Gaussian entanglement of symmetric two-mode Gaussian states

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A Gaussian degree of entanglement for a symmetric two-mode Gaussian state can be defined as its distance to the set of all separable two-mode Gaussian states. The principal property that enables us to evaluate both Bures distance and relative entropy between symmetric two-mode Gaussian states is the diagonalization of their covariance matrices under the same beam-splitter transformation. The multiplicativity property of the Uhlmann fidelity and the additivity of the relative entropy allow one to finally deal with a single-mode optimization problem in both cases. We find that only the Bures-distance Gaussian entanglement is consistent with the exact entanglement of formation.

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I. INTRODUCTION

Intense recent work on the entanglement of two-mode Gaussian states (TMGS’s) was stimulated by the important result that preservation of the nonnegativity of their density matrix under partial transposition [2] is not only a necessary, but also a sufficient condition for their separability. Using the Sp (2, R) x Sp (2, R) invariant form of this criterion written by Simon [3], one can easily check whether a two-mode Gaussian state is separable or not [4]. In spite of considerable effort in using some of the accepted measures of entanglement to the Gaussian-state case, the only exact evaluation at present appears to be the entanglement of formation (EoF) for a symmetric TMGS [5]. In this particular case the EoF proved to be a monotonous function of the smallest symplectic eigenvalue of the covariance matrix of the partially transposed (PT) state. This eigenvalue will hereafter be denoted by $\tilde{\kappa}_-$. A computable inseparability measure for an arbitrary bipartite state was proposed by Vidal and Werner [6] in terms of the sum of the negative eigenvalues of the PT- density matrix. For TMGS’s, the absolute value of this sum, called negativity [6], is an expression depending only on $\tilde{\kappa}_-$. It is thus consistent to the EoF. As proved by Vidal and Werner, the negativity is an entanglement monotone.

The possibility of identifying the set of separable TMGS’s [8] paved the way to the application of the distance-type proposal for quantifying entanglement made by Vedral and co-workers [2]. A class of distance-type Gaussian measures of entanglement was defined with respect to only the set of Gaussian states. To the best of our knowledge, the first authors who used and evaluated numerically a Gaussian measure of entanglement were Scheel and Welsch in Ref.[8]. In our paper [9] co-authored with H. Scutaru, an explicit analytic Gaussian amount of entanglement was calculated for two-mode squeezed thermal states (STS’s) by using the Bures distance. We then employed the Gaussian approximation for the entropic entanglement of a two-mode STS and evaluated it in the pure-state case. Comparison to the von Neumann entropy of the subsystems (reduced modes) which was known to be the exact relative entropy of entanglement in the pure-state case [7], indicated an encouraging accuracy of the Gaussian approach. Note that the STS’s are important non-symmetric TMGS that can be produced experimentally and are used in the protocols for quantum teleportation.

Another Gaussian measure of entanglement, the Gaussian entanglement of formation (EoF) for an arbitrary TMGS was defined with respect to its optimal decomposition in Gaussian pure states [10]. Analytically, the Gaussian EoF was evaluated for symmetric TMGS’s and was shown to coincide with the exact expression given in Ref.[5]. Following the prescription of Ref.[10], an evaluation of the Gaussian EoF for a STS was given in the paper [11]. In the general case an insightful formula was not yet written.

One can notice that, for a symmetric TMGS, the amount of entanglement is fairly well described by monotonous functions (negativity, EoF, and Gaussian EoF) depending on $\tilde{\kappa}_-$. However, the situation is different for other special TMGS’s. In the STS case, the Gaussian entanglement measured by Bures distance [8] and the Gaussian EoF [11] are found to be in agreement. They are nicely depending on the same parameter, the difference between the two-mode squeeze parameter $r$ and its value $r_s$ defining the separability threshold. The parameter $r - r_s$ cannot be expressed in terms of only $\tilde{\kappa}_-$. Therefore, the negativity of a STS is not equivalent to the two Gaussian measures of its entanglement evaluated at present [8,11]. A similar disagreement between the Gaussian EoF and the negativity of the Gaussian states having extremal negativity at fixed global and local purities [12] was recently noticed in Ref.[13].

