Algorithms of generating cuspidal edges of developable surfaces

N V Kovaleva, A V Fedorova, D A Pashyan
Department "Engineering geometry and computer graphics", Don State Technical University, 1 Gagarin sq., Rostov-on-Don 344000, Russia
E-mail: avf_belka@mail.ru

Abstract. Ruled surfaces (torses) are widely used when designing surfaces of machine parts, architectural projects, building structures, plow-bottom surfaces, etc. A ruled surface is generated by moving a straight line continuously in a straight line space in predetermined manner. Otherwise, a ruled surface is a one-parameter set of lines. Ruled surfaces are handy for practical application, since they can be manufactured from rectilinear elements widely provided by modern industry. All ruled surfaces are in divided into two types: developable and nondevelopable. Torses can be superposed on a plane by means of flexural deformation without strain and compression. This property of developable surfaces makes possible to easily produce them from flat materials. It is common knowledge that a torse is a surface which represents a set of lines being in contact with a given spatial line. A line which is tangent to all generatrices is called a cuspidal edge of the given developable surface. Thus, creation of spatial lines is a main point in designing developable surfaces. The article deals with algorithms of generating spatial lines by means of rolling machines for cylinders, cones and generalized torses that can be used hereafter as cuspidal edges when designing developable surfaces.

1. Introduction
Developable surfaces have become commonly used when designing surfaces of machine parts, building structures, architectural elements [1-6]. It is attributed to the easy manufacturing of such surfaces from flat materials as torses are surfaces which can be developed. Besides, developable surfaces are highly diverse, make possible their analytical description, and therefore algorithmization of their designing in computer graphics systems [7-12].

One of the main methods of generating developable surfaces is constructing of spatial lines[13-15] - cuspidal edges (edges of regression) of torses. This article gives a technique of determining generated spatial curves in some specific solutions. The technique is considered for cases of rolling cylinders, cones and general developable surfaces.

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2. Method of determining curves on cylindrical surfaces
For example, let the guide curve of a cylindrical surface be a spatial curve defined in the cylindrical coordinate system by equation [16-20]:
The direction of the generatrices is determined by the directing vector \( L \) with the direction cosines \( l_0, m_0, n_0 \).

Draw through the origin of coordinates a plane orthogonal to rectilinear generatrices of the cylindrical surface. The equation of such a plane will be as follows:

\[
l_0x + m_0y + n_0z = 0
\]

The generatrices of the cylindrical surface are defined by equations:

\[
\frac{x-x_0}{l_0} = \frac{y-y_A}{m_0} = \frac{z-z_A}{n_0},
\]

where \( x_0, y_A, z_A \) are coordinates of the current point of the curve defined by equation (1). From these equations it is possible to identify:

\[
y = \frac{m_0 \cdot (x-x_0)}{l_0} + y_A; \quad z = \frac{n_0 \cdot (x-x_0)}{l_0} + z_A
\]

Substitution of relations (4) in (2) will allow us to define parametric equations of the cylindrical surface normal section. They take the form:

\[
\begin{align*}
x_{ii} & = x_A - l_0 \cdot (l_0 x_A + m_0 y_A + n_0 z_A) ; \\
y_{ii} & = y_A - m_0 \cdot (l_0 x_A + m_0 y_A + n_0 z_A) ; \\
z_{ii} & = z_A - n_0 \cdot (l_0 x_A + m_0 y_A + n_0 z_A)
\end{align*}
\]

Develop the cylindrical surface into a flat system \( Oxz \) so that the initial generatrix coincides with the axis \( Oz \). Then the generatrices in development form an improper pencil of lines parallel to the axis \( Oz \) with equation:

\[
\begin{align*}
-x_i = s_i \int_0^\alpha \sqrt{(x_{ii})^2 + (y_{ii})^2 + (z_{ii})^2} \cdot d\omega
\end{align*}
\]

If a curve is given in the plane \( Oxz \) by equation:

\[
F(x, z) = 0,
\]

the value \( d = \overline{z} \) can be determined by simultaneous solution of (6) and (7). This value is equal to the distance from the curve points (7) to the points of the normal section. If the solution is multivalued, we can enter a logical sample \( d > 0 \) and/or \( d = \min \).

Returning to the cylinder formed in \( Oxyz \), we can get parametric equations of the curve \( r \) on its surface:

\[
\begin{align*}
x_r & = x_A - l_0 \cdot (l_0 x_A + m_0 y_A + n_0 z_A - d) ; \\
y_r & = y_A - m_0 \cdot (l_0 x_A + m_0 y_A + n_0 z_A - d) ; \\
z_r & = z_A - n_0 \cdot (l_0 x_A + m_0 y_A + n_0 z_A - d)
\end{align*}
\]

For the obtained curve, we can define a discrete continuous frame for the generatrices of developable surface so that the curve will be a cuspidal edge for them.

