CONFLICT-FREE INCIDENCE COLORING OF OUTER-1-PPLANAR GRAPHS

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ABSTRACT
An incidence of a graph $G$ is a vertex-edge pair $(v, e)$ such that $v$ is incidence with $e$. A conflict-free incidence coloring of a graph is a coloring of the incidences in such a way that two incidences $(u, e)$ and $(v, f)$ get distinct colors if and only if they conflict each other, i.e., (i) $u = v$, (ii) $uv$ is $e$ or $f$, or (iii) there is a vertex $w$ such that $uw = e$ and $vw = f$. The minimum number of colors used among all conflict-free incidence colorings of a graph is the conflict-free incidence chromatic number. A graph is outer-1-planar if it can be drawn in the plane so that vertices are on the outer-boundary and each edge is crossed at most once. In this paper, we show that the conflict-free incidence chromatic number of an outer-1-planar graph with maximum degree $\Delta$ is either $2\Delta$ or $2\Delta + 1$ unless the graph is a cycle on three vertices, and moreover, all outer-1-planar graphs with conflict-free incidence chromatic number $2\Delta$ or $2\Delta + 1$ are completely characterized. An efficient algorithm for constructing an optimal conflict-free incidence coloring of a connected outer-1-planar graph is given.

Keywords: outer-1-planar graph; incidence coloring; combinatorial algorithm; channel assignment problem.

1 Introduction

For groups of geographically separated people who need to keep in continuous voice communication, such as aircraft pilots and air traffic controllers, two-way radios are widely used [20]. This motivates us to investigate how to design a two-way radio network efficiently and economically.

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In a two-way radio network, each node represents a two-way radio that can both transmit and receive radio waves and there is a link between two nodes if and only if they may contact each other. Waves can transmit between two linked two-way radios in two different directions simultaneously. For a link $L$ connecting two nodes $N_i$ and $N_j$ in a two-way radio network, it is usually assigned with two channels $C(N_i, N_j)$ and $C(N_j, N_i)$. The former one is used to transmit waves from $N_i$ to $N_j$ and the latter one is used to transmit waves from $N_j$ to $N_i$. The associated channel box $B(N_i)$ of a node $N_i$ in a two-way radio network is a multiset of channels $C(N_i, N_j)$ and $C(N_j, N_i)$ such that $N_i$ is linked to $N_j$. An efficient way to avoid possible interference is to assign channels to links so that every radio receives a rainbow associated channel box (in other words, every two channels in $B(N_i)$ for every node $N_i$ in the network are apart). For the sake of economy, while assigning channels to a two-way radio network, the fewer channels are used, the better. This can be modeled by the conflict-free incidence coloring of graphs.

From now on, we use the language of graph theory and then define conflict-free incidence coloring. We consider finite graphs and use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of a graph $G$. The degree $d_G(v)$ of a vertex $v$ in a graph $G$ is the number of edges incident with $v$ in $G$. We use $d(v)$ instead of $d_G(v)$ whenever the graph $G$ is clear from the content. We call $\Delta(G) = \max\{d_G(v) \mid v \in V(G)\}$ and $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ the maximum degree and the minimum degree of a graph $G$. Other undefined notation is referred to [4].

Let $v$ be a vertex of $G$ and $e$ be an edge incident with $v$. We call the vertex-edge pair $(v, e)$ an incidence of $G$. For an edge $e = uv \in E(G)$, let $\text{Inc}(e) = \{(u, e), (v, e)\}$, and for a vertex $v \in V(G)$, let $\text{Inc}(v) = \cup_{e \ni v} \text{Inc}(e)$. For a subset $U \subseteq E(G)$, let $\text{Inc}(U) = \{\text{Inc}(e) \mid e \in U\}$. Two incidences $(u, e)$ and $(v, f)$ are conflicting if (i) $u = v$, (ii) $uw$ is $e$ or $f$, or (iii) there is a vertex $w$ such that $uw = e$ and $vw = f$. In other words, two incidences are conflicting if and only if there is a vertex $w$ such that both of them belong to $\text{Inc}(w)$.

A conflict-free incidence $k$-coloring of a graph $G$ is a coloring of the incidences using $k$ colors in such a way that every two conflicting incidences get distinct colors. The minimum integer $k$ such that $G$ has a conflict-free incidence $k$-colorable is the conflict-free incidence chromatic number of $G$, denoted by $\chi'_i(G)$. For a conflict-free incidence coloring $\varphi$ of a graph $G$ and an edge $e = uv \in E(G)$, we use $\varphi(\text{Inc}(e))$ to denote the set $\{\varphi(u, e), \varphi(v, e)\}$. For a subset $U \subseteq E(G)$, let $\varphi(\text{Inc}(U)) = \{\varphi(\text{Inc}(e)) \mid e \in U\}$.

We look back into the channel assignment problem of two-way radio networks and explain why the conflict-free incidence coloring of graphs can model it. Let $G$ be the graph representing the two-way radio network and let $L = N_iN_j$ be an arbitrary link, i.e, $L \in E(G)$. Assigning two channels $C(N_i, N_j)$ and $C(N_j, N_i)$ to $L$ is now equivalent to coloring the incidences $(N_i, L)$ and $(N_j, L)$. The goal of assigning every radio $N_i$ a rainbow associated channel box is translated to coloring the incidences of $G$ so that every two incidences in $\text{Inc}(N_i)$ receive distinct colors. This is exactly what we shall do while constructing a conflict-free incidence coloring of $G$.

From a theoretical point of view, one may be interested in a fact that the conflict-free incidence coloring relates to the $b$-fold edge-coloring, which is an assignment of sets of size $b$ to edges of a graph so that adjacent edges receive disjoint sets. An $(a : b)$-edge-coloring is a $b$-fold edge coloring out of $a$ available colors. The $b$-fold chromatic index $\chi'_b(G)$ is the least integer $a$ such that an $(a : b)$-edge-coloring of $G$ exists. It is not hard to check that $\chi'_i(G) = \chi'_2(G)$ for every graph $G$. However, there are hard problems related to $\chi'_2(G)$, among which the
most famous one is the Berge-Fulkerson conjecture [9], which states that every bridgeless cubic graph has a
collection of six perfect matchings that together cover every edge exactly twice. This is equivalent to conjecture
that every bridgeless cubic graph $G$ has a $(6 : 2)$-edge-coloring, i.e., $\chi'_2(G) \leq 6$. This conjecture is still widely
open [8, 10, 13, 15] and was generalized by Seymour [18] to $\gamma$-graphs.

