Wrong-Helicity Electrons in Radiative Muon Decay

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Abstract

We have studied in detail the spectrum of the decay $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu \gamma$ as a function of the electron helicity, and verify the prediction [1] that a significant fraction of the electrons in this decay is right-handed. These “wrong-helicity” electrons persist in the limit $\lambda = m_e/m_\mu \rightarrow 0$, and are connected with helicity-flip bremsstrahlung in QED. The longitudinal polarization of the electron is calculated as a function of the photon and electron energy, and deviates systematically from the naive V-A prediction $P_L = -1$. The right-handed component is concentrated in the collinear region $\theta \lesssim m_e/E_e$. In the limit $\lambda \rightarrow 0$, we reproduce the results obtained in [1] using the helicity-flip splitting function introduced by Falk and Sehgal.

1 Introduction

In a recent Letter [1] it was pointed out that as a consequence of helicity-flip bremsstrahlung, electrons in the radiative decay $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu \gamma$ are not purely left-handed, even in the limit $m_e \rightarrow 0$. The decay width into right-handed electrons was shown to be $\Gamma_R = \frac{\alpha^4}{4\pi} \Gamma_0$, where $\Gamma_0 \equiv G_F^2 m_\mu^3/(192\pi^3)$. The energy spectrum of these wrong-helicity electrons was calculated, as also that of the photons accompanying them. These results were obtained using the helicity-flip splitting function $D_{hf}(z) = \frac{\alpha^2}{2\pi} z$ introduced by Falk and Sehgal [2].

The appearance of right-handed electrons in $\mu$-decay is, at first sight, surprising since it goes against the conventional wisdom based on the V-A structure of weak interaction and the common assumption of helicity-conservation in massless QED. However, persistence of helicity-flip bremsstrahlung in the $m_e \rightarrow 0$ limit has been established in several calculations of radiative processes [3, 4, 5, 6, 7], going back to the seminal paper of Lee and Nauenberg [8]. In all cases, the origin of the effect is the characteristic behaviour of helicity-flip bremsstrahlung at small angles

$$d\sigma_{hf} \approx (\frac{m_e}{E_e})^2 \frac{d\theta^2}{\theta^2 + (\frac{m_e}{E_e})^2}$$

leading to a non-zero integrated rate that survives the limit $m_e \rightarrow 0$.

In the present paper, we derive the helicity-dependent spectrum of the decay $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu \gamma$ from first principles, without neglecting the electron mass. We obtain the longitudinal polarization $P_L$ of the electron as a function of the electron energy, photon energy and the angle between the electron and photon. We then consider the limit $m_e \rightarrow 0$ to demonstrate that the deviation from $P_L = -1$ is present even in the chiral limit of QED. The results are compared with those obtained in the equivalent particle approach based on the helicity-flip fragmentation function, thus providing a confirmation of the statements in Ref. [1].

2 Matrix Element and Phase Space for $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu \gamma$

The matrix element for the decay $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu \gamma$ may be obtained from the Feynman diagrams shown in Fig. [11]. Adopting the notation of Ref. [9, 10], we obtain the square of the matrix element, summed over the photon polarization, taking the muon to be unpolarized. The spectrum is derived in terms of the invariants

$$x := \frac{2p_\mu \cdot p_e}{m_\mu^2}, \quad y := \frac{2p_\mu \cdot k}{m_\mu^2}, \quad z := \frac{2p_e \cdot k}{m_e^2}$$

(2)
Figure 1: Feynman diagrams

and the mass ratio $\lambda := m_e/m_\mu$. In the rest frame of the muon,

$$x = \frac{2E_e}{m_\mu}, \quad y = \frac{2E_\gamma}{m_\mu}, \quad z = \frac{y}{2} \left[ x - \cos(\theta) \sqrt{x^2 - 4\lambda^2} \right],$$

(Eq. 3)

$E_e, E_\gamma$ being the energies of the electron, photon and $\cos(\theta)$ the angle between them. The boundary of phase space is determined by the condition $-1 \leq \cos(\theta) \leq 1$, and $0 \leq Q^2 \leq (1 - \lambda)^2 m_\mu^2$, $Q^2$ being the “missing mass” of the neutrino pair. This leads to the condition

$$0 \leq y \leq y_m(x), \quad 2 \lambda \leq x \leq 1 + \lambda^2,$$

(Eq. 4)

where

$$y_m(x) = 2 \frac{1 + \lambda^2 - x}{2 - x + \sqrt{x^2 - 4\lambda^2}}.$$  

(Eq. 5)

