Inversion and Symmetries of the Star Transform

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Abstract

The star transform is a generalized Radon transform mapping a function of two variables to its integrals along “star-shaped” trajectories, which consist of a finite number of rays emanating from a common vertex. Such operators appear in mathematical models of various imaging modalities based on scattering of elementary particles. The paper presents a comprehensive study of the inversion of the star transform. We describe the necessary and sufficient conditions for invertibility of the star transform, introduce a new inversion formula, discuss its stability properties and demonstrate its efficiency on numerical examples. As an unexpected bonus of our approach, we prove a long-standing conjecture from algebraic geometry about the zero sets of elementary symmetric polynomials.

1 Introduction

The star transform is a generalized Radon transform mapping a function in $\mathbb{R}^2$ to its weighted integrals along “star-shaped” trajectories, which consist of a finite number of rays emanating from a common vertex. Such operators appear in mathematical models of single-scattering optical tomography and single-scattering X-ray CT [9, 10, 11, 15, 21]. Particular attention has been drawn to the transforms with only two rays, usually called either broken ray transforms or V-line transforms [1, 3, 5, 6, 8, 11, 12, 14, 15, 17, 18, 20]. In addition to that, multiple authors have studied various generalizations of these transforms to higher dimensions (see the review article [2] and the references there).

All works cited above deal with cases, where the vertices of the stars or V-lines are inside the image domain. This paper also studies the star transform in that setup. A whole other world of applications and mathematical results exists in the case, when the vertices of the V-lines (or cones in higher dimensions) are restricted to the boundary of the image domain. To find out more about the latter, we refer the interested reader to the review article [19] and the references there.

While there are multiple interesting formulas and procedures for inversion of the V-line transform, the star transform with three or more branches has been studied only in the pioneering paper [21]. The method presented in that article
uses Fourier analysis techniques to reduce the inversion of the star transform to a problem of solving an infinite system of linear equations. The latter is solved using various truncations of the system and fast iterative computations of the Tikhonov-regularized pseudo-inverses of the resulting finite-size rectangular matrices. The paper also discusses the stability of the proposed inversion algorithms, formulating an algebraic condition that is necessary and sufficient for the stable reconstruction. Finally, several special geometric setups of the star transform are studied in relation to that condition.

This paper presents a comprehensive study of the inversion of the star transform using a totally different approach. We introduce a new exact closed-form inversion formula for the star transform, by establishing its connection with the (ordinary) Radon transform. It is much simpler than the inversion algorithm described in [21] and provides a very intuitive geometric insight to the problem. Our formula leads to a description of necessary and sufficient conditions for invertibility of the star transform based on its geometric configuration, which was not known before. It also naturally yields the necessary and sufficient condition for the stability of inversion described in [21]. A detailed analysis of this condition allows us to classify the geometric setups of the transform according to their stability and provides a recipe for building star configurations with the “most stable” inversions. As an unexpected bonus of our analysis, we prove a long-standing conjecture from algebraic geometry about the zero sets of elementary symmetric polynomials.

The rest of the article is organized as follows. In Section 2 we define the main concepts used in the paper. In Section 3 we present a new inversion formula for the star transform. In Section 4 we study the injectivity of that transform. In Section 5 we discuss the so-called singular directions of the star transform that play an important role in the stability of numerical reconstructions. In Section 6 we state and prove the aforementioned conjecture from algebraic geometry. In Section 7 we present various numerical simulations demonstrating the efficiency of our inversion method. We finish the paper with some additional remarks in Section 8 and acknowledgements in Section 9.

2 Definitions

Let \( f(x) \in C_c(\mathbb{R}^2) \) be a compactly supported continuous function, and let \( \gamma \) be a unit vector in the plane.

**Definition 1** The divergent beam transform \( \mathcal{X}_\gamma \) of \( f \) at \( x \in \mathbb{R}^2 \) is defined as:

\[
\mathcal{X}_\gamma f(x) = \int_0^\infty f(x + t\gamma) \, dt. \tag{1}
\]

One can use a linear combination of divergent beam transforms with a set of fixed distinct directions \( \gamma_i \in S^1 \) and non-zero constants \( c_i \in \mathbb{R}, \ i = 1, \ldots, m \) to define the star transform of \( f \). Namely,
Definition 2 The (weighted) star transform $S$ of $f$ at $x \in \mathbb{R}^2$ is defined as:

$$Sf(x) = \sum_{i=1}^{m} c_{i} \mathcal{X}_{\gamma_{i}} f(x) = \sum_{i=1}^{m} c_{i} \int_{0}^{\infty} f(x + t\gamma_{i}) dt.$$ (2)

In particular, if $c_{1} = \ldots = c_{m} = 1$, the star transform $S$ puts into correspondence to a given function $f(x)$ its integrals $Sf(x)$ along a two-parameter family of so-called “stars”. Each star consists of a union of rays emanating from a common vertex $x$ along fixed directions $\gamma_{i}$, $i = 1, \ldots, m$ (see Figure 1a).

![Image of a star](image_url)

(a) A star centered at point $x$ with rays emanating along fixed unit vectors $\gamma_{1}, \ldots, \gamma_{4}$.

(b) If $\text{supp} f \subseteq D_{1}$, then $\text{supp} Sf \subseteq \Omega$ and $Sf|_{\partial D_{2} \cap \Omega}$ uniquely defines $Sf$ in $\mathbb{R}^{2} \setminus D_{2}$.

Figure 1: (The color version is available online.) A star and the support of the corresponding transform.

Definition 3 Let the ray directions of the star be parameterized as

$$\gamma_{j} = (a_{j}, b_{j}) \in \mathbb{R}^{2}, \ j = 1, \ldots, m.$$ (3)

We call

$$a = (a_{1}, \ldots, a_{m}) \text{ and } b = (b_{1}, \ldots, b_{m})$$ (4)

the aperture vectors of that star. Since $\gamma_{j}$’s are unit vectors

$$a_{j}^{2} + b_{j}^{2} = 1, \ j = 1, \ldots, m.$$ (5)

Every star transform corresponds to an $m$-gon (polygon with $m$ sides) with vertices $c_{1}^{-1}\gamma_{1}, \ldots, c_{m}^{-1}\gamma_{m}$. Many properties of the star transform can be characterised by geometric features of the corresponding polygon. The following special star plays a prominent role in some of our proofs.

Definition 4 We call a star transform symmetric, if $m = 2k$ for some $k \in \mathbb{N}$ and (after possible re-indexing) $\gamma_{i} = -\gamma_{k+i}$ with $c_{i} = c_{k+i}$ for all $i = 1, \ldots, k$. The corresponding star is also called symmetric.
Definition 5 We call a star transform regular, if $c_1 = \ldots = c_m = 1$ and the ray directions $\gamma_i$, $i = 1, \ldots, m$ correspond to the radius vectors of the vertices of a regular $m$-gon (see Figure 4a). The corresponding star is also called regular.

We will show that the symmetric star transforms belong to the set of non-invertible configurations of $S$, while the regular star transforms with an odd number of vertices have the most stable inversions.

For regular stars with $m = 2k + 1$ rays, we parameterize the ray directions as follows (see Figure 4a):

$$\gamma_j = (a_j, b_j) = (\cos \alpha_j, \sin \alpha_j),$$

where

$$\alpha_1 = 0, \quad \alpha_{2j} = \frac{2\pi j}{2k + 1} \quad \text{and} \quad \alpha_{2j+1} = -\frac{2\pi j}{2k + 1}, \quad j = 1, \ldots, k.$$  \hspace{1cm} (7)

Since $a_{2j} = a_{2j+1}$ and $b_{2j} = -b_{2j+1}$ for $j = 1, \ldots, k$, the aperture vectors of the regular star with $2k + 1$ rays can be expressed as

$$a = (1, a_2, a_2, \ldots, a_{2k}, a_{2k}), \quad b = (0, b_2, -b_2, \ldots, b_{2k}, -b_{2k}).$$  \hspace{1cm} (8)

Notice that if aperture vectors are fixed, the star is uniquely defined by its vertex $x$. If one does not fix the directions of rays and allows arbitrary locations of its vertices, then the star transform will have $m + 2$ degrees of freedom. The problem of inverting such a transform will be overdetermined. We are interested in the inversion of the two-dimensional star transform $S$ with $m$ rays in fixed directions $\gamma_i$ and non-zero weights $c_i$, $i = 1, \ldots, m$.

