Reducing radicals in the spirit of Euclid*

Kurt Girstmair

Abstract
Let \( p \) be an odd natural number \( \geq 3 \). Inspired by results from Euclid’s *Elements*, we express the irrational
\[
y = \sqrt[p]{d + \sqrt{R}},
\]
whose degree is \( 2p \), as a polynomial function of irrationals of degrees \( \leq p \). In certain cases \( y \) is expressed by simple radicals. This reduction of the degree exhibits remarkably regular patterns of the polynomials involved. The proof is based on hypergeometric summation, in particular, on Zeilberger’s algorithm.

1. Introduction and main result

In the tenth book of Euclid’s elements, Proposition 54, one finds an answer to the following question: Suppose that the biquadratic radical
\[
y = \sqrt{d + \sqrt{R}}
\]
is given, where \( d \) and \( R \) are positive rational numbers, \( \sqrt{R} \not\in \mathbb{Q} \) and \( y \not\in \mathbb{Q}(\sqrt{R}) \). When is it possible to express \( y \) in terms of two simple square roots? Euclid’s answer is as follows: If \( d^2 - R \) is a square, i.e., \( d^2 - R = k^2, k \in \mathbb{Q} \), then
\[
y = \sqrt{d + k} + \sqrt{d - k}.
\]
(1)

But, to tell the truth, Euclid has no formulas, and his answer is phrased in purely geometric terms. So (1) is a modern algebraic interpretation of what Euclid did in the framework of his geometry (see [3, p. 119]).

Formula (1) has been considered as an example of denesting a nested radical (see [1], Th. 1). Our viewpoint, however, is different, as we illustrate with the example
\[
y = 4 \sqrt{d + \sqrt{R}}.
\]
(2)

Here we assume that \( y \) is an irrational of degree 8, which means that the polynomial \((Z^4 - d)^2 - R\) is irreducible. Provided that \( d^2 - R = k^4, k \in \mathbb{Q} \), we can apply Euclid’s result twice and obtain
\[
y = \sqrt[4]{\frac{d + k^2}{8} + \frac{k}{2}} + \sqrt[4]{\frac{d + k^2}{8} - \frac{k}{2}}.
\]
(3)

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So (3) does not denest the nested radical (2). Nevertheless, this identity can be considered as a reduction of the degree, inasmuch as it expresses an irrational of degree 8 as a sum of two irrationals of degree 4.

The present paper is devoted to this kind of reduction in the case of the radical

$$y = \sqrt[p]{d + \sqrt{R}},$$

where $p \geq 3$ is an odd natural number. To this end let $K$ be a field of characteristic 0 and let $d, R \in K \setminus \{0\}$. By $\overline{K}$ we denote an algebraic closure of $K$. An irrational is an element of $\overline{K} \setminus K$. The degree of an irrational is the degree of its minimal polynomial over $K$. Let $\sqrt{R}$ be an irrational. More precisely, we select one of the two possible values of $\sqrt{R}$, whereas the other value is denoted by $-\sqrt{R}$. This meaning of $\sqrt{R}$ shall be fixed throughout. Suppose that $y$ is an irrational of degree $2p$. This is the same as saying that the polynomial

$$g = (Z^p - d)^2 - R$$

is irreducible in the polynomial ring $K[Z]$. We are going to reduce $y$ to two irrationals of degree $\leq p$ together with $\sqrt{R}$, i.e., we express $y$ as a polynomial function (with coefficients in $\overline{K}$) of these quantities. Surprisingly, this can be done by means of explicit formulas valid for all odd natural numbers $p$.