Equivalently, the limits of phase space may be expressed as

$$2 \lambda \leq x \leq x_m(y), \quad 0 \leq y \leq 1 - \lambda,$$

(Eq. 6)

where

$$x_m(y) = \frac{\lambda^2 + (1 - y)^2}{1 - y}.$$  

(Eq. 7)

The differential decay width for $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu \gamma$ in the variables $x, y$ and $\cos(\theta)$ is then given by

$$\frac{d\Gamma}{dx \, dy \, d\cos(\theta)} |_{R,L} = (-1) \Gamma_0 \frac{\alpha}{2\pi} \sqrt{x^2 - 4 \lambda^2} \frac{1}{y^2} \frac{1}{2} (g_0 \pm g_1)$$  

(Eq. 8)

where $R$ and $L$ denote right-handed and left-handed electrons, corresponding to the polarization vectors

$$(s_e)_R = \left( \frac{p_e}{m_e}, \frac{E_e}{m_e} \hat{p}_e \right) \quad (s_e)_L = \left( \frac{p_e}{m_e}, -\frac{E_e}{m_e} \hat{p}_e \right).$$  

(Eq. 9)

The functions $g_0$ and $g_1$ are given by

$$g_0 := -2 \; x \; y \; z^2 + 6 \; x \; y \; z^3 - 6 \; x \; y^2 \; z - 8 \; x \; y^2 \; z^2 + 6 \; x \; y^3 \; z + 6 \; x \; z^2 + 8 \; x \; z^3 - 6 \; x^2 \; y \; z - 8 \; x^2 \; y \; z^2$$
$$+ 8 \; x^2 \; y^2 \; z - 4 \; x^2 \; z^2 + 4 \; x^3 \; y \; z + 6 \; y \; z^2 + 5 \; y \; z^3 - 2 \; y \; z^4 - 2 \; y^2 \; z^2 + 2 \; y^2 \; z^3 - 3 \; y^3 \; z - 2 \; y^3 \; z^2$$
$$+ 2 \; y^4 \; z - 6 \; z^3 - 4 \; z^4$$
$$+ \lambda^2 \; (8 \; x \; y \; z + 6 \; x \; y \; z^2 + 2 \; x \; y^2 \; z + 6 \; x \; y^2 - 8 \; x \; y^3 + 6 \; x \; z^2 - 6 \; x^2 \; y \; z - 4 \; x^2 \; y^2 + 6 \; y \; z^2 - 3 \; y \; z^3)$$
$$- 6 \; y^2 \; z - 2 \; y^2 \; z^2 + 5 \; y^3 \; z + 6 \; y^3 \; - 4 \; y^4 - 8 \; z^2 - 6 \; z^3)$$
$$+ \lambda^4 \; (6 \; x \; y^2 - 6 \; y^2 \; z - 8 \; y^2 + 6 \; y^3),$$

(Eq. 10)
\[ g_1 := \left\{6x^3yz - 4x^4yz + 6x^2y^2z - 8x^3y^2z + 3xy^3z - 6x^2y^3z - 2xy^4z - 6x^2z^2 + 4x^3z^2 - 6xyz^2 + 2x^2yz^2 + 8xyz^2 + 2x^2y^2z^2 + 8x^2y^2z^2 + 2x^2y^3z^2 + 6x^3z - 8x^2z^3 - 5x^2yz^3 - 2x^2y^3z^3 + 2x^2y^4z^3 + 2x^2y^3z^2 + 6x^3z - 8x^2z^3 + \lambda^2(-6x^2y^2 + 4x^3y^2 - 6xy^3 + 8x^2y^3 - 6y^4 + 8xy^4 - 24xy^2 + 16x^2yz - 4y^2z^3 + 4y^5 + 3x^3y z - 6x^2y^2 z + 3y^5 z - 2y^4 z + 24z^2 - 16x z^2 - 2x^2 z^2 - 16yz^2 - 18xy z^2 - 2x^2 y z^2 - 6y^2 z^2 + 10xyz^2 + 4y^3 z^2 + 16z^3 + 2xz^3 + xy z^3) + \lambda^4(24y^2 - 16x y^2 - 2x y^3 - 2y^4 - 8xyz + 16y^2 z + 2x y^2 z - 4y^3 z + 8z^2 - 2y^2 z^2) + 8\lambda^6y^2\right\} \cdot \frac{1}{2\sqrt{x^2 - 4\lambda^2}}. \] 

(11)

Adding the expressions for \(d\Gamma_R\) and \(d\Gamma_L\), we obtain the spectrum for unpolarized electrons, which depends on the function \(g_0\) only [10]. The specific effects of electron helicity are contained in the function \(g_1\).

