CHARACTERIZATION OF 9-DIMENSIONAL ANOSOV LIE ALGEBRAS

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Abstract. The classification of all real and rational Anosov Lie algebras up to dimension 8 is known in literature [14]. In this paper we study 9-dimensional Anosov Lie algebras by using the properties of very special algebraic numbers and Lie algebra classification tools. We prove that there exists a unique (up to a Lie algebra isomorphism) complex 3-step Anosov Lie algebra of dimension 9. In the 2-step case, we prove that a 2-step real 9-dimensional Anosov Lie algebra with no abelian factor must have a 3-dimensional derived algebra and we characterize these Lie algebras in terms of their Pfaffian forms. Among these Lie algebras, we have found a family of infinitely many complex nonisomorphic Anosov Lie algebras.

1. Introduction

A diffeomorphism \( f \) of a compact differentiable manifold \( M \) is called Anosov if the tangent bundle \( TM \) admits a continuous invariant splitting \( TM = E^+ \oplus E^- \) such that \( df \) expands \( E^+ \) and contracts \( E^- \) exponentially. A well-known class of Anosov diffeomorphisms arises as follows. Let \( N \) be a simply connected nilpotent Lie group and let \( \Gamma \) be a discrete subgroup of \( N \) such that \( N/\Gamma \) is compact in \( N \). We call \( N/\Gamma \) a nilmanifold. If \( f \) is a hyperbolic automorphism of \( N \) (i.e. no eigenvalue of the differential \( df \) is of absolute value 1) such that \( f(\Gamma) = \Gamma \), then the induced diffeomorphism \( f \) on \( N/\Gamma \), defined by \( f(x\Gamma) = f(x)\Gamma \) for all \( x \in N \), is an Anosov diffeomorphism of the nilmanifold \( N/\Gamma \). An Anosov diffeomorphism of a nilmanifold \( N/\Gamma \) arising in this way is called an Anosov automorphism of \( N/\Gamma \). More generally, one can get examples of Anosov diffeomorphisms on manifolds which are finitely covered by nilmanifolds in the following way. Let \( K \) be a finite group of automorphisms of a simply connected nilpotent Lie group \( N \) and let \( \Gamma \) be a discrete subgroup of \( K \times N \) such that the quotient \( N/\Gamma \) is compact. Here the action of \( \Gamma \) on \( N \) is given by \( x(\tau, y) = y\tau(x) \) for \( x \in N \) and \( (\tau, y) \in \Gamma \). We call the quotient space \( N/\Gamma \) an infranilmanifold. If \( g \) is a hyperbolic automorphism of \( N \) such that \( g \) normalizes the subgroup \( K \) in the group of automorphisms of \( N \) and \( g(\Gamma) = \Gamma \), then the induced diffeomorphism on \( N/\Gamma \) is called an Anosov automorphism of an infranilmanifold \( N/\Gamma \).

In [20], S. Smale raised the problem of classifying the compact manifolds admitting Anosov diffeomorphisms, and up to now the only known examples of Anosov diffeomorphisms are Anosov automorphisms of infranilmanifolds described as above. J. Franks [8] and A. Manning [15] proved that an Anosov diffeomorphism of a nilmanifold \( N/\Gamma \) is topologically conjugate to an Anosov automorphism of \( N/\Gamma \). This certainly highlights the problem of classifying all nilmanifolds which admit Anosov automorphisms.

The first example of a non-toral nilmanifold admitting an Anosov automorphism was described by S. Smale ([20]). For many years only relatively few examples appeared in the literature, but in the recent years families of nilmanifolds with Anosov automorphisms have been constructed showing that a complete classification does not seem to be possible (see [11] and also [2, 4, 5, 6, 7, 13, 14, 16, 18, 21]).

Any Anosov automorphism of a nilmanifold \( N/\Gamma \) gives rise to a hyperbolic automorphism \( \tau \) of the Lie algebra \( \mathfrak{n} \) of \( N \) (i.e. no eigenvalue of \( \tau \) is of absolute value 1) such that \( \tau \) stabilizes a \( \mathbb{Z} \)-subalgebra \( \Lambda \) of \( \mathfrak{n} \). Then the matrix of \( \tau \) with respect to a \( \mathbb{Z} \)-basis (i.e. with structure constants in \( \mathbb{Z} \)) of \( \Lambda \) lies

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in $GL_n(\mathbb{Z})$ where $n = \dim n$. Hence if a nilmanifold $N/T$ admits an Anosov automorphism, then the Lie algebra of $N$ is Anosov which is defined as follows.

**Definition 1.1.** (See [11]) A rational Lie algebra $n^\mathbb{Q}$ (i.e. with structure constants in $\mathbb{Q}$) of dimension $n$ is said to be Anosov if it admits a hyperbolic automorphism $\tau$ which is unimodular, i.e. $[\tau]_\beta \in GL_n(\mathbb{Z})$ for some basis $\beta$ of $n^\mathbb{Q}$, where $[\tau]_\beta$ denotes the matrix of $\tau$ with respect to the basis $\beta$. We will call a real or complex Lie algebra Anosov if it admits a rational form which is Anosov.

It can be seen that real Anosov Lie algebras give rise to nilmanifolds admitting Anosov automorphisms. Hence the problem of classifying nilmanifolds admitting Anosov automorphisms reduces to the classification of real Anosov Lie algebras. So far, in [14], all real and rational Anosov Lie algebras of dimension $\leq 8$ are classified up to isomorphism and we note that curiously enough, there are quite a few ones.

In this paper we consider the problem of classification of complex Anosov algebras of dimension 9. We note that in order to classify real Anosov Lie algebras, one has to find all the other real forms of the given complex one and study which of them are Anosov. Hence studying the structure of complex Anosov Lie algebras is the first step towards the classification. We study Anosov Lie algebras of dimension 9 by using some algebraic number theory. Very special algebraic numbers will play an important role in this study in the following way. The eigenvalues of the hyperbolic automorphism $\tau$ of an Anosov Lie algebra $n$ whose matrix with respect to the $\mathbb{Z}$-basis is in $GL(n, \mathbb{Z})$ where $n$ is the dimension of $n$, are algebraic units (algebraic integers whose reciprocals are also algebraic integers). The hyperbolicity of $\tau$ implies that none of these algebraic units is of absolute value 1. Hence the properties of these type of algebraic units can be used to give examples or to prove nonexistence of Anosov Lie algebras. This approach was introduced in [11] and have also been used in [13],[14] and [18]. The properties of the algebraic numbers arising from Anosov Lie algebras as described above are studied in [17].

To state our results, we recall a few definitions introduced in [13].

**Definition 1.2.** Let $n$ be a Lie algebra. An abelian factor of $n$ is an abelian (Lie) ideal $a$ of $n$ such that $n = m \oplus a$ for some ideal $m$ of $n$.

**Definition 1.3.** Let $n$ be an $r$-step nilpotent Lie algebra, i.e., the lower central series $\{C^i(n)\}$ (defined by $C^0(n) = n$ and $C^i(n) = [n, C^{i-1}(n)]$ for $i \geq 1$) satisfies $C^{r-1}(n) \neq 0$ and $C^r(n) = 0$. Then the type of $n$ is the $r$-tuple of positive integers $(n_1, \cdots, n_r)$, where $n_i = \dim C^{i-1}(n)/C^i(n)$.

In this paper we characterize all Anosov Lie algebras of dimension 9. First we prove that every 9-dimensional real Anosov Lie algebra without an abelian factor is either of type $(6,3)$ or $(3,3,3)$ using the results from [17]. Moreover, there is only one complex Anosov Lie algebra of type $(3,3,3)$ up to Lie algebra isomorphism. It can be seen that if $n$ is an Anosov Lie algebra of type $(3, n_2, n_3)$, then $n_2 = 3$ and $n_3 \geq 3$ (see [13, Proposition 2.3]). The example of $(3,3,3)$ is the first example (to our knowledge) of an Anosov Lie algebra of type $(3,3,*)$ showing that the condition in $n_3 \geq 3$ is in fact attained. To show that this algebra is actually an Anosov Lie algebra, we use the roots of unity in a way that seems to be generalizable. In the $(6,3)$ case, using the classification given by Galitzki and Timashev [9], we give a necessary condition on the Pfaffian form of a Lie algebra of type $(6,3)$ to be Anosov and then we give a list of possible candidates for $(6,3)$ type complex Anosov Lie algebras (Theorem 3.12).