For this purpose we work with the decomposition

$$g = h \cdot h' \text{ with } h = Z^p - d - \sqrt{R}, \ h' = Z^p - d + \sqrt{R},$$

which takes place in the polynomial ring $K(\sqrt{R})[Z]$. Now suppose that $y$ is a zero of $h$, whereas $y'$ ($\in \overline{K}$) is a zero of $h'$. We put

$$z = yy' \text{ and } u = z^{(p-1)/2}(y + y').$$

Then $z$ is a $p$th root of

$$D = d^2 - R,$$

i.e., $z^p = D \in K$. By our assumptions, $D \neq 0$. On the other hand, we will see that $u$ is a zero of the polynomial

$$f = D^{(p-1)/2} \sum_{k=0}^{(p-1)/2} \frac{c_{2k+1}}{D^k} Z^{2k+1} - 2dD^{(p-1)/2} \in K[Z],$$

where $D$ is defined by (6) and $c_{2k+1}$ by

$$c_{2k+1} = (-1)^{(p-1)/2-k} \frac{p}{2k+1} \left( \frac{p+1}{2} + k \right).$$

$k = 0, \ldots, (p-1)/2$. Here $\binom{m}{n}$ is the usual binomial coefficient. Note that $f$ is a polynomial of degree $p$ with leading coefficient $c_p = 1$. Define the polynomial $A \in K[Z]$ by

$$A = \frac{1}{2R} \sum_{k=0}^{(p-1)/2} \frac{a_{2k}}{D^k} Z^{2k} + (-1)^{(p+1)/2} \frac{d}{2RD} Z^{(p+1)/2}.$$
with

$$a_{2k} = (-1)^k \frac{p - 1}{\nu} + k \left( \frac{p-1}{2} + k \right),$$

(10)

$$k = 0, \ldots, (p-1)/2.$$ Then our main result reads as follows.

**Theorem 1** Let \( y, y', z = yy' \), and \( u = z^{(p-1)/2}(y + y') \) be as above. In particular, \( \sqrt{R} \notin K \), \( y, y' \) are irrationals of degree \( 2p \), and \( u \) is a zero of \( f \). Then

$$\{y, y'\} = \left\{ z^{(p+1)/2} \left( \frac{u}{2D} \pm A(u)\sqrt{R} \right) \right\}.$$  

(11)

**Remarks.** 1. In our setting, \( z \) is a \( p \)th root of \( D \) and \( u \) a zero of the polynomial \( f \) of degree \( p \) in \( K[Z] \). Therefore, \( (11) \) reduces the radical \( y \) of \( (10) \), whose degree is 2\( p \), to \( z, u \) and \( \sqrt{R} \), whose degrees are \( \leq p \) and 2, respectively.

2. Instead of choosing \( y, y' \) in the above way, one may choose a zero \( y \) of \( h \) and a \( p \)th root \( z \) of \( D \). Then \( y' = z/y \) is a zero of the polynomial \( h' \).

3. We will see below (Proposition 13) that for a given \( p \)th root \( z \) of \( D \) and a given zero \( u \) of \( f \) there is exactly one zero \( y \) of \( h \) such that for \( y' = z/y \) we have \( u = z^{(p-1)/2}(y + y') \).

The proof of Theorem 1 is based on techniques of hypergeometric summation, in particular, on Zeilberger’s algorithm. We think that one can hardly dispense with these techniques or, in other words, a proof without algorithmic methods of this kind seems to be out of reach.

**Example 1.** We consider the special case \( p = 5 \). Here we have

$$f = Z^5 - 5DZ^3 + 5D^2Z - 2dD^2 \quad \text{and} \quad A = \frac{1}{R} \left( \frac{Z^4}{2D} - \frac{2Z^2}{D} - \frac{dZ}{2D} + 1 \right).$$

In this case the polynomial \( f \) is called DeMoivre’s quintic (see [2], [11]). If, for instance \( d = 2 \) and \( R = 5 \), we obtain \( g = Z^{10} - 4Z^5 - 1 \), which is irreducible over \( K = \mathbb{Q} \). Moreover, \( D = -1 \), and we can choose \( z = -1 \) (in the sense of Remark 2). Finally, \( f = Z^5 + 5Z^3 + 5Z - 4 \) is irreducible over \( \mathbb{Q} \) and \( A = (Z^4 + 4Z^2 + 2Z + 2)/10 \).

