Pomeron with a running coupling constant in the nucleus

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November 8, 2021

Abstract

The running coupling is introduced into the equation for propagation of the pomeron in the nucleus via the bootstrap relation. The resulting equation coincides with the one obtained in the colour dipole formalism by summing contributions from quark-antiquark loops, with a general choice of the regularization scheme.

1 Introduction

Lately a renewed interest has been shown towards introduction of a running coupling into the BFKL dynamics. Summation of contributions from the quark-antiquark loops to the evolution of the gluon density has been used to restore the full dependence of the coupling on the running scale in the color dipole approach [1, 2, 3]. It turned out that the obtained kernel for the linear BFKL equation essentially coincides with the one which we found many years ago by imposing the bootstrap condition necessary for the fulfilment of unitarity [4, 5]. In this paper we draw attention to the fact that the bootstrap condition in fact allows to derive also the structure of the triple pomeron interaction and thus the form of the non-linear BFKL equation describing propagation of the pomeron in the nucleus. This is a simple consequence of the possibility to express the basic splitting kernel from 2 to 3 and 4 gluons (‘the Bartels vertex’, [6]) via the basic BFKL interaction. The relation between the splitting vertex and reggeized gluon interaction, together with the bootstrap, are the necessary ingredients of the pomeron interaction in the reggeized gluon approach. Indeed they allow to present all contributions to the 4-gluon amplitude in the standard form of a pomeron splitting into two pomerons. Thus preserving these two relations seems to be essential for the construction of pomeron interaction with the running coupling.

As in our earlier paper, we have to stress from the start that introduction of the running coupling into the BFKL formalism cannot be made rigorously and uniquely. The formalism admits transverse momenta of any magnitude, including very small ones, at which the concept of the gluon and its coupling looses any sense. The introduced running coupling has to be artificially continued to small momentum values, where it is completely undetermined. The only information one can get from the running of the coupling refers to the region of high momenta. It remains to be seen in what measure this information depends on the low momentum behaviour, which cannot be fixed theoretically in any reliable manner. It is known however that the non-linear BFKL equation, unlike the linear one, is not very sensitive to the infrared region. So hopefully this equation has a better chance to produce results weakly dependent on the assumed infrared behaviour of the running coupling.
The paper is organized as follows. The first section is dedicated to the derivation of the triple-pomeron vertex with the running coupling introduced via the bootstrap relation. After recalling this method to introduce the running coupling, we go along the same steps as in [7, 8], where the vertex was derived in the limit $N_c \to \infty$ for the fixed coupling. In Section 3 we construct the full amplitude with a single triple pomeron interaction, coupling the vertex with three pomerons. This result enables us to build the equation for the pomeron in the large nucleus in Section 4. Finally in Section 5 we compare our results with those obtained within the colour dipole approach.

Having in mind application of the formalism to the large nucleus as a target, we limit ourselves to the case of propagation of pomerons with zero total momenta. However generalization to non-zero total momenta is straightforward.

2 Triple-Pomeron vertex

2.1 Generalities

As mentioned in the Introduction, this paper follows the idea to introduce a running coupling via the bootstrap [4, 5]. Derivation of the triple-pomeron vertex in the limit $N_c \to \infty$ then goes as presented in [7, 8] for the fixed coupling case.

Basic formulas for the introduction of a running coupling via the bootstrap condition consist in expressing both the gluon trajectory $\omega$ and intergluon interaction in the vacuum channel $K$ in terms of a single function $\eta(q)$ of the gluon momentum, which then can be chosen to conform to the high-momentum behaviour of the gluon density with a running coupling. Explicitly

$$\omega(q) = -\frac{1}{2} N_c \int \frac{d^2 q_1}{(2\pi)^2} \eta(q) \eta(q_1) \eta(q_2),$$

$$K(q_1, q_2|q_1', q_2') = N_c \left[ \left( \frac{\eta(q_1)}{\eta(q_1')} + \frac{\eta(q_2)}{\eta(q_2')} \right) \frac{1}{\eta(q_1) - q_1'} - \frac{\eta(q_1 + q_2)}{\eta(q_1') \eta(q_2')} \right].$$

In these definitions it is assumed that $q_1 + q_2 = q_1' + q_2' = q$. For arbitrary $\eta(q)$ the following bootstrap relation is satisfied:

$$\frac{1}{2} \int \frac{d^2 q_1'}{(2\pi)^2} K(q, q_1, q_1') = \omega(q) - \omega(q_1) - \omega(q_2).$$

The fixed coupling corresponds to the choice

$$\eta(q) = \frac{2\pi}{g^2 q^2}.$$  

(4)

Then one finds the standard expression for the trajectory $\omega(q)$ and

$$K(q, q_1, q_1') = \frac{g^2 N_c}{2\pi} V(q, q_1, q_1'),$$

where $V$ is the standard BFKL interaction. Note that the extra $2\pi$ in the denominator corresponds to the BFKL weight $1/(2\pi)^3$ in the momentum integration, which we prefer to take standardly as $1/(2\pi)^2$.

