A new weak Galerkin (WG) finite element method for solving the second-order elliptic problems on polygonal meshes by using polynomials of boundary continuity is introduced and analyzed. The WG method is utilizing weak functions and their weak derivatives which can be approximated by polynomials in different combination of polynomial spaces. Different combination gives rise to different weak Galerkin finite element methods, which makes WG methods highly flexible and efficient in practical computation. This paper explores the possibility of certain combination of polynomial spaces that minimize the degree of freedom in the numerical scheme, yet without losing the accuracy of the numerical approximation. Error estimates of optimal order are established for the corresponding WG approximations in both a discrete $H^1$ norm and the standard $L^2$ norm. In addition, the paper also presents some numerical experiments to demonstrate the power of the WG method. The numerical results show a great promise of the robustness, reliability, flexibility and accuracy of the WG method.

Key words. weak Galerkin finite element methods, weak gradient, second-order elliptic equation, polygonal meshes.

AMS subject classifications. Primary, 65N30, 65N15, 65N12, 74N20; Secondary, 35B45, 35J50, 35J35

1. Introduction. Nowadays, finite element methods (FEMs) are widely used in almost every field of engineering and industrial analysis. Since R. Courant [7] formulated the essence of what is now called a finite element in 1943, this method has been getting more and more attractive with the development of computers and is now recognized as one of the most versatile and powerful methods for approximating the solutions of boundary-value problems, especially for problems over complicated domains. Among the different finite elements methods, the conforming finite element method [18] with continuous, piecewise polynomial approximating spaces, has long been employed to approximate solutions for partial differential equations. Within the past few decades, however, a number of researchers have investigated Galerkin methods based on fully discontinuous approximating spaces, such as the discontinuous Galerkin (DG) methods [5, 1, 2, 3], the Hybridized discontinuous Galerkin (HDG) methods [6, 8], the weak Galerkin (WG) methods [19, 20], etc.

The WG method was first introduced in 2012 [19] for the second order elliptic problem and further developed with other applications, such as the Stokes, Helmholtz, Maxwell, biharmonic [13, 12, 9, 10, 17, 16, 19, 20, 23], etc. Its central idea is that
the shape function in the interior of each element is simply polynomial. The weak finite element function could be totally discontinuous across elements. The continuity is compensated by the stabilizer through a suitable boundary integral defined on the boundary of elements. That is, we know more information on the shape function by sacrificing the continuity.

Comparing with the conforming FEMs, there are a lot of advantages for the discontinuous methods, for instance, high-order accuracy, multiphysics capability, the finite element partition can be of polygon or polyhedral type, the weak finite element space is easy to construct with any approximation requirement and suit for any given stability requirement. All of this, however, comes at a price: most notably through an increase in the total degrees of freedom (d.o.f.) as a direct result of the decoupling of the elements. For linear elements, this yields a doubling in the total number of degrees of freedom compared to the continuous FEM. For WG finite element methods, in order to reduce the d.o.f., people have made some efforts: in [14, 22], the possibility of optimal combination of polynomial spaces was explored to minimize the number of unknowns in the numerical scheme, yet without compromising the accuracy of the numerical approximation; in [22], some hybridization of finite element methods has been introduced by utilizing the Lagrange multiplier. The distinctive feature of the methods in this framework is that the only globally coupled degrees of freedom are those of an approximation of the solution defined only on the boundaries of the elements, then the global unknowns are the numerical traces of the field variables. Thus, one can reduce the number of the globally coupled degrees of freedom of WG methods.

The goal of this paper is to explore the possibility of certain combination of the polynomial spaces to reduce the number of unknowns without compromising the rate of convergence, which utilizes the boundary continuity for the WG finite element spaces. This idea is motivated by Chen [4] that was presented in ICIAM 2015. Combining with the Schur complement technique, we further eliminate the interior unknowns and produce a much reduced system of linear equations involving only the unknowns representing the interface variables. In fact, if we take the triangle mesh, the d.o.f. of the new scheme is the same with that of conforming FEMs.

Next, we introduce this WG methods. For the sake of simplicity and easy presentation of the main ideas, we restrict ourselves to the following model problem

\begin{align}
- \nabla \cdot (a \nabla u) &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align}

where \( \Omega \) is an open bounded polygonal domain in \( \mathbb{R}^2 \). We assume that \( f \) are given, sufficiently smooth functions, and \( a \) is a symmetric \( 2 \times 2 \) matrix-valued function. Suppose there exists a positive number \( \lambda \) such that

\[ \xi^t a \xi \geq \lambda \xi^t \xi, \quad \forall \xi \in \mathbb{R}^2, \]

where \( \xi \) is a column vector and \( \xi^t \) is the transpose of \( \xi \).

