Reply to the comment on "Exact Expression for Radiation of an Accelerated Charge in Classical Electrodynamics"

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Flaws and ambiguities are pointed out upon examining the comment attempting to solve a problem as raised recently — the currently accepted formulation of electromagnetic radiation of an accelerated charge violates the principle of conservation of energy. This problem is not solved by the comment, due to a misunderstanding in the meaning of the total radiated power crossing a sphere. An experiment is suggested to determine whether or not the currently accepted formulation is valid.

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In a recent literature [1], without any reason, Singal made a hasty conclusion that the newly derived exact expression for the electromagnetic radiation of an accelerated charge by Huang and Lu [2] is incorrect. In the comment [3], Singal attempted to resolve a serious problem — the currently accepted formulation of electromagnetic radiation of an accelerated charge violates the principle of conservation of energy, as raised recently [4]. With a view to making a comprehensible reply, relevant expressions of both currently accepted formulation [5–9] and the new formulation [2], as well as discrepancy between them are first presented.

I. THE CURRENTLY ACCEPTED FORMULATION VERSUS THE NEW FORMULATION

According to the currently accepted formulation, the energy flux density of radiation is

\[
S(r, t) = \frac{q^2}{16 \pi^2 \epsilon_0 c} \left( \hat{r} \times (\hat{r} - \beta) \times \dot{\beta} \right)^2 \frac{1}{r^2 (1 - \hat{r} \cdot \beta)^6} \hat{r}.
\]

(1)

The energy flux density of radiation at time \(t\) and at any position \(r\) on a sphere, Eq. (1), is due to the radiation emanating by an accelerated charge at an earlier time \(t_r = t - r/c\) and at the center of the sphere. The velocity \(\beta\) and the acceleration \(\dot{\beta}\) of the charge in the right-hand side of Eq. (1) are defined at the retarded time \(t_r = t - r/c\).

The energy radiated into a solid angle \(d\Omega\) in the direction \(\hat{r}\), and then measured at the position \(r\) and the time \(t\) is \(dW(r, t) = S \cdot \hat{r} r^2 d\Omega dt\), for an infinitesimal time interval \(dt\). Hence, \(S \cdot \hat{r}\) is the energy per unit area per unit time measured at the position \(r\) on the sphere. Consequently, the radiated power passing through the surface of a surrounding sphere per unit solid angle \(d\Omega\) in the direction \(\hat{r}\) is

\[
\frac{dP(\hat{r}, t)}{d\Omega} = \frac{dW(r, t)}{d\Omega dt} = \frac{q^2}{16 \pi^2 \epsilon_0 c} \frac{\left( \hat{r} \times (\hat{r} - \beta) \times \dot{\beta} \right)^2}{(1 - \hat{r} \cdot \beta)^6}.
\]

(2)

Eq. (2) is the angular distribution of radiated power per unit solid angle, as measured by observers on a surrounding sphere [1].

In contrast, according to the new formulation, the energy flux density of radiation is

\[
\tilde{S}(r, t) = \frac{q^2}{16 \pi^2 \epsilon_0 c} \gamma^2 \left( \hat{r} \times (\hat{r} - \beta) \times \dot{\beta} \right)^2 \frac{1}{r^2 (1 - \hat{r} \cdot \beta)^4} \hat{r}.
\]

(3)

Thus, the angular distribution of radiated power per unit solid angle, relative to the position of the charge at the retarded time \(t_r = t - r/c\), is

\[
\frac{d\tilde{P}(\hat{r}, t)}{d\Omega} = \frac{q^2}{16 \pi^2 \epsilon_0 c} \gamma^2 \frac{\left( \hat{r} \times (\hat{r} - \beta) \times \dot{\beta} \right)^2}{(1 - \hat{r} \cdot \beta)^4}.
\]

(4)

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By integrating Eq. (4) over a surrounding sphere, the total radiated power crossing any surrounding sphere is

\[ \bar{P} = \frac{q^2 \gamma^6}{6 \pi \epsilon_0 c} (\dot{\beta}^2 - (\beta \times \dot{\beta})^2). \]  

(5)

The total radiated power of radiation crossing any surrounding sphere is equal to the total power emitted by the charge at the retarded time \( t_r \), the so-called Liénard’s result,

\[ P(t_r) = \frac{q^2 \gamma^6}{6 \pi \epsilon_0 c} (\dot{\beta}^2 - (\beta \times \dot{\beta})^2), \]

(6)

as it should be to fulfill the principle of conservation of energy. Yet, according to the currently accepted formulation, by integrating Eq. (2) over a surrounding sphere, the total power of radiation crossing any surrounding sphere is not equal to the Liénard’s result [4]. The currently accepted formulation violates the principle of conservation of energy.

