KÄHLER TORI WITH ALMOST NON-NEGATIVE SCALAR CURVATURE

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Abstract. Motivated by the torus stability problem, in this work we study Kähler metrics with almost non-negative scalar curvature on complex torus. We prove that after passing to a subsequence, non-collapsing sequence of Kähler metrics with almost non-negative scalar curvature will converge to flat torus weakly.

1. Introduction

In conformal geometry, one can distinguish manifolds using the Yamabe type \cite{14,16}. It is well-known that a compact manifold is of positive Yamabe type if and only if $M$ admits a metric with positive scalar curvature. In the study of positive scalar curvature, one of the celebrated Theorem was proved by Schoen-Yau \cite{17,18} for $n \leq 7$ and Gromov-Lawson \cite{11} for general $n$ saying that torus $\mathbb{T}^n$ cannot admit metrics with positive scalar curvature. In particular, torus is of vanishing Yamabe type.

Motivated by the torus rigidity, Gromov \cite{10} conjectured a stability of metrics with almost non-negative scalar curvature on torus. Namely, if a sequence of metrics $g_i$ has uniform bound on diameter and volume, the scalar curvature is almost non-negative and is non-collapsing in appropriate sense, then after passing to a subsequence it will converge to a flat torus weakly. If the scalar curvature is strengthened to the Ricci curvature, the stability in the Gromov-Hausdorff topology follows from the celebrated work of Colding \cite{8} using volume lower bound as the non-collapsing condition. When $n = 3$, using the Ricci flow method of Hochard \cite{13} and Simon-Topping \cite{21}, a sequence of volume non-collapsed metrics $g_i$ on $\mathbb{T}^3$ with Ricci curvature bounded from below and almost non-negative scalar curvature will converge to a flat torus in the Gromov-Hausdorff topology. On the other hand, if one strengthens the assumption so that $g_i \to g_\infty$ in $C^0$, it was proved by Gromov \cite{10} and Bamler \cite{4} that $d_{g_\infty} = d_{g_{\text{flat}}}$ if $g_\infty$ is a-priori $C^2$. The $C^2$ assumption was later-on removed by Burkhardt-Guim \cite{5}. However with only upper bound on diameter and volume, it is possible that the sequence has increasingly many and increasingly thin wells. Hence one cannot expect the Gromov-Hausdorff convergence in general. When $n = 3$, Sormani \cite{23} proposed using the MinA
condition to prevent collapsing. She conjectures the convergence in the volume preserving intrinsic flat sense under the MinA condition. Progresses have been made towards the conjecture of Sormani under various settings, see [1, 2, 6, 7] and the references therein. We refer interested readers to the survey article [24] of Sormani for detailed exposition. See also [15] for the works on stability in different topologies.

In the above mentioned work, most of the results rely on special choice of gauge. To the best of authors’ knowledge, the limiting behaviour is still unclear even if the sequence is non-degenerating in the possible strongest sense, namely \( g_i \) are uniformly bi-Lipschitz on \( \mathbb{T}^n \), although it is known that the sequence is pre-compact in the Gromov-Hausdorff topology. In complex geometry, bi-Lipschitz Kähler metrics are usually well-behaved. Motivated by this, in this work we study the problem in the Kähler case of arbitrary complex dimension. Our main result is as follows:

**Theorem 1.1.** Let \( M^n = \mathbb{T}^{2n} \) be a complex torus and \( \omega_h \) be a Kähler metric on \( M \). Suppose \( \omega_{i,0} \) is a sequence of Kähler metrics such that for some \( \Lambda > 1 \) and \( p > n \), we have

