A dynamical study of quintom field cosmology with effects of curvature

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Abstract

The present study covers an interaction between the dark energy and dark matter with special emphasis on the effects of curvature in the realm of FLRW space-time. We investigate the system with the formation of dynamical set of equations for various critical points. Later, we study their stability characteristics and show that with the suitable choice of potential, the system gives rise to the late-time attractor solution in the expanding environment. Lastly, we present the cosmological compatibility of the model using the phase-space portrait tools.

1 Introduction

The cosmological and observational evidences that the universe is currently under an accelerated expansion has been put forward in various notable works [1, 2, 3, 4, 5, 6, 7]. To support such observational analysis, an ample of theoretical models have been thoroughly studied concerning the modern cosmology [8, 9, 10]. The mystery behind the unknown fundamental nature of the dark sector entities inspired many researchers to develop dark sector models in association with cosmological observations [11, 12, 13, 14, 15, 16, 17]. The cosmological framework where dark matter and dark energy, instead of evolving separately undergo interaction are considered to be useful to alleviate both, the cosmological constant problem and coincidence problem [18, 19, 20, 21, 22, 23, 24]. Several phenomenological fluid models [9, 25, 26, 27, 28, 29, 30] and scalar fields models [31, 32, 33, 34, 35, 36, 37, 38, 39] have extensively been studied in this realm.

Inspired from the above profound groundwork, we have constructed an interacting dark energy-dark matter model based upon the theory of chiral cosmology [40, 41, 42] in the earlier work [43]. There we have studied formation of an autonomous system from the coupled non-linear differential equations for quintom model through field-fluid analogy (see the references

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therein (44, 45)). We found the physically acceptable equilibrium points and examined their sta-
bility in detail. Our findings suggested that the stable attractor solution leading to accelerated
expanding universe with the crossing of phantom divide line is achievable through our model. It
perfectly pictures a sufficiently long, extended era of the matter-dominated universe following
the advancement of an accelerated expanding universe in the present time. Moving forward, we
extend the same analysis to see the effects of curvature on the cosmological dynamics. In the
present paper, we study an interacting quintom model with a potential of an exponential form
in the case of FLRW (homogeneous and isotropic) universe with non-zero curvature. Analogous
to the previous investigation, the qualitative behaviour of such models can be understood from
the dynamical study of non-linear differential equations transpiring during the analysis. Appar-
etently, many such theoretical models both in flat space-time (46, 47, 48, 49, 50, 51, 52, 53, 54, 55)
and curved space-time (56, 57, 58) have been built up and examined by the cosmologists to
reveal more about the dark sector phenomenon.

Our work is driven with the idea to comprehend whether the introduction of a curvature
term in the dynamical system would affect the local stability and late time cosmologies or not.
We noticed the significant changes in the results of some of the critical points due to the pres-
ence of curvature when compared to the flat case. These critical points have subsequently been
discussed in the manuscripts. The study also supports the physical point of view of the model
under analysis via geometrical display of the evolution of cosmological parameters.

We structure our work as follows: In Section 2, we set up an interacting two-field quintom
model with coupling in both the kinetic and potential terms and construct the gravitational field
equations. Section 3 comprises of the systematically constructed an autonomous system and
principal analysis of the critical points and their stability character. Also, the possible evolution
of key cosmological parameters is shown in the subsection. Finally, we discuss the results in
brief and propose the conclusion in Section 4.

2 Coupled dark sector model

The action integral for the quintom scalar field model is motivated from [43].

\[
S = \int d^4x \sqrt{-g} \left( \frac{R}{2} + \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} B(\phi) g^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi - V(\phi) - \alpha \phi^2 \psi^2 \right).
\] (1)

