High temperature QCD and QED with unstable excitations

P.A.Henning and R.Sollacher
Theoretical Physics, Gesellschaft für Schwerionenforschung GSI
P.O.Box 110552, D-64220 Darmstadt, Germany

March 16, 2017

Abstract

We consider the partition functions of QCD and QED at high temperature assuming small coupling constants, and present arguments in favor of an improved perturbative expansion in terms of unstable excitations. Our effective propagators are derived from spectral functions with a constant width. These spectral functions describe screening and damping of gluons (photons) as well as “Brownian” motion of quarks (electrons). BRST-invariance allows us to reduce the number of independent width parameters to three. These are determined in a self-consistent way from the one-loop self energy and polarization tensor in the infrared limit thus rendering this limit finite. All spectral width parameters are found to be proportional to $gT$. We reproduce the well known expression for the electric “Debye”-screening mass. The transverse (magnetic) gluons (photons) are found to interact only at nonzero momentum or energy, at least to leading order. As a consequence their spectral function acquires a width only away from the infrared limit. Finally, plasmon modes are determined and found to be strongly damped.
\section{Introduction}

It is well known that naive perturbation theory for gauge theories at finite temperature does not work. This is due to severe infrared divergences related to the approximation of massless gauge bosons having an infinite correlation length. The breakdown of perturbation theory can be understood qualitatively if one recalls that the average distance between "particles" in a heat bath is of order $1/T$, the inverse temperature. On the other hand, due to the uncertainty relation, each "particle" has a quantum mechanical correlation length $1/M$, where $M$ is its mass. This implies that the individual wave functions substantially overlap as soon as $T > M$. For massless fields this condition is met already at infinitesimally small temperature.

As an example providing a more concrete condition consider the gluon self energy at $\omega = 0, \mathbf{p} \to 0$. The $l$-loop contribution can be estimated as \cite[p.145]{1} (Lorentz indices are suppressed)

$$\Pi(\omega = 0, \mathbf{p} \to 0) \approx g^4 T^2 \left( \frac{g^2 T}{M} \right)^{l-2}$$  \hfill (1)

where $l > 2$, $g$ is the coupling constant and $M$ is a fictitious mass of the gluons serving as an infrared cutoff. Here, one can see the infrared divergences to become more and more severe the higher the loop order is.\par

There exists at least a partial solution to the infrared problem. This is the method of "hard thermal loops" \cite{2}. The essence of this method is the introduction of thermal screening masses. For electric gluons one finds a mass of the order of $gT$. This suggests that the loop terms in (1) are suppressed by factors of $g^{l-2}$ as compared to the leading terms. Unfortunately, one finds no mass of this order for the magnetic or transverse gluons. This seems to imply that QCD at high temperatures finally is highly nonperturbative \cite{3, 4}.

With the present work we are presenting a slightly modified approach which may overcome these problems and possibly will allow for an expansion in the coupling constant. The main difference to conventional approaches is that we do not rely on quasi-particles, i.e., particles that have a definite mass-shell only slightly perturbed by the interactions. There are several arguments against the quasi-particle concept at finite temperature:

1. A system at finite temperature does not exhibit the Poincaré symmetry of the vacuum state. What remains instead is a semi-direct product of the four-dimensional translation group $\mathbb{T}(4)$ and the three-dimensional rotation group $\mathbb{SO}(3)$. The irreducible representations of this group do not contain a Lorentz-invariant, mass-shell-like representation.
spectrum \[5\], and only these representations may be used in a perturbative expansion \[8\].

2. A self-consistent calculation of the spectral width of massive fermions at nonzero temperature \(T\) exhibits a nonanalytical behavior in \(T\) as \(T \to 0\) \[7\]. This is consistent with the picture of Poincaré symmetry restoration at \(T = 0\) as a “thermal phase transition”. It also tells us, that the basis for a perturbative expansion at \(T = 0\) is different from the basis at \(T > 0\) where we have to use a continuous mass spectrum \[6, 8\].

3. Consequences of the KMS-condition \[9, 10\] on quasi-particles have been investigated by Narnhofer, Requardt and Thirring \[11\]. Their theorem essentially states that quasi-particles can be used as an approximation only if one considers length and time scales larger than \(1/T\), i.e., the regime where the presence of thermal excitations is still negligible.

