NON VANISHING OF CENTRAL VALUES OF MODULAR L-FUNCTIONS FOR HECKE EIGENFORMS OF LEVEL ONE

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Abstract. Let \( F(z) = \sum_{n=1}^{\infty} a(n)q^n \) be a newform of weight \( 2k \) and level \( N \) with a trivial character, and assume that \( F(z) \) is a non-zero eigenform of all Hecke operators. For \( x > 0 \), let

\[
\mathcal{N}_F(x) := |\{ D \text{ fundamental } | |D| < x, (D, N) = 1, L(F, D, k) \neq 0 \}|.
\]

A based on the Goldfeld’s conjecture one expects to have \( \mathcal{N}_F(x) \gg x \ (x \to \infty) \). Kohnen [10] showed that if \( k \geq 6 \) is a even integer, then for \( x \gg 0 \) there is a normalized Hecke eigenform \( F \) of level 1 and weight \( 2k \) with the property that

\[
\mathcal{N}_F(x) \gg_k x \ (x \to \infty).
\]

In this paper, we extend the result in [10] to the case when \( k \) is any integer, in particular when \( k \) is odd. So, we obtain that, when the level is 1, for each integer \( 2k \) such that the dimension of cusp forms of weight \( 2k \) is not zero, there is a normalized Hecke eigenform \( F \) of weight \( 2k \) satisfying \( \mathcal{N}_F(x) \gg x \ (x \to \infty) \).

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1. Introduction and statement of result

Let \( F(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{2k}(N, \chi_0) \) be a newform of weight \( 2k \) and level \( N \) with a trivial character \( \chi_0 \), and assume that \( F(z) \) is a non-zero eigenform of all Hecke operators. For a fundamental discriminant \( D \), that is \( D = 1 \) or the discriminant of a quadratic field, we define the \( L \)-function \( L(F, D, s) \) of \( F \) twisted with the quadratic character \( (\frac{D}{\cdot}) \) by

\[
\sum_{n \geq 1} \left( \frac{D}{n} \right) \frac{a(n)}{n^s}.
\]

In this paper, we consider the central values of \( L \)-functions \( L(F, D, k) \).

It is well-known by Waldspurger [18] that the central critical values \( L(F, D, k) \) are essentially proportional to the squares of Fourier coefficients of the modular form of weight \( k + \frac{1}{2} \) corresponding to \( F \) under Shimura correspondence.

On the other hand, for \( x > 0 \), consider the set

\[
\mathcal{N}_F(x) := |\{ D \text{ fundamental } | |D| < x, (D, N) = 1, L(F, D, k) \neq 0 \}|.
\]

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A based on the Goldfeld’s conjecture one expected to have
\begin{equation}
\mathcal{N}_F(x) \gg x \quad (x \to \infty)
\end{equation}

(In \cite{5} Goldfeld conjectured that
\[
\sum_{D \text{ fundamental}, \ |D| < x, \gcd(D,N) = 1} \text{ord}_{s=k} L(F,D,s) \\
\approx \frac{1}{2} |\{D \text{ fundamental} \ | \ |D| < x, \gcd(D,N) = 1\}| \quad (x \to \infty).
\]

In \cite{14} using the theory of Galois representations of modular forms together with the results of Friedberg and Hoffstein \cite{4}, the following was proved
\[
\mathcal{N}_F(x) \gg \frac{x}{\log x} \quad (x \to \infty).
\]

In \cite{7} K. James gave an example of a form $F$, by looking at the difference of two special ternary theta series of level 56, satisfying $\mathcal{N}_F(x) \gg x \quad (x \to \infty)$. Furthermore, Kohnen \cite{10} showed that if $k \geq 6$ is an even integer, then there is a normalized Hecke eigenform $F$ of level 1 and weight $2k$ with the property that $\mathcal{N}_F(x) \gg k \cdot x \quad (x \gg 0)$; in particular, it was shown that $\mathcal{N}_\Delta(x) \gg x \quad (x \to \infty)$ where $\Delta$ is Ramanujan’s $\Delta$-function of weight 12. More precisely, Kohnen proved that if $N^+_{k,\Gamma_1}(x)$ is the number of fundamental discriminant $D$, $0 < D < x$, such that there exists a normalized Hecke eigenform $F \in S_{2k}$ satisfying $L(F,D,k) \neq 0$, then for any positive $\epsilon$
\[
N^+_{k,\Gamma_1}(x) \geq \left( \frac{9}{16\pi^2} - \epsilon \right) x \quad (x \gg 0).
\]

