Stable Flags and the Riemann-Hilbert Problem

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Abstract

We tackle the Riemann-Hilbert problem on the Riemann sphere as stalk-wise logarithmic modifications of the classical Röhrle-Deligne vector bundle. We show that the solutions of the Riemann-Hilbert problem are in bijection with some families of local filtrations which are stable under the prescribed monodromy maps. We introduce the notion of Birkhoff-Grothendieck trivialisation, and show that its computation corresponds to geodesic paths in some local affine Bruhat-Tits building. We use this to compute how the type of a bundle changes under stalk modifications, and give several corresponding algorithmic procedures.

Introduction

The Riemann-Hilbert problem (RHP) has a long and distinguished history, not even devoid of suspense, for it has been solved several times, using different tools, in a seemingly complete and positive way. It is finally A. A. Bolibrukh, in a celebrated series of papers at the beginning of the 1990’s who clarified the situation, by rigorously defining (and exhibiting a counter-example) to the strongest version of the RHP, thereby showing that people before him had either committed a mistake, or solved in reality a weaker problem.

The modern approach to the RHP was initiated by H. Röhrle in the 1950’s who used the theory of vector bundles in a way that has been conserved since. First, one constructs a vector bundle $E$ outside the singular points, whose cocycle mimicks the monodromy. We call this the topological RH problem, since the monodromy is so much encoded in the topology of the constructed bundle, that construction of the required connection becomes essentially trivial. The second step consists in extending the bundle (and the connection) to the singular points by means of a local solution to the inverse monodromy problem. It has been exposed in great generality in P. Deligne’s work ([D]) how to extend a holomorphic vector bundle $E$, defined over the complement of a divisor $D$ and endowed with a holomorphic connection $\nabla$ having a prescribed monodromy about $D$, into a logarithmic connection $(\overline{E}, \nabla)$ with singularities on the divisor, uniquely determined by a section of the natural projection $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}$. In this way, we get all logarithmic extensions of $E$ with non-resonant residue (the Deligne lattices). These two steps are sufficient to solve positively the weak Riemann-Hilbert problem (i.e. with regular singularities). Note, however, that in this second level, two different types of problems have been mixed. The connection constructed is essentially unique up to meromorphic equivalence (to be rigorously defined later) whereas the holomorphic vector bundle setting already introduces much.

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finer holomorphic equivalence problems. This fact can contribute to explain some of the confusions that have surrounded the precise formulation of the RHP.

The strong Riemann-Hilbert problem asks however for a logarithmic bundle (with the prescribed monodromy) which is moreover trivial.

So, to solve the Riemann-Hilbert problem in this way, one must modify the constructed Deligne bundle, over the support of the singular divisor exclusively (to keep the singular set invariant), while conserving its logarithmic character, until a trivial bundle is eventually found. Until A. Bolibrukh’s celebrated counter-example ([B]), it was widely acknowledged that this was possible, and it is indeed so in several “generic” instances, although some mistakes in the seemingly general solution by Plemelj had already been pointed out.

The counter-example found by A. A. Bolibrukh to the strong Riemann-Hilbert problem requires the knowledge of all the logarithmic extensions of a regular connection, in order to prove that none is trivial. Despite the production of both counter-examples and sufficient conditions for a positive answer, no general necessary and sufficient conditions for the solubility of the strong Riemann-Hilbert problem have been given in terms of the monodromy representation only.

As already stated, the strong Riemann-Hilbert problem admits a solution if and only if the stalks of the Deligne bundle over the singular set can be replaced by logarithmic lattices in such a way that the resulting bundle is trivial. To tackle this problem, it is logical to

a) determine the set of all logarithmic lattices above a given point,

b) get a criterion for the triviality of the modified bundle.

In this paper, we solve problem a by giving a complete description of the logarithmic lattices in terms of flags stabilised under the action of the residue of the connection. After finding such a characterisation, we became aware that such a description appears in [S] p., who attributes this result to Deligne-Malgrange. However, we preferred to state the whole result in the geometric terms of our paper. We also give a partial answer to problem b. In the case of \( \mathbb{P}^1(\mathbb{C}) \), the type of a vector bundle gives such a triviality criterion. In our selected approach, starting with the Deligne bundle \( D \), we perform a modification of a finite number of stalks, resulting in the bundle \( D^{\text{mod}} \). The question is then to compute the type of the modified bundle \( D^{\text{mod}} \). Generalising a result by Gabber and Sabbah (proposition 3), we show how to determine the type of \( D^{\text{mod}} \) from the type of \( D \). Thus, problem b is reduced to computing the type of the Deligne bundle. In a second step, we show that this problem in turn is reduced to the well-known problem of connection matrices.

The paper is organised as follows.

In a first section, we define the category in which we will work, and what we precisely mean by “modifying a bundle in one or several points”. In a second part, we describe the geometry on the local lattices involved. We describe this geometry in terms of the affine Bruhat-Tits building of \( \text{SL}_n \). This choice is justified by the fact that, more than the local lattices themselves, our description relies on the homothety class of such lattices, which are precisely the vertices of the considered Bruhat-Tits building. Several invariants attached to the underlying 1-skeleton, particularly the natural graph-theoretical distance, will play an important paper.

In a third part, we use this setting to give an effective method to compute how the type of an arbitrary bundle \( E \) is modified under a modification \( E^{\text{mod}} \) of \( E \). This algorithm can also be applied to compute the type of the bundle \( E \). This third
section concludes with a generalisation of an essential result due to Bolibrukh, the permutation lemma, for which we provide an interesting geometric interpretation.

The fourth section gives the complete description of the set of logarithmic lattices in terms of flags which are stable under the action of the residue of the connection on the Deligne bundle.

In the last part, after recalling the construction of the classical Röhr-Deligne bundle, we give a very concise proof of Plemelj’s theorem on the Riemann-Hilbert solubility. This well-known result becomes an immediate consequence of the geometrical interpretations of the permutation lemma and the set of logarithmic lattices. We describe all trivialisations of the Deligne bundle over an arbitrary point, and we establish a stronger inequality on the type of the weak solutions to Riemann-Hilbert in the irreducible case. Finally, we give algorithmically effective procedures that allow to search the space of weak solutions.

1 Holomorphic Vector Bundles

Let \( X \) be a compact Riemann surface and let \( \pi : E \rightarrow X \) be a rank \( n \) holomorphic vector bundle. The sheaf \( \mathcal{E} \) of holomorphic sections of \( E \) is a locally free sheaf of \( \mathcal{O}_X \)-modules of the same rank \( n \), where \( \mathcal{O}_X \) denotes as usual the sheaf of holomorphic functions on \( X \). There is a well-known equivalence between these two categories. However, it is also well known that this equivalence fails for sub-objects of the same rank. Any locally free subsheaf \( \mathcal{F} \subset \mathcal{E} \) of \( \mathcal{O}_X \)-modules of rank \( n \) on \( X \) can be seen as a sheaf locally generated over \( \mathcal{O}_X \) by holomorphic sections of \( E \), while the equivalence of categories allows us to call \( \mathcal{F} \) a holomorphic vector bundle. However, it is not possible to find an equivalent to \( \mathcal{F} \) as a sub-bundle of \( E \) since both have the same rank.

**Meromorphic Connections.** Let \( \mathcal{D} = \sum_{i=1}^{p} m_i x_i \) be a positive divisor on \( X \). Let \( \mathcal{O}_\mathcal{D} \) be the sheaf of meromorphic functions on \( X \) having pole orders bounded by \( \mathcal{D} \) (i.e. less than \( m_i \) at \( x_i \)). Let \( \mathcal{S}_\mathcal{D} = \{x_1, \ldots, x_p\} \) be its support. For any finite set \( \mathcal{S} = \{y_1, \ldots, y_t\} \), let \([\mathcal{S}] = y_1 + \cdots + y_t\).

Let \( \nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{O}_X \mathcal{O}_\mathcal{D} \) be a meromorphic connection with singular divisor \( \mathcal{D} \) on a vector bundle \( \mathcal{E} \) of rank \( n \). In the sequel, we will always assume that \( \mathcal{D} \) is the smallest possible. Sometimes for simplicity we’ll just say “connection” for the pair \((\mathcal{E}, \nabla)\). The Poincaré rank of \( \nabla \) at \( x \in X \) is the integer \( p_x(\nabla) = \max(0, m_x - 1) \). If \( p_x(\nabla) = 0 \), the sheaf \( \mathcal{E} \) is said to be logarithmic with respect to \( \nabla \) at \( x \). Let \( \mathcal{S} = |\mathcal{D}| \) be the singular, and \( \mathcal{S}_{\log} = \{x \in \mathcal{S} | p_x(\nabla) = 0\} \) the logarithmic singular sets of \( \nabla \). If \( \mathcal{S}_{\log} \neq \emptyset \), then one can define the residue map \( \text{Res}\nabla \in \text{End}(\mathcal{E}/\mathcal{E}_{-|\mathcal{S}_{\log}|}) \), where \([\{x_1, \ldots, x_m\}] = x_1 + \cdots + x_m \). If \( x \in \mathcal{S} \backslash \mathcal{S}_{\log} \) is not logarithmic, the residue of \( x^{p_x(\nabla)}\nabla \) for any local coordinate \( z \) induces the well-defined polar map \( PM\nabla \in \mathbb{P}\text{End}(\mathcal{E}/\mathcal{E}_{-|\mathcal{S}|}) \) of \( \nabla \) over \( \mathcal{E} \). We will specify in parentheses the bundle if necessary.

**The Meromorphic Bundle.** Let \( \mathcal{V} = \mathcal{E} \otimes \mathcal{O}_X \mathcal{M}_X \) be the sheaf of meromorphic sections of \( E \). A meromorphic connection \( \nabla \) on \( \mathcal{E} \) induces a canonical extension to \( \mathcal{V} \). Since the sheaf \( \mathcal{E} \) can be embedded into \( \mathcal{V} \), we consider henceforth only the set

\[
H = \{\mathcal{F} \subset \mathcal{V} | \mathcal{F} \overset{\text{loc.}}{\simeq} \mathcal{O}_X^n \}
\]

of holomorphic vector bundles of \( \mathcal{V} \). Each such bundle \( \mathcal{F} \) is automatically endowed with a meromorphic connection induced by \( \nabla \). By simplicity, we won’t make any notational difference between all these connections.
We say that \( \mathcal{F} \in H \) is trivial if \( \mathcal{F} \simeq \mathcal{O}_X^n \), or, equivalently, if \( \mathcal{F} \) is generated by its global sections. In this case, the set \( \Gamma(X, \mathcal{F}) \) of global sections is a \( \mathbb{C} \)-vector space of dimension \( n \), admitting as basis vectors global meromorphic sections of \( E \) with specific constraints on their divisor. Let \( H_0 \subset H \) be the subset of trivial holomorphic bundles in \( V \). The following result is well known ([S], p.?)

**Lemma 1.** Let \( \mathcal{F} \in H \) be a holomorphic vector bundle in \( (V, \nabla) \). If \( \mathcal{F} \) is trivial, then the space \( Y_\mathcal{F} = \Gamma(X, \mathcal{F}) \) of global sections is a \( \mathbb{C} \)-vector space of dimension \( n \). For any logarithmic singularity \( s \in \text{Sing}(\mathcal{F}) \), the residue \( \text{Res}_s \nabla \) induces a well-defined endomorphism \( \psi_s \in \text{End}_\mathbb{C}(Y_\mathcal{F}) \).

**Stalks and Lattices.** For any \( x \in X \), the stalk \( \mathcal{F}_x \) of a holomorphic vector bundle is a free \( (\mathcal{O}_X)_x \)-submodules of rank \( n \) (or lattice) of the stalk \( V_x = V_x \), which is a vector space of dimension \( n \) over the fraction field \( K_x \) of \( (\mathcal{O}_X)_x \). Let \( \mathbf{A}_x \) be the set of lattices of \( V_x \). We define an equivalence relation \( R_x \) on \( H \) as

\[
(\mathcal{F}, \tilde{\mathcal{F}}) \in R_x \text{ if and only if } \mathcal{F}|_{X\setminus\{x\}} = \tilde{\mathcal{F}}|_{X\setminus\{x\}}.
\]

For simplicity, we will drop the index \( x \) as soon as no ambiguity can be feared. Any coset of \( H/R_x \) can be identified with the set \( \mathbf{A}_x \), by identifying a vector bundle \( \mathcal{E} \) in a given coset of \( H/R_x \) with its stalk \( \mathcal{E}_x \in \mathbf{A}_x \) at \( x \).

Actually, since \( X \) is compact, two vector bundles \( \mathcal{E}, \mathcal{F} \in H \) have equal stalks outside a finite set \( \Delta(\mathcal{E}, \mathcal{F}) \).

**Lemma 2.** Let \( \mathcal{E} \in H \) be a holomorphic vector bundle. For any family of lattices \( M_x \in \mathbf{A}_x \) for \( x \) in a discrete set \( S \), there exists a unique vector bundle \( \mathcal{E}^M \in H \) such that

\[
(\mathcal{E}^M)_x = \begin{cases} 
\mathcal{E}_x & \text{if } x \notin S \\
M_x & \text{if } x \in S
\end{cases}
\]

Moreover, for any \( \mathcal{F} \in H \), there exists such a discrete set \( S \) and a family \( (M_x \in \mathbf{A}_x)_{x \in S} \) of lattices such that \( \mathcal{F} = \mathcal{E}^M \). If \( \mathcal{E} \) is endowed with a meromorphic connection \( \nabla \), there is a canonical extension \( \nabla^M \) of \( \nabla|_{X\setminus S} \) as a meromorphic connection on \( \mathcal{E}^M \).

The sheaf \( V \) is always trivial, and the group \( G \) of (meromorphic) automorphisms of the space \( \Gamma(X, V) \) is isomorphic to \( \text{GL}_n(\mathbb{C}(X)) \). Let \( \mathbf{A}_x^0 = \mathbf{A}_x \cap H_0 \) be the set of trivial bundles in the coset \( \mathbf{A}_x \). The subgroup \( G_x \subset G \) of automorphisms of \( \Gamma(X, V) \) that leave \( \mathbf{A}_x^0 \) globally invariant is called the group of monopole gauge transforms at \( x \). Each element of \( G_x \) sends a trivial sheaf \( \mathcal{F} \) to a trivial sheaf \( \tilde{\mathcal{F}} \) such that \( \mathcal{F}|_{X\setminus\{x\}} = \tilde{\mathcal{F}}|_{X\setminus\{x\}} \). An element of \( G_X \) modifies at most the stalk \( \mathcal{F}_x \).

## 2 Lattices and the Affine Building of \( \text{SL}_n \)

In this section, we fix a point \( x \in X \) and a coset \( \Lambda \in H/R_x \). We drop the index \( x \) for simplicity. The field \( K \) is local, and endowed with the discrete valuation \( v = \text{ord}_x \), whose valuation ring is \( \mathcal{O} \), whose maximal ideal is \( \mathfrak{m} \) and residue field \( k = \mathbb{C} \). As already mentioned, \( V \) is a \( K \)-vector space of dimension \( n \).

**Flags.** Here assume that \( V \) is a vector space over an arbitrary field of characteristic 0. Given a flag \( \mathcal{F} \) of vector spaces \( 0 = F_0 \subset F_1 \subset \cdots \subset F_s = V \) in \( V \), where \( s = |\mathcal{F}| \) is the length of the flag, the signature of \( \mathcal{F} \) is the integer sequence \( (n_1, \ldots, n_s) \) where \( n_i = \dim(F_i/F_{i-1}) \). A map \( u \in \text{End}_K(V) \) is said to stabilise the flag \( \mathcal{F} \) if \( u(F_i) \subset F_i \), etc.
for all $0 \leq i \leq |\mathcal{F}|$. Let $\mathcal{F}(V)$ for the set of flags of $V$, and $\mathcal{F}_u(V)$ for the subset of flags that are stabilised by $u$. Recall that a flag $\mathcal{F}' : 0 = F'_0 \subset F'_1 \subset \cdots \subset F'_s = V$ is said to be transversal to $\mathcal{F}$ if $F'_i \oplus F_{s-i} = V$ for $1 \leq i \leq s$. Note that in this case, the signature of $\mathcal{F}'$ is then equal to $(n_1, \ldots, n_s)$. For any subspace $M \subset V$, let $\mathcal{F} \cap M$ denote the flag of $M$ composed of the distinct subspaces among the $F_i \cap M$.

An $\mathcal{F}$-admissible sequence is an integer sequence

$$u = (k_1, \ldots, k_1, k_2, \ldots, k_2, \ldots, k_s, \ldots, k_s)$$

with $k_1 < \cdots < k_s$.

Let $\mathbb{Z}^n(\mathcal{F})$ be the set of integer $\mathcal{F}$-admissible sequences, and let

$$W(V) = \{(\mathcal{F}, \mathcal{K}) \mid \mathcal{F} \in \mathcal{F}(V) \text{ and } \mathcal{K} \in \mathbb{Z}^n(\mathcal{F})\}$$

be the set of admissible pairs of $V$. For any integer sequence $\mathcal{K} = (k_1, \ldots, k_n)$, let

$$\Delta \mathcal{K} = \max_i k_i - \min_i k_i \text{ and } i(\mathcal{K}) = \sum_{j=1}^{n} (\max_i k_i - k_j).$$

### Lattices.

Let $L(u)$ denote the free $\mathcal{O}$-module spanned by a family $(u)$ of vectors in $V$. An $\mathcal{O}$-module $M \subset V$ is a lattice if there exists a $K$-basis $(e)$ of $V$ such that $M = L(e)$. Let

$$v_\Lambda(x) = \min\{k \in \mathbb{Z} \mid m^k x \in \Lambda\},$$

be the natural valuation of $V$ induced by $\Lambda$. For any lattices $M \subset \Lambda$ in $V$, we define the interval $[M, \Lambda]$ as

$$[M, \Lambda] = \{N \in \Lambda \mid M \subset N \subset \Lambda\}.$$

### Elementary Divisors.

Let $z$ be a uniformising parameter of $K$. For any two lattices $\Lambda$ and $M$ in $V$, there exists a unique increasing sequence of integers $k_1 \leq \cdots \leq k_n$ (the elementary divisors of $M$ in $\Lambda$) and an $\mathcal{O}$-basis $(e_1, \ldots, e_n)$ of $\Lambda$ such that $(z^{k_1} e_1, \ldots, z^{k_n} e_n)$ is a basis of $M$. Such a basis $(e)$ is called a Smith basis of $\Lambda$ for $M$. We will write them $k_{1,\Lambda}(M)$ if we want to specify the respective lattices, and we write

$$\mathcal{K}_\Lambda(M) = (k_{1,\Lambda}(M), \ldots, k_{n,\Lambda}(M)).$$

Note that $k_{1,\Lambda}(M) = v_\Lambda(M)$, and let $M_\Lambda = z^{-v_\Lambda(M)}M$. It is convenient to be a bit more lax on the definition, and allow the elementary divisors to appear in another order.

The subgroup of the lattice stabiliser $GL_n(\mathcal{O})$ that acts on the set of Smith bases of $\Lambda$ for $M$ is the lattice $\mathcal{K}$-parabolic subgroup $\mathfrak{S}_\mathcal{K}$, whose intersection with $GL_n(\mathbb{C})$ is the $\mathcal{K}$-parabolic group $G_\mathcal{K}$ defined as

$$\mathfrak{S}_\mathcal{K} = \{P \in GL_n(\mathcal{O}) \mid v(P_{ij}) \geq k_i - k_j \} \text{ and } G_\mathcal{K} = \{P \in GL_n(\mathbb{C}) \mid P_{ij} \neq 0 \Rightarrow k_i \leq k_j\}.$$
which means that (e) and (e′) are two Smith bases of Λ for M. Note that in this case 
P ∈ GL_n(0) is 0_{2×2}-parabolic (and ˆP is 0_{2×2}-parabolic, symmetrically).

