Completing the solution for the $OSp(1|2)$ spin chain

Rafael I. Nepomechie
Physics Department, P.O. Box 248046
University of Miami, Coral Gables, FL 33124

Abstract

The periodic $OSp(1|2)$ quantum spin chain has both a graded and a non-graded version. Naively, the Bethe ansatz solution for the non-graded version does not account for the complete spectrum of the transfer matrix, and we propose a simple mechanism for achieving completeness. In contrast, for the graded version, this issue does not arise. We also explain (for both versions) the degeneracies and multiplicities of the transfer-matrix spectrum, and obtain conditions for selecting the physical singular solutions of the Bethe equations.
1 Introduction

The periodic $OSp(1|2)$ quantum spin chain was first formulated (as a graded model) and solved in [1]. A non-graded version of this model was formulated and solved in [2, 3]. These solutions have figured in various subsequent works, see e.g. [4, 5, 6, 7, 8, 9, 10] and references therein. In this note, we argue that – naively – the solution for the non-graded version does not account for the complete spectrum of the transfer matrix, and we propose a simple way to remedy the problem. In contrast, for the graded version, this issue does not arise. We also explain (for both versions) the degeneracies and multiplicities of the transfer-matrix spectrum, and obtain conditions for selecting the physical singular solutions of the Bethe equations. As a byproduct of our investigation, we clarify the signs in the Bethe equations, over which there has been some controversy, see e.g. [3, 9].

We treat the non-graded version of the $OSp(1|2)$ model in Sec. 2, and the graded version in Sec. 3. Our conclusions are in Sec. 4. The $OSp(1|2)$ symmetry of the graded transfer matrix is proved in the appendix.

2 The non-graded version

In Sec. 2.1, we review the construction of the transfer matrix for the non-graded version of the $OSp(1|2)$ quantum spin chain, and we note that it has only $SU(2)$ symmetry. In Sec. 2.2, we briefly review the solution [2, 3]. The degeneracies and multiplicities of the transfer-matrix spectrum are explained in Sec. 2.3. We give conditions for selecting the physical singular solutions of the Bethe equations in Sec. 2.4. The difficulty with completeness and its resolution are discussed in Sec. 2.5.

2.1 The transfer matrix and its symmetries

The $OSp(1|2)$ spin chain has a 3-dimensional vector space at each site. Following [3], we consider here the bosonic (non-graded) formulation of the model. The R-matrix is given by

\[
R(u) = \begin{pmatrix}
  a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & b & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & d & e & f & 0 & 0 & 0 & 0 & 0 \\
  0 & c & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & e & g & 0 & -e & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & b & 0 & c & 0 & 0 & 0 \\
  0 & 0 & f & 0 & -e & 0 & d & 0 & 0 & 0 \\
  0 & 0 & g & 0 & c & 0 & b & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0
\end{pmatrix},
\]

(2.1)

1See Eq. (5) in [3] with $\eta = i$. 

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where
\[
a = i - u, \quad b = u, \quad c = i, \quad d = -\frac{u(2u - i)}{2u - 3i}, \quad e = \frac{2iu}{2u - 3i}, \quad f = \frac{4iu + 3}{2u - 3i}, \quad g = u + \frac{3}{2u - 3i}.
\]

(2.2)

It has the regularity property \( R(0) = i\mathcal{P} \), where \( \mathcal{P} \) is the (non-graded) permutation matrix on \( \mathbb{C}^3 \otimes \mathbb{C}^3 \)
\[
\mathcal{P} = \sum_{a,b=1}^3 e_{ab} \otimes e_{ba}, \quad (e_{ab})_{ij} = \delta_{a,i} \delta_{b,j}.
\]

(2.3)

This \( R \)-matrix is a solution of the (non-graded) Yang-Baxter equation
\[
R_{12}(u - v) R_{13}(v) R_{23}(u) = R_{23}(v) R_{13}(u) R_{12}(u - v),
\]

(2.4)

where
\[
R_{12} = R \otimes I, \quad R_{13} = \mathcal{P}_{23} R_{12} \mathcal{P}_{23}, \quad R_{23} = \mathcal{P}_{12} R_{13} \mathcal{P}_{12}.
\]

(2.5)

The transfer matrix \( t(u) \) for a closed spin chain of length \( N \) with periodic boundary conditions, which is given as usual by the (non-graded) trace of the monodromy matrix
\[
t(u) = \text{tr}_0 R_{0N}(u) \cdots R_{01}(u),
\]