3 An Exact Inversion of the Star Transform

Let us assume that $\text{supp } f \subseteq D_1$, where $D_1$ is an open disc of radius $r_1$ centered at the origin. Then $Sf$ is supported in an unbounded domain $\Omega$ (see Figure 1b). However, there exists a closed disc $D_2$ of some finite radius $r_2 > r_1$ centered at the origin such that $Sf$ is constant along the directions of rays $\gamma_i$ inside the corresponding “strips” of $\mathbb{R}^2 \setminus D_2$. In other words, the restriction of $Sf$ to $D_2$ completely defines it everywhere else. Throughout the rest of the text we will assume that $Sf(x)$ is known for all $x \in D_2$.

In this section we formulate and prove an exact closed-form inversion formula for $S$ with arbitrary geometry and arbitrary non-zero weights. The intuitive idea of this inversion is the following. The Radon transform $R$ of $g = Sf$ at $(\psi, s)$ is the integral of the star transform data $g$ along the line $l(\psi, s)$. For a generic line, this gives a sum of weighted integrals of $f$ over half-planes in $\mathbb{R}^2$, with different weights on different sides of $l(\psi, s)$. By differentiating that quantity with respect to variable $s$, we recover the Radon data of $f$ along $l(\psi, s)$. The inversion of $S$ is then accomplished by inverting the Radon transform.
Before proving the theorem we discuss two auxiliary statements. They are geometric in nature and help with outlining the main ideas in the proof of Theorem 1.

3.1 Half Plane Transform and Its Derivative

For a unit vector $\psi \in \mathbb{R}^2$ and $s \in \mathbb{R}$ define the half plane transform of $f$ as

$$F_{\psi}(s) = \int_{\langle x, \psi \rangle \leq s} f \, dx.$$  \hspace{1cm} (9)

Geometrically, $F_{\psi}(s)$ is the integral of $f$ over the half plane on one side of the line $l(\psi, s) = \{ x \in \mathbb{R}^2 | \langle x, \psi \rangle = s \}$ with normal vector $\psi$ and distance $s$ from the origin (see Fig. 2).

![Figure 2: A setup for the half-plane, divergent beam and Radon transforms.](image)

Now, let’s look at the derivative of $F$ as a function of $s$

$$\frac{dF_{\psi}(s)}{ds} = \lim_{h \to 0} \frac{F_{\psi}(s + h) - F_{\psi}(s)}{h}.$$  \hspace{1cm} (10)

Note that $F_{\psi}(s + h) - F_{\psi}(s)$ is the integral of $f$ over the infinite strip between the lines $l(\psi, s)$ and $l(\psi, s + h)$, which plays a central role in our construction. Using the fundamental theorem of calculus, we get

$$\frac{dF_{\psi}(s)}{ds} = \frac{d}{ds} \int_{\langle x, \psi \rangle \leq s} f \, dx = \lim_{h \to 0} \frac{1}{h} \int_{s}^{s+h} \mathcal{R} f(\psi, t) \, dt = \mathcal{R} f(\psi, s),$$

where $\mathcal{R}$ is the standard Radon transform. Thus we proved the following

**Lemma 1** For function $f \in C_c(\mathbb{R}^2)$ we have

$$\frac{dF_{\psi}(s)}{ds} = \mathcal{R} f(\psi, s).$$  \hspace{1cm} (10)
A key step in our proof is the possibility of expressing the half plane transform $F_\psi$ of $f$ in terms of its star transform $\mathcal{S}f$. We show this by first finding a relation between $F_\psi$ and $\mathcal{X}_\gamma(f)$.

If $\langle \psi, \gamma \rangle \neq 0$ we can integrate $\mathcal{X}_\gamma(x)$ along the line $l(\psi, s)$ to get an expression in terms of $F_\psi(s)$. Namely

**Lemma 2** Let $f \in C_c(\mathbb{R}^2)$ and $\langle \psi, \gamma \rangle \neq 0$. Then

$$
\frac{d}{ds} \mathcal{R}(\mathcal{X}_\gamma f)(\psi, s) = \frac{-1}{\langle \psi, \gamma \rangle} \frac{dF_\psi(s)}{ds}. 
$$

**Proof.** Depending on whether $\langle \psi, \gamma \rangle$ is positive or negative, $\mathcal{R}(\mathcal{X}_\gamma f)(\psi, s)$ is integrating $f$ with the weight $\langle \psi, \gamma \rangle^{-1}$ over the half plane on one or the other side of the line $l(\psi, s)$ (see Figure 2). If $\langle \psi, \gamma \rangle > 0$ then we have

$$
\mathcal{R}(\mathcal{X}_\gamma f)(\psi, s) = \frac{1}{\langle \psi, \gamma \rangle} \left[ \int_{\mathbb{R}^2} f dx - F_\psi(s) \right]
$$

and if $\langle \psi, \gamma \rangle < 0$

$$
\mathcal{R}(\mathcal{X}_\gamma f)(\psi, s) = \frac{-1}{\langle \psi, \gamma \rangle} F_\psi(s).
$$

Taking the derivative of both sides in either case will finish the proof. □

3.2 Main Result and Some Corollaries

**Theorem 1** Consider the star transform $\mathcal{S}$ defined in (2) and let

$$
q(\psi) = \frac{-1}{m} \sum_{i=1}^{m} \frac{c_i}{\langle \psi, \gamma_i \rangle}, \quad \psi \in S^1. 
$$

Then the following is true for any unit vector $\psi$ in the domain of $q$:

$$
\mathcal{R}f(\psi, s) = q(\psi) \frac{d}{ds} \mathcal{R}(\mathcal{S}f)(\psi, s),
$$

where $\mathcal{R}$ denotes the (ordinary) Radon transform of a function in $\mathbb{R}^2$. Hence, if $q$ is defined almost everywhere, we can apply $\mathcal{R}^{-1}$ to recover $f$.

**Proof.** Using Lemmas 1 and 2 we get

$$
\frac{d}{ds} \mathcal{R}(\mathcal{X}_\gamma f)(\psi, s) = \frac{-1}{\langle \psi, \gamma \rangle} \mathcal{R}f(\psi, s).
$$
Now we write $S$ in terms of $\mathcal{X}_\gamma$'s

$$
\frac{d}{ds} \mathcal{R}(S)(\psi, s) = \frac{d}{ds} \mathcal{R} \left( \sum_{i=1}^{m} c_i \mathcal{X}_\gamma \right)(\psi, s) = \sum_{i=1}^{m} c_i \frac{d}{ds} \mathcal{R}(\mathcal{X}_\gamma f)(\psi, s) = \sum_{i=1}^{m} \frac{-c_i}{\langle \psi, \gamma \rangle} \mathcal{R} f(\psi, s),
$$

which ends the proof.

We finish this section with statements of two special cases of Theorem 1. In the case of $m = 2$, Theorem 1 yields a new and simple inversion for the V-line transform.

**Corollary 1** An inversion formula for the V-line transform $\mathcal{V} = c_1 \mathcal{X}_{\gamma_1} + c_2 \mathcal{X}_{\gamma_2}$ with fixed noncollinear ray directions $\gamma_1$, $\gamma_2$ and nonzero weights $c_1$, $c_2$ is given by

$$
f = \mathcal{R}^{-1} \left[ -\langle \psi, \gamma \rangle \frac{d}{ds} \mathcal{R}(\mathcal{V} f)(\psi, s) \right].
$$

(14)

A curious special case of Theorem 1 is the following.

