What are the Best Hierarchical Descriptors for Complex Networks?

Luciano da Fontoura Costa¹, Roberto Fernandes Silva Andrade²
¹Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, SP, PO Box 369, 13560-970, Brazil, luciano@if.sce.usp.br and
²Instituto de Física, Universidade Federal da Bahia, 40210-340, Salvador, Bahia, Brazil
(Dated: February 1, 2008)

This work reviews several hierarchical measurements of the topology of complex networks and then applies feature selection concepts and methods in order to quantify the relative importance of each measurement with respect to the discrimination between four representative theoretical network models, namely Erdös-Rényi, Barabási-Albert, Watts-Strogatz as well as a geographical type of network. The obtained results confirmed that the four models can be well-separated by using a combination of measurements. In addition, the relative contribution of each considered feature for the overall discrimination of the models was quantified in terms of the respective weights in the canonical projection into two dimensions, with the traditional clustering coefficient, hierarchical clustering coefficient and neighborhood clustering coefficient resulting particularly effective. Interestingly, the average shortest path length and hierarchical node degrees contributed little for the separation of the four network models.

PACS numbers: 89.75.Hc, 89.75.Fb, 89.75.-k

It is better to know some of the questions than all of the answers. (J. Thurber)

I. INTRODUCTION

A relevant analysis of several features of complex systems can be achieved through the recently developed complex network framework (e.g. [1, 2, 3, 4, 5]). Due to the large amount of variables normally involved in such dynamical systems, the set up of a interaction network, based on functional relationship among its degrees of freedom, offers a first picture to the actual internal structure of the system. This process requires the identification of the pertinent degrees of freedom as nodes, while the edges that connect them are defined by the mutual influence they are subject to. The ability of identifying nodes and edges in the appropriate way is a crucial step in this modeling.

The characterization of the so obtained networks constitute a second important step in this kind of analysis. In this process, a small number of features is chosen in order to measure, in an objective way, pertinent properties of the sets of nodes and edges. The choice of measurements used in the investigation constitutes the second key decision during the structural analysis of the networks, as it defines which information can be obtained (e.g. [5]). Nowadays, it is consensual that such a measure space whose axes are spanned to those in the basic set and among themselves. Therefore, new and distinct measures have been proposed in order to capture new aspects not covered by the set of four parameters listed above.

One particularly important aspect regards the characterization of individual nodes in the network, as this allows the identification of particularly distinct nodes such as hubs (e.g. [6]). While both the degree and clustering coefficient are defined for each individual node, they provide but a limited characterization of the connectivity around those nodes, with several nodes resulting with identical pairs of degree/clustering coefficient values even when they are placed at completely different contexts in the network. One interesting means to obtain a richer (i.e. less degenerate) set of measurements for each node is to consider subsequent hierarchical neighborhoods (e.g. [7, 8, 9, 10, 11, 12, 13, 14]) around each node, in addition to the immediate neighbors considered in the traditional degree and clustering coefficient.

Although new measurements provide extra information, it is important to understand how they are related to those in the basic set and among themselves. If one defines a measure space whose axes are spanned by the distinct parameters, one important issue regards the distribution of observations along each axis. A full answer to such task should include also an analytical relationship between the co-linear (correlated) measures. Another related issue refers to deciding, given a set of distinct measures, which of them are most effective in identifying and discriminating between distinct kinds of networks. The purpose of this work is to address these questions, by working with a set of hierarchical measures and using sound concepts and methods of multivariate statistics (e.g. [3, 15, 16]). We probe a large number of networks generated according to four representative theoretical models, Erdös-Rényi (ER), Barabási-Albert (BA), Watts-Strogatz (WS) as well as a geographical type of network (GG), which can be put in connection to
distinct complex network paradigmatic types, displaying feature that are associated to the random, scale-free and small-world behaviors. In this way, it becomes possible to quantify the relative importance of each measurement, with respect to the discrimination between the considered distinct network models. Although illustrated for these specific four types of networks, the reported methodology is completely general and can be applied virtually to any problem involving the choice of measurements given specific types of theoretical or real-world networks.

