Fractionally quantized angular momentum of impurities in Laughlin liquids

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The elementary excitations of a fractional quantum Hall liquid are quasiparticles or quasiholes which are neither bosons nor fermions, but so-called anyons. Here we study impurity particles immersed in a quantum Hall liquid which bind to the quasiholes via repulsive interactions with the liquid. We show that the angular momentum of an impurity is quantized at a fractional value and that this fractional quantization can directly be observed from the impurity density. In a system with several impurities bound to quasiholes, their total angular momentum interpolates between the values for free fermions and for free bosons. This interpolation is characterized by the fractional statistical parameter of the anyons which is typically defined via their braiding behavior.

Introduction. When quasiparticles emerge from strongly correlated quantum matter, their properties can be quite different from those of the matter particles themselves. A paradigm are the bulk excitations in fractional quantum Hall liquids: The liquid is made of interacting electrons, but its excitations can be viewed as fractional electrons, having fractional charge, fractional angular momentum, and fractional exchange statistics [1–4]. With this, they are neither bosons nor fermions, but so-called anyons. To date, the best experimental evidence of fractional quasiparticles is obtained by determining the fractional charge via shot noise measurements [5]. Signatures of fractional statistics have been seen via Fabry-Perot interferometry [6–8], and in a beamsplitter [9]. To improve the experimental evidence of anyons, there have been strong efforts to implement fractional quantum Hall physics in highly controllable quantum systems such as cold atoms [10, 11] or photonic quantum simulators [12]. Light-matter interactions can be used to create and trap fractional quasiparticles in atomic gases [13] or electronic systems [14], and may facilitate braiding operations [13,15,17]. It has also been suggested to observe the fractional exclusion principle via the energy spectrum of an atomic fractional quantum Hall droplet [18]. Moreover, it has been shown that signatures of fractional statistics are carried by the total angular momentum of a fractional quantum Hall system, and thus can be measured by time-of-flight imaging [19]. It has also been proposed to engineer anyonic systems through appropriately defined bath interactions [20,21]. Various works propose to use impurities which bind to fractional quasiparticles [22,25], and which then exhibit features such as fractional relative angular momentum [22], non-Abelian or Abelian statistics [23,24], or quantized transport properties [23,26].

Here, we take up the idea of binding impurities to quasiholes in a fractional quantum Hall liquid. First, we consider a single impurity and show that its angular momentum is fractional (in units $\hbar \equiv 1$). Then, by adding more impurities, taken as non-interacting fermions, we observe how the “anyon sea” is filled. Specifically, we show that the total angular momentum of the impurities matches neither the value from a fermionic construction, that is by filling the single-particle levels, nor the value of bosonic condensation. Instead, the total angular momentum is reproduced by a linear interpolation between a fermionic and a bosonic distribution, proportional to $\alpha = 1 - \nu$. Here, $\nu$ is the filling factor of the fractional quantum Hall liquid, and $\alpha$ equals the statistical parameter which describes the braiding phase of Laughlin quasi-hole bound to fermions.

While our results are obtained by numerically solving the underlying quantum Hall model, they can also be understood from fundamental theoretical concepts, and may thus serve as an illustration thereof. In fact, the relation of anyonic physics and fractional quantization of angular momentum dates back to earliest work on the subject: In Wilczek’s picture of anyonic statistics [2], the fractional behavior emerges through the attachment of matter particles to fluxes, i.e. vortex lines. In Laughlin’s wave function for fractional quantum Hall liquids [3], the matter particle appear as fluxes seen by the quasiholes, and on a mean-field level, this flux attachment re-defines the quasiholes’ effective vacuum, i.e., the effective magnetic field seen by the quasiholes. In Ref. [22], this reasoning has already been employed to explain the fractional relative angular momentum between two anyons, which can be measured via the correlation function of impurities bound to quasiholes. In the present paper, we demonstrate that the properties of the anyon vacuum and fractionally quantized angular momentum can even be probed with a single excitation. The fractional quantization of angular momentum can directly be inferred from the density of impurities bound to quasiholes, mak-
ing it easily accessible in experiment.

