A DEFINITION OF THE CRYSTAL COMMUTOR USING KASHIWARA’S INVOLUTION

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Abstract. Henriques and Kamnitzer defined and studied a commutor for the category of crystals of a finite dimensional complex reductive Lie algebra. We show that the action of this commutor on highest weight elements can be expressed very simply using Kashiwara’s involution on the Verma crystal.

1. Introduction

Let \( g \) be a complex reductive Lie algebra. If \( A \) and \( B \) are crystals of representations of \( g \), then \( A \otimes B \) and \( B \otimes A \) are isomorphic. However the map \((a, b) \mapsto (b, a)\) is not an isomorphism. In [HeKa], following an idea of Berenstein, A. Henriques and the first author construct an explicit isomorphism \( \sigma_{A,B} : A \otimes B \to B \otimes A \), which they call the commutor. This commutor is involutive and satisfies the cactus relation, a certain axiom involving triple tensor products (see section 5).

Consider the following alternative definition for a commutor. First notice that we only need to define \( \sigma_{A,B} \) when \( A \) and \( B \) are irreducible. Also, a crystal isomorphism is uniquely defined by the images of highest weight elements. So, for each highest weight element \( b_\lambda \otimes c \in B_\lambda \otimes B_\mu \), we need to specify its image \( b_\mu \otimes b \in B_\mu \otimes B_\lambda \). We do this using Kashiwara’s involution \( * \) on \( B_\infty \). By the properties of \( * \), if \( b_\lambda \otimes c \) is a highest weight element in \( B_\lambda \otimes B_\mu \), then \( *c \in B_\lambda \) (where we identify \( B_\lambda \) and \( B_\mu \) with their images in \( B_\infty \)), and \( b_\mu \otimes *c \) is highest weight. Therefore we can define a crystal commutor by specifying that each highest weight element \( b_\lambda \otimes c \) is taken to \( b_\mu \otimes *c \). In this note we show that this definition gives the same commutor as that studied by Henriques and Kamnitzer.

The original definition of the commutor used the Schützenberger involution on each \( B_\lambda \), while this definition uses Kashiwara’s involution on \( B_\infty \). Thus one way to interpret our result is that it gives a non-trivial relationship between these two involutions. The Schützenberger involution does not exist for crystals of non-finite symmetric Kac-Moody Lie algebras, however Kashiwara’s involution does. Hence this work extends the definition of the commutor to highest weight crystals of symmetric Kac-Moody Lie algebras.

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2. Background

2.1. Notation and terminology. We include only a brief review of some basic facts about crystals. For the most part we follow the conventions from the review article [Kas2], which we recommend for a more detailed overview of the subject.

- Let \( g \) be a complex reductive Lie algebra.
- Let \( I \) denote the set of vertices of the Dynkin diagram of \( g \).
Let \( \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I} \) denote the positive roots and coroots of \( g \).
Let \( \{s_i\}_{i \in I} \) denote the generators of the Weyl group.
Let \( w_0 \) denote the long element of the Weyl group.
Let \( \langle \cdot, \cdot \rangle \) denote the pairing between weight space and coweight space.
Let \( \Lambda \) denote the set of weights of \( g \), \( \{\Lambda_i\}_{i \in I} \) the set of fundamental weights, and \( \Lambda_+ \) the set of dominant weights.

A crystal for \( g \) is a finite set \( B \) along with maps \( e_i, f_i : B \to B \sqcup \emptyset \) for each \( i \in I \) and a map \( \text{wt} : B \to \Lambda \), satisfying a certain set of axioms. These axioms may be found in [Kas2, Section 7.2].

For \( \lambda \in \Lambda_+ \), let \( B_\lambda \) denote the crystal corresponding to the irreducible representation \( V_\lambda \) of \( g \).
An element of a crystal is called highest (resp. lowest) weight if it is killed by all \( e_i \) (resp. all \( f_i \)). We use \( b_\lambda \) and \( b_{\lambda_+}^{\text{cow}} \) to denote the unique highest and lowest weight elements of \( B_\lambda \).

