MAKING FIRST ORDER LINEAR LOGIC
A GENERATING GRAMMAR

SERGEY SLAVNOV

e-mail address: slavnovserge@gmail.com

ABSTRACT. It is known that different categorial grammars have surface representation in
a fragment of first order multiplicative linear logic (MLL1). We show that the fragment
of interest is equivalent to the recently introduced extended tensor type calculus (ETTC).
ETTC is a calculus of specific typed terms, which represent tuples of strings, more precisely
bipartite graphs decorated with strings. Types are derived from linear logic formulas, and
rules correspond to concrete operations on these string-labeled graphs, so that they can
be conveniently visualized. This provides the above mentioned fragment of MLL1 that
is relevant for language modeling not only with some alternative syntax and intuitive
geometric representation, but also with an intrinsic deductive system, which has been
absent.

In this work we consider a non-trivial notationally enriched variation of the previously
introduced ETTC, which allows more concise and transparent computations. We present
both a cut-free sequent calculus and a natural deduction formalism.

1. INTRODUCTION
The best known examples of categorial grammars are Lambek grammars, which are based
on Lambek calculus (LC) [Lam58], i.e. noncommutative intuitionistic linear logic (for
background on linear logic see [Gir87],[Gir95], for categorial grammars [MR12]). These,
however, have somewhat limited expressive power, and a lot of extensions/variations have
been proposed, using discontinuous tuples of strings and λ-terms, commutative and non-
commutative logical operations, modalities etc. Let us mention displacement grammars
[MVF11], abstract categorial grammars (ACG, also called λ-grammars and linear grammars)
[dG01], [Mus07], [MP12] and hybrid type logical categorial grammars [KL12].

It has been known for a while that different grammatical formalisms, such as those just
mentioned, have surface representation in the familiar first order multiplicative intuitionistic
linear logic (MILL1), which provides them in this way with some common ground. In
the seminal work [MP01] it was shown that LC translates into MILL1 as a conservative
fragment; later similar representations were obtained for other systems [Moo14a], [Moo14b]
(however, in the case of displacement calculus, the representation is not conservative, see
[Moo15]). This applies equally well to the classical setting of first order multiplicative linear
logic (MILL1), which is a conservative extension of MILL1 anyway.

Key words and phrases: categorial grammars, linear logic, Lambek calculus, diagrammatic reasoning.
The translations are based on interpreting first order variables as denoting string positions. Basically, this induces (at least on the intuitive level) an interpretation of certain well-formed formulas and sequents as expressing relations between strings, in particular, those encoded in formal grammars. Unfortunately, this intuitive interpretation (to the author’s knowledge) has never been formulated explicitly and rigorously, i.e. as a specific semantics for (a fragment of) MILL1. Therefore the underlying structures for constructions of the above cited works, arguably, remain somewhat unclear.

It should be emphasised that only a small fragment of MILL1, let us call it linguistically well-formed, is actually used in the above translations. It is this fragment that obtains in this way the above discussed “linguistic” interpretation in terms of relations between strings. As for the whole MILL1, the “linguistic” meaning of general formulas and sequents seems, at least, obscure. On the other hand, standard deductive systems of linear logic are not intrinsic to this fragment. Typically, when deriving, say, the MILL1 translation of an LC sequent, in general, we have to use at intermediate steps sequents that have no clear linguistic meaning, if any. It can be said that representation of formal grammars makes first order linear logic an auxiliary tool, but hardly something like a generating grammar. Derivations do not translate step-by-step to operations generating language elements.

(We should note, however, that the first order logic formalism has, in general, wider usage in linguistic applications. Apart from encoding string positions, first order constructions allow representing grammatical features, locality domains, tree depth levels etc. We do not discuss the latter applications in our paper.)

The system of extended tensor type calculus (ETTC) based on (classical) propositional MLL, recently proposed in [Sla21b] (elaborating on [Sla21a]), was directly designed for linguistic interpretation, namely for concrete geometric representation of ACG extended with constructions emulating noncommutative operations of LC. ETTC is a calculus of specific typed terms, which represent tuples of strings, more precisely bipartite graphs decorated with strings. Types are derived from MLL formulas, and rules correspond to concrete operations on these string-labeled graphs, so that they can be conveniently visualized. Typed terms become language elements, from which words are gradually assembled using these geometric operations.

It was shown in [Sla21b] that tensor grammars based on ETTC include both Lambek grammars and ACG as conservative fragments. Moreover, the representation of LC types in ETTC is very similar to that in MILL1. This led to the question (raised explicitly in that paper) about relations between the two systems. In this work we show that ETTC is equivalent precisely to the above discussed linguistically well-formed fragment of MILL1. (Actually, we consider a non-trivial notationally enriched variation of ETTC rather than the one proposed in [Sla21b]. The enriched system allows more concise and transparent computations.) Thus, it turns out that ETTC provides the linguistic fragment with an alternative syntax, together with an intrinsic deductive system and intuitive pictorial representation. It can be said that, in some sense, ETTC does “make the linguistic fragment a generating grammar”, as announced in the title. Speaking less ambitiously, it allows a different perspective on the MILL1 representations, which might be clarifying and useful.

In this paper we will work on the basis of classical linear logic, mostly because it suggests some nice symmetric notation, but all constructions automatically restrict to the intuitionistic setting.
2. Background: systems of linear logic

\( \text{Pred}_- = \{ \overline{p} \mid p \in \text{Pred}_+, \} \), \( p \) is \( n \)-ary \( \Rightarrow \) \( \overline{p} \) is \( n \)-ary, \( \text{Pred} = \text{Pred}_+ \cup \text{Pred}_- \),
\( \text{At} = \{ p(e_1, \ldots, e_n) \mid e_1, \ldots, e_n \in \text{Var}, p \in \text{Pred}, p \) is \( n \)-ary \},
\( \text{Fm} ::= \text{At}[(\text{Fm} \otimes \text{Fm})](\text{Fm} \otimes \text{Fm})[\forall x \text{Fm}][\exists x \text{Fm}], x \in \text{Var} \)
\( p(e_1, \ldots, e_n) = \overline{p}(e_n, \ldots, e_1), \ \overline{p}(e_1, \ldots, e_n) = p(e_n, \ldots, e_1) \) for \( P \in \text{Pred}_+, \)
\( A \otimes B = B \otimes A, \ \forall x A = \exists x \overline{A}, \ \exists x \overline{A} = \forall x (\overline{A}). \)

(A) MLL1 language

\[
\begin{align*}
&\vdash \Gamma, A, B \quad (\text{Id}) \\
&\vdash \Gamma, A, \overline{A} \quad (\Theta)
\end{align*}
\]

(b) MLL1 sequent calculus

\( \text{Fm} ::= \text{Prop}[(\text{Fm} \setminus \text{Fm})](\text{Fm} / \text{Fm})](\text{Fm} \bullet \text{Fm}) \)

(c) LC language

\[
\begin{align*}
|p|^{(l,r)} &= p(l, r) \text{ for } p \in \text{Prop}, \\
|A \bullet B|^{(l,r)} &= \exists x |A|^{(l,x)} \otimes |B|^{(x,r)}, \quad |A \setminus B|^{(l,r)} = \forall x |A|^{(x,l)} \rightarrow |B|^{(x,r)}, \\
A_1, \ldots, A_n \vdash_{\text{LC}} B &\iff |A_1|^{(c_0,c_1)}, \ldots, |A_n|^{(c_{n-1},c_n)} \vdash_{\text{MLL1}} |B|^{(c_0,c_n)}. \\
\end{align*}
\]

(d) Translating LC to MLL1

Figure 1. Systems of linear logic

Language and sequent calculus of first order multiplicative linear logic (MLL1) are shown in Figures 1a, 1b. We are given a set \( \text{Pred}_+ \) of positive predicate symbols with assigned arities and a set \( \text{Var} \) of variables. The sets \( \text{Pred}_- \) and \( \text{Pred} \) of negative and of all predicate symbols respectively are defined. The sets of first order atomic, respectively, linear multiplicative formulas are denoted as \( \text{At} \), respectively, \( \text{Fm} \). Connectives \( \otimes \) and \( \otimes \) are called respectively tensor (also times) and cotensor (also par). Linear negation \( (\_) \) is not a connective, but is definable by induction on formula construction, as in Figure 1a. Note that, somewhat non-traditionally, we follow the convention that negation flips tensor/cotensor factors, typical for noncommutative systems. This does not change the logic (the formulas \( A \otimes B \) and \( B \otimes A \) are provably equivalent), but is more consistent with our intended geometric interpretation.

A context \( \Gamma \) is a finite multiset of formulas, and a sequent is an expression of the form \( \vdash \Gamma \), where \( \Gamma \) is a context. The set of free variables of \( \Gamma \) is denoted as \( \text{FV}(\Gamma) \). In this work we will consider several systems of sequent calculus, and, in order to avoid confusion, we will sometimes use abbreviations like \( \vdash_{\text{MLL1}} \Gamma \) to say that the sequent \( \vdash \Gamma \) is derivable in MLL1, similarly for other systems. We also emphasize that a commutative system like MLL1 can be given an alternative ordered formulation, where contexts are defined as finite...
sequences (rather than multisets) of formulas and rules are supplemented with the explicit Exchange rule: \[ \Gamma, A, B, \Theta \vdash \Gamma, B, A, \Theta \] (Ex).

First order multiplicative intuitionistic logic (MILL1) is the fragment of MLL1 restricted to sequents of the form \( \vdash B, A_1, \ldots, A_n \), written in MILL1 in a two-sided form as \( A_1, \ldots, A_n \vdash B \), where \( A_1, \ldots, A_n, B \) are multiplicative intuitionistic formulas, which means constructed using only positive predicate symbols, quantifiers, tensor and linear implication defined by \( A \rightarrow B = \overline{A} \otimes B \). (Note that we can define linear implication differently as \( A \rightarrow B = B \otimes \overline{A} \) and get an equivalent system.)

The language of Lambek calculus (LC), i.e. noncommutative multiplicative intuitionistic linear logic, is summarised in Figure 1c. Noncommutative tensor is denoted as \( \otimes \), and the two directed implications caused by noncommutativity are denoted as slashes. Formulas of LC are often called types. A context in LC is, by definition, a sequence of types, and a sequent is an expression of the form \( \Gamma \vdash F \), where \( \Gamma \) is a context and \( F \) is a type. In LC there is no analogue of the Exchange rule or “unordered” formulation, i.e. the system is genuinely noncommutative. We will not reproduce the rules which can be found, for example, in [Lam58].

Given a finite alphabet \( T \) of terminal symbols, a Lambek grammar \( G \) (over \( T \)) is a pair \( G = (\text{Lex}, S) \), where Lex, the lexicon is a finite set of lexical entries, which are expressions \( t : A \), where \( t \in T \) and \( A \) is a type, and \( S \) is a selected atomic type. The language \( L(G) \) of \( G \) is the set of words

\[
\begin{align*}
L(G) = \left\{ t_1 \ldots t_n \left| \begin{array}{l}
t_1 : A_1, \ldots, t_n : A_n \in \text{Lex},
A_1, \ldots, A_n \vdash_{\text{LC}} S
\end{array} \right. \right\},
\end{align*}
\]

(2.1)

where \( t_1 \ldots t_n \) stands for the concatenation of strings \( t_1, \ldots, t_n \). More details on categorial grammars can be found in [MR12].

Given two variables \( l, r \), the first order translation \( ||F||^{(l,r)} \) of an LC formula \( F \) parameterized by \( l, r \) is shown in Figure 1d (LC propositional symbols are treated as binary predicate symbols). This provides a conservative embedding of LC into MILL1 [MP01].

3. MILL1 grammars and linguistic fragment

3.1. MILL1 grammars. Translation from LC suggests defining MILL1 grammars similar to Lambek grammars.

Let a finite alphabet \( T \) of terminal symbols be given. Assume for convenience that our first order language contains two special variables (or constants) \( l, r \).

Definition 3.1. Given a terminal alphabet \( T \), a MILL1 lexical entry (over \( T \)) is a pair \( (w, A) \), where \( w \in T^* \) is nonempty, and \( A \) is a MILL1 formula with one occurrence of \( l \) and one occurrence of \( r \) and no other free variables. For the formula \( A \) occurring in a lexical entry we will write \( A[x; y] = A[y/x][r/l] \) (so that \( A = A[l;r] \)).

A MILL1 grammar \( G \) (over \( T \)) is a pair \( (\text{Lex}, s) \), where Lex is a finite set of MILL1 lexical entries, and \( s \) is a binary predicate symbol. The language \( L(G) \) generated by \( G \) or, simply, the language of \( G \) is defined by

\[
\begin{align*}
L(G) = \left\{ w_1 \ldots w_n \left| \begin{array}{l}
w_1, A_1, \ldots, w_n, A_n \in \text{Lex},
A_1[c_0; c_1], \ldots, A_n[c_{n-1}; c_n] \vdash_{\text{MILL1}} s(c_0, c_n)
\end{array} \right. \right\}.
\end{align*}
\]

(3.1)
It seems clear that, under such a definition, Lambek grammars translate precisely to MILL1 grammars generating the same language (compare (3.1) with (2.1)).

It has been shown [Moo14a], [Moo14b] that MILL1 allows representing more complex systems such as displacement calculus, abstract categorial grammars, hybrid type-logical grammars. This suggests that the above definition should be generalized to allow more complex lexical entries, corresponding to word tuples rather than just words. (To the author’s knowledge, no explicit definition of a MILL1 grammar has been given in the literature, but this concept is implicit in the above cited works).

Now let us isolate the fragment of first order logic that is actually used in translations of linguistic systems.

3.2. Linguistic fragment.

Definition 3.2. A language of MILL1 is linguistically marked if each $n$-ary predicate symbol $p$ is equipped with valency $v(p) \in \mathbb{N}^{2}$, $v(p) = (v_l(p), v_r(p))$, where $v_l(p) + v_r(p) = n$, such that $v(\overline{p}) = (v_r(p), v_l(p))$. When $v(p) = (k, m)$ we say that first $k$ occurrences $x_1, \ldots, x_k$ in the atomic formula $p(x_1, \ldots, x_n)$ are left occurrences or have left polarity, and $x_{k+1}, \ldots, x_n$ are right occurrences, with right polarity.

For a variable occurrence $x$ in a compound formula in a linguistically marked language $F$ we define the polarity of $x$ in $F$ by induction. If $F = A \Box B$, where $\Box \in \{\otimes, \exists\}$, and $x$ occurs in $A$, respectively $B$, then the polarity of $x$ in $F$ is the same as the polarity of $x$ in $A$, respectively $B$. If $F = QxA$, where $Q \in \{\forall, \exists\}$, then the polarity of a variable occurrence $x$ in $F$ is the same as the polarity of $x$ in $A$. If $\Gamma = A_1, \ldots, A_n$ is a context, and $x$ is a variable occurrence in $A_i$ for some $i \in \{1, \ldots, n\}$, then the polarity of $x$ in the context $\Gamma$ and in the sequent $\vdash \Gamma$ is the same as the polarity of $x$ in $A_i$. For a MILL1 sequent $\Gamma \vdash F$, the polarity of its variable occurrences is determined by the translation to MILL1.

