RELATIONS BETWEEN TOPOLOGICAL AND METRICAL PROPERTIES OF SELF-AFFINE SIERPİŃSKI SPONGES

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Abstract. We investigate two Lipschitz invariants of metric spaces defined by $\delta$-connected components, called the maximal power law property and the perfectly disconnectedness. The first property has been studied in literature for some self-similar sets and Bedford-McMullen carpets, while the second property seems to be new. For a self-affine Sierpiński sponge $E$, we first show that $E$ satisfies the maximal power law if and only if $E$ and all its major projections contain trivial connected components; secondly, we show that $E$ is perfectly disconnected if and only if $E$ and all its major projections are totally disconnected.

1. Introduction

Let $d \geq 2$ and let $2 \leq n_1 < n_2 < \cdots < n_d$ be a sequence of integers. Let $\Lambda = \text{diag}(n_1, \ldots, n_d)$ be the $d \times d$ diagonal matrix. Let $D = \{i_1, \ldots, i_N\} \subset \prod_{j=1}^{d} \{0, 1, \ldots, n_j - 1\}$. For any $i \in D$ and $z \in \mathbb{R}^d$, we define $S_i(z) = \Lambda^{-1}(z + i)$, then $\{S_i\}_{i \in D}$ is an iterated function system (IFS). The unique non-empty compact set $E = K(\{n_j\}_{j=1}^{d}, D)$ satisfying

\begin{equation}
E = \bigcup_{i \in D} S_i(E)
\end{equation}

is called a $d$-dimensional self-affine Sierpiński sponge, see Kenyon and Peres [10] and Olsen [18]. In particular, if $d = 2$, then $E$ is called a Bedford-McMullen carpet.

There are a lot of works on dimensions, multifractal analysis and other topics of self-affine Sierpiński sponges, see for instance, McMullen [16], Bedford [2], King [11], Kenyon and Peres [10], Olsen [18], Barral and Mensi [1], Jordan and Rams [9], Mackay [15], Fraser and Howroyd [6]. Recently there are some works devoted to the Lipschitz classification of Bedford-McMullen carpets ([13, 17, 21, 23]). The goal of the present paper is to search new Lipschitz invariants of self-affine Sierpiński sponges, and investigate their relations with the topological properties of the major projections of such sponges.

Let $(E, \rho)$ be a metric space. Let $\delta > 0$. Two points $x, y \in E$ are said to be $\delta$-equivalent if there exists a sequence $\{x_1 = x, x_2, \ldots, x_{k-1}, x_k = y\} \subset E$ such that $\rho(x_i, x_{i+1}) \leq \delta$ for $1 \leq i \leq k - 1$. A $\delta$-equivalent class of $E$ is called a $\delta$-connected
component of $E$. We denote by $h_E(\delta)$ the cardinality of the set of $\delta$-connected components of $E$.

Two positive sequences $\{a_i\}_{i \geq 1}$ and $\{b_i\}_{i \geq 1}$ are said to be comparable, and denoted by $a_i \asymp b_i$, if there exists a constant $c > 1$ such that $c^{-1} \leq b_i/a_i \leq c$ for all $i \geq 1$. We define the maximal power law property as following.

**Definition 1.1 (Maximal power law).** Let $(E, \rho)$ be a compact metric space. Let $\gamma > 0$. We say $E$ satisfies the power law with index $\gamma$ if $h_E(\delta) \asymp \delta^{-\gamma}$; if $\gamma = \dim_B E$ in addition, we say $E$ satisfies the maximal power law.

The above definition is motivated by the notion of gap sequence. Gap sequence of a set on $\mathbb{R}$ was widely used by many mathematicians, for instance, [3, 5, 12]. Using the function $h_E(\delta)$, Rao, Ruan and Yang [20] generalized the notion of gap sequence to $E \subset \mathbb{R}^d$, denoted by $\{g_i(E)\}_{i \geq 1}$. It is shown in [20] that if two compact subsets $E, E' \subset \mathbb{R}^d$ are Lipschitz equivalent, then $g_i(E) \asymp g_i(E')$. Actually, the definition and result in [20] are also valid for metric spaces, see Section 2.

Miao, Xi and Xiong [17] observed that $E$ satisfies the power law with index $\gamma$ if and only if $g_i(E) \asymp i^{-1/\gamma}$ (see Lemma 2.2). Consequently, the (maximal) power law property is invariant under bi-Lipschitz maps.

Deng, Wang and Xi [4] proved that if $E \subset \mathbb{R}^d$ is a $C^{1+\alpha}$ conformal set satisfying the strong separation condition, then $E$ satisfies the maximal power law. Miao et al. [17] proved that a totally disconnected Bedford-McMullen carpet satisfies the maximal power law if and only if it possesses vacant rows. Liang, Miao and Ruan [14] completely characterized the gap sequences of Bedford-McMullen carpets. For higher dimensional fractal cubes (see Section 3 for the definition), we show that

**Theorem 1.1.** A fractal cube satisfies the maximal power law if and only if it has trivial points.

Let $E$ be a self-affine Sierpiński sponge defined in (1.1). A point $z \in E$ is called a trivial point of $E$ if $\{z\}$ is a connected component of $E$. For $x = (x_1, \ldots, x_d) \in E$ and $1 \leq j \leq d - 1$, we set

$$\pi_j(x) = (x_1, \ldots, x_j).$$

We call $\pi_j(E)$ the $j$-th major projection of $E$. We say $E$ is degenerated if $E$ is contained in a face of $[0,1]^d$ with dimension $d - 1$.

**Theorem 1.2.** Let $E$ be a non-degenerated self-affine Sierpiński sponge defined in (1.1). Then $E$ satisfies the maximal power law if and only if $E$ and all $\pi_j(E)(1 \leq j \leq d - 1)$ possess trivial points.

