Reliability of Solutions in Linear Ordering Problem: New Probabilistic Insight and Algorithms

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Abstract

In this work, our goal is to characterize the reliability of the solutions that can be obtained by the linear ordering problem (LOP) which is used to order $M$ objects from their pairwise comparisons. We adopt a probabilistic perspective, where the results of pairwise comparisons are modeled as Bernoulli variables with a common parameter which we estimate from the observed data. Estimation by brute-force enumeration has a prohibitive complexity of $O(M!)$ we thus reformulate the problem and introduce a concept of Slater’s spectrum which generalizes Slater’s index, and next, devising an efficient algorithm to find the spectrum, we lower the complexity to $O(M^{2^M})$ which is manageable for moderate-size LOPs. Furthermore, with a minor modification of the algorithm, we are able to find all solutions of the LOP. Numerical examples on synthetic and real-world data are shown and the Python-implemented algorithms are publicly available.

Keywords: Combinatorial optimization, Linear ordering problem, Ranking, Matrix triangulation, Bayesian estimation

1. Introduction

In this work, we address the problem of evaluating the reliability of the solutions in the linear ordering problem (LOP). This NP-hard combinatorial...
optimization problem consists in finding an order among objects from the results of objects’ pairwise comparisons. LOP has found wide use in many different areas, e.g., for aggregation of preferences of individual judgments or for ranking in sports where the teams (or players) are compared through the game result. Many more applications rely on the LOP formulation, see Ceberio et al. [1, Sec. 1], Martí et al. [2, Sec. 1], or Charon and Hudry [3]; so while the wording can change from one application to another, the fundamental problem remains the same.

The effort in the area of the LOP was mainly focused on devising efficient numerical solutions. The LOP can be solved exactly deCani [4] but, for large problems, approximations are unavoidable due to NP-hardness; see Charon and Hudry [3], Martí et al. [2], Ceberio et al. [1] for an overview of the literature on this issue.

In this work, we are not interested in finding solutions of LOP as such, but rather study the reliability of the solutions obtained. We use the probabilistic approach, explicitly modeling the results of pairwise comparisons as Bernoulli variables. This follows the spirit of early analysis in Slater [5] or Remage and Thompson [6]. In particular, Slater [5] proposed a hypothesis-testing approach, which assumes that all Bernoulli variables have the same distribution with parameter \( p \), and verifies if hypothesis \( p = 0 \) (meaning that the data is purely random and not related to the order) can be rejected. However, this method is too complex to be practical in most cases.

We also use the assumption that \( p \) is common to all variables, but, instead of testing the hypothesis, we estimate \( p \) from the data. The estimation problem relies on finding the so-called Slater’s spectrum, which generalizes the well-known Slater’s index, and tells us how “far” all possible orders are from the results we observe. We devise an algorithm that allows us to obtain Slater’s spectrum efficiently. This is our main contribution. Incidentally, the algorithm we develop also allows us to find all the solutions of LOP, which is itself an interesting contribution.

The remainder of this paper is organized as follows. In Sec. 2 we describe
the problem, define the model, show the relationship between the LOP and its probabilistic counterpart, and briefly describe the approach suggested in Slater [5]. The estimation problem is defined in Sec. 3 while Sec. 4 focuses on the proposed algorithm and evaluates the related numerical complexity. Interestingly, as a “byproduct” of the algorithm developed, we are able to recover all the solutions of the LOP and this comes with virtually no additional cost; the additional algorithmic steps required to do that are explained in Sec. 4.2. The algorithms implemented in Python are available at our GitHub repository. Numerical examples are shown in Sec. 5 where we use synthetic data, as well as the results of cricket games in the Indian Premier League (IPL) and the volleyball games of the Italian SuperLega. We conclude the work in Sec. 6.

2. Model and problem definition

Consider objects indexed with \( M \) integers from the set \( \mathcal{I} = \{1, \ldots, M\} \) and assume that they can be ordered (or ranked) e.g., \( 2 \prec 5 \prec 1 \prec \ldots \prec M \prec 4 \), where \( m \prec n \) means that object \( m \) is ordered/ranked before object \( n \).

The order is unknown and must be inferred from comparisons between pairs of objects; the total number of distinct pairs is given by \( \binom{M}{2} \). However, not all pairs need to be compared, and the comparison may be performed multiple times for the same pair.

