AN $l^p$-VERSION OF VON NEUMANN DIMENSION FOR BANACH SPACE REPRESENTATIONS OF SOFIC GROUPS II

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1. INTRODUCTION

This is intended as a follow up paper to [12]. Let us first recall some necessary notation and definitions.

Definition 1.1. Let $V$ be a Banach space. We shall use $\text{Isom}(V)$, for the group of all linear, surjective, isometric maps from $V$ to itself.

Definition 1.2. For $\sigma, \tau \in S_n$ (here $S_n$ is the group of self-bijections of $\{1, \cdots, n\}$) we define the Hamming distance by

$$d_{\text{Hamm}}(\sigma, \tau) = \frac{1}{n} |\{j : \sigma(j) \neq \tau(j)\}|,$$

can also be seen as the probability that $\sigma \neq \tau$, using the uniform probability measure on $\{1, \cdots, n\}$.

Definition 1.3. Let $\Gamma$ be a countable discrete group. A sofic approximation of $\Gamma$ is a sequence $\sigma_i : \Gamma \to S_{d_i}$ of functions, not assumed to be homomorphisms, with $\sigma_i(e) = 1$ such that

$$d_{\text{Hamm}}(\sigma_i(gh), \sigma_i(g)\sigma_i(h)) \to 0, \text{ for all } g, h \in \Gamma$$
$$d_{\text{Hamm}}(\sigma_i(g), \sigma_i(h)) \to 1, \text{ for all } g \neq h \in \Gamma$$

We say that $\Gamma$ is sofic, if it has a sofic approximation.

Definition 1.4. On $M_n(\mathbb{C})$ we shall use $\text{tr} = \frac{1}{n} \text{Tr}$, where $\text{Tr}$ is the usual trace. We shall use $\langle A, B \rangle = \text{tr}(B^*A)$, and $\|A\|_p = \text{tr}((A^*A)^{p/2})^{1/p}$. Finally, we shall use $\|A\|_{\infty}$ for the operator norm of $A$.

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In fact, if $M$ is a von Neumann algebra with a faithful normal tracial state $\tau$, we will use
\[ \|x\|_p = \tau((x^*x)^{p/2})^{1/p} \]
and $\|x\|_\infty$ for the operator norm of $x \in M$.

**Definition 1.5.** Let $\Gamma$ be a countable discrete group. An embedding sequence of $\Gamma$ is a sequence $\sigma_i: \Gamma \to U(d_i)$, (here $U(n)$ is the unitary group of $\mathbb{C}^n$) so that $\sigma_i(e) = \text{Id}$ and
\[ \|\sigma_i(gh) - \sigma_i(g)\sigma_i(h)\|_2 \to 0 \text{ for all } g, h \in \Gamma \]
\[ (\sigma_i(g), \sigma_i(h)) \to 0 \text{ for all } g \neq h \in \Gamma. \]

We say that $\Gamma$ is $R^\omega$-embeddable if it has an embedding sequence.

It is a simple exercise to show that every sofic group is $R^\omega$-embeddable. The class of sofic groups includes amenable groups, residually finite groups, and is closed under free products with amalgamation over amenable groups, increasing unions, and taking subgroups (see [8] for proofs of these facts). In particular, all linear groups are sofic. If the reader is knowledgeable about operator algebras, then it will not be difficult to show that $\Gamma$ is $R^\omega$-embeddable if the group von Neumann algebra $L(\Gamma)$ embeds into an (hence any) ultrapower of the hyperfinite $II_1$ factor. This is in explicitly shown in [12].

In [12] if we are given a countable discrete group $\Gamma$ and $\Sigma = (\sigma_i: \Gamma \to \text{Isom}(V_i))$ with $\dim(V_i) < \infty$, then to every uniformly bounded action of $\Gamma$ on a Banach space $V$, we have a number
\[ \dim_\Sigma(V, \Gamma) \in [0, \infty]. \]
when $\Gamma$ is a sofic group $V_i = l^p(d_i)$ and $\sigma_i$ comes from a sofic approximation (also denoted $\sigma_i: \Gamma \to S_{d_i}$) and the permutation action of $S_n$ on $l^p(n)$ we denote this number by
\[ \dim_{\Sigma, l^p}(V, \Gamma). \]
Similarly, if $\Gamma$ is $R^\omega$-embeddable and $\Sigma$ comes from a embedding sequence (also denoted $\sigma_i: \Gamma \to U(d_i)$) we let
\[ \dim_{\Sigma, S^p, multi}(V, \Gamma), \]
\[ \dim_{\Sigma, S^p, conj}(V, \Gamma) \]
be the dimensions coming from the left multiplication and conjugation actions of $U(n)$ on $S^p(n)$, respectively.

In [12], we proved the following properties of this dimension function

Property 1: $\dim_{\Sigma}(Y, \Gamma) \leq \dim_{\Sigma}(X, \Gamma)$ if there is an equivariant bounded linear map $X \to Y$ with dense image.

Property 2: $\dim_{\Sigma}(V, \Gamma) \leq \dim_{\Sigma}(W, \Gamma) + \dim_{\Sigma}(V/W, \Gamma)$, if $W \subseteq V$ is a closed $\Gamma$-invariant subspace.

Property 3: $\dim_{\Sigma, l^p}(Y \oplus W, \Gamma) \geq \dim_{\Sigma, l^p}(Y, \Gamma) + \dim_{\Sigma, l^p}(W, \Gamma)$ for $2 \leq p < \infty$, where $\dim$ is a “lower dimension,” and is also an invariant, further

Property 4: $\dim_{\Sigma, l^p}(l^p(\Gamma, V), \Gamma) = \dim_{\Sigma, l^p}(l^p(\Gamma, V)) = \dim(V)$ for $1 \leq p \leq 2$,

Property 5: $\dim_{\Sigma, l^p}(Y, \Gamma) \geq \dim_{l^p(\Gamma, Y)}(Y||Y)_2$ if $1 \leq p \leq 2$, $Y \subseteq l^p(N, l^p(\Gamma))$,

Property 6: $\dim_{\Sigma, S^p, conj}(Y, \Gamma) \geq \dim_{l^p(\Gamma, Y)}(Y||Y)_2$ if $1 \leq p \leq 2$, $Y \subseteq l^p(N, l^p(\Gamma))$,

Property 7: $\dim_{\Sigma, l^2}(H, \Gamma) = \dim_{\Sigma, l^2}(H, \Gamma) = \dim_{l^2(\Gamma)} H$ if $H \subseteq l^2(N, l^2(\Gamma))$. 

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Thus \( \dim_{\Sigma,t} \) can be seen as an extension of the von Neumann dimension of a \( \Gamma \) invariant subspace of \( l^2(\mathbb{N}, \Gamma) \) due to Murray and von Neumann. The above shows that \( \dim_{\Sigma,t} \) has many of the properties that the usual dimension in linear algebra and the von Neumann dimension have, and this it makes sense to think of \( \dim_{\Sigma,t} \) as a version of von Neumann dimension. Further in [13] it is shown that we cannot expect any invariant for \( \Gamma \) invariant subspaces of \( l^p(\Gamma)^{\otimes n} \) to have all the properties of von Neumann dimension. In particular, they show that if \( \Gamma \) contains an infinite elementary amenable subgroup and \( 2 < p < \infty \), then there exits closed \( \Gamma \)-invariants subspaces \( E_n \) and \( F \neq \{0\} \) of \( l^p(\Gamma) \) so that \( E_n \cap F = \{0\} \) for all \( n \), but
\[
l^p(\Gamma) = \bigcup_{n=1}^{\infty} E_n.
\]

This is impossible for \( p = 2 \), because of von Neumann dimension. Taking polar, if \( 1 < p < 2 \), then we can find a closed \( \Gamma \)-invariant subspace \( W \subseteq l^p(\Gamma) \), with \( W \neq l^p(\Gamma) \) and decreasing closed \( \Gamma \)-invariant subspaces \( V_n \subseteq l^p(\Gamma) \) such that \( \overline{W + \bigcap V_n} = l^p(\Gamma) \) and \( \bigcap_{n=1}^{\infty} V_n = \{0\} \). We thus have a \( \Gamma \)-equivariant map with dense image
\[
V_n \oplus W \to l^p(\Gamma),
\]
so
\[
1 \leq \dim_{\Sigma,t}(V_n, \Gamma) + \dim_{\Sigma,t}(W, \Gamma).
\]

Thus one of two things occurs. Either
\[
\lim_{n \to \infty} \dim_{\Sigma,t}(V_n, \Gamma) > 0
\]
or
\[
\dim_{\Sigma,t}(W, \Gamma) \geq 1
\]
and \( W \) is a proper \( \Gamma \)-invariant subspace of \( l^p(\Gamma) \).

Let us briefly recall our definition of dimension.

**Definition 1.6.** Let \( \Gamma \) be a countable discrete group and \( \Sigma = (\sigma_i : \Gamma \to \text{Isom}(V_i)) \). Let \( \Gamma \) have a uniformly bounded action on a Banach space \( X \), and let \( S = (x_j)_{j=1}^{\infty} \) be a bounded sequence in \( X \). For \( F \subseteq \Gamma \) finite, and \( m \in \mathbb{N} \), let \( X_{F,m} = \text{Span}\{g x_j : g \in F^k, 1 \leq j, k \leq m\} \). For \( M, \delta > 0 \), we let \( \text{Hom}_F(S, F, m, \delta, \sigma_i)_M \) consist of all bounded linear maps \( T : X_{F,m} \to V_i \) such that \( \|T\| \leq M \), and
\[
\|T(g_1 \cdots g_k x_j) - \sigma_i(g_1) \cdots \sigma_i(g_k) T(x_j)\| < \delta \quad \text{for all } g_1, \cdots, g_k \in F, 1 \leq j, k \leq m.
\]
We shall typical denote \( \text{Hom}_F(S, F, m, \delta, \sigma_i)_1 \) by \( \text{Hom}_F(S, F, m, \delta, \sigma_i) \). If \( \Sigma \) is a sofic approximation, we use \( \text{Hom}_F^p(S, F, m, \delta, \sigma_i) \) for the space of maps above using the permutation action of \( S_d \) on \( l^p(d_i) \).

**Definition 1.7.** Let \( V \) be a vector space with pseudonorm \( \rho \). For \( A \subseteq V \) and \( \varepsilon > 0 \) we say that a linear subspace \( W \subseteq V \varepsilon \)-contains \( A \), written \( A \subseteq_{\varepsilon} W \), if for all \( x \in A \) there is a \( w \in W \) so that \( \rho(x - w) < \varepsilon \). We set \( d_\varepsilon(A, \rho) \) to be the smallest dimension of a linear subspace which \( \varepsilon \)-contains \( A \).

**Definition 1.8.** A *product norm* on \( l^\infty(\mathbb{N}) \) is a norm \( \rho \) so that \( \rho(f) \leq \rho(g) \) if \( |f| \leq |g| \), and so that \( \rho \) induces the topology of pointwise convergence on \( \{f \in l^\infty(\mathbb{N}) : \|f\|_\infty \leq 1\} \). If \( \rho \) is a product norm, and \( V \) is a Banach space we define \( \rho_V \) on \( l^\infty(\mathbb{N}, V) \) by \( \rho_V(f) = \rho(\|f\|) \).
Definition 1.9. Let $\Gamma, S, \Sigma, X$ be as defined. For $F \subseteq \Gamma$ finite, and $m \in \mathbb{N}$ let $\alpha_S \colon B(X_{F,m}, V_1) \rightarrow l^\infty(\mathbb{N}, V_1)$ be given by $\alpha_S(T)(n) = \chi_{T^{-m}}(m)T(x_j)$. We define

$$\dim_{\Sigma}(S, F, m, \delta, \varepsilon, \rho) = \limsup_{i \to \infty} \frac{1}{\dim(V_i)} d_\varepsilon(\alpha_S(\text{Hom}_F(S, F, m, \delta, \sigma_i), \rho V_i)).$$

$$\dim_{\Sigma}(S, \varepsilon, \rho) = \limsup_{F, m, \delta} d_\varepsilon(\alpha_S(\text{Hom}_F(S, F, m, \delta, \sigma_i), \rho V_i)).$$

$$\dim_{\Sigma}(S, \rho) = \sup_{\varepsilon > 0} \dim_{\Sigma}(S, \varepsilon, \rho).$$

Here the triples $(F, m, \delta)$ are ordered by $(F, m, \delta) \leq (F', m', \delta')$ if $F \subseteq F'$, $m \leq m'$, $\delta' < \delta$. In [12] it is shown that

$$\dim_{\Sigma}(S, \rho) = \dim_{\Sigma}(S', \rho')$$

if $\text{Span}(\Gamma S) = \text{Span}(\Gamma S')$, and $\rho, \rho'$ are product norms. Because of this we call this number $\dim_{\Sigma}(X, \Gamma)$ if $\text{Span}(\Gamma S) = X$. Also we showed that

$$\dim_{\Sigma}(X, \Gamma) = \sup_{\varepsilon > 0} \liminf_{i \to \infty} \frac{1}{\dim(V_i)} d_\varepsilon(\text{Hom}_F(S, F, m, \delta, \sigma_i), \rho V_i)).$$

It is a simple exercise to show that $\text{Hom}_F(S, F, m, \delta, \sigma_i)$ may be replaced by $\text{Hom}_F(S, F, m, \delta, \sigma_i)_M$ for any $M > 0$.