The structure of this paper organizes as follows. In Section 2, we establish fundamental results for the conflict-
free incidence chromatic number of graphs. In Section 3, we investigate the conflict-free incidence coloring of
outer-1-planar graphs by showing that $2\Delta \leq \chi^c_i(G) \leq 2\Delta + 1$ for outer-1-planar graphs $G$ with maximum degree
$\Delta$ unless $G \sim C_3$, and moreover, characterizing outer-1-planar graphs $G$ with $\chi^c_i(G)$ equal to $2\Delta$ or $2\Delta + 1$. An
efficient algorithm for constructing an optimal conflict-free incidence coloring of a connected outer-1-planar graph
is also given. We end this paper with an interesting open problem relative to the complexity in Section 4.

2 Fundamental results

Let $\chi'(G)$ be the chromatic index of $G$, the minimum integer $k$ such that $G$ admits an edge $k$-coloring so that
adjacent edges receive distinct colors. The following is an interesting relationship between $\chi^c_i(G)$ and $\chi'(G)$.

Proposition 1. $2\Delta(G) \leq \chi^c_i(G) \leq 2\chi'(G)$.

Proof. Since $|\text{Inc}(v)| = 2\Delta(G)$ for a vertex $v$ with maximum degree, $\chi^c_i(G) \geq 2\Delta(G)$ for every graph $G$. If $\varphi$ is
a proper edge coloring of $G$ using the colors $\{1, 2, \ldots, \chi'(G)\}$, then one can construct a conflict-free incidence
$2\chi'(G)$-coloring of $G$ such that $\varphi(\text{Inc}(e)) = \{\varphi(e), \varphi(e) + \chi'(G)\}$ for every edge $e \in E(G)$. It follows that
$\chi^c_i(G) \leq 2\chi'(G)$.

The well-known Vizing’s theorem (see [4, p128]) states that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for every simple
graph $G$. This divides simple graphs into two classes. A simple graph $G$ belongs to class one if $\chi'(G) = \Delta(G)$,
and belongs to class two if $\chi'(G) = \Delta(G) + 1$. The following are immediate corollaries of Proposition 1.

Proposition 2. If $G$ is a class one graph, then $\chi^c_i(G) = 2\Delta(G)$.

Proposition 3. If $G$ is simple graph, then $\chi^c_i(G) \in \{2\Delta(G), 2\Delta(G) + 1, 2\Delta(G) + 2\}$.

The well-known Kőnig’s theorem (see [4, p127]) states that every bipartite graph is of class 1. So the following
is immediate by Proposition 1.

Theorem 2.1. If $G$ is a bipartite graph, then $\chi^c_i(G) = 2\Delta(G)$.

Now that we have Proposition 2, it would be worth determining the conflict-free incidence chromatic number
of a certain class of graphs of class two. We first look into a cycle $C_n$ of length $n$.

Theorem 2.2.

$$
\chi^c_i(C_n) = \begin{cases} 
4 & \text{if } n \text{ is even}, \\
5 & \text{if } n \geq 5 \text{ is odd}, \\
6 & \text{if } n = 3.
\end{cases}
$$
Algorithm 1: COLOR-CYCLE(n)

/* This algorithm constructs an optimal conflict-free incidence coloring of $C_n$ in linear time. */

Input: The length $n$ of a cycle $C_n$;
Output: A conflict-free incidence $\chi^c_n(C_n)$-coloring $\varphi$ of $C_n$.

/* Vertices of $C_n$ are $v_1, v_2, \ldots, v_n$ in this ordering. */

if $n = 3$ then
  $\varphi(\text{Inc}(v_1 v_2)) \leftarrow \{1, 2\}$;
  $\varphi(\text{Inc}(v_2 v_3)) \leftarrow \{3, 4\}$;
  $\varphi(\text{Inc}(v_3 v_1)) \leftarrow \{5, 6\}$;
  return;

$p \leftarrow$ the quotient of $n$ divided by 2;
$r \leftarrow$ the remainder of $n$ divided by 2;

if $r = 0$ then
  $v_{2p+1} \leftarrow v_1$;
  for $i = 1$ to $2p$ do
    if $i \equiv 1 \pmod{2}$ then
      $\varphi(\text{Inc}(v_i v_{i+1})) \leftarrow \{1, 2\}$;
    else
      $\varphi(\text{Inc}(v_i v_{i+1})) \leftarrow \{3, 4\}$;
  end
else
  for $i = 1$ to $2p - 2$ do
    if $i \equiv 1 \pmod{2}$ then
      $\varphi(\text{Inc}(v_i v_{i+1})) \leftarrow \{1, 2\}$;
    else
      $\varphi(\text{Inc}(v_i v_{i+1})) \leftarrow \{3, 4\}$;
  end
  $\varphi(\text{Inc}(v_{2p-1} v_{2p})) \leftarrow \{1, 5\}$;
  $\varphi(\text{Inc}(v_{2p} v_{2p+1})) \leftarrow \{2, 3\}$;
  $\varphi(\text{Inc}(v_{2p+1} v_1)) \leftarrow \{4, 5\}$;
Algorithm 2 is

Theorem 2.3.

To analyze the complexity of the algorithm, we need look into its lines 4 and 11. If

we first show that

Proof. One can easily see that $C_n$ admits neither a conflict-free incidence 3-coloring for any integer $n \geq 3$, and nor a conflict-free incidence 4-coloring for any odd $n \geq 3$. Moreover, $C_3$ does not admit a conflict-free incidence 5-coloring. Hence Algorithm 1 outputs a conflict-free incidence coloring of $C_n$ using the least number of colors in linear time and the result follows.

We now pay attention to the $n$-order complete graph $K_n$. The famous result of Fiorini and Wilson [6] states that $K_n$ is of class 1 provided $n$ is even. Hence Proposition 2 directly imply the following.

Proposition 4. $\chi^c_i(K_{2n}) = 2\Delta(K_{2n}) = 4n - 2$.

Fiorini and Wilson [6] also showed that $K_n$ is of class 2 provided $n$ is odd, and thus Proposition 2 cannot be applied to such a $K_n$. Nevertheless, we can determine the conflict-free incidence chromatic number of $K_n$ with $n$ being odd from another view of point.

Proposition 5. If $G$ is the graph derived from $K_{2n+1}$ by removing less than $n/2$ edges, then $\chi^c_i(G) = 2\Delta(G) + 2 = 4n + 2$.

Proof. We first show that $\chi^c_i(G) \geq 4n + 2$. Suppose for a contradiction that $\varphi$ is a conflict-free incidence $(4n + 1)$-coloring of $G$. Since $G$ totally has more than $4n^2 + 2n - n = (4n + 1)n$ incidences, there is a color of $\varphi$, say 1, that has been used at least $n + 1$ times. Since every two strong incidences of a vertex are differently colored, there are $n + 1$ vertices of $G$, say $v_1, v_2, \ldots, v_{n+1}$, such that for each $1 \leq i \leq n + 1$, $\varphi(v_i, v_i u_i) = 1$, where $u_i$ is one neighbor of $v_i$. Since every two weak incidences of a vertex are also differently colored, each $u_i$ is different from every $u_j$ with $j \neq i$. If $u_i$ coincides with some $v_j$ with $j \neq i$, then $\varphi(v_i, v_i u_i) = \varphi(u_i, u_i u_j)$, a contradiction as $(v_i, v_i u_i)$ conflicts $(u_i, u_i u_j)$. Hence each $u_i$ is different from every $v_j$ with $j \neq i$. It follows that $V(G) \supseteq \bigcup_{i=1}^{n+1} \{u_i, v_i\}$ and thus $|V(G)| \geq 2n + 2$, a contradiction. To show the equality, we apply proposition 3 to $G$. It follows that $\chi^c_i(G) \leq 2\Delta(G) + 2 = 4n + 2$, as desired.

