Integrability and chemical potential  
in the (3+1)-dimensional Skyrme model

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August 28, 2017

Abstract

Using a remarkable mapping from the original (3+1)dimensional Skyrme model to the Sine-Gordon model, we construct the first analytic examples of Skyrmions as well as of Skyrmions–anti-Skyrmions bound states within a finite box in 3+1 dimensional flat space-time. An analytic upper bound on the number of these Skyrmions–anti-Skyrmions bound states is derived. We compute the critical isospin chemical potential beyond which these Skyrmions cease to exist. With these tools, we also construct topologically protected time-crystals: time-periodic configurations whose time-dependence is protected by their non-trivial winding number. These are striking realizations of the ideas of Shapere and Wilczek. The critical isospin chemical potential for these time-crystals is determined.

1 Introduction

One of the most beautiful examples of the interplay between topology and field theory is provided by the Skyrme theory \cite{Skyrme}. Its importance in nuclear and particle physics (for instance see \cite{2,3,4,5,6,7}) arises from its relations with low energy QCD \cite{8} and it is the prototype of a non-integrable model. The non-trivial role of topology appears since the Skyrme model supports solitons, called Skyrmions, which are topologically stable and represent Baryonic degrees of freedom (see \cite{9,10,11,12,13,14,15} and references therein). The identification of the Baryon number in particle physics with the third homotopy class of the Skyrmion \cite{8} showed that the original Skyrme intuition was correct.

Moreover, the ideas of Skyrme have nontrivial applications not only in particles and nuclear physics but also in many other areas of physics. In \cite{16} it was found that skyrmion lattices exist in semiconductors. What is more, in an antiferromagnetic spinor Bose-Einstein condensate a two-dimensional skyrmion was observed \cite{17} to be stable on a short time. Applications of Skyrme model are also
encountered in gravitational physics, for instance in cosmology \cite{18, 19} or black holes physics where it has been found that the “no hair” conjecture can be violated \cite{20, 21, 22, 23, 24, 25}. A further field of research in which the role of Skyrmions is extremely relevant is the analysis of magnetic materials (a nice review being \cite{26}).

The original Skyrme model is very far from being integrable and only very few explicit analytic results are known. In particular, there is no explicit solution with non-vanishing Baryon number on flat space-times. Consequently, the Skyrme phase diagrams (which could provide very valuable informations on nuclear matter) are very difficult to approach with analytical methods.

By using the original spherical hedgehog ansatz of Skyrme, Klebanov proposed a phenomenological approach to the analysis of Skyrmions at finite density a long time ago \cite{27}. By following his point of view and using the technique of \cite{28, 29}, a non-vanishing isospin chemical potential was introduced in \cite{30, 31}. However, both finite volume effects and isospin chemical potential break spherical symmetry, and this fact makes difficult to apply the spherical hedgehog ansatz in these cases. Without an appropriate ansatz (which is both topologically non-trivial and non-spherically symmetric) it becomes very hard to derive analytic results on how the Isospin chemical potential as well as finite-volume effects affect the behavior of Skyrmions.

In recent years, a generalized hedgehog ansatz was proposed, which provides with a strategy to construct a non-spherical hedgehog-like ansatz with the right properties (see \cite{32, 33, 34, 35, 36, 37, 38, 39, 40, 41} and references therein) both in the Skyrme and in the Yang-Mills-Higgs cases. By using this technique, we construct analytic and topologically non-trivial solutions of the Skyrme model without spherical symmetry living within a finite box in flat space-times. We also construct analytic Skyrmions–anti-Skyrmions bound states. We derive a bound on the number of Skyrmions–anti-Skyrmions bound states in terms of the coupling constant. The isospin chemical potential can be included keeping, at the same time, the nice properties of the ansatz. The critical isospin chemical potential, beyond which the Skyrmion living in the box ceases to exist, can be explicitly determined.

Remarkably, the generalized hedgehog ansatz allowed us to construct novel types of topological configurations of the Skyrme model that can be defined as topologically protected time crystals, see below for more information.

The idea of time crystal was introduced by Wilczek and Shapere in \cite{42, 43, 44}, based on the following observations. Spontaneous Symmetry Breaking is a general property of nature that manifests itself in many different situations. It refers to situations where the observed configurations of a given system possess less symmetries than the corresponding action. They proposed the following very interesting and intriguing question: *is it possible to break spontaneously time translation symmetry?*

Powerful no-go theorems \cite{45, 46} ruled out the original proposals but they inspired a huge number of physicists to open new research lines (a nice review is \cite{47}). New types of time crystals in condensed matter physics have been proposed and realized in laboratories \cite{48, 49, 50, 51, 52, 53} (an up-to-date list of references can be found in \cite{47}). However, no example in nuclear and particles physics has been considered so far.

Here we show explicitly that, when finite volume effects are taken into account, the Skyrme model predicts the existence of a new type of time-crystal. These are exact time-periodic configurations of the
(3+1)-dimensional Skyrme model that cannot be deformed continuously to the trivial vacuum as they possess a non-trivial winding number extending along the time-direction (which is a sort of Lorentzian version of the Euclidean instanton number). Due to topological reasons, these time crystals can only decay into other time-periodic configurations: correspondingly, the time-periodicity is topologically protected. Hence the name *topologically protected time crystals*.

