The $O(N)$ Model at Finite Temperature: Renormalization of the Gap Equations in Hartree and Large-$N$ Approximation

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The temperature dependence of the sigma meson and pion masses is studied in the framework of the $O(N)$ model. The Cornwall–Jackiw–Tomboulis formalism is applied to derive gap equations for the masses in the Hartree and large-$N$ approximations. Renormalization of the gap equations is carried out within the cut-off and counter-term renormalization schemes. A consistent renormalization of the gap equations within the cut-off scheme is found to be possible only in the large-$N$ approximation and for a finite value of the cut-off. On the other hand, the counter-term scheme allows for a consistent renormalization of both the large-$N$ and Hartree approximations. In these approximations, the meson masses at a given nonzero temperature depend in general on the choice of the cut-off or renormalization scale. As an application, we also discuss the in-medium on-shell decay widths for sigma mesons and pions at rest.

I. INTRODUCTION

Although chiral symmetry is manifest in the Lagrangian of quantum chromodynamics (QCD) for vanishing quark masses, quantum effects break this symmetry spontaneously in the QCD vacuum. At temperatures of order 150 MeV, however, lattice QCD results indicate that chiral symmetry is restored $\hat{\text{3}}$. Such temperatures are expected to be reached in ultrarelativistic heavy-ion collisions at CERN-SPS, BNL-RHIC, and CERN-LHC energies $\hat{\text{3}}$. The restoration of chiral symmetry may lead to observable consequences, for instance, in the dilepton mass spectrum $\hat{\text{3}}$, or the formation of disoriented chiral condensates $\hat{\text{3}}$.

QCD with $N_f$ massless quark flavors has a $SU(N_f)_L \times SU(N_f)_R$ symmetry. The order parameter for the chiral transition is therefore $\chi^{ij} = \langle \bar{q}_L^i q_R^j \rangle$, $i,j = 1, \ldots, N_f$. For $N_f = 2$, $SU(2)_L \times SU(2)_R$ is isomorphic to $O(4)$. Consequently, the effective Lagrangian for the order parameter $\chi^{ij}$ falls in the same universality class as the $O(4)$ model, with order parameter $\phi^i$, $i = 1, \ldots, 4$. Thus, if the chiral transition is second order in QCD, the dynamics (and the critical exponents) will be the same as in the $O(4)$ model, provided one is sufficiently close to the transition temperature $\hat{\text{3}}$. This motivates the study of the $O(4)$ model $\hat{\text{3}}$ as an effective low-energy model for QCD. The study which will be presented here is based on the $O(N)$ model for arbitrary $N$. To make contact with QCD, however, we take $N = 4$ in all numerical calculations.

At finite temperature, the naive perturbative expansion in powers of the coupling constant breaks down, requiring resummation schemes to obtain reliable results $\hat{\text{3}}$. Typically, these schemes aim to include thermal fluctuations to all orders in the calculation of physical quantities. Over the years, many ways to achieve this have been pursued (some approaches are more rigorous, some are more
or less *ad hoc*; Refs. [10–18] constitute an incomplete list). The present study of the $O(N)$ model focuses on the Hartree approximation and its large-$N$ limit (the large-$N$ approximation).

The Hartree approximation is known from many-body theory and generically represents the self-consistent resummation of tadpole diagrams [19]. There is, however, no unique prescription to perform this resummation. This has led to the confusing situation that the same term “Hartree approximation” has been used for resummation schemes which actually differ in detail [10,14,15]. In this work, the so-called Cornwall–Jackiw–Tomboulis (CJT) formalism is applied to derive a Hartree approximation [20], and we shall use the term “Hartree approximation” exclusively for the resummation of tadpoles within the CJT formalism.

The CJT formalism can be viewed as a prescription for computing the effective action of a given theory. In general, the CJT formalism resums one-particle irreducible diagrams to all orders. The stationarity conditions for the effective action are nothing but Schwinger–Dyson equations for the one- and two-point Green’s functions of the theory. What is usually [14,21] referred to as the Hartree approximation in the context of the CJT formalism is the special case where only one-particle irreducible tadpole diagrams are included in the resummation. (To be precise, the original work [20] of Cornwall, Jackiw, and Tomboulis referred to this as the “Hartree–Fock approximation”.) In this case, the equations for the two-point Green’s functions simplify to self-consistency conditions, or “gap” equations, for the resummed masses of the quasi-particle excitations, i.e., in our case, the in-medium sigma and pion masses.

The $O(N)$ model has been previously studied using the CJT formalism by Amelino-Camelia [14], and Petropoulos [21]. The former work addressed renormalization using the cut-off renormalization scheme, but did not present solutions of the gap equations. In the latter work, the gap equations were numerically solved, but the issue of renormalization was not treated. In this paper, we complete these investigations by discussing the renormalization of the gap equations in the cut-off as well as the counter-term renormalization scheme and by presenting the corresponding numerical solutions.

Renormalization of expressions obtained in self-consistent approximation schemes is non-trivial. In perturbation theory, it is sufficient to perform renormalization in the vacuum, $T = 0$, order by order in the coupling constant, as a finite temperature does not introduce new ultraviolet singularities [22]. Consequently, the perturbative renormalization of the $O(N)$ model is straightforward [23] and has been known for a long time [24]. Self-consistent approximation schemes, however, in general resum only certain classes of diagrams. This has the consequence that performing renormalization after resummation may require renormalization constants that are no longer independent of the properties of the medium, see Ref. [10] and below. A resummation scheme which circumvents this problem by renormalizing prior to resummation is the so-called optimized perturbation theory [17]. This approach will not be discussed here.

Our main results are the following. In the cut-off renormalization scheme, we find that taking the cut-off, $\Lambda$, to infinity the masses of the sigma meson and the pions become identical [15], even in the phase where chiral symmetry is broken. This is clearly unphysical, as the pions are Goldstone bosons and thus much lighter than the sigma meson. Moreover, it is a well-known fact [25] that the $O(N)$ model becomes trivial in the limit $\Lambda \to \infty$. Consequently, renormalization of the $O(N)$ model within the cut-off scheme can only be meaningfully studied for a finite value of $\Lambda$. Even then, fixing $\Lambda$ to give the observed meson masses in the vacuum, we find that in the absence of explicit chiral symmetry breaking, the Hartree approximation requires $\Lambda$ to vanish. The large-$N$ approximation does not have this problem and allows for nonzero values of $\Lambda$.

In the counter-term scheme, renormalization can be performed within both the Hartree as well as the large-$N$ approximation. In the Hartree approximation, the value of the renormalization scale, $\mu$, is uniquely fixed by the vacuum mass of the sigma meson in the absence of explicit chiral symmetry breaking. In the large-$N$ approximation, this constraint does not exist, and $\mu$ can be chosen arbitrarily. As $\Lambda$ or $\mu$ are free parameters in the large-$N$ approximation, at any given temperature the values of the meson masses depend on the choice of these parameters. This is a consequence of the fact mentioned above that, for self-consistent resummation schemes, the renormalization constants may depend on the temperature.