For any crystal \( B \) and any \( b \in B \), let \( \varepsilon_i(b) = \max\{n : e_i^n(b) \neq 0\} \). Let \( \varepsilon(b) \in \Lambda_+ \) be the unique weight such that, for all \( i \in I \), \( (\varepsilon(b), \alpha_i) = \varepsilon_i(b) \).
Similarly, let \( \varphi_i(b) = \max\{n : f_i^n(b) \neq 0\} \) and \( \varphi(b) \in \Lambda_+ \) be the unique weight such that, for all \( i \in I \), \( (\varphi(b), \alpha_i^\vee) = \varphi_i(b) \).

The weight of \( b \in B \) is \( \text{wt}(b) = \varphi(b) - \varepsilon(b) \).

There is a tensor product rule for crystals corresponding to the tensor product for representations of \( g \). The underlying set of \( A \otimes B \) is \( A \times B \) (whose elements we denote \( a \otimes b \)) and the actions of \( e_i \) and \( f_i \) are given by the following rules:

\[
e_i(a \otimes b) = \begin{cases} e_i(a) \otimes b, & \text{if } \varphi_i(a) \geq \varepsilon_i(b) \\ a \otimes e_i(b), & \text{otherwise} \end{cases}
\]

\[
f_i(a \otimes b) = \begin{cases} f_i(a) \otimes b, & \text{if } \varphi_i(a) > \varepsilon_i(b) \\ a \otimes f_i(b), & \text{otherwise.} \end{cases}
\]

### 2.2. Kashiwara’s involution on \( B_\infty \)

For any dominant weights \( \gamma \) and \( \lambda \), there is an inclusion of crystals \( B_{\gamma + \lambda} \to B_{\lambda} \otimes B_\gamma \) which sends \( b_{\gamma + \lambda} \) to \( b_\gamma \otimes b_\lambda \). The following is immediate from the tensor product rule:

**Lemma 2.1.** The image of the inclusion \( B_{\lambda + \gamma} \to B_{\lambda} \otimes B_\gamma \) contains all elements of the form \( b \otimes b_\gamma \) for \( b \in B_\lambda \).

**Proof.**

Lemma 2.1 defines a map \( \iota_{\lambda + \gamma} : B_\lambda \to B_{\lambda + \gamma} \) which is \( e_i \) equivariant and takes \( b_\lambda \) to \( b_{\lambda + \gamma} \). These maps make \( \{B_\lambda\} \) into a directed system, and the limit of this system is \( B_\infty \). There are \( e_i \) equivariant maps \( \iota_\infty : B_\lambda \to B_\infty \). When there is no danger of confusion we denote \( \iota_\infty \) simply by \( \iota \).

The infinite set \( B_\infty \) has additional combinatorial structure. In particular, we will need:

(i) The map \( \tau : B_\infty \to \Lambda_+ \) defined by \( \tau(b) = \min\{\lambda : b \in \iota(B_\lambda)\} \).

(ii) The map \( \varepsilon : B_\infty \to \Lambda_+ \) given by, for any \( b \in B_\infty \) and any \( \lambda \) such that \( b \in \iota(B_\lambda) \), \( \varepsilon(b) = \varepsilon(\iota^{-1}(b)) \), where \( \varepsilon \) is defined on \( B_\lambda \) as in Section 2.1

(iii) Kashiwara’s involution \( * \) (for the construction of this involution see [Kas1, Theorem 2.1.1]).

These maps are related by the following result of Kashiwara.

**Proposition 2.2** ([Kas2, Prop. 8.2]). Kashiwara’s involution preserves weights and satisfies

\[ \tau(*) = \varepsilon(\iota) \quad \varepsilon(*) = \tau(\iota) \]

**Remark 2.3.** All of this combinatorial structure can be seen easily using the MV polytope model [Kam]. The inclusions \( \iota \) correspond to translating polytopes. The maps \( \tau \) and \( \varepsilon \) are given by counting...
the lengths of edges coming out of the top and bottom vertices. The involution \(*\) corresponds to negating a polytope. From this description, the proof of the above proposition is immediate.

2.3. The commutor. We now recall the definition of the commutor from [HeKa Section 2.2]. Let \( \theta : I \to I \) be the involution such that \(-w_0 \cdot \alpha_i = \alpha_{\theta(i)}\). Recall that each crystal \( B_\lambda \) comes with an involution \( \xi_\lambda \) which acts by \( w_0 \) on weights and exchanges the action of \( e_i \) and \( f_{\theta(i)} \). These involutions can be extended to a map \( \xi_B : B \to B \) for any crystal \( B \) and they lead to the definition of the commutor for crystals. Namely,

\[
\sigma_{B,C} : B \otimes C \to C \otimes B
\]

(3)

\[
b \otimes c \mapsto \xi_{B \otimes C}(\xi_C(c) \otimes \xi_B(b)) = \text{Flip} \circ \xi_B \otimes \xi_C(\xi_{B \otimes C}(b \otimes c)).
\]

The second expression here is just the inverse of the first expression, and the equality is proved in [HeKa Proposition 2].

