We study the elliptic algebras $Q_{n,k}(E, \tau)$ introduced by Feigin and Odesskii as a generalization of Sklyanin algebras. This is a family of quadratic algebras parametrized by coprime integers $n > k \geq 1$, an elliptic curve $E$, and a point $\tau \in E$. We compare several different definitions of these algebras and provide proofs of several statements about them made by Feigin and Odesskii. For example, we show that $Q_{n,k}(E,0)$, and $Q_{n,n-1}(E, \tau)$ for generic $\tau$, is a polynomial ring on $n$ variables. We also show that $Q_{n,k}(E, \tau + \zeta)$ is a Zhang twist of $Q_{n,k}(E, \tau)$ when $\zeta$ is an $n$-torsion point. This paper is the first of several we are writing about $Q_{n,k}(E, \tau)$.

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1. Introduction

1.1. Notation and conventions. Throughout this paper we use the notation $e(z) = e^{2\pi iz}$ for $z \in \mathbb{C}$.

We fix relatively prime integers, $n > k \geq 1$, and write $k'$ for the unique integer such that $n > k' \geq 1$ and $kk' = 1$ in $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

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We fix a point \( \eta \in \mathbb{C} \) lying in the upper half-plane, the lattice \( \Lambda = \mathbb{Z} + \mathbb{Z}\eta \), and the elliptic curve \( E = \mathbb{C}/\Lambda \). We write \( E[n] = \frac{1}{n}\Lambda/\Lambda \) for the \( n \)-torsion subgroup of \( E \).

We always work over the field \( \mathbb{C} \) of complex numbers unless otherwise specified. For an algebraic variety \( X \), \( x \in X \) means \( x \) is a closed point of \( X \).

1.2. The algebras \( Q_{n,k}(E, \tau) \). In 1989, Feigin and Odesskii defined a family of graded \( \mathbb{C} \)-algebras \( Q_{n,k}(E, \tau) \) depending on the data \( (n, k, E) \) and a point \( \tau \in \mathbb{C} - \frac{k}{n}\Lambda \). The algebras appear first in their manuscript [FO89] archived with the Academy of Science of the Ukrainian SSR (which we refer to as “the Kiev preprint”) and, almost simultaneously, in their published paper [OF89]. They defined \( Q_{n,k}(E, \tau) \) to be the free algebra \( \mathbb{C}\langle x_0, \ldots, x_{n-1} \rangle \) modulo the \( n^2 \) homogeneous quadratic relations\(^1\)

\[
R_{ij} = \sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r}(-\tau)\theta_{kr}(\tau)} x_{j-r} x_{i+r}
\]

where the indices \( i \) and \( j \) belong to \( \mathbb{Z}_n \) and \( \theta_0(z), \ldots, \theta_{n-1}(z) \) are certain theta functions of order \( n \), also indexed by \( \mathbb{Z}_n \), that are quasi-periodic with respect to the lattice \( \Lambda \). When \( \tau \in \frac{1}{n}\Lambda \), \( \theta_{kr}(\tau) = 0 \) for some \( r \) so the relations no longer make sense. Nevertheless, we show in §3.2 that there is a way to extend the definition to all \( \tau \in \mathbb{C} \) (Definition 3.2). In fact, \( Q_{n,k}(E, \tau) \) depends only on the image of \( \tau \) in \( E \), so for fixed \( (n, k, E) \) the algebras provide a family parametrized by \( E \).

A lot is known about the algebras \( Q_{n,1}(E, \tau) \). In [TVB96], Tate and Van den Bergh showed that \( Q_{n,1}(E, \tau) \) is a noetherian domain having the same Hilbert series and the same homological properties as the polynomial ring on \( n \) variables. The algebras \( Q_{3,1}(E, \tau) \) and \( Q_{4,1}(E, \tau) \) are very well understood due to the work of Artin-Tate-Van den Bergh ([ATVdB90, ATVdB91]), Smith-Stafford [SS92], and Levasseur-Smith [LS93]. For the most part though, the representation theory of \( Q_{n,1}(E, \tau) \) remains a mystery when \( n \geq 5 \).

Although the algebras \( Q_{n,k}(E, \tau) \) were defined almost 30 years ago they have not been studied much since then (with the exception of the case \( k = 1 \)). The algebras \( Q_{4,1}(E, \tau) \) were discovered by Sklyanin [Skl82] almost 40 years ago when he was studying questions arising from quantum physics. We endorse a sentiment he expressed in that paper:

> During our investigation it turned out that it is necessary to bring into the picture new algebraic structures, namely, the quadratic algebras of Poisson brackets and the quadratic generalization of the universal enveloping algebra of a Lie algebra. The theory of these mathematical objects is surprisingly reminiscent of the theory of Lie algebras, the difference being that it is more complicated. In our opinion, it deserves the greatest attention of mathematicians.

In investigating the algebras \( Q_{n,k}(E, \tau) \) one naturally encounters a fairly eclectic mix of mathematics. A few examples:

- The origin of these algebras in the study of elliptic solutions of the quantum Yang-Baxter equation is evident in the appearance and prevalence of \( R \)-matrices with spectral parameter defining the relations of \( Q_{n,k}(E, \tau) \).
- Theta functions and the sometimes mysterious identities they satisfy pervade the subject.
- When regarded as parametrized by \( \tau \), the family \( Q_{n,k}(E, \tau) \) “integrates” a natural Poisson structure on a moduli space of bundles on \( E \) of given rank and degree [FO98, Pol98].
- Understanding the point scheme for \( Q_{n,k}(E, \tau) \) is heavily reliant on the intricacies of the theory of bundles over abelian varieties.

We believe that this wide array of topics speaks to the depth of the subject and its richness as a source of problems, questions and perhaps answers. For that reason, we echo Sklyanin’s opinion that the algebras \( Q_{n,k}(E, \tau) \) deserve considerable attention.

\(^1\) The original definition uses \( x_{k(j-r)} x_{k(i+r)} \) instead of \( x_{j-r} x_{i+r} \); see §3.1.1.
1.3. The contents of subsequent papers. This paper is the first of several in which we examine the algebras $Q_{n,k}(E,\tau)$. One of them examines the characteristic variety $X_{n/k}$ for $Q_{n,k}(E,\tau)$, which is a subvariety of $\mathbb{P}^{n-1}$. Another will show that a certain quotient category of graded $Q_{n,k}(E,\tau)$-modules contains a “closed subcategory” that is equivalent to $\text{Qcoh}(X_{n/k})$, the category of quasi-coherent sheaves on $X_{n/k}$. This is done by exhibiting a homomorphism from $Q_{n/k}(E,\tau)$ to a “twisted homogeneous coordinate ring” of $X_{n/k}$ (defined in [AVdB90]). In many cases, $X_{n/k}$ is the $g$-fold product, $E^g$, of copies of $E$ where $g$ is the length of a certain continued fraction expression for the rational number $n/k$. For example, if $f_0 = f_1 = 1$ and $f_{i+1} = f_i + f_{i-1}$ and $(n,k) = (f_{2g+1},f_{2g-1})$, then $X_{n/k} \cong E^g$. If $k = 1$, then $g = 1$ and $X_{1/1} \cong E$. If $n \geq 5$ and $k = 2$, then $g = 2$ and $X_{n/k} \cong S^2E$ the $2^{nd}$ symmetric power of $E$.

Another paper will confirm the conjecture at [Ode02, §3, p. 1143], that the dimensions of the homogeneous components of $Q_{n,k}(E,\tau)$ are the same as those of the polynomial ring on $n$ variables. When $k = 1$, this was proved by Tate and Van den Bergh [TVdB96]. In fact, $Q_{n,k}(E,0)$ is the polynomial ring on $n$ variables so, for a fixed $(n,k,E)$ the algebras $Q_{n,k}(E,\tau)$ form a flat family of deformations of the polynomial ring on $n$ variables parametrized by the points of $E$.

1.4. The contents of this paper. The present paper is a prerequisite for our later papers.

In section 2 (see (2-6)) we fix a basis $\theta_0(z),\ldots,\theta_{n-1}(z)$ for a space $\Theta_n(\Lambda)$ of order-$n$ theta functions that are quasi-periodic with respect to $\Lambda$. We use this basis in the rest of this paper and in our subsequent papers. Theta functions are notorious for the fact that notation for them varies considerably from one source to another. Even when the same symbol appears in two different sources the reader must be alert to the possibility that the functions they denote are not the same. That is the case in Feigin and Odesskii’s various papers. For that reason, §2.2 makes a careful comparison of their various definitions and describes exactly how our $\theta_0(z),\ldots,\theta_{n-1}(z)$ relate to their functions labelled by the same symbols.

We then discuss the action of the Heisenberg group $H_n$ of order $n^2$ on $\Theta_n(\Lambda)$ and the canonical morphism $E = \mathbb{C}/\Lambda \to \mathbb{P}(\Theta_n(\Lambda)^*)$ to the projective space of 1-dimensional subspaces of the dual space $\Theta_n(\Lambda)^*$.

In section 3 we repeat the definition of $Q_{n,k}(E,\tau)$ for $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$ and show that the algebra depends only on the image of $\tau$ in the elliptic curve $E = \mathbb{C}/\Lambda$. Definition 3.2 defines $Q_{n,k}(E,\tau)$ for all $\tau \in E$.

In §3.3, we show that $Q_{n,k}(E,\tau) \cong Q_{n,k'}(E,\tau)$ for $k' = \{n > k' \geq 1 \text{ and } kk' = 1 \in \mathbb{Z}_n\}$. Feigin and Odesskii state this result but leave its proof to the reader. Feigin and Odesskii state several results without indicating how they might be proved. Some, like this isomorphism, are straightforward but we have had difficulty proving others. For that reason, and because the definition of the $\theta_i$’s in one of their papers is not always the same as in others, the proofs we provide should be useful to others.

One example of a result they state that we have been unable to verify is that the isomorphisms referred to in the previous paragraph are the only isomorphisms that occur among the algebras. Proposition 5.3 shows that $Q_{n,n-1}(E,\tau)$ is a polynomial ring for generic $\tau$ (and for all $\tau$ modulo the conjecture mentioned at the end of §1.3) so that non-isomorphism claim needs more precision. Perhaps it is the case that except for those isomorphisms the only isomorphisms are those mentioned in the previous paragraph.

In section 4 we show that $Q_{n,k}(E,\tau + \zeta)$ is isomorphic to a “twist” of $Q_{n,k}(E,\tau)$ for all $\zeta \in E[1]$. The Heisenberg group $H_n$ acts as degree-preserving algebra automorphisms of $Q_{n,k}(E,\tau)$. There is a surjective homomorphism $H_n \to E[n] = \mathbb{Z}_n \times \mathbb{Z}_n$ and the twist just referred to is induced by any one of the automorphisms in $H_n$ that is a preimage of $\zeta$. Since $Q_{n,k}(E,0)$ is a polynomial ring on $n$ variables (Proposition 5.1) this confirms Feigin and Odesskii’s statement [OF89, §1.2, Rem. 1] that $Q_{n,k}(E,\zeta)$ is isomorphic to an algebra of “skew polynomials” though they don’t define that term. The “twist” construction is quite general. Given any $\mathbb{Z}$-graded ring $A$ and a degree-preserving automorphism $\phi: A \to A$ the twist $A^\phi$ is the graded vector space $A$ endowed with multiplication $a * b := \phi^n(a)b$ when $b \in A_m$. There is an equivalence $\text{Gr}(A) \equiv \text{Gr}(A^\phi)$ between their categories of graded left modules.

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2Regarding the various notations for theta functions, the final paragraph of [AS64, §16.27] provides this warning: “There is a bewildering variety of notations ... so that in consulting books caution should be used”.
In section 5, we provide a proof of the assertion in [OF89, §1.2, Rem. 1] and [Ode02, §3] that $Q_{n,k}(E,0)$ is a polynomial ring on $n$ variables.

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2. Theta functions in one variable

In this section we collect some results on theta functions.

The results are “standard” but we could not find a single source that states them in quite the way we need them so we have included them here. Some of our proofs are given in more detail than strictly necessary but we did that because the calculations are often intricate and prone to error; the extra detail will make it easier for others to check our calculations.

2.1. The spaces $\Theta_n(\Lambda)$ and Riemann’s theta function. We fix an integer $n \geq 1$ and a point $c \in \mathbb{C}$.

In keeping with the notation in Odesskii’s survey article [Ode02, Appendix A], we write $\Theta_{n,c}(\Lambda)$ for the set of holomorphic functions $f(z)$ satisfying the quasi-periodicity conditions

$$f(z+1) = f(z),$$

$$f(z+\eta) = e(-nz + c + \frac{n^2}{2})f(z).$$

Proposition 2.1. $\Theta_{n,c}(\Lambda)$ is a vector space of dimension $n$.

Proof. This can be shown by looking at the Fourier expansions of elements of $\Theta_{n,c}(\Lambda)$. See [Mum07, I,§1], for example. □

In keeping with the notation in the Kiev preprint [FO89, p. 32] and in the first Odesskii-Feigin paper [OF89, §1.1], we will always use the notation

$$\Theta_n(\Lambda) := \Theta_{n,\frac{n-1}{2}}(\Lambda).$$

When $c = \frac{n-1}{2}$ the second quasi-periodicity condition becomes $f(z+\eta) = -e(-nz)f(z)$.