We give an infinite family of complex nonisomorphic Anosov Lie algebras of type $(6,3)$ (Proposition 3.7 and Remark 3.8). The real Anosov Lie algebras up to dimension 8 are classified in [14] and there are only finitely many (up to isomorphism) real (and hence complex) Anosov Lie algebras up to dimension 8 see [14, Table 3]. This shows that the least dimension in which there are infinitely many complex nonisomorphic Anosov Lie algebras is 9.
2. Preliminaries

In this section, in order to introduce the framework in which we are going to work. We will begin by recalling the following proposition which was proved in [13].

**Proposition 2.1.** Let \( n \) be a real \( r \)-step nilpotent Lie algebra. We define \( C^i(n) \) inductively by \( C^i(n) = \langle n, C^{i-1}(n) \rangle \) for \( i \geq 1 \), where \( C^0(n) = n \). If \( n \) is an Anosov Lie algebra then there exist a decomposition \( n = n_1 \oplus \cdots \oplus n_r \), satisfying \( C^i(n) = n_{i+1} \oplus \cdots \oplus n_r \), \( i = 0, \ldots, r \), and a hyperbolic \( \tau \in \text{Aut}(n) \) such that

1. \( \tau(n_i) = n_i \) for all \( i = 1, \ldots, r \).
2. \( \tau \) is semisimple (in particular \( \tau \) is diagonalizable over \( \mathbb{C} \)).
3. For each \( i \), there exists a basis \( \beta_i \) of \( n_i \) such that \( [\tau_i]_{\beta_i} \in SL_{n_i}(\mathbb{Z}) \), where \( n_i = \dim n_i \) and \( \tau_i = \tau|_{n_i} \).

Let \( n \) and \( \tau \) be as in Proposition 2.1, i.e. \( \tau \) is a hyperbolic automorphism of \( n \) and there exists a \( \mathbb{Z} \)-basis \( \beta \) (i.e. with integer structure constants) of \( n \) such that \( [\tau]_{\beta} \in SL_n(\mathbb{Z}) \) where \( n = \dim n \). We note that the eigenvalues of \( \tau \) are algebraic units, that is an eigenvalue of \( \tau \) satisfies a monic polynomial equation with integer coefficients and 1 constant term. This follows since there is a basis of \( \mathbb{Z} \) with respect to which the matrix of \( \tau \) is in \( SL_n(\mathbb{Z}) \) and hence the characteristic polynomial of \( \tau \) is a monic polynomial with integer coefficients and 1 constant term.

For an algebraic number \( \lambda \in \mathbb{C} \), we denote by \( \deg(\lambda) \) the degree of \( m_\lambda(x) \), the irreducible monic polynomial over \( \mathbb{Q} \) annihilated by \( \lambda \) and by the **conjugated numbers to \( \lambda \)** we will denote the conjugated algebraic numbers of \( \lambda \) over \( \mathbb{Q} \), that is, the other roots of \( m_\lambda(x) \). We then have the following lemma:

**Lemma 2.2.** Let \( n \) be a real \( r \)-step Anosov Lie algebra, and let \( \tau \) and \( n = n_1 \oplus n_2 \oplus \cdots \oplus n_r \) be as in Proposition 2.1. Let \( n_i = \dim n_i \) and \( \tau_i = \tau|_{n_i} \) for \( 1 \leq i \leq r \). Then every eigenvalue \( \lambda_i \) of \( \tau_i \) is an algebraic unit and \( 1 < \deg(\lambda_i) \leq n_i \) for all \( i \).

The following definition will be used in next sections.

**Definition 2.3.** Let \( V \) be a real vector space of dimension \( n \). Let \( \sigma \) be a linear automorphism of \( V \) whose characteristic polynomial has all integer coefficients. We say that \( \sigma \) has a splitting \( [k_1; \ldots; k_m] \), where \( k_i \in \mathbb{N}, k_i \geq k_{i+1} \), if the characteristic polynomial of \( \sigma \) can be written as a product of \( m \) irreducible polynomials (over \( \mathbb{Z} \)) \( f_1, f_2, \ldots, f_m \) such that \( \deg f_i = k_i \) for all \( i \).

For example if \( \sigma : \mathbb{R}^3 \to \mathbb{R}^3 \) is a linear automorphism given by \( \sigma(x, y, z) = (2x + y, x + y, 3z) \) for all \( x, y, z \in \mathbb{R} \), then the characteristic polynomial of \( \sigma \) is \( x^3 - 6x^2 + 10x - 3 \) whose irreducible factors over \( \mathbb{Z} \) are \( x^2 - 3x + 1 \) and \( x - 3 \). Hence the splitting of \( \sigma \) is \([2 : 1]\).

In our setup, if \( n, \tau, n_i \) and \( \tau_i \) are as in Proposition 2.1, we are going to look at the possible splittings of \( \tau_i \)'s. For a fixed \( i \), we note that the characteristic polynomial of \( \tau_i \) has all integer coefficients (Proposition 2.1 (iii)). If \( \tau_i \) has a splitting \( [k_1; \ldots; k_m] \), then \( k_1 + \cdots + k_m = n_i \) where \( n_i = \dim n_i \). Since \( \tau_i \) is hyperbolic and \( [\tau_i]_{\beta_i} \in SL_{n_i}(\mathbb{Z}) \), for some basis \( \beta_i \) of \( n_i \), we also have that \( k_j = 1 \) for all \( j \) (see also [13] Appendix).

We note that each eigenvalue of \( \tau_i \) is a product of certain eigenvalues of \( \tau_j \) with \( j < i \) and moreover, it is an algebraic unit. We will investigate how the product of such special algebraic numbers can behave. It is proven in [17] that this behavior can not be so wild. In fact, one has the following lemma that can be deduced from [17].

**Lemma 2.4.** Let \( \tau \) be an Anosov automorphism of a nilpotent Lie algebra \( n \). Let \( \alpha \) and \( \beta \) be two eigenvalues of \( \tau \). Then the following hold:

1. If \( \deg(\alpha) \) and \( \deg(\beta) \) are relatively prime then \( \deg(\alpha \beta) \) cannot be a prime.
2. If \( \alpha \beta \) is also an eigenvalue of \( \tau \) and \( \deg(\alpha) \leq \deg(\beta) \) then \( \gcd(\deg(\beta), \deg(\alpha \beta)) \neq 1 \).
Proof. If \( \deg(\alpha) \) and \( \deg(\beta) \) are relatively prime and \( \deg(\alpha \beta) \) is prime, then it follows from Corollary 2 of [17] that either \( |\alpha| = 1 \) or \( |\beta| = 1 \). This contradicts our assumption that both \( \alpha \) and \( \beta \) are eigenvalues of a hyperbolic automorphism \( \tau \).

Suppose that \( \alpha \beta \) is an eigenvalue of \( \tau \), \( \deg(\alpha) = \deg(\beta) \) and \( \text{g.c.d.}(\deg(\beta), \deg(\alpha \beta)) = 1 \). Then \( |\alpha \beta| = 1 \) by Corollary 1 of [17] which is a contradiction. Also if \( \deg(\alpha) < \deg(\beta) \), then by Lemma 2 of [17] we have \( \text{g.c.d.}(\deg(\beta), \deg(\alpha \beta)) \neq 1 \). \( \square \)

We recall [13, Theorem 3.1]: Let \( n \) be a rational Lie algebra and let \( n = \tilde{n} \oplus m \) be a Lie direct sum, where \( m \) is a maximal abelian factor of \( n \). Then \( n \) is Anosov iff \( \tilde{n} \) is Anosov and \( \dim m \geq 2 \). In view of this, we are interested in studying Anosov Lie algebras without an abelian factor.