The hierarchical measurements we take into account have been discussed in a series of previous investigations (e.g. [10, 11, 12, 13]), in which the authors have inquired how a node sees not only its immediate neighborhood, but also successive neighborhoods up to a maximal distance \( D \) from the reference node. The concepts of hierarchies and higher order neighborhoods, that have been independently introduced (e.g. [2, 3, 14, 15]), aim at providing a description of the relationship among given sets of nodes which are not necessarily linked by immediate edges, but for which the minimal distance along the network is bound to a value \( 1 \leq \ell \leq D \). \( \ell \) dependent clustering coefficients and node degrees have been investigated and compared for several sets of networks. In a second line of investigation, a recent contribution raises the issue of the interdependence among distinct measures, while reviewing the most relevant measures that have been introduced so far [5]. The results reported herein, heavily based on the ideas developed in the quoted references, are aimed at quantifying the role of the several hierarchical measurements while discriminating between the four considered theoretical network models. In order to quantify the influence of each measurement on the separation between the four classes of networks, we apply sound and objective concepts from multivariate statistics, namely standardization and canonical projections (e.g. [6, 16]).

This work is organized as follows: In Section II, we present the basic notions of complex networks and of the theoretical models used in our investigation. In Section III, we discuss the hierarchical measurements that will be taken into account for the selection method. These methods are presented and discussed in Section IV. Results from our analyzes are discussed in Section V, while Section VI closes the work with the concluding remarks.

II. BASIC CONCEPTS

This section introduces the main concepts used in our analysis, including network representation as well as the four theoretical models.

A. Complex Networks Basic Concepts

A non-weighted complex network \( \Gamma \) with \( N \) nodes and \( E \) edges can be fully specified in terms of its adjacency matrix \( K \), so that \( K(i,j) = 1 \) indicates the existence of an edge extending from node \( j \) to node \( i \). All networks considered in this work are undirected, which implies \( K \) to be symmetric. They are also devoid of multiple or self-connections. Networks whose nodes have well-defined spatial positions within an embedding space are called geographical networks.

The degree \( k_i \) of a node \( i \) is defined as the number of edges connected to it. In case a node \( j \) can be reached from a node \( i \), we can say that there is a path between these two nodes. Two nodes can be connected through more than one distinct path. The shortest path \( d_{i,j} \) between two nodes \( i \) and \( j \) corresponds to the path with the smallest number of edges connecting those nodes. The immediate neighborhood of a node \( i \) is the set of nodes which are directly connected to \( i \), i.e. the nodes \( j \) for which \( d_{i,j} = 1 \). The average shortest path \( d_i \) to a node \( i \) is the mean value of \( d_{i,j} \) over all nodes \( i \neq j \), while the network average shortest path \( (d) \) is obtained by taking the mean value of \( d_i \) over the whole set of network nodes.

The clustering coefficient \( C_i \) of node \( i \) can be calculated as the ratio between the number of edges among the immediate neighbors of \( i \) and the maximum possible number of edges between those nodes. Although measurements such as the node degree and clustering coefficient apply to individual nodes, it is common to take their average along the network, yielding the average node degree \( \langle k \rangle \) and the average clustering coefficient \( \langle C \rangle \).

B. Complex Networks Models

Several theoretical models of complex networks have been proposed (e.g. [1, 2, 3, 4, 5]). As announced in Section I, the current work considers four of such models (ER, BA, WS and GG), the most important features of which we briefly describe below. All networks used for the comparison of the hierarchical measurements in this work have the same number of nodes \( N \) and average node degrees \( \langle k \rangle \) as similar as possible.