Let \( z \) be a \( p \)th root of \( D \) and suppose that \( f \) has a zero \( u \) in the ground field \( K \). By Remark 3, there is a uniquely determined zero \( y \) of \( h \) such that \( u = z^{(p-1)/2}(y + y') \) for \( y' = z/y \). Then \( (11) \) expresses \( y \) and \( y' \) as polynomial functions of \( z \) and \( \sqrt{R} \) with coefficients in \( K \). In other words, the nested radicals \( y \) and \( y' \) are denested in this way.

**Example 2.** Let \( K = \mathbb{Q} \) and \( p = 7 \). Put \( d = -2158 \) and \( R = 6 \cdot 881^2 \). Thus, \( d + \sqrt{R} = -2158 + 881\sqrt{6} \) and \( D = -2 \). In this case the polynomial \( g \) equals \( Z^{14} + 4316Z^7 - 2 \) and is irreducible. It turns out that \( u = 4 \) is a zero of \( f \). We choose \( z = -2^{1/7} \), where \( 2^{1/7} \) is the real 7th root of 2. If the zero \( y \) of \( h \) is determined by \( u = z^{(p-1)/2}(y + y') \), \( y' = z/y \), then formula \( (11) \) gives

$$\{y, y'\} = \left\{ 2^{4/7} \left( -1 \pm \sqrt[6]{2}/2 \right) \right\}.$$
Section 5 contains the aforementioned Proposition 1. Moreover, we show how to find examples like the above Example 2 in a simple way.

In Section 6 we discuss the question whether \( f \) can be replaced by a polynomial of the form \( Z^p - a, a \in K \), which is the same as replacing \( u \) by the \( p \)th root of an element of \( K \) (Proposition 2). In this case our reduction could be replaced by a reduction which also denests the radical \( y \). In addition, we consider two cases where (11) is equivalent to expressing \( y, y' \) in terms of a \( \mathbb{Q} \)-basis of the field \( \mathbb{Q}(z, u, \sqrt{R}) \) (Proposition 3). These cases seem to be generic, inasmuch as most examples fall under them. We also get some insight into the structure of the splitting field of the polynomial \( g \).

2. The polynomial \( f \)

We adopt the above notation. In particular, \( f \) is defined by (7) and (8). We have to show that \( u = z^{(p-1)/2}(y + y') \) is a zero of \( f \).

For this purpose we look at the following expansion of the polynomial \( X^p + 1 \in \mathbb{Q}[X] \):

\[
X^p + 1 = \sum_{k=0}^{(p-1)/2} C_{p-2k} X^k (X + 1)^{p-2k}
\]

with rational coefficients \( C_{p-2k} \). We will see that this expansion exists and is unique. Indeed, if we apply the binomial formula to \((X + 1)^{p-2k}\) and consider only the coefficients of the monomials \( X^j, j = 0, \ldots, (p - 1)/2 \), we obtain

\[
\sum_{j=0}^{(p-1)/2} X^j \sum_{k=0}^j C_{p-2k} \binom{p-2k}{j-k} = 1. \tag{13}
\]

This gives the system of linear equations

\[
\begin{align*}
1 &= C_p, \\
0 &= C_p \binom{p}{1} + C_{p-2} \binom{p-2}{0}, \\
0 &= C_p \binom{p}{2} + C_{p-2} \binom{p-2}{1} + C_{p-4} \binom{p-4}{0}, \\
& \vdots \\
0 &= \sum_{k=0}^j C_{p-2k} \binom{p-2k}{j-k}.
\end{align*}
\]

One immediately sees that the coefficients \( C_{p-2k}, k = 0, \ldots, p - 1 \), are uniquely determined by (13).