Sometimes, we will find it more convenient to consider the elementary divisors with
their multiplicities. In this case, we will put k_1, . . . , k_s for the distinct elementary divisors of M in Λ and let n_j be their respective multiplicities. The set [n] = {1, . . . , n} of indices of ordinary (simple) elementary divisors is partitioned into the subsets I_j corresponding to a single value of the elementary divisors

I_j = \{1 ≤ ℓ ≤ n | k_ℓ = k_j\} for 1 ≤ j ≤ s.

2.1 The Affine Building of SL(V)
For this section, which is standard, good references are [Br, Ga]. The affine building B_n naturally attached to SL(V) is the following (n − 1)-dimensional simplicial complex. Two lattices Λ and M are homothetic if there exists α ∈ K∗ such that M = αΛ. Let [Λ] be the homothety class of the lattice Λ in V. The vertices of B_n are the homothety classes of lattices in V, and an edge connects two vertices L and L′ if and only if there exist representatives Λ of L and M of L′ such that mΛ ⊂ M ⊂ Λ.

The affine building B_n is the flag simplicial complex associated with this graph, or in other terms, its clique complex. A maximal simplex, or chamber in B_n, is an n-chain of vertices L_0, . . . , L_{n−1} with representatives Λ_i for 0 ≤ i ≤ n − 1 satisfying

mA_0 ⊂ Λ_1 ⊂ · · · ⊂ Λ_{n−1} ⊂ Λ_0.

The natural graph-theoretic distance d in B_n, that is the length of the shortest path between two lattice classes L and L′, and the index of L′ with respect to L, are given by

d(L, L′) = k_n,Λ(M) − k_1,Λ(M) and [L : L′] = \sum_{i=1}^{n} (k_i,Λ(M) − k_1,Λ(M)) (1)

for any representatives Λ, M of L, L′. Note that d(L, L′) = −v_Λ(M) − v_M(Λ) also holds.

Geodesics. A geodesic is a path Γ in B_n such that for any vertices L, L′ ∈ Γ, the length of the path between L and L′ induced by Γ is equal to d(L, L′). The following result explains how to construct a geodesic algebraically.

Proposition 1. Let L, L′ ∈ B_n, and let d = d(L, L′). For k ∈ N, let L_k = [Λ′ + m^kΛ],

where Λ ∈ L and Λ′ ∈ L′ are such that v_Λ(Λ′) = 0. Then d(L_k, L_{k+1}) = 1 for

0 ≤ k ≤ d − 1 and L_d = L′. The path

Γ(L, L′) = (L_0, L_1, . . . , L_d)

is called the geodesic path from L to L′. Moreover, the geodesic path Γ(L, L′) is the unique path of minimal length between L and L′.

Proof. The existential part of the lemma is easy to verify by using Smith bases of the representatives Λ and Λ′, and is left to the reader. Note that the geodesic interval is symmetric. Indeed, letting Γ(L′, L) = (L_0, L_1, . . . , L_d), we have L_k = [M_Λ + z^kΛ] and L_{d−k} = [M_Λ + z^{d−k}M]. By definition we have

Λ_M + z^{d−k}M = z^{−v_M(Λ)}Λ + z^{−v_M(Λ)−v_M(M)}M

= z^{−v_M(Λ)}(z^kΛ + z^{−v_M(M)}M).
Therefore $L'_{d-k} = L_k$.

Let us prove the uniqueness by induction on the distance $d = d(L, L')$. For convenience, let any path $([\Lambda] = L_0, L_1, \ldots, L_{d-1}, [M] = L_d)$ be represented by its normalised sequence $(\Lambda, M_1, \ldots, M_{d-1}, M_d)$ of lattices $M_i \in L_i$ such that $v_\Lambda(M_i) = 0$. We will first prove the following result: if $\Gamma' = (\Lambda = L_0, L_1, \ldots, L_{d-1}, M = L_d)$ is a path of minimal length, then the normalised sequence of lattices satisfies $\Lambda \supset M_1 \supset \cdots \supset M_d$. For $d = 1$, this is the very definition of adjacency in $B_n$. Assuming that the claim is established for any pair of lattices at distance $\leq d - 1$, we have $M_d \subset \Lambda \supset M_1 \supset \cdots \supset M_{d-1}$ for the normalised sequence of $\Gamma'$. Since $M_{d-1}$ and $M_d$ are adjacent, there exists a unique $k \in \mathbb{Z}$ such that $z^k M_{d-1} \supset M_d \supset z^{k+1} M_{d-1}$. We know that $d(\Lambda, M_{d-1}) = d - 1$, hence we have $\Lambda \supset M_{d-1} \supset z^{d-1} \Lambda$, therefore we get $z^k \Lambda \supset z^k M_{d-1} \supset M_d \supset z^{k+1} M_{d-1} \supset z^{d+k} \Lambda$.

If $k > 0$, then $v_\Lambda(M) \geq k > 0$, which was excluded by assumption. But if $k < 0$, then $d(\Lambda, M) < d$, which is also excluded. Thus we have $k = 0$, and the claim is proved.

Now we turn to the proof of the uniqueness of the geodesic. Since the claim is obvious for $d = 1$, let us suppose that there exists a unique geodesic between any pair of vertices in $B_n$ distant at most of $d - 1$. Suppose then that $d(\Lambda, M) = d$. Let $\Lambda = L_0 \supset L_1 \supset \cdots \supset L_{d-1} \supset M$ represent a path of minimal length, and $\Lambda = M_0 \supset M_1 \supset \cdots \supset M_{d-1} \supset M$ the geodesic path from $\Lambda$ to $M$. By assumption, $d(\Lambda, L_{d-1}) = d - 1$, therefore we have $\Lambda \supset L_{d-1} \supset z^{d-1} \Lambda$, and by definition, we have $M_{d-1} = M + z^{d-1} \Lambda$. Therefore, we get

$$M_{d-1} \cap L_{d-1} = (M + z^{d-1} \Lambda) \cap L_{d-1} = M + (z^{d-1} \Lambda \cap L_{d-1}) \text{ because } M \subset L_{d-1} = M + z^{d-1} \Lambda \text{ since } z^{d-1} \Lambda \subset L_{d-1} = M_{d-1}$$

Thus $M_{d-1} \cap L_{d-1}$ holds. On the other hand, $M_{d-1}$ is the largest lattice containing $M$, contained in $\Lambda$ and adjacent to $M$. Since $L_{d-1}$ also satisfies these conditions, we finally get $L_{d-1} = M_{d-1}$. By the induction assumption, the two geodesics coincide all along.

If $\mathcal{K} = (k_1 = 0, \ldots, k_n)$ represent the sequence of elementary divisors of $M$ in $\Lambda$, then the elements $M_k$ of the (normalised sequence of the) geodesic path from $\Lambda$ to $M$ have as elementary divisors in $\Lambda$ the sequence $\mathcal{K}_k \equiv (\min(k_i, k))$. The differences $T_k = \mathcal{K}_{k+1} - \mathcal{K}_k$ form what we will call the elementary splitting $\mathcal{K} = T_1 + \cdots + T_d$ of $\mathcal{K}$. We have then

$$\Lambda \xrightarrow{z^{T_1}} M_1 \xrightarrow{z^{T_2}} M_2 \cdots M_{d-1} \xrightarrow{z^{T_d}} M$$

Apartments. Let $\Phi = \{d_1, \ldots, d_n\}$ be an unordered set of one dimensional $K$-vector subspaces of $V$ such that $d_1 + \cdots + d_n = V$ (\Phi is called a frame). The set

$$\{\Lambda = \ell_1 + \cdots + \ell_n \mid \ell_i \text{ is a lattice in } d_i\}$$

of lattices spanned over multiples of the vectors in $\Phi$ induces a simplicial subcomplex in the affine building $B_n$ called the apartment spanned by $\Phi$. For any lattice $\Lambda \in \mathcal{A}$, a $\Lambda$-basis of the apartment $[\Phi]$ is a collection $(u_1, \ldots, u_n)$ of vectors such that $u_i$ spans $d_i$ and $v_\Lambda(u_i) = 0$. Such a family is unique up to permutation and to multiplication.
of each $u_i$ by a scalar $\lambda_i \in \mathbb{O}^*$. The lattice is an element of the apartment $[\Phi]$ if and only if the family $(u) = (u_1, \ldots, u_n)$ is actually a basis of the lattice $\Lambda$. Equivalently, and without reference to a basis, this means that

$$\Lambda = \bigoplus_{i=1}^{n} \Lambda \cap d_i.$$  

In the general case, the lattice $\Lambda_{\Phi} = \bigoplus_{i=1}^{n} \Lambda \cap d_i$ is the largest sublattice of $\Lambda$ in the apartment $[\Phi]$. The homothety class $L_{\Phi} = [\Lambda_{\Phi}]$ is therefore the closest point projection of $L = [\Lambda]$ on $[\Phi]$. Finally note that if $L, L' \in [\Phi]$, then $[\Phi]$ contains the whole geodesic path $\Gamma(L, L')$.

**Quotients.** For vertices $L, L' \in \mathcal{B}_n$, we define the quotient $L'/L$ as the quotient module $\Lambda'/\Lambda$, where $\Lambda \in L$ and $\Lambda' \in L'$ satisfy $v_\Lambda(\Lambda') = 0$. We will sometimes say, for shortness’ sake, that $\Lambda, \Lambda'$ are $L$-normalised representatives of $L, L'$. The quotient $L'/L$ is a well defined finite-dimensional $\mathbb{C}$-vector space. For any lattices satisfying $\Lambda' \subset N \subset \Lambda$, let

$$\Psi_{\Lambda,\Lambda'}(N) = ((N + \Lambda') \cap \Lambda)/\Lambda'.$$  

(2)

Let $E$ be the set of linear subspaces of $L'/L$, and

$$[L', L] = \{ M \in \mathcal{B}_n | \exists \Lambda \in L, \Lambda' \in L', N \in M, L,N$-normalised, such that $\Lambda' \subset N \subset \Lambda \}.$$  

Formula (2) defines a mapping $\Psi_{L, L'} : \mathcal{B}_n \rightarrow E$ which induces a poset isomorphism from $[L', L]$ to $F$, where $F = \{ G \in E | mG \subset G \} \subset E$. We will consider the space $L'/L$ as a sub-$\mathbb{C}$-vector space of $\mathcal{T}_d = \Lambda/m^d\Lambda$, where $d = d(L, L')$. This definition is independent of the choice of $\Lambda$ inasmuch as there is a canonical isomorphism between $\Lambda/m^d\Lambda$ and $m^d\Lambda/m^{d+k}\Lambda$. In the special case $d = 1$, the quotient space denoted with $\overline{L} = \Lambda/m\Lambda$ is a $\mathbb{C}$-vector space of dimension $n$. We write $\Psi_L$ for the isomorphism of simplicial complexes defined by relation (2) between the set of neighbours $\text{lk}(L)$ of $L = [\Lambda]$ in $\mathcal{B}_n$ (called the link of $L$) and the set $E$ of chains of linear subspaces of $\overline{L}$.

### 2.2 Relative Flag of a Lattice

Any lattice $M$ induces a natural flag in $\overline{M} = \Lambda/m\Lambda$ in the following way. For any $k \in \mathbb{Z}$, let

$$M_k = (m^{-k}M \cap \Lambda) + m\Lambda \in [m\Lambda, \Lambda].$$  

Let $(e)$ be a basis of elementary divisors of $\Lambda$ for $M$, and $I = \{ 1 \leq i \leq n | k_i \leq k \}$. Then $M_k$ admits $(u)$ as basis where $u_i = e_i$ if $i \in I$ and $u_i = ze_i$ if $i \notin I$. The spaces $M_k$ are thus embedded lattices, all belonging to the interval $[m\Lambda, \Lambda]$, so they take at most $n+1$ different values. Their images $\overline{M}_k$ in the quotient space $\overline{\Lambda}$ form a flag $F_\Lambda(M)$, and it is clear that $\overline{M}_{k-1} \subsetneq \overline{M}_k$ if and only if $k$ is an elementary divisor of $M$. Let therefore $k_1, \ldots, k_s$ be the distinct elementary divisors of $M$ in $\Lambda$, with multiplicities $n_i$. The subset of indices corresponding to $k_j$ can be written as

$$I_j = [n_1 + \cdots + n_{j-1}, n_1 + \cdots + n_{j+1} - 1].$$  

The lattices $M_k$ and $M_\ell$ coincide if and only if there exists $i$ such that $k_i \leq k, \ell < k_{i+1}$ (with the conventions $k_0 = -\infty$ and $k_{s+1} = +\infty$). Therefore the flag $F_\Lambda(M)$ has exactly length $s$, and its signature is equal to the sequence $(n_1, \ldots, n_s)$. Its components can be indexed either as $\overline{M}_{k_i}$, by the value of the elementary divisor $k_i$.
it is attached to (if known), or as $\overline{M}_i$ by its index in the flag (here $i$). In this latter case, we will also use the notation $F^i_\Lambda(M)$. It will hopefully be always clear what convention we are using.

Note that the flag $F_\Lambda(M)$ corresponds under the isomorphism $\Psi^{-1}_{[\Lambda]}$ to a canonical simplex in $\mathcal{B}_n$ containing the vertex $L = [\Lambda]$. Modulo homothety, one can define the flag $F_{\Lambda}(L')$ in the space $\overline{L}$ defined in section 2.1, and the following result holds.

**Lemma 3.** Let $L, L' \in \mathcal{B}_n$ be vertices in $\mathcal{B}_n$, let $d = d(L, L')$ and let $(L_0, \ldots, L_d)$ be the geodesic interval $\Gamma(L, L')$. For $0 \leq k \leq d$, the flag $F_{\Lambda}(L_k)$ is given by

$$F_{\Lambda}(L_k) : F^0_\Lambda(M) \subset \cdots \subset F^k_\Lambda(M) \subset \overline{L},$$

where $k$ is the index such that $k + v_\Lambda(M) \in I_k$, for any representatives $\Lambda \in L$ and $M \in L'$. Moreover, the flags $F_{\Lambda_k}(L)$ and $F_{\Lambda_k}(L')$ in $\overline{L_k}$ have supplementary first components if $k$ is a normalised elementary divisor of $L'$ in $L$.

**Proof.** Take representatives $\Lambda, M$ of $L, L'$, and a Smith basis $(e)$ of $\Lambda$ for $M$. Let $K_\Lambda(M) = (k_1, \ldots, k_n) = (k_1 I_n, \ldots, k_n I_n)$, and assume that $k_1 = 0$. Then suitable representatives of $L_k$ are the lattices $M^k = M + z^k \Lambda$, which admit as bases $(e^k) = (z^{\min(k, r)} e_i)_{i=1, \ldots, n}$. Therefore the elementary divisors of $\Lambda$ and $M$ in $M^k$ are respectively

$$K_1 = (\max(k - k_i, 0)) \quad \text{and} \quad K_2 = (\max(0, k_i - k)).$$

Let $j$ be the index such that $k_j \leq k < k_{j+1}$.

We must here distinguish two cases. If $k_j < k < k_{j+1}$, then we have

$$K_1 = (k I_{n_1}, (k - k_2) I_{n_2}, \ldots, (k - k_j) I_{n_j}, 0_{n_{j+1}}, \ldots, 0_{n_s})$$

and

$$K_2 = (0_{n_1}, 0_{n_2}, \ldots, 0_{n_j}, (k_{j+1} - k) I_{n_{j+1}}, \ldots, (k_s - k) I_{n_s}).$$

Then obviously the induced flags $\mathcal{F} = F_{\Lambda_k}(L)$ and $\mathcal{F}' = F_{\Lambda_k}(L')$ have respective signatures $(n_{j+1} + \cdots + n_s, n_{j+1}, \ldots, n_1)$ and $(n_1 + \cdots + n_j, n_{j+1}, \ldots, n_s)$. Their first components $\mathcal{F}_1$ and $\mathcal{F}'_1$ are supplementary subspaces of $L_k / mL_k$. If $k = k_j$, however, we have $\mathcal{F}_1 \cap \mathcal{F}'_1 = \{ z^k \overline{\tau}_{v_1}, \ldots, z^k \overline{\tau}_{v_{j+1} - 1} \}$. \hfill \Box

### 2.3 Forms

Fix a lattice $\Lambda$ and let for simplicity $\Phi_\Lambda : \Lambda \rightarrow W(\Lambda/m\Lambda), M \rightarrow (\mathcal{F}_\Lambda(M), \mathcal{K}_\Lambda(M))$. This map is clearly surjective, but as clearly not injective. The objective of this section is to show how to invert it.

Let a form in $\Lambda$ be a $\mathbb{C}$-vector subspace $Y$ of $\Lambda$ spanned by an $O$-basis $(e)$ of $\Lambda$. If we fix a form $Y$ in $\Lambda$ (that is a $\mathbb{C}$-linear section of the canonical projection $\pi : \Lambda \rightarrow \Lambda/m\Lambda$) then there is a unique way to lift the quotient module $\Lambda/m\Lambda$ in $Y$, that is, there is a well-defined isomorphism

$$\varphi_Y : Y \xrightarrow{\cong} \Lambda/m\Lambda.$$  

Let $\mathcal{B}(Y)$ be the sub-building of $\mathcal{B}_n$ composed of apartments $[\Phi]$ which are spanned by a basis of $Y$. For a given flag $F \in \mathcal{B}(\Lambda/m\Lambda)$, let us define the $Y$-fiber of $F$ as

$$\Psi_Y(F) = \{ M \in \mathcal{B}(Y) \mid F_\Lambda(M) = F \}$$

We will say that $Y$ is a *Smith form* for $M$ if $M \in \Psi_Y(F)$.
Lemma 4. Let $\Lambda$ be a lattice in $V$ and $Y$ be a form in $\Lambda$.

i) For any admissible pair $(F, \mathcal{X})$ of $\Lambda/m\Lambda$, there exists a unique lattice $M = \Psi_Y(F, \mathcal{X})$ in $\Psi_Y(F)$ such that $\mathcal{K}_\Lambda(M) = \mathcal{X}$.

ii) For any basis $(e)$ of the lattice $\Lambda$, there exists a unique $\mathbb{C}$-basis $(e_Y)$ of the form $Y$ whose image in $\Lambda/m\Lambda$ coincides with the image of $(e)$.

We call $(e_Y)$ the $Y$-basis of $(e)$.

Proof. For any $\mathbb{C}$-basis $(e)$ of $Y$ which respects the flag $F$, put $M = \bigoplus_{i=1}^n z^{k_i}e_i$. Let $(\tilde{e})$ be another basis of $Y$ and $\tilde{M} = \bigoplus_{i=1}^n z^{\delta_i}e_i$. The matrix of the change of basis $\Psi_Y(F, \mathcal{X})$ is equal to $P = z^X \tilde{C} z^{-X}$, where $C \in \text{GL}_n(\mathbb{C})$ is the matrix of the change of basis from $(e)$ to $(\tilde{e})$. By definition of the parabolic subgroup $P_F$, one has $z^X \tilde{C} z^{-X} \in \text{GL}_n(0) \iff C \in P_F$, hence $M = \tilde{M}$ if and only if $(e)$ and $(\tilde{e})$ both respect the flag $F$. The second claim is straightforward. Note that the gauge from the basis $(e)$ to its $Y$-basis is always of the form $P = I + zU \in \text{GL}_n(0)$. \hfill \Box

The correspondence $\Psi_Y$ is therefore a bijection between the set $W(Y)$ of admissible pairs of $Y$ and the sub-building $\mathcal{B}(Y)$. Let $F_Y(M) = \varphi_Y^{-1}(F_\Lambda(M))$ be the lifting of the relative flag of $M$ in $Y$, and define

$$\mathcal{F}_Y(M) = F_Y(M) \otimes_\mathcal{O} K.$$

The signatures of these three flags are all equal to the multiplicities $(n_1, \ldots, n_s)$ of the original lattice $M$. Putting all this together, we have the following definition.