(2.6)

has the commutativity property
\[
[t(u), t(v)] = 0.
\]

(2.7)

The corresponding Hamiltonian is given by
\[
H = -i \frac{d}{du} \log t(u) \bigg|_{u=0}.
\]

(2.8)

The transfer matrix has \( SU(2) \) symmetry\(^2\)
\[
[t(u), S^z] = [t(u), S^\pm] = 0,
\]

(2.9)

where
\[
S^z = \sum_{n=1}^N s^z_n, \quad S^\pm = \sum_{n=1}^N s^\pm_n,
\]

(2.10)

and
\[
s^z = \frac{1}{2} (e_{11} - e_{33}), \quad s^+ = e_{13}, \quad s^- = e_{31},
\]

(2.11)

with \( e_{ab} \) defined in (2.3). Indeed, the generators (2.10) obey the \( SU(2) \) algebra
\[
[S^z, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^z.
\]

(2.12)

\(^2\)The proof is similar to the one in Sec. A.1 for the graded transfer matrix.
2.2 The Bethe ansatz solution

Let $|\Lambda^{(M)}\rangle$ denote the simultaneous eigenvectors of the transfer matrix $t(u) \ (2.6)$ and of $S^z \ (2.10)$ that are $SU(2)$ highest-weight states

$$S^+ |\Lambda^{(M)}\rangle = 0, \quad (2.13)$$

and let $\Lambda^{(M)}(u)$ and $m$ denote the corresponding eigenvalues

$$t(u) |\Lambda^{(M)}\rangle = \Lambda^{(M)}(u) |\Lambda^{(M)}\rangle, \quad S^z |\Lambda^{(M)}\rangle = m |\Lambda^{(M)}\rangle. \quad (2.14)$$

It was argued in [3] that these eigenvalues are given by

$$m = \frac{1}{2} (N - M), \quad M = 0, 1, \ldots, N, \quad (2.15)$$

and

$$\Lambda^{(M)}(u) = (i - u)^N (-1)^M \frac{Q(u + \frac{i}{2})}{Q(u - \frac{i}{2})} + u^N \frac{Q(u) Q(u - \frac{3i}{2})}{Q(u - i) Q(u - \frac{i}{2})}$$

$$+ \left( \frac{u(i - u)}{u - \frac{3i}{2}} \right)^N (-1)^M \frac{Q(u - 2i)}{Q(u - i)}, \quad (2.16)$$

where

$$Q(u) = \prod_{k=1}^{M} (u - u_k). \quad (2.17)$$

The corresponding Bethe equations for the Bethe roots $\{u_1, \ldots, u_M\}$ are given by

$$\left( \frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}} \right)^N = -(-1)^{N-M} \frac{Q(u_j - i) Q(u_j + \frac{i}{2})}{Q(u_j + i) Q(u_j - \frac{i}{2})}, \quad j = 1, \ldots, M. \quad (2.18)$$

An unusual feature of the TQ-equation (2.16) is its explicit dependence (through the factors $(-1)^M$) on the number $M$ of Bethe roots; and there is a corresponding unusual factor $(-1)^{N-M}$ in the Bethe equations (2.18). It was implicitly assumed in [3] that all the Bethe roots are finite and pairwise distinct.

2.3 Degeneracies and multiplicities

The degeneracy (the number of times that a given eigenvalue $\Lambda^{(M)}(u)$ appears) is given by $N - M + 1$, since the eigenstates form $SU(2)$ irreducible representations with spin $s = m = (N - M)/2$, see (2.15), which have dimension $2s + 1$. Since the 3-dimensional vector space at

These vectors can be constructed by algebraic Bethe ansatz following [11], as sketched in appendix B of [3]. The highest-weight property can be checked for $M = 1, 2$, and we conjecture that it is generally true.
each site decomposes as spin-1/2 plus spin-0, the multiplicities in the spectrum follow from
the Clebsch-Gordan decomposition of \((2 \oplus 1)^{\otimes N}\). For example,

- For \(N = 2\):
  \((2 \oplus 1)^{\otimes 2} = 2 \cdot 1 \oplus 2 \cdot 2 \oplus 3\)

- For \(N = 3\):
  \((2 \oplus 1)^{\otimes 3} = 4 \cdot 1 \oplus 5 \cdot 2 \oplus 3 \cdot 3 \oplus 4\)

- For \(N = 4\):
  \((2 \oplus 1)^{\otimes 4} = 9 \cdot 1 \oplus 12 \cdot 2 \oplus 9 \cdot 3 \oplus 4 \cdot 4 \oplus 5\)

(2.19)

Hence, for \(N = 2\), there are two eigenvalues with degeneracy 1 \((M = 2)\), two eigenvalues
with degeneracy 2 \((M = 1)\), and one eigenvalue with degeneracy 3 \((M = 0)\); and similarly
for higher \(N\).