3 Unpolarized Spectra (summed over electron helicity)

Summing over the electron helicity, the differential decay rate of \(\mu^- \rightarrow e^-\nu_e\nu_\mu\gamma\) is

\[ \left(\frac{d\Gamma}{dx dy d\cos(\theta)}\right)_{R+L} = (-1)\Gamma_0 \frac{\alpha}{2\pi} \frac{1}{\sqrt{x^2 - 4\lambda^2}} \frac{1}{y z^2} g_0(x, y, z; \lambda). \] 

(12)

From this we can derive the following spectra.

3.1 Dalitz plot in electron and photon energies

\[ \left(\frac{d\Gamma}{dx dy}\right)_{R+L} = (-1)\Gamma_0 \frac{\alpha}{2\pi} \int_{-1}^{+1} d\cos(\theta) \sqrt{x^2 - 4\lambda^2} \frac{1}{y z^2} g_0(x, y, z; \lambda) \]

\[ = \Gamma_0 \frac{\alpha}{6\pi} \frac{1}{y} \left\{ A \left\{ 12y(-6 + 3y + y^2) + x(-72 + 78y + 33y^2 - 6y^3) + 2x^2(24 + 12y - 5y^2 + 2y^3) + \lambda^2[96 - 72y + 4y^2 - 4y^3 + 9x(-8 - 2y + y^2)] \right\} \right. \]

\[ \left. + 12L \left\{ -4x^3 + x^2(6 - 8y) + 6\lambda^4y - 6x(-1 + y)y + (3 - 2y)y^2 + \lambda^2[6x^2 + (6 - 5y)y - 2x(4 + y)] \right\} \right\}, \]

(13)

where we have defined

\[ A := \sqrt{x^2 - 4\lambda^2}, \quad L := \frac{1}{2} \log \left(\frac{x + A}{x - A}\right). \]

(14)

3.2 Photon Energy Spectrum

The photon energy spectrum \(d\Gamma/dy)_{R+L}\) is given explicitly in Eq. (A.1) of the Appendix The leading term for small \(y\) (small photon energies) is

\[ \left(\frac{d\Gamma}{dy}\right)_{R+L} \rightarrow \Gamma_0 \frac{\alpha}{6\pi} \frac{1}{y} \left\{ (-17 + 64\lambda^2 - 64\lambda^6 + 17\lambda^8) + (-12 + 144\lambda^4 - 12\lambda^8) \log(\lambda) \right\}. \]

(15)
In the limit of small $\lambda$, this reduces to the approximate form
\[
\left( \frac{d\Gamma}{dy} \right)_{R+L} \xrightarrow{\lambda \to 0} \Gamma_0 \frac{\alpha}{6 \pi} \frac{1}{y} \left[ -17 - 12 \log(\lambda) \right],
\] (16)
which agrees with the old calculation of Eckstein and Pratt [11] and Kinoshita and Sirlin [12], in which terms of order $\lambda^2$ and higher powers were neglected. The integrated rate for $y > y_0$ is
\[
\Gamma_{R+L}(y_0; \lambda) \xrightarrow{\lambda \to 0, y_0 \to 0} \Gamma_0 \frac{\alpha}{\pi} \left[ \log(\lambda) \left( 2 \log(y_0) + \frac{7}{3} \right) + \frac{17}{6} \log(y_0) - \frac{\pi^2}{3} + \frac{13409}{2520} \right],
\] (17)
which agrees with Ref. [11], except in the constant (non-logarithmic) term, which was $-\pi^2/6 + 601/144$ in the old calculation.

### 3.3 Electron energy spectrum

The exact result for the electron energy spectrum $\left( \frac{d\Gamma}{dx} \right)_{R+L}$ is given in Eq. (A.3) of the Appendix.

The normalized electron energy spectrum is shown in Fig. (2a). In the limit $\lambda \to 0$ we have for the

\[
\lim_{\lambda \to 0} (\Gamma_{R+L})^{-1} \left( \frac{d\Gamma}{dx} \right)_{R+L} = \left( \frac{1-y_0-x}{y_0} \right) \left( -5 - 5y_0 + 4y_0^2 + x(-17 + 14y_0) + 34x^2 \right)
+ 12x^2(-3 + 2x) \log \left( \frac{1-x}{y_0} \right)
\times \left[ (1-y_0)(7-3y_0+3y_0^2-y_0^3) + 6 \log(y_0) \right]^{-1}.
\] (18)

This is shown in Fig. (2b).

