KNOTTED NULLHOMOLOGOUS SURFACES IN 4-MANIFOLDS.

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Abstract: We point out that recent constructions of inequivalent smooth structures yield a manufacturing procedure of infinite sets of pairwise smoothly non-isotopic nullhomologous 2-tori and spheres inside a myriad of 4-manifolds. The corresponding infinite set consists of topologically isotopic surfaces that topologically bound a handlebody in several instances.

1. Introduction and main result

Among the many interesting four-dimensional phenomena is the existence of embedded surfaces $\Sigma^1, \Sigma^2 \hookrightarrow X$ that are topologically isotopic, but not smoothly isotopic. Fintushel-Stern introduced a surgery procedure on an embedded 2-torus that results in knotted examples of homologically essential 2-tori [17]; cf. Kim [13], Kim-Ruberman [14]. In [26, Theorem 1], Hoffman-Sunukjian exhibited infinite families of embedded nullhomologous 2-tori that are topologically isotopic but pairwise smoothly different. These homologically trivial 2-tori are the submanifolds employed in the production of inequivalent smooth structures on 4-manifolds through torus surgeries that are related to Fintushel-Stern’s knot surgery construction [18].

We point out a procedure to produce more nullhomologous examples of knotted spheres and 2-tori in our main theorem.

**Theorem A.** Let $X$ be a closed smooth connected 4-manifold that contains an embedded 2-torus $T \hookrightarrow X$ of self-intersection $[T]^2 = 0$ for which the inclusion induced homomorphism $\pi_1(X \setminus \nu(T)) \to \pi_1(X) = \{1\}$ is an isomorphism. Suppose that the Seiberg-Witten invariant $SW_X \neq 0$.

There is a an infinite set of topologically isotopic 2-tori

$$\{T_i : i \in \mathbb{N}\}$$

embedded in $X$ that satisfy $[T_i] = 0 \in H_2(X; \mathbb{Z})$ and $\pi_1(X \setminus \nu(T_i)) = \mathbb{Z}$ for every $i \in \mathbb{N}$, and which are pairwise smoothly non-equivalent.

There is a an infinite set of topologically isotopic 2-spheres

$$\{S_i : i \in \mathbb{N}\}$$

embedded in $X \# S^2 \times S^2$ that satisfy $[S_i] = 0 \in H_2(X \# S^2 \times S^2; \mathbb{Z})$ and 2-knot group $\pi_1(X \# S^2 \times S^2 \setminus \nu(S_i)) = \mathbb{Z}$ for every $i \in \mathbb{N}$, and which are pairwise smoothly non-equivalent.

Each element of (1.1) and (1.2) bounds a topologically embedded solid handlebody in $X$ and $X \# S^2 \times S^2$, respectively; cf. Sunukjian [32] and Hoffman-Sunukjian [26]. Moreover, surgery on these submanifolds yields infinitely many inequivalent smooth structures; see Section 3.1. The second part of Theorem A was kindly

1991 Mathematics Subject Classification. Primary 57K45, 57R55; Secondary 57R40, 57R52.
pointed out to us by Bob Gompf after an unsuccessful attempt to construct inequivalent smooth structures on 4-manifolds that lack an almost-complex structure à la Fintushel-Stern [16, §5].

The next theorem exemplifies several choices of ambient 4-manifolds that are obtained with these techniques.

**Theorem B.** Let $X \in \{S^2 \times S^2, S^1 \times S^3, S^2 \times T^2\}$. There is an infinite set

(1.3) \[ \{T_i : i \in \mathbb{N}\} \]

of pairwise smoothly non-equivalent 2-tori embedded in $X$ whose second homology class is $[T_i] = 0 \in H_2(X; \mathbb{Z})$.

There is a an infinite set

(1.4) \[ \{S_i : i \in \mathbb{N}\} \]

of pairwise smoothly non-equivalent 2-spheres embedded in $X \# S^2 \times S^2$ whose second homology class is $[S_i] = 0 \in H_2(X \# S^2 \times S^2; \mathbb{Z})$.

Our proofs of Theorem A and Theorem B build on the recent progress on the existence of inequivalent smooth structures on closed simply connected 4-manifolds $X$ with small second Betti number due to Akhmedov-Park [3, 4], Akhmedov-Baykur-Park [2], Baldridge-Kirk [7, 8], Fintushel-Stern [19, 20], Fintushel-Stern-Park [15]. The constructions of these authors are based on performing torus surgeries to a 4-manifold $Y$ with non-trivial fundamental group and non-trivial Seiberg-Witten invariants. Under the right conditions, the toughest step in all of these constructions is arguably the computation of the fundamental group. We observe that it is straight-forward to produce a 4-manifold with the desired fundamental group if one is willing to apply to $Y$ either a torus surgery of multiplicity zero or a surgery along a loop whose homotopy class corresponds to a generator of the fundamental group; Section 2.3 and Section 2.1. Performing either of these surgeries to $Y$ comes at the price of rendering useless the gauge theoretical invariants of the 4-manifold produced for the purposes of discerning smooth structures. Moreover, in a myriad of instances, the 4-manifold obtained after the surgeries is diffeomorphic to $X$ and no inequivalent smooth structure is unveiled. Nevertheless, our efforts are not entirely moot and we do obtain infinitely many pairwise smoothly inequivalent nullhomologous surfaces inside $X$ and $X \# S^2 \times S^2$ that are topologically trivial.