Combining Propositions 4 and 5 together, we conclude the following.

Theorem 2.3.

$$\chi^c_i(K_n) = \begin{cases} 2n - 2 & \text{if } n \text{ is even,} \\ 2n & \text{if } n \text{ is odd.} \end{cases}$$

We use the polygon method to construct an optimal conflict-free incidence coloring of $K_n$ by Algorithm 2. To analyze the complexity of the algorithm, we need look into its lines 4 and 11. If $n$ is even, then for each $1 \leq i \leq n - 1$, $E_i = \{v_{i-j}v_{i+j} | j = 1, \ldots, \frac{n-2}{2}\} \cup \{v_iv_n\}$ by line 4 where the subscripts are taken module $n$ and $v_0$ is recognized as $v_{n-1}$. If $n$ is odd, then for each $1 \leq i \leq n$, $E_i = \{v_{i-j}v_{i+j+1} | j = 0, 1, \ldots, \frac{n-2}{2}\}$ according to line 11 where the subscripts are taken module $n$ and $v_0$ is recognized as $v_n$. It follows that the complexity of Algorithm 2 is $O((n - 1)n/2) = O(n^2)$.
Algorithm 2: COLOR-COMPLETE-GRAPH(n)

/* This algorithm constructs an optimal conflict-free incidence coloring of \( K_n \) in quadratic time */

**Input:** The order \( n \) of a complete graph \( K_n \);

**Output:** A conflict-free incidence \( \chi^c_{ci}(K_n) \)-coloring \( \varphi \) of \( K_n \).

/* Vertices of \( K_n \) are \( v_1, v_2, \ldots, v_n \). */

1 if \( n \equiv 0 \pmod{2} \) then

2 \( G \leftarrow \) an \((n-1)\)-sided regular polygon formed by placing \( v_1, v_2, \ldots, v_{n-1} \) on a circle, with \( v_n \) at the center of the circle, and connecting every pair of vertices by straight line;

/* \( G \) now is a special drawing of \( K_n \) in the plane. */

3 for \( i = 1 \) to \( n - 1 \) do

4 \( E_i \leftarrow \) the set of all edges that lie on lines perpendicular to \( v_i v_n \) in \( G \) along with the edge \( v_i v_n \) itself;

5 for each edge \( e \in E_i \) do

6 \( \varphi(\text{Inc}(e)) \leftarrow \{2i - 1, 2i\} \);

7 else

8 \( G \leftarrow \) an \( n \)-sided regular polygon formed by placing \( v_1, v_2, \ldots, v_n \) on a circle and connecting every pair of vertices by straight line;

9 \( v_{n+1} \leftarrow v_1 \);

10 for \( i = 1 \) to \( n \) do

11 \( E_i \leftarrow \) the set of all edges that lie on lines parallel to \( v_i v_{i+1} \) in \( G \) along with the edge \( v_i v_{i+1} \) itself;

12 for each edge \( e \in E_i \) do

13 \( \varphi(\text{Inc}(e)) \leftarrow \{2i - 1, 2i\} \);


3 Outer-1-planar graphs

In this section we determine the conflict-free incidence chromatic numbers of outer-1-planar graphs, a subclass of planar partial 3-trees [1], which serve many applications ranging from network reliability to machine learning. Formally speaking, a graph is **outer-1-planar** if it can be drawn in the plane so that vertices are on the outer-boundary and each edge is crossed at most once. The notion of outer-1-planarity was first introduced by Eggleton [5] and outer-1-planar graphs are also known as **outerplanar graphs with edge crossing number one** [5] and **pseudo-outerplanar graphs** [19, 22, 27]. The coloring of outer-1-planar graphs were investigated by many authors including [3, 12, 14, 16, 19, 22–27].

The most popular result on the edge coloring of planar graphs is that planar graphs with maximum degree at least 7 is of class one [17, 21]. Since there exist class two planar graphs with maximum degree \( \Delta \) for each \( \Delta \leq 5 \),
the remaining problem is to determine whether every planar graph with maximum degree 6 is of class one, and this is still quite open (see survey [2]). Therefore, investigating the edge coloring of subclasses of planar graphs is natural and interesting. Fiorini [7] showed that every outerplanar graph is of class one if and only if it is not an odd cycle, and this conclusion had been generalized to the class of series-parallel graphs by Juvan, Mohar, and Thomas [11]. Zhang, Liu, and Wu [27] showed that outer-1-planar graphs with maximum degree at least 4 are of class one. The chromatic indexes of outer-1-planar graphs with maximum degree at most 3 was completely determined by Zhang [23].

We restate Zhang’s definition [23] as follows. Let $G_2, G_4, G_8$, and $H_t$ be configurations defined by Figure 1. For any solid vertex $v$ of a configuration and any graph $G$ containing such a configuration, the degree of $v$ in $G$ is exactly the number of edges that are incident with $v$ in the picture.

A graph belongs to the class $\mathcal{P}$, if it is isomorphic to $K_4^+$ (equal to $K_4$ with one edge subdivided) or derived from a graph $G \in \mathcal{P}$ by one of the following operations:

$G \sqcup_z G_t$ with $t = 2, 4, 8$ remove a vertex $z$ of degree two from $G$, and then paste a copy of $G_2$, or $G_4$, or $G_8$ on the current graph accordingly, by identifying $x$ and $y$ with $z_1$ and $z_2$, respectively, where $z_1$ and $z_2$ are the neighbors of $z$ (see Figure 2 for an example);

$G \vee_{z_1z_2} H_t$ with $t \geq 1$ remove an edge $z_1z_2$ from $G$, and then paste a copy of $H_t$ on the current graph by identifying $x_t$ and $y_t$ with $z_1$ and $z_2$, respectively (see Figure 3 for an example).

Let $\mathcal{P}^+$ be the class of connected outer-1-planar graphs with maximum degree 3 that contains some graph in $\mathcal{P}$ as a subgraph. Now we summarize the result of Zhang [23] and Zhang, Liu, and Wu [27] as follows.
Figure 2: The graph on the left shows $G$ and the one on the right shows $G \sqcup_2 G_2$

Figure 3: The graph on the left shows $G$ and the one on the right shows $G \vee_{z_1, z_2} H_1$

Theorem 3.1.
\[
\chi'(G) = \begin{cases} 
\Delta(G) & \text{if } G \notin \mathcal{P}^+ \text{ and } G \text{ is not an odd cycle}, \\
\Delta(G) + 1 & \text{otherwise},
\end{cases}
\]
if $G$ is a connected outer-1-planar graph.