We let $\dim_{\Sigma}(X, \Gamma)$ be the number obtained by replacing the first limit supremum with a limit infimum (again one can show this depends only on $\text{Span}(\Gamma S)$). Similar to above we showed that

$$\dim_{\Sigma}(X, \Gamma) = \sup_{\varepsilon > 0} \liminf_{i \to \infty} \frac{1}{\dim(V_i)} d_\varepsilon(\alpha_S(\text{Hom}_F(S, F, m, \delta, \sigma_i), \rho V_i)).$$

These definitions may seem quite technical and bizarre, but they are really inspired by ideas of Bowen [2], Kerr and Li [14], Gornay [10] and Voiculescu [25]. The main point is that one should view the usual von Neumann Dimension as a type of dynamical entropy, thinking of the action of $\Gamma$ on $l^2(\Gamma)$ as an analogue of a Bernoulli shift action of $\Gamma$. Bowen in [2], and Kerr and Li in [14] give a microstates version of dynamical entropy for sofic groups. Similar to Kerr and Li, we consider “almost structure-preserving maps” (in this case almost equivariant maps), and measure the growth rate of the size of the space of such maps. Here it makes sense to consider the linear growth rate, since $\varepsilon$-dimension can grow at most linearly.

We prove some of the conjectures stated in [12]. We show the following new properties of $p$-Dimension

Property 1: $\dim_{\Sigma,p}(H^p_0(\mathbb{F}_n), \mathbb{F}_n) = \dim_{\Sigma,p}(H^p_1(\mathbb{F}_n), \mathbb{F}_n) = n - 1$.

Property 2: $\dim_{\Sigma,p}(H^p_1(\mathbb{F}_n), \mathbb{F}_n) = \dim_{\Sigma,p}(H^p_0(\mathbb{F}_n), \mathbb{F}_n) = n - 1$.

Property 3: $\dim_{\Sigma, \text{Sp}, \text{multi}}(\bigoplus_{j=1}^n L^p(L(\Gamma)q_j, \tau)) = \sum_{j=1}^n \tau(q_j)$, for $1 \leq p < \infty$, where $q_1, \ldots, q_n$ are projections in $L(\Gamma)$ and $\tau$ is the group trace.

Property 4: $\dim_{\Sigma,p}(X, \Gamma) = 0$ if $X$ is finite-dimensional and $\Gamma$ is infinite.

Here $H^p_0, H^p_1$ are $p$-Homology and $p$-Cohomology spaces.

Our approach to proving the last property is to consider a free, ergodic, probability measure-preserving action of $\Gamma$, so that the associated equivalence relation $\mathcal{R}_\Gamma$ is sofic (the Bernoulli action for example). Because $\mathcal{R}_\Gamma$ contains an amenable equivalence relation we can try to adapt the proof of the last property in the case $\Gamma = \mathbb{Z}$. This suggests exploring $p$-Dimension for Representations of Equivalence Relations, which is part of ongoing research.
We also give an equivalent approach to $l^p$-Dimension defined by using vectors instead of almost equivariant operators.

2. Triviality In The Case of Finite-Dimensional Representations

In this section we prove the following.

**Theorem 2.1.** Let $\Gamma$ be a infinite sofic group, and $\Sigma$ a sofic approximation of $\Gamma$. Then for every $1 \leq p \leq \infty$,

$$\dim l^p_{\Sigma}(X, \Gamma) = 0.$$ 

Here is the outline of the proof. We will begin by studying $l^p$-dimension for amenable groups, using the standard technique of averaging over Følner sequences. Using these techniques, we show that for finite $\Gamma$,

$$\dim l^p_{\Sigma}(X, \Gamma) = \dim \mathcal{C}_{\Gamma}^X.$$ 

This easily implies proves the theorem when $\Gamma$ has finite subgroups of unbounded size. We then show that

$$\dim l^p_{\Sigma}(X, \Sigma) = 0,$$

if $X$ is finite-dimensional. We may thus assume that $\Gamma$ has no elements of infinite order, but that there is a uniform bound on the size of a finite subgroup of $\Gamma$, a compactness argument will show that $\Gamma$ has an infinite subgroup which acts on $X$ trivially, so we only have to show that

$$\dim l^p_{\Sigma}(\mathcal{C}, \Gamma) = 0,$$

where $\Gamma$ acts trivially on $\mathcal{C}$. To prove this last statement, we will pass to a sofic equivalence relation induced by the group, and use that the full group of such an equivalence relation contains $\mathbb{Z}/n\mathbb{Z}$ for every integer $n$.

We first show that in the case of an amenable group action, we may assume that the maps we use to compute dimension are only approximately equivariant after cutting down by certain subsets. We formalize this as follows.

**Definition 2.2.** Let $\Gamma$ be a sofic group with a uniformly bounded action on a Banach space $X$. Let $\sigma_i: \Gamma \to S_{d_i}$ be a sofic approximation, fix $S = (a_j)_{j=1}^\infty$ a bounded sequence in $X$. Let $A_i \subseteq \{1, \ldots, d_i\}$, for $F \subseteq \Gamma$ finite, $m \in \mathbb{N}, \delta > 0$, we set $\text{Hom}_{l^p}(S, F, m, \delta, \sigma_i)$ to be all linear maps $T: X_{F,m} \to l^p(d_i)$ such that $\|T\| \leq 1$, and for all $1 \leq j, k \leq m$, for all $s_1, \ldots, s_k \in F$ we have

$$\|T(s_1 \cdots s_k a_j) - \sigma_i(s_1) \cdots \sigma_i(s_k) T(a_j)\|_{l^p(A_i)} < \delta.$$ 

Set

$$\dim l^p_{\Sigma}(S, \Gamma, (A_i), \rho) = \sup_{\epsilon > 0} \inf_{\text{finite } F \subseteq \Gamma} \lim \sup_{m \in \mathbb{N}} \inf_{\delta > 0} \frac{1}{d_i} d_\epsilon(\alpha_S(\text{Hom}_{l^p}(S, F, m, \delta, \sigma_i)), \rho),$$

where $\rho$ is any product norm.

**Proposition 2.3.** Fix a product norm $\rho$ on $l^\infty(\mathbb{N})$. Let $\Gamma$ be a countable amenable group, and $\Sigma = (\sigma_i: \Gamma \to S_{d_i})$ a sofic approximation. Let $A_i \subseteq \{1, \ldots, d_i\}$ be such that

$$\frac{|A_i|}{d_i} \to 1.$$
Then for any uniformly bounded action of $\Gamma$ on a separable Banach space $X$, for every generating sequence $S$ in $X$, for every product norm $\rho$, and $1 \leq p < \infty$ we have

$$\dim_{\Sigma, tr}(X, \Gamma) = \dim_{\Sigma, \rho}(S, \Gamma, (A_i), \rho)$$

**Proof.** Fix $S = (x_j)_{j=1}^\infty$ a dynamically generating sequence for $X$. We first fix some notation, for $E, \in F$ finite subsets of $\Gamma$ containing the identity and $m \in \mathbb{N}$ define

$$P_i^{(E)} : B(X_{EF, m}, \|p\|_i(d_i)) \rightarrow B(X_{F, m}, \|p\|_i(d_i))$$

by

$$P_i^{(E)}(T) = \frac{1}{|E|} \sum_{s \in E} \sigma_i(s) \circ T \circ s^{-1},$$

then $\|P_i^{(E)}\| \leq 1$. Note that for $s_1, \ldots, s_k \in F$ and $T \in B(X_{F, k}, \|p\|_i(d_i))$ that

$$P_i^{(E)}(T)(s_1 \cdots s_k x) = \frac{1}{|E|} \sum_{s \in E} \sigma_i(s) T(s^{-1} s_1 \cdots s_k x) =$$

$$\frac{1}{|E|} \sum_{s \in s_1^{-1} \cdots s_k^{-1} E} \sigma_i(s_1 \cdots s_k s) T(s^{-1} x).$$

If $B_i \subseteq \{1, \ldots, d_i\}$ is the set of all $j$ such that

$$\sigma_i(s_1 \cdots s_k s)^{-1}(j) = \sigma_i(s)^{-1} \sigma_i(s_1 \cdots s_k)^{-1}(j),$$

for all $s \in E, s_1, \ldots, s_k \in F, 1 \leq k \leq m$, then the above shows that if $T \in B(X_{F, E, m}, \|p\|(B_i))$ then

$$\|\sigma_i(s_1 \cdot s_k) \circ P_i^{(E)}(T)(x_j) - P_i^{(E)}(T)(s_1 \cdots s_k j)(x_j)\| \leq \frac{2|E|s_k^{-1} \cdots s_1^{-1} E|}{|E|} \|T\| \|x_j\|,$$

for $1 \leq j \leq m$.

Suppose $T \in \text{Hom}_{\Gamma, tr}(A_i) (S, F, m, \delta, \sigma_i \in F$ is symmetric $m \geq 2$, and $E \supseteq F$, then

$$P_i^{(E)}(\chi_{B_i} T) = \frac{1}{|E|} \sum_{s \in E} \sigma_i(s) \chi_{B_i} T \circ s^{-1} =$$

$$\frac{1}{|E|} \sum_{s \in E} \chi_{\sigma_i(s) B_i} \sigma_i(s) T \circ s^{-1}.$$

Set $C_i = A_i \cap B_i \cap \bigcap_{s \in E} \sigma_i(s) (A_i \cap B_i)$, then $\frac{|C_i|}{d_i} \rightarrow 1$, and for $1 \leq j \leq m$

$$\|P_i^{(E)}(\chi_{B_i} T)(x_j) - T(x_j)\|_{\|p\|(C_i)} \leq \frac{1}{|E|} \sum_{s \in E} \|\sigma_i(s) T(s^{-1} x_j) - T(x_j)\|_{\|p\|(A_i)} < 2\delta.$$

The claim now easily follows by using a two-sided Følner sequence. 

**Corollary 2.4.** Let $\Gamma$ be an amenable group with a uniformly bounded action on a separable Banach space $X$. Let $\Sigma = (\sigma_i : \Gamma \rightarrow S_{d_i})$, $\Sigma' = (\sigma'_i : \Gamma \rightarrow S_{d_i})$ be two sofic approximations, then for all $1 \leq p \leq \infty$,

$$\dim_{\Sigma, \rho}(X, \Gamma) = \dim_{\Sigma', \rho}(X, \Gamma)$$

for all $1 \leq p < \infty$. 


Proof. Because any two sofic embeddings into $\prod d_i$ are conjugate (see [8]), a simple ultrafilter argument shows that we can find $\tau_i : S_{d_i} \to S_{d_i}$ such that

$$d_{\text{Hamm}}(\tau_i \sigma_i(s) \tau_i^{-1}, \sigma_i(s')) \to 0.$$  

Replacing $\sigma_i$ by $\tau_i \circ \sigma_i \circ \tau_i^{-1}$, we may assume that

$$d_{\text{Hamm}}(\sigma_i(s), \sigma'_i(s)) \to 0$$  

for all $s \in \Gamma$. In this case, we can find $A_i \subseteq \{1, \cdots, d_i\}$ such that

$$|A_i| \to 1$$  

and for all $s_1, \cdots, s_n \in \Gamma$, we have

$$\sigma_i(s_1 \cdots s_n)(j) = \sigma_i(s_1) \cdots \sigma_i(s_n)(j) = \sigma'_i(s_1) \cdots \sigma'_i(s_n)(j)$$  

for all $j \in A_i$ and all sufficiently large $i$. Thus if $F \leq \Gamma$ is finite, $m \in \mathbb{N}$, $\delta > 0$ then for all large $i$,

$$\text{Hom}_{\Gamma, lp(A_i)}(S, F, m, \delta, \sigma_i) = \text{Hom}_{\Gamma, lp(A_i)}(S, F, m, \delta, \sigma'_i).$$

\[\square\]

Proposition 2.5. Let $\Gamma$ be a finite group acting on a finite-dimensional vector space $X$. For $n \in \mathbb{N}$, let

$$n = q_n |\Gamma| + r_n$$  

where $0 \leq r_n < |\Gamma|$ and $q_n, r_n \in \mathbb{N}$. Let $A_n$ be a set of size $r_n$ and define a sofic approximation $\Sigma = (\sigma_n : \Gamma \to \text{Sym}(\Gamma \times \{1, \cdots, q_n\}) \prod A_n)$ by

$$\sigma_n(s)(g, j) = (sg, j) \text{ for } s \in \Gamma, 1 \leq j \leq q_n$$  

$$\sigma_n(s)(a) = a \text{ for } a \in A_n.$$  

Then for any $1 \leq p \leq \infty$

$$\dim_{\Sigma, lp}(X, \Gamma) = \dim_{\Sigma, l^p}(X, \Gamma) = \frac{\dim_{\Sigma} X}{|\Gamma|}.$$  

Proof. Fix a norm on $X$. By finite dimensionality we may use the operator norm on $B(X, l^p(d_i))$ as our pseudonorm, and we replace $\text{Hom}_\Gamma(S, \Gamma, m, \delta, \sigma_i)$ by the space $\text{Hom'}(\Gamma, m, \delta, \sigma_i)$ of all operators $T : X \to l^p(d_i)$ such that

$$||T \circ s_1 \cdots s_k - \sigma_i(s_1) \cdots \sigma_i(s_k) \circ T|| < \delta$$

for all $1 \leq k \leq m, s_1, \cdots, s_k \in \Gamma$.

