Exotic Black Holes?

Carl H. Brans

Institute for Advanced Study
Princeton, NJ 08540

and

Physics Department
Loyola University
New Orleans, LA 70118
e-mail:brans@loynovm.bitnet

March 24, 2022

Abstract

Exotic smooth manifolds, $\mathbb{R}^2 \times \Theta S^2$, are constructed and discussed as possible space-time models supporting the usual Kruskal presentation of the vacuum Schwarzschild metric locally, but not globally. While having the same topology as the standard Kruskal model, none of these manifolds is diffeomorphic to standard Kruskal, although under certain conditions some global smooth Lorentz-signature metric can be continued from the local Kruskal form. Consequently, it can be conjectured that such manifolds represent an infinity of physically inequivalent (non-diffeomorphic) space-time models for black holes. However, at present nothing definitive can be said about the continued satisfaction of the Einstein equations. This problem is also discussed in the original Schwarzschild $(t, r)$ coordinates for which the exotic region is contained in a world tube along the time-axis, so that the manifold is spatially, but not temporally, asymptotically standard. In this form, it is tempting to speculate that the confined exotic region might serve as a source for some exterior solution. Certain aspects of the Cauchy problem are also discussed in terms of $\mathbb{R}_\Theta^4$ models which are “half-standard”, say for all $t < 0$, but for which $t$ cannot be globally smooth.

PACS: 04.20.Cv, 02.40.+m
The process of extending investigations of space-time structures from local to global has clearly been of great value to theoretical physics. Standard physical theories use a smooth manifold model for space-time, $M$, and define physical fields as cross sections of bundles over this manifold. As a smooth manifold $M$ is locally Euclidean, $\mathbb{R}^4$, in its topological and differentiable properties, so expressions of theories as restrictions on cross sections by the imposition of differential equations can always be expressed locally as if space-time were standard $\mathbb{R}^4$. However, from the fairly early days of general relativity, it has become clear that the global properties of $M$ could have significant impact on physical implications of a theory. This progression has continued to the present, from wormholes to topological defects. Until fairly recently it seemed that the only way to impose these non-trivial and physically interesting global properties on $M$ is through the use of non-trivial topology. Tacitly then, physics has been assuming that different space-time manifold models which are homeomorphic are necessarily also diffeomorphic. Since diffeomorphisms are commonly accepted as the mathematical representation of generalized coordinate transformations, the fundamental principle of general relativity then implies that such models (i.e. diffeomorphic ones) are physically equivalent. However, physics must now face the fact that topologically identical manifolds need not be physically equivalent, especially in the physically distinguished case of dimension four.

In this paper, the end-sum techniques of Gompf are used to construct and discuss manifolds, $\mathbb{R}^2 \times \Theta S^2$, which have the same topology, $\mathbb{R}^2 \times S^2$, as the Kruskal space-time model, but which cannot be diffeomorphic to it. Thus,

**Theorem:** On some smooth manifolds which are topologically $\mathbb{R}^2 \times S^2$, the standard Kruskal metric cannot be smoothly continued over the full range, $u^2 - v^2 < 1$.

In order to provide the background for this result, let us begin with a brief review of the relevant mathematical facts. The apparently innocuous question of whether or not the set of differentiable structures (modulo diffeomorphisms) on $\mathbb{R}^n$ is trivial has long been of mathematical interest. As of about ten years ago, this question had been settled in the expected affirmative for all $n \neq 4$, and probably most people expected the exceptional case $n = 4$ to ultimately resolve to the same conclusion. After all, there is certainly no
interesting topology in \( \mathbb{R}^4 \) to provide a basis for any other expectation. It was thus of considerable interest when the existence of counter-examples began to appear around 1982, \[2,3,4\]. Our paper \[5\] provides a brief survey of this problem and some conjectures on the possible physical implications of these results. In this paper, certain questions raised in \[5\] are at least partially answered.

