THE ZETAFAST ALGORITHM FOR COMPUTING ZETA FUNCTIONS

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Abstract. We express the Riemann zeta function \( \zeta(s) \) of argument \( s = \sigma + i\tau \) with imaginary part \( \tau \) in terms of three absolutely convergent series. The resulting simple algorithm allows to compute, to arbitrary precision, \( \zeta(s) \) and its derivatives using at most \( C(\epsilon)|\tau|^{3/2} \) summands for any \( \epsilon > 0 \), with explicit error bounds. It can be regarded as a quantitative version of the approximate functional equation. The numerical implementation is straightforward. The approach works for any type of zeta function with a similar functional equation such as Dirichlet \( L \)-functions, or the Davenport-Heilbronn type zeta functions.

1. Results

Theorem 1. Zetafast algorithm for Riemann zeta function:

Given a positive integer \( v \), \( \zeta(s) \) has for \( \sigma \leq \sigma_0 < v \) and \( s \notin \{1, \ldots, v - 1\} \) the following representation in terms of three absolutely and uniformly converging series \( D(s) \), \( E_1(s) \), and \( E_{-1}(s) \),

\[
\zeta(s) = D(s) + \sum_{\mu = \pm 1} E_\mu(s) - \frac{\Gamma(1-s+v)}{(1-s)\Gamma(v)}N^{1-s}
\]

\( D(s) = \sum_{n=1}^\infty n^{-s}Q\left(v, \frac{n}{N}\right) \)

\[ E_\mu(s) = (2\pi)^{s-1}\Gamma(1-s)\frac{\epsilon^{\mu\pi(1-s)}}{2\pi N}\sum_{m=1}^\infty E_\mu(m, s) \]

\[ E_\mu(m, s) = m^{s-1} - \sum_{w=0}^{v-1} \binom{s-1}{w} \left( m + \frac{i\mu}{2\pi N} \right)^{s-1-w} \left( \frac{-i\mu}{2\pi N} \right)^w \]

where \( Q(v, m) \) is the normalized incomplete gamma function

\[ Q(v, m) = \sum_{w=0}^{v-1} \frac{m^w}{w!} e^{-m} \]

For positive integer \( s = k \) with \( k < v \) one has to take the limit \( \lim_{s \to k} E_\mu(s) \).

Corollary 2. The derivatives of \( \zeta(s) \) can be calculated by differentiating the above series termwise.

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We are mainly interested in the critical strip and its surroundings, for which we have the following rough but explicit estimate:

**Theorem 3.** Accuracy and speed of Zetafast algorithm:

For a given argument $s$ with

(1.6) \[ 0 \leq \sigma \leq 2 \]
(1.7) \[ 0 < \tau \]

and accuracy

(1.8) \[ \delta \leq 0.05 \]

choose $v$ as the next higher integer $\lceil x_0 \rceil$ of the unique solution $x_0$ of

(1.9) \[ x - \max \left( \frac{1 - \sigma}{2}, 0 \right) \ln \left( \frac{1}{2} + x + \tau \right) = \ln \frac{8}{\delta} \]

in the unknown $x$, and from this $M = \lceil N \rceil$ with

(1.10) \[ N = 1.11 \left( 1 + \frac{1}{2} + \tau \right)^{\frac{1}{2}} \]

as well as

(1.11) \[ \lambda = 3.151 \]

Then we have for some real number $c$ with $|c| < 1$ the approximation

(1.12) \[ \zeta(s) = D(N, s) + E_1(M, s) - \frac{\Gamma(1 - s)}{(1 - s) \Gamma(v)} N^{1-s} + c\delta \]

\[ D(N, s) = \sum_{n=1}^{\lceil \lambda v N \rceil} n^{-s} Q(v, \frac{n}{N}) \]

\[ E_1(M, s) = (2\pi)^{s-1} \Gamma(1-s) e^{i\frac{\pi}{2}(1-s)} \sum_{m=1}^{M} E_1(m, s) \]

Under the condition

(1.13) \[ \tau > \frac{5}{3} \left( \frac{3}{2} + \ln \frac{8}{\delta} \right) \]

we need at most $S$ summands to calculate $D(N, s)$ and $E_1(M, s)$, with

(1.14) \[ S = 2 + 8 \sqrt{1 + \ln \frac{8}{\delta} + \max \left( \frac{1 - \sigma}{2}, 0 \right) \ln (2\tau) \sqrt{\tau}} \]

For $\tau = 0$, the same estimates hold, but we can neglect $E_1(M, s)$ altogether.

