A formula for the coincidence Reidemeister trace of selfmaps on bouquets of circles

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Abstract
We give a formula for the coincidence Reidemeister trace of selfmaps on bouquets of circles in terms of the Fox calculus. Our formula reduces the problem of computing the coincidence Reidemeister trace to the problem of distinguishing doubly twisted conjugacy classes in free groups.

1 Introduction
Fadell and Husseini, in [2] proved the following:

Theorem 1. [Fadell, Husseini, 1983] Let \(X\) be a bouquet of circles, \(G = \pi_1(X)\), and let \(G_0\) be the set of generators of \(G\). If \(f : X \to X\) induces the map \(\varphi : G \to G\), then there is some lift \(\tilde{f} : \tilde{X} \to \tilde{X}\) to the universal covering space with

\[
RT(f, \tilde{f}) = \rho \left( 1 - \sum_{a \in G_0} \frac{\partial}{\partial a} \varphi(a) \right),
\]

where \(\rho : \mathbb{Z}G \to \mathbb{Z}R(f)\) is the linearization of the projection into twisted conjugacy classes, and \(\partial\) denotes the Fox derivative.

Theorem 1 reduces the calculation of the Reidemeister trace (and thus of the Nielsen number) in fixed point theory to the computation of twisted conjugacy classes. Our goal for this paper is to obtain a similar result in coincidence theory of selfmaps – a formula for the coincidence Reidemeister trace \(RT(f, \tilde{f}, g, \tilde{g})\) in terms of Fox derivatives which reduces the computation to twisted conjugacy decisions.

The proof of Theorem 1 given in [2] is brief, thanks to a natural trace-like formula for the Reidemeister trace in fixed point theory. No such formula exists.

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for the coincidence Reidemeister trace, and this will complicate our derivation
considerably. Our argument is based on first specifying a particular regular form
for maps in Section 3 and for pairs of maps in Section 4. In Section 5 we give
our main result.

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2 Preliminaries

Throughout the paper, let \( X \) be a bouquet of circles meeting at the base point
\( x_0 \). Let \( G = \pi_1(X) \), a free group, and let \( G_0 \) be the set of generators of \( G \).
Let \( \tilde{X} \) be the universal covering space of \( X \) with projection \( p_X : \tilde{X} \to X \), and
choose once and for all a base point \( \tilde{x}_0 \in \tilde{X} \) with \( p_X(\tilde{x}_0) = x_0 \).

Given maps \( f, g : X \to X \) and their induced homomorphisms \( \varphi, \psi : G \to G \),
we define an equivalence relation on \( G \) as follows: two elements \( \alpha, \beta \in G \) are
[doubly] twisted conjugate if

\[
\alpha = \varphi(\gamma) \beta \psi(\gamma)^{-1}.
\]

The equivalence classes with respect to this relation are the Reidemeister classes,
and we denote the set of such classes as \( \mathcal{R}(\varphi, \psi) \). Let \( \rho : G \to \mathcal{R}(\varphi, \psi) \) be the
projection into Reidemeister classes.

For any pair of maps \( f, g : X \to X \), denote their coincidence set by
\( \text{Coin}(f, g) = \{ x \in X \mid f(x) = g(x) \} \). The set of coincidence points are partitioned into coincidence classes of the form
\( \text{Coin}(\alpha^{-1} \tilde{f}, \tilde{g}) \), where \( \alpha \in G \) and \( \tilde{f}, \tilde{g} : \tilde{X} \to \tilde{X} \)
are specified lifts of \( f \) and \( g \). Lemma 9 of [5] shows that \( \text{Coin}(\alpha^{-1} \tilde{f}, \tilde{g}) = \text{Coin}(\beta^{-1} \tilde{f}, \tilde{g}) \)
if and only if \( \rho(\alpha) = \rho(\beta) \), and that \( \text{Coin}(\alpha^{-1} \tilde{f}, \tilde{g}) \) and \( \text{Coin}(\beta^{-1} \tilde{f}, \tilde{g}) \)
are disjoint if \( \rho(\alpha) \neq \rho(\beta) \). (The statement appears in a slightly different form
in [5] because a slightly different twisted conjugacy relation is used. Trivial
modifications will produce the version used here.)

Thus the coincidence classes are represented by Reidemeister classes in \( G \),
and so each particular coincidence point has an associated Reidemeister class.
For \( x \in \text{Coin}(f, g) \), let \( [x_{\tilde{f}, \tilde{g}}] \in \mathcal{R}(\varphi, \psi) \) denote the Reidemeister class \( \rho(\alpha) \)
for which \( x \in p_X(\text{Coin}(\alpha^{-1} \tilde{f}, \tilde{g})) \).

Let \( f, g : X \to X \) be mappings with isolated coincidence points, and for
each coincidence point \( x \) let \( U_x \subset X \) be a neighborhood of \( x \) containing no
other coincidence points. Then we define the coincidence Reidemeister trace as:

\[
RT(f, \tilde{f}, g, \tilde{g}) = \sum_{x \in \text{Coin}(f, g)} \text{ind}(f, g, U_x)[x_{\tilde{f}, \tilde{g}}],
\]

where ind denotes the coincidence index. Indeed we can define a local Reidemeister trace: for any open set \( U \), define

\[
RT(f, \tilde{f}, g, \tilde{g}, U) = \sum_{x \in \text{Coin}(f, g, U)} \text{ind}(f, g, U_x)[x_{\tilde{f}, \tilde{g}}],
\]
where \( \text{Coin}(f, g, U) = \text{Coin}(f, g) \cap U \). Clearly this local Reidemeister trace is equal to the nonlocal version if \( U \) is taken to be \( X \), and has the following additivity property: if \( V \) and \( W \) are disjoint subsets of \( U \) with \( \text{Coin}(f, g, U) \subset V \cup W \), then

\[
RT(f, \tilde{f}, g, \tilde{g}, U) = RT(f, \tilde{f}, g, \tilde{g}, V) + RT(f, \tilde{f}, g, \tilde{g}, W).
\]

The coincidence index above is well known in the setting of maps from one orientable manifold to another of the same dimension. In our setting, the space \( X \) is not a manifold, and so this index will in general be undefined on sets containing the point \( x_0 \) (or sets whose images under \( f \) and \( g \) contain \( x_0 \)). Many of our results (in particular Lemmas 4, 5, 6, and 7) are localized away from \( x_0 \), and as such will apply directly to the case where \( f, g : X \to Y \) with \( X \) and \( Y \) being two different bouquets of circles.

