A Topological Duality for Monotone Expansions of Semilattices

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Abstract
In this paper we provide a Stone style duality for monotone semilattices by using the topological duality developed in S. Celani, L.J. González (Appl Categ Struct 28:853–875, 2020) for semilattices together with a topological description of their canonical extension. As an application of this duality we obtain a characterization of the congruences of monotone semilattices by means of monotone lower-Vietoris-type topologies.

Keywords Semilattice · Duality · Monotone operator · Canonical extension

1 Introduction
In [5] Birkhoff proved that every finite distributive lattice is represented by the lower sets of a finite partial order. In [27] Stone generalized the results given by Birkhoff establishing a
categorical duality for bounded distributive lattices through spectral spaces with continuous functions. Later, in [25] Priestley proved another categorical duality for bounded distributive lattices by means of certain ordered topological spaces and continuous order-preserving maps. Along the years both Stone duality and Priestley duality have served as powerful tools not only for the study of bounded distributive lattices, but also as an inspiration for the study of many ordered algebraic structures associated with non-classical logics. Evidence of this is supplied by the several topological representations obtained for diverse classes of algebras with distributive semilattice reducts and semilattice reducts. This is the case of the Stone style duality for distributive semilattices developed in [7, 9], which extends Stone duality and completes the representation given by Grätzer in [18], the Priestley style duality developed by Bezhanishvili and Jansana in [2, 4] for the class of distributive semilattices and implicative semilattices, and more recently the Stone style duality for semilattices and lattices obtained by Celani and González in [10].

A particular case of interest raises when noticing that any of these dualities can be combined with some other tools coming from algebra. This is the case of canonical extensions ([13, 15–17]). The studies on this subject started with Jónsson and Tarski’s works for Boolean algebras with operators ([22, 23]). The most remarkable contribution of these papers was that they provided a procedure for transfer the benefits and the working methodology from the duality for Boolean algebras to several classes of algebras with additional operators. Along this paper we take these ideas and go one step further. Our methodology is based on the following crucial observation: consider a class of algebras with lattice reduct \( L \) and a suitable topological duality for \( L \). Then, it is possible to build new topological dualities for monotone expansions of \( L \), by using a topological description of the canonical extension of the members of \( L \), in terms of the given duality. This approach was used by Celani and Menchón in [11] to obtain a Stone style duality for monotone distributive semilattices. We stress that in [24] Moshier and Jipsen employed a duality different from the ones of Stone and Priestley to give a topological description of the canonical extension of a (not necessarily distributive) bounded lattice. Nevertheless, they do not apply such a description for providing a duality for monotone expansions of lattices.

The aim of this paper is to establish a Stone style duality for monotone semilattices. To do so, we provide a topological description of the canonical extension of a semilattice by using the duality obtained in [10]. We conclude by applying our duality to characterize the congruences of semilattices and monotone semilattices in terms of lower-Vietoris-type topologies ([21]). These results generalize the results obtained in [9] for the class of distributive semilattices. We stress that this work is based on the Stone style approach of ([10]) and not on a Priestley style approach. This is important since, as we know, the dualities are different, and therefore the results of their applications may also differ.

The paper is organized as follows. In Sect. 2 we give several notions and results which are needed for the rest of the paper. In Sect. 3 we present a topological description of the canonical extension of a semilattice in terms of the Stone style duality given in [10] and we explore the underlying connection of our results with the description of canonical extensions of semilattices given by Gouveia and Priestley in [17]. Further, we extend our results to a topological description of canonical extensions of monotone maps of semilattices. In Sect. 4 we establish our main result: a topological duality for the category of monotone semilattices and homomorphisms of monotone semilattices. Finally, in Sect. 5 as an application of the duality, we prove that the congruences of a semilattice and a monotone semilattice are in correspondence with the lower-Vietoris-type families and monotone lower-Vietoris-type topologies of their respective dual spaces.
2 Preliminaries

If \( f : X \to Y \) is a function and \( U \subseteq X \) we write \( f[U] \) for the direct image of \( U \) through \( f \). If \( V \subseteq Y \) we write \( f^{-1}[V] \) for the inverse image of \( V \) through \( f \). If \( g : Y \to Z \) is a function we write \( gf \) for the composition of \( g \) with \( f \). If \( R \subseteq X \times Y \) is a binary relation and \( x \in X \), then \( R(x) = \{ y \in Y : (x, y) \in R \} \). If \( T \subseteq Y \times Z \) is some other binary relation, then the composition of \( T \) with \( R \) is the set \( T \circ R = \{ (x, z) : \exists y \in Y(x, y) \in R \text{ and } (y, z) \in T \} \).

Let \( (X, \leq) \) be a poset. For each \( Y \subseteq X \), let \( [Y] = \{ x \in X : \exists y \in Y(y \leq x) \} \) and \( (Y) = \{ x \in X : \exists y \in Y(x \leq y) \} \). We will say that \( Y \) is an upset of \( X \) (a downset of \( X \)) if \( Y = [Y] \) (\( Y = (Y) \)). We write \( \mathcal{P}(X) \) for the set of all subsets of \( X \) and \( \text{Up}(X) \) for the set of all upsets of \( X \). The complement of \( Y \subseteq X \) will be denoted by \( Y^c \). A non-empty subset \( Z \subseteq X \) is called directed if for every \( x, y \in Z \), there exists \( z \in Z \) such that \( x \leq z \) and \( y \leq z \). Analogously, a non-empty \( Z \subseteq X \) is called dually directed if for every \( x, y \in Z \), there exists \( z \in Z \) such that \( z \leq x \) and \( z \leq y \). Two posets \( (X, \leq_X) \) and \( (Y, \leq_Y) \) are said to be dually isomorphic if there exists a onto function \( f : X \to Y \) such that\( x \leq_Y f(x) \) if and only if \( f(y) \leq_Y f(x) \). If the context is clear, the subscripts of the partial orders shall be omitted.

A meet-semilattice with greatest element, or semilattice for short, is an algebra \( A = (\langle A, \wedge \rangle, 1) \) of type \( (2, 0) \) such that the operation \( \wedge \) is idempotent, commutative, associative, and \( a \wedge 1 = a \) for all \( a \in A \). The partial order \( \leq \) on \( A \) is given by \( a \leq b \) if and only if \( a = a \wedge b \). For each poset \( (X, \leq) \), the structure \( \langle \text{Up}(X), \cap, X \rangle \) is a semilattice. If \( A \) is a semilattice, a subset \( F \subseteq A \) is a filter of \( A \) if it is an upset, \( 1 \in F \) and if \( a, b \in F \), then \( a \wedge b \in F \). The set of all filters of \( A \) will be denoted by \( \text{Fi}(A) \). The filter generated by the subset \( X \subseteq A \) will be denoted by \( F(X) \). If \( X = \{ a \} \), then \( F(\{ a \}) = \{ [a] \} \), or simply, \( [a] \). We say that a proper filter \( F \in \text{Fi}(A) \) is irreducible if for all \( F_1, F_2 \in \text{Fi}(A) \), if \( F = F_1 \cap F_2 \), then \( F = F_1 \) or \( F = F_2 \). We write \( \text{X}(A) \) the set of all irreducible filters of \( A \).

**Theorem 1** \((\cite{7})\) Let \( A \) be a semilattice. A proper filter \( F \) is irreducible if and only if for every \( a, b \notin F \), there exist \( c \notin F \) and \( f \in F \) such that \( a \wedge f \leq c \) and \( b \wedge f \leq c \).

A non-empty subset \( I \subseteq A \) is an order-ideal of \( A \) if it is a directed downset. We denote by \( \text{Id}(A) \) the set of all order-ideals of \( A \).

**Theorem 2** \((\cite{7})\) Let \( A \) be a semilattice. Let \( F \in \text{Fi}(A) \) and let \( I \in \text{Id}(A) \). If \( F \cap I = \emptyset \), then there exists \( P \in X(A) \) such that \( F \subseteq P \) and \( P \cap I = \emptyset \).

Let \( A \) and \( B \) be two semilattices. A map \( h : A \to B \) is a homomorphism if \( h(1) = 1 \) and \( h(a \wedge b) = h(a) \wedge h(b) \) for all \( a, b \in A \). If we consider the poset \( \langle X(A), \subseteq \rangle \) and the map \( \beta_A : A \to \text{Up}(X(A)) \) given by \( \beta_A(a) = \{ P \in X(A) : a \in P \} \), then it is proved that \( A \) is isomorphic to the subalgebra \( \beta_A[A] = \{ \beta_A(a) : a \in A \} \) of \( \langle \text{Up}(X(A)), \cap, X(A) \rangle \). Throughout the paper and to simplify notation, we will omit the subscript of \( \beta_A \) where appropriate.

By last, if \( \langle X, \tau \rangle \) is a topological space and \( Y \subseteq X \), we write \( \text{cl}(Y) \) for the topological closure of \( Y \). In particular, if \( Y = \{ y \} \), then \( \text{cl}(\{ y \}) = \text{cl}(y) \). It is known that every topological space can be endowed with a partial order \( \sqsubseteq \) defined by \( x \sqsubseteq y \) if and only if \( x \in \text{cl}(y) \). Such an order is called the specialization order of \( X \).

2.1 S-spaces

The authors in \cite{10} developed topological dualities for semilattices and lattices that are natural generalizations of the Stone duality for bounded distributive lattices through spectral spaces. To do so, they proposed the following definition.
Definition 1 Let $A$ be a semilattice and let $F \in \text{Fi}(A)$. A subset $I \subseteq A$ is an $F$-ideal if it is a downset and for every $a, b \in I$, there exist $c \in I$ and $f \in F$ such that $a \wedge f \leq c$ and $b \wedge f \leq c$.

It is clear that a proper filter $F$ is irreducible if and only if $F^c$ is an $F$-ideal. The following result generalizes Theorem 2.

Theorem 3 ([10]) Let $A$ be a semilattice. Let $F \in \text{Fi}(A)$ and let $I$ be an $F$-ideal. If $F \cap I = \emptyset$, then there exists $P \in X(A)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.

We stress that along this paper, by topological space we mean pairs $\langle X, K \rangle$ where $\langle X, \tau \rangle$ is a topological space and $K$ is a subbase for $\tau$ (i.e. $K \subseteq X$ and $X = \bigcup K$). We consider the following family of subsets of $X$:

$$S(X) = \{ U^c : U \in K \}.$$ 

Let $C_K(X)$ be the closure system on $X$ generated by $S(X)$, i.e., $C_K(X) = \{ \bigcap A : A \subseteq S(X) \}$. The elements of $C_K(X)$ are called subbasic closed subsets of $X$. Note that $S(X) \subseteq C_K(X) \subseteq C(X)$, where $C(X)$ is the collection of all closed subsets of the topology. A subset $Y \subseteq X$ is called saturated if it is an intersection of open sets.

Definition 2 Let $\langle X, K \rangle$ be a topological space. Let $Y \subseteq X$. A family $Z \subseteq S(X)$ is a $Y$-family if for every $A, B \in Z$, there exist $H, C \in S(X)$ such that $Y \subseteq H$, $C \in Z$, $A \cap H \subseteq C$ and $B \cap H \subseteq C$.

Definition 3 ([10]) An $S$-space is a topological space $\langle X, K \rangle$ satisfying the following conditions:

(S1) $\langle X, K \rangle$ is a $T_0$-space and $X = \bigcup K$.
(S2) $K$ is a subbase of compact open subsets, it is closed under finite unions and $\emptyset \in K$.
(S3) For every $U, V \in K$, if $x \in U \cap V$, then there exist $W, D \in K$ such that $x \notin W$, $x \in D$ and $D \subseteq (U \cap V) \cup W$.
(S4) If $Y \in C_K(X)$ and $\mathcal{J} \subseteq S(X)$ is a $Y$-family such that $Y \cap A^c \neq \emptyset$, for all $A \in \mathcal{J}$, then $Y \cap \bigcap \{ A^c : A \in \mathcal{J} \} \neq \emptyset$.

Let $\langle X, K \rangle$ be an $S$-space. Then the structure $S(X) = \langle S(X), \cap, X \rangle$ is a semilattice, called the dual semilattice of $\langle X, K \rangle$. Conversely, if $A$ is a semilattice, then $\langle X(A), \mathcal{K}_A \rangle$ is a topological space where $\mathcal{K}_A = \{ \beta(a)^c : a \in A \}$ is a subbase. It follows that $\langle X(A), \mathcal{K}_A \rangle$ is an $S$-space, called the dual $S$-space of $A$, and $\beta : A \rightarrow S(X(A))$ is an isomorphism between semilattices. If we consider the associated $S$-space $\langle X(S(X)), \mathcal{K}_{S(X)} \rangle$ of the dual semilattice $S(X)$, then the mapping $H_X : X \rightarrow X(S(X))$ given by $H_X(x) = \{ A \in S(X) : x \in A \}$ for all $x \in X$ is a homeomorphism between $S$-spaces.

Now, we mention the relationship existing between the subbasic closed subsets and the filters of the dual semilattice of certain topological spaces proved in [10].

Proposition 1 Let $\langle X, K \rangle$ be a topological space such that $K$ is a subbase of compact open subsets, and it is closed under finite unions and $\emptyset \in K$. Then:

1. For every $Y \in C_K(X)$, $F_Y = \{ A \in S(X) : Y \subseteq A \} \in \text{Fi}(S(X))$.
2. For every $F \in \text{Fi}(S(X))$, $Y_F = \bigcap F \in C_K(X)$.
3. $Y = Y_{F_Y}$ for all $Y \in C_K(X)$ and $F = F_{Y_F}$ for all $F \in \text{Fi}(S(X))$.