3. Main theorem

A crystal isomorphism \( B_\lambda \otimes B_\mu \to B_\mu \otimes B_\lambda \) is uniquely defined by the images of the highest weight elements in \( B_\lambda \otimes B_\mu \). These are all of the form \( b_\lambda \otimes c \), and must be sent to highest weight elements of \( B_\mu \otimes B_\lambda \), which in turn are of the form \( b_\mu \otimes b \). It follows from the tensor product rule for crystal that \( b_\lambda \otimes c \) is highest weight in \( B_\lambda \otimes B_\mu \) if and only if \( \varepsilon(c) \leq \lambda \).

As in Section 2.2, \( B_\lambda \) and \( B_\mu \) embed in \( B_\infty \). Let \( b_\lambda \otimes c \) be a highest weight element in \( B_\lambda \otimes B_\mu \). Since \( \varepsilon(c) \leq \lambda \), by Proposition 2.2 \( \tau(\ast c) \leq \lambda \), or equivalently \( \ast c \in \iota(B_\lambda) \). For this reason \( \ast c \) can be considered an element of \( B_\lambda \). Also \( \tau(c) \leq \mu \), which implies that \( \varepsilon(\ast c) \leq \mu \), and so \( b_\lambda \otimes \ast c \) is highest weight. So there is a unique isomorphism of crystals \( B_\lambda \otimes B_\mu \to B_\mu \otimes B_\lambda \) which takes each highest weight element \( b_\lambda \otimes c \) to \( b_\mu \otimes \ast c \). The following shows that this isomorphism is equal to the crystal commutor.

**Theorem 3.1.** If \( b_\lambda \otimes c \) is a highest weight element in \( B_\lambda \otimes B_\mu \), then \( \sigma_{B_\lambda,B_\mu}(b_\lambda \otimes c) = b_\mu \otimes \ast c \).

4. Proof

One of the main tools we will need is the notion of Kashiwara data (also called string data), first studied by Kashiwara (see for example [Kas2 section 8.2]). Fix a reduced word \( \mathbf{i} \) for \( w_0 \), by which we mean \( \mathbf{i} = (i_1, \ldots, i_m) \), where each \( i_k \) is a node of the Dynkin diagram, and \( w_0 = s_{i_1} \cdots s_{i_m} \).

The downward Kashiwara data for \( b \in B_\lambda \) with respect to \( \mathbf{i} \) is the sequence of non-negative integers \( (p_1, \ldots, p_m) \) defined by

\[
p_1 := \varphi_{i_1}(b), \quad p_2 := \varphi_{i_2}(f_{i_1}^{p_1}b), \quad \ldots, \quad p_m := \varphi_{i_m}(f_{i_{m-1}}^{p_{m-1}} \cdots f_{i_1}^{p_1}b).
\]

That is, we apply the lowering operators in the direction of \( i_1 \) as far as we can, then in the direction \( i_2 \), and so on. The following result is due to Littelmann [L section 1].

**Lemma 4.1.** After we apply these steps, we reach the lowest element of the crystal. That is:

\[
f_{i_m}^{p_m} \cdots f_{i_1}^{p_1} b = b^\text{low}_\lambda.
\]

Moreover, the map \( B_\lambda \to \mathbb{N}^m \) taking \( b \to (p_1, \ldots, p_m) \) is injective. \( \square \)

Similarly, the upwards Kashiwara data for \( b \in B_\lambda \) with respect to \( \mathbf{i} \) is the sequence \( (q_1, \ldots, q_m) \) defined by

\[
q_1 := \varepsilon_{i_1}(b) = \varphi_{i_1}(e_{i_1}^{q_1}b), \quad \ldots, \quad q_m := \varepsilon_{i_m}(e_{i_{m-1}}^{q_{m-1}} \cdots e_{i_1}^{q_1}b).
\]

We introduce the notation \( u_{i_k}^k := s_{i_1} \cdots s_{i_k} \).