In this paper we will compare two distance-type Gaussian entanglement measures to the exact EoF for a symmetric
TMGS, checking thus on the validity of the Gaussian approach. We recall in Section 2 several aspects of two-mode Gaussian states such as the diagonalization of the CM for a symmetric TMGS under the beam-splitter transformation. As distances we employ the Bures distance in Sec. 3 and the relative entropy in Sec. 4. In Sec. 3, by using the properties of the Uhlmann fidelity, we can restrict the reference set of all separable TMGS’s to its subset of only symmetric TMGS’s. Application of the beam-splitter transformation to both the given inseparable state and the set of symmetric separable TMGS’s enables us to evaluate and maximize just a product of one-mode fidelities. Inspired by the results obtained in Sec. 3, we define and calculate in Sec. 4 an entropic Gaussian entanglement as the minimal relative entropy between a symmetric TMGS and the set of all separable symmetric TMGS’s. Our final conclusions are presented in Sec. 5.

II. TWO-MODE GAUSSIAN STATES

An undisplaced TMGS is entirely specified by its covariance matrix (CM) denoted by $\mathcal{V}$ which determines the characteristic function of the state

$$\chi_G(x) = \exp\left(-\frac{1}{2} x^T \mathcal{V} x\right). \quad (2.1)$$

with $x^T$ denoting a real row vector $(x_1, x_2, x_3, x_4)$. The superscript $T$ stands for transpose. $\mathcal{V}$ is a symmetric and positive $4 \times 4$ matrix which has the following block structure:

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_1 & \mathcal{C} \\ \mathcal{C}^T & \mathcal{V}_2 \end{pmatrix}. \quad (2.2)$$

Here $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{C}$ are $2 \times 2$ matrices. Their entries are correlations of the canonical operators $q_j = (a_j + a_j^\dagger)/\sqrt{2}$, $p_j = (a_j - a_j^\dagger)/(\sqrt{2}i)$, where $a_j$ and $a_j^\dagger$, $(j = 1, 2)$, are the amplitude operators of the modes. $\mathcal{V}_1$ and $\mathcal{V}_2$ denote the symmetric covariance matrices for the individual reduced one-mode STS’s [15], while the matrix $\mathcal{C}$ contains the cross-correlations between modes. The Robertson-Schrödinger form of the uncertainty relations for the canonical variables reads

$$\mathcal{V} + \frac{i}{2} \Omega \geq 0. \quad (2.3)$$

Here $\Omega$ is the $4 \times 4$ fundamental symplectic block-diagonal matrix

$$\Omega := \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.4)$$

From Eq. (2.3) we have [3, 14]

$$\det(\mathcal{V} + \frac{i}{2} \Omega) = \det\mathcal{V} - \frac{1}{4} (\det\mathcal{V}_1 + \det\mathcal{V}_2 + 2\det\mathcal{C}) + \frac{1}{16} \geq 0. \quad (2.5)$$

A factorized form of the condition (2.5) in terms of the symplectic eigenvalues $\kappa_+$ and $\kappa_-$ of the CM,

$$\det(\mathcal{V} + \frac{i}{2} \Omega) = \left(\kappa_+^2 - \frac{1}{4}\right) \left(\kappa_-^2 - \frac{1}{4}\right) \geq 0, \quad (2.6)$$

shows that $\kappa_+ \geq \kappa_- \geq 1/2$.

As stated by the separability criterion [3], a TMGS is separable if and only if the uncertainty relation (2.5) is satisfied by the partially transpose state (PTS) $\rho^{PT}$ whose CM is hereafter denoted by $\tilde{\mathcal{V}}$. Hence the separability condition is

$$\det(\tilde{\mathcal{V}} + \frac{i}{2} \Omega) = \det\mathcal{V} - \frac{1}{4} (\det\mathcal{V}_1 + \det\mathcal{V}_2 + 2\det\mathcal{C}) + \frac{1}{16} \geq 0. \quad (2.7)$$

Equivalently, it can be written in terms of the smallest symplectic eigenvalue of $\tilde{\mathcal{V}}$ as $\tilde{\kappa}_- \geq 1/2$. 
A. Standard forms of the CM

