Toeplitz operators on generalized Bergman spaces

Kamthorn Chailuek and Brian C. Hall

Abstract. We consider the weighted Bergman spaces $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$, where we set $d\mu_\lambda(z) = c_\lambda (1 - |z|^2)\lambda \, d\tau(z)$, with $\tau$ being the hyperbolic volume measure. These spaces are nonzero if and only if $\lambda > d$. For $0 < \lambda \leq d$, spaces with the same formula for the reproducing kernel can be defined using a Sobolev-type norm. We define Toeplitz operators on these generalized Bergman spaces and investigate their properties. Specifically, we describe classes of symbols for which the corresponding Toeplitz operators can be defined as bounded operators or as a Hilbert–Schmidt operators on the generalized Bergman spaces.

Mathematics Subject Classification (2000). Primary 47B35; Secondary 32A36, 81S10.

Keywords. Bergman space; Toeplitz operator; quantization; holomorphic Sobolev space; Berezin transform.

1. Introduction

1.1. Generalized Bergman spaces

Let $\mathbb{B}^d$ denote the (open) unit ball in $\mathbb{C}^d$ and let $\tau$ denote the hyperbolic volume measure on $\mathbb{B}^d$, given by

$$d\tau(z) = (1 - |z|^2)^{-(d+1)} \, dz,$$

where $dz$ denotes the 2d-dimensional Lebesgue measure. The measure $\tau$ is natural because it is invariant under all of the automorphisms (biholomorphic mappings) of $\mathbb{B}^d$. For $\lambda > 0$, let $\mu_\lambda$ denote the measure

$$d\mu_\lambda(z) = c_\lambda (1 - |z|^2)^\lambda \, d\tau(z),$$

(1.1)

Supported in part by a grant from Prince of Songkla University.
Supported in part by NSF Grant DMS-0555862.
where $c_\lambda$ is a positive constant whose value will be specified shortly. Finally, let $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ denote the (weighted) **Bergman space**, consisting of those holomorphic functions on $\mathbb{B}^d$ that are square-integrable with respect to $\mu_\lambda$. (Often these are defined using the Lebesgue measure as the reference measure, but all the formulas look nicer if we use the hyperbolic volume measure instead.) These spaces carry a projective unitary representation of the group $SU(d, 1)$.

If $\lambda > d$, then the measure $\mu_\lambda$ is finite, so that all bounded holomorphic functions are square-integrable. For $\lambda > d$, we choose $c_\lambda$ so that $\mu_\lambda$ is a probability measure. Calculation shows that

$$c_\lambda = \frac{\Gamma(\lambda)}{\pi^d \Gamma(\lambda - d)}, \quad \lambda > d.$$  \hfill (1.2)

(This differs from the value in Zhu’s book [Z2] by a factor of $\pi^d/d!$, because Zhu uses normalized Lebesgue whereas we use un-normalized Lebesgue measure in (1.1).) On the other hand, if $\lambda \leq d$, then $\mu_\lambda$ is an infinite measure. In this case, it is not hard to show that there are no nonzero holomorphic functions that are square-integrable with respect to $\mu_\lambda$ (no matter which nonzero value for $c_\lambda$ we choose).

Although the holomorphic $L^2$ space with respect to $\mu_\lambda$ is trivial (zero dimensional) when $\lambda \leq d$, there are indications that life does not end at $\lambda = d$. First, the reproducing kernel for $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ is given by

$$K_\lambda(z, w) = \frac{1}{(1 - z \cdot \bar{w})^\lambda}$$

for $\lambda > d$. The reproducing kernel is defined by the property that it is anti-holomorphic in $w$ and satisfies

$$\int_{\mathbb{B}^d} K_\lambda(z, w) f(w) \, d\mu_\lambda(w) = f(z)$$

for all $f \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$. Nothing unusual happens to $K_\lambda$ as $\lambda$ approaches $d$. In fact, $K_\lambda(z, w) := (1 - z \cdot \bar{w})^{-\lambda}$ is a “positive definite reproducing kernel” for all $\lambda > 0$. Thus, it is possible to define a reproducing kernel Hilbert space for all $\lambda > 0$ that agrees with $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ for $\lambda > d$.

Second, in representation theory, one is sometimes led to consider spaces like $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ but with $\lambda < d$. Consider, for example, the much-studied metaplectic representation of the connected double cover of $SU(1, 1) \cong Sp(1, \mathbb{R})$. This representation is a direct sum of two irreducible representations, one of which can be realized in the Bergman space $\mathcal{H}L^2(\mathbb{B}^1, \mu_{3/2})$ and the other of which can be realized in (a suitably defined version of) the Bergman space $\mathcal{H}L^2(\mathbb{B}^1, \mu_{1/2})$. To be precise, we can say that the second summand of the metaplectic representation is realized in a Hilbert space of holomorphic functions having $K_\lambda, \lambda = 1/2$, as its reproducing kernel. See [13] Sect. 4.6].

Last, one often wants to consider the infinite-dimensional ($d \to \infty$) limit of the spaces $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$. (See, for example, [25] and [23].) To do this, one wishes to embed each space $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ isometrically into a space of functions on $\mathbb{B}^{d+1}$,
as functions that are independent of $z_{n+1}$. It turns out that if one uses (as we do) hyperbolic volume measure as the reference measure, then the desired isometric embedding is achieved by embedding $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ into $\mathcal{H}L^2(\mathbb{B}^{d+1}, \mu_\lambda)$. That is, if we use the same value of $\lambda$ on $\mathbb{B}^{d+1}$ as on $\mathbb{B}^d$, then the norm of a function $f(z_1, \ldots, z_d)$ is the same whether we view it as a function on $\mathbb{B}^d$ or as a function on $\mathbb{B}^{d+1}$ that is independent of $z_{d+1}$. (See, for example, Theorem 4, where the inner product of $z^m$ with $z^n$ is independent of $d$.) If, however, we keep $\lambda$ constant as $d$ tends to infinity, then we will eventually violate the condition $\lambda > d$.

Although it is possible to describe the Bergman spaces for $\lambda \leq d$ as reproducing kernel Hilbert spaces, this is not the most convenient description for calculation. Instead, drawing on several inter-related results in the literature, we describe these spaces as “holomorphic Sobolev spaces,” also called Besov spaces. The inner product on these spaces, which we denote as $H(\mathbb{B}^d, \lambda)$, is an $L^2$ inner product involving both the functions and derivatives of the functions. For $\lambda > d$, $H(\mathbb{B}^d, \lambda)$ is identical to $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ (the same space of functions with the same inner product), but $H(\mathbb{B}^d, \lambda)$ is defined for all $\lambda > 0$.

It is worth mentioning that in the borderline case $\lambda = d$, the space $H(\mathbb{B}^d, \lambda)$ can be identified with the Hardy space of holomorphic functions that are square-integrable over the boundary. To see this, note that the normalization constant $c_\lambda$ tends to zero as $\lambda$ approaches $d$ from above. Thus, the measure of any compact subset of $\mathbb{B}^d$ tends to zero as $\lambda \to d^+$, meaning that most of the mass of $\mu_\lambda$ is concentrated near the boundary. As $\lambda \to d^+$, $\mu_\lambda$ converges, in the weak-* topology on $\mathbb{B}^d$, to the unique rotationally invariant probability measure on the boundary. Alternatively, we may observe that the formula for the inner product of monomials in $H(\mathbb{B}^d, d)$ (Theorem 4 with $\lambda = d$) is the same as in the Hardy space.

1.2. Toeplitz operators

One important aspect of Bergman spaces is the theory of Toeplitz operators on them. If $\phi$ is a bounded measurable function, the we can define the Toeplitz operator $T_\phi$ on $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ by $T_\phi f = P_\lambda(\phi f)$, where $P_\lambda$ is the orthogonal projection from $L^2(\mathbb{B}^d, \mu_\lambda)$ onto the holomorphic subspace. That is, $T_\phi$ consists of multiplying a holomorphic function by $\phi$, followed by projection back into the holomorphic subspace. Of course, $T_\phi$ depends on $\lambda$, but we suppress this dependence in the notation. The function $\phi$ is called the (Toeplitz) symbol of the operator $T_\phi$. The map sending $\phi$ to $T_\phi$ is known as the Berezin–Toeplitz quantization map and it (and various generalizations) have been much studied. See, for example, the early work of Berezin [5, 6], which was put into a general framework in [26, 27], along with [22, 8, 7, 10], to mention just a few works. The Berezin–Toeplitz quantization may be thought of as a generalization of the anti-Wick-ordered quantization on $\mathbb{C}^d$ (see [15]).

When $\lambda < d$, the inner product on $H(\mathbb{B}^d, \lambda)$ is not an $L^2$ inner product, and so the “multiply and project” definition of $T_\phi$ no longer makes sense. Our strategy is to find alternative formulas for computing $T_\phi$ in the case $\lambda > d$, with the hope that these formulas will continue to make sense (for certain classes of symbols $\phi$).
for $\lambda \leq d$. Specifically, we will identify classes of symbols $\phi$ for which $T_\phi$ can be defined as:

- A bounded operator on $H(\mathbb{B}^d, \lambda)$ (Section 4)
- A Hilbert–Schmidt operator on $H(\mathbb{B}^d, \lambda)$ (Section 5).

We also consider in Section 3 Toeplitz operators whose symbols are polynomials in $z$ and $\bar{z}$ and observe some unusual properties of such operators in the case $\lambda < d$.

1.3. Acknowledgments

The authors thank M. Engliš for pointing out to them several useful references and B. Driver for useful suggestions regarding the results in Section 4. This article is an expansion of the Ph.D. thesis of the first author, written under the supervision of the second author. We also thank the referee for helpful comments and corrections.

2. $H(\mathbb{B}^d, \lambda)$ as a holomorphic Sobolev space

In this section, we construct a Hilbert space of holomorphic functions on $\mathbb{B}^d$ with reproducing kernel $(1 - z \cdot \bar{w})^{-\lambda}$, for an arbitrary $\lambda > 0$. We denote this space as $H(\mathbb{B}^d, \lambda)$. The inner product on this space is an $L^2$ inner product with respect to the measure $\mu_{\lambda+2n}$, where $n$ is chosen so that $\lambda + 2n > d$. The inner product, however, involves not only the holomorphic functions but also their derivatives. That is, $H(\mathbb{B}^d, \lambda)$ is a sort of holomorphic Sobolev space (or Besov space) with respect to the measure $\mu_{\lambda+2n}$. When $\lambda > d$, our space is identical to $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$—not just the same space of functions, but also the same inner product. When $\lambda \leq d$, the Hilbert space $H(\mathbb{B}^d, \lambda)$, with the associated projective unitary action of $SU(d, 1)$, is sometimes referred to as the analytic continuation (with respect to $\lambda$) of the holomorphic discrete series.

Results in the same spirit as—and in some cases almost identical to—the results of this section have appeared in several earlier works, some of which treat arbitrary bounded symmetric domains and not just the ball in $\mathbb{C}^d$. For example, in the case of the unit ball in $\mathbb{C}^d$, Theorem 3.13 of [30] would presumably reduce to almost the same expression as in our Theorem 4, except that Yan has all the derivatives on one side, in which case the inner product has to be interpreted as a limit of integrals over a ball of radius $1 - \varepsilon$. (Compare the formula for $D_\lambda^{13}$ on p. 13 of [30] to the formula for $A$ and $B$ in Theorem 4.) See also [2, 4, 21, 31, 32].

