Rate-independent linear operators on various function spaces

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Abstract. It is shown that the only rate-independent continuous linear operators acting on the Banach space $L^1$ of functions defined on the real line, are scalar multiples of the identity operator; and that the only such operators, acting on the corresponding Hilbert space $L^2$, are linear combinations of the Hilbert transform and the identity.

Keywords: rate-independent operators, hysteresis, function spaces, affine invariance, Hilbert transform

1. Introduction

An input-output relationship, $K$, which pairs functions defined on the real line, $\mathbb{R}$, is said to be rate-independent if, whenever $g = Kf$, the function $t \mapsto g(at + b)$ is the output corresponding to the input $t \mapsto f(at + b)$ for every positive number $a$ and every real number $b$. The importance of this concept in systems with hysteresis was recognised a long time ago by M. A. Krasnosel’skii and his co-workers. An account of its role in their development of a mathematical theory of hysteresis operators between suitable function spaces can be found in [4].

It’s an easy matter to provide examples of non-linear rate-independent operators. For instance, if $p$ is any polynomial with real or complex coefficients, then the composition map $f \mapsto p \circ f$ takes the space of all functions $f$ defined on $\mathbb{R}$ to itself, and is rate-independent to boot. However, unless $p$ is a multiple of the identity, this map is plainly non-linear. On the other hand, interesting examples of linear operators which are not of this kind, and possess the property, are hard to come by.

Several years ago, I asked myself the question: What reasonably behaved non-zero linear integral-operators are rate-independent? I discovered to my surprise that, essentially, only multiples of the Hilbert transform fit the bill, and outlined a proof of this to Alexei Pokrovskii, who ever since has attached the footnote\(^1\) to several of his publications [5, 6]! This statement can also be found on the website http://euclid.ucc.ie/hysteresis (which is worth visiting, as it contains a good introduction to the theory of hysteresis).

The purpose of this article is to provide a proper context within which to substantiate this footnote. Now, along with elephants, integral operators are easily recognised on sight, but they are not so easy to describe! For this reason, we have chosen instead to consider bounded linear transformations that act on appropriate normed vector spaces of functions, and to single

\(^1\) We note, in passing, a result of F. Holland(a personal communication): the only linear integral operator satisfying the rate independent property is the Hilbert transform.
out those that are rate-independent. To emphasise that we are excluding rate-independent non-linear transformations from our treatment, we will refer to the former as affine-invariant operators, a terminology that will be introduced and developed in the next sections.

2. Groups of affine functions
With each point \( z = x + iy \) in the open right half-plane
\[
\mathbb{C}_+ = \{ z \in \mathbb{C} : x = \Re z > 0 \},
\]
of the complex plane \( \mathbb{C} \), we associate the function \( z : \mathbb{R} \to \mathbb{R} \) defined by
\[
z(t) = xt + y, \quad -\infty < t < \infty.
\]
(It’s for notational convenience that we use the same letter for the function \( z \) and the point \( z \). Hopefully, this won’t cause any confusion.) This defines a homeomorphism of \( \mathbb{R} \) onto \( \mathbb{R} \). The collection \( \{ z : z \in \mathbb{C}_+ \} \) of such mappings is a subgroup of the group, under the operation of composition, of all homeomorphisms sending \( \mathbb{R} \) to itself. We call the members of this subgroup affine homeomorphisms.

Each affine homeomorphism, \( z \), induces a linear mapping \( \tau_z \) on the vector space of functions defined on the real line as follows: if \( f : \mathbb{R} \to \mathbb{C} \), \( \tau_z f \) is defined to be the composition \( f \circ z \), so that, for \( z \in \mathbb{C}_+ \),
\[
(\tau_z f)(t) = f(z(t)) = f(xt + y), \quad -\infty < t < \infty.
\]
Clearly, \( \tau_z \) is linear:
\[
\tau_z(g + \lambda g) = \tau_z f + \lambda \tau_z g,
\]
for all functions \( f, g \), and all scalars \( \lambda \). We call \( \tau_z \) an affine transformation. Plainly, the family of \( \{ \tau_z : z \in \mathbb{C}_+ \} \) is a non-commutative group under composition of operators.