1. \( \text{Vol}(M, \omega_{i,0}) \geq \Lambda^{-1} \);
2. \( \| \text{tr}_{\omega_h} \omega_{i,0} \|_{L^p(\omega_h)} \leq \Lambda \);
3. \( R(\omega_{i,0}) \geq -i^{-1} \) for all \( i \in \mathbb{N} \),

then there is a flat Kähler metric \( \omega_{\infty} \) on \( M \) such that after passing to a subsequence, \( \omega_{i,0} \to \omega_{\infty} \) in the sense of current and

\[
\lim_{i \to +\infty} \| v_{i,0} - 1 \|_{L^{q/n}(\omega_{\infty})} = 0
\]

for any \( q < p \), where \( v_{i,0} = \frac{\omega_{i,0}^n}{\omega_{\infty}^n} \) denotes the volume function. In particular, we have

\[
d\text{vol}_{g_{i,0}} \to d\text{vol}_{g_{\infty}} \text{ weakly on } M,
\]

where \( g_{i,0} \) and \( g_{\infty} \) denote the associated Riemannian metrics of \( \omega_{i,0} \) and \( \omega_{\infty} \) respectively. If in addition (ii) holds for \( p = +\infty \), then for any \( x, y \in M \),

\[
\limsup_{i \to +\infty} d_{\omega_{i,0}}(x, y) \leq d_{\omega_{\infty}}(x, y).
\]

The volume assumption is necessary to avoid collapsing while we impose \( L^p \) bound to ensure non-expanding. We refer readers to [3] for many interesting examples along this line. With this, we are able to show that the sequence sub-sequentially converges to some flat metric weakly without assumption on special gauge but the Kählerity. The distance estimate in principle says that potentially shorter path might be built along the convergence but not the other way round under almost non-negative scalar curvature, see [15] for examples in the Riemannian case.
Acknowledgement: J. Chu was partially supported by Fundamental Research Funds for the Central Universities (No. 7100603592). The authors would like to thank the referee for the useful comments.

2. Regularization using Ricci flows

To prove Theorem 1.1 we start with modifying the reference Kähler form using Calabi-Yau Theorem.

Lemma 2.1. Under the assumption of Theorem 1.1 there is a sequence $\alpha_i$ of flat Kähler metric inside the same Kähler class of $\omega_{i,0}$ such that $\alpha_i = \omega_{i,0} + \sqrt{-1} \partial \bar{\partial} u_i$ where

(a) $\sup_M u_i = 0$ and $\|u_i\|_\infty \leq L_1$;

(b) $L_1^{-1} \omega_h \leq \alpha_i \leq L_1 \omega_h$ on $M$;

(c) $\log \frac{\omega^n_{i,0}}{\alpha^n_i} \geq -L_1 i^{-1/2}$ on $M$

for some $L_1 > 1$ and for all $i \in \mathbb{N}$.

Proof. We may assume $\int_M \omega^n_h = 1$ by rescaling. By the solution to Yau’s solution to the Calabi conjecture [26], there is $u_i \in C^\infty(M)$ with $\sup_M u_i = 0$ such that

$$\alpha_i := \omega_{i,0} + \sqrt{-1} \partial \bar{\partial} u_i$$

is a Kähler Ricci flat metric on $M$. In particular, integrating by parts shows that for all $i \in \mathbb{N}$, we have

$$\text{Vol}(M, \alpha_i) = \text{Vol}(M, \omega_{i,0}) \geq \Lambda^{-1}$$

and

$$\int_M \alpha_i \wedge \omega^{n-1}_h = \int_M \omega_{i,0} \wedge \omega^{n-1}_h$$

$$= \frac{1}{n} \int_M \text{tr}_{\omega_h} \omega_{i,0} \cdot \omega^n_h$$

$$\leq \frac{1}{n} \cdot \| \text{tr}_{\omega_h} \omega_{i,0} \|_{L^p(\omega_h)}$$

$$\leq \frac{1}{n} \cdot \Lambda.$$

(2.3)

Since $M$ is a complex torus, then the Ricci flat metric $\alpha_i$ must be flat. Therefore, (2.2) and (2.3) imply that