Here, $S_{2k}$ is the space of cusp forms of weight 2$k$ on $\Gamma_1 = SL(2,\mathbb{Z})$. In this paper, we extend the result \cite{10} to the case when $k$ is any integer, in particular when $k$ is odd. To obtain these results, we refine the argument of Kohnen by using an isomorphism \cite{8} from the spaces of modular forms of integral weight to the Kohnen plus space of half integral weight modular forms.

Let $N_{k,\Gamma_1}(x)$ denote the number of fundamental discriminants $D$, $|D| < x$, such that there exists a normalized Hecke eigenform $F \in S_{2k}$ satisfying $L(F,D,k) \neq 0$, then we have the following:

**Theorem 1.1.** Suppose that $k \geq 9$ is odd. Then for any positive $\epsilon$

\[
N_{k,\Gamma_1}(x) \geq \left( \frac{9}{16\pi^2} - \epsilon \right) x \quad (x \gg 0).
\]

Let $d_k := \dim(S_{2k})$. With the result of Kohnen \cite{10} in which covers the case of $k \equiv 0 \pmod{2}$, we state the following result:
Theorem 1.2. Suppose that $k$ is a positive integer such that $d_k \geq 1$. Then for any $\epsilon > 0$

$$\mathcal{N}_{k,1}(x) \geq \left(\frac{9}{16\pi^2} - \epsilon\right) x \ (x \gg \epsilon 0).$$

Theorem 1.2 immediately implies that for each integer $k$ such that $d_k > 0$ there is a Hecke eigenform $F \in S_{2k}$ satisfying (1.1).

Corollary 1.3. Suppose that $k$ is an integer such that $d_k \geq 1$. Then there exists a Hecke eigenform $F(z) \in S_{2k}$ such that for any positive $\epsilon$

$$\mathcal{N}_F(x) \geq \frac{1}{d_k} \left(\frac{9}{16\pi^2} - \epsilon\right) x \ (x \gg \epsilon 0).$$

Recently, Farmer and James [3] proved that the characteristic polynomial of the Hecke operator $T_2$ on $S_{2k}$ is irreducible over $\mathbb{Q}$ for $k \leq 1,000$. If $K$ is the field obtained from $\mathbb{Q}$ by adjoining the eigenvalues of $T_2$, then the Galois group $G = Gal(K/\mathbb{Q})$ operates transitively on the set of normalized Hecke eigenforms in $S_{2k}$. Using Theorem 1.1 and the known fact [15] that $L(F^\sigma, D, k)_{alg} = L(F, D, k)_{alg}$ for all $\sigma \in G$, we have that every Hecke eigenform $F \in S_{2k}$ satisfies (1.1) for each integers $k, 6 \leq k \leq 1,000$. Here, “$alg$” means “algebraic part”.

Theorem 1.4. Suppose that $6 \leq k \leq 1,000$ is an integer. Then every normalized Hecke eigenform $F$ in $S_{2k}$ satisfies

$$\mathcal{N}_F(x) \geq \left(\frac{9}{16\pi^2} - \epsilon\right) x \ (x \gg \epsilon 0).$$

Remark 1.5. Maeda ([6] Conjecture 1.2) made a conjecture that the Hecke algebra of $S_{2k}$ over $\mathbb{Q}$ is simple, and that its Galois closure over $\mathbb{Q}$ has Galois group $G$ the full symmetric group. The conjecture implies that there is a single Galois orbit of Hecke eigenforms in $S_{2k}$. Thus, Maeda’s conjecture implies that every normalized Hecke eigenform $F$ in $S_{2k}$ satisfies (1.2).

