Nonlinear time-fractional dispersive equations

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Dedicated to Professor Francesco Mainardi
on the occasion of his retirement

Abstract

In this paper we study some cases of time-fractional nonlinear dispersive equations (NDEs) involving Caputo derivatives, by means of the invariant subspace method. This method allows to find exact solutions to nonlinear time-fractional partial differential equations by separating variables. We first consider a third order time-fractional NDE that admits a four-dimensional invariant subspace and we find a similarity solution. We also study a fifth order NDE. In this last case we find a solution involving Mittag-Leffler functions. We finally observe that the invariant subspace method permits to find explicit solutions for a wide class of nonlinear dispersive time-fractional equations.

Keywords: Nonlinear dispersion, Invariant subspace method, Fractional differential equations.

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1. Introduction.

In the present paper some exact solutions to time-fractional Nonlinear Dispersive Equations (NDEs) by means of the invariant subspace method (see e.g. [1]) are studied. In more details, we generalize to the fractional case some of the results presented in [1,2]. We discuss the peculiarity of the fractional cases, where the solutions can not be obtained as a trivial generalization of the ordinary one. In this context, we generalize to the time-fractional case the fifth order NDEs firstly considered by Dey in [3]. In this work the author considered the following family of nonlinear partial
differential equations, termed $K(m,n,p)$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^5 u^p}{\partial x^5} + \beta \frac{\partial^3 u^n}{\partial x^3} + \gamma \frac{\partial u^m}{\partial x}, \quad m, n, p > 1.$$ 

Thus, for the fractional case, we consider a new family of equations involving fractional derivatives in the Caputo sense, termed $K_{\alpha}(m,n,p)$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \nu \frac{\partial^5 u^p}{\partial x^5} + \beta \frac{\partial^3 u^n}{\partial x^3} + \gamma \frac{\partial u^m}{\partial x}, \quad \alpha \in (0,1).$$

We underline that now $m, n, p \in \mathbb{R}^+$. 

In more details we first consider the third order time-fractional NDE $K_{\alpha}(0,2,0)$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^3}{\partial x^3} \left( \frac{u^2}{2} \right),$$

that in the ordinary case $\alpha = 1$ was deeply studied in [2]. By means of the invariant subspace method, we find a generalization of the similarity solution.

The second case under consideration is the fifth-order NDE equation $K_{\alpha}(2,2,2)$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \nu \frac{\partial^5 u^2}{\partial x^5} + \beta \frac{\partial^3 u^2}{\partial x^3} + \gamma \frac{\partial u^2}{\partial x},$$

whose ordinary case was firstly studied in the framework of compacton solutions of NDEs (see [1,3]).

The main aim of this paper is to show the utility of the invariant subspace method in order to find exact solutions of time-fractional NDEs. In particular, also in the light of the literature on time-fractional NDEs (see e.g. [4–6]), this method provides rigorous and effective tools to generalize compact soliton solutions and similarity solutions for time-fractional NDEs.

This paper is organized as follows: in Section 2 we recall some preliminary definitions and results about fractional derivatives and invariant subspace method. In Section 3 we study the time-fractional third order NDE while in Section 4 the time-fractional fifth order NDE is considered. Finally we suggest further work to be done in the final Section 5.

2. Preliminaries.

2.1. Generalities about fractional calculus.

Here we recall definitions and basic results about fractional calculus, for more details we refer to [7–10].
Let $\gamma$ be a positive real number. The Riemann-Liouville fractional integral is defined by

$$J_0^\gamma t f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - \tau)^{\gamma - 1} f(\tau) d\tau,$$

where

$$\Gamma(\gamma) = \int_0^{+\infty} x^{\gamma - 1} e^{-x} dx,$$

is the Euler Gamma function. Note that, by definition, $J_0^0 t f(t) = f(t)$. Moreover it satisfies the semigroup property, i.e. $J_0^\alpha t J_0^\beta t f(t) = J_0^{\alpha + \beta} t f(t)$.