2.1.1. The Riemann theta function in one variable is the holomorphic function

$$\vartheta(z,\eta) := \sum_{n \in \mathbb{Z}} e(nz + \frac{1}{2}n^2\eta).$$

We will use the function $\theta(z) = \vartheta(z - \frac{1}{2} - \frac{1}{2}n\eta)$ as our basic (order one) theta function in one variable. Its Fourier expansion is given in (2-1).

Lemma 2.2. The function

$$\theta(z) := \sum_{n \in \mathbb{Z}} (-1)^n e(nz + \frac{1}{2}n(n-1)\eta)$$

has the following properties:

1. it is a basis for $\Theta_{1,0}(\Lambda)$;
2. $\theta(z+1) = \theta(z)$ and $\theta(z+\eta) = -e(-z)\theta(z)$;
3. $\theta(-z) = -e(-z)\theta(z)$;
4. $\theta(z) = 0$ if and only if $z \in \Lambda$. Each zero has order 1.

3 Usually $n$ is the integer fixed in §1.1 but we also allow $n = 1$ here.
Proof. Statement (1), and hence (2), follows from the fact that \( \vartheta(z, \eta) \) is a basis for \( \Theta_{1,\frac{1}{2}-\frac{1}{2}\eta}(\Lambda) \), which can be found in [Mum07, §1.1].

It follows from the definition of \( \theta(z) \) that
\[
\theta(-z) = \sum_{n \in \mathbb{Z}} (-1)^n e(-nz + \frac{1}{2}n(n-1)\eta)
\]
\[
= \sum_{n} (-1)^{-n} e(-nz + \frac{1}{2}(-n)(-n+1)\eta)
\]
\[
= \sum_{m} (-1)^{m-1} e(mz - z + \frac{1}{2}(m-1)m\eta) \quad \text{(after setting } m = -n + 1)\]
\[
= -e(-z) \sum_{m} (-1)^m e(mz + \frac{1}{2}m(m-1)\eta)
\]
\[
= -e(-z) \theta(z)
\]
as claimed in (3).

Statement (4) follows from [Mum07, Lem. 4.1]. Indeed, it is shown there that the zeroes of \( \vartheta_{00}(z) = \vartheta(z, \eta) \) are the points in \( \frac{1}{2} + \frac{1}{2}\eta + \Lambda \) and those zeroes have order 1. Thus the zeroes of \( \theta(z) = \vartheta(z - \frac{1}{2} - \frac{1}{2}\eta, \eta) \) are points in \( \Lambda \) and they also have order 1. \( \square \)

2.2. The standard basis for \( \Theta_n(\Lambda) \). In their various papers Feigin and Odesskii use a basis for \( \Theta_{n,c}(\Lambda) \) that is labelled \( \theta_0(z), \ldots, \theta_{n-1}(z) \). The functions they call \( \theta_\alpha(z) \) in one paper are not always the same as those called \( \theta_\alpha(z) \) in another paper. Nevertheless, in the papers [FO89, OF89, Ode02] the zeroes of \( \theta_\alpha(z) \) are always at
\[
\left\{ \frac{1}{n}(\alpha_\eta + m) \mid 0 \leq m \leq n - 1 \right\} + \Lambda.
\]
In particular, \( \theta_\alpha(z) \) has \( n \) distinct zeroes in the fundamental parallelogram
\[
[0,1) + (-1,0] \eta = \left\{ a + b\eta \mid 0 \leq a < 1, \ -1 < b \leq 0 \right\},
\]
each zero having multiplicity 1. Furthermore, their \( \theta_\alpha(z) \)'s, \( \alpha \in \mathbb{Z}_n \), always have the properties
\[
\theta_\alpha(z + \frac{1}{n}) = e\left( \frac{\alpha}{n} \right) \theta_\alpha(z),
\]
\[
\theta_\alpha(z + \frac{1}{n^2}) = C e(-z) \theta_{\alpha+1}(z),
\]
where \( C \) is a non-zero constant independent of \( \alpha \).

Since \( \theta(z) \) has a unique zero in the fundamental parallelogram, namely a simple zero at \( z = 0 \), the function
\[
\theta(z + \frac{a}{n} \eta) \theta(z + \frac{1}{n} + \frac{a}{n} \eta) \cdots \theta(z + \frac{n-1}{n} + \frac{a}{n} \eta)
\]
has exactly \( n \) zeroes in the fundamental parallelogram, namely \( \left\{ \frac{1}{n}(\alpha_\eta + m) \mid 0 \leq m \leq n - 1 \right\} \), each of which has order one. Thus, Feigin and Odesskii’s functions \( \theta_\alpha(z) \), \( \alpha \in \mathbb{Z}_n \), are multiples of the functions in (2-2) by nowhere vanishing functions.

Lemma 2.3. For each \( \alpha \in \mathbb{Z} \) let \( [\alpha] \in \mathbb{C} \) be an arbitrary complex number.\(^4\) The functions
\[
\theta_\alpha(z) := e(\alpha z + [\alpha]) \prod_{m=0}^{n-1} \theta(z + \frac{m}{n} + \frac{\alpha}{n} \eta),
\]
indexed by \( \alpha \in \mathbb{Z} \), have the following properties:
\begin{enumerate}
\item \( \theta_\alpha(z + 1) = \theta_\alpha(z) \) and \( \theta_\alpha(z + \eta) = -e(-nz) \theta_\alpha(z) \),
\item \( \theta_\alpha(z) \in \Theta_{n,\frac{n}{2n}}(\Lambda) \),
\item \( \theta_\alpha(z + \frac{1}{n}) = e\left( \frac{\alpha}{n} \right) \theta_\alpha(z) \),
\item \( \theta_\alpha(z + \frac{1}{n^2}) = e\left( \frac{\alpha}{n^2} \right) \theta_{\alpha+1}(z) \).
\end{enumerate}

\(^4\)Later we will make a judicious choice of \([\alpha]\).
(5) \( \theta_\alpha(-z) = -e(-nz + \alpha \eta + [\alpha] - [-\alpha]) \theta_{-\alpha}(z) \), and
(6) \( \theta_{\alpha+n}(z) = -e([\alpha + n] - [\alpha] - \alpha \eta) \theta_\alpha(z) \).

**Proof.** (1) First, the equality \( \theta_\alpha(z + 1) = \theta_\alpha(z) \) is immediate. Second, since \( \theta(z + \eta) = -e(-z) \theta(z) \),

\[
\theta_\alpha(z + \eta) = e(\alpha(z + \eta) + [\alpha]) \prod_{m=0}^{n-1} \theta(z + \frac{m}{n} + \frac{\alpha}{n} \eta)
\]

\[
= (-1)^n e(\alpha \eta) e(\alpha z + [\alpha]) e(-nz - \frac{1}{n} - \frac{2}{n} - \cdots - \frac{n-1}{n} - \alpha \eta) \prod_{m=0}^{n-1} \theta(z + \frac{m}{n} + \frac{\alpha}{n} \eta)
\]

\[
= (-1)^n e(-nz) e(\alpha z + [\alpha]) e(-\frac{1}{2}(n-1)) \prod_{m=0}^{n-1} \theta(z + \frac{m}{n} + \frac{\alpha}{n} \eta)
\]

\[
= -e(-nz) \theta_\alpha(z).
\]

(2) This is simply a restatement of (1).

(3) Since \( \theta(z + 1) = \theta(z) \),

\[
\theta_\alpha(z + \frac{1}{n}) = e(\alpha(z + \frac{1}{n}) + [\alpha]) \prod_{m=0}^{n-1} \theta(z + \frac{1+m}{n} + \frac{\alpha}{n} \eta)
\]

\[
= e(\alpha \eta) e(\alpha z + [\alpha]) \theta(z + \frac{1}{n} + \frac{\alpha}{n} \eta) \cdots \theta(z + \frac{n-1}{n} + \frac{\alpha}{n} \eta) \theta(z + \frac{n}{n} + \frac{\alpha}{n} \eta)
\]

\[
= e(\alpha \eta) e(\alpha z + [\alpha]) \prod_{m=0}^{n-1} \theta(z + \frac{m}{n} + \frac{\alpha}{n} \eta)
\]

\[
= e(\alpha \eta) \theta_\alpha(z),
\]

as claimed.

(4) Similarly,

\[
\theta_\alpha(z + \frac{1}{n} \eta) = e(\alpha(z + \frac{1}{n} \eta) + [\alpha]) \prod_{m=0}^{n-1} \theta(z + \frac{m}{n} + \frac{1+m}{n} \eta)
\]

\[
= e(\alpha(z + \frac{1}{n} \eta) + [\alpha]) e(-\alpha + 1) z - [\alpha + 1]) \theta_{\alpha+1}(z)
\]

\[
= e(\alpha \eta + [\alpha] - [\alpha + 1]) e(-z) \theta_{\alpha+1}(z),
\]

as claimed.

(5) Since \( \theta(-z) = -e(-z) \theta(z) \),

\[
\theta_\alpha(-z) = e(-\alpha z + [\alpha]) \prod_{m=0}^{n-1} \theta(-z + \frac{m}{n} + \frac{\alpha}{n} \eta)
\]

\[
= e(-\alpha z + [\alpha]) \prod_{m=0}^{n-1} (-1) e(-z + \frac{m}{n} + \frac{\alpha}{n} \eta) \theta(z - \frac{m}{n} - \frac{\alpha}{n} \eta)
\]

\[
= (-1)^n e(-\alpha z + [\alpha]) e(-nz + \alpha \eta) e(\frac{1}{n} + \cdots + \frac{n-1}{n}) \prod_{m=0}^{n-1} \theta(z - \frac{m}{n} - \frac{\alpha}{n} \eta).
\]

Let \( p(z) \) be the product of factors before the product symbol in the last formula. Then

\[
p(z) = (-1)^n e(-\alpha z + [\alpha]) e(-nz + \alpha \eta) e(\frac{1}{n} (n-1))
\]

\[
= (-1)^n e(-\alpha z + [\alpha]) e(-nz + \alpha \eta) (-1)^{n-1}
\]

\[
= -e(-nz + \alpha \eta + [\alpha]) e(-\alpha z).
\]
On the other hand,
\[
\prod_{m=0}^{n-1} \theta(z - \frac{m}{n} - \frac{\alpha}{n} \eta) = \prod_{m=0}^{n-1} \theta(z + \frac{n-m}{n} - \frac{\alpha}{n} \eta)
\]
\[
= \theta(z + \frac{n}{n} - \frac{\alpha}{n} \eta) \theta(z + \frac{n-1}{n} - \frac{\alpha}{n} \eta) \cdots \theta(z + \frac{1}{n} - \frac{\alpha}{n} \eta)
\]
\[
= e(\alpha z - [-\alpha]) \theta_{-\alpha}(z).
\]
Therefore
\[
\theta_\alpha(-z) = p(z) \prod_{m=0}^{n-1} \theta(z - \frac{m}{n} - \frac{\alpha+1}{n} \eta)
\]
\[
= -e(-nz + \alpha \eta + [\alpha]) e(-\alpha z) e(\alpha z - [-\alpha]) \theta_{-\alpha}(z)
\]
\[
= -e(-nz + \alpha \eta + [\alpha] - [-\alpha]) \theta_{-\alpha}(z),
\]
as claimed.

(6) Since \(\theta(z + \eta) = -e(-z)\theta(z)\),
\[
\theta_{\alpha+n}(z) = e((\alpha + n)z + [\alpha + n]) \prod_{m=0}^{n-1} \theta(z + \frac{m}{n} + \frac{\alpha+n}{n} \eta)
\]
\[
= e(nz + [\alpha + n] - [\alpha]) e(\alpha z + [\alpha]) \prod_{m=0}^{n-1} (-1) e(-z - \frac{n-m}{n} - \frac{\alpha}{n} \eta) \theta(z + \frac{m}{n} + \frac{\alpha}{n} \eta)
\]
\[
= (-1)^n e(nz + [\alpha + n] - [\alpha]) e(\alpha z + [\alpha]) e(-nz - \frac{1}{n} - \frac{2}{n} \cdots - \frac{n-1}{n} - \alpha \eta) \prod_{m=0}^{n-1} \theta(z + \frac{m}{n} + \frac{\alpha}{n} \eta)
\]
\[
= (-1)^n e([\alpha + n] - [\alpha]) (-1)^{n-1} e(-\alpha \eta) \theta_{\alpha}(z)
\]
\[
= -e([\alpha + n] - [\alpha] - \alpha \eta) \theta_{\alpha}(z),
\]
as claimed. \(\square\)

**Lemma 2.4.** The functions \(\theta_0(z), \ldots, \theta_{n-1}(z)\) in (2.3) are a basis for \(\Theta_n(\Lambda) = \Theta_{n, \frac{n-1}{2}}(\Lambda)\).

