Hypergraph Turán densities can have arbitrarily large algebraic degree

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Abstract

Grosu [Journal of Combinatorial Theory (B), 118 (2016) 137–185] asked if there exist an integer $r \geq 3$ and a finite family of $r$-graphs whose Turán density, as a real number, has (algebraic) degree greater than $r - 1$. In this note we show that, for all integers $r \geq 3$ and $d$, there exists a finite family of $r$-graphs whose Turán density has degree at least $d$, thus answering Grosu’s question in a strong form.

1 Introduction

For an integer $r \geq 2$, an $r$-uniform hypergraph (henceforth, an $r$-graph) $H$ is a collection of $r$-subsets of some finite set $V$. Given a family $F$ of $r$-graphs, we say $H$ is $F$-free if it does not contain any member of $F$ as a subgraph. The Turán number $ex(n, F)$ of $F$ is the maximum number of edges in an $F$-free $r$-graph on $n$ vertices. The Turán density $\pi(F)$ of $F$ is defined as $\pi(F) := \lim_{n \to \infty} ex(n, F)/\binom{n}{r}$; the existence of the limit was established in [12]. The study of $ex(n, F)$ is one of the central topics in extremal graph and hypergraph theory. For the hypergraph Turán problem (i.e. the case $r \geq 3$), we refer the reader to the surveys by Keevash [13] and Sidorenko [18].

For $r \geq 3$, determining the value of $\pi(F)$ for a given $r$-graph family $F$ is very difficult in general, and there are only a few known results. For example, the problem of determining $\pi(K^r_\ell)$ raised by Turán [19] in 1941, where $K^r_\ell$ is the complete $r$-graph on $\ell$ vertices, is wide open and the $500$ prize of Erdős for solving it for at least one pair $\ell > r \geq 3$ is still unclaimed.

For every integer $r \geq 2$, define

$$\Pi^{(r)}_{\text{fin}} := \{ \pi(F) : F \text{ is a finite family of } r\text{-graphs} \}, \quad \text{and}$$

$$\Pi^{(r)}_{\text{inf}} := \{ \pi(F) : F \text{ is a (possibly infinite) family of } r\text{-graphs} \}.$$
For \( r = 2 \) the celebrated Erdős–Stone–Simonovits theorem \([6, 7]\) determines the Turán density for every family \( \mathcal{F} \) of graphs; in particular, it holds that

\[
\Pi_{\infty}^{(2)} = \Pi_{\text{fin}}^{(r)} = \{1\} \cup \{1 - 1/k : \text{integer } k \geq 1\}.
\]

The problem of understanding the sets \( \Pi_{\text{fin}}^{(r)} \) and \( \Pi_{\infty}^{(r)} \) of possible \( r \)-graph Turán densities for \( r \geq 3 \) has attracted a lot of attention. One of the earliest results here is the theorem of Erdős \([5]\) from the 1960s that \( \Pi_{\infty}^{(r)} \cap (0, r!/r^r) = \emptyset \) for every integer \( r \geq 3 \). However, our understanding of the locations and the lengths of other maximal intervals avoiding \( r \)-graph Turán densities and the right accumulation points of \( \Pi_{\infty}^{(r)} \) (the so-called \textit{jump problem}) is very limited; for some results in this direction see e.g. \([1, 8, 9, 17, 21]\).

It is known that the set \( \Pi_{\infty}^{(r)} \) is the topological closure of \( \Pi_{\text{fin}}^{(r)} \) (and thus the former set is easier to understand) and that \( \Pi_{\infty}^{(r)} \) has cardinality of continuum (and thus is strictly larger than the countable set \( \Pi_{\text{fin}}^{(r)} \)), see respectively Proposition 1 and Theorem 2 in \([16]\).

For a while it was open whether \( \Pi_{\text{fin}}^{(r)} \) can contain an irrational number (see the conjecture of Chung and Graham in \([3, \text{Page 95}]\)), until such examples were independently found by Baber and Talbot \([2]\) and by the second author \([16]\). However, the question of Jacob Fox (\([16, \text{Question 27}]\)) whether \( \Pi_{\text{fin}}^{(r)} \) can contain a transcendental number remains open.