This paper is organized as follows: in section 2 we introduce the Skyrme action. In section 3, we discuss the sine-Gordon mapping and the effects of the chemical potential. In section 4, we describe the topologically protected time-crystals. In section 5, we draw some concluding ideas.

## 2 The Skyrme Model

We consider the $SU(2)$ Skyrme model in four dimensions. The action of the system is

\[
S = \frac{K}{2} \int d^4x \sqrt{-g} \left[ \frac{1}{2} \text{Tr} \left( R^\mu R_\mu \right) + \frac{\lambda}{16} \text{Tr} \left( G^\mu_\nu G^\nu_\mu \right) \right],
\]

(1)

\[
R_\mu = U^{-1} \nabla_\mu U, \quad G^\mu_\nu = [R_\mu, R_\nu],
\]

(2)

\[
U \in SU(2), \quad R_\mu = R^i_\mu t_j, \quad t_j = i \sigma_j,
\]

(3)

where $\sqrt{-g}$ is the (square root of minus) the determinant of the metric, the positive parameters $K$ and $\lambda$ are fixed experimentally and $\sigma_j$ are the Pauli matrices. In our conventions $c = \hbar = 1$, the space-time signature is $(-, +, +, +)$ and Greek indices run over space-time. The stress-energy tensor is

\[
T^\mu_\nu = -\frac{K}{2} \text{Tr} \left[ R_\mu R_\nu - \frac{1}{2} g^\mu_\nu R^\alpha R_\alpha + \frac{\lambda}{4} \left( g^{\alpha\beta} G^\alpha_\mu G^\beta_\nu - \frac{g_{\mu\nu}}{4} G^{\sigma\rho} G^\sigma_\mu G^\rho_\nu \right) \right],
\]

and the matter field equations are

\[
\nabla^\mu \left( R_\mu + \frac{\lambda}{4} [R^\nu, G^\nu_\nu] \right) = 0.
\]

(4)

We adopt a standard parametrization of the $SU(2)$-valued scalar $U(x^\mu)$

\[
U^{\pm 1}(x^\mu) = Y^0(x^\mu) I \pm Y^i(x^\mu) t_i, \quad (Y^0)^2 + Y^i Y_i = 1,
\]

(5)

where $I$ is the $2 \times 2$ identity and

\[
Y^0 = \cos C, \quad Y^i = n^i \cdot \sin C,
\]

\[
n^1 = \sin F \cos G, \quad n^2 = \sin F \sin G, \quad n^3 = \cos F.
\]

(6)

(7)

The Skyrme field possesses a non-trivial topological charge which, mathematically, is a suitable homotopy class or winding number: its explicit expression as an integral over a suitable three-dimensional
hypersurface $\Sigma$ is

$$W = -\frac{1}{24\pi^2} \int_{\Sigma} \varepsilon^{ijk} \text{Tr} \left( U^{-1} \partial_i U \right) \left( U^{-1} \partial_j U \right) \left( U^{-1} \partial_k U \right) = -\frac{1}{24\pi^2} \int_{\Sigma} \rho_B ,$$

where the baryon density is defined by $\rho_B = 12 \sin^2 C \sin F \ dC \wedge dF \wedge dG$. A necessary condition to have a non-vanishing baryon density is $dC \wedge dF \wedge dG \neq 0$.

When, in the above integral, the three-dimensional hypersurface $\Sigma$ is space-like then the topological charge is interpreted as Baryon number. However, due to the fact that $\rho_B$ does not depend on the metric, there are two further options: $\Sigma$ can be time-like or light-like. The last two possibilities have not been explored so far in the literature. In fact they are extremely interesting as whenever $W \neq 0$ (whether $\Sigma$ is space-like, time-like or light-like) the corresponding Skyrme configuration has a non-trivial homotopy and, consequently, cannot be deformed continuously into the trivial vacuum $U = I$. The cases in which $\Sigma$ is time-like and $W \neq 0$ correspond to topologically protected time crystals as it will be explained below. We will only consider an ansatz in which $\rho_B \neq 0$.

The natural generalization of the hedgehog ansatz introduced in [37] in the cases in which the metric is flat reads

$$G = \frac{\gamma + \phi}{2} , \quad \tan F = \frac{\tan H}{\cos A} , \quad \tan C = \frac{\sqrt{1 + \tan^2 F}}{\tan A} ,$$

where

$$A = \frac{\gamma - \phi}{2} , \quad H = H (r, z) .$$

It can be verified directly that, the topological density $\rho_B$ is non-vanishing. From the standard parametrization of $SU(2)$ ([54]) it follows that

$$0 \leq \gamma \leq 4\pi , \quad 0 \leq \phi \leq 2\pi ,$$

while the boundary condition for $H$ will be discussed below.