The ER model (e.g. [1]) is characterized by having constant probability \( \rho \) of connection between any possible pair of nodes. Its average degree is given as \( \langle k \rangle = 2E/N = 2\rho N \). The BA model can be obtained by starting with randomly interconnected \( m \) nodes. At each subsequent step, a new node is connected to \( m \) nodes in the current network such that each connection is preferential to the degree of the previous nodes. The average degree of a BA model is given as \( 2m \). Therefore, in order to have ER and BA networks with the same node degree, we need to enforce that \( m = (N - 1)\rho \). WS networks can be produced by starting with the \( N \) nodes distributed along a ring and connecting each node to its \( \langle k \rangle /2 \) clockwise neighbors and to the same number of counterclockwise neighbors, with \( \langle k \rangle \) being an even number. Then, a small percentage of edges are randomly rewired. Finally, the geographical model considered in this work is...
obtained by considering a Poisson spatial distribution of points with density $\gamma$ in a two-dimensional embedding space with uniform connecting all pair of nodes which are at Euclidean distance smaller than $\sqrt{\langle k \rangle / (\gamma \pi)}$.

### III. HIERARCHICAL MEASUREMENTS

Two of the most ubiquitously accepted network measures, namely the average number of links per node $\langle k \rangle$ and the clustering coefficient $C$, reflect the immediate landscape of the nodes, as they just consider, respectively, the number of neighbors each node is connected to by a direct edge, and how the neighbors of a node are connected among themselves. The hierarchical measurements introduced in [10, 11, 12, 13, 14] first require the identification of the sets named the hierarchical shells $H_i(\ell)$ or, alternatively, the neighborhoods $N_i(\ell)$, of order $\ell$ of a node $i$ as the nodes that lie at a minimal distance $\ell$ along the edges of the network of a given node $i$. For the sake of uniqueness, from now on we call these sets as $H_i(\ell)$. The hierarchical measurements result from the extension of the two basic concepts to the sets $H_i(\ell)$.

For the feature selection analysis we consider, respectively, two and three distinct types of node degrees and clustering coefficients, which are so defined. The average degree

$$\langle k(\ell) \rangle = \sum_{i=1}^{n} k_i(\ell),$$

where $k_i(\ell)$ counts the number of neighbors which are at a minimal distance $\ell$ of node $i$, indicates how the higher order neighborhoods of each node are populated. The average hierarchical degree

$$\langle k^H(\ell) \rangle = \sum_{i=1}^{n} k^H_i(\ell),$$

has a different meaning, as $k^H_i(\ell)$ counts the number of links between elements of the two sets $H_i(\ell)$ and $H_i(\ell+1)$. It expresses how deep connected are the nodes that lie in two successive hierarchical shells, namely $\ell$ and $\ell + 1$, of node $i$. Observe that we have $\langle k(\ell = 1) \rangle = \langle k^H(\ell = 0) \rangle = \langle k \rangle$, if we consider that the 0-th order neighborhood of a node is the node itself.

The three distinct $\ell$ dependent clustering coefficients coincide with the usual $C$ when $\ell = 1$. The hierarchical clustering coefficient $C^H(\ell)$ counts how many of the $k_i(\ell)(k_i(\ell) - 1)/2$ pairs of nodes formed the elements of the set $H_i(\ell)$ are directly linked by one edge. On the other hand, the neighborhood clustering coefficient $C^N(\ell)$ takes into account those pairs of the same set that are neighbors of order $\ell$. The original clustering coefficient $C$ is a direct measure of the presence of nearby triangles in a network, and it indirectly hints to the presence of connected structures as cliques. The higher order $C^M(\ell)$ and $C^N(\ell)$ give information on the how the nodes on more distinct hierarchical shells are related among themselves.

Finally, the hierarchical clustering coefficient by balls $C^B(\ell)$, which was also previously introduced, constitutes the third measurement that takes into account the hierarchical structure of neighbors of a node. This is a cumulative measure in the sense that, instead of considering the nodes in a single set $H_i(\ell)$, it considers all nodes in the set $\mathcal{H}_i(\ell) = \bigcup_{\ell=1}^{\ell} H_i(\ell)$.