There are different definitions of fractional derivative (see e.g [10]). In this paper we use the fractional derivatives in the sense of Caputo. Hereafter we denote by $AC^n([0, t])$, $n \in \mathbb{N}$, the class of functions $f(x)$ which are continuously differentiable in $[0, t]$ up to order $n - 1$ and with $f^{(n-1)}(x) \in AC([0, t])$. We recall the following Theorem ( [7, pagg. 92-93])

**Theorem 2.1.** Let $m - 1 < \gamma < m$, with $m \in \mathbb{N}$. If $f(t) \in AC^n([0, t])$, then the Caputo fractional derivative exists almost everywhere on $[0, t]$ and it is represented in the form

$$D_0^\gamma t f(t) = J_0^{m-\gamma} D_t^m f(t) = \frac{1}{\Gamma(m - \gamma)} \int_0^t (t - \tau)^{m-\gamma - 1} \frac{d^m}{dt^m} f(\tau) d\tau, \quad \gamma \neq m.$$

By definition the fractional derivative is a pseudodifferential operator given by the convolution of the ordinary derivative of the function with a power law kernel. So the reason why fractional derivatives introduce a memory formalism becomes evident.

The following properties of fractional derivatives and integrals (see e.g. [10]) will be used in the analysis:

$$D_0^\gamma J_0^\gamma f(t) = f(t), \quad \gamma > 0,$$

$$J_0^\gamma D_t^\gamma f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}, \quad \gamma > 0, \quad t > 0,$$

$$J_0^\gamma t^\delta = \frac{\Gamma(\delta + 1)}{\Gamma(\delta + \gamma + 1)} t^{\delta + \gamma} \quad \gamma > 0, \quad \delta > -1, \quad t > 0,$$

$$D_0^\gamma t^\delta = \frac{\Gamma(\delta + 1)}{\Gamma(\delta - \gamma + 1)} t^{\delta - \gamma} \quad \gamma > 0, \quad \delta \in (-1, 0) \cup (0, +\infty), \quad t > 0.$$

2.2. *Invariant subspace method.*

The invariant subspace method, as introduced by Galaktionov [1], allows to solve exactly nonlinear equations by separating variables.
Recently Gazizov and Kasatkin [11] suggested its application to nonlinear fractional equations.

We recall the main idea of this method: consider a scalar evolution equation

\[(10)\quad \frac{\partial u}{\partial t} = F[u],\]

where \( u = u(x, t) \) and \( F[u] \equiv F(u, \partial u/\partial x, \partial^2 u/\partial x^2, \ldots, \partial^k u/\partial x^k), \) \( k \in \mathbb{N}, \) is a nonlinear differential operator.

Given \( n \) linearly independent functions \( f_1(x), f_2(x), \ldots, f_n(x), \)

we call \( W_n \), the \( n \)-dimensional linear space

\[ W_n = \langle f_1(x), \ldots, f_n(x) \rangle. \]

This space is called invariant under the given operator \( F[u] \), if \( F[y] \in W_n \) for any \( y \in W_n \). This means that there exist \( n \) functions \( \Phi_1, \Phi_2, \ldots, \Phi_n \) such that

\[ F[C_1 f_1(x) + \cdots + C_n f_n(x)] = \Phi_1(C_1, \ldots, C_n) f_1(x) + \cdots + \Phi_n(C_1, \ldots, C_n) f_n(x), \]

where \( C_1, C_2, \ldots, C_n \) are arbitrary constants.

Once the set of functions \( f_i(x) \) forming the invariant subspace is given, we search an exact solution of (10) in the invariant subspace in the form

\[(11)\quad u(x, t) = \sum_{i=1}^{n} u_i(t) f_i(x).\]

where \( f_i(x) \in W_n \). In this way, we arrive to a system of ODEs. In many cases this is a simpler problem that allows to find exact solutions by just separating variables [1].

A relevant question in the theory of invariant subspace method is the following: how to find all the invariant subspaces admitted by a given differential operator \( F[u] \)? For the utility of the reader, we recall that a complete answer to this question is given by the following Proposition (see [1, sec. 2.1] and [11]).

**Proposition 2.1.** Let \( f_1(x), \ldots, f_n(x) \) form the fundamental set of solutions of a linear \( n \)-th order ordinary differential equation

\[(12)\quad L[y] = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0,\]

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and \( F[y] = F(x, y, y', \ldots, y^{(k)}) \) a given differential operator of order \( k \leq n - 1 \), then the subspace \( W_n = \langle f_1(x), \ldots, f_n(x) \rangle \) is invariant with respect to \( F \) if and only if
\[
L[F[y]] = 0,
\]
for all solutions \( y(x) \) of (12).

