Simple Properties of PUL-Stieltjes Integral in Banach Space

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Abstract

Using PUL integrals, Boonpogkrong in ² defined and discussed the Kurzweil-Henstock integral on manifolds. In this paper, we introduce the PUL-Stieltjes integral of Banach-valued functions and give some simple properties of this integral. Moreover, a characterization of PUL-Stieltjes integral is also given by establishing the Cauchy criterion.

1 Introduction

In ², Kurzweil-Henstock integral on manifolds is defined using partition of unity, a concept introduced by J. Kurzweil and J. Jarnik in ³. In the said paper, the authors defined the PUL integral and proved its equivalence to the Lebesgue integral in Rⁿ using lower and upper semi-continuous functions.

In classical theory, integration on manifolds is done by change of variables and the PUL integral can be used in such process. Although the PUL integral of a real-valued function f : M → R defined on compact differentiable r-manifold M is defined using an atlas Θ, its value is independent in the choice of Θ.

In this paper, we introduce and discuss some of the simple properties of the PUL-Stieltjes integral of functions taking values in a Banach space. Properties such as uniqueness, homogeneity, and linearity of both the integrands and integrators are proved. Moreover, Cauchy criterion for the PUL-Stieltjes integral is formulated and used to characterize such integrals.

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2 PUL-Stieltjes Integral in Banach Space

In what follows, we denote a compact interval in \( \mathbb{R}^n \) by \([a, b] = \prod_{k=1}^{n} [a_k, b_k] \) with \([a_k, b_k] \subseteq \mathbb{R}\) for each \( k = 1, 2, \ldots, n \) and \( \mu([a, b]) = \prod_{k=1}^{n}(b_k - a_k) \) be the volume of \([a, b]\). Moreover, \( \mathbb{R}^n \) is equipped with the norm \( \| \cdot \|_n \) defined by

\[
\|x\|_n = \max \{|x_i| : i = 1, 2, \ldots, n\}
\]

and for \( r > 0 \), we write \( B(x; r) = \{y \in \mathbb{R}^n : \|x - y\|_n < r\} \), where \( x - y = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n) \) for \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \).

For a smooth function \( \psi : [a, b] \to \mathbb{R} \), the support of \( \psi \), denoted by supp \( \psi \), is given by

\[
\text{supp } \psi = \{x \in [a, b] : \psi(x) \neq 0\},
\]

where \( \overline{A} \) denotes the closure of \( A \subseteq \mathbb{R}^n \). A gauge on \([a, b]\) is a positive function defined on \([a, b]\).

**Definition 2.1** A finite collection \( \{\psi_k\}_{k=1}^{m} \) of smooth functions defined on \([a, b]\) is said to be a partial partition of unity if the following holds:

1. \( \psi_k(\xi) \geq 0 \) for all \( \xi \in [a, b] \) and for all \( k \in \{1, 2, \ldots, m\} \) and
2. \( \sum_{k=1}^{m} \psi_k(\xi) \leq 1 \) for all \( \xi \in [a, b] \).

If \( \sum_{k=1}^{m} \psi_k(\xi) = 1 \) for all \( \xi \in [a, b] \), then \( \{\psi_k\}_{k=1}^{m} \) is said to be a partition of unity.

**Definition 2.2** Let \( \psi : [a, b] \to \mathbb{R} \) be a smooth function and \( \delta \) a gauge on \([a, b]\). A triple \( (\xi, I, \psi) \), with \( \xi \in [a, b] \) and \( I \subseteq [a, b] \), is said to be \( \delta \)-fine if

\[
\text{supp } \psi \subseteq I \subseteq B(\xi; \delta(\xi)).
\]

Note that \( \xi \) may not be in \( \text{supp } \psi \). If \( (\xi, I, \psi) \) is \( \delta \)-fine and \( x \notin I \), then \( \psi(x) = 0 \).