**Proof.** Since \(\theta_\alpha(z)\) is an eigenvector with eigenvalue \(e(\frac{\alpha}{n} \eta)\) for the linear transformation \(f(z) \mapsto f(z + \frac{1}{n})\), the functions \(\theta_0(z), \ldots, \theta_{n-1}(z)\) are linearly independent. But \(\Theta_n(\Lambda)\) has dimension \(n\), so they form a basis for it. \(\square\)

In §2.2.1 we consider how to choose \([\alpha]\) and hence \(\theta_\alpha(z)\). We then devote a single subsection to the definition of the functions \(\theta_\alpha(z)\) in each of the following papers of Feigin and Odesskii: the Kiev preprint [FO89]; their first published paper [OF89]; Odesskii’s survey [Ode02]. Eventually, in §2.2.5, we define the \(\theta_\alpha(z)\)'s that will be used in the rest of this paper and in our subsequent papers.

**2.2.1.** We now consider the choice of \([\alpha]\). First, we want the coefficient \(e(\frac{\alpha}{n} \eta + [\alpha] - [\alpha + 1])\) in **Lemma 2.3**(4) to be a constant \(C\) independent of \(\alpha\). Second, we want equalities \(\theta_{\alpha+n}(z) = \theta_\alpha(z)\) for all \(\alpha \in \mathbb{Z}\). Third, since adding a constant to \([\alpha]\) corresponds to multiplying all the \(\theta_\alpha(z)\)'s by a common scalar, we normalize the function \(\alpha \mapsto [\alpha]\) by requiring \([0] = 0\). In summary, we will choose the \([\alpha]\)'s so the following three conditions hold:

\[
\begin{aligned}
& e(\frac{\alpha}{n} \eta + [\alpha] - [\alpha + 1]) = C, \\
& -e([\alpha + n] - [\alpha] - \alpha \eta) = 1, \\
& [0] = 0.
\end{aligned}
\]
Taken together, the first and the third of these conditions imply that
\[ C^\alpha = \prod_{i=0}^{\alpha-1} e\left(\frac{i}{n} \eta + [i] - [i+1]\right) = e\left(\frac{\alpha(\alpha-1)}{2n} \eta - [\alpha]\right) \quad \text{and} \]
\[ C^\alpha = \prod_{i=-\alpha}^{-1} e\left(\frac{i}{n} \eta + [i] - [i+1]\right) = e\left(-\frac{(\alpha)(\alpha-1)}{2n} \eta + [-\alpha]\right) \]
for all integers \( \alpha > 0 \). Hence
\[ e([\alpha]) = C^{-\alpha} e\left(\frac{\alpha(\alpha-1)}{2n} \eta\right) \]
for all \( \alpha \in \mathbb{Z} \). Substituting this into the second condition, implies that
\[ 1 = -e([\alpha + n] - [\alpha] - \alpha \eta) = -C^{-n} e\left(\frac{(\alpha+n)(\alpha+n-1)}{2n} \eta - \frac{\alpha(\alpha-1)}{2n} \eta - \alpha \eta\right) = C^{-n} e\left(-\frac{1}{2} + \frac{n-1}{2} \eta\right). \]
Therefore \( C = e\left(\frac{r}{n} - \frac{1}{2n} + \frac{n-1}{2n} \eta\right) \) for some integer \( r \). It follows that
\[ e([\alpha]) = e\left(\frac{(1-2r)}{2n} + \frac{\alpha(n-n)}{2n} \eta\right). \]
Parts (4) and (5) of Lemma 2.3 now become
\[ \theta_\alpha(z + \frac{1}{n} \eta) = e\left(-z + \frac{2r-1}{2n} + \frac{n-1}{2n} \eta\right) \theta_{\alpha+1}(z), \]
\[ \theta_\alpha(-z) = -e\left(-nz + \frac{(1-2r)}{n}\right) \theta_{-\alpha}(z). \]
The next result summarizes these discussions.\(^5\)

**Lemma 2.5.** Let \( r \in \mathbb{Z} \) be any integer. The functions
\[ (2-4) \quad \theta_\alpha(z) := e\left(\alpha z + \frac{(1-2r)}{2n} + \frac{\alpha(n-n)}{2n} \eta\right) \prod_{m=0}^{n-1} \theta(z + \frac{m}{n} + \frac{\alpha}{n} \eta), \]
indexed by \( \alpha \in \mathbb{Z} \), have the following properties:

1. \( \theta_{\alpha+n}(z) = \theta_\alpha(z) \),
2. \( \{\theta_0(z), \ldots, \theta_{n-1}(z)\} \) is a basis for \( \Theta_\alpha(\Lambda) \),
3. \( \theta_\alpha(z + \frac{1}{n} \eta) = e\left(\frac{\alpha}{n}\right) \theta_\alpha(z) \),
4. \( \theta_\alpha(z + \frac{1}{n} \eta) = e\left(-z + \frac{2r-1}{2n} + \frac{n-1}{2n} \eta\right) \theta_{\alpha+1}(z), \) and
5. \( \theta_{\alpha}(-z) = -e\left(-nz + \frac{(1-2r)}{n}\right) \theta_{-\alpha}(z). \)

The key point in each of the next three subsections is how to choose the integer \( r \) (modulo \( n \)) so the functions \( \theta_\alpha(z) \) have the properties that Feigin and Odesskii ask of them.

2.2.2. The appendix of the Kiev preprint [FO89] says \( \Theta_\alpha(\Lambda) \) has a basis \( \{\theta_\alpha(z) \mid \alpha \in \mathbb{Z}_n\} \) such that

1. \( \theta_\alpha(z + \frac{1}{n} \eta) = e\left(\frac{\alpha}{n}\right) \theta_\alpha(z) \),
2. \( \theta_\alpha(z + \frac{1}{n} \eta) = -e\left(-z + \frac{n-1}{2n} \eta\right) \theta_{\alpha+1}(z) \),
3. \( \theta_{\alpha}(-z) = -e\left(-nz\right) \theta_{-\alpha}(z) \), and
4. \( \theta_{\alpha}(z) \) is zero exactly at the points in \( \{\frac{1}{n}(-\alpha \eta + m) \mid 0 \leq m \leq n - 1\} + \Lambda \).

This is not possible. There is no integer \( r \) such that the functions \( \theta_\alpha(z) \) defined by (2-4) satisfy these four properties: if there were, then (3) together with Lemma 2.5(5) would imply that \( \frac{(1-2r)}{2n} + \frac{1}{2} \) is an integer; that is not the case though when \( \alpha = 0 \), for example.

If (2) held, then Lemma 2.5(4) would imply that the number \( s := \frac{2r-1}{2n} - \frac{1}{2} \) is an integer so \( r = \frac{n+1}{2} + ns = \frac{n+1}{2} \pmod{n} \) which implies that \( n \) is odd. Thus, when \( n \) is odd we set \( r = \frac{n+1}{2} \); the functions \( \theta_\alpha(z) \) in (2-4) now satisfy (1), (2), and (4), but not (3) because \( \theta_{\alpha}(-z) = -e(-nz) \theta_{-\alpha}(z) \).

\(^5\)We note that \( e([\alpha]) \) depends only on the image of \( r \) in \( \mathbb{Z}_n \).
2.2.3. Let $2^p$ be the largest power of 2 dividing $n$. The paper [OF89, §1.1] says that $\Theta_n(\Lambda)$ has a basis \( \{ \theta_\alpha(z) \mid \alpha \in \mathbb{Z}_n \} \) such that

1. \( \theta_\alpha(z + \frac{1}{n}) = e\left(\frac{\alpha}{n}\right)\theta_\alpha(z) \),
2. \( \theta_\alpha(z + \frac{1}{n} \eta) = e\left(-z - \frac{1}{2n} + \frac{n-1}{2n} \eta\right)\theta_{\alpha+1}(z) \),
3. \( \theta_\alpha(-z) = -e(-nz)\theta_{-\alpha}(z) \) if $n$ is odd,
4. \( \theta_\alpha(-z) = -e(-nz + 2^{-p}\alpha)\theta_{-\alpha}(z) \) if $n$ is even, and
5. \( \theta_\alpha(z) \) is zero exactly at the points in \( \{ \frac{1}{n}(-\alpha \eta + m) \mid 0 \leq m \leq n-1 \} + \Lambda \).

If the functions $\theta_\alpha(z)$ defined by (2-4) satisfy these five properties, then (2) implies that the number $s := \frac{2r-1}{2n} - (2^{-p} - 1)$ is an integer and $r = \frac{1}{2}(-n \eta + \frac{n}{2} + 1) + ns = \frac{1}{2}(-n \eta + \frac{n}{2} + 1)$.

Conversely, set $r = \frac{1}{2}(-n \eta + \frac{n}{2} + 1)$, which is always an integer.\(^6\) Since \( \frac{1-2r}{n} = 2^{-p} \), the function $\theta_\alpha(z)$ in (2-4) now has the property that

\begin{equation}
\theta_\alpha(-z) = -e(-nz + 2^{-p}\alpha)\theta_{-\alpha}(z).
\end{equation}

Hence conditions (3) and (4) are satisfied, and so are (1), (2), and (5). We also note that

\[ e([\alpha]) = e\left(2^{-p-1} \alpha + \frac{\alpha(\alpha-n)}{2n} \eta\right) \]

in this case.

2.2.4. Odesskii’s survey [Ode02, Appendix A] says $\Theta_n(\Lambda)$ has a basis \( \{ \theta_\alpha(z) \mid \alpha \in \mathbb{Z}_n \} \) such that

1. \( \theta_\alpha(z + \frac{1}{n}) = e\left(\frac{\alpha}{n}\right)\theta_\alpha(z) \),
2. \( \theta_\alpha(z + \frac{1}{n} \eta) = e\left(-z - \frac{1}{2n} + \frac{n-1}{2n} \eta\right)\theta_{\alpha+1}(z) \), and
3. \( \theta_\alpha(z) = e^{\alpha z + \frac{\alpha}{2n} + \frac{\alpha(\alpha-n)}{2n} \eta} \prod_{\eta=0}^{n-1} \theta(z + \frac{m}{n} + \frac{\alpha}{n} \eta) \).

If the functions $\theta_\alpha(z)$ in (2-4) satisfy (2), then $r$ is divisible by $n$. If $r$ is divisible by $n$, then the functions $\theta_\alpha(z)$ in (2-4) have all three properties.

2.2.5. The “standard” definition of $\theta_\alpha(z)$. From now on, unless otherwise stated, $\theta_\alpha(z)$ denotes the function in (2-4) with $r = 0$ modulo $n.$\(^7\) We repeat this definition in (2-6) below. As remarked in §2.2.4, the function $\theta_\alpha(z)$ in (2-6) is the same as the function $\theta_\alpha(z)$ defined in Odesskii’s survey [Ode02, Appendix A].

**Proposition 2.6.** The functions

\begin{equation}
\theta_\alpha(z) := e\left(\alpha z + \frac{\alpha}{2n} + \frac{\alpha(\alpha-n)}{2n} \eta\right) \prod_{\eta=0}^{n-1} \theta(z + \frac{m}{n} + \frac{\alpha}{n} \eta),
\end{equation}

indexed by $\alpha \in \mathbb{Z}_n$, have the following properties:

1. $\theta_{\alpha+n}(z) = \theta_\alpha(z)$.
2. \( \{ \theta_0(z), \ldots, \theta_{n-1}(z) \} \) is a basis for $\Theta_n(\Lambda)$.
3. \( \theta_\alpha(z + \frac{1}{n}) = e\left(\frac{\alpha}{n}\right)\theta_\alpha(z) \).
4. \( \theta_\alpha(z + \frac{1}{n} \eta) = e\left(-z - \frac{1}{2n} + \frac{n-1}{2n} \eta\right)\theta_{\alpha+1}(z) \).
5. $\theta_\alpha(-z) = -e(-nz + \frac{\alpha}{n})\theta_{-\alpha}(z)$.
6. The zeroes of $\theta_\alpha(z)$ are \( \{ \frac{1}{n}(-\alpha \eta + m) \mid 0 \leq m \leq n-1 \} + \Lambda \), all of which have multiplicity one.
7. For all $r \in \mathbb{Z}$, $\theta_\alpha(z + \frac{r}{n} \eta) = e\left(-rz - \frac{r}{2n} + \frac{rn-r^2}{2n} \eta\right)\theta_{\alpha+r}(z)$.

**Proof.** All of this, with the exception of part (7) has been proved before. The formula in (7) is first proved by induction for all $r \geq 0$, then, by replacing $\alpha$ by $\alpha - r$ and $z$ by $z - \frac{r}{n} \eta$ in the formula, one sees that it holds for all $r \in \mathbb{Z}$.\(\square\)

The basis $\theta_0(z)$ for $\Theta_0(\Lambda)$ is the function $\theta(z)$ defined in (2-1).

\(^6\)If we write $n = 2^p(2l+1)$ as in [OF89], then $r = -t$ modulo $n$.

\(^7\)The function in (2-4) only depends on $r$ modulo $n$. 

2.2.6. A basis for $\Theta_{n,c}(\Lambda)$ can be constructed from the basis $\theta_\alpha(z)$ for $\Theta_n(\Lambda) = \Theta_{n,\frac{n-1}{2n}}(\Lambda)$.