We view “\(\phi\)” as a phantom field and “\(\psi\)” as a quintessence field. To achieve our aim to construct
a dark sector interacting (DSI) model, we assume phantom field “\(\phi\)” as a dark energy candidate
and quintessence field “\(\psi\)” as a dark matter candidate.
In FLRW space-time, the line element for isotropic and homogeneous universe is given as:

\[
d s^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-\epsilon r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right].
\] (2)

where, \(\epsilon\) is the Gaussian curvature of the space at the time when \(a(t)=1\) and \(a(t)\) is the scale factor
as a function of cosmic time. When \(\epsilon\) is zero, the universe is called a flat universe whereas non-
zero \(\epsilon\) produces curved space-time. We have a closed (spherical) universe for positive curvature
and an open (hyperbolic) universe for negative curvature. Unlike the earlier analysis, here, we
investigate the non-zero curvature case.
For the aforesaid model, the Friedmann equations and the conservation equations for non-zero curvature are given as:

\[
\frac{-\dot{\phi}^2}{2} + \frac{B(\phi)}{2}\psi^2 + V(\phi) + \alpha \phi^2 \psi^2 - \frac{3\epsilon}{a^2} = 3H^2, \tag{3}
\]

\[
-(\frac{-\dot{\phi}^2}{2} + \frac{B(\phi)}{2}\psi^2 - V(\phi) - \alpha \phi^2 \psi^2) - \frac{\epsilon}{a^2} = 2\dot{H} + 3H^2. \tag{4}
\]

\[
\ddot{\phi} - 3H \dot{\phi} - \frac{B_{,\phi}}{2} \dot{\psi}^2 + 2\alpha \phi \psi^2 + V_{,\phi} = 0, \tag{5}
\]

\[
\ddot{\psi} + 3H \dot{\psi} + \frac{B_{,\phi}}{B} \phi \dot{\psi} + 2\alpha \psi \phi^2 B^{-2} = 0. \tag{6}
\]

where, \(H = \frac{\dot{a}}{a}\) is the Hubble rate.

Later, we build up an analogy between the field and the fluid approach where the scalar field “\(\psi\)” has been considered as a dark matter fluid and phantom field “\(\phi\)” has been retained as a dark energy. In this case, the above gravitational field equations (3)-(6) can now be transformed as:

\[
\rho_{dm} - \frac{\dot{\phi}^2}{2} + V(\phi) - \frac{3\epsilon}{a^2} = 3H^2, \tag{7}
\]

\[
-(P_{dm} - \frac{\dot{\phi}^2}{2} - V(\phi)) - \frac{\epsilon}{a^2} = 2\dot{H} + 3H^2.
\]

\[
-\ddot{\phi} - 3H \dot{\phi}^2 + V_{,\phi} \dot{\phi} = Q, \tag{8}
\]

\[
\dot{\rho}_{dm} + 3H \rho_{dm} = -Q.
\]

‘\(Q\)’ is the interaction energy which is the result of continuous energy transfer between the dark sector components and is given as\(^1\):

\[
Q = \left(\frac{B_{,\phi}}{\phi} - \frac{2}{\dot{\phi}}\right) \rho_{dm} \dot{\phi}. \tag{9}
\]

### 3 Dynamical system approach

We formulate an autonomous set of equations to study the qualitative dynamics of the cosmological evolution. Accordingly, we introduce the following dimensionless variables

\[
x = \frac{\dot{\phi}}{\sqrt{6}H}, \quad y = \sqrt{\frac{V(\phi)}{3H}}, \quad z = \frac{\rho_{dm}}{3H^2}, \quad \Omega_k = -\frac{\epsilon}{a^2H^2}. \tag{10}
\]

satisfying the constraint equation

\[
-x^2 + y^2 + z + \Omega_k = 1. \tag{11}
\]

\(^1\)Interested reader can refer our previous work [43] for all detailed calculations.
Using Eq. 10, we write the cosmological parameters as expressed below:

\[ \Omega_\phi = -x^2 + y^2. \] (12)

Phantom field DE density parameter:

\[ \omega_\phi = \frac{-x^2 - y^2}{-x^2 + y^2}, \] (13)

\[ \omega_{\text{eff}} = \frac{1}{3} \left( -1 - 4x^2 - 2y^2 + z \right). \] (14)

