Comparison of two semiclassical expansions for a family of PT-symmetric oscillators

Francisco M. Fernández and Javier García
INIFTA (UNLP, CCT La Plata-CONICET), División Química Teórica,
Blvd. 113 S/N, Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina
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Abstract

We show that a recently developed semiclassical expansion for the eigenvalues of PT-symmetric oscillators of the form $V(x) = (ix)^{2N+1} + bix$ does not agree with an earlier WKB expression for $V(x) = -(ix)^{2N+1}$ the case $b = 0$. The reason is due to the choice of different paths in the complex plane for the calculation of the WKB integrals. We compare the Stokes and anti-Stokes lines that apply to each case for the quintic oscillator and derive a general WKB expression that contains the two earlier ones.

1 Introduction

In a recent paper Nanayakkara and Mathanaranjan obtained accurate high-energy expansions for the solutions to the Schrödinger equation with polynomial potentials of odd degree. By means of an alternative approach, the asymptotic-energy expansion (AEE) that is claimed to be simpler than the standard WKB method, the authors derived analytic expressions which allowed them to obtain accurate eigenvalues for several test examples. They compared their approximate eigenvalues with accurate ones provided by both the diagonalization
method and numerical integration. The agreement between the AEE and nu-
merical results is remarkable and increases with the quantum number as ex-
pected.

The first set of test examples chosen by Nanayakkara and Mathanaranjan \cite{1} is given by \( V(x) = (ix)^{2N+1} + bix \). It is surprising that they did not try to compare their results for \( b = 0 \) with those of Bender and Boettcher \cite{2}. Such comparison would have been interesting because the latter authors stated that the diagonalization method is unsuitable for the PT-symmetric oscillators \( V(x) = -(ix)^k \) when \( K \geq 4 \). The reason for this failure is that the Stokes wedges that lie in the lower half of the complex plane do not contain the real \( x \) axis. It is therefore striking at first sight that the AEE results of Nanayakkara and Mathanaranjan \cite{1} agree so accurately with the ones provided by the diagonalization method.

Recently, we studied the eigenvalues of some non-Hermitian operators by means of the complex-rotation diagonalization and Riccati-Padé methods \cite{3}. We verified that the diagonalization method is unable to yield the eigenvalues of the PT-symmetric oscillators for \( K \geq 4 \) and that it produces instead some real eigenvalues related to the resonances discussed more rigorously by Alvarez \cite{4,5} many years before.

It is clear from the results just discussed that the semiclassical approaches proposed by Nanayakkara and Mathanaranjan \cite{1} and Bender and Boettcher \cite{2} are not equivalent. The purpose of this paper is to compare them and make clear the discrepancy.

### 2 Comparison of the WKB approaches

Nanayakkara and Mathanaranjan \cite{1} considered non-Hermitian oscillators

\[
H = p^2 + V(x), \tag{1}
\]

with polynomial potentials of the form

\[
V(x) = (ix)^{2N+1} + P(x), \tag{2}
\]
where \( P(x) \) is a polynomial function of degree smaller than \( 2N + 1 \). For concreteness we focus present discussion on the first case studied by those authors: \( P(x) = bix \).

The authors calculated the integrals that appear in the AEE along a contour that encloses two of the \( 2N + 1 \) branch points. They stated that those particular branch points lie inside the Stokes wedges which are necessary for defining the “above non-Hermitian problem correctly as an eigenvalue problem” and made reference to Shin’s paper [6]. The latter author studied non-Hermitian oscillators with potentials \( V(z) = -(iz)^m + P(z) \), where \( P(z) \) is a polynomial function of degree smaller than \( m \) and the boundary conditions are such that the eigenfunctions tend to zero exponentially as \( |z| \to \infty \) along the rays

\[
\arg(z) = -\frac{\pi}{2} \pm \frac{2\pi}{m + 2}.
\]

This condition is exactly the one required by Bender and Boettcher and, consequently, the WKB expressions of Bender and Boettcher and Shin are exactly the same. Therefore, it is intriguing how Nanayakkara and Mathanaranjan based on the wedges for a slightly different problem (see the sign of the leading term) derived an expression that obviously yields different results but still seems to be sound. We analyze this question in detail in what follows.