In the limit $y_0 \to 0$ we reproduce the spectrum for non-radiative muon decay
\[
\lim_{\lambda \to 0, y_0 \to 0} (\Gamma_{R+L})^{-1} \left( \frac{d\Gamma}{dx} \right)_{R+L} = (\Gamma_{R+L})^{-1} \left( \frac{d\Gamma}{dx} \right)_{\text{non-rad}} = 2x^2(3 - 2x).
\] (19)

### 4 Spectra for Right- and Left-handed Electrons

The spectra when the final electron is polarized in the state R or L are obtained using Eq. (8). We will emphasize the case R, which is the one which behaves in an unexpected way in the $\lambda \to 0$ limit [1, 9]. The corresponding spectra for the state L are obtained by subtraction from the unpolarized result L+R discussed in Sec.3.
4.1 Dalitz plot for R-Electrons

\[
\left( \frac{d\Gamma}{dx \, dy} \right)_R = \Gamma_0 \frac{\alpha}{12 \pi} \frac{1}{A} \frac{1}{y} (36 \, A^2 \, y^2 + A(36 \, y^2 - 60 \, x \, y^2 - 24 \, y^3) + 12 \, A \lambda^4 (-8 + y^2))
\]

\[+ \lambda^2 \left[ -288 \, A + 96 \, A^2 + 192 \, A \, x - 72 \, A^2 \, x + 24 \, A \, x^2 + 192 \, A \, y \\
-72 \, A^2 \, y + 72 \, A \, x \, y - 18 \, A^2 \, x \, y + 6 \, A \, x^2 \, y + 48 \, A \, y^2 + 4 \, A^2 \, y^2 \\
-52 \, A \, x \, y^2 + 9 \, A^2 \, x \, y^2 - 3 \, A \, x^2 \, y^2 - 24 \, A \, y^3 - 4 \, A^2 \, y^3 + 16 \, A \, x \, y^3 \right] \\
+ A (A - x) \left[ -72 \, x + 48 \, x^2 - 72 \, y + 78 \, x \, y + 24 \, x^2 \, y + 33 \, y^2 \\
-10 \, x^2 \, y^2 + 12 \, y^3 - 6 \, x \, y^3 + 4 \, x^2 \, y^3 \right]
\]

\[+ \lambda^4 \left( 96 \, x - 192 \, y + 72 \, A \, y - 24 \, x \, y + 48 \, y^2 \right) \\
+ \lambda^2 \left( 288 \, x - 96 \, A \, x - 192 \, x^2 + 72 \, A \, x^2 - 24 \, x^3 + 72 \, A \, y - 264 \, x \, y - 24 \, A \, x \, y + 72 \, x^2 \, y - 48 \, y^2 - 60 \, A \, y^2 + 84 \, x \, y^2 + 48 \, y^3 \right) \}
\]

In the limit \( \lambda \rightarrow 0 \) we obtain

\[
\lim_{\lambda \rightarrow 0} \left( \frac{d\Gamma}{dx \, dy} \right)_R = \Gamma_0 \frac{\alpha}{\pi} \left( 3 - 2 \, x - 2 \, y \right).
\]

This limiting result coincides with what one obtains on the basis of the helicity-flip splitting function. Denoting the electron spectrum in the non-radiative decay \( \left( \frac{d\Gamma}{dx_0} \right)_{\text{non-rad}} \), the spectrum after helicity-flip bremsstrahlung is

\[
\left( \frac{d\Gamma}{dx_0} \right)_R = \left( \frac{d\Gamma}{dx_0} \right)_{\text{non-rad}} D_{\text{hf}}(z).
\]

Noting that \( y = x_0 \, z, \, x = x_0 (1 - z) \), and recalling that

\[
\left( \frac{d\Gamma}{dx_0} \right)_{\text{non-rad}} = \Gamma_0 2 \, x_0^2 (3 - 2 \, x_0), \quad D_{\text{hf}}(z) = \frac{\alpha}{2 \, \pi} \, z,
\]

we obtain

\[
\left( \frac{d\Gamma}{dx \, dy} \right)_R = \frac{1}{x_0} \left( \frac{d\Gamma}{dx_0} \right)_R = \Gamma_0 \frac{\alpha}{\pi} \left( 3 - 2 \, x - 2 \, y \right),
\]

exactly as in Eq. (21).