The outline of the paper is as follows. In Section II, we briefly discuss the effective potential
within the CJT formalism \[20\]. In Section III, this formalism is applied to derive the effective potential for the \( O(N) \) model in the Hartree approximation. Section IV is devoted to a discussion of the stationarity conditions for the effective potential, which lead to gap equations for the sigma and pion masses. The renormalization of the gap equations is then performed in Section V within the cut-off and the counter-term schemes. In Section VI we present numerical results. Section VII concludes this paper with a summary of our results. As an application we also compute the temperature dependence of the in-medium decay widths to one-loop order for on-shell \( \sigma \) and \( \pi \) mesons at rest.

We use the imaginary-time formalism to compute quantities at finite temperature. Our notation is

\[
\int_k f(k) \equiv T \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} f(2\pi inT, k) , \quad \int_x f(x) \equiv \int_0^{1/T} d\tau \int d^3x f(\tau, x) .
\]

We use units \( \hbar = c = k_B = 1 \). The metric tensor is \( g^{\mu\nu} = \text{diag}(+, -, -, -) \).

**II. THE EFFECTIVE POTENTIAL IN THE CORNWALL–JACKIW–TOMBOULIS FORMALISM**

The notion of an effective action is quite useful for studying theories with spontaneously broken symmetries. For translationally invariant systems, the effective action becomes the effective potential. At the classical level, the effective potential is given by the potential energy (density). The vacuum (ground) state is given by the minimum of the potential energy. For theories with spontaneously broken symmetry there may exist infinitely many equivalent (degenerate) minima. At the quantum level, there are additional terms in the effective potential, corresponding to quantum fluctuations. At finite temperature (and finite chemical potential), the minimum of the effective potential corresponds to the thermodynamic pressure \[26\].

The common way to compute the effective potential is via the loop expansion \[27\]. This approach, however, becomes problematic for theories with spontaneously broken symmetries. In particular, the energy of quasi-particle excitations with small 3-momenta becomes imaginary. The reason is that the requirement of convexity for the effective potential is violated. A way to salvage the loop expansion is to perform a Maxwell construction which restores the convexity of the effective potential \[26\]. Another way to compute the effective potential is via the CJT formalism \[20\]. As mentioned in the introduction, this method resums certain classes of diagrams and has the advantage that the energy of the quasi-particle excitations remains real for all values of 3-momentum.

Consider a scalar field theory with Lagrangian

\[
\mathcal{L}(\phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) ,
\]

for instance, \( \phi^4 \) theory where

\[
U(\phi) = \frac{1}{2} m^2 \phi^2 + \lambda \phi^4 .
\]

The generating functional for Green’s functions in the presence of sources \( J, K \) reads \[20\]:

\[
\mathcal{Z}[J, K] = e^{\mathcal{W}[J,K]} = \int \mathcal{D}\phi \exp \left\{ I[\phi] + \phi J + \frac{1}{2} \phi K \phi \right\} ,
\]

where \( \mathcal{W}[J,K] \) is the generating functional for connected Green’s functions, \( I[\phi] = \int_x \mathcal{L} \) is the classical action, and
The expectation values for the one-point function, $\bar{\phi}(x)$, and the connected two-point function, $G(x,y)$, in the presence of sources are given by

$$\frac{\delta W[J, K]}{\delta J(x)} = \bar{\phi}(x),$$

$$\frac{\delta W[J, K]}{\delta K(x,y)} = \frac{1}{2} \left[ G(x,y) + \bar{\phi}(x) \bar{\phi}(y) \right].$$

One now eliminates $J$ and $K$ in favor of $\bar{\phi}$ and $G$ via a double Legendre transformation to obtain the effective action

$$\Gamma[\bar{\phi}, G] = W[J, K] - \bar{\phi} J - \frac{1}{2} \bar{\phi} K \bar{\phi} - \frac{1}{2} G K,$$

where $G K = \int_{x,y} G(x,y) K(y,x)$. Thus,

$$\frac{\delta \Gamma[\bar{\phi}, G]}{\delta \bar{\phi}(x)} = J(x) - \int_y K(x,y) \bar{\phi}(y),$$

$$\frac{\delta \Gamma[\bar{\phi}, G]}{\delta G(x,y)} = -\frac{1}{2} K(x,y).$$

For vanishing sources, we find the stationarity conditions which determine the expectation value of the field $\phi(x)$ and the propagator $G(x,y)$ in the absence of sources:

$$\frac{\delta \Gamma[\bar{\phi}, G]}{\delta \phi(x)} \bigg|_{\bar{\phi}=\bar{\phi}, G=G} = 0,$$

$$\frac{\delta \Gamma[\bar{\phi}, G]}{\delta G(x,y)} \bigg|_{\bar{\phi}=\bar{\phi}, G=G} = 0.$$

Equation (10) corresponds to a Schwinger–Dyson equation for the full (dressed) propagator. It was shown in [20] that the effective action $\Gamma[\bar{\phi}, G]$ is given by

$$\Gamma[\bar{\phi}, G] = I[\bar{\phi}] - \frac{1}{2} \text{Tr} \left( \ln G^{-1} \right) - \frac{1}{2} \text{Tr} \left( D^{-1} G - 1 \right) + \Gamma_2[\bar{\phi}, G].$$

Here, $D^{-1}$ is the inverse of the tree-level propagator,

$$D^{-1}(x,y; \bar{\phi}) = -\frac{\delta^2 I[\phi]}{\delta \phi(x) \delta \phi(y)} \bigg|_{\phi=\bar{\phi}},$$

and $\Gamma_2[\bar{\phi}, G]$ is the sum of all two-particle irreducible diagrams where all lines represent full propagators $G$.

For constant fields $\bar{\phi}(x) = \bar{\phi}$, homogeneous systems, and for a Lagrangian of the type given by eq. (2), the effective potential $V$ is given by $V = -\frac{T}{\Omega} \Gamma_2[\bar{\phi}, G]$, where $\Omega$ is the 3-volume of the system, i.e.,

$$V[\bar{\phi}, G] = U(\bar{\phi}) + \frac{1}{2} \int_k \ln G^{-1}(k) + \frac{1}{2} \int_k \left[ D^{-1}(k; \bar{\phi}) G(k) - 1 \right] + V_2[\bar{\phi}, G],$$

with
\[ D^{-1}(k; \varphi) = -k^2 + u''(\varphi) \] (13)

and \( V_2[\varphi, G] = -T \Gamma_2[\varphi, G]/\Omega. \) The stationarity conditions are given by

\[ \frac{\delta V[\varphi, G]}{\delta \varphi} \bigg|_{\varphi = \varphi, G = G} = 0, \] (14a)

\[ \frac{\delta V[\varphi, G]}{\delta G(k)} \bigg|_{\varphi = \varphi, G = G} = 0. \] (14b)

With eq. (14), the latter can be written in the form

\[ G^{-1}(k) = D^{-1}(k; \varphi) + \Pi(k), \] (14c)

where

\[ \Pi(k) \equiv 2 \frac{\delta V[\varphi, G]}{\delta G(k)} \bigg|_{\varphi = \varphi, G = G}. \] (15)

is the self energy. Equation (14c) is the aforementioned Schwinger–Dyson equation. The thermodynamic pressure is then determined by

\[ p = -V[\varphi, G], \] (16)

which, in the absence of conserved charges, is (up to a sign) identical to the free energy density.