**Remark 4.** Estimate (1.14) shows that Zetafast allows for arbitrary precision, while being essentially as fast as the Riemann-Siegel formula [1]. Using similar arguments, we can obtain explicit error bounds for the derivatives of the Riemann zeta function, or for Dirichlet $L$-functions or its derivatives. What is more, there is room for further tightening the error bounds, or to accelerate the algorithm along the lines of [2], or [3] for multiple evaluations.
2.1. Proof of Theorem 1.

Proof of Theorem 1. For positive integer

\begin{equation}
(2.4)
\end{equation}

series (1.2) and express the cutoff function decreases exponentially with increasing \( \Gamma \) function.

Because the \( \Gamma \) function decreases exponentially with increasing \( \sigma \) and \( \zeta \) function increases at most algebraically \( [1] \), we can move the contour to \(-1 \leq x < -\sigma \) and pick up the residues at \( z + s = 1 \) and, for the case \( w = 0 \), at \( z = 0 \),

\begin{equation}
(2.1)
D (s) = \sum_{w=0}^{v-1} \frac{1}{w!} \int_{x>1-\sigma} \Gamma (z + w) \left( \frac{n}{N} \right)^{2^{-z}} \frac{dz}{2\pi i}
\end{equation}

where \( x > 1 - \sigma \) denotes that the integration contour is for this value of \( x \) along the vertical line from \( x-i\infty \) to \( x+i\infty \). We interchange the absolutely convergent series and integrals,

\begin{equation}
(2.2)
D (s) = \sum_{w=0}^{v-1} \frac{1}{w!} \int_{x>1-\sigma} \Gamma (z + w) N^z \zeta (z + s) \frac{dz}{2\pi i}
\end{equation}

Because the \( \Gamma \) function decreases exponentially with increasing \( |y| \) while the \( \zeta \) function increases at most algebraically \( [1] \), we can move the contour to \(-1 < x < -\sigma \) and pick up the residues at \( z + s = 1 \) and, for the case \( w = 0 \), at \( z = 0 \),

\begin{equation}
(2.3)
E (w, s) = \int_{-1 < x < -\sigma} \Gamma (z + w) N^z \zeta (z + s) \frac{dz}{2\pi i}
\end{equation}

The idea is now to use the functional equation for \( \zeta \),

\begin{equation}
(2.4)
\zeta (s + z) = (2\pi)^{s+z-1} \Gamma (1 - s - z) \zeta (1 - s - z) \sum_{\mu=\pm 1} e^{i\mu \pi (1-s-z)}
\end{equation}

For positive integer \( s = k \) with \( k < v \) one has to take the limit \( \lim_{k\rightarrow k} E_{\mu} (s, \chi) \).

2. Proofs

2.1. Proof of Theorem 1. We assume for the moment \( 0 < \sigma < 1 \), start with the series \( [1, 2] \) and express the cutoff \( Q \) in terms of its inverse Mellin transform,

\begin{equation}
(2.15)
L (s, \chi) = D (s, \chi) + (2\pi)^{s-1} \Gamma (1 - s) q^{-s} G (\chi) \sum_{\mu=\pm 1} \chi (-\mu) e^{i\mu \pi (1-s)} E_{\mu} (s, \chi)
\end{equation}

\begin{equation}
D (s, \chi) = \sum_{n=1}^{\infty} \chi (n) n^{-s} Q \left( v, \frac{n}{N} \right)
\end{equation}

\begin{equation}
E_{\mu} (s, \chi) = \sum_{m=1}^{\infty} \pi (m) \left[ m^{s-1} - \sum_{w=0}^{v-1} \left( s - 1 \right) w \right] \left( m + \frac{i\mu q}{2\pi N} \right)^{s-1-w} \left( -i\mu q \right)^w
\end{equation}

The same arguments show that linear combinations of Dirichlet \( L \)-functions \( L (\chi_k, s) \) for \( k = 1, 2, \ldots, l \) such as the Davenport-Heilbronn zeta function or Hurwitz zeta functions \( \zeta (s, r) \) for rational parameters \( r \), have a Zetafast algorithm, if they obey a functional equation expressing this linear combination in terms of a linear combination of functions \( r_k (1-s) e^{i\phi_k (1-s)} \Gamma (1 - s) L (\chi_k, 1 - s) \) for real numbers \( r_k \) and \( \phi_k \) with \( |\phi_k| < \pi \). We give here only one example:

