Duality and Recycling Computing in Quantum Computers

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Quantum computer possesses quantum parallelism and offers great computing power over classical computer [1, 2]. As is well-know, a moving quantum object passing through a double-slit exhibits particle wave duality. A quantum computer is static and lacks this duality property. The recently proposed duality computer has exploited this particle wave duality property, and it may offer additional computing power [3]. Simply put, it, a duality computer is a moving quantum computer passing through a double-slit. A duality computer offers the capability to perform separate operations on the sub-waves coming out of the different slits, in the so-called duality parallelism. Here we show that an n-dubit duality computer can be modeled by an \((n+1)\)-qubit quantum computer. In a duality mode, computing operations are not necessarily unitary. A n-qubit quantum computer can be used as an n-bit reversible classical computer and is energy efficient. Our result further enables a \((n+1)\)-qubit quantum computer to run classical algorithms in a \(O(2^n)\)-bit classical computer. The duality mode provides a natural link between classical computing and quantum computing. Here we also propose a recycling computing mode in which a quantum computer will continue to compute until the result is obtained. These two modes provide new tool for algorithm design. A search algorithm for the unsorted database search problem is designed.

In a duality computer, there exist two new computing gates in addition to the usual universal gates for quantum computers [3]. One such gate is the quantum wave divider (QWD), and a double-slit is an example of such gate. Suppose there is a complex Hilbert space \(H\), a QWD reproduces copies of the wave function with attenuated coefficient in \(m\) direct summed Hilbert spaces \(H^{\oplus m} = \sum_{i=1}^{m} \oplus H_i\), namely

\[
D_p(|\psi\rangle) = \sum_{i=1, m} \oplus p_i |\psi_i\rangle, \tag{1}
\]

where \(\sum_i p_i = 1\) and \(p_i\) is the strength of the sub-wave in the \(i\)-th slit, \(m\) is the number of slits. Another duality gate is the reverse of QWD, the quantum wave combiner (QWC), and it combines the sub-waves in \(H^{\oplus m}\) into a single Hilbert space \(H\),

\[
C(p_1 |\psi_1\rangle \oplus \cdots \oplus p_m |\psi_m\rangle) = \sum_{i=1}^{m} p_i |\psi_i\rangle. \tag{2}
\]

A duality computing process is described in the following

\[
|\psi\rangle \rightarrow \sum_i \oplus p_i |\psi_i\rangle \rightarrow \sum_i \oplus p_i U_i |\psi_i\rangle \rightarrow \sum_i \oplus p_i U_i |\psi\rangle. \tag{3}
\]

The general quantum gate \(\sum_i p_i U_i\), or duality gate, is no longer unitary. As a result, the final wave function in Eq. (3) is only part of a wave function. The power of duality computer depends sensitively on the result of measurement of part of a wave function. Three possibilities have been suggested [3]: 1) one will get a result immediately as if a whole wave function were measured; 2) one will get a result but with a longer time; 3) one sometimes gets a result and sometimes one does not. In the first two scenarios, a duality computer could solve NP-complete problems in polynomial time [3]. The mathematical description of duality computer with the first two scenarios has been given recently [4, 5]. In this work, we assume case 3, which comes out from the measurement postulate of quantum mechanics naturally.

A symmetric 2-routes duality computer has \(p_1 = p_2 = 1/2\). The complete wave function of a duality computer is \(|\psi\rangle = |\varphi\rangle|\kappa\rangle\) where \(|\varphi\rangle\) is the internal state and \(|\kappa\rangle\) is the center of mass translational motion wave function. When a QWD operation is performed, it changes the state to

\[
|\psi\rangle' = \sum_{i=1}^{2} \oplus \frac{1}{2} |\varphi\rangle|\kappa_i\rangle. \tag{4}
\]

One then performs different gate operations on the sub-waves,

\[
|\psi\rangle'' = \frac{1}{2} \sum_{i=1}^{2} \oplus U_i |\varphi\rangle|\kappa_i\rangle. \tag{5}
\]

Then a QWC operation is performed and changes the state to

\[
|\psi\rangle' = (p_1 U_1 + p_2 U_2) |\varphi\rangle|\kappa\rangle. \tag{6}
\]

A final measurement is performed on \(|\psi\rangle'\) to read-out the outcome. Under the 3rd assumption about partial measurement, the wave function should not renormalized, and a result to a conditional measurement is obtained only with some probability. Whereas in the other two scenarios, a result to a partial measurement is always obtained and the wave function should be renormalized.

The fundamental difference between duality computer and quantum computer is that duality gates need not be unitary. A quantum computer can not perform \(U_1 + U_2\), it can only perform \(U_1 U_2\) operation.

