Non–Perturbative Approach to Random Walk in Markovian Environment.

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NON-PERTURBATIVE APPROACH TO RANDOM WALK IN MARKOVIAN ENVIRONMENT.

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Abstract. We prove an averaged CLT for a random walk in a dynamical environment where the states of the environment at different sites are independent Markov chains.

1. Introduction

The study of random walk in random evolving environment has attracted much attention lately. The basic idea is that because of the time mixing properties of the environment the CLT should hold in any dimension. Many results have been obtained in the case of transition probabilities close to constant (\cite{5, 3, 4, 14, 11, 2, 7} etc) but there are still two open problems. On the one hand one would like to consider environments with weaker mixing properties (some results in this direction have been obtained in \cite{6, 7, 8}). On the other hand one would like to understand the case in which the dependence on the environment is not small. The present paper addresses the second issue presenting a new, non-perturbative approach to random walks in evolving environment.

To make the presentation as transparent as possible we consider a very simple environment: at each site of $\mathbb{Z}^d$ we have a finite state Markov chain and the chains at different sites are independent. However, the present argument applies to the situation where the evolution of each site is described by a Gibbs measure. In fact our approach relies on the fact that the environment seen from the particle is Gibbsian. We hope that this fact can be established for a wide class of mixing environments and so it will be useful for the first problem as well.

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Also, as we have already investigated in detail how quenched CLT can be obtained once the averaged (annealed) CLT is at hand, we limit ourself to the latter in order to explain the new approach in the simplest possible setting.

More precisely, at each site \( u \in \mathbb{Z}^d \) consider a Markov chain \( \{x_n^u\}_{n \in \mathbb{N}} \) with the set space \( \mathcal{A} \) and transition matrix \( p_{ab} > 0 \) for any \( a \) and \( b \). The chains at different sites are independent. Let \( p_{ab}(k) \) denote the \( k \) step transition probability and \( \pi_a \) denote the stationary distribution corresponding to \( p_{ab} \). Let \( \Lambda \) be a finite subset of \( \mathbb{Z}^d \). For each \( a \in \mathcal{A} \) let \( q_{a,v} \) be a probability distribution on \( \Lambda \). Consider a random walk \( S_n \) such that \( S_0 = 0 \) and \( S_{n+1} = S_n + v_n \) with probability \( q_{x_n^u,v_n} \). Let \( \mathbb{P} \) denote the measure of the resulting Markov process on \( \Omega := (\mathcal{A} \mathbb{Z}^d)^\mathbb{N} \times (\mathbb{Z}^d)^\mathbb{N} \) when the environment is started with the stationary measure \( \mathbb{E} \) and the walk starts from zero. We use \( \mathbb{E} \) to denote the associated expectation.

**Theorem 1.** For each \( d \in \mathbb{N} \) and random walk \( \mathbb{P} \) as above, there exists \( \nu \in \mathbb{R}^d \) and a \( d \times d \) matrix \( \Sigma \geq 0 \) such that
\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}(S_n) = \nu \\
\frac{1}{\sqrt{n}} [S_n - n\nu] \Rightarrow \mathcal{N}(0, \Sigma^2) \text{ under } \mathbb{P}.
\]
That is, \( S_n \) satisfies an averaged (annealed) Central Limit Theorem. In addition, \( \Sigma > 0 \) unless there exists a proper affine subspace \( \Pi \subset \mathbb{R}^d \) such that such that \( q_{a,z} = 0 \) for \( z \notin \Pi \).

2. Proof

To prove Theorem 1 we consider the history \( \omega \) of the local environment as seen from the particle. More precisely \( \omega_n \) is a pair \((x_n^u, v_n)\). Let \( \Omega^+ \) be the set of all possible histories. Thus \( \Omega^+ \) is a space of forward infinite sequences. Let \( \Omega \) be its natural extension to bi-infinite sequences and let \( \tau \) be the usual shift \((\tau \omega)_i = \omega_{i+1}\). Let \( \mathcal{F}_m \) denote the \( \sigma \)-algebra generated by \( \omega_0, \ldots, \omega_m \). Let \( \lambda \) be the second eigenvalue of \( p_{ab} \).