Let $V_n \subseteq B(X, l^p(n))$ be the linear subspace of all linear operators

$$T : X \to l^p(\Gamma \times \{1, \cdots, q_n\})$$

which are equivariant with respect to the $\Gamma$-action. Note that we have norm one projections

$$B(X, l^p(n)) \to B(X, l^p(\Gamma \times \{1, \cdots, q_n\}))$$

$$B(X, l^p(\Gamma \times \{1, \cdots, q_n\}) \to V_n,$$

given by multiplication by $\chi_{\{1, \cdots, q_n\}}$ and by

$$T \to \frac{1}{|\Gamma|} \sum_{s \in \Gamma} \sigma_n(s)^{-1} \circ T \circ s,$$
let $P_n$ denote the composition of these two projections. Since we have a norm one projection form $B(X, l^p(n)) \rightarrow V_n$, a quick application of the Riesz Lemma implies that

\[ d_c(V_n, \| \cdot \|) \geq \dim V_n = (\dim_{\mathbb{C}} X)q_n, \]

with the norm being the operator norm. Further for $T \in \text{Hom}_r^f(\Gamma, m, \delta, \sigma_i)$ we have

\[ \|P_n(T) - T\|_{B(X, l^p(n))} < \delta. \]

Thus

\[ d_c(\text{Hom}_r^f(\Gamma, m, \delta, \sigma_i)) \leq (\dim_{\mathbb{C}} X)q_n + r_n, \]

and (1), (2) are enough to imply the proposition. \hfill \Box

Corollary 2.6. Let $\Gamma$ be a finite group acting on a finite-dimensional vector space $X$. For any finite dimensional representation $X$ of $\Gamma$, for any sofic approximation $\Sigma = (\sigma_i : \Gamma \rightarrow S_{d_i})$ of $\Gamma$ and $1 \leq p \leq \infty$ we have

\[ \dim_{\Sigma, l^p}(X, \Gamma) = \dim_{\Sigma, l^p}(X, \Gamma) = \frac{\dim_{\mathbb{C}} X}{|\Gamma|}. \]

Proof. Take

\[ \Sigma' = (\rho_n : \Gamma \rightarrow S_{d_i}) \]

where $\rho_n$ is defined as in the previous proposition, then use the fact that two sofic approximations into the same size symmetric groups give the same dimension. \hfill \Box

Proposition 2.7. Let $X$ be a finite-dimensional Banach space with a uniformly bounded action of $\mathbb{Z}$. Let $\sigma_n : \mathbb{Z} \rightarrow \text{Sym}(\mathbb{Z}/n\mathbb{Z})$ be given by the quotient map $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$. Then for all $1 \leq p \leq \infty$,

\[ \dim_{\Sigma, l^p}(X, \mathbb{Z}) = 0. \]

Proof. By standard averaging tricks, we may assume that $X$ is a Hilbert space and that $\mathbb{Z}$ acts by unitaries. Since $X$ is now a Hilbert space, we will call it $H$ instead. Let $\pi : \mathbb{Z} \rightarrow U(H)$ be the representation given by the action of $\Gamma$, and set $U = \pi(1)$. By passing to direct sums, we may assume that $\pi$ is irreducible, so if we fix any $\xi \in H$ with $\|\xi\| = 1$, then $\xi$ is generating. We will take $S = (\xi)$, and as a pseudonorm we take

\[ \|T\| = \|T(\xi)\|. \]

Fix $1 > \varepsilon > 0$, and let $\varepsilon > \delta > 0$. Choose $k$ such that $k^p < \varepsilon$, (if $p = \infty$ then let $k$ be any integer.) Since $\pi(\mathbb{Z})$ is compact, we can find an integer $m$ such that

\[ \|U^m - 1\| < \delta, \]

for $1 \leq j \leq k$. We may assume that $m$ is large enough so that $\{U^j\xi : -m \leq j \leq -1\}$ spans $H$. Let $F = \{j \in \mathbb{Z} : |j| \leq m(2k + 1)\}$, finally let $q_n \in \mathbb{N} \cup \{0\}, 0 \leq r_n < k$ be the integers defined by

\[ n = q_n mk + r_n. \]

Define $Q_j, j = 0, \cdots, k - 1$ by

\[ Q_j = \bigcup_{l=1}^{m} \{jm + l + qm_k : 0 \leq q \leq q_n - 1\}, \]
pictorially, if we think of \(\{1, \ldots, q_n mk\}\) as a rectangle formed out of \(mk\) horizontal dots and \(q_n\) vertical dots, then \(Q_j\) is the rectangle from the \((jm + 1)^{st}\) horizontal dot to the \((j + 1)m^{th}\) horizontal dot. Let \(f_j: Q_j \to \mathbb{C}\) be given by

\[f_j(l) = T(\xi)(\sigma_i(mj)^{-1}(l)).\]

Note that for \(1 \leq p < \infty\),

\[
\left\| T(\xi) - \sum_{j=0}^{k-1} f_j \right\|_p^{p} = \sum_{j=0}^{k-1} \left\| T(\xi) - \sigma_i(mj)T(\xi) \right\|_p^{p} < \delta^p k + \sum_{j=0}^{k-1} \left\| T((U^{-mj} - 1)\xi) \right\|_p^{p} < 2\delta^p k < 2\varepsilon.
\]

Similarly for \(p = \infty\),

\[
\left\| T(\xi) - \sum_{j=0}^{k-1} f_j \right\|_\infty < 2\varepsilon.
\]

Finally note that \(\sum_{j=0}^{k-1} f_j\) is constant on \(\{i, i + m, \cdots, i + m(k - 1)\}\) for each \(i \in Q_0\). Thus

\[
\hat{d}_{2\varepsilon}(\text{Hom}_\Sigma((\xi), F, m, \delta, \sigma_i)) \leq q_n m + r_n.
\]

This implies that

\[
\text{dim}_{\Sigma, l^p}(S, F, m, \delta, \sigma_i) \leq \frac{1}{k},
\]

and since \(k\) becomes arbitrary large when \(\delta\) becomes small (or can be made arbitrarily large when \(p = \infty\)), this completes the proof.

We will now proceed to prove that if \(\Gamma\) is an infinite sofic group, and \(\Sigma\) is a sofic approximation of \(\Gamma\), then for any finite-dimensional representation \(V\) of \(\Gamma\) we have

\[
\text{dim}_{\Sigma, l^p}(V, \Gamma) = 0.
\]

The method is based on passing to an action of the group on a measure space, and then using that the corresponding equivalence relations contains an action of \(\mathbb{Z}\).

We shall first work with the trivial action of \(\Gamma\) on \(\mathbb{C}\). For this, fix a sofic group \(\Gamma\) and a sofic approximation \(\Sigma\), for \(S = \{1\}\), and the trivial action of \(\Gamma\) on \(\mathbb{C}\), note that \(T \to T(\{1\})\), identifies \(\text{Hom}_{\Gamma, l^p}(S, F, m, \delta, \sigma_i)\) with all vectors \(\xi \in l^p(d_i)\) such that

\[
\|\sigma_i(g)\xi - \xi\|_p < \delta
\]

for all \(g \in F\).

For the proof of the next Lemma, we will also need the concept of a sofic approximation of an equivalence relation. Let us recall some preliminary definitions.
Definition 2.8. A discrete equivalence relation, is a triple $(\mathcal{R}, X, \mu)$ where $X$ is a standard Borel space, $\mu$ is a Borel probability measure on $X$, $\mathcal{R} \subseteq X \times X$ is a Borel subset such that for all the relation $x \sim y$ if $(x, y) \in \mathcal{R}$ is an equivalence relation and such that for almost every $x \in X$, $\{y : (x, y) \in \mathcal{R}\}$ is countable. We say that $\mathcal{R}$ is measure-preserving if for all $A \subseteq \mathcal{R}$ Borel

$$\int_X |\{y \in X : (x, y) \in A\}| \, d\mu(x) = \int_X |\{x \in X : (x, y) \in A\}| \, d\mu(y).$$

We shall denote the above measure by $\overline{\mathcal{P}}(A)$. We say that $\mathcal{R}$ is ergodic if for all $f : X \to \mathbb{C}$ measurable with $f(x) = f(y)$ for $\mathcal{P}$-almost every $(x, y) \in \mathcal{R}$, there is a $\lambda \in \mathbb{C}$ so that $f(x) = \lambda$ almost everywhere with respect to $\mu$.

The main example of relevance for us is given by taking a countable discrete group $\Gamma$ with a measure-preserving action on a standard probability space $(X, \mu)$. In this case $\mathcal{R}_{\Gamma, (X, \mu)} = \{(x, gx) : g \in \Gamma\}$. Such an action is free if for all $g \in \Gamma \setminus \{e\}$, $\mu(\{x \in X : gx = x\}) = 0$.

Definition 2.9. Let $(\mathcal{R}, X, \mu)$ be a discrete, measure-preserving equivalence relation. A partial morphism is a bimeasurable bijection $\phi : \text{dom}(\phi) \to \text{ran}(\phi)$ where $\text{dom}(\phi), \text{ran}(\phi) \subseteq X$ are measurable and $(x, \phi(x)) \in \mathcal{R}$ for almost every $x \in \text{dom}(\phi)$. We let $\phi^{-1}$ be the partial morphism with $\text{dom}(\phi^{-1}) = \text{ran}(\phi), \text{ran}(\phi^{-1}) = \text{dom}(\phi)$ and $\phi(\phi^{-1}(x)) = x$ for $x \in \text{ran}(\phi)$. If $\phi, \psi \in \mathcal{R}$, we let $\phi \circ \psi$ be the partial morphism with $\text{dom}(\phi \circ \psi) = \{x \in \text{dom}(\psi) : \psi(x) \in \text{dom}(\phi)\}$, and $(\phi \circ \psi)(x) = \phi(\psi(x))$. If $\mathcal{A} \subseteq \mathcal{R}$ is measurable, we let $\text{Id}_A : \mathcal{A} \to A$ be the partial morphism which is the identity on $A$. We let $[\mathcal{R}]$ be the set of all partial morphisms of $\mathcal{R}$, we let $[\mathcal{R}]$ be the set of all partial morphisms of $\mathcal{R}$ so that $\mu(\text{dom}(\phi)) = 1$. For every $\phi \in [\mathcal{R}]$ define the operator $v_\phi \in B(L^2(\mathcal{R}, \overline{\mathcal{P}}))$ defined by $v_\phi f(x, y) = \chi_{\text{ran}(\phi)}(x) f(\phi^{-1} x, y)$. Let $L(\mathcal{R})$ be the von Neumann subalgebra of $B(L^2(\mathcal{R}, \overline{\mathcal{P}}))$ generated by $\{v_\phi : \phi \in [\mathcal{R}]\}$. If $\mathcal{R}$ is given by a free measure-preserving action of $\Gamma$, and $\alpha : X \to X$ for $g \in \Gamma$ is defined by $\alpha_g(x) = gx$, we let $u_g = v_{\alpha_g}$.

We let $\tau : L(\mathcal{R}) \to \mathcal{B}$ be defined by $\tau(x) = \langle x \chi_\Delta, \chi_\Delta \rangle$ where $\Delta = \{(x, x) : x \in X\}$.

For the next definition, we need to note the following example. For $n \in \mathbb{N}$, let $R_n$ be the equivalence relation on $\{1, \cdots, n\}$ declaring all points to be equivalent. Then $[\mathcal{R}_n]$ embeds into $B(\mathcal{P}(n)) \cong M_n(\mathbb{C})$, by the map $\phi \mapsto v_\phi$ defined by $(v_\phi \xi)(k) = \chi_{\text{ran}(\phi)}(k) \xi(\phi^{-1} k)$.

Definition 2.10. Let $\Gamma$ be a countable discrete sofic group with sofic approximation $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$. Let $\Gamma$ have a free, measure-preserving action on a standard probability space $(X, \mu)$. A sofic approximation of $\mathcal{R}$ extending $\Sigma$ is a sequence of maps $\Sigma' = (\rho_i : L(\mathcal{R}) \to M_{d_i}(\mathbb{C}))$ such that

$$\rho_i(u_g) = \sigma_i(g), \text{ for all } g \in \Gamma$$

$$\rho_i(\phi) \in [\mathcal{R}_n]$$

for all $A \subseteq X$ measurable, there exists $A_i \subseteq \{1, \cdots, d_i\}$ so that $\rho_i(\text{Id}_{A_i}) = \text{Id}_{A_i}$

$$\text{tr} \circ \rho_i(x) \to \tau(x) \text{ for all } x \in L(\mathcal{R})$$

$$\sup_i \|\rho_i(x)\|_\infty < \infty \text{ for all } x \in L(\mathcal{R})$$

$$\|P(\rho_i(x_1), \cdots, \rho_i(x_n)) - \rho_i(P(x_1, \cdots, x_n))\|_2 \to 0,$$

for all $x_1, \cdots, x_n \in L(\mathcal{R})$ and all $*$-polynomials in $n$ noncommuting variables.
Lemma 2.11. Let $\Gamma$ be a countable discrete sofic group with a sofic approximation $\Sigma$. Let $\Gamma \acts (X, \mu)$ be a free, ergodic, measure-preserving action on a standard probability space $(X, \mu)$ such that there is a sofic approximation (still denoted $\Sigma$) of $\mathcal{R}_{\Gamma \acts (X, \mu)}$ extending the sofic approximation of $\Gamma$. Fix $\phi \in [\mathcal{R}]$, and $\eta > 0$. Then there is a $F \subseteq \Gamma$ finite, $m \in \mathbb{N}$, $\delta > 0$ and $C_i \subseteq \{1, \cdots, d_i\}$ with $|C_i| \geq (1 - \eta)d_i$ so that for the trivial representation of $\Gamma$ on $\mathbb{C}$, and $T \in \text{Hom}_{\Sigma,p}(\{1\}, F, m, \delta, \sigma_i)$ with $\xi = T(1)$ we have

$$\|\sigma_i(\phi)\xi - \sigma_i(\text{Id}_{\text{ran}(\phi)})\xi\|_{p(C,i)} < \eta,$$

for all large $i$.