Our restriction to the case of selfmaps allows us to sidestep the complications at \( x_0 \) by reducing to the fixed point index, which is defined for any ANR (and in particular is well known for bouquets of circles). Obtaining a formula similar to our main result for non-selfmaps may require an extension of the coincidence index to certain non-manifold settings, which is in general a difficult problem. Gonçalves in [4] defines an index for maps from a complex into a manifold of the same dimension, but it is not integer-valued. No generalization of the index to maps from one complex to another is known.

Very little is known about the coincidence Reidemeister trace. The term “trace” is in reference to the construction in fixed point theory, in which the Reidemeister trace is given by the trace of a certain matrix (see [3]), but no trace-like formula is yet available in coincidence theory. The above definition suffices to define the Reidemeister trace only when the coincidence set is isolated. The main result of [5] shows that a local coincidence Reidemeister trace can be defined for general maps \( f : X \to Y \) (with perhaps nonisolated coincidence points) if \( X \) and \( Y \) are orientable manifolds of the same dimension, and that this local Reidemeister trace is uniquely characterized by five natural axioms, among them additivity and homotopy invariance.

Our setting (where \( X \) and \( Y \) are the same bouquet of circles) does not precisely fit the setting of [4] because \( X \) is not a manifold. We will apply certain of the results, however, to \( RT(f, \tilde{f}, g, \tilde{g}, U) \) in the case where the neither \( U \) nor \( f(U) \) nor \( g(U) \) contain the point \( x_0 \), as in that case \( U \) and its images can be regarded as manifolds in their own right. In particular, for such a set \( U_x \), we will make use of the fact that

\[
\text{ind}(f, g, U_x) = \text{sign} \det(dg_x - df_x),
\]

where as above \( U_x \) is some subset of \( U \) containing only one coincidence point \( x \), and \( dg_x \) and \( df_x \) denote the derivatives of \( f \) and \( g \) (assuming that they exist).

As for defining the Reidemeister trace for any pair of maps \( f, g : X \to X \) (perhaps having nonisolated coincidence points) with lifts \( \tilde{f}, \tilde{g} : \tilde{X} \to \tilde{X} \), we first change \( f \) and \( g \) by a homotopy to \( f', g' \) so that they have isolated coincidence points (that this is possible will be a consequence of our construction in Theorem
Now the homotopies of \( f, g \) to \( f', g' \) can be lifted to a homotopy of \( \tilde{f}, \tilde{g} \) to some lifts \( \tilde{f}', \tilde{g}' \). We then define

\[
RT(f, \tilde{f}, g, \tilde{g}) = RT(f', \tilde{f}', g', \tilde{g}').
\]

That this is well defined will be a consequence of the homotopy invariance of the coincidence index and the homotopy-relatedness of coincidence classes (see Lemma 14 of [5]).

We will now review the necessary properties of the Fox calculus (see e.g. [1]). If \( \{x_i\} \) are the generators of a free group \( G \), then the operators \( \frac{\partial}{\partial x_i} : G \to \mathbb{Z}G \) are defined by:

\[
\begin{align*}
\frac{\partial}{\partial x_i} 1 &= 0, \\
\frac{\partial}{\partial x_i} x_j &= \delta_{ij}, \\
\frac{\partial}{\partial x_i} (uv) &= \frac{\partial}{\partial x_i} u + u \frac{\partial}{\partial x_i} v,
\end{align*}
\]

where \( \delta_{ij} \) is the Kronecker delta, and \( u, v \in G \) are any words. Two important formulas can be obtained from the above:

\[
\begin{align*}
\frac{\partial}{\partial x_i} x_i^{-1} &= -x_i^{-1}, \\
\frac{\partial}{\partial x_i} (h_1 \ldots h_n) &= \sum_{k=1}^n h_1 \ldots h_{k-1} \frac{\partial}{\partial x_i} h_k.
\end{align*}
\]

3 A regular form for mappings

In this section we will describe a standard form for selfmaps of \( X \). Each circle of \( X \) is represented by some generator of the fundamental group. For each generator \( a \in G_0 \), let \( |a| \subset X \) be the circle represented by \( a \) (including the point \( x_0 \)).

For simplicity in our notation, we parameterize each circle by the interval \([0, 1]\) with endpoints identified. The circles of \( X \) will be parameterized so that the base point \( x_0 \) is identified with 0 (or equivalently, with 1). For any generator \( a \in G \) and any \( x \in [0, 1] \), let \([x]_a \) denote the point of \( |a| \subset X \) which has coordinate \( x \). Interval-like subsets of \( X \) will be denoted e.g. \((x_1, x_2)_a\) for the subset of points in \( |a| \) parameterized by the interval \((x_1, x_2)\).

Homotopy classes of mappings of \( X \) are characterized by their induced mappings on the fundamental groups. Consider the example where \( G = \langle a, b \rangle \) and the mapping \( f : X \to X \) induces \( \varphi : G \to G \) with

\[
\varphi(a) = ab^{-1}a^{-1}b^2.
\]

Geometrically speaking, the above formula for \( \varphi \) indicates that \( f \) is homotopic to a map which maps some interval \((0, x_1)_a \) bijectively onto \(|a| - x_0 \), maps...
some interval \((x_1, x_2)_a\) bijectively onto \(|b| - x_0\) (in the “reverse direction”), and so on.