Grosu \([11]\) initiated a systematic study of algebraic properties of the sets \( \Pi_{\text{fin}}^{(r)} \) and \( \Pi_{\infty}^{(r)} \). He proved a number of general results that, in particular, directly give further examples of irrational Turán densities.

Recall that the \textit{(algebraic) degree} of a real number \( \alpha \) is the minimum degree of a non-zero polynomial \( p \) with integer coefficients that vanishes on \( \alpha \); it is defined to be \( \infty \) if no such \( p \) exists (that is, if the real \( \alpha \) is transcendental). In the same paper, Grosu \([11, \text{Problem 3}]\) posed the following question.

\textbf{Problem 1.1} (Grosu). \textit{Does there exist an integer \( r \geq 3 \) such that \( \Pi_{\text{fin}}^{(r)} \) contains an algebraic number \( \alpha \) of degree strictly larger than \( r - 1 \)?}

Apparently, all \( r \)-graph Turán densities that Grosu knew or could produce with his machinery had degree at most \( r - 1 \), explaining this expression in his question. His motivation for asking this question was that if, on input \( \mathcal{F} \), we can compute an upper bound on the degree of \( \pi(\mathcal{F}) \) as well as on the absolute values of the coefficients of its minimal polynomial, then we can compute \( \pi(\mathcal{F}) \) exactly, see the discussion in \([11, \text{Page 140}]\).

In this short note we answer Grosu’s question in the following stronger form.

\textbf{Theorem 1.2.} \textit{For every integer \( r \geq 3 \) and for every integer \( d \) there exists an algebraic number in \( \Pi_{\text{fin}}^{(r)} \) whose minimal polynomial has degree at least \( d \).}

Our proof for Theorem 1.2 is constructive; in particular, for \( r = 3 \) we will show that the following infinite sequence is contained in \( \Pi_{\text{fin}}^{(3)} \):

\[
\frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3 - \frac{2}{\sqrt{3}}}}, \quad \frac{1}{\sqrt{3 - \frac{2}{\sqrt{3 - \frac{2}{\sqrt{3}}}}}}, \quad \frac{1}{\sqrt{3 - \frac{2}{\sqrt{3 - \frac{2}{\sqrt{3 - \frac{2}{\sqrt{3}}}}}}}}, \quad \ldots
\]

(1)
2 Preliminaries

In this section, we introduce some preliminary definitions and results that will be used later.

For an integer $r \geq 2$, an ($r$-uniform) pattern is a pair $P = (m, \mathcal{E})$, where $m$ is a positive integer, $\mathcal{E}$ is a collection of $r$-multisets on $[m] := \{1, \ldots, m\}$, where by an $r$-multiset we mean an unordered collection of $r$ elements with repetitions allowed. Let $V_1, \ldots, V_m$ be disjoint sets and let $V = V_1 \cup \cdots \cup V_m$. The profile of an $r$-set $R \subseteq V$ (with respect to $V_1, \ldots, V_m$) is the $r$-multiset on $[m]$ that contains element $i$ with multiplicity $|R \cap V_i|$ for every $i \in [m]$. For an $r$-multiset $S \subseteq [m]$, let $S((V_1, \ldots, V_m))$ consist of all $r$-subsets of $V$ whose profile is $S$. We call this $r$-graph the blowup of $S$ and the $r$-graph

$$\mathcal{E}((V_1, \ldots, V_m)) := \bigcup_{S \in \mathcal{E}} S((V_1, \ldots, V_m))$$

is called the blowup of $\mathcal{E}$ (with respect to $V_1, \ldots, V_m$). We say that an $r$-graph $H$ is a $P$-construction if it is a blowup of $\mathcal{E}$. Note that these are special cases of the more general definitions from [10].

