Modulation instability in quasi-two-dimensional spin–orbit coupled Bose–Einstein condensates

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Abstract
We theoretically investigate the dynamics of modulation instability (MI) in two-dimensional spin–orbit coupled Bose–Einstein condensates (BECs) with Rabi coupling. The analysis is performed for equal densities of pseudo-spin components. Different combination of the signs of intra- and intercomponent interaction strengths are considered, with a particular emphasis on repulsive interactions. We observe that the unstable modulation builds from originally miscible condensates, depending on the combination of the signs of the intra- and intercomponent interaction strengths. The repulsive intra- and intercomponent interactions admit instability and the MI immiscibility condition is no longer significant. The influence of interaction parameters such as spin–orbit and Rabi coupling on MI are also investigated. The spin–orbit coupling (SOC) inevitably contributes to instability regardless of the nature of the interaction. In the case of attractive interaction, SOC manifests in enhancing the MI. Thus, a comprehensive study of MI in two-dimensional spin–orbit coupled BECs of pseudo-spin components is presented.

Keywords: Bose–Einstein condensates, spin–orbit coupling, modulational instability

(Some figures may appear in colour only in the online journal)

1. Introduction

Study of spin–orbit (SO) coupled Bose–Einstein condensates (BECs) is one among the important topics of current research in the context of macroscopic quantum phenomena. Spin–orbit coupling (SOC) describes the interaction between the particle’s spin and orbital momentum and plays a crucial role in many physical phenomena in condensed matter systems including the spin-Hall effect, topological insulators, spintronics and so on [1]. The synthetic SOC in BECs was experimentally achieved very recently. In this realization technique, two Raman laser beams were used to couple with two-component BECs [2]. The momentum transfer between laser beams and atoms leads to synthetic SOC [3, 4]. SOC has been realized with cold atomic gases by designating the hyperfine atomic states as pseudo-spins and coupling them with Raman laser beams [5–7]. For instance, in the case of \(^{87}\text{Rb}\) the pseudo-spin states are \(|\uparrow\rangle = |F = 1, m_F = 0\rangle\) and \(|\downarrow\rangle = |F = 1, m_F = -1\rangle\), which are generated using pair of Raman laser beams.

The general properties of a BEC at absolute zero temperature can be well described, within the mean-field approximation, by the so-called Gross–Pitaevskii (GP) equation, a generalized form of the nonlinear Schrödinger equation for the condensate wavefunction. Many of the statics and dynamics of the condensates can be established by solving the GP equations both by analytical and numerical methods. One of the interesting dynamical features in the context of Bose–Einstein condensation which attracted a great deal of attention is the experimental observation of matter wave solitons [8]. On the other hand, in the case of multi-dimensional BEC geometry, there has been an intense theoretical and experimental activity on vortices and vortex structures in BECs with the self-repulsive intrinsic non-linearity [9]. This is due to the fact that vortices are intimately related to the superfluid properties of BECs, and play an...
important role in transport, dissipative dynamics and quantum turbulence [10].

SO-coupled BECs have been studied extensively in different contexts including phase separation, stripe phases [11], spotlighting the phase transition [12], vortices with or without rotations [13, 14], and so on. In addition, the study of topological excitations, for example, skyrmions, has also attracted much along these directions [15]. Further, matter wave solitons such as bright and dark solitons have been studied in quasi-one-dimensional with attractive and repulsive SO-coupled BEC [16, 17]. While these solitonic structures have been studied extensively in single and multi-component BEC [18, 19], SO coupling can significantly enrich their possibilities regarding the structural form of stability and dynamical properties. Recently, the solitonic structures studied in [20, 21] were embedded in different types of solitons; only possible due to the structure of the energy spectrum of SO-coupled BEC. In particular, positive mass dark and bright solitons were found for repulsive and attractive interactions, in either of the two regions of the energy spectrum, where the lowest energy band has either a single- or a double-well structure. It should be noted that most of the studies on SO-coupled BECs were primarily focused on quasi-one-dimensional systems. Only a few studies were devoted on multi-dimensional SO-coupled BECs. Recent studies on two-dimensional SO-coupled BECs of mixed Rashba–Dresselhaus type and Rabi couplings have earned particular interest [22–25]. Thus it is more appropriate and realistic to study SO-coupled BECs in two- and three-dimensional systems. Particularly, here we emphasize on the study of stability of plane waves SO-coupled BECs in two-dimensions (2D) [26].

The degree of instability in a BEC can be characterized by the modulation instability (MI), an instability process, and identified as a requisite mechanism to understand various physical effects in nonlinear systems. The phenomenon of MI was first observed in hydrodynamics by Benjamin and Feir [27]. Also, Ostrovskii predicted the possibility of MI in optics [28] and later explained in detail by Hasegawa et al in the context of optical fibers [29]. The MI is a general phenomenon occurring in many nonlinear systems and is of particular interest in dispersive nonlinear systems. In conventional dispersive nonlinear systems, MI manifests as a result of the constructive interplay between dispersion and nonlinearity such that any deviation from the steady state in the form of any weak perturbation leads to an exponential growth, resulting in a break-up of the carrier wave into trains of soliton-like pulses [30]. In addition, MI has been widely studied in various branches such as fluid dynamics [27], magnetism [31], plasma physics [32] and BEC [33].

