Non-extensive statistical mechanics of a self-gravitating gas

L. F. Escamilla-Herrera, C. Gruber, V. Pineda-Reyes, and H. Quevedo

1Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Mexico City 04510, México.
2Hanse-Wissenschaftskolleg Delmenhorst, Germany
3Institut für Physik, Universität Oldenburg, D-26111 Oldenburg, Germany
4Dipartimento di Fisica and ICRA, Università di Roma “La Sapienza”, I-00185 Roma, Italy
5Institute of Experimental and Theoretical Physics, Al-Farabi Kazakh National University, Almaty, Kazakhstan

(Dated: October 8, 2018)

The statistical mechanics of a cloud of particles interacting via their gravitational potentials is an old problem which encounters some issues when the traditional Boltzmann–Gibbs statistics is applied. In this article, we consider the generalized statistics of Tsallis and analyze the statistical and thermodynamical implications for a self-gravitating gas, obtaining analytical and convergent expressions for the equation of state and specific heat in the canonical as well as microcanonical ensembles. Although our results are comparable in both ensembles, it turns out that only in the canonical case the thermodynamic quantities depend explicitly on the non-extensivity parameter, indicating that the question of ensemble equivalence for Tsallis statistics must be further reviewed.

PACS numbers: 05.20.Jj, 95.30.Sf, 95.30.Tg
Keywords: Statistical Mechanics, Self-gravitating systems, Tsallis non-extensive statistics

I. INTRODUCTION

Statistical mechanics of many-body systems and the connection with principles and notions of thermodynamics have been a powerful tool in physics for a long time, and have established themselves as a very robust framework continuing to hold even in the face of big theoretical revolutions and paradigm shifts. However, systems where gravity is the dominant interaction have always posed a challenge even for thermodynamics, especially since the introduction and acceptance of general relativity. The inclusion of gravitational forces into statistical models, indeed representing only an example for the general problem of long-range forces in physics, has led very early on to puzzles or incompatibilities with well-established thermodynamic notions. The Tolman-Ehrenfest effect is a very good illustration – considering a compact object made up of a perfect fluid described by a spherical metric, it turns out that – supposing an isolated system – a stationary (non-changing) state can be achieved, but that thermodynamically this does not correspond to a state of constant temperature, as would intuitively be expected from a non-changing state of a system. Instead of a constant temperature throughout the object, a constant flow of temperature is achieved, an invariable gradient of temperature supported by the gravitational forces within the compact object, the inner parts being hotter than the outer parts. This corresponds to what is called the gravothermal catastrophe, which occurs in systems with negative heat capacities, such as is the case in gravitating fluids. The possibility of negative heat capacities was already considered very early by Maxwell. A negative heat capacity means that adding energy to the system leads to a cooling on it, and vice versa, a system becomes hotter by giving energy to a colder reservoir. Thus, the inner parts of a compact object heat up by losing energy to the cooler, outer parts, which can expand uncontrollably while the core becomes denser, and in principle there is nothing that can counteract this reaction until the gravitational collapse of the core, i.e., the gravothermal catastrophe. These considerations show that thermodynamical principles (and the statistical mechanical methods behind them) have to be reconsidered in the presence of gravitational (or generally, long-range) forces.

The problem of a self-gravitating gas has been considered in the literature before, within the framework of traditional Boltzmann-Gibbs (BG) statistics. In Ref. a very detailed discussion of the topic was presented by Padmanabhan. Ref. gives a broad and well-founded introduction to the problems arising in the analysis of the thermodynamics of gravitating systems as well, approaching the issue from several angles and introducing and evaluating several possible advances. One of the established problems of the thermodynamics of gravitational systems is with the thermodynamic limit, i.e.,

\[ N \to \infty, V \to \infty, \text{ and } N/V = \text{const}. \]  

In this limit, thermodynamic functions like the temperature or the heat capacity do not exist due to divergences, and thus the thermodynamics of the system cannot be calculated. A possible way to resolve the issues with the thermodynamic limit is to modify it. In Ref. de Vega and Sanchez extensively analyze the self-gravitating gas in BG statistics and show that thermodynamic functions

\[ ... \]
exist in the so-called dilute limit, where

\[ N \to \infty , V \to \infty , \text{ and } N/V^{1/3} = \text{const}. \]  
\[ (2) \]

In this case, the linear dimension \( L = V^{1/3} \) of the system scales with \( N \) instead of its volume, and thus the way in which \( N \) and \( V \) are taken to infinity with respect to each other is changed in the thermodynamic limit. However, while the dilute limit \( (2) \) is applied in \[ 13 \] in the definition of appropriate dimensionless variables it is disregarded in the equation of state, where the limit \( (1) \) is employed. Due to these conflicting applications of thermodynamic limits, and the general question as to what is the correct statistical mechanics picture to analyze a gravitating system, we would like to explore the option of alternative statistics.

In this work, we are thus going to describe a self-gravitating system using a generalized statistical approach as proposed by Tsallis, and explore the consequences for thermodynamic quantities and the thermodynamic limit of the system.

Tsallis statistics has first been introduced in the 1980’s (see e.g. \[ 14–17 \]) following the idea that for systems with long-range interactions such as gravity, but also complex systems with short-range interactions such as glasses or systems with dissipation, the statistical analysis within the framework of BG statistics should be replaced with a more suitable generalized statistics. In general, BG statistics works very well for systems with short-range interactions, Markovian processes and ergodic systems, with equal probability of each microstate in the system. For certain systems, as for example such involving long-range forces, however, ergodicity is not guaranteed, and strictly speaking, the BG ensemble statistics and its way to count microstates considering equiprobability of all states does not apply in these cases. Long-range interactions can cause the energy and the entropy of a system to be non-additive, as is well-known in the case of black holes, where the entropy scales with the surface area of the black hole, and not its volume, thus making the system non-extensive. A generalized statistics takes into account these peculiarities, and defines alternative notions of thermodynamic quantities, which then can be interpreted in the traditional BG sense. Also the definition of heat capacities, which turn out negative in conventional BG statistics, and other relevant thermodynamic response functions, should be revised. The statistics proposed by Tsallis is also known as non-extensive generalized statistics.

This paper is structured as follows. In Section \[ II \] we will introduce the physical system we are investigating, clarify important notions and quantities of a self-gravitating gas of free particles, and go on to review the basic ideas and properties of the Tsallis non-extensive statistics which will be used to analyze the thermodynamics of the system. We also comment on previous works presenting the case of an ideal gas within Tsallis statistics. In Section \[ III \] and \[ IV \] we will discuss in detail the statistical mechanics and thermodynamics of the system in the canonical and microcanonical ensembles, respectively, deriving the most important thermodynamic equation of state variables and response functions, and calculating them in the limit of a weak gravitational field, i.e., in the dilute regime. In Section \[ V \] we will comment on the connection between the two ensembles and the issue of ensemble equivalence, as well as conclude and summarize our work.

II. PRINCIPAL NOTIONS OF THE PHYSICAL SYSTEM AND THE STATISTICAL ANALYSIS

In this section, we will introduce the physical model of a self-gravitating gas as a system of \( N \) particles interacting with each other gravitationally; and we will define and clarify the basic notions and properties of such a system. We will also review the most important ideas and definitions of the Tsallis non-extensive statistics, such as the Tsallis probability distribution function, the generalized entropy \( S_q \), the definition of thermodynamic ensembles in this statistics and the physically relevant thermodynamic quantities and their interpretation.

