First quantum corrections for a hydrodynamics of a nonideal Bose gas

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Abstract

In the paper we consider a hydrodynamical description of a nonideal Bose gas in one-loop approximation. We calculate an effective action which consists of mean field contributions and first quantum correction. This provides the equations of motion for the density and velocity of the gas where both mean field contributions and fluctuations are presented. To fulfill the calculation we make use of the formalism of functional integrals to map the problem to a problem of quantum gravity to benefit from a well-developed technique in this field. This effective action provides all correlation functions for the system and is a basis for a consideration of dynamics of the gas. Response functions are briefly discussed. Applications to the trapped bosons are reviewed. Together with Ref.\textsuperscript{[9]} the paper provide complete description of the condensate fraction and the depletion in the case of Bose condensed gases.

1 Introduction

Recently a lot of attention was attracted to the theory of a nonuniform Bose gas and its collective excitations. The interest was stimulated by the success of the experimental observation of Bose-Einstein condensation systems of spin polarized magnetically trapped atomic gases at ultra-low temperatures \textsuperscript{[1,2,3]} and experimental studies of their collective properties \textsuperscript{[4]}. On the other hand, new exciting problems such as a description of the evolution of Bose condensate from relaxed trap, dynamics of a collapse of the condensate for Li\textsuperscript{7} atoms, heating-cooling phenomena, various coherence effects for the condensate and so on are being posed both theoretically and experimentally.

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Theoretically the evolution of the condensate and all related phenomena were investigated in number of papers [5, 6, 7, 15] but all these considerations were formulated in Hartree-Fock (or mean field) approximation. It means that the calculations were based on Nonlinear Schrodinger equation (NSE) or Ginzburg-Gross-Pitaevskii equation (GPE) [8]. It is obvious that the GPE describes strongly interacting Bose condensed gas well since it is possible to ignore non-condensed fraction. However, as it was noted in [9], this approximation fails to describe heating-cooling effects, fast dynamics or fast external potential variations, fluctuation effects around the edges of a trap and so on. To include the phenomena in the self-consistent consideration quantum corrections to the GPE should be calculated.

Let us note that in the previous papers GPE plays double role. First of all it describes the Bose condensate fraction. But it also gives the dynamics of full density since in this approximation the deplition was ignored. Based on GPE collective excitations and the deplition were considered in Refs [10, 11, 12, 13, 14] but the it was not self-consistent and the interaction of the condensate fraction with the noncondensed one have not been taken into account [16]. As a result, to leave the mean field approximation the GPE equation can be generalized in two different ways – we can write an equation for the condensate fraction as it was done in [9] and its analogue for the full gas density. In the paper [9] first (one-loop) quantum correction in the hydrodynamical approximation was calculated to generalize the GPE for the condensate fraction including effects of the interaction of the condensate with collective excitations (noncondensed fraction). To calculate the deplition in the framework we have to find Green functions for the corresponding effective action. Being principally straightforward this is not an easy technical problem. However there is a simpler way to find the deplition as a difference between full density and the condensate density if the first is calculated in the same approximation (it is not difficult to see that the mean field equation for the density is once again GPE as it was noted above). In the paper we solve the problem and give dynamical equations for the full density of the particles which generalizes GPE equation with an account of first quantum correction to the mean field picture. This allows us to calculate the deplition as well.

The paper is organized as follows. In the next section we introduce functional integration to calculate the effective hydrodynamic action for the system and define one-loop quantum corrections which are responsible for the fluctuation effects. These quantum corrections are evaluated using \( \zeta \)-function regularization for functional determinants and an explicit and very simple expression for the one-loop quantum corrections is presented. This permits in Section 3 the derivation of the quantum corrections to the equations of motion for the gas in closed and explicit form readily used in numerical calculations. Here corrections to the Nonlinear Schrödinger equation (due to the interaction of the mean field background with the fluctuations) are obtained. Section 4 is devoted to a consideration of Green functions and response functions. It is shown that all Green functions in the 1-loop approximation can be obtained from the effective action of Section 3 by functional differentiation. We conclude the paper with a discussion of the possible applications of our results. The analysis does not depend on the details of the confining potential and can be used in a variety of problems. The only simplification is common hydrodynamic approximation.
2 Hydrodynamical picture

As it was said above we consider nonideal Bose gas with point-like interaction in external potential. The Hamiltonian of the system then can be written as

\[ H = \frac{\hbar^2}{2m} \int dV \left[ \nabla \psi^+ \nabla \psi + \nu(x,t) \psi^+ \psi + 4\pi l \psi^+ \psi \psi \psi \right], \quad (1) \]

where \( \nu(x,t) \) is an external potential. For the case of experimentally trapped atoms the potential is supposed to be harmonic and can be varied or even switched off at some moment, say \( t = 0 \). More precisely it means that for the case of magnetically trapped atoms \( \nu = \theta(-t)a^-_\perp(t)(\rho^2 + \lambda^2(t)z^2) \), \( \rho^2 = x^2 + y^2 \), \( (a_\perp \text{ is oscillator lengths}) \) has to be put in Eqn(1). Another parameter in Eqn(1) is \( l \) — the s-wave scattering length for atoms in the system \[ 18 \]. The operators \( \psi^+, \psi \) are Bose creation and annihilation operators. This Hamiltonian is a basis for the calculation of quantum mechanical and thermodynamical quantities. For example, the corresponding vacuum-vacuum transition amplitude is given by the following functional integral:

\[ Z(\mu) = \int \mathcal{D}\psi^+(x,\tau) \mathcal{D}\psi(x,\tau) \exp[iS/\hbar], \quad (2) \]

with the action

\[ S = \int_{-\infty}^{\infty} d\tau \int dV \left\{ i\hbar \frac{\partial \psi}{\partial \tau} - \frac{\hbar^2}{2m} \left[ \nabla \psi^+ \nabla \psi + \nu \psi^+ \psi + 4\pi l \psi^+ \psi \right] \right\}. \quad (3) \]

where the parameter \( \mu \) is the chemical potential, controlling the number of particles in the system.

Now we will derive the hydrodynamic description directly from the functional integral formalism, which simplifies the description. We will consider zero temperature case but the generalisation for the case of finite temperatures is straightforward (though cumbersome) using Keldysh technique \[ 19 \]. As it is explained in Appendix A, we are looking for the effective action which describes all physical quantities for the system. For example, the effective action provides all Green’s functions in the same approximation used to calculate the effective action itself. We will obtain it in a hydrodynamic one-loop approximation. To clarify the description hydrodynamical variables, density and velocity, should be used. More precisely, we start with the action \[ 3 \]

\[ \frac{S}{\hbar} = \frac{1}{\hbar} \int_{-\infty}^{\infty} d\tau \int dV \left\{ i\hbar \frac{\partial \psi}{\partial \tau} - \frac{\hbar^2}{2m} \left[ \nabla \psi^+ \nabla \psi + \nu \psi^+ \psi + 4\pi l \psi^+ \psi \right] \right\}, \quad (4) \]

and rescale variables as \( \tau \to \frac{2m}{\hbar^2} \tau, \mu \to \frac{\hbar^2}{2m} \mu, l \to \frac{1}{4\pi} \) to get the following action form:

\[ \frac{S}{\hbar} = \int_{-\infty}^{\infty} d\tau \int dV \left\{ i\psi^+ \frac{\partial \psi}{\partial \tau} - \left[ \nabla \psi^+ \nabla \psi + (\nu - \mu) \psi^+ \psi + l \psi^+ \psi \right] \right\}. \quad (5) \]