In the context of BEC, MI has been given considerable importance over a long period of time, owing to its fundamental and applied interest in various aspects. In particular, MI has been found to be relevant in understanding the formation and propagation of solitonic waves [34], and also apparent in explaining the domain formation [35] and quantum phase transition [36]. MI has been studied extensively in BECs for both single [37] and two-component systems [38], and realized experimentally as well [39]. In the case of single-component BECs, MI has been found to be feasible only for attractive interaction, and in such case, the phase fluctuations caused by the MI leads to the formation of soliton trains. However, the breakthrough work by Goldstein and Meyrueis opens up the possibility of MI even for repulsive interactions [40], in similar lines to the case of cross-phase modulation induced instability in nonlinear optics [41, 42]. Thus, the two-component BEC finds particular interest in the study of MI, as it helps to achieve instability even in repulsive interactions. In the case of SO-coupled BEC, the MI in one-dimension was recently explored in [43], and the higher dimensional case is still an open problem. Thus, inspired by the special features of SO coupling and the physical relevance of a two-component BEC system, we intend to study the dynamical behavior of MI in SO-coupled BEC in two-dimensions. In this paper, we present a systematic study of MI in quasi-two-dimensional SO-coupled BECs with the inclusion of Rashba and Dresselhaus SO couplings. By considering a small perturbation approximation, we obtain a linearized GP equation. Further, the interplay between dispersion and nonlinear effects have been studied in terms of system parameters. We have also summarized the growth of MI gain for different combinations of intra- and inter-component of interaction strengths in the presence and absence of SO coupling.

The organization of the paper is as follows: after a detailed introduction in section 1, section 2 features the theoretical model for the case of SO-coupled BECs. In section 3, we present the MI dispersion relation through stability analysis, and systematically explained the effect of SOC and Rabi coupling for a different combination of intra- and inter-component interactions strength. Section 4, features the results and discussion followed by conclusion in section 5.

2. Theoretical model

We consider the SO-coupled BECs confined in a harmonic trap with equal Rashba and Dresselhaus couplings described, within the framework of mean-field theory by an energy functional of the following form [2]

\[ E = \int_{-\infty}^{\infty} d\tilde{x} d\tilde{y}, \]

where,

\[ \varepsilon = \frac{1}{2} (\Psi^* H_0 \Psi + \tilde{g}_1 |\psi|^4 + \tilde{g}_2 |\psi|^6 + 2\tilde{g}_12 |\psi|^4 |\psi|^2), \]

\[ \Psi = (\psi_1, \psi_2)^T \] is the condensate wavefunction, \( \psi_1 \) and \( \psi_2 \) are associated with the pseudo-spin components. The model Hamiltonian \( H_0 \) in (2) assumes the form,

\[ H_0 = \left[ \frac{\hat{\rho}^2}{2m} + V(\tilde{x}, \tilde{y}) \right] + \frac{\hbar \lambda}{2} \tilde{\sigma}_z - \frac{\hbar k_L}{m} \hat{\rho} \tilde{\sigma}_z \]

where, \( \hat{\rho} = -i\hbar (\partial_{\tilde{x}}, \partial_{\tilde{y}}) \) is the momentum operator,

\[ V(\tilde{x}, \tilde{y}) = \frac{1}{2} m [\omega_x^2 (\tilde{x}^2 + \tilde{y}^2) + \omega_y^2 \tilde{z}^2] \]

is a quasi-2D harmonic trapping potential, with \( \omega_x \gg \omega_y, \lambda \) is the frequency of
Raman coupling, \(a_{\sigma z}\) are Pauli spin matrices and \(\tilde{g}_L\) is the wavenumber of the Raman laser which couples the two hyperfine states. The effective two-dimensional coupling constant \(\tilde{g}_L = 4\pi\hbar^2 a_L/m\), \((i, j = 1, 2)\) represents the intracomponent \((\tilde{g}_{11}, \tilde{g}_{22})\) and intercomponent \((\tilde{g}_{12})\) interaction strengths, which are defined by the corresponding s-wave scattering lengths \(a_{ij}\) and atomic mass \(m\). Measuring energy in units of the radial trap frequency \(\omega_r\), i.e., \(\hbar/\omega_r\), length in units of harmonic oscillator length, \(a_0 = \sqrt{\hbar/(m\omega_r)}\), and time in units of \(\omega_r^{-1}\) the following dimensionless GP equations can be derived for different components of \(\psi_{1,2}\) from (2) as [44]

\[
\frac{i}{\hbar}\frac{\partial \psi_{1,2}}{\partial t} = \left[-\frac{1}{2}\nabla^2 + V(x, y) + g_{11}|\psi_{1,2}|^2 + g_{12}|\psi_{2,1}|^2\right]\psi_{1,2} + \imath k_L\frac{\partial \psi_{1,2}}{\partial x} + \Gamma \psi_{1,2},
\]

\[
(4a)
\]

where, \(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\), \(V(x, y) = (x^2 + y^2)/2\), \(x = x/a_0\), \(y = y/a_0\), \(\omega = \omega_r\), modified interactions strength \(g_{ij} = 4\pi N a_{ij}/a_0\), \(\psi_{1,2} = \psi_{1,2}(\sigma_{ij}a_{ij}^{1/2}/\sqrt{N})\). Here \(k_L = k_L a_0\) and \(\Gamma = \gamma/2\omega_r\) are the dimensionless SO and Rabi coupling parameters, respectively, and \(N\) is the total number of particles in the SO-coupled BECs, which is given by

\[
N = \int_{-\infty}^{\infty} \sum_{j=1,2} |\psi|^2 dx dy.
\]

In the following, we shall proceed with the study of modulational instability in the above two-dimensional model equation (4) for spin–orbit coupled BECs. The present results of the linear stability analysis can also be complemented with detailed numerical results, which can be done by extending the available numerical codes to solve the dimensionless GP equation in (4) without SO and Rabi coupling parameters [45]. However, we shall not carry out the numerical simulations in the present paper and will be considered for a future work.