A. Self-gravitating gas

The self-gravitating gas is a system of \( N \) particles interacting with each other only via Newtonian gravity, as analyzed e.g. in Ref. \[ 13 \]. We write the potential as the superposition of the interaction between pairs of particles, also known as pairwise approximation, thus treating the potential as a series of two-body problems instead of an \( N \)-body problem. At short distances, a repulsive particle interaction is assumed. The interaction potential between two single particles in the system is thus proposed as

\[-\frac{1}{|\mathbf{q}_i - \mathbf{q}_j|} = \begin{cases} \frac{1}{|\mathbf{q}_i - \mathbf{q}_j|}, & |\mathbf{q}_i - \mathbf{q}_j| \geq A, \\ +1/A, & |\mathbf{q}_i - \mathbf{q}_j| \leq A, \end{cases} \]
\[ (3) \]

and \( A \ll L \) is a repulsive short-distance cut-off, small compared to the size \( L \) of the system. The presence of the repulsive short-range interaction introduces a finite size of the particles, and prevents the unphysical case of the collapse of the system into one point. It was shown in previous works \[ 13 \] that the parameter \( A \) can be taken to zero safely in the quantities we consider.

The Hamiltonian of such a system is the sum of the kinetic term \( \mathcal{T} \) of all constituents and the interaction potential \( \mathcal{U} \),

\[ \mathcal{H} = \mathcal{T} + \mathcal{U} = \sum_{i=1}^{N} \frac{p_i^2}{2m} - Gm^2u(|\mathbf{q}_i - \mathbf{q}_j|); \]
\[ (4) \]

where \( G \) is the gravitational constant, and \( m \) the mass of an individual particle. The potential \( u(|\mathbf{q}_i - \mathbf{q}_j|) \) has
been defined as
\[ u(|q_i - q_j|) = \sum_{1 \leq i < j \leq N} \frac{1}{|q_i - q_j|}. \] (5)

For the sake of simplicity, we will assume the system to be contained in a cubic box of side length \( L \), which will ultimately be taken in the limit \( L \to \infty \), or \( V \to \infty \), respectively.

The long-range nature of the gravitational interactions leads to integrals which cannot be resolved analytically, and thus we will apply approximate methods for a weak gravitational field, considering a dilute gas cloud, which will enable us to obtain the results for thermodynamic quantities as a combination of the ideal gas contribution and small gravitational corrections.

### B. Tsallis statistics

The generalized statistics proposed by Tsallis [14–17] has been introduced in order to take into account systems with non-extensive thermodynamic behaviour. This generalization of the usual BG statistics revises, among other things, the notion of entropy by modifying the way of counting microstates. Let us first discuss the usual notion of BG entropy, and then show the generalization to Tsallis’ non-extensive entropy. Entropy in BG statistics is defined using Shannon’s formula as

\[ S_{BG} = -k_B \sum_{i=1}^{\Omega} p_i \ln p_i, \] (6)

where \( p_i \) is the probability of the \( i \)-th microstate, and the usual normalization condition applies to the sum of all probabilities,

\[ \sum_{i=1}^{\Omega} p_i = 1. \] (7)

Assuming equiprobability, i.e., that each microstate is equally probable, \( p_i = 1/\Omega \), the well-known BG entropy

\[ S_{BG} = k_B \ln \Omega \] (8)

is obtained. The BG entropy is, amongst other things, extensive, which means that when adding up the entropies of two independent systems \( A \) and \( B \), the result is the direct sum of both,

\[ S_{BG}(A + B) = S_{BG}(A) + S_{BG}(B). \] (9)

In Tsallis’ generalized statistics, the entropy is defined via a generalization of the logarithmic function, the so-called q-logarithm,

\[ \ln_q x = \frac{x^{1-q} - 1}{1 - q}. \] (10)

This function is actually a power-law of the variable \( x \), containing a free parameter \( q \), and reduces to the usual natural logarithm in the limit \( q \to 1 \). Using this q-logarithm, a generalized q-entropy can be defined as

\[ S_q = -k_B \sum_{i=1}^{\Omega} p_i^q \ln_q p_i = \frac{1 - \sum_{i=1}^{\Omega} p_i^q}{1 - q}. \] (11)

This entropy, in contrast to the BG entropy, is non-extensive. When adding the entropies of two independent systems \( A \) and \( B \), the result is not simply the sum of the two entropies, but also includes a cross-term,

\[ S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B). \] (12)

This cross-term is associated with the interaction between the two systems, is labeled by the so-called non-extensivity parameter \( q \) and makes the q-entropy non-extensive. Depending on the value of the parameter \( q \), the q-entropy is superextensive for \( q < 1 \), or subextensive for \( q > 1 \), while for \( q = 1 \) the extensivity of the BG form is recovered. Introduced as a free parameter, \( q \) should be assigned a specific value depending on the system in question, and its properties with respect to the type of interactions between the constituents. In the presence of gravitational interaction, \( q < 1 \) should be expected, i.e., a superextensive entropy. This can be illustrated in the simple case of a Schwarzschild black hole, where the entropy of a system with the mass \( M_1 + M_2 \) is higher than the sum of the entropies of two black holes with mass \( M_1 \) and \( M_2 \), and has as well been argued in the case of two particular astrophysical scenarios in Ref. [12].

As the generalized logarithm, also a generalized q-exponential function can be defined [16] as its inverse,

\[ \exp_q(x) = \left[ 1 + (1 - q)x \right]^{1/q}, \] (13)

which equally recovers the usual exponential function for \( q \to 1 \). Calculations become more involved for these generalized functions, as some of the usual identities for exponential and logarithm change, such as the q-exponential of a sum of arguments or the q-logarithm of a product of arguments. For a collection of identities regarding the q-calculus, see the appendix of Ref. [16].

Besides the generalized entropy, also other notions have to be generalized. Depending on the ensemble, different quantities such as the internal energy, the free energy, temperature or pressure are obtained in an analogous way to the BG statistics, but based on the Tsallis statistics and its respective distribution functions.

