RELATIONAL QUADRILATERAL LAND. II.
ANALOGUES OF ISOSPIN AND HYPERCHARGE

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Abstract

I consider the momenta and conserved quantities for \( \mathbb{CP}^2 \) interpreted as the space of quadrilaterals. This builds on Paper I’s kinematics via making use of MacFarlane’s work considering the \( SU(3) \)-like (and thus particle physics-like) conserved quantities that occur for \( \mathbb{CP}^2 \). I perform the additional step of further interpreting that as the configuration space of all relational quadrilaterals and thus an interesting toy model for whole-universe, relational and geometrodynamical-analogue physics. I also provide the Kuchař observables for the quadrilateral, which is a particular resolution of the Problem of Observables based on Paper I and II’s results. I study HO-like and highly symmetric potentials. I also some some exact solutions and qualitative behaviours for dynamics on \( \mathbb{CP}^2 \). In each case, I reinterpret the results in terms of quadrilaterals. This paves the way for the quantum mechanical study of the relational quadrilateral in Paper III and for investigations of a number of Problem of Time strategies (mostly in Paper IV) and of a number of other foundational and qualitative investigations of Quantum Cosmology.

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1 Introduction

Paper I [1] provided and interpreted shape coordinates and other coordinate systems for the relational particle mechanics (RPM) [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] up to quadrilateralland, whose shape space (relative configuration space) is $\mathbb{C}P^2$. See also Paper I for more details of what RPM’s are for Keys 1 to 8 for unlocking RPM’s up to quadrilateralland and for other applications of the $\mathbb{C}P^2$ geometry that quadrilateralland possesses.

In the present paper, in Sec 2, I provide and interpret the shape momenta conjugate to the shape coordinates for the various RPM theories up to quadrilateralland. In Sec 3, I consider the corresponding Hamiltonians. Secs 4 and 5 consider Key 9: the isometries for RPM’s up to quadrilateralland; this builds on earlier work by Smith [17], Franzen and I [8]. I already studied [7, 9] how triangleland has $SO(3) = SU(2)/\mathbb{Z}_2$ as its isometry group. On the other hand, $N$-stop metroland has $SO(N - 1)$ as its isometry group, among which 4-stop metroland’s is $SO(3)$ again. This led to spherical polar mathematics and various further analogies [8, 9] with Molecular Physics (rigid rotors, the Stark Effect, Pauling’s study of the spectra of crystals and the theory underlying Raman spectroscopy). In the present paper, I consider the conserved quantities for quadrilateralland, i.e. the quadrilateralland interpretation of $\mathbb{C}P^2$’s isometry group, $SU(3)/\mathbb{Z}_3$ [15]. The quadrilateralland interpretation of $\mathbb{C}P^2$’s ‘$SU(3)$’ of conserved quantities is one of the principal results of the present paper. Whilst this is no longer analogous to Molecular Physics, it is now analogous to Particle Physics. I give hypercharge and isospin [19] analogues in terms of Gibbons–Pope-type coordinates [20, 21] and interpret these in terms of quadrilateralland quantities.

In [21, 15], RPM’s are argued in detail to be good models of geometrodynamics; thus they are useful in a number of ways for the study of the Problem of Time and for various other foundational and qualitative issues in Quantum Cosmology (extending e.g. [13, 14, 15, 11]). One of these issues is the Problem of Observables; the present paper resolves this in the sense of Kuchar [22] for the quadrilateral, based on Papers I and II’s treatment of shape coordinates and shape momenta (Sec 6). More Problem of Time [23, 24, 25, 15] applications are in Papers III [26] (finishing the Naive Schrödinger Interpretation approach started in Paper I after obtaining wavefunctions) and, especially, IV [27] (Histories, Semiclassical, Records and Halliwell-type combined approaches). N.B. that the classical dynamics and QM are required prior to the Quantum Gravity and Quantum Cosmology applications; this is the main purpose of Papers II and III. In Sec 7, I give the classical equations of motion for quadrilateralland. In Sec 8, I consider HO-type potentials for quadrilateralland (Key 10) in quadrilateralland terms (i.e. as a sequence of quadrilaterals). In Sec 9, I interpret the geodesics on $\mathbb{C}P^2$ (Key 11) in quadrilateralland terms and, especially, IV [27] (Histories, Semiclassical, Records and Halliwell-type combined approaches). N.B. that the classical dynamics and QM are required prior to the Quantum Gravity and Quantum Cosmology applications; this is the main purpose of Papers II and III. In Sec 10 I consider HO dynamics on $\mathbb{C}P^2$ from a qualitative perspective (Key 12) in quadrilateralland terms. These free problem and HO problem cases are then considered at the quantum level in Paper III (see the Conclusion for an outline).

2 Physical interpretation of the shape momenta

I use relative angular momenta and relative distance momenta as names for the conjugates of relative angles and of ratios of relative separations respectively. I choose to use the first option in each case.

2.1 3- and 4-stop metroland

For 3-stop metroland in polar coordinates, the momenta are [dropping (a) labels and recycling the notation $p$ to mean the conjugate of $n^i$],

$$D := p_\varphi = \varphi^* = n_1 p_2 - n_2 p_1 = D_1 n_2 / n_1 - D_2 n_1 / n_1$$

for $D_i$ the partial dilations (in parallel to the $I_i$ being partial moments of inertia) $n_i p_i$ (no sum). The second form of this is manifestly a shape-weighted relative dilational quantity corresponding to a particular exchange of dilational momentum between the $\{bc\}$ and $\{a\}$ clusters. It is indeed conceptually clear that the conjugate to the non-angular length ratio $\varphi^{(a)}$ will be a relative distance momentum. I generally use the notation $D$ for whichever type of relative distance momenta.

For 4-stop metroland in spherical coordinates, the momenta are [dropping (Hb) or (Ka) labels]

$$D_\phi := p_\phi = \phi^* = n_1 p_2 - n_2 p_1 = D_3 n_2 / n_1 - D_4 n_1 / n_1 ,$$

i.e. a a weighted relative dilational quantity corresponding to a particular exchange of dilational momentum between the $\{ab\}$ and $\{cd\}$ clusters in the H-case or the $\{bc\}$ and $\{Ta\}$ clusters in the K-case, and

$$D_\theta := p_\theta .$$

2.2 Triangleland

The triangleland momenta in spherical coordinates and their interpretation are as follows (in terms of Dragt-type [28] coordinates and momenta):

$$D_\triangle := p_\Theta := \Theta^* = dra_1 \Pi_2^{dra} - dra_2 \Pi_1^{dra} ,$$

$$J := p_\phi := \sin^2 \Theta \Phi^* .$$
Here, and more generally, I use $J$ to denote angular momenta. This $J$, moreover, clearly cannot be an overall angular momentum since $L = 0$ applies. It is indeed a relative angular momentum [2]:

$$J = I_1 I_2 \Phi^*/I = I_1 I_2(\Theta_2^* - \Theta_1^*)/I = \{I_1 L_2 - I_2 L_1\}/I = L_2 = -L_1 = \{L_2 - L_1\}/2$$

[the fourth equality uses the zero angular momentum constraint (1.8)]. Thus this can be interpreted as the angular momentum of one of the two constituent subsystems, minus the angular momentum of the other, or half of the difference between the two subsystems' angular momenta. That is indeed a relative angular momentum also ought to also be clear from it being the conjugate of a relative angle. The $\Theta$ and $\Phi$ coordinates represent a clean split into pure non-angle ratios and pure angle ratios, by which they produce one relative dilational momentum and one relative angular momentum as their conjugates.

Franzen and I [8] termed the collective set of quantities of this nature relative rational momenta since they correspond to the general-ratio generalization of angle-ratio’s angular momenta. Franzen and I already noted that the rational momentum concept also naturally extends to include mixed dilational momentum and angular momentum objects in addition to the above examples of purely dilational and purely angular objects. Rational momenta was previously called generalized angular momenta by Smith [17]. Serna and I do not use this name since it is not conceptually descriptive; we rather unravel exactly what it means physically and thus call it by its ‘true name’ [29,30]. The rather conceptually-cleaner introduction of this at the level of the momenta rather than [8]’s at the level of the conserved quantities is new to the present paper. Our final proposal is to call them shape momenta, since what are mathematically ratio variables can also be seen to be dimensionless shape variables, and the quantity in question is the momentum conjugate to such a quantity. We celebrate this by passing from the notation $R$ for ‘rational’ to $S$ for ‘shape’.

This ‘true naming’ becomes clear in moving, away from the previous idea of interpreting in physical space the $SO(n)$ mathematics of the first few RPM models studied, to the following line of thought.

1) Scale–shape splits are well defined. Then there are corresponding splits into scale momenta and shape momenta.

