THE CHARACTERIZATION PROBLEM FOR ONE CLASS OF SECOND ORDER OPERATOR PENCIL WITH COMPLEX PERIODIC COEFFICIENTS.

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Abstract. The purpose of the present work is solving the characterization problem, which consists of identification of necessary and sufficient conditions on the scattering data ensuring that the reconstructed potential belongs to a particular class.

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1. Introduction

The purpose of the present work is solving the characterization problem, which consists of identification of necessary and sufficient conditions on the scattering data ensuring that the reconstructed potential belongs to a particular class. In our case the potential belongs to $Q^2_+$ consisting of functions

$$q(x) = \sum_{n=1}^{\infty} q_n \exp(inx),$$

which is a subclass of the class $Q^2$ of all $2\pi$ periodic complex valued functions on the real axis $\mathbb{R}$, belonging to $L^2[0,2\pi]$.

The object under consideration is the operator $L$ given by the differential expression

$$l \left( \frac{d}{dx}, \lambda \right) \equiv -\frac{d^2}{dx^2} + 2\lambda p(x) + q(x) - \lambda^2$$

in the $L_2(-\infty, \infty)$ with potentials $p(x) = \sum_{n=1}^{\infty} p_n \exp(inx)$, $q(x) = \sum_{n=1}^{\infty} q_n \exp(inx)$, for which fullfield

$$\sum_{n=1}^{\infty} n \cdot |p_n| < \infty, \sum_{n=1}^{\infty} |q_n| < \infty, \quad \lambda$$

a spectral parameter.

The inverse problem for the potentials (1.1) was formulated and solved in [1, 2], where is shown, that the equation $Ly = 0$, has the solution

$$e_{\pm}(x, \lambda) = e^{\pm i\lambda x} \left( 1 + \sum_{n=1}^{\infty} V_n \pm, e^{i\alpha x} + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{V_n^{\pm}}{n \pm 2\lambda} e^{i\alpha x} \right),$$

(1.3)

$V_n^\pm, V_n^{\pm}$ are some numbers and Wronskian of the system of solutions is equal to $2i\lambda$.

The limit

$$e_{\pm}^n(x) = \lim_{\lambda \to \pm n/2} (n \pm 2\lambda) e_{\pm}(x, \lambda) = \sum_{\alpha=n}^{\infty} V_n^{\pm, e^{i\alpha x} e^{-i\alpha x}}, \quad n \in \mathbb{N}$$

is also a solution of the equation $Ly = 0$, but is linearly dependent with $e_{\pm}(x, \pm n/2)$. Thus, there exist numbers $\hat{S}_n, n \in \mathbb{N}$, for which

$$e_{\pm}^n(x) = \hat{S}_n^{\pm} e_{\pm} \left( x, \pm \frac{n}{2} \right), \quad n \in \mathbb{N}.$$  

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From last relation one may obtain that $S_n^\pm = V_n^\pm$.

In the work [1] the spectral analysis of the operator pencil $L$ was carried out and sufficient condition for reconstruction $p(x), q(x) \in Q_+^2$ using the values $S_n^\pm, n \in N$ was found.

Note that some characterizations for the Sturm-Liouville operator in the class of real-valued potentials belonging to $L^1_R(R)$ is the class of measurable potentials satisfying the condition $\int dx (1+|x|)^\alpha |p_\gamma(x)| < \infty$, have been given by Melin [3] and Marchenko [4]. (See for details [5, 6, and 7]). For the potential $p(x) = 0, q(x) \in Q_+^2$ which in the nontrivial cases is complex valued the inverse problem was first formulated and solved by Gasymov M.G [8]. Later the complete solution of the inverse problem for the cases $p(x) = 0, q(x) \in Q_+^2$ was found by Pastur and Tkachenko [9].

Let us formulate now the basic result of the present work.

**Definition.** Constructed by the help of formula (1.4) sequence $\{\hat{S}_n^\pm\}_{n=1}^{\infty},$ is called as a set of spectral data for the operator $L$ given by the differential expression (1.2) with potential $p(x), q(x) \in Q_+^2$.