We organized the paper in the following way. The results that are involved in the proofs of our theorems are contained in Section 2. A description of the cut-and-paste constructions of 4-manifolds that are used is given in Section 2.1 Section 2.2 and Section 2.3. In particular, Proposition 1 relates Gluck twists and torus surgeries. Specific choices of surgeries as well as several diffeomorphisms that are useful for our proof of Theorem A are given in Section 2.4. The theorems stated in the introduction are proven in Section 3.

1.1. **Acknowledgement:** We are indebted to Bob Gompf for useful e-mail correspondence, which motivated the writing of this note. We thank Danny Ruberman for his suggestions that helped us improve the note.

2. **TOOLS AND TECHNOLOGY**

2.1. **Surgery.** Let $X$ be a closed smooth 4-manifold and let $\gamma \subset X$ be a closed simple loop; see [24, Section 5.2, Example 4.1.3]. If the orientation character of $\gamma \subset X$ is trivial, then its tubular neighborhood $\nu(\gamma)$ is diffeomorphic to $S^1 \times D^3$. 
Definition 1. [24] Definition 5.2.1. A surgery on $X$ along a closed simple loop with trivial orientation character is the cut-and-paste procedure that removes a copy of $S^1 \times D^3$ from $X$ and caps the boundary off with a copy of $D^2 \times S^2$ to produce a closed smooth 4-manifold

$$X_S := (X \setminus \nu(\gamma)) \cup_{id} (D^2 \times S^2).$$

Notice that the surgery of Definition 1 can be reversed, i.e.,

$$X = (X_S \setminus D^2 \times S^2) \cup_{id} (S^1 \times D^3).$$

We will always choose a framing such that the 4-manifold (2.1) has zero second Stiefel-Whitney class $w_2(X_S) = 0$ whenever $w_2(X) = 0$; cf. [24, Propositions 5.2.3 and 5.2.4].

2.2. Gluck twists. Let $S \hookrightarrow X$ be a smoothly embedded 2-sphere inside a smooth 4-manifold $X$. Suppose that $S$ has trivial normal bundle, i.e., the tubular neighborhood $\nu(S)$ of $S$ is diffeomorphic to $D^2 \times S^2$; this is always the case when $[S] = 0 \in H_2(X; \mathbb{Z})$. We identify the 2-sphere $S$ with the Riemann sphere $\hat{\mathbb{C}}$ and its tubular neighborhood with

$$\nu(S) = D^2 \times S^2 \cong D \times \hat{\mathbb{C}},$$

where $D \subset \mathbb{C}$ is the unit disk. The result of performing a Gluck twist to $X$ along $S$ is the 4-manifold

$$X_S := (X \setminus \nu(S)) \cup_{h} (D \times \hat{\mathbb{C}})$$

obtained by using a diffeomorphism to identify the common boundaries

$$\tau : \partial D \times \hat{\mathbb{C}} \to \partial D \times \hat{\mathbb{C}}$$

that is given by

$$\tau(v, u) = (v, uv),$$

where the map on the second factor is a rotation of the Riemann sphere

$$\hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

through an angle in $\partial D = S^1 = \mathbb{R}/\mathbb{Z}$ about a given axis [27], [21, Section 6].

2.3. Torus surgeries. Let $T \hookrightarrow X$ be a smoothly embedded 2-torus inside a closed smooth 4-manifold $X$. Suppose that $T$ has trivial normal bundle, i.e., the tubular neighborhood $\nu(T)$ is diffeomorphic to the thick torus $T^2 \times D^2$. Denote by $\{x, y\}$ isotopy classes of loops that carry the generators of the fundamental group $\pi_1(T) = \mathbb{Z}x \oplus \mathbb{Z}y$, and let $\{m, l\}$ be their respective framings in $\partial \nu(T) = T^3$. The meridian $\mu_T$ of the torus $T$ in the complement $M \setminus \nu(T)$ is a curve on the isotopy class of $\{t\} \times \partial D^2 \subset \partial \nu(T)$. The group $H_1(\partial \nu(T); \mathbb{Z}) = \mathbb{Z}^3$ is generated by

$$\{m, l, \mu_T\}.$$  

Starting with $X$, one can produce a 4-manifold

$$X_{T, \gamma}(p, q, r) := (X \setminus \nu(T)) \cup_{\varphi} (T^2 \times D^2)$$

by performing a $(p, q, r)$-torus surgery on the submanifold $T \subset X$ along the surgery curve $\gamma := m^a \ell^b \subset X$ for some relatively prime pair of integer numbers $a$ and $b$. The common boundaries are identified by a diffeomorphism

$$\varphi : T^2 \times \partial D^2 \to \partial(X \setminus \nu(T))$$
that satisfies
\[(2.11) \quad \varphi_*([pt] \times \partial D^2) = p[m] + q[l] + r[\mu_T]\]
in $H_1(\partial(X \setminus \nu(T)); \mathbb{Z})$ for integers $p$, $q$, and $r$; the integer $r$ is known as the multiplicity of the torus surgery. If the 2-torus $T$ and the surgery curve $\gamma$ are essential in homology in the construction (2.9), then the core 2-torus (2.12) is nullhomologous. Every surgery on a nullhomologous torus that is considered in this paper will be performed with respect to the nullhomologous framing; see [19, Section 2] for further details.

A torus surgery (2.9) can be undone. The 4-manifold $X$ is recovered by applying a torus surgery on the core torus (2.12)
\[T := T^2 \times \{0\} \subset T^2 \times D^2\]
along a surgery curve $\gamma' \subset X_{T,\gamma}(p, q, r)$ of the construction (2.9). See [8, 19, 24] more background on torus surgeries.