It is easy to see that the notion of a pattern is a generalization of a hypergraph, since every $r$-graph is a pattern in which $\mathcal{E}$ is a collection of (ordinary) $r$-sets. For most families $\mathcal{F}$ whose Turán problem was resolved, the extremal $\mathcal{F}$-free constructions are blowups of some simple pattern. For example, let $P_B := (2, \{\{1, 2, 2\}, \{1, 1, 2\}\})$, where we use $\{\}$ to distinguish multisets from ordinary sets. Then a $P_B$-construction is a 3-graph $H$ whose vertex set can be partitioned into two parts $V_1$ and $V_2$ such that $H$ consists of all triples that have nonempty intersections with both $V_1$ and $V_2$. A famous result in the hypergraph Turán theory is that the pattern $P_B$ characterizes the structure of all maximum 3-graphs of sufficiently large order that do not contain a Fano plane (see [4, 10, 14]).

For a pattern $P = (m, \mathcal{E})$, let the Lagrange polynomial of $\mathcal{E}$ be

$$\lambda_\mathcal{E}(x_1, \ldots, x_m) := r! \sum_{E \in \mathcal{E}} \prod_{i=1}^m \frac{x_i^{E(i)}}{E(i)!},$$

where $E(i)$ is the multiplicity of $i$ in the $r$-multiset $E$. In other words, $\lambda_\mathcal{E}$ gives the asymptotic edge density of a large blowup of $\mathcal{E}$, given its relative part sizes $x_i$.

The Lagrangian of $P$ is defined as follows:

$$\lambda(P) := \sup \{ \lambda_\mathcal{E}(x_1, \ldots, x_m) : (x_1, \ldots, x_m) \in \Delta_{m-1} \},$$

where $\Delta_{m-1} := \{(x_1, \ldots, x_m) \in [0,1]^m : x_1 + \ldots + x_m = 1\}$ is the standard $(m-1)$-dimensional simplex in $\mathbb{R}^m$. Since we maximise a polynomial (a continuous function) on a compact space, the supremum is in fact the maximum and we call the vectors in $\Delta_{m-1}$ attaining it $P$-optimal. Note that the Lagrangian of a pattern is a generalization of the well-known hypergraph Lagrangian that has been successfully applied to Turán-type problems (see e.g. [11, 9, 20]), with the basic idea going back to Motzkin and Straus [15].

For $i \in [m]$ let $P - i$ be the pattern obtained from $P$ by removing index $i$, that is, we remove $i$ from $[m]$ and delete all multisets containing $i$ from $E$ (and relabel the remaining indices to form the set $[m-1]$). We call $P$ minimal if $\lambda(P-i)$ is strictly smaller than $\lambda(P)$ for every $i \in [m]$, or equivalently if no $P$-optimal vector has a zero entry. For example, the 2-graph pattern $P := (3, \{\{1, 2\}, \{1, 3\}\})$ is not minimal as $\lambda(P) = \lambda(P-3) = 1/2$. 

3
Proof. Assume that $m \Delta$.

In particular, the real numbers $x$.

To prove the other direction of this inequality, observe that if we take ($x$s), then $E$.

Let $x$.

This and the AM-GM inequality give that $x$.

In [16], the second author studied the relations between possible hypergraph Turán densities and patterns. One of the main results from [16] is as follows.

**Theorem 2.1** ([16]). For every minimal pattern $P$ there exists a finite family $F$ of $r$-graphs such that $\pi(F) = \lambda(P)$, and moreover, every maximum $F$-free $r$-graph is a $P$-construction.

Let $r \geq 3$ and $s \geq 1$ be two integers. Given an $r$-uniform pattern $P = (m, E)$, one can create an $(r + s)$-uniform pattern $P + s := (m + s, \hat{E})$ in the following way: for every $E \in E$ we insert the $s$-set $\{m + 1, \ldots, m + s\}$ into $E$, and let $\hat{E}$ denote the resulting family of $(r + s)$-multisets. For example, if $P = (3, \{\{1, 2, 3\}, \{1, 3, 3\}, \{2, 3, 3\}\})$, then $P + 1 = (4, \{\{1, 2, 3, 4\}, \{1, 3, 3, 4\}, \{2, 3, 3, 4\}\})$.