Note, however, a number of these references give a construction that yields, for $\lambda > d$, the same space of functions as $\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)$ with a different but equivalent norm. Such an approach is not sufficient for our needs; we require the same inner product as well as the same space of functions.

Although our results in this section are not really new, we include proofs to make the paper self-contained and to get the precise form of the results that we want. The integration-by-parts argument we use also serves to prepare for our definition of Toeplitz operators on $H(\mathbb{B}^d, \lambda)$ in Section 4. We ourselves were introduced to this sort of reasoning by the treatment in Folland’s book [14] of the
disk model for the metaplectic representation. The paper \[16\] obtains results in the same spirit as those of this section, but in the context of a complex semisimple Lie group.

We begin by showing that for \( \lambda > d \), the space \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda) \) can be expressed as a subspace of \( \mathcal{H}L^2(\mathbb{B}^{d+2n}, \mu_{\lambda+2n}) \), with a Sobolev-type norm, for any positive integer \( n \). Let \( N \) denote the “number operator,” defined by

\[
N = \sum_{j=1}^d z_j \frac{\partial}{\partial z_j}.
\]

This operator satisfies \( Nz^m = |m|z^m \) for all multi-indices \( m \). If \( f \) is holomorphic, then \( Nf \) coincides with the “radial derivative” \( df(rz)/dr |_{r=1} \). We use also the operator \( \bar{N} = \sum_{j=1}^d \bar{z}_j \partial/\partial \bar{z}_j \).

A simple computation shows that

\[
(1 - |z|^2)^\alpha = \left( I - \frac{N}{\alpha + 1} \right) (1 - |z|^2)^{\alpha + 1} = \left( I - \frac{\bar{N}}{\alpha + 1} \right) (1 - |z|^2)^{\alpha + 1}.
\]  

We will use (2.1) and the following integration by parts result, which will also be used in Section 4.

**Lemma 1.** If \( \lambda > d \) and \( \psi \) is a continuously differentiable function for which \( \psi \) and \( N\psi \) are bounded, then

\[
c_\lambda \int_{\mathbb{B}^d} \psi(z)(1 - |z|^2)^{\lambda - d - 1} dz = c_{\lambda+1} \int_{\mathbb{B}^d} \left[ (I + \frac{N}{\lambda}) \psi \right] (z)(1 - |z|^2)^{\lambda - d} dz
\]

\[
= c_{\lambda+1} \int_{\mathbb{B}^d} \left[ (I + \frac{\bar{N}}{\lambda}) \psi \right] (z)(1 - |z|^2)^{\lambda - d} dz.
\]

Here \( dz \) is the 2d-dimensional Lebesgue measure on \( \mathbb{B}^d \).

**Proof.** We start by applying (2.1) and then think of the integral over \( \mathbb{B}^d \) as the limit as \( r \) approaches 1 of the integral over a ball of radius \( r < 1 \). On the ball of radius \( r \), we write out \( \partial/\partial z_j \) in terms of \( \partial/\partial x_j \) and \( \partial/\partial y_j \). For, say, the \( \partial/\partial x_j \) term we express the integral as a one-dimensional integral with respect to \( x_j \) (with limits of integration depending on the other variables) followed by an integral with respect to the other variables. We then use ordinary integration by parts in the \( x_j \) integral, and similarly for the \( \partial/\partial y_j \) term.

The integration by parts will yield a boundary term involving \( z_j \psi(z)(1 - |z|^2)^{\lambda - d} \); this boundary term will vanish as \( r \) tends to 1, because we assume \( \lambda > d \). In the nonboundary term, the operator \( N \) applied to \( (1 - |z|^2)^{\lambda - d} \) will turn into the operator \( -\sum_{j=1}^d \partial/\partial z_j \circ z_j = -(dI + N) \) applied to \( \psi \). Computing from (1.2) that \( c_{\lambda}/c_{\lambda+1} = (\lambda - d)/\lambda \), we may simplify and let \( r \) tend to 1 to obtain the desired result involving \( N \). The same reasoning gives the result involving \( \bar{N} \) as well. \( \square \)

We now state the key result, obtained from (2.1) and Lemma 1 relating the inner product in \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda) \) to the inner product in \( \mathcal{H}L^2(\mathbb{B}^{d+2n}, \mu_{\lambda+2n}) \) (compare \[14, p. 215\] in the case \( d = 1 \)).
Proposition 2. Suppose that \( \lambda > d \) and \( f \) and \( g \) are holomorphic functions on \( \mathbb{B}^d \) for which \( f, g, Nf, \) and \( Ng \) are all bounded. Then

\[
\langle f, g \rangle_{L^2(\mathbb{B}^d, \mu_\lambda)} = \left\langle f, \left( I + \frac{N}{\lambda}n \right)g \right\rangle_{L^2(\mathbb{B}^d, \mu_{\lambda+n})} = \left\langle \left( I + \frac{N}{\lambda}n \right) f, g \right\rangle_{L^2(\mathbb{B}^d, \mu_{\lambda+n})}.
\]  

(2.2)

Proof. Recalling the formula (1.1) for the measure \( \tau \), we apply Lemma 1 with \( \psi(z) = f(z)g(z) \) with \( f \) and \( g \) holomorphic. Observing that \( N(fg) = fNg \) gives the first equality and observing that \( \tilde{N}(fg) = (Nf)g \) gives the second equality. \( \square \)

Now, a general function in \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda) \) is not bounded. Indeed, the pointwise bounds on elements of \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda) \), coming from the reproducing kernel, are not sufficient to give a direct proof of the vanishing of the boundary terms in the integration by parts in Proposition 2. Nevertheless, (2.2) does hold for all \( f \) and \( g \) in \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda) \), provided that one interprets the inner product as the limit as \( r \) approaches 1 of integration over a ball of radius \( r \). (See [14, p. 215] or [30, Thm. 3.13].) We are going to iterate (2.2) to obtain an expression for the inner product on \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda) \) involving equal numbers of derivatives on \( f \) and \( g \). This leads to the following result.

Theorem 3. Fix \( \lambda > d \) and a non-negative integer \( n \). Then a holomorphic function \( f \) on \( \mathbb{B}^d \) belongs to \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda) \) if and only if \( N^l f \) belongs to \( \mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n}) \) for \( 0 \leq l \leq n \). Furthermore,

\[
\langle f, g \rangle_{\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)} = \langle Af,Bg \rangle_{\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})}
\]

for all \( f, g \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda) \), where

\[
A = \left( I + \frac{N}{\lambda+n} \right) \left( I + \frac{N}{\lambda+n+1} \right) \cdots \left( I + \frac{N}{\lambda+2n-1} \right)
\]

\[
B = \left( I + \frac{N}{\lambda} \right) \left( I + \frac{N}{\lambda+1} \right) \cdots \left( I + \frac{N}{\lambda+n-1} \right).
\]

Let us make a few remarks about this result before turning to the proof. Let \( \sigma = \lambda + 2n \). It is not hard to see that \( N^k f \) belongs to \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma) \) for \( 0 \leq k \leq n \) if and only if all the partial derivatives of \( f \) up to order \( n \) belong to \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma) \), so we may describe this condition as “\( f \) has \( n \) derivatives in \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma)\)” This condition then implies that \( f \) belongs to \( \mathcal{H}L^2(\mathbb{B}^d, \mu_{\sigma-2n}) \), which in turn means that \( f(z)/(1 - |z|^2)^n \) belongs to \( L^2(\mathbb{B}^d, \mu_\sigma) \). Since \( 1/(1 - |z|^2)^n \) blows up at the boundary of \( \mathbb{B}^d \), saying that \( f(z)/(1 - |z|^2)^n \) belongs to \( L^2(\mathbb{B}^d, \mu_\sigma) \) says that \( f(z) \) has better behavior at the boundary than a typical element of \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma) \). We may summarize this discussion by saying that each derivative that \( f \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma) \) has in \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma) \) results, roughly speaking, in an improvement by a factor of \( (1 - |z|^2)^n \) in the behavior of \( f \) near the boundary.

This improvement is also reflected in the pointwise bounds on \( f \) coming from the reproducing kernel. If \( f \) has \( n \) derivatives in \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\sigma) \), then \( f \) belongs to
\(\mathcal{H}L^{2}(\mathbb{B}^{d}, \mu_{\sigma-2n})\), which means that \(f\) satisfies the pointwise bounds
\[
|f(z)| \leq \|f\|_{L^{2}(\mathbb{B}^{d}, \mu_{\sigma-2n})} (K_{\sigma-2n}(z, z))^{1/2} = \|f\|_{L^{2}(\mathbb{B}^{d}, \mu_{\sigma-2n})} \left( \frac{1}{1 - |z|^2} \right)^{\sigma-n}.
\] (2.4)

These bounds are better by a factor of \((1 - |z|^2)^{n}\) than the bounds on a typical element of \(\mathcal{H}L^{2}(\mathbb{B}^{d}, \mu_{\sigma})\). See also [10] for another setting in which the existence of derivatives in a holomorphic \(L^{2}\) space can be related in a precise way to improved pointwise behavior of the functions.

The results of the two previous paragraphs were derived under the assumption that \(\lambda = \sigma - 2n > d\). However, Theorem 4 will show that (2.4) still holds under the assumption \(\lambda = \sigma - 2n > 0\).

**Proof.** If \(f\) and \(g\) are polynomials, then (2.3) follows from iteration of Proposition 2. Note that \(N\) is a non-negative operator on polynomials, because the monomials form an orthogonal basis of eigenvectors with non-negative eigenvalues. It is well known and easily verified that for any \(f\) in \(\mathcal{H}L^{2}(\mathbb{B}^{d}, \mu_{\lambda})\), the partial sums of the Taylor series of \(f\) converge to \(f\) in norm. We can therefore choose polynomials \(f_{j}\) converging in norm to \(f\). If we apply (2.3) with \(f = g = (f_{j} - f_{k})\) and expand out the expressions for \(A\) and \(B\), then the positivity of \(N\) will force each of the terms on the right-hand side to tend to zero. In particular, \(N^{l}f_{j}\) is a Cauchy sequences in \(\mathcal{H}L^{2}(\mathbb{B}^{d}, \mu_{\lambda+2n})\), for all \(0 \leq l \leq n\). It is easily seen that the limit of this sequence is \(N^{l}f\); for holomorphic functions, \(L^{2}\) convergence implies locally uniform convergence of the derivatives to the corresponding derivatives of the limit function. This shows that \(N^{l}f\) is in \(\mathcal{H}L^{2}(\mathbb{B}^{d}, \mu_{\lambda+2n})\). For any \(f, g \in \mathcal{H}L^{2}(\mathbb{B}^{d}, \mu_{\lambda})\), choose sequences \(f_{j}\) and \(g_{j}\) of polynomials converging to \(f, g\). Since \(N^{l}f_{j}\) and \(N^{l}g_{j}\) converge to \(N^{l}f\) and \(N^{l}g\), respectively, plugging \(f_{j}\) and \(g_{j}\) into (2.3) and taking a limit gives (2.3) in general.