(We will measure the action in terms of \( \hbar \) everywhere below). Now we change field variables to the hydrodynamic ones:

\[ \psi(x,\tau) = \sqrt{\rho(x,\tau)} e^{-i\varphi(x,\tau)} \quad , \quad \psi^+(x,\tau) = \sqrt{\rho(x,\tau)} e^{i\varphi(x,\tau)}. \]
Then the action (5) takes the form (up to a complete derivative term):

$$S = \int_{-\infty}^{\infty} d\tau \int dV \left\{ \frac{\partial \varphi}{\partial \tau} \rho - (\nabla \varphi)^2 - \rho(\nabla \varphi)^2 - (v - \mu)\rho - l\rho^2 \right\}. \quad (6)$$

It is not difficult to see that the classical equations of motion for the action lead to the equations:

$$\frac{\partial \varphi}{\partial \tau} - (\nabla \varphi)^2 - v + \mu - 2l\rho + \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} = 0, \quad (7)$$

$$- \frac{\partial \rho}{\partial \tau} + 2\nabla (\nabla \varphi \cdot \rho) = 0, \quad (8)$$

which are equivalent to the Gross-Pitaevskii equation [12] after the introduction of the velocity variable $c = -2\nabla \varphi$ instead of $\varphi$. In velocity-density variables these equations look as hydrodynamic equations for an irrotational compressible fluid:

$$\frac{\partial c}{\partial \tau} + \nabla \left( \frac{c^2}{2} + 2v - 2\mu + 4l\rho - \frac{2}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} \right) = 0, \quad (9)$$

$$\frac{\partial \rho}{\partial \tau} + \nabla (c \cdot \rho) = 0. \quad (10)$$

Now let us shift our variables by the zero-order mean field solution (see Appendix A for details):

$$\rho \rightarrow \rho + \sigma, \quad \varphi \rightarrow \varphi + \alpha$$

and we will keep up only terms including the square of $\sigma$ and $\alpha$ terms. Then the action transforms to $S(\rho, \varphi) + S_1(\rho, \varphi, \sigma, \alpha)$ where the action $S_1$ has the form:

$$S_1 = \int_{-\infty}^{\infty} d\tau \int dV \left\{ \frac{\partial \alpha}{\partial \tau} \sigma - \rho(\nabla \alpha)^2 - 2\sigma \nabla \cdot \varphi \nabla \alpha - l\sigma^2 - \frac{1}{4\rho} (\nabla \sigma)^2 - \frac{1}{8\rho} \left[ \Delta \rho + \frac{3(\nabla \rho)^2}{2\rho^2} \right] \sigma^2 \right\}. \quad (11)$$

Last two terms vanish in the hydrodynamical limit since $\rho$ is a large variable. Hence the action $S_1$ takes form:

$$S_1 = \int_{-\infty}^{\infty} d\tau \int dV \left\{ \frac{\partial \alpha}{\partial \tau} \sigma - \rho(\nabla \alpha)^2 - l\sigma^2 - 2\sigma \nabla \varphi \cdot \nabla \alpha \right\}. \quad (12)$$

Integrating in functional integral over the field $\sigma$ we get the action for $\alpha$ field only:

$$S_1 = \int_{-\infty}^{\infty} d\tau \int dV \left\{ -\rho(\nabla \alpha)^2 + \frac{1}{4l} \left( \frac{\partial \alpha}{\partial \tau} - 2\nabla \varphi \nabla \alpha \right)^2 \right\}. \quad (13)$$

The corresponding determinant is an inessential constant.

Now we will use the large parameter $\rho$ (for harmonic trap $\rho \sim N^{2/5}l^{-3/5}$ if co-ordinates and $l$ are measured in harmonic length units $a_\perp$). Let us introduce the following large number $\rho_0 \equiv \max \{\rho\}$ such that $\tilde{\rho} = \frac{\rho}{\rho_0} \sim 1$, $t \equiv \tau \sqrt{4l/\rho_0}$, $\tilde{\nabla} = \frac{\nabla \varphi}{\sqrt{\rho_0}} \sim 1$. Then
where the matrix $A$ has a form
\[
A = \begin{pmatrix}
-1 & \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 \\
\tilde{v}_1 & \tilde{\rho} - \tilde{v}_1^2 & -\tilde{v}_1 \tilde{v}_2 & -\tilde{v}_1 \tilde{v}_3 \\
\tilde{v}_2 & -\tilde{v}_1 \tilde{v}_2 & \tilde{\rho} - \tilde{v}_2^2 & -\tilde{v}_2 \tilde{v}_3 \\
\tilde{v}_3 & -\tilde{v}_1 \tilde{v}_3 & -\tilde{v}_2 \tilde{v}_3 & \tilde{\rho} - \tilde{v}_3^2
\end{pmatrix}
\]
with its determinant $\det A = -\tilde{\rho}^3$.

Our next step is to cast the action in the covariant form. To do this we introduce auxiliary metric $\tilde{g}_{\mu\nu}$ such that
\[
A_{\mu\nu} = \tilde{g}_{\mu\nu} \sqrt{-\det (||\tilde{g}||)}.
\]
One can easily find the covariant metric from the equation above:
\[
\tilde{g}^{\mu\nu} = \frac{A^{\mu\nu}}{-\det (||A||)}
\]
that leads to the form:
\[
||\tilde{g}|| = \tilde{\rho}^{-3/2} \begin{pmatrix}
-1 & \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 \\
\tilde{v}_1 & \tilde{\rho} - \tilde{v}_1^2 & -\tilde{v}_1 \tilde{v}_2 & -\tilde{v}_1 \tilde{v}_3 \\
\tilde{v}_2 & -\tilde{v}_1 \tilde{v}_2 & \tilde{\rho} - \tilde{v}_2^2 & -\tilde{v}_2 \tilde{v}_3 \\
\tilde{v}_3 & -\tilde{v}_1 \tilde{v}_3 & -\tilde{v}_2 \tilde{v}_3 & \tilde{\rho} - \tilde{v}_3^2
\end{pmatrix}.
\]
Finally we get the metric $||\tilde{g}||$:
\[
||\tilde{g}_{\mu\nu}|| = \tilde{\rho}^{1/2} \begin{pmatrix}
-\tilde{\rho} + \tilde{v}^2 & \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 \\
\tilde{v}_1 & 1 & 0 & 0 \\
\tilde{v}_2 & 0 & 1 & 0 \\
\tilde{v}_3 & 0 & 0 & 1
\end{pmatrix}
\]
with the determinant $\tilde{g} \equiv \det (||\tilde{g}||) = -\tilde{\rho}^3$.