Remark on Theorem 3.1: Zhang [23] claimed that every connected outer-1-planar graph with maximum degree 3 is of class one if and only if $G \notin \mathcal{P}$. However, this statement is incorrect. Indeed, Zhang showed that every graph in $\mathcal{P}$ is of class two. This further implies that every outer-1-planar graph with maximum degree 3 that contains some graph in $\mathcal{P}$ is of class two. In other words, every graph in $\mathcal{P}^+$ is of class two. Using the same proof of Theorem 3.3 in [23], one can show that if $G$ is a connected outer-1-planar graph with maximum degree 3 not in $\mathcal{P}^+$ then it is of class one (note that the minimal counterexample to this statement is 2-connected and thus Zhang’s original proof works now). Conclusively, every connected outer-1-planar graph with maximum degree 3 is of class one if and only if $G \notin \mathcal{P}^+$. Combining this with the result of Zhang, Liu, and Wu [27] that every outer-1-planar graph with maximum degree at least 4 is of class one, we have Theorem 3.1.

The following is an immediate corollary of Theorem 3.1 and Proposition 1.

Theorem 3.2. If $G$ is a connected outer-1-planar graph such that $G \notin \mathcal{P}^+$ and $G$ is not an odd cycle, then $\chi^c_i(G) = 2\Delta(G)$.

The next goal of this section is to prove $\chi^c_i(G) = 2\Delta(G) + 1$ if $G \in \mathcal{P}^+$ or $G$ is an odd cycle unless $G \cong C_3$. Theorem 2.2 supposes this conclusion while $G$ is an odd cycle of length at least 5. Hence in the following we assume that $G \in \mathcal{P}^+$. Note that $K^+_4$ is the smallest graph (in terms of the order) in $\mathcal{P}^+$. Now we prove $\chi^c_i(G) = 7$ for every graph $G \in \mathcal{P}^+$ by a series of lemmas.

Lemma 3.3. $\chi^c_i(K^+_4) = 7$. 

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Proof. Figure 4 shows a conflict-free incidence 7-colorable of $K^+_4$, so it is sufficient to show that 6 colors are not enough to create a conflict-free incidence coloring of $K^+_4$.

Figure 4: A conflict-free incidence 7-colorable of $K^+_4$

Suppose for a contradiction that $\varphi$ is a conflict-free incidence 6-coloring of $K^+_4$. Since $K^+_4$ has 7 edges and 14 incidences, there is a color, say 1, such that $\varphi(x_1, x_1x_1') = \varphi(x_2, x_2x_2') = \varphi(x_3, x_3x_3') = 1$. If $x_i = x_j$ or $x_i' = x_j'$ or $x_i = x_j'$ for some $1 \leq i < j \leq 3$, then $(x_i, x_i')$ and $(x_j, x_j')$ are conflicting and thus they cannot in a same color. Hence $|\{x_1, x_2, x_1', x_2', x_3, x_3'\}| = 6$, contradicting the fact that $|K^+_4| = 5$. □

From now on, if we say coloring a graph or a configuration we mean coloring its incidences so that every two conflicting ones receive distinct colors.

Lemma 3.4. If the configuration $G_2$ is colored with 6 colors under $\varphi$, then $\varphi(\text{Inc}(vx)) \cap \varphi(\text{Inc}(wy)) = \emptyset$.

Proof. If $\varphi$ is a conflict-free incidence 6-coloring of $G_2$, then $\varphi(u, uv), \varphi(v, uv), \varphi(u, uw), \varphi(w, uw), \varphi(v, vw)$ and $\varphi(w, vw)$ are pairwise distinct, so we assume, without loss of generality, that they are $1, 2, 3, 4, 5,$ and $6$, respectively. This forces that $\varphi(\text{Inc}(vx)) = \{3, 4\}$ and $\varphi(\text{Inc}(wy)) = \{1, 2\}$, as desired. □

Lemma 3.5. If the configuration $G_4$ is colored with 6 colors under $\varphi$, then $\varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset$.

Proof. If $\varphi$ is a conflict-free incidence 6-coloring of $G_4$, we have three cases: $\varphi(\text{Inc}(u_1x)) = \varphi(\text{Inc}(v_1y))$, or $\varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset$, or $|\varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y))| = 1$. If $\varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset$, then we win. So it is sufficient to show contradictions for another two cases. Without loss of generality, we assume $\varphi(\text{Inc}(u_1x)) = \{1, 2\}, \varphi(\text{Inc}(u_1v_0)) = \{3, 4\},$ and $\varphi(\text{Inc}(u_0u_1)) = \{5, 6\}$.

Case 1. $\varphi(\text{Inc}(u_1x)) = \varphi(\text{Inc}(v_1y))$.

Now $\varphi(\text{Inc}(v_1y)) \cup \varphi(\text{Inc}(u_0u_1)) = \{1, 2, 5, 6\}$ and $\varphi(\text{Inc}(v_1y)) \cup \varphi(\text{Inc}(v_0v_1)) = \{1, 2, 3, 4\}$ forces $\varphi(\text{Inc}(u_0v_1)) = \{3, 4\}$ and $\varphi(\text{Inc}(v_0v_1)) = \{5, 6\}$, respectively. It follows $\varphi(\text{Inc}(u_0u_1)) = \varphi(\text{Inc}(u_0v_0, v_0v_1)) = \{3, 4, 5, 6\}$ and thus $\varphi(\text{Inc}(v_0w)) = \varphi(\text{Inc}(v_0w)) = \{1, 2\}$, which is impossible.

Case 2. $|\varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y))| = 1$.

Assume, by symmetry, that $\varphi(\text{Inc}(v_1y)) = \{1, a\}$, where $a \in \{3, 4\}$. It follows that $\varphi(\text{Inc}(v_1y)) \cup \varphi(\text{Inc}(u_0u_1)) = \{1, a, 5, 6\}$, forcing $\varphi(\text{Inc}(v_0v_1)) = \{2, b\}$, $b \in \{3, 4\} \setminus \{a\}$. Now $\varphi(\text{Inc}(u_0v_1)) = \{2, b, 5, 6\}$ and $\varphi(\text{Inc}(v_1y)) \cup \varphi(\text{Inc}(u_0u_1)) = \{1, 2, 3, 4\}$, which implies $\varphi(\text{Inc}(v_0w)) = \{1, a\}$ and
\( \varphi(\text{Inc}(v_0v_1)) = \{5, 6\} \), respectively. It follows that \( \varphi(\text{Inc}(u_1v_0, v_0v_1, u_0w)) = \{1, 3, 4, 5, 6\} \) and thus \( \text{Inc}(uv_0) \)

have to be colored with 2, which is impossible. 

\[ \Box \]

**Lemma 3.6.** If the configuration \( G_8 \) is colored with 6 colors under \( \varphi \), then \( \varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y)) \) = \( \emptyset \).