Parametric equations of the generatrices will take the form:
From knowing the equations of the curve and of the discrete continuous frame for the generatrices of the ruled surface, it is possible to determine arbitrary bending, including flat bending. Equations (8) and (9) allow determining the differential and geometric characteristics of the curve: curvature, torsion, evolute, etc.

As an example, let us consider the simplest case when a right circular cylinder rolls along a circumference. The equation of the cylinder is as follows:

\[ x = a \cos \varphi, \quad y = a \sin \varphi, \quad z = 0, \quad (10) \]

where \( a \) is the circumference radius of a guide curve; \( \varphi \) is the rotation angle of the radius-vector, measured from axis \( O_x \) (Figure 1).

This equation is an equation of the normal section. In the plane \( Oxz \) select a circumference with equation \((x - \pi a)^2 + z^2 = \pi^2 a^2\). By solving simultaneously the circumference equation with the equation of pencil of lines parallel to the axis \( Oz \) we will get an expression to determine the distance from the points to the normal section: \( d = z = \sqrt{2\pi a^3 \cdot \varphi - a^2 \varphi_i^2} \). As a result, the parametric equations of the curve \( r \) will be as follows: \( x_i = a \cos \varphi_i, \quad y_i = a \sin \varphi_i, \quad z_i = a \sqrt{2\pi \varphi - \varphi_i^2} \).

3. Method of determining curves on conical surfaces
Suppose a guide curve of the conical surfaces is a curve determined by equation (1). The position of the conical surface center \( T \) shall be determined by the coordinates \( x_T, y_T, z_T \). The generatrices of the conical surface are defined by equations:

\[
\frac{x - x_T}{x_{A} - x_T} = \frac{y - y_T}{y_{A} - y_T} = \frac{z - z_T}{z_{A} - z_T},
\]

(11)
where \( x, y, z \) are determined from equation (1).

The direction cosines of the generatrices are defined by expressions:

\[
\begin{align*}
\alpha &= \frac{x_d - x_f}{\sqrt{(x_d - x_f)^2 + (y_d - y_f)^2 + (z_d - z_f)^2}}; \\
\beta &= \frac{y_d - y_f}{\sqrt{(x_d - x_f)^2 + (y_d - y_f)^2 + (z_d - z_f)^2}}; \\
\gamma &= \frac{z_d - z_f}{\sqrt{(x_d - x_f)^2 + (y_d - y_f)^2 + (z_d - z_f)^2}};
\end{align*}
\]

Let us assign the unit radius sphere with the center at the point \( T \) determined by the following equation:

\[
(x - x_f)^2 + (y - y_f)^2 + (z - z_f)^2 = 1 \tag{13}
\]

To determine a parametric equation of the curve \( c \), solve simultaneously generatrices in the intersection of the conical surface with the sphere: (11) and (13).

The parametric equations of the curve will be equal to:

\[
\begin{align*}
x &= x_f + \alpha l; \\
y &= y_f + \beta m; \\
z &= z_f + \gamma n
\end{align*}
\]

Let us introduce a spherical coordinate system with the coordinating angles determined by relations:

\[
\begin{align*}
\cos \phi &= n_0; \\
\gamma &= \arccos \frac{l_0}{\sqrt{l_0^2 + m_0^2}} \tag{15}
\end{align*}
\]

The arc length of the curve \( c \) on the surface of the sphere defines the angle \( \varphi \) formed by the generatrices of the conical surface in development, measured from a certain axis. This value is determined by expression:

\[
\varphi = \int_0^\alpha \sqrt{\sin^2 \varphi \left( \frac{dU}{d\omega} \right)^2 + \left( \frac{dV}{d\omega} \right)^2} \ d\omega \tag{16}
\]

If the center of the unfolded conical surface is placed at the origin of the coordinate system \( \bar{O}xz \) and the initial generatrix is combined with axis \( Ox \), the equations of the generatrices will be as follows:

\[
\bar{z} = t g \varphi \cdot x \tag{17}
\]

Let the plane \( \bar{O}xz \) have a plane curve \( \bar{k} \) defined by equation:

\[
F(x, \bar{z}) = 0 \tag{18}
\]

Solving simultaneously (17) and (18), we get intersection points of the generatrices in development with specified curve - \( x_k \) and \( z_k \). The distance from the origin of coordinates for the intersection
points will be determined by value: \( d = \sqrt{(\bar{x})^2 + (\bar{z})^2} \). Then, the parametric equations of the curve \( k \) on the conical surface are defined by expressions:

\[
x_k = x_0 + dl_0; \\
y_k = y_0 + dm_0; \\
z_k = z_0 + dn_0
\]

Further actions with equations (19) are similar to those given in the previous section.