**Lemma 4.2.** In the crystal \( B_\lambda \), we have the following:
(i) The downward Kashiwara data for \( b_\lambda \) is given by \( p_k = \langle w_{k-1}\cdot \alpha_{ik}^\vee, \lambda \rangle \).

(ii) For each \( k \), \( \varepsilon_{ik}(f^{p_{k-1}}_i \cdots f^1_i b_\lambda) = 0 \).

\[ \mu_k = s_{ik} \cdot \mu_{k-1} - a_k \alpha_{ik}, \]

where \( a_k = \varepsilon_{ik}(f^{p_{k-1}}_i \cdots f^1_i b_\lambda) \). Using this fact at each step,

\[ \mu_m = w_0 \cdot \lambda - \sum_{k=1}^{m} a_k s_{i_m} \cdots s_{i_{k+1}} \cdot \alpha_{ik}. \]

By Lemma 4.1, we know that \( f^{p_m}_i \cdots f^1_i b_\lambda = b^\text{low}_\lambda \), so that \( \mu_m = w_0 \cdot \lambda \). Hence

\[ \sum_{k=1}^{m} a_k s_{i_m} \cdots s_{i_{k+1}} \cdot \alpha_{ik} = 0. \]

Now, \( s_{i_m} \cdots s_{i_k} \) is a reduced word for each \( k \), which implies that \( s_{i_m} \cdots s_{i_{k+1}} \alpha_{ik} \) is a positive root. Thus each \( a_k \) is zero, proving part (ii).

Equation (4) now shows that \( \mu_k = s_{ik} \cdot \cdots \cdot s_{i_1} \cdot \lambda \), for all \( k \). In particular that \( f^{p_k}_i \) must perform the reflection \( s_{ik} \) on the weight \( s_{i_{k-1}} \cdots s_{i_1} \cdot \lambda \). Therefore,

\[ p_k = \langle \alpha_{ik}^\vee, s_{i_{k-1}} \cdots s_{i_1} \cdot \lambda \rangle = \langle w_{k-1}\cdot \alpha_{ik}^\vee, \lambda \rangle. \]

\[ \square \]

Lemma 4.3. Let \( b_\lambda \otimes c \) be a highest weight element of \( B_\lambda \otimes B_\mu \). Let \( b \otimes b^\text{low}_\mu \) be the lowest weight element of the component containing \( b_\lambda \otimes c \). Let \((p_1, \ldots, p_m)\) be the downward Kashiwara data for \( c \) with respect to \( 1 \), and \((q_1, \ldots, q_m)\) the upward Kashiwara data for \( b \) with respect to \( \nu^\text{rev} := (i_1, \ldots, i_1) \). Then, for all \( k \), \( p_k + q_{m-k+1} = \langle w_{k-1}\cdot \alpha_{ik}^\vee, \nu \rangle \), where \( \nu = \text{wt}(b_\lambda \otimes b) \).

Proof. Let \( r_k = \langle w_{k-1}\cdot \alpha_{ik}^\vee, \nu \rangle \). By part (i) of Lemma 4.2, \((r_1, \ldots, r_m)\) is the downward Kashiwara data for \( b_\lambda \otimes c \). Define \( b_k \in B_\lambda \) and \( c_k \in B_\mu \) by \( b_k \otimes c_k = f^{p_k}_i \cdots f^1_i (b_\lambda \otimes c) \). Part (ii) of Lemma 4.2, along with the definition of Kashiwara data, shows that, for each \( 1 \leq k \leq m \),

\[ e_{ik}(b_{k-1} \otimes c_{k-1}) = 0 \quad \text{and} \quad f_{ik}(b_k \otimes c_k) = 0. \]

In particular, the tensor product rule for crystals implies

\[ e_{ik}b_{k-1} = 0 \quad \text{and} \quad f_{ik}c_k = 0. \]