2) The shape momenta are conjugate to dimensionless variables, i.e. ratios (or functions of ratios), accounting for why the previously encountered objects were termed rational momenta.

3) Then in some cases, shape momentum mathematics coincides with (arbitrary-dimensional) angular momentum mathematics, and also some shapes/ratios happen to be physically angles in space, so the interpretation in space indeed is as angular momentum.

4) But in other cases, shapes can correspond physically to ratios other than those that go into angles in space, e.g. ratios of two lengths (then one’s momentum is a pure relative distance momentum) or a mixture of angle and non-angle in space ratios (in which case one has a general shape momentum). Moreover, there is no a priori association between shape momenta and $SO(n)$ groups; this happens to be the case for the first few examples encountered (N-stop metroland, triangleland) but ceases to be the situation for quadrilateralland (and N-a-gonlands beyond that).

2.3 Quadrilateralland

The Gibbons–Pope-type coordinates for quadrilateralland extend the above triangleland spherical polar coordinates in constituting a clean split into pure non-angle ratios and pure angle ratios (two of each). Thus their conjugates are again cleanly-split pure relative angular momenta and relative dilational momenta as their conjugates (two of each):

$$J_\psi =: p_\psi = \sin^2 \chi \cos^2 \chi (\psi^* + \cos \beta \phi^*)/4,$$
$$J_\phi =: p_\phi = \sin^2 \chi \{\cos^2 \chi (\phi^* + \cos \beta \psi^*) + \sin^2 \chi \sin^2 \beta \phi^*/4,$$
$$D_\beta =: p_\beta = \sin^2 \chi \beta^*/4,$$
$$D_\chi =: p_\chi = \chi^*.\)
3 Forms of the shape Hamiltonians

The corresponding Hamiltonians are

\[ H = p_\chi^2/2 + V = D^2/2 + V \text{ (3-stop metroland) ,} \]  
\[ H = p_\psi^2/2 + p_\phi^2/2 \sin^2 \theta + V = D_\theta^2/2 + D_\phi^2/2 \sin^2 \theta + V \text{ (4-stop metroland) ,} \]  
\[ H = p_\theta^2/2 + p_\phi^2/2 \sin^2 \Theta + V = D_\Delta^2/2 + J_2^2/2 \sin^2 \Theta + V \text{ (triangleland) and} \]

\[ H = \frac{p_\chi^2}{2} + \frac{2}{\sin^2 \chi} \left\{ \frac{1}{\sin^2 \beta} \left( p_\beta^2 + p_\psi^2 - 2 p_\beta p_\psi \cos \beta \right) \right\} + \frac{2}{\cos^2 \chi} p_\psi^2 + V \]

\[ = \frac{D^2_\beta}{2} + \frac{2}{\sin^2 \chi} \left\{ D_\beta^2 + \frac{1}{\sin^2 \beta} \left( J_\beta^2 + J_\psi^2 - 2 J_\beta J_\psi \cos \beta \right) \right\} + \frac{2}{\cos^2 \chi} J_\psi^2 + V \text{ (quadrilateralland) .} \]  

4 Physical interpretation of RPM’s relationalspace isometries/conserved quantities

For a dynamical system, conserved quantities correspond to isometries of the kinetic metric that are also respected by the potential. This Section and the next deal with isometries; the subdivision of the generators of these into conserved quantities and elsewise for various potentials is the subject of Sec 7.

\[ \text{Isom}(S(N, 1)) = \text{Isom}(S^{n-1}) = PSO(n) \text{ (the n-dimensional projective special orthogonal group) = } SO(n) . \]  
\[ \text{Isom}(S(N, 2)) = \text{Isom}(\mathbb{CP}^{n-1}) = PSU(n) \text{ (the n-dimensional projective special unitary group) = } SU(n)/Z_n . \]  

The SU(n) versus SU(n)/Z_n distinction does not affect the algebra involved, though it does matter as regards some further subtleties (along the lines of the much better-known SU(2) versus SU(2)/Z_2 = SO(3) distinction). The above is standard to fairly standard mathematics; moreover, the physical interpretation of the generators of these (which for various classes of potentials are to be interpreted as conserved quantities) is somewhat unusual, as I shall build up case-by-case below.

4.1 N-stop metroland cases

The pure-shape case of 3-stop metroland is relationally trivial as per [15], but it is part of dynamically nontrivial scaled 3-stop metroland problem. Here the generator of Isom(S(3, 1)) = Isom(S^1) = SO(2) = U(1) is just the above-described Dil. This is mathematically the ‘component out of the plane’ of ‘angular momentum’, albeit in configuration space, there clearly being no meaningful physical concept of angular momentum in 1-d space itself.

For 4-stop metroland, the three generators of Isom(S(4, 1)) = Isom(S^2) = SO(3) are

\[ D_i = \epsilon_{ij}^{\ k} n^j p_k . \]  
\[ (17) \]

\[ D_i \text{ are mathematically the three components of ‘angular momentum’ albeit again in configuration space. Their physical interpretation (for the moment in the setting of H-coordinates) in space is an immediate extension of that of the already-encountered 3-component of this object [2]:} \]
\[ D_i = D_k n^j/n^k - D_j n^k/n^j . \]  
\[ (18) \]

Moreover, this example’s interpretation relies, somewhat innocuously, on the three conserved quantities Dil, corresponding to three mutually perpendicular directions (the three DD axes picked out by using H-coordinates), as is brought out more clearly by the next example.

For 4-stop metroland in K-coordinates one has the above formulae again [dropping (Ka) labels instead of (Hb) ones]. They are clearly still all relative distance momenta, albeit corresponding to a different set of ratios. Then e.g. \( D_3 \) is a (weighted) relative dilational quantity corresponding to a particular exchange of dilational momentum between the \{12\} and \{T3\} clusters. Here, one needs to use an axis system containing only one T-axis, e.g. a \{T, M^*D, M^*D\} axis system (c.f. Fig I.4).

For 4-stop metroland the total shape momentum counterpart of the total angular momentum is \( D_{\text{Tot}} = \sum_{i=1}^{3} D_i^2 = D_\theta^2 + \sin^{-2}\theta D_\phi^2 = 2T \) in terms of momenta. For 3-stop metroland, this is just \( D_{\text{Tot}} = D^2 \). In each case, finally, \( H = D_{\text{Tot}}/2 + V \).

The above pattern repeats itself, giving, for N-stop metroland, \( n - 1 \) hyperspherical coordinates interpretable as a sequence of ratios of relative inter-particle cluster separations, shape space isometry group SO(n) and a set of \( n(n - 1)/2 \) isometry generators which are, mathematically, components of ‘angular momentum’ in configuration space.
4.2 Triangleland case

Here, the three Isom \((S^2) = \text{Isom}(3, 2) = SO(3) = SU(2)/\mathbb{Z}_2\) generators are given by

\[
S_A = \epsilon_{AB} c \delta^{B} \Pi^{a} \phi^{a},
\]

which are mathematically the three components of ‘angular momentum’ albeit yet again in configuration space rather than in space. Now on this occasion, there is a notion of relative angular momentum in space. There are even three natural such, one per clustering: \(J_m\). Are these the three components of \(S_A\)? No! These three are coplanar and at 120 degrees to each other, so only can only pick one of these for any given orthogonal coordinate basis, much as in the above K-coordinate example. The other components point in an E and an S direction (c.f. Fig I.5). E and D are then the two main useful choices of principal axes, furnishing the \(\{E, D, S\}\) and \(\{D, E, S\}\). Moreover the component pointing in the D direction has the form of a pure relative angular momenta, \(S_3 = J\) of the \(\{23\}\) subsystem relative to the 1 subsystem. The other two \(S_A\)'s are mixed dilational and angular momenta with shape-valued coefficient [dropping (a) labels]:

\[
\sin \Phi D_\Delta + \cos \Phi \cot \Theta J \quad \text{and} \quad -\cos \Phi D_\Delta + \sin \Phi \cot \Theta J.
\]

For triangleland, the total shape momentum counterpart of the total angular momentum is \(S_{\text{Tot}} = \sum_{A=1}^{3} S_A^2 = D_\Delta^2 + \sin^{-2} \Theta J^2 = 2T\). Finally, \(H = D_{\text{Tot}}/2 + V\).

5 Quadrilateralland case

Quadrilateralland’s isometry group is \(\text{Isom}(S(4, 2)) = \text{Isom}(\mathbb{CP}^2) = PSU(3) = SU(3)/\mathbb{Z}_3\), giving the same representation theory and mathematical form of conserved quantities as in the idealized flavour \(SU(3)\) or the colour \(SU(3)\) of Particle Physics [these also have this quotienting]. MacFarlane studied this and the difference between it and \(SU(3)\) in [31]; they share the same algebra, but some topological differences. There are some parallels with the extent of the similarities between \(SU(2)\) and \(SU(2)/\mathbb{Z}_2 = SO(3)\) [which itself is relevant to RPM’s via \(\text{Isom}(3, 2) = \text{Isom}(S^1) = \text{Isom}(\mathbb{CP}^1) = PSU(2) = SU(2)/\mathbb{Z}_2\)].

