\textbf{$\mathbb{Z}^2$-ALGEBRAS AS NONCOMMUTATIVE BLOW-UPS.}

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\textsc{Abstract.} The goal of this note is to first prove that for a well behaved $\mathbb{Z}^2$-algebra $R$, the category $\text{QGr}(R) := \text{Gr}(R)/\text{Tors}(R)$ is equivalent to $\text{QGr}(R_{\Delta})$ where $R_{\Delta}$ is a diagonal-like sub-$\mathbb{Z}$-algebra. Afterwards we use this result to prove that the $\mathbb{Z}^2$-algebras as introduced in \cite{12} are $\text{QGr}$-equivalent to a diagonal-like sub-$\mathbb{Z}$-algebra which is a simultaneous noncommutative blowup of a quadratic and a cubic Sklyanin algebra. As such we link the noncommutative birational transformation and the associated $\mathbb{Z}^2$-algebras in \cite{13} and \cite{12} with the noncommutative blowups as in \cite{14} and \cite{19}.

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\section{1. Introduction}
Throughout this note $R$ is a $\mathbb{Z}^2$-algebra over some algebraically closed field $k$. (See \cite{5} for the definition of $\mathbb{Z}^2$-algebras. We refer the interested reader to \cite{16} for a more thorough introduction to $G$-algebras). Following tradition \cite{5} we associate a noncommutative projective scheme $\text{Proj}(R)$ to a $\mathbb{Z}^2$-algebra $R$ whenever $R$ is sufficiently well behaved (e.g. a Noetherian $\mathbb{Z}^2$-algebra satisfying some analogue of generation in degree 1, see Definition 3.2). $\text{Proj}(R)$ is defined via its category of “quasicoherent sheaves”:

$$\text{Qcoh}(\text{Proj}(R)) := \text{QGr}(R) = \text{Gr}(R)/\text{Tors}(R)$$

where $\text{Gr}(R)$ is the category of graded right $R$-modules and $\text{Tors}(R)$ is the full subcategory of torsion $R$-modules. In this way these $\mathbb{Z}^2$-algebras give non-commutative generalizations of bi-homogeneous algebras.

Multi-homogeneous algebras (i.e. $\mathbb{Z}^n$-graded algebras with with $n > 1$) appear frequently in the literature \cite{6} \cite{7} \cite{9} \cite{17} \cite{18}. These multi-homogeneous algebras $S$ inherit many properties from diagonal subalgebras $S_{\Delta}$. Moreover to each multi-homogeneous algebra $S$ one associates a projective scheme $(\text{Multi})\text{Proj}(S)$ and given
suitable conditions on $S$ this projective scheme coincides with $(\text{Multi}\text{Proj}(S_\Delta))$ (see for example [18 Lemma 1.3]). For a bi-homogeneous algebra such a diagonal subalgebra is simply a $\mathbb{Z}$-graded algebra and as such $(\text{Multi}\text{Proj}(S))$ is isomorphic to the Proj of a graded algebra. In [38] we generalize this result to the level of $\mathbb{Z}^2$-algebras and prove (Theorem 3.5) that for sufficiently well behaved $\mathbb{Z}^2$-algebras and prove (Theorem 3.5) that for sufficiently well behaved $\mathbb{Z}^2$-algebras $\text{QGr}(R)$ is equivalent to $\text{QGr}(R_\Delta)$ where $R_\Delta$ is a diagonal-like sub-$\mathbb{Z}$-algebra. Using more abstract techniques, a similar result was obtained independently by Lowen, Ramos-González and Shokhet in [10].

Our main application of Theorem 3.5 is to the $\mathbb{Z}^2$-algebras appearing in [12]. These $\mathbb{Z}^2$-algebras were constructed when investigating the noncommutative versions of the standard birational transformation $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$ as in [13]. Such a noncommutative $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$ is obtained from an inclusion of $\mathbb{Z}$-algebras

$$A' \hookrightarrow ˇA^{(2)}$$

where $A'$ a quadratic Sklyanin algebra, $A$ is a cubic Sklyanin algebra, $A'$ and $\hat{A}$ are their associated $\mathbb{Z}$-algebras and $\hat{A}^{(2)}$ is the second Veronese algebra of $\hat{A}$ (i.e. $\hat{A}^{(2)}_{i,j} = \hat{A}_{2i,2j} =: A_{2i-2j}$).

Recall from [2] that quadratic and cubic Sklyanin algebras are classified using triples of geometric data $(Y, L, \sigma)$ where $Y$ is a smooth elliptic curve, $L$ is a line bundle on $Y$ and $\sigma \in \text{Aut}(Y)$. It was shown in [3] that there is a 1-1-correspondence between points of $Y$ and point modules of $A(Y, L, \sigma)$. The inclusion $A' \hookrightarrow ˇA^{(2)}$ is compatible with these geometric data in the sense that $Y = Y'$ and that the inclusion is constructed starting from some point $p \in Y$.

In [12] it was shown that there is an inclusion (constructed starting from two points $p', q'$)

$$\hat{A} \hookrightarrow A'$$

such that the composition $\hat{A} \hookrightarrow \hat{A}^{(2)}$ and $A' \hookrightarrow ˇA^{(2)}$ induce the identity on the associated function fields $\text{Frac}(A)_0$ and $\text{Frac}(A')_0$. This was done by investigating a $\mathbb{Z}^2$-algebra $\hat{A}$ “containing” both $A$ and $A'$ as respectively a column and row. In [4] and [5] we check that $\hat{A}$ satisfies the condition of Theorem 3.5. As such $\text{QGr}(\hat{A}) \cong \text{QGr}(\hat{A}_\Delta)$ for each diagonal-like $\mathbb{Z}$-algebra $\hat{A}_\Delta$.

Finally in [7] we focus on a specific $\Delta$. For this $\Delta$ the methods in [12] provide us with inclusions

$$(1) \quad \hat{A}_\Delta \hookrightarrow \hat{A}^{(4)}$$

and $\hat{A}_\Delta \hookrightarrow \hat{A}^{(3)}$.

We show that these inclusions are simultaneously compatible with the 1-periodicity of $\hat{A}^{(4)}$ and $\hat{A}^{(3)}$. As such there is a graded algebra $T$ for which $T \cong \hat{A}_\Delta$ and the inclusions in [11] give rise to inclusions $T \hookrightarrow \hat{A}^{(4)}$ and $T \hookrightarrow \hat{A}^{(3)}$. Finally we check that these inclusions give $T$ the construction of a noncommutative blowup $\hat{A}^{(4)}(p)$ and $\hat{A}^{(3)}(p' + q')$ as in [14], resulting in our main result:

**Theorem 1.1.** Let $A, A', p, p', q'$ and $\hat{A}$ be as above, then $A$ and $A'$ contain a common blowup $\hat{A}^{(4)}(p) \cong T \cong \hat{A}^{(3)}(p' + q')$ and there is an equivalence of categories

$$\text{QGr}(\hat{A}) \cong \text{QGr}(T)$$

**Remark 1.2.** With the exception of the equivalence of categories the above result was announced independently by Rogalski, Sierra and Stafford in [15, Theorem 1.7].
Remark 1.3. In [13] one considers a noncommutative version of the Cremona transform \( \mathbb{P}^2 \rightarrow \mathbb{P}^2 \). Such a noncommutative Cremona transform is given by an inclusion \( \tilde{A} \rightarrow \tilde{A}^{(2)} \) where \( A \) and \( A' \) are quadratic Sklyanin algebras. This inclusion is constructed with respect to some divisor \( d \). The methods in [12] provide an “inverse” inclusion \( \tilde{A} \rightarrow \tilde{A}^{(2)} \) with respect to some explicit divisor \( d' \). Moreover analogously to op.cit one can construct a \( \mathbb{Z}^2 \)-algebra \( \tilde{A} \) containing both \( A \) and \( A' \).

It is a straightforward check that the results in §5 and §6 are also applicable to these \( \mathbb{Z}^2 \)-algebras. If one choses \( \Delta(p_i q) = p_i q \) computations analogous to the ones in §7 show that \( A \) and \( A' \) contain a common blowup \( A^{(3)}(d) \cong T \cong A^{(3)}(d') \) and there is an equivalence of categories
\[
\text{QGr}(\tilde{A}) \cong \text{QGr}(T)
\]

In order to avoid unnecessary difficult notations, we have chosen to only focus on the proof of Theorem 1.1.

2. Acknowledgements

The author wishes to thank Shinnosuke Okawa for mentioning the idea that geometrically (at the level of QGr) the \( \mathbb{Z}^2 \)-algebras as in [12] are the same as the blowups in [14]. The author wishes to thank Wendy Lowen and Julia Ramos González for useful discussions about the more abstract nature of the equivalence of categories obtained in Theorem 3.5. The author further wishes to thank Michel Van den Bergh for providing insights in the most technical aspects of this paper as well as for reading through earlier versions.