Since the existence of non-trivial differentiable structures on topologically trivial spaces is so strikingly counter-intuitive, it is important to clarify several issues relating to differential topology. Specifically we must distinguish the case of merely different differentiable structures from non-diffeomorphic ones. The former are physically indistinguishable, but the latter are definitely not physically equivalent as space-time models. Consider a simple example, \( M \equiv \mathbb{R}^1 \), which is to be regarded as a point set with elements, \( p \), which happen to be the real numbers. \( M \) is turned into a topological manifold by imposing the standard real-number topology. Next, for physics, \( M \) must be supplied with a differentiable structure. This requires the definition of a coordinate patch structure. In this case, the standard structure \( D \), can be defined over the entire manifold by global coordinates, \( x(p) = p \). That is, the coordinates are simply the numerical values defining the point.\(^1\) \( M \) can be endowed with many other \( D \)'s. For example, let \( \bar{x}(p) = p^3 \). Clearly this structure, \( \bar{D} \), is not consistent with the first one since the map \( x \to \bar{x} = x^3 \) is not smooth at the origin. However, the homeomorphism, \( p \to p^{1/3} \) is actually a diffeomorphism, \( x \to \bar{x} = x \) when expressed in the local smooth coordinates. So, \( D \) and \( \bar{D} \), while different are actually diffeomorphic. Since diffeomorphisms are regarded as generalized coordinate transformations, this means that there is no new physics available in \( \bar{D} \) as compared to \( D \). In fact, it is easy to show that \( D \) on \( \mathbb{R}^1 \) is unique up to diffeomorphism, so any global vs local differences of physical significance can only be obtained by topological alterations to \( M \), for example replacing \( \mathbb{R}^1 \) by \( S^1 \).

However, precisely the opposite is the case for the surprising exotic structures. A smooth manifold homeomorphic to \( \mathbb{R}^4 \) but not diffeomorphic to it is called “exotic” (or “fake”) and denoted here by \( \mathbb{R}^4_{\Theta} \). Such a manifold

\(^1\)Technically \( D \) is the maximal atlas of all coordinate patch structures smoothly consistent with this one, but this issue is not significant in this paper.

\(^2\)In general, the subscript \( \Theta \) will indicate a non-standard object or process. So \( M \times_{\Theta} N \) means a smooth manifold which is the topological, but not smooth, cartesian product of the two manifolds.
consists of a set of points which can be globally topologically identified with the ordered set of four numbers, say \((t, x, y, z)\). While these may be smooth coordinates locally over some neighborhood, they cannot be globally continued as smooth functions. Furthermore, in no diffeomorphic image of this \(\mathbb{R}_\Theta^4\) can the global topological coordinates be extended as smooth beyond some compact set. This defining characteristic, which occurs only for dimension four, is in striking contrast to the case of \(\mathbb{R}^4\) discussed above. There the difference between the \(p\) and the \(p^3\) coordinates could be smoothed away by a diffeomorphism.

Also, note that certain \(\mathbb{R}_\Theta^4\) have the property that they contain compact sets which cannot themselves be contained in the interior of any smooth \(S^3\). Thus, for some \(R_0\), the topological three-sphere, \(t^2 + x^2 + y^2 + z^2 = R^2\), cannot be smooth if \(R > R_0\). This is illustrated in Figure 1. Notice that in Figures 1 through 5 one space dimension has been suppressed, so each point is actually a \(z\)-axis, while in Figure 6 two dimensions are suppressed and each point is an \(S^2\).

As interesting as these \(\mathbb{R}_\Theta^4\) are in their own right, a technique developed by Gompf allows the construction of a large topological variety of exotic four-manifolds, some of which would appear to have considerable potential for physics. Gompf’s “end-sum” process provides a straightforward technique for constructing an exotic version, \(M\), of any non-compact four-manifold whose standard version, \(M_0\), can be smoothly embedded in standard \(\mathbb{R}^4\). Recall that we want to construct \(M\) which is homeomorphic to \(M_0\), but not diffeomorphic to it. First construct a tubular neighborhood, \(T_0\), of a half ray in \(M_0\). \(T_0\) is thus standard \(\mathbb{R}^4 = [0, \infty) \times \mathbb{R}^3\). Now consider a diffeomorphism, \(\phi_0\) of \(T_0\) onto \(N_0 = [0, 1/2) \times \mathbb{R}^3\) which is the identity on the \(\mathbb{R}^3\) fibers. Do the same thing for some exotic \(\mathbb{R}_\Theta^4\) with the important proviso that it cannot be smoothly embedded in standard \(\mathbb{R}^4\). Such manifolds are known in infinite abundance \([1]\). Then construct a similar tubular neighborhood for this \(\mathbb{R}_\Theta^4\), \(T_1\), with diffeomorphism, \(\phi_1\), taking it onto \(N_1 = [1, 1/2) \times \mathbb{R}^3\). The desired exotic \(M\) is then obtained by forming the identification manifold structure