The paper is organized as follows. In Section 2, we present some standard notations in Sobolev spaces and preliminaries. A weakly-defined differential operator is also introduced. The weak Galerkin finite element scheme is developed and some properties for the error analysis are discussed in Section 3. In Section 4, we shall derive an error equation for the WG approximations. Optimal-order error estimates
of $H^1$ and $L^2$ for the WG finite element approximations are also derived in this Section. The equivalence of WG formulation and its Schur complement formulation is proved in Section 5. In Section 6, numerical experiments are conducted. Finally, we present some technical estimates for quantities related to the local $L^2$ projections into various finite element spaces and some approximation properties which are useful in the convergence analysis in Appendix.

2. Notations and preliminaries. In this section, we shall introduce some notations used in this paper.

We use the standard Soblev space notations. For an open set $D \in \mathbb{R}^d$, $\| \cdot \|_{s,D}$ and $(\cdot, \cdot)_{s,D}$ stand for the $H^s(D)$ norm and inner-product, namely. We shall drop the subscripts when $s = 0$ and $D = \Omega$.

Let $\mathcal{T}_h$ be a partition of the domain consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions specified in [20]. Denote by $\mathcal{E}_h$ the set of all edges or at faces in $\mathcal{T}_h$. For every element $T \in \mathcal{T}_h$, we denote by $h_T$ its diameter and mesh size $h = \max_{T \in \mathcal{T}_h} h_T$ for $\mathcal{T}_h$.

For a given partition $\mathcal{T}_h$, $P_k(T)$ denotes a piecewise polynomial on $\mathcal{T}_h$ whose degree is no more than $k$ on each $T \in \mathcal{T}_h$. Similarly, $P_k(e)$ denotes a piecewise polynomial on $\mathcal{E}_h$ whose degree is no more than $k$ on each $e \in \mathcal{E}_h$.

Now, we define the weak finite element space as follows:

$$V_h = \{ (v_0, v_b) : v_0|_T \in P_k(T), v_b|_e \in P_k(e), v_b \text{ is continuous on } \mathcal{E}_h, v_b = 0 \text{ on } \partial \Omega \}.$$  

It should be noticed that $v_b$ is single-valued on each edge, and $v_b$ is continuous on $\mathcal{E}_h$, which means that $v_b$ share the same value on each node of $\mathcal{T}_h$.

Similar to the definition in [19], we can define the following weak gradient operator on $V_h$.

**Definition 2.1.** For any $v_h \in V_h$, define the discrete weak gradient $\nabla_w v_h |_{T} \in [P_{k-1}(T)]^2$ satisfying

$$\nabla_w v_h, q_T = -(v_0, \nabla \cdot q)_T + \langle v_b, q \cdot n \rangle_{\partial T}, \quad \forall q \in [P_{k-1}(T)]^2,$$

where $n$ is the outward unit normal vector along $\partial T$, $(\cdot, \cdot)_T$ stands for the $L^2$-inner product in $L^2(T)$, and $(\cdot, \cdot)_{\partial T}$ is the inner product in $L^2(\partial T)$

3. A weak Galerkin finite element scheme. In this section, we shall propose a WG scheme for the second-order elliptic problem, and verify the wellposedness of the scheme.

For any $v_h, w_h \in V_h$, define the following bilinear forms

$$s(v_h, w_h) = \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle v_0 - v_b, w_0 - w_b \rangle_{\partial T},$$

$$a_w(v_h, w_h) = (a \nabla_w v_h, \nabla_w w_h) + s(v_h, w_h).$$

**Weak Galerkin Algorithm 1.** The weak Galerkin numerical solution of problem (1.1) - (1.2) can be obtained by seeking $u_h \in V_h$ such that

$$a_w(u_h, v_h) = (f, v_0) \quad \forall v_h \in V_h.$$
The following semi-norm can be inducted from $a_w(\cdot, \cdot)$ directly
\[ \| v_h \|^2 = a_w(v_h, v_h), \quad \forall v_h \in V_h. \]

We claim that $\| \cdot \|$ defines a norm on $V_h$ indeed.

Notice that when $\| v_h \| = 0$, we have $\nabla w v_h = 0$ on each $T \in T_h$ and $v_0 = v_b$ on each $e \in \mathcal{E}_h$, and it follows that $\forall T \in T_h$,
\[
0 = (\nabla w v_h, \nabla v_0)_T \\
= -(v_0, \nabla \cdot \nabla v_0)_T + (v_b, \nabla v_0 \cdot n)_{\partial T} \\
= (\nabla v_0, \nabla v_0)_T - (v_0 - v_b, \nabla v_0 \cdot n)_{\partial T} \\
= \| \nabla v_0 \|_T^2,
\]
which implies that $v_0$ is constant on $T$. With the condition that $v_b = v_0$ on each $e \in \mathcal{E}_h$ and $v_b = 0$ on $\partial \Omega$, we can conclude that $v_h = 0$, and thus $\| \cdot \|$ defines a norm on $V_h$. From this property we can arrive the wellposedness of WG scheme (3.1) directly.

