Representing a Profinite Group
as the Homeomorphism Group of a Continuum

by Karl H. Hofmann and Sidney A. Morris

Abstract. We contribute some information towards finding a general algorithm for constructing, for a given profinite group \(G\), a compact connected space \(X\) such that the full homeomorphism group \(\mathcal{H}(X)\) with the compact-open topology is isomorphic to \(G\) as a topological group. It is proposed that one should find a compact topological oriented graph \(\Gamma\) such that \(G \cong \text{Aut}(\Gamma)\). The replacement of the edges of \(\Gamma\) by rigid continua should work as is exemplified in various instances where discrete graphs were used.

It is shown here that the strategy can be implemented for profinite monothetic groups \(G\).

Mathematics Subject Classification 2010: 22C05, 22F50, 54H15, 57S10.
Key Words and Phrases: Homeomorphism group, compact group, profinite group, slice, \(G\)-space.

1. Introduction

One knows that the compact homeomorphism group \(\mathcal{H}(X)\) of a Tychonoff space has to be profinite ([16], [17]). In the converse direction GARTSIDE and GLYN [8] have established that every metric profinite group is the homeomorphism group of a continuum (i.e. a compact connected metric space).

For the goal of representing a given group as the homeomorphism group of a space, authors have pursued the following strategy:

Step (1): find some connected graph \(\Gamma\), usually oriented, and find an isomorphic representation \(\pi: G \to \text{Aut}(\Gamma)\); the standard attempt is to use some form of Cayley graphs (see [9], [8], [4])

Step (2): find a rigid continuum \(C\), that is, a continuum, that is, compact connected metric space, whose only continuous selfmaps are the identity and the constant function (see [7], [11]) and replace each of the directed edges of \(\Gamma\) by \(C\) or a variant obtaining a connected space \(X\); finally obtain an isomorphism \(\gamma: \text{Aut}(\Gamma) \to \mathcal{H}(X)\) (see [9], [11], [4]). Obtain an isomorphism \(\gamma \circ \pi: G \to \mathcal{H}(X)\).

All known variations of the strategy are highly technical, and different variations lead to rather different phase spaces \(X\). It would be nice to find a construction which is in some way canonical, perhaps even functorial. However, one of the major obstructions for representations of a profinite group in a combination with graph theoretical methods is that homeomorphism groups, like all automorphism groups in a category are, in no visible way, functorial.
We propose, that in Step (1) one should in fact go more than halfway and construct a compact connected directed graph $\Gamma$ and then apply Step (2) to achieve the final goal.

In the following we show that such constructions are possible in principle and yield for every profinite monothetic group $G$ a continuum $X$ such that $\mathcal{H}(X) \cong G$. While not all compact monothetic groups are metric, the profinite ones among them are.

Thus, in the vein of a general existence result, our construction yields nothing new beyond what Gartside and Glyn have shown in [8]. However, the continua we construct are completely different from those produced in [8] and the proposed construction may turn out to be useful in the future.

2. Directed topological graphs.

In order to construct topological spaces with prescribed homeomorphism groups we first construct directed topological graphs with prescribed automorphism groups.

**Definition 2.1.** A directed (topological) graph is a triple $\Gamma = (V, E, \eta)$ consisting of topological spaces $V$ and $E$ and a continuous function $\eta: E \to V \times V$ such that

(*) $V = \text{im}(\text{pr}_1 \circ \eta) \cup \text{im}(\text{pr}_2 \circ \eta)$.

The set $V$ is called the space of vertices and $E$ is called the space of (oriented) edges. If $e \in E$ we write $\eta(e) = (e^1, e^2) \in V \times V$, then $e^1$ is the origin and $e^2$ is the target of $e$. Condition (*) says

(**) $V = \bigcup_{e \in E} \{e_1, e_2\}$,

and this means that there is no vertex that is not an endpoint of an edge. Note that we allow a whole space $\eta^{-1}(v_1, v_2)$ of (directed) edges from $v_1$ to $v_2$. We shall, however, not use this fact in the sequel.

If the spaces $V$ and $E$ of a directed graph $\Gamma = (V, E, \eta)$ are discrete, we recover the more classical concept of a directed graph.