4.2 Photon-Spectra for R- and L-electrons

Of particular interest is the photon energy spectrum associated with the right-handed polarization state of the electron, which is naively forbidden in the \( \lambda \rightarrow 0 \) limit. The corresponding spectrum for L-electrons may be obtained by subtraction from Eq. (A.7). The exact result for \( \left( \frac{d\Gamma}{dy} \right)_R \) is stated in Eq. (A.8). In the limit \( m_e/m_\mu \rightarrow 0 \) one obtains

\[
\lim_{\lambda \rightarrow 0} \left( \frac{d\Gamma}{dy} \right)_R = \Gamma_0 \frac{\alpha}{\pi} \left( y \, (1 - y) \, (2 - y) \right),
\]

which is the result derived in [1] on the basis of the helicity-flip fragmentation function \( D_{\text{hf}}(z) \). The spectrum for \( \lambda \neq 0 \) is plotted in Fig. (3). The striking feature is the appearance of a divergence at \( y = 0 \). This is connected with the fact that for \( m_e \neq 0 \), helicity is not a good quantum number. As a consequence, ordinary (non-flip) bremsstrahlung, diverging as \( 1/y \) at small \( y \), is partly transmitted to the final state with electron polarization \( s_e = s_\mu^R \). This contribution vanishes in the limit \( \lambda \rightarrow 0 \), leaving behind the anomalous (helicity-flip) contribution in Eq. (25). This anomalous bremsstrahlung integrates to a decay width

\[
\Gamma_R = \Gamma_0 \frac{\alpha}{4 \, \pi} \left( 1 - y_0 \right) \left( 1 + y_0 - 3 \, y_0^2 + y_0^3 \right),
\]

which is the contribution of right-handed electrons to the muon decay width in the \( \lambda \rightarrow 0 \) limit [1]. The exact expression for \( \Gamma_R(y_0) \) and the corresponding left-handed rate \( \Gamma_L(y_0) \) are plotted in Fig. (4). From these, we obtain the longitudinal polarization

5
Figure 3: Photon energy spectrum $\left(\frac{d\Gamma}{dy}\right)_R$ for right-handed electrons in units of $\Gamma_0 \frac{\alpha}{4\pi}$ for a) $\lambda = 1/207$, b) $\lambda \to 0$.

Figure 4: Integrated rates for R- and L-electrons $\Gamma_{R,L}(y_0)$ in units of $\Gamma_0 \frac{\alpha}{4\pi}$ as function of minimum photon energy $y_0$.

\[ P_L(y_0) := \frac{\Gamma_R(y_0) - \Gamma_L(y_0)}{\Gamma_R(y_0) + \Gamma_L(y_0)} \]  

(27)

This figure demonstrates the presence of a significant right-handed electron component in radiative muon decay, which becomes as probable as the left-handed component at high photon energies.

To illustrate the approach to the chiral limit $\lambda \to 0$, we have also shown in Fig.5 the polarization of the electron in the decay $\tau^- \to e^- \bar{\nu}_e \nu_\tau \gamma$. In the chiral limit $\lambda \to 0$ the longitudinal polarization goes from $P_L = -1$ to $P_L = 0$ like a step-function at $y_0 = 1$.

### 4.3 Electron Energy Spectra for R- and L-electrons

The formula for the electron energy spectrum $(d\Gamma/dx)_R$ for right-handed electrons is given in Eq. (A.7), and may be compared with the unpolarized spectrum $(d\Gamma/dx)_{R+L}$ in Eq. (A.3). We have verified that in the limit $\lambda \to 0$, the right-handed spectrum takes the form

\[ \left(\frac{d\Gamma}{dx}\right)_R \xrightarrow{\lambda \to 0} \Gamma_0 \frac{\alpha}{6\pi} (1 - y_0 - x) [5 + 5y_0 - 4y_0^2 + x(-7 - 2y_0) + 2x^2]. \]  

(28)

This is exactly the result derived in Ref.[1] on the basis of the fragmentation function $D_{hf}(z)$. 
5 Discussion of the Collinear Region