**Theorem 3.1.** The WG scheme (3.1) has a unique solution.

Next, we shall discuss some properties for the error analysis.

Define the following $L^2$ projections operators:
\[
Q_0 : L^2(T) \rightarrow P_k(T), \quad \forall T \in T_h, \\
Q_b : L^2(e) \rightarrow P_k(e), \quad \forall e \in \mathcal{E}_h, \\
Q_h : [L^2(T)]^2 \rightarrow [P_{k-1}(T)]^2, \quad \forall T \in T_h.
\]

It is known that on a one-dimension edge $e$, for any smooth function $w$ we can get a polynomial interpolation of degree $k$ with $k + 1$ different points. Suppose the $k + 1$ interpolation points include the two endpoints of the edge, and we have the interpolation operator $I_b$.

Combining $Q_0$ and $I_b$, we can define
\[
\tilde{Q}_h = \{ Q_0, I_b \} : H^1(\Omega) \rightarrow V_h.
\]

In the rest of this paper, we denote $\bar{a}$ the local $P_0(T)$ projection of $a$. Obviously, $\bar{a}$ is symmetric and bounded. With these projections, it arrives at the following community property.

**Lemma 3.2.** For any $q \in [P_{k-1}(T)]^2$, $w \in H^1(\Omega)$,
\[
(\nabla w \tilde{Q}_h w, q) = (Q_h \nabla w, q) + \sum_{T \in T_h} (I_b w - w, q \cdot n)_{\partial T}.
\]

**Proof.** For any $T \in T_h$, it follows the definition (2.1) and the integration by parts that
\[
(\nabla_{\psi} \tilde{Q}_h w, q)_T
= -(Q_0 w, \nabla \cdot q)_T + (I_h w, q \cdot n)_{\partial T}
= -(Q_0 w, \nabla \cdot q)_T + (w, q \cdot n)_{\partial T} + (I_h w - w, q \cdot n)_{\partial T}
= (\nabla_\psi w, q)_T + (I_h w - w, q \cdot n)_{\partial T}
= (Q_h \nabla w, q)_T + (I_h w - w, q \cdot n)_{\partial T}.
\]

Summing over all \(T \in T_h\) and the proof is completed. \(\square\)

4. Error analysis. In this section, we shall present the \(H^1\) and \(L^2\) errors of the weak Galerkin numerical scheme (3.1) with optimal orders.

Denote the error \(e_h = \tilde{Q}_h u - u_h\), and we shall show the error equation for \(e_h\).

**Lemma 4.1.** Suppose \(u \in H^2(\Omega)\) is the solution of (1.1)-(1.2), and \(u_h\) is the WG numerical solution of (3.1). Then for any \(v_h \in V_h\),

\[
a_w(\tilde{Q}_h u - u_h, v_h) = l(u, v_h),
\]

where

\[
l(u, v_h) = (a \nabla w \tilde{Q}_h u - \tilde{Q}_h (a \nabla u), \nabla w v_h)
- \sum_{T \in T_h} \langle (\tilde{Q}_h (a \nabla u) - a \nabla u) \cdot n, v_0 - v_b \rangle_{\partial T} + s(\tilde{Q}_h u, v_h).
\]

**Proof.** For any \(v_h \in V_h\), it follows the integration by parts that

\[
a_w(u_h, v_h) = (f, v_0)
= (a \nabla u, \nabla v_0) - \sum_{T \in T_h} (a \nabla u \cdot n, v_0)_{\partial T}
= (a \nabla u, \nabla v_0) - \sum_{T \in T_h} (a \nabla u \cdot n, v_0 - v_b)_{\partial T}.
\]

(4.1)

From the definition (2.1), we can obtain

\[
(a \nabla u, \nabla v_0)
= (Q_h (a \nabla u), \nabla v_0)
= (Q_h (a \nabla u), \nabla w v_h) + \sum_{T \in T_h} \langle Q_h (a \nabla u) \cdot n, v_0 - v_b \rangle_{\partial T}
\]

(4.2)

Substituting (4.2) into (4.1), we get

\[
a_w(u_h, v_h) = (Q_h (a \nabla u), \nabla w v_h) + \sum_{T \in T_h} \langle (Q_h (a \nabla u) - a \nabla u) \cdot n, v_0 - v_b \rangle_{\partial T}.
\]

(4.3)

Notice that

\[
a_w(\tilde{Q}_h u, v_h) = (a \nabla w \tilde{Q}_h u, \nabla w v_h) + s(\tilde{Q}_h u, v_h).
\]

(4.4)
Subtracting (4.3) from (4.4), we have
\[ a_w(\tilde{Q}_h u - u_h, v_h) = (a\nabla_w \tilde{Q}_h u - \tilde{Q}_h (a\nabla u), \nabla_w v_h) \]
\[ - \sum_{T \in T_h} \langle (\tilde{Q}_h (a\nabla u) - a\nabla u) \cdot n, v_0 - v_b \rangle_{\partial T} + s(\tilde{Q}_h u, v_h), \]
which completes the proof. \(\square\)

With the error equation and the technique tools in Appendix, we can arrive at the estimate for the \(H^1\) error.

