THE WEINGARTEN MODEL
À LA POLYAKOV

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Abstract

The Weingarten lattice gauge model of Nambu-Goto strings is generalised to allow for fluctuations of an intrinsic worldsheet metric through a dynamical quadrilation. The continuum limit is taken for $c \leq 1$ matter, reproducing the results of hermitian matrix models to all orders in the genus expansion. For the compact $c = 1$ case the vortices are Wilson lines, whose exclusion leads to a theory of non-interacting fermions. As a by-product of the analysis one finds the critical behaviour of SOS and vertex models coupled to 2D quantum gravity.
1 Introduction

The motivation for the work presented in this note was to make the old analogy between string and (lattice) gauge theories an equivalence. This will be achieved through a modification à la Polyakov of Weingarten’s generalisation of lattice Yang-Mills theory [1]. Although the cut-off bosonic theory will be generally defined, the usual problems of finding a continuum limit for $c > 1$ matter will mean that quantitative results can only be obtained for $c \leq 1$. In fact work on this latter problem was initiated some time ago by T.R. Morris [2]. In this letter it is proved that there are continuum limits on surfaces of arbitrary genus equivalent to those of the multi-critical hermitian matrix (open) chain models [3, 4, 5]. The main reason for introducing a lattice gauge formulation however appears in the finite temperature theory, described by compact $c = 1$. In the new variables vortices are manifest, since they are just Wilson lines, and their exclusion leads to a simple proof of equivalence to non-interacting fermions at finite temperature (the singlet sector of hermitian matrix models). The phase transition due to a condensation of winding modes can now be couched in the lore of ‘deconfinement’. Dual to this phenomenon is a condensation of Kaluza-Klein momentum modes if the radius, for fixed number of target lattice sites, is made too large; the theory is equivalent to that on a continuous target circle provided the lattice spacing is less than its critical value [6]. As a subsidiary result, the analyses exhibit the critical behaviour of certain face and vertex models on random surfaces. In the discussion at the end, some speculations and suggestions for more detailed investigation are given.

2 The Weingarten Model

The original Weingarten model of Nambu-Goto strings was constructed as follows. On a $D$-dimensional hypercubic lattice with oriented links $\pm l$ one associates a complex $N \times N$ matrix $M(l)$ ($M(-l) = M^\dagger(l)$) to each link $l$. The partition function is;

$$Z_W = \int_{-\infty}^{+\infty} \prod_l \prod_{i,j=1}^N \frac{1}{\pi} d[\text{Re}M_{ij}(l)]d[\text{Im}M_{ij}(l)] \ e^{-S_W}$$

(1)

where

$$S_W = \sum_l \text{Tr}[M^\dagger(l)M(l)] - \frac{\lambda}{N} \sum_{\text{plaq}} \text{Tr}[M(l_1)M(l_2)M(l_3)M(l_4)]$$

(2)

These two terms in the action are illustrated in figure 1 for $D = 2$. ‘plaq’ indicates a sum over all oriented plaquettes on the lattice. The action has the usual gauge symmetry associated with unitary transformations at the sites connected by the links. When one expands the exponential in $\lambda$ and performs the $M$ integrals one finds the result [1];

$$Z_W = \sum_s \left( \frac{1}{N} \right)^x \lambda^n$$

(3)
where \( S \) denotes all closed surfaces made out of plaquettes. \( \chi \) is the Euler number and \( n \) the number of plaquettes (the area). The fact that complex rather than unitary matrices are link variables eliminates the higher contractions between plaquettes that one finds in the corresponding expansion of lattice Yang-Mills; the latter gives unwanted contact interactions between worldsheets. The expansion (3) is a discrete version of the partition function of 1st quantised Nambu-Goto strings. Evidently this model cannot be compared with recent results for \( c \leq 1 \) string theory since the plaquettes require an embedding dimension \( D \geq 2 \).