3. Third order nonlinear time-fractional dispersive equation.

In this section we consider the following third order nonlinear time-fractional dispersive equation
\[
(14) \quad \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^3}{\partial x^3} \left( \frac{u^2}{2} \right), \quad x \in \mathbb{R}, \ t \geq 0,
\]
where \( u = u(x, t) \) and \( \alpha \in (0, 1] \).

The original model equation (corresponding to \( \alpha = 1 \)) was deeply studied in [2]. This equation \( K_\alpha(0, 2, 0) \) belongs to the more general class of NDEs (1). Here we find an exact solution of (14) corresponding to a time-fractional generalization of the similarity solution discussed in [2]. First of all, we recall the following useful

**Lemma 3.1.** For \( \alpha \in (0, 1/2) \cup (1/2, 1) \) equation (14) admits
\[
(15) \quad W_4 = \{1, x, x^2, x^3\}
\]
as invariant subspace.

**Proof.** By direct calculation, for any
\[
(16) \quad g(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 \in W_4,
\]
we have that, being \( F[u] = \frac{\partial^3}{\partial x^3} \left( \frac{u^2}{2} \right) \)
\[
F[g] = 6(C_1 C_2 + C_0 C_3) + 12(C_2^2 + 2C_1 C_3)x + 60C_2 C_3 x^2 + 60C_3^2 x^3 \in W_4,
\]
as claimed.

The reason why we must exclude \( \alpha = 1/2 \) and \( \alpha = 1 \) will be clear in the following.

Thanks to Lemma 3.1, it is possible to express the solution of equation (14) in the form:
\[
(18) \quad u(x, t) = g_0(t) + g_1(t)x + g_2(t)x^2 + g_3(t)x^3.
\]
This leads us to the following system

\[
\begin{align*}
\frac{d^\alpha g_0}{dt^\alpha} &= 6(g_1g_2 + g_0g_3), \\
\frac{d^\alpha g_1}{dt^\alpha} &= 12(g_2^2 + 2g_1g_3), \\
\frac{d^\alpha g_2}{dt^\alpha} &= 60g_2g_3, \\
\frac{d^\alpha g_3}{dt^\alpha} &= 60g_3^2.
\end{align*}
\]

(19)

The solution \( u(x, t) \) will be found by solving one by one each equation of system (19).
The last of equations (19) is solved by

\[ g_3(t) = C_3t^\beta, \]

(20)

where \( C_3 \) is a real constant and the exponent \( \beta \) can be found by direct substitution in the equation.

It is

\[ \frac{d^\alpha}{dt^\alpha} C_3t^\beta = C_3 \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha} = 60C_3^2t^{2\beta}, \quad \beta > -1 \]

then, (20) is an effective solution if

\[
\beta = -\alpha, \quad C_3 = \frac{\Gamma(1 - \alpha)}{60\Gamma(1 - 2\alpha)}.
\]

Thus,

\[ g_3(t) = \frac{\Gamma(1 - \alpha)}{60\Gamma(1 - 2\alpha)} t^{-\alpha} \]

(22)

In the same way it is possible to find the solutions of the other equations in (19).
We obtain that all the solution are in the form

\[ g_i(t) = C_i t^{-\alpha}, \quad C_i \in \mathbb{R}, \ i = 0, \ldots, 3. \]

(23)

Below are reported the solutions \( g_i \):

\[
\begin{align*}
g_0(t) &= C_0t^{-\alpha} \\
g_1(t) &= C_1t^{-\alpha} \\
g_2(t) &= C_2t^{-\alpha} \\
g_3(t) &= C_3t^{-\alpha},
\end{align*}
\]

(24)
where
\[
\begin{align*}
C_0 &= \frac{400}{3} \left( \frac{\Gamma(1-2\alpha)}{60\Gamma(1-\alpha)} \right)^2, \\
C_1 &= 20 \frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)}, \\
C_2 &= 1, \\
C_3 &= \frac{\Gamma(1-\alpha)}{60\Gamma(1-2\alpha)}.
\end{align*}
\]

The complete solution \( u(x,t) \) results, then,
\[
(26) 
\quad u(x,t) = \frac{C_0}{t^\alpha} + C_1 \frac{x}{t^\alpha} + \frac{x^2}{t^\alpha} + C_3 \frac{x^3}{t^\alpha}.
\]

In order to understand the meaning of the above solution, we recall that the first results about shock and rarefaction waves for NDE (14) for \( \alpha = 1 \) have been discussed in [2] by analogy with the well known theory for first-order conservation laws (see e.g. [12]). The generalization of their analysis to the time-fractional case is not completely trivial but suggests the interpretation of the found solution (26) in the framework of the theory of global similarity solutions of third-order NDEs.