In this metric the action takes a covariant form
\[
S_1 = -\sqrt{\frac{\rho_0}{4l}} \int dx \sqrt{-\tilde{g}} \tilde{\alpha} \tilde{\square} \alpha = \sqrt{\frac{\rho_0}{4l}} \int dx \sqrt{-\tilde{g}} \alpha \frac{1}{\sqrt{-\tilde{g}}} \partial_{\mu} \tilde{g}^{\mu\nu} \sqrt{-\tilde{g}} \partial_{\nu} \alpha.
\]
This is the central equation of the paper.

Now the effective action is $\Gamma_1 = -\text{tr} \ln[(\rho_0/4l)^{1/2} \tilde{\square}]/2$ with $\tilde{\square} = -\frac{1}{\sqrt{-\tilde{g}}} \partial_{\mu} \tilde{g}^{\mu\nu} \sqrt{-\tilde{g}} \partial_{\nu}$. Let us again make use the existence of large parameter in the system. Indeed, the multiplier
$(\rho_0/4l)^{1/2}$ gives a possibility to express main contributions to the determinant such as (see Appendix B):

$$\text{tr} \ln[(\rho_0/4l)^{1/2}e] = \text{tr} \ln[e] + \frac{1}{2} \text{tr} \ln(\rho_0/4l)(\Phi_0(e) - L(e))$$

$$\sim \frac{1}{2} \text{tr} \ln(\rho_0/4l)(\Phi_0(e) - L(e))$$

since in our regularization $\text{tr} \ln[e]$ is order of Seeley coefficient $\Phi_0(e) \equiv \int dx \tau \sqrt{-g}\Psi_0$. Returning to the initial variables all curvature tensors and the metric are written in the “physical” variables (i.e. without tildes). Moreover $\ln(\rho_0/4l) \gg \ln(\rho)$ so that we can substitute $\ln(\rho/4l)$ instead of $\ln(\rho_0/4l)$. Summarizing, we obtain an expression for the first quantum correction to the effective action:

$$\Gamma_1 = \Gamma'_1 + \Gamma''_1$$

where

$$\Gamma'_1 = -\frac{1}{4} \int dx \tau \sqrt{-g}\ln(\rho/4l)\Psi_0(\Box) ,$$

$$\Gamma''_1 = \frac{1}{4} \ln(\rho_0/4l)L(\Box) .$$

with the metric

$$\|g_{\mu\nu}\| = \rho^{1/2} \begin{pmatrix} -\rho + v^2 & v_1 & v_2 & v_3 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix} .$$

and $v_i \equiv 2\partial_i\phi$.

### 3 Quantum corrections to equations of motion

In this section we derive from (16) required quantum corrections to equations of motion (see Appendix A). To make the consideration self-consistent we should not differentiate $\ln(\rho/4l)$ because in our approximation it behaves like a constant. In that case, the action we have to vary is covariant again. we find:

$$\frac{\delta \Gamma_1}{\delta \rho} = \frac{\delta g_{\mu\nu}}{\delta \rho} \frac{\delta \Gamma'_1}{\delta g_{\mu\nu}} + \frac{1}{4} \ln(\rho/4l) \frac{\delta L(\Box)}{\delta \rho} ,$$

$$\frac{\delta \Gamma_1}{\delta \phi} = \frac{\delta g_{\mu\nu}}{\delta \phi} \frac{\delta \Gamma'_1}{\delta g_{\mu\nu}} + \frac{1}{4} \ln(\rho/4l) \frac{\delta L(\Box)}{\delta \phi} .$$

Now,

$$\frac{\delta \Gamma'_1}{\delta g_{\mu\nu}} = -\frac{1}{4} \ln(\rho/4l) \int dx \sqrt{-g}\Psi_0(\Box) = -\frac{1}{4} \ln(\rho/4l) \frac{\delta}{\delta g_{\mu\nu}}\Phi_0(\Box)$$
Let us remember that
\[
\Phi_0(\Box) = \frac{1}{(4\pi)^2} \int dx \sqrt{-g} \left( -\frac{1}{30} \nabla^2 R + \frac{1}{72} R^2 - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{180} R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} \right)
\]

We note that one of the terms above is irrelevant:
\[
\int dx \sqrt{-g} \nabla^2 R = \int dx \sqrt{-g} \frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} g_{\mu\nu} \partial_{\nu} R = \int dx \partial_{\mu}(\sqrt{-g} \partial^{\mu} R)
\]
since this is just a complete derivative. Hence for our purposes it is sufficient to consider only
\[
\Phi_0(\Box) = \frac{1}{(4\pi)^2} \int dx \sqrt{-g} \left( \frac{1}{72} R^2 - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{180} R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} \right).
\]

Long but quite straightforward calculations lead us to the following result for the functional derivative (see Appendix C for details of the calculation):
\[
\frac{\delta}{\delta g_{\mu\nu}} \Phi_0(\Box) = \frac{1}{36(4\pi)^2} \sqrt{-g} \left\{ -\frac{1}{2} g_{\mu\nu} \left[ \frac{1}{2} R^2 - \frac{1}{5} R_{\sigma\rho} R^{\sigma\rho} + \frac{1}{5} R_{\sigma\rho\alpha\beta} R^{\sigma\rho\alpha\beta} \right] + R_{\mu\nu} R - \frac{2}{5} R_{\mu\alpha} R_{\nu}^{\alpha} + \frac{2}{5} R_{\mu\nu\alpha\beta} R_{\nu}^{\alpha\beta} - g_{\mu\nu} R + \frac{1}{2} \left\{ \nabla_{\mu}, \nabla_{\nu} \right\} R + \frac{1}{5} \left\{ -2 \nabla^{\sigma} \nabla_{\nu} R_{\mu}^{\sigma} + \Box R_{\mu\nu} + g_{\mu\nu} \nabla^{\sigma} \nabla_{\sigma} R_{\rho}^{\rho} \right\} + \frac{4}{5} \nabla^{\rho} \nabla_{\sigma} R_{\mu\rho\nu} \right\}
\]
taking into account
\[
\frac{\delta}{\delta g_{\mu\nu}} \Phi_0(\Box) = -g^{\mu\sigma} g^{\nu\rho} \frac{\delta}{\delta g_{\sigma\rho}} \Phi_0(\Box).
\]

This means that the equations of motions with 1-loop quantum correction take the form:
\[
\frac{\partial \varphi}{\partial \tau} - (\nabla \varphi)^2 - v + \mu - 2l \rho + \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} + \frac{\delta g_{\mu\nu}}{\delta \rho} \frac{\delta \Gamma_{\mu}}{\delta \rho} + \frac{1}{4} \ln(\rho/4l) \frac{\delta L(\Box)}{\delta \rho} = 0 , \quad (17)
\]
\[
- \frac{\partial \rho}{\partial \tau} + 2 \nabla(\nabla \varphi \cdot \rho) + \frac{\delta g_{\mu\nu}}{\delta \varphi} \frac{\delta \Gamma_{\mu}}{\delta \varphi} + \frac{1}{4} \ln(\rho/4l) \frac{\delta L(\Box)}{\delta \varphi} = 0 , \quad (18)
\]

These two equations (17) and (18) replace the mean field hydrodynamic equations (7,8) for the density and velocity of the Bose gas and are the main result of the paper. They contain contributions of fluctuations to the dynamics of the gas. For example, we note that the additional quantum pressure is due to dependence of the Bogoliubov particles determinant on the density of the condensate while local density changes (see continuity equation (18)) comes from phase-dependence of the determinant. There is no difficulty to see that the latter correction obeys particle conservation law (that means it is a divergence)[20].

