Yet again on polynomial convergence for SDEs with a gradient-type drift

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Abstract

Bounds on convergence rate to the invariant distribution for a class of stochastic differential equations (SDEs) are studied.

Key words: stochastic differential equation, invariant measure, convergence rate, gradient type drift.

1 Introduction

Let us consider a stochastic differential equation in $\mathbb{R}^d$

$$dX_t = dB_t - \nabla U(X_t) \, dt$$

with initial data

$$X_0 = x.$$ 

Here $B_t$, $t \geq 0$ is a $d$-dimensional Brownian motion, $X_t$ takes values in $\mathbb{R}^d$, $U$ is a non-negative function, $U(0) = 0$ and $\lim_{|x| \to \infty} U(x) = +\infty$. Function $U$ is assumed to be locally bounded and locally $C^1$. The aim of this paper is to establish ergodic properties of the Markov process $X_t$, namely, existence and uniqueness of its invariant probability measure, and to estimate convergence rate to the invariant measure which rate bound would not depend on the first derivatives of the function $U$. Such a problem — about bounds not depending explicitly on $\nabla U$ — was posed and in some

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particular case solved in [10]. Here we extend and relax some of the assumptions from [10]. It is widely known that the rate of convergence may be derived from the estimates of the type

\[ \mathbb{E}_x \tau^k \leq C(1 + |x|^m), \]  

\[ \sup_{t \geq 0} \mathbb{E}_x |X_t|^m \leq C(1 + |x|^{m'}), \]

for some \( k > 1, m, m', C > 0 \), where \( \tau = \inf(t \geq 0 : |X_t| \leq K) \) for some \( K > 0 \), see, e.g., [4, 8], et al. In particular, for SDEs (1) with a bounded \( \nabla U \) it can be derived from (3) and (4) that

\[ \text{var}(\mu_t^x - \mu_t^{inv}) \leq P(x)(1 + t)^{-k'}, \]

with any \( k' < k \) and some function \( P \) growing in \( x \) at infinity.

The bounds like (3) under various assumptions were obtained for various classes of processes by many authors, see, in particular, [1, 4, 5], [7] – [9] and the references therein; yet, for SDEs all assumptions were usually – except the paper [10] – stated in terms of \( \nabla U \). See also [3, 6] where stronger sub-exponential bounds were established under another standing assumption. In [8] and [9] a recurrence condition

\[ -p = \limsup (\nabla U(x), x) < 0 \]

was used to get bounds like (5). Here the goal is to use some analogue of the latter condition but in terms of the limiting behaviour of the function \( U \) itself, similar to [10] but under weaker assumptions.

# 2 Main results

## 2.1 Earlier results

Recall briefly some earlier results from [10] where, in fact, a little more general equation was considered. Assume

\[ \sup_{x, x', |x - x'| \leq 1} (U(x) - U(x')) < \infty \]

and let the structure of the function \( U \) be as follows:

\[ \text{var}(\mu_t^x - \mu_t^{inv}) \leq P(x)(1 + t)^{-k'}, \]
\[ U(x) = U^1(x) + U^2(x), \quad U^1(x) = V(|x|), \quad < U^2(x), x > \equiv 0. \] (7)

The function \( V \) here is assumed in the class \( C^1(0, \infty) \). In particular, the ”essential” divergent part \( U^1 \) of the drift has a central symmetry property while another divergent part \( U^2 \) is orthogonal to the direction \( x \) at any point \( x \). Let the following recurrence condition is satisfied,

\[ -\lim_{\xi \to \infty} \frac{V(\xi)}{\log \xi} + d = -p < 0. \] (8)

**Proposition 1** ([10]) Let (6)–(7) and (8) with \( p > 1/2 \) be satisfied. Then for any \( 0 < k < p + 1/2 \) and \( \varepsilon > 0 \) small enough the estimate (3) holds with \( m = 2k + \varepsilon \) and some \( C = C_\varepsilon \) and the estimate (2) is valid with any \( m < 2p - 1 \) and \( m' = m + 2\varepsilon \). Moreover, there exists a unique invariant measure for the Markov process \( X_t \).

