THE FIFTH MOMENT OF HECKE \( L \)-FUNCTIONS IN THE WEIGHT ASPECT

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Abstract. We prove an upper bound for the fifth moment of Hecke \( L \)-functions associated to holomorphic Hecke cusp forms of full level and weight \( k \) in a dyadic interval \( K \leq k \leq 2K \), as \( K \to \infty \). The bound is sharp on Selberg’s eigenvalue conjecture.

1. Introduction

Moments of \( L \)-functions, especially at the central point, are extensively studied. They yield valuable data about an \( L \)-function’s distribution, and can be used for example to infer information about the size, non-vanishing and symmetry-type of the central values.

This article is inspired by the recent works of Kiral and Young [14] and Blomer and Khan [2]. The former paper established, for the first time, an upper bound for the fifth moment of \( L \)-functions associated to holomorphic newforms of prime level \( q \) and fixed small weight, as \( q \to \infty \). The latter paper established a certain reciprocity-type formula for the twisted fourth moment of Hecke \( L \)-functions in the level aspect, which gave as a corollary an upper bound for the fifth moment, but with more general conditions and also allowing for Maass \( L \)-functions. In both papers, the upper bound for the fifth moment depends on the Ramanujan conjecture at the finite places, and when assuming the truth of this conjecture, the given upper bound is sharp (as strong as the Lindelöf bound on average).

The goal of the present paper is to fix the level (at 1) and prove a fifth moment estimate in the weight aspect (it should also be possible to work with Hecke Maass \( L \)-functions in the spectral aspect). Let \( H_k \) denote the orthonormal set of holomorphic Hecke cusp forms \( f \) of level 1 and weight \( k \). This has \( k/12 + O(1) \) elements and forms a basis of the space of cusp forms of level 1 and weight \( k \). Let \( \lambda_f(n) \) denote the (real) eigenvalue corresponding to \( f \in H_k \) of the \( n \)-th Hecke operator (which satisfies Deligne’s bound \( \lambda_f(n) \ll n^{\epsilon} \)). The \( L \)-function associated to \( f \) is defined for \( \Re(s) > 1 \) by

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}.
\]

The central point is \( s = \frac{1}{2} \) and by [15] the central value \( L(\frac{1}{2}, f) \) is known to be non-negative. Our main theorem is

**Theorem 1.1.** Let

\[
\mathcal{F} = \bigcup_{\substack{K \leq k \leq 2K \\text{mod} \ 2}} H_k,
\]

a set of \( O(K^2) \) elements. For any \( \epsilon > 0 \), we have

\[
\sum_{f \in \mathcal{F}} L\left(\frac{1}{2}, f\right)^5 \ll K^{2+2\theta+\epsilon}
\]

as \( K \to \infty \), where \( \theta = \frac{7}{64} \) is the current best bound towards the Selberg eigenvalue conjecture [13, Appendix 2].

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The “log of conductor to log of family size” ratio in [14] is 5/2, the same as in the level aspect fifth moment considered in [14] and [2]. Thus our result should be considered an analogue of the level aspect estimate. Assuming the Selberg eigenvalue conjecture (which is a part of the Ramanujan conjecture at the infinite place), our bound is sharp. This seems to be the first time that a sharp bound has been proven (conditionally) for any moment higher than the fourth in the archimedian (weight or spectral) aspect. Jutila [12] proved a good upper bound for the twelfth moment of Hecke Maass $L$-functions in the spectral aspect, but that is not sharp.

Other authors [8,11,17] have proven sharp bounds for the third and fourth moments over smaller families. For example, in [17] Peng proved a sharp bound for the third moment over $H_k$, which yields the Weyl-quality bound $L(\frac{1}{2}, f) \ll k^{\frac{1}{2}+\epsilon}$. Since such a strong bound already exists, we do not pursue a twisted fourth moment and amplification, although our methods would permit it. The goal is not to obtain individual bounds, although our main theorem already implies a weaker subconvexity bound.

Our main ideas have a similar flavour to those of [14, 2, 16], but our method is different – for example, we apply “reciprocity” twice, while the other papers apply it once. Compared to [14], our proof is simpler and shorter, and as already noted above, our method could also be used to prove a bound for the twisted fourth moment, while this is not the case in [14] (because as explained in section 2 of that paper, the assumption $m_1 \leq m_2$ is made at the outset and cancellation in the $m_1$ sum is used to deal with “fake main terms”). We cannot really compare with [2] because that paper was after a more general result. It might be possible to derive our result from [2] by first understanding the relevant integral transforms in terms of the weight, but our paper is self-contained and has the advantage (depending on taste) of being more “classical” in its approach.

Throughout the paper, we will use the convention that $\epsilon$ denotes an arbitrarily small positive constant, but not necessarily the same one from one occurrence to the next.

2. Rough Sketch

The purpose of this sketch is to explain the main ideas, ignoring all technicalities. We will consider only the generic ranges of all sums.

Using approximate function equations, we can write the fifth moment as

$$\frac{1}{K^2} \sum_{f \in F} L(\frac{1}{2}, f)^5 \approx \frac{1}{K^2} \sum_{f \in F} \sum_{n_1 \leq K} \lambda_f(n_1) \sum_{n_2, n_3, n_4, n_5 \geq K} \lambda_f(n_2) \lambda_f(n_3) \lambda_f(n_4) \lambda_f(n_5) \frac{S(n_1, n_2 n_3 n_4 n_5, c)}{c} J_{k-1} \left( 4\pi \frac{\sqrt{n_1 n_2 n_3 n_4 n_5}}{c} \right).$$

We need an upper bound of $O(K^{2g+r})$. We will in fact find that this kind of grouping with $n_1$ on one side and $n_2, n_3, n_4, n_5$ on the other leads to cleaner calculations. Applying Petersson’s trace formula, the off-diagonal part of this is

$$\frac{1}{K^2} \sum_{K \leq k \leq K} \sum_{c \geq 1} \sum_{n_1, n_2, n_3, n_4, n_5 \geq K} 2\pi k S(n_1, n_2 n_3 n_4 n_5, c) \frac{S(n_1, n_2 n_3 n_4 n_5, c)}{c} J_{k-1} \left( 4\pi \frac{\sqrt{n_1 n_2 n_3 n_4 n_5}}{c} \right).$$

Summing over $k$ first, we will get that this is

$$\frac{1}{K^2} \sum_{n_1, n_2, n_3, n_4, n_5 \geq K} \sum_{c \leq K} \frac{S(n_1, n_2 n_3 n_4 n_5, c)}{c} \left( \frac{2\sqrt{n_1 n_2 n_3 n_4 n_5}}{c} \right) \approx \frac{1}{K^4} \sum_{n_1, n_2, n_3, n_4, n_5 \geq K} \sum_{c \leq K} \frac{S(n_1, n_2 n_3 n_4 n_5, c)}{c} \left( \frac{2\sqrt{n_1 n_2 n_3 n_4 n_5}}{c} \right).$$
where as usual $e(z)$ denotes $e^{2\pi iz}$. Splitting the sum over $n_1$ into residue classes mod $c$ and applying Poisson summation (denote the dual variable by $m_1$) we get

$$\frac{1}{K^{\frac{1}{2}}} \sum_{n_2,n_3,n_4,n_5 \cong K} \sum_{c \cong K^{\frac{1}{2}}} \sum_{m_1 \cong K^{\frac{1}{2}}} S(a,n_2n_3n_4n_5,c)e\left(\frac{am_1}{c}\right) \int_{x=1} e\left(\frac{2\sqrt{x}Kn_2n_3n_4n_5}{c}\right)e\left(-\frac{-xKm_1}{c}\right) dx.$$ 

The complete sum over residue classes evaluates to $ce\left(\frac{-n_2n_3n_4n_5c}{c}\right)$, and the integral is evaluated using the stationary phase method. We get

$$\frac{1}{K^{\frac{1}{2}}} \sum_{n_2,n_3,n_4,n_5 \cong K} \sum_{c \cong K^{\frac{1}{2}}} \sum_{m_1 \cong K^{\frac{1}{2}}} e\left(\frac{-n_2n_3n_4n_5m_1}{c}\right)e\left(\frac{n_2n_3n_4n_5}{m_1}\right) = \frac{1}{K^{\frac{1}{4}}} \sum_{n_2,n_3,n_4,n_5 \cong K} \sum_{c \cong K^{\frac{1}{2}}} \sum_{m_1 \cong K^{\frac{1}{2}}} e\left(\frac{n_2n_3n_4n_5c}{m_1}\right),$$

by reciprocity.

Next we apply Poisson summation (mod $m_4$) to the $n_2$ and $n_3$ sums (in the actual proof, we will apply Voronoi summation once instead of Poisson summation twice). Note that if we were following [13] step by step, we would have applied Poisson summation to $n_2$, $n_3$ and $n_4$, but this is not how we proceed. We get

$$\frac{1}{K^{\frac{1}{2}}} \sum_{n_4,n_5 \cong K} \sum_{c \cong K^{\frac{1}{2}},m_3 \cong K^{\frac{1}{2}}} e\left(\frac{m_2m_3n_4n_5}{m_1}\right).$$

This sum displays only the generic ranges of $m_2$ and $m_3$ (the dual variables). The zero frequencies $m_2 = 0$ or $m_3 = 0$, which are omitted, are in fact quite troublesome. For example, return to (2.1) and consider the terms with $m_1 | n_3n_4n_5$ (these terms correspond to $m_2 = 0$). The contribution of such terms is

$$\frac{1}{K^{\frac{1}{4}}} \sum_{n_2,n_3,n_4,n_5 \cong K} \sum_{c \cong K^{\frac{1}{2}},m_1 \cong K^{\frac{1}{2}}} 1 \cong K^{\frac{1}{2}},$$

while we need to prove a bound of $K^{2\theta + \epsilon}$. It seems that we cannot do better because there are no harmonics present to produce further cancellation. Of course, it is not possible (by the Lindelöf hypothesis) for the fifth moment to be so large, so a careful evaluation of the fifth moment must show that these “fake main terms” should cancel out somehow. But there is a shortcut. The weight functions from the approximate functional equations have been suppressed in (2.3). If we take them into account, there is a way to design them carefully so that (2.3) is not so large. This idea was used in [11] and [14], and section 2 of the latter paper contains a nice heuristic about how the idea works.