**Theorem 1.** In order the given sequence of complex numbers $\{\hat{S}_n^\pm\}_{n=1}^{\infty}$ to be a spectral data for the operator $L$ given by the differential expression (1.2) with potentials $p(x), q(x) \in Q_+^2$ it is necessary and sufficient, that the conditions are fulfilled at the same time:

1) $$\{n^2\hat{S}_n^\pm\}_{n=1}^{\infty} \in l_1;$$ (1.5)

2) Infinite determinant

$$D(z) \equiv \det \left[ \delta_{nm} - \sum_{k=1}^{\infty} \frac{4\hat{S}_m^\pm \hat{S}_k^\pm}{(m+k)(n+k)} e^{\frac{(m+k)z}{2}} e^{\frac{(n+k)z}{2}} \right]_{n,m=1}^{\infty}$$ (1.6)

exists, is continuous, is not equal to zero in the closed semi plane $\overline{C_+} = \{z : Imz \geq 0\}$ and is analytical inside open semi plane $C_+ = \{z : Imz > 0\}$.

2. **ON AN INVERSE PROBLEM OF THE SCATTERING THEORY IN THE SEMIAxis**

On the base of the proof of the Theorem 2, lies investigation of the equation $Ly = 0$. Taking in it

$$x = it, \lambda = -ik, y(x) = Y(t)$$ (2.1)

we get

$$-Y''(t) + 2\mu \bar{p}(it) Y(t) + \bar{q}(it) Y(t) = \mu^2 Y(t)$$ (2.2)

where

$$\bar{p}(t) = ip(it) = i \sum_{n=1}^{\infty} p_n e^{-nt}, \sum_{n=1}^{\infty} n \cdot |p_n| < \infty, \quad \bar{q}(t) = -q(it) = -\sum_{n=1}^{\infty} q_n e^{-nt} \sum_{n=1}^{\infty} |q_n| < \infty.$$ (2.3)

As a result the equation (2.2) is obtained, the potential of which decreases at $t \to \infty$. The specification of the considered inverse problem is defined by the fact that the potentials belong to the class $Q_+^2$. In this section we suppose $t \in R^+$.

The procedure of analytic continuation that allows from the result for the equation (2.2) to get corresponding results for the equation (1.2) will be investigated in next section.

The equation (2.2) with potentials (2.3) has the solution

$$f_\pm(t, \mu) = e^{\pm i\mu t} \left( 1 + \sum_{n=1}^{\infty} V_n^\pm e^{-nt} + \sum_{n=1}^{\infty} \sum_{\alpha=\pm} V_n^\alpha_{\alpha n} e^{-\alpha t} \right)$$ (2.4)
and the numbers $V_n^\pm, V_{n\alpha}^\pm$ are defined by the following recurrent formulae:

\[
\alpha^2 V_n^\pm + \alpha \sum_{n=1}^{\alpha} V_{n\alpha}^\pm + \sum_{s=1}^{\alpha-1} (q_{\alpha-s} V_s^\pm \pm p_{\alpha-s} \sum_{n=1}^{s} V_{n\alpha}^\pm) + q_{\alpha} = 0 \quad (2.5)
\]

\[
\alpha(\alpha - n)V_{n\alpha}^\pm + \sum_{s=n}^{\alpha-1} (q_{\alpha-s} \mp n \cdot p_{\alpha-s}) V_{n\alpha}^\pm = 0 \quad (2.6)
\]

\[
\alpha V_{n\alpha}^\pm \pm \sum_{s=1}^{\alpha-1} V_s^\pm p_{\alpha-s} \mp p_{\alpha} = 0 \quad (2.7)
\]

and the sequence (2.4) admits double termwise differentiation. Then under the conditions (2.3) we get

\[
f_{\pm} (t, \mu) = \Psi_{\pm} (t) e^{\pm i\mu t} + \int_{t}^{\infty} K_{\pm} (t, u) e^{\pm i\mu u} du , \quad (2.8)
\]

where $K_{\pm} (t, u), \Psi_{\pm} (t)$ have a form

\[
K_{\pm} (t, u) = \frac{1}{2t} \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} V_{n\alpha}^\pm \cdot e^{-\alpha t} \cdot e^{-(u-t)n/2}, \Psi_{\pm} (t) = 1 + \sum_{n=1}^{\infty} V_{n\alpha}^\pm e^{-\alpha t} , \quad (2.9)
\]

