REFINEMENT OF THE CLASSICAL BOHR INEQUALITY

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Abstract. The classical inequality of Bohr asserts that if a power series converges in the unit disk and its sum has modulus less than or equal to 1, then the sum of absolute values of its terms is less than or equal to 1 for the subdisk $|z| < 1/3$ and $1/3$ is the best possible constant. Recently, there has been a number of investigations on this topic. In this article, we present a refined version of Bohr’s inequality along with few other related improved versions of previously known results.

1. Introduction and two Main results

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk and $B$ denote the class of all analytic functions in $D$ such that $|f(z)| \leq 1$ in $D$. Let us first recall the theorem of H. Bohr [11] in 1914 which inspired a lot in the recent years.

Theorem A. If $f \in B$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

\[ \sum_{n=1}^{\infty} |a_n| r^n \leq 1 - |a_0| \text{ for } r \leq 1/3 \]

and the constant $1/3$ cannot be improved.

The inequality (1) is known as the classical Bohr inequality and the number $1/3$ is called the Bohr radius for the family $B$. It is worth pointing out that if $|a_0|$ in (1) is replaced by $|a_0|^2$, then the constant $1/3$ could be replaced by $1/2$. Moreover, if $a_0 = 0$ in Theorem A then the sharp Bohr radius is known to be $1/\sqrt{2}$ (see for example [19] and [23, Corollary 2.9]). Extensions, modifications and improvements of this result can be found in many recent papers [3, 5–7, 9, 10, 16, 19, 21, 22, 24, 25]. For example, several improved versions of Theorem A are established recently in [8, 20]. Bohr’s theorem attracted the attention of many after the appearance of the article of Dixon [15] who has used Bohr’s original theorem to construct a Banach algebra which is not an operator algebra, yet satisfies the non-unital von Neumann’s inequality. Paulsen et al. [23] have applied operator-theoretic techniques to extend it to Banach algebras and obtain multidimensional generalizations of Bohr’s inequality. Further investigations such as interconnections among multidimensional Bohr radii and local Banach space theory may be obtained from the survey articles of Abu-Muhanna et al. [2], Defant and Prengel [14], Garcia et al. [17]. See also Defant et al. [13] and the references therein.

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The proof of the inequality (1) used the sharp coefficient inequalities which may be obtained as an application of Pick’s invariant form of Schwarz’s lemma for $f \in \mathcal{B}$:

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}.$$ 

In particular, $|f'(0)| = |a_1| \leq 1 - |f(0)|^2 = 1 - |a_0|^2$ from which sharp inequalities $|a_n| \leq 1 - |a_0|^2$ ($n \geq 1$, $f \in \mathcal{B}$) follow. As remarked in [19], we were not able to obtain sharp result due to the fact that in the extremal case $|a_0| < 1$. Lately in [8], a sharp version of Theorem A has been achieved and this works for any individual function from $\mathcal{B}$. Closer examination of different proofs of Theorem A shows that the determination of the Bohr radius for geometric subfamilies of the class of univalent functions are easy to establish because in these cases the corresponding bounds for such families are well-known with extremal functions (with the replacement of $\text{dist}(f(0), \partial \mathbb{D}) = 1 - |a_0|$ by $\text{dist}(f(0), \partial f(\mathbb{D}))$) as initiated by Aizenberg [4] and studied later by Abu-Muhanna [1] using the concept of subordination). For instance, it is an easy exercise to derive the Bohr radius for families such as the family of convex functions of order $\alpha$ ($-1/2 \leq \alpha < 1$), the family of starlike functions of order $\alpha$ ($0 \leq \alpha < 1$), and family of functions with real part bigger than $\alpha$ ($0 \leq \alpha < 1$). We refer to [18] for the definition of these families and many other related families of analytic functions in the unit disk. In all these cases analogs of Theorem A are easy to present, and thus we are not dealing with such situations.

In view of recent developments on Bohr’s inequalities in various settings with improved formulation, it is therefore natural to ask whether we could establish a refined version of the coefficient inequality for the family $\mathcal{B}$ and if so, is it possible to use it to establish a more refined and improved version of the Bohr inequality (1)? The answer is yes and we present such a possibility. Here is set of two main results whose proofs will be presented in Section 2. Several consequences of this idea will be dealt elsewhere.