**Theorem 5. Zetafast algorithm for Dirichlet \( L \)-functions:**

A Dirichlet \( L \)-function with primitive, non-principal character \( \chi \) and Gauss sum \( G (\chi) \) can be calculated for \( \sigma \leq \sigma_0 < v \) and \( s \notin \{1, \ldots, v-1\} \) by the following absolutely converging series,

\begin{equation}
(1.15)
L (s, \chi) = D (s, \chi) + (2\pi)^{s-1} \Gamma (1 - s) q^{-s} G (\chi) \sum_{\mu=\pm 1} \chi (-\mu) e^{i\mu \pi (1-s)} E_{\mu} (s, \chi)
\end{equation}

\begin{equation}
D (s, \chi) = \sum_{n=1}^{\infty} \chi (n) n^{-s} Q \left( v, \frac{n}{N} \right)
\end{equation}

\begin{equation}
E_{\mu} (s, \chi) = \sum_{m=1}^{\infty} \pi (m) \left[ m^{s-1} - \sum_{w=0}^{v-1} \left( s - 1 \right) w \right] \left( m + \frac{i\mu q}{2\pi N} \right)^{s-1-w} \left( -i\mu q \right)^w
\end{equation}

For positive integer \( s = k \) with \( k < v \) one has to take the limit \( \lim_{k\rightarrow k} E_{\mu} (s, \chi) \).
and to re-express the $\zeta$ function in terms of its then absolutely convergent Dirichlet series, because $1 - \sigma - x > 1$, so that

$$E(w, s) = (2\pi)^{s-1} \Gamma(1-s) \sum_{\mu = \pm 1} e^{i\mu \pi (1-s)} \sum_{m=1}^{\infty} E_\mu(w, m, s)$$

$$E_\mu(w, m, s) = m^{s-1} \int_{-1 < x < -\sigma} \frac{\Gamma(z + w) \Gamma(1-s-z)}{\Gamma(1-s)} \left( \frac{i\mu}{2\pi N m} \right)^{-z} \frac{dz}{2\pi i}$$

This Mellin-Barnes integral is the well-known inverse of the beta function integral ([4] 5.12.3 and 5.13.1), thus

$$E_\mu(w, m, s) = \frac{\Gamma(1-s+w)}{\Gamma(1-s)} \left( m + \frac{i\mu}{2\pi N} \right)^{s-1-w} \left( \frac{i\mu}{2\pi N} \right)^w - \delta_{w0} m^{s-1}$$

From (2.2), (2.3), (2.5) and (2.6) and the elementary identities,

$$\sum_{w=0}^{v-1} \frac{\Gamma(1-s+w)}{w!} = \frac{\Gamma(1-s+v)}{(1-s) \Gamma(v)}$$

$$\frac{\Gamma(1-s+w)}{\Gamma(1-s) w!} = \left( \frac{s-1}{w} \right) (-1)^w$$

and setting

$$-\sum_{w=0}^{v-1} \frac{1}{w!} E_\mu(w, m, s) = E_\mu(m, s)$$

we arrive at our representation (1.1)-(1.4).

Finally, absolute and uniform convergence is obvious for $D(s)$, and follows for $E_\mu(s)$ in the region $\sigma < \alpha_0 < v$ by analytic continuation because by the binomial theorem $\Gamma(1-s) E_\mu(m, s) \sim (1-s+v) m^{s-1-v}$ as $m \to \infty$. This shows also that for $s = k$ for a positive integer $k < v$, the poles of $\Gamma(1-s+w)$ for $w = 0, 1, \ldots, s-1$ cancel each other, so that we can take the limit $\lim_{s \to k} E_\mu(s)$.

2.2. Proof of Theorem [3]

2.2.1. Choosing $v$. Because we assume (1.6) and $\sigma < v$, we are free to restrict $v$ to

$$v \geq 5$$

First we prove that equation (1.9) has a unique solution $x_0 > 5$.

For the case $\sigma \geq 1$, the unique solution is $\ln \frac{8}{5}$. Because by assumption (1.8)

$$5 < \ln \frac{8}{5}, \text{ hence } x_0 > 5.$$ 

For $\sigma < 1$, equation (1.9) has at most one solution because the left-hand side is growing monotonically with $x$ and is unbounded from above. However, because the highest possible value of the left-hand side for $x = 5$, which is realized for $\sigma = \tau = 0$, is smaller then the right hand side, $5 - \frac{1}{2} \ln 5.5 < \ln \frac{8}{5}$, there is exactly one solution $x_0 > 5$. 