**Theorem 4.2.** Suppose \(u \in H^1_0(\Omega) \cap H^{k+1}(\Omega)\) is the solution of (1.1)- (1.2), and \(u_h \in V_h\) is the WG numerical solution of (3.1). The following estimate holds true,
\[ \| \tilde{Q}_h u - u_h \| \leq C h^k \| u \|_{k+1}. \]

**Proof.** Take \(v_h = \tilde{Q}_h u - u_h\) in Lemma 4.1, then from Lemma 7.6 we can obtain
\[ \| \tilde{Q}_h u - u_h \|^2 = l(u, \tilde{Q}_h u - u_h) \]
\[ \leq C h^k \| \tilde{Q}_h u - u_h \|, \]
which completes the proof. \(\square\)

Next, we shall show the optimal \(L^2\) error order using the well-known Nitsche’s argument. Consider the following dual problem: find \(\varphi \in H^1_0(\Omega)\) satisfying
\[ - \nabla \cdot (a\nabla \varphi) = e_0. \]
(4.5)
Assume the dual problem (4.5) has \(H^2\) regularity, i.e.
\[ \| \varphi \|_2 \leq C \| e_0 \|. \]
(4.6)

**Theorem 4.3.** Suppose \(u \in H^1_0(\Omega) \cap H^{k+1}(\Omega)\) is the solution of (1.1)- (1.2), \(u_h \in V_h\) is the WG numerical solution of (3.1), and the dual problem (4.5) satisfies the regularity assumption (4.6). The following estimate holds true,
\[ \| Q_0 u - u_0 \| \leq C h^{k+1} \| u \|_{k+1}. \]

**Proof.** Testing equation (4.5) by \(e_0\) and we can obtain
\[ \| e_0 \|^2 \]
\[ = (a\nabla \varphi, \nabla e_0) - \sum_{T \in T_h} \langle \nabla \varphi \cdot n, e_0 \rangle_{\partial T} \]
\[ = (Q_h (a\nabla \varphi), \nabla e_0) - \sum_{T \in T_h} \langle \nabla \varphi \cdot n, e_0 - e_b \rangle_{\partial T} \]
\[ = (Q_h (a\nabla \varphi), \nabla e_0) + \sum_{T \in T_h} \langle (Q_h (a\nabla \varphi) - a\nabla \varphi) \cdot n, e_0 - e_b \rangle_{\partial T}. \]
(4.7)
Similar to the derivative of formula \((4.2)\), we have

\[
a_w(Q_h \varphi, e_h) = (a \nabla_w Q_h \varphi, \nabla_w e_h) + s(Q_h \varphi, e_h).
\]

Subtracting \((4.8)\) from \((4.7)\) yields

\[
\|e_0\|^2 = a_w(Q_h \varphi, e_h) - l(\varphi, e_h).
\]

By substituting \(v_h = \tilde{Q}_h \varphi\) in Lemma \(4.1\), we get

\[
a_w(e_h, \tilde{Q}_h \varphi) = l(u, \tilde{Q}_h \varphi),
\]

which implies

\[
\|e_0\|^2 = l(u, \tilde{Q}_h \varphi) - l(\varphi, e_h).
\]

From Lemma \(7.6\) and \((4.9)\), we derive that

\[
l(\varphi, e_h) \leq Ch\|e_h\|\|\varphi\|_2
\]

\[
\leq Ch^{k+1}\|u\|_{k+1}\|e_0\|.
\]

Now we just need to estimate the following three terms of \(l(u, \tilde{Q}_h \varphi)\). As to the first term, we split it into two formulas and estimate them separately as follows

\[
\begin{align*}
&= (a \nabla_w \tilde{Q}_h u - Q_h(a \nabla u), \nabla_w \tilde{Q}_h \varphi) \\
&= (\nabla_w \tilde{Q}_h u - \nabla u, a \nabla_w \tilde{Q}_h \varphi) \\
&= (\nabla_w \tilde{Q}_h u - \nabla u, a \nabla_w \tilde{Q}_h \varphi - Q_h(a \nabla_w \tilde{Q}_h \varphi)) \\
&\quad + (\nabla_w \tilde{Q}_h u - \nabla u, Q_h(a \nabla_w \tilde{Q}_h \varphi)) \\
&= (\nabla_w \tilde{Q}_h u - \nabla u, a \nabla_w \tilde{Q}_h \varphi - Q_h(a \nabla_w \tilde{Q}_h \varphi)) \\
&\quad + (\nabla_w \tilde{Q}_h u - Q_h \nabla u, Q_h(a \nabla_w \tilde{Q}_h \varphi)).
\end{align*}
\]