If \( \delta_1 \) and \( \delta_2 \) are gauges on \([a, b]\) such that \( \delta_1(\xi) \geq \delta_2(\xi) \) and \( (\xi, I, \psi) \) is \( \delta_2 \)-fine, then \( (\xi, I, \psi) \) is also \( \delta_1 \)-fine.
Throughout this paper, a division of \([a, b]\) is a finite collection \(D = \{I_k\}_{k=1}^m\) of subintervals \(I_k = \prod_{i=1}^{n} [a_i^{(k)}, b_i^{(k)}]\) of \([a, b]\) such that \(\text{int}(I_k) \cap \text{int}(I_j) = \emptyset\) for \(k \neq j\) and \(\bigcup_{k=1}^{m} I_k = [a, b]\). A division \(D = \{I_k\}_{k=1}^m\) of \([a, b]\) is a net if for each \(k = 1, 2, \ldots, m\) there exists a division \(D_k\) of \([a_k, b_k]\) such that
\[
D = \left\{ \prod_{k=1}^{m} [s_k, t_k] : [s_k, t_k] \in D_k \text{ for } k = 1, 2, \ldots, m \right\}.
\]

**Definition 2.3** A finite collection \(D = \{(\xi_k, I_k, \psi_k)\}_{k=1}^m\) is said to be a \(\delta\)-fine partial division of \([a, b]\) if the collection \(\{\psi_k\}_{k=1}^m\) is a partition of unity and every \((\xi_k, I_k, \psi_k)\) is \(\delta\)-fine. If \(\{\psi_k\}_{k=1}^m\) is a partition of unity, then \(D\) is said to be a \(\delta\)-fine division of \([a, b]\).

The existence of \(\delta\)-fine divisions of \([a, b]\) is guaranteed by the open covering theorem and the existence of a partition of unity.

Let \(D = \{(\xi_k, I_k, \psi_k)\}_{k=1}^m\) be a \(\delta\)-fine division of \([a, b]\), and \(g : [a, b] \to \mathbb{R}\) be a function. Suppose that for each \(k \in \{1, 2, \ldots, m\}\), the Riemann-Stieltjes integral \(\int_{I_k} \psi_k \, dg\) exists. Define the PUL-Stieltjes sum of \(f\) with respect to \(g\) over \(D\) by
\[
S(f, g, D) = \sum_{k=1}^{m} f(\xi_k) \int_{I_k} \psi_k(x) \, dg(x) = \sum_{k=1}^{m} f(\xi_k) \int_{I_k} \psi_k \, dg.
\]

For brevity, we write a \(\delta\)-fine division of \([a, b]\) by \(D = \{\xi, I, \psi\}\) and a PUL-Stieltjes sum of \(f\) with respect to \(g\) over \(D\) by
\[
S(f, g, D) = (D) \sum_{\xi} f(\xi) \int_{I} \psi \, dg = \sum_{D} f(\xi) \int_{I} \psi \, dg.
\]

**Remark 2.4** If \(D_1 = \{(\xi_k, I_k, \varphi_k)\}_{k=1}^m\) and \(D_2 = \{(\xi_j, I_j, \psi_j)\}_{j=1}^m\) are two \(\delta\)-fine divisions of \([a, b]\), then \(S(f, g, D_1) = S(f, g, D_2)\).

**Proof:** Let \(D_1 = \{(\xi_k, I_k, \varphi_k)\}_{k=1}^m\) and \(D_2 = \{(\xi_j, I_j, \psi_j)\}_{j=1}^m\) be \(\delta\)-fine divisions of \([a, b]\). Note that \(D_1\) and \(D_2\) differ only by the partition of unity. Then for each \(k = 1, 2, \ldots, m\), \(\psi_j(x) = 0\), for all \(j \neq k\) and \(x \notin I_k\). Thus, for each \(k = 1, 2, \ldots, m\) and for all \(x \in I_k\)
\[
\sum_{j=1}^{m} \psi_j(x) = \varphi_k(x).
\]