**Theorem 1.** Suppose that $f \in \mathcal{B}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $f_0(z) = f(z) - a_0$, and $\|f_0\|_r$ denotes the quantity defined by

$$\|f_0\|_r = \sum_{n=1}^{\infty} |a_n|^2 r^{2n}.$$ 

Then

$$\sum_{n=0}^{\infty} |a_n| r^n + \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \|f_0\|_r \leq 1 \quad \text{for} \quad r \leq \frac{1}{2 + |a_0|}$$

and the numbers $\frac{1}{2 + |a_0|}$ and $\frac{1}{1 + |a_0|}$ cannot be improved. Moreover,

$$|a_0|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \|f_0\|_r \leq 1 \quad \text{for} \quad r \leq \frac{1}{2}$$

and the numbers $\frac{1}{2}$ and $\frac{1}{1 + |a_0|}$ cannot be improved.
We remark that $\frac{1}{3} \leq \frac{1}{2 + |a_0|} \leq \frac{1}{2}$. In particular, in the case of $a_0 = 0$ the conclusion of Theorem 1, namely, the inequality (2) gives that

$$\sum_{n=1}^{\infty} |a_n|r^n + \frac{1}{1-r} \| f \|_r \leq 1 \text{ for } r \leq \frac{1}{2}. $$

Unfortunately, the direct substitution for $a_0$ would not yield sharp Bohr radii. In the following, we prove two different improved versions of the last inequality.

**Theorem 2.** Suppose that $f \in \mathcal{B}$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n$. Then we have the following:

(a) $\sum_{n=1}^{\infty} |a_n|r^n + \left( \frac{1}{1 + |a_1|} + \frac{r}{1 - r} \right) \sum_{n=2}^{\infty} |a_n|^2 r^{2n-1} \leq 1 \text{ for } r \leq \frac{3}{5}.$

The number $3/5$ is sharp.

(b) $\sum_{n=1}^{\infty} |a_n|r^n + \left( \frac{r^{-1}}{1 + |a_1|} + \frac{1}{1 - r} \right) \| f \|_r \leq 1 \text{ for } r \leq \frac{5 - \sqrt{17}}{2}.$

The number $\frac{5 - \sqrt{17}}{2}$ is sharp.

(c) Let $|a_1| = \frac{1}{\sqrt{2}}$. Then $\sum_{n=1}^{\infty} |a_n|r^n + \left( \frac{r^{-1}}{1 + |a_1|} + \frac{1}{1 - r} \right) \| f \|_r \leq 1 \text{ for } r \leq \frac{1}{2}.$

The number $\frac{1}{2}$ is sharp.

(d) Let $|a_1| =: a \in [0,1) \setminus \left\{ \frac{1}{\sqrt{2}} \right\}$. Then $\sum_{n=1}^{\infty} |a_n|r^n + \left( \frac{r^{-1}}{1 + |a_1|} + \frac{1}{1 - r} \right) \| f \|_r \leq 1 \text{ for } r \leq \frac{(1 + 2a + 2a^2) - \sqrt{4a^4 + 8a + 5}}{2(2a^2 - 1)}.$

The radius $r(a)$ is sharp for any $a$ in question.

2. Proofs of Theorems 1 and 2

The following lemma due to Carlson [12] is key for the proofs of Theorems 1 and 2. It seems to us that this result was unnoticed by many at least in the context of refining the Bohr inequality. For the sake of completeness and because of its independent interest in establishing improved versions of the classical Bohr inequality, we include its proof in a simplified form (without the proof of the sharpness part).

**Lemma B.** Suppose that $f \in \mathcal{B}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then the following inequalities holds.

(a) $|a_{2n+1}| \leq 1 - |a_0|^2 - \cdots - |a_n|^2$, $n = 0, 1, \ldots$

(b) $|a_{2n}| \leq 1 - |a_0|^2 - \cdots - |a_{n-1}|^2 - \frac{|a_n|^2}{1 + |a_0|}$, $n = 1, 2, \ldots$

Further, to have equality in (a) it is necessary that $f$ is a rational function of the form

$$f(z) = \frac{a_0 + a_1z + \cdots + a_n z^n + \epsilon z^{2n+1}}{1 + (a_n z^n + \cdots + a_0 z^{2n+1}) \epsilon}, \quad |\epsilon| = 1,$$
and to have equality in (b) it is necessary that $f$ is a rational function of the form

$$f(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n + \epsilon z^{2n}}{1 + (\frac{a_n}{1+|a_0|} z^n + \cdots + \frac{a_0}{1} z^{2n})}, \quad |\epsilon| = 1,$$

where the condition $a_0 a_n^{-2} \epsilon$ is non-positive real.