**Proof.** If \( \varphi \) is a conflict-free incidence 6-coloring of \( G_8 \), we have three cases: \( \varphi(\text{Inc}(u_2x)) = \varphi(\text{Inc}(v_1y)) \), or \( \varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset \), or \( |\varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y))| = 1 \). If \( \varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset \), then we win. So it is sufficient to show contradictions for another two cases. Without loss of generality, we assume \( \varphi(\text{Inc}(v_1y)) = \{1, 2\}, \varphi(\text{Inc}(u_0v_1)) = \{3, 4\} \), and \( \varphi(\text{Inc}(v_0v_1)) = \{5, 6\} \).

**Case 1.** \( \varphi(\text{Inc}(u_2x)) = \varphi(\text{Inc}(v_1y)) \).

Now \( \varphi(\text{Inc}(u_2x)) \cup \varphi(\text{Inc}(v_0v_1)) = \{1, 2, 5, 6\} \) and \( \varphi(\text{Inc}(u_0v_0, v_0v_1)) = \{3, 4\} \) forces \( \varphi(\text{Inc}(u_2v_0)) = \{3, 4\} \) and \( \varphi(\text{Inc}(u_0v_0)) = \{1, 2\} \), respectively. It follows \( \varphi(\text{Inc}(u_0v_0, u_0v_1)) = \varphi(\text{Inc}(u_2x)) \cup \varphi(\text{Inc}(u_2v_0)) = \{1, 2, 3, 4\} \) and thus \( \varphi(\text{Inc}(u_0u_1)) = \varphi(\text{Inc}(u_1u_2)) = \{5, 6\} \), which is impossible.

**Case 2.** \( |\varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y))| = 1 \).

Assume, by symmetry, that \( \varphi(\text{Inc}(u_2x)) = \{1, a\} \), where \( a \in \{5, 6\} \). It follows that \( \varphi(\text{Inc}(u_0v_1)) \cup \varphi(\text{Inc}(v_0v_1)) = \{3, 4, 5, 6\} \) and \( \varphi(\text{Inc}(v_0v_1)) = \{3, 4\} \) forces \( \varphi(\text{Inc}(u_2v_0)) = \{1, 2\} \) and \( \varphi(\text{Inc}(u_2v_0)) = \{3, 4\} \). Now \( \varphi(\text{Inc}(u_0v_0, u_0v_1)) = \{1, 2, 3, 4\} \) which implies \( \varphi(\text{Inc}(u_0u_1)) = \{5, 6\} \). It follows that \( \varphi(\text{Inc}(u_0u_1, u_2x, u_2v_0)) = \{1, 3, 4, 5, 6\} \) and thus \( \text{Inc}(u_1u_2) \) have to be colored with 2, which is impossible. 

\[ \Box \]

**Lemma 3.7.** If the configuration \( H_t \) with some \( t \geq 1 \) is colored with 6 colors under \( \varphi \), then \( \varphi(\text{Inc}(x_{t-1}x_t)) = \varphi(\text{Inc}(y_{t-1}y_t)) \).

**Proof.** We prove it by induction on \( t \). If \( \varphi \) is a conflict-free incidence 6-coloring of \( H_1 \), then we assume, without loss of generality, \( \varphi(x', x'y'), \varphi(y', x'y'), \varphi(x', x'y'), \varphi(y_0, x'y_0), \varphi(x', x'x_0), \) and \( \varphi(x_0, x'x_0) \) are \( 1, 2, 3, 4, 5, \) and 6, respectively. Since \( \varphi(\text{Inc}(x'y', x'x_0)) = \{1, 2, 5, 6\} \) and \( \varphi(\text{Inc}(x'y', x'y_0)) = \{1, 2, 3, 4\} \), we have \( \varphi(\text{Inc}(x_0y')) = \{3, 4\} \) and \( \varphi(\text{Inc}(y_0y')) = \{5, 6\} \), which imply \( \varphi(\text{Inc}(x_0x_1)) = \varphi(\text{Inc}(y_0y_1)) = \{1, 2\} \). This completes the proof of the base case. Now suppose that the lemma holds for \( H_{t-1} \) with some \( t \geq 2 \) and prove that it also holds for \( H_t \). By the induction hypothesis, \( \varphi(\text{Inc}(x_{t-2}x_{t-1})) = \varphi(\text{Inc}(y_{t-2}y_{t-1})) \). This implies \( \varphi(\text{Inc}(x_{t-1}x_t)) = \{1, 2, 3, 4, 5, 6\} \setminus \{\varphi(\text{Inc}(x_{t-2}x_{t-1})) \cup \varphi(\text{Inc}(x_{t-1}y_{t-1}))\} \) and \( \varphi(\text{Inc}(y_{t-1}y_t)) = \{1, 2, 3, 4, 5, 6\} \setminus \{\varphi(\text{Inc}(y_{t-2}y_{t-1})) \cup \varphi(\text{Inc}(x_{t-1}y_{t-1}))\} \), and thus \( \varphi(\text{Inc}(x_{t-1}x_t)) = \varphi(\text{Inc}(y_{t-1}y_t)) \), as desired. 

\[ \Box \]

**Lemma 3.8.** If \( \varphi \) is a partial incidence coloring of the configuration \( G_2 \) such that \( \varphi(\text{Inc}(vx)) \cap \varphi(\text{Inc}(wy)) = \emptyset \), then \( \varphi \) can be extended to a conflict-free incidence 6-coloring of the configuration \( G_2 \).

**Proof.** Suppose \( \varphi(\text{Inc}(vx)) = \{1, 2\} \) and \( \varphi(\text{Inc}(wy)) = \{3, 4\} \). It is easy to see that we can extend \( \varphi \) to a conflict-free incidence 6-coloring of \( G_2 \) by coloring \( \text{Inc}(uw, uw, vw) \) so that \( \varphi(\text{Inc}(uv)) = \{3, 4\} \), \( \varphi(\text{Inc}(uw)) = \{1, 2\} \) and \( \varphi(\text{Inc}(vw)) = \{5, 6\} \). 

\[ \Box \]
Lemma 3.9. If \( \varphi \) is a partial incidence coloring of the configuration \( G_4 \) such that \( \varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset \), then \( \varphi \) can be extended to a conflict-free incidence 6-coloring of the configuration \( G_4 \).