III. THE O(N) MODEL

Let us now turn to the discussion of the \( O(N) \) model. Its Lagrangian is given by

\[ \mathcal{L} = \frac{1}{2} \left( \partial \mu \phi \cdot \partial \nu \phi - m^2 \phi \cdot \phi \right) - \frac{\lambda}{N} (\phi \cdot \phi)^2 + H \phi_1, \] (17)

where \( \phi \) is an \( N \)-component scalar field. For \( H = 0 \) and \( m^2 > 0 \), the Lagrangian is invariant under \( O(N) \) rotations of the fields. For \( H = 0 \) and \( m^2 < 0 \), this symmetry is spontaneously broken down to \( O(N-1) \), with \( N-1 \) Goldstone bosons (the pions). The phenomenological explicit symmetry breaking term, \( H \), is introduced to yield the observed finite masses of the pions. Spontaneous symmetry breaking leads to a non-vanishing vacuum expectation value for \( \phi \):

\[ \langle \phi \rangle = \phi > 0. \] (18)

(\( \phi \) assumes the role of \( \tilde{\phi} \) in section II.) At tree level,

\[ \phi \equiv f_\pi = \sqrt{-\frac{N m^2}{4 \lambda}} \frac{2}{\sqrt{3}} \cos \frac{\theta}{3}, \quad \theta = \arccos \left[ \frac{H N}{8 \lambda} \left( \frac{-12 \lambda}{N} \right)^{3/2} \right]. \] (19)

For \( H = 0 \), \( \cos(\theta/3) = \sqrt{3}/2 \). The inverse tree-level sigma and pion propagators are given by

\[ D^{-1}_\sigma(k; \phi) = -k^2 + m^2 + \frac{12 \lambda}{N} \phi^2, \] (20a)

\[ D^{-1}_\pi(k; \phi) = -k^2 + m^2 + \frac{4 \lambda}{N} \phi^2. \] (20b)

This leads to the zero-temperature tree-level masses
\[ m_\sigma^2 = m^2 + \frac{12 \lambda f_\sigma^2}{N}, \]  
\[ m_\pi^2 = m^2 + \frac{4 \lambda f_\pi^2}{N}, \]  
for the sigma meson and the pion. At tree level, the parameters of the Lagrangian are fixed such that these masses agree with the observed values of \( m_\sigma = 600 \text{ MeV} \) and \( m_\pi = 139 \text{ MeV} \). Then, the coupling constant is
\[ \lambda = \frac{N (m_\sigma^2 - m_\pi^2)}{8 f_\pi^2}, \]
where \( f_\pi = 93 \text{ MeV} \) is the pion decay constant and
\[ m^2 = -\frac{m_\sigma^2 - 3 m_\pi^2}{2}. \]
The explicit symmetry breaking term is \( H = m_\pi^2 f_\pi \). These tree-level results may change upon renormalization.

The CJT effective potential for the \( O(N) \) model is obtained from eq. (12) as
\[
V(\phi, G_\sigma, G_\pi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{N} \phi^4 - H \phi \\
+ \frac{1}{2} \int_k \left[ \ln G_\sigma^{-1}(k) + D_\sigma^{-1}(k; \phi) G_\sigma(k) - 1 \right] \\
+ \frac{N - 1}{2} \int_k \left[ \ln G_\pi^{-1}(k) + D_\pi^{-1}(k; \phi) G_\pi(k) - 1 \right] \\
+ V_2(\phi, G_\sigma, G_\pi),
\]
where \( V_2(\phi, G_\sigma, G_\pi) \) denotes the contribution from two-particle irreducible diagrams. In the following we include only the two-loop diagrams shown in Fig. 1 in \( V_2 \). These diagrams have no explicit \( \phi \) dependence. Then, using eq. (15) only tadpole diagrams (with resummed propagators) contribute to the self energies. As explained in the introduction, this corresponds to the Hartree approximation. The Schwinger–Dyson equations for the full propagators contain no momentum dependence. Thus, these equations are simply gap equations for the masses of the sigma meson and pion.

On the two-loop level there exist, however, two more diagrams, cf. Fig. 2, which will not be taken in our analysis. They depend explicitly on \( \phi \) and introduce an additional momentum dependence in the Schwinger–Dyson equations, which makes their solution more complicated. However, in the large-\( N \) limit these terms are \textit{a priori} absent, because they are of order \( 1/N \).

In the Hartree approximation,
\[
V_2(\phi, G_\sigma, G_\pi) = 3 \frac{\lambda}{N} \left[ \int_k G_\sigma(k) \right]^2 \\
+ (N + 1)(N - 1) \lambda \frac{N}{N} \left[ \int_k G_\pi(k) \right]^2 \\
+ 2 (N - 1) \lambda \frac{N}{N} \left[ \int_k G_\sigma(k) \right] \left[ \int_k G_\pi(k) \right].
\]
The coefficients in this equation are chosen such that, when computing the self energies from eq. (15) and replacing the dressed propagators by the tree-level propagators, one obtains the standard results for the perturbative one-loop self energies.

6
IV. THE STATIONARITY CONDITIONS FOR THE EFFECTIVE POTENTIAL

The stationarity conditions (14a), (14b) read

\[ 0 = m^2 \varphi + \frac{4\lambda}{N} \varphi^3 - H + \frac{4\lambda}{N} \varphi \int_q \left[ 3 G_\sigma(q) + (N - 1) G_\pi(q) \right], \]  

\[ (26a) \]

\[ G_\sigma^{-1}(k) = D_\sigma^{-1}(k; \varphi) + \frac{4\lambda}{N} \int_q \left[ 3 G_\sigma(q) + (N - 1) G_\pi(q) \right], \]  

\[ (26b) \]

\[ G_\pi^{-1}(k) = D_\pi^{-1}(k; \varphi) + \frac{4\lambda}{N} \int_q \left[ G_\sigma(q) + (N + 1) G_\pi(q) \right]. \]  

\[ (26c) \]

The integrals on the right-hand side of the last two equations correspond to the sigma meson and pion self energies. According to eq. (15), they originate from the diagrams of Fig. 1 via cutting one of the two loops in these diagrams. As one observes, these terms are independent of the momentum \( k^\mu \) appearing the propagator. The only \( k \) dependence on the right-hand side enters through \( D_\sigma^{-1} \) and \( D_\pi^{-1} \), cf. eqs. (20a) and (20b). Therefore, one is allowed to make the following ansatz for the full propagators:

\[ G_{\sigma,\pi}(k) = \frac{1}{-k^2 + M_{\sigma,\pi}^2}, \]  

\[ (27) \]
where now $M_{\sigma}$ and $M_{\pi}$ are the masses dressed by interaction contributions from the diagrams of Fig. 1. Note that the diagrams in Fig. 2 have an explicit dependence on the external momentum; including them would invalidate the ansatz (27).

The dressed sigma and pion masses are then determined by the following gap equations

\[ M_{\sigma}^2 = m^2 + \frac{4\lambda}{N} \left[ 3\varphi^2 + 3 Q(M_{\sigma}, T) + (N - 1) Q(M_{\pi}, T) \right], \]  
\[ M_{\pi}^2 = m^2 + \frac{4\lambda}{N} \left[ \varphi^2 + Q(M_{\sigma}, T) + (N + 1) Q(M_{\pi}, T) \right]. \]  

(28a)  
(28b)

Here we introduced the function

\[ Q(M, T) = \int \frac{1}{k^2 + M^2} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\epsilon_k(M)} \left\{ \frac{1}{\exp[\epsilon_k(M)/T] - 1} + \frac{1}{2} \right\}, \]  

(29)

where $\epsilon_k(M) \equiv \sqrt{k^2 + M^2}$. The last term in the integral is divergent and requires renormalization. This will be discussed in section 5. The standard practice, however, is to ignore this term, claiming it is independent of temperature. This is wrong, because $\epsilon_k(M)$ depends on $T$ through the gap equation for $M$. As is shown below, the correct renormalization procedure changes the results.