3 Sine-Gordon and Skyrmions

Let us consider the following flat metric

$$ds^2 = -dz^2 + \ell^2 \left( dr^2 + d\gamma^2 + d\phi^2 \right) ,$$

(in this section $z$ is the time variable). The length $\ell$ represents the size of the box where the Skyrmion lives. The coordinates $r$, $\gamma$ and $\phi$ are angular coordinates; the domain of $\gamma$ and $\phi$ is given by [11], while for $r$ we choose the finite interval $0 \leq r \leq 2\pi$.

The full Skyrme field equations (4) with the generalized hedgehog ansatz in Eqs. (6), (7), (9) and
reduce to just one scalar differential equation for the profile $H$

$$\Box H - \frac{\lambda}{8 \ell^2 (\lambda + 2 \ell^2)} \sin (4H) = 0,$$

where $\Box$ is the two-dimensional D’Alambert operator.

The energy of the configuration is given by

$$E = \int \ell^3 T^{00} dr d\gamma d\phi,$$

where

$$T^{00} = \frac{K}{64 \ell^4} \left[ 16(\lambda + 2 \ell^2) \left( (\partial_r H)^2 + \ell^2 (\partial_z H)^2 \right) + \lambda (1 - \cos(4H)) + 16 \ell^2 \right].$$

The topological Baryon charge $B$ and charge density $\rho_B$ become respectively

$$B = -\frac{1}{24\pi^2} \int_{t=\text{const}} \rho_B , \quad \rho_B = 3 \sin(2H)dHd\gamma d\phi .$$

If we replace the topologically non-trivial ansatz in Eqs. (6), (7), (9) and (10) into the original action (1) we obtain an effective action given by

$$\mathcal{L}(H) = 16 \ell^2 (\lambda + 2 \ell^2) \nabla_\mu H \nabla^\mu H - \lambda \cos(4H),$$

which reproduces equation of motion (13). The boundary conditions for the function $H$ are

$$H(0) = 0 , \quad H(2\pi) = \pm \frac{\pi}{2},$$

which corresponds to $B = \pm 1$ and

$$H(0) - H(2\pi) = 0 ,$$

which corresponds to $B = 0$. The sector $B = 0$ is relevant in the construction of Skyrmion anti-Skyrmion bound states.

Taking into account that the Skyrme model in (3+1) dimensions is the prototype of non-integrable systems, the above results in Eqs. (13), (15) and (17) are quite remarkable since they show that the full (3+1)-dimensional Skyrme field equations, energy density and effective action in a topologically non-trivial sector (as $\rho_B \neq 0$) can be reduced to the corresponding quantities of the (1+1)-dimensional sine-Gordon model. The latter is a well known example of integrable models, see [55] for a detailed review. In particular, it is trivial to construct kink-like solutions of Eq. (13) satisfying the boundary conditions in Eq. (18) (see [55] and references therein) and which (due to Eq. (16)) represent analytic (anti)Skyrmions living in the finite flat box defined above.

Since Eqs. (13), (15) and (17) allow to use all the available results in Sine-Gordon theory to analyze the (3+1)-dimensional Skyrme model at finite density, it is useful to follow the conventions of
The effective action for a rescaled the Skyrmion profile $\Phi$ is

$$ S = \ell^3 \int d\gamma d\phi \int dt dr L(\Phi), \quad H = \frac{\ell}{(\lambda + 2\ell^2)^{1/2}} \Phi, $$

where $\ell^3$ comes from the square root of the determinant of the metric. Thus the effective Lagrangian for $\Phi$ reads

$$ L(\Phi) = -\frac{1}{2} \nabla^\mu \Phi \nabla^\nu \Phi + \frac{\alpha}{\beta^2} (\cos (\beta \Phi) - 1), \quad (20) $$

$$ \alpha = \frac{\lambda}{2\ell^2 (\lambda + 2\ell^2)}, \quad \beta = \frac{4\ell}{\sqrt{\lambda + 2\ell^2}}. \quad (21) $$

The effective sine-Gordon coupling constant that appears from the Skyrme model always satisfies the Coleman bound $\beta^2 < 8\pi$.

The mapping presented above allows to construct analytic Skyrmion–anti-Skyrmion bound states. Namely, the breather-like solutions of Eq. (13) satisfying the boundary conditions in Eq. (19) (which correspond to kink anti-kink bound states) correspond to analytic Skyrmion–anti-Skyrmion bound states. To the best of authors knowledge, this is the first analytic construction of Skyrmions–anti-Skyrmions bound states in the original (3+1)dimensional Skyrme model. In particular the number $n_B$ of bound states satisfies $n_B \leq \frac{8\pi}{\beta^2} - 1$. Note that already Skyrme and Perring [57] used sine-Gordon in 1 + 1 dimensions as a “toy model” for the 3 + 1 dimensional Skyrme model. What is remarkable about the present treatment is that we found a nontrivial topological sector of the full Skyrme model in which they are exactly equivalent.

The semi-classical quantization in the present sector of the Skyrme model can be analyzed following [14] [13]. One first has to identify the (classical) low energy modes and then it is necessary to quantize such modes. In the present case, the task is simplified by one of the results mentioned above: namely, not only the Skyrme field equations with the generalized hedgehog ansatz in Eqs. (6), (7), (9) and (10) reduce to the sine-Gordon equation but also the full Skyrme action reduces to the corresponding sine-Gordon action in 1 + 1 dimensions with the coupling constants defined in Eq. (21). Thus, the principle of symmetric criticality [58] applies in the present case. Consequently, the low energy semi-classical fluctuations of the Skyrme model in the sector described by Eqs. (6), (7), (9) and (10) are described by the reduced action itself (which is nothing but the sine-Gordon action). Thus, all the known semi-classical results on the sine-Gordon theory hold.