One can endow the worldsheets of the Weingarten model with an intrinsic metric by adding the following piece to the action;

\[
S_C = -\sum_{l} \frac{\kappa_1}{N} \text{Tr}[M^\dagger(l)M(l)M^\dagger(l)M(l)] + \frac{\kappa_2}{N} \text{Tr}[M^\dagger(l)M(l)M(l+1)M^\dagger(l+1)] \\
-\frac{\kappa_3}{N} \sum_{L} \text{Tr}[M(l_1)M^\dagger(l_1)M(l_2)M^\dagger(l_2)]
\]

(4)

whose terms are also illustrated in figure 1. \( \text{L} \) indicates all L-shaped terms, as shown. \( S_C \) thus consists of ‘collapsed’ plaquettes. Note that orientation is irrelevant for terms contributing to (3); reversing the link arrows makes no difference. The model with action \( S_W + S_C \) is remarkably similar to the light-cone lattice string model studied by Klebanov and Susskind [7] (see also [8]), although at present the precise connection is still missing.

If one now expands in \( \lambda \) and the \( \kappa_i \) the effect of the new terms from \( S_C \) is to produce insertions on the links of the surfaces of operators coupling to the \( \kappa_i \), each insertion giving a factor of \( \kappa_i \). One may interpret the latter as the insertion of a surface element of extrinsic area 0 but intrinsic area 1. The intrinsic and extrinsic areas of plaquettes in \( S_W \) are both 1. By allowing collapsed plaquettes one has generated a dynamical quadrilation (square simplices), representing an intrinsic metric on the surface [3], since a vertex on the surface may be surrounded by a variable number of plaquettes for given extrinsic geometry. Put another way, starting from a dynamical quadrilation one defines the matter field at each vertex to be a height variable \((h_1, \ldots, h_D)\), where \( h_j \) is an integer and neighbouring vertices have \( \Delta h_j = 0, \pm 1 \). These height variables are reminiscent of those of the Solid-On-Solid (SOS) models in statistical mechanics [10], where neighbouring vertices differ in height by 1. The correspondence can be made precise by transforming to an auxiliary target lattice. For example if \( D = 2 \), by drawing the plaquettes on the diagonal lattice of the square height lattice (figure 2) one finds that \( \Delta h_j = \pm 1 \). In higher dimensions the construction becomes increasingly more complicated. The links of the auxiliary lattice on which one draws the plaquettes form the body diagonals of the hypercubes of the height lattice. In other words it is generated by the (overcomplete) set of vectors \( \{(\pm 1, \pm 1, \ldots, \pm 1)\} \). Note that plaquettes on this lattice need not be planar. However it is always possible to break them up into triangles so that the surface element interpretation is not lost. In this way one sees that the expansion of the new Weingarten model is equivalent to \( D \) coupled SOS models coupled to

\[^{1}\text{For a general introduction see e.g. [1].}\]
two-dimensional discrete quantum gravity. Only in the continuum limit might one expect to see a decoupling between the models.

One must now ask for the meaning of the various bare couplings in the model and what sort of critical behaviour one can expect to obtain. $\frac{1}{\alpha'}$ is of course the string coupling. Since every plaquette represents a unit element of intrinsic area, the parameter $-\log s$ obtained by scaling $\lambda \rightarrow s \lambda$, $\kappa_i \rightarrow s \kappa_i$ plays the rôle of worldsheet cosmological constant. Another parameter should set the target-space scale, the fundamental string tension $\frac{1}{\alpha'}$. This leaves two couplings in the model $S_W + S_C$. It is easy to see that these are not couplings to intrinsic geometry alone since for $D = 1$ $\lambda$ and $\kappa_3$ do not exist. Neither do they provide coupling to extrinsic worldsheet curvature (e.g. one cannot even define normals to the plaquettes of $S_C$). This only leaves terms in the worldsheet action consisting of higher derivatives in the embedding co-ordinate, which are irrelevant. In fact there are of course an infinite number of naively irrelevant terms one could put into the model by including plaquettes of length greater than four. The fact that they can still affect the continuum limit under appropriate tuning is most clearly illustrated by Kazakov’s multicritical points [12], which will be mentioned in section 2. One is especially interested in reproducing Polyakov’s formulation of bosonic strings in the continuum [13]. In that case the ‘link factor’ in a dynamical quadrilation is gaussian;