Let \( F \) denote the polynomial on the right hand side of (12). We obtain

\[
F(y/y') \cdot y'^p = \sum_{k=0}^{(p-1)/2} C_{p-2k} z^k (y + y')^{p-2k} = y^p + y'^p = 2d, \tag{14}
\]
since \( y^p = d + \sqrt{R}, y'^p = d - \sqrt{R} \) (see \cite{5}). If we multiply the identity (14) by \( D^{(p-1)/2} = z^{p(p-1)/2} \), we have

\[
2dD^{(p-1)/2} = \sum_{k=0}^{(p-1)/2} C_{p-2k} D^k z^{(p-1)/2-k} (y + y')^{p-2k}.
\]

In other words, \( u = z^{(p-1)/2}(y + y') \) is a zero of \( f \), provided that the coefficients \( C_{p-2k} \) of (12) coincide with the coefficients \( c_{p-2k} \) of \( f \), \( k = 0, \ldots, (p-1)/2 \).

We will show that the coefficients \( c_{p-2k} \) satisfy the system (13) of linear equations. By (8),

\[
c_{p-2k} = (-1)^k \frac{p}{p-k} \binom{p-k}{k}.
\]

\( k = 0, \ldots, (p-1)/2 \). We recall that \( c_p = 1 \) and observe that the right hand side of (13) reads, for these values of \( c_{p-2k} \),

\[
\sum_{k=0}^{j} (-1)^k \frac{p}{p-k} \binom{p-k}{k} \binom{p-2k}{j-k}, \tag{15}
\]

\( j = 1, \ldots, (p-1)/2 \). This is a typical example of a hypergeometric summation (see \cite{7}, chap. 2). The summand of (15) is defined for all integers \( k \in \mathbb{Z} \), since \( \binom{p-k}{p-k}/(p-k) = \binom{p-k-1}{k} \) for all \( k \neq 0 \). In particular, it takes the value 0 for all \( k > j \) and all \( k < 0 \). For our purpose it is advisable to change the summation order, i.e., we consider

\[
\sum_{k=0}^{j} (-1)^{j-k} \frac{p}{p-j+k} \binom{p-j+k}{j-k} \binom{p-2j+2k}{k}. \tag{16}
\]

Let \( b_k \) denote the summand of (16). Hypergeometric summation requires considering \( b_{k+1}/b_k, k = 0, \ldots, j \), and \( b_0 \). We have

\[
\frac{b_{k+1}}{b_k} = (k-j)(k+p-j) \frac{(k+p-2j+1)(k+1)}{(k+p-2j+1)(k+1)}.
\]

Since \( b_0 = (-1)^j \frac{p}{p-j} \binom{p-j}{j} \), we obtain that (15) equals

\[
(-1)^j \frac{p}{p-j} \binom{p-j}{j} 2F_1 \left( -j, p-j \mid \frac{p-2j+1}{p-2j+1} \right),
\]

\( j = 1, \ldots, (p-1)/2 \), where \( 2F_1 \) denotes Gauss’ hypergeometric function. A theorem of Gauss (see \cite{7}, p. 32) says

\[
2F_1 \left( -j, b \mid c \right) = \frac{(c-b)_j}{(c)_j},
\]

here \((a)_j\) is given by \((a)_j = a \cdot (a+1) \cdots (a+j-1)\) and \( b, c \in \mathbb{Z}, c > 0 \). In our case, \( c = p-2j+1 > 0 \) and, thus, \((c)_j \neq 0\). On the other hand, \( c-b = -j+1 \), which means \((c-b)_j = 0\). Hence the sum (15) vanishes for all \( j = 1, \ldots, (p-1)/2 \).

We remark that the coefficients of the monomials \( X^j, j = (p+1)/2, \ldots, p \), in (12) also yield the system (13) for the numbers \( C_{p-2k} \).
3. The basic identity

We return to Theorem 1. Indeed, (11) is the same as saying

\[ \pm z^{(p+1)/2} A(u) \sqrt{R} = \frac{y - y'}{2}. \]  

(17)

For instance, the plus-sign on the left hand side of (17) gives

\[ y = \frac{y + y'}{2} + \frac{y - y'}{2} = \frac{u}{2z^{(p-1)/2}} + z^{(p+1)/2} A(u) \sqrt{R}, \]

which obviously has the form of (11). However, (17) is equivalent to

\[ z^{p+1} A(u)^2 R = \frac{(y + y')^2 - 4yy'}{4}. \]