Proof. Let $\{A_g : g \in \Gamma\}$ be a partition of ran$(\phi)$ so that

$$v_\phi = \sum_{g \in \Gamma} \text{Id}_{A_g} u_g.$$

Choose $F \subseteq \Gamma$ finite so that

$$\left\| v_\phi - \sum_{g \in F} \text{Id}_{A_g} u_g \right\|_2 < \eta.$$

For all large $i$, we may find $|C_i|$ with $|C_i| \geq (1 - 2\eta)d_i$ so that

$$\chi_{C_i}\sigma_i(\text{Id}_{A_g})\chi_{C_i}\sigma_i(\text{Id}_{A_h})\chi_{C_i} = 0, \text{ for } g \neq h \in F$$

$$\sigma_i(\phi) = \sum_{g \in F} \sigma_i(\text{Id}_{A_g})\sigma_i(g) \text{ on } C_i.$$

Thus on $C_i$,

$$\sigma_i(\phi)\xi = \sum_{g \in F} \sigma_i(\text{Id}_{A_g})\sigma_i(g)\xi,$$

$$\sigma_i(\text{Id}_{\text{ran}(\phi)})\xi = \sum_{g \in F} \sigma_i(\text{Id}_{A_g})\xi,$$

so

$$\|\sigma_i(\phi)\xi - \sigma_i(\text{Id}_{\text{ran}(\phi)})\xi\|_{p(C,i)} \leq |F|\delta,$$

so if $\delta < \frac{\eta}{|F|}$, our claim is proved. \hfill $\Box$

Lemma 2.12. Let $\Gamma$ be a countably infinite discrete sofic group with sofic approximation $\Sigma$. Then for the trivial representation of $\Gamma$ on $\mathbb{C}$, we have

$$\dim_{\Sigma,p}(\mathbb{C}, \Gamma) = 0.$$

Proof. Let $\mathcal{R}$ be the equivalence relation induced by the Bernoulli action of $\Gamma$ on $(X, \mu) = (\{0, 1\}, u)^\Gamma$, $u$ being the uniform measure. Extend $\Sigma$ to a sofic approximation of $[\mathcal{R}]$, (this is essentially possible by [2] Theorem 8.1, see also [6] Proposition 7.1, [3] Theorem 5.5, [19] Theorem 2.1). Since $\Gamma$ is an infinite group, by [?] 9.3.2 we know that for all $n \in \mathbb{N}$, there is a subequivalence relation $\mathcal{R}_n$, generated by a free, measure-preserving action of $\mathbb{Z}/n\mathbb{Z}$ on $(X, \mu)$. Let $\alpha \in [\mathcal{R}_n]$ generate the action of $\mathbb{Z}/n\mathbb{Z}$ on $(X, \mu)$. Fix $\eta > 0$, choose a finite subset $F \subseteq \Gamma$, $\delta > 0$ and subsets $C_i \subseteq \{1, \cdots, d_i\}$ with $|C_i| \geq (1 - d_i)\eta$ so that if $T \in \text{Hom}_{\Gamma}(\{1\}, F, 1, \delta, \sigma_i)$ and $\xi = T(1)$, then

$$\|\sigma_i(\alpha^j)\xi - \xi\|_{p(C,i)} < \eta, \text{ for } 1 \leq j \leq n - 1.$$
for all large $i$. We may assume that there are $A_i \subseteq \{1, \ldots, d_i\}$ with $\frac{|A_i|}{d_i} \to \frac{1}{n}$, so that

$$\{\sigma_i(\alpha)^j(A_i) : 0 \leq j \leq n - 1\},$$

are a disjoint family.

$$\sigma_i(\alpha)|_{\{1, \ldots, d_i\} \setminus \bigcup_{j=0}^{n-1} \sigma_i(\alpha)^j(A_i)} = \text{Id}.$$ 

Let

$$\eta = \sum_{i=1}^{n} \sigma_i(\alpha)^j \chi_{A_i} \xi = \sum_{i=1}^{n} \chi_{\sigma_i(\alpha)^j(A_i)} \sigma_i(\alpha)^j \xi.$$ 

Set $D_i = C_i \cap \bigcup_{j=0}^{n-1} \sigma_i(\alpha)^j(A_i)$, then

$$\chi_{D_i} \eta - \chi_{D_i} \xi = \sum_{i=1}^{n} \chi_{D_i \cap \sigma_i(\alpha)^j(A_i)} (\sigma_i(\alpha)^j \xi - \xi),$$

so

$$\|\chi_{D_i} \eta - \chi_{D_i} \xi\|_p \leq \eta n.$$ 

Thus

$$\dim_{\Sigma}({\{1\}, \eta n, \Gamma}) \leq \frac{1}{n} + 2\eta n.$$ 

Letting $\eta \to 0$, and then $n \to \infty$ completes the proof.

\[\square\]

**Theorem 2.13.** Let $\Gamma$ be a countably infinite sofic group with sofic approximation $\Sigma$. Then, for any representation of $\Gamma$ on a finite-dimensional vector space $V$, and for all $1 \leq p < \infty$,

$$\dim_{\Sigma,p}(V, \Gamma) = 0$$

**Proof.** By Corollary 2.6 and Proposition 2.7 we may assume that

$$\{|\Lambda| : \Lambda \subseteq \Gamma \text{ is finite}\},$$

is bounded, and that every element of $\Gamma$ has finite order. Also, the usual tricks imply that we may assume that $V$ is a Hilbert space and $\Gamma$ acts by unitaries. Let $M$ be greater than $|\Lambda|$ for any finite subgroup of $\Gamma$. Choose $\varepsilon > 0$ so that if $U$ is a unitary on a Hilbert space and

$$\|U - 1\| < \varepsilon,$$

then $U^M \neq 1$ unless $U = 1$. Since $\pi(\Gamma)$ is compact, we may find an infinite sequence $g_n$ of distinct elements of $\Gamma$ so that

$$\|\pi(g_n) - 1\| < \varepsilon.$$ 

If

$$\Lambda = \langle g_n : n \in \mathbb{N} \rangle,$$

our assumptions then imply that $\Lambda$ is an infinite subgroup of $\Gamma$ which acts trivially. Thus by the preceding Lemma and subadditivity under exact sequences,

$$\dim_{\Sigma,p}(V, \Gamma) \leq \dim_{\Sigma,p}(V, \Lambda) = 0.$$

\[\square\]
3. A Complete Calculation In The Case of $\bigoplus_{j=1}^{n} L^p(L(\Gamma))q_j$.

In this section, we show that if $\Gamma$ is $\mathcal{R}^\omega$-embeddable, $\Sigma$ is an embedding sequence and $q_1, \cdots, q_n \in \text{Proj}(L(\Gamma))$, then

$$\dim_{\Sigma, \text{Sp}, \text{multi}} \left( \bigoplus_{j=1}^{n} L^p(L(\Gamma), \tau)q_j, \Gamma \right) = \dim_{\Sigma, \text{Sp}, \text{multi}} \left( \bigoplus_{j=1}^{n} L^p(L(\Gamma), \tau)q_j, \Gamma \right) = \sum_{j=1}^{n} \tau(q_j)$$

where $\tau$ is the group trace.

We shall frequently use functional calculus throughout the proof. For notation, if $T$ is a bounded operator on a Hilbert space $H$, then $|T| = (T^*T)^{1/2}$. We use $\{u_g : g \in \Gamma\}$ for the canonical unitaries generating $L(\Gamma)$. We use $\text{tr}$ for the linear functional on $M_n(\mathbb{C})$ equal to $\frac{1}{n} \text{Tr}$, and for $A \in M_n(\mathbb{C})$, we use $\|A\|_p = \text{tr}(|A|^p)$, as before $\|\|_{\infty}$ is defined to be the operator norm.

We introduce some background.

**Definition 3.1.** A subalgebra $M$ of $B(H)$ is a **von Neumann algebra** if it is closed under adjoints, limits in the weak operator topology and contains the identity of $H$. A **faithful normal tracial state on $M$** is a linear functional $\tau : M \to \mathbb{C}$ so that $\tau(xy) = \tau(yx)$, for $x, y \in M$, $\tau(x^*x) \geq 0$ for all $x \in M$, with equality if and only if $x = 0$, and $\tau\{x \in M : \|x\|_{\infty} \leq 1\}$ is weak operator topology continuous. Here, as always, $\|\|_{\infty}$ is the operator norm of $x$.

If $M \subseteq H$ is a von Neumann algebra and $\tau : M \to \mathbb{C}$ is a faithful normal tracial state, we define $L^p(M, \tau)$ to be all closed densely-defined operators $x$ on $H$ affiliated to $M$, (i.e. commuting with all the unitaries in $M' = \{T \in B(H) : TX = XT \text{ for all } x \in M\}$) so that if $|x| = \int_{[0, \infty)} t \, dE(t)$ is the spectral decomposition of $x$, then

$$\|x\|_p = \int_{[0, \infty)} t^p \, d\tau \circ E(t) < \infty.$$ 

For any closed densely-defined operator $T$ on $H$ we will use $\text{dom}(T), \text{ran}(T)$ for its domain and range.

Further $\|\|_p$ is a norm on $L^p(M, \tau)$ which makes $L^p(M, \tau)$ into a Banach space, with the sum being the closure of the operator $x + y$ with domain $\text{dom}(x) \cap \text{dom}(y)$. and we have the inequalities

$$\|xy\|_r \leq \|x\|_p \|y\|_q$$

if

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Here we are using $xy$ for the closure of the (densely-defined) operator with domain $y^{-1}(\text{dom}(x))$ and defined by $xy(\xi) = x(y(\xi))$. 
Lemma 3.2. (a) Let $n \in \mathbb{N}$, suppose that $A, B \in M_n(\mathbb{C})$ are such that $|A| \leq |B|$, then for all $\beta > 0$,
\[
\text{tr}(|A|^\beta) \leq \text{tr}(|B|^\beta).
\]
(b) Suppose that $A, B \in M_n(\mathbb{C})$ and $Q$ is a orthogonal projection in $M_n(\mathbb{C})$. Fix $1 \leq p < \infty$, suppose that $\delta, \eta > 0$ are such that
\[
\|(A - 1)B\|_p < \delta, \quad \|A - Q\|_p < \eta.
\]
Then
\[
\|B - \chi_{(0,\sqrt{\delta})}(|A - 1|)B\|_p < \sqrt{\delta},
\]
and
\[
\text{tr}(\chi_{(0,\sqrt{\delta})}(|A - 1|)) \leq \text{tr}(Q) + \left(\frac{\eta}{1 - \sqrt{\delta}}\right)^p.
\]

Proof. We first make the following preliminary observation: if $P, Q$ are orthogonal projections in $M_n(\mathbb{C})$ with
\[
PC^n \cap QC^n = \{0\},
\]
then
\[
\text{tr}(P) \leq 1 - \text{tr}(Q).
\]
This follows directly from the fact that $1 - Q$ is injective on $PC^n$.

(a) First note that
\[
\text{tr}(T^\alpha) = \alpha \int_0^\infty t^{\alpha - 1} \text{tr}(\chi_{(t,\infty)}(T)) \, dt
\]
if $T \geq 0$. If $0 \leq T \leq S$, and
\[
\xi \in \chi_{(t,\infty)}(T)(\mathbb{C}^n) \cap \chi_{[0,t]}(S)(\mathbb{C}^n)
\]
and $\xi \neq 0$, then
\[
t\|\xi\|^2 < \langle T \xi, \xi \rangle \leq \langle S \xi, \xi \rangle \leq t\|\xi\|^2,
\]
which is a contradiction. Hence
\[
\chi_{(t,\infty)}(T)(\mathbb{C}^n) \cap \chi_{[0,t]}(S)(\mathbb{C}^n) = \{0\},
\]
so the above integral formula and our preliminary observation prove (a).

(b) Note that
\[
|\chi_{[\sqrt{\delta},\infty)}(|A - 1|)B|^2 = B^*\chi_{[\sqrt{\delta},\infty)}(|A - 1|)B \leq \frac{1}{\delta}B^*|A - 1|^2B = \left|\frac{1}{\sqrt{\delta}}(A - 1)B\right|^2,
\]
thus by (a)
\[
\|B - \chi_{(0,\sqrt{\delta})}(|A - 1|)B\|_p = \|\chi_{[\sqrt{\delta},\infty)}(|A - 1|)B\|_p < \sqrt{\delta}.
\]
Further if $\xi \in \chi_{(0,\sqrt{\delta})}(|A - 1|)(\mathbb{C}^n) \cap (1 - Q)(\mathbb{C}^n) \cap \chi_{[0,1-\sqrt{\delta}])(|A - Q|)(\mathbb{C}^n)$, is nonzero, then
\[
(1 - \sqrt{\delta})^2\|\xi\|^2 \geq \langle |A - Q|^2 \xi, \xi \rangle = \|A\xi\|^2 > (1 - \sqrt{\delta})^2\|\xi\|^2,
\]
which is a contradiction. Thus
\[
\text{tr}(\chi_{(0,\sqrt{\delta})}(|A - 1|)) \leq \text{tr}(Q) + \text{tr}(\chi_{(1-\sqrt{\delta},\infty)}(|A - Q|)).
\]
Let $0 < \varepsilon, \kappa < 1$. By subadditivity of dimension, it suffices to handle the case of

$$L^p(M, \tau)q.$$ 

Let $M = L(\Gamma)$ and $\tau$ the canonical group trace on $M$. Then, for all $1 \leq p < \infty$ and for every $q \in \text{Proj}(M)$ we have

$$\dim_{\Sigma, sp, \text{multi}} \left( \bigoplus_{j=1}^{n} L^p(M, \tau)q_j, \Gamma \right) \leq \sum_{j=1}^{n} \tau(q_j).$$

**Proof.** By subadditivity of dimension, it suffices to handle the case of $L^p(M, \tau)q$. Let $0 < \varepsilon, \kappa < 1/2$. Let $A$ be the $\ast$-algebra in $L(\Gamma)$ generated by $q$ and $\Gamma$, let

$$\tilde{\sigma}_i : A \to M_{d_i}(\mathbb{C})$$

be an embedding sequence which extends $\sigma_i$, and choose projections $q_i$ so that $\|q_i - \sigma_i(q_i)\|_p \to 0$. Choose $f \in c_0(\Gamma)$ so that

$$\left\| q - \sum_{s \in \Gamma} f(s)u_s \right\|_p < \kappa.$$

If $T : L^p(M, \tau)q \to L^p(M_{d_i}(\mathbb{C}), \tau)$, define

$$\tilde{T}(x) = T(xq).$$

Let $F$ be the support of $f$, then if $m \in \mathbb{N}, \kappa, \delta > 0$ are sufficiently small we have

$$\left\| \left( \sum_{s \in \Gamma} f(s)\tilde{\sigma}_i(s) - 1 \right) \tilde{T}(q) \right\|_p < \varepsilon^2,$$

for all $\tilde{T} \in \text{Hom}_F(S, F, m, \delta, \sigma_i)$. Thus the proceeding lemma implies that if

$$e_i = \chi_{(\varepsilon, \infty)} \left( \sum_{s \in \Gamma} f(s)\tilde{\sigma}_i(s) - 1 \right),$$

then for all large $i$, we have

$$\| T(q) - e_i T(q) \|_p < \varepsilon,$$

$$\text{tr}(e_i) \leq \text{tr}(q_i) + 2\varepsilon \kappa^p \to \tau(q) + 2\varepsilon \kappa^p.$$

Since $\kappa > 0$ is arbitrary, this proves the claim.