For an Anosov Lie algebra \( n \) without an abelian factor, we will study the properties of eigenvalues of the Anosov automorphism of \( n \) and then using these properties we will deduce the Lie bracket structure on \( n_C = n \otimes \mathbb{C} \). Moreover, we also note that if \( n_C \) has an abelian factor, then \( n \) must also carry an abelian factor. In fact, \( \tilde{\mathfrak{g}}(n_C) \cap [n_C, n_C] = (\tilde{\mathfrak{g}}(n) \cap [n, n])_C \) and hence if \( \tilde{\mathfrak{g}}(n_C) \cap [n_C, n_C] \neq \tilde{\mathfrak{g}}(n_C) \) then \( (\tilde{\mathfrak{g}}(n) \cap [n, n])_C \neq \tilde{\mathfrak{g}}(n) \) where \( \tilde{\mathfrak{g}}(n) \) denotes the center of \( n \).

Finally, we will state the following lemma that can be proved by using Proposition 2.1.

**Lemma 2.5.** Let \( n \) be an Anosov Lie algebra of type \((n_1, \ldots, n_r)\) and let \( n = n_1 \oplus \ldots \oplus n_r \) be the decomposition of \( n \) given in Proposition 2.1. Then \( \tilde{n} = n/n_r \) is Anosov.

**Proof.** If \( n \) is an Anosov Lie algebra, \( \tau \) and \( n = n_1 \oplus \ldots \oplus n_r \) are as in Proposition 2.1 then, since \( \tau n_i = n_i \) for all \( i = 1, \ldots, r \), it is easy to see that it induces an automorphism of \( \tilde{n} = n/n_r \simeq n_1 \oplus \ldots \oplus n_{r-1} \). Also this automorphism is hyperbolic and to see that it is unimodular, recall that for each \( i \), there exists a basis \( \beta_i \) of \( n_i \) such that \( [\tau_i]_{\beta_i} \in SL_{n_i}(\mathbb{Z}) \), where \( n_i = \dim n_i \) and \( \tau_i = \tau|_{n_i} \). \( \square \)

Note that this argument is valid not only at real or complex level but also in the rational case. We also note that if \( n \) is a three step nilpotent Anosov Lie algebra, then \( \tilde{n} \) is 2-step and the decomposition \( \tilde{n} = n/n_3 \simeq n_1 \oplus n_2 \) gives the type of \( \tilde{n} \).

### 3. Dimension 9

In this section we will study Anosov Lie algebras of dimension 9. We will prove that an Anosov Lie algebra without an abelian factor must be of type \((6, 3)\) or \((3, 3, 3)\). Moreover, we will prove that there is only one complex Anosov Lie algebra (up to a Lie algebra isomorphism) without an abelian factor of type \((3, 3, 3)\). The \((6, 3)\) case is the hardest. Using the classification theory of Lie algebras from [9], we obtain a necessary condition on the Pfaffian form of the Lie algebra of type \((6, 3)\) to be Anosov.

Using [13, Proposition 2.3] we can see that the possible types for a 9 dimensional Anosov Lie algebra are \((7, 2), (6, 3), (5, 4), (4, 5), (5, 2, 2), (4, 3, 2), (4, 2, 3)\) and \((3, 3, 3)\). As a corollary of Lemma 2.5 and the fact that there is no non-toral 7 dimensional Anosov Lie algebra (see [14]), there are no Anosov Lie algebras of type \((5, 2, 2)\) and \((4, 3, 2)\). It is not hard to see, by using Lemma 2.4, that there are no Anosov Lie algebras with no abelian factor of type \((7, 2), (5, 4), (4, 5)\) and \((4, 2, 3)\). As an example we will illustrate \((5, 4)\) case.

In the rest of this section, we will use the following notation:

**Notation 3.1.** If \( n \) is an Anosov Lie algebra and \( \tau, \tau_i \) and \( n_i \) are as in Proposition 2.1, we will denote the eigenvalues of \( \tau_1 \) by \( \lambda_i \)'s, eigenvalues of \( \tau_2 \) by \( \mu_i \)'s and eigenvalues of \( \tau_3 \) by \( \nu_i \)'s, and the corresponding eigenvectors by \( X_i \)'s, \( Y_i \)'s and \( Z_i \)'s (in \( n_C \)) respectively.

**Proposition 3.2.** There is no Anosov Lie algebra of type \((5, 4)\).
Proof. Suppose that there exists an Anosov Lie algebra \( \mathfrak{n} \) of type \((5, 4)\), and let \( \tau \) be an Anosov automorphism as in Proposition 2.1. We note that the possible splittings (see Definition 2.3) \( \tau_1 \) are \([5]\) and \([3; 2]\). In the first case, for each nonzero bracket among the eigenvectors of \( \tau_1 \) we get an eigenvector of \( \tau_2 \) (of degree 2 or 4). That is if \([X_i, X_j] \neq 0\) for some \(1 \leq i, j \leq 5\), we then have that \( \lambda_i \lambda_j = \mu_k \) has degree 2 or 4. It is easy to see that either way this contradicts Lemma 2.4 since \( \text{g.c.d}(5, 4) = \text{g.c.d}(5, 2) = 1 \). In the second one, by the same argument, we get that \([X_i, X_j] = 0\) for \(i, j \in \{1, 2, 3\}\), and if \( \mu_k = \lambda_i \lambda_j \) with \( i \leq 3 \) and \( j = 4 \) or 5, then \( \text{g.c.d.}(\text{deg} \lambda, \text{deg} \mu_k) = 1 \), contradicting again Lemma 2.4. This means that \([X_i, X_j] = 0\) for all \(i, j\). This is a contradiction because \( \mathfrak{n} \) is of type \((5, 4)\). \(\square\)

**Proposition 3.3.** There is only one (up to a Lie algebra isomorphism) complex Anosov Lie algebra of type \((3, 3, 3)\).

**Proof.** Let \( \mathfrak{n} \) be a nilpotent Lie algebra of type \((3, 3, 3)\) that admits an Anosov automorphism \( \tau \) as in Proposition 2.1. In this case we note that the characteristic polynomials of \( \tau_1, \tau_2 \) and \( \tau_3 \) are all irreducible degree 3 polynomials over \( \mathbb{Q} \). Following Notation 3.1, we can assume the following Lie bracket structure in \( \mathfrak{n}_C \) (complexification of \( \mathfrak{n} \)):

\[
[X_1, X_2] = Y_1, \quad [X_2, X_3] = Y_2, \quad [X_1, X_3] = Y_3,
\]

and since \( \lambda_1 \lambda_2 \lambda_3 = \pm 1 \) we also have that

\[
[X_3, Y_1] = 0, \quad [X_1, Y_2] = 0, \quad [X_2, Y_3] = 0.
\]

Let \( \mathfrak{n}(a, b, c) \) denotes the Lie algebra given by

\[
[X_1, X_2] = Y_1, \quad [X_2, X_3] = Y_2, \quad [X_1, X_3] = Y_3,
\]

\[
[X_1, Y_1] = a \ Z_1, \quad [X_2, Y_2] = b \ Z_2, \quad [X_3, Y_3] = c \ Z_3,
\]

for \( a, b, c \in \mathbb{C} \). If \( abc \neq 0 \), it can be seen that \( \mathfrak{n}(a, b, c) \) is isomorphic to \( \mathfrak{n}(1, 1, 1) \) by changing the basis in the center.

