DECOMPOSABLE CYCLES AND NOETHER-LEFSCHETZ LOCI

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0. Introduction

Let $X$ be a smooth complex surface: a rational equivalence class of 0-cycles on $X$ is decomposable if it is the intersections of two divisor classes. Let $DCH_0(X) \subset CH_0(X)$ be the subgroup generated by decomposable 0-cycles. Beauville and Voisin [1] proved that if $X$ is a $K3$ surface then $DCH_0(X) \cong \mathbb{Z}$. What can be said of the group $DCH_0(X)$ in general? A projective irregular surface $X$ with non-zero map $\bigwedge^2 H^0(\Omega_X^1) \to H^0(\Omega_X^2)$ provides an example with group of decomposable 0-cycles that is not finitely generated, even after tensorization with $\mathbb{Q}$. Let us assume that $X$ is a projective regular surface: then $DCH_0(X)$ is finitely generated because $\text{CH}^1(X)$ is finitely generated, and we may ask what is the rank. The blow-up of a regular surface with non-zero geometric genus at $(r-1)$ generic points gives examples of projective regular surfaces with $DCH_0(X)$ of rank at least $r$ (see Example 1.3 b) of [2]). What about a less artificial class of surfaces, such as (smooth) surfaces in $\mathbb{P}^3$? If the rank of $DCH_0(X)$ is to be larger than 1 then the rank of $\text{CH}^1(X)$ must be larger than 1 but the latter condition is not sufficient, for example curves on $X$ whose canonical line-bundle is a (fractional) power of the hyperplane bundle do not increase the rank of $DCH_0(X)$, see Subsection 1.2. (The papers [13, 4] provide examples of smooth surfaces in $\mathbb{P}^3$ with Picard group of large rank and generated by lines: it follows that the group spanned by decomposable 0-cycles of such surfaces has rank 1.) On the other hand Lie Fu proved that there exist degree-8 surfaces $X \subset \mathbb{P}^3$ such that $DCH_0(X)$ has rank at least 2, see 1.4 of [6]. In the present paper we will prove the result below.

Theorem 0.1. There exist smooth surfaces $X \subset \mathbb{P}^3$ of degree $d$ such that the rank of $DCH_0(X)$ is at least $\lfloor \frac{d-1}{3} \rfloor$.

In particular the rank of the group of decomposable 0-cycles of a smooth surface in $\mathbb{P}^3$ can be arbitrarily large.

The paper is organized as follows. In Section 1 we examine surfaces in a smooth 3-fold $V$ that contain the disjoint union $C = C_1 \cup \ldots \cup C_n$ of smooth closed curves, more precisely surfaces in the linear system $|\mathcal{F}_C(H)|$, where $H$ is a sufficiently ample divisor on $V$. We assume that $V$ has trivial Chow group. We prove that if the curves $C_i$ are not rationally canonical (see Item (1) of Proposition 1.6 for the precise hypothesis), and a certain Noether-Lefschetz condition is satisfied by surfaces in $|\mathcal{F}_C(H)|$, then the classes of $C_1^*, \ldots, C_n^*$ on a very general $X \in |\mathcal{F}_C(H)|$ are linearly independent, and they span a subgroup with no non-trivial intersection with the image of $\text{CH}^2(V) \to \text{CH}^2(X)$. The idea behind our result is that of spread. In Section 2 the 3-fold $V$ is assumed to be projective 3-space; applying Joshi’s results [9] and the idea of Griffiths-Harris as developed by Lopez and Brevik-Nollet [8, 12, 5], we prove a Noether-Lefschetz result for surfaces containing fixed curves that will suffice to apply the main result of Section 1. In Section 3 we prove Theorem 0.1.

Conventions and notation: We work over $\mathbb{C}$. Points are closed points.

Let $X$ be a variety: “If $x$ is a generic point of $X$, then...” is shorthand for “There exists an open dense $U \subset X$ such that if $x \in U$ then...”. Similarly the expression “If $x$ is a very general point of $X$, then...” is shorthand for “There exists a countable collection of closed nowhere dense $Y_i \subset X$ such that if $x \in (X \setminus \bigcup_i Y_i)$ then...”.

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From now on we will denote by $\text{CH}(X)$ the group of rational equivalence classes of cycles with rational coefficients. Thus if $Z_1, Z_2$ are cycles on $X$ then $Z_1 \equiv Z_2$ means that for some non-zero integer $\ell$ the cycles $\ell Z_1, \ell Z_2$ are integral and rationally equivalent. If $Z$ is a cycle on $X$ we will often use the same symbol (i.e. $Z$) for the rational equivalence class represented by $Z$.

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1. The family of surfaces containing given curves

1.1. Threefolds with trivial Chow groups. Throughout the paper $V$ is an integral smooth projective threefold.

Hypothesis 1.1. The cycle class map $cl: \text{CH}(V) \rightarrow H(V; \mathbb{Q})$ is an isomorphism.

The archetypal such $V$ is $\mathbb{P}^3$. A larger class of examples is given by 3-folds with an algebraic cellular decomposition (see Ex. 1.9.1 of [7]), and conjecturally the above assumption is equivalent to vanishing of $H^{p,q}(V)$ for $p \neq q$. An integral smooth projective threefold has trivial Chow group if Hypothesis 1.1 holds.

Claim 1.2. Let $V$ be as above, in particular it has trivial Chow group. The natural map

$$S^2 \text{CH}^1(V) \rightarrow \text{CH}^2(V)$$

(1.1)

is surjective.

Proof. The natural map $S^2 H^2(V; \mathbb{Q}) \rightarrow H^4(V; \mathbb{Q})$ is surjective by Hard Lefschetz. The claim follows because of Hypothesis 1.1. \hfill \Box

1.2. Standard relations. Let $V$ be an integral smooth projective 3-fold with trivial Chow group. Let $X \subset V$ be a closed surface (pure 2-dimensional subscheme) and $i: X \hookrightarrow V$ be the inclusion map. Let $R^s(X) \subset \text{CH}^s(X)$ be the image of the restriction map

$$\text{CH}^s(V) \rightarrow \text{CH}^s(X) \quad \xi \mapsto i^* \xi$$

(1.2)

Notice that $R^2(X) \subset \text{DCH}_0(X)$ by Claim 1.2. Suppose that $C \subset X$ is an integral smooth curve. We will assume that $C \cdot C$ makes sense in $\text{CH}_0(X)$, for example that will be the case if $X$ is $\mathbb{Q}$-Cartier. Let us list some elements of the kernel of the map

$$R^2(X) \oplus R^1(X) \oplus R^0(X) \rightarrow \text{DCH}_0(X) \quad (\alpha, \beta, \gamma) \mapsto \alpha + \beta + \gamma \cdot C$$

(1.3)

Let $j: C \hookrightarrow V$ be the inclusion map. By Cor. 8.1.1 of [7] the following relation holds in $\text{CH}_0(X)$:

$$i^*(j_*[C]) = C \cdot c_1(J_{V/X}) = C \cdot i^* \theta_V(X).$$

(1.4)

Thus

$$\alpha_C - C \cdot i^* \theta_V(X) = 0,$$

(1.5)

where $\alpha_C := i^*(j_*[C]) \in R^2(X)$. Equation (1.5) is the first standard relation.

Now suppose that there exists $\xi \in \text{CH}^1(V)$ such that

$$c_1(K_C) = \xi |_C.$$

(1.6)

(Recall that Chow groups are with $\mathbb{Q}$-coefficients, thus (1.6) means that there exists an integer $n > 0$ such that $K_C^{\otimes n}$ is the pull-back of a line bundle on $V$.) Adjunction for $X \subset V$ and for $C \subset X$ give

$$C \cdot C + C \cdot (i^* K_V + i^* \theta_X(X)) \equiv C \cdot i^* \xi.$$

(1.7)

Thus there exists $\beta_C \in R^1(X)$ such that

$$\beta_C \cdot C - C \cdot C = 0.$$

(1.8)

The above is the second standard relation (it holds assuming (1.6)).

Example 1.3. Let $V = \mathbb{P}^3$, let $X \subset \mathbb{P}^3$ be a smooth surface of degree $d$, and $C \subset X$ a smooth curve. The subgroup of $\text{CH}_0(X)$ spanned by intersections of linear combinations of $H := c_1(\mathcal{O}(1))$ and $C$ has rank at most 2. In fact the first standard relation reads $dC \cdot H = (\deg C)H \cdot H$. If $c_1(K_C) = mC \cdot H$, where $m \in \mathbb{Q}$, then the second standard relation reads $C \cdot C = (m + 4 - d)C \cdot H$, and hence $C \cdot C, C \cdot H, H \cdot H$ span a rank-1 subgroup.
1.3. **Surfaces containing disjoint curves.** Let $V$ be a smooth projective 3-fold with trivial Chow group and $C_1,\ldots,C_n \subset V$ be pairwise disjoint integral smooth projective curves. Let $C := C_1 \cup \ldots \cup C_n$, and let $\pi : W \to V$ be the blow-up of $C$. Let $E$ be the exceptional divisor of $\pi$, and let $E_k$, for $k \in \{1,\ldots,n\}$, be the irreducible component of $E$ mapping to $C_k$. Let $H$ be a divisor on $V$. For $k \in \{1,\ldots,n\}$ we let

$$\Sigma_k := \{ S \in |\pi^*(H) - E| \mid \pi(S) \text{ is singular at some point of } C_k \}, \quad \Sigma := \cup_{k=1}^n \Sigma_k. \quad (1.9)$$

Let $S \in |\pi^*(H) - E|$, and let $X := \pi(S)$. Then $S \in \Sigma_k$ if and only if $S$ contains one (at least) of the fibers of $E_k \to C_k$, or, equivalently, the map $S \to X$ given by restriction of $\pi$ is not an isomorphism over $C_k$. We will always assume that $(\pi^*(H) - E)$ is very ample on $W$; with this hypothesis $\Sigma_k$ is irreducible of codimension 1, or empty (compute the codimension of the loci of $S \in |\pi^*(H) - E| \mid \text{which contain one or two fixed fibers of } E_k \to C_k$). Suppose that $H$ is sufficiently ample: then, in addition, if $S \in \Sigma_k$ is generic the surface $X = \pi(S)$ is smooth except for one ODP (ordinary double point) belonging to $C_k$, and the set of reducible $S \in |\pi^*H - E|$ has codimension at least 3 in $|\pi^*H - E|$. We will assume that both of these facts hold (but we do not assume that $H$ is “sufficiently ample”, because we want to prove effective results).

**Hypothesis 1.4.** Let $C_1,\ldots,C_n \subset V$ and $H$ be as above, and suppose in addition that

1. for $k \in \{1,\ldots,n\}$, and $S \in \Sigma_k$ generic, the surface $\pi(S)$ is smooth except for one ODP (ordinary double point) belonging to $C_k$, and
2. the set of reducible $S \in |\pi^*H - E|$ has codimension at least 3 in $|\pi^*H - E|$.

Assume Hypothesis [1.4] and let $S \in \Sigma_k$ be generic. Let $x$ the unique singular point of $\pi(S)$; then $S$ contains the line $\pi^{-1}(x)$, and the class of $\pi^{-1}(x)$ in $CH^1(S)$ is not in the image of the restriction map $CH^1(W) \to CH^1(S)$. The hypothesis below contains the key Noether-Lefschetz assumption.

**Hypothesis 1.5.** Let $C_1,\ldots,C_n \subset V$ and $H$ be as above, suppose that Hypothesis [1.4] holds, and that in addition the following Noether-Lefschetz statements hold:

1. If $A \subset |\pi^*(H) - E|$ is an integral closed codimension-1 subset, not equal to one of $\Sigma_1,\ldots,\Sigma_n$, and $S \in A$ is very general, the restriction map $CH^1(W) \to CH^1(S)$ is surjective.
2. For $k \in \{1,\ldots,n\}$, $S \in \Sigma_k$ very general, and $x$ the unique singular point of $\pi(S)$ (an ODP belonging to $C_k$, by Hypothesis [1.4], $CH^1(S)$ is generated by the image of the restriction map $CH^1(W) \to CH^1(S)$ together with the class of $\pi^{-1}(x)$).