The following observation follows easily from the definitions.

**Observation 2.2.** If $P$ is a minimal pattern, then $P + s$ is a minimal pattern for every integer $s \geq 1$.

For the Lagrangian of $P + s$ we have the following result.

**Proposition 2.3.** Suppose that $r \geq 2$ is an integer and $P$ is an $r$-uniform pattern. Then for every integer $s \geq 1$ we have

$$
\lambda(P + s) = \frac{r^r(s + r)!}{(r + s)^{r + s} r!} \lambda(P).
$$

In particular, the real numbers $\lambda(P + s)$ and $\lambda(P)$ have the same degree.

**Proof.** Assume that $P = (m, E)$. Let $\hat{P} := P + s = (m + s, \hat{E})$. Let $(x_1, \ldots, x_{m + s}) \in \Delta_{m + s - 1}$ be a $P$-optimal vector. Note from the definition of Lagrange polynomial that

$$
\lambda(\hat{P}) = \lambda_\hat{E}(x_1, \ldots, x_{m + s}) = \frac{(r + s)!}{r!} \lambda_\hat{E}(x_1, \ldots, x_m) \prod_{i = m + 1}^{m + s} x_i.
$$

Let $x := \frac{1}{r} \sum_{i = m + 1}^{m + s} x_i$ and note that $\sum_{i = 1}^{m} x_i = 1 - sx$. Since $\lambda_\hat{E}$ is a homogenous polynomial of degree $r$, we have

$$
\lambda_\hat{E}(x_1, \ldots, x_m) = \lambda_\hat{E} \left( \frac{x_1}{1 - sx}, \ldots, \frac{x_m}{1 - sx} \right) (1 - sx)^r \leq \lambda(P)(1 - sx)^r.
$$

This and the AM-GM inequality give that

$$
\lambda(\hat{P}) = \frac{(r + s)!}{r!} \lambda_\hat{E}(x_1, \ldots, x_m) \prod_{i = m + 1}^{m + s} x_i \leq \frac{(r + s)!}{r!} \lambda(P)(1 - sx)^r x^s.
$$

For $x \in [0, 1/s]$, the function $(1 - sx)^r (rx)^s$, as the product of $s + r$ non-negative terms summing to $r$, is maximized when all terms are equal, that is, at $x = \frac{1}{r + s}$. So

$$
\lambda(\hat{P}) \leq \frac{(r + s)!}{r!} \lambda(P)(1 - sx)^r x^s \leq \frac{r^r(s + r)!}{(r + s)^{r + s} r!} \lambda(P).
$$

To prove the other direction of this inequality, observe that if we take $(x_1, \ldots, x_m) = \frac{r}{r + s} (y_1, \ldots, y_m)$, where $(y_1, \ldots, y_m) \in \Delta_{m - 1}$ is $P$-optimal, and take $x_{m + 1} = \cdots = x_{m + s} = \frac{1}{r + s}$, then all inequalities above hold with equalities. \qed
3 Proof of Theorem 1.2

In this section we prove Theorem 1.2. By Theorem 2.1, it suffices to find a sequence of \(r\)-uniform minimal patterns \((P_k)_{k=1}^{\infty}\) such that the degree of the real number \(\lambda(P_k)\) goes to infinity as \(k\) goes to infinity. Furthermore, by Observation 2.2 and Proposition 2.3 it suffices to find such a sequence for \(r = 3\). So we will assume that \(r = 3\) in the rest of this note.

To start with, we let \(P_1 := (3, \{\{1,2,3\}, \{1,3,3\}, \{2,3,3\}\})\). Recall that a 3-graph \(H\) is a \(P_1\)-construction (see Figure 1) if there exists a partition \(V(H) = V_1 \cup V_2 \cup V_3\) such that the edge set of \(H\) consists of

(a) all triples that have one vertex in each \(V_i\),

(b) all triples that have one vertex in \(V_1\) and two vertices in \(V_3\), and

(c) all triples that have one vertex in \(V_2\) and two vertices in \(V_3\).