**Proposition 2** ([10]) Let (6)–(7) and (8) with \( p > 1/2 \) be satisfied. Then the bound (5) holds true with any \( k' < k < p + 1/2 \) and \( \tilde{P}(x) = C_\varepsilon(1 + |x|^m), m = 2k + \varepsilon \) with any \( \varepsilon > 0 \) small enough and some \( C \). If, moreover, \( p > 3/2 \) then the bound (10) holds true with any \( k' < k < p - 1/2 \) and \( \tilde{P}(x) = C_\varepsilon(1 + |x|^m), m = 2k + \varepsilon \) with any \( \varepsilon > 0 \) small enough and some \( C \).

The assumption \( p > 3/2 \) relates to the critical value 3/2 in [6].

### 2.2 New results

Below \( [a] \) denotes the integer value of \( a \in \mathbb{R}^1 \).

**Theorem 1** Let there exist \( 1/2 < p_2 \leq p_1 \) such that

\[ 0 < p_2 \leq \frac{V(\xi)}{\log \xi} - d \leq p_1, \] (9)

for all \( \xi > 0 \) which are large enough by the absolute value. Then, the bound (3) holds true with \( m' = m + 2(p_1 - p_2) \) and \( m = 2k_1(1 + p_1 - p_2) \). Moreover, for any positive integer value of \( k < 1 + \frac{2p_2 - 1}{2(1 + p_1 - p_2)} \) and \( m = 2k_1(1 + p_1 - p_2) \), the bound (3) holds. Moreover there is a unique invariant probability measure \( \mu_{inv} \), and for any \( 0 < k' < k \), and for any \( t \geq 0 \),

\[ \text{var}(\mu^x_t - \mu_{inv}) \leq P(x)(1 + t)^{-k'}, \] (10)

with some polynomial function \( P(x) \).
Remark 1 Note that $k = 1$ is included in the range of values for which the bound \( b \) will be established. The assumption \( (7) \) may be replaced by a similar one with $\limsup_{|\xi| \to \infty}$ and $\liminf_{|\xi| \to \infty}$ instead of exact inequalities which may or may not change slightly the resulting statement depending on whether or not the value $\frac{2p_2 - 1}{2(1 + p_1 - p_2)}$ is integer. Also, depending on whether the same value is integer, the range of $k$ for which the bound \( b \) holds true may change a bit. We do not pursue the inspection of all these possible changes here. Let us mention that the assumption \( (6) \) is needed for the “local mixing” which explanation may be read in \( [10] \) in detail.

3 Proof

1. As in \( [10] \), due to comparison theorems for SDEs with reflection and the assumption on the structure of the drift one gets,

$$ |X_t| \leq y_t, $$

$$ dy_t = d\tilde{w}_t + \left( \frac{d}{y_t} - V'(y_t) \right) dt + d\varphi_t \equiv d\tilde{w}_t - \tilde{V}'(y_t) dt + d\varphi_t, \quad (11) $$

where $\tilde{w}$ is a 1-dimensional Wiener process, $y$ is a solution of the SDEs above with a non-sticky boundary condition at (any) point $K > 0$, $\varphi$ is its local time at $K$, $\tilde{V}'(y) = V'(y) - d/y$; in other words, we let

$$ \tilde{V}(y) = V(y) - d \ln y, \quad y > 0. $$

Condition \( (6) \) can be rewritten in the form

$$ \xi^{2p_2} \leq \exp(2\tilde{V}(\xi)) \leq \xi^{2p_1}, \quad \xi \geq K. $$

2. The invariant density of the process $\xi_t$ with $K = |x|$ has a form

$$ C(|x|) \exp \left( -2\tilde{V}(y) \right), \quad y > |x|. $$

The normalizing identity implies the estimation from above (under $2p_2 > 1$),

$$ C(|x|) \leq \left( \int_{|x|}^{\infty} \exp(-2\tilde{V}(y)) dy \right)^{-1} \leq \left( \int_{|x|}^{\infty} \xi^{-2p_2} dy \right)^{-1} = (2p_2 - 1)|x|^{2p_2 - 1}, $$