In the other direction, suppose that \(N^{l}f\) belongs to \(\mathcal{H}L^{2}(\mathbb{B}^{d}, \mu_{\lambda+2n})\) for all \(0 \leq l \leq n\). Let \(f_{j}\) denote the \(j\)th partial sum of the Taylor series of \(f\). Then since \(Nz^{m} = |m|z^{m}\) for all multi-indices \(m\), the functions \(N^{l}f_{j}\) form the partial sums of a Taylor series converging to \(N^{l}f_{j}\), and so these must be the partial sums of the Taylor series of \(N^{l}f\). Thus, for each \(l\), we have that \(N^{l}f_{j}\) converges to \(N^{l}f\) in \(\mathcal{H}L^{2}(\mathbb{B}^{d}, \mu_{\lambda+2n})\). If we then apply (2.3) with \(f = g = f_{j} - f_{k}\), convergence of each \(N^{l}f_{j}\) implies that all the terms on the right-hand side tend to zero. We conclude that \(f_{j}\) is a Cauchy sequence in \(\mathcal{H}L^{2}(\mathbb{B}^{d}, \mu_{\lambda})\), which converges to some \(\hat{f}\). But \(L^{2}\) convergence of holomorphic functions implies pointwise convergence, so the limit in \(\mathcal{H}L^{2}(\mathbb{B}^{d}, \mu_{\lambda})\) (i.e., \(\hat{f}\)) coincides with the limit in \(\mathcal{H}L^{2}(\mathbb{B}^{d}, \mu_{\lambda+2n})\) (i.e., \(f\)). This shows that \(f\) is in \(\mathcal{H}L^{2}(\mathbb{B}^{d}, \mu_{\lambda})\).

Now, when \(\lambda \leq d\), Proposition 2.2 no longer holds. This is because the boundary terms, which involve \((1 - |z|^2)^{\lambda-d}\), no longer vanish. This failure of equality is
actually a good thing, because if we take \( f = g \), then
\[
c_{\lambda} \int_{\mathbb{B}^d} |f(z)|^2 \left(1 - |z|^2\right)^\lambda \, d\tau(z) = +\infty
\]
for all nonzero holomorphic functions, no matter what positive value we assign to \( c_{\lambda} \). (Recall that when \( \lambda > d \), \( c_{\lambda} \) is chosen to make \( \mu_{\lambda} \) a probability measure, but this prescription does not make sense for \( \lambda \leq d \).) Although the left-hand side of (2.2) is infinite when \( f = g \) and \( \lambda \leq d \), the right-hand side is finite if \( \lambda + 1 > d \) and, say, \( f \) is a polynomial.

More generally, for any \( \lambda \leq d \), we can choose \( n \) big enough that \( \lambda + 2n > d \). We then take the right-hand side of (2.3) as a definition.

**Theorem 4.** For all \( \lambda > 0 \), choose a non-negative integer \( n \) so that \( \lambda + 2n > d \) and define
\[
H(\mathbb{B}^d, \lambda) = \{ f \in \mathcal{H}(\mathbb{B}^d) \mid |N^k f| \in \mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n}), \ 0 \leq k \leq n \}.
\]
Then the formula
\[
\langle f, g \rangle_{\lambda} = \langle Af, Bg \rangle_{\mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n})}
\]
where
\[
A = \left( I + \frac{N}{\lambda + n} \right) \left( I + \frac{N}{\lambda + n + 1} \right) \cdots \left( I + \frac{N}{\lambda + 2n - 1} \right),
\]
\[
B = \left( I + \frac{N}{\lambda} \right) \left( I + \frac{N}{\lambda + 1} \right) \cdots \left( I + \frac{N}{\lambda + n - 1} \right)
\]
defines an inner product on \( H(\mathbb{B}^d, \lambda) \) and \( H(\mathbb{B}^d, \lambda) \) is complete with respect to this inner product.

The monomials \( z^m \) form an orthogonal basis for \( H(\mathbb{B}^d, \lambda) \) and for all multi-indices \( l \) and \( m \) we have
\[
\langle z^l, z^m \rangle_{\lambda} = \delta_{l,m} \frac{m!\Gamma(\lambda)}{\Gamma(\lambda + |m|)}.
\]

Furthermore, \( H(\mathbb{B}^d, \lambda) \) has a reproducing kernel given by
\[
K_{\lambda}(z, w) = \frac{1}{(1 - z \cdot \bar{w})^\lambda}.
\]

Using power series, it is easily seen that for any holomorphic function \( f \), if \( N^k f \) belongs to \( \mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n}) \), then \( N^k f \) belongs to \( \mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda+2n}) \) for \( 0 \leq k < n \).

Note that the reproducing kernel and the inner product of the monomials are independent of \( n \). Thus, we obtain the same space of functions with the same inner product, no matter which \( n \) we use, so long as \( \lambda + 2n > d \).

From the reproducing kernel we obtain the pointwise bounds given by
\[
|f(z)|^2 \leq \|f\|_{\lambda}^2 \left(1 - |z|^2\right)^{-\lambda}.
\]
Theorem 2.2 of [Z2] with our \( \lambda \) have bounded inverses. We need that

\[ \langle \chi^\lambda, f \rangle_{L^2(B^d, \mu_{\lambda+2n})} = f(z) \]

for all \( f \) in \( \mathcal{H}L^2(B^d, \mu_{\lambda+2n}) \). Explicitly, \( \chi^\lambda(w) = (1 - \bar{z} \cdot w)^{(\lambda+2n)} \). (This is Theorem 2.2 of [Z2] with our \( \lambda \) corresponding to \( n + \alpha + 1 \) in [Z2].) Now, a simple

Proof. Using a power series argument, it is easily seen that if \( f \) and \( N^k f \) belong to \( \mathcal{H}L^2(B^d, \mu_{\lambda+2n}) \), then \( \langle f, N^k f \rangle_{L^2(B^d, \mu_{\lambda+2n})} \geq 0 \). From this, we obtain positivity of the inner product \( \langle \cdot, \cdot \rangle_\lambda \). If \( f_j \) is a Cauchy sequence in \( H(B^d, \lambda) \), then positivity of the coefficients in the expressions for \( A \) and \( B \) imply that for \( 0 \leq k \leq n \), \( N^k f_j \) is a Cauchy sequence in \( \mathcal{H}L^2(B^d, \mu_{\lambda+2n}) \), which converges (as in the proof of Theorem 3) to \( N^k f \). This shows that \( N^k f \) is in \( \mathcal{H}L^2(B^d, \mu_{\lambda+2n}) \) for each \( 0 \leq k \leq n \), and so \( f \in H(B^d, \lambda) \). Further, convergence of each \( N^k f_j \) to \( N^k f \) implies that \( f_j \) converges to \( f \) in \( H(B^d, \lambda) \).

To compute the inner product of two monomials in \( H(B^d, \lambda) \), we apply the definition. Since \( N z^m = |m| z^m \), we obtain

\[
\langle z^l, z^m \rangle_\lambda = \delta_{l,m} \frac{\lambda + |m|}{\lambda} \frac{\lambda + 1 + |m|}{\lambda + 1} \cdots \frac{\lambda + 2n - 1 + |m|}{\lambda + 2n - 1} \frac{m! \Gamma(\lambda + 2n + |m|)}{\Gamma(\lambda + 2n + |m|)}
\]

where we have used the known formula for the inner product of monomials in \( \mathcal{H}L^2(B^d, \mu_{\lambda+2n}) \) (e.g., [Z2]).

Completeness of the monomials holds in \( H(B^d, \lambda) \) for essentially the same reason it holds in the ordinary Bergman spaces. For \( f \in \mathcal{H}L^2(B^d, \mu_\lambda) \), expand \( f \) in a Taylor series and then consider \( \langle z^m, f \rangle_\lambda \). Each term in the inner product is an integral over \( B^d \) with respect to \( \mu_{\lambda+2n} \), and each of these integrals can be computed as the limit as \( r \) tends to 1 of integrals over a ball of radius \( r < 1 \). On the ball of radius \( r \), we may interchange the integral with the sum in the Taylor series. But distinct monomials are orthogonal not just over \( B^d \) but also over the ball of radius \( r \), as is easily verified. The upshot of all of this is that \( \langle z^m, f \rangle_\lambda \) is a nonzero multiple of the \( m \)th Taylor coefficient of \( f \). Thus if \( \langle z^m, f \rangle_\lambda = 0 \) for all \( m \), \( f \) is identically zero.

Finally, we address the reproducing kernel. Although one can use essentially the same argument as in the case \( \lambda > d \), using the orthogonal basis of monomials and a binomial expansion (see the proof of Theorem 12), it is more enlightening to relate the reproducing kernel in \( H(B^d, \lambda) \) to that in \( \mathcal{H}L^2(B^d, \mu_{\lambda+2n}) \). We require some elementary properties of the operators \( A \) and \( B \); since the monomials form an orthogonal basis of eigenvectors for these operators, these properties are easily obtained. We need that \( A \) is self-adjoint on its natural domain and that \( A \) and \( B \) have bounded inverses.

Let \( \chi_\lambda^{\lambda+2n} \) be the unique element of \( \mathcal{H}L^2(B^d, \mu_{\lambda+2n}) \) for which

\[ \langle \chi_\lambda^{\lambda+2n}, f \rangle_{L^2(B^d, \mu_{\lambda+2n})} = f(z) \]

for all \( f \) in \( \mathcal{H}L^2(B^d, \mu_{\lambda+2n}) \). Explicitly, \( \chi_\lambda^{\lambda+2n}(w) = (1 - \bar{z} \cdot w)^{(\lambda+2n)} \). (This is Theorem 2.2 of [Z2] with our \( \lambda \) corresponding to \( n + \alpha + 1 \) in [Z2].) Now, a simple
calculation shows that
\[(I + N/a)(1 - \bar{z} \cdot w)^{-a} = (1 - \bar{z} \cdot w)^{-(a+1)}, \tag{2.5}\]
where $N$ acts on the $w$ variable with $z$ fixed. From this, we see that $N^k \chi_z^{\lambda+2n}$ is a bounded function for each fixed $z \in \mathbb{B}^d$ and $k \in \mathbb{N}$, so that $\chi_z^{\lambda+2n}$ is in $H(\mathbb{B}^d, \lambda)$.

For any $f \in H(\mathbb{B}^d, \lambda)$ we compute that
\[
\langle f, (AB)^{-1} \chi_z^{\lambda+2n} \rangle_{\lambda} = \langle Af, B(AB)^{-1} \chi_z^{\lambda+2n} \rangle_{L^2(\mathbb{B}^d, \mu_{\lambda+2n})}
= \langle f, \chi_z^{\lambda+2n} \rangle_{L^2(\mathbb{B}^d, \mu_{\lambda+2n})} = f(z).
\]
This shows that the reproducing kernel for $H(\mathbb{B}^d, \lambda)$ is given by $K_\lambda(z, w) = \langle (AB)^{-1} \chi_z^{\lambda+2n} \rangle(w)$. Using (2.5) repeatedly gives the desired result. \hfill \Box

We conclude this section with a simple lemma that will be useful in Section 5.

**Lemma 5.** For all $\lambda_1, \lambda_2 > 0$, if $f$ is in $H(\mathbb{B}^d, \lambda_1)$ and $g$ is in $H(\mathbb{B}^d, \lambda_2)$ then $fg$ is in $H(\mathbb{B}^d, \lambda_1 + \lambda_2)$.