### 3.0.1 An interesting function

Here we consider an interesting function $\Delta$ of the Skyrmions with charge $\pm 1$ defined above which encodes the information about how close they can get to saturate the Skyrme-BPS bound (which, as already emphasized, cannot be saturated on flat space-times). Nevertheless, it is interesting to analyze

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1Indeed, under the present mapping from the (3+1)-dimensional Skyrme model into the (1+1)-dimensional Sine-Gordon model, the (anti)Skyrmion is mapped into the (anti)kink. Consequently, kink-antikink bound states correspond to Skyrmion-antiSkyrmion bound states.
the following function defined\(^2\) as

\[ \Delta = E - 12\sqrt{2} \pi^2 |B| = E - 12\sqrt{2} \pi^2 \]  \hspace{1cm} (22)

where \(E\) is the energy of the (anti)Skyrmion defined above and \(B\) is its baryon charge. This relation is nothing but the Bogomol’ny bound, as can be found in [59], expressed in our conventions. It is worth to emphasize that in the usual case of the spherical Skyrmion found by Skyrme himself, the energy exceeds the topological charge by 23%.

In this case, the above difference \(\Delta\) for static configurations \(H = H(r)\) can be evaluated explicitly in terms of elliptic integral as follows. From Eq. (15) one gets the following expression for the energy-density

\[ T_{00} = \frac{K}{64 \ell^4} \left[ 16(\lambda + 2\ell^2)(\partial_r H)^2 + \lambda (1 - \cos(4H)) + 16\ell^2 \right] . \]  \hspace{1cm} (23)

The field equations Eq. (13) can be reduced to

\[ (1 + 2\ell^2) \left( \frac{dH}{dr} \right)^2 + \frac{1}{32} \cos(4H) = I_0 \Rightarrow \]  \hspace{1cm} (24)

\[ \frac{dH}{dr} = \pm \sqrt{\frac{Q(H)}{H^1/2}} = \pm \frac{1}{\sqrt{1 + 2\ell^2}} \left( 2I_0 - \frac{\cos 4H}{16} \right)^{1/2} . \]  \hspace{1cm} (25)

In the above equations \(I_0\) is an integration constant satisfying the following condition:

\[ \int_0^{\pi/2} \frac{dH}{Q(H)^{1/2}} = \pm 2\pi , \]  \hspace{1cm} (26)

arising from the requirement to have Baryon charge \(\pm 1\) (in this subsection, from now on we will consider the + sign). The above condition fixes the integration constant \(I_0\) as a function of \(\ell\):

\[ \int_0^{\pi/2} \frac{dH}{Q(H)^{1/2}} = 2\pi \Rightarrow I_0 = I_0 (\ell) . \]  \hspace{1cm} (27)

An explicit expression however cannot be extracted since,

\[ \int_0^{\pi/2} \frac{dH}{Q(H)^{1/2}} = 2\sqrt{2}(1 + 2\ell^2)\pi K (-x^2) \]  \hspace{1cm} (28)

where \(x^2 = \frac{2}{32I_0 - 1} > 0\) and \(K(x)\) is the complete elliptic integral of the first kind.

We can evaluate the energy using Eq. (26)

\[ E = \pi^2 \int_0^{\pi/2} \frac{dH}{Q(H)^{1/2}} \left( 4(1 + 2\ell^2) - \frac{1}{\ell} Q(H) + \frac{(1 - \cos(4H))}{4\ell} + 4\ell \right) = E (\ell) . \]  \hspace{1cm} (29)

\(^2\)Only in this subsection, we will adopt the convention that \(K = 2\) and \(\lambda = 1\) which means, roughly (see page 25 of [11], taking into account that the authors use the opposite convention for the space-time metric with respect to our), that we are measuring lengths in fm and energy in MeV.
Thus, the natural question is: how far is the energy of the Skyrmions from saturating the topological bound? The answer depends on the size of the box $\ell$. From Eqs. (26) and (29) one can write the energy $E(\ell)$ as combinations of elliptic integrals.

\[
E(x) = \left( \frac{2}{32I_0 - 1} \right)^{1/2}.
\]

Figure 1: The size of the box $\ell$ and the energy $E$ as functions of $x = \left( \frac{2}{32I_0 - 1} \right)^{1/2}$.

Although relation (27) cannot provide us with a closed form relation for $I_0(\ell)$, we can invert it and express $\ell$ as a function of $I_0$ or of $x$. In this manner we can also express the energy with respect to $x$. Thus we obtain the two graphs that we see in Fig. 1. From the first graph we can see that $\ell$ and $x$ are in inverse proportion with each other. At the same time there is a cutoff at a finite value of $x$, this is owed to taking a finite bound in the integration of the $r$ variable in (26). As can be seen, by combining (27) and (28) to solve algebraically for $\ell$; for $x$ above a certain value, $\ell$ becomes imaginary. The extension of integration over the full real line should be seen as a pushing of this bound to infinity. It can be checked arithmetically that the most economic energy configuration corresponds to a size of the box of $\ell \simeq 0.32$ where $E \simeq 181.312$ which is approximately 8.25% above the lower bound $12\sqrt{2}\pi^2$ as seen in (22).