**Lemma 3.4.** Fix $1 \leq p \leq \infty$, and a sequence of positive integers $d(n) \to \infty$, and let $\mu_n$ be the Lebesgue measure on $L^p(M_{d(n)}(\mathbb{C}), \frac{1}{d(n)} \text{Tr})$ normalized so that $\mu_n(\text{Ball}(L^p(M_{d(n)}(\mathbb{C}), \frac{1}{d(n)} \text{Tr}))) = 1$. Let $A_n \subseteq \text{Ball}(V_n)$ and suppose that there is an $\alpha > 0$ so that

$$\limsup_{n \to \infty} \mu_n(\text{Ball}(V_n))^{1/2d(n)^2} \geq \alpha > 0.$$
Further, let \( q_n \in \text{Proj}(M_d(n)(\mathbb{C})) \) be such that \( \frac{1}{d(n)} \text{Tr}(p_n) \) converges to a positive real number. Then for every \( \varepsilon > 0 \),

\[
\limsup_{n \to \infty} \frac{1}{d(n) \text{Tr}(p_n)} d_{\varepsilon}(A_n q_n, \| \cdot \|_p) \geq \kappa(\alpha, \varepsilon)
\]

with

\[
\lim_{\varepsilon \to 0} \kappa(\alpha, \varepsilon) = 1 \text{ for all fixed } \alpha > 0.
\]

**Proof.** Fix \( 1 > \varepsilon > 0 \), and suppose that

\[
\limsup_{n \to \infty} \frac{1}{d(n) \text{Tr}(p_n)} d_{\varepsilon}(A_n q_n, \| \cdot \|_p) < \kappa,
\]

then for all large \( n \),

\[
d_{\varepsilon}(A_n q_n, \| \cdot \|_{V_n}) < d(n) \kappa \text{Tr}(q_n).
\]

Let \( W_n \) be a subspace of dimension at most \( d(n) \kappa \text{Tr}(q_n) \) which \( \varepsilon \)-contains \( A_n q_n \), thus

\[
A_n q_n \subseteq (1 + \varepsilon) \text{Ball}(W_n) + \varepsilon \text{ Ball}(L^p(M_d(n)(\mathbb{C}), \text{tr})q_n).
\]

Let \( S \subseteq (1 + \varepsilon) \text{Ball}(W_n) \) be a maximal family of \( \varepsilon \)-separated vectors, i.e. for all \( x, y \in S \) with \( x \neq y \) we have \( \|x - y\| \geq \varepsilon \). Then the \( \varepsilon/3 \) balls centered at points in \( S \) are disjoint so by a volume computation

\[
|S| \leq \left( \frac{3 + 3\varepsilon}{\varepsilon} \right)^{2 \dim(W_n)}.
\]

By maximality, \( S \) is \( \varepsilon \)-dense in \( (1 + \varepsilon) \text{Ball}(W_n) \). Thus

\[
A_n q_n \subseteq \bigcup_{x \in S} x + 2\varepsilon \text{ Ball}(L^p(M_d(n)(\mathbb{C}), \text{tr})q_n),
\]

so

\[
\text{vol}(A_n q_n) \leq 2^{2d(n) \text{Tr}(p_n)} \varepsilon^{2d(n) \text{Tr}(q_n) - 2 \dim(W_n) \text{Tr}(q_n)} (3 + 3\varepsilon)^{2 \dim(W_n)} V_p(q_n),
\]

where for \( q \in \text{Proj}(M_d(n)(\mathbb{C})) \) we use

\[
V_p(q) = \text{Ball}(L^p(M_d(n)(\mathbb{C}), \text{tr})q).
\]

Since \( A_n \subseteq A_n q_n \times \text{Ball}(L^p(M_d(n)(\mathbb{C}), \text{tr}), 1) \), we have

\[
\alpha \leq \limsup_{n \to \infty} 6 \cdot 2^{\pi/2(n) \text{Tr}(q_n) - 1 - \kappa} \left( \frac{V(q_n) V(1 - q_n)}{V(\text{Id}_{d(n)})} \right)^{1/2d(n)^2}.
\]

Hence it suffices to show that

\[
\limsup_{n \to \infty} \left( \frac{V_p(q_n) V_p(1 - q_n)}{V_p(\text{Id}_{d(n)})} \right)^{1/2d(n)^2} < \infty.
\]

For this, we know by [24] that there is a constant \( C > 0 \) so that

\[
\left( \frac{V_p(q_n) V_p(1 - q_n)}{V_p(\text{Id}_{d(n)})} \right)^{1/2d(n)^2} \leq C \left( \frac{V_p(q_n) V_p(1 - q_n)}{V_2(\text{Id}_{d(n)})} \right)^{1/2d(n)^2}.
\]

Further by the Santolo inequality, we may reduce to the case that \( p \geq 2 \). Since \( \|A\|_2 \leq \|A\|_p \), we then have that

\[
\left( \frac{V_p(q_n) V_p(1 - q_n)}{V_2(\text{Id}_{d(n)})} \right)^{1/2d(n)^2} \leq \left( \frac{V_2(q_n) V_2(1 - q_n)}{V_2(\text{Id}_{d(n)})} \right)^{1/2d(n)^2}.
\]
Since

\[ V_2(q) = \frac{\pi^{\text{Tr}(q)}}{\text{Tr}(q)!} d(n)^{-d(n)} , \]

it follows from Stirling’s formula and the fact that \( \frac{1}{d(n)} \text{Tr}(q_n) \) converges that

\[ \left( \frac{V_2(q_n)V_2(1-q_n)}{V_2(\text{Id}_{d(n)})} \right)^{1/2d(n)^2} \]

is bounded.

\[ \square \]

To complete the calculation, it suffices to prove the following Theorem.

**Theorem 3.5.** Let \( \Gamma \) be an \( R^\omega \)-embeddable group and \( \Sigma \) an embedding sequence. Let \( M = L(\Gamma) \) and \( \tau \) the canonical group trace on \( M \). Then, for all \( 1 \leq p < \infty \) and for every \( q \in \text{Proj}(M) \) we have

\[
\dim_{\Sigma,S^p,\text{multi}} \left( \bigoplus_{j=1}^{n} L^p(M, \tau q_j, \Gamma) \right) = \dim_{\Sigma,S^p,\text{multi}} \left( \bigoplus_{j=1}^{n} L^p(M, \tau q_j, \Gamma) \right) \\
= \sum_{j=1}^{n} \tau(q_j). 
\]

**Proof.** We use the generating sequence \( S = (q_1, \cdots, q_n, 0, \cdots) \) to do the calculation. By Proposition 3.3, we have the upper bound. So it suffices to prove the lower bound. By Lemma 5.5 in \[12\], we can find a sequence of asymptotically linear, asymptotically trace-preserving, asymptotic \(*\)-homomorphisms

\[
\rho_i: L(\Gamma) \to M_{d_i}(\mathbb{C})
\]

such that

\[
\rho_i(u_g) = \sigma_i(g),
\]

and

\[
\sup_i \| \rho_i(x) \|_\infty < \infty
\]

for all \( x \in R \).

Let \( p \in L(\Gamma) \) be any orthogonal projection, then

\[
\| \rho_i(p) - \rho_i(p)^* \rho_i(p) \|_2 \to 0
\]

\[
\| \rho_i(p)^* \rho_i(p) - (\rho_i(p)^* \rho_i(p))^2 \|_2 \to 0.
\]

By functional calculus, for any \( \varepsilon < 1/2 \),

\[
\| \chi_{[1-\varepsilon,1+\varepsilon]}(\rho_i(p)^* \rho_i(p)) - \rho_i(p)^* \rho_i(p) \|_2 \leq \| \chi_{[0,\infty)}(1-\varepsilon,1+\varepsilon)(\rho_i(p)^* \rho_i(p))\rho_i(p)^* \rho_i(p)\|_2
\]

\[
+ \| \chi_{[1-\varepsilon,1+\varepsilon]}(\rho_i(p)^* \rho_i(p))(1-\rho_i(p)^* \rho_i(p))\|_2
\]

\[
\leq \frac{1}{1-\varepsilon} \| \rho_i(p)^* \rho_i(p) - (\rho_i(p)^* \rho_i(p))^2 \|_2
\]

\[
+ \frac{1}{\varepsilon} \| \rho_i(p)^* \rho_i(p) - (\rho_i(p)^* \rho_i(p))^2 \|_2.
\]

Thus for all \( \varepsilon < 1/2 \),

\[
\| \rho_i(p) - \chi_{[1-\varepsilon,1+\varepsilon]}(\rho_i(p)^* \rho_i(p)) \|_2 \to 0.
\]
By the above argument, we may assume that $\rho_i(q_j)$ is an orthogonal projection for all $i, j$.

Fix $F \subseteq \Gamma$ finite $m \geq n$ in $\mathbb{N}$, $\delta > 0$. Let $E \subseteq \Gamma$ be a finite set which is sufficiently large in a manner to be determined later. Let

$$V_E^{(j)} = \text{Span}\{u_gq : g \in E\}.$$ 

For $A \in M_d(\mathbb{C})$ $E \subseteq \Gamma$ finite define

$$T_A^{(j)} \left( \sum_{g \in E} c_g u_g q \right) = \sum_{g \in E} c_g \sigma_i(g) \rho_i(q_j) A.$$ 

Note that

$$\left\| T_A^{(j)} \left( \sum_{g \in E} c_g u_g q \right) \right\|_p \leq \|A\|_\infty \left\| \sum_{g \in E} c_g \sigma_i(g) \rho_i(q_j) \right\|_p.$$ 

Since $\sigma_i$ is an embedding sequence, we know that

$$\left\| \sum_{g \in E} c_g \sigma_i(g) \rho_i(q_j) \right\|_p \rightarrow \left\| \sum_{g \in E} c_g u_g q \right\|_p,$$

pointwise. As $V_E^{(j)}$ is finite-dimensional,

$$\left\| \sum_{g \in E} c_g \sigma_i(g) \rho_i(q_j) \right\|_p \rightarrow \left\| \sum_{g \in E} c_g u_g q \right\|_p,$$

uniformly on the $\| \cdot \|_p$ unit ball of $V_E^{(j)}$.

Also if $E$ is sufficiently large, then for all $g_1, \cdots, g_k \in F$,

$$\|T_A^{(j)}(g_1 \cdots g_k q_j) - \sigma_i(g_1) \cdots \sigma_i(g_k) T_A^{(j)}(q_j)\|_p = \|\sigma_i(g_1) \cdots \sigma_i(g_k) \rho_i(q_j) A - \sigma_i(g_1) \cdots \sigma_i(g_k) \rho_i(q_j) A\|_p \leq \|A\|_\infty \|\sigma_i(g_1) \cdots \sigma_i(g_k) - \sigma_i(g_1) \cdots \sigma_i(g_k)\|_p \rightarrow 0.$$ 

Thus if $E$ is sufficiently large, depending upon $F, m, \delta$ then for all $A_1, \cdots, A_n \in M_d(\mathbb{C})$ with $\|A_j\|_\infty \leq 1$,

$$T_{A_1}^{(1)} \oplus \cdots \oplus T_{A_n}^{(n)} \in \text{Hom}_F(S, F, m\delta, \sigma_i)_n.$$ 

$$\left( \frac{\text{vol}(\text{Ball}(M_d(\mathbb{C}), \| \cdot \|_\infty))}{\text{vol}(\text{Ball}(M_d(\mathbb{C}), \| \cdot \|_{L^p(1/d, \Gamma)})} \right)^{1/2d^2}.$$ 

The theorem follows from Lemma 3.4.

We can prove an analouge for the action of $\Gamma$ on its reduced $C^*$-algebra but first we need a Lemma.

**Lemma 3.6.** Let $\Gamma$ be a countable discrete group, and $X \subseteq L^p(L(\Gamma), \tau_\Gamma)$ a closed $\Gamma$-invariant subspace (for the action of left multiplication by elements of $\Gamma$). Then there is an orthogonal projection $q \in L(\Gamma)$ so that $X = L^p(L(\Gamma), \tau_\Gamma)$. 

