UNIQUENESS ON RECOVERY OF PIECEWISE CONSTANT CONDUCTIVITY AND INNER CORE WITH ONE MEASUREMENT

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Abstract. We consider the recovery of piecewise constant conductivity and an unknown inner core in inverse conductivity problem. We first show the unique recovery of the conductivity in a one layer structure without inner core by one measurement on any surface enclosing the unknown medium. Then we recover the unknown inner core in a one layer structure. We then show that in a two layer structure, the conductivity can be uniquely recovered by using one measurement.

1. Introduction.

1.1. Background and motivation. Consider the conductivity problem in $\mathbb{R}^d$, $d = 2, 3$

$$
\begin{cases}
\nabla \cdot \left( \chi(\mathbb{R}^d \setminus B) + \sigma \chi(B) \right) \nabla u = 0 \quad \text{in} \quad \mathbb{R}^d, \\
u(x) - h(x) = O(|x|^{1-d}) \quad \text{as} \quad |x| \to \infty,
\end{cases}
$$

where $B$ is the inclusion embedded in $\mathbb{R}^d$ with a $C^{1,\alpha}$ ($0 < \alpha < 1$) smooth boundary, $\chi(B)$ (resp. $\chi(\mathbb{R}^d \setminus B)$) is the characteristic function of $B$ (resp. $\mathbb{R}^d \setminus B$). The parameter $\sigma$ stands for the conductivity of the inclusion, which is supposed to be different from the background conductivity 1. We also call $\sigma$ the contrast. The function $h$ is a harmonic function in $\mathbb{R}^d$ representing the background electrical potential, and $u$ represents the perturbed electrical potential.

The inverse conductivity problem can be defined as finding the inclusion $B$ (and its conductivity $\sigma$) given $h$ and boundary measurement $u|_{\partial \Omega}$. There are lots of existing works on recovery of the inclusion $B$ by using finite measurements or infinitely many measurements. The global uniqueness results with finite many measurements

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are only obtained when the inclusion is restricted to convex polyhedrons and balls in three-dimensional space and polygons and disks in the plane (see [8, 13, 16, 17, 23] and the references therein). For uniqueness of finding the inclusion with infinitely many measurements, there is a well-established theory. We refer to survey papers and books [15, 24, 25, 26, 27]. We also refer to [3, 1, 9, 10, 14, 20, 22] for uniqueness results in recovering of obstacles in acoustic scattering problems.

In previous works on recovery the inclusion by one measurement, they all assume that the contrast \( \sigma \) is a given constant in advance. The main focus of those works is on how to recover the shape of the inclusion. As far as now, only few special types of inclusion, like disk and ball, polyhedral and polygon, has been proved to be reconstructed by using one measurement. In this paper, instead of considering the recovery of the shape, we consider the recovery of the piecewise constant conductivity together with the inner core. We mainly use the Green function theory, transmission condition and unique continuation theorem to prove the uniqueness by using one measurement. We first prove that the conductivity in a one layer structure without inner core can be recovered by using one measurement. Then we prove that the unknown inner core in a one layer structure can also be recovered by using one measurement. Finally, we show that in a two layer structure, the conductivity can be uniquely recovered by using one measurement. The method in fact can be extended to the recovery of piecewise constant material parameters and obstacle in acoustic scattering problem and electromagnetic scattering problem. We mention that our method can also be used to recover piecewise constant conductivities in multi-layered structures, which will be our future work. On one hand, this method may be used to construct the generalized polarization tensors vanishing structures, which were widely studied in [5, 6, 7], for enhancement of approximating cloaking effects. On the other hand, the method can also be used for devising nano-shell structures in the application related to plasmon resonances [11].

1.2. Integral representation. Let \( \Gamma(x) \) be the fundamental solution to the Laplacian

\[
\Gamma(x) = \begin{cases}
\frac{1}{2\pi} \ln|x|, & d = 2, \\
-\frac{1}{4\pi}|x|^{-1}, & d = 3.
\end{cases}
\]

In what follows, we denote by \( S_B : H^{-1/2}(\partial B) \to H^1(\mathbb{R}^d \setminus \partial B) \) the single layer potential operator given by

\[
S_B[\phi](x) := \int_{\partial B} \Gamma(x - y)\phi(y)dy, \quad x \in \mathbb{R}^d \setminus \partial B,
\]

and the double layer potential \( D_B : H^{1/2}(\partial B) \to H^1(\mathbb{R}^d \setminus \partial B) \) given by

\[
D_B[\varphi](x) := \int_{\partial B} \frac{\partial}{\partial n_y} \Gamma(x - y)\varphi(y)dy, \quad x \in \mathbb{R}^d \setminus \partial B,
\]

and \( K_B : H^{1/2}(\partial B) \to H^{1/2}(\partial B) \) the Neumann-Poincaré operator

\[
K_B[\phi](x) := \text{p.v.} \int_{\partial B} \frac{\partial \Gamma(x - y)}{\partial n_y} \phi(y)dy,
\]

where \( \text{p.v.} \) stands for the Cauchy principle value. In (1.4) and (1.5), \( \nu \) signifies the exterior unit normal vector to \( B \). It is well-known that the single layer potential
$S_B$ is continuous across $\partial B$. Furthermore, the single layer potential operator $S_B$ and the double layer potential operator $D_B$ satisfy the trace formula

$$\frac{\partial}{\partial \nu} S_B[\phi]_\pm = (\pm \frac{1}{2} I + (K_B)^*)[\phi] \quad \text{on} \quad \partial B,$$