### 2.4 Physical singular solutions of the Bethe equations

As in the periodic Heisenberg spin chain [12, 13, 14], the Bethe equations (2.18) have many
“singular” solutions containing \(\pm \frac{i}{2}\); only a subset of these singular solutions, which we call
“physical”, correspond to actual eigenvalues and eigenstates of the transfer matrix. Starting
from the Bethe equations for the model with twisted boundary conditions (see appendix C in
[3]) and following [14], we find that the condition for the singular solution \(\{\frac{i}{2}, -\frac{i}{2}, u_3, \ldots, u_M\}\)
to be physical is

\[
\left(\prod_{j=3}^{M} \frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}}\right)^N = (-1)^{(N-M+1)M}.
\]

(2.20)

Moreover, \(\{u_3, \ldots, u_M\}\) must also obey

\[
\left(\frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}}\right)^{N-1} \left(\frac{u_j + \frac{3i}{2}}{u_j - \frac{3i}{2}}\right) \left(\frac{u_j - i}{u_j + i}\right) = (-1)^{N-M+1} \prod_{k=3}^{M} \frac{(u_j - u_k - i)(u_j - u_k + \frac{i}{2})}{(u_j - u_k + i)(u_j - u_k - \frac{i}{2})},
\]

\(j = 3, \ldots, M\).

(2.21)

### 2.5 Completing the Bethe ansatz solution

It was argued in Sec. 3 of [3], by considering all finite solutions of the Bethe equations
(2.18) for the case \(N = 4\), that the Bethe ansatz solution (2.16)-(2.18) accounts for all the
eigenvalues of the Hamiltonian (2.8).

However, we find that this Bethe ansatz solution cannot account in this way for all the
eigenvalues of the transfer matrix (2.6). Indeed, for \(N = 2\), there is an eigenvalue with
\(M = 1\) that cannot be described by this solution, namely4

\[
\Lambda^{(1)}(u) = \frac{-4u^4 + 12iu^3 + 29u^2 - 30iu - 9}{(2u - 3i)^2}.
\]

(2.22)

4The Bethe equation (2.18) with \(N = 2\) and \(M = 1\) has only one finite solution, namely \(u_1 = 0\), which
describes through (2.16) a transfer-matrix eigenvalue with \(M = 1\) that is different from (2.22).
For $N = 3$, there are 2 such eigenvalues (with $M = 2$); and for $N = 4$, there are 7 such eigenvalues (1 eigenvalue with $M = 1$, and 6 eigenvalues with $M = 3$).

We propose a simple way to account for the missing transfer-matrix eigenvalues: admit precisely one infinite Bethe root. Since the Q-functions appear in the expression for the eigenvalues (2.16) only as ratios, the effect in (2.16) of one infinite Bethe root ($u_M = \infty$) is to replace the expression for $Q(u)$ (2.17) by a product over only the finite roots, i.e.

$$Q(u) = \prod_{k=1}^{M-1} (u - u_k),$$

where \{u_1, \ldots, u_{M-1}\} are finite. Moreover, the Bethe equations for the finite roots are again given by (2.18) except with $Q(u)$ given by (2.23).

We have explicitly verified that all the missing transfer-matrix eigenvalues for $N = 2, 3, 4$ can be obtained in this way, with $u_M = \infty$. The Bethe roots corresponding to all of the transfer-matrix eigenvalues for $N = 2, 3$ are given in Table 1. Note that the degeneracies and multiplicities are in accordance with the group-theory predictions in Sec. 2.3. For $N = 4$, we report only the Bethe roots corresponding to the “missing” transfer-matrix eigenvalues, and the physical singular solutions, see Table 2. The physical singular solutions in Tables 1 and 2 satisfy (2.20)-(2.21).