Back to (2.2), we can apply reciprocity again to get

$$\frac{1}{K^{\frac{1}{2}}} \sum_{n_4,n_5 \cong K} \sum_{c \cong K^{\frac{1}{2}},m_2 \cong K^{\frac{1}{2}}} e\left(\frac{m_2m_3n_4n_5}{m_1}\right) e\left(\frac{-m_2m_3c}{m_1n_4n_5}\right) \approx \frac{1}{K^{\frac{1}{2}}} \sum_{n_4,n_5 \cong K} \sum_{c \cong K^{\frac{1}{2}},m_3 \cong K^{\frac{1}{2}}} e\left(\frac{m_2m_3c}{m_1n_4n_5}\right).$$

Applying Poisson summation (mod $n_4n_5$) to the $m_1$ sum (denote the dual variable by $l_1$), this is

$$\frac{1}{K^{\frac{1}{4}}} \sum_{n_4,n_5 \cong K} \sum_{c \cong K^{\frac{1}{2}},m_2,l_1 \cong K^{\frac{1}{2}}} S(m_2m_3c,l_1,n_4n_5).$$
Now we can sum over $n_4$ using Kuznetsov’s formula. The sum of Kloosterman sums is in the Linnik range as $n_4n_5 \geq \sqrt{m_2m_3c_1}$. This leads to

$$
\frac{1}{K^2} \sum_{n_5 \geq K} \left( \sum_{c,m_2,m_3,l_1} \sum_{l_2 < 1} \frac{\lambda_j(m_2m_3c)\lambda_j(l_1)}{\sqrt{m_2m_3c_1}} + \ldots \right)
$$

which the sum is over an orthonormal basis of Maass cusp forms $\{u_j\}$ of level $n_5$ and (essentially bounded) Laplacian eigenvalue $\frac{1}{4} + t_j^2$, and the ellipsis denotes the contribution of the Eisenstein series and holomorphic forms. Actually we lose $O(K^{2\theta+\epsilon})$ here due to the possibility of exceptional eigenvalues, but for the purposes of this sketch we ignore this issue.

The inner sum of (2.4), given within the parentheses, looks like the fourth moment of $L(\frac{1}{2}, u_j)$ in the level aspect, provided that we can decompose $\lambda_j(m_2m_3c)$ by multiplicativity. For this, we need to work with a basis comprising of lifts of newforms; such a basis is given in [4] or [2]. Then the expected bound for the fourth moment, which can be proved using the spectral large sieve, gives

$$
\frac{1}{K^2} \sum_{n_5 \geq K} \approx \frac{1}{K^{\epsilon}}
$$

as desired. We never need any cancellation from the $n_5$-sum, which is why a twisted fourth moment bound would probably be possible in place of the main theorem.

3. Background

3.1. Approximate functional equations. For $f \in H_k$ we have the functional equation [9, Theorem 14.7],

$$
\Lambda(s, f) := (2\pi)^{-s} \Gamma(s + \frac{k-1}{2})L(s, f) = i^k \Lambda(1 - 2, f).
$$

Let $\tau(m)$ denote the number of divisors of $m$. We will use the following standard approximate functional equations. For any $f \in H_k$, we have

$$
L(\frac{1}{2}, f)^2 = 2 \sum_{m \geq 1} \frac{\lambda_f(m)\tau(m)}{\sqrt{m}} V_k(m),
$$

where

$$
V_k(x) = \frac{1}{2\pi i} \int_{(A)} x^{-s} G(s) \frac{\Gamma(s + \frac{k}{2})^2}{\Gamma(\frac{k}{2})^2} \zeta(1 + 2s) \frac{ds}{s}
$$

for any $A > 0$ and

$$
G(s) = 2e^{x^2}(\frac{1}{2} - s^2).
$$

This follows from the functional equation [3.1] and [9, Theorem 5.3]. As explained in that theorem, we may insert in the integrand above any even function which is bounded in a fixed horizontal strip about $\Re(s) = 0$, and has value 1 at $s = 0$. Our function $G(s)$ satisfies these properties and is chosen to decay exponentially in the vertical direction (this is convenient for convergence) and to vanish at $s = \frac{1}{2}$ (this will be needed later to deal with the “fake main terms”).

For $k \equiv 0 \mod 4$, the root number in the functional equation is 1, and we have

$$
L(\frac{1}{2}, f) = 2 \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} W_k(n),
$$

where

$$
W_k(x) = \frac{1}{2\pi i} \int_{(A)} x^{-s} e^{x^2} \frac{\Gamma(s + \frac{k}{2})}{\Gamma(\frac{k}{2})} \frac{ds}{s}
$$
We have
\[(3.5) \quad V_k^{(j)}(x), W_k^{(j)}(x) \ll x^{-j}(1 + x)^{-A}\]
for any $A > 0$ and integer $j \geq 0$. Using this for $j = 0$, large $A$ and Stirling's estimates for the gamma function, the sums \[(3.2)\] and \[(3.4)\] may be restricted to $m \ll k^{2+\varepsilon}$ and $n \ll k^{1+\varepsilon}$ respectively, up to an error of $O(k^{-100}).$ Taking $j = 0$ and $A = \epsilon$ shows that $|V_k(x)|, |W_k(x)| < k^{\epsilon}$.

3.2. Summation formulae. We will need the Voronoi summation formula and the Poisson summation formula.

**Lemma 3.1. Voronoi summation.** Given a compactly supported smooth function $\Phi$ with bounded derivatives, and coprime integers $h$ and $\ell$, we have
\[
\sum_{m \geq 1} \frac{\tau(m)}{m} e\left(\frac{mh}{\ell}\right) \Phi\left(\frac{m}{M}\right) = \frac{1}{\ell} \int_{-\infty}^{\infty} \log \left(\frac{x^2 + 2\gamma}{\ell^2}\right) \Phi\left(\frac{x}{M}\right) dx + \sum_{r \geq 1} \frac{1}{r} \sum_{r \geq 1} \tau(r) e\left(\frac{\pm rh}{\ell}\right) \tilde{\Phi}_\pm\left(\frac{Mr}{\ell^2}\right),
\]
where
\[
\tilde{\Phi}_\pm(x) = \frac{1}{2\pi i} \int_{(A)} H_1^{\pm}(s) \hat{\Phi}(-s)x^{-s} ds,
\]
\(\hat{\Phi}\) is the Mellin transform of $\Phi$,
\[
H_1^{\pm}(s) = 2(2\pi i)^{-2s}\Gamma(s)^2 \cos\left(\frac{1+1}{2}\pi s\right),
\]
and $A > 0$.

**Proof.** See [1, section 2.3]. We can take any $A > 0$ because $\hat{\Phi}(-s) \ll (1 + |s|)^{-B}$ for any $B \geq 0$ by integration by parts. $\square$

**Lemma 3.2. Poisson summation.** Given a compactly supported smooth function $\Phi$ with bounded derivatives, and an arithmetic function $S_q(n)$ with period $q$, we have
\[
\sum_{-\infty < n < \infty} \Phi\left(\frac{n}{N}\right) S_q(n) = \frac{N}{q} \sum_{-\infty < l < \infty} \hat{\Phi}\left(\frac{lN}{q}\right) \sum_{a \mod q} S_q(a)e\left(\frac{al}{q}\right)
\]
\[
= \frac{N}{q} \hat{\Phi}(0) \sum_{a \mod q} S_q(a) + \frac{N}{q} \sum_{-\infty < l < \infty} \sum_{a \mod q} S_q(a)e\left(\frac{al}{q}\right) \int_{-\infty}^{\infty} \frac{1}{2\pi i} \int_{(A)} \left(-\frac{2\pi x l N}{q}\right)^{-s} \Phi(x) H_2(s) ds dx,
\]
where $\hat{\Phi}$ denotes the Fourier transform of $\Phi$,
\[
H_2(s) = \Gamma(s) \exp\left(\frac{i\pi s}{2}\right)
\]
and $A > 0$.

**Proof.** For the second line of \[(3.7)\], separate the $n$ sum into sums over residue classes $a$ modulo $q$ and apply the usual Poisson summation formula to each sum. For the third line we keep aside the contribution of $l = 0$, and for $l \neq 0$ we first compute the Mellin transform
\[
\int_{-\infty}^{\infty} \hat{\Phi}(y) y^{s-1} dy = \int_{0}^{\infty} \int_{-\infty}^{\infty} \Phi(x) e(-yx) y^{s-1} dx dy = \int_{-\infty}^{\infty} \Phi(x) (-2\pi x)^{-s} H_2(s) dx.
\]
This follows by swapping the order of integration, which we can do by the compact support of \( \Phi \), and then using the Mellin transform
\[
\int_0^\infty e^{iy}y^{s-1}dy = H_2(s)
\]
which holds for \( 0 < \Re(s) < 1 \). But since \( \hat{\Phi}(y) \ll (1 + |y|)^{-B} \) for any \( B \geq 0 \) by integration by parts, we have that \( \int_0^\infty \hat{\Phi}(y)y^{s-1}dy \) converges absolutely for \( \Re(s) > 0 \). Thus the Mellin transform given in (3.8) analytically continues to \( \Re(s) > 0 \), and by the Mellin inversion theorem we have
\[
\hat{\Phi}\left(\frac{1}{Nq}\right) = \frac{1}{2\pi i} \int_{(A)} \frac{1}{q} \left. s \int_{-\infty}^\infty \Phi(x)(-2\pi x)^{-s}H_2(s)dxds \right|_{s=0}
\]
for any \( A > 0 \).

3.3. An average of the J-Bessel function. The following result can be found in [10, Corollary 8.2].

**Lemma 3.3.** Let \( x > 0 \) and let \( h \) be a smooth function compactly supported on the positive reals and possessing bounded derivatives. We have
\[
\frac{1}{K} \sum_{k \equiv 0 \text{ mod } 2} 2^k h\left(\frac{k-1}{K}\right) J_{k-1}(x) = -\frac{1}{\sqrt{x}} \Im e^{-2\pi i/8} e^{ix} h\left(\frac{K^2}{2\pi}\right) + O\left(\frac{x}{K^5} \int_{-\infty}^\infty v^4 |\hat{h}(v)|dv\right),
\]
where for real \( v \),
\[
\hat{h}(v) := \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} e^{iu} du
\]
and \( \hat{h} \) denotes the Fourier transform of \( h \). The implied constant is absolute.

By integrating by parts several times we get that \( \hat{h}(v) \ll |v|^{-B} \) for any \( B \geq 0 \). Thus the main term of (3.9) is not dominant if \( x < K^{2-\epsilon} \).

For future use, define for any complex number \( s \) the more general function
\[
\hat{h}_s(v) := \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} u^s e^{iu} du.
\]
Integrating by parts, we get
\[
\hat{h}_s^{(j)}(v) \ll_{\Re(s)} (1 + |s|)^B |v|^{-B}
\]
for any \( B \geq 0 \). Thus the Mellin transform
\[
\hat{h}_s(w) = \int_0^\infty \hat{h}_s(v)v^{w-1}dv
\]
is holomorphic in the half plane \( \Re(w) > 0 \), and we have by integrating by parts \( j \) times:
\[
\hat{h}_s(w) \ll_{\Re(s)} (1 + |s|)^{j+\Re(w)+1}(1 + |w|)^{-j}.
\]

4. Hecke relations

Define
\[
\sum_{f \in H_k}^P \gamma_f := \sum_{f \in H_k} \frac{2\pi^2}{(k-1)L(1, \text{sym}^2 f)} \gamma_f
\]
for any complex numbers \( \gamma_f \) depending on \( f \). The average \( \sum_{f \in H_k}^P \) arises in the Petersson trace formula [9, Proposition 14.5]:
\[
\sum_{f \in H_k}^P \lambda_f(n) \lambda_f(m) = \delta_{m,n} + 2\pi i \sum_{c=1}^{\infty} \frac{S(n,m,c)}{c} J_{k-1}\left(\frac{4\pi \sqrt{mn}}{c}\right),
\]
where the value of \( \delta_{m,n} \) is 1 if \( m = n \) and 0 otherwise, \( S(n,m,c) \) is the Kloosterman sum, and \( J_{k-1}(x) \) is the J-Bessel function.

