Holographic transform for tensor product of holomorphic
discrete series

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Abstract

We study holographic operators associated with Rankin-Cohen brackets which are symmetry
breaking operators for the restriction of tensor products of holomorphic discrete series of $\text{SL}_2(\mathbb{R})$.
Furthermore, we investigate a geometrical interpretation of these operators and their relations
to classical Jacobi polynomials.

1 Introduction

Let $G$ be a real reductive Lie group, $G'$ a Lie subgroup of $G$ and $\pi$ a irreducible unitary representation
of $G$ on a vector space $V$. The decomposition into irreducible representations $(\rho, W)$ of $G'$ of the
restriction $\pi|_{G'}$ of $\pi$ to $G'$ is called the branching rule. Symmetry breaking operators are defined as
elements of the space of linear continuous maps $V \rightarrow W$ that intertwine $\pi|_{G'}$ and $\rho$, denoted by
$\text{Hom}_{G'}(\pi|_{G'}, \rho)$, for any given irreducible representation $(\rho, W)$ of $G'$. Analogously, elements of the
space $\text{Hom}_{G'}(\rho, \pi|_{G'})$ are called holographic operators (see for example (2.9)).

In the article [KP20], the authors investigate holographic operators in two different geometric
settings : $(G, G') = (\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}), \text{SL}_2(\mathbb{R}))$ referred to as the diagonal case,
and $(G, G') = (\text{SO}_0(2, n), \text{SO}_0(2, n - 1))$ referred to as the conformal case. In both situations, they consider
the restriction of holomorphic discrete series representations of $G$ to $G'$ and develop two explicit
approaches to study corresponding holographic operators:

- The first method is based on the Laplace transform for tube domains, and leads to a new
transform involving integration along a line segment in the complex upper half-plane $\Pi$ (see
2.6) for the diagonal case.

- The second one uses the reproducing kernel technics for weighted Bergman spaces, and leads
to the construction of a relative reproducing kernel in the conformal case (see Thm 3.10 in
[KP20]).

It is known (see [DP07] for instance) that the symmetry breaking operators for the decomposition
of the tensor products of holomorphic discrete series of $\text{SL}_2(\mathbb{R})$ (what we call the diagonal case) are
proportional to Rankin-Cohen brackets $RC^{\lambda''}_{\lambda', \lambda}$ (see (2.4)).

In this work, we extend the method of the relative reproducing kernel to the diagonal case and
describe the corresponding holographic $(RC^{\lambda''}_{\lambda', \lambda})^*$ operator as follows.

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Theorem 1.1. Suppose $\lambda', \lambda'', \lambda''' \in \mathbb{N}\{0; 1\}$ such that $l = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. Let $w_1, w_2 \in \Pi$, and $g \in H^{2}_{\lambda'''}(\Pi)$. Then we have:

\[
(\mathcal{R}^{\lambda'''}_{\lambda', \lambda''} g)(w_1, w_2) = C(\lambda', \lambda'') \int_{\Pi} g(z) K^{\lambda'''}_{\lambda', \lambda''}(z, w_1, w_2) d\mu(z),
\]

(1.1)

where $K^{\lambda'''}_{\lambda', \lambda''}(z, w_1, w_2) = (w_2 - w_1)^l \left( \frac{w_2 - z}{2i} \right)^{-(\lambda' + l)} \left( \frac{w_1 - z}{2i} \right)^{-(\lambda'' + l)}$, and $C(\lambda', \lambda'') = \frac{(\lambda' - 1)_l (\lambda'' - 1)_l}{2^{2l} l!}.

Notice that this kind of operators has already been considered by H. Rosengren in [Ros99]. We give two different proofs of this theorem, the first one is inspired by the proof of Theorem 3.10 in [KP20], and the second one, based on the Laplace transform, is new and can also be used in the conformal case.

Our second point (see Fact 4.2) explore a conceptual interpretation of the link between orthogonal polynomials and branching rules for restriction of discrete series representations. More precisely, Rankin-Cohen brackets can be expressed in terms of Jacobi polynomials as showed in [KP16] (see Thm.8.1). This is due to the fact that the equivariance condition for symmetry breaking operators can be reduced to the Jacobi ordinary differential equation for the symbols of such operators. In [KP20] p.15, the authors give yet another interpretation of the link between Rankin-Cohen brackets and Jacobi polynomials based on the fact that holographic transform can be expressed using the inversion of the usual Jacobi transform. In this work, we interpret this statement emphasising the fact that Jacobi polynomials form an orthogonal basis for the Hilbert space $L^2((−1, 1), (1−v)^{\alpha}(1+v)^{\beta} \, dv)$ (see [KP20], section 5.1). For this, we use another realization of the holomorphic discrete series in which symmetry breaking operators are expressed in terms of the classical Jacobi transform.

Notation: We use the Pochammer symbol, defined for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ by :

\[
(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n).
\]

and set the condition

\[
\lambda', \lambda'', \lambda''' \in \mathbb{N} \text{ such that } l := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}. \quad (C1)
\]

2 Setting

In this section, we describe two different models for the holomorphic discrete series representations of the Lie group $SL_2(\mathbb{R})$, and their tensor products.

2.1 Holomorphic model

We denote by $\Pi$ the Poincar upper half-plane endowed with the hyperbolic metric, and by $H^2_\lambda(\Pi)$ the weighted Bergman space defined by:

\[
H^2_\lambda(\Pi) = \mathcal{O}(\Pi) \cap L^2(\Pi, y^{\lambda-2} \, dx dy).
\]

It is known that $H^2_0(\Pi) = \{0\}$ for $\lambda \leq 1$, so we suppose that $\lambda', \lambda'', \lambda''' \in \mathbb{N}\{0,1\}$. In this case $H^2_\lambda(\Pi)$ admits a reproducing kernel, also called the Bergman kernel, given by (see [FK94], Prop.XIII.1.2, p.261):
We set $H_{\lambda',\lambda''}^2(\Pi \times \Pi) \simeq H_{\lambda'}^2(\Pi) \hat{\otimes} H_{\lambda''}^2(\Pi)$, where $\hat{\otimes}$ denotes the completion of the tensor product. This space also admits a reproducing kernel:

$$K_\lambda(z, w) = \frac{\lambda - 1}{4\pi} \left( \frac{z - \bar{w}}{2i} \right)^{-\lambda}. \quad (2.1)$$

The holomorphic discrete series representations $\pi_\lambda$ of $SL_2(\mathbb{R})$ can be realized on $H_{\lambda}^2(\Pi)$ by the following formula:

$$(\pi_\lambda(g) f)(z) = (cz + d)^{-\lambda} f \left( \frac{az + b}{cz + d} \right), \quad (2.3)$$

where $g^{-1} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{R})$, and $f \in H_{\lambda}^2(\Pi)$.

