Multivalued Monotone Stochastic Differential Equations with Jumps

Lucian Maticiuc\textsuperscript{a,1}, Aurel Răşcanu\textsuperscript{a,1} and Leszek Słomiński\textsuperscript{b,*}

\textsuperscript{a} Faculty of Mathematics, “Alexandru Ioan Cuza” University, Carol 1 Blvd., no. 11, Iaşi, Romania,
\textsuperscript{b} Faculty of Mathematics and Computer Science, Nicolaus Copernicus University,
ul. Chopina 12/18, 87-100 Toruń, Poland

Abstract

We study existence and uniqueness of a solution for stochastic differential equations driven by a maximal monotone operator and by a general semimartingale with jumps. Efficient methods of approximations of the solutions based on discretization of processes and the Yosida approximations of the monotone operator are considered in detail. Existence of a weak solution is also envisaged.

AMS Classification subjects: Primary: 60H20; Secondary: 60G17.

Keywords or phrases: Stochastic Differential Equations with Jumps; Maximal Monotone Operators; Yosida Approximations.

1 Introduction

Let $A : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ be a maximal monotone multivalued operator on $\mathbb{R}^d$ with the domain $D(A) = \{ z \in \mathbb{R}^d : A(z) \neq \emptyset \}$ and its graph

$$\text{Gr}(A) = \{ (z, y) \in \mathbb{R}^{2d} : z \in \mathbb{R}^d, y \in A(z) \}.$$ 

Let $\Pi : \mathbb{R}^d \to \overline{D(A)}$ be a \textit{generalized projection} on $\overline{D(A)}$ (in the sense that $\Pi(x) = x$ for all $x \in \overline{D(A)}$ and $\Pi$ is a non–expansive map).

We consider the following $d$–dimensional multivalued stochastic differential equation (SDE) driven by the operator $A$ and associated to the projection $\Pi$:

$$X_t + K_t = H_t + \int_0^t \langle f(X_{s-}), dZ_s \rangle, \quad t \in \mathbb{R}^+, \quad (1)$$

where $Z$ is a $d$–dimensional semimartingale with $Z_0 = 0$, $H$ is a càdlàg adapted process with $H_0 \in \overline{D(A)} = D(A) \cup \text{Bd}(D(A))$ and $f : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a Lipschitz continuous function.

---

\textsuperscript{*}Corresponding author. Research supported by Polish NCN grant no. 2012/07/B/ST1/03508
\textsuperscript{1}The work of these authors was supported by the project ERC-like, code 1ERC/02.07.2012.

\textit{E-mail addresses:} lucian.maticiuc@ymail.com (Lucian Maticiuc), aurel.rascanu@uaic.ro (Aurel Răşcanu), leszek@mat.uni.torun.pl (Leszek Słomiński)
By a solution of (1) we understand a pair \((X, K)\) of càdlàg adapted processes such that \(X_t \in \overline{D(A)}\) for any \(t \in \mathbb{R}^+\), \(K\) is a locally bounded variation process with \(K_0 = 0\) such that for any \((\alpha, \beta) \in \text{Gr}(A)\),
\[
\int_0^t \langle X_u - \alpha, dK^c_u - \beta du \rangle \geq 0, \quad 0 \leq s < t, \quad s, t \in \mathbb{R}^+,
\]
where \(K^c_t := K_t - \sum_{s \leq t} \Delta K_s, t \in \mathbb{R}^+\) and if \(|\Delta K_t| > 0\) then
\[
X_t = \Pi(X_t - \Delta H_t + \langle f(X_t), \Delta Z_t \rangle), t \in \mathbb{R}^+
\]
(for the precise definition see Section 4).

Particular cases of the above type of SDEs were considered earlier in many papers. For instance, the existence and uniqueness of solutions of (1) in the case of Itô diffusions was proved independently in E. Cépa [7] and A. Răşcanu [23] (in the infinite dimensional framework). SDEs with subdifferential operator (i.e., the maximal monotone operator is \(A = \partial \varphi\), where \(\varphi\) is a proper convex and lower semicontinuous function; see Remark 1–a from the next section) were introduced by A. Răşcanu in [24]. More recently, L. Maticiuc et al. [5] obtained an extension to the non–convex setup by proving the existence and uniqueness results both for the Skorokhod problem and for the associated SDE driven by the Fréchet subdifferential \(\partial^{-}\varphi\) of a semiconvex function \(\varphi\).

In the case of Itô diffusions, conditions ensuring existence, uniqueness and convergence of approximation schemes for such equations were given in I. Asiminoaei, A. Răşcanu [1], V. Barbu, A. Răşcanu [2], A. Bensoussan, A. Răşcanu [3] and R. Pettersson [20]. SDEs with subdifferential operator driven by general continuous semimartingale were considered in A. Storm [33]. It is worth pointing out that the authors of all the mentioned above papers restricted themselves to processes with continuous trajectories.

It is well known that for every nonempty closed convex set \(D \subset \mathbb{R}^d\) its indicator function \(\varphi = I_D\) is a convex and proper lower semicontinuous function (see Remark 1–b)). This implies that equations (1) are strongly connected with SDEs with reflecting boundary condition in convex domains. Such type of equations were introduced by A.V. Skorokhod [26, 27] in one-dimensional case and \(D = \mathbb{R}^+\). The case of reflecting Itô diffusions in convex domains \(D\) was studied in detail by T. Tanaka [34] and for generally, not necessary convex, domains by P.–L. Lions, A.S. Sznitman [11] and Y. Saisho [25]. L. Slomiński [28, 30] and W. Laukajtys [12] considered SDEs with reflecting boundary conditions in convex domain driven by a semimartingale. Approximations of solutions of SDEs with reflecting boundary condition were studied in D. Lépingle [15], J.L. Menaldi [17], R. Pettersson [19], L. Slomiński [29, 31] and W. Laukajtys, L. Slomiński [13, 14]. In all cited papers devoted to reflecting SDEs in convex domains the projection used is the classical one, i.e. if \(|\Delta K_t| > 0\) then
\[
X_t = \Pi_{\overline{D(A)}}(X_t - \Delta H_t + \langle f(X_t), \Delta Z_t \rangle),
\]
where \(\Pi_{\overline{D(A)}}\) denotes the classical projection on \(\overline{D(A)}\):
\[
x := \Pi_{\overline{D(A)}}(z) \Leftrightarrow |z - x| = \inf\{|z - x' : x' \in \overline{D(A)}\}.
\]
In the present paper we study existence, uniqueness and approximations of solutions of (1) driven by semimartingale with jumps, provided that \(\text{Int } (D(A)) \neq \emptyset\) and the projections \(\Pi\)
are non-expansive. Since we consider generalized projections, our results are new even in the case of SDEs with reflecting boundary condition in convex domains. Our results are based on the previous paper of the same authors [16] concerning the deterministic Skorokhod problem

\[ x_t + k_t = y_t, \quad t \in \mathbb{R}^+ \]  

(2)

associated to the maximal monotone operator \( A \) and the projection \( \Pi \), with \( y \) a given càdlàg function such that \( y_0 \in \overline{D(A)} \).

The paper is organized as follows: in next Section we recall the deterministic results proved in L. Maticiuc et al. [16] which are essential in order to study the associated multivalued SDE. In Section 3 we present the existence and uniqueness result for the multivalued SDE with additive noise. Section 4 is devoted to the study of SDEs (1): we prove the existence and uniqueness of a solution for (1) and we propose two practical methods of their approximations. The first one is based on discrete approximations of processes \( H \) and \( Z \) and is constructed with the analogy to the Euler scheme. We prove its convergence in probability in the Skorokhod topology \( J_1 \). The second method is a modification of the Yosida approximations and has a form

\[ X^n_t + \int_0^t A_n(X^n_s)ds = H_t + \int_0^t \langle f(X^n_s), dZ_s \rangle, \quad t \in \mathbb{R}^+, n \in \mathbb{N}^*. \]  

(3)

We prove that for any stopping time \( \tau \) such that

\[ \mathbb{P}(\tau < +\infty) = 1 \quad \text{and} \quad \mathbb{P}(\Delta H_\tau = \Delta Z_\tau = 0) = 1, \]

we have the convergence in probability \( X^n_\tau \to X_\tau \), where \( X \) is a solution of (1) associated to the maximal monotone operator \( A \) and the classical projection \( \Pi_{\overline{D(A)}}(X^n \text{ need not to converge in probability in the Skorokhod topology } J_1) \). We also show that a slightly modified Yosida type approximation converges to solutions of (1) with general non-expanding projections \( \Pi \).

In the paper we use the following notations: \( \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \) is the space of all mappings \( y : \mathbb{R}^+ \to \mathbb{R}^d \) which are càdlàg (right continuous and admit left-hand limits) endowed with the Skorokhod topology \( J_1 \).

Let \( T > 0, k : [0, T] \to \mathbb{R}^d \) and \( \mathcal{D} \) be the set of the partitions of the interval \([0, T]\). Set

\[ V_\Delta(k) = \sum_{i=0}^{n-1} |k(t_{i+1}) - k(t_i)| \text{ and} \]

\[ \nabla_{k,T} := \sup_{\Delta \in \mathcal{D}} V_\Delta(k), \]

where \( \Delta : 0 = t_0 < t_1 < \cdots < t_n = T \).

Write \( \text{BV}([0, T]; \mathbb{R}^d) = \{ k : [0, T] \to \mathbb{R}^d : \nabla_{k,T} < \infty \} \). We will say that a function \( k \in \text{BV}_{loc}(\mathbb{R}^+; \mathbb{R}^d) \) if, for every \( T > 0, k \in \text{BV}([0, T]; \mathbb{R}^d) \).

If \( k \) is a function with locally bounded variation then \( \nabla_{k,T} \) stands for its variation on \((t, T]\) and

\[ k^c_t := k_t - \sum_{s \leq t} \Delta k_s \quad \text{and} \quad k^d_t = k_t - k^c_t, \quad t \in \mathbb{R}^+. \]

Set \( \|x\|_{[s,t]} := \sup_{r \in [s,t]} |x_r| \) and \( \|x\|_{t} := \|x\|_{[0,t]} \).
Let \( Y = (Y_t) \) be an \( \mathcal{F}_t \) adapted process and \( \tau \) be an \( \mathcal{F}_t \) stopping time. We write \( Y^\tau \) and \( Y^{\tau-} \) to denote the stopped processes \( Y_{\tau\wedge t} \) and \( Y_{\tau-\wedge t} \), respectively. Given a semimartingale \( Y \) we denote by \( [Y] \) its quadratic variation process and by \( \langle Y \rangle \) the predictable compensator of \( [Y] \).

