LIPSCHITZIAN SOLUTIONS TO INHOMOGENEOUS LINEAR ITERATIVE EQUATIONS

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Abstract. We study the problems of the existence, uniqueness and continuous dependence of Lipschitzian solutions \( \varphi \) of equations of the form

\[
\varphi(x) = \int_{\Omega} g(\omega)\varphi(f(x,\omega))\mu(d\omega) + F(x),
\]

where \( \mu \) is a measure on a \( \sigma \)-algebra of subsets of \( \Omega \).

1. Introduction

Fix a measure space \( (\Omega, \mathcal{A}, \mu) \) and a separable metric space \( (X, \rho) \).

Motivated by appearance of the equation

\[
\varphi(x) = \int_{A_1} \varphi(f(x,\omega))\mu(d\omega) + c - \int_{A_2} \varphi(f(x,\omega))\mu(d\omega)
\]

with disjoint \( A_1, A_2 \in \mathcal{A} \) in the theory of perpetuities and of refinement equations, see section 3.4 of the survey paper [3], we consider problems of the existence, uniqueness and continuous dependence of Lipschitzian solutions \( \varphi \) to the equation

\[
\varphi(x) = \int_{\Omega} g(\omega)\varphi(f(x,\omega))\mu(d\omega) + F(x).
\]

Concerning the given functions \( f, g \) and \( F \) we assume the following hypotheses in which \( B \) stands for the \( \sigma \)-algebra of all Borel subsets of \( X \) and \( K \in \{ \mathbb{R}, \mathbb{C} \} \).

(H1) Function \( f \) maps \( X \times \Omega \) into \( X \) and for every \( x \in X \) the function \( f(x,\cdot) \) is \( \mathcal{A} \)-measurable, i.e.,

\[
\{ \omega \in \Omega : f(x,\omega) \in B \} \in \mathcal{A} \quad \text{for all} \ x \in X \ \text{and} \ B \in B.
\]

(H2) Function \( g : \Omega \rightarrow \mathbb{K} \) is integrable,

\[
\int_{\Omega} |g(\omega)|\rho(f(x,\omega), x)\mu(d\omega) < \infty \quad \text{for every} \ x \in X,
\]

and

\[
\int_{\Omega} |g(\omega)|\rho(f(x,\omega), f(z,\omega))\mu(d\omega) \leq \lambda \rho(x, z) \quad \text{for all} \ x, z \in X
\]

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with a $\lambda \in [0, 1)$.

(H$_3$) Function $F$ maps $X$ into a separable Banach space $Y$ over $\mathbb{K}$ and

(3) \[ \|F(x) - F(z)\| \leq L\rho(x, z) \quad \text{for all } x, z \in X \]

with an $L \in [0, +\infty)$.

As emphasized in [4, section 0.3] iteration is the fundamental technique for solving functional equations in a single variable, and iterates usually appear in the formulae for solutions. However, as it seem, Lipschitzian solutions are examined rather by the fixed-point method (cf. [4, section 7.2D]). We iterate the operator which transforms a Lipschitzian $F: X \to Y$ into \( \int_\Omega g(\omega)F(f(x, \omega))\mu(d\omega) \); cf. formulas (6) and (8) below. The special case where $g(\omega) = -1$ for every $\omega \in \Omega$ and $\mu(\Omega) = 1$ was examined in [2] on a base of iteration of random-valued functions.

Integrating vector functions we use the Bochner integral.

2. Existence and uniqueness

Putting

(4) \[ \gamma = \int_\Omega g(\omega)\mu(d\omega), \]

we start with two simple lemmas.

Lemma 2.1. Assume (H$_1$) and let $g: \Omega \to \mathbb{K}$ be integrable with $\gamma \neq 1$. If (2) holds with a $\lambda \in [0, 1)$, then for any $F$ mapping $X$ into a normed space $Y$ over $\mathbb{K}$ equation (1) has at most one Lipschitzian solution $\varphi: X \to Y$.