It is known (see [Knu01], p.35) that this representation is irreducible and unitary for $\lambda \in \mathbb{N} \setminus \{0; 1\}$. The outer product representation $\pi_{\lambda'} \otimes \pi_{\lambda''}$ of the Lie group $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ acts on the space $H_{\lambda',\lambda''}^2(\Pi \times \Pi)$. This representation is also irreducible and unitary for $\lambda', \lambda'' > 1$.

### 2.2 Rankin-Cohen operators

Let $\lambda', \lambda'', \lambda'''$ verify (CI), and define a differential operator $R_{\lambda',\lambda''}^{\lambda'''}$ from $\mathcal{O}(\Pi \times \Pi)$ to $\mathcal{O}(\Pi \times \Pi)$ by

$$R_{\lambda',\lambda''}^{\lambda'''}(f)(w_1, w_2) = \sum_{j=0}^l \frac{(-1)^j(\lambda' + l - j)(\lambda'' + j)(\lambda'''+j-1)!}{j!(l-j)!} \partial_{\bar{z}_1}^{-j-1} \partial_{\bar{z}_2}^j f(w_1, w_2). \quad (2.4)$$

The Rankin-Cohen operator is a map from $\mathcal{O}(\Pi \times \Pi)$ to $\mathcal{O}(\Pi)$ defined by:

$$RC_{\lambda',\lambda''}^{\lambda'''} = Rest \circ R_{\lambda',\lambda''}^{\lambda'''} \quad (2.5)$$

where $Rest$ is the restriction to the diagonal of $\Pi \times \Pi$. The Rankin-Cohen operator $RC_{\lambda',\lambda''}^{\lambda'''}$ is a symmetry breaking operator for the outer product representation $\pi_{\lambda'} \otimes \pi_{\lambda''}$, more precisely it generates the space $\text{Hom}_{SL_2(\mathbb{R})}(\pi_{\lambda'} \otimes \pi_{\lambda''}, \pi_{\lambda'''} \mid SL_2(\mathbb{R}), \pi_{\lambda'''}{})$, for $\lambda', \lambda'', \lambda'''$ satisfying (CI) (see [KP16] Cor. 9.3 for the precise statement).

The goal of this paper is to study the holographic operators associated to the Rankin-Cohen operators. In order to do so, we compute the adjoint operators $(RC_{\lambda',\lambda''}^{\lambda'''})^* : H_{\lambda'''}^2(\Pi \times \Pi) \rightarrow H_{\lambda',\lambda''}^2(\Pi \times \Pi)$ which give, in the unitary case, the holographic operators.

### 2.3 Holographic operators for the diagonal case

Let $\lambda', \lambda'', \lambda'''$ verify (CI), and define the operators $\Psi_{\lambda',\lambda''}^{\lambda'''} : H_{\lambda'''}^2(\Pi \times \Pi) \rightarrow H_{\lambda',\lambda''}^2(\Pi \times \Pi)$ by

$$\Psi_{\lambda',\lambda''}^{\lambda'''}(g)(w_1, w_2) = \frac{(w_1 - w_2)^l}{2^{\lambda' + \lambda'' + 2l-1} l!} \int_{-1}^1 g(w(v))(1 - v)^{\lambda' + l-1}(1 + v)^{\lambda'' + l+1} dv, \quad (2.6)$$

where $w(v) = \frac{1}{2}((w_2 - w_1)v + (w_2 + w_1))$.

In [KP20], the authors prove that this is a holographic operator from $H_{\lambda'''}^2(\Pi)$ to $H_{\lambda',\lambda''}^2(\Pi \times \Pi)$. They
also establish a Parseval-Plancherel type theorem for the corresponding Rankin-Cohen transform and its associated holographic transform (see [KP20] Thm. 2.7). Finally, they show (Prop 2.23) that:

\[(RC^{λ''}_{N,λ''})^* = CΨ^{λ''}_{N,λ''}.\]  

(2.7)

where \(C = \frac{Γ(λ'+λ''+2l-1)}{2^{2l+2}Γ(λ'-1)Γ(λ''-1)}.\)

However, they also used another method to get a different expression for this map (see lemma 5.1 below). This approach leads to a different type of integral transformations based on the relative reproducing kernel \(K^{λ''}_{N,λ''}(z, w_1, w_2)\), defined for \(w_2, w_1, z \in \Pi\) by:

\[K^{λ''}_{N,λ''}(z, w_1, w_2) = (w_2 - w_1)^l \left(\frac{w_1 - \bar{z}}{2i}\right)^{(λ'+l)} \left(\frac{w_2 - \bar{z}}{2i}\right)^{(λ''+l)}.\]  

(2.8)

This relative reproducing kernel gives an explicit expression for the adjoint of the Rankin-Cohen operator as follows

**Theorem 2.1.** Suppose \(λ', λ'' \in \mathbb{N}\{0; 1\}\) such that \(l = \frac{1}{2}(λ'' - λ' - λ'') \in \mathbb{N}\). Let \(w_1, w_2 \in \Pi\), and \(g \in H^2_{π''}(\Pi)\). Then we have:

\[(RC^{λ''}_{N,λ''})^* g(w_1, w_2) = C(λ', λ'') \int_{\Pi} g(z)K^{λ''}_{N,λ''}(z, w_1, w_2)dµ(z),\]  

(2.9)

where \(C(λ', λ'') = \frac{(λ'-1+l)(λ''-l+1)}{2^{2l+2}π2l!}\).