We can represent the action of this map on \(|a|\) pictorially as in Figure 1 where in this case \(x_i = [i/5]_a\), and the label on each interval indicates that the interval is being mapped bijectively onto the corresponding circle. Note that sliding the points \(x_i\) around the circle will not change the homotopy class of \(f\), provided that \(x_0\) does not move, no \(x_i\) ever moves across another, and the ordering of the labels is preserved.

Thus any map \(f : X \to X\) is homotopic to a map which is characterized as follows: for each generator \(a \in G_0\), specify \(n_a\) intervals \(I^a_1, \ldots, I^a_{n_a}\) together with labels \(\{h^a_1, \ldots, h^a_{n_a}\}\), where each of \(h^a_i\) are letters of \(G\) (a letter of \(G\) is an element which is either a generator or the inverse of a generator of \(G\)). These intervals will be called intervals of \(f\), and the labels will be called the labels of \(f\).

Since we are concerned only with homotopy classes of maps, we may assume that \(f\) maps \((x_i, x_{i+1})_a\) affine linearly onto \((0, 1)_b\) for some generator \(b \in G\). A map specified by intervals and labels which is linear on each interval in this way will be called regular.

Specifying a map \(f\) by intervals and labels gives precise information which can be used to compute the derivatives of \(f\) at any point. For any interval \(I = (x_i, x_{i+1})_a\), define \(w(I)\), the width of \(I\), as the real number \(x_{i+1} - x_i\).

**Lemma 2.** Let \(f : X \to X\) be a regular map, let \(I\) be an interval of \(f\) labeled by \(b \in G\), and let \(x\) be any point of \(I\). Then we have:
• If $b$ is a generator of $G$, then
  \[ df_x = \frac{1}{w(I)} \]

• If $b$ is the inverse of a generator of $G$, then
  \[ df_x = -\frac{1}{w(I)} \]

where we have abused our notation slightly, writing the derivatives (typically represented as $1 \times 1$ matrices) as real numbers.

Proof. Let $I = (x_i, x_{i+1})_a$. In the case where $b$ is a generator of $G$, we can compute the restriction $f : I \to (0,1)_b$ as
  \[ f(x) = \frac{1}{w(I)}(x - x_i), \]
and the derivative is as desired. In the case where $b$ is the inverse of a generator, $f$ will reverse orientation, and the restriction will be
  \[ f(x) = \frac{1}{w(I)}(x_{i+1} - x) \]
again giving the desired derivative. 

4 Regular pairs of mappings

Let $G_0$ be the set of generators of $G$, and let $\Phi = \{u_a \mid a \in G_0\}$, $\Psi = \{v_a \mid a \in G_0\}$ be ordered sets of unreduced words in $G$ of length at least 2. We will construct from these sets two maps $f$ and $g$ in a standardized way.

We construct $g$ from $\Psi$ as follows, each $v_a$ determining $g$’s behavior on the circle $|a|$: Let $m_a$ be the length of $v_a$. Let $g$’s first interval on $|a|$ be $(0,1/2)_a$, and let the remaining $m_a - 1$ intervals be equally spaced over $(1/2,1)_a$. These $m_a$ intervals are to be labeled respectively by the $m_a$ letters of $v_a$.

We will construct $f$ from $\Phi$ similarly, this time using a single wide interval on $(1/2,1)_a$ and several evenly spaced intervals on $(0,1/2)_a$. Our construction of $f$ is slightly complicated by the fact that we will also require $f$ to be constant in small neighborhoods of $x_0$ and $[1/2]_a$. On each circle $|a|$, define $f$ as follows: let $n_a$ be the length of $u_a$. Fixing some small $\epsilon > 0$, our first interval in $|a|$ will be $(0,\epsilon)_a$, followed by $n_a - 1$ evenly spaced intervals on over $(\epsilon,1/2 - \epsilon)_a$, followed by the intervals $(1/2 - \epsilon,1/2 + \epsilon)_a$, $(1/2 + \epsilon,1 - \epsilon)_a$, and $(1 - \epsilon,1)_a$. These intervals will be labeled, respectively, by 1, the letters of $u_a$ except the last, 1, the last letter of $u_a$, and 1. (Labeling an interval by 1 indicates that $f$ is constant on that interval.)

A diagram of typical mappings constructed above is given in Figure 2. We say that the $f$ and $g$ constructed above are the regular pair given by $\Phi$ and $\Psi$. 

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Figure 2: Typical action of a regular pair on a single circle $|a|$, where $u_a = h_1 \ldots h_n$ and $v_a = l_1 \ldots l_m$. Intervals and labels for $f$ are marked on the outside, and intervals and labels for $g$ are marked on the inside.

Typically we think of our sets $\Phi$ and $\Psi$ as coming from a pair of homomorphisms $\varphi, \psi : G \to G$, letting $\Phi = \{ \varphi(a) \mid a \in G_0 \}$ and $\Psi = \{ \psi(a) \mid a \in G_0 \}$. In such a case we will say that $f$ and $g$ are the regular pair given by $\varphi$ and $\psi$. By our construction it is clear that any pair of maps are homotopic to the regular pair given by their induced homomorphisms on the fundamental group.

For example, if $\varphi, \psi : G \to G$ are homomorphisms with $G = \langle a, b, c \rangle$ such that

$$\varphi(a) = aba^{-1}bc, \quad \psi(a) = c^2ab^{-1}a,$$

then the regular pair $f, g$ given by $\varphi$ and $\psi$ is pictured in Figure 3.

The slight generality obtained by allowing $\Phi$ and $\Psi$ to include nonreduced words is an important one which we use below.

