INTEGRABILITY OF ANTI-SELF-DUAL VACUUM EINSTEIN EQUATIONS WITH NONZERO COSMOLOGICAL CONSTANT: AN INFINITE HIERARCHY OF NONLOCAL CONSERVATION LAWS

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ABSTRACT. We present an infinite hierarchy of nonlocal conservation laws for the Przanowski equation, an integrable second-order PDE locally equivalent to anti-self-dual vacuum Einstein equations with nonzero cosmological constant. The hierarchy in question is constructed using a nonisospectral Lax pair for the equation under study. As a byproduct, we obtain an infinite-dimensional differential covering over the Przanowski equation.

INTRODUCTION

Integrable systems play an important role in modern theoretical and mathematical physics, see e.g. [1 11 10 15 21 27 28 29], and this is particularly true for integrable systems in four independent variables a.k.a. 3+1 dimensional. Moreover, a number of integrable (3+1)-dimensional systems of immediate relevance for physics arises within general relativity upon imposition of (anti)self-duality conditions, see, for instance, [1 3 20] and references therein, as is the case for the Przanowski equation studied below.

Namely, Przanowski [23] has shown that locally every anti-self-dual Einstein four-manifold \((M,g)\) admits a compatible complex structure and the metric has the form

\[
g = 2 \left( u_{w \tilde{w}} \, dw \, d\tilde{w} + u_{w \bar{z}} \, dw \, d\bar{z} + u_{z \bar{w}} \, dz \, d\tilde{w} + \left( u_{z \bar{z}} + \frac{2}{\Lambda} \exp(\Lambda u) \right) dz \, d\bar{z} \right).\]

Here \(u = u(w, \tilde{w}, z, \bar{z})\) is a real function on \(M, w\) and \(z\) are holomorphic coordinates on \(M\) and \(\tilde{w}\) and \(\bar{z}\) denote their complex conjugates; \(\Lambda \neq 0\) is the cosmological constant, cf. e.g. [6] and references therein. As usual, the subscripts indicate partial derivatives, e.g. \(u_w = \frac{\partial u}{\partial w}\) etc.

The metric (1) is, see [23], an anti-self-dual Einstein metric if and only if \(u\) satisfies the Przanowski equation

\[
u_{z \bar{w}}u_{w \bar{z}} - u_{w \tilde{w}} \left( u_{z \bar{z}} + \frac{2}{\Lambda} \exp(\Lambda u) \right) + u_w u_{\bar{w}} \exp(\Lambda u) = 0,
\]

which is a subject of intense research, see e.g. [2 11 16] and references therein.

We leave aside the case of \(\Lambda = 0\) as then, upon having imposed (anti)self-duality on the metric one has to use a normal form of the metric different from (1) and arrives instead of (2) at a different PDE, cf. e.g. [1 16 20 22 25 26] and references therein.

Hoegner [9] has established integrability of (2) by constructing a nonisospectral Lax pair for (2) of the form

\[
l_i \psi = 0, \quad i = 1, 2,
\]

where \(\psi = \psi(z, w, \bar{z}, \tilde{w}, \xi), Q = -u_w u_{\bar{w}} \exp(\Lambda u)\), and

\[
l_1 = \partial_w - \frac{\xi}{Q} u_{w \bar{z}} Q \partial_{\bar{z}} + \frac{\xi u_{w \bar{w}}}{Q} \partial_{\bar{w}} + \left( \frac{\partial_{\bar{z}} Q + \exp(\Lambda u) u_w u_{w \bar{w}}}{Q} - \frac{u_{w \bar{w}}}{u_{\bar{w}}} \right) \xi \partial_{\xi},
\]

\[
l_2 = \partial_{\bar{z}} - \frac{\xi}{Q} \left( u_{z \bar{z}} + \frac{2}{\Lambda} \exp(\Lambda u) \right) \partial_{w} + \frac{u_{z \bar{w}}}{Q} \partial_{\bar{w}} + \left( \frac{\partial_{\bar{z}} Q + \exp(\Lambda u) (u_w u_{z \bar{w}} - u_{\bar{w}} \xi)}{Q} - \frac{u_{z \bar{w}}}{u_{\bar{w}}} + \frac{\xi}{u_w} \right) \xi \partial_{\xi}.
\]

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Here $\xi$ is an additional independent variable which plays the role of (variable) spectral parameter, see [7 10 21 29] and references therein; we stress that $u_\xi = 0$.

The presence of this Lax pair makes it possible, at least in principle, to obtain exact solutions for (2) using the inverse scattering transform, cf. e.g. [19] and references therein, or the twistorial methods, see, for example, [3 9 20] and references therein.

Existence of infinite hierarchies of conservation laws is a well-known feature of integrable systems, cf. e.g. [15 21] and references therein, and we show below how to construct such a hierarchy for the Przanowski equation using a modification of the above Lax pair, and prove that the conservation laws in question are nontrivial and linearly independent.

The rest of the article is organized as follows. In Section 1 we construct an infinite hierarchy of the nonlocal conservation laws in question. In Section 2 we set the stage for Section 3 and our main result, Theorem 1 establishing linear independence and nontriviality for the conservation laws from the hierarchy in question.

1.-infinitely-many-nonlocal-conservation-laws-for-the-przanowski-equation

**Proposition 1.** Equation (2) admits a Lax pair $L_j \chi = 0$, $j = 1, 2$, where $\chi = \chi(z, \bar{z}, w, \bar{w}, p)$ and

\[
L_1 = \partial_w - \frac{pu_{w\bar{w}}}{u_{\bar{w}}} \partial_{\bar{z}} + \frac{pu_{w\bar{z}}}{u_{\bar{w}}} \partial_{\bar{w}} + \left( \frac{u_{w\bar{w}} u_{w\bar{z}} - u_{w\bar{z}} u_{w\bar{w}} + \Lambda u_{w\bar{w}} (u_{\bar{w}w} - u_{\bar{w}w})}{u_{\bar{w}}} \right) p^2 \partial_p,
\]

\[
L_2 = \partial_{\bar{z}} - \frac{pu_{\bar{w}z}}{u_{\bar{w}}} \partial_{\bar{w}} + \left( \frac{u_{\bar{w}z} u_{w\bar{z}} + \exp(\Lambda u) u_{w\bar{w}}}{u_{\bar{w}}^2} u_{w\bar{w}} \right) \partial_{\bar{w}} - \left( \frac{u_{\bar{w}z} (u_{w\bar{w}} u_{w\bar{z}} - u_{w\bar{z}} u_{w\bar{w}} - \Lambda u_{\bar{w}} (u_{\bar{w}w} - u_{\bar{w}w}))}{u_{\bar{w}}^3} u_{w\bar{w}} \right) p^2 \partial_p.
\]

**Proof.** It suffices to observe that $L_i$ are related to $l_i$ by the change of variables $p = \xi \exp(\Lambda u) u_{w}/u_{\bar{w}}$.  