To evaluate all the above hierarchical measures we profited from the formalism introduced in [14], which amounts to first identifying all the higher order neighborhoods of the networks and storing the information in a single matrix

$$\mathbf{\hat{M}} = \sum_{\ell=0}^{D} \ell M(\ell).$$

All distinct hierarchical measures can be easily defined in terms of the elements of $\mathbf{\hat{M}}$.

### IV. FEATURE SELECTION METHODS

Given $L$ classes of networks (in the case of the current article the four theoretical models ER, BA, WS and GG) and $Q$ respective measurements of their topology, an important question is: which subset of measurements is more effective for discriminating between such classes? Such a problem provides a good example of feature selection.

Two main approaches have been considered for feature selection: filter and wrapper. The difference between these two families of methods is that the latter evaluates the features by considering the results obtained after feeding them into a classifier, while the former methods investigate the intrinsic relationship between the measurements between and/or within the classes (e.g. [16, 17, 18]). For instance, the canonical projection method used in this work provides an example of a filter approach to feature selection.

It should be observed that none of the feature approaches are absolutely optimal. While wrapper methods will select features which are most effective for given classifiers, filter approaches will depend on the definition of some optimality criterion. For instance, the canonical projection method adopted in this work quantifies the separation between the classes by maximizing the distance between the classes and minimizing the dispersion inside each class (see Section [IVC]). Because our interest in the current work is to characterize the discrimination power of the several hierarchical measurements, we limit our attention to filter feature selection methods.

The following subsections present the basic concepts from multivariate statistics as well as the principal component analysis and the canonical projection methodologies.
A. Basic Concepts in Multivariate Statistics

Let each of the $Q$ objects of interest (e.g. networks) be characterized in terms of $R$ measurements $x(i)$, $i = 1, 2, \ldots, R$. It is convenient to organize the set of measurements obtained for each object $p = 1, 2, \ldots, Q$ into the respective feature vector

$$\vec{v}_p = [x_p(1), x_p(2), \ldots, x_p(R)]^T.$$  

(4)

The mean feature vector $\vec{\mu}$ can be calculated as

$$\mu(i) = \frac{1}{Q} \sum_{p=1}^{Q} x_p(i).$$  

(5)

The elements $C(i, j)$ of the covariance matrix $C$ of the measurements of the objects can be estimated as

$$C(i, j) = \frac{1}{Q - 1} \sum_{p=1}^{Q} (x_p(i) - \mu(i))(x_p(j) - \mu(j)).$$  

(6)

The standardized feature vector can be obtained as

$$s_p = \left[ \frac{x_p(1) - \mu(1)}{\sigma(1)}, \frac{x_p(2) - \mu(2)}{\sigma(2)}, \ldots, \frac{x_p(R) - \mu(R)}{\sigma(R)} \right]^T,$$

(7)

where $\sigma(i)$ is the standard deviation of measurement $x(i)$. Note that each normalized measurement $s(i)$ has zero mean and unity standard deviation.

The Pearson Correlation Coefficient between two measurements $x(i)$ and $x(j)$ can be given by the covariance between the standardized measurements $s(i)$ and $s(j)$.

B. Principal Component Analysis — PCA

The multivariate statistical method known as principal component analysis (PCA) allows dimensionality reduction while maximizing the data variance along the first projected axes (e.g. $\frac{1}{Q}$). Because the class of each point is not taken into account in this method, it corresponds neither to filter nor wrapper feature selection. This method is considered in this work for two reasons. First, it can be used to obtain preliminary visualizations of the distribution of points and classes. Second, it provides an introduction and a comparison standard to the more sophisticated canonical projections methodology, to which it is related.