To complete this section we consider the stationary situation when the equilibrium condensate velocity is equal to zero there is no local particle transfer. However we think that equations (17) and (18) are of the main interest in non-equilibrium problems such as evolution of the gas under changing of trap shapes and so on. In equilibrium case of trapped atomic gases, the velocity field is equal to zero as are all time derivatives in the equations.
of the previous section. This allows us to greatly simplify the equations and find quantum corrections in closed and compact form. Indeed, it is cumbersome but straightforward to check that in the regime there is no corrections to continuity equation \( (11) \), \( L(\square) = 1 \), and there is only the contribution to the quantum pressure term in equation \( (9) \). This means that the equation of motion take the form:

\[
\frac{c^2}{4} + v - \mu + 2l\rho - \frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} + \delta P = 0, \tag{19}
\]

where the correction to quantum pressure \( \delta P \) has the form:

\[
\delta P = \frac{\ln(\rho/4l)}{48(4\pi)^2 \sqrt{\rho}} \left( \frac{43}{20} \Delta^2 \ln \rho + \frac{43}{20} (\partial_i \ln \rho)(\partial_i \Delta \ln \rho) \right.
\]

\[
+ \frac{27}{10} (\partial_i \partial_k \ln \rho)(\partial_i \partial_k \ln \rho) - \frac{87}{80} (\Delta \ln \rho)^2
\]

\[
- \frac{67}{80} (\partial_i \ln \rho)(\partial_k \partial_k \ln \rho)(\partial_i \partial_k \ln \rho)
\]

\[
- \frac{197}{160} (\partial_i \ln \rho)^2 \Delta \ln \rho - \frac{417}{1280} [((\partial_i \ln \rho)^2]^2 \right) \tag{20}
\]

where roman indices correspond to spatial derivatives.

This implies the mean field approximation is applicable at the center of the trap where the correction \( \delta P \) is negligible. On Fig.1 we plot the contribution in the central region of the trap. Since the contribution is particularly important near to the classical turning points we do not make use the Thomas-Fermi approximation and exploit more correct analytic expression. The profile of trapped condensate was obtained in Ref. [21] in the mean field approximation as an interpolation between small density and high density expansions; this demonstrates very good agreement with numerical solution of the corresponding mean field equation [22]. The expression for the profile with \( N \) particles has the form:

\[
\rho(r) = \frac{1}{2l} W \left[ 2l N \exp(4C - r^2) \right], \tag{21}
\]

where \( W(x) \), defined as the principal branch (regular at the origin) solution to the Eq. \( We^W = x \) [23], is a standard MAPLE function [24]; Constant \( C \) in Eqn. (21) is to be determined from the normalization condition \( \int d^3x \rho(r) = N \) [25]. The profile of the condensate density calculated from eq.(21) is shown in Fig.1 for \( N = 10^5 \) atoms.

Two points should be noted. Firstly, the magnitude of the correction in the central region of the trap is of order of \( 10^{-6} \) relative to the main interaction term (see Figs 2) (the contribution in the intermediate region is shown on Fig.3). Secondly, the sign of the correction term changes as can be seen from Fig.4. This means that the character of exchange interaction due to excited states depends on the density of the condensate and leads to an effective attraction for density higher than some value and to an effective repulsion for a smaller density. However we have to remind that the leading contributions in the region are much larger than the corrections and after a self-consistent solution of the corrected
Nonlinear Schrödinger equation is performed the profile of the full density will not change significantly in this region of the trap.

Since the sign of the contribution $\delta P$ became positive, the value of this term starts to increase and becomes quite considerable. However, at some point, our hydrodynamic approximation has to fail and we need other correction terms which will stop the increase. Although we cannot treat them properly it is possible to give an approximation for this region based on the low density expansion. Up to now we dealt with a high density expansion denoted $\delta P_h$. It is easy to calculate the leading contribution in low density expansion that controls the behavior of depletion interaction correction at large distances. In section 2 we saw that this correction corresponds to sum of all the one-point diagrams for the system with the action $(2)$. In the low density regime the main contribution $\delta P_l$ is due to a one loop diagram. It means that the correction to eqn. $(19)$ $\delta P_l$ is

$$\delta P_l = 4l\delta \rho .$$  \hspace{1cm} (22)

It is hard to give exact expression for the depletion in this region since the comprehensive solution is required. We estimate the depletion as one fourth of the local density of the particles and use this estimate in Fig.5. It shows plots of $\delta P_l$ and $\delta P_h$ which let us estimate the contribution in the intermediate density region. It is easy to see that the two curves meet at point $r \sim 6.13$ where the density of the condensate $\rho \sim 1$ and both (high and low density) approximations fails. This however let us estimate the magnitude of the correction. It is obvious that the crack at the figure should be smoothed and this comes from high order corrections. We cannot calculate them explicitly and the only interpolation should be used. The magnitude of the correction to the Nonlinear Schrödinger equation is $\sim 1$ and in the region is compared with both kinetic and repulsion energies as it is shown in Fig.6.

We see that the only region where the corrections are not small is the boundary region. Here the corrections are order of both kinetic and interaction energies and can play significant role in the determination of the profile of the condensate close to classical turning points. The Thomas–Fermi approximation does not work in the region and hence both Stringari [12] and Wu and Griffin [14] approximations are inadequate. This explain the discrepancy of our results.

Other point to note is that the form of Eqn. (19) coincides with the form of the corresponding equation in the case of the condensate of the gas [1], i.e. there is only one generalization of GP equation for the stationary case. However, solutions for the condensate density and full density can be different since the generalized GP equations is highly nonlinear and its solution crucially depends on initial conditions.

4 Green functions, response functions and the like

Equations of motion and quantum corrections for them are the central questions of previous consideration. However in many applications related to experiment it is very important to analyze various response functions and form-factors. They can be expressed in terms of some combinations of Green functions of the theory. That is why in this section we shortly consider a calculation of Green functions in the effective action formalism.
As it is shown in Appendix A the effective action (in any self-consistent approximation) allows the evaluation of all Green functions in the same approximation by just taking of variational derivatives. Since the effective action in 1-loop approximation was obtained the problem of Green function calculation is the problem of functional differentiation only [26].