Hence,
\[
S(f, g, D_1) = \sum_{k=1}^{m} f(\xi_k) \int_{I_k} \varphi_k(x) \, dg(x) = \sum_{k=1}^{m} f(\xi_k) \int_{I_k} 1 \cdot \varphi_k(x) \, dg(x)
= \sum_{k=1}^{m} f(\xi_k) \int_{I_k} \left( \sum_{j=1}^{m} \psi_j(x) \right) \cdot \varphi_k(x) \, dg(x) = \sum_{k=1}^{m} f(\xi_k) \int_{I_k} \psi_k(x) \cdot \varphi_k(x) \, dg(x).
\]
Similarly,
\[ S(f, g, D_2) = \sum_{j=1}^{m} f(\xi_k) \int_{I_j} \psi_j(x) \, dg(x) = \sum_{j=1}^{m} f(\xi_j) \int_{I_j} 1 \cdot \psi_j(x) \, dg(x) \]
\[ = \sum_{j=1}^{m} f(\xi_j) \int_{I_j} \left[ \sum_{k=1}^{m} \phi_k(x) \right] \cdot \psi_j(x) \, dg(x) = \sum_{j=1}^{m} f(\xi_j) \int_{I_j} \varphi_j(x) \cdot \psi_j(x) \, dg(x). \]

Consequently, \( S(f, g, D_1) = S(f, g, D_2). \)

**Definition 2.5** Let \((X, \| \cdot \|)\) be a Banach space. A function \( f : [a, b] \to X \) is said to be \textit{PUL-Stieltjes integrable to} \( A \in X \) with respect to \( g : [a, b] \to \mathbb{R} \) if for every \( \epsilon > 0 \), there exists a gauge \( \delta(\xi) \) on \([a, b]\) such that for every \( \delta\)-fine division \( D = \{(\xi_k, I_k, \psi_k)\}_{k=1}^{m} \) of \([a, b]\), we have
\[ \|S(f, g, D) - A\| < \epsilon. \]

If \( A \) is the PUL-Stieltjes integral of \( f \) with respect to \( g \), then we write
\[ A = \int_{[a,b]} f \, dg. \]

Note that Remark 2.4 means that a PUL-Stieltjes sum is independent of the choice of the partition of unity. Consequently, the value the PUL-Stieltjes integral is independent of the choice of the partition of unity.

**Theorem 2.6** The PUL-Stieltjes integral of \( f \) with respect to \( g \) is unique.

**Proof**: Let \( A \) and \( B \) be PUL-Stieltjes integrals of \( f \) with respect to \( g \). Then there is a gauge \( \delta_1(\xi) > 0 \) on \([a, b]\) such that for any \( \delta_1\)-fine division \( D \) of \([a, b]\), we have
\[ \|S(f, g, D) - A\| < \frac{\epsilon}{2}. \]

Similarly, there exists a gauge \( \delta_2(\xi) \) on \([a, b]\) such that for any \( \delta_2\)-fine division \( D' \) of \([a, b]\), we have
\[ \|S(f, g, D') - B\| < \frac{\epsilon}{2}. \]

Take \( \delta = \min\{\delta_1, \delta_2\} \). Then \( \delta \) is a gauge in \([a, b]\). Now, fix a \( \delta\)-fine division \( D \) of \([a, b]\). Then \( D \) is both \( \delta_1\)-fine and \( \delta_2\)-fine. Thus,
\[ \|A - B\| \leq \|A - S(f, g, D)\| + \|S(f, g, D) - B\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]

Since \( \epsilon > 0 \) is arbitrary, \( \|A - B\| = 0 \). Hence, \( A = B \).
3 Simple Properties of PUL-Stieltjes Integral

Here, we show that the PUL-Stieltjes integral has the homogeneity and linearity properties over both the integrands and integrators.

