Explosion of smoothness for conjugacies between multimodal maps

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Abstract

Let $f$ and $g$ be smooth multimodal maps with no periodic attractors and no neutral points. If a topological conjugacy $h$ between $f$ and $g$ is $C^1$ at a point in the nearby expanding set of $f$, then $h$ is a smooth diffeomorphism in the basin of attraction of a renormalization interval of $f$. In particular, if $f: I \to I$ and $g: J \to J$ are $C^r$ unimodal maps and $h$ is $C^1$ at a boundary of $I$, then $h$ is $C^r$ in $I$.

1. Introduction

There is a well-known theory in hyperbolic dynamics that studies properties of the dynamics and of the topological conjugacies that lead to additional regularity for the conjugacies. Mostow [21] proved that if $\mathbb{H}/\Gamma_X$ and $\mathbb{H}/\Gamma_Y$ are two closed hyperbolic Riemann surfaces covered by finitely generated Fuchsian groups $\Gamma_X$ and $\Gamma_Y$ of finite analytic type, and $\phi: \mathbb{H} \to \mathbb{H}$ induces the isomorphism $i(\gamma) = \phi \circ \gamma \circ \phi^{-1}$, then $\phi$ is a Möbius transformation if, and only if, $\phi$ is absolutely continuous. Shub and Sullivan [24] proved that for any two analytic orientation-preserving circle-expanding endomorphisms $f$ and $g$ of the same degree, the conjugacy is analytic if, and only if, the conjugacy is absolutely continuous. Furthermore, they proved that if $f$ and $g$ have the same set of eigenvalues, then the conjugacy is analytic. De la Llave [11] and Marco and Moriyon [14, 15] proved that if Anosov diffeomorphisms have the same set of eigenvalues, then the conjugacy is smooth. For maps with critical points, Lyubich and Milnor [12] proved that $C^2$ unimodal maps with Fibonacci combinatorics and the same eigenvalues are $C^1$ conjugate. De Melo and Martens [17] proved that if topological conjugate unimodal maps, whose attractors are cycles of intervals, have the same set of eigenvalues, then the conjugacy is smooth. Dobbs [2] proved that if a multimodal map $f$ has an absolutely continuous invariant measure, with a positive Lyapunov exponent, and $f$ is absolutely continuous conjugate to another multimodal map, then the conjugacy is $C^r$ in the domain of some induced Markov map of $f$.

Here, we study the explosion of smoothness for topological conjugacies, that is, the conditions under which the smoothness of the conjugacy in a single point extends to an open set. Tukia [27] extended the result above of Mostow proving that if $\mathbb{H}/\Gamma_X$ and $\mathbb{H}/\Gamma_Y$ are two closed hyperbolic Riemann surfaces covered by finitely generated Fuchsian groups $\Gamma_X$ and $\Gamma_Y$ of finite analytic type, and $\phi: \mathbb{H} \to \mathbb{H}$ induces the isomorphism $i(\gamma) = \phi \circ \gamma \circ \phi^{-1}$, then $\phi$ is a Möbius transformation if, and only if, $\phi$ is differentiable at one radial limit point with non-zero
derivative. Sullivan [26] proved that if a topological conjugacy between analytic orientation-preserving circle-expanding endomorphisms of the same degree is differentiable at a point with non-zero derivative, then the conjugacy is analytic. Extensions of these results for Markov maps and hyperbolic basic sets on surfaces were developed by Faria [3], Jiang [8, 10] and Pinto, Rand and Ferreira [4, 22], among others. For maps with critical points, Jiang [5–7, 9] proved that quasi-hyperbolic one-dimensional maps are smooth conjugated in an open set with full Lebesgue measure if the conjugacy is differentiable at a point with uniform bound. In this paper, we define the nearby expanding set $NE(f)$ of a multimodal map $f$ and characterize $NE(f)$ in terms of the basins of attraction of renormalization intervals. We prove that if a topological conjugacy between multimodal maps is $C^1$ at a point in the nearby expanding set $NE(f)$ of $f$, then the conjugacy is a smooth diffeomorphism in the basin of attraction of a renormalization interval.

2. Explosion of smoothness

Let $I$ be a compact interval and $f : I \rightarrow I$ a $C^{1+}$ map. By $C^{1+}$, we mean that $f$ is a differentiable map whose derivative is Hölder. We say that $c$ is a non-flat turning point of $f$, if there exist $\alpha > 1$ and a $C^\alpha$ diffeomorphism $\phi$ defined in a small neighbourhood $K$ of 0 such that

$$f(c + x) = f(c) + \phi(|x|^\alpha) \quad \text{for every } x \in K.$$  \hspace{1cm} (2.1)

We say that $\alpha$ is the order of the turning point $c$ and denote it by $\text{ord}_f(c)$. We say that $f$ is a multimodal map if the next three conditions hold: (i) $f(\partial I) \subset \partial I$; (ii) $f$ has a finite number of turning points that are all non-flat; and (iii) $\# \text{Fix}(f^n) < \infty$ for all $n \in \mathbb{N}$. A unimodal map $f : I \rightarrow I$ is a non-flat multimodal map with a unique turning point $c \in I$.

The non-critical backward orbit $O^-_{nc}(p)$ of $p$ is the set of all points $q$ such that there is $n = n(q) \geq 0$ with the property that $f^n(q) = p$ and $(f^n)'(q) \neq 0$. The non-critical alpha limit set $\alpha_{nc}(p)$ of $p$ is the set of all accumulation points of $O^-_{nc}(p)$. A periodic point $p$ with period $n \in \mathbb{N}$ is a repeller if $|Df^n(p)| > 1$. Let us denote by $\text{PR}(f)$ the set of all repeller periodic points of $f$. Let $O^-_{nc}(\text{PR}(f))$ be the union $\bigcup_{p \in \text{PR}(f)} O^-_{nc}(p)$ of the non-critical backward orbits $O^-_{nc}(p)$ for all repeller-periodic points of $p \in \text{PR}(f)$. Let $\alpha_{nc}(\text{PR}(f))$ be the union $\bigcup_{p \in \text{PR}(f)} \alpha_{nc}(p)$ of the non-critical alpha limit sets $\alpha_{nc}(p)$ for all repeller periodic points of $p \in \text{PR}(f)$.

A set $A \subset J$ is said to be forward invariant if $f(A) \subset A$. The basin $B(A)$ of a forward invariant set $A$ is the set of all points $x \in A$ such that its omega limit set $\omega(x)$ is contained in $A$. An invariant compact set $A \subset J$ is called a (minimal) attractor, in Milnor’s sense [19, 20], if the Lebesgue measure of its basin is positive and there is no forward invariant compact set $A'$ strictly contained in $A$ such that $B(A')$ has non-zero measure. The attractors of a $C^r$ non-flat multimodal map are of one of the following three types: (i) a periodic attractor; (ii) a minimal set with zero Lebesgue measure; or (iii) a cycle of intervals such that the omega limit set of almost every point in the cycle is the whole cycle (see [25]). According to van Strien and Vargas [25], if $f : I \rightarrow I$ is a $C^r$ non-flat multimodal map, then there is a finite set of attractors $A_1, \ldots , A_t \subset I$ such that the union of their basins has full Lebesgue measure in $I$.

An open interval $J(c)$ containing a critical point $c$ is a renormalization interval of a multimodal (respectively, unimodal) map $f$, if there is $n = n(J(c)) \geq 1$ such that $f^n(J(c))$ is also a multimodal (respectively, unimodal) map. Hence, the forward orbit of $J(c)$ is a positive invariant set. A multimodal map $f$ is not renormalizable inside a renormalization interval $J(c)$, if there is not a renormalization interval strictly contained in $J$. A multimodal map $f$ is infinitely renormalizable around a critical point $c$ if there is an infinite sequence of renormalization intervals $J_1(c), J_2(c), \ldots$ such that $J_{n+1}(c)$ is strictly contained in $J_n(c)$ and
Definition 2.1 (Expanding and nearby expanding points). A point \( p \in I \) is called nearby expanding if there are

1. a sequence of points \( p_n \) converging to \( p \),
2. a sequence of open intervals \( V_n \) containing \( p_n \),
3. a sequence of positive integers \( k_n \) tending to infinity, and
4. \( \delta = \delta(p) > 0 \), with the following properties:

   i. \( f^{k_n} \big| V_n \) is a diffeomorphism and
   ii. \( f^{k_n} (V_n) = B_\delta(f^{k_n}(p_n)) \).

Furthermore, a point \( p \in I \) is called expanding if \( p \in I \) is a nearby expanding point with \( p_n = p \) for every \( n \in \mathbb{N} \).

The nearby expanding set \( \text{NE}(f) \) is the set of all nearby expanding points of \( f \) and the expanding set \( E(f) \) is the set of all expanding points of \( f \).

Lemma 2.2 (Fatness of \( E(f) \) and \( \text{NE}(f) \)). Let \( f \) be \( C^r \) a multimodal map with \( r \geq 3 \) and no periodic attractors or neutral periodic points. Then the following conditions are satisfied.

1. \( E(f) \supset O_{nc}(\text{PR}(f)) \) and \( \text{NE}(f) \supset \alpha_{nc}(\text{PR}(f)) \);
2. if \( f \) is infinitely renormalizable around a critical point \( c \), then there is a renormalization interval \( J(c) \) such that \( E(f) \) and \( \text{NE}(f) \) are dense in \( B(J(c)) \);
3. if \( f \) is not renormalizable inside a renormalizable interval \( J \), then \( E(f) \) is dense in \( B(J) \) and \( \text{NE}(f) \) contains \( B(J) \).

If \( f : I \to I \) is a unimodal map, then for every renormalization interval \( J \), \( \partial B(J) \) is uniformly expanding, \( \partial I \subset \partial B(J) \) and \( B(J) \) is an open set with full Lebesgue measure. Hence, by Lemma 2.2, if \( f \) is a unimodal map whose attractor is a cycle of intervals, then \( E(f) \) is dense in \( I \) and \( \text{NE}(f) = I \). Furthermore, if \( f \) is a unimodal map that is infinitely renormalizable, then \( E(f) \) and \( \text{NE}(f) \) are dense in \( I \).

Proof. Let \( f \) be infinitely renormalizable around a critical point \( c \). By Lemma A.5, there is a renormalization interval \( J(c) \) such that \( O_{nc}(\text{PR}(f)) \) is a dense set in \( J(c) \). Since \( E(f) \supset O_{nc}(\text{PR}(f)) \), we obtain that \( E(f) \) and \( \text{NE}(f) \) are dense in \( J(c) \).

Let \( f \) be not renormalizable inside a renormalizable interval \( J \). By Lemma A.5, \( \alpha_{nc}(\text{PR}(f)) \) contains \( J \). Hence, \( E(f) \) is dense in \( J \) and \( \text{NE}(f) \) contains \( J \).

Definition 2.3 (Puncture set \( P(J) \)). Let \( C_P(I) \) be the set of all critical points \( c \) whose non-critical alpha limit sets \( \alpha_{nc}(c) \) do not intersect the interior of \( I \). The puncture set \( P(I) \) of \( I \) is \( P(I) = \bigcup_{c \in C_P(I)} \partial O_{nc}(c) \). Let \( J \) be a renormalization interval and \( n \) the smallest integer such that \( F = f^n \big| J \) is a renormalization of \( f \). Let \( C_P(J) \) be the set of all critical points \( c \) whose non-critical alpha limit sets \( \alpha_{nc}(c) \) with respect to \( F \big| J \) do not intersect the interior of \( J \). The puncture set \( P(J) \) of \( J \) is \( P(J) = \bigcup_{c \in C_P(J)} \partial O_{nc}(c) \).