We now formalize all said above and give formulas to evaluate Green functions in effective action formalism. Let $W(J)$ being generating functional of connected Green functions. Then quantities

$$W_n(x_1, \ldots, x_n) = \frac{\delta}{\delta J(x_1)} \ldots \frac{\delta}{\delta J(x_n)} W(J)$$

are connected Green functions in the external field $J$. The effective action is defined then by the Legendre transformation of $W$:

$$\Gamma(\alpha) = W(J(\alpha)) - \alpha J(\alpha) , \quad \alpha = \frac{\delta W(J)}{\delta J}$$

where the function $\alpha(x) = \langle \hat{\phi}(x) \rangle = W_1(x; J)$ is the first connected Green function [23]. It is easy to see that the functions $\alpha$ and $A$ are related by the second relation (24). Then the quantities

$$\Gamma_n(x_1, \ldots, x_n) = \frac{\delta}{\delta \alpha(x_1)} \ldots \frac{\delta}{\delta \alpha(x_n)} \Gamma(\alpha)$$

define 1PI (one-particle-irreducible) Green functions for the theory with mean field $\alpha(x) = \langle \hat{\phi}(x) \rangle$. Knowledge of 1PI Green function is equivalent to knowledge of any (whole or connected) Green functions for the corresponding system. For the first 1PI Green function we have from (24)

$$\Gamma_1(x) = -J(x) .$$

All connected Green functions can be expressed in terms of 1PI ones. Indeed, differentiating (24) $J$ we obtain

$$W_2 \Gamma_2 = -1 , \quad W_2 = -\Gamma_2^{-1}$$

or, in expanded form,

$$\int dz W_2(x,z) \Gamma_2(z,y) = -\delta(x-y) .$$

Differentiating now the second relation in (27) on $J$ and using the following rule

$$\frac{\delta}{\delta J} = -\Gamma_2^{-1} \frac{\delta}{\delta \alpha}$$

we can derive expressions for all the higher connected Green functions through the 1PI ones. For example, for the third connected function we have

$$W_3 = -\left[ \Gamma_2^{-1} \right]^3 \Gamma_3$$

or

$$W_3(x_1, x_2, x_3) = - \int dy_1 dy_2 dy_3 \Gamma_2^{-1}(x_1, y_1) \Gamma_2^{-1}(x_2, y_2) \Gamma_2^{-1}(x_3, y_3) \Gamma_3(y_1, y_2, y_3)$$

(29)
and so on. It means that since we know 1PI functions we have to solve the differential equation \( \Box \) for the two-point correlation function and then find all other connected Green functions by integration of 1PI functions with two-point correlators as in Eqn.(29).

Putting \( J = 0 \) in (26) we get equations of motion for \( \rho \) and \( \varphi \) (17,18). To obtain, for example, the Green function \( \langle (\rho(x) - \langle \rho(x) \rangle)(\rho(y) - \langle \rho(y) \rangle) \rangle \) we twice differentiate the effective action on \( \rho \), substitute solution of equations of motion and finally invert the result in the sense of the kernel of an integral operator.

As an example of usage of this technique we calculate two-point Green function in the mean field approximation (tree or 0-loop approximation). In this approximation the effective action has the form:

\[
\Gamma = \int_{-\infty}^{\infty} d\tau \int dV \left\{ \frac{\partial \varphi}{\partial \tau} \rho - (\nabla \sqrt{\rho})^2 - \rho (\nabla \varphi)^2 - (v - \mu) \rho - l \rho^2 \right\} .
\]  
(30)

such that, in the equilibrium hydrodynamic picture, the matrix of second variational derivatives of the effective action can be written as

\[
\delta^2 \Gamma = \begin{pmatrix} \frac{\delta^2 \Gamma}{\delta \rho(x) \delta \rho(y)} & \frac{\delta^2 \Gamma}{\delta \rho(x) \delta \varphi(y)} \\ \frac{\delta^2 \Gamma}{\delta \varphi(x) \delta \rho(y)} & \frac{\delta^2 \Gamma}{\delta \varphi(x) \delta \varphi(y)} \end{pmatrix} = \begin{pmatrix} -2l & -\frac{\partial}{\partial t} \\ \frac{\partial}{\partial t} & 2\nabla \cdot \rho \nabla \end{pmatrix} .
\]

where \( \rho \equiv \langle \hat{\rho} \rangle \). The matrix of two-point connected correlation functions of fluctuations \( \hat{\rho}(x) = \hat{\rho}(x) - \langle \hat{\rho}(x) \rangle \) and \( \hat{\varphi}(x) = \hat{\varphi}(x) - \langle \hat{\varphi}(x) \rangle \)

\[
G = \begin{pmatrix} \langle \hat{\rho}(x) \hat{\rho}(y) \rangle & \langle \hat{\rho}(x) \hat{\varphi}(y) \rangle \\ \langle \hat{\varphi}(x) \hat{\rho}(y) \rangle & \langle \hat{\varphi}(x) \hat{\varphi}(y) \rangle \end{pmatrix}
\]

is defined as a solution of the equation

\[
\delta^2 \Gamma \cdot G = -I .
\]

Solving the equation we get the following expressions for the correlators

\[
\langle \hat{\rho}(x) \hat{\rho}(y) \rangle = \delta(x - y) - \frac{1}{2l} \frac{\partial}{\partial t} \langle \hat{\varphi}(x) \hat{\rho}(y) \rangle
\]

\[
\langle \hat{\rho}(x) \hat{\varphi}(y) \rangle = -\frac{1}{2l} \frac{\partial}{\partial t} \langle \hat{\varphi}(x) \hat{\varphi}(y) \rangle
\]

and the equation for the phase-phase correlator \( \langle \hat{\varphi}(x) \hat{\varphi}(y) \rangle \):

\[
\left( \frac{1}{2l} \frac{\partial^2}{\partial t^2} - 2\nabla \cdot \rho \nabla \right) \langle \hat{\varphi}(x) \hat{\varphi}(y) \rangle = \delta(x - y)
\]

(here \( x \) denotes both space and time variables). It means that the phase-phase correlator is the Green function of the wave equation.

In summary, various response functions, form-factors and Green functions are evaluated from effective action in any given approximation taking variational derivatives and integrating them with the two-point correlation function. This can be done in general using formulas of this section.
In the paper a self-consistent approach to the calculation of 1-loop quantum corrections to the Gross-Pitaevskii equation for a Bose gas density due to fluctuations around mean field configuration are calculated. To do this the hydrodynamic approximation was used. In this approximation excitations are equivalent to sound waves on a mean field background. This opens the possibility to use methods of quantum gravity and theory of quantum gauge fields where the problem of calculation of effective action for gravitational and gauge backgrounds (or more precise, quantum corrections due to other quantum fields) is common problem.

Many of the methods to approach the problem were developed during the last four decades. They are $\zeta$-function regularization for determinants of operators, Schwinger-De-Witt-Seeley expansion of heat kernels \cite{27, 28, 29}, covariant methods of calculation of Seeley’s coefficients and the covariant perturbation technique for effective actions \cite{30, 31}. Although the methods are well-known in field theory they are not so familiar in condensed matter physics and the theory of coherent systems. An accurate account of corrections is very complicated even for a few first orders and the solutions are obtained by using the covariant perturbation technique and curvature expansions. On the way effects nonlocal in space and time appeared. In quantum gravity framework such effects play significant role in the gravitational collapse problem, Hawking radiation and so on.

However to benefit from it sometimes very general and only basic information about principle scales in systems in question is required. Indeed, this is the only information needed to extract leading logarithmic contributions while the calculation of other corrections is much more complicated. That is why we think that the developed approach can be widely used in many problems which have not much to do with Bose-condensation of trapped atoms or liquid Helium.