The Lax operators $L_i$ enjoy a simpler structure than $l_i$. Indeed, they can be written as

\[
L_1 = \partial_w + p \left( \frac{1}{u_{\bar{w}}} \right) \partial_{\bar{z}} - p(\omega_1)_w \partial_{\bar{w}} + p^2 \left( \frac{u_{\bar{w}} (2 \Lambda u_z + (\omega_1)_w) - u_{\bar{w}z}}{2 u_{\bar{w}}^2} \right) \partial_p,
\]

\[
L_2 = \partial_{\bar{z}} + p \left( \frac{1}{u_{\bar{w}}} \right) \partial_w - p(\omega_1)_w \partial_{\bar{w}} + p^2 \left( \frac{u_{\bar{w}} (2 \Lambda u_z + (\omega_1)_w) - u_{\bar{w}z}}{2 u_{\bar{w}}^2} \right) \partial_p,
\]

where $\omega_1$ is defined by the formulas

\[
(\omega_1)_w = \frac{u_{w\bar{z}}}{u_{\bar{w}}},
\]

\[
(\omega_1)_z = \frac{u_{\bar{w}z} u_{w\bar{z}} + \exp(\Lambda u) u_{w\bar{w}}}{u_{\bar{w}}^2} u_{w\bar{w}},
\]

i.e., the quantity $\omega_1$ is a nonlocal variable, namely, a potential for the following local conservation law for (2):

\[
\left( \frac{u_{w\bar{z}}}{u_{\bar{w}}} \right)_w = \left( \frac{u_{\bar{w}z} u_{w\bar{z}} + \exp(\Lambda u) u_{w\bar{w}}}{u_{\bar{w}}^2} \right)_w u_{w\bar{w}}.
\]

Substituting a formal Taylor expansion $\chi = \sum_{i=0}^{\infty} \chi_i p^i$ into the equations $L_j \chi = 0$ shows that $\chi_0 = \chi_0^0(w, \bar{z})$ is an arbitrary smooth function of $w$ and $\bar{z}$, and $\chi_1$ satisfies the equations

\[
(\chi_1)_w = \frac{u_{w\bar{z}}}{u_{\bar{w}}} (\chi_0^0)_z - \frac{u_{w\bar{z}}}{u_{\bar{w}}} (\chi_0^0)_{\bar{w}},
\]

\[
(\chi_1)_z = \frac{u_{\bar{w}z}}{u_{\bar{w}}} (\chi_0^0)_z - \frac{u_{\bar{w}z} u_{w\bar{z}} + \exp(\Lambda u) u_{w\bar{w}}}{u_{\bar{w}}^2 u_{w\bar{w}}} (\chi_0^0)_{\bar{w}},
\]

whence

\[
\chi_1 = -\frac{(\chi_0^0)_z}{u_{\bar{w}}} - \omega_1 (\chi_0^0)_{\bar{w}} + \chi_1^0,
\]

where $\chi_1^0(w, \bar{z})$ is again an arbitrary smooth function of $w$ and $\bar{z}$.
We now see that \( \chi_i \) for \( i = 2, 3, \ldots \) satisfy the recursion relations

\[
(\chi_i)_w = - \left( \frac{1}{u_{\bar{w}}} \right)_w (\chi_{i-1})_\bar{w} + (\omega_1)_w (\chi_{i-1})_\bar{w} - \left( \frac{u_{\bar{w}}(2\Lambda u_{\bar{z}} + (\omega_1)_{\bar{w}}) - u_{\bar{w}}}{2u_{\bar{w}}} \right)_w (i - 1)\chi_{i-1},
\]

\[
(\chi_i)_z = - \left( \frac{1}{u_{\bar{w}}} \right)_z (\chi_{i-1})_\bar{w} + (\omega_1)_z (\chi_{i-1})_\bar{w} - \left( \frac{u_{\bar{w}}(2\Lambda u_{\bar{z}} + (\omega_1)_{\bar{w}}) - u_{\bar{w}}}{2u_{\bar{w}}} \right)_z (i - 1)\chi_{i-1}
\]

(7)

Thus, we have an infinite-dimensional (differential) covering over (2) defined by (7).

System (7) gives rise to an infinite hierarchy of nonlocal conservation laws for (2):

\[
(A_{k,r,s})_z = (B_{k,r,s})_w,
\]

where \( k = 1, 2, 3, \ldots, r, s = 0, 1, 2, 3, \ldots \), and \( A_{k,r,s} \) and \( B_{k,r,s} \) are defined by the following relations:

\[
A_{k,r+1,s} = (A_{k,r,s})_\bar{w}, \quad A_{k,r,s+1} = (A_{k,r,s})_\bar{w}, \quad B_{k,r+1,s} = (B_{k,r,s})_\bar{w}, \quad B_{k,r,s+1} = (B_{k,r,s})_\bar{w}, \quad k, r, s = 1, 2, 3, \ldots
\]

and for \( k = 1, 2, 3, \ldots \) we set

\[
A_{k,0,0} = - \left( \frac{1}{u_{\bar{w}}} \right)_w (\chi_k)_\bar{z} + (\omega_1)_w (\chi_k)_\bar{w} - \left( \frac{u_{\bar{w}}(2\Lambda u_{\bar{z}} + (\omega_1)_{\bar{w}}) - u_{\bar{w}}}{2u_{\bar{w}}} \right)_w k\chi_k,
\]

\[
B_{k,0,0} = - \left( \frac{1}{u_{\bar{w}}} \right)_z (\chi_k)_\bar{z} + (\omega_1)_z (\chi_k)_\bar{w} - \left( \frac{u_{\bar{w}}(2\Lambda u_{\bar{z}} + (\omega_1)_{\bar{w}}) - u_{\bar{w}}}{2u_{\bar{w}}} \right)_z k\chi_k.
\]

The conservation laws (8) are linearly independent and nontrivial, as we are going to establish in Theorem 1 below. Notice that this implies \textit{inter alia} nontriviality of the covering (7).