To begin with, we compare the leading term of the AEE when \( b = 0 \):

\[
E_{n}^{NM} = \frac{\sqrt{\pi}(2N + 3)\Gamma\left(\frac{3}{2} + \frac{1}{2N+1}\right)}{2 \cos\left(\frac{\pi}{2N+1}\right) \Gamma\left(\frac{1}{2N+1}\right)} \left( n + \frac{1}{2} \right)^{\frac{4N+2}{2N+3}},
\]

and the WKB expression adapted to the present problem

\[
E_{n}^{BB} = \frac{\sqrt{\pi}\Gamma\left(\frac{3}{2} + \frac{1}{2N+1}\right)}{\sin\left(\frac{\pi}{2N+1}\right) \Gamma\left(1 + \frac{1}{2N+1}\right)} \left( n + \frac{1}{2} \right)^{\frac{4N+2}{2N+3}},
\]

where NM and BB stand for Nanayakkara and Mathanaranjan and Bender and Boettcher, respectively. Before proceeding any further, it is worth noting that the asymptotic eigenvalue \( \lambda_j \) discussed by Shin [6] (see also the references therein) agrees with the latter expression when \( j = n + 1 \) and \( m = 2N + 1 \).
Straightforward calculation shows that $E_{BB}^n > E_{NM}^n$ for all $N > 1$ and that the discrepancy increases with $N$ as shown by $E_{BB}^n / E_{NM}^n = 1, 1.988629015, 3.523156867$ for $N = 1, 2, 3$, respectively. We appreciate that both approaches agree only in the case that the Stokes wedges chosen by Bender and Boettcher and Shin contain the real $x$ axis ($N = 1$).

Another noticeable difference is that Bender and Boettcher chose all the the turning points on the lower half of the complex plane, whereas those of Nanayakkara and Mathanaranjan appear alternatively in the upper and lower half of the complex plane for $N = 1, 2, \ldots$.

We can obtain accurate eigenvalues of the PT-symmetric oscillators $V(x) = -(ix)^{2N+1}$ by means of the Riccati-Padé method (RPM) applied recently to this kind of problems [3] (and references therein). Table 1 shows the first eigenvalues of the PT-symmetric oscillator $V(x) = -(ix)^5$ calculated by the RPM and the WKB approach of Bender and Boettcher. Table 2 shows the first eigenvalues of the oscillator $V(x) = (ix)^5$ calculated by means of the diagonalization method, numerical integration (see below) and the leading term of the AEE [1]. We appreciate that the results of both tables do not agree as expected from the discussion above.

In order to understand the discrepancy between the AEE and WKB we carried out numerical integrations of the Schrödinger equation with $V(x) = (ix)^5$ along different anti-Stokes lines. Our results are consistent with the AEE ones when we choose the anti-Stokes lines shown in figure 1 (A). In this case the Stokes wedges contain the real axis and these results agree with those coming from the diagonalization method as indicated above. We have recently shown that the optimal angle in the complex-rotation method is the one that converts a non-Hermitian oscillator in either a Hermitian or a PT-symmetric one [3]. In the present case, since we are dealing with PT-symmetric oscillators, the rotation angle is expected to be exactly zero. Therefore, the straightforward diagonalization with a real basis set (for example, harmonic oscillator eigenfunctions) should give reasonable results as already shown in Table 2. On the other hand, Bender and Boettcher chose the Stokes wedges and anti-Stokes lines shown in
In this case the Stokes wedges do not contain the real axis and the eigenvalues cannot be obtained by diagonalization. In order to obtain similar results for the potential $V(x) = (ix)^5$ one should choose the lines shown in figure 1 (B) in which case the Stokes wedges do not contain the real axis. From the Schrödinger equation with $V(x) = -(ix)^5$ we can also obtain the results of Nanayakkara and Mathanaranjan if we choose the wedges indicated in 2 (B).