The simplest case occurs if the result of the comparison between the objects \( m \) and \( n \) is binary, i.e., either \( m \) wins against \( n \), or \( n \) wins, that is, the ties are not allowed. In this case, the results can be stored in the observation matrix \( W \) with the entries \( w_{m,n} \in \mathbb{N} \) indicating how many times \( m \) won against \( n \). Diagonal entries in the matrix are ignored or, equivalently, set to zero, \( w_{m,m} = 0 \). The number of comparisons between \( m \) and \( n \) is given by \( K_{m,n} = K_{n,m} = w_{m,n} + w_{n,m} \), and the comparison scheduling consists in defining \( K_{m,n} \) for each pair \((m, n)\). Therefore, the total number of comparisons is \( T = \sum_{m,n} w_{m,n} \).

If knowing \( m \prec n \) implies that \( w_{n,m} = 0 \), the pairwise comparison perfectly reflects the underlying order of the objects. In practice, however, we may have \( w_{m,n} > 0 \) and \( w_{n,m} > 0 \) which means that, in some comparisons (between \( m \)
and \( n \) \( m \) wins and, in others, \( n \) does. This inconsistency of observations with the underlying order may be interpreted as a manifestation of randomness. For example, we may use the following Bernoulli model for the results:

\[
\begin{align*}
\Pr \{ m \text{ wins against } n | m \prec n \} &= p \\
\Pr \{ m \text{ wins against } n | m \succ n \} &= 1 - p,
\end{align*}
\]

where \( p \in [0.5, 1] \) and \( p = 1 \) mean that observations are always in agreement with the underlying order and \( p = 0.5 \) means that the win of \( m \) over \( n \) is independent of the order. In the latter case, searching for the order is obviously useless.

The ranking \( \rho \) is a permutation of the set \( I \), that is, \( \rho = [\rho_1, \ldots, \rho_M] \) where \( \rho_m \in I \) and \( \rho_m \neq \rho_n \) if \( m \neq n \). Essentially, it indicates that the object indexed with \( \rho_l \) is ranked at the position \( l \).

Assuming that all possible rankings are a priori equiprobable, we may estimate the ranking via maximum likelihood (ML)

\[
\hat{\rho} = \arg\max_{\rho \in \mathcal{P}_I} J(\rho) \tag{3}
\]

\[
J(\rho) = \Pr \{ W | \rho \} \tag{4}
\]

where \( \mathcal{P}_I \) is the set of all possible permutations of the set \( I \), \( |\mathcal{P}_I| = M! \), and, in general, \( \rho \) may have many equivalent solutions \( \hat{\rho} \).

Further, assuming that the results of comparisons are independent when conditioned on the ranking, we may write the following:

\[
J(\rho) = \prod_{m \neq n} \Pr \{ w_{m,n} | \rho \} \tag{5}
\]

\[
= p^{T - S(\rho)} (1 - p)^{S(\rho)}, \tag{6}
\]

\[
S(\rho) = T - C(\rho), \tag{7}
\]

\[
C(\rho) = \sum_{m=1}^{M} \sum_{n=m+1}^{M} w_{\rho_m, \rho_n} \tag{8}
\]

where the notation \( \Pr \{ w_{m,n} | \rho \} \) should be read as the probability of observing
$w_{m,n}$ wins of $m$ over $n$ conditioned on the given ranking $\rho$, and $S(\rho)$ is the Slater index of the ranking/order $\rho$ which has the meaning of the number of observations that are inconsistent with the ranking $\rho$. For example, in sport ranking, $S(\rho)$ is the number of games in which the favorite teams looses against the underdogs, where the favorite is ranked before the underdog (according to $\rho$).

Similarly, if convenient, we may use the consistency index, $C(\rho)$ which measures the number of observations in agreement with the order $\rho$. In fact, an alternative formulation of the [LOP] known as matrix triangulation, sees [9] as a search for a permutation of rows and columns of the matrix $W$, which maximizes the consistency index $C(\rho)$ which is the sum of elements in the upper triangular part of the permuted matrix $W$; this can be seen directly in [8].

As long as $p > 0.5$, we may write [3] as

$$\hat{\rho} = \text{argmin}_{\rho \in P_T} S(\rho), \quad (9)$$
$$\hat{S} = S(\hat{\rho}), \quad (10)$$

that is, the optimal ranking $\hat{\rho}$ will have minimum Slater index $\hat{S}$.

Thus, for $p > \frac{1}{2}$, there is a one-to-one mapping between problems [6] and [3]. However, for $p = \frac{1}{2}$, we have $J(\rho) \equiv 2^{-T}$ for all $\rho \in P_T$. Then, although solutions can be found for [6] they do not solve [3] in any meaningful way.