So, it is proved the following

**Lemma 1:** The function $\Psi_{\pm} (t)$ and the kernel of the transformation operator of the equation \[ (2.10) \] $K (t, u), u \geq t$, attached to $+\infty$, with the potentials \[ (2.11) \] permits the representation \[ (2.12) \], in which the series

\[
\sum_{n=1}^{\infty} n^2 |V_{n\alpha}^\pm| + \sum_{\alpha=n+1}^{\infty} (\alpha n) |V_{n\alpha}^\pm| + \sum_{n=1}^{\infty} n \cdot |V_{n\alpha}^\pm|
\]

are convergent.

Remark: In our case the kernel of the operator of transformation $K_{\pm} (t, u), u \geq t$, at $+\infty$, and function $\Psi_{\pm} (t)$ are constructed effectively.

It is possible to get the equality \[ (1) \]

\[
f_{n}^\pm (t) = S_{n}^\pm f_{n}^\pm \left( t, \mp \frac{\mu}{2} \right) , \quad (2.10)
\]

where $f_{n}^\pm (t) = \lim_{\mu \to \pm i n/2} (in \pm 2\mu) f_{n}^\pm (t, \mu)$.

Rewriting the equality \[ (2.10) \] in the form

\[
\sum_{\alpha=n}^{\infty} V_{n\alpha}^\pm e^{-\alpha t} \cdot e^{nt/2} = S_{n}^\pm e^{-nt/2} \left( 1 + \sum_{m=1}^{\infty} V_{m\alpha}^\pm e^{-\alpha t} \cdot e^{-m\alpha t} \right) \quad (2.11)
\]

and denoting by

\[
z_{\pm} (t+s) = \sum_{m=1}^{\infty} S_{m}^\pm e^{-(t+s)m/2} \quad (2.12)
\]

from \[ (2.11) \] we obtain the Marchenko type equation

\[
K_{\pm} (t, s) = \Psi_{\pm} (t) z_{\pm} (t+s) + \int_{t}^{\infty} K_{\mp} (t, u) z_{\pm} (u+s) du. \quad (2.13)
\]

So, the following is proved

**Lemma 2:** If the coefficients $\overline{\mathbf{f}} (t), \mathbf{f} (t)$ of the equation \[ (2.2) \] have the form \[ (2.3) \], then at every $t \geq 0$, the kernel of the transformation operator \[ (2.9) \] satisfies to the equation of the Marchenko type...
in which the transition function $z^\pm(t)$ has the form (2.12) and the numbers $S^\pm_m$ are defined by the equality (2.10), from which it is obtained, that $S^\pm = V^\pm_{mm}$.

Note that from relation (2.7) it is easy to obtain useful in further formulas $\Psi^+(t) \cdot \Psi^-(t) = 1$ and $\lim_{x \to \infty} \Psi^\mp(x) = 1$. The coefficients are reconstructed by the kernel of the transformation operator and function with the help of the formulas

$$\Psi^\pm(t) = J \pm i \int \limits_{t}^{\infty} \mathcal{P}(u) \Psi^\pm(u) \, du,$$

(2.14)

$$K^\pm(t, t) = \pm \frac{1}{2} \int \limits_{t}^{\infty} \mathcal{Q}(u) \Psi^\pm(u) \, du \mp i \mathcal{P}(t) \Psi^\pm(t) \pm i \int \limits_{t}^{\infty} \mathcal{P}(u) K^\pm(u, u) \, du.$$

(2.15)

Hence the basic equation (2.13) and the form of the transition function (2.12) make natural the formulation of the inverse problem of reconstruction potentials of the equation (2.2) by numbers $S^\pm_m$.