3. Analysis of modulation instability

The fundamental framework of MI analysis relies on the stability analysis, such that the steady state solution is perturbed by a small amplitude/phase, and then study whether the perturbation amplitude grows or decays [30]. For this purpose, we consider a continuous wave state of miscible SO-coupled BECs with the two-dimensional density \(n_{j0} = |\psi_{j0}|^2\) of the form

\[
\psi_j(x, y, t) = \sqrt{n_{j0}} e^{-\imath \omega t}.
\]

Then the stability of the SO-coupled BECs can be examined by assuming perturbations in the wavefunctions as

\[
\psi_j(x, y, t) = (\sqrt{n_{j0}} + \delta \psi_j) e^{-\imath \omega t},
\]

\[
(6)
\]

A set of linearized equations for the perturbation can be obtained by using equation (6) in (4)

\[
i\frac{\partial (\delta \psi_j)}{\partial t} = -\frac{1}{2} \left(\frac{\partial^2 (\delta \psi_j)}{\partial x^2} + \frac{\partial^2 (\delta \psi_j)}{\partial y^2}\right) - i k_L \frac{\partial (\delta \psi_j)}{\partial x} + \Gamma \delta \psi_j + g_{11} n_{10} (\delta \psi_1 + \delta \psi_1^*) + g_{12} n_{20} (\delta \psi_2 + \delta \psi_2^*),
\]

\[
(7a)
\]

where the symbol * denotes complex conjugate.

In order to analyze the modulation instability, we assume an ansatz for the perturbations of the form

\[
\delta \psi_j = \zeta_j \cos(k_x x + k_y y - \Omega t) + \imath \eta_j \sin(k_x x + k_y y - \Omega t),
\]

\[
(8)
\]

where \(k_x\) and \(k_y\) are the wavenumbers and \(\zeta_j\) and \(\eta_j\) \((j = 1, 2)\) are the amplitudes of wavefunction, and \(\Omega\) is the eigen-frequency. We also assume that two pseudo-spin states of equal density \(n_{10} = n_{20} = n\). A straightforward substitution of equation (8) in (7) yields the following dispersion relation for \(\Omega\)

\[
\Omega^4 - \Omega^2 \left[\frac{1}{4} (k_x^2 - 2\Gamma) (k_y^2 + G_1 + G_2) + k_x^2 k_L^2 + 2\Gamma G_{12}\right] + \frac{\Omega}{2} k_x k_L (k_y^2 - 2\Gamma) (G_1 - G_2) + k_x^2 k_L^2 + 2\Gamma G_{12}
\]

\[
- \frac{1}{4} (k_x^2 - 2\Gamma) (k_y^2 + G_1 + G_2) + \left[\frac{k_x^2}{4} - \Gamma \left(\frac{1}{4} (k_x^2 + G_1) (k_y^2 + G_2) - G_{12}^2\right)\right]
\]

\[
- \frac{1}{2} k_x^2 k_L^2 \left[\frac{2\Gamma - k_x^2}{2} (G_1 + G_2 - 4\Gamma) + 2k_x^2 k_L^2 - k_x^4 + 4\Gamma (G_{12} + k_x^2 - \Gamma)\right] = 0,
\]

\[
(9)
\]

where \(k_x^2 = k_x^2 + k_y^2\) and \(G_1 = 4g_{11} n - 2\Gamma, G_2 = 4g_{22} n - 2\Gamma, G_{12} = 2g_{12} n + \Gamma\) are the modified intracomponent interaction strengths, respectively. For equal strengths of intracomponent interactions, i.e., \(a_{11} = a_{22}\) \((g_{11} = g_{22} = g)\), the dispersion relation recast into a simpler
where, \( G = G_1 = G_2 \). The above equation (10) is the dispersion relation corresponding to the stability of the miscible SO-coupled BECs. As is known from the theory of MI, the system exhibits a stable configuration for all real values of \( k_x \) and \( k_y \), if \( \Omega_+^2 \) is positive (\( \Omega_+^2 > 0 \)). If \( \Lambda > 0 \), the eigen-frequency \( \Omega_+ \) is always real but \( \Omega_- \) may be real or imaginary, which is dependent on \( \Delta \). If the eigenfrequencies \( \Omega_\pm \) have an imaginary part, the spatially modulated perturbation become exponential with time, as is obvious from the form of \( \delta\phi_j \). On the other hand, for negative value of \( \Lambda (\Lambda < 0) \), \( \Omega_+^2 \) does not need to be positive. In such case, \( \Omega_+^2 \) is characterized by the values of \( \Delta \). For \( \Lambda > 0 \) the value of lower branch \( \Omega_-^2 \) is negative if, \( \Delta < 0 \). Similarly for \( \Lambda > 0 \) the value of upper branch \( \Omega_+^2 \) is negative when \( \Delta < 0 \). Regardless of anything, \( \Omega_-^2 \) is always negative, and therefore, the MI sets in via the exponential growth of the weak perturbations. The MI gain is defined as \( \xi \equiv |\Im(\Omega_\pm)| \), where \( \Im(\ldots) \) denotes the imaginary part. It is quite obvious from the dispersion relation (10), characterizing MI, the instability/stability is extremely sensitive to the system parameters. This has been quite comprehensively illustrated both theoretically as well as experimentally in the literature. Depending on the nature of the system, different physical effects will come into play and influence the instability. In the current context, one can straightforwardly notice from the equation (10), that the dispersion relation and the instability spectra are substantially influenced by SO- and Rabi-coupling strengths. Also, the nature of the interaction strength is found to be fundamental to the existence of MI and plays a crucial role in characterizing the instability spectra. In this paper, we would like to discuss the interplay between different physical effects in the instability spectrum. In order to highlight the importance of the system parameters, we consider different combinations and presented the results in a self-explanatory manner. Thus, following the mathematical calculation pertaining to the dispersion relation corresponding to the instability/stability of the system, we dedicate the subsequent sections to the study of various effects in the instability spectra of SO-coupled BEC.