2. Preliminaries

Let $q := e^{2\pi iz}$, where $z$ is in the complex upper half plane $\mathbb{H}$. For an integer $k \geq 2$ recall the normalized Eisenstein series $E_{2k}(z) := 1 - \frac{4k}{B_{2k}} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$ of weight $2k$, and for an integer $r \geq 1$ let $H_{r+\frac{1}{2}}(z) = \sum_{N=0}^{\infty} H(r, N) q^N$ be the Cohen-Eisenstein series of weight $r + \frac{1}{2}$ on $\Gamma_0(4)$ (see [1]). Here, for each positive integer $N$, define

$$h(r, N) = \begin{cases} (-1)^{\lfloor \frac{r}{2} \rfloor} (r-1)! N^{r-\frac{1}{2}} 2^{1-r} \pi^{-r} L(r, \chi_{(-1)^r} N) & \text{if } (-1)^r N \equiv 0 \text{ or } 1 \pmod{4}, \\
0 & \text{if } (-1)^r N \equiv 2 \text{ or } 3 \pmod{4}, \end{cases}$$

where $L(s, \chi)$ is the normalized $L$-function of a Dirichlet character $\chi$. The function $h(r, N)$ is used to define the Hecke operators on $S_{2k}$.
where $\chi_D$ denotes the character $\chi_D(d) = (\frac{D}{d})$. Furthermore, for $N \geq 1$, define

$$H(r, N) = \begin{cases} 
\sum_{d | N} h(r, \frac{N}{d}) & \text{if } (-1)^r N \equiv 0 \text{ or } 1 \pmod{4}, \\
\zeta(1 - 2r) & \text{if } N = 0, \\
0 & \text{otherwise.}
\end{cases}$$

The followings are proved in [1]:

**Proposition 2.1.** (1) For $r \geq 2$, Cohen-Eisenstein series $H_{r+1/2}(z)$ is a modular form of weight $r + \frac{1}{2}$ on $\Gamma_0(4)$ and it is in Kohnen plus condition, that is,

$$H(r, N) = 0 \text{ if } (-1)^r N \not\equiv 0, 1 \pmod{4}.$$ 

(2) Let $a$ and $b$ be integers with $a \geq 1$. Suppose that $-b$ is a quadratic non residue of $a$. Then the function

$$G_{a,b}(z) := \sum_{N \equiv b \pmod{a}} H(1, n)q^n$$

is a modular form of weight $\frac{3}{2}$ and character $a$ over $\Gamma_0(A)$, where we can take $A = 4a^2$, and furthermore $A = a^2$ if $a$ is even.

For a nonnegative integer $k$ denote $M_{k+1/2}(\Gamma_0(4))$ as the usual complex vector space of cusp forms of weight $k + \frac{1}{2}$ on $\Gamma_0(4)$ with the trivial character. In [8], Kohnen introduces the plus space $M_{k+1}^+(\Gamma_0(4))$ of modular forms $g(z)$ of weight $k + \frac{1}{2}$ on $\Gamma_0(4)$ with a Fourier expansion of the form

$$g(z) = \sum_{(-1)^k n \equiv 0, 1 \pmod{4}} c(n)q^n.$$

and proved the following isomorphism (Proposition 1 in [8]).

**Proposition 2.2.** Let $M_k$ be the space of modular forms of weight $k$ on $\Gamma_1$. If $k$ is even, then the spaces $M_k \bigoplus M_{k-2}$ and $M_{k+1}^+(\Gamma_0(4))$ are isomorphic under the map

$$(f(z), h(z)) \mapsto f(4z)\theta(z) + h(4z)H_{1/2}(z),$$

where $\theta(z) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$. If $k$ is odd, then the spaces $M_{k-3} \bigoplus M_{k-5}$ and $M_{k+1}^+(\Gamma_0(4))$ are isomorphic under the map

$$(f(z), h(z)) \mapsto f(4z)H_{3/2}(\tau) + h(4z)H_{11/2}(z).$$

For $k \geq 2$ we have $M_{k+1}^+(\Gamma_0(4)) = \mathbb{C}H_{k+1/2} \bigoplus S_{k+1}^+(\Gamma_0(4)).$

The results of [11], [9], and [18] connect the coefficients of Hecke eigenforms of half-integral weight to the central $L$-values of twists of integral weight Hecke eigenforms. More precisely, suppose that $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{2k}$ is a normalized Hecke eigenform and that $g(z) = \sum_{n=1}^{\infty} c(n)q^n \in S_{k+1}^+(\Gamma_0(4))$ is a Hecke eigenform with the same Hecke eigenvalues as those of $f$. Here,
Theorem 1 of [11] states the following.