5 The Reidemeister trace of a regular pair

Let $f, g : X \to X$ be a regular pair of maps, and choose lifts $\tilde{f}$ and $\tilde{g}$ of $f$ and $g$ respectively so that $\tilde{f}(\tilde{x}_0) = \tilde{g}(\tilde{x}_0)$. In this section we will describe a method for calculating $RT(f, \tilde{f}, g, \tilde{g})$.

Since $f$ and $g$ are linear on intervals which never have the same endpoints (except at $x_0$), we know that their coincidence set consists of isolated points. We consider first the points $x_0$ and $[1/2]a$, which are coincidence points by construction.
Lemma 3. Let \( f, g : X \to X \) be the regular pair given by \( \Phi = \{u_a\} \) and \( \Psi = \{v_a\} \). Let \( I_0 \) be the union of the point \( x_0 \) and all intervals of \( f \) having \( x_0 \) as an endpoint. If \( v_a = aw_a \) for a word \( w_a \in G \), then

\[
\text{ind}(f, g, I_0) = 1.
\]

Proof. Since \( v_a \) begins and ends with \( a \) and \( g(x_0) = x_0 \), the map \( g \) will map some initial segment of \( |a| \) to itself, and also will map some terminal segment of \( |a| \) to itself. Thus there is some \( U \subset I_0 \) on which \( g \) is homotopic to the identity map.

Thus we have

\[
\text{ind}(f, g, I_0) = \text{ind}(f, \text{id}, U),
\]

and this is simply the fixed point index of \( f \) at \( x_0 \). The fixed point index at the “wedge point” of a bouquet of circles is well studied (see e.g. \cite{7}), and is equal to 1. \( \square \)

Lemma 4. Let \( f, g : X \to X \) be the regular pair given by \( \Phi = \{u_a\} \) and \( \Psi = \{v_a\} \). Let \( I_a \) be the interval of \( f \) containing \([1/2]\]a. If \( v_a = b_ab_a^{-1}w_a \) for some generator \( b_a \in G \) and a word \( w_a \in G \), then

\[
\text{ind}(f, g, I_a) = 0
\]

for any generator \( a \in G \).
Proof. Since the intervals of \(g\) on either side of \([1/2]_a\) are labeled by inverse elements, our map \(g\) can be changed by a homotopy in a neighborhood of \([1/2]_a\) to be constant, with constant value some \(x \in |a| - x_0\). But \(f\) will have constant value \(x_0\) on \(I_a\), and so will be coincidence free with the resulting map homotopic to \(g\). Thus the index on \(I_a\) is zero as desired.

Now we turn to the coincidence points other than \(x_0\) and \([1/2]_a\).

**Lemma 5.** If \(f, g\) is the regular pair given by \(\Phi = \{u_a\}\) and \(\Psi = \{v_a\}\) and \(x \in |a|\) is a coincidence point other than \(x_0\) or \([1/2]_a\) lying in an interval labeled \(h_i\) by \(f\) and \(l_j\) by \(g\), then

\[
[x_{f, g}] = \begin{cases} 
\rho(h_1 \ldots h_{i-1}(l_1 \ldots l_{j-1})^{-1}) & \text{if } h_i = l_j, \\
\rho(h_1 \ldots h_{i-1}(l_1 \ldots l_j)^{-1}) & \text{if } h_i = l_j^{-1},
\end{cases}
\]

where

\[u_a = h_1 \ldots h_{n_a}, \quad v_a = l_1 \ldots l_{n_a}.\]

Proof. Suppose that \(x\) lies in the intervals \(I = (x_i, x_{i+1})_a\) and \(J = (z_j, z_{j+1})_a\) of \(f\) and \(g\) respectively. Since \(x\) is a coincidence point, we know that \(f(I) = g(J)\), which means that either \(h_i = l_j\) or \(h_i = l_j^{-1}\), and thus our cases in the statement of the Lemma are exhaustive.

Define a path by a positively oriented arc from \(x_0\) to \(x_i\), and lift this path to \(\tilde{X}\), starting at the initial point \(\tilde{x}_0\). Define \(\tilde{x}_i\) as the terminal point of this path, and we have

\[
\tilde{f}(\tilde{x}_i) = \alpha \tilde{y}_0,
\]

where \(\alpha\) is the covering transformation corresponding to the word \(h_1 \ldots h_{i-1} \in G\), and \(\tilde{y}_0 = \tilde{f}(\tilde{x}_0) = \tilde{g}(\tilde{x}_0)\). Similarly defining \(\tilde{z}_j\), we have

\[
\tilde{g}(\tilde{z}_j) = \beta \tilde{y}_0,
\]

where \(\beta\) is the covering transformation corresponding to the word \(l_1 \ldots l_{j-1} \in G\). Thus we have

\[
\tilde{g}(\tilde{z}_j) = \beta \alpha^{-1} \tilde{f}(\tilde{x}_i).
\]

In the case where \(h_i = l_j\), we have \(f(x_i) = g(z_j)\) and \(f(x_{i+1}) = g(z_{j+1})\). Thus any group element \(\sigma \in G\) with \(x \in p_X(\text{Coin}(\sigma^{-1} f, g))\) must have \(\sigma^{-1} \tilde{f}(\tilde{x}_i) = \tilde{g}(\tilde{z}_j)\). Then by the above formula we have \(\sigma^{-1} = \beta \alpha^{-1}\), and thus that \([x_{f, g}] = \rho(h_1 \ldots h_{i-1}(l_1 \ldots l_{j-1})^{-1})\) as desired.