**Proof.** At first, we consider the integral

$$I = \frac{1}{2\pi i} \int_{|z|=r} f(z) \left(1 + \frac{a_n}{1+|a_0|} z^n + \cdots + \frac{a_0}{1} z^{2n}\right)^2 \frac{dz}{z^{2n+2}}$$

which may be conveniently rewritten as

$$I = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{2n+2}} dz + \frac{2}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} \left(\sum_{k=0}^{n} \frac{a_{n-k}}{1} z^k\right)^2 dz + \frac{1}{2\pi i} \int_{|z|=r} f(z) \left(\sum_{k=0}^{n} \frac{a_{n-k}}{1} z^k\right)^2 dz.$$

As an application of Cauchy integral formula and Cauchy theorem, we obtain

$$I = a_{2n+1} + 2 \sum_{k=0}^{n} |a_{n-k}|^2.$$

Without loss of generality, assume that $a_{2n+1}$ is real and non-negative. Moreover, because $|f(z)| \leq 1$ for $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, it follows from the definition of $I$ and (4) that

$$|I| \leq r^{-(2n+1)} \int_{0}^{2\pi} \left|1 + \sum_{k=0}^{n} \frac{a_{n-k}}{1} r^{n+k+1} e^{i(n+k+1)\theta}\right|^2 d\theta$$

$$= r^{-(2n+1)} \left[1 + \sum_{k=0}^{n} |a_{n-k}|^2 r^{2(n+k+1)}\right].$$

Allowing $r \to 1^-$ and using (5) give the desired inequality (a).

To prove (b), we consider the integral

$$J = \frac{1}{2\pi i} \int_{|z|=r} f(z) \left(1 + \frac{a_n}{1+|a_0|} z^n + \frac{a_{n-1}}{1} z^{n+1} + \cdots + \frac{a_0}{1} z^{2n}\right)^2 \frac{dz}{z^{2n+1}}$$

and proceeding as above we find that

$$J = a_{2n} + 2 \left[\frac{|a_n|^2}{1+|a_0|} + \sum_{k=1}^{n} |a_{n-k}|^2\right] + \left(\frac{a_n}{1+|a_0|}\right)^2 a_0.$$

Again without loss of generality, we may assume that $a_{2n}$ is real and non-negative. Then by using the triangle inequality we have

$$|J| \geq |a_{2n}| + 2 \left[\frac{|a_n|^2}{1+|a_0|} + \sum_{k=1}^{n} |a_{n-k}|^2\right] - \left(\frac{|a_n|}{1+|a_0|}\right)^2 |a_0|.$$
Again $|J| \leq 1 + \left(\frac{|a_n|}{1 + |a_0|}\right)^2 + \sum_{k=1}^n |a_{n-k}|^2$. Comparing the last two inequalities, we have

$$|a_{2n}| \leq 1 - \sum_{k=1}^n |a_{n-k}|^2 + \left(\frac{|a_n|}{1 + |a_0|}\right)^2 - 2 \frac{|a_n|^2}{1 + |a_0|} + \left(\frac{|a_n|}{1 + |a_0|}\right)^2 |a_0|$$

which is same as the inequality (b).

**2.1. Proof of Theorem 1.** Let $M_f(r) = \sum_{n=0}^\infty |a_n|r^n$ denote the associated majorant series for $f(z) = \sum_{n=0}^\infty a_n z^n$ which is analytic and $|f(z)| \leq 1$ in $\mathbb{D}$. Using Lemma B, one has

$$M_f(r) = |a_0| + \sum_{n=1}^\infty |a_{2n}|r^{2n} + \sum_{n=0}^\infty |a_{2n+1}|r^{2n+1}$$

$$\leq |a_0| + \sum_{n=1}^\infty \left[1 - \sum_{k=0}^{n-1} |a_k|^2 - \frac{|a_n|^2}{1 + |a_0|}\right]r^{2n} + \sum_{n=0}^\infty \left[1 - \sum_{k=0}^{n} |a_k|^2\right]r^{2n+1}$$

$$= |a_0| + \sum_{n=1}^\infty r^n - |a_0|^2 \left(\sum_{n=1}^\infty r^{2n}\right) - \left(\frac{1}{1 + |a_0|} + r \frac{r}{1-r}\right)\sum_{n=1}^\infty |a_n|^2 r^{2n}$$