From the high-momentum behaviour of the gluon distribution with a running coupling one finds

$$\eta(q) = \frac{1}{2\pi} b q^2 \ln \frac{q^2}{\Lambda^2}, \quad q^2 >> \Lambda^2,$$
where $\Lambda$ is the standard QCD parameter and
\[
\frac{b}{12} = 11N_c - 2N_f. \tag{6}
\]
As to the behaviour of $\eta(q)$ at small momenta, we shall assume
\[
\eta(0) = 0, \tag{7}
\]
which guarantees that the gluon trajectory $\omega(q)$ passes through zero at $q = 0$ in accordance with the gluon properties. The asymptotic (5) and condition (7) are the only properties of $\eta(q)$ which follow from the theoretical reasoning. A concrete form of $\eta(q)$ interpolating between (7) and (5) may be chosen differently. One hopes that the following physical results will not strongly depend on the choice.

Our old derivation in [7] of the triple-pomeron vertex was actually based on the property (7) obviously valid for (4), the bootstrap relation and the relation between the Bartels transition vertex $K_{2\to3}$ and intergluon BFKL interaction $V$ (Eq. (12) in [7])
\[
K_{2\to3}(1,2,3|1',3') = V(2,3|1' - 1,3') - V(12,3|1'3'). \tag{8}
\]
Here and in the following we frequently denote gluon momenta just by numbers: $1 \equiv q_1$, $1' \equiv q_1'$ etc. Also we use $12 \equiv q_1 + q_2$. All the rest conclusions were obtained from these three relations in a purely algebraic manner.

Our idea is that if we define the transition vertex by a similar relation in terms of the new intergluon vertex $K$, Eq. (2), then all the derivation will remain valid also for arbitrary $\eta(q)$ satisfying (7) and thus for a running coupling, provided $\eta(q)$ is chosen appropriately. In the next subsection we briefly recapitulate successive stages of the derivation for arbitrary $\eta(q)$ with $\eta(0) = 0$.

\subsection*{2.2 Leading order in $N_c \to \infty$}

The changes necessary to pass to arbitrary $\eta(q)$ are minimal. Obviously in our old formulas we have to drop the coupling constant $g$ factors, since now $g$ is provided by $1/\eta$. We also drop factors $N_c$ in the interaction because we prefer to include it into the kernel $K$. With these comments, the two-gluon equation becomes
\[
S_{20}D_2 = D_{20} + K_{12}D_2, \tag{9}
\]
where
\[
S_{20} = j - 1 - \omega(1) - \omega(2), \tag{10}
\]
$j$ is the angular momentum variable,
\[
D_{20} = N_c\left(f(0,q) - f(q_1,q_2)\right) \tag{11}
\]
and $f(q_1,q_2)$ is just the quark-antiquark loop with, say, the gluon with momentum $q_1$ coupled to the quark and the other, with momentum $q_2$, coupled to the antiquark. With the running coupling $D_{20}$ does not change its form, but both $\omega$ in (10) and $K$ in (9) are now given by Eqs. (11) and (2) respectively.

The three-gluon system exists in two colour states, which differ in the ordering of the three gluons along the loop, 123 and 213. For each order the equation is
\[
S_{30}D_3 = D_{30} + D_{2\to3} + \frac{1}{2}(K_{12} + K_{23} + K_{31})D_3, \tag{12}
\]
where now
\[ S_{30} = j - 1 - \sum_{j=1}^{3} \omega(j) \]  
(13)

and
\[ D_{30}^{(123)} = -D_{30}^{(213)} = \sqrt{\frac{N_c}{8}} \left(D_{20}(2) - D_{20}(1) - D_{20}(3)\right). \]  
(14)

The new element is the term \( D_{2\to3} \) which corresponds to transitions of the initial two-gluon system into the final three-gluon system. This transition is accomplished by the \( 2\to3 \) vertex \( W \) which, as explained, we define via the new intergluon interaction by a relation similar to (8)
\[ W(1, 2, 3|1', 3') = K(2, 3|1' - 1, 3') - K(12, 3|1'3'). \]  
(15)

Explicitly, in terms of \( \eta \)
\[ \frac{1}{N_c} W(1, 2, 3|1', 3') = \frac{\eta(2)}{\eta(1')\eta(3-3')} - \frac{\eta(23)}{\eta(1')\eta(3-3')} - \frac{\eta(12)}{\eta(1')\eta(3'-3')} + \frac{\eta(123)}{\eta(1')\eta(3')} . \]  
(16)

Note that, as with a fixed coupling constant,
\[ W(1, 2, 3|1', 3') = W(3, 2, 1|3', 1'). \]  
(17)

We find in full analogy with [7]
\[ D_{2\to3} = \sqrt{\frac{N_c}{8}} W(1, 2, 3|1', 3') \otimes D_{2}(1', 3') \equiv \sqrt{\frac{N_c}{8}} W(1, 2, 3), \]  
(18)

where \( \otimes \) means integration over the intermediate gluon momentum with weight \( 1/(2\pi)^2 \).