The following lemma explains how we will group together variables in the fifth moment.

**Lemma 4.1.** To prove the main theorem, it suffices to prove that for any smooth functions \( h, U_1, U_2, U_3 \) compactly supported on \((\frac{1}{2}, \frac{5}{2})\) with bounded derivatives, and any

\[
\alpha, \beta, \beta_1, \beta_2 \geq 1, \quad 1 \leq N_1, N_2, N_3 < K^{1+\epsilon},
\]

we have

\[
\frac{1}{K} \sum_{k \equiv 0 \mod 2} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} L^{(\frac{1}{2}, f)} \ll K^{2+\epsilon},
\]

where

\[
S_f := \sum_{n_1, n_2, n_3, m \geq 1} \frac{\lambda_f(n_1)}{\sqrt{n_1 n_2 n_3}} \frac{\lambda_f(n_2)}{\sqrt{n_1 n_2 n_3}} \frac{\lambda_f(n_3)}{\sqrt{n_1 n_2 n_3}} W_k(n_1) W_k(n_2) W_k(n_3) U_1\left(\frac{n_1}{N_1}\right) U_2\left(\frac{n_2}{N_2}\right) U_3\left(\frac{n_3}{N_3}\right).
\]

*Proof.* To prove the main theorem, it suffices to prove that

\[
\frac{1}{K} \sum_{k \equiv 0 \mod 2} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} L^{(\frac{1}{2}, f)} \ll K^{2+\epsilon},
\]

because we have \( L^{(\frac{1}{2}, f)} \geq 0 \) by [15] and \( k^{-\epsilon} < L(1, \text{sym}^2 f) < k^\epsilon \) by [7, Appendix].

We claim that

\[
L^{(\frac{1}{2}, f)} = \frac{1}{2} \left( \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} W_k(n) \right)^3 L^{(\frac{1}{2}, f)}.
\]

This holds by [3.4] when \( k \equiv 0 \mod 4 \). But when \( k \equiv 2 \mod 4 \), it also holds because then \( L^{(\frac{1}{2}, f)} = 0 \) by the functional equation [3.1], so both sides of [3.3] vanish. Now we can insert the approximate functional equation for \( L^{(\frac{1}{2}, f)} \) given in [3.2] to get that

\[
L^{(\frac{1}{2}, f)} = 16 \left( \sum_{n \geq 1} \frac{\lambda_f(n)}{\sqrt{n}} W_k(n) \right)^3 \left( \sum_{m \geq 1} \frac{\lambda_f(m) \tau(m)}{\sqrt{m}} V_k(m) \right).
\]

Expanding the cube and working in dyadic intervals, to establish the main theorem it suffices to prove that

\[
\frac{1}{K} \sum_{k \equiv 0 \mod 2} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} S_1 \ll K^{2+\epsilon},
\]

where

\[
S_1 := 3 \prod_{i=1}^3 \left( \sum_{n_i \geq 1} \frac{\lambda_f(n_i)}{\sqrt{n_i}} W_i\left(\frac{n_i}{N_i}\right) \right) \left( \sum_{m \geq 1} \frac{\lambda_f(m) \tau(m)}{\sqrt{m}} V_k(m) \right)
\]

for

\[
W_i(n_i) := W_k(n_i) U_i\left(\frac{n_i}{N_i}\right)
\]
and $1 \leq N_1, N_2, N_3 < K^{1+\varepsilon}$. By symmetry, we can suppose that $N_3 \geq N_2$. By Hecke multiplicativity, we have
\[ \lambda_f(m)\lambda_f(n_1) = \sum_{d|(m,n_1)} \lambda_f\left(\frac{mn_1}{d^2}\right), \]
so replacing $m$ by $md$ and $n_1$ by $n_1 d$, we get
\[ S_1 = \sum_{n_1, n_2, n_3, m, d \geq 1} \frac{\lambda_f(mn_1)\lambda_f(n_2)\tau(md)\lambda_f(n_3)\tau(mn_1 n_2 n_3 m)}{db_2 b_3 b_4 \sqrt{m_1 n_1 n_3 m}} \times \frac{W_1\left(\frac{n_1 b_2 b_3 d}{N_1}\right)W_2\left(\frac{b_1 b_2 b_3}{N_2}\right)W_3\left(\frac{n_3}{N_3}\right)}{V_k(m b_3 b_4 d)}. \]

Now we combine
\[ \lambda_f(mn_1)\lambda_f(n_2) = \sum_{b|(mn_1, n_2)} \lambda_f\left(\frac{mn_1 n_2}{b^2}\right) = \sum_{n_2 = b_1 b, b_1 \in \mathbb{N}} \lambda_f\left(\frac{mn_1 b_1}{b}\right). \]

Ordering by the gcd of $n_1$ and $b$, we have the disjoint union
\[ (4.4) \quad \{n_1, m : b|n_1 m\} = \bigcup_{b = b_2 b'} \{n_1, m : b|n_1 m\} = \bigcup_{b = b_2 b'} \{n_1, m : b_2|n_1, b'|m, \left(\frac{n_1}{b_2}, b'\right) = 1\}, \]
and $\left(\frac{m_1}{b_2}, b'\right) = 1$ can be detected using the Mobius function:
\[ (4.5) \quad \sum_{b' = b_2 b_4 \mid b_3} \mu(b_3) = \begin{cases} 1 & \text{if } \left(\frac{m_1}{b_2}, b'\right) = 1 \\ 0 & \text{otherwise.} \end{cases} \]

Thus replacing $b$ by $b_2 b_3 b_4$, $n_2$ by $b_1 b_2 b_3 b_4$, $n_1$ by $n_1 b_3 b_4$, and $m$ by $m b_3 b_4$, we get
\[ S_1 = \sum_{n_1, b_1, n_3, m \geq 1} \frac{\lambda_f(m b_3 b_4 d)\tau(m b_3 b_4 d)\lambda_f(n_3)\tau(m n_1 n_2 n_3 m)}{db_2 b_3 b_4 \sqrt{m_1 n_1 n_3 m}} \times \frac{W_1\left(\frac{n_1 b_2 b_3 d}{N_1}\right)W_2\left(\frac{b_1 b_2 b_3}{N_2}\right)W_3\left(\frac{n_3}{N_3}\right)}{V_k(m b_3 b_4 d)}. \]

Splitting the divisor function
\[ \tau(m b_3 b_4 d) = \sum_{r|m b_3 b_4 d} \mu(r)\tau\left(\frac{m}{r}\right)\tau\left(\frac{b_3 b_4 d}{r}\right), \]
replacing $m$ by $mr$, and renaming $b_1$ to $n_2$, we have
\[ S_1 = \sum_{n_1, n_2, n_3, m \geq 1} \frac{\lambda_f(m n_2 b_3 r)\lambda_f(n_3)\tau(m)\tau\left(\frac{b_3 b_4 d}{r}\right)\mu(b_3)\mu(r)}{db_2 b_3 b_4 \sqrt{n_1 n_2 n_3 m r}} \times \frac{W_1\left(\frac{n_1 b_2 b_3 d}{N_1}\right)W_2\left(\frac{n_2 b_2 b_3}{N_2}\right)W_3\left(\frac{n_3}{N_3}\right)}{V_k(m b_3 b_4 dr)}. \]

We plan to find cancellation in the sum over $n_1, n_2, n_3, m$ and to sum trivially over the remaining parameters $b_2, b_3, b_4, r, d$. Thus it suffices to prove that
\[ (4.6) \quad \sum_{b_2, b_3, b_4, d \geq 1} \frac{1}{db_2 b_3 b_4 \sqrt{r}} \left| \frac{1}{K} \sum_{k \equiv 0 \mod 2} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} S_2 \right| \ll K^{2g + \varepsilon}, \]
where

\[ S_2 := \sum_{n_1, n_2, n_3, m \geq 1} \frac{\lambda_f(n_1 n_2 m) \lambda_f(n_3) \tau(m)}{\sqrt{n_1 n_2 n_3}} W_1 \left( \frac{n_1 b_2 b_3 d}{N_1} \right) W_2 \left( \frac{n_2 b_2 b_3 d}{N_2} \right) W_3 \left( \frac{n_3}{N_3} \right) V_k(mb_3 b_4 dr). \]

For (4.6) it suffices to show that

\[ \frac{1}{K} \sum_{k \equiv 0 \mod 2} h \left( \frac{k - 1}{K} \right) S_2 \ll K^{20+\epsilon} \sqrt{b_3 r}. \]

This is given by (4.2), once in \( S \) we replace \( N_1 \) by \( N_1/b_2 b_3 d \) and \( N_2 \) by \( N_2/b_2 b_3 b_4 \), and take \( \alpha = b_3 r \), \( \beta = b_3 b_4 d \), \( \beta_1 = b_2 b_3 d \), \( \beta_2 = b_2 b_3 b_4 \). Note that these substitutions lead to a smaller value of \( N_2 \), so that \( N_3 \geq N_2 \) still holds. Since \( \beta_1 \beta_2 \geq b_3(b_4 r) \geq \alpha \), we have \( N_1 N_2 < K^{2+\epsilon}/\alpha \). Also note that \( \beta \geq \alpha \).