2 Preliminaries. The Skorokhod problem

In this section we introduce the assumptions and we recall the main results from [16] (the deterministic case).

A set–valued operator \( A \) on \( \mathbb{R}^d \) is said to be monotone if
\[
\langle y - y', z - z' \rangle \geq 0, \quad \forall (z, y), (z', y') \in \text{Gr}(A)
\]
and \( A \) is said to be maximal monotone if the condition \( \langle y - v, z - u \rangle \geq 0, \forall (u, v) \in \text{Gr}(A) \) implies that \((z, y) \in \text{Gr}(A)\).

Remark 1 (see [4])

(a) Let \( \varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\} \) be a proper convex and lower semicontinuous function. The subdifferential operator of \( \varphi \) is defined by
\[
\partial \varphi(z) := \{ y \in \mathbb{R}^d : \langle y, z' - z \rangle + \varphi(z) \leq \varphi(z'), \forall z' \in \mathbb{R}^d \}, \quad z \in \mathbb{R}^d
\]
and it is a maximal monotone operator on \( \mathbb{R}^d \).

(b) Let \( D \) be a closed convex nonempty subset of \( \mathbb{R}^d \) and let \( I_D : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\} \) be the convexity indicator function (i.e. \( I_D(z) = 0 \) if \( z \in D \) and \(+\infty\), otherwise). Then \( I_D \) is convex lower semicontinuous and proper. Moreover, \( \partial I_D(z) = \emptyset \) if \( z \notin D \) and
\[
\partial I_D(z) = \{ \langle y, x - z \rangle \leq 0, x \in D \}, \quad z \in D,
\]
which implies that
\[
\partial I_D(z) = \begin{cases} 
\{0\}, & \text{if } z \in \text{Int}(D), \\
N_D(z), & \text{if } z \in \text{Bd}(D).
\end{cases}
\]
Here \( N_D(z) \) denotes the closed external cone normal to \( D \) at \( z \in \text{Bd}(D) \).

In the rest of the paper we will restrict our attention on (H1) the maximal monotone operators \( A \) such that
\[
\text{Int}(D(A)) \neq \emptyset
\] 
(H2) and on the generalized projections \( \Pi : \mathbb{R}^d \to \overline{D(A)} \) defined by:
\[
\left\{ \begin{array}{c}
\Pi(z) = z, \quad \forall z \in \overline{D(A)} \quad \text{and} \\
|\Pi(z) - \Pi(z')| \leq |z - z'|, \quad \forall z, z' \in \mathbb{R}^d.
\end{array} \right.
\]
It is well known that \( \overline{D(A)} \) is convex (see, e.g., [4]). Let \( \Pi_{D(A)} \) denotes the classical projection on \( D(A) \) with convention \( \Pi_{\overline{D(A)}}(z) = z, \forall z \in \overline{D(A)} \). One can check that
\[
x = \Pi_{D(A)}(z) \iff \langle z - x, x' - x \rangle \leq 0, \forall x' \in \overline{D(A)}
\]
and
\[ |\Pi_{D(A)}(z) - \Pi_{D(A)}(z')|^2 \leq \langle \Pi_{D(A)}(z) - \Pi_{D(A)}(z'), z - z' \rangle, \quad \forall z, z' \in \mathbb{R}^d \] (6)
which implies (5).

There exist important examples of other non-expanding projections on \( D(A) \) connected with the elasticity condition (introduced in one-dimensional case in [6] and [32]): let \( c \in [0,1] \) and \( \Pi^c : \mathbb{R}^d \rightarrow \mathbb{R}^d \) given by
\[
\Pi^c(z) := \Pi_{D(A)}(z) - c(z - \Pi_{D(A)}(z)), \quad z \in \mathbb{R}^d
\]
and its compositions \( \Pi^{c,n} : \mathbb{R}^d \rightarrow \mathbb{R}^d \) of the form
\[
\Pi^{c,n}(z) = \Pi_n \circ \ldots \circ \Pi_1(z), \quad z \in \mathbb{R}^d, \text{ where } \Pi_1 = \ldots = \Pi_n = \Pi^c, \ n \in \mathbb{N}^*.
\]
It can be shown that (see [16, Proposition 9]) there exists the limit
\[
\Pi^c(z) := \lim_{n \to \infty} \Pi^{c,n}(z)
\]
and \( \Pi^c(z) \) is a generalized projection (satisfies (5)).

**Definition 2** Let \( y \in D(\mathbb{R}^+, \mathbb{R}^d) \) be such that \( y_0 \in \overline{D(A)} \). We say that a pair \((x,k) \in D(\mathbb{R}^+, \mathbb{R}^{2d})\) is a solution of the Skorokhod problem (2) associated to \( y \), the maximal monotone operator \( A \) and the projection \( \Pi \) (written for short \((x,k) = \mathcal{SP}(A,\Pi;y)\)) if

(i) \( x_t = y_t - k_t \in \overline{D(A)}, t \in \mathbb{R}^+ \),

(ii) \( k \) is a function with locally bounded variation such that \( k_0 = 0 \) and, for any \((\alpha, \beta) \in \text{Gr}(A)\)
\[
\int_s^t \langle x_u - \alpha, dk_u^c - \beta du \rangle \geq 0, \quad 0 \leq s < t, \ s, t \in \mathbb{R}^+,
\]

(iii) if \( |\Delta k_t| > 0 \) then
\[
x_t := \Pi(x_{t^-} + \Delta y_t), \quad t \in \mathbb{R}^+.
\]

**Remark 3** (a) Note that
\[
\Delta k_t = \Delta y_t - \Delta x_t = x_{t^-} + \Delta y_t - \Pi(x_{t^-} + \Delta y_t), t \in \mathbb{R}^+.
\]

Since \( x_{t^-} \in \overline{D(A)} \) and \( \Pi \) is non–expansive, we have
\[
|\Delta k_t| \leq 2|\Delta y_t|, \quad t \in \mathbb{R}^+.
\]

(b) Let \((x,k) = \mathcal{SP}(A,\Pi;y)\). We will use the notation \( x = \mathcal{SP}^{(1)}(A,\Pi;y) \) and \( k = \mathcal{SP}^{(2)}(A,\Pi;y) \).

If \( y \) is continuous then \((x,k)\) is also continuous and does not depend on \( \Pi \). In this case we will write \((x,k) = \mathcal{SP}(A;y) \) and \( x = \mathcal{SP}^{(1)}(A;y), k = \mathcal{SP}^{(2)}(A;y) \).

**Lemma 4** Assume that \( y, y' \in D(\mathbb{R}^+, \mathbb{R}^d) \) such that \( y_0, y'_0 \in \overline{D(A)} \). If \((x,k) = \mathcal{SP}(A,\Pi;y)\) and \((x',k') = \mathcal{SP}(A,\Pi;y')\) then
(i) for all $0 \leq s < t, s, t \in \mathbb{R}^+$

$$\int_s^t \langle x_u - x'_u, dk_u - dk'_u \rangle + \frac{1}{2} \sum_{s < u \leq t} |\Delta k_u - \Delta k'_u|^2 \geq 0,$$

(ii) for all $t \in \mathbb{R}^+$

$$|x_t - x'_t|^2 \leq |y_t - y'_t|^2 - 2 \int_0^t \langle y_{t'} - y_s, dk_s - dk'_{s} \rangle.$$

In Cépa [7] and Răşcanu [23] it is proved that for any continuous $y$ such that $y_0 \in \overline{D(A)}$ there exists a unique solution $SP(A, y)$. In particular, for any constant $\alpha \in \overline{D(A)}$ there exists a unique solution $SP(A, \alpha)$ and therefore:

**Lemma 5 ([16, Lemma 20])** For any step function $y \in D(\mathbb{R}^+, \mathbb{R}^d)$ such that $y_0 \in \overline{D(A)}$ there exists a unique $SP(A, \Pi; y)$. Moreover if $y$ has the form $y_t = \alpha_k, t \in [t_k, t_{k+1}), k \in \mathbb{N},$ where $0 = t_0 < t_1 < \ldots$ and $\lim_{k \to \infty} t_k = +\infty$ then $(x, k) = SP(A, \Pi; y)$ is given by

$$x_t = \begin{cases} SP^{(1)}(A, \alpha_0)_t, & t \in [0, t_1) \\ SP^{(1)}(A, \Pi(x_{t_k} + \alpha_k - \alpha_{k-1}))_{t-t_k}, & t \in [t_k, t_{k+1}), k \in \mathbb{N}^* \end{cases}$$

(7)

and

$$k_t = \begin{cases} SP^{(2)}(A, \alpha_0)_t, & t \in [0, t_1) \\ SP^{(2)}(A, \Pi(x_{t_k} + \alpha_k - \alpha_{k-1}))_{t-t_k}, & t \in [t_k, t_{k+1}), k \in \mathbb{N}^*. \end{cases}$$

(8)

**Proof.** Of course $x_t = y_t - k_t \in \overline{D(A)}, t \in \mathbb{R}^+$, and $k$ is a function with locally bounded variation with $k_0 = 0$ and such that for any $\alpha, \beta \in \Gr(A)$

$$\int_s^t \langle x_u - \alpha, dk_u - \beta du \rangle \geq 0, \quad 0 \leq s < t, s, t \in \mathbb{R}^+.$$ 

Moreover, for any $k \in \mathbb{N}^*, x_{t_k} = \Pi(x_{t_k} + \alpha_k - \alpha_{k-1}) = \Pi(x_{t_k} + \Delta y_{t_k})$ and the proof is complete. $\blacksquare$

**Theorem 6 ([16, Theorems 22–23])** I. Assume that $y, y^n \in D(\mathbb{R}^+, \mathbb{R}^d)$ are such that $y^n \in \overline{D(A)}, \forall n \in \mathbb{N}^*$. Let $(x, k) = SP(A, \Pi; y)$ and $(x^n, k^n) = SP(A, \Pi; y^n)$, $\forall n \in \mathbb{N}^*$.