Given the set of $Q$ objects, characterized by $R$ measurements, it is possible to project such measurements into a reduced space with $W < Q$ dimensions. In order to do so, the covariance matrix $C$ of the measurements is estimated as described in Section IV A and its eigenvalues and respective eigenvectors are calculated. The eigenvectors corresponding to the $W$ largest eigenvalues (in decreasing order of absolute values) are organized into a matrix $A$ such that each line corresponds to an eigenvector. The matrix $A$ defines the statistical linear transformation of the original set of data that maximizes variances along the first new axes. Provided $W$ is equal to 2 or 3, the so-transformed data can now be visualized as a 2D or 3D distribution of points. The original classes of each point can be visualized with different marks.

C. Canonical Projections

The method of canonical projections, also called canonical analysis or canonical variables (e.g. $\frac{1}{Q}$), also performs a projection of the original measurement space, but now considering explicitly the original classes of each object. The projection is performed not in order to maximize the variances along the first new axes, but so as to obtain maximum separation of the classes, quantified by an optimality index $\xi$ reflecting the distribution of the data both inside and among classes. More specifically, $\xi$ will favor well-separated classes, with small dispersions of the respective objects. The inter- and intra-class dispersion matrix, respectively $D_e$ and $D_a$, can be calculated as described in $\frac{1}{Q}$. The eigenstructure of the matrix $(D_a)^{-1}D_e$ provide the basis for the sought linear transformation (as in the PCA, the eigenvectors associated to the largest absolute eigenvalues are stacked as lines in the transformation matrix) maximizing the separation between the classes. For instance, in the case of canonical projections into two-dimensional spaces, the eigenvectors $v1$ and $v2$ associated to the largest and second largest absolute eigenvalues are used to define the projection linear transformation. Similarly to the PCA, the contribution of each original measurement to the projection can be quantified in terms of the absolute value of the weights defined by the respective eigenvector transformation matrix. Therefore, the measurements yielding the largest absolute weights for the first axes can be understood as those which are more important for the separation between the classes. In this work we define the importance of each measurement $i$ as the sum of the absolute values of the respective weights in $v1$ and $v2$, i.e.

$$I(i) = |v1(i)| + |v2(i)|.$$  

(8)

In order to avoid intrinsic biases implied by the relative amplitude of each measurement, it is interesting to perform the canonical projections on standardized versions of the measurements.

V. RESULTS AND DISCUSSION

In order to investigate, in a comparative fashion, the relative contributions of each measurement for the char-
acterization and discrimination between the four considered complex network models, 30 realizations of each model, all with mean degree equal to 6 and sizes \( N \) of 100, 200 and 300 nodes, were first obtained. Two types of WS networks were obtained, considering 0.1\( N \) and 0.1\( E \) connection rewirings, where \( E \) is the overall number of connections. These two types of WS networks are henceforth abbreviated as WS-R and WS-S. Observe that the former type corresponds to almost regular networks (i.e. similar node degrees throughout), while the latter type presents the small world property. All considered models had their average traditional and hierarchical measurements (for \( \ell = 1, 2, \) and 3) calculated and used as feature vectors. Table I lists the considered measurements as well as their respective symbols and abbreviations.

| Measurement                                      | Symbol   | Abbreviation |
|--------------------------------------------------|----------|--------------|
| Hierarchical clustering coefficient by balls \( C^B(\ell) \) | \( C^B \) | \( cb \)      |
| Hierarchical clustering coefficient \( C^H(\ell) \) | \( C^H \) | \( ch \)      |
| Neighborhood clustering coefficient \( C^n(\ell) \) | \( C^n \) | \( cn \)      |
| Average number of nodes \( (k(\ell)) \)         | \( k \)  | \( n \)       |
| Average shortest path \( (d) \)                 | \( d \)  | \( sp \)      |
| Average hierarchical degree \( (k^H(\ell)) \)   | \( k^H \) | \( hd \)      |

**TABLE I:** The symbols and abbreviations of the considered hierarchical measurements defined in Sections II and III. The respective hierarchical level \( (\ell) \) is henceforth represented in front of each abbreviation, e.g. the hierarchical node degree at level \( \ell = 3 \) is abbreviated as \( hd^3 \).

