Vacuum solutions with nontrivial boundaries for the Einstein-Gauss-Bonnet theory

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The classification of certain class of static solutions for the Einstein-Gauss-Bonnet theory in vacuum is presented. The spacelike section of the class of metrics under consideration is a warped product of the real line with a nontrivial base manifold. For arbitrary values of the Gauss-Bonnet coupling, the base manifold must be Einstein with an additional scalar restriction. The geometry of the boundary can be relaxed only when the Gauss-Bonnet coupling is related with the cosmological and Newton constants, so that the theory admits a unique maximally symmetric solution. This additional freedom in the boundary metric allows the existence of three main branches of geometries in the bulk, containing new black holes and wormholes in vacuum.

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The basic principles of General Relativity in dimensions higher than four give rise to the Lovelock theories of gravity [1], with the most general covariant, divergence free second order field equations for the metric. In three and four dimensions, the Lovelock theory reduces to General Relativity with cosmological constant, while in five and six dimensions it corresponds to the so-called Einstein-Gauss-Bonnet (EGB) theory, whose field equations contain quadratic powers of the curvature in a very precise combination. In \(d\) dimensions, the field equations of the EGB theory read

\[
\mathcal{E}_a := \epsilon_{ab_1...b_{d-1}} \left[ (d-4) \alpha_2 R^{b_1b_2} R^{b_3b_4} + (d-2) \alpha_1 R^{b_1b_2} e^{b_3} e^{b_4} + d \alpha_0 e^{b_1} e^{b_2} e^{b_3} e^{b_4} \right] e^{b_5}...e^{b_{d-1}} = 0 \quad (1)
\]

where \(e^a\) is the vielbein and \(R^{ab}\) stands for the curvature two-form. The last two terms in (1) correspond to the Einstein tensor and the cosmological term, respectively, so that General Relativity is recovered when \(\alpha_2 = 0\). This theory possesses a wide spectrum of solutions in vacuum [2,3], including static wormholes [4,5] and black holes.
with nontrivial boundaries, black p-branes, solutions with NUT charge, nontrivial torsion, spontaneous compactifications and metrics with a nontrivial jump in the extrinsic curvature.

For generic values of the couplings, the theory admits two maximally symmetric solutions, which merge to a single one provided the Gauss-Bonnet coupling in Eq. (1) fulfills

$$\alpha_2 = \frac{(d-2)^2}{4d(d-4)} \alpha_1^2 \alpha_0 .$$

In Ref. 5, vacuum solutions for the five-dimensional EGB theory were classified within the following class of metrics:

$$ds^2 = -f^2(r) dt^2 + \frac{dr^2}{g^2(r)} + r^2 d\Sigma_{d-2}^2 ,$$

where $d\Sigma_{d-2}$ is the line element of an arbitrary $(d-2)$-dimensional base manifold which determines the geometry at the boundary. Here we extend the analysis of Ref. 5 to dimensions greater than five. For simplicity we focus on the six-dimensional case, which captures the main features of the entire classification. The extension of these results to higher dimensions is performed in detail in Ref. 23.

1. Classification of the six-dimensional case

1.1. Generic case

For arbitrary values of the coupling constants, the most general solution in vacuum for the EGB equation in six dimensions within the family of metrics given by

$$f^2(r) = g^2(r) = \gamma + \frac{\alpha_1}{\alpha_2} \left[ 1 \pm \sqrt{\left( 1 - 3 \frac{\alpha_2 \alpha_0}{\alpha_1^2} \right) + \frac{\mu}{r^2} + \frac{\alpha_2^2}{\alpha_1^2} \left( \gamma^2 + \xi \right) } \right] ,$$

where the base manifold $\Sigma_4$ must be Einstein, i.e. $\tilde{R}^{ij} = 3 \gamma \delta^{ij}$, with the following scalar condition:

$$\tilde{R}^{ijkl} \tilde{R}_{ijkl} - 4 \tilde{R}^{ij} \tilde{R}_{ij} + \tilde{R}^2 + 24 \xi = 0 ,$$