Next step is to show that Eq. (12) for the three-gluon system is solved by the reggeized zero term ansatz:
\[ D_{3}^{(123)} = -D_{3}^{(213)} = \sqrt{\frac{N_c}{8}} \left(D_{2}(2) - D_{2}(1) - D_{2}(3)\right), \]  
(19)

that is by (14) in which the loops are substituted by the full solutions of the two-gluon equation (9). The proof is wholly based on the bootstrap and relation (15) and literally repeats the corresponding proof in [7].

Passing to the 4-gluon system, in the limit \( N_c \to \infty \) we find two configurations differing by the order of gluons along the quark-antiquark loop: 1234 and 2134. The equation governing the 4-gluon system is
\[ S_{40}D_{4} = D_{40} + D_{2\to4} + D_{3\to4} + \frac{1}{2}(K_{12} + K_{23} + K_{34} + K_{41})D_{4}, \]  
(20)

where
\[ S_{40} = j - 1 - \sum_{j=1}^{4} \omega(j). \]  
(21)

The inhomogeneous terms are
\[ D_{40}^{(1234)} = \frac{1}{4} N_c \left(D_{20}(1) + D_{20}(4) - D_{20}(14)\right), \]  
(22)
\[ D_{40}^{(2134)} = \frac{1}{4} N_c \left( D_{20}(2) + D_{20}(3) - D_{20}(12) - D_{20}(13) \right), \]  
\[ D_{2 \rightarrow 4}^{(1234)} = -\frac{1}{4} N_c W(1, 23, 4), \quad D_{2 \rightarrow 4}^{(2134)} = 0 \]

(the definition of \( W(1, 2, 3) \) is given by the 2nd equality in (\ref{18})),

\[ D_{3 \rightarrow 4}^{(1234)} = \sqrt{\frac{N_c}{8}} \left( W(2, 3, 4|2', 4') \otimes D_{3}^{(124)}(1', 2', 4') + W(1, 2, 3|1', 3') \otimes D_{3}^{(134)}(1', 3', 4) \right), \]

and

\[ D_{3 \rightarrow 4}^{(2134)} = -\sqrt{\frac{N_c}{8}} \left( W(1, 2, 4|1', 4') \otimes D_{3}^{(134)}(1', 3, 4') + W(1, 3, 4|1', 4') \otimes D_{3}^{(124)}(1', 2, 4') \right). \]

Repeating the corresponding derivation in \cite{7} we demonstrate that in the limit \( N_c \to \infty \) the solution of the 4-gluon equation is again given by the reggeized zero-order terms:

\[ D_{4}^{(1234)} = \frac{1}{4} N_c \left( D_{2}(1) + D_{2}(4) - D_{2}(14) \right) \]  
and

\[ D_{4}^{(2134)} = \frac{1}{4} N_c \left( D_{2}(2) + D_{2}(3) - D_{2}(12) - D_{2}(13) \right). \]

The proof is purely algebraic and is wholly based on the bootstrap and relations (\ref{15}) and (\ref{7}) valid for any choice of function \( \eta(q) \) with \( \eta(0) = 0 \).

### 2.3 The triple pomeron configuration

Next step is to study the next-to-leading configuration in \( N_c \to \infty \) corresponding to the triple pomeron interaction. Again the derivation practically literally repeats our old one for a fixed coupling constant. The governing 4-gluon equation is similar to Eq. (\ref{20})

\[ S_{40} D_{4} = D_{40} + D_{2 \rightarrow 4} + D_{3 \rightarrow 4} + D_{4 \rightarrow 4} + (K_{12} + K_{34}) D_{4}. \]

The formal difference is in the absence of interaction between the two final pomerons, which are assumed to be made of gluon pairs (1,2) and (3,4) doubling of the rest interactions acting in the vacuum colour channels and in the appearance of the term \( D_{4 \rightarrow 4} \), which describes transitions from the leading colour configuration to the subleading one corresponding to two pomerons.

The four inhomogeneous terms are

\[ D_{40} = \frac{1}{2} \left( \sum_{j=1}^{4} D_{20}(j) - \sum_{j=2}^{4} D_{20}(1j) \right), \]

\[ D_{2 \rightarrow 4} = -W(1, 23, 4), \]