Iwase [27, Proposition 3.5] showed that there is a 2-torus $T$ inside the 4-sphere such that the homotopy 4-sphere $\Sigma_S$ that is obtained by applying a Gluck twist to an embedded 2-sphere $S \hookrightarrow S^4$ is diffeomorphic to $S^4_{T,\gamma}(p, 0, 1)$ for some surgery curve $\gamma \subset T$ and $p = \pm 1$. We point out that Iwase’s result can be extended to every closed smooth 4-manifold.

**Proposition 1.** Let $X$ be a closed connected smooth 4-manifold and let $X_S$ be the 4-manifold that is obtained by performing a Gluck twist along a smoothly embedded 2-sphere $S \hookrightarrow X$ with trivial normal bundle $\nu(S) = D^2 \times S^2$. There exists a smoothly embedded 2-torus
\[(2.13) \quad T \hookrightarrow X\]
and a surgery curve $\gamma \subset T$ such that there is a diffeomorphism
\[(2.14) \quad X_S \approx X_{T,\gamma}(p, 0, 1),\]
where $X_{T,\gamma}(p, 0, 1)$ is the 4-manifold obtained from $X$ by performing a $(p, 0, 1)$-torus surgery on $T$ along $\gamma$ for $p = \pm 1$.

**Proof.** The first step is to build the 2-torus (2.13) along with its tubular neighborhood by attaching a trivial handle
\[(2.15) \quad H : [0, 1] \times D^3 \to M \setminus \nu(S)\]
to the 2-sphere $S$ and $\nu(S)$, respectively [11, Section 2], [10, Section 2.1]. We use the identification of the 2-sphere with the Riemann sphere and its tubular neighborhood (2.3) that was discussed in Section 2.2. Consider the subsets
\[(2.16) \quad D_0 := \{ |z| \leq \frac{1}{9} \}\]
and
\[(2.17) \quad D_\infty := \{ |z| \geq 9 \}\]
of the Riemann sphere $\hat{C}$. Embed
\[(2.18) \quad (D^3 \times D^1, \partial D^1 \times D^3) \hookrightarrow (X \setminus \nu(S), \partial \nu(S))\]
such that
\[(2.19) \quad D^3 \times \{-1\} \subset \partial D^2 \times D_0\]
and
\[(2.20)\quad D^3 \times \{1\} \subset \partial D^2 \times D_\infty.\]

Denote the image of (2.13) by \(H\), which is a 1-handle attached to \(\nu(S)\) [11, Definition 1]. We build the 2-torus \((2.13)\) by gluing pieces
\[(2.21)\quad T := \left(\{0\} \times (\hat{C} \setminus (\text{int } D_0 \cup \text{int } D_\infty))\right) \cup A_0 \cup (U \times D^1) \cup A_\infty\]
that consist of annuli \(A_0 \subset D^2 \times D_0\) and \(A_\infty \subset D^2 \times D_\infty\), and \(U \times D^1\) is a cylinder where \(U\) is the unknot in a \(D^3\). This construction yields a tubular neighborhood of \(T\)
\[(2.22)\quad \nu(T) \subset \nu(S) + H\]
with the properties
\[(2.23)\quad \nu(T)|_{\hat{C} \setminus (\text{int } D_0 \cup \text{int } D_\infty)} = \nu(S)|_{\hat{C} \setminus (\text{int } D_0 \cup \text{int } D_\infty)}\]
and
\[(2.24)\quad \nu(T)|_{U \times \{pt\}} = \nu_0(U) \times \{pt\},\]
where \(\nu_0(U)\) is a tubular neighborhood of the unknot \(U\) in \(D^3\).

Let us pin down the surgery curve \(\gamma \subset T\). Related to the diffeomorphism \((2.0)\)
involved in the Gluck surgery construction, we have a circle action, i.e., the rotation \((2.7)\) of the 2-sphere \(\{pt\} \times \hat{C} \subset D \times \hat{C}\). Consider an orbit of this circle action
\[(2.25)\quad \tau(\partial D \times \{|u| = c\}) = \partial D \times \{|u| = c\}\]
that intersect transversally both subsets \((2.16)\) and \((2.17)\), and let \(\gamma_{\hat{C}}\) be the closed simply curve traced by \(\tau\). We denote the restriction of this curve to \((\hat{C} \setminus (\text{int } D_0 \cup \text{int } D_\infty))\) by \(\gamma_{\hat{C}}^0\). Let \(\gamma_0 \subset A_0\) be an untwisted path that connects a pair of points each of which is located at a different boundary component \(A_0\). Analogously, let \(\gamma_\infty \subset A_\infty\) be an untwisted path that connects a pair of points each of which is located at a different boundary component \(A_\infty\), and let \(\gamma_{U \times D^1} \subset U \times D^1\) be an untwisted path that connects a pair of points each of which is located at a different boundary component of \(U \times D^1\). The surgery curve is given by concatenation of these four paths as
\[(2.26)\quad \gamma := \gamma_{\hat{C}}^0 \cup \gamma_0 \cup \gamma_{U \times D^1} \cup \gamma_\infty.\]

Let
\[(2.27)\quad \varphi : \partial \nu(T) \to \partial \nu(T)\]
be the gluing diffeomorphism that corresponds to a \((\pm 1, 0, 1)\)-torus surgery on \(T\) along \(\gamma = U\). The diffeomorphism \((2.27)\) satisfies
\[(2.28)\quad \varphi(\{pt\} \times D^3 \cap \partial \nu(T)) = \{pt\} \times D^3 \cap \partial \nu(T)\]
and
\[(2.29)\quad \varphi|_{\partial D \times \hat{C} \setminus (\text{int } D_0 \cup \text{int } D_\infty)} = \tau|_{\partial D \times \hat{C} \setminus (\text{int } D_0 \cup \text{int } D_\infty)}.\]