Hence, the puncture set \( P \) is either empty or a discrete set. Furthermore, we observe that the puncture set is not located in the central part of the dynamics, that is, (i) if \( f \) is infinitely renormalizable, then there is a renormalization interval \( J(c) \) such that \( P \cap J(c) = \emptyset \) and (ii) if the Milnor’s attractor \( A \) of \( f \) is a cycle of intervals, then \( P \cap A = \emptyset \), because \( \alpha_{nc}(c) \) is dense in \( A \) for every critical point \( c \) in the interior of \( A \).
Given a renormalization interval $J$, let $I(J)$ be the set of all points $x \in I$ whose forward orbit intersects $J$. Let $D(J)$ be the set of all connected components $G$ of $I(J)$, that is,

$$I(J) = \bigcup_{G \in D(J)} G.$$  

The open intervals $G \in D(J)$ are called the gaps of $I(J)$. We note that the boundary $\partial I(J)$ of $I(J)$ is totally disconnected. For every connected component $G \in D(J)$, let $m = m(G)$ be the smallest integer such that $f^m(G) \subset J$. If $m = 0$, then the puncture set $G_P \subset G$ of $G$ is $G_P = P(J)$, and if $m > 0$, then the puncture set $G_P \subset G$ is the union of all points $x \in G$ such that (i) $(f^m)'(x) = 0$ or (ii) $(f^m)'(x) \in P(J)$. We observe that $G_P \cap G$ is either a discrete set or empty. The punctured basin of attraction $\mathcal{B}_P(J)$ of $J$ is the union $\bigcup_{G \in D(J)} G \setminus G_P$. A renormalization domain $J = \bigcup_{c \in CR} J(c)$ of a multimodal map $f$ is the union of renormalization intervals $J(c)$ for a given subset $CR \subset C_f$. Set $\mathcal{B}_P(J) = \bigcup_{c \in CR} \mathcal{B}_P(J(c))$. We observe that $\mathcal{B}_P(J) = \mathcal{B}(J)$.

**Definition 2.4 (C^1 at a point).** We say that a map $h : I \to I'$ is C^1 at a point $p \in I$, if

$$\lim_{x,y \to p, x \neq y} \frac{h(x) - h(y)}{x - y} = h'(p) \neq 0.$$  

We observe that $h$ is C^1 at every point belonging to an interval $K \subset I$ if, and only if, $f$ is a C^1 local diffeomorphism in that interval $K$.

We say that a topological conjugacy $h : I \to L$ between $f : I \to I$ and $g : I' \to I'$ preserves the order of the critical points, if $\text{ord}_f(c) = \text{ord}_g(h(c))$ for every critical point $c \in C_f$.

**Theorem 2.5 (Explosion of smoothness).** Let $f$ and $g$ be $C^r$ multimodal maps with $r \geq 3$ and no periodic attractors or neutral periodic points. Let $h$ be a topological conjugacy between $f$ and $g$ preserving the order of the critical points. If $h$ is C^1 at a point $p \in \text{NE}(f)$, then one of the two following conditions holds.

1. $h$ is a $C^r$ diffeomorphism in $I \setminus P(I)$; or
2. there is a unique maximal renormalization domain $J$ such that $h$ is a $C^r$ diffeomorphism in $J \setminus P(J)$. Furthermore, we have the following properties.
   a. $h$ is a $C^r$ diffeomorphism in the punctured basin of attraction $\mathcal{B}_P(J)$;
   b. $h$ is not $C^r$ at any open interval contained in $I \setminus \mathcal{B}(J)$;
   c. $h$ is not C^1 at any point in $E(f) \cap \partial \mathcal{B}(J)$.

We observe that Theorem 2.5 still holds if we replace the hypothesis of $h$ being C^1 at a point $p \in E(f)$ by $h$ being $C^r$ in an open set. Dobbs [2] proved that if (i) a multimodal map $f$ has an absolutely continuous invariant measure with a positive Lyapunov exponent and (ii) the conjugacy $h$ between $f$ and another multimodal map $g$ is absolutely continuous, then $h$ is $C^r$ in an open set. Hence, Theorem 2.5 applies to this case.

The proof of Theorem 2.5 is given at the end of Section 6.

**Corollary 2.6 (Full measure explosion of smoothness for unimodal maps).** Let $f$ and $g$ be $C^r$ unimodal maps with $r \geq 3$ and no periodic attractors or neutral periodic points. Let $h$ be a topological conjugacy between $f$ and $g$ preserving the order of the critical points. If $h$ is C^1 at a point $p \in \text{NE}(f)$, then one of the two following conditions holds.

1. $h$ is a $C^r$ diffeomorphism in the full interval $I$; or
(2) there is a unique maximal renormalization interval \( J \subseteq I \) such that we have the following properties.
(a) \( h \) is a \( C^r \) diffeomorphism in the basin \( B(J) \), and
(b) \( h \) is not \( C^1 \) at any point in \( \partial B(J) \).

We observe that if \( f : I \to I \) is a unimodal map, then (i) \( \partial B(J) \) is uniformly expanding, (ii) \( \partial I \subset \partial B(J) \) and (iii) \( B(J) \) is an open set with full Lebesgue measure in \( I \). By Corollary 2.6, the map \( h \) is \( C^1 \) at a point \( p \in \partial I \) if, and only if, \( h \) is a \( C^r \) diffeomorphism in \( I \).

3. Zooming pairs

In Theorem 5.6 and in its two corollaries, we will prove that the hypothesis that \( h \) is \( C^1 \) at a point \( p \) can be weakened to \( h \) being (u.a.a.) uniformly asymptotically affine at \( p \). We will define the zooming pairs that we will use to show that if \( h \) is u.a.a. at a point, then \( h \) and \( h^{-1} \) are \( C^r \) in small open sets.

Let \( h : I \to I' \) be a homeomorphism. For every \((x, y, z)\) of points \( x, y, z \in I \), such that \( x < y < z \), we define the logarithmic ratio distortion \( lrd(y, z) \) with respect to a sequence \( x, y \)\( \in \mathbb{R} \times \mathbb{R} \) such that \( y < z \) for all such \( p \in C \) are asymptotically affine for all \( \theta \) (3.4).

**Definition 3.1** (u.a.a.). Let \( h : I \to I' \) be a homeomorphism. The map \( h \) is uniformly asymptotically affine (u.a.a.) at a point \( p \) if, for every \( C > 1 \), there is a continuous function \( \epsilon_C : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) with \( \epsilon_C(0) = 0 \), such that
\[
\operatorname{lrd}_h(x, y, z) \leq \epsilon_C(|x - p|) \tag{3.1}
\]
for all \( x < y < z \) with \( C^{-1} < |z - y|/|y - x| < C \).

**Lemma 3.2** (\( C^1 \) implies u.a.a.). Let \( h : I \to I' \) be a homeomorphism. If \( h \) is \( C^1 \) at a point \( p \in I \), then \( h \) is u.a.a. at \( p \).

**Proof.** If \( h \) is \( C^1 \) at \( p \), then there is a sequence \( \theta_m \) converging to 0, when \( m \) tends to \( \infty \), such that
\[
\left| \log \frac{|h(y) - h(x)|}{|y - x|} - h'(p) \right| \leq O \left( \frac{1}{m} \right) \tag{3.2}
\]
for all \( x, y \in B_{\theta_m}(p) \). Hence, for all \( x, y, z \in B_{\theta_m}(p) \), we obtain
\[
\left| \log \frac{|h(z) - h(y)|}{|y - x|} - h'(p) \right| \leq O \left( \frac{1}{m} \right), \tag{3.3}
\]
and so \( h \) is u.a.a. at \( p \).

**Definition 3.3** (\( \alpha \)-bounded distortion). We say that a \( C^r \) multimodal map \( f \) has \( \alpha \)-bounded distortion with respect to a sequence \( V_1, V_2, \ldots \) of intervals and a sequence of integers \( k_n \) tending to \( \infty \), if there is \( C \geq 1 \) such that
\[
\operatorname{lrd}_{f^{k_n}}(x, y, z) \leq C|f^{k_n}(z) - f^{k_n}(x)|^\alpha \tag{3.4}
\]
for all \( x, y, z \in V_n \), with \( x < y < z \), and all \( n \geq 1 \).

**Definition 3.4** (Zooming pair \((p, V)\)). Let \( f : I \to I \) and \( g : I' \to I' \) be \( C^r \) maps, with \( r \geq 2 \), and \( h : I \to I' \) a topological conjugacy between \( f \) and \( g \). An \( \alpha \)-zooming pair \((p, V)\)
consists of a point \( p \in I \) and an open interval \( V \subset I \) such that (1) there is a sequence \( V_1, V_2, \ldots \) of intervals in \( I \) and (2) a sequence of integers \( k_n \) tending to \( \infty \), with the following properties:

1. \( \sup_{x \in V_n} |x - p| \to 0 \) when \( n \to \infty \);
2. \( f^{k_n}|V_n \) and \( g^{k_n}|h(V_n) \) are diffeomorphisms onto the intervals \( V \) and \( h(V) \), respectively;
3. \( f \) has \( \alpha \)-bounded distortion with respect to the sequences \( V_1, V_2, \ldots \) and \( k_1, k_2, \ldots \);
4. \( g \) has \( \alpha \)-bounded distortion with respect to the sequences \( h(V_1), h(V_2), \ldots \) and \( k_1, k_2, \ldots \).

An \( \alpha \)-central zooming pair \((p, V)\) is an \( \alpha \)-zooming pair \((p, V)\) with the property that \( p \in V_n \) for some \( n \in \mathbb{N} \).

**Proposition 3.5 (Explosion of smoothness from \( p \) to \( V \)).** Let \( f \) and \( g \) be \( C^r \) maps, with \( r \geq 3 \), topologically conjugated by a homeomorphism \( h \). Assume that \((p, V)\) is an \( \alpha \)-zooming pair for some \( 0 < \alpha < 1 \). If \( h \) is u.a.a. at \( p \), then \( h|V \) is a \( C^{1+\alpha} \) diffeomorphism onto its image. Furthermore, if \((p, V)\) is an \( \alpha \)-central zooming pair, then \( h|V_0 \) is a \( C^{1+\alpha} \) diffeomorphism onto its image, for some open interval \( V_0 \) containing \( p \).