\[ D_{3 \rightarrow 4} = \sqrt{\frac{2}{N_c}} \left( W(1, 2, 3|1', 3') \otimes D_{3}^{(134)}(1', 3', 4) - W(1, 2, 4|1', 4') \otimes D_{3}^{(134)}(1', 3, 4') 
+ W(2, 3, 4|2', 4') \otimes D_{3}^{(124)}(1', 2', 4) - W(1, 3, 4|1', 4') \otimes D_{3}^{(124)}(1', 2, 4) \right) \]
and
\[ D_{4 \to 4} = \frac{1}{N_c} (K_{23} + K_{14} - K_{13} - K_{24})(D_{4}^{(1234)} - D_{4}^{(2134)}), \]
where the \( D_4 \)'s on the right-hand side are given by Eqs. (27) and (28). As with a fixed coupling constant, all terms except \( D_{40} \) can be presented as a result of action of a certain operator \( Z \) on the two gluon state, so that Eq. (29) can be rewritten as
\[ S_{40} D_4 = D_{40} + Z \otimes D_2 + (K_{12} + K_{34}) D_4. \]
The explicit form of this operator can be expressed in terms of a function
\[ G(1, 2, 3) = -W(1, 2, 3) - D_2(1, 23)(\omega(2) - \omega(23)) - D_2(12, 3)(\omega(2) - \omega(12)). \]
Then one finds
\[ Z \otimes D_2 = \frac{1}{2} \left( 2 G(1, 34, 2) + 2 G(3, 12, 4) + G(1, 23, 4) + G(1, 24, 3) + G(2, 13, 4) + G(2, 14, 3) \right) \]
\[ -G(1, 3, 24) - G(1, 4, 23) - G(2, 3, 14) - G(2, 4, 13) - G(3, 2, 14) \]
\[ -G(3, 1, 24) - G(4, 2, 13) - G(4, 1, 23) + G(23, 0, 14) + G(13, 0, 24) \].
This formula is identical to the old one in [7], but with new expressions for both \( W \) and \( \omega \) in terms of function \( \eta \).

2.4 The triple pomeron vertex

Corresponding to the two inhomogeneous terms in Eq. (34) its solution is split in two terms, the double pomeron exchange term, generated by \( D_{40} \), and the triple pomeron interaction term \( Z \otimes D_2 \). However one can simplify the solution transferring the part of the double pomeron exchange term leading in the high-energy limit into the triple interaction part (9, 8). This is achieved by separating from the total solution the reggeized \( D_{40} \) term:
\[ D_4 = D_{40}(D_{20} \to D_2) + D_4^I. \]
The irreducible part \( D_4^I \) proves to be a pure triple pomeron interaction, which satisfies
\[ D_4^I = Y \otimes D_2 + (K_{12} + K_{34}) D_4^I. \]
The derivation again uses only the bootstrap, relation (15) between \( W \) and \( K \) and property (7). The explicit form for the new triple-pomeron vertex \( Y \) turns out to be
\[ Y \otimes D_2 = \frac{1}{2} \left( G(1, 23, 4) + G(1, 24, 3) + G(2, 13, 4) + G(2, 14, 3) + G(12, 0, 34) \right) \]
\[ -G(1, 2, 34) - G(2, 1, 34) - G(3, 4, 12) - G(4, 3, 12) \].
For a fixed coupling constant this expression was found long ago in [9]. In our approach it remains true also for arbitrary function \( \eta(q) \) and thus for a running coupling introduced by means of this function.
3 Coupling to pomerons

3.1 Momentum space approach

In the momentum space coupling the vertex $Y$ to two outgoing pomerons is straightforward. The two pomerons are described by a product $P(1,2)P(3,4)$ and all one is to do is to integrate this product with $Y \otimes D_{2}$ over the gluon momenta 1,2,3 and 4 with 12 and 34 fixed and 1234 = 0 from the momentum conservation.

As with a fixed coupling constant, there are certain properties of the wave function and the vertex which simplify the resulting expression. First we expect that the pomeron wave function in the coordinate space $P(r_{1},r_{3})$ vanishes if the two gluons are located at the same spatial point: $P(r,r) = 0$. This property is well-known for the BFKL pomeron with a fixed coupling constant and is related to the behaviour of (4) at $q = 0$. With a running coupling this behaviour does not change, so we expect that the coordinate wave function will continue to vanish at $r_{1} = r_{2}$. As a result the last five terms in the sum (39) will give no contribution, since they depend only on the sum of the momenta in one of the pomerons and so put the two gluons at the same spatial point in it. If the mentioned property of the pomeron wave function is violated with the introduction of a running coupling, we still can drop the five last terms once we restrict ourselves to the case when the two pomerons are taken at zero total momentum: 12=34=0, which is the only case relevant for the pomeron propagation through the nucleus. Indeed direct inspection shows that

$$G(q_{1},q_{2},q_{3}) = 0, \text{ if } q_{1} = 0, \text{ or } q_{3} = 0. \quad (40)$$

