Rotationally symmetric tilings with convex pentagons belonging to both the Type 1 and Type 7 families

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Abstract

Rotationally symmetric tilings by a convex pentagonal tile belonging to both the Type 1 and Type 7 families are introduced. Among them are spiral tilings with two-fold and four-fold rotational symmetry. Those rotationally symmetric tilings are connected edge-to-edge and have no axis of reflection symmetry.

Keywords: pentagon, octagon, tiling, rotational symmetry, monohedral, spiral

1 Introduction

To date, fifteen families of convex pentagonal tiles\textsuperscript{1),2)} each of them referred to as a “Type,” are known. For example, if the sum of three consecutive angles in a convex pentagonal tile is 360\degree, the pentagonal tile belongs to the Type 1 family. Convex pentagonal tiles belonging to some families also exist. Known convex pentagonal tiles can form periodic tiling\textsuperscript{[2–4, 7]}. In [6], we introduced rotationally symmetric tilings and rotationally symmetric tiling-like patterns with a regular convex polygonal hole at the center, using convex pentagonal tiles. Then, it showed that the convex pentagonal tile that belongs to both the Type 1 and Type 7 families (see Figure 1) can form a rotationally symmetric tiling-like pattern with a regular convex octagonal hole at the center, and spiral tilings with two-fold rotational symmetry. Note that the tiling-like pattern is not considered tiling due to the presence of a gap, but is simply called tiling in this study. Those rotationally symmetric tilings are connected in an edge-to-edge\textsuperscript{3)} manner and have no axis of reflection symmetry\textsuperscript{4).}

The convex pentagonal tile that belongs to both the Type 1 and Type 7 families is the only convex pentagon that satisfies the conditions “\(A = 90\degree\), \(B = C = 135\degree\), \(D = \ldots\)\textsuperscript{1)} A tiling (or tessellation) of the plane is a collection of sets that are called tiles, which covers a plane without gaps and overlaps, except for the boundaries of the tiles. The term “tile” refers to a topological disk, whose boundary is a simple closed curve. If all the tiles in a tiling are of the same size and shape, then the tiling is monohedral\textsuperscript{2)}\textsuperscript{2)}. In this study, a polygon that admits a monohedral tiling is called a polygonal tile\textsuperscript{3)}\textsuperscript{3)}. Note that, in monohedral tiling, it admits the use of reflected tiles.

\textsuperscript{2)} In May 2017, Michaël Rao declared that the complete list of Types of convex pentagonal tiles had been obtained (i.e., they have only the known 15 families), but it does not seem to be fixed as of March 2020\textsuperscript{7)\textsuperscript{7)}. A tiling by convex polygons is edge-to-edge if any two convex polygons in a tiling are either disjoint or share one vertex or an entire edge in common. Then other case is non-edge-to-edge\textsuperscript{2)}\textsuperscript{2)}. \textsuperscript{3)} Hereafter, a figure with \(n\)-fold rotational symmetry without reflection is described as \(C_{n}\) symmetry. “\(C_{n}\)” is based on the Schoenflies notation for symmetry in a two-dimensional point group\textsuperscript{8}\textsuperscript{8).}
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Figure 1: Convex pentagonal tile $P(T_1 \cap T_7)$ that belongs to both the Type 1 and Type 7 families $67.5^\circ$, $E = 112.5^\circ$, $a = b = c = d$ shown in Figure 1 [3–5]. Hereafter, let $P(T_1 \cap T_7)$ be the convex pentagonal tile that belongs to both the Type 1 and Type 7 families. $P(T_1 \cap T_7)$ can generate the representative tiling of Type 1 (see Figure 2(a)), variations of Type 1 tilings (i.e., tilings whose vertices are formed only by the relations of $A + B + C = 360^\circ$ and $D + E = 180^\circ$) as shown in Figures 2(b)–2(e), and the representative tiling of Type 7 (see Figure 3(a)). It was previously recognized that $P(T_1 \cap T_7)$ could generate other tilings as shown in Figure 3(b) [3]. From the research in [6], we noticed that it is possible to generate more tilings with $P(T_1 \cap T_7)$. In this study, some of the tilings generated by $P(T_1 \cap T_7)$ are introduced.