Definition 1. Let $\Lambda$ be a lattice in $V$, and $Y$ a form in $\Lambda$. Let $M$ be a lattice in $V$, let $F = F_\Lambda(M)$ be the induced flag and $\mathcal{K} = K_\Lambda(M)$ its elementary divisors.

i) The flag $\mathcal{F}_Y(M)$ of $K$-vector spaces in $V$ is called the $Y$-flag of $M$.

ii) The lattice $M_Y = \Psi_Y(F, \mathcal{X}) \in \Psi_Y(F)$ is called the $Y$-lattice of $M$.

For any two forms $Y$ and $\tilde{Y}$, the set of gauges between bases of $Y$ and $\tilde{Y}$ is an element of the double coset $\text{GL}_n(\mathbb{C})/\text{GL}_n(0)/\text{GL}_n(\mathbb{C})$. Let $z$ be a uniformising parameter. With the convention that $\text{deg}_z P = \infty$ if $P \in \text{GL}_n(0) \setminus \text{gl}(\mathbb{C}[z])$, the following definition makes sense.

Definition 2. Let $Y, \tilde{Y}$ two forms in $\Lambda$. The $z$-distance $\delta_z(Y, \tilde{Y})$ is defined as $\min(\text{deg}_z P, \text{deg}_z P^{-1}) \in \mathbb{N} \cup \{\infty\}$ for any gauge $P$ from a basis of $Y$ to a basis of $\tilde{Y}$.

Lemma 5. If $d = d(\Lambda, M)$, then for any form $Y$ of $\Lambda$, and any uniformising parameter $z$, there exists a Smith form $Y'$ of $\Lambda$ for $M$ at a $z$-distance $\delta_z(Y, \tilde{Y}) \leq d - 1$.

Proof. There exists a Smith form $Y'$ of $\Lambda$ for $M$. Let $P = P_0 + P_1z + \cdots \in \text{GL}_n(0)$ be a gauge corresponding to a basis change from $Y$ to $Y'$. Let $\bar{P} = P_0 + \cdots + P_1z^t$, and let $Y$ be the form obtained by this gauge transformation, as explained in the following scheme.

$$
\begin{array}{c}
\tilde{Y} \xrightarrow{\bar{P}} Y \xrightarrow{P} Y' \\
M \xrightarrow{Q} M
\end{array}
$$

$$
\begin{array}{c}
\text{deg}_z P, \text{deg}_z P^{-1} \in \mathbb{N} \cup \{\infty\}
\end{array}
$$
We have $Q = z^{-X} P^{-1} T z^X = (P_{ij} z^{k_j - b_i})$ where $P = P^{-1} T$. By construction, we have $P = P^{-1} (P - (P - T)) = I + z^{t+1} U$ with $U \in gl(0)$. As soon as $t \geq d - 1$, we have $Q \in GL_n(0)$, hence the form $Y$ is a Smith form for $M$.

The definitions of distance and index in the Bruhat-Tits building suggest the following.

**Definition 3.** Let $E, F \in H$ be two holomorphic vector bundles. The distance between $E$ and $F$ and the index of $F$ with respect to $E$ are defined as the integers

$$d(E, F) = \max_{x \in X} d(E_x, F_x) \quad \text{and} \quad [E : F] = \sum_{x \in X} [E_x : F_x]$$

where quantities on the right-hand side denote those defined in the local Bruhat-Tits building at $x$ by relation (1).

### 3 Birkhoff-Grothendieck Trivialisations

The central result in the theory of holomorphic vector bundles on $X = \mathbb{P}^1(\mathbb{C})$ is the Birkhoff-Grothendieck theorem, which states that any such bundle is isomorphic to a direct sum of line bundles. In this section, we investigate what properties of the vector bundle can be retrieved by considering only the Bruhat-Tits building at a point $x \in X$. In what follows, we take $X = \mathbb{P}^1(\mathbb{C})$.

#### 3.1 The Birkhoff-Grothendieck Property

According to section 1, a holomorphic vector bundle $E \in H$ is completely described by the coset $\Lambda = [E] \in H/R$ and the lattice $\Lambda = E_x \in \Lambda$. Let us take up the notations of section 2 again. Let $V$ denote the meromorphic stalk $V_x$ and let $B$ be the corresponding Bruhat-Tits building. Let $\mathcal{B}_0$ the subset of trivialising lattices of $\mathcal{B}$. Strictly speaking, these are the lattices $M \in \Lambda$ such that the extension $E^M$ gives a quasi-trivial vector bundle, but we will not bother much to make the difference, since we will get a trivial bundle by simply tensoring by a line bundle. The space $\Gamma(X, \mathcal{F})$ of sections of a trivial bundle $\mathcal{F} = E^M$ induces, by taking stalks at $x$, a form $Y_M$ in the corresponding lattice $M = \mathcal{F}_x$, that we call the global form of $M$.

It follows from the Birkhoff-Grothendieck theorem that the set $\mathcal{B}_0$ is always non-empty. We do not know if this is actually a weaker result. However, if we admit this possibly weaker result, we can deduce from it an elementary algebraic proof of the Birkhoff-Grothendieck theorem that displays quite nicely the geometric properties of the local Bruhat-Tits building. First we start by making the link between the Birkhoff-Grothendieck theorem and the algebraic structure of the local lattices. Let us say that $E$ has the Birkhoff-Grothendieck property if $E \simeq \bigoplus_{i=1}^{n} L_i$ where $L_i$ are holomorphic line bundles. Then the following characterisation is straightforward.

**Lemma 6.** A vector bundle $E \in H$ has the Birkhoff-Grothendieck property if and only if there exists a trivialising lattice $M \in \mathcal{B}_0$ and a Smith basis $(e)$ of $M$ with respect to $\Lambda = E_x$ that is simultaneously a $\mathbb{C}$-basis of the global form $Y_M$ of $M$.

The previous result can be understood in the following sense: if we put

$$E \simeq \bigoplus_{i=1}^{n} O(a_i) \quad \text{with} \quad a_1 \geq \cdots \geq a_n,$$
then there is a basis \((e)\) of \(E_x\) such that the matrix of the change of basis to \(M = \mathcal{F}_x\) is given by the diagonal matrix \(T = \text{diag}(a_1, \ldots, a_n)\) of elementary divisors, where \(a_i \geq a_{i+1}\). We sum this situation by the diagram \(E_x \xrightarrow{z^T} \mathcal{F}_x\), where \(z\) is a local coordinate at \(x\). Note that Bolibrukh uses the inverse convention with types \(O(-c_1)\) and \(c_1 \leq \cdots \leq c_n\).

In this case, we say that \(\mathcal{F}\) is a Birkhoff-Grothendieck trivialisation of \(E\) at \(x\), the basis \((e)\) a Birkhoff-Grothendieck basis of \(\mathcal{F}\) for \(E\), and the apartment \([\Phi]\) spanned by \((e)\), a Birkhoff-Grothendieck apartment for \(E\). To avoid multiplying definitions, we will say that a basis \((e)\) of \(V_x\) is Birkhoff-Grothendieck if there is a local coordinate \(z\) at \(x\) and a diagonal integer matrix \(T = \text{diag}(a_1, \ldots, a_n)\) such that \(z^T(e)\) is a basis of a trivialisation of the coset \(\Lambda \in H/R_x\).

**Note 3.1.** When \(X = \mathbb{P}^1(\mathbb{C})\), a line bundle \(\mathcal{L}\) is characterised by its degree. Recall that if the integers \(a_i = \deg \mathcal{L}_i\) satisfy \(a_1 \geq \cdots \geq a_n\), then the sequence \(T(\mathcal{E}) = (a_1, \ldots, a_n)\) is unique and called the type of \(E\). The group of monopole gauges is described by the group of unimodular polynomial matrices \(GL_n(\mathbb{C}[T])\), that is matrices of the form

\[
P = P_0 + P_1 T + \cdots + P_k T^k \text{ where } \exists \alpha \in \mathbb{C} - \{0\}, \det P = \alpha \text{ for all } T.
\]

We state now the following result separately for further reference.

**Proposition 2.** Assume \(X = \mathbb{P}^1(\mathbb{C})\), and let \(E \in H\) be a holomorphic vector bundle. The type of the bundle \(E\) is equal to the sequence of elementary divisors \(\kappa_{E_x}(\mathcal{F}_x)\) (in reverse order) of the stalk \(E_x\) with respect to \(\mathcal{F}_x\) (viewed as lattices in \(V_x\)), for any Birkhoff-Grothendieck trivialisation \(\mathcal{F}\) of \(E\) at any \(x \in X\).

Let \(E\) have type \(T(\mathcal{E}) = (a_1, \ldots, a_n)\). The triviality index \(i(\mathcal{E}) = \sum_{i=1}^n (a_1 - a_i)\) measures how far \(E\) is from being quasi-trivial. In a more “intrinsic” way, we can define it as the sum of the indices of the dual bundles \(i(\mathcal{E}) = [E^* : \mathcal{F}^*]\) for any Birkhoff-Grothendieck trivialisation \(\mathcal{F}\) of \(E\).

The Birkhoff-Grothendieck trivialisations of a bundle \(E\) are as a rule not unique.

**Lemma 7.** Let \(\Lambda \in \Lambda \subseteq H/R_x\) represent a bundle of type \(\mathcal{K} = (k_1, \ldots, k_n)\). Then the set of Birkhoff- Grothendieck bases of \(\Lambda\) is the orbit of any Birkhoff-Grothendieck basis \((e)\) under the \(\mathcal{K}\)-staged parabolic group \(G_{\mathcal{K}} = \{P \in GL_n(\mathbb{C}) \mid \deg(P_{ij}) \leq k_i - k_j\}\).

**Proof.** Consider two Birkhoff-Grothendieck trivialisations \(M, \tilde{M}\) of \(\Lambda\), like in the following diagram:

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{z^X} & Y_M \\
\downarrow & & \downarrow \Pi \\
\Lambda & \xrightarrow{z^X} & Y_{\tilde{M}}
\end{array}
\]

Since \(v(P_{ij}) \geq 0\), the gauge \(\Pi = z^{-K} P z^X\) is a monopole gauge if and only if \(\deg(P_{ij}) \leq k_i - k_j\). \(\square\)

Since any \(\mathcal{K}\)-staged parabolic group in dimension 1 is equal to \(\mathbb{C}^*\), a line bundle \(\mathcal{L}\) does however admit a unique trivialisation \(T_x(\mathcal{L})\) at \(x\).

**Note 3.2.** If the type is ordered by decreasing values, then the matrix \(P\) is in block-upper-triangular form with respect to the blocks of equal elements of \(\mathcal{K}\) (that is, \(\mathcal{K}\)-parabolic), and so is \(\Pi\).
Then the following hold: the Grothendieck trivialisation relation (3) is independent of the Birkhoff-Grothendieck trivialisation filtration can be defined as follows. Let $(e_1, \ldots, e_n)$ be a basis of global sections of $\mathcal{E}$ and an integer sequence $\mathcal{K} = (k_1, \ldots, k_n)$, such that $(e) = (t^{-k_1}\sigma_1, \ldots, t^{-k_n}\sigma_n)$ spans the stalk $\mathcal{E}_x$ over the local ring $\mathcal{O} = (\mathcal{O}_X)_x$, where $t$ is a local coordinate at $x$. This coordinate $t$ can be arbitrarily chosen, since the local behaviour of $\mathcal{E}$ only depends on the local ring $\mathcal{O}$. However, if we choose as coordinate $t$ a meromorphic function on $X$, then the sections $(e)$ form a basis of global (meromorphic) sections of $\mathcal{V}$. The $\mathcal{O}_X$-module $\mathcal{F}$ spanned by $(e)$ in this case does coincide with $\mathcal{E}$ at $x$, and differs from it at most on the support of the divisor of the function $t$. When $X = \mathbb{P}^1(\mathbb{C})$, we can obviously find a function $t$ with divisor $(t) = x - y$ for any arbitrary point $y \neq x$. In this case, the bundle $\mathcal{F}$ is a Birkhoff-Grothendieck trivialisation of $\mathcal{E}$ at $y$. It is clearly independent of the global basis $(\sigma)$ of $\mathcal{F}$, which is defined up to a $(-K)$-parabolic constant matrix $C \in \text{GL}_n(\mathbb{C})$, and of the specific meromorphic function $t$, which is only defined up to a non-zero constant. We call $t_y(\mathcal{F}) = \mathcal{F}$ the transport at $y$ of the Birkhoff-Grothendieck trivialisation $\mathcal{F}$ of $\mathcal{E}$ at $x$.

Understood otherwise, this is the description of a non-trivial bundle $\mathcal{E}$ by means of two trivial bundles $\mathcal{F}$ and $\tilde{\mathcal{F}}$ coinciding outside $\{x, y\}$, and glued along the cocycle $g = t^X$, where $(t) = x - y$.

### 3.1.2 The Harder-Narasimhan Flag

The Harder-Narasimhan filtration $HN(\mathcal{E})$ of $\mathcal{E}$ over $\mathbb{P}^1(\mathbb{C})$ can be obtained easily (see [S] p. 65) from a decomposition $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{L}_i$ of $\mathcal{E}$, as a direct sum of line bundles $\mathcal{L}_i \cong \mathcal{O}(a_i)$ of the appropriate degree, by

$$F^k(\mathcal{E}) = \bigoplus_{i \mid a_i \geq k} \mathcal{L}_i.$$ 

Note that such a direct sum $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_n)$ induces at $x$ a canonical Birkhoff-Grothendieck trivialisation $\mathcal{L}_x(\mathcal{E}) = \bigoplus_{i=1}^n T_x(\mathcal{L}_i)$. Locally, the Harder-Narasimhan filtration can be defined as follows. Let $(e)$ be a Birkhoff-Grothendieck basis of $\mathcal{E}_x$. The Harder-Narasimhan flag $HN_{\Lambda}$ of $V_x$ is defined by

$$F^k = \bigoplus_{i \mid a_i \geq k} Ke_i$$

(3)

**Lemma 8.** Let $\mathcal{E}$ be a holomorphic vector bundle over $X = \mathbb{P}^1(\mathbb{C})$. For $x \in X$, let $V = V_x$ and $\Lambda = \mathcal{E}_x$. Then the Harder-Narasimhan flag $HN_{\Lambda}$ of $V$ defined by relation (3) is independent of the Birkhoff-Grothendieck trivialisation $\mathcal{F}$ and basis $(e)$ appearing in the definition. Moreover, let $\pi^\mathcal{E}_x$ be the projection $\mathcal{E} \to E = \mathcal{E}_x/\mathfrak{m}_x\mathcal{E}_x$. Then the following hold:

i) The $\mathfrak{O}$-flag $HN_{\Lambda} \cap \Lambda$ coincides with the stalk $HN(\mathcal{E})_x$ of the Harder-Narasimhan filtration of $\mathcal{E}$.

ii) For any Birkhoff-Grothendieck trivialisation $\mathcal{F}$ of $\mathcal{E}$ at $x$, letting $M = \mathcal{F}_x$, the flag $\pi^\mathcal{E}_x(HN_{\Lambda} \cap M)$ coincides with the relative flag $F_M(\Lambda)$ defined in section 2.2.
iii) Conversely, for any flag $F'$ which is transversal to the flag $\pi_E^x(HN(E)_x)$ in $E = \Lambda/m_x\Lambda$, there exists a Birkhoff-Grothendieck trivialisation $\mathcal{F}'$ of $E$ at $x$, such that the relative flag $F_x(\mathcal{F}'_x)$ in $E$ coincides with $F'$.

Proof. The two first assertions are straightforward enough. Let us prove the third one. Let $T = \text{diag}(a_1 I_{n_1}, \ldots, a_s I_{n_s})$ with $a_i > a_{i+1}$ be the matrix of elementary divisors corresponding to the transformation $\Lambda = \mathcal{E}_x z^T M = \mathcal{F}_x$ in the basis $(e)$, and let $(\varepsilon)$ be the basis $z^T(e)$. Let for simplicity of notation $\nu_i = \sum_{1 \leq k \leq i} n_i$. The $(n-i+1)$-th component $F_i$ of the flag $F_\Lambda(M)$ induced by $M$ in $E = \Lambda/m_x\Lambda$ is spanned by $(\overline{v}_i+1, \ldots, \overline{v}_n)$, where $\overline{v}_k$ is the image of the basis vector $e_k$ in $E$, whereas, according to what has just been established, the image of the Harder-Narasimhan flag has its $i$-th component spanned by $(\overline{v}_1, \ldots, \overline{v}_i)$, hence both flags are transversal to each other. Any other Birkhoff-Grothendieck trivialisation $\tilde{M}$ is obtained from $(\varepsilon)$ by a monopole gauge transform $\Pi$ such that $P = z^T \Pi z^{-T} \in \text{GL}_n(\mathbb{C})$. According to Note 3.2, $\Pi$ is block-upper-triangular with respect to the blocks of equal elements of $T$, hence so is $P$. For any such $P \in \text{GL}_n(\mathbb{C})$, the matrix $z^T P z^{-T}$ is a monopole. The orbit of $(\varepsilon)$ under the set of the constant $T$-parabolic matrices covers the set of all flags in $E$ which are transversal to the image of $HN(E)_x$ in $E$. \qed

For any Birkhoff-Grothendieck trivialisation $\mathcal{F}$ of $E$ at $x$, let $Y = \Gamma(X, \mathcal{F})$ be the $\mathbb{C}$-vector space of global sections of $\mathcal{F}$. The Harder-Narasimhan filtration $HN(E)$ also induces a canonical filtration $HN_X(Y)$ of $\mathbb{C}$-vector spaces of $Y$. To avoid defining new concepts, we will also refer to this filtration as the Harder-Narasimhan filtration of $Y$. Note that it depends completely on the lattice $\Lambda \in \Lambda$.

### 3.2 Modification of the type

We wish to answer algebraically the following question: “What does the type of $E$ become when the stalk $\mathcal{E}_x = \Lambda$ at $x$ is replaced by another lattice $\tilde{\Lambda}$?” It turns out that the question can be very explicitly answered when the lattice $\tilde{\Lambda}$ is not too far from $\Lambda$, namely at distance 1 in the graph-theoretic distance of the Bruhat-Tits building. The following proposition generalizes a result of Gabber and Sabbah.

**Proposition 3.** Let $E \cong \bigoplus_{i=1}^n \mathcal{O}(a_i)$ be a holomorphic vector bundle on $X = \mathbb{P}^1(\mathbb{C})$, with $a_1 \geq \cdots \geq a_n$, and let $x \in X$. Let $\Lambda \in \mathcal{A}_x$ be a lattice such that $m_x \mathcal{E}_x \subset \tilde{\Lambda} \subset \mathcal{E}_x$. Let $E = \mathcal{E}_x/m_x \mathcal{E}_x$ be the local fiber at $x$, let $F : F_0 = 0 \subset F_1 \subset \cdots \subset F_s = E$ be the flag induced in $E$ by the Harder-Narasimhan filtration of $E$, and $W = \Lambda/m_x \mathcal{E}_x$ be the image of $\Lambda$. Assume that the type of $E$ is written as

$$a = (a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_s, \ldots, a_s).$$

Then the modified bundle $\mathcal{F} = E^{\tilde{\Lambda}}$ has type

$$\tilde{a} = (a_1, \ldots, a_1, a_2 - 1, \ldots, a_i - 1, \ldots, a_s, \ldots, a_s, a_{s+1} - 1, \ldots, a_s - 1)$$

where $m_i = \text{dim}_\mathbb{C} F_i \cap W - \text{dim}_\mathbb{C} F_{i-1} \cap W$.

Proof. This is explained in the following scheme. Let $\Lambda = \mathcal{E}_x$, and let $t$ be a local coordinate at $x$. Let $K = \text{diag}(a_1, \ldots, a_n)$ be the elementary divisors of the Birkhoff-
Grothendieck trivialisation $M$ in $\Lambda$ (or, in this case, the type of $E$).