In the present section we will prove the following result.

**Proposition 1.6.** Let $C_1,\ldots,C_n \subset V$ and $H$ be as above, and assume that Hypothesis [1.5] holds. Suppose also that for $k \in \{1,\ldots,n\}$ there does not exist $\xi \in CH^1(V)$ such that $c_1(K_{C_k}) = \xi|_{C_k}$. (Recall that Chow groups are with coefficients in $\mathbb{Q}$.) Then for very general smooth $X \in |\mathcal{I}_C(H)|$ the following hold:

1. The map $CH^2(V) \to CH_0(X)$ is injective.
2. Let $\{\zeta_1,\ldots,\zeta_m\}$ be a basis of $CH^1(V)$. Suppose that for very general smooth $X \in |\mathcal{I}_C(H)|$

$$0 = P(\zeta_1|X,\ldots,\zeta_m|X) + r_1C_1^2 + \ldots + r_nC_n^2,$$

where $P \in \mathbb{Q}[x_1,\ldots,x_m][2]$ is a homogeneous quadratic polynomial. Then $0 = P(\zeta_1,\ldots,\zeta_m) = r_1 = \ldots = r_n$.

The proof of Proposition [1.6] will be given in Subsection [1.7]. Throughout the present section we let $V,C,W,E$ and $H$ be as above, and we assume that Hypothesis [1.4] holds.

1.4. **The universal surface.** Let

$$\Lambda := |\pi^*(H) - E| \quad (1.10)$$

$$\mathcal{I} := \{(x,S) \in W \times \Lambda \mid x \in S\}. \quad (1.11)$$

Let $p_W : \mathcal{I} \to W$ and $p_{\Lambda} : \mathcal{I} \to \Lambda$ be the forgetful maps. Thus we have

$$V \xrightarrow{p_W} W \xrightarrow{p_{\Lambda}} \Lambda. \quad (1.12)$$
Let \( N := \dim \Lambda \). Since \((\pi^*(H) - E)\) is very ample it is globally generated and hence the map \( p_W \) is a \( \mathbb{P}^{N-1} \)-fibration. It follows that \( \mathcal{S} \) is smooth and
\[
\dim \mathcal{S} = (N + 2). \tag{1.13}
\]

**Definition 1.7.** Let \( \text{Vert}^q(\mathcal{S}/\Lambda) \subset \text{CH}^q(\mathcal{S}) \) be the subspace spanned by codimension-\( q \) integral closed subsets \( Z \subset \mathcal{S} \) such that the dimension of \( p_\Lambda(Z) \) is strictly smaller than the dimension of \( Z \).

The result below is an instance of the spreading principle.

**Claim 1.8.** Keep notation and assumptions as above, in particular **Hypothesis [1.4]** holds. Let \( Q \in \mathbb{Q}[x_1, \ldots, x_m, y_1, \ldots, y_n]_2 \) be a homogeneous polynomial of degree 2 and let \( \zeta_1, \ldots, \zeta_m \in \text{CH}^1(V) \). Then
\[
p_W^*Q(\pi^*\zeta_1, \ldots, \pi^*\zeta_m, E_1, \ldots, E_n) \in \text{Vert}^2(\mathcal{S}/\Lambda) \tag{1.14}
\]
if and only if for all smooth \( X \in |\mathcal{F}_C(H)| \) the relation
\[
Q(\zeta_1|x, \ldots, \zeta_m|x, c_1(\mathcal{O}_X(C_1)), \ldots, c_1(\mathcal{O}_X(C_n))) = 0 \tag{1.15}
\]
holds in \( \text{CH}_0(X) \).

**Proof.** Suppose that (1.14) holds. Then, by definition, there exists an open dense \( \mathcal{U} \subset \Lambda \) such that the restriction of \( p_W^*Q(\pi^*\zeta_1, \ldots, \pi^*\zeta_m, E_1, \ldots, E_n) \) to \( p_\Lambda^{-1}\mathcal{U} \) vanishes. By shrinking \( \mathcal{U} \) we may assume that for \( S \in \mathcal{U} \) the surface \( X := \pi(S) \) is smooth. Let \( S \in \mathcal{U} \): then \( 0 = p_W^*Q(\pi^*\zeta_1, \ldots, \pi^*\zeta_m, E_1, \ldots, E_n)|_S \), and since \( X \cong S \) it follows that (1.14) holds for \( X = \pi(S) \). On the other hand the locus of smooth \( X \in |\mathcal{F}_C(H)| \) such that (1.15) holds is a countable union of closed subsets of \( \Lambda_{nm} \) (the open dense subset of \( \Lambda \) parametrizing smooth surfaces); since it contains an open dense subset of \( \Lambda_{nm} \) it is equal to \( \Lambda_{nm} \). Now suppose that (1.15) holds for all smooth \( X \in |\mathcal{F}_C(H)| \). Let \( S \in \Lambda \) be generic, \( X := \pi(S) \). Then \( X \) is smooth and the restriction of \( \pi \) to \( S \) defines an isomorphism \( \varphi: S \xrightarrow{\sim} X \), thus by our assumption
\[
p_W^*Q(\pi^*\zeta_1, \ldots, \pi^*\zeta_m, E_1, \ldots, E_n)|_S = 0.
\]
Since \( S \) is generic in \( \Lambda \) it follows (see [3] [13]) that there exists an open dense subset \( \mathcal{U} \subset \Lambda \) such that
\[
p_W^*Q(\pi^*\zeta_1, \ldots, \pi^*\zeta_m, E_1, \ldots, E_n)|_{p_\Lambda^{-1}\mathcal{U}} = 0. \tag{1.16}
\]
(We recall that Chow groups are with rational coefficients, if we consider integer coefficients then (1.16) holds only up to torsion.) Let \( B := (\Lambda \setminus \mathcal{U}) \). By the localization exact sequence
\[
\text{CH}^2(p_\Lambda^{-1}B) \to \text{CH}^2(\mathcal{S}) \to \text{CH}^2(p_\Lambda^{-1}\mathcal{U}) \to 0
\]
we get (1.14) because the fibers of \( p_\Lambda \) have dimension 2. \( \square \)

1.5. **The Chow groups of \( \mathcal{S} \) and \( W \).** Let \( \xi \in \text{CH}^1(\mathcal{S}) \) be the pull-back of the hyperplane class on \( \Lambda \) via the map \( p_\Lambda \) of (1.12). Since \( p_W \) is the projectivization of a rank-\( N \) vector-bundle on \( W \) and \( \xi \) restricts to the hyperplane class on each fiber of \( p_W \) the Chow ring \( \text{CH}(\mathcal{S}) \) is the \( \mathbb{Q} \)-algebra generated by \( p_W^* \text{CH}(W) \) and \( \xi \), with ideal of relations generated by a single relation in codimension \( N \). We have \( N \geq 3 \) because \((\pi^*H - E)\) is very ample by **Hypothesis [1.4]** thus we have an isomorphism
\[
\begin{array}{ccc}
Q \oplus \text{CH}^1(W) \oplus \text{CH}^2(W) & \xrightarrow{\sim} & \text{CH}^2(\mathcal{S}) \\
(a_0, a_1, a_2) & \mapsto & a_0\xi^2 + p_W^*(a_1) \cdot \xi + p_W^*(a_2)
\end{array} \tag{1.17}
\]
The Chow groups \( \text{CH}_q(W) \) are computed by first describing \( \text{CH}_q(E_k) \) for \( k \in \{1, \ldots, n\} \), and then considering the localization exact sequence
\[
\bigoplus_k \text{CH}_q(E_k) \to \text{CH}_q(W) \to \text{CH}_q(W \setminus \{E_1 \cup \ldots \cup E_n\}) \to 0.
\]
One gets an isomorphism
\[
\begin{array}{ccc}
\text{CH}^1(V) \oplus \mathbb{Q}^n & \xrightarrow{\sim} & \text{CH}^1(W) \\
(a, t_1, \ldots, t_n) & \mapsto & \pi^*a + \sum_{k=1}^n t_k E_k
\end{array} \tag{1.18}
\]
and an exact sequence
\[
0 \to \text{CH}^2(W)_{\text{hom}} \to \text{CH}^2(W) \to H^4(W; \mathbb{Q}) \to 0 \tag{1.19}
\]
where \( \text{CH}^2(W)_{\text{hom}} \) is described as follows. Let \( \rho_k : E_k \to C_k \) be the restriction of the blow-up map \( \pi \), and \( \sigma_k : E_k \hookrightarrow W \) be the inclusion map; then we have an Abel-Jacobi isomorphism
\[
AJ: \text{CH}^2(W)_{\text{hom}} \xrightarrow{\sim} \bigoplus_{k=1}^n \text{CH}_q(C_k)_{\text{hom}} \\
\alpha & \mapsto (\rho_1, \sigma_1^* \alpha, \ldots, \rho_n, \sigma_n^* \alpha) \tag{1.20}
\]
Let $AJ_k$ be the $k$-th component of the map $AJ$.

**Lemma 1.9.** Assume that Hypothesis [1.4] holds. Let

$$\omega := \pi^* \alpha + \sum_{k=1}^{n} E_k \cdot \pi^* \beta_k + \sum_{k=1}^{n} \gamma_k E_k \cdot E_k,$$

where $\alpha \in CH^2(V)$, $\beta_k \in CH^1(V)$, and $\gamma_k \in \mathbb{Q}$ for $k \in \{1, \ldots, n\}$. Then the following hold:

1. The cohomology class of $\omega$ vanishes if and only if
   $$\alpha = \sum_{k=1}^{n} \gamma_k C_k,$$
   and for all $k \in \{1, \ldots, n\}$
   $$\deg(\beta_k \cdot C_k) = -\gamma_k \deg(\mathcal{N}_{C_k/V}).$$
2. Suppose that (1.21) and (1.22) hold. Then for $k \in \{1, \ldots, n\}$
   $$AJ_k(\omega) = -\gamma_k c_1(\mathcal{N}_{C_k/V}) - c_1(\beta_k |_{C_k}).$$

**Proof.** Since the cohomology class map $cl: CH^1(V) \to H^2(V; \mathbb{Q})$ is a surjection (by hypothesis), also the cohomology class map $cl: CH^1(W) \to H^2(W; \mathbb{Q})$ is surjective. Thus, by Poincaré duality, $cl(\omega) = 0$ if and only if $deg(\omega \cdot \xi) = 0$ for all $\xi \in CH^1(W)$. By (1.18) we must test $\xi = \pi^* \zeta$ with $\zeta \in CH^1(V)$ and $\xi = E_j$ for $k \in \{1, \ldots, n\}$. We have

$$\deg(\omega \cdot \pi^* \zeta) = \deg \left( \left( \alpha - \sum_{k=1}^{n} \gamma_k C_k \right) \cdot \zeta \right).$$

Thus $\deg(\omega \cdot \pi^* \zeta) = 0$ for all $\zeta \in CH^1(V)$ if and only if (1.21) holds. Next, in $CH_0(C_j)$ we have the relation

$$\rho_j \cdot c_1(\mathcal{N}_{C_j/E_j}) = -c_1(\mathcal{N}_{C_j/V}),$$

and hence

$$\deg(\omega \cdot E_j) = -\deg(\beta_j \cdot C_j) - \gamma_j \deg(\mathcal{N}_{C_j/V}).$$