In this formulation, which employs the transformation operator method, an important moment is the proof of uniqueness solvability of the basic equation (2.13).

Lemma 3. The homogenous equation

$$g^\pm(s) - \int \limits_{0}^{\infty} z^\pm(u + s) g^\mp(u) \, du = 0$$

(2.16)

has only trivial solution in the space $L_2(R^+)$. The proof of the lemma 3 is analogous to [5, p.198].

Lemma 4. For each fixed $a, (Ima \geq 0)$ the homogenous equation

$$g^\pm(s) - \int \limits_{t}^{\infty} z^\pm(u + s - 2ai) g^\mp(u) \, du = 0$$

(2.17)

has only trivial solution in the space $L_2(R^+)$. The proof: In the equation (1.2) we substitute $x$ by $x + a$, where $Ima \geq 0$. Then we obtain the same equation with the coefficients $p_a(x) = p_a(x + a), \quad q_a(x) = q_a(x + a)$ belonging to $Q^2_+$. Let us remark, that the function $e^\pm(x + a, \lambda)$ are solutions of the equation

$$-y'' + 2\lambda p_a(x)y + q_a(x)y = \lambda^2 y$$

that at $x \to \infty$ have the form

$$e_\pm(x + a, \lambda) = e^{\pm i\lambda x} e^{\pm i\lambda a} + o(1).$$

Therefore the function $e_\pm^a(x, \lambda) = e^{x/a} e_\pm(x + a, \lambda)$ is also solution of the type (1.3).

Then let us denote by $\{\hat{S}_n^\pm(a)\}_{n=1}^{\infty}$ the spectral data of the operator $L \equiv -\frac{d^2}{dx^2} + 2\lambda p_a(x) + q_a(x) - \lambda^2$. According to (1.4)

$$\hat{S}_n^\pm(a) e_\pm^a(x, \pm \eta/2) = \lim_{\lambda \to \pm \eta/2} (n + 2\lambda) e_\pm^a(x, \lambda) =$$

$$= \lim_{\lambda \to \pm \eta/2} (n + 2\lambda) e^{\pm i\lambda x} e_\pm(x + a, \lambda) = e^{i\eta/n/2} \hat{S}_n^\pm e_\pm(x + a, \pm \eta/2) =$$

$$= e^{i\eta/n/2} \hat{S}_n^\pm e^{i\eta/2} e_\pm(x, \pm \eta/2) = \hat{S}_n^\pm e^{i\eta} e_\pm(x, \pm \eta/2)$$

Hence $\hat{S}_n^\pm(a) = S_n^\pm e^{i\eta}.$

Now discussing as in the above, we obtain the basic equation of the form (2.13) with the transition function

$$Z^\pm_a(t) = \sum_{n=1}^{\infty} S_n^\pm(a) e^{-nt/2} = \sum_{n=1}^{\infty} S_n^\pm e^{i\eta} e^{-nt/2} = Z^\pm(t - 2ia)$$
From lemmas 3 and 4 follows

**Theorem 3:** The potentials \( \overline{\varphi}(t), \overline{\varphi}(t) \) of the equation (2.13), satisfying to the condition (2.8) are uniquely defined by the numbers \( S_n^\pm \).

### 3. III. Proof of Theorem 2.

**Necessity:** The necessity of the condition (1) has been proved in the [1].

To prove the necessity of the condition (2) we firstly show, that from the trivial solvability of the main equation (2.13) at \( t = 0 \) in the class of functions satisfying to inequality \( \| g(u) \| \leq Ce^{-\frac{u}{T}}, u \geq 0 \), follows trivial solvability in \( l_2(R^+) \) of the infinite system of equations

\[
g_n^\pm = - \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{2S_m^\pm g_k^\pm}{(m+k)(n+k)} g_m^\pm = 0, \tag{3.1}
\]

where \( g_n^\pm \in l_2(R) \); \( S_n^\pm \in l_1 \).