The characterization of the single-particle levels provides crucial information which we then use to reveal the anyonic quantum statistics of the impurities. Usually, the statistical behavior of anyons is defined by considering adiabatic braiding operations \[1\]. In contrast, this Letter makes a different approach: we examine how the many-body angular momentum of several impurities is composed of the single-particle values. This composition provides an immediate fingerprint of the distribution function describing the anyon statistics, and in accordance with the general expectation \[27\], it can be understood as an interpolation between Bose and Fermi statistics. Strikingly, even for very small systems the interpolation yields a statistical parameter which is in almost perfect agreement with the predictions from an effective impurity Hamiltonian derived in Ref. \[24\].

**System.** The system we consider consists of two types of particles \(a\) and \(b\). The majority of particles is of the \(a\) type, and forms a fractional quantum Hall liquid and Landau filling fraction \(\nu = 1/q\). The \(b\) particles are impurities. For simplicity, we assume similar single-particle physics for both species: Specifically, both species of equal mass \(M\) are trapped in the \(xy\)-plane by harmonic potentials of frequencies \(\omega_a\) and \(\omega_b\), and brought into the lowest Landau level by a sufficiently strong gauge potential \(A = \frac{\mathbf{B}}{2}(y, x, 0)\). In this gauge, the Fock-Darwin functions are characterized by a quantum number \(m\) for the angular momentum along the \(z\)-direction, and read \(\varphi_m(z) = (2\pi m!m^m)^{-1/2}e^{-|z|^2/4}\). The corresponding single-particle energies are \(\epsilon_{m,s} = \hbar \omega z_s\), with \(s \in \{a, b\}\), and \(\Omega_a \equiv \sqrt{\omega_a^2 + \omega_B^2}\). Here, \(\omega_B = eB/M\) is the cyclotron frequency, with \(e\) the electric or synthetic charge of the particle, parametrizing the coupling strength with the \(B\)-field. The coordinates \(z \in (x+iy)/l_B\) are given in units of the oscillator length \(l_B = (\hbar /M\Omega_B)^{1/2}\), which depend on the trapping frequency. We assume \(\omega_a \approx \omega_b \ll \omega_B\), such that \(\Omega_a \approx \Omega_b\), and the length scale \(l_B\) takes the same value for both \(a\) and \(b\).

To make the \(a\) particles form a fractional quantum Hall liquid, we consider repulsive interactions. Conveneiently, interactions are expressed by Haldane pseudopotentials \(U_t\), which parametrize the strength of interactions of a pair of particles in the lowest Landau level at fixed relative angular momentum \(\ell \equiv 2\mathbf{B}/l_B\). By truncating the pseudopotential expansion at \(\ell = q\) (i.e. by setting \(U_{\ell} = 0\) for \(\ell \geq q\)), we obtain parent Hamiltonians for Laughlin liquids at \(\nu = 1/q\). The ground state is then exactly given by the Laughlin wave function \(\Psi_q \sim \prod_{i<j \in \mathcal{Q}}(z_i - z_j)^q e^{-\sum_i |z_i|^2/4}\), and it is further characterized as a state of zero interaction energy. The total angular momentum of the Laughlin ground state is \(L_q = \frac{q}{2}N_a(N_a - 1)\), with \(N_a\) the number of \(a\) particles. No eigenstates of zero interaction energy are possible for \(L_a < L_q\), and within the Hilbert space with \(L_a = L_q\), the Laughlin wave function \(\Psi_q\) is non-degenerate. Laughlin liquids can be formed either by fermionic or bosonic \(a\) particles, depending on whether \(q\) is odd or even.

When the angular momentum of the liquid is increased above \(L_q\), i.e. for \(L_a = L_q + d\) with \(d > 0\), the liquid can accommodate for a characteristic number \(N_d\) of zero-energy modes. Their wave function is of the form \(\Psi_{q,d} = f_d^a(\{z_i\})\Psi_q\), where \(f_d^a(\{z_i\})\) is an arbitrary symmetric polynomial of degree \(d\). The index \(\alpha\) runs from 1 to \(N_d\), and \(N_d\) equals the number of partitions of the positive integer \(d\). These states describe deformations at the edge, when \(d \sim 1\), but for \(d \sim N_a\) they may also describe quasiholes in the liquid. Specifically, the function \(\Psi_{q,\alpha b} \sim \Pi_i(w - z_i)\Psi_q\) describes a quasihole at position \(w\). Note that for \(w = 0\) the factor \(\Pi_i(w - z_i)\) becomes a symmetric polynomial of degree \(d = N_a\), and thus, a quasihole in the center belongs to the manifold of zero-energy solution at \(L_a = L_q + N_a\).

