REVISITING \( T \)-NORMS FOR TYPE-2 FUZZY SETS

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Abstract. Let \( L \) be the set of all normal and convex functions from \([0,1]\) to \([0,1]\). This paper proves that \( t \)-norm in the sense of Walker-and-Walker is strictly stronger than \( t_r \)-norm on \( L \), which is strictly stronger than \( t \)-norm on \( L \). Furthermore, let \( \triangledown \) and \( \triangleright \) be special convolution operations defined by
\[
(f \triangledown g)(x) = \sup \{ f(y) \star g(z) : y \triangle z = x \},
\]
\[
(f \triangleright g)(x) = \sup \{ f(y) \star g(z) : y \triangledown z = x \},
\]
for \( f, g \in \text{Map}([0,1],[0,1]) \), where \( \triangle \) and \( \triangledown \) are respectively a \( t \)-norm and a \( t \)-conorm on \([0,1]\) (not necessarily continuous), and \( \star \) is a binary operation on \([0,1]\). Then, it is proved that if the binary operation \( \triangledown \) is a \( t_r \)-norm (resp., \( \triangleright \) is a \( t_r \)-conorm), then \( \triangle \) is a continuous \( t \)-norm (resp., \( \triangledown \) is a continuous \( t \)-conorm) on \([0,1]\), and \( \star \) is a \( t \)-norm on \([0,1]\).

1. Introduction

Type-2 fuzzy sets (T2FSs), which are originated from the work of Zadeh \[26\], extend the notions of type-1 fuzzy sets (T1FSs) and interval-valued fuzzy sets (IVFSs), and are equivalently expressed in different forms by Mendel et al. \[10, 13, 14, 12\]. Briefly speaking, the membership degrees of every element in a T1FS and an IVFS are respectively a point and a closed subinterval of the unite interval \( I = [0,1] \), while the membership degree of every element in a T2FS is a T1FS in \( I \). Because of the fuzzy properties of the membership degree of T2FSs, they are more suitable for describing uncertainty and complexity than T1FSs and IVFSs, and play an increasingly important role in applications \((1, 2, 3, 4, 9, 17, 18, 22)\).

Throughout this paper, let \( \text{Map}(X,Y) \) be the set of all mappings from \( X \) to \( Y \), and ‘\( \leq \)’ denote the usual order relation in the lattice of real numbers, \( M = \text{Map}(I, I) \). Let \( \triangledown \) and \( \wedge \) be the maximum and minimum operations, respectively, on a lattice.

Definition 1. \[25\] A type-1 fuzzy set \( A \) in space \( X \) is a mapping from \( X \) to \( I \), i.e., \( A \in \text{Map}(X,I) \).

Definition 2. \[20\] A fuzzy set \( A \in \text{Map}(X,I) \) is normal if \( \sup \{ A(x) : x \in I \} = 1 \).

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Definition 3. [20] A function \( f \in \mathbf{M} \) is convex if, for any \( 0 \leq x \leq y \leq z \leq 1 \), \( f(y) \geq f(x) \land f(z) \).

Let \( \mathbf{N} \) and \( \mathbf{L} \) denote the set of all normal functions in \( \mathbf{M} \) and the set of all normal and convex functions in \( \mathbf{M} \), respectively.

For any subset \( B \) of \( X \), a special fuzzy set \( \mathbf{1}_B \), called the characteristic function of \( B \), is defined by

\[
\mathbf{1}_B(x) = \begin{cases} 
1, & x \in B, \\
0, & x \in X \setminus B. 
\end{cases}
\]

Let \( J = \{ \mathbf{1}_{\{x\}} : x \in I \} \) and \( K = \{ \mathbf{1}_{[a,b]} : 0 \leq a \leq b \leq 1 \} \).

Definition 4. [20] A type-2 fuzzy set \( A \) in space \( X \) is a mapping \( A : X \rightarrow \mathbf{M} \), i.e., \( A \in \text{Map}(X, \mathbf{M}) \).

Triangular norm (t-norm) and triangular conorm (t-conorm) on the unit interval \( I \) were introduced by Menger [15] in 1942 and by Schweizer and Sklar [19] in 1961, respectively, extending the logic connective conjunction and disjunction in classical two-valued logic.

Definition 5. [11] A binary operation \( * : I^2 \rightarrow I \) is a t-norm on \( I \) if it satisfies the following axioms:

1. \((\text{commutativity})\) \( x * y = y * x \) for \( x, y \in I \);
2. \((\text{associativity})\) \( (x * y) * z = x * (y * z) \) for \( x, y, z \in I \);
3. \((\text{increasing})\) \( * \) is increasing in each argument;
4. \((\text{neutral element})\) \( 1 * x = x * 1 = x \) for \( x \in I \).

A binary operation \( * : I^2 \rightarrow I \) is a t-conorm on \( I \) if it satisfies axioms (T1), (T2), and (T3) above, and moreover axiom (T4‘): \( 0 * x = x * 0 = x \) for \( x \in I \).

In 2006, Walker and Walker [21] extended t-norm and t-conorm on \( I \) to the algebra of truth values on T2FSs and IVFSs (see Definition 8 below). Then, Hernández et al. [8] modified the definition of Walker and Walker, and introduced the notions of \( t_r \)-norm and \( t_r \)-conorm by adding some “restrictive axioms” (see Definition 8 below).

Definition 6. [20] The operations of \( \sqcup \) (union), \( \sqcap \) (intersection), \( \neg \) (complementation) on \( \mathbf{M} \) are defined as follows: for \( f, g \in \mathbf{M} \),

\[
(f \sqcup g)(x) = \sup \{ f(y) \land g(z) : y \lor z = x \},
\]

\[
(f \sqcap g)(x) = \sup \{ f(y) \land g(z) : y \land z = x \},
\]

and

\[
(\neg f)(x) = \sup \{ f(y) : 1 - y = x \} = f(1 - x).
\]

From [20], it follows that \( \mathbb{M} = (\mathbf{M}, \sqcup, \sqcap, \neg, \mathbf{1}_{\{0\}}, \mathbf{1}_{\{1\}}) \) does not have a lattice structure, although \( \sqcup \) and \( \sqcap \) satisfy the De Morgan’s laws with respect to the complementation \( \neg \).

Walker and Walker [20] introduced the following partial orders \( \subseteq \) and \( \preceq \) on \( \mathbf{M} \).