A question of interest is the behaviour of helicity-flip and helicity-non-flip bremsstrahlung in the collinear region, \( \theta \approx 0 \), where \( \theta \) is the angle between the photon and electron in the final state. To this end, we have calculated exactly the distribution \((d\Gamma/d\cos(\theta))_{R,L}\). The analytic results are, however, too lengthy to be written down here. The left- and right-handed spectra for \( \lambda = 1/207 \), in the small angle region, are plotted in Fig. 6. The longitudinal polarization of the electron as a function of \( \theta \) is shown in Fig. 7, and, changes from the V-A value \( P_L \approx -1 \) to the value \( P_L \approx +1 \) in the forward direction. From the exact expressions we have derived the behaviour in the chiral limit \( \lambda \to 0 \). For \( \theta \neq 0 \) and \( \lambda \to 0 \), we find

\[
\left( \frac{d\Gamma}{d\cos(\theta)} \right)_R \xrightarrow{\lambda \to 0} \Gamma_0 \frac{\alpha}{6\pi} \frac{\lambda}{\theta^3} \left( 1 - y_0 \right) (5 + 5 y_0 - 4 y_0^2),
\]

\[
\left( \frac{d\Gamma}{d\cos(\theta)} \right)_L \xrightarrow{\lambda \to 0} \Gamma_0 \frac{\alpha}{3\pi} \frac{1}{\theta^2} \left[ -7 + 10 y_0 - 6 y_0^2 + 4 y_0^3 - y_0^4 - 6 \log(y_0) \right].
\]
Figure 7: Longitudinal polarization $P_L$ for $\lambda = 1/207$ and $y_0 = 0.189$.

Note that, in the chiral limit, the $\lambda/\theta^3$ behaviour of $(d\Gamma/d\cos(\theta))_R$ refers to the opening angle of the electron and photon in the final state. The corresponding $\lambda^2/\theta^4$ behaviour of the helicity-flip scattering cross section (Eq. (1)) refers to the angle between the photon and the incident electron.

We have also derived the value of $(d\Gamma/d\cos(\theta))$ in the forward direction ($\theta = 0$), for small $\lambda$:

\[
\left. \left( \frac{d\Gamma}{d\cos(\theta)} \right) \right|_{\theta=0}^R = \Gamma_0 \frac{\alpha}{\pi} \frac{1}{360} \frac{1}{\lambda^2} (1 - y_0)^4 (4 + 16 y_0 - 5 y_0^2)
\]

(30)

We have found that the angular distribution $(d\Gamma/d\cos(\theta))_R$ connected with helicity-flip bremsstrahlung can be simulated by the following ansatz for the differential decay spectrum (where the label “APP” connotes “approximate”):

\[
\left. \left( \frac{d\Gamma}{dx\,dy\,d\cos(\theta)} \right) \right|_{\theta=0}^{APP} = \Gamma_0 \frac{2\alpha}{\pi} \lambda^2 y (3 - 2x - 2y) \frac{1}{(x - \cos(\theta)\,A)^2}.
\]

(31)

When integrated over $\cos(\theta)$, this spectrum, in the limit $\lambda \to 0$, yields the Dalitz pair distribution given in Eq. (21). After integration over $x$ and $y$ we obtain an angular distribution $(d\Gamma/d\cos(\theta))_{R\,APP}^{\lambda\to0}$ which agrees with the exact calculation over a wide range of angles. This formula is particularly simple for $y_0 = 0$:

\[
\left. \left( \frac{d\Gamma(y_0 = 0)}{d\cos(\theta)} \right) \right|_{\theta=0}^{\lambda\to0} = \Gamma_0 \frac{\alpha}{\pi} \frac{5}{6} \lambda \left[ 1 + (\pi - \theta) \frac{\cos(\theta)}{\sin(\theta)} \right] \frac{1}{\sin^2(\theta)},
\]

(32)

which agrees with the small angle result given in Eq. (29). Similarly, in the forward direction ($\theta = 0$), for small values of $\lambda$, the approximate representation Eq. (31) reproduces the result given in Eq. (30).

6 Conclusions

The purpose of this paper was to examine the dependence of the radiative decay $\mu^- \to e^-\bar{\nu}_e\nu_\mu\gamma$ on the helicity of the final electron, in order to demonstrate the presence of right-handed electrons, even in the chiral limit $m_e \to 0$. This non-intuitive feature is a consequence of helicity-flip bremsstrahlung in QED, and was discussed in [1] with the help of the helicity-flip splitting function derived in [2]. We have now calculated the complete helicity-dependent spectrum of this decay from first principles, without neglecting the electron mass, and have shown that the results of [1] are reproduced in the limit $\lambda \to 0$. The presence
of right-handed electrons manifests itself in the longitudinal polarization $P_L = (\Gamma_R - \Gamma_L) / (\Gamma_R + \Gamma_L)$ which deviates from the naive V-A value $P_L = -1$ even in the limit $\lambda \to 0$. The dependence of the polarization on the photon energy cut $y_0$, Fig. 3, shows that right-handed electrons occur predominantly with high energy photons. A characteristic signature of wrong-helicity electrons is also present in the opening-angle distribution $d\Gamma/d\cos(\theta)$, where the collinear region $\theta \lesssim \lambda/(1 - y_0)$ is dominated by right-handed electrons. The fact that helicity-flip bremsstrahlung in the chiral limit $m_e \to 0$ can be exactly reproduced by the simple and universal splitting function $D_{\text{uf}}(z)$ in many different processes, is suggestive of a relationship to the chiral anomaly in QED [13, 14, 15].