The \(b\) species are taken to be non-interacting fermions. In order to bind to quasiholes of the Laughlin liquid, we consider a sufficiently strong repulsive contact interaction between \(a\) and \(b\) particles. This interaction allows for exchange of angular momentum between the species, but the joint angular momentum \(L = L_a + L_b\) remains a conserved quantity. For the case of a single impurity, the quasihole state \(\Psi_{q,\alpha b} \sim \Pi_i(w - z_i)\Psi_q\) is a state of zero interaction energy, where \(w\) now is taken as a dynamical variable representing the position of the impurity. In fact, the presence of interspecies repulsion makes this state non-degenerate at \(L = L_q + N_a\), and no zero-energy states exist at \(L < L_q + N_a\). For \(L = L_q + N_a + d\), zero-energy solutions with a single impurity have the form \(\Psi_{q,m_1,m_2} \sim w^{m_1}f_{m_2}^a(\{z_i\})\Pi_i(w - z_i)\Psi_q\), where \(m_1\) and \(m_2\) are positive integers with \(m_1 + m_2 = d\). Thus, the number of zero-energy modes at \(L = L_q + N_a + d\) is given by \(N_{d,imp} = \sum_{m_2 = 0}^d N_{m_2}\), see Table 1 for explicit numbers.

**Results for a single impurity.** The Laughlin state \(\Psi_q\) can be seen as an effective impurity vacuum, and the states \(\Psi_{q,\alpha}\) and \(\Psi_{q,m_1,m_2}\) define the ground and excited state of a single impurity. These two states have total angular momentum \(L = L_q + N_a\) and \(L = L_q + N_a + m_1 + m_2\), but it is not immediately clear how the angular momentum is distributed between the two species. Let \(L_b^0\) denote the average angular momentum of the impurity in its ground state, i.e. \(L_b^0 \equiv \langle \Psi_{q,\alpha b}| \hat{L}_b | \Psi_{q,\alpha b} \rangle\) with \(\hat{L}_b\) the angular momentum operator for the \(b\) particles. Naively, one may expect that the angular momentu

| \(d\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| \(N_a\) | 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| \(N_{d,imp}\) | 1 | 2 | 4 | 7 | 12 | 19 | 30 |

**TABLE I.** Number \(N_d\) of edge modes in the Laughlin liquid of degree \(d\), and number \(N_{d,imp}\) of zero-energy modes of degree \(d\) in the presence of an impurity.
turn $L_b^m$ of an impurity in its $m$th excited state, i.e. in $L_b^m \equiv \langle \Psi_{q,m,d-m} | L_b | \Psi_{q,m,d-m} \rangle$, is given by $L_b = L_b^0 + m$. However, as we show below, this is not the case. Instead, the angular momentum of the impurity is quantized to fractional values.

Before we turn to an explicit numerical demonstration of the fractional quantization of the impurity’s angular momentum, we provide analytical arguments for such behavior. Specifically, we note that the impurity at $w$ “sees” the majority particles at $z_i$ as fluxes, thus the gauge field seen by the impurity is reduced to $B^* = B - \rho_a B = B(1 - \nu)$, where $\rho_a$ is the density of the majority particles [22]. This leads to an increased magnetic length scale $l_B^\nu = l_B / \sqrt{1 - \nu}$. Thus, the normalized wave functions for a single impurity are given by

$$\tilde{\varphi}_m(w) = \sqrt{\frac{(1 - \nu)^{m+1}}{2\pi 2^m m!}} w^m e^{- (1 - \nu)|w|^2/2}. \quad (1)$$

In the limit of $\nu = 0$, this wave function is identical to the unrenormalized wave function $\varphi_m(w)$, the density corresponding to $\tilde{\varphi}_m$ is given by

$$\tilde{\rho}_m(w) = |\tilde{\varphi}_m(w)|^2 = \frac{(1 - \nu)^{m+1}}{2\pi 2^m m!} |w|^{2m} e^{- (1 - \nu)|w|^2/2} = \sum_{n=0}^\infty \rho_{m+n}(w) \nu^n (1 - \nu)^{m+n+1} \frac{(m+n)!}{m!n!}. \quad (2)$$