(i) If $\|y^n - y\|_T \to 0, T \in \mathbb{R}^+$ then

$$\|x^n - x\|_T \to 0 \quad \text{and} \quad \|k^n - k\|_T \to 0, T \in \mathbb{R}^+.$$

(ii) If $y^n \longrightarrow y$ in $D(\mathbb{R}^+, \mathbb{R}^d)$ then

$$(x^n, k^n, y^n) \longrightarrow (x, k, y) \quad \text{in} \quad D(\mathbb{R}^+, \mathbb{R}^{3d}).$$

II. If assumptions (H1–H3) are satisfied and $y \in D(\mathbb{R}^+, \mathbb{R}^d)$ such that $y_0 \in \overline{D(A)}$ then there exists a unique solution $(x, k) = SP(A, \Pi; y)$. 

6
**Definition 7** 1. We say that \( \pi = \{t_0, t_1, t_2, \ldots \} \) is a partition of \( \mathbb{R}^+ \) if

\[
0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots \quad \text{and} \quad t_k \to +\infty.
\]

2. We write

\[
\|\pi\| := \sup \left\{ r' - r : r \in \pi \right\}.
\]

The set of all partitions of \( \mathbb{R}^+ \) will be denoted \( \mathcal{P}_{\mathbb{R}^+} \).

**Corollary 8** Let \( y \in \mathcal{D}(\mathbb{R}^+, \mathbb{R}^d) \) be such that \( y_0 \in \overline{\mathcal{D}(A)} \) and let \( \pi_n = \{0 = t_{n0} < t_{n1} < \ldots < t_{nk} < \ldots \} \in \mathcal{P}_{\mathbb{R}^+}, n \in \mathbb{N}^* \), be a sequence of partitions of \( \mathbb{R}^+ \) such that \( \lim_{n \to \infty} \|\pi_n\| = 0 \). If \( y^{(n)}_{t_k} := y_{t_k}, t \in [t_n, t_{n+1}), k \in \mathbb{N}, n \in \mathbb{N}^* \) denotes the sequence of discretizations of \( y \) and \( (x^n, k^n) = \mathcal{SP}(A, \Pi; y^{(n)}), n \in \mathbb{N}^* \) then

(i) \( (x^n, k^n, y^{(n)}) \to (x, k, y) \) in \( \mathcal{D}(\mathbb{R}^+, \mathbb{R}^{3d}) \),

(ii) for any \( T \in \mathbb{R}^+ \)

\[
\sup_{t \leq T, t \in \pi_n} |x^n_t - x_t| \to 0 \quad \text{and} \quad \sup_{t \leq T, t \in \pi_n} |k^n_t - k_t| \to 0,
\]

where \( (x, k) = \mathcal{SP}(A, \Pi; y) \).

**Proof.** (i) It is sufficient to observe that in this case \( y^{(n)} \to y \) in \( \mathcal{D}(\mathbb{R}^+, \mathbb{R}^d) \) (see, e.g., [8, Chapter 3, Proposition 6.5]) and to apply Theorem 6–(ii).

(ii) From (i) we deduce that

\[
(x^n, x^{(n)}, k^n, k^{(n)}) \to (x, x, k, k) \quad \text{in} \quad \mathcal{D}(\mathbb{R}^+, \mathbb{R}^4),
\]

where \( x^{(n)}_t := x_{t_n}, k^{(n)}_t := k_{t_n} \), for \( t \in [t_n, t_{n+1}), k \in \mathbb{N}, n \in \mathbb{N}^* \). Consequently, by [9, Chapter VI, Proposition 1.17],

\[
\sup_{t \leq T, t \in \pi_n} |x^n_t - x_t| = ||x^n - x^{(n)}||_T \to 0, \quad T \in \mathbb{R}^+
\]

and

\[
\sup_{t \leq T, t \in \pi_n} |k^n_t - k_t| = ||k^n - k^{(n)}||_T \to 0, \quad T \in \mathbb{R}^+,
\]

which completes the proof. ■

**Remark 9 ([16, Remark 24])** If \( y \) is continuous then the solution of the Skorokhod problem \( (x, k) = \mathcal{SP}(A, \Pi; y) \) is also continuous and is not depending on projections \( \Pi \). By Theorem 6–(ii) for any projection \( \Pi \), if \( (x^n, k^n) = \mathcal{SP}(A, \Pi; y^n), n \in \mathbb{N} \) and \( \|y^n - y\|_T \to 0, \quad T \in \mathbb{R}^+ \) then

\[
\|x^n - x\|_T \to 0 \quad \text{and} \quad \|k^n - k\|_T \to 0, \quad T \in \mathbb{R}^+.
\]
3 Stochastic models with additive noise

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ be a stochastic basis, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and \{\mathcal{F}_t\}_{t \geq 0} is a filtration (an increasing collection of completed $\sigma$–algebras of $\mathcal{F}$).

A random variable $\tau : \Omega \rightarrow [0, \infty]$ is a stopping time if \[
\{ \omega \in \Omega : \tau(\omega) \leq t \} \in \mathcal{F}_t, \quad \forall \ t \in [0, \infty].
\]

We point out some notions. Let $X : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ be a stochastic process.

- $X$ is a continuous stochastic process (respectively càdlàg stochastic process) if the paths $X(\omega, .) : \mathbb{R}^+ \rightarrow \mathbb{R}^d$ are continuous (respectively càdlàg) $\mathbb{P}$–a.s. $\omega \in \Omega$.

- $X$ is a locally bounded variation stochastic process if $X(\omega, .) : \mathbb{R}^+ \rightarrow \mathbb{R}^d$ are locally bounded functions $\mathbb{P}$–a.s. $\omega \in \Omega$.

- $X$ is $\mathcal{F}_t$–adapted (adapted to the history $\mathcal{F}_t$, $t \geq 0$) if $\omega \mapsto X_t(\omega) : \Omega \rightarrow \mathbb{R}^d$ is $\mathcal{F}_t \otimes \mathcal{B}_{\mathbb{R}^d}$–measurable for all $t \geq 0$.

- $X$ is a measurable stochastic process if $X$ is $(\mathcal{F} \otimes \mathcal{B}_{\mathbb{R}^+}, \mathcal{B}_{\mathbb{R}^d})$–measurable.

- $X : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ is $\mathcal{F}_t$–progressively measurable stochastic process if $(\omega, s) \mapsto X(\omega, s) : \Omega \times [0, t] \rightarrow \mathbb{R}^d$ is $\mathcal{F}_t \otimes \mathcal{B}_{[0, t]}, \mathcal{B}_{\mathbb{R}^d})$–measurable for all $t \geq 0$.

**Remark 10** If both $X, Y : \mathbb{R}^+ \rightarrow \mathbb{R}^d$ are right (or left) continuous stochastic processes, then

\[
(X_t = Y_t, \ \forall t \in \mathbb{R}^+) \ \mathbb{P}–a.s. \iff (X_t = Y_t, \ \mathbb{P}–a.s.) \ \forall t \in \mathbb{R}^+.
\]

It is easy to prove that:

- every progressively measurable stochastic process is adapted.

- if $X$ is an adapted right (or left) continuous stochastic process, then $X$ is progressively measurable.

- if $\{X_t\}_{t \geq 0}$ is progressively measurable then $\{X_{t\wedge \tau}\}_{t \geq 0}$ is progressively measurable for every $\tau : \Omega \rightarrow [0, \infty]$ a stopping time.

Let now $Y$ be an $\mathcal{F}_t$–adapted and càdlàg stochastic process such that $Y_0 \in \overline{D(A)}$.

**Definition 11** We say that a pair $(X, K)$ of $\mathcal{F}_t$–adapted càdlàg stochastic processes is a solution of the SDE

\[
\begin{cases}
X_t + K_t = Y_t, \quad t \in \mathbb{R}^+, \\
X_0 = Y_0,
\end{cases}
\tag{9}
\]

associated to the maximal monotone operator $A$ and the generalized projection $\Pi$ (and will be denoted by $(X, K) = S\mathcal{P}(A, \Pi; Y))$, if $(X(\omega), K(\omega)) = S\mathcal{P}(A, \Pi; Y(\omega)), \mathbb{P}–a.s. \ (see \ Definition \ 2)$.

From the deterministic part we deduce:
Proposition 12 Under the assumptions \((H_1-H_2)\), if \(Y : \Omega \times \mathbb{R}^+ \to \mathbb{R}^d\) is a \(\mathcal{F}_t\)-adapted càdlàg stochastic process such that \(Y_0 \in \overline{D(A)}\), then there exists a unique couple \((X,K)\) of \(\mathcal{F}_t\)-adapted càdlàg stochastic processes which is solution of (9).

Proof. By Theorem 6 we deduce that for each \(\omega \in \Omega\) fixed there exists a unique couple \((X(\omega), K(\omega)) = SP(A, \Pi; Y(\omega))\) with \(X(\omega) \in \mathbb{D}\left(\mathbb{R}^+, \mathbb{R}^d\right)\) and \(K(\omega) \in \mathbb{D}\left(\mathbb{R}^+, \mathbb{R}^d\right) \cap BV\left([0,T]\times \mathbb{D}\left(\mathbb{R}^+, \mathbb{R}^d\right)\right)\). It is left to show that \((X,K)\) is adapted. Let \(\pi_n = \{0 = t_{n0} < t_{n1} < \ldots < t_{nk} < \ldots\}\), \(n \in \mathbb{N}^*\), be a sequence of partitions of \(\mathbb{R}^+\) such that \(\lim_{n \to \infty} \|\pi_n\| = 0\).