For the first formula, from Lemma \(7.4\) and Lemma \(7.5\), we can obtain

\[
\begin{align*}
&|\langle \nabla_w \tilde{Q}_h u - \nabla u, a \nabla_w \tilde{Q}_h \varphi - Q_h(a \nabla_w \tilde{Q}_h \varphi) \rangle| \\
&\leq \|\nabla_w \tilde{Q}_h u - \nabla u\|\|a \nabla_w \tilde{Q}_h \varphi - Q_h(a \nabla_w \tilde{Q}_h \varphi)\| \\
&\leq Ch^{k}\|u\|_{k+1}\|\nabla_w \tilde{Q}_h \varphi\|_1 \\
&\leq Ch^{k+1}\|u\|_{k+1}\|\varphi\|_2.
\end{align*}
\]

As to the second formula, it follows Lemmas \(3.2\) and \(7.3\) that

\[
\begin{align*}
&= \sum_{T \in \mathcal{T}_h} \langle I_b u - u, Q_h(a \nabla_w \tilde{Q}_h \varphi) \cdot n \rangle_{\partial T} \\
&= \sum_{T \in \mathcal{T}_h} \langle I_b u - u, Q_h(a \nabla_w \tilde{Q}_h \varphi - a \varphi) \cdot n \rangle_{\partial T} \\
&\quad + \sum_{T \in \mathcal{T}_h} \langle I_b u - u, (Q_h(a \nabla \varphi) - a \nabla \varphi) \cdot n \rangle_{\partial T} \\
&\leq Ch^{k+1}\|u\|_{k+1}\|\varphi\|_2.
\end{align*}
\]
For the second term, from Lemma 7.3 and the trace inequality we can obtain
\[
\sum_{T \in \mathcal{T}_h} \langle (Q_h(a \nabla u) - a \nabla u) \cdot n, Q_0 \varphi - I_b \varphi \rangle_{\partial T} \leq C \left( \sum_{T \in \mathcal{T}_h} h_T \| Q_h(a \nabla u) - a \nabla u \|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \| Q_0 \varphi - \varphi \|_{\partial T}^2 \right)^{1/2} + C \left( \sum_{T \in \mathcal{T}_h} \| Q_h(a \nabla u) - a \nabla u \|_{T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \| \varphi - I_b \varphi \|_{\partial T}^2 \right)^{1/2} \leq C h^{k+1} \| u \|_{k+1} \| \varphi \|_2.
\]
Similarly, for the third term we have
\[
s(\tilde{Q}_h u, \tilde{Q}_h \varphi) = \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_0 u - I_b u, Q_0 \varphi - I_b \varphi \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_0 u - u, Q_0 \varphi - \varphi \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle u - I_b u, Q_0 \varphi - \varphi \rangle_{\partial T} + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle u - I_b u, \varphi - I_b \varphi \rangle_{\partial T} \leq C h^{k+1} \| u \|_{k+1} \| \varphi \|_2,
\]
which leads to
\[
I(u, \tilde{Q}_h \varphi) \leq C h^{k+1} \| u \|_{k+1} \| \varphi \|_2 \leq C h^{k+1} \| u \|_{k+1} \| e_0 \|.
\]
By substituting (4.10) and (4.11) into (4.9), the proof is completed.

5. Schur Complement. In this section, we shall show the technique in practical which can eliminate the degree of freedoms of \( v_0 \), and then only the degree of freedoms of \( v_b \) are involved in the total linear system.

Let \( u_h = \{ u_0, u_b \} \) be the solution of the WG scheme (3.1), then \( u_h \) satisfies the following equation
\[
a_w(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h.
\]
This equation can be rewritten as the following form
\[
a_w(u_h, v_h) = (f, v_h), \quad \forall v_h = \{ v_0, 0 \} \in V_h,
\]
\[
a_w(u_h, v_h) = 0, \quad \forall v_h = \{ 0, v_b \} \in V_h.
\]
For given \( u_b \) on \( \partial T \), one can solve equation (5.1) for the \( u_0 \), and note
\[
u_0 = D(u_b, f).
\]
It should be noticed that \( D(u_b, f) \) can be calculated elementwisely. We can solve the local problem
\[
a_w(u_h, v_0) = (f, v_0), \quad \forall v_0 \in P_k(T),
\]
and get
\[ u_0 = D_T(u_b, f). \]  

Then we substitute (5.3) into (5.2) and get the following equation
\[ a_w(\{ D(u_b, f) \}, v_h) = 0, \quad \forall v_h = \{ 0, v_b \} \in V_h. \]

Thus we can conclude a complement for the WG scheme as follows.