II. SINGAL’S RESOLUTION OF THE PROBLEM — THE CURRENTLY ACCEPTED FORMULATION VIOLATES THE PRINCIPLE OF CONSERVATION OF ENERGY

In the comment, Singal first claimed that our reasoning above is fallacious, because we equate the evaluated result of the total radiated power of radiation crossing a sphere to the total power emitted by the charge. Then, he presented a resolution to the problem. Referring to Fig. 1 at time \( t \) the radiation in the region enclosed by spheres \( S' \) and \( S \) is due to the radiation emitted by the charge during the time interval from 0 to \( dt_r \). The radiation region is not spherically symmetric around \( O \), since the charge moves a distance \( v dt_r \) from \( O \) to \( O' \). Thus, the radiation emitted by the charge during the time interval \( dt_r \) does not cross the sphere \( S \) at all points in the time interval \( dt_r \), rather it takes \( dt = dt_r (1 - \hat{r} \cdot \beta) \).

![Diagram](image)

FIG. 1: Suppose at time \( t = 0 \), an accelerated charge is at the origin \( O \) and moves with a velocity \( v \) along the z-axis. In a time interval \( dt_r \), the charge moves a distance \( v dt_r \) from \( O \) to \( O' \) in the z-axis. At time \( t \), the radiation in the region (shade area) enclosed by spheres \( S' \) and \( S \) is the radiation emitted by the charge during the time interval \( dt_r \). Therefore, from Eq. (2), the total energy of radiation \( \mathcal{W} \) enclosed by spheres \( S' \) and \( S \) is evaluated as

\[
\mathcal{W} = \int dP(\hat{r}, t) \, dt_r (1 - \hat{r} \cdot \beta) \, d\Omega \\
= dt_r \times \int \frac{q^2 \gamma^6}{6 \pi \epsilon_0 c} \frac{((\hat{r} \times (\hat{r} - \beta)) \times \dot{\beta})^2}{(1 - \hat{r} \cdot \beta)^5} d\Omega \\
= dt_r \times \frac{q^2 \gamma^6}{6 \pi \epsilon_0 c} (\beta^2 - (\beta \times \dot{\beta})^2). 
\]

(7)
The total energy of radiation $W$ is due to the total energy emitted by the charge during the time interval $dt_r$. $W/dt_r$ is equal to the total power emitted by the charge at the retarded time $t_r$, that is,

$$\frac{W}{dt_r} = P(t_r) = \frac{q^2}{6\pi\epsilon_0}\epsilon(\beta^2 - (\beta \times \hat{\beta})^2). \quad (8)$$

Therefore, the currently accepted formulation does not violate the principle of conservation of energy. Yet, it should be noted that $W/dt_r$ is not the total radiated power crossing the sphere $S$ at time $t$. According to Singal’s reasoning, what is the total radiated power crossing the sphere $S$?

III. THE PROBLEM OF THE CURRENTLY ACCEPTED FORMULATION REMAINS UNSOLVED

In our reasoning [2, 4], we do not derive the Liénard’s result, and not equate the evaluated result of the total radiated power crossing a sphere to the Liénard’s result. Instead, we compare the evaluated result with the Liénard’s result to see which formulation satisfies the principle of conservation of energy. It turns out that currently accepted formulation, instead of the new formulation, violates the principle of conservation of energy.

There are problems in the currently accepted formulation. Since the charge is accelerated, it does not move uniformly during the time interval $dt_r$. Thus, the angular distribution of radiated power in the time interval is not exactly in accordance with Eq. (2) as evaluated at merely one retarded time $t_r$, because the time-retarded positions and velocities $\beta$ of the charge change during the time interval. The total power emitted by the charge as evaluated in accordance with Singal’s reasoning should be only approximately valid, as noticed by Panofsky and Jackson [5, 6]. Therefore, it is very unlikely that the exact Liénard’s result is obtained simply by an approximation approach, unless Eq. (2) is only approximately correct.

According to Singal, the factor $(1 - \hat{r} \cdot \beta)$ in the time interval $dt = dt_r (1 - \hat{r} \cdot \beta)$ is interpreted as just a matter of simple geometry. Yet this interpretation negates the existent interpretations: a Lorentz transformation of time between the charge’s frame and the observer’s frame [6], or something similar to the Doppler effect [7]. Which interpretation is correct?

Foremost, Singal does not resolve the problem that the currently accepted formulation violates the principle of conservation of energy. The total radiated power crossing a surrounding sphere at time $t$ is not equal to $W/dt_r$. Suppose that one wants to evaluate the total radiated power $P$ crossing a sphere. One first measures the total radiated energy $W$ crossing the sphere in a time interval $\Delta t (\Delta t << 1)$. Then, the total radiated power crossing the sphere is given as $P = W/\Delta t$. It should be emphasized that the time interval $\Delta t$ must be the same, as the measurement of the total radiated energy is carried out at all points on the sphere. However, according to Singal’s reasoning, the measurement of the total radiated energy $W$ crossing the sphere $S$ is carried out with different time intervals $dt = dt_r (1 - \hat{r} \cdot \beta)$ at different points on the sphere. $W/dt$ is meaningless, and $W/dt_r$ is not the total radiated power crossing the sphere. If Singal thinks that the total radiated power crossing a surrounding sphere is $W/dt_r$, and it is equal to the total power emitted by the charge at the retarded time, then he might make a mistake in the meaning of the total radiated power crossing a sphere. Hence, the problem that the currently accepted formulation violates the principle of conservation of energy remains unsolved.