We choose \( v = \lfloor x_0 \rfloor \), and determine \( N \) using (1.10). Because the solutions of (1.9) increase monotonically with \( x \), we have

\[
\begin{align*}
\max & \left( \frac{1 - \sigma}{2}, 0 \right) \left[ \ln \left( \frac{1}{2} + v + \tau \right) + \ln \frac{8}{\delta} \right] \leq v \\
\max & \left( \frac{1 - \sigma}{2}, 0 \right) \left[ \ln \left( \frac{1}{2} + v + \tau \right) + \ln \frac{8}{\delta} + 1 \right] \geq v
\end{align*}
\]

We now determine \( E_1(s) \) and \( D(s) \) to accuracy \( \frac{\delta}{4} \) and show that we can neglect within this accuracy \( E - 1(s) \), so that we can calculate \( \zeta(s) \) to accuracy \( \delta \).

2.2.2. Upper bound for \( E_\mu(m,s) \). We express \( E_\mu(m,s) \) as the remainder of a Taylor expansion ([4], 1.4.35, 1.4.37), setting \( f(z) = z^{s-1} \), \( a = m + \frac{\mu}{\pi N} \) and \( b = m \), so that

\[
E_\mu(m,s) = f(b) - \sum_{w=0}^{v-1} \frac{f(w)(a)}{w!} (b-a)^w = \int_a^b (b-z)^{v-1} \left( \frac{f(v)(z)}{(v-1)!} \right) dz
\]

Here we can assume that the integration runs over a straight line segment from \( a \) to \( b \). The triangle inequality for integrals yields,

\[
|\Gamma(1-s)E_\mu(m,s)| \leq \left( \frac{2\pi N}{{\Gamma(v)}} \right) |\Gamma(1-s+v)| \max_{\mu \in [0;1]} \left( m + i\frac{\mu u}{2\pi N} \right)^{s-1-v} \]

The last term is \( \exp[\eta_\mu(m,s)] \) where the real number \( \eta_\mu(m,s) \) is at most

\[
\eta_\mu(m,s) = \max_{\mu \in [0;1]} \Re \left[ (\sigma - 1 - v + i\tau) \left( \ln \left| m + i\frac{\mu u}{2\pi N} \right| + i\mu \arctan \frac{u}{2\pi m N} \right) \right] < (\sigma - 1 - v) \ln m + \max_{\mu \in [0;1]} \left( -\mu \tau \arctan \frac{u}{2\pi m N} \right) < (\sigma - 1 - v) \ln m + \frac{1}{4} \pi \mu
\]

We have because of (1.6) and (2.9),

\[
|1-s+v| \geq 4
\]

Therefore, using the upper bounds ([4], 5.6.1 and 5.6.9),

\[
(2\pi N)^{-1} \leq \left( 2\pi \right)^{-\frac{1}{2}} v^{\frac{1}{2}-v} e^{v}
\]

\[
|\Gamma(1-s+v)| < \left( e^{-\frac{1}{2\pi+\pi^2}} \sqrt{2\pi} \right) \left| \frac{1}{2} + v + \tau \right|^{\frac{1}{2}-\sigma+v} e^{-\frac{1}{2} \tau}
\]