Moreover, the posets $\langle C_K(X), \subseteq \rangle$ and $\langle \text{Fi}(S(X)), \subseteq \rangle$ are dually isomorphic.
We conclude this part by making explicit Proposition 1 in terms of semilattices and their dual S-spaces. Such a restatement will be extensively used along the rest of the paper.

Let $A$ be a semilattice. Then the mapping $\varphi: \text{Fi}(A) \to C_{K_A}(X(A))$ defined by

$$\varphi(F) = \bigcap \{\beta(a): a \in F\} = \{P \in X(A): F \subseteq P\}$$

is a dual isomorphism, whose inverse $\psi: C_{K_A}(X(A)) \to \text{Fi}(A)$ is given by

$$\psi(Y) = \{a \in A: Y \subseteq \beta(a)\}.$$  

2.2 Canonical Extension

Following [17], we now recall the basic definitions of the theory of canonical extensions of semilattices. It is worth noticing that these definitions also coincide with the definition of canonical extension when restricted to bounded distributive lattices and Boolean algebras ([15, 16, 22, 23]).

**Definition 4** Let $A$ be a semilattice. A completion of $A$ is a pair $E = \langle E, e \rangle$ where $E$ is a complete lattice and the map $e: A \to E$ is an embedding.

Let $A$ be a semilattice and let $E$ be a completion of $A$. An element $x \in E$ is called closed if there is $F \in \text{Fi}(A)$ such that $x = \bigwedge e(F)$. An element $x \in E$ is called open if there is $I \in \text{Id}(A)$ such that $x = \bigvee e(I)$. Let us denote the collection of all closed elements of $E$ by $K(E)$ and the collection of all open elements of $E$ by $O(E)$. The completion $E$ is dense if $x = \bigwedge\{y \in O(E): x \leq y\}$ and $x = \bigvee\{y \in K(E): y \leq x\}$ for all $x \in E$. The completion $E$ is compact if for every non-empty dually directed subset $D$ of $A$ and every non-empty directed subset $U$ of $A$ such that $\bigwedge e(D) \leq \bigvee e(U)$, then there exist $x \in D$ and $y \in U$ such that $x \leq y$.

From now on we will suppress the embedding $e$ and we assume that $A$ is a subalgebra of a completion $E$ of $A$.

**Definition 5** Let $A$ be a semilattice. A completion $E$ of $A$ is a canonical extension if it is dense and compact.

The following result can be considered as a summary of Theorem 2.8, Proposition 2.9 and Corollary 2.11 of [17].

**Theorem 4** Let $A$ be a semilattice. Then $A$ has a canonical extension and it is unique up to isomorphism.

3 Canonical Extensions of Semilattices

The aim of this section is to exhibit a topological description of the canonical extension of a semilattice in terms of the duality obtained in [10]. As a result we obtain that the canonical extension of a semilattice is determined by a closure system generated by the set of subbasic saturated subsets of its dual space. Our motivations differ from those of [17] because we are interested in producing a duality for monontone semilattices. Further details of the relation between our construction and the Gouveia and Priestley construction will be provided in Sect. 3.1.
Definition 6 Let \( (X, \mathcal{K}) \) be an S-space. A subset \( Z \subseteq X \) is a \textit{subbasic saturated subset} if there exists a dually directed family \( \mathcal{L} \subseteq \mathcal{K} \) such that \( Z = \bigcap \{ W : W \in \mathcal{L} \} \).

We denote by \( \mathcal{Z}(X) \) the set of all subbasic saturated subsets of an S-space \( (X, \mathcal{K}) \). Note that every subbasic saturated subset is a saturated set.

Lemma 1 Let \( (X, \mathcal{K}) \) be an S-space. Then:

1. If \( \mathcal{L} \subseteq \mathcal{K} \) is a dually directed family, then \( \mathcal{A}_\mathcal{L} = \{ U^c : U \in \mathcal{L} \} \subseteq S(X) \) is a Y-family for all \( Y \in C_\mathcal{K}(X) \).
2. If \( \mathcal{A} \subseteq S(X) \) is a Y-family for all \( Y \in C_\mathcal{K}(X) \), then \( \mathcal{L}_\mathcal{A} = \{ U^c : U \in \mathcal{A} \} \subseteq \mathcal{K} \) is a dually directed family.

Moreover, the assignments \( \mathcal{L} \mapsto \mathcal{A}_\mathcal{L} \) and \( \mathcal{A} \mapsto \mathcal{L}_\mathcal{A} \) establish a bijection between the set of dually directed families of \( \mathcal{K} \) and the families of \( C_\mathcal{K}(X) \) which are a Y-family for all \( Y \in C_\mathcal{K}(X) \).

Proof 1. Let \( Y \in C_\mathcal{K}(X) \) and \( U, V \in \mathcal{L} \). Since \( \mathcal{L} \) is a dually directed family, there exists \( W \in \mathcal{L} \) such that \( W \subseteq U \cap V \). From \( X \in S(X) \), we get \( Y \subseteq X \), \( U^c \cap X = U^c \subseteq W^c \) and \( V^c \cap X = V^c \subseteq W^c \). So, \( \mathcal{A}_\mathcal{L} = \{ U^c : U \in \mathcal{L} \} \) is a Y-family.

2. Let \( \mathcal{A} \subseteq S(X) \) be a Y-family for all \( Y \in C_\mathcal{K}(X) \). As \( X \in C_\mathcal{K}(X) \), \( \mathcal{A} \) is a Y-family, therefore, from Definition 2, \( \mathcal{A} \) must be directed, so \( \mathcal{A} \neq \emptyset \) and consequently \( \mathcal{L}_\mathcal{A} \) must be non-empty too. Now, let \( U^c, V^c \in \mathcal{L}_\mathcal{A} \), thus \( U, V \in \mathcal{A} \). As \( \mathcal{A} \) is directed, there exists \( W \in \mathcal{A} \) such that \( U \subseteq W \) and \( V \subseteq W \). Then, there exists \( W^c \in \mathcal{L}_\mathcal{A} \) such that \( W^c \subseteq U^c \) and \( W^c \subseteq V^c \). Therefore, \( \mathcal{L}_\mathcal{A} = \{ U^c : U \in \mathcal{A} \} \subseteq \mathcal{K} \) is a dually directed family. For the moreover part, observe that \( \mathcal{L}_\mathcal{A}_\mathcal{L} = \mathcal{L} \) and \( \mathcal{A}_\mathcal{L}_\mathcal{A} = \mathcal{A} \).

Remark 1 Notice that in particular, if \( Z = \bigcap \{ W : W \in \mathcal{L} \} \in \mathcal{Z}(X) \) for a dually directed family \( \mathcal{L} \) of members of \( \mathcal{K} \), by Lemma 1 (1), the collection \( \mathcal{A}_\mathcal{L} = \{ W^c : W \in \mathcal{L} \} \) is a Y-family for all \( Y \in C_\mathcal{K}(X) \).

Proposition 2 Let \( (X, \mathcal{K}) \) be an S-space. The subbasic saturated subsets are compact saturated subsets of the topology.

Proof Let \( Z \in \mathcal{Z}(X) \). We prove that \( Z \) is compact. By the Alexander’s subbasis Theorem, we only need to prove that for every cover \( \{ O_i : i \in I \} \) of \( Z \) by elements of \( \mathcal{K} \), there is a finite subcover. By assumption, there exists a dually directed family \( \mathcal{L} \subseteq \mathcal{K} \) such that \( Z = \bigcap \{ W : W \in \mathcal{L} \} \). Since \( Z \subseteq \bigcup \{ O_i : i \in I \} \), we have
\[
\bigcap \{ W : W \in \mathcal{L} \} \subseteq \bigcap \{ O_i : i \in I \} = \emptyset.
\]

Let \( Y = \bigcap \{ O_i^c : i \in I \} \). Notice that it is the case that \( Z = \bigcap \{ (W^c)^c : W^c \in \mathcal{A}_\mathcal{L} \} \) and \( Y \in C_\mathcal{K}(X) \). Since by Remark 1, \( \mathcal{A}_\mathcal{L} \) is a Y-family, then by the contrapositive of (S4), there exists \( W^c \in \mathcal{A}_\mathcal{L} \) such that \( Y \cap (W^c)^c = \emptyset \). Thus there exists \( W \in \mathcal{L} \) such that \( Y \cap W = \emptyset \). Whence, \( W \subseteq \bigcup \{ O_i : i \in I \} \) and as \( W \) is a compact subset by assumption, there exist \( O_{i_1}, \ldots, O_{i_n} \) such that \( W \subseteq O_{i_1} \cup \ldots \cup O_{i_n} \). So, \( Z \subseteq W \subseteq O_{i_1} \cup \ldots \cup O_{i_n} \) and \( Z \) is compact, as desired.

\( \square \)

Let \( \mathbf{A} \) be a semilattice and let \( I \in \text{Id}(\mathbf{A}) \). We consider the following subset
\[
\alpha(I) = \bigcap \{ \beta(a)^c : a \in I \} = \{ P \in X(\mathbf{A}) : I \cap P = \emptyset \}.
\]
Then $\alpha(I)$ is a subbasic saturated subset of the dual $S$-space $\langle X(\mathbf{A}), K_\mathbf{A} \rangle$. Conversely, let $Z \subseteq X(\mathbf{A})$ be a subbasic saturated subset and consider

$$I_\mathbf{A}(Z) = \{ a \in A : \beta(a) \cap Z = \emptyset \}.$$  \hfill (4)

It is easy to see that $I_\mathbf{A}(Z)$ is a downset of $\mathbf{A}$.

**Proposition 3** Let $\langle X, K \rangle$ be an $S$-space and let $Z \subseteq X$ be a saturated set such that $Z = \bigcap \{ W : W \in \mathcal{J} \}$ for some family $\mathcal{J} \subseteq K$. Then $Z$ is a subbasic saturated subset if and only if the family $\{ U \in K : Z \subseteq U \}$ is dually directed.

**Proof** If $Z$ is a subbasic saturated subset, then there exists a dually directed family $\mathcal{L} \subseteq K$ such that $Z = \bigcap \{ W : W \in \mathcal{L} \}$. Let $S, T \in \{ U \in K : Z \subseteq U \}$. Thus, $Z \cap S^c = \emptyset$ and $Z \cap T^c = \emptyset$. Now, by applying the same argument we used in the Proof of Proposition 2, by the contrapositive of (S4), there exist $S', T' \in \mathcal{L}$ such that $S' \cap S^c = \emptyset$ and $T' \cap T^c = \emptyset$. As $\mathcal{L}$ is dually directed, there is $W \in \mathcal{L}$ such that $W \subseteq S'$ and $W \subseteq T'$, i.e., $W \subseteq S$ and $W \subseteq T$. So, $\{ U \in K : Z \subseteq U \}$ is dually directed.

Conversely, suppose that the family $\{ U \in K : Z \subseteq U \}$ is dually directed. Since $Z = \bigcap \{ W : W \in \mathcal{J} \}$ for some family $\mathcal{J} \subseteq K$, then $Z \subseteq \bigcap \{ U \in K : Z \subseteq U \} \subseteq \bigcap \{ W : W \in \mathcal{J} \} = Z$ and $Z$ is a subbasic saturated subset. \hfill $\square$

In the following result we see that order-ideals are in one to one correspondence with the subbasic saturated subsets.

**Theorem 5** Let $\mathbf{A}$ be a semilattice. Then the posets $\langle \text{Id}(\mathbf{A}), \subseteq \rangle$ and $\langle \mathcal{Z}(X(\mathbf{A})), \subseteq \rangle$ are dually isomorphic.

**Proof** Let $Z \in \mathcal{Z}(X(\mathbf{A}))$. We see that $I_\mathbf{A}(Z)$ is an order-ideal of $\mathbf{A}$ and $Z = \alpha(I_\mathbf{A}(Z))$. Moreover, if $I \in \text{Id}(\mathbf{A})$, then we prove that $I = I_\mathbf{A}(\alpha(I))$.

It is clear that $I_\mathbf{A}(Z)$ is a downset of $\mathbf{A}$. Let $a, b \in I_\mathbf{A}(Z)$. So, $\beta(a) \cap Z = \emptyset$ and $\beta(b) \cap Z = \emptyset$. Since $Z = \bigcap \{ \beta(x)^c : \beta(x)^c \in \mathcal{L} \}$ for some dually directed family $\mathcal{L} \subseteq K_\mathbf{A}$ and $\beta(a), \beta(b) \in C_{K_\mathbf{A}}(X(\mathbf{A}))$, then from Remark 1 \{ $\beta(x)^c : \beta(x)^c \in \mathcal{L} \}$ is a $\beta(a)$-family and a $\beta(b)$-family. Whence, by (S4) there exist $\beta(c)^c, \beta(d)^c \in \mathcal{L}$ such that $\beta(c)^c \cap \beta(a) = \emptyset$ and $\beta(d)^c \cap \beta(b) = \emptyset$. Thus, there is $\beta(e)^c \in \mathcal{L}$ such that $\beta(e)^c \subseteq \beta(c)^c, \beta(d)^c$. So $\beta(e)^c \cap \beta(a) = \emptyset, \beta(e)^c \cap \beta(b) = \emptyset$ and $Z \cap \beta(e) = \emptyset$, i.e., $a, b \leq c$ and $c \in I_\mathbf{A}(Z)$. Therefore, $I_\mathbf{A}(Z)$ is an order-ideal of $\mathbf{A}$ and

$$\alpha(I_\mathbf{A}(Z)) = \bigcap \{ \beta(a)^c : Z \subseteq \beta(a)^c \} \subseteq \bigcap \{ \beta(x)^c : \beta(x)^c \in \mathcal{L} \} = Z.$$  The other inclusion is immediate.