To see that \( \mathfrak{n}_C \) is isomorphic to \( \mathfrak{n}(1, 1, 1) \), suppose on the contrary in that we have (in \( \mathfrak{n}_C \))

\[
[X_1, Y_j] = a \ Z_k, \quad \text{and} \quad [X_1, Y_l] = b \ Z_r,
\]

for some non zero \( a, b \in \mathbb{C} \) and some \( i \) that we can assume to be 1. It is clear from (1) that \( j, l \neq 2 \) so we may assume that \( j = 1 \) and \( l = 3 \). Hence, if \( k = r \), (3) implies that \( \lambda_1^2 \lambda_2 = \lambda_1^2 \lambda_3 \) and then \( \lambda_2 = \lambda_3 \). This is a contradiction because \( \lambda \)'s are roots of the characteristic polynomial of \( \tau_1 \) and they have to be distinct because the characteristic polynomial of \( \tau_1 \) is irreducible over \( \mathbb{Q} \). On the other hand, if \( k \neq r \), with no loss of generality we can assume that

\[
[X_1, Y_1] = a \ Z_1, \quad [X_1, Y_3] = b \ Z_2.
\]

Since \( Z_3 \) is an eigenvector of \( \tau_3 \) corresponding to \( \nu_3 \) (see Notation 3.1), \( Z_3 = [X_1, Y_j] \) for some \( i, j \).

Using 1 and that \( \nu \)'s are all distinct, the possibilities for \((i, j)\) are \((2, 1), (2, 2), (3, 2), \) and \((3, 3)\) but all of them lead us to the same kind of contradiction by checking that the product of the eigenvalues in the center equals \( \nu_1^2, \nu_1, \nu_2, \nu_2^2 \) respectively. Indeed, if for example we consider case \((2, 2)\), we obtain

\[
[X_1, Y_1] = a \ Z_1, \quad [X_1, Y_3] = b \ Z_2, \quad [X_2, Y_2] = Z_3,
\]

and therefore

\[
1 = \lambda_1^2 \lambda_2, \lambda_1^2 \lambda_3, \lambda_2^2 \lambda_3 = \lambda_2^2 \lambda_2 = \nu_1, \quad \text{contradicting the fact that} \quad \tau_3 \quad \text{is hyperbolic.}
\]

This shows that \( \mathfrak{n}_C \) is isomorphic to \( \mathfrak{n}(1, 1, 1) \).

We will construct a hyperbolic automorphism \( \sigma \) and a \( \mathbb{Z} \)-basis of \( \mathfrak{n}(1, 1, 1) \) preserved by \( \sigma \) such that the matrix of \( \sigma \) in that basis has integer entries. Consider the polynomial in \( \mathbb{Z}[x] \) given by

\[
f(X) = x^3 - 3x + 1.
\]

Then its roots are given by

\[
\lambda_1 = \xi + \xi^8, \quad \lambda_2 = \xi^2 + \xi^7, \quad \lambda_3 = \xi^4 + \xi^5,
\]
where $\xi = e^{2i\pi/9}$ a ninth root of unity. In this case we have that the extension $\mathbb{Q}(\lambda_1)$ is a cyclic extension of degree 3 over $\mathbb{Q}$ (see [1, p. 543]), and moreover straightforward calculation shows that

$$\lambda_1 = \lambda_2^2 - 2, \quad \lambda_2 = \lambda_3^2 - 2, \quad \lambda_3 = \lambda_2^2 - 2.$$  

Let

$$\mu_1 = \lambda_1\lambda_2, \quad \mu_2 = \lambda_2\lambda_3, \quad \mu_3 = \lambda_1\lambda_3,$$

$$\nu_1 = \lambda_1\mu_1, \quad \nu_2 = \lambda_2\mu_2, \quad \nu_3 = \lambda_3\mu_3.$$  

We consider now the automorphism $\sigma$ as above, corresponding to these $\lambda_i$'s, that is, defined by $\sigma(X_i) = \lambda_iX_i$, $\sigma(Y_i) = \mu_iY_i$ and $\sigma(Z_i) = \nu_iZ_i$ for $i = 1, 2, 3$.

Note that, due to our choice of $\lambda_i$, $\deg(\mu_i) = 3 = \deg(\nu_i)$ for all $1 \leq i \leq 3$.

Concerning the new basis, for $i = 1, 2, 3$ let us denote by

$$\mathcal{X}_i = \sum_{j=1}^{3} \lambda_j^{-1} X_j,$$

$$\mathcal{Y}_i = \lambda_3^{-1}((\lambda_2 - \lambda_1)Y_1 + \lambda_1^{-1}(\lambda_3 - \lambda_2)Y_2 + \lambda_2^{-1}(\lambda_3 - \lambda_1)Y_3),$$

$$\mathcal{Z}_i = (\lambda_1)^{-1}(\lambda_2 - \lambda_1)Z_1 + (\lambda_2)^{-1}(\lambda_3 - \lambda_2)Z_2 + (\lambda_3)^{-1}(\lambda_3 - \lambda_1)Z_3.$$  

Let $\beta_1 = \{\mathcal{X}_i, 1 \leq i \leq 3\}$, $\beta_2 = \{\mathcal{Y}_i, 1 \leq i \leq 3\}$ and $\beta_3 = \{\mathcal{Z}_i, 1 \leq i \leq 3\}$. It is easy to see that $\beta_i$'s are linearly independent sets. Moreover, we can see that $\sigma(\mathcal{X}_i) = \mathcal{X}_{i+1}$, $\sigma(\mathcal{Y}_i) = \mathcal{Y}_{i+1}$ for $i = 1, 2$, $\sigma(\mathcal{X}_3) = 3\mathcal{X}_2 - \mathcal{X}_1$ and $\sigma(\mathcal{Y}_3) = 3\mathcal{Y}_2 - \mathcal{Y}_1$ by using that $f(\lambda_i) = 0$ and therefore $\lambda_i^{-1} = 3 - \lambda_i^2$. Also, by using (6) one can see that $\sigma(\mathcal{Z}_i)$ is an integer linear combination of the $\mathcal{Z}_i$'s for all $i$. In fact, for example

$$\sigma(\mathcal{Z}_1) = \lambda_2^2\lambda_2(\lambda_2 - \lambda_1)Z_1 + \lambda_3^2\lambda_3(\lambda_3 - \lambda_2)Z_2 + \lambda_1^2\lambda_1(\lambda_3 - \lambda_1)Z_3$$

$$= \lambda_2^2(\lambda_2^2 - 2)(\lambda_2 - \lambda_1)Z_1 + \lambda_3^2(\lambda_3^2 - 2)(\lambda_3 - \lambda_2)Z_2 + \lambda_1^2(\lambda_3^2 - 2)(\lambda_3 - \lambda_1)Z_3$$

$$= (\lambda_2^3 - \lambda_2)(\lambda_2 - \lambda_1)Z_1 + (\lambda_3^3 - \lambda_3)(\lambda_3 - \lambda_2)Z_2 + (\lambda_1^3 - \lambda_1)(\lambda_3 - \lambda_1)Z_3$$

$$= \mathcal{Z}_3 - \mathcal{Z}_2.$$  

Therefore, $\beta = \{\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_k : i, j, k = 1, 2, 3\}$ is a basis of $\mathfrak{n}(1, 1, 1)$ preserved by $\sigma$ and moreover $[\sigma]_\beta \in GL(9, \mathbb{Z})$. To finish, it remains to show that $\beta$ is also a $\mathbb{Z}$-basis. We first note that since $\lambda_1\lambda_2\lambda_3 = 1$, $\lambda_1 + \lambda_2\lambda_3 = 0$ and $\lambda_i^{-2} = 3\lambda_i^{-1} - \lambda_i$, we have that

$$[\mathcal{X}_1, \mathcal{X}_2] = \mathcal{Y}_1, \quad [\mathcal{X}_1, \mathcal{X}_3] = \mathcal{Y}_3 - 3\mathcal{Y}_2.$$  

Moreover, by using (6) as above, we have

$$[\mathcal{X}_1, \mathcal{Y}_1] = \mathcal{Z}_1, \quad [\mathcal{X}_1, \mathcal{Y}_2] = \mathcal{Z}_2 - \mathcal{Z}_1, \quad [\mathcal{X}_1, \mathcal{Y}_3] = \mathcal{Z}_3 - 2\mathcal{Z}_2 + \mathcal{Z}_1.$$

Note that since $\sigma$ is an automorphism of $\mathfrak{n}$ and by our construction of the basis, these are the only Lie brackets one needs to check. Hence $\beta$ is a $\mathbb{Z}$-basis of $\mathfrak{n}(1, 1, 1)$ preserved by the hyperbolic automorphism $\sigma$, such that $[\sigma]_\beta \in GL(9, \mathbb{Z})$, and therefore $\mathfrak{n}(1, 1, 1)$ is an Anosov Lie algebra. \[\square\]

**Remark 3.4.** We will use the above proof as a model proof to show that a given Lie algebra is Anosov in some of the following cases where we will just exhibit the automorphism and the $\mathbb{Z}$-basis.