In closing note that (2) enjoys an obvious discrete symmetry \( w \leftrightarrow \bar{w}, z \leftrightarrow \bar{z} \) which however does not extend to its Lax operators \( \lambda_i \) or \( \hat{\lambda}_i \). This implies that there exists another infinite hierarchy of nonlocal conservation laws for (2) obtained from (3) by the simultaneous swap \( w \leftrightarrow \bar{w}, z \leftrightarrow \bar{z} \).

2. Nontriviality of conservation laws: preliminaries

2.1. Simplifications. Before proceeding further, notice that the problem under study admits some useful simplifications. First of all, note that

(1) Under the rescaling \( u \mapsto \Lambda u \) equation (2) transforms into

\[
u_{\bar{w}} u_{\bar{w}} - u_{\bar{w}} (u_{\bar{z}} + 2e^u) + u_w u_{\bar{w}} e^u = 0
\]

and thus we can set \( \Lambda = 1 \) in all subsequent computations without loss of generality.

(2) Coverings (4) and (5) are equivalent in the sense of (13), which \textit{inter alia} implies that we can set without loss of generality \( \chi_0^0 = -\bar{w}, \chi_1^0 = 0 \).

Hence, the infinite-dimensional covering (7) can be rewritten as

\[
\chi_{1,w} = \frac{u_{\bar{w}} u_{\bar{z}}}{u_{\bar{w}}^2}, \quad \chi_{1,z} = \frac{(u_{\bar{w}} u_{\bar{z}} + u_w u_{\bar{w}} e^u)}{u_{\bar{w}}^2 u_{\bar{w}}}
\]

and

\[
\chi_{i,w} = - \left( \frac{1}{u_{\bar{w}}} \right)_w (\chi_{i-1})_\bar{w} + (\chi_1)_w (\chi_{i-1})_\bar{w} - \left( \frac{u_{\bar{w}}(2u_{\bar{z}} + (\chi_1)_\bar{w}) - u_{\bar{w}}}{2u_{\bar{w}}} \right)_w (i - 1)\chi_{i-1},
\]

\[
\chi_{i,z} = - \left( \frac{1}{u_{\bar{w}}} \right)_z (\chi_{i-1})_\bar{w} + (\chi_1)_z (\chi_{i-1})_\bar{w} - \left( \frac{u_{\bar{w}}(2u_{\bar{z}} + (\chi_1)_\bar{w}) - u_{\bar{w}}}{2u_{\bar{w}}} \right)_z (i - 1)\chi_{i-1}
\]

for \( i > 1 \).
2.2. Coordinates and total derivatives. We rewrite (2), where we set $\Lambda = 1$ as per Section 2.1, in the form

\begin{equation}
 u_{22} = \frac{u_{w2}u_{z\bar{w}} - (u_{w1}u_{\bar{w}} - 2u_{w\bar{w}})e^u}{u_{w\bar{w}}},
\end{equation}

and choose internal coordinates on the associated diffiety $\mathcal{D}$, i.e., the infinite prolongation of (9), as follows (see e.g. [3] for geometry of diffieties):

\begin{equation}
 u_{w^i\bar{w}^j} = \frac{\partial^{i+j}u}{\partial w^i\partial \bar{w}^j}, \quad u_{w^i\bar{w}^jz^k} = \frac{\partial^{i+j+1}u}{\partial w^i\partial \bar{w}^j\partial z^k}, \quad u_{w^i\bar{w}^jz^{k}} = \frac{\partial^{i+j+k}u}{\partial w^i\partial \bar{w}^j\partial z^k},
\end{equation}

where $i, j \geq 0, k > 0$.

Then the total derivatives on $\mathcal{D}$ read

\begin{align*}
 D_w &= \frac{\partial}{\partial w} + \sum_{i,j,k} \left( u_{w^{i+1}\bar{w}^j} \frac{\partial}{\partial u_{w^{i+1}\bar{w}^j}} + u_{w^i\bar{w}^{j+1}z^k} \frac{\partial}{\partial u_{w^i\bar{w}^{j+1}z^k}} + u_{w^i\bar{w}^jz^{k+1}} \frac{\partial}{\partial u_{w^i\bar{w}^jz^{k+1}}} \right), \\
 D_{\bar{w}} &= \frac{\partial}{\partial \bar{w}} + \sum_{i,j,k} \left( u_{w^{i+1}\bar{w}^j} \frac{\partial}{\partial u_{w^{i+1}\bar{w}^j}} + u_{w^i\bar{w}^{j+1}z^k} \frac{\partial}{\partial u_{w^i\bar{w}^{j+1}z^k}} + u_{w^i\bar{w}^jz^{k+1}} \frac{\partial}{\partial u_{w^i\bar{w}^jz^{k+1}}} \right), \\
 D_z &= \frac{\partial}{\partial z} + \sum_{i,j,k} \left( u_{w^i\bar{w}^{j+1}z^k} \frac{\partial}{\partial u_{w^i\bar{w}^{j+1}z^k}} + u_{w^{i+1}\bar{w}^jz^{k+1}} \frac{\partial}{\partial u_{w^{i+1}\bar{w}^jz^{k+1}}} + D_w D_{\bar{w}} D_z^{-1}(R) \frac{\partial}{\partial u_{w^i\bar{w}^{j+1}z^k}} \right), \\
 D_{\bar{z}} &= \frac{\partial}{\partial \bar{z}} + \sum_{i,j,k} \left( u_{w^i\bar{w}^{j+1}z^k} \frac{\partial}{\partial u_{w^i\bar{w}^{j+1}z^k}} + D_w D_{\bar{w}} D_{\bar{z}}^{-1}(R) \frac{\partial}{\partial u_{w^i\bar{w}^{j+1}z^k}} + u_{w^{i+1}\bar{w}^jz^{k+1}} \frac{\partial}{\partial u_{w^{i+1}\bar{w}^jz^{k+1}}} \right),
\end{align*}

where

\begin{equation}
 R = \frac{u_{w2}u_{z\bar{w}} - (u_{w1}u_{\bar{w}} - 2u_{w\bar{w}})e^u}{u_{w\bar{w}}}
\end{equation}

is the right-hand side of (10).