Notions change only slightly for the microcanonical ensemble. The entropy of the system is calculated from Eq. (11). After the maximization of the entropy under usual constraints such as constant internal energy \( E \), volume \( V \) and number of particles \( N \), the equilibrium distribution turns out to be a constant, i.e., an equiprobability distribution, and so in analogy to the BG case the entropy is [16]

\[ S_q = k_B \ln_q \Omega, \] (14)
with \( \Omega \) being the number of possible microstates of the system, just as in BG statistics. From this entropy, the thermodynamic equations of state are calculated and lead to the Tsallis temperature as
\[
T_q = \frac{1}{k_B \beta} = \left( \frac{\partial S_q}{\partial E} \right)_{V,N}^{-1},
\] (15)
and the Tsallis pressure as
\[
P_q = T_q \left( \frac{\partial S_q}{\partial V} \right)_{E,N}. \] (16)

Here, \( \beta \) is the inverse temperature directly derived from the Tsallis temperature \( T_q \). However, based on considerations of a generalized notion of equilibrium (thermal and mechanical), derived from the generalized rule of addition, it is argued that instead of the intensive Tsallis variables, more physical physical definitions of intensive variables should be interpreted as the true intensive quantities of the system, i.e.,
\[
T^* = \left( 1 + \frac{1 - q}{k_B} S_q \right) \left( \frac{\partial S_q}{\partial E} \right)_{V,N}^{-1} =: \frac{1}{k_B \beta_q}, \] (17)
and
\[
P^* = \frac{T^*}{1 + [(1 - q)/k_B] S_q} \left( \frac{\partial S_q}{\partial V} \right)_{E,N}. \] (18)

Here, the inverse physical temperature is defined as \( \beta_q \), and the factor of proportionality between \( \beta \) and \( \beta_q \) is commonly referred to as \( c \), i.e.,
\[
c = 1 + \frac{1 - q}{k_B} S_q, \] (19)
and
\[
\beta_q = \frac{\beta}{c}. \] (20)

We will thus calculate these physical (starred) thermodynamic variables instead of the original Tsallis ones in our analyses.

In contrast to the microcanonical ensemble, where everything is derived from the entropy and only thereafter a partition function is introduced, in the canonical ensemble everything originates from the canonical partition function \( Z_q \). These partition functions for microcanonical or canonical ensembles are defined in analogy to their respective BG partition functions, with slight modifications, and will be introduced in the analyses in sections and . In the canonical ensemble, also extensive variables are referred to as Tsallis variables, e.g. the free energy \( F_q \). The entropy is connected to the canonical partition function via
\[
S_q = k_B \ln q \, Z_q, \] (21)
and also the Tsallis extensive variables have definitions involving their physical intensive counterparts, such as the physical free energy \( F^* = -k_B T^* \ln q \) (22). From this definition, the physical pressure can be derived in the case of the canonical ensemble as
\[
P^* = -\left( \frac{\partial F^*}{\partial V} \right)_T, \] (23)
i.e., the derivative of the physical free energy with respect to the volume at constant physical temperature. This is the alternative derivation of the pressure for the canonical ensemble. In Section V we will comment on this question in the context of our own calculations for microcanonical and canonical ensemble.

Additionally, the thermodynamic limit is modified in Tsallis’ statistics. In usual BG statistics, extensive quantities are scaled with respect to the total number of particles according to, and intensive quantities do not scale at all, but remain constant in all of the system for the thermodynamic limit \( N, V \to \infty \). In Tsallis statistics, the notions of extensive and intensive variables are generalized to three classes: extensive variables should be differentiated into real extensive quantities, such as entropy and volume, which indeed scale with the number of particles in the system, and pseudo-extensive ones, like the internal energy or the free energy, which should be scaled with a modified factor of \( N N^* \) in the thermodynamic limit. Here, \( N^* \propto N^{1 - 1/\alpha} \) with \( \alpha \) being the exponent of the interaction potential (in the case of gravity \( \alpha = 1 \)), and \( d \) the spatial dimension of the system (in this case \( d = 3 \)), and the proportionality constant is a combination of \( \alpha \) and \( d \). The scaling of pseudo-extensives thus takes into account the particular type of long-range interaction and other properties of the system. In the case of the self-gravitating gas, \( N^* \propto N^{2/3} \). Intensive quantities should then be considered pseudo-intensive, and scale with the factor \( N^* \). The thermodynamic limit in our case for a large system with \( N \to \infty \) thus will be carried out for the extensive as
\[
\lim_{N, V \to \infty} \frac{S}{N} \frac{V}{N} = \text{const.}, \] (24)
for the pseudo-extensives as
\[
\lim_{N \to \infty} \frac{F^*}{N N^*} = \text{const}, \] (25)
and for the pseudo-intensives as
\[
\lim_{N \to \infty} \frac{T^*}{N^*}, \frac{P^*}{N^*} = \text{const.} \] (26)

Previous works exist on the case of ideal gas of particles analyzed via Tsallis statistics. In the case of the canonical ensemble of the ideal gas, we obtain coincidence between their results and ours in the limit.
of zero gravity. In the case of the microcanonical ensemble however, we differ from [21] in methodology. In order to achieve convergent results, [21] promoted the parameter \( q \) to a thermodynamic variable, scaling as \( 1/N \) in the thermodynamic limit. We do not adopt this idea, but retain \( q \) as a non-determined constant parameter in the theory, and thus do not reproduce the results of [21] for the microcanonical ensemble in the limit of zero gravity.

### III. Tsallis Non-Extensive Statistical Mechanics of the Self-Gravitating Gas: The Canonical Ensemble

We first consider the canonical ensemble for the Tsallis analysis of the self-gravitating gas, and start with the Tsallis canonical partition function. The generalization to \( q \)-statistics consists in the use of the \( q \)-exponential of the Hamiltonian [19] in the integral over coordinate and momentum space,

\[
Z_q = \frac{1}{N! h^{3N}} \int d^{3N} q d^{3N} p \exp_q (-\beta_q \mathcal{H}(p, q)), \tag{27}
\]

where \( \beta_q \), the inverse physical temperature, is used as the Lagrange multiplier associated with the energy constraint in Tsallis’ nonextensive statistical mechanics [18]. This quantity is functionally different from \( \beta \), the Lagrange multiplier associated with the energy constraint in BG statistical mechanics, and was introduced in [20].

The integral can be evaluated via the introduction of the complex plane representation of the Gamma function [22],

\[
\frac{1}{\Gamma(y)} = \frac{i}{2\pi} x^{1-y} \int_C (-t)^{-y} e^{-it} dt, \tag{28}
\]

for non-integer \( y \), with the identifications

\[
x = 1 - (1 - q) \beta_q \mathcal{H} \quad \text{and} \quad y - 1 = \frac{1}{1 - q}. \tag{29}
\]

This substitution holds for \( q < 1 \), i.e., superextensive entropies such as is the case for a self-gravitating gas. The integral over \( d^{3N} p \) can then be calculated using \( DN \)-dimensional polar coordinates, and leads to

\[
Z_q = \frac{\Gamma\left(\frac{3-q}{1-q}\right)}{N! h^{3N}} \left(\frac{2\pi m}{(1-q)\beta_q}\right)^{3N/2} \frac{i}{2\pi} \int_C dt (-t)^{-3N/2} e^{-t \int d^{3N} q e^{\ell(1-q)\beta_q \mathcal{U}}}, \tag{30}
\]

where \( \mathcal{U} \) has been defined in Eq. (4). In order to simplify the integral over the coordinates \( q_i \), they are rescaled such that the limits of the integration do not depend on the size of the box, i.e., \( q_i = L r_i \). Thus, the integral measure becomes

\[
d^3 q_i = L^3 d^3 r_i. \tag{31}
\]

Further, we will introduce a new dimensionless variable \( \eta \),

\[
\eta = \frac{G m^2 N \beta_q}{L}, \tag{32}
\]

which is constructed to be a truly intensive quantity remaining constant in the Tsallis thermodynamic limit as defined in (23)–(26). It is the same variable \( \eta \) as used in [13], and under their modified thermodynamic limit, \( \eta \) remains a constant as well.