$$G(X_i - X_j) = \exp \left( -\frac{1}{2}(X_i - X_j)^2 \right)$$

where $X_i$ and $X_j$ are the target space co-ordinates of neighbouring vertices. For SOS-type matter the link factor is;

$$\delta(X_i - X_j - 1) + \delta(X_i - X_j + 1)$$

In momentum $p$ space (5) is again gaussian $\exp(-\frac{p^2}{2})$ while the fourier transform of (6) gives $\cos p$. These agree to order $p^2$ and so if the critical behaviour is governed by the small $p$ IR properties one expects to end up in the same universality class [14]. This picture could only be upset by UV divergences, in which case the higher dimension terms in the Weingarten model acquire renewed importance.

In the next section the continuum limit of the model will be taken for $D = 1$, in which there are no UV divergences. On a flat worldsheet the critical behaviour of the SOS model is that of one free, massless boson [11]. Given all this it will be no surprise to learn that on a fluctuating worldsheet the critical behaviour is that of a free boson coupled to 2D gravity, as specified by the hermitian matrix model formulation [3]. However the reformulation as a lattice gauge theory has certain conceptual advantages which will become apparent in section 3.
3 Continuum Limit for $c \leq 1$

In $D = 1$ one retains only the collapsed plaquettes coupling to $\kappa_1$ and $\kappa_2$, so the action for an $L$-link lattice is:

$$S_{D=1} = \sum_{l=1}^{L} \text{Tr}[M(l)M(l)] - \frac{\kappa_1}{N} \sum_{l=1}^{L} \text{Tr}[M(l)M(l)M^\dagger(l)M(l)]$$

$$- \frac{\kappa_2}{N} \sum_{l=1}^{L-1} \text{Tr}[M^\dagger(l)M(l)M(l+1)M^\dagger(l+1)]$$

(7)

One can decompose $M(l)$ by a bi-unitary transformation $M(l) = U(l)x(l)V(l)$ where $U$, $V$ are unitary matrices and $x$ is a diagonal matrix with non-negative elements. The measure becomes [13]:

$$\mathcal{D}U\mathcal{D}V \prod_{i=1}^{N} dy_i \Delta^2(y)$$

(8)

where $y_i = x^2_i$ is the $i^{th}$ diagonal element, $\Delta$ is the Van-der-Monde determinant $\prod_{i<j}(y_i - y_j)$ and $\mathcal{D}U$ is the Haar measure for the unitary group. Defining $\Omega(l) = U(l)V^\dagger(l+1)$ the action (7) becomes:

$$\sum_{l=1}^{L} \text{Tr} \left[ y(l) - \frac{\kappa_1}{N} y^2(l) \right] - \frac{\kappa_2}{N} \sum_{l=1}^{L-1} \text{Tr}[\Omega^\dagger(l)y(l)\Omega(l)y(l+1)]$$

(9)

and the measure is:

$$\prod_{l=1}^{L} dy(l) \Delta^2(y(l)) \mathcal{D}U(l) \mathcal{D}V(1) \prod_{l=1}^{L-1} \mathcal{D}\Omega(l)$$

(10)

The presence of $L + 1$ redundant angular integrals is a reflection of gauge invariance. The remaining angular integrals can be performed using the result of Itzykson and Zuber [10] to give the partition function:

$$Z_{D=1} \propto \int_{0}^{\infty} \prod_{l=1}^{L} dy(l) \Delta(y(1)) \Delta(y(L)) \exp \left( \text{Tr} \left[ -y(l) + \frac{\kappa_1}{N} y^2(l) \right] \right)$$

$$\prod_{l=1}^{L-1} \text{det}_{ij} \left[ \exp \left( \frac{\kappa_2}{N} y_i(l)y_j(l+1) \right) \right]$$

(11)