\[
M = M_0 \cup_{\phi_0} ([0, 1] \times \mathbb{R}^3) \cup_{\phi_1} \mathbb{R}_\Theta^4
\]  

(1)

The techniques of forming tubular manifolds and defining identification manifolds can be found in standard differential topology texts, such as \([4]\) or \([5]\).

Informally, what is being done is that the tubular neighborhoods are being smoothly glued across their “ends”, each \(\mathbb{R}^3\). The proof that the resulting
$M$ is indeed exotic is then easy: $M$ contains $\mathbb{R}_\Theta^4$ as a smooth sub-manifold. If $M$ were diffeomorphic to $M_0$ then $M$, and thus $\mathbb{R}_\Theta^4$, could be smoothly embedded in standard $\mathbb{R}^4$, contradicting the assumption on $\mathbb{R}_\Theta^4$. Finally, it is clear that the constructed $M$ is indeed homeomorphic to the original $M_0$ since all that has been done topologically is the extension of $T_0$. See figure 2 for a visualization of this process when $M_0$ is $\mathbb{R}^4$. Smoothly “stuffing” the upper $\mathbb{R}_\Theta^4$ into the tube results in another visualization of the new manifold as shown in figure 3. A natural doubling of this process leads to figure 4. Finally, smoothly spreading out the exotic tube in figure 3 leads to figure 5.

This is clearly a powerful technique for generating exotic manifolds. Using it we are able to generate an infinity of non-diffeomorphic manifolds, $\mathbb{R}^2 \times_\Theta S^2$, each having the topology of the Kruskal presentation of the Schwarzschild metric. Using the standard Kruskal notation $\{(u, v, \omega); u^2 - v^2 < 1, \omega \in S^2 \}$ constitute global topological coordinates, but $(u, v)$ cannot be continued as smooth functions over the entire range: $u^2 - v^2 < 1$. However, by techniques discussed in [3], these coordinates can be smooth over some closed submanifold, say $A$, as illustrated in Figure 6. Over $A$ then we can solve the vacuum Einstein equations as usual to get the Kruskal form,

$$ds^2 = \frac{32M^3e^{-r/2M}}{r}(-du^2 + dv^2) + r^2d\Omega^2,$$

(2)

where $d\Omega^2$ is the standard spherical metric and

$$(\frac{r}{2M} - 1)(e^{r/2M}) = v^2 - u^2.$$  

(3)

This metric form is thus valid over $A$, but cannot be extended beyond it, not for any reasons associated with the development of singularities in the coordinate expression of the metric, or for any topological reasons, but simply because the coordinates, $(u, v, \omega)$, cannot be continued smoothly beyond some proper subset, $A$, of the full manifold, thus establishing the theorem stated earlier.

However, given any Lorentzian metric on a closed submanifold, $A$, some smooth continuation of the metric to all of $M$ can be guaranteed to exist under certain conditions. For example, we have

**Lemma 1** If $M$ is any smooth connected 4-manifold and $A$ is a closed sub-manifold for which $H^4(M, A; \mathbb{Z}) = 0$, then any smooth Lorentz signature metric defined over $A$ can be smoothly continued to all of $M$. 