Definition 7. [20] \( f \sqsubseteq g \) if \( f \sqcap g = f \); \( f \preceq g \) if \( f \sqcup g = g \).
It follows from [20, Proposition 14] that both \( \sqsubseteq \) and \( \preceq \) are partial orders on \( M \). Generally, the partial orders \( \sqsubseteq \) and \( \preceq \) do not coincide. In [20, 7, 6, 16], it was proved that \( \sqsubseteq \) and \( \preceq \) coincide on \( L \), and the subalgebra \( L = (L, \sqcup, \sqcap, \neg, 1_{\{0\}}, 1_{\{1\}}) \) is a bounded complete lattice. In particular, \( 1_{\{0\}} \) and \( 1_{\{1\}} \) are the minimum and maximum of \( L \), respectively. For systematic study on the truth value algebra of T2FSs, one is referred to [5].

**Definition 8.** [8, 21] A binary operation \( T : L^2 \to L \) is a \( t_r \)-norm (\( t \)-norm according to the restrictive axioms), if

1. \( T \) is commutative, i.e., \( T(f, g) = T(g, f) \) for \( f, g \in L \);
2. \( T \) is associative, i.e., \( T(T(f, g), h) = T(f, T(g, h)) \) for \( f, g, h \in L \);
3. \( T(f, 1_{\{1\}}) = f \) for \( f \in L \) (neutral element);
4. \( T \) for \( f, g, h \in L \) such that \( f \sqsubseteq g \), \( T(f, h) \sqsubseteq T(g, h) \) (increasing in each argument);
5. \( T(1_{\{0,1\}}, 1_{\{a,b\}}) = 1_{\{0\}} \); \( T \) is closed on \( J \);
6. \( T \) is closed on \( K \).

A binary operation \( S : L^2 \to L \) is a \( t_r \)-conorm if it satisfies axioms (O1), (O2), (O4), (O6), and (O7) above, axiom (O3'): \( S(f, 1_{\{0\}}) = f \), and axiom (O5'): \( S(1_{\{0,1\}}, 1_{\{a,b\}}) = 1_{\{a,1\}} \). Axioms (O1), (O2), (O3), (O3'), and (O4) are called “basic axioms”, and an operation that complies with these axioms will be referred to as \( t \)-norm and \( t \)-conorm, respectively.

According to Walker and Walker [21], a binary operation \( R : L^2 \to L \) is a \( t \)-norm in the sense of Walker-and-Walker (\( t \)-norm for short) if it satisfies axioms (O1), (O2), (O3), (O5), (O6), and (O7) above, axiom (O4'): \( R(f, g \sqcup h) = R(f, g) \sqcup R(f, h) \), and axiom (O4''): \( R(f, g \sqcap h) = R(f, g) \sqcap R(f, h) \).

**Definition 9.** [5, Definition 1.3.3] Let \( \circ \) and \( \triangleright \) be two binary operations defined on \( X \) and \( Y \), respectively, and \( \triangledown \) be an appropriate operation on \( Y \). If \( \circ \) is a surjection, define a binary operation \( \bullet \) on the set \( Map(X, Y) \) by

\[
(f \bullet g)(x) = \triangledown \{ f(y) \triangleright g(z) : y \circ z = x \}.
\]

This rule for defining a binary operation on \( Map(X, Y) \) is called *convolution*.

**Definition 10.** [8] Let \( \star \) be a binary operation on \( I \), \( \triangle \) be a \( t \)-norm on \( I \), and \( \triangledown \) be a \( t \)-conorm on \( I \). Define the binary operations \( \wedge \) and \( \vee : M^2 \to M \) as follows: for \( f, g \in M \),

\[
(f \wedge g)(x) = \sup \{ f(y) \star g(z) : y \triangle z = x \}, \tag{1.1}
\]

and

\[
(f \vee g)(x) = \sup \{ f(y) \star g(z) : y \triangledown z = x \}. \tag{1.2}
\]

Recently, Hernández et al. [8] proved that the binary operation \( \wedge \) (resp., \( \vee \)) is a \( t_r \)-norm (resp., a \( t_r \)-conorm) on \( L \), if \( \triangle \) is a continuous \( t \)-norm on \( I \) (resp., \( \triangledown \) is a continuous \( t \)-conorm on \( I \)) and \( \star \) is a continuous \( t \)-norm on \( I \). We [24] answered negatively an open problem posed by Hernández et al. [8], proving that the binary operation \( \ast \), which ensures that \( \wedge \) is a \( t_r \)-norm on \( L \) or \( \vee \) is a \( t_r \)-conorm on \( L \), is a \( t \)-norm, provided that \( \triangle \) and \( \triangledown \) are a continuous \( t \)-norm and a continuous \( t \)-conorm respectively. Then, we [23] constructed a \( t_r \)-norm and a \( t_r \)-conorm.
on \( L \), which cannot be obtained by the formulas that define the operations ‘\( \land \)' and ‘\( \vee \)’.

Inspired by the above research progress, in this paper we further study \( t \)-norms for T2FSs. We first obtain the implication relations among three notions of \( t \)-norms for T2FSs, proving that \( t \)-norm in the sense of Walker-and-Walker is strictly stronger than \( t_r \)-norm on \( L \), which is strictly stronger than \( t \)-norm on \( L \). Moreover, we characterize the restrictive axioms (O5), (O5’), and (O6) for the binary operation \( \land \) and \( \lor \). In particular, we prove that if the binary operation \( \land \) is a \( t_r \)-norm (resp., \( \lor \) is a \( t_r \)-conorm) on \( L \), then \( \triangle \) is a continuous \( t \)-norm (resp., \( \nabla \) is a continuous \( t \)-conorm) on \( I \), and \( \star \) is a \( t \)-norm on \( I \).

2. \( t_w \)-Norm is Strictly Stronger than \( t_r \)-Norm

This section reveals the relation between \( t_w \)-norm and \( t_r \)-norm on \( L \). In particular, it is shown that \( t_w \)-norm is strictly stronger than \( t_r \)-norm on \( L \), by constructing a \( t_w \)-norm which is not a \( t_r \)-norm.