Definition 3.3. Given a linguistically marked MILL1 language, a formula, context or sequent is linguistically marked if every quantifier binds exactly one left and one right variable occurrence. An MILL1 derivation $\pi$ is linguistically marked if all sequents occurring in $\pi$ are linguistically marked.

A linguistically marked formula or context is linguistically well-formed if, furthermore, it has at most one left and at most one right occurrence of any free variable. A linguistically marked sequent is linguistically well-formed if each of its free variables has exactly one left and one right occurrence. A MILL1 grammar is well-formed if the formula in every lexical entry is linguistically well-formed with $l$ and $r$ occurring with left and right polarity respectively.

Proposition 3.4. Any cut-free derivation of a linguistically marked MILL1 sequent is linguistically marked. $\Box$

Evidently, if we understand MILL1 as a fragment of MILL1, then the translation of LC in Figure 1d has precisely the linguistically well-formed fragment as its target (with $v(p) = (1, 1)$ for all $p \in \text{Pred}$), and Lambek grammars translate to linguistically well-formed ones. Similar observations apply to translations in [Moo14a], [Moo14b].

However, the standard sequent calculus formulation of MILL1 is not intrinsic to the linguistic fragment. For an illustration, a basic LC derivable sequent $A, B \vdash A \bullet B$ translates to MILL1 as $A(c_0, c_1), B(c_1, c_2) \vdash \exists x A(c_0, x) \otimes B(x, c_2)$. The latter, obviously, is derivable in MILL1 by the (\exists) rule applied to the sequent $A(c_0, c_1), B(c_1, c_2) \vdash A(c_0, c_1) \otimes B(c_1, c_2)$,
which itself is not in the linguistic fragment. Thus, in order to derive a linguistically well-formed sequent in MLL1 we have to use “linguistically ill-formed” ones at intermediate steps.

Our goal is to provide the linguistic fragment with an intrinsic deductive system.

\[ \Gamma \vdash a(e, t), \exists x(t, x) \equiv a(e, s), \exists x(s, e) \]

**Figure 2.** Occurrence net example

### 3.2.1. Occurrence nets. Let us introduce some convenient format and terminology for derivations of linguistically well-formed sequents.

**Definition 3.5.** An occurrence net of a linguistically marked MLL1 sequent \( \vdash \Gamma \) is a perfect matching \( \sigma \) between left and right free occurrences in \( \Gamma \), such that each pair (link) in \( \sigma \) consists of occurrences of the same variable.

Basically, occurrence nets are rudiments of proof-nets. To each cut-free linguistically marked derivation \( \pi \) with conclusion \( \vdash \Gamma \) we will assign an occurrence net \( \sigma(\pi) \), which is an occurrence net of \( \vdash \Gamma \). (Note that for a linguistically well-formed sequent there is only one occurrence net possible.)

**Definition 3.6.** The occurrence net \( \sigma(\pi) \) of a cut-free linguistically marked derivation \( \pi \) is constructed by induction on \( \pi \).

- If \( \pi \) is the axiom \( \vdash X, X \), where \( X = p(e_1, \ldots, e_n) \), the net is defined by matching each occurrence \( e_i \) in \( X \) with the occurrence \( e_i \) in \( X = p(e_n, \ldots, e_1) \).
- If \( \pi \) is obtained from a derivation \( \pi' \) by the (\( \exists \)) rule, then \( \sigma(\pi) = \sigma(\pi') \).
- If \( \pi \) is obtained from derivations \( \pi_1, \pi_2 \) by the (\( \otimes \)) rule, then \( \sigma(\pi) = \sigma(\pi_1) \cup \sigma(\pi_2) \).
- If \( \pi \) is obtained from some \( \pi' \) by the (\( \forall \)) rule applied to a formula \( A' = A[v/x] \) and introducing the formula \( \forall x.A \), then \( \sigma(\pi) = \sigma(\pi') \setminus \{ (v_l, v_r) \} \), where \( v_l, v_r \) are the two occurrences of \( v \) in \( A' \). (The variable \( v \) has no free occurrences in the premise other than those in \( A' \), and, since all sequents are linguistically marked, there must be precisely one left and one right occurrence of \( v \) in \( A' \). It follows that \( (v_l, v_r) \in \sigma(\pi') \).)
- If \( \pi \) is obtained from \( \pi' \) by the (\( \exists \)) rule applied to a formula \( A' = A[v/x] \) and introducing the formula \( \exists x.A \), there are two cases depending on \( \sigma(\pi') \). Let \( v_l \) and \( v_r \) be, respectively, the left and the right occurrence of \( v \) in \( A' \) corresponding to the two occurrences of \( x \) in \( A \) bound by the existential quantifier.
  - If \( (v_l, v_r) \in \sigma(\pi') \), then \( \sigma(\pi) = \sigma(\pi') \setminus \{ (v_l, v_r) \} \).
  - Otherwise let \( v'_l, v'_r \) be such that \( (v'_l, v_r), (v_l, v'_r) \in \sigma(\pi') \). Then \( \sigma(\pi) = \sigma(\pi') \setminus \{ (v'_l, v_r), (v_l, v'_r) \} \) \( \cup \{ (v'_l, v'_r) \} \).

(If we see occurrence nets geometrically as bipartite graphs, then the (\( \forall \)) rule does not change the graphs, the (\( \otimes \)) rule takes the disjoint union of two graphs. The (\( \exists \)) rule erases the link corresponding to the two occurrences that become bound by the universal quantifier. The (\( \exists \)) rule has two cases: it erases a link in the first case and glues two links into one in the second case. An example for the second case of the (\( \exists \)) rule is shown in Figure 2.)
In what follows we will use a number of times the operation of replacing a free variable occurrence in an expression with another variable. So we introduce some notation generalizing familiar notation for substitution. Let \( \Phi \) be a context or a formula, let \( x \) be a free variable occurrence in \( \Phi \) and \( v \in \text{Var} \). Then \( \Phi[v/x] \) is the expression obtained from \( \Phi \) by replacing \( x \) with \( v \). We will also use the notation \( \Phi[v_1/x_1, \ldots, v_n/x_n] = \Phi[v_1/x_1] \ldots [v_n/x_n] \) for iterated substitutions, where it is assumed implicitly that \( x_1, \ldots, x_n \) are pairwise distinct occurrences, so that the substitutions commute with each other.

Finally, we will use for an induction parameter the size of a cut-free derivation defined in the following natural way. If the derivation \( \pi \) is an axiom then the size \( \text{size}(\pi) \) of \( \pi \) is 1. If \( \pi \) is obtained from derivations \( \pi_1, \pi_2 \) by a two-premise rule then \( \text{size}(\pi) = \text{size}(\pi_1) + \text{size}(\pi_2) + 1. \)

**Proposition 3.7.** If \( \Gamma, \Gamma' \) are linguistically marked contexts differing from each other only by renaming bound variables and the sequent \( \vdash \Gamma \) is MLL1 derivable with a cut-free derivation \( \pi \), then \( \vdash \Gamma' \) is derivable with a cut-free derivation \( \pi' \) of the same size and with the same occurrence net, \( \text{size}(\pi') = \text{size}(\pi), \sigma(\pi') = \sigma(\pi) \).  

**Proof.** Induction on \( \pi \). \hfill \Box

**Proposition 3.8.** Let \( \pi \) be a linguistically marked cut-free MLL1 derivation with conclusion \( \vdash \Gamma \), and assume that \((e_l, e_r) \in \sigma(\pi)\). Let \( v \in \text{Var} \) be such that \( e_l, e_r \) are not in the scope of a quantifier \( Qv, Q \in \{\forall, \exists\} \), and let \( \tilde{\Gamma} = \Gamma[v/e_l, v/e_r] \). Then \( \vdash \tilde{\Gamma} \) is derivable in MLL1 with a linguistically marked cut-free derivation \( \tilde{\pi} \) of the same size as \( \pi \). Moreover, if \( v_l, v_r \) are occurrences of \( v \) in \( \tilde{\Gamma} \) replacing, respectively, the occurrences \( e_l, e_r \) in \( \Gamma \), then \( \tilde{\sigma}(\tilde{\pi}) = (\sigma(\pi) \setminus \{(e_l, e_r)\}) \cup \{(v_l, v_r)\} \).

**Proof.** Induction on \( \text{size}(\pi) \). The most involved step is when \( \pi \) is obtained from a derivation \( \pi' \) of some sequent \( \vdash \Gamma' \) by the \((\exists)\) rule and \((e_l, e_r) \notin \sigma(\pi')\).

This means that \( \Gamma' \) contains some formula \( A' \), while \( \Gamma \) contains the formula \( \exists x A' \), where \( A = A'[e_l/e, e_r/e] \), and the occurrences \( e_l', e_r' \) of \( e \) in \( A \) that correspond to the two occurrences of \( x \) in \( A' \) are linked in \( \sigma(\pi') \) to, respectively, \( e_r, e_l \), i.e. \((e_l', e_r'), (e_r', e_l') \in \sigma(\pi') \). Note that

\[
A' = A[x/e_l, x/e_r]. \tag{3.2}
\]

Observe from (3.2) that the context \( \tilde{\Gamma} = \Gamma[v/e_l, v/e_r] \) can be obtained as follows.

Let \( \Theta = \Gamma[v/e_l', v/e_r', v/e_l, v/e_r'] \), and let \( v_l', v_r, v_l, v_r' \) be the occurrences of \( v \) in \( \Theta \) replacing, respectively, the occurrences \( e_l', e_r, e_l, e_r' \) in \( \Gamma' \). Let \( B \) be the formula in \( \Theta \) that corresponds to \( A \) in \( \Gamma' \). Put

\[
\tilde{A} = \exists x B[x/v_l', x/v_r']. \tag{3.3}
\]

Then \( \tilde{\Gamma} \) is obtained from \( \Theta \) by replacing \( B \) with \( \tilde{A} \).

Moreover, the sequent \( \vdash \tilde{\Gamma} \) is derivable from \( \vdash \Theta \) by the \((\exists)\) rule. It follows that, if the occurrences \( e_l', e_r, e_l, e_r' \) in \( \Gamma' \) are not in the scope of a quantifier \( Qv, Q \in \{\forall, \exists\} \), then, by the induction hypothesis (applied twice), we have that \( \vdash_{\text{MLL1}} \Theta \), hence \( \vdash_{\text{MLL1}} \tilde{\Gamma} \). However, the above condition may fail, and then we cannot apply the induction hypothesis directly. Therefore, in a general case, we need some more work.

By renaming bound variables if necessary, we can obtain from \( \Gamma' \) a linguistically marked context \( \Gamma'' \) such that the occurrences \( e_l', e_r, e_l, e_r' \) are not in the scope of a quantifier \( Qv, Q \in \{\forall, \exists\} \). By the preceding proposition the sequent \( \vdash \Gamma'' \) is derivable with a cut-free derivation \( \pi'' \), where \( \text{size}(\pi'') = \text{size}(\pi'), \sigma(\pi'') = \sigma(\pi') \). Then we put \( \Theta'' = \Gamma''[v/e_l', v/e_r', v/e_l, v/e_r'] \). Now
the induction hypothesis can be applied for sure and the sequent \( \vdash \Theta'' \) is derivable with a linguistically marked cut-free derivation \( \rho \), and size\((\rho) = \text{size}(\Pi'') = \text{size}(\Pi') \). Moreover,

\[
\sigma(\rho) = (\sigma(\Pi') \setminus \{(e'_l, e_r), (e_l, e'_r)\}) \cup \{(v'_l, v_r), (v_l, v'_r)\}, \tag{3.4}
\]

where \( v'_l, v_r, v_l, v'_r \) are the occurrences of \( v \) in \( \Theta'' \) replacing, respectively, the occurrences \( e'_l, e_r, e_l, e'_r \) in \( \Gamma'' \).

Let \( A'' \) be the formula in \( \Gamma'' \) corresponding to \( A \) in \( \Gamma' \), and \( B'' \) be the formula in \( \Theta'' \) obtained from \( A'' \) in \( \Gamma'' \). Let \( \hat{B} = \exists x B''[x/v'_l, x/v'_r] \). The formula \( \hat{A} \) in (3.3) may differ from \( \hat{B} \) only by renaming bound variables. Let \( \hat{\Theta} \) be the context obtained from \( \Theta'' \) by replacing \( B'' \) with \( \hat{B} \). Then \( \hat{\Theta} \) and \( \hat{\Gamma} \), as well, may differ only by renaming bound variables. But \( \vdash \hat{\Theta} \) is derivable from \( \vdash \Theta'' \) by the \((\exists)\) rule, i.e. the derivation \( \hat{\rho} \) of \( \vdash \hat{\Theta} \) is obtained from \( \rho \) by attaching the \((\exists)\) rule. Since size\((\rho) = \text{size}(\Pi') \), it follows that size\((\hat{\rho}) = \text{size}(\Pi') \), and the occurrence net \( \sigma(\hat{\rho}) \) is computed from (3.4). Applying Proposition 3.7 once more, we obtain the desired result.

\[ \square \]

3.2.2. Linguistic derivations.

**Definition 3.9.** Let \( \vdash \Gamma \) be a linguistically well-formed MLL1 sequent containing a formula \( A \) and let \( s, t \in \text{Var}, s \neq t \), have, respectively, a left and a right free occurrences in \( A \), denoted as \( s_l \) and \( t_r \). Let \( s_r \) be the unique right occurrence of \( s \) in \( \Gamma \), and \( t_l \) be the unique left occurrence of \( t \). Let \( x, v \in \text{Var} \) be such that \( x \) does not occur in \( A \) freely, and \( s_r, t_l \) do not occur in \( \Gamma \) in the scope of a quantifier \( Qv, v \in \{\forall, \exists\} \).

Let \( A' = \exists x A[x/t_l, x/s_l] \), and the context \( \Gamma' \) be obtained from \( \Gamma \) by replacing \( A \) with \( A' \). Finally, let \( \hat{\Gamma} = \Gamma'[v/t_l, v/s_l] \). Then the sequent \( \vdash \hat{\Gamma} \) is obtained from \( \vdash \Gamma \) by the \((\exists')\) rule.