We prove this theorem in the following sections 3.1 and 3.2 and make explicit the link with the operator \(Ψ^{λ''}_{N,λ''}\) introduced in (2.6). For this we need another model for the holomorphic discrete series representations of \(SL_2(\mathbb{R})\).

### 2.4 \(L^2\)-model of holomorphic discrete series

For \(λ > 1\), the Laplace transform defined by:

\[Fg(z) = \int_0^{\infty} g(t)e^{itz}dt,\]  

(10.10)

is a one-to-one isometry (up to a constant) from \(L^2_λ(\mathbb{R}^+) := L^2(\mathbb{R}^+, t^{1-λ}dt)\) to \(H^2_λ(\Pi)\) (see [FK92], Thm. XII.1.1). More precisely:

\[\|Fg\|^2_{H^2_λ(\Pi)} = b(λ)\|g\|_2^2_{L^2_λ(\mathbb{R}^+)},\]

for every \(g \in L^2_λ(\mathbb{R}^+)\), where \(b(λ) = 2^{2-λ}πΓ(λ-1)\).

The inverse Laplace transform is given, for \(f \in H^2_λ(\Pi) \cap H^2_λ(\Pi)\), by the following formula:

\[F^{-1}f(t) = \frac{1}{2π} \int_{\mathbb{R}} f(x+iy)e^{-i(x+iy)t}dx.\]  

(11.11)

Using this transform, we consider another realization for the holomorphic discrete series representations of \(SL_2(\mathbb{R})\) on \(L^2_λ(\mathbb{R}^+)\), an call it the \(L^2\)-model for \(π_λ\).

For \(λ', λ'' > 1\), we define

\[L^2_{N,λ''}(\mathbb{R}^+ × \mathbb{R}^+) := L^2_λ(\mathbb{R}^+)⊗L^2_λ(\mathbb{R}^+) \simeq L^2(\mathbb{R}^+ × \mathbb{R}^+, x^{1-λ'}y^{1-λ''}dxdy).\]

We denote \(F_2 = F ⊗ F\) the Laplace transform from \(L^2_{N,λ''}(\mathbb{R}^+ × \mathbb{R}^+)\) to \(H^2_λ(\Pi) × \Pi\).
2.5 Holographic operators in the $L^2$-model

Set $\lambda', \lambda'', \lambda'''$ satisfying (C1). Using the Laplace transform (2.10), we define the analogue of the Rankin-Cohen operators in the $L^2$-model by:

$$RC_{\lambda', \lambda''}^{\lambda'''} := F^{-1} \circ RC_{\lambda', \lambda''} \circ F_2.$$

(2.12)

It is a symmetry breaking operator for the $SL_2(\mathbb{R})$-action in the $L^2$-model.

In [KP20], the authors prove that $RC_{\lambda', \lambda''}^{\lambda'''}$ is given by the following integral formula for $F \in L^2_{\lambda', \lambda''}(\mathbb{R}^+ \times \mathbb{R}^+)$ (see [KP20], Prop. 2.13):

$$RC_{\lambda', \lambda''}^{\lambda'''}F(t) = \frac{t^{l+1}}{2^d} \int_{-1}^{1} P_{l}^{\lambda'-1, \lambda''-1}(v) F \left( \frac{t}{2}(1-v), \frac{t}{2}(1+v) \right) dv,$$

(2.13)

where $P_{l}^{\lambda'-1, \lambda''-1}$ denotes the Jacobi polynomials (see [KP20], section 5.1).

We define the following operator which associates to a function $g(t)$ defined on $\mathbb{R}^+$ a function of two variables $\Phi_{\lambda', \lambda''}g$ on $\mathbb{R}^+ \times \mathbb{R}^+$:

$$\Phi_{\lambda', \lambda''}g(x, y) := \frac{x^{\lambda'-1}y^{\lambda''-1}}{(x+y)^{\lambda'+\lambda''+l-1}} P_{l}^{\lambda'-1, \lambda''-1} \left( \frac{y-x}{x+y} \right) g(x+y).$$

(2.14)

Once again, it is shown in [KP20], that $\Phi_{\lambda', \lambda''}$ is a holographic operator between $L^2_{\lambda', \lambda''}(\mathbb{R}^+)$ and $L^2_{\lambda', \lambda''}(\mathbb{R}^+ \times \mathbb{R}^+)$. We summarize the framework in the two following commutative diagrams

Symmetry breaking operators in the holomorphic and $L^2$-models

2.6 Inverse Laplace transform of the reproducing kernel

We compute the inverse Laplace transform of the Bergman kernels $K_\lambda(z, w)$. The following lemma can be proved using the Cauchy residue theorem.
Lemma 2.2. For \( x > 0, v \in \mathbb{R} \) and \( \lambda \in \mathbb{N}\backslash\{0; 1\} \), we have:

\[
\int_{\mathbb{R}} (u + iv - \overline{z})^{-\lambda} e^{-i(u+iv)x} du = \frac{2\pi i^{-\lambda}}{(\lambda - 1)!} x^{\lambda - 1} e^{-ix\overline{z}}.
\]

\( (2.15) \)

Remark: One should notice in the previous lemma that the right hand side does not depend on \( v \).

Then the following corollary gives the inverse Laplace transform for the Bergman kernels \( K_\lambda(\cdot, w) \).