Definition 11. For \( f \in M \), define

\[
\begin{align*}
    f^L(x) &= \sup \{ f(y) : y \leq x \}, \\
    f^{I_w}(x) &= \begin{cases} 
        \sup \{ f(y) : y < x \}, & x \in (0, 1], \\
        f(0), & x = 0,
    \end{cases}
\end{align*}
\]

and

\[
\begin{align*}
    f^R(x) &= \sup \{ f(y) : y \geq x \}, \\
    f^{I_w}(x) &= \begin{cases} 
        \sup \{ f(y) : y > x \}, & x \in [0, 1), \\
        f(1), & x = 1.
    \end{cases}
\end{align*}
\]

Clearly, (1) \( f^L \), \( f^{I_w} \) and \( f^R \), \( f^{I_w} \) are monotonically increasing and decreasing, respectively; (2) \( f^L(x) \lor f^R(x) = f^L(x) \lor f^{I_w}(x) = \sup_{z \in I} \{ f(z) \} \) and \( f^R(x) \lor f^{I_w}(x) = \sup_{z \in I} \{ f(z) \} \) for all \( x \in I \). The following properties of \( f^L \) and \( f^R \) are obtained by Walker et al. [20, 7, 6].

Proposition 12. [20] For \( f, g \in M \),

1. \( f \leq f^L \land f^R \);
2. \( (f^L)^L = f^L \), \( (f^R)^R = f^R \);
3. \( (f^L)^R = (f^R)^L = \sup_{x \in I} \{ f(x) \} \);
4. \( f \subseteq g \) if and only if \( f^R \land g \leq f \leq g^R \);
5. \( f \preceq g \) if and only if \( f \land g^L \leq g \leq f^L \);
6. \( f \) is convex if and only if \( f = f^L \land f^R \).

Theorem 13. [7, 6] Let \( f, g \in L \). Then, \( f \subseteq g \) if and only if \( g^L \leq f^L \) and \( f^R \leq g^R \).

Lemma 14. [23, Lemma 17] For \( f \in N \), \( \inf \{ x \in I : f^L(x) = 1 \} \leq \sup \{ x \in I : f^R(x) = 1 \} \).

Definition 15. [23, Definition 27] Define a binary operation \( \star : L^2 \to M \) as follows: for \( f, g \in L \),

1. \( f = 1_{(1)}, f \star g = g \star f = g \);
(2) \( g = 1_{\{1\}} \), \( f \star g = g \star f = f \);
(3) \( f \neq 1_{\{1\}} \) and \( g \neq 1_{\{1\}} \),

\[
(f \star g)(t) = \begin{cases}
  f^L(t) \lor g^L(t), & t \in [0, \eta), \\
  1, & t \in [\eta, \xi), \\
  f^R(\xi) \land g^R(\xi), & t = \xi, \\
  0, & t \in (\xi, 1],
\end{cases}
\]  

(2.1)

where \( \eta = \inf\{x \in I : f^L(x) = 1\} \land \inf\{x \in I : g^L(x) = 1\} \) and \( \xi = \sup\{x \in I : f^R(x) = 1\} \land \sup\{x \in I : g^R(x) = 1\} \).

Clearly, \( f \star g \) is increasing on \([0, \eta)\). Meanwhile, applying Lemma 14 yields that \( \eta \leq \xi \).

**Theorem 16.** [23, Theorem 34] The binary operation \( \star \) is a \( t_r \)-norm on \( L \).

**Lemma 17.** [20, Corollary 5, Proposition 6] For \( f, g \in L \),

1. \( (f \cap g)^L = f^L \lor g^L \) and \( (f \cap g)^R = f^R \land g^R \);
2. \( (f \sqcap g)^L = f^L \land g^L \) and \( (f \sqcap g)^R = f^R \lor g^R \).

**Theorem 18.** [23, Theorem 30] For \( f, g \in L \setminus \{1_{\{1\}}\} \),

\[
(f \star g)^L(t) = \begin{cases}
  f^L(t) \lor g^L(t), & t \in [0, \eta), \\
  1, & t \in [\eta, 1],
\end{cases}
\]

(2.2)

\[
(f \star g)^R(t) = \begin{cases}
  1, & t \in [0, \xi), \\
  f^R(\xi) \land g^R(\xi), & t = \xi, \\
  0, & t \in (\xi, 1],
\end{cases}
\]

(2.3)

where \( \eta = \inf\{x \in I : f^L(x) = 1\} \land \inf\{x \in I : g^L(x) = 1\} \), and \( \xi = \sup\{x \in I : f^R(x) = 1\} \land \sup\{x \in I : g^R(x) = 1\} \).

**Proposition 19.** [23, Proposition 31] For \( f, g \in L \setminus \{1_{\{1\}}\} \), \( f \star g \neq 1_{\{1\}} \).

**Lemma 20.** [8, Proposition 14]

1. \( \sqcap \) is a \( t_r \)-norm on \( L \);
2. \( \sqcup \) is a \( t_r \)-conorm on \( L \).

The following theorem shows that the binary operation \( \star \) satisfies the distributive law for the binary operation \( \sqcap \).

**Theorem 21.** For \( f, g, h \in L \), \( f \star (g \cap h) = (f \star g) \cap (f \star h) \).

**Proof.** Consider the following two cases:

Case 1. If one of \( f, g, \) and \( h \) is equal to \( 1_{\{1\}} \), it is clear that \( f \star (g \cap h) = (f \star g) \cap (f \star h) \);
Case 2. If none of \( f, g, \) and \( h \) are equal to \( 1_{\{1\}} \), from Theorem 28 and Proposition 19, it follows that

\[
(f \star g)^L(t) = \begin{cases} 
    f^L(t) \lor g^L(t), & t \in [0, \eta_1), \\
    1, & t \in [\eta_1, 1],
\end{cases}
\]

\hspace{1cm}

\[
(f \star g)^R(t) = \begin{cases} 
    f^R(\xi_1) \land g^R(\xi_1), & t = \xi_1, \\
    0, & t \in (\xi_1, 1],
\end{cases}
\]

where \( \eta_1 = \inf\{x \in I : f^L(x) = 1\} \land \inf\{x \in I : g^L(x) = 1\} \) and \( \xi_1 = \sup\{x \in I : f^R(x) = 1\} \land \sup\{x \in I : g^R(x) = 1\} \); and

\[
(f \star h)^L(t) = \begin{cases} 
    f^L(t) \lor h^L(t), & t \in [0, \eta_2), \\
    1, & t \in [\eta_2, 1],
\end{cases}
\]

\hspace{1cm}

\[
(f \star h)^R(t) = \begin{cases} 
    f^R(\xi_2) \land h^R(\xi_2), & t = \xi_2, \\
    0, & t \in (\xi_2, 1],
\end{cases}
\]

where \( \eta_2 = \inf\{x \in I : f^L(x) = 1\} \land \inf\{x \in I : h^L(x) = 1\} \) and \( \xi_2 = \sup\{x \in I : f^R(x) = 1\} \land \sup\{x \in I : h^R(x) = 1\} \). This, together with Lemma 17, implies that

\[
((f \star g) \cap (f \star h))^L(t) = (f \star g)^L \lor (f \star h)^L
\]

\hspace{1cm}

\[
= \begin{cases} 
    f^L(t) \lor g^L(t) \lor h^L(t), & t \in [0, \eta_1 \land \eta_2), \\
    1, & t \in [\eta_1 \land \eta_2, 1].
\end{cases}
\]

Claim 1. \( (f \star g)^R(\xi_1 \land \xi_2) \land (f \star h)^R(\xi_1 \land \xi_2) = f^R(\xi_1 \land \xi_2) \land g^R(\xi_1 \land \xi_2) \land h^R(\xi_1 \land \xi_2) \).