**Example 2.2.** (i) Let $n$ be natural number $n > 2$ and let $\mathbb{Z}(n) = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of $n$ elements. Define

$V = \mathbb{Z}(n) = \{m + \mathbb{Z} \in \mathbb{Z}/n\mathbb{Z} : m = 0, 1, \ldots, n - 1\},$

$E = V,$

$\eta(m + n\mathbb{Z}) = (m + n\mathbb{Z}, m + 1 + n\mathbb{Z}) \in V \times V.$
The directed graph $C(\mathbb{Z}(n)) \stackrel{\text{def}}{=} (V, E, \eta)$ is the $n$-cycle. It is the Cayley-graph of the pair $(\mathbb{Z}(n), \{1 + n\mathbb{Z}\})$ consisting of the cyclic group of order $n$ and the singleton generating set containing the element $1 + n\mathbb{Z}$.

(ii) More generally, let $G$ be a topological group and let $g \in G$. We set
\[
V = G, \\
E = V, \\
\eta(x) = (x, xg) \in V \times V.
\]

The directed graph $C(G) \stackrel{\text{def}}{=} (V, E, \eta)$ is the topological Cayley graph of $(G, \{g\})$.

(iii) Taking $\mathbb{Z}$ with the discrete topology we obtain the Cayley graph $C(\mathbb{Z})$ of $(\mathbb{Z}, \{1\})$, the chain $\mathbb{Z}$ with its natural order-orientation.  

**Definition 2.3.** A doubly pointed connected topological space $L = (L, b_1, b_2)$ with $b_1 \neq b_2$ is called a link.

Typically $I= ([0, 1], 0, 1)$ is a link: an interval joining its endpoints. If $L= (L, b_1, b_2)$ is a link and a Tychonoff space, then there is a continuous function $F: L \rightarrow I$, $F: L \rightarrow I$, $F(b_1) = 0$ and $F(b_2) = 1$, called a morphism of links.

**Construction 2.4.** Let $\Gamma = (V, E, \eta)$ be a topological directed graph and $L= (L, b_1, b_2)$ a link. We construct a space $|\Gamma|_L$ from these data by “inserting into each oriented edge $e = \eta(v) = (e^1, e^2)$ a copy of $\{e\} \times L$ of the link $L$ such that $(e, b_1)$ is identified with $e^1$ and $v$ while $(e^2, b_2)$ is identified with $e^2$ and $v$.” Indeed we let $X = E \times L$ and define an equivalence relation $\rho$ on $X$ with the following equivalence classes:

\[
\rho(e, x) = \begin{cases} 
\{(e, x)\} & \text{if } x \neq b_1, b_2, \\
\{(e', b_1) : \eta(e') = (v, (e')^2) \} \cup \{v\} & \text{if } \eta(e) = (v, e^2), \ x = b_1, \\
\{(e'', b_2) : \eta(e'') = ((e'')^1, v) \} \cup \{v\} & \text{if } \eta(e) = (e^1, v), \ x = b_2, \\
\{(e^*, x) : \eta(e^*) = (v, v) \} \cup \{v\} & \text{if } \eta(e) = (v, v), \ x \in \{b_1, b_2\}.
\end{cases}
\]

Then let $|\Gamma|_L \stackrel{\text{def}}{=} X/\rho$. The space $|\Gamma|_L$ is called the topological realisation of $\Gamma$ via the link $L= (L, b_1, b_2)$. 

3
If $L$ is a Tychonoff link and $F: L \to I$ a morphism of links then our construction obviously induces a morphism $F^*: \Gamma_L \to \Gamma_I$ of topological realisations. 

Notice that in the case of a Cayley graph of a group $G$ with an element $g \in G$, the quotient space $((E \times L))/\rho$ can be expressed in the form

$$(*) \quad |C(G)|_L = (G \times L)/\rho.$$ 

and that there is a morphism $F^*: |C(G)|_L \to |C(G)|_I$ of topological realisations given by $F^*(\rho_L(a, x)) = \rho_I(a, f(x))$.

The verification of the details of the following examples is straightforward.