and the triangle inequality, we arrive with (2.12) at the upper bound

\[
(2\pi)^{s-1} \Gamma(1-s) e^{i\mu \pi (1-s)} E_\mu(m,s) < e^{\frac{1}{2\pi \sigma-1-v}} N^{-v} v^{\frac{1}{2}-v} e^{v} \left( \frac{1}{2} + v + \tau \right)^{\frac{1}{2}-\sigma+v} \exp \left[ \frac{\mu - 1}{4\pi \tau} \right] m^{\sigma-v-1}
\]
2.2.3. Upper bound for $E_{-1}(s)$. (1.10) implies $N > 1$ and therefore
\[ M \geq 2 \]
For $\mu = -1$ and because of
\[ \sum_{m=1}^{\infty} m^{\sigma - 1 - v} \leq \zeta(4) \]
we can sum over all $m$ and have from (2.15) and (2.16) the upper bound,
\[ |E_{-1}(s)| < \left[ e^{\frac{\pi^2}{16}} \right] (2\pi)^{\sigma - 1 - v} \left( e^{\frac{\pi}{4\sigma}} \left( v + \frac{1}{2} \right)^{\frac{s}{\sigma}} \right) \]
Because the term in square brackets decreases monotonically with $\tau$, we choose its maximum value at $\tau = 0$,
\[ |E_{-1}(s)| < \left[ e^{\frac{\pi^2}{16}} \right] (2\pi)^{\sigma - 1 - v} \left( e^{\frac{\pi}{4\sigma}} \left( v + \frac{1}{2} \right)^{\frac{s}{\sigma}} \right) \]
The last term is less than $\sqrt{e}$, so that
\[ 3 |E_{-1}(s)| < \left[ e^{\frac{\pi^2}{16}} \right] (2\pi)^{\sigma - 1 - v} \left( e^{\frac{\pi}{4\sigma}} \left( v + \frac{1}{2} \right)^{\frac{s}{\sigma}} \right) (8e^{-v}) \]
The term in square brackets is always less than one, because for $v = 5$ it has because of $5.5 < 2\pi$ its maximum $< 1$ for $\sigma = 2$, and for $v \geq 6$ we have because of $6.5 > 2\pi$ an upper bound $< 1$ by setting $\sigma = 0$ and $v = 6$.
Therefore we can neglect $E_{-1}(s)$ up to accuracy $\frac{\delta}{3}$, because we see from (2.10) that
\[ \ln \frac{8}{\delta} \leq v \]
However, because for real argument $s$, $E_{1}(s)$ and $E_{-1}(s)$ are complex conjugates, this shows that we can neglect in this case both.

2.2.4. Accuracy of $E_{1}(s)$. In (1.12), we cut off the series for $E_{1}(s)$ at $m = M$. We estimate now the rest $r_E$. Using
\[ \sum_{m=1}^{\infty} m^{\sigma - 1 - v} < \int_{M}^{\infty} x^{\sigma - 1 - v} dx < M^{\sigma - 1 - v} \left( \frac{M^{\sigma - v}}{v - 2} \right) \]
and the upper bound (2.15) we have
\[ |r_E| < \left[ e^{\frac{\pi^2}{16}} \right] (2\pi)^{\sigma - 1 - v} \left( e^{\frac{\pi}{4\sigma}} \left( v + \frac{1}{2} \right)^{\frac{s}{\sigma}} \right) \left( \frac{1}{2 + v + \tau} \right) \ln \frac{8}{\delta} \]
Because of $1 - \frac{\sigma}{\tau} \leq \max \left( 1 - \frac{\sigma}{\tau}, 0 \right)$ and (2.10), we can replace the term in square brackets by its upper bound $1$. Inserting $N$ of (1.10), we have of
\[ 1.112^{v - \sigma} > \left( e^{\frac{\pi}{2\sigma}} \right)^{\frac{1}{2} (2v - \sigma)} \geq \left( e^{\frac{\pi}{2\sigma}} \right)^{v} \]
the upper bound
\[ |r_E| < \left[ \frac{3 e^{\frac{\pi^2}{16}}}{8 v - 2 \left( \frac{v}{4\sigma^2} \right)^{\frac{1}{2}}} \right] \frac{\delta}{3} \]
The term in square brackets has for \( v \leq 4\pi^2 \) its maximum at \( \sigma = 2 \) and \( v = 5 \) as well as for \( v \geq 4\pi^2 \) for \( \sigma = 0 \) and \( v = 40 \). In both cases this is less than one. Hence the value (1.10) for \( N \) is sufficient for the desired accuracy \( |r_E| < \frac{\delta}{3} \).

2.2.5. **Accuracy of \( D(s) \).** At first we determine an upper bound for \( N^{1-\sigma} \), using (1.10),

\[
\ln \left( \frac{N}{\delta} \right)^{1-\sigma} = (-\ln 8 + \frac{\sigma - 1}{2} \ln v) + \left( \ln \frac{8}{\delta} + \frac{1-\sigma}{2} \ln \left( \frac{1}{2} + v + \tau \right) \right)
\]

The first term on the right hand side is at most

\[-\ln 8 + \frac{1}{2} \ln v < 0.003 \cdot v\]

The second term is because of (2.10) and \( \frac{1-\sigma}{2} \leq \max \left( \frac{1-\sigma}{2}, 0 \right) \) at most \( v \). Hence we have the upper bound

\[N^{1-\sigma} < 1.11 \cdot e^{1.003v}\]