**Duality Mode in a Quantum Computer**—Now we give a quantum computer simulation of the duality computer. To simulate an \(n\)-dubit duality computer, we...
need an \((n + 1)\)-qubit quantum computer, and one qubit
is used as auxiliary qubit and \(n\) qubits are used as work
qubits. Let’s make the following correspondence
\[
|\varphi\rangle|\kappa\rangle_a \leftrightarrow |\varphi\rangle|0\rangle,
|\varphi\rangle|\kappa\rangle_d \leftrightarrow |\varphi\rangle|1\rangle,
\]
(7)
namely, when the auxiliary qubit is in \(|0\rangle\) (\(|1\rangle\)), it resem-
bles a duality computer sub-wave from the upper (lower)
slit. We ascribe the initial and final wave function of the
duality computer to the wave function when the auxiliary
is in state \(|0\rangle\). Thus, before the QWD, the state of duality
computer is \(|\varphi\rangle|0\rangle\), after the QWD, the wave-function
becomes
\[
|\varphi\rangle\frac{|0\rangle + |1\rangle}{\sqrt{2}},
\]
(8)
namely, the QWD is simulated by a W alsch-Hadamard
transformation in the auxiliary qubit. Thus the wave-
function describes the \(n\)-bit quantum computer simulta-
neeously in two slits, \(|0\rangle\) and \(|1\rangle\). Different gate operations
on different routes can be simulated using conditional
gates, as shown in Fig. 1.

![Diagram](image)

FIG. 1: An \(n\)-dubit duality computer can be simulated by an
\((n + 1)\) qubit quantum computer. The W alsch-Hadamard
gates can be replaced by other unitary gates to simulate more
complicated slits. One can also use more qubits as auxiliary
to simulate more slits.

Then we have
\[
\frac{U_0|\varphi\rangle|0\rangle + U_1|\varphi\rangle|1\rangle}{\sqrt{2}}.
\]
(9)
The quantum combiner operation is again a W alsch-
Hadamard transformation, after which the wave-function
becomes
\[
\frac{U_0 + U_1|\varphi\rangle|0\rangle + U_0 - U_1|\varphi\rangle|1\rangle}{2}.
\]
(10)
We make a measurement on condition that the auxiliary
qubit is \(|0\rangle\). The probability to obtain a result is
the square of the norm of the wave function in \(|0\rangle\) state,
namely \(P_0 = \langle\varphi|(U_0 + U_1)^\dagger(U_0 + U_1)|\varphi\rangle/4\). With proba-
bility \(1 - P_0\), the conditional measurement will not obtain
a result, and if this occurs, the state of the wave function
is collapsed in a normalized state \(N(U_0 - U_1)|\varphi\rangle|1\rangle\). In
this simulation, the conditional measurement simulates a
measurement on a part of a wave function. The result is
clear from the basic postulate in quantum mechanics.

The scheme can be generalized by replacing the two
W alsch-Hadamard gates with other unitary gates to sim-
ulate more complicated slits, such as asymmetric slits,
slits with different phases and so on. By adding more
auxiliary qubits, one can simulate a duality computer with
more slits.

We now give the mathematical description for such du-
ality mode. Some of the result in Ref. \[2\] can be used
directly, though the results regarding QWD and QWC
need modification. Denoting the set of duality gates on
\(H\) as \(G(H)\), there are four theorems and corollaries re-
garding duality gates in Ref. \[3\] and we directly copy them
here. For proof of these results, see Ref. \[2\].

**Theorem 1.** The identity \(I_H\) is an extreme point of
\(G(H)\).

**Corollary 2.** The extreme points of \(G(H)\) are pre-
cisely the unitary operators in \(H\).

The theorem and corollary tell us that the duality gate
is unitary only when \(U_0 = U_1\), and in particular, when
\(U_0 = U_1 = I_H\), the duality gate is the identity gate.

Let \(B(H)\) be the set of bounded linear operators on \(H\)
and let \(\mathbb{R}^+G(H)\) be the positive cone generated by \(G(H)\),
that is
\[
\mathbb{R}^+G(H) = \{\alpha A : A \in G(H), \alpha \geq 0\}.
\]
(11)
Then the next theorem tells that any operator on \(H\) can
be simulated.

**Theorem 3.** If \(\text{dim } H < \infty\), then \(B(H) = \mathbb{R}^+G(H)\).

The next corollary is about normal operators.

**Corollary 4.** If \(\text{dim } H < \infty\), then \(A \in B(H)\) is normal
if and only if \(A = \alpha \sum p_i U_i\) where \(\alpha \geq 0\), \(p_i > 0\), \(\sum p_i = 1\)
and \(U_i\) are unitary operators that mutually commute.