5.1 Particle Physics analogues

Analogy 1) Flavour symmetry (constitution of hadrons in terms of up, down and strange quarks)\(^{[1]}\). Our use of 1, 2, 3, +, and – is the standard one of \(SU(2)\) mathematics. \(SU(3)\) contains three overlapping such ladders (in fact three overlapping \(SU(2)\times U(1)\)'s, with the \(SU(2)\)'s being isospins \(I_+\), \(I_-\), \(I_3\), \(V_+\), \(V_-\), \(V_3\) and \(U_+\), \(U_-\), \(U_3\) and the \(U(1)\)'s being hypercharges \(Y, Y_3\) and \(Y_0\)). The usual set of independent such objects, \(I_+, I_-\), \(I_3\), \(V_+\), \(V_-\), \(V_3\) and \(Y, Y_3\) are then represented by the Gell-Mann \(\lambda\)-matrices up to proportion. Then one can obtain \(V_3, U_3, V_0\) and \(Y_0\) in terms of these, these other quantities being useful on grounds of even-handedness between the three \(SU(2)\times U(1)\)'s (see the next subsection). One can then define \(I_{\text{Tot}} = \sum_{A=1}^{3} I_A^2\), \(U_{\text{Tot}} = \sum_{A=1}^{3} U_A^2\) and \(V_{\text{Tot}} = \sum_{A=1}^{3} V_A^2\). In total, \(SU(3)\) has 3 independent commuting quantities, which are usually taken to be \(I_{\text{Tot}}, I_3, Y\).

Note 1) Flavour symmetry is broken by mass differences, as it is only an approximate symmetry.

Note 2) The word ‘hypercharge’ leaves something to be desired via not being particularly descriptive. In flavour physics, it is an ‘extra charge’ that partly contains strangeness, unlike the isospin which is pure up and down. Thus Serna and I prefer ‘strange charge’ and ‘extra charge’ as names for it (taking due note that the charge is used to imply the more common \(U(1)\) symmetry rather than generalized non-abelian symmetry). In particular, we like ‘extra charge’ due to it coming picked out alongside, but not within, the \(SU(2)\) in the \(SU(2)\times U(1)\) combination selected by the basis.

Analogy 2) Colour symmetry. This use of \(SU(3)\) differs in being postulated to be exact, and in the red, green and blue labels being frivolous choices, so that one really has \(SU(3)/\mathbb{Z}_3\).

5.2 Key 9: Quadrilateralland’s conserved quantities

I calligraphize all of the above symbols in the quadrilateralland case, to distinguish these quantities clearly from their particle physics analogues. This application is fact more like colour physics than approximate flavour physics, in that the symmetry is exact. However, whilst for uninterpreted \(\mathbb{CP}^2\) one can take the three types of ladder to be frivolous labels and so involve \(SU(3)/\mathbb{Z}_3\), the quadrilateralland interpretation pins distinction upon the three ladders, so that one wishes for the whole \(\mathbb{CP}^2\) with its three uniform states per hemi-\(\mathbb{CP}^2\) rather than a folded-up version in which the three coincide.

On the basis of the above discussion, Serna and I call \(Z_3\) the angular charge and \(Y\) the extra angular charge due to its coming alongside the usual angle charge’s \(SU(2)\) but not within it, as a picked out \(SU(2)\times U(1)\). In the \(\mathbb{CP}^2\) realization of \(SU(3)\), this is \([32]\) not only picked out by the basis but also by the Gibbons–Pope-type coordinates in use.

\(^{[1]}\) Hence going from 1-d to 2-d or (3, 2) to \((N > 3, 2)\) parallels the transition of theoretical physics from the theoretical chemistry of Mendeleev through to the 1930’s to the particle physics of the 60’s through to the present day (including GUT’s).
\begin{align}
\mathcal{Y} &= 2p_\psi = 2\mathcal{J}_\psi, \quad \mathcal{I}_3 = p_\phi = \mathcal{J}_\phi.
\end{align}

In terms of the quadrilateralland-significant inhomogeneous bipolar coordinates, these are then
\begin{align}
\mathcal{Y} &= -2\{p_{\phi_1} + p_{\phi_2}\}, \quad \mathcal{I}_3 = p_{\phi_2} - p_{\phi_1}.
\end{align}

The meanings of \( \mathcal{Y} \) and \( \mathcal{I} \) are immediately inherited from those of \( p_\psi \) and \( p_\phi \) given in Sec 2.4.

### 5.3 Generators of the isometries in \( \mathbb{Z}^p \) coordinates from Noether’s theorem

Conserved quantities in terms of \( \mathbb{Z}^p \) and \( \Pi_p \) are presented below. MacFarlane [18] derived these from the Euler–Lagrange action. The present paper uses instead the Jacobi-type action, the outcome from which is equivalent to MacFarlane’s result by the following Lemma.

**Lemma.** The quantities arising from Noether’s theorem as applied to a Jacobi-type action are equivalent to those arising from the corresponding Euler–Lagrange-type action.

Thus MacFarlane’s results carry over to the context of relational Jacobi-type actions, and provide the following conserved quantities.
\begin{align}
2i\mathcal{I}_3 &= \mathbf{\Pi} \tau_3 \mathbf{Z} - \mathbf{\Pi} \mathbf{Z} \tau_3 \mathbf{Z}, \quad i\mathcal{Y} = \mathbf{\Pi} \cdot \mathbf{Z} - \mathbf{\Pi} \cdot \mathbf{Z} \quad (23)
\end{align}
\begin{align}
i\mathcal{I}_+ &= \mathbf{\Pi}_1 \mathbf{Z}^2 - \mathbf{\Pi}_2 \mathbf{Z}^1, \quad i\mathcal{I}_- = \mathbf{\Pi}_2 \mathbf{Z}^1 - \mathbf{\Pi}_1 \mathbf{Z}^2 \quad (24)
\end{align}
\begin{align}
i\mathcal{V}_+ &= \mathbf{\Pi}_1 + \mathbf{\Pi} \cdot \mathbf{Z} \mathbf{Z}^1, \quad -i\mathcal{V}_- = \mathbf{\Pi}_1 + \mathbf{\Pi} \cdot \mathbf{Z} \mathbf{Z}^1 \quad (25)
\end{align}
\begin{align}
i\mathcal{U}_+ &= \mathbf{\Pi}_2 + \mathbf{\Pi} \cdot \mathbf{Z} \mathbf{Z}^2, \quad -i\mathcal{U}_- = \mathbf{\Pi}_2 + \mathbf{\Pi} \cdot \mathbf{Z} \mathbf{Z}^2 \quad (26)
\end{align}
where \( \tau_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \) (the third Pauli matrix).

Note 1) The above constitutes a nonlinear realization of the \( SU(3) \).

Note 2) For the triangland counterpart, \( \mathcal{Y}_U = -\frac{1}{2}\{\mathcal{I}_3 + 3\mathcal{Y}\}, \quad \mathcal{I}_3 = \frac{1}{2}\{\mathcal{I}_3 - \mathcal{Y}\} \) and \( \mathcal{V}_3 = \frac{1}{2}\{\mathcal{I}_3 + \mathcal{Y}\} \).

The generators are of types
\begin{align}
\Pi \mathbf{Z} - \Pi \mathbf{Z}, \quad \Pi + \mathbf{Z}^2 \Pi, \quad \Pi + \mathbf{Z}^2 \Pi, \quad (27)
\end{align}
i.e., respectively, what \( \mathcal{I}_3 \) and \( \mathcal{Y} \), \( \mathcal{I}_+ \) and \( \mathcal{V}_+ \), and \( \mathcal{I}_- \) and \( \mathcal{V}_- \) pairwise collapse to: \( \mathcal{I}_\pm \) cease to exist at all. Quantities proportional to these generators are then a \( \mathcal{I}_3 \) and \( \mathcal{I}_\pm \) for the triangle.

Note 3) The three \( SU(2) \) ladders correspond to the three triangles (or two triangles and a rhombus) of coarse-graining in Fig I.8 [each of which, of course, is associated with a coarse-grained shape space sphere whose isometry group is the corresponding \( SO(3) \).] Furthermore, each ladder is paired with a hypercharge-type quantity to form three overlapping embedded \( SU(2) \times U(1) \) ‘s. \( \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{Y} \) is one of the embedded \( SU(2) \times U(1) \) groups within the \( SU(3) \), the others being the \( U \) and \( V \) counterparts of this.