**Proposition 2.7.** For $\alpha \in \mathbb{Z}$, let $\theta_\alpha(z)$ be the function defined in (2-6). The functions

$$\theta_{\alpha,c}(z) := \theta_\alpha(z - \frac{1}{n}c + \frac{n-1}{2n})$$

have the following properties:

1. $\{\theta_{0,c}(z), \ldots, \theta_{n-1,c}(z)\}$ is a basis of $\Theta_{n,c}(\Lambda)$.
2. $\theta_{\alpha,c}(z + \frac{1}{n}) = e(\frac{\alpha}{n})\theta_{\alpha,c}(z)$.
3. $\theta_{\alpha,c}(z + \frac{1}{n}\eta) = -e(-z + \frac{1}{n}c + \frac{n-1}{2n}\eta)\theta_{\alpha+1,c}(z)$.
4. $\theta_{\alpha,\frac{n-1}{2n}}(z) = \theta_\alpha(z)$.

**Proof.** It is clear that (4) holds.

The properties $\theta_{\alpha,c}(z + 1) = \theta_{\alpha,c}(z)$ and $\theta_{\alpha,c}(z + \frac{1}{n}) = e(\frac{\alpha}{n})\theta_{\alpha,c}(z)$ follow from the same properties of $\theta_\alpha(z)$. Let $d := \frac{1}{n}c - \frac{n-1}{2n}$. Then

$$\theta_{\alpha,c}(z + \eta) = \theta_\alpha(z + \eta - d)$$

$$= -e(-nz + nd)\theta_\alpha(z - d)$$

$$= -e(-nz + c - \frac{n-1}{2})\theta_{\alpha,c}(z)$$

$$= (-1)^n e(-nz + c)\theta_{\alpha,c}(z).$$

Hence $\theta_{\alpha,c}(z) \in \Theta_{n,c}(\Lambda)$. Since the $\theta_{0,c}(z), \ldots, \theta_{n-1,c}(z)$ are eigenvectors for the linear operator $f(z) \mapsto f(z + \frac{1}{n})$ with different eigenvalues and the dimension of $\Theta_{n,c}(\Lambda)$ is $n$, they are a basis for $\Theta_{n,c}(\Lambda)$.

Statement (3) holds because

$$\theta_{\alpha,c}(z + \frac{1}{n}\eta) = \theta_\alpha(z + \frac{1}{n}\eta - d)$$

$$= e(-z + d - \frac{1}{2n} + \frac{n-1}{2n}\eta)\theta_{\alpha+1}(z - d)$$

$$= e(-z + \frac{1}{n}c - \frac{1}{2} + \frac{n-1}{2n}\eta)\theta_{\alpha+1,c}(z)$$

$$= -e(-z + \frac{1}{n}c + \frac{n-1}{2n}\eta)\theta_{\alpha+1,c}(z).$$

The proof is now complete. \(\Box\)

In [Ode02, Appendix A], Odesskii considered another basis $\{\theta_\alpha(z - \frac{1}{n}c - \frac{n-1}{2n}) \mid \alpha \in \mathbb{Z}_n\}$ for $\Theta_{n,c}(\Lambda)$. It is a basis because

$$\theta_\alpha(z - \frac{1}{n}c - \frac{n-1}{2n}) = \theta_\alpha(z - \frac{1}{n}c + \frac{n-1}{2n} - 1 + \frac{1}{n})$$

$$= e(\frac{\alpha}{n})\theta_\alpha(z - \frac{1}{n}c + \frac{n-1}{2n})$$

$$= e(\frac{\alpha}{n})\theta_{\alpha,c}(z).$$

2.3. $\Theta_n(\Lambda)$ as a representation of the Heisenberg group. Fix $d \in \mathbb{C}$. Let $S$ and $T$ be the operators on the space of meromorphic functions on $\mathbb{C}$ defined by

$$(S \cdot f)(z) = f(z + \frac{1}{n})$$

$$(T \cdot f)(z) = e(z + d)f(z + \frac{1}{n}\eta).$$

Both $S$ and $T$ are invertible and satisfy $ST = e(\frac{1}{n})TS$. 
It is clear that $\Theta_n(\Lambda)$ is stable under the action of $S$ and $T$ and that $S^n$ acts as the identity on $\Theta_n(\Lambda)$. When $d = \frac{1}{2n} - \frac{n-1}{2n}\eta$ the operator $T^n$ also acts as the identity on $\Theta_n(\Lambda)$ because

\[
(T^n \cdot f)(z) = e(z + d)(T^{n-1} \cdot f)(z + \frac{1}{n}\eta) = e(z + d)e(z + \frac{1}{n}\eta + d)(T^{n-2} \cdot f)(z + \frac{2}{n}\eta) = \ldots = e(z + d)e(z + \frac{1}{n}\eta + d) \cdots e(z + \frac{n-1}{n}\eta + d)f(z + \frac{n}{n}\eta) = e(nz + nd + \frac{n-1}{2}\eta)f(z + \eta) = -e(nd + \frac{n-1}{2}\eta)f(z) = f(z).
\]

This leads to a representation of the Heisenberg group of order $n^3$ on $\Theta_n(\Lambda)$. This group is

\[
H_n := \langle S, T, \epsilon \mid S^n = T^n = \epsilon^n = 1, \epsilon = [S, T], [S, \epsilon] = [T, \epsilon] = 1 \rangle.
\]

**Lemma 2.8.** The space $\Theta_n(\Lambda)$ is an irreducible representation of $H_n$ via the actions

\[
(S \cdot f)(z) = f(z + \frac{1}{n}), \quad (T \cdot f)(z) = e(z + \frac{1}{2n} - \frac{n-1}{2n}\eta)f(z + \frac{1}{n}\eta).
\]

The action on the basis $\theta_\alpha(z)$ in (2-6) is given by

\[
(S \cdot \theta_\alpha)(z) = e\left(\frac{\alpha}{n}\right)\theta_\alpha(z), \quad (T \cdot \theta_\alpha)(z) = \theta_{\alpha + 1}(z).
\]

**Proof.** The action of $S$ and $T$ on the $\theta_\alpha$’s is as claimed because $\theta_\alpha\left(z + \frac{1}{n}\right) = e\left(\frac{\alpha}{n}\right)\theta_\alpha(z)$ and

\[
(T \cdot \theta_\alpha)(z) = e(z + \frac{1}{2n} - \frac{n-1}{2n}\eta)\theta_\alpha(z + \frac{1}{n}\eta) = e(z + \frac{1}{2n} - \frac{n-1}{2n}\eta)e(-z - \frac{1}{2n} + \frac{n-1}{2n}\eta)\theta_{\alpha + 1}(z) = \theta_{\alpha + 1}(z).
\]

Because the $\theta_\alpha$’s are $S$-eigenvectors with different eigenvalues, every subspace of $\Theta_n(\Lambda)$ that is stable under the action of $S$ is spanned by some of the $\theta_\alpha$’s. Since $T \cdot \theta_\alpha = \theta_{\alpha + 1}$ the only non-zero subrepresentation of $\Theta_n(\Lambda)$ is $\Theta_n(\Lambda)$ itself. Hence $\Theta_n(\Lambda)$ is an irreducible representation of $H_n$. \qed

### 2.4. Embedding $E$ in $\mathbb{P}^{n-1}$ via $\Theta_n(\Lambda)$.

Evaluation at a point $z \in \mathbb{C}$ provides a surjective linear map $\Theta_n(\Lambda) \to \mathbb{C}$. The kernel of this evaluation map depends only on the coset $z + \Lambda$ so there is a well-defined map from $\mathbb{C}/\Lambda$ to the set of codimension-one subspaces of $\Theta_n(\Lambda)$ or, what is essentially the same thing, a holomorphic map

\[
(\theta_\alpha(z), \ldots, \theta_{\alpha - 1}(z)) \mapsto \mathbb{P}(\Theta_n(\Lambda)^*)
\]

to the projective space of 1-dimensional subspaces of $\Theta_n(\Lambda)^*$. Since $E$ and $\mathbb{P}(\Theta_n(\Lambda)^*)$ are smooth projective varieties, $\iota$ is a morphism of algebraic varieties [GH78, p. 170].

Since the $\theta_\alpha$’s are a basis for $\Theta_n(\Lambda)$ they form a system of homogeneous coordinate functions on $\mathbb{P}(\Theta_n(\Lambda)^*)$. With respect to this system of homogeneous coordinates the map in (2-8) is

\[
z \mapsto (\theta_\alpha(z), \ldots, \theta_{\alpha - 1}(z)).
\]

Suppose $n \geq 3$. Since the pullback $\iota^*\mathcal{O}(1)$ of the twisting sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\Theta_n(\Lambda)^*)$ has degree $n$, [Har77, Cor. IV.3.2] implies that $\iota^*\mathcal{O}(1)$ is very ample, and hence $\iota$ is a closed immersion. We will often identify $E$ with its image under $\iota$. Each linear form on $\mathbb{P}(\Theta_n(\Lambda)^*)$ vanishes at exactly $n$ points of $E$ counted with multiplicity and the sum of those points is the identity element 0 in the group $(E, +)$. Conversely, if $p_1, \ldots, p_n$ are points on $E$ whose sum is 0 there is a function $f \in \Theta_n(\Lambda)$, unique up to non-zero scalar multiples, whose divisor of zeroes on $E$ is $(p_1) + \cdots + (p_n)$. 

Thus, for fixed \((n, k, E)\), the contragredient action \((g \cdot \varphi)(f) = \varphi(g^{-1} \cdot f)\) for \(g \in H_n, \varphi \in \Theta_n(\Lambda)^*\), and \(f \in \Theta_n(\Lambda)\). Thus \(H_n\) acts as linear automorphisms of \(\mathbb{P}(\Theta_n(\Lambda))^*\). For example, if \(z \in E\), then
\[
S : (\theta_0(z), \ldots, \theta_{n-1}(z)) = (\theta_0(z - \frac{1}{n}), \ldots, \theta_{n-1}(z - \frac{1}{n})).
\]

Since the commutator \([S, T]\) acts on \(\Theta_n(\Lambda)\) as multiplication by \(e \left(\frac{1}{n}\right)\), it acts trivially on \(\mathbb{P}(\Theta_n(\Lambda))^*\). Thus, the action of \(H_n\) factors through the quotient of \(H_n\) by the subgroup generated by \([S, T]\). This quotient is isomorphic to \(\mathbb{Z}_n \times \mathbb{Z}_n\).

### 2.5. Another basis for \(\Theta_n(\Lambda)\)

As we discussed in §2.2.2, the characterization of the basis for \(\Theta_n(\Lambda)\) in the Kiev preprint [FO89] is not compatible with (2-4) and, even after removing condition (3) in §2.2.2, it is only compatible when \(n\) is odd, and in that case, the integer \(r\) (modulo \(n\)), and hence the definition of the basis, coincides with that of [OF89] described in §2.2.3.

We denote that basis by \(\psi_0(z), \ldots, \psi_{n-1}(z)\). Explicitly, we assume that \(n\) is odd, and the \(\psi_\alpha(z)\)'s are the functions in (2-4) with \(r = -\frac{n+1}{2}\) (modulo \(n\)); i.e.,
\[
\psi_\alpha(z) = e \left(\frac{\alpha(n-1)}{2n}\right) \theta_\alpha(z).
\]

The bases \(\{\theta_\alpha(z)\}\) and \(\{\psi_\alpha(z)\}\) do not coincide unless \(n = 1\).

For some purposes the \(\psi_\alpha(z)\)'s provide a “better” basis than the \(\theta_\alpha(z)\)'s. Define the automorphism \(\iota\) of \(\mathbb{P}^{n-1}\) by
\[
\iota(x_0, x_1, \ldots, x_{n-1}) = (x_0, x_{n-1}, \ldots, x_1)
\]
as in [Fis10, Lem. 3.5]. Since property (3) in §2.2.3 says
\[
\psi_\alpha(-z) = -e(-nz)\psi_\alpha(z),
\]
the closed immersion \(\psi : E \to \mathbb{P}^{n-1}\) given by \(\psi(z) = (\psi_0(z), \ldots, \psi_{n-1}(z))\) fits into the commutative diagram
\[
\begin{array}{ccc}
E & \xrightarrow{\psi} & \mathbb{P}^{n-1} \\
\downarrow & & \downarrow \\
E & \xrightarrow{\psi} & \mathbb{P}^{n-1}
\end{array}
\]
where \([-] : E \to E\) is the automorphism that sends \(z\) to \(-z\).

We will not refer to the \(\psi_\alpha(z)\)'s again in this paper.

### 3. The algebras \(Q_{n,k}(E, \tau)\)

From now on \(n > k \geq 1\) are relatively prime integers.

For the remainder of this paper the \(\theta_\alpha\)'s are the functions defined in (2-6).

