New exactly solvable systems with Fock symmetry

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Abstract

New superintegrable systems are presented which, like the hydrogen atom, possess a dynamical symmetry w.r.t. algebra o(4). One system simulates a neutral fermion with non-trivial dipole moment, interacting with the external e.m. field. This system is presented in both non-relativistic and relativistic formulations. Another recently discovered system (see Désilets et al 2012 arXiv:1208.2886v1) is non-relativistic and includes minimal and spin–orbit interaction with the external electric field. It is shown that all the systems considered are shape invariant. Applying this quality, these systems are integrated using the tools of SUSY quantum mechanics.

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1. Introduction

The hydrogen atom (HA) is one of the most important systems in quantum mechanics. This is a perfect physical model with a nice symmetry. Namely, in addition to its transparent invariance with respect to the rotation group, the HA possesses a hidden (Fock [1]) symmetry w.r.t. group O(4) whose generators are the orbital momentum and Runge–Lenz vector [2]. Moreover, this system is also supersymmetric and can be solved algebraically using the tools of SUSY quantum mechanics [3].

To the best of my knowledge, until the first version of this paper appeared in arXiv:1205.3094v1, there were only two known generalizations of the Runge–Lenz vector for 3D QM systems with spin. The first was discovered by Johnson and Lippman as far back as in 1950 [4] as a hidden symmetry of the relativistic Dirac equation with a Coulomb potential. A contemporary treatment of the Johnson–Lippman constant of motion in arbitrary dimensional space is presented in [5]. The other generalization was proposed in 1985 by D’Hoker and Vinet who proved that the Schrödinger–Pauli equation for electron interacting with the dyon field admits a vector integral of motion depending on spin [6]. However, the analogy of these symmetries with the Runge–Lenz vector is rather poor. Indeed, the Johnson–Lippman integral of motion is a scalar which is decoupled to three vector components only.
in the non-relativistic limit. The constant of motion found in [6] includes the same higher
derivative terms as the Runge–Lenz vector, but it does not generate the dynamical symmetry
w.r.t. group O(4).

In a very recent paper [9], superintegrable QM systems with spin–orbit interaction and
second-order integrals of motion are classified. One of these systems, which includes the
generalized Runge–Lenz vector, is discussed briefly in section 6 below.

Notice that the spin dependent Runge–Lenz vector was also discovered for the spinning
Taub-NUT space [7], but it was done on the pseudo classical level.

In this paper a new QM system with spin 1/2, which admits a hidden symmetry w.r.t.
group O(4), is discussed. Its integrals of motion are the total orbital momentum and the
generalized Runge–Lenz vector dependent on spin. Like the HA, this system also appears to
be supersymmetric, which makes it possible to find its exact solutions using the regular SUSY
approach.

Mathematically, the models presented in the following are interesting new examples of
superintegrable and supersymmetric systems with spin, which are exactly solvable. These
systems include shape invariant matrix potentials, which are particular cases of potentials
recently classified in [10, 11]. In addition, these examples corroborate the conjecture that all
maximally superintegrable systems are exactly solvable [12] (see [13–15] for discussion) and
present a new field for studying the relations between supersymmetry and superintegrability.

Physically, the models discussed in sections 3–5 simulate a neutral fermion with a non-
trivial dipole moment (e.g., the neutron), interacting with the external field. Moreover, a
relativistic version of this model is also presented. It is shown that the neutron can be trapped
by specific external fields. Potentially, these systems and their exact solutions can have a wide
spectrum of applications, from using them as tutorial examples and to understanding their
relevance to the problems of the security of nuclear reactors. In any case, superintegrable
systems like the HA and the isotropic harmonic oscillator are extremely important in physical
applications. Thus we can hope that, to some degree, this would be the case for the systems
discussed in this paper.

2. Symmetries of the HA

Let us recall the main symmetry properties of the HA which will be used as a standard for the
construction of the model with spin.