Therefore, before using any of the solutions $\hat{\rho}$, it may be useful to know the value of $p$, or at least to know if $p > \frac{1}{2}$ because this will determine how useful/reliable the ranking $\hat{\rho}$ is. This is the focus of this work.

The probabilistic model we adopt is not new, of course, and has been used, e.g., by deCani [4], Flueck and Korsh [7], or Tiwisina and Külmann [8]. In general, each comparison $m$ against $n$ may have its own parameters $p_{m,n}$. However, estimating $M(M-1)/2$ parameters $p_{m,n}$ is not obvious and may be unreliable, especially when the number of comparisons $T$ is not very large. Furthermore, ensuring the transitivity of probabilities in a model (i.e., the condition $m \prec k \prec n \implies p_{m,n} \geq p_{m,k} \wedge p_{m,n} \geq p_{k,n}$) is not trivial, as noted in Ti-
wisina and Külpman [8]. Therefore, using a common $p$, as we do, may be seen as a pragmatic modeling middle ground and, as we already said, actually corresponds to the LOP as it is often defined.

2.1. Reliability testing: Slater’s method

This reliability issue has been addressed in Slater [5] via hypothesis testing where the hypothesis $H_0$ is that all $T$ observations are obtained with probability $p = \frac{1}{2}$, that is, are drawn uniformly from the space of $2^T$ elements, each represented by the observation matrix which we may treat as a random variable $W$.

The rankings obtained from random observations $W$ are random as well

$$\hat{\rho}_W = \arg\min_{\rho \in \mathcal{P}} S(\rho; W)$$

$$\hat{S} = S(\hat{\rho}_W; W),$$

where $S(\rho, W)$ is the same as (7) with the dependence on the observation matrix indicated explicitly.

Hypothesis $H_0$ is rejected if Slater’s index $\hat{S}$ in (10) is unlikely to be obtained from the outcomes generated randomly; then we treat $\hat{\rho}$ as a meaningful solution of (3). More precisely,

$$\Pr\{\hat{S} \leq \hat{S}\} < p_{\text{val}} = \Rightarrow \text{reject } H_0,$$

where the threshold p-value is arbitrarily set, e.g., commonly used $p_{\text{val}} = 0.05$.

We note that Slater [5] suggested calculating the probability in (13) by brute force, i.e., by (i) enumerating all possible outcomes, (ii) forming the corresponding observation matrices $W$, (iii) solving (9) for each of the matrices, and (iv) counting all solutions whose Slater’s index is smaller than $\hat{S}$.

The major problem lies in steps (i)-(iii): solving the LOP for all realizations of the matrices $W$, is too complex to be practical. For example, with the simplest scheduling $K_{m,n} = 1$ (i.e., one comparison per pair), we have $T = \binom{M}{2} = M(M-1)/2$ and thus we have to solve the LOP for $2^T$ matrices $W$. Even if $M$ is moderate, e.g., for $M = 20$, when (9) is easily solved, solving it
2^T \approx 10^{57} \text{ times is an impossible challenge. Thus, without workarounds that would allow us to efficiently calculate the probability in (13), Slater’s method is difficult to apply in practice.}

Another issue with Slater’s method is that it does not provide us with the value of $p$ that would characterize the problem at hand well. The objective of our work is to address the latter problem and propose numerically efficient methods to solve it.

3. Bayesian inference

The approach we propose may be seen as an estimation of the Bernoulli distribution parameter, $p$, from the observation matrix $W$. We quickly note that it is tempting to address this issue by solving the joint ML problem

$$\hat{p}, \hat{\rho} = \arg\max_{p, \rho} J_p(\rho),$$

where $J_p(\rho)$ is the same as (6) with the dependence on $p$ explicitly indicated.

But since $\hat{\rho}$ does not depend on $p$, we have $\hat{S} = S(\hat{\rho})$ so

$$\hat{p} = \arg\max_p p^T - \hat{S} (1 - p) \hat{S}$$

$$= 1 - \frac{\hat{S}}{T}. \quad (16)$$

The solution $\hat{p}$ thus obtained is clearly not very useful, as it is simply a linear function of $\hat{S}$. In particular, (15) always yields $\hat{p} > \frac{1}{2}$ and thus we would always conclude that the solution $\hat{p}$ is reliable.