3. Diagonal-like subalgebras

Throughout this section \( R \) is a \( \mathbb{Z}^2 \)-algebra over some field \( k \). I.e. \( R \) is a \( k \)-algebra together with a decomposition
\[
R = \bigoplus_{(i,j),(m,n) \in \mathbb{Z}^2} R_{(i,j),(m,n)}
\]
such that addition is degree-wise and multiplication satisfies
\[
R_{(a,b),(i,j),(m,n)} \subset R_{(a,b),(m,n)} \quad \text{and} \quad R_{(a,b),(c,d),(i,j),(m,n)} = 0 \quad \text{whenever} \quad (c,d) \neq (i,j)
\]
Moreover there are local units \( e_{(i,j)} \in R_{(i,j),(i,j)} \) such that for each \( x \in R_{(a,b),(m,n)} \):
\[
e_{(a,b)x} = x = xe_{(m,n)}
\]
A graded \( R \)-module is an \( R \)-module \( M \) together with a decomposition
\[
M = \bigoplus_{(i,j) \in \mathbb{Z}^2} M_{(i,j)}
\]
such that the \( R \)-action on \( M \) satisfies
\[
M_{(i,j)}R_{(i,j),(m,n)} \subset M_{(m,n)} \quad \text{and} \quad M_{(i,j)}R_{(a,b),(m,n)} = 0 \quad \text{if} \quad (a,b) \neq (i,j)
\]
We denote \( \text{Gr}(M) \) for the category of graded \( R \)-modules. In this section we also make the assumption that \( R \) is Noetherian in the sense that \( \text{Gr}(R) \) is locally Noetherian, moreover we assume that each \( R_{(i,j),(m,n)} \) is a finite dimensional vectorspace and \( R_{(i,j),(i,j)} = k \).
Notation 3.1. Let $R$ be a $\mathbb{Z}^2$-algebra. Then we denote $R_+$ for the $\mathbb{Z}^2$-subalgebra:

$$(R_+)_{(i,j),(m,n)} = \begin{cases} R_{(i,j),(m,n)} & \text{if } i \leq m \text{ and } j \leq n \\ 0 & \text{else} \end{cases}$$

Definition 3.2. Let $R$ be a $\mathbb{Z}^2$-algebra and let $R_+$ be as above. We say that $R_+$ is generated in degree $(0,1)$ and $(1,0)$ if each homogeneous element in $R_+$ can be written as a product of elements of degree $(0,1)$ or $(1,0)$. I.e.:

$$\forall i, j, m, n \in \mathbb{Z}, i < m, j \leq n : R_{(i,j),(i+1,j)} \otimes R_{(i+1,j),(m,n)} \to R_{(i,j),(m,n)}$$

is surjective and

$$\forall i, j, m, n \in \mathbb{Z}, i \leq m, j < n : R_{(i,j),(i,j+1)} \otimes R_{(i,j+1),(m,n)} \to R_{(i,j),(m,n)}$$

is surjective.

Definition 3.3. Let $R$ be a $\mathbb{Z}^2$-algebra such that $R_+$ is generated in degree $(0,1)$ and $(1,0)$ and let $M$ be a graded $R$-module. We say $M$ is right-upper-bounded if there exist $i_0, j_0 \in \mathbb{Z} : \forall i \geq i_0, j \geq j_0 : M_{(i,j)} = 0$. $M$ is said to be torsion if it is a direct limit of right-upper-bounded modules. We denote $\text{Tors}(R)$ for the full subcategory of torsion modules in $\text{Gr}(R)$.

The assumption that $R$ is Noetherian implies that $\text{Tors}(R)$ is a Serre subcategory of $\text{Gr}(R)$ and as such we can define a quotient category

$$\text{QGr}(R) := \text{Gr}(R)/\text{Tors}(R)$$

The main result of this section is that we can understand $\text{QGr}(R)$ in terms of diagonal-like sub-$\mathbb{Z}$-algebras of $R$.

Theorem 3.5. Let $R$ be a $\mathbb{Z}^2$-algebra and let $R_\Delta$ be a diagonal-like sub-$\mathbb{Z}$-algebra. Then there is an equivalence of categories

$$(2) \quad \text{QGr}(R) \cong \text{QGr}(R_\Delta)$$

Proof. There is a restriction functor $F : \text{Gr}(R) \to \text{Gr}(R_\Delta)$ defined by $F(M)_i := M_{\Delta(i)}$. $F$ obviously maps torsion modules to torsion modules and hence induces a functor $F : \text{QGr}(R) \to \text{QGr}(R_\Delta)$.

Next we define a right exact functor $G : \text{Gr}(R_\Delta) \to \text{Gr}(R)$ by setting $G(e_{\Delta(i)}R) = e_{\Delta(i)}R$ at the level of objects. As $\text{Hom}_{R_\Delta}(e_iR_\Delta, e_jR_\Delta)$ and $\text{Hom}_R(e_{\Delta(i)}R, e_{\Delta(j)}R)$ are both canonically isomorphic to $R_{\Delta(j),\Delta(i)}$, $G : \text{Hom}_{R_\Delta}(e_iR_\Delta, e_jR_\Delta) \to \text{Hom}_R(e_{\Delta(i)}R, e_{\Delta(j)}R)$ is chosen to be the identity at the level of homomorphisms. We now claim that $G$ sends torsion modules to torsion modules. As $G$ is right exact and commutes with direct sums, it is compatible with direct limits. As such, it suffices to check that $G$ sends finitely generated, right bounded $R_\Delta$-modules to right-upper-bounded $R$-modules. For this let $M$ be a finitely generated, right bounded $R_\Delta$-module. There is a resolution

$$\bigoplus_m e_{i_m}R_\Delta \xrightarrow{f} \bigoplus_{n=0} e_{j_n}R_\Delta \to M \to 0$$
and there is a \( u_0 \in \mathbb{Z} \) such that \( f \) is surjective in all degrees \( u \) with \( u \geq u_0 \). Moreover we can assume \( u_0 \geq j_n \) for all \( n \). Now write \( \Delta(u_0) = (a_0, b_0) \). The fact that \( R_+ \) is assumed to be generated in degree \((0, 1)\) and \((1, 0)\) implies that the induced map

\[
G(f) : \bigoplus_{m} e_{\Delta((m))} R \rightarrow \bigoplus_{n=0}^{n_0} e_{\Delta((j_n))} R
\]

is surjective in all degrees \((a, b)\) with \( a \geq a_0, b \geq b_0 \). This implies \( G(M)_{(a,b)} = 0 \) for all such \( a, b \), hence \( G(M) \) is right-upper-bounded. In particular the functor \( G: \text{Gr}(R_\Delta) \rightarrow \text{Gr}(R) \) induces a functor \( G: \text{QGr}(R_\Delta) \rightarrow \text{QGr}(R) \).

It is immediate that \( G \circ F = \text{Id} \). By Lemma 3.6 below it now suffices to check that \( F(G(e_{\Delta(i)} R)) = e_{\Delta(i)}(R) \). This is however obvious.

\[\square\]

**Lemma 3.6.** The collection \( \{e_{\Delta(i)} R \mid i \in \mathbb{Z}\} \) (or rather the set of corresponding objects in \( \text{QGr}(R) \)) forms a set of generators for \( \text{QGr}(R) \).

*Proof.* As the collection \( \{e_{(m,n)} R \mid m, n \in \mathbb{Z}\} \) forms a set of generators for \( \text{Gr}(R) \) and hence also for \( \text{QGr}(R) \) it suffices to show that every \( e_{(m,n)} R \) is a quotient of a direct sum of objects \( e_{\Delta(i)} R \) in \( \text{QGr}(R) \). For this fix some \( m, n \in \mathbb{Z} \). We claim that there are surjective maps (in \( \text{QGr}(R) \))

\[
(3) \quad e_{(m+1,n)} R^\oplus N \rightarrow e_{(m,n)} R
\]

and

\[
(4) \quad e_{(m,n+1)} R^\oplus N' \rightarrow e_{(m,n)} R
\]

Assume these claims. As we \( \Delta \) was assumed to be diagonal-like, there exist integers \( a, b, i \in \mathbb{Z} \) such that \( a, b \geq 0 \) and \( \Delta(i) = (m + a, n + b) \). In particular the surjective maps \( (3) \) and \( (4) \) give rise to a surjective map \( e_{\Delta(i)} R^\oplus N'' \rightarrow e_{(m,n)} R \). Hence the lemma follows from the claims.

We now prove the claims. As both claims are similar we only prove \( (3) \). For this let \( N = \dim_k(\text{Hom}(e_{m+1,n} R, e_{m,n} R)) = \dim_k(R(\text{m,n}_{(m+1,n)})) \). As we assumed \( R \) to be positively generated in degree \((1,0)\) and \((0,1)\) there is a map

\[
e_{(m+1,n)} R^\oplus N \rightarrow e_{(m,n)} R
\]

in \( \text{Gr}(R) \) whose cokernel lives in degrees \((x, y)\) with either \( x \leq m \) or \( y < n \). As such this cokernel is torsion and the induced map in \( \text{QGr}(R) \) is surjective. \[\square\]

### 4. Summary of the Constructions in [13] and [12]

Throughout the following sections \( k \) is assumed to be an algebraically closed field of characteristic zero.