| $N$ | $M$ | degeneracy | $\{u_1, \ldots, u_M\}$ |
|-----|-----|------------|------------------|
| 2   | 0   | 3          |                  |
| 2   | 1   | 2          | $\infty$         |
| 2   | 1   | 2          | 0                |
| 2   | 2   | 1          | $\pm \sqrt{2}/4$|
| 2   | 2   | 1          | $\pm i/2$        |
| 3   | 0   | 4          |                  |
| 3   | 1   | 3          | $\sqrt{3}/2$     |
| 3   | 1   | 3          | $-\sqrt{3}/2$    |
| 3   | 1   | 3          | 0                |
| 3   | 2   | 2          | $\sqrt{3}/6, \infty$ |
| 3   | 2   | 2          | $-\sqrt{3}/6, \infty$ |
| 3   | 2   | 2          | $(3\sqrt{3} \pm i\sqrt{13})/8$ |
| 3   | 2   | 2          | $(-3\sqrt{3} \pm i\sqrt{13})/8$ |
| 3   | 2   | 2          | $\pm \sqrt{5}/10$ |
| 3   | 3   | 1          | 0.507442, -0.253721 ± 0.505591i |
| 3   | 3   | 1          | -0.507442, 0.253721 ± 0.505591i |
| 3   | 3   | 1          | 0, ±0.550503     |
| 3   | 3   | 1          | 0, ±0.963355i    |

Table 1: Bethe roots for $N = 2, 3$ corresponding to all eigenvalues in the non-graded version
Table 2: Bethe roots for $N = 4$ corresponding to “missing” eigenvalues, and physical singular solutions, in the non-graded version

3 The graded version

We turn now to the graded version of the periodic $OSp(1|2)$ quantum spin chain [1]. We review the construction of the transfer matrix and its symmetries in Sec. 3.1. We review the solution in Sec. 3.2, explain the degeneracies and multiplicities of the transfer-matrix spectrum in Sec. 3.3, and check completeness in Sec. 3.4.

3.1 The transfer matrix and its symmetries

The graded R-matrix is given by

$$R(u) = \begin{pmatrix}
  a & 0 & 0 & 0 & 0 & 0 & e & 0 & -e & 0 \\
  0 & b & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & b & 0 & 0 & c & 0 & 0 & 0 & 0 \\
  0 & c & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\
  -e & 0 & 0 & 0 & 0 & g & 0 & f & 0 & 0 \\
  0 & 0 & c & 0 & 0 & 0 & b & 0 & 0 & 0 \\
  e & 0 & 0 & 0 & 0 & f & 0 & g & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0
\end{pmatrix}, \quad (3.1)$$

where

$$a = u + i - \frac{2iu}{2u - 3i}, \quad b = u, \quad c = i, \quad d = u - i,$$

$$e = \frac{2iu}{2u - 3i}, \quad f = -\frac{4iu + 3}{2u - 3i}, \quad g = \frac{u(2u - i)}{2u - 3i}. \quad (3.2)$$

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5See Eq. (2.20) in [1] with $\Delta = \frac{3}{2}$; up to an overall factor, our R-matrix is $iR_{Kulish}(u/i)$. 
It has the regularity property \( R(0) = i\mathcal{P} \), where \( \mathcal{P} \) is now the graded permutation matrix

\[
\mathcal{P} = \sum_{a,b=1}^{3} (-1)^{p(a)p(b)} e_{ab} \otimes e_{ba}, \tag{3.3}
\]

where the gradings are given by \( p(1) = 0, p(2) = p(3) = 1 \). This R-matrix is a solution of the graded Yang-Baxter equation, which is given by (2.4) and (2.5), but with the graded permutation matrix. The transfer matrix \( t(u) \) satisfying the commutativity property (2.7) is given by (2.6), but with the factors \( R_{0j}(u) \) constructed as in (2.5) using the graded permutation matrix, and with the graded trace (supertrace) \( \text{str} X = \sum_{a=1}^{3} (-1)^{p(a)} X_{aa} \).

The transfer matrix now has \( OSp(1|2) \) symmetry, in contrast with the non-graded version that has only \( SU(2) \) symmetry. Indeed, we show in the appendix that

\[
[t(u), S^z] = [t(u), S^\pm] = [t(u), J^\pm] = 0, \tag{3.4}
\]

where the generators are given by

\[
S^z = \sum_{n=1}^{N} s^z_n, \quad S^\pm = \sum_{n=1}^{N} s^\pm_n, \quad J^\pm = \sum_{n=1}^{N} j^\pm_n P_{n+1} \cdots P_N, \tag{3.5}
\]

with

\[
s^z = \frac{1}{2} (e_{33} - e_{22}), \quad s^+ = e_{32}, \quad s^- = e_{23}, \tag{3.6}
\]

and

\[
j^+ = -e_{12} - e_{31}, \quad j^- = -e_{13} + e_{21}, \quad P = e_{11} - e_{22} - e_{33}. \tag{3.7}
\]