**Theorem 3.1** If \( f_1 : [a, b] \rightarrow X \) and \( f_2 : [a, b] \rightarrow X \) are PUL-Stieltjes integrable with respect to \( g : [a, b] \rightarrow \mathbb{R} \) and \( c \in \mathbb{R} \), then \( cf_1 \) and \( f_1 + f_2 \) is PUL-Stieltjes integrable with respect to \( g \) on \([a, b]\) and

\[
\int_{[a, b]} cf_1 \, dg = c \int_{[a, b]} f_1 \, dg \quad \text{and} \quad \int_{[a, b]} (f_1 + f_2) \, dg = \int_{[a, b]} f_1 \, dg + \int_{[a, b]} f_2 \, dg.
\]

**Proof:** Let \( c \in \mathbb{R} \) and let \( \epsilon > 0 \). Then there exists a gauge \( \delta(\xi) \) on \([a, b]\) such that for any \( \delta \)-fine division \( D \) of \([a, b]\), we have

\[
\left\| S(f, g, D) - \int_{[a, b]} f_1 \, dg \right\| < \frac{\epsilon}{1 + |c|}.
\]

Thus, for any \( \delta \)-fine division \( D \) of \([a, b]\) we have

\[
\left\| S(cf, g, D) - \int_{[a, b]} cf_1 \, dg \right\| = |c| \cdot \left\| S(f, g, D) - \int_{[a, b]} f_1 \, dg \right\| < |c| \cdot \frac{\epsilon}{1 + |c|} < \epsilon.
\]

Hence, \( cf_1 \) is PUL-Stieltjes integrable with respect to \( g \) and

\[
\int_{[a, b]} cf_1 \, dg = c \int_{[a, b]} f_1 \, dg.
\]

For the remaining part, let \( \epsilon > 0 \). Then there exists a gauge \( \delta_1(\xi) \) on \([a, b]\) such that for any \( \delta_1 \)-fine division \( D \) of \([a, b]\), we have

\[
\left\| S(f_1, g, D) - \int_{[a, b]} f_1 \, dg \right\| < \frac{\epsilon}{2}.
\]

Also, there exists a gauge \( \delta_2(\xi) \) on \([a, b]\) such that for any \( \delta_2 \)-fine division \( D' \) of \([a, b]\), we have

\[
\left\| S(f_2, g, D') - \int_{[a, b]} f_2 \, dg \right\| < \frac{\epsilon}{2}.
\]

Let \( \delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\} \) for all \( \xi \in [a, b] \). Then \( \delta \) is a gauge on \([a, b]\). Let \( D \) be a \( \delta \)-fine of \([a, b]\). Then \( D \) is both \( \delta_1 \)-fine and \( \delta_2 \)-fine of \([a, b]\). Thus,

\[
\left\| S(f_1 + f_2, g, D) - \left[ \int_{[a, b]} f_1 \, dg + \int_{[a, b]} f_2 \, dg \right] \right\| \\
\leq \left\| S(f_1, g, D) - \int_{[a, b]} f_1 \, dg \right\| + \left\| S(f_2, g, D) - \int_{[a, b]} f_2 \, dg \right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Thus, \( f_1 + f_2 \) is PUL-Stieltjes integrable with respect to \( g \) on \([a, b]\) and

\[
\int_{[a, b]} (f_1 + f_2) \, dg = \int_{[a, b]} f_1 \, dg + \int_{[a, b]} f_2 \, dg.
\]

\[\blacksquare\]
Theorem 3.2 If \( f : [a, b] \rightarrow X \) is PUL-Stieltjes integrable with respect to both \( g_1 : [a, b] \rightarrow \mathbb{R} \) and \( g_2 : [a, b] \rightarrow \mathbb{R} \) on \([a, b]\) and \( c \in \mathbb{R} \), then \( f \) PUL-Stieltjes integrable with respect to both \( cg_1 \) and \( g_1 + g_2 \) on \([a, b]\) and
\[
\int_{[a,b]} f \, d(cg_1) = c \int_{[a,b]} f \, dg_1 \quad \text{and} \quad \int_{[a,b]} f \, d(g_1 + g_2) = \int_{[a,b]} f \, dg_1 + \int_{[a,b]} f \, dg_2.
\]
The proof is similar to Theorem 3.1.