In this section we briefly recall the constructions in [13] and [12] necessary to understand the proof of Theorem [14]. We refer to these papers for all unexplained notations. Three dimensional Artin-Schelter regular algebras generated in degree 1 provide important examples of noncommutative surfaces. They were defined in [14] and it was shown that they are either generated by 3 elements satisfying 3 relations of degree 2 (the quadratic case) or 2 generators satisfying 2 relations of degree 3 (the cubic case). For use below we define \((r, s)\) to be respectively the number of generators and the degrees of the relations. Thus \((r, s) = (3, 2)\) or \((2, 3)\) depending on whether the algebra is quadratic or cubic.
Three dimensional Artin-Schelter regular algebras were classified in [2] in terms of geometric triples \((Y, \mathcal{L}, \sigma)\). Our main focus lies on quadratic and cubic “Sklyanin algebras” of infinite order. In this case \(Y\) is a smooth elliptic curve, \(\mathcal{L}\) is a line bundle of degree \(s\) and \(\sigma : Y \rightarrow Y\) is an infinite order morphism given by a translation. It is customary to write \(\tau := \sigma^{s+1}\). Moreover for each Sklyanin algebra \(A = A(Y, \mathcal{L}, \sigma)\) there exists a central element \(g \in A_{s+1}\) such that \(A/gA \cong B(Y, \mathcal{L}, \sigma)\) where \(B(Y, \mathcal{L}, \sigma)\) is the twisted homogeneous coordinate ring \((\mathcal{H})\) with respect to \((Y, \mathcal{L}, \sigma)\).

In [13] a noncommutative version of the classical birational transformation \(\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2\) was constructed as an inclusion
\begin{equation}
\gamma : \tilde{A} \hookrightarrow \tilde{A}^{(2)}
\end{equation}
where \(A = A(Y, \mathcal{L}, \sigma)\) is a cubic Sklyanin algebra, \(A'\) a quadratic Sklyanin algebra and \(\tilde{A}\) and \(\tilde{A'}\) are their associated \(\mathbb{Z}\)-algebras. (i.e. \(\tilde{A}_{i,j} := A_{j-i}\) and multiplication in \(\tilde{A}\) is defined in the obvious way). The inclusion in (5) is constructed starting from a point \(p \in Y\). More concretely: one defines \(X = \text{QGr}(A)\) and introduces a category of bimodules \(\text{Bimod}(X - X)\). Of particular importance are the bimodules \(\sigma_1(1)\) (corresponding to degree shifting in \(A\)) and \(m_p\) (the ideal “sheaf” of the point \(p\)). We refer the interested reader to [13] Chapter 3 for a thorough introduction to bimodules. The inclusion (5) is then obtained by the following identifications
\begin{align}
\tilde{A} &\cong \bigoplus_{m,n \in \mathbb{Z}} \text{Hom}_X(O_X(-n), O_X(-m)) \\
\tilde{A}' &\cong \bigoplus_{m,n \in \mathbb{Z}} \text{Hom}_X(O_X(-2n), O_X(-2m) \otimes m_{\tau-m} \cdots m_{\tau-n+1})
\end{align}
The identification in (7) is obtained by noticing that the right hand side is generated in degree 1, has the correct Hilbert series and has a quotient isomorphic to \(B(Y, \mathcal{L} \otimes \sigma^s \mathcal{L}(-p), \psi)\) where \(\psi : Y \rightarrow Y\) is some automorphism such that \(\psi^3 = \sigma^4\).

In [12] it was shown that there is an inclusion
\begin{equation}
\delta : \tilde{A} \hookrightarrow \tilde{A'}
\end{equation}
such that the composition \(\gamma \circ \delta : \tilde{A} \hookrightarrow \tilde{A}^{(2)}\) and \(\delta \circ \gamma : \tilde{A'} \hookrightarrow \tilde{A'}^{(2)}\) induce the identity on the associated function fields. This was done by investigating the following \(\mathbb{Z}^2\)-algebra \(\tilde{A}\):
\[\tilde{A}_{(i,j),(m,n)} := \begin{cases} \text{Hom}_X(O_X(-m - 2n), O_X(-(i - 2j)m_{\tau-1} \cdots m_{\tau-n+1}) & \text{if } n > j \\ \text{Hom}_X(O_X(-m - 2n), O_X(-(i - 2j)) & \text{if } n \leq j \end{cases}\]
The construction of the inclusion (8) is then based upon the following observations:
1) \(\tilde{A}_{(i,0),(m,0)} \cong A_{i,m}\)
2) \(\tilde{A}_{(0,j),(0,n)} \cong A_{j,n}\)
3) \(\tilde{A}_{(i,j),(m,n)} \subset A_{(i+2j,m+2n)}\), hence \(\tilde{A}\) contains no non-trivial zero-divisors (because \(A\) is a domain).
4) \(\text{dim}_k \left( \tilde{A}_{(1,j),(i+1,j-1)} \right) = 1\)
We let \(\delta_{(i,j)}\) be a nonzero element in \(\tilde{A}_{(i,j),(i+1,j-1)}\). Then (8) is of the form
\begin{equation}
\tilde{A}_{(i,0),(m,0)} \rightarrow \delta_{(1,i-1)}^{-1} \cdots \delta_{(1,0)}^{-1} \tilde{A}_{(i,0),(m,0)} \delta_{(m,0)} \cdots \delta_{(1,m-1)} \subset \tilde{A}_{(0,i),(0,m)}
\end{equation}
The fact that $\gamma \circ \delta$ and $\delta \circ \gamma$ induce the identity on the function fields of $A$ and $A'$ is based upon the following observations: One can fix nonzero elements $\gamma_{(i,j)}$ in the 1-dimensional vectorspace $\tilde{A}_{(i,j),(i+2,j+1)}$ such that (13) is actually of the form
\begin{align}
\tilde{A}_{(0,0),(0,m)} \to \gamma_{(2i-1)}^{-1} \cdots \gamma_{(0,0)}^{-1} \tilde{A}_{(0,0),(0,m)} \gamma_{(0,m)} \cdots \gamma_{(2m-1,0)}^{-1} \subset \tilde{A}_{(2i,0),(2m,0)}
\end{align}
Conversely it was shown that
\begin{align}
\tilde{A} \cong \bigoplus_{(i,j)} \text{Hom}_{X'}(O'_X(-2j), O'_X(-2i) \otimes m_{d_i} \cdots m_{d_j})
\end{align}
where $X' = \text{QGr}(A')$ and
\begin{align}
d_i = \begin{cases}
\tau^{-j}p' & \text{if } i = 2j \\
\tau^{-j}q' & \text{if } i = 2j + 1
\end{cases}
\end{align}
where $p', q' \in Y$ are defined by the following relations in $\text{Pic}(Y)$:
\begin{align}
p + \tau q' &\sim [L] \\
p + p' &\sim [\sigma^*L]
\end{align}
Moreover the obvious inclusion (induced by (11))
\begin{align}
\tilde{A} \cong \bigoplus_{(i,j)} \text{Hom}_{X'}(O'_X(-2j), O'_X(-2i) \otimes m_{d_i} \cdots m_{d_j}) \subset \bigoplus_{(i,j)} \text{Hom}_{X'}(O'_X(-2i), O'_X(-2i)) \cong \tilde{A}'
\end{align}
coincides with (11). These facts are combined in (12) to conclude that $\gamma \circ \delta$ and $\delta \circ \gamma$ induce the identity on the function fields of $A$ and $A'$.

Finally for further use in this paper we recall from (12) that the $\mathbb{Z}^2$-algebra $\tilde{B}$ was defined as
\begin{align}
\tilde{B}_{(i,j),(m,n)} := \begin{cases}
\Gamma(Y, \sigma^*(i+2j)L)\sigma^*(i+2j+1) \cdots \sigma^*(m+2n-1)L(-\tau^{-j}p - \cdots - \tau^{-n+1}p) & \text{if } n > j \\
\Gamma(Y, \sigma^*(i+2j)L)\sigma^*(i+2j+1) \cdots \sigma^*(m+2n-1)L & \text{if } n \leq j
\end{cases}
\end{align}
and there is a morphism $\tilde{A} \to \tilde{B}$ which is an epimorphism in the first quadrant (12, Lemma 8.9))

5. The $\mathbb{Z}^2$-algebras as in (12) are Noetherian.

In this section we show that the $\mathbb{Z}^2$-algebras $\tilde{A}$ as introduced above are Noetherian (in the sense that $\text{Gr}(\tilde{A})$ is locally Noetherian). In order to prove that $\tilde{A}$ is Noetherian we work through $\tilde{B}$. By construction there is a surjective $\mathbb{Z}^2$-algebra morphism $\tilde{A}_+ \to \tilde{B}_+$ (recall Notation 3.1). We first check that in sufficiently high degrees this map is given by killing a certain collection of “normalizing” elements $\{g_{(i,j)}\}$. I.e. we prove the following

Lemma 5.1. With the notations as above:
\begin{enumerate}
\item For all $i, j \in \mathbb{Z}$ the map $\tilde{A}_{(i,j),(i+2,j+1)} \to \tilde{B}_{(i,j),(i+2,j+1)}$ has a one dimensional kernel. Let $g_{(i,j)}$ be a nonzero element in this kernel.
\item For all $a \geq 2, b \geq 1$ the kernel of $\tilde{A}_{(i,j),(i+a,j+b)} \to \tilde{B}_{(i,j),(i+a,j+b)}$ is given by $g_{(i,j)} \tilde{A}_{(i+2,j+1),(i+a,j+b)} = \tilde{A}_{(i,j),(i+a-2,j+b-1)}g_{(i+a-2,j+b-1)}$.
\end{enumerate}
Proof: (1) follows from Lemma 5.3.
In order to prove (2) we only prove the first equality. The second equality is analogous. Now for each \(i, j \in \mathbb{Z}\) take
\[
g_{(i,j)} \in \ker \left( \tilde{A}_{(i,j), (i+2,j+1)} \to \tilde{B}_{(i,j), (i+2,j+1)} \right) \setminus \{0\}
\]
The following diagram shows \(g_{(i,j)} \tilde{A}_{(i+2,j+1), (i+a,j+b)} \subset \ker \left( \tilde{A}_{(i,j), (i+2,j+1)} \to \tilde{B}_{(i,j), (i+2,j+1)} \right)\)

\[
\begin{array}{c}
0 \\
g_{(i,j)}k \otimes \tilde{A}_{(i+2,j+1), (i+a,j+b)} \\
\tilde{A}_{(i,j), (i+2,j+1)} \otimes \tilde{A}_{(i+2,j+1), (i+a,j+b)} \\
\tilde{B}_{(i,j), (i+2,j+1)} \otimes \tilde{A}_{(i+2,j+1), (i+a,j+b)} \\
0
\end{array}
\]

Hence it suffices to show that the alternating sum of the dimensions of the spaces in the right column equals zero. This again follows from Lemma 5.3.