These generators satisfy the \( OSp(1|2) \) algebra

\[
[S^z, S^\pm] = \pm S^\pm, \quad [S^z, J^\pm] = \pm \frac{1}{2} J^\pm, \quad [S^+, S^-] = 2S^z, \quad \{ J^+, J^- \} = 2S^z, \quad \{ J^\pm, J^\pm \} = \pm 2S^\pm, \quad [S^\pm, J^\mp] = -J^\pm, \tag{3.8}
\]

and all other (anti-)commutators vanish. Note that the fermionic generators \( J^\pm \) have a non-trivial coproduct (3.5) involving the grading involution \( P \), see e.g. [15, 16].

### 3.2 The Bethe ansatz solution

Let us denote by \( |\Lambda^{(M)}\rangle \) the simultaneous eigenvectors of \( t(u) \) and \( S^z \) that are \( OSp(1|2) \) highest-weight states\(^6\)

\[
S^+ |\Lambda^{(M)}\rangle = 0, \quad J^+ |\Lambda^{(M)}\rangle = 0, \tag{3.9}
\]

with corresponding eigenvalues \( \Lambda^{(M)}(u) \) and \( m \)

\[
t(u) |\Lambda^{(M)}\rangle = \Lambda^{(M)}(u) |\Lambda^{(M)}\rangle, \quad S^z |\Lambda^{(M)}\rangle = m |\Lambda^{(M)}\rangle. \tag{3.10}
\]

\(^6\)As in the non-graded version, the highest-weight property can be checked for small values of \( M \), and is conjectured to be generally true.
These eigenvalues are given by [1]

\[ m = \frac{1}{2} (N - M), \quad M = 0, 1, \ldots, N, \quad (3.11) \]

and

\[ \Lambda^{(M)}(u) = -(u - i)^N \frac{Q(u + \frac{i}{2})}{Q(u - \frac{i}{2})} + u^N \frac{Q(u) Q(u - \frac{3i}{2})}{Q(u - i) Q(u - \frac{i}{2})} - \left( \frac{u(u - \frac{i}{2})}{u - \frac{3i}{2}} \right)^N \frac{Q(u - 2i)}{Q(u - i)}, \quad (3.12) \]

where \( Q(u) \) is again given by (2.17). The corresponding Bethe equations are given by

\[ \left( \frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}} \right)^N = \frac{Q(u_j - i) Q(u_j + \frac{i}{2})}{Q(u_j + i) Q(u_j - \frac{i}{2})}, \quad j = 1, \ldots, M. \quad (3.13) \]

In contrast with the non-graded version, the TQ-equation and Bethe equations do not depend explicitly on the number \( M \) of Bethe roots.

We remark that the reference state (i.e., the state \( |\Lambda^{(M)}\rangle \) with \( M = 0 \)) in the graded version is given by \( \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \otimes N \), while in the non-graded version it is given by \( \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) \otimes N \).

The condition for the singular solution \( \{ \frac{i}{2}, -\frac{i}{2}, u_3, \ldots, u_M \} \) to be physical is now

\[ \left( -\prod_{j=3}^{M} \frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}} \right)^N = 1, \quad (3.14) \]

together with

\[ \left( \frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}} \right)^{N-1} \left( \frac{u_j + \frac{3i}{2}}{u_j - \frac{3i}{2}} \right) \left( \frac{u_j - i}{u_j + i} \right) = \prod_{k=3}^{M} \left( \frac{u_j - u_k - i}{u_j - u_k + i} \right) \left( \frac{u_j - u_k + \frac{i}{2}}{u_j - u_k - \frac{i}{2}} \right), \quad j = 3, \ldots, M, \quad (3.15) \]

cf. (2.20)-(2.21).

### 3.3 Degeneracies and multiplicities

The degeneracy (the number of times that a given eigenvalue \( \Lambda^{(M)}(u) \) appears) is given by \( 2N - 2M + 1 \), since the eigenstates form \( OSp(1|2) \) irreducible representations with spin \( s = m = (N - M)/2 \), see (3.11), which have dimension \( 4s + 1 \). Indeed, these irreps, which we denote by \([s]\), consist of a pair of \( SU(2) \) irreps that have highest weights \( |\Lambda^{(M)}\rangle \) (with spin \( s \) and dimension \( 2s + 1 \)) and \( J^- |\Lambda^{(M)}\rangle \) (with spin \( s - \frac{1}{2} \) and dimension \( 2s \)).