Finally, $\varphi$ is determined by

\[ H = \varphi \left\{ m^2 + \frac{4\lambda}{N} \left[ \varphi^2 + 3 Q(M_{\sigma}, T) + (N - 1) Q(M_{\pi}, T) \right] \right\}. \]  

(30a)

Using eq. (28a) this can be written in the compact form

\[ H = \varphi \left[ M_{\sigma}^2 - \frac{8\lambda}{N} \varphi^2 \right]. \]  

(30b)

Note that this equation does not require explicit renormalization, and therefore is valid independent of the renormalization scheme. Equations (28a), (28b), and (30b) are the stationarity conditions in the Hartree approximation. In the case where chiral symmetry is not explicitly broken, $H = m_{\pi} = 0$, they imply the following:

1. $\varphi \neq 0$. This is the phase where chiral symmetry is spontaneously broken. From eq. (30b) follows

\[ M_{\sigma} = \sqrt{\frac{8\lambda}{N}} \varphi. \]  

(31)

On the other hand, eqs. (28a) and (30a) can be combined to give

\[ M_{\pi}^2 = \frac{8\lambda}{N} \left[ Q(M_{\pi}, T) - Q(M_{\sigma}, T) \right]. \]  

(32)

This implies that Goldstone’s theorem cannot be satisfied in the Hartree approximation at all temperatures: $M_{\sigma} \neq 0$ on account of (31), therefore $M_{\pi} = 0$ is not a solution of (32). (Note, however, that after proper renormalization of the function $Q(M, T)$, $M_{\pi}$ can be chosen to be zero at one particular temperature, for instance $T = 0$, but then will be nonzero for other values of $T$.)

2. $\varphi = 0$. In this phase chiral symmetry is restored and eqs. (28a) and (28b) can be combined to give

\[ M_{\sigma}^2 - M_{\pi}^2 = \frac{8\lambda}{N} \left[ Q(M_{\sigma}, T) - Q(M_{\pi}, T) \right], \]  

(33)

which has the solution $M_{\sigma} = M_{\pi}$; the masses become degenerate.
Let us now turn to the discussion of the large-$N$ approximation which is in fact the $N \gg 1$ limit of the Hartree approximation. To derive the large-$N$ limit from the previous results, one simply neglects all contributions of order $1/N$. Note, however, that $\phi^2 \sim N$, cf. eq. (19). Therefore, in the large-$N$ limit, the stationarity conditions read

$$M^2_\sigma = m^2 + \frac{4\lambda}{N} [3\phi^2 + N Q(M_\pi, T)] ,$$

$$M^2_\pi = m^2 + \frac{4\lambda}{N} [\phi^2 + N Q(M_\pi, T)] ,$$

$$H = \phi \left\{ m^2 + \frac{4\lambda}{N} [\phi^2 + N Q(M_\pi, T)] \right\}.$$  

This leads to

$$M^2_\sigma = M^2_\pi + \frac{8\lambda}{N} \phi^2 ,$$

and

$$M^2_\pi \phi = H .$$

These two equations are valid independent of the renormalization scheme. The latter equation implies the following in the case that chiral symmetry is not explicitly broken, $H = m_\pi = 0$:

1. $\phi \neq 0$. In this phase of spontaneously broken chiral symmetry $M_\pi$ has to vanish, i.e., Goldstone’s theorem is respected in the large-$N$ limit. $M_\sigma$ obeys the same relation (31) as in the Hartree approximation.

2. $\phi = 0$. In this phase of restored chiral symmetry we again have $M_\sigma = M_\pi$, as in the Hartree case.

V. THE RENORMALIZED GAP EQUATIONS

As mentioned above, the last term in the integrand in (29) is divergent and requires renormalization. In the following we discuss renormalization with a three-dimensional momentum cut-off (CO) and via the counter-term (CT) renormalization scheme.

In the literature one often encounters the argument that this divergent term does not depend on temperature and can therefore either be absorbed in the definition of the renormalized vacuum mass in the CO scheme, or it is completely cancelled by a counter term in the CT scheme. This argument is correct to one-loop order in perturbation theory, since then the mass $M$ in this term is simply the bare mass and independent of temperature. However, in a self-consistent approximation scheme, like the Hartree approximation, this argument is incorrect, since the mass $M$ is the resummed mass, which becomes a function of the temperature through the self-consistent solution of the gap equation. Therefore, removing the divergence in either the CO or CT scheme may leave a finite, temperature-dependent contribution. Another way of stating this fact is that, as mentioned in the introduction, the renormalization constants may have to be chosen such that they depend on properties of the medium, like the temperature.

A. CO scheme

The simplest way to regularize the divergent integral is to introduce a three-dimensional ultraviolet momentum cutoff, $\Lambda$. Computing the divergent integral then proceeds as follows,
\[ Q_M(\lambda) = \int_0^\Lambda \frac{d^3 k}{(2\pi)^3} \frac{1}{2\epsilon_k(M)} = \frac{1}{4\pi^2} \int_0^\Lambda dk \frac{k^2}{\epsilon_k(M)} = \frac{1}{8\pi^2} \left[ \Lambda \epsilon_M(\lambda) - M^2 \ln \frac{\Lambda + \epsilon_M(\lambda)}{M} \right] \] . \tag{35}

In the limit \( \Lambda \to \infty \), this yields
\[ Q(M, 0) = \lim_{\lambda \to \infty} Q_M(\lambda) = \int_0^\Lambda \frac{d^3 k}{(2\pi)^3} \frac{1}{2\epsilon_k(M)} = I_1 - M^2 I_2 + \frac{M^2}{16\pi^2} \ln \frac{M^2}{\mu^2} \] , \tag{36}

where we have introduced a renormalization scale, \( \mu \), and following \[27\] we have defined
\[ I_1 \equiv \lim_{\lambda \to \infty} \frac{\Lambda^2}{8\pi^2} \] , \tag{37a}
\[ I_2 \equiv \lim_{\lambda \to \infty} \frac{1}{16\pi^2} \ln \frac{4\Lambda^2}{\mu^2} \] . \tag{37b}

The renormalization is carried out by introducing new parameters \[14\]
\[ \frac{m^2_R}{\lambda_R} = \frac{m^2}{\lambda} + \frac{4(N + 2)}{N} I_1 \] , \tag{38a}
\[ \frac{1}{\lambda_R} = \frac{1}{\lambda} + \frac{4(N + 2)}{N} I_2 \] , \tag{38b}

where \( m^2_R \) and \( \lambda_R \) are the finite, renormalized mass and coupling constant.