Proof. Suppose \( \varphi(\text{Inc}(u_1x)) = \{1, 2\} \) and \( \varphi(\text{Inc}(v_1y)) = \{3, 4\} \). It is easy to see that we can extend \( \varphi \) to a conflict-free incidence 6-coloring of \( G_4 \) by coloring \( \text{Inc}(u_0v_1, v_0w, u_0w, u_1v_0, u_0u_1, v_0v_1) \) so that \( \varphi(\text{Inc}(u_0v_1)) = \varphi(\text{Inc}(v_0w)) = \{1, 2\}, \varphi(\text{Inc}(u_0w)) = \varphi(\text{Inc}(u_1v_0)) = \{3, 4\}, \) and \( \varphi(\text{Inc}(u_0u_1)) = \varphi(\text{Inc}(v_0v_1)) = \{5, 6\} \).

Lemma 3.10. If \( \varphi \) is a partial incidence coloring of the configuration \( G_8 \) such that \( \varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset \), then \( \varphi \) can be extended to a conflict-free incidence 6-coloring of the configuration \( G_8 \).

Proof. Suppose \( \varphi(\text{Inc}(u_2x)) = \{1, 2\} \) and \( \varphi(\text{Inc}(v_1y)) = \{3, 4\} \). We can extend \( \varphi \) to a conflict-free incidence 6-coloring of \( G_4 \) by coloring the incidences on \( v_0v_1, u_0u_1, u_0v_0, u_1u_2, u_0v_1, \) and \( u_2v_0 \) so that \( \varphi(\text{Inc}(u_0v_1)) = \varphi(\text{Inc}(u_0u_1)) = \{1, 2\}, \varphi(\text{Inc}(u_0v_0)) = \varphi(\text{Inc}(u_1u_2)) = \{3, 4\}, \) and \( \varphi(\text{Inc}(u_0v_1)) = \varphi(\text{Inc}(u_2v_0)) = \{5, 6\} \).

Lemma 3.11. If \( \varphi \) is a partial incidence coloring of the configuration \( H_t \) with some \( t \geq 1 \) such that \( \varphi(\text{Inc}(x_{t-1}x_t)) = \varphi(\text{Inc}(y_{t-1}y_t)) \), then \( \varphi \) can be extended to a conflict-free incidence 6-coloring of the configuration \( H_t \).

Proof. We prove it by induction on \( t \). If \( \varphi \) is a partial incidence coloring of the configuration \( H_1 \) such that \( \varphi(\text{Inc}(x_0x_1)) = \varphi(\text{Inc}(y_0y_1)) = \{1, 2\} \), then \( \varphi \) can be extended to a conflict-free incidence 6-coloring of \( H_t \) by coloring \( \text{Inc}(x'y', x'y_0, x_0y', x'x_0, y'y_0) \) so that \( \varphi(\text{Inc}(x'y')) = \{1, 2\}, \varphi(\text{Inc}(x'y_0)) = \varphi(\text{Inc}(x_0y')) = \{3, 4\}, \) and \( \varphi(\text{Inc}(x'x_0)) = \varphi(\text{Inc}(y'y_0)) = \{5, 6\} \). This completes the proof of the base case. Now suppose that the lemma holds for \( H_{t-1} \) with some \( t \geq 2 \) and prove that it also holds for \( H_t \). Assume, without loss of generality, that \( \varphi(\text{Inc}(x_{t-1}x_t)) = \varphi(\text{Inc}(y_{t-1}y_t)) = \{1, 2\} \). We extend \( \varphi \) by coloring \( \text{Inc}(x_{t-2}x_{t-1}, y_{t-2}y_{t-1}, x_{t-1}y_{t-1}) \) so that \( \varphi(\text{Inc}(x_{t-2}x_{t-1})) = \varphi(\text{Inc}(y_{t-2}y_{t-1})) = \{3, 4\} \) and \( \varphi(\text{Inc}(x_{t-1}y_{t-1})) = \{5, 6\} \). This constructs a partial incidence coloring of the configuration \( H_{t-1} = H_t - \{x_{t-1}y_{t-1}, x_{t-1}x_t, y_{t-1}y_t\} \) such that \( \varphi(\text{Inc}(x_{t-2}x_{t-1})) = \varphi(\text{Inc}(y_{t-2}y_{t-1})) \). Since any incidence of \( \text{Inc}(x_{t-1}y_{t-1}, x_{t-1}x_t, y_{t-1}y_t) \) is conflict-free to any incidence of \( \text{Inc}(H_{t-1}) \), by the induction hypothesis, the extended \( \varphi \) can be further extended to a conflict-free incidence 6-coloring of the configuration \( H_t \).

Proposition 6. If \( G \in \mathcal{P} \), then \( \chi^*_5(G) = 7 \).

Proof. We proceed by induction on \( |G| \). Since the smallest graph in \( \mathcal{P} \) is \( K_4^+ \), and \( \chi^*_5(K_4^+) = 7 \) by Lemma 3.3, the proof of the base case has been done. Now assume \( |G| > 5 \). By the construction of \( \mathcal{P} \), we meet four cases. Here and elsewhere, once \( G \) contains a configuration as shown in Figure 11, we use the same labelling of any vertex appearing on the configuration as the one marked in the corresponding picture.

Case 1. There is a graph \( G' \in \mathcal{P} \) and a degree 2 vertex \( z \) of \( G' \) such that \( G = G' \cup z G_2 \) (or \( G = G' \cup z G_4 \), or \( G = G' \cup z G_8 \), respectively).
By the induction hypothesis, $\chi^c_i(G') = 7$. Let $z_1, z_2$ be two neighbors of $z$ in $G'$ and let $\varphi$ be a conflict-free incidence 7-coloring of $G'$. Clearly, $\varphi(\text{Inc}(zz_1)) \cap \varphi(\text{Inc}(zz_2)) = \emptyset$. We construct a conflict-free incidence 7-coloring $\phi$ of $G$ as follows. Let $\phi(\text{Inc}(vx)) = \varphi(\text{Inc}(zz_1))$ and $\phi(\text{Inc}(wy)) = \varphi(\text{Inc}(zz_2))$ (or $\phi(\text{Inc}(u_1x)) = \varphi(\text{Inc}(zz_1))$ and $\phi(\text{Inc}(v_1y)) = \varphi(\text{Inc}(zz_2))$, respectively). This makes a partial incidence coloring of the configuration $G_2$ (or $G_4$, or $G_8$, respectively) such that $\varphi(\text{Inc}(vx)) \cap \varphi(\text{Inc}(wy)) = \emptyset$ (or $\varphi(\text{Inc}(u_1x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset$, or $\varphi(\text{Inc}(u_2x)) \cap \varphi(\text{Inc}(v_1y)) = \emptyset$, respectively). By Lemma 3.8 (or Lemma 3.9 or Lemma 3.10, respectively), $\phi$ can be extended to a conflict-free incidence 7-coloring of the incidences of $I(E(G) \setminus E(G'))$ receive distinct colors. Now for every edge $e \in E(G) \cap E(G')$, let $\phi(\text{Inc}(e)) = \varphi(\text{Inc}(e))$. This completes a 7-coloring of the incidences of $G$ and it is easy to check that this coloring is conflict-free.