Now we turn to the case where \(h_i = l_{j}^{-1}\), in which \(f(x_i) = g(z_{j+1})\) and \(f(x_{i+1}) = g(z_j)\). Similar to \(\tilde{z}_j\) above, we can define \(\tilde{z}_{j+1}\) as the terminal point of a lifted path from \(x_0\) to \(z_{j+1}\), and we have

\[
\tilde{g}(\tilde{z}_{j+1}) = \gamma \alpha^{-1} \tilde{f}(\tilde{x}_i),
\]

where \(\gamma\) is the covering transformation corresponding to \(l_1 \ldots l_j \in G\). Thus any group element \(\sigma\) with \(x \in p_X(\text{Coin}(\sigma^{-1} f, g))\) must have \(\sigma^{-1} \tilde{f}(\tilde{x}_i) = \tilde{g}(\tilde{z}_{j+1})\), and we have \(\sigma^{-1} = \gamma \alpha^{-1}\), and thus that \([x_{f, g}] = \rho(h_1 \ldots h_{i-1}(l_1 \ldots l_j)^{-1})\) as desired. \(\square\)
Lemma 6. Let \( f \) and \( g \) be the regular pair given by \( \Phi = \{ u_a \} \) and \( \Psi = \{ v_a \} \), with \( v_a \) having a generator as its first letter, and let \( I \) be an interval of \( f \) with \( I \subset (0, 1/2) \) for some circle \( |a| \subset X \). Then

\[
RT(f, \tilde{f}, g, \tilde{g}, I) = -\rho(h_1 \ldots h_{i-1} \frac{\partial}{\partial l_1} h_i),
\]

where \( u_a = h_1 \ldots h_n \), the interval \( I \) is labeled by \( h_i \), and \( l_1 \) is the first letter of \( v_a \).

Proof. Since \( I \) contains at most one coincidence point (both \( f \) and \( g \) are affine linear on \( I \)), we know that \( RT(f, \tilde{f}, g, \tilde{g}, I) \) is zero if there is no coincidence point in \( I \), and otherwise

\[
RT(f, \tilde{f}, g, \tilde{g}, I) = \text{ind}(f, g, I)[x_{f, \tilde{g}}],
\]

where \( x \) is the unique coincidence point in \( I \).

We prove the lemma in three cases, treating the various possible relationships between \( h_i \) and \( l_1 \). Either:

- \( h_i \) is not \( l_1 \) or its inverse, or they are equal, or they are inverses.

In the first case, \( h_i \neq l_1 \), and so the maps \( f \) and \( g \) have no coincidence point in \( I \). Thus

\[
RT(f, \tilde{f}, g, \tilde{g}, I) = 0 = -\rho(h_1 \ldots h_{i-1} \frac{\partial}{\partial l_1} h_i).
\]

In the second case, \( h_i = l_1 \), and so \( \frac{\partial}{\partial l_1} h_i = 1 \). Thus we must show that

\[
RT(f, \tilde{f}, g, \tilde{g}, I) = -\rho(h_1 \ldots h_{i-1}).
\]

Since \( h_i = l_1 \), the maps \( f \) and \( g \) have one coincidence point \( x \) in \( I \). By our calculation in Lemma 5 we know that \( df_x > 2 \), since the width of the interval of \( f \) containing \( x \) is less than 1/2. Since \( g \)'s interval containing \( x \) is of width exactly 1/2, we know that \( dg_x = 2 \), and thus

\[
\text{ind}(f, g, I) = \text{sign}(\det(dg_x - df_x)) = -1.
\]

By Lemma 5 we have that \( [x_{f, \tilde{g}}] = \rho(h_1 \ldots h_{i-1}) \). Thus we have

\[
RT(f, \tilde{f}, g, \tilde{g}, I) = \text{ind}(f, g, I)[x_{f, \tilde{g}}] = -\rho(h_1 \ldots h_{i-1})
\]

as desired.

In the third case we assume \( h_i = l_1 \). In this case, we have \( \frac{\partial}{\partial l_1} h_i = -h_i \), and so we must show that

\[
RT(f, \tilde{f}, g, \tilde{g}, I) = \rho(h_1 \ldots h_i).
\]

As above, we have one coincidence point \( x \), but this time the derivatives are: \( df_x < -2 \) and \( dg_x = 2 \) as before. Thus \( \text{ind}(f, g, I) = 1 \). By Lemma 5 we have \( [x_{f, \tilde{g}}] = \rho(h_1 \ldots h_i) \), and thus that

\[
RT(f, \tilde{f}, g, \tilde{g}, I) = \text{ind}(f, g, x)[x_{f, \tilde{g}}] = \rho(h_1 \ldots h_i)
\]

as desired. □
Lemma 7. With notation as in Lemma 4 let $I$ be some interval of $g$ with $J \subset (1/2, 1)_a$ for some circle $|a| < X$, and assume that $h_n$ is a generator of $G$. Then

$$RT(f, \tilde{f}, g, \tilde{g}, J) = \rho \left( h_1 \ldots h_{n-1}(l_1 \ldots l_{i-1}) \frac{\partial}{\partial h_n}(l_i)^{-1} \right),$$

where $v_a = l_1 \ldots l_m$, and the interval $J$ is labeled by $l_i$.

Proof. The proof is very similar to that of Lemma 6. Again we split the argument into three cases: either $l_i$ is not $h_n$ or its inverse, or $l_i = h_n$, or $l_i = h_n^{-1}$.

In the first case, $l_i \neq h_n^{\pm 1}$ and so $f$ and $g$ have no coincidences in $J$. Thus we have

$$RT(f, \tilde{f}, g, \tilde{g}, J) = 0 = \rho \left( h_1 \ldots h_{n-1}(l_1 \ldots l_{i-1}) \frac{\partial}{\partial h_n}(l_i)^{-1} \right),$$

as desired.

In the second case, $l_i = h_n$, and so $\frac{\partial}{\partial h_n}l_i = 1$, and we must show

$$RT(f, \tilde{f}, g, \tilde{g}, J) = \rho(h_1 \ldots h_{n-1}(l_1 \ldots l_{i-1})^{-1}).$$

In this case $f$ and $g$ have a single coincidence point $x \in J$. As in our arguments for Lemma 6 we can compute that $df_x = 2 + \epsilon$ for some arbitrarily small $\epsilon$, and $dg_x > 2$, and thus that $\text{ind}(f, g, J) = 1$. By Lemma 5 we have $[x_{\tilde{f}, \tilde{g}}] = \rho(h_1 \ldots h_{n-1}(l_1 \ldots l_{i-1})^{-1})$. Thus

$$RT(f, \tilde{f}, g, \tilde{g}, J) = \text{ind}(f, g, J)[x_{\tilde{f}, \tilde{g}}] = \rho(h_1 \ldots h_{n-1}(l_1 \ldots l_{i-1})^{-1})$$

as desired.