Let \(Y_t^{(n)} : = Y_{t_{nk}}, t \in [t_{nk}, t_{nk+1}), k \in \mathbb{N}, n \in \mathbb{N}^*\) denotes the sequence of discretizations of \(Y\) and \(X^{(n)} = SP(A_n, \Pi; Y^{(n)})\). Observe now that by Lemma 5 \(X^n\) is given by the following formula

\[
X_t^{(n)} = \begin{cases} 
SP^{(1)}(A, Y_0)_t, & t \in [0, t_{n1}), \\
SP^{(1)}(A, \Pi(X_{t_{nk+}}^{(n)} + Y_{t_{nk}} - Y_{t_{nk-}}))_{t-t_{nk}}, & t \in [t_{nk}, t_{nk+1}), k \in \mathbb{N}^*
\end{cases}
\]

and hence is \(\mathcal{F}_t\)-adapted. Since by Corollary 8–(i)

\[X^{(n)} \rightarrow X\quad \text{in } \mathbb{D}\left(\mathbb{R}^+, \mathbb{R}^d\right), \quad \mathbb{P}\text{-a.s.,}\]

the limit process \(X\) is \(\mathcal{F}_t\)-adapted as well. Finally, \(K = Y - X\) is also \(\mathcal{F}_t\)-adapted. 

4 SDEs with maximal monotone operators

Let \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})_{t \geq 0}\) be a stochastic basis. Throughout this section we will use assumptions \((H_1-H_2)\) and let

\[(H_3) \ f : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d\text{ for which there exists a constant } L > 0 \text{ such that } \|f(x) - f(y)\| \leq L|x - y|, \ x, y \in \mathbb{R}^d.

Let \(Y, \hat{Y}\) be two \(\mathcal{F}_t\)-adapted processes with trajectories in \(\mathbb{D}\left(\mathbb{R}^+, \mathbb{R}^d\right)\) admitting decompositions

\[Y_t = Y_0 + H_t + M_t + V_t, \quad \hat{Y}_t = \hat{Y}_0 + H_t + \hat{M}_t + \hat{V}_t, \quad t \in \mathbb{R}^+,
\]

with \(Y_0 + H_0, \hat{Y}_0 + H_0 \in \overline{D(A)}\), where \(H\) is an \(\mathcal{F}_t\)-adapted process with trajectories in \(\mathbb{D}\left(\mathbb{R}^+, \mathbb{R}^d\right)\), \(M, \hat{M}\) are \(\mathcal{F}_t\)-adapted local martingales and \(V, \hat{V}\) are \(\mathcal{F}_t\)-adapted processes with bounded variation such that \(M_0 = \hat{M}_0 = V_0 = \hat{V}_0 = 0\).

Lemma 13 Let \((X,K) = SP(A, \Pi; Y)\) and \((\hat{X}, \hat{K}) = SP(A, \Pi; \hat{Y})\). Then, for any \(p \in \mathbb{N}^*\), there exists a constant \(C_p > 0\) such that, for every stopping time \(\tau\),

\[
(i) \quad \mathbb{E}\|X - \hat{X}\|^{2p} \leq C_p \mathbb{E}\left(\|Y_0 - \hat{Y}_0\|^{2p} + \|M - \hat{M}\|_{\tau}^p + \mathcal{H}V - \mathcal{H}\hat{V}\|^{2p}\right),
\]

\[
(ii) \quad \mathbb{E}\sup_{t < \tau} |X_t - \hat{X}_t|^{2p} \leq C_p \mathbb{E}\left(\|Y_0 - \hat{Y}_0\|^{2p} + \|M - \hat{M}\|_{\tau}^p + \mathcal{H}V - \mathcal{H}\hat{V}\|^{2p}\right).
\]
Proof. (i) By Lemma 4–(ii),

\[ |X_t - \hat{X}_t|^2 \leq |Y_t - \hat{Y}_t|^2 - 2 \int_0^t \langle Y_t - \hat{Y}_t - Y_s + \hat{Y}_s, dK_s - d\hat{K}_s \rangle, \quad \forall t \in \mathbb{R}^+. \tag{11} \]

Using equality

\[ \int_0^t \langle Y_s - \hat{Y}_s, d(K_s - \hat{K}_s) \rangle = \int_0^t \langle Y_{s-} - \hat{Y}_{s-}, d(K_s - \hat{K}_s) \rangle + [Y - \hat{Y}, K - \hat{K}]_t \tag{12} \]

and the integration by parts formula

\[ \int_0^t \langle Y_{s-} - \hat{Y}_{s-}, d(K_s - \hat{K}_s) \rangle + \int_0^t \langle K_s - \hat{K}_s, d(Y_s - \hat{Y}_s) \rangle = \int_0^t d\langle Y_s - \hat{Y}_s, K_s - \hat{K}_s \rangle - [Y - \hat{Y}, K - \hat{K}]_t, \]

we deduce

\[ \int_0^t \langle Y_t - \hat{Y}_t - Y_s + \hat{Y}_s, dK_s - d\hat{K}_s \rangle = \langle Y_t - \hat{Y}_t, K_t - \hat{K}_t \rangle - \int_0^t \langle Y_s - \hat{Y}_s, dK_s - d\hat{K}_s \rangle \]

\[ = \int_0^t \langle K_s - \hat{K}_s, dY_s - d\hat{Y}_s \rangle. \]

Hence, using equation (9), we see that

\[ 2 \int_0^t \langle Y_t - \hat{Y}_t - Y_s + \hat{Y}_s, dK_s - d\hat{K}_s \rangle = 2 \int_0^t \langle Y_{s-} - \hat{Y}_{s-}, dY_s - d\hat{Y}_s \rangle - 2 \int_0^t \langle X_{s-} - \hat{X}_{s-}, dY_s - d\hat{Y}_s \rangle \]

\[ = 2 \int_0^t \langle Y_s - \hat{Y}_s, dY_s - d\hat{Y}_s \rangle - 2[Y - \hat{Y}, Y - \hat{Y}]_t - 2 \int_0^t \langle X_{s-} - \hat{X}_{s-}, dY_s - d\hat{Y}_s \rangle \]

\[ = |Y_t - \hat{Y}_t|^2 - |Y_0 - \hat{Y}_0|^2 - [Y - \hat{Y}, Y - \hat{Y}]_t - 2 \int_0^t \langle X_{s-} - \hat{X}_{s-}, dY_s - d\hat{Y}_s \rangle. \]

Inequality (11) becomes

\[ |X_t - \hat{X}_t|^2 \leq |Y_0 - \hat{Y}_0|^2 + [Y - \hat{Y}, Y - \hat{Y}]_t + 2 \int_0^t \langle X_{s-} - \hat{X}_{s-}, dY_s - d\hat{Y}_s \rangle, \]

therefore, for any \( p \in \mathbb{N}^* \) and any stopping time \( \tau \), there exists \( c_p > 0 \) such that

\[ \mathbb{E} ||X - \hat{X}||_\tau^{2p} \leq c_p \left( \mathbb{E}[Y_0 - \hat{Y}_0]^{2p} + \mathbb{E}[Y,Y]^{2p}_\tau + \mathbb{E} \sup_{t \leq \tau} \left| \int_0^t \langle X_{s-} - \hat{X}_{s-}, dM_s - d\hat{M}_s \rangle \right|^p \right) \]

\[ + \mathbb{E} \sup_{t \leq \tau} \left| \int_0^t \langle X_{s-} - \hat{X}_{s-}, dV_s - d\hat{V}_s \rangle \right|^p \].

By Burkholder-Davis-Gundy inequality, there exists \( c'_p > 0 \) such that

\[ \mathbb{E} \sup_{t \leq \tau} \left| \int_0^t \langle X_{s-} - \hat{X}_{s-}, dM_s - d\hat{M}_s \rangle \right|^p \leq c'_p \mathbb{E} \left( \int_0^\tau |X_{s-} - \hat{X}_{s-}|^2 d|M - \hat{M}, M - \hat{M}|_\tau \right)^{p/2} \]

\[ \leq c'_p \left( \mathbb{E} \sup_{s \leq \tau} |X_{s-} - \hat{X}_{s-}|^{2p} \right)^{1/2} \left( \mathbb{E}[M - \hat{M}, M - \hat{M}]_\tau^{p} \right)^{1/2}. \]
Using now Young’s inequality and Gronwall’s inequality, we see that, for some $c > 0$ (if not we consider stopping times $\tau^c = \inf\{t > 0 : |H_t| > c\}$ or $|Z_t| > c$) and let $H$ be an $\mathcal{F}_t$–adapted process with trajectories in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ and let $Z$ be an $\mathcal{F}_t$–adapted semimartingale such that $Z_0 = 0$.

**Definition 14** We say that a pair $(X, K)$ of $\mathcal{F}_t$–adapted processes with trajectories in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ is a solution of the SDE (1) if $\mathbb{P}$–a.s., $X_t(\omega) \in \overline{\mathbb{D}(A)}$, $\forall \, t \geq 0$, $K_t(\omega) \in BV_{loc}(\mathbb{R}^+;\mathbb{R}^d)$ with $K_0(\omega) = 0$ and

(i) $X_t(\omega) + K_t(\omega) = H_t(\omega) + \int_0^t \langle f(X_s(\omega)), dZ_s(\omega) \rangle, \ \forall \, t \geq 0$,

(ii) $K_t(\omega) = K_t^c(\omega) + K_t^d(\omega), \ \ K_t^d(\omega) = \sum_{0 \leq s \leq t} \Delta K_s(\omega), \ \forall \, t \geq 0$,

(iii) $dK_t^c(\omega) \in A(X_r(\omega))(dr)$,

(iv) $X_t(\omega) = \Pi\left( X_{t-}(\omega) + \Delta[H_t(\omega) + \int_0^t \langle f(X_s(\omega)), dZ_s(\omega) \rangle] \right), \ \forall \, t \geq 0^+$

(definition $dK_t^c(\omega) \in A(X_r(\omega))(dr)$ means that $\int_{s}^{t} \langle X_r(\omega) – z, dK_t^c(\omega) – \hat{z}dr \rangle \geq 0, \ \forall \, (z, \hat{z}) \in Gr(A), \ \forall 0 \leq s < t$).

**Remark 15** Hence $(X, K) = \mathcal{SP}(A, \Pi; Y)$ where

$Y_t = H_t + \int_0^t \langle f(X_s), dZ_s \rangle, \ t \in \mathbb{R}^+$.

**Theorem 16** Suppose that conditions (H₁ – H₃) are satisfied. Let $H$ be an $\mathcal{F}_t$–adapted process with trajectories in $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ such that $H_0 \in \overline{\mathbb{D}(A)}$, and let $Z$ be an $\mathcal{F}_t$–adapted semimartingale with $Z_0 = 0$. Then there exists a unique strong solution $(X, K)$ of SDE (1).