Let $I \in \text{Id}(\mathbf{A})$. If $b \in I_\mathbf{A}(\alpha(I))$, then $\beta(b) \cap \alpha(I) = \beta(b) \cap \bigcap \{ \beta(a)^c : a \in I \} = \emptyset$. Since $\beta(b) \in C_{K_\mathbf{A}}(X(\mathbf{A}))$ and the family $\{ \beta(a)^c : a \in I \}$ is dually directed, by Remark 1 and (S4) we get that there is $c \in I$ such that $\beta(b) \cap \beta(c)^c = \emptyset$, i.e., $\beta(b) \subseteq \beta(c)$. So, $b \leq c$ and as $I$ is a downset, we have $b \in I$. The other inclusion is straightforward.

Then we have a mapping $\alpha : \text{Id}(\mathbf{A}) \to \mathcal{Z}(X(\mathbf{A}))$ with inverse $I_\mathbf{A} : \mathcal{Z}(X(\mathbf{A})) \to \text{Id}(\mathbf{A})$. Let $I_1, I_2 \in \text{Id}(\mathbf{A})$. If $I_1 \subseteq I_2$ and $P \in \alpha(I_2)$, then $P \cap I_2 = \emptyset$. It follows that $P \cap I_1 = \emptyset$ and $P \in \alpha(I_1)$. Reciprocally, assume that $\alpha(I_1) \subseteq \alpha(I_2)$. Let $a \in I_2$ and suppose $a \notin I_1$. By Theorem 2 there exists $P \in X(\mathbf{A})$ such that $a \in P$ and $P \cap I_1 = \emptyset$. So, $P \in \alpha(I_1)$ and $P \notin \alpha(I_2)$ which is a contradiction. Then $a \in I_1$ and $I_2 \subseteq I_1$. Therefore, $\langle \text{Id}(\mathbf{A}), \subseteq \rangle$ and $\langle \mathcal{Z}(X(\mathbf{A})), \subseteq \rangle$ are dually isomorphic. \hfill $\square$

**Lemma 2** Let $\mathbf{A}$ be a semilattice. If $Y \in C_{K_\mathbf{A}}(X(\mathbf{A}))$ and $Z \in \mathcal{Z}(X(\mathbf{A}))$, then $\psi(Y) \cap I_\mathbf{A}(Z) \neq \emptyset$ if and only if $Y \cap Z = \emptyset$.  

$\square$
Proof Suppose $Y \cap Z = \emptyset$. Since $Z = \bigcap \{ \beta(a)^c : \beta(a)^c \in \mathcal{L} \}$ for some dually directed family $\mathcal{L} \subseteq \mathcal{K}_A$, by Remark 1 and (S4) there is $\beta(c)^c \in \mathcal{L}$ such that $Y \cap \beta(c)^c = \emptyset$. So, there exist $c \in A$ such that $c \in \psi(Y)$ and $c \in I_A(Z)$. Hence, $\psi(Y) \cap I_A(Z) \neq \emptyset$. The other implication is straightforward.

Remark 2 Observe that Lemma 2 can be rephrased in terms of order-ideals and filters of $A$ as follows: Let $A$ be a semilattice. If $F \in \text{Fi}(A)$ and $I \in \text{Id}(A)$, then $\varphi(F) \cap \alpha(I) = \emptyset$ if and only if $F \cap I \neq \emptyset$.

Let $A$ be a semilattice and let us consider the following closure operator $\Lambda : \text{Up}(X(A)) \to \text{Up}(X(A))$ given by

$$\Lambda(Y) = \bigcap \{ U^c : U \in \mathcal{Z}(X(A)) \text{ and } Y \subseteq U^c \}. \tag{5}$$

We write $E(X(A))$ for the closure system on $\text{Up}(X(A))$ generated by $\mathcal{Z}(X(A))$, i.e.:

$$E(X(A)) = \left\{ \bigcap \{ U^c : U \in \mathcal{B} \} : \mathcal{B} \subseteq \mathcal{Z}(X(A)) \right\}.$$

Notice that the structure $(E(X(A)), \lor, \cap, \emptyset, X(A))$ is a complete lattice, where the join of an arbitrary family $\{ C_i : i \in I \}$ of $E(X(A))$, $\lor \{ C_i : i \in I \}$ is given by

$$\bigcap \left\{ G \in E(X(A)) : \bigcup \{ C_i : i \in I \} \subseteq G \right\} = \Lambda \left( \bigcup \{ C_i : i \in I \} \right). \tag{6}$$

Lemma 3 Let $A$ be a semilattice. Then:

1. $\beta(a) \in E(X(A))$ for all $a \in A$ and $C_K(X(A)) \subseteq E(X(A))$.
2. If $a, b \in A$ such that $a \lor b$ exists, then $\beta(a \lor b) = \beta(a) \lor \beta(b)$.
3. Let $D \subseteq A$ be a directed subset. Then

$$\lor \{ \beta(a) : a \in D \} = \bigcup \{ \beta(a) : a \in D \}.$$  

In particular, if $U \in \mathcal{Z}(X(A))$, we have

$$U^c = \bigcup \{ \beta(a) : a \in I_A(U) \} = \lor \{ \beta(a) : a \in I_A(U) \}.$$  

Proof 1. It follows from the fact that $\beta(a)^c \in \mathcal{Z}(X(A))$ for all $a \in A$.

2. Let $a, b \in A$ be such that $a \lor b$ exists. It is clear that $\beta(a) \lor \beta(b) \subseteq \beta(a \lor b)$. On the other hand, let $P \in \beta(a \lor b)$ and $U \in \mathcal{Z}(X(A))$ be such that $\beta(a) \cup \beta(b) \subseteq U^c$. Then $\beta(a) \cap U = \emptyset$ and $\beta(b) \cap U = \emptyset$. So, $a, b \in I_A(U)$ and $a \lor b \in I_A(U)$. Thus, $P \cap I_A(U) \neq \emptyset$ and we get $P \notin U$, i.e., $P \subseteq U^c$. Therefore, $P \in \beta(a) \lor \beta(b)$. Then $P \cap U = \emptyset$ and $U \cap I_A(U) = \emptyset$. Thus, $P \neq \emptyset$ and $P \cap I_A(U) = \emptyset$. Therefore, $P \in \beta(a) \lor \beta(b)$.

3. Let $D \subseteq A$ be a directed subset. Then $\lor \{ \beta(a) : a \in D \} \subseteq \lor \{ \beta(a) : a \in D \}$. Since $\beta(a) \subseteq \alpha((D)^c)$ for all $a \in D$, then $\lor \{ \beta(a) : a \in D \} \subseteq \alpha((D)^c)$. Thus, $P \notin \alpha((D)^c)$ and $P \cap (D) \neq \emptyset$. It follows that there is $a \in D$ such that $a \in P$ which implies $P \subseteq \lor \{ \beta(a) : a \in D \}$. 

Now we are ready to prove the main result of this section.

Theorem 6 Let $A$ be a semilattice. Then the pair $(E(X(A)), \beta)$ is a canonical extension of $A$.  

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Proof From Lemma 3, $\beta(a) \in E(X(A))$ for all $a \in A$ and $\beta$ is an embedding.

In order to prove that $E(X(A))$ is dense, let $V \in E(X(A))$. Then there exists $B \subseteq Z(X(A))$ such that $V = \bigcap \{ U^c : U \in B \}$. From Theorem 5 (3), it follows that
\[ U^c = \left( \bigcap \{ \beta(a)^c : a \in I_A(U) \} \right)^c = \bigcup \beta[I_A(U)] = \bigvee \beta[I_A(U)] \]
and
\[ V = \bigcap \{ \bigvee \beta[I_A(U)] : U \in B \}. \]

On the other hand, since $V \in \text{Up}(X(A))$, then $V = \bigcup \{ \phi(P) : P \in V \}$. So from (1) it follows that $V = \bigcup \{ \phi(P) : P \in V \}$. Since $E(X(A))$ is complete, we get $V = \bigcap \{ G \in E(X(A)) : V \subseteq G \}$. Thus from the latter description of $V$ and (6) we can conclude that
\[ V = \bigcup \{ \phi(P) : P \in V \} = \bigvee \{ \bigcap \beta[P] : P \in V \}. \]

Finally, we prove that $E(X(A))$ is compact. Let $F \in \text{Fi}(A)$ and $I \in \text{Id}(A)$ be such that $\bigcap \beta[F] \subseteq \bigcup \beta[I]$. Then $\bigcap \beta[F] \cap \left( \bigcup \beta[I] \right)^c = \emptyset$ so, from (1) and (3) we obtain $\phi(F) \cap \alpha(I) = \emptyset$. Hence, by Remark 2, we have $F \cap I \neq \emptyset$. Therefore, there exists $a \in A$ such that $\beta(a) \in \beta[F]$ and $\beta(a) \in \beta[I]$. The result follows. \(\square\)

As usual, we write $A^\delta$ to denote the canonical extension of a semilattice $A$.

Remark 3 According to Theorem 6 and (5) we may say that the construction of $A^\delta$ of a semilattice $A$, which we have presented in this section, is in fact the semilattice of $\text{Up}(X(A))$ of closed elements of $A$. I.e. elements $Y \in \text{Up}(X(A))$ such that $A(Y) = Y$.

Remark 4 Let $A$ be a semilattice and $\langle X(A), K_A \rangle$ its dual $S$-space. As an immediate consequence of Theorem 5 and Proposition 3.2 of [10], we have
\[ K(A^\delta) = C_{K_A}(X(A)) \quad \text{and} \quad O(A^\delta) = \{ Z^c : Z \in Z(X(A)) \}, \]
i.e., the closed elements of the canonical extension of $A$ are exactly the subbasic closed subsets of $\langle X(A), K_A \rangle$ and the open elements of the canonical extension of $A$ are exactly the complements of the subbasic saturated subsets of $\langle X(A), K_A \rangle$.

3.1 Connection with the Construction of Gouveia and Priestley

Canonical extensions of semilattices have been studied in detail by Gouveia and Priestley in [17]. This work is connected with the topological construction of canonical extensions made by Moshier and Jipsen [24]. Their approach places strong emphasis on the iterated filter completion and the canonical extension: they consider a semilattice $A$ and the lattice of filters of filters of $A$, denoted by $\text{Fi}^2(A) = \text{Fi}(\text{Fi}(A))$. The family of sets $\text{Fi}^2(A)$ is an algebraic closure system and forms an algebraic lattice in which arbitrary meet and directed join are given, respectively, by intersection and directed union. The join of an arbitrary family of elements $\{ F_i : i \in I \}$ is given by
\[ \bigcap \{ G \in \text{Fi}^2(A) : \bigcup \{ F_i : i \in I \} \subseteq G \}. \]

In [17], the authors define an order embedding $e : A \rightarrow \text{Fi}^2(A)$ given by $e(a) = \{ F \in \text{Fi}(A) : a \in F \}$ for all $a \in A$ and the pair $(\text{Fi}^2(A), e)$ is a completion of $A$. Since $\text{Fi}^2(A)$ may not be a canonical extension of $A$, to construct a canonical extension the authors consider
the subset $C$ of $\text{Fi}^2(A)$ consisting of those of its elements which are meets of directed joins from $e(A)$, i.e.,

$$C = \left\{ \bigcap \bigcup e(I_j): \{I_j: j \in J\} \subseteq \text{Id}(A) \right\}.$$ 

Since the canonical extension is unique up to an isomorphism that fixes $A$, there is an isomorphism between the canonical extension given in [17] and the one we have developed in this work from the topological duality introduced in [10]. We now proceed to the details. We know by Theorem 6 that $\langle E(X(A)), \beta \rangle$ is a canonical extension for $A$, where

$$E(X(A)) = \left\{ \bigcap \{U^c: U \in \mathcal{B}\}: \mathcal{B} \subseteq \mathcal{Z}(X(A)) \right\}$$

and $\beta: A \to E(X(A))$ is given by $\beta(a) = \{P \in X(A): a \in P\}$. So, if we define the mapping $i: C \to E(X(A))$ by

$$i(X) = \{P \in X(A): P \in X\} = X \cap X(A)$$

then it is easy to prove that $i(X) = \bigcap \{\alpha(I)^c: I \in \text{Id}(A)\}$ and $X \subseteq \bigcup e(I)$ and $i(e(a)) = \beta(a)$. We get that $i$ is an isomorphism of lattices that fixes $A$ with inverse $j: E(X(A)) \to C$ given by

$$j(Y) = \bigcap \left\{ \bigcup e(I): I \in \text{Id}(A) \text{ and } Y \subseteq \alpha(I)^c \right\}.$$ 

It is also worth noting that $i(\bigcup e(I)) = \alpha(I)^c$ for all $I \in \text{Id}(A)$ and $i(\bigcap e(F)) = \varphi(F) = Y_F$ for all $F \in \text{Fi}(A)$.