**Remark 3.5.** Note that with this result we have shown that there is only one $k$-step nilpotent complex Anosov Lie algebra with $k > 2$ (see the begining of this section).

**Type (6, 3).** To study Anosov Lie algebras of type (6, 3), we will use the classification of nilpotent Lie algebras of type (6, 3) by Galitzki and Timashev [9]. First, we will find the possibilities for the
For example, spitting of an Anosov automorphism (Definition 2.3). Then using classification from [9], we will give a list of nonisomorphic Lie algebras of type (6, 3) as possible candidates for Anosov algebras of type (6, 3). In most cases we will give a rational basis to show that they are Anosov Lie algebras.

**Notation 3.6.** We will refer to the classification of nilpotent Lie algebras of type (6, 3) given in [9]. Here we will introduce notation for some Lie algebras from [9, Section 4] which will be used in this section. For \( t, s, r \in \mathbb{C} \), we will denote by \( \mathcal{U}_1 + s_2 \mathcal{U}_2 + s_3 \mathcal{U}_3 \) a 2-step nilpotent Lie algebra of type (6, 3) with a basis \( \{ v_i : 1 \leq i \leq 6 \} \cup \{ w_j : 1 \leq j \leq 3 \} \) and nonzero Lie brackets given by

\[
\begin{align*}
[ v_1, v_2 ] &= s_1 w_1 & [ v_3, v_4 ] &= s_1 w_2 & [ v_5, v_6 ] &= s_1 w_3 \\
[ v_5, v_4 ] &= s_2 w_1 & [ v_1, v_6 ] &= s_2 w_2 & [ v_3, v_2 ] &= s_2 w_3 \\
[ v_3, v_6 ] &= s_3 w_1 & [ v_5, v_2 ] &= s_3 w_2 & [ v_1, v_4 ] &= s_3 w_3.
\end{align*}
\]

(8)

Here, \( \mathcal{U}_i \) is a 9-dimensional nilpotent Lie algebra with Lie bracket given by the \( i \)th row of (8) and with \( s_i = 1 \).

Let \( V \) be a 6-dimensional vector space and \( W \) be a 3-dimensional vector space. The classification of nilpotent Lie algebras of type (6, 3) has been given in [9] by classifying the tensors in \( \wedge^2 V \otimes W \) under the action of \( SL(V) \times SL(W) \). This classification is given in 7 families (see [9, Section 4]) by decomposing every tensor into a semisimple and nilpotent part under this action. The semisimple parts are given in terms of \( s_1 \mathcal{U}_1 + s_2 \mathcal{U}_2 + s_3 \mathcal{U}_3 \) for some \( s_1, s_2, s_3 \) and nilpotent parts are listed in tables which we are going to denote by \( \mathfrak{m}_j \), where \( j \) is its number in the corresponding table. For example, by \( \mathcal{U}_1 + \mathfrak{m}_{11} \) from Family 6 (and Table 7) from [9, Section 4], we mean the 9-dimensional Lie algebra with a basis \( \{ v_i, w_j : 1 \leq i \leq 6, 1 \leq j \leq 3 \} \) and nonzero Lie brackets given by

\[
\begin{align*}
[ v_1, v_2 ] &= w_1 & [ v_3, v_4 ] &= w_2 & [ v_5, v_6 ] &= w_3 \\
[ v_1, v_5 ] &= w_2 & [ v_3, v_6 ] &= w_1.
\end{align*}
\]

(9)

Since we will come across the following \( \mathfrak{m}_j \)'s in this section, we will list them by specifying nonzero Lie brackets of their basis vectors \( \{ v_i, w_j : 1 \leq i \leq 6, 1 \leq j \leq 3 \} \) for our reference. From Family 6 and Table 7 [9, Section 4]:

\[
\begin{align*}
(\mathfrak{m}_6) & \quad [ v_1, v_4 ] = w_3 & [ v_1, v_6 ] = w_2 & [ v_3, v_5 ] = w_1 \\
(\mathfrak{m}_7) & \quad [ v_1, v_3 ] = w_3 & [ v_1, v_5 ] = w_2 & [ v_3, v_6 ] = w_1 \\
(\mathfrak{m}_{10}) & \quad [ v_1, v_3 ] = w_3 & [ v_1, v_5 ] = w_2 & [ v_3, v_5 ] = w_1 \\
(\mathfrak{m}_{11}) & \quad [ v_1, v_5 ] = w_2 & [ v_3, v_6 ] = w_1 \\
(\mathfrak{m}_{12}) & \quad [ v_1, v_3 ] = w_3 & [ v_1, v_5 ] = w_2 \\
(\mathfrak{m}_{14}) & \quad [ v_1, v_3 ] = w_3
\end{align*}
\]

From Family 4 and Table 5 [9, Section 4]:

\[
\begin{align*}
(\mathfrak{m}_3) & \quad [ v_5, v_3 ] = w_1 & [ v_1, v_5 ] = w_2 & [ v_3, v_1 ] = w_3
\end{align*}
\]

We will use the classification given in [9] to conclude that certain Lie algebras are nonisomorphic. For example, \( \mathcal{U}_1 + \mathfrak{m}_{11} \) and \( \mathcal{U}_4 + \mathfrak{m}_{14} \) are nonisomorphic.
Note that since we are interested in (6, 3) Lie algebras up to isomorphism, we need orbits of tensors under the action of $GL(V) \times GL(W)$, and therefore to get this classification, the canonical form for the semisimple part of the tensors can be reduced by multiplying by a nonzero scalar. Then, for example we have that $s\mathcal{U}_2 + r\mathcal{U}_3 \simeq s'\mathcal{U}_2 + \mathcal{U}_3$ for any $r \neq 0$ and the semisimple part of Family 4 (and 5) are all isomorphic (see [9]).

**Anosov Lie algebras of type (6, 3).** Let $\mathfrak{n}$ be an Anosov Lie algebra of type (6, 3) with no abelian factor, and let $\tau, \tau_1, \tau_2$, be as in Proposition 2.1. As in the previous cases, we will denote by $X_i$ and $Y_j$ the eigenvectors of $\tau$ and $\tau_2$ respectively with corresponding eigenvalues $\lambda_i$ and $\mu_j$. Note that, by using Lemma 2.4, it is easy to see that the splitting of $\tau_1$ is [6] or [3;3] (see Definition 2.3). We are going to study now each one of these cases separately.

**Case i:** The splitting of $\tau_1$ is [6] or equivalently, the characteristic polynomial of $\tau_1$ is irreducible over $\mathbb{Z}$. We note that, in particular, this implies that $\lambda_i \neq \lambda_j$ for all $i \neq j$. From this, it can be shown that one may assume that

$$[X_1, X_2] = Y_1, \quad [X_3, X_4] = Y_2, \quad [X_5, X_6] = Y_3. \quad (10)$$

In fact, if this is not the case we can reorder the basis so that

$$[X_1, X_2] = Y_1, \quad [X_3, X_3] = Y_2, \quad (11)$$

and in this situation one would have to consider three possibilities for $Y_3$,

$$\text{(I)} [X_1, X_4] = Y_3, \quad \text{(II)} [X_2, X_3] = Y_3, \quad \text{(III)} [X_4, X_5] = Y_3. \quad (12)$$

Note that each one of these situations represents a few others that are totally equivalent to the considered one.