**Proof.** We will assume without restrict our generality that $|H_t|, |Z_t| \leq c$ for some constant $c > 0$ (if not we consider stopping times $\tau = \inf\{t > 0 : |H_t| > c\}$ or $|Z_t| > c$) and stopping processes $H^{\tau^-}, Z^{\tau^-}$ and we prove existence first on interval $[0, \tau)$.

Since $|\Delta Z| \leq 2c$, from [22, Chapter III, Theorem 32] we see that $Z$ is a special semimartingale and admits a unique decomposition $Z_t = M_t + V_t, \ t \in \mathbb{R}^+$, where $M$ is a local
square-integrable martingale with $|\Delta M| \leq 4c$ and $V$ is a predictable process with locally bounded variation with $|\Delta V| \leq 2c$.

Let $L, C_1$ be constants from (H$_3$) and Lemma 13 respectively.

Set

$$\tau' = \inf \{ t : |H_t + \langle f(H_0), Z_t \rangle - H_0| \geq \gamma/2 \} \wedge 1.$$ 

In the first step of the proof we will show the existence and uniqueness of a solution of the SDE (1) on the interval $[0, \tau)$, where

$$\tau = \inf \{ t > 0 : \max([M]_t, \langle M \rangle_t, \|V\|_t^2) > b \} \wedge \tau'.$$

We define

$$Y^\tau = \lim_{\tau_n \uparrow \tau} Y_{\tau_n \wedge \tau}.$$ 

Set now

$$S^2 = \{ Y : Y \text{ is } \mathcal{F}_t \text{-adapted}, Y_0 = H_0, Y_t = Y_t^{\tau-}, \mathbb{E} \sup_{t \geq 0} |Y_t|^2 < \infty \}$$

and define mapping $\Phi : S^2 \rightarrow S^2$ by putting $\Phi(Y)$ to be the first coordinate of the solution of the Skorokhod problem associated with $H^{\tau-} + \int_0^\tau \langle f(Y_{s-}), dZ_s^\tau \rangle$.

We will show that $\Phi$ is a contraction mapping on $S^2$. Let first remark that

$$\Phi(H_0) = H^{\tau-} + \langle f(H_0), Z^{\tau-} \rangle - K^{\tau-}$$

and therefore $\Phi(H_0) \in S^2$.

Furthermore, by Lemma 13-(ii), for any $Y, Y' \in S^2$

$$\mathbb{E} \sup_{t < \tau} |\Phi(Y)_t - \Phi(Y')_t|^2 \leq C_1 \left\{ E \int_0^{\tau-} |f(Y_{s-}) - f(Y'_{s-})|^2 d([M]_s + \langle M \rangle_s) + \mathbb{E} \left( \int_0^{\tau-} |f(Y_{s-}) - f(Y'_{s-})| d\|V\|_s^2 \right) \right\} \leq 3C_1 bL^2 \mathbb{E} \sup_{t < \tau} |Y_t - Y'_t|^2 = \frac{1}{2} \mathbb{E} \sup_{t < \tau} |Y_t - Y'_t|^2.$$ 

Hence $[\Phi(Y) - \Phi(H_0)] \in S^2$ and consequently $\Phi(Y) \in S^2$; moreover we see that $\Phi : S^2 \rightarrow S^2$ is a contraction. Therefore, by the Banach contraction principle, there exists a fixed point $X^1$, which is a unique solution of (1) on $[0, \tau)$. Moreover, putting $X^1_t = \Pi(X^1_{\tau-} + \Delta H_t + \langle f(X^1_{\tau-}), \Delta Z_t \rangle)$ we obtain a solution on $[0, \tau]$.

Now, we define sequence of stopping times $\{\tau_k\}$ by putting $\tau_1 = \tau$ and

$$\tau_{k+1} = \tau_k + \inf \{ t > 0 : \max([\hat{M}]_t, \langle \hat{M} \rangle_t, \|\hat{V}\|_t^2) > b \} \wedge \tau'_k, \quad k \in \mathbb{N},$$

where $\hat{M}_t = M_{\tau_k + t} - M_{\tau_k}$, $\hat{V}_t = V_{\tau_k + t} - V_{\tau_k}$, $\tau'_k = \inf \{ t : |H_{\tau_k + t} + \langle f(H_{\tau_k}), \hat{Z}_t \rangle - H_{\tau_k}| \geq \gamma/2 \} \wedge 1$. Arguing as above, one can obtain a solution $X^{k+1}$ of (1) on $[\tau_k, \tau_{k+1}]$. Since $\tau_k \uparrow +\infty$, we obtain a solution $X$ on $\mathbb{R}^+$ by putting together the solutions $X^k$ on $[\tau_k, \tau_{k+1}]$, $k \in \mathbb{N}$.

Now, we will consider discrete approximation schemes, which are constructed with the natural analogy to the Euler
scheme. Let \( \pi_n = \{0 = t_{n0} < t_{n1} < ... < t_{nk} < ..\} \in \mathcal{P}_{\mathbb{R}^+}, n \in \mathbb{N}^* \) be a sequence of partitions of \( \mathbb{R}^+ \) such that \( \|\pi_n\| \to 0 \) (see Definition 7).

Set

\[
\begin{align*}
\bar{X}_t^n &= \begin{cases} 
SP^{(1)}(A; H_0)t, & t \in [0, t_{n1}), \\
SP^{(1)}(A; \Pi(X^n_{t_{n,k-1}} + (H_{t_{nk}} - H_{t_{n,k-1}}) \
+ (f(X^n_{t_{n,k-1}}), (Z_{t_{nk}} - Z_{t_{n,k-1}}))))_{t-t_{n,k}} & t \in [t_{nk}, t_{n,k+1}), k \in \mathbb{N}^*.
\end{cases}
\end{align*}
\]

Let \( (\mathcal{F}_t^n) \) denotes the discretization of \( (\mathcal{F}_t) \), i.e., \( \mathcal{F}_t^n := \mathcal{F}_{t_{nk}} \), if \( t \in [t_{nk}, t_{n,k+1}) \) and let \( H_t^{(n)} = H_{t_{nk}}, Z_t^{(n)} = Z_{t_{nk}}, \) if \( t \in [t_{nk}, t_{n,k+1}), k \in \mathbb{N}, n \in \mathbb{N}^* \).

Set

\[
\hat{Y}_t^n = H_t^{(n)} + \int_0^t \langle f(X^n_{s-}), dZ_t^{(n)} \rangle, \quad t \in \mathbb{R}^+, n \in \mathbb{N}^*
\]

and note that \( H^{(n)}, Z^{(n)} \) and \( \bar{X}^n, \bar{K}^n = \bar{X}^n - \hat{Y}^n \) are \( \mathcal{F}_t^n \)-adapted processes such that \( (\bar{X}^n, \bar{K}^n) = SP(A; \Pi; \hat{Y}^n), n \in \mathbb{N}^* \).

**Theorem 17** Under the assumptions of Theorem 16,

(i) \( (\bar{X}^n, \bar{K}^n, H^{(n)}, Z^{(n)}) \xrightarrow{p} (X, K, H, Z) \) in \( \mathbb{D}(\mathbb{R}^+; \mathbb{R}^{4d}) \).

(ii) for any \( T \in \mathbb{R}^+ \)

\[
\sup_{t \leq T, t \in \pi_n} |\bar{X}_t^n - X_t| \xrightarrow{p} 0 \quad \text{and} \quad \sup_{t \leq T, t \in \pi_n} |\bar{K}_t^n - K_t| \xrightarrow{p} 0.
\]

(iii) for any \( t \in \mathbb{R}^+ \) such that \( \mathbb{P}(\Delta H_t = \Delta Z_t = 0) = 1 \) or \( t \in \liminf_{n \to +\infty} \pi_n \)

\[
\bar{X}_t^n \xrightarrow{p} X_t \quad \text{and} \quad \bar{K}_t^n \xrightarrow{p} K_t,
\]

where \( (X, K) \) is a solution of stochastic differential equation (1).

**Proof.** (i) Set

\[
Y_t = H_t + \int_0^t \langle f(X_{s-}), Z_s \rangle, \quad t \in \mathbb{R}^+
\]

and \( Y_t^{(n)} = Y_{t_{nk}}, \) if \( t \in [t_{nk}, t_{n,k+1}), k \in \mathbb{N}, n \in \mathbb{N}^* \).

Let \( (X^n, K^n) = SP(A; \Pi; Y^{(n)}), n \in \mathbb{N}^* \). Due to Corollary 8,

\[
(X^n, K^n, H^{(n)}, Z^{(n)}) \xrightarrow{p} (X, K, H, Z), \quad \mathbb{P}\text{-a.s. in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}). \tag{14}
\]

Moreover, let \( (\bar{X}^n, \bar{K}^n) = SP(A; \Pi; \hat{Y}^n) \), where

\[
\hat{Y}_t^n = H_t^{(n)} + \int_0^t \langle f(X^n_{s-}), dZ_t^{(n)} \rangle, \quad t \in \mathbb{R}^+, \quad n \in \mathbb{N}^*.
\]

By (14) and the theorem on functional convergence of stochastic integrals (see, e.g., [10, Theorem 2.11])

\[
(\hat{Y}^n, X^n, H^{(n)}, Z^{(n)}) \xrightarrow{p} (Y, X, H, Z) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{4d}).
\]
Therefore, using Theorem 6,
\[ (\hat{X}^n, \hat{K}^n, X^n, H^{(n)}, Z^{(n)}) \xrightarrow{p} (X, K, H, Z) \text{ in } D(\mathbb{R}^+, \mathbb{R}^{5d}). \] (15)
Combining (14) with (15) implies that
\[ (\hat{X}^n, X^n, \hat{K}^n, K^n) \xrightarrow{p} (X, X, K, K) \text{ in } D(\mathbb{R}^+, \mathbb{R}^{4d}). \]
Therefore, \(|X^n - \hat{X}^n|_T \xrightarrow{p} 0 \) and \(|K^n - \hat{K}^n|_T \xrightarrow{p} 0, \forall T \in \mathbb{R}^+\), and in order to complete the proof of (i) it is sufficient to show that, for any \( T \in \mathbb{R}^+ \),
\[ ||\hat{X}^n_i - \hat{X}^n||_T \xrightarrow{p} 0 \text{ and } ||\hat{K}^n - \hat{K}^n||_T \xrightarrow{p} 0. \] (16)
The proof of (16) runs as the proof of (3.8) in [32, Theorem 3.5].
(ii) See the proof of Corollary 8–(ii).
(iii) Follows easily from (i) and (ii).