Case 2-1. If \( \xi_1 = \xi_2 \), then \( (f \star g)^R(\xi_1 \land \xi_2) \land (f \star h)^R(\xi_1 \land \xi_2) = f^R(\xi_1) \land g^R(\xi_1) \land f^R(\xi_2) \land h^R(\xi_2) = f^R(\xi_1) \land g^R(\xi_1) \land h^R(\xi_1) \).

Case 2-2. If \( \xi_1 < \xi_2 \), then \( (f \star g)^R(\xi_1) = f^R(\xi_1) \land g^R(\xi_1) \) and \( (f \star h)^R(\xi_1) = 1 \), implying that

\[
(f \star g)^R(\xi_1) \land (f \star h)^R(\xi_1) = f^R(\xi_1) \land g^R(\xi_1).
\]

From \( \xi_1 < \xi_2 \leq \sup\{x \in I : h^R(x) = 1\} \), it follows that there exists \( \xi_1 < \hat{x} \leq \sup\{x \in I : h^R(x) = 1\} \) such that \( h^R(\hat{x}) = 1 \). Thus,

\[
h^R(\xi_1) \geq h^R(\hat{x}) = 1.
\]

This, together with (2.9), implies that

\[
(f \star g)^R(\xi_1) \land (f \star h)^R(\xi_1) = f^R(\xi_1) \land g^R(\xi_1) \land h^R(\xi_1).
\]

Case 2-3. If \( \xi_2 < \xi_1 \), similarly to the proof of Case 2-2, it can be verified that

\[
(f \star g)^R(\xi_2) \land (f \star h)^R(\xi_2) = f^R(\xi_2) \land g^R(\xi_2) \land h^R(\xi_2).
\]
Combining Lemma 17, (2.5), (2.7), and Claim 1 yields that

\[(f \star g) \cap (f \star h) = (f \star g)^R(t) \wedge (f \star h)^R(t) = \begin{cases} 1, & t \in [0, \xi_1 \wedge \xi_2), \\ f^R(t) \wedge g^R(t) \wedge h^R(t), & t = \xi_1 \wedge \xi_2, \\ 0, & t \in (\xi_1 \wedge \xi_2, 1]. \end{cases} \quad (2.10)\]

**Claim 2.** \(\inf \{x \in I : (g \cap h)^L(x) = 1\} = \inf \{x \in I : g^L(x) = 1\} \wedge \inf \{x \in I : h^L(x) = 1\} \). 

It is clear that \(\{x \in I : g^L(x) = 1\} \cup \{x \in I : h^L(x) = 1\} = \{x \in I : (g^L \cup h^L)(x) = 1\} \). Applying Lemma 17 yields that 

\(\{x \in I : (g \cap h)^L(x) = 1\} = \{x \in I : (g^L \cap h^L)(x) = 1\} \).

Thus,

\[\inf \{x \in I : (g \cap h)^L(x) = 1\} = \inf \{(x \in I : g^L(x) = 1\} \cup \{x \in I : h^L(x) = 1\}\) 

\[= \inf \{x \in I : g^L(x) = 1\} \wedge \inf \{x \in I : h^L(x) = 1\}. \]

**Claim 3.** \(\sup \{x \in I : (g \cap h)^R(x) = 1\} = \sup \{x \in I : g^R(x) = 1\} \wedge \sup \{x \in I : h^R(x) = 1\}. \)

From Lemma 17, it follows that \(\{x \in I : (g \cap h)^R(x) = 1\} = \{x \in I : (g^R \cap h^R)(x) = 1\} = \{x \in I : g^R(x) = 1\} \cap \{x \in I : h^R(x) = 1\} \). This implies that 

\[\sup \{x \in I : (g \cap h)^R(x) = 1\} \leq \sup \{x \in I : g^R(x) = 1\} \wedge \sup \{x \in I : h^R(x) = 1\}. \quad (2.11)\]

Since \(g^R\) and \(h^R\) are decreasing, one has 

\[\{x \in I : g^R(x) = 1\} \supset \{0, \sup \{x \in I : g^R(x) = 1\}\}, \]

and 

\[\{x \in I : h^R(x) = 1\} \supset \{0, \sup \{x \in I : h^R(x) = 1\}\}. \]

Thus, \(\{x \in I : (g \cap h)^R(x) = 1\} \supset \{0, \sup \{x \in I : g^R(x) = 1\} \wedge \sup \{x \in I : h^R(x) = 1\}\}\), implying that \(\sup \{x \in I : (g \cap h)^R(x) = 1\} \leq \sup \{x \in I : g^R(x) = 1\} \wedge \sup \{x \in I : h^R(x) = 1\}\). This, together with (2.11), implies that \(\sup \{x \in I : (g \cap h)^R(x) = 1\} = \sup \{x \in I : g^R(x) = 1\} \wedge \sup \{x \in I : h^R(x) = 1\}\).