**Lemma 2.** There exists a shift invariant measure \( \mu \) on \( \Omega \) and \( \theta \in (0, 1) \) such that for any \( n \in \mathbb{N}, k \geq m \in \mathbb{N} \), and any \( \mathcal{F}_k \) measurable function

1In fact our main result holds if the environment is started from any initial measure and the proof requires very little change. We assume that the initial measure is stationary since this allows us to simplify the notation a little.

2That is, we record only the environment at the visited sites, contrary to the usual strategy of considering all the environment.
Proof of Lemma 2. Let $\mathcal{B} = \mathcal{A} \times \Lambda$ be the alphabet for $\Omega$. If $g : \Omega^+ \to \mathcal{B}$ is a $\mathcal{F}_{j+1}$ measurable function, then

\[
\mathbb{E}(g|\mathcal{F}_j)(\omega_0, \ldots, \omega_j) = \sum_{b \in \mathcal{B}} \mathbb{P}(\{\omega_{j+1} = b\}|\mathcal{F}_j)g(\omega_0, \ldots, \omega_j, b).
\]

We need a little digression. Let $\Omega^- \subset \mathcal{B}^{\mathbb{Z}_-}$, where $\mathbb{Z}_- := \{\ldots, -1, 0\}$, be the space of backward admissible sequences. For $\xi \in \Omega^-$, with $\xi_i = (z_i, v_i)$, and $s \in \mathbb{N}$ let

\[
e^\phi(\xi) = \begin{cases} p_{z_i, z_0}(-l) \cdot q_{z_0 v_0} & \text{if } l = l(\xi) \neq -\infty \\ \pi_{z_0} \cdot q_{z_0 v_0} & \text{if } l(\xi) = -\infty \end{cases}
\]

where $l(\xi)$ is the last time before 0 such that $S_l := \sum_{i=-l}^{1} v_i = 0$.

We can then define the following transfer operators acting on $\mathcal{C}(\Omega^-, \mathbb{R})$:

\[
(\mathcal{L} h)(\xi) = \sum_{\tau^{-1} \xi = \xi} e^{\phi(\xi)} h(\xi); \quad (\mathcal{L}_n h)(\xi) = \sum_{\tau^{-1} \xi = \xi} e^{\phi_n(\xi)} h(\xi).
\]

Note that $\mathcal{L}_n 1 = \mathcal{L}^n 1 = 1$.

For each $b \in \mathcal{B}$ let $A_b = \{\xi \in \Omega^- : \xi_0 = b\}$ and choose a sequence $\tilde{\xi}_b^b$. Also, let $\xi_b^l = \tau^{-1} \tilde{\xi}_b^l$. Observe that

\[
\mathbb{P}(\{\omega_{j+1} = b\}|\mathcal{F}_j)(\omega_0, \ldots, \omega_j) = \exp(\phi_j(\tau^{-j-1}(\xi_0^b, \omega)b)))
\]

where $\delta_j = \mathcal{O}(\lambda^j)$.

By hypotheses there exists $\tilde{f} : \mathcal{B}^{k+1} \to \mathbb{R}$ such that $f(\omega) = \tilde{f}(\omega_0, \ldots, \omega_k)$. Remembering (iii), for each $l \in \{0, \ldots, n+k-m\}$ and $\omega^m = (\omega_0, \ldots, \omega_m)$ we have

\[
\mathbb{E}(f \circ \tau^n | \mathcal{F}_m)(\omega^m) = (\mathcal{L}_m \cdots \mathcal{L}_{n+k} \tilde{f}_k)(\xi_0^\omega \omega^m)
\]

\[
= \mathcal{L}_m \cdots \mathcal{L}_{n+k-l} \mathcal{L}_l \tilde{f}_k + \mathcal{O}(\lambda^{n+k-l}|f|_\infty),
\]

\[\text{In other words we extend } S_n \text{ to negative } n \text{ using } \xi_i \text{ with negative indexes and look at the last negative time then the walks visits zero.}\]

\[\text{In fact } \delta_j = 0 \text{ unless } l(\tau^{-j-1}(\xi_0^b, \omega)b)) < -j \text{ (see equation (i), in which case the distribution of } x_j^b \text{ is } \mathcal{O}(\lambda^j) \text{-close to } \pi).\]
where \( \hat{f}_k(\xi) = \hat{f}(\xi_{-k}, \ldots, \xi_0) \), for \( \xi \in \Omega^- \).