Rewrite (3.1) in the form

\[
g_n^\pm = - \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{4S_m^\pm g_k^\pm}{(m+k)(n+k)} g_m^\pm = 0. \tag{3.2}
\]

Really, if \( \{g_n\}_{n=1}^\infty \in l_2 \) is a solution of this system, then the function

\[
g^\pm(u) = - \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{2S_m^\pm g_k^\pm}{m+k} e^{-ku/2} g_m^\pm \tag{3.3}
\]

is defined for all \( u \geq 0 \), satisfies inequality

\[|g^\pm(u)| \leq c \cdot e^{-u/2}; \quad u \geq 0 \]

and is a solution of (2.13), as

\[
g^\pm(s) - \int \int z^\pm(u+s) z^\mp(u+s_1) g^\pm(s_1) ds_1 du =
\]

\[
- \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{2S_m^\pm S_k^\pm}{m+k} e^{-ks/2} g_m^\pm + \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} S_m^\pm S_k^\pm e^{-(u+s)k/2} \cdot e^{-(u+s_1)m/2} \times
\]

\[
\times \left( \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} \frac{2S_m^\pm S_r^\pm}{m+r} e^{-s_1/k/2} g_k^\pm \right) ds_1 du = - \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{2S_m^\pm S_k^\pm}{m+k} e^{-ks/2} g_m^\pm +
\]

\[
+ \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{2S_m^\pm S_k^\pm}{m+k} e^{-ks/2} \left( \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} \frac{4S_n^\pm S_r^\pm}{(m+r)(n+r)} g_n^\pm \right) = 0.
\]

Since \( g^\pm(u) = 0 \) therefore, \( S_m^\pm S_k^\pm g_m^\pm = 0 \) for all \( m \geq 1, k \geq 1 \), and \( g_m^\pm = 0, m \geq 1 \) according to (3.2). Let us introduce in the space \( l_2 \) operator \( F_2^\pm(t) \), given by matrix

\[
F_{mn}^\pm(t) = \sum_{k=1}^{\infty} \frac{4S_k^\pm S_m^\pm}{(m+k)(n+k)} e^{-k(m+k)t/2} e^{-(n+k)t/2}, n, m \in N. \tag{3.4}
\]

Then, from \( n^2 S_n^\pm \in l_1 \), we get that \( \sum_{j,k=1}^{\infty} \left| \left( F_2^\pm \varphi_j, \varphi_k \right) \right|_2 < \infty \), i.e. \( F(t) \) is a kernel operator [10].

So there exists the determinant \( \Delta^\pm(t) = \det \left( E - F_2^\pm(t) \right) \) of the operator \( E - F_2^\pm(t) \) related, as it is not difficult to see, with the determinant \( D^\pm(z) \) from the condition 2) of the theorem 2, by relation \( \Delta^\pm(-iz) = \det(E - F_2^\pm(-iz)) \equiv D^\pm(z) \).

The determinant of the system (3.1) is \( D^\pm(0) \), and the determinant of analogous system with potential \( p_z(x) = p(x+z), q_z(x) = q(x+z), Imz > 0 \) is

\[
D^\pm(z) = \det \left| \delta_{mn} - \sum_{k=1}^{\infty} \frac{4S_k^\pm S_m^\pm(z)}{(m+k)(n+k)} \right|_{m,n=1}^{\infty} = \delta_{mn} - \sum_{k=1}^{\infty} \frac{4S_k^\pm S_m^\pm}{(m+k)(n+k)} e^{\frac{m+k}{2}z} e^{\frac{n+k}{2}z} \right|_{m,n=1}^{\infty}. \]
Therefore to prove the necessity of the condition 2) of Theorem 2 one should check that $\Delta^\pm(0) = D^\pm(0) \neq 0$. Really, the system (3.1) may be written in $l_2$ as 
\[ g^\pm - F^\pm_2(0)g^\pm = 0 \]
As $F^\pm_2(0)$ is a kernel operator, the Fredholm theory is applicable to it. In accordance with this theory trivial solvability of the last equation is equivalent to the fact that $\det(E + F^\pm_2(0))$ is not equal to zero [10]. Necessity of the condition 2) is proved.