Operating derivative with respect to number of e-foldings \( N = \ln(a) \) on above variables (Eq.10), we can derive the following autonomous system

\[ x' = -\sqrt{\frac{3}{2}} \left( \lambda y^2 + \frac{I}{x} \right) - \frac{3}{2} x \left( 1 + x^2 + y^2 + \frac{\Omega_k}{3} \right), \]

\[ y' = -\sqrt{\frac{3}{2}} (\lambda xy) + \frac{3}{2} y \left( 1 - x^2 - y^2 - \frac{\Omega_k}{3} \right), \]

\[ z' = -\sqrt{6} I - 3z \left( x^2 + y^2 + \frac{\Omega_k}{3} \right), \]

\[ \Omega_k' = \Omega_k (1 - 3x^2 - 3y^2 - \Omega_k), \]

\[ \lambda' = \sqrt{6} \lambda^2 x (1 - \Gamma_\lambda), \]

\[ k' = \sqrt{6} k^2 x (1 - \Gamma_k), \]

\[ \beta' = \sqrt{6} \beta^2 x (1 - \Gamma_\beta), \]

\[ \alpha' = -\sqrt{6} \alpha \beta x. \] (15)

where, 'I' represents the scaled interaction term

\[ I = \frac{(2\beta - k)}{2} x z = \frac{Q}{3 \sqrt{6} H^3}. \] (16)

with \( \lambda, k, \beta \) and \( \alpha \) are defined as follows

\[ \lambda(\phi) = -\frac{V_{\phi\phi}}{V(\phi)}, \quad k(\lambda) = -\frac{B_{\phi\phi}}{B(\phi)}, \quad \beta(\phi) = -\frac{\alpha_{\phi\phi}}{\alpha(\phi)}, \quad \alpha = \alpha(\phi). \]

whereas the functions \( \Gamma_\lambda, \Gamma_k, \Gamma_\beta \) are defined as

\[ \Gamma_\lambda = \frac{V_{\phi\phi} V}{V_{\phi\phi}^2}, \quad \Gamma_k = \frac{B_{\phi\phi} B}{B_{\phi\phi}^2}, \quad \Gamma_\beta = \frac{\alpha_{\phi\phi}}{\alpha_{\phi\phi}^2}. \]

The appearance of extra free parameters makes the system difficult to handle. To avoid these complications, we assume an exponential potential \( V(\phi) \propto e^{a(\phi)} \) driving \( \lambda' \) to disappear since, \( \Gamma_\lambda = 1 \). Similarly, the coupling function \( B(\phi) \propto e^{\lambda(\phi)} \) implies \( k' = 0 \). However, respecting \( \alpha(\phi) \) as a non-zero constant additionally reduces the dimensionality of the system by making \( \alpha' = 0 \) and \( \beta' = 0 \). Thus, the autonomous system 15 finally results into a 4D system for which we present the stability analysis of the equilibrium points in detail in the next section.
3.1 Critical points and phase space analysis

The equilibrium points for the reduced 4D dynamical system are manifested in table 1.

Table 1: Table constitutes critical points and cosmological parameters relative to the reduced 4D autonomous system.

| Critical Points | x        | y        | z        | Ω_k     | ω_{eff} |
|-----------------|----------|----------|----------|---------|---------|
| A               | 0        | 0        | 0        | 1       | −1/3    |
| B               | −\sqrt{2}/(2β−k) | 0        | 8/(3(2β−k)^2) | 1−2/(2β−k)^2 | −1/3    |
| C               | (2β−k)/\sqrt{6} | 0        | 1+(2β−k)^2/6   | 0       | −1/6(2β−k)^2 |
| D               | −\sqrt{5}/(2β−k−2λ) | \sqrt{-6/(2β−k−2λ)^2+(2β−k)/(2β−k−2λ)} | 12/(2β−k−2λ)^2−2λ/(2β−k−2λ) | 0       | −(2β−k)/(2β−k−2λ) |
| E               | −\lambda/\sqrt{6} | \sqrt{1+λ^2/6} | 0        | 0       | −1−λ^2/3 |

In the following we inspect each equilibrium point and discuss their stability and other physical attributes.

**Point A** is of hyperbolic nature with eigenvalues \(\left(\frac{5}{3}, \frac{5}{3}, \frac{-4}{3}, 1\right)\). The point is a saddle-node, attracting nearby trajectories in some directions and repelling them along the others. The condition, \(\omega_{eff} = −\frac{1}{3}\) implies the solution that describes the Milne-like universe. The flow of the vector fields on 3D phase-space in all possible respective coordinate systems is displayed in figure 1. The point is representative of a curvature dominated epoch. Nonetheless, the saddle nature of this point reflects the fact that the curvature domination period is just a transitory stage in the history of cosmology.