The Hamiltonian of the HA is as follows:

$$H = \frac{p^2}{2m} - \frac{q}{x}$$

(1)

where \(p^2 = p_1^2 + p_2^2 + p_3^2, \quad p_1 = -\frac{i}{\hbar} \frac{\partial}{\partial x_1}, \quad x = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad q > 0.

Operator (1) is manifestly invariant w.r.t. rotation group O(3) and so it commutes with the
angular momentum vector:

$$L = x \times p.$$  

(2)

In addition, it commutes with the Runge–Lenz vector

$$R = \frac{1}{2m} (p \times L - L \times p) + xV$$

(3)

where \(V = -\frac{q}{x^2}.

Components of vectors (2) and (3) satisfy the following commutation relations

$$[L_a, L_b] = i\epsilon_{abc} L_c, \quad [R_a, L_b] = i\epsilon_{abc} R_c, \quad [R_a, R_b] = -\frac{2i}{m} \epsilon_{abc} L_c H.$$  

(4)
On the set of eigenvectors of Hamiltonian $H$, corresponding to coupled states, algebra (4) is isomorphic to the Lie algebra of group $O(4)$.

Thus there are six integrals of motion for the Hamiltonian (1) (we do not take into account the Hamiltonian itself). However, we can select maximally four such integrals which are algebraically independent and include a pair of commuting representatives. This means that the HA is a maximally superintegrable system: the number of its algebraically independent constants of motion, including the Hamiltonian, is equal to $2n - 1$, i.e. is maximal for a system with $n = 3$ degrees of freedom.

The other (discrete) symmetry of the Hamiltonian (1) is the space reflection $P$. Indeed, this Hamiltonian is transparently invariant w.r.t. the change $x \rightarrow -x$.

In addition, operator (1) admits a hidden supersymmetry [3]. In other words, its radial component is shape invariant with respect to the special Darboux transformation which will be specified in section 4.

3. Runge–Lenz vector for fermions

By construction, operators (2) and (3) correspond to a spinless system. A straightforward way to generalize them for the case of a fermion system is to change the orbital momentum $L$ by the total angular momentum:

$$L \rightarrow J = L + S, \quad R \rightarrow \hat{R} = \frac{1}{2m} (p \times J - J \times p) + x\hat{V}$$

(5)

where $S = \frac{1}{2}\sigma$ is the spin vector and $\sigma$ is the matrix vector whose components are Pauli matrices. The potential $\hat{V}$ should commute with $J$ and is, in general, spin dependent.

Operators (5) should satisfy relations (4) where $La \rightarrow Ja$, $Ra \rightarrow \hat{Ra}$:

$$[Ja, J_b] = i\epsilon_{abc}J_c, \quad [\hat{Ra}, J_b] = i\epsilon_{abc}\hat{R}_c,$$

$$[\hat{Ra}, \hat{R}_b] = -\frac{2i}{m}\epsilon_{abc}J_cH.$$  

(6)

and commute with a Hamiltonian which we will search for in the form

$$H = \frac{p^2}{2m} + \hat{V}.$$  

(7)

It can be verified by direct calculation that, up to the constant multiplier $\alpha$, we have a unique choice for $\hat{V}$:

$$\hat{V} = \alpha \frac{\sigma \cdot x}{x^2}.$$  

(8)

Thus it is the Hamiltonian (7) with a ‘matrix Coulomb potential’ (8) which admits the integrals of motion (5). This Hamiltonian is not invariant w.r.t. space inversion $x \rightarrow -x$ since $\sigma$ is a pseudo-vector. However, the space inversion is an admissible transformation provided the wave function $\psi$ co-transforms in a non-standard way:

$$\psi(t, x) \rightarrow B\psi(t, -x)$$

(9)

where $B = \frac{\sigma L + 1}{|\sigma L + 1|}$ is the Biedenharn operator [16]. More exactly, the following condition is satisfied:

$$BH(p, x)B = H(-p, -x).$$

A natural question whether this Hamiltonian succeeds another symmetry of (1), i.e. the shape invariance, is discussed in the following section.
4. Supersymmetry and exact solutions

Consider the eigenvalue problem for the Hamiltonian (7):
\[
\left( \frac{p^2}{2m} + \alpha \sigma \cdot x \right) \psi = E \psi. \tag{10}
\]

Introducing the rescaled independent variables \( r = 2m \alpha x \) it is possible to rewrite (10) in a more compact form:
\[
\left( -\Delta + \frac{\sigma \cdot r}{r^2} \right) \psi = \varepsilon \psi \tag{11}
\]
where \( \varepsilon = \frac{E}{2m^2} \).