The following definition characterizes a class of fractals that all $\delta$-connected components are small. Let $\text{diam } U$ denote the diameter of a set $U$.

**Definition 1.2 (Perfectly disconnectedness).** Let $(E, \rho)$ be a compact metric space. We say $E$ is perfectly disconnected, if there is a constant $M_0 > 0$ such that for any $\delta$-connected component $U$ of $E$ with $0 < \delta < \text{diam}(E)$, $\text{diam } U \leq M_0 \delta$.

It is clear that perfectly disconnectedness implies totally disconnectedness, and the perfectly disconnectedness property is invariant under bi-Lipschitz maps.
Remark 1.1. It is essentially shown in Xi and Xiong [22] that a fractal cube is perfectly disconnected if and only if it is totally disconnected. We guess this may be true for a large class of self-similar sets.

**Theorem 1.3.** Let $E$ be a non-degenerated self-affine Sierpiński sponge defined in (1.1). Then $E$ is perfectly disconnected if and only if $E$ and all $\pi_j(E)(1 \leq j \leq d-1)$ are totally disconnected.

As a consequence of Theorem 1.2 and Theorem 1.3, we obtain

**Corollary 1.3.** Suppose $E$ and $E'$ are two non-degenerated self-affine Sierpiński sponges in $\mathbb{R}^d$. If $E$ and $E'$ are Lipschitz equivalent, then

(i) if $E$ and all $\pi_j(E)$, $1 \leq j \leq d-1$ possess trivial points, then so do $E'$ and $\pi_j(E')$, $1 \leq j \leq d-1$;

(ii) if $E$ and all $\pi_j(E)$, $1 \leq j \leq d-1$ are totally disconnected, so are $E'$ and $\pi_j(E')$, $1 \leq j \leq d-1$.

This paper is organized as follows: In Section 2, we give some basic facts about gap sequences of metric spaces. Then we prove Theorem 1.1, Theorem 1.2 and Theorem 1.3 in Section 3, 4, 5 respectively.

2. Gap sequences of metric spaces

Let $(E, \rho)$ be a metric space. Recall that $h_E(\delta)$ is the cardinality of the set of $\delta$-connected components of $E$. It is clear that $h_E : (0, +\infty) \to \mathbb{Z}_{\geq 1}$ is non-increasing. Let $\{\delta_k\}_{k \geq 1}$ be the set of discontinuous points of $h_E$ in decreasing order. Then $h_E(\delta) = 1$ on $[\delta_1, \infty)$, and is constant on $[\delta_{k+1}, \delta_k)$ for $k \geq 1$. We call $m_k = h_E(\delta_{k+1}) - h_E(\delta_k)$ the multiplicity of $\delta_k$ and define the gap sequence of $E$, denoted by $\{g_i(E)\}_{i \geq 1}$, to be the sequence

$$\delta_1, \ldots, \underbrace{\delta_1, \ldots, \delta_1}_{m_1}, \underbrace{\delta_2, \ldots, \delta_2}_{m_2}, \ldots, \underbrace{\delta_k, \ldots, \delta_k}_{m_k}, \ldots$$

In other words,

$$(2.1) \quad g_i(E) = \delta_k, \quad \text{if} \ h_E(\delta_k) \leq i < h_E(\delta_{k+1}).$$

**Lemma 2.1.** If two compact metric spaces $(E, \rho)$ and $(E', \rho')$ are Lipschitz equivalent, then $g_i(E) \sim g_i(E')$.

**Proof.** The lemma is proved in [20] in the case that both $E$ and $E'$ are subsets of $(\mathbb{R}^d, | \cdot |)$. Their proof works for metric spaces without any change. \hfill \Box

Miao et al. [17] gave the following criterion for power law property in the case that $E \subset \mathbb{R}^d$, but the conclusion and proof work for metric spaces. Here we give an alternative proof for the reader’s sake.

**Lemma 2.2 (17).** Let $(E, \rho)$ be a compact metric space and let $\gamma > 0$. Then

$$g_i(E) \sim i^{-1/\gamma} \iff h_E(\delta) \sim \delta^{-\gamma}.$$

Remark 2.1. Let \( \{\delta_k\}_{k \geq 1} \) be the set of discontinuous points of \( h_E \) in decreasing order. First, we show that either \( g_i(E) \approx i^{-1/\gamma}, i \geq 1 \) or \( h_E(\delta) \approx \delta^{-\gamma}, \delta \in (0, 1) \) will imply that

\[
M = \sup_{k \geq 1} \frac{\delta_k}{\delta_{k+1}} < \infty.
\]

If \( g_i(E) \approx i^{-1/\gamma}, i \geq 1 \), holds, then for any \( k \geq 1 \), there exists \( i \) such that \( g_i(E) = \delta_k \) and \( g_{i+1}(E) = \delta_{k+1} \), so \((2.2)\) holds in this case. If \( h_E(\delta) \approx \delta^{-\gamma}, \delta \in (0, 1) \), holds, then this together with \( \lim_{\delta \to (\delta_{k+1})^+} h_E(\delta) = h_E(\delta_k) \) imply \((2.2)\) again.

Finally, using \((2.1)\) and \((2.2)\), we obtain the lemma by a routine estimation. \( \square \)

Remark 2.1. Let \( m > 1 \) be an integer and \( \gamma > 0 \). Since \( h_E(\delta) \) is non-increasing, we see that \( h_E(m^{-k}) \approx m^{-k\gamma}(k \geq 0) \), implies that \( E \) satisfies the power law property.