$$C_1^{-1} \omega_h \leq \alpha_i \leq C_1 \omega_h$$

for some $C_1 > 1$ and for all $i \in \mathbb{N}$. From this, (b) follows immediately.
To see (a), we use (2.4) to see that
\[
\|\Delta \omega_h u_i\|_{L^p(\omega_h)} = \|\text{tr}_{\omega_h} (\alpha_i - \omega_{i,0})\|_{L^p(\omega_h)}
\leq \|\text{tr}_{\omega_h} \alpha_i\|_{L^p(\omega_h)} + \|\text{tr}_{\omega_h} \omega_{i,0}\|_{L^p(\omega_h)}
\leq C_2
\]
for some $C_2 > 0$ and for all $i \in \mathbb{N}$. We claim that
\[
\|u_i\|_{L^\infty} \leq C \|\Delta \omega_h u_i\|_{L^p(\omega_h)}
\]
for some constant $C$ and all $u \in C^\infty(M)$ with $\sup_M u = 0$. Then (a) follows from (2.6) immediately. To prove (2.6), we argue by contradiction. Suppose that (2.6) is not true, then for some $u_i \in C^\infty(M)$ with $\sup_M u_i = 0$, we have
\[
\|u_i\|_{L^\infty} > i \|\Delta \omega_h u_i\|_{L^p(\omega_h)}.
\]
Write $v_i = u_i/\|u_i\|_{L^\infty}$. Recalling $\sup_M u_i = 0$,
\[
\sup_M v_i = 0, \quad \inf_M v_i = -1, \quad \|\Delta \omega_h v_i\|_{L^p(\omega_h)} < i^{-1}.
\]
Using $p > n$, $W^{2,p}$ estimate and Sobolev embedding (note that the real dimension of $M$ is $2n$), and passing to a subsequence again, we may assume
\[
v_i \to v_\infty \quad \text{in the } C^0 \text{ sense},
\]
where $v_\infty$ is a weak solution of the Laplacian equation. The standard elliptic theory shows that $v_\infty$ is smooth. By the strong maximum principle, $v_\infty$ is a constant function. This contradicts to (2.8) and (2.9).

It remains to establish the lower bound of the volume form, i.e. (c). Following the argument of [12, Theorem 1.9], we consider the function
\[
F_i = \log \frac{\omega_{i,0}^n}{\alpha_i^n} - \delta u_i,
\]
where $\delta$ is a constant to be determined later. At its minimum $x_0 \in M$, we have
\[
0 \leq \Delta_{\omega_{i,0}} F_i
\leq -R(\omega_{i,0}) - \delta \left(\text{tr}_{\omega_{i,0}} \alpha_i - n\right)
\leq (i^{-1} + n\delta) - \delta \text{tr}_{\omega_{i,0}} \alpha_i.
\]
Combining this with AM-GM inequality,
\[
\left(\frac{\alpha_i^n}{\omega_{i,0}^n}\right)^{1/n} \leq \frac{1}{n} \text{tr}_{\omega_{i,0}} \alpha_i \leq 1 + \frac{1}{\delta n} \quad \text{at } x_0.
\]
Hence for all \( x \in M \),

\[
e^{F_i(x)} \geq e^{F_i(x_0)}
\]

(2.13)

\[
= \frac{\omega_{i,0}}{\alpha_i^n} e^{-\delta u_i} \big|_{x_0}
\]

\[
\geq \left( 1 + \frac{1}{\delta ni} \right)^{-n} e^{-\delta \|u_i\|_{L^\infty}}.
\]

By taking \( \delta = i^{-1/2} \) and using the definition of \( F_i \),

(2.14)

\[
\log \frac{\omega_{i,0}}{\alpha_i^n} \geq -n \log \left( 1 + \frac{1}{ni^{1/2}} \right) - 2i^{-1/2} \|u_i\|_{L^\infty} \geq -Ci^{-1/2},
\]

where we have used (a) in the last inequality. \( \square \)

To obtain the stability, we will make use of the Kähler-Ricci flow to regularize the sequence \( \omega_{i,0} \). We let \( \omega_i(t) \) be the solution to the Kähler-Ricci flow starting from \( \omega_i(0) = \omega_{i,0} \), i.e.,

(2.15)

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \omega_i(t) = -\text{Ric}(\omega_i(t)); \\
\omega_i(0) = \omega_{i,0}.
\end{array} \right.
\]

Since \( M \) is a complex torus, then the first Chern class \( c_1(M) \) is zero. By [25, Section 1], it is known that each \( \omega_i(t) \) exists for \( t \geq 0 \). We now obtain estimates by re-writing the flow as the parabolic Monge-Ampère equation:

(2.16)

\[
\left\{ \begin{array}{l}
\dot{\phi}_i(t) = \log \frac{\omega_i(t)^n}{\alpha_i^n}; \\
\omega_i(t) = \omega_{i,0} + \sqrt{-1} \partial \bar{\partial} \varphi_i(t); \\
\varphi_i(0) = 0,
\end{array} \right.
\]

where \( \dot{\phi}_i = \frac{\partial \varphi_i}{\partial t} \) and \( \alpha_i \) is the flat metric obtained in Lemma 2.1. We begin with the zeroth order estimates of \( \varphi_i \).

