u'_i = u_i\prod_{i \in I} H_i$.

A paring is nonempty if $H_i \neq \emptyset$ for each $i \in I$.

Definition 5. Given a paring $H$ of $G$, the strict dominance relation $\succ_H$ on $G_i$ can be defined:

for $x, y \in G_i$, $y \succ_H x$ if $H_{-i} \neq \emptyset$ and $u_i(y, s_{-i}) > u_i(x, s_{-i})$ for each $s_{-i} \in H_{-i}$.

Remark 1. $\succ_H$ is transitive.

Let us consider parings $G, H$ with the property that $H_i \subseteq G_i$ for each $i \in I$. We give here the definition of game reduction used by Dufwenberg and Stegeman (2002), in order to generalize their main results, following that, in the next section we will introduce other types of reduction and discuss the relationships amongst them.

Definition 6. i) $G \rightarrow H$ is called a reduction if for each $x \in G_i \setminus H_i$, there exists $y \in G_i$ such that $y \succ_G x$.

ii) the reduction $G \rightarrow H$ is called fast if $y \succ_G x$ for some $x, y \in G_i$ implies $x \notin H_i$.

iii) the reduction $G \rightarrow^* H$ is defined by the existence of (finite or countable infinite) sequence of parings $A^t$ of $G$, $t = 0, 1, 2, \ldots$, such that $A^0 = G$, $A^t \rightarrow A^{t+1}$ for each $t \geq 0$ and $H_i = \cap_t A^t_i$ for each $i \in I$.

iv) $H$ is said to be a maximal ($\rightarrow^*$)-reduction of $G$ if $G \rightarrow^* H$ and $H \rightarrow H'$ only for $H = H'$.
2.2. Transfer upper continuity

Tian and Zhou (1995) relaxed the continuity assumptions on functions and correspondences which can be used in some economic models. Their work was motivated by questions concerning the minimal conditions under which a function reaches its maximum on a compact set or the set of maximum points of a function defined on a compact set is non-empty and compact. Tian and Zhou introduced the transfer continuities and generalized the Weierstrass Theorem by giving a necessary and sufficient condition for a function $f$ to reach its maximum on a compact set.

We are providing here the concepts of transfer upper semicontinuity and transfer weakly upper continuity for functions, the concept of transfer closed-valuedness for correspondences and some of their properties.

Let $X, Y$ be subsets of topological spaces.

**Definition 7.** A function $f : X \to \mathbb{R}$ is said to be upper semicontinuous on $X$ if $\{x \in X : f(x) \geq r\}$ is closed in $X$ for all $r \in \mathbb{R}$.

**Definition 8.** (Tian and Zhou, 1995) A function $f : X \to \mathbb{R} \cup \{-\infty\}$ is said to be transfer upper continuous on $X$ if for points $x, y \in X$, $f(y) < f(x)$ implies that there exists a point $x' \in X$ and a neighborhood $\mathcal{N}(y)$ of $y$ such that $f(z) < f(x')$ for all $z \in \mathcal{N}(y)$.

**Definition 9.** (Tian and Zhou, 1995) A correspondence $F : X \to 2^Y$ is said to be transfer closed-valued on $X$ if for every $x \in X$, $y \notin F(x)$ implies that there exists $x' \in X$ such that $y \notin \text{cl}F(x')$.

**Remark 2.** (Tian and Zhou, 1995) It is clear that, for any function $f : X \to \mathbb{R} \cup \{-\infty\}$, the correspondence $F : X \to 2^X$ defined by $F(x) = \{y \in X : f(y) \geq f(x)\}$ for all $x \in X$ is transfer closed-valued on $X$ if and only if $f$ is transfer upper continuous on $X$.

The next lemma characterizes the correspondences which have transfer closed-values.

**Lemma 1.** (Tian and Zhou, 1995) Let $X$ and $Y$ be two topological spaces, and let $F : X \to 2^Y$ be a correspondence. Then, $\cap_{x \in X} \text{cl}F(x) = \cap_{x \in X} F(x)$ if and only if $F$ is transfer closed-valued on $X$. 


The next property is a necessary condition for a function to have a maximum on a choice set $G$.

**Definition 10.** (Tian and Zhou, 1995) A function $f : X \to \mathbb{R} \cup \{-\infty\}$ is said to be transfer weakly upper continuous on $X$ if, for points $x, y \in X$, $f(y) < f(x)$ implies that there exists a point $x' \in X$ and a neighbourhood $\mathcal{N}(y)$ of $y$, such that $f(z) \leq f(x')$ for all $z \in \mathcal{N}(y)$.

Theorem 1 generalizes the Weierstrass theorem.

**Theorem 2.** (Tian and Zhou, 1995) Let $X$ be a compact subset of a topological space and let $f : X \to \mathbb{R} \cup \{-\infty\}$ be a function. Then $f$ reaches its maximum on $X$ if and only if $f$ is transfer weakly upper continuous on $X$.

Morgan and Scalzo (2007) defined the upper pseudocontinuity and proved the existence of Nash equilibrium for economic models with payoff functions having this property.

**Definition 11.** (Morgan and Scalzo, 2007) Let $X$ be a topological space and $f : X \to \mathbb{R}$. $f$ is said to be upper pseudocontinuous at $z_0 \in X$ such that $f(z_0) < f(z)$, we have $\limsup_{y \to z_0} f(y) < f(z_0)$.

**Remark 3.** The class of upper pseudocontinuous functions is strictly included in the class of transfer upper continuous functions introduced by Tian and Zhou.

3. Generalizations of Dufwenberg-Stegeman Lemma

The following lemma is due to Dufwenberg and Stegeman (2002).

**Lemma 3.** If $G \to^* H$ for some compact and own-uppersemicontinuous game $G$, and $y \succ^H x$ for some $x, y \in G_i$ and $i \in I$, then there exists $z^* \in H_i$ such that $z \not\succ^H z^* \succ^H x$ for each $z \in G_i$.

Let $G \to H$ be a game reduction. We introduce the following definition.

**Definition 12.** $\succ^H$ has property $K$ if for each $i \in I$ and for each $y \in G_i$, there exists $z_0 \in G_i$ with $z_0 \succeq^H y$ such that $\{z \in G_i : z \succeq^H z_0\}$ is compact.
Lemma 3 generalizes the Dufwenberg-Stegeman Lemma by relaxing the continuity assumption on the payoff functions of the game. We use the notion of transfer upper continuity due to Tian and Zhou (1995). Note that $G$ may not be compact.

Before stating the lemma, we define two types of discontinuous games.