**Proof.** Given \( a, b, c \in V \), with \( a < b < c \), let \( a_n, b_n, c_n \in V_n \) be such that \( f^{k_n}(a_n) = a \), \( f^{k_n}(b_n) = b \) and \( f^{k_n}(c_n) = c \). Since \( f \) has \( \alpha \)-uniformly bounded distortion,

\[
|\text{lrd}_{f^{k_n}}(a_n, b_n, c_n)| \leq O(|c - a|^{\alpha}).
\]

(3.5)

Hence, there is \( C > 1 \) such that \( C^{-1} < |c_n - b_n|/|b_n - a_n| < C \) for every \( n \geq 1 \). Since \( g \) has \( \alpha \)-uniformly bounded distortion, we obtain

\[
|\text{lrd}_{g^{k_n}}(h(a_n), h(b_n), h(c_n))| \leq O(|h(c) - h(a)|^{\alpha}).
\]

(3.6)

By the definition of zooming, there is a sequence \( \sigma_n \to 0 \) such that, for all \( x \in V_n \),

\[
|x - p| < \sigma_n.
\]

(3.7)

Since \( h \) is (u.a.a.) at \( p \), by (3.1), we have

\[
|\text{lrd}_h(a_n, b_n, c_n)| \leq c_n(\sigma_n).
\]

Hence, by (3.7), there is \( n \) large enough such that

\[
|\text{lrd}_h(a_n, b_n, c_n)| \leq |c - a|.
\]

(3.8)

Combining (3.5), (3.6) and (3.8), we have

\[
|\text{lrd}_h(a, b, c)| \leq |\text{lrd}_{g^{k_n}}(h(a_n), h(b_n), h(c_n))| + |\text{lrd}_h(a_n, b_n, c_n)| + |\text{lrd}_{f^{k_n}}(a_n, b_n, c_n)| \leq O(|c - a|^{\alpha} + |h(c) - h(a)|^{\alpha}).
\]

(3.9)

Therefore, the homeomorphism \( h \) is quasi-symmetric in \( V \). Hence, there is \( \gamma > 0 \), such that \( h|V \) is \( \gamma \)-Hölder continuous. Thus, we obtain that (3.9) is bounded by \( C_1|c - a|^{\alpha \gamma} \) for some \( C_1 > 1 \). Hence, by Pinto and Sullivan [23], we obtain that \( h|V \) and \( h^{-1}|h(V) \) are \( C^{1+\alpha} \) maps. Therefore, \(|h(c) - h(a)| \leq O(|c - a|)\) and, so (3.9) is also bounded by \( C_2|c - a|^{\alpha} \) for some \( C_2 > 1 \). Hence, again by Pinto and Sullivan [23], we obtain that \( h|V \) and \( h^{-1}|h(V) \) are \( C^{1+\alpha} \) maps.

Furthermore, if \((p, V)\) is a central zooming pair, then there is an open interval \( V_0 \) containing \( p \) and an integer \( n \) such that \( f^n|V_0 \) is a \( C^r \) diffeomorphism and \( f^n(V_0) \subset V \). Hence, \( h|V_0 = (g^n|h(V_0))^{-1} \circ h \circ f^n \) is a \( C^{1+\alpha} \) diffeomorphism.

**Lemma 3.6 (Building up smoothness from \( C^{1+\alpha} \) to \( C^r \)).** Let \( f \) and \( g \) be \( C^r \) maps, with \( r \geq 3 \), topologically conjugated by a homeomorphism \( h \). If \( h|V \) is a \( C^{1+\alpha} \) diffeomorphism in some open set \( V \), then \( h|W \) is a \( C^r \) diffeomorphism for some open set \( W \subset V \).
Proof. By Lemma A.5, there is a repeller \( p \in I \) and integers \( m \) and \( l \) such that \( p \in \text{int}(f^m(V)) \) and \( f^l(p) = p \). Since \( p \) is a repeller there is an open interval \( W \subset \text{int}(f^n(V)) \) with \( p \in W \) such that \( |(f^n)'(x)| > \lambda > 1 \) for all \( x \in W \). Let \( W_0, W_1, \ldots \) be a sequence of open intervals contained in \( W \) such that (i) \( f^l(W_{n+1}) = W_n \), (ii) \( W_{n+1} \subset W_n \) and (iii) \( |W_n| \to 0 \) for every \( n \geq 0 \). Let \( i_n : W_n \to (0,1) \) be the affine map with the property that \( i_n(W_n) = (0,1) \) and let \( f_n = i_0 \circ f^{nl} \circ i_n^{-1} \). By Pinto, Rand and Ferreira [22, Lemma E13], there is \( b > 0 \) such that \( \| \ln d_{\gamma}(n) \|_{C^{\gamma}} \leq b \) for every \( n \geq 0 \). Hence, by Pinto, Rand and Ferreira [22, Lemma E15], there is a small \( \epsilon > 0 \) and a subsequence \( _{m}k_n \) converging to a \( C^{\gamma} \) diffeomorphism \( h \) in the \( C^{\gamma-\epsilon} \) norm.

Let \( W'_n = h(W_n) \) and let \( j_n : W'_n \to (0,1) \) be the affine map with the property that \( j_n(W'_n) = (0,1) \) for every \( n \geq 1 \). Let \( g_n = j_0 \circ g^{nl} \circ j_n^{-1} \). By Pinto, Rand and Ferreira [22, Lemma E13], there is \( b > 0 \) such that \( \| \ln d_{\gamma}(n) \|_{C^{\gamma}} \leq b \) for all \( n \geq 1 \). Hence, by Pinto, Rand and Ferreira [22, Lemma E15], there is a small \( \epsilon > 0 \) and a subsequence \( m_n \) of the sequence \( k_n \) such that \( g_{m_n} \) converges to a \( C^{\gamma} \) diffeomorphism \( g \) in the \( C^{\gamma-\epsilon} \) norm.

Let \( h_n = j_n \circ h \circ j_n^{-1} \). Since \( h \) is a \( C^{\gamma+\alpha} \) diffeomorphism, there is a sequence \( \lambda_n \) tending to 1 such that
\[
\frac{|h_n(z) - h_n(y)|}{|y - x|} \leq \lambda_n
\]
for all \( x, y, z \in (0,1) \). Hence, \( h = \lim h_n \) is an affine map.

We note that \( h|W_0 = j_0^{-1} \circ g_n \circ h_n \circ f_n^{-1} \circ i_0 \) for every \( n \geq 1 \). Hence,
\[
h|W_0 = \lim j_0^{-1} \circ g_{m_n} \circ h_{m_n} \circ f_{m_n}^{-1} \circ i_0 = j_0^{-1} \circ g \circ h \circ f^{-1} \circ i_0.
\]
Since, \( g, h \) and \( f \) are \( C^\gamma \) diffeomorphisms, we obtain that \( h|W_0 \) is a \( C^\gamma \) diffeomorphism. \( \square \)

4. Nearby expanding set

We will prove that for every nearby expanding point \( p \in \text{NE}(f) \) there is an open set \( V \) such that \( (p, V) \) is a 1-zooming pair.

Given any \( K \subset \mathbb{R} \) and \( r > 0 \), set \( B_r(K) = \bigcup_{p \in K} B_r(p) \), where \( B_r(p) = (p-r, p+r) \).

Recall that the Schwarzian derivative of \( f \) in the complement of the critical points is defined by
\[
Sf := \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2.
\]

**Proposition 4.1** (Nearby expanding point originates a zooming pair). Let \( f \) and \( g \) be \( C^3 \) multimodal maps topologically conjugated by \( h \), with no periodic attractors and no neutral periodic points. For every \( x \in \text{NE}(f) \), there is an interval \( V \) such that \( (x, V) \) is a 1-zooming pair. Furthermore, for every \( x \in E(f) \), there is an interval \( V \) such that \( (x, V) \) is a central 1-zooming pair.

**Proof.** By van Strien and Vargas [25], there is \( \gamma > 0 \) such that, for every point \( x \in I \), with
\[
f^n(x) \in \bigcup_{c \in C(f)} B_\gamma(c) \quad \text{and} \quad g^n(h(x)) \in \bigcup_{c \in C(f)} h(B_\gamma(c)),
\]
we have \( Sf^{n+1}(x) < 0 \) and \( Sg^{n+1}(h(x)) < 0 \).

By Lemma A.4, one finds \( \gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 \) and nice sets \( J_0, J_1, J_2 \) such that
\[
B_{\gamma_0}(C_f) \subset J_0 \subset B_{\gamma_1}(C_f) \subset B_{\gamma_2}(C_f) \subset J_1 \subset B_{\gamma_3}(C_f) \subset B_{\gamma_4}(C_f) \subset J_2 \subset B_{\gamma}(C_f).
\]

Let \( J_i = \bigcup_{c \in C_i} J_i(c), c \in J_i(c) = (a_i(c), b_i(c)) \) for every \( c \in C_f \) and \( i = 0, 1, 2 \).
Given $x \in \text{NE}(f)$, for some small $\delta > 0$, take a sequence of points $x_j \to x$ and intervals $W_j^0 \ni x_j$ such that $f^{m_j}(W_j^0)$ is a diffeomorphism and $f^{m_j}(W_j^0) = B_{2\delta}(f^{m_j}(x_j))$ for $m_j \to \infty$. Let $W_j \subset W_j^0$ be the interval such that $f^{m_j}(W_j) = B_\delta(x_j)$ and let $L_j^0, R_j^0$ be the connected components of $W_j^0 \setminus W_j$.

For every $j \geq 1$, define $n_j$ as follows: if $f^i(x_j) \notin J_1$ for every $0 \leq i < m_j$, then take $n_j = -1$; otherwise, take $n_j < m_j$ as the biggest integer such that $f^{n_j}(x_j) \in J_1$.

Our goal is to obtain a sequence $j_i \to +\infty$ and intervals $V_{j_i} \subset W_j^0$ containing $x_{j_i}$, with the following properties: $\inf_j |f^{m_j}(V_{j_i})| > 0$ and the ratio distortion of $f^{m_j}|V_{j_i}$ is uniformly bounded. If $n_j = -1$, then take $V_j = W_j$. In this case, $|f^{m_j}(V_j)| = 2\delta$ and the boundedness of the ratio distortion follows from Theorem A.1, because $J_1 \supset B_{2\gamma_2}(C_f)$. Thus, we assume from now on that $n_j \neq -1$.

If $\lim \inf_j m_j - n_j < \infty$, then let $V_j$ be the maximal interval such that $x_j \in V_j \subset W_j$ and $f^{n_j}(V_j) \subset J_2$. Taking a subsequence, we assume that there is $K > 0$ such that $m_j - n_j \leq K$ for every $j$.

Since $Df^{m_j} \neq 0$ in $W_j$ and $f^{n_j}(W_j) \cap J_1 \neq \emptyset$ and by maximality of $V_j$, if $W_j \neq V_j$, then

$$f^{m_j}(V_j) \supset (c_j - \gamma_4, c_j - \gamma_3) \quad \text{or} \quad f^{m_j}(V_j) \supset (c_j + \gamma_3, c_j + \gamma_4)$$

for some $c_j \in C_f$. In particular, $|f^{n_j}(V_j)| \geq \gamma_4 - \gamma_3$. Thus, there is $\varepsilon > 0$ such that, for every $j$, $|f^{m_j}(V_j)| > \varepsilon > 0$, $|f^{m_j}(V_j)| = |f^{m_j}(W_j)| = 2\delta$ or $f^{m_j}(V_j)$ is a finite iteration of an interval with length greater than $\gamma_4 - \gamma_3$. Furthermore, since

$$|f^{m_j}(L_j)|/|f^{m_j}(W_j)| = |f^{ni}(R_j)|/|f^{m_j}(W_j)| = \frac{1}{2} \quad \text{for every } j,$$  

we obtain

$$\frac{|f^{n_j+1}(L_j)|}{|f^{n_j+1}(V_j)|} \geq \frac{|f^{n_j+1}(L_j)|}{|f^{n_j+1}(W_j)|}$$

and

$$\frac{|f^{n_j+1}(R_j)|}{|f^{n_j+1}(V_j)|} \geq \frac{|f^{n_j+1}(R_j)|}{|f^{n_j+1}(W_j)|}$$

are bounded away from zero. Since $Sf^{n_j+1}(z) < 0$ for every

$$z \in f^{-n_j}(B_{\gamma_5}(C_f)) \supset f^{-n_j}(B_{\gamma_5}(C_f)) \supset f^{-n_j}(J_2) \supset V_j,$$

then the ratio distortion of $f^{n_j+1}(V_j)$ is uniformly bounded ($V_j \subset W_j$). Thus, the ratio distortion of $f^{m_j}(V_j)$ is also uniformly bounded and $|f^{m_j}(V_j)| > \varepsilon > 0$ for every $j$.