Let $(e)$ be a basis of $\Lambda$, such that $(\sigma) = (t^K e)$ is a basis of the form $Y_M$. Under the canonical projection $\pi : \Lambda \to E = \Lambda/t\Lambda$, the HN filtration of $\Lambda$ descends to a flag of $C$-vector spaces $F : 0 = F_0 \subset \cdots \subset F_n = E$, and the quotient basis $(\tau)$ is a basis respecting this flag. Let $t\Lambda \subset \tilde{\Lambda} \subset \Lambda$ be the new lattice, and let $W \subset E$ be the subspace it is projected upon by $\pi$. Let $(u)$ be a basis respecting both $W$ and the flag $F$, and let $P_0$ be a change of basis from $(e)$ to $(u)$. Consequently, the matrix $P_0$ belongs to the parabolic subgroup $P_F$ stabilising the flag $F$, therefore it is block-upper-triangular, with blocks given by the equal elements among the $a_i$. Define now the basis $(\varepsilon)$ of $\Lambda$ as the image of $(e)$ under the constant gauge $P_0$. Here is where $d(\Lambda, \tilde{\Lambda}) \leq 1$ is important: the basis $(\varepsilon)$ is a Smith basis of $\Lambda$ (this would be not necessarily true if the lattices were further apart). Let $T = \text{diag}(t_1, \ldots, t_n)$ be the diagonal matrix such that $t_i = 0$ if $\pi(\varepsilon_i) \in W$ and $t_i = 1$ otherwise. Then $(\bar{\varepsilon}) = t^T(\varepsilon)$ is a basis of $\tilde{\Lambda}$. Let now $(\bar{\varepsilon}) = t^K (\varepsilon)$ be the basis of $\tilde{M}$ deduced from $(\varepsilon)$. The matrix of the basis change from $\Lambda$ to $\tilde{M}$ corresponding to the bases $(\sigma)$ and $(\bar{\varepsilon})$ is equal to $Q = t^{-K} P_0 t^K = (P_0)_{ij} t^{k_j-k_i}$. Now, since $P_0 \in P_F$, we have $(P_0)_{ij} = 0$ whenever $k_i - k_j < 0$. Therefore this gauge $Q = \sum Q_k + \cdots + Q_0$ is a Laurent polynomial in $t$ with only non-positive terms, where moreover $Q_0 \in \text{GL}_n(C)$. Since $X = \mathbb{P}^1(C)$, it is possible to choose as local coordinate at $\infty$ a meromorphic function with divisor $(\infty) - (0)$, namely $t = 1/z$. Accordingly, $Q$ is a polynomial in $z$, whereas $\det Q = \det P_0 \in C^*$. Hence $Q \in \text{GL}_n(C[z])$ is a monopole gauge. Since $(\sigma)$ was a basis of global meromorphic sections of $E$, then $(\bar{\varepsilon})$ also is. Therefore $\tilde{M} \in B_0$ is a trivialising lattice. Moreover, $\tilde{M}$ is a Birkhoff-Grothendieck trivialisation of both $E$ and $\mathcal{F} = E^{\tilde{\Lambda}}$, because the basis $(\bar{\varepsilon})$ is a Smith basis for $\Lambda$ and $\tilde{\Lambda}$. Therefore, we can explicitly compute the new elementary divisors of $\tilde{\Lambda}$ in $\tilde{M}$, which are given by the matrix $K - T$. Summing up, we see that the change of lattice has subtracted 1 to all the elementary divisors corresponding to the vectors of the basis $(\varepsilon)$ whose image under $\pi$ do not fall into the subspace $W$. We obtain the Harder- Narasimhan filtration of the modified bundle by reordering the type by decreasing values.

This generalises the construction given by Sabbah based on an idea of O. Gabber in [S], prop. 4.11 (where only the case where $W$ is 1-dimensional is tackled). Based on this result, the Birkhoff-Grothendieck theorem would get an immediate proof.

**Corollary 1** (Birkhoff-Grothendieck theorem). Any vector bundle $\mathcal{E} \in H$ over $\mathbb{P}^1(C)$ has the Birkhoff-Grothendieck property.

**Proof.** According to proposition 3, if a bundle $\mathcal{E}$ has the Birkhoff-Grothendieck property, so does $\mathcal{E}^M$ for any lattice $M \in [\mathcal{E}]_x$ which is adjacent to $\mathcal{E}_x$. However, according to lemma 1, two lattices are always connected by a path of adjacent lattices. Since a trivial bundle obviously has the Birkhoff-Grothendieck property, the result is established.
Note that an arbitrary trivialisation $M$ at $x$ of a vector bundle $\mathcal{E}$ is not necessarily a Birkhoff-Grothendieck one. Another obvious but useful remark is that, if $M$ is a Birkhoff-Grothendieck trivialisation of $\Lambda$, then it also is for any lattice $\Lambda'$ on the geodesic path $\Gamma(\Lambda, M)$.

Proposition 3 allows to construct effectively from an arbitrary trivialisation $M$ a Birkhoff-Grothendieck one, by following geodesics in the Bruhat-Tits building from $M$ to $\mathcal{E}_x$. The following result shows how to start the construction.

**Corollary 2.** Let $M \in \Lambda$ be a trivialising lattice in $\mathcal{B}_n$. Then any adjacent lattice $\Lambda$ admits $M$ as Birkhoff-Grothendieck trivialisation. More precisely, let $Y \subset M$ be the global form of $M$. For any basis $(e)$ respecting $W = \Lambda/mM$, the $Y$-basis $(e_Y)$ is a Smith basis for $\Lambda$.

**Proof.** Let $W = \Lambda_M/mM$ and let $T = \begin{pmatrix} 0_r & 0 \\ 0 & I_{n-r} \end{pmatrix}$ be the elementary divisors of $\Lambda_M$ with respect to $M$. Assume that $(e)$ satisfies the assumptions of the corollary. Then, according to lemma 4, the $Y$-basis $(e_Y)$ is obtained by a gauge $P = I + tU \in \text{GL}_n(\mathbb{Q})$. Putting $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, we have

$$
Y : (e_Y) \xrightarrow{t^T} \Lambda_M \xrightarrow{t^T(\Lambda)} \Lambda \\
I + tU \downarrow \uparrow \tilde{U} \in \text{GL}_n(\mathbb{Q})
$$

$$
M : (e) \xrightarrow{t^T} \Lambda_M
$$

since $\tilde{U} = t^{-T}(I + tU)t^T = \begin{pmatrix} I_r + tU_{11} & t^2U_{12} \\ U_{12} & I_{n-r} + tU_{22} \end{pmatrix}$. The basis $(e_Y)$ is therefore indeed a Smith basis of $M$ for $\Lambda$. Since it is a basis of the global form of $M$, the result follows, and in particular, the Harder-Narasimhan filtration of the corresponding bundle is equal to the $Y$-lifting of the flag $(0 \subset W \subset \mathbb{M}/mM)$. \qed

### 3.2.1 An Algorithm to compute a Birkhoff-Grothendieck Trivialisation

Let $x \in X$, and let $\Lambda = [\mathcal{E}]_x$ be the $R_x$-equivalence class of $\mathcal{E}$. Let $\Lambda = \mathcal{E}_x$ and $M = \mathcal{F}_x \in \Lambda$ where $\mathcal{F}$ is an arbitrary trivialisation of $\mathcal{E}$ at $x$. In this local setting, we “see” the global sections of $\mathcal{F}$ as the *global form* $Y \subset M$. According to lemma 1, putting $\Lambda^k = \Lambda_M + m^kM$ for $k \in \mathbb{Z}$, the sequence $(\Lambda^0, \ldots, \Lambda^d)$, where $d = d(\Lambda, M)$, forms a chain of adjacent lattices from $M$ to $\Lambda_M$. By successive applications of proposition 3, we construct a Birkhoff-Grothendieck trivialisation $M_k$ of $\Lambda^k$. Let us explain this precisely.

If there existed a Smith basis of $M$ for $\Lambda$ which spans simultaneously $Y_M$, the lattice $M$ would be a Birkhoff-Grothendieck trivialisation of $\Lambda$, and the sequence $\mathcal{K} = -T$ would represent up to homothety the type of $\mathcal{E}$. This is generally not the case.

**Lemma 9.** Let $N \in \Lambda$ be a lattice admitting $M \in \Lambda^0$ as Birkhoff-Grothendieck trivialisation. If $v_N(\Lambda) = 0$, then there exists a trivial lattice $M$ which is a Birkhoff-Grothendieck trivialisation of both $N$ and $\Lambda + mN$.

**Proof.** By assumption, there exists a Birkhoff-Grothendieck basis $(e)$ of $M$ for $N$. Let $(y)$ be the corresponding basis of $N$, and let $(\varepsilon)$ be a Smith basis of $N$ for $\Lambda$. We also
assume that the elementary divisors $T$ of $N$ in $M$ and $T'$ of $\Lambda$ in $N$ are ordered by increasing values. The gauge $U$ from $(y)$ to $(\varepsilon)$ can be factored as

$$U = U_0(I + tU')$$

with $U_0 \in \text{GL}_n(\mathbb{C})$.

According to lemma 8, the Harder-Narasimhan filtration of $N$ induces in $E = N/mN$ a flag $F$ spanned by the basis $(y)$, whereas the flag $F'$ induced by $\Lambda M$ in $E$ is spanned by $(\varepsilon)$. Let $B$ be the standard Borel subgroup of $\text{GL}_n(\mathbb{C})$. By the Bruhat decomposition, the group $\text{GL}_n(\mathbb{C})$ is a disjoint union of double cosets

$$\text{GL}_n(\mathbb{C}) = \coprod_{w \in W} BwB$$

where $W$ is the Weyl group $W = S_n$. The constant term $U_0$ of the gauge $U$ belongs to only one such cell: let $w \in S_n$ be the label of the corresponding Schubert cell. We have a decomposition

$$U_0 = QP_wQ'\ell$$

where $Q, Q' \in B$, and $P_w$ is the matrix representation of the permutation $w$. Accordingly, the gauge transforms $Q$ and $Q'$ respect respectively the flags $F$ and $F'$. In the quotient space $E = N/mN$, we have:

$$E : (y) \xrightarrow{Q} E : (y')$$

$$E : (\varepsilon) \xrightarrow{Q'} E : (\varepsilon')$$

The gauge $U_0$ represents geometrically the change of a basis that spans the Harder-Narasimhan flag $\text{HN}_N$ to one that spans the flag $\text{HN}_{\Lambda}$ induced by $\Lambda$.

Let $T' = T'_1 + \cdots + T'_k$ be the elementary splitting of $T'$. Since $(\varepsilon)$ respects the flag $F'$, it will in particular respect the trace of the first element $N_1 = \Lambda + mN$ of the geodesic $\Gamma(N, \Lambda)$, therefore any lifting of $(\varepsilon)$ will be a Smith basis of $N_1$ with elementary divisors $T'_1$. Put $T'' = T' - T'_1$. The previous scheme gets thus lifted to the following complete picture.

As a result, the elementary divisors of $N_1$ with respect to the common Birkhoff-Grothendieck trivialisation $\tilde{M}$ of $N$ and $N_1$ are not $T + T'_1$ (as with respect to $M$), but $T + w(T'_1)$, namely the elements of $T'_1$ have been twisted according to the permutation indexing the Bruhat cell that contains the matrix $U_0 \in \text{GL}_n(\mathbb{C})$. 
Note that to we have to perform an additional permutation $\sigma$ to ensure that $T + w(T_i')$ is ordered by increasing values: the resulting ordered diagonal is then $\sigma(T + w(T_i'))$.

Let $\Lambda \in \mathbf{A}$ and $M \in \mathbf{A}_0$ be an arbitrary trivialising lattice of $\Lambda$. Let $\Gamma = (\Lambda^0 = M, \Lambda^1, \ldots, \Lambda^d = \Lambda)$ be (a normalised representative of) the geodesic through $[\Lambda], [M]$. Let $(e)$ be a Smith basis of $M$ for $\Lambda$, and let the elementary divisors $T$ of $\Lambda_M$ in $M$ be written as $T = (t_1 I_{n_1}, \ldots, t_s I_{n_s})$ where $t_1 = 0 < \cdots < t_s$. Consider the elementary splitting of $T$

$$T = T_1 + \cdots + T_d \text{ where } T_i = (0_{\nu_i}, I_{n - \nu_i})$$

for a non-decreasing sequence $(\nu_i)$. Recall that each partial sum $T_1 + \cdots + T_k$ represents the elementary divisors of $\Lambda^k$ in $M$. The basis $(e)$ respects the flag $T_M(\Lambda)$ in the quotient $M/mM$, and in fact, if we let $(e^k)$ be a $\Lambda^k$-basis of the apartment $[\Phi]$ spanned by $(e)$, then $(e^k)$ respects both flags $F_M(\Lambda)$ and $F_M(\Lambda)$ in $\mathcal{T}_\kappa = \Lambda^k/m\Lambda^k$ for any $k$. With the help of lemma 9, we can construct a Birkhoff-Grothendieck trivialisation $M_k$ of the $k$-th element $\Lambda^k$ of the geodesic $\Gamma$, which is simultaneously a Birkhoff-Grothendieck trivialisation of the lattice $\Lambda_M + m\Lambda^k = \Lambda_M + m(\Lambda_M + m\Lambda^k) = \Lambda^{k+1}$. At the end of at most $d$ steps, the lattice $M_d$ is a Birkhoff-Grothendieck trivialisation of $\Lambda_M$, thus of $\Lambda$. To get the actual type, we only need to subtract $v_M(\Lambda)$.

By the way, we have proved the following result.

**Proposition 4.** Let $M$ be an arbitrary trivialising lattice of $E$ at $x$. Let $T = (t_1 I_{n_1}, \ldots, t_s I_{n_s}) = T_1 + \cdots + T_d$ with $t_1 < \cdots < t_s$ be the elementary splitting of the normalised elementary divisors $K = \min_{x_k} k_i$ of $E_x$ in $M$. There exists a sequence of permutations $w_k \in S_n$ such that the type $T(E)$ of $E$ is equal (up to permutation) to $-(T_1 + w_2(T_2) + \cdots + w_d(T_d))$.

### 3.2.2 The Abacus

Proposition 4 corresponds to a combinatorial interpretation of elementary divisors, and some manipulation of Young tableaux. Let $M^{\times n}_x \Lambda$ represent the elementary divisors of $\Lambda$ in $M$ such that $v_M(\Lambda) = 0$, and let $K = T_1 + \cdots + T_d$ be the elementary splitting of $K = (k_1, \ldots, k_n)$. Recall that all the sequences $T_i$ have the form $(0_{m_i}, I_{n-m_i})$, and that the sequence of the $m_i$ is non-decreasing. For $w = (w_2, \ldots, w_d) \in S_{n-1}^d$, let $w(K) = T_1 + w_2(T_2) + \cdots + w_d(T_d)$.

Let $Y(K)$ be the Young tableau containing whose $n$ rows have respective lengths the elements of $K$ (by decreasing order). Then we have $T_i = (0_{m_i}, I_{n-m_i})$ where $m_i$ is the number of boxes in the $i$-th column. Said otherwise, the sequence $(n-m_1, \ldots, n-m_d)$ corresponds to the Young tableau which is dual to $Y(K)$.

Let us define the abacus $ab(K)$ of $K$ as the set of box diagrams obtained from $Y(K)$ by allowing to move some boxes only vertically inside the whole corresponding column of length $n$ (like in a chinese abacus), except in the first column. As a matter of fact, we could allow to move the boxes in the first column, but, in this way, we stick to proposition 4. The diagram thus obtained can have non-adjacent boxes. To any diagram in the abacus, we attach the sequence $(a_1, \ldots, a_n)$ of number of boxes contained in each of the $n$ rows. Then we have the following result.

**Lemma 10.** Let $Y(K)$ be the Young tableau containing whose $n$ rows have respective lengths the elements of $K$ (by decreasing order). The set of sequences $w(K)$ for $w = (w_2, \ldots, w_d) \in S_{n-1}^d$ is in bijective correspondence with the abacus of $K$. Moreover, for any sequence $w(K) \in ab(K)$, we have $\Delta w(K) \leq \Delta K$ and $i(w(K)) \leq i(K)$.
Proof. We will only prove the claim on $i(\mathcal{K})$, since the other two are clear by definition. We proceed by induction on the number $d$ of columns in the Young tableau $Y(\mathcal{K})$. The Young tableau $Y$ for $\mathcal{K} = (k_1, \ldots, k_n)$ can be described unequivocally by its dual $T = (T_1, \ldots, T_d)$. First note that the diagram obtained from $Y$ by erasing the last column is again a Young tableau $Y'$, corresponding to the sequence $T' = (T_1, \ldots, T_{d-1})$. Let $\mathcal{K}' = (k'_1, \ldots, k'_n)$ be the associated sequence. Then we have $k_i = k'_i$ for $1 \leq i \leq n - T_d$ and $k_i = k'_i + 1$ for $n - T_d + 1 \leq i \leq n$. Therefore, we get $i(\mathcal{K}) = i(\mathcal{K}') + n - T_d$. In fact, an element $N \in \text{ab}(\mathcal{K})$ given, say, by the permutations $w = (w_2, \ldots, w_d)$ corresponds uniquely to the pair $(N', w_d)$ where $N' \in \text{ab}(\mathcal{K}')$ is given by the restriction $w' = (w_2, \ldots, w_{d-1})$.

For $d = 1$, the claim is clear, for $i(w(\mathcal{K})) = |\{j \mid k_j = 0\}| = i(\mathcal{K})$. Assume then that for any tableau $Y' = Y(\mathcal{K}')$ with at most $d - 1$ columns, we have $i(w(\mathcal{K}')) \leq i(\mathcal{K}')$ for $w(\mathcal{K}') \in \text{ab}(\mathcal{K}')$. Let $Y = Y(\mathcal{K})$ have $d$ columns. Let $N \in \text{ab}(\mathcal{K})$ be described by the number $t_i$ of boxes in the $i$-th row for $1 \leq i \leq n$, and let $N' = (t'_1, \ldots, t'_n)$ be the restriction of $N$ to the $d - 1$ first columns. Let $J = \{i \mid t_i = t'_i + 1\}$. Note that $|J| = T_d$. Then $i(N) = \sum_{i=1}^n (\max_j t_j - t_i)$. We distinguish two cases:

1) If $\max t_i = \max t'_i = t'_{i_0}$, then we get

$$i(N) = \sum_{i \notin J} (t'_{i_0} - (t'_i + 1)) + \sum_{i \notin J} (t'_{i_0} - t'_i)$$

$$= \sum_{i=1}^n (t'_{i_0} - t'_i) - |J| = i(N') + T_d$$

By the induction assumption, we have $i(N') \leq i(\mathcal{K}')$, therefore we get $i(N) \leq i(\mathcal{K}') - T_d = i(\mathcal{K}) - n \leq i(\mathcal{K})$.

2) Otherwise, we have $\max t_i = \max t'_i + 1 = t'_{i_0} + 1$. Then we get

$$i(N) = \sum_{i \in J} (t'_{i_0} + 1 - (t'_i + 1)) + \sum_{i \notin J} (t'_{i_0} + 1 - t'_i)$$

$$= \sum_{i=1}^n (t'_{i_0} - t'_i) + n - T_d$$

$$\leq i(\mathcal{K}') + n - T_d = i(\mathcal{K})$$

Therefore the result is established. $\square$

The Birkhoff-Grothendieck trivialisations satisfy thus a local criterion.

Proposition 5. Let $\Lambda \in \mathcal{A}$ be a lattice. For any Birkhoff-Grothendieck trivialisaton $M$ of $\Lambda$, we have

$$d(\Lambda, M) = \min_{\tilde{M} \in \mathcal{A}^0} d(\Lambda, \tilde{M}).$$

Moreover, the triviality index of $\Lambda$ is $i(\Lambda) = \min_{\tilde{M} \in \mathcal{A}^0} i(\mathcal{X}_\Lambda(\tilde{M}))$.