### 5.4 Generators of the isometries in terms of Gibbons–Pope-type momenta

As well as
\begin{align}
\mathcal{Y} &= 2p_\psi = 2\mathcal{J}_\psi = -2\{p_{\phi_1} + p_{\phi_2}\}, \quad \mathcal{I}_3 = p_\phi = \mathcal{J}_\phi = p_{\phi_2} - p_{\phi_1},
\end{align}
one has
\begin{align}
\mathcal{I}_1 &= -\sin \phi p_\beta + \frac{\cos \phi}{\sin \beta} \{p_\psi - \cos \beta p_\phi\} = -\sin \phi D_\beta + \frac{\cos \phi}{\sin \beta} \{\mathcal{J}_\psi - \cos \beta \mathcal{J}_\phi\},
\end{align}
\begin{align}
\mathcal{I}_2 &= \cos \phi p_\beta + \frac{\sin \phi}{\sin \beta} \{p_\psi - \cos \beta p_\phi\} = \cos \phi D_\beta + \frac{\sin \phi}{\sin \beta} \{\mathcal{J}_\psi - \cos \beta \mathcal{J}_\phi\}.
\end{align}
Finally,
\begin{align}
\mathcal{I}_{tot} := \mathcal{I}_2^2 + \frac{1}{\sin^2 \beta} \{p_\psi^2 - 2\cos \beta p_\psi p_\phi + p_\phi^2\} = D_\beta^2 + \frac{1}{\sin^2 \beta} \{\mathcal{J}_\psi^2 - 2\cos \beta \mathcal{J}_\psi \mathcal{J}_\phi + \mathcal{J}_\phi^2\}.
\end{align}

Note 1) Thus, whether for \( H \)’s or for \( K \)’s there is also a pair of coordinates \( \beta \) and \( \chi \): additionally dependent on only one corresponding ratio of relative separations, i.e. the \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) depend on \( \beta \) alone rather than on \( \chi \). These are conjugate to quantities that involve relative distance momenta in addition to relative angular momenta.

Note 2) The other expressions (\( \mathcal{U}_\pm, \mathcal{V}_\pm \)) are much more complex and less insightful in these particular \( \mathcal{I} \)-adapted Gibbons–Pope type coordinates. Of course, \( U \) and \( V \) adapted Gibbons–Pope type coordinates exist as well, via omitting in each case a different choice of Jacobi vector. E.g. in the Jacobi \( H \) case, \( \mathcal{I} \) is tied to the collapse to the rhombus, with \( U \) and \( V \) corresponding to the two one-post collapse triangles. In terms of each of these coordinate systems, the corresponding sets of picked-out \( SU(2) \times U(1) \) quantities (i.e. \( \{\mathcal{U}_\pm, \mathcal{U}_3, \mathcal{Y}_U, \mathcal{U}_{tot}\} \) and \( \{\mathcal{V}_\pm, \mathcal{V}_3, \mathcal{V}_U, \mathcal{V}_{tot}\} \) have the same expressions as above (with \( U \), \( V \) labels, respectively, understood but dropped on the Gibbons–Pope type coordinates in use.)
5.5 Quantum-mechanical operator expressions for these isometry generators

See [8 7 9 15] for the forms these take in metroland and triangleland. For quadrilateraland (using \( h = 1 \)),

\[
\mathcal{Y} = -2i \frac{\partial}{\partial \psi}, \quad \hat{I}_3 = -i \frac{\partial}{\partial \phi}.
\]  

(32)

In terms of the quadrilateraland-significant inhomogeneous bipolar coordinates, these are

\[
\mathcal{Y} = 2i \left\{ \frac{\partial}{\partial \Phi_1} + \frac{\partial}{\partial \Phi_2} \right\}
\]  

and \( \hat{I}_3 = -i \left\{ \frac{\partial}{\partial \Phi_2} - \frac{\partial}{\partial \Phi_1} \right\} \).

(33)

Also,

\[
i\hat{I}_1 = -\sin \phi \frac{\partial}{\partial \beta} \cos \phi \frac{\partial}{\sin \beta}
\]  

\[
\left\{ \frac{\partial}{\partial \psi} - \cos \beta \frac{\partial}{\partial \phi} \right\}, \quad i\hat{I}_2 = \cos \phi \frac{\partial}{\partial \beta} + \sin \phi \frac{\partial}{\sin \beta} \left\{ \frac{\partial}{\partial \psi} - \cos \beta \frac{\partial}{\partial \phi} \right\}.
\]

(34)

Finally,

\[
\hat{I}^2 = - \left\{ \frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \sin \beta \frac{\partial}{\partial \beta} + \frac{1}{\sin \beta} \frac{\partial^2}{\partial \psi^2} - 2 \cos \beta \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} + \frac{\partial^2}{\partial \phi^2} \right\}
\]

(35)

is also needed for the subsequent QM application [18 26]. I give this in the operator-ordering that is relevant for Paper III’s time-independent Schrödinger equation to be in terms of the Laplacian. This is motivated by essentially amounting to constructing the argued-for (Sec I.1) conformal operator ordering, since the two are out by just a constant.

5.6 Interpretation: the collapse of the above to the usual \( SU(2) \) operators for the “\( I \)” coarse-graining

For \( \rho_3 = 0 \), i.e. \( \chi = \pi/2 \), one recovers the usual \( I_\Gamma \), \( \Gamma = 1 \) to 3, of \( SU(2) \) with \( \beta \) playing the role of \( \theta \). I.e. at the QM level at which these expressions are most familiar,

\[
I_1 = i \left\{ \sin \Phi \frac{\partial}{\partial \Theta} + \cos \Phi \cot \Theta \frac{\partial}{\partial \Phi} \right\} = iS_2, \quad I_2 = i \left\{ -\cos \Phi \frac{\partial}{\partial \Theta} + \sin \Phi \cot \Theta \frac{\partial}{\partial \Phi} \right\}, \quad I_3 = -i \frac{\partial}{\partial \Phi}.
\]

(36)

[The slight disalignment is due to different axis conventions and coefficients between the inherited-from-\( SU(3) \) case and the straight \( SU(2) \) case.] N.B. that this case is not a triangle; it is the rhombic coarse-graining.

5.7 Interpretation: quadrilateraland isometry generators

I mention the parallel with \( J = -i \partial/\partial \phi \) pure relative angular momentum in in triangleland whilst \( R_1 \) and \( R_2 \) are mixtures of relative angular momentum and relative dilational momentum. There is a looser parallel with \( D = -i \partial/\partial \phi \) in 4-stop metroland which has, however, a different meaning.

In \( H[\text{H}(\text{DD})] \) coordinates, the meaning of \( I_1 \) and \( I_2 \) coordinates is that of mixed relative angular momentum and relative dilation of the \( \beta \) type, i.e. a rate of change in the contents inhomogeneity (the ratio of the sizes of the two constituent subclusters). The meaning of \( I_\text{Tot} \) is the total angular momentum of the third, rhombic, coarse-graining triangle of the \( H \) in Fig I.8. In each case, changing which ratios one regards as primary gives similar presentations for the \( U \)'s and \( V \)'s. In \( H(M^3D) \) coordinates, the meaning of \( I_1 \) and \( I_2 \) is that of mixed relative angular momentum and relative dilation of the \( \beta \) type, i.e. a rate of change in the ratio of the selected post to the crossbar. The meaning of \( I_\text{Tot} \) is the total angular momentum of the first or second coarse-graining of \( H \) in Fig I.8 (depending on which post is selected).

In \( K(=K(T)) \) coordinates, the meaning of \( I_1 \) and \( I_2 \) is that of mixed relative angular momentum and relative dilation of the \( \beta \) type, i.e. a rate of change in the ratio of the back to the seat (ie a sharpness/flatness shape quantity for the obvious triangle subsystem). The meaning of \( I_\text{Tot} \) is the total angular momentum of the second coarse-graining triangle of \( K \) in Fig I.8.

In \( K(M^3D) \)-coordinates, the meaning of \( I_1 \) and \( I_2 \) is that of mixed relative angular momentum and relative dilation of the \( \beta \) type, a rate of change of the ratio of the seat to the leg The meaning of \( I_\text{Tot} \) in \( K(M^3D) \)-coordinates is the total angular momentum of the third coarse-graining of \( K \) in Fig I.8.

The meaning of \( I_1 \) and \( I_2 \) in \( H(M^3D) \) coordinates is that of mixed relative angular momentum and relative dilation of the \( \beta \) type, i.e. a rate of change in the ratio of the back to the leg. The meaning of \( I_\text{Tot} \) in \( H \)-coordinates is the total angular momentum of the first coarse-graining triangle of \( K \) in Fig I.8. In each case, changing which ratios one regards as primary gives similar presentations for the \( U \)'s and \( V \)'s.