Let us consider the case $\lim \inf_j m_j - n_j = \infty$. Taking a subsequence, if necessary, we assume $\lim_j m_j - n_j = \infty$.

**Claim 4.2.** $f^{n_j}(W_j^0) \subset J_2$ for every $j \in \mathbb{N}$.

**Proof of the claim.** Let $V_j^0$ be the maximal interval such that

$$x_j \in V_j^0 \subset W_j^0 \quad \text{and} \quad f^{n_j}(V_j^0) \subset J_2.$$

We will show $W_j^0 = V_j^0$.

By the maximality of $V_j^0$, if $W_j^0 \neq V_j^0$, then there is $p_{2,j} \in \partial J_2 \cap \partial(f^{n_j}(V_j^0))$. On the other hand, since $f^{n_j}(x_j) \in J_1$, there is $p_{1,j} \in \partial J_1$ such that

$$f^{n_j}(V_j^0) \supset (p_{1,j}, p_{2,j}) \quad \text{or} \quad f^{n_j}(V_j^0) \supset (p_{2,j}, p_{1,j}).$$

If $p_{1,j} < p_{2,j}$, then take $T_j = (p_{1,j}, p_{2,j})$; otherwise, take $T_j = (p_{2,j}, p_{1,j})$. Since $J_1$ and $J_2$ are nice sets with $J_1 \subset J_2$, it follows that $f^k(\partial T_j) \cap J_1 = \emptyset$ for every $k \geq 0$. Hence, if $\ell_j \geq 0$ is the smaller integer such that $f^{\ell_j}(T_j) \cap J_1 \neq \emptyset$, then $f^{\ell_j}(T_j) \cap J_1(c_j) \neq \emptyset$ for some $c_j \in C_f$.  


Furthermore, \( f^{\ell_i}(T_j) \supset J_i(c_j) \). However, since \( Df^{m_j} \neq 0 \) on \( W_j^0 \), we obtain \( \ell_j \geq m_j - n_j \). Thus, it follows from Theorem A.1 that

\[
4\delta = |f^{m_j}(W_j^0)| \geq |f^{m_j}(V_j^0)| \geq |f^{m_j-n_j}(T_j)| \geq C\lambda^{m_j-n_j}|T_j| \\
\geq C\lambda^{m_j-n_j}(|\gamma_4 - \gamma_3|) \to \infty \text{ (for a subsequence)}.
\]

Hence, we get a contradiction. \( \square \)

By Theorem A.1, if \( f^i(W_j^0) \cap J_0 = \emptyset \) for every \( n_j < i < m_j \), then \( f^{m_j-(n_j+1)} \) has uniformly bounded distortion on \( f^{n_j+1}(W_j^0) \) not dependent upon \( j \). In particular,

\[
|f^{n_j+1}(L_j)|/|f^{n_j+1}(W_j)| \text{ and } |f^{n_j+1}(R_j)|/|f^{n_j+1}(W_j)|
\]

are bounded away from zero. Since \( f^{m_j}(W_j^0) \subset B_j(C_f) \) and \( Sf^{n_j+1}(z) < 0 \) for every \( z \in W_j^0 \), the ratio distortion of \( f^{n_j+1}|W_j \) is uniformly bounded. Thus, taking \( V_j = W_j \), the ratio distortion of \( f^{m_j}|V_j \) is uniformly bounded and \( |f^{m_j}(V_j)| = 2\delta \) for every \( j \).

From now on, we will assume not only that \( m_j - n_j \to \infty \), but also that \( f^i(W_j^0) \cap J_0 \neq \emptyset \) for some \( n_j < i < m_j \).

Let \( k_j \) be the smallest integer \( \ell > n_j \) such that \( f^\ell(W_j^0) \cap J_0 \neq \emptyset \), that is,

\[
k_j = \min\{\ell > n_j : f^\ell(W_j^0) \cap J_0 \neq \emptyset\}.
\]

**Claim 4.3.** There is \( K > 0 \) such that \( m_j - k_j \leq K \) for every \( j \in \mathbb{N} \).

**Proof of the claim.** Since \( f^\ell(x_j) \notin J_1 \) for all \( n_j < j < m_j \), there is a connected component \( T_j \) of \( J_1 \setminus \bar{J}_0 \) such that \( T_j \subset f^{k_j}(W_j^0) \). Since \( J_0 \) and \( J_1 \) are nice sets with \( J_0 \subset J_1 \), it follows that

\[
f^i(\partial T_j) \cap J_0 = \emptyset
\]

for all \( i \geq 0 \). Let \( \ell_j \geq 0 \) be the smaller integer such that

\[
f^{\ell_j}(T_j) \cap J_0 \neq \emptyset,
\]

that \( f^{\ell_j}(T_j) \cap J_0(c_j) \neq \emptyset \), for some \( c_j \in C_f \). Thus, \( f^{\ell_j}(T_j) \supset J_0(c_j) \). Since \( f^{m_j}|W_j^0 \) is a diffeomorphism, we obtain \( \ell_j \geq m_j - k_j \). Thus, from Theorem A.1, it follows that

\[
4\delta = |f^{m_j}(W_j^0)| \geq |f^{m_j-n_j}(T_j)| \geq C\lambda^{m_j-n_j} |T_j| \geq C\lambda^{m_j-n_j} (|\gamma_2 - \gamma_1|)
\]

for every \( j \in \mathbb{N} \). Since \( \lambda > 1 \), we necessarily have \( m_j - k_j \) bounded. \( \square \)

Using Theorem A.1, we conclude that \( f^{k_j-(n_j+1)} \) has uniformly bounded distortion on \( f^{n_j+1}(W_j^0) \) (not dependent upon \( j \)). Since \( 0 \leq m_j - k_j \leq K \) and \( f^{m_j}|W_j^0 \) is a diffeomorphism, we obtain that \( f^{m_j-(n_j+1)} \) has uniformly bounded distortion on \( f^{n_j+1}(W_j^0) \) (also not dependent upon \( j \)). Thus,

\[
|f^{n_j+1}(L_j)|/|f^{n_j+1}(W_j)| \text{ and } |f^{n_j+1}(R_j)|/|f^{n_j+1}(W_j)|
\]

are bounded away from zero. Since \( Sf^{n_j+1}(z) < 0 \), the ratio distortion of \( f^{n_j+1}|W_j \) is uniformly bounded for all \( z \in W_j^0 \). Again, taking \( V_j = W_j \), the ratio distortion of \( f^{m_j}|V_j \) is uniformly bounded and \( |f^{m_j}(V_j)| = 2\delta \) for all \( j \).

Thus, replacing \( j \) by a subsequence, we get intervals \( V_j \subset W_j^0 \) containing \( x_j \) with the following properties: \( \inf_j |f^{m_j}(V_j)| > 0 \), the ratio distortion of \( f^{m_j}|V_j \) is uniformly bounded and the ratio distortion of \( g^{m_j}|h(V_j) \) is also uniformly bounded.

By compactness, taking a subsequence, there is an open interval \( V \) and a sequence of intervals \( x_j \in V_j \) such that \( f^j(V_j) = V \) for all \( j \). Thus, \((x, V)\) is a 1-zooming pair. Similarly, if \( x \in E(f) \), then there is an interval \( V \) such that \((x, V)\) is a central 1-zooming pair. \( \square \)
Lemma 4.4 (Explosion of smoothness at expanding points). Let $f$ and $g$ be $C^3$ multimodal maps topologically conjugated by $h$, with no periodic attractors and no neutral periodic points. Let the conjugacy $h$ be $C^3$ at a point $x$. If $x \in \text{NE}(f)$, then there is an open interval $V$ such that $h|V$ is $C^r$.

Proof. By Proposition 4.1, if $x \in \text{NE}(f)$, then there is an interval $V$ such that $(x, V)$ is a 1-zooming pair. Since $h$ is $C^3$ at $x$, then by Lemma 3.2 we have that $h$ is u.a.a. at $x$. Thus, it follows from Proposition 3.5 that $h|V$ is a $C^{3+\alpha}$ diffeomorphism. Hence, by Lemma 3.6, $h|W$ is a $C^r$ diffeomorphism for some $W \subset V$.

5. Smooth conjugacy and renormalization intervals

In this section, we assume that $f$ and $g$ are $C^r$ multimodal maps with $r \geq 3$ and no periodic attractors or neutral periodic points. Furthermore, we assume that $h$ is a topological conjugacy between $f$ and $g$ preserving the order of the critical points. We define

$$s = \min_{\{c \in C_f\}} \{\text{ord}_f(c), r\}.$$

Definition 5.1 (Smooth conjugacy domain). For $s \leq t \leq r$, the $t$-smooth conjugacy interval $V$ is an open set $V$ such that $h|V$ is a $C^t$ diffeomorphism. The set $C^t_f \subseteq C_f$ consists of all critical points $c$ such that there is a $t$-smooth conjugacy open interval $V$ containing $c \in V$. For every $c \in C^t_f$, the $s$-smooth conjugacy maximal interval $J^s(c)$ of $c$ is the maximal open interval $J^s(c)$ containing $c$ such that $h$ is $C^s$ in $J^s(c)$. The $s$-smooth conjugacy domain $J^s$ is

$$J^s = \bigcup_{c \in C^t_f} J^s(c).$$

We say that a critical point $c \in C_f$ is $s$-recurrent if there is $n = n(c, s) \geq 1$ such that $J^s(c) \cap f^n J^s(c) \neq \emptyset$. Let $CR^s \subset C_f$ be the set of all $s$-recurrent critical points and let

$$J^s_R = \bigcup_{c \in CR^s} J^s(c).$$

Lemma 5.2 (Spreading smooth conjugacy intervals). Let $h$ be a topological conjugacy between $f$ and $g$ and let $s \leq t \leq r$. Then we have the following properties.

1. if $V$ is a $t$-smooth conjugacy interval, then $\text{int}(f(V))$ is a $t$-smooth conjugacy interval;
2. if $V$ is a $t$-smooth conjugacy interval, then the connected components of $f^{-1}(V) \setminus (f^{-1}(V) \cap C_f)$ are $t$-smooth conjugacy intervals;
3. if $V$ is an $s$-smooth conjugacy interval, then $f^{-1}(V)$ is an $s$-smooth conjugacy interval;
4. if $c \in C_f$ and, for some small open interval $V$ containing $c$ and some $n$ such that $f^n(V) \subset J^s$, then $c \in C^s_f$ and
5. if $c \in C_f$ and, for some small open interval $V \subset C^r$ and some $n$, $c \in \text{int}(f^n(V))$, then $c \in C^r_f$.