To introduce nonlocal variables, we, for convenience of notation, do some relabeling, namely, let $\chi^1 = \omega_1$ and $\chi^s = \chi_{s-1}$, $s \geq 2$. Then the nonlocal variables used below are

\begin{equation}
 \chi_{\bar{w}^kz^l} = \frac{\partial^{k+l}\chi^s}{\partial \bar{w}^k\partial z^l}, \quad i \geq 1, \quad k, l \geq 0.
\end{equation}

The total derivatives lifted to the covering equation are

\begin{align*}
 \tilde{D}_w &= D_w + W, \quad \tilde{D}_{\bar{w}} = D_{\bar{w}} + W, \quad \tilde{D}_z = D_z + Z, \quad \tilde{D}_{\bar{z}} = D_{\bar{z}} + Z,
\end{align*}

where the nonlocal tails

\begin{align*}
 W &= \sum W_{s}^{k,l} \frac{\partial}{\partial \chi_{\bar{w}^kz^l}^s}, \quad \bar{W} = \sum \bar{W}_{s}^{k,l} \frac{\partial}{\partial \chi_{\bar{w}^kz^l}^s}, \quad Z = \sum Z_{s}^{k,l} \frac{\partial}{\partial \chi_{\bar{w}^kz^l}^s}, \quad \bar{Z} = \sum \bar{Z}_{s}^{k,l} \frac{\partial}{\partial \chi_{\bar{w}^kz^l}^s}
\end{align*}

are defined by the formulas

\begin{align*}
 W_{s}^{k,l} &= \bar{D}_{\bar{w}}^{k} D_{\bar{z}}^{l}(\chi^s), \quad \bar{W}_{s}^{k,l} = \chi_{\bar{w}^kz^l+1}^s, \quad Z_{s}^{k,l} = \tilde{D}_{\bar{w}}^{k} D_{\bar{z}}^{l}(\chi^s), \quad \bar{Z}_{s}^{k,l} = \chi_{\bar{w}^kz^l+1}^s
\end{align*}

and $\chi_{\bar{w}^kz^l}^1$, $\chi_{\bar{w}^kz^l}^2$ are as defined above and take the form

\begin{equation}
\begin{align*}
 \chi^1_{\bar{w}^kz^l} &= \frac{u_{w2}}{u_{\bar{w}^{k+1}}} + \frac{u_{w1}}{u_{\bar{w}^{k+1}}} u_{w2} u_{z\bar{w}} + u_{w1} u_{\bar{w}} e^u, \\
 \chi^1_{\bar{w}^kz^l} &= \frac{u_{w2}}{u_{\bar{w}^{k+1}}} + \frac{u_{w1}}{u_{\bar{w}^{k+1}}} u_{w2} u_{z\bar{w}} + u_{w1} u_{\bar{w}} e^u, \\
 \chi^s_{\bar{w}^kz^l} &= -A_w \chi_{\bar{w}^kz^l-1}^s + \chi_{\bar{w}^{k+1}z^l}^s - (s-1)B_w \chi_{\bar{w}^kz^l-1}^{s-1}, \\
 \chi^s_{\bar{w}^kz^l} &= -A_w \chi_{\bar{w}^kz^l-1}^s + \chi_{\bar{w}^{k+1}z^l}^s - (s-1)B_w \chi_{\bar{w}^kz^l-1}^{s-1},
\end{align*}
\end{equation}

here $s \geq 2$ and

\begin{equation}
 A = \frac{1}{u_{\bar{w}}}, \quad B = \frac{u_{\bar{w}}(2u_{z} + \chi_{\bar{w}})}{2u_{\bar{w}}^2},
\end{equation}

while

\begin{equation}
 (\chi_{\bar{w}^kz^l}^s)_{w} = \bar{D}_{\bar{w}}^{k} D_{\bar{z}}^{l}(RW^s), \quad (\chi_{\bar{w}^kz^l}^s)_{z} = \tilde{D}_{\bar{w}}^{k} D_{\bar{z}}^{l}(RZ^s),
\end{equation}

where $RW^s$, $RZ^s$ are the right-hand sides in (10).
In what follows, we shall need the following presentation of the coefficients \( RW_s \):

\[
RW^1 = \frac{u_{w \bar{z}}}{u_{w}} , \\
RW^2 = \frac{1}{2} \left( \frac{1}{u_{w}^2} - \frac{1}{u_{w}^3} \right) u_{w w \bar{z}} \chi^1 + \left( \frac{u_{w \bar{z}}}{u_{w}^2} + \frac{1}{2} u_{w w \bar{z}} \right) \chi_{1w} + \frac{u_{w w \bar{z}}}{u_{w}^2} \chi_{2w} + o, \\
RW^s = \frac{1}{2} \left( \frac{1}{u_{w}^2} - \frac{1}{u_{w}^3} \right) u_{w w \bar{z}} (s-1) \chi^{s-1} + \frac{u_{w \bar{z}}}{u_{w}^2} \chi^{s-1} + \frac{u_{w w \bar{z}}}{u_{w}^2} \chi^{s-1} + o,
\]

where \( s > 2 \) and \( o \) denotes terms of lower jet order both in \( u \) and \( \chi^s \) and inessential for the subsequent computations.

3. Nontriviality of conservation laws: the proof

Equations (10)–(11) define an infinite family of (nonlocal) conservation laws

\[
\omega^{i}_{k,l} = \bar{D}_{\bar{w}}^k (R W^i) \, dw + \bar{D}_{\bar{w}}^k (R Z^i) \, dz, \quad i \geq 1, \quad k, l \geq 0,
\]

for equation (9).

In other words, on \( E \) we have

\[
\bar{D}_{\bar{w}} (\rho_{k,l}^i) - \bar{D}_{w} (\sigma_{k,l}^i) = 0
\]

where \( \rho_{k,l}^i = \bar{D}_{\bar{w}}^k (R W^i) \) and \( \sigma_{k,l}^i = \bar{D}_{\bar{w}}^k (R Z^i) \).