An example of the \((\exists')\) rule is the following

\[
\begin{align*}
\vdash \overline{a}(t, z), &a(y, s) \not\exists (\overline{a}(s, y) \otimes a(z, t)) \\
\vdash \overline{a}(v, z), &\exists x(a(y, v) \not\exists (\overline{a}(x, y) \otimes a(z, x)))\end{align*} \tag{3.5}
\]

where the valency \( v(a) = (1, 1) \). In the notation of the above definition, the sequents in (3.5) have the following structure:

\[
\begin{align*}
A &= a(y, s) \not\exists (\overline{a}(s, y) \otimes a(z, t)), & \Gamma &= \pi(t, z), a(y, s) \not\exists (\overline{a}(s, y) \otimes a(z, t)) \\
A' &= \exists x(a(y, s) \not\exists (\overline{a}(x, y) \otimes a(z, x))), & \Gamma' &= \pi(t, z), \exists x(a(y, s) \not\exists (\overline{a}(x, y) \otimes a(z, x))) \\
\hat{A} &= \exists x(a(y, v) \not\exists (\overline{a}(x, y) \otimes a(z, x))), & \hat{\Gamma} &= \pi(v, z), \exists x(a(y, v) \not\exists (\overline{a}(x, y) \otimes a(z, x)))
\end{align*}
\]

where we also indicated the formula \( \hat{A} \) in the context \( \hat{\Gamma} \) that corresponds to the formula \( A \) in the context \( \Gamma \).

**Proposition 3.10.** If \( \vdash \Gamma \) is an MLL1 derivable linguistically well-formed sequent and the sequent \( \vdash \hat{\Gamma} \) is obtained from \( \vdash \Gamma \) by the \((\exists')\) rule, then \( \vdash \hat{\Gamma} \) is linguistically well-formed and MLL1 derivable.

\[ \text{Proof.} \] Let the notation be as in the definition above. By Proposition 3.4 any cut-free derivation \( \pi \) of \( \vdash \Gamma \) is linguistically marked so it has an occurrence net \( \sigma(\pi) \). Moreover, since \( \vdash \Gamma \) is linguistically well-formed, there are only two occurrences of \( s \) and \( t \) in \( \Gamma \), which implies \( (s_l, s_r), (t_l, t_r) \in \sigma(\pi) \).

By renaming, if necessary, bound variables of \( A \) we can obtain from \( \Gamma \) a linguistically well-formed context \( \Theta \) such that \( s_l, t_r \) in \( \Theta \) are not in the scope of a quantifier \( Qv, Q \in \{\forall, \exists\} \). By
Proposition 3.7, the sequent $\vdash \Theta$ is cut-free derivable with a linguistically marked derivation $\rho$ such that $\sigma(\rho) = \sigma(\pi)$, i.e. $(s_l, s_r), (t_l, t_r) \in \sigma(\rho)$. It follows from Proposition 3.8 that $\vdash^{\text{MLL1}} \Theta'$, where $\Theta' = \Theta[v/t_1, v/t_2, v/s_l, v/s_r]$.

Let $B$ be the formula in $\Theta$ corresponding to $A$ in $\Gamma$ and $B'$ be the formula in $\Theta'$ corresponding to $B$ in $\Theta$. Finally, let $v_l, v_r', v', v_r$ be the occurrences of $v$ in $\Theta'$ replacing the occurrences $t_l, t_r, s_l, s_r$ in $\Theta$ respectively, and let $\tilde{B} = \exists x B'[x/v', x/v_r']$. It follows from the definition that the formulas $\tilde{B}$ and $\tilde{A}$, where $\tilde{A}$ is the formula in $\Gamma$ corresponding to $A'$ in $\Gamma'$, may differ only by renaming bound variables. Let $\tilde{\Theta}$ be obtained from $\Theta'$ by replacing $B'$ with $\tilde{B}$. Then the sequent $\vdash \tilde{\Theta}$ is derivable from $\vdash \Theta'$ by the $(\exists)$ rule. But $\tilde{\Theta}$, again, may differ from $\tilde{\Gamma}$ only by renaming bound variables, and the statement follows from Proposition 3.7. That $\vdash \tilde{\Gamma}$ is linguistically well-formed is obvious from counting free left and right occurrences.

Note that on the level of occurrence nets, seen as bipartite graphs, the $(\exists)$ rule does the same gluing as $(\exists)$; only vertex labels are changed.

**Definition 3.11.** Derivations of linguistically well-formed sequents using rules of MLL1 and the $(\exists)$ rule and involving only linguistically well-formed sequents are **linguistic derivations**.

**Lemma 3.12.** Any cut-free MLL1 derivation $\pi$ of a linguistically well-formed sequent $\vdash \Gamma$ translates to a linguistic derivation.

**Proof.** By Proposition 3.4 the derivation $\pi$ is linguistically marked. We use induction on $\text{size}(\pi)$.

The main case is when the last step in $\pi$ is the $(\exists)$ rule applied to a cut-free linguistically marked derivation $\pi'$ of some non-linguistically well-formed $\vdash \Gamma'$. This means that $\Gamma'$ has four free occurrences of some free variable $v$ (two of which are renamed when the quantifier is introduced in $\Gamma$). Let $v_l', v_r'$ be, respectively, the left and the right occurrences of $v$ in $\Gamma'$ that become bound in $\Gamma$, and let $v_l, v_r$ be, respectively, the remaining left and right occurrence of $v$ in $\Gamma'$ (inherited by $\Gamma$). We have that $\Gamma'$ contains a formula $A$ in which $v_l', v_r'$ are located and $\Gamma$ is obtained from $\Gamma'$ by replacing $A$ with the formula $A' = \exists x A[x/v', x/v_r']$.

The derivation $\pi'$ still has an occurrence net $\sigma(\pi')$. If $(v_l', v_r') \in \sigma(\pi')$, then, by Proposition 3.8, the sequent $\vdash \Gamma''$, where $\Gamma'' = \Gamma'[v_l'/v_l', e'/v_r']$ and $e \in \text{Var}$ is fresh, is derivable with the derivation of the same size as $\pi'$. But $\vdash \Gamma$ is equally well derivable from $\vdash \Gamma''$ by the $(\exists)$ rule, and the statement follows from the induction hypothesis. Otherwise we have $(v_l, v_r'), (v_l', v_r) \in \sigma(\pi')$. Let $u, w \in \text{Var}$ be fresh and let $\Gamma'' = \Gamma'[u/v_l, w/v_r']$, where $\Theta$ is cut-free derivable with a linguistically marked derivation.

Thus, adding the $(\exists)$ rule, we obtain a kind of intrinsic deductive system for the linguistically well-formed fragment. It might seem though that the usual syntax of first order sequent calculus is not very natural for such a system. It is not even clear how to write the $(\exists)$ rule in the sequent calculus format concisely. Arguably, some other representation might be desirable.

4. **Tensor type calculus**

We assume that we are given an infinite set $\text{Ind}$ of indices. They will be used in all kinds of syntactic objects (terms, types, typing judgements) that we consider.
Indices will have upper or lower occurrences, which may be free or bound. (The exact meaning of free and bound occurrences for different syntactic objects will be defined below.) Following the practice of first order logic, we will write \( e^{[i/j]} \), respectively \( e_{[i/j]} \), for the expression obtained from \( e \) by replacing the upper, respectively lower, free occurrence(s) of the index \( j \) with \( i \), implicitly stating by this notation that the substituted occurrences of \( i \) are free in the resulted expression. We will write \( I^*(e) \) and \( I_*(e) \) for the sets of upper and lower indices of \( e \) respectively, and \( I(e) \) for the pair \( (I^*(e), I_*(e)) \); similarly, \( FI(e) = (FI^*(e), FI_*(e)) \) for sets of free indices.

For pairs of index sets \( W_i = (U_i, V_i) \), \( i = 1, 2 \), we write binary operations componentwise, e.g. \( W_1 \cup W_2 = (U_1 \cup U_2, V_1 \cup V_2) \). We write \( W_1 \perp W_2 \) for \( U_1 \cap U_2 = V_1 \cap V_2 = \emptyset \). We also write \( (i, j) \perp (U, V) \) for \( i \not\in U, j \not\in V \) and \( (i, j) \in (U, V) \) for \( i \in U, j \in V \). Finally, for \( W = (U, V) \), we write \( W^\dagger = (V, U) \) and \( |W| = U \cup V \).

4.1. Tensor terms.

**Definition 4.1.** Given an alphabet \( T \) of terminal symbols, tensor terms (over \( T \)) are the elements of the free commutative monoid (written multiplicatively with the unit denoted as 1) generated by the sets

\[
\{[w]_i^j \mid i, j \in \text{Ind}, w \in T^* \} \text{ and } \{[w] \mid w \in T^* \}
\]

satisfying the constraint that any index has at most one lower and one upper occurrence.

Generating set (4.1) elements of the form \([w]_i^j\) are elementary regular tensor terms and those of the form \([w]\) are elementary singular tensor terms.

(The adjective “tensor” will often be omitted in the following.)

It follows from the definition, that an index in a term cannot occur more than twice.

**Definition 4.2.** An index occurring in a term \( t \) is free in \( t \) if it occurs in \( t \) exactly once. An index occurring in \( t \) twice (once as an upper one and once as a lower one) is bound in \( t \). A term is \( \beta \)-reduced if it has no bound indices. Terms \( t, t' \) that can be obtained from each other by renaming bound indices are \( \alpha \)-equivalent, \( t \equiv_\alpha t' \).

**Definition 4.3.** \( \beta \)-Reduction of tensor terms is generated by the relations

\[
[u]_i^j \cdot [v]_j^k \rightarrow_\beta [w]_i^k, \quad [w]_i^j \rightarrow_\beta [w], \quad [e] \rightarrow_\beta 1,
[a_1 \ldots a_n] \rightarrow_\beta [a_n a_1 \ldots a_{n-1}] \text{ for } a_1, \ldots, a_n \in T,
\]

where \( \epsilon \) denotes the empty word. Terms related by \( \beta \)-reduction are \( \beta \)-equivalent, notation: \( t \equiv_\beta s \).

The meaning of \( \beta \)-reduction will be discussed shortly in Section 4.1.1.

It is easy to see that free indices are invariant under \( \beta \)-equivalence and that any term is \( \beta \)-equivalent to a \( \beta \)-reduced one. Note also that \( \alpha \)-equivalent terms are automatically \( \beta \)-equivalent, because their \( \beta \)-reduced forms coincide.

**Definition 4.4.** A \( \beta \)-reduced term is regular if it is the product of elementary regular terms. A general tensor term is regular if it is \( \beta \)-equivalent to a \( \beta \)-reduced regular term. A tensor term that is not regular is singular.

The elementary term \([\epsilon]_i^j\) is denoted as \( \delta_i^j \), more generally, we write \( \delta_{i_1 \ldots i_k}^{j_1 \ldots j_k} \) for the term \( \delta_{i_1}^{j_1} \cdots \delta_{i_k}^{j_k} \). When all indices \( i_1, \ldots, i_k, j_1, \ldots, j_k \) are pairwise distinct, the term \( \delta_{i_1 \ldots i_k}^{j_1 \ldots j_k} \) is called Kronecker delta.
A term $t$ is *lexical* if it is regular and not $\beta$-equivalent to some term of the form $\delta^i_j \cdot t'$ with $(i, j) \in FI(t)$.

Multiplication by Kronecker deltas amounts to renaming indices: if $t$ is a term with $(i, j) \in FI(t)$ and $(i', j') \perp I(t)$, then $\delta^i_j \cdot t \equiv \beta t'[\{i'/i\}]$ and $\delta^j_i \cdot t \equiv \beta t'[\{j'/j\}]$. For the case of $i = j$, we have from (4.2) that $\delta^i_i \equiv \beta 1$.

As for lexical terms they can be equivalently characterized as $\beta$-equivalent to regular $\beta$-reduced ones of the form $[w_1]^{i_1}_{j_1} \cdots [w_k]^{i_k}_{j_k}$ where all $w_1, \ldots, w_k$ are nonempty.

### 4.1.1. Geometric representation

We think of regular terms as bipartite graphs having indices as vertices and edges labeled with words, the direction of edges being from lower indices to upper ones.

A regular term $[w]_i^j$, $i \neq j$, corresponds to a single edge from $i$ to $j$ labeled with $w$, the product of two terms without common indices is the disjoint union of the corresponding graphs, and a term with repeated indices corresponds to the graph obtained by gluing edges along matching vertices. The unit 1 corresponds to the empty graph. As for singular terms, such as $[w]_i^i$, they correspond to closed loops (with no vertices) labeled with cyclic words (this explains the last relation in (4.2)). These arise when edges are glued cyclically. Singular terms are pathological, but we need them for consistency of definitions. Note however our convention that when there are no labels, loops evaporate ($\delta^i_i \equiv \beta 1$), so that the singularity does not arise. The correspondence between terms and edge-labeled graphs is illustrated in Figure 3. We emphasize that this geometric representation is an invariant of $\beta$-equivalence.

Finally, we note that a Kronecker delta is a bipartite graph with no edge labels, and a lexical term, vice versa, is a graph whose every edge has a nonempty label.

(We also remark that in the system introduced in [Sla21b], which preceded this work, empty loops were not factored out. This was adequate for problems considered there.)

### 4.1.2. Remarks on binding and multiplication

We emphasize that binding in terms is *global*, not restricted to some “scope". If $t$ is a subterm (a factor) in a term $s$ and $t$ has a bound index $i$, then $i$ is bound everywhere in $s$. In particular, if $t$ and $t'$ both have a bound index $i$ then the expression $tt'$ simply is not a term (it has four occurrences of $i$). In general, because term multiplication is only partially defined (concretely, the expression $ts$ is a term if $I(t) \perp I(s)$), we have to be careful when commuting multiplication with $\beta$- or $\alpha$-equivalence. It can be that $t \equiv \beta t'$, but for some term $s$ the product $ts$ is well-defined, while $t's$ is not a term at all because of index collisions.

As a compensation, multiplication of terms is strictly associative whenever defined: if $(ts)k$ is a term then $t(sk)$ is also a term and the two are strictly equal. Also, if $t \rightarrow \beta t'$, $s \rightarrow \beta s'$, $k \rightarrow \beta k'$ and the expression $tsk$ is a term then $t's'k'$ is also a term and $tsk \rightarrow \beta t's'k'$.

Passing to equivalence classes we have the following: if $t \equiv \beta t'$, $s \equiv \beta s'$, $k \equiv \beta k'$ and both
expressions \textit{tsk}, \textit{t's}'\textit{k} are well-defined terms, then \textit{tsk} \equiv_\beta \textit{t's}'\textit{k}; the same property holds for \(\alpha\)-equivalence.

(On the other hand, one could argue that some amount of non-associativity, in the case of linguistic applications, might be an \textit{advantage} rather than a drawback. Such an observation might suggest possible directions for refinements and modifications for the system introduced in this paper.)

4.2. Tensor formulas and types.

\textbf{Definition 4.5.} Given a set \(\text{Lit}_+\) of \textit{positive literals}, where every element \(p \in \text{Lit}_+\) is assigned a \textit{valency} \(v(p) \in \mathbb{N}^2\), the set \(\overline{Fm}\) of \textit{tensor pseudoformulas} is built according to the grammar in Figure 4, where \(\text{Lit}_-\) and \(\text{Lit}\) are, respectively, the set of \textit{negative literals} and of all literals. The convention for negative literals is that \(v(\overline{p}) = (m,n)\) if \(v(p) = (n,m)\). The set \(\overline{At}\) is the set of tensor pseudoformulas.