**Corollary 2** An inversion of the divergent beam transform $\mathcal{X}_\gamma$ is given by the formula

$$
f = \mathcal{R}^{-1} \left[ -\langle \psi, \gamma \rangle \frac{d}{ds} \mathcal{R}(\mathcal{X}_\gamma f)(\psi, s) \right].
$$

(15)

Let $D_\gamma$ denote the directional derivative in $\mathbb{R}^2$, i.e.

$$
D_\gamma h(x) = \gamma \cdot \nabla h(x).
$$

(16)

$D_\gamma$ is the inverse of $\mathcal{X}_\gamma$ by a straightforward application of the Fundamental Theorem of Calculus. Using this fact and the above corollary, for any $h \in U_\gamma = \{ \mathcal{X}_\gamma(f) : f \in C_c(\mathbb{R}^2) \}$ in the range of $\mathcal{X}_\gamma$, one can obtain

$$
- D_\gamma h(x) = \mathcal{R}^{-1} \left[ \langle \psi, \gamma \rangle \frac{d}{ds} \mathcal{R} h(x) \right].
$$

(17)

In other words, the directional derivative $-D_\gamma$ in the image domain translates into $d/ds$ differentiation with a multiplication factor of $\langle \psi, \gamma \rangle$ in the Radon data domain.
4 Injectivity of the Star Transform

In certain configurations of the star transform, the function $q(\psi)$ is not defined for any $\psi$. For example, this happens when $m = 2$, $c_1 = c_2 = 1$, and $\gamma_1 = -\gamma_2 = (1,0)$. In this case $\langle \psi, \gamma_1 \rangle + \langle \psi, \gamma_2 \rangle \equiv 0$ for any $\psi \in S^1$.

Consider a similar configuration with more rays: $m = 4$, $c_1 = \ldots = c_4 = 1$, $\gamma_1 = -\gamma_3$ and $\gamma_2 = -\gamma_4$. While it is not obvious that $q(\psi)$ is not defined at any $\psi$, it is easy to notice that in this case the star transform is not injective, since it provides less information than the ordinary Radon transform restricted to two directions. Interestingly enough, such configurations are the only ones, for which the star transform is not injective.

**Theorem 2** The star transform $S = \sum_{i=1}^{m} c_i X_{\gamma_i}$ is invertible if and only if it is not symmetric.

An important object in our inversion is the function $q(\psi)$. It leads to the geometric condition that is necessary for the star transform to be non-invertible. To describe the connection between $q(\psi)$ and the geometric condition we need to consider the elementary symmetric polynomial of degree $m - 1$ in $m$ variables $y = (y_1,\ldots,y_m)$ (see [16])

$$e_{m-1}(y_1,\ldots,y_m) = \sum_{i=1}^{m} \prod_{j \neq i} y_j.$$  \hspace{1cm} (18)

Using the above notation, one can re-write formula (12) as

$$q(\psi) = \frac{c_1^{-1}\ldots c_m^{-1}}{e_{m-1}(\langle \psi, c_1^{-1}\gamma_1 \rangle,\ldots,\langle \psi, c_m^{-1}\gamma_m \rangle)} \langle \psi, \gamma_1 \rangle \ldots \langle \psi, \gamma_m \rangle.$$  \hspace{1cm} (19)

The proof of Theorem 2 is based on the description of zero sets of $e_{m-1}$ and the fact that the star transform $S$ is invertible if $e_{m-1}(\langle \psi, c_1^{-1}\gamma_1 \rangle,\ldots,\langle \psi, c_m^{-1}\gamma_m \rangle)$ is not identically zero as a function of $\psi$.

If $\hat{\psi} = (r,s) \in \mathbb{R}^2$ (not necessarily on $S^1$) and $c = c_1\ldots c_m$, we define a polynomial in two variables $(r,s)$ as follows:

$$P_2(\hat{\psi}) = c e_{m-1}(\langle \hat{\psi}, c_1^{-1}\gamma_1 \rangle,\ldots,\langle \hat{\psi}, \gamma_m \rangle).$$  \hspace{1cm} (20)

**Lemma 3** The polynomial $P_2(r,s)$ either has finitely many zeros on $S^1$ or it is identically zero.

**Proof.** Let us start with noticing that $e_{m-1}(y)$ is a homogeneous polynomial of degree $d - 1$, i.e. $e_{m-1}(\lambda y) = \lambda^{m-1} e_{m-1}(y)$. Hence, $P_2(r,s)$ is also a homogeneous polynomial of degree $m - 1$. As a result, the zero set of $P_2$ on $S^1$ corresponds to the zero set of a polynomial of one variable

$$Q(t) \doteq P_2(t,1)$$  \hspace{1cm} (21)
defined on the projective space of all one-dimensional subspaces of $\mathbb{R}^2$ (see Figure 3). The latter, either has finitely many (projective) roots or is identically zero.

Another proof of Lemma 3 can be obtained by a direct application of Bézout’s Theorem to algebraic curves defined by $P_2(r, s) = 0$ and $r^2 + s^2 - 1 = 0$, and homogeneity of $P_2$. ■

Proof (of Theorem 2). Assume that for a fixed choice of $\gamma_1, \ldots, \gamma_m$ there is no inversion for the corresponding star transform. By Theorem 1, $P_2(\psi)$ has to be zero on an infinite subset of $S^1$. Therefore, by Lemma 3, $P_2(\psi) \equiv 0$.

Writing in components of $\psi = (r, s)$ and aperture vectors $a$ and $b$, we have

$$P_2(\psi) = P_2(r, s) = \sum_{i=1}^{m} \left[ c_i \prod_{j \neq i} (ra_j + sb_j) \right] = 0. \quad (22)$$

If we take $r = b_1$ and $s = -a_1$, then

$$0 = \prod_{j \neq 1} (ra_j + sb_j) = (a_2b_1 - a_1b_2) \cdots (a_mb_1 - a_1b_m),$$

which implies that for some index $\sigma(1)$ we are required to have $a_{\sigma(1)}b_1 = a_1b_{\sigma(1)}$ or equivalently $\frac{a_{\sigma(1)}}{b_1} = \frac{a_1}{b_{\sigma(1)}}$. Given the assumption that $\gamma_i$’s are distinct unit vectors, we conclude that $\gamma_1 = -\gamma_{\sigma(1)}$. Applying this procedure with $r = a_j$ and $s = -a_j$ repeatedly for all $j$, we conclude that in order for the star transform to be non-invertible, its ray directions have to come in opposite pairs.

Now, we prove the equality of the corresponding weights $c_i$ and $c_{k+i}$ for each pair. Without loss of generality, let us assume that $m = 2k$, $\gamma_i = -\gamma_{k+i}$ for $i = 1, \ldots, k$ and no pair of vectors $\gamma_1, \ldots, \gamma_k$ are collinear. Then we can re-write formula (22) as

$$P_2(r, s) = (-1)^k \prod_{j=1}^{k} (ra_j + sb_j) \sum_{i=1}^{k} \left[ (c_i - c_{k+i}) \prod_{j=1}^{k} (ra_j + sb_j) \right] = 0, \ \forall \ r, s \in \mathbb{R}.$$
Since $\prod_{j=1}^{k}(r_{j} + sb_{j}) = \langle \psi, \gamma_{1} \rangle \cdots \langle \psi, \gamma_{m} \rangle$, we have

$$q(\psi) = \frac{1}{P_{2}(r, s)} = \frac{(-1)^{k}}{k} \sum_{i=1}^{k} (c_{i} - c_{k+i}) \prod_{j=1}^{k \neq i}(r_{j} + sb_{j})$$

(23)

for all $\psi \in S^{1}$ outside of the finite set $\{\psi : \langle \psi, \gamma_{i} \rangle = 0, \ i = 1, \ldots, m\}$.

Hence, in order for $S$ to be non-invertible, we must have

$$\sum_{i=1}^{k} (c_{i} - c_{k+i}) \prod_{j=1 \neq i}^{k}(r_{j} + sb_{j}) \equiv 0.$$ 

Following the argument from the first part of the proof, if we take $r = b_{1}$ and $s = -a_{1}$, then

$$0 = (c_{1} - c_{k+1}) \prod_{j=2}^{k}(r_{j} + sb_{j}) = (c_{1} - c_{k+1}) (a_{2}b_{1} - a_{1}b_{2}) \cdots (a_{k}b_{1} - a_{1}b_{k})$$

Since all vectors $\gamma_{1}, \ldots, \gamma_{k}$ are pairwise linearly independent, $a_{j}b_{1} - a_{1}b_{j} \neq 0$ for $j = 2, \ldots, k$. Hence, the last equation implies $c_{k+1} = c_{1}$.

