Soft breaking in Hořava Gravitational theory

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Abstract. We consider anisotropic gravity at the conformal point of the kinetic term of the
Hořava Gravity. This value is protected under quantum corrections by the constraints of the
theory. The symmetry of the \( z = 3 \) interacting terms, \( z \) being the time scale, which ensure
the renormalizability of the theory is softly broken by \( z = 1 \) interacting terms. We show that
the resulting theory has the same number of physical degrees of freedom as General Relativity.
Moreover the linearized theory around the Minkowski solution is exactly equivalent to General
Relativity. We also study the spherically symmetric solutions to the fields equations. We find
a close expression for the solutions. It terms out that the solutions are structurally stable from
the right and unstable from the left around the value of the coupling constant corresponding to
the Schwarzschild solution of General Relativity.

1. Introduction

The main idea of the Hořava Gravity [1] theory is to consider an anisotropic behaviour of space
and time. The scaling being of the form

\[
\begin{align*}
t & \sim [b]^z \\
x & \sim [b]^1.
\end{align*}
\]

The scaling breaks the relativistic symmetry, with the hope of recovering it at low energies or
equivalently at large distances. The main benefit would be, to obtain a renormalizable theory,
and this goal may be achieved by considering a suitable scaling \( z \). If at low energy one recovers
General Relativity the Hořava theory could be regarded as the \( \text{UV} \) completion of it.

The Hořava proposal considered the foliation-preserving diffeomorphism of the 3-dimensional
space as a gauge symmetry of the theory and the hope is to have an enhancement of it to space-
time diffeomorphisms at low energies. The action of the non-projectable Hořava theory written
in terms of the \( \text{ADM} \) variables \( g_{ij}, N, N_i \) is

\[
S = \int dt d^3x \sqrt{g} N \left( G^{ijkl} K_{ij} K_{kl} - \nu \right)
\]

where

\[
K_{ij} = \frac{1}{2N} \left( \dot{g}_{ij} - 2 \nabla_{(i} N_{j)} \right)
\]

\[
G^{ijkl} = \frac{1}{2} \left( g^{ik} g^{jl} + g^{il} g^{jk} \right) - \lambda g^{ij} g^{kl}
\]
and the potential density \( \nu = \nu(g_{ij}, a_i, \ldots) \) is the most general combination of the spatial metric, its curvature tensor, the vector \( a_i \equiv \frac{N_i}{N} \) and covariant spatial derivatives of these objects transforming as a scalar under spatial diffeomorphisms. In order to have power-counting renormalizativity the potential must include terms up to sixth order in spatial derivatives. The potential may include a cosmological constant but we do not consider it here.

The potential including the sixth order terms is given by

\[
\nu = -R - \alpha a_i a^i + w C_{ij} C^{ij}
\]

where \( \alpha \) and \( w \) are coupling constants. The second term on the right hand member was introduced in [2]. \( C_{ij} \) is the Cotton tensor

\[
C_{ij} = \frac{1}{\sqrt{g}} \epsilon^{kl} (\nabla_k R_{lj} - \nabla_l R_{kj})
\]

The sixth order term \( \sqrt{g} C_{ij} C^{ij} \) was introduced in Hořava [1] and it is the unique interacting term of that order arising from the detailed balance principle proposed by Hořava. It transforms homogeneously, with weight \( \frac{3}{2} \), under conformal transformations of the metric. The lower order differential terms in the potential violate the detailed balance principle but the action is still renormalizable. In this sense the inclusion of these terms is a soft-breaking mechanism.

Most of the analysis of the theory has been performed for \( \lambda \neq \frac{1}{3} \) [3, 4, 5, 6, 7, 8, 9, 10, 11]. In that case the theory contains an additional degree of freedom compared to GR. In this work we consider \( \lambda = \frac{1}{3} \). It is the particular case where the tensor \( C_{ijkl} \) is not invertible. It turns out that this case is quite interesting because the number of physical degrees of freedom of the theory is the same as in GR.

2. The hamiltonian

From the action (1) we may obtain the hamiltonian of the theory [12]. Its expression is

\[
\int d^3 x \left( \frac{N}{\sqrt{g}} \pi_{ij} \pi_{ij} + \sqrt{g} N \nu + N_i \mathcal{H}^i + \sigma \phi + \mu \pi \right) + E_{ADM}
\]

where

\[
\mathcal{H}^i \equiv -2 \nabla_j \pi^{ij} + \phi \partial^i \mathcal{N}.
\]

\( \phi \) is the conjugate momentum associated to \( N \) and \( \sigma, \mu \) are Lagrange multipliers. The primary constraint

\[
\pi = 0
\]

is a particular property of the Hořava theory formulated for \( \lambda = \frac{1}{3} \). This constraint ends up being a second class one and it is preserved on the quantum formulation of the theory. In that sense the \( \lambda = \frac{1}{3} \) value is protected from quantum corrections of the theory. The constraint \( \phi = 0 \) arises as a consequence of the absence of \( \dot{N} \) terms in the hamiltonian.