**Definition 13.** The game $G$ is called

i) own transfer upper continuous if $u_i(\cdot, s_{-i})$ is transfer upper continuous for each $i \in I$ and for each $s_{-i} \in G_{-i}$;

ii) own transfer weakly upper continuous if $u_i(\cdot, s_{-i})$ is transfer weakly upper continuous for each $i \in I$ and for each $s_{-i} \in G_{-i}$;

**Lemma 4.** Let us assume that $G \rightarrow^* H$ for an own-transfer weakly upper continuous game $G$ and $\succ_H$ has property $K$. If $y \succ_H x$ for some $x, y \in G_i$ and $i \in I$, then there exists $z^* \in H_i$ such that $z^* \not\succ_H z$ for each $z \in G_i$.

**Proof.** Since $G \rightarrow^* H$, there exists a sequence of parings $A^t$, $t = 0, 1, 2...$ such that $A^0 = G$, $A^t \rightarrow A^{t+1} \forall t \geq 0$ and $H_i = \cap_t A_i^t$, $\forall i \in I$.

Let $Z := \{z \in G_i : u_i(z, s_{-i}) \geq u_i(y, s_{-i}) \forall s_{-i} \in H_{-i}\}$. According to property $K$ of $\succ_H$, it follows that there exists $z_0 \in G_i$ such that $z_0 \succeq_H y$ and $U := \{z \in G_i : z \succeq_H z_0\}$ is compact. Since $y \succ_H x$, we have that $H_{-i} \neq \emptyset$. Let us define $f : U \rightarrow \mathbb{R}$ by $f(z) = u_i(z, s^*_{-i})$, where $s^*_{-i} \in H_{-i}$ is fixed.

Since $f$ is transfer weakly upper continuous on $U$, $f$ reaches its maximum in $z^* \in U \subset Z$. We note that $z^* \in Z$ and $y \succ_H x$ imply $z^* \succ_H x$. If $z \succ_H z^*$ for some $z \in G_i$, then $u_i(z, s_{-i}) > u_i(z^*, s_{-i}) \forall s_{-i} \in H_{-i}$, implying that $z \in U$ and $f(z) > f(z^*)$, contradiction. Therefore, $z \not\succ_H z^* \forall z \in G_i$, so that $z \not\succ_A, z^* \forall z \in G_i \forall t \geq 0$ implying that $z^* \in A_i^t \forall t \geq 0$. It follows that $z^* \in H_i$.

**Example 1.** Let $I = \{1, 2\}$, $G_1 = G_2 = [0, 2]$, $u_i : G_i \times G_j \rightarrow \mathbb{R}$,

$$u_i(x, y) = \begin{cases} 1 & \text{if } x = 0; \\ 2 & \text{if } x \in (0, 1); \\ x + 1 & \text{if } x \in [1, 2]. \end{cases}$$

Let $H = (H_1, H_2)$, $H_1 = H_2 = [0, 1]$.

We notice that, for each $y \in G_2$, $u_i(\cdot, y)$ is transfer weakly upper continuous on $[0, 2]$ and $u_i(\cdot, y)$ is not upper semicontinuous at $x = 0$.

We prove that $\succ_H$ has property $K$:
If $y = 0$, there exists $z_0 = 0$ such that $U(0) = \{z \in [0, 2] : u_1(z, s) \geq u_1(0, s) \text{ for each } s \in H_2\} = [0, 2]$ is a compact set.

If $y \in (0, 1)$, there exists $z_0 = \frac{3}{2}$ such that $U\left(\frac{3}{2}\right) = \{z \in [0, 2] : u_1(z, s) \geq u_1\left(\frac{3}{2}, s\right) \text{ for each } s \in H_2\} = [\frac{3}{2}, 1]$ is a compact set.

If $y \in [1, 2]$, there exists $z_0 = y$ such that $U(z_0) = \{z \in [0, 2] : u_1(z, s) \geq u_1(z_0, s) \text{ for each } s \in H_2\} = [y, 2]$ is a compact set.

We have that for any $x, y \in [0, 2]$ such that $y \succeq_H x$, there exists $z^* \in [0, 2]$ such that $z^* \succeq_H x$ and $z \not\succeq_H z^* \succeq x$ for each $z \in H_i$.

If $H = G$, we obtain the following corollary.

Corollary 5. Let assume that $G$ is an own-transfer weakly upper continuous game $G$ and $\succ_G$ has property $K$. If $y \succ_G x$ for some $x, y \in G_i$ and $i \in I$, then there exists $z^* \in G_i$ such that $z \not\succ_G z^* \succ_G z$ for each $z \in G_i$.

If in the last corollary, the game $G$ is transfer upper semicontinuous and compact, we obtain the following result.

Corollary 6. Let assume that $G$ is a compact, own transfer upper semicontinuous game $G$. If $y \succ_G x$ for some $x, y \in G_i$ and $i \in I$, then there exists $z^* \in G_i$ such that $z \not\succ_G z^* \succ_G z$ for each $z \in G_i$.

In order to obtain other generalization of Dufwenberg-Stegeman Lemma (2002), we further define the property $\mathcal{M}$ for a function $u$.

Definition 14. Let $X$ be a subset of a topological space. The function $u : X \rightarrow \mathbb{R} \cup \{-\infty\}$ has the property $\mathcal{M}$ on $X$ if for each $y \in X$, $x \in \text{cl}\{z \in X : u(z) \geq u(y)\}$, $\{z \in X : u(z) \geq u(y)\}$, implies there exists $x' \in X$ such that $u(x') > u(x)$.

We provide an example of transfer weakly upper continuous function which verifies the property $\mathcal{M}$.

Example 2. $u : [0, 1] \rightarrow \mathbb{R}$, $u(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number;} \\ 0, & \text{otherwise}. \end{cases}$

First, let $y \in Q$. For example, let $y = \frac{1}{2}$, $u(y) = 1$. Let $U = \{z \in [0, 1] : u(z) \geq 1\} = [0, 1] \cap Q$. The set $\text{cl}U = [0, 1]$. If $x \in \text{cl}U \setminus U = [0, 1] \cap (\mathbb{R} \setminus Q)$, $u(x) = 0$ and there exists $x' \in [0, 1]$ such that $u(x') = 1 > u(x) = 0$.

The property $\mathcal{M}$ is also verified for $y \in \mathbb{R} \setminus Q$.

Lemma 4 also generalizes Dufwenberg-Stegeman Lemma (2002).
Lemma 7. Let us assume that $G \to^* H$ for a compact and own-transfer weakly upper continuous game $G$ and for each $i \in I$ and for each $s_{-i} \in H_{-i}$, the function $u_i(\cdot, s_{-i})$ has property $\mathcal{M}$. If $y \succ_H x$ for some $x, y \in G_i$ and $i \in I$, then there exists $z^* \in H_i$ such that $z \not\succ_H z^* \succ_H z$ for each $z \in G_i$.