6 Problem of Time application: set of Kuchař observables

(\text{Dirac}) Observables \([33]\) alias constants of the motion alias \textit{perennials} \([22 34 35]\) are any function(al)s of the canonical variables \( D\{Q,P\} \) of the canonical variables (see footnote 1 of Paper I for this notation) such that, at the classical level, their Poisson brackets with all the constraint functions vanish (perhaps weakly \([24]\)). For a theory with
total constraint set \( \{ \mathcal{C}_\Lambda \} \), Dirac observables \( O = D(\mathbf{Q}, \mathbf{P}) \) obey

\[
\{ \mathcal{C}_\Lambda, O \} = 0 . \tag{37}
\]

Thus, for geometrodynamics

\[
\{ \mathcal{H}(x), O \} = 0 , \tag{38}
\]

\[
\{ \mathcal{H}(x), O \} = 0 . \tag{39}
\]

Justification of the name ‘constants of the motion’ conventionally follows from the total Hamiltonian being \( H | \Lambda^\Lambda \rangle = \int_\Sigma \Lambda^\Lambda \mathcal{C}_\Lambda \), so that (37) implies

\[
\frac{dO}{dt}[Q(t), P(t)] = 0 . \tag{40}
\]

**Alternative Frozen Formalism Facet.** The operator-and-commutator counterparts of the above are then another manifestation of the Frozen Formalism Problem of classical canonical GR. [This is some sort of ‘Heisenberg’ counterpart of the ‘Schrödinger’ Wheeler–DeWitt equation being frozen.]

**Kuchař’s Unicorn** I take this to be a sufficient set of Dirac observables/perennials to describe one’s theory is termed. This follows from his quotation “Perennials in canonical gravity may have the same ontological status as unicorns – a priori, these are possible animals, but a posteriori, they are not roaming on the earth” [22].

Replace (37) with split conditions

\[
\{ \text{Quad}, O \} = 0 , \tag{41}
\]

\[
\{ \text{Lin}_{\Sigma}, O \} = 0 . \tag{42}
\]

Usually there is but one quadratic constraint (per space point), though one could index it if needs be (see e.g. [36]).

Kuchař observables [22] are then as above except that only their brackets with the linear constraints (42) need vanish.

Kuchař then argued [22] for only the former needing to hold, in which case I denote the objects not by \( O = D[\mathbf{Q}, \mathbf{P}] \) but by \( K[\mathbf{Q}, \mathbf{P}] \), with the K standing for ‘Kuchař observable’. See also [22, 35].

Beyond these arguments of sufficiency, I also use these in this series of papers as a technical half-way concept/construct in the formal and actual construction of Dirac observables.

As regards partial observables, one usually starts this discussion with true observables (Rovelli 1991 [37], see also [38]) alias complete observables (Rovelli 2002 [39], and which at least Thiemann [40] also calls evolving constant of the motion) classically involve operations on a system each of which produces a number that can be predicted if the state of the system is known. This conceptualization of observables is related to the above Dirac observables and should then be contrasted with the following much more cleanly distinct conception.

Then Partial observables themselves (Rovelli 1991 [37]) classically involve operation on the system that produces a number that is possibly totally unpredictable even if the state is perfectly known.

While the above definitions were more or less in place by 1991, the early 1990’s and 2000’s forms of the Problem of Time strategies that use these do themselves in part differ. Since these approaches will largely not play a further role in the present article, I refer to [39, 41, 42, 25] for their further characterization and remaining difficulties.

Quantum-mechanically, each of the above two notions of observables carry over except that the entities whose predictabilities enter the definitions become quantum mechanical, the brackets become commutators and, in Rovelli’s approach, the states are now taken to be specifically Heisenberg states.

I view this as a major first application of the understanding gained in Paper I about the shape variables and in Paper II about their conjugates to Problem of Time issues.

Now, shape variables and shape momenta have the additional interpretation as Kuchař observables. Then that the shape variables lucidly correspond to/are centred about geometrically significant configurations and their momenta lucidly correspond to changes of these acquires further significance. One gets the sense that these are practically interesting observables and, at least sometimes, correspond to localized clusters.

Now, for the pure-shape case, (41) and (42) become

\[
\{ \mathcal{H}, O \} = 0 , \tag{43}
\]

\[
\{ \mathcal{L}_\mu, O \} = 0 , \tag{44}
\]

\[
\{ \mathcal{D}, O \} = 0 . \tag{45}
\]
Then Kuchar observables $O = K(\mathbf{Q}, \mathbf{P})$ solve \(14\) for the scaled case, and \(14\) for the pure-shape case. Dirac observables $O = D(\mathbf{Q}, \mathbf{P})$ solve \(13\) for the scaled case and \(13\) for the pure-shape case. This is because the Best Matching problem \(15\) (the geometrodynamical case of which is the Thin Sandwich Problem, and which is a further facet of the Problem of Time) is solved for 1- and 2-d RPM’s, whether pure-shape or scaled, by \(6\) \(10\) \(15\). And that straightforwardly amounts to a construction and interpretation of a resolution of the problem of Kuchar observables for that. This occurs in pure-shape RPM for precisely the set of all functions of the shape variables and the shape momenta, $K(\mathbf{S}, \mathbf{P}_S)$. Likewise, the set of Kuchar observables for pure-shape RPM is precisely the set of all functions of the scale and shape variables and the scale and shape momenta, $K(\sigma, \mathbf{S}, \mathbf{P}_\sigma, \mathbf{P}_S)$. The quantum counterpart of the above then ‘straightforwardly’ involves some operator form for the canonical variables and commutators in place of Poisson brackets.

Note that here the best-matching problem is solved for 1- and 2-d RPM’s, whether pure-shape or scaled by results summarized in Paper I. And have been able to straightforwardly construct and interpret a resolution of the problem of observables in the sense of Kuchar. The corresponding Kuchar observables are those quantities whose brackets with the linear constraints vanish. This occurs in pure-shape RPM for precisely the set of all functions of the shape variables and the shape momenta. Likewise, the set of Kuchar observables for pure-shape RPM is precisely the set of all functions of the scale and shape variables and the scale and shape momenta.

I can spell out what all of these are for pure-shape and scaled RPM’s in 1- and 2-d. The 1-d pure-shape r-configuration spaces are \(4\) $S^{N-2}$ and suitable shape variables thereupon are the (ultra)spherical angles \(8\), interpreted as functions of ratios of relative separations. This is as exemplified in Sec I.4 for 3- and 4-stop metroland cases. The corresponding shape momenta are then as per Sec 2.

The 2-d pure-shape r-configuration spaces are $\mathbb{CP}^{N-2}$ and suitable shape variables fore these are the inhomogeneous coordinates $Z^\rho$. To interpret these complex coordinates in terms of the N-a-gons, it is useful to pass to their polar forms, $Z^\rho = R^\rho \exp(\psi^\rho)$. Then the moduli are, again, ratios of relative separations, and the phases are now relative angles. In the specific case of the scalefree triangle, there is one of each, e.g. in coordinates based around the 1,23 clustering, these are \[\Theta = 2 \arctan(\rho_2/\rho_1) \text{ and } \arccos(\rho_1 \cdot \rho_2/\rho_1 \rho_2).\] The shape momenta for the N-a-gon are \[15\]

$$
P^R_\rho = \left\{ \frac{\delta_{pq}}{1 + ||R||^2} \right\} R^q \quad P^\Theta_\rho = \left\{ \frac{\delta_{pq}}{1 + ||R||^2} \right\} R^q_\pi \Theta^p_\pi.$$

I gave the triangle in \(10\) as a specific 2-d example; in the present paper I give the quadrilateral case as a larger and new specific example. This puts the program in \(43\) into a whole-universe, nontrivially linearly constrained context. The Gibbons–Pope version of this gets a fivefold interpretation in terms of quadrilaterals. All of these are intuitive and, for some configuration space regions, local, conditions. Namely, that of Sec I.5 for shapes and Sec 2 for momenta. Then Kuchar observables for this problem are functions of the form $K(\chi, \beta, \phi, \psi, \pi_\chi, \pi_\beta, \pi_\phi, \pi_\psi$ alone).

These all make for geometrically (in space) meaningful propositions and some are sometimes locally determinable/locally observable. Actual propositions involve approximate values of quantities, and this then rests on configuration space regions as studied in Paper I.

As regards the use of conserved quantities in preference to/alongside the momenta,

1) these, or functions thereof, commute also with the Hamiltonian constraint and are thus Dirac Observables. They manage to be this way via not encountering an obstruction from the potential term in $H$. 
2) They feature in the kinematical quantization procedure, making them even more natural at the quantum level. For the sphere, these are the $SU(2)$ quantities $\mathcal{S}_X$ for the quadrilateral, these are the $SU(3)$ quantities, especially the $\mathcal{I}_A$ and $\mathcal{Y}$ that remain conserved quantities for a wider range of potentials. Here also e.g. for the sphere, $\Phi$ and $\Theta$ are not good operators, it is the unit Cartesian vectors that are.