At the end let us stop on other applications of the presented method. It is easy to imagine other condensed matter examples of problems treatable by the same technique. Clearest and simplest of them are antiferromagnets, Josephson arrays and superconductors in nonuniform (in space and time) external magnetic fields. However many of other physical systems in a non-equilibrium background are potential field of applications.

Returning to Bose condensation, the paper together with our previous work \cite{9} where the generalization of Gross-Pitaevskii equation for the condensate fraction was given allows to consider the condensate fraction and the depletion on the same basis since the depletion is a difference of the full density and the condensate density. However, the results of this paper can be used even without reffering to the Bose condensation phenomena.

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Appendices

A Effective action

In this Appendix we state some basic information concerning the formalism of an effective action following [32]. Let us consider a set of fields $\phi^i \equiv \varphi^A(x) = \{\psi(x), \psi^+(x)\}$. The classical dynamics of these fields is described by the classical equations of motion

$$\frac{\delta S}{\delta \varphi^A(x)} = 0 \quad \text{(A.1)}$$

derived from the classical action functional

$$S(\varphi) = \int dx \ L(\varphi(x), \partial_\mu \varphi(x)) \ . \quad \text{(A.2)}$$

After quantization the fields become field operators $\hat{\phi}$ acting on some state vectors of Fock space. Let us suppose that there exist some initial $|\text{in}\rangle$ and final $\langle \text{out}|$ vacuum states defined in some appropriate way. The vacuum-vacuum transition amplitude can be presented as a Feynman path integral

$$\langle \text{out}|\text{in}\rangle \equiv Z(J) = \int D\varphi \ \exp \left\{ \frac{i}{\hbar} \left( S(\varphi) + J_i \varphi^i \right) \right\} \quad \text{(A.3)}$$

Here we introduced the classical sources $J_i$ to investigate the linear reaction of the system on the external perturbation. All the content of quantum field theory with all quantum effects is contained in the following Green functions

$$\langle \hat{\phi}^{i_1} \cdots \hat{\phi}^{i_n} \rangle \equiv \frac{\langle \text{out}|T(\hat{\phi}^{i_1} \cdots \hat{\phi}^{i_n})|\text{in}\rangle}{\langle \text{out}|\text{in}\rangle} \quad \text{(A.4)}$$

where $T$ means the chronological ordering operator, i.e. the fields must be arranged from left to right in order of decreasing time arguments.

It is easy to show that all these Green functions can be obtained by the functional differentiation of the functional $Z(J)$, that is called, therefore, the generating functional. Moreover, factorizing the unconnected contributions one comes to generating functional for connected Green functions $W(J) = -i\hbar \ln Z(J)$

$$\langle \hat{\phi}^{i_1} \cdots \hat{\phi}^{i_n} \rangle = \left(\frac{\hbar}{i}\right)^n \exp\left(-\frac{i}{\hbar} W\right) \frac{\delta^n}{\delta J_{i_1} \cdots \delta J_{i_n}} \exp\left(\frac{i}{\hbar} W\right). \quad \text{(A.5)}$$

The lowest connected Green functions have special names: the mean field

$$\langle \varphi^i \rangle \equiv \phi^i(J) = \frac{\delta W}{\delta J_i} \quad \text{(A.6)}$$

and the propagator

$$\langle (\hat{\phi}^i - \phi^i)(\hat{\phi}^k - \phi^k) \rangle \equiv -i\hbar G^{ik}(J) = -i\hbar \frac{\delta^2 W}{\delta J_i \delta J_k} \quad \text{(A.7)}$$
Further, the connected Green functions are expressed in terms of vertex functions. The generating functional for vertex functions is defined then by the Legendre functional transform

\[ \Gamma(\phi) = W(J(\phi)) - J_k(\phi)\phi^k \] (A.8)

This is the most important object in quantum field theory. It contains all the information about the quantized fields.

1. First, one can show that it satisfies the equation

\[ \Gamma_{,i}(\phi) \equiv \frac{\delta \Gamma}{\delta \phi^i} = -J_i(\phi) \] (A.9)

and, therefore, gives the effective equations for the mean field. These equations replace the classical equations of motion and describe the effective dynamics of background fields taking into account all quantum corrections. Thus \( \Gamma(\phi) \) is called usually the effective action.

2. Second, it determines the full or exact propagator of quantized fields

\[ -\Gamma_{,ik}G^{kn} = \delta^n_i \] (A.10)

where \( \delta^n_i \equiv \delta^A_B \delta(x - y) \), and the vertex functions

\[ \Gamma_n \equiv \Gamma_{,i_1\ldots i_n}, \quad (n \geq 3) \] (A.11)

This means that any \( S \)-matrix amplitude, or any Green function, is expressed in terms of propagator and vertex functions that are determined by the effective action.

3. At last, when the test sources vanish the effective action is just the vacuum amplitude

\[ \langle \text{out} | \text{in} \rangle \bigg|_{J=0} = \exp \left( \frac{i}{\hbar} \Gamma \right) \bigg|_{J=0} \] (A.12)

Let us rewrite the definition of the effective action. To define the path integral it is convenient to make a so-called Wick rotation, or Euclidization, i.e. one replaces the real time coordinate to the purely imaginary one \( t \rightarrow -it \) and singles out the imaginary factor also from the action \( S \rightarrow -iS \) and the effective action \( \Gamma \rightarrow -i\Gamma \). Then the metric becomes Euclidean, i.e. positive definite, and the classical action in all good field theories becomes positive definite functional. So, the Euclidean effective action is defined to satisfy the equation

\[ \exp \left( \frac{1}{\hbar} \Gamma(\phi) \right) = \int \mathcal{D}\phi \ \exp \left\{ \frac{1}{\hbar} \left[ S(\varphi) - (\varphi^i - \phi^i)\Gamma_{,i}(\phi) \right] \right\} \] (A.13)

This path integral is still not well defined. There is not any reasonable method, except for lattice theories, to calculate this integral in general case. The only path integrals that can be well defined are the Gaussian ones. Then the full path integral can be well defined as an asymptotic series of Gaussian ones. This is just the quasiclassical, or WKB, approximation.
in the usual quantum mechanics. We decompose the fields in the classical and quantum parts
\[ \varphi = \phi + \sqrt{\hbar} \]
and look for a solution of the equation for the effective action in form of an asymptotic series in powers of Planck constant.

\[ \Gamma(\phi) = S(\phi) + \sum_{n \geq 1} \hbar^n \Gamma(n)(\phi) \]  

(A.15)

Then all the coefficients of this expansion can be found. They are expressed in terms of the well-known Feynman diagrams. The number of loops in these diagrams correspond to the power of the Planck constant. We will be interested below in the so called one-loop effective action

\[ \Gamma^{(1)} = -\frac{1}{2} \ln \text{Det} F \]  

(A.16)

where \( F_{ik} = S_{ik} \).

**B Determinants of elliptic operators**

Although Eqn. (A.16) seems to be very easy it is still ill defined. The point is it is divergent. This is just the well-known ultraviolet divergence of the quantum field theory. Indeed, we can rewrite the functional determinant as

\[ \ln \text{Det} A = \text{Tr} \ln A = \ln \prod_n \lambda_n = \sum_n \ln \lambda_n \]  

(B.1)

where \( \lambda_n \) are the eigenvalues of the operator \( A \). This series is easy to show to be divergent.