Duality \(\overline{\cdot}\) is not a connective or operator, but is definable. The symbols \(\nabla, \Delta\) are \textit{binding operators}. They bind indices exactly in the same way as quantifiers bind variables. The operator \(Q \in \{\nabla, \Delta\}\) in front of an expression \(Q^i_j A\) has \(A\) as its scope and binds all lower occurrences of \(i\), respectively, upper occurrences of \(j\) in \(A\) that are not already bound by some other operator.

\textbf{Definition 4.6.} A pseudoformula \(A\) is \textit{well-formed} when any index has at most one free upper and one free lower occurrence in \(A\), and every binding operator binds exactly one lower and one upper index occurrence. A well-formed pseudoformula \(A\) is a \textit{tensor formula} or a \textit{pseudotype}. If moreover \(FI^\ast(A) \cap FI_\ast(A) = \emptyset\) then \(A\) is a \textit{tensor type}.

Note that definitions of free and bound indices for tensor terms and tensor formulas are \textit{different}. In particular, unlike terms, general tensor formulas (that are not tensor types) may have repeated free indices (i.e. an index may have both an upper and a lower free occurrence). Also, unlike the case of terms, binding in tensor formulas is \textit{local}, visible only in the scope of a binding operator.

Tensor formulas that can be obtained from each other by renaming bound indices are \(\alpha\)-\textit{equivalent}. The set of atomic types (i.e. coming from atomic pseudoformulas) is denoted as \(At\).

\textbf{Definition 4.7.} A \textit{tensor pseudotype context} \(\Gamma\), or, simply, a \textit{tensor context} is a finite multiset of pseudotypes such that for any two distinct \(A, B \in \Gamma\) we have that \(FI(A) \perp FI(B)\). The pseudotype context \(\Gamma\) is a \textit{type context} if \(FI^\ast(\Gamma) \cap FI_\ast(\Gamma) = \emptyset\), where \(FI(\Gamma) = \bigcup_{A \in \Gamma} FI(A)\).
An ordered tensor context $\Gamma$ is defined identically with the difference that $\Gamma$ is a sequence rather than a multiset.

Again, in a pseudotype context an index can have two free occurrences, once as a lower one and once as an upper one. For example the expression $a^i, \overline{a^i}$ is a legitimate pseudotype context, but not a type context.

4.3. Sequent and typing judgements.

**Definition 4.8.** A pseudotyping judgement $\Sigma$ is an expression of the form $t :: \Gamma$, where $\Gamma$ is a tensor context and $t$ is a tensor term such that

$$I(t) \perp FI(\Gamma), \quad FI(t) \cup FI(\Gamma) = (FI(t) \cup FI(\Gamma))^\dagger.$$  \hspace{1cm} (4.3)

When $\Gamma$ is a type (not just pseudotype) context, we say that $\Sigma$ is a tensor typing judgement.

A tensor sequent is an expression of the form $\vdash \Sigma$, where $\Sigma$ is a pseudotyping judgement.

An ordered pseudotyping judgement and ordered tensor sequent are defined identically with the difference that the tensor context should be ordered.

(In the paper [Sla21b] preceding this work we used a slightly different notation for tensor sequents with the term to the left of the turnstile.)

Spelling out defining relation (4.3): if we erase from $\Sigma$ all bound indices of $\Gamma$, then every remaining index has exactly one upper and one lower occurrence. When $\Sigma$ is a tensor typing judgement, we have $FI(t) = FI(\Gamma)^\dagger$, i.e. every index occurring in $\Sigma$ freely has exactly one free occurrence in $\Gamma$ and one in $t$. In this case we read $\Sigma$ as “$t$ has type $\Gamma$”. A pseudotyping judgement is not a genuine typing judgement if there are repeated free indices in the tensor context (i.e. some index in the tensor context has both an upper and a lower free occurrence). When $t = 1$ we write the judgement $\Sigma$ simply as $\Gamma$. When $\Gamma$ consists of a single formula $F$, we write $\Sigma$ as $t : F$. When $t$ is $\beta$-reduced or lexical we say that $\Sigma$ is, respectively, $\beta$-reduced or lexical.

The ordered and unordered contexts and sequents correspond to two possible formulations of sequent calculus. We will generally use the unordered version, but for geometric representation of tensor sequents and sequent rules we need to consider explicit Exchange rule. The definitions below apply to both versions.

**Definition 4.9.** If $t, t'$ are terms with $t \equiv_\beta t'$ then the pseudotyping judgements $t :: \Gamma$ and $t' :: \Gamma$ are $\beta$-equivalent, $t :: \Gamma \equiv_\beta t' :: \Gamma$.

$\alpha$-Equivalence of pseudo-typing judgements is generated by the following:

$t \equiv_\alpha t', F \equiv_\alpha F' \Rightarrow t :: \Gamma, F \equiv_\alpha t' :: \Gamma, F'$,

$i \in FI^*(t) \cup FI^*(\Gamma), \quad j \text{ fresh } \Rightarrow t :: \Gamma \equiv_\alpha (t :: \Gamma)^{[j/i]}.$

$\eta$-Expansion of pseudo-typing judgements is the transitive closure of the relation defined by

$i \in FI^*(\Gamma) \cap FI_\gamma(\Gamma), \quad j \text{ fresh } \Rightarrow t :: \Gamma \rightarrow_\eta \delta^i_j t :: \Gamma^{[j/i]}, \quad t :: \Gamma \rightarrow_\eta \delta^i_j t :: \Gamma^{[j/i]}.$

$\eta$-Reduction of judgements is the opposite relation of $\eta$-expansion, $\eta$-equivalence ($\equiv_\eta$) is the symmetric closure of $\eta$-expansion.

Tensor sequents $\vdash \Sigma, \vdash \Sigma'$ are, respectively $\alpha$-, $\beta$- or $\eta$-equivalent iff $\Sigma$ and $\Sigma'$ are.

$\eta$-Expansion removes from the tensor context free index repetitions by renaming them and, simultaneously, adds Kronecker deltas to the term as a compensation. Typing judgements have no repeated free indices in the tensor context and are, thus, $\eta$-long in the sense
that they cannot be $\eta$-expanded. A general pseudotyping judgement could be thought as a shorthand notation for its $\eta$-expansion; the sequent $\vdash a^i, \pi_j$ “morally” is a short for $\vdash \delta^j_i :: a^i, \pi_j$. Lexical pseudotyping judgements, on the contrary, are $\eta$-reduced in the sense that they are not $\eta$-expansions of anything, for having no Kronecker deltas in the term. Lexical typing judgements, at once $\eta$-long and $\eta$-reduced, will have natural interpretation as nonlogical axioms. It is easy to see that any pseudotyping judgement has (many $\alpha\beta$-equivalent) $\eta$-long and $\eta$-reduced forms.

![Figure 5. Tensor sequent](image_url)

4.3.1. Geometric representation. If tensor terms can be thought as edge-labeled graphs, then ordered pseudotyping judgements or tensor sequents correspond to particular pictorial representations of these graphs. Especially natural the representation is for genuine typing judgements, so we discuss it first.

Let $\Sigma = t :: \Gamma$, where $t$ is $\beta$-reduced, be an ordered typing judgement, i.e. such that $FI(t) = FI(\Gamma)^l$. (When the term is not $\beta$-reduced, we replace it with its $\beta$-reduced form.) We interpret free indices of $\Gamma$ as vertices. For a pictorial representation we equip the set of vertices with a particular ordering corresponding to their positioning in $\Gamma$. Say, indices occurring in the same formula are ordered from left to right, from top to bottom (first come the upper indices, then come the lower ones), and the whole set of free indices occurring in $\Gamma$ is ordered lexicographically according to the ordering of formulas in $\Gamma$, i.e. indices occurring in $A$ come before indices occurring in $B$ if $A$ comes before $B$ in $\Gamma$. We depict them aligned, say, horizontally in this order.

The edge-labeled graph on these vertices is constructed as follows. Free indices in $\Gamma$ are in bijection with free indices/vertices of $t$ and we connect them with labeled edges corresponding to factors of $t$. That is, for any index $\mu \in FI^*(\Gamma)$ we have that $\mu \in FI^*(t)$ and there is unique $\nu \in FI^*(t)$ such that $t$ has the form $[w]_{\mu}^{\nu}t'$, so that $t$, seen as a graph, contains an edge from $\mu$ to $\nu$ labeled with the word $w$. It follows that $\nu \in FI^*(\Gamma)$, and we draw an edge from $\mu$ to $\nu$ with the label $w$. In this way every index/vertex in $FI(\Gamma)$ becomes adjacent to a (unique) edge.

The constructed graph is a specific geometric representation of the graph corresponding to $t$, the representation being induced by a particular ordering of vertices. Note that the direction of edges is from upper indices of $\Gamma$ to lower ones (upper indices of $\Gamma$ correspond to lower indices of $t$ and vice versa). Also, note that bound indices of $\Gamma$ are not in the picture.

When $\Sigma = t :: \Gamma$ is only a pseudo-typing judgement, i.e. there are repeated free indices in $\Gamma$, we treat it as a short expression for its $\eta$-long expansion $t' :: \Gamma'$. Any pair of repeated free index occurrences in $\Gamma$ corresponds to a Kronecker delta, i.e. an edge in $t'$, connecting two distinct vertices and carrying no label. In this case, for geometric representation of $\Sigma$, we take as vertices the set of free index occurrences rather than indices in $\Gamma$ and order them in the same way as above. In the picture, we connect every pair of repeated free indices/vertices...
The prescription for the remaining indices has already been described. An example of a concrete pseudotyping judgement representation is given in Figure 5.

Observe that $\alpha \beta \eta$-equivalent pseudo-typing judgements or sequents have identical geometric representation (up to vertex labeling). Usually it is convenient to erase indices from the picture, avoiding notational clutter. For example, the pseudotyping judgement $[xy]_{lm} \cdot [ba]_{ij} : a_l \otimes b_k i j, c_{km} \cdot \delta_{\mu l}$ is $\alpha$-equivalent to the one in Figure 5, while $[xy]_{lm} \cdot [ba]_{ij} \cdot \delta_{\mu l} : a_i \otimes b_{\mu l} i j, c_{km}$ is an $\eta$-expansion of the latter. Also, we note that lexical typing judgements correspond to pictures where every edge has a non-empty label.

4.4. Sequent calculus.

4.4.1. Rules. We defined tensor sequents in order to define tensor grammars, which generate languages from a given lexicon of typing judgements. But at first, we introduce the underlying purely logical system of extended tensor type calculus (ETTC). (The title “extended”, introduced in [Sla21b], refers to usage of binding operators, which extend plain types of MLL.) The system of ETTC, being cut-free, does not use any non-logical axioms and is independent from terminal alphabets.

Our default formulation involves unordered tensor sequents. For geometric representation of the rules one needs an ordered formulation with the explicit rule for exchange; this will be discussed in the next subsection.

Definition 4.10. The system of extended tensor type calculus (ETTC) is given by the rules in Figure 6a, where sequents are unordered, and it is assumed that all expressions are well-formed, i.e. there are no forbidden index repetitions and upper occurrences match lower ones.

The requirement that all expressions in Figure 6a must be well-formed is not to be overlooked. It imposes severe restrictions on the rule premises. These are spelled out in Figure 6b (recall our shorthand notation for pairs of sets introduced in the beginning of this section).

Proposition 4.11. The rules in Figure 6a transform well-formed tensor sequents to well-formed tensor sequents iff the premises satisfy the conditions in Figure 6b.

Proof.

$\bullet\ (\alpha \eta \rightarrow)$, $(\alpha \eta \rightarrow)$: Sufficiency is obvious. The rule adds two matching occurrences of the fresh index $j$, and one free occurrence of $i$ in the tensor context gets moved to the term without changing the polarity. Let us check necessity. For definiteness, consider the $(\alpha \eta \rightarrow)$ rule, the other case being identical.

There are two possibilities for the term in the conclusion of the rule to be well-formed. Either $i \in FI_\bullet(t), i \notin I^\bullet(t)$ or $i \notin |I(t)|$. If the first possibility holds, then it must be that $i \in FI^\bullet(\Gamma)$ for the premise to be well-formed. Assume that the second possibility holds. Then it must be that $i \in FI_\bullet(\Gamma^{[i/d]}) = FI_\bullet(\Gamma)$ for the conclusion of the rule to be well-formed. But $i \in FI_\bullet(\Gamma)$ and $i \notin |I(t)|$ implies, again, $i \in FI^\bullet(\Gamma)$ or the premise is not well-formed. Thus, in both cases the condition $i \in FI^\bullet(\Gamma)$ must hold.

Consider the second condition, $j \notin |I(t)| \cup |FI(\Gamma)|$. We note that the conclusion of the rule has a free upper occurrence of $j$ located in the tensor context (the one replacing $i$ in
\[
A \in At \quad \vdash t :: \Gamma, A \vdash s :: \overline{A}, \Theta \quad \text{(Cut)} \quad \vdash t :: \Gamma, t \equiv t' \quad \text{(\equal)}
\]

\[
\vdash t :: \Gamma
\]

\[
\vdash \delta^j t :: \Gamma^{[j]} \quad \vdash t :: \Gamma
\]

\[
\vdash \delta^j t :: \Gamma^{[j]} \quad \vdash t :: \Gamma \quad \vdash \delta^j t :: \Gamma^{[j]} \quad \vdash t' :: \Gamma
\]

\[
\vdash t :: \Gamma, A, B \vdash \exists \quad \vdash t :: \Gamma, A \vdash s :: B, \Theta \quad \vdash \delta^j \mu t :: \Gamma, A \quad \vdash t :: \Gamma, \nabla^\mu A (\nabla) \quad \vdash t :: \Gamma, A, \Gamma
\]

(A) Rules

\[\text{(\alpha \eta \rightarrow)} : i \in FI^*(\Gamma), j \notin |I(t)| \cup |FI(\Gamma)|. \quad \text{(\alpha \eta \rightarrow)} : i \in FI_\#(\Gamma), j \notin |I(t)| \cup |FI(\Gamma)|.\]

\[\text{(\otimes)} : |I(t)| \cap |I(s)| = |FI(\Gamma, A)| \cap |FI(B, \Theta)| = \emptyset, I(t) \perp FI(B, \Theta), I(s) \perp FI(\Gamma, A).\]

\[\text{(Cut)} : I(t) \perp I(s), FI(\Gamma) \perp FI(\Theta), FI(t) \cap FI(s)^! \subseteq FI(\Gamma)^! \cap FI(\Theta) \subseteq FI(A).\]

\[\text{(\nabla), (\triangle)} : (\nu, \mu) \in FI(A).\]

(b) Restrictions on rules

\[\Gamma^! \text{ and an occurrence of } j \text{ in the term located in the factor } \delta^j_j. \text{ If } j \notin |FI(\Gamma)| \text{ then there is a third occurrence of } j \text{ in the conclusion located in the tensor context, and if } j \in |I(t)| \text{ then there is a third occurrence of } j \text{ in the conclusion located in the term. In both cases the conclusion is not well-formed.} \]

- \text{\textbullet} (\otimes) : Sufficiency is very easy. We also need to check that all conditions are indeed necessary. Consider, for example, the first one that \(|I(t)| \cap |I(s)| = \emptyset, I(t) \perp FI(B, \Theta), I(s) \perp FI(\Gamma, A).\)
  - (Cut): Similar to the (\otimes) case.
  - (\nabla), (\triangle): The condition is necessary simply for the formula in the conclusion to be well-formed. For sufficiency we observe that there are also an upper and a lower free occurrence of \(\mu\) and \(\nu\) respectively in the premise, and a simple analysis shows that indices in the conclusion match correctly. \(\square\)

4.4.2. Geometric meaning. In order to give geometric representation of the ETTC rules we need to consider ordered sequents and enrich the system with the rule of exchange

\[
\vdash t :: \Gamma, A, B, \Theta \quad \vdash t :: \Gamma, B, A, \Theta \quad \text{\text{(Ex)}}
\]

Clearly, this results in an equivalent formulation. Then, using geometric representation of tensor sequents, the rules of ETTC can be illustrated as in Figure 7.