### 3.2 Extension of Monotone Maps

In this section we study canonical extensions of monotone maps of semilattices by using Theorem 6. We start by recalling some results about canonical extensions of posets. For further details the reader may consult [13].

Let $P$ and $Q$ be two posets and $f: P \to Q$ be an order preserving map. Let $P^\delta$ and $Q^\delta$ be the canonical extensions of $P$ and $Q$, respectively. We define the maps $f^\sigma, f^\pi: P^\delta \to Q^\delta$ by

$$f^\sigma(u) = \bigvee \left\{ \bigwedge \{f(p): x \leq p \in P\}: u \geq x \in K(P^\delta) \right\},$$  

$$f^\pi(u) = \bigwedge \left\{ \bigvee \{f(p): y \geq p \in P\}: u \leq y \in O(P^\delta) \right\}. $$

For every order preserving map $f: P \to Q$, both $f^\sigma$ and $f^\pi$ are order preserving extensions of $f$. In addition, $f^\sigma \leq f^\pi$ with equality holding on both $K(P^\delta)$ and $O(P^\delta)$. For every $u \in P^\delta, x \in K(P^\delta)$ and $y \in O(P^\delta)$ we have

$$f^\sigma(x) = \bigwedge \{f(p): x \leq p \in P\},$$

$$f^\sigma(u) = \bigvee \{f^\sigma(x): u \geq x \in K(P^\delta)\},$$

$$f^\pi(y) = \bigvee \{f(p): y \geq p \in P\},$$

$$f^\pi(u) = \bigwedge \{f^\pi(y): u \leq y \in O(P^\delta)\}.$$ 

Also, $f^\sigma$ and $f^\pi$ map closed elements to closed elements and open elements to open elements. In general the extensions $f^\sigma$ and $f^\pi$ do not coincide.
Since every semilattice is a poset, we can borrow the properties described before to this context. The following result is a straightforward consequence of Lemma 3.4 of [13], Theorem 6 and Remark 4.

**Lemma 4** Let $A$ and $B$ be two semilattices and $f : A \rightarrow B$ be an order preserving map. Consider the mappings $f^\sigma, f^\pi : A^\delta \rightarrow B^\delta$ defined in (7) and (8). Then for every $a \in A$, $f^\sigma(\beta_A(a)) = f^\pi(\beta_A(a)) = \beta_B(f(a))$. Moreover, $f^\sigma \preceq f^\pi$ with the equality holding in $K(A^\delta) \cup O(A^\delta)$. For every $Y \in C_{K_A}(X(A))$, $Z \in \mathcal{Z}(X(A))$ and $V \in E(X(A))$ the following hold:

1. $f^\sigma(Y) = \bigcap \{ \beta_B(f(a)) : a \in A \text{ and } Y \subseteq \beta_A(a) \}$,
2. $f^\pi(V) = \bigvee \{ f^\sigma(Y) : V \supseteq C_{K_A}(X(A)) \}$,
3. $f^\pi(Z^c) = \bigvee \{ \beta_B(f(a)) : a \in A \text{ and } \beta_A(a) \subseteq Z^c \}$,
4. $f^\pi(V) = \bigcap \{ f^\pi(Z^c) : V^c \supseteq Z \in \mathcal{Z}(X(A)) \}.

**Proof** It follows from the fact that $I_A(Z)$ is directed and both $f$ and $\beta_B$ are monotone. □

**Lemma 5** Let $A$ and $B$ be two semilattices and $f : A \rightarrow B$ be an order preserving map. Then for every $Z \in \mathcal{Z}(X(A))$ the subset $\{ \beta_B(f(a)) : a \in I_A(Z) \}$ is directed.

**Proof** Let $Z \in \mathcal{Z}(X(A))$. Notice that from (4) and Lemmas 4, 5 and 3(3), we can write

$$ f^\pi(Z^c) = \bigvee \{ \beta_B(f(a)) : a \in I_A(Z) \} = \bigcup \{ \beta_B(f(a)) : a \in I_A(Z) \}. $$

Now take $P \in X(A)$. Straightforward calculations show that $P \in f^\pi(Z^c)$ if and only if $f^{-1}[P] \cap I_A(Z) \neq \emptyset$. This concludes the proof. □

Let $A$ and $B$ be two semilattices and $f : A \rightarrow B$ be an order preserving map. We define $R_f \subseteq X(B) \times \mathcal{Z}(X(A))$ by

$$ (P, Z) \in R_f \iff f^{-1}[P] \cap I_A(Z) = \emptyset. $$

**Proposition 4** Let $A$ and $B$ be two semilattices and $f : A \rightarrow B$ be an order preserving map. Then for every $V \in E(X(A))$ we have

$$ f^\pi(V) = \{ P \in X(B) : \forall Z \in R_f(P)[Z \cap V \neq \emptyset] \}. $$

**Proof** Let $V \in E(X(A))$. In order to prove the first inclusion, let $P \in f^\pi(V)$ and $Z \in R_f(P)$. If $Z \cap V = \emptyset$, then $Z \subseteq V^c$. By Lemma 4, it follows that $P \in f^\pi(Z^c)$. So, by Lemma 6, $f^{-1}[P] \cap I_A(Z) \neq \emptyset$ which contradicts our assumption. Hence, $Z \cap V \neq \emptyset$. For the remaining inclusion, let $P \in X(B)$ be such that for every $Z \in R_f(P)$ we have $Z \cap V \neq \emptyset$. Let $Z \in \mathcal{Z}(X(A))$ be such that $Z \subseteq V^c$. We prove that $f^{-1}[P] \cap I_A(Z) \neq \emptyset$. If we assume the contrary, then $Z \in R_f(P)$ and $Z \cap V \neq \emptyset$ which gives us a contradiction. Therefore, $P \in f^\pi(V)$. □

Finally, we analyze $f^\sigma$. Note that for every $P \in X(B)$ and every $Y \in C_{K_A}(X(A))$ we have $P \in f^\sigma(Y)$ if and only if $\psi(Y) \subseteq f^{-1}[P]$ where $\psi(Y) = \{ a \in A : Y \subseteq \beta_A(a) \}$. So, we can define the relation $G_f \subseteq X(B) \times C_{K_A}(X(A))$ by

$$ (P, Y) \in G_f \iff \psi(Y) \subseteq f^{-1}[P]. $$

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Proposition 5 \ Let \( A \) and \( B \) be two semilattices and \( f : A \to B \) be an order preserving map and let \( \Lambda \) be the closure operator defined in (5). Then for every \( V \in E(X(A)) \) we have
\[
f^\sigma(V) = \Lambda \left( \bigcup \{ G_f^{-1}(Y) : V \supseteq Y \in C_{\mathcal{K}_A}(X(A)) \} \right).
\]

Proof \ Let \( V \in E(X(A)) \) and \( P \in f^\sigma(P) \). Since \( f^\sigma(Y) = G_f^{-1}(Y) \) for all \( Y \in C_{\mathcal{K}_A}(X(A)) \), we have
\[
\bigcup \{ f^\sigma(Y) : V \supseteq Y \in C_{\mathcal{K}_A}(X(A)) \} = \bigcup \{ G_f^{-1}(Y) : V \supseteq Y \in C_{\mathcal{K}_A}(X(A)) \}
\]
so the result follows. \( \square \)

The reader may notice that the topological description of \( f^\sigma \), obtained in Proposition 5, it may not be quite easy to handle. This is not the case when considering monotone distributive semilattices (see [11] for details). Motivated by this fact, in the next section we will develop our duality by means of the topological description of \( f^\pi \) obtained in Proposition 4.

4 Topological Duality

As we saw along Sect. 3, the topological description of canonical extensions of semilattices can be used to obtain topological descriptions of the canonical extensions of monotone maps of semilattices, namely, the extensions \( \pi \) and \( \sigma \). In this section we take advantage of these results in order to define a suitable category of multirelational topological spaces which will serve as the categorical dual of the category of monotone semilattices. The construction of such multirelations strongly relies in a modeling process raising from the \( f^\pi \) extension of a monotone function \( f \) between semilattices.

Our motivation for employing multirelations instead of binary relations in our topological spaces relies on the fact that the latter provides a more expressive framework to model monotone operators. This was proved in [8]. In this paper, it was shown that a Boolean algebra with a monotone operator (in the case of a unary operator) is an operator that preserves supreme if and only if the associated frame is point-wise equivalent to a Kripke frame. With that result, it is clear that if binary relations are used, then this forces the operator to be a normal operator. Therefore, since in our work we deal with monotone operators which are not necessarily normal, we must use multirelations to model them suitably.

Definition 7 \ Let \( A \) be a semilattice. A map \( m : A \to A \) is said to be monotone if it is an order preserving map, i.e, it satisfies the following condition: if \( a \leq b \), then \( ma \leq mb \) for all \( a \in A \). Then a monotone semilattice is a pair \( \langle A, m \rangle \) where \( A \) is a semilattice and \( m \) is a monotone operation on \( A \).

Remark 5 \ It is immediate that for every semilattice \( A \) a map \( m : A \to A \) is monotone if and only if \( m(a \land b) \leq ma \land mb \) for all \( a, b \in A \).

It is clear from Remark 5 that monotone semilattices form a variety. We write \( \mathcal{MS} \) for the variety of monotone semilattices. Let \( \langle A, m \rangle \) and \( \langle B, n \rangle \) be two monotone semilattices. We say that a homomorphism \( h : A \to B \) is a monotone homomorphism if \( h(ma) = nh(a) \) for all \( a \in A \). We write \( \text{mMS} \) for the category of monotone semilattices and monotone homomorphisms.

A multirelation on set \( X \) is a subset of the cartesian product \( X \times \mathcal{P}(X) \), that is, a set of ordered pairs \( (x, Y) \) where \( x \in X \) and \( Y \subseteq X \) ([14, 26]). Recall that in classical monotone
modal logic a neighborhood frame is a pair \( \langle X, R \rangle \) where \( X \) is a set and \( R \subseteq X \times \mathcal{P}(X) \), i.e., \( R \) is a multirelation ([12, 19]).

We will represent a monotone operator \( m \) on a semilattice \( \mathbf{A} \) as a multirelation on the dual space of \( \mathbf{A} \), where the canonical extension offers an advantageous point of view. To do so, we will apply the results we obtained in Sect. 3.2.

From Lemmas 3, 4 and 5 we obtain a topological description of the monotone operators on a semilattice.

**Lemma 7** Let \( \langle \mathbf{A}, m \rangle \in \mathcal{MS} \). Then \( \langle \mathbf{A}^\#, m^\# \rangle \in \mathcal{MS} \) and for every \( a \in A \), \( m^\# (\beta(a)) = \beta(m(a)) \). Moreover, for every \( Z \in \mathcal{F}(\mathbf{A}) \) and \( V \in E(\mathbf{A}) \) the following hold:

1. \( m^\# (Z^c) = \bigcup \{ \beta(\mathbf{a}) : Z \subseteq \beta(\mathbf{a})^c \} \).
2. \( m^\# (V) = \bigcap \{ m^\# (Z^c) : V^c \supseteq Z \in \mathcal{F}(\mathbf{A}) \} \).

Let \( \langle \mathbf{A}, m \rangle \in \mathcal{MS} \). We consider the multirelation \( R_m \subseteq X(\mathbf{A}) \times \mathcal{F}(\mathbf{A}) \) given by

\[
(P, Z) \in R_m \iff m^{-1}[P] \cap I_A(Z) = \emptyset.
\]

As a particular case of Proposition 4 we obtain the following result.

**Proposition 6** Let \( \langle \mathbf{A}, m \rangle \in \mathcal{MS} \). Then for every \( V \in E(\mathbf{A}) \) we have

\[
m^\# (V) = \{ P \in X(\mathbf{A}) : \forall Z \in R_m(P)[Z \cap V \neq \emptyset] \}.
\]

Let \( \langle X, \mathcal{K} \rangle \) be an \( S \)-space. For each \( U \in S(X) \) we define the subset \( L_U \) of \( \mathcal{F}(X) \) as follows:

\[
L_U = \{ Z \in \mathcal{F}(X) : Z \cap U \neq \emptyset \}.
\]

**Definition 8** An \( mS \)-space is a structure \( \langle X, \mathcal{K}, R \rangle \) where \( \langle X, \mathcal{K} \rangle \) is an \( S \)-space and \( R \subseteq X \times \mathcal{F}(X) \) a multirelation such that:

1. \( m_R(U) = \{ x \in X : \forall Z \in R(x)[Z \cap U \neq \emptyset] \} \in S(X) \) for all \( U \in S(X) \),
2. \( R(x) = \bigcap \{ L_U : U \in S(X) \text{ and } x \in m_R(U) \} \) for all \( x \in X \).

Notice that condition 1. of Definition 8 can be restated as

\[
m_R(U) = \{ x \in X : R(x) \subseteq L_U \} \in S(X)
\]

for all \( U \in S(X) \). Furthermore, this guarantees that for every \( mS \)-space \( \langle X, \mathcal{K}, R \rangle \), the map \( m_R : S(X) \rightarrow S(X) \) is well defined and it is monotone.