Taking into account the invariance of equation (10) w.r.t. the rotation group, it is convenient to rewrite it in the spherical coordinates and expand \( \psi \) via the spherical spinors \( \Omega_{j,j-\lambda,\kappa} (\psi, \theta) \):
\[
\psi = \frac{1}{r} \sum_{j,\lambda,\kappa} \psi_{j,\lambda,\kappa} (r) \Omega_{j,j-\lambda,\kappa} (\psi, \theta). \tag{12}
\]

Here \( j = \frac{1}{2}, \frac{3}{2}, \ldots, \kappa = -j, -j+1, \ldots, j \), and \( \lambda, \kappa = \pm \frac{1}{2} \) are quantum numbers labelling eigenvalues of the commuting operators \( J^2, L^2 \) and \( J_z \):
\[
J^2 \Omega_{j,j-\lambda,\kappa} = j(j+1) \Omega_{j,j-\lambda,\kappa},
L^2 \Omega_{j,j-\lambda,\kappa} = (j-\lambda)(j+\lambda) \Omega_{j,j-\lambda,\kappa},
J_z \Omega_{j,j-\lambda,\kappa} = k \Omega_{j,j-\lambda,\kappa}.
\]
The explicit form of the spherical spinors is given by the following formula [17]:
\[
\Omega_{j,j-\lambda,\kappa} = \begin{pmatrix} \sqrt{\frac{j+\kappa}{2j} Y_{j-\lambda+\frac{1}{2}, \kappa-\frac{1}{2}}} \\ \sqrt{\frac{j-\kappa}{2j} Y_{j-\lambda+\frac{1}{2}, \kappa+\frac{1}{2}}} \end{pmatrix}, \quad \Omega_{j,j+\lambda,\kappa} = \begin{pmatrix} -\sqrt{\frac{j+\kappa+1}{2j+2} Y_{j+\lambda+\frac{1}{2}, \kappa-\frac{1}{2}}} \\ \sqrt{\frac{j+\kappa+1}{2j+2} Y_{j+\lambda+\frac{1}{2}, \kappa+\frac{1}{2}}} \end{pmatrix}
\]
where \( Y_{j+\frac{1}{2}, \kappa \pm \frac{1}{2}} \) are spherical functions.

Substituting (12) into (11) we come to the following equations
\[
\mathcal{H}_j \Phi_{j,\kappa} = \left( -\frac{\partial^2}{\partial r^2} + V_j \right) \Phi_{j,\kappa} = \varepsilon \Phi_{j,\kappa} \tag{13}
\]
where
\[
\Phi_{j,\kappa} = \begin{pmatrix} \psi_{j-\lambda,\kappa} (r) \\ \psi_{j+\lambda,\kappa} (r) \end{pmatrix}
\]
and \( V_j \) is the matrix potential:
\[
V_j = \left( j(j+1) + \frac{1}{4} - \sigma_3 \left( j + \frac{1}{2} \right) \right) \frac{1}{r^2} - \sigma_1 \frac{1}{r}. \tag{15}
\]
Equation (13) appears to be supersymmetric since \( V_j \) belongs to the list of shape invariant potentials classified in [10], see equation (5.11) for \( \kappa = \frac{1}{2}, \mu = j \) there. Indeed, the Hamiltonian \( \mathcal{H}_j \) can be factorized as
\[
\mathcal{H}_j = a_j^+ a_j + c_j \tag{16}
\]
where
\[
a_j = \frac{\partial}{\partial r} + W_j, \quad a_j^+ = -\frac{\partial}{\partial r} + W_j, \quad c_j = -\frac{1}{4(j+1)^2}
\]
and \( W_j \) is the matrix superpotential
\[
W_j = \left( \frac{1}{2} \sigma_3 - j - 1 \right) \frac{1}{r} + \frac{1}{2(j+1)} \sigma_1. \tag{18}
\]
Moreover, the Hamiltonians $\mathcal{H}_j$ and $\mathcal{H}_{j+1}$ satisfy the following intertwining relations