**Lemma 2.2.** There is \( L_2 > 0 \) such that for all \( i \in \mathbb{N} \), we have

\[
\sup_{M \times [0, +\infty)} |\varphi_i(t)| \leq L_2.
\]

**Proof.** To obtain the upper bound of \( \varphi_i(t) \), we consider the function \( F_i = \varphi_i - u_i - \varepsilon t \) for \( \varepsilon > 0 \). For any \( T > 0 \), let \( (x_0, t_0) \) be the maximum point of \( F \) on \( M \times [0, T] \). By the maximum principle, at \( (x_0, t_0) \), we have \( \sqrt{-1} \partial \bar{\partial} \varphi_i \leq 0 \).
\[ \sqrt{-1} \partial \bar{\partial} u_i \text{ and hence} \]
\[ \partial_t F_i = \dot{\varphi}_i - \varepsilon \]
\[ = \log \left( \frac{(\omega_{i,0} + \sqrt{-1} \partial \bar{\partial} \varphi_i)^n}{\alpha_i^n} \right) - \varepsilon \]
\[ \leq \log \left( \frac{(\omega_{i,0} + \sqrt{-1} \partial \bar{\partial} u_i)^n}{\alpha_i^n} \right) - \varepsilon \]
\[ = -\varepsilon, \]
where we have used \( \alpha_i = \omega_{i,0} + \sqrt{-1} \partial \bar{\partial} u_i \) in the last equality. This is impossible for \( t_0 > 0 \). Then \( t_0 = 0 \). Combining this with \( \varphi_i(0) = 0 \), we see that
\[ \sup_{M \times [0,T]} F_i \leq \sup_M F_i(\cdot, 0) = -\inf_M u_i. \]

By letting \( \varepsilon \to 0 \) and followed by letting \( T \to +\infty \), we obtain the upper bound of \( \varphi_i \) using (a) of Lemma 2.1. The lower bound is similar by applying the minimum principle to the function \( G = \varphi_i - u_i + \varepsilon t \) for \( \varepsilon > 0 \). \( \square \)

Next, we derive the estimate on the volume form \( \dot{\varphi}_i = \log \frac{\omega_i^n}{\alpha_i^n} \). This follows from a slight modification of standard argument.

**Lemma 2.3.** There is \( L_3 > 0 \) such that for all \( i \in \mathbb{N} \) and \( (x, t) \in M \times [0, +\infty) \), we have
\[ (a) \ \dot{\varphi}_i \geq -L_3 i^{-1/2}; \]
\[ (b) \ \dot{\varphi}_i(t) \leq L_3 t^{-1} + n. \]

**Proof.** We begin with the lower bound of \( \dot{\varphi}_i \) as it is relatively simpler. By differentiating \( \dot{\varphi}_i \) with respect to time \( t \), we have
\[ \left( \frac{\partial}{\partial t} - \Delta_{\omega(t)} \right) \dot{\varphi}_i(t) = 0 \]
and hence the minimum principle implies
\[ \inf_{M \times [0, +\infty)} \dot{\varphi}_i(x, t) \geq \inf_M \dot{\varphi}_i(0) = \inf_M \log \frac{\omega_{i,0}^n}{\alpha_i^n} \geq -L_1 i^{-1/2}, \]
by using (c) of Lemma 2.1.