Proof. Since $G \to^* H$, there exists a sequence of parings $A^t, t = 0, 1, 2...$ such that $A^0 = G, A^t \to A^{t+1} \forall t \geq 0$ and $H_i = \bigcap_t A^t_i, \forall i \in I$.

Since $y \succ_H x$, we have that $H_{-i} \neq \emptyset$. For each $s_{-i} \in H_{-i}$, let $Z(s_{-i}) := \{z \in G_i : u_i(z, s_{-i}) \geq u_i(y, s_{-i})\}$ and $Z := \cap_{s_{-i} \in H_{-i}} \text{cl}(Z(s_{-i}))$. The set $Z$ is compact. Let us define $f : Z \to \mathbb{R}$ by $f(z) = u_i(z, s^*_{-i})$, where $s^*_{-i} \in H_{-i}$ is fixed.

Since $f$ is transfer weakly upper continuous on $Z$, $f$ attains its maximum in $z^* \in Z$. Each $u_i$ has property $\mathcal{M}$ and then we conclude that for each $s_{-i} \in H_{-i}$, $z^* \in Z(s_{-i})$. We have $y \succ_H x$ and this fact implies $z^* \succ_H x$. If $z \succ_H z^*$ for some $z \in G_i$, then $u_i(z, s_{-i}) > u_i(z^*, s_{-i}) \forall s_{-i} \in H_{-i}$, implying that $z \in Z$ and $f(z) > f(z^*)$, contradiction. Therefore, $z \not\succ_H z^* \forall z \in G_i$, so that $z \not\succ_{A_i} z^* \forall z \in G_i \forall t \geq 0$ implying $z^* \in A^t_i \forall t \geq 0$. It follows that $z^* \in H_i$.

If $H = G$, we obtain the following corollary.

Corollary 8. Let us assume that $G$ is a compact and own-transfer weakly upper continuous game $G$ and for each $i \in I$ and for each $s_{-i} \in G_{-i}$, the function $u_i(\cdot, s_{-i})$ has property $\mathcal{M}$. If $y \succ_G x$ for some $x, y \in G_i$ and $i \in I$, then there exists $z^* \in G_i$ such that $z \not\succ_G z^* \succ_G z$ for each $z \in G_i$.

The next theorem is Theorem 1 in Dufwenberg and Stegeman (2002). It is the main result concerning the existence and uniqueness of nonempty maximal reductions of compact and continuous games.

Theorem 9. a) If a game $G$ is compact and own-uppersemicontinuous, then any nonempty maximal ($\rightarrow^*$) reduction of $G$ is the unique maximal ($\rightarrow^*$) reduction of $G$.

b) If a game $G$ is compact and continuous, then $G$ has a unique maximal ($\rightarrow^*$) reduction $M$; furthermore, $M$ is nonempty, compact and continuous.

We generalize the theorem above by weakening the continuity conditions on payoff functions which describe the game model. In order to do this, we introduce the following definition.
Definition 15. The reduction $G \rightarrow^{**} H$ is defined by the existence of (finite or countable infinite) sequence of parings $A^t$ of $G$, $t = 0, 1, 2, \ldots$, such that $A^0 = G$, $A^t \rightarrow A^{t+1}$ for each $t \geq 0$ and $H_i = \cap t A^t_i$ for each $i \in I$ and by the consistency with the continuity of the utility functions, which means that for each $i \in I$, the payoff function $u_i$ maintains the same continuity property on each set $\prod_{i \in I} A^t_i$, $t = 0, 1, 2, \ldots$, as it has on $\prod_{i \in I} G_i$.

Definition 16. The function $u_i : G \rightarrow \mathbb{R}$ has the intersection property with respect to the $i^{th}$ variable if there exists $S_{-i} \subset G_{-i}$ such that $Z_{-i}(x) = S_{-i}$ for each $x \in G_i$ and $Z_i(x) = \cap s_{-i} \in S_{-i} F_i(x, s_{-i})$, where $Z(x) = \{ (s_i, s_{-i}) \in G : u_i(x, s_{-i}) \leq u_i(s_i, s_{-i}) \}$, $Z_i(x) = pr_i Z(x)$, $Z_{-i}(x) = pr_{-i} Z(x)$ and $F_i(x, s_{-i}) = \{ s_i \in G_i : u_i(x, s_{-i}) \leq u_i(s_i, s_{-i}) \}$.

Example 3. Let $G = [0, 1] \times [0, 1]$ and let $u_1 : G \rightarrow \mathbb{R}$ be defined by

$$u_1(x, y) = \begin{cases} 1 + x + y & \text{if } x \in Q; \\ x & \text{if } x \in \mathbb{R} \setminus Q. \end{cases}$$

For each $y \in [0, 1]$, the function $u_1(\cdot, y)$ is not upper semicontinuous, but it is transfer upper continuous since, for a neighborhood $\mathcal{N} \subset [0, 1]$, we may choose any $x'$ rational such that $\sup \{ x : x \in \mathcal{N} \} < x t \leq 1$.

We prove that $u_1$ fulfills the intersection property with respect to $x$.

We have that $Z(x) = \left\{ \begin{array}{ll} [x, 1] \cap Q \times [0, 1] & \text{if } x \in Q; \\ \{ [0, 1] \cap Q \} \times [0, 1] \cup \{ [x, 1] \cap Q \} \times [0, 1] & \text{if } x \in \mathbb{R} \setminus Q, \end{array} \right.$

$Z_1(x) = \left\{ \begin{array}{ll} [x, 1] \cap Q & \text{if } x \in Q; \\ \{ [0, x] \cap Q \} \cup [x, 1] & \text{if } x \in \mathbb{R} \setminus Q, \end{array} \right.$ and $Z_2(x) = [0, 1] = G_2$ and

$F_1(x, s_2) = \left\{ \begin{array}{ll} [x, 1] \cap Q & \text{if } x \in Q; \\ \{ [0, x] \cap Q \} \cup [x, 1] \cap \mathbb{R} \setminus Q & \text{if } x \in \mathbb{R} \setminus Q. \end{array} \right.$

It follows that $Z_1(x) = \cap s_2 \in G_2 F_1(x, s_2)$.

Definition 17. The game $G$ has the intersection property if $u_i : G \rightarrow \mathbb{R}$ has the intersection property with respect to the $i^{th}$ variable for each $i \in I$.