In the second line, we have expanded the renormalized density $\tilde{\rho}_m$ in terms of unrenormalized densities $\rho_{m+n} = |\varphi_{m+n}|^2$, corresponding to angular momentum $m+n$. Thus, the average angular momentum $L_b^m$ of an impurity in level $m$ is given by

$$L_b^m = \sum_{n=0}^\infty (n+m) \nu^n (1 - \nu)^{m+n+1} \frac{(m+n)!}{m!n!} = \frac{m + \nu}{1 - \nu}. \quad (3)$$

In its ground state ($m = 0$), the impurity has average angular momentum value $L_b^0 = \nu/(1 - \nu)$, and exciting the impurity by one unit (from $m$ to $m+1$) changes the average angular momentum by $\Delta L_b = 1/(1 - \nu) > 1$. For the standard deviation, we find $\delta L_b^m = \sqrt{\nu(m+1)/(1 - \nu)}$, so the relative deviation $\delta L_b^m / L_b^0 \rightarrow 0$ for large $m$.

We have used different methods to verify these results numerically: (i) Applying numerical diagonalization to the pseudopotential Hamiltonian at fixed total angular momentum $L_b$, the analytical construction of the zero-energy modes can be verified, and in particular the counting of Table I. We lift the ground state degeneracy $N_{d_{\text{imp}}}$ by $L_b = L_a + N_a$ by choosing the trap frequency $\omega_a$ slightly larger than $\omega_b$. The states within the quasi-degenerate manifold are then energetically ordered decreasingly with the excitation level $m$ of the impurity; the unique ground state is $\Psi_{q,a,0}$, followed by $\Psi_{q,a,1}$, and subsequently two degenerate states $\Psi_{q,a,2}$, etc. The corresponding impurity angular momentum $\langle L_b \rangle$ is immediately obtained from the numerical solution, and for each $m \geq d$, we find $N_{d-m}$ degenerate states, in which the impurity’s angular momentum matches very well with the theoretically expected value $L_b^m = L_b^0 + m \Delta L_b$. This behavior is exemplified in Fig. 3 for two cases corresponding to Laughlin filling factors $\nu = 1/q$ with $q = 3$ and $q = 5$. In this example, we have chosen $d = 4$ yielding twelve quasi-degenerate states (left of the red-dotted vertical line).

(ii) Another way of checking Eq. (3) is by evaluating the impurity angular momentum from the first-quantized wave functions. For small system sizes, this evaluation can be done by symbolical operations (cf. Ref. [29]), but numerical values can be produced for much larger systems (e.g. $N_a \sim 40$) using a Monte Carlo sampling method. Specifically, we have used this method to determine the impurity angular momentum of $\Psi_{q,1,0}$ for $2 \leq q \leq 6$, which is accurately given by $L_b^0$.

The fractional quantization of angular momentum is reflected by the impurity density, plotted in Fig. 3a. Higher orbitals correspond to larger angular momenta and are characterized by broader density profiles. More quantitatively, in the absence of a liquid (i.e. for $\nu = 0$), the form of the Fock-Darwin functions yields a linear relation between the mean square of the radial position, $\langle r^2 \rangle$, and the angular momentum $m$: $\langle r^2 \rangle_m \equiv \int_0^\infty dr r^2 |\varphi_m(r)|^2 = 2m + 2$. Numerically, we find such a linear relation also at finite $\nu$. In this case, let $m$ denote the excitation level of the impurity, i.e. we consider a many-body state $\Psi_{q,a,d-m}$. Note that the cor-

![FIG. 1. We plot the angular momentum $\langle L_b \rangle$ of an impurity in a Laughlin liquid at (a) $\nu = 1/3$ ($N_a = 8$ particles at $L = L_a + N_a = 4 + 96$), and at (b) $\nu = 1/5$ ($N = 6$ particles at $L = L_a + N_a = 4 + 85$). The twelve lowest states (on the left to the red-dotted line) are states of zero interaction energy. On average, the impurity takes fractionally quantized values $L_b = \frac{m + \nu}{1 - \nu}$ (indicated through the blue-dashed lines).](image-url)
corresponding impurity density $\rho^{m,q}_b(|w|)$ is essentially independent from the concrete choice of $d$ and $\alpha$. As seen in Fig. 2(b), the corresponding mean square of position, $(r^2)_{m,q} = \int_0^{\infty} dr r^2 \rho^{m,q}_b(r)$, scales linearly with $m$. However, as compared to $\nu = 0$, the slope is changed. In the absence of a majority liquid, a slope of value 2 corresponds to integer quantization of angular momentum. At $\nu = 1/3$ and $\nu = 1/5$, we find the slopes increased by a factor 3/2 and 5/4, in full accordance with the corresponding quantization of angular momentum.