(1.6)

$$D_B[\varphi](x) \mid_{\pm} = (\mp \frac{1}{2} I + K_B)[\varphi] \quad \text{on} \quad \partial B,$$

(1.6)

where $(K_B)^*$ is the adjoint operator of $K_B$. Here and throughout this paper, the subscripts $\pm$ indicate the limits from the outside and inside of a given inclusion $B$, respectively. We mention that for Lipschitz domain $B$, the eigenvalues of $K_B$ lies in $(-1/2, 1/2]$, where for $C^{1,\alpha}$ $(0 < \alpha < 1)$ domain $B$ the corresponding eigenfunction associated with eigenvalue 1/2 is 1, i.e., $K_B[1] = 1/2$. We refer to [4, 12, 18] for relevant discussion. In what follows, we suppose that all bounded domains which are mentioned in the sequel are $C^{1,\alpha}$ smooth domains. We mention that if $B$ is a ball then $K_B^*$ is a self-adjoint operator. On the other hand, if $B$ is not a ball then $K_B^*$ is generally not a self-adjoint operator, but $S_B K_B^*$ is self-adjoint and satisfies the Calderón’s identity

$$S_B K_B^* = K_B S_B.$$

(1.7)

In the following let $L^2_0(\partial B)$ be the set of functions which are in $L^2(\partial B)$ and with zero average on $\partial B$. We recall that $-S_B$ is a positive definite operator on $L^2(\partial B)$ for $d = 3$ and on $L^2_0(\partial B)$ for $d = 2$ (see, e.g. [4] for details). Furthermore, there exists a self-adjoint operator $A_B : L^2(\partial B) \rightarrow L^2(\partial B)$ such that

$$A_B \sqrt{-S_B} = \sqrt{-S_B K_B^*}.$$

(1.8)

The proof can be seen from the proof of Theorem 1 in [19] (see also (34) in [2]). In the sequel, we shall only consider the uniqueness results in three dimensional case. However, we want to point out that, most of the results also work in the two dimensional case, and some results only need to have some mild modifications in the two dimensional case.

2. **Unique recovery results.**

2.1. **One-layer medium.** Suppose that $\sigma = k$ in $B$, where $k$ is a constant, and $B$ is a simply connected inclusion which has $C^{1,\alpha}$ $(0 < \alpha < 1)$ smooth boundary. Note that (1.1) is equivalent to

$$\begin{cases}
\Delta u = 0 & \text{in} \quad \mathbb{R}^3 \setminus \partial B, \\
\frac{\partial u}{\partial \nu} \bigg\rvert_+ = k \frac{\partial u}{\partial \nu} \bigg\rvert_- & \text{on} \quad \partial B, \\
u \bigg\rvert_+ = u \bigg\rvert_- & \text{on} \quad \partial B, \\
\nu(x) - h(x) = O(|x|^{-1}) & \text{as} \quad |x| \to \infty.
\end{cases}
$$

(2.1)

It can be found (see, e.g., [4]) that the solution to (2.1) can be represented by

$$u = h + S_B[\phi] \quad \text{in} \quad \mathbb{R}^3,
$$

(2.2)

where $\phi$ has the form

$$\phi = \left(\lambda I - K_B^*\right)^{-1} \frac{\partial h}{\partial \nu},
$$

(2.3)

where $\lambda$ is defined by

$$\lambda := \frac{k + 1}{2(k - 1)}.
$$

(2.4)
We shall present some unique recovery results. Firstly we show the simplest example which has no core inside the inclusion $B$.

**Theorem 2.1.** Let $u_j$ be the solution to (1.1) with conductivity $k_j$, $j = 1, 2$, respectively. Let $Ω$ be a bounded domain enclosing $B$, i.e., $B ⊂ Ω$. If $u_1 = u_2$ on $∂Ω$, then $k_1 = k_2$.