5. Application of the Trace Formula

Applying the Petersson trace formula to Lemma 4.1, we need to prove that

\[ D + OD < \sqrt{\alpha} K^{20+\epsilon}, \]

where the diagonal

\[ D := \sum_{n_1, n_2, n_3, m \geq 1} \frac{\lambda_f(n_1 n_2 m) \lambda_f(n_3) \tau(m)}{\sqrt{n_1 n_2 n_3}} U_1 \left( \frac{n_1}{N_1} \right) U_2 \left( \frac{n_2}{N_2} \right) U_3 \left( \frac{n_3}{N_3} \right) W_k(n_1 \beta_1) W_k(n_2 \beta_2) W_k(n_3) V_k(m \beta) \]

trivially satisfies the required bound, and the off-diagonal is

\[ OD := \sum_{n_1, n_2, n_3, m, c \geq 1} \frac{S(n_1 n_2 m, n_3, c) \tau(m)}{c \sqrt{n_1 n_2 n_3}} U_1 \left( \frac{n_1}{N_1} \right) U_2 \left( \frac{n_2}{N_2} \right) U_3 \left( \frac{n_3}{N_3} \right) \]

\[ \frac{1}{K} \sum_{k \equiv 0 \mod 2} h \left( \frac{k - 1}{K} \right) 2\pi i^k \rho_{-1} \left( \frac{4\pi \sqrt{n_1 n_2 n_3 m \alpha}}{c} \right) W_k(n_1 \beta_1) W_k(n_2 \beta_2) W_k(n_3) V_k(m \beta). \]

At this point, we cannot absorb the \( W_k \) functions into the arbitrary weight functions \( U_i \) because \( W_k \) depends on \( k \) and we still need to average over \( k \), which is what we do next. Applying Lemma 3.3 the contribution of its error term is bounded by

\[ \frac{1}{K^b-\epsilon} \sum_{n_1 < K^{b+1}/\alpha} \sum_{n_2 < K^{1+\epsilon}/\alpha} \sum_{n_3 < K^{1+\epsilon}/\alpha} \sum_{c \geq 1} \frac{|S(n_1 n_2 m, n_3, c)|}{c \sqrt{n_1 n_2 n_3 \alpha}} \ll K^\epsilon, \]

on using Weil’s bound for the Kloosterman sum. Thus we need only consider the main term of Lemma 3.3 and it suffices to prove

\[ OD_1 := \sum_{n_1, n_2, n_3, m, c \geq 1} \frac{S(n_1 n_2 m, n_3, c) \tau(m)}{\sqrt{c(n_1 n_2 n_3)} \sqrt{n_1 n_2 n_3 \alpha}} \left( \frac{2\pi \sqrt{n_1 n_2 n_3 m}}{c} \right) \left( \frac{K^2 c}{8\pi \sqrt{n_1 n_2 n_3 m \alpha}} \right) \prod_{j=1}^{\beta} U_j \left( \frac{n_j}{N_j} \right) \ll \alpha^2 K^{20+\epsilon}, \]
where

\[ \Psi_K(x_1, x_2, x_3, x_4, v) := \frac{1}{(2\pi i)^4} \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} \int_{(A_4)} \frac{\zeta(1 + 2s_4) G(s_4)}{x_4^{s_4}} \frac{\Gamma\left(\frac{\sqrt{u} K + s_4 + \frac{1}{2}}{2}\right)}{\Gamma\left(\frac{\sqrt{u} K + s_4 + \frac{1}{2}}{2}\right)} \prod_{j=1}^3 e^{\frac{\pi}{4} i} \frac{\Gamma\left(\frac{\sqrt{u} K + s_j + \frac{1}{2}}{2}\right)}{\Gamma\left(\frac{\sqrt{u} K + s_j + \frac{1}{2}}{2}\right)} \frac{ds_4}{s_4} e^{iuvdu} \]

By the rapid decay of the integrands in vertical lines, we may effectively truncate the integrals to \(|s_1|, |s_2|, |s_3|, |s_4| < K^\epsilon\). For \(|s| < K^\epsilon\), by Stirling’s approximation we have

\[ \frac{\Gamma\left(\frac{\sqrt{u} K + s + \frac{1}{2}}{2}\right)}{\Gamma\left(\frac{\sqrt{u} K + s + \frac{1}{2}}{2}\right)} = \left(\frac{\sqrt{u} K}{2}\right)^s (1 + \frac{P(s)}{\sqrt{uK}} + O\left(\frac{1}{K^{2-\epsilon}}\right)) \]

for some polynomial \(P\). Thus

\[ \Psi_K(x_1, x_2, x_3, x_4, v) = \Psi\left(\frac{x_1}{K}, \frac{x_2}{K}, \frac{x_3}{K}, \frac{x_4}{K^2}, v\right) + \frac{1}{K} \Psi_0\left(\frac{x_1}{K}, \frac{x_2}{K}, \frac{x_3}{K}, \frac{x_4}{K^2}, v\right) + O(K^{-2+\epsilon}) \]

where for \(\xi_i > 0\) and real \(v\) we define

\[ \Psi(\xi_1, \xi_2, \xi_3, \xi_4, v) := \frac{1}{(2\pi i)^4} \int_{(A_4)} \int_{(A_3)} \int_{(A_2)} \int_{(A_1)} e^{s_1^2 + s_2^2 + s_3^2} \zeta(1 + 2s_4) G(s_4) \prod_{j=1}^3 e^{\xi_j s_j} \frac{ds_1 ds_2 ds_3 ds_4}{s_1 s_2 s_3 s_4} \]

and \(W_0\) has the same definition except for the presence of an extra factor \(\frac{P(s)}{\sqrt{u}}\) in the integrand for some polynomial \(P\). It suffices to treat only the contribution of \(\Psi\), as the treatment of the secondary term \(\Psi_0\) will be similar. Thus we need to prove

\[ OD_2 := \sum_{n_1, n_2, m, c \geq 1} \frac{S(n_1 n_2 m \alpha, n_3, c) \tau(m)}{\sqrt{c(n_1 n_2 n_3 m)^4}} e\left(\frac{2\pi \sqrt{n_1 n_2 n_3 m \alpha}}{c}\right) \Psi\left(\frac{n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K^2}, \frac{n_3}{K^2}, \frac{m \beta}{8\pi \sqrt{n_1 n_2 n_3 m}}, \frac{K^2 c}{8}\right) \prod_{j=1}^3 U_j \left(\frac{n_j}{N_j}\right) \ll \alpha^2 K^{2+\epsilon} \]

By 3.10, we may assume (up to negligible error) that

\[ c \ll \frac{\sqrt{n_1 n_2 n_3 m \alpha}}{K^{2-\epsilon}}. \]

By 3.35 and 3.10, we have that

\[ \frac{\partial^j_1 \xi_1 \partial^j_2 \xi_2 \partial^j_3 \xi_3 \partial^j_4 \xi_4}{\partial^j_1 \xi_1^{j_1} \partial^j_2 \xi_2^{j_2} \partial^j_3 \xi_3^{j_3} \partial^j_4 \xi_4^{j_4}} \Psi(\xi_1, \xi_2, \xi_3, \xi_4, v) \ll K^\epsilon \xi_1^{-j_1 - A_1} \xi_2^{-j_2 - A_2} \xi_3^{-j_3 - A_3} \xi_4^{-j_4 - A_4} v^{-j - B} \]

for \(\xi_1, \xi_2, \xi_3, \xi_4, v > 0\), any integers \(j_i, B \geq 0\) and any real \(A_i > 0\).
6. Poisson summation and reciprocity

In (5.3), we sum over $n_3$ in residue classes mod $c$ and apply Poisson summation (Lemma 5.2), getting

$$
(6.1) \quad OD_2 = \sum_{n_1, n_2, m, c \geq 1, -\infty < \ell < \infty} \frac{N_3}{c} \sum_{a \mod c} S(a, n_1 n_2 m, c) e\left(\frac{a\ell}{c}\right) \frac{\tau(m)}{\sqrt{c(n_1 n_2 N_3 m)}} U_1 \left(\frac{n_1}{N_2}\right) U_2 \left(\frac{n_2}{N_2}\right)
$$

$$
\int_{-\infty}^{\infty} \frac{e^{-\ell N_3 x}}{c} e\left(2\sqrt{n_1 n_2 N_3 m\alpha}\right) \psi\left(\frac{n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K}, \frac{x N_3}{K}, \frac{\beta}{K^2}, \frac{K^2 c}{8\pi \sqrt{n_1 n_2 N_3 m}}\right) U_3(x) \, dx.
$$

Call the integral above $I$. We will evaluate it using stationary phase approximation.

**Lemma 6.1.** We have that $I \ll K^{-1000}$ unless $|\ell| \leq K^{100}$, in which case

$$
I = \sqrt{\frac{2\ell c}{n_1 n_2 m\alpha}} e\left(\frac{n_1 n_2 m\alpha}{\ell c} - \frac{\pi}{8}\right) \psi\left(\frac{n_1 \beta_1}{N_1}, \frac{n_2 \beta_2}{N_2}, \frac{n_1 n_2 m\alpha}{\ell^2 K}, \frac{\beta}{K^2}, \frac{K^2 \ell c}{8\pi \sqrt{n_1 n_2 N_3 m}}\right) U_3 \left(\frac{n_1 n_2 m\alpha}{\ell^2 N_3}\right) \left(\frac{n_1 n_2 m\alpha}{\ell^2 N_3}\right)^{\frac{1}{2}}
$$

$$
+ O(K^{-1000}) + O\left(\frac{1}{K^{1-\epsilon}}\right) \delta_{|\ell| > \sqrt{\frac{n_1 n_2 N_3 m\alpha}{\beta}}},
$$

with the understanding that the main term vanishes if $\ell = 0$. The delta function $\delta_{|\ell|}$ equals 1 if the statement $P$ holds and 0 otherwise.

**Proof.** Suppose first that $\ell = 0$. Then integrate by parts $j$ times the integral $I$ given in (6.1). Here we repeatedly integrate $e\left(\frac{2\sqrt{n_1 n_2 N_3 m\alpha}}{\ell c}\right)$ after substituting $y = \sqrt{x}$ and differentiate the rest of the integrand. Using (6.3) and (6.4) we have that

$$
I \ll K^\epsilon \left(\frac{\sqrt{n_1 n_2 N_3 m\alpha}}{c}\right)^{-j} \ll K^\epsilon \left(\frac{1}{K^{2-\epsilon}}\right)^j.
$$

Taking $j$ large enough, this is $O(K^{-1000})$.

Now suppose that $\ell \neq 0$ and $|\ell| > K^{100}$. We integrate by parts $j$ times the integral $I$ given in (6.1). This time we repeatedly integrate $e\left(-\frac{\ell N_3 x}{c}\right)$ and differentiate the rest of the integrand. We again get that $I \ll K^{-1000}$.

Henceforth assume $\ell \neq 0$ and $|\ell| \leq K^{100}$. Making the substitution

$$
y = \frac{\ell^2 N_3}{n_1 n_2 m\alpha} x,
$$

the integral is

$$
I = \int_{-\infty}^{\infty} e\left(\frac{n_1 n_2 m\alpha}{\ell c} (2\sqrt{y} - y)\right) \psi\left(\frac{n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K}, \frac{y n_1 n_2 m\alpha}{\ell^2 K}, \frac{\beta}{K^2}, \frac{K^2 \ell c}{8\pi \sqrt{n_1 n_2 N_3 m}}\right) \left(\frac{n_1 n_2 m\alpha}{\ell^2 N_3}\right)^{\frac{1}{2}} U_3 \left(\frac{n_1 n_2 m\alpha}{\ell^2 N_3} y\right) \, dy.
$$

Define $U_0(x) = x^{\frac{1}{2}} U_3(x)$, so that

$$
(6.2) \quad I = \int_{-\infty}^{\infty} e\left(\frac{n_1 n_2 m\alpha}{\ell c} (2\sqrt{y} - y)\right) \psi\left(\frac{n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K}, \frac{y n_1 n_2 m\alpha}{\ell^2 K}, \frac{\beta}{K^2}, \frac{K^2 \ell c}{8\pi \sqrt{n_1 n_2 N_3 m}}\right) U_0 \left(\frac{n_1 n_2 m\alpha}{\ell^2 N_3} y\right) \, dy.
$$

The stationary point occurs at $y = 1$. Let $\Omega(y)$ be a smooth function with bounded derivatives which is equal to 1 on $(1/2, 3/2)$ and 0 on $(-\infty, 1/4) \cup (2, \infty)$. We write

$$
I = I_1 + I_2,
$$
where $I_1$ is defined as in (6.2) except that its integrand has an extra factor $1 - \Omega(y)$, and $I_2$ is defined as in (6.2) except that its integrand has an extra factor $\Omega(y)$.