The problem with the ML estimation is that, focusing on the mode, it ignores the form of the underlying distribution. Therefore, instead of the joint ML we
formulate the problem of finding the posterior distribution of $p$

$$f(p|W) \propto \Pr(W|p)f(p)$$  (17)

$$\propto \Pr(W|p)$$  (18)

$$= \sum_{\rho \in \mathcal{P}} \Pr(W|\rho)p^{T-S(\rho)}(1-p)^{S(\rho)}$$  (19)

$$\propto \sum_{\rho \in \mathcal{P}} p^{T-S(\rho)}(1-p)^{S(\rho)}$$  (20)

$$= \phi(p),$$  (21)

where $f(p)$ is the prior distribution of the parameter $p$ and, in the absence of any prior knowledge, we may assume a uniform distribution, i.e., $f(p) = 2, p \in [0.5, 1]$ which explains (18). In (19), we do not use the optimal solution $\hat{\rho}$, but rather marginalize over all $\rho \in \mathcal{P}$ treating the ranking $\rho$ as a random variable uniformly distributed over the set of permutations $\mathcal{P}$ which explain (20).

The function $\phi(p)$ is not normalized (hence the symbol $\propto$) and we obtain the posterior probability density function (PDF) as follows:

$$f(p|W) = \frac{\phi(p)}{\int_{0.5}^{1} \phi(u) \, du}.$$  (22)

Our preference for the estimation strategy should be clear: we obtain much stronger information about the ordering problem than in the case of Slater’s method as the latter only allows us to reject the hypothesis $H_0 : p = 0.5$. In particular, we can now use the PDF to obtain the maximum a posteriori (MAP) estimate of $p$ (which, with non-informative prior we use, is equivalent to the ML) as follows:

$$\hat{p} = \arg\max_{p} \phi(p).$$  (23)

4. Algorithm and complexity evaluation

The complexity of explicit ranking enumeration in (20) has a complexity of $M!$ that is prohibitive even for moderate $M$. Our goal is to reduce the complexity by exploiting the structure of the solutions.
First, we note that, because $S(\rho) \in \{0, \ldots, T\}$, i.e., Slater’s index is an integer which cannot be greater than $T$, it is possible to rewrite (20) as follows:

$$\phi(p) = \sum_{t=0}^{T} a_t p^{T-t} (1-p)^t$$

(24)

where

$$a_t = \sum_{\rho \in P_I} \mathbb{I}[S(\rho) = t], \quad t = 0, \ldots, T$$

(25)

is the number of the rankings $\rho$ (in the set $P_I$) with Slater’s index $S(\rho) = t$, where $t = 0, \ldots, T$. The sequence $\{a_t\}_{t=0}^T$ will be called Slater’s spectrum and, by construction, it satisfies the conditions: $\sum_{t=0}^{T} a_t = M!$. Moreover, for any order $\rho$, it is enough to reverse it (i.e., enumerate its elements backward) $\rho' = [\rho_M, \rho_{M-1}, \ldots, \rho_1]$, to transform all inconsistencies into agreements with the order and vice versa. Thus, $S(\rho) = T - S(\rho')$ and all $\rho$ that are taken into account in $a_t$ have reversed versions that will produce $a_{T-t} = a_t$. Combining the latter with (8) we have

$$a_t = \sum_{\rho \in P_I} \mathbb{I}[C(\rho) = t], \quad t = 0, \ldots, T;$$

(26)

in other words, Slater’s spectrum and consistency spectrum are the same and the choice of using (25) or (26) is a matter of convenience and the choice of the name is due to historical reasons.

Once Slater’s spectrum is known, the posterior PDF (22) is trivially obtained from (24), so the problem is now reduced to finding $\{a_t\}_{t=0}^T$ and the algorithm we propose to this end is explained below.

4.1. Recursive calculation of Slater’s spectrum

For any subset $I' \subset I$, $\rho' \in P_{I'}$ is a ranking of the elements in $I'$ and $C(\rho')$ is the associated consistency index, where, to take into account the cardinality of $I'$, we redefine (7) as follows:

$$C(\rho') = \sum_{m=1}^{|I'|} \sum_{n=m+1}^{|I'|} w_{\rho'_m, \rho'_n}.$$ 

(27)
Let us define a mapping from the ranking $\rho'$ onto the polynomial in $u$, which is an auxiliary variable, i.e.,

$$\rho' \mapsto u^{C(\rho')}$$  \hspace{1cm} (28)

and, since the order of the polynomial in \ref{28} gives us the consistency index of the ranking $\rho'$, then