Next, suppose $U ⊂⊂ B$ is an inner core inside the inclusion $B$, which has vanishing boundary conditions. In this case, the conductivity problem can be formulated as

\[
\begin{align*}
\Delta u &= 0, \quad \text{in } \mathbb{R}^3 \setminus (∂B ∪ ∂D), \\
\frac{∂u}{∂ν} &= \sigma^{(1)} \frac{∂u}{∂ν} - \sigma^{(2)} \frac{∂u}{∂ν}, \quad \text{on } ∂B, \\
\frac{∂u}{∂ν} &= \sigma^{(1)} \frac{∂u}{∂ν} - \sigma^{(2)} \frac{∂u}{∂ν}, \quad \text{on } ∂D, \\
u(x) - h(x) &= O(|x|^{-1}), \quad \text{as } |x| → ∞,
\end{align*}
\]

where $\mathcal{B}[u]$ stands for Dirichlet or Neumann boundary conditions. Under this circumstance, we also have the uniqueness result on recovering both the core $U$ and the contrast $k$.

**Theorem 2.2.** Let $u_j$ be the solution to (2.5) with conductivity $k$ and inner core $U_j$, $j = 1, 2$, respectively. Let $U_{12}$ be the unbounded connected component of $\mathbb{R}^3 \setminus U_1 \cup U_2$ and $U_{12} := \mathbb{R}^3 \setminus U_{12}$. If $\mathcal{B}[u]$ stands for the Neumann condition, then suppose that $U_{12}$ is still a $C^{1,α}$ smooth domain. Suppose that $h$ is not a constant in $\mathbb{R}^3$. Let $Ω$ be a bounded domain enclosing $B$, i.e., $B ⊂ Ω$. If $u_1 = u_2$ on $∂Ω$, then $U_1 = U_2$.

2.2. Two-layer medium. In this subsection, we suppose that $σ$ satisfies

\[
σ = \sigma^{(1)} \chi(B) + \sigma^{(2)} \chi(D),
\]

where $B$ and $D$ are simply connected inclusions which have $C^{1+α}$, $0 < α < 1$, smooth boundary, denoted by $∂B$ and $∂D$, respectively. Suppose $D ⊂⊂ B$. Then (1.1) is equivalent to

\[
\begin{align*}
\frac{∂^2 u}{∂x^2} + \frac{∂^2 u}{∂y^2} + \frac{∂^2 u}{∂z^2} &= 0, \quad \text{in } \mathbb{R}^3 \setminus (∂B ∪ ∂D), \\
\frac{∂u}{∂x} &= 0, \quad \text{on } ∂B, \\
\frac{∂u}{∂y} &= 0, \quad \text{on } ∂D, \\
u(x) - h(x) &= O(|x|^{-1}), \quad \text{as } |x| → ∞.
\end{align*}
\]

It can be seen that the solution to (2.7) can be represented by (see [4])

\[
u = h + \mathcal{S}_B[φ_1] + \mathcal{S}_D[φ_2], \quad \text{in } \mathbb{R}^3.
\]

By using the uniqueness of harmonic function in $D$ with the same boundary condition on $∂D$, the solution to (2.7) can then be represented by

\[
\begin{align*}
u = \begin{cases}
\begin{align*}
h + \mathcal{S}_B[φ_1] + \mathcal{S}_D[φ_2], & \text{in } \mathbb{R}^3 \setminus \overline{D} \\
\mathcal{S}_D[φ_3], & \text{in } \overline{D}
\end{align*}
\end{cases}
\end{align*}
\]

where $φ_1$, $φ_2$ and $φ_3$ satisfy

\[
\begin{align*}
\begin{cases}
\begin{align*}
-\frac{I}{2} + \mathcal{K}_B^*[φ_1] + \frac{∂}{∂ν}\mathcal{S}_D[φ_2] &= ν \cdot ∇(u - h)|_+, \quad \text{on } ∂B, \\
\frac{∂}{∂ν}\mathcal{S}_B[φ_1] + \left(\frac{I}{2} + \mathcal{K}_D^*[φ_2]\right)[φ_2] &= ν \cdot ∇(u - h)|_+, \quad \text{on } ∂D, \\
\left(-\frac{I}{2} + \mathcal{K}_D^*[φ_3]\right) &= ν \cdot ∇u|_-, \quad \text{on } ∂D,
\end{align*}
\end{cases}
\end{align*}
\]
and
\begin{equation}
(2.10) \quad h + S_B[\phi_1] + S_D[\phi_2] = S_D[\phi_3], \quad \text{on} \quad \partial D.
\end{equation}

The transmission condition further shows that
\begin{equation}
(2.11) \quad \sigma^{(1)} \nu \cdot \nabla u_- = \nu \cdot \nabla u_+, \quad \text{on} \quad \partial B.
\end{equation}

Before proceeding, we make some remarks on the notation of single layer potential \(S_B[\phi]\), for any \(C^{1,\alpha}\) smooth domain \(B\) and density \(\phi \in L^2(\partial B)\). As we have mentioned that \(S_B\) is defined on \(\mathbb{R}^3 \setminus \partial B\), and is continuous across \(\partial B\), we shall still use \(S_B[\phi]\) to denote for the trace of \(S_B[\phi]\) on \(\partial B\). Recall that \(S_B\) is invertible from \(L^2(\partial B)\) to \(H^1(\partial B)\), we denote by \(S_B^{-1}\) the inverse of \(S_B\) from \(H^1(\partial B)\) to \(L^2(\partial B)\).