#### 3.1. The definition of \(Q_{n,k}(E, \tau)\) and \(R_{n,k}(E, \tau)\) when \(\tau \notin \frac{1}{n}\Lambda\)

Fix \(\tau \in \mathbb{C} - \frac{1}{n}\Lambda\). Let \(V\) be a \(\mathbb{C}\)-vector space with basis \(\{x_i \mid i \in \mathbb{Z}_n\}\). The algebra \(Q_{n,k}(E, \tau)\) is the quotient of the free algebra \(TV = \mathbb{C}(x_0, \ldots, x_{n-1})\) by the \(n^2\) relations
\[
(3-1) \quad R_{ij} = \sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r}(-\tau)\theta_{kr}(\tau)} x_{j-r}x_{i+r}, \quad i, j \in \mathbb{Z}_n.
\]

We will write \(R_{ij}(\tau)\) when we want to emphasize \(\tau\). Let
\[
R_{n,k}(E, \tau) = \text{span}\{R_{ij}(\tau) \mid i, j \in \mathbb{Z}_n\} \subseteq V \otimes V.
\]

If \(\zeta \in \Lambda\), then the quasi-periodicity properties of the \(\theta_\alpha\)'s imply that \(R_{ij}(\tau + \zeta)\) and \(R_{ij}(\tau)\) are non-zero scalar multiples of each other. Thus the linear span of \(R_{ij}(\tau)\) depends only on the image of \(\tau\) in \(E = \mathbb{C}/\Lambda\). In particular, \(R_{n,k}(E, \tau)\), and therefore \(Q_{n,k}(E, \tau)\), depends only on the image of \(\tau\) in \(E\). Thus, for fixed \((n, k, E)\) we think of the algebras \(Q_{n,k}(E, \tau)\) as a family of algebras over \(E - E[n]\).
3.1. The term $x_{j-r}x_{i+r}$ in the definition of $R_{ij}$ is replaced by $x_{k(j-r)}x_{k(i+r)}$ in [OF89, OF93, FO98, Ode92]. The difference is due to the change of variables $x_i \mapsto x_{ki}$. Our definition of $R_{ij}$ is the same as the one in [Ode02, §3], which is also used in [OR08, ORTP11a, ORTP11b].

3.1.2. Suppose $k = 1$. Then $R_{ij} = 0$ for all $i$ because $\theta_0(0) = 0$. Thus, whenever we speak of $R_{ij}$ when $k = 1$ we will assume that $i \neq j$. (When $k \neq 1$ all $R_{ij}$ are non-zero.) When $i \neq j$ all the structure constants in $R_{ij}$ have the same numerator so $R_{ij}$ can be replaced by the relation

$$
(3-2) \quad \sum_{r \in \mathbb{Z}_n} \frac{x_{j-r}x_{i+r}}{\theta_{j-i-r}(-\tau)\theta_r(\tau)} = 0.
$$

3.1.3. In the Kiev preprint [FO89, §3], $Q_{n,1}(E, \tau)$ is defined for odd $n \geq 3$ as the free algebra $\mathbb{C}(x_0, \ldots, x_{n-1})$ modulo the $n(n-1)$ relations

$$
(3-3) \quad \frac{x_i^2}{\theta_j(\tau)\theta_{-j}(\tau)} + \frac{x_{i-1}x_{i+1}}{\theta_{j+1}(\tau)\theta_{-j+1}(\tau)} + \cdots + \frac{x_{i-(n-1)}x_{i+n-1}}{\theta_{j+n-1}(\tau)\theta_{-j+n-1}(\tau)} = 0
$$

indexed by $(i, j) \in \mathbb{Z}_n \times (\mathbb{Z}_n - \{0\})$. This is not compatible with our definition (3-1) because our basis $\theta_\alpha(z)$ differs from that in [FO89, §3] (as discussed in §2.2.2). In Proposition 3.10, we provide a similar description of the relations for $Q_{n,k}(E, \tau)$ using our $\theta_\alpha$’s.

3.2. The definition of $Q_{n,k}(E, \tau)$ and $R_{n,k}(E, \tau)$ when $\tau \in \frac{1}{n}\Lambda$. We now extend the definition of $R_{n,k}(E, \tau)$ to all $\tau \in E$.

**Proposition 3.1.** Fix $(i, j) \in \mathbb{Z}_n^2$ such that $R_{ij}(\tau)$ is not identically zero on $E - E[n]$. When $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$, let $L_{ij}(\tau)$ be the 1-dimensional subspace of $V \otimes V$ spanned by the element $R_{ij}(\tau)$ in (3-1). The map

$$
(3-4) \quad \varphi_{ij} : E - E[n] \to \mathbb{P}(V \otimes V), \quad \tau \mapsto L_{ij}(\tau),
$$

is a morphism of algebraic varieties and extends in a unique way to a morphism $\bar{\varphi}_{ij} : E \to \mathbb{P}(V \otimes V)$.

**Proof.** Since the zeroes of the $\theta_\alpha$’s belong to $\frac{1}{n}\Lambda$, the hypothesis that $\tau$ is not in $\frac{1}{n}\Lambda$ ensures that the coefficient of every $x_{j-r} \otimes x_{i+r}$ in $R_{ij}(\tau)$ is a complex number. By hypothesis, at least one of those coefficients is non-zero so $R_{ij}(\tau) \neq 0$ for all $\tau \in E - E[n]$. As remarked in §3.1, the subspace $L_{ij}(\tau)$ depends only the image of $\tau$ in $E - E[n]$. Hence $\varphi_{ij}$ is a well-defined map from $E - E[n]$.

Since the map $E \to \mathbb{P}^{n-1}$ given by $z \mapsto (\theta_0(z), \ldots, \theta_{n-1}(z))$ is a morphism of algebraic varieties, the ratios $\theta_\alpha(z)/\theta_\beta(z)$ are rational functions on $E$ and therefore regular functions on $E - E[n]$. Thus, since $\theta_\alpha(-\tau) = -e(-n\tau + \frac{a}{n})\theta_\alpha(\tau)$, the ratio of any two of the coefficients of $R_{ij}(\tau)$ is a regular function on $E - E[n]$. Hence $\varphi_{ij}$ is a morphism of algebraic varieties.

Since $E$ is a non-singular curve, this morphism extends in a unique way to a morphism $\bar{\varphi}_{ij} : E \to \mathbb{P}(V \otimes V)$ by using [Har77, Prop. I.6.8] repeatedly.

**Definition 3.2.** Fix $\tau \in E$. For each $(i, j) \in \mathbb{Z}_n^2$ for which $R_{ij}(\tau)$ is not identically zero on $E - E[n]$, we define the 1-dimensional subspace

$$
L_{ij}(\tau) := \bar{\varphi}_{ij}(\tau) \subseteq V^\otimes 2
$$

where $\bar{\varphi}_{ij}$ is the morphism obtained in Proposition 3.1. Let $R_{n,k}(E, \tau)$ be the subspace of $V^\otimes 2$ spanned by those $L_{ij}(\tau)$’s, and define

$$
Q_{n,k}(E, \tau) := \frac{\mathbb{C}(x_0, \ldots, x_{n-1})}{(R_{n,k}(E, \tau))},
$$

i.e., the quotient of the free algebra $TV$ by the ideal generated by $R_{n,k}(E, \tau)$.

**Proposition 3.3.** For all $\tau \in E$, $Q_{2,1}(E, \tau) = \mathbb{C}[x_0, x_1]$. 
Proof. First we consider the case $\tau \notin E[2]$. Since $\theta_0(0) = 0$, $R_{00} = R_{11} = 0$. The other relations in (3-1) are

$$R_{01} = \theta_1(0) \left( \frac{x_1x_0}{\theta_1(-\tau)\theta_0(\tau)} + \frac{x_0x_1}{\theta_0(-\tau)\theta_1(\tau)} \right), \quad \text{and}
$$

$$R_{10} = \theta_1(0) \left( \frac{x_0x_1}{\theta_1(-\tau)\theta_0(\tau)} + \frac{x_1x_0}{\theta_0(-\tau)\theta_1(\tau)} \right)$$

in $\mathbb{C}\langle x_0, x_1 \rangle$. Since $n = 2$,

$$\theta_\alpha(-z) = -e(-2z + \frac{\alpha}{2}) \theta_{-\alpha}(z).$$

In particular, $\theta_0(-z) = -e(-2z)\theta_0(z)$ and $\theta_1(-z) = e(-2z)\theta_1(z)$ so

$$R_{01} = -\frac{\theta_1(0)}{e(-2\tau)\theta_0(\tau)\theta_1(\tau)} (x_0x_1 - x_1x_0) = -R_{10}.$$

Let $(i, j) = (0, 1)$ or $(1, 0)$. The morphism $\varphi_{ij}: E - E[2] \to \mathbb{P}(V \otimes V)$ is constant with value $x_0x_1 - x_1x_0$ so it extends to the constant morphism $\tilde{\varphi}_{ij}: E \to \mathbb{P}(V \otimes V)$ with the same value. Therefore $R_{21}(E, \tau) = \mathbb{C} \cdot (x_0x_1 - x_1x_0)$ and $Q_{2,1}(E, \tau) = \mathbb{C}[x_0, x_1]. \square$

3.2.1. Families of subspaces. It is stated in [Ode02, §3] that the Hilbert series of $Q_{n,k}(E, \tau)$ is the same as that of the polynomial ring on $n$ variables for generic $\tau \in E$, and it is conjectured that this is true for all $\tau \in E$.\footnote{We will prove this conjecture in another paper.} We now give an alternative definition of $Q_{n,k}(E, \tau)$ and $R_{n,k}(E, \tau)$ under the assumption that the conjecture is true.

Although the following result is well known we include a proof for convenience of the reader:

**Lemma 3.4.** Let $X$ be a variety over an algebraically closed field $\mathbb{k}$. Let $V$ be a finite dimensional $\mathbb{k}$-vector space $V$ with basis $\{v_1, \ldots, v_n\}$. Fix an integer $m \geq 0$ and let $\lambda_{ij}, 1 \leq i \leq m, 1 \leq j \leq n$, be regular functions on $X$. For each closed point $x \in X$, define

$$r_i(x) := \sum_{j=1}^{n} \lambda_{ij}(x)v_j$$

for $1 \leq i \leq m$, $R(x) := \text{span}\{r_i(x) \mid 1 \leq i \leq m\}$, and $d := \max\{\dim(R(x)) \mid x \in X\}$.

1. $U := \{x \in X \mid \dim(R(x)) = d\}$ is a non-empty Zariski-open subset of $X$.
2. The map $f: U \to \text{Grass}(d, V)$, $x \mapsto R(x)$, is a morphism of algebraic varieties.
3. If $X$ is a non-empty Zariski-open subset of a non-singular curve $\overline{X}$, then $f$ extends in a unique way to a morphism $\overline{X} \to \text{Grass}(d, V)$.

**Proof.** (1) The dimension of $R(x)$ is the rank of the matrix $M(x) = (\lambda_{ij}(x))$. Let $s$ be a positive integer. The rank of a matrix is $< s$ if all its $s \times s$ minors vanish so the set of matrices having rank $< s$ is a Zariski-closed subset of the space $M_{m,n}(\mathbb{k})$ of all $m \times n$ matrices. Since the map $X \to M_{m,n}(\mathbb{k})$ given by $x \mapsto M(x)$ is a morphism of algebraic varieties, the sets

$$Z_s := \{x \in X \mid \text{rank}(M(x)) < s\}$$

are Zariski-closed subsets of $X$. The sets $U_s := \{x \in X \mid \dim(R(x)) \geq s\}$ are therefore open subsets of $X$. Since $\dim_k(V) < \infty$, $\max\{\dim(R(x)) \mid x \in X\}$ exists and $U = U_d$ is a non-empty open subset of $X$.

(2) For a given set of indices $1 \leq i_1, \ldots, i_d \leq m$, the set

$$U_{i_1, \ldots, i_d} := \{x \in U \mid \{r_{i_1}(x), \ldots, r_{i_d}(x)\}\}$$

is a Zariski-open subset of $U$ and the function $x \mapsto r_{i_1}(x) \wedge \ldots \wedge r_{i_d}(x)$ is a morphism $U_{i_1, \ldots, i_d} \to \wedge^d V - \{0\} \to \mathbb{P}(\wedge^d V)$ whose image is contained in the image of the Plücker embedding

$$p: \text{Grass}(d, V) \to \mathbb{P}(\wedge^d V), \quad p(\text{span}\{v_1, \ldots, v_d\}) := v_1 \wedge \ldots \wedge v_d.$$
Thus the restriction of $f$ to $U_{i_1,\ldots,i_d}$ is a morphism. Since $\{U_{i_1,\ldots,i_d} \mid 1 \leq i_1,\ldots,i_d \leq m\}$ is an open covering of $U$, $f$ is a morphism.

(3) This is an immediate consequence of [Har77, Prop. I.6.8].

Proposition 3.5. Let $d := \max\{\dim(R(E,\tau)) \mid \tau \in E - E[n]\}$.

1. There is a unique morphism $f : E \to \text{Grass}(d, V \otimes V)$ such that $f(\tau) = R_{n,k}(E,\tau)$ on a non-empty Zariski-open subset of $E - E[n]$.
2. For all $\tau \in E$, $R_{n,k}(E,\tau) \subseteq f(\tau)$.
3. The set $U := \{\tau \in E \mid \dim R_{n,k}(E,\tau) = d\}$ is a non-empty Zariski-open subset of $E$.

We will use the notation $\overline{R}_{n,k}(E,\tau) := f(\tau)$ for all $\tau \in E$.

Proof. (1) The existence and uniqueness of $f$ can be seen by applying Lemma 3.4 to $X = E - E[n] \subseteq E = \overline{X}$, the function $\tau \mapsto R_{n,k}(E,\tau) \subseteq V \otimes V$, and the integer $d$.

(2) It suffices to prove that $L_{ij}(\tau) \subseteq f(\tau)$ for all $(i,j)$ and $\tau \in E$. We can assume that $R_{ij}(\tau)$ is not identically zero.