4
Proof: This is basically a question of the continuation of cross sections on fiber bundles. Standard obstruction theory is usually done in the continuous category, but it has a natural extension to the smooth class, as described on page 25 in [9]. First, we note that any Lorentz metric is decomposable into a Riemannian one, $g$, plus a non-zero vector field, $v$. The continuation of $g$ follows from the fact that the fiber, $Y$, of non-degenerate symmetric four by four matrices is $q$-connected for all $q$. From the theorem stated on page 149 of [9], this means that $g$ can be continued from $A$ to all of $M$ without any topological restrictions. On the other hand, the fiber of non-zero vector fields is the three-sphere which is $q$-connected for all $q < 3$, but certainly not $3$-connected ($\pi_3(S^3) = \mathbb{Z}$). From the theorem and corollary on page 178 of [9], any obstruction to a continuation of $v$ from $A$ to all of $M$ is an element of $H^4(M, A; \mathbb{Z})$. Thus, the vanishing of this group is a sufficient condition for the continuation of $v$, establishing the Lemma.

In the applications in this paper, $M$ is non-compact, so $H^4(M; \mathbb{Z}) = 0$. Using the exact cohomology sequence generated by the inclusion $A \rightarrow M$,

$$\cdots \rightarrow H^3(M; \mathbb{Z}) \rightarrow H^3(A; \mathbb{Z}) \rightarrow H^4(M, A; \mathbb{Z}) \rightarrow H^4(M; \mathbb{Z}) \rightarrow \cdots \quad (4)$$

we see that one way to guarantee the condition of the Lemma is to have $H^3(A; \mathbb{Z}) = 0$. Another would be to establish that the map, $H^3(M; \mathbb{Z}) \rightarrow H^3(A; \mathbb{Z})$ is an epimorphism. For example, if $A$ is simply a closed miniature version of $\mathbb{R}^2 \times S^2$ itself, i.e., $A = D^2 \times S^2$, then $H^3(A; \mathbb{Z}) = 0$ so the continuation of a smooth Lorentzian metric is ensured. Whatever this metric is, it cannot be the Kruskal one, since otherwise the manifold would be diffeomorphic to standard $\mathbb{R}^2 \times S^2$. An interesting variation of the situation described in Figure 6 occurs when $A$ intersects the horizon. Thus it contains a trapped surface, so a singularity will inevitably develop from well-known theorems. However, if $A$ does not contain a trapped surface what will happen is not known.

What is missing from this result, of course, is that the continued metric satisfy the vacuum Einstein equations and that it be complete in the Lorentian sense. Of course, any smooth Lorentzian metric satisfies the Einstein equation for some stress-energy tensor, but this tensor must be shown to be physically acceptable. Unfortunately, these issues cannot be resolved without more explicit information on the global exotic structure than is presently available.
Another way to study this metric is in terms of the original Schwarzschild \((r, t)\) coordinates, as seen in figure 4. For this model the coordinates \((t, r, \omega)\) are smooth for all of the closed sub-manifold \(A\) defined by \(r \geq R_0 > 2M\) but cannot be continued as smooth over the entire \(M\) or over any diffeomorphic (physically equivalent) copy. In this case \(A\) is topologically \(\mathbb{R}^1 \times [R_0, \infty) \times S^2\), so again \(H^3(A; \mathbb{Z}) = 0\) and the conditions of lemma 1 are met. Hence there is some smooth continuation of any exterior Lorentzian metric in \(A\), in particular, the Schwarzschild metric, over the full \(\mathbb{R}^4_\Theta\). Whatever this metric is, it cannot be Schwarzschild since the manifolds are not diffeomorphic. An interesting feature of this model is that the manifold is “asymptotically” standard in spite of the well known fact that exotic manifolds are badly behaved “at infinity”. However, we note that this model is asymptotically standard only as \(r \to \infty\), but certainly not as \(t \to \infty\).

These models, especially as visualized in figures 3 and 4 are clearly highly suggestive for investigation of alternative continuation of exterior solutions into the tube near \(r = 0\). We often discover an exterior, vacuum solution, and look to continue it back to some source. This is a standard problem. In the stationary case, we typically have a local, exterior solution to an elliptic problem, and try to continue it into origin but find we can’t as a vacuum solution unless we have a topology change (e.g., a wormhole), or unless we add a matter source, changing the equation. Now, looking at figures 3 and 4, we are led to consider a third alternative.