Sufficiency: Let us study (2.13) in detail. It is known [5] that $K^\pm(t,s)$ can be expressed by $\Psi^\pm(t)$ and solutions $P^\pm(t,s),Q^\pm(t,s)$ of the Marchenko type equations from (2.13) by the replacement of $\Psi^\pm(t)$ by 1 and $\pm i$.

Then
\[ K^\pm(t,s) = \Psi^\pm(t) \alpha^\pm(t,s) + \Psi^\pm(t) \beta^\pm(t,s), \]  
where
\[ \alpha^\pm(t,s) = \frac{1}{2} [P^\pm(t,s) \mp iQ^\pm(t,s)] \]  
\[ \beta^\pm(t,s) = \frac{1}{2} [P^\pm(t,s) \pm iQ^\pm(t,s)] \]

Thus for further studies we should consider the following equations
\[ P^\pm(t,s) = z^\pm(t+s) + \int_t^\infty P^\pm(t,u) z^\pm(u+s) du \]  
\[ Q^\pm(t,s) = \pm iz^\pm(t+s) + \int_t^\infty Q^\pm(t,u) z^\pm(u+s) du \]

Rewrite (3.9) in the form
\[ P^\pm(t,s) = z^\pm(t+s) + \int_t^\infty z^\pm(u+t) z^\pm(u+s) du + \int_t^\infty \int_t^\infty P^\pm(t,\tau) z^\pm(u+\tau) z^\pm(u+s) d\tau du \]  

Let us introduce in the space $l_2$ operator $F^\pm_1(t)$ defined by the matrix
\[ F^\pm_1 \equiv \frac{2S^\pm}{m+n} e^{-(m+n)t/2}, \quad \text{Re} t > 0 \]  

Multiplying the equation (3.11) by $e^{-\mu u/2}$ and integrating it over $s \in [t, \infty)$ we obtain
\[ p^\pm(t) = F^\pm_1(t) e(t) + F^\pm_2(t) e(t) + p^\pm(t) F^\pm_2(t), \]
where $F^\pm_1(t), F^\pm_2(t)$ is defined by the matrix (3.4), (3.12) and
\[ e(t) = \{ e^{-nt/2}E_n \}_{n=1}^\infty; \quad p^\pm(t) = \{ \int_t^\infty P^\pm(t,u) e^{-\mu u/2} du \}_{n=1}^\infty; \]
As $F^\pm_2(t)$ is kernel operator for $t \geq 0$ and the condition $\det(E - F^\pm_2(t)) \neq 0$ is satisfied, there exists bounded in $l_2$ inverse operator $R^\pm_2(t) = (1 - F^\pm_2(t))^{-1}$.

As $F^\pm_1(t)e(t), F^\pm_2(t)e(t) \in l_2$, from (3.13) we get

$$p^\pm(t) = R^\pm(t) [F^\pm_1(t) + F^\pm_2(t)] e(t).$$  \hspace{1cm} (3.14)

Now, let’s take $f, g = \sum_{n=1}^\infty f_n g_n$. Then (3.11) gives

$$p^\pm(t, s) = <e(t), B^\pm(s)> + <e(t), A^\pm(s, t)> + <p^\pm(t), A^\pm(s, t)> =$$

$$= <e(t), B^\pm(s)> + <e(t), A^\pm(s, t)> + <R^\pm(t)(F^\pm_1(t) + F^\pm_2(t))e(t), A^\pm(s, t)> =$$

$$= <e(t), B^\pm(s)> + <R^\pm(t)(F^\pm_1(t) + F^\pm_2(t)) + 1)e(t), A^\pm(s, t)>$$

where $B^\pm(s) = \{B^\pm_m(s) = S^\pm_m e^{-ms/2}, s > 0\}^{\infty}_{m=1}$

and $A^\pm(s, t) = \{A^\pm_m(s, t) = \sum_{k=1}^\infty 2S^\pm_m e^{-ks/2} \cdot e^{-(m+k)t/2}; s, t > 0\}$