Since the 3-dimensional vector space at each site forms an irrep \([\frac{1}{2}]\), the multiplicities in the spectrum follow from the \( OSp(1|2) \) decomposition of \([\frac{1}{2}] \otimes N \), which can be easily computed from the fact (see e.g. [17] and references therein)

\[ [s_1] \otimes [s_2] = [[s_1 - s_2]] \oplus [s_1 - s_2] + \frac{1}{2} \oplus \cdots \oplus [s_1 + s_2] - \frac{1}{2} \oplus [s_1 + s_2]. \quad (3.16) \]
For example, in terms of the dimensions of the irreps,

\[ N = 2 : \quad 3^\otimes 2 = 1 \oplus 3 \oplus 5 \]
\[ N = 3 : \quad 3^\otimes 3 = 1 \oplus 3 \cdot 3 \oplus 2 \cdot 5 \oplus 7 \]
\[ N = 4 : \quad 3^\otimes 4 = 3 \cdot 1 \oplus 6 \cdot 3 \oplus 6 \cdot 5 \oplus 3 \cdot 7 \oplus 9 \] (3.17)

Hence, for \( N = 2 \), there is one eigenvalue with degeneracy 1 (\( M = 2 \)), one eigenvalue with degeneracy 3 (\( M = 1 \)), and one eigenvalue with degeneracy 5 (\( M = 0 \)); and similarly for higher \( N \). Evidently, the patterns (3.17) differ significantly from those in the non-graded version (2.19).

### 3.4 Checking completeness

We have explicitly verified for \( N = 2, 3, 4 \) that the Bethe ansatz solution (3.11)-(3.13) accounts for all the eigenvalues of the transfer matrix, where all the Bethe roots are finite and pairwise distinct. In contrast with the non-graded version, an infinite Bethe root is not necessary, which is consistent with the fact that the TQ-equation and Bethe equations do not depend explicitly on \( M \).

The Bethe roots for \( N = 2, 3, 4 \) are given in Table 3. Note that the degeneracies and multiplicities are in accordance with the group-theory predictions in Sec. 3.3. The physical singular solutions in Table 3 satisfy (3.14)-(3.15).

### 4 Conclusions

We have argued that, by allowing for the possibility of an infinite Bethe root (and all other Bethe roots finite and pairwise distinct), the Bethe ansatz solution for the non-graded version of the periodic \( OSp(1|2) \) spin chain (2.16)-(2.18) can account for all the distinct eigenvalues of the transfer matrix (2.6).

We emphasize the difference with respect to, say, the Bethe ansatz solution for the periodic \( SU(2) \)-invariant Heisenberg (XXX) spin chain, which (as far as has been checked [13]) accounts for all the transfer-matrix eigenvalues by means of finite Bethe roots.\(^7\)

The reason that, in the non-graded version, an infinite Bethe root can help give a “missing” transfer-matrix eigenvalue (i.e., an eigenvalue that cannot be obtained with exclusively

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\(^7\)As is well known, in the algebraic Bethe ansatz approach for the Heisenberg spin chain, the Bethe states are given by \( B(u_1) \cdots B(u_M) |0\rangle \), where \( B(u) \) is a certain creation operator and \( |0\rangle \) is a reference state. If all the Bethe roots \( \{u_1, \ldots, u_M\} \) are finite, then the Bethe state is an \( SU(2) \) highest-weight state; and the lower-weight states (which have the same transfer-matrix eigenvalue as the Bethe state) can be obtained by repeatedly acting with the spin-lowering operator \( S^- \) on the Bethe state. Since \( B(\infty) \sim S^- \), some authors prefer to describe such lower-weight states of the Heisenberg spin chain in terms of (multiple) infinite Bethe roots. In contrast, for the non-graded version of the \( OSp(1|2) \) spin chain, we find that one infinite Bethe root is necessary to construct certain Bethe states. (As already noted, the algebraic Bethe ansatz for the \( OSp(1|2) \) spin chain is sketched in appendix B of [3]; the renormalized creation operator \( u^{-N+1} B_1(u) \) is finite in the limit \( u \to \infty \).)
Table 3: Bethe roots for $N = 2, 3, 4$ corresponding to all eigenvalues in the graded version