3 General WKB formula

In this section we will derive a general WKB expression that contains both Bender and Boettcher’s and Nanayakkara and Mathanaranjan’s equations. To this end we write the potential as

$$V(x) = x^{2M}(ix)^\epsilon,$$

where $M = 1, 2, \ldots$ and $\epsilon$ is chosen conveniently. The leading term of the WKB energy is given by

$$\left( n + \frac{1}{2} \right) \pi = \int_{x_-}^{x_+} \sqrt{E - x^{2M}(ix)^\epsilon} \, dx,$$

where the turning points $x_\pm$ are two roots of

$$E - x_j^{2M}(ix_j)^\epsilon = 0,$$

that are given by

$$x_j = e^{-\frac{i\pi}{2}} \left( \frac{\epsilon - 4j}{2M + \epsilon} \right) E^{\frac{1}{2M + \epsilon}}.$$

Obviously, if $x_j$ is a root then $-x_j^*$ is also a root; therefore, if we choose $x_+ = x_0$ and $x_- = -x_0^*$ then we obtain

$$E_n \sim \left[ \frac{\sqrt{\pi} \Gamma \left( \frac{3}{2} + \frac{1}{2M + \epsilon} \right) \left( n + \frac{1}{2} \right)^{\frac{1}{2M + \epsilon}}}{\sin \left( \frac{\pi M}{2M + \epsilon} \right) \Gamma \left( 1 + \frac{1}{2M + \epsilon} \right)} \right]^{1/2M + \epsilon}.$$

If $M = 1$ and $\epsilon = 2N - 1$ we have $V(x) = x^2(ix)^{2N-1} = -(ix)^{2N+1}$ that is exactly the PT-symmetric potential chosen by Bender and Boettcher for the
particular case of odd oscillators. It is clear that for those values of \( M \) and \( \epsilon \) the formula (10) becomes (5). Besides, the corresponding turning points are also in complete agreement.

In order to obtain the equation of Nanayakkara and Mathanaranjan we have to proceed more carefully. If we choose \( M = 2L \) and \( \epsilon = \pm 1 \) we have \( V(x) = x^{4L} (ix)^\epsilon = (ix)^{4L+\epsilon} \), where \( L \) and \( \epsilon \) are determined by the relation \( 4L + \epsilon = 2N + 1 \). Alternatively, we may set \( M = N \) and \( \epsilon = 1 \) so that equation (10) becomes (4). It seems surprising at first sight that we obtain the correct eigenvalues when the potential parameters lead to \( V(x) = x^{2N} (ix) = (-1)^N (ix)^{2N+1} \). The reason is that in this case all the turning points are on the lower half of the complex plane. In the former case the form of the potential is kept fixed while the turning points shift from one half of the plane to the other as \( N \) changes by unity (exactly as in Nanayakkara and Mathanaranjan’ approach). On the other hand, in the latter case the turning points are kept on the lower half of the complex plane while the sign of the potential changes as \( N \) changes. Both strategies lead to the same eigenvalues and are embodied in the general expression (10).

4 Conclusions

The purpose of this paper is to compare the AEE proposed by Nanayakkara and Mathanaranjan [1] and an earlier WKB approach derived Bender and Boettcher [2] whose expression was already confirmed by Shin [6] (see also references therein). We have shown that those approaches are not equivalent and yield completely different results, except for \( N = 1 \). The reason lies on the choice of the Stokes wedges within which the eigenfunction vanishes exponentially. While the Stokes wedges chosen by Bender and Boettcher do not contain the real axis when \( N > 1 \) those chosen by Nanayakkara and Mathanaranjan contain it for all \( N \). For this reason the results of the latter authors agree so accurately with the eigenvalues given by the diagonalization method. Both semiclassical eigenvalues agree exactly for \( N = 1 \) because the corresponding
Stokes wedges contain the real axis \cite{2}. Under such condition the diagonalization method with a real basis set is known to be exactly equivalent to the complex-rotation method because the optimal rotation angle is zero \cite{3}.

Nanayakkara and Mathanaranjan \cite{1} also compared their AEE eigenvalues with accurate results provided by numerical integration of the differential equations. Unfortunately, they did not give any detail that enables one to understand the nature of their results. We think that present paper makes this point clear by showing explicitly the Stokes and anti-Stokes lines that define each of the alternative eigenvalue problems. Bender and Boettcher \cite{2} also carried out a numerical integration of the Schrödinger equation which confirmed the accuracy of their WKB eigenvalues. They found that convergence is most rapid when one integrates along the anti-Stokes lines that they showed explicitly in their paper together with the corresponding Stokes lines.