Now suppose that the conditions of the theorem are satisfied. Let us define the function $P^\pm(t, s)$ by the equality (3.15) at $0 \leq t \leq u$ according to given above considerations. Then for $u \geq t$ we obtain

$$P^\pm(t, s) - \int_t^\infty \int_t^\infty P^\pm(t, \tau)z^\pm(u + \tau)z^\pm(u + s)d\tau d\tau =$$

$$<e(t), B^\pm(s)> + <R^\pm(t)(F^\pm_1(t) + F^\pm_2(t)) + 1)e(t), A^\pm(s, t)> -$$

$$- \int_t^\infty <e(t), B^\pm(\tau)> + <R^\pm(t)(F^\pm_1(t) + F^\pm_2(t)) + 1)e(t), A^\pm(\tau, t)> (e(\tau), A^\pm(s, t)> d\tau =$$

$$= <e(t), B^\pm(s)> + <e(t), A^\pm(s, t)> + <R^\pm(t)(F^\pm_1(t) + F^\pm_2(t))e(t), A^\pm(s, t)> -$$

$$- <F^\pm_1(t)e(t), A^\pm(s, t)> - <F^\pm_2(t)e(t), A^\pm(s, t)> -$$

$$= <e(t), B^\pm(s)> + <e(t), A^\pm(s, t)> = z^\pm(t + s) + \int_t^\infty z^\pm(u + t)z^\pm(u + s)du$$

For the $Q^\pm(t, s)$ we analogously obtain, that $Q^\pm(t, s) = \pm i < e(t), B^\pm(s) > \mp i < e(t), A^\pm(s, t) > + < Q^\pm(t), A^\pm(s, t) >$

where

$$Q^\pm(t) = \pm i R^\pm(t) [F^\pm_1(t) - F^\pm_2(t)] e(t), \quad Q^\pm(t) = \{\int_t^\infty Q^\pm(t, u)e^{-nu/2}du\}^{\infty}_{n=1}$$

Thus the following lemma was proved.

**Lemma 5:** At any $t \geq 0$ the kernel $K^\pm(t, s)$ of the transformation operator and function $\Psi^\pm(t)$ satisfies the main equation

$$K^\pm(t, s) = \Psi^\pm(t) z^\pm(t + s) + \int_t^\infty K^\pm(t, u) z^\pm(u + s) du.$$  \hspace{1cm} (3.15)

The uniquely solvability of the main equation follows from Lemma 3. By substitution it is not difficult to calculate, that the solution of the main equation, indeed, is

$$K^\pm(t, u) = \frac{1}{2i} \sum_{n=1}^\infty \sum_{\alpha=0}^\infty V^\pm_{\alpha n} e^{-\alpha t} \cdot e^{-(u-t)n/2}, \quad \Psi^\pm(t) = 1 + \sum_{n=1}^\infty V^\pm_n e^{-nt}$$

where the numbers $V^\pm_{\alpha n}, V^\pm_n$ are defined by the numbers $S^\pm_n, n \in N$ from recurrent relations

$$V^\pm_{mm} = S^\pm_m, \quad V^\pm_{m, \alpha + m} = S^\pm_m \left(V^\pm_\alpha + \sum_{n=1}^\infty \frac{V^\pm_n}{n + m}\right)$$

Now we are in position to prove the main statement of the theorem, namely that the coefficient $\Psi(t)$ and $\bar{q}(t)$ have the form (2.23). First from the formula (2.14) and (2.15) we find that the potentials $\bar{p}(t), \bar{q}(t)$ have the form
\[ \tilde{p}(t) = \sum_{n=1}^{\infty} ip_n e^{-nt}, \quad \tilde{q}(t) = \sum_{n=1}^{\infty} -q_n e^{-nt}, \]

where the numbers \( p_n, q_n \) are defined by the relations (2.6) - (2.8). After, in order to prove that the numbers \( p_n, q_n \) satisfy the condition \( \sum_{n=1}^{\infty} n \cdot |p_n| < \infty, \sum_{n=1}^{\infty} |q_n| < \infty \) we demonstrate for the matrix elements \( R_{mn}(t) \), of the operator \( R(t) \) correctness of the estimations