Next, the Ruelle-Perron-Frobenius Theorem for \( \mathcal{L} \) implies (15) that there are a Gibbs measure \( \mu^- \) on \( \Omega^- \), and a \( \hat{\theta} \in (0, 1) \) such that
\[
\mathbb{E}(f \circ \tau^n | \mathcal{F}_m) = \mathcal{L}_m \cdots \mathcal{L}_{n+k-1} 1 \cdot \mu^-(\hat{f}_k) + O(\lambda^{n+k-l} + C_k \hat{\theta}^l)|f|_\infty
\]
\[
= \mu^-(\hat{f}_k) + O(C_k \theta^n)|f|_\infty,
\]
where we have chosen \( l = n/2 \) and defined \( \theta^2 = \max \{\lambda, \hat{\theta}\} \).

Let \( \mu \) be the natural extension of \( \mu^- \), that is \( \mu \) is a \( \tau \) (or, more precisely, \( \tau^{-1} \)) invariant measure on \( \Omega \) whose projection to \( \Omega^- \) equals \( \mu^- \). Then \( \mu \) (and hence \( \mu^+ \)) is Gibbs since \( \mu^- \) is Gibbs.

To prove the absolute continuity note that equations (2), (4) imply for \( f \geq 0 \) and \( \mathcal{F}_k \) measurable\(^5\)
\[
\mathbb{E}(f | \mathcal{F}_0)(b) = \inf_{\xi \in A_b} L_k \cdots L_1 \hat{f}_k(\xi) = \inf_{\xi \in A_b} \frac{L_k \cdots L_1 \hat{f}_k(\xi)}{L_k \hat{f}_k(\xi)} L_k \hat{f}_k(\xi)
\]
\[
\leq C \inf_{\xi \in A_b} L_k \hat{f}_k(\xi) \leq C \frac{\mu^-(1_{A_b} L_k \hat{f}_k)}{\mu^-(A_b)} \leq C' \mu(f).
\]
Thus \( \mathbb{E}(f) \leq C' \mu(f) \), which implies the absolute continuity. \( \square \)

Proof of Theorem \( \square \) It suffices to prove the CLT for \( \hat{S}_n := \sum_{j=0}^{n-1} \hat{v}_j \). If the initial conditions are chosen according to \( \mu^+ \), then the result follows from the CLT for Gibbs measures (13, 10). Moreover, by Eagleson (4 section 4.2a) (see also (11 section 3.6), (15)) it follows that the Central Limit Theorem holds in the sense of Rényi (12), that is it holds for any measure absolutely continuous with respect to \( \mu^+ \) (with the same mean and variance). Hence, the CLT with respect to \( \mathbb{P}^+ \) follows by Lemma \( \square \)

It remains to analyze the possibility that there exists \( w \in \mathbb{R}^d \) such that \( \Sigma^2 w = 0 \). In this case (11) implies that \( f(\xi) := \langle \hat{v}_j, w \rangle \) is a coboundary, that is there is a continuous function \( g : \Omega \rightarrow \Omega \) such that \( f(\xi) = g(\xi) - g(\tau^n \xi) \). Then \( \langle \hat{S}_n, w \rangle = g(\xi) - g(\tau^n \xi) \) is uniformly bounded. Now, suppose there exist pairs \( (a_1, v_1) \) and \( (a_2, v_2) \) such that \( q_{a_j, v_j} > 0 \) and \( \langle v_1, w \rangle \neq \langle v_2, w \rangle \). Then \( \langle \hat{S}_n, w \rangle \) cannot be bounded along both the orbits defined by the sequences \( \xi^{(j)} = (a_j, v_j)^\infty, j = 1, 2 \). This completes the proof of Theorem \( \square \)

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\(^5\)The first equality follows since the RHS of (11) does not depend on the choice of \( \xi \in A_b \) and the last inequality is true since \( \mu^-(A_b) \) takes only finitely many values.
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