(6) $$\leq |a_0| + \frac{r}{1-r} (1 - |a_0|^2) - \left(\frac{1}{1 + |a_0|} + r \frac{r}{1-r}\right)\|f_0\|_{r}.$$

Thus, since $r/(1-r)$ is an increasing function of $r \in (0, 1)$, it follows from the last relation that for $r \leq 1/(2 + |a_0|)$

$$M_f(r) + \left(\frac{1}{1 + |a_0|} + \frac{r}{1-r}\right)\|f_0\|_{r} \leq |a_0| + \frac{r}{1-r} (1 - |a_0|^2) \Psi(r)$$

$$\leq \Psi\left(\frac{1}{2 + |a_0|}\right) = 1$$

which implies the inequality (2).

To prove that the radius is sharp, we consider the function $f = \varphi_a$ given by

$$\varphi_a(z) = \frac{a - z}{1 - az} = a - (1 - a^2) \sum_{k=1}^\infty a^{k-1} z^k, \quad z \in \mathbb{D},$$

where $a \in (0, 1)$. For this function, with $a_0 = a$ and $a_n = -(1 - a^2)a^{n-1}$, straightforward calculations shows that

$$M_{\varphi_a}(r) = \sum_{n=0}^\infty |a_n|r^n = a + \frac{1 - a^2}{1 - ar}$$

and

$$\left(\frac{1}{1 + a} + \frac{r}{1-r}\right)\|\varphi_a - a\|_{r} = \frac{1 + ar}{1-r} (1 - a^2)^2 r^2 \frac{2}{(1-r)(1+a) 1 - a^2 r^2} = \frac{(1 - a)^2(1 + a)r^2}{(1-r)(1-ar)}$$

so that

$$M_{\varphi_a}(r) + \left(\frac{1}{1 + a} + \frac{r}{1-r}\right)\|\varphi_a - a\|_{r}.$$
\[ f = a + r \frac{1 - a^2}{1 - ar} + \frac{(1 - a)^2(1 + a)r^2}{(1 - r)(1 - ar)} \]
\[ = 1 - (1 - a) \left[ \frac{1 - (1 + 2a)r}{1 - ar} - \frac{(1 - a^2)r^2}{(1 - r)(1 - ar)} \right] \]
\[ = 1 + \frac{(1 - a)[(2 + a)r - 1]}{1 - r} \]

which shows that the left hand side is bigger than 1 whenever \( r > 1/(2 + a) \).

The second part, namely, the inequality (3), is clear if we replace \( |a_0| \) by \( |a_0|^2 \) in the majorant sum, and thus, from (6), we see that
\[ |a_0|^2 + \sum_{n=1}^{\infty} |a_n|r^n + \left( \frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \|f_0\|_r \leq |a_0|^2 + (1 - |a_0|^2) \frac{r}{1 - r} \]

which is obviously less or equal to 1 whenever \( r \leq 1/2 \). The sharpness of the constant 1/2 can be established as in the previous case. Indeed, for the function \( f = \varphi_a \), we have by (7) that
\[ M_{\varphi_a}(r) + \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \|\varphi_a - a\|_r - a + a^2 = -a(1 - a) + 1 + \frac{(1 - a)[(2 + a)r - 1]}{1 - r} \]
\[ = 1 + \frac{(1 - a^2)(2r - 1)}{1 - r} \]

which is bigger than 1 whenever \( r > 1/2 \). The proof of the theorem is complete. \( \square \)

2.2. **Proof of Theorem 2.** Let \( f(z) = \sum_{n=1}^{\infty} a_n z^n \), where \( |f(z)| \leq 1 \) for \( z \in \mathbb{D} \). At first, we remark that the function \( f \) can be represented as \( f(z) = zg(z) \), where \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) is analytic in \( \mathbb{D} \) with \( b_n = a_{n+1} \) and \( |g(z)| \leq 1 \) for \( z \in \mathbb{D} \). Let \( |b_0| = a \) and \( g_0(z) = g(z) - b_0 \). Then it follows from the proof of Theorem 1 that
\[ \sum_{n=1}^{\infty} |a_n|r^n = \sum_{n=0}^{\infty} |b_n|r^{n+1} \leq r \left[ a + \frac{r}{1 - r} \left( 1 - a^2 \right) - \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \|g_0\|_r \right] \]
and thus, we have
\[ \sum_{n=1}^{\infty} |a_n|r^n + \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \sum_{n=1}^{\infty} |b_n|^2 r^{2n+1} \leq \Psi(a; r), \]

where
\[ \Psi(x; r) = rx + \frac{r^2}{1 - r} \left( 1 - x^2 \right) \text{ for } x \in [0, 1] \text{ and } r \in [0, 1). \]