Applying this procedure with $r = a_{j}$ and $s = -a_{j}$ repeatedly for all $j$, we conclude that in order for the star transform to be non-invertible, we must have $c_{k+i} = c_{i}$ for all $i = 1, \ldots, k$. ■

Theorem 2 immediately implies the following.

**Corollary 3** Any star transform with an odd number of rays is invertible.

### 5 Singular Directions of the Star Transform

It is easy to notice that if the star transform is injective, then the singularities appearing in the right hand side of inversion formula (13) are removable. Namely,

**Remark 1** Let $\{\psi_{i}\}_{i=1}^{M}$, $M \geq m$ be such that

$$\langle \psi_{i}, \gamma_{j} \rangle = 0, \ \text{for some} \ j \in \{1, \ldots, m\}$$

(24)

and/or

$$\sum_{j=1}^{m} c_{j} \prod_{i \neq j}^{m} \langle \psi_{i}, \gamma_{j} \rangle = 0,$$

(25)

then

$$\lim_{\psi \to \psi_{i}} \left[ q(\psi) \frac{d}{ds} \mathcal{R}(Sf)(\psi, s) \right] = \mathcal{R}f(\psi_{i}, s) < \infty.$$ 

(26)
Definition 6 We call \( \psi_i \)'s satisfying condition (24) singular directions of Type 1 and those satisfying condition (25) singular directions of Type 2. These two types are not mutually exclusive.

The number and location of singular directions affect the quality of numerical reconstructions. The singular directions of Type 2 correspond to “division by zero” of the processed data \( \frac{d}{dr} R(Sf) \), while those of Type 1 correspond to “multiplication by zero”. Hence, it is natural to expect that singular directions of Type 2 will create instability and adversely impact the reconstruction. Our numerical experiments confirm these expectations (see Section 7). It is also interesting, that the (totally different) algorithm for inversion of the star transform obtained in [21] produces a relation equivalent to (25) as a necessary and sufficient condition for the instability of that algorithm (see formula (51) on page 18 in [21]).

While the geometric meaning of singular directions of Type 1 is obvious for any \( m \), there is no easy interpretation of condition (25) for \( m \geq 3 \). However, the singular directions of Type 2 are more crucial for the quality of reconstruction. In Section 5.1 we discuss the existence of singular directions of Type 2 in the star transforms with uniform weights. In section 5.2 we show that absence of Type 2 singularities in regular star transforms with an odd number of rays. The case of star transforms with non-uniform weights is discussed in Section 5.3.

5.1 Star Transforms with Uniform Weights

Theorem 3 Consider the star transform \( S = \sum_{i=1}^{m} X_{\gamma_i} \) with uniform weights.

1. If \( m \) is even, \( S \) must contain a singular direction of Type 2.

2. When \( m \) is odd, there exist configurations of \( S \) that contain singular directions of Type 2, as well as configurations that do not contain them.

![Stars of 2k+1 rays with and without singular directions of Type 2.](image)

Figure 4: Stars of 2k+1 rays with and without singular directions of Type 2.
Proof (of Part 1). Let \( m = 2k \). Recall the polynomial \( P_2(\hat{\psi}) \) defined in (20) for \( \hat{\psi} = (r, s) \in \mathbb{R}^2 \):

\[
P_2(\hat{\psi}) = e_{m-1}(\langle \hat{\psi}, \gamma_1 \rangle, \ldots, \langle \hat{\psi}, \gamma_m \rangle) = e_{m-1}(ra + sb).
\]

Along the line \( s = 1 \) it becomes a polynomial in one variable \( P_2(r, 1) \) of order \( 2k - 1 \), unless the coefficient of the highest order term is 0. That coefficient is equal to \( e_{m-1}(a) = P_2(1, 0) \).

If the highest order term of the polynomial is odd, then it will have to become 0 at some point along the line \( s = 1 \), since it has different algebraic signs when \( r \to \infty \) and \( r \to -\infty \). Thus, \( P_2(r, s) = 0 \) either when \( r = 1 \) and \( s = 0 \), or for some value of \( r \) when \( s = 1 \). By homogeneity of \( P_2(r, s) \) it will also have to be zero at some point on \( S^1 \).

Proof (of Part 2a).

Now let us show that for an odd number of rays, there exist star transforms with singularities of Type 2. In other words, for any \( m = 2k + 1 \), \( k \in \mathbb{N} \) there exist configurations of ray directions \( \gamma_i, i = 1, \ldots, m \), such that \( P_2(\psi) = 0 \) for some \( \psi \in S^1 \), where \( P_2 \) is the polynomial defined in (20).

A trivial example is when a pair of rays point to opposite directions, i.e. \( \gamma_i = -\gamma_j \) for some indices \( i \) and \( j \). Then it is easy to see that the direction normal to them is a singular direction of Type 2. A more interesting example is discussed below.

Consider an arbitrary open half-plane \( H = \{ x : \langle x, n \rangle < 0 \} \), where \( n \) is the outward normal to its boundary passing through the origin. Choose a set of distinct ray directions \( \{ \gamma_i \}_{i=1}^m \) so that \( \gamma_i \in H \) for all \( i = 1, \ldots, m \). Without loss of generality, let us assume that \( \gamma_i \)'s are indexed according to the growth of their polar angle (see Figure 4b).

Consider \( \psi = \gamma_2^\perp \) oriented so that \( \langle \psi, \gamma_1 \rangle > 0 \). Then \( \langle \psi, \gamma_2 \rangle = 0 \) and \( \langle \psi, \gamma_i \rangle < 0 \) for all \( i > 2 \). Estimating the terms of the sum in (18), it is easy to notice that \( \prod_{j \neq 2} (\psi, \gamma_j) < 0 \), while all of the other terms are 0. Thus, for \( \psi = \gamma_2^\perp \) we get \( P_2(\psi) < 0 \).

On the other hand, if \( \psi = n \), then all terms of the sum in (18) are positive and \( P_2(\psi) > 0 \). By continuity of \( P_2(\psi) \), there has to exist a value of \( \psi \), for which \( P_2(\psi) = 0 \).

Proof (of Part 2b). The regular star transform \( S \) with \( m = 2k + 1 \) rays does not have a singular direction of Type 2. See Section 5.2.

5.2 Regular Star Transforms with an Odd Number of Rays

In this section we show that any regular star transform with an odd number of rays (see Figure 4a) has a stable inversion. More specifically,

Theorem 4 The regular star transform \( S \) with \( m = 2k + 1 \) rays does not have a singular direction of Type 2.
Proof. We need to show that for the regular star with \( m = 2k + 1 \) rays, \( P_2(\psi) \neq 0 \) for any \( \psi \in S^1 \), where \( P_2 \) is the polynomial defined in (20).

Using the parameterization \( \psi(\alpha) = (\cos \alpha, \sin \alpha) \), we denote

\[
F(\alpha) = P_2(\psi(\alpha)).
\]

We will show that \( F(\alpha) \neq 0 \) for any \( \alpha \in [0, 2\pi] \).

Referring to formulas (6) and (7) let us define

\[
\beta_j = \pi + \alpha_j, \quad j = 1, \ldots, 2k + 1.
\]

Due to the symmetry of \( P_2(\psi) \) with respect to \( \gamma_i \)'s and the symmetric distributions of \( \gamma_i \)'s on \( S^1 \), it follows that

\[
F(\alpha_j + \varepsilon) = F(\alpha_j - \varepsilon) \quad \text{and} \quad F(\beta_j + \varepsilon) = F(\beta_j - \varepsilon) \quad \text{for} \quad \forall \varepsilon > 0.
\]

Equations (29) imply that either all values in the set

\[
S = \{\alpha_j\}_{j=1}^{2k+1} \cup \{\beta_j\}_{j=1}^{2k+1}
\]

are points of extrema for \( F(\alpha) \), or \( F(\alpha) \equiv \text{const.} \).