Case (III) is the simplest one because directly from (11) it follows that $\lambda_1^2 \lambda_2 \lambda_3 \lambda_4 \lambda_5 = 1$ and therefore we obtain the contradiction $\lambda_1 = \lambda_6$.

Cases (I) and (II) can be done in a very similar way. That is, by considering the possibilities for the other brackets among the $X_i$ we get either a contradiction or (10). Therefore, we can assume (10):\n
$$[X_1, X_2] = Y_1, \quad [X_3, X_4] = Y_2, \quad [X_5, X_6] = Y_3. \quad (13)$$

If there are no more nontrivial Lie brackets but these, then $\mathfrak{n} \simeq \mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathfrak{h}_3 \simeq \mathcal{U}_1$ which is known to be Anosov (see [14]). If there are more nontrivial Lie brackets, without any loss of generality, we can assume that $[X_3, X_2] = c Y_4$ and moreover, using that there is no abelian factor, one can see that $\mathfrak{n}_C$ is isomorphic to $\mathfrak{n}'(a, b, c)$, given by

$$[X_1, X_2] = a Y_1, \quad [X_3, X_4] = b Y_2, \quad [X_5, X_4] = c Y_3, \quad [X_5, X_6] = Y_3, \quad (14)$$

for some $a, b, c \in \mathbb{C}$. It is easy to see that we can not add more nontrivial Lie brackets by using that $\tau$ is hyperbolic and $\prod_{i=1}^{6} \lambda_i = 1$. To know which nonisomorphic Lie algebras we obtain from (13), we start by noting that if $abc = 0$, from [9] we get that we have only two nonisomorphic Lie algebras:

$$\mathfrak{n}'(a, b, 0) \simeq \mathfrak{n}'(1, 1, 0) \simeq \mathcal{U}_1 + \mathfrak{m}_{11}, \quad \text{and} \quad \mathfrak{n}'(0, 0, c) \simeq \mathfrak{n}'(0, 0, 1) \simeq \mathcal{U}_1 + \mathfrak{m}_{14},$$

(see Notation 3.6).

On the other hand, if $abc \neq 0$, by changing the basis to

$$\beta' = \{X_1, aX_2, \frac{1}{a}X_3, \frac{1}{c}X_4, cX_5, X_6, bY_1, \frac{1}{bc}Y_2, cY_3\},$$

we have that $\mathfrak{n}'(a, b, c) \simeq \mathfrak{n}'(1, abc, 1)$, and then (if $abc \neq 0$)

$$\mathfrak{n}'(a, b, c) \simeq \mathfrak{n}'(s, s, s) \simeq \mathcal{U}_1 + s\mathcal{U}_2 \simeq s\mathcal{U}_2 + \mathcal{U}_3,$$
where $s^3 = abc \neq 0$. Hence, from [9] we get the following nonisomorphic Lie algebras

\begin{itemize}
  \item $s \mathcal{U}_2 + \mathcal{U}_3$, corresponding to $s^3 \neq \pm 1$ (Family 2)
  \item $\mathcal{U}_2 + \mathcal{U}_3$, corresponding to $s^3 = 1$ (Family 4)
  \item $-\mathcal{U}_2 + \mathcal{U}_3$, corresponding to $s^3 = -1$ (Family 5)
\end{itemize}

(14)

Here the families are referred to the families from [9, Section 4].

We will consider now the ones corresponding to $s \in \mathbb{Q}$ which includes, in particular, the algebras $\mathcal{U}_2 + \mathcal{U}_3$ and $-\mathcal{U}_2 + \mathcal{U}_3$.

**Proposition 3.7.** If $s \in \mathbb{Q}$, then a Lie algebra $s \mathcal{U}_2 + \mathcal{U}_3$ is Anosov.

**Proof.** Let $s = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $n = s \mathcal{U}_2 + \mathcal{U}_3$. Then $n \simeq p \mathcal{U}_2 + q \mathcal{U}_3$. We recall that $p \mathcal{U}_2 + q \mathcal{U}_3$ is a 2-step nilpotent Lie algebra in which the nonzero Lie brackets on its basis vectors $\{X_1, Y_1: 1 \leq i \leq 6, 1 \leq j \leq 3\}$ are given by

\begin{equation}
\begin{bmatrix}
X_5, X_4 = p Y_1 & X_1, X_6 = p Y_2 & X_3, X_2 = p Y_3 \\
X_3, X_6 = q Y_1 & X_5, X_2 = q Y_2 & X_1, X_4 = q Y_3.
\end{bmatrix}
\end{equation}

(15)

Let $\lambda_1, \lambda_2, \lambda_3$ be the roots of the polynomial $x^3 - 3x + 1$. Note that we used these algebraic units in the proof of Proposition 3.3. Let $\sigma$ be the automorphism of $n$ whose matrix with respect to a basis $\{X_1, \ldots, X_6, Y_1, Y_2, Y_3\}$ is a diagonal matrix $D(\lambda_3, \lambda_3, \lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_2, \lambda_3, \lambda_3, \lambda_3)$. Let

\begin{equation}
\begin{align*}
X_l &= \lambda_3^{l-1} X_1 + \lambda_1^{l-1} X_3 + \lambda_2^{l-1} X_5, \quad l = 1, 2, 3, \\
Y_k &= \lambda_3^{k-1} X_2 + \lambda_1^{k-1} X_4 + \lambda_2^{k-1} X_6, \quad k = 1, 2, 3, \\
Z_r &= \lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3, \quad r = 1, 2, 3.
\end{align*}
\end{equation}

(16)

Hence,

\begin{equation}
\beta' = \{X_l, Y_k, Z_r \mid l, k, r = 1, 2, 3\}.
\end{equation}

(17)

is a basis of $n$. Using the properties of $\lambda_1, \lambda_2, \lambda_3$ it can be checked that $[\sigma], \beta' \in GL(9, \mathbb{Z})$. For example, as $\lambda_1 \lambda_2 = \lambda_1 - 1$ (see (5)) we get that $\sigma(Z_1) = Z_2 - Z_1$.

To see that $\beta'$ is a $\mathbb{Z}$-basis, we note the following:

\begin{align*}
[X_i, X_j] &= [Y_i, Y_j] = [Z_i, Z_j] = 0 \quad \forall \ 1 \leq i, j \leq 3, \\
[X_1, Y_1] &= (p + q)Z_1, \\
[X_2, Y_1] &= -2pZ_1 + qZ_2 + pZ_3, \\
[X_3, Y_1] &= 2pZ_1 + pZ_2 + qZ_3.
\end{align*}

\[ \square \]

**Remark 3.8.** Using the above proposition we get that $\{s \mathcal{U}_2 + \mathcal{U}_3 : s \in \mathbb{Q}\}$ is an infinite family of nonisomorphic Anosov Lie algebras of type $(6, 3)$. We know that there are, up to isomorphism, only finitely many (real) Anosov Lie algebras up to dimension 8 (see [14, Table 3]) and hence this is the smallest dimension in which there are infinitely many nonisomorphic Anosov Lie algebras. We also note that since $\{s \mathcal{U}_2 + \mathcal{U}_3 : s \not\in \mathbb{Q}, s^3 \neq \pm 1\}$ is an uncountable family of nonisomorphic Lie algebras, not all of them can be Anosov. Moreover, if $s^3 \not\in \mathbb{Q}$ it is not even clear if $s \mathcal{U}_2 + \mathcal{U}_3$ admits a rational form.

**Case ii:** The splitting of $\tau_1$ is $[3, 3]$ (see Definition 2.3). Let $\lambda_1, \ldots, \lambda_6$ denote the eigenvalues of $\tau_1$ such that $\lambda_1, \lambda_2, \lambda_3$ are conjugates over $\mathbb{Q}$ and $\lambda_4, \lambda_5, \lambda_6$ are conjugates over $\mathbb{Q}$. We will follow Notation 3.1.