**Point B** represents a hyperbolic point. The behaviour of this point is same as the fixed point A with \(\omega_{eff} = −\frac{1}{3}\). Eigenvalues (See 4), in this case, are unsuitable to carry out further analysis and know the stability of the point. Nevertheless, to intact the analysis, in figure 2, we showcase the region of parameter space \((β, k)\) where the point B exhibits the stability attributes.

**Point C** is real and physically acceptable for any real value of parameters \(β\) and \(k\). In the parameter space constraint, \(\sqrt{2} < (2β−k) < −\sqrt{2}\) with \(β = k \neq 0\), the point illustrates an accelerated solution of the universe. The eigenvalues respective to the Jacobian matrix of the fixed point C are:

\[
\left(−\frac{1}{2}(2β−k)^2, \quad 1−\frac{1}{2}(2β−k)^2, \quad −\frac{3}{2}−\frac{1}{4}(2β−k)^2, \quad \frac{3}{2}−\frac{1}{4}(2β−k)(2β−k−2λ)\right).
\]

The point C is a hyperbolic equilibrium point. It characterizes stable behaviour when \((2β−k−2λ) < 0\).
Figure 1: Three dimensional phase portrait corresponding to critical point A when $\beta = k = 1$. 
(a) portrays projection of vector fields on $\mathbf{x-y-z}$-plane (saddle node), (b) portrays projection of vector fields on $\mathbf{x-y-\Omega_k}$-plane (unstable node), (c) portrays projection of vector fields on $\mathbf{x-z-\Omega_k}$-plane (unstable node) and (d) portrays projection of vector fields on $\mathbf{y-z-\Omega_k}$-plane (unstable node). Arrows represent the direction of the flow along the trajectories.

$2\lambda \geq 3\sqrt{2}$ and $(2\beta - k) > \sqrt{2}$ or $(2\beta - k) < -\sqrt{2}$. Similarly it characterizes saddle behaviour when $(2\beta - k - 2\lambda) \leq 3\sqrt{2}$ and $(2\beta - k) < \sqrt{2}$ or $(2\beta - k) > -\sqrt{2}$.

Point D is valid for $-2 < \frac{2\lambda}{(2\beta - k)} < 1$ in the phase-space and substantiates the possibility of the
accelerated universe within the aforementioned limit for $\frac{2\lambda}{(2\beta - k)}$. The ratio of DE and DM density parameter describes the scaling solution in the phase-space.

$$\frac{\Omega_\phi}{\Omega_m} = \frac{-3 + \beta^2 - \beta \lambda}{3 + \lambda^2 - \beta \lambda}.$$  

For $0 < \frac{2\lambda}{(2\beta - k)} < 1$, point D features a phantom field dominated universe with $\omega_{eff} < -1$. On the other hand, the constraint, $-2 < \frac{2\lambda}{(2\beta - k)} < 0$ describes a quintessence dominated universe with $-1 < \omega_{eff} < -\frac{1}{3}$.

Accordingly, when $\frac{2\lambda}{(2\beta - k)} \to 0$, the point D describes the solution corresponds to an accelerated de Sitter expansion dominated by a cosmological constant with $\omega_{eff} = -1$.

Moving right along the stability note, we found that the eigenvalues corresponding to the Jacobian matrix of point D are of the similar form as of the point B (See 4). Therefore, here as well, to intact the analysis, the region on the parameter space $(\beta, k)$ where equilibrium point D is an attractor (stable) point is depicted in figure 3.

**Point E** is physically acceptable for all real values of $\lambda$ and describes an accelerating solution for parameter constraint $\lambda^2 > -2$. It also pictures the absolute dominance by scalar field
Figure 3: The blue shaded region of the parameter space \((\beta, k)\) shows the existence of stable features for equilibrium point D. The phase plane is plotted with \(\lambda = 0.4\).

\(\phi\). The point exhibits various (quintessence, phantom, cosmological constant) field dominated behaviour. For \(\lambda^2 > 0\) and \(-2 < \lambda^2 < 0\), we achieve a phantom field and quintessence field dominated universe while in the limit \(\lambda \to 0\), scalar field behaves as a cosmological constant.