The pattern \(P_1\) was studied by Yan and Peng in [20], where they proved that there exists a single 3-graph whose Turán density is given by \(P_1\)-constructions which, by \(\lambda(P_1) = \frac{1}{\sqrt{3}}\), is an irrational number. It seems that some other patterns could be used to prove Theorem 1.2; however, the obtained sequence of Turán densities (i.e. the sequence in (1)) produced by using \(P_1\) is nicer than those produced by the other patterns that we tried.

Next, we define the pattern \(P_{k+1} = (2k + 3, \mathcal{E}_{k+1})\) for every \(k \geq 1\) inductively. It is easier to define what a \(P_{k+1}\)-construction is rather than to write down the definition of \(P_{k+1}\): for every integer \(k \geq 1\) a 3-graph \(H\) is a \(P_{k+1}\)-construction if there exists a partition \(V(H) = V_1 \cup V_2 \cup V_3\) such that

(a) the induced subgraph \(H[V_3]\) is a \(P_k\)-construction, and

(b) \(H \setminus H[V_3]\) consists of all triples whose profile is in \(\{\{1,2,3\}, \{1,3,3\}, \{2,3,3\}\}\).

The pattern \(P_k\) can be written down explicitly, although this is not necessary for our proof.
later. For example, \( P_2 = (5, E_2) \) (see Figure 1), where
\[
E_2 = \{ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{2, 5, 4\}, \{2, 5, 5\}, \{3, 4, 5\}, \{3, 5, 4\}, \{4, 5, 3\} \}.
\]

Our first result determines the Lagrangian of \( P \) for every integer \( k \geq 1 \). For convenience, we set \( \lambda_0 := 0 \).

**Proposition 3.1.** For every integer \( k \geq 0 \), we have \( \lambda(P_{k+1}) = 1/\sqrt{3 - 2\lambda(P_k)} \) and the pattern \( P_{k+1} \) is minimal. In particular, \( (\lambda(P_k))_k^{\infty} \) is the sequence in (1).

**Proof.** We use induction on \( k \) where the base \( k = 0 \) is easy to check directly (or can be derived by adapting the forthcoming induction step to work for \( k = 0 \)). Let \( k \geq 1 \).

Let us prove that \( \lambda(P_{k+1}) = 1/\sqrt{3 - 2\lambda(P_k)} \). Recall that \( P_k = (2k + 1, E_k) \) and \( P_{k+1} = (2k+3, E_{k+1}) \). Let \( (x_1, \ldots, x_{2k+3}) \in \Delta_{2k+2} \) be a \( P_{k+1} \)-optimal vector. Let \( x := \sum_{i=3}^{2k+3} x_i = 1 - x_1 - x_2 \). It follows from the definitions of \( P_{k+1} \) and the Lagrange polynomial that
\[
\lambda(P_{k+1}) = \lambda_{E_{k+1}}(x_1, \ldots, x_{2k+3}) = 6 \left( x_1 x_2 + (x_1 + x_2) \frac{x^2}{2} \right) + \lambda_{E_k}(x_3, \ldots, x_{2k+3}). \quad (2)
\]

Since \( \lambda_{E_k}(x_3, \ldots, x_{2k+3}) \) is a homogeneous polynomial of degree 3, we have
\[
\lambda_{E_k}(x_3, \ldots, x_{2k+3}) = \lambda_{E_k}(x_3^2 \frac{x}{x}, \ldots, x_{2k+3} \frac{x^2}{x}) x^3 \leq \lambda(P_k)x^3.
\]