### 3.1. Effect of Rabi coupling in the MI of SO-coupled BECs

In order to study the effect of Rabi coupling in the MI, we turn off the SOC by making \( k_L = 0 \). For a better insight, we consider two special cases, (i) one without Rabi coupling \((\Gamma = 0)\) and (ii) another in the presence of Rabi coupling \((\Gamma \neq 0)\).

#### 3.1.1. Zero Rabi coupling

In absence of Rabi coupling \((\Gamma = 0)\), the eigenfrequencies of the system for \( k_x \neq k_y \) assumed the form,

\[
\Omega_\pm^2 = \frac{1}{2}[k^2(k^2 + 2n(g \pm g_{12}))]
\]

One can infer from the above equation (12), that based on the sign/nature of the interaction strength, \( \Omega_\pm \) may be real or imaginary. It is obvious from the combination of signs of intra- and intercomponent interactions, \( \Omega_\pm \) is found to be real in the following cases:

(i) both intra- \((g)\) and inter- \((g_{12})\) component interactions are repulsive \((g > 0 \text{ and } g_{12} > 0)\),

(ii) attractive intracomponent \((g < 0)\) and repulsive intercomponent interactions \((g_{12} > 0)\) with \(|g| \leq |g_{12}|\), and

(iii) repulsive intracomponent \((g > 0)\) and attractive intercomponent interactions \((g_{12} < 0)\) with \(|g| \geq |g_{12}|\).

For attractive intra- and intercomponent interactions \( \Omega_\pm \) becomes imaginary and thereby inevitably contributes to MI. However, \( \Omega_- \) becomes imaginary for all cases. Thus, as far as MI is concerned, \( \Omega_- \) contributes better to the instability in all means than the \( \Omega_+ \) counterpart. It is worth mentioning that, at \( k_x = 0 \), our results completely agree with the [43], and could reproduce the results of the MI in the conventional two-component system as in [38].

#### 3.1.2. Non-zero Rabi coupling

Next, we study the effect of Rabi coupling on MI by considering any finite value for \( \Gamma \) \((\Gamma \neq 0)\). Here the dispersion relation as given by the equation (10) can be modified as follows

\[
\Omega_\pm^2 = \frac{1}{2}[k^2(k^2 + 2n(g + g_{12}))]
\]

with \( k_x \neq k_y \), and in order to highlight the effect of Rabi coupling, the coefficient of SOC is turned off, i.e., \( k_L = 0 \). It is straightforward to notice that \( \Omega_\pm^2 \) given by the equation (13) for \( \Gamma = 0 \) is similar to the equation (12) for the case of zero Rabi coupling. Therefore, the instability/stability condition as defined by the zero Rabi coupling in the earlier section is completely applicable here as well. Hence, for non-zero Rabi coupling, \( \Omega_\pm^2 \) is not different from that of zero Rabi coupling, which implies that \( \Omega_\pm^2 \) is independent of \( \Gamma \). On the other hand, \( \Omega_\pm^2 \) is found to be significantly influenced by Rabi coupling and can be expressed as

\[
\Omega_\pm^2 = \frac{k^4}{4} + n(k^2 - 4\Gamma)(g - g_{12}) + 2\Gamma(2\Gamma - k^2).
\]

The effect of Rabi coupling from (14) can be better explored for three representative cases of \( \Gamma \), namely (i) \( \Gamma = 0 \), (ii)
$\Gamma > 0$, and (iii) $\Gamma < 0$. When $\Gamma = 0$, equation (14) reverts to the expression for $\Omega_\omega$. As given by the equation (12), and therefore, the analysis for zero Rabi coupling holds here also. On the other hand, when $\Gamma < 0$, $\Omega_\omega$ is purely imaginary for repulsive intra- and attractive intercomponent interactions, and for all other cases $\Omega_\omega$ does not contribute to MI, as it is real. However, the effect of Rabi coupling is more pronounced for $\Gamma > 0$, as the instability/stability conditions qualitatively differ from the previous cases. It is found that the $\Omega_\omega$ is unstable for all combination interactions, except the repulsive intra- and attractive intercomponent interactions. As the magnitude of intra- and intercomponent interaction strengths are identified to be crucial for defining MI. It is observed from equations (12)–(14), that for repulsive intra- and intercomponent interactions, the instability is possible only when the condition $|g| \geq |g_{ij}|$ is satisfied. Similarly, for attractive interaction, the condition for MI can be modified as $|g_{ij}| \geq |g|$.