4
for the values of \(|x|\) large enough. For smaller values of \(|x|\), convergence of the integral cannot be destroyed because in some bounded neighbourhood of zero the function \(\exp(-2\bar{V}(y))\) is bounded. Note that for small values of \(|x|\) the expressions
\[
\left(\int_{|x|}^{\infty} \exp(-2\bar{V}(y) \, dy)\right)^{-1}
\]
are smaller, which means that in all cases for some \(C_0\),
\[
C(|x|) \leq (2p_2 - 1)|x|^{2p_2 - 1} \wedge C_0.
\]

3. The inequality \([4]\) with any real value \(m < 2p_2 - 1\) and with \(m' = m + 2(p_1 - p_2)\) (where \(m'\) may not be necessarily integer either) follows from a direct calculation,
\[
\mathbb{E}_x|X_t|^m \leq \mathbb{E}_x|y_t|^m \leq C(|x|) \int_{|x|}^{\infty} \xi^m \exp(-2\bar{V}(\xi)) \, d\xi
\]
\[
\leq (C|x|^{2p_1 - 1} \wedge C_0) \int_{|x|}^{\infty} \xi^{m-2p_2} \, d\xi \leq C|x|^{m+2(p_1 - p_2)}
\]
(here the constants \(C\) may be different on different lines and even on the same line), which is true for any \(x\) large enough, due to comparison theorems for the processes \(y_t\) with different initial data \(y_0\). For any \(x\) – not necessarily small – this implies the bound \([4]\), as required.