Second, due to the symmetry of the pomeron wave function, each of the four remaining terms in (39) gives the same contribution. So, as for the fixed coupling, the triple pomeron vertex effectively reduces to

$$Y \otimes D_{2} = 2G(1,23,4). \quad (41)$$

Coupling this vertex to two forward pomerons and using Eq. (35) we get an expression for the triple pomeron interaction amplitude $T$ in terms of function $\eta$ (suppressing the dependence on rapidities)

$$T = 2N_{c} \int \frac{d^{2}q_{1}d^{2}q_{3}d^{2}q_{4}}{(2\pi)^{6}}P(1,2)P(3,4)$$

$$\left\{ - D_{2}(1',-1') \left( \frac{\eta(14)}{\eta(1-1')\eta(4-4')} - \frac{\eta(1)}{\eta(1-1')\eta(4')} \right) + \frac{1}{2} D_{2}(1,-1) \left( \frac{\eta(14)}{\eta(1')\eta(14-1')} - \frac{\eta(1)}{\eta(1')\eta(1-1')} \right) + \frac{1}{2} D_{2}(4,-4) \left( \frac{\eta(14)}{\eta(1')\eta(14-1')} - \frac{\eta(4)}{\eta(1')\eta(4-1')} \right) \right\}$$

(42)

with 12 = 34 = 1’4’ = 0. In this formula actually both $D_{2}$ and $P$ are the pomeron wave functions for the incoming and outgoing pomerons respectively. Function $D_{2}$ is the amputated wave function related to $P$ by the relation

$$P(1,-1) = \frac{1}{\eta^{2}(1)} D_{2}(1,-1). \quad (43)$$

Expression (42) is rather cumbersome. A simpler expression is obtained in the coordinate representation, which will be presently derived.
3.2 Coordinate space approach

We present

\[(2\pi)^2 \delta^2(q_{12} - q_1 - q_2)P(q_1, q_2) = \int d^2r_1 d^2r_2 P_{q_{12}}(r_1, r_2)e^{iq_1r_1+iq_2r_2}, \quad (44)\]

where \(P_{q_{12}}(r_1, r_2)\) is the coordinate wave function of the pomeron with the total momentum \(q_{12}\). Similarly we present the second pomeron via \(P_{q_{34}}(r_3, r_4)\). Finally

\[G(q_1, q_2 + q_3, q_4) = \int d^2r_1 d^2r_2 d^2r_3 e^{-iq_1r_1-i(q_2+q_3)r_2-iq_4r_3}G(r_1, r_2, r_3). \quad (45)\]

Then integration over \(q_1, ...q_4\) gives

\[(2\pi)^2 \delta^2(q_{12} + q_{34})T = 2 \int d^2r_1 d^2r_2 d^2r_3 P_{q_{12}}(r_1, r_2)P_{q_{34}}(r_3, r_2)G(r_1, r_2, r_3) \quad (46)\]

We have

\[P_{q_{12}}(r_1, r_2) = e^{\frac{i}{2}iq_{12}(r_1+r_2)}P(r_{12}), \quad (47)\]

where \(r_{12} = r_1 - r_2\). So (46) becomes

\[(2\pi)^2 \delta^2(q_{12} + q_{34})T = 2 \int d^2r_1 d^2r_2 d^2r_3 e^{\frac{i}{2}[q_{12}r_1+(q_{12}+q_{34})r_2+q_{34}r_3]}P(r_{12})P(r_{32})G(r_1, r_2, r_3). \quad (48)\]

At this stage we note that if we drop \(P(r_{12})\) from the integrand we get

\[\int d^2r_1 d^2r_2 d^2r_3 e^{\frac{i}{2}[q_{12}r_1+(q_{12}+q_{34})r_2+q_{34}r_3]}P(r_{32})G(r_1, r_2, r_3) \quad (49)\]

If \(q_{12} = 0\) then this expression vanishes due to property (40). As a result one can substitute in (48) \(P(r_{12}) \rightarrow P(r_{12}) - P(0), \quad P(r_{32}) \rightarrow P(r_{32}) - P(0). \quad (50)\)

So whether \(P(0)\) is equal to zero or not, for the forward case one can always make it equal to zero by substitution (50). So in the following we assume \(P(0) = 0\).

As follows from the translational invariance, for the overall zero total momentum \(G(r_1, r_2, r_3) = G(r_{12}, r_{32})\). So taking as integration variables \(r_2, r_{12}\) and \(r_{32}\) we finally obtain

\[T = 2 \int d^2r_{12} d^2r_{32} P(r_{12})P(r_{32})G(r_1, r_2, r_3). \quad (51)\]

Now we have to calculate \(G(r_1, r_2, r_3)\). Due to \(P(0) = 0\) we may drop all terms containing \(\delta^2(r_{12})\) and/or \(\delta^2(r_{23})\). The total contribution consists of two parts, the first one coming from the term \(-W\) in (55) and the second one from the rest. As for the fixed coupling constant case (see [8]) in the first part only the first term in (10) gives a contribution which does not contain \(\delta^2(r_{12})\) nor \(\delta^2(r_{23})\) nor both. Direct calculation gives for this contribution

\[G_1(r_1, r_2, r_3) = -N_c D_2(r_1, r_3) \int d^2\rho \eta(\rho)\xi(r_{12} - \rho)\xi(r_{32} - \rho), \quad (52)\]
where $\tilde{\eta}(r)$ is the Fourier transform of $\eta(q)$ and $\xi(r)$ is the Fourier transform of $1/\eta(q)$. From the four terms in the second part the contribution which does not contain $\delta^2(r_{12})$ nor $\delta^2(r_{23})$ nor both comes from the first and third terms. Its calculation gives

$$G_2(r_1, r_2, r_3) = \frac{1}{2} N_c D_2(r_1, r_3) \int d^2 \rho \tilde{\eta}(\rho) \left( \xi^2(r_{12} - \rho) + \xi^2(r_{32} - \rho) \right).$$