3. Proof of Theorem 1.1

Let \( m \geq 2 \) be an integer. Let \( D_F = \{j_1, \ldots, j_r\} \subset \{0, 1, \ldots, m-1\}^d \). For \( j \in D_F \) and \( y \in \mathbb{R}^d \), we define \( \varphi_j(y) = \frac{1}{m}(y + j) \), then \( \{\varphi_j\}_{j \in D_F} \) is an IFS. The unique non-empty compact set \( F = F(m, D_F) \) satisfying

\[
F = \bigcup_{j \in D_F} \varphi_j(F)
\]

is called a \( d \)-dimensional fractal cube, see [22].

For \( \sigma = \sigma_1 \ldots \sigma_k \in D_F^k \), we define \( \varphi_\sigma(z) = \varphi_{\sigma_1} \circ \cdots \circ \varphi_{\sigma_k}(z) \). We call

\[
F_k = \bigcup_{\sigma \in D_F^k} \varphi_\sigma([0, 1]^d)
\]

the \( k \)-th approximation of \( F \). We call \( \varphi_\sigma([0, 1]^d) \) a \( k \)-th basic cube, and call it a \( k \)-th boundary cube if in addition \( \varphi_\sigma([0, 1]^d) \cap \partial[0, 1]^d \neq \emptyset \). Clearly, \( F_k \subset F_{k-1} \) for all \( k \geq 1 \) and \( F = \bigcap_{k=0}^{\infty} F_k \).

Recall that \( F \) is degenerated if \( F \) is contained in a face of \([0, 1]^d\) with dimension \( d-1 \). A connected component \( C \) of \( F_k \) is called a \( k \)-th island of \( F_k \) if \( C \cap \partial[0, 1]^d = \emptyset \), see [7]. Huang and Rao ([7],Theorem 3.1) proved that

**Proposition 3.1** ([7]). Let \( F \) be a \( d \)-dimensional fractal cube which is non-degenerated. Then \( F \) has trivial points if and only if there is an integer \( p \geq 1 \) such that \( F_p \) contains an island.

For \( A \subset \mathbb{R}^d \), we denote by \( N_c(A) \) the number of connected components of \( A \).

**Lemma 3.2.** Let \( F \) be a \( d \)-dimensional fractal cube defined in \((3.1)\). If \( F \) has no trivial point, then for any \( \delta \in (0, 1) \) we have

\[
h_F(\delta) \leq c_1 \delta^{-\log_m(r-1)},
\]

where \( c_1 > 0 \) is a constant.
Proof. Since a degenerated fractal cube \( F = F(m, \mathcal{D}_F) \) is always isometric to a non-degenerated fractal cube \( F' = F(m, \mathcal{D}_{F'}) \) such that \(#\mathcal{D}_F = #\mathcal{D}_{F'}\), so we only need to consider the case that \( F \) is non-degenerated.

Let \( q = \lfloor \log_m \sqrt{d} \rfloor + 1 \), where \([a]\) denotes the greatest integer no larger than \( a\). We will prove
\[
(3.3) \quad h_F(m^{-k}) \leq 2d(r-1)^{k+q}, \quad \text{for all } k \geq 1
\]
by induction on \( d\). Notice that \( r \geq m \) since \( F \) has no trivial point.

If \( d = 1 \), then \( r = m, \, F = [0,1] \) and \( h_F(m^{-k}) = 1 \) for all \( k \geq 1 \), so \((3.3)\) holds in this case.

Assume that \((3.3)\) holds for all \( d'\)-dimensional fractal cubes which have no trivial point, where \( d' < d\). Now let \( F \) be a \( d\)-dimensional fractal cube which has no trivial point. Denote the \((d-1)\)-faces of \([0,1]^d\) by \( \Omega_1, \ldots, \Omega_{2d} \). Let
\[
r_i = \# \{ j \in \Sigma; \, \varphi_j([0,1]^d) \cap \Omega_i \neq \emptyset \},
\]
be the number of 1-th boundary cube intersecting the face \( \Omega_i \). Since \( F \) is non-degenerated, we have \( r_i \leq r-1 \) for all \( 1 \leq i \leq 2d \). Clearly, the number of \( k\)-th boundary cubes which intersect \( \Omega_i \) is \( r_i^k \); so the number of \( k\)-th boundary cubes of \( F_k \) are at most \( 2d(r-1)^k \). Notice that \( F \) has no trivial point, then \( F_k \) has no island by Proposition 3.1. Thus each connected component of \( F_k \) contains at least one \( k\)-th boundary cube. By the choice of \( q \) we see that the diameter of a \((k+q)\)-th basic cube is less than \( m^{-k} \), then the points of \( F \) in a connected component of \( F_{k+q} \) are contained in a \( m^{-k}\)-connected component of \( F \). Therefore, we have
\[
h_F(m^{-k}) \leq N_c(F_{k+q}) \leq 2d(r-1)^{k+q},
\]
This completes the proof of \((3.3)\). Finally, by an argument similar to Lemma 3.2 we obtain \((3.2)\).

Proof of Theorem 1.1. Let \( F \) be a fractal cube defined by \((3.1)\). Notice that \( \dim_B F = \log_m r \). The necessity of the theorem is guaranteed by Lemma 3.2.

Now we prove the sufficiency. Suppose that \( F \) has trivial points. By Remark 2.1 it is sufficient to show
\[
(3.4) \quad h_F(m^{-k}) \propto m^{k \dim_B F}, \quad k \geq 0.
\]
As before, we only need to consider the case that \( F \) is non-degenerated.