\( E_{ADM} \) denotes the ADM energy

\[
E_{ADM} = \oint d\Sigma_i (\partial_j g_{ij} - \partial_i g_{jj})
\]

It has to be included in the hamiltonian, following [13], in order to obtain the equations of motion compatible with the boundary conditions from the most general variations of the hamiltonian.

The preservation of the primary constraints give rise to two new constraints, the hamiltonian constraint

\[
\mathcal{H} \equiv \frac{1}{\sqrt{g}} \pi_{ij} \pi_{ij} - \sqrt{g} R + \alpha \sqrt{g} (2 \nabla_i a^i + a_i a^i) + w \sqrt{g} C_{ij} C^{ij} = 0
\]
and
\[ C \equiv \frac{3N}{2\sqrt{g}} \pi^{ij} \pi_{ij} + \frac{1}{2} \sqrt{g} NR - \sqrt{g} N \left( 2N_i a^i + \left( 2 - \alpha \right) a a^i \right) + \frac{3w}{2} \sqrt{g} NC_{ij} C^{ij} = 0. \] (11)

The conservation of these new constraints provides two elliptic partial differential equations for the Lagrange multipliers \( \sigma \) and \( \mu \). The Dirac algorithm ends up at this stage.

From (10) and (11) we obtain
\[ \left( 1 - \frac{\alpha}{2} \right) \nabla^2 N = N \left( \frac{1}{g} \pi^{ij} \pi_{ij} + w C_{ij} C^{ij} \right). \] (12)

The right hand member is positive provided \( w > 0 \). If, in addition, \( \alpha < 2 \) then the results in [11] imply the existence and uniqueness of a globally defined solution \( N \), moreover it is guaranteed that \( N \geq 0 \) over all the foliation.

3. The energy of the theory
The Hamiltonian (6) may also be rewritten as a sum of constraints plus boundary terms
\[ H = \int d^3 x \left( H_0 + N_i \dot{H}^i + \sigma \phi + \mu \pi \right) + E_{ADM} - 2\alpha \Phi_N \] (13)
where
\[ \Phi_N \equiv \int d\Sigma_i \partial_i N. \] (14)

The role of the boundary terms ensures the existence of the Gateaux derivative of the Hamiltonian under variations of \( N \) that behave as \( \delta N = O(r^{-1}) \) at infinity. The energy of the theory is given by the value of the Hamiltonian on the constrained submanifold and it is thus equal to \( [E_{ADM} - 2\alpha \Phi_N] \).

The positive mass theorem in [14, 15] may be used to analyze the positivity of \( E_{ADM} - 2\alpha \Phi_N \). A strong result was obtained in [12] where we show that if \( -\frac{2}{3} \leq \alpha \leq 0 \) then \( R \geq 0 \) and using the positivity mass theorem one gets \( E_{ADM} - 2\alpha \Phi_N \geq 0 \). Besides, a new result follows by considering a conformal transformation on the three dimensional leaves of the foliation, see also [16]. If we take \( g_{ij} \to \tilde{g}_{ij} = g_{ij} e^{\alpha N} \) then \( \tilde{E}_{ADM} = E_{ADM} - 2\alpha \Phi_n \) and
\[ \tilde{R} = e^{-\alpha N} \left[ R - 2\alpha \nabla_i \nabla^i N - \frac{\alpha^2}{4} (\nabla N)^2 \right] \]
\[ = e^{-\alpha N} \left[ R - \frac{4\alpha}{(2-\alpha)} N \left( \frac{1}{g} \pi^{ij} \pi_{ij} + w C^{ij} C_{ij} \right) - \frac{\alpha^2}{4} (\nabla N)^2 \right]. \]

Hence if \( R \geq \frac{4\alpha}{(2-\alpha)} N \left( \frac{1}{g} \pi^{ij} \pi_{ij} + w C^{ij} C_{ij} + \frac{\alpha^2}{4} (\nabla N)^2 \right) \) then \( E_{ADM} - 2\alpha \Phi_N \geq 0 \).

4. The physical degrees of freedom
From the analysis of the constraints it follows that the number of degrees of freedom of the theory is the same as in General Relativity. Moreover a perturbative analysis around a Minkowski background
\[ g_{ij} = \delta_{ij} + h_{ij} \ , \ \pi^{ij} = p_{ij} \ , \ N = 1 + n \] (15)
yields
\[ h^T = n = 0 \] (16)
and
\[ H = \int d^3 x \left( p^T_{ij} p^T_{ij} + \frac{1}{4} \partial_i h^{TT}_{jk} \partial_j h^{TT}_{ik} \right) \] (17)
which is exactly the Hamiltonian of linearized General Relativity.
5. The static, spherical symmetric solutions

It turns out that the relevant coupling in the analysis of the static, spherically symmetric solutions is $\alpha$, the solutions depend only on $\alpha$. When $\alpha = 0$, one obtains the Schwarzchild solution of General Relativity. In terms of the spherical coordinates, the metric has the form

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2$$

where $N = N(r)$ and $f = f(r)$.