**Theorem 2.3.** Suppose that $f$ and $g$ are as above, $D$ is a fundamental discriminant with $(-1)^k D > 0$, and $L(f, D, s)$ is the twisted $L$-series

$$L(f, D, s) = \sum_{n=1}^{\infty} \left( \frac{D}{n} \right) a(n) n^{-s}.$$ 

Then

$$(g, g)_D = \left( \frac{k-1)!}{\pi^k} \frac{|D|^{k-1/2}}{|D|^{k-1}} L(f, D, k) \langle f, f \rangle.$$ 

Here, $(g, g)$ and $(f, f)$ are the normalized Petersson scalar products

$$(g, g) = \frac{1}{6} \int_{\Gamma_0(4) \backslash \mathbb{H}} |g(z)|^2 y^{k-3/2} dx \, dy$$

$$(f, f) = \int_{\Gamma_1 \backslash \mathbb{H}} |f(z)|^2 y^{2k-2} dx \, dy.$$ 

### 3. Proof of Theorem 1.1

For any function $f(z)$ on $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ and any positive integer $d$ we define the operator $U_d$

$$\langle f \mid U_d \rangle(z) := \frac{1}{d} \sum_{j=0}^{d-1} f \left( \frac{z + j}{d} \right).$$

Suppose that $f$ has a Fourier expansion $f(z) = \sum a(n) q^n$. Then $\langle f \mid U_d \rangle(z) = \sum a(nd) q^n$. If $g(z) = \sum_{n=0}^{\infty} c(n) q^n$ is in $M_{k+\frac{1}{2}}(\Gamma_0(4N))$, then the Hecke operator with the trivial character on $M_{k+\frac{1}{2}}(\Gamma_0(4N))$ is defined for odd primes $\ell$ by

$$\langle g \mid T(\ell^2, k) \rangle(z) := \sum_{n=0}^{\infty} \left( c(\ell^2 n) + \ell^{k-1} \left( \frac{-1}{\ell^2} \right) c(n) \right) + \left( \frac{-1}{\ell^2} \right) \ell^{2k-1} c \left( \frac{n}{\ell^2} \right) \right) q^n,$$

where $\left( \frac{\ell}{\ell^2} \right)$ and $\left( \frac{n}{\ell^2} \right)$ are Jacobi symbols, and $c \left( \frac{n}{\ell^2} \right) := 0$ if $\ell^2 \nmid n$. If $g$ has integral coefficients, then one also has that

$$(g \mid U_\ell \equiv g^\ell \mid T(\ell^2, \ell k + \frac{(\ell - 1)}{2}) \pmod{\ell}).$$

For any positive integer $d$ we define the operator $V_d$

$$(g \mid V_d)(z) := \sum_{n=0}^{\infty} c(n) q^{dn}.$$
Note that if $\ell$ is a prime, then

\[(g|V_\ell)(z) \equiv g(z)^\ell \pmod{\ell}.\]

For a Dirichlet character $\chi$ let

\[g \otimes \chi := \sum_{n=0}^{\infty} \chi(n)c(n)q^n.\]

The following proposition immediately implies our main theorems.

**Proposition 3.1.** Suppose that $k \geq 8$ is an integer.

1. If $k$ is odd, then for any positive $\epsilon$

\[N_{k,\Gamma_1}(x) \geq \left( \frac{9}{16\pi^2} - \epsilon \right) x \quad (x \gg \epsilon).\]

2. If $k$ is an even integer such that $d_k > 1$ or $k = 10$, then for any positive $\epsilon$

\[N_{k,\Gamma_1}(x) \geq \left( \frac{9}{16\pi^2} - \epsilon \right) x \quad (x \gg \epsilon).\]

**Proof** For a positive even positive integer $t$, let

\[m(t) := \frac{1}{2} \left( t - 4 \left[ \frac{t}{4} \right] \right).\]

For a non negative integer $t$ we define a modular form $R_t(z)$:

\[R_t(z) := \begin{cases} E_4(4z)^{\left[\frac{t}{4}\right]-m(t)}E_6(4z)^{m(t)} & \text{if } t > 0, \\ 1 & \text{if } t = 0. \end{cases}\]

For any even positive integer $t$, we have

\[(3.5) \quad R_t(z) \equiv 1 \pmod{3}.\]

This is from the fact that if $k \geq 4$ is an even integer, then $E_{p-1}(z) \equiv 1 \pmod{p}$ for any prime $p$ such that $k \equiv 0 \pmod{p-1}$ (see [12]).