A further issue here is what is the extent of overlap between kinematical quantization’s \(14\) object selection and selection of observables. One’s classical notion of observable is in each of the above cases to be replaced with the quantum one tied to a suitable commutation algebra in place of the classical Poisson algebra; this correspondence is however nontrivial (e.g. the two algebras may not be isomorphic) due to global considerations \(14\).

7 Equations of motion and conserved quantities for various potentials

7.1 N-stop metrolands and triangledland

For triangleland [and suppressing (a)-labels]: $\Phi$-independence in the potential corresponds to there being no means for angular momentum to be exchanged between the subsystem composed of particles 2, 3 and that composed of particle 1. This corresponds to having an $SO(2)$ invariance (‘special case’) If the potential is additionally $\Theta$-independent and so constant, one has the full $SO(3)$ invariance (‘very special case’) For N-stop metroland, there is likewise a sequence of special, very special, ... very $Y^{-2}$ special potentials corresponding to $SO(2)$, $SO(3)$, ... $SO(Y)$. These above observations are useful as regards finding a nice range of analytic solutions of increasing complexity \[12\] \[8\] \[11\] \[13\]. Moreover, e.g. for triangleland, there are in fact three particular $SO(2)$’s, corresponding to the three $\Phi$’s defined relative to the three DM axes present, albeit only one of these may be realized in any given model. I will next consider the quadrilateral counterparts of these statements.
7.2 Equations of motion for quadrilateralland in Gibbons–Pope type coordinates

The \( \psi \)-, \( \phi \)-, \( \beta \)- and \( \chi \)-equations are, respectively,

\[
\{ \sin^2 \chi \cos^2 \chi \{ \psi^* + \cos \beta \phi^* \} / 4 \}^* = -\partial \mathcal{V} / \partial \psi ,
\]

\[
\{ \sin^2 \chi \{ \cos^2 \chi \{ \phi^* + \cos \beta \psi^* \} + \sin^2 \chi \sin^2 \beta \phi^* \} / 4 \}^* = -\partial \mathcal{V} / \partial \phi ,
\]

\[
\{ \sin^2 \beta \phi^* / 4 \}^* = \sin^2 \chi \sin \beta \{ \sin^2 \chi \cos \beta \phi^* - \cos^2 \chi \psi^* \} \phi^* / 4 - \partial \mathcal{V} / \partial \beta , \quad \text{and}
\]

\[
\chi^* = \sin \chi \cos \chi \{ \beta^* + \cos 2 \chi \{ \phi^* + \psi^* + 2 \phi^* \psi^* \cos \beta \} + 2 \sin^2 \chi \sin^2 \beta \phi^* \} / 4 - \partial \mathcal{V} / \partial \chi .
\]

One of these can be supplanted by the energy first-integral,

\[
\chi^* = \sin \chi \cos \chi \{ \beta^* + \cos 2 \chi \{ \phi^* + \psi^* + 2 \phi^* \psi^* \cos \beta \} + 2 \sin^2 \chi \sin^2 \beta \phi^* \} / 4 - \partial \mathcal{V} / \partial \chi .
\]

(See [8] for the 4-stop metroland equations of motion, and [7] for the triangleland ones.)

7.3 Which potentials realize which subgroups?

i) For \( V \) explicitly dependent on all of \( \chi, \beta, \phi, \psi \), no isometry generator survives as a conserved quantity.

ii) For \( V \) \( \psi \)-independent, \( \psi \) is a cyclic coordinate and yields one constant of the motion,

\[
\sin^2 \chi \cos^2 \chi \{ \psi^* + \cos \beta \phi^* \} = C .
\]

This corresponds to a \( U(1) \) symmetry. I identify this \( C \) as \( 2 \mathcal{V} \).

iii) For \( V \) \( \phi \)-independent, \( \phi \) is a cyclic coordinate and yields another constant of the motion,

\[
\sin^2 \chi \{ \cos^2 \chi \{ \phi^* + \cos \beta \psi^* \} + \sin^2 \chi \sin^2 \beta \phi^* \} = K .
\]

This also corresponds to a \( U(1) \) symmetry. I identify this constant \( K \) as \( 4 \mathcal{I}_3 \).

iv) Potentials independent of both \( \psi \) and \( \phi \) yield both of these at once, corresponding to a \( U(1) \times U(1) \) symmetry.

v) Potentials independent of both \( \beta \) and \( \phi \) yield three conserved quantities, corresponding to \( SU(2) \) symmetry.

vi) Potentials independent of all of \( \beta, \phi \) and \( \psi \) yield four, corresponding to \( SU(2) \times U(1) \) symmetry.

vii) If the potential is constant, one has all eight conserved quantities corresponding to the full \( SU(3) \) isometry group.

There are also \( U \) and \( V \) counterparts of all of the above, so that there are 3 versions of all the partial symmetries.

---

7.4 Hamiltonians with conserved quantities back-substituted in

For 3-stop metroland with \( \phi \)-independent potential, \( H = \mathcal{D}_{\text{Tot}} + V \), constant. For 4-stop metroland, with \( \phi \)-independent potentials,

\[
H = p_{\phi}^2 / 2 + \mathcal{D}_{\phi}^2 / 2 \sin^2 \theta + \mathcal{V}(\theta) .
\]

For triangleland, with \( \Phi \)-independent potentials [\( U(1) \)-symmetric case] one has \[3 [7

\[
H = p_{\phi}^2 / 2 + \mathcal{J}^2 / 2 \sin^2 \Theta + \mathcal{V}(\Theta) .
\]

Each of these two has a constant case for the full \( SO(3) \) symmetry. The question then is what is the quadrilateralland counterpart of these simplified (partly) symmetric cases.

One can now use the conserved quantity equations to write down Hamiltonians with more constants of the motion inside instead of momenta.

---

Figure 1: Flow diagram for the breakdown of the \( SU(3) \) symmetry group due to various different potentials. The coordinates listed are those required to be absent from the potential. In fact, there are 3 different \( SU(2) \times U(1) \) possibilities, each adapted to one of \( \mathcal{I}, \mathcal{V} \) and \( \mathcal{U} \).
If there is a $U(1)$ symmetry of the $\psi$ type,

$$H = \frac{p_\psi^2}{2} + \frac{2}{\sin^2 \chi} \left\{ p_\beta^2 + \frac{1}{\sin^2 \beta} \left( \frac{p_\phi^2 + \frac{\gamma^2}{4} - p_\phi \gamma \cos \beta}{2 \cos^2 \chi} \right) \right\} + \frac{\gamma^2}{2 \cos^2 \chi} + V(\chi, \beta, \phi), \ \gamma \text{ constant} .$$

(56)

If there is a $U(1)$ symmetry of the $\phi$ type,

$$H = \frac{p_\chi^2}{2} + \frac{2}{\sin^2 \chi} \left\{ p_\beta^2 + \frac{1}{\sin^2 \beta} \left( \frac{\mathcal{I}_3^2 + p_\phi^2 - 2 \mathcal{I}_3 p_\phi \cos \beta}{2 \cos^2 \chi} \right) \right\} + \frac{2p_\phi}{2 \cos^2 \chi} + V(\chi, \beta, \psi), \ \mathcal{I}_3 \text{ constant} .$$

(57)

If there is a $U(1) \times U(1)$ symmetry,

$$H = \frac{p_\chi^2}{2} + \frac{2}{\sin^2 \chi} \left\{ p_\beta^2 + \frac{1}{\sin^2 \beta} \left( \frac{\mathcal{I}_3^2 + \frac{\gamma^2}{4} - \mathcal{I}_3 \gamma \cos \beta}{2 \cos^2 \chi} \right) \right\} + \frac{\gamma^2}{2 \cos^2 \chi} + V(\chi, \beta), \ \mathcal{I}_3, \gamma \text{ constant} .$$

(58)

If there is a $SU(2)$ symmetry,

$$H = \frac{p_\chi^2}{2} + 2 \mathcal{I}_3^2 / \sin^2 \chi + 2 p_\phi^2 / \cos^2 \chi + V(\chi, \psi), \ \mathcal{I} \text{ constant} .$$

(59)

If there is an $SU(2) \times U(1)$ symmetry,

$$H = \frac{p_\psi^2}{2} + 2 \mathcal{I}_3^2 / \sin^2 \chi + \frac{\gamma^2}{2 \cos^2 \chi} + V(\chi), \ \gamma \text{ constant} .$$

(60)

The $SU(3)$ symmetry has $H$ constant.