| $N$ | $M$ | degeneracy | $\{u_1, \ldots u_M\}$ |
|-----|-----|------------|------------------------|
| 2   | 0   | 5          | $-$                    |
| 2   | 1   | 3          | 0                      |
| 2   | 2   | 1          | $\pm i/2$              |
| 3   | 0   | 7          | $-$                    |
| 3   | 1   | 5          | $\sqrt{3}/6$           |
| 3   | 1   | 3          | $-\sqrt{3}/6$          |
| 3   | 2   | 3          | $(3\sqrt{3} \pm i\sqrt{13})/8$ |
| 3   | 2   | 3          | $(-3\sqrt{3} \pm i\sqrt{13})/8$ |
| 3   | 3   | 1          | $\pm \sqrt{5}/10$      |
| 3   | 3   | 1          | $0, \pm i/2$           |
| 4   | 0   | 9          | $-$                    |
| 4   | 1   | 7          | $1/2$                  |
| 4   | 1   | 7          | $-1/2$                 |
| 4   | 1   | 7          | 0                      |
| 4   | 2   | 5          | $1.06752 \pm 0.421806i$ |
| 4   | 2   | 5          | $-1.06752 \pm 0.421806i$ |
| 4   | 2   | 5          | $0.0366242, 0.431751$   |
| 4   | 2   | 5          | $-0.0366242, -0.431751$ |
| 4   | 2   | 5          | $\pm i/2$              |
| 4   | 2   | 5          | $\pm \sqrt{5}/28$      |
| 4   | 3   | 3          | $0.106997, 0.696501 \pm 0.464584i$ |
| 4   | 3   | 3          | $-0.106997, -0.696501 \pm 0.464584i$ |
| 4   | 3   | 3          | $0.367029, -0.795887 \pm 0.448098i$ |
| 4   | 3   | 3          | $-0.367029, 0.795887 \pm 0.448098i$ |
| 4   | 3   | 3          | $0, \pm 1.01641i$      |
| 4   | 3   | 3          | $0, \pm 0.364822$      |
| 4   | 4   | 1          | $\pm 1.67174i, \pm i/2$ |
| 4   | 4   | 1          | $\pm 0.211488, \pm i/2$ |
| 4   | 4   | 1          | $0.589529 \pm 0.471746i, -0.589529 \pm 0.471746i$ |

finite Bethe roots) is that the TQ-equation depends explicitly on the number $M$ of Bethe roots. Indeed, if a (homogeneous) TQ-equation does not depend explicitly on $M$, then taking one Bethe root to infinity simply gives the same TQ-equation with one less Bethe root, so nothing new can be obtained. We expect that a similar mechanism may be necessary for achieving completeness of the transfer-matrix spectrum in other integrable models whose TQ-equation depends explicitly on the number of Bethe roots, e.g. [4].

By verifying the completeness of the Bethe ansatz solutions for both the non-graded and graded versions of the model for small values of $N$, we have seen that both Bethe equations (2.18) and (3.13) appear to correctly describe the corresponding spectra. However, for the non-graded version, $SU(2)$ symmetry alone completely accounts for the degeneracies and multiplicities of the transfer-matrix spectrum. Only the graded transfer matrix has
OSp(1|2) symmetry, which is reflected in the degeneracies and multiplicities of its spectrum. This result is in agreement with the observation in [9], and in disagreement with [3], that the Bethe equations for the model with OSp(1|2) symmetry are (3.13), and not (2.18).

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A Proof of OSp(1|2) symmetry

We show here that the graded transfer matrix has OSp(1|2) symmetry (3.4). We first consider the bosonic generators $S^z, S^\pm$ in Sec. A.1, and then consider the fermionic generators $J^\pm$ in Sec. A.2. For later reference, we introduce the monodromy matrix

$$T_0(u) = R_{0N}(u) \cdots R_{01}(u) = \begin{pmatrix} A_1(u) & B_1(u) & B_2(u) \\ C_1(u) & A_2(u) & B_3(u) \\ C_2(u) & C_3(u) & A_3(u) \end{pmatrix}, \quad (A.1)$$

in terms of which the transfer matrix is given by

$$t(u) = \text{str}_0 T_0(u) = A_1(u) - A_2(u) - A_3(u). \quad (A.2)$$

A.1 Bosonic generators

The proof for the bosonic generators is similar to the classic proof of SU(2) symmetry for the Heisenberg spin chain [18]. The SU(2) symmetry of the R-matrix (3.1) means that