For a better understanding of the effect of Rabi coupling, as a representative case, we have shown in figure 1, the MI gain corresponding to the repulsive intra- and intercomponent interactions with $k_1 = 0$, $\Gamma = 1$, $g = 2$, $g_{ij} = 1$ and $n = 1$. It should be noted that the condition for instability $(|g| \geq |g_{ij}|)$ in repulsive interactions is true for the above choice of parameters. It is evident from the existence of an instability region that the MI is caused by Rabi coupling for repulsive intra- and intercomponent interactions. It should be noted that the instability region is symmetric in momentum space on either side of the wavenumbers, $k_x$ and $k_y$.

Overall, it is apparent from the above discussion on the influence of Rabi coupling in the instability that out of the different choices of Rabi coupling strengths, the condition $\Gamma > 1$ is found to carry more information about the MI. Therefore, in the subsequent section we shall study the effect of SO coupling by fixing the Rabi coupling strength as $\Gamma = 1$. Following section we would like to briefly emphasize the effect of different combinations of intra- and intercomponent interaction strength with the inclusion of both Rabi ($\Gamma = 0$) and SOC ($k_1 = 0$). We consider following four representative cases to study MI in the SO-coupled BEC system.

1. Both repulsive intra- and intercomponent interactions ($g > 0$, $g_{ij} > 0$).
2. Attractive intra- and repulsive intercomponent interactions ($g < 0$, $g_{ij} > 0$).
3. Repulsive intra- and attractive intercomponent interactions ($g > 0$, $g_{ij} < 0$).
4. Both attractive intra- and intercomponent interactions ($g < 0$, $g_{ij} < 0$).

4.1. Repulsive intra- and intercomponent interactions

Here, we consider self-repulsive intra- ($g > 0$) and repulsive inter- ($g_{ij} > 0$) components of modified interactions $G_1$, $G_2$, and $G_{ij}$. Our investigation follows from the general dispersion relation for non-zero SO and Rabi coupling as given by the equation (10). It is apparent from equation (10) that the expression $\Omega_\omega$ can be real or complex depending on the sign of $\Delta > 0$ and $\Delta^2 + 4\Delta$. For $\Delta > 0$, the upper branch $\Omega_\omega$ will be imaginary only for $\Delta^2 + 4\Delta < 0$, and therefore contribute to MI. Figure 2 shows the MI gain for $\Omega_\omega$ as a function of one of the momentum component ($k_z$) for different values of intra-component ($g$) at fixed intercomponent interaction strength ($g_{ij} = 1$). The choice of parameters are $k_1 = \Gamma = 1$, $n = 1$, $g_{ij} = 1$ and $k_z = 1$. It is evident from figure 2 that there are two symmetrical instability regions on either side of the zeros of $k_z$ and $k_y$. As the intercomponent interaction strength increases further, the two instability region approaches to the zero wavenumber and merges into a single coalesced instability region with elevated gain at higher values of $g$.

On the other hand, $\Omega_\omega$ from equation (10) is more interesting, since $\Omega_\omega$ leads to an unstable region even for $\Delta^2 + 4\Delta > 0$. Figure 3 shows the MI gain for $\Omega_\omega$ as a function of momentum component for similar values as used for $\Omega_\omega$. One can easily see that there are two pairs of instability regions for $\Omega_\omega$, as against the single pair of

**Figure 1.** (a) Three-dimensional (3D) surface plot showing the MI gain, $\xi = |\Omega_\omega|$, and (b) the corresponding two-dimensional (2D) contour plot for the parameters $k_1 = 0$, $\Gamma = 1$, $n = 1$, $g = 2$ and $g_{ij} = 1$. **Figure 2.** Plots showing MI gain, $\Omega_\omega$, as a function of several values of intra-component ($g$) at fixed intercomponent interaction strength ($g_{ij} = 1$). The choice of parameters are $k_1 = \Gamma = 1$, $n = 1$, $g_{ij} = 1$ and $k_z = 1$. It is evident from figure 2 that there are two symmetrical instability regions on either side of the zeros of $k_z$ and $k_y$. As the intercomponent interaction strength increases further, the two instability region approaches to the zero wavenumber and merges into a single coalesced instability region with elevated gain at higher values of $g$. On the other hand, $\Omega_\omega$ from equation (10) is more interesting, since $\Omega_\omega$ leads to an unstable region even for $\Delta^2 + 4\Delta > 0$. Figure 3 shows the MI gain for $\Omega_\omega$ as a function of momentum component for similar values as used for $\Omega_\omega$. One can easily see that there are two pairs of instability regions for $\Omega_\omega$, as against the single pair of
Figure 2. Plot of the MI gain, $\xi = |\mathcal{J}(\Omega_\perp)|$, as a function of $k_y$ for different intracomponent interaction strengths with $k_x = 1$, $\Gamma = 1$, $n = 1$, $g_{12} = 1$ and $k_y = 1$.