2 Octa-unit and Hexa-unit

$P(T_1 \cap T_7)$ can also form non-edge-to-edge tilings as shown in Figures 2(b), 2(d), and 2(e) by using the relation $D + E = 180^\circ$, but the tilings by $P(T_1 \cap T_7)$ introduced in this study are edge-to-edge. The edge $e$ of $P(T_1 \cap T_7)$ is the only edge of different length. As shown in Figure 4(a), an equilateral concave octagon formed by two convex pentagons, connected through a line symmetry whose axis is edge $e$, is referred to as the Octa-unit. As shown in Figure 4(b), a convex hexagon formed by two convex pentagons, connected with rotational symmetry on edge $e$, is referred to as the Hexa-unit.

When variations of Type 1 tilings are generated by $P(T_1 \cap T_7)$, as shown in Figure 2, $P(T_1 \cap T_7)$ can generate tilings with only Octa-units, tilings with only Hexa-units, or tilings with both, Octa-units and Hexa-units. The representative tiling of Type 7 shown in Figure 3(a) and other tiling shown in Figure 3(b) are tilings with only Octa-units.

Note that the candidates of concentration relations that will be used in tilings with $P(T_1 \cap T_7)$ are as follows.

$D + E = 180^\circ$, $A + B + C = 360^\circ$, $2B + A = 360^\circ$, $2C + A = 360^\circ$, $2E + B = 360^\circ$, $2E + C = 360^\circ$, $4A = 360^\circ$, $2A + D + E = 360^\circ$, $2D + A + B = 360^\circ$, $2D + A + C = 360^\circ$, $2D + 2E = 360^\circ$, $4D + A = 360^\circ$.

Edge-to-edge tilings with "$2E + B = 360^\circ$, $2E + C = 360^\circ$, $2D + A + B = 360^\circ$, $2D + A + C = 360^\circ$, $2D + E = 360^\circ", or $4D + A = 360^\circ" require the use of Octa-units, and edge-to-edge tilings with "$2A + D + E = 360^\circ" require the use of Hexa-units.
First, let us introduce rotationally symmetric tilings with only Octa-units. In Figure 5, a rotationally symmetric tiling with $C_8$ symmetry, with a regular octagonal hole at the center, is depicted as shown in [6]. In Figures 6 and 7, respectively, a spiral tiling with $C_2$ symmetry...
and tilings that maintain the spiral structure and extend in one direction, are depicted as shown in [6]. Although a tiling is not rotationally symmetric, it is also possible to remove one spiral structure and to extend the belts wherein the Octa-units are arranged (see Figure 32 in [6]).

As mentioned above, $P(T_1 \cap T_7)$ can use the concentration $4A = 360^\circ$ for tiling. When the concentration $4A = 360^\circ$ is formed only by an Octa-unit, there are two types of concave 16-gons (hexadecagon) as shown in Figures 8(a) and 8(b), which are reflection images. Hereinafter, the concave 16-gon formed of four Octa-units, as shown in Figure 8, will be referred to as a Hexadeca-4oc-unit. Hexadeca-4oc-units have four-fold rotational symmetry and four axes of reflection symmetry passing through the center of the rotational symmetry (hereafter, this property is described as $D_4$ symmetry).

David Dailey presented tilings with “Dented octagons” on his site [1]. His dented octagons correspond to the Octa-unit in Figure 4(a). The figures that he presented are those of very interesting tilings; however, they also include patterns that are not expandable without leaving gaps. From his results, we can form a spiral tiling with $C_4$ symmetry, the center of which is the Hexadeca-4oc-unit by $P(T_1 \cap T_7)$, as shown in Figure 9 (The other tilings with $P(T_1 \cap T_7)$ that can be obtained from his results are introduced in the next section.) In addition, we found that $P(T_1 \cap T_7)$ can form a spiral tiling with $C_4$ symmetry, the center of which is the Hexadeca-4oc-unit, as shown in Figure 10. The tilings presented in Figures 9 and 10 have the relation of a spiral with reverse rotation, and the central Hexadeca-4oc-unit is the same (corresponding to Figure 8(a)). As mentioned above, the Hexadeca-4oc-units have $D_4$ symmetry; therefore, they can be reversed freely. Considering this property, we can determine that when the central Hexadeca-4oc-unit in Figure 10 is replaced with that in Figure 8(b) and the entire tiling is reversed, the tiling matches that shown in Figure 9.

In Figures 11 and 12, non-spiral rotationally symmetric tilings with $C_4$ symmetry with centers that are Hexadeca-4oc-units, are illustrated. The tilings depicted in Figures 11 and 12 have a relationship in which the central Hexadeca-4oc-unit is reversed.