Appendix A

$$
\left( \frac{d\Gamma}{dy} \right)_{R+L} = \int_{2\lambda}^{x_{\text{in}}(y)} dx \left( \frac{d\Gamma}{dx dy} \right)_{R+L} = \Gamma_0 \frac{\alpha}{36 \pi} \frac{1}{y} \left\{ (-102 + 46 y - 95 y^2 + 109 y^3 - 45 y^4 + 21 y^5 - 6 y^6) (1 - y) 
+ 6 \lambda^2 (64 - 250 y + 386 y^2 - 240 y^3 + 67 y^4 - 20 y^5 + 5 y^6) (1 - y)^{-1}
+ 18 \lambda^4 y (18 + 12 y - 40 y^2 + 11 y^3 + 3 y^4) (1 - y)^{-2}
+ 2 \lambda^6 (-192 + 622 y - 856 y^2 + 552 y^3 - 105 y^4 - 21 y^5) (1 - y)^{-4}
+ 3 \lambda^8 (34 - 64 y - 13 y^2 + 4 y^3) (1 - y)^{-4}
+ 24 (-3 + 2 y - 4 y^2 + 2 y^3) (1 - y) + 72 \lambda^2 y (-13 + 19 y - 7 y^2) (1 - y)^{-1}
+ 72 \lambda^4 (12 - 27 y + 18 y^2 - 5 y^3) (1 - y)^{-3} + 24 \lambda^6 y (5 + 15 y - 18 y^2) (1 - y)^{-3}
+ 72 \lambda^8 (-1 + 2 y) (1 - y)^{-4} \log \left( \frac{\lambda}{1 - y} \right) \right\}.
$$

(A.1)

$$
\left( \frac{d\Gamma}{dx} \right)_{R+L} = \Gamma_0 \frac{\alpha}{\pi A} u^{-4} (u^2 - \lambda^2) \left[ k_1 + \log \left( \frac{u}{A} \right) k_2 + \log \left( \frac{1 - u}{y_0} \right) k_3 + \log \left( \frac{1 - u}{y_0} \right) k_4 \right],
$$

(A.2)

where $u = (x + A)/2$, and

$$
k_1 := \frac{1}{36} (u^2 - \lambda^2) (1 - y_0 - u) \left[ u^2 \left( -300 + 399 u + 71 u^2 + 2 u^3 + 8 u^4 + 132 y_0 + 63 u y_0 - 10 u^2 y_0 
- 8 u^3 y_0 + 24 y_0^2 - 12 u y_0^2 + 8 u^2 y_0^2 \right)
+ \lambda^2 u (555 - 259 u - 61 u^2 - 19 u^3 + 87 y_0 - 28 u y_0 + 19 u^2 y_0
- 12 y_0^2 + 8 u y_0^2)
+ \lambda^4 (122 - 67 u - 19 u^2 - 22 y_0 + 19 u y_0 + 8 y_0^2) \right],
$$

$$
k_2 := \frac{1}{3} u (1 - y_0 - u) \left[ u^2 (5 + 17 u - 34 u^2 + 5 y_0 - 14 u y_0 - 4 y_0^2) + 3 \lambda^2 u (6 + 19 u + u^2 - 6 y_0 - 5 u y_0
+ 12 \lambda^4 (-4 - u + 3 u^2) \right],
$$

$$
k_3 := 4 (\lambda^2 - u^2) \left[ u^2 (3 - 2 u) + \lambda^2 u (3 - 8 u + 3 u^2) + \lambda^4 (-2 + 3 u) \right],
$$

(A.3)

$$
k_4 := 4 (\lambda^2 + u^2) \left[ u^2 (3 - 2 u) + \lambda^2 u (3 - 8 u + 3 u^2) + \lambda^4 (-2 + 3 u) \right],
$$
\[
\frac{d\Gamma}{dy}_R = \int_{j2\lambda}^{x_m(y)} dx \left( \frac{d\Gamma}{dx dy}_R \right)_R = \Gamma_0 \frac{\alpha}{\pi} \left[ y(1 - y)(2 - y) + h_1 + \log \left( \frac{1 - y}{\lambda} \right) h_2 + \log^2 \left( \frac{1 - y}{\lambda} \right) h_3 \right],
\]