$$G(u; I) = \sum_{\rho \in P_I} u^{C(\rho)} = \sum_{t=0}^{T} a_{\rho} u^t$$  \hspace{1cm} (29)

is a polynomial whose coefficients $a_{\rho} \equiv a_t$ tell us how many rankings $\rho$ with consistency index $C(\rho) = t$ there are in the set $P_I$; there is the same number of rankings with Slater’s index $S(\rho) = t$ so we naturally call $G(u; I)$ Slater’s spectrum (in the polynomial domain). Of course, the spectrum can be defined for any subset $I' \subset I$, i.e.,

$$G(u; I') = \sum_{\rho' \in P_{I'}} u^{C(\rho')} = \sum_{t=0}^{T} a_{\rho'} u^t.$$  \hspace{1cm} (30)

The idea is to use the relationship between the spectra of the subsets recursively and arrive at the spectrum \ref{29}. So we start with the latter and make the generalization next.

The summation over all the rankings in $P_I$, required to calculate \ref{29}, can be written as

$$\sum_{\rho \in P_I} u^{C(\rho)} = \sum_{i=1}^{M} \sum_{\rho' \in P_{I \setminus i}} u^{C([i, \rho'])},$$  \hspace{1cm} (31)

where $\rho = [i, \rho']$ is a concatenation of the element $i$, with the partial rankings $\rho'$ obtained by permutation of the indices in the set $I \setminus i$, which is a subset of $I$ with $M - 1$ elements, obtained by removing the element $i$ from $I$.

The consistency index of the ranking $\rho = [i, \rho']$ in \ref{31} is calculated as
follows:

\[
C([i, \rho']) = \sum_{m=1}^{M} \sum_{n=m+1}^{M} w_{\rho_m, \rho_n} \\
= \sum_{n=1}^{M-1} w_{i, \rho'_n} + \sum_{m=1}^{M-1} \sum_{n=m+1}^{M-1} w_{\rho'_m, \rho'_n} \\
= C(\rho') + C_i(\mathcal{I}),
\]

where

\[
C_i(\mathcal{I}) = \sum_{n \in \mathcal{I}} w_{i, n}
\]

is the sum of the elements of the matrix \( W \) in the row indexed with \( i \) and the columns indexed with all the indices from \( \mathcal{I} \) (\( M - 1 \) elements are added because \( w_{i, i} = 0 \)).

We can now rewrite (29) using (31) and (34)

\[
G(u; \mathcal{I}) = \sum_{i=1}^{M} u^{C_i(\mathcal{I})} \sum_{\rho' \in \mathcal{P}_{\mathcal{I} \setminus i}} u^{C(\rho')} = \sum_{i=1}^{M} u^{C_i(\mathcal{I})} G(u; \mathcal{I} \setminus i),
\]

where, in (37) we use the definition of the spectrum in (30) valid for any subset of \( \mathcal{I} \).

Each spectrum is a polynomial in \( u \) and the multiplication by \( u^{C_i(\mathcal{I})} \) corresponds to increasing the degree of all terms in \( G(u; \mathcal{I} \setminus i) \) by \( C_i(\mathcal{I}) \). Therefore, to calculate the spectrum \( G(u; \mathcal{I}) \) we need to know the spectra \( G(u; \mathcal{I} \setminus i) \) of all subsets \( \mathcal{I} \setminus i, i = 1, \ldots, M \).

Similarly, to calculate a spectrum \( G(u; \mathcal{I} \setminus i), i = 1, \ldots, M \) we need to know the spectra for all subsets with \( M - 2 \) elements, but before we generalize this approach, we need some additional definitions.

Let all \( k \)-elements distinct subsets of \( \mathcal{I} \) be denoted by \( \mathcal{I}^{[k]}_l = \{e^{[k]}_{l,1}, \ldots, e^{[k]}_{l,k}\}, l = 1, \ldots, \binom{M}{k} \), where \( e^{[k]}_{l,i}, i = 1, \ldots, k \) is the \( i \)-th element of the subset \( \mathcal{I}^{[k]}_l \). The order of the elements in \( \mathcal{I}^{[k]}_l \) does not matter, but to identify the sets, we
enumerate their elements in lexicographic order. For example, for $M = 4$, 
$I_3^1 = \{1, 2, 3\}$, $I_3^2 = \{1, 2, 4\}$, $I_3^3 = \{1, 3, 4\}$, $I_3^4 = \{2, 3, 4\}$. Similarly, 
$I_2^1 = \{1, 2\}$, $I_2^2 = \{1, 3\}$, $I_2^3 = \{1, 4\}$, $I_2^4 = \{2, 3\}$, $I_2^5 = \{2, 4\}$, and $I_2^6 = \{3, 4\}$. Then, the elements are obtained as $e_{3,3}^{2,3} = 4$, or $e_{5,1}^{2,1} = 1$.