Theorem 10. a) Let $G$ be an own-transfer weakly upper continuous game which has also the intersection property, such that $>_H$ has property $K$ for every $G \rightarrow H$. Then, any nonempty maximal reduction $G \rightarrow^{**} M$ is the unique maximal reduction.
b) If $G$ is a compact, own-transfer upper continuous game such that $\succ_H$ has property K for every $G \to H$, then it has a nonempty compact own-transfer upper semicontinuous maximal $\to^{**}$ reduction $M$ and this reduction is unique.

Proof. a) The proof follows the same line as Theorem 1 of Dufwenberg and Stegeman (2002).

b) We prove that if $G \to H$ fast, then $H$ is compact and nonempty. Since $y \succ_G x$ for some $x, y \in G_i$, then $H_i \neq \emptyset$.

We will show further that $H_i$ is compact. Let $Z(x) = \{(s_i, s_{-i}) \in G : u_i(x, s_{-i}) \leq u_i(s_i, s_{-i})\}$, $Z_i(x) = \text{pr}_iZ(x)$ and $Z_{-i}(x) = \text{pr}_{-i}Z(x) = S_{-i}$ for each $x \in S_{-i}$. Since $x \in Z_i(x)$, it follows that $Z_i(x) \neq \emptyset$. We first prove that $H_i = \cap_{x \in H_i} Z_i(x)$. Let us choose an arbitrary element $z$ of $G_i$. If $z \notin Z_i(x)$, for each $x \in H_i$, then $u_i(x, s_{-i}) > u_i(z, s_{-i})$ for each $s_{-i} \in G_{-i}$. It follows that $x \succ_H z$ and therefore, $z \notin H_i$. This fact implies that $H_i \subseteq \cap_{x \in H_i} Z_i(x)$. If $z \notin H_i$, there exists $x \in G_i$ such that $x \succ_G z$ and according to Corollary 2, it follows that there exists $x^* \in G_i$ such that $x^* \succ_G z$. The last assertion implies that $z \notin Z_i(x^*)$ and therefore, $z \notin \cap_{x \in H_i} Z_i(x)$. We have $\cap_{x \in H_i} Z_i(x) \subseteq H_i$ and the equality $H_i = \cap_{x \in H_i} Z_i(x)$ follows from the above assertions.

Now let us define $F_i(x, s_{-i}) = \{s_i \in G_i : u_i(x, s_{-i}) \leq u_i(s_i, s_{-i})\}$ and then, $Z_i(x) = \cap_{s_{-i} \in S_{-i}} F_i(x, s_{-i})$. Since we have the reduction $G \to^{**} H$, the function $u_i(., s_{-i})$ is transfer upper continuous on $H_i$ for $s_{-i}$ fixed, and, according to Lemma 1, it follows that $\cap_{x \in H_i} F_i(x, s_{-i}) = \cap_{x \in H_i} \text{cl} F_i(x, s_{-i})$.

Therefore, $H_i = \cap_{x \in H_i} Z_i(x) = \cap_{s_{-i} \in S_{-i}} F_i(x, s_{-i}) = \cap_{s_{-i} \in S_{-i}} \cap_{x \in H_i} \text{cl} F_i(x, s_{-i})$, then $H_i$ is a closed set. $H_i$ is closed, $H_i \subseteq G_i$, $G_i$ is compact, then $H_i$ is compact.

We consider $C(t) \equiv (0, 1, \ldots$ the unique sequence of subgames of $G$ such that $C(0) = G$ and $C(t) \to C(t + 1)$ is fast for each $t \geq 0$. The set $C(t)$ is compact and nonempty for each $t \geq 0$. The game $M_t = \cap_{t \geq 0} C(t)$ is compact, transfer upper semicontinuous and nonempty. We show that $M$ is a maximal $\to^{**}$-reduction of $G$. Consider any player $i$ and $x, y \in M_i$. Let $X(t) := \{s_{-i} \in (C(t))_{-i} : u_i(y, s_{-i}) \leq u_i(x, s_{-i})\}$. We claim that $X(t) \neq \emptyset$. If not, for each $s_{-i} \in (C(t))_{-i}$, it follows that $u_i(y, s_{-i}) > u_i(x, s_{-i})$, so that $y \succ_{C(t)} x$, contradicting $x \in M_i$. $(C(t))_{-i}$ is compact and $\cap_{t \geq 0} (C(t))_{-i}$ is nonempty and compact.

Let $X' = \{s_{-i} \in \cap_{t \geq 0} (C(t))_{-i} : u_i(y, s_{-i}) \leq u_i(x, s_{-i})\}
= \{s_{-i} \in M_{-i} : u_i(y, s_{-i}) \leq u_i(x, s_{-i})\}$.

Since $M_{-i} \neq \emptyset$, it follows that $X' \neq \emptyset$ and therefore $y \not\succ_M x$ and $M$ is maximal.
Corollary 11. The results also maintain for the class of upper pseudocontinuous games.

By applying Lemma 4, we obtain the following result.

Theorem 12. a) Let $G$ be a compact and own-transfer weakly upper continuous game which has also the intersection property, such that for each $i \in I$, the payoff function $u_i$ has property $M$. Then, any nonempty maximal reduction $G \to^{**} M$ is the unique maximal reduction.

b) If $G$ is a compact, own-transfer upper continuous game such that for each $i \in I$, the payoff function $u_i$ has property $M$, then it has a nonempty compact own-transfer upper semicontinuous maximal ($\to^{**}$) reduction $M$. The reduction $M$ is unique.

4. The Mixed Strategies Case

In Subsection 6.2 Dufwenberg and Stegeman (2002) approached the issue of mixed strategy dominance. They distinguished between the case in which a pure strategy is dominated by a pure strategy and the case in which it is dominated by a mixed strategy. The main result is obtained by applying Theorem 1 to the mixed extensions of finite games. We will extend Dufwenberg and Stegeman’s research by taking into consideration several types of dominance relations and game reductions.

For the reader’s convenience, we review here a few basic notions and notations which deal with measurability. For an overview, please see Parthasarathy (2005).

4.1. Measurable spaces

Suppose that $(G, \mathcal{G})$ is a measurable space and $H \in \mathcal{G}$. Let us define $\mathcal{H} = \{H \cap A : A \in \mathcal{G}\}$. Then $\mathcal{H}$ is a $\sigma$–algebra of subsets of $H$ and $(H, \mathcal{H})$ is a measurable space.

Definition 18. Given a measurable space $(G, \mathcal{G})$ and $x \in G$, define the probability measure $\delta_x$ as

$$\delta_x(H) = \begin{cases} 1 & \text{if } x \in H; \\ 0 & \text{if } x \notin H \end{cases} \text{ for each } H \in \mathcal{G}.$$ 

$\delta_x$ is called the Dirac measure with unit mass at $x$. 