Proof. Fix a function $F$ mapping $X$ into a normed space $Y$ over $\mathbb{K}$, let $\varphi_1, \varphi_2: X \to Y$ be Lipschitzian solutions of (1), and put $\varphi = \varphi_1 - \varphi_2$. Then $\varphi$ is a Lipschitzian solution of (1) with $F = 0$, and denoting by $L_\varphi$ the smallest Lipschitz constant for $\varphi$, by (2) for all $x, z \in X$ we have

\[ \|\varphi(x) - \varphi(z)\| \leq \int_\Omega |g(\omega)|\|\varphi(f(x, \omega)) - \varphi(f(z, \omega))\|\mu(d\omega) \leq L_\varphi\lambda\rho(x, z), \]

whence $L_\varphi = 0$ and $\varphi$ is a constant function. Since $\gamma$ defined by (4) is different from 1, the only constant solution of (1) with $F = 0$ is the zero function. \qed

Lemma 2.2. Under the assumptions (H$_1$)–(H$_3$) for every $x \in X$ the function

\[ \omega \mapsto g(\omega)F(f(x, \omega)), \quad \omega \in \Omega, \]

is Bochner integrable and

(5) \[ \left\| \int_\Omega g(\omega)F(f(x, \omega))\mu(d\omega) - \int_\Omega g(\omega)F(f(z, \omega))\mu(d\omega) \right\| \leq L\lambda\rho(x, z) \]

for all $x, z \in X$. 

The function considered is $A$-measurable, for every $\omega \in \Omega$ we have
\[
\|g(\omega)F(f(x,\omega))\| \leq L|g(\omega)|\rho(f(x,\omega),x) + L|g(\omega)||F(x)|,
\]
and (5) holds for all $x, z \in X$.

Assuming (H$_1$)–(H$_3$) and applying Lemma $2.2$ we define
\[
F_0(x) = F(x), \quad F_n(x) = \int \Omega g(\omega)F_{n-1}(f(x,\omega))\mu(d\omega)
\]
for all $x \in X$ and $n \in \mathbb{N}$, and we see that
\[
\|F_n(x) - F_n(z)\| \leq L\lambda^n \rho(x,z) \quad \text{for all } x, z \in X \text{ and } n \in \mathbb{N}.
\]

Our main result reads.

**Theorem 2.3.** Assume (H$_1$)–(H$_3$). If $\gamma \neq 1$ then equation (11) has exactly one Lipschitzian solution $\varphi: X \to Y$; it is given by the formula
\[
\varphi(x) = \frac{1}{1 - \gamma} \left( \sum_{n=1}^{\infty} (F_n(x) - \gamma F_{n-1}(x)) + F(x) \right) \quad \text{for every } x \in X,
\]
and
\[
\|\varphi(x) - \varphi(z)\| \leq \frac{L(1 + |\gamma|)}{|1 - \gamma|(1 - \lambda)} \rho(x,z) \quad \text{for all } x, z \in X,
\]
and
\[
\|\varphi(x)\| \leq \frac{1}{|1 - \gamma|} \left( \frac{L}{1 - \lambda} \int \Omega |g(\omega)|\rho(f(x,\omega),x)\mu(d\omega) + \|F(x)\| \right)
\]
for every $x \in X$.

**Proof.** For the proof of the existence observe first that by (11), (6) and (7) for all $x \in X$ and $n \in \mathbb{N}$ we have
\[
\|F_n(x) - \gamma F_{n-1}(x)\| = \left\| \int \Omega g(\omega)F_{n-1}(f(x,\omega))\mu(d\omega) - \int \Omega g(\omega)F_{n-1}(x)\mu(d\omega) \right\|
\]
\[
\leq L\lambda^{n-1} \int \Omega |g(\omega)|\rho(f(x,\omega),x)\mu(d\omega).
\]
Consequently (8) defines a function $\varphi: X \to Y$. Routine calculations, (8), (7), (2) and (11) show that this function satisfies (9) and (10).