Write $W := V \otimes V$ and consider the Plücker embedding $p : \text{Grass}(d, W) \to \mathbb{P}\left(\bigwedge^d W\right)$. Let $Y$ be the zero locus of the linear map

$$\left(\bigwedge^d W\right) \otimes W \to \bigwedge^{d+1} W, \quad \omega \otimes v \mapsto \omega \wedge v$$

inside $\mathbb{P}(\left(\bigwedge^d W\right) \otimes W)$. Then the set $Z := \{\tau \in E \mid L_{ij}(\tau) \subseteq f(\tau)\}$ is the inverse image of $Y$ with respect to the composition

$$E \xrightarrow{(f,\tilde{\phi}_i)} \text{Grass}(d, W) \times \mathbb{P}(W) \xrightarrow{p \times \text{id}} \mathbb{P}\left(\bigwedge^d W\right) \times \mathbb{P}(W) \xrightarrow{\iota} \mathbb{P}\left(\left(\bigwedge^d W\right) \otimes W\right)$$

where $\iota$ is the Segre embedding. Thus $Z$ is a Zariski-closed subset of $E$. If $\tau \in U \cap (E - E[n])$, then

$$L_{ij}(\tau) \subseteq R_{n,k}(E,\tau) = f(\tau)$$

so $Z \supseteq U \cap (E - E[n])$. Since $U \cap (E - E[n])$ is a Zariski-dense subset of $E - E[n]$, $Z = E$.

(3) Since $U$ contains $\{\tau \in E - E[n] \mid \dim R_{n,k}(E,\tau) = d\}$ which is a non-empty Zariski-open subset of $E - E[n]$, $U$ is co-finite in $E$ and hence a Zariski-open subset of $E$. □

Corollary 3.6. If $Q_{n,k}(E,\tau)$ has the same Hilbert series as the polynomial ring in $n$ variables, as conjectured at [Ode02, §3, p. 1143], then $R_{n,k}(E,\tau) = \overline{R}_{n,k}(E,\tau)$ for all $\tau \in E$.

Proof. The hypothesis implies that $\dim(R_{n,k}(E,\tau)) = \binom{n}{2}$ for all $\tau \in E$. The number $d$ in Proposition 3.5 is therefore $\binom{n}{2}$. Hence $\dim(\overline{R}_{n,k}(E,\tau)) = \binom{n}{2}$ for all $\tau \in E$ so the inclusion in Proposition 3.5(2) implies that $R_{n,k}(E,\tau) = \overline{R}_{n,k}(E,\tau)$ for all $\tau \in E$. □

In the rest of this paper, we make no use of Corollary 3.6 or the space $\overline{R}_{n,k}(E,\tau)$.

3.3. Isomorphisms and anti-isomorphisms. The next result is stated in [OF89, §1, Rem. 3].

Proposition 3.7. There is an isomorphism $\Phi : Q_{n,k}(E,\tau) \to Q_{n,k}(E,\tau)$ given by $\Phi(x_i) = x_{ki}$.

Proof. Let $\Phi$ be the automorphism of $\mathbb{C}\langle x_0,\ldots,x_{n-1}\rangle$ defined by $\Phi(x_i) = x_{ki}$ for all $i \in \mathbb{Z}_n$. We will show that $\Phi$ sends the relations for $Q_{n,k}(E,\tau)$ bijectively to the relations for $Q_{n,k}(E,\tau)$.

Assume $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$. For all $i,j,r \in \mathbb{Z}_n$, let

$$c_{ijkr}(\tau) = \frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r}(-\tau)\theta_{k,r}(\tau)}$$

and

$$R_{ijk}(\tau) = \sum_{r \in \mathbb{Z}_n} c_{ijkr}(\tau)x_{j-r}x_{i+r}.$$
Let \( i' = kj, j' = ki \), and \( r' = -k(j - i - r) \). Because \( \theta_\alpha(-z) = -e(-nz + \frac{2\alpha}{n})\theta_\alpha(z) \),

\[
c_{ijkr}(\tau) = \frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r}(-\tau)\theta_{kr}(\tau)} \frac{-e\left(\frac{j-i+(k-1)r}{n}\right)\theta_{-(j-i+(k-1)r)}(0)}{(-e\left(-n\tau + \frac{j-i-(k-1)r}{n}\right)\theta_{-(j-i-r)}(\tau) \cdot (-e(n\tau + \frac{k\tau}{n}))\theta_{-kr}(\tau)}
\]

\[
= -\frac{\theta_{j'-i'+(k'-1)r'}(0)}{\theta_{j'-i'-r'}(\tau)\theta_{k'r'}(\tau)}
= -c_{ij'k'r'}(\tau).
\]

Hence

\[
\Phi(R_{ijk}(\tau)) = \sum_{r \in \mathbb{Z}_n} c_{ijkr}(\tau) x_{k(j-r)} x_{k(i+r)}
= -\sum_{r' \in \mathbb{Z}_n} c_{ij'k'r'}(\tau) x_{j'-r'} x_{i'+r'}
= -R_{ij'k'r'}(\tau).
\]

This implies that \( \Phi(R_{n,k}(E,\tau)) = R_{n,k'}(E,\tau) \) for all \( \tau \in E \). Thus \( \Phi \) descends to an isomorphism \( Q_{n,k}(E,\tau) \to Q_{n,k'}(E,\tau) \).

**Proposition 3.8.** There is an equality of \( \mathbb{C} \)-algebras, \( Q_{n,k}(E,\tau)^{op} = Q_{n,k}(E,-\tau) \).

**Proof.** We will use the fact that \( \theta_\alpha(-z) = -e\left(-nz + \frac{2\alpha}{n}\right)\theta_\alpha(z) \).

Assume \( \tau \in \mathbb{C} - \frac{1}{n}\Lambda \). The defining relations for \( Q_{n,k}(E,\tau)^{op} \) are

\[
R_{ij}^{op}(\tau) := \sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r}(-\tau)\theta_{kr}(\tau)} x_{i+r} x_{j-r}, \quad (i, j) \in \mathbb{Z}_n^2.
\]

The defining relations for \( Q_{n,k}(E,-\tau) \) are

\[
R_{\alpha\beta}(-\tau) = \sum_{s \in \mathbb{Z}_n} \frac{\theta_{\beta-\alpha+(k-1)s}(0)}{\theta_{\beta-\alpha-s}(\tau)\theta_{ks}(-\tau)} x_{\beta-s} x_{\alpha+s}
= \sum_{s \in \mathbb{Z}_n} \frac{-e\left(\frac{\beta-\alpha+(k-1)s}{n}\right)\theta_{-(\beta-\alpha-s)}(0)}{e(n\tau + \frac{\beta-\alpha-s}{n})\theta_{-(\beta-\alpha-s)}(-\tau) e\left(-n\tau + \frac{ks}{n}\right)\theta_{-ks}(\tau)} x_{\beta-s} x_{\alpha+s}
= -\sum_{s \in \mathbb{Z}_n} \frac{\theta_{\beta-\alpha-1}(0)}{\theta_{\beta-\alpha}(\tau)\theta_{-ks}(\tau)} x_{\beta-s} x_{\alpha+s}
= -\sum_{s' \in \mathbb{Z}_n} \frac{\theta_{\beta-\alpha-(k-1)s'}(0)}{\theta_{\beta-\alpha-s'}(\tau)\theta_{ks'}(\tau)} x_{\beta+s'} x_{\alpha-s'} \quad \text{(after setting } s' = -s) \]

\[
= -R_{\beta\alpha}^{op}(\tau).
\]

This means that the two morphisms

\[
E \xrightarrow{[-]} E \xrightarrow{\bar{\varphi}_{\alpha\beta}} \mathbb{P}(V \otimes V) \quad \text{and} \quad E \xrightarrow{\bar{\varphi}_{\beta\alpha}} \mathbb{P}(V \otimes V) \xrightarrow{\mathbb{P}(V \otimes V)} \mathbb{P}(V \otimes V)
\]

\[
z \mapsto -z \quad \text{and} \quad x_i \otimes x_j \mapsto x_j \otimes x_i
\]

coincide on \( E - E[n] \), and hence on \( E \). Therefore \( Q_{n,k}(E,\tau)^{op} = Q_{n,k}(E,-\tau) \) for all \( \tau \in E \).
3.4. The Heisenberg group acts as automorphisms of \( Q_{n,k}(E, \tau) \). As observed in Lemma 2.8, the Heisenberg group generators act on the basis for \( \Theta_n(\Lambda) \) as \( S \cdot \theta \alpha = e(\frac{\alpha}{n}) \theta \alpha \), and \( T \cdot \theta \alpha = \theta \alpha + 1 \), and the commutator \( \epsilon = [S, T] \) acts as multiplication by

\[
\omega := e\left(\frac{1}{n}\right).
\]

We now identify the vector space \( V = \text{span}\{x_0, \ldots, x_{n-1}\} \) generating \( Q_{n,k}(E, \tau) \) with \( \Theta_n(\Lambda) \) by identifying \( x_\alpha \) with \( \theta \alpha \). Thus, \( V \) also becomes a representation of \( H_n \) with the action given by (3-5) below. We extend the action of \( H_n \) on \( V \) to \( TV \) in the natural way.

**Proposition 3.9.** The Heisenberg group \( H_n \) acts as degree-preserving \( \mathbb{C} \)-algebra automorphisms of \( Q_{n,k}(E, \tau) \) by

\[
(3-5) \quad S \cdot x_i = \omega^j x_i, \quad T \cdot x_i = x_{i+1}, \quad \epsilon \cdot x_i = \omega x_i.
\]

**Proof.** It is easy to show that \( S \cdot R_{ij} = \omega^{i+j} R_{ij} \) and \( T \cdot R_{ij} = R_{i+1,j+1} \). Hence \( R_{n,k}(E, \tau) \) is an \( H_n \)-subrepresentation of \( V \otimes V \) for all \( \tau \in E \) and therefore \( H_n \) acts as degree-preserving \( \mathbb{C} \)-algebra automorphisms of \( TV/(R_{n,k}(E, \tau)) \). \( \square \)

3.5. **Another set of relations for** \( Q_{n,k}(E, \tau) \). One drawback to the presentation of \( Q_{n,k}(E, \tau) \) via the relations in (3-1) is that both \( i \) and \( j \) appear in the indices of the monomials \( x_{j-r}x_{i-r} \) and in the indices of the structure constants that are the coefficients of those monomials. In particular, if \( j - i = j' - i' \), then \( R_{ij} \) and \( R_{i'j'} \) involve the same monomials but it is not immediately clear which coefficients occur before the same monomial; for example, if \( j - i = j' - i' = 0 \) some calculation is required to compare the coefficients of \( x_{2}^2 \) in each relation. There is, however, a different set of relations for \( Q_{n,k}(E, \tau) \) with the property that the new relation indexed by \( (i, j) \) has the following property: only \( i \) is involved in indices of the structure constants and only \( j \) is involved in the indices on the quadratic monomials \( x_\alpha x_\beta \). Ultimately, one sees there are row vectors \( A_0, \ldots, A_{n-1} \) in \( \mathbb{C}^n \) and column vectors \( B_0, \ldots, B_{n-1} \) of quadratic monomials such that the new relation indexed by \( (i, j) \) is the product \( A_iB_j \).

We are grateful to Kevin De Laet for allowing us to include the next result.

**Proposition 3.10 (De Laet).** Assume \( \tau \in \mathbb{C} - \frac{1}{n}\Lambda \). For each \( (i, j) \in \mathbb{Z}_n^2 \), let

\[
r_{ij} := \sum_{r \in \mathbb{Z}_n} e\left(\frac{2r}{n}\right) \theta_{-(k+1)i+(k-1)r}(0) \frac{\theta_{r+i}(\tau)\theta_{k(r-i)}(\tau)}{\theta_{r+i}(\tau)\theta_{k(r-i)}(\tau)} x_{j-r}x_{j+r},
\]

and

\[
r'_{ij} := \sum_{r \in \mathbb{Z}_n} e\left(\frac{2r}{n}\right) \theta_{k(r+1)i+(k-1)r}(0) \frac{\theta_{r+i}(\tau)\theta_{k(r-i+1)}(\tau)}{\theta_{r+i}(\tau)\theta_{k(r-i)}(\tau)} x_{j-r}x_{j+r+1}.
\]

1. \( S \cdot r_{ij} = e\left(\frac{2i}{n}\right)r_{ij} \) and \( S \cdot r'_{ij} = e\left(\frac{2i+1}{n}\right)r'_{ij} \).
2. \( T \cdot r_{ij} = r_{i,j+1} \) and \( T \cdot r'_{ij} = r'_{i,j+1} \).
3. If \( n \) is odd, then \( R_{n,k}(E, \tau) = \text{span}\{r_{ij} \mid i, j \in \mathbb{Z}_n\} = \text{span}\{r'_{ij} \mid i, j \in \mathbb{Z}_n\} \).
4. If \( n \) is even, then \( R_{n,k}(E, \tau) = \text{span}\{r_{ij}, r'_{ij} \mid i, j \in \mathbb{Z}_n\} \).
5. If \( n \) is even, then \( r_{i,j+\frac{1}{2},j+\frac{1}{2}} = -r_{ij} \) and \( r'_{i,j+\frac{1}{2},j+\frac{1}{2}} = -r'_{ij} \).