Eigenvalues corresponding to the Jacobian matrix of fixed point E are:

\[
\begin{pmatrix}
-2 - \lambda^2, & -3 - \lambda^2, & -3 - \frac{\lambda^2}{2}, & -3 - \frac{k\lambda}{2} + (\beta - \lambda)\lambda
\end{pmatrix}
\]

The point E is of hyperbolic critical point. The point exhibits the stable attractor for \((\beta - \lambda)\lambda < (3 + \frac{k\lambda}{2})\). However it exhibits unstable features for \((\beta - \lambda)\lambda > (3 + \frac{k\lambda}{2})\). With the help of these constraints, our objective to feature the late-time cosmological behaviour of the universe can be achieved. As a result, we observe the dark energy driven accelerated cosmological solution from the three dimensional time evolution plot of critical point E in figure 4. All the trajectories approaching the point E indicate the stable (attractor) behaviour of the fixed point E. It must be noted that we have considered "\(C = \frac{(2\beta - k)}{2} z\)" as a coupling constant as described in [43]. Also, we chose the values of \(\beta\) and \(k\) in such a way that the "C" put up a positive number.
The scalar field dark energy dominated solution for the critical point E is also represented via 2D phase space portrait in figure 5. Here, the point E is a phantom dominated attractor and the red/dashed line which connects the point A with point E features the transition from curvature dominated epoch to phantom accelerated epoch of the universe. The internal light blue region of the phase space portrait of figure 5 display the part where the universe undergoes acceleration with $-1 < \omega_{\text{eff}} < -\frac{1}{3}$, whereas the external dark blue region display the phantom dominated behaviour ($\omega_{\text{eff}} < -1$) of the point E where it exhibits future attractor solution. That is, it is evident that the figure exhibits an everlasting late-time accelerated nature of the universe for the parameter constraint $\lambda^2 > -2$ for point E. The equilibrium points, B, C and D act as saddle points in the phase plane. The movement of the trajectories is such that after crossing the neighbourhood of saddle points, they all finally converge towards the attractor point E. In such a case, the equilibrium points, A, B, C and D pose past attractor features in the specified phase-space of the system.

Figure 4: Three-dimensional phase space portrait of the dynamical system in the plane $(x,y,\Omega_k)$ for an equilibrium point E when $\lambda = 0.4$. The third axis represents $\Omega_k$. 
3.2 Cosmological significance of the model

Any model to be of cosmologically significant, solution of the reduced dynamical system should exhibit a current era of an accelerated expansion of the universe preceded by long-enough matter dominated epoch. To achieve this, we perform the numerical simulations where we choose the following initial conditions:

\[ x_i = 1.5 \times 10^{-4}, \quad y_i = 2.5 \times 10^{-4}, \quad z_i = 0.99, \quad \Omega_{k_i} = 1 + x_i^2 - y_i^2 - z_i. \]

We study the case where \( \lambda \) and coupling constant \( C \) are retained to have positive values. With the employment of the aforesaid constraints, we attempt to substantiate the above qualitative analysis. We display the qualitative evolution of the key cosmological parameters relative to the reduced 4D autonomous system in figure 6.

The inclusion of curvature term makes the attributes of the cosmological parameters to match with the current observed data at different values of \( \lambda \) and coupling constant \( C \) than what it was in the curvature zero case. In figure 6, for \( \lambda = 0.7, \) \( \omega_{\text{eff}} \) clearly portrays the transition from quintessence behaviour to phantom field behaviour in current scenario. Meaning that it crosses the phantom divide line pertaining to the proposal of quintom model and retains the behaviour in the future as well (Fig. 6a). On the other hand, for \( \lambda = 0.3, \) \( \omega_{\text{eff}} \) limits the behaviour to quintessence kind. Nevertheless, it does cross the phantom divide line in near future (Fig. 6b). Furthermore, depending on these choices of parameters and initial conditions the DE and DM energy densities correspond to the present time measurements obtained from the various observational analysis [59, 60, 61]. That being the case, we infer that though the presence of non-zero curvature alters the parameters constraints at which the viable cosmological solution is being attained, it does not alter the overall evolution of the universe much. Here as well, we notice a matter dominated evolutionary period, long enough for structure formation...
to take place accompanied by accelerated expansion driven by dark energy (displaying both, quintessence and phantom character for certain parameter values) at present time.