Using these substitutions, the partition function can be written as

\[
Z_q = \frac{V^N \Gamma\left(\frac{2-q}{1-q}\right)}{N! h^{3N}} \left(\frac{2\pi m}{(1-q)\beta_q}\right)^{3N/2} \frac{i}{2\pi} \int_C dt (-t)^{-3N/2} e^{-t \int d^{3N} r e^{\eta u(|r_1 - r_j|)}}, \tag{33}
\]

where \( u(|r_1 - r_j|) \) is the interaction potential [3] as defined before, but in terms of the new variable \( r \), and for brevity we have introduced

\[
\eta_q = \left(-\frac{2m}{N}\right) \frac{1}{1-q} \eta. \tag{34}
\]

In order to be able to further process this expression, in particular to carry out the coordinate integrals, we now resort to an approximation approach, considering the regime of weak gravity, i.e., a dilute self-gravitating gas, to expand the interaction potential \( u(|r_1 - r_j|) \) and the exponential term.

#### A. Dilute regime

The dilute regime assumes a weakly gravitating gas with low density. The introduced quantity \( \eta_q \) remains constant in the Tsallis thermodynamic limits [23]–[26], but can be assumed to be small in the dilute regime, i.e., \( \eta_q \ll 1 \). In that case, a Taylor expansion of the interaction potential can be carried out, i.e., up to second order,

\[
e^{\eta_q u(|r_1 - r_j|)} \approx 1 + \frac{\eta_q u + \frac{1}{2} \eta_q^2 u^2}{2}. \tag{35}
\]

Furthermore, restricting the potential interactions between \( N \) particles to a sum of identical two-body interactions, the potential can be written as

\[
u(|r_1 - r_j|) = \frac{1}{|r_1 - r_2|} [(N - 1) + (N - 2) + \ldots + 1] \quad = \frac{1}{|r_1 - r_2|} \sum_{k=1}^N (N - k). \tag{36}
\]

In the limit \( N \to \infty \) this results in

\[
u(|r_1 - r_j|) = \frac{N(N - 1)}{2|r_1 - r_2|}. \tag{37}
\]
The quadratic term in Eq. (35) can be simplified in a similar manner, assuming that the contributing terms stem from the interaction between two independent particles squared, from interactions between one particle separately with two others, or from interactions between two pairs of particles with four independent particles. Thus, the second order interaction in Eq. (35) can be written as

\[
u(|r_1 - r_2|)^2 = \frac{N(N-1)}{2|1 - r_1|} + \frac{N(N-1)(N-2)}{|1 - r_1| |1 - r_2|} + \frac{N(N-1)(N-2)(3-N)}{4|1 - r_1| |1 - r_2| |1 - r_3| |1 - r_4|}. \tag{38}\]

Plugging all of these things into the coordinate integral of Eq. (33), up to second order we end up with

\[e^\eta \nu(|r_1 - r_2|) \approx 1 - \beta N^2 + t^2 N^2 B, \tag{39}\]

where we introduced

\[A = \frac{\eta(1-q)}{2N} \int \left( 1 - \frac{1}{N} \right), \tag{40}\]

\[B = \frac{\eta^2(1-q)^2}{2N^2} \left[ b_0 \left( 1 - \frac{1}{N} \right) \left( 1 - \frac{2}{N} \right) \left( 1 - \frac{3}{N} \right) + b_1 \left( 1 - \frac{1}{N} \right) + b_2 \left( 1 - \frac{1}{N} \right) \left( 1 - \frac{2}{N} \right) \right]. \tag{41}\]

We are now interested in the thermodynamic limit \(\nu \rightarrow \infty\) under consideration of the Tsallis rules for the limit \((q \rightarrow 1)\). As mentioned before, the quantity \(\eta\) was constructed to remain constant in this limit. Considering the arguments of the Gamma functions, we identify

\[b_0 = \int_0^1 d^3r_1 d^3r_2 \frac{1}{|r_1 - r_2|}, \tag{42}\]

\[b_0^2 = \int_0^1 d^3r_1 d^3r_2 d^3r_3 d^3r_4 \frac{1}{|r_1 - r_2| |r_3 - r_4|}, \tag{43}\]

\[b_1 = \int_0^1 d^3r_1 d^3r_2 d^3r_3 \frac{1}{|r_1 - r_2| |r_3 - r_4|}, \tag{44}\]

\[b_2 = \int_0^1 d^3N r_1 d^3N r_2 \frac{1}{|r_1 - r_2|^2}. \tag{45}\]

These coefficients are simple expressions which can be computed numerically and depend on the symmetry of the box containing the system. They do not carry any dependence on variables anymore. The short-distance cutoff as introduced in [22] needs to be considered only in these coefficients, and it has been shown in [13] that its influence on the numerical results can be neglected. With these substitutions, we can compute the integral over the auxiliary variable \(t\) as in [22], leading to the partition function in the dilute regime up to second order as

\[Z_q \approx \left[ \frac{2\pi m}{(1-q)b_q} \right]^{3N/2} \frac{V^N}{N! h^{3N}} \frac{\Gamma \left( \frac{2-q}{1-q} \right)}{\Gamma \left( \frac{2-q}{1-q} + \frac{3N}{2} \right)} \left[ 1 + AN^2 \frac{\Gamma \left( \frac{2-q}{1-q} + \frac{3N}{2} \right)}{\Gamma \left( \frac{2-q}{1-q} + 3 \frac{3N}{2} \right)} + BN^2 \frac{\Gamma \left( \frac{2-q}{1-q} + \frac{3N}{2} \right)}{\Gamma \left( \frac{2-q}{1-q} + 3 \frac{3N}{2} \right)} \right]. \tag{46}\]

Already here we can see that in the limit of \(\eta = 0\), i.e., in the absence of gravitational interaction, \(A = B = 0\), the partition function recovers the result of the ideal gas in Tsallis statistics according to [24].

\[Z_q^{(IG)} = \left[ \frac{2\pi m}{(1-q)b_q} \right]^{3N/2} \frac{V^N}{N! h^{3N}} \frac{\Gamma \left( \frac{2-q}{1-q} \right)}{\Gamma \left( \frac{2-q}{1-q} + 2 \right)}. \tag{47}\]

Note that the result does not recover the partition function of the ideal gas in BG statistics – there is an additional fraction of two \(\Gamma\)-functions. The limit \(q \rightarrow 1\) of the above expression [24] does not straightforwardly reproduce the desired result. We thus argue that the limit \(q \rightarrow 1\) can not be taken at any point in these calculations, particularly since the introduction of the integral transformations according to [22] is not applicable in the case \(q = 1\). Our result for \(Z_q^{(IG)}\) does however coincide with the result in [20] for the ideal gas in the \(q\)-calculus, and further defined the gravitational integrals as

\[b_0 = \int_0^1 d^3r_1 d^3r_2 \frac{1}{|r_1 - r_2|}, \tag{42}\]

\[b_0^2 = \int_0^1 d^3r_1 d^3r_2 d^3r_3 d^3r_4 \frac{1}{|r_1 - r_2| |r_3 - r_4|}, \tag{43}\]

\[b_1 = \int_0^1 d^3r_1 d^3r_2 d^3r_3 \frac{1}{|r_1 - r_2| |r_3 - r_4|}, \tag{44}\]

\[b_2 = \int_0^1 d^3N r_1 d^3N r_2 \frac{1}{|r_1 - r_2|^2}. \tag{45}\]
to second order as
\[ \ln(1 + x) \approx x - \frac{x^2}{2!}. \] (51)

With this, we can express the thermodynamic limit of the expression \( \ln Z_q^{(grav)}/N \) as
\[ \lim_{N \to \infty} \frac{1}{N} \ln Z_q^{(grav)} \simeq \frac{\eta(1 - q)b_0}{2} + \eta^2(1 - q)^2 \left( \frac{b_1}{2} - \frac{b_0^2}{2} \right). \] (52)

In the following, we calculate thermodynamically relevant quantities, such as the equation of state and heat capacity in the dilute regime.