Note that all these conservation laws are two-component in the sense that expressions (14) involve only two total derivatives, \( \bar{D}_{\bar{w}} \) and \( \bar{D}_{w} \), out of four.

It could be of interest to find out whether [9] also has three- or four-component conservation laws, cf. e.g. [17, 18] and references therein, and to explore nonlocal symmetries for [9] involving nonlocal variables \( \chi_{k+l}^{s} \) being potentials for the conservation laws (13).

We intend to prove that the system of the conservation laws in question is nontrivial. Let us clarify this claim.

The system of conservation laws \( \omega_{k,l}^{i} \) defines the tower of coverings

\[
E = E^0 \leftarrow E^1 \leftarrow E^1_1 \leftarrow \ldots \leftarrow E^1_i \leftarrow \ldots \leftarrow E^1,
\]

where \( \varepsilon \) is the infinite prolongation of the Przanowski equation, while the covering equations \( \varepsilon^1 \) contain the nonlocal variables \( \chi_{k+l}^{1} \), \( 0 \leq k + l \leq i \), and \( \varepsilon^1 \) being the inverse limit. In a similar way, we define by induction the towers

\[
E^s \leftarrow E^{s+1} \leftarrow E^{s+1}_1 \leftarrow \ldots \leftarrow E^{s+1}_i \leftarrow \ldots \leftarrow E^{s+1},
\]

and

\[
E \leftarrow E^1 \leftarrow E^2 \leftarrow \ldots \leftarrow E^s \leftarrow \ldots \leftarrow E^s.
\]

We are going to prove the following

**Theorem 1.** For any \( i \geq 0 \), an arbitrary finite system of conservation laws \( \omega_{k,l}^{i+1} \) of the equation \( \varepsilon^i \) is linearly independent.

The nontriviality of these conservation laws is, in view of their structure, see [8], a straightforward consequence of their linear independence.

The proof of Theorem 1 will be based on the following

**Proposition 2** (see [11], cf. [13, 12]). Let \( E \) be a differentially connected equation [1]. The conservation laws \( \omega_{k,l}^{i} \) are mutually independent in the sense of Theorem 1 if and only if \( \varepsilon^{*} \) is differentially connected as well, i.e., the only solutions of the system

\[
\bar{D}_{\bar{w}} (f) = \bar{D}_{w} (f) = \bar{D}_{\bar{z}} (f) = \bar{D}_{z} (f) = 0, \quad f \in C^\infty (\varepsilon^*),
\]

are constants.

**Proof of Theorem 1.** We begin the proof with an easy observation:

**Lemma 1.** The Przanowski equation (2) is differentially connected.

\footnote{Recall that an equation is called differentially connected if the only functions that are invariant with respect to all total derivatives on \( \varepsilon \) are constants.}
Before proceeding with the proof of the theorem, let us briefly describe its outline. We prove that the space \( \ker \tilde{D}_w \) consists of functions that depend on \( \bar{w}, z, \) and \( \bar{z} \) only, from where the desired result follows immediately. To do this, we perform double induction: on \( i \) in (16) and on \( s \) in (13) for each \( i \). The case \( i = 1 \) is special and is considered separately. So, the base of induction is \( i = 2 \).

Let us employ the notation \( \mathcal{F}(\mathcal{U}, n_1, \ldots, n_s) \) for the space of functions that depend on a finite set \( \mathcal{U} \) of internal coordinates in \( \mathcal{E} \) and nonlocal variables \( \chi_{\bar{w}^k \bar{z}^l}^\alpha, \alpha = 1, \ldots, s, k + l \leq n_\alpha \).

**Lemma 2.** If \( F \in \mathcal{F}(\mathcal{U}, n_1, \ldots, n_s) \), then \( n_s > n_{s-1} > \cdots > n_1 \).

**Proof of Lemma** The fact is a straightforward consequence of the defining equations (10) and (11).

Let us now pass to the proof of the theorem.

**Step 1** \((s = 1)\). We prove here by induction on \( n_1 \) that the conservation laws \( \omega^1_{k,l} \) are linearly independent. Denote \( n_1 = n \). Let \( n = 1 \) and perform induction on \( n \). Let \( n = 0 \). Then

\[
\tilde{D}_w(F) = D_w(F) + \frac{u_{w\bar{z}}}{u_{\bar{w}}} \frac{\partial F}{\partial \chi^1} = 0.
\]

This means that the set \( \mathcal{U} \) may consist of the variables \( w, \bar{w}, z, \bar{z}, \) and \( u_\bar{z} \) only, i.e.,

\[
\frac{\partial F}{\partial w} + u_{w\bar{z}} \frac{\partial F}{\partial z} + u_{w\bar{z}} \frac{\partial F}{\partial \chi^1} = 0
\]

and consequently

\[
\frac{\partial F}{\partial u_{\bar{z}}} + \frac{1}{u_{\bar{w}}} \frac{\partial F}{\partial \chi^1} = 0.
\]

Thus, \( F = F(\bar{w}, z, \bar{z}) \).

Let now \( n > 0 \). Then

\[
\tilde{D}_w = D_w + \sum_{k+l \leq n} \tilde{D}_{\bar{w}^k} \tilde{D}_{\bar{w}^l} \left( \frac{u_{w\bar{z}}}{u_{\bar{w}}} \right) \frac{\partial}{\partial \chi_{\bar{w}^k \bar{z}^l}^1}.
\]

In this expression, the variables \( u_\sigma \) of the maximal jet order are \( u_{w\bar{w}^k \bar{z}^l}^1, k + l = n \). Consequently, \( F \) is invariant with respect to the fields

\[
Z_{k,l} = \frac{\partial}{\partial u_{w\bar{w}^k \bar{z}^l}^1} + \frac{1}{u_{\bar{w}}} \frac{\partial}{\partial \chi_{\bar{w}^k \bar{z}^l}^1}, \quad k + l = n.
\]

Note now that the coefficients of \( \tilde{D}_w \) at \( \partial/\partial \chi_{\bar{w}^k \bar{z}^l}^1 \) are independent of the variables \( u_{w\bar{w}^i}^1, i > 0 \). From this fact in immediately follows that \( F \) cannot depend on \( u_{\bar{w}^i}^1, i > 0 \). Hence, taking commutators of the fields \( Z_{k,l} \) with \( \partial/\partial u_{\bar{w}^i}^1 \) we obtain that \( F \) is invariant with respect to the derivations \( \partial/\partial \chi_{\bar{w}^k \bar{z}^l}^1 \), and this finishes the induction step.