To finish the proof, it is sufficient to show that:

- \( F(\alpha) \) can have at most \( 2m - 2 \) points of extrema. Hence, \( F(\alpha) \equiv \text{const.} \).
- \( F(\alpha_1) \neq 0 \).

Those statements are deduced from the following two lemmas.

**Lemma 4** In the symmetric setup described above, there exists a polynomial of one variable \( P_1 \) of degree \( \leq m - 1 \), such that

\[
F(\alpha) = P_1(\cos \alpha),
\]

Proof. Recall Definition 3 and Equation (8). Using our previous notations \( \psi(\alpha) = (\cos \alpha, \sin \alpha) = (r, s) \) we can write

\[
F(\alpha) = P_2(\psi(\alpha)) = e_{m-1}(\langle \psi, \gamma_1 \rangle, \ldots, \langle \psi, \gamma_m \rangle) = e_{m-1}(ra + sb)
\]

\[
= \sum_{i=1}^{m} \prod_{j \neq i} (ra_j + sb_j) = \prod_{j \neq 1} (ra_j + sb_j) + \prod_{j \neq 2} (ra_j + sb_j) + \prod_{j \neq 3} (ra_j + sb_j) + \ldots + \prod_{j \neq 2k} (ra_j + sb_j) + \prod_{j \neq 2k+1} (ra_j + sb_j)
\]

It easy to notice, that after appropriate cancellations inside the brackets using (8), the last expression does not contain any odd degree of \( s \). For example:

\[
\prod_{j \neq 1} (ra_j + sb_j) = [(ra_2 + sb_2)(ra_3 + sb_3)] \ldots [(ra_{2k} + sb_{2k})(ra_{2k+1} + sb_{2k+1})]
\]

\[
= [(ra_2 + sb_2)(ra_2 - sb_2)] \ldots [(ra_{2k} + sb_{2k})(ra_{2k+1} - sb_{2k})]
\]

\[
= (r^2a_2^2 - s^2b_2^2) \ldots (r^2a_{2k}^2 - s^2b_{2k}^2).
\]
Similarly,
\[
\prod_{j \neq 2} (ra_j + sb_j) + \prod_{j \neq 3} (ra_j + sb_j) = (ra_3 + sb_3 + ra_2 + sb_2) \prod_{j \neq 2,3} (ra_j + sb_j)
\]
\[
= (ra_2 - sb_2 + ra_2 + sb_2) r(ra_4 + sb_4)(ra_4 - sb_4) \ldots (ra_{2k} + sb_{2k})(ra_{2k} - sb_{2k})
\]
\[
= 2r^2a_2(r^2a_4^2 - s^2b_4^2) \ldots (r^2a_{2k}^2 - s^2b_{2k}^2).
\]
Substituting \(s^2 = 1 - r^2\) we obtain the desired result.

The degree \(m - 1\) polynomial \(P_1(r)\) can have at most \(m\) points of extrema on the interval \([-1,1]\), where \(m - 2\) correspond to the critical points (i.e. the derivative is 0) and 2 more for the endpoints of the interval.

As a result, the function \(F(\alpha)\) can have at most \(2m - 2\) points of extrema. Indeed, at most \(2m - 4\) points will correspond to the critical points of \(P_1 (r = \cos \alpha\) defines \(s = \sin \alpha\) up to a sign) and the “endpoints” \((1,0), (-1,0)\). Since the set \(S\) contains \(2m\) distinct points, it can not be the set of extrema of \(F\). Thus, \(F(\alpha) \equiv \text{const.}\)

**Lemma 5** Consider the regular star transform with \(m = 2k + 1\) rays defined in (6) and (7). If \(k\) is odd, then \(F(\alpha_1) < 0\). If \(k\) is even, then \(F(\alpha_1) > 0\).

**Proof.** We present below a rigorous algebraic argument using the cosines of polar angles of ray directions. It will be very intuitive and easy to follow if the reader draws a sketch similar to Figure 4a for each case and visualizes the relations between the rays and corresponding cosines there.

Let \(l\) denote the number of rays \(\gamma_j, j > 1\) in the first quadrant (see Figure 4a for reference). It is easy to see that
\[
l = \left\lfloor \frac{k}{2} \right\rfloor, \quad (31)
\]
where \(\lfloor z \rfloor\) denotes the integer part of \(z\). Notice, that it means the following vectors are in the first quadrant: \(\gamma_2, \gamma_4, \ldots, \gamma_{2l}\), while \(\gamma_{2(l+1)}, \ldots, \gamma_{2k}\) are in the second. Moreover, if \(k > 1\) then
\[
0 < \cos \alpha_{2+2i} < -\cos \alpha_{2k-2i}, \quad \forall i = 0, \ldots, l, \quad (32)
\]
and
\[
0 < -\cos \alpha_{2k-2(i+1)} < \cos \alpha_{2+2i}, \quad \forall i = 0, \ldots, l. \quad (33)
\]
Since \(\alpha_1 = 0\) and \(\psi(\alpha_1) = \gamma_1 = (1,0)\) we have
\[
F(\alpha_1) = P_2(\alpha_1) = e_{m-1}((\alpha_1, \alpha_1), \ldots, (\alpha_1, \alpha_m)) = \sum_{i=1}^{m} \prod_{j \neq i} \cos \alpha_j \quad (34)
\]
We will group the terms of the sum in (34) in different ways, depending on whether \(k\) is odd or even.

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Let $k$ be even. Due to the symmetry of the setup with respect to $\gamma_i$’s with odd and even indices $i > 1$, we have

$$ F(\alpha_1) = \sum_{i=1}^{m} \prod_{j \neq i} \cos \alpha_j = \prod_{j \neq 1} \cos \alpha_j + 2 \left\{ \prod_{j \neq 2} \cos \alpha_j + \prod_{j \neq 2k} \cos \alpha_j \right\} $$

$$ + \left[ \prod_{j \neq 4} \cos \alpha_j + \prod_{j \neq 2k-2} \cos \alpha_j \right] + \ldots + \left[ \prod_{j \neq 2l} \cos \alpha_j + \prod_{j \neq 2k-2l} \cos \alpha_j \right] $$

Since $\alpha_{2j} = -\alpha_{2j+1} \neq \frac{\pi}{2}$ for all $j = 1, \ldots, k$, we have

$$ \prod_{j \neq 1} \cos \alpha_j = \prod_{1 \leq j \leq k} \cos^2 \alpha_{2j} > 0. $$

At the same time,

$$ \prod_{j \neq 2} \cos \alpha_j + \prod_{j \neq 2k} \cos \alpha_j = (\cos \alpha_{2k} + \cos \alpha_2) \prod_{j \neq 2, 2k} \cos \alpha_j > 0, $$

since both multipliers in the last expression are negative by formula (32). Evaluating the remaining terms of $F(\alpha_1)$ in the same way, we get $F(\alpha_1) > 0$.

Now, let $k$ be odd.

$$ F(\alpha_1) = \sum_{i=1}^{m} \prod_{j \neq i} \cos \alpha_j = \left[ \prod_{j \neq 1} \cos \alpha_j + \prod_{j \neq 2k} \cos \alpha_j + \prod_{j \neq 2k+1} \cos \alpha_j \right] $$

$$ + 2 \left\{ \prod_{j \neq 2} \cos \alpha_j + \prod_{j \neq 2k-2} \cos \alpha_j \right\} + \ldots + \left[ \prod_{j \neq 2l} \cos \alpha_j + \prod_{j \neq 2k-2l} \cos \alpha_j \right] $$

Using formula (33) we obtain

$$ \prod_{j \neq 1} \cos \alpha_j + \prod_{j \neq 2k} \cos \alpha_j = (\cos \alpha_{2k} + \cos \alpha_1) \prod_{j \neq 1, 2k} \cos \alpha_j < 0, $$

and

$$ \prod_{j \neq 2k+1} \cos \alpha_j < 0. $$

Hence the first bracket in the formula for $F(\alpha_1)$ is negative. At the same time

$$ \prod_{j \neq 2} \cos \alpha_j + \prod_{j \neq 2k-2} \cos \alpha_j = (\cos \alpha_{2k-2} + \cos \alpha_2) \prod_{j \neq 2, 2k-2} \cos \alpha_j < 0, $$

by formula (33). Applying this approach to all other brackets in the equation for $F(\alpha_1)$, we conclude that $F(\alpha_1) < 0$. ■
Combining the previous two lemmas and the discussion after Lemma 4 we complete the proof of Theorem 4.