We stress that \(S_B S_B^{-1}[\phi]\) is generally not \(\phi\) if it does not denote for the trace on \(\partial B\).

**Definition 2.1.** For any \(\varphi \in H^1(\partial D)\), define
\begin{equation}
(2.12) \quad \begin{aligned}
& w^{(1)}(\varphi) := (-I + S_B S_B^{-1} S_D S_D^{-1}) \left(-\frac{I}{2} + K_D\right)[\varphi], \quad \text{on} \quad \partial D, \\
& w^{(2)}(\varphi) := \left(\frac{I}{2} + K_D - S_B S_B^{-1} D_D\right)[\varphi], \quad \text{on} \quad \partial D, \\
& w^{(3)}(\varphi) := -S_B^{-1} S_D S_D^{-1} \left(-\frac{I}{2} + K_D\right)[\varphi] - S_B^{-1} \left(-\frac{I}{2} + K_B\right)[w^{(1)}]|_{\partial B}, \\
& w^{(4)}(\varphi) := S_B^{-1} D_D[\varphi] - S_B^{-1} \left(-\frac{I}{2} + K_B\right)[w^{(2)}]|_{\partial B}.
\end{aligned}
\end{equation}

We want to stress that in (2.12), the formulae should be understood in the domain of the related operator. For example, the operators \(S_B S_B^{-1} S_D S_D^{-1}\) appears in the first equation of (2.12) is short for
\[\{S_B S_B^{-1}[(S_D S_D^{-1})]|_{\partial B}]\]|_{\partial D}.

In other words, for \(\varphi \in H^1(\partial D)\), \(S_D^{-1}[\varphi] \in L^2(\partial D)\) and \(S_D S_D^{-1}[\varphi] \in H^{3/2}(\mathbb{R}^3 \setminus D)\) and we restrict the values on \(\partial B\), then \(S_B^{-1} S_D S_D^{-1}[\varphi] \in L^2(\partial B)\). Finally by restricting \(S_B S_B^{-1} S_D S_D^{-1}[\varphi]\) on \(\partial D\), we obtain the desired result.

**Theorem 2.3.** Let \(u_j\) be the solution to (1.1) with conductivity \(\sigma_j\), \(j = 1, 2\), in the following form
\begin{equation}
(2.13) \quad \sigma_j = \sigma_j^{(1)} \chi(B) + \sigma_j^{(2)} \chi(D),
\end{equation}
respectively. Let \(\Omega\) be a bounded domain enclosing \(B\), i.e., \(B \subset \Omega\). Suppose \(u_1 = u_2\) on \(\partial \Omega\). If there exist \(\varphi_1, \varphi_2, \varphi_3 \in H^1(\partial D)\) such that the matrix \(M\) defined by
\begin{equation}
(2.14) \quad M := \begin{pmatrix}
T_1^{(1)} & T_2^{(1)} & T_3^{(1)} \\
T_1^{(2)} & T_2^{(2)} & T_3^{(2)} \\
T_1^{(3)} & T_2^{(3)} & T_3^{(3)}
\end{pmatrix},
\end{equation}
is invertible then \(\sigma_1 = \sigma_2\). The elements \(T_i^{(n)}\), \(i, n = 1, 2, 3\) are defined by
\begin{equation}
(2.15) \quad \begin{aligned}
&T_1^{(n)} := \int_{\partial B} w^{(1)}(\varphi_n) \nu \cdot \nabla u_1|_+, \quad T_2^{(n)} := \int_{\partial B} w^{(2)}(\varphi_n) \nu \cdot \nabla u_1|_+, \\
&T_3^{(n)} := \int_{\partial B} w^{(3)}(\varphi_n) u_1|_+, \quad n = 1, 2, 3,
\end{aligned}
\end{equation}
where \(w^{(1)}\), \(w^{(2)}\) and \(w^{(3)}\) are defined in (2.12).
We mention that the assumption of matrix $M$ defined in (2.14) is indeed restriction of measurement $u$ on $\partial \Omega$, or essentially the restriction on the background potential $h$. To ensure the uniqueness of recovering the two-layer piecewise constant conductivity from one measurement, some restriction on $h$ is quite necessary. In fact, if $h$ is chosen specially, then there is possibility that two different piecewise constant conductivities can generate the same boundary data. One important application of such phenomenon is for enhancement of near cloaking effect (see [5] for details).

3. **Proof of the main results.** In this section, we shall prove the main theorems stated in the last section.