Figure 1 shows the two-dimensional phase spaces obtained by PCA projection of the original 13-dimensional phase spaces for \( N = 100 \) and 300. Each point in this phase space corresponds to a specific network realization. The ER and BA clusters resulted near one another, which was also obtained for the WS/GG pair of clusters. The GG networks resulted in the most dispersed clusters.

The phase spaces obtained while considering all 13 measurements were also projected into two dimensions by using the canonical methodology described in Section IV C. Figure 2 shows the projected phase spaces obtained for networks with size of \( N = 100 \) (a) and 300 (b) nodes, respectively. It is clear that the separation between the four networks modes is much better than that obtained by using PCA (Fig. 1). It is also clear from the two dimensional spaces in Fig. 2 that the four models could be very well separated as a consequence of using such a comprehensive set of features. Interestingly, the ER/BA and WS/GG models again tended to cluster together. Observe that the dispersion of the points for all distinct classes decreased for larger \( N \). The relative separation between the ER and BA models along the \( v_2 \) direction also decreased for this case. Also, the WS-R and WS-S families of networks resulted near one another.

The seven most important measurements considering all the three network sizes, in decreasing order, were: \( cb1 = ch1 = cn1 = C, \) \( cb3, \) \( cn2, \) \( cb2, \) \( cb3 \) and \( hd2 \). Figure 3 shows the importance of each of these measurements obtained for each of the network sizes (i.e. 100, 200 and 300). It is noteworthy the absence of two of the most used measurements, \((k)\) and \((d)\), as well as of the higher order hierarchical node degrees, for the purpose of identifying the distinct network classes. The fact that all networks considered in this work had nearly the same average degree explains why \((k)\) had little contribution for the discrimination. However, the fact that the distribution of average hierarchical degrees have been found to vary between different network models should, at least in principle, imply better discrimination potential for those measurements. The relatively minor contribution provided by the average shortest path is also surprising.

The traditional clustering coefficient, \( C \) resulted the most important measurement in all cases, but the relevance of considering higher hierarchies for the character-
FIG. 2: The distribution, in the projected phase space, of the four theoretical complex network models obtained for networks with 100 (a) and 300 (b) nodes. The axes $v_1$ and $v_2$ correspond to the two main projection orientations as provided by the eigenvectors associated to the largest absolute eigenvalues.

FIG. 3: The importance of the most relevant measurements for each of the considered three network sizes, i.e. 100, 200 and 300 nodes.

The two measurements are said to be correlated. A similar tendency was observed for the same hierarchical measurements with distinct values of $\ell$, as in the panels for $C^N(2) \times C^N(3)$ (Fig. 4) and $\langle k^H(2) \rangle \times \langle k^H(3) \rangle$ (Fig. 4). Note, however, that the points in these cases are aligned in a less clear way in comparison to $C^H(2) \times C^H(3)$ (Fig. 4). The panels which combine hierarchical measures of distinct classes ($C^H$, $\langle k^H \rangle$, $C^N$) have the points away from the diagonal (e.g. Fig. 4c). This is an indication that these measurements are uncorrelated, tending to provide non-redundant information and, consequently, enhanced discrimination power. It is however important to stress that the overall discrimination can not be fully predicted simply from pairwise relationships between the measurements, such as those illustrated above. Observe also that none of the two-dimensional scatterplots in Figure 4 provide separation between the four models as good as that shown in Figure 2b. That is because the latter scatterplot was obtained from the much higher dimensional phase space by projecting into the plane allowing the best separation between the four models. Such a result clearly corroborates the increased separability allowed by the consideration of a comprehensive set of distinct hierarchical measurements.