Proof. If $\tilde{M} \in \mathcal{A}^0$ is a trivialisaton of $\Lambda$ with elementary divisors $\mathcal{K}$, then, by proposition 4, the elementary divisors $\mathcal{K}$ of the Birkhoff-Grothendieck trivialisation $M$ found by the algorithm above are up to permutation equal to an element $w(\mathcal{K})$ of the abacus of $\mathcal{K}$. Therefore, lemma 10 implies directly the claimed result. $\square$
3.3 The Permutation Lemma

In the local approach that we are using, the global information on the vector bundle is carried by the global form \( Y_M \) that sits inside any given trivial lattice \( M \in \Lambda^0 \). However, any trivial lattice is not necessarily a Birkhoff-Grothendieck trivialising lattice, therefore the corresponding elementary divisors do not always give the type of the corresponding bundle. Here, we establish the relevant results, that are based on the following remarkable lemma.

**Lemma 11** (Permutation lemma). Let \( \mathcal{K} = (k_1, \ldots, k_n) \) be an integer sequence and \( P \in \text{GL}_n(\mathbb{C}[[t]]) \) a lattice gauge. Then

1) (Bolibrukh) there exist a permutation \( \sigma \in S_n \) and a lattice gauge \( \tilde{P} \in \text{GL}_n(\mathbb{C}[t^{-1}]) \) such that

\[
\Pi = t^{-\mathcal{K}}P^{-1}t^{\mathcal{K}_\sigma}\tilde{P} \in \text{GL}_n(\mathbb{C}[[t]])
\]

where \( t^{\mathcal{K}} = \text{diag}(t^{k_1}, \ldots, t^{k_n}) \) and \( \mathcal{K}_\sigma = (k_{\sigma(1)}, \ldots, k_{\sigma(n)}) \).

2) there exists moreover a lattice gauge \( Q \in \text{GL}_n(\mathcal{O}) \) such that

\[
t^{\mathcal{K}}\Pi = Q t^{\mathcal{K}}.
\]

We will give a self-contained proof of this result, following for the first item basically the same lines as the proof of this lemma given by Ilyashenko and Yakovenko [IY]. The second part of this lemma is, up to our knowledge, new.

The proof proceeds by induction, using the following simple lemma.

**Lemma 12.** Let \( k \leq n \) and \( T = \begin{pmatrix} I_k & 0 \\ 0 & 0_{n-k} \end{pmatrix} \). Let \( H = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}_n(\mathbb{C}[[t]]) \) be a lattice gauge matrix, decomposed as a 2 \times 2 block matrix according to the blocks of \( T \). If \( \det A(0) \neq 0 \), then there exists a monopole gauge matrix \( \Pi = \begin{pmatrix} I_k & t^{-1}\tilde{\Pi} \\ 0 & I_{n-k} \end{pmatrix} \) with \( \tilde{\Pi} \) a constant matrix, such that \( \tilde{H} = t^{-T}Ht^T\Pi \) is a lattice gauge matrix, that is \( \tilde{H} \in \text{GL}_n(\mathcal{O}) \).

**Proof.** Put for simplicity \( M_0 = M(0) \) for a holomorphic matrix \( M \). One checks that putting \( \tilde{\Pi} = -A_0^{-1}B_0 \), we have

\[
\tilde{H} = t^{-T}Ht^T\Pi = \begin{pmatrix} A & \tilde{B} \\ tC & D \end{pmatrix},
\]

where \( \tilde{B} = t^{-1}(B + A\tilde{\Pi}) \) and \( \tilde{D} = D + C\tilde{\Pi} \). By construction, the residue of \( \tilde{B} \) is equal to \( B_0 - A_0A_0^{-1}B_0 = 0 \), hence \( \tilde{B} \) is holomorphic; therefore \( \tilde{H} \) also is. To check that \( \tilde{H} \in \text{GL}_n(\mathcal{O}) \), it is sufficient to check the invertibility of

\[
\tilde{H}_0 = \begin{pmatrix} A_0 & \tilde{B}_0 \\ 0 & D_0 - C_0A_0^{-1}B_0 \end{pmatrix}.
\]

By assumption \( A_0 \) is invertible, and it is a simple exercise in linear algebra to show that \( D - CA^{-1}B \) is invertible when \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}_n(\mathbb{C}) \) is.

Note that the upper-left block of \( H \) appears unchanged in \( \tilde{H} \). Note also that

\[
\overline{H} = t^T\Pi = \begin{pmatrix} tk & \tilde{\Pi} \\ 0 & I_{n-k} \end{pmatrix}.
\]

Geometrically, we can summarize the construction of
lemma 12 as the following scheme.

\[
\Lambda \xrightarrow{H} \Lambda \xrightarrow{t^T} M
\]

\[
\Lambda \xrightarrow{t^T} \tilde{H} \xrightarrow{\Pi} \Lambda
\]

We only need a small technical lemma before giving the actual proof of the permutation lemma. Let \( K \) denote an integer sequence \((k_1, \ldots, k_{s+1}, k_s)\) with \( k_i > k_{i+1} \). We say that a matrix \( H \) is strongly \( K \)-parabolic if it has the following form

\[
H = \begin{pmatrix}
 t^{k_1} I_{n_1} & \cdots & P_{ij} \\
0 & \ddots & 0 \\
0 & & t^{k_s} I_{n_s}
\end{pmatrix},
\]

where \( P_{ij} \) is a \( n_i \times n_j \) polynomial matrix satisfying \( \deg P_{ij} < k_i \) and \( v(P_{ij}) \geq k_j \).

**Lemma 13.** Let \( H \) be strongly \( K \)-parabolic, and let \( H' = \begin{pmatrix} tI_m & \tilde{\Pi} \\ 0 & I_{n-m} \end{pmatrix} \), where \( \tilde{\Pi} \) is a constant matrix and \( m \leq n_1 \). Then the product \( HH' \) is strongly \( K' \)-parabolic, where \( K' = (k_1 + 1, \ldots, k_1 + 1, k_1, \ldots, k_1, \ldots, k_s, \ldots, k_s) \).

**Proof.** Let \( \tilde{K} = (k_2, \ldots, k_s, \ldots, k_s) \). The matrix \( H \) can be written as \( H = \begin{pmatrix} t^{k_1} I_{n_1} & P \\ 0 & \tilde{H} \end{pmatrix} \), where \( \tilde{H} \) is strongly \( K \)-parabolic, and \( P = (P_2, \cdots, P_s) \) where the blocks \( P_i \) satisfy \( \deg P_i < k_1 \) and \( v(P_i) \geq k_i \). Then, if \( m = n_1 \), the product \( HH' \) is simply

\[
HH' = \begin{pmatrix} t^{k_1+1} I_{n_1} & t^{k_1} \tilde{\Pi} + P \\ 0 & \tilde{H} \end{pmatrix}.
\]

Otherwise, we split the matrices in 3 \( \times \) 3-blocks, as

\[
HH' = \begin{pmatrix} t^m I_m & 0 & P_1 \\ 0 & t^{k_1} I_{n_1-m} & P_2 \\ 0 & 0 & \tilde{H} \end{pmatrix} \begin{pmatrix} tI_m & \tilde{\Pi} & \tilde{\Pi}_2 \\ 0 & I_{n_1-m} & 0 \\ 0 & 0 & I_{n-n_1} \end{pmatrix} = \begin{pmatrix} t^{k_1+1} I_m & t^{k_1} \tilde{\Pi}_1 & t^{k_1} \tilde{\Pi}_1 + P_1 \\ 0 & t^{k_1} I_{n_1-m} & P_2 \\ 0 & 0 & \tilde{H} \end{pmatrix}
\]

In both cases, we see that the product \( HH' \) is strongly \( K' \)-parabolic as requested. \( \square \)

**Proof of lemma 11.** Assume for simplicity that \( K = \text{diag}(k_1 I_{n_1}, \ldots, k_s I_{n_s}) \) is written by blocks, and that \( k_1 > k_2 > \ldots > k_s \). Then there exist \( m = k_1 - k_s \) matrices \( T_1, \ldots, T_m \) of the type \( T_i = \begin{pmatrix} b_i & 0 \\ 0 & 0 \end{pmatrix} \), where every \( b_i \) is equal to some \( n_1 + \cdots + n_t \), for some decreasing sequence \( t_i \), such that \( K = T_1 + \cdots + T_m \). Secondly, assume that all left-upper square sequence \( t_i \) of sizes \( b_i \) are invertible. Letting \( H = H_1 \), according to lemma 12, there exists a sequence of monopole matrices \( \Pi_i = \begin{pmatrix} t^{k_1} I_{n_1} & P_{ij} \\ 0 & t^{k_2} I_{n_2} \\ \vdots \\ 0 & t^{k_s} I_{n_s} \end{pmatrix} \).
\[
\begin{pmatrix}
I_b & t^{-1} \tilde{\Pi}_i \\
0 & I_{n-b_i}
\end{pmatrix}
\] with a constant matrix \( \tilde{\Pi}_i \), and a sequence of lattice gauge transforms \( H_i \in \text{GL}_n(\mathbb{O}) \) such that
\[
H_{i+1} = t^{-T_i} H_i t^{T_i} \tilde{\Pi}_i.
\]

Let \( \tilde{H}_i = t^{T_i} \tilde{\Pi}_i = \begin{pmatrix} t I_b & \tilde{\Pi}_i \\ 0 & I_{n-b_i} \end{pmatrix} \). It follows from lemma 13 that \( \tilde{H} = \tilde{H}_1 \cdots \tilde{H}_m \) is strongly \( K \)-parabolic. It follows then, as a remarkable consequence, that the diagonal matrix \( t^K \) can be both factored from the matrix \( \tilde{H} \) both on the left as \( \tilde{H} = t^K \Pi \) with a monopole matrix \( \Pi \), and simultaneously from the right as \( \tilde{H} = P t^K \) with a lattice gauge \( P \in \text{GL}_n(\mathbb{O}) \).

\section*{Note 3.3.} It results from the previous proof that the monopole gauge \( \Pi \) is block-upper-triangular according to \( K \), and that its block matrices \( \Pi_{ij} \) satisfy
\[
k_j - k_i \leq v(\Pi_{ij}) \leq \deg \Pi_{ij} \leq 0.
\]

As stated in [B] or [IY], one can assume that \( \sigma = \text{id} \) if all leading principal minors of \( P \) are holomorphically invertible (which can always be ensured by a permutation of the columns of \( P \)).

Geometrically, the picture obtained is very evocative.

The first row corresponds to a geodesic \( \Gamma = (\Lambda, M_1, \ldots, M) \) from \( \Lambda \) to a given Birkhoff-Grothendieck trivialisation \( M \). This path \( \Gamma \) is included in an apartment \( [\Psi] \), namely the one spanned by a Birkhoff-Grothendieck basis \( (e) \) of \( \Lambda \) corresponding to the trivialisation \( M \). By definition, the apartment \( [\Psi] \) goes through the global form \( Y \) of \( M \). The gauge \( H^{-1} \) does not map the geodesic \( \Gamma \) onto anything special. However, if we call \( [\Phi] = H^{-1}([\Psi]) \) the image of the apartment spanned by \( (e) \), the permutation lemma tells us how to construct a geodesic \( \Gamma' \) in \( [\Phi] \) whose end point is also a
Birkhoff-Grothendieck trivialisation of \( \Lambda \). Lemma 12 gives the step-by-step modification of the geodesic \( \Gamma \). Row \( i \) of the diagram corresponds indeed to a partial geodesic \( \Gamma_i = (\tilde{M}_1, \ldots, \tilde{M}_m) \) whose end-point is a Birkhoff-Grothendieck trivialisation of the \( i \)-th element \( M_i \) of the geodesic \( \Gamma' = (\Lambda, \tilde{M}_1, \ldots, \tilde{M}_m) \). Even if the end-point \( \tilde{M}_m \) is a Birkhoff-Grothendieck trivialisation of \( \Lambda \), note that the apartment \([\Phi]\) does not contain the global form \( \tilde{Y} \) of \( \tilde{M}_m \), and that we still need the gauge transform \( \tilde{H}_m \) to obtain it.

Since a permutation leaves a frame unchanged, we can deduce the following result.

**Theorem 1.** Let \( \mathcal{E} \) be a holomorphic vector bundle over \( X \), and let \( \Lambda = \mathcal{E}_x \in \Lambda \) be its stalk at \( x \in X \). For any apartment \([\Phi]\) in the Bruhat-Tits building \( \mathcal{B} \) at \( x \) such that \([\Lambda]\) \( \in \) \([\Phi]\), there exists a Birkhoff-Grothendieck trivialisation of \( \Lambda \) in \([\Phi]\).

**Proof.** Let \( (e) \) be a Birkhoff-Grothendieck basis of \( \Lambda \), and \( M \) be a Birkhoff-Grothendieck trivialisation of \( \Lambda \). Let \( (\varepsilon) \) be a basis of the lattice \( \Lambda \) which spans the apartment \([\Phi]\). Since \([\Phi]\) is invariant under \( S_n \), we can assume that the matrix \( P \in \text{GL}_n(\mathbb{O}) \) of the basis change from \( (\varepsilon) \) to \( (e) \) has invertible principal leading minors. According to the permutation lemma, there exists a matrix \( \tilde{P} \in \text{GL}_n(\mathbb{O}) \) such that \( \Pi = z^{-K} P^{-1} z^K \tilde{P} \in \text{GL}_n(\mathbb{C}[z^{-1}]) \). The gauge \( \Pi \) sends the basis of global sections \( (\sigma) = (z^K e) \) of the Birkhoff-Grothendieck trivialisation of \( \mathcal{E} \), given at \( x \) by \( M \), into a basis \( (\tilde{e}) \) of \( \tilde{M} \). Since \( \Pi \) is a monopole, the basis \( (\tilde{e}) \) is also a global basis of sections, but spans another trivialising bundle, namely \( \mathcal{F} = \mathcal{E}_{\tilde{M}} \). Therefore the arbitrary apartment \([\tilde{\Phi}]\) spanned by \( (\varepsilon) \) indeed contains a trivial bundle. Now the matrix \( \mathcal{H} = z^K \Pi \) admits a right factorisation \( \mathcal{H} = Q z^K \). As a consequence, if we let \( (\tilde{\varepsilon}) \) be the basis of \( \Lambda \) obtained from \( (e) \) by the matrix \( Q \), then \( z^K (\tilde{\varepsilon}) \) is also a basis of \( Y_{\tilde{M}} \). The following scheme sums up the situation.

\[
\begin{array}{ccc}
\Lambda : (\varepsilon) & \xrightarrow{z^K} & \Lambda : (\tilde{\varepsilon}) \\
\downarrow Q & & \downarrow \mathcal{H} = Q z^K \\
\Lambda : (e) & \xrightarrow{z^K} & Y_M \subset M \\
\downarrow P & & \downarrow \tilde{P} \\
\Lambda : (\varepsilon) & \xrightarrow{z^K} & \tilde{M}
\end{array}
\]

Therefore the lattice \( \tilde{M} \) is also a Birkhoff-Grothendieck trivialisation of \( \Lambda \).

**Corollary 3.** Let \( M \) be a Birkhoff-Grothendieck trivialisation of the lattice \( \Lambda = \mathcal{E}_x \), and \( K = (k_1, \ldots, k_n) \) the type of the corresponding bundle \( \mathcal{E} \). For any form \( Y \) in \( \Lambda \), the lattice \( \hat{M} = \Psi_Y(\mathcal{F}_\Lambda(M), K) \) is also a Birkhoff-Grothendieck trivialisation of \( \Lambda \).

**Proof.** Let \( (e) \) be a Smith basis of \( \Lambda \) for \( M \). The lattice \( \hat{M} \) is spanned by \( z^K (e_Y) \) where \( (e_Y) \) is the \( Y \)-basis of \( (e) \). The only thing to check is that the gauge from \( (e) \) to \( (e_Y) \) has invertible principal leading minors, but this is obvious since the gauge is tangent to \( I \).

The permutation lemma is in fact a sort of converse to the Birkhoff-Grothendieck theorem. Indeed, this theorem asserts that for any lattice \( \Lambda \) in the Bruhat-Tits building at infinity there exists a trivialising lattice \( \tilde{M} \) such that there is a Smith basis for \( \Lambda \) sitting inside the global form \( Y_{\tilde{M}} \). The problem then amounts to, given a lattice gauge \( P \) and a diagonal \( K \), find \( Q \) and \( \tilde{K} \) such that \( z^{-\tilde{K}} Q z^K P \in \text{GL}_n(\mathbb{C}[z]) \),
whereas in the permutation lemma, the input data would be the matrix $Q$ and the diagonal $\tilde{K}$. Schematically, the picture would be like this:

$$
\Lambda \xrightarrow{z^K} M \xrightarrow{P} Y_M \subset M
$$

$$
\Lambda \xrightarrow{z^K} \tilde{M} \supset Y_{\tilde{M}}
$$

4 Local Meromorphic Connections

Let $D = \text{Der}_C(K)$ be the $K$-vector space of dimension 1 of $C$-derivations of $K$ and $\Omega = \Omega^1_C(K)$ the dual composed of differentials of $K$. The valuation $v$ extends naturally to these spaces by the formulæ $v(\vartheta) = v(f)$ and $v(\omega) = v(g)$ if $\vartheta = f \frac{d}{dz}$ and $\omega = gdz$ for any uniformising parameter $z$ of $K$. The space $\Omega$ is naturally filtered by the rank 1 free $O$-modules $\Omega(\lambda) = \{ \omega \in \Omega | v(\omega) \geq -k \}$.

Let $V$ be a $K$-vector space of finite dimension $n$ and let $\Omega(V) = V \otimes_K \Omega^1_C(K)$. We fix a meromorphic connection $\nabla$ on $V$. This is an additive map $\nabla : V \rightarrow \Omega(V)$ satisfying the Leibniz rule

$$
\nabla(fv) = v \otimes df + f \nabla \vartheta
$$

for all $f \in K$ and all $\vartheta \in V$.

For any basis $(e) = (e_1, \ldots, e_n)$ of $V$, the matrix $\text{Mat}(\nabla, (e))$ of the connection $\nabla$ in the basis $(e)$ is the matrix $A = (A_{ij}) \in M_n(\Omega)$ such that

$$
\nabla e_j = - \sum_{i=1}^n e_i \otimes A_{ij} \text{ for all } j = 1, \ldots, n.
$$

If the matrix $P = \text{Mat}(\text{id}_V, (\varepsilon), (e)) \in \text{GL}_n(K)$ is the basis change from $(e)$ to any other basis $(\varepsilon)$, then the matrix of $\nabla$ in $(\varepsilon)$ is given by the gauge transform of $A$

$$
A_{[P]} = P^{-1}AP - P^{-1}dP.
$$

For any derivation $\tau \in \text{Der}(K/C)$, the contraction of $\nabla$ with $\tau$ induces a differential operator $\nabla_\theta$ on $V$. The connection $\nabla$ is regular whenever the set of logarithmic lattices

$$
\Lambda_{\log} = \{ \Lambda \in \Lambda | \nabla(\Lambda) \subset \Lambda \otimes C, \Omega(1) \}
$$

is non-empty. For any logarithmic lattice $\Lambda \in \Lambda_{\log}$, the connection $\nabla$ induces a well-defined residue endomorphism $\text{Res}_\Lambda \nabla \in \text{End}_{C}(\Lambda/m\Lambda)$. Note that, since the set $\Lambda_{\log}$ is closed under homothety and module sums ([C2], lemma 2.5), it induces a geodesically convex subset of the Bruhat-Tits building: if $L, L' \in \Lambda_{\log}$, then $\Gamma(L, L') \subset \Lambda_{\log}$.