According to the important Lemma 1 in Ref.[14], the $4 \times 4$ covariance matrix of a Gaussian state may be cast into a standard form $V(I)$ by local symplectic transformations such that the submatrices $V_1$, $V_2$ are multiples of the $2 \times 2$ identity matrix $I$ and $C$ is diagonal. We have

$$V_1 = b_1 I, \quad V_2 = b_2 I, \quad C = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \quad \left( b_1 \geq \frac{1}{2}, \quad b_2 \geq \frac{1}{2} \right). \quad (2.8)$$

An obvious one-to-one correspondence can be found between the set of the four standard-form parameters $b_1$, $b_2$, $c$, $d$ appearing as entries in $V(I)$ and the set of the $\text{Sp}(2, \mathbb{R}) \times \text{Sp}(2, \mathbb{R})$ invariants ($\det V_1$, $\det V_2$, $\det C$, and $\det V$). According to Simon [3], entangled TMGS’s should have a negative $d$ parameter.

Another important form of the CM achieved by local squeezing transformations of the standard CM $V(I)$ was discovered by Duan et al. [14] and termed the standard form II, hereafter denoted by $V_{II}$. It describes a TMGS for which the separability and classicality conditions coincide. Generally, the classicality condition (existence of a well-behaved $P$ representation) is stronger than the separability one, Eq. (2.7). See our paper [4] for a more detailed analysis on this issue. It was proved in Ref.[14] that the squeezing factors $v_1, v_2$ defining the standard form II satisfy the algebraic system

$$\frac{b_1 (v_1^2 - 1)}{2b_1 - v_1} = \frac{b_2 (v_2^2 - 1)}{2b_2 - v_2}, \quad (2.9)$$

$$b_1 b_2 (v_1^2 - 1)(v_2^2 - 1) = (cv_1 v_2 - |d|)^2. \quad (2.10)$$

The solution of the system (2.9)–(2.10) for an arbitrary TMGS arises finally from a still unsolved eighth-order one-variable algebraic equation. However, it is possible to find $v_1, v_2$ in some particular cases.

B. Symmetric TMGS’s

When having $\det V_1 = \det V_2 = b^2$ we are dealing with symmetric TMGS’s. The standard parameters of the CM’s for symmetric TMGS’s are denoted as $b := b_1 = b_2$, $c > |d|$, $d = -|d|$. The symplectic eigenvalues of the CM are found to be

$$\kappa_+ = \sqrt{(b - |d|)(b + c)}, \quad \kappa_- = \sqrt{(b + |d|)(b - c)}. \quad (2.11)$$

Equations (2.9) and (2.10) can be solved for a symmetric TMGS. We readily get the squeezed factors in the standard form II

$$v_1 = v_2 = \sqrt{\frac{b - |d|}{b - c}}. \quad (2.12)$$

Equation (2.7) factorizes

$$\det(\tilde{V} + \frac{i}{2} \Omega) = \left[ (b - |d|)(b - c) - \frac{1}{4} \right] \left[ (b + |d|)(b + c) - \frac{1}{4} \right] \geq 0, \quad (2.13)$$

leading to the separability condition [14]

$$(b - |d|)(b - c) - \frac{1}{4} \geq 0. \quad (2.14)$$

Remark that

$$\tilde{\kappa}_- = \sqrt{(b - |d|)(b - c)} \quad (2.15)$$

is the smallest symplectic eigenvalue of the CM for the PTS.

The most important property of the CM of a symmetric TMGS is its diagonalization under a beam-splitter transformation. The possibility of using this nice property to evaluate a distance-type Gaussian entanglement was first
pointed out by de Oliveira in Ref. [16]. The optical effect of a lossless beam splitter is described by the wave mixing operator [17, 18]

$$B(\theta, \phi) = \exp \left[ -\frac{\theta}{2} (e^{i\phi} a_1^\dagger a_2 - e^{-i\phi} a_1 a_2^\dagger) \right]$$

(2.16)

with $\theta \in [0, \pi]$, $\phi \in (-\pi, \pi)$. Transformation of an arbitrary CM is governed by a $4 \times 4$ symplectic and orthogonal matrix $M(\theta, \phi) \in \text{SO}(4) \cap \text{Sp}(4, \mathbb{R})$

$$\tilde{V} = M^T \gamma M.$$  

(2.17)