The identity/structure group \{\text{(Id)}, \text{(Cut)}, \text{(Ex)}\} is schematically illustrated in Figure 7A. As for the multiplicative group, the (\otimes) rule puts two graphs together in the disjoint union and
the \((\forall)\) rule does nothing in the picture. The \((\equiv_\beta)\) and \((\alpha\eta)\) rules do not change the picture either (except for vertex labels); they could be thought of as coordinate transformations.

Finally, the \((\triangle)\) rule glues together two indices/vertices, and the \((\forall)\) rule is applicable only in the case when the corresponding indices/vertices are connected with an edge carrying no label. Then this edge is erased from the picture completely (the information about the erased edge is stored in the introduced type). This is illustrated in Figures 7b, 7c (the lines of dots denote that there might be other vertices around).

4.4.3. Towards first order translation. We will discuss the exact relationship of ETTC with the linguistic fragment of MLL1 in due course, but the outline of their correspondence is the following. Indices in tensor formulas correspond to variables in predicate formulas, with upper/lower polarities of indices corresponding to left/right polarities of variables. Multiplicative connectives translate to themselves, while the binding operators \(\nabla/\triangle\) of ETTC correspond to the \(\forall/\exists\) quantifiers in the linguistic fragment. A quantifier in the linguistic fragment binds exactly two variable occurrences of opposite polarities. These two bound occurrences in a predicate formula translate to two different indices bound by the corresponding operator in the tensor formula: \(\forall x.A(x, x)\) translates to \(\nabla^i_j A^j_i\).

Tensor sequents may also contain terms, and these do not translate to the first order language directly. However, we note that Kronecker deltas behave much like (some rudimentary versions of) equalities to the left of the turnstile. Typically, the tensor sequent \(\vdash \delta^i_1; \ldots; \delta^i_k \Gamma \vdash \Gamma\) could be thought intuitively as the sequent

\[ j_1 = i_1, \ldots, j_k = i_k \vdash \Gamma \]

in a first order language with equality. At least, the \((\alpha\eta)\) rules are consistent with such an interpretation. And the \((\equiv_\beta)\) rule, in the absence of terminal symbols, amounts to the equivalences \(\delta^i_j \delta^j_k \equiv_\beta \delta^i_k\) and \(\delta^i_i \equiv_\beta 1\), which, basically correspond to transitivity and
reflexivity of equality (if we translate the term 1 as the empty context). This intuition, probably, could be developed more formally.

4.4.4. Properties. The crucial property of cut-elimination in \textbf{ETTC} will be established in a separate section below. Here we collect some simple observations about the system.

**Proposition 4.12.** In \textbf{ETTC}:

(i) the sequent $\vdash F, F$ is derivable for any formula $F$;

(ii) $\alpha$-equivalent formulas are provably equivalent, i.e. if $F \equiv_{\alpha} F'$ then the sequent $\vdash F, F'$ is derivable;

(iii) $\beta\eta$-equivalent sequents are cut-free derivable from each other;

(iv) $\alpha$-equivalent sequents are derivable from each other.

**Proof.** Claim (i) is easy. Claim (ii) is established by induction on $A, A'$, the base cases being $F = Q^\nu_A, F' = Q^\nu_y(A^{[y/\nu]}_x), Q \in \{\nabla, \triangle\}$. A derivation for this case is shown in Figure 8a. For claim (iii) note that $\beta$-equivalent sequents are derivable from each other by the $(\equiv_{\beta})$ rule, and if $\Sigma'$ is a one-step $\eta$-expansion of $\Sigma$ then $\Sigma'$ is derivable from $\vdash \Sigma$ by the $(\alpha\eta^{-})$ rule, and $\vdash \Sigma$ from $\vdash \Sigma'$ by the $(\alpha\eta_{-})$ rule.

Finally, consider claim (iv). There are two relations generating $\alpha$-equivalence of sequents. The first one involves sequents of the form $\vdash t : \Gamma, F$ and $\vdash t' : \Gamma, F'$, where $t \equiv_{\alpha} t', F \equiv_{\alpha} F'$. The statement follows from (ii), (iii). The second relation involves sequents of the form $\vdash \Sigma$ and $\vdash \Sigma^{[y/\nu]}_{[y/\nu]}$. This has two subcases: either the index $i$ has two free occurrences in the pseudotype context in $\Sigma$, or it has one free occurrence in the pseudotype context and one in the term. Derivations for both situations are shown in Figure 8b. \hfill \Box

4.4.5. Implicational types. The implicational types of \textbf{LC} can be translated to \textbf{ETTC} as

$$\left(b/a\right)_{y}^{\nu} = \nabla_{y}^{\mu}(b^{\mu}_{x}y^{2}_{x}), \quad (a/b)_{y}^{\nu} = \nabla_{y}^{\mu}(a^{\mu}_{x}y^{b}_{x}) \quad (4.4)$$

(compare with translation to \textbf{MILL1} in Figure 1d). Let us discuss the geometric meaning of this.

Terms of (pseudo-)type $\nabla_{y}^{\mu}A$ encode a particular “subtype” of $A$: consisting of terms whose indices/vertices corresponding to $\mu$ and $\nu$ are connected with an edge carrying no
Let \( A, B \) be types with exactly one free upper and one free lower index. Then regular terms of type \( A \) or \( B \) are, essentially, strings. As for the implication type \( A \rightarrow B \), which we encode in \( \text{ETTC} \) as \( \overline{A} \rightarrow B \) for definiteness (the alternative encoding is \( B \rightarrow \overline{A} \)), its terms are rather pairs of strings, as Figure 9a suggests. There are two “degenerate subtypes”, shown in Figure 9b, where one of the two strings is empty. (They are degenerate, for example, for having no lexical terms). These are precisely “subtypes” of the kind encoded by \( \Rightarrow \). Now, it is easy to see that terms of the first “subtype” act on type \( A \) terms by multiplication (concatenation) on the left, and terms of the second one act by multiplication on the right (see Figure 9c). That is, the two “degenerate subtypes” of the general, “undirected” implication type correspond precisely to two implicational types of \( \text{LC} \). Note, however, that the translations in (4.4) use both possible encodings of the undirected implication, one for the left \( \text{LC} \) implication and the other one for the right implication. This choice seems most natural.

5. Cut-elimination

5.1. \( \eta \)-Long fragment. The \( \eta \)-long fragment \( \text{ETTC}_{\eta \rightarrow} \) is \( \text{ETTC} \) restricted to \( \eta \)-long sequents, i.e. to typing judgements. The fragment can be given a separate axiomatization.

**Definition 5.1.** The system \( \text{ETTC}_{\eta \rightarrow} \) is obtained from \( \text{ETTC} \) by removing the \((\alpha \eta)\) rules and replacing the \((\text{Id})\) axioms with their \( \eta \)-long forms \((\text{Id}_{\eta \rightarrow})\) and the \((\langle \rangle), (\langle \rangle)\) rules with their closures \((\langle \rangle), (\langle \rangle)\) under \( \alpha \)-equivalence shown in Figure 10a.

**Proposition 5.2.** If a typing judgement \( \Sigma \) is derivable in \( \text{ETTC}_{\eta \rightarrow} \) then it is derivable in \( \text{ETTC} \). If the \( \text{ETTC}_{\eta \rightarrow} \) derivation of \( \Sigma \) is cut-free, then \( \Sigma \) is cut-free derivable in \( \text{ETTC} \).

**Proof.** The \((\text{Id}_{\eta \rightarrow})\) axioms of \( \text{ETTC}_{\eta \rightarrow} \) are \( \text{ETTC} \) derivable from \((\text{Id})\) axioms using \((\alpha \eta)\) rules. The \((\langle \rangle), (\langle \rangle)\) rules are cut-free emulated in \( \text{ETTC} \) as shown in Figures 10b, 10c.

(It was precisely the \( \eta \)-long fragment that was introduced and discussed in [Sla21b] under the name \( \text{ETTC} \), with the exception that in [Sla21b] we did not have the term reduction \( \delta^1 \rightarrow \beta \). Also, in [Sla21b] types were defined as quotiented under \( \alpha \)-equivalence, so that the rules \((\langle \rangle), (\langle \rangle)\) in those definitions would be identical to the original \((\langle \rangle), \).)
\[ p \in \text{Lit} \]
\[ \vdash \delta_{i_1 \ldots i_m}^{n_1 \ldots n_{k_1} \ldots k_m} : p_{j_1 \ldots j_n}, \Gamma_{k_1 \ldots k_n}^{n_1 \ldots n_{k_1} \ldots k_m} (\text{Id}_{\eta \to}) \]
\[ \vdash \delta_{j}^{i_1 \ldots i_m} : \Gamma, A_{[j/i_1]}^{[i_1/i_2]} (\Delta_{\eta \to}) \]
\[ \vdash \delta_{l}^{j_1 \ldots j_n} : \Gamma, A_{[
u/l]}^{[
u/l]} (\Delta_{\eta \to}) \]
\[ \vdash \delta_{m}^{j_1 \ldots j_n} : \Gamma, A_{[\mu/l]}^{[\mu/l]} (\Delta_{\eta \to}) \]
\[ \vdash \delta_{n}^{j_1 \ldots j_n} : \Gamma, A_{[\nu/\mu]}^{[\nu/\mu]} (\Delta_{\eta \to}) \]
\[ \vdash \delta_{s}^{j_1 \ldots j_n} : \Gamma, A_{[\nu/\mu]}^{[\nu/\mu]} (\Delta_{\eta \to}) \]

(A) \textbf{ETTC}_{\eta \to} \text{ rules}

\[ \vdash \delta_{i}^{i_1 \ldots i_m} : \Gamma, A_{[i/i]}^{[i/i]} (\Delta_{\eta \to}) \]
\[ \vdash \delta_{l}^{j_1 \ldots j_n} : \Gamma, A_{[
u/l]}^{[
u/l]} (\Delta_{\eta \to}) \]
\[ \vdash \delta_{m}^{j_1 \ldots j_n} : \Gamma, A_{[\mu/l]}^{[\mu/l]} (\Delta_{\eta \to}) \]
\[ \vdash \delta_{s}^{j_1 \ldots j_n} : \Gamma, A_{[\nu/\mu]}^{[\nu/\mu]} (\Delta_{\eta \to}) \]

(B) Emulating (\Delta_{\eta \to}) in ETTC

(C) Emulating (\Delta_{\eta \to}) in ETTC

\textbf{Figure 10. ETTC}_{\eta \to}

(\Delta) versions. The enriched ETTC of this work has the advantage of more concise notation; derivations in the ETTC_{\eta \to} tend to become overloaded with indices and barely readable.

The \eta\text{-long fragment is notationally cumbersome, but is easy to analyze. In particular, cut-elimination in ETTC_{\eta \to} is more or less routine. For the full system ETTC of this work, cut-elimination is trickier because of (a\eta) rules.

5.2. Cut-elimination for ETTC_{\eta \to}.

**Proposition 5.3.** If \Sigma, \Sigma' are \alpha-equivalent typing judgements and \pi is an ETTC_{\eta \to} derivation of \vdash \Sigma then there exists an ETTC_{\eta \to} derivation \pi' of \vdash \Sigma', which is obtained from \pi by renaming some indices.

**Proof.** Easy induction on \pi. \hfill \Box

We define the size of a derivation in ETTC_{\eta \to}, as follows. If the derivation \pi is an axiom then the size size(\pi) of \pi is 1. If \pi is obtained from a derivation \pi' by a single-premise rule size(\pi) = size(\pi') + 1. If \pi is obtained from derivations \pi_1, \pi_2 by a two-premise rule then size(\pi) = size(\pi_1) + size(\pi_2) + 1.

Given an ETTC_{\eta \to} derivation with two subderivations \pi_1, \pi_2 of the sequents \vdash t :: \Gamma, A and \vdash s :: A, \Theta respectively followed with the Cut rule, we say that this application of a cut rule is a principal cut if \pi_1, respectively \pi_2, ends with a rule introducing the formula A, respectively \overline{A}. An application of the Cut rule that is not principal is a side cut.

**Lemma 5.4.** There is an algorithm transforming an ETTC_{\eta \to} derivation \pi of a sequent \vdash \Sigma to a derivation \pi' of the same sequent, where size(\pi') = size(\pi) and all cuts in \pi' principal.

**Proof.** Let \pi_1, \ldots, \pi_n be all subderivations of \pi ending with side cuts and let \( N = \sum \text{size}(\pi_i) \).

We will use N as the induction parameter for proving termination of the algorithm.

The algorithm consists in permuting a side cut with the preceding rule introducing a side formula. We will consider only one case that seems most involved, namely, permutation with the (\Delta_{\eta \to}) rule when there is a possibility of index collision. Other cases are similar or easier.
Figure 11. Cut-elimination $\Delta/\nabla$

Let a side cut be between sequents $\vdash \Sigma_1, \vdash \Sigma_2$ of the forms
$$\vdash t :: \nabla^{\mu}_\nu B, \Phi, A \quad \vdash s :: \overline{A}, \Theta$$
respectively, where $\vdash \Sigma_1$ is obtained by the $(\nabla_{\equiv, a})$ rule from the sequent $\vdash \Sigma_0$ of the form
$$\vdash \delta_j^i t :: B_{[\mu/\nu]}^{[i/j]}, \Phi, A.$$

The transformation of derivations is shown in Figure 11a, where in order to avoid possibility of forbidden index repetitions, we use Proposition 5.3 and replace the derivation of $\vdash \Sigma_0$ with a derivation with the same size and an $\alpha$-equivalent conclusion obtained from $\Sigma_0$ by renaming $i, j$ with fresh indices. The induction parameter $N$ for the new derivation is smaller at least by 1.