4 Cauchy Criterion

We the Cauchy criterion for the PUL-Stieltjes integral.

Theorem 4.1 (Cauchy Criterion) A function \( f : [a, b] \rightarrow X \) is PUL-Stieltjes integrable with respect to \( g : [a, b] \rightarrow \mathbb{R} \) on \([a, b]\) if and only if for any \( \epsilon > 0 \), there exists a gauge \( \delta(\xi) \) on \([a, b]\) such that for any pair of \( \delta \)-fine divisions \( D_1 \) and \( D_2 \) of \([a, b]\), we have
\[
\| S(f, g, D_1) - S(f, g, D_2) \| < \epsilon.
\]

Proof: \((\Rightarrow)\) Let \( \epsilon > 0 \). Then there is a gauge \( \delta(\xi) \) on \([a, b]\) such that for any \( \delta \)-fine division \( D \) of \([a, b]\), we have
\[
\left\| S(f, g, D) - \int_{[a,b]} f \, dg \right\| < \frac{\epsilon}{2}.
\]
Let \( D_1 \) and \( D_2 \) be any two \( \delta \)-fine divisions of \([a, b]\). Then
\[
\left\| S(f, g, D_1) - S(f, g, D_2) \right\|
\leq \left\| S(f, g, D_1) - \int_{[a,b]} f \, dg \right\| + \left\| \int_{[a,b]} f \, dg - S(f, g, D_2) \right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
\((\Leftarrow)\) By hypothesis, for each \( n \in \mathbb{N} \), there exists a gauge \( \delta_n(\xi) \) on \([a, b]\) such that for any \( \delta_n \)-fine divisions \( D_n \) and \( D'_n \) of \([a, b]\), we have
\[
\| S(f, g, D_n) - S(f, g, D'_n) \| < \frac{1}{n}.
\]
We may assume that \( \delta_n(\xi) \geq \delta_{n+1}(\xi) \) for each \( \xi \in [a, b] \) and for all \( n \in \mathbb{N} \).

For every \( n \in \mathbb{N} \), let \( D_n \) be a fix \( \delta_n \)-fine division of \([a, b]\) and consider its corresponding PUL-Stieltjes sum \( s_n = S(f, g, D_n) \). We show that the sequence \( \{s_n\}_{n=1}^{+\infty} \) is Cauchy in \( X \).

Let \( \epsilon > 0 \) and let \( N \in \mathbb{N} \) with \( \frac{1}{N} < \epsilon \). Suppose that \( n, m \geq N \). Then \( \delta_n(\xi) \leq \delta_N(\xi) \) and \( \delta_m(\xi) \leq \delta_N(\xi) \) for all \( \xi \in [a, b] \). Thus, \( D_n \) and \( D_m \) are both \( \delta_N \)-fine division of \([a, b]\). Hence,
\[
\| s_n - s_m \| = \| S(f, g, D_n) - S(f, g, D_m) \| < \frac{1}{N} < \epsilon.
\]
This shows that $\langle s_n \rangle_{n=1}^{+\infty}$ is Cauchy in $X$. Since $X$ is complete, $\langle s_n \rangle_{n=1}^{+\infty}$ converges in $X$, say, $\lim_{n \to \infty} s_n = A$.