Corollary 2.3. Suppose \( \lambda, \lambda', \lambda'' \in \mathbb{N}\backslash\{0; 1\} \). The inverse Laplace transform of the reproducing kernel \( K_\lambda \) is given by the following formula:

\[
\mathcal{F}^{-1} K_\lambda(\cdot, z)(t) = \frac{2^{\lambda-1}}{2\pi(\lambda - 2)!} t^{\lambda - 1} e^{-it\overline{z}}.
\]

\( (2.16) \)

Consequently :

\[
\mathcal{F}^{-1} K_{\lambda', \lambda''}(\cdot, (w_1, w_2))(x, y) = \frac{2^{\lambda'+\lambda''-2}}{4\pi^2(\lambda' - 2)!(\lambda'' - 2)!} e^{x'\lambda' - 1} y^{\lambda'' - 1} e^{-i(x, y)\cdot(w_1, w_2)}.
\]

\( (2.17) \)

Proof. According to [God15] (p.107) \( K_\lambda(\cdot, z) \in H^1_{\lambda}(\Pi) \cap H^2_{\lambda}(\Pi) \). Then, the first equality is due to lemma (2.2) and the second, to the fact that \( K_{\lambda', \lambda''}((w_1, w_2), (z, z)) = K_{\lambda'}(w_1, z) \cdot K_{\lambda''}(w_2, z) \).

3 Computation of the relative reproducing kernel
The goal of this section is to give two different proofs of Theorem 1.1. For this, we use Lemma 3.13 from [KP20]:

Lemma 3.1. Let \( D_j (j = 1, 2) \) be some complex manifolds, and \( H_j \) some Hilbert spaces of holomorphic functions on \( D_j \) with reproducing kernels \( K^{(j)}(\cdot, \cdot) \). If \( R : H_1 \to H_2 \) is a continuous linear map, then :

1. \( RK^{(1)}(\cdot, \zeta)(\tau') = (R^* K^{(2)}(\zeta, \cdot))(\tau') \) for \( \zeta \in D_1, \tau' \in D_2 \).
2. \( (R^* g)(\zeta) = (g, RK^{(1)}(\cdot, \zeta))_{H_2} \) for \( g \in H_2, \zeta \in D_1 \).

Thanks to this lemma, we only need to compute \( R\lambda_{\lambda', \lambda''}(K_{\lambda', \lambda''}(\cdot, (w_1, w_2))) \) in order to find an explicit formula for the holographic operator. We give two different ways for this computation. First, we compute it directly and in a second time we use the Laplace transform to get our result.

3.1 First proof of Theorem 1.1
Our first proof of Theorem 1.1 reduces to a direct computation.

Lemma 3.2. For \( l \in \mathbb{N} \) and \( \lambda \in \mathbb{N}\backslash\{0; 1\} \), we have :

\[
\frac{\partial^l}{\partial z^l} K_\lambda(z, w) = \frac{(-1)^l(\lambda - 1)_{l+1}}{4\pi^2(2i)^l} (\frac{z - \bar{w}}{2i})^{-(\lambda+l)}.
\]

\( (3.1) \)

This statement can be proved by induction on \( l \). Then we are ready to prove Theorem 1.1.
Proof of Theorem 1.1

\[ R_{\lambda''}^{\lambda''}K_{\lambda',\lambda''}(\cdot,(w_1,w_2))(z_1,z_2) \]
\[ = \sum_{j=0}^{l} \frac{(-1)^j}{j!(l-j)!} \left( \frac{z_1 - \overline{w}_1}{2i} \right)^{(\lambda'+l)} \left( \frac{z_2 - \overline{w}_2}{2i} \right)^{(\lambda''+l)} \]
\[ \times \sum_{j=0}^{l} \frac{(-1)^j(\lambda' + l - j)(\lambda'' + j)_{l-j+1}(\lambda'' - 1)_{j+1}}{(4\pi)^2l!(2l)^2} \left( \frac{z_1 - \overline{w}_1}{2i} \right)^{-(\lambda'+l)} \left( \frac{z_2 - \overline{w}_2}{2i} \right)^{-(\lambda''+l)} \]
\[ = \frac{1}{(4\pi)^2l!(2l)^2} \left( \frac{z_1 - \overline{w}_1}{2i} \right)^{-(\lambda'+l)} \left( \frac{z_2 - \overline{w}_2}{2i} \right)^{-(\lambda''+l)} \]
\[ \times \sum_{j=0}^{l} \left( \frac{1}{j!(l-j)!} \right) (z_1 - \overline{w}_1)^{l-j} (z_2 - \overline{w}_2)^{(l-j)} \]
\[ = \frac{(\lambda' - 1)_{l+1}(\lambda'' - 1)_{l+1}}{(4\pi)^2l!(2l)^2} \left( \frac{z_1 - \overline{w}_1}{2i} \right)^{-(\lambda'+l)} \left( \frac{z_2 - \overline{w}_2}{2i} \right)^{-(\lambda''+l)} \]
\[ (z_1 - z_2 + \overline{w}_2 - \overline{w}_1)^l. \]

Finally, we get by restriction to the diagonal \( z_1 = z_2 = z \):

\[ RC_{\lambda',\lambda''}^{\lambda''}K_{\lambda',\lambda''}(\cdot,(w_1,w_2))(z) = \text{Re}st|_{z_1=z_2=z} \circ R_{\lambda',\lambda''}^{\lambda''}K_{\lambda',\lambda''}(\cdot,(w_1,w_2))(z) \]
\[ = \frac{(\lambda' - 1)_{l+1}(\lambda'' - 1)_{l+1}}{(4\pi)^2l!(2l)^2} \left( \frac{w_1 - \overline{w}}{2i} \right)^{-(\lambda'+l)} \left( \frac{w_2 - \overline{w}}{2i} \right)^{-(\lambda''+l)} \]
\[ (w_2 - w_1)^l. \]

Thus Lemma 3.1 implies the statement. \( \square \)

3.2 Second proof of Theorem 1.1

Here we give a second proof of Theorem 1.1 using the Laplace transform (2.10).

Proposition 3.3. Set \( \lambda', \lambda'', \lambda''' \in \mathbb{N} \setminus \{0;1\} \) satisfying \( (C1) \).

\[ RC_{\lambda',\lambda''}^{\lambda''}K_{\lambda',\lambda''}(\cdot,(w_1,w_2))(z) = C''(w_2 - w_1)^l \int_{0}^{\infty} \mathcal{F}^{-1}K_{\lambda''}(\cdot,z)(t) \, 1F_1(\lambda' + l, \lambda' + \lambda'' + 2l; -i(w_2 - w_1)t)e^{iwt} \, dt, \]

where \( C'' = \frac{\Gamma(\lambda' + \lambda'' + 2l + 1)\Gamma(\lambda' - 1)}{2^{\lambda' + \lambda'' + 2l + 1}\Gamma(\lambda' - 1)\Gamma(\lambda'' - 1)} \) and \( 1F_1 \) is the Kummer function (see Appendix 2).