For the third case we have $l_i = h_n^{-1}$, and so $\frac{\partial}{\partial h_n}l_i = -l_i$, and we must show

$$RT(f, \tilde{f}, g, \tilde{g}, J) = -\rho(h_1 \ldots h_{n-1}(l_1 \ldots l_i)^{-1}).$$

As usual we compute derivatives and find that $df_x = 2 + \epsilon$ and $dg_x < -2$ and thus that $\text{ind}(f, g, J) = -1$. Lemma 5 gives $[x_{\tilde{f}, \tilde{g}}] = \rho(h_1 \ldots h_{n-1}(l_1 \ldots l_i)^{-1})$ as desired. \qed

Let $i : ZG \to ZG$ be the involution defined by

$$i \left( \sum_k c_k g_k \right) = \sum_k c_k g_k^{-1}.$$

Theorem 8. Let $f, g : X \to X$ be maps which induce the homomorphisms $\varphi, \psi : G \to G$. Then there are lifts $\tilde{f}$ and $\tilde{g}$ such that for any particular generator $b \in G$, we have

$$RT(f, \tilde{f}, g, \tilde{g}) = \rho \left( 1 - \sum_{a \in G_0} \frac{\partial}{\partial \alpha} \varphi(a) + \varphi(a)\psi(a)^{-1} - \varphi(a)^{-1}i(\varphi(a) \frac{\partial}{\partial \alpha} \psi(a)) \right).$$

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Proof. For each circle $|a| \subset X$, define unreduced words $u_a = \varphi(a) a^{-1} a$ and $v_a = a a^{-1} \psi(a) a^{-1} a$. Now let $f', g' : X \to X$ be the regular pair given by $\Phi = \{ u_a \}$ and $\Psi = \{ v_a \}$ which is homotopic to the original pair $f, g$. By construction this regular pair satisfies the hypotheses of Lemmas 3, 4, 6, and 7. Let $\tilde{f}$ and $\tilde{g}$ be the lifts of $f$ and $g$ which fix $\tilde{x}_0$. Now lifting the homotopies of $f$ and $g$ to $f'$ and $g'$ gives lifts $\tilde{f}'$ and $\tilde{g}'$ with $
abla(i, j, k) = \nabla(i_0, j_0, k_0).$

Write $\varphi(a) = h_1^a \ldots h_n^a$ and $\psi(a) = l_1^a \ldots l_m^a$. A diagram of the action of $f'$ and $g'$ on a typical circle $|a|$ is given in Figure 4.

In order to compute the Reidemeister trace, we partition $X$ into several intervals: let $I_{a_1}^a, \ldots, I_{a_n}^a$ be the intervals of $f'$ in $(0, 1/2)_a$ labeled by $h_1^a, \ldots, h_n^a$, and let $J_{a_1}^a, \ldots, J_{a_m}^a$ be the intervals of $g'$ in $(1/2, 1)_a$ labeled by $l_1^a, \ldots, l_m^a$. Let $I_0$ and $I_a$ be as in Lemmas 3 and 4.

We define four more intervals to cover the remaining portions of $X$: let $K_{a_1}^a$ be the interval of $f$ in $(0, 1/2)_a$ which follows $I_{a_n}^a$ and is labeled by $a^{-1}$. Let $V^a$ be the first interval of $g$ in $(1/2, 1)_a$ (this interval is labeled by $a^{-1}$) and we set $K_{a_2}^a$ to be the interior of $V^a - I_a$. Let $K_{a_3}^a$ be the interval of $g$ following $J_{a_m}^a$, this interval is labeled $a^{-1}$ by $g$. Finally let $U^a$ be the last interval of $g$, and let $K_{a_4}^a$ be the interior of $U^a - I_0$. Since our regular maps are constructed to have no coincidences at interval endpoints except for $x_0$ and the $[1/2]_a$, we have

$$\text{Coin}(f', g') \subset I_0 \cup \bigcup_{a \in G_0} \left( I_a \cup \bigcup_{i=1}^{a} I_{a_1}^a \cup \bigcup_{j=1}^{a} J_{a_1}^a \cup \bigcup_{k=1}^{a} K_{a_1}^a \right),$$

and so we can compute $RT(f', \tilde{f}', g', \tilde{g}')$ as a sum of the Reidmeister traces over
the various intervals in the above union.

By Lemma 4, the index (and thus the Reidemeister trace) will be zero on each $I_α$. Thus our summation simplifies to

$$RT(f', \tilde{f}', g', \tilde{g}') = RT(I_0) + \sum_{α ∈ G_0} \left( \sum_{i=1}^{n_α} RT(I^α_i) + \sum_{j=1}^{m_α} RT(J^α_j) + \sum_{k=1}^{4} RT(K^α_k) \right),$$

where for brevity we write $RT(\cdot)$ for $RT(f', \tilde{f}', g', \tilde{g}', \cdot)$.