We now show that $f$ is PUL-Stieltjes integrable with respect to $g$ on $[a, b]$ and

$$\int_{[a,b]} f \, dg = A.$$ 

Let $\epsilon > 0$. Since $\lim_{n \to \infty} s_n = A$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\|S(f, g, D_n) - A\| = \|s_n - A\| < \frac{\epsilon}{2}.$$ \hfill (2)

We chose $N \in \mathbb{N}$ in which $\frac{1}{N} < \epsilon$. Put $\delta(\xi) = \delta_N(\xi)$ for all $\xi \in [a, b]$. Let $D$ be any $\delta$-fine division of $[a, b]$. Then $D$ is $\delta_N$-fine division of $[a, b]$. Since $N \geq N_2$, by (1) we have

$$\|S(f, g, D) - S(f, g, D_N)\| < \frac{1}{N} < \frac{\epsilon}{2}.$$ \hfill (3)

Also, since $N \geq N_1$, inequality (2) for $n = N$; i.e.

$$\|S(f, g, D_N) - A\| < \frac{\epsilon}{2}.$$ \hfill (4)

Hence, by (3) and (4) we have

$$\|S(f, g, D) - A\| \leq \|S(f, g, D) - S(f, g, D_N)\| + \|S(f, g, D_N) - A\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $f$ is a PUL-Stieltjes integrable with respect to $g$ on $[a, b]$ and

$$\int_{[a,b]} f \, dg = A.$$ \hfill $\blacksquare$

In what follows, we denote the set of all compact subintervals of $[a, b] \subseteq \mathbb{R}^n$ by $\mathcal{I}_n([a, b])$.

**Corollary 4.2** If $f : [a, b] \to X$ is PUL-Stieltjes integrable with respect to $g : [a, b] \to \mathbb{R}$ on $[a, b]$ and $I \in \mathcal{I}_n([a, b])$, then $f$ is PUL-Stieltjes integrable with respect to $g$ on $I$.

**Proof:** Let $\epsilon > 0$. By Theorem 4.1, there exists a gauge $\delta(\xi)$ on $[a, b]$ such that for any $\delta$-fine divisions $D_1$ and $D_2$ of $[a, b]$, we have

$$\|S(f, g, D_1) - S(f, g, D_2)\| < \epsilon.$$ \hfill (5)

If $I = [a, b]$, then we are done. Suppose that $I \subset [a, b]$. Then there is a finite collection $\mathcal{J} \subseteq \mathcal{I}([a, b])$ such that $I \notin \mathcal{J}$ and $\mathcal{J} \cup \{I\}$ is a net of $[a, b]$. For each $J \in \mathcal{J} \cup \{I\}$, fix a $\delta$-fine division $D_J$ of $J$. Let $D_I^{(1)}$ and $D_I^{(2)}$ be two $\delta$-fine divisions of $I$. Let

$$D_1 = D_I^{(1)} \cup \bigcup_{J \in \mathcal{J}} D_J \quad \text{and} \quad D_2 = D_I^{(2)} \cup \bigcup_{J \in \mathcal{J}} D_J.$$
Then $D_1$ and $D_2$ are $\delta$-fine divisions of $[a, b]$ and

$$S(f, g, D_I^{(1)}) = S(f, g, D_1) - \sum_{J \in J} S(f, g, D_J) \quad \text{and} \quad S(f, g, D_I^{(2)}) = S(f, g, D_2) - \sum_{J \in J} S(f, g, D_J).$$

Thus, by (5)

$$\|S(f, g, D_I^{(1)}) - S(f, g, D_I^{(2)})\| = \|S(f, g, D_1) - S(f, g, D_2)\| < \epsilon.$$  

The desired result now follows from Theorem 4.1. \hfill \qed

In what follows, let $\mathcal{V}[u, v]$ be the collection of all the vertices of an interval $[u, v]$.

**Definition 4.3** \textsuperscript{2} Let $g : [a, b] \to \mathbb{R}$. The total variation of $g$ over $[a, b]$ is given by

$$\text{Var}(g, [a, b]) = \sup \left\{ \sum_{[u, v] \in D} |\Delta_g([u, v])| : D \text{ is a division of } [a, b] \right\}$$

where $\Delta_g([u, v]) = \sum_{t \in \mathcal{V}[u, v]} g(t) \prod_{k=1}^{m} (-1)^{x_{(uk)}(tk)}$.