Corollary 5.2. The collection \(\{g_{(i,j)}\}_{i,j \in \mathbb{Z}}\) is normalizing in the sense that if \(a \in \tilde{A}_{(i,j), (m,n)}\) then there exists a unique \(a' \in \tilde{A}_{(i,j), (m-2,n-1)}\) such that \(g_{(i,j)}a = a'g_{(m-2,n-1)}\). As such, for every right \(\tilde{A}\)-module \(M\) one can consider the right \(\tilde{A}\)-module \(Mg\) defined by \((Mg)_{(i,j)} := M_{(i-2,j-1)}g_{(i-2,j-1)}\).

Lemma 5.3.
\[
\dim_k \left( \tilde{A}_{(i,j), (i+a,j+b)} \right) = \begin{cases} 
\frac{a^2 + 4ab + 2b^2 + 4a + 6b + 4}{4} & \text{for } a, b \geq 0, \ a \text{ even} \\
\frac{a^2 + 4ab + 2b^2 + 4a + 6b + 3}{4} & \text{for } a, b \geq 0, \ a \text{ odd}
\end{cases}
\]
and
\[
\dim_k \left( \tilde{B}_{(i,j), (i+a,j+b)} \right) = \begin{cases} 
2a + 3b & \text{for } a, b \geq 0, \ (a, b) \neq (0, 0) \\
1 & \text{for } (a, b) = (0, 0)
\end{cases}
\]

Proof: For \(\tilde{B}\) the equality is trivial as this corresponds to calculating sections of line bundles on an elliptic curve. For \(\tilde{A}\) the computation is done in a similar way as in [4, §6].

In §6 we will slightly generalize the arguments in [4, §3] to show

Theorem 5.4. \(\tilde{B}_+\) is Noetherian.

Using this result as well as Lemma 5.1 we can prove
Theorem 5.5. \( \hat{A} \) is Noetherian

Proof. Obviously the objects \( e_{(i,j)} \hat{A} \) generated \( \text{Gr}(\hat{A}) \), hence it suffices to show that these are Noetherian. Let

\[
M_0 \subset M_1 \subset \ldots \subset e_{(i,j)} \hat{A}
\]

be an ascending chain of right \( \hat{A} \) modules. We need to show that this sequence stabilizes. This is done in two steps. First we show the existence of an \( N_0 \in \mathbb{N} \) such that \( (M_n)_{(i+a,j+b)} = (M_{N_0})_{(i+a,j+b)} \) holds for all \( n \geq N_0 \), \( ab \leq 0 \). This is based on Noetherianity of \( A \) and \( A' \). Next we show the existence of an \( N_1 \in \mathbb{N} \) such that \( (M_n)_{(i+a,j+b)} = (M_{N_1})_{(i+a,j+b)} \) holds for \( a, b \geq 0 \). Which is heavily based on Theorem 5.4 (This also explains why we only need Noetherianity of \( \hat{B} \) in the first quadrant in Theorem 5.4).

Step 1: Convergence of \( \left( (M_n)_{(i+a,j+b)} \right)_{n \in \mathbb{N}} \) for \( ab \leq 0 \)

As the cases \( a \leq 0 \) and \( b \leq 0 \) are analogous, we only prove the case \( a \leq 0 \). Moreover without loss of generality we can assume \( i = j = 0 \). Recall that \( \hat{A}_{(m,n),c} \) is a 1-dimensional vector space and that \( \delta_{(m,n)} \) is a nonzero element in this vector space. Moreover as \( \hat{A} \) has no non-trivial zero divisors in the sense that

\[
\forall a, b, i, j, m, n \in \mathbb{Z}, x, y \in \hat{A}_{(a,b),(i,j)}, \, y \in \hat{A}_{(i,j),c} : xy = 0 \Rightarrow x = 0 \lor y = 0
\]

multiplication by these elements defines embeddings of vector spaces. By the dimension count in Lemma 5.3 the embeddings

\[
\hat{A}_{(0,0),(a,b)} \hookrightarrow \hat{A}_{(0,0),(a-1,b+2)} : x \mapsto x\delta_{(a,b)}
\]

are in fact isomorphisms. Moreover these isomorphisms induce embeddings \( (M_n)_{(a,b)} \hookrightarrow (M_n)_{(a-1,b+2)} \). In particular if we define \( M_c^0 \subset e_0 A' \) via

\[
(M_c^0)_n := \{ x \in A'_{(0,m)} \cong \hat{A}_{(0,0),(0,m)} \mid x\delta^c \in (M_n)_{(-c,m+2c)} \}
\]

(\( \delta^c = \delta_{(0,m)} \delta_{(-1,m+2)} \cdots \delta_{(-c+1,m+2c-2)} \)) we find an ascending chain of \( A' \)-submodules of \( e_0 A' \). (The \( A' \)-structure follows from the fact that \( A'_{m,m'} = \hat{A}_{(m,m'),(0,0)} \) is isomorphic to \( \hat{A}_{(-c,m+2c),(-c,m'+2c)} \)) via \( \delta^{-c}(\cdots \delta^c) \). As \( A' \) is Noetherian this sequence must stabilize. I.e. for each \( n \) there is a natural number \( c_n \) such that \( M_c^{c_n} = M_n^c \) holds for all \( c \geq c_n \). Moreover [14] induces

\[
M_0^c \subset M_1^c \subset \ldots \subset e_0 A'
\]

This chain must stabilize as well. In particular there is some \( N \) such that \( M_n^{c_n} = M_N^c \) holds for all \( n \geq N \). Going back to [14] and the definition of \( M_c^c \) one sees that \( c_n \geq c_0 \) must hold for all \( n \geq N \). Hence \( M_n^c = M_N^c \) for all \( n \geq N, c \geq c_N \). Similarly the (finitely many!) chains of inclusions

\[
M_0^c \subset M_1^c \subset \ldots \subset e_0 A'
\]

must simultaneously stabilize. Hence there exists a \( N_0 \in \mathbb{N} \) such that \( M_n^c = M_{N_0}^c \) holds for all \( n \geq N_0 \) and all \( c \in \mathbb{N} \). But this is exactly the same as

\[
\forall n \geq N_0, a \leq 0, b \in \mathbb{Z} : (M_n)_{(a,b)} = (M_{N_0})_{(a,b)}
\]
Step 2: Convergence of \( (M_n(a,b))_{n \in \mathbb{N}} \) for \( a, b \geq 0 \)

Let \( \tilde{A}_+ \) be defined as in Notation 3.1 and define \( M_{n, (\geq 0, \geq 0)} \) via

\[
(M_{n, (\geq 0, \geq 0)}) = \begin{cases} M_{(i,j)} & \text{if } i \geq a, j \geq b \\ 0 & \text{else} \end{cases}
\]

then \( (M_{n, (\geq 0, \geq 0)})_{n \in \mathbb{N}} \) defines an ascending chain of \( \tilde{A}_+ \)-submodules of \( e_{(0,0)} \tilde{A}_+ \).