Finally, we proceeded with a further test to investigate the ability of the used feature section methods to uncover network specificities, by adding one extra network to two of the previous groups of 120 specimens. We considered the Apollonian network (AN) [20, 21], a geometrical assembly of nodes and links, which is defined on the basis of the classical problem of finding the optimal covering of a plane by circles. As they are defined in a recursive way, we considered two successively AN generations, respectively with $N = 124$ and 367 nodes, and included them into the groups consisting of networks with $N = 100$ and 300. It is important to recall that AN shows several features that are typical both of small world (small $D$ and $\langle d \rangle$), and scale free scenarios ($p(k) \sim k^{-\gamma}$). The feature selection method can help to identify whether AN
FIG. 4: Scatterplots respective to pairwise combinations of the adopted measurements for $N = 300$. Measurements $cn2$ and $hd2$ provided the best discrimination between the network models when taken individually (a,b). Measurements taken at successive hierarchies tended to be moderately correlated (c), while distinct types of measurements presented very little correlation (d-h).
lies closer to the WS or BA clusters and, hence, to indicate which of the quoted scenarios it stays closer to. The results \((v_1, v_2) = (-0.90, 2.53)\) for \(N = 124\), and \((-0.24, 0.96)\) for \(N = 367\), indicate that, for both generations, the AN is mapped away from all four clusters corresponding to each of the considered theoretical models. This is, indeed, a very interesting result, as it shows that our method is able to identify that this type of network presents quite distinct topological features, albeit it shares some common properties with those that fit into those the small world and scale free scenarios. In other words, the very fact that several features of a given network coincide with those typical for a large network class, does not automatically implies that it belongs to the same set.

VI. CONCLUDING REMARKS

Much of the advances in science have only been allowed by ability of researchers to focus attention on the most important features and variables in each problem. Because human beings have a rather limited ability to cope with a large numbers of measurements, it becomes critical to devise and apply methods which can possibly identify the most relevant features. Fortunately, sound and objective concepts and methods — defining the research area called feature selection — have been developed which can help us in such tasks. Perhaps for historical reasons, such methods are not so widely known and used by the Physics community.

Because highly structured complex networks can only be comprehensively characterized by considering several measurements, the application of feature selection methods presents great potential for helping researchers in that area. In a recent work \(8\), canonical variables projections and Bayesian decision theory were applied in order to classify complex networks and to investigate measurements. The current work has unfolded such a possibility with respect to the discriminative potential of a comprehensive set of hierarchical measurements.

While the traditional node degree, clustering coefficient and shortest path provide quantifications of important features of the networks under analysis, they are degenerated in the sense that several networks may map into the same measurement values. The extension of such concepts to reflect also the progressive neighborhoods around each node has been proposed (e.g. \(10, 11, 12, 13, 14\)) in order to obtain enhanced, less degenerated, characterizations of complex networks. Each of such measurements are defined for a series of hierarchical levels \(\ell\), yielding a high dimensional measurement space. Actually, the values of such measurements taken at each level can be understood as a measurement in itself. Given such a large number of features, it becomes important to identify which measurements are potentially more effective in providing discriminative descriptions of specific types of networks under analysis. In this work, we applied standardization and canonical projections in order to identify the most important measurements in Table \(\text{T}\) with respect to four representative complex networks models, namely Erdös-Rényi, Barabási-Albert, Watts-Strogatz and a geographical type of network. Each hierarchical measurement was calculated along three successive neighborhood levels. The traditional average degree, clustering coefficient and shortest path lengths were also considered. A total of 13 measurements were considered in our investigation.