The explicit form of $M(\theta, \phi)$ is given in Refs. [17, 18]. The CM of a symmetric state having equal local squeezing factors ($u = u_1 = u_2$) is diagonalized by the transformation (2.16) having the angles $\phi = 0$ and $\theta = \pi/2$. We obtain in a straightforward manner

$$\tilde{V}(u, u) = \text{diag}[(b + c)u, (b - |d|)/u, (b - c)u, (b + |d|)/u].$$

(2.18)

In the particular case of symmetric TMGS’s having the CM’s in the standard form II we get

$$\tilde{V}(II) = \text{diag} \left[ (b + c)\sqrt{\frac{b - |d|}{b - c}}, \tilde{\kappa}_-, \tilde{\kappa}_-, (b + |d|)\sqrt{\frac{b - c}{b - |d|}} \right].$$

(2.19)

### III. GAUSSIAN ENTANGLEMENT BY BURES METRIC

Vedral and co-workers [7] characterized the degree of inseparability of any bipartite state by its distance to the set of all separable states of the given system. Although the distance-type definition is an ideal measure of inseparability, one is usually forced to modify it by restricting the set of all separable states to a relevant one identified by a separability criterion. For the continuous-variable two-mode systems, a separability criterion was proved only for TMGS’s [3, 14]. We find thus natural to use the separable TMGS’s as reference set when defining an entanglement measure for a symmetric TMGS. All the states sharing the same local symplectic invariants have the same entanglement. For later convenience, we choose to evaluate the entanglement of a symmetric TMGS $\rho_s$ whose CM is in the standard form II. Its parameters are denoted by $b, c, d = -|d|$ and the standard-form II squeezing factors by $v_1 = v_2 = \sqrt{(b - |d|)/(b - c)}$. Among the defined distances [19] we concentrate now on those providing the best distinguishability of quantum states $\rho$. From this point of view, the strongest candidates are the Bures distance [20] and the relative entropy [7, 21]. We give here a short account of the results on considering the Bures metric as a measure of entanglement for symmetric Gaussian states recently obtained in our paper [24]. Recall that the Bures distance $d_B(\rho, \rho')$ between the density operators $\rho$ and $\rho'$ acting on a Hilbert space $\mathcal{H}_A$ originally introduced on mathematical grounds [20] was then written by Uhlmann [22] as

$$d_B(\rho, \rho') := [2 - 2\sqrt{\mathcal{F}(\rho, \rho')}]/2.$$  

(3.1)

In Eq. (3.1), $\mathcal{F}(\rho, \rho')$ is the Uhlmann fidelity [22, 23] of the two states. Uhlmann also derived an intrinsic formula of the fidelity [22]:

$$\mathcal{F}(\rho, \rho') = \left\{ \text{Tr}[\sqrt{\rho} \rho']^{1/2} \right\}^2.$$  

(3.2)

Following [7] we define the Bures-metric entanglement of the symmetric TMGS $\rho_s$

$$E_B(\rho_s) := \min_{\rho' \in \mathcal{D}^{sep}} \frac{1}{2} d_B^2(\rho_s, \rho') = 1 - \max_{\rho' \in \mathcal{D}^{sep}} \sqrt{\mathcal{F}(\rho_s, \rho')}.$$  

(3.3)

In Eq. (3.3) we have introduced the set $\mathcal{D}^{sep}$ of all separable scaled standard TMGS which is included in the set of all separable TMGS. The states belonging to the set $\mathcal{D}^{sep}$ have their CM’s of the type

$$V'(u_1', u_2') = \begin{pmatrix} b'_1 u'_1 & 0 & c' \sqrt{u'_1 u'_2} & 0 \\ 0 & b'_1 u'_1 & 0 & d' \sqrt{u'_1 u'_2} \\ c' \sqrt{u_1 u_2} & 0 & b_2 u'_2 & 0 \\ d' \sqrt{u_1 u_2} & 0 & 0 & b_2 u'_2 \end{pmatrix}, \quad (b'_1 \geq 1/2, b_2 \geq 1/2).$$  

(3.4)
Our task is to maximize the fidelity between the entangled symmetric TMGS \( \rho_s \) and a state \( \rho' \in D^{sep}_s \). As discussed in our paper \cite{24}, the closest separable state, say \( \rho'' \), has the property
\[
\kappa'' = 1/2. \tag{3.5}
\]

Among the remarkable general properties of the fidelity listed and largely discussed in Refs.\cite{10, 22, 23}, the following two ones proved to be especially important to our problem:

P1. \[ F(U \rho U^\dagger, U \rho' U^\dagger) = F(\rho, \rho'), \] (invariance under unitary transformations \( U \)).