4. Probably the most severe objection against the use of quasi-particles arises in the long-time limit, i.e., for times larger than \(\approx 1/T\). For such long times the assumption of a quasi-particle with an infinite life-time no longer holds because the presence of thermal excitations necessarily leads to scattering or absorption. Attempts to calculate the damping rates in a perturbative manner from the imaginary part of the self energy or polarization on-shell must fail if the mass of the quasi-particle is smaller than \(T\). Indeed, calculations of damping rates indicate that one has to determine such rates in a self-consistent manner, although one frequently makes the assumption that the damping rate is much smaller than the mass of the quasi-particle \[12, 13, 17\].

It is this breakdown of the quasi-particle picture for temperatures higher than the relevant masses on which we base our procedure. We are taking this feature into account establishing simple ansätze for spectral functions with a finite width. This will be the content of section 2. In section 3 we calculate the self energies and polarizations to one-loop order and determine from them the width parameters in a self-consistent manner. We show these results as well as those for plasmon modes and their damping rates for QCD, and in section 4 also for QED. Section 5 is devoted to a summary and conclusions.
2 Spectral functions and imaginary time Green’s functions

The safest method for the determination of Green’s functions uses dispersion relations. They enforce the proper analyticity in the complex energy plane by removing unphysical poles from the physical Riemann sheet \[7\]. The central quantity of this procedure is a spectral function \(A(E, p)\) which must be known for real \(E\). For arbitrary complex \(E\), the retarded and advanced Green’s functions read

\[
S_{R,A}(E, p) = \int_{-\infty}^{\infty} dE' A(E', p) \frac{1}{E - E' \pm i\epsilon} .
\]

This automatically implies that the spectral function is proportional to the imaginary part of the retarded respectively the advanced Green’s function:

\[
A(E, p) = \mp \frac{1}{\pi} \text{Im}(S_{R,A}(E, p)) = \frac{1}{\pi} \frac{1}{2} \left( S_A(E, p) - S_R(E, p) \right) .
\]

For perturbative calculations of the partition function one usually needs causal Green’s functions in euclidean time respectively their Fourier transform \(S(p_0, p)\) with the Matsubara frequencies \[14\]

\[
p_0 = \left\{ \begin{array}{ll}
\nu_n^+ = 2n \pi T & \text{(bosons)} \\
\nu_n^- = (2n + 1) \pi T & \text{(fermions)} 
\end{array} \right. , n = 0, \pm 1, \ldots .
\]

These functions can be obtained from the retarded or advanced Green’s functions by analytic continuation in a unique manner \[15, 16\], using the relations

\[
S_R(E, p) = \pm \frac{1}{\pi} \text{Im}(S_{R,A}(E, p)) \quad \text{and} \quad S_A(E, p) = \pm \frac{1}{\pi} \text{Im}(S_{R,A}(E, p)) .
\]

With these relations we can now define \(S(p_0, p)\) via dispersion integral:

\[
S(p_0, p) = \pm \int_{-\infty}^{\infty} dE A(E, p) \frac{1}{E - ip_0} .
\]

The upper sign in \(\text{\textcircled{1}}\) and \(\text{\textcircled{2}}\) is valid for fermions, the lower one for bosons; they account for the signs when switching to euclidean space-time and euclidean \(\gamma\)-matrices.

We are interested in spectral functions with a finite constant width. For bosons, we abbreviate the spectral width as \(\gamma_B\) for the moment, and use the spectral function

\[
A_B(E, p) = \frac{1}{2\pi} \frac{2E\gamma_B}{\sqrt{E^2 - \gamma_B^2}} \quad \text{for} \quad E = p_0 .
\]
where \( \omega_p \) denotes the energy-momentum dispersion relation. We will neglect masses and simply choose \( \omega_p^2 = p^2 \). The spectral function (7) is normalized according to the commutation relations for bosonic fields:

\[
\int_{-\infty}^{\infty} dE' \ A(E', p) = 1.
\]

The corresponding euclidean Green’s function can now be obtained using eqns. (6) and (7):

\[
S_B(p_0, p) = \frac{1}{(|p_0| + \gamma_B)^2 + p^2}.
\]

In the limit of vanishing width, \( \gamma_B \to 0 \), one recovers the usual perturbative Matsubara Green’s function \( 1/(p_0^2 + p^2) \). A remarkable feature is that the Green’s function (9) is real, even with \( \gamma_B > 0 \).