We just need to maximize \( \Psi(x; r) \) (with respect to \( x \)) over the interval \([0, 1]\). We see that \( \Psi \) has a critical point at \( x_0 = (1 - r)/(2r) \) and obtain that the maximum occurs at this point so that
\[ \Psi(x; r) \leq \Psi(x_0; r) = \frac{1 - r}{2} + \frac{(3r - 1)(1 + r)}{4(1 - r)} = 1 - \frac{(3 - 5r)(1 + r)}{4(1 - r)}. \]
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which is less than or equal to 1 whenever \( r \leq 3/5 \). The first inequality (a) follows from (9) and (10).

To discuss the sharpness in (a), we consider the function \( f = \varphi_a \) given by

\[
\varphi_a(z) = z \left( \frac{a - z}{1 - az} \right) = az - (1 - a^2) \sum_{n=1}^{\infty} a^{n-1} z^{n+1}, \quad z \in \mathbb{D},
\]

where \( a \in (0, 1) \). For this function, with \( a_1 = a \) and \( a_n = -(1 - a^2)a^{n-2} \) for \( n \geq 2 \), straightforward calculations show that

\[
M_{\varphi_a}(r) = \sum_{n=1}^{\infty} |a_n|r^n = ar + (1 - a^2) \frac{r^2}{1 - ar}
\]

and

\[
\left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \sum_{n=2}^{\infty} |a_n|^2 r^{2n-1} = \frac{(1 - a^2)(1 - a)r^3}{(1 - r)(1 - ar)}
\]

so that the sum gives

\[
M_{\varphi_a}(r) + \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \sum_{n=2}^{\infty} |a_n|^2 r^{2n-1} = ar + (1 - a^2) \frac{r^2}{1 - r} = \frac{\Phi(a, r)}{1 - r},
\]

where \( \Phi(a, r) = ar(1 - r) + (1 - a^2)r^2 \). We want to show that \( \Phi(a, r) > 1 - r \) for any \( r > 3/5 \) and an appropriate number \( a \in (0, 1) \). Since

\[
\frac{\partial \Phi(a, r)}{\partial a} = r(1 - r) - 2ar^2,
\]

we see that \( \Phi \) is an increasing function of \( a \) for \( a \leq (1 - r)/(2r) =: c(r) \). For \( r \in (1/3, 1) \), we have \( c(r) \in (0, 1) \). Now, we calculate

\[
\Phi \left( \frac{1 - r}{2r}, r \right) = \frac{(1 - r)^2 + 4r^2}{4},
\]

and we see that this quantity is bigger than \( 1 - r \) (which is equivalent to \( 5r^2 + 2r - 3 = (5r - 3)(r + 1) > 0 \)) for \( r > 3/5 \). This completes the proof of the sharpness of the constant \( 3/5 \) in (a).