Define \( p_k \) to be the number of times \( f_{ik} \) acts on \( c_{k-1} \) to go from \( b_{k-1} \otimes c_{k-1} \) to \( b_k \otimes c_k \), and \( q_{m-k+1} \) to be the number of times \( f_{ik} \) acts on \( b_{k-1} \). Since \( f_{ik}c_k = 0 \), we see that \( \varphi_{ik}(c_{k-1}) = p_k \). Hence, by definition \((p_1, \ldots, p_m)\) is the downward Kashiwara data for \( c \) with respect to \( i \). Similarly, \( e_{ik}b_{k-1} = 0 \), so \( \varepsilon_{ik}(c_k) = q_{m-k+1} \). By Lemma 4.1, \( b_m = b \), so this implies that \((q_1, \ldots, q_m)\) is the upward Kashiwara data for \( b \) with respect to \( \nu^\text{rev} \). Since \( p_k + q_{m-k+1} \) is the number of times that \( f_{ik} \) acts on \( b_{k-1} \otimes c_{k-1} \) to reach \( b_k \otimes c_k \), we see that \( p_k + q_{m-k+1} = r_k \).

Let \( b_\lambda \otimes c \) be a highest weight element in \( B_\lambda \otimes B_\mu \). As discussed in section 3.1, \( c \) can be considered as an element of \( B_\lambda \).

Lemma 4.4. Define \( \nu = \text{wt}(b_\lambda \otimes c) \). Let \((p_1, \ldots, p_m)\) be the downward Kashiwara data for \( c \) in \( B_\mu \) with respect to \( i \). Let \((q_1, \ldots, q_m)\) be the downward Kashiwara data for \( c \) in \( B_\lambda \) with respect to the decomposition \( \theta(\nu^\text{rev}) := (\theta(i_1), \ldots, \theta(i_1)) \) of \( w_0 \). Then, for all \( k \), \( p_k + q_{m-k+1} = \langle w_{k-1}\cdot \alpha_{ik}^\vee, \nu \rangle \).
Proof. The proof will depend on results from [Kam] on the MV polytope model for crystals. In particular, within this model it is easy to express Kashiwara data and the Kashiwara involution.

Let $P = P(M_\bullet)$ be the MV polytope of weight $(\nu - \lambda, \mu)$ corresponding to $c$. Then by Theorem 6.6 of [Kam],

$$p_k = M_{w_k^{i-1} \cdot \Lambda_{ik}} - M_{w_k^{i} \cdot \Lambda_{ik}}.$$  

Now, consider $P$ as a stable MV polytope (recall that this means that we only consider it up to translation). Then by Theorem 6.2 of [Kam], we see that $*(P) = -P$.

The element $i_{\mu}^{-1} \ast \iota_\lambda(c) \in B_\lambda$ corresponds to the MV polytope $\nu - P$ and hence has BZ datum $N_\bullet$, where $M_\bullet$ and $N_\bullet$ are related by

$$M_\gamma = \langle \gamma, \nu \rangle + N_{-\gamma}.$$  

Let $i' = \theta(\overline{\nu}^{\text{rev}})$. Then,

$$-w_{k-1}^i \cdot \Lambda_{ik} = w_{m-k+1}^{i'} \cdot \Lambda_{i'}_{m-k+1} \quad \text{and} \quad -w_k^i \cdot \Lambda_{ik} = w_{m-k}^{i'} \cdot \Lambda_{i'}_{m-k+1}.$$  

Combining the last 3 equations, we see that

$$p_k = \langle w_{k-1}^i \cdot \Lambda_{ik}, \nu \rangle + N_{-w_{k-1}^i \cdot \Lambda_{ik}} - \langle w_k^i \cdot \Lambda_{ik}, \nu \rangle + N_{-w_k^i \cdot \Lambda_{ik}}$$

$$= N_{w_{m-k+1}^{i} \cdot \Lambda_{i'}_{m-k+1}} - N_{w_{m-k}^{i'} \cdot \Lambda_{i'}_{m-k+1}} + \langle w_k^{i'} \cdot \Lambda_{ik} - w_k^i \cdot \Lambda_{ik}, \nu \rangle.$$  

Applying Theorem 6.6 of [Kam] again,

$$q_k = N_{w_k^{i'}} \cdot \Lambda_{i'}_{k} - N_{w_k^{i} \cdot \Lambda_{i}^{k}}.$$  

We now add equation (5) and (6), substituting $m - k + 1$ for $k$ in the second equation, to get

$$p_k + q_{m-k+1} = \langle w_{k-1}^i \cdot \Lambda_{ik} - w_k^i \cdot \Lambda_{ik}, \nu \rangle = \langle w_k^{i'} \cdot \Lambda_{i'}_{m-k}, \nu \rangle = \langle w_k^{i} \cdot \alpha_{ik}, \nu \rangle.$$  