(53)

Summing we get the final expression

$$G(r_1, r_2, r_3) = \frac{1}{2} N_c D_2(r_{13}) F(r_{12}, r_{32}).$$

(54)

where

$$F(r_1, r_2) = \int d^2 \rho \tilde{\eta}(\rho) \left( \xi(r_1 - \rho) - \xi(r_2 - \rho) \right)^2$$

(55)

and we also used that for the forward pomeron $D_2(r_1, r_3) = D_2(r_{13})$. The triple pomeron contribution to the amplitude is then obtained as

$$T(Y) = N_c \int_0^Y dy \int d^2 r_{12} d^2 r_{32} F(r_{12}, r_{32}) P(Y - y, r_{12}) P(Y - y, r_{32}) D_2(y, r_{13}),$$

(56)

where we restored the $y$-dependence, suppressed up to now.

Note that for a fixed coupling constant we have (see Appendix and also [8])

$$G^{fix}(r_1, r_2, r_3) = \frac{g^2 N_c}{8\pi^3} \frac{r_{13}^2}{r_{12} r_{32}^2} \left( g^2 D_2(r_{13}) \right).$$

(57)

We can consider our new expression as a result of changing the fixed $g^2$ to a running quantity

$$g^2 \to -\frac{4\pi^3 r_{12} r_{32}^2}{r_{13}^2} F(r_{12}, r_{32}).$$

(58)

Function $F(r_1, r_2)$ can be presented in a different form, which demonstrates absence of ultraviolet divergency coming from the singular behavior of $\tilde{\eta}(\rho)$ at $\rho \to 0$. We have

$$\int d^2 \rho \tilde{\eta}(\rho) \xi^2(r_1 - \rho) = \int d^2 \rho \frac{d^2 q}{(2\pi)^2} \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} \frac{\eta(q)}{\eta(q_1) \eta(q_2)} e^{i\rho q + i q_1 (r_1 - \rho) + i q_2 (r_1 - \rho)}$$

$$= \int \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} \frac{\eta(q_1 + q_2)}{\eta(q_1) \eta(q_2)} e^{i r_1 (q_1 + q_2)}$$

(59)

and similarly

$$\int d^2 \rho \tilde{\eta}(\rho) \xi(r_1 - \rho) \xi(r_2 - \rho) = \int d^2 \rho \frac{d^2 q}{(2\pi)^2} \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} \frac{\eta(q)}{\eta(q_1) \eta(q_2)} e^{i\rho q + i q_1 (r_1 - \rho) + i q_2 (r_2 - \rho)}$$

$$= \int \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} \frac{\eta(q_1 + q_2)}{\eta(q_1) \eta(q_2)} e^{i r_1 q_1 + i r_2 q_2}.$$ 

(60)

So we find

$$F(r_1, r_2) = \int \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} \frac{\eta(q_1 + q_2)}{\eta(q_1) \eta(q_2)} \left( e^{i q_1 r_1} - e^{i q_1 r_2} \right) \left( e^{i q_2 r_1} - e^{i q_2 r_2} \right).$$

(61)

In this form it is clear that $F(r_1, r_2)$ is a well defined function which does not contain ultraviolet nor infrared divergency.

For further use note the identity

$$\int d^2 r_1 F(r_1 - r, r_1) = \int \frac{d^2 q_1}{(2\pi)^2} \frac{d^2 q_2}{(2\pi)^2} \frac{\eta(q_1 + q_2)}{\eta(q_1) \eta(q_2)} \left( e^{-i q_1 r} - 1 \right) \left( e^{-i q_2 r} - 1 \right) \int d^2 r_1 e^{i r_1 (q_1 + q_2)} = 0,$$

(62)

since $\eta(0) = 0$. 