The rotationally symmetric tilings introduced at the end are tilings with $C_2$ symmetry as

\[D_4\]

is based on the Schoenflies notation for symmetry in a two-dimensional point group [8, 9]. “$D_n$” represents an n-fold rotation axis with n reflection symmetry axes.
shown in Figures 13 and 14 in which an Octa-unit and a Hexa-unit are mixed. The center of the tiling depicted in Figure 13 is the concentration $4A = 360^\circ$ that was formed with four Hexa-units, and the tiling depicted in Figure 14 has the two concentrations $4A = 360^\circ$ that were formed with two Octa-units and two Hexa-units.

4 Other tilings

Using the concentration $4A = 360^\circ$ formed by mixing an Octa-unit and a Hexa-unit, we found the tiling depicted in Figure 15(a). We also found a tiling, as shown in Figure 15(b), in which Hexadeca-4oc-units are arranged side by side in one row and the top and bottom are filled with Octa-units.

Collections of cases similar to those in Dailey’s results, where tilings would not be broken by expansion, are shown in Figure 16. In Figure 16(c), a fusion of the tiling methods shown in Figures 16(a) and 16(b) is depicted, showing that similar tiling can be extended vertically and horizontally.

5 Conclusions

In this study, we mainly introduced rotationally symmetric tilings that would not have been known but for using the convex pentagonal tile $P(T_1 \cap T_7)$ belonging to both the Type 1 and Type 7 families. We think that some of them can be properly called spiral tiling. In this study, we only introduce those tilings with $P(T_1 \cap T_7)$, and do not fully consider the properties of the convex pentagon and other such tilings. (At present, there are many aspects that we have not been able to consider; hence, in this article, we focused on an introduction of the tiling that we discovered.)

$P(T_1 \cap T_7)$ can also generate tilings other than those introduced in this study. In particular, Yoshiaki ARAKI, of Japan Tessellation Design Association, found several interesting tilings using a concave octagon called an Octa-unit. They produce four-fold rotationally symmetric tilings that correspond to extended tilings such as those shown in Figure 7 and spiral tilings other than the ones described in this study. They will be introduced in a separate study.

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References

[1] D. Dailey, A Page for David Dailey [http://srufaculty.sru.edu/david.dailey/] >tiling [http://cs.sru.edu/~ddailey/tiling/tilingNew.html] >Dented octagons [http://cs.sru.edu/~ddailey/tiling/octagons.svg] (accessed on 28 March 2020).

[2] B. Grünbaum, G.C. Shephard, *Tilings and Patterns*. W. H. Freeman and Company, New York, 1987, pp.15–35 (Chapter 1), pp.471–518 (Chapter 9).

[3] T. Sugimoto, Convex pentagons for edge-to-edge tiling, I, *Forma*, 27 (2012) 93–103. Available online: [https://forma.katachi-jp.com/abstract/2701/27010093.html](https://forma.katachi-jp.com/abstract/2701/27010093.html) (accessed on 4 March 2022).
Figure 5: Eight-fold rotationally symmetric tiling with a regular convex octagonal hole at the center by a convex pentagonal tile $P(T_1 \cap T_7)$
Figure 6: Spiral tiling with two-fold rotational symmetry by a convex pentagonal tile $P(T_1 \cap T_7)$
Figure 7: Spiral tilings that maintain the spiral structure and extend in one direction by a convex pentagonal tile $P(T1 \cap T7)$

Figure 8: Concave 16-gons (Hexadeca-4oc-units) that formed of four Octa-units
Figure 9: Spiral tiling with four-fold rotational symmetry by a convex pentagonal tile $P(T_1 \cap T_7)$ based on Dailey’s tiling
Figure 10: Spiral tiling with four-fold rotational symmetry by a convex pentagonal tile $P(T1 \cap T7)$
Figure 11: Four-fold rotationally symmetric tiling by a convex pentagonal tile $P(T_1 \cap T_7)$, Part 1

Figure 12: Four-fold rotationally symmetric tiling by a convex pentagonal tile $P(T_1 \cap T_7)$, Part 2
Figure 13: Two-fold rotationally symmetric tiling by a convex pentagonal tile $P(T_1 \cap T_7)$, Part 1

Figure 14: Two-fold rotationally symmetric tiling by a convex pentagonal tile $P(T_1 \cap T_7)$, Part 2
Rotationally symmetric tilings with convex pentagons belonging to both the Type 1 and Type 7 families

Figure 15: Other tilings with $4A = 360^\circ$ by a convex pentagonal tile $P(T1 \cap T7)$

Figure 16: Convex pentagonal tilings with $P(T1 \cap T7)$ based on Dailey's other tilings