By Proposition 3.1 \( F_p \) contains a \( p\)-th island \( C \) for some \( p \geq 1 \). Fix \( k > p \). Clearly any \( m^{-k}\)-connected component of \( F \) is contained in a connected component of \( F_{k-1} \). It is easy to see that \( \varphi_{\sigma}(C) \) is a \((k-1)\)-th island of \( F_{k-1} \) for any \( \sigma \in \mathcal{D}_F^{k-p-1} \), and the distance of any two distinct \((k-1)\)-th islands of the form \( \varphi_{\sigma}(C) \) is no less than \( m^{1-k} \), so
\[
(3.5) \quad h_F(m^{-k}) \geq N_c(F_{k-1}) \geq (\#\mathcal{D}_F)^{k-p-1} = r^{k-p-1}.
\]
Let \( q = \lfloor \log_m \sqrt{d} \rfloor + 1 \). Then the points of \( F \) in a \((k+q)\)-th basic cube is contained in a \( m^{-k}\)-connected component of \( F \), which implies that \( h_F(m^{-k}) \leq r^{k+q} \). This together with \((3.5)\) imply \( h_F(m^{-k}) \propto r^k = m^{k \dim_B F} \). The theorem is proved.
Remark 3.1. It is shown in [7] that if a fractal cube \( F \) has a trivial point, then the Hausdorff dimension of the collection of its non-trivial points is strictly less than \( \dim_H F \).

4. Proof of Theorem 1.2

In this section, we always assume that \( E \) is a self-affine Sierpiński sponge defined in (1.1). We call

\[
E_k = \bigcup_{\omega \in D^k} S_\omega([0,1]^d)
\]

the \( k \)-th approximation of \( E \), and call each \( S_\omega([0,1]^d) \) a \( k \)-th basic pillar of \( E_k \).

Recall that \( \pi_j(x_1, \ldots, x_d) = (x_1, \ldots, x_j) \), \( 1 \leq j \leq d-1 \) and by convention we set \( \#\pi_0(D) = 1 \) and \( \pi_d = \text{id} \). Note that \( \pi_j(E) \) is a self-affine Sierpiński sponge which determined by \( \{n_\ell\}_{\ell=1}^j \) and \( \pi_j(D) \). By [10], the box-counting dimension of \( E \) is

\[
\dim_B E = \sum_{j=1}^d \frac{1}{\log n_j} \log \frac{\#\pi_j(D)}{\#\pi_{j-1}(D)},
\]

Recall that \( \Lambda = \text{diag}(n_1, \ldots, n_d) \). A \( k \)-th basic pillar of \( E \) can be represented by

\[
S_{\omega_1 \ldots \omega_k}([0,1]^d) = \sum_{l=1}^k \Lambda^{-\ell_l} \omega_l + \prod_{j=1}^d [0, n_j^{-\ell_j(k)}],
\]

where \( \omega_1 \ldots \omega_k \in D^k \). For \( 1 \leq j \leq d \), denote \( \ell_j(k) = \lfloor k \log n_d / \log n_j \rfloor \). Clearly \( n_j^{-\ell_j(k)} \approx n_d^{-k} \) and \( \ell_1(k) \geq \ell_2(k) \geq \cdots \geq \ell_d(k) = k \). We call

\[
Q_k := \left( \sum_{l=1}^{\ell_1(k)} \frac{i_1(\omega_l)}{n_1^{\ell_1(k)}}, \ldots, \sum_{l=1}^{\ell_d(k)} \frac{i_d(\omega_l)}{n_d^{\ell_d(k)}} \right) + \prod_{j=1}^d [0, n_j^{-\ell_j(k)}],
\]

a \( k \)-th approximate box of \( E \), if \( \omega_l = (i_1(\omega_l), \ldots, i_d(\omega_l)) \in D \) for \( 1 \leq l \leq \ell_1(k) \). Let \( \tilde{E}_k \) be the union of all \( k \)-th approximate boxes. It is clear that \( \tilde{E}_k \subset E_k \).

Let \( \mu \) be the uniform Bernoulli measure on \( E \), that is, every \( k \)-th basic pillar has measure \( 1/N^k \) in \( \mu \). The following lemma illustrates a nice covering property of self-affine Sierpiński sponges; it is contained implicitly in [10] and it is a special case of a result in [8].

Lemma 4.1 ([8,10]). Let \( E \) be a self-affine Sierpiński sponge. Let \( R \) be a \( k \)-th basic pillar of \( E \). Then the number of \( (k+p) \)-th approximate boxes contained in \( R \) is comparable to

\[
\frac{n_d^{(k+p)\dim_B E}}{N^k}, \quad p \geq 1.
\]

Corollary 4.2. Let \( V \) be a union of some \( k \)-th cylinders of \( E \). Then there exists a constant \( M_1 > 0 \) such that

\[
h_V(\delta) \leq M_1 \mu(V)\delta^{-\dim_B E}, \quad \delta \in (0, n_d^{-k}).
\]
Proof. Let \( p \geq 1 \) and \( \delta = n_d^{-k-p} \). By Lemma 4.1, the number of \((k+p)\)-th approximate boxes contained in the union of the corresponding \(k\)-th basic pillars of \( V \) is comparable to \( \mu(V)n_d^{(k+p)\dim_{B}E} \). Since every \((k+p)\)-th approximate box can intersect a bounded number of \( \delta \)-connected component of \( E \), we obtain the lemma. \( \square \)

Similar as Section 3, a connected component \( C \) of \( E_k \) (resp. \( \tilde{E}_k \)) is called a \( k \)-th island of \( E_k \) (resp. \( \tilde{E}_k \)) if \( C \cap \partial[0,1]^d = \emptyset \). It is easy to see that \( E_k \) has islands if and only if \( \tilde{E}_k \) has islands. Recall that \( E \) is said to be degenerated if \( E \) is contained in a face of \([0,1]^d\) with dimension \( d-1 \). Zhang and Xu [24] Theorem 4.1] proved that

**Proposition 4.3.** Let \( E \) be a non-degenerated self-affine Sierpiński sponge. Then \( E \) has trivial points if and only if there is an integer \( q \geq 1 \) such that \( E_q \) has islands.