Note that, if we restrict \( z \) to the line \( x = 1 \), the resulting mappings generate the familiar commutative group of translation transformations indexed on \( \mathbb{R} \):
\[
(\tau_{(1,y)} f)(t) = f(t + y), \quad -\infty < t < \infty, \quad -\infty < y < \infty.
\]

3. Affine-invariant vector spaces
Definition 1 We call a vector space \( E \) of complex functions defined on \( \mathbb{R} \) affine-invariant if, for all \( z \in \mathbb{C}_+ \), \( \tau_z f \in E \) whenever \( f \in E \).

Of course, the vector space of complex functions defined on the real line is affine-invariant. But certain metrizable subspaces of this are also affine-invariant. In particular, if \( 0 < p \leq \infty \), the \( L^p \)-spaces are. (Recall that, if \( 0 < p < \infty \), \( L^p \) consists of Lebesgue measurable functions \( f \) for which
\[
\|f\|_p = \left( \int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{1/p} < \infty;
\]
while \( L^\infty \) is the space of bounded Lebesgue measurable functions \( f \) with
\[
\|f\|_\infty = \sup \{|f(t)| : -\infty < t < \infty \}.)
\]
Indeed, it’s easily verified that, if \( 0 < p \leq \infty \), then
\[
\|\tau_z f\|_p = x^{-1/p} \|f\|_p, \quad \forall f \in L^p, \forall z \in \mathbb{C}_+.
\]
In case $1 \leq p \leq \infty$, $L^p$ is a Banach space with norm $f \rightarrow ||f||_p$. In what ensues, only the Banach spaces $L^1$, $L^2$ and $L^\infty$ will be of major interest to us, and we observe especially that $L^2$ is a Hilbert space endowed with its usual inner-product

$$<f, g> = \int_{-\infty}^{\infty} f(t)g(t) \, dt,$$

so that

$$||f||_2 = \sqrt{<f, f>}, \forall f \in L^2.$$

Remark. The notion of affine invariance can be extended to any collection of functions defined on $\mathbb{R}$, as follows: A collection $X$ of functions defined on $\mathbb{R}$ is said to be affine-invariant if, for all $z \in \mathbb{C}^+$, $\tau_z f \in X$ whenever $f \in X$.

**Definition 2** Suppose $E$ is an affine-invariant function space on $\mathbb{R}$. We say that a linear transformation $A$ on $E$ to $E$ is an affine-invariant transformation if it commutes with all members of the family $\{\tau_z : z \in \mathbb{C}^+\}$, i.e., if

$$A\tau_z = \tau_z A, \forall z \in \mathbb{C}^+.$$ 

More generally, if $X, Y$ are two affine-invariant collections of functions on $\mathbb{R}$, and $K : X \rightarrow Y$, we say that $K$ is affine-invariant if it belongs to the commutant of the family $\{\tau_z : z \in \mathbb{C}^+\}$, i.e., commutes with all members of it. Thus, such mappings—in general non-linear—are what we previously described as rate-independent; but from now on we’ll adhere to the terminology just introduced.

The identity transformation on any vector space of functions defined on $\mathbb{R}$ is affine-invariant. Note too that an affine-invariant transformation $A$ commutes with translations:

$$A\tau_{(1,y)} = \tau_{(1,y)} A, \forall y \in \mathbb{R}.$$ 

The aim of this note is to characterise those affine-invariant transformations on the spaces $L^1$ and $L^2$, introduced above, that are, in addition, continuous; we refer to these as affine-invariant operators.

**4. Affine-invariant operators on $L^1$.**

**Theorem 1** Suppose $A$ is an affine-invariant operator on $L^1$. Then $A$ is a scalar multiple of the identity.

Proof. To begin with, since $A$ commutes with all translations, and belongs to $B(L^1)$, the normed vector space of continuous linear transformations on $L^1$ to itself, with the operator norm, there is a finite measure $\mu$ on $\mathbb{R}$ such that

$$Af = f * \mu, \forall f \in L^1,$$

where $*$ denotes convolution (see [7], p. 75). We claim that $\mu$ is concentrated at the origin. To see this, we apply the Fourier transform to the last identity, noting that, for $f \in L^1$, and all $z \in \mathbb{C}^+$, the Fourier transform of $\tau_z f$, evaluated at any real number $\omega$, is given by