These results have a quite natural interpretation. When the size of the box is small (as the order of $fm$) one should expect strong deviations from spherical symmetry as extended objects feel strongly the presence of boundaries. Indeed, the present nonspherical skyrmion is closer to the topological bound than the usual spherical skyrmion (which exceeds the topological bound by about 23%). On the other hand, when the box size increases, it should be expected that the presence of the box itself becomes less relevant. The above plot shows that this is the case since the energy of the present nonspherical skyrmion grows very rapidly (well above the topological bound) as the size of the box increases. Thus, when the box is large enough only small deviations from the spherical skyrmion should be expected. Consequently, the type of Skyrmions analyzed here is expected to be favoured at small volumes (or high densities).
3.1 Inclusion of chemical potential

The effects of the isospin chemical potential can be taken into account by using the following covariant derivative (see [28] [29]):

\[ D_\mu = \nabla_\mu + \bar{\mu}[t_3, ]\delta_{\mu 0}. \] (30)

Thus \( R_\mu \) becomes \( \bar{R}_\mu = U^{-1}D_\mu U \) and the equations of motion read

\[ D^\mu \left( \bar{R}_\mu + \frac{\lambda}{4}[\bar{R}^\nu, \bar{G}_{\mu \nu}] \right) = 0, \] (31)

where \( \bar{G}_{\mu \nu} = [\bar{R}_\mu, \bar{R}_\nu] \). For static configurations \( H(r, z) = H(r) \), the good property of the hedgehog ansatz is not lost: the full Skyrme field equations with isospin chemical potential in Eq. (31) reduce to just one scalar ODE for \( H \):

\[
\begin{align*}
(\lambda + 2\ell^2 - 8\lambda \ell^2 \bar{\mu}^2 \sin^2(H(r))) H''(r) - 4\lambda \ell^2 \bar{\mu}^2 \sin(2H(r))H^2 \\
+ \lambda \left( \bar{\mu}^2 \ell^2 - \frac{1}{8} \right) \sin(4H(r)) + 4 \bar{\mu}^2 \ell^4 \sin(2H(r)) = 0.
\end{align*}
\] (32)

This is a quite non-trivial technical achievement in itself (see, for instance, [30] [31]). Moreover, the above differential equation can be reduced to

\[
Y(H) \frac{(H')^2}{2} + V(H) = E_0,
\] (33)

where

\[
\begin{align*}
Y(H) &= \lambda + 2\ell^2 - 8\lambda \ell^2 \bar{\mu}^2 \sin^2(H), \\
V(H) &= -\frac{\lambda}{4} \left( \bar{\mu}^2 \ell^2 - \frac{1}{8} \right) \cos(4H) - 2 \bar{\mu}^2 \ell^4 \cos(2H).
\end{align*}
\]

\( E_0 \) is an integration constant to be determined imposing the physical boundary condition:

\[
\int_0^{\pi/2} \frac{[Y(H)]^{1/2}}{[E_0 - V(H)]^{1/2}} dH = \sqrt{22\pi}.
\] (34)

Thus, we can compute the critical isospin chemical potential \( \bar{\mu}_c \) as the one for which the above boundary condition cannot be satisfied anymore as \( Y \) can be negative\(^3\) when \( \bar{\mu} \geq \bar{\mu}_c \):

\[
(\bar{\mu}_c)^2 = \frac{\lambda + 2\ell^2}{8\lambda \ell^2}.
\]

\(^3\)When \( Y(H) \) becomes negative, the numerator in the left hand side of Eq. (34) develops an imaginary part which cannot be compensated by the denominator.
4 Time crystals

Obviously, not any time-periodic solution of the Skyrme model is a time-crystal. For instance, the Skyrmion–anti-Skyrmion bound state constructed above are time-periodic. However they are not topologically protected since, if one ‘pays’ the corresponding binding energies, they decay into the trivial vacuum.

Here we adopt the flat space line element

$$ds^2 = -d\gamma^2 + \ell^2 (dz^2 + dr^2 + d\phi^2) ,$$

where $\gamma$ plays the role of time. We have to make the following modification to ansatz $^9$, $^10$

$$A = \frac{\omega \gamma - \phi}{2}, \quad G = \frac{\omega \gamma + \phi}{2},$$

where $0 \leq \omega \gamma \leq 4\pi$ and the frequency $\omega$ is necessary to keep $A$ and $G$ dimensionless.