For all \( n \in \mathbb{N}^* \) and \( z \in \mathbb{R}^d \) let us define now
\[ J_n(z) = (I + \frac{A}{n})^{-1}(z), \quad A_n(z) = n(z - J_n(z)). \]
\( A_n \) is called the Yosida approximation of the operator \( A \).

**Remark 18** It is well known (see, e.g., [4]) that \( A_n \) is a maximal monotone operator such that for all \( z, z' \in \mathbb{R}^d \) and \( n \in \mathbb{N}^* \)
1. \(|J_n(z) - J_n(z')| \leq |z - z'|\),
2. \(|A_n(z) - A_n(z')| \leq n|z - z'|\),
3. \(\lim_{n \to \infty} J_n(z) = \Pi_{D(A)}(z)\),
4. \(\langle z - z', A_n(z) - A_n(z') \rangle \geq \frac{1}{n}(|A_n(z)|^2 + |A_n(z')|^2 - 2(A_n(z), A_n(z'))) \geq 0\).

Since \( A_n \) is Lipschitz continuous it is well known that there exists a unique solution \( X^n = \mathcal{S}P^{(1)}(A_n, \Pi_{D(A_n)}, Y) \). We will call \( X^n \) the solution of the Yosida problem and we will denote for short by \( X^n = \mathcal{Y}P(A_n; Y) \), \( n \in \mathbb{N}^* \). We remark that, in fact, \( \mathcal{Y}P(A_n; Y) = \mathcal{S}P^{(1)}(A_n; Y), \) since the domain of \( A_n \) is \( \mathbb{R}^d \) and the generalized projection \( \Pi_{D(A_n)} \) becomes the identity.

**Lemma 19** Let \( Y, \hat{Y} \) be two processes admitting decompositions (10) and \( X^n := \mathcal{Y}P(A_n; Y) \)
and \( \hat{X}^n = \mathcal{Y}P(A_n; Y), n \in \mathbb{N}^* \). For any \( p \in \mathbb{N}^* \) there exists a constant \( C_p > 0 \) such that for any stopping time \( \tau \) and any \( n \in \mathbb{N}^* \),
1. \( \mathbb{E}|X^n - \hat{X}^n|^2_\tau^2 \leq C_p \mathbb{E} \left[ |Y_0 - \hat{Y}_0|^2 + |M - \hat{M}|_\tau^p + \langle M - \hat{M}, \hat{V} \rangle_\tau^2 \right] \),
2. \( \mathbb{E} \sup_{t < \tau} |X^n_t - \hat{X}^n_t|^2 \leq C_p \mathbb{E} \left[ |Y_0 - \hat{Y}_0|^2 + |M - \hat{M}|_\tau^p + \langle M - \hat{M}, \hat{V} \rangle_\tau^2 \right] \).
Proof. (i) Set $K^n = Y - X^n$, $\hat{K}^n = \hat{Y} - \hat{X}^n$, $n \in \mathbb{N}^*$. By Lemma 4–(ii)

$$|X^n_t - \hat{X}^n_t|^2 \leq |Y_t - \hat{Y}_t|^2 - 2\int_0^t (Y_t - \hat{Y}_t - Y_s + \hat{Y}_s, dK^n_s - d\hat{K}^n_s), \quad t \in \mathbb{R}^+.$$ 

The rest of the proof runs as in the proof of Lemma 13. □

Theorem 20 Let $\{X^n\}$ be the solution of (3). Under the assumptions of Theorem 16 the following assertions hold:

(i) for any stopping time $\tau$ such that $\mathbb{P}(\tau < +\infty) = 1,$

$$X^n_\tau \xrightarrow{p} \bar{X}_\tau = X_\tau + \Delta H_\tau + \langle f(X_\tau), \Delta Z_\tau \rangle,$$

and, in particular, $X^n_\tau \xrightarrow{p} X_\tau$ provided that $\mathbb{P}(\Delta H_\tau = \Delta Z_\tau = 0) = 1$.

(ii) for any $T \in \mathbb{R}^+$

$$||J_n(X^n) - X||_T \xrightarrow{p} 0,$$

where $(X, K)$ is a solution of the SDE (1) with $\Pi = \Pi_{\tilde{D}(A)}$.

Proof. Set

$$Y_t = H_t + \int_0^t \langle f(X_s), dZ_s \rangle, \quad t \in \mathbb{R}^+.$$ 

Let $\hat{X}^n = \mathcal{Y}^n(A^n; Y)$, $n \in \mathbb{N}^*$. By Theorem [16, Theorem 29], (j) and (jjj), for any stopping time $\tau$ such that $\mathbb{P}(\tau < +\infty) = 1,$

$$\hat{X}^n_\tau \xrightarrow{p} \bar{X}_\tau = X_\tau + \Delta H_\tau + \langle f(X_\tau), \Delta Z_\tau \rangle \quad (17)$$

and

$$\hat{X}^n_{t-} \xrightarrow{p} X_{t-}, \quad t \in \mathbb{R}^+. \quad (18)$$

In order to check that

$$||\hat{X}^n - X^n||_T \xrightarrow{p} 0, \quad T \in \mathbb{R}^+,$$

without loss of the generality, we will assume that there is a constant $c > 0$ such that $|Z_t| \leq c$. Using notation from the proof of Theorem 16, $Z = M + V$, where $M$ is a local square–integrable martingale with $|\Delta M| \leq 4c$ and $V$ is a predictable process with locally bounded variation with $|\Delta V| \leq 2c$, $M_0 = V_0 = 0$. For $b > 0$ let us denote

$$\tau^b_n = \inf\{t > 0 : \max(\|M\|_t, \langle M \rangle_t, \|V\|^\uparrow_t, ||\hat{X}^n_t||, |X_t|) > b\}.$$ 

By Theorem [16, Theorem 30]–(i), for any $T \in \mathbb{R}^+$, the family $\{||\hat{X}^n||_T\}$ is bounded in probability, which implies that

$$\lim_{b \to \infty} \limsup_{n \to \infty} \mathbb{P}(\tau^b_n \leq T) = 0, \quad T \in \mathbb{R}^+. \quad (20)$$
By Lemma 19–(i), with \( p = 1 \), we see that, for any stopping time \( \sigma_n \),
\[
\mathbb{E} \sup_{t < \sigma_n \wedge \tau^n_b} |X^n_t - \hat{X}^n_t|^2 \leq C_1 \left[ \mathbb{E} \int_0^{(\sigma_n \wedge \tau^n_b)} \|f(X^n_{s-}) - f(X_{s-})\|^2 d([M]_s + (M)_s) + b^2 \mathbb{E} \int_0^{(\sigma_n \wedge \tau^n_b)} \|f(X^n_{s-}) - f(X_{s-})\|^2 d \uparrow V_{s+} \right]
\leq 2C_1L^2 \left[ \sup_{u \leq s} |X^n_{u-} - \hat{X}^n_{u-}|^2 d([M]_s + < M >_s + b \uparrow V_{s+}) + \epsilon_n \right],
\]
where
\[
\epsilon_n = \int_0^{(\sigma_n \wedge \tau^n_b)} |\hat{X}^n_{s-} - X_{s-}|^2 d([M]_s + < M >_s + b \uparrow V_{s+}), \quad n \in \mathbb{N}^*.
\]
Hence by Gronwall's Lemma,
\[
\mathbb{E} \sup_{t < \tau^n_b} |X^n_t - \hat{X}^n_t|^2 \leq 2C_1L^2 \epsilon_n \exp \{2C_1L^2(2b + b^2)\}.
\]
Since (18) implies that \( \epsilon_n \to 0 \) and (19) follows from (20) and the proof of (i) is complete.

(ii) By Theorem [16, Theorem 29]–(jj)
\[
\|J_n(\hat{X}^n) - X\|_T \to 0, \quad T \in \mathbb{R}^+.
\]
By the Lipschitz property of operator \( J_n \) we deduce that
\[
\|J_n(X^n) - X\|_T \leq \|J_n(\hat{X}^n) - X\|_T + \|\hat{X}^n - X^n\|_T, \quad T \in \mathbb{R}^+.
\]
which, combined with (19) completes the proof.

Now, we consider SDE of the form
\[
X^n_t + K^n_t = H_t + \int_0^t \langle f(X^n_{s-}), dZ_s \rangle, \quad t \in \mathbb{R}^+.
\]
where
\[
K^n_t = \int_0^t A_n(X^n_s) ds - \sum_{s \leq t} [X^n_{s-} + \Delta Y^n_s - \Pi(X^n_{s-} + \Delta Y^n_s)] 1_{\{\max(|\Delta H_s|, |\Delta Z_s|) > 1/n\}}
\]
and let
\[
Y^n_t := H_t + \int_0^t \langle f(X^n_{s-}), dZ_s \rangle, \quad t \in \mathbb{R}^+.
\]
If we set \( \sigma_0 = 0 \) and
\[
\sigma_{k+1} = \inf\{t > \sigma_k : \max(|\Delta H_t|, |\Delta Z_t|) > 1/n\}, \quad k \in \mathbb{N},
\]
then, on every stochastic interval \([\sigma_k, \sigma_{k+1})\), \( X^n \) satisfies the equation
\[
X^n_t = \Pi(X^n_{\sigma_k-} + \Delta H_{\sigma_k} + \langle f(X^n_{\sigma_k-}), \Delta Z_{\sigma_k} \rangle) + H_t - H_{\sigma_k}
+ \int_{\sigma_k}^t \langle f(X^n_{s-}), dZ_s \rangle - \int_{\sigma_k}^t A_n(X^n_s) ds.
\]
Since \( f, A_n \) are Lipschitz continuous, it is well known that there exists a unique solution of (21) and, if \( Y := H_t + \int_0^t \langle f(X^n_s), dZ_s \rangle \) we shall denote \( X^n = \mathcal{P}(A_n, \Pi; Y) \).