9
4 Pomeron in the nucleus

With the triple Pomeron vertex known, it is straightforward to obtain the equation which sums fan diagrams describing propagation of the pomeron in the nucleus. Repeating the derivation for the fixed coupling constant in [10] we find for this sum \( \Phi(y, r, b) \) at fixed impact parameter \( b \):

\[
\Phi(y, r, b) = \Phi_1(y, r, b) + \frac{1}{2} N_c \int_0^\infty dy' \prod_{j=1}^3 d^2 r_j \delta^2(r_1 - r_2 + r_3)
\]

\[
F(r_2, r_3)\eta^2(-i\nabla)G(y - y', r, r_1)\Phi(y', r_2, b)\Phi(y', r_3, b).
\]

(63)

Here \( G(y, r, r') \) is the pomeron forward Green function satisfying the equation

\[
\left( \frac{\partial}{\partial y} + H \right) G(y, r, r') = \delta(y)\eta^{-1}(-i\nabla)\eta^{-1}(-i\nabla')\delta^2(r - r'),
\]

(64)

with the Hamiltonian \( H \) for the non-amputated forward wave function given by

\[
H = 2\omega + K^1,
\]

(65)

where \( \omega \) is expressed via function \( \eta \) according to [11] and

\[
K^1(q_1 | q'_1) = 2N_c \frac{\eta(q'_1)}{\eta(q_1)\eta(q_1 - q'_1)}.
\]

(66)

The inhomogeneous term \( \Phi_1 \) corresponds to the single pomeron exchange:

\[
\Phi_1(y, r, b) = \frac{1}{2} N_c T(b) \int d^2 r' G(y, r, r')\rho(r'),
\]

(67)

where \( T(b) \) is the nuclear profile function and \( \rho(r) \) is the color density of the nucleon. Applying operator \( \partial/\partial y + H \) to Eq. (63) we find the evolution equation for the pomeron in the nucleus

\[
\left( \frac{\partial}{\partial y} + H \right) \Phi(y, r, b) = \delta(y)\Phi_0(r, b) + \frac{1}{2} N_c \int \prod_{j=2}^3 d^2 r_j \delta^2(r - r_2 + r_3)F(r_2, r_3)\Phi(y, r_2, b)\Phi(y, r_3, b),
\]

(68)

where \( \Phi_0(r, b) \), playing the role of the initial condition, is given by

\[
\Phi_0(r, b) = \frac{1}{2} A T(b)\eta^{-2}(-i\nabla)\rho(r).
\]

(69)

Unlike the case of the fixed coupling constant this equation is not simplified in the momentum space.

To compare with the dipole approach, we rewrite our Eq. (68) as a whole in the coordinate space. To do this we have to rewrite action of the Hamiltonian (65) on the wave function in the coordinate space. In transforming the pomeron amplitude \( \Phi(y, q, b) \) to the coordinate space we have to take into account condition

\[
\Phi(y, r = 0, b) = 0
\]

(70)

which we have extensively used. Technically it means that we have to add to \( \Phi(y, q, b) \) a term proportional to \( \delta^2(q) \) which guarantees this property. This is essential to obtain the correct form for the linear part of the evolution equation in the coordinate space.

---

1 We are greatly indebted to Yu. Kovchegov who pointed out this circumstance.
It is convenient to split (55) into three terms
\[ F(r_1, r_2) = f(r_1, r_1) + f(r_2, r_2) - 2f(r_1, r_2), \]  
where
\[ f(r_1, r_2) = \int d^2\rho \tilde{\eta}(\rho) \xi(r_1 - \rho)\xi(r_2 - \rho). \]  
In terms of this function one easily finds
\[ 2 \int \frac{d^2q}{(2\pi)^2} e^{iqr} \omega(q) \Phi(y, q, b) = N_c \int d^2r_1 f(r_1 - r, r_1 - r) \Phi(y, r_1, b) \]  
and
\[ \int \frac{d^2q}{(2\pi)^2} \frac{d^2q'}{(2\pi)^2} e^{iqq'} K(q, q') \Phi(y, q', b) = 2N_c \int d^2r_1 f(r_1 - r, r_1) \Phi(y, r_1, b). \]  
Thus in the coordinate space we get
\[ H\Phi(y, r, b) = -N_c \int d^2r_1 \left( 2f(r_1 - r, r_1) - f(r_1 - r, r_1 - r) \right) \Phi(y, r_1) + \text{const}, \]  
where const should be taken to ensure property (70). As a result we find
\[ H\Phi(y, r, b) = N_c \int d^2r_1 F(r_1 - r, r_1) \Phi(y, r_1, b) = \frac{1}{2} N_c \int d^2r_1 F(r_1 - r, r_1) \left( \Phi(y, r_1, b) + \Phi(y, r_1 - r, b) \right). \]  
Using identity (52) we may add to the bracket any function independent of \( r_1 \), to finally obtain
\[ H\Phi(y, r, b) = \frac{1}{2} N_c \int d^2r_1 F(r_1 - r, r_1) \left( \Phi(y, r_1, b) + \Phi(y, r_1 - r, b) - \Phi(y, r_1 - r, b) \right). \]  
In this form the linear part of the evolution equation acquires the standard colour dipole structure (see e.g. [2]) and the whole evolution equation becomes
\[ \frac{\partial}{\partial y} \Phi(y, r) = -\frac{1}{2} N_c \int d^2r_1 F(r_1 - r, r_1) \left( \Phi(y, r_1, b) + \Phi(y, r_1 - r, b) - \Phi(y, r_1 - r, b) \right). \]  
In the limit of the fixed coupling constant we find, dropping the infrared regularization terms,
\[ f^{fix}(r_1, r_2) = \frac{\alpha_s}{\pi^2} \frac{r_1 r_2}{r_1^2 + r_2^2}, \]  
where at \( r_1 = r_2 = r \) one should understand \( 1/r^2 \) as regularized in the ultraviolet (see [8]):
\[ \frac{1}{r^2} \equiv \frac{1}{r^2 + \epsilon^2} + 2\pi \delta^2(r) \ln \epsilon, \quad \epsilon \to 0. \]  
So in analogy with [1] we may define 3 running coupling constants by
\[ f(r_1, r_2) = -\frac{\alpha_s(r_1) \alpha_s(r_2)}{\pi^2} \frac{r_1 r_2}{r_1^2 + r_2^2}, \]  
with the additional condition \( \alpha_s(r, r) = \alpha_s(r) \), and rewrite Eq. (78) as
\[ \frac{\partial}{\partial y} \Phi(y, r, b) = \frac{1}{2\pi^2} N_c \int d^2r_2 d^2r_3 \delta(r - r_1 + r_2) F(r_1, r_2) \left( \frac{\alpha_s(r_1)}{r_1^2} + \frac{\alpha_s(r_2)}{r_2^2} - 2 \frac{\alpha_s(r_1) \alpha_s(r_2)}{\alpha_s(r_1, r_2)} \frac{r_1 r_2}{r_1^2 + r_2^2} \right) \left( \Phi(y, r_1, b) + \Phi(y, r_2, b) - \Phi(y, r, b) - \Phi(y, r_1, b) \Phi(y, r_2, b) \right). \]
5 Discussion