**Example 2.5.** $|C(\mathbb{Z}(n))|_I$ is a circle and $|C(\mathbb{Z})|_I$ is homeomorphic to $\mathbb{R}$. 

For the following, let $R$ be a compact connected space and pick two different points $b_1, b_2 \in E$. and $R = (R, b_1, b_2)$ the corresponding link.

**Lemma 2.6.** Let $(X, x_0)$ be a compact connected pointed space and $(R, b_1, b_2)$ a doubly pointed de Groot-continuum. Assume that $X$ and $R$ are disjoint with the exception of $b_2$ and $x_0$ which are assumed to be equal. Then a continuous function $f: R \to X \cup R$ is exactly one of the following kind

(i) $f$ is constant.
(ii) $f(R) = R$ and its corestriction $R \to f(R)$ is the identity map of $R$.
(iii) $f(R) \subseteq X$ and $f(b_2) = b_2$.
(iv) $f(R) \cap R = \emptyset$.

**Proof.** The function $\pi: R \cup X \to R$ defined by $\pi(X) = \{b_2\}$ is continuous; hence the continuous self-map $\pi \circ f: R \to R$ is either the identity or is constant with image $\{b_2\}$. 

**Example 2.7.** Let $R = (R, b_1, b_2)$ be a doubly pointed de Groot-continuum and $F: R \to I$ a morphism of links. Then

(i) $|C(\mathbb{Z}(n))|_R$ is a one-dimensional continuum $X$ whose homeomorphism group $\mathcal{H}(X)$ is isomorphic to the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Moreover $F^*: |C(\mathbb{Z}(n))|_L \to |C(\mathbb{Z}(n))|_I$ is a morphism of realisations onto, or “over”, a circle.

(ii) $|C(\mathbb{Z})|_R$ is a one-dimensional connected locally compact space $X$ whose homeomorphism space $\mathcal{H}(X)$ is isomorphic to the infinite cyclic group $\mathbb{Z}$. Moreover $F^*: |C(\mathbb{Z})|_L \to |C(\mathbb{Z})|_I$ is a morphism of realisations over a line.
Proof. From (*) above recall that for a cyclic group \( Z = \mathbb{Z}/m\mathbb{Z}, \ n = 0, 1, \ldots \) we have \(|C(Z)|_{\mathbb{R}} = (Z \times R)/\rho\) and note via Lemma 2.6, that the action \((n, (k + m\mathbb{Z}, x)) \mapsto (n + k + m\mathbb{Z}, x) : Z \times Z \times R \rightarrow Z \times R\) gives a unique action \((n, \rho(k + m\mathbb{Z}, x)) \mapsto \rho(n + k + m\mathbb{Z}, x) : Z \times |C(Z)|_{\mathbb{R}} \rightarrow |C(Z)|_{\mathbb{R}}\) representing the action of \(\mathcal{H}(|C(Z)|_{\mathbb{R}})\) on \(|C(Z)|_{\mathbb{R}}\).

We now generalize Examples 2.7(i) to monothetic compact groups by utilizing Example 2.7(ii).

Main Lemma 2.8. Let \( G \) be a compact group with a nonidentity element \( g \). Let \( C(G) \) be the topological Cayley graph of \((G, g)\). Let \( L = (L, b_1, b_2) \) be a compact link. Let \( F: L \rightarrow I \) be a morphism of links onto the interval. Then the following conclusions hold:

(i) \( \mathbb{Z} \) acts freely with discrete orbits on \( G \times |C(Z)|_{L} \) via \( m \cdot (a, x) = (a^m, -m \cdot x) \) and the compact orbit space \( (G \times |C(Z)|_{L})/\mathbb{Z} \) is naturally isomorphic to \(|C(G)|_{L}|. It is locally homeomorphic to \( G \times |C(Z)|_{L} \) under the orbit map of the \( \mathbb{Z} \)-action.

(ii) If \( G \) is monothetic with generator \( g \), then \(|C(G)|_{L}| is a compact connected Hausdorff space. If \( G \) is profinite monothetic, then \(|C(G)|_{I}| is homeomorphic to a solenoid (\( p \)-adic if \( G = \mathbb{Z}_p \), the additive group of \( p \)-adic integers), and \( F^*: |C(G)|_{L} \rightarrow |C(G)|_{I}| is a morphism of realisations over a solenoid.