Each determinant is antisymmetric in the $y_i$’s and in the $y_j$’s but the whole integrand is symmetric in these variables since $\Delta$ is also antisymmetric. Thus one can replace each determinant by $N!$ times the first ordering in its expansion. The only requirement on the range of integration is that it be non-zero. This is the content of the result of Mehta [17]:

$$Z_{D=1} \propto \int_{0}^{\infty} \prod_{l=1}^{L} dy(l) \Delta(y(1)) \Delta(y(L)) \exp \left( \sum_{l=1}^{L} \text{Tr}[-y(l) + \frac{\kappa_1 + \kappa_2}{N} y^2(l)] \right)$$

$$- \frac{\kappa_2}{2N} \text{Tr}[y^2(1) + y^2(L)] - \sum_{l=1}^{L-1} \frac{\kappa_2}{2N} \text{Tr}[(y(l+1) - y(l))^2]$$

(12)
This can be interpreted as the discrete-time quantum mechanical partition function of \( N \) non-interacting particles (on \([0, \infty)\) in the present case) which are in addition fermionic, on account of the \( \Delta_s \) antisymmetrising the final states at ‘time’ \( L \) with respect to the initial states at ‘time’ 1. For \( c = 1, \ L \to \infty \). The saddle point distribution of eigenvalues \( y_i \) that one is interested in is sketched in figure 3. The non-negativity restriction is denoted by an infinite barrier and the eigenvalues fill states up to a Fermi level coinciding with the top of the quadratic maximum by adjusting \( \kappa_1 + \kappa_2 \). One now proceeds to the continuum limit by taking a double scaling limit in \( N \) and \( \kappa_1 + \kappa_2 \) in the usual way. Lack of space here prevents a detailed account and the reader is referred to the review example. The only novel feature is the presence of the infinite barrier; but this does not affect the universal critical properties to all orders in the \( \frac{1}{N} \) (genus) expansion at least, since these are governed purely by the occurrence of a quadratic maximum.

It was shown by Gross and Klebanov that in the continuum limit of the \( c = 1 \) hermitian matrix chain, the transfer matrix between neighbouring sites in the chain is equivalent to that of the theory on \( \mathbb{R} \), between those same points, after a redefinition of \( \alpha' \) and the string coupling dependent on the target chain spacing. The same is true here, identifying those chain sites with the centres of the links of the present model. There is a range of lattice spacing from zero to \( \pi \sqrt{\alpha'} \) for which one obtains the critical behaviour of one free boson coupled to 2D gravity. This is the quantum gravity version of the flat-space fact that, amongst other \( c = 1 \) models, the SOS model renormalises onto a Gaussian fixed line on which it has continuously varying critical exponents. This remarkable short distance property of string theory is nothing more nor less. The transition point of ‘SOS-strings’ has been studied in detail by I.Kostov in recent times.

To end this section, consider the finite chain models (\( c < 1 \)). The \( L \)-link linear lattice describes the \( L + 1 \)st restricted solid-on-solid (RSOS) model coupled to 2D gravity. This target lattice is precisely that of the \( A_{L+1} \) Dynkin diagram (figure 4) and on a flat worldsheet it is well-known that these RSOS models at criticality provide particular realisations of the unitary discrete series with \( c = 1 - 6/L(L + 1) \). Coupling to 2D gravity should again leave them in the same universality classes as those of the finite hermitian matrix chains, for reasons already explained. In particular the \( A_2 \) model;

\[
S_{A_2} = \text{Tr}[M^\dagger M] - \frac{\kappa_1}{N} \text{Tr}[M^\dagger MM^\dagger M]
\]

should describe pure gravity. This has previously been verified in detail, as have Kazakov’s multicritical points obtained by tuning couplings to plaquettes of length \( > 4 \), \( \text{Tr}[M^\dagger M]^p \). Lastly, the reader will have noted the omission of discussion on correlation functions at \( c \leq 1 \). It is easy to see however, from the identification with hermitian matrix eigenvalues, that the equivalence of correlators punctual in target space carries over also since these involve only eigenvalues. A proof that appropriately defined string states on the lattice can yield the standard correlators on a continuous target space, in any context, is still lacking.
4 Continuum limit on $S^1$