Step 1. Solve equation (5.4) on each element \( T \in T_h \) and get \( u_0 = D(u_b, f) \).

Step 2. Solve the global system (5.2) for \( u_b \).

Step 3. Recover \( u_0 \) by \( u_0 = D_T(u_b, f) \) on each element.

When we take the uniform triangle mesh on the unit square and set \( k = 1 \), the degree of freedom of different schemes are listed in Table 5.

| \( h \)       | \( WG \) | \( CWG \) | \( WG \text{ Schur} \) | \( CWG \text{ schur} \) | \( CG \) |
|-------------|--------|---------|---------------------|--------------------|-------|
| 1/8         | 592    | 465     | 208                 | 81                 | 81    |
| 1/16        | 2336   | 1825    | 800                 | 289                | 289   |
| 1/32        | 9280   | 7233    | 3136                | 1089               | 1089  |
| 1/64        | 36992  | 28801   | 12416               | 4225               | 4225  |
| 1/128       | 147712 | 114945  | 49408               | 16641              | 16641 |

\( WG \) stands for the degree of freedom in the scheme in [15], \( CWG \) stands for scheme (3.1), \( WG \text{ Schur} \) stands for the scheme in [15] with Schur complement, \( CWG \text{ Schur} \) stands for scheme (5.2) with Schur complement, and \( CG \) stands for the conforming Galerkin method.

6. Numerical experiment. In this section, we shall present some numerical results to show the efficiency and accuracy of the WG scheme.

Example 1 Consider the problem (1.1) - (1.2) on the unit square \( \Omega = [0, 1] \times [0, 1] \). The analytic solution \( u \) is set to be \( u = \sin(\pi x)\sin(\pi y) \), the right side and the Dirichlet boundary condition is computed to match the exact solution. In this numerical experiment, the uniform rectangle mesh is employed. We can conclude from Table 6 that the convergence rate for \( H^1 \) and \( L^2 \) errors are of \( O(h) \) and \( O(h^2) \), respectively. This is coincide with the theoretical analysis in this paper.

Example 2 Consider the problem (1.1)- (1.2) on the unit square \( \Omega = [0, 1] \times [0, 1] \). The analytic solution \( u \) is set to be \( u = x(1 - x)y(1 - y) \), the right-hand side side and the Dirichlet boundary condition is computed to match the exact solution. In this numerical experiment, the uniform rectangle mesh is employed. We can conclude from Table 6 that the convergence rate for \( H^1 \) and \( L^2 \) errors are of \( O(h) \) and \( O(h^2) \), respectively. This is coincide with the theoretical results.

7. Appendix. In this section, we shall give some technique tools which are applied in the error estimate.
Table 6.1
\(k=1\) uniform convergence rates.

| \(h\)  | \(dof\)  | \(|Q_h u - u_h|\) | order | \(|Q_0 u - u_0|\) | order |
|-------|----------|---------------------|-------|---------------------|-------|
| 1/8   | 4.6500e+02 | 3.8193e-01          |       | 2.6130e-02          |       |
| 1/16  | 1.8250e+03 | 1.9065e-01          | 1.0024| 6.5871e-03          | 1.9880|
| 1/32  | 7.2330e+03 | 9.5281e-02          | 1.0006| 1.6503e-03          | 1.9969|
| 1/64  | 2.8801e+04 | 4.7635e-02          | 1.0002| 4.1281e-04          | 1.9992|
| 1/128 | 1.1495e+05 | 2.3817e-02          | 1.0000| 1.0322e-04          | 1.9998|

Table 6.2
\(k=1\) not uniform convergence rates.

| \(h\)  | \(dof\)  | \(|Q_h u - u_h|\) | order | \(|Q_0 u - u_0|\) | order |
|-------|----------|---------------------|-------|---------------------|-------|
| 1/8   | 7.5900e+02 | 2.7721e-01          |       | 9.9566e-03          |       |
| 1/16  | 3.0850e+03 | 1.3806e-01          | 1.0056| 2.5059e-03          | 1.9903|
| 1/32  | 1.2259e+04 | 7.0399e-02          | 0.9717| 6.0741e-04          | 2.0446|
| 1/64  | 4.9521e+04 | 3.5100e-02          | 1.0041| 1.5030e-04          | 2.0149|
| 1/128 | 1.9978e+05 | 1.7449e-02          | 1.0084| 3.6918e-05          | 2.0254|

Table 6.3
\(k=1\) uniform rectangle mesh convergence rates.