IV. THE ISSUE OF THE RELATIVISTIC TRANSFORMATION OF ELECTROMAGNETIC FIELDS

Another issue in the comment is "While deriving expressions for the electric and magnetic fields, $E$ and $B$, Huang and Lu [2] in their Eqs. (20)-(25) simply replaced $r'$ with $r$ which is not correct as these two quantities are actually related by $r' = r/\delta$, where $\delta = 1/\gamma(1 - \hat{r} \cdot \beta)$ is the Doppler factor [3]. Thus their transformed electric and magnetic fields are wrong." The newly derived expressions for the electromagnetic fields become the currently accepted expressions, if $r' = r\gamma(1 - \hat{r} \cdot \beta)$, instead of $r' = r$, is employed in the transformation of electric and magnetic fields between inertial frames.

First, that the new formulation, rather than the currently accepted formulation, fulfills the principle of conservation of energy reinforces the validity of the replacement $r' = r$ in the transformation. Furthermore, Maxwell’s equations of electrodynamics are shown form-invariant via a novel perspective on relativistic transformation — transformation of physical quantities, instead of space-time coordinates [10]. An extra transformation of spacial coordinate such as $r' = r\gamma(1 - \hat{r} \cdot \beta)$ is not necessary in the transformation of electric and magnetic fields to render Maxwell’s equations form-invariant among inertial frames.

According to Singal, the expression $r' = r\gamma(1 - \hat{r} \cdot \beta)$ is considered as due to the Doppler effect. Yet, the Doppler effect is the transformation of physical quantities of waves such as frequency $\nu$ and wave vector $k$ relative to inertial
frames, instead of space-time coordinates \((t, \mathbf{r})\). The expression \(r' = r \gamma (1 - \hat{r} \cdot \beta)\) has nothing to do with the Doppler effect, since it involves spatial coordinates only, without frequency and wave vector of waves. A systematic method to derive the Doppler effect, without involving transformation of space-time coordinates, was presented in the literature \[1\] \[11\]. Even more, in a certain case an anomaly — the problem of negative frequency of waves, was found by applying the invariance of the phase of waves which is equivalent to relativistic transformation of both physical quantities \((\nu, \mathbf{k})\) and space-time coordinates \((t, \mathbf{r})\) simultaneously \[12\]. This indicates that the invariance of the phase of waves is invalid. Therefore, the Doppler effect should be related to the transformation of physical quantities \((\nu, \mathbf{k})\) and space-time coordinates \((t, \mathbf{r})\) simultaneously \[12\]. This indicates that the invariance of the phase of waves is invalid. Therefore, the Doppler effect should be related to the transformation of physical quantities \((\nu, \mathbf{k})\) of waves only. In Singal’s interpretation, the factor \((1 - \hat{r} \cdot \beta)\) in \(dt = dt' (1 - \hat{r} \cdot \beta)\) is a matter of simple geometry, whereas the factor \(\gamma (1 - \hat{r} \cdot \beta)\) in \(r' = r \gamma (1 - \hat{r} \cdot \beta)\) is due to the Doppler effect. The two interpretations seem incompatible.

V. CONCLUSION

Ambiguities on the meaning of \(dt = dt' (1 - \hat{r} \cdot \beta)\) and \(r' = r \gamma (1 - \hat{r} \cdot \beta)\) still exist in the currently accepted formulation of electromagnetic radiation of an accelerated charge. Owing to a misunderstanding in the meaning of the total radiated power crossing a sphere, the serious problem that currently accepted formulation violates the principle of conservation of energy remains unsolved. Such controversies as paradoxes in special relativity are hardly resolved just by theoretical reasoning. In order to convincingly determine the validity of the currently accepted formulation, an experimental test on the angular distribution of radiated power was proposed \[4\]. Nonetheless, it is necessary to clarify which the angular distribution of radiated power is: Eq. 2, or from Eq. 7.

\[
\frac{d P (\mathbf{r}, t)}{d \Omega} = \frac{q^2}{16 \pi^2 \varepsilon_0 c} \frac{\left( \mathbf{r} \times ((\mathbf{r} - \beta) \times \hat{\beta}) \right)^2}{(1 - \hat{r} \cdot \beta)^5}.
\]

Further theoretical and experimental examinations on the currently accepted formulation and the new formulation should be highly welcome.

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