$$\hat{(\tau_z f)}(\omega) = \int_{-\infty}^{\infty} f(z(t))e^{-i\omega t} \, dt = \int_{-\infty}^{\infty} f(xt + y)e^{-i\omega t} \, dt = 1/x e^{i\omega y} \hat{f}(\omega/x).$$
Bearing in mind that the Fourier transform of a convolution of an $L^1$-function and a complex measure is the product of their Fourier transforms, it follows that $\hat{f} * \mu = \hat{f} \hat{\mu}$, for all $f \in L^1$. Hence, for all $x > 0$, and all $f \in L^1$,

$$
\frac{1}{x} \hat{f}(\frac{\omega}{x}) \hat{\mu}(\omega) = (\tau_{(x,0)} f)(\omega) \hat{\mu}(\omega)
= (\tau_{(x,0)} f * \mu)(\omega)
= A(\tau_{(x,0)} f)(\omega)
= \tau_{(x,0)} (Af)(\omega)
= \tau_{(x,0)} (f * \mu)(\omega)
= \frac{1}{x} \hat{f} * \mu(\frac{\omega}{x})
= \frac{1}{x} \hat{f}(\frac{\omega}{x}) \hat{\mu}(\frac{\omega}{x}).
$$

Hence, for all $x > 0$ and all real $\omega$,

$$
\hat{\mu}(\omega) = \hat{\mu}(\frac{\omega}{x}).
$$

Fixing $\omega \neq 0$ and letting $x \to \infty$, we deduce that

$$
\hat{\mu}(\omega) = \lim_{x \to \infty} \frac{\omega}{x} \hat{\mu}(\frac{\omega}{x}) = \hat{\mu}(0),
$$

by continuity of $\hat{\mu}$. In other words, $\hat{\mu}$ is a constant $\lambda$, say, i.e.,

$$
\lambda = \int_{\mathbb{R}} e^{-i\omega t} d\mu, \forall \omega \in \mathbb{R}.
$$

Thus, $\mu$ is concentrated at the origin, i.e., $\mu(\mathbb{R}) = \mu(\{0\}) = \lambda$, and hence $Af = \lambda f$, $\forall f \in L^1$. This ends the proof.

5. The Hilbert transform

If $1 \leq p < \infty$, and $f \in L^p$, the limit

$$
\lim_{\delta \to 0^+} \frac{1}{\pi} \int_{|s-t| \geq \delta} \frac{f(t)}{s-t} dt
$$

exists and is finite for almost all real $s$ [8, 3]. The limit function is denoted by $\mathbb{H}f$, and called the Hilbert transform of $f$. Clearly, the correspondence between $f$ and $\mathbb{H}f$ is linear.

Moreover, arguing formally, we see that, if $x + iy = z \in \mathbb{C}_+$, then

$$
(\mathbb{H} \tau_z f)(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(xt + y)}{s - t} dt
= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(xt + y)}{xs + y - (xt + y)} d(xt + y)
= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{xs + y - t} dt
= \mathbb{H}f(xs + y)
= (\tau_z \mathbb{H}f)(s).
$$
In other words, \( \mathbb{H} \) is an affine-invariant linear transformation.

(Incidentally, the same heuristic argument may lead one to believe that the “convolution-type integral operator” that sends \( f \) to \( g \), where

\[
g(y) = \int_{-\infty}^{\infty} \frac{f(x)}{|y-x|} \, dx, \quad -\infty < y < \infty,
\]

is also of this kind. However, unlike the Hilbert transform, the latter may be infinite on a set of positive measure for functions \( f \) that belong to every \( L^p \)-space. For instance, if \( 0 \leq y \leq 1 \),

\[
\int_0^1 \frac{1}{|y-x|} \, dx = \infty.
\]

Hence, it is not an operator on such spaces in the accepted sense.)