**Proof.** If \( v \) and \( w \) are non-zero scalar multiples of each other we write \( v \equiv w \).

Statements (1) and (2) are immediate.
Since $\theta_\alpha(-z) = -e\left(-nz + \frac{2}{n}\right)\theta_\alpha(z),$

$$R_{i,-i} = \sum_{r \in \mathbb{Z}_n} \frac{\theta_{-2i+(k-1)r}(0)}{\theta_{-2i-r}(-\tau)\theta_k(\tau)} x_{-i-r} x_{i+r}$$

$$= \sum_{r \in \mathbb{Z}_n} \frac{\theta_{-2i+(k-1)r}(0)}{\theta_{-2i-r}(-\tau)\theta_k(\tau)} x_{-i-r} x_{i+r}$$

$$= \sum_{r \in \mathbb{Z}_n} e\left(\frac{r}{n}\right) \frac{\theta_{-2i+(k-1)r}(0)}{\theta_{-2i-r}(-\tau)\theta_k(\tau)} x_{-i-r} x_{i+r}$$

$$= \sum_{r' \in \mathbb{Z}_n} e\left(\frac{r'-i}{n}\right) \frac{\theta_{-(k+1)i+(k-1)r'}(0)}{\theta_{-r-i}(-\tau)\theta_k(\tau)} x_{-r-i} x_{r'+1} \quad \text{(after setting } r' = i + r)$$

$$= r'_{i0}.$$

Using (2) and $T \cdot R_{ij} = R_{i+1,j+1},$ we obtain $r_{ij} = T^j \cdot r_{i0} \equiv T^j \cdot R_{i,-i} = R_{j+i,j-i}.$ Therefore

$$\text{span}\{r_{ij} \mid i, j \in \mathbb{Z}_n\} = \text{span}\{R_{j+i,j-i} \mid i, j \in \mathbb{Z}_n\}$$

$$= \text{span}\{R_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{Z}_n, \alpha + \beta \in 2\mathbb{Z}_n\}.$$ 

Similarly,

$$R_{i,1-i} = \sum_{r \in \mathbb{Z}_n} \frac{\theta_{1-2i+(k-1)r}(0)}{\theta_{1-2i-r}(-\tau)\theta_k(\tau)} x_{1-i-r} x_{i+r}$$

$$= \sum_{r \in \mathbb{Z}_n} \frac{\theta_{1-2i+(k-1)r}(0)}{\theta_{1-2i-r}(-\tau)\theta_k(\tau)} x_{1-i-r} x_{i+r}$$

$$= \sum_{r' \in \mathbb{Z}_n} e\left(\frac{r'-1}{n}\right) \frac{\theta_{k-(k+1)i+(k-1)r'}(0)}{\theta_{r+i}(-\tau)\theta_k(\tau)} x_{-r-i} x_{r'+1} \quad \text{(after setting } r' = i + r - 1)$$

$$= r'_{i0}$$

which implies that $r'_{ij} = T^j \cdot r'_{i0} \equiv T^j \cdot R_{i,1-i} = R_{j+i,j-i+1}.$

$$\text{span}\{r'_{ij} \mid i, j \in \mathbb{Z}_n\} = \text{span}\{R_{j+i,j-i+1} \mid i, j \in \mathbb{Z}_n\}$$

$$= \text{span}\{R_{\alpha,\beta} \mid \alpha, \beta \in \mathbb{Z}_n, \alpha + \beta + 1 \in 2\mathbb{Z}_n\}.$$ 

If $n$ is odd, then $2\mathbb{Z}_n = \mathbb{Z}_n$ so $\text{span}\{r_{ij}\} = \text{span}\{r'_{ij}\} = R_{n,k}(E, \tau).$ If $n$ is even, then $\text{span}\{r_{ij}, r'_{ij}\} = R_{n,k}(E, \tau).$ Hence (3) and (4) hold.

(5) Assume $n$ is even. The relation $r_{ij}$ is a linear combination of terms of the form $x_{j-r} x_{j+r},$ $r \in \mathbb{Z}_n,$ and $r_{i+\frac{n}{2},j+\frac{n}{2}}$ is a linear combination of terms of the form $x_{j+\frac{n}{2}-r} x_{j+\frac{n}{2}+r},$ $r' \in \mathbb{Z}_n.$ Now $x_{j-r} x_{j+r} = x_{j+\frac{n}{2}-r} x_{j+\frac{n}{2}+r}$ if and only if $r' = r + \frac{n}{2}.$ Let $r' = r + \frac{n}{2}.$ The coefficient of $x_{j-r} x_{j+r}$ in $r_{i+\frac{n}{2},j+\frac{n}{2}}$ is

$$e\left(\frac{r'}{n}\right) \frac{\theta_{-(k+1)i+(k-1)r'}(0)}{\theta_{r+i}(-\tau)\theta_k(\tau)}$$

which is equal to the coefficient of $x_{j-r} x_{j+r}$ in $-r_{ij}.$ Thus $r_{i+\frac{n}{2},j+\frac{n}{2}} = -r_{ij}$ as claimed. A similar argument shows that $r'_{i+\frac{n}{2},j+\frac{n}{2}} = -r'_{ij}.$
4. Twisting \(Q_{n,k}(E, \tau)\)

4.1. Twists. Given a degree-preserving automorphism \(\phi: A \to A\) of a \(\mathbb{Z}\)-graded algebra over a field \(k\), the Zhang twist, or simply the twist, \(A^{\phi}\) is the graded vector space \(A\) endowed with the associative multiplication

\[ a \ast b = \phi^m(a)b \]

when \(b \in A_m\). There is an equivalence \(\text{Gr}(A) \equiv \text{Gr}(A^{\phi})\) between their categories of graded left modules [Zha96, Thm. 1.1].

Suppose \(A = TV/I\) is the tensor algebra of a vector space \(V\) modulo a graded ideal \(I\) in \(TV\). The restriction of \(\phi\) to \(V\) extends to a degree-preserving automorphism of \(TV\) that we also denote by \(\phi\). Since \(\phi\) descends to \(A\), \(\phi(I) = I\).

A presentation of \(A^{\phi}\) is obtained as follows:

**Lemma 4.1.** Let \((-)^{\phi}: TV \to TV\) be the linear map \(\text{id}_V \otimes \phi \otimes \cdots \otimes \phi^{m-1}\) on each \(V^\otimes m\). The identity map \(V \to V\) extends to a graded algebra isomorphism

\[ \frac{TV}{I^{\phi}} \to \left(\frac{TV}{I}\right)^{\phi} \]

where \(I^{\phi}\) is the image of \(I\) under the map \((-)^{\phi}\).

**Proof.** Since \((TV/I)^{\phi}\) is generated by \(V\) as a \(k\)-algebra, the identity \(V \to V\) extends to a graded algebra homomorphism \(\rho: TV \to (TV/I)^{\phi}\). We show that \(\ker(\rho) = I^{\phi}\).

Let \(f \in V^\otimes m\) and write \(f = \sum c_i x_{i_1} \cdots x_{i_m}\) where \(c_i \in k\) for each \(i = (i_1, \ldots, i_m)\). The image of \(f\) by \(\rho\) is

\[ g := \sum_i c_i x_{i_1} \ast \cdots \ast x_{i_m} \in (TV/I)^{\phi}. \]

Thus \(\rho(f) = 0\) if and only if \(g \in I\), that is,

\[ \sum_i c_i \phi^{m-1}(x_{i_1})\phi^{m-2}(x_{i_2}) \cdots \phi(x_{i_{m-1}})x_{i_m} \in I. \]

Since \(I\) is stable under \(\phi\), this is also equivalent to the statement that \(I\) contains

\[ \phi^{-(m-1)} \left( \sum_i c_i \phi^{m-1}(x_{i_1})\phi^{m-2}(x_{i_2}) \cdots \phi(x_{i_{m-1}})x_{i_m} \right) = \sum_i c_i x_{i_1} \phi(x_{i_2}) \cdots \phi(x_{i_{m-1}})\phi^{m-1}(x_{i_m}) \]

\[ = (\text{id}_V \otimes \phi \otimes \cdots \otimes \phi^{m-1})^{-1}(f). \]

Therefore \(\ker(\rho) = I^{\phi}\). \(\square\)

Consider, for example, a degree-preserving automorphism, \(\phi\), of the polynomial ring \(\mathbb{C}[x_0, \ldots, x_{n-1}]\) with its standard grading. If \(a\) and \(b\) are homogeneous elements of degree 1, then

\[ a \ast \phi(b) = \phi(a)b = \phi(b)a = b \ast \phi(a) \]

so \(\mathbb{C}[x_0, \ldots, x_{n-1}]^{\phi}\) is the free algebra \(\mathbb{C}[x_0, \ldots, x_{n-1}]\) modulo the ideal generated by the elements \(x_i \otimes \phi(x_j) - x_j \otimes \phi(x_i)\) for \(0 \leq i < j \leq n-1\).

4.2. The twists of \(Q_{n,k}(E, \tau)\) induced from translations by \(n\)-torsion points. In this subsection, we prove that for each \(\zeta \in E[n]\), \(Q_{n,k}(E, \tau + \zeta)\) is a twist of \(Q_{n,k}(E, \tau)\) with respect to an automorphism that is in the image of the map \(H_n \to \text{Aut}(Q_{n,k}(E, \tau))\) (see Proposition 3.9).

Recall that \(R_{n,k}(E, \tau)\) is the sum of the 1-dimensional subspaces \(L_{ij}(\tau) \subseteq V \otimes V\) defined in Definition 3.2. If \(\tau \in \mathbb{C} - \frac{1}{n}\Lambda\), then \(L_{ij}(\tau)\) is spanned by \(R_{ij}(\tau)\).

For a degree-preserving automorphism \(\phi: Q_{n,k}(E, \tau) \to Q_{n,k}(E, \tau)\), the automorphism

\[ (-)^{\phi} = 1 \otimes \phi: V \otimes V \to V \otimes V \]

descends to \((-)^{\phi}: \mathbb{P}(V \otimes V) \to \mathbb{P}(V \otimes V)\).
Lemma 4.2.

(1) \( L_{ij}(\tau + \frac{1}{n}) = L_{ij}(\tau)^{S-k^{-1}} \) and

\[ Q_{n,k}(E, \tau + \frac{1}{n}) = Q_{n,k}(E, \tau)^{S-k^{-1}}. \]

(2) \( L_{ij}(\tau + \frac{1}{n} \eta) = L_{i+1,j+k'}(\tau)^{T-k'^{-1}} \) and

\[ Q_{n,k}(E, \tau + \frac{1}{n} \eta) = Q_{n,k}(E, \tau)^{T-k'^{-1}}. \]

Proof. First we assume \( \tau \in \mathbb{C} - \frac{1}{n} \Lambda \). Since

\[
R_{ij}(\tau + \frac{1}{n}) = \sum_{r \in \mathbb{Z}_n} e\left(-\frac{i-j-r}{n}\right) \theta_{j-i+(k-1)r}(0) x_{j-r} x_{i+r}
= \sum_{r \in \mathbb{Z}_n} e\left(-\frac{i-j-(k-1)r}{n}\right) \theta_{j-i-r}(0) x_{j-r} x_{i+r}
= e\left(\frac{ki+j}{n}\right) \sum_{r \in \mathbb{Z}_n} e\left(-\frac{1}{n}(i+j+r)\right) \theta_{j-i-r}(0) x_{j-r} x_{i+r}
= e(2\tau + \frac{1}{n} \eta) \sum_{r \in \mathbb{Z}_n} \theta_{j-i-r}(0) x_{j-r} x_{i+r},
\]
statement (1) holds for all \( \tau \in \mathbb{C} - \frac{1}{n} \Lambda \). The first step towards proving (2) is the calculation

\[
R_{ij}(\tau + \frac{1}{n} \eta) = \sum_{r \in \mathbb{Z}_n} e\left(-\tau - \frac{1}{n} \eta + \frac{n-1}{2n} \eta \theta_{j-i-r-1}(0) e(-\tau - \frac{1}{n} \eta) \theta_{kr+1}(\tau) x_{j-r} x_{i+r}
= e(2\tau + \frac{1}{n} \eta) \sum_{r \in \mathbb{Z}_n} \theta_{j-i-r}(0) x_{j-r} x_{i+r}.
\]

Given \((i, j, r)\), there is a unique solution \((i', j', r')\) to the system of equations

\[
\begin{align*}
  j - i - r - 1 &= j' - i' - r', \\
  kr + 1 &= kr', \\
  j - r &= j' - r',
\end{align*}
\]

namely \((i', j', r') = (i + 1, j + k', r + k')\). Hence

\[
\begin{align*}
  \frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r-1}(0)} x_{j-r} x_{i+r} &= \frac{\theta_{j'-i'+(k-1)r'}(0)}{\theta_{j'-i'-r'}(0)} x_{j'-r'} x_{i'+r'-k'-1},
\end{align*}
\]

Therefore

\[
R_{ij}(\tau + \frac{1}{n} \eta) = e(2\tau + \frac{1}{n} \eta) R_{i+1,j+k'}(\tau)^{T-k'^{-1}}.
\]

Hence (2) holds for all \( \tau \in \mathbb{C} - \frac{1}{n} \Lambda \).