We now expose the significance of duality mode in a
quantum computer. First, an \((n+1)\)-qubit quantum com-
puter can simulate an \(n\)-dubit duality computer. This
allows a quantum computer to perform any operation
in the \(n\)-qubit Hilbert space. Hence, classical algorithms
can be translated into quantum algorithms in this duality
mode. This is significant because this is not a direct use
of a quantum computer as a classical computer using a
qubit as a classical bit, and the qubit resource in duality
mode is exponentially small. For instance for an unsorted
database search problem with \(N = 2^n\) items, a classical
computer needs at least \(n2^n = Nn\) bits to express the
database. However in a quantum computer running in
duality mode, it needs only \((n + 1)\) qubits to express and
manipulate the database. This also provides a natural
bridge between classical and quantum computing. It was
suggested that future quantum computer may be used
as special-purpose processor to perform task that needs
quantum acceleration, and then the result is returned to
classical computer for further processing. Using this
duality mode, one may perform both tasks in a quantum
computer.

Secondly, the duality mode has provided a new av-
enance for algorithm design. Let’s study the unsorted
database search problem \[2\]. Quantum algorithms find
a marked item from an unsorted database with \(O(\sqrt{N})\)
steps \[2, 3, 5, 10\]. However these quantum algorithms are
not fixed point search algorithm, the success probability
is a periodic function of the number of searching steps. A

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\[\text{Corollary 4. If } \text{dim } H < \infty, \text{ then } A \in B(H) \text{ is normal if and only if } A = \alpha \sum p_i U_i \text{ where } \alpha \geq 0, p_i > 0, \sum p_i = 1 \text{ and } U_i \text{ are unitary operators that mutually commute.}
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recent fixed point quantum search algorithm uses $O(3^n)$ number of queries in a quantum computer [11]. Though the number of queries is more than $O(2^n)$ in a classical computer, it is still advantageous because it is an algorithm in a quantum computer using only $n$-qubit. Here we give a duality mode algorithm that uses only $O(2^n)$ steps while being still a quantum computer algorithm. This is a modified duality algorithm in Ref. [3]. One starts from the evenly distributed state, and switch onto the duality mode to give

$$
\frac{1}{\sqrt{2N}} \sum_{i=0}^{N-1} |i\rangle (|0\rangle + |1\rangle).
$$

(12)

Performing on the upper slit the following gate operation

$$
\frac{1}{\sqrt{2N}}(|0\rangle + \ldots + |\tau\rangle + \ldots) \rightarrow \frac{1}{\sqrt{2N}}(|0\rangle - \ldots + |\tau\rangle - \ldots),
$$

(13)

and leaves the lower-slit sub-state unchanged. Then recombine them through a Hadamard gate, we obtain

$$
\frac{1}{\sqrt{N}} |\tau\rangle |0\rangle + \frac{1}{\sqrt{N}} \sum_{i \neq \tau} |i\rangle |1\rangle.
$$

(14)

Then conditionally measure on the auxiliary state being in $|0\rangle$, one obtains $\tau$ with probability $1/N$. Conditional measurement is simply achieved by measuring the auxiliary qubit. If it is 0, then measure the working $n$ qubit to read out $\tau$. If it is 1, repeat the process again. Repeating this process $O(N = 2^n)$ number of times, one will get the desired result.

The algorithm can be further speeded up by first performing the quantum amplitude amplification a number of times before switching to duality mode. After $j$ iteration, the wave function becomes

$$
|\psi_j\rangle = \sin((2j + 1)\beta)|\tau\rangle + \cos((2j + 1)\beta)|c\rangle,
$$

(15)

where

$$
|c\rangle = \sqrt{\frac{1}{N-1}} \sum_{i \neq \tau} |i\rangle.
$$

(16)

Then switching to the duality mode, it gives for the wave function in the upper slit

$$
|\psi_u\rangle = \sin((2j + 1)\beta)|\tau\rangle.
$$

(17)

Conditionally measure it one gets $\tau$ with probability $\sin^2((2j + 1)\beta)$. Repeating this $O(1/\sin((2j + 1)\beta)$ times, the marked state will be found. When $j$ is small, the number of repetitions is about $N/(2j + 1)$. When $j$ approaches $\tau\sqrt{N}/4$, it finds the marked state with only single query. A schematic plot for $N = 2^{10}$ is given in Fig. 2. When one knows the number of marked state in an unsorted database, one can optimize the number of repetition. If one does not know this information, one can simply switch to the duality mode, and repeat the process until a result is obtained in the conditional measurement. This algorithm works also for an arbitrary database where the coefficient of each item is arbitrary. Using the recycling mode, the repetition process is performed automatically.

Thirdly, this formulation of duality computer has provided a way for error correction in duality computer. Quantum error correction has been solved successfully in quantum computer [12], this analogy easily gives the scheme of error correction in a duality computer. All the good quantum error codes, can be translated into duality computer.