Lemma 5.5. There is an algorithm transforming an ETTC$_{\eta\rightarrow}$ derivation $\pi$ of a sequent $\Sigma$ with cuts into an ETTC$_{\eta\rightarrow}$ derivation $\pi'$ of some sequent $\Sigma'$ such that $\Sigma' \equiv_\beta \Sigma$ and $size(\pi') < size(\pi)$, provided that all cuts in $\pi$ are principal.

Proof. If there is a cut
$$\vdash \delta_{k_n \ldots k_1, j_m \ldots j_1} :: p_{j_1 \ldots j_m, k_1 \ldots k_n}^{i_1 \ldots i_n} \vdash \delta_{s_n \ldots s_1, l_m \ldots l_1} :: p_{l_1 \ldots l_m, k_1 \ldots k_n}^{i_1 \ldots i_n} \vdash \delta_{s_n \ldots s_1, j_m \ldots j_1} :: p_{j_1 \ldots j_m, k_1 \ldots k_n}^{i_1 \ldots i_n} \vdash \delta_{s_n \ldots s_1, l_m \ldots l_1} :: p_{l_1 \ldots l_m, k_1 \ldots k_n}^{i_1 \ldots i_n}$$
between (Id$_{\eta\rightarrow}$) axioms then its conclusion is $\beta$-equivalent to the (Id$_{\eta\rightarrow}$) axiom
$$\vdash \delta_{s_n \ldots s_1, l_m \ldots l_1} :: p_{l_1 \ldots l_m, k_1 \ldots k_n}^{i_1 \ldots i_n}.$$

The transformation of a principal cut between a $\Delta$- and a $\nabla$-formula is shown in Figure 11b, where, similarly to Proposition 5.4, we use Proposition 5.3 and replace derivations of the premises with derivations of $\alpha$-equivalent sequents. The conclusion of the new derivation is $\beta$-equivalent to the conclusion of the original one, they are obtained from each other by renaming bound indices in the terms. (That is, the terms are $\alpha$-equivalent, but $\alpha$-equivalent terms are automatically $\beta$-equivalent).
The case of a \( \otimes\)- and a \( \exists\)- formula is indistinguishable from the familiar case of multiplicative linear logic.

**Corollary 5.6.** The system \( \text{ETTC}_{\eta\rightarrow} \) is cut-free.

**Proof.** Combining Lemma 5.4 and Lemma 5.5 we prove by induction on the size of an \( \text{ETTC}_{\eta\rightarrow} \) derivation \( \pi \) of a sequent \( \vdash \Sigma \) that \( \pi \) transforms to a cut-free derivation of some \( \beta\)-equivalent sequent \( \vdash \Sigma' \). But \( \vdash \Sigma' \) is cut-free derivable from \( \vdash \Sigma \) in \( \text{ETTC}_{\eta\rightarrow} \) by the \((\equiv_\beta)\) rule.

\( \square \)

### 5.3. Conservativity of \( \text{ETTC}_{\eta\rightarrow} \).

**Definition 5.7.** If \( \Gamma \) is a pseudotype context, the degeneracy \( D(\Gamma) \) of \( \Gamma \) is the index set

\[ D(\Gamma) = \{ i \mid i \in FI^*(\Gamma) \cap FI_\bullet(\Gamma) \}. \]

If \( \Sigma \) is a pseudotyping judgement of the form \( \vdash t : \Gamma \), then the degeneracy \( D(\Sigma) \) of \( \Sigma \) is the set \( D(\Sigma) = D(\Gamma) \). A typing judgement \( \Sigma \) is an \( \eta\)-long representative of \( \Sigma \) if \( \Sigma \) has the form

\[ \delta_{i_1\ldots i_n} \vdash t :: \Gamma^{[\nu_1/\eta_1,\ldots,\nu_n/\eta_n]}, \]

where \( \{i_1, \ldots, i_n\} = D(\Sigma) \).

We will also use the following notation for pairs of contexts: if \( \Gamma, \Theta \) are pseudotype contexts then

\[ D(\Gamma; \Theta) = \{ i \mid i \in FI^*(\Gamma) \cap FI_\bullet(\Theta) \}, \]

so that

\[ D(\Gamma, \Theta) = D(\Gamma) \cup D(\Theta) \cup D(\Gamma; \Theta) \cup D(\Theta; \Gamma). \]

**Lemma 5.8.** If \( \Sigma \) is a pseudotyping judgement and \( \vdash \text{ETTC} \Sigma \), then for any \( \eta\)-long representative \( \Sigma \) of \( \Sigma \) it holds that \( \vdash \text{ETTC}_{\eta\rightarrow} \Sigma \).

**Proof.** Induction on the \( \text{ETTC} \) derivation of \( \Sigma \).

For the base we have that \( \eta\)-long representatives of (Id) axioms of \( \text{ETTC} \) are (Id_{\eta,\rightarrow}) axioms of \( \text{ETTC}_{\eta\rightarrow} \). If \( \Sigma \) is obtained from a sequent \( \Sigma' \) by an \((\eta\eta)\) or the \((\equiv_\beta)\) rule then any \( \eta\)-long representative of \( \Sigma \) is \( \alpha\)-equivalent to some \( \eta\)-long representative of \( \Sigma' \) and the statement follows from the induction hypothesis. The case of the \((\otimes\) or the \((\exists)\) rule is very easy.

Let \( \vdash \Sigma \) of the form \( \vdash \delta' t :: \Delta\mu A, \Gamma \) be obtained from the sequent \( \vdash \Sigma' \) of the form \( \vdash t :: A, \Gamma \), where \( (\nu, \mu) \in FI(A) \), by the \((\Delta)\) rule. There are different possibilities on the structure of \( D(\Sigma') \); we will consider the most involved case, when \( \mu, \nu \in D(\Sigma') \). In this case we have \( \mu, \nu \notin D(\Sigma) \) (since repetitions of \( \mu, \nu \) in the context are eliminated by the \( \Delta \) operator) and, in fact, \( D(\Sigma') = D(\Sigma) \cup \{ \mu, \nu \} \). Any \( \eta\)-long representative \( \Sigma \) of \( \Sigma \) has the form

\[ \delta_{i_1\ldots i_n} \cdot \delta' \mu t :: (\Delta\nu A, \Gamma)^{[\nu_1/\eta_1,\ldots,\nu_n/\eta_n]}, \]

where \( \{i_1, \ldots, i_n\} = D(\Sigma) \). It follows that the typing judgement \( \Sigma' \) of the form

\[ \delta_{i_1\ldots i_n} \cdot \delta' \mu t :: (A^{[\nu/\eta]}, \Gamma)^{[\nu_1/\eta_1,\ldots,\nu_n/\eta_n]} \]
is an \(\eta\)-long representative of \(\Sigma'\), and, by the induction hypothesis, \(\vdash_{\text{ETTC}_{\eta}} \Sigma'\). But \(\vdash \Sigma\) is \(\text{ETTC}_{\eta}\)-derivable from \(\vdash \Sigma'\) by the \(\langle \triangle \equiv_\eta \rangle\) rule followed by the \(\langle \equiv_\beta \rangle\) rule. Other possibilities of the \(\langle \triangle \rangle\) rule are similar or easier. The case of the \(\langle \triangledown \rangle\) rule is analogous to the \(\langle \triangle \rangle\) rule.

The most involved case is that of the Cut rule. Let \(\vdash \Sigma\) of the form \(\vdash ts :: \Gamma, \Theta\) be obtained from \(\vdash \Sigma_1, \vdash \Sigma_2\) of the forms \(\vdash t :: \Gamma, A, \vdash s :: A, \Theta\) respectively by the Cut rule.

Any \(\eta\)-long representative \(\Sigma\) of \(\Sigma\) can be written as

\[
\delta_\Gamma \delta_\Theta \delta_\Gamma A \delta_\Theta t s :: \Gamma', \Theta',
\]

(5.3)

where \(\Gamma', \Theta'\) are obtained from \(\Gamma, \Theta\) respectively by renaming repeated free indices, and \(\delta_\Gamma\), \(\delta_\Theta\), \(\delta_\Gamma A\), \(\delta_\Theta A\) are Kronecker delta corresponding to this renaming, respectively, in \(D(\Gamma)\), \(D(\Theta)\), \(D(\Theta; \Gamma)\) (see Figure 12 for explicit expressions with indices).

We are going to construct \(\eta\)-long representatives \(\Sigma_1, \Sigma_2\) of \(\Sigma_1, \Sigma_2\) respectively such that \(\Sigma_1\), up to \(\beta\)-equivalence, is obtained from them by a cut.

Note that we have the equalities

\[
D(\Gamma; \Theta) = D(\Gamma; A) = D(A; \Theta), \quad D(\Theta; \Gamma) = D(\Theta; A) = D(A; \Gamma).
\]

Indeed, if an index \(i\) occurs freely both in \(\Gamma\) and \(\Theta\), then, in order for \(\Sigma\) to be well-formed, it should have just one free occurrence in \(\Gamma\) and one in \(\Theta\), and no occurrences in \(ts\) whatsoever. Then in order for \(\Sigma_1, \Sigma_2\) to be well-formed, there must be a free occurrence of \(i\) in \(A\) (hence in \(A\)) to match the occurrence in \(\Gamma\) for \(\Sigma_1\) and in \(\Theta\) for \(\Sigma_2\). With this observation is to construct desired \(\Sigma_1, \Sigma_2\), respectively of the forms

\[
\delta_\Gamma \delta_\Theta A \delta_\Gamma t s :: \Gamma', A', \quad \delta_\Theta \delta_\Theta A \delta_\Theta t s :: A', \Theta',
\]

where we use the same notational convention for Kronecker deltas as in (5.3), so that the terms satisfy the relations

\[
\delta_\Gamma A \delta_\Theta A \equiv_\beta \delta_\Theta A, \quad \delta_\Theta A \delta_\Theta A \equiv_\beta \delta_\Theta A, \quad \delta A \delta_\Theta A \equiv_\beta 1.
\]

Explicit formulas are given in Figure 12.

Then the cut between \(\vdash \Sigma_1, \vdash \Sigma_2\) yields a sequent \(\beta\)-equivalent to \(\vdash \Sigma\). Since derivability in \(\text{ETTC}_{\eta}\), is closed under \(\beta\)-equivalence, the statement follows from the induction hypothesis.

\[\square\]

**Corollary 5.9.** \(\text{ETTC}_{\eta}\) is a conservative fragment of \(\text{ETTC}\). \(\square\)

**Corollary 5.10.** \(\text{ETTC}\) is cut-free.
Proof. Assume that ⊢_{\text{ETTC}} \Sigma. Let \tilde{\Sigma} be an \eta-long form of \Sigma. In particular, \tilde{\Sigma} is an \eta-long representative of \Sigma, and, using the preceding lemma and cut-elimination for \text{ETTC}_{\eta \rightarrow}, the sequent ⊢ \tilde{\Sigma} in cut-free derivable in \text{ETTC}_{\eta \rightarrow}. Then, by Proposition 5.2, the sequent ⊢ \tilde{\Sigma} is cut-free derivable in \text{ETTC}. But ⊢ \Sigma is cut-free derivable from ⊢ \tilde{\Sigma} in \text{ETTC} using (\alpha \eta) rules.

6. Natural deduction and grammars

For derivations from nonlogical axioms it seems more convenient to allow variables standing for tensor terms, in the style of natural deduction (abbreviated below as “n.d.”). N.d. formulation might also be more convenient if we want to consider the purely intuitionistic fragment of \text{ETTC} that is restricted to types built using only binding operators, tensor and linear implication (the latter defined as \( A \to B = A \Join B \) or \( A \to B = B \Join A \)).

Definition 6.1. Given a terminal alphabet \( T \) and a countable set \( \text{var} \otimes \) of tensor variable symbols of different valencies, where valency \( v(x) \) of the symbol \( x \) is a pair of nonnegative integers, the set \( \text{Var} \otimes \) of tensor variables is the subset of \( \{ x_{i_1}^{m_1}, \ldots, x_{i_m}^{m_m} | x \in \text{var} \otimes, v(x) = (m, n), i_1, \ldots, i_m, j_1, \ldots, j_n \in \text{Ind} \} \) satisfying the constraint that there are no repeated indices. N.d. tensor terms are defined same as tensor terms in Definition 4.1 except that the set \( \text{Var} \otimes \) is added to sets (4.1) of generators.

β-Reduction of n.d. tensor terms is defined same as β-reduction of tensor terms in Definition 4.3. Lexical, n.d. terms are defined as in Definition 4.4 with the addition that they contain no tensor variables.

Definition 6.2. An n.d. tensor pseudotyping, respectively typing, judgement is an expression of the form \( t : C \) where \( t \) is an n.d. tensor term and \( C \) a tensor pseudotype, respectively type, satisfying the same relation (4.3) as \( t \) and \( \Gamma \) in Definition 4.8 of usual tensor pseudotyping judgement. Lexical, β-reduced and α-, β- or η-equivalent pseudotyping judgements are defined as previously in Definition 4.9.

A variable declaration is an n.d. typing judgement \( X : A \), where \( X \) is a tensor variable. An n.d. typing context \( \Gamma \) is a finite set (rather than a multiset) of variable declarations that have no common variable symbols and such that for any two distinct variables \( X, Y \) in \( \Gamma \) we have \( I(X) \perp I(Y) \).

An n.d. tensor sequent is an expression of the form \( \Gamma \vdash \Sigma \), where \( \Gamma \) is an n.d. typing context, and \( \Sigma \) an n.d. pseudotyping judgement. α-, β-, η-Equivalence of n.d. sequents are generated by the relation
\[
\Sigma \equiv_\xi \Sigma' \implies (\Gamma \vdash \Sigma) \equiv_\xi (\Gamma \vdash \Sigma'), \quad \text{where } \xi \in \{ \alpha, \beta, \eta \}.
\]

Definition 6.3. Natural deduction system \text{ETTC}_{n.d.} is given by the rules in Figure 13 with the usual implicit restriction that all expressions are well-formed.

Note that there are two pairs of introduction/elimination rules for the \( \Join \) connective. They correspond to two different encodings of implication. If we are interested only in the intuitionistic fragment then one of the two pairs (depending on the chosen encoding) is not needed.

Proposition 6.4. In \text{ETTC}_{n.d.}, αβη-equivalent sequents are derivable from each other.
Proof. Similar to Proposition 4.12.

Proposition 6.5. The system ETTC<sub>n.d.</sub> is closed under the following substitution rule

$$
\Gamma \vdash t : A, \ X : A, \Theta \vdash X \cdot s : B \\
\Gamma, \Theta \vdash ts : B
$$

Proof. The rule is emulated with the (\(\forall I_i\)) rule followed by the (\(\forall E_l\)) rule.

We defined typing contexts in n.d. sequents as unordered sets. However, notationally it will be convenient to consider them as “vectors” with enumerated components as in the following.