Let \( \mathbf{Rel} \) be the category of sets and binary relations and \( \mathbf{Set} \) be the category of sets and functions. It is well known that there exists a faithful functor \( \mathcal{F} : \mathbf{Rel}^{op} \rightarrow \mathbf{Set} \) defined for every set \( X \) by \( \mathcal{F}(X) = \mathcal{P}(X) \), and for every binary relation \( T \subseteq X \times Y \) and every \( U \in \mathcal{P}(Y) \), \( \mathcal{F}(T) \) is defined by

\[
\mathcal{F}(T)(U) = \{ x \in X : T(x) \subseteq U \}.
\]

A meet-relation between two \( S \)-spaces \( \langle X_1, \mathcal{K}_1 \rangle \) and \( \langle X_2, \mathcal{K}_2 \rangle \) was defined in [10] as a relation \( T \subseteq X_1 \times X_2 \) satisfying the following conditions:

1. \( \mathcal{F}(T)(U) \subseteq S(X_1) \) for all \( U \in S(X_2) \),
2. \( T(x) = \bigcap \{ U \in S(X_2) : T(x) \subseteq U \} \) for all \( x \in X_1 \).
Let \( (X_i, \mathcal{K}_i) \), with \( i = 1, 2, 3 \) be \( S \)-spaces and \( R \subseteq X_1 \times X_2 \) and \( T \subseteq X_2 \times X_3 \) be meet-relations. Consider now the following relation
\[ T \ast R = \{(x, z) \in X_1 \times X_3 : \forall U \in S(X_3) [(T \circ R)(x) \subseteq U \Rightarrow z \in U]\}. \]

In [9, 10] it was proved that \( T \ast R \) is a meet-relation, \( \ast \) is associative and if \( \supseteq \) is the dual of the specialization order of \( X_2 \), then \( T \ast \supseteq = T \) and \( \supseteq \ast R = R \). Observe that the latter can be rephrased by means that \( S \)-spaces and meet-relations form a category in which the composition is given by \( \ast \) and the identity arrow is given by the dual of the specialization order. We write \( m\text{Rel} \) for such a category. Let \( MS \) be the category of semilattices and homomorphisms. Let us assign to each \( S \)-space \( \langle X, \mathcal{K} \rangle \) the semilattice \( S(X) \) and to each meet-relation \( T \) the map \( F(T) \). In Proposition 3.16 of [10] it was proved that \( F(T \ast R) = F(T \circ R) \) and it is no hard to see that \( F(\supseteq) \) is the identity map of \( S(X) \). Moreover, from the definition of meet-relation, \( F(T) \) and Proposition 3.19 of [10], it follows that these assignments define a functor \( \Box : m\text{Rel}\text{op} \to MS \) which is faithful by Lemma 3.21 of [10]. We stress that this is an important operational advantage taking into account that \( (T \ast R)(x) = \text{cl}_{X_3}(\{T \circ S\}(x)) \) for all \( x \in X_1 \). In the literature ([9–11]) \( \Box(T) \) is denoted by \( \Box_T \). So for the sake of readability we will keep such a notation along this paper.

**Definition 9** Let \( \langle X_1, \mathcal{K}_1, R_1 \rangle \) and \( \langle X_2, \mathcal{K}_2, R_2 \rangle \) be two \( mS \)-spaces. A meet-relation \( T \subseteq X_1 \times X_2 \) is a monotone meet-relation if the following diagram commutes:

\[
\begin{array}{ccc}
S(X_2) & \xrightarrow{\Box_T} & S(X_1) \\
m_{R_2} \downarrow & & \downarrow m_{R_1} \\
S(X_2) & \xrightarrow{\Box_T} & S(X_1)
\end{array}
\]

The following result is a topological characterization of monotone meet-relations. Its proof is essentially the same as that of Lemma 44 in [11] so we will omit it.

**Proposition 7** Let \( \langle X_1, \mathcal{K}_1, R_1 \rangle \) and \( \langle X_2, \mathcal{K}_2, R_2 \rangle \) be two \( mS \)-spaces. A meet-relation \( T \subseteq X_1 \times X_2 \) is monotone if and only if for every \( x \in X_1 \) and every \( U \in S(X_2) \) we have
\[
U^c \in R_2[T(x)] \iff T^{-1}[U^c] \in R_1(x)
\]
where \( R_2[T(x)] = \{Z \in Z(X_2) : \exists y \in T(x)[(y, Z) \in R_2]\} \).

**Proposition 8** Let \( \langle X_1, \mathcal{K}_1, R_1 \rangle \), \( \langle X_2, \mathcal{K}_2, R_2 \rangle \) and \( \langle X_2, \mathcal{K}_3, R_3 \rangle \) be \( mS \)-spaces. Let \( L \subseteq X_1 \times X_2 \) and \( T \subseteq X_2 \times X_3 \) be monotone meet-relations. Then \( T \ast L \) is a monotone meet-relation. Moreover, \( \ast \) is associative.

**Proof** Consider the diagram

\[
\begin{array}{ccc}
S(X_3) & \xrightarrow{\Box_T} & S(X_2) & \xrightarrow{\Box_T} & S(X_1) \\
m_{R_3} \downarrow & & \downarrow m_{R_2} & & \downarrow m_{R_1} \\
S(X_3) & \xrightarrow{\Box_T} & S(X_2) & \xrightarrow{\Box_T} & S(X_1)
\end{array}
\]

Since \( \Box \) is contravariant and each of the squares is commutative by assumption, then from the following calculation
\[
m_{R_1}(\Box_T m_{R_3}) = m_{R_1}(\Box_L \Box_T) = (\Box_L m_{R_2}) \Box_T = \Box_L (\Box_T m_{R_3}) = \Box_T m_{R_3}
\]

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we obtain that $T \ast L$ is a meet-relation. The proof of the associativity is analogue. \hfill \Box

**Proposition 9** Let $(X, \mathcal{K}, R)$ be an $mS$-space. The dual of the specialization order $\square \subseteq X \times X$ is a monotone meet-relation.

**Proof** Recall that from the functoriality of $\square$ it follows that $\square \supseteq$ is in fact the identity map of $S(X)$. So, it is clear that $\square$ satisfies Definition 9. \hfill \Box

By Propositions 8 and 9 we get that $mS$-spaces and monotone meet-relations form a category in which the identity map is given by the dual of the specialization order. We write $mSp$ for such a category.

**Lemma 8** Let $(X_1, \mathcal{K}_1, R_1)$ and $(X_2, \mathcal{K}_2, R_2)$ be two $mS$-spaces and $f : X_1 \to X_2$ be a bijective map such that

1. $\mathcal{K}_2 = \{ f[V] \mid V \in \mathcal{K}_1 \}$,
2. $(x, Z) \in R_1$ if and only if $(f(x), f[Z]) \in R_2$ for all $x \in X_1$ and all $Z \in \mathcal{Z}(X_1)$.

Then $f$ induces an isomorphism of $mS$-spaces.

**Proof** Since $f$ is bijective, we can consider the relations $R_f \subseteq X_1 \times X_2$ and $T_f \subseteq X_2 \times X_1$ which are given by

$$(x, f(y)) \in R_f \iff f(x) \supseteq_2 f(y)$$

and

$$(f(y), x) \in T_f \iff f(y) \supseteq_2 f(x).$$

We prove that $R_f$ and $T_f$ are monotone-meet relations. We only exhibit the proof for the case of $R_f$ because the proof for $T_f$ is similar. Since $f$ is bijective and from 1. we obtain

$$\mathcal{Z}(X_2) = \{ f[Z] \mid Z \in \mathcal{Z}(X_1) \}, \quad S(X_2) = \{ f[W] \mid W \in S(X_1) \} \quad \text{and} \quad \square_{R_f}[f[U]] = U,$$

for every $U \in S(X_1)$. Whence, it follows that for every $x, y \in X_1$, $f(x) \supseteq_2 f(y)$ if and only if $x \supseteq_1 y$. Notice that from these facts it is straightforward to see that $R_f$ is a meet-relation. In order to proof that $R_f$ is monotone we need to check that the following diagram commutes:

$$
\begin{array}{ccc}
S(X_2) & \xrightarrow{\square_{R_f}} & S(X_1) \\
m_{R_2} \downarrow & & \downarrow m_{R_1} \\
S(X_2) & \xrightarrow{\square_{R_f}} & S(X_1)
\end{array}
$$

Now we prove that $m_{R_1}(U) = \square_{R_f}(m_{R_2}(f[U]))$ for all $U \in S(X_1)$. Let $x \in \square_{R_f}(m_{R_2}(f[U]))$, then $R_f(x) \subseteq m_{R_2}(f[U])$. Thus, for every $y \in X_1$ such that $f(x) \supseteq_2 f(y)$ we have $f(y) \in m_{R_2}(f[U])$. Recall that from (10), $f(y) \in m_{R_2}(f[U])$ if and only if $R_2(f(y)) \subseteq L_f[U]$, so from (9), for every $Z \in \mathcal{Z}(X_1)$ such that $(f(y), f[Z]) \in K_2$, we have $f[Z] \cap f[U] \neq \emptyset$. Hence, from 2. and since $f$ is bijective by assumption, we get $y \in m_{R_1}(U)$. I.e., for every $y \in X_1$, if $x \supseteq_1 y$, then $y \in m_{R_1}(U)$. But this is equivalent to say that $x \in m_{R_1}(U)$. Thus, $\square_{R_f}(m_{R_2}(f[U])) \subseteq m_{R_1}(U)$. The proof of the reverse inclusion is analogous.

Finally, we prove that $T_f$ is the inverse of $R_f$ in $mSp$. It is clear from the definitions of $R_f$ and $T_f$ that $R_f \circ T_f = \supseteq_2$ and $T_f \circ R_f = \supseteq_1$. Then, since $\supseteq_2$ is a meet-relation, for every $y \in X_1$ we have

$$(R_f \ast T_f)(f(y)) = \text{cl}_{K_2}((R_f \circ T_f)(f(y))) = \text{cl}_{K_2}(\supseteq_2 (f(y))) = \supseteq_2 (f(y)).$$
Therefore, \( R_f \ast T_f = \exists 2 \). The proof for \( T_f \ast R_f = \exists 1 \) is similar. \( \square \)

**Corollary 1** Let \( \langle X_1, K_1, R_1 \rangle \) be an \( mS \)-space and \( \langle X_2, K_2 \rangle \) be an \( S \)-space. If \( f : X_1 \to X_2 \) is a bijective map such that \( K_2 = \{ f[V] : V \in K_1 \} \), then there exists a multirelation \( R_2 \subseteq X_2 \times \mathcal{Z}(X_2) \) such that \( \langle X_1, K_1, R_1 \rangle \) and \( \langle X_2, K_2, R_2 \rangle \) are isomorphic \( mS \)-spaces.

**Proof** We start by noticing that from the hypothesis on \( f \) it is immediate that \( f[U] \in S(X_2) \) for all \( U \in S(X_1) \) and \( f[Z] \in \mathcal{Z}(X_2) \) for all \( Z \in \mathcal{Z}(X_1) \). Therefore, it easily follows that \( L_{f[U]} = f[L_U] \) for all \( U \in S(X_1) \). In addition, it is also the case that for every \( V \in S(X_2) \) there exists a unique \( U \in S(X_1) \) such that \( V = f[U] \). We define \( R_2 \subseteq X_2 \times \mathcal{Z}(X_2) \) as

\[
(f(x), W) \in R_2 \iff (x, f^{-1}[W]) \in R_1. \tag{11}
\]

From the bijectivity of \( f \) and the facts of above it follows that \( R_2 \) is well defined. In order proof that \( \langle X_2, K_2, R_2 \rangle \) is an \( mS \)-space, note that since \( f^{-1}[f[Z]] = Z \) for all \( Z \in R_1(x) \), then we get \( f[R_1(x)] = R_2(f(x)) \) for all \( x \in X \). Now we proof that \( m_{R_1}(U) = f^{-1}[m_{R_2}(f[U])] \) for all \( U \in S(X_1) \). If \( x \in m_{R_1}(U) \), then from (10), \( R_1(x) \subseteq L_U \), so \( f[R_1(x)] = R_2(f(x)) \subseteq L_{f[U]} = f[L_U] \), thus \( (f(x)) \in m_{R_2}(f[U]) \) and consequently \( x \in f^{-1}[m_{R_2}(f[U])] \). The proof of reverse inclusion is similar but takes into account the bijectivity of \( f \). If \( V \in S(X_2) \), then

\[
m_{R_2}(V) = m_{R_2}(f[U]) = f[m_{R_1}(U)]. \tag{12}
\]

Since \( \langle X_1, K_1, R_1 \rangle \) is an \( mS \)-space, then \( m_{R_1}(U) \in S(X_1) \) and from (12) we conclude \( m_{R_2}(V) \in S(X_2) \). This proofs 1. of Definition 8. On the other hand, as \( \langle X_1, K_1, R_1 \rangle \) is an \( mS \)-space and \( f \) is bijective, we get

\[
R_2(f(x)) = f[R_1(x)] = f \left[ \bigcap \{ L_U : U \in S(X_1) \text{ and } x \in m_{R_1}(U) \} \right] = \bigcap \{ L_{f[U]} : U \in S(X_1) \text{ and } f(x) \in m_{R_2}(f[U]) \}.
\]

So, 2. of Definition 8 holds. Finally, it is clear from the definition of \( R_2 \) that condition 2. of Lemma 8 holds. Therefore, \( \langle X_1, K_1, R_1 \rangle \) and \( \langle X_2, K_2, R_2 \rangle \) are isomorphic \( mS \)-spaces, as granted. \( \square \)

We recall that if \( \langle X, K \rangle \) is an \( S \)-space, then we have that \( \langle X(S(X)), K_{S(X)} \rangle \) is the dual \( S \)-space associated to the semilattice \( S(X) \). In [10] it was proved that the map

\[
H_X : X \to X(S(X))
\]

given by \( H_X(x) = \{ U \in S(X) : x \in U \} \) is a homeomorphism between \( S \)-spaces such that \( K_{S(X)} = \{ H_X[U] : U \in K \} \). Therefore, as a straightforward consequence of Corollary 1 we obtain:

**Lemma 9** Let \( \langle X, K, R \rangle \) be a \( mS \)-space. Then the map \( H_X : X \to X(S(X)) \) defined by \( H_X(x) = \{ U \in S(X) : x \in U \} \) induces an isomorphism of \( mS \)-spaces.