**Step 2** \((s = 2)\), the base of induction on \( s \). Let now \( F \in \mathcal{F}(\mathcal{U}, n_1, n_2) \).

**Lemma 3.** \( n_1 = n_2 + 1 \).

**Proof of Lemma** Consider the total derivative

\[
\tilde{D}_w(F) = \frac{\partial F}{\partial w} + \sum_{\mathcal{U}} u_{w\sigma} \frac{\partial F}{\partial u_\sigma} + \sum_{k+l \leq n_1} \tilde{D}_{\bar{w}^k} \tilde{D}_{\bar{w}^l} (RW^1) \frac{\partial F}{\partial \chi_{\bar{w}^k \bar{z}^l}^1} + \sum_{k+l \leq n_2} \tilde{D}_{\bar{w}^k} \tilde{D}_{\bar{w}^l} (RW^2) \frac{\partial F}{\partial \chi_{\bar{w}^k \bar{z}^l}^2}.
\]

Assume that \( n_1 > n_2 + 1 \). Then differentiating (17) with respect to \( u_{w\bar{w}^k \bar{z}^l+1}^1 \) we get that \( F \) is invariant with respect to the field

\[
Z = \frac{\partial}{\partial u_{w\bar{w}^k \bar{z}^l+1}^1} + \frac{1}{u_{\bar{w}}} \frac{\partial}{\partial \chi_{w\bar{w}^k \bar{z}^l}^1}.
\]

Consequently, it is invariant with respect to the commutator

\[
[\tilde{D}_w, Z] = \tilde{D}_w \left( \frac{1}{u_{\bar{w}}} \right) \frac{\partial}{\partial \chi_{w\bar{w}^k \bar{z}^l}^1},
\]

i.e., \( F \) does not depend on \( \chi_{w\bar{w}^k \bar{z}^l}^1 \) for \( k + l = n_1 \). \( \square \)
Set \( n_2 = n \); then \( n_1 = n + 1 \) and let \( F \in \mathcal{F}(\mathcal{W}, n + 1, n) \). The internal coordinates of maximal order on which the coefficients at \( \partial / \partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1 \) and \( \partial / \partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2 \) in \( D_w(F) \) depend are

\[
\begin{align*}
&u_{w\bar{w}^n+1}, u_{w\bar{w}^n+2}, \ldots, u_{w\bar{w}^{n+2}} \quad \text{for} \quad \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1}, \\
&u_{w\bar{w}^{n+1}}, u_{w\bar{w}^n+2}, \ldots, u_{w\bar{w}^{n+1}} \quad \text{for} \quad \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2}.
\end{align*}
\]

Differentiating \( D_w(F) \) with respect to these coordinates, we see that \( F \) must be invariant with respect to the fields

\[
\begin{align*}
Z_0 &= \frac{\partial}{\partial u_{w\bar{w}^n+2}} + \frac{1}{2u_{\bar{w}}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1}, \\
Z_1 &= \frac{\partial}{\partial u_{w\bar{w}^n+1}} + \frac{1}{2u_{\bar{w}}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1} + \frac{1}{2} \left( \frac{1}{u_{\bar{w}}} - \frac{1}{u_{\bar{w}}} \right) \chi \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2}, \\
\cdots \\
Z_i &= \frac{\partial}{\partial u_{w\bar{w}^{n-i+2}}} + \frac{1}{2u_{\bar{w}}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1} + \frac{1}{2} \left( \frac{1}{u_{\bar{w}}} - \frac{1}{u_{\bar{w}}} \right) \chi \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2}, \\
\cdots \\
Z_{n+1} &= \frac{\partial}{\partial u_{w\bar{w}^n+1}} + \frac{1}{2u_{\bar{w}}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1} + \frac{1}{2} \left( \frac{1}{u_{\bar{w}}} - \frac{1}{u_{\bar{w}}} \right) \chi \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2}.
\end{align*}
\]

Note that \([Z_i, Z_j] = 0\). We use the induction on \( n \). The base is \( n = 0 \). Compute the commutator

\[
[D_w, Z_0] = -2u_{w\bar{w}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1} + \frac{1}{2u_{\bar{w}}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2}.
\]

Hence, \( F \) is invariant with respect to

\[
Z_0^1 = 2u_{\bar{w}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1} + \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2}.
\]

Now,

\[
[D_w, Z_0^1] = 2u_{w\bar{w}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1} - 2u_{\bar{w}} \frac{u_{w\bar{w}}}{u_{\bar{w}}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2} = 2u_{w\bar{w}} \left( u_{\bar{w}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1} - \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2} \right)
\]

and thus the field

\[
Z_0^2 = u_{\bar{w}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1} - \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2}
\]

annihilates \( F \) as well. But

\[
\frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2} = \frac{1}{2} (Z_0^1 - Z_0^2)
\]

which means that \( F \) does not depend on \( \chi_{\mathcal{W}^{k \mathcal{W}}}^2 \), i.e., we find ourselves in the situation of Step 1.

Now pass to the induction step and assume \( n > 0 \). Similar to the previous computations, we see that the commutator \([D_w, Z_0] \) is proportional to the field

\[
Z_0^1 = 2u_{\bar{w}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1} + \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2},
\]

while the commutator \([D_w, Z_0^1] \) is proportional to

\[
Z_0^2 = u_{\bar{w}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1} - \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2}
\]

Thus, \( F \) is independent of \( \chi_{\mathcal{W}^{k \mathcal{W}}}^2 \), while the field \( Z_1 \) we change to new

\[
Z_1 = \frac{\partial}{\partial u_{w\bar{w}^n+1}} + \frac{1}{2u_{\bar{w}}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1}.
\]

Then \([D_w, Z_1] \) is proportional to the field

\[
Z_1^1 = 2u_{\bar{w}} \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^1} + \frac{\partial}{\partial \chi_{\mathcal{W}^{k \mathcal{W}}}^2},
\]
while the commutator $[\tilde{D}_w, Z_1]$ equals, up to a factor, the field
\[
Z_1^2 = u_w \frac{\partial}{\partial \chi_{\omega^2 n}} + \frac{\partial}{\partial \chi_{\omega^2 n-1}}.
\]
Hence, $F$ is independent of $\chi_{\omega^2 n-1}$ and instead of $Z_2$ we can take the field
\[
\frac{\partial}{\partial u_{\omega^2 n}} + \frac{1}{u_w^2} \frac{\partial}{\partial \chi_{\omega^2 n-1}},
\]
etc. Eventually, we shall arrive to the independence of $F$ on all the variables $\chi_{\omega^2 n-1}$, and this completes the induction step for $s = 2$.