This proves Item (1). Item (2) follows from Equation (1.25). \qed

**Remark 1.10.** By Lemma 1.9 the kernel of the map

$$CH^2(V) \oplus \bigoplus_{k=1}^{n} CH^1(V) \oplus \bigoplus_{k=1}^{n} \mathbb{Q} \to CH^2(W) \quad \to \quad \pi^* \alpha + \sum_{k=1}^{n} E_k \cdot \pi^* \beta_k + \sum_{k=1}^{n} \gamma_k E_k \cdot E_k$$

is generated over $\mathbb{Q}$ by the classes $E_k \cdot \pi^* \beta$, where $\beta \in CH^1(V)$ and $\beta |_{C_k} = 0$, together with the classes

$$\pi^* [C_k] + E_k \cdot \pi^* \beta + E_k \cdot E_k, \quad (1.28)$$

where $\beta \in CH^1(V)$, $deg(\beta \cdot C_k) = -deg(\mathcal{N}_{C_k/V})$, and $0 = -c_1(\mathcal{N}_{C_k/V}) - c_1(\beta_{C_k})$. Next notice that $0 = -c_1(\mathcal{N}_{C_k/V}) - c_1(\beta_{C_k})$ if and only if $c_1(K_{C_k})$ is equal to the restriction of a class in $CH^1(V)$ i.e. (1.10) holds. Assume that this is the case, and that $X \in |\mathcal{I}_C(H)|$ is a smooth surface at all points of $C_k$. Let $S \in |\pi^* H - E|$ be the strict transform of $S$. Then $S$ is isomorphic to $X$ over $C_k$, and restricting to $S$ the equation $\pi^* [C_k] + E_k \cdot \pi^* \beta + E_k \cdot E_k = 0$ we get the second standard relation (1.8), see Subsection 1.2

We record for later use the following formulae:

$$\sigma_k \cdot \rho^*_k c_1(\mathcal{N}_{C_k/V}) = \pi^* C_k + E_k \cdot E_k,$$

$$p_{W,*}(\pi^* H - E).$$

The first formula follows from the “Key formula” for $\pi^* C_k$, see Prop. 6.7 of [7]. The second formula is immediate (recall that $N = \dim \Lambda$).
1.6. A vertical cycle on $\mathcal{F}$. Let $k \in \{1, \ldots, n\}$. By Hypothesis 1.4 there exists an open dense $U \subset \Sigma_k$ such that, if $S \in U$, then $S \cdot E_k = L_x + Z$, where $x \in C_k$ is the unique singular point of $\pi(S)$, $\Lambda := \pi^{-1}(x)$, and $Z$ is the residual divisor (which does not contain $L_x$). It follows that

$$E_k \cap p_\Lambda^{-1}(U) = \mathcal{F}_k + \mathcal{Z}_k,$$

where $\mathcal{F}_k$, $\mathcal{Z}_k$ restricted to $E_k \cap S$, are equal to $L_x$ and $Z$, respectively. We let

$$\Theta_k := \mathcal{F}_k.$$

Thus $p_\Lambda(\Theta_k) = \Sigma_k$, and the generic fiber of $\Theta_k \to \Sigma_k$ is a projective line. By Hypothesis 1.4 $\Theta_k$ is of pure codimension 2 in $\mathcal{F}$ (or empty). On the other hand $p_\Lambda(\Theta_k) = \Sigma_k$ is of pure codimension 1 in $\Lambda$, hence

$$\Theta_k \in \text{Vert}^2(\mathcal{F}/\Lambda).$$

The result below will be instrumental in writing out the class of $\Theta_k$ in $\text{CH}^2(\mathcal{F})$ according to Decomposition (1.17).

**Proposition 1.11.** Let $k \in \{1, \ldots, n\}$. Then

$$pw_*(\Theta_k \cdot \xi^{N-1}) = 2E_k \cdot \pi^*H - E_k \cdot E_k - \pi^*C_k.$$  \hfill (1.34)

**Proof.** Let $\alpha, \beta \in H^0(W, \pi^*(H) - E)$ be generic. Then $\text{div}(\alpha|_{E_k})$ and $\text{div}(\beta_{E_k})$ are smooth divisors intersecting transversely at points $p_1, \ldots, p_s$. Let $q_i := \pi(p_i)$ for $i \in \{1, \ldots, s\}$. Let $R = \mathbb{P}((\alpha, \beta)) \subset \Lambda$; thus $p_\Lambda^{-1}R$ represents $\xi^{N-1}$. Given $p_i$, there exists $[\lambda, \mu_i] \in \mathbb{P}^1$ such that $\text{div}(\lambda \alpha + \mu_i \beta)$ contains $\pi^{-1}(q_i)$, and hence $[\lambda, \alpha + \mu_i \beta] \in R \cap \Sigma_k$. Conversely, every point of $R \cap \Sigma_k$ is of this type. Thus in order to compute $pw_*(\Theta_k \cdot \xi^{N-1})$ we must determine the class of the 0-cycle $q_1 + \ldots + q_s$. Let $\phi: C_k \times R \to C_k$ and $\psi: C_k \times R \to R$ be the projections and $\mathcal{F}$ the rank-2 vector-bundle on $C_k \times R$ defined by

$$\mathcal{F} := \phi^*(\mathcal{N}_{C_k/V} \otimes \mathcal{O}_{C_k}(H)) \otimes \psi^*\mathcal{O}_R(1).$$

The composition of the natural maps

$$(\alpha, \beta) \mapsto H^0(W, \pi^*H - E) \to H^0(E_k, E_k(\pi^*H - E)) \to H^0(C_k, \mathcal{N}_{C_k/V} \otimes \mathcal{O}_{C_k}(H))$$

defines a section $\tau \in H^0(\mathcal{F})$ whose zero-locus consists of points $p'_1, \ldots, p'_{s_1}$ such that $\pi(p'_i) = q_i$. By the discussion above

$$pw_*(\Theta_k \cdot \xi^{N-1}) = \sigma_{s_{\phi}}(\rho_{s_{\phi}}^*(\phi_*\mathcal{F}(\tau))).$$

The formula

$$c_1(\mathcal{F}) = \phi^*(2c_1(\mathcal{O}_{C_k}(H)) - c_1(\mathcal{N}_{C_k/V})) \cdot \psi^*c_1(\mathcal{O}_R(1)).$$

gives that

$$pw_*(\Theta_k \cdot \xi^{N-1}) = 2E_k \cdot \pi^*H - \sigma_{s_{\phi}}\left(\rho_{s_{\phi}}^*(\phi_*\mathcal{F}(\tau))\right).$$

Then (1.34) follows from the above equality together with (1.29). \hfill $\Box$

**Corollary 1.12.** Let $k \in \{1, \ldots, n\}$. Then

$$\Theta_k = \xi \cdot p_W^*E_k + p_W^*(E_k \cdot \pi^*H - \pi^*C_k).$$

**Proof.** By (1.17) there exist $\beta_0 \in \text{CH}^0(W)$ for $h = 0, 1, 2$ such that

$$\Theta_k = \xi^2 \cdot p_W^*\beta_0 + \xi \cdot p_W^*\beta_1 + p_W^*\beta_2.$$\hfill (1.37)

Restricting $p_W$ to $\Theta_k$ we get a $\mathbb{P}^{N-2}$-fibration $\Theta_k \to E_k$: it follows that $\beta_0 = 0$ and $\beta_1 = E_k$. By (1.30)

$$pw_*(\Theta_k \cdot \xi^{N-1}) = pw_*(\xi \cdot p_W^*E_k + \xi \cdot \xi^{N-1} \cdot p_W^*\beta_2) = (E_k \cdot \pi^*H - E_k \cdot E_k + \beta_2).$$

On the other hand $pw_*(\Theta_k \cdot \xi^{N-1})$ is equal to the right-hand side of (1.34) equating that expression and the right-hand side of (1.33) we get $\beta_2 = (E_k \cdot \pi^*H - \pi^*C_k)$. \hfill $\Box$

**Corollary 1.13.** Let $k \in \{1, \ldots, n\}$. Then $p_W^*(E_k \cdot \pi^*H - \pi^*C_k) \in \text{Vert}^2(\mathcal{F}/\Lambda)$.\hfill (1.38)

**Proof.** By Corollary 1.12 we have

$$p_W^*(E_k \cdot \pi^*H - \pi^*C_k) = (\Theta_k - \xi \cdot p_W^*E_k).$$

Now $\Theta_k \in \text{Vert}^2(\mathcal{F}/\Lambda)$ (see (1.33)) and $\xi \cdot p_W^*E_k \in \text{Vert}^2(\mathcal{F}/\Lambda)$ because it is supported on the inverse image of a hyperplane via $p_\Lambda$; thus $p_W^*(E_k \cdot \pi^*H - \pi^*C_k) \in \text{Vert}^2(\mathcal{F}/\Lambda)$. \hfill $\Box$
Remark 1.14. By Claim 1.8 the relation \( p_W^*(E_k \cdot \pi^*H - \pi^*C_k) \in \text{Vert}^2(\mathcal{S}/\Lambda) \) gives a relation in \( CH(X) \) for an arbitrary smooth \( X \in |\mathcal{S}_C(H)| \). In fact it gives the first standard relation (1.5), see Subsection 1.2.

1.7. Proof of the main result of the section.

Lemma 1.15. Assume that Hypothesis 1.5 holds. Then the projection \( CH^2(\mathcal{S}) \to CH^2(W) \) determined by \( p_W^* \) maps \( \text{Vert}^2(\mathcal{S}/\Lambda) \) to the subspace spanned by

\[
(E_1 \cdot \pi^*H - \pi^*C_1), \ldots, (E_k \cdot \pi^*H - \pi^*C_k), \ldots, (E_n \cdot \pi^*H - \pi^*C_n).
\]

Proof. Let \( Z \subset \mathcal{S} \) be an irreducible closed codimension-2 subset of \( \mathcal{S} \) such that the dimension of \( p_\Lambda(Z) \) is strictly smaller than the dimension of \( Z \). Since the fibers of \( p_\Lambda \) are 2-dimensional the generic fiber of \( Z \to p_\Lambda(Z) \) has dimension 1 or 2. Suppose that the latter holds. We claim that

\[
Z = p_\Lambda^{-1}(p_\Lambda(Z)).
\] (1.39)

In fact \( \text{cod}(p_\Lambda(Z), \Lambda) = 2 \), hence by Hypothesis 1.4 there exists an open dense \( U \subset p_\Lambda(Z) \) such that any \( S \in U \) is irreducible. Let \( S \in U \); since \( Z \cap S \) is closed of dimension 2, it follows that \( Z \cap S = S \). Thus \( Z \cap p_\Lambda^{-1}(U) = p_\Lambda^{-1}(U) \). On the other hand every irreducible component of \( p_\Lambda^{-1}(p_\Lambda(Z)) \) has dimension at most 2 in \( \mathcal{S} \) (because \( \Lambda \) is smooth), and every fiber of \( p_\Lambda \) has dimension 2. It follows that \( p_\Lambda^{-1}(p_\Lambda(Z)) \) is irreducible, hence equal to the closure of \( p_\Lambda^{-1}(U) \), and this proves (1.39). Since \( \Lambda \) is a projective space the class of \( p_\Lambda(Z) \) is a multiple of \( c_1(\Theta_1(1))^2 \). It follows that the class of \( Z \) is a multiple of \( \xi^2 \) and hence the projection \( CH^2(\mathcal{S}) \to CH^2(W) \) maps it to 0. Now assume that the generic fiber of \( Z \to p_\Lambda(Z) \) has dimension 1. We distinguish between the two cases:

1. \( \lambda_\Lambda(Z) \notin \{\Sigma_1, \ldots, \Sigma_n\} \).
2. There exists \( k \in \{1, \ldots, n\} \) such that \( \lambda_\Lambda(Z) = \Sigma_k \).

Suppose that Item (1) holds. By Hypothesis 1.5 there exist an open dense subset \( U \subset p_\Lambda(Z) \), and a class \( \Gamma \in CH^1(W) \), such that, for \( S \in U \), the intersection \( S \cap Z \) has dimension 1, and the class in \( CH^1(S) \) represented by \( S \cap Z \) is equal to \( \Gamma \). It follows that there exists an open dense \( U_0 \subset U \) such that on \( p_\Lambda^{-1}(U_0) \) the cycle \( Z \) is rationally equivalent to the restriction of \( p_W^* \Gamma \). There exists \( r \in \mathbb{N}^+ \) such that \( p_\Lambda(Z) = r \cdot c_1(\Theta_1(1)) \). This proves that there exists \( x \in \mathbb{Q} \) such that

\[
Z = p_\Lambda^{-1}(p_\Lambda(Z)).
\] (1.40)

It follows that the projection \( CH^2(\mathcal{S}) \to CH^2(W) \) maps \( Z \) to 0. Lastly suppose that Item (2) holds. By Hypothesis 1.5 there exist an open dense subset \( U \subset p_\Lambda(Z) \), and a class \( \Gamma \in CH^1(W) \) and \( a \in \mathbb{Z} \), such that, for \( S \in U \), the intersection \( S \cap Z \) has dimension 1, and represents \( \Gamma \cap S + a(\Theta_k \cdot S) \). It follows that there exists an open dense \( U_0 \subset U \) such that

\[
Z \cdot p_\Lambda^{-1}U_0 = (p_W^* \Gamma + a(\Theta_k)) \cdot p_\Lambda^{-1}U_0.
\] (1.40)

By Corollary 1.12 it follows that the projection \( CH^2(\mathcal{S}) \to CH^2(W) \) maps \( Z \) to \( (E_k \cdot \pi^*H - \pi^*C_k) \).