where $\tilde{R}^{ijkl}$ and $\tilde{R}_{ij}$ are the Riemann and Ricci tensors of $\Sigma_4$, respectively, and $\tilde{R}$ corresponds to its Ricci scalar. This last condition means that the Euler density of $\Sigma_4$ must be constant. Thus, assuming the base manifold $\Sigma_4$ to be compact without boundary, integration of Eq. 5 on $\Sigma_4$ gives a topological restriction on the base manifold, which reads $\xi = -\frac{1}{2} \frac{\pi^2}{12} \frac{\chi(\Sigma_4)}{\chi}$. Here $\chi(\Sigma_4)$ is the Euler characteristic of the base manifold and $\chi$ stands for its volume. Note that the term proportional to $r^{-4}$ inside the square root of 4 does not appear in the Boulware-Deser solution since it vanishes if and only if the base manifold is of constant curvature. It is worth pointing out that this term severely modifies the asymptotic behavior of the metric. Depending on the value of the parameters, this spacetime can describe black holes being asymptotically locally (A)dS or flat.
1.2. Special case

In six dimensions, in the special case in which the Gauss-Bonnet coupling is given by (2), the solutions split into four main branches according to the geometry of the base manifold:

**First branch:** The base manifold $\Sigma_4$ has the same restrictions as in the generic case, i.e., it is Einstein and satisfies (5). The entire metric can describe black holes with nontrivial boundaries. The results of Ref[9] are recovered if we further restrict to base manifolds of constant curvature.

The special case, however, allows the possibility of relaxing the condition that the base manifold be Einstein, as long as

$$g^2(r) = \sigma r^2 + \gamma, \quad \sigma := \frac{3a_0}{\alpha_1},$$

$$\tilde{R}^{ij}_{\ kl} \tilde{R}^{kl}_{\ ij} - 4 \tilde{R}^{ij} \tilde{R}^{ij} + \tilde{R}^2 - 4 \gamma \tilde{R} + 24 \gamma^2 = 0,$$

case in which we get the three remaining branches:

**Second branch (Special class of black holes):** If we do not impose further requirements on $\Sigma_4$ besides Eq.(7), the solution is given by Eqs. (3) and (6) with $f^2 = g^2$.

**Third branch (Wormholes and spacetime horns):** For compact base manifolds without boundary satisfying (7) and having a constant Ricci scalar, $\tilde{R} = 12 \gamma$, the integration of (7) on $\Sigma_4$ assuming $\gamma = \pm 1$ gives $\chi(\Sigma_4) = \frac{3}{4} V_4$, where $\chi(\Sigma_4)$ is the Euler characteristic of the base manifold, which, being proportional to its volume $V_4$, cannot be negative. In this case the metric is given by Eq.(3) with $g^2(r) = \sigma r^2 + \gamma$ and

$$f^2(r) = \begin{cases} 
\left(a \sqrt{\sigma r^2 - 1} + 1 - \sqrt{\sigma r^2 - 1} \tan^{-1}\left(\frac{1}{\sqrt{\sigma r^2 - 1}}\right)\right)^2 : \gamma = -1 \\
\left(a \sqrt{\sigma r^2 + 1} + 1 - \sqrt{\sigma r^2 + 1} \tanh^{-1}\left(\frac{1}{\sqrt{\sigma r^2 + 1}}\right)\right)^2 : \gamma = 1 
\end{cases},$$

where $a$ is an integration constant. It is simple to show that, for negative cosmological constant ($\sigma > 0$) and $\gamma = -1$, the spacetime can be extended to describe a static wormhole solution in vacuum$^{23}$, provided $a^2 < \frac{\gamma}{\sigma}$. In the case $\gamma = 0$, $\tilde{R} = 0$ and, for compact $\Sigma_4$ without boundary, $\chi(\Sigma_4) = 0$. In this case the solution is given by (3), with $g^2(r) = \sigma r^2$ and $f(r) = \left(a \sqrt{\sigma r} + \frac{1}{\sqrt{\sigma r}}\right)$. If $\sigma > 0$ and $a \geq 0$ this looks like a “spacetime horn”.

**Fourth branch:** If $\Sigma_4$ is a constant curvature manifold, the field equations degenerate and $f$ can be an arbitrary function.

In sum, we have classified the solutions of the EGB theory in vacuum within the class of metrics (3). In the generic case, the base manifold must be Einstein with constant Ricci curvature. This scalar condition is written in (5), and it can be seen as the six-dimensional analogue of the restriction on the squared Weyl tensor found in Ref. [24]. It was shown that the base manifold acquires additional freedom...
only for the special case (2), where new wormholes in vacuum and black holes with nontrivial boundaries are found. The extension of these results to higher dimensions involves, among further details, a new parameter that characterizes the geometry of the boundary.

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