| \(h\)  | \(dof\)  | \(|Q_h u - u_h|\) | order | \(|Q_0 u - u_0|\) | order |
|-------|----------|---------------------|-------|---------------------|-------|
| 1/8   | 2.7300e+02 | 2.9292e-02          |       | 1.8766e-03          |       |
| 1/16  | 1.0570e+03 | 1.4587e-02          | 1.0059| 4.8858e-04          | 1.9415|
| 1/32  | 4.1610e+03 | 7.2859e-03          | 1.0015| 1.1926e-04          | 2.0344|
| 1/64  | 1.6513e+04 | 3.6420e-03          | 1.0004| 3.1107e-05          | 1.9389|
| 1/128 | 6.5793e+04 | 1.8209e-03          | 1.0001| 7.5782e-06          | 2.0373|

The trace inequality and the inverse inequality for the weak Galerkin method are proved in [20].

**Lemma 7.1.** Assume that the partition \(T_h\) satisfies the assumptions (A1), (A2), and (A3) as specified in [20]. Then, there exists a constant \(C\) such that for any \(T \in T_h\) and edge/face \(e \in \partial T\), we have for any \(\theta \in H^1(T)\),

\[
\|\theta\|^p_e \leq C h^{-1}_T (\|\theta\|^p_T + h^p_T \|\nabla \theta\|^p_T).
\]

**Lemma 7.2.** Assume that the partition \(T_h\) satisfies the assumptions (A1), (A2), (A3) and (A4) as specified in [20]. Then, there exists a constant \(C(k)\) such that for any \(T \in T_h\), we have for any \(\varphi \in P_k(T)\),

\[
\|\nabla \varphi\|^p_T \leq C(n) h^{-1}_T \|\varphi\|^p_T.
\]

The following estimate of interpolation error plays an essential role in the error estimate.

**Lemma 7.3.** Assume that the partition \(T_h\) satisfies the assumptions (A1), (A2), (A3) and (A4) as specified in [20]. For any \(w \in H^{k+1}(\Omega)\), the following estimate...
holds true
\[ \sum_{T \in T_h} \| w - I_b w \|^2_{\partial T} \leq C h^{2k+1} \| w \|^2_{k+1}. \]

Proof. For any edge \( e \in \partial T \), according to assumption (A3), there exists a triangle \( P_e \), whose base is identical to \( e \) and height is proportional to \( h_T \). Then there exists an interpolation operator \( I_{P_e} \) onto \( P_k(P_e) \) which contains all the interpolation points of \( I_b \). It follows the trace inequality (7.1) and the property of interpolation operator that for any \( w \in H^{k+2}(T) \),
\[
\| w - I_b w \|^2_e = \| w - I_{P_e} w \|^2_e \\
\leq C (h_{P_e}^{-1} \| w - I_{P_e} w \|^2_{P_e} + h_{P_e} \| \nabla (w - I_{P_e} w) \|^2_{P_e}) \\
\leq C h_{P_e}^{2k+1} \| w \|^2_{k+1,P_e} \\
\leq C h_T^{2k+1} \| w \|^2_{k+1,T}.
\]
Summing over all \( T \in T_h \) and the proof is completed. \( \square \)

With Lemma 7.3, we can describe the difference between \( \nabla w \tilde{Q}_h w \) and \( Q_h \nabla w \) in the following lemma.

**Lemma 7.4.** Assume that the partition \( T_h \) satisfies the assumptions (A1), (A2), (A3) and (A4) as specified in [20]. For any \( w \in H^{k+1}(\Omega) \), the following estimate holds true
\[ \| \nabla w \tilde{Q}_h w - Q_h \nabla w \| \leq C h^{k} \| w \|_{k+1}. \]

Proof. It follows Lemma 3.2 that for any \( q \in [P_{k-1}(T)]^2 \),
\[
(\nabla w \tilde{Q}_h w - Q_h \nabla w, q) = \sum_{T \in T_h} (I_b w - w, q \cdot n)_{\partial T} \\
\leq C \left( \sum_{T \in T_h} \| I_b w - w \|^2_{\partial T} \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} \| q \|^2_{\partial T} \right)^{\frac{1}{2}}.
\]
From the trace inequality and the inverse inequality, we can obtain
\[
\sum_{T \in T_h} \| q \|^2_{\partial T} \\
\leq C \sum_{T \in T_h} h_T^{-1} \| q \|^2_{T} + C \sum_{T \in T_h} h_T \| \nabla q \|^2_{T} \\
\leq C \sum_{T \in T_h} h_T^{-1} \| q \|^2_{T}.
\]
Taking \( q = \nabla w \tilde{Q}_h w - Q_h \nabla w \) and applying Lemma 7.3 yield
\[
\| \nabla w \tilde{Q}_h w - Q_h \nabla w \|^2 \leq C h^k \| w \|_{k+1} \| \nabla w \tilde{Q}_h w - Q_h \nabla w \|.
\]
which completes the proof. \(\Box\)