In the case of \( m = 3 \), there is a simpler proof of Theorem 4. That proof also shows that the transform \( S \) corresponding to the regular star has (in some sense) the most stable inversion. It is based on the following result from \([7]\): 

**Theorem** The zero set of \( e_2(y_1, y_2, y_3) \) is the circular cone
\[
e_2^{-1}(0) = \left\{ y \in \mathbb{R}^3 : \cos^2 (y, u) = \frac{1}{3} \right\},
\]
where \( u = (1, 1, 1) \) and \((y, u)\) denotes the angle between vectors \( y \) and \( u \).

**An Alternative Proof of Theorem 4 when** \( m = 3 \).  
We need to show that \( e_2(ra + sb) \neq 0 \), where \( a \) and \( b \) are the aperture vectors of the star transform defined by (6), (7), (8) as
\[
a = \left( 1, -\frac{1}{2}, -\frac{1}{2} \right), \quad b = \left( 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \right),
\]
and \( r^2 + s^2 = 1 \). In light of (35), it is equivalent to proving that the plane \( T \) spanned by aperture vectors \( a \) and \( b \) intersects the circular cone defined in (35) only at the origin. Since
\[
\langle a, u \rangle = 0 \quad \text{and} \quad \langle b, u \rangle = 0,
\]
it is clear that \( u \perp T \). Hence, \( T \cap e_2^{-1}(0) = \{0\} \). Moreover, the aperture vectors \( a \) and \( b \) corresponding to the regular star, span the plane that is “as far as possible” from the zero set of \( e_2 \).

5.3 Star Transforms with Non-Uniform Weights

The regular stars with an odd number of rays are not the only configurations of the transform \( S \) that do not have a singular direction of Type 2. Due to continuous dependence of \( P_2(\psi) \) on \( \alpha_i \)'s, small perturbations of \( \gamma_i \)'s from symmetry positions will not introduce a singular direction of Type 2. Similarly, small perturbations of uniform weights \( c_1 = \ldots = c_m = 1 \) to non-uniform values will not introduce singular directions of Type 2. Moreover,

**Remark 2** The polynomial \( P_2(\psi) \) does not change if one simultaneously replaces \( \gamma_i \) by \(-\gamma_i \) and \( c_i \) by \(-c_i \) for any \( i \). This provides a recipe for creating star transforms with weights of mixed algebraic signs that have stable inversions.

For example, modifying the regular star transform of 3 rays (defined by formula (36)) through replacement of \( \gamma_2 \) by \(-\gamma_2 \) and changing the sign of \( c_2 = 1 \), we get a stable configuration with weights \( c_1 = 1, c_2 = -1, c_3 = 1 \) and aperture vectors:
\[
a = \left( 1, \frac{1}{2}, -\frac{1}{2} \right), \quad b = \left( 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \right).
\]
In general, one can get stable setups when the absolute values of weights are close to each other.

The star transforms with weights of mixed algebraic signs satisfying the condition

\[ \sum_{i=1}^{m} c_i = 0 \]  

play an important role in simultaneous reconstruction of scattering and absorption coefficients in single scattering tomography [9, 10, 13, 21]. Simultaneously satisfying the above condition and keeping the absolute values of weights as close to each other as possible, suggest choosing the weights (up to a proportionality coefficient and re-indexing) as follows:

\[
\begin{align*}
c_1 &= \ldots = c_k = -\frac{1}{k}, & c_{k+1} &= \ldots = c_{2k+1} = \frac{1}{k+1}, & m &= 2k+1. \\
\end{align*}
\]

In the case of \( m = 3 \), this implies choosing the weights proportional to \( c_1 = c_2 = 1 \) and \( c_3 = -2 \). Such a setup was analyzed in detail in [21].

We finish this section with generalizations of the statements of Section 5.1 to the case of transforms with non-uniform weights. For the stars with an even number of rays, both the result and its proof are almost identical. Namely,

**Theorem 5** Consider the star transform \( S = \sum_{i=1}^{m} c_i x_{\gamma_i} \). If \( m \) is even, \( S \) must contain a singular direction of Type 2.

In the case of star transforms with an odd number of rays, it should be clarified in what sense we want to generalize Part 2 of Theorem 3. If one is allowed to choose both the weights \( c_i \) and the ray directions \( \gamma_i \) to claim existence of setups with and without Type 2 singularities, then the statement has a trivial proof by choosing uniform weights and referring to Theorem 3. Hence, we consider the following question. Are there configurations (i.e. ray directions \( \gamma_1, \ldots, \gamma_m \)) with/without Type 2 singularities for a specified set of weights \( c_1, \ldots, c_m \)?

**Theorem 6** Consider a star transform \( S = \sum_{i=1}^{m} c_i x_{\gamma_i} \), where \( m = 2k+1 \).

(a) For any set of specified weights \( c_1, \ldots, c_m \), there exist \( \gamma_1, \ldots, \gamma_m \) such that \( S \) contains singular directions of Type 2. In other words, for any set of weights there are configurations of \( S \) with unstable inversion.

(b) Let \( m = 3 \). For any set of specified weights \( c_1, c_2, c_3 \), there exist \( \gamma_1, \gamma_2, \gamma_3 \) such that \( S \) does not contain singular directions of Type 2. In other words, for any set of weights there are configurations of \( S \) with a stable inversion.

**Proof of (a): Existence of configurations with Type 2 singularities.** Both examples provided in the proof of Theorem 3 work here with small modifications. If \( \gamma_i = -\gamma_j \) for some indices \( i \) and \( j \), then the direction normal to them is a singular direction of Type 2.
A more interesting example is the case when all $\gamma_i$’s belong to the same half-plane. Recall the corresponding setup from the proof of Theorem 3. Without loss of generality, let us assume that the $c_1c_2 > 0$, i.e. the weights $c_1$ and $c_2$ have the same sign. Since we consider the case of $m = 2k + 1$ for $k \geq 1$, the assumption above is just a matter of re-indexing the weights (if necessary).

Let $\psi_1 = \gamma_1^\perp$ and $\psi_2 = \gamma_2^\perp$ be vectors obtained by a clockwise rotation by $\pi/2$ from $\gamma_1$ and $\gamma_2$ correspondingly. Then, it is easy to verify that $P_2(\psi_1)$ and $P_2(\psi_2)$ have different signs. Hence, by continuity of $P_2$ there exists some $\psi$ s.t. $P_2(\psi) = 0$. ■

Before proceeding to the proof of the second statement of Theorem 6, we define a new class of planes in $\mathbb{R}^3$ and discuss their properties.

**Definition 7** We call a plane $T \subseteq \mathbb{R}^3$ "spannable by star aperture vectors" or SSAV if $T = \text{span}\{a, b\}$, where $a, b \in \mathbb{R}^3$ are aperture vectors of some star with $m = 3$ rays. In particular this means that $(a_i, b_i)$ are unit vectors in $\mathbb{R}^2$ for $i = 1, 2, 3$.

Not every plane (containing the origin) in $\mathbb{R}^3$ is SSAV. The following lemma gives a complete description of admissible normal vectors to an SSAV plane.

**Lemma 6** A vector $n = (n_1, n_2, n_3) \in \mathbb{R}^3$ is normal to an SSAV plane if and only if $|n_1|$, $|n_2|$ and $|n_3|$ correspond to side lengths of some triangle. We will denote the set of all such vectors by $A$.