Applying Claims 2–3 leads to that \(\inf \{x \in I : f^L(x) = 1\} \wedge \inf \{x \in I : (g \cap h)^L(x) = 1\} = \inf \{x \in I : f^L(x) = 1\} \wedge \inf \{x \in I : g^L(x) = 1\} \wedge \inf \{x \in I : h^L(x) = 1\} = \eta_1 \wedge \eta_2\) and \(\sup \{x \in I : f^R(x) = 1\} \wedge \sup \{x \in I : (g \cap h)^R(x) = 1\} = \sup \{x \in I : f^R(x) = 1\} \wedge \sup \{x \in I : g^R(x) = 1\} \wedge \sup \{x \in I : h^R(x) = 1\} = \xi_1 \wedge \xi_2\). This, together with Theorem 28 and Proposition 19, implies that

\[(f \star (g \cap h))^L(t) = \begin{cases} f^L(t) \lor (g \cap h)^L(t), & t \in [0, \eta_1 \wedge \eta_2), \\ 1, & t \in [\eta_1 \wedge \eta_2, 1]. \end{cases}\]
and

\[
(f \star (g \sqcap h))^R(t) = \begin{cases} 
1, & t \in [0, \xi_1 \land \xi_2), \\
f^R(t) \land (g \sqcap h)^R(t), & t = \xi_1 \land \xi_2, \\
0, & t \in (\xi_1 \land \xi_2, 1].
\end{cases}
\]

Combining this with Lemma 17, (2.8), and (2.10) yields that

\[
(f \star (g \sqcap h))^L = ((f \star g) \sqcap (f \star h))^L,
\]

and

\[
(f \star (g \sqcap h))^R = ((f \star g) \sqcap (f \star h))^R.
\]

Therefore,

\[
f \star (g \sqcap h) = (f \star g) \sqcap (f \star h).
\]

□

Lemma 22. [20, Theorem 4] Let \( f, g \in M \). Then,

1. \( f \sqcup g = (f \lor g) \land (f^L \land g^L) \),
2. \( f \sqcap g = (f \lor g) \land (f^R \land g^R) \).

The following theorem shows that the binary operation \( \star \) does not satisfy the distributive law for the binary operation \( \sqcup \).

Theorem 23. There exist \( f, g, h \in L \) such that \( f \star (g \sqcup h) \neq (f \star g) \sqcup (f \star h) \).

Proof. Choose respectively \( f, g, h \in L \) as follows:

\[
f(x) = 1_{[0, 0.75)}(x), \quad x \in [0, 1],
g(x) = \begin{cases} 
0, & x \in [0, 0.5], \\
2(1 - x), & x \in (0.5, 1],
\end{cases}
h(x) = x, \quad x \in [0, 1].
\]
(i) It can be verified that

\[
\begin{align*}
    f^L(x) &= \begin{cases} 
    0, & x \in [0, 0.75), \\
    1, & x \in [0.75, 1], 
    \end{cases} \\
    f^R(x) &= \begin{cases} 
    1, & x \in [0, 0.75], \\
    0, & x \in (0.75, 1], 
    \end{cases} \\
    h^L(x) &= x, \ x \in [0, 1], \\
    h^R &\equiv 1, \\
    g^L(x) &= \begin{cases} 
    0, & x \in [0, 0.5], \\
    1, & x \in (0.5, 1], 
    \end{cases} \\
    g^R &= \begin{cases} 
    1, & x \in [0, 0.5], \\
    2(1 - x), & x \in (0.5, 1], 
    \end{cases} \\
    (g \lor h)(x) &= \begin{cases} 
    x, & x \in [0, 0.5], \\
    2(1 - x), & x \in [0.5, 2/3], \\
    x, & x \in (2/3, 1]. 
    \end{cases}
\end{align*}
\]

These, together with (2.1), imply that

\[
\begin{align*}
    (f \star g)(x) &= \begin{cases} 
    0, & x \in [0, 0.5), \\
    1, & x = 0.5, \\
    0, & x \in (0.5, 1], 
    \end{cases} \\
    (f \star h)(x) &= \begin{cases} 
    x, & x \in [0, 0.75), \\
    1, & x = 0.75, \\
    0, & x \in (0.75, 1]. 
    \end{cases}
\end{align*}
\]

Combining this with Lemma 22, it follows that

\[
((f \star g) \sqcup (f \star h))(0.5) = (1 \lor 0.5) \land (1 \land 0.5) = 0.5. \tag{2.12}
\]

(ii) Applying Lemma 22 yields that

\[
\begin{align*}
    (g \sqcup h)(x) &= ((g \lor h) \land (g^L \land h^L))(x) \\
    &= \begin{cases} 
    0, & x \in [0, 0.5], \\
    x, & x \in (0.5, 1]. 
    \end{cases} \tag{2.13}
\end{align*}
\]

Then,

\[
\begin{align*}
    (g \sqcup h)^L(x) &= \begin{cases} 
    0, & x \in [0, 0.5], \\
    x, & x \in (0.5, 1], 
    \end{cases} \\
    (g \sqcup h)^R(x) &\equiv 1.
\end{align*}
\]
(iii) Applying (2.1) and (2.13) yields that
\[
(f \star (g \sqcup h))(x) = \begin{cases} 
0, & x \in [0, 0.5], \\
x, & x \in (0.5, 0.75), \\
1, & x = 0.75, \\
0, & x \in (0.75, 1]. 
\end{cases}
\]

In particular, \((f \star (g \sqcup h))(0.5) = 0\). This, together with (2.12), implies that
\[f \star (g \sqcup h) \neq (f \star g) \sqcup (f \star h).\]

Theorem 24. Let \(T\) be a \(t_w\)-norm on \(L\). Then, it is a \(t_r\)-norm on \(L\).

Proof. It suffices to check that axiom (O4”) implies axiom (O4). For \(f, g, h \in L\) with \(f \sqsubseteq g\), \(T(f, h) \sqcap T(g, h) = T(f \sqcap g, h) = T(f, h)\). This, together with Definition 7, implies that \(T(f, h) \sqsubseteq T(g, h)\). \(\square\)

Remark 25. Applying Theorems 16 and 23 yields that \(\star\) is a \(t_r\)-norm but not a \(t_w\)-norm on \(L\). This, together with Theorem 24, implies that \(t_w\)-norm is strictly stronger than \(t_r\)-norm.

3. \(t_r\)-Norm is Strictly Stronger than \(t\)-Norm

Clearly, \(t_r\)-norm is stronger than \(t\)-norm. This section gives an example to show that this is strict.

Definition 26. Define a binary operation \(\blacklozenge : L^2 \to M\) as follows: for \(f, g \in L\),

(1) \(f = 1_{\{1\}}, f \blacklozenge g = g \blacklozenge f = g\);
(2) \(g = 1_{\{1\}}, f \blacklozenge g = g \blacklozenge f = f\);
(3) \(f \neq 1_{\{1\}}\) and \(g \neq 1_{\{1\}}\),

\[
(f \blacklozenge g)(t) = \begin{cases} 
1, & t \in [0, \xi), \\
f^R(\xi) \land g^R(\xi), & t = \xi, \\
0, & t \in (\xi, 1], 
\end{cases}
\]  
\(\text{(3.1)}\)

where \(\xi = \sup\{x \in I : f^R(x) = 1\} \land \sup\{x \in I : g^R(x) = 1\}\).