The adoption of line-element (35) means that in this case the profile $H$ depends on two space-like coordinates. The Skyrme configurations $U$ defined in Eqs. (6), (7), (9), (10) and (36) are necessarily time-periodic. The full Skyrme field equations (4) reduce in this case to

$$\Delta H - \frac{\lambda \omega^2}{4(\ell^2(\lambda \omega^2 - 4) - \lambda)} \sin(4H) = 0 ,$$

$$\omega^2 \neq \omega_c^2 = \frac{\lambda + 4\ell^2}{\ell^2\lambda} ,$$

where $\Delta$ is the two-dimensional Laplacian in $z$ and $r$. Eq. (37) is the Euclidean sine-Gordon equation.\(^4\) Exact solutions of Eq. (37) can easily be constructed taking, for instance, $H = H(r)$.

To construct a time crystal configuration, we firstly need to find stable kinks satisfying Eq. (37). As in the previous section, it is useful to obtain the reduced action $\mathcal{L}$ corresponding to the Eq. (37),

$$\mathcal{L}(H) = \nabla_\mu H \nabla^\mu H + \frac{\lambda \omega^2}{4(\ell^2(\lambda \omega^2 - 4) - \lambda)} \sin^2(2H) .$$

This allows to use well known results on quantization of sine-Gordon theory also in this sector, for instance, a sine-Gordon kink with $H = H(r)$.

Secondly, the configuration has to have a non-trivial winding number. The topological density is given by $\rho_B = 3\sin(2H)dH \wedge d(\omega \gamma) \wedge d\phi$, and thus the winding number can be evaluated to

$$W = -\frac{\omega}{8\pi^2} \int_{z=\text{const}} \sin(2H)dHd\gamma d\phi = \pm 1.$$\(^5\)

\(^4\)It is worth to note that there is a critical value $\omega_c$ for the frequency $\omega$ of the time crystal (defined in Eq. (38)) at which Eq. (37) becomes degenerate. On the other hand, in the case of the Skyrmions described in the previous section the theory is defined for all values of the parameters of the model.

\(^5\)Previous literature on the analogies between sine-Gordon and Skyme models can be found in $^60$, $^61$, $^62$, $^63$ and references therein. As it has been emphasized previously, sine-Gordon theory was believed to be just a “toy model” for the 3+1 dimensional Skyrme model. In fact, we proved that in a nontrivial topological sector they exactly coincide.
This is one of the main results of the paper. We have shown that there are smooth time-periodic regular configurations of the Skyrme model living at finite volume possessing a non-trivial winding number along a three-dimensional time-like surface. Since the winding number is invariant under any continuous deformation, these configurations can only decay into other configurations which are also time-periodic (as for static configurations the above winding number vanishes). Thus, the time periodicity of these configurations is topologically protected. These are classical topologically protected time-crystals in the sense of [43]. Interestingly enough, the principle of symmetric criticality [58] can be applied to time-crystal as well since, with the above time-crystal ansatz not only the Skyrme field equations reduce to the sine-Gordon system but also the full Skyrme action reduces to the corresponding sine-Gordon action. Thus, the low energy semi-classical fluctuations around the time-crystals constructed here are described by the semi-classical analysis of sine-Gordon theory. Hence, well-known results on sine-Gordon theory suggest that such time-crystals should also be present at semi-classical level.

4.1 The chemical potential

We can introduce the chemical potential as in the previous section. The full Skyrme field equations with isospin chemical potential reduce to just one scalar partial differential equation for the profile $H$,

$$
\begin{align*}
&\left(\lambda + 4\ell^2 - \lambda \ell^2 \omega^2 + 8 \lambda \ell^2 \bar{\mu}(\omega - 2\bar{\mu}) \sin^2(\lambda H)\right) \Delta H + 4\lambda \ell^2 \bar{\mu}(\omega - 2\bar{\mu}) \sin(2H) \nabla_{\mu} H \nabla^\mu H \\
&+ 4\ell^2 \bar{\mu}(2\bar{\mu} - \omega) \sin(2H) + \frac{\lambda}{4} (\omega - 2\bar{\mu})^2 \sin(4H) = 0 .
\end{align*}
$$

(41)

In this case the critical chemical potential $\mu^*$ corresponding to time crystal can be found easily as in the previous section (see the comments below Eqs. (32) and (33)). In particular, let us consider a kink-like solution of Eq. (41) in which the profile only depends on one coordinate $H = H(r)$ and satisfying the boundary condition in Eq. (18). Then, Eq. (41) reads

$$
Y_1(H) \left(\frac{H'}{2}\right)^2 + V_1(H) = E_0 ,
$$

(42)

where

$$
Y_1(H) = \lambda + \ell^2 \left[4 - \lambda \omega^2 + 8 \lambda \bar{\mu} \left(\omega - 2\bar{\mu}\right) \sin^2(H)\right] ,
$$

$$
V_1(H) = -\frac{\lambda}{16} (\omega - 2\bar{\mu})^2 \cos(4H) - 2\ell^2 \bar{\mu} \left(2\bar{\mu} - \omega\right) \cos(2H) .
$$

Thus, the critical chemical potential $\mu^*$ can be determined by requiring

$$
\lambda + \ell^2 \left[4 - \lambda \omega^2 + 8 \lambda \mu^* \left(\omega - 2\mu^*\right)\right] \leq 0 ,
$$