Proof of Proposition 1.6. Let \( P \in \mathbb{Q}[x_1, \ldots, x_m] \) be homogeneous of degree 2 and \( r_1, \ldots, r_n \in \mathbb{Q} \). The set of smooth \( X \in |\mathcal{S}_C(H)| \) such that

\[
0 = P(\zeta_1 | X, \ldots, \zeta_m | X) + r_1 C_1^2 + \ldots + r_n C_n^2
\] (1.41)

is a countable union of closed subsets of the open dense subset of \( |\mathcal{S}_C(H)| \) parametrizing smooth surfaces. It follows that if the proposition is false then there exist \( P \) and \( r_1, \ldots, r_n \), not all zero, such that (1.41) holds for all smooth \( X \in |\mathcal{S}_C(H)| \). Now we argue by contradiction. By Claim 1.8

\[
p_W^*(P(\pi^*C_1, \ldots, \pi^*C_m) + \sum_{i=1}^n r_i E_i) \in \text{Vert}^2(\mathcal{S}/\Lambda).
\] (1.42)

By Lemma 1.15 it follows that there exist rationals \( s_1, \ldots, s_n \) such that

\[
P(\pi^*C_1, \ldots, \pi^*C_m) + \sum_{i=1}^n r_i E_i = \sum_{i=1}^n s_i (E_i \cdot \pi^*H - \pi^*C_i),
\]
i.e.,

\[
0 = \pi^*(P(\zeta_1, \ldots, \zeta_m) + \sum_{i=1}^n s_i C_i) - \sum_{i=1}^n s_i E_i \cdot \pi^*H + \sum_{i=1}^n r_i E_i^2. \] (1.43)
Let $\omega$ be the right hand side of (1.33); then the homology class of $\omega$ vanishes, and also the Abel-Jacobi image $AJ(\omega)$, notation as in (1.20). Item (2) of Lemma 1.9, together with our hypothesis that there does not exist $\xi \in \text{CH}^1(V)$ such that $c_1(K_{C_k}) = \xi|_{C_k}$, gives $r_k = 0$ for $k \in \{1, \ldots, n\}$. Then it follows easily that $s_k = 0$ for $k \in \{1, \ldots, n\}$, and hence $P(\zeta_1, \ldots, \zeta_m) = 0$ by Item (1) of Lemma 1.9. \hfill $\square$

2. NOETHER-LEFSCHETZ LOCI FOR LINEAR SYSTEMS OF SURFACES IN $\mathbb{P}^3$ WITH BASE-LOCUS

2.1. The main result. In the present section we let $V = \mathbb{P}^3$. Thus $C_1, \ldots, C_n \subset \mathbb{P}^3$, and $\pi: W \to \mathbb{P}^3$. We let $\Lambda(d) := |\pi^*\mathcal{O}_\mathbb{P}^3(-d)|$. For $k \in \{1, \ldots, n\}$ let $\Sigma_k(d) \subset \Lambda(d)$ be the subset $\Sigma_k$ considered in Section 1.5 thus $\Sigma_k(d)$ parametrizes surfaces $S$ such that $\pi(S)$ is singular at some point of $C_k$. Let $\Sigma(d) := \Sigma_1(d) \cup \ldots \cup \Sigma_n(d)$. We denote the tangent sheaf of a smooth variety $X$ by $T_X$. Below is the main result of the present section.

Theorem 2.1. Suppose that $d \geq 5$, and that the following hold:

1. $\pi^*\mathcal{O}_\mathbb{P}^3(d-3)(-E)$ is very ample.
2. $H^1(C, T_C(d-4)) = 0$.
3. The sheaf $\mathcal{I}_C$ (on $\mathbb{P}^3$) is $(d-2)$-regular.
4. The curves $C_1, \ldots, C_n$ are not planar.

Then Hypothesis 1.5 holds for $H \in |\mathcal{O}_\mathbb{P}^3(d)|$.

Recall that Hypothesis 1.3 states that Hypothesis 1.4 holds, and that Items (1) and (2) (our Noether-Lefschetz hypotheses) of Hypothesis 1.5 hold. The proof that Hypothesis 1.4 holds is elementary, and will be given in Subsection 2.2. We will prove that Items (1) and (2) of Hypothesis 1.5 hold by applying Joshi’s main criterion (Prop. 3.1 of [9]), and also the idea of Griffiths-Harris [8], as further developed by Lopez [12] and Brevik-Nollet [5]. The proof will be outlined in Subsection 2.3 details are given in the remaining subsections.

Remark 2.2. Choose disjoint integral smooth curves $C_1, \ldots, C_n \subset \mathbb{P}^3$ such that for each $k \in \{1, \ldots, n\}$ there does not exist $r \in \mathbb{Q}$ such that $c_1(K_{C_k}) = r c_1(\mathcal{O}_{C_k}(1))$. Let $d \gg 0$: the hypotheses of Theorem 2.1 are satisfied, and hence a very general $X \in |\mathcal{I}_C(d)|$ has group of decomposable 0-cycles of rank at least $n + 1$ by Proposition 1.6. In Section 3 we will prove Theorem 0.1 by giving an explicit example.

2.2. Dimension counts. We will prove that, if the hypotheses of Theorem 2.1 are satisfied, then Hypothesis 1.4 holds for $H \in |\mathcal{O}_\mathbb{P}^3(d)|$. Let $\Delta(r) \subset \Lambda(r)$ be the discriminant hypersurface parametrizing singular surfaces.

Proposition 2.3. Suppose that $\pi^*\mathcal{O}_\mathbb{P}^3(r-1)(-E)$ is very ample. Then the following hold:

1. Let $x \in C$. The linear system $|\mathcal{I}_C^2(r)| \cap |\mathcal{I}_C(r)|$ has base locus equal to $C$, and codimension 2 in $|\mathcal{I}_C(r)|$. If $X$ is generic in $|\mathcal{I}_C^2(r)| \cap |\mathcal{I}_C(r)|$ then it has an ODP at $x$ and no other singularity.
2. Given $x \in W \setminus E$ there exists $S \in \Delta(r)$ which has an ODP at $x$ and is smooth away from $x$. Moreover $\Delta(r)$ is irreducible of codimension 1.

Proof. Let $q \in \mathbb{P}^3 \setminus C$. Since $\pi^*\mathcal{O}_\mathbb{P}^3(r-1)(-E)$ is very ample there exists $X \in |\mathcal{I}_C(r-1)|$ such that $q \notin X$. Let $P \subset \mathbb{P}^3$ be a plane containing $x$ but not $q$: then $X + P \in |\mathcal{I}_C^2(r)| \cap |\mathcal{I}_C(r)|$ does not pass through $q$, and this proves that $|\mathcal{I}_C^2(r)| \cap |\mathcal{I}_C(r)|$ has base locus equal to $C$. Since $\pi^*\mathcal{O}_\mathbb{P}^3(r-1)(-E)$ is very ample there exist $F, G \in H^0(\mathbb{P}^3, \mathcal{I}_C(r-1))$ and $q_1, \ldots, q_m \in (C \setminus \{x\})$ such that $V(F), V(G)$ are smooth and transverse at each point of $C \setminus \{q_1, \ldots, q_m\}$. Let $P \subset \mathbb{P}^3$ be a plane not passing through $x$: the pencil in $|\mathcal{I}_C(r)|$ spanned by $V(F) + P$ and $V(G) + P$ does not intersect $|\mathcal{I}_C^2(r)| \cap |\mathcal{I}_C(r)|$, and hence $|\mathcal{I}_C^2(r)| \cap |\mathcal{I}_C(r)|$ has codimension at least 2 in $|\mathcal{I}_C(r)|$. The codimension is equal to 2 because imposing on $X \in |\mathcal{I}_C(r)|$ to be singular at $x \in C_j$ we get 2 linear conditions. In order to show that the singularities of a generic element of $|\mathcal{I}_C^2(r)| \cap |\mathcal{I}_C(r)|$ are as claimed we consider the embedding

$$
\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{I}_C(1)) \oplus H^0(\mathbb{P}^3, \mathcal{I}_x(1))) \rightarrow \Sigma(r) \\
[A, B] \mapsto V(AF + BG)
$$

(2.1)

The image is a sublinear system of $|\mathcal{I}_C^2(r)| \cap |\mathcal{I}_C(r)|$ whose base locus is $C$, hence the generic $V(A \cdot F + B \cdot G)$ is smooth away from $C$ by Bertini’s Theorem. A local computation shows that the projectivized tangent cone of $V(AF + BG)$ at $x$ is a smooth conic for generic $A, B$. Lastly let $q \in C \setminus \{x\}$. The set
Proposition 2.6.

Remark Suppose that \( \pi^* \Theta_p(r - 1)(-E) \) is very ample. Then the following hold:

1. \( \Delta(r) \) is irreducible of codimension 1 in \( \Lambda(r) \), and the generic \( S \in \Delta(r) \) has a unique singular point, which is an ODP.
2. Let \( k \in \{1, \ldots, n\} \). If \( S \) is a generic element of \( \Sigma_k(r) \), then \( \pi(S) \) has a unique singular point \( x \), which is an ODP.

Corollary 2.4 proves that, if the hypotheses of Theorem 2.1 are satisfied, then Item (1) of Hypothesis [1,4] holds with \( H \in [[\Theta_p(d)]]. \)

Remark 2.5. Let \( x \in C \). The proof of Proposition 2.3 shows that the projectivized tangent cone at \( x \) of the generic \( X \in |{\mathcal{I}}^2(r)| \cap |{\mathcal{I}}_3(r)| \) is the generic conic in \( \mathbb{P}(T_x \mathbb{P}^3) \) containing the point \( P(T_x C) \).

Proposition 2.6. Suppose that \( \pi^* \Theta_p(r)(E) \) is very ample and that \( \pi^* \Theta_p(r - 3)(E) \) is base point free. Then the locus of non-integral surfaces \( S \in |\Lambda(r)| \) has codimension at least 3.

Proof. Let \( k \in \{1, \ldots, n\} \). Since \( \pi^* \Theta_p(r)(-E) \) is very ample, and \( E_k \) is a \( \mathbb{P}^1 \)-bundle, the image of the restriction map \( H^0(W, \pi^* \Theta_p(r)(-E)) \rightarrow H^0(E_k, \pi^* \Theta_p(r)(-E)|_{E_k} \) has dimension at least 4, and hence the locus of \( S \in |\pi^* \Theta_p(r)(-E)| \) which contain \( E_k \) has codimension at least 4. Let \( S \in |\Lambda(r)| \) be a generic non-integral surface, and \( X := \pi(S) \): it follows from the above discussion that \( X \) contains no \( E_k \), and hence \( X \in (\mathbb{P}^3 \setminus C) \) has dimension at least 1. In particular there exists a couple of distinct \( p, q \in (X \setminus C) \) such that \( X \) is singular at \( p \) and \( q \), with quadratic term which is degenerate (in fact the set of such couples is infinite). Thus, by a parameter count it suffices to prove that the subset \( \Omega_{p,q} \subset |{\mathcal{I}}_C(r)| \) of \( X \) singular at \( p, q \), with degenerate quadratic term has codimension 10 (as expected).