**Proof.** If, say, $\lambda_1 > d$, then we have the following simple argument:
\[
\|fg\|^2_{\lambda_1+\lambda_2} = c_{\lambda_1+\lambda_2} \int_{\mathbb{B}^d} |f(z)|^2 |g(z)|^2 (1 - |z|^2)^{\lambda_1+\lambda_2} \ d\tau(z)
\leq c_{\lambda_1+\lambda_2} \|g\|^2_{\lambda_2} \int_{\mathbb{B}^d} |f(z)|^2 (1 - |z|^2)^{-\lambda_2} (1 - |z|^2)^{\lambda_1+\lambda_2} \ d\tau(z)
= \frac{c_{\lambda_1+\lambda_2}}{c_{\lambda_1}} \|f\|^2_{\lambda_1} \|g\|^2_{\lambda_2}.
\]
Unfortunately, $c_{\lambda_1+\lambda_2}/c_{\lambda_1}$ tends to infinity as $\lambda_1$ approaches $d$ from above, so we cannot expect this simple inequality to hold for $\lambda_1 < d$.

For any $\lambda_1, \lambda_2 > 0$, choose $n$ so that $\lambda_1 + n > d$ and $\lambda_2 + n > d$. Then $fg$ belongs to $H(\mathbb{B}^d, \lambda_1 + \lambda_2)$ provided that $N^n(fg)$ belongs to $HL^2(\mathbb{B}^d, \lambda_1 + \lambda_2 + 2n)$.

But
\[
N^n(fg) = \sum_{k=0}^{n} \binom{n}{k} N^k f N^{n-k} g. \tag{2.6}
\]
Using Theorem 4 it is easy to see that if $f$ belongs to $H(\mathbb{B}^d, \lambda_1)$ then $N^k f$ belongs to $H(\mathbb{B}^d, \lambda_1 + 2k)$. Thus,
\[
|N^k f(z)|^2 \leq a_k (1 - |z|^2)^{-(\lambda_1+2k)}.
\]

Now, for each term in (2.6) with $k \leq n/2$, we then obtain the following norm estimate:
\[
c_{\lambda_1+\lambda_2+2n} \int_{\mathbb{B}^d} |N^k f(z) N^{n-k} g(z)|^2 (1 - |z|^2)^{\lambda_1+\lambda_2+2n} \ d\tau(z)
\leq c_{\lambda_1+\lambda_2+2n} a_k \int_{\mathbb{B}^d} |N^{n-k} g(z)|^2 (1 - |z|^2)^{\lambda_2+2n-2k} \ d\tau(z). \tag{2.7}
\]
Since \( k \leq n/2 \), we have \( \lambda_2 + 2n - 2k \geq \lambda_2 + n > d \). We are assuming that \( g \) is in \( H(\mathbb{B}^d, \lambda_2) \), so that \( N^{n-k}g \) is in \( H(\mathbb{B}^d, \lambda_2 + 2n - 2k) \), which coincides with \( \mathcal{H}L^2(\mathbb{B}^d, \mu_{\lambda_2+2n-2k}) \). Thus, under our assumptions on \( f \) and \( g \), each term in (2.6) with \( k \leq n/2 \) belongs to \( \mathcal{H}L^2(\mathbb{B}^d, \lambda_1 + \lambda_2 + 2n) \). A similar argument with the roles of \( f \) and \( g \) reversed takes care of the terms with \( k \geq n/2 \).

3. Toeplitz operators with polynomial symbols

In this section, we will consider our first examples of Toeplitz operators on generalized Bergman spaces, those whose symbols are (not necessarily holomorphic) polynomials. Such examples are sufficient to see some interesting new phenomena, that is, properties of ordinary Toeplitz operator that fail when extended to these generalized Bergman spaces. The definition of Toeplitz operators for the case of polynomial symbols is consistent with the definition we use in Section 4 for a larger class of symbols.

For \( \lambda > d \), we define the Toeplitz operator \( T_\phi \) by

\[
T_\phi f = P_\lambda(\phi f)
\]

for all \( f \) in \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda) \) and all bounded measurable functions \( \phi \). Recall that \( P_\lambda \) is the orthogonal projection from \( L^2(\mathbb{B}^d, \tau) \) onto the holomorphic subspace. Because \( P_\lambda \) is a self-adjoint operator on \( L^2(\mathbb{B}^d, \mu_\lambda) \), the matrix entries of \( T_\phi \) may be calculated as

\[
\langle f_1, T_\phi f_2 \rangle_{\mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda)} = \langle f_1, \phi f_2 \rangle_{L^2(\mathbb{B}^d, \mu_\lambda)}, \quad \lambda > d,
\]

(3.1) for all \( f_1, f_2 \in \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda) \). From this formula, it is easy to see that \( T_\phi = (T_\phi)^* \).

If \( \psi \) is a bounded holomorphic function and \( \phi \) is any bounded measurable function, then it is easy to see that \( T_\phi \psi = T_\phi M_\psi \). Thus, for any two multi-indices \( m \) and \( n \), we have

\[
T_{z^m \bar{z}^n} = (M_{z^m})^*(M_{z^n}).
\]

(3.2)

We will take (3.2) as a definition for \( 0 < \lambda \leq d \). Our first task, then, is to show that \( M_{z^n} \) is a bounded operator on \( \mathcal{H}L^2(\mathbb{B}^d, \mu_\lambda) \) for all \( \lambda > 0 \).

**Proposition 6.** For all \( \lambda > 0 \) and all multi-indices \( n \), the multiplication operator \( M_{z^n} \) is a bounded operator on \( H(\mathbb{B}^d, \lambda) \). Thus, for any polynomial \( \phi \), the Toeplitz operator \( T_\phi \) defined in (3.2) is a bounded operator on \( H(\mathbb{B}^d, \lambda) \).

**Proof.** The result is a is a special case of a result of Arazy and Zhang [3] and also of the results of Section 4 but it is easy to give a direct proof. It suffices to show that \( M_{z^j} \) is bounded for each \( j \). Since \( M_{z^j} \) preserves the orthogonality of the monomials, we obtain

\[
\|M_{z^j}\| = \sup_m \frac{\|z_j z^m\|_\lambda}{\|z^m\|_\lambda} = \sup_m \frac{m_j + 1}{|m| + \lambda}.
\]

Note that \( m_j \leq |m| \) with equality when \( m_k = 0 \) for \( k \neq j \). Thus the supremum is finite and is easily seen to have the value of 1 if \( \lambda \geq 1 \) and \( 1/\lambda \) if \( \lambda < 1 \). □
We now record some standard properties of Toeplitz operators on (ordinary) Bergman spaces. These properties hold for Toeplitz operators (defined by the “multiply and project” recipe) on any holomorphic $L^2$ space. We will show that these properties do not hold for Toeplitz operators with polynomial symbols on the generalized Bergman spaces $H(B^d, \lambda)$, $\lambda < d$.

**Proposition 7.** For $\lambda > d$ and $\phi(z)$ bounded, the Toeplitz operator $T_\phi$ on the space $HL^2(B^d, d\mu_\lambda)$, which is defined by $T_\phi f = P_\lambda(\phi f)$, has the following properties.

1. $\|T_\phi\| \leq \sup_z |\phi(z)|$
2. If $\phi(z) \geq 0$ for all $z$, then $T_\phi$ is a positive operator.

Both of these properties fail when $\lambda < d$.

In fact, for $\lambda < d$, there is no constant $C$ such that $\|T_\phi\| \leq C \sup_z |\phi(z)|$ for all polynomials $\phi$.

As we remarked in the introduction, when $\lambda = d$, the space $H(B^d, \lambda)$ may be identified with the Hardy space. Thus Properties 1 and 2 in the proposition still hold when $\lambda = d$, if, say, $\phi$ is continuous up to the boundary of $B^d$ (or otherwise has a reasonable extension to the closure of $B^d$).

**Proof.** When $\lambda > d$, the projection operator $P_\lambda$ has norm 1 and the multiplication operator $M_\phi$ has norm equal to $\sup_z |\phi(z)|$ as an operator on $L^2(B^d, \mu_\lambda)$. Thus, the restriction to $HL^2(B^d, \mu_\lambda)$ of $P_\lambda M_\phi$ has norm at most $\sup_z |\phi(z)|$.

Meanwhile, if $\phi$ is non-negative, then from (3.1) we see that $\langle f, T_\phi f \rangle \geq 0$ for all $f \in HL^2(B^d, \mu_\lambda)$.

Let us now assume that $0 < \lambda < d$. Computing on the orthogonal basis in Theorem 4, it is a simple exercise to show that

$$T_{\bar{z}_j z_j}(z^m) = \frac{\Gamma(\lambda + |m|)}{m!} \frac{(m + e_j)!}{\Gamma(\lambda + |m| + 1)} z^m = \frac{1 + m_j}{\lambda + |m|} z^m. \quad (3.3)$$

If we take $\phi(z) = |z|^2$, then summing $(3.3)$ on $j$ gives

$$T_\phi z^m = \frac{d + |m|}{\lambda + |m|} z^m.$$

Since $\lambda < d$, this shows that $\|T_\phi\| > 1$, even though $|\phi(z)| < 1$ for all $z \in B^d$. Thus, Property 1 fails for $\lambda < d$. (From this calculation it easily follows that if $\phi(z) = (1 - |z|^2)/(\lambda - d)$, then $T_\phi$ is the bounded operator $(\lambda I + N)^{-1}$, for all $\lambda \neq d$.)

For the second property, we let $\psi(z) = 1 - \phi(z) = 1 - |z|^2$ which is positive. From the above calculation we obtain

$$\langle T_\psi z^m, z^m \rangle_{H_\lambda} = \|z^m\|^2_{H_\lambda} - \left(\frac{d + |m|}{\lambda + |m|}\right) \|z^m\|^2_{H(B^d, \lambda)},$$

which is negative if $0 < \lambda < d$. 


We now show that there is no constant $C$ such that $\|T_\phi\| \leq C \sup_z |\phi(z)|$.

Consider $\phi_k(z) := (|z|^2)^k = \left(\sum_{i=1}^{d} |z_i|^2\right)^k = \sum_{|i|=k} \frac{k!}{i!} (|z_1|^2)^{i_1} (|z_2|^2)^{i_2} \cdots (|z_d|^2)^{i_d} = \sum_{|i|=k} \frac{k!}{i!} i! z_i$.

Computing on the orthogonal basis in Theorem 4 we obtain

$$T_{\phi_k} 1 = \sum_{|i|=k} \frac{k!}{i!} (T_z z_i \cdot 1) = \sum_{|i|=k} \frac{k!}{i!} i! \Gamma(\lambda) \Gamma(\lambda + k) 1 = \mathcal{I} \frac{k! \Gamma(\lambda)}{\Gamma(\lambda + k)} 1,$$

where $1$ is the constant function. Here, $\mathcal{I}$ is the number of multi-indices $i$ of length $d$ such that $|i| = k$, which is equal to $\binom{k+d+1}{d-1}$. Thus

$$T_{\phi_k} 1 = \frac{(k + d - 1)! \Gamma(\lambda)}{(d-1)! \Gamma(\lambda + k)} 1 = \frac{(d + k - 1) \cdots (d)}{(\lambda + k - 1) \cdots (\lambda)} 1 = \prod_{j=0}^{k-1} \frac{d + j}{\lambda + j} 1.$$

Consider $\prod_{j=0}^{k-1} \frac{d + j}{\lambda + j} = \prod_{j=0}^{k-1} \left(1 + \frac{d - \lambda}{\lambda + j}\right)$. Since $d > \lambda$, the terms $\frac{d - \lambda}{\lambda + j}$ are positive and $\sum_{j=0}^{\infty} \frac{d - \lambda}{\lambda + j}$ diverges. This implies $\prod_{j=0}^{\infty} \frac{d + j}{\lambda + j} = \infty$. Since $\sup_z |\phi_k(z)| = 1$ for all $k$, there is no a constant $C$ such that $\|T_\phi\| \leq C \sup_z |\phi(z)|$. □

Remark 8. For $\lambda < d$, there does not exist any positive measure $\nu$ on $\mathbb{B}^d$ such that $\|f\|_\lambda = \|f\|_{L^2(\mathbb{B}^d, \nu)}$ for all $f$ in $H(\mathbb{B}^d, \lambda)$. If such a $\nu$ did exist, then the argument in the first part of the proof of Proposition 4 would show that Properties 1 and 2 in the proposition hold.