Let \( Q_k \in \tilde{E}_k \), we call \( Q_k \) a \( k \)-th boundary approximate box of \( E \) if \( Q_k \cap \partial[0,1]^d \neq \emptyset \).

Let \( W \) be a \( k \)-th cylinder of \( E \). Write \( W = f(E) \), then \( f([0,1]^d) \) is the corresponding \( k \)-th basic pillar of \( W \). A \( \delta \)-connected component \( U \) of \( W \) is called an inner \( \delta \)-connected component, if

\[
\text{dist} \left( U, \partial(f([0,1]^d)) \right) > \delta,
\]

otherwise, we call \( U \) a boundary \( \delta \)-connected component. We denote by \( h_W^b(\delta) \) the number of boundary \( \delta \)-connected components of \( W \), and by \( h_W^i(\delta) \) the number of inner \( \delta \)-connected components of \( W \).

**Lemma 4.4.** Let \( E \) be a non-degenerated self-affine Sierpiński sponge.

(i) Let \( W \) be a \( k \)-th cylinder of \( E \). Then

\[
h_W^b(\delta) = o(\mu(W)\delta^{-\dim_{B}E}), \quad \delta \to 0.
\]

(ii) If \( E \) satisfies the maximal power law, then \( E \) possesses trivial points; moreover, there exists an integer \( p_0 \geq 1 \) and a constant \( c' > 0 \) such that for any \( k \geq 1 \), any \( k \)-th cylinder \( W \) of \( E \) and any \( \delta \leq n_d^{-(k+p_0)} \),

\[
h_W^i(\delta) \geq c' h_W(\delta).
\]

**Proof.** (i) Write \( W = f(E) \). Let \( W_p^* \) be the union of \((k+p)\)-th cylinders of \( W \) whose corresponding basic pillars intersecting the boundary of \( f([0,1]^d) \). Since \( E \) is non-degenerated, we have

\[
\mu(W_p^*) = o(\mu(W)), \quad p \to \infty.
\]

Set \( \delta = n_d^{-(k+p)} \). If \( U \) is a boundary \( \delta \)-connected component of \( W \), then \( U \cap W_p^* \neq \emptyset \). Therefore,

\[
h_W^b(\delta) \leq h_{W_p^*}(\delta) \leq M_1\mu(W_p^*)\delta^{-\dim_{B}E},
\]

where the last inequality is due to Corollary 4.2. This together with (4.7) imply (4.6).

(ii) The assumption that \( E \) satisfies the maximal power law implies that there exists \( M_2 > 0 \) such that

\[
h_E(\delta) \geq M_2 \cdot \delta^{-\dim_{B}E}, \quad \text{for any } \delta \in (0,1).
\]
If $E$ does not possess trivial points, then for all $k \geq 1$, $E_k$ does not contain any $k$-th island by Proposition 4.3 furthermore, $E_k$ does not contain any $k$-th island. Thus each $\delta$-connected component of $E$ contains points of $E \cap \partial([0, 1]^d)$, so $h_E(\delta) = h_E^b(\delta)$. On the other hand, by (i) we have $h_E^b(\delta) = o(\delta^{-\dim_B E})$ as $\delta \to 0$, which contradicts (1.8). This proves that $E$ possesses trivial points.

Let $W$ be a $k$-th cylinder of $E$. Since $h_{A \cup B}(\delta) \leq h_A(\delta) + h_B(\delta)$, by (4.8) we have

\begin{equation}
(4.9) \quad h_W(\delta) \geq N^{-k}h_E(\delta) \geq M_2\mu(W)\delta^{-\dim_B E}, \quad \text{for any } \delta \in (0, 1).
\end{equation}

Take $\varepsilon \leq M_2/2$. By (4.6), there exists an integer $p_0 \geq 1$ such that

\begin{equation}
(4.10) \quad h_W^b(\delta) \leq \varepsilon\mu(W)\delta^{-\dim_B E} \quad \text{for } \delta \leq n_d^{-(k+p_0)}.
\end{equation}

This together with (4.9) and (4.10) imply that for $\delta \leq n_d^{-(k+p_0)},$

\begin{equation}
(4.11) \quad h_W^i(\delta) = h_W(\delta) - h_W^b(\delta) \geq \frac{M_2}{2}\mu(W)\delta^{-\dim_B E} \geq c'h_W(\delta),
\end{equation}

where $c' = M_2/(2M_1)$ and $M_1$ is the constant in Corollary 4.2. The lemma is proved. \hfill \Box

**Proof of Theorem 1.2.** We will prove this theorem by induction on $d$. Notice that $E$ is a 1-dimensional fractal cube if $d = 1$, and in this case the theorem holds by Theorem 1.1. Assume that the theorem holds for all $d'$-dimensional self-affine Sierpiński sponge with $d' \leq d - 1$.

Now we consider the $d$-dimensional self-affine Sierpiński sponge $E$. Denote $G := \pi_{d-1}(E)$. First, $G$ is non-degenerated since $E$ is. Secondly, by (1.2) we have

\begin{equation}
(4.12) \quad \log \frac{N}{\# \pi_{d-1}(D)} + \dim_B G = \dim_B E.
\end{equation}

"$\Rightarrow$": Suppose $E$ and all $\pi_j(E)$ ($1 \leq j \leq d - 1$) possess trivial points, then $G = \pi_{d-1}(E)$ satisfies the maximal power law by induction hypothesis. We will show that $E$ satisfies the maximal power law.

Firstly, by Corollary 4.2

\begin{equation}
(4.13) \quad h_E(\delta) \leq M_1\delta^{-\dim_B E} \quad \text{for all } \delta \in (0, 1).
\end{equation}

Now we consider the lower bound of $h_E(\delta)$.