Proof. Since $f$ is a multimodal map, the interior of $f(V)$ is an open interval and for every $x \in f(V)$ there is an open interval $W$ such that $x \in f(W)$ and $f|W$ is a $C^r$ diffeomorphism. Hence, $h|f(W) = g \circ h \circ (f|W)^{-1}$ is a $C^r$ diffeomorphism.

For every $x \in f^{-1}(V) \setminus (f^{-1}(V) \cap C_f)$, there is an open interval $W$ such that $x \in W$ and $f|W$ is a $C^r$ diffeomorphism. Hence, $h|W = (g|h(W))^{-1} \circ h \circ f$ is a $C^r$ diffeomorphism.
Let \( c \in f^{-1}(V) \cap C_f \). Recall that in a small open neighbourhood \( V \) of \( h(c) \), for every \( h(c) + x \in V \),
\[
g(h(c) + x) = g(h(c)) + \psi(|x|^\alpha).
\]
For every \( y \in g(V) \), we define
\[
g_-(y) = h(c) - [\psi^{-1}(y - g(h(c)))]^{1/\alpha}
\]
and
\[
g_+(y) = h(c) + [\psi^{-1}(y - g(h(c)))]^{1/\alpha}.
\]
Hence,
\[
g^{-1}(y) = \{g_-(y), g_+(y)\}.
\]
Recall that in a small open neighbourhood \( W \) of \( c \), for every \( c + x \in W \),
\[
f(c + x) = f(c) + \phi(|x|^\alpha).
\]
Hence,
\[
h(c + x) = \begin{cases} g_- \circ h \circ f(c + x) = h(c) - [\psi^{-1}(-g(c') + h(f(c) + \phi(|x|^\alpha)))]^{1/\alpha}, & x \leq 0, \\ g_+ \circ h \circ f(c + x) = h(c) - [\psi^{-1}(-g(c') + h(f(c) + \phi(|x|^\alpha)))]^{1/\alpha}, & x \geq 0. \end{cases}
\]
The map \( \psi^{-1}(-g(h(c)) + h(f(c) + w)) \) is a \( C^r \) diffeomorphism with \( \psi^{-1}(-g(h(c)) + h(f(c))) = 0 \). Hence, by Taylor’s theorem, there is a constant \( \epsilon \) and a \( C^r \) diffeomorphism \( \theta \) such that
\[
\psi^{-1}(-g(h(c)) + h(f(c) + w)) = w(\epsilon + \theta(w)).
\]
Therefore,
\[
h(c + x) = h(c) + x(\epsilon + |x|^\alpha \theta(|x|^\alpha))^{1/\alpha}.
\]
Hence, \( h|W \) is a \( C^r \) diffeomorphism. \(\)

Following Martens [16], a union \( J = \bigcup J_i \) of pairwise disjoint open intervals \( J_1, J_2, \ldots \) is a nice set if the forward orbit of the boundaries \( \bigcup_{i=1}^j \partial J_i \) of \( J \) do not intersect \( J \). Denoting by \( p \) and \( q \) the points that form the boundary of an interval \( V \), the interval \( V \) is dynamically symmetric if either \( f(p) = f(q) \) or \( f(p) \) and \( f(q) \) form the boundary of \( f(V) \).

**Lemma 5.3 (Nice \( J^* \)).** If the topological conjugacy \( h \) is a \( C^s \) diffeomorphism in an open set \( V \), then the \( s \)-conjugacy maximal domain \( J^* \) is non-empty and there is \( n \geq 0 \) such that \( \text{int}(f^n(V)) \subset J^* \). Furthermore, the following conditions are satisfied:

1. if \( c \in C_f^* \) then the set \( J^*(c) \) is dynamically symmetric;
2. for all \( c_1, c_2 \in C_f^* \) the sets \( J^*(c_1) \) and \( J^*(c_2) \) are either disjoint or equal; and
3. the \( s \)-conjugacy maximal domain \( J^* \) is a nice set.

**Proof.** Let us assume that \( h \) is a \( C^s \) diffeomorphism in an open set \( V \). It follows from Lemma A.2 that there is an \( n \in \mathbb{N} \) and \( c \in C_f \) such that \( f^n|V \) is a diffeomorphism \( C^r \) and \( c \in \text{int}(f^n(V)) \). Hence, by Lemma 5.2(1), \( h \) is a \( C^s \) diffeomorphism in \( f^n(V) \). Therefore, \( J^*(c) \supset f^n(V) \) and so \( J^*(c) \) is a non-empty closed interval.

Let us denote \( J^* \) by \( J \). Let us denote by \( p \) and \( q \) the boundary points of \( J(c) \). Let us prove that the interval \( J(c) \) is dynamically symmetric, that is, either \( f(p) = f(q) \), or \( f(p) \) and \( f(q) \) form the boundary of \( f(J(c)) \). Let us suppose, in contradiction, that there is \( z \in \text{int} J(c) \) that is not a critical point such that \( f(z) = f(q) \) (or, similarly, \( f(z) = f(p) \)). Let \( V_z \) and \( V_q \) be small neighbourhoods of \( z \) and \( q \), respectively, such that \( f|V_z \) is a \( C^r \) diffeomorphism and
Let $G \cap J$ is a small open interval by Lemma 5.2, then $\partial G = J(1)$. Let us prove that the set $J$ is nice. Let us suppose, in contradiction, that there is a point $p \in \partial J(c)$ and $n \geq 0$ such that $f^n(p) \in J$ and $f^m(p) \notin J$ for all $0 < m < n$. Hence, there is a small neighbourhood $V$ of $p$ such that $f^n(V) \subset J$. By Lemma 5.2(1) and (3), $h$ is a $C^s$ diffeomorphism in $V$, which is absurd.

The proof that for all $c_1, c_2 \in C'_f$ the sets $J^{s}(c_1)$ and $J^{s}(c_2)$ are either disjoint or equal follows from a similar argument to the one above.

Given a nice set $J$, let $I(J)$ be the set of all points $x \in I$ whose forward orbit intersects $J$. Let $D(J)$ be the set of all connected components $G$ of $I(J)$, that is,

$$I(J) = \bigcup_{G \in D(J)} G.$$  

The open intervals $G \in D(J)$ are called the gaps of $I(J)$. We note that the boundary $\partial I(J)$ of the basin $I(J)$ is totally disconnected.

**Lemma 5.4 (The basin of attraction of $J^s$).** Let $\emptyset \neq J^s \subset \text{int}(I)$. For every $G \in D(J^s)$ with $G \cap J^s = \emptyset$, there is $n = n(G) \geq 1$ such that the following properties hold.

1. $f^n|G$ is a diffeomorphism;
2. there is $c \in C^s_f$ such that $f^n(G) = J^s(c)$;
3. $f^j(G) \cap J^s = \emptyset$ for every $0 \leq j < n$.

**Proof.** For every $x \in I(J) \setminus J$, let $n(x) > 1$ be such that $f^n(x) \in J$ and $f^j(x) \notin J$ for every $0 \leq j < n$. Let $E = \{x, \ldots, f^{-1}(x)\}$. By Lemma 5.3, $E \cap C_f = \emptyset$ and so there is a small open set $V$ such that $f^n|V$ is a $C^r$ diffeomorphism and $f^n(V) \subset J$. Let us prove, by contradiction, that there is a small open interval $W \subset V$ containing $x$ such that $n(y) = n(x)$ for every $y \in W$. If there is not a small open interval $W \subset V$ containing $x$ such that $n(y) = n(x)$ for every $y \in W$, then there is a sequence of points $x_n \in V$ converging to $x$ with $n(x_n) = j < n(x)$. Hence, $f^j(x) \notin \partial J$. Since $J$ is nice, $f^{n-j}(f^j(x)) \cap J = \emptyset$, which is a contradiction. Let $V = (x, a)$ be the maximal open interval containing $x$ such that $n(y) = n(x)$ for every $y \in V$. Let us prove, by contradiction, that $f^n(a) \in \partial J$. By the above argument, if $f^n(a) \in J$, then there is an open interval $W_a$ such that $n(y) = n(a)$ for every $y \in W_a$ which is absurd by maximality of $V$. Hence, for every $x \in I(J) \setminus J$, there is a maximal open interval $G$ such that $n(y) = n(x)$ for every $y \in G$, and $f^n(G) \subset \partial J$. Hence, $f^n|G$ is a $C^r$ diffeomorphism and $f^n(G) = J(c)$ for some $c$.

Recall from Definition 5.1, the definition of the domain $J^s_R$.

**Lemma 5.5 ($J^s_R$ is a renormalization domain).** Let $\emptyset \neq J^s \subset \text{int}(I)$. For every $c \in C^s_f$, there is $n(c)$ and $c' \in C^s_f$ with the following properties:

1. $f^n(c)(J^s(c)) \subset (J^s(c'))$;
2. $\partial f^n(c)(J^s(c)) \subset \partial J^s(c')$;
3. $f^i(J^s(c)) \cap J^s = \emptyset$ for every $1 \leq i < n(c)$;
4. $J^s_R$ is a renormalization domain;
5. $\overline{B(J^s_R)} \subset I(J^s)$ and $\overline{B(J^s_R)} = I(J^s)$. 

Proof. By Lemma 5.4, for every gap \( G \in D(J) \), there are \( n = n(G) \geq 1 \) and \( c(G) \in C_f \cap J \) such that \( f^n(G) = J(c(G)) \), \( f^n|G \) is a \( C^r \) diffeomorphism and \( f^n(G) \cap J = \emptyset \) for every \( 0 \leq i < n \).

For every \( c \in C_f \), either (A) \( \text{int}(f(J(c))) \cap \partial I(J) = \emptyset \); or (B) \( \text{int}(f(J)) \cap \partial I(J) \neq \emptyset \).

Case (I). Since \( \text{int}(f(J(c))) \cap \partial I(J) = \emptyset \), there is an open interval \( K \), that is, \( (i) \) an interval \( J(c') \) or \( (ii) \) a gap \( G \), such that \( f(J(c)) \subset K \). In case (i), this lemma follows from noting that \( J \) is nice, and so \( \partial f(J(c)) \subset \partial J(c') \). In case (ii), there is \( n = n(G) \geq 1 \) such that \( f^{n+1}(J(c)) \subset J(c(G)) \) and \( f^{i+1}(J(c)) \cap J = \emptyset \) for every \( 0 \leq i < n \). Furthermore, since \( J \) is nice, \( f^{n+1}(\partial J(c)) \subset \partial J(c(G)) \) which proves this lemma in case (ii).