Figure 3. Plot showing the MI gain, $\xi = |\mathcal{J}(\Omega_\perp)|$ as a function of $k_y$ for different $g$ with $k_x = 1$, $\Gamma = 1$, $n = 1$, $g_{12} = 1$ and $k_y = 1$.

instability regions observed for the case $\Omega_\perp$. As the strength of the intracomponent interaction increases, the instability region at the center unifies into a single band (similar to the case of $\Omega_\parallel$), while the other pair of instability regions at higher values of $k_y$ substantially decreases in gain and width of the instability region. For insight, we plot in figure 4 the 3D variation of MI gain for a range of $k_x$ and $g$ with the intercomponent interaction fixed ($g_{12}$). It is obvious that the two instability regions merge into a single instability region with elevated gain. To explore the effect of intercomponent interaction in the instability, in figure 5 we depict the MI gain for a range of $g_{12}$ with fixed $g$. It is apparent from figure 5 that, for smaller values of $g_{12}$, the gain in the inner instability band decreases gradually to zero, while the gain in the instability region at higher values of $k_y$ grows with an increase in $g_{12}$. In order to explore the effect of the wavenumbers, $k_x$ and $k_y$, we plot the MI gain as a function of $k_x$ for different values of $k_y$ and vice versa. Figures 6 and 7 show that the MI bands drift towards the center and coalesced into single instability band with the increase in the wavenumbers. Thus, there are no changes in the general trend of shifting of MI band for both cases, however the peak gain and the width of instability region substantially differs. One can infer that the maximum gain is observed for $k_x$, as evident from figure 6 in comparison to the plot of MI gain for $k_y$ in figure 7. Figure 8 shows the instability gain on the momentum space as a function of $k_x$ and $k_y$ for some representative values of intra- and intercomponent interaction strength. It is observed that there are two instability bands corresponding to $k_x$ and $k_y$. The inner pair of bands corresponds to $k_x$ with slightly higher gain than the outer band as a result of $k_y$. This combination of intra- and intercomponent interaction is of particular interest, because the instability is generally not feasible, as both interaction components are repulsive and therefore does not contribute to MI. However, the above results suggest that the MI is still possible even in the repulsive two-component BEC with the aide of SOC.

4.2. Attractive intracomponent and repulsive intercomponent interactions

This situation corresponds to that of BECs with attractive intra- and repulsive intercomponent interactions, which is subject to MI even in the absence of the SOC. Although SOC is not fundamental to the occurrence of MI in this particular case, it significantly affects the instability. The MI corresponding to $\Omega_\parallel$ produces the same number of bands as in the previous case for repulsive interaction. However, the MI corresponding to $\Omega_\perp$ qualitatively differs, and it can be better explained in the following two combination: (i) MI gain as a function of $g$ for fixed $g_{12}$; and (ii) variation of MI gain as a function of $g_{12}$ at constant value of $g$. Figure 9 shows the possibility of three pairs of instability bands for attractive intracomponent and constant $g_{12}$, and the instability bands grow in gain with the increase in $g$. Figure 10 depicts the MI gain for a range of repulsive intercomponent interaction strength ($g_{12}$) at constant $g$. It is obvious that there are two pairs of instability bands on either side of $k_y$, the inner one decreases with increase in $g_{12}$, while the outer one grow with the increase in $g_{12}$. The instability gain in momentum space for some representative values of intra- and intercomponent interaction strength is shown in figure 11. It is observed that there are three symmetric instability bands corresponding to $k_x$, while only two for $k_y$. Unlike the previous case, the instability gain is maximum for the bands corresponding to $k_y$ as shown in figure 11.

4.3. Repulsive intracomponent and attractive intercomponent interactions

Here, we consider the BECs with repulsive intracomponent ($g > 0$) and attractive intercomponent interaction ($g_{12} < 0$). It may be obvious from the earlier discussion that, in the absence of Rabi and SO coupling, both of them (through $\Omega_\perp$)
Figure 4. 3D surface plot of the MI gain, $\xi = |\mathcal{D}(\Omega_-)|$ in the $k_x$-$g$ plane and (b) the corresponding 2D contour plot for $k_L = 1$, $\Gamma = 1$, $n = 0.3$, $g_{12} = 1$ and $k_y = 1$.

Figure 5. 3D surface plotting showing the MI gain, $\xi = |\mathcal{D}(\Omega_-)|$, in the $k_x$-$g_{12}$ plane and (b) the corresponding 2D contour plot for $k_L = 1$, $\Gamma = 1$ for $n = 0.3$, $g = 1$ and $k_y = 1$.

Figure 6. Plot showing the MI gain, $\xi = |\mathcal{D}(\Omega_-)|$ as a function of $k_x$ for different $k_y$ values with $k_L = 1$, $\Gamma = 1$, $n = 0.3$, $g = 1$ and $g_{12} = 2$.

Figure 7. Plot of the MI gain, $\xi = |\mathcal{D}(\Omega_-)|$ as a function of $k_y$ for different $k_x$ values with $k_L = 1$, $\Gamma = 1$, $n = 0.3$, $g = 1$ and $g_{12} = 2$. 
are always stable. However, in the presence of SO and Rabi couplings, that is, $k_L = 0$ and $\Gamma > 0$, MI is found to occur regardless of the sign of the interaction strengths and one does not need to impose any further conditions. On the other hand, when $\Gamma < 0$, MI is observed provided that the condition $|g| > |g_{12}|$ is satisfied. In similar lines with the previous section, we discuss MI in the two particular cases, i.e. constant intracomponent interaction strength with varying intercomponent strength and vice versa. Figure 12 shows the MI gain at constant $g$ as a function of $g_{12}$. As the strength of the intercomponent interaction increases, the outer instability band grows and merges with the inner instability band of

Figure 8. (a) 3D surface plot showing the MI gain, $\xi = |\Im(\omega)|$, and (b) the corresponding 2D contour plot for the parameters $k_L = 1, \Gamma = 1, n = 0.3, g = 5$ and $g_{12} = 3$.