It remains to prove that $\varphi$ solves (11). To this end define $M: X \to [0,\infty)$ by
\[
M(x) = L\int \Omega |g(\omega)|\rho(f(x,\omega),x)\mu(d\omega)
\]
and fix $x_0 \in X$. An obvious application of (12), (H$_2$), (10) and (3) gives
\[
M(x) \leq c_1 \rho(x, x_0) + c_2, \quad \|\varphi(x)\| \leq c_1 \rho(x, x_0) + c_2 \quad \text{for every } x \in X
\]
with some constants $c_1, c_2 \in [0,\infty)$. 
Fix $x \in X$. According to Lemma 2.2 the function
\[ \omega \mapsto g(\omega)\phi(f(x, \omega)), \quad \omega \in \Omega, \]
is Bochner integrable. Moreover, by (11)–(13),
\[ \|g(\omega)(F_n(f(x, \omega)) - \gamma F_{n-1}(f(x, \omega)))\| \leq \lambda^{n-1}|g(\omega)|M(f(x, \omega)) \]
\[ \leq \lambda^{n-1}|g(\omega)|(c_1\rho(f(x, \omega), x_0) + c_2) \]
\[ \leq \lambda^{n-1}|g(\omega)|(c_1\rho(f(x, \omega), x) + c_1\rho(x, x_0) + c_2) \]
for all $n \in \mathbb{N}$ and $\omega \in \Omega$. Hence, making use of (H2), the dominated convergence theorem and (6) we see that
\[ \int_\Omega \sum_{n=1}^\infty g(\omega)(F_n(f(x, \omega)) - \gamma F_{n-1}(f(x, \omega))) \mu(d\omega) \]
(14)
\[ = \sum_{n=1}^\infty \int_\Omega g(\omega)(F_n(f(x, \omega)) - \gamma F_{n-1}(f(x, \omega))) \mu(d\omega) \]
\[ = \sum_{n=1}^\infty (F_{n+1}(x) - \gamma F_n(x)). \]
Applying now (8), (14) and (6) we get
\[ \int_\Omega g(\omega)\phi(f(x, \omega)) \mu(d\omega) \]
\[ = \frac{1}{1-\gamma} \int_\Omega \left[ \sum_{n=1}^\infty g(\omega)(F_n(f(x, \omega)) - \gamma F_{n-1}(f(x, \omega))) \right] \]
\[ + g(\omega)F(f(x, \omega)) \right] \mu(d\omega) \]
\[ = \frac{1}{1-\gamma} \left[ \sum_{n=1}^\infty (F_{n+1}(x) - \gamma F_n(x)) + F_1(x) \right] \]
\[ = \frac{1}{1-\gamma} \left[ \sum_{n=1}^\infty (F_n(x) - \gamma F_{n-1}(x)) + \gamma F(x) \right] \]
\[ = \phi(x) - F(x). \]
The proof is complete. \(\Box\)

3. Examples

Example 3.1. Given $\lambda \in (0, 1)$ and an integrable $\xi : \Omega \to \mathbb{R}$ consider the equation
\[ \phi(x) = \lambda^2 \int_\Omega \phi \left( \frac{1}{\lambda} x + \xi(\omega) \right) \mu(d\omega) \]
with $\mu(\Omega) = 1$. According to Lemma 2.1 the zero function is the only its Lipschitzian solution $\varphi: \mathbb{R} \to \mathbb{R}$. Note however that if
\[
\int_{\Omega} \xi(\omega) \mu(d\omega) = 0 \quad \text{and} \quad \int_{\Omega} \xi(\omega)^2 \mu(d\omega) < \infty,
\]
then this equation solves also the function $\varphi: \mathbb{R} \to \mathbb{R}$ given by
\[
\varphi(x) = x^2 + \frac{\lambda^2}{1-\lambda^2} \int_{\Omega} \xi(\omega)^2 \mu(d\omega).
\]

**Example 3.2.** Given $\lambda \in (0, 1)$ consider the equation
\[
\varphi(x) = 2\varphi(\lambda \sqrt{x} + 1 - \lambda) + \log \frac{x}{(\lambda \sqrt{x} + 1 - \lambda)^2}.
\]
According to Lemma 2.1 (in this case $f(x, \omega) = \lambda \sqrt{x} + 1 - \lambda$, $g(\omega) = 2$ and $F(x) = \log \frac{x}{(\lambda \sqrt{x} + 1 - \lambda)^2}$ for all $x \in [1, \infty)$ and $\omega \in \Omega$, $\mu(\Omega) = 1$) the logarithmic function restricted to $[1, \infty)$ is the only Lipschitzian solution $\varphi: [1, \infty) \to \mathbb{R}$ to this equation, and it is unbounded in spite of the fact that $F$ is bounded.