Hence it suffices to show that every \( \tilde{A}_+ \)-submodule of \( e_{(0,0)} \tilde{A}_+ \) is finitely generated in the sense that it can be written as a quotient of some finite direct sum of projective generators \( \bigoplus_{n=1}^{n_0} e_{(i_n,j_n)} \tilde{A}_+ \). This is shown in a similar way as in [2, Lemma 8.2]:

By way of contradiction suppose that there is some submodule \( L \subset e_{(0,0)} \tilde{A}_+ \) which is not finitely generated. Using Zorn’s Lemma \( L \) can be chosen maximal with respect to the inclusion of submodules. The quotient \( \overline{A} = e_{(0,0)} \tilde{A}_+ / L \) as well as all its submodules must hence be finitely generated. Now consider the following diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & e_{(0,0)} \tilde{A}_+ & \longrightarrow & \overline{A} & \longrightarrow & 0 \\
\end{array}
\]

Applying the Snake Lemma (and the fact that the \( g_{(i,j)} \) are normalizing, non-zero divisors) provides an exact sequence

\[
0 \to K \to L / Lg \to e_{(0,0)} \tilde{A}_+ / g_{(0,0)} \tilde{A}_+ \to 0
\]

where \( K = \ker(\overline{A} \twoheadrightarrow \overline{A}) \). As mentioned above, all submodules of \( \overline{A} \) are finitely generated, hence \( K \) is finitely generated. We also claim \( \im(\epsilon) \subset e_{(0,0)} \tilde{A}_+ / g_{(0,0)} \tilde{A}_+ \)

is finitely generated. From this claim it follows that \( L / Lg \) is finitely generated as an extension of finitely generated modules and hence \( L \) is finitely generated as well (as the elements \( g_{(i,j)} \) live in strictly positive degrees). Thus it suffices to prove the claim. First we prove the existence of a map

\[
\bigoplus_{n=1}^{n_1} e_{(i_n,j_n)} \tilde{A}_+ \to \im(\epsilon)
\]

which is surjective in degrees \( (a, b) \) with \( a \geq 2, b \geq 1 \). Note that the elements in \( \im(\epsilon) \) living in these degrees actually form a \( \tilde{B}_+ \)-submodule by Lemma 5.1. By Theorem 5.3 it must be finitely generated as a \( \tilde{B}_+ \)-module and hence also as an \( \tilde{A}_+ \)-module. Next we prove the existence of a map

\[
\bigoplus_{n=1}^{n_2} e_{(i_n,j_n)} \tilde{A}_+ \to \im(\epsilon)
\]

which is surjective in degrees \( (a, b) \) with \( a < 2 \) or \( b = 0 \). Note that \( \im(\epsilon)_{(a,0)}_{a \in \mathbb{N}} \) forms an \( A \)-submodule of \( e_{(0,0)} A \). Hence there is an epimorphism

\[
\bigoplus_{m=1}^{m_2} e_{(i_m)} A \to (\im(\epsilon)_{(a,0)})_{a \in \mathbb{N}}
\]
We also associate a sheaf of algebras $\mathcal{B}_{(i,j), (m,n)}$ to it via
\begin{equation}
\mathcal{B}_{(i,j), (m,n)} = \begin{cases}
\mathcal{L}_{(i,j)} \mathcal{L}_{(i+1,j)} \cdots \mathcal{L}_{(m-1,j)} \mathcal{G}_{(m,j)} \mathcal{G}_{(m,j+1)} \cdots \mathcal{G}_{(m,n-1)} & \text{if } i \leq m, j \leq n \\
(\mathcal{L}_{(m,j)} \mathcal{L}_{(m+1,j)} \cdots \mathcal{L}_{(i-1,j)})^{-1} \mathcal{G}_{(m,j)} \mathcal{G}_{(m,j+1)} \cdots \mathcal{G}_{(m,n-1)} & \text{if } i > m, j \leq n \\
\mathcal{L}_{(i,j)} \mathcal{L}_{(i+1,j)} \cdots \mathcal{L}_{(m-1,j)} (\mathcal{G}_{(m,n)} \mathcal{G}_{(m,n+1)} \cdots \mathcal{G}_{(m,j-1)})^{-1} & \text{if } i \leq m, j > n \\
(\mathcal{L}_{(m,j)} \mathcal{L}_{(m+1,j)} \cdots \mathcal{L}_{(i-1,j)})^{-1} (\mathcal{G}_{(m,n)} \mathcal{G}_{(m,n+1)} \cdots \mathcal{G}_{(m,j-1)})^{-1} & \text{if } i > m, j > n
\end{cases}
\end{equation}

We also associate a $\mathbb{Z}^2$-algebra $B = B\left(\{\mathcal{L}_{(i,j)}, (i,j)\}_{i,j \in \mathbb{Z}}, \{\mathcal{G}_{(i,j)}, (i,j)\}_{i,j \in \mathbb{Z}}\right)$ to the sequence via
\begin{equation}
B_{(i,j), (m,n)} = \begin{cases}
\Gamma(Y, \mathcal{B}_{(i,j), (m,n)}) & \text{if } i \leq m, j \leq n \\
0 & \text{else}
\end{cases}
\end{equation}
**Definition 6.3.** Let \( \{L_{i,j}\}_{i,j \in \mathbb{Z}} \), \( \{G_{i,j}\}_{i,j \in \mathbb{Z}} \) be a projective \( \mathbb{Z}^2 \)-sequence and let \( B \) be the associated sheaf of \( \mathbb{Z}^2 \)-algebras, then we say the sequence is an ample sequence if for each coherent sheaf \( F \) on \( Y \) and for each \( i, j \in \mathbb{Z} \) there exist \( i_0, j_0 \in \mathbb{Z} \) such that

\[
H^q(Y, F \otimes_Y B_{i,j},(m,n)) = 0
\]

holds for all \( q > 0 \) and \( m \geq i_0, n \geq j_0 \).

The following is immediate

**Proposition 6.4.** With the notation as in \([4]\) define \( L_{i,j} = \sigma^{(i+2)}L \) and \( G_{i,j} = \sigma^{(i+2)}L^r \sigma^{(i+2)}L \). Then \( \{L_{i,j}\}_{i,j \in \mathbb{Z}}, \{G_{i,j}\}_{i,j \in \mathbb{Z}} \) is an ample \( \mathbb{Z}^2 \)-sequence and \( B \{\{L_{i,j}\}_{i,j \in \mathbb{Z}}, \{G_{i,j}\}_{i,j \in \mathbb{Z}}\} = B_+ \).

**Proof.** It is obvious that \( B_+ \) equals the \( \mathbb{Z}^2 \)-algebra associated to the sequence and that \([20]\) is satisfied. It only remains to show that the associated sheaf of \( \mathbb{Z}^2 \)-algebras \( B \) satisfies the condition in Definition 6.3. For this let \( F \) be a coherent sheaf on \( Y \). As \( Y \) is a nonsingular elliptic curve, \([8, \text{Exercice II.6.11(c)}]\) gives us the existence of a line bundle \( L \) and a torsion sheaf \( T \) together with a short exact sequence:

\[
0 \to L^{(g)} \to F \to T \to 0
\]

where \( r \) is the rank of \( F \). As \( T \) is torsion, its support is zero dimensional and hence \( \mathcal{T} \otimes B_{i,j},(m,n) \) has no higher cohomology. In particular

\[
H^q(Y, F \otimes B_{i,j},(m,n)) \cong H^q(Y, L \otimes B_{i,j},(m,n))^{(g)}
\]

Again using the fact that \( Y \) is one dimensional, we only need to show \( H^1(Y, L \otimes B_{i,j},(m,n)) = 0 \) for sufficiently high \( m, n \). By Riemann-Roch the latter follows when \( \text{deg}(L \otimes B_{i,j},(m,n)) \geq 2g - 2 \) where \( g \) is the genus of \( Y \). Finally this condition is satisfied for \( m, n \) big enough as \( \text{deg}(B_{i,j},(m,n)) \) is a strictly increasing in function of the variables \( m \) and \( n \). \( \square \)

In particular Theorem 6.5 follows from Proposition 6.4 and the following:

**Theorem 6.5.** Let \( \{L_{i,j}\}_{i,j \in \mathbb{Z}}, \{G_{i,j}\}_{i,j \in \mathbb{Z}} \) be an ample \( \mathbb{Z}^2 \)-sequence on a Noetherian, projective scheme \( Y \), then \( B := B \{\{L_{i,j}\}_{i,j \in \mathbb{Z}}, \{G_{i,j}\}_{i,j \in \mathbb{Z}}\} \) is Noetherian in the sense that \( \text{Gr}(B) \) is a locally Noetherian category.

**Remark 6.6.** It was shown in \([6]\) that, under suitable conditions, twisted bi-homogeneous coordinate rings are Noetherian. The \((-)-\)construction which turns graded algebras into \( \mathbb{Z} \)-algebras can be generalized to turn bi-homogeneous algebras into \( \mathbb{Z}^2 \)-algebras, tri-homogeneous algebras into \( \mathbb{Z}^3 \)-algebras, etc. Moreover it is not hard to see that in this way we can turn a twisted bi-homogeneous coordinate ring into a \( \mathbb{Z}^2 \)-algebra of the form \([22]\). As such Theorem 6.5 is a generalization of \([6, \text{Theorem 5.2}]. \)

The proof of Theorem 6.5 follows from a chain of lemmas:

**Lemma 6.7.** Let \( F \) be a coherent sheaf on \( Y \), then for all \( i, j \in \mathbb{Z} : F \otimes B_{i,j},(m,n) \) is generated by its global sections for sufficiently large \( m, n \), in the sense that there exist \( i_0, j_0 \in \mathbb{Z} \) such that \( F \otimes_Y B_{i,j},(m,n) \) is generated by global sections whenever \( m \geq i, n \geq j \) or \( m \geq i_0, n \geq j_0 \).