**B. Thermodynamic equation of state and heat capacity**

In the canonical ensemble, the physical temperature \( T^* \), or equivalently, the inverse physical temperature \( \beta_\eta \), is fixed due to the energy constraint, and enters the calculation as a constant Lagrange multiplier. In contrast, the physical pressure \( P^* \) can be calculated from the physical free energy \( F^* \), by derivation of the canonical partition function as obtained in the dilute regime, Eq. (46). The physical pressure divided by the physical temperature is thus
\[ \frac{P^*}{k_B T^*} = \left( \frac{\partial \ln Z_q}{\partial V} \right)_{T^*}, \] (53)
i.e., the derivative of the logarithm of the partition function, taken at constant \( \beta_\eta \). The derivative of the ideal gas part of the partition function yields
\[ \frac{\partial \ln Z_q^{(IG)}}{\partial V} = \frac{N}{V}, \] (54)
thus recovering the analog of the ideal gas equation of state in the limit of zero gravitational interactions, i.e., the equation of state of the ideal gas in the Tsallis statistics, using the Tsallis variables \( T^* \) and \( P^* \). The equation of state of the gravitating gas can be written as
\[ \frac{P^* V}{Nk_B T^*} = 1 - \eta \frac{\partial}{\partial \eta} \ln Z_q^{(grav)} \]. (55)

Substituting Eq. (52) into Eq. (55), interchanging the thermodynamic limit and the derivative with respect to \( \eta \), allows us to compute the thermodynamic limit of the equation of state. This limit can be taken without any additional factors of \( N \) on the left hand side, since the weights of \( P^* \) and \( T^* \) cancel each other, and also the ratio \( V/N \) represents already the correct thermodynamic limit. The equation of state up to second order thus results in
\[ \frac{P^* V}{Nk_B T^*} \simeq 1 - \frac{\eta(1 - q)b_0}{6} - \frac{\eta^2(1 - q)^2}{3} \left( b_1 - b_0^2 \right). \] (56)

This result is the same as obtained in [13], considering the differences in the definition of the \( b_i \) and the use of the Tsallis variables instead of the usual BG ones. Moreover, there is an explicit dependence on the parameter \( q \).

While the equation of state is useful in order to study the particular features of any thermodynamic system, relevant information often appears when studying the second derivatives of a certain thermodynamic potential, the so-called response functions. Thus it is also instructive to consider the specific heat capacity at constant volume, defined in the Tsallis statistics as
\[ \frac{(c_V)_q}{k_B} \simeq \frac{3}{2} + \eta^2(1 - q)^2 \left( b_1 - b_0^2 \right). \] (57)

In the case of zero gravity, \( \eta = 0 \), the result of the ideal gas is reproduced in the analogous Tsallis framework, since the definition of the canonical ensemble includes the use of Tsallis temperature. The specific heat capacity for the ideal gas is positive, and the gravitational corrections show that it remains positive also in the presence of gravitational forces. This is qualitatively the same result as obtained in [13], where the heat capacity was not calculated explicitly, but it was shown that it is always positive for the canonical ensemble.

**IV. TSALLIS NON-EXTENSIVE STATISTICAL MECHANICS OF THE SELF-GRAVITATING GAS: THE MICROCANONICAL ENSEMBLE**

To have a more complete perspective of the self-gravitating gas in Tsallis non-extensive statistical mechanics, it is desirable to explore other ensembles in order to compare the obtained thermodynamic properties. As in regular BG statistics, the microcanonical ensemble is the most fundamental one, where the system is considered to be closed in all regards. The microcanonical entropy of the system is given as in Eq. (14), as the q-logarithm of the number of accessible microstates of the system,
\[ S_q = k_B \ln_q \Omega(E, V, N), \] (59)
for a given energy \( E \), volume \( V \) and particle number \( N \) of the gas [16]. These microstates are restricted by the energy constraints imposed by the Hamiltonian \( \mathcal{H} \).

We follow the standard procedure to obtain the microstates of the system. In general, the number of total microstates for a specific energy constraint is given as
\[ \omega(E) = \int_{\mathcal{H} \leq E} d^N q \, d^N p. \] (60)
Thus, the density of microstates at an energy $E$ is given by

$$
\Omega(E) = \frac{1}{N! h^{3N}} \frac{\partial \omega(E)}{\partial E},
$$

(61)

Using the Hamiltonian $H$, the momentum integral can be calculated as

$$
\int_{H \leq E} d^{3N} p = \frac{\pi^{3N/2}}{\Gamma\left(\frac{3N}{2} + 1\right)} \left[2m(E-U)\right]^{3N/2},
$$

(62)

which is the volume of a $3N$-dimensional hypersphere of radius $2m(E-U)$, and $U$ has been defined in $[4]$. The number of microstates for an energy $E$ is then given by derivation of (62) as

$$
\Omega = \frac{(2\pi m)^{3N/2}}{N! h^{3N} \Gamma\left(\frac{3N}{2} + 1\right)} \int d^{3N} q \left[E - U\right]^{3N/2-1}.
$$

(63)

For an ideal gas, i.e., without any particle interactions, the number of microstates can be computed from Eq. (63) assuming $U = 0$ as

$$
\Omega^{(IG)} = \frac{(2\pi m)^{3N/2} V N}{N! h^{3N} \Gamma\left(\frac{3N}{2} + 1\right)} E^{3N/2-1}.
$$

(64)

Therefore, it is useful to separate the number of microstates in two different contributions,

$$
\Omega = \Omega^{(IG)} \cdot Z^{(grav)},
$$

(65)

one from the ideal gas behaviour of the kinetic term $\Omega^{(IG)}$, and one due to the gravitational interactions. $Z^{(grav)}$ is the configuration integral containing the gravitational interaction, a sort of microcanonical partition function, given by

$$
Z^{(grav)} = \frac{1}{V N E^{3N/2-1}} \int d^{3N} q \left[E - U((q_i - q_j))\right]^{3N/2-1}.
$$

(66)

In order to carry out the integral over the coordinates $q_i$, we introduce the same rescaled coordinate $r_i$ as in the case of the canonical ensemble, c.f. Eq. (61). Further, we will introduce a new dimensionless variable $\chi$,

$$
\chi = \frac{EL}{Gm^2 N^2},
$$

(67)

which is an intensive quantity constructed in such a way that it remains constant in the Tsallis thermodynamic limit as defined by Eqs. (24)–(26). Again, it is the same variable as $\chi$ in Ref. [13], where it was a constant under their modified thermodynamic limit. With these substitutions, the configuration integral can be rewritten as

$$
Z^{(grav)} = \chi^{3N/2} \int_0^1 \cdots \int_0^1 d^{3N} r_i \left[\chi + \frac{1}{N^2} u(|q_i - q_j|)\right]^{3N/2-1},
$$

(68)

leading to the Tsallis entropy

$$
S_q = k_B \ln \left[\Omega^{(IG)} \cdot Z^{(grav)}\right].
$$

(69)

We will now introduce again the case of weak gravity, before we then calculate thermodynamic state functions such as the equation of state and the heat capacity.