P2. \[ F(\rho_1 \otimes \rho_2, \rho'_1 \otimes \rho'_2) = F(\rho_1, \rho'_1) F(\rho_2, \rho'_2), \] (multiplicativity).

In our paper \cite{24}, we have considerably simplified the minimization procedure required by Eq. (3.3) by showing that the closest separable scaled standard state \( \rho'' \) to a given symmetric scaled standard state having equal local squeeze factors \( u_1 = u_2 = u \) is a similar symmetric scaled standard state observing the threshold condition (3.5). Therefore, the amount of Gaussian entanglement for a symmetric TMGS can be calculated in a simpler way, because the separable reference set \( D^{sep}_s \) used in Eq. (3.3) is in fact restricted to the set \( D^{sep}_s \) of symmetric scaled standard states. We have then used the property P1. of the fidelity with respect to the beam-splitter transformation (2.16) at the angles \( \phi = 0 \) and \( \theta = \pi/2 \). The CM’s of the given state \( \rho_s \) and any equally scaled symmetric state \( \rho' \in D^{sep}_s \) became diagonal. The multiplicativity property P2. allowed us to reduce the evaluation of fidelity to a single-mode problem. The maximal fidelity was finally obtained in an elegant manner due to our choice for the given state \( \rho_s \) (namely the symmetric TMGS having the CM in the standard form II):

\[
\max_{\rho'' \in D^{sep}_s} F(\rho_s, \rho'') = \frac{2\kappa_-}{(\kappa_- + 1/2)^2} \tag{3.6}
\]

We found that the Gaussian degree of entanglement measured by the Bures distance,

\[
E_B(\rho_s) = \frac{(\sqrt{2\kappa_-} - 1)^2}{2\kappa_- + 1}, \quad \kappa_- < 1/2, \tag{3.7}
\]

depends only on the smallest symplectic eigenvalue \( \kappa_- \) of the covariance matrix of the PTS. It is thus in agreement with the exact expression of the entanglement of formation for symmetric TMGS’s.

IV. GAUSSIAN RELATIVE ENTROPY OF ENTANGLEMENT

The relative entropy of a state \( \rho' \) with respect to the state \( \rho \) is defined as

\[
S(\rho'/\rho) := \text{Tr}[\rho (\ln \rho - \ln \rho')]. \tag{4.1}
\]

It is evident that the relative entropy is not a true metric because it lacks for symmetry. Among the important properties of the relative entropy proved and discussed in the classic paper of Wehrl \cite{21} and the more recent ones of Vedral et. al. \cite{22}, we shall use here the following ones:

\begin{enumerate}
  \item[II1:] \( S(\rho'/\rho) = S(U \rho U^\dagger/U \rho' U^\dagger) \), (invariance under unitary transformations \( U \))
  \item[II2:] \( S(\rho_1 \otimes \rho_2/\rho_1 \otimes \rho_2) = S(\rho_1/\rho_1) + S(\rho_2/\rho_2) \). (additivity)
\end{enumerate}

A. Defining Gaussian relative entropy of entanglement

In Ref.\cite{7}, the minimal relative entropy between a state of a two-component system and the set of all separable states, now called the relative entropy of entanglement, was proved to be a good measure of entanglement. The minimization process was realized in the important case of the pure states. For mixed ones no exact result could be found so far. In order to perform a comparison to the Bures-metric entanglement, we consider the same reference set \( D^{sep}_s \) of separable states and the same given entangled state \( \rho_s \) and define the Gaussian relative entropy of entanglement

\[
E_S(\rho_s) := \min_{\rho'' \in D^{sep}_s} S(\rho'/\rho_s). \tag{4.2}
\]