Let \( Y \in |{\mathcal{I}}_C(r - 3)| \) be a surface not containing \( p \) nor \( q \) (it exists because \( \pi^* \Theta_p(r - 3)(-E) \) is base point free), and consider the subset

\[
\Omega_Y := \{ Y + Z \mid Z \subset |{\mathcal{I}}_3(3)| \}.
\]

An explicit computation shows that the codimension of the set of \( Z \subset |{\mathcal{I}}_3(3)| \) singular at \( p, q \), with degenerate quadratic term has codimension 10: it follows that \( \Omega_{p,q} \cap \Omega_Y \) has codimension 10, and hence \( \Omega_{p,q} \) has codimension 10 in \( |{\mathcal{I}}_C(r)| \).

Proposition 2.6 proves that, if the hypotheses of Theorem 2.1 are satisfied, then Item (2) of Hypothesis [1,4] holds for \( H \in ([\Theta_p(d)] \). This proves that, if the hypotheses of Theorem 2.1 are satisfied, then Hypothesis [1,4] holds for \( H \in ([\Theta_p(d)] \).

2.3. Outline of the proof that the Noether-Lefschetz hypothesis holds. We must control \( \text{CH}^1(S) \) for \( S \) a general element of an integral closed codimension-1 subset \( A \) of \( \Lambda(d) \). Let \( A^\dual \subset \Lambda(d)^\dual \) the projective dual, i.e. the closure of the locus of projective tangent hyperplanes \( T_A \) for \( A \) in the smooth locus \( A^{sm} \) of \( A \). Since \( \pi^* \Theta_p(d)(-E) \) is very ample we have the natural embedding \( W \hookrightarrow \Lambda(d)^\dual \), thus it makes sense to distinguish between the following two cases:

1. \( A^\dual \) is not contained in \( W \).
2. \( A^\dual \) is contained in \( W \).

Thus (I) holds if and only if, for the generic \( S \in A^{sm} \), the projective tangent hyperplane \( T_A \) is a base point free linear subsystem of \( \Lambda(d) \). Examples of codimension-1 subsets of \( \Lambda(d) \) for which (II) holds are given by \( \Delta(d) \) and \( \Sigma_k(d) \) for \( k \in \{1, \ldots, n\} \). In fact \( \Delta(d)^\dual = W \) and \( \Sigma_k(d)^\dual = E_k \). (The second equality follows from the tautological equality \( \Sigma_k(d) = E_k^\vee \) and projective duality.)

Let \( \text{NL}(\Delta(d)) \) be the Noether-Lefschetz locus, i.e. the set of those smooth surfaces \( S \in \Lambda(d) \) such that the restriction map \( \text{Pic}(W) \rightarrow \text{Pic}(S) \) is not surjective. As is well-known \( \text{NL}(\Delta(d)) \) is a countable union of closed subsets \( \Lambda(d) \) in \( \Delta(d) \). In Subsection 2.5 we will apply Joshi’s criterion (Proposition 3.1 of [9]) in order to prove the following result.

Proposition 2.7. Suppose that \( d \geq 5 \) and that the following hold:

1. \( \pi^* \Theta_p(d)(-E) \) is ample.
2. \( H^1(C, T_C(d - 4)) = 0 \).
The above result deals with codimension-1 subsets $A \subset \Lambda(d)$ for which (I) above holds. Thus, in order to finish the proof of \textbf{Theorem 2.1}, it remains to deal with those $A$ such that (II) above holds.

\textbf{Definition 2.8.} Given $p \in W$ and $F \subset T_pW$ a vector subspace, we let
\[ \Lambda_{p,F}(d) := \{ S \in |\mathcal{F}_C| \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(-E) | \ F \subset T_pS \}. \] (2.2)

Let $\Gamma(d) := |\mathcal{F}_C|$. We have a tautological identification $\Lambda(d) \cong \Gamma(d)$: we let $\Gamma_{p,F}(d) \subset \Gamma(d)$ be the image of $\Lambda_{p,F}(d)$, and for $k \in \{1, \ldots, n\}$ we let $\Pi_k(d) \subset \Gamma(d)$ be the image of $\Lambda_k(d)$.

Notice that $\Lambda_{p,F}(d)$ and $\Gamma_{p,F}(d)$ are sub linear systems of $\Lambda(d)$ and $\Gamma(d)$ respectively. In \textbf{Subsection 2.6} we will prove the result below by applying an idea of Griffiths-Harris \S{} as further developed by Lopez \cite{6} and Brevik-Nollet \cite{5}.

\textbf{Proposition 2.9.} Suppose that the following hold:
\begin{enumerate}[1.)]
\item $d \geq 4$ and $\pi^* \mathcal{O}_\mathbb{P}^3(d-3)(-E)$ is very ample.
\item The curves $C_1, \ldots, C_n$ are not planar.
\end{enumerate}

Let $p \in W$, $F \subset T_pW$ a subspace, and assume that if $p \in E$ then $F \subset T_pE$ and $T_p(-\pi^{-1}(x)) \not\subset F$. Let $k \in \{1, \ldots, n\}$. Then, if $X$ is a very general element of $\Gamma_{p,F}(d)$ or of $\Pi_k(d)$, the Chow group $\text{CH}^1(X)_\mathbb{Q}$ is generated by $c_1(\mathcal{O}_X(1))$ and the classes of $C_1, \ldots, C_n$.

Granting \textbf{Proposition 2.9}, let us prove that the statement of \textbf{Theorem 2.1} holds for $A$ such that $A^\vee$ is contained in $W$. Let us distinguish the following two sub-cases of (II):
\begin{enumerate}[\text{IIa}.)]
\item $A \not\subset \{\Sigma_1(d), \ldots, \Sigma_n(d)\}$.
\item $A \in \{\Sigma_1(d), \ldots, \Sigma_n(d)\}$.
\end{enumerate}

Suppose that (IIa) holds. By projective duality $A$ is the closure of
\[ \bigcup_{p \in (A^\vee)^{\text{sm}}} \Lambda_{p,T_pA^\vee} \] (2.3)

Let $p \in (A^\vee)^{\text{sm}}$ be generic. We claim that the hypotheses of \textbf{Proposition 2.9} hold for $p$ and $F = T_pA^\vee$. In fact if $A^\vee \not\subset E$ then $p \not\in E$ by genericity. If $A^\vee \subset E$ then $T_pA^\vee \subset T_pE$, it remains to check that $T_p(-\pi^{-1}(x)) \not\subset T_pA^\vee$. In fact suppose that $T_p(-\pi^{-1}(x)) \subset T_pA^\vee$ for generic $p \in A^\vee \subset E$ (and a priori variable $x$). Then there exists $x_0 \in C$ such that $A^\vee = -\pi^{-1}(x_0)$, and since $A$ is the closure of (2.2) we get that the generic $S$ is $A$ is tangent to $-\pi^{-1}(x_0)$. Since $S \cdot E$ is a section of $E \to C$ it follows that $S$ contains $-\pi^{-1}(x_0)$ and therefore is tangent to $E$, i.e., $A \in \Sigma(d)$, contradicting the hypothesis.

Thus the hypotheses of \textbf{Proposition 2.9} hold for $p \in (A^\vee)^{\text{sm}}$ generic and $F = T_pA^\vee$, and hence if $S \in \Lambda_{p,T_pA^\vee}(d)$ is very general, then $\text{CH}^1(X)_\mathbb{Q}$ is generated by $c_1(\mathcal{O}_X(1))$ and the classes of $C_1, \ldots, C_n$.

On the other hand, since $A \not\subset \Sigma(d)$, the linear system $\Lambda_{p,T_pA^\vee}(d)$ is not contained in $\Sigma(d)$ i.e. $S$ intersects transversely $E$, and hence the restriction of $\pi$ to $S$ is an isomorphism $S \cong X$. It follows that $\text{CH}^1(S)_\mathbb{Q}$ is equal to the image of $\text{CH}^1(W)_\mathbb{Q} \to \text{CH}^1(S)_\mathbb{Q}$. This proves that there exists $S \in A$ such that $\text{CH}^1(S)_\mathbb{Q}$ is equal to the image of $\text{CH}^1(W)_\mathbb{Q} \to \text{CH}^1(S)_\mathbb{Q}$. Actually our argument proves that there exists such an $S$ which is smooth if $A \neq \Delta(d)$, and that if $A = \Delta(d)$ there exists such an $S$ whose singular set consists of a single ODP. If the former holds, then we are done because $\text{NL}(A \setminus \Delta(d))$ is a countable union of closed subsets of $A \setminus \Delta(d)$, and we have shown that the complement is non-empty. If the latter holds, let $\Delta_0(d) \subset \Delta(d)$ be the open dense subset parametrizing surfaces with an ODP and no other singular point, then the set of $S \in \Delta_0(d)$ such that $\text{CH}^1(W) \to \text{CH}^1(S)_\mathbb{Q}$ is not surjective is a countable union of closed subsets of $\Delta_0(d)$ (take a simultaneous resolution), and we are done because we have shown that the complement is non empty.

Lastly suppose that (IIb) holds, say $A = \Sigma_k(d)$. Let $S \in \Sigma_k(d)$ be generic, and let $X := \pi(S)$. By \textbf{Corollary 2.4} the restriction of $\pi$ to $S$ is the blow-up of the unique singular point of $X$, which is an ordinary node, call it $x$. Now suppose that $S$ is very general, then by \textbf{Proposition 2.9} the Chow group $\text{CH}^1(S)_\mathbb{Q}$ is generated by the image of $\text{CH}^1(W)_\mathbb{Q} \to \text{CH}^1(S)_\mathbb{Q}$ and the class of $-\pi^{-1}(x)$. Of course $S$ is smooth. Now notice that the set of $S \in \Sigma_k(d) \setminus \Delta(d)$ such that $\text{CH}^1(S)$ is not generated by the
image of $\text{CH}^1(W) \cap$ together with $\pi^{-1}(x)$ is a countable union of closed subsets of $\Sigma_k(d) \setminus \Delta(d)$; we have proved that the complement is non empty, and hence we are done. □

Summing up: we have shown that in order to prove Theorem 2.1 it suffices to prove Proposition 2.7 and Proposition 2.9. The proofs are in the following subsections.

2.4. Infinitesimal Noether-Lefschetz results. Let $U \subset H^0(W, \pi^* T_{\Theta^3}(d)(-E))$ be a subspace and $\sigma \in U$ be non-zero, we let $S := V(\sigma)$, and we assume $S$ is smooth. Let $m_{\sigma,U} \subset \mathcal{O}_{\mathcal{U},[\sigma]}$ be the maximal ideal and $T_{\sigma,U} := \text{Spec}(\mathcal{O}_{\mathcal{U},[\sigma]}/m_{\sigma,U}^2)$ be the first-order infinitesimal neighborhood of $[\sigma]$ in $\mathcal{U}$. We let $\mathcal{T}_{\sigma,U} \to T_{\sigma,U}$ be the family of surfaces obtained by pulling back the family $\mathcal{H}_A \to \Lambda$ via the inclusion $T_{\sigma,U} \hookrightarrow \mathcal{U}$. The Infinitesimal Noether Lefschetz (INL) Theorem is valid at $(U, \sigma)$ (see Section 2 of [1]) if the group of line-bundles on $\mathcal{T}_{\sigma,U}$ is equal to the image of the composition

$$
\text{Pic}(W) \to \text{Pic}(W \times C \mathcal{T}_{\sigma,U}) \to \text{Pic}(\mathcal{T}_{\sigma,U}).
$$

(2.4)

Let $A \subset \Lambda$ be an integral closed subset. Let $[\sigma] \in A$, where $\sigma \in H^0(W, \pi^* T_{\Theta^3}(d)(-E))$, and suppose that $S := V(\sigma)$ is smooth and $A$ is smooth at $[\sigma]$. Let $T(P)$ be the projective tangent space to $A$ at $[\sigma]$. If the INL Theorem holds for $(U, \sigma)$ then $A \setminus \Delta$ does not belong to $NL(A \setminus \Delta)$. Joshi [2] gave a cohomological condition which suffices for the validity of the INL Theorem. Suppose that $U \subset H^0(W, \pi^* T_{\Theta^3}(d)(-E))$ generates $\pi^* T_{\Theta^3}(d)(-E)$; we let $M(U)$ be the locally-free sheaf on $W$ fitting into the exact sequence

$$
0 \to M(U) \to U \otimes \mathcal{O}_W \to \pi^* T_{\Theta^3}(d)(-E) \to 0.
$$

(2.5)

Proposition 2.10 (K. Joshi, Prop. 3.1 of [1]). Let $0 \neq \sigma \in U$ such that $S = V(\sigma)$ is smooth, and suppose that

(a) $H^1(W, \Omega_{\mathcal{O}_W}^1 \otimes \pi^* T_{\Theta^3}(d)(-E)) = 0$.