![Graph](image)

Figure 6: Qualitative evolution of cosmological parameters for reduced 4D autonomous system for (a) $\lambda = 0.7$ and (b) $\lambda = 0.3$ with coupling constant 'C' = 0.7 in late cosmology during matter to dark energy transition era.

### 4 Conclusion

In this work, we examined the dynamical system formulation of coupled DE-DM quintom model developed by operating the field-fluid analogy but in this instance in curved space-time. The fixed point analysis and stability analysis revealed that the equilibrium points of interest are hyperbolic and differ in their stability properties (Table 1) from the flat space-time case. The inclusion of curvature term produces the results for different values of parameter constraints in the current work.

Point A and point B both show Milne-like solutions. In case of equilibrium point A, all possible 3D phase portraits display saddle (unstable manifold) behaviour. However, in case of equilibrium point B, we painted the stability features as shown in figure 2. Critical point C also illustrates an accelerated expanding universe solution under certain constraints on parameters $\beta$ and $k$. The equilibrium point D has equal significance as the point details about the late-time accelerated scaling solution that can help to alleviate one of the leading problems in cosmology i.e. the coincidence problem. It also illustrates the solution for a phantom-dominated and quintessence-dominated accelerated universe under different constraints on parameters $\beta$, $\lambda$ and $k$. In addition to above, the curvature part plays its main role when the analysis is pertained to an equilibrium point E. The presence of curvature slightly alters the eigenvalues of point E, hence, the behaviour and hence, the corresponding analysis. Unlike in the flat case [43], where the critical point A represents a matter-dominated point, figure 5 features the transition from curvature dominated point, point A to phantom dominated point, point E. Here, the course of action of all the trajectories clearly explains the future attractor nature of point E impelling the other equilibrium points, A, B, C and D, to act as past attractors.
Later, we discussed the physical significance of the working model in curved cosmologies via geometrical presentation in figure 6, where we found out that the present cosmologically viable universe can be realized in such a scenario. However, this can be accomplished for different set of values for $\beta$, $\lambda$ and $k$ in view of curved space-time. In the future, we plan to examine the compatibility of the model with the recent observational data, in detail.

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**Appendix**

Since the eigenvalues of the critical point B and critical point D are too complex and are difficult to arrange them in full size, we struck out a large number of terms and simply intended to impart the mannerism in which they emerge out during the course of action. The complexity of the eigenvalues indicates their unsuitability to perform the stability analysis of the corresponding fixed points.

**Critical Point B Eigenvalues:**

\[
\begin{align*}
1 + \lambda \frac{1}{2\beta - k} - \frac{1}{12(2\beta - k)^3} \sqrt{(6912k^7 - 3456k^9 - 497664k^7\beta^2 + ...) + (1152k^2 - 4608k\beta + ...),} \\
- \frac{1}{12(2\beta - k)^3} \sqrt{(6912k^7 - 96768k^6\beta - ...) + (1152k^2 + ...) + (-216k^2\beta + 432k\beta^2 - ...),} \\
- \frac{1}{12(2\beta - k)^3} \sqrt{(6912k^7 - 3456k^9 - 497664k^7\beta^2 + ...) + (-48k + 36k^3 + 96\beta - ...).}
\end{align*}
\]

**Critical Point D Eigenvalues:**

\[
\begin{align*}
- \frac{2(18 + (2\beta - k - \lambda)(2\beta - k + 2\lambda))}{(2\beta - k + 2\lambda)^2}, \\
- \frac{-1}{(2\beta - k + 2\lambda)^3} \sqrt{(248832k^5 - 27869184k^2\beta^5 + ...) + (4320k^4 + 288k^6 - 228096k\beta + ...),} \\
- \frac{-1}{(2\beta - k + 2\lambda)^3} \sqrt{(41472k^7 - 288k^{10}\lambda - ...) + (57024k^2 + 55296k\beta^5\lambda^2 + ...),} \\
- \frac{-1}{(2\beta - k + 2\lambda)^3} \sqrt{(2322432k^6\beta^2\lambda - 5308416\beta^7 + ...) + (432k + 24k^3 + 432\beta^2\lambda + ...).}
\end{align*}
\]
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