**Proof.** Let $n$ be normal to a plane $T \in \mathbb{R}^3$ containing the origin. $T$ is SSAV if and only if there exists a star with aperture vectors $a, b$ such that $\langle n, a \rangle = \langle n, b \rangle = 0$. By definition of the aperture vectors, the last relation implies

$$n_1 \gamma_1 + n_2 \gamma_2 + n_3 \gamma_3 = 0,$$

where $\gamma_1$, $\gamma_2$ and $\gamma_1$ are the ray directions of the corresponding star. Since $\gamma_i$’s are unit vectors, the statement of the lemma follows from the above equality interpreting it as a sum of vectors tracing the perimeter of a triangle. ■

The set $A$ of admissible normals to an SSAV plane has an interesting geometric description due to the three triangle inequalities

$$|n_1| + |n_2| \leq |n_3|, \quad |n_2| + |n_3| \leq |n_1|, \quad |n_1| + |n_3| \leq |n_2|.$$

In the octant $n_1 > 0$, $n_2 > 0$, $n_3 > 0$ it coincides with an infinite polyhedral cone with a vertex at the origin and infinite edges along vectors $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$. We denote the portion of $A$ in this first octant by $A^+$. In all other octants $A$ coincides with a similar tetrahedron with the edges along appropriately signed versions of the same vectors.

Formula (35) shows that every plane with a normal vector $n \in A^+$ intersects $e_2^{-1}(0)$ only at the origin. Moreover, among the SSAV planes, those with a normal vector in $A^+$ are the only ones that intersect $e_2^{-1}(0)$ only at the origin.
Hence, the set $A^+$ contains normals to SSAV planes that correspond (through the spanning aperture vectors) to star transforms with uniform weights $c_1 = c_2 = c_3 = 1$ and a stable inversion, i.e. without a singular direction of Type 2.

Let

$$W = \text{diag}(c_1, c_2, c_3)$$

denote the diagonal matrix of weights.

**Lemma 7** For any set of specified weights $c_1, c_2, c_3 > 0$ there exist SSAV planes $T_1, T_2$ such that $T_2 = W^{-1}T_1$ and $T_i \cap c_2^{-1}(0) = \{0\}$ for $i = 1, 2$.

**Proof.** Let $n^{(1)}, n^{(2)}$ denote some vectors normal to the planes $T_1$ and $T_2$ correspondingly. The proof will follow immediately if we show that for any $c_1, c_2, c_3 > 0$ there exist $n^{(1)}, n^{(2)} \in A^+$ such that $n^{(2)} = W^{-1}n^{(1)}$.

We first note that the problem is invariant with respect to multiplication of all weights $c_1, c_2, c_3 > 0$ by a positive constant, as well as their permutation. Hence, without loss of generality we will assume that $0 < c_1 \leq c_2 = 1 \leq c_3$.

A choice of desired pair of vectors $n^{(1)}, n^{(2)}$ is given by

$$n^{(1)} = (c_1, 1, 1) \in A^+, \quad n^{(2)} = W^{-1}n^{(1)} = (1, 1, 1/c_3) \in A^+,$$

which ends the proof.

**Proof of Theorem 6 (b): Configurations without Type 2 singularities.** Assume $m = 3$ and let $c_1, c_2, c_3 \neq 0$ be specified weights. By Remark 2, each stable configuration of $S$ with signed weights $c_1, c_2, c_3$ corresponds to another stable configuration of $S$ with positive weights $|c_1|, |c_2|, |c_3|$ and the other way around. Hence, without loss of generality we will prove our statement for $c_1, c_2, c_3 > 0$.

We will use our knowledge of existence of stable configurations for every star transform with uniform weights (marked by superscript (2) in the proof below) to show existence of stable configurations for star transforms with positive weights (marked by superscript (1) in the proof below).

By Lemma 7 there exist SSAP planes $T_1, T_2$ such that $T_2 = W^{-1}T_1$ and $T_i \cap c_2^{-1}(0) = \{0\}$ for $i = 1, 2$. Let $a^{(1)}, b^{(1)}$ and $a^{(2)}, b^{(2)}$ be the star aperture vector pairs spanning correspondingly $T_1$ and $T_2$.

To prove the theorem, it is enough to show that for ray directions $\gamma_1^{(1)}, \gamma_2^{(1)}, \gamma_3^{(1)}$ corresponding to the aperture vectors $a^{(1)}, b^{(1)}$ the following relation holds

$$P_2^{(1)}(\psi) \equiv c_1(\psi, \gamma_2^{(1)})(\psi, \gamma_3^{(1)}) + c_2(\psi, \gamma_1^{(1)})(\psi, \gamma_3^{(1)}) + c_3(\psi, \gamma_1^{(1)})(\psi, \gamma_2^{(1)}) \neq 0$$

for all $\psi \in S^1$.

Due to the properties of selected planes and corresponding aperture vectors

$$P_2^{(2)}(\psi) \equiv e_2\left(ra^{(2)} + sb^{(2)}\right) \neq 0 \text{ for all } \psi = (r, s) \in S^1.$$

At the same time for any $\psi = (r, s) \in S^1$ there exists $\hat{\psi} = (\hat{r}, \hat{s}) \in S^1$ such that

$$P_2^{(1)}(\psi) = c e_2\left(W^{-1}(ra^{(1)} + sb^{(1)})\right) = c e_2\left(\hat{r}a^{(2)} + \hat{s}b^{(2)}\right) = c P_2^{(2)}(\hat{\psi}) \neq 0,$$