Several interesting findings have been obtained by the applied methodology. First, four types of networks were well-separated even in the two-dimensional canonical projected phase space (considerably worse separations were obtained by PCA), with the pairs of models ER/BA and WS/GG forming superclusters. By taking into account the respective weights of each measurement in the canonical projections, it was possible to associate an overall importance value to each measurement. Such values were calculated for three network sizes \((N = 100, 200 \text{ and } 300)\). While the traditional clustering coefficient was identified as contributing more intensely for the separation between the network types, several hierarchical measurements resulted in relatively high complementary contributions, with the hierarchical clustering coefficient by balls \((C^B(\ell))\) and neighborhood clustering coefficient \((C^N(\ell))\) providing particularly relevant contributions. The node degree (traditional and hierarchical), as well as the average shortest path length, did not contribute significantly to the separation of the network models. It is important to recall that such results are, in principle, specific to the separation of the four considered types of networks, in the sense that different results may be obtained when considering other networks models.

In addition to providing an objective means for selecting measurements for characterization and discrimination of complex networks models, the multivariate approach considered in this work can also provide valuable insights about the structural differences between distinct types of networks. For instance, the fact that the clustering coefficient resulted more relevant than the shortest path length suggests that the four considered models present local connectivity (expressed in the clustering coefficient) even more distinct than shortest path length distribution. It would be interesting to apply such multivariate methods to the characterization of other types of networks, especially those involving community structure.

Acknowledgment: Luciano da F. Costa is grateful to FAPESP (05/00587-5) and CNPq (301303/06-1) for financial support. Roberto F. S. Andrade acknowledges support by FAPESB (012/05) and CNPq (306369/04-4).
[1] R. Albert and A. L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[2] S. N. Dorogovtsev and J. F. F. Mendes, Advances in Physics **51**, 1079 (2002).
[3] M. E. J. Newman, SIAM Review **45**, 167 (2003), cond-mat/0303516.
[4] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Physics Reports **424**, 175 (2006).
[5] L. da F. Costa, F. A. Rodrigues, G. Travieso, and P. V. Boas, Adv. Phys. **56**, 167 (2007).
[6] L. da F. Costa, M. Kaiser, and C. Hilgetag (2006), physics/060727.
[7] M. Faloutsos, P. Faloutsos, and C. Faloutsos, Comp. Comm. Rev. **29**, 251 (1999).
[8] M. E. Newman, cond-mat/0111070 (2001).
[9] T. Kalisker, R. Cohen, O. Mokryn, D. Dolev, Y. Shavitt, and S. Havlin, Phys. Rev. E **74**, 066108 (2006), cond-mat/0305582.
[10] L. da F. Costa, Phys. Rev. Letts. **93**, 098702 (2004).
[11] L. da F. Costa and L. E. C. da Rocha, Eur. Phys. J. B **50**, 237 (2005).
[12] L. da F. Costa and F. N. Silva, J. Stat. Phys. **125**, 845 (2006).
[13] L. da F. Costa and O. Sporns, Eur. Phys. J. B **48**, 567 (2006), q-bio.NC/0508007.
[14] R. F. S. Andrade, J. G. V. Miranda, and T. P. Lobão, Phys. Rev. E **73**, 046101 (2006).
[15] R. O. Duda, P. E. Hart, and D. G. Stork, *Pattern Classification* (Wiley-Interscience, 2001).
[16] G. J. McLachlan, *Discriminant Analysis and Statistical Pattern Recognition* (Wiley, 2004).
[17] J. Han and M. Kamber, *Data Mining: Concepts and Techniques* (Morgan Kaufmann, 2001).
[18] D. Hand, H. Mannila, and P. Smyth, *Data Mining* (The MIT Press, 2001).
[19] L. da F. Costa and R. M. C. Jr., *Shape Analysis and Classification: Theory and Practice* (CRC Press, 2001).
[20] J. S. Andrade, H. J. Herrmann, R. F. S. Andrade, and L. R. da Silva, Phys. Rev. Lett. **94**, 018702 (2005).
[21] J. P. K. Doye and C. P. Massen, Phys. Rev. E **71**, 016128 (2005).