To obtain the upper bound of \( \dot{\varphi}_i \) for \( t > 0 \), we consider the function
\[ F_i = t \dot{\varphi}_i - \varphi_i + u_i - nt. \]
It is clear that
\[ \Delta_{\omega(t)}(\varphi_i - u_i) = \text{tr}_{\omega(t)}(\omega_i(t) - \alpha_i) = n - \text{tr}_{\omega(t)} \alpha_i. \]
Combining this with (2.19), we have
\[
\left( \frac{\partial}{\partial t} - \Delta_{\omega_i(t)} \right) F_i = \dot{\varphi}_i - \dot{\varphi}_i + \Delta_{\omega_i(t)} \varphi_i - \Delta_{\omega_i(t)} u_i - n \\
= - \text{tr}_{\omega_i(t)} \alpha_i
\]
(2.23)

Then the required estimate follows from the maximum principle, (a) of Lemma 2.1 and Lemma 2.2.

We will show that \( \omega_i(t) \) is compact in the \( C^\infty_{\text{loc}} \) topology. To do this, we need to obtain an upper and lower bounds of \( \omega_i(t) \) for \( t > 0 \).

**Lemma 2.4.** There is \( L_4 > 0 \) such that for all \( i \in \mathbb{N} \) and \( (x,t) \in M \times (0,2] \), we have
\[
\text{tr}_{\alpha_i} \omega_i(t) \leq L_4 t^{1-n} e^{\dot{\varphi}_i(t)}.
\]
(2.24)

In particular, \( \omega_i(t) \) is uniformly equivalent to \( \omega_h \) for \( t \in (0,2] \), i.e.,
\[
e^{-C_4 t^{-1}} \cdot \omega_h \leq \omega_i(t) \leq e^{C_4 t^{-1}} \cdot \omega_h
\]
(2.25)

for some \( C_4 > 0 \) and for all \( i \in \mathbb{N} \).

**Proof.** We first show how to use (2.24) to deduce (2.25). By (b) of Lemma 2.3,
\[
\text{tr}_{\alpha_i} \omega_i \leq L_4 t^{1-n} e^{\dot{\varphi}_i(t)}.
\]
(2.26)

Combining this with the following elementary inequality
\[
\text{tr}_{\omega_i} \alpha_i \leq \left( \frac{\alpha_i^n}{\omega_i^n} \right) \cdot (\text{tr}_{\alpha_i} \omega_i)^{-n-1} = e^{-\dot{\varphi}_i} \cdot (\text{tr}_{\alpha_i} \omega_i)^{-n-1}
\]
(2.27)

and (a) of Lemma 2.3 we have
\[
\text{tr}_{\omega_i} \alpha_i \leq e^{L_3 (n-1) C t^{-1}}.
\]
(2.28)

Then (2.25) follows from (2.26), (2.28) and (b) of Lemma 2.1.

It suffices to prove (2.24). When \( n = 1 \), (2.24) is trivial as \( \text{tr}_{\alpha_1} \omega_1 = e^{\dot{\varphi}_1} \). We may assume \( n \geq 2 \). We consider the following function
\[
F_i(t) = \log \text{tr}_{\alpha_i} \omega_i(t) + (n-1) \log t + \Lambda \cdot (\varphi_i(t) - u_i) - \dot{\varphi}_i(t),
\]
(2.29)

where \( \Lambda \) is a constant to be determined later. Let \( (x_0,t_0) \) be the maximum point of \( F_i \) on \( M \times [0,2] \). Since \( F_i \to -\infty \) as \( t \to 0 \), we have \( t_0 > 0 \). At \( (x_0,t_0) \), by the standard parabolic Schwarz’s lemma (see e.g. [22 Proposition 3.2.4]) and the fact that \( \alpha_i \) is flat, we have
\[
\left( \frac{\partial}{\partial t} - \Delta_{\omega_i(t_0)} \right) \log \text{tr}_{\alpha_i} \omega_i(t_0) \leq 0.
\]
(2.30)
Combining this with (2.19) and (2.22),

\[ R(t) = \frac{\omega(t)^n}{\alpha^i} \cdot (\tr\omega(t) \alpha_i)^{n-1} \leq e^{\hat{\phi}(t)} \cdot (\tr\omega(t) \alpha_i)^{n-1}, \]

which implies

\[ \log \tr\omega(t) \leq \hat{\phi}(t) - (n-1) \log t + C(n, L_3, \Lambda). \]

Choosing \( \Lambda = 1 \) and using the definition of \( F \), we conclude that

\[ \sup_{M \times [0,2]} F_i = F_i(x_0, t_0) \leq C(n, L_3) + \sup_{M \times [0,2]} |\varphi_i - u_i|. \]