Using this bound and the triangle inequality, we get an upper bound for \( |r_D| \),

\[
|r_D| \leq 1.11 \cdot e^{1.003v} \cdot (v)^{-1} \sum_{n=\lceil \lambda v N \rceil}^{\infty} \left( \frac{N}{\lambda N} \right)^{-\sigma} \int_{\lambda N}^{\infty} t^{v-1} e^{-t} dt
\]

Because of \( \sigma \geq 0 \), the integral is a decreasing function of \( \frac{N}{\lambda N} \), so that using inequality (2.13), we can bound the sum by the double integral

\[
|r_D| \leq 1.11 \cdot e^{1.003v} \left( 2\pi \right)^{-\frac{1}{2}} v^{\frac{1}{2}} e^{v} \int_{\lambda v}^{\infty} du \int_{u}^{\infty} t^{v-1} e^{-t} dt
\]

We have for \( \lambda > 1 \) and \( t \geq u \geq \lambda v \),

\[1 < \frac{1 - \frac{v-1}{1 - \frac{1}{\lambda}}}\]

and therefore

\[\int_{u}^{\infty} t^{v-1} e^{-t} dt < \frac{\lambda}{\lambda - 1} \int_{u}^{\infty} (t^{v-1} - (v-1) t^{v-2}) e^{-t} dt = \frac{\lambda}{\lambda - 1} u^{v-1} e^{-u}\]

and by the same argument

\[\int_{\lambda v}^{\infty} du \int_{u}^{\infty} t^{v-1} e^{-t} dt < \frac{\lambda}{\lambda - 1} \int_{\lambda v}^{\infty} du u^{v-1} e^{-u} < \frac{\lambda}{(\lambda - 1)^2} \lambda v^{v-1} e^{-\lambda v}\]

Hence inequality (2.23) becomes

\[
|r_D| < \frac{\delta}{3} \left[ \frac{\lambda}{(\lambda - 1)^2} \sqrt{2\pi v} \right] e^{-(\lambda - 2.003 - \ln \lambda)}
\]

Assuming \( \lambda \geq 3 \), the term in square brackets is always smaller than one. Hence it suffices for an accuracy \( \frac{\delta}{3} \) to choose \( \lambda = 3.151 \) because then

\[
\lambda - 2.003 - \ln \lambda > 0
\]
2.2.6. Estimating the total number of summands. We have $\hat{\lambda v N} < \lambda v N + 1$ summands for $D(N,s)$ and $(v + 1) M < (v + 1) (N + 1)$ summands for $E_1(M,s)$. Therefore we have for the total number of summands the upper bound

$$S < (\lambda v + v + 1) N + v + 2$$

Because of $v \geq 5$ we have $1.11 (\lambda v + v + 1) < 4.83 v$ and hence from (1.10) the upper bound

$$S < 4.83 \left[ v \left( \frac{1}{2} + v + \tau \right) \right]^{\frac{1}{2}} + v + 2$$

We have for all $\tau \geq 0$

$$\tau - \frac{1}{2} \ln (2 \tau) > \frac{3}{5} \tau$$

and therefore because of (1.13) for all $\sigma \geq 0$,

$$\tau - \frac{3}{2} - \max \left( \frac{1 - \sigma}{2}, 0 \right) \ln \left( \frac{1}{2} + \tau - \frac{3}{2} + \tau \right) > \ln \frac{8}{\delta}$$

Because the solutions of (1.9) increase monotonically with $x$, it follows

$$\tau > \frac{3}{2} + x_0 > \frac{1}{2} + v$$

thus also $v < (v \tau)^{\frac{1}{2}}$, and therefore from (2.27),

$$S < \left( 4.83 \sqrt{\tau} + 1 \right) \sqrt{\tau} \sqrt{\tau} + 2$$

Using (2.11) and (2.29) we arrive at the upper bound (1.14).

2.3. Proof of Theorem 5. A Dirichlet $L$-function with primitive character $\chi$ mod $q$ is given for $\sigma > 0$ by

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

and fulfills the functional equation ([4], 25.15.5 and 25.15.6),

$$L(s,\chi) = G(\chi) q^{-s} (2\pi)^{s-1} \Gamma(1-s) \sum_{\mu=\pm 1} L(1-s,\chi) \chi(-\mu) e^{i\mu \frac{\pi}{q}(1-s)}$$

where

$$G(\chi) = \sum_{p=1}^{q} \chi(p) e^{2\pi pi/q}$$

is the Gauss sum. Repeating the arguments of section 2.1 and assuming that $\chi$ is not the principal character, we have at once its Zetafast algorithm (1.15).

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