(b) $H^1(W, M(U) \otimes K_W \otimes \pi^* T_{\Theta^3}(d)(-E)) = 0$.

Then the INL Theorem holds at $(U, \sigma)$.

2.5. The generic tangent space is a base-point free linear system. The aim of the present subsection is to prove Proposition 2.7. First we go through a couple of auxiliary results.

Lemma 2.11. Suppose that

$$
0 = H^1(\mathbb{P}^3, \mathcal{I}_C \otimes T_{\Theta^3}(d-4)) = H^1(C, T_C(d-4)).
$$

(2.6)

Then $H^1(W, \Omega_{\mathcal{O}_W}^1 \otimes \pi^* T_{\Theta^3}(d)(-E)) = 0$.

Proof. Since $\Omega_{\mathcal{O}_W}^1 \cong T_W \otimes K_W$ it is equivalent to prove that

$$
0 = H^1(W, T_W \otimes K_W \otimes \pi^* T_{\Theta^3}(d)(-E)) = H^1(W, T_W \otimes \pi^* T_{\Theta^3}(d-4)).
$$

(2.7)

Let $\iota: E \hookrightarrow W$ be the inclusion of the exceptional set of $\pi$ and $\xi$ be the natural quotient line-bundle on $E$. Thus letting $\rho: E \to C$ be the restriction of $\pi$ we have an exact sequence

$$
0 \to \mathcal{O}_W(E)|_E \to \mathcal{O}_C \to \xi \to 0.
$$

(2.8)

The differential of $\pi$ gives the exact sequence

$$
0 \to T_W \otimes \pi^* T_{\Theta^3}(d-4) \to \pi^* T_{\Theta^3}(d-4) \to \iota_* (\xi \otimes \rho^* \mathcal{O}_C(d-4)) \to 0.
$$

(2.9)

Below is a piece of the associated long exact sequence of cohomology:

$$
H^0(W, \pi^* T_{\Theta^3}(d-4)) \to H^0(W, \pi^* T_{\Theta^3}(d-4)) \to H^1(W, T_W \otimes \pi^* T_{\Theta^3}(d-4)) \to H^1(W, \pi^* T_{\Theta^3}(d-4)).
$$

(2.10)

We claim that $H^1(W, \pi^* T_{\Theta^3}(d-4)) = 0$. In fact the spectral sequence associated to $\pi$ and abutting to the cohomology $H^1(W, \pi^* T_{\Theta^3}(d-4))$ gives an exact sequence

$$
0 \to H^1(\mathbb{P}^3, \pi_* \pi^* T_{\Theta^3}(d-4)) \to H^1(W, \pi^* T_{\Theta^3}(d-4)) \to H^0(\mathbb{P}^3, R^1 \pi_* \pi^* T_{\Theta^3}(d-4)) \to 0.
$$

(2.11)

Now $\pi_* \pi^* T_{\Theta^3}(d-4) \cong T_{\Theta^3}(d-4)$ and hence $H^1(\mathbb{P}^3, \pi_* \pi^* T_{\Theta^3}(d-4)) = 0$. Moreover $R^1 \pi_* \pi^* T_{\Theta^3}(d-4) = 0$ because $R^1 \pi_* \mathcal{O}_W = 0$, and hence $H^1(W, \pi^* T_{\Theta^3}(d-4)) = 0$. Thus in order to complete the proof it suffices to show that the map

$$
H^0(W, \pi^* T_{\Theta^3}(d-4)) \to H^0(W, \xi \otimes \rho^* \mathcal{O}_C(d-4))
$$

is surjective. We have an isomorphism $H^0(W, \xi \otimes \rho^* \mathcal{O}_C(d-4)) \cong H^0(C, \mathcal{O}_C(d-4))$, and moreover Map (2.12) is identified with the composition

$$
H^0(\mathbb{P}^3, T_{\Theta^3}(d-4)) \xrightarrow{\alpha} H^0(C, T_{\Theta^3}(d-4)) \xrightarrow{\beta} H^0(C, \mathcal{O}_C(d-4)).
$$

(2.13)
The map $\alpha$ is surjective by the first vanishing in (2.6), while $\beta$ is surjective by the second vanishing in (2.6). This proves that (2.7) holds.

Let $U \subset H^0(\mathbb{P}^3, \mathcal{J}(d))$ be a subspace which generates $\mathcal{J}(d)$; we let $\overline{\mathcal{M}}(U)$ be the sheaf on $\mathbb{P}^3$ fitting into the exact sequence

$$0 \to \overline{\mathcal{M}}(U) \to U \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{J}(d) \to 0. \quad (2.14)$$

The curve $C$ is a local complete intersection because $C$ is smooth, and hence $\overline{\mathcal{M}}(U)$ is locally-free.

**Lemma 2.12.** Suppose that the hypotheses of Lemma 2.11 hold and that in addition the sheaf $\mathcal{J}$ is $d$-regular. Let $U \subset H^0(\mathbb{P}^3, \mathcal{J}(d))$ be a subspace which generates $\mathcal{J}(d)$, and let $c$ be its codimension. Then $\nabla^p \overline{\mathcal{M}}(U)$ is $(p+c)$-regular.

**Proof.** Let $\overline{\mathcal{M}} := \overline{\mathcal{M}}(H^0(\mathcal{J}(d)))$. Then $\overline{\mathcal{M}}$ is 1-regular: in fact $H^1(\mathbb{P}^3, \overline{\mathcal{M}}) = 0$ because the exact sequence induced by (2.14) on $H^0$ is exact by definition, and $H^i(\mathbb{P}^3, \overline{\mathcal{M}}(1-i)) = 0$ for $i \geq 2$ because $\mathcal{J}$ is $d$-regular. It follows that $\nabla^p \overline{\mathcal{M}}$ is $p$-regular (Corollary 1.8.10 of [11]). Then, arguing as in the proof of the Lemma on p. 371 of [10] (see also Example 1.8.15 of [11]) one gets that $\nabla^p \overline{\mathcal{M}}(U)$ is $(p+c)$-regular. □

**Proof of Proposition 2.7.** By hypothesis there exists $[\sigma] \in A$, where $\sigma \in H^0(\pi^* \mathcal{O}_{\mathbb{P}^3}(-E))$, such that $A$ is smooth at $[\sigma]$, and the projective tangent space $T_S A$ is a base-point free codimension-1 linear subsystem of $\Lambda$. We have $T_S A = \mathbb{P}(U)$, where $U \subset H^0(\pi^* \mathcal{O}_{\mathbb{P}^3}(-E))$ is a codimension-1 subspace which generates $\mathcal{O}_{\mathbb{P}^3}(-E)$. We will prove that the INL Theorem holds for $(U, \sigma)$, and Proposition 2.7 will follow. By Joshi’s Proposition 2.10 it suffices to prove that the following hold:

(a) $H^1(W, \Omega^1_{\mathbb{P}^3} \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(d)(-E)) = 0.
$

(b) $H^1(W, M(U) \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(d)(-E)) = 0.
$

We start by noting that, since $T_{\mathbb{P}^3}$ is $-1$-regular, and by hypothesis $\mathcal{J}$ is $(d-2)$ regular, the sheaf $\mathcal{J} \otimes T_{\mathbb{P}^3}$ is $(d-3)$-regular, see Proposition 1.8.9 and Remark 1.8.11 of [11]. Thus the hypotheses of Lemma 2.11 are satisfied, and hence Item (a) holds. Let us prove that Item (b) holds. Tensoring (2.5) by $K_W \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(d)(-E) \cong \pi^* \mathcal{O}_{\mathbb{P}^3}(d-4)$ and taking cohomology we get an exact sequence

$$0 \to H^0(W, M(U) \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(d-4)) \to U \otimes H^0(W, \pi^* \mathcal{O}_{\mathbb{P}^3}(d-4)) \to H^0(W, \pi^* \mathcal{O}_{\mathbb{P}^3}(2d-4)(-E)) \to H^1(W, M(U) \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(d-4)) \to U \otimes H^1(W, K_W \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(d)(-E)). \quad (2.15)$$

Now $H^1(W, K_W \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(d)(-E)) = 0$ because by hypothesis $\pi^* \mathcal{O}_{\mathbb{P}^3}(d)(-E)$ is ample. Thus it suffices to prove that $U \otimes H^0(W, \pi^* \mathcal{O}_{\mathbb{P}^3}(d-4)) \to H^0(W, \pi^* \mathcal{O}_{\mathbb{P}^3}(2d-4)(-E))$ is surjective. We have an identification $H^0(W, \pi^* \mathcal{O}_{\mathbb{P}^3}(d)(-E)) \cong H^0(\mathbb{P}^3, \mathcal{J}(d))$, and hence $U$ is identified with a codimension-1 subspace of $H^0(\mathbb{P}^3, \mathcal{J}(d))$ that we will denote by the same symbol. Clearly it suffices to prove that the natural map

$$U \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-4)) \to H^0(\mathbb{P}^3, \mathcal{J}(2d-4)) \quad (2.16)$$

is surjective. Tensorize Exact Sequence (2.14) by $\mathcal{O}_{\mathbb{P}^3}(d-4)$ and take the associated long exact sequence of cohomology: then (2.10) appears in that exact sequence, and hence it suffices to prove that $H^1(\mathbb{P}^3, \overline{\mathcal{M}}(U))(d-4) = 0$. By Lemma 2.12 the sheaf $\overline{\mathcal{M}}(U)$ is 2-regular, and by hypothesis $d \geq 5$: the required vanishing follows. □

2.6. **All tangent spaces at smooth points are linear systems with a base-point.** We will prove Proposition 2.9. We start with an elementary result.

**Lemma 2.13.** Assume that $\pi^* \mathcal{O}_{\mathbb{P}^3}(r-2)(-E)$ is very ample. Let $p \in W$ and $F \subset T_p W$ be a subspace such that one of the following holds:

(1) $p \notin E$ and $F \neq T_p W$,

(2) $p \notin E$ and $F = T_p W$,

(3) $p \in E$, $F \subset T_p E$ and $T_p(\pi^{-1}(q)) \not\subset F$, where $q = \pi(p)$.

If $X \in \Gamma_{\pi, F}(r)$ (see Definition 2.5) is generic then, corresponding to (1) - (3) above we have the following:

(1') $X$ is smooth,

(2') $X$ has an ODP at $q = \pi(p)$ and is smooth elsewhere,

(3') $X$ is smooth.
Proof. Suppose first that (1) or (2) holds, i.e. \( p \notin E \), and let \( q := \pi(p) \). The linear system \( \Gamma_{p,F}(r) \)
has base locus \( C \cup \{q\} \): in fact let \( z \in \{p^3 \setminus C \setminus \{q\}\} \), there exists \( Y \in |\mathcal{I}_C(r-2)| \) not containing \( z \)
because \( \pi^* \mathcal{O}_{\mathbb{P}^3}(r-2)(-E) \) is very ample, and also a quadric \( Q \) not containing \( z \) and such that \( p \in Q \)
and \( F \subset T_pQ \), and hence \( Y + Q \) is an element of \( \Gamma_{p,F}(r) \) which does not contain \( z \). By Bertini the
generic \( X \in \Gamma_{p,F}(r) \) is smooth away from \( C \cup \{q\} \). Considering \( Y + Q \in \Gamma_{p,F}(r) \) as above we also get
that the behaviour in \( q \) of the generic element of \( \Gamma_{p,F}(r) \) is as claimed.