Case (II). Let us suppose that there is a point \( z \in \partial I(J) \cap \text{int}(f(J(c))) \). Let \( V \) be a small neighbourhood contained in \( J(c) \) such that \( f|V \) is a \( C^r \) diffeomorphism and \( x \) is contained in the interior of \( f(V) \). Since \( \partial I(J) \) is a totally disconnected set, there are gaps \( G_y \) and \( G_y' \) with a boundary point \( y \in f(V) \). Let \( z \in V \) be such that \( f(z) = y \) and take a smaller neighbourhood \( V_0 \subset V \) of \( z \) such that \( f(V_0 \setminus \{z\}) \subset G_y \cup G_y' \). By Lemma 5.2, if there is

\[
w \in f^{n(G_y)+1}(V_0 \setminus \{z\}) \cap \partial J(c(G_y))\]

then there is an open interval \( W \subset f^{n(G_y)+1}(V_0 \setminus \{z\}) \) containing \( w \) such that \( h|W \) is a \( C^r \) diffeomorphism. Since \( w \in \partial J(c(G_y)) \), we obtain a contradiction. Hence, for some \( 0 \leq i < n(G_y) \), there is a critical point \( c_y \in C_f \) such that \( c_y = f^i(y) \). Therefore, \( J(c(G_y)) = J(c(G_y')) \) and \( n(G_y) = n(G_y') \). Since the set of critical points is finite, \( \partial I(J) \cap \text{int}(f(J(c))) \) is also finite and for every \( w \in \partial I(J) \cap \text{int}(f(J(c))) \), there are gaps \( G_w \) and \( G_w' \) with \( w \in \partial G_w \cap \partial G_w' \) such that

\[
J(c(G_w)) = J(c(G_w')) = J(c(G_y)) \quad \text{and} \quad n(G_w) = n(G_y).
\]

Furthermore, since \( J \) is nice, \( f^{n(G_y')}(\partial J(c)) \subset \partial J(c(G_y)) \), which proves this lemma in case (B).

Hence, Lemma 5.5(1) and (2) hold. Therefore, \( J_h \) is a renormalization domain. Lemma 5.5(1) and (2) also imply that for every gap \( G \subset I(J^s) \) there is a gap \( G' \subset B(J_h) \) such that \( G \setminus G' \) is either empty or it is a finite set of points \( S_G = G \setminus G' \) with the following properties: for every \( x \in S_G \) there is \( i = i(x) \) and \( j = j(x) \) such that (1) \( 0 \leq i < j \), (2) \( f^i(x) \in C_f^s \), (3) \( f^i(x) \notin J_h \), and (4) \( f^j(x) \in \partial J_h \). Hence, Lemma 5.5(4) holds.

**Theorem 5.6 (Explosion of smoothness).** Let \( f \) and \( g \) be \( C^r \) multimodal maps with \( r \geq 3 \) and no periodic attractors and no neutral periodic points. Let \( h \) be a topological conjugacy between \( f \) and \( g \) preserving the order of the critical points. If \( h \) is \( C^1 \) at a point \( p \in \text{NE}(f) \), then one of the following two conditions holds.

1. \( h \) is a \( C^s \) diffeomorphism in the full interval \( I \) or in its interior \( \text{int}(I) \); or
2. there is a unique maximal renormalization domain \( J \subset I \) such that \( h \) is a \( C^s \) diffeomorphism in \( J \) Furthermore,
   (a) \( h \) is a \( C^s \) diffeomorphism in the basin of attraction \( B(J) \);
   (b) \( h \) is not \( C^s \) at any open interval contained in \( I \setminus B(J) \);
   (c) \( h \) is not \( C^1 \) at any point in \( E(f) \cap \partial B(J) \).

Proof. By Lemma 4.4, there is an open interval \( W \) such that \( h|W \) is \( C^s \) and so the \( s \)-smooth conjugacy maximal domain \( J^s \neq \emptyset \). If \( h \) is not a \( C^s \) diffeomorphism in \( I \) or \( \text{int}(I) \), then, by Lemma 5.5, there is a renormalization domain \( J_h^s \) such that (i) \( h|B(J_h^s) \) is a \( C^s \) diffeomorphism and (ii) there is no open interval \( V \subset I \setminus B(J_h^s) = I \setminus \text{int}(J^s) \) such that \( h|V \) is a \( C^s \) diffeomorphism. Let us prove, by contradiction, that \( h \) is not \( C^1 \) at any point in \( E(f) \cap \partial B(J) \). By Lemma 4.4, if \( h \) is \( C^1 \) at some point \( x \in E(f) \cap \partial B(J) \), then there is an open interval \( W \) containing \( x \) such that \( h|W \) is \( C^s \), which is a contradiction.
Theorem 5.7 gives a criterion for non-smoothness of the conjugacy when the conjugacy does not preserve the order of the critical points. The non-critical forward orbit $O^+_nc(p)$ of $p$ is the set of all points $q$ such that there is $n = n(q) \geq 0$ with the property that $f^n(p) = q$ and $(f^n)'(p) \neq 0$. The non-critical omega limit set $\omega_{nc}(p)$ of $p$ is the set of all accumulation points of $O^+_nc(p)$.

Theorem 5.7 (Implosion of non-smoothness). Let $f$ and $g$ be $C^r$ multimodal maps with $r \geq 3$ and no periodic attractors and no neutral periodic points. Let $h$ be a topological conjugacy, between $f$ and $g$, not preserving the order of the critical points $c_f$ and $c_g = h(c_f)$. The conjugacy $h$ is not $C^1$ simultaneously at (i) a point belonging to $E(f) \cap \alpha_{nc}(c_f)$ and (ii) a point belonging to $E(f) \cap \omega_{nc}(c_f)$.

If $f$ is a Collet–Eckmann map with negative Schwarzian derivative, then $E(f) \cap \omega_{nc}(c_f) \neq \emptyset$ and $\alpha_{nc}(c_f)$ contains Milnor’s attractor cycle.

Proof. Let us prove, by contradiction, that $h$ is not $C^1$ at any point belonging to $E(f) \cap \alpha(c_f)$. If $h$ is $C^1$ at a point $x \in E(f) \cap \alpha_{nc}(c_f)$, then, by Lemma 4.4, there is an open interval $V_1$ containing $x$ such that $h|V_1$ is $C^r$. Since $x \in \alpha_{nc}(c_f)$, there is an integer $n$ such that $c \in \text{int}(f^n(V_1))$. Hence, by Lemma 5.2, $h$ is a $C^r$ diffeomorphism in an open set $V_c$ containing $c$.

If $h$ is $C^1$ at a point $x \in E(f) \cap \omega_{nc}(c_f)$, then, by Lemma 4.4, there is an open interval $W_1$ containing $x$ such that $h|W_1$ is $C^r$. Since $x \in \omega_{nc}(c_f)$, there is an open set $W_{f(c)}$ containing $f(c)$ and an integer $n$ such that $f^n(W_{f(c)}) \subset W_1$ and $f^n|W_{f(c)}$ is a $C^r$ diffeomorphism. Hence, by Lemma 5.2, $h|W_{f(c)}$ is a $C^r$ diffeomorphism.

Since $h$ does not preserve the order of the critical points $c_f$ and $c_g = h(c_f)$, $h$ cannot be $C^1$ at $c_f$ and $f(c_f)$ simultaneously, which is absurd.

\[ 6. \quad C^r \text{ smoothness of the conjugacy} \]

In this section, we prove Theorem 2.5.

Lemma 6.1 (K(c) $\subseteq J^c_R$ is a renormalization interval). Let $h$ be a $C^r$ diffeomorphism in an open set $V_1$. There is a maximal renormalization interval $K(c) \subseteq J^c_R$ and a puncture set $P(c) \subset K(c)$ such that

1. $h$ is a $C^r$ diffeomorphism in $K(c) \setminus P(c)$, and
2. $\text{int}(V_1 \cap \mathcal{B}(K(c))) \neq \emptyset$.

Furthermore, $\partial K(c) \subset E(f)$ and $h$ is not $C^1$ at the boundary $\partial K(c)$.

Proof. Using Lemma A.2, there is a sequence of open sets $V_1, V_2, V_3, \ldots$ such that (i) $V_{i+1} \cap C_f \neq \emptyset$; (ii) $f^m(V_i) \supset V_{i+1}$ and (iii) $|V_i| \to 0$. Since $C_f$ is finite, (i) there is $c' \in C_f \cap J$ and (ii) a subsequence $V_{n_1}, V_{n_2}, V_{n_3}, \ldots$ such that $f^m_i(V_{n_i}) \supset V_{n_i+1}$, where $m_i = \sum_{j=n_i}^{n_i+1-1} n_j$ and (iii) $c' \in V_{n_i}$ for every $i \geq 1$. By Lemma 5.2, $h|\text{int}(f^m_i(V_{n_i}))$ is a $C^r$ diffeomorphism and so $h|V_{n_i+1}$ is also a $C^r$ diffeomorphism. By Lemma 5.5, there is a non-empty maximal renormalization interval $J = J^c(c) \subseteq J^c_R$ containing $V_{n_i}$ for all $i$. Let $l$ be the smallest integer such that $F = f^l|J$ is a renormalization of $f$ restricted to $J$.

Let $C$ (possibly empty) be the set of all critical points $c \in C_F$ of $F|J$ such that there is no open interval $V_c \subset J$ with the property that $c \in V_c$ and $h|V_c$ is a $C^r$ diffeomorphism. For every $c \in C$, the $\alpha_{nc}(c)$ is the non-critical alpha limit set of $c$ with respect to $F|J$. Set $\alpha_{nc}(C) = \bigcup_{c \in C} \alpha_{nc}(c)$.

Let us prove that the open connected component $H$ of $J \setminus \alpha_{nc}(C)$ we have containing $c'$ is a renormalization interval for $F$. Let us start proving, by contradiction, that $H$ is non-empty. If $H = \emptyset$, then there are (i) $c_1 \in C$, (ii) an open interval $U \subset V_{n_1}$ and (iii) an integer $l$
such that $c_1 \in \text{int} F^l(U)$. By Lemma 5.2, $c_1 \in C_F^r$ which is absurd. Take $i_0$ large enough such that, for every $i \geq i_0$, $c' \in V_{n_i} \subset H$ and $c' \in V_{n_{i+1}} \subset H$. Since $f^{m_i}(V_{n_i}) \supset V_{n_{i+1}}$, there is $l_i$ such that (i) $F^{l_i}(V_{n_i}) = f^{m_i}(V_{n_i})$ and (ii) $F^{l_i}(V_{n_i}) \cap H \neq \emptyset$. Since $\alpha_{nc}(C)$ is forward invariant, $\partial F^{l_i}(H) \subset \alpha_{nc}(C)$. Let us prove, by contradiction, that (i) $\partial F^{l_i}(H) \subset \partial H$ and (ii) $F^{l_i}(H) \subset \bar{H}$. If $F^{l_i}(H) \not\subset \bar{H}$, then there is $x \in \partial H$ such that $x \in \text{int}(F^{l_i}(H))$. Hence, by Lemma 5.2, $h$ is $C^r$ in an open set containing $x$ which is a contradiction. Hence, $F^{l_i}(H) \subset \bar{H}$ and, by forward invariance of $\alpha_{nc}(C)$, we have $\partial F^{l_i}(H) \subset \partial H$. Thus, $H$ is a renormalization interval for $F$. Take the smallest integer such that $F_1 = F^k|H$ is a renormalization of $F$ restricted to $H$.