Figure 9. 3D surface plot of the MI gain, $\xi = |\Im(\omega)|$ in the $k_x-g$ plane and (b) the corresponding 2D contour plot for fixed $g_{12}$ with $k_L = 1, \Gamma = 1, n = 0.3, g_{12} = 1$ and $k_y = 1$.

Figure 10. 3D surface plot of the MI gain, $\xi = |\Im(\omega)|$ in the $k_x-g_{12}$ plane and (b) the corresponding 2D contour plot for fixed $g$ with $k_L = 1, \Gamma = 1, n = 0.3, g = -1$ and $k_y = 1$. 

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higher gain. The variation of MI gain for $g_{12}$ at constant $g$ shows a similar trend, except the changes in the numerical value of gain as shown in figure 13. Figure 14 depicts the MI gain in momentum space for $k_x$ and $k_y$. Unlike the earlier cases, the gain of the inner band is quantitatively the same for both bands corresponding to $k_x$ and $k_y$. However, the instability gain of the outer band corresponding to $k_y$ is slightly larger than the outer band of $k_x$.

4.4. Attractive intra- and intercomponent interactions

In this region, both intra- and intercomponent interactions are attractive, i.e., $g < 0$ and $g_{12} < 0$, and therefore, MI occurs naturally even without the aide of Rabi and SO coupling. This case has already been discussed thoroughly in the context of MI in SO-coupled BECs, and hence, an extensive investigation is not required. However, for the sake of completeness, we
focus on the effect of SOC in the instability. Figure 15 shows that the growth of MI gain with the variation of $g_{12}$ for constant $g = -1$. The current case completely concurs with our earlier discussion, and the MI becomes independent of the $g$ for the strong $g_{12}$ interaction. The variation of MI gain in momentum space is shown in figure 16. Like in the previous section, the MI bands are symmetric across the zero wavenumber, and the maximum gain occurs for the bands corresponding to $k_y$.

4.5. Results and discussion

For ease of understanding and to make the analysis self-explanatory, we summarize our results of MI in the two-
Table 1. Summary of MI in SO-coupled two-dimensional BECs.

| $\kappa$ | $\Gamma$ | MI gain ($\xi$) | Different combinations | Inference |
|---------|---------|-----------------|-----------------------|-----------|
|         |         | $[\delta(\Omega_0)]$ | both interactions are repulsive | Always stable |
|         |         | $[\delta(\Omega_-)]$ | $g < 0, g_{12} > 0$ | Always stable |
|         |         |                    | $g > 0, g_{12} < 0$ | Always stable |
|         |         | $[\delta(\Omega_+)]$ | both interactions are attractive | Always unstable |
|         |         | $[\delta(\Omega_-)]$ | both interactions are repulsive | Stable for all cases |
|         |         |                    | $g < 0, g_{12} > 0$ | Similar to the case |
|         |         | $[\delta(\Omega_+)]$ | both interactions are attractive | with $\Gamma = 0$ and $\Omega_+$ |
|         | $\Gamma = 0$ | $[\delta(\Omega_-)]$ | both interactions are repulsive | Unstable if $|g| > |g_{12}|$ |
|         |         |                    | $g < 0, g_{12} > 0$ | Always stable |
|         |         | $[\delta(\Omega_+)]$ | both interactions are attractive | Always unstable |
|         |         | $[\delta(\Omega_-)]$ | both interactions are repulsive | Stable for all cases |
|         |         |                    | $g < 0, g_{12} > 0$ | Similar to the case |
|         |         | $[\delta(\Omega_+)]$ | both interactions are attractive | with $\Gamma = 0$ and $[\delta(\Omega_-)]$ |
|         | $\Gamma < 0$ | $[\delta(\Omega_-)]$ | both interactions are repulsive | Unstable if $|g| > |g_{12}|$ |
|         |         |                    | $g < 0, g_{12} > 0$ | Always stable |
|         |         | $[\delta(\Omega_+)]$ | both interactions are attractive | Always unstable |
|         |         | $[\delta(\Omega_-)]$ | both interactions are repulsive | Unstable if $|g_{12}| > |g|$ |
|         |         |                    | $g < 0, g_{12} > 0$ | Always stable |
|         |         | $[\delta(\Omega_+)]$ | both interactions are attractive | Always stable |
|         | $\Gamma = 0$ | $[\delta(\Omega_-)]$ | both interactions are repulsive | Unstable if $|g_{12}| > |g|$ |
|         |         |                    | $g < 0, g_{12} > 0$ | Always stable |
|         |         | $[\delta(\Omega_+)]$ | both interactions are attractive | Always stable |
|         |         | $[\delta(\Omega_-)]$ | both interactions are repulsive | Unstable if $|g| > |g_{12}|$ |
|         |         |                    | $g < 0, g_{12} > 0$ | Always stable |
|         |         | $[\delta(\Omega_+)]$ | both interactions are attractive | Always stable |
|         | $\Gamma > 0$ | $[\delta(\Omega_-)]$ | both interactions are repulsive | Unstable if $|g| > |g_{12}|$ |
|         |         |                    | $g < 0, g_{12} > 0$ | Always stable |
|         |         | $[\delta(\Omega_+)]$ | both interactions are attractive | Always stable |
|         |         | $[\delta(\Omega_-)]$ | both interactions are repulsive | Unstable if $|g_{12}| > |g|$ |
|         |         |                    | $g < 0, g_{12} > 0$ | Always stable |
|         |         | $[\delta(\Omega_+)]$ | both interactions are attractive | Always stable |
|         |         | $[\delta(\Omega_-)]$ | both interactions are repulsive | Unstable if $|g| > |g_{12}|$ |
|         |         |                    | $g < 0, g_{12} > 0$ | Always stable |
|         |         | $[\delta(\Omega_+)]$ | both interactions are attractive | Always stable |
dimensional SO-coupled BECs in table 1. We systematically discussed the presence/absence of SO/Rabi coupling under different combinations of signs of intra- and intercomponent interactions strength. As is evident from our extensive investigation, SO coupling inevitably destabilizes the initial steady state for equal densities of BECs and thereby makes the system unstable for all combinations of interaction strength. Also, we have shown that the conventional MI immiscibility condition, \( g_{12} > g \), for a repulsive two-component BEC system is no longer significant for MI. Our particular focus is on repulsive intra- and intercomponent interactions, as it is proven to be stable against the perturbation, and therefore, MI is generally not feasible. However, we have shown that MI can be achieved with the effect of SOC, as demonstrated through figures 2–8. We discussed the MI gain in momentum space as a function of \( k_x \) and \( k_y \) and emphasized the variation of gain over the wavenumbers in the two directions. We noted that the MI gain is not identical on \( k_x \) and \( k_y \), and significant changes in MI gain, width of instability region and the number of instability bands are readily observed.