Since the right hand side of this identity equals \((u^2 - 4z^p)/(4z^{p-1})\), we see that (17) is equivalent to

\[ 4D^2 A(u)^2 R = u^2 - 4D. \]  

(18)

We define

\[ f' = \frac{1}{RD^{(p-3)/2}} \sum_{j=0}^{(p-3)/2} \frac{c'_{2j+1}}{D^j} Z^{2j+1} - \frac{2d}{RD^{(p-3)/2}} \]

with

\[ c'_{2j+1} = (-1)^{p-3/2-j} \frac{p - 2}{\frac{p-1}{2} + j} \left( \frac{\frac{p-1}{2} + j}{2j + 1} \right). \]

We are going to prove the identity of polynomials

\[ 4D^2 A^2 R = f \cdot f' + Z^2 - 4D. \]  

(19)

If we insert \(u\) for the variable \(Z\) in (19) and observe \(f(u) = 0\), we obtain (18). Hence (19) can be considered as the fundamental identity of this paper. The proof of this identity consists in comparing the coefficients of the monomials \(Z^m\) on both sides.

The following three cases have to be distinguished. First, \(m \in \{0, 2\}\), second, \(m\) odd, and third, \(m = 2k; k = 2, \ldots, p - 1\). The case \(m \in \{0, 2\}\) may be checked by the reader. In the remaining cases, we write \(\alpha_m\) for the coefficient of \(Z^m\) on the left hand side of (19) and \(\beta_m\) for the coefficient of \(Z^m\) on the right hand side. In view of (9), (11), we obtain

\[ RD^{k-1} \alpha_{2k+1} = (-1) \frac{\frac{p-1}{2} + k}{2} \cdot \frac{p - 1}{\frac{p-1}{2} + k} \left( \frac{\frac{p-1}{2} + k}{2k} \right) \]

and

\[ RD^{k-1} \beta_{2k+1} = (-1) \frac{\frac{p-1}{2} + k}{2} \cdot \left( \frac{p}{\frac{p-1}{2} + k} \left( \frac{\frac{p-1}{2} + k}{2k + 1} \right) - \frac{p - 2}{\frac{p-1}{2} + k} \left( \frac{\frac{p-1}{2} + k}{2k + 1} \right) \right), \]

\(k = 0, \ldots, (p - 1)/2\) (observe that \(\binom{p-1}{p} = 0\)). Using elementary identities of binomial coefficients, one sees that \(\alpha_{2k+1} = \beta_{2k+1}\).
The remaining case is the most difficult one. We obtain
\[ RD^{k-2} \alpha_{2k} = (-1)^k \sum_{j=0}^{k} \frac{(p-1)^2}{(\frac{k-1}{2} + j)(\frac{k-1}{2} + k - j)} \left( \frac{p-1}{2} + j \right) \left( \frac{p-1}{2} + k - j \right) \]  
(20)

and
\[ RD^{k-2} \beta_{2k} = (-1)^k \sum_{j=0}^{k-1} \frac{p(p-2)}{(\frac{k+1}{2} + j)(\frac{k+1}{2} + k - j - 1)} \left( \frac{p+1}{2} + j \right) \left( \frac{p+1}{2} + k - j - 1 \right) \]  
(21)

for \( k = 2, \ldots, (p-1) \). Here we observe that the left binomial coefficient in (20) as well as in (21) equals 0 if \( j > (p-1)/2 \). In the same way the right binomial coefficient vanishes in both identities if \( k - j > (p-1)/2 \). We also observe that the sum on the right hand side of (21) may be extended to the upper bound \( k \) (instead of \( k-1 \)) since the respective summand is 0. In the next section we show that \( \alpha_{2k} = \beta_{2k}, k = 2, \ldots, p-1 \).