Obviously, there is only one subset with $M$ elements: the set $I$ itself, thus $I_{M}^1 = I$; also, $e_{1,i}^{M} = i, i = 1, \ldots, M$.

By removing an element from the subset $I_{l}^{k}$ we obtain another subset $I_{j}^{[k-1]}$:

$$I_{l}^{[k]} \setminus e_{i,j}^{[k]} = I_{j}^{[k-1]}, \quad i = 1, \ldots, k,$$

(38)

where we do not specify how the indices $l, i, and j$ are related because it depends on the enumeration strategy. For example, $I_{3}^{[3]} \setminus e_{3,2}^{[3]} = \{2, 3, 4\} \setminus \{3\} = \{2, 4\} = I_{2}^{[2]}$.

With this notation, we generalize (37) as follows:

$$G(u; I_{l}^{[k]}) = \sum_{i=1}^{k} u C_{i}(x_{i,j}^{[k]}) G(u; I_{l}^{[k]} \setminus e_{i,j}^{[k]}) \quad l = 1, \ldots, \binom{M}{k}$$

(39)

where

$$C_{i}(I_{l}^{[k]}) = \sum_{n \in I_{l}^{[k]}} w_{e_{i,j}^{[k]}} n.$$

(40)

We start the recursion by calculating the spectra for all one-element subsets $I_{l}^{[1]}, l = 1, \ldots, M$ which is trivial: $G(u; I_{l}^{[1]}) \equiv 1^{1}$ Then we apply (39) for $k = 2, 4, \ldots, M$, where the final step $k = M$ is exactly the same as (37).

The complexity of (39) is thus due to the enumeration of $\binom{M}{k}$ terms in (39), where $k = 2, \ldots, M$. For each enumerated term, we must (i) calculate $k$ times the value of (40), each of which requires a summation of $k - 1$ elements of the matrix $W$, and (ii) add $k - 1$ polynomials with $T$ elements (many of the coefficients are zero; this property is not exploited here but may be used to

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1This is why we preferred to work with the consistency index.
reduce the complexity); this may be summarized as follows:

\[
\text{CMPLX} = \sum_{k=2}^{M} \binom{M}{k} (k + T)(k - 1)
\]

\[
= O\left(M^2 2^M\right),
\]

where we exploit the fact that \( T \sim M^2 \), e.g., if we make \( K_{m,n} = K \) comparisons for each pair, then \( T = KM(M - 1)/2 \).

For example, with \( M = 20 \), we have the complexity CMPLX \( \approx 10^8 \) which is clearly within reach of any modern computer, while the explicit enumeration in [20] with the complexity of \( M! \approx 2 \cdot 10^{18} \) is not.

The algorithm implemented in Python is available at our GitHub repository

An interesting feature of the algorithm is that even though we do not explicitly find the solution \( \hat{\rho} \) to the ordering problem, we immediately know how many such solutions there are: this is the coefficient \( a_{I,T} = a_{I,T-\hat{S}} \), where \( \hat{S} \) is the optimal Slater’s index in [10]. Furthermore, with a minor modification of the above algorithm, we are able to recover all, \( a_{I,S} \) solutions \( \hat{\rho} \); this is shown below.

4.2. Recovering all solutions \( \hat{\rho} \)

Let \( q(I') \) be the order of the spectrum polynomial \( G(u; I') \), that is, the maximal \( t \) indexing a non-zero coefficient \( a_{I',t} \)

\[
q(I') = \arg\max_t \{a_{I',t} > 0\},
\]

where, of course, \( q(I) = T - \hat{S} \).

From (39) and (40) we know that

\[
q(I_{[k]}) = \max_{i \in 1,\ldots,k} \{C_i(I_{[k]}) + q(I_{[k]} \backslash e_{i,[k]})\},
\]

and we can keep track of the polynomial-orders when executing the steps of the recursion (39).
Once we arrive at the final step, \( k = M \), and find

\[
q(I) = \max_{i \in 1, \ldots, M} \{ C_i(I) + q(I \setminus i) \},
\]

(45)

we can start the “backward” search and identify all the indices \( i \) and corresponding sets \( I_i^{[M-1]} \) which are “order-compatible” with (45), that is, which satisfy the following:

\[
q(I_i^{[M-1]}) + C_i(I) = q(I);
\]

(46)

and this order-compatible index \( i \) is the first element of the optimal order \( \hat{\rho} \); in fact, we should talk about order-compatible indices as there may be more than one \( i \) satisfying (46).