First, we assume that $k$ is odd. For each odd integers $k \geq 9$ define

\[(3.6) \quad \Phi_{k+\frac{1}{2}}(z) = 28H_{3+\frac{1}{2}}(z)R_{k-3}(z) - \frac{44}{3}H_{5+\frac{1}{2}}(z)R_{k-5}(z) := \sum_{n=1}^{\infty} \beta_k(n)q^n.\]

Then $\Phi_{k+\frac{1}{2}}(z)$ is in $S_{k+\frac{1}{2}}(\Gamma_0(4))$ by Proposition [22] and the Fourier coefficients of $\Phi_{k+\frac{1}{2}}(z)$ are 3-integral. On the other hand, for every odd $k \geq 9$, we have

\[\Phi_{k+\frac{1}{2}}(z) \equiv \Phi_{9+\frac{1}{2}}(z) := 28H_{3+\frac{1}{2}}(z)R_6(z) - \frac{44}{3}H_{5+\frac{1}{2}}(z)R_4(z) \pmod{3}.\]
Let
\[
F(z) := (\Phi_{g+\frac{1}{2}}(z) - (\Phi_{g+\frac{1}{2}}(z)|U_3|V_3)) + (\Phi_{g+\frac{1}{2}}(z) - (\Phi_{g+\frac{1}{2}}(z)|U_3|V_3)) \otimes \left( \frac{1}{3} \right)
\]
\[= \sum_{n \equiv 1 \pmod{3}} \beta_9(n)q^n.
\]

Recall that
\[G_{3,1}(z) = \sum_{n \equiv 1 \pmod{3}} H(1, n)q^n.
\]

Proposition 2.1 implies that \(G_{3,1}(z) = \sum_{n \equiv 1 \pmod{3}} H(1, n)q^n\) is a modular form of weight \(\frac{3}{2}\) on \(\Gamma_0(36)\) such that its coefficients are 3-integral.

Thus, by computing a few coefficient modulo 3 and using Sturm’s bound in (16) we have
\[
F(z) \equiv \sum_{n \equiv 1 \pmod{3}} \beta_9(n)q^n \\
\equiv 2q^4 + q^7 + q^{19} + 2q^{28} + 2q^{40} + q^{43} + 2q^{49} + q^{52} + \cdots \\
\equiv \sum_{n \equiv 1 \pmod{3}} H(1, n)q^n \equiv G_{3,1}(z) \pmod{3}.
\]

On the other hand, let \(h(D) = H(1, D)\) be the class number of \(\mathbb{Q}(\sqrt{D})\). It is known that for \(D < 0\), \(h(D) = -B_{1,1}(\frac{D}{4})\) (apart from \(D = -3\) and \(-4\)) (for example, see [17]). Thus, we have
\[
\beta_k(D) \equiv B_{1,1}(\frac{-D}{4}) \equiv h(-D) \pmod{3},
\]
for a fundamental discriminant \(D > 1\) such that \(D \equiv 1 \pmod{3}\).

Now let \(m\) and \(N\) be positive integers satisfying the condition:

\[**\quad \text{If an odd prime } p \text{ is a common divisor of } m \text{ and } N, \text{ then } p \mid N \text{ and } p^2 \nmid m. \text{ Further if } N \text{ is even, then } (i) \ 4 \mid N \text{ and } m \equiv 1 \pmod{4} \text{ or } (ii) \ 16 \mid N \text{ and } m \equiv 8 \text{ or } 12 \pmod{16}.
\]

We denote by \(N_2^-(x, m, N)\) the number of fundamental discriminants \(D\) with \(-x < D < 0\) and \(D \equiv m \pmod{N}\). The results of Davenport-Heilbronn [2] and Nakagawa-Horie [13] imply that for any positive number \(\epsilon\)
\[
|\{ \text{fundamental discriminants } D \equiv 1 \pmod{3} \mid 0 < D < x \text{ and } 3 \mid h(-D)\}| \\
\gg (\frac{1}{2} - \epsilon) (N_2^-(x, 1, 3)).
\]
Since \(N_2^-(x, 1, 3) \sim \frac{9}{8\pi}x\) for \(x \to \infty\) (see Proposition 2. in [13]), for odd integers \(k \geq 7\) we have

\[
(3.9) \quad |\{ \text{fundamental discriminants } D \mid 0 < D < x \text{ and } 3 \nmid \beta_k(D) \}| \gg \left(\frac{9}{16\pi^2} - \epsilon\right)x.
\]