The set $I_0$ contains a single coincidence point $x_0$. We know that $x_0 = p_X(\tilde{x}_0)$, with $\tilde{x}_0 ∈ Coin(f', \tilde{g}')$. Thus $[x_0, \tilde{g}'] = \rho(1)$, and by Lemma 3 the index on $I_0$ is 1. Thus we have

$$RT(I_0) = \text{ind}(f', g', I_0)(x_0, \tilde{g}') = \rho(1).$$

Now by Lemma 6, we can compute the Reidemeister traces on the $I^α_i$:  

$$RT(I^α_i) = -\rho \left( h^α_1 \ldots h^α_{i-1} \frac{∂}{∂α} h^α_i \right),$$

and so

$$\sum_{i=1}^{n_α} RT(I^α_i) = -\rho \left( \frac{∂}{∂α} (h^α_1 \ldots h^α_{n_α}) \right) = -\rho \left( \frac{∂}{∂α} ϕ(α) \right).$$

We can also apply Lemma 6 to compute $RT(K^α_k)$, which is labeled by $f$ with $a^{-1}$ and by $g$ with $a$. We obtain

$$RT(K^α_k) = -\rho(ϕ(a) \frac{∂}{∂α} a^{-1}) = ρ(ϕ(a)a^{-1}).$$

Similarly, we can use Lemma 7 to compute the Reidemeister traces on the remaining intervals. For the $J^α_j$, we have

$$RT(J^α_j) = \rho \left( ϕ(a)a^{-1}(aa^{-1}l^α_1 \ldots l^α_{j-1} \frac{∂}{∂α} l^α_j)^{-1} \right),$$

and thus

$$\sum_{j=1}^{m_α} RT(J^α_j) = ρ(ϕ(a)a^{-1}i_i \frac{∂}{∂α} ψ(α)).$$

Both $f$ and $g$ map $K^α_2$ to some initial segment of $|a|$, and so there will be one coincidence point in $K^α_2$, and Lemma 4 gives

$$RT(K^α_2) = ρ(ϕ(a)a^{-1}(a \frac{∂}{∂α} a^{-1})^{-1}) = -ρ(ϕ(a)a^{-1}).$$

On $K^α_3$, we have $g(K^α_3) = |a|$ and $f(K^α_3) ⊂ |a|$, so there will be one coincidence point, and again Lemma 4 gives

$$RT(K^α_3) = ρ(ϕ(a)a^{-1}(aa^{-1}ψ(a) \frac{∂}{∂α} a^{-1})^{-1}) = -ρ(ϕ(a)a^{-1}(ψ(a)a^{-1})^{-1})$$

$$= -ρ(ϕ(a)ψ(a)^{-1}).$$

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On $K_4^*$, we can change $g$ by a homotopy to make it coincidence free with $f$, and thus

$$RT(K_4^*) = 0.$$ 

Now summing (2), (3) and (4) with our formulas for the $K_a^k$, our sum (1) gives

$$RT(f', f', g', g') = \rho \left( 1 + \sum_{a \in G_0} -\frac{\partial}{\partial a} \varphi(a) + \varphi(a)a^{-1}i\left(\frac{\partial}{\partial a} \psi(a)\right) - \varphi(a)\psi(a)^{-1} \right)$$

$$= \rho \left( 1 - \sum_{a \in G_0} \frac{\partial}{\partial a} \varphi(a) + \varphi(a)\psi(a)^{-1} - \varphi(a)a^{-1}i\left(\frac{\partial}{\partial a} \psi(a)\right) \right).$$

We end this section with an interpretation of the above formula with respect to a variation on the Fox calculus. For generators $\{x_i\}$ of $G$, define the operators $\Delta = \Delta x_i : G \to ZG$ as follows:

$$\Delta x_i 1 = 0$$

$$\Delta x_i x_j = \delta_{ij}$$

$$\Delta x_i (uv) = \left( \frac{\Delta}{\Delta x_i} u \right) v + \frac{\Delta}{\Delta x_i} v$$

Analogous to the properties given for the Fox calculus we can derive:

**Lemma 9.** If $\Delta$ denotes the operator above and $h_i$ are letters of $G$, we have:

$$\frac{\Delta}{\Delta x_i} x_i^{-1} = -x_i,$$

$$\frac{\Delta}{\Delta x_i} (h_n \ldots h_1) = \sum_{k=1}^{n} \left( \frac{\Delta}{\Delta x_i} h_k \right) h_{k-1} \ldots h_1$$

**Proof.** The first statement follows by the computation:

$$0 = \frac{\Delta}{\Delta x_i} (x_i x_i^{-1}) = \left( \frac{\Delta}{\Delta x_i} x_i \right) x_i^{-1} + \frac{\Delta}{\Delta x_i} x_i^{-1} = x_i^{-1} + \frac{\Delta}{\Delta x_i} x_i^{-1}.$$

The second statement is proved by induction on $n$. For $n = 1$ the statement is clear. For the inductive case, we have

$$\frac{\Delta}{\Delta x_i} (h_n \ldots h_1) = \frac{\Delta}{\Delta x_i} (h_n \ldots h_2 h_1) = \left( \frac{\Delta}{\Delta x_i} (h_n \ldots h_2) \right) h_1 + \frac{\Delta}{\Delta x_i} h_1$$

$$= \sum_{k=2}^{n} \left( \frac{\Delta}{\Delta x_i} h_k \right) h_{k-1} \ldots h_2 h_1 + \frac{\Delta}{\Delta x_i} h_1$$

$$= \sum_{k=1}^{n} \left( \frac{\Delta}{\Delta x_i} h_k \right) h_{k-1} \ldots h_1$$

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We will use one further property, relating our new operator to the ordinary Fox calculus operator:

**Lemma 10.** For any word \( w \in G \), we have

\[
\frac{\Delta}{\Delta x_j} w = x_j^{-1} i \left( \frac{\partial}{\partial x_j} w \right) w. \tag{5}
\]

**Proof.** Our proof is by induction on the length of \( w \). If \( w \) is the trivial element, then we have \( \frac{\Delta}{\Delta x_j} w = 0 \) by definition. Note also that

\[
x_j^{-1} i \left( \frac{\partial}{\partial x_j} 1 \right) 1 = 0,
\]

as desired.