Similarly, we proceed for fermions, where we abbreviate the spectral width as \( \gamma_F \) to distinguish it from the Dirac \( \gamma \)-matrices \( (\gamma_0, \gamma) \). As spectral function we take

\[
\mathcal{A}_F(E, p) = \frac{\gamma_F}{\pi} \frac{\gamma_0 (E^2 + \omega_p^2 + \gamma_F^2)}{(E^2 - \omega_p^2 - \gamma_F^2)^2 + 4E^2 \gamma_F^2} + i2E \gamma_p
\]

\[
= \frac{1}{4 \pi i \omega_p} \left( \frac{\gamma_0 \omega_p + i\gamma_p}{E - \omega_p + i\gamma_F} - \frac{\gamma_0 \omega_p - i\gamma_p}{E - \omega_p - i\gamma_F} \right) \cdot
\]

with euclidean \( \gamma \)-matrices, see appendix A. This spectral function is normalized such that

\[
\int_{-\infty}^{\infty} dE' \ Tr [\gamma_0 \mathcal{A}(E', p)] = 4.
\]

The corresponding Matsubara Green’s function with euclidean \( \gamma \)-matrices and \( \omega_p^2 = p^2 \) simply reads

\[
S_F(p_0, p) = -i \frac{\gamma_0 \text{sign}(p_0) (|p_0| + \gamma_F) + \gamma_p}{(|p_0| + \gamma_F)^2 + p^2}.
\]

Obviously, the only modification as compared to the free case is a shift of the absolute value of the Matsubara frequencies by the spectral width parameter \( \gamma_F \).

For illustration let us consider the spatial and temporal behavior of a retarded real-time propagator as defined in (2). With a boson spectral function according to (7) we get

\[
S^R(t, p) = \int_{-\infty}^{\infty} dE \ exp(-iEt) \ S^R(E, p)
\]
\[\propto \Theta(t) \exp(-\gamma_B t) \frac{\sin(\omega_p t)}{\omega_p}, \quad (13)\]

whereas the \(\gamma_0\) component of the fermion propagator with spectral function (10) becomes

\[
\text{Tr} \left[ \gamma_0 S_F^R(t, p) \right] = \int_{-\infty}^{\infty} dE \exp(-iEt) \text{Tr} \left[ \gamma_0 S_F(E, p) \right] \\
\propto \Theta(t) \exp(-\gamma_F t) \cos(\omega_p t). \quad (14)
\]

This shows that \(\gamma\) describes the damping of the corresponding wave-function. In order to investigate the spatial behavior we consider the time average of (13):

\[
\int_{-\infty}^{\infty} dt \, S_B^R(t, p) = -i S_B(0, p) = \frac{-1}{p^2 + \gamma_B^2}. \quad (15)
\]

Obviously, for these zero frequency bosonic modes one cannot distinguish between a mass and a spectral width. This implies that a spectral width not only describes damping but also screening.

Another interpretation of these spectral functions, which is easier to accept in the fermionic case, is the one of Brownian motion \([17, \text{eq. (6.33)}]\). In this case, (13) resp. (14) represent the decaying probability amplitude to find a particle with momentum \(p\) in the same state after time \(t\), when it is subject to thermal fluctuations of the surrounding medium.

### 3 The partition function of hot QCD

We now consider the partition function of QCD represented as a functional integral,

\[Z = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}c \mathcal{D}\bar{c} \exp(-S_{QCD}) \quad . \quad (16)\]

For convenience we consider a family of \(O(3)\)-covariant gauges \([18]\). As we are interested in high temperatures we neglect current quark masses. The action reads

\[
S_{QCD} = \int d^4x \left[ \bar{\psi}_{l,\alpha} (x) \gamma_\mu \left( \partial_\mu \delta_{\alpha\beta} - i g A^a_\mu(x) T^a_{\alpha\beta} \right) \psi_{l,\beta} (x) + \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \\
+ \frac{1}{2} (\Lambda_\mu A^a_\mu(x))^2 + \Lambda_\mu \bar{c}^a(x) \left( \partial_\mu c^a(x) + g f^{abc} A^b_\mu(x) c^c(x) \right) \right] \quad (17)
\]