4. Two hypergeometric summations

In this section we denote the right hand side of (20) by \( s_k \) and the right hand side of (21) by \( t_k \) for \( k = 0, \ldots, p - 1 \). Further, we introduce
\[ u_k = \frac{(-1)^k(p-1)}{k} \left( \frac{p+k-2}{2k-1} \right), \]

\( k = 1, \ldots, p - 1 \). We will show that \( s_k = u_k \) and \( t_k = u_k \) for all \( k = 2, \ldots, p - 1 \). In this way we also exhibit the value of the coefficient \( \alpha_{2k} \) of \( 4D^2A^2R \).

Zeilberger’s algorithm yields recursion formulas for \( s_k \) and \( t_k \) (see [7, chap. 7]). In the case of \( s_k \), this formula reads
\[ (4k^2 + 6k + 2)s_{k+1} + (-k^2 - 2p + 1 + p^2)s_k = 0. \]  
(22)

Note that \( 4k^2 + 6k + 2 \neq 0 \) for \( k \geq 0 \). Now \( u_k \neq 0 \) for \( k = 1, \ldots, p - 2 \) and
\[ \frac{u_{k+1}}{u_k} = \frac{-k^2 - 2p + 1 + p^2}{4k^2 + 6k + 2}. \]

In particular, \( u_k \) satisfies formula (22). Moreover, \( u_1 = s_1 = -(p-1)^2 \). Hence \( u_k = s_k \) for all \( k = 1, \ldots, p - 1 \).

In the case of \( t_k \) Zeilberger’s algorithm yields
\[ a \cdot t_{k+2} + b \cdot t_{k+1} + c \cdot t_k = 0, \]

with
\[ a = 16k^3 + 64k^2 + 76k + 24, \quad b = -8k^3 - 12k^2 - 8pk + 4p^2k + 2p^2 - 4p + 2, \]
\[ c = k^3 - k^2 + p^2k + 2pk - k + p^2 - 2p + 1. \]

Again, \( a \neq 0 \) for all \( k \geq 0 \). It is not hard to check that
\[ a \cdot \frac{u_{k+2}}{u_k} + b \cdot \frac{u_{k+1}}{u_k} + c = 0 \]
for \( k = 2, \ldots, p - 3 \). Since \( t_2 = u_2 = p(p-1)^2(p-2)/12 \) and \( t_3 = u_3 = -p(p-1)^2(p-2)(p-3)(p+1)/360 \), we obtain \( u_k = t_k \) for all \( k = 2, \ldots, p - 1 \).
5. Some additional observations

Let the above assumptions hold, in particular, \( d \neq 0 \) and \( \sqrt{R} \notin K \).

**Proposition 1** Let \( z \) be an arbitrary \( p \)th root of \( D \) and \( u \) a zero of \( f \). Then there is a uniquely determined zero \( y \) of \( h \) such that for \( y' = z/y \) we have \( u = z^{(p-1)/2}(y+y') \).

**Proof.** Let \( V(h) \) and \( V(f) \) denote the sets of the zeros (in \( \overline{K} \)) of \( h \) and \( f \), respectively. Then the map
\[
V(h) \to V(f) : y \mapsto z^{(p-1)/2}(y+z/y)
\] (23)
is well defined (recall Remark 2 in Section 1). This map is injective. Indeed, if \( u \in V(f) \) equals \( z^{(p-1)/2}(y+z/y) \) for some \( y \in V(h) \), then the quadratic equation
\[
z^{(p-1)/2}(x+z/x) = u
\] (24)
has at most two solutions, namely, \( x = y \) and \( x = z/y \). However, \( z/y \) is a zero of \( h' \), and \( h \) and \( h' \) have no common zero since \( \sqrt{R} \neq 0 \). Hence \( y \) is the only solution of this equation in \( V(h) \).