**Proof of Theorem 2.1.** Suppose $k_1 \neq k_2$. Firstly, from unique continuation of harmonic function, it is easy to see that $u_1 = u_2$ in $\mathbb{R}^3 \setminus \overline{B}$. Thus there holds

$$
\frac{\partial u_1}{\partial \nu} \bigg|_+ = \frac{\partial u_2}{\partial \nu} \bigg|_+ \quad \text{on} \quad \partial B.
$$

In what follows, we define $g := \frac{\partial u_1}{\partial \nu} \bigg|_+$ and $f := \frac{\partial h}{\partial \nu}$ on $\partial B$. By using transmission condition on $\partial B$, there holds

$$
k_1 \frac{\partial u_1}{\partial \nu} \bigg|_+ = g = k_2 \frac{\partial u_2}{\partial \nu} \bigg|_- \quad \text{on} \quad \partial B.
$$

By using (2.2) and the trace formula (1.6) one can find that

$$
g = k_j f + k_j \left(\frac{I}{2} + K_B^* (\lambda_j I - K_B^*)^{-1} \right) [f] = \frac{k_j}{k_j - 1} (\lambda_j I - K_B^*)^{-1} [f],
$$

where $j = 1, 2$. Then by using (3.2) and (3.3) there holds

$$
(\lambda_1 - \lambda_2) g = \left( \frac{k_1}{k_1 - 1} - \frac{k_2}{k_2 - 1} \right) [f], \quad \text{on} \quad \partial B.
$$

By straightforward computation one has

$$
k_2 - k_1 \frac{k_2 - k_1}{(k_1 - 1)(k_2 - 1)} g = \frac{k_2 - k_1}{(k_1 - 1)(k_2 - 1)} f,
$$

and thus by assumption $k_1 \neq k_2$ again one has $g = f$. By definition of $g$ and $f$, one immediately has

$$
\left( \frac{I}{2} + K_B^* (\lambda_j I - K_B^*)^{-1} \right) [f] = 0.
$$

Recall that $\lambda I + K_B^*$ is invertible on $L^2(\partial B)$ for any $|\lambda| > 1/2$ and $\lambda = 1/2$, thus from (3.6) one has $\frac{\partial h}{\partial \nu} = f = 0$. Due to the fact that $h$ is harmonic function in $\mathbb{R}^3$ one has $h = c$ and so $u_1 = u_2 = 0$, which is a contradiction. Thus $k_1 = k_2$.

The proof is complete.

**Proof of Theorem 2.2.** Without loss of generality we suppose that $U^* := U_{12} \setminus \overline{U_1}$ and $U^*$ is nonempty. We mention that the method is similar to the obstacle recovery result in [21]. We first consider the Neumann condition case, i.e., $B[u] = \frac{\partial u}{\partial \nu}$ in (2.5).

Since $k_1 = k_2$ in $B \setminus U_{12}$ and $u_1 = u_2$ on $\partial B$, by unique continuation theorem one
can find that \( u_1 = u_2 \) and \( \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \) on \( \partial U_{12} \). Together with the conditions that \( \frac{\partial u_1}{\partial \nu} = 0 \) on \( \partial U^* \cap \partial U_1 \) and \( \frac{\partial u_2}{\partial \nu} = 0 \) on \( \partial U^* \cap \partial U_2 \), one can see that \( u_1 \) satisfies

\[
\begin{cases}
\Delta u_1 = 0, & \text{in } U^*, \\
\frac{\partial u_1}{\partial \nu} = 0, & \text{on } \partial U^*.
\end{cases}
\]

(3.7)

From (3.7) one immediately has that \( u_1 = C \) in \( U^* \). By using unique continuation theorem one thus has \( u_1 = C \) in \( \mathbb{R}^3 \setminus \overline{U_{12}} \), which is a contradiction. Thus \( U_1 = U_2 \).

For Dirichlet boundary condition, one can follow exactly the same strategy one can prove that \( u_1 \) is harmonic in \( U^* \) and has vanishing boundary condition on \( \partial U^* \), by using min-max value principle, where one can avoid the smoothness restriction on \( U_{12} \), one can thus show that \( u_1 = 0 \) in \( \mathbb{R}^3 \setminus \overline{U_{12}} \), which makes a contradiction and thus \( U_1 = U_2 \).

The proof is complete. \( \square \)

**Proof of Theorem 2.3.** Since \( u_1 = u_2 \) on \( \partial \Omega \), by using unique continuation one can easily find that \( u_1 = u_2 \) in \( \mathbb{R}^3 \setminus \overline{B} \). Define \( f := u_1|_{\partial B} = u_2|_{\partial B} \) and \( g := \nu \cdot \nabla u_1|_+ = \nu \cdot \nabla u_2|_+ \) on \( \partial B \). Then \( u_1 \) can be represented by (2.8). Let \( H = L^2(\partial B) \times L^2(\partial D) \) and the Neumann-Poincaré-type operator \( K^* : H \to H \) by

\[
K^* := \left[ \begin{array}{cc} -K_B & -\frac{\partial}{\partial \nu} S_D \\ \frac{\partial}{\partial \nu} S_D & K_D \end{array} \right].
\]