The proofs of the following two Lemmas are similar to Propositions 30 and 46 of [11] with the only difference that in our proofs, Lemma 7 and Proposition 6 are involved. We leave the details to the reader.

**Lemma 10** Let \( \langle A, m \rangle \in \mathcal{MS} \). Then \( \langle X(A), K_A, R_m \rangle \) is an \( mS \)-space. Moreover, \( m_{R_m} = m^\pi \).
Let $A$ and $B$ be two semilattices. Let $h : A \to B$ be a homomorphism. In [10] it was proved that the binary relation $R_h \subseteq X(B) \times X(A)$ defined by

$$(P, Q) \in R_h \iff h^{-1}[P] \subseteq Q$$

(13)
is a meet-relation such that $\beta_B h = \Box_{R_h} \beta_A$.

**Lemma 11** Let $\langle A, m \rangle, \langle B, n \rangle \in \mathcal{MS}$ and $h : A \to B$ be a homomorphism. Then $h$ is a monotone homomorphism if and only if $R_h$ is a monotone meet-relation. In particular, if $h = id_A$, then $R_h = \Box_{X(A)}$.

**Lemma 12** Let $\langle A, m \rangle, \langle B, n \rangle, \langle C, l \rangle \in \mathcal{MS}$. If $h : A \to B$ and $g : B \to C$ be monotone homomorphisms, then $R_{gh} = R_h \ast R_g$.

**Proof** From Lemma 11 $R_{gh}$, $R_h$ and $R_g$ are monotone meet-relations. Then, since $\Box$ is faithful, for proving our claim it is enough to check $\Box_{R_{gh}} = \Box_{R_h \circ R_g}$. But this is essentially the Proposition 3.22 of [10].

Notice that Lemmas 10 and 11 allow to define a functor $G : m\mathcal{MS}^{op} \to m\mathcal{Sp}$ as follows:

$$\langle A, m \rangle \mapsto \langle X(A), K_A, R_m \rangle, \quad h \mapsto R_h$$

On the other hand, observe that the assignments

$$\langle X, K, R \rangle \mapsto \langle S(X), m_R \rangle, \quad R \mapsto \Box_R$$

also define a functor $H : m\mathcal{Sp} \to m\mathcal{MS}^{op}$. Now we state our main theorem.

**Theorem 7** The categories $m\mathcal{Sp}$ and $m\mathcal{MS}$ are dually equivalent.

**Proof** Let $\langle A, m \rangle \in \mathcal{MS}$. From Theorem 3.24 of [10] and Lemma 7 it follows that the map $\beta : A \to S(X(A))$ defines a natural isomorphism between $H \circ G$ and $Id_{m\mathcal{MS}^{op}}$. On the other hand, let $\langle X, K, R \rangle$ be a $m\mathcal{S}$-space. Then, from Lemma 9 and again from Theorem 3.24 of [10], we get that the map $H_X : X \to X(S(X))$ leads to a natural isomorphism between $G \circ H$ and $Id_{m\mathcal{Sp}}$. This concludes the proof.

**5 Characterization of Homomorphic Images and Congruences by Means of Lower-Vietoris-Type Topologies**

In [9] Celani and Calomino proved that homomorphic images of a distributive semilattice are characterized by one-to-one meet-relations and by lower Vietoris families on the dual space. This characterization was motivated by the results given in [3] and by the fact that the class of distributive semilattices is not a variety. Of course, this is not the case of semilattices and monotone semilattices. In this section we characterize congruences of semilattices and monotone semilattices by means of lower-Vietoris-type topologies and monotone lower-Vietoris-type topologies of their respective associated dual spaces. To achieve that goal, we start by studying the existing relation between homomorphic images and lower-Vietoris-type topologies in the variety of semilattices and later we extend this study to the variety of monotone semilattices.
Definition 10 Let \( (X_1, \mathcal{K}_1) \) and \( (X_2, \mathcal{K}_2) \) be two \( S \)-spaces. A meet-relation \( R \subseteq X_1 \times X_2 \) is \textit{one-to-one} if for each \( x \in X_1 \) and \( U \in S(X_1) \) with \( x \notin U \), there exists \( V \in S(X_2) \) such that \( U \subseteq \square_R(V) \) and \( x \notin \square_R(V) \).

The following result is the \( S \)-space version of Theorem 36 (2) of [9]. Since its proof is similar, we omit it.

Theorem 8 Let \( (X_1, \mathcal{K}_1) \) and \( (X_2, \mathcal{K}_2) \) be two \( S \)-spaces and \( R \subseteq X_1 \times X_2 \) be a meet-relation. Then the homomorphism \( \square_R : S(X_2) \to S(X_1) \) is onto if and only if \( R \) is one-to-one.

Corollary 2 Let \( \mathbf{A} \) and \( \mathbf{B} \) be two semilattices and \( h : \mathbf{A} \to \mathbf{B} \) be a homomorphism. Then \( h \) is onto if and only if the meet-relation \( R_h \subseteq X(\mathbf{B}) \times X(\mathbf{A}) \) is one-to-one.

Let \( \mathbf{A} \) be a semilattice and \( (X, \mathcal{K}) \) be an \( S \)-space. Consider a one-to-one meet-relation \( R \subseteq X \times X(\mathbf{A}) \) and let \( \mathcal{F}_R = \{ R(x) : x \in X \} \). It is clear that \( \mathcal{F}_R \subseteq \mathcal{C}_\mathcal{K}(X(\mathbf{A})) \). For every \( a \in \mathbf{A} \) we consider the set
\[
H_a = \{ R(x) : R(x) \cap \beta(a)^c \neq \emptyset \}.
\]

Remark 6 Notice that for every \( a \in \mathbf{A} \) and \( x \in X \) the following holds
\[
R(x) \in H_a \iff x \in (\square_R(\beta(a)))^c.
\]

In [21], Ivanova-Dimova introduces a new lower-Vietoris-type hypotopology that is a generalized version of the lower Vietoris topology.

Definition 11 ([21]) Let \( X \) be a set. Let \( \mathcal{F} \) be a family of non-empty subsets of \( X \) and \( \mathcal{O} \) be a topology on \( \mathcal{F} \). For each \( U \subseteq X \) let us consider
\[
U^\mathcal{F} = \{ Y \in \mathcal{F} : Y \cap U \neq \emptyset \}.
\]

Let \( \mathcal{M}_\mathcal{F} = \{ U^\mathcal{F} : U \subseteq X \} \). We say that \( \mathcal{O} \) is a \textit{lower-Vietoris-type topology} on \( \mathcal{F} \) if \( \mathcal{O} \cap \mathcal{M}_\mathcal{F} \) is a subbase for \( \mathcal{O} \).

Proposition 10 Let \( \mathbf{A} \) be a semilattice and \( (X, \mathcal{K}) \) be an \( S \)-space. Let \( R \subseteq X \times X(\mathbf{A}) \) be a one-to-one meet-relation. Then:

1. \( R(x) \neq \emptyset \) for all \( x \in X \).
2. The family \( \mathcal{M} = \{ H_a : a \in \mathbf{A} \} \) is a subbasis for a topology on \( \mathcal{F}_R \).
3. The topology \( \tau_\mathcal{M} \) generated by \( \mathcal{M} \) is a lower-Vietoris-type topology on \( \mathcal{F}_R \).

Proof We only need to prove 1. since the proof of 2. is analogue to the proof of Lemma 38 of [9] and 3. is a consequence of 2. and Corollary 2.11 of [21]. Let \( x \in X \). Since \( \mathcal{K} \) is a subbasis, i.e. \( X = \bigcup \mathcal{K} \) and \( S(X) = \{ U^c : U \in \mathcal{K} \} \), there exists \( U \in S(X) \) such that \( x \notin U \). As \( R \) is a one-to-one meet-relation, there exists \( a \in \mathbf{A} \) such that \( U \subseteq \square_R(\beta(a)) \) and \( x \notin \square_R(\beta(a)) \).

Therefore, from Remark 6, \( R(x) \in H_a \) and this implies that \( R(x) \neq \emptyset \).

Remark 7 We would like to stress that although the Proof of Proposition 10 (2) is similar to the proof of Lemma 38 in [9] when considering \( S \)-spaces, the family \( \mathcal{M} \) in general is not a basis. To check this, let us consider the following non-distributive lattice \( \mathbf{L} \) together with the identity map \( \text{id}_\mathbf{L} : \mathbf{L} \to \mathbf{L} \):
We prove the conditions of Definition 3. By Proposition 10 (2) the family $\mathcal{M}$ is a subbasis for a topology on $\mathcal{F}_R$. We see that $(\mathcal{F}_R, \mathcal{M})$ is $T_0$. Let $x, y \in X$ be such that $R(x) \neq R(y)$. Since $(X, \mathcal{K})$ is an S-space, there exists $U \in S(X)$ such that $x \notin U$ and $y \notin U$. As $R$ is a one-to-one meet-relation, there exists $a \in A$ such that $U \subseteq \mathcal{R}(\beta(a))$ and $y \notin \mathcal{R}(\beta(a))$. From Remark 6 it follows that $R(y) \notin H_a$ and $R(x) \notin H_a$. So, $(S_1)$ is satisfied.

We see $(S_2)$. It is clear that $H_1 = \emptyset$ and $\mathcal{M}$ is closed under finite unions. Let $a \in A$ and consider $H_a$. Observe that as a consequence of Remark 6, we have $H_a \subseteq \bigcup\{H_b : b \subseteq A\}$ if and only if $\mathcal{R}(\beta(a))^c \subseteq \bigcup\{\mathcal{R}(\beta(b))^c : b \subseteq A\}$. Since $\mathcal{R}(\beta(a))^c$ is a compact subset, by Alexander's subsbasis Theorem we get that $H_a$ is compact.

Notice that $(S_3)$ follows from Remark 6 and from the assumption that $(X, \mathcal{K})$ is an S-space.

Finally, we prove $(S_4)$. Let $Y \in C_X(\mathcal{F}_R)$ and $\mathcal{J} \subseteq S(\mathcal{F}_R)$ be a $\mathcal{F}$-family such that $Y \cap A^c \neq \emptyset$ for all $A \in \mathcal{J}$. We show that $Y \cap \bigcap\{A^c : A \in \mathcal{J}\} \neq \emptyset$. By Lemma 13 there exists $F \in \text{Fi}(A)$ such that $Y = \{R(x) : R(x) \subseteq \varphi(F)\}$. We consider $G = \bigcap\{\mathcal{R}(\beta(f)) : f \in F\}$ which is a subbasis closed subset of $S(X(A))$ and $I = \{\mathcal{R}(\beta(a)) : H_a \in \mathcal{J}\}$.

Let us see that $I$ is a $\mathcal{F}$-family of $S(X)$. Let $H_{a}^c, H_{b}^c \in \mathcal{J}$ be such that $\mathcal{R}(\beta(a)), \mathcal{R}(\beta(b)) \in \mathcal{I}$. Since $\mathcal{J}$ is a $\mathcal{F}$-family, there exist $d, e \in A$ such that $H_{e}^c \in \mathcal{J}, Y \subseteq H_{d}^c, H_{a}^c \cap H_{d}^c \subseteq H_{e}^c$ and $H_{b}^c \cap H_{d}^c \subseteq H_{e}^c$. From $Y \subseteq H_{d}^c$ we get that $\bigcap\{\mathcal{R}(\beta(f)) : f \in F\} \subseteq \mathcal{R}(\beta(d))$, i.e., $G \subseteq \mathcal{R}(\beta(d))$. Moreover, $\mathcal{R}(\beta(e)) \in \mathcal{I}$. By Remark 6, $\mathcal{R}(\beta(a)) \cap \mathcal{R}(\beta(b)) \subseteq \mathcal{R}(\beta(e))$ and we get that $I$ is a $\mathcal{F}$-family. Now we prove $G \cap \bigcap\{A^c : A \in \mathcal{I}\} \neq \emptyset$. Suppose that $G \cap \bigcap\{A^c : A \in \mathcal{I}\} = \emptyset$. So, there exists
$H_a^c \in \mathcal{J}$ such that $G \cap \Box_R (\beta(a))^c = \emptyset$. Then $G \subseteq \Box_R (\beta(a))$ and again by Remark 6 we have $Y \cap H_a = \emptyset$ which is a contradiction. So, $G \cap \bigcap \{A^c : A \in \mathcal{I}\} \neq \emptyset$.