Proposition 27. \(\blacklozenge (L^2) \subset L\).

Theorem 28. For \(f, g \in L \setminus \{1_{\{1\}}\}\),

\[
(f \blacklozenge g)^L(t) = 1, 
\]  
\(\text{(3.2)}\)

\[
(f \blacklozenge g)^R(t) = \begin{cases} 
1, & t \in [0, \xi), \\
f^R(\xi) \land g^R(\xi), & t = \xi, \\
0, & t \in (\xi, 1], 
\end{cases}
\]  
\(\text{(3.3)}\)

where \(\xi = \sup\{x \in I : f^R(x) = 1\} \land \sup\{x \in I : g^R(x) = 1\}\).

Proposition 29. The binary operation \(\blacklozenge\) does not satisfy (O6).
Proof. For \( x_1, x_2 \in (0, 1) \), form (3.1), it follows that
\[
(1_{\{x_1\}} \diamond 1_{\{x_2\}})(t) = \begin{cases} 
1, & x \in [0, x_1 \land x_2], \\
0, & x \in (x_1 \land x_2, 1], 
\end{cases}
\]

\[
= 1_{[0, x_1 \land x_2]}(t) \in J.
\]
This implies that \( \diamond \) does not satisfy (O6). \(\square\)

Similarly to the proofs of A–D in [24], applying Theorem 28 and Proposition 29 leads to the following result.

**Theorem 30.** The binary operation \( \diamond \) is a t-norm but not a \( t_\ast \)-norm on \( L \).

4. Some Further Results on the Binary Operations \( \wedge \) and \( \gamma \)

The following lemma, originated from [11, Proposition 1.19], shows that the continuity in one component is sufficient for the continuity of t-norms.

**Lemma 31.** [11, Proposition 1.19] A binary operation \( T : I^2 \rightarrow I \), which satisfies (T3), is continuous if and only if it is continuous in each component, i.e., for all \( x_0, y_0 \in I \), both the vertical section \( T(x_0, \cdot) : I \rightarrow I \) and the horizontal section \( T(\cdot, y_0) : I \rightarrow I \) are continuous functions in one variable.

**Proposition 32.** (1) Let \( \ast \) be a t-norm on \( I \). Then, \( x \ast y = 1 \) if and only if \( x = y = 1 \).

(2) Let \( \ast \) be a \( t_\ast \)-conorm on \( I \). Then, \( x \ast y = 0 \) if and only if \( x = y = 0 \).

**Proposition 33.** Let \( \triangledown \) be a t-norm on \( I \) and \( \ast \) be a binary operation on \( I \) satisfying that \( 0 \ast 0 = 0 \ast 1 = 1 \ast 0 = 0 \). If \( \wedge \) satisfies axiom (O6), then, for \( x_1, x_2 \in I \), one has \( 1 \ast 1 = 1 \) and \( 1_{\{x_1\}} \wedge 1_{\{x_2\}} = 1_{\{x_1 \land x_2\}} \).

**Proof.** From \( 0 \ast 0 = 0 \ast 1 = 1 \ast 0 = 0 \), it follows that

(a) for \( y, z \in I \), \( 1_{\{x_1\}}(y) \ast 1_{\{x_2\}}(z) \in \{0, 1 \ast 1\} \);

(b) \( 1_{\{x_1\}}(y) \ast 1_{\{x_2\}}(z) = 1 \ast 1 \) if and only if \( y = x_1 \) and \( z = x_2 \).

This, together with

\[
(1_{\{x_1\}} \wedge 1_{\{x_2\}})(x) = \text{sup} \{1_{\{x_1\}}(y) \ast 1_{\{x_2\}}(z) : y \triangledown z = x\},
\]

implies that

\[
(1_{\{x_1\}} \wedge 1_{\{x_2\}})(x) = \begin{cases} 
1 \ast 1, & x = x_1 \triangledown x_2, \\
0, & x \in I \setminus \{x_1 \triangledown x_2\}.
\end{cases}
\]

Since \( \wedge \) satisfies the axiom (O6), one has \( 1 \ast 1 = 1 \) and

\[
1_{\{x_1\}} \wedge 1_{\{x_2\}} = 1_{\{x_1 \land x_2\}}.
\]

\(\square\)

**Theorem 34.** Let \( \ast \) be a binary operation on \( I \) and \( \triangledown \) be a t-norm on \( I \). Then, the following statements are equivalent:

(1) \( \wedge \) satisfies axiom (O5);

(2) \( 1 \ast 0 = 0, 1 \ast 1 = 1, \) and the function \( \triangledown(\cdot, b) \) is continuous for all \( b \in I \);

(3) \( 1 \ast 0 = 0, 1 \ast 1 = 1, \) and \( \triangledown \) is a continuous t-norm.
Proof. (1) $\implies$ (2).

(a) $1 \star 0 = 0$.

Suppose, on the contrary, that $1 \star 0 > 0$, and fix a closed interval $[0, 0.5] \subset I$. For $z \in [0, 1] \setminus [0, 0.5] = (0.5, 1]$, since $1 \triangle z = z$, it follows from the definition of $\triangle$ that

$$(1_{[0,1]} \land 1_{[0,0.5]})(z) \geq 1_{[0,1]}(1) \star 1_{[0,0.5]}(z) = 1 \star 0 > 0.$$ 

Since $\land$ satisfies axiom (O5), one has $1_{[0,0.5]}(z) = (1_{[0,1]} \land 1_{[0,0.5]})(z) > 0$. This, together with $1_{[0,0.5]}(z) \in \{0, 1\}$, implies that $1_{[0,0.5]}(z) = 1$. This means that $z \in (0, 0.5]$, which contradicts with $z \in (0.5, 1]$.

(b) $1 \star 1 = 1$.

Clearly, $1 = 1_{[0,0.5]}(0) = (1_{[0,1]} \land 1_{[0,0.5]})(0) \in \{0, 1 \star 1 \} = \{0, 1 \star 1 \}$. This implies that $1 \star 1 = 1$.

(c) $\triangle(\_b)$ is continuous for all $b \in I$.