**Example 3.3.** To see that assumptions (H$_1$)–(H$_3$) do not guarantee the existence of a continuous solution $\varphi: X \to Y$ to equation (11), given $\alpha \in (-1, 1)$, a bounded and $\mathcal{A}$-measurable $\xi: \Omega \to \mathbb{R}$, and a Lipschitzian $F: \mathbb{R} \to [0, \infty)$ such that $F^{-1}(\{0\})$ is a singleton, consider the equation
\[
(15) \quad \varphi(x) = \int_{\Omega} \varphi(\alpha x + \xi(\omega)) \mu(d\omega) + F(x)
\]
with $\mu(\Omega) = 1$. Assume a continuous $\varphi: \mathbb{R} \to \mathbb{R}$ solves it. We shall see that then $\xi$ is a.e. constant. To this end fix an $M \in (0, \infty)$ such that $|\xi(\omega)| \leq M$ for every $\omega \in \Omega$, and a real number $a \geq \frac{M}{1-|\alpha|}$ such that $F^{-1}(\{0\}) \subset [-a, a]$. Then
\[
|\alpha x + \xi(\omega)| \leq a \quad \text{for all } x \in [-a, a] \text{ and } \omega \in \Omega
\]
and so $\varphi|_{[-a,a]}$ is a continuous, hence also bounded, solution of (15). According to [1, Corollary 4.1(ii) and Example 4.1] it is possible only if $\xi$ is a.e. constant.

4. **Continuous dependence**

Given a normed space $(Y, \| \cdot \|)$ consider now the linear space $\text{Lip}(X, Y)$ of all Lipschitzian functions mapping $X$ into $Y$, and its linear subspace $\text{BL}(X, Y)$ of all Lipschitzian and bounded functions mapping $X$ into $Y$. Fix $x_0 \in X$ and define $\| \cdot \|_{\text{Lip}}: \text{Lip}(X, Y) \to [0, \infty)$ by
\[
\|u\|_{\text{Lip}} = \|u(x_0)\| + \|u\|_L,
\]
where $\|u\|_L$ stands for the smallest Lipschitz constant for $u$. Clearly $\| \cdot \|_{\text{Lip}}$ is a norm in $\text{Lip}(X, Y)$. It depends on the fixed point $x_0$, but for different points such norms are equivalent. It is well known that if $(Y, \| \cdot \|)$ is Banach, then so is $(\text{Lip}(X, Y), \| \cdot \|_{\text{Lip}})$. In the linear space $\text{BL}(X, Y)$ we consider the norm $\| \cdot \|_{\text{BL}}$ given by
\[
\|u\|_{\text{BL}} = \sup \{ \|u(x)\| : x \in X \} + \|u\|_L.
\]
It is also well known that if \((Y, \| \cdot \|)\) is Banach, then so is \((BL(X, Y), \| \cdot \|_{BL})\).

Assume \((H_1)\) and \((H_2)\), \(\gamma \neq 1\), and let \(Y\) be a separable Banach space over \(\mathbb{K}\).