Theorem 13. Let $X$ be a finite set with a discrete $\sigma-$algebra. Then, every probability $\mu$ on this measurable space can be uniquely represented in the form $\mu = \sum_{x \in X} c_x \delta_x$, where $c_x \in [0, 1]$ $\forall x \in X$, $\sum_{x \in X} c_x = 1$, thus $\mu(E) = \sum_{x \in E} c_x$ for all $E \subset X$.

Notation If $(G, \mathcal{G})$ is a measurable space, we will denote by $\Delta(G)$ the set of probability measures defined on $G$.

Let $I = \{1, 2, \ldots, n\}$ and the game $G = (G_i, u_i)_{i \in I}$.

Assume that for each $i \in I$, $G_i$ is a compact subset in a metric space $X$ and $u_i(., s_{-i}) : G_i \rightarrow \mathbb{R}$ is upper semicontinuous for each $s_{-i} \in G_i$.

Each $u_i$ is measurable since it is upper semicontinuous and since it is also bounded, it is integrable. We denote by $\Delta(G_i)$ the set of probability measure on the set of Borel sets on $G_i$. $\Delta(G_i)$ will be equipped with the weak topology.

Theorem 14. Let $G$ be a subset of a metric space. Then, $G$ is compact if and only if $\Delta(G)$ is compact.

A mixed strategy for player $i$ is an element $\mu_i \in \Delta(G_i)$.

Definition 19. (Billingsley (1968), p 7). Suppose $\{\mu_n\}_{n \geq 1}$, $\mu_n$ belong to $\Delta(G)$, the set of probability measures on the Borel sets of some compact metric space $G$. Then $\mu_n$ weakly converges to $\mu$, written $\mu_n \xrightarrow{w} \mu$ iff $\int f d\mu_n \rightarrow \int f d\mu$ for all $f : G \rightarrow \mathbb{R}$, $f$ continuous. This topology is consistent with Prohorov metric.

Lemma 15. (Robson 1990). Consider $u : G \rightarrow \mathbb{R}$ an upper semicontinuous function, where $G$ is a compact metric space. It follows that $\int u d\mu$ is upper semicontinuous in $\mu : \limsup_n \int u d\mu_n \leq \int u d\mu$ if $(\mu_n)_n, \mu \in \Delta(G)$, the set of probability measures on Borel sets of $G$ and $\mu_n \xrightarrow{w} \mu$.

Corollary 16. If for the game $G$, $u_i(., s_{-i}^*) : G_i \rightarrow \mathbb{R}$ is upper semicontinuous, then the function $V_i(., s_{-i}^*) : \Delta(G_i) \rightarrow \mathbb{R}$, defined by $V_i(\mu_i, s_{-i}^*) = \int u_i(\mu_i, s_{-i}^*) d\mu_i(s_i)$ is upper semicontinuous.

We define the following extension of $\succ_H$:

Definition 20. Let $G \rightarrow H$, $G = (G_i, u_i)_{i \in I}$, $I$ finite, $G_i$ is a subset of a metric space $X$ for each $i \in I$. Let $\Delta(G_i)$ be the set of probability measures on Borel sets of $G_i$ and $V_i(., s_{-i}) : \Delta(G_i) \rightarrow \mathbb{R}$, defined by $V_i(\mu_i, s_{-i}) = \int u_i(\mu_i, s_{-i}) d\mu_i(s_i)$ for each $s_{-i}$ fixed. Given $x \in G_i$ and $\mu_i \in \Delta(G_i)$, we say that $\mu_i \succ_H x$ if $H_{-i} \neq \emptyset$ and $V_i(\mu_i, s_{-i}) > u_i(x, s_{-i})$ for each $s_{-i} \in H_{-i}$.
Lemma 17. If \( G \to^* H \) for some compact and own-upper semicontinuous game \( G \) and \( y \succ_H x \) for some \( x, y \in G_i \) and \( i \in I \), then, there exists \( z^* \in H_i \) such that \( \mu \not\succ_H z^* \succ_H x \) for each \( \mu \in \Delta(G_i) \).

Proof. The assumptions of Dufwenberg-Stegeman Lemma are fulfilled. Then there exists \( z^* \in H_i \) such that \( z \not\succ_H z^* \succ_H x \) for each \( z \in G_i \). We prove that, in addition, \( \mu \not\succ_H z^* \) for each \( \mu \in \Delta(G_i) \).

If \( \mu \succ_H z^* \) for some \( \mu \in \Delta(G_i) \), then \( \int u_i(s_i, s_{-i})d\mu_i(s_i) > u_i(z^*, s_{-i}) \) for each \( s_{-i} \in H_{-i} \), implying \( \int u_i(s_i, s^*_{-i})d\mu_i(s_i) > u_i(z^*, s^*_{-i}) \) for some \( s^*_{-i} \) fixed in \( H_{-i} \).

We note that \( z^* = \arg\max_{s_i \in Z} u_i(s_i, s^*_{-i}) = \arg\max_{s_i \in G_i} u_i(s_i, s^*_i) \), where \( Z = \{ z \in G_i : u_i(z, s_{-i}) \geq u_i(y, s_{-i}) \forall s_{-i} \in H_{-i} \} \). It follows that \( u_i(z^*, s^*_{-i}) \geq u_i(s_i, s^*_{-i}) \) for each \( s_{-i} \in G_{-i} \), and therefore, \( u_i(z^*, s^*_{-i}) \geq \int u_i(s_i, s^*_{-i})d\mu_i(s_i) \), relation which contradicts (1). Therefore, \( \mu \not\succ_H z^* \) for each \( \mu \in \Delta(G_i) \).

4.2. Types of dominance relations and reductions

Let \( I \) be a finite set. For each \( i \in I \), let \((G_i, G_i)\) be a measurable space, \( H_i \in G_i, H_i = \{ H_i \cap A : A \in G_i \} \) and \( u_i : \prod_{i \in I} G_i \to \mathbb{R} \) be a \( \otimes G_i \)-measurable and bounded. Let \( G = \prod_{i \in I} G_i \).