Since \(\Phi_{k+\frac{1}{2}}(z) \in S_{k+\frac{1}{2}}^+(\Gamma_0(4))\), the cusp form \(\Phi_{k+\frac{1}{2}}(z)\) is a linear combination of Hecke eigenforms \(g_\ell(z) = \sum_{n=1}^{\infty} c_\ell(n)q^n \in S_{k+\frac{1}{2}}^+(\Gamma_0(4))\) for \(1 \leq \ell \leq d_k\). By Theorem 2.3 we complete the proof of Theorem.

From now on, suppose that \(k\) is even, and that \(d_k > 1\) or \(k = 10\). Let

\[
(3.10) \quad \Psi_{k+\frac{1}{2}}(z) := \Delta(4z)R_{k-12}(z)\theta(z) = \sum_{n=1}^{\infty} \alpha_k(n)q^n \quad \text{for } k > 10
\]

and

\[
(3.11) \quad \Psi_{10+\frac{1}{2}}(z) := -(\theta(z)E_4(4z)E_6(4z) - H_{\frac{1}{2}}(4z)^2) \otimes \chi_3 + (\theta(z)E_4(4z)E_6(4z) - H_{\frac{1}{2}}(4z)^2) \otimes \chi_3^2,
\]

where \(\chi_3(n) = \left(\frac{n}{3}\right)\). We have by the Sturm’s bound and (3.5)

\[
\sum_{n \equiv 2 \pmod{3}} \alpha_k(n)q^n \equiv \sum_{n \equiv 2 \pmod{3}} \alpha_{12}(n)q^n \equiv 2q^8 + 2q^{17} + q^{20} + 2q^{41} + q^{44} + q^{53} + q^{56} + 2q^{68} + 2q^{80} + 2q^{89} + 2q^{92} \cdots 
\]

\[
\equiv \sum_{n \equiv 2 \pmod{3}} H(1, 3n)q^n \quad \pmod{3}.
\]

The remained part of the proof can be completed in a similar way as before, so we omit the details. \(\square\)

**Remark 3.2.** The argument given in [10], p. 186 bottom] in the case where \(k\) is even and \(k \equiv 1 \pmod{3}\) is not correct, since it would require that all the coefficients of \(\delta_{k-4}\) are 3-integral which in general is not the case.

## 4. Conclusion

In this paper, we extend the result in [10] to the case when \(k\) is any integer, in particular when \(k\) is odd. So, we obtain that, for each integer \(2k\) such that \(\dim S_{2k} \geq 1\), there is a normalized Hecke eigenform \(F\) in \(S_{2k}\) satisfying \(N_F(x) \gg x \quad (x \to \infty)\). We conclude this paper with the following remark:
**Remark 4.1.**

1. For each odd \( k \geq 9 \) and even \( \lambda \) such that \( \lambda \geq 12 \), all the coefficients of \( \Phi_{k+\frac{1}{2}} \) and \( \Psi_{\lambda+\frac{1}{2}} \) are 3-integral and a positive portion of these coefficients \( \beta_k(n) \) and \( \alpha_\lambda(n) \) is not vanishing modulo 3.

2. Note that \( k = 9 \) is the minimum odd integer such that \( \dim(S_{2k}) > 0 \). Let

\[
\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.
\]

If we take \( F(z) = \Delta(z)E_6(z) \), then \( F(z) \) is the unique normalized Hecke eigenform in \( S_{18} \). Corollary[1.3] implies that

\[
\mathcal{N}_F(x) \geq \left( \frac{9}{16\pi^2} - \epsilon \right) x \quad (x \gg \epsilon 0).
\]

3. The direct computation shows that if \( f \in S_{k+\frac{1}{2}}^+(\Gamma_0(4)) \) has integral coefficients for even \( k \) such that \( d_k = 1 \), then

\[
f \equiv c \sum_{n \geq 1 \atop \not\equiv 3m} q^{n^2} \quad (mod \ 3)
\]

for some \( c \in \{-1, 1\} \).

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