The above argument also shows that, imposing to \( X \in |\mathcal{I}_C(r)| \) that it contains \( q \) and that \( F \subset T_qX \)
(since \( q \notin C \) the differential \( d\pi(p) \) identifies \( T_pW \sim \rightarrow T_q\mathbb{P}^3 \)), one gets \( F + 1 \) linearly independent conditions, i.e.
\[
\dim \Gamma_{p,F}(r) = \dim |\mathcal{I}_C(r)| - r - 1. \tag{2.17}
\]

It remains to prove that the generic \( X \in \Gamma_{p,F}(r) \) is smooth at every point of \( C \), i.e. that \( \Sigma(r) \cap \Gamma_{p,F}(r) \)
is a proper closed subset of \( \Gamma_{p,F}(r) \). Since
\[
\Sigma(r) \cap \Gamma_{p,F}(r) = \bigcup_{x \in C} |\mathcal{I}_x^2(r)| \cap \Gamma_{p,F}(r) \tag{2.18}
\]
it suffices by (2.17) to prove that
\[
\dim |\mathcal{I}_x^2(r)| \cap \Gamma_{p,F}(r) \leq \dim |\mathcal{I}_C(r)| - r - 3. \tag{2.19}
\]

By Item (1) of Proposition 2.2, we have \( \dim |\mathcal{I}_x^2(r)| \cap |\mathcal{I}_C(r)| = \dim |\mathcal{I}_C(r)| - 2 \), and hence (2.19) is equivalent to
\[
\codim(|\mathcal{I}_x^2(r)| \cap \Gamma_{p,F}(r), |\mathcal{I}_x^2(r)| \cap |\mathcal{I}_C(r)|) = r + 1. \tag{2.20}
\]

Thus we must check that requiring that \( X \in |\mathcal{I}_x^2(r)| \cap |\mathcal{I}_C(r)| \) contains \( q \) and \( F \subset T_qX \) imposes \( r + 1 \) linearly independent conditions. This is proved by choosing \( Y \in |\mathcal{I}_x^2(r-2)| \cap |\mathcal{I}_C(r-2)| \) not containing \( q \) (see Item (1) of Proposition 2.2), and considering the surfaces \( Y + Q \in |\mathcal{I}_x^2(r)| \cap |\mathcal{I}_C(r)| \) where \( Q \in |\mathcal{O}_{\mathbb{P}^3}(2)| \): in this sublinear system it is clear that the requirements discussed above impose \( r + 1 \) linearly independent conditions, and (2.20) follows. This finishes the proof that if (1) holds then (1’) holds, and that if (2) holds then (2’) holds.

Now suppose that (3) holds. If \( F = \{0\} \) then \( \Gamma_{p,F}(r) \) consists of \( |\mathcal{I}_p \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \). Let \( S \in |\mathcal{I}_p \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \) be generic. Then \( S \) is clearly smooth at \( p \), and by Bertini’s Theorem it is smooth away from \( p \) as well. In order to prove that \( X = \pi(S) \) is smooth we must check that \( S \) does not contain any of the lines \( \mathbf{L}_x := \pi^{-1}(x) \) for \( x \in C \). Now
\[
\codim(|\mathcal{I}_x \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \cap |\mathcal{I}_p \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)|, |\mathcal{I}_p \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)|) = \begin{cases} 1 & \text{if } x = q, \\ 2 & \text{if } x \neq q. \end{cases} \tag{2.21}
\]

It follows that a generic \( S \in |\mathcal{I}_p \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \) does not contain any \( \mathbf{L}_x \).

We are left to deal with the case of a 1-dimensional \( F \subset T_pE \) transverse to \( T_p(\pi^{-1}(q)) \). Let \( Z \subset W \)
be the 0-dimensional scheme of length 2 supported at \( p \), with tangent space \( F \); thus \( Z \subset C \). We must prove that a generic \( S \in |\mathcal{I}_Z \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \) is smooth and contains no line \( \mathbf{L}_x \) where \( x \in C \).

We claim that the base-locus of \( |\mathcal{I}_Z \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \) is \( p \). In fact no \( z \in (\mathbf{L}_q \setminus \{p\}) \) is in the base-locus of \( |\mathcal{I}_Z \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \) because \( \mathbf{L}_q \) is a line and there exists \( S \in |\mathcal{I}_Z \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \) which is not tangent to \( \mathbf{L}_q \) at \( p \). Moreover no \( z \in (W \setminus \mathbf{L}_q \setminus \mathbf{L}_p) \) is in the base-locus of \( |\mathcal{I}_Z \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \) because of the following argument. There exist \( T \in |\mathcal{I}_p \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r-1)(-E)| \) not containing \( z \), and a plane \( P \subset \mathbb{P}^3 \) containing \( q \) and not containing \( p \); then \( (T + P) \in |\mathcal{I}_Z \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \) does not contain \( z \). This proves that the base-locus of \( |\mathcal{I}_Z \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \) is \( p \); it follows that the generic \( S \in |\mathcal{I}_Z \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \) is smooth.

We finish by showing that (2.21) holds with \( \mathcal{I}_p \) replaced by \( \mathcal{I}_Z \). The case \( x = q \) is immediate. If \( x \in C \setminus \{q\} \) we get the result by considering elements \( (T + P) \in |\mathcal{I}_Z \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r)(-E)| \) where \( P \) is a fixed plane containing \( q \) and not containing \( x \), and \( T \in |\mathcal{I}_p \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(r-1)(-E)| \).

\[ \square \]

Remark 2.14. The proof of Lemma 2.13 shows that, if Item (2) holds, the projectivized tangent cone at \( q \) of the generic \( X \in \Gamma_{p,F}(r) \) is the generic conic in \( P(T_q\mathbb{P}^3) \).

Next we prepare the stage for the promised application of the Griffiths-Harris method. Let \( r \in \{d-1, d\} \). Suppose that \( p \in W \) and \( F \subset T_pW \) satisfy the hypotheses of Proposition 2.9. By Lemma 2.13 there exists an open dense subset \( \mathcal{V}_{p,F}(r) \subset \Gamma_{p,F}(r) \) such that for \( X \in \mathcal{V}_{p,F}(r) \) the following holds:

(a) \( X \) is smooth if \( p \notin E \) and \( F \neq T_pW \), or \( p \in E \).
(b) $X$ has an ODP at $q = \pi(p)$, and is smooth elsewhere, if $p \notin E$ and $F = T_pW$.

Similarly, if $q \in C_k$, then by Proposition 2.3 there exists an open dense subset $\mathcal{U}_{q,k}(r) \subset \Sigma_k(r) \cap |\mathcal{F}_q^r(r)|$ such that $X \in \mathcal{U}_{q,k}(r)$ has an ODP at $q$ and is smooth elsewhere.

It will suffice to prove that if $X$ is a very general element of $\mathcal{U}_{p,F}(r)$ or of $\mathcal{U}_{q,k}(r)$, then $CH^1(X)\mathbb{Q}$ is generated by $c_1(\mathcal{O}_X(1))$ and the classes of $C_1, \ldots, C_n$. Notice that if $X$ is an element of $\mathcal{U}_{p,F}(r)$ or of $\mathcal{U}_{q,k}(r)$, then $X$ is $\mathbb{Q}$-factorial. More precisely: if $D$ is a Weil divisor on $X$ then $2D$ is a Cartier divisor.

Let $NL(\mathcal{U}_{p,F}(d)) \subset \mathcal{U}_{p,F}(d)$ be the subset of $X$ such that $\text{Pic}(X) \otimes \mathbb{Q}$ is not generated by $c_1(\mathcal{O}_X(1))$ and $c_1(\mathcal{O}_X(2C_1), \ldots, c_1(\mathcal{O}_X(2C_n))$, and define similarly $NL(\mathcal{U}_{q,k}(d)) \subset \mathcal{U}_{q,k}(d)$.

Hence it suffices to prove that $\mathcal{U}_{p,F}(d) \setminus NL(\mathcal{U}_{p,F}(d))$ and $\mathcal{U}_{q,k}(d) \setminus NL(\mathcal{U}_{q,k}(d))$ are not empty.

Proof of Proposition 2.3 Let $Y$ be an element of $\mathcal{U}_{p,F}(d-1)$ of $\mathcal{U}_{q,k}(d-1)$, and let $X$ be a generic element of $\mathcal{U}_{p,F}(d)$ or of $\mathcal{U}_{q,k}(d)$. Since $\pi^*|\mathcal{O}_Y(\mathcal{O}_X(-E))$ is very ample and $X$ is generic, the intersection of $X$ and $Y$ is reduced, and its irreducible decomposition is

$$X \cap Y = C_0 \cup C_1 \cup \ldots \cup C_n. \quad (2.22)$$

Now let $P \subset \mathbb{P}^3$ be a generic plane, in particular transverse to $C_0 \cup C_1 \cup \ldots \cup C_n$. Let $X = V(f)$, $Y = V(g)$ and $P = V(l)$, and $t$ an affine coordinate on $\mathbb{A}^1$. Let

$$\mathcal{Z} := V(g \cdot l + tf) \subset \mathbb{P}^3 \times \mathbb{A}^1. \quad (2.23)$$

The projection $\mathcal{Z} \to \mathbb{A}^1$ is a family of degree-$d$ surfaces, and it gives a degeneration of $X$ to $Y + P$. The 3-fold $\mathcal{Z}$ is singular, we desingularize as follows. First $\mathcal{Z}$ is singular at the points $(x,0)$ such that $x \in X \cap Y \cap P$, and it has an ODP at each of these points because $P$ is transverse to $C_0 \cup C_1 \cup \ldots \cup C_n$. Moreover

(I) $\mathcal{Z}$ has no other singularities if we are dealing with $\mathcal{U}_{p,F}(d)$ and $F \neq T_pW$.

(II) $\mathcal{Z}$ is also singular at $\{q\} \times \mathbb{A}^1$ if we are dealing with $\mathcal{U}_{p,F}(d)$ and $F = T_pW$, or if we are dealing with $\mathcal{U}_{q,k}(d)$.

The isolated ODP’s are eliminated by a small resolution (we follow p. 35 of [8], and choose a specific small resolution among the many possible ones), while to desingularize $\{q\} \times \mathbb{A}^1$ we blow-up that curve: let $\mathcal{F} \to \mathcal{Z}$ be the birational morphism. Then $\mathcal{F}$ is smooth (if $p \notin E$ and $F = T_pW$, or if we are dealing with $\mathcal{U}_{q,k}(d)$), then $\mathcal{F}$ is smooth over $\{q\} \times \mathbb{A}^1$ by Remark 2.5 and Remark 2.14.

Composing $\mathcal{F} \to \mathcal{Z}$ with the projection $\mathcal{Z} \to \mathbb{A}^1$ we get a flat family $\varphi: \mathcal{F} \to \mathbb{A}^1$ of surfaces.

The surface $\tilde{Z}_0 := \varphi^{-1}(0)$ has normal crossings, obtained from the union of $Y$ and the blow-up $\tilde{P}$ of $P$ at the points of $X \cap Y \cap P$ by gluing the curve $Y \cap P$ with its strict transform in $\tilde{P}$. There will be an open dense $B \subset \mathbb{A}^1$ containing $0$ such that $\tilde{Z}_t := \varphi^{-1}(t)$ is smooth for $t \in B \setminus \{0\}$, and it is isomorphic to $Z_t := V(g \cdot l + tf)$. In Case (I), while it is the blow-up of $Z_t$ at $q$ (an ODP) in Case (II).

We replace $\mathcal{F}$ by $\varphi^{-1}(B)$ but we do not give it a new name. Then $\mathcal{F} \to B$ is a flat family of surfaces, smooth for $t \neq 0$, and the “central” fiber $\tilde{Z}_0$ has normal crossings.

The idea is to prove that if $P$ is very general then the following hold:

(I') In Case (I), if $t$ is very general in $B \setminus \{0\}$, then Pic($\tilde{Z}_t) \otimes \mathbb{Q}$ is generated by the classes of $\mathcal{O}_{\tilde{Z}_t}(1), \mathcal{O}_{\tilde{Z}_t}(C_1), \ldots, \mathcal{O}_{\tilde{Z}_t}(C_n)$. (Notice that $\tilde{Z}_t = Z_t$ because we are in case (I).)