In this section the continuum limit of the model defined on a periodic target lattice will be explored. One may think of these lattices as the extended Dynkin diagrams $\hat{A}_{L-1}$ (figure 4). Recalling the expression (9), there will now be a coupling:

$$-\frac{\kappa_2}{N} \text{Tr}[\Omega^\dagger(L)y(L)\Omega(L)y(1)]$$

(14)

where $\Omega(L) = U(L)V^\dagger(1)$. However the measure (10), unlike that of the hermitian matrix model, allows one to again perform all angular integrals using Itzykson and Zuber’s formula to derive;

$$Z_{S^1} = \text{const} \int_0^\infty \prod_{l=1}^L dy(l) \exp \left( \text{Tr} \left[ -y(l) + \frac{\kappa_1}{N} y^2(l) \right] \right) \det_{ij} \left[ \exp \left( \frac{\kappa_2}{N} y_i(l)y_j(l+1) \right) \right]$$

(15)

where $y(L+1) \equiv y(1)$. This represents $N$ non-interacting non-relativistic fermions on $[0, \infty)$ at finite temperature. The angular degrees of freedom could be eliminated because the action (7) explicitly excludes vortex configurations, which are the plaquettes which wind around $S^1$, $\text{Tr}[\prod_{l=1}^L M(l)]^p$. These statements will now be justified in more detail.

In order to show equivalence to non-interacting fermions it is convenient to take the simplest case of $L = 1$ i.e. a single periodic link. The argument generalises easily to $L > 1$ and could equally have been used in the previous section. Note that the target lattice $\hat{A}_0$ (which is not a Dynkin diagram) is usually omitted from the $\hat{A}\hat{D}\hat{E}$ classification [11], and has no analogue in the hermitian matrix models. Periodic $L = 1$ is the complex matrix model:

$$Z_{\hat{A}_0} = \int dM \exp (-\text{Tr}[M^\dagger M] + \frac{\kappa_1}{N} \text{Tr}[M^\dagger M M^\dagger M] + \frac{\kappa_2}{N} \text{Tr}[M^\dagger M M M^\dagger])$$

(16)

$$= \text{const} \int_0^\infty \prod_{k=1}^N dy_k \exp \left( -y_k + \frac{\kappa_1}{N} y^2_k \right) \det_{ij} \left[ \exp \left( \frac{\kappa_2}{N} y_i y_j \right) \right]$$

(17)

The restriction to a periodic gauge invariance $M \rightarrow U^\dagger MU$ has meant that one cannot transform to eigenvalues directly as happens for (13). As an aside, one notes that $Z_{\hat{A}_0}$ describes the vertex model of statistical mechanics on a dynamical quadrilation. One might expect this from the SOS $\leftrightarrow$ vertex correspondence. More directly, if one expands $Z_{\hat{A}_0}$ in Feynman diagrams using double-line notation, the propagators carry an orientation since the matrices are complex (figure 5). In fact these are the dual diagrams to the those of the plaquettes on the worldsheet. The notion of orientation of one vertex with respect to its neighbour is significant in defining the regular lattice vertex model, but this distinction is lost when coupling to gravity. For this reason the random lattice version (16) is the analogue of a restricted type of vertex model known as the ferro-electric or F-model [11].

The clearest way to show that (17) represents non-interacting non-relativistic fermions at finite temperature is to introduce a complete set of antisymmetric wavefunctions$^\dagger$. In

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$^\dagger$I thank I.Klebanov for suggesting to argue in this way.
detail, if \{\psi_\alpha\} are a complete set of one-body wavefunctions (the precise hamiltonian being unimportant initially) then \(\sum_\alpha \psi_\alpha^\dagger(y)\psi_\alpha(z) = \delta(y - z)\) and for any function \(S(y) \equiv \prod_i S(y_i)\):

\[
\int \prod_k dy_k \det_{ij} \left[ \exp \left( \frac{\kappa_2^2}{N} y_i y_j \right) \right] S(y) = \sum_{\Pi(j_1\ldots j_N)} \epsilon_{j_1\ldots j_N} \int \prod_k dy_k dz_k \sqrt{S(y)} \sqrt{S(z)} \exp \left( \frac{\kappa_2^2}{N} y_k z_k \right) \delta(y_k - z_{j_k})
\]