$$[\vec{s}^1, R_{12}(u)] = -[\vec{s}^2, R_{12}(u)], \quad (A.3)$$

where $\vec{s} = (s^x, s^y, s^z)$ are the 1-site spin operators, see (3.6). Using the fact $\vec{S} = \sum_{n=1}^N \vec{s}_n$, we obtain

$$[\vec{S}, T_0(u)] = \sum_{n=1}^N [\vec{s}_n, T_0(u)] = \sum_{n=1}^N [\vec{s}_n, R_{0N}(u) \cdots R_{01}(u)]$$

$$= \sum_{n=1}^N \sum_{k=1}^N R_{0N}(u) \cdots [\vec{s}_n, R_{0k}(u)] \cdots R_{01}(u)$$

$$= \sum_{n=1}^N R_{0N}(u) \cdots [\vec{s}_n, R_{0m}(u)] \cdots R_{01}(u)$$
\[
\mathcal{S}_0 \cdot \mathcal{S}_0 \cdot R_0(u) = \mathcal{S}_0 \cdot \mathcal{S}_0 \cdot R_0(u),
\tag{A.4}
\]
where we have used (A.3) to pass to the final line. We therefore arrive at the identity
\[
\left[ \mathcal{S}, T_0(u) \right] = -\left[ \mathcal{S}_0, T_0(u) \right].
\tag{A.5}
\]
The RHS of (A.5) can be readily evaluated using the expressions (3.6) for the 1-site spin operators and the expression (A.1) for the monodromy matrix. In particular, we obtain
\[
\begin{align*}
[J^z, A_i(u)] &= 0, & i &= 1, 2, 3, \\
[J^+, A_1(u)] &= 0, & [J^+, A_2(u)] &= B_3(u), & [J^+, A_3(u)] &= -B_3(u) \\
[J^-, A_1(u)] &= 0, & [J^-, A_2(u)] &= -C_3(u), & [J^-, A_3(u)] &= C_3(u).
\end{align*}
\tag{A.6}
\]
In view of the expression (A.2) for the transfer matrix, we conclude that
\[
\begin{align*}
[J^z, t(u)] &= 0, & [J^\pm, t(u)] &= 0.
\end{align*}
\tag{A.7}
\]

A.2 Fermionic generators

The proof for the fermionic generators \( J^\pm \) is similar, except that the coproduct is no longer trivial (3.5). We start from the symmetry of the R-matrix (3.1)
\[
\left[ j^+_1 P_2, R_{12}(u) \right] = -\left[ j^+_2 P_2, R_{12}(u) \right],
\tag{A.8}
\]
where the 1-site operators are defined in (3.7). Proceeding as before, we obtain a result analogous to (A.5)
\[
\left[ J^\pm, T_0(u) \right] = -\left[ j^\pm_0 \Pi, T_0(u) \right],
\tag{A.9}
\]
where we have introduced the quantum-space operator
\[
\Pi = P_1 \cdots P_N.
\tag{A.10}
\]
This operator commutes with the monodromy matrix’s bosonic elements
\[
\begin{align*}
[\Pi, A_i(u)] &= 0, & i &= 1, 2, 3, \\
[\Pi, B_3(u)] &= 0, & [\Pi, C_3(u)] &= 0,
\end{align*}
\tag{A.11}
\]
and anticommutes with its fermionic elements
\[
\begin{align*}
\{\Pi, B_i(u)\} &= 0, & \{\Pi, C_i(u)\} &= 0, & i &= 1, 2.
\end{align*}
\tag{A.12}
\]
It is now straightforward to evaluate the RHS of (A.9), and we obtain
\[
\begin{align*}
[J^+, A_1(u)] &= -(B_2(u) + C_1(u)) \Pi, & [J^+, A_2(u)] &= -C_1(u) \Pi, & [J^+, A_3(u)] &= -B_2(u) \Pi, \\
[J^-, A_1(u)] &= (B_1(u) - C_2(u)) \Pi, & [J^-, A_2(u)] &= B_1(u) \Pi, & [J^-, A_3(u)] &= -C_2(u) \Pi.
\end{align*}
\tag{A.13}
\]
In view of the expression (A.2) for the transfer matrix, we conclude that
\[
\begin{align*}
[J^\pm, t(u)] &= 0.
\end{align*}
\tag{A.14}
\]
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