Definition 21. We define the followings types of dominance relations.

i) \( \mu \succ_{\Delta(H)} m \) for \( \mu, m \in \Delta(G_i) \) if \( \int_{G_i \times H_{-i}} u_i(s_i, s_{-i})d\mu_1 \times \ldots \times d\mu_{i-1} \times d\mu \times d\mu_{i+1} \times \ldots \times d\mu_n > \int_{G_i \times H_{-i}} u_i(s_i, s_{-i})d\mu_1 \times \ldots \times d\mu_{i-1} \times dm \times d\mu_{i+1} \times \ldots \times d\mu_n, \forall \mu_1 \times \ldots \times \mu_{i-1} \times \mu_{i+1} \times \ldots \mu_n \in \Delta(H_{-i}) \).

ii) \( \mu \succ_H m \) for \( \mu, m \in \Delta(G_i) \) if \( \int_{G_i} u_i(s_i, s_{-i})d\mu > \int_{G_i} u_i(s_i, s_{-i})dm \forall s_{-i} \in H_{-i} \).

iii) \( \mu \succ_H x \) for \( \mu \in \Delta(G_i) \) and \( x \in G_i \) if \( \mu \succ_H \delta_x \), which is equivalent with \( \int_{G_i} u_i(s_i, s_{-i})d\mu > u_i(x, s_{-i}) \forall s_{-i} \in H_{-i} \).

We obtain the following theorem.

Theorem 18. With the notations above, we have the following relations amongst the former types of dominance:

i) \( \mu \succ_{\Delta(H)} m \) for \( \mu, m \in \Delta(G_i) \) \( \Rightarrow \) \( \mu \succ_H m \)
To prove this fact, we take \( \mu = \delta_{s_j} \) for \( j \neq i, s_j \in H_j \).

ii) \( \mu \succ_H m \) for \( \mu, m \in \Delta(G_i) \) \( \Rightarrow \) \( \mu \succ_H x \) \( \Leftrightarrow \) \( \mu \succ_H \delta_x \), \( x \in G_i \).
Since \( \delta_x \in \Delta(G_i) \) for \( x \in G_i \), ii) can be easily checked.
Let us consider pairings $G, H$ with the property that $H_i \subseteq G_i$ for each $i \in I$. In addition to the game reduction used by Dufwenberg and Stegeman (2002), we present the following ones.

**Definition 22.**

i) (Gilboa, Kalai and Zemel 1990) $G \Rightarrow H$ if, for each $x \in G_i \setminus H_i$, there exists $y \in H_i$ such that $y \succ_H x$.

ii) $G \mapsto H$ if, for each $x \in G_i \setminus H_i$, there exists $\mu \in \Delta(G_i)$ such that $\mu \succ_H x$.

iii) $G \Rightarrow H$ if, for each $x \in G_i \setminus H_i$, there exists $\mu \in \Delta(H_i)$ such that $\mu \succ_H x$.

iv) $\Delta(G) \multimap \Delta(H)$ if, for each $m \in \Delta(G_i) \setminus \Delta(H_i)$, there exists $\mu \in \Delta(G_i)$ such that $\mu \succ_H m$.

v) $\Delta(G) \multimap \Delta(H)$ if, for each $m \in \Delta(G_i) \setminus \Delta(H_i)$, there exists $\mu \in \Delta(H_i)$ such that $\mu \succ_H m$.

We will need the following theorem.

**Theorem 19.** There are the following relations amongst the former types of reductions.

i) $(G \Rightarrow H) \implies (G \mapsto H)$

ii) $(G \mapsto H) \implies (G \Rightarrow H)$

$(\Delta(G) \Rightarrow \Delta(H)) \iff (\Delta(G) \multimap \Delta(H))$

iii) $(\Delta(G) \multimap \Delta(H)) \implies (\Delta(G) \multimap \Delta(H)) \implies (G \mapsto H)$

**Proof.**

i) The proof is obvious.

ii) Suppose $(\Delta(G) \Rightarrow \Delta(H))$. It follows that $\Delta(H_i) \subseteq \Delta(G_i)$ and for each $m \in \Delta(G_i) \setminus \Delta(H_i)$, there exists $\mu \in \Delta(H_i)$ such that $\mu \succ_{\Delta(H)} m$. According to Theorem 7, it follows that $H_i \subseteq G_i$ for each $i \in I$ and for each $m \in \Delta(G_i) \setminus \Delta(H_i)$, there exists $\mu \in \Delta(H_i)$ such that $\mu \succ_H m \iff \Delta(G) \multimap \Delta(H)$.

If $m = \delta_x$ with $x \in G_i \setminus H_i$, we have that $H_i \subseteq G_i$ for each $i \in I$ and for each $x \in G_i \setminus H_i$, there exists $\mu \in \Delta(H_i)$ such that $\mu \succ_H x$, which is equivalent with $G \mapsto H$.

iii) The implication are true from i) and ii).

**Theorem 20.** If $G$ is a finite game, then $(\Delta(G) \Rightarrow \Delta(H)) \iff (G \Rightarrow H)$. 

15
Proof. The direct implication "\( \Rightarrow \)" comes from Theorem 8, ii).

We prove "\( \Leftarrow \)". Let \( x \in G_i \setminus H_i \). Since \( G \Rightarrow H \), there exists \( \mu_x \in \Delta(H_i) \) such that \( \mu_x \succ \Delta(H) \rightarrow \mu_x \succ \Delta(H) \).

Let \( m \in \Delta(G_i) \setminus \Delta(H_i) \). According to Theorem 5, \( m \) can be uniquely represented as a convex combination of Dirac measures \( \delta_x \), \( x \in G_i \setminus H_i \). Then, there exists unique \( c_x \in [0, 1] \), \( \sum_{x \in G_i \setminus H_i} c_x = 1 \) such that \( m = \sum_{x \in G_i \setminus H_i} c_x \delta_x \).

But, as we noted above, for each \( \delta_x \) with \( x \in G_i \setminus H_i \), there exists \( \mu_x \in \Delta(H_i) \) such that \( \mu_x \succ H \delta_x \). Therefore, \( \mu = \sum_{x \in G_i \setminus H_i} c_x \mu_x \) is a probability measure on \( G_i \setminus H_i \) and \( \mu_x \succ H \).

**Theorem 21.** Let \( G \Rightarrow H \). If there exists \( x^* \in G_i \setminus H_i \) such that \( u_i(x^*, s_{-i}) \geq u_i(x, s_{-i}) \) for each \( x \in G_i \setminus H_i \) and \( s_{-i} \in H_{-i} \), then \( \Delta(G) \Rightarrow \Delta(H) \).

Proof. Let \( x^* \) be such that \( x^* \in G_i \setminus H_i \) and \( u_i(x^*, s_{-i}) \geq u_i(x, s_{-i}) \) for each \( x \in G_i \setminus H_i \) and \( s_{-i} \in H_{-i} \). Then, \( \int_{G_i \setminus H_i} u_i(x, s_{-i}) dm \leq u_i(x^*, s_{-i}) \) for each \( m \in \Delta(G_i \setminus H_i) \).