That means that one needs a regularization. This point was investigated very thoroughly by many authors and it is found that in quantum gravity and gauge theories the most appropriate regularizations are the analytical ones. The functional determinants can be well defined in terms of the so called \( \zeta \)-function.

At first let us use a Wick rotation to produce a Laplace operator from the wave operator. It allows to make use of methods for the evaluation of the determinants of elliptic operators [33]. Now we are ready to introduce the \( \zeta \)-function of an elliptic operator \( A \):

\[ \zeta(s, A) = \sum_i \lambda_i^{-s} \]

where \( \{\lambda_i\} \) are eigenvalues of the operator \( A \). Then

\[ \text{Tr} \ln A = -\frac{d}{ds} \zeta(s, A) \bigg|_{s=0} \]

To study the behavior of the \( \zeta \)-function it is common to use the formula

\[ \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \exp(-t\lambda_i) . \]
Summing over all nonzero eigenvalues of $A$ we get

$$\zeta(s, A) = \frac{1}{\Gamma(s)} \int_0^\infty \! dt \ t^{s-1} \bigl( \text{Tr} \exp(-tA) - L(A) \bigr)$$  \hspace{1cm} (B.2)$$

where $L(A)$ is a number of zero-modes of $A$. Representation (B.2) is valid when the integral converges. For nonnegative operator it always does converge as $t \to +\infty$. Conditions of the convergency of the integral as $t \to +0$ depend on details of the operator $A$. For Laplace operator in 4D it converges as $t \to +0$ if $\text{Re} \ s > 2$.

There exists the well-known Seeley expansion for the $\text{Tr} \bigl( \exp(-tA) \bigr)$:

$$\text{Tr} \exp(-tA) = \sum_{k \geq 0} \Phi_{-k}(A) t^{-k} + \rho(t) \ ,$$

where $|\rho(t)|$ is bounded by a constant times $t$ as $t \to +0$. The coefficients and play important role in the investigation of elliptic operators and their topological properties (for example in the Index theory). For example, for 4D Laplace operator $\Phi_{-k} = 0$ for $k \geq 3$, and

$$\Phi_{-2} = (4\pi)^{-2} \ ,$$

$$\Phi_{-1} = -(4\pi)^{-2} \frac{1}{6} R \ ,$$

where $R$ — scalar curvature.

Here $R_{\mu\nu\rho\sigma}$ and $R_{\mu\nu}$ are Riemann and Ricci tensors correspondingly. They are defined by a metric of a curved space. The latter coefficient $\Phi_0$ is particular important for the calculation of determinants.

Splitting (B.2) into an integral over $[0, 1]$ and one over $[1, +\infty]$ we get

$$\zeta(s) = \frac{1}{\Gamma(s)} \left( \sum_{k>0} \frac{\Phi_{-k}(A)}{s - k} + \frac{\Phi_0(A) - L(A)}{s} \right) +$$

$$+ \int_1^\infty \! dt \ \text{Tr} \exp(-tA)t^{s-1} + \int_0^1 \! dt \ \rho(t)t^{s-1} \right)$$

The singularity at $s = 0$ turns out to be removable since $\lim_{s \to 0} s\Gamma(s) = 1$ and $\zeta(s)$ is defined by analytical continuation from the half-plane $\text{Re} \ s > 0$.

From the relation $\ln \det A = -\zeta'(0)$ we find that

$$\ln \det A = \sum_{k>0} \frac{\Phi_{-k}(A)}{k} + \frac{\Phi_0(A) - L(A)}{1} - \int_1^\infty \! \frac{dt}{t} \ \text{Tr} \exp(-tA) -$$

$$- \int_0^1 \! \frac{dt}{t} \left( \text{Tr} \exp(-tA) - \sum_{k<0} \Phi_{-k}(A)t^{-k} \right) .$$

This is $\zeta$-regularized $\ln \det A$ which we use in the paper.

To conclude this section we give the relation between $\ln \det A$ and $\ln \det \alpha A$ where $\alpha$ is a number parameter. It is easy to see that

$$\zeta(s, \alpha A) = \zeta(s, A) \cdot \alpha^{-s}$$
because \( \lambda_i(\alpha A) = \alpha \lambda_i(A) \). This leads to the relation:

\[
\ln \det \alpha A = \ln \det A + \ln \alpha \cdot \zeta(0, A) = \ln \det A + \ln \alpha \cdot \left( \Phi_0(A) - L(A) \right)
\]

(B.3)

which we use intensively in the paper.

### C Variational derivative of the effective action

In this appendix we give details of the calculation omitted in section 3 to do not interrupt the main stream of the consideration. To make the calculation more efficient we make use covariant form of the quantum correction since there exists well-known way to simplify covariant variation calculation [34].

We are interested in the following variational derivative

\[
\frac{\delta}{\delta g^{\mu\nu}} \Phi_0(\Box)
\]

(C.1)

where

\[
\Phi_0(\Box) = \frac{1}{(4\pi)^2} \int dx \sqrt{-g} \left( \frac{1}{12} R^2 - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right)
\]

(C.2)

For variation of the volume element we have

\[
\delta \sqrt{-g} = -\frac{1}{2\sqrt{-g}} \delta g_{\mu\nu} \Delta^{\mu\nu} = \frac{\sqrt{-g}}{2} \delta g_{\mu\nu} g^{\mu\nu} = -\frac{\sqrt{-g}}{2} g_{\mu\nu} \delta g^{\mu\nu}
\]

where \( \Delta^{\mu\nu} \) is defined by as

\[
\Delta^{\mu\nu} = g_{\mu\nu} g^{\mu\nu}
\]

and the following relation was used

\[
\delta g_{\mu\nu} g^{\mu\nu} = -g_{\mu\nu} \delta g^{\mu\nu}.
\]

For variations \( R^2, R_{\mu\nu} R^{\mu\nu}, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) one can obtain

\[
\delta R^2 = 2 \delta R \cdot R = 2 \delta g^{12} R_{12} R + 2 g^{12} \delta R_{12} \cdot R
\]

\[
\delta R_{12} R^{12} = \delta (g^{13} g^{24} R_{12} R_{34}) = 2 \delta g^{12} R_{13} R_{2}^3 + 2 \delta R_{12} R_{12}^{12}
\]

\[
\delta R_{1234} R^{1234} = \delta (g^{15} g^{26} g^{37} g^{48} R_{1234} R_{5678}) = 4 \delta g^{12} R_{1345} R_{2}^{345} + 2 \delta R_{1234} R_{1234}^{1234}
\]

where numbers 1, 2, 3, . . . mean indices \( \mu_1, \mu_2, \mu_3, \ldots \) respectively. Hence,

\[
\delta \Phi_0 = \frac{1}{36(4\pi)^2} \int dx \sqrt{-g} \left( \delta \Phi_0 \left[ \frac{1}{2} g^{\mu\nu} \left( \frac{1}{2} R^2 - \frac{1}{5} R_{12} R^{12} + \frac{1}{5} R_{1234} R^{1234} \right) \right]
\]

\[
+ R_{\mu\nu} R - \frac{2}{5} R_{\mu1} R_{\nu}^1 + \frac{4}{5} R_{\mu123} R_{\nu}^{123} \right]
\]

\[
+ \frac{1}{36(4\pi)^2} \int dx \sqrt{-g} \left[ g^{12} \delta R_{12} R - \frac{2}{5} \delta R_{12} R^{12} + \frac{2}{5} \delta R_{1234} R^{1234} \right]
\]

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So we need to calculate last three terms.