4. Bounded Toeplitz operators

In this section, we will consider a class of symbols $\phi$ for which we will be able to define a Toeplitz operator $T_\phi$ as a bounded operator on $H(\mathbb{B}^d, \lambda)$ for all $\lambda > 0$. Our definition of $T_\phi$ will agree (for the relevant class of symbols) with the usual “multiply and project” definition for $\lambda > d$. In light of the examples in the previous section, we cannot expect boundedness of $\phi$ to be sufficient to define $T_\phi$ as a bounded operator. Instead, we will consider functions $\phi$ for which $\phi$ and a certain number of derivatives of $\phi$ are bounded.

Our strategy is to use integration by parts to give an alternative expression for the matrix entries of a Toeplitz operator with sufficiently regular symbol, in the case $\lambda > d$. We then take this expression as our definition of Toeplitz operator in the case $0 < \lambda \leq d$. 
Theorem 9. Assume \( \lambda > d \) and fix a positive integer \( n \). Let \( \phi \) be a function that is \( 2n \) times continuously differentiable and for which \( \bar{N}^k N^l \phi \) is bounded for all \( 0 \leq k, l \leq n \). Then

\[
(f, T_\phi g)_{H L^2(\mathbb{B}^d, \mu_\lambda)} = c_{\lambda+2n} \int_{\mathbb{B}^d} C \left[ \left( \overline{f(z)} \phi(z) g(z) \right) \right] (1 - |z|^2)^{\lambda + 2n} \, dz
\]

for all \( f, g \in H L^2(\mathbb{B}^d, \mu_\lambda) \), where \( C \) is the operator given by

\[
C = \left( I + \frac{N}{\lambda + 2n} - 1 \right) \cdots \left( I + \frac{N}{\lambda + n} \right) \left( I + \frac{N}{\lambda + n - 1} \right) \cdots \left( I + \frac{N}{\lambda} \right). \tag{4.1}
\]

Thus, there exist constants \( A_{jklm} \) (depending on \( n \) and \( \lambda \)) such that

\[
(f, T_\phi g)_{H L^2(\mathbb{B}^d, \mu_\lambda)} = \sum_{j,k,l,m=1}^n A_{jklm} \langle N^j f, (\bar{N}^k N^l \phi) N^m g \rangle_{L^2(\mathbb{B}^d, \mu_{\lambda+2n})}. \tag{4.2}
\]

Proof. Assume at first that \( f \) and \( g \) are polynomials, so that \( f \) and \( g \) and all of their derivatives are bounded. We use (3.1) and apply the first equality in Lemma 1 with \( \psi = f \phi g \). We then apply the first equality in the lemma again with \( \psi = (I + N/\lambda) [f \phi g] \). We continue on in this fashion until we have applied the first equality in Lemma 1 \( n \) times and the second equality \( n \) times. This establishes the desired equality in the case that \( f \) and \( g \) are polynomials. For general \( f \) and \( g \) in \( H L^2(\mathbb{B}^d, \mu_\lambda) \), we approximate by sequences \( f_a \) and \( g_a \) of polynomials. From Theorem 3 we can see that convergence of \( f_a \) and \( g_a \) in \( H L^2(\mathbb{B}^d, \mu_\lambda) \) implies convergence of \( N^j f_a \) and \( N^k g_a \) to \( N^j f \) and \( N^k g \), so that applying (4.2) to \( f_a \) and \( g_a \) and taking a limit establishes the desired result for \( f \) and \( g \). \( \square \)

Definition 10. Assume \( 0 < \lambda < d \) and fix a positive integer \( n \) such that \( \lambda + 2n > d \). Let \( \phi \) be a function that is \( 2n \) times continuously differentiable and for which \( \bar{N}^k N^l \phi \) is bounded for all \( 0 \leq k, l \leq n \). Then we define the Toeplitz operator \( T_\phi \) to be the unique bounded operator on \( H(\mathbb{B}^d, \lambda) \) whose matrix entries are given by

\[
(f, T_\phi g)_{H(\mathbb{B}^d, \lambda)} = c_{\lambda+2n} \int_{\mathbb{B}^d} C \left[ \left( \overline{f(z)} \phi(z) g(z) \right) \right] (1 - |z|^2)^{\lambda + 2n} \, dz, \tag{4.3}
\]

where \( C \) is given by (4.1).

Note that from Theorem 4 \( N^j f \) and \( N^m g \) belong to \( L^2(\mathbb{B}^d, \mu_{\lambda+2n}) \) for all \( 0 \leq j, m \leq n \), for all \( f \) and \( g \) in \( H L^2(\mathbb{B}^d, \mu_\lambda) \). Furthermore, \( \|N^j f\|_{L^2(\mathbb{B}^d, \mu_{\lambda+2n})} \) and \( \|N^m g\|_{L^2(\mathbb{B}^d, \mu_{\lambda+2n})} \) are bounded by constants times \( \|f\|_\lambda \) and \( \|g\|_\lambda \), respectively. Thus, the right-hand side of (4.3) is a continuous sesquilinear form on \( H(\mathbb{B}^d, \lambda) \), which means that there is a unique bounded operator \( T_\phi \) whose matrix entries are given by (4.3).

If \( \lambda = d \), then (as discussed in the introduction) the Hilbert space \( H(\mathbb{B}^d, \lambda) \) is the Hardy space of holomorphic functions that are square-integrable over the boundary. In that case, the Toeplitz operator \( T_\phi \) will be the zero operator whenever \( \phi \) is identically zero on the boundary of \( \mathbb{B}^d \). If \( \lambda = d - 1, d - 2, \ldots \), then the inner product on \( H(\mathbb{B}^d, \lambda) \) can be related to the inner product on the Hardy space. It is
not hard to see that in these cases, \( T_\phi \) will be the zero operator if \( \phi \) and enough of its derivatives vanish on the boundary of \( \mathbb{B}^d \).

Let us consider the case in which \( \phi(z) = \overline{\psi_1(z)} \psi_2(z) \), where \( \psi_1 \) and \( \psi_2 \) are holomorphic functions such that the function and the first \( n \) derivatives are bounded. Then when applying \( C \) to \( f(z) \phi(z) g(z) \), all the \( N \)-factors go onto the expression \( \psi_2(z) g(z) \) and all the \( N \)-factors go onto \( f(z) \psi_1(z) \). Recalling from Theorem \( 4 \) the formula for the inner product on \( H(\mathbb{B}^d, \lambda) \), we see that

\[
\langle f, T_\phi g \rangle_{H^2(\mathbb{B}^d, \mu_\lambda)} = \langle \psi_1 f, \psi_2 g \rangle_{H^2(\mathbb{B}^d, \lambda)},
\]
as expected. This means that in this case, \( T_{\psi_1 \psi_2} = (M_{\psi_1})^*(M_{\psi_2}) \), as in the case \( \lambda > d \). In particular, Definition \( 10 \) agrees with the definition we used in Section \( 3 \) in the case that \( \phi \) is a polynomial in \( z \) and \( \bar{z} \).

5. Hilbert–Schmidt Toeplitz operators

5.1. Statement of results

In this section, we will give sufficient conditions under which a Toeplitz operator \( T_\phi \) can be defined as a Hilbert–Schmidt operator on \( H(\mathbb{B}^d, \lambda) \). Specifically, if \( \phi \) belongs to \( L^2(\mathbb{B}^d, \tau) \) then \( T_\phi \) can be defined as a Hilbert–Schmidt operator, provided that \( \lambda > d/2 \). Meanwhile, if \( \phi \) belongs to \( L^1(\mathbb{B}^d, \tau) \), then \( T_\phi \) can be defined as a Hilbert–Schmidt operator for all \( \lambda > 0 \). In both cases, we define \( T_\phi \) in such a way that for all bounded functions \( f \) and \( g \) in \( H(\mathbb{B}^d, \lambda) \), we have

\[
\langle f, T_\phi g \rangle_\lambda = c_\lambda \int_{\mathbb{B}^d} f(z) \phi(z) g(z) (1 - |z|^2)^\lambda \, d\tau(z), \tag{5.1}
\]

where \( c_\lambda \) is defined by \( c_\lambda = \Gamma(\lambda)/((\pi^d \Gamma(\lambda - d)) \). This expression is identical to (3.1) in the case \( \lambda > d \). The value of \( c_\lambda \) should be interpreted as 0 when \( \lambda - d = 0, -1, -2, \ldots \). This means that for \( \phi \) in \( L^2(\mathbb{B}^d, \tau) \) or \( L^1(\mathbb{B}^d, \tau) \) (but not for other classes of symbols!), \( T_\phi \) is the zero operator when \( \lambda = d, d - 1, \ldots \). This strange phenomenon is discussed in the next subsection. Note that we are not claiming \( T_\phi = 0 \) for arbitrary symbols when \( \lambda = d, d - 1, \ldots \), but only for symbols that are integrable or square-integrable with respect to the hyperbolic volume measure \( \tau \). Such functions must have reasonable rapid decay (in an average sense) near the boundary of \( \mathbb{B}^d \).

In the case \( \phi \in L^2(\mathbb{B}^d, \tau) \), the restriction \( \lambda > d/2 \) is easy to explain: the function \((1 - |z|^2)^\lambda \) belongs to \( L^2(\mathbb{B}^d, \tau) \) if and only if \( \lambda > d/2 \). Thus, if \( f \) and \( g \) are bounded and \( \phi \) is in \( L^2(\mathbb{B}^d, \tau) \), then (5.1) is absolutely convergent for \( \lambda > d/2 \).

In this subsection, we state our results; in the next subsection, we discuss some unusual properties of \( T_\phi \) for \( \lambda < d \); and in the last subsection of this section we give the proofs.

We begin by considering symbols \( \phi \) in \( L^2(\mathbb{B}^d, \tau) \).
Theorem 11. Fix $\lambda > d/2$ and let $c_\lambda = \Gamma(\lambda)/(\pi^d \Gamma(\lambda - d))$. (We interpret $c_\lambda$ to be zero if $\lambda$ is an integer and $\lambda \leq d$.) Then the operator $A_\lambda$ given by

$$A_\lambda \phi(z) = c_\lambda \int_{\mathbb{B}^d} \frac{(1 - |z|^2)(1 - |w|^2)}{(1 - w \cdot z)(1 - w \cdot \bar{z})} \lambda^\lambda \phi(w) d\tau(w)$$

is a bounded operator from $L^2(\mathbb{B}^d, \tau)$ to itself.