Definition (4.2) allows us to use the simultaneous diagonalization of the CM’s under the beam-splitter transformation at the angles \( \phi = 0 \) and \( \theta = \pi/2 \) as a consequence of the property II1. The transformed state of \( \rho_s \) will be denoted
by \( \hat{\rho}_s \) and has the diagonal CM written as Eq. (2.19). The transformation of an arbitrary state \( \rho' \in D_s^{\text{rep}} \) leads us to the state \( \hat{\rho}' \) which is described by the CM

\[
V_{\hat{\rho}'} = \text{diag} [2(b' + c')(b' - |d'|), 1/2, 1/2, 2(b' + |d'|)(b' - c')] ,
\]

where the separability threshold condition (3.5) was inserted. Equation (4.2) becomes

\[
E_S(\rho_s) = E_S(\hat{\rho}_s) = \min_{\rho' \in D_s^{\text{rep}}} S(\rho' / \hat{\rho}_s). \tag{4.4}
\]

A diagonal \( 4 \times 4 \) CM describes in fact a product-state. Let us denote by \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) the reduced one-mode states of \( \hat{\rho}_s \). According to Eq. (2.19), the CM’s of the states \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) are, via Eqs. (2.11) and (2.15),

\[
V_{\hat{\rho}_1} = \begin{pmatrix} \kappa_+^2 & 0 \\ \kappa_- & 0 \end{pmatrix}, \quad V_{\hat{\rho}_2} = \begin{pmatrix} \kappa_- & 0 \\ 0 & \kappa_+^2 \end{pmatrix}. \tag{4.5}
\]

Equation (4.5) nicely depends on both symplectic eigenvalues of the CM and also on the smallest symplectic eigenvalue of the partially transposed CM. Similarly, the state \( \rho' \) has the structure \( \rho' = \hat{\rho}_1 \otimes \hat{\rho}_2 \) with

\[
V_{\rho'} = \begin{pmatrix} 2(\kappa_+^2) & 0 \\ 0 & 1/2 \end{pmatrix}, \quad V_{\rho'} = \begin{pmatrix} 1/2 & 0 \\ 0 & 2(\kappa_-)^2 \end{pmatrix}. \tag{4.6}
\]

We apply now the additivity property \( \pi 2 \) of the relative entropy and get the interesting result

\[
S(\rho' / \hat{\rho}_s) = S(\rho'_1 / \hat{\rho}_1) + S(\rho'_2 / \hat{\rho}_2). \tag{4.7}
\]

Therefore, as in the case of Bures-metric entanglement discussed in Sec.3, evaluation of the entropic Gaussian entanglement of symmetric TMGS's is a one-mode problem. We now take advantage of having previously derived a general formula for the relative entropy between two one-mode Gaussian states. The derivation was rigorously performed in our paper (25) co-authored with H. Scutaru in order to define an entropic degree of nonclassicality for one-mode Gaussian states.

### B. Evaluating Gaussian relative entropy of entanglement

We adapt Eq. (A 14) from the Appendix of Ref. (25) to the present case of two undisplaced one-mode states having diagonal \( 2 \times 2 \) CM’s of the type

\[
V := \begin{pmatrix} \sigma_{qq} & 0 \\ 0 & \sigma_{pp} \end{pmatrix}, \quad V' := \begin{pmatrix} \sigma'_{qq} & 0 \\ 0 & \sigma'_{pp} \end{pmatrix}. \tag{4.8}
\]

In Eq. (4.8), the entries of the CM’s are expectation values of the canonical operators such as \( \sigma_{pp} = \langle \hat{p}^2 \rangle \). We readily get

\[
S(\rho' / \rho) = S_N(\rho) + \frac{1}{2} \ln \left[ \frac{\sqrt{\det V'}}{\sqrt{\det V}} \right] \left[ 1 + \frac{\sigma_{qq}\sigma'_{pp} + \sigma_{pp}\sigma'_{qq}}{\sqrt{\det V'}} \right] \left[ 1 - \frac{\sigma_{qq}\sigma'_{pp} + \sigma_{pp}\sigma'_{qq}}{\sqrt{\det V'}} \right]. \tag{4.9}
\]

Here we have introduced the von Neumann entropy \( S_N(\rho) := -\text{Tr}(\rho \ln \rho) \). For a one-mode Gaussian state we have

\[
S_N(\rho) = \left( \sqrt{\det V + 1/2} \right) \ln \left( \sqrt{\det V + 1/2} \right) - \left( \sqrt{\det V - 1/2} \right) \ln \left( \sqrt{\det V - 1/2} \right). \tag{4.10}
\]