Proof. By definition (2.12), we have:

\[ \text{Lemma 2.3 and (5.1) and formula (2.13) imply:} \]
\[ \frac{1}{2^{\lambda' + \lambda'' + l - 1}} \int_{-1}^{1} P_{l}^{\lambda' - 1, \lambda'' - 1}(v)e^{-it\lambda' - 1(1 + v)\lambda'' - 1} \, dv \]
\[ = \frac{2^{\lambda' + \lambda'' - l - 1}}{\pi^2(\lambda' - 2)!(\lambda'' - 2)!} (w_2 - w_1)^l \int_{0}^{\infty} \mathcal{F}^{-1}K_{\lambda''}(\cdot,z)(t) \, dt. \]
As \( RC_{\lambda', \lambda''}^{\lambda''} = F \circ RC_{\lambda', \lambda''} \circ F^{-1} \) (see (2.10)), it gives:

\[
RC_{\lambda', \lambda''}^{\lambda''}K_{\lambda', \lambda''}(\cdot; (w_1, w_2))(z) = C'(w_2 - w_1) \int_{\mathbb{R}_{+}} \frac{2^{\lambda''-1}}{2\pi(\lambda'' - 2)!} t^{\lambda''-1}e^{itz} F_1 (\lambda' + l, \lambda' + \lambda'' + 2l; i(w_1 - w_2)t) e^{-itw_2} dt.
\]

Corollary (2.3) gives \( \frac{2^{\lambda''-1}}{2\pi(\lambda'' - 2)!} t^{\lambda''-1}e^{-itz} = F^{-1}K_{\lambda''}(\cdot, t) \), which ends our proof.

An alternative proof of Theorem 1.1 is based on the properties of Kummer special functions as follows.

**Second proof of Theorem 1.1** First, remark that, for every \( n \in \mathbb{N} \), we have:

\[
\left( \frac{\partial}{\partial w} \right)^n K_{\lambda''}(w, z) = \frac{(-1)^n(\lambda'' - 1)(\lambda'')}n (2i)^{\lambda''}(w - z)^{-\lambda''-n}.
\]

Then, we use the power series expansion for Kummer functions (5.1) to get:

\[
\int_{0}^{\infty} F^{-1}(K_{\lambda''}(\cdot, z))(t) e^{itz} F_1 (\lambda' + l, \lambda' + \lambda'' + 2l; -i(w_2 - w_1)t) dt
\]

\[
= \sum_{n=0}^{\infty} (\lambda' + l)_n (\lambda' + \lambda'' + 2l)_n \int_{0}^{\infty} F^{-1}(K_{\lambda''}(\cdot, z))(t) e^{itz} (-i(w_2 - w_1)t)^n e^{itz} dt
\]

\[
= \sum_{n=0}^{\infty} (\lambda' + l)_n (\lambda' + \lambda'' + 2l)_n \left( \frac{\partial}{\partial w} \right)^n K_{\lambda''}(w, z) \bigg|_{w=w_2}
\]

\[
= \lambda'' - 1 \sum_{n=0}^{\infty} (\lambda' + l)_n (-1)^n (2i)^{\lambda''}(w_2 - z)^{-\lambda''-n} (w_2 - w_1)^n
\]

\[
= \lambda'' - 1 \sum_{n=0}^{\infty} (\lambda' + l)_n (-1)^n \left( \frac{w_2 - w_1}{w_2 - z} \right)^{\lambda'' - l} (\frac{w_2 - w_1}{w_2 - z})^{-\lambda'' - l}
\]

Notice that the result is valid for \( |\frac{w_2 - w_1}{w_2 - z}| < 1 \), and use the analytic continuation to extend it for arbitrary \( w_1, w_2, z \in \Pi \). Proposition 3.3 implies the result. Finally, for the constant \( C(\lambda', \lambda'') \), we have:

\[
C(\lambda', \lambda'') = C(\lambda'' - 1)B(\lambda' + l, \lambda' + l) = \frac{(\lambda'' - 1)\Gamma(\lambda'' - 1)\Gamma(\lambda' + l)\Gamma(\lambda' + l)}{2^{2l}4\pi^2 l!\Gamma(\lambda' - 1)\Gamma(\lambda'' - 1)\Gamma(\lambda'' - 1)\Gamma(\lambda' - 1)\Gamma(\lambda')}
\]

**3.3 Link with the operator \( \Psi_{\lambda', \lambda''}^{\lambda''} \)**

We give another characterization of the operators \( \Psi_{\lambda', \lambda''}^{\lambda''} \) (see (2.6)) as follows.
Proposition 3.4. Suppose $\lambda', \lambda'', \lambda''' \in \mathbb{N} \setminus \{0; 1\}$ such that $l = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. Then we have the following identity:

$$\Psi_{\lambda', \lambda''}(K_{\lambda'''}(\cdot, z))(w_1, w_2) = \frac{(-1)^l(\lambda''' - 1)B(\lambda' + l, \lambda'' + l)}{4\pi!} K_{\lambda'''}(z, w_1, w_2).$$

(3.3)

where $K_{\lambda'''}$ is the relative reproducing kernel \(2.3\) given by

$$K_{\lambda'''}(z, w_1, w_2) = (w_1 - w_1)^l \left(\frac{w_1 - z}{2t}\right)^{-(\lambda' + l)} \left(\frac{w_2 - z}{2t}\right)^{-(\lambda'' + l)}.$$