Theorem 12. Fix $\lambda > d/2$. Then for each $\phi \in L^2(\mathbb{B}^d, \tau)$, there is a unique Hilbert–Schmidt operator on $H(\mathbb{B}^d, \lambda)$, denoted $T_\phi$, with the property that

$$\langle f, T_\phi g \rangle_\lambda = c_\lambda \int_{\mathbb{B}^d} \overline{f(z)} \phi(z)(1 - |z|^2)^\lambda d\tau(z)$$

(5.2)

for all bounded holomorphic functions $f$ and $g$ in $H(\mathbb{B}^d, \lambda)$. The Hilbert–Schmidt norm of $T_\phi$ is given by

$$\|T_\phi\|_{HS}^2 = \langle \phi, A_\lambda \phi \rangle_{L^2(\mathbb{B}^d, \tau)}.$$

If $\lambda > d$ and $\phi \in L^2(\mathbb{B}^d, \tau) \cap L^\infty(\mathbb{B}^d, \tau)$, then the definition of $T_\phi$ in Theorem 12 agrees with the “multiply and project” definition; compare (3.1).

Applying Lemma 5 with $\lambda_1 = \lambda_2 = \lambda$ and $\lambda > d/2$, we see that for all $f$ and $g$ in $H(\mathbb{B}^d, \lambda)$, the function $z \to \overline{f(z)} g(z)(1 - |z|^2)^\lambda$ is in $L^2(\mathbb{B}^d, \tau)$. This means that the integral on the right-hand side of (5.2) is absolutely convergent for all $f, g \in H(\mathbb{B}^d, \lambda)$. It is then not hard to show that (5.2) holds for all $f, g \in H(\mathbb{B}^d, \lambda)$.

The operator $A_\lambda$ coincides, up to a constant, with the Berezin transform. Let $\chi_\lambda(z, w) := K_\lambda(z, w)$ be the coherent state at the point $z$, which satisfies $f(z) = \langle \chi_\lambda^z, f \rangle_\lambda$ for all $f \in H(\mathbb{B}^d, \lambda)$. Then one standard definition of the Berezin transform $B_\lambda$ is

$$B_\lambda \phi = \frac{\langle \chi_\lambda^z, T_\phi \chi_\lambda^z \rangle_\lambda}{\langle \chi_\lambda^z, \chi_\lambda^z \rangle_\lambda}.$$

The function $B_\lambda \phi$ may be thought of as the Wick-ordered symbol of $T_\phi$, where $T_\phi$ is thought of as the anti-Wick-ordered quantization of $\phi$. Using the formula (Theorem 4) for the reproducing kernel along with (5.2), we see that $A_\lambda = c_\lambda B_\lambda$. (Note that $\chi_\lambda^z(w)$ is a bounded function of $w$ for each fixed $z \in \mathbb{B}^d$ and that $\langle \chi_\lambda^z, \chi_\lambda^z \rangle_\lambda = K_\lambda(z, z)$.)

Note that $\tau$ is an infinite measure, which means that if $\phi$ is in $L^2(\mathbb{B}^d, \tau)$ or $L^1(\mathbb{B}^d, \tau)$, then $\phi$ must tend to zero at the boundary of $\mathbb{B}^d$, at least in an average sense. This decay of $\phi$ is what allows (5.2) to be a convergent integral. If, for example, we want to take $\phi(z) \equiv 1$, then we cannot use (5.2) to define $T_\phi$, but must instead use the definition from Section 3 or Section 4.

Note also that the space of Hilbert–Schmidt operators on $H(\mathbb{B}^d, \lambda)$ may be viewed as the quantum counterpart of $L^2(\mathbb{B}^d, \tau)$. It is thus natural to investigate the question of when the Berezin–Toeplitz quantization maps $L^2(\mathbb{B}^d, \tau)$ into the Hilbert–Schmidt operators.

We now show that if one considers a symbol $\phi$ in $L^1(\mathbb{B}^d, \tau)$, then one obtains a Hilbert–Schmidt Toeplitz operator $T_\phi$ for all $\lambda > 0$. 

16 Kamthorn Chailuek and Brian C. Hall
Theorem 13. Fix \( \lambda > 0 \) and let \( c_\lambda \) be as in Theorem 12. Then for each \( \phi \in L^1(\mathbb{B}^d, \tau) \), there exists a unique Hilbert–Schmidt operator on \( H(\mathbb{B}^d, \lambda) \), denoted \( T_\phi \), with the property that
\[
\langle f, T_\phi g \rangle_\lambda = c_\lambda \int_{\mathbb{B}^d} \overline{f(z)} \phi(z) g(z) (1 - |z|^2)^\lambda \, d\tau(z)
\]
for all bounded holomorphic functions \( f \) and \( g \) in \( H(\mathbb{B}^d, \lambda) \). The Hilbert–Schmidt norm of \( T_\phi \) satisfies
\[
\| T_\phi \|_{HS} \leq c_\lambda \| \phi \|_{L^1(\mathbb{B}^d, \tau)}.
\]

Using the pointwise bounds on elements of \( H(\mathbb{B}^d, \lambda) \) coming from the reproducing kernel, we see immediately that for all \( f, g \in H(\mathbb{B}^d, \lambda) \), the function \( z \mapsto \overline{f(z)} g(z) (1 - |z|^2)^\lambda \) is bounded. It is then not hard to show that (5.3) holds for all \( f, g \in H(\mathbb{B}^d, \lambda) \).

We have already remarked that the definition of \( T_\phi \) given in this section agrees with the “multiply and project” definition when \( \lambda > d \) (and \( \phi \) is bounded). It is also easy to see that the definition of \( T_\phi \) given in this section agrees with the one in Section 4 when \( \phi \) falls under the hypotheses of both Definition 10 and either Theorem 12 or Theorem 13. For some positive integer \( n \), consider the set of \( \lambda \)'s for which \( \lambda + 2n > d \) and \( \lambda > d/2 \), i.e., \( \lambda > \max(d - 2n, d/2) \). Now suppose that \( \phi \) belongs to \( L^2(\mathbb{B}^d, \tau) \) and that \( \tilde{N}^k \tilde{N}^l \phi \) is bounded for all \( 0 \leq k, l \leq n \). It is easy to see that the matrix entries \( \langle f, T_\phi g \rangle_\lambda \) depend real-analytically on \( \lambda \) for fixed polynomials \( f \) and \( g \), whether \( T_\phi \) is defined by Definition 10 or by Theorem 12. For \( \lambda > d \), the two matrix entries agree because both definitions of \( T_\phi \) agree with the “multiply and project” definition. The matrix entries therefore must agree for all \( \lambda > \max(d - 2n, d/2) \). Since polynomials are dense in \( H(\mathbb{B}^d, \lambda) \) and both definitions of \( T_\phi \) give bounded operators, the two definitions of \( T_\phi \) agree. The same reasoning shows agreement of Definition 10 and Theorem 13.

5.2. Discussion

Before proceeding on with the proof, let us make a few remarks about the way we are defining Toeplitz operators in this section. For \( \lambda > d \), \( c_\lambda \) is the normalization constant that makes the measure \( \mu_\lambda \) a probability measure, which can be computed to have the value \( \Gamma(\lambda)/(\pi^d \Gamma(\lambda - d)) \). For \( \lambda \leq d \), although the measure \( (1 - |z|^2)^\lambda \, d\tau(z) \) is an infinite measure, we simply use the same formula for \( c_\lambda \) in terms of the gamma function. We understand this to mean that \( c_\lambda = 0 \) whenever \( \lambda \) is an integer in the range \( \{0, d\} \). It also means that \( c_\lambda \) is negative when \( d - 1 < \lambda < d \) and when \( d - 3 < \lambda < d - 2 \), etc.

In the cases where \( c_\lambda = 0 \), we have that \( T_\phi = 0 \) for all \( \phi \in L^1(\mathbb{B}^d, \tau) \) or \( L^2(\mathbb{B}^d, \tau) \). This first occurs when \( \lambda = d \). Recall that for \( \lambda = d \), the space \( H(\mathbb{B}^d, \lambda) \) can be identified with the Hardy space of holomorphic functions square-integrable over the boundary. Meanwhile, having \( \phi \) being integrable or square-integrable with respect to \( \tau \) means that \( \phi \) tends to zero (in an average sense) at the boundary, in which case it is reasonable that \( T_\phi \) should be zero as an operator on the Hardy
space. For other integer values of $\lambda \leq d$, the inner product on $H(\mathbb{B}^d, \lambda)$ can be expressed using the methods of Section 2 in terms of integration over the boundary, but involving the functions and their derivatives. In that case, we expect $T_\phi$ to be zero if $\phi$ has sufficiently rapid decay at the boundary, and it is reasonable to think that having $\phi$ in $L^1$ or $L^2$ with respect to $\tau$ constitutes sufficiently rapid decay. Note, however, that the conclusion that $T_\phi = 0$ when $c_\lambda = 0$ applies only when $\phi$ is in $L^1$ or $L^2$; for other classes of symbols, such as polynomials, $T_\phi$ is not necessarily zero. For example, $T_{z^m}$ is equal to $M_{z^m}$, which is certainly a nonzero operator on $H(\mathbb{B}^d, \lambda)$, for all $\lambda > 0$.

Meanwhile, if $c_\lambda < 0$, then we have the curious situation that if $\phi$ is positive and in $L^1$ or $L^2$ with respect to $\tau$, then the operator $T_\phi$ is actually a negative operator. This is merely a dramatic example of a phenomenon we have already noted: for $\lambda < d$, non-negative symbols do not necessarily give rise to non-negative Toeplitz operators. Again, though, the conclusion that $T_\phi$ is negative for $\phi$ positive applies only when $\phi$ belongs to $L^1$ or $L^2$. For example, the constant function $1$ always maps to the (positive!) identity operator, regardless of the value of $\lambda$.

5.3. Proofs

As motivation, we begin by computing the Hilbert–Schmidt norm of Toeplitz operators in the case $\lambda > d$. For any bounded measurable $\phi$, we extend the Toeplitz operator $T_\phi$ to all of $L^2(\mathbb{B}^d, \mu_\lambda)$ by making it zero on the orthogonal complement of the holomorphic subspace. This extension is given by the formula $P_\lambda M_\phi P_\lambda$. Then the Hilbert–Schmidt norm of the operator $T_\phi$ on $H(L^2(\mathbb{B}^d, \mu_\lambda))$ is the same as the Hilbert–Schmidt norm of the operator $P_\lambda M_\phi P_\lambda$ on $L^2(\mathbb{B}^d, \mu_\lambda)$. Since $P_\lambda$ is computed as integration against the reproducing kernel, we may compute that

$$P_\lambda M_\phi P_\lambda f(z) = \int_{\mathbb{B}^d} K_\phi(z, w) f(w) d\mu_\lambda(w),$$

where

$$K_\phi(z, w) = \int_{\mathbb{B}^d} K(z, u) \phi(u) K(u, w) d\mu_\lambda(u).$$

If we can show that $K_\phi$ is in $L^2(\mathbb{B}^d \times \mathbb{B}^d, \mu_\lambda \times \mu_\lambda)$, then it will follow by a standard result that $T_\phi$ is Hilbert–Schmidt, with Hilbert–Schmidt norm equal to the $L^2$ norm of $K_\phi$. For sufficiently nice $\phi$, we can compute the $L^2$ norm of $K_\phi$ by rearranging the order of integration and using twice the reproducing identity $\int K(z, w) K(w, u) d\mu_\lambda(w) = K(z, u)$. (This identity reflects that $P_\lambda^2 = P_\lambda$.) This yields

$$\int_{\mathbb{B}^d \times \mathbb{B}^d} |K_\phi(z, w)|^2 d\mu_\lambda(z) d\mu_\lambda = \langle \phi, A\phi \rangle_{L^2(\mathbb{B}^d, \tau)},$$
where $A_\lambda$ is the integral operator given by

$$A_\lambda \phi(z) = c_\lambda^2 \int_{\mathbb{B}^d} |K(z, w)|^2 (1 - |z|^2)\lambda |(1 - |w|^2)\lambda \phi(w) \, d\tau(w)$$

$$= c_\lambda^2 \int_{\mathbb{B}^d} \left[ \frac{(1 - |z|^2)(1 - |w|^2)}{(1 - \bar{w} \cdot z)(1 - z \cdot w)} \right]^\lambda \phi(w) \, d\tau(w). \quad (5.4)$$

In the case $d/2 < \lambda \leq d$, it no longer makes sense to express $T_\phi$ as $P_\lambda M_\phi P_\lambda$. Nevertheless, we can consider an operator $A_\lambda$ defined by $(5.4)$. Our goal is to show that for all $\lambda > d/2$, (1) $A_\lambda$ is a bounded operator on $L^2(\mathbb{B}^d, \tau)$ and (2) if we define $T_\phi$ by $(5.1)$, then the Hilbert–Schmidt norm of $T_\phi$ is given by $(\phi, A_\lambda \phi)_{L^2(\mathbb{B}^d, \tau)}$. We will obtain similar results for all $\lambda > 0$ if $\phi \in L^1(\mathbb{B}^d, \tau)$.