We first show that

$$I_1 \ll K^{-1000}.$$ 

For this we will use [3, lemma 8.1] with

$$h(y) = \frac{2 \pi n_1 n_2 m \alpha}{\ell c} (2 \sqrt{y} - y),$$

$$w(y) = \frac{1 - \Omega(y)}{y} \Psi \left( \frac{n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K}, \frac{y n_1 n_2 m \alpha}{\ell^2 K}, -\frac{m \beta}{K^2}, \frac{K^2 \ell c}{8 \pi \sqrt{y n_1 n_2 m \alpha}} \right) U_0 \left( \frac{n_1 n_2 m \alpha}{\ell^2 N_3} y \right).$$

The parameters in this lemma are

$$R = \frac{n_1 n_2 m \alpha}{|\ell| c} \left( \left( \frac{n_1 n_2 m \alpha}{\ell^2 N_3} \right)^{\frac{1}{2}} + 1 \right), \quad U = Q = \frac{\ell^2 N_3}{n_1 n_2 m \alpha}, \quad Y = \frac{n_1 n_2 m \alpha}{|\ell| c} \left( \frac{\ell^2 N_3}{n_1 n_2 m \alpha} \right)^{\frac{1}{2}}.$$ 

This is because

$$h'(y) = \frac{n_1 n_2 m \alpha}{|\ell| c} \left( \frac{1}{\sqrt{y}} - 1 \right), \quad h^{(j)}(y) \ll \frac{n_1 n_2 m \alpha}{|\ell| c} y^{\frac{j}{2}} y^{-j}$$

for $j \geq 2$, and we can assume that $|y - 1| \gg 1$ by the support of $1 - \Omega(y)$, and that $y \approx \frac{\ell^2 N_3}{n_1 n_2 m \alpha}$ by the support of $U_0$. Further, by [3, lemma 8.1], we have

$$w^{(j)}(y) \ll \frac{K^\epsilon}{y^{j+1}}.$$ 

We don’t need to specify the remaining parameters $\alpha, \beta, X$ given in [3, lemma 8.1], apart from noting that they are bounded by some power of $K$. The result of the lemma is

$$I_1 \ll (\beta - \alpha) X ((QR/\sqrt{Y})^{-A} + (RU)^{-A})$$

for any $A \geq 0$. Thus it suffices to show that $QR/\sqrt{Y} > K^\epsilon$ and $RU > K^\epsilon$, and then to take $A$ large enough.

**Case 1.** Suppose that $\frac{\ell^2 N_3}{n_1 n_2 m \alpha} \geq 1$. Then $R \gg \frac{n_1 n_2 m \alpha}{|\ell| c}$ and so

$$QR/\sqrt{Y} \gg \left( \frac{n_1 n_2 m \alpha}{|\ell| c} \right)^{\frac{1}{2}} \left( \frac{\ell^2 N_3}{n_1 n_2 m \alpha} \right)^{\frac{1}{2}} \gg \left( \frac{\ell^2 N_3}{n_1 n_2 m \alpha} \right)^{\frac{1}{2}} = \left( \frac{\sqrt{N_2 N_3 m \alpha}}{c} \right)^{\frac{1}{2}},$$

$$RU \gg \frac{n_1 n_2 m \alpha}{|\ell| c} \left( \frac{\ell^2 N_3}{n_1 n_2 m \alpha} \right)^{\frac{1}{2}} = \sqrt{N_2 N_3 m \alpha},$$

By [3, lemma 8.1], we have $\sqrt{\frac{\ell^2 N_3 m \alpha}{c}} \gg K^{2-\epsilon}$.

**Case 2.** Suppose that $\frac{\ell^2 N_3}{n_1 n_2 m \alpha} < 1$. Then $R \gg \frac{n_1 n_2 m \alpha}{|\ell| c} \left( \frac{\ell^2 N_3}{n_1 n_2 m \alpha} \right)^{\frac{1}{2}}$ and so

$$QR/\sqrt{Y} \gg \left( \frac{n_1 n_2 m \alpha}{|\ell| c} \right)^{\frac{1}{2}} \left( \frac{\ell^2 N_3}{n_1 n_2 m \alpha} \right)^{\frac{1}{2}} = \left( \frac{\sqrt{N_2 N_3 m \alpha}}{c} \right)^{\frac{1}{2}},$$

$$RU \gg \frac{n_1 n_2 m \alpha}{|\ell| c} \left( \frac{\ell^2 N_3}{n_1 n_2 m \alpha} \right)^{\frac{1}{2}} = \sqrt{N_2 N_3 m \alpha},$$

and the conclusion is the same.

Now consider $I_2$. We have

$$I_2 = \int_{-\infty}^{\infty} e \left( \frac{n_1 n_2 m \alpha}{\ell c} (2 \sqrt{y} - y) \right) U \left( \frac{n_1 \beta_1}{N_1}, \frac{n_2 \beta_2}{N_2}, \frac{y n_1 n_2 m \alpha}{\ell^2 N_3}, -\frac{m \beta}{K^2}, \frac{K^2 \ell c}{8 \pi \sqrt{y n_1 n_2 m \alpha}} \right) U_0 \left( \frac{n_1 n_2 m \alpha}{\ell^2 N_3} y \right) \Omega(y) \, dy.$$
We apply Proposition 8.2, with
\[ h(y) = \frac{2\pi n_1 n_2 \alpha}{\ell c} (2\sqrt{y} - y), \]
\[ w(y) = \frac{\Omega(y)}{y} \Psi \left( \frac{n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K}, \frac{y n_1 n_2 \alpha}{\ell^2 K}, \frac{m \beta}{K^2}, -\frac{K^2 \ell c}{8\pi \sqrt{y} n_1 n_2 \alpha} \right) U_0 \left( \frac{n_1 n_2 \alpha}{\ell^2 N_3} y \right), \]
\[ X = K^\eps, V = V_1 = Q = 1, Y = \frac{n_1 n_2 \alpha}{|\ell| c} \times \sqrt{\frac{n_1 n_2 N_3 \alpha}{c}}. \]

The approximation to \( Y \) is given by \( \ell^2 \approx \frac{n_1 n_2 \alpha}{N_3} \) and \( y \approx 1 \) by the support of \( U_0 \) and \( \Omega \) respectively. By (5.4), we have that \( Y \gg K^{2-\eps} \). Thus the conditions [8 line (8.7)] are satisfied for \( \delta = 1/5 \) say, and we get (we have a factor of \( e(-1/8) \) instead of \( e(1/8) \) because the second derivative of \( h \) is negative)
\[ I_2 = \sqrt{\frac{2\ell c}{n_1 n_2 \alpha}} e \left( \frac{n_1 n_2 \alpha}{\ell c} - \frac{\pi}{8} \right) \Psi \left( \frac{n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K}, \frac{n_1 n_2 \alpha}{\ell^2 K}, \frac{m \beta}{K^2}, -\frac{K^2 \ell c}{8\pi n_1 n_2 \alpha} \right) U_0 \left( \frac{n_1 n_2 \alpha}{\ell^2 N_3} \right) + \text{error}, \]
where
\[ \text{error} = O\left( \frac{1}{\sqrt{Y}} \sum_{1 \leq n \leq 10^6} p_n(1) + K^{-100} \right), \]
where
\[ p_n(1) = \frac{1}{n!} \left[ \frac{G^{(2n)}(1)}{Y^n} \right], \quad G(t) = w(t)e(H(t)), \quad H(t) = h(t) - h(1) - \frac{1}{2} h''(1)(t - 1)^2. \]

Note that \( H(1) = H'(1) = H''(1) = 0 \), and so \( G^{(2n)}(1) \ll Y^{\frac{2n}{2n+1}} \ll Y^{n-1} \). Thus
\[ \text{error} = O\left( Y^{-3/2} Y^{\frac{1}{|\ell| \sqrt{\pi n_1 n_2 \alpha}}} \right) = O\left( \frac{1}{K^{3-\eps}} Y^{\frac{1}{100 |\ell| \sqrt{n_1 n_2 \alpha}}} \right). \]

Now we are ready to return to (6.1). We evaluate the \( a \)-sum there as
\[ \sum_{a \mod c} S(a, n_1 n_2 \alpha, c) e \left( \frac{a \ell}{c} \right) = \begin{cases} \epsilon c (-\frac{n_1 n_2 \alpha}{c} \ell) & \text{if } (\ell, c) = 1 \\ 0 & \text{otherwise,} \end{cases} \]
and then apply Lemma 6.1 for the integral. The error term of this lemma contributes, using (6.1) and (5.4), at most
\[ \frac{1}{K^{3-\eps}} \sum_{n_1 \cong N_1, n_2 \cong N_2} \frac{N_3^{\frac{1}{2}}}{c^\eps \langle n_1 n_2 \alpha \rangle^{\frac{1}{2}}} \ll K^{\eps}. \]

Thus we only need to consider the contribution of the main term. It suffices to prove (we only treat the terms with \( \ell > 0 \))
\[ (6.3) \quad \text{OD}_3 := \sum_{n_1, n_2, m, e, \ell \geq 1} e \left( \frac{-n_1 n_2 \alpha}{c} \ell \right) e \left( \frac{n_1 n_2 \alpha}{\ell c} \right) \frac{\tau(m)}{n_1 n_2 m} \\Psi \left( \frac{n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K}, \frac{n_1 n_2 \alpha}{\ell^2 K}, \frac{m \beta}{K^2}, -\frac{K^2 \ell c}{8\pi n_1 n_2 \alpha} \right) U_1 \left( \frac{n_1}{N_1} \right) U_2 \left( \frac{n_2}{N_2} \right) U_3 \left( \frac{n_1 n_2 \alpha}{\ell^2 N_3} \right) \ll \alpha K^{\eps}, \]
where it is understood that the sum is restricted to \((\ell, c) = 1\). By the reciprocity relation for exponentials, we have

\[
OD_3 = \sum_{n_1, n_2, m, c, \ell \geq 1} \frac{e\left(\frac{n_1 n_2 m}{\ell}\right)}{n_1 n_2 m} \frac{\tau(m)}{b_1 b_2 \ell} \Psi\left(\frac{n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K}, \frac{n_1 n_2 m}{\ell}, \frac{m \beta}{K^2}, \frac{K^2 \ell c}{8 \pi n_1 n_2 m a}\right) U_1\left(\frac{n_1}{N_1}\right) U_2\left(\frac{n_2}{N_2}\right) U_3\left(\frac{n_1 n_2 m}{\ell \ell N_3}\right).
\]