Since \( x^* \in G_i \setminus H_i \) and \( G \Rightarrow H \), it follows that there exists \( \mu \in \Delta(H_i) \) such that \( \mu \succ H x^* \), that is \( \int_{H_i} u_i(x, s_{-i}) d\mu > u_i(x^*, s_{-i}) \) for each \( s_{-i} \in H_{-i} \).

(2)

From 1) and 2), it follows that for \( m \in \Delta(G_i) \setminus \Delta(H_i) \), there exists \( \mu \in \Delta(H_i) \) such that \( \int_{H_i} u_i(x, s_{-i}) d\mu > \int_{G_i \setminus H_i} u_i(x, s_{-i}) dm \) for each \( s_{-i} \in H_{-i} \), that is \( \mu \succ H m \). Therefore, \( \mu \succ H m \).

**Corollary 22.** Let \( G \mapsto H \). If there exists \( x^* \in G_i \setminus H_i \) such that \( u_i(x^*, s_{-i}) \geq u_i(x, s_{-i}) \) for each \( x \in G_i \setminus H_i \) and \( s_{-i} \in H_{-i} \), then \( \Delta(G) \mapsto \Delta(H) \).

4.3. Dufwenberg-Stegeman-like Lemma

We study first the case of the game reduction \( G \mapsto H \).

**Lemma 23.** In the case of a finite game, Lemma Dufwenberg-Stegeman remains true for the game reduction \( G \mapsto H \).

Proof. If \( G \mapsto H \), then \( \Delta(G) \mapsto \Delta(H) \), according to Theorem 9 and Theorem 8. Let \( \mu' \succ H x \) for some \( x \in G_i \) and \( \mu' \in \Delta(G_i) \). Then, \( \mu' \succ H \delta_x \).

By applying Lemma Dufwenberg-Stegeman to \( \Delta(G) \mapsto \Delta(H) \), we obtain that there exists \( \mu^* \in \Delta(H_i) \) such that \( \mu \nRightarrow H \mu^* \succ H \delta_x \) for each \( \mu \in \Delta(G_i) \).

Therefore, there exists \( \mu^* \in \Delta(H_i) \) such that \( \mu \nRightarrow H \mu^* \succ H x \) for each \( \mu \in \Delta(G_i) \).

We also obtain the next result concerning the game reduction \( G \mapsto H \).
Lemma 24. Let $I$ be a finite set. For each $i \in I$, let $G_i$ be a compact subset of a metric space $X$ considered with its borelian sets, $H_i \subset G_i$ and $u_i : \prod_{i \in I} G_i \to \mathbb{R}_+$ uppersemicontinuous in each argument. Let $G \mapsto H$ and suppose that $\Delta(G) \mapsto \Delta(H)$. If $\mu \succ_{H} x$ for some $x \in G_i$ and $\mu' \in \Delta(G_i)$, $i \in I$, then there exists $\mu^* \in \Delta(H_i)$ such that $\mu \not\succ_{H} \mu^* \succ_{H} x$ for each $\mu \in \Delta(G_i)$.

Proof. According to Theorem 6, if $G$ is compact, $\Delta(G)$ is also compact. According to Corollary 2, if $u_i(., s_{-i})$ is upper semicontinuous for each $s_{-i} \in G_{-i}$, then $V_i(., s_{-i})$ is also upper semicontinuous on $\Delta(G_i)$ for each $s_{-i} \in G_{-i}$, where $V_i(\mu, s_{-i}) = \int_{G_i} u_i(s, s_{-i})d\mu$.

Since $G \mapsto H$, there exists a sequence of parings $A^t$, $t = 0, 1, 2...$ such that $A^0 = G$, $A^t \mapsto A^{t+1}$ for each $t \geq 0$ and $H_i = \bigcap_t A_i^t$, $\forall i \in I$. Let $Z = \{\mu \in \Delta(G_i) : V_i(\mu, s_{-i}) \geq V_i(\mu', s_{-i})$ for each $s_{-i} \in H_{-i}\}$. $Z$ is a nonempty set, since $\mu' \in Z$. $Z$ is also closed (as intersection of the closed sets $Z(s_{-i}) = \{\mu \in \Delta(G_i) : V_i(\mu, s_{-i}) \geq V_i(\mu', s_{-i})$, $s_{-i} \in H_{-i}\}$) and therefore compact. Let us define $f : Z \to \mathbb{R}$, $f(\mu) = V_i(\mu, s_{-i}^*)$ for $s_{-i}^* \in H_{-i}$ fixed. The function $f$ is uppersemicontinuous and it reaches its maximum on the compact set $Z$. Denote by $\mu^* = \arg \max_{\mu \in Z} f(\mu)$.

It follows that, there exists $\mu^* \in \Delta(G_i)$ such that $\mu \not\succ_{H} \mu^* \succ_{H} x$ for each $\mu \in \Delta(G_i)$. Therefore, $\mu \not\succ_{H} \mu^*$ for each $\mu \in \Delta(G_i)$ and then $\mu \not\succ_{A^t} \mu^*$ for each $\mu \in \Delta(G_i)$ and $t \geq 0$. Since $\Delta(G) \mapsto \Delta(H)$, we conclude that $\mu^* \in \Delta(A^t)$ for each $t \geq 0$, and therefore, $\mu^* \in \Delta(H_i)$.

For $H = G$, we obtain the following corollary.

Corollary 25. Let $I$ be a finite set. For each $i \in I$, let $G_i$ be a compact subset of a metric space $X$ considered with its borelian sets and $u_i : \prod_{i \in I} G_i \to \mathbb{R}_+$ upper semicontinuous in each argument. If $\mu' \succ_{G} x$ for some $x \in G_i$ and $\mu' \in \Delta(G_i)$, $i \in I$, then there exists $\mu^* \in \Delta(G_i)$ such that $\mu \not\succ_{G} \mu^* \succ_{G} x$ for each $\mu \in \Delta(G_i)$.

Corollary 26. Lemma 8 is true if, instead of having the assumption $\Delta(G) \mapsto \Delta(H)$, we have the following one: there exists $x^* \in G_i \setminus H_i$ such that $u_i(x^*, s_{-i}) \geq u_i(x, s_{-i})$ for each $x \in G_i \setminus H_i$ and $s_{-i} \in H_{-i}$.

Proof. The proof of the corollary comes from Theorem 6.
4.4. Existence and uniqueness of maximal reductions

The main result of Section 4 is Theorem 11.