Suppose now that $Y \cap \bigcap \{A^c : A \in \mathcal{J}\} = \emptyset$. Then $Y \subseteq \bigcup \{H_a^c : H_a^c \in \mathcal{J}\}$. By Remark 6 we get $\bigcap \{\Box_R (\beta(f)) : f \in F\} \subseteq \bigcup \{\Box_R (\beta(a)) : H_a^c \in \mathcal{J}\}$, i.e., $G \subseteq \bigcup \{A : A \in \mathcal{I}\}$ which is a contradiction. Thus, $Y \cap \bigcap \{A^c : A \in \mathcal{J}\} \neq \emptyset$. Therefore, $(\mathcal{F}_R, \mathcal{M})$ is an S-space as claimed. □

**Corollary 3** Let $A$ and $B$ be two semilattices and $h : A \to B$ be an onto homomorphism. Then $(\mathcal{F}_{Rh}, \mathcal{M})$ is an S-space which is homeomorphic to $(X(B), K_B)$.

**Proof** It follows from Corollary 2 and Theorem 9 that the pair $(\mathcal{F}_{Rh}, \mathcal{M})$ is an S-space. If we define the map $\lambda : X(B) \to \mathcal{F}_{Rh}$ by

$$\lambda(P) = R_h(P)$$

then by arguments similar to the proof of Lemma 40 of [9] it is proved that $\lambda$ is a homomorphism. □

**Lemma 14** Let $(X, \mathcal{K})$ be an S-space and $\mathcal{F}$ be a family of non-empty subbasic closed subsets of $(X, \mathcal{K})$. Then the family $\mathcal{M}_\mathcal{F}$ is a subbasis for a topology on $\mathcal{F}$.

**Proof** Let $Y \in \mathcal{F}$. Then there exists $x \in X$ such that $x \in Y$. Since $\mathcal{K}$ is a subbase, there exists $U \in \mathcal{K}$ such that $x \in U$. Then $Y \in U^\mathcal{K} \cap \mathcal{F} = \bigcup \{U^\mathcal{F} : U \in \mathcal{K}\}$. □

**Definition 12** Let $(X, \mathcal{K})$ be an S-space. A family $\mathcal{F}$ of non-empty subbasic closed subsets of $(X, \mathcal{K})$ is a lower-Vietoris-type family if the pair $(\mathcal{F}, \mathcal{M}_\mathcal{F})$ is an S-space where $\mathcal{M}_\mathcal{F} = \{U^\mathcal{F} : U \in \mathcal{K}\}$.

We write $\mathcal{V}(X)$ for the set of lower-Vietoris-type families of $(X, \mathcal{K})$.

**Remark 8** If $\mathcal{F}$ is a lower-Vietoris-type family of an S-space $(X, \mathcal{K})$, then $S(\mathcal{F}) = \langle S(\mathcal{F}), \cap, \mathcal{F} \rangle$ is a semilattice where $S(\mathcal{F}) = \{ (U^\mathcal{F})^c : U \in \mathcal{K} \}$.

Let $A$ be a semilattice. If $\mathcal{F} \subseteq C_K(X(A))$ is a lower-Vietoris-type family of the dual S-space $(X(A), \mathcal{K}_A)$, then we can define a relation $R_\mathcal{F} \subseteq \mathcal{F} \times X(A)$ by

$$(\mathcal{Y}, \mathcal{P}) \in R_\mathcal{F} \iff \mathcal{P} \in \mathcal{Y}.$$ 

We stress that the proofs of the following three results can be performed in the same way that the proofs of Lemmas 42, 43 and Theorem 44 of [9]. Therefore we omit them.

**Lemma 15** Let $A$ be a semilattice and $\mathcal{F} \subseteq C_K(X(A))$ be a lower-Vietoris-type family of the dual S-space of $A$. Then $R_\mathcal{F} \subseteq \mathcal{F} \times X(A)$ is a one-to-one meet-relation.

**Lemma 16** Let $A$ be a semilattice and $(X, \mathcal{K})$ be an S-space. Then:

1. If $R \subseteq X \times X(A)$ is a one-to-one meet-relation, then for every $x \in X$ and $\mathcal{P} \in X(A)$ we have

$$(x, \mathcal{P}) \in R \iff (R(x), \mathcal{P}) \in R_{\mathcal{F}_R}.$$ 

2. If $\mathcal{F} \subseteq C_K(X(A))$ is a lower-Vietoris-type family of the dual S-space of $A$, then $\mathcal{F} = \mathcal{F}_{R_{\mathcal{F}}}$.

\(\square\) Springer
Since the class of semilattices is a variety, it is fairly known that homomorphic images are in correspondence with congruences. In what follows, we present an explicit characterization of congruences of semilattices as lower-Vietoris-type families: let $A$ be a semilattice and $\theta \subseteq A \times A$ be a congruence on $A$. Let us consider the natural homomorphism $q_\theta : A \to A/\theta$ assigning to every $a \in A$ the equivalence class $q_\theta(a) = a/\theta$. Since $q_\theta$ is onto, by Corollary 2 we have that $R_{q_\theta} \subseteq X(A/\theta) \times X(A)$ is a meet-relation one-to-one. Then, by Corollary 3

$$\mathcal{F}_\theta = \{R_{q_\theta}(Q) : Q \in X(A/\theta)\} \subseteq C_K(X(A))$$

(14)

is a lower-Vietoris-type family of the dual $S$-space $\langle X(A), K_A \rangle$, i.e., the structure $\langle \mathcal{F}_\theta, M \rangle$ is an $S$-space.

**Theorem 10** Let $A$ be a semilattice and $\mathcal{F} \subseteq C_K(X(A))$ be a lower-Vietoris-type family of the dual $S$-space of $A$. Consider the relation

$$\theta_{\mathcal{F}} = \{(a, b) \in A^2 : \beta(a)^c = \beta(b)^c\}.$$ 

(15)

1. Then $\theta_{\mathcal{F}}$ is a congruence on $A$. In particular, we have that $\theta_{\mathcal{F}}$ is the kernel of the homomorphism $\square_{R_{\mathcal{F}}} : A \to S(\mathcal{F})$.

2. Then $Y \in \mathcal{F}$ if and only if there exists $Q \in X(A/\theta_{\mathcal{F}})$ such that $Y = R_{q_\theta A}(Q)$. Moreover, $\mathcal{F} = \mathcal{F}_{\theta_{\mathcal{F}}}$.

**Proof.** 1. It is routine.

2. Let $Y \in \mathcal{F}$ and let $\psi$ be the map defined in (2). Let us consider the set $Q = q_{\theta_{\mathcal{F}}}[\psi(Y)]$. Then $q_{\theta_{\mathcal{F}}}(a) \in Q$ if and only if $a \in \psi(Y)$. It follows that $Q \in \text{Fi}(A/\theta_{\mathcal{F}})$ and since $\mathcal{F}$ is a lower-Vietoris-type family of $\langle X(A), K_A \rangle$, we have that $\langle \mathcal{F}, M \rangle$ is an $S$-space and satisfies condition (S3). Then it follows $Q \in X(A/\theta_{\mathcal{F}})$ and $P \in R_{q_\theta A}(Q)$ if and only if $\psi(Y) \subseteq P$. Then $P \in Y$ and we get that $R_{q_\theta A}(Q) = Y$. Conversely, suppose there exists $Q \in X(A/\theta_{\mathcal{F}})$ such that $Y = R_{q_\theta A}(Q)$. We consider the homomorphism $H_{\mathcal{F}} : \mathcal{F} \to X(S(\mathcal{F}))$ between $S$-spaces and let $\chi : A/\theta_{\mathcal{F}} \to S(\mathcal{F})$ be the isomorphism between semilattices given by $\chi(a/\theta_{\mathcal{F}}) = \{Y \in \mathcal{F} : Y \cap \beta(a)^c = \emptyset\}$. Therefore, $Y = H_{\mathcal{F}}^{-1}[\chi(Q)] \in \mathcal{F}$. \hfill $\Box$

**Theorem 11** Let $A$ be a semilattice and $\theta$ be a congruence on $A$. Then $\theta = \theta_{\mathcal{F}_\theta}$.

**Proof.** Let $(a, b) \in \theta$. If $Y \in [\beta(a)^c]_{\mathcal{F}_\theta}$, then there exists $Q \in X(A/\theta)$ such that $Y = R_{q_\theta}(Q)$ and $R_{q_\theta}(Q) \cap \beta(a)^c = \emptyset$. So, there is $P \in X(A)$ such that $(Q, P) \in R_{q_\theta}$ and $P \not\in \beta(a)$, i.e., $q_\theta^{-1}(Q) \subseteq P$ and $a \not\in P$. It follows that $q_\theta(a) \not\in Q$ and since $(a, b) \in \theta$, we have $q_\theta(b) \not\in Q$. Then $b \not\in q_\theta^{-1}(Q)$ and by Theorem 2 there exists $P' \in X(A)$ such that $q_\theta^{-1}(Q) \subseteq P'$ and $b \not\in P'$. Thus, $P' \in R_{q_\theta}(Q) \cap \beta(b)^c$ and $R_{q_\theta}(Q) = Y = [\beta(b)^c]_{\mathcal{F}_\theta}$. The other inclusion is similar. Hence, $[\beta(a)^c]_{\mathcal{F}_\theta} = [\beta(b)^c]_{\mathcal{F}_\theta}$ and $(a, b) \in \theta_{\mathcal{F}_\theta}$. Conversely, let $(a, b) \in \theta_{\mathcal{F}_\theta}$. Suppose that $(a, b) \not\in \theta$. Then $q_\theta(a) \not\in q_\theta(b)$ and by Theorem 2 there exists $Q \in X(A/\theta)$ such that $a \in q_\theta^{-1}(Q)$ and $b \not\in q_\theta^{-1}(Q)$. Again by Theorem 2, there exists $P \in X(A)$ such that $q_\theta^{-1}(Q) \subseteq P$ and $b \not\in P$, i.e., $P \in R_{q_\theta}(Q) \cap \beta(b)^c$ and $R_{q_\theta}(Q) = [\beta(b)^c]_{\mathcal{F}_\theta}$. Then, by assumption, we have $R_{q_\theta}(Q) \in [\beta(a)^c]_{\mathcal{F}_\theta}$ and $R_{q_\theta}(Q) \cap \beta(a)^c = \emptyset$. So, there is $P' \in X(A)$ such that $q_\theta^{-1}(Q) \subseteq P'$ and $a \not\in P'$. On the other hand, since $a \in q_\theta^{-1}(Q)$ it follows that $a \in P'$, which is a contradiction. Therefore, $(a, b) \in \theta$ and $\theta = \theta_{\mathcal{F}_\theta}$, as desired. \hfill $\Box$

Now we can make explicit the existing relation between congruences of semilattices and lower-Vietoris-type families of $S$-spaces.
Theorem 12 Let $A$ be a semilattice. Then there is a one-to-one correspondence between the congruences of $A$ and the lower-Vietoris-type families of the dual $S$-space of $A$.

Proof Let $A$ be a semilattice. From (14) and (15) we can consider the assignments $\text{Con}(A) \rightarrow \mathcal{V}(X(A))$ and $\mathcal{V}(X(A)) \rightarrow \text{Con}(A)$, defined as $\theta \mapsto F_{\theta}$ and $F \mapsto \theta_F$, respectively. Observe that Corollary 2 and Theorem 10(1) guarantee that such assignments are well defined. Moreover, Theorem 10(2) and 11 grant that they are in fact, the inverse of each other. This concludes the proof.

Something more can be said about the structure of the lower-Vietoris-type families of an $S$-space. Let $(X, K)$ be an $S$-space and let $F_1, F_2 \in \mathcal{V}(X)$. Now let us regard the following relation on $\mathcal{V}(X)$:

$$F_1 \leq F_2 \iff \forall U, V \in K[U_{F_2}^- = V_{F_2}^- \Rightarrow U_{F_1} = V_{F_1}^-].$$

Theorem 13 Let $(X, K)$ be an $S$-space. Then $(\mathcal{V}(X), \leq)$ is a complete lattice which is dually isomorphic to the set of all congruences of $S(X)$.

Proof Let $\text{Con}(S(X))$ be the set of all congruences of $S(X)$. We start by recalling that from Theorem 12, the maps $\text{Con}(S(X)) \rightarrow \mathcal{V}(X)$, defined by $\theta \mapsto F_{\theta}$ and $\mathcal{V}(X) \rightarrow \text{Con}(S(X))$, defined by $F \mapsto \theta_F$, are mutually inverse of each other. Then from Theorem 10, it is immediate that $F_1 \leq F_2$ if and only if $\theta_{F_2} \subseteq \theta_{F_1}$. Hence, by general reasons $(\mathcal{V}(X), \leq)$ is a poset which is dually isomorphic to the poset $(\text{Con}(S(X)), \subseteq)$. Moreover, since $\text{Con}(S(X))$ is a complete lattice then if $\{F_i : i \in I\}$ is a family of lower-Vietoris-type families of $(X, K)$, the latter allows us to define:

$$\bigvee \{F_i : i \in I\} = F_{\bigcap \{\theta_{F_i} : i \in I\}}$$

and

$$\bigwedge \{F_i : i \in I\} = F_{\bigvee \{\theta_{F_i} : i \in I\}}.$$  

This concludes the proof.

We conclude this part with an interesting application of Theorem 12. It roughly says that every semilattice can be represented as a semilattice associated to a lower-Vietoris-type family which is determined by a freely generated semilattice.