4. Denote \(v^q(\xi) = \mathbb{E}_x\gamma^q\) for any integer \(q \geq 0\), \(\gamma = \inf(t : y_t \leq K)\) and let \(L\) denote the generator of \(y_t\). By virtue of the identity
\[
\left(\int_{0}^{\gamma} 1 \, dt\right)^{q} = q \int_{0}^{\gamma} \left(\int_{s}^{\gamma} 1 \, ds\right)^{q-1} \, dt,
\]
it follows,
\[
v^q(y_0) = q\mathbb{E}_{y_0} \int_{0}^{\gamma} v^{q-1}(y_t) \, dt,
\]
for any \(q\) such that the integral in the right hand side converges. In turn, this implies an equation (for example, by Itô’s or Dynkin’s formula)
\[
Lv^q = -qv^{q-1}, \quad (q \geq 1)
\]
(cf. with \([2]\) theorem 13.17 where the equation is explained differently and under another stronger assumption). Evidently, one boundary value for the latter equation is \(v^q(K) = 0\). Concerning the “second boundary value” usual for a PDE of the second
order, it is seemingly missing here. The justification of the formula for solution
below can be done by the following limiting procedure. Let \( N > K \) be the second
boundary (later on \( N \) would go to infinity). Let \( \nu^\gamma_N(\xi) = \mathbb{E}\xi\gamma^\gamma_N \) for any integer \( q \geq 0 \),
\( \gamma_N = \inf(t: y_t^N \leq K) \), where the process \( y_t^N \) is a solution of the equation similar to (11) but with another non-sticky reflection at \( N \). Note that all solutions are strong
and, hence, may be constructed on the same probability space; see, e.g., [11] for SDEs
with one boundary, and results from this paper are easily extended on the case with
two finite boundaries. Apparently, \( y_t^N \leq y_t \) for any \( t \) and \( N \), and \( \gamma_N \uparrow \gamma \) as \( N \uparrow \infty \).
So, by the monotone convergence, \( \nu^\gamma_N \uparrow \nu^\gamma \) for all values of \( q \) (even if the limit \( \nu^q \)
is not finite). Then the sequence of the functions \( \nu^\gamma_N(\xi) \) satisfies the equations (12)
with boundary conditions
\[
\nu^\gamma_N(K) = 0, \quad (\nu^\gamma_N)'(N) = 0.
\]
The formula for solution of such an equation reads,
\[
\nu^\gamma_N(\xi) = 2q\int_K^\xi \exp(2\bar{V}(y_1)) \, dy_1 \int_{y_1}^N \nu^\gamma_{N-1}(y_2) \exp(-2\bar{V}(y_2)) \, dy_2, \quad K \leq \xi \leq N,
\]
which may be verified by a direct calculation. Hence, by induction, the function
\( \nu^q(\xi) \) is given by the formula via the function \( \nu^{q-1} \),
\[
\nu^q(\xi) = 2q\int_K^\xi \exp(2\bar{V}(y_1)) \, dy_1 \int_{y_1}^\infty \nu^{q-1}(y_2) \exp(-2\bar{V}(y_2)) \, dy_2. \tag{13}
\]
By another induction this implies the inequalities (assuming \( \nu^0 \equiv 1 \)):
\[
\nu^1(\xi) = 2\int_K^\xi \exp(2\bar{V}(y_1)) \, dy_1 \int_{y_1}^\infty \nu^0(y_2) \exp(-2\bar{V}(y_2)) \, dy_2
\]
\[
= 2\int_K^\xi \exp(2\bar{V}(y_1)) \, dy_1 \int_{y_1}^\infty \exp(-2\bar{V}(y_2)) \, dy_2 \leq 2\int_K^\xi y_1^{2p_1} \, dy_1 \int_{y_1}^\infty y_2^{-2p_2} \, dy_2
\]
\[
= C\int_K^\xi y_1^{2p_1-2p_2+1} \, dy_1 = C(\xi^{2(p_1-p_2)+2} - K^{2(p_1-p_2)+2}) \leq C\xi^{2(p_1-p_2)+2},
\]
under the condition that \( p_2 > 1/2 \) (otherwise the inner integral diverges). Further,
\[
\nu^2(\xi) = 4\int_K^\xi \exp(2\bar{V}(y_1)) \, dy_1 \int_{y_1}^\infty \nu^1(y_2) \exp(-2\bar{V}(y_2)) \, dy_2
\]
\[ \leq C \int_{\xi}^{\infty} \exp(2V(y_1)) \, dy_1 \int_{y_1}^{\infty} y_2^{2(p_1 - p_2) + 2} \exp(-2V(y_2)) \, dy_2 \]
\[ \leq C \int_{\xi}^{\infty} y_1^{2p_1} \, dy_1 \int_{y_1}^{\infty} y_2^{2(p_1 - p_2) + 2 - 2p_2} \, dy_2 \]
\[ = C \int_{\xi}^{\infty} y_1^{2p_1} \, dy_1 \int_{y_1}^{\infty} y_2^{4(p_1 - p_2) + 3} = C(\xi^{4(p_1 - p_2) + 4} - K^{4(p_1 - p_2) + 4}) \]
\[ \leq C\xi^{4(p_1 - p_2 + 1)}, \]

where in the calculus it was assumed that \( 2p_1 - 4p_2 + 2 < -1 \), that is, that \( p_1 < 2p_2 - 3/2 \), otherwise the inner integral in the calculus diverges. Since from the beginning \( p_1 \geq p_2 \), for the value of \( p_2 \) this means that compulsory \( p_2 > 3/2 \).