Of course, the discussion of stationary solutions involves the idea of time foliations, which cannot exist globally for these exotic manifolds, at least not into standard factors. In fact,

**Lemma 2** \(\mathbb{R}^4_\Theta\) cannot be written as a smooth product, \(\mathbb{R}^1 \times_{\text{smooth}} \mathbb{R}^3\). Similarly \(\mathbb{R}^2 \times_{\Theta} S^2\) cannot be written as \(\mathbb{R}^1 \times_{\text{smooth}} (\mathbb{R}^1 \times S^2)\).

Clearly, if either factor decomposition were smooth, the original manifold would be standard, since the factors are necessarily standard from known lower dimensional results, establishing the lemma. I am indebted to Robert Gompf and Duane Randall for pointing out to me that because of still open questions it is not now possible to establish the more general result for which the second factor is simply some smooth three manifold without restriction. Also, note that the question of factor decomposition of \(\mathbb{R}^4\) into Whitehead spaces was considered by McMillan\[10\].
Of course, the lack of a global time foliation of these manifolds means that such models are inconsistent with canonical approach to gravity, quantum theory, etc. However, it is worth noting that all experiments yield only local data, so we have no a priori basis for excluding such manifolds.

These discussions lead naturally to a consideration of what can be said about Cauchy problems. Consider then the manifold in figure 5. The global \((t, x, y, z)\) coordinates are smooth for all \(t < 0\) but not globally. Now consider, the Cauchy problem \(R_{\alpha\beta} = 0\), with flat initial data on \(t = -1\). This is guaranteed to have the complete flat metric as solution in the standard, \(\mathbb{R}^4\) case. However, the similar problem cannot have a complete flat solution for \(\mathbb{R}^4_\Theta\) since then the exponential geodesic map would be a diffeomorphism of \(\mathbb{R}^4_\Theta\) onto its tangent space, which is standard \(\mathbb{R}^4\). This is discussed in [5]. What must go wrong in the exotic case, of course, is that \(t = -1\) is no longer a Cauchy surface. However, Lemma 1 can again be applied here to guarantee the continuation of some Lorentzian metric over the full manifold since here \(A = (-\infty, -1] \times \mathbb{R}^3\) so clearly \(H^3(A; \mathbb{Z}) = 0\).

Finally, consider the cosmological model, \(\mathbb{R}^1 \times_\Theta \mathbb{S}^3\) discussed in [3]. In this case, assume a standard cosmological metric for some time, so here \(A = (-\infty, 1] \times \mathbb{S}^3\). Clearly, \(H^3(A; \mathbb{Z})\) does not vanish in this case, but it can be shown that the inclusion induced map \(H^3(M; \mathbb{Z}) \to H^3(A; \mathbb{Z})\) is onto, so the conditions of Lemma 1 are met. Thus some smooth Lorentzian continuation will indeed exist, leading to some exotic cosmology on \(\mathbb{R}^1 \times \mathbb{S}^3\).

I am very grateful to Duane Randall and Robert Gompf for their invaluable assistance in this work.

References

[1] Robert E. Gompf, *J. Differential Geometry*, 37, 199 (1993).
[2] M. H. Freedman, *J. Differential Geometry*, 17, 357 (1982).
[3] S. K. Donaldson, *J. Differential Geometry*, 18, 269 (1983).
[4] Robert E. Gompf, *J. Differential Geometry*, 18, 317 (1983).
[5] Carl H. Brans and Duane Randall, *Gen. Rel. Grav.*, 25, 205 (1993).
[6] Robert E. Gompf, *J. Differential Geometry*, 21, 283 (1985).
[7] Th. Bröcker and K. Jänisch, *Introduction to Differential Topology*, Cambridge University Press (1987).

[8] Morris W. Hirsch, *Differential Topology*, Springer-Verlag (1988).

[9] Norman Steenrod, *The Topology of Fiber Bundles*, Princeton University Press (1951).

[10] D. R. McMillan, Jr., *Bull. Am. Math. Soc.*, 67, 510 (1961).