Combining this with (a) of Lemma 2.1 and Lemma 2.2, we have

\[ \sup_{M \times [0,2]} (e^{-\hat{\phi}_i t^{n-1}} \tr\omega_i) \leq C(n, L_1, L_2, L_3). \]

3. Proof of Main Theorem

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. For the smooth solution \( \omega_i(t) \) of Kähler-Ricci flow, it is well-known that (see e.g. [22, (3.56)])

\[ \left( \frac{\partial}{\partial t} - \Delta_{\omega_i(t)} \right) R(\omega_i(t)) \geq 0. \]

Then \( R(\omega_i(0)) = R(\omega_{i,0}) \geq -i^{-1} \) and the maximum principle imply

\[ R(\omega_i(t)) \geq -i^{-1} \]

for all \( i \in \mathbb{N} \) and \((x, t) \in M \times [0, +\infty)\). By Lemma 2.4 and [19, Theorem 1], \( \omega_i(t) \) is bounded in \( C^k_{\text{loc}} \) for all \( k \in \mathbb{N} \) for \( t \in (0, 2] \). After passing to subsequence, we may assume that \( \omega_i(t) \rightarrow \omega_{i,\infty}(t) \) in \( C^k_{\text{loc}}(M \times (0, 2]) \) as \( i \rightarrow +\infty \). Moreover, for any \( t \in (0, 2] \), \( R(\omega_{i,\infty}(t)) \geq 0 \) by (3.2).

Let \( \hat{\omega} \) be a Kähler-Ricci flat metric on \( M \). Since \( \omega_{i,\infty}(t) \) is Kähler, the non-negativity of scalar curvature implies that

\[ \Delta_{\omega_{i,\infty}} \left( \log \frac{\omega_{i,\infty}(t)^n}{\hat{\omega}^n} \right) = -R(\omega_{i,\infty}(t)) \leq 0. \]
By the strong maximum principle, the function \( \log \frac{\omega_\infty(t)^n}{\tilde{\omega}_n} \) is constant, and then \( \omega_\infty(t) \) is Ricci flat. As \( M \) is a complex torus, \( \omega_\infty(t) \) must be flat. Moreover, as \( \omega_\infty(t) \) is a smooth solution to the Kähler-Ricci flow, uniqueness implies that \( \omega_\infty(t) \) is independent of time \( t \), i.e., \( \omega_\infty(t) \equiv \tilde{\omega}_\infty \) for \( t \in (0, 2] \) for some flat Kähler metric \( \tilde{\omega}_\infty \) on \( M \).

It remains to show that \( \omega_{i,0} \) converges to \( \tilde{\omega}_\infty \) weakly. We first show the convergence in current sense. Since \( \omega_{i,0} \) and \( \alpha_i \) are in the same Kähler class, by Lemma 2.1 and the Banach-Alaoglu Theorem [9, Chapter III Proposition 1.23], we may assume that \( \omega_{i,0} \) converge weakly to a Kähler current \( \omega_\infty \). It suffices to show that \( \omega_\infty \equiv \tilde{\omega}_\infty \) as a current.

Let \( \eta \) be an arbitrary \((n - 1, n - 1)\) test form on \( M \). Using integration by parts,

\[
\int_M \eta \wedge \omega_i(1) = \int_M \eta \wedge (\omega_{i,0} + \sqrt{-1} \bar{\partial} \partial \varphi_i(1)) = \int_M \eta \wedge \omega_{i,0} + \int_M \varphi_i(1) \cdot \sqrt{-1} \bar{\partial} \partial \eta = \int_M \eta \wedge \omega_{i,0} + \mathbf{E}.
\]

Since \( \eta \) is a fixed \((n - 1, n - 1)\) form on \( M \), there is \( C_\eta > 0 \) such that

\[
|\sqrt{-1} \bar{\partial} \partial \eta| \leq C_\eta \cdot \omega_n^h.
\]