(iii) The group \( G \) acts on \( G \times |C(Z)|_{L} \) by the left regular action on the left factor. This action commutes with the \( \mathbb{Z} \)-action. It thus induces an action

\[(a, Z \cdot (b, x)) \mapsto Z \cdot (a + b, x) : G \times (G \times |C(Z)|_{L})/\mathbb{Z} \rightarrow G \times |C(Z)|_{L}/\mathbb{Z}.

The orbit space \(|C(G)|_{L}/G| is homeomorphic to the space \( L/\{b_1, b_2\} \) arising from \( L \) by identifying the two points \( \{b_1, b_2\} \).

Proof. (i) The assertions on the action are straightforward. Now

\[|C(Z)|_{L} = (Z \times L)/\rho\]

has \( \hat{L} \overset{\text{def}}{=} \rho^{-1}(\{0\} \times L)/\rho \cong L \) as a fundamental domain for the \( \mathbb{Z} \)-action, i.e. each orbit meets \( \hat{L} \) only once except the orbit of \( b_1 \) which meets \( \hat{L} \) in \( \{b_1, b_2\} \). Now we deduce that in this spirit \( G \times \hat{L} \) is a fundamental domain of the \( \mathbb{Z} \)-action on \( G \times |C(Z)|_{L}|. Thus the orbit space \((G \times |C(Z)|_{L})/\mathbb{Z}\) is a continuous image of \( G \times \hat{L} \) and therefore is compact.

Next we denote by \( \sigma \) the equivalence relation on \( \mathbb{Z} \times L \) which identifies \((n, b_2)\) and \((n + 1, b_1)\) for all \( n \in \mathbb{Z} \) so that \(|C(Z)|_{L} = (Z \times L)/\sigma|\. We let \( q: G \times |C(G)|_{L} \rightarrow \)
Let $\rho Z \in \mathbb{N}$ and $a, \beta \in \mathbb{N}$. We see at once that $\gamma$ since (1) and (2) are obviously equal, the commuting of the middle rectangle $(2)$ follows. We then have a commutative diagram

\[
\begin{array}{cccccc}
G \times \mathbb{Z} \times L & \overset{Q}{\longrightarrow} & \frac{G \times \mathbb{Z} \times L}{\rho} & \overset{\Gamma}{\longrightarrow} & G \times \mathbb{Z} \times L & \overset{\pi \times \text{id}_L}{\longrightarrow} & G \times L \\
\text{id}_G \times \delta & \downarrow & \alpha & \downarrow & \beta & \downarrow & \rho \\
G \times |C(\mathbb{Z})|_L & \overset{\varphi}{\longrightarrow} & \frac{G \times |C(\mathbb{Z})|_L}{\rho} & \overset{\gamma}{\longrightarrow} & G \times L & \overset{\rho}{\longrightarrow} & |C(G)|_L \\
\end{array}
\]

Indeed the commuting of the first rectangle is clear from the definition of the action of $\mathbb{Z}$ on $G \times |C(\mathbb{Z})|_L$, and the commuting of the right rectangle is an immediate consequence of the definitions.

The middle rectangle, however, commutes since for all $a \in G$ and $m \in \mathbb{Z}$ we have

\[
\alpha \circ \Gamma((ag^n, m - n) : n \in \mathbb{Z}), b_2) = \alpha((ag^n, m - n, b_2) : n \in \mathbb{N})
\]

(1) $= \{(ag^n, \sigma(m - n, b_2)) : n \in \mathbb{Z}\} = \{(ag^n, \sigma(m - n + 1, b_1)) : n \in \mathbb{Z}\}$

while

\[
\gamma \circ \beta((ag^n, m - n) : n \in \mathbb{Z}), b_2) = \gamma(\rho(a, b_2) = \gamma(\rho(a, b_1))
\]

(2) $= \{(ag^{p+1}, \sigma(m - p, b_1)) : p \in \mathbb{Z}\} = \{(ag^n, \sigma(m - (n - 1), b_1)) : n \in \mathbb{Z}\}$.