The simplest nontrivial case of $SU(2) \times U(1)$ symmetry can be straightforwardly represented as (via $u = \cos^2 \chi$)

$$t^e - t_0^e = -\frac{1}{2} \int \frac{u}{\sqrt{-\mathcal{Y}^2 + \{I \mathcal{Y} + 2 \mathcal{W}(u) - 4\mathcal{I}_3^2\}u - 2\mathcal{W}(u)u^2}} .$$

(61)

8 Key 10: HO-type potentials

As explained in e.g. [15] these are not HO’s per se in the pure-shape case, since they have to be homogeneous of degree zero in order to be consistent. This is attained by dividing the usual HO expression for the potential by the moment of inertia of the system (which subsequently turns out to be a constant). The scaled case has the usual HO potentials.

For 3-stop metroland [15], $V = K_1 n_1^2 / 2 + K_2 n_2^2 / 2 + L_1 n_1 n_2 = A + B \cos 2 \varphi + C \sin 2 \varphi$

(62)

for $A = \{K_1 + K_2\}/2, \ B = \{K_2 - K_1\}/2, \ C = L/2$ .

(63)

There is a special case with $SO(2) = U(1)$ symmetry, for $B = 0 = C$ i.e. $L = 0$ and $K_1 = K_2$, so that cluster 1 and cluster 2 have the same ‘constitution’: the same Jacobi–Hooke coefficient per Jacobi cluster mass; here the ‘constituent springs’ potential contributions balance out to produce the constant potential, $V = A$. This is a kind of ‘homogeneity requirement’ on the ‘structure’ of the model universe. For 4-stop metroland [8] [11],

$$V = \sum_{a=1}^{3} \{ K_a n_a^2 / 2 + L_a n_a^2 \} = A + B \cos 2 \Theta + C \sin^2 \Theta \cos 2 \phi + D \sin^2 \Theta \sin 2 \phi + E \sin 2 \Theta \cos \phi + F \sin 2 \Theta \sin \phi$$

(64)

for $A = \frac{1}{4} \left\{ K_3 + K_1 + K_2 \right\}/2, \ B = \frac{1}{4} \left\{ K_3 - K_1 + K_2 \right\}/2, \ C = \frac{K_1 - K_2}{4}$ .

(65)

This has a special case with $U(1)$ symmetry, for $L_a = 0$ and $K_1 = K_2$ [i.e. $C, D, E, F = 0$], $V = A + B \cos 2 \Theta$. It also has a very special case with $SO(3)$ symmetry, for $B, C, D, E, F = 0$, i.e. $L_a = 0$ and $K_1 = K_2 = K_3$, for which high-symmetry situation the various potential contributions balance out to produce the constant, $V = A$.

For trianguland [9] [13],

$$V = K_1 n_1^2 / 2 + K_2 n_2^2 / 2 + L n_1 n_2 = A + B \cos \Theta + C \sin \Theta \cos \Phi$$

(66)

This has a special case with $U(1)$ symmetry, for $L = 0$ (i.e. $C = 0$), $V = A + B \cos \Theta$. It also has a very special case with $SO(3)$ symmetry, for $L = 0$, $K_1 = K_2$, i.e. $B, C = 0$, $V = A$.

Then for quadrilateralland, a parametrization of the HO-type potential at level of Jacobi vectors is

$$V = \{ K_1 n_1^2 + K_2 n_2^2 + K_3 n_3^2 \}/2 + L_1 n_1 \cdot n_3 + L_2 n_2 \cdot n_1 + L_3 n_3 \cdot n_2$$

(67)

I firstly note that Kuiper coordinates are very HO-adapted:

$$V = \{ K_1 N^1 + K_2 N^2 + K_3 N^3 \}/2 + L_1 \text{aniso}(23) + L_2 \text{aniso}(31) + L_3 \text{aniso}(12)$$

(68)

However, redundancy and non-adaptation of the kinetic term limit the usefulness of this expression.
The first 3 terms of this in Gibbons–Pope-type coordinates form the combination as the first three terms of (64), since it involves solely the real parts for which the 2-d 4 particle problem reduces to the 1-d 4 particle one (in fact the mirror image identified version of this). This rearrangement uses linear combinations of eqs (I.70-I.72). For the other 3 cross-terms, however, the analysis has specific 2-d character in contrast to (64)’s 1-d character. These are, for H and K(M∗D) coordinates, and using eqs (I.73-I.75),

\[ V = \left\{ L_3 \sin \beta \sin^2 \chi \cos f_3 + L_1 \sin \beta \sin 2\chi \cos f_1 + L_2 \cos \beta \sin 2\chi \cos f_2 \right\} / 2. \] (69)

On the other hand, for K(T) coordinates, one has the 2 ↔ 3 of the above, and likewise by Sec I.7’s transpositions argument for the other choices of tree and of ratios. As regards HO-like potentials possessing particular symmetries, see Fig 2.

Figure 2: Flow diagram of the breakdown of the kinetic term’s SU(3) symmetry group due to various different HO potentials. In the SU(2) × U(1) symmetric: if realized alongside at most the first term, it is the partial balance \( K_1 = K_2 \) of particular significance in the H-case as universe contents homogeneity, and in which case \( B = \{ K_3 - K_1 \} / 4. \)

9 Classical solutions for quadrilateralland

9.1 Geodesics of \( \mathbb{CP}^k \) and their \( N \)-a-gonland interpretation

For the general \( \mathbb{CP}^k \) \[45\], geodesics through the origin are particularly simply expressed in complex form,

\[ Z^I = \tau C^I \] (70)

for \( \tau \) a parameter and \( C^I \) a constant vector.

9.2 Triangleland geodesics

(This is new to this program, insofar as \[7\] treated this as a real manifold.) Eliminating \( \tau \) from (70) in this case, and passing to the convenient spherical coordinates amounts to \( \Phi \) being constant whilst \( \Theta \) varies. The first of these conditions means that \( \mathcal{J} = 0 \). These are the set of meridians with the origin being the D-pole corresponding to the underlying choice of clustering and the infinity being the M-pole antipodal to this. These are indeed a subset of the great circles that are well-known to be the geodesics in this case, and which were interpreted in terms of quadrilaterals in \[7, 9\]. A particular such is the meridian of collinearity and another such is the meridian of isoscelesness. For later comparison, I furthermore note that these run between the clustering’s two notions of collapse: \( \rho_1 \to 0 \) and \( \rho_2 \to 0 \). I.e. from the arbitrarily sharp triangle to the arbitrarily flat one. The \( N \)-stop metroland spheres give (generalized) great circles but with complex formulations essentially absent (only present for \( N = 3 \)).

9.3 Key 11: Quadrilateralland geodesics

In the quadrilateralland case, eliminating \( \tau \) from (70) and passing to the useful Gibbons–Pope coordinates amounts to \( \psi, \phi \) and \( \beta \) being constant whilst \( \chi \) varies. The first three of these conditions imply that \( \mathcal{Y} \) and \( \mathcal{I}_{\text{Tot}} \) are both zero. These also run between two collapsed cases, although now there is a diversity of such collapses available and of interpretations for these geodesics, according to the choices H or K and then of which common denominator to pick in making the subsequent two ratios (the usual five choices of Sec I.7). The 0 end is the 2 Jacobi distance collapse case and the \( \infty \) end is the 1 Jacobi distance collapse complementary to it. Thus, these motions pick out the following. For the usual H, this geodesic family corresponds to the posts to crossbar ratio increasing from ‘both posts collapsed to form a DD’ of Fig I.8 o) to crossbar collapsed to form the rhombus of Fig I.8 g).

For H(M∗D), this geodesic family corresponds to the crossbar and one post to the other post ratio increasing from ‘one post and the crossbar collapsed to form M∗D’ of e.g. Fig I.8.m) to the other post collapsing to form the triangle of Fig I.8.e). (This one covers 2 cases at once, corresponding to symmetry of either post collapsing).
For the usual K, this geodesic family corresponds to the back and seat to leg ratio increasing from ‘back and seat collapsed to form a T’ of Fig I.8.q) to ‘leg-collapsed particle 3 onto T’ triangle of Fig I.8.k).

For K(M^T D), this geodesic family corresponds to the back and leg to seat ratio increasing from ‘back and leg collapsed to form a M^T and a D’ of Fig I.8.r) to ‘seat-collapsed T onto +’ triangle of Fig I.8.l).

For K(M^* D), this geodesic family corresponds to the leg and seat to back ratio increasing from ‘leg and seat collapsed to form a M^* D’ of Fig I.8.p) to ‘back-collapsed DD’ triangle of Fig I.8.j).

9.4 Time-traversal formulae

Some simple time-traversal cases that have analytical integrals (in terms of the emergent time) are as follows [from 61].