□

Proof of Theorem 3.1. We know that $\sigma_{B_\lambda, B_\mu}(b_\lambda \otimes c) = c_\mu \otimes b$ for some $b \in B_\lambda$. By the definition of $\sigma_{B_\lambda, B_\mu}$ (see Section 2.3), we see that

$$\xi(b_\lambda \otimes c) = (\xi \circ \xi)(\text{Flip}(\sigma(b_\lambda \otimes c))) = \xi(b) \otimes b^\text{low}_\mu.$$  

In particular, $\xi(b) \otimes b^\text{low}_\mu$ is the lowest weight element of the component of $B_\lambda \otimes B_\mu$ containing $b_\lambda \otimes c$.

Fix a reduced expression $i = (i_1, \ldots, i_m)$ for $w_0$. Let $(p_1, \ldots, p_m)$ be the downward Kashiwara data for $c$ with respect to $i$, and let $(q_1, \ldots, q_m)$ be the downward Kashiwara data for $b$ with respect to $\theta(\overline{\nu}^{\text{rev}}) := (\theta(i_m), \ldots, \theta(i_1))$. Notice that $(q_1, \ldots, q_m)$ is also the upward Kashiwara data for $\xi(b)$ with respect to $\overline{\nu}^{\text{rev}} := (i_m, \ldots, i_1)$, since $\xi$ interchanges the action of $f_i$ and $e_{\theta(i)}$. Hence by Lemma 4.3

$$p_k + q_{m-k+1} = \langle w_k^{i} \cdot \alpha_{ik}, \nu \rangle$$

for all $k$, where $\nu$ is the weight of $b_\lambda \otimes c$.

As discussed in Section 3, $*c \in \iota(B_\mu^*)$, and so can be considered as an element of $B_\mu^*$. Let $(q'_1, \ldots, q'_m)$ by the downward Kashiwara data for $*c \in B_\mu^*$ with respect to $\theta(\overline{\nu}^{\text{rev}})$. By Lemma 4.4 we have

$$p_k + q'_{m-k+1} = \langle w_k^{i} \cdot \alpha_{ik}, \nu \rangle.$$  

Comparing equations (7) and (8), $q_k = q'_k$ for each $1 \leq k \leq m$. That is, the downward Kashiwara data for $b$ and $*c$ with respect to $\theta(\overline{\nu}^{\text{rev}})$ are identical. Hence by Lemma 4.1 $b = *c$.  

□
5. Questions

The involution $*$ gives $B_\infty$ an additional crystal structure, defined by $f_i^* \cdot b := * \circ f_i \circ *(b)$. Let $B^{i}_{\infty}$ denote the crystal with vertex set $\mathbb{Z}_{\geq 0}$, where $e_j, f_j$ act trivially for $j \neq i$ and $e_i, f_i$ act as they do on the usual $B_\infty$ for $\mathfrak{sl}_2$. Kashiwara [Kas1, Theorem 2.2.1] showed that the map 

$$B_\infty \rightarrow B_\infty \otimes B^i_{\infty}$$

$$b \mapsto (e_i^*)^{\varepsilon_i(*b)}(b) \otimes \varepsilon_i(*b)$$

is a morphism of crystals with respect to the usual crystal structures on each side. We can think of this fact as an additional property of $*$. On the other hand, the commutor $\sigma$ also has an additional property, which is called the cactus relation. This relation states that if $A, B, C$ are crystals, then

$$\sigma_{A,C \otimes B} \circ (1 \otimes \sigma_{B,C}) = \sigma_{B \otimes A,C} \circ (\sigma_{A,B} \otimes 1).$$

(See [HeKa, Theorem 3]).

**Question 1.** Is there a relation between this additional property of Kashiwara's involution $*$ and the cactus relation for the commutor $\sigma$?

Another direction is to consider the generalization beyond finite dimensional reductive Lie algebras. We can define a crystal commutor for any the crystals of highest weight representations of any symmetrizable Kac-Moody algebra by $\sigma(b_\lambda \otimes c) = b_\mu \otimes *c$ whenever $b_\lambda \otimes c$ is a highest weight element. This will be well defined by the analysis given in section 3.

**Question 2.** Does this commutor satisfy the cactus relation?\footnote{This question has recently been answered in the affirmative by Savage [S, Theorem 6.4]}

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