$$\mathcal{H}_j a_j^+ = a_j^+ \mathcal{H}_{j+1}. \quad (19)$$

Thus equation (13) can be easily solved using the tools of SUSY quantum mechanics. In fact it has been already done in [10]. The ground state $\Phi_{0j}^0$ should solve the first order equation $a_j \Phi_{0j}^0 = 0$. Thus, using definitions (17), we obtain:

$$\Phi_{0j}^0 = c_k \left( \frac{r^{j+\frac{1}{2}}K_1 \left( \frac{r}{2(j+1)} \right)}{r^{j+\frac{1}{2}}K_0 \left( \frac{r}{2(j+1)} \right)} \right) \quad (20)$$

where $K_1$ and $K_0$ are the modified Bessel functions and $c_k$ are arbitrary constants. The corresponding eigenvalue $\epsilon_0$ is equal to $c_j$, i.e.

$$\epsilon_0 = -\frac{1}{4(j+1)^2}. \quad (21)$$

The solution $\Phi_{nk}^0$ confirming to the $n$th excited state and the corresponding eigenvalue $\epsilon_n$ are:

$$\Phi_{nk}^n = a_j^+ a_{j+1}^+ \cdots a_{j+n-1}^+ \Phi_{0j}^0, \quad \epsilon_n = -\frac{1}{4(j+n+1)^2}. \quad (22)$$

The related energy value in (10) is given by the following equation:

$$E = -\frac{m \alpha^2}{2N^2}. \quad (23)$$

The energy levels (22) depend on the main quantum number $N$ (23) which can take the same value for different pairs of $n$ and $j$. Namely, for a fixed $N$ there are $N - \frac{1}{2}$ of such pairs. In addition, the quantum number $k$ which labels the eigenvectors in (13) is not present in (22) and (23). Thus, as in the HA problem, the energy levels are highly degenerated, and this degeneration is caused by the hidden symmetry w.r.t. group O(4).

To end this section we recall that the radial component of the HA Hamiltonian (1) is also shape invariant and can be factorized like (16) where $j \to \ell$, $W \to \frac{a}{2\ell+1} + \frac{\ell+1}{2}$ and $\ell$ is the quantum number labelling eigenvalues of angular momentum $L$. In other words, our model succeeds in both the hidden symmetry and supersymmetry of the HA.

5. Relativistic system

Let us show that the eigenvalue problem (11) admits a relativistic formulation. To demonstrate that we start with the Dirac equation for a neutral particle having non-trivial dipole moments:

$$\left( \gamma^\mu p_\mu - m - \alpha \sigma^{\mu\nu} F_{\mu\nu} + \tilde{\alpha} \gamma^5 \gamma^\mu F_\mu \right) \Psi = 0. \quad (24)$$

In addition to the standard Pauli term $\alpha \sigma^{\mu\nu} F_{\mu\nu}$ equation (24) includes an additional term $\tilde{\alpha} \gamma^5 \gamma^\mu F_\mu$ with an external pseudo-vector field $F_\mu$. For convenience, the following realization of Dirac matrices will be used:

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \gamma_a = \begin{pmatrix} i\sigma_a & 0 \\ 0 & -i\sigma_a \end{pmatrix} \quad (25)$$

where $I$ and $0$ are the $2 \times 2$ unit and zero matrices correspondingly, $a = 1, 2, 3$. Choosing in (24)

$$F_{ab} = F_0 = 0, \quad F_a = \frac{\alpha}{\tilde{\alpha}} F_{0a} = \frac{\chi_a}{\chi^2} \quad (26)$$
and representing $\Psi$ as:

$$\Psi = \exp(-iE_0) \begin{pmatrix} \psi(x) \\ \xi(x) \end{pmatrix}$$  \hspace{1cm} (27)

with the two-component functions $\psi$ and $\xi$, we come to the following system:

$$ (i\sigma \cdot \mathbf{p} - m)\psi + E\xi = 0, \hspace{1cm} (28) $$

$$ \left( E - \frac{2\alpha}{x^2} \right) \psi - (m + i\sigma \cdot \mathbf{p})\xi = 0. \hspace{1cm} (29) $$