4.1 The Deligne Lattice

As is well known, the choice of a matrix logarithm corresponds to fixing a special lattice in the space $V$. More precisely, let $V^\nabla \subset V \otimes_K H$ be the $C$-vector space of horizontal sections on any Picard-Vessiot extension $H$ of $K$. Let $g = g_u g_s \in \text{End}(V^\nabla)$ be the multiplicative Jordan decomposition of the corresponding local monodromy map. Then the logarithm of the unipotent part $g_u$ is canonically defined (by the Taylor expansion formula for $\log(1 + x)$), but there are several ways to define the logarithm of the semi-simple part $g_s$. Namely, one must fix a branch of the complex logarithm for every distinct eigenvalue of $g_s$. 
A classical result (variously attributed to Deligne, Manin..., see [S]) says that this choice uniquely defines a lattice in $V$. In Deligne’s terms, for any section $\sigma$ of $C \to \mathbb{C}/\mathbb{Z}$, there is a unique logarithmic lattice $\Delta_\sigma$ such that the eigenvalues of the residue map $\text{Res}_{\Delta_\sigma} \nabla$ are in the image $\text{Im} \sigma$ of $\sigma$. As a habit, one usually takes $\text{Re}(\text{Im} \sigma) \subset [0,1]$. In fact, such a habit is not as arbitrary as it seems.

**Proposition 6.** Assume that the connection $\nabla$ admits an apparent singularity (i.e., the monodromy map is trivial). Then the matrix $\text{Mat}(\nabla, (e))$ is holomorphic if and only if the lattice spanned by $(e)$ is equal to the Deligne lattice $\Delta$ attached to $\text{Re}(\text{Im} \sigma) \subset [0,1]$.

**Proof.** Since the monodromy map is trivial, its normalised logarithm with respect to $\Delta$ is 0. Hence, there is a basis of $\Delta$ where the connection has matrix 0. In any other basis $(e)$ of $\Delta$, the connection has matrix $A = P^{-1}dP$ with $P \in \text{GL}_n(\mathbb{O})$, which is holomorphic. Let $M$ be another lattice, and let $(e)$ be a Smith basis of $\Delta$ for $M$. Then the matrix in a basis of $M$ is given by the gauge equation

$$\tilde{A} = z^{-K} A z^K - z^{-K} d(z^K) = (A_{ij} z^{k_j - k_i}) - K \frac{dz}{z}.$$

The non-zero diagonal terms of the matrix $K$ of elementary divisors of $M$ give necessarily rise to a pole of order 1 in $\tilde{A}$. Therefore, $\Delta$ is the only lattice where the connection has a holomorphic matrix. $\square$

As a result, we will call $\Delta$ the Deligne lattice of $V$.

### 4.1.1 Birkhoff Forms

According to a very classical result (see e.g. [G], p. 150) if

$$\Omega = \text{Mat}(\nabla, (e)) = \sum_{k \geq 0} A_k z^k \frac{dz}{z}$$

is the series expansion in $z$ of the matrix of $\nabla$ in a basis $(e)$ of $\Delta$, the gauge $P = \sum_{k \geq 0} P_k z^k \in \text{GL}_n(\mathbb{O})$ defined recursively by

$$\begin{cases}
    P_0 = I \\
    P_k = \Phi_{A_0^{-1} A_{0} - k I}^{-1}(Q_k) \text{ where } Q_k = \sum_{i=1}^k A_i P_{k-i}
\end{cases} \tag{8}$$

transforms $\Omega$ into $A_0 dz/z$. Here we put $\Phi_{U,V}(X) = UX - VX$. Recall that the map $\Phi_{U,V}$ is an automorphism of $\mathfrak{gl}(\mathbb{C})$ when the spectra of $U$ and $V$ are disjoint. The gauge $P$ thus defined is uniquely determined; moreover, the set of bases where $\nabla$ has matrix $L \frac{dz}{z}$ where $L \in \text{M}_n(\mathbb{O})$ is a constant matrix spans a form $\Upsilon_z$ of $\Delta$, that we call the Birkhoff form of the Deligne lattice $\Delta$. The gauge transform $P$ sends in fact the basis $(e)$ to its $\Upsilon_z$-basis, that we denote here for simplicity $(e_z)$.

As it results from the proof of proposition 6, when the singularity is apparent, the Birkhoff form is uniquely defined. Otherwise, however, the form $\Upsilon_z$ depends on the choice of the local coordinate $z$. Two Birkhoff forms are nevertheless canonically isomorphic.

**Lemma 14.** Let $z,t$ be two local coordinates, and let $\alpha \in \mathbb{O}^*$ such that $z = \alpha t$. Let $P_z$ and $P_t$ be the gauge transforms that send $(e)$ to $(e_z)$ and $(e_t)$ respectively. There is a unique gauge transform $\tilde{P}$ that sends $(e_z)$ to $(e_t)$. 
Proof. One has \( \frac{dz}{z} = u \frac{dt}{t} \) with \( u = 1 + \frac{\theta t}{t} \alpha \). Put \( u = \sum_{i=0}^{\infty} u_i t^i \).

Accordingly, the matrix of the connection in \((e_z)\) satisfies

\[
\text{Mat}(\nabla, (e_z)) = A_0 \frac{dz}{z} = A_0 \left( \sum_{i=0}^{\infty} u_i t^i \right) \frac{dt}{t}.
\]

There exists therefore a uniquely defined gauge transform \( \tilde{P} = \sum_{i=0}^{\infty} \tilde{P}_i t^i \) that transforms the expression \( A_0 \frac{dz}{z} \) into \( A_0 \frac{dt}{t} \), as explained in the following scheme.

\[
\begin{array}{c}
\Omega \quad \xrightarrow{P_i} \quad A_0 \frac{dz}{z} = \sum_{i=0}^{\infty} u_i A_0 t^i \frac{dt}{t} \\
A_0 \frac{dt}{t} \quad \xrightarrow{\tilde{P}} \quad \tilde{P}
\end{array}
\]

The matrix series \( \tilde{P} \) is determined recursively by the equations (8) applied to the series \( \sum_{i=0}^{\infty} A_0 u_i t^i \). The coefficients \( \tilde{P}_i \) are even polynomials in \( A_0 \), defined by the following induction rule

\[
\begin{cases}
\tilde{P}_0 = I \\
\tilde{P}_k = \frac{1}{k} \sum_{i=1}^{k} u_i A_0 \tilde{P}_{k-i}
\end{cases}
\]

\[\square\]

4.2 Logarithmic Lattices and Stable Flags

When two lattices \( \Lambda, M \) are adjacent, all the relevant information on \( M \) can be retrieved from the quotient \( M/m \Lambda \). This is also true in presence of a connection.

Lemma 15. Let \( \Lambda \in \Lambda_{\log} \) be a logarithmic lattice. For any adjacent lattice \( M \in [m \Lambda, \Lambda] \), we have \( M \in \Lambda_{\log} \) if and only if \( M/m \Lambda \) is \( \text{Res}_{\Lambda} \nabla \)-stable.

Proof. In any basis \((e)\) of \( \Lambda \) such that the images of the first \( m = \dim W \) vectors span \( W = M/m \Lambda \), the connection matrix \( \Omega = \text{Mat}(\nabla, (e)) \) has a residue of the form \( \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in M_m(\mathbb{C}) \), where \( A \in M_m(\mathbb{C}) \). Putting \( T = \text{diag}(0_m, I_{n-m}) \), the basis \((e) = z^T(e)\) spans \( M \). It is then straightforward that the matrix \( z^{-T} \Omega z^T - T \frac{dz}{z} \) of \( \nabla \) in \((e)\) has a simple pole. \[\square\]

When the lattices are further apart, this correspondence fails. However, there is also a complete description of the logarithmic lattices as follows. Let \( \Delta \) be the Deligne lattice, and let \( \delta_{\Delta} = \text{Res}_\Delta \nabla \) be the residue \( \mathbb{C} \)-endomorphism on \( D = \Delta/m \Delta \). Let \( \Upsilon \) be the Birkhoff form of \( \Delta \) attached to a uniformising parameter \( z \). Logarithmic lattices can then be characterised as stable flags (as already remarked by Sabbah [S], th. 1.1).

Proposition 7. The set \( \Lambda_{\log} \) of logarithmic lattices is in bijection with the subset \( W_0(\Upsilon) \) of \( W(\Upsilon) \) defined by

\[ W_0(\Upsilon) = \{ (F, \mathcal{K}) \in W(\Upsilon) \mid F \text{ stabilised by } \delta_{\Delta} \}. \]

Proof. According to a classical, although not so well known, result (which can be found for instance in [BV, B, C]), a lattice \( \Lambda \in \Lambda \) is logarithmic if and only if
i) There exists a basis \((e)\) of \(\Upsilon\) such that \((z^K e)\) is a basis of \(\Lambda\), with \(K = K_\Delta(\Lambda)\),

ii) \(z^{-K} L z^K \in M_n(O)\), where \(L = \text{Mat}(\nabla_\theta, (e))\).

It results from (ii) that in this case, the matrix \(L\) is \(K\)-parabolic. Since the flag \(T_\Delta(\Lambda)\) induced by \(\Lambda\) on \(D = \Delta/m_\Delta\) is stable under \(\delta_\Delta\). Conversely, it is simply a matter of computation to show that any lattice in the \(\Upsilon\)-fibre of a \(\delta_\Delta\)-stable flag of \(D\) is logarithmic.

A difference between our result and Sabbah’s is that he only states this result as an equivalence of categories between the set of stable filtrations of \(D\) and the logarithmic lattices, whereas we give the explicit correspondence based on the lifting of \(D\) to a Birkhoff form. Although it would seem that the previous result has little value to effectively determine all logarithmic lattices, it is always possible to determine them in finite terms.

**Lemma 16.** Let \(M \in \Lambda_{\log}\) and let \((F, K) = \Phi_\Delta(M)\). Let \(Y\) be a form of \(\Delta\), and let \((e)\) be a basis of \(Y\) respecting the flag \(F\). Fix a coordinate \(z\), and let \(P = I + P_1 z + \cdots\) be the gauge from \((e)\) to its \(\Upsilon_z\)-basis \((e_z)\). Then the Laurent polynomial gauge \(Q \in \mathfrak{gl}((\mathbb{C}[z, z^{-1}])\) defined by

\[
Q = (I + \cdots + P_{d-1} z^{d-1}) z^K
\]

where \(d = d(\Delta, M)\) sends the basis \((e)\) of \(\Delta\) to a basis of \(M\).

**Proof.** This is an almost direct consequence of lemma 5.

Note that the polynomial gauge \(Q\) can be explicitly computed from formula (8). On the other hand, one can also explicitly describe the set \(W_0(\Upsilon)\). For a linear map \(f \in \text{End}(\mathbb{C}^n)\), let \(B_f\) be the set of complete flags that are stable under \(f\), and say that an apartment \([\Phi]\) is a diagonalising apartment of \(f\) if the frame \(\Phi\) is composed of eigenlines of \(f\). Then we have the following.

**Lemma 17.** Let \(\delta_\Delta = \mathfrak{d} + \mathfrak{n}\) be the additive Jordan decomposition of the residue map \(\delta_\Delta = \text{Res}_\Delta \nabla\). The pair \((F, K) \in W(\Upsilon)\) is an element of \(W_0(\Upsilon)\) if and only if \(F\) admits a complete flag refinement \(\hat{F}\) such \(\hat{F} \in B_n\) and there is a diagonalising apartment \([\Phi]\) for \(\mathfrak{d}\) that respects the flag \(\hat{F}\).

**Proof.** A flag \(F\) is \(\delta\)-stable if and only if it is stable under both \(\mathfrak{d}\) and \(\mathfrak{n}\). It is known that \(F\) is stable under \(\mathfrak{d}\) if and only if every component \(F_i\) of \(F\) is a direct sum of \(\mathfrak{d}\)-stable lines, and under \(\mathfrak{n}\) if and only if it admits a complete flag refinement \(\hat{F}\) that belongs to the flag subvariety \(B_n\) of complete flags which are preserved by the action of \(\mathfrak{n}\).

**5 The Riemann-Hilbert Problem**

This problem is by now very well-known, so we will just state the necessary notations and definitions, and refer to the classical paper of Bolibrukh [B] and to the account he gives of the construction of the Deligne bundle (see also [S] and [IY]).

Let \(S = \{s_1, \ldots, s_p\}\) a prescribed set of singular points, \(z_0 \notin S\) be an arbitrary base point, and let a representation

\[
\chi : \pi_1(X \backslash S, z_0) \longrightarrow \text{GL}_n(\mathbb{C}).
\]
The Riemann-Hilbert problem asks informally for a linear differential system having \( \chi \) as monodromy representation. In the terms used in this paper, it asks for a regular meromorphic connection \( \nabla \) with singular set \( S \) and monodromy \( \chi \) on a holomorphic vector bundle \( \mathcal{E} \). If the bundle is required to be logarithmic with respect to \( \nabla \) one speaks of a weak solution to RH. In its strongest form, the Riemann-Hilbert problem asks for a differential system \( Y' = A(z)Y \) having simple poles on \( S \) as only singularities, and whose monodromy representation is globally conjugate to \( \chi \). This amounts to asking for a weak solution \((\mathcal{E}, \nabla)\) which is moreover trivial.

### 5.1 The Röhrl-Deligne Construction

We briefly recall H. Röhrl’s construction (as presented for instance in [B, BMM]). Let \( \mathfrak{U} = (U_i)_{i \in I} \) be a finite open cover of \( X^* = X \setminus S \) by connected and simply connected open subsets \( U_i \subset X^* \) such that their intersection has the same property, and all triple intersections are empty. Let arbitrary points \( z_i \in U_i \) and \( z_{ij} \in U_i \cap U_j \), and paths \( \gamma_i : z_0 \rightarrow z_i \) and \( \gamma_{ij} : z_i \rightarrow z_{ij} \), so that \( \delta_{ij} = \gamma_i \gamma_{ij}^{-1} \gamma_j^{-1} \) is a positively-oriented loop around \( z_j \) having winding number 1. Then the cocycle \( g = (g_{ij}) \) defined over \( \mathfrak{U} \) by the constant functions \( g_{ij} = \chi([\delta_{ij}]) \) defines a flat vector bundle \( \mathcal{F} \) over \( X^* \). Define the connection \( \nabla \) over \( U_i \) by the \((0)\) matrix in the basis of sections corresponding to the cocycle \( g \). The \( \nabla \)-horizontal sections of \( \mathcal{F} \) have by construction the prescribed monodromy behaviour. This solves what we called the topological Riemann-Hilbert problem in our introduction.

Add now a small neighbourhood \( D \) of each singular point \( s \in S \) to the cover \( \mathfrak{U} \), in such a way that \( D \setminus \{s\} \) is covered by \( k \) pairwise overlapping sectors \( \Sigma_1 = D \cap U_{j_1}, \ldots, \Sigma_k = D \cap U_{j_k} \). On an arbitrarily chosen sector among the \( \Sigma_i \), say, \( \Sigma_1 \), let \( \tilde{g}_{s1} = z^L \) where \( z \) is a local coordinate at \( s \) and \( L = \frac{1}{2\pi i} \log \chi(\delta) \) normalised with eigenvalues having their real part in the interval \([0,1]\). Since the open subset \( \Sigma_1 \) only intersects \( \Sigma_2 \) and \( \Sigma_k \), the only necessary cocycle relations to satisfy are \( \tilde{g}_{s2} = \tilde{g}_{s1} g_{12} \) and \( \tilde{g}_{sk} = \tilde{g}_{s1} g_{1k} \), that we take as definition of the cocycle elements \( \tilde{g}_{s2} \) and \( \tilde{g}_{sk} \). Define in this way the remaining elements of the cocycle \( \tilde{g} \) on \( D \setminus \{s\} \). By construction, the result defines a holomorphic vector bundle \( \mathcal{D} \) on the whole of \( X \), and the connection \( \nabla \) can be extended as \( L \frac{dz}{z} \) in the basis of sections \( (\sigma) \) of \( \mathcal{D} \) over \( D \) chosen to construct \( \tilde{g}_{s1} \). The pair \((\mathcal{D}, \nabla)\) is called the Deligne bundle of \( \chi \). This construction solves simultaneously the meromorphic and the weak Riemann-Hilbert problem.

**Note 5.1.** The basis \((\sigma)\) is, in our terms, a basis of the Birkhoff form attached to the coordinate \( z \) at \( s \).

### 5.2 Weak and Strong Solutions

The Riemann-Hilbert problem can be seen as involving three different levels. The topological level is only governed by the (analytic) monodromy around the prescribed singular set. The meromorphic level is essentially based on the solution of the local inverse problem. The third one, that we call holomorphic is global and asks for the existence of a trivial holomorphic vector bundle. In fact, separating these three aspects is not so easy to do, because the Röhrl-Deligne construction in fact yields a particular holomorphic vector bundle \( \mathcal{E} \) with a connection \( \nabla \) that already respects the holomorphic prescribed behaviour.

What makes the strong Riemann-Hilbert problem a difficult one is precisely this third level. The local meromorphic invariants added to the topological solution of the inverse monodromy specify up to meromorphic equivalence class the connection \( \nabla \) on
In this respect, the natural category to state this construction is not the category of holomorphic vector bundles with meromorphic connections, but the meromorphic vector bundles, that is, pairs \((V, \nabla)\) where \(V\) is locally (but in fact globally) isomorphic to \(M_X^n\). This is why we call the second step *meromorphic*. The Riemann-Hilbert problem with the given data here corresponds to the *very weak* Riemann-Hilbert problem (as coined by Sabbah [S]): any subsheaf \(\mathcal{F}\) of locally free \(\mathcal{O}_X\)-modules contained in the (trivial) meromorphic bundle \(V\) is endowed naturally with the connection \(\nabla\), and therefore is a holomorphic vector bundle with a regular connection having the prescribed monodromy. As stated by the next result (and is otherwise well known), all solutions to the weak problem are obtained as local modifications of the Deligne bundle.

**Proposition 8.** Let \(\tilde{\pi} : \tilde{E} \to X\) and \(\tilde{\nabla} : \tilde{E} \to \tilde{E} \otimes \mathcal{O}_X\Omega\) be a weak solution to the Riemann-Hilbert problem. Then there exist a finite set \(S \subset X\), and local lattices \(M_x\) for \(x \in S\) such that the pair \((\tilde{E}, \tilde{\nabla})\) is holomorphically isomorphic to \((\mathcal{O}^M, \nabla)\).

The last step of the strong Riemann-Hilbert problem consists in searching the set of holomorphic vector bundles endowed with the connection \(\nabla\) for a bundle which at the same time has the required holomorphic invariants and is holomorphically trivial. A negative answer requires to know all the holomorphic vector bundles with this prescribed logarithmic property. Note that up to this point, the discussion presented in this section holds over an arbitrary compact Riemann surface.

### 5.2.1 Plemelj’s Theorem

In 1908, the Slovenian mathematician J. Plemelj (see [Pl]) proved a first version of the strong Riemann-Hilbert problem, under the assumption that at least one monodromy is diagonalisable. Whereas his first proof used an analytic approach (Fredholm integrals) to construct the actual matrix of solutions, to thence deduce the differential system and prove that it has only simple poles, the general framework of vector bundles recalled so far allows to establish this fact in an amazingly concise way.

**Theorem 2** (Plemelj). If one of the elementary monodromy maps from representation \(\chi : \pi_1(X\setminus S, z_0) \to \text{GL}_n(\mathbb{C})\) is diagonalisable, then the Riemann-Hilbert problem has a strong solution.

**Proof.** Let \((\mathcal{O}, \nabla)\) be the Röhrl-Deligne bundle attached to the representation \(\chi\). Let, say \(G = \chi(\gamma)\) around \(s \in S\), be diagonalisable. Let \(\Upsilon\) be a Birkhoff form at \(s\), and let \((e)\) be a basis of \(\Upsilon\) where \(G\) is diagonal. According to condition ii) in section 4.2, the whole apartment \([\Phi]\) spanned by \((e)\) consists of logarithmic lattices, whereas theorem 1 implies that \([\Phi]\) contains a trivialising lattice \(M\). The vector bundle \(\mathcal{O}^M\) is therefore both logarithmic and trivial.

**Note 5.2.** Here we have a solution by modifying the Deligne bundle only at one point. Note that the lattice \(M\) corresponds to a Birkhoff-Grothendieck trivialisation of \(\mathcal{O}\) (see theorem 3 below). Also note that this result also holds replacing \(\mathcal{O}\) with any other weak solution to Riemann-Hilbert.