1. Hartree approximation

In the Hartree approximation, this leads to the following renormalized gap equations for the sigma and pion masses:
\[ M^2 = m^2_R + \frac{4\lambda_R}{N} \frac{N + 2}{N} \left[ \phi^2 + P(M_\pi, T) + (N - 1) P(M_\pi, T) \right] - 2 \frac{\lambda}{N \lambda_R} \left( M^2 - m^2_R + \frac{4\lambda_R}{N} \frac{N + 2}{N} \left[ \phi^2 + P(M_\pi, T) \right] \right) \] , \tag{39a}
\[ M^2 = m^2_R + \frac{4\lambda_R}{N} \frac{N + 2}{N} \left[ \phi^2 + P(M_\pi, T) + (N - 1) P(M_\pi, T) \right] - 2 \frac{\lambda}{N \lambda_R} \left( M^2 - m^2_R + \frac{4\lambda_R}{N} \frac{N + 2}{N} \left[ \phi^2 + P(M_\pi, T) \right] \right) \] , \tag{39b}

where the function \( P(M, T) \) is defined as
\[ P(M, T) = \frac{M^2}{16\pi^2} \ln \frac{M^2}{\mu^2} + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\epsilon_k(M)} \frac{1}{\exp[\epsilon_k(M)/T] - 1} \] . \tag{40}

Equations (39a) and (39b) are equivalent to eqs. (13), (14) in \[14\] after the replacements \( \lambda \to \lambda/6 \), \( \phi^2 \to N \phi^2 \), \( P(M, T) \to P(T) \]. [Note that the terms \( \sim M^2 - M^2 \) in eqs. (13), (14) of \[14\] can be eliminated by taking the difference of eqs. (13) and (14).]

In the limit \( \Lambda \to \infty \), \( \lambda \to 0^+ \) in order to have a finite \( \lambda_R \), and the (bare) theory becomes unstable (see also Ref. \[28\]). Also, the (renormalized) masses obey \( M^2 = M^2 \), cf. \(39a\), \(39b\), which is undesirable. It would imply that chiral symmetry is unbroken, even when \( \phi \neq 0 \). This problem was also addressed by the authors of \[13\]. On the other hand, for \( 0 < \lambda < \infty \), \( \lambda_R \to 0^+ \) in the limit \( \Lambda \to 0 \), indicating that the (renormalized) theory becomes trivial \[22\].

Therefore, in the CO scheme the gap equations can only be meaningfully studied for finite \( \Lambda \). In this case, in the original gap equations \(28a\), \(28b\) we replace
\[ Q(M, T) \rightarrow Q_{\Lambda}(M) + Q_{T}(M), \]  
(41)

where

\[ Q_{T}(M) \equiv Q(M, T) - Q(M, 0) = \int \frac{d^{3}k}{(2\pi)^{3}} \frac{1}{\epsilon_{k}(M)} \exp[\epsilon_{k}(M)/T] - 1. \]  
(42)

The integral \( Q_{T}(M) \) is UV-finite and does not require the introduction of a momentum cut-off. Consequently, the gap equations read

\[ M_{\sigma}^{2} = m^{2} + \frac{4\lambda}{N} \{ 3 \phi^{2} + 3 [Q_{T}(M_{\sigma}) + Q_{\Lambda}(M_{\sigma})] + (N - 1) [Q_{T}(M_{\pi}) + Q_{\Lambda}(M_{\pi})] \}, \]  
(43a)

\[ M_{\pi}^{2} = m^{2} + \frac{4\lambda}{N} \{ \phi^{2} + [Q_{T}(M_{\sigma}) + Q_{\Lambda}(M_{\sigma})] + (N + 1) [Q_{T}(M_{\pi}) + Q_{\Lambda}(M_{\pi})] \}. \]  
(43b)

The cut-off \( \Lambda \) has to be determined from the values of \( M_{\sigma} \) and \( M_{\pi} \) at \( T = 0 \):

\[ m_{\sigma}^{2} = m^{2} + \frac{4\lambda}{N} \{ 3 f_{\sigma}^{2} + 3 Q_{\Lambda}(m_{\sigma}) + (N - 1) Q_{\Lambda}(m_{\pi}) \}, \]  
(44a)

\[ m_{\pi}^{2} = m^{2} + \frac{4\lambda}{N} \{ f_{\pi}^{2} + Q_{\Lambda}(m_{\sigma}) + (N + 1) Q_{\Lambda}(m_{\pi}) \}, \]  
(44b)

where we have used \( \phi(T = 0) = f_{\pi} \). In the chiral limit \((H = m_{\pi} = 0)\), the difference of (44a) and (44b) reads

\[ m_{\sigma}^{2} = \frac{8\lambda}{N} \{ f_{\pi}^{2} + Q_{\Lambda}(m_{\sigma}) - Q_{\Lambda}(0) \}. \]  
(45)

However, from the stationarity condition (30b) we conclude that \( m_{\sigma}^{2} = 8\lambda f_{\pi}^{2}/N \) in the chiral limit. This immediately leads to

\[ Q_{\Lambda}(m_{\sigma}) = Q_{\Lambda}(0), \]  
(46)

which for finite \( m_{\sigma} \) can only be fulfilled if \( \Lambda = 0 \). This, however, is exactly the case treated in [21], without renormalization.

In conclusion, the CO scheme fails to provide a consistent renormalization of infinities in the phase of broken chiral symmetry in the Hartree approximation when \( H = m_{\pi} = 0 \). Note that the same conclusion can be reached with a four-dimensional momentum cut-off. This failure can be traced to the fact that in the Hartree approximation diagrams of the type shown in Fig. 2 are not included (cf. the perturbative renormalization of the linear sigma model [23], see also the discussion in [10]).

Due to this failure, no results will be shown for the Hartree approximation with CO renormalization. However, we note that the case of explicitly broken symmetry, \( H \neq 0, m_{\pi} > 0 \), is free of this problem. Then, the difference of eqs. (44a) and (44b) determines the coupling constant as

\[ \lambda \equiv \lambda(\Lambda) = \frac{N}{8} \frac{m_{\sigma}^{2} - m_{\pi}^{2}}{f_{\pi}^{2} + Q_{\Lambda}(m_{\sigma}) - Q_{\Lambda}(m_{\pi})}. \]  
(47)

The mass parameter is given by

\[ m_{\pi}^{2} = -\frac{m_{\sigma}^{2} - 3 m_{\pi}^{2}}{2} - \frac{4\lambda}{N} (N + 2) Q_{\Lambda}(m_{\pi}). \]  
(48)

\( H \) is determined from (40) to be \( H = f_{\pi} [m_{\sigma}^{2} - 8\lambda(\Lambda)f_{\pi}^{2}/N] \).
2. Large-\(N\) approximation

In the large-\(N\) limit, the renormalized gap equation for the pion mass reads
\[
M_\pi^2 = m_R^2 + \frac{4\lambda R}{N} \left[ \phi^2 + N P(M_\pi, T) \right],
\]
while \(M_\sigma^2\) is still given by (34d). This again has the consequence that \(M_\sigma = M_\pi\) in the limit \(\Lambda \to \infty\), i.e., \(\lambda \to 0^-\), which as discussed above is an unwanted feature. On the other hand, there is no inconsistency in the large-\(N\) approximation for finite \(\Lambda\). \(\Lambda\) is a free parameter and the gap equations to be solved are (34d) for the sigma mass and
\[
M_\sigma^2 = \frac{4\alpha}{N} \left[ \phi^2 + N \left[ Q_T(M_\pi) + Q_\Lambda(M_\pi) \right] \right],
\]
for the pion mass. The parameters are again determined from \(M_\sigma(T = 0) = m_\sigma, M_\pi(T = 0) = m_\pi\), and \(\phi(T = 0) = f_\pi\). From these conditions we derive that the coupling constant is still given by its tree-level value, eq. (22), but \(m^2\) is now determined from
\[
m^2 = -\frac{3m_\pi^2}{2} - 4\lambda Q_\Lambda(m_\pi).
\]
\(H\) retains its tree-level value on account of (34e).