Solving (28) for $\xi$ and substituting this solution into (29), we come to the eigenvalue problem (11) where $r = 2\alpha E x$ and $\varepsilon = \frac{E^2 - m^2}{4\alpha^2}$. Thus, in accordance with (21), the relativistic energies are also discrete and can be given by the following formula:

$$ E^2 = \frac{m^2}{1 + \frac{x^2}{3\alpha}} $$  \hspace{1cm} (30)

with $N$ being a natural number which can be represented in the form of (23). The corresponding (non-normalized) wave functions are given by equation (27) where $\psi(x)$ is defined in (12), (14), (20), (21) and $\xi(x)$ can be found from (28).

Notice that for the small coupling constants $\alpha$, the positive energy values (30) are reduced to the non-relativistic form (22) up to the constant internal energy term $m$ and the terms of order $\alpha^4$.

6. Superintegrable and supersymmetric system with spin–orbit interaction

In a very recent paper [9], one more QM model with a hidden symmetry w.r.t. group O(4) is presented. The Hamiltonian of this model is as follows:

$$ H = \frac{\mathbf{p}^2}{2m} + \hat{V} - \frac{1}{8\alpha^2} $$  \hspace{1cm} (31)

where

$$ \hat{V} = \frac{1}{2mx^2} \left( \sigma \cdot \mathbf{L} + 1 \right) - \frac{\alpha}{x}. $$  \hspace{1cm} (32)

In contrast with (7), the Hamiltonian (31) does not include the dipole interaction term but involves a scalar potential together with the spin–orbit interaction. Nevertheless, it also commutes with the total orbital momentum and generalized Runge–Lenz vector (5) where $\hat{V}$ is given by equation (32) (in paper [9] another representation for $\hat{R}$ is used which is equivalent to (5), (32)). This result was predictable, since the Hamiltonian (31) can be reduced to the direct sum of the two Hamiltonians of the HA via the gauge transformation [8].

It can be proven by direct verification, that the operators (5), (32) and (31) satisfy the commutation relations (6) and so they generate a hidden symmetry with respect to group O(4). Thus the QM system with the Hamiltonian (31) admits a generalized Fock symmetry.

The Hamiltonian (31) succeeds in a further symmetry of the systems considered in the above, i.e. the shape invariance. Indeed, starting with (31) and repeating all the steps following equation (11), one comes to the radial equation (13) where

$$ V_j = j(j + 1) - \frac{1}{r}. $$  \hspace{1cm} (33)
Here and in the next equation each term in the r.h.s. is multiplied by a $2 \times 2$ identity matrix.

Like (15), potential (33) is shape invariant. The related superpotential is given by the following formula:

$$W = \frac{1}{2(j+1)} + \frac{j+1}{r}$$

while the eigenvalues of the Hamiltonian (31) are given by equation (22) where $n$ and $j$ are positive integers and half integers, correspondingly. The corresponding wave function is easily calculated in a complete analogy with section 4. It is given by equation (12) where

$$\psi_{j,\lambda,k} \rightarrow \psi^{n}_{j,\lambda,k}$$

and

$$\psi^{n}_{j,\lambda,k} = c_{\lambda,k} y^{j+1} \exp \left( -\frac{y^2}{2} \right) L_{n}^{2j+1}(y).$$

Here $y = \frac{m_{\sigma} \cdot r}{\pi^{1/2} \alpha}$, and $L_{n}^{2j+1}(y)$ are Laguerre polynomials and $c_{\lambda,k}$ are constants, satisfying the normalizing condition $\sum_{\lambda,k} c_{\lambda,k} c^{*}_{\lambda,k} = 1$.