### 5.2.2 Trivialisations of Weak Solutions

Let \(\mathcal{E}\) be a weak solution of the Riemann-Hilbert problem, and let \(\mathcal{F}\) be a trivialisation of \(\mathcal{E}\) at \(x \notin S\). In a global basis of sections \((e)\) of the bundle \(\mathcal{F}\), the connection \(\nabla\) is expressed by the matrix of global meromorphic 1-forms \(\Omega\), which has a simple pole at
every \( s \in S \), and an \textit{a priori} uncontrolled pole at \( x \). Assuming for simplicity that \( x \notin S \) is the point at infinity \( \infty \in \mathbb{P}^1(\mathbb{C}) \), there exist matrices \( A_i \in M_s(\mathbb{C}) \) for \( 1 \leq i \leq p \) and a matrix

\[
B(z) = B_0 + \cdots + B_hz^h
\]
such that the connection has the following matrix

\[
\Omega = \left( \sum_{i=1}^{p} \frac{A_i}{z - s_i} + B(z) \right) \, dz.
\]

The most surprising consequence of the permutation lemma, as we state it, concerns the analytic invariants of the weak solutions to the Riemann-Hilbert problem.

**Theorem 3.** Let \( E \) be a weak solution to the Riemann-Hilbert problem for \( \chi \). Then, for any \( x \notin S \), there exists a Birkhoff-Grothendieck trivialisation \( F \) of \( E \) at \( x \) which is also logarithmic at \( x \). Let \( Y = \Gamma(X,F) \) and let \( \psi_s = \text{Res}_x^\psi \, \nabla \in \text{End}(Y) \). Then we have the following.

1. The map \( \Psi = \sum_{s \in S} \psi_s = -\text{Res}_x \nabla \) is semi-simple, and has integer eigenvalues, which are equal to the type of the bundle \( E \).

2. The image of the Harder-Narasimhan filtration of \( E \) in \( Y \) is equal to the flag induced by the eigenspaces of \( \Psi \) ordered by increasing values.

**Proof.** If \( x \notin S \), the monodromy at \( x \) is trivial, and the stalk \( E_x \) of \( E \) coincides with \( \mathfrak{D}_x \). The Birkhoff form \( \Upsilon \) of \( D \) (which is then unique) is equal to the space \( V^\nabla \) of horizontal sections at \( x \). All flags in \( D = \mathfrak{D}_x/\mathfrak{m}_x \mathfrak{D}_x \) are stable under \( \text{Res}_x^{\mathfrak{D}} \nabla = 0 \). According to corollary 3, the \( \Upsilon \)-lifting of the flag induced by any Birkhoff-Grothendieck trivialisation of \( E \) at \( x \) is a logarithmic Birkhoff-Grothendieck trivialisation of \( E \) at \( x \).

In a global basis of sections \((e)\) of \( F \), the connection has the following matrix

\[
A = \sum_{s \in S \setminus \{\infty\}} \frac{A_s}{z - s} + \frac{B}{z - x} \quad \text{where} \quad B = -\sum_{s \in S} A_s \text{ if } x \neq \infty \quad (10)
\]

\[
= \sum_{s \in S} \frac{A_s}{z - s} \text{ if } x = \infty \notin S \quad (11)
\]

since \( \nabla \) has no other singularities outside \( S \cup \{x\} \). The eigenvalues of \(-B = \sum_{s \in S} A_s\) are therefore equal to the type of \( E \), and the Harder-Narasimhan filtration is defined by the blocks of equal eigenvalues ordered by increasing values.

As a consequence, we deduce the following new sufficient condition for the solubility of the strong Riemann-Hilbert problem.

**Corollary 4.** Let \( E \subset H \) be a holomorphic vector bundle in \((V, \nabla)\), and let \( \mathfrak{D} \) be the Deligne lattice of \( V \). Let \( x \in X \), such that \( E_x = \mathfrak{D}_x \). If the flag \( F \) induced in \( D = \mathfrak{D}_x/\mathfrak{m}_x \mathfrak{D}_x \) by the stalk \( \mathfrak{F}_x \) of a Birkhoff-Grothendieck trivialisation \( F \) of \( E \) at \( x \) is stable under the residue map \( \text{Res}_x^{\mathfrak{D}} \nabla \in \text{End}(D) \), then there exists a Birkhoff-Grothendieck trivialisation \( \tilde{F} \) of \( E \) at \( x \) which is moreover logarithmic at \( x \).

**Proof.** Let \( \tilde{M} \) be the \( \Upsilon \)-lifting of the flag \( F \), where \( \Upsilon \) is a Birkhoff form of the local stalk \( \mathfrak{D}_x \) of the Deligne bundle at \( x \). According to proposition 7, the lattice \( \tilde{M} \) is logarithmic, and by the permutation lemma, it is a Birkhoff-Grothendieck trivialising lattice. Therefore, the bundle \( E^{\tilde{M}} \) satisfies the conclusions of the corollary.
At this point, we would like to sum up our findings about trivial bundles in the following proposition.

**Proposition 9.** Let $F \in H_0$ be a trivial bundle in $\mathcal{V}$, and let $Y = \Gamma(X, F)$ be the $\mathbb{C}$-vector space of global sections. Let $x \in X$, and $E \in H$ such that $(F, E) \in R_x$.

i) $Y$ admits a well-defined flag $HN$ induced by the Harder-Narasimhan filtration of $E$.

ii) If $F$ is a Birkhoff-Grothendieck trivialisation of $E$ at $x$, then the flag $HN$ is obtained from a Smith basis of $Y$ for the stalk $E_x$, according to the elementary divisors $\mathcal{K}^{E_x}(F_x)$, which give moreover the type $T(E)$ of the bundle $E$.

iii) If $F$ is additionally logarithmic at $x$, and the stalk $E_x$ coincides with the Deligne lattice $\mathcal{D}_x$, then the type $T(E)$ is given by the integer parts of the eigenvalues of the residue $\text{Res}_x F \nabla \in \text{End}(Y)$, that is, of the exponents of $\nabla$ on $F$ at $x$.

iv) Finally, if $E \in \text{RH}_x$ is moreover a weak solution to Riemann-Hilbert, then the following relation holds

$$\sum_{x \in X} \text{Res}_x F \nabla = 0.$$ 

When $(E, F)$ satisfy i) to iv), we say that they form a weak RH-pair at $x$.

Let $(E, F)$ be a weak RH-pair at $x \in S$. Let $(\sigma)$ be any basis of $Y = \Gamma(X, F)$. In $(\sigma)$, the connection has a matrix of the form (13). The identification of $Y$ to $\mathbb{C}^n$ by means of $(\sigma)$ endows $\mathbb{C}^n$ with $p + 1$ linear maps $\psi_s$ for $s \in S^* = S \cup \{x\}$, that we can identify with the matrices $\tilde{L}_s$ for $s \in S$ and $-\sum_{s \in S} \tilde{L}_s$ for $s = x$. With these notations, we set the following definition.

**Definition 4.** The space $\mathbb{C}^n$, endowed with the maps $\psi_s$ for $s \in S^*$ is called a linear Fuchsian model of $E$.

With this notion, we can reduce some questions about vector bundles to linear algebra statements. For instance we can give the following computable version of a criterion for the reducibility of the triviality index (originally appearing in Sabbah [S], cor. 7), that we state here only for the case of a logarithmic modification.

**Corollary 5.** Let $E \in \text{RH}_x$ be a weak solution, and consider a linear Fuchsian model at $x \in S$, given by $p$ matrices $A_s$ for $s \in S$ such that

$$\sum_{s \in S} A_s = X = \text{diag}(k_1 I_{n_1}, \ldots, k_s I_{n_s})$$

where the integers $k_i$ satisfy $k_i > k_{i+1}$, in such a way that the flag $HN$ is the flag $0 = F_0 \subset F_1 \subset \cdots \subset F_s = \mathbb{C}^n$ having signature $(n_1, \ldots, n_s)$ in the canonical basis of $\mathbb{C}^n$. There exists a weak solution $E'$ which is adjacent to $E$ at $s \in X$ if and only if there exists an $A_s$-stable subspace $W \subset \mathbb{C}^n$ such that $W \cap F_1 = (0)$.

**Proof.** The triviality index of $E$ is equal to $i(E) = \sum_{i=1}^s n_i (k_1 - k_i)$. According to propositions 3, any adjacent weak solution $E'$ is given by an $A_s$-stable subspace $W \subset \mathbb{C}^n$. For any basis $(e)$ of $\mathbb{C}^n$ respecting the flag $HN$, the bundle $E'$ has type $K' = X - T$ where $T_i = 0$ when $e_i \in W$ and $T_i = 1$ otherwise, therefore $i(E') = \sum_{i=1}^s (\max(k_i -
we have \( i(E) - i(E') = \sum_{i=1}^{n} (k_i - T_i - \max(k_i - T_i)) \).

Now, if there exists \( i \) such that \( k_i = k_1 \) and \( T_i = 0 \), then \( \max(k_i - T_i) = k_1 \), thus \( i(E) - i(E') = \sum_{i=1}^{n} -T_i < 0 \) (because we exclude the trivial case \( W = \mathbb{C}^n \)). Otherwise we have \( \max(k_i - T_i) = k_1 - 1 \), and then \( i(E) - i(E') = \sum_{i=1}^{n} (1 - T_i) > 0 \). Therefore \( E' \) exists if and only if there exists \( W \) stable under some \( A_s \) such that \( W \cap F_1 = 0 \). \( \square \)

**Proposition 10.** Let \( \mathcal{F} \) be a Birkhoff-Grothendieck trivialisation of \( \mathcal{D} \) at \( x \notin S \). If there exists a flag \( F \) in \( Y_\mathcal{F} \) which is transversal to HN, and is moreover stable under the action of one of the maps \( \psi_s \) for \( s \in S \), then the strong Riemann-Hilbert problem has a solution, which moreover coincides with \( \mathcal{D} \) outside \( s \).

**Proof.** Let \( F \) be a flag of \( Y_\mathcal{F} \), which is stable under \( \psi_s \). Taking stalks at \( x \) of a \( \mathbb{C} \)-basis of \( F \), we can see the flag \( F \) in \( D = \mathcal{D}_s / m_s \mathcal{D}_s \). According to lemma 8, iii), there exists a Birkhoff-Grothendieck trivialisation \( \mathcal{E} \) of \( \mathcal{D} \) at \( x \), whose image in \( D = \mathcal{D}_s / m_s \mathcal{D}_s \) is \( F \). Let \( (e) \) be a Birkhoff-Grothendieck basis of \( \mathcal{D}_s \) with respect to \( \mathcal{E}_s \). Consequently, its image in \( D \) respects the flag \( F \). Let \( \Upsilon \) be a Birkhoff form of \( \mathcal{D}_s \), and let \( (e_\Upsilon) \) be the \( \Upsilon \)-basis of \( (e) \). Since the gauge from \( (e) \) to \( (e_\Upsilon) \) is tangent to \( I \), the lattice \( M \) induced from \( (e_\Upsilon) \) by the elementary divisors \( K \) of \( \mathcal{E}_s \) in \( \Lambda \) is also a trivialising Birkhoff-Grothendieck lattice for \( \mathcal{D} \) at \( s \). However, the lattice \( M \) is also logarithmic, since by construction it induces in \( D \) the \( \psi_s \)-stable flag \( F \), and moreover sits inside an apartment that contains the Birkhoff form \( \Upsilon \). Hence, the bundle \( \mathcal{D}^M \) is both trivial and logarithmic. \( \square \)

We have represented the weak solutions to the Riemann-Hilbert problem as points in a product of subvarieties of stable flags.

**Theorem 4.** Let \( \mathcal{D} \) be the Deligne bundle, and \( \mathcal{F} \) a Birkhoff-Grothendieck trivialisation at an apparent singularity \( x \notin S \). The set of weak solutions to the Riemann-Hilbert problem for \( \chi \) is parameterised by the set

\[
\text{RH}_\chi = \{(F_s, K_s)_{s \in S} \mid F_s \in \mathcal{M}_\psi(Y), K_s \in \mathbb{Z}^n(F_s)\},
\]

where \( Y = \Gamma(X, \mathcal{F}) \) and \( \psi_s = \text{Res}_{s}^\mathcal{F} \nabla \in \text{End}_\mathcal{C}(Y) \) for \( s \in S \).

### 5.3 The Type of the Deligne Bundle

The strong version of the Riemann-Hilbert would directly have a solution if the Deligne bundle were trivial. However, this is not the case, unless all singular points are apparent, since the exponents of \( \nabla \) are normalised in such a way that their sum is non-negative. This means that the type of the Deligne bundle as a rule is not trivial.

We have seen several ways to characterise this non-triviality. The type characterises the isomorphism classes of holomorphic vector bundles, so it would seem possible to work with this sole information. However, we are not in the right category to do so, since we consider holomorphic bundles with an embedding in a meromorphic one, denoted with \( \mathcal{V} \). This is the reason for which there are several trivial bundles in \( \mathcal{V} \).

From another point of view, it is not possible to determine on the sole basis of the sequence \( T = (a_1, \ldots, a_n) \), what the effect of changing the stalk of \( \mathcal{D} \) at \( x \) will be. Obviously the geometry of the Harder-Narasimhan filtration will play a decisive role.
5.3.1 Trivialisations of the Deligne Bundle

Let us examine in further detail the case of the Deligne bundle $\mathcal{D}$. Let us say that $\delta_i$ is an elementary generator of the homotopy group $G = \pi_1(X \setminus \mathcal{S}, z_0)$, if $\delta_i$ is a closed path based at $z_0$, having winding number +1 around the singularity $s_i$ and 0 around the others. Let $G_i = \chi(\delta_i)$ and $L_i = \frac{1}{2\pi i} \log G_i$ normalised as for the Deligne lattice. Let $(\sigma_i)$ be a basis of the Birkhoff form $\mathcal{T}_i$ at $s_i$ described in remark 5.1, such that the connection has locally as matrix $\Omega_i = L_i \frac{dz}{z}$, on a neighbourhood, say $D_i$ of $s_i$. On the other hand, let $D_0$ be a neighbourhood of $z_0$, and consider a basis $(\sigma_0)$ of the local Birkhoff form. According to what precedes, $(\sigma_0)$ is a basis of local $\nabla$-horizontal sections of $\mathcal{D}$ over $D_0$. One can moreover choose this basis in such a way that the monodromy of $(\sigma_0)$ around $s_i$ is exactly given by the matrix $G_i$.

Assume now for simplicity that $x \notin \mathcal{S}$ is the point at infinity $\infty \in \mathbb{P}^1(\mathbb{C})$, and let $\mathcal{F}$ be a trivialisation of $\mathcal{D}$ at $x$. In a global basis of sections ($e$) of the bundle $\mathcal{F}$, there exist matrices $B_i \in M_n(\mathbb{C})$ and a matrix

$$B(z) = B_0 + \cdots + B_t z^t$$

and $C_i \in \text{GL}_n(\mathbb{C})$ for $1 \leq i \leq p$ such that the connection has the following matrix

$$\Omega = \left( \sum_{i=1}^{p} \frac{C_i^{-1}L_iC_i}{z-s_i} + B(z) \right) dz.$$

**Note 5.3.** If the bundle $\mathcal{F}$ is moreover logarithmic at $\infty$ which can be achieved, e. g. by Plemelj’s theorem – then $B = 0$ and the residue at infinity $L_\infty = -\sum_{i=1}^{p} C_i^{-1}L_iC_i$ is semi-simple with integer eigenvalues (ssie). At the cost of a (harmless) global conjugation, we can already assume that

$$L_\infty = \text{diag}(b_1 I_{n_1}, \ldots, b_s I_{n_s}) \text{ with } b_1 < \ldots < b_s.$$

Note that the sequence $\mathcal{B} = (b_1 I_{n_1}, \ldots, b_s I_{n_s})$ coincides with the elementary divisors of the stalk $\mathcal{F}_\infty$ in $\mathcal{D}_\infty$.

**Definition 5.** We say that $(C_1, \ldots, C_p) \in \text{GL}_n(\mathbb{C})^p$ is a normalising $p$-tuple for $\chi$ if $\sum_{i=1}^{p} C_i^{-1}L_iC_i$ is ssie for some (and therefore any) normalised logarithms $L_i$ of the generators $\chi(\gamma_i)$ of the monodromy group.

Normalising $p$-tuples always exist. Putting $t$ as the coordinate $1/z$ at infinity, the Taylor expansion of $\nabla\chi$ at $x = \infty$ has then the following nice expression

$$\Omega = -\sum_{k \geq 0} \sum_{i=1}^{p} s_i^k \bar{L}_i t^k \frac{dt}{t} \text{ with } \bar{L}_i = C_i^{-1}L_iC_i.$$

(12)

We have thus reduced the computation of the type of the Deligne bundle to the computation of the matrices $C_i$ (the so-called connection matrices, because they connect the different local expressions of $\nabla\chi$ on the local Birkhoff forms). It is however well known that the computation of the connection matrices is difficult. Any other trivialisation of $\mathcal{D}$ at infinity is given by a monopole gauge ([1Y]), namely a unimodular polynomial matrix $\Pi \in \text{GL}_n(\mathbb{C}[z])$, that is, a matrix satisfying

$$\Pi = P_0 + P_1 z + \cdots + P_k z^k$$

such that $\det \Pi(z) = \text{cst} \in \mathbb{C}^*$. 

**Proposition 11.** Given a family of points $s_1, \ldots, s_p \in \mathbb{C}$ and invertible matrices $C_1, \ldots, C_p \in \text{GL}_n(\mathbb{C})$ all having the same determinant, there exists a monopole gauge $\Pi \in \text{GL}_n(\mathbb{C}[z])$ such that $\Pi(s_i) = C_i$ for $1 \leq i \leq p$. 


Proof. The group $\text{SL}_n(R)$ on a ring is generated by transvections $T_{ij}(\lambda) = I + \lambda E_{ij}$ where $\lambda \in R$ and $E_{ij}$ is the $(i,j)$ element of the canonical basis of the vector space $\mathfrak{gl}_n$. Factoring out the common value of $\det C_i$, one can assume that the matrices $C_i$ are in $\text{SL}_n(\mathbb{C})$, and that they appear as a product of transvections. At the cost of introducing the trivial transvections $T_{ij}(0) = I$, one can even assume that all are factored as a product of the same transvections with different parameters

$$C_i = T_1(\mu_1^i) \cdots T_s(\mu_s^i) \text{ with } \mu_i^i \in \mathbb{C}.$$ 

Define then $\lambda_k \in \mathbb{C}[z]$ such that $\lambda_k(s_i) = \mu_k^i$ for $1 \leq i \leq p$. By construction, the product $\tilde{\Pi} = T_1(\lambda_1) \cdots T_s(\lambda_s) \in \text{SL}_n(\mathbb{C}[z])$ indeed interpolates the matrices $C_i$ at the points $s_i$. The general case is obtained by multiplying $\tilde{\Pi}$ by the common value of $\det$.

As a consequence of this result, one can find a trivialisation $\mathcal{E}$ at infinity of the Deligne bundle such that the residues of the connection $\nabla$ are expressed in a basis of $Y = \Gamma(X, \mathcal{E})$ as the actual matrices $L_i$ (and not conjugated to them). Although the point at infinity of $\mathcal{E}$ is still an apparent singularity, we have no control on the Poincaré rank of $\nabla$ at $\infty$.

The results of this section also hold (with the adequate modifications) if the apparent singularity is assumed to be located at $z_0 \notin S \cup \{\infty\}$. We will refer to the trivialisation $\mathcal{E}$ as an adapted trivialisation of $\mathcal{D}$ at $z_0$.