Rewriting the \(\delta\) functions using the fact that:

\[
\sum_{\Pi} \epsilon_{j_1\ldots j_N} \prod_{k=1}^N \psi_{\alpha_k}^\dagger(y_k) \psi_{\alpha_k}(z_{j_k}) = \frac{1}{N!} \sum_{\alpha_1\ldots \alpha_N} \left( \sum_{\Pi} \epsilon_{\alpha_1\ldots \alpha_N} \prod_{k} \psi_{\alpha_k}^\dagger(y_k) \right) \left( \sum_{\Pi} \epsilon_{\alpha_1\ldots \alpha_N} \prod_{k} \psi_{\alpha_k}(z_k) \right)
\]

\[
\equiv \sum_A <\Psi_A|y><z|\Psi_A>
\]

where \(\Psi_A\) are a complete set of Slater determinants (by definition essentially), (18) becomes;

\[
\int dydz \sum_A <\Psi_A|y> \sqrt{S(y)} \exp \left( \frac{\kappa_2^2}{N} \sum_k y_k z_k \right) \sqrt{S(z)} <z|\Psi_A>
\]

\[
= \int dydz \sum_A <\Psi_A|y><y|e^{-\beta H}|z><z|\Psi_A> \equiv \text{Tr}_A e^{-\beta H}
\]

for an appropriate sum of one-body hamiltonians \(H = \sum_k h_k\). The transfer matrix elements for each \(h_k\) in the present case are;

\[
\exp \left( -\frac{\kappa_2^2}{2N} (y_k - z_k)^2 - \frac{1}{2} \left( y_k - \frac{\kappa_1 + \kappa_2}{N} y_k^2 \right) - \frac{1}{2} \left( z_k - \frac{\kappa_1 + \kappa_2}{N} z_k^2 \right) \right)
\]

The analysis now proceeds much as in ref.[6]. (23) is a particular example of a class of transfer matrices which are equivalent, in the double scaling limit, to that of the harmonic oscillator hamiltonian. The only requirement is that \(V(y) = y - \frac{\kappa_1 + \kappa_2}{N} y^2\) (figure 3) possess a quadratic maximum. Thus \(Z_{L_0}\) is the partition function of fermions at temperature \(1/\epsilon\), where \(\epsilon\) is the length of a single link. The equivalence with the harmonic oscillator breaks down for \(\epsilon > \pi \sqrt{\alpha'}\) when there is presumably a momentum-mode condensation in the same manner as that of the regular lattice vertex model. Generalising the previous analysis to \(\hat{A}_{L-1}\) shows that this describes a compact boson with radius \(0 < r < L \sqrt{\alpha'}/2\) coupled to 2D gravity in accordance with the hermitian matrix models. On a circle of circumference \(\epsilon L\) there are Kaluza-Klein momentum modes \(\cos (2\pi n X/\epsilon L)\) of conformal weight \(\pi^2 n^2 \alpha'/\epsilon^2 L^2\). For \(c = 1\) the conformal weight \(\Delta_0\) of an operator and its gravitational scaling dimension \(\Delta\) are simply related, \(\Delta^2 = \Delta_0\). Given that \(\gamma_{str} = 0\), this means that coupling to worldsheet

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3The temperature seems to be out by a factor of 2 which the author does not fully understand. A similar factor of 2 arises in the Klebanov-Susskind model due to a mismatch of Hilbert spaces [8].
gravity should not change the radius at which operators become relevant. Thus it would seem that the lowest dimension operator in the action for the model on an $L$-link periodic lattice corresponds to momentum $n = L$. At the critical radius this corresponds to a rather trivial example of perturbation by one of the so-called special operators of $c = 1$ string theory.