**Theorem 27.** Let \( G = (I, (G_i)_{i \in I}, (u_i)_{i \in I}) \) be a strategic game such that \( I \) is a finite set and for each \( i \in I \), \( G_i \) is a nonempty compact subset of a metric space, \( u_i : \prod_{i \in I} G_i \to \mathbb{R} \) is upper semicontinuous in each argument and for each \( G \mapsto H \), \( \Delta(G) \mapsto \Delta(H) \) (or there exists \( x^* \in G_i \setminus H_i \) such that \( u_i(x^*, s_{-i}) \geq u_i(x, s_{-i}) \) for each \( x \in G_i \setminus H_i \) and \( s_{-i} \in H_{-i} \)). Then, \( G \) has a unique nonempty maximal (\( \longrightarrow^* \)) reduction \( M \) and \( M \) is nonempty, compact and upper semicontinuous.

**Proof.** The game \( (I, (\Delta(G_i))_{i \in I}, (V_i)_{i \in I}) \) is also compact and own-upper-semicontinuous. According to Lemma 8, we have that if \( \mu' >_H x \) for some \( x \in G_i \) and \( \mu' \in \Delta(G_i) \), \( i \in I \), then there exists \( \mu^* \in \Delta(H_i) \) such that \( \mu \not> H \mu^* >_H x \) for each \( \mu \in \Delta(G_i) \). The set \( \Delta(H_i) \) is nonempty since \( H_i \) is nonempty.

The proof of the uniqueness of \( M \) follows the same line as the proof of Theorem 1a) of Dufwenberg-Stegeman.

Now we are proving that, if \( G \) is compact and own-upper semicontinuous and \( G \mapsto H \) fast, then \( H \) is compact and nonempty.

Choose \( i \in I \) such that \( H_i \neq G_i \). Since \( x \ni G \mu > G x \) for some \( x \in G_i \), \( \mu \in \Delta(G_i) \), according to Corollary 4, we have that \( H_i \neq \emptyset \). It remains to show that \( H_i \) is compact. Choose \( \mu \in \Delta(H_i) \) and let \( Z(\mu) = \{ (s_i, s_{-i}) \in G : V_i(\mu, s_{-i}) \leq u_i(s_i, s_{-i}) \} \), \( Z_1(\mu) = \text{pr}_1 Z(\mu) \) and \( Z_{-1}(\mu) = \text{pr}_1 Z(\mu) \).

The set \( Z_i(\mu) \) is nonempty. In order to prove this fact, we will assume the opposite: \( Z_i(\mu) = \emptyset \). In this case, \( V_i(\mu, s_{-i}) > u_i(s, s_{-i}) \) for each \( s \in G_i \) and for each \( s_{-i} \in G_{-i} \), and it follows that \( \mu > G s \) for each \( s \in G_i \). We can conclude that \( \mu > H s \) for each \( s \in G_i \). Since \( G \Rightarrow H \) fast, we have that \( (\text{for each} \ s \in G_i \Rightarrow \ s \notin H_i \) and, then, \( H_i \) is an empty set, and we reached a contradiction.

Now let us define \( F_i(\mu, s_{-i}) = \{ s_i \in G_i : V_i(\mu, s_{-i}) \leq u_i(s_i, s_{-i}) \} \) and then, \( Z_i(\mu) = \bigcap_{s_{-i} \in Z_{-1}(\mu)} F_i(x, s_{-i}) \).

Since \( u_i(\cdot, s_{-i}) \) is upper semicontinuous for each \( s_{-i} \in G_{-i} \), we have that \( Z_i(\mu) \) is closed as being an intersection of closed subsets. We will show that \( H_i = \bigcap_{\mu \in \Delta(H_i)} Z_i(\mu) \).

Let us consider \( x \in G_i \). For any \( \mu \in \Delta(H_i) \), if \( x \notin Z(\mu) \), we have that \( V_i(\mu, s_{-i}) > u_i(s, s_{-i}) \) for each \( s_{-i} \in G_{-i} \) and, therefore \( \mu > G x \). Then \( x \notin H_i \) and \( H_i \subseteq \bigcap_{\mu \in \Delta(H_i)} Z(\mu) \).
If \( x \notin H_i \), then \( \mu \succ_G x \) for some \( \mu \in \Delta(G_i) \) and Lemma 8 implies that there exists \( \mu^* \in \Delta(G_i) \) such that \( \mu^* \succ_G x \) and therefore, \( x \notin Z(\mu^*) \) and we can conclude that \( x \notin \cap_{\mu \in \Delta(H_i)} Z(\mu) \). Therefore, \( H_i \supseteq \cap_{\mu \in \Delta(H_i)} Z(\mu) \).

The equality \( H_i = \cap_{\mu \in \Delta(H_i)} Z(\mu) \) holds and, since \( Z_i(\mu) \) is closed for all \( \mu \), \( H_i \) is also closed and therefore compact.

Let \( C(t) \), \( t = 0, 1, \ldots \) denote the unique sequence of subgames of \( G \) such that \( C(0) = G \) and \( C(t) \mapsto C(t + 1) \) is fast for each \( t \geq 0 \). We have that \( C(t) \) is compact and nonempty for each \( t \geq 0 \). It follows that \( M_i = \cap_{t \in C(t)} M_i \) is compact, nonempty for each \( i \in I \).

We will show that \( M \) is a maximal (\( \mapsto^* \)) reduction of \( G \). Let \( x \in M_i, \mu \in \Delta(M_i) \). Let \( X(t) = \{ s_{-i} \in C(t)_{-i} : V_i(\mu, s_{-i}) \leq u_i(x, s_{-i}) \} \). If \( X(t) = \emptyset \) for each \( t \) such that \( C(t) \neq M \), then \( \mu \succ_{C(t)} x \), contradiction. Therefore, \( X(t) \neq \emptyset \). The set \( C(t)_{-i} \) is compact for each \( t \) such that \( C(t) \neq M \). Then, \( M_{-i} \neq \emptyset \) and it follows that the set \( X = \{ s_{-i} \in M_{-i} : V_i(\mu, s_{-i}) \leq u_i(x, s_{-i}) \} \) is nonempty. We conclude that \( \mu \not\in_{M} x \).

5. Concluding remarks

We identified a class of discontinuous games for which the iterated elimination of strictly dominated strategies produce a unique maximal reduction that is nonempty. We also provided conditions under which order independence remains valid for the case that the pure strategies are dominated by mixed strategies. Our results expel M. Dufwenberg and M. Stegeman’s idea in [6] that ‘the proper definition and the role of iterated strict dominance is unclear for games that are not compact and continuous’. G. Tian and J. Zhou’s notion of transfer upper continuity proved to be a suitable assumption for the payoff functions of a game in order to obtain our results. Their Weierstrass-like theorem for transfer weakly upper continuous functions defined on a compact set was the key of the proofs of Lemma 3 and Lemma 4. We can conclude and emphasize that, even outside the continuous class of games, the iterated elimination of strictly dominated strategies remains an interesting procedure.

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