So it follows from (2) and the 2-variable AM-GM inequality that
\[
\lambda(P_{k+1}) \leq 6 \left( \frac{x_1 + x_2}{2} \right)^2 x + (x_1 + x_2) \frac{x^2}{2} + \lambda(P_k)x^3
\]
\[
= 6 \left( \frac{1 - x}{2} \right)^2 x + (1 - x) \frac{x^2}{2} + \lambda(P_k)x^3 = \frac{3x - (3 - 2\lambda(P_k)) x^3}{2}.
\]

Since \( 0 \leq \lambda(P_k) \leq 1 \), one can easily show by taking the derivative that the maximum of the function \( (3x - (3 - 2\lambda(P_k)) x^3)/2 \) on \([0,1]\) is achieved if and only if \( x = 1/\sqrt{3 - 2\lambda(P_k)} \), and the maximum value is \( 1/\sqrt{3 - 2\lambda(P_k)} \). This proves that \( \lambda(P_{k+1}) \leq 1/\sqrt{3 - 2\lambda(P_k)} \).

To prove the other direction of this inequality, one just need to observe that when we choose
\[
x_1 = x_2 = \frac{1}{2} - \frac{1}{2\sqrt{3 - 2\lambda(P_k)}} \quad \text{and} \quad (x_3, \ldots, x_{2k+3}) = \frac{1}{\sqrt{3 - 2\lambda(P_k)}} (y_1, \ldots, y_{2k+1})
\]
where \((y_1, \ldots, y_{2k+1}) \in \Delta_{2k}\) is a \( P_k \)-optimal vector, then all inequalities above hold with equality. Therefore, \( \lambda(P_{k+1}) = 1/\sqrt{3 - 2\lambda(P_k)} \).

To prove that \( P_{k+1} \) is minimal, take any \( P_{k+1} \)-optimal vector \((x_1, \ldots, x_{2k+3}) \in \Delta_{2k+2}\); we have to show that it has no zero entries. This vector attains equality in all our inequalities above, which routinely implies that \((x_1, \ldots, x_{2k+3})\) must satisfy (3), for some \( P_k \)-optimal vector \((y_1, \ldots, y_{2k+1})\). We see that \( x_1 = x_2 \) are both non-zero because the sequence \((\lambda(P_0), \ldots, \lambda(P_{k+1}))\) is strictly increasing (since \( x < 1/\sqrt{3 - 2x} \) for all \( x \in [0,1] \)) and thus \( \lambda(P_k) < 1 \). The remaining conclusion that \( x_3, \ldots, x_{2k+3} \) are non-zero follows from the induction hypothesis on \((y_1, \ldots, y_{2k+1})\). \( \square \)
In order to finish the proof of Theorem 1.2, it suffices to prove that the degree of \( \mu_k := \lambda(P_k) \) goes to infinity as \( k \to \infty \). This is achieved by the last claim of the following lemma.

**Lemma 3.2.** Let \( p_1(x) := 3x^2 - 1 \) and inductively for \( k = 1, 2, \ldots \) define

\[
p_{k+1}(x) = (2x^2)^{2^k} p_k \left( \frac{3x^2 - 1}{2x^2} \right), \quad \text{for } x \in \mathbb{R}.
\]

Then the following claims hold for each \( k \in \mathbb{N} \):

(a) \( p_k(\mu_k) = 0 \);

(b) \( p_k \) is a polynomial of degree at most \( 2^k \) with integer coefficients: \( p_k(x) = \sum_{i=0}^{2^k} c_{k,i} x^i \) for some \( c_{k,i} \in \mathbb{Z} \);

(c) the integers \( b_{k,i} := c_{k,i} \) for even \( k \) and \( b_{k,i} := c_{k,2^k-i} \) for odd \( k \) satisfy the following:

\( c.i \) for each integer \( i \) with \( 0 \leq i \leq 2^k \), \( 3 \) divides \( b_{k,i} \) if and only if \( i \neq 2^k \);

\( c.ii \) \( 9 \) does not divide \( b_{k,0} \);

(d) the polynomial \( p_k \) is irreducible of degree exactly \( 2^k \);

(e) the degree of \( \mu_k \) is \( 2^k \).