Eqs. (78) and (82) present our final result for the non-linear BFKL equation with the running coupling. We stress that function $\eta(q)$ in them is determined only by its asymptotic form (5) together with the requirement (7). A simple possibility is to choose

$$
\eta(q) = \frac{1}{2\pi} b q^2 \ln \left( a + \frac{q^2}{\Lambda^2} \right),
$$

(83)

with $b$ given by (6) and arbitrary $a > 1$. Also one has to remember that the equations are derived only in the leading order in the running coupling. Already subleading terms of the relative order $1/\ln(q^2/\Lambda^2)$ remain undetermined, since they correspond to the next-to-leading order in the running coupling.

Our final coordinate space equation (82) fully coincides with Eq. (101) in [1] obtained in the dipole formalism (for the forward case). However in our approach the running couplings $\alpha_s(r)$ and $\alpha_s(r_1, r_2)$ are defined by Eq. (81) in a general manner, irrespective to any regularization procedure and are determined by the concrete choice of function $\eta(q)$. In fact they are fixed only in as far the high-momentum behaviour of this function is known and so admit a high degree of arbitrariness. It remains to be seen how this arbitrariness influence concrete results which follow from the solution of the evolution equation (82).

6 Appendix. Function $F(r_1, r_2)$ with a fixed coupling constant

We check that for a fixed coupling constant function $F(r_1, r_2)$ indeed passes into the expression corresponding to Eq. (57). In the fixed coupling case $\eta(k)$ is given by (4). Then we find

$$
\tilde{\eta}(\rho) = -\frac{2\pi}{g^2} \nabla^2 \delta^2(\rho)
$$

(84)

and

$$
\xi(r) = -\frac{g^2}{(2\pi)^2} (\ln r - c),
$$

(85)

where $c = \ln(2/m) + \psi(1)$ and $m$ is the infrared regularizer (gluon mass). Terms containing $c$ and thus depending on the infrared regularization cancel in the final result. Performing the integration over $\rho$ with the help of the $\delta$-function we find

$$
f^{fix}(r_1, r_2) = -\frac{g^2}{(2\pi)^3} \left( (\ln r_1 - c) \nabla_2^2 \ln r_2 + (\ln r_2 - c) \nabla_1^2 \ln r_1 + 2 \nabla_1 \ln r_1 \nabla_2 \ln r_2 \right).
$$

(86)

Taking into account that

$$
\nabla^2 \ln r = 2\pi \delta^2(r),
$$

(87)

we find

$$
f^{fix} = -\frac{g^2}{8\pi^3} \left( \frac{(r_1 r_2)^2}{r_1^2 r_2^2} + 2\pi \delta^2(r_1)(\ln r_2 - c) + 2\pi \delta^2(r_2)(\ln r_1 - c) \right).
$$

(88)

Forming $F(r_1, r_2)$ according to (71) we see that terms with $c$ cancel and dropping terms proportional to $\delta^2(r_1)$ or $\delta^2(r_2)$ we find

$$
F^{fix}(r_1, r_2) = -\frac{g^2}{4\pi^3} \frac{(r_1 - r_2)^2}{r_1^2 r_2^2},
$$

(89)

in full correspondence with (57).
7 Acknowledgements

The author would like to thank Yu. Kovchegov for numerous informative discussions. This work has been supported by grants RNP 2.1.1.1112 and RFFI 06-02-16115a of Russia.

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