Let \( Y, \hat{Y} \) be processes admitting decompositions (10). Arguing as in the proof of Lemma 19 we obtain the following result:

**Lemma 21** Let \( X^n = \mathcal{P}(A_n, \Pi; Y) \) and \( \hat{X}^n = (A_n, \Pi, \hat{Y}) \), \( n \in \mathbb{N}^* \). For any \( p \in \mathbb{N}^* \) there exists a constant \( C_p > 0 \) such that for any stopping time \( \tau \) and any \( n \in \mathbb{N}^* \) the estimates \((i - ii)\) from Lemma 19 hold true.

**Theorem 22** Let \( \{X^n\} \) be a sequence of solutions of (21). Under the assumptions of Theorem 16 we have

\[
\|X^n - X\|_T \longrightarrow 0, \quad T \in \mathbb{R}^+,
\]

where \( (X, K) \) is a solution of the SDE (1).

**Proof.** Let

\[
Y_t = H_t + \int_0^t \langle f(X^n_s), dZ_s \rangle, \quad t \in \mathbb{R}^+
\]

and let

\[
\hat{X}^n = \mathcal{P}(A_n, \Pi; Y), \quad n \in \mathbb{N}^*.
\]

By Theorem [16, Theorem 34]–(i)

\[
\|\hat{X}^n - X\|_T \longrightarrow 0, \quad \mathbb{P}\text{-a.s.}, \quad T \in \mathbb{R}^+.
\]

We remark now that similarly as in the proof of (19) from Theorem 20 (using Lemma 21–(ii) instead of Lemma 19–(ii)) it follows that

\[
\|\hat{X}^n - X\|_T \longrightarrow 0, \quad T \in \mathbb{R}^+,
\]

which completes the proof.

In the sequel we will replace assumption (H_3) with assumption (H_4) \( f : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \) is a continuous function such that

\[
\|f(x)\| \leq L(1 + |x|), \quad x \in \mathbb{R}^d
\]

and for any \( N \in \mathbb{N}^* \) there is \( K_N > 0 \) such that

\[
\|f(x) - f(y)\| \leq K_N|x - y|, \quad x, y \in B(0, N).
\]

**Corollary 23** Under the assumptions (H_1 – H_2) and (H_4), there exists a unique solution of the SDE (1).

**Proof.** Clearly, for any \( N \in \mathbb{N}^* \) there exists a Lipschitz continuous function \( f_N \) satisfying (23–24) and such that \( f_N(x) = f(x), x \in B(0, N) \) and \( f_N(x) = 0, x \in B^c(0, N + 1) \). By Theorem 16, for any \( N \in \mathbb{N}^* \), there exists a unique strong solution of the equation of the form

\[
X^n_t + K^n_t = H_t + \int_0^t \langle f_N(X^n_s), dZ_s \rangle, \quad t \in \mathbb{R}^+.
\]
Set $\gamma_0 = 0$ and 

$$
\gamma_N = \inf \{ t : |X_t^N| > N \}, \ N \in \mathbb{N}^*.
$$

Since $f_N(x) = f_{N+1}(x)$, $x \in B(0, N)$, $X_t^N = X_t^{N+1}$, $t \leq \gamma_N$ and $\gamma_N \leq \gamma_{N+1}$. In order to finish the proof it is sufficient to show that 

$$
\gamma_N \nearrow +\infty, \ \mathbb{P}\text{-a.s.} \quad (26)
$$

and to observe that the unique solution of (1) has the form 

$$
X_t = X_t^N, \ t \in [\gamma_{N-1}, \gamma_N), \ N \in \mathbb{N}^*.
$$

We assume w.r.g. that $|Z_t| \leq c$ for some constant $c > 0$. Then $|\Delta Z| \leq 2c$, $Z$ is a special semimartingale and admits a unique decomposition $Z_t = M_t + V_t$, $t \in \mathbb{R}^+$, where $M$ is a local square–integrable martingale and $V$ is a predictable process with locally bounded variation, such that $|\Delta M| \leq 4c$, $|\Delta V| \leq 2c$. Let $(X', K')$ denotes the solution of the Skorokhod problem with $Y' = H$.

Define 

$$
\tau_k = \inf \left\{ t : |X_t'| \lor |V_t| \lor [M]_t \lor \langle M \rangle_t > k \right\} \land k, \ k \in \mathbb{N}^*.
$$

It is clear that 

$$
\tau_k \nearrow +\infty, \ \mathbb{P}\text{-a.s.} \quad (27)
$$

By Lemma 13–(ii) with $p = 1$, and by (23) for every stopping time $\sigma$

$$
\mathbb{E} \sup_{t < \sigma \land \tau_k} |X_t^N - X_t'|^2 \leq C_1 \mathbb{E} \left[ \int_0^{(\sigma \land \tau_k)^-} ||f(X_s^N)||^2 d[M]_s + \int_0^{(\sigma \land \tau_k)^-} ||f(X_s^-)||^2 d\langle M \rangle_s 
\right.
\left. + k \int_0^{(\sigma \land \tau_k)^-} ||f(X_s^N)||^2 d\downarrow V\downarrow_s \right]
\leq C(k, L) \left[ 1 + \mathbb{E} \int_0^{(\sigma \land \tau_k)^-} \sup_{u \leq s} |X_u^N - X_u'|^2 d(\downarrow V\downarrow + [M] + \langle M \rangle)_s \right].
$$

Therefore for every stopping time $\sigma$

$$
\mathbb{E} \sup_{t < \sigma} |X_{t \land \tau_k}^N - X_{t \land \tau_k}'|^2 \leq C(k, L) \left[ 1 + \mathbb{E} \int_0^{\sigma^-} \sup_{u \leq s} |X_u^N - X_u'|^2 d(\downarrow V\downarrow + [M] + \langle M \rangle)_s \right].
$$

Consequently, by Gronwall’s lemma

$$
\mathbb{E} \sup_{t < \tau_k} |X_t^N - X_t'|^2 \leq C(k, L) \exp\{3k C(k, L)\},
$$

which implies that for any $k \in \mathbb{N}^*$

$$
\sup_N \mathbb{E} \sup_{t < \tau_k} |X_t^N|^2 \leq C'(k, L).
$$
Hence and by Chebyshev’s inequality

\[ P(\gamma_N < \tau_k) \leq P(\sup_{t<\tau_k} |X_N^t| \geq N) \leq \frac{C'(k, L)}{N^2} \to 0, \text{ as } N \to \infty \]

and using (27), condition (26) follows.

**Corollary 24** Under the assumptions \((H_1 - H_2)\) and \((H_4)\) the conclusions of Theorems 17, 20 and 22 hold true.

**Proof.** It is sufficient to define \(\gamma_N = \inf\{t : |X_t| > N\}, N \in \mathbb{N}^*\) and remark that from Theorems 17, 20 and 22, for any \(N \in \mathbb{N}^*\), one can deduce the convergence of approximating sequences on all sets \(\{T \leq \gamma_N\}, T \in \mathbb{R}^+\). Since \(\gamma_N \nearrow +\infty, P\text{-a.s.},\) the result follows.

We say that SDE (1) has a weak solution if there exists a probability space \((\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t), \hat{\mathbb{P}})\) and \(\hat{\mathcal{F}}_t\)-adapted processes \(\hat{H}, \hat{Z}\) and \((\hat{X}, \hat{K})\) such that \(L(\hat{H}, \hat{Z}) = L(H, Z)\) and \((\hat{X}, \hat{K})\) is a solution of the Skorokhod problem associated with \(\hat{Y}_t = \hat{H}_t + \int_0^t f(\hat{X}_s) d\hat{Z}_s, t \in \mathbb{R}^+\).

**Lemma 25** Let \(\{Y^n\}\) be a given sequence of processes such that \(Y^n_0 \in D(A), n \in \mathbb{N}^*\) and let \(\{(X^n, K^n)\}\) be a sequence of solutions of the Skorokhod problem associated with \(\{Y^n\}\).

(i) For any sequence of processes \(\{Z^n\}, \{H^n\}\) if

\[ \{(Y^n, H^n, Z^n)\} \text{ is tight in } D(\mathbb{R}^+, \mathbb{R}^{2d}) \]

then

\[ \{(X^n, Y^n, H^n, Z^n)\} \text{ is tight in } D(\mathbb{R}^+, \mathbb{R}^{4d}). \]

(ii) For any sequences of processes \(\{Z^n\}, \{H^n\}\), if

\[ (Y^n, H^n, Z^n) \longrightarrow_D (Y, H, Z) \text{ in } D(\mathbb{R}^+, \mathbb{R}^{3d}) \]

then

\[ (X^n, Y^n, H^n, Z^n) \longrightarrow_D (X, Y, H, Z) \text{ in } D(\mathbb{R}^+, \mathbb{R}^{4d}), \]

where \((X, K)\) is a solution of the Skorokhod problem associated with a process \(Y\).

**Proof.** In the proof it suffices to combine the deterministic results given in Theorem [16, Theorem 34] with the Skorokhod representation theorem.

**Theorem 26** Under the assumptions \((H_1 - H_2)\), if \(f\) is continuous and satisfies (23) then there exists a weak solution \((X, K)\) of the SDE (1).
Proof. It is well known that one can construct a sequence \( \{f^n\} \) of functions such that \( f^n \in C^2 \) and satisfies (23), \( n \in \mathbb{N}^* \) and
\[
\sup_{x \in K} \|f^n(x) - f(x)\| \to 0,
\]
for any compact subset \( K \subset \mathbb{R}^d \).

By Corollary 23, for any \( n \in \mathbb{N}^* \), there exists a unique strong solution of the equation of the form
\[
X^n_t = H_t + \int_0^t \langle f^n(X^n_{s-}), dZ_s \rangle - K^n_t, \quad t \in \mathbb{R}^+.
\]
Let \( \gamma^N_n = \inf \{ t : |X^n_t| > N \}, n, N \in \mathbb{N}^* \). Similarly to the proof of (27) we check that
\[
\lim_{N \to \infty} \limsup_{n \to \infty} \mathbb{P}(\gamma^n_{\gamma^n_N} \leq T) = 0, \quad T \in \mathbb{R}^+,
\]
which is equivalent to the fact that for every \( T \in \mathbb{R}^+ \), the family \( \{ \sup_{t \leq T} |X^n_t| \} \) is bounded in probability.