On the other hand, we show that $G$ admits no conflict-free incidence 6-coloring. Suppose, for a contradiction, that $\phi$ is a conflict-free incidence 6-coloring of $G$. By Lemma 3.4 (or Lemma 3.5 or Lemma 3.6, respectively), $\phi(\text{Inc}(vx)) \cap \phi(\text{Inc}(wy)) = \emptyset$ (or $\phi(\text{Inc}(u_1x)) \cap \phi(\text{Inc}(v_1y)) = \emptyset$, or $\phi(\text{Inc}(u_2x)) \cap \phi(\text{Inc}(v_1y)) = \emptyset$, respectively). This makes us possible to construct a conflict-free incidence 6-coloring $\varphi$ of $G'$ by setting $\varphi(\text{Inc}(zz_1)) = \phi(\text{Inc}(vx)), \varphi(\text{Inc}(zz_2)) = \phi(\text{Inc}(wy))$, (or $\varphi(\text{Inc}(zz_1)) = \phi(\text{Inc}(u_1x)), \varphi(\text{Inc}(zz_2)) = \phi(\text{Inc}(v_1y))$, or $\varphi(\text{Inc}(zz_1)) = \phi(\text{Inc}(u_2x)), \varphi(\text{Inc}(zz_2)) = \phi(\text{Inc}(v_1y))$, respectively) and $\varphi(\text{Inc}(e)) = \phi(\text{Inc}(e))$ for every edge $e \in E(G') \setminus E(G)$. This is a contradiction.

Case 2. There is a graph $G' \in \mathcal{P}$ and an edge $z_1z_2$ of $G'$ such that $G = G' \cup z_1z_2 H_i$.

By the induction hypothesis, $\chi^c_i(G') = 7$. Let $\varphi$ be a conflict-free incidence 7-coloring of $G'$. We construct a conflict-free incidence 7-coloring $\phi$ of $G$ as follows. Let $\phi(\text{Inc}(x_{i-1}x_i)) = \phi(\text{Inc}(y_{i-1}y_i)) = \varphi(\text{Inc}(z_1z_2))$. This makes a partial incidence coloring of the configuration $H_i$ such that $\phi(\text{Inc}(x_{i-1}x_i)) = \phi(\text{Inc}(y_{i-1}y_i))$. By Lemma 3.11, $\phi$ can be extended to a conflict-free incidence 7-coloring of the configuration $H_i$. Now for every edge $e \in E(G) \cap E(G')$, let $\phi(\text{Inc}(e)) = \varphi(\text{Inc}(e))$. This completes a 7-coloring of the incidences of $G$ and it is easy to check that this coloring is conflict-free.

On the other hand, we show that $G$ admits no conflict-free incidence 6-coloring. Suppose, for a contradiction, that $\phi$ is a conflict-free incidence 6-coloring of $G$. By Lemma 3.7, $\phi(\text{Inc}(x_{i-1}x_i)) = \phi(\text{Inc}(y_{i-1}y_i))$. This makes us possible to construct a conflict-free incidence 6-coloring $\varphi$ of $G'$ by setting $\varphi(\text{Inc}(z_1z_2)) = \phi(\text{Inc}(x_{i-1}x_i))$ and $\varphi(\text{Inc}(e)) = \phi(\text{Inc}(e))$ for every edge $e \in E(G') \setminus E(G)$. This is a contradiction.

Algorithm 3 summarises the idea of proving Theorem 6, showing how we can construct a conflict-free incidence 7-coloring of a graph in $\mathcal{P}$ efficiently. Now we are ready to prove a more general result as follows.

**Theorem 3.12.** If $G \in \mathcal{P}^+$, then $\chi^c_i(G) = 7$.

**Proof.** We proceed by induction on $|G|$. Note that the base case is supported by Lemma 3.3. By the definition of $\mathcal{P}$, every graph in $\mathcal{P}$ has exactly one vertex of degree 2, besides which all vertices are of degree 3. By Proposition 6, we assume $G \in \mathcal{P}^+ \setminus \mathcal{P}$.
Algorithm 3: COLOR-CLASS-P(G)

**Input:** A graph $G \in \mathcal{P}$;

**Output:** A conflict-free incidence 7-coloring $\varphi$ of $G$.

1. $i \leftarrow 0;
2. G_0 \leftarrow G; 
3. \textbf{while } G_i \not\simeq K_4^- \textbf{ do }
4. \quad \textbf{if there is a graph } G' \in \mathcal{P} \text{ with a degree 2 vertex } z \text{ such that } G_i = G' \sqcup_z G_t \text{ for some } t \in \{2, 4, 8\}; 
5. \quad \text{ then }
6. \quad \quad G_{i+1} \leftarrow G';
7. \quad \quad \text{sign}_i \leftarrow t;
8. \quad \textbf{else }
9. \quad \quad \text{Find a graph } G' \in \mathcal{P} \text{ with an edge } z_1 z_2 \text{ such that } G_i = G' \lor z_1 z_2 \text{ for some integer } t; 
10. \quad \quad G_{i+1} \leftarrow G'; 
11. \quad \quad \text{sign}_i \leftarrow 0; 
12. \quad i \leftarrow i + 1;

/* We obtain a series $G_0, G_1, \ldots, G_i$ of graphs in $\mathcal{P}$ where $G_0 = G$ and $G_i = K_4^-$. */

13. Construct a conflict-free 7-coloring $\varphi_i$ of $G_i$ by Lemma 3.3;
14. \textbf{for } j = i - 1 \textbf{ to } 0 \textbf{ do }
15. \quad Extend $\varphi_{j+1}$ to a conflict-free 7-coloring $\varphi_j$ of $G_j$ by Lemma 3.8, 3.9, 3.10, or 3.11 whenever sign$_j$ equals to 2, 4, 8, or 0, respectively;
16. \quad $\varphi \leftarrow \varphi_0$;

Suppose that $G$ contains a graph $H \in \mathcal{P}$ as a proper subgraph. Let $u$ be the unique vertex of degree 2 of $H$ and let $v$ and $w$ be the two neighbors of $u$ in $H$. Since $\Delta(G) \leq 3$ and $G$ is connected, the degree of $u$ in $G$ must be 3. Let $x$ be the third neighbor of $u$ in $G$. Since every vertex in $V(H) \setminus \{u\}$ has degree 3 in $H$ (and thus in $G$), $u$ is a cut-vertex of $G$.