**Remark 3.1.** It is important to note that this solution is valid for \( \alpha \in (0, 1/2) \cup (1/2, 1) \). Indeed for \( \alpha = 1/2 \) and \( \alpha = 1 \), coefficients appearing in (26) can be not defined, due to the singularity in the Gamma coefficients. The reason why the found solution is not valid for \( \alpha = 1 \) is simply given by the definition of Caputo derivatives. Indeed for an integer \( \alpha \) the Caputo derivative coincides with an ordinary derivative and we must consider directly the ordinary equation
\[
(27) 
\quad \frac{\partial u}{\partial t} = \frac{\partial^3}{\partial x^3} \left( \frac{u^2}{2} \right), \quad x \in \mathbb{R}, \ t \geq 0,
\]
deiely studied by Galaktionov and Pohozaev in [2], where blowing-up and global similarity solutions have been considered.

A further remark regards the critical case \( \alpha = 1/2 \). From direct calculations we have shown that in this case the found solution (26) is divergent. On the other hand this critical value of \( \alpha \) plays a significant role to discriminate different regimes. Indeed the sign of the found solutions is positive for \( 0 < \alpha < 1/2 \) and negative for \( 1/2 < \alpha < 1 \). This non-trivial result should be object of further research about similarity solutions of time-fractional NDEs.
4. Fifth order nonlinear time-fractional dispersive equation.

In this section we study the following fifth order nonlinear time-fractional dispersive equation

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \nu \frac{\partial^5 u^2}{\partial x^5} + \beta \frac{\partial^3 u^2}{\partial x^3} + \gamma \frac{\partial u^2}{\partial x}, \quad t \geq 0, \ x \in \mathbb{R}, \ \alpha \in (0, 1].
\]

The ordinary case \( \alpha = 1 \) was studied by Dey in [3], where compacton solutions of NDEs were considered.

In order to find an exact solution of equation (28), by the invariant subspace method, we make use of the following Lemma (see [1, p. 165]).

**Lemma 4.1.** Equation (28) admits \( W_3 = \{1, \cos x, \sin x\} \) as invariant subspace if and only if

\[
16\nu - 4\beta + \gamma = 0.
\]

The proof of the above Lemma is based on the same arguments used in Lemma 3.1. Under the assumption of lemma 4.1, it is possible to express the solution of equation (28) in the following form:

\[
u(x, t) = g_1(t) + g_2(t) \cos x + g_3(t) \sin x, \quad t \geq 0.
\]

This leads us to following system

\[
\begin{cases}
\frac{d^\alpha g_1}{dt^\alpha} = 0, \\
\frac{d^\alpha g_2}{dt^\alpha} = \mu g_1 g_3, \\
\frac{d^\alpha g_3}{dt^\alpha} = -\mu g_1 g_2,
\end{cases}
\]

where \( \mu = 2(\nu - \beta + \gamma) \neq 0 \) is a real constant. We assume the condition \( \mu \neq 0 \) in order to get a non trivial solution of system (31).

Now, from the first of equations (31), we get

\[
g_1(t) = \text{const} =: C
\]

Then, defining \( \bar{\mu} = C\mu \) and by applying the time-fractional derivative \( d^\alpha/dt^\alpha \) to the second of equations (31), we get the new system

\[
\begin{cases}
g_1 = C \\
\frac{d^\alpha}{dt^\alpha} \frac{d^\alpha g_2}{dt^\alpha} = -\bar{\mu}^2 g_2 \\
\frac{d^\alpha g_3}{dt^\alpha} = -\bar{\mu} g_2.
\end{cases}
\]
The main effort in order to solve equation (33) is the solution of the third equation. To this aim we will study the following Cauchy problem

\[
\begin{align*}
\frac{d^\alpha}{dt^\alpha} \frac{d^\alpha}{dt^\alpha} f(t) &= -\bar{\mu}^2 f(t), \\
f(0) &= 1 \\
\left. \frac{d^\alpha}{dt^\alpha} f(t) \right|_{t=0} &= 0.
\end{align*}
\]