We have to continue the recursion keeping track of the order-compatible indices and of the order-compatible sets \( I_i^{[k]} \) which we gather in “super-sets” \( A^{[k]} \). We obviously have \( A^{[M]} = I \), and then, for all \( I_i^{[k]} \in A^{[k]} \), we add sets to a super-set \( A^{[k-1]} \) as follows:

\[
q(I_i^{[k]}) = q(I_i^{[k]} \setminus e_i^{[k]}) + C_i(I_i^{[k]}) \implies I_i^{[k]} \setminus e_i^{[k]} \in A^{[k-1]},
\]

(47)

where the first condition generalizes order-compatibility condition of (46).

In simple words, if the set \( I_i^{[k]} \) is order-compatible and if, by removing an index \( e_i^{[k]} \), we obtain a set \( I_i^{[k]} \setminus e_i^{[k]} \) that satisfies the order-compatibility condition of (47), then the element \( e_i^{[k]} \) will be a \( (M - k) \)-th order-compatible index.

We have to keep track of all the order-compatible sets, each of which will have a list of order-compatible indices associated with it (in fact, there may be many of such lists associated with a particular order-compatible set \( I_i^{[k]} \)).

We continue in this way till we arrive at \( k = 1 \) and thus find all lists of order-compatible indices.

We should also consider the complexity of the backward phase in which we recover the solutions. However, since we only need the subsets \( I_i^{[k]} \), \( k = M - 1, \ldots, 1 \) that are order-compatible, i.e., which are in the super-sets \( A^{[k]} \), the complexity is similar to \( Ma_{x, 5} \). In practice, for \( M > 10 \), we observed
that the execution time of the backward phase was smaller by many orders of magnitude compared to the forward phase (in which we calculate the orders \( q(T_q^{[k]}) \)).

Note that the entire algorithm we described does not make any reference to the spectrum, which is because it only uses the orders of the polynomials. So, this algorithm is truly an aside result obtained thanks to the particular formulation of the problem.

Again, the algorithm is available at [our GitHub repository](https://github.com/our-repository).

5. Numerical examples

5.1. Synthetic data

To better understand the estimation problem at hand, we randomly generate the data with the value of the parameter \( p = \bar{p} \) where we control \( \bar{p} \). We use \( M = 10 \) and \( K_{m,n} = 2 \) so that \( T = M(M - 1) \), which is similar to the sport league scenario in which teams play against each other twice (typically once at home and once as a visitor).

Fig. 1 shows the posterior distributions \( f(p|\mathbf{W}) \) for ten random realizations of the matrix \( \mathbf{W} \). These are, of course, only examples, because the function \( f(p|\mathbf{W}) \), depending on randomly generated \( \mathbf{W} \), is also random.

We observe in Fig. 1 that even if the data are generated with \( \bar{p} = 0.6 \), it is still possible to obtain the estimate \( \hat{p} = 0.5 \) (the mode of the PDF). Similarly, by generating data with \( \bar{p} = 0.5 \) whose results are plotted in Fig. 1, we see that despite the fact that the results are entirely random (and therefore not related to the ranking), we can still obtain estimates \( \hat{p} > 0.5 \).

This problem looks very similar to what is proposed in Slater’s method; the difference is that instead of \( \hat{S} \), we use \( \hat{p} \) as a statistic for testing. Finding the relationship between both methods may be an interesting research problem.

5.2. Real-world data

We will now use real-world data obtained from sports competitions with binary outcomes (wins/losses) and we consider ten seasons of cricket games in
the Indian Premier League (IPL) and ten seasons in the Italian professional volleyball SuperLega. The summary of the data and the results obtained are shown in Table 1. In principle, there are \( K = 2 \) games per pair of teams, and thus \( T = M(M - 1) \) but in some seasons of the IPL, the games were abandoned and therefore \( T \) is different as we can see in Table 1.

We show an example of Slater’s spectrum in Fig. 2 obtained in 2016 season of Superlega where the symmetric form of the spectrum may be well appreciated and the non-zero coefficients of the spectrum are found only for \( \hat{S} \leq t \leq T - \hat{S} \).