**Step 3** (the induction step). Let $F \in \mathcal{F}(\mathcal{U}, n_1, \ldots, n_s)$

**Lemma 4.** One has $n_j = n_{j+1} + 1$, $j = 1, \ldots, s - 1$, i.e., $n_j = n + s - j$, where $n = n_s$.

**Proof of Lemma 4.** Similar to the proof of Lemma 3. $\square$

Thus, $F \in \mathcal{F}(\mathcal{U}, n + s - 1, \ldots, n)$ and the internal coordinates of maximal jet order on which the coefficients in the total derivatives at $\partial/\partial \chi_{w^k \bar{z}^l}$ that act nontrivially on $F$ may depend are
\[
u_{w^{n+s-1} \bar{z}}^2, u_{w^{n+s-2} \bar{z}^2}, \ldots, u_{w^{n+s}} \quad \text{for} \quad \frac{\partial}{\partial \chi_{\omega^k \bar{z}^l}},
\]
\[
u_{w^{n+s-1} \bar{z}}^2, u_{w^{n+s-2} \bar{z}^2}, \ldots, u_{w^{n+s-1}} \quad \text{for} \quad \frac{\partial}{\partial \chi_{\omega^k \bar{z}^l}}.
\]
Consequently, $F$ is invariant with respect to the vector fields
\[
Z_0 = \frac{\partial}{\partial u_{\omega^2 n+s}} + \frac{1}{u_w^2} \frac{\partial}{\partial \chi_{\omega^2 n+s-1}},
\]
\[
Z_1 = \frac{\partial}{\partial u_{\omega^2 n+s-1}} + \frac{1}{u_w^2} \frac{\partial}{\partial \chi_{\omega^2 n+s-2}} + \frac{1}{2} \left( \frac{1}{u_w^2} - \frac{1}{u_w^3} \right) \chi^1 \frac{\partial}{\partial \chi_{\omega^2 n+s-2}},
\]
\[
\vdots
\]
\[
Z_i = \frac{\partial}{\partial u_{\omega^2 n+s-i-1}} + \frac{1}{u_w^2} \frac{\partial}{\partial \chi_{\omega^2 n+s-i-1}} + \frac{1}{2} \left( \frac{1}{u_w^2} - \frac{1}{u_w^3} \right) \chi^1 \frac{\partial}{\partial \chi_{\omega^2 n+s-i-1}},
\]
\[
\vdots
\]
\[
Z_{n+s-1} = \frac{\partial}{\partial u_{\omega^2 n+s-1}} + \frac{1}{u_w^2} \frac{\partial}{\partial \chi_{\omega^2 n+s-2}} + \frac{1}{2} \left( \frac{1}{u_w^2} - \frac{1}{u_w^3} \right) \chi^1 \frac{\partial}{\partial \chi_{\omega^2 n+s-2}}.
\]
We first fix $s > 2$ and set $n = 0$ (the base of induction). Then
\[
Z_0 = \frac{\partial}{\partial u_{\omega^2 z}} + \frac{1}{u_w^2} \cdot \frac{\partial}{\partial \chi_{\omega^2 z-1}}.
\]
Hence the commutator $[\tilde{D}_w, Z_0]$ is proportional to
\[
Z_0^1 = 2u_w \frac{\partial}{\partial \chi_{\omega^2 z-1}} + \frac{\partial}{\partial \chi_{\omega^2 z-2}},
\]
for the commutator $[\tilde{D}_w, Z_0^1]$ we get
\[
Z_0^2 = 3u_w \frac{\partial}{\partial \chi_{\omega^2 z-2}} + \frac{\partial}{\partial \chi_{\omega^2 z-3}},
\]
etc., and $[\tilde{D}_w, Z_0^{s-2}]$ equals, up to a functional factor, to the vector field
\[
Z_0^{s-1} = s u_w \frac{\partial}{\partial \chi_{\omega^2 z-1}} + \frac{\partial}{\partial \chi^s}.
\]
Finally, for $[\tilde{D}_w, Z_0^{s-1}]$ we get
\[
Z_0^{s} = u_w \frac{\partial}{\partial \chi^{s-1}} - \frac{\partial}{\partial \chi^s}.
Proceeding in a similar fashion, we shall obtain the fields

\[ Z_0^i = [\hat{D}_w, Z_0^{i-1}] \sim (i + 1)u_{\bar{w}} \frac{\partial}{\partial \chi_{\bar{w}z^{n+i-1}}} + \frac{\partial}{\partial \chi_{\bar{w}z^{n+i-2}}} \]

where \( \sim \) denotes proportionality. In particular,

\[ Z_0^{s-1} = su_{\bar{w}} \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-1}}} + \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-2}}} \]

Then we have

\[ [\hat{D}_w, Z_0^{s-1}] \sim u_{\bar{w}} \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-1}}} - \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-2}}} = Z_0^s. \]

This leads to the invariance of \( F \) with respect to the fields \( \partial/\partial \chi_{\bar{w}z^{n+s-1}} \) and \( \partial/\partial \chi_{\bar{w}z^{n+s-2}} \). In addition, using equations (19) we obtain by induction independence of \( F \) on all the variables \( \chi_{\bar{w}z^{n+i}} \). In particular, this means that instead of \( Z_1 \) in equations (18) we can consider the field

\[ Z_1 = \frac{\partial}{\partial u_{\bar{w}z^{n+s-1}}} + \frac{\partial}{u_{\bar{w}}^2 \partial \chi_{\bar{w}z^{n+s-2}}} \]

Using (12) we get

\[ [\hat{D}_w, Z_1] = -2u_{\bar{w}w} \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-2}}} - \frac{1}{u_{\bar{w}}^2} \left( u_{\bar{w}w} \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-3}}} + u_{\bar{w}w} \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-2}}} \right). \]