### 7. Voronoi summation and fake main terms

The next goal is to perform Voronoi summation on \(m\) but we cannot do so immediately because in the exponential \(e\left(\frac{n_1 n_2 m}{\ell}\right)\), the integers \(n_1 n_2\) and \(\ell\) may not be coprime. We first prepare by eliminating any common factors. Re-ordering the sum \(OD_3\) by \(b_1 = (n_1, \ell)\), and replacing \(n_1\) by \(b_1 n_1\) and \(\ell\) by \(b_1 \ell\), we have

\[
OD_3 = \sum_{n_1, n_2, m, c, \ell \geq 1 \atop b_1 \geq 1 \atop (n_1, \ell) = 1} \frac{e\left(\frac{n_1 n_2 m}{\ell}\right)}{b_1 n_1 n_2 m} \frac{\tau(m)}{b_1 b_2 n_1 n_2 m} \Psi\left(\frac{b_1 n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K}, \frac{n_1 n_2 m}{b_1 b_2 \ell^2 K}, \frac{m \beta}{K^2}, \frac{K^2 \ell c}{8 \pi n_1 n_2 m a}\right) U_1\left(\frac{b_1 n_1}{N_1}\right) U_2\left(\frac{n_2}{N_2}\right) U_3\left(\frac{n_1 n_2 m}{b_1 b_2 \ell^2 N_3}\right).
\]

Next we re-order the sum by \(b_2 = (n_2, \ell)\), and replace \(n_2\) by \(b_2 n_2\) and \(\ell\) by \(b_2 \ell\), then re-order the result by \(b_3 = (\alpha, \ell)\), and replace \(\ell\) by \(b_3 \ell\) and \(\alpha\) by \(b_3 \alpha\). In this way, the conditions \((4.11)\) become

\[
N_3 \geq N_2, \quad N_1 N_2 < \frac{K^{2+\epsilon}}{b_3 \alpha}, \quad \beta \geq b_3 \alpha,
\]

and

\[
OD_3 = \sum_{n_1, n_2, m, c, \ell \geq 1 \atop b_1, b_2 \geq 1 \atop (n_1, n_2, \alpha, \ell) = 1} \frac{e\left(\frac{n_1 n_2 m}{\ell}\right)}{b_1 b_2 b_3 n_1 n_2 \ell} \frac{\tau(m)}{b_1 b_2 \ell^2 \ell} \Psi\left(\frac{b_1 n_1 \beta_1}{K}, \frac{b_2 n_2 \beta_2}{K}, \frac{n_1 n_2 m}{b_1 b_2 b_3 \ell^2 K}, \frac{m \beta}{K^2}, \frac{K^2 \ell c}{8 \pi n_1 n_2 m a}\right) U_1\left(\frac{b_1 n_1}{N_1}\right) U_2\left(\frac{n_2}{N_2}\right) U_3\left(\frac{n_1 n_2 m}{b_1 b_2 b_3 \ell^2 N_3}\right),
\]

for which the required bound \((6.3)\) becomes

\[
OD_3 \ll b_3 \alpha K^{2\theta+\epsilon}.
\]

Working in dyadic intervals of \(m\) by taking a partition of unity, it suffices to show that

\[
OD_3 = \sum_j \sum_{n_1, n_2, m, c, \ell \geq 1 \atop b_1, b_2 \geq 1 \atop (n_1, n_2, \alpha, \ell) = 1} \frac{e\left(\frac{n_1 n_2 m}{\ell}\right)}{b_1 b_2 b_3 n_1 n_2 \ell} \frac{\tau(m)}{b_1 b_2 \ell} \Psi\left(\frac{b_1 n_1 \beta_1}{K}, \frac{b_2 n_2 \beta_2}{K}, \frac{n_1 n_2 m}{b_1 b_2 b_3 \ell^2 K}, \frac{m \beta}{K^2}, \frac{K^2 \ell c}{8 \pi n_1 n_2 m a}\right) U_4_j\left(\frac{m}{M_j}\right) \ll b_3 \alpha K^{\theta+\epsilon}
\]

for some smooth functions \(U_4_j\) compactly supported on \((\frac{1}{2}, \frac{5}{2})\), for \(j \ll \log K\), and \(M_j \asymp 2^j\). We now apply the Voronoi summation formula (Lemma \((3.1)\)) to the \(m\) sum, getting

\[
OD_3 := FM + OD_4,
\]
where the “fake main term” is

$$FM := \int_{-\infty}^{\infty} \left( \log \frac{x}{\ell^2} + 2\gamma \right) \sum_{j} U_{4j} \left( \frac{x}{M_{j}} \right) \sum_{\substack{n_1, n_2, c, \ell \geq 1 \\ell_1 \ell_2 \mid b_1 b_2 b_3 \ell^2 K \\ell_1 \ell_2 \mid \alpha \\ell_1 \ell_2 \mid \alpha \\ell_1 \ell_2 \mid \ell \\alpha}} \frac{1}{\ell b_1 b_2 n_1 n_2}$$

$$\psi \left( \frac{b_1 n_1 \beta_1}{K}, \frac{b_2 n_2 \beta_2}{K}, \frac{n_1 n_2 \alpha x}{\ell^2 K}, \frac{K^2 \ell \alpha}{8\pi n_1 n_2 x} \right) U_1 \left( \frac{b_1 n_1}{N_1} \right) U_2 \left( \frac{n_2 n_3 \alpha}{N_2} \right) U_3 \left( \frac{b_1 b_2 b_3 \ell^2 N_3}{x} \right),$$

and $OD_1$ is given in the next section. In the sum $FM$, we may re-patch the partition of unity and reverse the steps which led to (1.2), getting that

$$FM = \int_{-\infty}^{\infty} \left( \log \frac{x}{\ell^2} + 2\gamma \right) \sum_{n_1, n_2, c, \ell \geq 1} \frac{1}{\ell n_1 n_2}$$

$$\psi \left( \frac{n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K}, \frac{n_1 n_2 \alpha x}{\ell^2 K}, \frac{K^2 \ell \alpha}{8\pi n_1 n_2 x} \right) U_1 \left( \frac{n_1}{N_1} \right) U_2 \left( \frac{n_2}{N_2} \right) U_3 \left( \frac{n_1 n_2 \alpha}{\ell^2 N_3} \right) \frac{dx}{x}.$$

The trivial bound for $M$ is $O(K^{4+\epsilon})$, from the length of the $c$-sum given by (5.4). It seems like we cannot do better because there are no exponentials or other harmonics present which may produce further cancellation (hence the name “fake main term”). However we can exploit our judicious choice of weight function in the approximate functional equation, as follows.

Making the substitution $y = \frac{2n_1 n_2 \alpha}{N_3}$, we have

$$FM = \int_{-\infty}^{\infty} \sum_{n_1, n_2, c, \ell \geq 1} \left( \log \frac{y N_3}{n_1 n_2 \alpha} + 2\gamma \right) \frac{1}{\ell n_1 n_2}$$

$$\psi \left( \frac{n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K}, \frac{y N_3}{K}, \frac{y \ell^2 N_3 \alpha}{K^2 n_1 n_2 \alpha}, \frac{K^2 \ell \alpha}{8\pi \ell N_3 y} \right) U_1 \left( \frac{n_1}{N_1} \right) U_2 \left( \frac{n_2}{N_2} \right) U_3 (y) \frac{dy}{y}.$$

It suffices to show that

$$FM' := \sum_{c, \ell \geq 1} \frac{1}{\ell} \psi \left( \frac{n_1 \beta_1}{K}, \frac{n_2 \beta_2}{K}, \frac{y N_3}{K}, \frac{y \ell^2 N_3 \alpha}{K^2 n_1 n_2 \alpha}, \frac{K^2 \ell \alpha}{8\pi \ell N_3 y} \right) \ll K^\epsilon,$$

for any $n_1 \simeq N_1, n_2 \simeq N_2, y \simeq 1$. Using (5.2) and Mellin inversion, we have

$$FM' = \frac{1}{(2\pi i)^4} \int_{(1+\epsilon)} \int_{(2+\epsilon)} \int_{(3+\epsilon)} \int_{(4+\epsilon)} \frac{e^{s_1 + s_2 + s_3} \zeta(1 + 2s_4) \zeta(s_4)}{2^{s_1 + s_2 + s_3} 4^{s_4}} \zeta(1 + 2s_4 - w) \zeta(w)$$

$$\left( \frac{K}{n_1 \beta_1} \right)^{s_1} \left( \frac{K}{n_2 \beta_2} \right)^{s_2} \left( \frac{y N_3}{K} \right)^{s_3} \left( \frac{K^2 n_1 n_2 \alpha}{K} \right)^{s_4} \left( \frac{8\pi N_3 y}{K^2} \right)^w h_{s_1 + 2s_2 + 3s_3} (w) \frac{ds_1 ds_2 ds_3 ds_4}{s_1 s_2 s_3 s_4}.$$

Here $\zeta(1 + 2s_4 - w)$ comes from the $c$-sum (and $\zeta(w)$ comes from the $\ell$-sum). We must initially keep the lines of integration at $\Re(w) = 1 + \epsilon$ and $\Re(s_4) = \frac{3}{2} + \epsilon$ in order to stay in the region of absolute convergence. The goal is to move all the lines of integration to $(e)$, and this would prove the claim. We first move the $w$-integral to $\Re(w) = \epsilon$. This crosses a simple pole at $w = 1$, with residue

$$FM'' := \frac{1}{(2\pi i)^4} \int_{(2+\epsilon)} \int_{(3+\epsilon)} \int_{(4+\epsilon)} \frac{e^{s_1 + s_2 + s_3} \zeta(1 + 2s_4) \zeta(s_4)}{2^{s_1 + s_2 + s_3} 4^{s_4}} \zeta(2s_4)$$

$$\left( \frac{K}{n_1 \beta_1} \right)^{s_1} \left( \frac{K}{n_2 \beta_2} \right)^{s_2} \left( \frac{y N_3}{K} \right)^{s_3} \left( \frac{K^2 n_1 n_2 \alpha}{K} \right)^{s_4} \left( \frac{8\pi N_3 y}{K^2} \right)^w h_{s_1 + 2s_2 + 3s_3} (1) \frac{ds_1 ds_2 ds_3 ds_4}{s_1 s_2 s_3 s_4}.$$

On the shifted integral at $\Re(w) = \epsilon$, which is not displayed, we may move the $s_4$ integral to $\Re(s_4) = \epsilon$ and then estimate (this does not cross any pole of $\zeta(1 + 2s_4 - w)$ so this straight forward). Thus the shifted integral is $O(K^\epsilon)$ and we are left to estimate $FM''$. In the integral $FM''$, we moved the line of integration to $\Re(s_4) = \epsilon$. This does not cross any poles because the simple pole of $\zeta(2s_4)$
at $s_4 = \frac{1}{4}$ is cancelled out by the zero at $s_4 = \frac{1}{2}$ of $\mathcal{G}(s_4)$. See the definition (3.3). Thus $FM''$ is $O(K^{-1+\varepsilon})$.