According to Theorem 2.3 for every \(F \in Lip(X, Y)\) the formula

\[
\varphi^F(x) = \frac{1}{1 - \gamma} \left( \sum_{n=1}^{\infty} \left( F_n(x) - \gamma F_{n-1}(x) \right) + F(x) \right)
\]

for every \(x \in X\), defines the only Lipschitzian solution \(\varphi^F\) of equation (1),

\[
\|\varphi^F\|_{Lip} \leq \frac{1 + |\gamma|}{|1 - \gamma|} \|F\|_{Lip}
\]

and

\[
\|\varphi^F(x)\| \leq \frac{1}{|1 - \gamma|} \left( \frac{\|F\|_{Lip} L}{1 - \lambda} \int_\Omega |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega) + \|F(x)\| \right)
\]

for every \(x \in X\). Putting

\[
c_0 = \frac{1}{1 - \lambda} \left( \int_\Omega |g(\omega)| \rho(f(x_0, \omega), x_0) \mu(d\omega) + 1 + |\gamma| \right), \quad c = \max\{1, c_0\},
\]

and applying (17) and (18) we see that if \(F \in Lip(X, Y)\), then \(\varphi^F \in BL(X, Y)\) as well and

\[
\|\varphi^F\|_{BL} \leq \frac{c}{|1 - \gamma|} \|F\|_{Lip}.
\]

Moreover, if \(d_0\) defined by

\[
d_0 = \sup \left\{ \int_\Omega |g(\omega)| \rho(f(x, \omega), x) \mu(d\omega) : x \in X \right\}
\]

is finite, then putting

\[
d = \max \left\{ 1, \frac{d_0 + 1 + |\gamma|}{1 - \lambda} \right\}
\]

and applying (18) and (17) again we see also that if \(F \in BL(X, Y)\), then \(\varphi^F \in BL(X, Y)\) as well and

\[
\|\varphi^F\|_{BL} \leq \frac{1}{|1 - \gamma|} \left( \frac{d_0 + 1 + |\gamma|}{1 - \lambda} \|F\|_{Lip} + \sup \{ \|F(x)\| : x \in X \} \right) \leq \frac{d}{|1 - \gamma|} \|F\|_{BL}.
\]

**Theorem 4.1.** Assume \((H_1), (H_2)\) and let \(\gamma\) defined by (4) be different from 1. If \(Y\) is a separable Banach space over \(\mathbb{K}\) then:

(i) for any \(F \in Lip(X, Y)\) the function \(\varphi^F : X \to Y\) defined by (16) and (19) is the only Lipschitzian solution of (1), the operator

\[
F \mapsto \varphi^F, \quad F \in Lip(X, Y),
\]
is a linear homeomorphism of \((\text{Lip}(X,Y), \| \cdot \|_{\text{Lip}})\) onto itself and
\[
\|\varphi^F\|_{\text{Lip}} \leq \frac{c}{|1 - \gamma|} \|F\|_{\text{Lip}} \quad \text{for every } F \in \text{Lip}(X,Y)
\]
with \(c\) given by (19);

(ii) if additionally \(d_0\) defined by (20) is finite, then the restriction of the operator (22) to \(\text{BL}(X,Y)\) is a linear homeomorphism of \((\text{BL}(X,Y), \| \cdot \|_{\text{BL}})\) onto itself and
\[
\|\varphi^F\|_{\text{BL}} \leq \frac{d}{|1 - \gamma|} \|F\|_{\text{BL}} \quad \text{for every } F \in \text{BL}(X,Y)
\]
with \(d\) given by (21).

Proof. By the above considerations and the Banach inverse mapping theorem it remains to show that operator (22) is one-to-one, maps \(\text{Lip}(X,Y)\) onto \(\text{Lip}(X,Y)\) and \(\text{BL}(X,Y)\) onto \(\text{BL}(X,Y)\).

The first property follows from the fact that for any \(F \in \text{Lip}(X,Y)\) the function \(\varphi^F\) is a solution of (11): if \(\varphi^F = 0\), then \(F = 0\). To get the next two observe that if \(\psi \in \text{Lip}(X,Y)\), then by Lemma 2.2 the function \(F: X \rightarrow Y\) given by
\[
F(x) = \psi(x) - \int_{\Omega} g(\omega)\psi(f(x,\omega))\mu(d\omega)
\]
belongs to \(\text{Lip}(X,Y)\), if \(\psi\) is also bounded, then so is \(F\), and, since both \(\psi\) and \(\varphi^F\) solve (11), \(\psi = \varphi^F\) by Lemma 2.1. \(\square\)

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