**Proof.** Let us use induction on \( k \). All stated claims are clearly satisfied for \( k = 1 \), when \( p_1(x) = 3x^2 - 1 \) and \( \mu_1 = 1/\sqrt{3} \). Let us prove them for \( k + 1 \) assuming that they hold for some \( k \geq 1 \).

For Part [a], we have by Proposition 3.1 that

\[
\frac{3\mu_{k+1}^2 - 1}{2\mu_{k+1}} = \frac{3/(3 - 2\mu_k) - 1}{2/(3 - 2\mu_k)} = \mu_k
\]

and thus \( p_{k+1}(\mu_{k+1}) = (2\mu_{k+1}^2)^{2^k} p_k(\mu_k) \), which is 0 by induction.

Part [b] also follows easily from the induction assumption:

\[
p_{k+1}(x) = (2x^2)^{2^k} \sum_{i=0}^{2^k} c_{k,i} \left( \frac{3x^2 - 1}{2x^2} \right)^i = \sum_{i=0}^{2^k} c_{k,i} (3x^2 - 1)^i (2x^2)^{2^k-i}.
\]

Let us turn to Part [c]. The relation in (4) when taken modulo 3 reads that

\[
\sum_{j=0}^{2^{k+1}} c_{k+1,j} x^j \equiv \sum_{i=0}^{2^k} c_{k,i} x^{2^{k+1}-2i} \quad (\text{mod } 3).
\]

Thus, \( c_{k+1,j} \equiv c_{k,2^{k-j/2}} \) (mod 3) for all even \( j \) between 0 and \( 2^{k+1} \), while \( c_{k+1,j} \equiv 0 \) (mod 3) for odd \( j \) (in fact, \( c_{k+1,j} = 0 \) for all odd \( j \) since \( p_{k+1} \) is an even function). In terms of the sequences \( (b_{k,j}) \), this relation states that

\[
b_{k+1,j} \equiv b_{k,j/2} \quad (\text{mod } 3) \quad \text{for all even } j \text{ with } 0 \leq j \leq 2^k,
\]

while \( b_{k+1,j} \equiv 0 \) (mod 3) for all odd \( j \). This implies Part [c.i]. For Part [c.ii], the relation in (4) when taken modulo 9 gives that \( c_{k+1,0} \equiv c_{k,2^k} \) and \( c_{k+1,2^k+1} \equiv c_{k,0} 2^{2^k} + c_{k,1} 3 \cdot 2^{2^k-1} \).
Since \( c_{k,1} \) is divisible by 3, we have in fact that \( c_{k+1,2k+1} \equiv c_{k,0} \cdot 2^{2k} \equiv c_{k,0} \pmod{9} \). By the induction hypothesis, this implies that 9 does not divide \( b_{k+1,0} \).

By the argument above, \( c_{k+1,2k+1} \) is non-zero module 3 for odd \( k \) and non-zero module 9 for even \( k \). Thus, regardless of the parity of \( k \), the degree of the polynomial \( p_{k+1} \) is exactly \( 2^{k+1} \). Moreover, \( p_{k+1} \) satisfies Eisenstein’s criterion for prime \( q = 3 \) (namely, that \( q \) divides all coefficients, except exactly one at the highest power of \( x \) or at the constant term while the other of the two is not divisible by \( q^2 \)). By the criterion (whose proof can be found in e.g. [16, Section 4]), the polynomial \( p_{k+1} \) is irreducible, proving Part (d).

By putting the above claims together, we see that \( \mu_{k+1} \) is a root of an irreducible polynomial of degree \( 2^{k+1} \), establishing Part (e). This completes the proof the lemma (and thus of Theorem 1.2).

4 Concluding remarks

Our proof of Theorem 1.2 shows that for every integer \( d \) which is a power of 2 there exists a finite family \( \mathcal{F} \) of \( r \)-graphs such that \( \pi(\mathcal{F}) \) has algebraic degree \( d \). It seems interesting to know whether this is true for all positive integers.