Next,
\[ v^3(\xi) = 6 \int_{K}^{\xi} \exp(2\bar{V}(y_1)) \, dy_1 \int_{y_1}^{\infty} v^2(y_2) \exp(-2\bar{V}(y_2)) \, dy_2 \]
\[ \leq C \int_{K}^{\xi} \exp(2\bar{V}(y_1)) \, dy_1 \int_{y_1}^{\infty} y_2^{4(p_1 - p_2 + 1)} \exp(-2\bar{V}(y_2)) \, dy_2 \]
\[ \leq C \int_{K}^{\xi} y_1^{2p_1} \, dy_1 \int_{y_1}^{\infty} y_2^{4(p_1 - p_2 + 1) - 2p_2} \, dy_2 \]
\[ = C \int_{K}^{\xi} y_1^{2p_1} \, dy_1 \int_{y_1}^{\infty} y_2^{4p_1 - 6p_2 + 5} = C(\xi^{6(p_1 - p_2 + 1)} - K^{6(p_1 - p_2 + 1)}) \leq C\xi^{6(p_1 - p_2 + 1)}. \]

For the inner integral to converge, the values of \( p_1, p_2 \) must satisfy \( 4p_1 - 6p_2 + 4 < -1 \), that is, \( p_1 < \frac{3}{2}p_2 - \frac{5}{4} \). Due to the condition \( p_1 \geq p_2 \), for \( p_2 \) this compulsory implies \( p_2 > \frac{5}{2} \). Note that, as usual, constants \( C \) may be different for any \( q \) and even from line to line. It looks plausible that the general formula – as long as the integrals converge – reads,
\[ v^q(\xi) \leq C_q\xi^{2q(1 + p_1 - p_2)}. \quad (14) \]

The base being already established, let us show the induction step. Assume that for \( q = n - 1 \) the formula is valid with some constant \( C_{n-1} \), that is,
\[ v^{n-1}(\xi) \leq C_{n-1}\xi^{2(n-1)(1 + p_1 - p_2)}. \]
Then for \( q = n \) (as long as the integrals in the calculus below converge) we have,

\[
v^n(\xi) = 2n \int_{K}^{\xi} \exp(2\bar{V}(y_1)) \, dy_1 \int_{y_1}^{\infty} v^{n-1}(y_2) \exp(-2\bar{V}(y_2)) \, dy_2 \\
\leq 2n \int_{K}^{\xi} y_1^{2p_1} \, dy_1 \int_{y_1}^{\infty} C_{n-1} y_2^{2(n-1)(p_1-p_2+1)} y_2^{-2p_2} \, dy_2 \\
= C_n \int_{K}^{\xi} y_1^{2p_1} y_1^{2n-1+2(n-1)p_1-2np_2} \, dy_1 = C_n \int_{K}^{\xi} y_1^{2n-1+2np_1-2np_2} \, dy_1 \leq C_n \xi^{2n(p_1-p_2+1)}.
\]

Hence, indeed, by induction the formula (14) is established. The values of \( q \) for which the integrals in the calculus converge must satisfy the bound

\[
2(q - 1)(1 + p_1 - p_2) - 2p_2 < -1,
\]
that is,

\[
q < q_0 := 1 + \frac{2p_2 - 1}{2(1 + p_1 - p_2)} = \frac{1 + 2p_1}{2(1 + p_1 - p_2)}.
\]

As a consequence, it is compulsory that \( p_2 > q - 1/2 \). Recall that in this paper only integer values of \( q \) are used; however, \( q_0 \) introduced above may not be necessarily integer, but in any case \( q_0 > 1 \). Also, note that if \( p_1 = p_2 = p \) as in [10], then the latter inequality \( q < q_0 \) reduces to \( q < p + 1/2 \), precisely as in [10].

5. By virtue of the established bounds (3)–(4), the bound (5) on convergence towards the stationary measure follows from various sources (cf., e.g., [9, 10], et al.) and, hence, in this brief presentation we skip the details of this step. The existence of the invariant probability measure may be justified via the Harris–Khasminsky principle based on (3) with any \( k \geq 1 \). Its uniqueness follows, for example, from the bound (4). This completes the proof of the Theorem 1.

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