**Results for multiple impurities.** Having established the quantization of angular momentum for a single impurity, Eq. (3), it is interesting to ask how $\langle L_\nu \rangle$ behaves in the presence of $N_b$ impurities. To obtain states of zero interaction energy, the total angular momentum needs to accommodate the anticorrelations of the majority liquid, the presence of $N_b$ quasiholes, and, in case of fermionic impurities, a Vandermonde determinant $\prod_{i<j}(w_i - w_j)$ which renders the wave function antisymmetric. Thus, the zero-energy ground state occurs at $L = L_q + N_b N_a + \frac{1}{2} N_b (N_b - 1)$, and its wave function reads:

$$\Psi_{F,\text{qhs}} \sim \prod_{1<j} (w_i - w_j) \cdot \prod_{i=1}^{N_b} \prod_{j=1}^{N_a} (z_i - w_j) \prod_{\alpha}^{N_b} \Psi_\alpha.$$ (4)

Naively, one might expect that the total angular momentum of the impurities is equal to the value obtained from filling the single-particle levels, $L_{b,\text{Fermi}} = \sum_{\alpha=0}^{N_b-1} \frac{m+\nu}{\nu}$. However, this expectation is not correct: Table II presents our numerical results for $N_b$ fermionic impurities interacting with a fermionic liquid at $\nu = 1/3$ and $\nu = 1/5$, at different system sizes. Table III presents analog results for fermionic impurities interacting with a bosonic liquid at $\nu = 1/2$ and $\nu = 1/4$. The impurity angular momentum is always found between the value of a condensate of $N_b$ impurities in the effective impurity ground state, $L_{b,\text{Fermi}} = N_b L^0_0 = N_b \nu \nu$, and the fermionic value $L_{b,\text{Fermi}}$. More precisely, the impurity angular momentum matches well with the following interpolation formula:

$$L_{b,\text{any}} = (1 - \nu) L_{b,\text{Fermi}} + \nu L_{b,\text{Bose}}.$$ (5)

This formula suggests that the statistical parameter $\alpha$, which interpolates from Bose statistics ($\alpha = 0$) to Fermi statistics ($\alpha = 1$), is given by $\alpha = 1 - \nu$. This is in agreement with the effective Hamiltonian derived in Ref. [24] for impurities coupled to fractional quasiholes (see also Refs. [30, 31]), and with the general expectation for a Laughlin quasihole ($\alpha = -\nu$) bound to fermion ($\alpha = 1$).

**Summary and Outlook.** We have shown fractional quantization of angular momentum for impurities bound to quasiholes of a fractional quantum Hall liquid, and
TABLE III. Filling the anyon sea: Same as in Table II, but for impurities (fermionic) bound to quasiholes of bosonic Laughlin liquids.

| \(\nu = 1/2\) | \(\nu = 1/4\) |
|----------------|----------------|
| \(N_a\) | \(N_b\) | \(L_{b,\text{any}}\) | (\(L_o\)) | \(N_a\) | \(N_b\) | \(L_{b,\text{any}}\) | (\(L_o\)) |
| 2 | 20 | 421 | 3 | 2081 | 10/6 | 1.65(1) | 3 | 20 | 801 | 234 | 4 | 4.00(2) |
| 3 | 20 | 443 | 6 | 6.26(4) | 4 | 3 | 20 | 823 | 3 | 2046 | 10 | 10.47(5) |
| 4 | 20 | 466 | 10 | 10.47(5) | 3 | 20 | 846 | 22/3 | 7.35(2) |

anyonic filling of these quantized levels. This establishes a relation between anyonic properties and the impurity density, providing a way to detect fractional statistics without braiding. More generally, the impurities realize a non-interacting gas of anyons, and future theoretical and experimental work may study the intricate thermodynamics of the system. Further insight might also be obtained by generalizing this study to other pseudopotentials and/or phases (including non-Abelian ones), or to similar models describing electrons interacting with excitons, and/or to interacting impurities which might themselves form fractional quantum Hall liquids.

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