In this case one may either have

(a) $[X_i, X_j] = Y_k$ for some $1 \leq i, j \leq 3$ (or equivalently $4 \leq i, j \leq 6$) or

(b) $[X_i, X_j] = 0$ for all $1 \leq i, j \leq 3$ and $[X_k, X_l] = 0$ for all $4 \leq k, l \leq 6$. 
In (a), with no loss of generality we may assume \( [X_1, X_2] = Y_1 \). At the eigenvalue level, this means that \( \lambda_1 \lambda_2 = \lambda_3^{-1} \), \( \lambda_1 \lambda_3 = \lambda_2^{-1} \) and \( \lambda_2 \lambda_3 = \lambda_1^{-1} \) must be the eigenvalues of \( \tau_2 (\tau \text{ on } [n, n]) \). On the other hand, the absence of abelian factor implies that

\[
[X_4, X_j] \neq 0 \quad \text{for some } 1 \leq j \leq 6.
\]

Hence \( \lambda_4 \lambda_j = \lambda_1^{-1} \) for some \( j \) and \( i, 1 \leq j \leq 6, 1 \leq i \leq 3 \). It is not hard to see that from here we can either have

\[
\{\lambda_4, \lambda_5, \lambda_6\} = \{\lambda_1^2, \lambda_2^2, \lambda_3^2\} \quad \text{or} \quad \{\lambda_4, \lambda_5, \lambda_6\} = \{\lambda_1, \lambda_2, \lambda_3\}.
\]

If \( \{\lambda_4, \lambda_5, \lambda_6\} = \{\lambda_1^2, \lambda_2^2, \lambda_3^2\} \), we will prove that \( n \) is Anosov. In this case, we may rearrange the basis so that \( \lambda_4 = \lambda_1^2 \), \( \lambda_5 = \lambda_2^2 \) and \( \lambda_6 = \lambda_3^2 \). Hence we can assume that the nonzero Lie brackets in \( n \) are given by

\[
\begin{align*}
[X_1, X_2] &= Y_1, \\
[X_2, X_3] &= Y_2, \\
[X_1, X_3] &= Y_3, \\
[X_3, X_6] &= c Y_1, \\
[X_1, X_4] &= a Y_2, \\
[X_2, X_5] &= b Y_3,
\end{align*}
\]

for some \( a, b, c \in \mathbb{C} \). Here we need to use the special properties of \( \lambda_i \)'s like \( |\lambda_i| \neq 1 \), \( \lambda_i \)'s are all distinct for \( 1 \leq i \leq 3 \) etc.

Let \( \tilde{n}(a, b, c) \) denote the Lie algebra defined by (18). Due to our assumption of no abelian factor we have that \( abc \neq 0 \) and then

\[
n \simeq \tilde{n}(a, b, c) \simeq \tilde{n}(1, 1, 1) \simeq U_4 + m_{10}
\]

(see Notation 3.6 and \( m_{10} \)).

**Proposition 3.9.** A Lie algebra \( \tilde{n}(1, 1, 1) \) defined by (18) with \( a = b = c = 1 \) is Anosov.

**Proof.** We will use very similar arguments as used in the case of type \((3, 3, 3)\) (proof of Proposition 3.3). Let \( \lambda_1, \lambda_2, \lambda_3 \) be the roots of \( x^3 - 3x + 1 \). Let \( \sigma \) be the automorphism of \( \tilde{n}(1, 1, 1) \) whose matrix with respect to a basis \( \{X_1, \ldots, X_6, Y_1, Y_2, Y_3\} \) is a diagonal matrix

\[
D(\lambda_1, \lambda_2, \lambda_3, \lambda_1^{-2}, \lambda_2^{-2}, \lambda_3^{-2}, \lambda_1^{-1}, \lambda_2^{-1}).
\]

Let

\[
X_i = \lambda_1^{i-1} X_1 + \lambda_2^{i-1} X_2 + \lambda_3^{i-1} X_3,
\]

\[
X'_j = \lambda_3^{1-j}(\lambda_2 - \lambda_1)X_3 + \lambda_3^{1-j}(\lambda_3 - \lambda_2)X_4 + \lambda_2^{1-j}(\lambda_3 - \lambda_1)X_5,
\]

\[
Y_k = \lambda_3^{1-k}(\lambda_2 - \lambda_1)Y_1 + \lambda_1^{1-k}(\lambda_3 - \lambda_2)Y_2 + \lambda_2^{1-k}(\lambda_3 - \lambda_1)Y_3.
\]

It can be shown that \( \tilde{\beta} = \{X_i, X'_j, Y_k \mid i, j, k = 1, 2, 3\} \) is a \( \mathbb{Z} \)-basis of \( \tilde{n}(1, 1, 1) \) such that \( [\sigma]_{\tilde{\beta}} \in GL(9, \mathbb{Z}) \). Note that due to the similarities with the \((3, 3, 3)\)-case one only needs to check brackets that involve some \( X'_j \) (see [18]).

On the other hand, if \( \{\lambda_4, \lambda_5, \lambda_6\} = \{\lambda_1, \lambda_2, \lambda_3\} \), then by rearranging the basis vectors we assume that \( \lambda_4 = \lambda_1, \lambda_5 = \lambda_2, \lambda_6 = \lambda_3 \) and the nonzero Lie brackets on \( n \) are given by

\[
\begin{align*}
[X_1, X_2] &= a Y_1, \\
[X_2, X_3] &= b Y_2, \\
[X_1, X_3] &= c Y_3, \\
[X_4, X_5] &= d Y_1, \\
[X_5, X_6] &= e Y_2, \\
[X_4, X_6] &= f Y_3, \\
[X_1, X_5] &= g Y_1, \\
[X_2, X_6] &= h Y_2, \\
[X_4, X_3] &= i Y_3, \\
[X_4, X_2] &= j Y_1, \\
[X_3, X_6] &= k Y_2, \\
[X_1, X_6] &= l Y_3,
\end{align*}
\]

for some \( a, b, c, d, e, f, g, h, i, j, k, l \in \mathbb{C} \). Note that the Pfaffian form of these Lie algebras (see [13]) is given by

\[
Pf(x, y, z) = -xyz(a f k - a e i + b d l - b g f + c j e - c d h + i h g - l j k).
\]

Calculating the Pfaffian forms of the algebras listed in the classification given in [9, Section 4], one can see that \( n \) should be isomorphic to one of the following (see Notation 3.6):
Moreover, we don’t get any new Lie algebras in this case.

Remark 3.10. Note that the algebras in [9, Section 4, Family 7] with 0 Pfaffian form have always an abelian factor and therefore are not included in our list.

The Lie algebras $su_2 + u_3$ have been considered in case i (see (14), Proposition 3.7). Also $u_1 + m_i$ for $i = 6, 7, 10, 11, 12, 14, 15$ have been considered in case i. We don’t know if $u_1 + m_i$ is Anosov for $i = 6, 7, 12$. However we prove the following:

Proposition 3.11. $u_2 + u_3 + m_3$ is an Anosov Lie algebra.

Proof. Let $n = u_2 + u_3 + m_3$. We recall that the nonzero brackets in $n$ on the basis vectors are given by (see Notation 3.6):

\[
\begin{align*}
[X_5, X_4] &= Y_1, & [X_1, X_6] &= Y_2, & [X_3, X_2] &= Y_3, \\
[X_3, X_6] &= Y_1, & [X_5, X_2] &= Y_2, & [X_1, X_4] &= Y_3, \\
[X_5, X_3] &= Y_1, & [X_1, X_5] &= Y_2, & [X_3, X_1] &= Y_3.
\end{align*}
\]

Once again we will take $\lambda_1, \lambda_2, \lambda_3$ to be the roots of $x^3 - 3x + 1$. Let $\sigma$ be the automorphism of $n$ whose matrix with respect to the basis $\{X_i, Y_j : 1 \leq i \leq 6, 1 \leq j \leq 3\}$ is a diagonal matrix $D(\lambda_3, \lambda_3, \lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_2, \lambda_2, \lambda_3, \lambda_3)$. Then $\sigma$ is a hyperbolic automorphism because of our choice of $\lambda_1, \lambda_2, \lambda_3$.