(43)

since, when this happens, the boundary condition in Eq. (18) cannot be satisfied anymore. It is also interesting to note that high values (compared to $\lambda$) of the time-crystal frequency $\omega^2$ decrease the critical chemical potential: hence, low values of $\omega^2$ are favoured from the thermodynamical point of
There is a further special value for the chemical potential (which does not coincide with the one defined in Eq. (43)) for time-crystal configurations. Indeed, for $\mu = \omega/2$, the non-linear partial differential equation Eq. (41) reduces to the linear $\triangle H = 0$, which, obviously, possesses more symmetries than the generic one for $\mu < \omega/2$. Since this value $\mu = \omega/2$ of the isospin chemical potential corresponds to a symmetry enhancement of the field equations, it is natural to wonder whether it is related to some phase transition of the system. We hope to come back on this interesting issue in a future publication.

5 Conclusions

We constructed the first analytic examples of Skyrmions as well as of Skyrmions–anti-Skyrmions bound states on flat spaces at finite volume. We have derived an analytic upper bound for the number of Skyrmion–anti-Skyrmion bound states in terms of the parameters of the model. The critical isospin chemical potential can also be computed. With the same formalism, one can build topologically protected time-crystals: these are exact configurations of the Skyrme model whose time-dependence is topologically protected by the non-vanishing winding number. We computed the corresponding critical isospin chemical potential and determined a possible experimental signature of these time-crystals.

The present construction answers positively to the question posed in [43] on the existence of a classical time crystal in systems possessing non-vanishing topological charges. In fact, as these classical configurations are topologically protected, the presence of quantum fluctuations cannot destroy them. Using the results presented above it is easy to show that these configurations are also present in the semi-classical quantization of the model. The reason why the powerful no-go theorems in [45] [46] do not apply in the present case is that, in these theorems (both explicitly and implicitly) it is assumed that the ground state of the theory is static. On the other hand, in theories with non-Abelian internal symmetries each non-trivial topological sector has its own ground state.

For instance, in the case of non-Abelian gauge theories admitting BPS monopoles, the ground state in the sector with unit non-Abelian magnetic charge is the well-known BPS monopole which cannot be deformed continuously to the trivial vacuum. Such a ground state is not invariant under the full symmetry group of the trivial ground state $|0\rangle$. In particular, it is not invariant under spatial rotations (unless they are compensated by internal rotations). This fact, is the origin of the “spin from isospin effect” discovered in the seventies. In our case, in the sectors we have named time-crystal, the ground state is time-periodic and consequently the theorems in [45] and [46] do not apply, as we discussed above.

In the present paper, we focused on the $SU(2)$ Skyrme model, but our results can be generalized to any theory with $SU(N)$ internal symmetry.
Acknowledgements

This work has been funded by the Fondecyt grants 1160137, 1161150, 3150016 and 3160121. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt.

References

[1] T. Skyrme, Proc. R. Soc. London A 260, 127 (1961); Proc. R. Soc. London A 262, 237 (1961); Nucl. Phys. 31, 556 (1962).
[2] T.S. Walhout, Nucl. Phys. A 531, 596 (1991)
[3] C. Adam, M. Haberichter and A. Wereszczynski, Phys. Rev. C 92, 055807 (2015)
[4] J.-i Fukuda and S. Žumer, Nature Comm. 2, 246 (2011)
[5] H. Stefan et al. Nature Physics 7, 713 (2011)
[6] D. Fostar and S. Krusch, Nuc. Phys. B 897, 697 (2015)
[7] M. Gillard, Nucl. Phys. B 895, 272 (2015)
[8] E. Witten, Nucl. Phys. B 223 (1983), 422; Nucl. Phys. B 223 (1983), 433.
[9] D. Finkelstein, J. Rubinstein, J. Math. Phys. 9, 1762–1779 (1968).
[10] N. Manton and P. Sutcliffe, Topological Solitons, (Cambridge University Press, Cambridge, 2007).
[11] V. G. Makhanov, Y. P. Rybakov, V. I. Sanyuk, The Skyrme model, Springer-Verlag (1993).
[12] D. Giulini, Mod. Phys.Lett. A8, 1917–1924 (1993).
[13] A.P. Balachandran, A. Barducci, F. Lizzi, V.G.J. Rodgers, A. Stern, Phys. Rev. Lett. 52 (1984), 887.
[14] G. S. Adkins, C. R. Nappi, E. Witten, Nucl. Phys. B 228 (1983), 552-566.
[15] E. Guadagnini, Nucl. Phys. B 236 (1984), 35-47.
[16] W. Münzer et al. Phys. Rev. B 81, 041203 (2010)
[17] J.-y. Choi, W.J. Kwon and Y.-i. Shin, Phys. Rev. Lett. 108, 035301 (2012)
[18] K. Benson and M. Bucher, Nucl. Phys. B 406, 355 (1993)
[19] L. Parisi, N. Radicella and G. Vilakis, Phys. Rev. D 91 063533 (2015)
[20] S. Droz, M. Heusler and N. Straumann, Phys. Lett. B 268 371 (1991)
[21] H. Luckock and I. Moss, *Phys. Lett.* B, **176** 341 (1986)

[22] N. Shiki and N. Sawado, *Phys. Rev.* D **71**, 104031 (2005)

[23] T. Ioannidou, B. Kleihaus and J. Kunz, Phys. Lett. B **643** 213 (2006)

[24] S.B. Gudnason, M. Nitta and N. Sawado, JHEP **1609** 055 (2016).