Let $H'$ be the subgraph of $G$ containing $u$ such that $V(H') \cap V(H) = \{u\}$ and $V(H') \cup V(H) = V(G)$. Since $u$ has degree 1 in $H'$, $H'$ is not an odd cycle. Therefore, if $H' \in \mathcal{P}^+$, then $\chi_c^G(H') = 7$ by the induction hypothesis, and if $H' \not\in \mathcal{P}^+$, then $\chi_c'(H') = \Delta(H') \leq 3$ by Theorem 3.1 and thus $\chi_c^G(H') \leq 6$ by Proposition 1. In each case, there is a conflict-free incidence 7-coloring $\phi'$ of $H'$.

Since $H \in \mathcal{P}$, there is a conflict-free incidence 7-coloring $\phi$ of $H$ by Proposition 6. We permute (if necessary) the colors of $\phi$ so that $\phi(Inc(uv)), \phi(Inc(uw))$, and $\phi'(Inc(ux))$ are pairwise disjoint, and then obtain a conflict-free incidence 7-coloring of $G$ by combining $\phi'$ with $\phi$. This implies $\chi_c^G(H) \leq 7$.

On the other hand, $\chi_c^G(G) \geq \chi_c^G(H) = 7$. Hence $\chi_c^G(G) = 7$. \qed
Algorithm 4: COLOR-CLASS-P-PLUS($G$)

**Input:** A graph $G \in \mathcal{P}^+$;  
**Output:** A conflict-free incidence 7-coloring $\varphi$ of $G$.

1. if $G \in \mathcal{P}$ then  
2. \hspace{1em} COLOR-CLASS-P($G$);  
   \hspace{1em} /* The coloring outputted by line 2 is denoted by $\varphi$. */  
3. else  
4. \hspace{1em} Find a subgraph $H \in \mathcal{P}$ of $G$ with a vertex $u$ that has exactly two neighbors $v$ and $w$ in $H$;  
5. \hspace{1em} $H' \leftarrow$ the graph with vertex set $V(G) \setminus (V(H) \setminus \{u\})$ and edge set $(E(G) \setminus E(H)) \cup \{ux\}$;  
6. \hspace{1em} $x \leftarrow$ the unique neighbor of $u$ in $H'$;  
7. \hspace{1em} if $H' \in \mathcal{P}^+$ then  
8. \hspace{2em} COLOR-CLASS-P-PLUS($H'$);  
   \hspace{2em} /* The coloring outputted by line 8 is denoted by $\varphi'$. */  
9. else  
10. \hspace{2em} Find a proper edge 3-coloring $\varphi'$ of $H'$ by Theorem 3.1;  
11. \hspace{2em} for each edge $e \in H'$ do  
12. \hspace{3em} $\varphi'(\text{Inc}(e)) \leftarrow \{\varphi'(e), \varphi'(e) + 3\}$;  
13. \hspace{1em} COLOR-CLASS-P($H$);  
   \hspace{1em} /* The coloring outputted by line 13 is denoted by $\phi$. */  
14. \hspace{1em} Exchange (if necessary) the colors of $\phi$ so that $\phi(\text{Inc}(uw)), \phi'(\text{Inc}(uw))$, and $\varphi'(\text{Inc}(ux))$ are pairwise disjoint;  
15. \hspace{1em} $\varphi \leftarrow$ the coloring obtained via combing $\phi'$ with $\phi$;

Algorithm 4 shows the idea of constructing a conflict-free incidence 7-coloring of a given graph in $\mathcal{P}^+$. Now that we have Theorems 2.2, 3.2, and 3.12 the conflict-free incidence chromatic number of connected outer-1-planar graphs (and thus all outer-1-planar graphs) can be completely determined by Theorem 3.13. Algorithm 5 shows an approach to efficiently construct a conflict-free incidence $\chi_i(G)$-coloring $\varphi$ of a connected outer-1-planar graph $G$.

**Theorem 3.13.**

$$
\chi_i^c(G) = \begin{cases} 
6 & \text{if } G \cong C_3, \\
2\Delta(G) & \text{if } G \not\in \mathcal{P}^+ \text{ and } G \text{ is not an odd cycle}, \\
2\Delta(G) + 1 & \text{otherwise}
\end{cases}
$$

for every connected outer-1-planar graph $G$. 

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Algorithm 5: COLOR-O1P($G$)

**Input:** A connected out-1-planar graph $G$;

**Output:** A conflict-free incidence $\chi_i(G)$-coloring $\varphi$ of $G$.

/* This algorithm constructs an optimal conflict-free incidence coloring of a connected outer-1-planar graph $G$. */

1. if $G$ is a cycle then
   2. COLOR-CYCLE($|G|$);
3. else
4.   if $G \in \mathcal{P}^+$ then
5.     COLOR-CLASS-P-PLUS($G$);
6.   else
7.     Find a proper edge $\Delta(G)$-coloring $\phi$ of $G$ by Theorem 3.1
8.     for each edge $e \in G$ do
9.        $\varphi(\text{Inc}(e)) \leftarrow \{\phi(e), \phi(e) + \Delta(G)\}$;

4 Open problem

To end this paper, we leave an open problem relative to the complexity of the conflict-free incidence coloring. As one can know from Proposition 3 that $\chi^c_i(G) \in \{2\Delta(G), 2\Delta(G) + 1, 2\Delta(G) + 2\}$ for every simple graph $G$, an interesting problem is to investigate the complexity of the following question.

**CONFLICT-FREE INCIDENCE COLORING PROBLEM (CFICP)**

Input: A graph $G$ and a positive integer $k$.

Question: Is there a conflict-free incidence $k$-coloring of $G$?

We conjecture that CFICP is NP-Complete.

References

[1] C. Auer, C. Bachmaier, F. J. Brandenburg, A. Gleißner, K. Hanauer, D. Neuwirth, and J. Reislhuber, Recognizing outer 1-planar graphs in linear time, Graph drawing, 2013, pp. 107–118. MR3162015

[2] Y. Cao, G. Chen, G. Jing, M. Stiebitz, and B. Toft, Graph edge coloring: a survey, Graphs Combin. 35 (2019), no. 1, 33–66. MR3898375

[3] Q. Chen, Adjacent vertex distinguishing total colorings of outer 1-planar graphs, J. Combin. Math. Combin. Comput. 108 (2019), 221–230. MR3967165

[4] R. Diestel, Graph theory (fifth edition), Springer Berlin Heidelberg, Berlin, Heidelberg, 2017. ↑2, 3

[5] R. B. Eggleton, Rectilinear drawings of graphs, Utilitas Math. 29 (1986), 149–172. MR846198

[6] S. Fiorini and R. J. Wilson, Edge-colourings of graphs, Research Notes in Mathematics, No. 16, Pitman, London; distributed by Fearon-Pitman Publishers, Inc., Belmont, Calif., 1977. MR0543798

[7] S. Fiorini, On the chromatic index of outerplanar graphs, J. Combinatorial Theory Ser. B 18 (1975), 35–38. MR0366724