Proof. Similar argument as in the proof of Proposition 3.3 gives:

$$\Psi_{\lambda', \lambda''}(K_{\lambda'''}(\cdot, z))(w_1, w_2)$$

$$= \frac{(w_1 - w_2)^l}{2^{\lambda'''} - 1} \int_1^\infty \int_0^{\lambda'''} f^{-1}K_{\lambda'''}(\cdot, z)(t)e^{itw(v)} dt \left(1 - v\right)^{\lambda'' + l - 1}(1 + v)^{\lambda'' + l - 1} dv$$

$$= \frac{2^{l+2\pi}(w_1 - 1)!^{(\lambda'' - 2)!}}{(\lambda''' - 2)!} RC_{\lambda', \lambda''}(K_{\lambda'''}(\cdot, (w_1, w_2)))\Pi.$$

\(\square\)

Remark: This proposition gives another proof of the link between $\Psi_{\lambda', \lambda''}$ and $(RC_{\lambda', \lambda''})^*$ (see formula (3.3)). Indeed, for every $g \in H_{\lambda'''}(\Pi)$:

$$\Psi_{\lambda', \lambda''}(g)(w_1, w_2)$$

$$= \frac{(w_1 - w_2)^l}{2^{\lambda'''} + \lambda'' + 2l - 1} \int_1^\infty g(w(v))\left(1 - v\right)^{\lambda'' + l - 1}(1 + v)^{\lambda'' + l - 1} dv$$

$$= \frac{(w_1 - w_2)^l}{2^{\lambda'''} + \lambda'' + 2l - 1} \int_{\Pi} g(z) K_{\lambda'''}(z, (w(v))))^l(1 - v)^{\lambda'' + l - 1}(1 + v)^{\lambda'' + l - 1} dxdydv$$

$$= \int_{\Pi} g(z) \left(\int_1^\infty g(w(v), z)\left(1 - v\right)^{\lambda'' + l - 1}(1 + v)^{\lambda'' + l - 1} dv\right) y^{\lambda'' - 2} dxdy$$

$$= \int_{\Pi} g(z) \Psi_{\lambda', \lambda''}(K_{\lambda'''}(\cdot, z))(w_1, w_2) y^{\lambda'' - 2} dxdy.$$

4 Geometrical interpretation

4.1 Cone stratification

Let $\sigma$ be an involutive automorphism of a connected semi-simple Lie group $G$, which commutes with the Cartan involution $\theta$ of $G$. We use the same letters $\sigma$ and $\theta$ to denote their differentials. Define a $\theta$-stable subgroup of $G$ by

$$G^\theta = \{g \in G | \sigma(g) = g\}.$$

(4.1)

Consider $K$ a maximal compact subgroup of $G$ and $g = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $g = \text{Lie}(G)$. The space $\mathfrak{p}$ is $\sigma$-stable, so we have the following diffeomorphism for the symmetric space $G/K$:

$$G/K \simeq \mathfrak{p} \oplus \mathfrak{p}^{-\sigma}.$$

(4.2)
where $p^{\pm \sigma} = \{ X \in p \mid \sigma(X) = \pm X \}$. This diffeomorphism is explicit using the exponential map (see [Hel01, Thm.1.1, p.252]):

$$X + Y \in p^\sigma \oplus p^{-\sigma} \mapsto \exp(X + Y)K \in G/K.$$  

(4.3)

The automorphism $\sigma_\theta$ is also involutive, so one checks that:

$$p^\sigma \simeq G^\sigma/K^\sigma \text{ and } p^{-\sigma} \simeq G^\sigma/K^{\sigma_\theta}.$$  

(4.4)

Finally, (4.2) and (4.4) gives the following diffeomorphism for $G/K$:

$$G/K \simeq G^\sigma/K^\sigma \times G^{\sigma_\theta}/K^{\sigma_\theta},$$  

(4.5)

where the diffeomorphism for the right-hand side is explicitly given by

$$X + Y \in p^\sigma \oplus p^{-\sigma} \mapsto (\exp(X)K^\sigma, \exp(Y)K^{\sigma_\theta}) \in G^\sigma/K^\sigma \times G^{\sigma_\theta}/K^{\sigma_\theta}.$$  

(4.6)

In the diagonal case, the following diffeomorphism $\iota$ from $\mathbb{R}^+ \times (-1; 1) \to \mathbb{R}^+ \times \mathbb{R}^+$

$$\iota : (t, v) \mapsto \iota(t, v) \equiv \left( \frac{t}{2}(1 - v), \frac{t}{2}(1 + v) \right)$$  

(4.7)

is an explicit realization of the decomposition (1.5) for $G/K = \mathbb{R}^+ \times \mathbb{R}^+$ and $\sigma(x, y) = (y, x)$.

This diffeomorphism corresponds to the diagonal embedding of $\mathbb{R}^+ \times \mathbb{R}^+$ into $\mathbb{R}^+ \times \mathbb{R}^+$, and where the cone $\mathbb{R}^+ \times \mathbb{R}^+$ has a stratification by the segments orthogonal to the diagonal. In his recent preprint [Cle20], J-L. Clerc gives a generalization of this diffeomorphism for the diagonal embedding of a symmetric cone $\Omega$ into the direct product $\Omega \times \Omega$ (see [Cle20]).

Let $\alpha, \beta > 1$, and $d\mu_{\alpha, \beta}(v) := (1 - v)^{\alpha - 1}(1 + v)^{\beta - 1} \, dv$ be a measure on the segment $(-1, 1)$. We define the Hilbert spaces

$$L^2_{\alpha, \beta}(\mathbb{R}^+ \times (-1; 1)) := L^2(\mathbb{R}^+ \times (-1; 1), t^{\alpha + \beta - 1} \, dtd\mu_{\alpha, \beta}(v))$$

. For a function $f$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$, we define a function $T_\iota(f)$ on $\mathbb{R}^+ \times (-1, 1)$ by

$$T_\iota(f)(t, v) = f \circ \iota(t, v) \left( \frac{t}{2} \right)^{2-\lambda'-\lambda''} (1 - v)^{1-\lambda'}(1 + v)^{1-\lambda''}.$$  

(4.8)