For every open interval $H_1 \subset H$, let $C_{H_1}$ be the set of all critical points $c \in H_1$ of $F_1|H$ such that there is no open interval $V_c \subset H$ with the property that $c \in V_c$ and $h|V_c$ is a $C^r$ diffeomorphism. For every $c \in C_{H_1}$, let $\mathcal{O}_{nc}(c)$ be the non-critical backward orbit of $c$ with respect to $F_1|H$. Set $\mathcal{O}_{nc}(C_{H_1}) = \bigcup_{c \in C_{H_1}} \mathcal{O}_{nc}(c)$. Since the accumulation set of $\mathcal{O}_{nc}(C_{H_1})$ is contained in $\alpha_{nc}(C)$, the set $\mathcal{O}_{nc}(C_{H_1})$ is a discrete set of $H$, for every open interval $H_1 \subset H$.

Now, let $H_1 \subset H$ be the maximal open set such that $h|H_1 \setminus \mathcal{O}_{nc}(C_{H_1})$ is $C^r$. Either (i) $H_1 = H$, or (ii) $H_1 \neq H$ is non-empty.

Case (i). The interval $K(c') = H$ is the maximal interval of renormalization containing $c'$ and $P(c') = \mathcal{O}_{nc}(C_{H_1})$ is the punctured set of $K(c')$ with the property that $h|K(c') \setminus P(c')$ is $C^r$. Furthermore, $\text{int}(V_1 \cap \mathcal{B}(K(c'))) \neq \emptyset$.

Case (ii). There is a large enough such that $V_{n_1} \subset H_1$ and $F_1(V_{n_1}) \cap H_1 \neq \emptyset$.

Let us prove, by contradiction, that $\partial H_1 \cap \mathcal{O}_{nc}(C_{H_1}) = \emptyset$. If $x \in \partial H_1 \cap \mathcal{O}_{nc}(C_{H_1})$, then take the smallest $m$ such that $F^m_1(x) \in C_{H_1}$. Let $a$ and $b$ be close enough to $x$ such that (i) either $(a, x)$ or $(x, b)$ is contained in $H_1$, (ii) $F^{m+1}(a) = F^{m+1}(b)$, (iii) $F^m(a, b), F^{m+1}(a, x)$ and $F^{m+1}(x, b)$ are diffeomorphisms. Hence, $(a, b) \subset H_1$, which is a contradiction.

Let us prove, by contradiction, that if $x \in \partial H_1$, then $x$ is not contained in the pre-orbit of a critical point. Take the smallest $m$ such that $F^m_1(x) = c$ is a critical point. Since $c \notin \mathcal{O}_{nc}(C_{H_1})$, there is a small open set $W$ containing $c$ such that $h|W$ is a $C^r$ diffeomorphism. Furthermore, there is a small enough open set $V$ such that $V$ contains $x$, (i) $F^m_1|V$ is a diffeomorphism and (iii) $F^m_1(V) \subset W$. Thus, by Lemma 5.2, $h|V$ is also a $C^r$ diffeomorphism, which is a contradiction.

Let us prove, by contradiction, that $F_1(\partial H_1) \cap H_1 = \emptyset$. If $x \in \partial H_1$ and $F_1(x) \in H_1$, then there are small enough open sets $V$ and $W$ such that (i) $V$ contains $x$, (ii) $F^m_1|V$ is a diffeomorphism because $x$ is not a critical point of $F_1$, (iii) $F_1(V) = W_i$, (iv) $W_i \subset H_1$ and (v) $h|W_i$ is a $C^r$ diffeomorphism. Hence, $h|V$ is also a $C^r$ diffeomorphism, which is a contradiction.

There is $x$ large enough such that $V_{n_1} \subset H_1$ and $c' \in F^r_1(V_{n_1})$, for some $k$, and so $F^r_1(H_1) \cap H_1 \neq \emptyset$. Hence, to prove that $H_1$ is a renormalization maximal interval it is enough to prove, by contradiction, that $F_1(\partial H_1) \subset \partial H_1$. If (i) $F_1(x) \notin \partial H_1$ and so $F_1(x) \notin H_1$, then (i) there is $y \in H_1$ such that $F_1(y) = x$ and (ii) open intervals $V$ and $W$ with the following properties: (i) $V$ contains $x$, (ii) $W \subset H_1$ contains $y$, (iii) $F_1|W$ is a diffeomorphism, (iv) $F_1(W) = V$. Since $h|W$ is a $C^r$ diffeomorphism, by Lemma 5.2, we get that $h|V$ is also a $C^r$ diffeomorphism, which is a contradiction. Therefore, $K(c') = H_1$ is a renormalization interval containing $c'$ and $P(c') = \mathcal{O}_{nc}(C_{H_1})$ is the punctured set of $K(c')$ such that $h|K(c') \setminus P(c')$ is $C^r$. Furthermore, $\text{int}(V_1 \cap \mathcal{B}(K(c'))) \neq \emptyset$. \hfill \Box

Proof of Theorem 2.5. By Lemma 4.4, there is an open interval $V_1$ such that $h|V_1$ is $C^r$. If $h$ is not a $C^r$ diffeomorphism in $I \setminus P$, then, by Lemma 6.1, there is a maximal renormalization interval $K(c')$ and a punctured set $P(c') \subset K(c')$ such that $h|K(c') \setminus P(c')$. By Lemma 5.2, $h$ is a $C^r$ diffeomorphism in the punctured basin of attraction $\mathcal{B}_P(J(c'))$.

Let $C_f$ be the union of all critical points $c \in C_f$ such that $K(c) \neq \emptyset$ is a maximal renormalization interval and $P(c) \subset K(c)$ is a punctured subset such that $h$ is a $C^r$ diffeomorphism in $K(c) \setminus P(c)$. Let $J = \bigcup_{c \in C_f} K(c)$ be the maximal renormalization domain and $P = \bigcup_{c \in C_f} P(c)$.
the punctured set of $J$. By Lemma 5.2, $h$ is a $C^r$ diffeomorphism in the punctured basin of attraction $\mathcal{B}_p(J) = \bigcup_{c \in C_f^r} \mathcal{B}_p(J(c'))$.

Let us prove, by contradiction, that $h$ is not a $C^r$ diffeomorphism at any open interval $V \subset I \setminus \overline{\mathcal{B}(J)}$. If $h$ is a $C^r$ diffeomorphism at $V$, then, by Lemma 6.1, there is $c \in C_f^r$ such that $\text{int}(V \cap \mathcal{B}(K(c))) \neq \emptyset$, which is a contradiction.

Let us prove, by contradiction, that $h$ is not $C^1$ at any point in $E(f) \cap \partial \mathcal{B}(J)$. By Lemma 4.4, if $h$ is $C^1$ at some point $x \in E(f) \cap \partial \mathcal{B}(J)$, then there is an open interval $W$ containing $x$ such that $h|W$ is $C^1$, which is a contradiction. ■

Appendix. Properties of multimodal maps

A periodic point $p$ with period $n \in \mathbb{N}$ is called a periodic attractor if there is an open set $V$ with $p \in \partial V$ such that $\lim_{j \to +\infty} f^j(V) = p$. A periodic point $p$ with period $n \in \mathbb{N}$ is called neutral if $|Df^n(p)| = 1$. A periodic point $p$ with period $n \in \mathbb{N}$ is weakly repelling if $p$ is neutral and there is an open set $V$ with $p \in V$ such that $f^n|V$ is a diffeomorphism and $\lim_{j \to +\infty} (f^j|V)^{-j}(x) = p$ for all $x \in V$. Recall that a periodic point $p$ with period $n \in \mathbb{N}$ is a repeller if $|Df^n(p)| > 1$. Let us denote by $\text{PR}(f)$ the set of all repeller periodic points of $f$.

**Theorem A.1** (Mañé). Let $f : I \to I$ be a $C^2$ map without weak repelling periodic points and such that $\# \text{Fix}(f^n) < \infty$ for all $n \in \mathbb{N}$. For every $\gamma > 0$, there are $C > 0$ and $\lambda > 1$ with the following property:

1. If $J \subset I$ is an interval whose $\omega(J)$ does not intersect any periodic attractor, and
2. If $n \in \mathbb{N}$ is such that, for every $0 \leq j \leq n$, $f^j(J) \cap B_\gamma(C_f) = \emptyset$,

then

$$\text{Ird} f^n(x, y, z) \leq C |f^n(z) - f^n(x)| \quad \text{and} \quad |f^n(J)| \geq C \lambda^n |J|,$$

for every $x, y, z \in J$ with $x < y < z$.

**Proof.** It follows from Mañé’s theorem [13] and the fact that the logarithm of a $C^2$ map is locally Lipschitz outside the critical set. ■

**Lemma A.2** (Forward capture of a critical point). Let $f : I \to I$ be a $C^2$ map and $\# \text{Fix}(f^n) < \infty$ for every $n \in \mathbb{N}$. For each interval $J \subset I$, whose $\omega(J)$ does not intersect a periodic attractor, there is $n \in \mathbb{N}$ such that the interior of $f^n(J)$ contains a critical point.

**Proof.** Let us suppose, in contradiction, that $f^n|\text{int}(J)$ is a diffeomorphism onto its image for every $n \in \mathbb{N}$. Since $\omega(J)$ does not intersect a periodic attractor and a $C^2$ map does not admit a wandering interval (see [1, 18]), there is $k > l > 0$ such that $f^k(J) \cap f^l(J) \neq \emptyset$. The closure $D$ of the set $\bigcup_{n \geq 0} f^{kn-l}(J)$ is a forward invariant interval for $f^{k-l}$. Thus, $g = f^{2(k-l)}|D$ is a monotone map of $D$ into itself. Thus, $\omega_g(x) \subset \text{Fix}(g)$ for every $x \in D$. Since $\# \text{Fix}(g) < \infty$, we get that there is an attracting fixed point $p \in D$ for $g$. Hence, $O_f^+(p)$ is an attracting periodic orbit for $f$ intersecting $\omega_f(J)$, contradicting our hypothesis. ■

**Lemma A.3** (Domain shrinking for iterated local diffeomorphisms). Let $f : I \to I$ be a $C^2$ map and $\# \text{Fix}(f^n) < \infty$ for every $n \in \mathbb{N}$. If $J_1, J_2, \ldots \in I$ is a sequence of open intervals such that $\bigcup_{n \geq 1} \omega(J_n)$ does not intersect a periodic attractor and $f^{m_n}|J_n$ are diffeomorphisms, with $m_n$ tending to $\infty$, then $|J_n| \to 0$ when $n$ tends to infinity.
Proof. Let us suppose, in contradiction, that there is $\delta > 0$ such that $|J_n| > \delta$ for every $n \geq 1$. Since $I$ is compact, there is an interval $L$ and an infinite subsequence $J_{m_1}, J_{m_2}, \ldots$ of intervals such that $L \subset J_{m_n}$ for every $n \geq 1$. Hence, $f^\ell|L$ is a diffeomorphism, for every $\ell \geq 1$, which, by Lemma A.2, is a contradiction.

Following Martens [16], recall that a union $J = \bigcup_i J_i$ of pairwise disjoint open intervals $J_1, J_2, \ldots$ is a nice set if the forward orbit of the boundaries $\bigcup_{i=1}^l \partial J_i$ of $J$ do not intersect $J$.