(3.8)

It is shown in [2] that the \( L^2 \)-adjoint of \( K^* \), \( K \), is given by

\[
K := \left[ \begin{array}{cc} -K_B & D_D \\ -D_B & K_D \end{array} \right],
\]

(3.9)

Then from the first two equations in (2.9), (2.10) and (2.11) one can get

\[
\left( \frac{1}{2} I + K^* \right) [p] = (\sigma_1^{(1)})^{-1} f - h,
\]

(3.10)

where \( p := (\phi_1, \phi_2)^T \), \( f := (-g, \sigma_2^{(2)} \tilde{g})^T \), where \( \tilde{g} := \nu \cdot \nabla u_1 \) on \( \partial D \) and \( h := (-\nu \cdot \nabla h|_{\partial B}, \nu \cdot \nabla h|_{\partial D})^T \). \( I \) is the identity operator on \( H \). The superscript \( T \) means transpose of a vector. Define

\[
S := \left[ \begin{array}{cc} S_B & S_D \\ S_B & S_D \end{array} \right],
\]

(3.11)

then there holds the Calderón’s identity \( SK^* = KS \) (see [2]). By applying \( S \) on both sides of (3.10) and using the Calderón’s identity, one thus has

\[
\left( \frac{1}{2} I + K \right) S[p] = (\sigma_1^{(1)})^{-1} S[f] - S[h].
\]

(3.12)

Similarly, by applying \( S_D \) on both sides of the third equation in (2.9) one has

\[
\left( -\frac{I}{2} + K_D \right) S_D[\phi_3] = S_D[\nu \cdot \nabla u_1|_-].
\]

(3.13)

By definition of \( f \) and (2.10), one additionally has

\[
S[p] = (f - h|_{\partial B}, S_D[\phi_3] - h)^T = ((f - h)|_{\partial B}, (u_1 - h)|_{\partial D})^T.
\]

(3.14)
By using Green’s formula and transmission condition there holds for $x \in \partial D$ that
\[
S_B[\nu \cdot \nabla h](x) - D_B[h](x) = S_D[\nu \cdot \nabla h](x) - D_D[h](x).
\]
(3.15)

Similarly, for $x \in \partial B$, one has
\[
S_D[\nu \cdot \nabla h](x) - D_D[h](x) = S_B[\nu \cdot \nabla h](x) - D_B[h](x).
\]
(3.16)

Combining (3.15) and (3.16), there holds
\[
(\frac{1}{2} I + K)[(h|_{\partial B}, h|_{\partial D})^T] = S[h].
\]
(3.17)

Thus (3.12) and (3.13) can be rewritten by
\[
S^{-1}\left(\frac{1}{2} I + K\right) \begin{bmatrix} f \\ u_1 \end{bmatrix} = (\sigma_1^{(1)})^{-1} \begin{bmatrix} \frac{g}{\sigma_1^{(2)}} S_D^{-1} \left(- \frac{I}{2} + \cal{K}_D\right) \end{bmatrix} \begin{bmatrix} u_1 \end{bmatrix}.
\]
(3.18)

Next, suppose $\varphi \in L^2_0(\partial D)$ and define $\xi_1 \in L^2(\partial D)$ by
\[
\xi_1 := (\sigma_1^{(1)})^{-1} \sigma_1^{(2)} (-I + S_B S^{-1}_D S_D^{-1}) \left(- \frac{I}{2} + \cal{K}_D\right) \varphi
\]
\[
+ \left(\frac{I}{2} + \cal{K}_D - S_B S^{-1}_D \cal{D}_D\right) \varphi,
\]
(3.19)

Let $w_1$ be a harmonic function in $B$ which satisfies $w_1|_{\partial D} = \xi_1$. Then by straightforward calculations, one has
\[
S^{-1}\left(\frac{1}{2} I + K\right) \begin{bmatrix} w_1|_{\partial D} \\ \varphi \end{bmatrix} = \begin{bmatrix} l_1 \\ \xi_1 \end{bmatrix},
\]
(3.20)

where $\xi_1$ and $l_1$ are given by
\[
\xi_1 = (\sigma_1^{(1)})^{-1} \sigma_1^{(2)} S_D^{-1} \left(- \frac{I}{2} + \cal{K}_D\right) \varphi, \quad \text{on} \quad \partial D,
\]
\[
l_1 = -S_B^{-1}[(S_D[\xi_1] - \cal{D}_D[\varphi])|_{\partial B}] - S_B^{-1} \left(- \frac{I}{2} + \cal{K}_B\right) |w_1|_{\partial B}, \quad \text{on} \quad \partial B.
\]
(3.21)

By taking inner product of both sides of (3.18) with $(w|_{\partial B}, \varphi)$ and using (3.20), and some straightforward calculations one then has
\[
(\sigma_1^{(1)})^{-1} \int_{\partial B} w_1 g - \int_{\partial B} l_1 f = 0.
\]
(3.22)