Since (1) and (2) are obviously equal, the commuting of the middle rectangle follows. We see at once that $\gamma$ is surjective since $\alpha$ is surjective. We notice that $\gamma(\rho(a, x)) = \gamma(\rho(a', x'))$ iff $(ag^n, \sigma(m - n, x)) : n \in \mathbb{Z} = \{(a'g^n, \sigma(m - n'), x') : n' \in \mathbb{Z}\}$ if $x \notin \{b_1, b_2\}$ these $\mathbb{Z}$-orbits on agree iff and only if their intersections with the fundamental domain $G \times L$ agree. But $\sigma(k, x) \in L$ iff one of the three possibilities apply: (i) $k = 0$, (ii) $k = 1$ and $x = b_1$, or (iii) $k = -1$ and $x = b_2$. In the first case, $n = m = n'$ and $a = a'$, $x = x'$ follow. In the second case $n = m + 1 = n' + 1$ is a possibility, whence $ag^{m+1} = a'g^m$ that is $a' = ag$, $x' = b_1$ and $x = b_2$, which implies that $\rho(a, x) = \rho(a', x')$. The other cases are discussed similarly and likewise yield $\rho(a, x) = \rho(a', x')$. Therefore the continuous function $\gamma$
between compact Hausdorff spaces is bijective and therefore is a homeomorphism. This completes the proof of (i)

(ii) First we have to prove connectivity of \(|C(G)|_L\). The space \(A = |C(\mathbb{Z})|_L\) is arcwise connected. Since \(g^2\) is dense in \(G\), the image \(\mathbb{Z} \cdot |C(\mathbb{Z})|_L / \mathbb{Z}\) of \(A\) is dense in \((G \times |C(\mathbb{Z})|_L) / \mathbb{Z}\) which is naturally homeomorphic \(|C(G)|_L\). Hence the latter space is connected.

Now let \(A\) be a subgroup of the discrete group \(\mathbb{Q}\) containing \(\mathbb{Z}\) such that \(T \text{ def } A / \mathbb{Z}\) is infinite. Then the character group \(S \text{ def } \hat{A}\) is a compact connected one-dimensional abelian group called a solenoid. The character group \(G = \hat{T}\) is profinite monothetic.

By 2.7(ii) and (i) we know that \(|C(G)|_I\) is homeomorphic to the quotient group \((G \times \mathbb{R}) / \Delta\) for the subgroup \(\Delta = \{(n, -n) : n \in \mathbb{Z}\}\). This quotient is one of the well-known representations of the solenoid \(S\). (See e.g. [15], Exercise E1.11., Theorem 8.22.)

(iii) We identify \(|C(G)|_L\) with \((G \times |C(\mathbb{Z})|_L) / \mathbb{Z}\). The assertions are straightforward. The orbit space \(|C(G)|_L / G = ((G \times |C(\mathbb{Z})|_L) / \mathbb{Z}) / G\) is isomorphic to \(((G \times |C(\mathbb{Z})|_L) / G) / \mathbb{Z} \cong |C(\mathbb{Z})|_L / \mathbb{Z} \cong L / \{b_1, b_2\}\).

\[\square\]

**Theorem 2.9.** For any profinite monothetic group \(G\) there is a compact connected 1-dimensional space \(X\) such that \(\mathcal{H}(X) \cong G\).

**Proof.** Let \(g \in G\) be a generator of \(G\). If \(g\) has finite order, there is nothing to prove because the assertion was established in Example 2.7(i). We therefore assume for the rest of the proof that \(g\) has infinite order. We apply Main Lemma 2.8 with \(X = |C(G)|_R\) for a doubly pointed de Groot continuum \(R\). By the Main Lemma 2.8 we know that \(X\) is a compact connected space which is locally homeomorphic to the space \(G \times |C(\mathbb{Z})|_R\) which is one-dimensional since \(R\) is one-dimensional. Therefore \(X\) is one-dimensional.