We will show that \( \mathbf{E} \to 0 \) as \( i \to +\infty \). Using \( \varphi_i(0) = 0 \), (b) of Lemma 2.1 and almost non-negativity of \( \dot{\varphi}_i \) from (a) of Lemma 2.3, we have

\[
|\mathbf{E}| = \left| \int_0^1 \left( \int_M \dot{\varphi}_i(t) \cdot \sqrt{-1} \bar{\partial} \partial \eta \right) dt \right| \\
\leq C \int_0^1 \left( \int_M |\dot{\varphi}_i(t)| \alpha_i^n \right) dt \\
\leq C \int_0^1 \left( \int_M \dot{\varphi}_i(t) \alpha_i^n \right) dt + Ci^{-1/2}.
\]

Using the elementary inequality \( x \leq e^x - 1 \) for all \( x \in \mathbb{R} \), \( \dot{\varphi}_i = \log \frac{\omega_i^n}{\alpha_i^n} \) and the fact that \( \omega_i(t) \) and \( \alpha_i \) are in the same class,

\[
|\mathbf{E}| \leq C_\eta \int_0^1 \left( \int_M (e^{\dot{\varphi}_i(t)} - 1) \alpha_i^n \right) dt + Ci^{-1/2} = C_\eta \int_0^1 \left( \int_M (\omega_i(t)^n - \int_M \alpha_i^n) \right) dt + Ci^{-1/2} = Ci^{-1/2}.
\]
By letting $i \to +\infty$ in (3.4), we have
\[
\int_M \eta \wedge \tilde{\omega}_\infty = \int_M \eta \wedge \omega_\infty
\]
for any test form $\eta$ on $M$. Hence, $\omega_\infty = \tilde{\omega}_\infty$ and is smooth on $M$.

Thanks to (3.2), we have (see e.g. [22, (3.65)])
\[
\frac{\partial}{\partial t} \left( \log \frac{\omega_i(t)}{\omega_i^0} \right) = -\mathcal{R}(\omega_i(t)) \leq i^{-1}.
\]
Then for all $i \in \mathbb{N}$, we have
\[
e^{-1/i} \cdot \omega_i(1)^n \leq \omega_i^0.
\]
On the other hand, by the fact that $\omega_i(1)$ and $\omega_i^0$ are in the same class,
\[
\int_M \omega_i(1)^n = \int_M \omega_i^0.
\]
Let $f_i = \frac{\omega_i(1)^n}{\omega_i^0}$. Since $v_{i,0} = \frac{\omega_i^0}{\omega_i^0}$, then (3.10) and (3.11) show
\[
e^{-1/i} \cdot f_i \leq v_{i,0}, \quad \int_M f_i \omega_\infty^n = \int_M v_{i,0} \omega_\infty^n.
\]
We compute
\[
\int_M |v_{i,0} - f_i| \omega_\infty^n \leq \int_M |v_{i,0} - e^{-1/i} \cdot f_i| \omega_\infty^n + (1 - e^{-1/i}) \int_M |f_i| \omega_\infty^n
\]
\[
= \int_M (v_{i,0} - e^{-1/i} \cdot f_i) \omega_\infty^n + (1 - e^{-1/i}) \int_M f_i \omega_\infty^n
\]
\[
= 2(1 - e^{-1/i}) \int_M f_i \omega_\infty^n.
\]
Since $\omega_i(1) \to \omega_\infty$ in the $C^\infty$ sense, then $f_i \to 1$ in the $C^\infty$ sense. By letting $i \to +\infty$, we obtain
\[
\lim_{i \to +\infty} \|v_{i,0} - 1\|_{L^1(\omega_\infty)} = 0.
\]
In particular, $v_{i,0}$ converges to 1 point-wise almost everywhere on $M$. By assumption (ii) and AM-GM inequality, we obtain
\[
\int_M v_{i,0}^{p/n} \omega_\infty^n \leq C(n, \omega_h, \omega_\infty) \cdot \int_M \left( \frac{\omega_i^0/\omega_h^n}{\omega_i^0/\omega_h^n} \right)^{p/n} \omega_h^n
\]
\[
\leq C(n, \omega_h, \omega_\infty) \cdot \int_M (\text{tr} \omega h \omega_i^0)^{p/n} \omega_h^n
\]
\[
\leq C(n, \omega_h, \omega_\infty) \cdot \Lambda^p.
\]
Then the $L^{q/n}$ convergence follows from the interpolation inequality.
If \( p = +\infty \) in (ii), then (b) and (c) of Lemma 2.1 imply that \( \omega_{i,0} \) are uniformly equivalent to the fixed metric \( \omega_h \), i.e.,