B. CT scheme

In the CT scheme, counter terms are introduced to subtract the UV divergences in \(Q(M, T)\). Rewrite
\[
\int \frac{d^4k}{(2\pi)^4} \frac{1}{2\epsilon_k(M)} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + M^2},
\]
where \(k_0 \in \mathbb{R}\) with \(k^2 = k_0^2 + k^2\), \(d^4k = d^3k\,dk_0\). To determine the counter terms, expand the integrand in a Taylor series around \(M^2 = \mu^2\), where \(\mu\) is the renormalization scale.
\[
\frac{1}{k^2 + M^2} = \frac{1}{k^2 + \mu^2} \sum_{n=0}^{\infty} \left( \frac{\mu^2 - M^2}{k^2 + \mu^2} \right)^n.
\]
The \(d^4k\) integral over the \(n = 0\) term in this expansion is quadratically divergent, while the integral over the \(n = 1\) term diverges logarithmically. The counter terms are chosen to remove these two terms, such that the renormalized result for the divergent integral is
\[
\int \frac{d^4k}{(2\pi)^4} \left[ \frac{1}{k^2 + M^2} - \frac{1}{k^2 + \mu^2} - \frac{\mu^2 - M^2}{(k^2 + \mu^2)^2} \right] = \sum_{n=2}^{\infty} (\mu^2 - M^2)^n \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + \mu^2)^{n+1}}.
\]
Note that the second counter term depends on the temperature through \(M\). This fact represents the aforementioned possibility of having temperature-dependent counter terms in self-consistent approximation schemes, and was already discussed by the authors of [10]. They also pointed out that this problem does not occur in less than three spatial dimensions. This is obvious from eq. (54), because then the second counter term is finite, and thus not required. In contrast, either in ordinary perturbation theory or in optimized perturbation theory [17] renormalization at \(T = 0\) is sufficient to remove all divergences.

The last integral in (54) is finite and equal to
\[ \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} = \frac{1}{(4\pi)^2} \mu^{2(1-n)} . \]  

(55)

Expression (54) can be rearranged to give the final result

\[ \int \frac{d^4k}{(2\pi)^4} \left[ \frac{1}{k^2 + \mu^2} - \frac{\mu^2 - M^2}{(k^2 + \mu^2)^2} \right] = \frac{1}{(4\pi)^2} \left[ M^2 \ln \frac{M^2}{\mu^2} - M^2 + \mu^2 \right] . \]  

(56)

To obtain the renormalized gap equations, simply replace \(Q(M, T)\) as given in (29) by

\[ Q(M, T) = Q_T(M) + Q_\mu(M) , \]  

(57)

where

\[ Q_\mu(M) = \frac{1}{(4\pi)^2} \left[ M^2 \ln \frac{M^2}{\mu^2} - M^2 + \mu^2 \right] . \]  

(58)

The renormalization scale \(\mu\) is chosen to give the correct values for sigma and pion mass at \(T = 0\).

As an alternative to the above procedure, one can also compute (52) in dimensional regularization, i.e., in \(d\) space-time dimensions, where the coupling constant \(g\) is replaced by \(g\bar{\mu}^\epsilon\). Here, \(\bar{\mu}\) is the renormalization scale in dimensional regularization and \(\epsilon = 4 - d\). In order to obtain (50), one has to add a counter term \(M^2/(8\pi^2\epsilon) + \mu^2/(16\pi^2)\). Here, \(\mu\) is the renormalization scale from the previous treatment and related to \(\bar{\mu}\) by \(\bar{\mu} = 4\pi\epsilon^{-1}\mu^2\), where \(\gamma\) is the Euler-Mascheroni constant. Note again, that the counter term depends implicitly on the temperature through the resummed mass \(M\). In the Appendix, we furthermore show that the CO and CT schemes are equivalent for unbroken \(O(N)\) symmetry.

1. Hartree approximation

In the Hartree approximation, the gap equations read

\[ M_\sigma^2 = m^2 + \frac{4\lambda}{N} \left\{ 3 \varphi^2 + 3 [Q_T(M_\sigma) + Q_\mu(M_\sigma)] + (N - 1) [Q_T(M_\pi) + Q_\mu(M_\pi)] \right\} , \]  

(59a)

\[ M_\pi^2 = m^2 + \frac{4\lambda}{N} \left\{ \varphi^2 + [Q_T(M_\sigma) + Q_\mu(M_\sigma)] + (N + 1) [Q_T(M_\pi) + Q_\mu(M_\pi)] \right\} . \]  

(59b)

The renormalization scale \(\mu\) is determined from the vacuum values for the sigma and pion masses:

\[ m_\sigma^2 = m^2 + \frac{4\lambda}{N} \left[ 3 f_\pi^2 + 3 Q_\mu(m_\sigma) + (N - 1) Q_\mu(m_\pi) \right] , \]  

(60a)

\[ m_\pi^2 = m^2 + \frac{4\lambda}{N} \left[ f_\pi^2 + Q_\mu(m_\sigma) + (N + 1) Q_\mu(m_\pi) \right] . \]  

(60b)

In the chiral limit, the difference of these two equations reads

\[ m_\sigma^2 = \frac{8\lambda}{N} \left[ f_\pi^2 + \frac{m_\pi^2}{16\pi^2} \ln \frac{m_\pi^2}{\mu^2} \right] . \]  

(61)

However, in order to be consistent with the (generally valid) equation (30a), there is only a single choice for the renormalization scale, \(\mu^2 \equiv m_\sigma^2/e\). Then, the coupling constant is given by its classical value, \(\lambda = N m_\sigma^2/(8 f_\pi^2)\), while

\[ m^2 = \frac{-m_\sigma^2}{2} - \frac{4\lambda}{N} (N + 2) \frac{\mu^2}{16\pi^2} . \]  

(62)
In the case that chiral symmetry is explicitly broken, the difference of (60a) and (60b) yields the following equation for the coupling constant:

\[
\lambda = \frac{N}{8} f^2 + \frac{m^2}{f^2} \ln(m^2/\mu^2 e) - \frac{m^2}{f^2} \ln(m^2/\mu^2 e) \bigg/ 16\pi^2 \equiv \lambda(\mu),
\]

i.e., \( \lambda \) runs with the renormalization scale. However, there is one value for the renormalization scale, where \( \lambda \) retains its tree-level (i.e. classical) value,

\[
\mu^2 \equiv \mu^2_{cl} = \exp \left[ \frac{m^2 (\ln m^2 - 1) - m^2 (\ln m^2 - 1)}{m^2 - m^2} \right].
\]

The results for the Hartree case with explicitly broken symmetry presented in the next section will exclusively employ this value of \( \mu \). The mass parameter is determined from

\[
m^2 = - \frac{m^2 - 3 m^2}{2} - 4 \lambda N (N + 2) Q_{\mu}(m_{\pi}).
\]

\( H \) can be obtained from (34e) at \( T = 0 \).