Since $E$ possesses trivial points, by Proposition 4.3 there exists an integer $q_0 \geq 1$ such that $E_{q_0}$ has a $q_0$-th island, which we denote by $I$. Let $p_0$ be the constant in Lemma 4.3 (ii). Let $k \geq q_0$ and $\delta = n_d^{-(k+p_0)}$. Since $G$ satisfies the maximal power law, there exists a constant $c > 0$ such that

\begin{equation}
(4.14) \quad c^{-1}\delta^{-\dim_B G} \leq h_G(\delta) \leq c\delta^{-\dim_B G}.
\end{equation}

It is easy to see that $S_\tau(I)$ is a $k$-th island of $E$ for any $\tau \in \mathcal{D}^{k-q_0}$. So $E_k$ has $N^{k-q_0}$ number of $k$-th islands like $I' := S_\omega(I)$ for some $\omega \in \mathcal{D}^{k-q_0}$. Since the distance of any two $k$-th islands of $E_k$ is at least $n_d^{-k}$, we have

\begin{equation}
(4.15) \quad h_E(\delta) \geq N^{k-q_0} \cdot h_{E \cap I'}(\delta).
\end{equation}
Let $W$ be any $k$-th cylinder of $E$ contained in $I'$. Since $\pi_{d-1}$ is contractive, we obtain
\begin{equation}
 h_{E\cap I'}(\delta) \geq h_{G\cap \pi_{d-1}(I')}((\delta) \geq c' h_{\pi_{d-1}(W)}(\delta),
\end{equation}
where the last inequality is due to Lemma 4.4(ii) with the constant $c'$ depends on $G$. Furthermore, from $h_{A\cup B}(\delta) \leq h_A(\delta) + h_B(\delta)$ we infer that
\begin{equation}
(\# \pi_{d-1}(D))^k \cdot h_{\pi_{d-1}(W)}(\delta) \geq h_G(\delta).
\end{equation}
By (4.12), we have
\begin{align*}
h_E(\delta) & \geq N^{k-\eta_0} \cdot \frac{c'h_G(\delta)}{(\# \pi_{d-1}(D))^k} \geq c'c^{-1}N^{-\eta_0} \cdot \left( \frac{N}{\# \pi_{d-1}(D)} \right)^k \cdot \delta^{-\dim B}
\end{align*}
where the last equality holds by (4.10). This together with (4.11) imply that $G$ satisfies the maximal power law.

$\implies$: Suppose $E$ satisfies the maximal power law. Then $E$ possesses trivial points by Lemma 4.4(ii). So it is sufficient to show that $G = \pi_{d-1}(E)$ satisfies the maximal power law by induction hypothesis.

Suppose on the contrary this is false. Then given $\epsilon > 0$, there exists $\delta$ as small as we want, such that
\begin{equation}
 h_G(\delta) \leq \epsilon \delta^{-\dim B}. \tag{4.16}
\end{equation}

Let $W$ be a $k$-th cylinder of $E$, then $V := \pi_{d-1}(W)$ is a $k$-th cylinder of $G$. Let $\mu'$ be the uniform Bernoulli measure on $G$. By Lemma 4.4 (i), there exists an integer $p_1 \geq 1$ such that
\begin{equation}
 h_{\pi_{d-1}}^b(\eta) \leq \epsilon \mu'(V) \eta^{-\dim B} \text{ for } \eta \leq \pi_{d-1}^{-(k+p_1)}.
\end{equation}
We choose $\delta$ small and thus $k$ large so that $n^{-(k+1)} \leq \delta < n^{-k} < n^{-(k+p_1)}$, then
\begin{equation}
 h_{\pi_{d-1}}^b(\delta) \leq \epsilon \delta^{-\dim B} \left( \frac{1}{(N')^k} \right),
\end{equation}
where $N' = \# \pi_{d-1}(D)$. On the other hand, by (4.16),
\begin{equation}
(N')^k h_{\pi_{d-1}}^b(\delta) \leq h_G(\delta) \leq \epsilon \delta^{-\dim B} \tag{4.16'}
\end{equation}
So we obtain
\begin{equation}
 h_{\pi_{d-1}}^b(\delta) = h_{\pi_{d-1}}^i(\delta) + h_{\pi_{d-1}}^i(\delta) \leq \frac{2\epsilon \delta^{-\dim B}}{(N')^k}. \tag{4.17}
\end{equation}

Now we estimate $h_W(\delta')$, where $\delta' = \sqrt{n^{-k}} > \sqrt{\delta^2 + (n^{-k})^2}$. Since $W$ is contained in $\pi_{d-1}(W) \times [b, b + n^{-k}]$ for some $b \in [0, 1]$, we deduce that if two points $\pi_{d-1}(x)$ and $\pi_{d-1}(y)$ belong to a same $\delta$-connected component of $G$, then $x$ and $y$ belong to a same $\delta'$-connected component of $E$. Therefore,
\begin{equation}
 h_W(\delta') \leq h_{\pi_{d-1}}^i(\delta), \tag{4.18}
\end{equation}
and consequently,

$$h_E(\delta') \leq N^k \cdot h_W(\delta') \leq N^k \cdot \frac{2\varepsilon \delta^{-\dim B} G}{(N')^k} \leq M' \varepsilon (\delta')^{-\dim B E}$$

for $M' = 2(\sqrt{2})^{\dim B E} n_d^{\dim B G}$. This is a contradiction since $E$ satisfies the maximal power law. The theorem is proved.

\[\square\]

5. Proof of Theorem 1.3

Before proving Theorem 1.3 we prove a finite type property of totally disconnected self-affine Sierpiński sponge. The proof is similar to [22] and [17], which dealt with fractal cubes and Bedford-McMullen carpets, respectively.