The dual operators which also govern the phase structure are the vortices (winding mode condensates). These are given by the Wilson line operators for (anti) vortex number $(q)p$;

$$L_p = \text{Tr} \left[ \prod_{l=1}^{L} M(l) \right]^p, \quad L_{-q} = \text{Tr} \left[ \prod_{l=1}^{L} M^\dagger(l) \right]^q$$

which were explicitly omitted from the action up to now. Although their inclusion leaves one with angular degrees of freedom which cannot be eliminated with standard methods, their expression in the familiar form (24) represents a considerable simplification over the hermitian matrix formulation which organises the contribution of vortices unusually. For example the simplest two-point function, which can be evaluated in the $Z_{A_0}$ theory without loss of generality, involves the angular integral;

$$\int d\Omega \text{Tr}[\Omega \sqrt{y}] \text{Tr}[\Omega^\dagger \sqrt{y}] \exp(\Omega y \Omega^\dagger y)$$

which arises in models of unitary matrices in external fields. Since one expects duality to be a good symmetry in the $1/N$ expansion even after coupling to worldsheet gravity [6], the vortex operators should become relevant at the appropriate dual radii. This locates the first possible transition at $r = 2\sqrt{\alpha'}$ [24] where $<L_1L_{-1}>$ will scale correctly.

5 Discussion

To summarise, the main result of this letter has been to rewrite non-critical bosonic string theory in the form of a lattice gauge theory. There are many infamous parallels between the confining phase of Yang-Mills and string theory. In particular the finite temperature deconfining transition of pure Yang-Mills with $U(1)$ group centre is analogous to a phase transition conjectured to lie at (or near) the Hadgedorn temperature $T_H$ of critical string theory [24], interpreted as a condensation of winding modes. It is natural to ask what the equivalence proved in this letter can tell one about the nature of this transition in general, since in higher dimensional string theories one hopes to expose the underlying degrees of freedom in the high-temperature phase [26]. In Yang-Mills theory this further tier of simplification reveals the glue.

It is the presence of a global $U(1)$ symmetry of the local action, whose charge is winding number, which ties together seemingly disparate models in the picture of deconfinement [23]. Under $M \rightarrow e^{i\theta} M$, for link matrices around a compact direction, the Wilson lines are

\footnote{For an introduction to aspects of deconfinement in (lattice) gauge theories see e.g. [23].}
not invariant, $L_p \rightarrow e^{ipL_\theta}L_p$. $< L_1 >$ is then an order parameter for the transition to a phase in which the vacuum breaks this $U(1)$ symmetry spontaneously. This picture is only strictly correct in target spaces of dimension $> 2$ due to usual the prohibition of spontaneous breakdown of continuous symmetry. In the case of section 4, the effective action for the field $L_1$, defined by;

$$e^{-S[L_1]} = \int dM \delta \left( L_1 - \text{Tr} \left[ \prod_{i=1}^{L} M(l_i) \right] \right) \ e^{-S_W - S_C}$$  \hspace{1cm} (26)$$
is zero dimensional. $S[L_1]$ is symmetric under $U(1)$ and it alone is a reliable guide to the phase structure. The transition is then of Kosterlitz-Thoules-Berezinski type \cite{25}, where $< |L_1| >$ is the order parameter and $U(1)$ winding symmetry remains unbroken. From the worldsheet point of view the transition would always be of BKT type whatever the target space dimension. Simply speaking one cannot span a winding loop with a surface. Thus there is no particularly fundamental difference between the target space and worldsheet descriptions in low dimensions. One expects the high temperature phase to also be a random surface theory. In higher dimensions it is possible to break $U(1)$ and have a true deconfining transition, where $< L_1 > \sim e^{-N}$. Unfortunately it is precisely in these higher dimensions that the simplest interacting bosonic theory does not present one with a conventional random surface representation at zero temperature. Nevertheless it may be fruitful (if a little naive) to re-examine the one-dimensional Weingarten model as a more or less conventional gauge theory. While the effectively 2 field theoretic degrees of freedom argued to underlie the $c = 26$ critical string \cite{26} are realised trivially in the $c = 1$ non-critical string, such an examination may shed light on a non-perturbative framework for string theory.

**Acknowledgements:** I am indebted to Tim Morris for arousing my (belated) interest in this problem and for generously sharing with me his results \cite{2}. I also thank Mike Newman and especially Igor Klebanov for valuable conversations. This work was supported by S.E.R.C.(U.K.) post-doctoral fellowship RFO/B/91/9033.
Figure Captions

Figure 1: Terms in $S_W$ (the propagator and $\lambda$-plaquette) and $S_C$ (the $\kappa_1, \kappa_2, \kappa_3$-collapsed plaquettes).