**Proof.** This is a straightforward generalization of \([4, \text{Proposition 3.2.ii and Lemma 3.3}]. \) \( \square \)
Lemma 6.8. We have two pairs of functors between $\text{Gr}(B)$, $\text{Gr}(B)$ and $\text{Qcoh}(Y)$:

\[
\begin{array}{ccc}
\text{Gr}(B) & \overset{(-)}{\longrightarrow} & \text{Gr}(B) \\
\downarrow & & \downarrow \\
\Gamma_* := \bigoplus_{n \in \mathbb{Z}} \Gamma(Y, (-)_n) & \overset{(-)(0,0)}{\longrightarrow} & - \otimes_Y e_{(0,0)}B \\
\end{array}
\]

Moreover these satisfy the following properties:

1. $(-)(0,0)$ and $- \otimes_Y e_{(0,0)}B$ are quasi-inverses and define an equivalence of categories.
2. $(-)$ is left adjoint to $\Gamma_*$. 
3. $\Gamma_*$ is exact modulo torsion modules.
4. Let $M \in \text{Gr}(B)$ and define $\overline{M} := \Gamma_* (\overline{M})$. Then there is a natural map $M \to \overline{M}$, moreover the kernel and cokernel of this map are torsion.
5. $(-)$ is exact.

Proof. (1) This is standard and follows from the fact that $B$ is strongly graded (i.e. $B_{(a,b),(i,j),(m,n)} = B_{(a,b),(m,n)}$ holds for all $a, b, i, j, m, n \in \mathbb{Z}$).

(2) The functor $(-)$ is defined by considering $B$ as well as each graded $B$-module as constant sheaves on $Y$. As such $B$ obtains a natural left $B$-structure. The fact that $(-)$ is left adjoint to $\Gamma_*$ follows immediately from this construction.

(3) Analogous to [4] Lemma 3.7.(ii).

(4) The existence of the natural map $M \to \overline{M}$ follows from the adjunction in (2). The fact that the kernel and cokernel are torsion is a straightforward generalization of [4] Lemma 3.13.(i) and (iii).

(5) Analogous to [4] Lemma 3.13.(iv) (also using [4] Lemma 3.7.(iii)).

Remark 6.9. The above lemma implies that up to isomorphism any graded $B$-module $\mathcal{M}$ can be written as $\mathcal{F} \otimes_Y e_{(0,0)}B$ for some quasi-coherent sheaf $\mathcal{F}$. In the special case that $\mathcal{F}$ is coherent, $\mathcal{M}$ is called coherent as well.

Corollary 6.10. Let $\mathcal{M}$ be a coherent $B$-module. Then there exist $i, j, n \in \mathbb{Z}$, $n \geq 0$ such that $\mathcal{M}$ is a quotient of $(e_{(i,j)}B)^{\otimes n}$.

Proof. There is a coherent sheaf $\mathcal{F}$ such that $\mathcal{M} = \mathcal{F} \otimes (e_{(0,0)}B)$. By Lemma 6.7 there are $i, j \in \mathbb{Z}$ such that $\mathcal{F} \otimes B_{(0,0),(i,j)}$ is generated by global sections. As such $\mathcal{M}$ is a quotient of $\Gamma(Y, \mathcal{F} \otimes B_{(0,0),(i,j)}) \otimes_k e_{(i,j)}B$.

Lemma 6.11. For each $i, j \in \mathbb{Z}$, $\forall a, b \in \mathbb{Z} \cup \{-\infty\}$ we have that $(e_{(i,j)}B)^{\geq a, \geq b}$ is finitely generated. (Recall: the notation $M_{\geq a, \geq b}$ was introduced in [15].)

Proof. (inspired by [4] p.261-262]) Without loss of generality we assume $i = j = 0$. Moreover replacing $a, b$ by $\max(a, 0)$, $\max(b, 0)$ we can assume $a, b \geq 0$. By Lemma 6.7 there is an $m_0 \geq a$ such that $B_{(0,0),(m_0,b)}$ is generated by sections. I.e. there is a short exact sequence

\[
0 \to \mathcal{F} \to B_{(0,0),(m_0,b)} \otimes_k \mathcal{O}_Y \to B_{(0,0),(m_0,b)} \to 0
\]
As \( \{ \mathcal{L}_{(i,j)} \}_{i,j \in \mathbb{Z}} \), \( \{ \mathcal{G}_{(i,j)} \}_{i,j \in \mathbb{Z}} \) is an ample sequence (see Definition 6.3) there is \( m_1 > m_0 \) such that \( H^1(Y, F \otimes_Y B_{(m_0,b),(m,n)}) = 0 \) for all \( m \geq m_1 \) and \( n \geq b \). In particular, applying \( \Gamma_k \) to the above sequence provides a surjective morphism for all \( m \geq m_1 \) and \( n \geq b \)
\[
B_{(0,0),(m_0,b)} \otimes_k B_{(m_0,b),(m,n)} \rightarrow B_{(0,0),(m,n)}
\]
I.e. the natural map
\[
B_{(0,0),(m_0,b)} \otimes_k e_{(m_0,b)}B \rightarrow \left( e_{(0,0)}B \right)_{(\geq a, \geq b)}
\]
is surjective in degrees \((m, n)\) with \( m \geq m_1 \) and \( n \geq b \). Similarly there is a \( n_0 \geq b \) and \( n_1 > n_0 \) such that
\[
B_{(0,0),(a,n_0)} \otimes_k e_{(a,n_0)}B \rightarrow \left( e_{(0,0)}B \right)_{(\geq a, \geq b)}
\]
is surjective in degrees \((m, n)\) with \( m \geq a \) and \( n \geq n_0 \). Combining the above two morphisms we find a surjective morphism
\[
\bigoplus_{\substack{0 \leq m < m_1 \\ b \leq n < n_1}} B_{(0,0),(m,n)} \otimes_k e_{(m,n)}B \rightarrow \left( e_{(0,0)}B \right)_{(\geq a, \geq b)}
\]
As \( \dim_k(B_{(0,0),(m,n)}) < \infty \) for all \( m, n \) this finishes the proof. \( \square \)

**Lemma 6.12.** Let \( 0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0 \) be an exact sequence in \( \text{Gr}(B) \).

1. If \( M \) and \( M'' \) are finitely generated, then so is \( M' \).
2. If \( M' \) is finitely generated and there exist \( a, b, c, d \in \mathbb{Z} \) such that \( M''_{(i,j)} = 0 \) if \( i \geq c \) and \( j \geq d \) or if \( i < a \) or if \( j < b \) (i.e. there exists a \( N \in \text{Gr}(B) \) such that \( M'' = N_{(\geq a, \geq b)} / N_{(\geq c, \geq d)} \)), then \( M \) is finitely generated.

**Proof.**

1. This is standard.

2. By induction it suffices to show the lemma in case \( c = a, d = b + 1 \) or \( c = a + 1, d = b \). Without loss of generality we may assume the latter, i.e. \( M''_{(i,j)} = 0 \) if \( i \neq a \) or \( j < b \). As \( M' \) is finitely generated there exists as surjective morphism
\[
\bigoplus_{n=1}^{N} e_{(i_n,j_n)}B \rightarrow M'
\]
For each \( n \), consider the composition
\[
e_{(i_n,j_n)}B \rightarrow M''
\]
As this is a morphism of \( B \)-modules, it can only be nonzero if \( i_n = a, j_n \geq b \). Moreover for each such \( n \), \( e_{(a,j_n)}B \) is finitely generated by Lemma 6.11 say
\[
\bigoplus_{n=1}^{x_{n,0}} e_{(i_{x_n},j_{x_n})}B \rightarrow \left( e_{(a,j_n)}B \right)_{(\geq a + 1, \geq j_n)}
\]
Combining the above we obtain a morphism
\[
\bigoplus_{\substack{n=1 \\ i_n \neq a \ or \ j_n < b}}^{N} e_{(i_n,j_n)}B \oplus \bigoplus_{\substack{n=1 \\ i_n = a, j_n \geq b}}^{x_{n,0}} e_{(i_{x_n},j_{x_n})}B \rightarrow M
\]
and by construction this morphism is surjective in all degrees \((i, j)\) for which \( i \neq a \) or \( j < b \).
Now consider the \(\mathbb{Z}\)-algebra \(B'\) defined by

\[
B'_{n,m} := B_{(a,b+n),(a,b+m)}
\]

By construction \(B'_{n,m}\) is the twisted homogeneous coordinate ring with respect to the ample sequence \((G_{(a,b+n)})_{n\in\mathbb{Z}}\). It is standard that \(B'\) is a Noetherian \(\mathbb{Z}\)-algebra (see for example [4] or [11]). Moreover define \(B'\)-modules \(L\) and \(L'\) by

\[
L_{m} := M_{(a,b+n)} \quad L'_{m} := M'_{(a,b+m)}
\]

Recall that \(M'\) was finitely generated as in (23). For each \(n\) we know that \((e_{(i_n,j_n)}B)_{(\geq a, \geq b)}\) is finitely generated, say

\[
\bigoplus_{n=0}^{\infty} e_{(i_n,j_n)}B \twoheadrightarrow (e_{(i_n,j_n)}B)_{(\geq a, \geq b)}
\]

As such we obtain that \(L'\) is finitely generated. For the interested reader we mention that the explicit surjective morphism is given by

\[
\bigoplus_{n=1}^{N} \left( \bigoplus_{n=0}^{\infty} e_{(i_n,j_n)}B \twoheadrightarrow (e_{(i_n,j_n)}B)_{(\geq a, \geq b)} \right) \rightarrow L'
\]

As \(B'\) is Noetherian, \(L \subset L'\) is finitely generated as well, say

\[
\bigoplus_{v=1}^{v_0} e_{j,v}B' \rightarrow L
\]

The induced morphism

\[
(26) \quad \bigoplus_{v=1}^{v_0} e_{(a,b+j,v)}B \rightarrow M
\]

is surjective in all degrees \((i,j)\) with \(i = a, j \geq b\). Combining (25) and (26) we obtain that \(M\) is finitely generated as required.