**Recycling quantum computing**—In Eq. (14), the probability of obtaining a result in a conditional measurement is small. However it is different from an evenly distributed state $\frac{1}{\sqrt{N}} \sum_i |i\rangle$. When one measures the evenly distributed state, one always obtains a result, say $|x\rangle$, however the result to be $\tau$ has only a probability of $1/N$, afterwards the state collapses to the eigenstate of that measured eigenvalue. However, in Eq. (14), after the measurement, one has two possibilities: 1) The state collapses into state $|\tau\rangle$ with probability $1/N$; 2) No result is obtained, however the state has collapsed out from (14), and the state becomes

$$
\frac{1}{\sqrt{N-1}} \sum_{i \neq \tau} |i\rangle |1\rangle.
$$

This happens with probability $(N - 1)/N$. This state can be reused again as input after flipping the auxiliary qubit and apply a recovering unitary operation to the original input state. Then the calculating process is recycled again and again until a final result is obtained. Using this observation, we propose a recycling quantum computing mode as shown in Fig. 3.

Namely, before the conditional measurement, suppose the wave-function of the duality computer is

$$
\frac{U_0 + U_1}{2} |\psi\rangle |0\rangle + \frac{U_0 - U_1}{2} |\psi\rangle |1\rangle.
$$
FIG. 3: Schematic illustration of a recycling duality computation. After the auxiliary-qubit-conditioned measurement, if a result is obtained, the calculation is completed and the process is stopped. If no result is obtained, it destroys the state in |0⟩ auxiliary qubit, and leaves a state where the auxiliary is in state |1⟩. The state is restored to the input state by a unitary operation V and guided to the input end. The calculation process is repeated again and again until the final result is obtained.

After the conditional-measurement, if a result is obtained, the wave-function is collapsed, and the (U₀ + U₁)|φ⟩ result is read out. If no-result is obtained, then the state in |0⟩ collapses out, and the wave-function becomes

|ψ′⟩ = N′U₀ − U₁|φ⟩|1⟩,

where N′ is a renormalization constant. Then performing a unitary operation V on the n qubits, the initial input state is restored. Meanwhile flipping |1⟩ to |0⟩ in the auxiliary qubit. The (n + 1) qubits are guided into the beginning of the circuit, as input state. The process will continue until a result is read-out in the conditional measurement device.

This simulation of duality computer also provides a relativity view regarding quantum computer. In a duality computer, a double-slit is located statically, and a quantum computer is moving. However in the duality mode, a quantum computer is static, and the double-slit, the auxiliary qubit, changes to simulate the motion of the double-slit. To make a quantum computer moving is very difficult, however it is much easier to add one additional qubit to an n-qubit quantum computer.

In summary we have given a quantum computer realization of the duality computer. This realization itself serves as a new mode of quantum computing, the duality mode. This provides a way to run classical algorithm in quantum computers using much reduced qubit resources. It also provides a method for error correction in a duality computer. With conditional measurement, the recycling quantum computing mode is also proposed. These two modes provide new ways and flexibility in quantum algorithm designs.

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[1] Shor, P W. Algorithms for quantum computation: discrete logarithms and factoring. In: Proceedings of the Symposium on the Foundations of computer Science. New York: IEEE Computer Society Press, 1994, 124-134.
[2] Grover, L K, A fast quantum mechanical algorithm for database search. In: Proceedings of 28th Annual ACM Symposium on Theory of Computing. New York: ACM, 212-219 (1996).
[3] Long, G L, The general quantum interference principle and the duality computer. Commun. Theor. Phys. 45, 825-844 (2006).
[4] Wang, W Y, Shang, B, Wang, C and Long, G L, Prime factorization in the duality computer. Commun. Theor. Phys. 47, 471-473 (2007).
[5] Gudder, S, Duality quantum computers. Quantum Information Processing, 6 (1): 49-54, (2007).
[6] Long, G L, Mathematical theory of duality computer in the density matrix formalism. Quantum Information Processing, 6 (1): 49-54, (2007).
[7] Long, G L and Liu, Y, Search an unsorted database with quantum mechanics. Front. Comput. Sci. China, 1 (3): 247-271 (2007)
[8] Brassard, G, Hoyer, P, Mosca, M and Tapp, A, Quantum amplitude amplification and estimation. AMS Contemporary Mathematics Series Vol. 305, eds. S. J. Lomonaco and H. E. Brandt, AMS (Providence), p.53, (2002)
[9] Hoyer, P, Arbitrary phases in quantum amplitude amplification. Phys. Rev. A 62, 052304 (2001).
[10] Long, G L, Grover algorithm with zero failure rate, Phys. Rev. A 64, 022307 (2001).
[11] Grover, L K, Fixed-point quantum search, Phys. Rev. Lett. 95, 150501 (2005).
[12] Shor, P W, Scheme for reducing decoherence in quantum computer memory. Phys. Rev. A. 52, R2493 (1995).