Along with the choice of the Stokes wedges there is also the question of the choice of the turning points for the calculation of the WKB integrals. Whereas Bender and Boettcher chose all the turning points in the lower half of the complex plane, those of Nanayakkara and Mathanaranjan appear alternatively either in the upper or lower half. From the comparison of both approaches we have been able to derive the WKB expression \cite{10} that contains the equations derived by both Bender and Boettcher and Nanayakkara and Mathanaranjan. It is undoubtedly the most important point of present paper.

References

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\cite{2} Bender C M and Boettcher S 1998 \textit{Phys. Rev. Lett.} \textbf{80} 5243.

\cite{3} Fernández F M and Garcia J 2013 \textit{J. Phys. A} \textbf{46} 195301. [\texttt{arXiv:1301.1676}] [\texttt{math-ph}]

\cite{4} Alvarez G 1988 \textit{Phys. Rev. A} \textbf{37} 4079.
Table 1: Eigenvalues of the anharmonic oscillator $V(x) = (ix)^5$ calculated by means of the Riccati-Padé method (RPM) and the WKB expression of Bender and Boettcher [2] ($E_{n}^{BB}$)

| $n$ | RPM                      | $E_{n}^{BB}$          |
|-----|--------------------------|-----------------------|
| 0   | 1.908264578170777079714407742647568531562 | 1.771244715           |
| 1   | 8.58722083620722180027956616257834275867345 | 8.509035978           |
| 2   | 17.710809011731145002460444521074221024  | 17.65253759           |
| 3   | 28.595103311735974787298524540082589714  | 28.54706617           |

[5] Alvarez G 1995 *J. Phys. A* **28** 4589.

[6] Shin K C 2005 *J. Phys. A* **38** 6147.
Table 2: Eigenvalues of the anharmonic oscillator $V(x) = (ix)^5$ calculated by means of diagonalization (DM), numerical integration along the anti-Stokes lines (NI), and the leading term of the AEE [1]

| $n$ | DM            | NI            | $E^{NM}_n$  |
|-----|---------------|---------------|-------------|
| 0   | 1.16477040794341 | 1.164771     | 0.8906863480 |
| 1   | 4.36378436771211 | 4.363785     | 4.278845331  |
| 2   | 8.95516699824067 | 8.955167     | 8.876737420  |
| 3   | 14.4177548302741 | 14.417755   | 14.3514917   |
| 4   | 20.6101375100489 | 20.610138   | 20.55551587  |
| 5   | 27.4284077210062 | 27.428408   | 27.37969662  |
| 6   | 34.8037156407346 | 34.803715   | 34.75941365  |
| 7   | 42.6845638108818 | 42.684564   | 42.64372812  |
| 8   | 51.030837828189  | 51.030837   | 50.99281286  |
| 9   | 59.81014759020   | 59.810150   | 59.77445901  |
| 10  | 68.9956534721    | 68.995644   | 68.96194510  |
Figure 1: Some Stokes (dashed, blue) and anti-Stokes (solid, red) lines for the PT-symmetric oscillator $V(x) = (ix)^5$. The lines are located at (A): $\frac{\pi}{14}, \frac{13\pi}{14}, \frac{15\pi}{14}, \frac{17\pi}{14}, \frac{25\pi}{14}, \frac{27\pi}{14}$, (B): $\frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}, \frac{9\pi}{14}, \frac{11\pi}{14}, \frac{13\pi}{14}$
Figure 2: Some Stokes (dashed, blue) and anti-Stokes (solid, red) lines for the PT-symmetric oscillator \( V(x) = -(ix)^5 \). The lines are located at (A): \( \frac{15\pi}{14}, \frac{17\pi}{14}, \frac{19\pi}{14}, \frac{23\pi}{14}, \frac{25\pi}{14}, \frac{27\pi}{14} \). (B): \( \frac{\pi}{14}, \frac{3\pi}{14}, \frac{11\pi}{14}, \frac{13\pi}{14}, \frac{15\pi}{14}, \frac{27\pi}{14} \).