Proof. We always have the inequality
\[ \|xy\|_p \leq \|x\|_\infty \|y\|_p. \]

Note that if \( x_n \in L^p(\Gamma), \sup_n \|x_n\|_\infty < \infty, \) and \( x_n \to x \) in the strong operator topology on \( L^p(\Gamma) \), then \( x_n y \to xy \). Indeed, this follows by the above inequality and the density of \( L^2(\Gamma) \) in \( L^p(\Gamma, \tau_\Gamma) \). Thus a closed \( \Gamma \)-invariant subspace is the same as a \( L(\Gamma) \)-invariant subspace.

It suffices to prove the following two claims.

**Claim 1.** If \( x \in L^p(\Gamma, \tau_\Gamma) \), then \( L(\Gamma)x^\|p = L^p(\Gamma, \tau_\Gamma)\chi_{[0,\infty)}(\|x\|) \).

**Claim 2.** If \( e, f \) are orthogonal projections in \( L(\Gamma) \), then
\[ L^p(\Gamma, \tau_\Gamma)e + L^p(\Gamma, \tau_\Gamma)f = L^p(\Gamma, \tau_\Gamma)(e \vee f). \]

Indeed, if we grant the two claims, then by separability, we can find increasing subspaces \( X_n \) of \( \Gamma \) of the form \( L^p(\Gamma, \tau_\Gamma)q_n \) for some orthogonal projection \( q_n \). Setting \( q = \sup q_n \) we see that
\[ X = L^p(\Gamma, \tau_\Gamma)q. \]

For claim 2, it suffices to note that by functional calculus,
\[ 1 - (e \vee f) = 1 - (1 - e) \land (1 - f) = 1 - \lim_{n \to \infty} ((1 - e)(1 - f)(1 - e))^n, \]
the limit in \( \| \cdot \|_p \). As
\[ 1 - [(1 - e)(1 - f)(1 - e)]^n \in L^p(\Gamma, \tau_\Gamma)e + L^p(\Gamma, \tau_\Gamma)f \]
for all \( n \), this implies that
\[ L^p(\Gamma, \tau_\Gamma)(e \vee f) \subseteq L^p(\Gamma, \tau_\Gamma)e + L^p(\Gamma, \tau_\Gamma)f, \]
the reverse inclusion being trivial, this proves claim 2.

For claim 1, let \( x = v |x| \) be the polar decomposition. Since \( |x| = v^*x, \)
\[ L(\Gamma)x^\|p = L(\Gamma)|x|^\|p. \]

Let
\[ y_n = \chi_{[\varepsilon, \infty)}(\|x\|)|x|^{-1}, \]
then by functional calculus,
\[ \|y_n|x| - \chi_{[0, \infty)}(\|x\|)\|_p \to 0. \]

Thus
\[ L(\Gamma)|x|^\|p \supseteq L^p(\Gamma, \tau_\Gamma)\chi_{[0, \infty)}(\|x\|), \]
the reverse inclusion being trivial, we are done.

If \( \Gamma \) is a countable discrete group we use \( C^*_\Lambda(\Gamma) \) for \( \| \cdot \|_\infty \), with the closure taken in the left regular representation. As a corollary of the above Theorem, we deduce one of conjectures stated in [12].

**Corollary 3.7.** Let \( \Gamma \) be an \( \mathcal{R}^\omega \)-embeddable group and \( 1 \leq p < \infty \). Let \( I \subseteq C^*_\Lambda(\Gamma) \) be a norm closed left-ideal. Let \( T^\mathbb{R}^\omega = L(\Gamma)q \) (with the closure taken in \( L(\Gamma) \)). Then
\[ \dim_{\Sigma,S^p,\text{multi}}(I, \Gamma) \geq \tau(q). \]
Proof. It suffices to show that the inclusion $I \subseteq L^p(L(\Gamma), \tau)q$ has dense image. By the previous Lemma, let $q' \in \text{Proj}(L(\Gamma))$ be such that

$$T^p = L^p(L(\Gamma), \tau)q'.$$

By the argument in the previous Lemma,

$$q' = \sup_{x \in \mathcal{I}} \chi(0, \infty)(|x|).$$

So it suffices to prove the following two claims.

Claim 1. If $x \in C^*_\lambda(\Gamma)$, then $\chi(0, \infty)(|x|) \in T^{wk*}$.

Claim 2. If $e, f \in \text{Proj}(T^{wk*})$, then $e \lor f \in \text{Proj}(T^{wk*})$.

For the proof of claim 1, let $x = v|x|$ be the polar decomposition. By the Kaplansky Density Theorem, we can find $v_n \in C^*_\lambda(\Gamma)$ so that $\|v_n\|_\infty \leq 1$ and $\|v_n - v\|_2 \to 0$. But then $\|v_n^*x - |x|\|_2 \to 0$, so $|x| \in T^{wk*}$. Since

$$\chi_{(\epsilon, \infty)}(|x|) = |x|^{-1} \chi_{(\epsilon, \infty)}(|x|)|x|,$$

we find that $\chi(0, \infty)(|x|) \in T^{wk*}$.

For the proof of claim 2, we use the formula (proved by functional calculus):

$$e \lor f = 1 - \lim_{n \to \infty} \left(\left(1 - e\right)(1 - f)(1 - e)\right)^n$$

where the limit is in $\|\cdot\|_2$. Since $e, f \in L(\Gamma)q$, a little calculation shows that

$$1 - \left(\left(1 - e\right)(1 - f)(1 - e)\right)^n \in L(\Gamma)q,$$

this proves the corollary.

We can also handle the case $p = \infty$ if we assume a little more.

Definition 3.8. A $C^*$-algebra $A$ is said to be a matricial field algebra if there is an injective $*$-homomorphism

$$\sigma: A \to \left\{(A_n)_{n=1}^\infty : A_n \in M_{d(n)}(\mathbb{C}), \sup_n \|A_n\|_\infty < \infty\right\}$$

for some $d(n) \in \mathbb{N}$ and $d(n) \to \infty$. A sequence $\sigma_n: A \to M_{d_n}(\mathbb{C})$, of potentially nonmultiplicative, nonlinear maps, such that $\sigma_n(a)$ is the image of $(\sigma_n(a))$ is called a norm microstates sequence.

Theorem 3.9. Let $\Gamma$ be a countable discrete group. Assume that there are norm microstates $\sigma_i: C^*_\lambda(\Gamma) \to M_{d_i}(\mathbb{C})$ such that

$$\text{tr}(\sigma_i(x)) \to \tau(x)$$

for all $x \in \mathbb{C}[\Gamma]$. Let $I \subseteq C^*_\lambda(\Gamma)$ be a norm-closed left ideal, and let $I^{wk*} = L(\Gamma)q$, with $q \in \text{Proj}(L(\Gamma))$. Then,

$$\dim_{\text{S*}, \text{multi}}(I, \Gamma) \geq \tau(q).$$

Proof. Let

$$A = \left\{(x_i)_{i=1}^\infty \in \prod_{i=1}^\infty M_{d_i}(\mathbb{C}) : \sup_i \|\sigma_i(x_i)\|_\infty < \infty\right\}$$

then our hypothesis implies that there is an isometric $*$-homomorphism

$$\sigma: C^*_\lambda(\Gamma) \to A,$$
such that
\[ \sigma(u_g) = \pi(\sigma_1(g), \sigma_2(g), \cdots) \]
where
\[ \pi: \left\{ (x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} M_{d_i}(\mathbb{C}) : \sup_i \| \sigma_i(x) \|_\infty < \infty \right\} \to A, \]
is the quotient map.

As before, we may extend \( \phi_i \) to an embedding sequence
\[ \psi_i: L(\Gamma) \to M_{d_i}(\mathbb{C}). \]

Now let \( \varepsilon > 0 \), and choose a finite subset \( E \subseteq \Gamma, l \in \mathbb{N}, \) and \( c_{gj} \in \mathbb{C}, \) for \( (g,j) \in E \times \{1, \cdots, l\} \) so that
\[ \left\| q - \sum_{g \in E, 1 \leq j \leq l} c_{gj} u_g x_j \right\|_2 < \varepsilon. \]

Fix \( E \subseteq F \subseteq \Gamma \) finite, \( l \leq m \in \mathbb{N}, \delta > 0. \) Since all injective \( * \)-homorphisms defined on \( C^* \)-algebras are isometric, it is easy to see that if we define \( \rho_i = \frac{\phi_i|_{I_{F,m}}}{\| \phi_i|_{I_{F,m}} \|} \), then
\[ \left\| \rho_i - \phi_i \right\|_{I_{F,m}} \to 0. \]

For \( B \in M_{d_i}(\mathbb{C}) \) define
\[ T_B: I_{F,m} \to M_{d_i}(\mathbb{C}), \]
by
\[ T_B(x) = \rho_i(x)B. \]
If \( \|B\|_{\infty} \leq 1, \) then
\[ \|T_B(x)\| \leq \|B\|_{\infty}. \]
Further if \( \|B\|_{\infty} \leq 1, \) and \( 1 \leq j, k \leq m, \) and \( g_1, \cdots, g_k \in F, \) then
\[ \left\| T_B(g_1 \cdots g_k x_j) - \sigma_i(g_1) \cdots \sigma_i(g_k) T_B(x_j) \right\| \leq \left\| \phi_i(g_1 \cdots g_k x_j) - \sigma_i(g_1) \cdots \sigma_i(g_k) \phi_i(x_j) \right\| \to 0 \]
using that
\[ \pi((\phi_i(g_1 \cdots g_k x_j))_{i=1}^{\infty}) = \pi((\sigma_i(g_1) \cdots \sigma_i(g_k) \phi_i(x_j))_{i=1}^{\infty})). \]

Now suppose \( V \subseteq I^{\infty}(\mathbb{N}, M_{d_i}(\mathbb{C})) \) \( \varepsilon \)-contains \( \{(\rho_i(x_j)B)_{j=1}^{\infty} : \|B\|_{\infty} \leq 1\}. \) Define a map \( \Phi: I^{\infty}(\mathbb{N}, M_{d_i}(\mathbb{C})) \to L^2(M_{d_i}(\mathbb{C}), \text{tr}) \) by
\[ \Phi(f) = \sum_{g \in E, 1 \leq j \leq l} c_{gj} \sigma_i(g) f(j), \]
then our hypotheses imply that for all large \( i, \)
\[ \Phi(V) \supseteq \{ qB : B \in \text{Ball}(M_{d_i}(\mathbb{C}), \| \cdot \|_{\infty}) \}. \]

Our methods to prove Theorem 6.5 can be used to complete the proof. \( \square \)
4. Definition of $l^p$-Dimension Using Vectors

In this section, we give a definition of the extended von Neumann dimension using vectors instead of almost equivariant operators. Thus may be conceptually simpler, as we do not have to deal with the technicalities involving changing domains inherit to the definition of Hom$_\Gamma(\cdots)$. The definition is much simpler and requires fewer preliminaries as well. However, for many theoretical purposes it will still be easier to use the notion of almost equivariant operators. We will give this alternate definition after the following Lemma.

**Lemma 4.1.** Let $V$ be a finite-dimensional Banach space spanned by vectors $v_1, \ldots, v_n$. Then for any $\varepsilon > 0$, there is a $\delta > 0$ so that if $Y$ is a Banach space and $\xi_1, \ldots, \xi_n$ have the property that for all $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{k=1}^n |c_k| \leq 1$,

$$\left\| \sum_{k=1}^n c_k \xi_k \right\| \leq \delta + \left\| \sum_{k=1}^n c_k v_k \right\|,$$

then there is a $T : V \to Y$ with $\|T\| \leq 1$, such that

$$\|T(v_j) - \xi_j\| < \varepsilon.$$

**Proof.** Let $A \subseteq \{1, \ldots, n\}$ be such that $\{v_j : j \in A\}$ is a basis for $V$. Let $\widetilde{T} : V \to Y$ be defined by

$$\widetilde{T}(v_j) = \xi_j,$$

for $j \in A$. By finite-dimensionality, there is a $C(V) > 0$ so that

$$\sum_{k=1}^n |c_j| \leq C(V) \left\| \sum_{j \in A} c_j v_j \right\|.$$

Thus our hypothesis implies that

$$\|\widetilde{T}\| \leq C(V) \delta + 1.$$

Set $T = \frac{1}{1 + C(V)\delta} \widetilde{T}$. For each $j \in \{1, \cdots, n\} \setminus A$ choose $c_j^{(j)}, k \in A$ so that

$$v_j = \sum_{k \in A} c_k^{(j)} v_j.$$

Then

$$\|T(v_j) - \xi_j\| = \left\| \frac{1}{1 + C(V)\delta + 1} \sum_{k \in A} c_k^{(j)} \xi_k - \xi_j \right\| \leq \sup_j |c_k^{(j)}| \left( 1 - \frac{1}{C(V)\delta + 1} \right) + \left\| \sum_{k \in A} c_k^{(j)} \xi_k - \xi_j \right\| \leq \sup_j \left( |c_k^{(j)}| \left( 1 - \frac{1}{C(V)\delta + 1} \right) + \delta \right).$$

Clearly the right hand side can be made smaller than $\varepsilon$ by choosing $\delta$ small. \qed
**Definition 4.2.** Let $X$ be a Banach space with a uniformly bounded action of a countable discrete group $\Gamma$ and $\sigma_i: \Gamma \to \text{Isom}(V_i)$ with $V_i$ finite-dimensional. We let $\text{Vect}_\Gamma(S, F, m, \delta, \sigma_i)$ be all vectors $(\xi_j)_{j=1}^m$ such that for all $c_{g_1, \ldots, g_m,j}$ with $\sum_{1 \leq j \leq m} |c_{g_1, \ldots, g_m,j}| \leq 1$, we have

$$\left\| \sum_{g_1, \ldots, g_m,j \in F} c_{g_1, \ldots, g_m,j} \sigma_i(g_1) \cdots \sigma_i(g_m) \xi_j \right\| \leq \delta + \left\| \sum_{g_1, \ldots, g_m,j \in F} c_{g_1, \ldots, g_m,j} g_1 \cdots g_m x_j \right\|.$$