The effective potential (33) and superpotential (34) are similar to those of the HA. However, there are some differences, namely:

- Potential (33) includes the quantum number $j$ which takes half-integer values while the potential of the HA depends on the orbital quantum number $l$ which is integer.
- In contrast with the HA, potential (33) is a matrix. More exactly, it is a direct sum of two scalar potentials. That leads to the additional two-fold degeneration of the energy spectrum of the Hamiltonian (31). This degeneration is caused by the additional integral of motion

$$C = \frac{1}{\alpha} \mathbf{J} \cdot \hat{\mathbf{R}} = \frac{\sigma \cdot \mathbf{x}}{2\alpha}$$

whose eigenvalues are not included in the spectrum formula (22).
- Matrix (36) is proportional to the Casimir operator $\hat{C} = \mathbf{J} \cdot \hat{\mathbf{R}}$ of group O(4), and its eigenvalues are equal to $\pm \frac{1}{2}$. For the HA, this Casimir operator is trivial.
- The additional symmetry operator (36) extends the number of algebraically independent constants of motion (including the Hamiltonian) to 6. This number is maximal for a 3D system with the additional (spin) degree of freedom, thus the discussed system is maximally superintegrable.

The matrix (36) and many other integrals of motion for the Hamiltonian (31) were represented in [8]. These integrals of motion are algebraic functions of the basic symmetry operators (5) where $\hat{V}$ is the matrix given in (32).

7. Discussion

The Hamiltonian model (7) represents a new, exactly solvable QM system which admits extended symmetries. Like the HA, this system admits the hidden (Fock) symmetry w.r.t. group O(4) whose generators are the total angular momentum and Runge–Lenz vector. In addition, (and again like the HA) this system is supersymmetric and can be easily solved using the tools of SUSY QM, see section 4. Moreover, this is a fermionic system with spin $\frac{1}{2}$, while the non-relativistic model of the HA ignores the spin of the electron.

A new feature of the supersymmetry of Hamiltonian (7) in comparison with the HA, is that it is realized in terms of matrix superpotentials. Such (one dimensional) superpotentials have been classified in papers [10, 11], and it was very inspiring for us to search for multi-dimensional matrix systems which can be solved using their matrix supersymmetry in separated variables. As a result, the Hamiltonian (7) has been discovered. The other results of such searches are presented in papers [18, 19].
The operator (7) can be interpreted as the Hamiltonian of a neutral particle with spin 1/2 which has a non-trivial dipole moment. The matrix potential $\alpha \sigma \cdot x = \alpha \sigma \cdot E$ represents a Pauli type interaction with the external vector field $E \sim x^2$.

The latter can be interpreted as the electric field. Moreover, such a field can be realized experimentally at least on the finite interval $a < x < b$, $a > 0$, see, e.g., problem 1018 in [20]. Notice that functions (37) also solve nonlinear Maxwell equations including the additional vector field, [21]. They also solve the equations of axion electrodynamics [22]. The same is true for fields (26) involved in the relativistic equation (24).

Thus the QM systems whose Hamiltonians are given by equations (7) and (31), posses all the symmetries and supersymmetries admitted by the HA. However, these symmetries are generalized by introducing the spin. In addition, there exists a relativistic counterpart of system (10) given by equations (24) and (26) which is also exactly solvable.

The presented non-relativistic system (10) can be treated as a 3D generalization of the solvable planar system proposed by Pron’ko and Stroganov [23]. It would be interesting to study its possible generalizations in multidimensional spaces. One more interesting task is to construct solvable systems with Fock symmetry for particles with arbitrary spin. This work is in progress.

It is well-known that a tight coupling exists between the hidden symmetry of the HA and its supersymmetry [24, 25]. The same is true for the Pron’ko–Stroganov problem [26]. A contemporary discussion of the relations between supersymmetry and superintegrability can be found in [27].

All systems discussed in this paper are both supersymmetric and maximally superintegrable. Surely, these combinations of symmetries are not accidental, and it would be interesting to extend the results [25, 26] to the 3D matrix systems presented here.

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