A. Dilute regime

As in the canonical ensemble, we will now define the regime of weak gravity, i.e., the case of low densities, and then consider the thermodynamic limit. Let us rewrite $Z^{(grav)}$ in the form

$$
Z^{(grav)} = \int_0^1 \cdots \int_0^1 d^{3N} r_i \left[1 + \frac{1}{\chi N^2} \sum_{1 \leq i < j \leq N} \frac{1}{|r_i - r_j|}\right].
$$

(70)

The dilute limit, where $\chi \gg 1$, implies $1/\chi \ll 1$, and thus we use a Taylor expansion up to second order as

$$
(1 + x)^{3N-1} \approx 1 + \left(\frac{3N}{2} - 1\right) x + \left(\frac{3N}{2} - 1\right) \left(\frac{3N}{2} - 2\right) \frac{x^2}{2!},
$$

(71)

for the argument

$$
x = \frac{1}{\chi N^2} u(|r_i - r_j|).
$$

(72)

Thus, we have up to second order,

$$
\left[1 + \frac{u(\cdot)}{\chi N^2}\right]^{3N-1} \approx 1 + \left(\frac{3N}{2} - 1\right) \frac{u(\cdot)}{\chi N^2} + \frac{1}{2} \left(\frac{3N}{2} - 1\right) \left(\frac{3N}{2} - 2\right) \frac{u(\cdot)^2}{\chi^2 N^4}.
$$

(73)

The potential $u(\cdot) = u(|r_i - r_j|)$, as defined in [5], can be reduced in this regime using the same symmetry arguments presented for the canonical ensemble from Eq. (36) through Eq. (38), in order to obtain the corresponding contribution to each term of the expansion up to the second order in inverse $\chi$. Therefore, the same virial coefficients $b_0, b_2, b_4$ and $b_6$ appear for the microcanonical ensemble, as defined earlier. The logarithm of $Z^{(grav)}$ in the limit $N \to \infty$ up to second order in $1/\chi$ is then written as

$$
\lim_{N \to \infty} \frac{1}{N} \ln Z^{(grav)} \simeq -\frac{9b_0}{2\chi} + \frac{9}{8\chi^2} (b_1 - 42b_2^2).
$$

(74)

This result can be used in the following in order to simplify the computation of the thermodynamic state functions.
B. Thermodynamic equation of state and heat capacity

We proceed to compute thermodynamic quantities from the entropy of the system according to the definitions of physical thermodynamic quantities given in the introduction [13, 20] – the temperature [17] and the pressure [18], to be combined to yield the equation of state, as done in the canonical ensemble. We will first calculate the equations of state in an abstract form, before investigating it in the dilute regime to obtain concrete expressions to compare to the results of the canonical ensemble.

In order to calculate the physical temperature, it is necessary to compute the derivative of the microcanonical Tsallis entropy \( S_q \) (75), given by

\[
\left( \frac{\partial S_q}{\partial E} \right)_{V,N} = k_B \left[ \Omega^{(IG)} \cdot Z^{(grav)} \right]^{1-q} \cdot \left[ \frac{\partial}{\partial E} \ln \Omega^{(IG)} + \frac{\chi}{E} \frac{\partial}{\partial \chi} \ln Z^{(grav)} \right].
\]

Due to the properties of the q-logarithm, the prefactor in these expressions can be identified as

\[
\left[ \Omega^{(IG)} \cdot Z^{(grav)} \right]^{1-q} = 1 + \frac{1 - q}{k_B} S_q = c,
\]

i.e., the prefactor from the derivation cancels with the prefactor that should be included into the physical thermodynamic variables [17] and [18]. Substituting relation (76) into Eq. (17) and simplifying leads to

\[
\frac{1}{k_B T^*} = \left( \frac{3N}{2} - 1 \right) \frac{\chi}{E} \left( \frac{1}{\chi + u(\cdot)/N^2} \right),
\]

where

\[
\left( \frac{1}{\chi + u(\cdot)/N^2} \right) = \left( \frac{1}{\chi + u(\cdot)/N^2} \right)^{1_q} = \int_0^1 \cdots \int_0^1 d\Omega_r \int_{E_0}^{E_1} \frac{d\Omega}{\Omega} \left[ \chi + u(\cdot)/N^2 \right]^{3N/2 - 1} \left[ \chi + u(\cdot)/N^2 - 1 \right]^{3N/2 - 1}
\]

(78)
can be interpreted as some sort of average, and still contains the integrals over the coordinates to be carried out, to be computed in the following. This result coincides with the one obtained in [13] for the BG temperature in the microcanonical ensemble, i.e., it seems that the physical quantities in Tsallis statistics recover the results obtained in BG statistics, however with different thermodynamic limits. It is important to note that the physical temperature thus implicitly contains the information introduced by the Tsallis distribution, even though the resulting expression does not seem to differ from the original BG case. Already here we can see that in the limit of zero gravity, the average (78) becomes

\[
\left( \frac{1}{\chi + u(\cdot)/N^2} \right)^{1_q} = \frac{1}{\chi},
\]

(79)

and thus the analog of the ideal gas result for the Tsallis temperature is recovered in (77).

Using the thermodynamic limit for the gravitational configuration integral [14] in (75), and interchanging derivatives and thermodynamic limit significantly simplifies the expressions, and the temperature becomes, up to second order,

\[
\frac{E}{Nk_B T^*} \approx \frac{3}{2} - \frac{9b_0}{2\chi} - \frac{9}{4\chi^2} (b_1 - 42b_0^2).
\]

(80)

In the case of zero gravity, 1/\( \chi \) = 0, and thus the result for the ideal gas case in Tsallis statistics is obtained. Note that it is not identical to the ideal gas in BG statistics, since the temperature is the one defined for a Tsallis statistics, and not the conventional temperature of BG.

For the calculation of the physical pressure, we take the derivative of \( S_q \) with respect to the volume,

\[
\left( \frac{\partial S_q}{\partial V} \right)_{E,N} = k_B \left[ \Omega^{(IG)} \cdot Z^{(grav)} \right]^{1-q} \cdot \left[ \frac{\partial}{\partial V} \ln \Omega^{(IG)} + \frac{\chi}{3V} \frac{\partial}{\partial \chi} \ln Z^{(grav)} \right].
\]

Substituting this and using the physical temperature (77), we obtain

\[
\frac{P^*}{k_B T^*} = \frac{1}{3V} \left( \frac{3N}{2} + 1 \right) + \frac{\chi}{3V} \left( \frac{3N}{2} - 1 \right) \left( \frac{1}{\chi + u(\cdot)/N^2} \right),
\]

(82)

which is the corresponding Tsallis result to the one obtained in the BG statistics in Ref. [13]. Again, it is not identical to the BG result, due to the use of the Tsallis variables instead of the conventional BG ones. Using (74) in (81) and applying the same procedure as in the case of the temperature, the equation of state is obtained, up to second order, as

\[
\frac{P^* V}{Nk_B T^*} \approx 1 - \frac{3b_0}{2\chi} - \frac{3}{4\chi^2} (b_1 - 42b_0^2).
\]

(83)

Also for this equation of state, the Tsallis analog of the ideal gas is recovered considering (70), or equivalently, 1/\( \chi \) = 0. This result again reproduces the result of [13], considering the different definitions of the \( b_i \), and the use of the Tsallis variables instead of the conventional BG ones.