(II') In Case (II), if $t$ is very general in $B \setminus \{0\}$, letting $\mu_t: \tilde{Z}_t \to Z_t$ be the blow-up of $q$ and $R_t \subset \tilde{Z}_t$ the exceptional curve, the group Pic($\tilde{Z}_t) \otimes \mathbb{Q}$ is generated by the classes of $\mu_t^*\mathcal{O}_{Z_t}(1), \mu_t^*\mathcal{O}_{Z_t}(2C_1), \ldots, \mu_t^*\mathcal{O}_{Z_t}(2C_n)$ and $\mathcal{O}_{\tilde{Z}_t}(R_t)$.

One does this by controlling the Picard group of the degenerate fiber $\tilde{Z}_0$. As proved in [8] [12] [5] it suffices to show that the following hold:

(a) Let $\mathcal{Y} \subset |\mathcal{O}_Y(1)|$ be the open subset of planes intersecting transversely $C_0 \cup \ldots \cup C_n$, let $I \subset (C_0 \cup \ldots \cup C_n) \times \mathcal{Y}$ be the incidence subset and $\rho: I \to \mathcal{Y}$ be the natural finite map: then the mododromy of $\rho$ acts on a fiber $(D_0, \ldots, D_n, P)$ as the product of the symmetric groups $\mathfrak{S}_{\text{deg} C_0} \times \ldots \times \mathfrak{S}_{\text{deg} C_n}$.

(b) For $k \in \{0, \ldots, n\}$, a very general plane $P \subset \mathbb{P}^3$, and distinct points $x, y \in C_k \cap P$, the divisor class $x - y$ on the curve $Y \cap P$ is not torsion.
Now Item (a) is Proposition II.2.6 of [12]. It remains to prove that (b) holds. To this end we will show that $C_0$ is not planar and we will control the set of planes $P$ such that $P_2 \cap Y$ is reducible (see the proof of Item (b) of Lemma 3.4 of [5]).

**Claim 2.15.** The curve $C_0$ (see (2.22)) is not planar.

**Proof.** Assume the contrary. Then the restriction map $\alpha: H^0(\mathcal{H}_C(d)) \to H^0(\mathcal{H}_Y(d))$ has image of dimension at most 4. On the other hand the kernel of $\alpha$ has dimension 4 because $Y \in |\mathcal{H}_C(d - 1)|$, and hence $h^0(\mathcal{H}_C(d)) \leq 8$. By hypothesis $\pi^*\mathcal{H}_p(d - 3)(-E)$ is very ample, in particular it has a non-zero section: multiplying that section by sections of $\mathcal{H}_p(3)$ we get that that $h^0(\mathcal{H}_C(d)) = h^0(\pi^*\mathcal{H}_p(d)(-E)) \geq 20$, and that is a contradiction. □

Thus none of the curves $C_0, C_1, \ldots, C_n$ is planar.

**Lemma 2.16.** Let $Y \subset \mathbb{P}^3$ be a surface which is either smooth or has ODP’s. The set of planes $P$ such that $P \cap Y$ is reducible is the union of a finite set and the collection of pencils through lines of $Y$ (if there are any).

**Proof.** Suppose the contrary. Then there exists a 1-dimensional family of planes $P$ such that $P \cdot Y = C_1 + C_2$ with $C_1, C_2$ divisors which intersect properly, supp $C_1$ is irreducible, and $\deg C_1 > 1$, $\deg C_2 > 1$, where the degree is with respect to $\mathcal{H}_p(1)$. We will get a contradiction. We distinguish between the two cases:

1. The generic $P$ does not contain any singular point of $Y$.
2. The generic $P$ contains a single point $a \in \text{sing} Y$, or two points $a, b \in \text{sing} Y$.

Assume that (1) holds. Let $m_i := \deg C_i$ for $i = 1, 2$. Then

$$m_1 m_2 = (C_1 \cdot C_2)_P = (C_1 \cdot C_2)_Y = (C_1 \cdot (P - C_1))_Y = m_1 - (C_1 \cdot C_1)_Y \quad (2.24)$$

where $(C_1 \cdot C_2)_P$ is the intersection number of $C_1, C_2$ in the plane $P$, $(C_1 \cdot C_2)_Y$ is the intersection number of $C_1, C_2$ in the surface $Y$ etc. The first equality of (2.23) holds by Bézout, the second equality is proved by a local computation of the multiplicity of intersection at each point of $C_1 \cap C_2$ (one needs the hypothesis that $Y$ is smooth at each such point). Thus (2.24) gives that $(C_1 \cdot C_1)_Y = m_1(1 - m_2) < 0$, and this contradicts the hypothesis that $C_1$ moves in $Y$. If (2) holds one argues similarly. Let $\mathbb{P}^3 \to \mathbb{P}^3$ be the blow of $a$ (respectively $\{a, b\}$), and $\tilde{Y}, \tilde{P} \subset \mathbb{P}^3$ be the strict transforms of $Y$ and $P$ respectively. By hypothesis $Y$ has an ODP at each of its singular points and hence $\tilde{Y}$ is smooth, and of course $\tilde{P}$ is smooth. Let $\tilde{C}_i$ be the strict transform of $C_i$ in $\mathbb{P}^3$. The computation in the case that $P$ contains two singular points is as follows. Let $r_i := \text{mult}_a C_i, s_i := \text{mult}_b C_i$. Then the equality

$$(\tilde{C}_1 \cdot \tilde{C}_2)_\tilde{P} = (\tilde{C}_1 \cdot \tilde{C}_2)_\tilde{P} \quad (2.25)$$

gives the equality

$$(\tilde{C}_1 \cdot \tilde{C}_2)_\tilde{P} = -(m_1 m_2 - m_1 - r_1 r_2 - s_1 s_2 + r_1 + s_1). \quad (2.26)$$

Now $r_i + s_i \leq m_i$ for $i = 1, 2$, or else the line $\{a, b\}$ is contained in $Y \cap C_i$, and hence we are considering the curves residual to a line in $Y$, against the hypothesis. Since $r_i + s_i \leq m_i$ for $i = 1, 2$ the right-hand side of (2.25) is strictly negative, and this is a contradiction. □

Now we prove that Item (b) holds. Let $k \in \{0, \ldots, n\}$. Let $a, b \in C_k$ be generic, in particular they are smooth points of $Y$. By Lemma 2.16 every plane containing $a, b$ intersects $Y$ in an irreducible curve. Let $\tilde{Y} \to Y$ be the blow-up of the base-locus of the pencil of plane sections of $Y$ containing $a, b$. Then $\tilde{Y}$ has at most ADE singularities, and hence is $\mathbb{Q}$-factorial. Let $E, F$ be the exceptional sets over $a$ and $b$ respectively, both have strictly negative self-intersection. Let $i > 0$ be such that $iE$ and $iF$ are Cartier. Let $\varphi: \tilde{Y} \to \mathbb{P}^1$ be the regular map defined by the pencil of plane sections of $Y$ containing $a, b$; for $s \in \mathbb{P}^1$ we let $D_s := \varphi^{-1}(s)$. It suffices to prove that, given $r > 0$, the set of $s \in \mathbb{P}^1$ such that $\mathcal{H}_s(r_i E - r_i F)|_{D_s}$ is trivial is finite. Assume the contrary: then $\mathcal{H}_s(r_i E - r_i F) \cong \varphi^* \mathcal{H}_p(\ell)$ for some $\ell \in \mathbb{Z}$ because every plane containing $a, b$ intersects $\tilde{Y}$ in an irreducible curve (see the proof of Item (b) of Lemma 3.4 of [5]). It follows that the degrees of $\mathcal{H}_s(r_i E - r_i F)$ on $E$ and $F$ are both equal to $\ell$, and that is absurd because they have opposite signs. □
3. Proof of the main result

We will prove Theorem \ref{Thm1} Let \( Q \subset \mathbb{P}^3 \) be a smooth quadric and choose an isomorphism \( \varphi: Q \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{P}^1 \): we let \( \mathcal{O}_Q(a,b) := \varphi^* \mathcal{O}_{\mathbb{P}^1}(a) \boxtimes \mathcal{O}_{\mathbb{P}^1}(b) \).

**Proposition 3.1.** Let \( C \subset Q \) be a smooth curve in \( |\mathcal{O}_Q(2,3)| \). Then \( C \) is 3-regular.

**Proof.** Considering the exact sequence \( 0 \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0 \) we see right away that if \( i = 2,3 \), then \( H^i(\mathbb{P}^3, \mathcal{F}(3-i)) = 0 \). In order to prove that \( H^1(\mathbb{P}^3, \mathcal{O}_C(2)) = 0 \) we must show that \( H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(C, \mathcal{O}_C(2)) \) is surjective. By surjectivity of \( H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2,2)) \to H^0(Q, \mathcal{O}_Q(2)) \) it suffices to prove that \( H^0(Q, \mathcal{O}_Q(2,2)) \to H^0(C, \mathcal{O}_C(2)) \) is surjective. We have an exact sequence

\[ 0 \to \mathcal{O}_Q(0,1) \to \mathcal{O}_Q(2,2) \to \mathcal{O}_C(2) \to 0, \]

and since \( H^1(Q, \mathcal{O}_Q(0,1)) = 0 \) the map \( H^0(Q, \mathcal{O}_Q(2,2)) \to H^0(C, \mathcal{O}_C(2)) \) is surjective. \( \square \)

**Proof of Theorem \ref{Thm1}** If \( d \leq 6 \) there is nothing to prove, hence we may assume that \( d \geq 7 \). Let \( n := \lfloor \frac{d-1}{3} \rfloor \). Choose disjoint smooth curves \( C_1, \ldots, C_n \) such that each \( C_k \) is a \( (2,3) \)-curve on a smooth quadric, and let \( C := C_1 \cup \ldots \cup C_n \). We may assume that for \( k \in \{1, \ldots, n\} \) the degree-0 class in \( \text{CH}_0(C_k) \) given by \( 5c_1(K_{C_k}) - 2c_1(\mathcal{O}_{C_k}(1)) \) is not zero. Let us show that the hypotheses of Theorem \ref{Thm2} are satisfied. Let \( k \in \{1, \ldots, n\} \), let \( \pi_k: W_k \to \mathbb{P}^3 \) be the blow-up of \( C_k \), and let \( E_k \subset W_k \) be the exceptional divisor. Then \( \pi_k^* \mathcal{O}_{\mathbb{P}^3}(3)(-E_k) \) is globally generated, and \( \pi_k^* \mathcal{O}_{\mathbb{P}^3}(4)(-E_k) \) is very ample: since \( d-3 \geq 3(n-1) + 4 \) it follows that \( \pi^* \mathcal{O}_{\mathbb{P}^3}(d-3)(-E) \) is very ample. Let \( k \in \{1, \ldots, n\} \) since \( d \geq 7 \) the cohomology group \( H^1(C_k, T_{C_k}(d-4)) \) vanishes, and hence \( H^1(C, T_C(d-4)) = 0 \). By Proposition \ref{Prop3} and Example 1.8.32 of [11] the curve \( C \) is 3n-regular, and since \( 3n \leq (d-4) \) the curve \( C \) is \((d-2)\)-regular. Lastly, by construction no curve \( C_k \) is planar. We have shown that the hypotheses of Theorem \ref{Thm2} are satisfied, and hence Hypothesis \ref{Hyp1} holds for \( H \in |\mathcal{O}_{\mathbb{P}^3}(d)| \). Let \( X \in |\mathcal{F}(d)| \) be smooth and very generic: since for \( k \in \{1, \ldots, n\} \) the class \( 5c_1(K_{C_k}) - 2c_1(\mathcal{O}_{C_k}(1)) \) is not zero, the decomposable classes \( H^2, C^2_1, \ldots, C^2_n \) on \( X \) are linearly independent by Proposition \ref{Prop1}. Thus \( \text{DCH}_0(X) \) has rank at least \( n+1 = \lfloor \frac{d-1}{3} \rfloor \). \( \square \)

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