Let \(\vec{X} = \{X(1), \ldots, X(n)\}\) be a sequence of tensor variables and \(\Gamma = A(1), \ldots, A(n)\) be an ordered tensor type context (we put sequence numbers in brackets in order to avoid confusion with indices). We denote the n.d. typing context \(\vec{X}_n : A(1), \ldots, X(n) : A(n)\) as \(\vec{X} : \Gamma\). Given a term \(t\), we denote the term \(tX_1 \cdots X_n\) as \(t\vec{X}\). Finally, we write \(\Gamma\) for the ordered type context \(\vec{A}(n), \ldots, \vec{A}(1)\). (As usual, it is implicitly assumed that all these expressions are well-formed).

Definition 6.6. An n.d. tensor sequent is standard if it has the form \(\vec{X} : \Gamma \vdash t\vec{X} : B\).

Proposition 6.7. Any sequent derivable in ETTC<sub>n.d.</sub> is standard.

Proposition 6.8 ("Deduction theorem"). Consider a finite set

$$
\Xi = \{\tau(1) : A(1), \ldots, \tau(n) : A(n)\}
$$

of n.d. lexical typing judgements with \(FI(A(i)) \perp FI(A(j))\) for \(i \neq j, \ i, j = 1, \ldots, n\).
An n.d. sequent $\Sigma$ of the form $\Gamma \vdash t : B$ is $\text{ETTC}_{n.d.}$ derivable from $\Xi$ using each element of $\Xi$ exactly once if and only if there exist an n.d. sequent $\Sigma'$ of the form

$$X(1) : A(1), \ldots, X_n : A(n), \Gamma \vdash t'X(1) \ldots X(n) : B$$

(6.1)
derivable in $\text{ETTC}_{n.d.}$ without nonlogical axioms such that

$$t' \cdot \tau(1) \cdot \ldots \cdot \tau(n) \equiv_\beta t.$$ 

(6.2)

Proof. If there is $\Sigma'$ satisfying (6.1), (6.2), then $\Sigma$ is $\text{ETTC}_{n.d.}$ derivable from $\Xi$ and $\Sigma'$ by a series of substitution rules using each element of $\Xi$ exactly once. Thus, if $\vdash_{\text{ETTC}_{n.d.}} \Sigma'$ then $\Sigma$ is $\text{ETTC}_{n.d.}$ derivable from $\Xi$ as required. The other direction is proven by an easy induction on the n.d. derivation of $\Sigma$.

6.1. Translation to sequent calculus.

**Proposition 6.9.** Given a standard n.d. sequent $\Xi$ of the form $\vec{\Xi} : \Gamma \vdash t\vec{\Xi} : A$, the expression $\Xi'$ of the form $t :: A, \Gamma$ is a well-defined tensor pseudotyping judgement.

Proof. Let us denote $\vec{t} = t\vec{\Xi}$ and $\Theta = A, \Gamma$.

The n.d. type context $\vec{\Xi} : \Gamma$, by definition, consists of variable declarations. In any variable declaration $Y : F$ we have that $FI(Y) = FI(\vec{F})$, because it is an n.d. typing judgement, and we have $I(Y) = FI(Y)$, because $Y$, being a tensor variable, has no repeated indices. It follows that, for $FI(\vec{\Xi}) = \bigcup_{Y \in \vec{\Xi}} FI(Y)$, we have $I(\vec{\Xi}) = FI(\Gamma) = FI(\Gamma)$.

Since $\vec{t}$ is a well-formed term we have that $I(t) \downarrow I(\vec{\Xi}), I(Y) \downarrow I(Y') \forall Y, Y' \in \vec{\Xi}$. Since $\Sigma$ is a well-defined n.d. pseudotyping judgement we have $I(\vec{t}) \downarrow FI(A), FI(\vec{\Xi}) \downarrow FI(A)$. It follows that $FI(\Gamma) \downarrow FI(A), FI(F) \downarrow FI(F') \forall F, F' \in \vec{\Gamma}, I(t) \downarrow FI(\Gamma)$, and $I(t) \downarrow FI(A)$. The first two conditions mean that $\Theta$ is a well-defined pseudotype context, and the last two mean that $I(t) \downarrow FI(\Theta)$.

In order for $\Xi'$ to be a pseudotyping judgement it remains to show that $I^\ast(t) \cup FI^\ast(\Theta) = I^\ast(\vec{t}) \cup FI^\ast(\Theta)$. Again, since $\Sigma$ is an n.d. pseudotyping judgement, we have that $I^\ast(\vec{t}) \cup FI^\ast(A) = I^\ast(\vec{t}) \cup FI^\ast(\Theta)$. But $I(\vec{t}) = I(t) \cup I(\vec{\Xi}) = I(t) \cup FI(\Gamma)$ and $FI(\Theta) = FI(\Gamma) \cup FI(A)$. The desired equality follows.

**Definition 6.10.** In the notation of Proposition 6.9, the tensor sequent $\vdash \Xi'$ is the sequent calculus translation of a standard n.d. sequent $\Xi$.

**Lemma 6.11.** A standard n.d. sequent is derivable in $\text{ETTC}_{n.d.}$ whenever its sequent calculus translation is derivable in $\text{ETTC}$.

Proof. Let the n.d. sequent in question be $\vec{\Xi} : \Gamma \vdash t\vec{\Xi} : A$. Use induction on the $\text{ETTC}$ derivation of the translation $\vdash t :: A, \Gamma$. A rule introducing or modifying the “principal” formula $A$ directly translates to the corresponding n.d. rule (with two variations for the $(\forall \eta)$ rule), the same applies to the $(\equiv_\beta)$ rule. A “side” rule, i.e. modifying a formula in $\Gamma$, typically, is emulated using the substitution rule and an elimination rule for the dual connective. Basic cases of the $(\alpha \eta \forall), (\forall), (\Delta)$ and $(\otimes)$ rules are collected in Figure 14, the remaining ones are treated similarly or easier. Note that for the $(\otimes)$ rule there is also a case mirror-symmetric to that in the figure, namely, when the “principal” formula $A$ in the
⊢ δ^i_j t :: A, Θ, (B)  (αη→)  \implies

\[ Z : B \vdash Z : B \quad (\alphaη) \]
\[ Z : B \vdash \delta^i_j t : B_{[j/i]} \quad \Gamma Z \vdash \delta^i_j tYX : A \quad (Sb) \]
\[ Z : B, \overrightarrow{X} : \Theta \vdash \delta^i_j tYX : A \quad (\equiv) \]
\[ Z : B, \overrightarrow{X} : \Theta \vdash t : A \quad (\triangle) \]
\[ Z : \triangle^\mu_\nu B \vdash Z : \triangle^\mu_\nu B \quad (\triangle) \]
\[ Z : \triangle^\mu_\nu B, \overrightarrow{X} : \Theta \vdash \delta^\mu_\nu tYX : A \quad (Sb) \]
\[ Z : \triangle^\mu_\nu B, \overrightarrow{X} : \Theta \vdash t : A \quad (\triangle) \]
\[ Z : \triangle^\mu_\nu B, \overrightarrow{X} : \Theta \vdash \delta^\mu_\nu t : A \quad (\triangle) \]
\[ Z : \triangle^\mu_\nu B, \overrightarrow{X} : \Theta \vdash t \vdash C, \overrightarrow{X} : \Theta \quad (\otimes) \quad \Gamma \]
\[ T : B, \overrightarrow{Z} : \Theta \vdash tT \overrightarrow{Z} : A \quad (Sb) \]

**Figure 14.** From ETTC to ETTC\textsubscript{\textit{n.d.}}: “side” rule cases

conclusion comes from the right premise of the rule. In this situation the n.d. translation is obtained using the (\alpha\eta) rule.

**Lemma 6.12.** If a standard sequent is derivable in ETTC\textsubscript{\textit{n.d.}} then its sequent calculus translation is derivable ETTC.

**Proof.** For the axioms of ETTC\textsubscript{\textit{n.d.}} use Proposition 4.12, claim (i). The introduction rules, (\alpha\eta) rules and the (\equiv) rule of ETTC\textsubscript{\textit{n.d.}} translate to ETTC directly. The elimination rules (\triangle) and (\otimes) are emulated in ETTC using cuts with ETTC\textsubscript{\textit{n.d.}} derivable sequents \vdash \delta^\mu_\nu t : A, A and \vdash B \otimes A, A, B respectively. The (\otimes) rule is similar to the (\equiv) rule. The remaining elimination rules are emulated very easily, using ETTC rules for the corresponding dual connectives and the Cut rule.

**Corollary 6.13.** A standard sequent is derivable in ETTC\textsubscript{\textit{n.d.}} iff its sequent calculus translation is derivable ETTC. □
6.2. **Geometric representation.** Translating to sequent calculus we can obtain geometric representation for n.d. sequents and derivations. Namely, the standard n.d. sequent $\overrightarrow{X} : \Gamma \vdash t \overrightarrow{X} : F$ can be depicted as the ordered tensor sequent $\Gamma \vdash t :: F, \Gamma$.

In this case, however, it seems natural to deform the graph of the sequent calculus translation, so that indices corresponding to the left and to the right of the turnstile in the n.d. sequent appear on two parallel vertical lines. That is, we depict indices of $F$ aligned vertically, from the top to the bottom on one line, and indices of $\Gamma$ aligned from the bottom to the top on another line to the left. (This is the same as to say that the indices occurring in $F$ are depicted on the right vertical line, aligned from the top to the bottom, and those occurring in $\Gamma$ depicted on the left vertical line, also ordered from the top to the bottom, but with opposite polarities.) The prescription for drawing labeled edges is the same as in the sequent calculus. An example of such a representation is shown in Figure 15a.

In Figure 15b we show schematically geometric representations of some n.d. rules, namely introduction/elimination for the $\vee$ connective and elimination rules for the binding operators. Note that $\vee$-introduction rules do not change the graph, but change its pictorial representation by a continuous deformation.

6.3. **Example: slash elimination rules of LC.** Recall that we introduced a translation of LC directional implications into the language of ETTC, see (4.4).

**Proposition 6.14.** In ETTC$_{n.d.}$ there are derivable sequents

\[
y_m^i : (b/a)_j^m, x_j^i : a_j^i \vdash y_m^j x_j^i : b_j^m, \quad x_j^i : a_j^i, y_i^m : (a \backslash b)_j^m \vdash x_j^i y_i^m : b_j^m.
\]

**Proof.** Derivation of the first sequent together with is geometric representation is shown in Figure 16a. The case of the second one is similar. \(\square\)

**Corollary 6.15.** The rules

\[
\frac{\Gamma \vdash t : (b/a)_j^i \quad \Theta \vdash s : a_j^i}{\Gamma, \Theta \vdash ts : b_k^j} (/E) \quad \frac{\Gamma \vdash s : a_j^i \quad \Theta \vdash t : (a \backslash b)_j^i}{\Gamma, \Theta \vdash st : b_k^i} (\backslash E)
\]

are admissible in ETTC$_{n.d.}$. \(\square\)

6.4. **Grammars.**

**Definition 6.16.** Given a terminal alphabet $T$, a tensor lexical entry (over $T$), is an $\alpha$-equivalence class of $\beta$-reduced lexical typing judgements (over $T$). A tensor grammar $G$ (over $T$) is a pair $G = (\text{Lex}, s)$, where $\text{Lex}$, the lexicon, is a finite set of lexical entries (over $T$) and $s$ is an atomic type symbol of valency $(1,1)$.

For a tensor grammar $G$ we write $\vdash_G \sigma : A$ to indicate that the sequent $\vdash \sigma : A$ is derivable in ETTC$_{n.d.}$ from elements of $\text{Lex}$.

**Definition 6.17.** The language $L(G)$ of a tensor grammar $G$ is the set

\[
L(G) = \{ w \mid \vdash_G [w]_j^i : s_i^j \}.
\]

Since ETTC$_{n.d.}$ derivability is closed under $\alpha$-equivalence, nothing will change if we require lexicons to be finite only modulo $\alpha$-equivalence. In fact, it is more convenient to consider lexicons closed under $\alpha$-equivalence and consisting only of finitely many equivalence classes. Given a tensor grammar $G = (\text{Lex}, s)$, we write $\text{Lex} \equiv_{\alpha}$ for the closure of $\text{Lex}$ under $\alpha$-equivalence.
Proposition 6.18. Given a tensor grammar $G = (\text{Lex}, s)$ and a typing judgement $\Sigma$ of the form $t : F$, we have $\vdash_G \Sigma$ iff there are lexical entries $\tau_1 : A_1, \ldots, \tau_n : A_n \in \text{Lex}_{\equiv_{\alpha}}$ satisfying
\[ FI(A_i) \perp FI(A_j) \text{ for } i \neq j \] and a term $t'$, which is the product of Kronecker deltas, such that
\[ \vdash_{\text{ETTC}} t' :: F, \overline{A_n}, \ldots, \overline{A_1}, \quad t' \cdot \tau_1 \ldots \tau_n \equiv_{\beta} t. \]

Proof. Assume that $\vdash_G \Sigma$.

Since $\alpha$-equivalent n.d. sequents in $\text{ETTC}_{n.d.}$ are derivable from each other, we may assume that the entries from $\text{Lex}_{\equiv_{\alpha}}$ used for deriving $\vdash \Sigma$ satisfy (6.4). Then the statement is a direct corollary of Proposition 6.8 ("Deduction theorem") and Corollary 6.13 on the relation between derivability in $\text{ETTC}_{n.d.}$ and $\text{ETTC}$. That $t'$ is the product of Kronecker deltas follows from a simple observation that $\text{ETTC}$ is independent from terminal alphabets, hence $\vdash_{\text{ETTC}} t' :: \Gamma$ implies that $t'$ has no terminal symbols.

The other direction follows from Corollary 6.13 and Proposition 6.8 in a similar way. 

\[ \vdash_G \Sigma \]

\[ \vdash_{\text{ETTC}} t' :: \Gamma \]

\[ t' \cdot \tau_1 \ldots \tau_n \equiv_{\beta} t. \]
A linguistically motivated example of tensor grammar is given in Figure 17, where we use notation for LC slashes as in (4.4). For better readability, we omit free indices in formulas, whenever they are uniquely determined by free indices in terms. Figure 17a shows the lexicon, which we assume closed under $\alpha$-equivalence of typing judgements, and in Figure 17b we derive the noun phrase “Mary who John loves madly”.

The grammar, obviously, is an extension of a one adapted from LC. However, the lexical entry for “who”, which allows medial extraction, is not a translation from LC, as is manifested by the binding operator in the type. Using the implicational notation, the above lexical entry can be written as

$$[\text{who}]_{ij}^j : (\text{np}\setminus\text{np})/\Delta^u_i ((\text{np})^u - o s)).$$

It might be entertaining to reproduce the derivation in the geometric language.