**Theorem 5.1.** Let $E$ be a totally disconnected self-affine Sierpiński sponge, then there is an integer $M_3 > 0$ such that for every integer $k \geq 1$, each connected component of $E_k$ contains at most $M_3$ $k$-th basic pillars.

Denote $d_H(A, B)$ the Hausdorff metric between two subsets $A$ and $B$ of $\mathbb{R}^d$. The following lemma is obvious, see for instance [22].

**Lemma 5.1.** Suppose $\{X_k\}_{k \geq 1}$ is a collection of connected compact subsets of $[0, 1]^d$. Then there exist a subsequence $\{k_i\}_{i \geq 1}$ and a connected compact set $X$ such that $X_{k_i} \rightarrow E \times i \rightarrow \infty$.

**Proof of Theorem 5.1.** Let $E$ be a totally disconnected self-affine Sierpiński sponge. We set

$$\Xi_k = \bigcup_{h \in \{-1,0,1\}^d} (E_k + h).$$

First, we claim that there exists an integer $k_0$ such that for any connected component $X$ of $\Xi_{k_0}$ with $X \cap [0, 1]^d \neq \emptyset$, $X$ is contained in $(-1, 2)^d$.

Suppose on the contrary that for any $k$ there are connected components $X_k \subset \Xi_k$ and points $x_k \in [0, 1]^d \cap X_k$ and $y_k \in \partial [-1, 2]^d \cap X_k$. By Lemma 5.1, we can take a subsequence $\{k_i\}_i$ such that $x_{k_i} \rightarrow x^* \in [0, 1]^d$; $y_{k_i} \rightarrow y^* \in \partial [-1, 2]^d$ and $X_{k_i} \rightarrow E \times i \rightarrow \infty$ for some connected compact set $X$ with

$$X \subset \bigcup_{h \in \{-1,0,1\}^d} (E + h), \ x^* \in X \cap [0, 1]^d, \text{ and } y^* \in X \cap \partial [-1, 2]^d.$$

Clearly $\bigcup_{h \in \{-1,0,1\}^d} (E + h)$ is totally disconnected since it is a finite union of totally disconnected compact sets. (A carefully proof of this fact can be found in [22]). This contradiction proves our claim.

For any $k \geq 1$, let $U$ be a connected component of $E_{k+k_0}$ such that $U \cap S_\omega([0, 1]^d) \neq \emptyset$ for some $\omega \in D^k$. Notice that $S_\omega([1, -2]^d) \cap E_{k+k_0} \subset S_\omega(\Xi_{k_0})$, then $[-1, 2]^d \cap S_\omega^{-1}(U) \subset \Xi_{k_0}$. By the claim above, every connected component of $[-1, 2]^d \cap S_\omega^{-1}(U)$ which intersects $[0, 1]^d$ is contained in $(-1, 2)^d$. This implies that

$$S_\omega^{-1}(U) \subset (-1, 2)^d$$
since $U$ is connected. It follows that $U \subset S_\omega((-1,2)^d)$, so $U$ contains at most $3^dN^{k_0}$ $(k + k_0)$-th basic pillars of $E$. Thus $E$ is of finite type. □

Recall that $\pi_j(x_1, \ldots, x_d) = (x_1, \ldots, x_j)$ for $1 \leq j \leq d-1$. Denote $\hat{\pi}_d(x_1, \ldots, x_d) = x_d$. Let $z_1, \ldots, z_p \in E$. For $0 < \delta < 1$, we call $z_1, \ldots, z_p$ a $\delta$-chain of $E$ if $|z_{i+1} - z_i| \leq \delta$ for $1 \leq i \leq p-1$, and define the size of the chain as

$$L = \frac{\text{diam}\{z_1, \ldots, z_p\}}{\delta}.$$  

**Proof of Theorem 1.3.** We will prove the theorem by induction on $d$. If $d = 1$, $E$ is a 1-dimensional fractal cube, and the theorem holds by Remark 1.1.

**Totally disconnected $\Rightarrow$ perfectly disconnected:** Suppose that $E$ and $\pi_j(E)$, $j = 1, \ldots, d-1$, are totally disconnected. By induction hypothesis, $\pi_{d-1}(E)$ is perfectly disconnected. We will show that $E$ is perfectly disconnected.

Take $\delta \in (0, 1)$. Let $U$ be a $\delta$-connected component of $E$. Let $k$ be the integer such that $n_d^{-(k+1)} \leq \delta < n_d^{-k}$. Then $U$ is contained in a connected component $V$ of $E_k$. Let $M_3$ be a constant such that Theorem 5.1 holds for $E$ and $\pi_{d-1}(E)$ simultaneously, then $V$ contains at most $M_3$ $k$-th basic pillars. Denote $L = \text{diam}(U)/\delta$.

Let $z_1, \ldots, z_p$ be a $\delta$-chain in $U$ such that $|z_1 - z_p| \geq L\delta/2$. Then $\pi_{d-1}(z_1), \ldots, \pi_{d-1}(z_p)$ is also a $\delta$-chain in $\pi_{d-1}(E)$. Since $|\hat{\pi}_d(x) - \hat{\pi}_d(y)| \leq M_3n_d^{-k}$ for any $x, y \in V$, we have

$$|\pi_{d-1}(z_1) - \pi_{d-1}(z_p)| \geq \sqrt{L^2\delta^2/4 - M_3^2n_d^{-2k}} \geq n_d^{-k}\sqrt{\frac{L^2}{4n_d^2} - M_3^2},$$

so

$$L' = \frac{|\pi_{d-1}(z_1) - \pi_{d-1}(z_p)|}{\delta} \geq \sqrt{\frac{L^2}{4n_d^2} - M_3^2}.$$

If $E$ is not perfectly disconnected, then we can choose $U$ such that $L$ is arbitrarily large, so $\text{diam}(\pi_{d-1}(U))/\delta \geq L'$ can be arbitrarily large, which contradicts the fact that $\pi_{d-1}(E)$ is perfectly disconnected.