\[ \| R_{mn}(t) \| \leq \delta_{mn} + C_0 |S_n^*|, \quad \left\| \frac{d}{dt} R_{mn}(t) \right\| \leq C_j |S_n^*|, \]

where \( S_n^* = \max (S_n^-, S_n^+) \) and \( C_j, j \in N \) are some constants. Really, from the equality \( R^\pm(t) = E + R^\pm(t)F_2^\pm(t) \) follows that

\[ \| R_{mn}(t) \| \leq \delta_{mn} + \left( \sum_{p=1}^{\infty} \| R_{mp}^\pm(t) \| ^2 \right) \left( \sum_{p=1}^{\infty} \| F_{pn}^\pm(t) \| ^2 \right)^{\frac{1}{2}} \leq \]

\[ \leq \delta_{mn} + 2 \cdot (\left( R^\pm(t)R^\pm(t) \right)_{pp} \sum_{p=1}^{\infty} \left( \sum_{k=1}^{4S_n^\pm} \frac{S_n^\pm}{(p+k)(n+k)} \right)^2 )^{\frac{1}{2}} \leq \]

\[ \leq \delta_{mn} + 2 \cdot (\left( R^\pm(t)R^\pm(t) \right)_{pp} \sum_{p=1}^{\infty} \left( \sum_{k=1}^{4S_n^\pm} \frac{S_n^\pm}{|S_k^\pm|} \right)^2 )^{\frac{1}{2}} \leq |S_n^\pm| \]

\[ \leq \delta_{mn} + \| R(t) \| _{l_2 \to l_2} |S_n^*|. \]

From other hand, as it was noted, the operator-function \( R^\pm(t) \) exists and is bounded in \( l_2 \) (as \( F_2^\pm(t) \) is kernel operator by \( t \geq 0 \) and \( \Delta^\pm(t) = \det (E + F_2^\pm(t)) \neq 0 \) that proves the first inequality of (3.9).

To prove the estimation \( \left\| \frac{d}{dt} R_{mn}(t) \right\| \leq C_5 |S_n^*| \) can be proved analogously.

Using these estimations, from (2.13) and (2.14) one can demonstrate the correctness of the estimations

\[ \left\| \frac{d^2}{dt^2} P^\pm(t, s) \right\| \leq \infty, \quad \left\| \frac{d^2}{dt^2} Q^\pm(t, s) \right\| \leq \infty. \]

Thus, the functions \( K^\pm(t, s) \) and \( \Psi^\pm(t) \) have the second derivatives over \( t \). From this we conclude that the series \( \sum_{n=1}^{\infty} \alpha^2 |V^\pm_\alpha| \) and \( \sum_{n=1}^{\infty} n \sum_{\alpha=1}^{n} (\alpha + n) |V^\pm_{\alpha n}| \) are convergence. The forms of the coefficients \( \overline{p}(t) \) and \( \overline{q}(t) \) are directly determined from the form of the functions \( K^\pm(t, s) \), \( \Psi^\pm(t) \) employing the formulae (1.15), (1.16). We get that for the numbers \( p_n, q_n \) the recurrent relations (2.6) - (2.8) are correct and hence the series \( \sum_{n=1}^{\infty} n \cdot |p_n| < \infty, \sum_{n=1}^{\infty} |q_n| < \infty \) are converges.

Let, finally, \( \{ S_n^\pm \}_{n=1}^{\infty} \) be a spectral data set of the operator \( L \) with the constructed coefficients \( p(x), q(x) \in Q^\pm_2 \). For completing of the proof it remains to show, that \( \{ S_n^\pm \}_{n=1}^{\infty} \) coincides with the initial numbers \( \{ \hat{S}_n^\pm \}_{n=1}^{\infty} \). This follows from the equality \( S_n^\pm = V^\pm_{mn} = \hat{S}_n^\pm \).

The theorem is proved.

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