**Lemma 7.5.** Assume that the partition \(T_h\) satisfies the assumptions (A1), (A2), (A3) and (A4) as specified in [20]. For any \(w \in H^2(\Omega)\), the following estimate holds true

\[ \| \nabla w \tilde{Q}_h w \|_1 \leq C \| w \|_2. \]

**Proof.** When \(w \in P_1(T)\), from definition 2.1 we know \(w = \tilde{Q}_h w\) and \(\nabla w \tilde{Q}_h w = \nabla w\), so that

\[ \| \nabla w \tilde{Q}_h w - \nabla w \|_1 = 0, \quad \forall w \in P_1(T). \]

Then, from Bramble-Hilbert Theorem it follows that

\[ \| \nabla w \tilde{Q}_h w - \nabla w \|_1 \leq C \| w \|_2, \quad \forall w \in H^2(T). \]

From the triangle inequality we can obtain

\[
\begin{align*}
\| \nabla w \tilde{Q}_h w \|_1 & \\
& \leq \| \nabla w \tilde{Q}_h w - \nabla w \|_1 + \| \nabla w \|_1 \\
& \leq C \| w \|_2,
\end{align*}
\]

which completes the proof. \(\Box\)

In order to give the error estimate, we need to estimate the remainder of the error equation in Lemma 4.1.

**Lemma 7.6.** Suppose \(w \in H^{k+1}(\Omega)\) and \(v_h \in V_h\), then the following estimates hold true

\[
\begin{align*}
(a \nabla w \tilde{Q}_h w - Q_h(a \nabla w), \nabla w v_h) & \leq Ch^k \| w \|_{k+1} \| v_h \|, \\
\sum_{T \in T_h} \langle (Q_h(a \nabla w) - a \nabla w) \cdot n, v_0 - v_b \rangle_{\partial T} & \leq Ch^k \| w \|_{k+1} \| v_h \|, \\
s(\tilde{Q}_h w, v_h) & \leq Ch^k \| w \|_{k+1} \| v_h \|.
\end{align*}
\]

Moreover, we have

\[ l(w, v_h) \leq Ch^k \| w \|_{k+1} \| v_h \|. \]

**Proof.** For the first inequality, from the triangle inequality and Lemma 7.3 it follows that

\[
\begin{align*}
(a \nabla w \tilde{Q}_h w - Q_h(a \nabla w), \nabla w v_h) & \\
& \leq \| a \nabla w \tilde{Q}_h w - Q_h(a \nabla w) \| \| \nabla w v_h \| \\
& \leq \| a \nabla w \tilde{Q}_h w - a(Q_h \nabla w) \| \| v_h \| + \| a(Q_h \nabla w) - a \nabla w \| \| v_h \| \\
& \quad + \| a \nabla w - Q_h(a \nabla w) \| \| v_h \| \\
& \leq Ch^k \| w \|_{k+1} \| v_h \|.
\end{align*}
\]
As the second inequality, from the properties of projection operator and the trace inequality we can obtain

\[
\sum_{T \in T_h} \langle (Q_h(a \nabla w) - a \nabla w) \cdot n, v_0 - v_b \rangle_{\partial T} \\
\leq C \left( \sum_{T \in T_h} h_T \| Q_h(a \nabla w) - a \nabla w \|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-1} \| v_0 - v_b \|_{\partial T}^2 \right)^{\frac{1}{2}} \\
\leq C \left( \sum_{T \in T_h} h_T \| Q_h(a \nabla w) - a \nabla w \|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-1} \| v_0 - v_b \|_{\partial T}^2 \right)^{\frac{1}{2}} \\
\leq C h^k \| w \|_{k+1} \| v_h \|.
\]

Similarly, for the last inequality we have

\[
s(Q_h w, v_h) \\
= \sum_{T \in T_h} h_T^{-1} \langle Q_0 w - I_h w, v_0 - v_b \rangle_{\partial T} \\
= \sum_{T \in T_h} h_T^{-1} \langle Q_0 w - w, v_0 - v_b \rangle_{\partial T} + \sum_{T \in T_h} h_T^{-1} \langle w - I_h w, v_0 - v_b \rangle_{\partial T} \\
\leq C \left( \sum_{T \in T_h} h_T^{-1} \| Q_0 w - w \|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-1} \| v_0 - v_b \|_{\partial T}^2 \right)^{\frac{1}{2}} \\
+ C \left( \sum_{T \in T_h} h_T^{-1} \| w - I_h w \|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-1} \| v_0 - v_b \|_{\partial T}^2 \right)^{\frac{1}{2}} \\
\leq C h^k \| w \|_{k+1} \| v_h \|.
\]

\[
\square
\]

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