As in the canonical ensemble, we will further consider the specific heat capacity as defined in Eq. (57) for the canonical ensemble, derived from physical quantities as proposed in [18]. It is important to remark that in the microcanonical ensemble \((cV)_q\) can take positive as well as negative values. In [18] it was shown that if fluctuations in the system are sufficiently large, this response function becomes negative, implying the existence of a
phase transition for the self-gravitating gas. For the microcanonical ensemble, the expression \( 57 \) can be reformulated in more simple terms as

\[
(c_V)_q = \frac{1}{N} \left( \frac{\partial T^*}{\partial E} \right)^{-1} \left. \right|_{V,N} .
\]

(84)

It is useful, as done in \([13]\), to define a new function in order to calculate the specific heat capacity,

\[
g(\chi) = \frac{\chi}{N} \frac{\partial \ln \left[ \chi^{3N/2-1} Z^{(grav)} \right]}{\partial \chi} = \frac{\chi}{N} \left( 3 \frac{N}{2} - 1 \right) \left( \frac{1}{\chi + u(\gamma)/N^2} \right)^2 .
\]

Substituting into \( 73 \), the microcanonical physical temperature can be expressed as

\[
T^* = \frac{E}{Nk_B g(\chi)} ,
\]

(86)

and thus

\[
\frac{\partial T^*}{\partial E} = \frac{1}{Nk_B} \frac{\partial}{\partial E} \left[ E \right] g(\chi) = \frac{1}{Nk_B} \left[ 1 - \frac{\chi}{g(\chi)} \right] ,
\]

(87)

where \( \frac{\partial g(\chi)}{\partial \chi} = \frac{\partial g(\chi)}{\partial \chi} \). Therefore, the specific heat capacity is given by

\[
(c_V)_q = \frac{k_B g(\chi)}{1 - \chi g'(\chi)/g(\chi)} , \quad \text{or} \quad \frac{1}{c_{qV}} = \frac{d}{d\chi} \left( \frac{\chi}{g(\chi)} \right) .
\]

(88)

The factor inside the derivative on the right can be written as

\[
\frac{\chi}{g(\chi)} = \frac{2}{3} \int_0^1 \int_0^1 d^3N \int_0^1 d^3N' \left[ \chi + u(\gamma)/N^2 \right]^{3N/2-1} \left[ \chi + u(\gamma)/N^2 \right]^{3N/2-2} .
\]

(89)

Calculating the derivative with respect to \( \chi \), and dropping terms of the order of \( O(1/N) \), leads to

\[
\frac{1}{c_{qV}} \simeq \frac{2}{3} - \frac{N}{3} \left[ \left\langle \frac{1}{\chi + u(\gamma)/N^2} \right\rangle^2 - \left\langle \frac{1}{\chi + u(\gamma)/N^2} \right\rangle^2 \right] ,
\]

(90)

which is the analogous result for the Tsallis statistics as found for the BG microcanonical ensemble in \([13]\). This relation can be put in terms of the fluctuation for inverse of the physical temperature Eq. \( 74 \) using \( \beta_q \),

\[
\frac{1}{(c_V)_q} = \frac{2}{3} - \left( \frac{\Delta \beta_q}{\beta_q} \right)^2 .
\]

(91)

If the fluctuations are large, the specific heat capacity can become negative, and the system thus unstable, as was shown in \([13]\) as well. This holds in general, i.e., not only for weak gravity, since it was obtained from the exact expressions before applying the dilute regime.

An explicit expression for the specific heat capacity \( c_{qV} \) can be found from Eq. \( 88 \). In order to calculate the function \( g(\chi) \) from Eq. \( 55 \), we use the thermodynamic limit in the dilute regime of \( Z^{(grav)} \) given from Eq. \( 71 \), to second order resulting in

\[
\lim_{N \to \infty} g(\chi) \simeq \frac{3}{2} + \frac{9b_0}{2\chi} - \frac{9}{4\chi^2}(b_1 - 42b_0^2) .
\]

(92)

The specific heat is then obtained from Eq. \( 88 \) by derivation with respect to \( \chi \), resulting in

\[
\frac{(c_V)_q}{k_B} = \frac{3}{2} \left[ \frac{1}{[1 + 6b_0\chi^{-1} - 3/2\chi^{-2}(b_1 - 48b_0^2)]} \right] .
\]

(93)

Also here we see that in the limit of zero gravity, i.e., \( 1/\chi = 0 \), the specific heat also recovers the analog of the ideal gas case in the Tsallis statistics. From this result, it seems that the heat capacity in the microcanonical ensemble does not become negative for any value of \( \chi \). However, this expression only holds in the weak gravitational regime, and thus it cannot capture effects that happen in the presence of strong gravitational forces, where fluctuations might be large, as indicated by \( 91 \).

V. COMPARISONS AND CONCLUSIONS

In this work we have applied the framework of Tsallis generalized statistics and all its principles to analyze a system of self-gravitating particles, calculating thermodynamic state and response functions of the system in the appropriate thermodynamic limits and investigating their properties. After the analyses of canonical and microcanonical ensembles of the self-gravitating gas, respectively, we would like to comment on a few points concerning comparison and peculiarities of the results.

The application of Tsallis statistics was done in the hopes of addressing the question of ensemble equivalence and obtaining a consistent description of the self-gravitating gas throughout the different ensembles, in particular retaining a traditional scaling behaviour for \( N \) and \( V \). In \([13]\), ensemble equivalence has been achieved, but only by the application of a modified “dilute” thermodynamic limit, in which for \( N/V \to \infty \), the ratio \( N/V^{1/3} \) remains constant, instead of the ratio \( N/V \) as usual. Moreover, closer inspection reveals that the limits were indeed applied inconsistently – the constant ratio \( N/V^{1/3} \) was required in order to have the variable \( \eta \) constant, but at the same time terms such as \( N/V \) have been used to express equations of state, corresponding to the conventional thermodynamic limit instead of the dilute one.