8. Second application of reciprocity

We now return to (3.3) and give the definition of $OD_4$ corresponding to the sum on the right hand side of (3.6). We have ($r$ is the dual variable)

$$OD_4 := \sum_j \sum_{\pm} \frac{1}{2\pi i} \int_0^\infty \sum_{n_1,n_2,r,c,\ell \geq 1 \atop (c,\ell)=1} e\left(\frac{\pm r cn_1n_2 \alpha}{\ell}\right) \tau(r) \frac{M_j r x}{\ell^2} w H_1(w) U_1 \left( b_1n_1 \left/ N_1 \right. \right) \frac{1}{b_1b_2n_2}$$

$$U_2 \left( n_2 b_2 \right) U_3 \left( n_1 n_2 x M_j \alpha \right) U_4 (x) \Psi \left( b_1 n_1 \beta_1 \left/ K \right. , b_2 n_2 \beta_2 \left/ K \right. , n_1 n_2 x M_j \alpha x M_j \beta \left/ K^2 \right. , 8 \pi n_1 n_2 M_j \alpha \right) \frac{dx}{x} \frac{dw}{dw},$$

where it is understood that the sum is restricted to $(n_1 n_2, \ell) = 1$ and we need $OD_4 \ll b_2 \alpha K^{2+\varepsilon}$.

We first simplify the notation a bit (we did not do this earlier because we needed the exact form of the weight functions in order to deal with fake main terms). First, we observe that since there are $O(K^\varepsilon)$ dyadic intervals, it is enough to consider any one smooth function $U_{4j} = U_4$ and $M_j = M$.

From the fourth component of $\Psi$ and the assumption $\beta \geq b_3 \alpha$ from (4.1), we can assume

$$M < \frac{K^{2+\varepsilon}}{b_3 \alpha}.$$

We can also consider the sum in dyadic intervals $r \approx R$ by inserting a smooth bump function $U_5(x)$, where $U_5$ is supported on $(\frac{1}{2}, \frac{5}{2})$. We can assume that

$$R < \frac{K \varepsilon \ell^2}{M}$$

because the contribution of $r \geq \frac{K \varepsilon \ell^2}{M}$ is $O(K^{-100})$ say. This can be seen by moving the $w$-integral in $OD_4$ far to the right (taking $A$ large). By repeatedly integrating by parts the $x$-integral, we may restrict the $w$-integral to $|3w| < K^\varepsilon$ (the real part is already fixed at $A$). Doing so, we may absorb $r^{-w}$ and $(\ell^2)^{-w}$ into $U_5$ and $U_3$ respectively. Similarly we may expand the function $\Psi$ using (3.2), truncate the integrals there to $|3s_1|, |3s_2|, |3s_3|, |3s_4| < K^\varepsilon$ (with $\Re(s_1)$ fixed of course) and absorb part of this function into the bump functions $U_1, U_2, U_3, U_4$. Thus it suffices to prove (we do not seek cancellation in the sum over $b_1, b_2, b_3$)

$$\sum_{n_1,n_2,r,c,\ell \geq 1 \atop (c,\ell)=1} e\left(\frac{\pm r cn_1n_2 \alpha}{\ell}\right) \tau(r) \frac{M_j r x}{\ell^2} U_1 \left( b_1n_1 \left/ N_1 \right. \right) U_2 \left( n_2 b_2 \left/ N_2 \right. \right) U_3 \left( n_1 n_2 M \left/ b_1b_2b_3 \ell^2 \right. \right) U_5 \left( r \left/ R \right. \right) h^s \left( \frac{K^2 \ell c}{8 \pi n_1 n_2 M j \alpha} \right)$$

$$\ll b_2 \alpha K^{\theta + \varepsilon}$$

for any $b_1, b_2, b_3 \geq 1$, $x \approx 1, |s| < K^\varepsilon$ and any compactly supported functions $U_j$ with $j$-th derivative bounded by $(K^\varepsilon)^j$. We simplify the notation a bit more. We suppress the factor $\pi x$ in $h^s$, rename $b_1b_2$ to $a$, $b_3 = b$, $M \alpha$ to $M$, $N_1/b_1$ to $N_1$ and $N_2/b_2$ to $N_2$. Thus it suffices to prove

$$\sum_{n_1,n_2,r,c,\ell \geq 1 \atop (c,\ell)=1} e\left(\frac{\pm r cn_1n_2 \alpha}{\ell}\right) \tau(r) \frac{M_j r x}{\ell^2} U_1 \left( n_1 \left/ N_1 \right. \right) U_2 \left( n_2 \left/ N_2 \right. \right) U_5 \left( \frac{n_1 n_2 M}{ab\ell^2 N_3} \right) U_5 \left( r \left/ R \right. \right) h^s \left( \frac{K^2 \ell c}{n_1 n_2 M} \right) \ll b \alpha K^{\theta + \varepsilon},$$

for any integers $a, b, \alpha$ and

(8.1) $N_1, N_2, N_3 < K^{1+\varepsilon}, \quad N_1 N_2 < \frac{K^{2+\varepsilon}}{ab}, \quad M < \frac{K^{2+\varepsilon}}{b}, \quad R < \frac{K^{\varepsilon} \ell^2 \alpha}{b} \ll \frac{K^\varepsilon N_1 N_2 \alpha}{ab N_3}, \quad N_3 \geq N_2.$
The approximation \( \frac{K^\epsilon \ell^2}{M} \approx \frac{K^\epsilon N_2 n_2}{ab\ell} \) follows from the support of \( U_3 \). This updates \( (7.1) \).

Now using reciprocity for exponentials, we have
\[
e \left( \pm rc_{1}n_2\alpha \right) = e \left( \mp r \alpha \ell \right) e \left( \pm r \alpha \ell \right) = e \left( \mp r \alpha \ell \right) \left( 1 + O \left( \pm r \alpha \ell \right) \right).
\]
The contribution to \( (8.1) \) of this error term is less than
\[
\sum_{n_1 \geq n_2} K^\epsilon \sum_{r \leq K^\epsilon} \frac{r c_{1} n_2 \alpha}{\ell n_1 n_2 \alpha} \frac{N_2^2 M^2 \alpha}{K^{4-\epsilon} N_3^2} \ll \frac{1}{K^{1-\epsilon}},
\]
by \( (8.2) \). So in \( (8.1) \) we can replace the exponential with \( e \left( \mp r \alpha \ell \right) \) and detect the condition \( (\ell, c) = 1 \) using the Möbius function:
\[
\sum_{l \mid (\ell, c)} \mu(l) = \begin{cases} 1 & \text{if } (\ell, c) = 1 \\ 0 & \text{otherwise}. \end{cases}
\]
Thus replacing \( \ell \) by \( \ell l \) and \( c \) by \( cl \), it suffices to prove
\[
\sum_{n_1, n_2, r, c, l \geq 1} e \left( \mp rlc_{1}n_2\alpha \right) \frac{\mu(l) \tau(r)}{\ell n_1 n_2} U_1 \left( \frac{n_1}{N_1} \right) U_2 \left( \frac{n_2}{N_2} \right) U_3 \left( \frac{n_1 n_2 M}{ab \ell^2 N_3} \right) U_5 \left( \frac{r}{R} \right) h_s \left( \frac{K^2 \ell c}{n_1 n_2 M} \right) \ll \alpha K^{\theta + \epsilon}.
\]
We do not seek cancellation over the \( l \)-sum, so it suffices to prove
\[
OD_5 := \sum_{n_1, n_2, r, c, l \geq 1} e \left( \mp rlc_{1}n_2\alpha \right) \frac{\tau(r)}{\ell n_1 n_2} U_1 \left( \frac{n_1}{N_1} \right) U_2 \left( \frac{n_2}{N_2} \right) U_3 \left( \frac{n_1 n_2 M}{ab \ell^2 N_3} \right) U_5 \left( \frac{r}{R} \right) h_s \left( \frac{K^2 \ell c}{n_1 n_2 M} \right) \ll \alpha K^{\theta + \epsilon}
\]
for any integer \( l \geq 1 \) and assuming \( (8.2) \). Also keep in mind that it is understood that the sum is restricted to \((\ell, n_1 n_2 \alpha) = 1\).

9. Second Poisson summation

Now we split the \( l \)-sum in \( OD_5 \) into (primitive) residue classes mod \( n_1 n_2 \alpha \) and apply Poisson summation (Lemma 3.3). Note that \( \ell \) is supported in compact interval of size \( \sqrt{n_1 n_2 M} \). The result is that (the dual variable is \( d \))
\[
(9.1) \quad OD_5 = \sum_{n_1, n_2, r, c, l \geq 1} S(\mp r c_{1}d, n_1 n_2 \alpha) \frac{\tau(r)}{n_1 n_2} U_1 \left( \frac{n_1}{N_1} \right) U_2 \left( \frac{n_2}{N_2} \right) U_3 \left( \frac{r}{R} \right) \\
\times \left( \int_{-\infty}^{\infty} U_3 \left( \frac{1}{y} \right) h_s \left( \frac{y K^2 \ell c}{\sqrt{ab n_1 n_2 N_3 M}} \right) dy \right) \\
+ \sum_{d \neq 0} \frac{1}{2 \pi i} \int_{(A)} \int_{0}^{\infty} H_2(z) \left( -\frac{2 \pi y d \sqrt{M}}{I_{c} \sqrt{ab n_1 n_2 N_3}} \right) \left( \frac{z}{ab n_1 n_2 N_3} \right)^{\frac{1}{2}} \left( \frac{1}{y} \right) h_s \left( \frac{y K^2 \ell c}{\sqrt{ab n_1 n_2 N_3 M}} \right) dy dz.
\]
We first consider the contribution of the second line of \( (9.1) \). This is the zero frequency contribution, and it is bounded by
\[
\sum_{n_1 \leq N_1} \sum_{n_2 \leq N_2} \frac{K^\epsilon |S(\mp r c_{1}, 0, n_1 n_2 \alpha)|}{n_1 n_2 \alpha} \ll \sqrt{N_1 N_2 M} \frac{\sqrt{K^{2-\epsilon} N_3}}{K^{2-\epsilon} \sqrt{N_3}} \ll \frac{1}{K^{2-\epsilon}},
\]
on using \( N_3 \geq N_2 \) and that the Ramanujan sum is \( O(K^\epsilon) \) on average.
Now we consider the contribution of the third line of (9.1), arising from the sum over $d \neq 0$. We consider this sum in dyadic intervals $d \asymp D$ (for simplicity, we restrict to only positive values of $d$) and $c \asymp C$ by inserting smooth bump functions $U_6\left(\frac{d}{D}\right)$ and $U_7\left(\frac{c}{C}\right)$ say. We can assume that