**Definition 4.4** \textsuperscript{8} A function $g : [a, b] \to \mathbb{R}$ is said to be of bounded variation on $[a, b]$ if $\text{Var}(g, [a, b])$ is finite. We denote $BV([a, b])$ to be the collection of functions of bounded variation on the interval $[a, b]$.

**Theorem 4.5** If $f : [a, b] \to X$ is continuous on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, then $f$ is PUL-stieltjes integrable on $[a, b]$ with respect to $g$.

**Proof:** Let $\epsilon > 0$. Note that the Riemann-stieltjes integral $\int_{[a,b]} \varphi dg$ exists, whenever $\varphi$ is a partition of unity and $g$ is of bounded variation on $[a, b]$. Since $g \in BV([a, b])$, $M = V(g, [a, b]) \in \mathbb{R}$. Since $f$ is continuous on $[a, b]$, then $f$ is uniformly continuous on $[a, b]$. Thus, there exists a $\delta > 0$ such that for any $x, y \in [a, b]$ with $\|x - y\|_{\mathbb{R}} < \delta(x)$, we have

$$\|f(x) - f(y)\|_X < \frac{\epsilon}{2[M + 1]}.$$  

Let $D_1 = \{(\xi, I, \varphi)\}$ and $D_2 = \{(\zeta, J, \psi)\}$ be any two $\delta$-fine divisions of $[a, b]$. Let $D_3 = \{\gamma, K, \sigma\}$ be a $\delta$-fine division of $[a, b]$, where $K = I \cap J$ with $I \in D_1$ and $J \in D_2$. Observe that

$$S(f, g, D_1) = \sum_{I \in D_1} f(\xi) \int_I \varphi dg = \sum_{I \in D_1} f(\xi) \left[ \sum_{J \in D_2} \int_{I \cap J} \varphi dg \right] = \sum_{K \in D_3} f(\xi) \int_K \sigma dg$$

and

$$S(f, g, D_2) = \sum_{J \in D_2} f(\zeta) \int_J \psi dg = \sum_{J \in D_2} f(\zeta) \left[ \sum_{I \in D_1} \int_{J \cap I} \psi dg \right] = \sum_{K \in D_3} f(\zeta) \int_K \sigma dg.$$
Then
\[
\|S(f,g,D_1) - S(f,g,D_2)\| \leq \|S(f,g,D_1) - S(f,g,D_3)\| + \|S(f,g,D_3) - S(f,g,D_2)\|
\]
\[
= \left\| \sum_{K \in D_3} f(\xi) \int_{K} \sigma dg - \sum_{K \in D_3} f(\gamma) \int_{K} \sigma dg \right\| + \left\| \sum_{K \in D_3} f(\gamma) \int_{K} \sigma dg - \sum_{K \in D_3} f(\zeta) \int_{K} \sigma dg \right\|
\]
\[
\leq \sum_{K \in D_3} \left\| f(\xi) - f(\gamma) \right\| \left\| \int_{K} \sigma dg \right\| + \sum_{K \in D_3} \left\| f(\gamma) - f(\zeta) \right\| \left\| \int_{K} \sigma dg \right\|
\]
\[
< \sum_{K \in D_3} \left[ \frac{\epsilon}{2(M+1)} \left\| \int_{K} \sigma dg \right\| \right] + \sum_{K \in D_3} \left[ \frac{\epsilon}{2(M+1)} \left\| \int_{K} \sigma dg \right\| \right]
\]
\[
= \frac{\epsilon}{M+1} \sum_{K \in D_3} \left\| \int_{K} \sigma dg \right\| = \frac{\epsilon}{M+1} \sum_{K \in D_3} \Delta_g(K) \leq \frac{\epsilon}{M+1} \cdot M < \epsilon.
\]
Therefore, \( f \) is PUL-Stieltjes integrable on \([a,b]\) with respect to \( g \).

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