**Proof of Theorem 11.** We give two proofs of this result; the first generalizes more easily to other bounded symmetric domains, whereas the second relates $A_\lambda$ to the Laplacian for $\mathbb{B}^d$ (compare [13]).

**First Proof.** We let

$$F_\lambda(z, w) = c_\lambda^2 \left[ \frac{(1 - |z|^2)(1 - |w|^2)}{(1 - \bar{w} \cdot z)(1 - z \cdot w)} \right]^\lambda,$$

i.e., $F_\lambda$ is the integral kernel of the operator $A_\lambda$. A key property of $F_\lambda$ is its invariance under automorphisms: $F_\lambda(\psi(z), \psi(w)) = F_\lambda(z, w)$ for each automorphism (biholomorphism) $\psi$ of $\mathbb{B}^d$ and all $z, w \in \mathbb{B}^d$. To establish the invariance of $F_\lambda$, let

$$f_\lambda(z) = c_\lambda^2 (1 - |z|^2)\lambda.$$  \hspace{1cm} (5.5)

According to Lemma 1.2 of [22], $F_\lambda(z, w) = f_\lambda(\phi_w(z))$, where $\phi_w$ is an automorphism of $\mathbb{B}^d$ taking 0 to $w$ and satisfying $\phi_w^0 = I$. Now, if $\psi$ is any automorphism, the classification of automorphisms (Theorem 1.4 of [22]) implies that $\psi \circ \phi_w = \phi_{\psi(w)} \circ U$ for some unitary matrix $U$. From this we can obtain $\phi_{\psi(w)} = U \circ \phi_w \circ \psi^{-1}$, and so

$$f_\lambda(\phi_{\psi(w)}(\psi(z))) = f_\lambda(U(\phi_w(\psi^{-1}(\psi(z)))) = f_\lambda(\phi_{\psi(w)}(z)),$$

i.e., $F_\lambda(\psi(z), \psi(w)) = F_\lambda(z, w)$.

The invariance of $F_\lambda$ under automorphisms means that $A_\lambda \phi$ can be thought of as a convolution (over the automorphism group $\text{PSU}(d, 1)$) of $\phi$ with the function $f_\lambda$. What this means is that

$$A_\lambda \phi(z) = \int_G f_\lambda(gh^{-1} \cdot 0) \phi(h \cdot 0) \, dh,$$

where $g \in G$ is chosen so that $g \cdot 0 = z$. Here $G = \text{PSU}(d, 1)$ is the group of automorphisms of $\mathbb{B}^d$ (given by fractional linear transformations) and $dh$ is an appropriately normalized Haar measure on $G$. Furthermore, $L^2(\mathbb{B}^d, \tau)$ can be identified with the right-$K$-invariant subspace of $L^2(G, dg)$, where $K := U(d)$ is the stabilizer of 0.

If $\lambda > d$, then $f_\lambda$ is in $L^1(\mathbb{B}^d, \tau)$, in which case it is easy to prove that $A_\lambda$ is bounded; see, for example, Theorem 2.4 in [5]. This argument does not work if
\( \lambda \leq d. \) Nevertheless, if \( \lambda > d/2, \) an easy computation shows that \( f_\lambda \) belongs to \( L^2(\mathbb{B}^d, \tau) \) and also to \( L^p(\mathbb{B}^d, \tau) \) for some \( p < 2. \) We could at this point appeal to a general result known as the Kunze–Stein phenomenon [24]. The result states that convolution with a function in \( L^p(G, dg) \), \( p < 2, \) is a bounded operator from \( L^2(G, dg) \) to itself. (See [11] for a proof in this generality.) However, the proof of this result is simpler in the case we are considering, where the function in \( L^p(G, dg) \) is bi-\( K \)-invariant and the other function is right-\( K \)-invariant. (In our case, the function in \( L^p(G, dg) \) is the function \( g \to f_\lambda(g \cdot 0) \) and the function in \( L^2(G, dg) \) is \( g \to \phi(g \cdot 0). \)) Using the Helgason Fourier transform along with its behavior under convolution with a bi-\( K \)-invariant function ([19] Lemma III.1.4), we need only show that the spherical Fourier transform of \( f_\lambda \) is bounded. (Helgason proves Lemma III.1.4 under the assumption that the functions are continuous and of compact support, but the proof also applies more generally.) Meanwhile, standard estimates show that for every \( \varepsilon > 0, \) the spherical functions are in \( L^{2+\varepsilon}(G/K) \), with \( L^{2+\varepsilon}(G/K) \) norm bounded independent of the spherical function. (Specifically, in the notation of [18] Sect. IV.4, for all \( \lambda \in \mathfrak{a}^* \), we have \( |\phi_\lambda(g)| \leq \phi_0(g), \) and estimates on \( \phi_0 \) (e.g., [1] Prop. 2.2.12) show that \( \phi_0 \) is in \( L^{2+\varepsilon} \) for all \( \varepsilon > 0. \))

Choosing \( \varepsilon \) so that \( 1/p + 1/(2+\varepsilon) = 1 \) establishes the desired boundedness.

Second proof. If \( c_\lambda = 0 \) (i.e., if \( \lambda \in \mathbb{Z} \) and \( \lambda \leq d \)), then there is nothing to prove. Thus we assume \( c_\lambda \) is nonzero, in which case \( c_{\lambda+1} \) is also nonzero. The invariance of \( F_\lambda \) under automorphisms together with the square-integrability of the function \( (1-|z|^2)^\lambda \) for \( \lambda > d/2 \) show that the integral defining \( A_\lambda f(z) \) is absolutely convergent for all \( z. \)

We introduce the (hyperbolic) Laplacian \( \Delta \) for \( \mathbb{B}^d \), given by

\[
\Delta = (1-|z|^2) \sum_{j,k=1}^d (\delta_{jk} - \bar{z}_j z_k) \frac{\partial^2}{\partial z_j \partial \bar{z}_k}.
\]  

(This is a negative operator.) This operator commutes with the automorphisms of \( \mathbb{B}^d \). It is known (e.g., [28]) that \( \Delta \) is an unbounded self-adjoint operator on \( L^2(\mathbb{B}^d, \tau) \), on the domain consisting of those \( f \)'s in \( L^2(\mathbb{B}^d, \tau) \) for which \( \Delta f \) in the distribution sense belongs to \( L^2(\mathbb{B}^d, \tau) \). In particular, if \( f \in L^2(\mathbb{B}^d, \tau) \) is \( C^2 \) and \( \Delta f \) in the ordinary sense belongs to \( L^2(\mathbb{B}^d, \tau) \), then \( f \in \text{Dom}(\Delta). \)

We now claim that

\[
\Delta_z F_\lambda(z, w) = \lambda(\lambda - d)(F_\lambda(z, w) - F_{\lambda+1}(z, w)),
\]  

where \( \Delta_z \) indicates that \( \Delta \) is acting on the variable \( z \) with \( w \) fixed. Since \( \Delta \) commutes with automorphisms, it again suffices to check this when \( w = 0, \) in which case it is a straightforward algebraic calculation. Suppose, then, that \( \phi \) is a \( C^\infty \) function of compact support. In that case, we are free to differentiate under the integral to obtain

\[
\Delta A_\lambda \phi = \lambda(\lambda - d)A_\lambda \phi - \lambda(\lambda - d)A_{\lambda+1} \phi.
\]
Now, the invariance of \( F_\lambda \) tells us that \( L^2(\mathbb{B}^d, \tau) \) norm of \( F_\lambda(z, w) \) as a
function of \( z \) is finite for all \( w \) and independent of \( w \). Putting the \( L^2 \) norm inside the
integral then shows that \( A_\lambda \phi \) and \( A_{\lambda+1} \phi \) are in \( L^2(\mathbb{B}^d, \tau) \). This shows that \( A_\lambda \phi \) is
in \( \text{Dom}(\Delta) \). Furthermore, the condition \( \lambda > d/2 \) implies that \( \lambda(\lambda - d/2) > -d^2/4 \).
It is known that the \( L^2 \) spectrum of \( \Delta \) is \((-\infty, -d^2/4]\). For general symmetric
space of the noncompact type, the \( L^2 \) spectrum of the Laplacian is \((-\infty, -\|\rho\|^2]\),
where \( \rho \) is half the sum of the positive (restricted) roots for \( G/K \), counted with
their multiplicity. In our case, there is one positive root \( \alpha \) with multiplicity \((2d-2)\)
and another positive root \( 2\alpha \) with multiplicity \(1\). (See the entry for “A IV” in Table VI of Chapter X of \[17\].) Thus, \( \rho = d\alpha \).
It remains only to check that if the metric is normalized so that the Laplacian comes out as in
\((5.9)\), then \( \|\alpha\|^2 = 1/4 \).
This is a straightforward but unilluminating computation, which we omit.

Since \( \lambda(\lambda - d) \) is in the resolvent set of \( \Delta \), we may rewrite \((5.8)\) as
\[
A_\lambda \phi = -\lambda(\lambda - d)[\Delta - \lambda(\lambda - d)I]^{-1}A_{\lambda+1} \phi.
\]
Suppose now that \( \lambda + 1 > d \), so that (as remarked above) \( A_{\lambda+1} \) is bounded. Since
\([\Delta - cI]^{-1} \) is a bounded operator for all \( c \) in the resolvent of \( \Delta \), we see that \( A_\lambda \)
has a bounded extension from \( C_c^\infty(\mathbb{B}^d) \) to \( L^2(\mathbb{B}^d, \tau) \). Since the integral computing
\( A_\lambda \phi(z) \) is a continuous linear functional on \( L^2(\mathbb{B}^d, \tau) \) (integration against an element
of \( L^2(\mathbb{B}^d, \tau) \)), it is easily seen that this bounded extension coincides with the original definition of \( A_\lambda \).