**Note 5.4.** We know that there exists a family of invertible matrices $(C_i)$ such that $\sum_{i=1}^p C_i^{-1} L_i C_i$ is semi-simple with integer eigenvalues and that these eigenvalues are equal to the type of the Deligne bundle. This raises two questions:

1. Does there exist a logarithmic trivialisation of $\mathcal{D}$ for any such family $(C_i)$?

2. If there exist several families with this property, how to recognize those that indeed give the type of the Deligne bundle?

### 5.4 Reducibility of the Monodromy Representation

We establish now an improvement of a result of Bolibrukh [AB] (prop. 4.2.1, p. 84).

**Proposition 12.** If the representation $\chi$ is irreducible, then for any weak solution $\mathcal{E} \in \text{RH}_\chi$, the type $(k_1, \ldots, k_n)$ of $\mathcal{E}$ satisfies $k_i - k_j \leq p - 2$.

**Proof.** Assume here for simplicity that $x = \infty \notin S$, and consider again the setting of section 5.3.1. Let $\mathcal{E}$ be any weak solution to Riemann-Hilbert, and $\mathcal{F}$ be a logarithmic Birkhoff-Grothendieck trivialisation of $\mathcal{E}$ at $x$. Let $K = (k_1, \ldots, k_n)$ be the type of $\mathcal{E}$. In a basis $(e)$ of global sections of $\mathcal{F}$, there exist constant matrices $\bar{L}_a$ for $a \in S$ such that the connection $\nabla$ has in $(e)$ the following matrix

$$\Omega = \sum_{a \in S} \frac{\bar{L}_a}{z-a} dz = -\frac{dt}{t} \sum_{k \geq 0} \Omega_k t^k \text{ with } \Omega_k = \sum_{a \in S} a^k \bar{L}_a \text{ and } t = \frac{1}{z}. \quad (13)$$

By lemma 6, the shearing $t^{-K}$ suppresses the singularity at $x$, since the basis $t^{-K}(e)$ spans the Deligne lattice. As a consequence, $\tilde{\Omega} = t^K \Omega(t)t^{-K} + K dt$ must satisfy $v(\tilde{\Omega}) \geq 0$. Therefore, the residue matrix $B = -\sum_{a \in S} \bar{L}_a$ of $\Omega$ at $x$ is diagonal and equal to $-K$. We can assume further that

$$B = \text{diag}(b_1 I_{n_1}, \ldots, b_s I_{n_s}) \text{ with } b_1 = -k_1 < \cdots < b_s = -k_s$$
where \((k_1 I_{n_1}, \ldots, k_s I_{n_s})\) represents the type of \(\mathcal{E}\) with multiplicities. Partition any matrix \(M\) according to the eigenvalue multiplicities of \(B\), as \((M_{\ell,m})\) for \(1 \leq \ell, m \leq s\). Then the matrix of the connection can be rewritten by blocks as

\[
\tilde{\Omega}_{\ell,m} = \Omega_{\ell,m}^t(k_{\ell}-k_m) + K \frac{dt}{t} = \left( - \sum_{j \geq 0} \Omega_{\ell,m}^{(j)} t^{j+k_{\ell}-k_m} + \delta_{\ell,m} k_{\ell} I_{n_\ell} \right) \frac{dt}{t}.
\]

For each \((\ell, m)\) block, this series must have strictly positive valuation. The sum \(\sum_{\ell \in S} \tilde{L}_a = K\) imposes conditions on all blocks of the residues \(\tilde{L}_a\), while when \(\ell > m\) we get the following equations.

\[
\Omega_{\ell,m}^{(j)} = \sum_{a \in S} a^j(\tilde{L}_a)_{\ell,m} = 0 \text{ for } 0 \leq j \leq k_m - k_{\ell} \text{ when } \ell > m. \tag{14}
\]

For a fixed pair \((\ell, m)\), let \(k = \max(0, k_m - k_{\ell})\), and let \(X_i \in \mathbb{C}^{n_i \times n_m}\) be the \((\ell, m)\)-block of the matrix \(\tilde{L}_a\), for \(1 \leq i \leq p\). For \(1 \leq \alpha \leq n_\ell\) and \(1 \leq \beta \leq n_m\), let \(v_{\alpha, \beta} \in \mathbb{C}^p\) be the vector constructed by taking the coefficient of index \((\alpha, \beta)\) of \(X_i\), for \(1 \leq i \leq p\). Then, the equations (14) can be reformulated as

\[
v_{\alpha, \beta} \in \ker M_k(\mathcal{s}) \text{ where } M_k(\mathcal{s}) = \begin{pmatrix} 1 & \cdots & 1 \\ s_1 & \cdots & s_p \\ \vdots & \ddots & \vdots \\ s_{k_1} & \cdots & s_{k_p} \end{pmatrix}.
\]

The matrix \(M_k(\mathcal{s})\) is an upper-left submatrix of a Vandermonde matrix with coefficients

\[
\mathcal{s} = (s_1, \ldots, s_p) \in \mathbb{C}^p \setminus \bigcup_{i \neq j} \{x_i \neq x_j\}.
\]

Since all the \(s_i\) are distinct, this matrix has always full rank. In particular, as soon as \(k_m - k_{\ell} \geq p - 1\), it has a null kernel, and so all the blocks \(X_i\) are zero. Due to the ordering of the \(k_i\), we also have \(k_m' - k_{\ell'} \geq p - 1\) for \(m' \leq m\) and \(\ell' \geq \ell\), thus all matrices \(\tilde{L}_a\) have a lower-left common zero block. This means that the representation \(\chi\) is reducible. \(\square\)

### 5.5 Testing the Solubility of the Riemann-Hilbert Problem

In this section, we apply the results of this paper to the experimental investigation of the solubility of the Riemann-Hilbert problem. We present two ways to search the space of weak solutions, which are completely effective (up to the known problem of connection matrices): one that follows paths of adjacent logarithmic lattices, based on lemma 15, the other that uses the characterisation as stable flags given in proposition 7. Note that, if any (not necessarily logarithmic) trivial holomorphic bundle of the meromorphic solution to Riemann-Hilbert is explicitly given, the procedures that we present, coupled with classical Poincaré rank reduction methods, implemented on a computer algebra system, allow to make the actual computations. We however do not know if this bypasses the problem of the connection matrices.

Let \(\mathcal{D}\) be the Deligne bundle of the representation \(\chi\). Let \(x \notin \mathcal{S}\), and consider a logarithmic Birkhoff-Grothendieck trivialisation \(\mathcal{F}\) of \(\mathcal{D}\) at \(x\). Let \(Y = \Gamma(X, \mathcal{F})\) and choose a basis \((\sigma)\) of \(Y\) in which the residue matrix at \(x\) is equal to the diagonal that represents the type of \(\mathcal{D}\)

\[
\text{Mat}(\text{Res}_x^\mathcal{F} \nabla, (\sigma)) = -\mathcal{K} = \text{diag}(-k_1 I_{n_1}, \ldots, -k_s I_{n_s}) \text{ where } k_1 > \cdots > k_s.
\]
In the basis \((\sigma)\), the connection has a matrix of the form (13), and the Harder-Narasimhan filtration is expressed as the flag \(\text{HN}_Y\) of signature \((n_1, \ldots, n_s)\) of \(Y\).

Let \(V = \Gamma(X, \mathcal{V})\) be the \(\mathcal{R}\)-vector space of meromorphic sections of \(\mathcal{V}\), where \(\mathcal{R} = \Gamma(X, \mathcal{M}_X)\) is the field of meromorphic functions on \(X\).

For \(s \in S\), let \(t\) be a coordinate at \(x\) with divisor \((t) = x - s\), and \((\bar{\sigma}) = t^{-\mathcal{X}}(\sigma)\). Recall that \(t^{-1}\) is a coordinate at \(s\). For clarity’s sake, we will put \(t_x = t\) and \(t_s = t^{-1}\) when we are dealing with local sections. Let \(\mathcal{F} = t_s(\mathcal{F})\) be the transport of \(\mathcal{F}\) at \(s\) and \(\tilde{Y} = \Gamma(X, \mathcal{F})\). We regard \(Y\) and \(\tilde{Y}\) as sub-\(\mathcal{C}\)-vector spaces of \(V\), spanned respectively by the \(\mathcal{R}\)-bases \((\sigma)\) and \((\bar{\sigma})\) of \(V\). The relation \((\bar{\sigma}) = t^{-\mathcal{X}}(\sigma)\) induces a well-defined fixed isomorphism between \(Y\) and \(\tilde{Y}\).

**Claim 1:** The trivial bundle \(\tilde{\mathcal{F}}\) is a Birkhoff-Grothendieck trivialisation of \(\mathcal{D}\) at \(s\).

**Claim 2:** The flag \(\text{HN}_Y\) is the flag of signature \((n_1, \ldots, n_s)\) spanned by \((\bar{\sigma})\).

**Claim 3:** The germ \((\sigma_s)\) of the global basis of \(Y\) at \(s\) is a local basis of \(\mathcal{D}_s\).

Indeed, we have the two dual schematic representations, where \((\sigma_x) : \mathcal{E}_x\) means that \((\sigma)\) is a local basis of \(\mathcal{E}\) at \(x\) and \((\sigma) : Y\) means that \((\sigma)\) is a global basis of the form \(Y\)

\[
(\bar{\sigma}_x) : \mathcal{D}_x \xrightarrow{t^X_x} (\sigma) : Y \quad \text{and} \quad (\sigma_s) : \mathcal{D}_s \xrightarrow{t^X_s} (\bar{\sigma}) : \tilde{Y}.
\]

### 5.5.1 Adjacent Lattices

In this section, we consider more generally a weak solution \(\mathcal{E} \in \text{RH}_Y\). In the following proposition, we describe a procedure which allows to read off at an apparent singularity \(x \notin S\), fixed once and for all, the effect on the weak solution \(\mathcal{E}\) of a change of logarithmic adjacent lattice at any singularity \(s \in S\). More precisely, let \((\sigma)\) be a global basis of a logarithmic Birkhoff-Grothendieck trivialisation of \(\mathcal{E}\) at \(x\), and \(\Omega\) the matrix in Fuchsian form (13) of the connection \(\nabla\) in \((\sigma)\), whose residue at \(x\) gives precisely the type of \(\mathcal{E}\). Let \(M\) be a logarithmic lattice at \(s\) that is adjacent to \(\mathcal{E}_s\). We determine explicitly a gauge transform \(\Pi_M\) which is a monopole at \(x\), such that \(\Omega_{[\Pi_M]}\) has again Fuchsian form (13). From its semi-simple residue at \(x\) we read directly the type of the modified bundle \(\mathcal{E}^M\), equal to the eigenvalues, and the Harder-Narasimhan filtration of \(\mathcal{E}^M\), spanned by the eigenspaces ordered by increasing values.

This procedure is completely effective once the connection matrices \(C_s\) that relate the local residue matrices \(L_s = \frac{1}{t^X_s} \log G_s\) in the Birkhoff form at \(s\) and the global residue matrices \(\bar{L}_s = C^{-1}_s L_s C_s\) in the basis \((\sigma)\), have been determined.

Let \(M\) be a lattice at \(s\) that is adjacent to \(\mathcal{E}_s\). This lattice is uniquely characterised by its image \(W = M/m_s \mathcal{E}_s\), that can be seen as a sub-\(\mathcal{C}\)-vector space \(W \subset Y\). It is logarithmic if and only if \(W\) is stable under the map \(\text{Res}_s^{\mathcal{E}} \nabla\).

According to proposition 3, a Birkhoff-Grothendieck trivialisation of \(\mathcal{E}^M\) is obtained from a basis of \(\mathcal{E}_s\) that simultaneously respects the space \(W\) and the flag \(\text{HN}\). Moreover, we can choose \((\varepsilon)\) in the \(\text{GL}_n(\mathcal{C})\)-orbit of \((\sigma)\).

**Claim 4:** There exists a basis \((\varepsilon)\) of \(Y\) such that \(t^X_s(\varepsilon)\) spans a Birkhoff-Grothendieck trivialisation of both \(\mathcal{E}\) and \(\mathcal{E}^M\) at \(s\).

**Claim 5:** The matrix \(P \in \text{GL}_n(\mathcal{C})\) of the basis change from \((\sigma)\) to \((\varepsilon)\) is \((-\mathcal{X})\)-parabolic.

**Claim 6:** The gauge \(t^{-\mathcal{X}}_s P t^X_s = t^{-\mathcal{X}}_s P t^X_s\) is a monopole at \(s\) and an element of \(\text{GL}_n(\mathcal{O}_x)\).
**Claim 7:** The basis \((\sigma')\) generates \(M\) at \(s\) and \(E_y\) at \(y \neq x\).

**Claim 8:** The trivial bundle \(\mathcal{F}'\) spanned by \((\sigma')\) is a Birkhoff-Grothendieck trivialisation of \(E^M\) at \(x\).

**Claim 9:** The gauge transform from \((\sigma)\) to \((\sigma')\) is \(P t_x^T = P t_x^{-T}\).

**Claim 10:** The Harder-Narasimhan filtration of \(E^M\) is given by the flag of \(Y'\) spanned by \((\sigma')\) according to \(K - T\).

Indeed, the last arrow on the right implies that at \(x\), we have

\[
(\tilde{\varepsilon}_x) : E_x = E_x^M \xrightarrow{t_x^{K-T}} (\sigma') : Y' \quad \text{where} \quad Y' \subset V \text{ is spanned over } \mathbb{C} \text{ by } (\sigma').
\]

Therefore the type of \(E^M\) is, as expected, equal to \(K - T\).

**Proposition 13.** Assume that \(S \subset \mathbb{C}\) and \(x = \infty\). Let \(\varepsilon \in \text{RH}_x\) be a weak solution to the Riemann-Hilbert problem. Let the connection \(\nabla\) have a matrix \(\Omega\) of the form (13) in a basis \((\sigma)\) of a logarithmic Birkhoff-Grothendieck trivialisation \(\mathcal{F}\) of \(E\) at \(x\). Then, for any \(\bar{L}_s\)-stable subspace \(W_s\) of \(\mathbb{C}^n\), there exists a computable monopole gauge \(\Pi \in \text{GL}_n(\mathbb{C}[z])\), a constant matrix \(P_0 \in \text{GL}_n(\mathbb{C})\) and a diagonal matrix \(T\) with only 0,1 elements such that \(\Omega|_{\varepsilon_0(x,s)\mathcal{F}}\Pi\) has again a form (13) corresponding to the modification \(E^M\), where \(M\) is the lattice of \(V_s\) adjacent to \(E_s\) canonically defined by \(W_s\).

**Proof.** We identify \(\Gamma(X,\mathcal{F})\) with \(\mathbb{C}^n\) by means of the basis \((\sigma)\). The residue of \(\nabla\) at \(s\) is then equal to the matrix \(L = \bar{L}_s\) of formula (13). A logarithmic adjacent lattice \(M\) is uniquely defined by an \(L\)-stable subspace \(W \subset \mathbb{C}^n\). Let \((\varepsilon)\) be a basis respecting both \(W\) and the Harder-Narasimhan flag \(F\), and let \(P \in \text{GL}_n(\mathbb{C})\) be the basis change from \((\sigma)\) to \((\varepsilon)\). Assume for simplicity that we have ordered the vectors \(\varepsilon_1, \ldots, \varepsilon_n\) in such a way that if \(\varepsilon_i \in F_k \cap W\) and \(\varepsilon_i+1 \notin W\) then \(\varepsilon_i+1 \notin F_k\). Let \(T = \text{diag}(t_1, \ldots, t_n)\) be the diagonal integer matrix defined by \(t_i = 1\) if and only if \(\varepsilon_i \notin W\). With the simplifying assumption, the type of \(E^M\) is equal to \(K - T\), including the ordering condition, and the Harder-Narasimhan filtration is exactly obtained by putting together the groups of vectors corresponding to equal values of \(K - T\). Therefore the basis \((\sigma') = (z - s)^{K-T}(\varepsilon)\) spans a Birkhoff-Grothendieck \(\mathcal{F}'\) trivialisation of \(E^M\) at \(s\), and it is simultaneously a global basis of \(V\). The transport \(t_x(\mathcal{F}')\) is again a Birkhoff-Grothendieck trivialisation of \(E^M\) at \(x\), but it needs not be logarithmic anymore. Since \(\varepsilon\) is a weak solution, we have \(E_x = E_x^M = D_x\). Therefore, there exists a lattice gauge transformation \(P = I + P_1 t_x + P_2 t_x^2 + \cdots\) which sends the basis \((\sigma')\) into its \(\Upsilon\)-basis \((\varepsilon)\), where \(\Upsilon\) is the Birkhoff form at \(x\). The lattice \(M'\) spanned by \(t_x^{K-T}(\sigma')\) is then necessarily logarithmic, according to
proposition 5. We can effectively determine $M'$ by truncating the gauge $P$ at order $d(M', D) - 1 = k_n - k_1 - 2$, and then applying Gantmacher’s classical recursive formula (8). Then, the permutation lemma yields a monopole gauge transform $\Pi$ at $x$ so that the resulting trivialisation $\mathcal{T}$ is both Birkhoff-Grothendieck and logarithmic. In this last basis, the connection has again a form (13), where the spectrum of the residue at $x$ gives the type of the modified logarithmic bundle $\mathcal{E}^M$. 

It results from proposition 8 that iterated applications of this procedure will describe the set of all weak solutions to the Riemann-Hilbert problem, and the strong problem will be solvable if under the orbit of these transformations, one of the bundles $\mathcal{T}$ has a 0 residue at $x$.

5.5.2 The General Case

For the general case, we start with the Deligne bundle $\mathfrak{D}$, for we only have the complete description of the local logarithmic lattices from the Deligne lattice.

According to the description given in proposition 7, any logarithmic lattice $N \in \Lambda_z$ is given by an admissible pair $(F, T)$ where $F$ is a $\text{Res}_z^D \nabla$-stable flag. If we put us in the situation of section 5.5.1, and consider a logarithmic Birkhoff-Grothendieck trivialisation $\mathcal{T}$ of $\mathfrak{D}$ at $x$, and identifying $\Gamma(X, \mathcal{T})$ to $\mathbb{C}^n$ by means of the basis $(\sigma)$, then the flag $F$ can be viewed as a flag in $\mathbb{C}^n$ stable under the matrix $\tilde{L}_z$. In order to actually construct the lattice $N$, one should in principle reach first a Birkhoff form $\Upsilon_z$ in $\Lambda = \mathfrak{D}_z$. We know from lemma 2 that if we put $d = \max(t_i - t_j)$, a gauge $P$ of $z$-degree $d - 1$ is already sufficient, as remarked in the proof of proposition 13. Let $(\varepsilon)$ the basis obtained by $P$. In the apartment spanned by the basis $(\varepsilon)$ of $\Lambda$ there is a Birkhoff-Grothendieck trivialisation $\tilde{M}$ of $\Lambda$, as shown in the following scheme.

Here we cannot avoid the permutation $\tau \in S_n$, because we can’t ensure that the Birkhoff gauge $P$ satisfies the principal minors condition from the permutation lemma: the constant term $P_0$ sends the basis $(\sigma)$ onto a basis that respects both the Harder-Narasimhan flag of $\Lambda$ and the flag $F$, but not as an ordered basis. Actually, the permutation $\tau \in S_n$ is the label of the Schubert cell of $\text{GL}_n(\mathbb{C})$ that contains the matrix $P_0$.

Although we can explicitly determine the diagonal $\mathcal{K}_\tau - T$, there is no reason that these integers give the type of $\mathfrak{D}^M$, nor that $M$ is a Birkhoff-Grothendieck trivialisation of $N$. The lattice $\tilde{M}$ is nevertheless a trivialising lattice, therefore it is possible to compute a Birkhoff-Grothendieck trivialisation $M'$ by means of the algorithm described in section 3.2.1. Indeed, the gauge $\tilde{P}$, and therefore the monopole $\Pi$, have polynomial coefficients that can be effectively computed.

We would also like to note that very recently and independently, in the arXiv paper [Bo], P. Boalch has taken a similar view on local logarithmic lattices, in terms of stable filtrations and Bruhat-Tits buildings.
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