Since \( h \) has no multiple zeros, the set \( V(h) \) has \( p \) elements and \( V(f) \) at most \( p \). By the injectivity, \( V(f) \) also has \( p \) elements and the map of (23) is bijective. This proves our assertion. \( \square \)

Our next aim is a simple construction of examples like Example 2 in Section 1. Suppose that \( D \in K \setminus \{0\} \) and \( u \in K \) are given. Then the equation \( f(u) = 0 \) holds if, and only if,
\[
d = \frac{1}{2} \sum_{j=0}^{(p-1)/2} \frac{e_{2j+1}}{D^j} u^{2j+1}
\]
(recall (7), (8)). For instance, if \( D = -2 \) and \( u = 4 \), we obtain \( d = -2158 \) (see the aforementioned example). Then we determine \( R \) by \( R = d^2 - D \). In the case \( K = \mathbb{Q} \) it frequently happens that \( \sqrt{R} \notin \mathbb{Q} \) and that \( g = (Z^p - d)^2 - R \) is irreducible in \( \mathbb{Q}[Z] \). Choose a \( p \)th root \( z \) of \( D \) (in \( \overline{K} \)). Then the zero \( y \) of \( h \) is uniquely determined as a solution of the quadratic equation (24); and \( y \) and \( y' = z/y \) can be obtained by (11). In general, however, it is simpler to obtain \( y, y' \) as solutions of the quadratic equation — provided that \( u \) is known. In this case
\[
\{y, y'\} = \left\{ \frac{1}{2z^{(p-1)/2}} \left( u \pm \sqrt{u^2 - 4z^p} \right) \right\}.
\] (25)
Since \( u \) and \( z^p = D \in K \), we have \( u^2 - 4z^p \in K \). Of course, (25) is equivalent to (11) in this context.

6. Two further results

The question arises whether the zero \( u \) of \( f \) can be replaced by the \( p \)th root of an element of \( K \). This would imply that the splitting field \( L \) of \( f \) is also the splitting field of a polynomial \( P \) of the form \( P = Z^p - a \) for some \( a \in K \).
In order to obtain a partial answer, we suppose that \( p \geq 3 \) is a prime and \( K = \Q \). Further, we assume that \( f \) is irreducible over \( \Q \). If \( L \) is also the splitting field of \( P \), then \( P \) is irreducible over \( \Q \) and \( L \) contains a primitive \( p \)th root of unity \( \zeta_p \). Indeed, \( L = \Q(v, \zeta_p) \), where \( v \) is a zero of \( P \).

**Proposition 2** As above, let \( K = \Q \), \( p \geq 3 \) a prime and \( f \) irreducible in \( \Q[Z] \). If \( \Q(\sqrt[1]{R}) \neq \Q(\sqrt[1]{-(1)^{(p-1)/2}p}) \), then \( \zeta_p \notin L \). In particular, \( L \) is not the splitting field of a polynomial \( P \) as above.

**Proof.** Let \( y \) be a zero of \( h \), \( y' \) a zero of \( h' \), and \( z = yy' \). The bijection \((23)\) implies that the numbers \( u_k = z^{(p-1)/2}(y\zeta_p^k + y'\zeta_p^{-k}) \), \( k = 0, \ldots, p-1 \), are exactly the zeros of \( f \). The Lagrange resolvent
\[
\sum_{k=0}^{p-1} \zeta_p^{-k}u_k = z^{(p-1)/2}yp + z^{(p-1)/2}y'\sum_{k=0}^{p-1} \zeta_p^{-2k} = z^{(p-1)/2}yp
\]
shows \( z^{(p-1)/2}y \in L(\zeta_p) \) and, thus, \( D^{(p-1)/2}yp = D^{(p-1)/2}(d + \sqrt[1]{R}) \in L(\zeta_p) \). In particular, \( \sqrt[1]{R} \in L(\zeta_p) \). Observe that all elements of \( L \) of a degree different from \( p \) are contained in a uniquely determined subfield \( L_1 \), whose degree (over \( \Q \)) divides \( p-1 \) (see [11 p. 163]).

Suppose that \( \zeta_p \in L \). Then \( \zeta_p \in L_1 \), and, therefore, \( L_1 = \Q(\zeta_p) \) since \( \zeta_p \) has the degree \( p-1 \) over \( \Q \). Moreover, \( \sqrt[1]{R} \in L_1 \). The field \( \Q(\zeta_p) \) has a cyclic Galois group of order \( p-1 \) over \( \Q \). Accordingly, it contains a uniquely determined quadratic subfield, namely, \( \Q(\sqrt[1]{-(1)^{(p-1)/2}p}) \) (see [6 p. 71]). This implies \( \Q(\sqrt[1]{R}) = \Q(\sqrt[1]{-(1)^{(p-1)/2}p}) \).