By using Definition 2.1 one can further decompose $w_1$ and $l_1$ by
\[
w_1 = (\sigma_1^{(1)})^{-1} \sigma_1^{(2)} w^{(1)} + w^{(2)},
\]
and
\[
l_1 = (\sigma_1^{(1)})^{-1} \sigma_1^{(2)} w^{(3)} + w^{(4)}.
\]
Recovery of conductivity and core respectively. Thus (3.22) can be rewritten by
\[
(\sigma_1^{(1)})^{-1} \sigma_1^{(2)} \left( (\sigma_1^{(1)})^{-1} \int_{\partial B} w^{(1)} g - \int_{\partial B} l^{(1)} f \right) \\
+ (\sigma_1^{(1)})^{-1} \int_{\partial B} w^{(2)} g - \int_{\partial B} l^{(2)} f = 0.
\]
(3.23)

Similarly, by considering \( u_2 \) in the same way one has
\[
(\sigma_2^{(1)})^{-1} \sigma_2^{(2)} \left( (\sigma_2^{(1)})^{-1} \int_{\partial B} w^{(1)} g - \int_{\partial B} l^{(1)} f \right) \\
+ (\sigma_2^{(1)})^{-1} \int_{\partial B} w^{(2)} g - \int_{\partial B} l^{(2)} f = 0.
\]
(3.24)

By subtracting (3.23) from (3.24) and some elementary calculations one finally obtains
\[
t_1 \int_{\partial B} w^{(1)} g + t_2 \int_{\partial B} l^{(1)} f + t_3 \int_{\partial B} w^{(2)} g = 0,
\]
(3.25)

where
\[
t_1 = (\sigma_1^{(1)} \sigma_2^{(1)})^{-1} \left( (\sigma_2^{(1)} - \sigma_2^{(2)}) (\sigma_2^{(1)})^2 - (\sigma_1^{(1)} - \sigma_1^{(2)}) (\sigma_2^{(1)} + \sigma_1^{(1)}) \right),
\]
\[
t_2 = (\sigma_2^{(1)} - \sigma_1^{(2)}) (\sigma_2^{(1)} - \sigma_1^{(1)}) \sigma_2^{(2)},
\]
\[
t_3 = (\sigma_2^{(1)} - \sigma_1^{(1)}).
\]
(3.26)

By using the assumption that \( M \), defined in (2.14) and (2.15), is invertible, one thus has \( t_1 = t_2 = t_3 = 0 \) and so \( \sigma_1 = \sigma_2 \).

The proof is complete. \( \square \)

4. Remarks and future works. In this paper, we consider the inverse medium problem of recovering the unknown piecewise constant conductivity and inner core. We have presented the uniqueness for such recovery. We want to mention that the uniqueness does not just work for two-layer medium case, but also work for piecewise constant case. This will be our forthcoming work. We also stress that the uniqueness can also be, in some intuitive way, used to numerically recover the contrast and the inner core. In fact, from (3.3) we have
\[
(\lambda I - K_B^*) \left[ \frac{\partial u}{\partial \nu} \right]_+ = \frac{k}{k-1} \frac{\partial h}{\partial \nu}, \quad \text{on} \quad \partial B.
\]
(4.1)

If \( B \) is a disk or a ball, then it is quite easy to recover \( k \) by using spectral expansion of \( K_B^* \). Otherwise, suppose \( \hat{h} \in H^{1/2}(\partial B) \) such that
\[
\int_{\partial B} \frac{\partial h}{\partial \nu} \hat{h} = 0, \quad \text{and} \quad \int_{\partial B} \frac{\partial u}{\partial \nu} \bigg|_+ \hat{h} \neq 0.
\]
Then by multiplying both sides of (4.1) with \( \hat{h} \) and integrating on \( \partial B \), there holds
\[
\lambda = \int_{\partial B} \frac{\partial u}{\partial \nu} \bigg|_+ K_B \hat{h} / \int_{\partial B} \frac{\partial u}{\partial \nu} \bigg|_+ \hat{h},
\]
and \( k \) can be recovered. For recovery of the inner core, the numerical method can be applied by using the same scheme as in [21].
For two-layer structure, from (3.23) one can derive

\[
(\sigma^{(1)})^{-2} \sigma^{(2)} \int_{\partial B} w^{(1)} g - (\sigma^{(1)})^{-1} \sigma^{(2)} \int_{\partial B} w^{(3)} f \\
+ (\sigma^{(1)})^{-1} \int_{\partial B} w^{(2)} g = \int_{\partial B} w^{(4)} f.
\]

By choosing \( \varphi_1, \varphi_2, \varphi_3 \in H^1(\partial D) \) such that the matrix \( M \) defined in (2.14) is invertible one can thus uniquely recover \( \sigma^{(1)} \) and \( \sigma^{(2)} \).