2. Large-\( N \) approximation

In the large-\( N \) limit, the gap equations to be solved are (34d) for the sigma meson and

\[
M^2_{\pi} = m^2 + 4\lambda N \left\{ \phi^2 + N [Q_{\tau}(M_{\pi}) + Q_{\mu}(M_{\pi})] \right\}
\]

for the pion. In this case, \( \mu \) is a free parameter, and cannot be fixed by the vacuum values for the sigma and pion masses. \( \lambda \) and \( H \) are always given by their tree-level values. The mass parameter is determined from

\[
m^2 = - \frac{m^2 - 3 m^2}{2} - 4 \lambda Q_{\mu}(m_{\pi}).
\]

VI. RESULTS

In this section, we discuss numerical solutions of the gap equations for the meson masses and the stationarity condition on \( \phi \). Three different cases are considered: the large-\( N \) approximation in (a) the CO scheme, (b) the CT scheme, and (c) the Hartree approximation in the CT scheme. The Hartree approximation in the CO scheme will not be discussed, due to the problems exhibited in section V. We focus separately on the cases \( m_{\pi} = 0 \) and \( m_{\pi} > 0 \).

A. \( m_{\pi} = 0 \)

Figures 3 (a,c,e) show the meson masses and (b,d,f) \( \phi \) as functions of temperature. Results for the large-\( N \) approximation with CO renormalization are shown in parts (a,b), and with CT renormalization in (c,d). Results for the Hartree approximation with CT renormalization are shown in (e,f). For comparison, the dashed lines in each figure correspond to the unrenormalized results of [21].

In Figs. 3 (a,b), in the phase of spontaneously broken symmetry, there is no difference between the unrenormalized and renormalized cases. To understand this, first remember that \( M_\pi = 0 \), cf. the
discussion following eq. (34e). Therefore, on account of (34d), $M_\sigma$ is simply given by $(8\lambda/N)^{1/2}\varphi$.

In turn, $\varphi$ is determined by (34c). However, for $M_\pi = 0$, this has the simple form

$$0 = m^2 + \frac{4\lambda}{N} \varphi^2 + 4\lambda \left[ \frac{T^2}{12} + Q_\Lambda(0) \right].$$

(68)

Using (51) for $m_\pi = 0$, this becomes

$$0 = -\frac{m_\sigma^2}{2} + \frac{4\lambda}{N} \varphi^2 + 4\lambda \frac{T^2}{12},$$

(69)

which is the same condition as in the unrenormalized case (where $Q_\Lambda$ is absent). Since the coupling constant is given by its tree-level value (22), this immediately leads to the conclusion that the temperature for chiral symmetry restoration is

$$T^* = \sqrt{3} f_\pi.$$  

(70)

In the restored phase, $\varphi = 0$, sigma and pion masses are equal, and given by eq. (34a) or (34b). These equations are cut-off dependent, on account of (41). The mass is decreasing for increasing $\Lambda$.

In Figs. 3 (c,d), we show results for the large-$N$ approximation in the CT scheme. In the broken phase, $\varphi > 0$, renormalization again does not affect the masses or $\varphi$. In the phase of restored symmetry, $\varphi = 0$, the sigma and pion masses are degenerate, but depend on the renormalization scale. They decrease for increasing $\mu$. Note the similarity between the masses in the CO and the CT scheme when choosing the same value for the cut-off $\Lambda$ and the renormalization scale $\mu$. Considering that both renormalization schemes are fundamentally different, this similarity is quite surprising. Another important conclusion is that renormalization of the gap equations does not destroy the second-order nature of the transition.

FIG. 3. The meson masses and $\varphi$ as a function of $T$ for $m_\pi = 0$. 

15
The results in the Hartree approximation, eqs. (28a) – (30b), are displayed in Figs. 3 (e,f). As in the unrenormalized case, we obtain a first order transition, with a transition temperature that appears to be slightly higher than in the unrenormalized case. To determine this temperature, however, one would have to analyze the shape of the effective potential, which is outside the scope of this paper.

B. $m_\pi = 139$ MeV

Fig. 4 (a,c,e) shows the temperature dependence of the meson masses and Fig. 4 (b,d,f) the function $\phi(T)$ in the case of explicit symmetry breaking. As in Fig. 3, large-$N$ results are shown in (a,b) for the CO scheme and in (c,d) for the CT scheme. Parts (e,f) show our results for the Hartree approximation with CT renormalization. As already observed in the chiral limit, there is a striking similarity between the results in the CO and the CT schemes when choosing $\Lambda = \mu$. Also, increasing the cut-off or the renormalization scale tends to increase the temperature at which (approximate) symmetry restoration takes place.

Baym and Grinstein [10] noted that the additional terms originating from renormalization have the effect that the gap equations do not have a solution beyond a certain temperature (see also [17,28]). We found evidence for this in the CT scheme at temperatures above 400 MeV. In the CO scheme with a finite $\Lambda$, this phenomenon does not occur.

FIG. 4. The meson masses and $\phi$ as a function of $T$ for $m_\pi = 139$ MeV.
VII. CONCLUSIONS AND OUTLOOK

In this paper, we have studied the temperature dependence of sigma and pion masses in the framework of the $O(N)$ model. The Cornwall–Jackiw–Tomboulis formalism was applied to derive gap equations for the masses in the Hartree and large-$N$ approximations. Renormalization of the gap equations was carried out within the cut-off and counter-term renormalization schemes. In agreement with [14], it was found that the cut-off scheme is flawed when the cut-off $\Lambda \to \infty$. We therefore studied this renormalization scheme for $\Lambda < \infty$. For the Hartree approximation we found that, in the chiral limit ($m_\pi = 0$), there is no finite value for the cut-off, which is consistent with the set of stationarity conditions for the effective potential; $\Lambda = 0$ is the only possible choice. This problem was not encountered in the large-$N$ approximation; here any choice for $\Lambda$ is possible. In the counter-term renormalization scheme, the Hartree approximation can be consistently renormalized, but in the chiral limit, the renormalization scale is restricted to a unique value in order to achieve consistency with the stationarity conditions for the effective potential. In the large-$N$ limit, the renormalization scale can be chosen arbitrarily. Changing the cut-off in the cut-off scheme or the renormalization scale in the counter-term scheme changes the meson masses at a given temperature. The reason is that, in the self-consistent approximation schemes considered here, the renormalization constants (or counter terms, respectively) may depend implicitly on temperature. This does not occur when renormalizing ordinary perturbation theory.

Our results can be compared to those of Roh and Matsui [15] and Chiku and Hatsuda [17]. The authors of [15] computed the sigma and pion masses from the second derivative of an effective potential which was determined via the standard loop expansion approach. Being aware that this approach fails for theories with spontaneously broken symmetry, they corrected the resulting expressions to obtain gap equations which look similar to the ones in the Hartree approximation. (They are identical to Baym and Grinstein’s modified Hartree approximation [10]). The stationarity condition for $\varphi$, however, was taken to be the same as in the large-$N$ approximation. Thus, their solutions respect Goldstone’s theorem in the phase of spontaneously broken symmetry, similar to the large-$N$ approximation discussed here, while the transition in their model is first order (in the chiral limit), like in the Hartree approximation.

The authors of [17] employ optimized perturbation theory to compute the sigma and pion masses. This approach has the advantage that renormalization is straightforward. The results are similar to those of [15].