(34)

The choice of these special initial conditions will be clear in the following. The solution of equation (34) can be obtained by using the Laplace transform method. To this purpose we here recall that for \( \alpha \in (0, 1) \), (see e.g. [7])

\[
\mathcal{L}\left\{ \frac{d^\alpha}{dt^\alpha} f(t) \right\} = s^\alpha \tilde{f}(s) - s^{\alpha-1} f(0),
\]

(35)

where

\[
\mathcal{L}\{ f(t) \} = \tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt.
\]

(36)

Thus, by setting \( h(t) := \frac{d^\alpha}{dt^\alpha} f(t) / \frac{d^\alpha}{dt^\alpha} \), by formula (35), we obtain

\[
\mathcal{L}\left\{ \frac{d^\alpha}{dt^\alpha} \frac{d^\alpha}{dt^\alpha} f(t) \right\} = s^{2\alpha} \tilde{f}(s) - s^{2\alpha-1},
\]

(37)

where we used both initial conditions of the Cauchy problem (34). We now make the Laplace transform in both terms of the first of equations (34), we get

\[
s^{2\alpha} \tilde{f}(s) = -\bar{\mu}^2 \tilde{f} + s^{2\alpha-1}
\]

(38)

thus

\[
\tilde{f} = \frac{s^{2\alpha-1}}{s^{2\alpha} + \bar{\mu}^2}
\]

whose inverse Laplace transform is given by

\[
f(t) = E_{2\alpha,1}(-\bar{\mu}^2 t^{2\alpha}), \quad \alpha \in (0, 1],
\]

(39)

where \( E_{2\alpha,1}(\cdot) \) is the Mittag-Leffler function

\[
E_{2\alpha,1}(-\bar{\mu} t^{2\alpha}) = \sum_{k=0}^{\infty} \frac{(-1)^k \bar{\mu}^{2k} t^{2\alpha k}}{\Gamma(2\alpha k + 1)}.
\]

(40)
We remind that Mittag-Leffler functions play a fundamental role in the theory of fractional differential equations, see for example [13–15].

Going back to the system of equations (33), we have that
\[
g_2(t) = E_{2\alpha,1}(-\tilde{\mu}^2 t^{2\alpha}),
\]
and, by substitution in the second equation of (33), we have
\[
g_3(t) = -J_t^\alpha \tilde{\mu} g_2(t) = -\tilde{\mu} t^\alpha E_{2\alpha,\alpha+1}(-\tilde{\mu}^2 t^{2\alpha}),
\]
up to an integration additive constant, that we set equal to zero for sake of simplicity. By the above calculations we are allowed to write the solution \( u(x,t) \) of equation (28) in the form
\[
u(x,t) = C + E_{2\alpha,1}(-\tilde{\mu}^2 t^{2\alpha}) \cos x - \tilde{\mu} t^\alpha E_{2\alpha,\alpha+1}(-\tilde{\mu}^2 t^{2\alpha}) \sin x.
\]

We observe that, for \( \alpha = 1 \) we retrieve the solution in [1, p. 167]. Indeed, for \( \alpha = 1 \)
\[
E_{2,1}(-\tilde{\mu}^2 t^2) = \sum_{k=0}^{\infty} \frac{(-1)^k \tilde{\mu}^{2k} t^{2k}}{2k!} = \cos(\tilde{\mu} t),
\]
\[
\tilde{\mu} t E_{2,2}(-\tilde{\mu}^2 t^2) = \sum_{k=0}^{\infty} \frac{(-1)^k \tilde{\mu}^{2k+1} t^{2k+1}}{(2k + 1)!} = \sin(\tilde{\mu} t),
\]
so that the solution (43) becomes
\[
u(x,t) = C + \cos(\tilde{\mu} t) \cos x - \sin(\tilde{\mu} t) \sin x = C + \cos(x + \tilde{\mu} t),
\]
that can be written as a compacton (see [1]).
Figure 1. Mittag-Leffler function, appearing in (43) for different values of $2\alpha$.