$$D < \frac{K'\alpha \sqrt{abN_1N_2N_3}}{\sqrt{M}}$$

(9.2)

because the contribution of $d > \frac{\alpha \sqrt{abN_1N_2N_3}}{\sqrt{M}}$ is $O(K^{-100})$ by moving $z$-integral in (9.1) far to the right. Restricting to $|\Im z| < K\epsilon$ and $\Re z$ fixed, which we may do up to negligible error by repeatedly by parts with respect to $y$, we may absorb $d^{-\alpha}, n_1^2, n_2^2$ into the existing weight functions. We can also assume that

$$C < \frac{\sqrt{abN_1N_2N_3M}}{K^{2-\epsilon}}$$

(9.3)

and absorb the function $h_s$ into $U_7$, by using Mellin inversion and separating variables as above. Thus it suffices to prove

$$OD_6 := 1_{N_1N_2\alpha} \sum_{n_1, n_2, r, c, d \geq 1} S(\pm rc, d, n_1n_2\alpha) Y_1\left(\frac{4\pi X}{\sqrt{rcd}}\right) Y_2\left(\frac{d}{D}\right) Y_3\left(\frac{r}{R}\right) Y_4\left(\frac{c}{C}\right) \ll b\alpha K^{2\theta + \epsilon},$$

where $Y_i$ are smooth functions compactly supported on $\left(\frac{1}{2}, \frac{5}{2}\right)$ with $\|Y_j^{(j)}\|_\infty \ll (K^*)^j$ and we assume (8.2), (9.2) and (9.3).

10. **Kuznetsov’s formula**

The goal now is to prove the required bound for $OD_6$ using Kuznetsov’s formula and the spectral large sieve. We consider only the case of positive sign; the negative sign case is similar. By [9, Theorem 16.5], we have that

$$\sum_{n_2 \geq 1} \frac{S(rc, d, n_1n_2\alpha)}{n_1n_2\alpha} Y_1\left(\frac{4\pi X}{\sqrt{rcd}}\right) = \text{Maass} + \text{Eis} + \text{Hol},$$

(10.1)

where these are the contributions of the Maass cusp forms, Eisenstein series, and holomorphic forms as given in the referenced theorem. For the Maass forms we have

$$\text{Maass} = \sum_{j \geq 1} \mathcal{M}_j Y_1(t_j) \frac{\rho_j(rc) \eta_j(d)}{\cosh(\pi t_j)},$$
where the sum is over an orthonormal basis of Maass cusp forms of level \( n_1 \alpha \) with Fourier coefficients \( \rho_j(n) \) and Laplacian eigenvalue \( \frac{1}{4} + t_j^2 \), and

\[
\mathcal{M}_{Y_1}(t) = \frac{\pi i}{2 \sinh(\pi t)} \int_0^\infty (J_{2\mu t}(x) - J_{-2\mu t}(x))Y_1(x) \frac{dx}{x}
\]

By \([3] \text{ Lemma } 3.6\), for example, we have that \( \mathcal{M}_{Y_1}(t) \ll K^{-100} \) if \(|t| \geq K^\varepsilon \), so we can restrict the sum Maass to \(|t_j| < K^\varepsilon \), in which range \( \mathcal{M}_{Y_1}(t_j) \ll X^{2\beta + \varepsilon} \), by the same Lemma. We have \( X \ll K^{1-\varepsilon} \) by \([10,2]\), so it suffices to prove that

\[
\sum_{n_1 \gg N_1} \frac{1}{N_1 N_2} \sum_{|t_j| < K^\varepsilon} \frac{K'(uv)}{N_1 \alpha} \sum_{r, d, c \geq 1} \rho_j(rc) \frac{\bar{\pi}_j(d)}{\cosh(\pi t_j)} \tau(r) Y_2 \left( \frac{d}{D} \right) Y_3 \left( \frac{r}{R} \right) Y_4 \left( \frac{c}{C} \right) \ll b \alpha K^\varepsilon.
\]

Now we would like to decompose \( \rho_j(rc) \), so that Cauchy-Schwarz and the spectral large sieve may be applied. To do this we need to work with newforms, whose Fourier coefficients are multiplicative. We consult \([2] \text{ section 3}\) to see how to choose a basis consisting of lifts of newforms. By \([2] \text{ equation } (3.10)\], and the \( \cosh(\pi t_j) \frac{\pi}{2} \) normalization from the first display of \([2] \text{ section } 3.2\), it suffices to prove that

\[
\sum_{n_1 \gg N_1} \frac{1}{N_1 N_2} \sum_{|t_j| < K^\varepsilon} \frac{K'(uv)}{N_1 \alpha} \sum_{r, d, c \geq 1} \lambda_j \left( \frac{r c}{u} \right) \lambda_j \left( \frac{d}{v} \right) \tau(r) Y_2 \left( \frac{d}{D} \right) Y_3 \left( \frac{r}{R} \right) Y_4 \left( \frac{c}{C} \right) \ll b \alpha K^\varepsilon
\]

for any integers \( u, v \geq 1 \) and \( N_0 | n_1 \alpha \), where \( \lambda_j(n) \) are the Hecke eigenvalues corresponding to newforms of level \( N_0 \). We now replace \( d \) by \( dv \) and, proceeding exactly like in steps \([1.1]\) to \([1.5]\), we can write \( u = u_1 u_2 u_3 \) and replace \( r \) by \( ru_1 u_2 \) and \( c \) by \( cu_2 u_3 \) to see that it suffices to prove

\[
\sum_{n_1 \gg N_1} \frac{1}{N_1 N_2} \sum_{|t_j| < K^\varepsilon} \frac{K'(u_1 u_2 u_3 v)}{N_1 \alpha} \left| \sum_{r, d, c \geq 1} \lambda_j \left( ru_1 u_2 \right) \lambda_j \left( d \right) \mu(u_2) \tau \left( ru_1 u_2 \right) Y_2 \left( \frac{d v}{D} \right) Y_3 \left( \frac{r u_1 u_2}{R} \right) Y_4 \left( \frac{c u_2 u_3}{C} \right) \right| \ll b \alpha K^\varepsilon.
\]

To simplify notation, we may replace \( r \) by \( ru_2 \). Thus it suffices to prove

\[
\sum_{n_1 \gg N_1} \frac{1}{N_1 N_2} \sum_{|t_j| < K^\varepsilon} \frac{K'(u_1 u_2 u_3 v)}{N_1 \alpha} \left| \sum_{r \propto R/u_1} \sum_{c \propto C/u_2 u_3} \sum_{d \propto D/v} \lambda_j \left( r c u_2 \right) \lambda_j \left( d \right) \gamma_r \gamma_d \gamma_c \right| \ll b \alpha K^\varepsilon
\]

for any \( \gamma_r, \gamma_c, \gamma_d \ll K^\varepsilon \). By Hecke multiplicativity, we have

\[
\lambda_j \left( r c \right) \lambda_j \left( d \right) = \sum_{s | (r, c)} \mu \left( s \right) \lambda_j \left( \frac{r}{s} \right) \lambda_j \left( \frac{d}{s} \right) = \sum_{s | (r, c)} \mu \left( s \right) \lambda_j \left( \frac{r}{s} \right) \lambda_j \left( \frac{d c}{s w^2} \right)
\]

and so after replacing \( r \) by \( rs \), \( c \) by \( cs w \), and \( d \) by \( dw \), it suffices to prove

\[
OD_7 := \sum_{n_1 \gg N_1} \frac{1}{N_1 N_2} \sum_{|t_j| < K^\varepsilon} \frac{K'(u_1 u_2 u_3 v)}{N_1 \alpha} \left| \sum_{r \propto R/u_1} \sum_{c \propto C/D} \sum_{d \propto D/u_2 u_3 u s w^2} \lambda_j \left( r c d \right) \gamma_r \gamma_c \gamma_d \right| \ll b \alpha K^\varepsilon,
\]
for any \( \gamma_r, \gamma_{cd} \ll K^\epsilon \). By the Cauchy-Schwarz inequality and the spectral large sieve [6, Theorem 2], we have that \( OD_T \) is bounded by

\[
\sum_{n_1, N_2} \frac{1}{N_1 \alpha} \sum_{u_1, u_2, u_3} K^\epsilon \left( \sum_{|r| < K^\epsilon} \sum_{cd \leq |CD|/u_2u_3} \lambda_j(r) \gamma_r^2 \right)^{\frac{1}{2}} \left( \sum_{|r| < K^\epsilon} \sum_{cd \leq |CD|/u_2u_3} \lambda_j(cd) \gamma_{cd}^2 \right)^{\frac{1}{2}} < \sum_{n_1, N_2} \frac{1}{N_1 \alpha} \frac{K^\epsilon}{N_1 N_2} \left( N_1 \alpha + R \right)^{\frac{1}{2}} \left( N_1 \alpha + CD \right)^{\frac{1}{2}} \lesssim b_\alpha K^\epsilon.
\]

Thus it suffices to prove

\[
\frac{K^\epsilon}{N_1 N_2} \left( N_1 \alpha + R \right)^{\frac{1}{2}} \left( N_1 \alpha + CD \right)^{\frac{1}{2}} \lesssim b_\alpha K^\epsilon.
\]

By [82, 92], and (9.3), we have

\[
(RCD)^{\frac{1}{2}} \lesssim \frac{\alpha N_1 N_2}{K^{1 - \epsilon}},
\]

so it suffices to prove

\[
\frac{1}{K} \left( N_1 \alpha + R \right)^{\frac{1}{2}} \left( N_1 \alpha + CD \right)^{\frac{1}{2}} \lesssim b_\alpha K^\epsilon.
\]

We have

\[
\frac{N_1 \alpha}{K} \ll K^\epsilon \alpha,
\]

\[
\frac{(N_1 \alpha CD)^{\frac{1}{2}}}{K} \ll \frac{N_1 \alpha (abN_2 N_3)^{\frac{1}{2}}}{K^2} \ll \frac{(N_1 N_3 \alpha)^{\frac{1}{2}} (ab \alpha N_1 N_2)^{\frac{1}{2}}}{K^2} \ll K^\epsilon \alpha^{\frac{1}{2}},
\]

\[
\frac{(N_1 \alpha R)^{\frac{1}{2}}}{K} \ll \frac{N_1 \alpha N_2^{\frac{1}{2}}}{KN_3^{\frac{1}{2}}} \ll K^\epsilon \alpha,
\]

where in the last inequality we use crucially that \( N_3 \geq N_2 \). This establishes (10.3).

It remains to consider \( \text{Eis} \) and \( \text{Hol} \) in (10.1). These are similarly treated using the large sieve, once we use the multiplicative Fourier coefficients provided explicitly in [2, section 3].

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