Recent dilepton experiments at CERN-SPS energies [3] have generated interest in medium modifications of meson properties such as their mass and decay width. In general, the meson mass (squared) is given by the inverse propagator at $k = 0$, $M^2 \equiv G^{-1}(0)$. The decay width of a particle with energy $\omega$ at rest, $k = 0$, is given by $\gamma(\omega) \equiv -\text{Im}\Pi(\omega, 0)/\omega$ [29], where $\Pi(\omega, k)$ is the self energy. In the CJT formalism, $G^{-1}(k) = D^{-1}(k; \varphi) + 2\Pi(k)$, cf. eq. (14c). In the Hartree or large-$N$ approximation studied here, the self energies do not acquire an imaginary part, because they are simply constants, and thus only shift the mass of the particles. Therefore, in these approximations, the particles are true quasi-particles with vanishing decay width. This would change if we included the diagrams of Fig. 2 in the effective potential, because, as is well-known [29], the imaginary part of these diagrams corresponds to decay and scattering processes.

To include these diagrams in the above treatment, however, is prohibitively difficult, because then the simple momentum dependence of the propagators $G_{\sigma, \pi}(k)$ in eq. (27) changes, since the self energies become explicitly momentum dependent. Then, instead of simple gap equations for the meson masses, the stationarity condition (14b) becomes an (infinite) set of coupled integral equations for the propagators $G_{\sigma, \pi}(k)$.

Therefore, as a first approximation, we compute the decay widths from the self energies corresponding to these diagrams, but with internal lines given by the Hartree or large-$N$ propagators (27). This is equivalent to computing the decay width to one-loop order in perturbation theory, but taking the medium-modified masses of the particles computed above instead of the vacuum masses. The on-shell decay width of $\sigma$ and $\pi$ mesons at rest is then given by the following expressions [30], valid for $2M_\pi \leq M_\sigma$:
\[ \gamma_\sigma = \left( \frac{4\lambda_\phi}{N} \right)^2 \frac{N - 1}{16 \pi M_\sigma} \sqrt{1 - \frac{4M_\sigma^2}{M_\sigma^2}} \coth \frac{M_\sigma}{4T}, \]  

(71a)

\[ \gamma_\pi = \left( \frac{4\lambda_\phi}{N} \right)^2 \frac{M_\sigma^2}{8 \pi M_\pi^2} \sqrt{1 - \frac{4M_\pi^2}{M_\sigma^2}} \frac{1 - \exp[-M_\sigma/T]}{1 - \exp[-M_\pi^2/2m_\pi T]} \frac{1}{\exp[(M_\sigma^2 - 2M_\pi^2)/2M_\pi T] - 1}. \]  

(71b)

These quantities are shown in Fig. 5 for \( m_\pi = 0 \), and in Fig. 6 for \( m_\pi = 139 \) MeV, for the cases discussed in Figs. 3 and 4. For \( m_\pi = 0 \) and in the large-\( N \) approximation, pions are true Goldstone bosons, and therefore their decay width vanishes below the temperature corresponding to chiral symmetry restoration, see Figs. 5 (b,d). This is different in the Hartree approximation, where Goldstone’s theorem is violated, cf. Fig. 5 (f), and when chiral symmetry is explicitly broken, Figs. 6 (b,d,f). The reason is that, because pions have a finite mass, they can acquire a finite decay width on account of the absorption processes \( \pi \sigma \rightarrow \pi \) and \( \pi \pi \rightarrow \sigma \). For massless particles, these processes are kinematically forbidden.

Sigma mesons, however, can always decay into two pions, and therefore acquire a large decay width, cf. Figs. 5 and 6 (a,c,e). All decay widths vanish above the temperature where \( M_\sigma \) becomes smaller than 2 \( M_\pi \). This, however, is an artefact of the one-loop approximation. In two-loop order, the scattering processes \( \sigma \sigma \rightarrow \sigma \sigma, \sigma \sigma \rightarrow \pi \pi, \sigma \pi \rightarrow \sigma \pi \), and \( \pi \pi \rightarrow \pi \pi \) lead to a finite decay width for all particles even above this threshold.

The decay widths and masses computed here are relevant for the formation of disoriented chiral condensates [30], since they enter the evolution equations of the long-wavelength modes. This will be the subject of a subsequent investigation [31].

Acknowledgements

We thank T. Appelquist, J. Berges, S. Gavin, M. Gyulassy, T. Hatsuda, Y. Kluger, J. Knoll, L. McLerran, E. Mottola, B. Müller, and R. Pisarski for valuable discussions. D.H.R. thanks Columbia University’s Nuclear Theory group for continuing access to their computational facilities. J.T.L. is supported by the Director, Office of Energy Research, Division of Nuclear Physics of the Office of High Energy and Nuclear Physics of the U.S. Department of Energy under Contract No. DE-FG-02-91ER-40609.
FIG. 5. The sigma and pion decay widths as a function of $T$ for $m_\pi = 0$.

FIG. 6. The sigma and pion decay widths as a function of $T$ for $m_\pi = 139$ MeV.
APPENDIX A: SCHEME EQUIVALENCE FOR UNBROKEN $O(N)$ SYMMETRY

In this Appendix, we show that the CO and the CT schemes are equivalent when the $O(N)$ symmetry is not broken ($\varphi = 0$). In this case, the gap equations become degenerate,

$$M^2 = m^2 + \frac{4\lambda}{N} (N + 2) Q(M, T). \tag{A1}$$

(We only discuss the Hartree case here, for the large-$N$ approximation, simply replace $N + 2$ by $N$.) In the CO scheme, using eqs. (36) and (38), this becomes

$$M^2 = m^2_R + \frac{4\lambda R}{N} (N + 2) \left[ Q_T(M) + \frac{M^2}{16\pi^2} \ln \frac{M^2}{\mu^2} \right]. \tag{A2}$$

The renormalization scale $\mu^2$ can be determined from the $T = 0$ limit of this equation. For unbroken $O(N)$ symmetry, $M(T = 0) = m_R$, which then yields $\mu = m_R$.

On the other hand, in the CT scheme, we have

$$M^2 = m^2_R + \frac{4\lambda R}{N} (N + 2) \left[ Q_T(M) + \frac{M^2}{16\pi^2} \ln \frac{M^2}{\mu^2} - \frac{M^2_1}{16\pi^2} \right]. \tag{A3}$$

Here, we made the finiteness of $m$ and $\lambda$ explicit by replacing them with $m_R$ and $\lambda_R$. Again, the condition $M(T = 0) = m_R$ yields $\mu = m_R$.

The last term in (A3) leads to an apparent difference between the two schemes. However, shifting the coupling constant by a finite amount,

$$\frac{1}{\lambda_R} \to \frac{1}{\lambda_R} - \frac{4(N + 2)}{16\pi^2 N}, \tag{A4}$$

one obtains (A2), which proves the equivalence of both schemes after properly redefining the coupling constant. The same conclusion can be reached starting from (A1) and using instead of (38) the modified renormalization conditions

$$m^2_R \left[ \frac{1}{\lambda_R} + \frac{4(N + 2)}{16\pi^2 N} \right] = \frac{m^2}{\lambda} + \frac{4(N + 2)}{N} I_1, \tag{A5a}$$

$$\frac{1}{\lambda_R} + \frac{4(N + 2)}{16\pi^2 N} = \frac{I_1}{\lambda} + \frac{4(N + 2)}{N} I_2, \tag{A5b}$$

which then leads to (A3).

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