The following lemma will be crucial in the proof of Theorem 6.5.

**Lemma 6.13.** Let \(M \in \text{Gr}(B)\) be such that \(\tilde{M}\) is coherent, then for each \(a, b \in \mathbb{Z}\): \(\overline{M}_{(\geq a, \geq b)}\) is a finitely generated graded \(B\)-module.

**Proof.** (inspired by [4] Lemma 3.17) As \(\tilde{M}\) is coherent, Corollary 6.10 provides us with \(i,j,n \in \mathbb{Z}\), \(n \geq 0\) and a surjective morphism

\[
(e_{(i,j)}B)^{\otimes n} \twoheadrightarrow \tilde{M}
\]

Let \(K\) be the kernel of this morphism. Then we have a long exact sequence

\[
0 \rightarrow \Gamma_{*}(K) \overset{f}{\rightarrow} e_{(i,j)}B^{\otimes n} \rightarrow \overline{M} \rightarrow H^{1}(K)
\]

Truncation turns this into an exact sequence

\[
0 \rightarrow \text{coker}(f)_{(\geq a, \geq b)} \rightarrow \overline{M}_{(\geq a, \geq b)} \rightarrow H^{1}(K)_{(\geq a, \geq b)}
\]

\(\text{coker}(f)_{(\geq a, \geq b)}\) is finitely generated as a quotient of \(e_{(i,j)}B^{\otimes n}_{(\geq a, \geq b)}\), which in turn is finitely generated by Lemma 6.11. \(K\) is coherent as a \(B\)-submodule of \(e_{(i,j)}B^{\otimes n}\), hence by the definition of an ample sequence, \(H^{1}(K)_{(\geq a, \geq b)}\) is concentrated in
as a set of generators for the modules $M$ are finitely generated. The result now follows from Lemma 6.12. □

We can now finish the proof of the main theorem of this section.

**Proof of Theorem 6.5.** As $e_{i,j} B$ obviously serves as a set of generators for $\text{Gr}(B)$ it suffices to show that these modules are Noetherian. Hence let $M$ be a submodule of some $e_{i,j} B$. By Lemma 6.8(5) this induces an embedding

$$\tilde{M} \hookrightarrow e_{i,j} B$$

As such $\tilde{M}$ is coherent and Lemma 6.13 implies $\tilde{M}$ is a finitely generated module. As $M = M_{i,j} B$ the natural map $M \rightarrow \tilde{M}$ factors through $\tilde{M}_{i,j} B$. Now consider the diagram

$$
\begin{array}{ccc}
M & \longrightarrow & e_{i,j} B \\
\downarrow & & \downarrow \\
\tilde{M}_{i,j} B & \longrightarrow & e_{i,j} B
\end{array}
$$

As the upper horizontal arrow and the right vertical arrow are injective, so is the left vertical arrow. We obtain a short exact sequence

$$(27) \quad 0 \rightarrow M \rightarrow \tilde{M}_{i,j} B \rightarrow \left(\frac{\tilde{M}}{M}\right)_{i,j} B \rightarrow 0$$

By Lemma 6.8(4) we have that $\tilde{M}/M$ and hence also $\left(\frac{\tilde{M}}{M}\right)_{i,j} B$ is torsion. Being a quotient of a finitely generated module, $\left(\frac{\tilde{M}}{M}\right)_{i,j} B$ is also finitely generated and hence concentrated in finitely many degrees. In particular there exist $c, d \in \mathbb{Z}$ such that $\left(\frac{\tilde{M}}{M}\right)_{i,j} B = 0$ for $i > c, j > d$. As such (27) satisfies the assumptions in Lemma 6.12, implying $M$ is finitely generated. □

7. **Proof of Theorem 1.1**

In this section we give the proof of Theorem 1.1. For this we fix the notations $A, A', \gamma, \delta, Y, L, \sigma, p, p', q', \tilde{A}$ and $\tilde{B}$ as in [11]. As a result of Theorem 5.5 we can apply Theorem 6.3 to the $\mathbb{Z}^2$-algebra $\tilde{A}$. Hence for each diagonal-like $\Delta : \mathbb{Z} \rightarrow \mathbb{Z}^2$ there is an equivalence of categories $\text{QGr}(\tilde{A}) \cong \text{QGr}(\tilde{A}_\Delta)$. Throughout this section we focus on a specific choice of $\Delta$: $\Delta(i) = (2i, i)$. For this choice of $\Delta$ we have the following:

$$\left(\tilde{A}_\Delta\right)_{i,j} = \tilde{A}_{(2i, i), (2j, j)} = \begin{cases} 
\text{Hom}_X(O_X(-4j), O_X(-4i)m_{i-4p} \cdots m_{j-4p} & \text{if } j \geq i \\
0 & \text{else}
\end{cases}$$

In particular there is an obvious inclusion

$$(28) \quad \tilde{A}_\Delta \hookrightarrow \tilde{A}^{(4)}$$

Moreover $\tilde{A}^{(4)}$ is 1-periodic as it is induced by the graded algebra $A^{(4)}$. The equality

$$o_X(4)m_d = m_{r,d}o_X(4)$$

implies that \( \tilde{A}_\Delta \) is compatible with this 1-periodicity. i.e. there is a graded algebra \( T \) such that \( \tilde{A}_\Delta = \tilde{T} \) and (29) is induced by an inclusion

\[
T \hookrightarrow A^{(4)}
\]

By construction \( T \) is the subalgebra of \( A^{(4)} \) generated by the elements in \( A^{(4)}_1 = A_4 \) whose images in \( A^{(4)}/g \) lie in \( \Gamma(Y, \mathcal{L} \sigma^* \mathcal{L} \sigma^* \mathcal{L}(-p)) \). This is exactly the definition of the noncommutative blow-up \( A^{(4)}(p) \) as in [14].

Next we need to show that \( \tilde{A}_\Delta = \tilde{T} \) can also be seen as a noncommutative blow-up of \( A' \). i.e. we need to show the existence of a divisor \( d' \) such that \( T = A^{(3)}(d') \).

We claim that \( d' = p' + q' \) with \( p' \) and \( q' \) as in [14] will do the job. First note that

\[
\delta^{-2i}(-)^{d_{2j}} : (\tilde{A}_\Delta)_{i,j} = \tilde{A}_{(2i,i),(2j,j)} \rightarrow \tilde{A}_{(0,3i),(0,3j)} \cong A'_{3i,3j}
\]

defines an inclusion \( \tilde{T} = \tilde{A}_\Delta \hookrightarrow \tilde{A}^{(3)} \). We need to show that (up to replacing the \( \delta_{(i,j)} \) by some scalar multiples) (29) is compatible with the 1-periodicity of \( \tilde{A}^{(3)} \) and \( \tilde{T} \) such that there is an induced inclusion \( T \hookrightarrow A^{(3)} \). To this we need to show that the periodicity isomorphisms \( \tilde{A}_{(0,3i),(0,3j)} \cong \tilde{A}_{(0,3i+3),(0,3j+3)} \) and \( \tilde{A}_{(2i,i),(2j,j)} \cong \tilde{A}_{(2i+2,i+1),(2j+2,j+1)} \) induced by \( A^{(3)} \) and \( T \) respectively, are compatible in the sense that the following diagram commutes

\[
\begin{array}{c}
\tilde{A}_{(2i,i),(2j,j)} \\
\cong \\
\tilde{A}_{(2i+2,i+1),(2j+2,j+1)} \\
\delta^{-2i}(-)^{d_{2j}} \\
\tilde{A}_{(0,3i),(0,3j)} \\
\cong \\
\tilde{A}_{(0,3i+3),(0,3j+3)} \\
\delta^{-2i}(-)^{d_{2j+2}}
\end{array}
\]

(30)

As all morphisms in the above diagram are compatible with the algebra structure of \( \tilde{A} \) and as \( T \) and \( A' \) are generated in degree 1, it suffices to check that the above diagram commutes for \( j = i + 1 \).