1) For constant potential, \( \mathcal{I}_{\text{Total}} \) and \( \mathcal{Y} \) are both 0, and then

\[
\chi = \sqrt{2W \{t^{em} - t^{em}_0\}}.
\]

(71)

The shapes here are just \( \chi \) changes as per above, and the time traversal confirms that these do not turn around, so the \( \chi \) runs from one extreme value to the other.

2) Formulae such as (precise trig/hyp functions involved depend on the signs of the various coefficients involved)

\[
\chi = \arccos \left( \sqrt{W - 2\mathcal{I}_{\text{Total}}} \sin \left( -\sqrt{2W \{t^{em} - t^{em}_0\}} \right) \right)
\]

(72)

for nonzero \( \mathcal{I}_{\text{Total}} \) and zero \( \mathcal{Y} \) and still having constant potential.

3) Formulae such as

\[
\chi = \arccos \left( \sqrt{1 - 2Y^2}/2W \sin \left( \sqrt{2W \{t^{em} - t^{em}_0\}} \right) \right)
\]

(73)

for \( \mathcal{I}_{\text{Total}} \rightarrow \mathcal{Y} \) of the above.

4) \[
\chi = \arccos \left( \frac{\exp(C \{t^{em} - t^{em}_0\} - B) \{\exp(C \{t^{em} - t^{em}_0\} - B - 2A\sqrt{C})}{2\sqrt{C} \{B + 2\sqrt{C}\}} \right)
\]

(74)

for both of these conserved quantities being nonzero, and where \( A = -Y^2, B = 2W - 4I^2 + Y^2 \) and \( C = -2W \). 4-stop metroland and triangleland are analogous to zero extra charge cases of the above.

9.5 Further solutions

Unlike for 4-stop metroland and triangleland, where there is a range of further classically-tractable solutions [8, 7, 15], further cases for quadrilateraland, and the simplest HO counterparts, give at best combinations including elliptic functions (checked with Maple). The QM of these isn’t analytically tractable either. However, the direction I will take is treating the HO potentials as small perturbations about the free case at the quantum level.

The following Figure provides useful qualitative analysis of \( \mathcal{Y} = 0 \) and \( \neq 0 \) and \( \mathcal{I}_{\text{Total}} = 0 \) and \( \neq 0 \) (Key 12).

10 Conclusion

10.1 Contents summary

The shape momenta conjugate to the shape variables studied in Paper I were considered. Physically, these are relative dilational momenta, relative angular momenta and mixtures of these. Conserved quantities for RPM’s in 1-d were also considered; these are particular combinations of the preceding. Functions of the shape variables and shape momenta can be used to resolve the Problem of Observables in the sense of Kuchař for these models. This paper consolidates the above for small RPM’s and is the first place to extend as far as the explicit quadrilateraland examples of these things.

This is a substantial extension due to how quadrilateraland is the smallest RPM to simultaneously possess nontrivial subsystem structure and linear constraints in the quantum cosmologically significant case. It also has more general and typical mathematics for an N-a-gonland than triangleland does (since its shape space is \( \mathbb{CP}^2 \) whilst triangleland’s is \( \mathbb{CP}^1 \), which is atypical through also being \( S^2 \)).

Quadrilateraland’s isometry group is then (Key 9), more or less, \( SU(3) \); more precisely, it is \( SU(3)/\mathbb{Z}_3 \). Thus quadrilateraland exhibits a number of parallels with the Particle Physics of the strong force, though I use the more descriptive name ‘extra charge’ instead of hypercharge. A new question addressed in this paper then is how to interpret the \( SU(3) \) quantities in terms of the quadrilateral. Gibbons–Podpe-type coordinates are useful in this investigation - they have clear geometrical interpretation in quadrilateraland terms, and they are additionally cyclic coordinates for some of the simpler potentials I gave the geometric interpretations of the momenta and conserved quantities in terms of these, for Jacobi H coordinates and Jacobi K coordinates for the various ratio choices, and then what various of the \( SU(2) \) quantities are in terms of these. The quantum form of \( \mathcal{I}_{\text{Total}} \) that I provide in this paper is useful in building Paper III’s time-independent Schrödinger equation.
Figure 3: Qualitative behaviour of some of the simpler HO dynamics: for the potential \( A + B \cos 2\chi \) for various ratios of \( A \) and \( B \). The Quadrilateralland case of this (column 2) is usefully compared with the corresponding potentials from two existing works and a further variant. Namely, triangleland (column 1 [9]), 4-stop metroland (column 4, [8]) and O4-stop metroland (column 3, also new to this Paper). Furthermore, a centrifugal spike as felt for \( L_{\text{Tot}} \neq 0 \) in ordinary mechanics or spherical mechanics, \( S_{\text{Tot}} \neq 0 \) triangleland and \( D_{\text{Tot}} \neq 0 \) for 4-stop metroland. That the on-sphere cases of this have a spike at each pole is standard. This occurs for the triangleland case and the 4-stop metroland case. The O4-stop metroland case has just the North Pole spike. A feature that first occurs in this paper’s quadrilateralland case is having independent spikes at each ‘pole’. Namely, a spike felt for \( I_{\text{Total}} \neq 0 \) at the ‘North pole’ \( \chi = 0 \) and a different spike felt by \( Y \neq 0 \) at the ‘South pole’ \( \chi = \pi/2 \). I note that the O4-stop metroland case is recovered in the study of the collinear submanifold of the quadrilateralland case. For contrast, I also note that 4-stop metroland itself is the odd one out in having peanut/egg/tyre shaped potentials, with the tyre alone possessing two wells, whereas all the other cases are heart/egg shaped potentials with at most one well.

In this paper’s quadrilateralland case, the motions are qualitatively as follows. In the cross-section for conserved quantities being 0, and ‘round and round’ in the other cases. That the \( SU(2) \) round and round is some higher-d \( S^2 \) analogue of axisymmetry. The \( U(1) \) of the \( \psi \) is standard axisymmetry. [Note: each of these corresponds to going round-and-round in different suppressed dimensions.]

I considered the multi HO-like potential for quadrilateralland in terms of the useful intrinsic Gibbons–Pope coordinates (Key 10). I interpreted a geodesics result for \( CP^2 \) (Key 11) in terms of quadrilaterals, and link it to the various collapses of the quadrilateralland trees. In the qualitative dynamics figure for the simpler HO-like potential (Key 12), I provide a nice extension of work in [8, 9] with one extra case necessitated so as to parallel the quadrilateral better: the qualitative analysis of HO-type potentials on O4-stop metroland (i.e. the mirror-image-identified version). The new feature in quadrilateralland is two repulsive spikes that are felt independently of each other according to whether the system in question has each of the two types of charge: angular charge, paralleling isospin, and extra angular charge, paralleling hypercharge. These qualitative dynamics figures also combine nicely with the tessellation interpretations of the 4-stop metroland and triangleland shape spheres and of the characterized submanifolds of \( CP^2 \) as provided in Paper I.

10.2 Outline of further applications of this paper’s quadrilateralland work

The \( N \)-a-gonland interpretation of conserved quantities and of corresponding quantum numbers remains open. For higher-\( N \) cases, \( SU(N) \) mastery including first parts of [46] but there is now no ready analogue of Gibbons–Pope coordinates available... The geodesics result used [45] is general-\( N \). Some further results on the classical dynamics on \( CP^N \) can be found in e.g. [47].

QM needs conserved quantity study and classical solutions as back-up. Some problem of time strategies require classical solutions (semiclassical approach) or work at the classical level (internal time), some observables approaches.
The Schrödinger equation for quadrilateralland is treated in Paper III. The equation itself is Key 13, and is built from this paper’s expression for $T_{\text{Total}}$ and using conformal invariance. This is then solvable in terms of Jacobi polynomials and Wigner D-functions for the free case (Key 14), and a perturbative scheme can be set up within this sort of methods of mathematical physics so as to study small multi-HO like potentials (Key 15). These are quite clearly extensions of importance of the Keys provided in the present paper.

Papers I to III are then useful for subsequent investigations of Problem of Time in Quantum Gravity strategies and various other quantum-cosmological issues. These are mostly in Paper IV, with a bit about regions and uniformity in this paper, Kuchař observables in Paper II and the Naïve Schrödinger Interpretation and peakedness in Paper III. Particular such application for Quadrilateralland and $N$-a-gonland are to timeless approaches to the Problem of Time in Quantum Gravity – semiclassical, histories, records, observables and combined approaches (including Halliwell’s [43, 16]), qualitative models of structure formation in Quantum Cosmology, and to the robustness study based on the $\{N-1\}$-a-gon model lying inside the $N$-a-gon one. I note that complex projective mathematics (the present paper involving the simplest RPM model paper with nontrivial such) will underlie this robustness study.

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