But thanks to the previous remark, the last summand can be skipped and we can set

\[ Z_1^i = 2u_{\bar{w}} \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-2}}} + \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-3}}} \]

Proceeding in a similar fashion, we shall obtain the fields

\[ Z_1^s = (i + 1)u_{\bar{w}} \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-1}}} + \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-2}}} \]

In particular,

\[ Z_1^{s-1} = su_{\bar{w}} \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-1}}} + \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-2}}} \]

and

\[ Z_1^s = u_{\bar{w}} \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-1}}} - \frac{\partial}{\partial \chi_{\bar{w}z^{n+s-2}}} \]

Using the same reasoning as for the fields \( Z_0^i \), we deduce that \( F \) does not depend on the variables \( \chi_{\bar{w}z^{n+i-1}} \), etc. Eventually, we shall arrive at the independence of \( F \) of all variables \( \chi_{\bar{w}z^{n+i}} \). This completes the induction step.

Theorem 1 is proved. \( \square \)

**References**

[1] M. Ablowitz, P.A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge University Press, Cambridge, 1991.

[2] S. Alexandrov, B. Pioline, and S. Vandoren, Self-dual Einstein spaces, heavenly metrics, and twistors, J. Math. Phys. 51 (2010), 073510, arXiv:0912.3406.

[3] M. Atiyah, M. Dunajski, L.J. Mason, Twistor theory at fifty: from contour integrals to twistor strings, Proc. R. Soc. A 473 (2017), no. 2206, art. 20170530.

[4] O. Babelon, D. Bernard, M. Talon, *Introduction to classical integrable systems*, Cambridge University Press, Cambridge, 2003.

[5] A.V. Bocharov et al., *Symmetries of Differential Equations in Mathematical Physics and Natural Sciences*, edited by A.M. Vinogradov and I.S. Krasil’shchik. Factorial Publ. House, 1997 (in Russian). English translation: Amer. Math. Soc., 1999.

[6] P. Bull et al., Beyond ΛCDM: Problems, solutions, and the road ahead, Physics of the Dark Universe 12 (2016), 56–99, arXiv:1512.05356.

[7] S.P. Burtsev, V.E. Zakharov, A.V. Mikhailov, Inverse scattering method with variable spectral parameter, Theoret. and Math. Phys. 70 (1987), no. 3, 227–240.
[8] F. Calogero, Why are certain nonlinear PDEs both widely applicable and integrable?, in What is integrability?, ed. by V.E. Zakharov, Springer, Berlin, 1991, 1–62.

[9] M. Hoegner, Quaternion-Kähler four-manifolds and Przanowski’s function, J. Math. Phys. 53, 103517 (2012). arXiv:1205.3977

[10] V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, Quantum inverse scattering method and correlation functions, Cambridge University Press, Cambridge, 1993.

[11] I.S. Krasil’schik, Integrability in differential coverings, J. Geom. Phys. 87 (2015) 296–304. arXiv:1310.1189

[12] I.S. Krasil’schik, A. Sergyeyev, Integrability of S-deformable surfaces: Conservation laws, Hamiltonian structures and more, J. Geom. Phys. 97 (2015), 266–278. arXiv:1511.09430

[13] I.S. Krasil’schik, A. Sergyeyev, O.I. Morozov, Infinitely many nonlocal conservation laws for the ABC equation with \( A + B + C \neq 0 \), Calc. Var. PDE 55 (2016), 1–12. arXiv:1511.09430

[14] I.S. Krasil’schik and A.M. Vinogradov, Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations, Acta Appl. Math. 15 (1989) 1–2, 161–209.

[15] J. Krasil’schik, A. Verbovetsky, and R. Vitolo, The symbolic computation of integrability structures for partial differential equations, Springer, Texts & Monographs in Symbolic Computation, 2017.

[16] K. Krasnov, Self-dual gravity, Class. Quantum Grav. 34 (2017) 095001. arXiv:1610.01457

[17] A. Lelito, O.I. Morozov, Three-component nonlocal conservation laws for Lax-integrable 3D partial differential equations. J. Geom. Phys. 131 (2018), 89–100.

[18] Z. Makridin, An effective algorithm for finding multidimensional conservation laws for integrable systems of hydrodynamic type, Theor. Math. Phys. 194 (2018), no. 2, 274–283.

[19] S.V. Manakov, P.M. Santini, Integrable dispersionless PDEs arising as commutation condition of pairs of vector fields, J. Phys. Conf. Ser. 482 (2014), 012029. arXiv:1312.2740

[20] L.J. Mason, N.M.J. Woodhouse, Integrability, self-duality, and twistor theory, Clarendon & Oxford Univ. Press, N.Y., 1996.

[21] P.J. Olver, Applications of Lie groups to differential equations, 2nd ed., Springer, N.Y., 1993.

[22] J.F. Plebański, Some solutions of complex Einstein equations, J. Math. Phys. 16 (1975), no. 12, 2395–2402.

[23] M. Przanowski, Locally Hermit-Einstein, self-dual gravitational instantons, Acta Phys. Polon. B14 (1983), 625–627.

[24] A. Sergyeyev, New integrable (3+1)-dimensional systems and contact geometry, Lett. Math. Phys. 108 (2018), no. 2, 359–376. arXiv:1401.2122

[25] M.B. Sheftel, A.A. Malych, Partner symmetries, group foliation and ASD Ricci-flat metrics without Killing vectors, SIGMA 9 (2013), 075 arXiv:1306:3195

[26] I.A.B. Strachan, The symmetry structure of the anti-self-dual Einstein hierarchy, J. Math. Phys. 36 (1995), 3566–3573, arXiv:hep-th/9410047.

[27] E. Witten, Integrable lattice models from gauge theory, Adv. Theor. Math. Phys. 21 (2017), no. 7, 1819–1843. arXiv:1611.00592

[28] V.E. Zakharov, Integrable systems in multidimensional spaces, in Mathematical problems in theoretical physics (Berlin, 1981), Springer, Berlin, 1982, 190–216.

[29] V.E. Zakharov, Dispersionless limit of integrable systems in 2+1 dimensions, in Singular limits of dispersive waves (Lyon, 1991), Plenum, New York, 1994, 165–174.

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