Here, \(l, m\) are flavor indices, \(\alpha, \beta\) are color indices and the field strength tensor is defined as

\[F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \quad (18)\]

The gauge fixing involves a “vector” \(\Lambda_\mu\) defined as

\[\Lambda_\mu = (\lambda \partial_\mu \frac{1}{\theta}). \quad (19)\]
with gauge parameters $\xi$ and $\lambda$. The usual perturbative 2-point Green’s functions are defined as follows:

$$
\langle \psi_{l,\alpha}(x) \bar{\psi}_{m,\beta}(y) \rangle_0 = \delta_{lm} \delta_{\alpha\beta} \int \frac{d^4 p^-}{(2\pi)^4} S_{F0}(p) \exp(ip(x-y))
$$

$$
S_{F0}(p) = (i\gamma_0 p_0 + i\gamma p)^{-1}
$$

for the quarks,

$$
\langle A_{\mu}^a(x) A_{\nu}^b(y) \rangle_0 = \delta^{ab} \int \frac{d^4 p^+}{(2\pi)^4} \exp(ip(x-y))
$$

$$
(S_{T0}(p)\mathcal{P}_T^{\mu\nu} + S_{E0}(p)\mathcal{P}_E^{\mu\nu} + S_{L10}(p)\mathcal{P}_L^{1\mu\nu} + S_{L20}(p)\mathcal{P}_L^{2\mu\nu})
$$

$$
S_{T0}(p) = \frac{1}{p_0^2 + p^2}
$$

$$
S_{E0}(p) = \frac{p_0^2 + \frac{1}{\xi}p^2}{[\lambda p_0^2 + \frac{1}{\sqrt{\xi}}p^2]^2}
$$

$$
S_{L10}(p) = \frac{\lambda^2 p_0^2 + p^2}{[\lambda p_0^2 + \frac{1}{\sqrt{\xi}}p^2]^2}
$$

$$
S_{L20}(p) = \left(1 - \frac{\lambda}{\sqrt{\xi}}\right) \frac{p_0 |p|}{[\lambda p_0^2 + \frac{1}{\sqrt{\xi}}p^2]^2}
$$

(21)

for the gluons and

$$
\langle c^a(x)c^b(y) \rangle_0 = \delta^{ab} \int \frac{d^4 p^+}{(2\pi)^4} S_{gho}(p) \exp(ip(x-y))
$$

$$
S_{gho}(p) = \frac{1}{\lambda p_0^2 + \frac{1}{\sqrt{\xi}}p^2}
$$

(22)

for the ghost fields. The symbol $\langle \ldots \rangle_0$ denotes functional integration with measure $\exp(-S_0)$, taking into account only the bilinear part of the action. The integration over $d^4p^\pm$ is defined in appendix A. Recall, that for the corresponding objects in operator formalism time ordering is implied.

In (21) we have introduced four symmetric, $O(3)$-covariant tensors,

$$
\mathcal{P}_T^{\mu\nu} = \delta_{\mu i} (\delta_{ij} - \frac{p_i p_j}{p^2}) \delta_{j\nu}
$$

$$
\mathcal{P}_E^{\mu\nu} = \delta_{\mu 0} \delta_{0\nu}
$$

$$
\mathcal{P}_L^{1\mu\nu} = \delta_{\mu i} \frac{p_i p_j}{p^2} \delta_{j\nu}
$$

$$
\mathcal{P}_L^{2\mu\nu} = \delta_{\mu i} \delta_{0\nu} + \delta_{\mu 0} \delta_{i\nu}
$$

(23)
Among these, $P^T, P^E$ and $P^{L1}$ are projectors with

$$P^T + P^E + P^{L1} = 1_4.$$  \hspace{1cm} (24)

The square of the traceless tensor $P^{L2}$ is the projector onto the “longitudinal” subspace:

$$(P^{L2})^2 = P^E + P^{L1}.$$ \hspace{1cm} (25)

Our aim now is to provide propagators having finite spectral widths. These spectral widths are assumed to be constant and will be determined self-consistently from the lowest order self energies and polarizations at zero momentum and energy. This means that we separate the unperturbed part of the action from the interaction part in the following way:

$$S_{QCD} = S_0 + S_{int}$$
$$= S_0 + S_\gamma + S_{int} - S_\gamma$$
$$= \tilde{S}_0 + \tilde{S}_{int}$$

$$\tilde{S}_0 = S_0 + S_\gamma.$$ \hspace{1cm} (26)

Perturbation theory is now done with respect to $\tilde{S}_{int}$, and $\tilde{S}_0$ is the action of generalized free fields, which already contain the full two-point correlations present in the theory. In particular, these generalized free fields may have a continuous mass spectrum, and therefore avoid the pitfalls of the quasi-particle picture we outlined in the introduction. This method has been introduced by Licht \[19\], its application to systems at finite temperature is discussed in refs. \[3, 8\].