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**REFERENCES**

[1] G. Alessandrini and L. Rondi, Determining a sound-soft polyhedral scatterer by a single far-field measurement, *Proc. Amer. Math. Soc.*, 35 (2005), 1685–1691.

[2] H. Ammari, G. Ciraolo, H. Kang, H. Lee and G. Milton, Spectral analysis of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance, *Arch. Ration. Mech. Anal.*, 208 (2013), 667–692.

[3] H. Ammari and H. Kang, Reconstruction of Small Inhomogeneities from Boundary Measurements Lecture Notes in Mathematics, 1846. Springer-Verlag, Berlin Heidelberg, 2004.

[4] H. Ammari and H. Kang, *Polarization and Moment Tensors With Applications to Inverse Problems and Effective Medium Theory*, Applied Mathematical Sciences, Springer-Verlag, Berlin Heidelberg, 2007.

[5] H. Ammari, H. Kang, H. Lee and M. Lim, Enhancement of near cloaking using generalized polarization tensors vanishing structures. Part I: The conductivity problem, *Comm. Math. Phys.*, 317 (2013), 253–266.

[6] H. Ammari, H. Kang, H. Lee and M. Lim, Enhancement of near cloaking. Part II: The Helmholtz equation, *Comm. Math. Phys.*, 317 (2013), 485–502.

[7] H. Ammari, H. Kang, H. Lee, M. Lim and Y. Sanghyeon, Enhancement of near cloaking for the full Maxwell equations, *SIAM J. Appl. Math.*, 73 (2013), 2055–2076.

[8] B. Barceló, E. Fabes and J. K. Seo, The inverse conductivity problem with one measurement: Uniqueness for convex polyhedra, *Proc. Am. Math. Soc.*, 122 (1994), 183–189.

[9] E. Blåsten and H. Liu, Recovering piecewise constant refractive indices by a single far-field pattern, preprint, arXiv:1705.00815

[10] D. Colton and B. D. Sleeman, Uniqueness theorems for the inverse problem of acoustic scattering, *IMA J. Appl. Math.*, 31 (1983), 253–259.

[11] Y. Deng, X. Fang and J. Li, Plasmon resonance and heat generation in nanostructures, *Math. Method Appl. Sci.*, 38 (2015), 4663–4672.

[12] L. Escauriaza, E. B. Fabes and G. Verchota, On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries, *Proc. Amer. Math. Soc.*, 115 (1992), 1069–1076.

[13] A. Friedman and V. Isakov, On the uniqueness in the inverse conductivity problem with one measurement, *Indiana Univ. Math. J.*, 38 (1989), 553–579.

[14] G. Hu, M. Salo and E. V. Vesalainen, Shape identification in inverse medium scattering, *SIAM J. Math. Anal.*, 48 (2016), 152–165.

[15] V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer, New York, 1998.

[16] V. Isakov and J. Powell, On the inverse conductivity problem with one measurement, *Inverse Problem*, 6 (1990), 311–318.
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[17] H. Kang and J. K. Seo, Inverse conductivity problem with one measurement: Uniqueness of balls in $\mathbb{R}^3$, SIAM J. Appl. Math., 59 (1990), 1533–1539.

[18] O. D. Kellogg, Foundations of Potential Theory, Reprint from the first edition of 1929. Die Grundlehren der Mathematischen Wissenschaften, Band 31 Springer-Verlag, Berlin-New York, 1967.

[19] D. Khavinson, M. Putinar and H. S. Shapiro, Poincaré’s variational problem in potential theory, Arch. Ration. Mech. Anal., 185 (2007), 143–184.

[20] J. Li, H. Liu, Z. Shang and H. Sun, Two single-shot methods for locating multiple electromagnetic scatterers, SIAM J. Appl. Math., 73 (2013), 1721–1746.

[21] H. Liu and X. Liu, Recovery of an embedded obstacle and its surrounding medium from formally determined scattering data, Inverse Problems, 33 (2017), 065001, 20pp.

[22] H. Liu and J. Zou, Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers, Inverse Problems, 22 (2006), 515–524.

[23] J. K. Seo, A uniqueness result on inverse conductivity problem with two measurements, J. Fourier Anal. Appl., 2 (1996), 227–235.

[24] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. of Math., 125 (1987), 153–169.

[25] J. Sylvester and G. Uhlmann, The Dirichlet to Neumann map and applications, Inverse Problems in Partial Differential Equations (Arcata, CA, 1989), SIAM, Philadelphia, (1990), 101–139.

[26] G. Uhlmann, Inverse boundary value problems for partial differential equations, Proceedings of the International Congress of Mathematicians, (Berlin, 1998) Documenta Mathematica, 3 (1998), 77–86.

[27] G. Uhlmann, Developments in inverse problems since Calderón’s foundational paper, Harmonic Analysis and Partial Differential Equations, M. Christ, C. Kenig and C. Sadosky, eds., University of Chicago Press, (1999), 295–345.

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