By using Eq. (4.9) via Eqs. (4.5) and (4.5), we have for the two-mode relative entropy, Eq. (4.7),

\[
S(\rho' / \hat{\rho}_s) = -S_N(\rho_1) - S_N(\rho_2)
\]

\[
+ \frac{1}{2} \left[ \ln \left( x_1 + \frac{1}{2} \right) \left[ 1 + \frac{\kappa_+^2 + 4x_1^2(\kappa_-)^2}{2x_1\kappa_-} \right] + \ln \left( x_1 - \frac{1}{2} \right) \left[ 1 - \frac{\kappa_+^2 + 4x_1^2(\kappa_-)^2}{2x_1\kappa_-} \right] \right]
\]

\[
+ \frac{1}{2} \left[ \ln \left( x_2 + \frac{1}{2} \right) \left[ 1 + \frac{\kappa_-^2 + 4x_2^2(\kappa_-)^2}{2x_2\kappa_-} \right] + \ln \left( x_2 - \frac{1}{2} \right) \left[ 1 - \frac{\kappa_-^2 + 4x_2^2(\kappa_-)^2}{2x_2\kappa_-} \right] \right]. \tag{4.11}
\]
In Eq. (4.11) we have denoted

\[ x_1 = \kappa'_+, \quad x_2 = \kappa'_- \]  \hspace{1cm} (4.12)

the symplectic eigenvalues of the separable state \( \tilde{\rho} \). The minimization of the relative entropy required by the definition of the symplectic eigenvalues of the separable state \( \tilde{\rho} \) will be performed with respect to the variables \( x_1 \) and \( x_2 \) which are independent and separate in the expression (4.11) as a consequence of the additivity property (4.7). Therefore, we can formulate the following statement:

The Gaussian relative entropy of entanglement of a TMGS which can be unitarily transformed in a product–state equals the sum of the nonclassicality degrees of the transformed one-mode reductions:

\[ E_S(\rho_s) = Q_S(\tilde{\rho}_1) + Q_S(\tilde{\rho}_2), \]  \hspace{1cm} (4.13)

where \( Q_S(\rho) \) is the entropic nonclassicality degree of the one-mode Gaussian state \( \rho \) defined and evaluated in our paper [25]. Note that in order to find the absolute minimum of the relative entropy one has to solve a transcendental equation, which does not have an exact analytic solution, but can be analyzed graphically as in Ref. [25]. Equation (4.11) displays a dependence on the symplectic eigenvalues \( \kappa_-, \kappa_+ \) and the smallest symplectic eigenvalue \( \tilde{\kappa}_- \) of the PTS. The transcendental nature of the minimization condition suggests that the Gaussian relative entropy of entanglement will not depend only on \( \tilde{\kappa}_- \). Note finally that in Ref.[26], a general formula for the relative entropy of TMGS’s was given by exploiting the exponential form of their density operators.

V. CONCLUSIONS

The symmetric TMGS is the only continuous-variable state for which an exact measure of entanglement is evaluated at present. Its EoF was found as depending only on \( \tilde{\kappa}_- \) [5]. In this paper we have analyzed two distance-type Gaussian degrees of entanglement for a symmetric TMGS in order to compare our results to the exact EoF and check on the validity of the Gaussian approach. The principal property that enabled us to evaluate both Bures distance and relative entropy between two symmetric TMGS is the possibility of diagonalizing their CM’s under the same beamsplitter transformation. Calculation was simplified by considering the given state with its CM in the standard form II. Notice that this form of the CM is involved in giving an inseparability criterion for TMGS [14]. The remarkable multiplicativity property of Uhlmann fidelity and the additivity of the relative entropy allowed us to deal with a single-mode optimization problem in both cases. Our result for the Gaussian degree of entanglement measured by Bures distance depends only on the smallest symplectic eigenvalue of the covariance matrix of the PTS. Thus, it is in agreement with the exact EoF found in Ref[5] and enforces our previous idea [9] that the Bures distance is a reliable measure of entanglement in the Gaussian approximation.

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