For variations of Ricci and Riemann tensors we have \[34\]:

\[
\delta R_{\mu\nu} = \nabla_{\nu}(\delta \Gamma_{\mu}^{1}) - \nabla_{1}(\delta \Gamma_{\mu\nu}^{1})
\]

\[
= \frac{1}{2}g^{12}[\nabla_{\mu} \nabla_{\nu}\delta g_{12} - \nabla_{1} \nabla_{\nu}\delta g_{2\mu} - \nabla_{1} \nabla_{\nu}\delta g_{2\mu} + \nabla_{1} \nabla_{\nu}\delta g_{\mu\nu}] \tag{C.3}
\]

\[
\delta R_{\mu\nu\sigma\rho} = -g_{\mu\nu}\delta g^{12}R_{2\sigma\rho} + g_{\mu\nu}\nabla_{\rho}(\delta \Gamma_{\nu}^{1}) - g_{\mu\nu}\nabla_{\sigma}(\delta \Gamma_{\nu}^{1})
\]

\[
= -g_{\mu\nu}\delta g^{12}R_{2\sigma\rho} + \frac{1}{2}[\nabla_{\rho}\nabla_{\sigma}\delta g_{\mu\nu} + \nabla_{\rho}\nabla_{\nu}\delta g_{\mu\sigma} - \nabla_{\rho}\nabla_{\nu}\delta g_{\mu\sigma}]
\]

\[= \frac{1}{2}\nabla_{\rho}\nabla_{\sigma}\delta g_{\mu\nu} - \nabla_{\sigma}\nabla_{\nu}\delta g_{\mu\rho} + \nabla_{\sigma}\nabla_{\mu}\delta g_{\nu\rho}] \tag{C.4}
\]

Let us consider covariant divergence of some vector \( T^k \):

\[
\nabla_{\mu}T^\mu = \nabla^\mu T_\mu = \partial_\mu T^\mu + \Gamma^\nu_{\mu\nu}T^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}T^\mu)
\]

Hence

\[
\sqrt{-g}\nabla_\mu T^\mu = \partial_\mu T^\mu
\]

is a pure divergence and can be dropped under integration. Using this we can easily derive the following expression for variations of the integrals we are interested in

\[
\int d^4x \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}R = \int d^4x \sqrt{-g}\delta g^{\mu\nu}\left[-\frac{1}{2}\{\nabla_\mu, \nabla_\nu\}R + g_{\mu\nu}\Box R\right]
\]

\[
\int d^4x \sqrt{-g}\delta R_{\mu\nu}R^{\mu\nu} = \frac{1}{2}\int d^4x \sqrt{-g}\delta g^{\mu\nu}\left[-\Box R_{\mu\nu} - g_{\mu\nu}\nabla_1 \nabla_2 R^{12} + \nabla_1 \nabla_\mu R_\nu^{1} + \nabla_1 \nabla_\nu R_\mu^{1}\right]
\]

\[
\int d^4x \sqrt{-g}\delta R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} = \int d^4x \sqrt{-g}\delta g^{\mu\nu}\left[-R_{\mu123}R_{\nu}^{123} + 2\nabla^2\nabla^{1}R_{\mu12\nu}\right]
\]

Summing all terms we obtain

\[
\frac{\delta}{\delta g^{\mu\nu}} \Phi_0(\Box) = \frac{1}{36(4\pi)^2}\sqrt{-g}\left\{-\frac{1}{2}g_{\mu\nu}\left[\frac{1}{2}R^2 - \frac{1}{5}R_{\sigma\rho}R^{\sigma\rho} + \frac{1}{5}R_{\sigma\rho\alpha\beta}R^{\sigma\rho\alpha\beta}\right]
\]

\[
+ R_{\mu\nu}R - \frac{2}{5}R_{\mu\sigma}R_\nu^{\sigma} + \frac{2}{5}R_{\mu\nu\sigma\rho}R_{\nu\sigma\rho} - g_{\mu\nu}\Box R + \frac{1}{2}\{\nabla_\mu, \nabla_\nu\}R
\]

\[
+ \frac{1}{5}\left[-2\nabla^\sigma\nabla_\nu R_{\mu\sigma} + \Box R_{\mu\nu} + g_{\mu\nu}\nabla^\sigma\nabla^\rho R_{\sigma\rho}\right] + \frac{4}{5}\nabla^\rho\nabla^\sigma R_{\mu\sigma\rho\nu}\right\}
\]

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[25] Although in the section the isotropic case is only considered it is easy and straightforward to give the corresponding consideration for anisotropic case. To do this the only change is to make use the analytic expression for anisotropic condensate profile which can be found in [21].

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1. Fig. 1 Bose condensate density profile obtained in mean field approximation. The curve is drawn using the analytic Ansatz of Ref. [21]. Lengths are measured in units of the oscillator length \( a_\perp \). The multiplier is defined as \( \lambda = 4\pi l/a_\perp \).

2. Fig. 2 The calculated one-loop quantum correction to the Nonlinear Schrödinger equation in the central region of the isotropic trap. Lengths are measured in units of the oscillator length. \( N \) is the number of particles (for the picture \( N = 10^5 \)) and \( \delta P \) is defined by eq. (20).

3. Fig. 3 The calculated one-loop quantum correction to the Nonlinear Schrödinger equation at the intermediate region of the isotropic trap. Lengths are measured in units of the oscillator length. \( N \) is the number of particles (for the picture \( N = 10^5 \)) and \( \delta P \) is defined by eq. (20).

4. Fig. 4 Change in sign of the one-loop contribution in intermediate region of the trap. Lengths are measured in units of the oscillator length. \( N \) is the number of particles (for the picture \( N = 10^5 \)) and \( \delta P \) is defined by eq. (20).

5. Fig. 5 Merging of High density approximation for the quantum correction (solid line) and the corresponding Low density one (dashed line). Lengths are measured in units of the oscillator length. In the region to the left to the point \( r \sim 6.13 \) High density approximation produces the correction \( (20) \). In the region to the right to the point \( r \sim 6.13 \) Low density approximation \( (22) \) for the correction should be used. In the region of the boundary point both approximations fail and other quantum contributions should be kept.

6. Fig. 6 Comparison of various terms in the equation for the condensate density. The interpolating quantum correction (solid line) of Fig. 5, nonlinear interaction term (dashed line) and kinetic energy (circles) are pictured. Lengths are measured in units of the oscillator length.
Fig. 1. Mean field Bose condensate profile
Fig. 2. Quantum correction: center of the trap
Fig. 3. Quantum correction: intermediate region

\[ \frac{N \delta p}{10} \]
Fig. 4. Quantum correction: change of the sign
Fig. 5. High and Low density expansions
Fig. 6. Comparison of contributions