Clearly, both $\triangle(\_0)$ and $\triangle(\_1)$ are continuous. For $b \in I$, since $1_{[0,1]}(y) = 1$ and $1_{(y)}(z) \in \{0, 1\}$ for $y, z \in I$, it follows from (b) that, for $y \in I$,

$$(1_{[0,1]} \land 1_{[b,b]})(y \triangle b) = 1.$$ 

(4.1)

For $x \in I \setminus \{y \triangle b : y \in I\}$, if $y \triangle z = x$, then $z \neq b$. This, together with (a), implies that

$$(1_{[0,1]} \land 1_{[b,b]})(x) = \sup \{1_{[0,1]}(y) \star 1_{[b,b]}(z) : y \triangle z = x\} = 0.$$ 

(4.2)

Since $\land$ satisfies axiom (O5), applying (4.1) and (4.2) yields that

$$1_{[0, b]} = 1_{[0,1]} \land 1_{[b,b]} = 1_{\{y \triangle z : y \in I, z = b\}},$$

i.e.,

$$[0, b] = \{y \triangle z : y \in I, z = b\} = I \triangle b.$$ 

(4.3)

Claim 4. $\triangle(\_b)$ is continuous for all $b \in (0, 1)$.

Suppose, on the contrary, that $\triangle(\_b)$ is not continuous for some $b \in (0, 1)$. Then, there exists $z \in (0, 1)$ such that $\triangle(\_b)$ is not continuous at $z$.

Case 1. $z = 0$. Since $\triangle(\_b)$ is increasing, the right-limit of $\triangle(\_b)$ at 0 exists. Let $\xi = \lim_{x \to 0^+} \triangle(x, b)$. Clearly, $\xi > \triangle(0, 0) = 0$, since $\triangle(x, b)$ is not continuous at 0. This, together with (4.3) and the fact that $\triangle(\_b)$ is increasing, implies that

$$[0, b] = I \triangle b$$

$$= \{0 \triangle b\} \cup \{y \triangle b : y \in (0, 1]\}$$

$$\subset \{0\} \cup [\xi, 1 \triangle b]$$

$$= \{0\} \cup [\xi, b]$$

$$\subset [0, b],$$

which is a contradiction.

Case 2. $z \in (0, 1)$. Since $\triangle(\_b)$ is increasing, both the left-limit and the right-limit of $\triangle(\_b)$ at $z$ exist. Let $\eta = \lim_{x \to z^-} \triangle(x, b)$ and $\xi = \lim_{x \to z^+} \triangle(x, b)$. Clearly, $\eta < \xi$, since $\triangle(x, b)$ is not continuous at $z$. This, together with (4.3) and the fact
that $\triangle(\cdot, b)$ is increasing, implies that

$$[0, b] = I \triangle b$$

$$= \{y \triangle b : y \in [0, z)\} \cup \{z \triangle b\} \cup \{y \triangle b : y \in (z, 1]\}$$

$$\subset [0 \triangle b, \eta] \cup \{z \triangle b\} \cup [\xi, 1 \triangle b]$$

$$= [0, \eta] \cup \{z \triangle b\} \cup [\xi, b]$$

$$\subset [0, b],$$

which is a contradiction.

Case 3. $z = 1$. Similarly to the proof of Case 1, it can be verified that this is true.

(2) $\Rightarrow$ (3). Since $\triangle$ is commutative and satisfies (T3), by applying Lemma 31, this can be verified immediately.

(3) $\Rightarrow$ (1). For a closed interval $[a, b] \subset I$, similarly to the proofs of (4.1) and (4.2), it follows from $1 \star 0 = 0$ and $1 \star 1 = 1$ that

$$1_{[0,1]} \land 1_{[a,b]} = 1_{\{y \in [0,1], z \in [a,b]\}} = 1_{I \triangle [a,b]}. \quad (4.4)$$

Since $\triangle$ is a continuous $t$-norm, it can be verified that

$$[0, b] = I \triangle b \subset I \triangle [a, b] \subset [0 \triangle a, 1 \triangle b] = [0, b].$$

This, together with (4.4), implies that

$$1_{[0,1]} \land 1_{[a,b]} = 1_{[0, b]}.$$

□

**Theorem 35.** Let $\star$ be a binary operation on $I$ and $\triangle$ be a $t$-norm on $I$. Then, the following statements are equivalent:

1. $\land$ satisfies axiom (O6);
2. $0 \star 0 = 0 \star 1 = 1 \star 0 = 0, 1 \star 1 = 1$;
3. for $x_1, x_2 \in I$, $1_{\{x_1\}} \land 1_{\{x_2\}} = 1_{\{x_1 \triangle x_2\}}$.

**Proof.** (2) $\Rightarrow$ (3). For $x_1, x_2 \in I$, from (2), it follows that $\{1_{\{x_1\}}(y), 1_{\{x_2\}}(z) : y, z \in I\} \subset \{0, 1\}$. This, together with (2), implies that

$$(1_{\{x_1\}} \land 1_{\{x_2\}})(x)$$

$$= \sup\{1_{\{x_1\}}(y) \star 1_{\{x_2\}}(z) : y \triangle z = x\}$$

$$= \begin{cases} 
0, & x \neq x_1 \triangle x_2, \\
1, & x = x_1 \triangle x_2,
\end{cases}$$

i.e.,

$$1_{\{x_1\}} \land 1_{\{x_2\}} = 1_{\{x_1 \triangle x_2\}}.$$

(3) $\Rightarrow$ (1). This holds trivially.

(1) $\Rightarrow$ (2). Fix $x_1 \in I$. It is clear that $1_{\{x_1\}} \land 1_{\{0.5\}} \in J$.

**Claim 5.** $0 \star 0 = 0$. 

Suppose, on the contrary, that \(0 \star 0 > 0\), and fix \(x_1 \in I\). From \(1_{\{x_1\}} \land 1_{\{0.5\}} \in J\), it follows that there exists \(a \in I\) such that \(1_{\{a\}} = 1_{\{x_1\}} \land 1_{\{0.5\}}\). This implies that, for \(x \in I \setminus \{a, x_1\}\),
\[
0 = 1_{\{a\}}(x) = (1_{\{x_1\}} \land 1_{\{0.5\}})(x) \\
= \sup\{1_{\{x_1\}}(y) \land 1_{\{0.5\}}(z) : y \triangle z = x\} \\
\geq 1_{\{x_1\}}(x) \land 1_{\{0.5\}}(1) \quad \text{(as } x \triangle 1 = x) \\
= 0 \star 0 > 0,
\]
which is a contradiction.