Next, we prove the inequalities (b), (c), and (d). In order to do this, we use the abbreviation \( |a_1| = a \), and we make minor changes in the proof of (a). Accordingly, from (9), we find that
\[
\sum_{n=1}^{\infty} |a_n| r^n + \frac{1}{1 - r} \|f\|_r = \\
\leq r \left[ a + \frac{r}{1 - r} (1 - a^2) - \left( \frac{1}{1 + a} + \frac{r}{1 - r} \right) \sum_{n=2}^{\infty} |a_n|^2 r^{2(n-1)} \right] + \frac{r^2}{1 - r} \sum_{n=1}^{\infty} |a_n|^2 r^{2(n-1)} \\
= ra + \frac{r^2}{1 - r} - \frac{r^{-1}}{1 + a} \sum_{n=2}^{\infty} |a_n|^2 r^{2n} \\
= ra + \frac{r^2}{1 - r} + \frac{ra^2}{1 + a} - \frac{r^{-1}}{1 + a} \sum_{n=1}^{\infty} |a_n|^2 r^{2n}
\]
and thus, we have
\[
(11) \quad \sum_{n=1}^{\infty} |a_n| r^n + \left( \frac{r^{-1}}{1 + |a_1|} + \frac{1}{1 - r} \right) \|f\|_r \leq \Xi(a; r),
\]
where
\[
\Xi(a; r) = ra + \frac{r^2}{1 - r} + \frac{ra^2}{1 + a} = 1 - \frac{B(a, r)}{(1 - r)(1 + a)},
\]
where \(B(a, r) = r^2(2a^2 - 1) - r(1 + 2a + 2a^2) + 1 + a\). Since
\[
\frac{\partial \Xi(a, r)}{\partial a} = r + \frac{2ar + a^2r}{(1 + a)^2},
\]
the function \(\Xi\) is a strictly monotonic increasing function of \(a\). Therefore
\[
\Xi(a, r) \leq \Xi(1, r) = \frac{3r - r^2}{2(1 - r)} \leq 1,
\]
if \(2 - 5r + r^2 \geq 0\). This is the case for \(r \in [0, (5 - \sqrt{17})/2]\). To see that this radius is sharp, we consider the function \(f(z) = z\). For this function the left hand side of (b) takes the value
\[
\frac{3r - r^2}{2(1 - r)}
\]
which is less than or equal to unity if and only if \(r\) takes the values indicated above. This completes the proof of part (b).

To prove (c) and (d) we consider
\[
\frac{\partial \Xi(a, r)}{\partial r} = a + \frac{r(2 - r)}{(1 - r)^2} + \frac{a^2}{1 + a} > 0.
\]
Therefore \(\Xi(a, r)\) is a strictly monotonic increasing function of \(r\). Hence, for a fixed value of \(a\), the desired inequalities are valid for \(r \in [0, r(a)]\), where \(r(a)\) is the first positive zero of the function \(B(a, r)\). If \(a = 1/\sqrt{2}\), the only zero of the function \(B\) is \(r = 1/2\). This proves the inequality (c).
For \( a \in (1/\sqrt{2}, 1) \), the solution \( r_\pm \) of the equation \( B(a, r) = 0 \) given by

\[
r_\pm = \frac{(1 + 2a + 2a^2) \pm \sqrt{(1 + 2a + 2a^2)^2 - 4(2a^2 - 1)(1 + a)}}{2(2a^2 - 1)}
\]

are both positive. Hence, we can take \( r(a) = r_- \).

Finally, for \( a \in (0, 1/\sqrt{2}) \), we have

\[
\frac{\partial B(a, r)}{\partial r} \bigg|_{r=0} < 0.
\]

This proves that one of the zeros is negative. The other one is again \( r_- \) such that we have \( r(a) = r_- \). This completes the proof of the inequalities in (c) and (d).

Now, it remains to prove the sharpness in the assertions (c) and (d). To that end, we again consider the function \( f = \varphi_a \) from part (a) for the values \( a \in [0, 1) \). If we add

\[
M_{\varphi_a}(r) + \left( \frac{r^{-1}}{1 + a} + \frac{1}{1 - r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} = \left( \frac{1 + ar}{r(1 + a)(1 - r)} \right) \left( a^2 r^2 + \frac{(1 - a^2)^2 r^4}{1 - a^2 r^2} \right),
\]

then we get

\[
M_{\varphi_a}(r) + \left( \frac{r^{-1}}{1 + a} + \frac{1}{1 - r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n}
\]

\[
= ar + r^2 \left( \frac{1 - a^2}{1 - ar} \right) + \left( \frac{1 + ar}{r(1 + a)(1 - r)} \right) \left( a^2 r^2 + \frac{(1 - a^2)^2 r^4}{1 - a^2 r^2} \right)
\]

\[
= ar + r^2 \left( \frac{1 - a^2}{1 - ar} \right) \left( 1 + \frac{r(1 - a)}{1 - r} \right) + a^2 r \left( \frac{1 + ar}{(1 + a)(1 - r)} \right)
\]

which is greater than 1 whenever \( B(a, r) < 0 \).

Therefore, the above discussion of \( B \) as a function of \( r \) for a fixed value of \( a \) immediately delivers that the radii in (c) and (d) are sharp. This completes the proof of Theorem 2.

\[\square\]

Remark 1. It may be interesting that for \( a_1 = 0 \) the sharp radius in the second part of Theorem 2 has the value \( (\sqrt{5} - 1)/2 \) and that in this case the extremal function is given by \( \varphi_a(z) = z^2 \).

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