**Perfectly disconnected $\Rightarrow$ totally disconnected:** Assume that $E$ is perfectly disconnected. Clearly $E$ is totally disconnected. Hence, by induction hypothesis, we only need to show that $G := \pi_{d-1}(E)$ is perfectly disconnected.

Firstly, we claim that $G$ must be totally disconnected.

Suppose on the contrary that the claim is false. Let $\Gamma$ be a connected subset of $(0,1)^{d-1} \cap G$. Fix two points $a, b \in \Gamma$. Take any $\omega \in (\pi_{d-1}(\mathcal{D}))^k$, then $S'_{\omega}(\Gamma)$ is contained in the interior of the $k$-th basic pillar $V = S'_{\omega}([0,1]^{d-1})$, where $\{S'_{\omega}\}_{i \in \pi_{d-1}(\mathcal{D})}$ is the IFS of $G$. Denote $a' = S'_{\omega}(a)$, $b' = S'_{\omega}(b)$. Then $|b' - a'| \geq |b - a|/n_{d-1}^k$.

Let $\delta = n_d^{-k}$. Let $z_1 = a', \ldots, z_p = b'$ be a $\delta$-chain in $S'_{\omega}(\Gamma) \subset V$. Let $W$ be a $k$-th cylinder of $E$ such that $\pi_{d-1}(W) = V \cap G$. Let $y_1, \ldots, y_p$ be a sequence in $W$ such that $\pi_{d-1}(y_j) = z_j$, $1 \leq j \leq p$, then it is a $(\sqrt{2}\delta)$-chain. Since

$$L = \frac{|y_1 - y_p|}{\sqrt{2}\delta} \geq \frac{|z_1 - z_p|}{\sqrt{2}\delta} \geq \frac{|b - a|}{\sqrt{2}} \cdot \left(\frac{n_d}{n_{d-1}}\right)^k$$

Proof of Theorem 1.3.
can be arbitrarily large when \( k \) tends to \( \infty \), we conclude that \( E \) is not perfectly disconnected, which is a contradiction. The claim is proved.

Secondly, suppose on the contrary that \( G \) is not perfectly disconnected. Take \( \delta \in (0, 1) \). Let \( U \) be a \( \delta \)-connected component of \( G \). Denote \( L = \text{diam}(U)/\delta \). Let \( k \) be the integer such that \( n_d^{-k-1} \leq \delta < n_d^{-k} \). Then \( U \) is contained in a connected component \( V \) of \( G_k \). Since \( G \) is totally disconnected, by Theorem 5.1 \( V \) contains at most \( M_3 \) \( k \)-th basic pillars of \( G \), and we denote the corresponding \( k \)-th cylinders by \( V_1, \ldots, V_h \) where \( h \leq M_3 \).

Let \( z_1, \ldots, z_p \) be a \( \delta \)-chain of \( U \) such that \( |z_1 - z_p| = L\delta \). Without loss of generality, we may assume that the diameter of \( \{z_1, \ldots, z_p\} \cap V_j \) attains the maximality when \( j = 1 \). Let \( \{x_j\}_{j=1}^\ell \) be the subsequence of \( \{z_j\}_{j=1}^p \) located in \( V_1 \), then \( |x_1 - x_\ell| \geq L\delta/h \).

Let \( \Delta = \max_{1 \leq j \leq \ell} |x_j - x_{j+1}| \), then \( x_1, \ldots, x_\ell \) is a \( \Delta \)-chain in \( V_1 \).

Let \( W \) be a \( k \)-th cylinder of \( E \) such that \( \pi_{d-1}(W) = V_1 \). Then the pre-image of \( x_j \) in \( W \), which we denote by \( y_1, \ldots, y_\ell \), is a \( \sqrt{\Delta^2 + (n_d\delta)^2} \)-chain. The size of this chain is

\[
\tilde{L} \geq \frac{|y_1 - y_\ell|}{\sqrt{\Delta^2 + (n_d\delta)^2}} \geq \frac{|x_1 - x_\ell|}{\sqrt{\Delta^2 + (n_d\delta)^2}} \geq \frac{L\delta}{h\sqrt{\Delta^2 + (n_d\delta)^2}}.
\]

**Case 1.** \( \Delta \leq \sqrt{L}\delta \).

In this case we have \( \tilde{L} \geq \frac{L}{M_3\sqrt{L+n_d^2}} \). Since \( G \) is not perfectly disconnected, \( \tilde{L} \) can be arbitrarily large when \( L \to \infty \), we deduce that \( E \) is not perfectly disconnected, which is a contradiction.

**Case 2.** \( \Delta > \sqrt{L}\delta \).

Let \( 1 \leq j^* + 1 \leq \ell - 1 \) be the index such that \( |x_{j^*+1} - x_{j^*}| = \Delta \). Denote the sub-chain of \( z_1, \ldots, z_p \) between \( x_{j^*} \) and \( x_{j^*+1} \) by

\[
z'_1 = z_{m+1}, \ldots, z'_s = z_{m+s}.
\]

Then \( (z'_j)_{j=1}^s \) belong to \( V_2 \cup \cdots \cup V_h \) and \( |z'_1 - z'_s| \geq (\sqrt{L} - 2)\delta \).

Now we repeat the above argument by considering the \( \delta \)-chain \( (z'_j)_{j=1}^s \) in \( V_2 \cup \cdots \cup V_h \). In at most \( M_3 - 1 \) steps, we will obtain a \( \Delta' \)-chain in \( E \) with arbitrarily large size when \( L \to \infty \). This finishes the proof of Case 2 and the theorem is proved. \( \square \)

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