**Problem 4.1.** Let \( r \geq 3 \) be an integer. Is it true that for every positive integer \( d \) there exists a finite family \( \mathcal{F} \) of \( r \)-graphs such that \( \pi(\mathcal{F}) \) has algebraic degree exactly \( d \)?

By considering other patterns, one can get Turán densities in \( \Pi^{(r)}_{\text{fin}} \) whose algebraic degrees are not powers of 2. For example, the pattern \( ([3], \{1, 2, 3\}, \{1, 2\}) \) with recursive parts 1 and 2 (where we can take blowups of the single edge \( \{1, 2, 3\} \) and recursively repeat this step inside the first and the second parts of each added blowup) gives a Turán density in \( \Pi^{(3)}_{\text{fin}} \) (by [16, Theorem 3], a generalisation of Theorem 2.1) whose degree can be computed to be 3. However, we did not see any promising way of how to produce a pattern whose Lagrangian has any given degree \( d \).

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References

[1] R. Baber and J. Talbot. Hypergraphs do jump. *Combin. Probab. Comput.*, 20(2):161–171, 2011.
[2] R. Baber and J. Talbot. New Turán densities for 3-graphs. *Electron. J. Combin.*, 19(2):Paper 22, 21, 2012.
[3] F. Chung and R. Graham. *Erdős on graphs: his legacy of unsolved problems*. A K Peters, Ltd., Wellesley, MA, 1998.
[4] D. De Caen and Z. Füredi. The maximum size of 3-uniform hypergraphs not containing a Fano plane. *J. Combin. Theory Ser. B*, 78(2):274–276, 2000.
[5] P. Erdős. On extremal problems of graphs and generalized graphs. *Israel J. Math.*, 2:183–190, 1964.
[6] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar.*, 1:51–57, 1966.

[7] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946.

[8] P. Frankl, Y. Peng, V. Rödl, and J. Talbot. A note on the jumping constant conjecture of Erdős. *J. Combin. Theory Ser. B*, 97(2):204–216, 2007.

[9] P. Frankl and V. Rödl. Hypergraphs do not jump. *Combinatorica*, 4(2-3):149–159, 1984.

[10] Z. Füredi and M. Simonovits. Triple systems not containing a Fano configuration. *Combin. Probab. Comput.*, 14(4):467–484, 2005.

[11] C. Grosu. On the algebraic and topological structure of the set of Turán densities. *J. Combin. Theory Ser. B*, 118:137–185, 2016.

[12] G. Katona, T. Nemetz, and M. Simonovits. On a problem of Turán in the theory of graphs. *Mat. Lapok*, 15:228–238, 1964.

[13] P. Keevash. Hypergraph Turán problems. In *Surveys in combinatorics 2011*, volume 392 of *London Math. Soc. Lecture Note Ser.*, pages 83–139. Cambridge Univ. Press, Cambridge, 2011.

[14] P. Keevash and B. Sudakov. The Turán number of the Fano plane. *Combinatorica*, 25(5):561–574, 2005.

[15] T. S. Motzkin and E. G. Straus. Maxima for graphs and a new proof of a theorem of Turán. *Canadian J. Math.*, 17:533–540, 1965.

[16] O. Pikhurko. On possible Turán densities. *Israel J. Math.*, 201(1):415–454, 2014.

[17] O. Pikhurko. The maximal length of a gap between $r$-graph Turán densities. *Electron. J. Combin.*, 22(4):Paper 4.15, 7, 2015.

[18] A. Sidorenko. What we know and what we do not know about Turán numbers. *Graphs Combin.*, 11(2):179–199, 1995.

[19] P. Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[20] Z. Yan and Y. Peng. An irrational Lagrangian density of a single hypergraph. *SIAM J. Discrete Math.*, 36(1):786–822, 2022.

[21] Z. Yan and Y. Peng. Non-jumping Turán densities of hypergraphs. *Discrete Math.*, 346(1):Paper No. 113195, 11, 2023.