(A.4)

where

\[ h_1 := 4 \lambda y (-3 + 2y + \frac{1}{18} \lambda^2 \frac{1}{y} \frac{1}{1 - y} (-248 + 176 y + 489 y^2 - 304 y^3 - 53 y^4 - 6 y^5) \]
\[ + \frac{8}{9} \lambda^3 \frac{1}{y} (48 - 10 y + 15 y^2 + 3 y^3) + \frac{1}{4} \lambda^4 \frac{1}{y} \frac{1}{(1 - y)^2} (-150 + 320 y - 209 y^2 + 16 y^3 + 29 y^4 - 2 y^5) \]
\[ + \frac{4}{9} \lambda^5 \frac{1}{y} (32 - 12 y - 3 y^2) + \frac{1}{18} \lambda^6 \frac{1}{y} \frac{1}{(1 - y)^3} (-120 + 412 y - 465 y^2 + 114 y^3 + 18 y^4) \]
\[ + \frac{1}{36} \lambda^8 \frac{1}{y} \frac{1}{(1 - y)^4} (38 - 140 y - 21 y^2 + 6 y^3), \]

\[ h_2 := \frac{2}{3} \lambda^2 \frac{1}{y} \frac{1}{1 - y} (8 + 13 y - 30 y^2 + 7 y^3 + 5 y^4) + \lambda^4 \frac{1}{y} \frac{1}{1 - y} (-16 + 22 y - 19 y^2 + 3 y^3) \]
\[ + \frac{4}{3} \lambda^6 \frac{1}{(1 - y)^3} (-1 - 6 y + 6 y^2) + \frac{2}{3} \lambda^8 \frac{1}{y} \frac{1}{(1 - y)^4} (1 - 4 y), \]

\[ h_3 := 2 \lambda^2 (1 - y) (-3 - y) + 2 \lambda^4 \frac{1}{y} (-2 - y + y^2). \]

(A.5)

\[
\frac{d\Gamma}{dx}_R = \Gamma_0 \frac{\alpha}{\pi} \left[ j_1 + \log \left( \frac{u}{\lambda} \right) j_2 + \log \left( \frac{1 - u}{y_0} \right) j_3 + \log \left( \frac{u}{\lambda} \right) \log \left( \frac{1 - u}{y_0} \right) j_4 \right],
\]

(A.6)

where

\[ j_1 := \frac{1}{36} \frac{1}{u^3} (1 - y_0 - u) \left\{ u^3 \left[ 30 + 12 u^2 + u (-42 - 12 y_0) + 30 y_0 - 24 y_0^2 \right] \right. \]
\[ + \lambda^2 u^2 \left[ 264 - 168 y_0 - 24 y_0^2 + u (261 - 15 y_0 - 12 y_0^2) \right] \]
\[ + \lambda^2 u (-155 - 26 y_0 + 4 y_0^2) + u^3 (-5 + 5 y_0) + u^4 (-5) \]
\[ + \lambda^4 u \left[ -555 - 87 y_0 + 12 y_0^2 + u (199 - 32 y_0 + 4 y_0^2) \right] \]
\[ + u^2 (73 + 5 y_0) + u^3 (-5) \]
\[ + \lambda^6 \left[ -122 + 22 y_0 - 8 y_0^2 + 10 u (4 - y_0) + 10 u^2 \right] \right\}, \]

\[ j_2 := \frac{1}{3} \frac{1}{u^2} \left( \lambda^2 \frac{u}{u - \lambda} \right) \left\{ u^2 (1 - y_0 - u) \left[ -5 - 5 y_0 + 4 y_0^2 + u (-19 + 12 y_0 + 4 y_0^2) + u^2 (-13 - y_0) + 13 u^3 \right] \right. \]
\[ + 6 \lambda^2 u (1 - y_0 - u) \left[ -3 + 3 y_0 + u (2 + 3 y_0) + u^2 (-4 + y_0) + u^3 \right] \]
\[ + 24 \lambda^4 (2 - u) (1 + u) (1 - y_0 - u) \right\}, \]

\[ j_3 := 4 \frac{1}{u^3} \lambda^2 (1 - u) (\lambda^2 - u^2) (-2 \lambda^2 + 3 u - u^2), \]

\[ j_4 := 4 \frac{1}{u^3} \lambda^2 (1 - u) (\lambda^2 + u^2) (-2 \lambda^2 + 3 u - u^2). \]

(A.7)
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