7. Correspondence with first order logic

7.1. Translating formulas. Given a linguistically marked first order language, we identify predicate symbols with literals of the same valencies and define the $\eta$-reduced and $\eta$-long translations, denoted respectively as $||.||$ and $||.||_{\eta \rightarrow}$, of linguistically well-formed formulas and contexts to tensor formulas and contexts. For the $\eta$-reduced translation we identify first order variables with indices, left occurrences with upper ones and right occurrences with lower ones. For the $\eta$-long translation, on the other hand, we identify free variable occurrences...
\[ \text{Definition 7.1.} \] The \( \eta \)-long translation of linguistically well-formed MLL1 formulas, contexts and sequents to ETTC is given in Figure 18a. The \( \eta \)-reduced translation of linguistically well-formed MLL1 formulas, contexts and sequents to ETTC is defined up to \( \alpha \)-equivalence of tensor formulas and is given in Figure 18b, where it is assumed that the indices \( u, v \) in the translation of quantified formulas are such that they result in well-defined expressions.

The need to choose bound indices \( u, v \) in the \( \eta \)-reduced translation of quantifiers makes the latter translation ambiguous. However, different choices of indices result in \( \alpha \)-equivalent tensor formulas, which are provably equivalent in ETTC.

\[ \text{Proposition 7.2.} \] For any linguistically well-formed MLL1 sequent \( \vdash \Gamma \), its \( \eta \)-reduced and \( \eta \)-long translations are \( \alpha \eta \)-equivalent, \[ || \vdash \Gamma ||_{\eta \rightarrow} \equiv_{\alpha \eta} || \vdash \Gamma ||. \] \( \Box \)

In terms of the geometric representation, the \( \eta \)-long translation \( \vdash \pi(\Gamma) :: ||\Gamma||_{\eta \rightarrow} \) of a linguistically well-formed MLL1 sequent \( \vdash \Gamma \) is the occurrence net of \( \vdash \Gamma \), encoded as a \( \beta \)-reduced \( \eta \)-long typing judgement (thinking both of the occurrence net and of the term
\[\|p(x_1, \ldots, x_n)\|_{\eta \rightarrow} = p^{x_1, \ldots, x_n}_{x_{k+1}, \ldots, x_{n}},\quad \text{for } p \in \text{Lit}, v(p) = (k, n - k),\]
\[\|A \otimes B\|_{\eta \rightarrow} = \|A\|_{\eta \rightarrow} \otimes \|B\|_{\eta \rightarrow},\quad \|A \not\exists B\|_{\eta \rightarrow} = \|A\|_{\eta \rightarrow} \not\exists \|B\|_{\eta \rightarrow},\]
\[\|\forall x A\|_{\eta \rightarrow} = \nabla^{x_i}_{x'}\|A\|_{\eta \rightarrow},\quad \|\exists x A\|_{\eta \rightarrow} = \Delta^{x_i}_{x'}\|A\|_{\eta \rightarrow},\]
\[\|A_1, \ldots, A_n\|_{\eta \rightarrow} = \|A_1\|_{\eta \rightarrow}, \ldots, \|A_n\|_{\eta \rightarrow},\quad \| \vdash \Gamma \|_{\eta \rightarrow} = (\vdash \pi(\Gamma) :: \|\Gamma\|_{\eta \rightarrow}),\quad \text{where } \pi(\Gamma) = \prod_{x \in \text{FV}(\Gamma)} \delta_{x_j}^{x'}_{x_j}.
\]

\[\text{(A) } \eta\text{-Long translation}\]
\[\|p(x_1, \ldots, x_n)\| = p^{x_1, \ldots, x_k}_{x_{k+1}, \ldots, x_n},\quad \text{for } p \in \text{Lit}, v(p) = (k, n - k),\]
\[\|A \otimes B\| = \|A\| \otimes \|B\|,\quad \|A \not\exists B\| = \|A\| \not\exists \|B\|,\]
\[\|\forall x A\| = \nabla^{u_i}_{u'}\|A\|^{[v/x]}_{[v/x]'},\quad \|\exists x A\| = \Delta^{u_i}_{u'}\|A\|^{[v/x]}_{[v/x]'},\]
\[\|A_1, \ldots, A_n\| = \|A_1\|, \ldots, \|A_n\|,\quad \| \vdash \Gamma \| = (\vdash \|\Gamma\|).
\]

\[\text{(B) } \eta\text{-Reduced translation}\]

**Figure 18.** Translations of linguistic fragment to ETTC

\[\pi(\Gamma) \text{ as bipartite graphs}.\] The ETTC\textsubscript{\eta\rightarrow} rules correspond to transformations of occurrence nets under linguistic derivation rules. The (\forall) rule, which erases a link from the occurrence net, naturally corresponds to the (\forall) rule, and the (\exists) rule, which glues links together, to the (\Delta) rule. The (\exists) rule applied to a linguistically well-formed sequent erases a link in the same way as the (\forall) rule. This corresponds to the situation when the (\Delta) rule is applied to a sequent of the form \(\vdash \delta_{j}^{t} \coloneq \Theta \) and binds the pair of occurrences \((j, i) \in F I(\Theta)\), introducing the term \(\delta_{i}^{j} \cdot \delta_{j}^{t} \equiv \beta t\), basically, erasing \(\delta_{j}^{t}\) from the premise. After these observations we state the main lemma.

**Lemma 7.3.** A linguistically well-formed sequent is derivable in MLL1 if and only if its \(\eta\)-long and \(\eta\)-reduced translations are derivable in ETTC. Conversely, any ETTC derivable sequent is \(\alpha \beta \eta\)-equivalent to a translation of an MLL1 derivable linguistically well-formed sequent.

**Proof.** Given an MLL1 derivable linguistically well-formed sequent \(\vdash \Gamma\), we prove by induction on linguistic derivation of \(\vdash \Gamma\) that the tensor sequent \(\| \vdash \Gamma \|_{\eta \rightarrow}\) is derivable in ETTC\textsubscript{\eta\rightarrow}, hence in ETTC. Then Proposition 7.2 implies that \(\| \vdash \Gamma \|\) is ETTC derivable as well since \(\alpha \eta\)-equivalent sequents are ETTC derivable from each other.

Given an ETTC derivable tensor sequent \(\vdash \Sigma\), we consider its \(\beta\)-reduced \(\eta\)-long form \(\vdash \Sigma'\), which is derivable in ETTC\textsubscript{\eta\rightarrow} by Lemma 5.8. By induction on ETTC\textsubscript{\eta\rightarrow} derivation of \(\vdash \Sigma'\) we prove that \(\vdash \Sigma'\) is the translation of an MLL1 derivable linguistically well-formed sequent.

Restricting to the intuitionistic fragments we obtain an analogous correspondence. Note that the \(\eta\)-reduced fragment of ETTC and the linguistically well-formed fragment of MLL1 become literally the same thing, only written in different notation (up to bound indices/variables renaming).
7.2. Translating grammars.

Definition 7.4. Given a well-formed MILL1 grammar \( G = (\text{Lex}, s) \), its tensor translation is the tensor grammar \( \bar{G} = (\text{Lex}, s) \), where \( \text{Lex} = \{ [w]_{\mu}^r : |A|, (w, A) \in \text{Lex} \} \).

Lemma 7.5. For any well-formed MILL1 grammar \( G \), its tensor translation \( \bar{G} \) generates the same language: \( L(G) = L(\bar{G}) \).

Proof. Let \( \text{Lex} \) be the lexicon of \( G \).

Let \( w \in L(\bar{G}) \), so that \( \vdash_{\bar{G}} [w]_{\mu}^r : s_{r}^l \). Obviously \( w \) is the concatenation

\[
w = w_0 \ldots w_{n-1}
\]

of all words occurring in lexical entries used in the derivation of the latter sequent. By Proposition 6.18 there are lexical entries

\[
[w]_{\mu}^r : (a(\mu))_{\mu}^r \in \widetilde{\text{Lex}_{\equiv}},
\]

where

\[
(a(\mu))_{\mu}^r = |A_{\mu}|_{[r][j]}, (w_{\mu}, A_{\mu}) \in \text{Lex}, \mu = 0, \ldots, n-1,
\]

satisfying (6.4) and a term \( t' \), which is a product of Kronecker deltas, such that

\[
\vdash_{\text{ETTC}} t' :: s_{r}^l, (a_{(n-1)})_{j_{n-1}}^{i_{n-1}}, \ldots, (a_{(0)})_{j_{0}}^{i_{0}} \text{ and } t'[w]_{j_{0}}^{i_{0}} \ldots [w_{n-1}]_{j_{n-1}}^{i_{n-1}} \equiv \beta |w|_{l}^r.
\]

It follows from (7.1) that \( t' \equiv \beta \delta_{i_{0}}^{l_{0}} \delta_{r_{1}}^{j_{1}} \cdots \delta_{r_{n-2}}^{j_{n-2}} \delta_{r_{n-1}}^{j_{n-1}} \). and we have

\[
\vdash_{\text{ETTC}} \delta_{i_{0}}^{l_{0}} \delta_{r_{1}}^{j_{1}} \cdots \delta_{r_{n-2}}^{j_{n-2}} \delta_{r_{n-1}}^{j_{n-1}} :: s_{r}^l, (a_{(n-1)})_{j_{n-1}}^{i_{n-1}}, \ldots, (a_{(0)})_{j_{0}}^{i_{0}}.
\]

The above is just (\( \alpha \)-equivalent to) the \( \eta \)-long translation of

\[
\vdash s[l, r], A_{n-1}[r, c_{n-1}], A_{n-2}[c_{n-1}, c_{n-2}], \ldots, A_{0}[c_{0}, t].
\]

The latter is identified as the image of a MILL1 sequent expressing, by definition, that \( w \in L(G) \). The opposite inclusion is similar. \( \square \)

We note that, compared with tensor grammars of this paper, (provisional) Definition 3.1 of MILL1 grammars is too restrictive. It allows only lexical entries corresponding to single words. Tensor grammars, on the contrary, can have lexical entries corresponding to word tuples.

In particular, we do not have an inverse translation from tensor grammars to MILL1 grammars. However, this does not mean that ETTC is more expressive than MILL1. In fact, Lemma 7.3 shows that ETTC is strictly equivalent to the linguistic fragment of MILL1, and the intuitionistic fragment of ETTC equivalent to the linguistic fragment of MILL1. We discussed in Section 3 that the simple definition of MILL1 grammars should be generalized to allow more complex lexical entries. Tensor grammars of this paper can be considered as such a generalization, only expressed in a different syntax. The advantage of the ETTC syntax, arguably, is that ETTC contains the term component for explicit manipulations with strings. Using this component we managed to give a concise definition for tensor grammars. Translating this definition back to MILL1 might be cumbersome.
8. Conclusion

We defined and studied the system of extended tensor type calculus (ETTC), a non-trivial notationally enriched variation of the system previously introduced in [Sla21b]. We presented a cut-free sequent calculus formalisation and a natural deduction version of the system. We also discussed geometric representation of ETTC derivations. We showed that the system of ETTC is strictly equivalent to the fragment of first order linear logic relevant for language modeling, that is, to representing categorial grammars, used in [MP01], [Moo14a], [Moo15], [Moo14b]. In this way we provided the fragment in question with an alternative syntax, intuitive geometric representation and an intrinsic deductive system, which has been absent.

We leave for future research open questions such as: what are proof-nets for ETTC, is there a complete semantics, what is the complexity of the calculus? We also think that the system can be further enriched/modified to go beyond first order linear logic. This is the subject of on-going work.

References

[SG01] Philippe de Groote. Towards abstract categorial grammars. In Proceedings of 39th Annual Meeting of the Association for Computational Linguistics, pages 148–155, 2001. doi:10.3115/1073012.1073045.

[Gir87] Jean-Yves Girard. Linear logic. Theoretical Computer Science, 50:1–102, 1987. doi:10.1016/0304-3975(87)90045-4.

[Gir95] Jean-Yves Girard. Linear logic: its syntax and semantics. In Jean-Yves Girard and Laurent Lafont, Yves and. Regnier, editors, Advances in Linear Logic, pages 1–42. Cambridge University Press, 1995. Proceedings of the Workshop on Linear Logic, Ithaca, New York, June 1993. doi:10.1017/CBO9780511629150.002.

[KL12] Yusuke Kubota and Robert Levine. Gapping as like-category coordination. In Denis Béchet and Alexander Dilovskiy, editors, Logical Aspects of Computational Linguistics, pages 135–150, Berlin, Heidelberg, 2012. Springer Berlin Heidelberg. doi:10.1007/978-3-642-31262-5_9.

[Lam58] Joachim Lambek. The mathematics of sentence structure. The American Mathematical Monthly, 65(3):154–170, 1958. doi:10.1080/00029890.1958.11989160.

[Moo14a] Richard Moot. Extended lambek calculi and first-order linear logic. In Claudia Casadio, Bob Coecke, Michael Moortgat, and Philip Scott, editors, Categories and Types in Logic, Language, and Physics: Essays Dedicated to Jim Lambek on the Occasion of His 90th Birthday, pages 297–330. Springer Berlin Heidelberg, Berlin, Heidelberg, 2014. doi:10.1007/978-3-642-54789-8_17.

[Moo14b] Richard Moot. Hybrid type-logical grammars, first-order linear logic and the descriptive inadequacy of lambda grammars, 2014. arXiv:1405.6678.

[Moo15] Richard Moot. Comparing and evaluating extended lambek calculi. CoRR, abs/1506.05561, 2015. URL: http://arxiv.org/abs/1506.05561, arXiv:1506.05561.

[MP01] Richard Moot and Mario Piazza. Linguistic applications of first order intuitionistic linear logic. J. Log. Lang. Inf., 10(2):211–232, 2001. doi:10.1023/A:1008399708659.

[MP12] Vedrana Mihalíček and Carl Pollard. Distinguishing phenogrammar from tectogrammar simplifies the analysis of interrogatives. In Philippe de Groote and Mark-Jan Nederhof, editors, Formal Grammar, pages 130–145, Berlin, Heidelberg, 2012. Springer Berlin Heidelberg. doi:10.1007/978-3-642-32024-8_9.

[MR12] Richard Moot and Christian Retoré. The Logic of Categorial Grammars - A Deductive Account of Natural Language Syntax and Semantics, volume 6850 of Lecture Notes in Computer Science. Springer, 2012. doi:10.1007/978-3-642-32024-8_9.

[Mus07] Reinhard Muskens. Separating syntax and combinatorics in categorial grammar. Research on Language and Computation, 5(3):267–285, 2007. Pagination: 17. doi:10.1007/s11168-007-9035-1.

[MVF11] Glyn Morrill, Oriol Valentín, and Mario Fadda. The displacement calculus. Journal of Logic, Language and Information, 20(1):1–48, 2011. doi:10.1007/s10849-010-9129-2.
[Sla21a] Sergey Slavnov. Cobordisms and commutative categorial grammars. *Journal of Cognitive Science*, 22(2):68–91, 2021. doi:10.17791/jcs.2021.22.2.68.

[Sla21b] Sergey Slavnov. On embedding Lambek calculus into commutative categorial grammars. *Journal of Logic and Computation*, 32(3):479–517, 10 2021. arXiv:https://academic.oup.com/logcom/article-pdf/32/3/479/43365809/exab065.pdf, doi:10.1093/logcom/exab065.