The PDFs obtained for the IPL data are shown in Fig. 3a and those for Superlega in Fig. 3b.

| season | \( M \) | \( T \) | \( \hat{S} \) | \( a_{\hat{S}} \) | \( \hat{\rho} \) |
|--------|--------|--------|--------|--------|--------|
| 2011   | 10     | 68     | 20     | 15     | 0.50   |
| 2012   | 9      | 70     | 19     | 6      | 0.65   |
| 2013   | 9      | 72     | 19     | 52     | 0.68   |
| 2014   | 8      | 56     | 14     | 38     | 0.71   |
| 2015   | 8      | 53     | 16     | 21     | 0.50   |
| 2016   | 8      | 56     | 17     | 12     | 0.50   |
| 2017   | 8      | 55     | 15     | 33     | 0.65   |
| 2018   | 8      | 56     | 17     | 14     | 0.50   |
| 2019   | 8      | 55     | 17     | 14     | 0.50   |
| 2020   | 8      | 56     | 18     | 10     | 0.50   |

| season | \( M \) | \( T \) | \( \hat{S} \) | \( a_{\hat{S}} \) | \( \hat{\rho} \) |
|--------|--------|--------|--------|--------|--------|
| 2009   | 15     | 210    | 32     | 102    | 0.84   |
| 2010   | 14     | 182    | 40     | 5      | 0.75   |
| 2011   | 14     | 182    | 41     | 564    | 0.76   |
| 2012   | 12     | 132    | 26     | 86     | 0.79   |
| 2013   | 12     | 132    | 32     | 420    | 0.73   |
| 2014   | 13     | 156    | 25     | 20     | 0.83   |
| 2015   | 12     | 132    | 23     | 51     | 0.81   |
| 2016   | 14     | 182    | 37     | 3584   | 0.79   |
| 2017   | 14     | 182    | 25     | 93     | 0.86   |
| 2018   | 14     | 182    | 22     | 8      | 0.88   |

Table 1: IPL and Superlega: data: note that in the IPL, for the same \( M \) we may have different values of \( T \) due to canceled games; results: \( \hat{S} \) is the optimal Slater’s index from (10), \( a_{\hat{S}} \) is the number of optimal solutions \( \hat{\rho} \) in (9), and \( \hat{\rho} \) is the MAP estimate from (23).
In the IPL seasons 2012, 2013, 2014, and 2017, we obtain the ML estimate $\hat{p} = 0.5$ (this is the mode of the PDF), that is, the most likely explanation for the data is that the results are generated independently of any order. On the one hand, this is an interesting conclusion, which means that the league is well “balanced” and thus, the games cannot be predicted and are interesting to watch. On the other hand, from the ranking perspective, this means that the solutions of the LOP cannot be meaningfully interpreted.

The results obtained in the SuperLega are quite different and consistently show the estimate $\hat{p} > 0.7$. This may be treated as confirmation that the ranking makes sense under the model (1)-(2).

Note also that even if we obtain $\hat{p} > 0.5$, it is not clear how multiple solutions should be interpreted. For example, in the SuperLega 2016 season, we obtained 3584 solutions $\hat{\rho}$ with the same optimal Slater’s index $\hat{S} = 37$. Using the algorithm shown in Sec. 4.2, we can easily recover all of these solutions, but it is not obvious how to use them to rank the teams.

However, our goal is not to determine whether or how the ranking should be
6. Conclusions

In this work we addressed the issue of estimating reliability in the linear ordering problem (LOP). Our approach is probabilistic: we model the observations in the pairwise comparison setup as Bernoulli variables characterized by a common parameter $p$.

To find an estimate of $p$ we defined the so-called Slater’s spectrum which tells us how many ranking share a given value of Slater’s index; this is done for all possible values of the index. We then developed an efficient algorithm to find Slater’s spectrum that allows us to calculate the posterior distribution of the parameter $p$, which in turn tells us whether the LOP solution can be treated as a meaningful answer to the ordering problem.

As an additional result, thanks to the new formulation of the estimation problem, with a small modification of the algorithms we proposed, and with a negligible increase in complexity, we are able to find all optimal solutions of LOP. Python-implemented algorithms are available at our GitHub repository.

We show numerical examples of estimation using synthetic and real-world data. Our examples indicate that, indeed, in some practical cases, the most likely explanation for the observed data is that it is independent of any under-
lying order, which, of course, puts a serious question mark on the very meaning of ranking in these cases.

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