Let

\[
\begin{align*}
\mathcal{X}_l &= \lambda_3^{l-1} X_1 + \lambda_1^{l-1} X_3 + \lambda_2^{l-1} X_5 & l &= 1, 2, 3, \\
\mathcal{Y}_k &= \lambda_3^{k-1} X_2 + \lambda_1^{k-1} X_4 + \lambda_2^{k-1} X_6 & k &= 1, 2, 3, \\
\mathcal{Z}_r &= \lambda_3^{r-1} Y_1 + \lambda_1^{r-1} Y_2 + \lambda_2^{r-1} Y_3 & r &= 1, 2, 3.
\end{align*}
\]

We note that a similar construction of $\mathcal{X}_l$’s, $\mathcal{Y}_k$’s and $\sigma$ have been used in the proof of Proposition 3.7. Let $\beta' = \{\mathcal{X}_l, \mathcal{Y}_k, \mathcal{Z}_r | l, k, r = 1, 2, 3\}$. It can seen that $[\sigma]_{\beta'} \in GL(9, \mathbb{Z})$ To show that $\beta'$ is a $\mathbb{Z}$-basis of $n$, we note that

\[
\lambda_1 = \xi + \xi^8, \quad \lambda_2 = \xi^2 + \xi^7, \quad \lambda_3 = \xi^4 + \xi^5,
\]

where $\xi = e^{2i\pi/9}$ a ninth root of unity.

Hence if $P(\lambda) = 2\lambda^2 + \lambda - 4$, then

\[
\lambda_2 - \lambda_3 = P(\lambda_1), \quad \lambda_3 - \lambda_1 = P(\lambda_2), \quad \lambda_1 - \lambda_2 = P(\lambda_3).
\]

Moreover, $P(\lambda_i)\lambda_i = \lambda_i^2 + 2\lambda_i - 2$ for $1 \leq i \leq 3$. Bearing all this in mind, straightforward calculation shows that

\[
\begin{align*}
[X_j, X_k] &= [Z_j, Z_k] = 0 & 1 \leq j, k \leq 3, \\
[X_1, X_2] &= -4Z_1 + Z_2 + 2Z_3, & [X_1, Y_3] &= [X_3, Y_1] = -Z_2, \\
[X_1, X_3] &= 2Z_1 - 2Z_2 - Z_3, & [X_1, Y_2] &= [X_2, Y_1] = 6Z_1 - Z_2.
\end{align*}
\]

Hence $u_2 + u_3 + m_3$ is an Anosov Lie algebra.

To conclude, let us study Case (b). We recall that in this case, our assumption is

\[
[X_i, X_j] = 0 = [X_k, X_l] \quad \text{for} \quad 1 \leq i, j \leq 3, \quad 4 \leq k, l \leq 6.
\]

We will show that $n$ is isomorphic to one of the Lie algebras we came across in Case (a). In other words, we don’t get any new Lie algebras in this case.

We note that we can assume

\[
[X_1, X_4] = Y_1, \quad [X_2, X_5] = Y_2, \quad [X_3, X_6] = Y_3.
\]

In fact, if this is not the case, rearranging a basis we can only assure that

\[
[X_1, X_4] = Y_1, \quad [X_i, X_j] = Y_2, \quad [X_k, X_l] = Y_3,
\]

and we are done. \(\square\)
\[ i, k \leq 3 \text{ and } i \text{ or } k = 1. \]

Let us say that \( [X_1, X_3] = Y_2 \) and then it is not hard to check that \( k \neq 1 \) so we may assume that \( k = 2 \). Also, since \( l \neq 6 \) \( (l = 6 \) implies the contradiction \( \lambda_1 = \lambda_3 \) \), say \( l = 5 \) \( (l = 4 \) is similar). Then we get at least the following nonzero Lie brackets:

\[ [X_1, X_4] = Y_1, \quad [X_1, X_5] = Y_2 \quad [X_2, X_3] = Y_3. \]

Therefore \( \lambda_2^2 \lambda_3 \lambda_4 \lambda_5^2 = 1 \) or equivalently, \( \lambda_1 \lambda_3 = \lambda_3 \lambda_5 \) and from this, since \( n \) has no abelian factor, by considering all possibilities for \( [X_r, X_l] = Y_k \) and \( [X_3, X_4] = Y_l \) it is not hard to check that we should have \( [X_3, X_6] = Y_2 \), as desired, since any other possibility leads to a contradiction.

If \( n \) has only the nonzero Lie brackets given in (21) then \( n \simeq U_l \) which is Anosov. If there are more nonzero Lie brackets, with no loss of generality, we may assume that \( [X_3, X_5] = a Y_1 \), and hence we have that

\[ \mu_1 = \lambda_1 \lambda_4 = \lambda_3 \lambda_5. \]

Note that since \( \lambda_i \)'s are distinct for \( 1 \leq i \leq 3 \) and \( \lambda_j \)'s are distinct for \( 4 \leq j \leq 6 \), if \( [X_i, X_j] = Y_k = [X_r, X_l] \) then \( i \neq l \) and \( j \neq r \). From this, it can be seen that the nonzero Lie brackets in \( n \) are given by

\[ [X_1, X_4] = Y_1, \quad [X_2, X_3] = Y_2 \quad [X_3, X_6] = Y_3, \]

\[ [X_3, X_5] = a Y_1, \quad [X_1, X_6] = b Y_2 \quad [X_2, X_4] = c Y_3, \]

(22)

for some constants \( a, b, c \in \mathbb{C} \). By reordering the basis

\[ \beta' = \{ X_1, X_4, X_2, X_5, X_3, X_6, Y_1, Y_2, Y_3 \} \]

one can see that this algebra is isomorphic to \( n(a, b, c) \) given in (13) (from Case (a)) that has already been studied.

Finally, if \( n \) is Anosov and has an abelian factor, and if \( n = \bar{n} \oplus m \), where \( m \) is a maximal abelian factor of \( n \), according to [13, Theorem 3.1] one has that \( \dim m \geq 2 \). Moreover, \( \bar{n} \) is also an Anosov Lie algebra and \( \dim \bar{n} \leq 7 \). Since there are no 7 dimensional nonabelian Anosov Lie algebras, we then have that \( n \) is isomorphic to one of the following (see [14]):

- \( \mathbb{R}^9 \)
- \( h_3 \oplus h_3 \oplus \mathbb{R}^3 \)
- \( f_3 \oplus \mathbb{R}^3 \),

where \( h_3 \) is the 3-dimensional Heisenberg algebra, and \( f_3 \) is the free 2-step nilpotent Lie algebra on 3 generators.

Summarizing the results of this section we have the following Theorem.

**Theorem 3.12.** If \( n \) is a 9-dimensional Anosov Lie algebra then

i: if it has no abelian factor, then one of the following holds

- it is a type \((3, 3, 3)\) nilpotent Lie algebra isomorphic to \( n(1, 1, 1) \) given in (2) or
- it is a two step nilpotent Lie algebra of type \((6, 3)\) and its Pfaffian form is projectively equivalent to \( xyz \) or 0. Moreover it is isomorphic to one of the following nonisomorphic Lie algebras: \( U_4, U_4 + m_i \) for \( i = 6, 7, 10, 11, 12, 14, sU_2 + U_5 \) \( s \in \mathbb{C} \) or \( U_2 + U_3 + m_3 \).

ii: If \( n \) has an abelian factor then it is isomorphic to \( \mathbb{R}^9, h_3 \oplus h_3 \oplus \mathbb{R}^3 \) or \( f_3 \oplus \mathbb{R}^3 \).

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