Note, too, that \( \mathbb{H} \) is not a multiple of the identity—for example, the function \( s \rightarrow -s(1+s^2)^{-1} \) is the Hilbert transform of \( t \rightarrow (1+t^2)^{-1} \). It therefore follows from the previous section that \( \mathbb{H} \) is not continuous on \( L^1 \). (Of course, the example just given shows this directly.) We record this fact as

**Corollary 1** The restriction of \( \mathbb{H} \) to \( L^1 \) is unbounded.

However, according to a celebrated theorem of Marcel Riesz, \( \mathbb{H} \) is a bounded transformation on \( L^p \) if \( 1 < p < \infty \) [8, 3]. Hence, if \( 1 < p < \infty \), the Hilbert transform is an example of a nontrivial affine-invariant operator on \( L^p \). In particular, it is an example of an interesting affine-invariant operator acting on \( L^2 \).

We note, for future reference, a few additional facts about the restriction of \( \mathbb{H} \) to \( L^2 \). In particular, if \( f \in L^2 \), then, for almost all \( \omega \in \mathbb{R} \),

\[
\hat{\mathbb{H}}f(\omega) = \text{isgn}(\omega)\hat{f}(\omega),
\]

whence

\[
\hat{\mathbb{H}}^2f(\omega) = \text{isgn}(\omega)\hat{\mathbb{H}}f(\omega) = (\text{isgn}(\omega))^2\hat{f}(\omega) = -\hat{f}(\omega).
\]

In other words, applying the inverse Fourier transform, we see that \( \mathbb{H}^2 = -I \). Thus, \( \mathbb{H} \) is a square-root of the identity operator \( I \).

6. Affine-invariant operators on \( L^2 \).

**Theorem 2** Suppose \( A \) is an affine-invariant operator on \( L^2 \). Then \( A \) is a linear combination of the identity \( I \), and the Hilbert transform \( \mathbb{H} \).

Proof. We set the problem in the frequency domain, and, to do so, introduce the following family of mappings \( \{ \zeta_z : z \in \mathbb{C}_+ \} \) on \( L^2 \): if \( g \in L^2 \) and \( z = x + iy \in \mathbb{C}_+ \), then

\[
(\zeta_z g)(\omega) = \frac{1}{x} e^{-y\omega} g\left(\frac{\omega}{x}\right) = \frac{1}{x} e_y\left(\frac{\omega}{x}\right) g\left(\frac{\omega}{x}\right),
\]

where, for \( y \in \mathbb{R} \), the exponential function \( e_y \) is defined by

\[
e_y(t) = e^{iyt}, \quad -\infty < t < \infty.
\]

Define \( T \) on \( L^2 \) as follows: if \( f \in L^2 \), set

\[
Tf = \hat{A}f.
\]
Since the correspondence between a function and its Fourier transform is linear, and $A$ is linear, then $T$ is, and, furthermore,

$$||T\hat{f}||_2 = ||Af||_2 \leq ||A|| ||f||_2 = ||A|| ||\hat{f}||_2,$$

since the Fourier transform is a unitary operator on $L^2$, and $A$ is bounded with norm $||A||$. It follows that $T$ is bounded on $L^2$, and its norm doesn’t exceed that of $A$. Note next that

$$T(\zeta_z \hat{f}) = T\tau_z f = \tau_z Af = \zeta_z \hat{Af} = \zeta_z T\hat{f}.$$

In other words, $T$ commutes with the family of linear operators $\{\zeta_z : z \in \mathbb{C}_+\}$.

In particular, $T\zeta_{(1,y)} g = \zeta_{(1,y)} Tg, \forall g \in L^2$, and all real $y$, i.e.,

$$T(e_y g) = e_y Tg, \forall g \in L^2, \forall y \in \mathbb{R}.$$

Hence, for every trigonometric polynomial $p$ of the form

$$p(\omega) = \sum c_k e^{y_k \omega},$$

we have $T(pg) = pTg$. Choose any non-zero function $g_0 \in L^2$. Then $\{e_y g_0 : y \in \mathbb{R}\}$ is dense in $L^2$. (Because, if, for some $f \in L^2$, $0 = <e_y g_0, f>, \forall y \in \mathbb{R}$, the Fourier transform of $g_0 \hat{f}$ is the zero function, whence $f = 0$, since $g_0 \neq 0$ by hypothesis.) In other words, the subspace $\{p g_0 : p$ a trigonometric polynomial $\}$ is dense in $L^2$. Hence, given any $g \in L^2$, there is a sequence of trigonometric polynomials $p_n$ such that $||g - p_n g_0||_2 \to 0$ as $n \to \infty$. From the continuity of $T$, and the fact that it commutes with polynomials, we deduce that