Theorem 14 Let $A$ be a semilattice and let $F(A)$ be the semilattice freely generated by $A$. Let $h : F(A) \rightarrow A$ be the onto homomorphism which extends the identity map on $A$. If $\theta_h$ is the kernel of $h$ and $F_{\theta_h}$ is the lower-Vietoris-type family associated to $\theta_h$, then $A$ is isomorphic to the semilattice $S(F_{\theta_h})$.

Proof Since the class of semilattices is a variety, from II.10.9 in [6] it is the case that $A$ is isomorphic to $F(A)/\theta_h$. Now, by Theorem 12 and Corollary 3, $F_{\theta_h}$ is a lower-Vietoris-type family of $X(F(A))$ such that the $S$-spaces $(X(A), K_A)$ and $(F_{\theta_h}, M)$ are homeomorphic. Therefore, by Theorem 3.24 of [10] and Remark 8, $A$ must be isomorphic to the semilattice $S(F_{\theta_h})$, as claimed.

Next we provide a characterization of the congruences of monotone semilattices by means of lower-Vietoris-type families. We stress that the most of the proofs of the following results will be omitted. This is motivated by the fact that the ideas we will employ are a mere adaptation of the ideas we used for the case of semilattices.
Definition 13 Let \( (X_1, \mathcal{K}_1, R_1) \) and \( (X_2, \mathcal{K}_2, R_2) \) be two \( mS \)-spaces. A monotone meet-relation \( T \subseteq X_1 \times X_2 \) is one-to-one if it is one-to-one as a meet-relation.

Theorem 15 Let \( (X_1, \mathcal{K}_1, R_1) \) and \( (X_2, \mathcal{K}_2, R_2) \) be two \( mS \)-spaces. Let \( T \subseteq X_1 \times X_2 \) be a monotone meet-relation. Then the monotone homomorphism \( \Box_T : S(X_2) \to S(X_1) \) is onto if and only if \( T \) is a one-to-one monotone meet-relation.

Corollary 4 Let \( (A, m), (B, n) \in \mathcal{MS} \) and \( h : A \to B \) be a monotone homomorphism. Then \( h \) is onto if and only if \( R_h \) is a one-to-one monotone meet-relation.

\[ \lambda(P) = R_h(P) \]

for all \( P \in X(B) \).

Theorem 16 Let \( (A, m), (B, n) \in \mathcal{MS} \) and \( h : A \to B \) be an onto monotone homomorphism. Let \( T \subseteq \mathcal{F}_{R_h} \times \mathcal{Z}(\mathcal{F}_{R_h}) \) be the relation given by

\[ (R_h(Q), Z) \in T \iff (Q, \lambda^{-1}(Z)) \in R_n \]

where \( Q \in X(B) \) and \( Z \in \mathcal{Z}(\mathcal{F}_{R_h}) \). Then \( \langle \mathcal{F}_{R_h}, \mathcal{M}, T \rangle \) is an \( mS \)-space which is homeomorphic to \( \langle X(B), \mathcal{K}_B, R_n \rangle \) and it satisfies the following condition

\[ \beta(a)^c \in R_m[R_h(P)] \iff H_a \in T(R_h(P)) \]

for all \( a \in A \) and \( P \in X(B) \).

\[ \Box \therefore \quad \text{Springer} \]
Now, we proceed with the study on congruences of monotone semilattices. To achieve this goal, let \((X, \mathcal{K}, R)\) be an \(mS\)-space and let \(F \in \mathcal{V}(X)\). For a subset \(H \subseteq \mathcal{Z}(X)\) we consider the set

\[
[H \cap \mathcal{K}]_F = \{V \in \mathcal{K} : \exists U \in H \cap \mathcal{K} [U_F \subseteq V_F]\}.
\]

We say that a subset \(H \subseteq \mathcal{Z}(X)\) is \(\mathcal{M}_F\)-increasing if \([H \cap \mathcal{K}]_F = H \cap \mathcal{K}\).

**Definition 14** Let \((X, \mathcal{K}, R)\) be an \(mS\)-space. A family \(\mathcal{F}\) of non-empty subbasic closed subsets of \((X, \mathcal{K})\) is a monotone lower-Vietoris-type family if \(\langle \mathcal{F}, \mathcal{M}_F \rangle\) is an \(S\)-space and for every \(Y \in \mathcal{F}\) we have that

\[
R[Y] = \{Z \in \mathcal{Z}(X) : \exists y \in Y[(y, Z) \in R]\}
\]

is a \(\mathcal{M}_F\)-increasing subset.

We write \(\mathcal{V}_m(X)\) for the set of monotone lower-Vietoris-type families of an \(mS\)-space \((X, \mathcal{K}, R)\). Note that as a consequence of Theorem 16, if \((\mathcal{A}, m), (\mathcal{B}, n) \in \mathcal{MS}\) and \(h : A \rightarrow B\) is an onto monotone homomorphism, we get that \(R_m[R_h(P)]\) is \(\mathcal{M}\)-increasing for all \(P \in X(\mathcal{B})\). \(\mathcal{F}_T\) is a monotone lower-Vietoris-type family.

**Theorem 17** Let \((\mathcal{A}, m) \in \mathcal{MS} and \(\langle X, \mathcal{K}, R \rangle\) be an \(mS\)-space. Let \(T \subseteq X \times X(\mathcal{A})\) be a one-to-one monotone meet-relation. Then \(\mathcal{F}_T\) is a monotone lower-Vietoris-type family of \((X(\mathcal{A}), \mathcal{K}_A, R_m)\).

**Proof** From Theorem 9 we have that \(\langle \mathcal{F}_T, \mathcal{M} \rangle\) is an \(S\)-space. Let \(x \in X\), we will prove that \(R_m[T(x)]\) is \(\mathcal{M}\)-increasing. Let \(a, b \in A\) be such that \(H_a \subseteq H_b\) and suppose that \(\beta(a)^c \in R_m[T(x)]\). We will see that \(\beta(b)^c \in R_m[T(x)]\). Since \(T\) is a monotone meet-relation we have \(T^{-1}[\beta(a)^c] \subseteq R(x)\). It is easy to see that \(y \in T^{-1}[\beta(a)^c]\) if and only if \(R(y) \in H_a\). From \(H_a \subseteq H_b\) we get that \(T^{-1}[\beta(a)^c] \subseteq T^{-1}[\beta(b)^c]\). So, \(T^{-1}[\beta(b)^c] \subseteq R(x)\) and this implies that \(\beta(b)^c \in R_m[T(x)]\).

**Lemma 17** Let \((\mathcal{A}, m) \in \mathcal{MS} and \mathcal{F} \subseteq C_K(X(\mathcal{A}))\) be a monotone lower-Vietoris-type family of the dual \(S\)-space of \(\mathcal{A}\). Let us consider the relation \(R \subseteq \mathcal{F} \times \mathcal{Z}(\mathcal{F})\) given by

\[
(Y, Z) \in R \iff Z \in \bigcap \{L_{H_a} : a \in A and \beta(a)^c \notin R_m[Y]\}.
\]

Then \(\langle \mathcal{F}, \mathcal{M}, R \rangle\) is an \(mS\)-space and \(R_F \subseteq \mathcal{F} \times X(\mathcal{A})\) is a one-to-one monotone meet-relation.

**Proof** We know that \(\mathcal{F}\) is an \(S\)-space. Let \(a \in A\). We will see that \(m_R[H_a^c] \in S(\mathcal{F})\). Since

\[
m_R[H_a^c] = \{Y \in \mathcal{F} : \forall Z \in R(Y)[Z \cap H_a^c \neq \emptyset]\},
\]

we have \(Y \notin m_R[H_a^c]\) if and only if there exists \(Z \in R(Y)\) such that \(Z \notin L_{H_a}^c\). So, by definition of \(R\) we get \(\beta(a)^c \in R_m[Y]\).

On the other hand, let \(\beta(a)^c \in R_m[Y]\) and suppose that \(Y \in m_R(H_a^c)\). We have \(H_a \notin R(Y)\). So, there exists \(b \in A\) such that \(\beta(b)^c \notin R_m[Y]\) and \(H_a \cap H_b^c = \emptyset\). Thus, \(H_a \subseteq H_b^c\) and since \(R_m[Y]\) is \(\mathcal{M}\)-increasing we get a contradiction. Therefore, \(Y \notin m_R[H_a^c]\) if and only if \(\beta(a)^c \in R_m[Y]\) and it is easy to see that \(\beta(a)^c \in R_m[Y]\) if and only if \(Y \cap m_R(\beta(a)^c) \neq \emptyset\). As \(m_R(\beta(a)^c) = \beta(ma)^c\), we finally get that \(m_R[H_a^c] = H_{ma}\) and then \(m_R[H_a^c] \in S(\mathcal{F})\). It is immediate to see that \(R(Y) = \bigcap \{L_{H_a} : a \in A and Y \in m_R[H_a^c]\}\). So, \(\langle \mathcal{F}, \mathcal{M}, R \rangle\) is an \(mS\)-space.
We will see now that $R_{\mathcal{F}}$ is a monotone one-to-one relation. Let $a \in A$. We only need to prove that
\[ \beta(a)^c \in R_m[Y] \iff R_{\mathcal{F}}^{-1}[\beta(a)^c] \in R(Y). \]
Note that $R_{\mathcal{F}}^{-1}[\beta(a)^c] = H_a$. Then
\[ \beta(a)^c \in R_m[Y] \iff Y \notin m_R(H_a^c) \iff H_a \in R(Y) \]
and the assertion follows. \(\square\)

At this stage, we are able to provide a characterization of the congruences on monotone semilattices. The proofs of the following two results are similar to the proofs of Theorems 10, 11 and 12 respectively, so we omit them.

**Theorem 18** Let $\langle A, m \rangle \in \mathcal{MS}$.

1. If $\mathcal{F} \subseteq C_K(X(A))$ is a monotone lower-Vietoris-type family of the dual $mS$-space of $\langle A, m \rangle$, then $\theta_{\mathcal{F}}$ given by (15) is a congruence on $\langle A, m \rangle$ such that $\mathcal{F} = \mathcal{F}_{\theta_{\mathcal{F}}}$.
2. If $\theta$ is a congruence on $\langle A, m \rangle$, and $\mathcal{F}_{\theta}$ is the monotone lower-Vietoris-type family of the dual $mS$-space of $\langle A, m \rangle$ given by (14), then $\theta = \theta_{\mathcal{F}_{\theta}}$.

**Theorem 19** Let $\langle A, m \rangle \in \mathcal{MS}$. Then there is a one-to-one correspondence between the congruences of $\langle A, m \rangle$ and the monotone lower-Vietoris-type families of the dual $mS$-space of $\langle A, m \rangle$.

As we saw on Remark 5, the class of monotone semilattices is a variety so as an application of Theorem 19 we obtain a representation theorem analogue to Theorem 14. Before we state our result, we need to recall some facts. Let $\langle A, m \rangle \in \mathcal{MS}$ and $\mathcal{F} \subseteq C_K(X(A))$ be a monotone lower-Vietoris-type family of the dual $S$-space of $A$. Recall that from Lemma 17, $\langle \mathcal{F}, \mathcal{M}, T \rangle$ is an $mS$-space, so from Theorem 7, $\langle S(\mathcal{F}), m_T \rangle$ is a monotone semilattice.

**Theorem 20** Let $\langle A, m \rangle \in \mathcal{MS}$ and let $\langle F(A), n' \rangle$ be the monotone semilattice freely generated by $A$. Let $h : \langle F(A), n' \rangle \rightarrow \langle A, m \rangle$ be the onto homomorphism which extends the identity map on $A$. If $\theta_h$ is the kernel of $h$ and $\mathcal{F}_{\theta_h}$ is the monotone lower-Vietoris-type family associated to $\theta_h$, then $\langle A, m \rangle$ is isomorphic to the monotone semilattice $\langle S(\mathcal{F}_{\theta_h}), m_T \rangle$.

**Proof** As $\mathcal{MS}$ is a variety, by II.10.9 in [6] it follows that $\langle A, m \rangle$ is isomorphic to $\langle F(A)/\theta_h, n', \rangle$, where $n'(t/\theta_h) = n(t)/\theta_h$. Notice that by Theorems 18 and 19, $\mathcal{F}_{\theta_h}$ is a monotone lower-Vietoris-type family of the dual $mS$-space of $\langle F(A)/\theta_h, n' \rangle$ and by Theorem 16, the $mS$-spaces $\langle X(A), K_A, R_m \rangle$ and $\langle \mathcal{F}_{\theta_h}, \mathcal{M}, T \rangle$ are homemorphic. Hence, by Theorem 7 and taking into account Remark 8, we conclude that $\langle A, m \rangle$ is isomorphic to the semilattice $\langle S(\mathcal{F}_{\theta_h}), m_T \rangle$, as desired. \(\square\)

Finally, let $\langle X, K, R \rangle$ be an $mS$-space. If we endow $\mathcal{V}_m(X)$ with the induced order on $\mathcal{V}(X)$ then we obtain an analogue of Theorem 13 for $mS$-spaces.

**Theorem 21** For every $mS$-space $\langle X, K, R \rangle$, the poset $\langle \mathcal{V}_m(X), \leq \rangle$ is a complete lattice which is dually isomorphic to the set of all congruences of $\langle S(X), m_R \rangle$.

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