We have to prove that \(\mathcal{H}(X) \cong G\). This is the most delicate portion of the proof. We identify \(X\) with \((G \times |C(\mathbb{Z})|_R) / \mathbb{Z}\). For \(h \in G\) let \(\gamma_h : X \to X\) be defined by \(\gamma_h(\xi) = h \cdot \xi\) for the \(G\)-action on \(P\). Let \(\varphi\) be a homeomorphism of \(X\). We claim that \(\varphi\) is of the form \(\gamma_h\) for some \(h \in G\), that is \(\gamma_h(\mathbb{Z} \cdot (a, x)) = \mathbb{Z} \cdot (h + a, x)\).

The path components of \(G \times |C(\mathbb{Z})|_R\) are the spaces \(\{a\} \times |C(\mathbb{Z})|_R\) since \(G\) is totally disconnected, and they are permuted by the action of \(G\) on the left factor by the regular representation. We may therefore consider \(a = 0\) without loss of generality. The orbit \(O \text{ def } \mathbb{Z} \cdot (0, \sigma(0, b_1)) = \{(n \cdot g, \sigma(-n, b_1)) : n \in \mathbb{Z}\}\) meets \(\{0\} \times |C(\mathbb{Z})|_R\) in an element \((n \cdot g, \sigma(-n, b_1))\) only if \(n \cdot g = 0\) iff \(n = 0\) since \(g\) has
infinite order. It follows that the orbit map $q: G \times |C(Z)|_R \to X$ maps each set \( \{a\} \times |C(Z)|_R \) continuously and bijectively onto

$$Z_a \overset{\text{def}}{=} \frac{\mathbb{Z}.\{\{a\}\times |C(Z)|_R\}}{\mathbb{Z}} = \frac{\mathbb{Z}\cdot a \times |C(Z)|_R}{\mathbb{Z}},$$

for $a \in G$. In particular, each of the sets $Z_a$ is arcwise connected. We claim that the $Z_a$ are the arc components. This is well-known in the case of the solenoid $S = |C(G)|_I \cong (G \times \mathbb{R})/\Delta \cong \hat{A}$ (cf. 2.8(ii) and its proof and [15], Theorem 8.30).

We have the morphism $F^*: |C(G)|_R \to |C(G)|_I = S$ mapping $Z_0$ to

\[
\{(0) \times \mathbb{R}\}/\Delta = (\mathbb{Z} \times \mathbb{R})/\Delta
\]

the identity arc component of the solenoid $S$. Different sets $Z_a$ and $Z_b$ are mapped to different arc components in $S$ and so there can be no arc connecting a point in $Z_a$ to a point in $Z_b$. Hence the $Z_a$ are precisely the arc components of $|C(G)|_R$.

Thus the homeomorphism $\varphi$ permutes the sets $Z_a$.

We consider the particular arc component

$$Z = Z_0 = \mathbb{Z}.\{\{0\} \times |C(Z)|_R\}/\mathbb{Z}.$$ 

Assume that $\varphi(Z) = Z_a$. Then $\gamma_a^{-1}\varphi$ is a homeomorphism of $X$ which maps the arc component $Z$ into itself.

The function $\varepsilon: |C(Z)|_R \to Z$, $\varepsilon(x) = \mathbb{Z}\cdot\sigma(0, x)$ is a continuous bijection.

We shall now invoke the arc component topology attached functorially to a topological space as summarized in [15], Appendix 4, p. 781. Since $R$ and thus $|C(Z)|_R$ are locally arcwise connected, the bijective function $\varepsilon: |C(Z)|_R \to Z$ is the universal map $\varepsilon_Z: Z^a \to Z$ of Lemma A4.1 of [15], p. 781. Therefore, by Lemma A4.1(iv) every homeomorphism of $Z$ lifts uniquely to a homeomorphism of $|C(Z)|_R$ and thus is an action of $m \in \mathbb{Z}$ on $|C(Z)|_R$ by Example 2.7(ii).