Considering the outcomes for the thermodynamic equation of state in our calculations using the Tsallis statistics, it turns out that there is a certain equivalence between microcanonical and canonical ensemble, up to a
factor of $(1 - q)$. Leaving aside this factor, we can refer to \[13\], where an equivalence was shown between the two working variables $\chi$ and $\eta$ of the two ensembles, and argue that our results equal those of \[13\], at least in their dependence on $\chi$ and $\eta$. We thus achieve equivalence for the equation of state between microcanonical and canonical ensemble, up to the additional factor of $(1 - q)$ which is present in the result for the canonical ensemble. The presence of this factor however seems like a shortcoming of the theory itself – both ensembles yield results that are of Tsallis nature, by the use of Tsallis intensive variables instead of the conventional BG ones, but only in the canonical ensemble there is an explicit presence of the non-extensivity parameter. We suspect that such an explicit dependence should also occur in the microcanonical case, and that thus the microcanonical ensemble should be re-evaluated. For example, we would expect the entropy in the microcanonical ensemble to be formulated purely in terms of Tsallis variables, yet a conventional energy $E$ is used to calculate the sum over possible microstates. Another way to investigate the equivalence of ensembles in a general form is to check whether the probability distribution functions of microcanonical and canonical ensemble are related by a Laplace transformation \[22\], a standard tool of conventional thermodynamics.

Unfortunately, the microcanonical ensemble is a rarely used and somewhat neglected tool in order to describe thermodynamic systems, and even more so in the case of Tsallis statistics. There are very few instances of the application of the Tsallis microcanonical ensemble in the literature, and thus it has rarely been put to the test in well-known physical systems. We attribute the non-achievement of ensemble equivalence in our work to built-in non-equivalences in the ensembles in Tsallis statistics, and suspect that perhaps by according all variables between the ensembles, and putting them onto the same footing, ensemble equivalence might be reached.

A different case is the heat capacity. We reproduce the results of \[13\] in both ensembles, up to the factor of $(1 - q)$ in the canonical case, and they are not equivalent – not only for the explicit $q$-dependence, but also qualitatively. The heat capacity stays strictly positive in the canonical ensemble, but may become negative in the microcanonical. This fundamental difference has also been obtained in \[13\], and is physically reasonable, since second derivatives of thermodynamic state variables are related to the fluctuations of a system, and these fluctuations are not necessarily equivalent between ensembles. The final results for the heat capacity has not been explicitly calculated in \[13\], but we do achieve agreement qualitatively.

Another question is the limit of zero gravity, i.e., the ideal gas case. We have obtained the correct analog of the ideal gas contribution in all thermodynamic equations of state and response functions calculated, for the case when the gravitational interaction is turned off – analog, because in all the expressions the variables are the Tsallis variables, but in the functional form they obey the same laws as in the BG case of the ideal gas. In the microcanonical ensemble, the zero gravity case corresponds to $1/\chi = 0$, and all the results for an ideal gas are recovered analogously for this case. In the canonical ensemble, zero gravity is mathematically equivalent to the case of $q \to 1$, but it is not implied, and should not be confused. Zero gravity corresponds to $\eta = 0$, so all terms containing $\eta$ are eliminated – which incidentally eliminates most of the explicit $q$-dependence in the thermodynamic functions, seemingly resulting in the exact BG ideal gas results. However, $q$-dependence is retained, namely in the Tsallis intensive variables, which are used instead of the BG intensive variables. The zero gravity limit of the thermodynamic state functions is thus the $q$-analog of the ideal gas in BG statistics, and not the exact BG result.

Something similar happens with the specific heat in the canonical ensemble. The zero gravity case, i.e., $\eta = 0$, is mathematically equivalent to the case $q \to 1$, but the result in the zero gravity case is not equivalent to the ideal gas in BG statistics. The difference lies in the definition of the temperature used – our result for the specific heat is valid for a canonical ensemble with the Tsallis temperature $T^*$ kept fixed, while the usual BG statistics use the conventional notion of temperature.

In general, it is mathematically unsound to take the limit $q \to 1$ in the results for the thermodynamic functions. The computation of the final expressions requires the integration method described by \[22\], which is valid in the region $0 < q < 1$ only, not including the case $q = 1$. Thus taking the limit $q \to 1$ is technically not permitted in the results obtained after the integral transformations of \[22\], since these two mathematical steps do not commute.

In summary, we have obtained reasonable and physically sound results from the application of Tsallis statistics to the case of the self-gravitating gas, achieving convergence of important thermodynamic functions and state equations under the assumption of a thermodynamic limit consistent with the statistical framework used. Our results have shown that Tsallis statistics is a viable tool for the description of systems with long-range forces, but that its application has to be carried out with care, and that its results have to be understood in the context of the modified statistical framework, and all it implies. We think that the question of ensemble equivalence merits more detailed investigations, especially on the exact and correct formulation of the microcanonical ensemble in Tsallis statistics, where we see the potential for consistency between ensembles.

Acknowledgments

The authors would like to thank F. Nettel for many fruitful discussions. L. F. Escamilla-Herrera thanks the financial support from the Consejo Nacional de Ciencia y Tecnología (CONACyT, México). C. Gruber ac-
knowledges support by a Junior Fellowship of the Hanse-Wissenschaftskolleg Delmenhorst, and from the University of Oldenburg and the Research Training Group “Models of Gravity”. This work was partially supported by UNAM-DGAPA-PAPIIT, Grant No. 111617.

[1] A. Campa, T. Dauxois, and S. Ruffo, Phys. Rep. 480, 57 (2009).
[2] J. Oppenheim, Phys. Rev. E 68, 016108 (2003).
[3] J. Barré, D. Mukamel, and S. Ruffo, Phys. Rev. Lett. 87, 030601 (2001).
[4] R. C. Tolman, Phys. Rev. 35, 904 (1930).
[5] R. C. Tolman and P. Ehrenfest, Phys. Rev. 36, 1791 (1930).
[6] R. C. Tolman, Phys. Rev. 35, 875 (1930).
[7] D. Lynden-Bell and R. Wood, Mon. Not. Roy. Astron. Soc. 138, 495 (1968).
[8] D. Lynden-Bell and P. P. Eggleton, Mon. Not. Roy. Astron. Soc. 191, 483 (1980).
[9] V. A. Antonov, in Dynamics of Star Clusters, edited by J. Goodman and P. Hut (1985), vol. 113 of IAU Symposium, p. 525.
[10] J. Maxwell, Cambridge Philosophical Societys Trans. XII, 90 (1876).
[11] T. Padmanabhan, Phys. Rep. 5, 285 (1990).
[12] L. Velazquez, J. Stat. Mech. 2016 (2016).
[13] H. J. de Vega and N. Sanchez, Nuclear Physics B 625, 409 (2002), astro-ph/0101568.
[14] C. Tsallis, Journal of Statistical Physics 52, 479 (1988).
[15] C. Tsallis, Brazilian Journal of Physics 29, 1 (1999), ISSN 0103-9733.
[16] C. Tsallis, Introduction to nonextensive statistical mechanics: approaching a complex world (Springer, New York, 2009).
[17] C. Tsallis and E. Brigatti, Continuum Mechanics and Thermodynamics 16, 223 (2004).
[18] S. Abe, S. Martinez, F. Pennini, and A. Plastino, Physics Letters A 281, 126 (2001).
[19] R. Toral, Physica A: Statistical Mechanics and its Applications 317, 209 (2003).
[20] S. Abe, Physica A: Statistical Mechanics and its Applications 269, 403 (1999).
[21] A. Parvan, Physics Letters A 350, 331 (2006).
[22] D. Prato, Physics Letters A 203, 165 (1995).
[23] W. Greiner, L. Neise, and H. Stöcker, Thermodynamics and Statistical Mechanics (Springer, New York, 1995).