where $c 

```
where \( c = c_1 c_2 c_3. \)

The generalization of Theorem 6 (b) to the case of \( m = 2k + 1 > 3 \) is technically more complicated and we plan to address in a future publication. So we finish this sections with the following.

**Conjecture 1** Let \( m = 2k + 1 \). For any set of specified weights \( c_1, \ldots, c_m \), there exist \( \gamma_1, \ldots, \gamma_m \) such that \( S \) does not contain singular directions of Type 2. In other words, for any set of weights there are configurations of \( S \) with a stable inversion.

## 6 Zeros of \( e_{m-1}(y_1, \ldots, y_m) \) for Odd \( m \).

In paper [7] published in 2006, the author formulated the following conjecture:

**Conjecture 2** If \( r \) is even then \( e_\gamma^{-1}(0) \) contains no real vector subspace of dimension \( r \).

Furthermore, it is stated there that one of the extreme cases is “the case \( e_{m-1}(y_1, \ldots, y_m), m \equiv 1 \pmod{2} \), which becomes a task quite hard to tackle”.

Our proof of Theorem 4 includes a proof of the aforementioned extreme case. Namely,

**Theorem 7** Let \( m = 2k + 1 \) for some \( k \in \mathbb{N} \). Then \( e_{m-1}^{-1}(0) \) contains no real vector subspace of dimension \( m - 1 \).

Indeed, let \( a, b \in \mathbb{R}^m \) be the (linearly independent) aperture vectors of the regular star with \( m = 2k + 1 \) rays. It was shown that \( e_{m-1}^{-1}(\psi) = 0 \) for any \( \psi = (r,s) \in S^1 \). The set \( C = \{ ra + sb : (r,s) \in S^1 \} \) represents a closed contour around the origin in the 2-dimensional subspace \( T \subset \mathbb{R}^m \) spanned by the aperture vectors \( a \) and \( b \). Due to homogeneity of \( e_{m-1}(y) \), the fact that it has no zeros on \( C \), implies that \( e_{m-1}^{-1}(0) \cap T = \{ 0 \} \). Since \( \dim(T) = 2 \), the zero set \( e_{m-1}^{-1}(0) \) can not contain a real vector subspace of dimension \( m - 1 \).

## 7 Numerical Simulations

In this section we present some examples of numerical reconstructions of the standard Shepp-Logan phantom from its star transforms of various configurations. The reconstruction algorithm is based on Theorem 1 and filtered back-projection algorithm of the Radon transform.

It is easy to notice that the setups corresponding to an even number of rays have severe artifacts propagating in the direction of Type 2 singularities. The setups with an odd number of rays that do not have singular directions of Type 2 are of much better quality and do not have such artifacts.

The resolution of all images is 400x400 pixels.
(a) Shepp-Logan phantom used in all reconstructions

(b) Reconstruction from $S$ with $\gamma_1 = (1, 0), \gamma_2 = (\cos \frac{2\pi}{3}, \sin \frac{2\pi}{3})$

Figure 5

(a) Reconstruction from regular $S$ with $\gamma_1 = (1, 0), \gamma_2 = (\cos \frac{2\pi}{3}, \sin \frac{2\pi}{3}), \gamma_3 = (\cos \frac{4\pi}{3}, \sin \frac{4\pi}{3})$

(b) Reconstruction from $S$ with $\gamma_1 = (\cos \frac{2\pi}{3}, \sin \frac{2\pi}{3}), \gamma_2 = (\cos \frac{4\pi}{3}, \sin \frac{4\pi}{3}), \gamma_3 = (\cos \frac{6\pi}{3}, \sin \frac{6\pi}{3})$

Figure 6

(a) Reconstruction from $S$ with $\gamma_1 = (1, 0), \gamma_2 = (0, 1), \gamma_3 = (\cos \frac{2\pi}{3}, \sin \frac{2\pi}{3})$

(b) Reconstruction from regular $S$ with $\gamma_1 = (1, 0), \gamma_2 = (\cos \frac{2\pi}{3}, \sin \frac{2\pi}{3}), \gamma_3 = (\cos \frac{4\pi}{3}, \sin \frac{4\pi}{3}), \gamma_4 = (\cos \frac{6\pi}{3}, \sin \frac{6\pi}{3}), \gamma_5 = (\cos \frac{8\pi}{3}, \sin \frac{8\pi}{3})$

Figure 7
8 Additional Remarks

1. The statement of a special case of Theorem 1 was published without proof in the abstract [4].

2. We did not aim to formulate the most general conditions under which Theorem 1 holds. The requirement of \( f \in C_c(\mathbb{R}^2) \) is sufficient, but not necessary. The proof will hold under weaker assumptions, e.g. when \( f \) decays sufficiently fast at infinity.

3. In the case of \( m = 1 \), Theorem 1 establishes a relation between the divergent beam transform and the (ordinary) Radon transform. A different relation between these transforms was obtained in [13], where the divergent beam transform is integrated over all possible angles in \( S^{n-1} \) to generate the integral of the function over the whole space \( \mathbb{R}^n \). The latter can also be obtained by integrating the Radon transform over all hyperplanes normal to a given direction.

Similarly, in the case of \( m = 2 \), our Theorem 1 establishes a relation between the V-line transform and the Radon transform. A different relation between these transforms was obtained in [18] using an overdetermined setup of the V-line transform. It is similar in the spirit to the formula for the divergent beam transform in [13], as it uses all possible angles for the branch directions of the V-line transform.

Our formulas derived in Theorem 1 are essentially different from those in [13] and [18], as we do not require overdetermined data and use only a finite number of branch directions.

4. A known application of the star transform is related to imaging of a physical medium with a variable scattering coefficient. One can use the star transform to recover the scattering coefficient simultaneously with the absorption coefficient from the same data set.

Every pair of unit vectors \( \gamma_i \) and \( \gamma_j \) corresponds to a source and a detector direction respectively. At each vertex \( x \in \mathbb{R}^2 \), one measures the data

\[
\phi_{ij}(x) = \mathcal{X}_i f(x) + \mathcal{X}_j f(x) + \eta(x),
\]

where \( f(z) \) and \( \eta(x) \) represent correspondingly the absorption and scattering coefficient at \( x \). To get the appropriate linear combination of data at each vertex and cancel the contribution from the \( \eta \) term, one can choose a symmetric matrix \( \omega \) with zeros on the main diagonal satisfying \( \sum_{i,j} \omega_{ij} = 0 \). Writing this linear combination, we get

\[
\sum_{i=1}^{m} \sum_{j=1}^{m} \omega_{ij} \phi_{ij}(x) = \sum_{i=1}^{m} \left( \sum_{j=1}^{m} \omega_{ij} \right) \mathcal{X}_i f(x) = \sum_{i=1}^{m} c_i \mathcal{X}_i f(x),
\]

where \( c_i \) are determined by the chosen matrix \( \omega \).
where
\[ c_i = \sum_{j=1}^{m} \omega_{ij} \]

After applying the inversion of the star transform and finding \( f \), we can recover the scattering term using (39).

It is easy to notice that when \( m > 3 \), for any predefined set of weights \( c_i \neq 0, i = 1, \ldots, m \) there are infinitely many choices of \( \omega \) satisfying the described requirements. In the case of \( m = 3 \), not all choices of \( c_1, c_2, c_3 \) work, but there are still infinitely many admissible options.

We refer the interested reader to [21] for more details on physical aspects and imaging settings.

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References

[1] G. Ambartsoumian, *Inversion of the V-line Radon transform in a disc and its applications in imaging*, Computers & Mathematics with Applications, 64, 3 (2012), pp 260—265.

[2] G. Ambartsoumian, *V-line and conical Radon transforms with applications in imaging*, Chapter 7 in “The Radon Transform: The First 100 Years and Beyond”, edited by R. Ramlau and O. Scherzer, Radon Series on Computational and Applied Mathematics, De Gruyter, 2019.

[3] G. Ambartsoumian and M. J. Latifi Jebelli, *The V-line transform with some generalizations and cone differentiation*, Inverse Problems, 35 3 (2019), 034003.

[4] G. Ambartsoumian and M. J. Latifi Jebelli, *Inversion of the star transform*, an abstract in “Tomographic Inverse Problems: Theory and Applications”, edited by M. Burger, B. Hahn and E. T. Quinto, Oberwolfach Reports, EMS, 2019.

[5] G. Ambartsoumian and S. Moon, *A series formula for inversion of the V-line Radon transform in a disc*, Computers & Mathematics with Applications, 66 9 (2013), pp 1567–1572.

[6] G. Ambartsoumian and S. Roy, *Numerical inversion of a broken ray transform arising in single scattering optical tomography*, IEEE Transactions on Computational Imaging, 2 2 (2016), pp 166–173.
[7] A. Conflitti, *Zeros of real symmetric polynomials*, Applied Mathematics E-Notes, 6 (2006), pp 219–224.

[8] L. Florescu, V. Markel and J. Schotland, *Inversion formulas for the broken ray Radon transform*, Inverse Problems, 27 2 (2011) 025002.

[9] L. Florescu, J. Schotland and V. Markel, *Single scattering optical tomography*, Physical Review E, 79 3 (2009), 036607.

[10] L. Florescu, V. Markel and J. Schotland, *Single scattering optical tomography: simultaneous reconstruction of scattering and absorption* Physical Review E, 81 1 (2010), 016602.

[11] L. Florescu, V. Markel and J. Schotland, *Nonreciprocal broken ray transforms with applications to fluorescence imaging*, Inverse Problems, 34 9 (2018) 094002.

[12] R. Gouia-Zarrad and G. Ambartsoumian, *Exact inversion of the conical Radon transform with a fixed opening angle*, Inverse Problems, 30 4 (2014), 045007.

[13] C. Hamaker, K. T. Smith, D. C. Solomon and S. I. Wagner, *The divergent beam x-ray transform*, Rocky Mountain J. Math., 10 1 (1980), pp 253–284.

[14] A. Katsevich and R. Krylov, *Broken ray transform: inversion and a range condition*, Inverse Problems, 29 7 (2013), 075008.

[15] R. Krylov and A. Katsevich, *Inversion of the broken ray transform in the case of energy dependent attenuation*, Physics in Medicine & Biology, 60 11 (2015), pp 4313–4334.

[16] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2-nd Edition, Clarendon Press, Oxford 1998.

[17] B. Sherson, *Some results in single-scattering tomography*, PhD Thesis, Oregon State University, 2015.

[18] F. Terzioglu, *Some inversion formulas for the cone transform*, Inverse Problems, 31 11 (2015), 115010.

[19] F. Terzioglu, P. Kuchment and L. Kunyansky, *Compton camera imaging and the cone transform: a brief overview*, Inverse Problems, 34 5 (2018), 054002.

[20] M. R. Walker and J. A. O’Sullivan, *The broken ray transform: additional properties and new inversion formula*, Inverse Problems, 35 11 (2019), 115003.

[21] Z. Zhao, J. Schotland and V. Markel, *Inversion of the star transform*, Inverse Problems, 30 10 (2014), 105001.