We can now discuss the physical meaning of the introduction of memory effects by means of time-fractional derivatives in the fifth order NDE (28). As it can be seen in Figure 1, fractional derivatives introduce damping effects on the time-evolution depending on the real parameter $\alpha$. We recall that the Mittag-Leffler function of parameter $\alpha \in (1, 2)$ is related to the so-called fractional oscillation (see e.g. [13,16]), that is a damped oscillation where the damping effect depends on the order of fractionality.

**Remark 4.1.** We now explain the reason of the choice of the initial conditions in the Cauchy problem (34). It is known that, in general,

$$\frac{d^\alpha}{dt^\alpha} \frac{d^\alpha f}{dt^\alpha} \neq \frac{d^{2\alpha} f}{dt^{2\alpha}}$$

so that we can not write $d^{2\alpha}g_2/dt^{2\alpha}$ instead of $d^{\alpha}/dt^{\alpha}d^{\alpha}g_2/dt^{\alpha}$ in the second equation of (33). This substitution would have been very convenient to find the solution without considering an initial condition on the time-fractional derivative without a clear physical meaning (a recent discussion about this point can be find in [17]). In light of this fact, in order to find the explicit solution of (34) by using the Laplace transform method, we must give the initial condition on the time-fractional derivative, that we take null for simplicity.
Remark 4.2. In the case $\nu = 0$, $\beta = \gamma = 1$ in (28) the time-fractional Rosenau-Hyman equation $K_\alpha(2, 2)$ is obtained:
\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^3 u^2}{\partial x^3} + \frac{\partial u^2}{\partial x}.
\]
The Rosenau-Hyman equation (see [18]) plays a relevant role in the theory of solitary waves with compact support. A detailed study about the fractional Rosenau-Hyman equation traveling wave solutions should be object of further studies.

5. Conclusions and remarks.

In recent papers time-fractional NDEs have been studied by different authors with semi-analytical methods. In some cases they find exact solutions that can be recovered by the invariant subspace method. For example Odibat in [5] has considered variants of the KdV equation involving Caputo time-fractional derivatives, such as
\[
\left(\frac{\partial^\alpha u}{\partial t^\alpha} + a \frac{\partial u^2}{\partial x} + \frac{\partial}{\partial x} \left[ u \frac{\partial^2 u}{\partial x^2} \right] \right) = 0, \quad t, a > 0, \alpha \in (0, 1].
\]
The author finds an explicit solution to (45) by means of the homotopy perturbation method, that is
\[
(46) \quad u(x, t) = \begin{cases} \frac{\mu}{\sqrt{a}} \sin^2(\sqrt{a}x, \sqrt{a}ct^\alpha, \alpha), & |x - ct^\alpha| < \frac{\pi}{\mu}, \\ 0 & \text{otherwise}, \end{cases}
\]
where $\mu = \frac{\sqrt{\pi}}{2}$,
\[
(47) \quad \sin^2(x, t, \alpha) = 1 - \cos x \cos(t, \alpha) - \sin x \sin(t, \alpha),
\]
and
\[
\cos(t, \alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{\Gamma(2k\alpha + 1)} = E_{2\alpha, 1} (-t^2),
\]
\[
\sin(t, \alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{\Gamma(2k\alpha + \alpha + 1)} = tE_{2\alpha, \alpha+1} (-t^2).
\]
The solution (46) can be found in a direct way by using the invariant subspace method. Indeed this solution generalizes the compact wave solution, but it is in separating variable form. This means that (45) admits as invariant subspace
\[
(48) \quad W_3 = \{1, \cos x, \sin x\}.
\]
In the literature about time-fractional NDEs these exact solutions with separating variables are found also in other recent works, even if with different methods. In [6] the author has considered time-fractional $K(n, m)$ equations. Also in this case, some exact results can be recovered by the invariant subspace method. It can be proved that the fractional equations discussed in [6] admit as invariant subspace $W_3 = \{1, \cosh \frac{x}{2}, \sinh \frac{x}{2}\}$.

We can conclude that, considering the literature on time-fractional NDEs, the invariant subspace method can provide effective and rigorous tools to find exact solutions to a wide class of nonlinear fractional equations, avoiding the use of perturbative or approximate methods. Moreover, from the physical point of view, it allows to find relevant compacton-like solutions and rarefaction wave solutions to time-fractional NDEs. The meaning of these generalized equations is explained in Section 4, where we have shown that the role of fractionality is to introduce damping effects in the evolution of compacton solutions of NDEs.

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