$$Tg = \lim_n T(p_n g_0) = \lim_n p_n Tg_0,$$

in the $L^2$-norm, whence

$$\int_{-\infty}^{\infty} Tg(\omega) g_0(\omega) \, d\omega = <Tg, \hat{g}_0> = \lim <p_n Tg_0, \hat{g}_0> = \lim <p_n g_0, T\hat{g}_0> = <g, T\hat{g}_0> = \int_{-\infty}^{\infty} g(\omega) T\hat{g}_0(\omega) \, d\omega.$$

More generally, in a similar manner, we see that the Fourier transforms of the $L^1$-functions $g_0 Tg$ and $g Tg_0$ agree on $\mathbb{R}$. Hence, they are equal a.e. In particular, if $h_0 \in L^2, h_0 \neq 0$ a.e.,

$$h_0 Tg_0 = g_0 Th_0.$$

Accordingly, setting

$$\phi = \frac{Tg_0}{g_0},$$

we see that $\phi$ is measurable, independent of $g_0$ and $Tg = \phi g, \forall g \in L^2$; and $\phi g \in L^2$, whenever $g \in L^2$. Moreover,

$$\int_{-\infty}^{\infty} |\phi(\omega) g(\omega)|^2 \, d\omega = ||\phi g||_2^2 = ||Tg||_2^2 \leq ||T|| ||g||_2^2, \forall g \in L^2.$$
Iterating this we infer that
\[ \int_{-\infty}^{\infty} |\phi(\omega)|^{2n} h(\omega) \, d\omega \leq ||T||^{2n} ||h||_1, \quad n = 1, 2, \ldots, \]
for all non-negative \( h \in L^1 \).
Hence, if \( ||h||_1 = 1 \), and \( h \geq 0 \), then
\[ \sqrt[n]{\int_{-\infty}^{\infty} |\phi(\omega)|^{2n} h(\omega) \, d\omega} \leq ||T||^{2n}, \quad n = 1, 2, \ldots \]
It follows that \( \phi \in L^\infty \) and \( ||\phi||_\infty \leq ||T|| \). In other words, \( T \) is multiplication by \( \phi \), and \( ||T|| = ||\phi||_\infty \).

To finish off, the further condition that \( T\zeta(x,0) = \zeta(x,0)T \), \( \forall x > 0 \), implies that, for any pair of functions \( f, g \in L^2 \),
\[ \int_{-\infty}^{\infty} \phi(\omega) g(\omega) \bar{f}(\omega) \, d\omega = \langle \phi g, f \rangle = \langle Tg, f \rangle = x \langle \zeta(x,0)g, \zeta(x,0)f \rangle = x \langle \phi \zeta(x,0)g, \zeta(x,0)f \rangle = \int_{-\infty}^{\infty} \phi(x\omega) g(\omega) \bar{f}(\omega) \, d\omega, \]
for all \( x > 0 \). Hence, for all \( h \in L^1 \),
\[ \int_{-\infty}^{\infty} \phi(\omega) h(\omega) \, d\omega = \int_{-\infty}^{\infty} \phi(x\omega) h(\omega) \, d\omega, \]
for all \( x > 0 \). Choosing \( h \) to be the characteristic (or indicator) function of the interval \((0, 1)\), we deduce that
\[ \int_{0}^{x} \phi(\omega) \, d\omega = x \int_{0}^{1} \phi(\omega) \, d\omega, \]
whence, by Lebesgue’s theorem on differentiation [9],
\[ \phi(x) = \int_{0}^{1} \phi(\omega) \, d\omega = a, \]
say, for almost all \( x > 0 \). Similarly,
\[ \phi(x) = \int_{-1}^{0} \phi(\omega) \, d\omega = b, \]
say, for almost all \( x < 0 \). Letting
\[ \lambda = \frac{a + b}{2}, \quad \mu = \frac{a - b}{2i}, \]
we can say that
\[ \phi(\omega) = \lambda + i\mu \text{sgn}(\omega), \quad \text{a.e. on } (-\infty, \infty). \]
Hence, reverting to $A$,