Next we investigate the connection between formula \((11)\) and \( \Q \)-bases.

**Proposition 3** As above, let \( K = \Q \), \( p \geq 3 \) a prime and \( f \) irreducible in \( \Q[Z] \). In addition, suppose that \( \zeta_p \) is not contained in the splitting field of \( f \).

(a) If \( D = z^p \) for some \( z \in \Q \), then \( u^k\sqrt[1]{R}^l \), \( k = 0, \ldots, p-1 \), \( l = 0, 1 \), is a \( \Q \)-basis of \( \Q(y) = \Q(y') = \Q(u, \sqrt[1]{R}) \).

(b) Suppose that \( D \) does not have this form. Let \( z \) be a \( p \)th root of \( D \). Then \( z^ju^k\sqrt[1]{R}^l \), \( j, k = 0, \ldots, p-1 \), \( l = 0, 1 \), is a \( \Q \)-basis of \( \Q(z, u, \sqrt[1]{R}) \).

In both cases, \((11)\) expresses \( y, y' \) in terms of the respective basis.

**Proof.** Assertion (a) is obvious. Suppose that \( D \) does not have the form of (a). Then \( Z^p - D \) is irreducible over \( \Q \) (see [10 p. 221]). Let \( z \) be a \( p \)th root of \( D \). Suppose that \( Z^p - D \) has a zero in \( \Q(u) \). This means that \( z\zeta_p^k \in \Q(u) \) for some \( k \in \{0, \ldots, p-1\} \). But then \( \Q(u) = \Q(z\zeta_p^k) \), because both \( f \) and \( Z^p - D \) are irreducible. Since the splitting field \( L \) of \( f \) is a Galois extension of \( \Q \), this implies that \( Z^p - D \) splits into linear factors over \( L \). In particular, \( \zeta_p \in L \), which we have excluded. Hence \( Z^p - D \) has no zero in \( \Q(u) \), and, by the cited argument, \( Z^p - D \) is irreducible over \( \Q(u) \). Accordingly, \( \Q(z, u) \) has the degree \( p^2 \) over \( \Q \). This field does not contain \( \sqrt[1]{R} \) for reasons of degree. Altogether, the field \( \Q(z, u, \sqrt[1]{R}) \) has the degree \( 2p^2 \) over \( \Q \) and the \( \Q \)-basis of (b). Now it is clear that \((11)\) expresses \( y, y' \) in terms of this basis.

Note that in case (b) neither \( y \in \Q(u, \sqrt[1]{R}) \) nor \( y \in \Q(z, \sqrt[1]{R}) \). Observe, further, that in this case only \( (p+1)/2 + 2 \) of a total of \( 2p^2 \) basis vectors actually occur in \((11)\).
seems that Propositions 2 and 3 cover the generic case, i.e., most examples satisfy the assumptions of these propositions.

Let us briefly look at the splitting field $M$ of the polynomial $g$ in case (b). If $L$ denotes the splitting field of $f$, then $M$ is the composite of the Galois extensions $\mathbb{Q}(z, \zeta_p)$ and $L$ of $\mathbb{Q}$ (recall that $\sqrt{R} \in L(\zeta_p)$). The degrees of these extensions over $\mathbb{Q}$ are $p(p-1)$ and $pq$, $q \mid p - 1$, respectively. The structure of their Galois groups is well known (see [4, p. 163]). The intersection of these Galois extensions is a subfield of $\mathbb{Q}(\zeta_p)$.

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Kurt Girstmair
Institut für Mathematik
Universität Innsbruck
Technikerstr. 13/7
A-6020 Innsbruck, Austria
Kurt.Girstmair@uibk.ac.at