The above argument shows that \( A_\lambda \) is bounded if \( \lambda > d/2 \) and \( \lambda + 1 > d \).
Iteration of the argument then shows boundedness for all \( \lambda > d/2 \).
\( \square \)

**Proof of Theorem 12.** We wish to show that for all \( \lambda > d/2 \), if \( \phi \) is in \( L^2(\mathbb{B}^d, \tau) \),
then there is a unique Hilbert–Schmidt operator \( T_\phi \) with matrix entries given in
\((5.1)\) for all polynomials, and furthermore, \( \|T_\phi\|_{HS} = (\phi, A_\lambda \phi)_\lambda \). At the beginning
of this section, we had an calculation of \( \|T_\phi\|_2 \) in terms of \( A_\lambda \), but this argument relied
on writing \( T_\phi \) as \( P_\lambda M_\lambda P_\lambda \), which does not make sense for \( \lambda \leq d \).

We work with an orthonormal basis for \( H(\mathbb{B}^d, \lambda) \) consisting of normalized monomials, namely,
\[
e_m(z) = z^m \sqrt{\frac{\Gamma(\lambda + |m|)}{m!\Gamma(\lambda)}},
\]
for each multi-index \( m \). Then we want to establish the existence of a Hilbert–
Schmidt operator whose matrix entries in this basis are given by
\[
A_{lm} := c_\lambda \int_{\mathbb{B}^d} \overline{c_l(z)} \phi(z) e_m(z)(1 - |z|^2)^\lambda \ d\tau(z). \tag{5.9}
\]
There will exist a unique such operator provided that \( \sum_{l,m} |A_{lm}|^2 < \infty \).
If we assume, for the moment, that Fubini’s Theorem applies, we obtain
\[
\sum_{l,m} |a_{lm}|^2 = c_2^2 \int_{\mathbb{B}^d} \int_{\mathbb{B}^d} \sum_{l,m} \frac{\Gamma(\lambda + |l|)}{l!\Gamma(\lambda)} \frac{\Gamma(\lambda + |m|)}{m!\Gamma(\lambda)} |z|^l |w|^m \bar{z}^m \bar{w}^m
\times \phi(z)\overline{\phi(w)} (1 - |z|^2)^\lambda (1 - |w|^2)^\lambda \, d\tau(z) \, d\tau(w),
\] (5.10)
where \( l \) and \( m \) range over all multi-indices of length \( d \).

We now apply the binomial series
\[
\frac{1}{(1 - r)^\lambda} = \sum_{k=0}^{\infty} \binom{\lambda + k - 1}{k} r^k
\]
for \( r \in \mathbb{C} \) with \( |r| < 1 \), where
\[
\binom{\lambda + k - 1}{k} = \frac{\Gamma(\lambda + k)}{k!\Gamma(\lambda)}.
\]
(This is the so-called negative binomial series.) We apply this with \( r = \sum_j \bar{z}_j w_j \), and we then apply the (finite) multinomial series to the computation of \( (\bar{z} \cdot w)^k \).

The result is that
\[
\sum_l \frac{\Gamma(\lambda + |l|)}{l!\Gamma(\lambda)} |z|^l |w|^l = \frac{1}{(1 - \bar{z} \cdot w)^\lambda},
\] (5.11)
where the sum is over all multi-indices \( l \). Applying this result, (5.10) becomes
\[
\sum_{l,m} |a_{lm}|^2 = \langle \phi, A_\lambda \phi \rangle_\lambda,
\] (5.12)
which is what we want to show.

Assume at first that \( \phi \) is “nice,” say, continuous and supported in a ball of radius \( r < 1 \). This ball has finite measure and \( \phi \) is bounded on it. Thus, if we put absolute values inside the sum and integral on the right-hand side of (5.10), finiteness of the result follows from the absolute convergence of the series (5.11).

Thus, Fubini’s Theorem applies in this case.

Now for a general \( \phi \in L^2(\mathbb{B}^d, \tau) \), choose \( \phi_j \) converging to \( \phi \) with \( \phi_j \) “nice.” Then (5.12) tells us that \( T_{\phi_j} \) is a Cauchy sequence in the space of Hilbert–Schmidt operators, which therefore converges in the Hilbert–Schmidt norm to some operator \( T \). The matrix entries of \( T_{\phi_j} \) in the basis \( \{e_m\} \) are by construction given by the integral in (5.9). The matrix entries of \( T \) are the limit of the matrix entries of \( T_{\phi_j} \), hence also given by (5.9), because \( e_l \) and \( e_m \) are bounded and \( (1 - |z|^2)^\lambda \) belongs to \( L^2(\mathbb{B}^d, \tau) \) for \( \lambda > d/2 \).

We can now establish that (5.2) in Theorem 12 holds for all bounded holomorphic functions \( f \) and \( g \) in \( H(\mathbb{B}^d, \lambda) \) by approximating these functions by polynomials.\[\square\]
Proof of Theorem 13. In the proof of Theorem 12, we did not use the assumption \( \lambda > d/2 \) until the step in which we approximated arbitrary functions in \( L^2(\mathbb{B}^d, \tau) \) by “nice” functions. In particular, if \( \phi \) is nice, then (5.9) makes sense for all \( \lambda > 0 \), and (5.12) still holds. Now, since \( F_\lambda(z, w) = f_\lambda(\phi_w(z)) \), where \( f_\lambda \) is given by (5.5), we see that \( |F_\lambda(z, w)| \leq c_2^2 \) for all \( z, w \in \mathbb{B}^d \). Thus,

\[
\langle \phi, A_\lambda \phi \rangle \leq c_2^2 \| \phi \|_{L^1(\mathbb{B}^d, \tau)}^2
\]

for all nice \( \phi \). An easy approximation argument then establishes the existence of a Hilbert–Schmidt operator with the desired matrix entries for all \( \phi \in L^1(\mathbb{B}^d, \tau) \), with the desired estimate on the Hilbert–Schmidt norm. \( \square \)

References

[1] J.-P. Anker and L. Ji, Heat kernel and Green function estimates on noncompact symmetric spaces. Geom. Funct. Anal. 9 (1999), 1035-1091.
[2] J. Arazy, Integral formulas for the invariant inner products in spaces of analytic functions on the unit ball. Function spaces (Edwardsville, IL, 1990), 9–23, Lecture Notes in Pure and Appl. Math., 136, Dekker, New York, 1992.
[3] J. Arazy and G. Zhang, Homogeneous multiplication operators on bounded symmetric domains. J. Funct. Anal. 202 (2003), 44–66.
[4] F. Beatrous Jr. and J. Burbea, Holomorphic Sobolev spaces on the ball. Dissertationes Math. (Rozprawy Mat.) 276 (1989), 60 pp.
[5] F. A. Berezin, Quantization. Math. USSR Izvestija, 8 (1974), 1109-1165.
[6] F. A. Berezin, Quantization in complex symmetric spaces. Math. USSR Izvestija 9 (1976), 341-379.
[7] M. Bordemann, E. Meinrenken, and M. Schlichenmaier, Toeplitz quantization of Kähler manifolds and \( gl(N), N \rightarrow \infty \) limits. Comm. Math. Phys. 165 (1994), 281–296.
[8] D. Borthwick, A. Lesniewski, and H. Upmeier, Nonperturbative deformation quantization of Cartan domains. J. Funct. Anal. 113 (1993), 153–176.
[9] D. Borthwick, T. Paul, and A. Uribe, Legendrian distributions with applications to relative Poincaré series. Invent. Math. 122 (1995), 359–402.
[10] L. A. Coburn, Deformation estimates for the Berezin-Toeplitz quantization. Comm. Math. Phys. 149 (1992), 415–424.
[11] M. Cowling, The Kunze–Stein phenomenon. Ann. Math. 107 (1978), 209-234.
[12] P. Duren and A. Schuster, Bergman spaces. Mathematical surveys and monographs; no.100, American Mathematical Society, 2004.
[13] M. Engliš, Berezin transform and the Laplace-Beltrami operator. Algebra i Analiz 7 (1995), 176–195; translation in St. Petersburg Math. J. 7 (1996), 633–647.
[14] G. Folland, Harmonic analysis in phase space. Princeton University Press, 1989.
[15] B. C. Hall, Holomorphic methods in analysis and mathematical physics. In: First Summer School in Analysis and Mathematical Physics (S. Pérez-Esteve and C. Villegas-Blas, Eds.), 1–59, Contemp. Math., 260, Amer. Math. Soc., 2000.
[16] B. C. Hall and W. Lewkeeratiyutkul, *Holomorphic Sobolev spaces and the generalized Segal–Bargmann transform*. J. Funct. Anal. 217 (2004), 192–220.

[17] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*. Academic Press, 1978.

[18] S. Helgason, *Groups and geometric analysis: Integral geometry, invariant differential operators, and spherical functions*, corrected reprint of the 1984 edition. Amer. Math. Soc., 2000.

[19] S. Helgason, *Geometric analysis on symmetric spaces*. Amer. Math. Soc., 1994.

[20] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman spaces*. Springer-Verlag, 2000.

[21] H. T. Kaptanoğlu, *Besov spaces and Bergman projections on the ball*. C. R. Math. Acad. Sci. Paris 335 (2002), 729–732.

[22] S. Klimek and A. Lesniewski, *Quantum Riemann surfaces I. The unit disc*. Commun. Math. Phys., 46 (1976) 103-122.

[23] A. Konechny, S. G. Rajeev, and O. T. Turgut, *Classical mechanics on Grassmannian and disc*. In: Geometry, integrability and quantization (Varna, 2000), 181–207, Coral Press Sci. Publ., Sofia, 2001.

[24] R. A. Kunze and E. M. Stein, *Uniformly bounded representations and harmonic analysis of the 2 × 2 unimodular group*. Amer. J. Math. 82 (1960), 1-62.

[25] S. G. Rajeev and O. T. Turgut, *Geometric quantization and two-dimensional QCD*. Comm. Math. Phys. 192 (1998), 493–517.

[26] J. H. Rawnsley, *Coherent states and Kähler manifolds*. Quart. J. Math. Oxford Ser. (2) 28 (1977), 403–415.

[27] J. H. Rawnsley, M. Cahen, S. Gutt, *Quantization of Kähler manifolds. I. Geometric interpretation of Berezin’s quantization*. J. Geom. Phys. 7 (1990), 45–62.

[28] R. Strichartz, *Harmonic analysis as spectral theory of Laplacians*. J. Funct. Anal. 87 (1989), 51–148.

[29] A. Unterberger and H. Upmeier, *The Berezin transform and invariant differential operators*. Commun. Math. Phys., 164 (1994), 563-597.

[30] Z. Yan, *Invariant differential operators and holomorphic function spaces*. J. Lie Theory 10 (2000), 31 pp.

[31] R. Zhao and K. H. Zhu, *Theory of Bergman spaces in the unit ball of \( \mathbb{C}^n \)*. Preprint, arxiv.org/abs/math/0611093.

[32] K. Zhu, *Holomorphic Besov spaces on bounded symmetric domains*. Quart. J. Math. Oxford Ser. (2) 46 (1995), 239–256.

Kamthorn Chailuek
Department of Mathematics
Prince of Songkla University
Hatayai, Songkhla, Thailand 90112

Kamthorn Chailuek
Department of Mathematics
Prince of Songkla University
Hatayai, Songkhla, Thailand 90112
e-mail: kamthorn.c@psu.ac.th
Brian C. Hall  
Department of Mathematics  
University of Notre Dame  
255 Hurley Building  
Notre Dame IN 46556-4618 USA  
e-mail: bhall@nd.edu