**Lemma A.4** (Nice infinitesimal neighbourhoods of critical points). Let $f : I \to I$ be a multimodal map without periodic attractors. For every small $\varepsilon > 0$, there is a nice set $J = \bigcup_{c \in \mathcal{C}_f}(p_c, q_c)$ such that $c \in (p_c, q_c) \subset B_{\varepsilon}(c)$ for all $c \in \mathcal{C}_f$.

We note that, if $\{J_k\}$ is the set of connected components of a nice set $J$, then

$$J' = \bigcup_{J_k \cap \mathcal{C}_f \neq \emptyset} J_k$$

is also a nice set. Let $\mathcal{N}$ be the collection of all nice sets $J = \bigcup_k (p_k, q_k)$ such that $\mathcal{C}_f \subset J$ and $(p_k, q_k) \cap \mathcal{C}_f \neq \emptyset$ for all $k$. We note that if $U, V \in \mathcal{N}$, then $U \cap V \in \mathcal{N}$.

**Proof.** First, let us show that there is a nice set $J$ such that $\mathcal{C}_f \subset J$. Consider the compact positive invariant set

$$\Lambda = \{x \in I; f^j(x) \notin B_{\varepsilon}(\mathcal{C}_f) \ \forall j \geq 0\}.$$

For every $c \in \mathcal{C}_f$, there is a connected component $J_{c, \Lambda} \supset B_{\varepsilon}(c)$ of $I \setminus \Lambda$. Let $J = \bigcup_{c \in \mathcal{C}_f} J_{c, \Lambda}$. Since $\partial J = \bigcup_{c \in \mathcal{C}_f} \partial J_{c, \Lambda} \subset \Lambda$, we get $f^j(\partial J) \subset \Lambda$ for every $j \geq 0$. Hence, $f^j(\partial J) \cap J = \emptyset$ for every $j \geq 0$, that is, $J$ is a nice set and contains $\mathcal{C}_f$. Thus, $\mathcal{N}$ is not an empty collection.

If $c \in \mathcal{C}_f$, then either $V \supset B_{\varepsilon}(c)$, for all $V \in \mathcal{N}$, or there exists $V(c) = \bigcup_{c \in \mathcal{C}_f} V_c(c) \in \mathcal{N}$ such that $V_c(c) \subset B_{\varepsilon}(c)$ and $\tilde{c} \in V_c(c)$ for all $\tilde{c} \in \mathcal{C}_f$.

Let $\mathcal{C}_f^j$ be the set of $c \in \mathcal{C}_f$ such that $V \supset B_{\varepsilon}(c)$ for all $V \in \mathcal{N}$. For every $c \in \mathcal{C}_f^j$, let $H(c) = \text{int} \bigcap_{J \in \mathcal{N}} J_c$, where $J_c$ is the connected component of $J$ containing $c$. Hence, $H(c)$ is a nice interval and

$$H(c) \subset W$$

for all $W \in \mathcal{N}$. (A.1)

**Claim A.1.** If $c_0 \in \mathcal{C}_f$ is non-wandering, then $c_0 \notin \mathcal{C}_f^j$ for all $\varepsilon > 0$.

**Proof of the claim.** Let $\varepsilon > 0$ and $c_0 \in \mathcal{C}_f$ be a non-wandering point. Hence, take the smallest $n \geq 1$ such that $f^n(H(c_0)) \cap H(c_0) \neq \emptyset$.

Either (i) $f^n(H(c_0)) \not\subset H(c_0)$ or (ii) $f^n(H(c_0)) \subset H(c_0)$.

Case (i). Take $q \in H(c_0)$ such that $f^n(q) \in f^n(H(c_0)) \cap H(c_0)$ and there is a small interval $V_q$ containing $q$ such that $f^n|V_q$ is a diffeomorphism. For every $c \in \mathcal{C}_f$, let $U_c$ be the connected component of $\text{int}(I) \setminus \{q, \ldots, f^{n-1}(q)\}$ containing $c$. We get that $U = \bigcup_{c \in \mathcal{C}_f} U_c$ belongs to $\mathcal{N}$ and $H(c_0) \not\subset U_c$, because $q \in H(c_0)$ but $q \notin U_{c_0}$, contradicting (A.1).

Case (ii). Since $f^n(H(c_0)) \subset H(c_0)$, then $q = f^n|H(c_0)$ is a multimodal map and $q(\partial H(c_0)) \subset \partial H(c_0)$. Since there is no periodic attractor for $g$, there is a periodic point $q \in H(c_0)$ for the map $g$. For every $c \in \mathcal{C}_f$, let $U_c$ be the connected component of $\text{int}(I) \setminus \{q, \ldots, f^{m-1}(q)\}$ containing $c$, where $m$ is the period of $q$ with respect to $f$. We obtain that $U = \bigcup_{c \in \mathcal{C}_f} U_c$ belongs to $\mathcal{N}$ and $H(c_0) \not\subset U_c$, because $q \in H(c_0)$ but $q \notin U_{c_0}$, contradicting (A.1).
Now, we consider the case of the wandering critical points. Let \( \varepsilon > 0 \) and \( q_0 \) be a wandering critical point. From Lemma A.2, there is \( n \geq 1 \) and a non-wandering \( \tilde{c} \in C_f \) such that \( \tilde{c} \in f^n(H(q_0)) \). By the claim above, \( \tilde{c} \notin C_f^\varepsilon \). Thus, there is \( V = \bigcup_{c \in C_f^\varepsilon} V_c \in \mathcal{N} \) such that \( \partial V_c \cap f^n(H(q_0)) \neq \emptyset \). Let \( q \in H(q_0) \) be such that \( f^n(q) \in \partial V_c \) and there is a small interval \( V_q \) containing \( q \) such that \( f^n|V_q \) is a diffeomorphism. For every \( c \in C_f \) consider \( U_c \) the connected component of \( V_c \setminus \{ q, \ldots, f^n(q) \} \) containing \( c \). Thus, \( U = \bigcup_{c \in C_f} U_c \in \mathcal{N} \) and \( H(q_0) \not\subseteq U_{c_0} \), contradicting (A.1).

**Lemma A.5** (Fatness of repellers). Let \( f \) be a \( C^r \) multimodal map with \( r \geq 3 \) and no periodic attractors and no neutral points.

1. If \( f \) is infinitely renormalizable around a critical point \( c \), then there is a renormalization interval \( J(c) \) such that \( O_{nc}(\text{PR}(f)) \) is dense in \( B(J(c)) \).
2. If \( f \) is not renormalizable inside a renormalizable interval \( J \), then \( \alpha_{nc}(\text{PR}(f)) \) contains \( B(J) \).

**Proof.** Let us prove (1). Since \( f \) is infinitely renormalizable around \( c \), there is an infinite sequence of intervals \( J_1, J_2, \ldots \) such that \( J_{n+1} \) is strictly contained in \( J_n \) and there is a sequence \( m_1, m_2, \ldots \) such that \( f^{m_n}|J_n \) is a multimodal map and \( \alpha \in f^{m_n}(J_n) \). By taking \( J_1 \) sufficiently small, we assume that for every critical point \( c' \in J_1 \) with \( c' \neq c \), there is a sequence \( l_1, l_2, \ldots \) such that \( m_n l_n < m_{n+1} \) and \( c' \in f^{m_n l_n}(J_{n+1}) \). Let \( p_n \) be a periodic point contained in the boundary \( \partial J_n \) of \( J_n \). Hence, \( p_n \) is a repeller and the set \( S = \bigcup_{n \geq 1} \alpha_{nc}(p_n) \) contains \( c \in \partial S \). Let us prove that \( S \) is dense in the smallest interval set that contains \( S \). In contradiction, suppose that \( S \) is not a dense set. Hence, there is an open interval \( K \) such that \( K \subseteq J_1 \setminus S \) and \( \partial K \subseteq \partial S \).

By forward invariance of \( S \) under \( f^{m_1} \), then \( f^{m_1 k_1}(K) \subseteq J_1 \setminus S \) and \( f^{m_1 k_1} \subseteq \partial f^{m_1 k_1}(K) \subseteq S \) for every \( k \). By Lemma A.2, there is \( k_1 \) such that \( f^{m_1 k_1}(K) \) contains some critical point \( c' \in J_1 \). Hence, \( S \) is large enough and \( l_n \) such that \( f^{m_n l_n}(J_{n+1}) \subseteq f^{m_1 k_1}(K) \). Hence, there is \( k_2 \) such that \( f^{m_n l_n}(J_{n+1}) \subseteq f^{m_1 k_2}(K) \). Since \( c \in f^{m_n l_n}(J_{n+1}) \), we obtain \( c \in f^{m_1 k_2}(K) \). Noting that \( p_n \) converges to \( c \), we obtain that \( f^{m_1 k_2}(K) \) contains some \( p_n \), for \( n \) large, which contradicts that \( f^{m_1 k_2}(K) \subseteq J_1 \setminus S \). Hence, \( S \) is dense in the smallest interval set that contains \( S \). Since \( c \in \partial S \) is a turning point, \( S \) is dense in a small neighbourhood of \( c \). Hence, there is a renormalization interval \( J(c) \), small enough, containing \( c \), that is contained in the closure of \( S \).

Let us prove (2). Since \( J \) is a renormalization interval, there is \( m \) such that \( f^m|J \) is a multimodal map. Let \( p \in J \) be a periodic repeller with period \( k \) of the map \( f^m|J \). Since \( \alpha_{nc}(p) \) is a closed set, it is enough to prove that \( \alpha_{nc}(p) \) is dense in \( J \). In contradiction, suppose that \( \alpha_{nc}(p) \) is not a dense set. Hence, there is an open interval \( K \) such that \( K \subseteq J \setminus \alpha_{nc}(p) \) and \( \partial K \subseteq \alpha_{nc}(p) \). By forward invariance of \( \alpha_{nc}(p) \) under \( f^m \) then \( f^{mk}(K) \subseteq J_1 \setminus \alpha_{nc}(p) \) and \( \partial f^{mk}(K) \subseteq \alpha_{nc}(p) \) for every \( k \). By Lemma A.2, there is a sequence \( k_1, k_2, \ldots \) such that \( K_n = f^{mk_n}(K) \) contains some critical point \( c_n \in J \). Since the set of critical points in \( J \) is finite, there is a critical point \( c \in J \) and \( k_i < k_{i+1} \) such that \( K_{k_i} \) and \( K_{k_{i+1}} \) contain the critical point \( c \in J \). Hence, \( K_{k_i} \cap K_{k_{i+1}} = \emptyset \). Since

\[
\partial K_{k_i} \subseteq \alpha_{nc}(p), \quad \partial K_{k_{i+1}} \subseteq \alpha_{nc}(p), \quad K_{k_i} \cap \alpha_{nc}(p) = \emptyset \quad \text{and} \quad K_{k_{i+1}} \cap \alpha_{nc}(p) = \emptyset,
\]

we obtain \( K_{k_i} = K_{k_{i+1}} \). In particular, \( f^{mk_{i+1}}(K_{k_i}) \) is a multimodal map and \( K_{k_i} \) is strictly contained in \( J \), which contradicts that \( f \) is not renormalizable inside the renormalizable interval \( J \). Hence, \( \alpha_{nc}(p) \) contains the closure of \( J \). Hence, by definition of alpha limit, \( \alpha_{nc}(p) \) contains \( B(J) \).

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