The argument in the proof of Proposition 3.8 then shows that (1) and (2) hold for all \( \tau \in E \). □

Let \( \psi : H_n \to \frac{1}{n} \Lambda \) be the group homomorphism defined by

\[
\psi(S) := -\frac{1}{n}, \quad \psi(T) := -\frac{2}{n} \eta, \quad \psi(\epsilon) := 0.
\]

It induces an isomorphism \( H_n/\epsilon H_n \to E[n] = \frac{1}{n} \Lambda/\Lambda \).
Theorem 4.3. For all $\tau \in E$ and all $\sigma \in H_n$,

$$Q_{n,k}(E, \tau + \psi(\sigma)) = Q_{n,k}(E, \tau)^{\sigma^{k+1}}.$$  

If $a, b \in \mathbb{Z}$, then $Q_{n,k}(E, \tau + \frac{a}{n} + \frac{b}{n} \eta)$ is the twist of $Q_{n,k}(E, \tau)$ by the automorphism defined by

$$x_i \mapsto e(-\frac{(k+1)ai}{n})x_{i-(k'+1)b}.$$  

Proof. Let $\sigma = T^bS^a$ in $H_n/\epsilon H_n$. By Lemma 4.2,

$$Q_{n,k}(E, \tau + \psi(\sigma)) = Q_{n,k}(E, \tau - \frac{a}{n} - \frac{b}{n} \eta)$$

$$= Q_{n,k}(E, \tau)^{-b(k' - 1)}S^{-a(-k - 1)}$$

$$= Q_{n,k}(E, \tau)^{(T^bS^a)^{k+1}}$$

$$= Q_{n,k}(E, \tau)^{\sigma^{k+1}}.$$  

Here the order of $S$ and $T$ does not matter because the twist by $\epsilon$ does not change the algebra. The second statement in the proposition is obtained from the first with $\sigma = T^{-bk'}S^{-a}$. \qed

Remark 4.4. The element $k + 1 \in \mathbb{Z}_n$ need not be invertible. (Notice that $k + 1$ is invertible if and only if $k' + 1$ is since $k' + 1 = k(k + 1)$.) We will see in Proposition 5.3 that $Q_{n,n-1}(E, \tau) = \mathbb{C}[x_0, \ldots, x_{n-1}]$ for generic $\tau$. In that case $k + 1 = 0$ in $\mathbb{Z}_n$ so adding an $n$-torsion point to $\tau$ does not change the relations. However, twisting $\mathbb{C}[x_0, \ldots, x_{n-1}]$ by $S$ (or $T$) does change the relations.

5. $Q_{n,k}(E, \tau)$ when $\tau \in E[n]$  

The first result in this section provides a proof of the assertion in [OF89, §1.2, Rem. 1] and [Ode02, §3] that $Q_{n,k}(E, 0)$ is the polynomial ring on $n$ variables. We then show that $Q_{n,k}(E, \tau)$ is a twist of that polynomial ring when $\tau \in E[n]$.

Proposition 5.1.

1. If $i \neq j$, then $L_{ij}(0) = \mathbb{C} \cdot [x_i, x_j]$.
2. If $R_{ii}(\tau)$ is not identically zero on $E - E[n]$, then

$$L_{ii}(0) = \mathbb{C} \cdot \sum_{r=1}^{\left[\frac{n}{2}\right]-1} \theta_{(k-1)r}(0) \frac{\theta_{(k-1)r}(0)}{\theta_{-r}(0)}[x_{i-r}, x_{i+r}].$$

3. $Q_{n,k}(E, 0) = \mathbb{C}[x_0, \ldots, x_{n-1}]$.

Note that

$$\left[\frac{n}{2}\right] - 1 = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} - 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. When taking limits in this proof, we give $E$, $V \otimes V$, and $\mathbb{P}(V \otimes V)$ the analytic topologies.

(1) Assume $i \neq j$. We first show that

$$(5-1) \quad \lim_{\tau \to 0} \theta_0(\tau)R_{ij}(\tau) = -[x_i, x_j]$$

in $V \otimes V$.

Let $\tau \in \mathbb{C} - \frac{1}{n}A$. If $\alpha \in \mathbb{Z}_n$, then $\theta_\alpha(0) = 0$ if and only if $\alpha = 0$. Among the terms

$$\theta_0(\tau)\frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r}(0)\theta_{kr}(\tau)} x_{j-r}x_{i+r}$$
appearing in $\theta_0(\tau)R_{ij}(\tau)$, we only have to look at those with $r$ satisfying $\theta_{j-i-r}(0) = 0$ or $\theta_{kr}(0) = 0$, or equivalently, with $r = 0$ or $r = j - i$, since all other terms approach zero as $\tau \to 0$. Therefore the left-hand side of (5-1) is equal to

$$\lim_{\tau \to 0} \theta_0(\tau) \left( \frac{\theta_{j-i}(0)}{\theta_{j-i}(-\tau)\theta_0(\tau)} x_j x_i + \frac{\theta_{k(j-i)}(0)}{\theta_0(-\tau)\theta_{k(j-i)}(\tau)} x_i x_j \right)$$

$$= \lim_{\tau \to 0} \left( \frac{\theta_{j-i}(0)}{\theta_{j-i}(-\tau)} x_j x_i + \frac{\theta_0(\tau)}{-e(-n\tau)\theta_0(\tau)} \cdot \frac{\theta_{k(j-i)}(0)}{\theta_{k(j-i)}(\tau)} x_i x_j \right)$$

$$= -[x_i, x_j].$$

Here we used $\theta_\alpha(-z) = -e(-n z + \alpha) \theta_\alpha(z)$.

Since $[x_i, x_j] \neq 0$ in $V \otimes V$ and $\theta_0(\tau) \neq 0$ on a punctured open neighborhood of 0, we can rephrase (5-1) as $L_{ij}(\tau) \to \mathbb{C} \cdot [x_i, x_j]$ in $\mathbb{P}(V \otimes V)$ as $\tau \to 0$ in $E$. On the other hand, the morphism of algebraic varieties

$$\tilde{\varphi}_{ij} : E \to \mathbb{P}(V \otimes V), \quad \tau \mapsto L_{ij}(\tau),$$

is continuous with respect to the analytic topologies. So $L_{ij}(\tau) \to L_{ij}(0)$ as $\tau \to 0$. The uniqueness of the limit implies the desired conclusion.

(2) Assume $R_{ii}(\tau)$ is not identically zero. In a similar way to (1), it suffices to prove

$$\lim_{\tau \to 0} R_{ii}(\tau) = \sum_{r=1}^{[\frac{n}{2}]-1} \frac{\theta_{(k-1)r}(0)}{\theta_{-r}(0)\theta_{kr}(0)} [x_{i-r}, x_{i+r}]$$

in $V \otimes V$. By definition,

$$R_{ii}(\tau) = \sum_{r \in \mathbb{Z}_n} \frac{\theta_{(k-1)r}(0)}{\theta_{-r}(0)\theta_{kr}(\tau)} x_{i-r}x_{i+r}.$$

Since $\theta_0(0) = 0$, the $r = 0$ summand in $R_{ii}(\tau)$ is zero on a punctured open neighborhood of 0. When $r \neq 0$, the limit as $\tau \to 0$ of that summand is obtained by substituting $\tau = 0$.

Assume $n$ is even. Since $k$ is coprime to $n$, $k$ is odd and $(k-1)\frac{n}{2} = 0$ in $\mathbb{Z}_n$; the $r = \frac{n}{2}$ summand is therefore zero.

Therefore, in general, $\lim_{r \to 0} R_{ii}(\tau)$ is equal to

$$\sum_{r=1}^{[\frac{n}{2}]-1} \left( \frac{\theta_{(k-1)r}(0)}{\theta_{-r}(0)\theta_{kr}(0)} x_{i-r}x_{i+r} + \frac{\theta_{(k-1)(-r)}(0)}{\theta_{-(-r)}(0)\theta_{k(-r)}(0)} x_{i-(-r)}x_{i+(-r)} \right)$$

$$= \sum_{r=1}^{[\frac{n}{2}]-1} \left( \frac{\theta_{(k-1)r}(0)}{\theta_{-r}(0)\theta_{kr}(0)} x_{i-r}x_{i+r} + \frac{-e(-\frac{(k-1)r}{n})\theta_{(k-1)r}(0)}{(-e(-\frac{n}{n}))\theta_{-r}(0)(-e(-\frac{k}{n}))\theta_{kr}(0)} x_{i+r}x_{i-r} \right)$$

$$= \sum_{r=1}^{[\frac{n}{2}]-1} \frac{\theta_{(k-1)r}(0)}{\theta_{-r}(0)\theta_{kr}(0)} [x_{i-r}, x_{i+r}].$$

(3) This is immediate from (1) and (2). \hfill \Box

**Corollary 5.2.** If $\zeta \in E[n]$, then $Q_{n,k}(E, \zeta)$ is the twist of the polynomial ring $\mathbb{C}[x_0, \ldots, x_{n-1}]$ by the automorphism $s^{k+1}$ where $s = \psi^{-1}(\zeta) \in H_n$ and $\psi$ is the homomorphism, in (4-1).

**Proof.** This is a consequence of Theorem 4.3 and Proposition 5.1. \hfill \Box

It is stated at [OF89, §1.2, Rem. 1] and in [Ode02, §3] that $Q_{n,n-1}(E, \tau)$ is a polynomial ring in $n$ variables for all $\tau$. As an application of Proposition 5.1, we confirm this for generic $\tau$. If, as we will show in a later paper, all $Q_{n,k}(E, \tau)$ have the same Hilbert series as the polynomial ring on $n$ variables, then $Q_{n,n-1}(E, \tau)$ is a polynomial ring in $n$ variables for all $\tau \in E$.

**Proposition 5.3.**
Corollary 5.2 implies that Proposition 3.10. Proposition 3.5.

Proof. (1) If $\tau \in \frac{1}{n}\Lambda$, then Corollary 5.2 implies that $R_{n,n-1}(E, \tau) = \bigwedge^2 V$. We assume that $\tau \in \mathbb{C} - \frac{1}{n}\Lambda$ for the rest of the proof.

Suppose $n$ is odd. The relations $r_{ij}$ in Proposition 3.10 are

$$r_{ij} = \sum_{r \in 2\mathbb{Z}} e\left(\frac{z}{n}\right) \frac{\theta_{-2r}(0)}{\theta_{i+r}(\tau)\theta_{i-r}(\tau)} x_{j-r} x_{j+r}.$$ 

Since $\theta_0(0) = 0$, the coefficient of $x_j^2$ in $r_{ij}$ is equal to 0. The coefficient of $x_{j+r} x_{j-r}$ is

$$e\left(\frac{z}{n}\right) \frac{\theta_{2r}(0)}{\theta_{i+r}(\tau)\theta_{i-r}(\tau)} = -e\left(\frac{z}{n}\right) \frac{\theta_{-2r}(0)}{\theta_{i+r}(\tau)\theta_{i-r}(\tau)},$$

which is the negative of the coefficient of $x_{j-r} x_{j+r}$. Hence $r_{ij} \in \bigwedge^2 V$.

Suppose $n$ is even. As in the odd case, the coefficient of $x_j^2$ in $r_{ij}$ is zero and so is the coefficient of $x_{j+r} x_{j-r}$. The “same” computation shows that $r_{ij} \in \bigwedge^2 V$. The coefficient of $x_{j-r} x_{j+r+1}$ in $r_{ij}$ is

$$e\left(\frac{z}{n}\right) \frac{\theta_{-2r-1}(0)}{\theta_{i+r}(\tau)\theta_{i-r-1}(\tau)}$$

and the coefficient of $x_{j+r+1} x_{j-r} = x_{j-(-r-1)} x_{j+(-r-1)+1}$ is

$$e\left(\frac{-r-1}{n}\right) \frac{\theta_{-2(-r-1)-1}(0)}{\theta_{i+(-r-1)-1}(\tau)\theta_{i-(-r-1)-1}(\tau)} = e\left(\frac{-r-1}{n}\right) \frac{\theta_{2r+1}(0)}{\theta_{i-1}(\tau)\theta_{i+r}(\tau)} = -e\left(\frac{-r-1}{n}\right) \frac{e\left(\frac{2r+1}{n}\right) \theta_{-2r-1}(0)}{\theta_{i-r-1}(\tau)\theta_{i+r}(\tau)} = -e\left(\frac{z}{n}\right) \frac{\theta_{-2r-1}(0)}{\theta_{i-r-1}(\tau)\theta_{i+r}(\tau)}.$$ 

Hence $r_{ij} \in \bigwedge^2 V$.

(2) Since equality holds in (1) for $\tau \in E[n]$, the largest dimension of $R_{n,n-1}(E, \tau)$ when $\tau$ runs over $E$ is $\binom{n}{2}$. Proposition 3.5(3) therefore implies that $\dim R_{n,k}(E, \tau) = \binom{n}{2}$ on a non-empty Zariski-open subset $U$ of $E$. Therefore the inclusion in (1) is an equality when $\tau \in U$.  

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