**Claim 6.** \(0 \star 1 = 0\).

Suppose, on the contrary, that \(0 \star 1 > 0\), and fix \(x_1 \in I\). From \(1_{\{x_1\}} \land 1_{\{1\}} \in J\), it follows that there exists \(a \in I\) such that \(1_{\{a\}} = 1_{\{x_1\}} \land 1_{\{1\}}\). This implies that, for \(x \in I \setminus \{a, x_1\}\),
\[
0 = 1_{\{a\}}(x) = (1_{\{x_1\}} \land 1_{\{1\}})(x) \\
= \sup\{1_{\{x_1\}}(y) \land 1_{\{1\}}(z) : y \triangle z = x\} \\
\geq 1_{\{x_1\}}(x) \land 1_{\{1\}}(1) \quad \text{(as } x \triangle 1 = x) \\
= 0 \star 1 > 0,
\]
which is a contradiction.

Similarly, the following claims can be verified.

**Claim 7.** \(1 \star 0 = 0\).

**Claim 8.** \(1 \star 1 = 1\).

Applying Proposition 33 and Claims 5-8, this holds trivially. \(\Box\)

**Corollary 36.** Let \(\star\) be a binary operation on \(I\) and \(\triangle\) be a \(t\)-norm on \(I\). If \(\land\) satisfies axioms (O5) and (O6), then it satisfies axiom (O7).

**Proof.** Take two closed intervals \([a_1, b_1]\), \([a_2, b_2]\) \(\subseteq I\). Applying Theorems 34 and 35, it can be verified that
\[
1_{[a_1, b_1]} \land 1_{[a_2, b_2]} = 1_{\{x \in [a_1, b_1] \cup [a_2, b_2]\}} = 1_{[a_1 \land a_2, b_1 \land b_2]}.
\]
Noting that \([a_1, b_1] \times [a_2, b_2]\) is a compact and connected subset of \(\mathbb{R}^2\) and \(\triangle\) is continuous, one has that \([a_1, b_1] \triangle [a_2, b_2]\) is a compact and connected subset of \(\mathbb{R}\). This, together with the increasing property of \(\triangle\) in each argument, implies that
\[
[a_1, b_1] \triangle [a_2, b_2] = [a_1 \triangle a_2, b_1 \triangle b_2].
\]
Therefore,
\[
1_{[a_1, b_1]} \land 1_{[a_2, b_2]} = 1_{[a_1 \triangle a_2, b_1 \triangle b_2]} \in K.
\]
\(\Box\)

**Lemma 37.** [24, Theorem 21] Let \(\triangle\) be a continuous \(t\)-norm on \(I\) and \(\star\) be a binary operation on \(I\). If \(\land\) is a \(t\)-norm on \(L\), then \(\star\) is a \(t\)-norm.

**Theorem 38.** [24, Theorem 22] Let \(\triangledown\) be a continuous \(t\)-conorm on \(I\) and \(\star\) be a binary operation on \(I\). If \(\triangledown\) is a \(t\)-conorm on \(L\), then \(\star\) is a \(t\)-norm.
Theorem 39. Let $\star$ be a binary operation on $I$ and $\triangle$ be a $t$-norm on $I$. If the binary operation $\land$ is a $t_r$-norm on $L$, then $\triangle$ is a continuous $t$-norm and $\star$ is a $t$-norm.

Proof. Since $\land$ is a $t_r$-norm on $L$, from Theorem 34, it follows that $\triangle$ is a continuous norm. This, together with Lemma 37, implies that $\star$ is a $t$-norm on $I$. $\square$

Some slight changes in the proofs of Proposition 33, Theorems 34, 35, and 39, and Corollary 36 lead to the following.

Proposition 40. Let $\triangledown$ be a $t$-conorm on $I$ and $\star$ be a binary operation on $I$ satisfying that $0 \star 0 = 0 \star 1 = 1 \star 0 = 0$. If $\wp$ satisfies axiom (O6), then, for $x_1, x_2 \in I$, one has $1 \star 1 = 1$ and $1_{\{x_1\}} \wp 1_{\{x_2\}} = 1_{\{x_1 \triangledown x_2\}}$.

Theorem 41. Let $\star$ be a binary operation on $I$ and $\triangledown$ be a $t$-conorm on $I$. Then, the following statements are equivalent:

1. $\wp$ satisfies axiom (O5');
2. $1 \star 0 = 0, 1 \star 1 = 1$, and the function $\triangledown(\_, b)$ is continuous for all $b \in I$;
3. $1 \star 0 = 0, 1 \star 1 = 1$, and $\triangledown$ is a continuous $t$-conorm.

Theorem 42. Let $\star$ be a binary operation on $I$ and $\triangledown$ be a $t$-conorm on $I$. Then, the following statements are equivalent:

1. $\wp$ satisfies axiom (O6);
2. $0 \star 0 = 0 \star 1 = 1 \star 0 = 0, 1 \star 1 = 1$;
3. For $x_1, x_2 \in I$, $1_{\{x_1\}} \wp 1_{\{x_2\}} = 1_{\{x_1 \triangledown x_2\}}$.

Corollary 43. Let $\star$ be a binary operation on $I$ and $\triangledown$ be a $t$-conorm on $I$. If $\wp$ satisfies axioms (O5') and (O6), then it satisfies axiom (O7).

Theorem 44. Let $\star$ be a binary operation on $I$ and $\triangledown$ be a $t$-conorm on $I$. If the binary operation $\wp$ is a $t_r$-conorm on $L$, then $\triangledown$ is a continuous $t$-conorm and $\star$ is a $t$-norm.

5. Conclusion

The notions of $t$-norm, $t_r$-norm, and $t_w$-norm on $L$ were introduced by Walker and Walker [21] in 2006 and modified by Hernández et al. [8]. This paper first shows that $t_w$-norm on $L$ is strictly stronger than $t_r$-norm on $L$, which is strictly stronger than $t$-norm on $L$. Moreover, some equivalent conditions on restrictive axioms (O5), (O5'), and (O6) for the binary operation $\land$ and $\wp$ are obtained. As corollaries, it is proved that

1. For the binary operation $\land$, axioms (O5) and (O6) imply axiom (O7);
2. For the binary operation $\wp$, axioms (O5') and (O6) imply axiom (O7).

Furthermore, it is proved that, if the binary operation $\land$ is a $t_r$-norm (resp., $\wp$ is a $t_r$-conorm) on $L$, then $\triangle$ is a continuous $t$-norm (resp., $\triangledown$ is a continuous $t$-conorm) on $I$, and $\star$ is a $t$-norm on $I$.

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