To finish the proof of Proposition 1 we construct a diffeomorphism
\[(2.30)\quad F : X_S \to X_{T, \gamma}(p, 0, 1)\]
for \(p = \pm 1\). Given conditions \((2.28)\) and \((2.29)\) relating the diffeomorphism \(\tau\) of the Gluck twists and the diffeomorphism \((2.27)\) of the \((p, 0, 1)\)-torus surgery, we need
to define (2.30) on (2.24), and on $D^2 \times D_0$ and $D^2 \times D_\infty$. Consider the submanifold $X_* \subset X_{T,\gamma}(p,0,1)$ defined by

\begin{equation}
X_* := (D^3 \setminus \text{int} \nu_0(U)) \times \{pt\} \cup (\nu_0(U) \times \{pt\})
\end{equation}

where $\varphi_p : \partial \nu_0(U) \to \partial \nu_0(U)$ is the gluing map that corresponds to performing a $p$-Dehn surgery along $U \subset D^3$ for $p = \pm 1$. There is a diffeomorphism

\begin{equation}
F_* : X_* \to D^3 \times \{pt\}
\end{equation}

such that

\begin{equation}
F_*|_{\partial X_*} = \text{id}.
\end{equation}

We set

\begin{equation}
F|_{X_*} = F_*.
\end{equation}

We then can extend $F|_{\partial(D^2 \times D_0)}$ to $D^2 \times D_0$ and $F|_{\partial(D^2 \times D_\infty)}$ to $D^2 \times D_\infty$ canonically and concludes the construction of (2.30).

There are annuli $A_0$ and $A_\infty$ in $\nu(S)$ such that

\begin{equation}
A_0 = H(\{i\} \times \partial D^2) \cup p(H(\{i\} \times \partial D^2))
\end{equation}

Baykur-Sunukjian have shown that the known inequivalent smooth structures on closed simply connected 4-manifolds are related by a series of torus surgeries [9, Corollary 11], answering a problem stated by Stern [31, Problem 12]. Along with work of Akbulut [1], Proposition \ref{prop:1} has the following consequence regarding inequivalent smooth structures on non-orientable 4-manifolds.

Corollary 1. There is an embedded 2-torus

\begin{equation}
T \mapsto S^3 \# S^2 \times S^2
\end{equation}

such that performing a $(1,0,1)$-torus surgery on it yields an inequivalent smooth structure on the connected sum of the non-orientable 3-sphere bundle over the circle with a copy of $S^2 \times S^2$.

2.4. Three pairs of geometrically dual 2-tori in the 4-torus and some useful diffeomorphisms. Let $\{x, y\}$ and $\{a, b\}$ be pair of curves, each of which form a standard set of generators of the fundamental group of the 2-torus $\pi_1(T^2) = \mathbb{Z}^2$; in particular, the 2-torus can be written as $x \times y = T^2 = a \times b$. Denote the 4-torus by

\begin{equation}
T^4 = x \times y \times a \times b,
\end{equation}

where the notation emphasizes the curves that generate the free abelian group

\begin{equation}
\pi_1(T^4) = \mathbb{Z}^4 = \mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}a \oplus \mathbb{Z}b.
\end{equation}

We abuse notation and define three disjoint submanifolds of (2.37) as

\begin{equation}
T_1 := x \times \{pt\} \times a \times \{pt\},
\end{equation}

\begin{equation}
T_2 := \{pt\} \times y \times \{pt\} \times b
\end{equation}

and

\begin{equation}
T_3 := \{pt\} \times \{pt\} \times a \times b.
\end{equation}
The 2-tori (2.39) (2.40) (2.41), and their geometrically dual 2-tori generate the second homology group $H_2(T^4; \mathbb{Z}) = \mathbb{Z}^6$. We specify the framings for the 2-tori that were discussed in Section (2.23) by equipping the 4-torus with a product symplectic structure. The 2-torus (2.41) is a symplectic submanifold, while the 2-tori (2.39) and (2.40) are Lagrangian with respect to this choice of symplectic structure. Torus surgeries on (2.39) and/or (2.40) are performed with respect to the Lagrangian structure. The 2-torus (2.41) is a symplectic submanifold, while the 2-tori (2.39) were discussed in Section 2.3 by equipping the 4-torus with a product symplectic structure. The loop $m$ for every $n$ for every $n$ for every $n$.

**Lemma 1.** Let $n \in \mathbb{N}$. Let $W_n$ be the 4-manifold that is obtained from the 4-torus by applying a $(1,0,n)$-torus surgery on $T_1$ along $m = x$ and a $(0,1,0)$-torus surgery on $T_2$ along $l = y$. There is a diffeomorphism

$$W_n \cong S^2 \times T^2$$

for every $n \in \mathbb{N}$.

Let $Z_n$ be the 4-manifold that is obtained from the 4-torus by applying a $(1,0,n)$-torus surgery on $T_2$ along $m = x$ and a surgery along the loop $l = y$. Suppose that its second Stiefel-Whitney class satisfies $w_2(Z_n) = 0$. There is a diffeomorphism

$$Z_n \cong S^2 \times T^2 \# S^2 \times S^2$$

for every $n \in \mathbb{N}$.