Figure 2: The auxiliary lattice for $D = 2$, such that $\Delta h_{1,2} = \pm 1$. Only terms from $S_W$ are shown.

Figure 3: The single-eigenvalue potential for $S_{D=1}$ near criticality.

Figure 4: The (extended) Dynkin diagram target lattices.

Figure 5: Feynman diagram vertices for $Z_{\hat{A}_0}$. The propagator is $<M_{ab}M_{cd}^\dagger> = \delta_{ad}\delta_{bc}$.
References

[1] D.Weingarten, Phys. Lett. 90 (1980) 280.

[2] T.R.Morris, unpublished (Spring 1990).

[3] M.R.Douglas and S.H.Shenker, Nucl. Phys. B335 (1990) 135.
    D.J.Gross and A.A.Migdal, Phys. Rev. Lett. 64 (1990) 127.
    E.Brézin and V.A.Kazakov, Phys. Lett. B236 (1990) 144.

[4] M.R.Douglas, Phys. Lett. B238 (1990) 176.

[5] D.J.Gross and N.Miljković, Phys. Lett. B238 (1990) 217.
    E.Brézin, V.A.Kazakov and A.Zamolodchikov, Nucl. Phys. B333 (1990) 673
    G.Parisi, Phys. Lett. B238 (1990) 209.
    P.Ginsparg and J.Zinn-Justin, Phys. Lett. B240 (1990) 333.

[6] D.J.Gross and I.R.Klebanov, Nucl. Phys. B344 (1990) 475.

[7] I.Klebanov and L.Susskind, Nucl. Phys. B309 (1988) 175.

[8] S.Dalley and T.R.Morris, Int. J. Mod. Phys. A5 (1990) 3929.

[9] J.Ambjorn, B.Durhuus and J.Fröhlich, Nucl. Phys. B257 (1985) 433.
    F.David, Nucl. Phys. B257 (1985) 45.
    V.A.Kazakov, Phys. Lett. B150 (1985) 282.

[10] G.E.Andrews, R.J.Baxter and J.P.Forrester, J. Stat. Phys. 35 (1984) 193.

[11] P.Ginsparg, HUTP-89/A027, Lectures at Trieste Spring School, April 1989.

[12] V.A.Kazakov, Mod. Phys. Lett. A4 (1989) 2125.

[13] A.M.Polyakov, Phys. Lett. B103 (1981) 207.

[14] V.A.Kazakov and A.A.Migdal, Nucl. Phys. B311 (1989) 171.

[15] T.R.Morris, Nucl. Phys. B356 (1991) 703.

[16] C.Itzykson and J-B.Zuber, J. Math. Phys. 21 (1980) 411.

[17] M.Mehta, Comm. Math. Phys. 79 (1981) 327.

[18] E.Brézin, C.Itzykson, G.Parisi and J-B.Zuber, Comm. Math. Phys. 59 (1978) 35.

[19] I.R.Klebanov, PUPT-1271, Lectures at Trieste Spring School, April 1991.

[20] I.Kostov, Saclay preprint SPhT/91-142.
[21] J.Ambjorn, J.Jurkiewicz and Y.Makeenko, Phys. Lett. B251 (1990) 517.

[22] S.Dalley, C.V.Johnson and T.R.Morris, Nucl. Phys. B368 (1992) 627.

[23] B.Svetitsky, Phys. Rep. 132 (1986) 1.

[24] B.Sathiapalan, Phys. Rev D35 (1987) 3277.
    J.J.Atick and E.Witten, Nucl. Phys. B310 (1988) 291.

[25] D.J.Gross and I.R.Klebanov, Nucl Phys. B359 (1991) 3.
    V.A.Kazakov and D.Boulatov, Ecole Normale Superieure preprint LPTENS-91-24.

[26] E.Witten, Royal Society Lecture, London (1988), IASSNS-HEP-88/55.