[25] G. Dvali and A. Gußmann, Nucl. Phys. B **913** 1001 (2016).

[26] S. Seki, M. Mochizuki, *Skyrmions in magnetic materials*, Springer, 2016.

[27] I. Klebanov, *Nucl. Phys.* B **262** (1985) 133.

[28] A. Actor, *Phys. Lett.* B **157** (1985) 53.

[29] H. A. Weldon, *Phys. Rev.* D **26** (1982) 1394.

[30] M. Loewe, S. Mendizabal, J.C. Rojas, *Phys. Lett.* B **632** (2006) 512.

[31] J. A. Ponciano, N. N. Scoccola, *Phys. Lett.* B **659** (2008) 551.

[32] F. Canfora, P. Salgado-Rebolledo, *Phys. Rev.* D **87**, 045023 (2013).

[33] F. Canfora, H. Maeda, *Phys. Rev.* D **87**, 084049 (2013).

[34] F. Canfora, *Phys. Rev.* D **88**, 065028 (2013).

[35] F. Canfora, F. Correa, J. Zanelli, *Phys. Rev.* D **90**, 085002 (2014).

[36] S. Chen, Y. Li, Y. Yang, *Phys. Rev.* D **89** (2014), 025007.

[37] E. Ayon-Beato, F. Canfora, J. Zanelli, *Phys. Lett.* B **752**, (2016) 201-205.

[38] F. Canfora, G. Tallarita, JHEP 1409 (2014) 136.

[39] F. Canfora, G. Tallarita, Phys. Rev. D 91 (2015), 085033

[40] F. Canfora, G. Tallarita, Phys. Rev. D 94 (2016), 025037

[41] F. Canfora, G. Tallarita, *Nucl. Phys.* B **921** (2017) 394.

[42] F. Wilczek, *Phys. Rev. Lett.* **109**, 160401 (2012).

[43] A. Shapere, F. Wilczek, *Phys. Rev. Lett.* **109**, 160402 (2012).

[44] F. Wilczek, *Phys. Rev. Lett.* **111**, 250402 (2013).

[45] P. Bruno, *Phys. Rev. Lett.* **111**, 070402 (2013); *Phys. Rev. Lett.* **110**, 118901 (2013); *Phys. Rev. Lett.* **111**, 029301 (2013).
[46] H. Watanabe, M. Oshikawa, *Phys. Rev. Lett.* **114**, 251603 (2015).

[47] K. Sacha, J. Zakrzewski, ”Time crystals: a review” arXiv: 1704.03735.

[48] K. Sacha, *Phys. Rev. A* **91**, 033617 (2015).

[49] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani, C. von Keyserlingk, N. Y. Yao, E. Demler, M. D. Lukin, *Nature* **543** (7644), 221–225 (2017), letter.

[50] Zhang, J, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I.-D. Potirniche, A. C. Potter, A. Vishwanath, N. Y. Yao, C. Monroe, *Nature* **543** (7644), 217–220 (2017), letter.

[51] N. Y. Yao, A. C. Potter, I.-D. Potirniche, A. Vishwanath, *Phys. Rev. Lett.* **118**, 030401 (2017).

[52] D. V. Else, B. Bauer, C. Nayak, *Phys. Rev. Lett.* **117**, 090402 (2016); *Phys. Rev. Lett.* **118**, 030401 (2017).

[53] V. Khemani, A. Lazarides, R. Moessner, S. L. Sondhi, *Phys. Rev. Lett.* **116**, 250401 (2016).

[54] Y. M. Shnir, *Magnetic Monopoles* (Springer-Verlag, Berlin, Heidelberg, 2005) pp.500

[55] Jesús Cuevas-Maraver (Editor), Panayotis G. Kevrekidis (Editor), Floyd Williams (Editor), *The sine-Gordon Model and its Applications: From Pendula and Josephson Junctions to Gravity and High-Energy Physics* (Springer, 2014).

[56] S. Coleman, *Phys. Rev. D* **11** (1975) 2088.

[57] J. K. Perrin and T. H. R. Skyrme, *Nucl. Phys.* **31** (1962) 550-555.

[58] R. S. Palais, *Comm. Math. Phys.* **69** (1979), 19–30.

[59] I. Zahed and G. E. Brown, *Physics Reports* **142** Nos. 1 & 2 (1986) 1-102.

[60] B. Piette and W. J. Zakrzewski, Nontopological structures in the baby-Skyrme model, Proceedings of the Soliton conference (Kingston 1997), [hep-th/9710012](https://arxiv.org/abs/hep-th/9710012).

[61] S.B. Gudnason and M. Nitta, *Phys. Rev. D* **90** (2014) 085007.

[62] M. Nitta, *Nucl. Phys. B* **895** (2015) 288.

[63] M. Eto and M. Nitta, *Phys. Rev. D* **91** (2015) 085044.