Set

$$\text{vdim}_\Sigma(S, F, m, \delta, \varepsilon, \rho) = \limsup_{i \to \infty} \frac{1}{\dim V_i} d_\varepsilon(\text{Vect}_\Gamma(S, F, m, \delta, \sigma_i), \rho V_i),$$

$$\text{vdim}_\Sigma(S, \varepsilon, \rho) = \inf_{F, m, \delta} \text{vdim}_\Sigma(S, F, m, \delta, \varepsilon, \rho),$$

$$\text{vdim}_\Sigma(S, \rho) = \sup_{\varepsilon > 0} \text{vdim}_\Sigma(S, \varepsilon, \rho).$$

**Proposition 4.3.** Let $X$ be a Banach space with a uniformly bounded action of a countable discrete group $\Gamma$ and $\sigma_i: \Gamma \to \text{Isom}(V_i)$ with $V_i$ finite-dimensional. Then for any dynamically generating sequence $S$, and any product norm $\rho$,

$$\dim_\Sigma(X, \Gamma) = \text{vdim}_\Sigma(S, \rho).$$

**Proof.** Fix $\varepsilon \in F \subseteq \Gamma$ finite, $m \in \mathbb{N}$, $\delta > 0$. Let $\delta' > 0$ to be determined. Suppose that $T \in \text{Hom}_\Gamma(S, F, m, \delta, \sigma_i)$ and set $\xi_j = T(x_j)$. Then for all $c_{g_1, \ldots, g_m,j}$ with

$$\sum_{1 \leq j \leq m} |c_{g_1, \ldots, g_m,j}| \leq 1,$$

we have

$$\left\| \sum_{g_1, \ldots, g_m,j \in F} c_{g_1, \ldots, g_m,j} \sigma_i(g_1) \cdots \sigma_i(g_m) \xi_j \right\| \leq \delta + \left\| T \left( \sum_{g_1, \ldots, g_m,j \in F} c_{g_1, \ldots, g_m,j} g_1 \cdots g_m \xi_j \right) \right\| \leq \delta + \left\| \sum_{g_1, \ldots, g_m,j \in F} c_{g_1, \ldots, g_m,j} g_1 \cdots g_m \xi_j \right\|.$$

So $(\xi_j)_{j=1}^m \in \text{Vect}_\Gamma(S, F, m, \delta, \sigma_i)$ and $\text{vdim} \leq \dim$.

For the opposite inequality, fix $\varepsilon \in F \subseteq \Gamma$ finite $m \in \mathbb{N}, \delta' > 0$. By the previous lemma, find $\delta > 0$ for $V = X_{F,m}, v_{g_1, \ldots, g_m,j} = g_1 \cdots g_m \xi_j$, and $\varepsilon = \delta'$. If $(\xi_j)_{j=1}^m \in \text{Vect}_\Gamma(S, F, m, \delta, \sigma_i)$, then we can find a $T: X_{F,m} \to V_i$ with $\|T\| \leq 1$ and $\|T(g_1 \cdots g_m x_j) - \sigma_i(g_1) \cdots \sigma_i(g_m) \xi_j\| < \delta'$. Thus

$$\|T(g_1 \cdots g_m x_j) - \sigma_i(g_1) \cdots \sigma_i(g_m) T(x_j)\| < 2\delta',$$

and this proves the opposite inequality. \qed
5. \(l^p\)-Betti Numbers of Free Groups

Let \(X\) be a CW complex and let \(\Delta_\infty\) be the \(n\)-simplices of \(X\). Suppose that \(\Gamma\) acts properly on \(X\) with cocompact quotient, preserving the simplicial structure. For \(v_0, \cdots, v_n \in X\), let 
\[
[v_0, v_1, \cdots, v_n]
\]
be the simplex spanned by \(v_0, \cdots, v_n\). Let 
\[
V_n(X) = \{(v_0, \cdots, v_n) \in X : [v_0, \cdots, v_n] \in \Delta_\infty\}.
\]
Let \(l^p(\Delta_\infty(X))\) be all functions \(f : V_n(X) \to \mathbb{C}\) such that 
\[
f(v_{\sigma(0)}, \cdots, v_{\sigma(n)}) = (\text{sgn } \sigma)f(v_0, \cdots, v_n) \text{ for } \sigma \in \text{Sym}(\{0, \cdots, n\})
\]
by our antisymmetry condition the above sum is unchanged if we use a different representative for \([v_0, \cdots, v_n]\). On \(l^p(\Delta_\infty(X))\) we use the norm 
\[
\|f\|^p_p = \sum_{v \in \Delta_\infty(X)} |f(v_0, \cdots, v_n)|^p.
\]
Define the discrete differential \(\delta : l^p(\Delta_{n-1}(X)) \to l^p(\Delta_n(X))\) by 
\[
(\delta f)(v_0, \cdots, v_n) = \sum_{j=0}^n (-1)^j f(v_0, \cdots, \hat{v}_j, \cdots, v_n),
\]
where the hat indicates a term omitted, note that \(\delta f\) satisfies the appropriate antisymmetry condition. Define the \(n\)th \(l^p\)-Cohomology space of \(X\) by 
\[
H^p_n(X) = \frac{\ker(\delta) \cap l^p(\Delta_n(X))}{\ker(l^p(\Delta_{n-1}(X))}. 
\]
We define the \(l^p\)-Betti numbers of \(X\) with respect to \(\Gamma\) by 
\[
\beta^{(p)}_{\Sigma, n}(X, \Gamma) = \text{dim}_{\Sigma, l^p}(H^p_n(X), \Gamma).
\]
It is known that if \(X\) is contractible and \(\pi_1(X/\Gamma) \cong \Gamma\), then the \(l^p\)-cohomology space only depends upon \(\Gamma\), thus we may define 
\[
H^p_n(\Gamma) = H^p_n(X, \Gamma),
\]
\[
\beta^{(p)}_{\Sigma, n}(\Gamma) = \beta^{(p)}_{\Sigma, n}(X, \Gamma),
\]
for such \(X\).

We also consider \(l^p\)-Homology. Define \(\partial : l^p(\Delta_n(X)) \to l^p(\Delta_{n-1}(X))\) by 
\[
\partial f(v_0, \cdots, v_{n-1}) = \sum_{x \in \{v_0, \cdots, v_{n-1}, x\} \in \Delta_n(X)} f(v_0, \cdots, v_{n-1}, x),
\]
by direct computation 
\[
(\partial : l^p(\Delta_n(X)) \to l^p(\Delta_{n-1}(X))) = (\delta : l^p(\Delta_{n-1}(X)) \to l^p(\Delta_n(X)))^t,
\]
when \(\frac{1}{p} + \frac{1}{q} = 1\). Define the \(l^p\)-Homology of \(X\) by 
\[
H^p_n(X) = \frac{\ker(\partial) \cap l^p(\Delta_n(X))}{\ker(l^p(\Delta_{n+1}(X))}. 
\]
We shall be interested in the \(l^p\)-Betti numbers of free groups.
Fix $n \in \mathbb{N}$, and consider the free group $\mathbb{F}_n$ on $n$ letters $a_1, \ldots, a_n$. Let $G$ be the Cayley graph of $\mathbb{F}_n$ with respect to $a_1, \ldots, a_n$, we regard the edges of $G$ as oriented. Then the topological space $X$ associated to $G$ is contractible, since $G$ is a tree, and has $\pi_1(X/\mathbb{F}_n) \cong \mathbb{F}_n$, so the $l^p$-cohomology of $G$ is the $l^p$-cohomology of $\mathbb{F}_n$. Let $E(\mathbb{F}_n)$ denote the edges of $\mathbb{F}_n$. Then $l^p(E(\mathbb{F}_n))$ as defined above is given by all functions $f : E(\mathbb{F}_n) \to \mathbb{C}$ such that

$$f(x, s) = -f(s, x) \text{if } (s, x) \in E(\mathbb{F}_n),$$

$$\sum_{j=1}^n \sum_{x \in \mathbb{F}_n} |f(x, xa_j)|^p < \infty.$$ 

With the norm

$$\|f\|_p^p = \sum_{j=1}^n \sum_{x \in \mathbb{F}_n} |f(x, xa_j)|^p.$$ 

Note that this is indeed a norm on $l^p(E(\mathbb{F}_n))$, and that $\mathbb{F}_n$ acts isometrically on $l^p(E(\mathbb{F}_n))$ by left translation. Also $l^p(E(\mathbb{F}_n))$ is isomorphic to $l^p(\mathbb{F}_n)$ with respect to this action. If $(x, s) \in E(\mathbb{F}_n)$, we let $\mathcal{E}_{x, s}$ be the function on $E(\mathbb{F}_n)$ such that

$$\mathcal{E}_{x, s}(y, t) = 0 \text{ if } \{x, s\} \neq \{y, t\}$$

$$\mathcal{E}_{x, s}(x, s) = 1$$

$$\mathcal{E}_{x, s}(s, x) = -1.$$ 

We think of $\mathcal{E}_{x, s}$ as representing the edge going from $x$ to $s$.

Then the discrete differential we defined above

$$\delta : l^p(\mathbb{F}_n) \to l^p(E(\mathbb{F}_n))$$

is given by

$$(\delta f)(x, s) = f(s) - f(x) \text{ if } (x, s) \in E(\mathbb{F}_n)).$$

And the corresponding $l^p$-Cohomology space is given by

$$H^1_{\delta}l^p(\mathbb{F}_n) = l^p(E(\mathbb{F}_n))/\delta(l^p(\mathbb{F}_n)).$$

Also $\partial : l^p(E(\mathbb{F}_n)) \to l^p(\mathbb{F}_n)$ is given by

$$(\partial f)(x) = \sum_{j=1}^n f(x, xa_j) - \sum_{j=1}^n f(xa_j^{-1}, x).$$

Since $\mathbb{F}_n$ is non-amenable, we know $l^p(\mathbb{F}_n)$ does not have almost invariant vectors under the translation action of $\mathbb{F}_n$. Thus, there is a $C > 0$ such that

$$\|\delta f\|_p \geq C\|f\|_p.$$

So $\delta(l^p(\mathbb{F}_n))$ is closed in $l^p(E(\mathbb{F}_n))$.

In this section, we compute the $l^p$-Betti numbers

$$\beta^{(p)}_{\Sigma, 1}(\mathbb{F}_n),$$

for $1 \leq p \leq 2$.

**Lemma 5.1.** Fix $n \in \mathbb{N}$, $1 \leq p < \infty$. Then the image of the elements $\mathcal{E}_{e, a_1}, \ldots, \mathcal{E}_{e, a_{n-1}}$ are dynamically generating for $H^1_{\delta}l^p(\mathbb{F}_n)$. 
Proof. For this, it suffices to show that

\[ W = \delta(l^p(\mathbb{F}_n)) + \text{Span}\{\mathcal{E}_{(s,a_j)} : s \in \mathbb{F}_n, 1 \leq j \leq n - 1\} \]

is norm dense in \( l^p(E(\mathbb{F}_n)) \).

It is enough to show that \( \mathcal{E}_{(e,a_n)} \in W \), and by convexity it is enough to show that \( \mathcal{E}_{(e,a_n)} \) is in the weak closure of \( W \).

To do this, we shall prove by induction on \( k \) that

\[ \mathcal{E}_{(a_n^{k+1},a_n^k)} \equiv \mathcal{E}_{(a_n^k,a_n^{k+1})} \mod W, \]

this is enough since

\[ \mathcal{E}_{(a_n^k,a_n^{k+1})} \rightarrow 0 \]

weakly.

The base case \( k = 0 \) is trivial, so assume the result true for some \( k \). Then

\[
\mathcal{E}_{(a_n^k,a_n^{k+1})} - \delta(\chi_{(a_n^{k+1})}) = \sum_{j=1}^n \mathcal{E}_{(a_n^{k+1},a_n^{k+1}a_j)} + \sum_{j=1}^{n-1} \mathcal{E}_{(a_n^{k+1},a_n^{k+1}a_j^{-1})}
\]

\[
= \mathcal{E}_{(a_n^{k+1},a_n^{k+2})} + \sum_{j=1}^{n-1} a_n^{k+1} \mathcal{E}_{(e,a_j)} - \sum_{j=1}^{n-1} a_n^{k+1} a_j^{-1} \mathcal{E}_{(e,a_j)}
\]

\[
\equiv \mathcal{E}_{(a_n^{k+1},a_n^{k+2})}.
\]

Here is a graphical explanation of the above calculation. If we think of the elements of \( l^p(E(\mathbb{F}_n)) \) as formal sums of oriented edges, then \(-\delta(\chi_{a_n^{k+1}})\) is a “source” at \( a_n^{k+1} \), it is the sum of all edges adjacent to \( a_n^{k+1} \), directed away from \( a_n^{k+1} \). Pictured below:

![Graphical explanation of the calculation](image)

The above computation can be phrased as follows:

\[-\delta(\chi_{a_n^{k+1}}) + \mathcal{E}_{(a_n^k,a_n^{k+1})} = \]


and the second term on the right-hand side is easily seen to be in the span of translates of \( \mathcal{E}_{(e, a_j)}, j = 1, \ldots, n - 1. \)

This completes the induction step. □

We shall prove the analogous claim for \( l^p \)-Homology of free groups, but we need a few preliminary results. These next few results must be well known, but we include proofs for completeness.

**Lemma 5.2.** Let \( \Gamma \) be a non-amenable group with finite-generating set \( S \). Let \( A: l^p(\Gamma) \to l^p(\Gamma) \) be defined by

\[
Af = \frac{1}{|S \cup S^{-1}|} \sum_{s \in S \cup S^{-1}} \rho(s),
\]

then for \( 1 < p < \infty \), there is a constant \( C_p < 1 \) so that \( \|Af\|_p < C_p \|f\|_p \).