\[
C_1^{-1} \omega_h \leq \omega_{i,0} \leq C_1 \omega_h
\]

for some \( C_1 > 1 \) and for all \( i \in \mathbb{N} \). The standard argument of Kähler-Ricci flow (see e.g. [22, Corollary 3.3.5]) shows that for all \( i \in \mathbb{N} \) and \( (x, t) \in M \times [0, 2] \),

\[
C_2^{-1} \omega_h \leq \omega_i(t) \leq C_2 \omega_h
\]

for some \( C_2 > 1 \). It follows from [19, Theorem 1.1] that

\[
|\text{Rm}(\omega_i(t))| \leq L^{-1} t
\]

on \( M \times (0, 2] \) for some \( L > 0 \). By [20, Corollary 3.3], for all \( x, y \in M \) and \( t \in [0, 1] \),

\[
d_{\omega_i,0}(x, y) \leq d_{\omega_i(t)}(x, y) + C(n) \sqrt{Lt}.
\]

Recall that \( \omega_i(t) \to \omega_\infty \) in \( C^\infty_{\text{loc}}(M \times (0, 2]) \) as \( i \to +\infty \). In (3.18), by letting \( i \to +\infty \) and followed by letting \( t \to 0 \), we obtain the required distance estimate (1.3). □

As an application of the method, the following stability is also proved along the same line.

**Theorem 3.1.** Let \( (M, \omega_h) \) be a compact Kähler manifold with \( c_1(M) = 0 \). Suppose \( \omega_{i,0} = \alpha_i + \sqrt{-1} \partial \bar{\partial} u_i \) is a sequence of Kähler metrics such that for some \( \Lambda > 0 \) and \( p > n \), we have

(i) \( \Lambda^{-1} \omega_h \leq \alpha_i \leq \Lambda \omega_h \);

(ii) \( \| \text{tr} \omega_i,0 \|_{L^p(\omega_h)} \leq \Lambda \);

(iii) \( \mathcal{R}(\omega_i) \geq -i^{-1} \) for all \( i \in \mathbb{N} \),

then there is a Kähler Ricci flat metric \( \omega_\infty \) on \( M \) such that after passing to a subsequence, \( \omega_{i,0} \to \omega_\infty \) in the sense of current and

\[
\lim_{i \to +\infty} \| v_{i,0} - 1 \|_{L^{n/n}(\omega_\infty)} = 0
\]

for any \( q < p \), where \( v_{i,0} = \frac{\omega_{i,0}^n}{\omega_\infty^n} \) denotes the volume function. In particular, we have

\[
d\text{vol}_{g_{i,0}} \to d\text{vol}_{g_\infty} \quad \text{weakly on } M,
\]

where \( g_{i,0} \) and \( g_\infty \) denote the associated Riemannian metrics of \( \omega_{i,0} \) and \( \omega_\infty \) respectively. If in addition (ii) holds for \( p = +\infty \), then for any \( x, y \in M \),

\[
\lim_{i \to +\infty} \sup_{i} d_{\omega_{i,0}}(x, y) \leq d_{\omega_\infty}(x, y).
\]

**Sketch of Proof.** Since the proof is almost identical to that of Theorem 1.1, we only point out the necessary modifications. Since \( \alpha_i \) is uniformly equivalent to \( \omega_h \), the proof of Lemma 2.1 shows that we may assume \( u_i \) to be bounded uniformly and \( \alpha_i \) is Ricci flat by Yau’s Theorem [26] with uniformly bounded geometry. Then the proof of Lemma 2.2 and Lemma 2.3 can be carried over. To obtain the trace estimate, i.e. Lemma 2.4, we modify the
test function (2.29) with $\Lambda$ chosen to be large depending only on the uniform curvature bound of $\alpha_i$. With this modification, we have the same estimates as in Lemma 2.4. The proof of convergence in Theorem 1.1 can now be carried over directly. □

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