In section 4.2, we discussed the MI condition for attractive intra- and repulsive intercomponent interactions. Figures 9–11 show the instability gain as a function of \( g \) and \( g_{12} \). One can straightforwardly observe the emergence of new instability bands in the MI gain plot, which is identified to be the consequence of the incorporation of SOC. Following that, we discussed in section 4.3 the MI scenario in the case of repulsive intra- and attractive intercomponent interactions. Figures 12–14 portray the variation of MI gain for \( g \) and \( g_{12} \). Along similar lines with the earlier cases, the SOC results in new instability bands and thereby help in enhancing the MI in such systems. Finally, we studied MI in attractive intra- and intercomponent interactions in section 4.4. It is very well known from the theory of MI in BEC that attractive interactions naturally support MI. Although, SOC is not fundamental for the origin of MI; however, SOC significantly influences the instability region in terms of peak gain and width, as evident from figures 15 and 16. Overall, the effect of SOC can be understood as a means to achieve MI in repulsive interactions, and also enhance instability in the system.

Last but not least, it is also important to see the impact of SO and Rabi couplings on MI gain for fixed wavenumbers \( k_x \) and \( k_y \). Figure 17 depicts the MI gain as a function of SO and Rabi coupling. As it is evident from our choice of parameters, the instability bands are symmetric for both positive and negative values of Rabi and SO coupling. One can also infer, that MI is possible even for zero SOC, provided the Rabi coupling is \( \Gamma > 0 \).

It is always informative to discuss the possible experimental realization of the proposed theoretical results. Ever since the experimental realization of the matter wave solitons [46] and MI [39], BEC has seen dramatic developments in the past few decades. Courtesy of the sophisticated technologies, there were numerous studies performed in different settings [47]. SO-coupled BEC was one such recent example, where two Raman laser beams were used to couple with two-component BECs [2] and the momentum transfer between the laser beams and atoms leads to synthetic SOC [4]. As SO-coupled BEC is still in the early stages of investigation, most of the studies on SO-coupled BEC are at a theoretical level. On a par with the recent advancements in advanced technologies and the growing interest in the SOC–BEC, we believe our theoretical results could be of interest and can possibly set guidelines to realize MI experimentally in such systems.

5. Conclusions

To summarize, we have investigated the modulation instability in spin–orbit-coupled Bose–Einstein condensates with equal densities of pseudo-spin components in a quasi-two-dimensional setting. The dispersion relation corresponding to the instability of the flat continuous wave background against small perturbation was studied. For a comprehensive study, we have considered all the possible combination of signs of intra- and intercomponent interactions, with a particular emphasis on repulsive interactions.
Our analysis illustrates that spin–orbit coupling inevitably contributes to instability, regardless of the nature of the interaction strength. With detailed interpretation, we have shown that the repulsive intra- and intercomponent interactions admit instability and the MI immiscibility condition $g_{2} > g$ is no longer essential for MI. We also have shown, for the strong attractive intercomponent interaction, the nature of the intracomponent interaction is immaterial for constant spin–orbit and Rabi coupling. We also analyzed the variation of instability domain in momentum space for $k_{x}$ and $k_{y}$. The MI gain is not identical on $k_{x}$ and $k_{y}$, and significant changes in MI gain and a number of bands are observed. In the case of systems naturally admitting MI (attractive interactions), the spin–orbit and Rabi coupling manifest in the generation of new instability bands, thereby enhancing the MI. Thus, we presented a critical analysis with detailed interpretation and graphical illustration of MI in two-dimensional spin–orbit coupled BECs. The present results could potentially provide new ways to generate and manipulate MI and solitons in spin–orbit coupled BECs.

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