One checks that it is a one-to-one isometry from $L^2(\mathbb{R}^+ \times \mathbb{R}^+, x^{1-\lambda'}y^{1-\lambda''} \, dx\, dy)$ to $L^2_{\lambda', \lambda''}((\mathbb{R}^+ \times (-1; 1)).$ The inverse map is given by the following formula:

$$T_\iota^{-1}(h)(x, y) = h \circ \iota^{-1}(x, y) \, x^{\lambda'-1}y^{\lambda''-1}.$$  

(4.9)

We use this operator in order to transfer the restriction of the outer product of holomorphic discrete series representations of $SL_2(\mathbb{R})$ on the space $L^2_{\lambda', \lambda''}(\mathbb{R}^+ \times (-1; 1))$ and define, for every $g \in SL_2(\mathbb{R})$,

the following operator:

$$\pi_{\lambda', \lambda''}(g) = T_\theta \circ \pi_{\lambda', \lambda''}(g) \circ T_\theta^{-1},$$  

(4.10)

where $\pi_{\lambda', \lambda''}(g) = F_2^{-1} \circ \pi_{\lambda', \lambda''}(g, g) \circ F_2$. We call this model the stratified $L^2$-model.
Finally, we introduce the map $\Theta$ for any function defined on $\mathbb{R}^+$:

$$\Theta(h)(t, v) = t^{-(\lambda^\prime + \lambda^\nu + t - 1)} \frac{P_t^{\lambda^\prime - 1, \lambda^\nu - 1}(v)}{\|P_t^{\lambda^\prime - 1, \lambda^\nu - 1}\|} h(t). \quad (4.11)$$

This is a one-to-one isometry from $L^2_{\lambda^\nu}(\mathbb{R}^+)$ to the Hilbert space

$$V_t^{\lambda^\prime, \lambda^\nu} := L^2\left(\mathbb{R}^+, t^{\lambda^\prime + \lambda^\nu - 1} \, dt\right) \otimes \mathbb{C} \cdot P_t^{\lambda^\prime - 1, \lambda^\nu - 1}(v), \quad (4.12)$$

whose inverse is given by:

$$\Theta^{-1}(f \times P_t^{\lambda^\prime - 1, \lambda^\nu - 1})(t) = \|P_t^{\lambda^\prime - 1, \lambda^\nu - 1}\| t^{\lambda^\prime + \lambda^\nu + t - 1} f(t). \quad (4.13)$$

where $\|P_t^{\lambda^\prime - 1, \lambda^\nu - 1}\| = \left(\frac{2^\alpha \beta + 1 \Gamma((\lambda^\prime + 1) \Gamma((\lambda^\nu + 1))}{\Gamma((\lambda^\prime + \alpha + 1) \Gamma((\lambda^\nu + \beta + 1))}\right)^{\frac{1}{2}}$ is the norm of Jacobi polynomials in the Hilbert space $L^2((-1; 1), d\mu_{\lambda^\prime, \lambda^\nu}(v))$.

Analogously to (4.10), we use this map to transfer the holomorphic discrete series representation of $SL_2(\mathbb{R})$ and define, for every $g \in SL_2(\mathbb{R})$, the following operator on $V_t^{\lambda^\prime, \lambda^\nu}$:

$$\widetilde{\pi_{\lambda^\nu}}(g) = \Theta \circ \pi_{\lambda^\nu}(g) \circ \Theta^{-1}. \quad (4.14)$$

where $\pi_{\lambda^\nu}(g) = \mathcal{F}^{-1} \circ \pi_{\lambda^\nu}(g) \circ \mathcal{F}$.

### 4.2 Symmetry breaking and holographic operator

According to the orthogonality of Jacobi polynomials, we have the following isomorphisms of Hilbert spaces:

$$L^2_{\lambda^\prime, \lambda^\nu}(\mathbb{R}^+ \times (-1; 1)) \approx L^2\left(\mathbb{R}^+, t^{\lambda^\prime + \lambda^\nu - 1} \, dt\right) \otimes L^2((-1; 1), d\mu_{\lambda^\prime, \lambda^\nu}(v))$$

$$\approx \sum_{l \geq 0} \otimes L^2\left(\mathbb{R}^+, t^{\lambda^\prime + \lambda^\nu - 1} \, dt\right) \otimes \mathbb{C} \cdot P_t^{\lambda^\prime - 1, \lambda^\nu - 1}(v) \quad (4.15)$$

where $\sum_{\otimes}$ stands for an orthogonal Hilbert sum. The last isomorphism is due to the fact that the Jacobi polynomials form a Hilbert basis for $L^2((-1; 1), d\mu_{\lambda^\prime, \lambda^\nu}(v))$ if $\lambda^\prime, \lambda^\nu > 1$.

This allows us to consider the orthogonal projection $\mathcal{J}_{t}^{\lambda^\prime, \lambda^\nu}$ from $L^2_{\lambda^\prime, \lambda^\nu}(\mathbb{R}^+ \times (-1; 1))$ on the Hilbert subspace $V_t^{\lambda^\prime, \lambda^\nu}$ which corresponds to the usual Jacobi transform:

$$\mathcal{J}_{t}^{\lambda^\prime, \lambda^\nu}(h)(t, v) = \frac{P_t^{\lambda^\prime - 1, \lambda^\nu - 1}(v)}{\|P_t^{\lambda^\prime - 1, \lambda^\nu - 1}\|} \int_{t-1}^{t} h(t, u) P_t^{\lambda^\prime - 1, \lambda^\nu - 1}(u) \, d\mu_{\lambda^\prime, \lambda^\nu}(u). \quad (4.16)$$

We summarize the situation in the following Proposition.
Proposition 4.1. Let $\lambda', \lambda'', \lambda''' \in \mathbb{N}$ such that $l = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. The following diagram is commutative

\[
\begin{array}{c}
L^2_{\lambda',\lambda''}(\mathbb{R}^+ \times \mathbb{R}^+) \xrightarrow{c_1 RC_{\lambda',\lambda''}} L^2_{\lambda''}(\mathbb{R}^+) \xrightarrow{c_2 \Phi_{\lambda''}} L^2_{\lambda',\lambda''}(\mathbb{R}^+ \times \mathbb{R}^+)
\end{array}
\]

where $c_1 = \frac{2^{\lambda'+\lambda''-3}l}{\|P^1_{\lambda'-1,\lambda''-1}\|$ and $c_2 = \frac{1}{\|P^1_{\lambda'-1,\lambda''-1}\|}$.