The exact solution for $\alpha < 2$ has the expression

$$r = \beta GM e^{\frac{\chi}{2}}$$

where $\chi > 0$ and $\beta = \sqrt{1 - \frac{\alpha}{2}}$,

$$N = e^{-\frac{\chi}{2}},$$

$$f = \left(\cosh \chi - \frac{\sinh \chi}{\beta}\right)^2.$$ 

For $0 \leq \alpha < 2$ the metric presents an horizon (for relativistic matter) at $\hat{\chi}$ where $\tanh \hat{\chi} = \beta$ and a essential singularity at $\chi \to \infty$.

For $\alpha < 0$ the solution does not present an horizon, it has a naked essential singularity at $\chi \to \infty$.

The solution for $\alpha > 2$ has a different expression but it always presents an horizon and an essential singularity.

The asymptotic behaviour of the solution is

$$N = 1 - 2\frac{GM}{r}$$

$$f = 1 - 2\frac{GM}{r}$$

as in Schwarzchild solution for General Relativity.

It is remarkable that the exact solution at $\alpha = 0$, corresponding to the Schwarzchild solution, is stable from the left and unstable from the right when we vary the coupling constant $\alpha$. The solution corresponding to $\alpha > 0$ approaches the Schwarzchild solution, in a suitable norm, when $\alpha \to 0$. In distinction when $\alpha < 0$, the solution is qualitatively different to the one at $\alpha = 0$.

Perturbation analysis may be performed for $\alpha > 0$ around $\alpha = 0$, for any $r$. The explicit solution in the asymptotic regime $r \to \infty$ agrees with the already reported result in [17].

6. Conclusions

The Hořava theory for $\lambda = \frac{1}{3}$ whose hamiltonian formalism is given by (6) is a good candidate for a quantum renormalizable gravity theory. The value $\lambda = \frac{1}{3}$ is protected by the constraints of the theory and does not acquire quantum corrections. The theory has the same number of physical degrees of freedom as General Relativity. Moreover at the linearized level around Minkowski space-time it exactly agrees with General Relativity. The static spherically symmetric solution can be obtained explicitly in a closed form. The solutions for $\alpha > 0$ converge smoothly to the Schwarzchild solution, corresponding to $\alpha = 0$. When $\alpha < 0$ the solutions are qualitatively different from the one at $\alpha = 0$. They do not have an horizon, only a naked essential singularity.

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References

[1] P. Hořava, Phys. Rev. D 79, 084008 (2009).
[2] D. Blas, O. Pujolas, and S. Sibiryakov, Phys. Rev. Lett. 104, 181302 (2010).
[3] C. Charmousis, G. Niz, A. Padilla, and P. M. Saffin, J. High Energy Phys. 08 (2009) 070; X. Gao, Y. Wang, R. Brandenberger, and A. Riotto, Phys. Rev. D 81, 083508 (2010); J.-O. Gong, S. Koh, and M. Sasaki, Phys. Rev. D 81, 084053 (2010); A. Kehagias and K. Sfetsos, Phys. Lett. B 678, 123 (2009); Y.W. Kim, H.W. Lee, and Y. S. Myung, Phys. Lett. B 682, 246 (2009); M.-i. Park, Classical Quantum Gravity 28, 015004 (2011); Gen. Relativ. Gravit. 43, 2979 (2011).
[4] D. Blas, O. Pujolas, and S. Sibiryakov, J. High Energy Phys. 10 (2009) 029.
[5] M. Li and Y. Pang, J. High Energy Phys. 08 (2009) 015.
[6] A. Papazoglou and T. P. Sotiriou, Phys. Lett. B 685, 197 (2010); D. Blas, O. Pujolas, and S. Sibiryakov, Phys. Lett. B 688, 350 (2010); I. Kimpton and A. Padilla, J. High Energy Phys. 07 (2010) 014; A. Cerioni and R. H. Brandenberger, arXiv:1008.3589; R.-G. Cai, B. Hu, and H.-B. Zhang, Phys. Rev. D 83, 084009 (2011).
[7] J. Bellorin and A. Restuccia, Int. J. Mod. Phys. D 21, 1250029-1 (2012).
[8] Bellorin and A. Restuccia, Phys. Rev. D 84, 104037 (2011).
[9] J. Bellorin and A. Restuccia, Phys. Rev. D 83, 044003 (2011).
[10] W. Donnelly and T. Jacobson, Phys. Rev. D 84, 104019 (2011).
[11] J. Bellorin, A. Restuccia and A. Sotomayor, Phys. Rev. D 85, 124060 (2012).
[12] J. Bellorin, A. Restuccia and A. Sotomayor, Phys. Rev. D 87, 084020 (2013).
[13] T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) 88, 286 (1974).
[14] R. Schoen and S-T Yau, Commun. Math. Phys. 79, 47 (1981); 79, 231 (1981).
[15] E. Witten, Commun. Math. Phys. 80, 381 (1981).
[16] D. Garfinkle and T. Jacobson, Phys. Rev. Lett. 107, 191102 (2011).
[17] E. Kiritsis, Phys. Rev. D 81:044009, (2010).