For example, consider rolling a right circular cone along a plane curve—a circumference. The cone is defined by a plane guide (Figure 2) lying in the plane \( xOy \) and having equations:

\[
x = r \cdot \cos \gamma; \quad y = r \cdot \sin \gamma.
\]

The vertex of the cone is placed at the point \( T (0, 0, z_T) \), so it is located on the axis \( Oz \). As you know, the development of such a cone will represent a sector of the circle with radius \( R = \sqrt{a^2 + z_T^2} \).

![Figure 2. Line on the surface of the cone.](image)

4. Method of determining curves on developable surfaces

Let the edge of developable surface be given in formula (1). The tangents to the developable surface are determined by equations:

\[
\frac{x-x_A}{x'_A} = \frac{y-y_A}{y'_A} = \frac{z-z_A}{z'_A},
\]

where \( x_A, y_A, z_A \) are coordinates of the point on the curve defined by equation (1); \( x'_A, y'_A, z'_A \) are derivatives of the parameter, calculated at the point \( A \).

The direction cosines of the torse rectilinear generatrices are defined by expressions:
Coordinating angles in the spherical coordinate system for the director cone are determined by relations: \( \cos V = n_r \); \( U = \arccos \frac{l_r}{\sqrt{l_r^2 + m_r^2}} \). The equations of cuspidal edge of unfolded developable surface are defined by expressions:

\[
x_p = \int_0^\pi \left( \frac{ds}{d\omega} \right) \cdot \cos \left( \int_0^\pi \sqrt{\cos^2 V \left( \frac{dU}{d\omega} \right)^2 + \left( \frac{dV}{d\omega} \right)^2} \, d\omega \right) \, d\omega,
\]

\[
y_p = \int_0^\pi \left( \frac{ds}{d\omega} \right) \cdot \sin \left( \int_0^\pi \sqrt{\cos^2 V \left( \frac{dU}{d\omega} \right)^2 + \left( \frac{dV}{d\omega} \right)^2} \, d\omega \right) \, d\omega,
\]

where \( \frac{ds}{d\omega} = \sqrt{\left( \frac{dx}{d\omega} \right)^2 + \left( \frac{dy}{d\omega} \right)^2 + \left( \frac{dz}{d\omega} \right)^2} \).

The equation for the generatrices of unfolded developable surface will be as follows:

\[
\bar{y} = \tan \varphi (x - x_p) + y_p,
\]

where \( \varphi = \int_0^\pi \sqrt{\cos^2 V \left( \frac{dU}{d\omega} \right)^2 + \left( \frac{dV}{d\omega} \right)^2} \, d\omega \).

Let the plane \( Oxz \) have the plane curve \( \bar{k} \) defined by equations:

\[
\bar{x}_k = \bar{\rho} \cos \bar{\omega}, \quad \bar{z}_k = \bar{\rho} \sin \bar{\omega}
\]

Solving simultaneously (23) and (24), define \( x_{\bar{\rho}} \) and \( z_{\bar{\rho}} \). Then the distance value for the points of the curve is \( d = \sqrt{(x_p - x_{\bar{\rho}})^2 + (z_p - z_{\bar{\rho}})^2} \).

The curve equation on the spatial developable surface will be determined:

\[
x_{\bar{\rho} \bar{t}} = x_\rho + l_r \cdot d;
\]

\[
y_{\bar{\rho} \bar{t}} = y_\rho + m_r \cdot d;
\]

\[
z_{\bar{\rho} \bar{t}} = z_\rho + n_r \cdot d
\]

The obtained equations can be used to determine all the differential and geometric properties of the curve and can be included as cuspidal edges for developable surfaces that are used to compose axode pairs.

Figure 3 shows an example of drawing the curve \( k \) obtained by rolling the developed helicoid along the circumference \( k_0 \). For this purpose, a helix is drawn on a right circular cylinder with a normal section in the shape of a circumference with radius \( a \) and tangents to it are outlined.
The helicoid segment has been unfolded in the coordinate system $\mathbf{xOy}$, where the cuspidal edge $r$ represents the circumference $r_0$ (Figure 3).

At corresponding points $r_0$ tangents are constructed to it (rectilinear generatrices of unfolded developable surface) and is applied circumference $r_0$ with the center on the axis $\mathbf{Oy}$ and being tangent to $\mathbf{Ox}$. By fixing distances values from points $r_0$ to points of intersection of generatrices in development with $k_0$, we have transferred points of intersection onto the helicoid generatrices. The obtained curve can be used as a cuspidal edge.

5. Conclusion
In this article a technique for constructing spatial curves when rolling any developable surfaces along arbitrary plane curves has been developed. The resulting spatial curves can be used as cuspidal edges of toruses, what is a new method to specify developable surfaces. The developed algorithms can be useful for generating CAD modules to form developable surfaces with the given edges of regression.

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