Let $M_n$ be the 4-manifold that is obtained from the 4-torus by applying a $(0,1,1)$-torus surgery on $T_2$ along $l = y$, a $(1,0,n)$-torus surgery on $T_1$ along $m = x$, and a $(0,1,0)$-torus surgery on $T_3$ along $m = a$. There is a diffeomorphism

$$M_n \cong S^1 \times S^3$$

for every $n \in \mathbb{N}$.

Let $F_n$ be the 4-manifold that is obtained from the 4-torus by applying a $(0,1,1)$-torus surgery on $T_2$ along $l = y$, a $(1,0,n)$-torus surgery on $T_1$ along $m = x$ and a surgery along the loop $m = a$. Suppose that its second Stiefel-Whitney class satisfies $w_2(F_n) = 0$. There is a diffeomorphism

$$F_n \cong S^1 \times S^3 \# S^2 \times S^2$$

for every $n \in \mathbb{N}$.

**Proof.** Recall that a diffeomorphism $M^3 \to M^3$ of a 3-manifold $M^3$ extends to a diffeomorphism $M^3 \times S^1 \to M^3 \times S^1$. To show the existence of the diffeomorphism (2.42), after fixing an $n \in \mathbb{N}$, we see both four-dimensional torus surgeries that are applied to the 4-torus $T^4 = T^3 \times S^1$ as a pair of Dehn twists times $S^1$ applied to the 3-torus times $S^1$. More precisely, the 4-manifold $W_n$ is the product of the circle and the 3-manifold $M^3$ that is obtained by applying a $(1,n)$-Dehn surgery on $T_1$ along $m = x$ and a $(1,0)$-Dehn surgery on $T_2$ along $l = y$ to the 3-torus. A standard handlebody argument reveals that $M^3$ is diffeomorphic to $S^2 \times S^1$; see [35, Section 3]. The loop $a$ generates the fundamental group $\pi_1(M^3) = \mathbb{Z}a$. We conclude $W_n \cong (S^2 \times S^1) \times S^1$. Since the choice of $n$ was arbitrary, the existence of the diffeomorphism (2.42) follows. Adapting the pervious argument to the construction of $M_n$ allows us to conclude the existence of the diffeomorphism (2.44).

We now show the existence of the diffeomorphism (2.43). Fix $n \in \mathbb{N}$. A classical argument due to Moishezon implies that the 4-manifold $Z_n$ is obtained by performing a surgery to $W_n$ along a nullhomotopic loop [28, Lemma 13]; cf. Gompf’s.
description in [22] Lemma 3]. By (2.12) and the hypothesis $w_2(Z_n) = 0$, we conclude that $Z_n$ is diffeomorphic to $S^2 \times T^2 \# S^2 \times S^2$. Since $n$ was arbitrary, we conclude that there is a diffeomorphism (2.43) for every $n \in \mathbb{N}$. Adapting the previous argument to the construction of $F_n$ allows us to conclude the existence of the diffeomorphism (2.49). This concludes the proof of Lemma 1. 

We now point out two straightforward properties of generalized fiber sums that we will use in Section 3.1.

**Lemma 2.** Let $X$ be a closed smooth 4-manifold that contains a smoothly embedded 2-torus $T \hookrightarrow X$ of self-intersection $[T]^2 = 0$ and let

$$X \#_T S^2 \times T^2 := (X \setminus \nu(T)) \cup (S^2 \times T^2 \setminus \nu(\{pt\} \times T^2)).$$

There is a diffeomorphism

$$X \#_T S^2 \times T^2 \approx X.$$

Let

$$\tilde{T} := \{pt\} \times T^2 \subset (S^2 \times T^2 \setminus D^4) \subset (S^2 \times T^2 \# S^2 \times S^2)$$

be a canonical homologically essential 2-torus in the connected sum, and let

$$X \#_{\tilde{T}} (S^2 \times T^2 \# S^2 \times S^2) := (X \setminus \nu(T)) \cup ((S^2 \times T^2 \# S^2 \times S^2) \setminus \nu(\tilde{T})).$$

There is a diffeomorphism

$$X \#_{\tilde{T}} (S^2 \times T^2 \# S^2 \times S^2) \approx X \# S^2 \times S^2.$$

**Proof.** The observation $S^2 \times T^2 \approx (D^2 \times T^2) \cup_{id} (D^2 \times T^2)$ implies the existence of the diffeomorphism (2.47). The existence of the diffeomorphism (2.47) imply the existence of the diffeomorphism (2.50). 

2.5. **Examples out of Reverse-Engineering of 4-manifolds.** We now describe how Fintushel-Park-Stern’s work can be used to produce examples of infinitely many smoothly non-isotopic nullhomologous 2-tori inside $S^2 \times S^2$ and nullhomologous 2-spheres inside $S^2 \times S^2 \# S^2 \times S^2$. In [15, Section 4], Fintushel-Park-Stern construct an infinite set

$$\{X_n : SW_{X_n} = n \in \mathbb{N}\}$$

of pairwise non-diffeomorphic closed smooth 4-manifold such that $X_n$ is homologically equivalent to $S^2 \times S^2$ for every $n \in \mathbb{N}$. We briefly recall their construction and set-up some notation. Let $\{a_1, b_1, a_2, b_2\}$ and $\{c_1, d_1, c_2, d_2\}$ be loops whose homotopy classes form a standard set of generators for the fundamental group $\pi_1(\Sigma_2) \times \pi_1(\Sigma_2) = \pi_1(\Sigma_2 \times \Sigma_2)$. Push-off of a loop $a_i \subset \Sigma_2$ are denoted by $a_i'$ and $a_i''$. Once the 4-manifold $\Sigma_2 \times \Sigma_2$ is equipped with a product symplectic form, Fintushel-Park-Stern choose the following eight disjoint homologically essential Lagrangian 2-tori and surgery curves:

- $T_1 := a_1' \times c_1'$, $m_1 = a_1'$,
- $T_2 := a_2' \times c_1'$, $l_2 = c_1'$,
- $T_3 := a_2' \times d_1', l_3 = d_1'$,
- $T_4 := b_1' \times c_1'$, $m_4 = b_1'$,
- $T_5 := a_1'' \times c_2'$, $m_5 = a_2'$,
- $T_6 := a_1'' \times c_2'$, $l_6 = c_2'$.
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• \( T_7 := a_1' \times d_2', m_5 = d'_2 \), and
• \( T_8 := b_2' \times c_2', m_1 = b'_2 \).

Fintushel-Park-Stern construct an infinite set

\[
\{ X(1, n) : SW_{X(1, n)} = n \in \mathbb{N} \}
\]

of pairwise non-diffeomorphic closed smooth 4-manifolds that are homologically equivalent to the connected sum

\[
2(S^2 \times S^2) \# S^1 \times S^3\]

with infinite cyclic first homology group

\[
H_1(X(1, n); \mathbb{Z}) = \mathbb{Z} b_2
\]
by applying to \( \Sigma_2 \times \Sigma_2 \) the following torus surgeries with respect to the Lagrangian framing

• \((1, 0, -1)\)-torus surgery on \( T_1 \) along \( \gamma_1 = a_1' \),
• \((0, 1, 1)\)-torus surgery on \( T_2 \) along the surgery curve \( \gamma_2 = c_1' \),
• \((0, 1, -1)\)-torus surgery on \( T_3 \) along the surgery curve \( \gamma_3 = d_1'' \),
• \((1, 0, 1)\)-torus surgery on \( T_4 \) along the surgery curve \( \gamma_4 = b_1' \),
• \((1, 0, -1)\)-torus surgery on \( T_5 \) along the surgery curve \( \gamma_5 = a_2' \),
• \((0, 1, 1)\)-torus surgery on \( T_6 \) along the surgery curve \( \gamma_6 = c_2' \), and
• \((0, 1, n)\)-torus surgery on \( T_7 \) along the surgery curve \( \gamma_7 = d_2' \).

For each \( n \in \mathbb{N} \), the corresponding 4-manifold contains the eighth 2-torus

\[
T_8 \subset X(1, n),
\]
which carries a loop whose homotopy/homology class corresponds to the generator \( b_2 \) in (2.54).

We build a 4-manifold \( X(0, n) \) by applying to a

• \((1, 0, 0)\)-torus surgery to \( X(1, n) \) on \( T_8 \) along the surgery curve \( \gamma_8 = b_2' \) to (2.52).

Similarly, build a 4-manifold \( Y(0, n) \) by applying

• a surgery to \( X(1, n) \) along the curve \( \gamma_8 = b'_2 \).

These constructions belong to the following diffeomorphism types.

**Lemma 3.** There are diffeomorphisms

\[
X(0, n) \approx S^2 \times S^2
\]
and

\[
Y(0, n) \approx S^2 \times S^2 \# S^2 \times S^2
\]
for every \( n \in \mathbb{N} \).

**Proof.** An argument to prove Lemma 3 can be found in [16, §5], [28, Lemma 13] and [22, Lemma 3, Lemma 4], and we now sketch it; cf. [19, Proposition 6], [34].

Fix an \( n \in \mathbb{N} \). There is a pair of geometrically dual genus two surfaces

\[
\Sigma, \Sigma^D \subset X(1, n)
\]
of self-intersection zero, as well as a pair of geometrically dual 2-tori

\[
T_8, T_8^D \subset X(1, n).
\]

Since torus surgeries are local surgical operations, each genus two surface in (2.58) comes from a factor genus two surface in the product \( \Sigma_2 \times \Sigma_2 \). The spherical
modification along \( \gamma_8 = b_2 \) that is used to construct \( Y(0, n) \) turns each pair of submanifolds \((2.58)\) and \((2.59)\) \( T_8 \subset X(1, n) \) into a pair of geometrically dual 2-spheres, i.e., an \( S^2 \times S^2 \) summand in a connected sum. We conclude the existence of the diffeomorphism \((2.57)\). The existence of the diffeomorphism \((2.56)\) follows from an argument due to Moishezon \([28, \text{Lemma 13}]\) and \([22, \text{Lemma 3}]\), where he shows the 4-manifold that is obtained by applying a \((1, 0, 0)\)- or a \((0, 1, 0)\)-torus surgeries to a given 4-manifold \( X \) is diffeomorphic to the 4-manifold that is obtained from \( X \) by performing the cut-and-paste construction of Definition \([1]\) along a loop and then doing surgery along a 2-sphere \([24, \text{Section 5.2}]\).

\[
\square
\]

3. Proof of main results.

3.1. **Proof of Theorem A**

The proof consists of eight steps. The first step is to construct a closed 4-manifold \( X(2) \) with four properties:

1. it has non-trivial Seiberg-Witten invariant and
2. its fundamental group is isomorphic to \( \mathbb{Z}^2 = \mathbb{Z}x \oplus \mathbb{Z}y \).
3. There are two disjoint homologically essential 2-tori of self-intersection zero \( T_1 \) and \( T_2 \) embedded in \( X(2) \) such that the inclusion induced homomorphism
   \[
   \pi_1(X(2) \setminus (\nu(T_1) \cup \nu(T_2))) \to \pi_1(X(2))
   \]
   is an isomorphism. The torus \( T_1 \) contains a loop whose homotopy class corresponds to \( x \) and the torus \( T_2 \) contains a loop whose homotopy class corresponds to \( y \), where \( x \) and \( y \) are the generators of the group \((3.1)\).
4. The Euler characteristic satisfies \( \chi(X(2)) = \chi(X) \), the signature \( \sigma(X(2)) = \sigma(X) \), and the second Stiefel-Whitney class \( w_2(X(2)) = w_2(X) \); cf. Szabó \([36, \text{Section 2}]\).