By the definition of the $\mathbb{Z}$-action on $G \times |C(Z)|_R$ according to 2.8(i), the action by $m$ on $|C(Z)|_R$ when pushed down to $Z$ is induced by the action of

$$\gamma_m: X \to X, \quad X = (G \times |C(Z)|_R)/\mathbb{Z}.$$ 

Therefore the homeomorphism $\gamma_{m-a}\varphi = \gamma_m\gamma_a^{-1}\varphi$ fixes the arc component $Z$ elementwise. However, $Z$ is dense in $X$. Hence $\gamma_{m-a}\varphi = \text{id}_X$. Thus $\varphi = \gamma_{a-m}$ and this had to be shown. \[\square\]

As we have noted in the proof of 2.8(ii), a compact profinite group $G$ is monothetic, iff its character group $\hat{G}$ is isomorphic to a subgroup $A/\mathbb{Z}$ of the group
The solenoid attached to this monothetic group is the character group $\hat{A}$ of $A \subseteq \mathbb{Q}$.

Our feeling is that the occurrence of compact homeomorphism groups, given certain restrictions, is not so rare as one might surmise initially even though the construction of compact spaces having a given homeomorphism group requires work. Ideally, one should be able to prove the following

Conjecture. Let $G$ be a compact group. Then the following conditions are equivalent:

1. There is a compact connected space $X$ such that $G \cong \mathcal{H}(X)$.
2. There is a compact space $X$ such that $G \cong \mathcal{H}(X)$.
3. $G$ is profinite.

Note that we have (1) $\implies$ (trivially) (2) $\implies$ (3); the implication (3) $\implies$ (1) we have proved only for compact monothetic groups $G$ and GARTSIDE and GLYN have proved it for arbitrary metric profinite groups. Cantor groups $\mathbb{Z}(2)^X$ (with arbitrary exponent $S$) Keesling [18] was able to represent as homeomorphism groups of one-dimensional metric spaces.

3. References

[1] Anderson, R. D., The algebraic simplicity of certain groups of homeomorphisms, Amer. J. Math. 80 (1958), 955-963.
[2] Arens, R. F., Topologies for homeomorphism groups, Amer. J. Math. 68 (1946), 593-610.
[3] Bourbaki, N., Topologie générale, many publishers from Hermann, Paris, ca 1950, to Springer Berlin, etc., ca 2000.
[4] Bredon, G., Introduction to Compact Transformation Groups, Academic Press, New York, 1972.
[5] Cook, H., Continua which admit only the identity mapping onto non-degenerate subcontinua, Fund. Math. 60 (1967), 241-249.
[6] Gartside, P. and A. Glyn, Autohomeomorphism groups, Topology Appl. 129 (2003), 103–110.
[7] de Groot, J., Groups represented by homeomorphism groups. Math. Ann. 138 (1959), 80–102.
[10] de Groot, J., and R. H. McDowell, Autohomeomorphism groups of 0-dimensional spaces. Compositio Math. 15 (1963), 203–209.

[11] de Groot, J., and R. J. Wille, Rigid continua and topological group-pictures. Archiv d. Math. 9 (1958), 441–446.

[12] Dijkstra, J. J., and J. van Mill, On the group of homeomorphisms of the real line that map the pseudoboundary onto itself, Canad. J. Math. 58 (2006), 529–547.

[13] Droste, M., and R. Göbel, On the Homeomorphism Groups of Cantor’s Discontinuum and the Spaces of Rational and Irrational Numbers, Bulletin of the London Mathematical Society 34 (2002), 474–478.

[14] Ellis, R., Locally compact transformation groups, Duke Math. J. 24 (1957), 119–126.

[15] Hofmann, K. H., and S. A. Morris, The Structure of Compact Groups, Verlag Walter De Gruyter Berlin, 1998, xvii+834pp. Second Revised and Augmented Edition 2006, xviii+858pp.

[16] —, Compact Homeomorphism Groups are Profinite, Preprint

[17] Keesling, J., Locally compact full homeomorphism groups are zero dimensional, Proc. Amer. Math. Soc. 29 (1971), 390–396.

[18] —, The group of homeomorphisms of a solenoid, Trans. Amer. Math. Soc. 172 (1972), 390–396.

[19] Rybicky, T., Commutators of homeomorphisms of a manifold, Universitatis Jagellonicae Acta Math 23 (1996), 153–1960.

[20] tom Dieck, T., Transformation Groups, Verlag Walter De Gruyter Berlin, 1987, x+312pp.