Recall that there is a central element \( g' \in A'_3 \) such that \( A'/g'A' \cong B' \) where \( B' \) is the twisted homogeneous coordinate ring of \( (Y, \mathcal{L} \otimes \sigma^* \mathcal{L}(-p), \psi) \). Let \( g'_i \) be the corresponding element in \( \tilde{A}'_{j,i+j+3} \). Then for each \( i \) there is an exact sequence

\[
0 \rightarrow kg'_i \rightarrow \tilde{A}_{(0,3i),(0,3i+3)} \rightarrow \tilde{B}_{(0,3i),(0,3i+3)} \rightarrow 0
\]

Similarly by Lemma [5.1] for each \( i \) there is an element \( g_{(2i,i)} \in \tilde{A}_{(2i,i),(2i+2,i+1)} \) together with an exact sequence

\[
0 \rightarrow kg_{(2i,i)} \rightarrow \tilde{A}_{(2i,i),(2i+2,i+1)} \rightarrow \tilde{B}_{(2i,i),(2i+2,i+1)} \rightarrow 0
\]

Now if we let \( \tilde{\delta}_{(i,j)} \) be the image of \( \delta_{(i,j)} \) under \( \tilde{A}_{(i,j),(i-1,j+1)} \rightarrow \tilde{B}_{(i,j),(i-1,j+1)} \) then we see that (29), (31) and (32) are compatible in the sense that there is a commutative diagram:
\[
\begin{array}{ccccccc}
0 & \xrightarrow{k g'_{3i}} & \hat{A}_{(0,3i),(0,3i+3)} & \xrightarrow{\delta^{-2i}(-\delta^j)} & \hat{B}_{(0,3i),(0,3i+3)} & \xrightarrow{\delta^{-2i}(-\delta^j)} & 0 \\
\delta^{-2i}(-\delta^j) & & \delta^{-2i}(-\delta^j) & & \delta^{-2i}(-\delta^j) & & \\
0 & \xrightarrow{k g_{(2i,i)}} & \hat{A}_{(2i,i),(2i+2,i+1)} & \xrightarrow{\delta^{-2i}(-\delta^j)} & \hat{B}_{(2i,i),(2i+2,i+1)} & \xrightarrow{\delta^{-2i}(-\delta^j)} & 0 \\
\end{array}
\]

Hence in order to prove commutativity of (30) it suffices to prove commutativity of the following two diagrams:

\[
\begin{array}{ccc}
\hat{B}_{(2i,i),(2i+2,i+1+1)} & \xrightarrow{\delta^{-2i}(-\delta^j)} & \hat{B}_{(0,3i),(0,3i+3)} \\
\mathcal{B}_{(2i+2,i+1+1),(2j+2,j+1)} & \xrightarrow{\delta^{-2i}(-\delta^j)} & \hat{B}_{(0,3i+3),(0,3i+6)} \\
\end{array}
\]

(33)

and

\[
\begin{array}{ccc}
k g_{(2i,i)} & \xrightarrow{\delta^{-2i}(-\delta^j)} & k g'_{3i} \\
\mathcal{B}_{(2i+2,i+1+1)} & \xrightarrow{\delta^{-2i}(-\delta^j)} & k g'_{3i+3} \\
\end{array}
\]

(34)

We first focus on (33). The left vertical arrow is given by

\[
\tau^* : \Gamma(Y, \sigma^{4i-1} \mathcal{L} \ldots \sigma^{4i+3} \mathcal{L}(-\tau^{-i} p)) \rightarrow \Gamma(Y, \sigma^{4i+4} \mathcal{L} \ldots \sigma^{4i+7} \mathcal{L}(-\tau^{-i-1} p))
\]

A closer investigation of the 1-periodicity of $\mathcal{B}^{(3)}$ shows that the right vertical arrow in (33) factors as

\[
\begin{array}{cccc}
\hat{B}_{(0,3i),(0,3i+3)} & \xrightarrow{\tau^*} & \hat{B}_{(2,3i+1),(2,3i+4)} & \xrightarrow{\varphi} & \hat{B}_{(0,3i+3),(0,3i+6)} \\
\end{array}
\]

where $\varphi$ is given by multiplication by a nonzero section of

\[
\left(\sigma^{(6i+4)} \mathcal{L} \sigma^{(6i+5)} \mathcal{L}(-\tau^{-3i-1} p - \tau^{-3i-2} p)\right)^{-1} \left(\sigma^{(6i+11)} \mathcal{L} \sigma^{(6i+12)} \mathcal{L}(-\tau^{-3i-4} p - \tau^{-3i-5} p)\right)
\]

As $\delta_{(m,n)}$ is a nonzero section of $\sigma^{(m+2n)} \mathcal{L}(-\tau^{-j} p)$, we can (after inductively changing the $\delta_{(1,a)}$ by a scalar multiple) assume that the morphism

\[
\left(\delta_{(1,3i+2)}(2,3i+1)\right)^{-1} \left(-\delta_{(2,3i+4)}(1,3i+5)\right) : \hat{B}_{(2,3i+1),(2,3i+4)} \rightarrow \hat{B}_{(0,3i+3),(0,3i+6)}
\]

coincides with $\varphi$ for all $i$. Next note that $\tau^*$ maps $k g_{(m,n)} = \hat{B}_{(m,n),(m-1,n+1)}$ to $\hat{B}_{(m+2,n+1),(m+1,n+2)} = k g_{(m+2,n+1)}$. Hence (after changing the $\delta_{(1+2b,a+b)}$ and $\delta_{(2+2b,a+b)}$ by a scalar multiple by induction on $b$) we can assume

\[
\tau^* \delta_{(m,n)} = \delta_{(m+2,n+1)}
\]

(36)

In particular (33) commutes as it factors as follows:
Next we focus on commutativity of (34). The right vertical arrow in (34) can be described as follows: write $g_{3i}$ as a product of elements in $A'_{3i,3i+1} = B'_{3i,3i+1}$, $A_{3i+1,3i+2} = B'_{3i+1,3i+2}$ and $A_{3i+2,3i+3} = B'_{3i+2,3i+3}$, apply $\varphi \circ \tau^*$ : $B'_{(m,m+1)} \to B'_{(m+3,m+4)}$ on each of the 3 factors, then $g'_{3i+3}$ is the product of these 3 new elements. This remark together with (36) shows that the commutativity of (34) reduces to the following claim:

if $g(2i,j)\delta(2i-1,i-1) = a_0 \cdot a_1 \cdot a_2$ for $a_n \in A_{(2i,i+n),(2i,i+n+1)} = \tilde{B}_{(2i,n+n)}$, then $g(2i+1,j)\delta(2i-1,i+1) = \tau^* a_0 \cdot \tau^* a_1 \cdot \tau^* a_2$.

As each there are embeddings (37) it suffices to check the claim in $\tilde{A}$. For this denote $x$ for the image of some $x \in \tilde{A}$ under the embedding (37). Consider the equality

$$g(2i,j) \cdot \delta(2i,j) \cdot \delta(2i-1,j+1) = a_0 \cdot a_1 \cdot a_2$$

As $\delta(2i,j), \delta(2i-1,j+1), a_0, a_1$ and $a_2$ lie in $\tilde{B}$ the 4-periodicity morphism $\tilde{A}_{m,n} \to \tilde{A}_{m+4,n+4}$ sends them to $\tau^* \delta(2i,j), \tau^* \delta(2i-1,j+1), \tau^* a_0 = \tau^* a_0, \tau^* a_1 = \tau^* a_1$ and $\tau^* a_2 = \tau^* a_2$. Moreover by (36)

$$\tau^* \delta(i,j) = \tau^* \delta(i,j) = \delta(i+2,j+1)$$

Finally notice that by construction $g(2i,j)$ is the element in $\tilde{A}_{4i-4i+4}$ corresponding to the central element $g \in A_4$, as such $\tilde{A}_{4i,4i+4} \to \tilde{A}_{4i+4,4i+8}$ sends it to $g(2i+2,j+1)$.

In particular the 4-periodicity of $\tilde{A}$ turns the equality (38) into

$$g(2i+2,j+1) \cdot \delta(2i+2,j+1) \cdot \delta(2i+1,j+1) = \tau^* a_0 \cdot \tau^* a_1 \cdot \tau^* a_2$$

proving our claim and hence showing commutativity of (34) and hence of (30). As mentioned above this implies that the inclusion (29) induces an inclusion $T \hookrightarrow A'(3)$.

Our goal is to understand the image of this inclusion. As $T$ is generated in degree 1, it suffices to understand the image of

$$T_1 \cong \tilde{A}_{(0,0),(2,1)} \xrightarrow{\delta^2} \tilde{A}_{(0,0),(0,3)} \cong A'_3$$

As was mentioned above this image contains $g'$ and the image of $T_1 \to A'_3 \to A'_3 / g'$ is the same as the image of

$$\delta(2,1) \delta(1,2) : \tilde{B}_{(0,0),(2,1)} \to \tilde{B}_{(0,0),(0,3)} \cong B'_3$$

Following the computations in [12 §8.2] $\delta(2,1) \delta(1,2)$ is a nonzero global section of $\sigma^* (-1) \mathcal{L} \sigma^* (-2) \mathcal{L}(-\tau^{-1} p - \tau^{-2} p - q' - q')$.
where $p', q'$ are as in [13].

Hence $T$ is the subalgebra of $A^{(3)}$ generated by the elements of $A_3'$ whose image in $B_3' = \Gamma(Y, \mathcal{L} \sigma^{-(1)} \mathcal{L} \sigma^{-(2)} \mathcal{L}(-p - \tau^{-1}p - \tau^{-2}p))$

lies in $B_3' = \Gamma(Y, \mathcal{L} \sigma^{-(1)} \mathcal{L} \sigma^{-(2)} \mathcal{L}(-p - \tau^{-1}p - \tau^{-2}p - q' - p'))$

Hence $T$ is isomorphic to the noncommutative blow up $A^{(3)}(p' + q')$ as desired.

This finishes the proof of Theorem 1.1.

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