$$\hat{A}f(\omega) = (T\hat{f})(\omega) = \phi(\omega)\hat{f}(\omega) = \lambda\hat{f}(\omega) + i\mu\text{sgn} (\omega)\hat{f}(\omega) = \lambda\hat{f}(\omega) + i\mu\hat{H}f,$$

whence, by the Fourier Inversion Theorem,

$$Af = \lambda f + \mu\hat{H}f, \forall f \in L^2.$$ 

In other words, $A$ is the sum of a multiple of the identity operator and a multiple of the Hilbert transform. This ends the proof.

7. Some additional remarks

It may be of interest to workers in the field of hysteresis to point out that we can now use the Hilbert transform to manufacture non-linear affine-invariant transformations that send $L^2$ to itself, (or, indeed, more generally, that map $L^p$ to $L^p$, if $1 < p < \infty$). For instance, the following mappings are examples of this kind:

$$f \rightarrow |Hf|, \quad f \rightarrow H|f|, \quad f \rightarrow Hf + |Hf|.$$ 

It may be worth remarking as well that $H$ preserves the Lip$_\alpha$ classes [8]. More precisely, if $f \in L^p$, for $p > 1$, and satisfies the Lipschitz condition

$$||\tau_{(1,y)}f - f||_\infty = O(|y|^{\alpha}) \quad (0 < \alpha < 1),$$

then $g$, the Hilbert transform of $f$, is defined on all of $\mathbb{R}$, belongs to $L^p$, and satisfies the same Lipschitz condition:

$$||\tau_{(1,y)}g - f||_\infty = O(|y|^{\alpha}).$$

Hence, since $|f|$ belongs to Lip$_\alpha$ whenever $f$ does, it’s clear, from what we’ve just said, that each of the displayed non-linear operators leave the intersection of $L^p$ and Lip$_\alpha$ invariant if $p > 1$.

8. Open problems

As we’ve previously remarked, the Hilbert transform is an example of an affine-invariant operator that maps $L^p$ to $L^p$, for every $p \in (1, \infty)$. This raises the question: If $1 < p < \infty$, but $p \neq 2$, what are the affine-invariant operators that map $L^p$ to $L^p$? It’s to be expected that the analogue of Theorem 2 holds in this case. However, the methods used to establish this theorem are heavily reliant on the properties of the Fourier transform—which do not extend to all the $L^p$ spaces; for instance, it’s not even definable in the ordinary sense if $p > 2$—and a different approach is needed to tackle the question. The first hurdle that must be overcome, is a description of the translation-invariant operators acting on an arbitrary $L^p$ space, and a solution to this problem is furnished by Schwarz’s theory of distributions [2]. Indeed, this problem was explicitly solved by Schwarz himself [1]. I suspect that one can build on his result, and confirm my belief that the only affine-invariant operators on $L^p$ are linear combinations of the identity and the Hilbert transform, but I’ve not carried out the analysis.

What about $L^1$? As we know, the Hilbert transform is defined on this space, and is affine-invariant as a linear transformation from this space to the space of measurable functions whose distributions functions are small at infinity. (More precisely, there is an absolute constant $c > 0$, such that, if $f \in L^1$, and $\lambda > 0$, then,

$$m (\{ x \in \mathbb{R} : |Hf(x)| \geq \lambda \}) \leq \frac{c||f||_1}{\lambda},$$
where \( m \) denotes Lebesgue measure on \( \mathbb{R} \), a result due to Kolmogorov [3].) The difficulty, however, is that \( \mathbb{H} \) is not continuous on this space. But it does leave the subspace
\[
M = \{ f \in L^1 : ||Hf||_1 < \infty \}
\]
invariant. If we give \( M \) the norm \( f \rightarrow ||f||_1 + ||Hf||_1 \), are linear combinations of the identity and the Hilbert transform the only affine-invariant operators on it?

Finally, we ask: what are the affine-invariant operators between different \( L^p \) spaces?

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