**Proof.** For \( p = 2 \), this is automatic from non-amenability of \( \Gamma \). Since \( \|Af\|_{\infty} \leq \|f\|_{\infty}, \|Af\|_1 \leq \|f\|_1 \), the lemma follows by interpolation. □

**Lemma 5.3.** Let \( \Gamma \) be a non-amenable group with finite generating set \( S \). For \( 1 < p < \infty \), the operator \( \partial \circ \delta: l^p(\Gamma) \to l^p(\Gamma) \), is invertible.

**Proof.** We have that

\[
\partial(\delta f)(x) = \sum_{s \in S \cup S^{-1}} f(x) - f(xs) = |S \cup S^{-1}| \left( f(x) - \frac{1}{|S \cup S^{-1}|} \sum_{s \in S \cup S^{-1}} \rho(s)f(x) \right).
\]

By the previous lemma,

\[
\left\| \frac{1}{|S \cup S^{-1}|} \sum_{s \in S \cup S^{-1}} \rho(s) \right\|_{l^p \to l^p} < 1,
\]
for $1 < p < \infty$, so this proves that $\partial(\delta)$ is invertible for $1 < p < \infty$. \hfill \square

**Corollary 5.4.** Let $\Gamma$ be a non-amenable group with finite generating set $S$. For $1 < p < \infty$, we have the following Hodge Decomposition:

$$L^p(E(\Gamma)) = \ker(\partial): L^p(E(\Gamma)) \rightarrow L^p(\Gamma)) + \delta(L^p(\Gamma)).$$

**Proof.** If $f \in \ker(\partial): L^p(E(\Gamma)) \rightarrow L^p(\Gamma)) \cap \delta(L^p(\Gamma))$, write $f = \delta(g)$, then

$$0 = \partial(f) = \partial(\delta(g)),$$

so by the preceding lemma we have that $g = 0$.

If $f \in L^p(E(\Gamma))$, then by the preceding lemma we can find a unique $g$ so that $\partial(f) = \partial(\delta(g))$. Then $f - \delta(g) \in \ker(\partial)$, and

$$f = f - \delta(g) + \delta(g).$$

\hfill \square

**Proposition 5.5.** Let $n \in \mathbb{N}$, and $1 < p < \infty$, then $H^1_p(F_n)$ can be generated by $n-1$ elements.

**Proof.** The claim for $n = 1$ is clear since $H^1_p(\mathbb{Z}) = 0$. We claim that it suffices to do the proof for $n = 2$. For this, let $n > 2$, and let $a_1, \ldots, a_n$ be the generators of $F_n$. Consider the injective homomorphisms $\phi_j : F_2 \rightarrow F_n$ for $1 \leq j \leq n - 1$ given by $\phi_j(a_i) = a_{i+j}$. Let $f$ be an element in $L^p(E(F_2))$ so that Span$(F_2f)$ is dense in $\ker(\partial) \cap L^p(E(F_2))$. Let $f_j \in L^p(E(F_n))$ be the element defined by

$$f_j(x,y) = \begin{cases} 0, & \text{if one of } x, y \notin \phi_j(F_2) \\ f(\phi_j^{-1}(x), \phi_j^{-1}(y)), & \text{otherwise}. \end{cases}$$

Then $f_j \in \ker(\partial)$. It is easy to see from the preceding corollary and the fact that $f$ generates $\ker(\partial) \cap L^p(E(F_2))$, that

$$E_{(e,a_j)} \in \ker(\partial) + \delta(L^p(F_n)),$$

again by the preceding corollary we find that $f_1, \ldots, f_{n-1}$ generate $\ker(\partial)$. Thus it suffices to handle the case $n = 2$.

We now concentrate on the case $n = 2$, and we use $a, b$ for the generators of $F_2$. Let $f : E(F_2) \rightarrow \mathbb{R}$ defined by the following inductive procedure. Set

$$f_1 = E_{(e,a)} + E_{(e,b)} + E_{(a^{-1}, e)} + E_{(b^{-1}, e)}.$$

Having constructed $f_1, \ldots, f_n$ so that $f_j$ is supported on the pairs of edges which have word length at most $j$, define $f_{n+1}$ as follows. For each word $w$ of length $n$, let $e_1, e_2, e_3$ be the three oriented edges which have their terminal vertex $w$ and the initial vertex a word of length $n + 1$, and let $e$ be the oriented edge which has its initial vertex $w$ and its terminal vertex a word of length $n - 1$. Define for $j = 1, 2, 3$

$$f_{n+1}(e_j) = \frac{1}{3} f_n(e),$$

and define

$$f_{n+1}(e) = f_n(e)$$

if both vertices of $e$ have length at most $n$. It is easy to see that the $f_n$’s as constructed above converge pointwise to a function $f$ in $L^p(E(F_2)) \cap \ker(\partial)$ for $1 < p \leq \infty$.

The function $f$ is pictured below:
Set \( V = \text{Span}(F_2 f) + \delta(l^p(F_2)) \). To show that \( f \) generates \( \ker(\partial) \) it suffices, by the preceding corollary to show that

\[ E(e,a_1), E(e,a_2) \in V. \]

Let \( B_n = \{(x,y) \in G : \|x\|, \|y\| \leq n\} \). For \( n \geq 0 \), let \( g_n : E(F_n) \to \mathbb{C} \), be the function defined by

\[
\chi_{B_n} g_n = \left( \sum_{k=0}^{n-1} (1/3)^n \right) \left( E(e,a) + E(e,b) + E(a^{-1},e) + E(b^{-1},e) \right),
\]

\[
(1 - \chi_{B_n}) g_n = (1 - \chi_{B_n}) f,
\]

we first show that \( g_n \in \text{Span}(F_2 f) + \delta(l^p(F_2)) \), for all \( n \). We prove this by induction on \( n \), the case \( n = 1 \) being clear since \( g_1 = f \). Suppose the claim true for some \( n \). Then for each word \( w \) of length \( n \), we can add either \((1/3)^n \delta(\chi(w))\) or \(- (1/3)^n \delta(\chi(w))\) to \( f_n \) to make the value on every edge from \( w \) to a word of length \( n + 1 \) zero. This now adds a value of \( \pm (1/3)^n \) to every edge going from a word of length \( n \) to a word of length \( n - 1 \). Now repeat for every word of length \( n - 1 \) : add on \( \pm (1/3)^n \delta(\chi(w)) \) for every word \( w \) of length \( n - 1 \) to force a value of 0 on every edge going from a word of length \( n - 1 \) to a word of length \( n \). Repeating this inductively until we get to words of length 1, we find by construction of \( f \) that
$g_n \in \text{Span}(\mathbb{F}_2 f) + \delta(lP(\mathbb{F}_2))$. The first two steps of this process are pictured below:
Since \( \sup_n \|g_n\|_p < \infty \) we find that \( g_n \) converges weakly to

\[
\frac{3}{2}(\mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)} + \mathcal{E}_{(b^{-1},e)} + \mathcal{E}_{(a^{-1},e)})
\]

Rescaling we find that

\[
\mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)} + \mathcal{E}_{(b^{-1},e)} + \mathcal{E}_{(a^{-1},e)} \in V.
\]

By adding \( \pm \delta(\chi_{\{e\}}) \) and scaling we find that

\[
\mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)} \in V,
\]

\[
\mathcal{E}_{(e,b^{-1})} + \mathcal{E}_{(e,a^{-1})} \in V.
\]

Inductively, we now see that

\[
\mathcal{E}_{(e,a)} + \mathcal{E}_{((ba^{-1})^{-1} b, (ba^{-1})^n)} \in V,
\]

and taking weak limits proves that

\[
\mathcal{E}_{(e,a)} \in V.
\]
Subtracting $\mathcal{E}_{(e,a)}$ from $\mathcal{E}_{(e,a)} + \mathcal{E}_{(e,b)}$ we find that
$$\mathcal{E}_{(e,a)}, \mathcal{E}_{(e,b)} \in \mathcal{V}$$
and thus by $\mathbb{F}_2$-invariance that $V = l^p(E(\mathbb{F}_2))$, this completes the proof.

\[ \square \]

**Theorem 5.6.** Fix $n \in \mathbb{N}$, and a sofic approximation $\Sigma$.

(a) The dimension of the $l^p$-cohomology groups of $\mathbb{F}_n$ satisfy
$$\dim_{\Sigma, l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) = \dim_{\Sigma, l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) = n - 1, \text{ for } 1 \leq p \leq 2,$$
$$H^m_{l^p}(\mathbb{F}_n) = \{0\} \text{ for } m \geq 2.$$

(b) The dimension of the $l^p$-homology groups of $\mathbb{F}_n$ satisfy:
$$\dim_{\Sigma, l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) = \dim_{\Sigma, l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) = n - 1, \text{ for } 1 < p < 2,$$
$$H^1_{l^p}(\mathbb{F}_n) = \ker(\partial) \cap l^1(E(\mathbb{F}_n)) = \{0\},$$
$$H^m_{l^p}(\mathbb{F}_n) = 0 \text{ for } m \geq 2.$$

**Proof.** The statements about higher-dimensional homology or cohomology are clear, since we know that the Cayley graph of $\mathbb{F}_n$ is contractible and one-dimensional.

Since the image of $\delta$ is closed, the sequence
$$0 \longrightarrow l^p(\mathbb{F}_n) \xrightarrow{\delta} l^p(E(\mathbb{F}_n)) \longrightarrow H^1_{l^p}(\mathbb{F}_n) \longrightarrow 0$$
is exact. Subadditivity under exact sequences, and the computation for $l^p$-spaces implies that
$$n = \dim_{\Sigma, l^p}(l^p(E(\mathbb{F}_n)), \mathbb{F}_n)$$
$$\leq \dim_{\Sigma, l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) + \dim_{\Sigma, l^p}(l^p(\mathbb{F}_n))$$
$$= \dim_{\Sigma, l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) + 1.$$  

Thus
$$\dim_{\Sigma, l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) \geq n - 1.$$  

On the other hand, by the Lemma 5.1, $H^1_{l^p}(\mathbb{F}_n)$ can be generated by $n - 1$ elements, so
$$\dim_{\Sigma, l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) \leq n - 1,$$
which proves the first claim.

For the second claim, by surjectivity of $\partial$ for $1 < p \leq 2$, the sequence
$$0 \longrightarrow H^1_{l^p}(\mathbb{F}_n) \longrightarrow l^p(E(\mathbb{F}_n)) \xrightarrow{\partial} l^p(\mathbb{F}_n) \longrightarrow 0$$
is exact. As in the first half this implies that
$$\dim_{\Sigma, l^p}(H^1_{l^p}(\mathbb{F}_n), \mathbb{F}_n) \geq n - 1,$$
for $1 < p \leq 2$. The upper bound for $1 < p \leq 2$ also holds by the preceding proposition.

We turn to the last claim. If $x \in \mathbb{F}_n$, because the Cayley graph of $\mathbb{F}_n$ is a tree we can define $\gamma_x$ to be the unique geodesic path from $e$ to $x$. Let $|x| = d(x, e)$, and define
$$A : \mathbb{C}^E(\mathbb{F}_n) \to \mathbb{C}^{\mathbb{F}_n}.$$
by
\[(Af)(x) = \sum_{j=1}^{[x]} f(\gamma_x(j-1), \gamma_x(j)),\]

note that \(\delta(Af) = f\). A direct computation verifies that
\[A(E(x, xa_j)) \in l^\infty(F_n),\]
thus \(\delta(l^\infty(F_n))\) is weak\(^*\) dense in \(l^\infty(E(F_n))\). By duality \(\ker(\partial) \cap l^1(E(F_n)) = \{0\}\), this completes the proof.

\[\square\]

6. Closing Remarks

Here are some natural conjectures based on our work in this paper and [12].

**Conjecture 1.** Let \(\Gamma\) be a amenable group, and \(Y \subseteq l^p(\Gamma)^{\oplus n}\), for some \(n \in \mathbb{N}\). Let \(\dim_{l^p}(Y, \Gamma)\) be \(l^p\)-Dimension as defined by Gornay. Then for any sofic approximation \(\Sigma\) of \(\Gamma\) we have
\[\dim_{l^p}(Y, \Gamma) = \dim_{\Sigma, l^p}(Y, \Gamma).\]

**Conjecture 2.** Let \(2 < p < \infty\), and let \(\Gamma\) be a countable discrete sofic group with sofic approximation \(\Sigma\). Then for all \(n \in \mathbb{N}\),
\[\dim_{\Sigma, l^p}(l^p(\Gamma)^{\oplus n}, \Gamma) = \lim_{n \to \infty}(l^p(\Gamma)^{\oplus n}, \Gamma)\]

Little progress has been made on Conjectures 1,2. It may be quite possible that our definition is simply not the right way to look at von Neumann dimension for the action of \(\Gamma\) on \(l^p(\Gamma)^{\oplus n}\) if \(2 < p < \infty\).

Another natural conjecture based on the techniques in section 2 is the following.

**Conjecture 3.** Let \(\Gamma\) be an amenable group. If \(\Sigma, \Sigma'\) are two sofic approximations of \(\Gamma\) and \(X\) is a uniformly bounded representation of \(\Gamma\), then
\[\dim_{\Sigma, l^p}(X, \Gamma) = \dim_{\Sigma', l^p}(X, \Gamma).\]

Because of the techniques in [2] if \(\Sigma = (\sigma_i: \Gamma \to S_{d_i})\), it suffices to assume \(\Sigma = (\sigma^{\oplus k_i})\), for a sequence of integers \(k_i\).

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