Define the 4-manifold as \( X(2) := X \setminus \nu(T) \cup T^2 \times T^2 \setminus \nu(T_3) \), where we follow the notation of Section \((2.4)\).

We argue that \( SW_{X(2)} \neq 0 \) as follows. The value of the Seiberg-Witten invariant of the 4-torus follows from a well-known result of Taubes \([33, \text{Main Theorem}]\). The claim regarding the Seiberg-Witten invariant of the 4-manifold \( SW_{X(2)} \neq 0 \), follows from a product formula of Morgan-Szabó-Taubes \([30, \text{Theorem 3.1}]\) since we assumed \( SW_X \neq 0 \). If we were to assume that \( X \) has a symplectic structure, we could then appeal to a construction of Gompf \([23, \text{Chapter 10.2}]\) along with the result of Taubes that we mentioned in order to conclude that \( SW_{X(2)} \neq 0 \). Hence, \( X(2) \) does satisfy Property (1).

We now argue that \((3.3)\) satisfies Property (2) and Property (3). Let \( \mu_3 \) be the homotopy class of the meridian of the torus \((2.41)\). The hypothesis on the existence of an isomorphism \( \pi_1(X \setminus \nu(T)) \to \pi_1(X) = \{1\} \) implies that the relations

\[
\mu_3 = 1 = a = b
\]

hold in the fundamental group of \( X(2) \) and we conclude that the group \( \pi_1(X(2)) \) is the rank two free abelian group on the generators \( x \) and \( y \) using the Seifert-van Kampen theorem. Regarding Property (3), notice that the 2-tori \((2.39)\) and \((2.40)\) are disjoint from the 2-torus \((2.41)\) that was employed in the construction of \( X(2) \).
Therefore, both (2.39) and (2.40) are contained in $X(2)$. To show that the group isomorphism (3.2) exists, we show that the meridians $\mu_1$ and $\mu_2$ of the two 2-tori are nullhomotopic. Following the calculations of Baldridge-Kirk [7, p. 92–2], with our choice of framings for $T_1$ and $T_2$, we can conclude that the meridian of $T_1$ is given by $[\tilde{b}, \tilde{y}]$ and the meridian of $T_2$ is $[\tilde{a}, \tilde{x}]$ where $\tilde{g}$ denotes a conjugate of the element $g$. By (3.4), we conclude that the relations (3.5) hold in the group $\pi_1(X(2) \setminus (\nu(T_1) \sqcup \nu(T_2)))$ and the existence of the isomorphism (3.2) follows. A Mayer-Vietoris sequence reveals that $\chi(X(2)) = \chi(X)$. Novikov additivity implies $\sigma(X(2)) = \sigma(X)$. An argument guaranteeing that the second Stiefel-Whitney class of $X(2)$ is zero whenever $w_2(X) = 0$ was given by Gompf in [23, Proposition 1.2].

The second step is to construct an infinite set (3.6)
$$\{X(1)_n : n \in \mathbb{N}\}$$

of closed 4-manifolds with infinite cyclic fundamental group $\mathbb{Z}_y$ that satisfy three properties:

1' $X(1)_n$ is homeomorphic to $X \# S^2 \times S^2 \# S^1 \times S^1$ for ever $n \in \mathbb{N}$.
2' $X(1)_{n_1}$ is not diffeomorphic to $X(1)_{n_2}$ if $n_1 \neq n_2$.
3' For every $n \in \mathbb{N}$, there is a homologically essential 2-torus of self-intersection zero $T_2$ embedded in $X(1)_n$ such that the inclusion induced homomorphism (3.7) $\pi_1(X(1)_n \setminus (\nu(T_2))) \to \pi_1(X(1)_n) = \mathbb{Z}$ is an isomorphism. The torus $T_2$ contains a loop whose homotopy class $y$ is the generator of the fundamental group $\pi_1(X(1)_n) = y$.

Fix an $n \in \mathbb{N}$ and define a closed 4-manifold (3.8)
$$X(1)_n := X(2)_{T_1, S^1}(1, 0, n),$$
i.e., the 4-manifold obtained from by applying a $(1, 0, n)$-torus surgery to $X$ on the 2-torus $T_1$ along the surgery curve $\gamma = S^1_c$.

Since the meridian $\mu_1$ is nullhomotopic in the complement (cf. 3.2 and 3.5), the Seifert-van Kampen theorem imply (3.9)$\pi_1(X(1)_n) = \langle x, y : [x, y] = 1 = x \rangle = \mathbb{Z}_y$; see [7, Lemma 4].