Proof. Let $f \in L^2_{\lambda',\lambda''}(\mathbb{R}^+ \times \mathbb{R}^+)$. On one hand side we have:

\[
\Phi_{\lambda',\lambda''} \circ RC_{\lambda',\lambda''} f(x,y) = \frac{x^{\lambda'-1}y^{\lambda''-1}}{2i(l(x+y))^{\lambda'+\lambda''-2}} p^1_{\lambda'-1,\lambda''-1} \left( \frac{y-x}{x+y} \right) \left( \int_{-1}^{1} p^1_{\lambda'-1,\lambda''-1}(u) F(t(x+y),u) du \right).
\]

On the other hand side we have:

\[
T^{-1}_i \circ \mathcal{J}_{\lambda',\lambda''} \circ T_i(f)(x,y) = \left( \frac{l}{2} \right)^{2-\lambda'-\lambda''} \left( \int_{-1}^{1} f(t(u)) p^1_{\lambda'-1,\lambda''-1}(u) du \right) = \frac{2^{\lambda'+\lambda''-2}}{\|P^1_{\lambda'-1,\lambda''-1}\|} \frac{x^{\lambda'-1}y^{\lambda''-1}}{(x+y)^{\lambda'+\lambda''-2}} p^1_{\lambda'-1,\lambda''-1} \left( \frac{y-x}{x+y} \right) \left( \int_{-1}^{1} p^1_{\lambda'-1,\lambda''-1}(u) F(t(x+y),u) du \right).
\]

A direct computation shows that $T^{-1}_i \circ \Theta = c_2 \Phi_{\lambda',\lambda''}$.

The fact that $\widetilde{RC}_{\lambda',\lambda''}$ and $\Phi_{\lambda',\lambda''}$ are intertwining operators (see [KP20], thm. 2.11), and Proposition 4.1 lead to the following:

**Fact 4.2.** The orthogonal projection $\mathcal{J}_{\lambda',\lambda''}$ is a symmetry breaking operator for the $SL_2(\mathbb{R})$ action \[4.10\] in the stratified $L^2$-model and the canonical injection corresponds to the associated holographic operator.

In [KP20], the authors explore the link between the symmetry breaking operator and Jacobi transform in this geometric setting (see Rmk.2.15). The Fact 4.2 directly relates the $L^2$ space associated with Jacobi polynomials to the construction of symmetry breaking operators for the holomorphic discrete series representations of $SL_2(\mathbb{R})$ and explains the structure of the corresponding branching rules.

**Remark:** We should mention that we have similar result in the conformal case $(G,G') = (SO_0(2,n),SO_0(2,n-1))$ (see [KP20] for more details). For this we use the stratification of the time-like cone $\Omega(n)$ given by the diffeomorphism

\[
\epsilon : \Omega(n-1) \times (-1,1) \rightarrow \Omega(n),
\]

\[
(y',v) \rightarrow (y', -vQ_{1,n-2}(y')^{\frac{1}{2}})
\]

(4.17)
Appendix: a lemma about Kummer function

The Kummer function, denoted $1F_1$ is a special case of generalized hypergeometric functions defined for complex parameter $a, b$ where $b$ is not a negative integer by

$$1F_1(a, b; x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b)_n n!}. \quad (5.1)$$

For more results about Kummer function, we refer to [BW10], for example. The Kummer function admits the following integral representation in terms of Jacobi polynomials:

**Lemma 5.1.** Let $\alpha, \beta > 1, l \in \mathbb{N}$ and $x \in \mathbb{R}^+$. Then:

$$C_{\alpha, \beta} x^l e^{ix} 1F_1(\alpha + l, \alpha + \beta + 2l; -2ix) = \int_{-1}^{1} P^{\alpha-1,\beta-1}_{l-1}(v)e^{ix} (1-v)^{\alpha-1}(1+v)^{\beta-1} \, dv, \quad (5.2)$$

where $C_{\alpha, \beta} = \frac{2^{\alpha+\beta+l-1} l!}{l!} B(\alpha + l, \beta + l)$, and $B$ is the usual Euler beta function.

**Proof.** The following integral representation for the Kummer function, for $\text{Re}(c) > \text{Re}(a) > 0$ is classical (see [BW10] p.190):

$$1F_1(a, c; x) = \frac{1}{B(a, c - a)} \int_0^1 s^{a-1}(1-s)^{c-a-1} e^{sx} \, ds = \frac{e^x}{B(a, c - a) 2^{-a-1}} \int_{-1}^{1} e^{-ixv} (1-v)^{a-1}(1+v)^{c-a-1} \, dv, \quad (5.3)$$

where we put $v = 1 - 2s$.

Then, the Rodrigues formula for Jacobi polynomials (see [KP20], section 5.1) and integration by part give:

$$\int_{-1}^{1} P_l^{\alpha-1,\beta-1}(v)e^{ix} (1-v)^{\alpha-1}(1+v)^{\beta-1} \, dv$$

$$= \frac{(-1)^l}{2^l l!} \int_{-1}^{1} e^{ixv} \left( \frac{d}{dv} \right)^l ((1-v)^{a+l-1}(1+v)^{\beta+l-1}) \, dv$$

$$= \frac{1}{2^l l!} \int_{-1}^{1} \left( \frac{d}{dv} \right)^l e^{ixv} (1-v)^{a+l-1}(1+v)^{\beta+l-1} \, dv$$

$$= \frac{i^l x^l}{2^l l!} \int_{-1}^{1} e^{ixv} (1-v)^{a+l-1}(1+v)^{\beta+l-1} \, dv.$$

The result follows from the integral representation (5.3) for the Kummer function with $c = \alpha + \beta + 2l > \alpha + 2l > 0$. 

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