For the inductive case, write \( w = uv \). Then the left hand side of (5) is

\[
\frac{\Delta}{\Delta x_j} (uv) = \left( \frac{\Delta}{\Delta x_j} u \right) v + \frac{\Delta}{\Delta x_j} v = x_j^{-1} i \left( \frac{\partial}{\partial x_j} u \right) uv + x_j^{-1} i \left( \frac{\partial}{\partial x_j} v \right) v.
\]

while the right hand side of (5) becomes

\[
x_j^{-1} i \left( \frac{\partial}{\partial x_j} (uv) \right) uv = x_j^{-1} i \left( \frac{\partial}{\partial x_j} u + u \frac{\partial}{\partial x_j} v \right) uv
\]

\[
= x_j^{-1} i \left( \frac{\partial}{\partial x_j} u \right) uv + x_j^{-1} i \left( \frac{\partial}{\partial x_j} v \right) u^{-1} uv
\]

as desired. \( \square \)

Theorem 8 takes a nice form when expressed in terms of the above operators:

**Corollary 11.** With notation as in Theorem 8, we have

\[
RT(\tilde{f}, \tilde{f}, g, \tilde{g}) = \rho \left( 1 - \sum_{a \in G_0} \frac{\partial}{\partial a} \varphi(a) - \frac{\Delta}{\Delta a} \psi(a) + \varphi(a) \psi(a)^{-1} \right).
\]

**Proof.** It suffices to show that

\[
\rho \left( \varphi(a) a^{-1} i (\frac{\partial}{\partial a} \psi(a)) \right) = \rho \left( \frac{\Delta}{\Delta a} \psi(a) \right).
\]

Noting that \( \varphi(x)y = \varphi(x)(y \psi(x)) \psi(x)^{-1} \) for any \( x, y \in G \) gives the well known identity \( \rho(\varphi(x)y) = \rho(y \psi(x)) \). This identity, together with Lemma 10 gives

\[
\rho \left( \varphi(a) a^{-1} i (\frac{\partial}{\partial a} \psi(a)) \right) = \rho \left( a^{-1} i (\frac{\partial}{\partial a} \psi(a)) \psi(a) \right) = \rho \left( \frac{\Delta}{\Delta a} \psi(a) \right).
\]

\( \square \)
6 Some examples

First we will show how Theorem 1 is a simple consequence of our main result. Letting $\psi$ be the identity map gives $\rho(\varphi(a)a^{-1}) = \rho(a^{-1}\psi(a)) = \rho(1)$, and Theorem 8 gives

$$RT(f, \tilde{f}) = \rho \left( 1 - \sum_{a \in G_0} \frac{\partial}{\partial a} \varphi(a) - \frac{\Delta}{\Delta a} a + \varphi(a)a^{-1} \right)$$

$$= \rho \left( 1 - \sum_{a \in G_0} \frac{\partial}{\partial a} \varphi(a) - 1 + 1 \right)$$

$$= \rho \left( 1 - \sum_{a \in G_0} \frac{\partial}{\partial a} \varphi(a) \right),$$

as desired.

Our formula also gives the classical formula for the coincidence Nielsen number on the circle:

**Example 12.** Let $G = \langle a \rangle$, and let $f$ and $g$ be maps which induce the homomorphisms

$$\varphi(a) = a^n, \quad \psi(a) = a^m.$$

Without loss of generality, we will assume that $n \geq m$. Our formula then gives

$$RT(f, \tilde{f}, g, \tilde{g}) = \rho(1 - \frac{\partial}{\partial a} a^n + \frac{\Delta}{\Delta a} a^m - a^n a^{-m})$$

$$= \rho(1 - (1 + \cdots + a^{n-1}) + (1 + \cdots + a^{m-1}) - a^{n-m})$$

$$= \rho(1 - a^m - \cdots - a^{n-1} - a^{n-m})$$

A simple calculation shows that $\rho(a^i) = \rho(a^j)$ if and only if $i \equiv j \mod n - m$. Thus $\rho(1) = \rho(a^{n-m})$, and all other terms in the above sum are in distinct Reidemeister classes. Thus the Nielsen number is $n - m$, as desired.

We conclude with one nontrivial computation of a coincidence Reidemeister trace of two selfmaps of the bouquet of three circles.

**Example 13.** Let $X$ be a space with fundamental group $G = \langle a, b, c \rangle$, and let $f$ and $g$ be maps which induce homomorphisms as follows:

- $a \mapsto acb^{-1}, \quad a \mapsto a^{-1}cb^{-1}$
- $\varphi: b \mapsto ab, \quad \psi: b \mapsto c$
- $c \mapsto b, \quad c \mapsto b^{-1}a$

Our formula gives

$$RT(f, \tilde{f}, g, \tilde{g}) = \rho(1 - (1 + a^{-1}cb^{-1} + a^2) - (a + abc^{-1}) - (ba^{-1}b))$$

$$= \rho(-a - a^2 - abc^{-1} - ba^{-1}b - a^{-1}cb^{-1}).$$
and we must decide the twisted conjugacy of the above elements. We use the technique from [6] of abelian and nilpotent quotients.

Checking in the abelianization suffices to show that \( a \) is not twisted conjugate to any of the other terms. We also see that \( a^2 \) and \( ba^{-1}b \) are twisted conjugate in the abelianization, and our computation reveals that \( \rho(a^2) = \rho(ba^{-1}b) \) with conjugating element \( \gamma = ac^{-1} \). Similarly, we find that \( \rho(a^2) = \rho(abc^{-1}) \) by the element \( \gamma = ab^{-1} \).

It remains to decide whether or not \( a^2 \) and \( a^{-1}cb^{-1} \) are twisted conjugate, and a check in the class 2 nilpotent quotient shows that they are not. Thus
\[
RT(f, \tilde{f}, g, \tilde{g}) = -\rho(a) - 3\rho(a^2) - \rho(a^{-1}cb^{-1})
\]
is a fully reduced expression for the Reidemeister trace, and so in particular the Neilsen number is 3.

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