The claim that the 4-manifolds (3.8) satisfies Property (1') follows from the identities (3.10)$\chi(X(1)_n) = \chi(X(2)) = \chi(X)$,

(3.11)$\sigma(X(1)_n) = \sigma(X(2)) = \sigma(X)$,
and (3.12)$w_2(X(1)_n) = w_2(X(2)) = w_2(X)$

for arbitrary $n \in \mathbb{N}$, and the homeomorphism criteria of Hambleton-Teichner [25] and Kawauchi [12]. An argument to prove the claim that the 4-manifold (3.8) satisfies Property (3') is obtained by a small tweak to the argument used to the prove of Property (3) for $X(2)$. Indeed, the 2-torus $T_2 \subset X(2)$ is disjoint from the
surgery and is contained in $X(1)_n$. The existence of the isomorphism (3.7) follows from (3.5). Property (2') holds since the Seiberg-Witten invariants satisfy
\[
\text{SW}_{X(1)_n} \neq \text{SW}_{X(1)_{n_2}}
\]
for $n_1 \neq n_2$ as follows from the Morgan-Mrowka-Szabó formula [29, Theorem 1.1]. For details on the computation of the Seiberg-Witten invariants of $X(1)_n$, the reader is directed to [36, Section 3] or [15, Corollary 1]. This concludes the construction of the infinite set (3.6) of pairwise non-diffeomorphic 4-manifolds in the homeomorphism class of $X \# S^2 \times S^2 \# S^1 \times S^3$.

The third step is to construct an infinite set of nullhomologous 2-tori as in (1.1). We now fix $n \in \mathbb{N}$ and apply a $(0,1,0)$-torus surgery to $X(1)_n$ on $T_2$ along $\gamma = S^1_y$ and denote the resulting 4-manifold by $X_n$. That is,
\[
X_n := X(1)_{nT_2,S^1_y}(0,1,0).
\]
In particular, $X_n$ is simply connected. Let
\[
T_n := T^2 \times \{0\} \subset X_n
\]
be the core torus of the surgery (3.14). Notice that $T_n$ is nullhomologous 2-torus since $T_2$ was an essential 2-torus. Since $n \in \mathbb{N}$ was arbitrary, this concludes the proof of the existence of the infinite set (1.1).

The fourth step is to argue that these 2-tori are topologically equivalent and smoothly inequivalent. We can undo the construction (3.14) by applying a torus surgery to $X_n$ on $T_n$ and obtain $X(1)_n$ back for any $n \in \mathbb{N}$; see [9, Figure 1] for a handlebody calculus argument. Since the set of 4-manifolds (3.6) corresponds to a unique homeomorphism class, any two 2-tori (3.15) are topologically equivalent. Since the set (3.6) consists of pairwise non-diffeomorphic 4-manifolds, any two 2-tori (3.15) are smoothly inequivalent.

In the fifth step we show that there is a diffeomorphism
\[
X_n \approx X
\]
for every $n \in \mathbb{N}$. The 4-manifold $X_n$ is diffeomorphic to a generalized fiber sum of $X$ and the 4-manifold that is obtained from the 4-torus by applying a $(1,0,n)$-torus surgery on $T_1$ and a $(0,1,0)$-torus surgery on $T_2$ for a fixed $n \in \mathbb{N}$. The latter is diffeomorphic to the product of a 2-sphere and 2-torus by the diffeomorphism (2.42) of Lemma 1. Diffeomorphism (2.47) of Lemma 2 imply that $X_n$ is diffeomorphic to $X$. Since the choices of $n$ was arbitrary, we conclude that there is a diffeomorphism (3.16). This finishes the proof of existence of the infinite set (1.1).

The sixth step consists of producing the nullhomologous 2-spheres of (1.2). Fix an $n \in \mathbb{N}$ and apply a surgery to $X(1)_n$ along the curve that generates its infinite cyclic fundamental group
\[
Y_n := (X(1)_n \cup (y)) \cup (D^2 \times S^2)
\]
(cf. [16, §5]). A similar argument to the one used in the third step implies the existence of a nullhomologous 2-sphere $S_n \subset Y_n$ for any $n \in \mathbb{N}$.

We discern these submanifolds in the seventh step. Similar to what was done in the fourth step, we use the fact that reversing the surgery (3.17) as in (2.2) yields $X(1)_n$. Since every element in the set of 4-manifolds (3.6) corresponds to the same homeomorphism class, any two 2-spheres are topologically equivalent. Given that the element in the set of 4-manifolds (3.6) are pairwise non-diffeomorphic, any two 2-spheres are smoothly inequivalent.
The eight and final step consists of showing that there is a diffeomorphism
\[(3.18) \quad Y_n \cong X \# S^2 \times S^2.\]
for every \(n \in \mathbb{N}\). In order to pin down the diffeomorphism type, we can modify the argument that was used in the fifth step. Lemma [1] and the diffeomorphism \((2.43)\) along with the diffeomorphism \((2.50)\) of Lemma [1] imply the existence of the diffeomorphism \((3.18)\). This concludes the construction of the infinite set \((1.2)\) as well as the proof of the theorem.

\[\square\]

Remark 1. For further results regarding handle addition to the knotted embedded surfaces and stabilizations of the ambient 4-manifolds, the reader is directed towards Baykur-Sunukjian [10], Auckly-Kim-Melvin-Ruberman [5], and Auckly-Kim-Melvin-Ruberman-Schwartz [6].

3.2. Proof of Theorem B. The constructions of the 4-manifolds and the surfaces \((1.3)\) and \((1.4)\) were described in Section 2.4. The argument to discern these submanifolds was given in Section 3.1. The corresponding diffeomorphisms are established in \((2.42), (2.43), (2.44)\), and \((2.45)\) of Lemma [1].

\[\square\]

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