By (23) also for every \( T \in \mathbb{R}^+ \)
\[
\left\{ \sup_{t \leq T} |f(X^n_t)| \right\} \text{ is bounded in probability.}
\]
Note that for every \( Z \) there is a unique decomposition \( Z_t = J_t + M_t + V_t, t \in \mathbb{R}^+ \), where
\[
J_t = \sum_{s \leq t} \Delta Z_s 1_{\{ |\Delta Z_s| > 1 \}}, \quad t \in \mathbb{R}^+,
\]
\( M \) is a local square-integrable martingale and \( V \) is a predictable process with locally bounded variation, such that \( |\Delta M| \leq 2, |\Delta V| \leq 1 \) and \( M_0 = V_0 = 0 \).

Moreover, \( M = M^c + M^d \), where \( M^c \) denotes the continuous martingale part of \( Z \) and \( \Delta M^d = \Delta M \). It is well known that there exists a sequence \( \{M^i\} \) of locally square integrable martingales with locally bounded variation with \( |\Delta M^i| \leq 2, i \in \mathbb{N}^* \) and such that
\[
|M^i - M^d|_{T \to \infty} \to 0, \quad T \in \mathbb{R}^+.
\]
Hence for any \( i \in \mathbb{N}^* \), \( Z \) admits the decomposition
\[
Z_t = (J_t + V_t + M^i_t) + M^c_t + (M^d_t - M^i_t), \quad t \in \mathbb{R}^+,
\]
where \( W^i = J + V + M^i \) is a process with locally bounded variation. By (29) for every \( i \in \mathbb{N}^* \)
\[
\left\{ \left( \int_0^t \langle f^n(X^n_{s-}), dW^i \rangle, H, Z \right) \right\} \text{ is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})
\]
and
\[
\left\{ \left( \int_0^t \langle f^n(X^n_{s-}), dM^c_s \rangle, H, Z \right) \right\} \text{ is tight in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}),
\]
where, in addition,
\[
\Delta \int_0^t \langle f^n(X^n_{s-}), dM^c_s \rangle = 0, \quad n \in \mathbb{N}^*.
\]
Moreover,
\[
\sup_{t \leq T} \left| \int_0^t \langle f^n(X^n_s), d(M^d - M^i)_s \rangle \right| \to 0, \quad T \in \mathbb{R}^+.
\]

Therefore, by [9, Chapter VI, Lemma 3.32] we also deduce that
\[
\left\{ \left( \int_0^t \langle f^n(X^n_s), dZ_s \rangle, H, Z \right) \right\}
\]
is tight in \( \mathcal{D}(\mathbb{R}^+, \mathbb{R}^{3d}) \).

Hence and by Lemma 25–(i)
\[
\left\{ (X^n, H, Z) \right\}
\]
is tight in \( \mathcal{D}(\mathbb{R}^+, \mathbb{R}^{3d}) \).

Assume that there exists a subsequence \( \{n'\} \subset \{n\} \) such that \((X^{n'}, H^{n'}, Z^{n'}) \xrightarrow{D} (\hat{X}, \hat{H}, \hat{Z})\) in \( \mathcal{D}(\mathbb{R}^+, \mathbb{R}^{3d}) \), where \( \mathcal{L}(\hat{H}, \hat{Z}) = \mathcal{L}(H, Z) \). Then
\[
(X^{n'}, \int_0^t \langle f^{n'}(X^{n'}_s), dZ_s \rangle, H^{n'}, Z^{n'}) \xrightarrow{D} (\hat{X}, \int_0^\tau \langle f(\hat{X}_s), d\hat{Z}_s \rangle, \hat{H}, \hat{Z}) \quad \text{in} \quad \mathcal{D}(\mathbb{R}^+, \mathbb{R}^{4d}).
\]

By Lemma 25–(ii), \( \hat{X} \) is a first coordinate of the solution of the Skorokhod problem associated with
\[
\hat{Y} = \hat{H} + \int_0^\tau \langle f(\hat{X}_s), d\hat{Z}_s \rangle,
\]
which implies that \( \hat{X} \) is a weak solution of the SDE (1).

References

[1] I. Asiminoaei, A. Răşcanu, Approximation and simulation of stochastic variational inequalities-splitting up method, Numer. Funct. Anal. Optim. 18 (1997), no. 3&4, 251–282.

[2] V. Barbu, A. Răşcanu, Parabolic variational inequalities with singular inputs, Differ. Integral. Equ. 10 (1997), no. 1, 67–83.

[3] A. Bensoussan, A. Răşcanu, Stochastic variational inequalities in infinite-dimensional spaces, Numer. Funct. Anal. Funct. Anal. Optim 18 (1997), no. 1&2, 19–54.

[4] H. Brezis, Opérateurs Maximaux Monotones, North Holland, Amsterdam, 1973.

[5] R. Buckdahn, L. Maticiuc, E. Pardoux, A. Răşcanu, Stochastic Variational Inequalities on Non-Convex Domains, preprint 2013.

[6] M. Chaleyat–Maurel, N. El–Karoui, B. Marchal, Reflexion discontinue et systemes stochastiques, Ann. Probab. 5 (1980), 1049–1067.

[7] E. Cépa, Problème de Skorohod multivoque, Ann. Probab. 26 (1998), no. 2, 500–532.

[8] S. Ethier, T. Kurtz, Markov Processes. Characterization and Convergence, John Wiley & Sons, New York, 1986.
[9] J. Jacod, A. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer Verlag, Berlin, 1987.

[10] A. Jakubowski, J. Mémin, G. Pages, *Convergence en loi des suites d’intégrales stochastiques sur l’espace $D^1$ de Skorokhod*, Probab. Th. Rel. Fields 81 (1989), 111–137.

[11] P.L. Lions, A.S. Sznitman, *Stochastic Differential Equations with Reflecting Boundary Conditions*, Comm. Pure and Appl. Math. XXXVII (1983), 511–537.

[12] W. Laukajtys, *On Stochastic Differential Equations with Reflecting Boundary Condition in Convex Domains*, Bull. Pol. Acad. Sci. Math. 52 (2004), no. 4, 445–455.

[13] W. Laukajtys, L. Słomiński, *Penalization methods for reflecting stochastic differential equations with jumps*, Stoch. Stoch. Rep. 75 (2003), no. 5, 275–293.

[14] W. Laukajtys, L. Słomiński, *Penalization methods for the Skorokhod problem and reflecting SDEs with jumps*, Bernoulli Bernoulli 19 (2013), no. 5A, 1750-1775.

[15] D. Lépingle, *Euler scheme for reflected stochastic differential equations*, Mathematics and Computers in Simulations 38 (1995), 119–126.

[16] L. Maticiuc, A. Răşcanu, L. Słomiński, M. Topolewski, *Càdlàg Skorokhod problem driven by a maximal monotone operator*, preprint, 2013 ([http://arxiv.org/abs/1306.1686](http://arxiv.org/abs/1306.1686)).

[17] J.L. Menaldi, *Stochastic variational inequality for reflected diffusion*, Indiana Univ. Math. J. 32 (1983), 733–744.

[18] M. Metivier, J. Pellaumail, *On a stopped Doob’s inequality and general stochastic equations*, Ann. Probab. 8 (1980), no. 1, 96–114.

[19] R. Pettersson, *Approximations for stochastic differential equations with reflecting convex boundaries*, Stochastic Process. Appl. 59 (1995), 295-308.

[20] R. Pettersson, *Projection scheme for stochastic differential equations with convex constraints*, Stochastic Process. Appl. 88 (2000), 125-134.

[21] M. Pratelli, *Majorations dans $L^p$ du type Metivier–Pellaumail pour les semimartingales*, Séminaire de Probabilités XVII 1981/82, Lect. Notes Math. 986 (1983) 125-131.

[22] P. Protter, *Stochastic Integration and Differential Equations* (2nd ed.), Springer-Verlag, Berlin Heidelberg, 2004.

[23] A. Răşcanu, *Deterministic and stochastic differential equations in Hilbert spaces involving multivalued maximal monotone operator*, PanAmerican Math. J. 6 (1996), no. 3, 83–119.

[24] A. Răşcanu, *Existence for a class of stochastic parabolic variational inequalities*, Stochastics 5 (1981), no. 3, 201–239.

[25] Y. Saisho, *Stochastic Differential Equations for Multidimensional Domain with Reflecting Boundary*, Probab. Theory Related Fields 74 (1987), no. 3, 455–477.

[26] A.V. Skorokhod, *Stochastic Equations for Diffusion Processes in a Bounded Region I*, Theory of Probability & Its Applications 6 (1961), no. 3, 264–274.
[27] A.V. Skorokhod, *Stochastic Equations for Diffusion Processes in a Bounded Region. II*, Theory of Probability & Its Applications 7 (1962), no. 1, 3–23.

[28] L. Słomiński, *On existence, uniqueness and stability of solutions of multidimensional SDE's with reflecting boundary conditions* Ann. Inst. H. Poincaré 29 (1993), no. 2, 163–198.

[29] L. Słomiński, *On approximation of solutions of multidimensional SDEs with reflecting boundary conditions*, Stochastic Process. Appl. 50 (1994), 197-219.

[30] L. Słomiński, *Stability of stochastic differential equations driven by general semimartingales*, Diss. Math. 349 (1996), 1–113.

[31] L. Słomiński, *Euler’s approximations of solutions of SDEs with reflecting boundary*, Stochastic Process. Appl. 94 (2001), 317–337.

[32] L. Słomiński, T Wojciechowski, *Stochastic differential equations with jump reflection at time-dependent barriers*, Stoch. Process. Appl. 120 (2010), 1701-1721.

[33] A. Storm, *Stochastic differential equations with convex constraint*, Stochastics Stochastics Rep. 53 (1995), 241–274.

[34] T. Tanaka, *Stochastic Differential Equations with Reflecting Boundary Condition in Convex Regions*, Hiroshima Math. J. 9 (1979), 163–177.