As this procedure constitutes a certain approximation scheme we have to worry about gauge dependence. The appropriate tool in this context are BRST-Ward identities. After gauge fixing the partition function of QCD has a global symmetry with Grassmann character, the so-called BRST-symmetry \[20\]:

$$\delta\bar{\psi}_{i,\alpha}(x) = -i\bar{\psi}_{i,\beta}(x)T^{a}_{\beta,\alpha}c^a(x)$$
$$\delta\psi_{i,\alpha}(x) = ic^a(x)T^{a}_{\alpha,\beta}\psi_{i,\beta}(x)$$
$$\delta A^{a}_{\mu}(x) = \frac{1}{g}\partial_{\mu}c^a(x) + f^{abc}A^{b}_{\mu}(x)c^c(x)$$
$$\delta c^a(x) = -\frac{1}{2}f^{abc}c^b(x)c^c(x)$$

$$\delta\bar{c}^a(x) = \frac{1}{g}\Lambda A^a_{\mu}(x).$$ \hspace{1cm} (27)
This symmetry allows to derive the following Ward identity:

\[
\langle \delta [g A^a_\mu(x) c^b(y)] \rangle = 0
\]

\[
= \langle A^a_\mu(x) \Lambda_\nu A^b_\nu(y) + \partial_\mu c^a(x) \bar{c}^b(y) + g f^{acd} A^c_\mu(x) c^d(x) \bar{c}^b(y) \rangle.
\] (28)

At leading order in an expansion in powers of the coupling constant this identity implies the following relations among the perturbative propagators, which also hold for the effective propagators of the generalized free fields:

\[
\lambda p_0 S_E(p) + \frac{1}{\sqrt{\xi}} |p| S_{L2}(p) = p_0 S_{gh}(p)
\]

\[
\lambda p_0 S_{L2}(p) + \frac{1}{\sqrt{\xi}} |p| S_{L1}(p) = |p| S_{gh}(p)
\] (29)

These identities are fulfilled by the expressions in (20) – (22) as one can see by insertion.

In order to simplify calculations we choose \( \lambda = \sqrt{\xi} \) in the following. In use of the simple expressions we derived in section 2 for propagators with a constant spectral width, we now provide the following improved propagators for QCD:

\[
S_F(p) = \left( i \gamma_0 \text{sign}(p_0) (|p_0| + \gamma_F) + i \gamma_p \right)^{-1} = \gamma_\mu S_{F,\mu}(p)
\]

\[
S_T(p) = \frac{1}{(|p_0| + \gamma_T)^2 + p^2}
\]

\[
S_E(p) = \frac{1}{\xi (|p_0| + \gamma_E)^2 + \frac{1}{\xi} p^2}
\]

\[
S_{L1}(p) = \frac{1}{(|p_0| + \gamma_E)^2 + \frac{1}{\xi} p^2}
\]

\[
S_{L2}(p) = 0
\]

\[
S_{gh}(p) = \frac{1}{\sqrt{\xi} (|p_0| + \gamma_E)^2 + \frac{1}{\xi} p^2}.
\] (30)

As these propagators describe generalized free fields \[19\], they fulfill the tree-level BRST identities \[23\] by construction. We consider this a crucial point of the present paper, as also shown in the importance of the BRST symmetry over particularities of...
\( \Pi(p_0, p) = + \sum F(p_0, p) = + \) 

Figure 1: Feynman diagrams calculated in this work.

the approximation scheme: From eqn. (24) follows, that the identities (29) should hold in every order of the resummed perturbation expansion in powers of \( \tilde{S}_\gamma \).

We now have to determine the three unknown parameters \( \gamma_F, \gamma_T \) and \( \gamma_E \). For that purpose we calculate the self energy of the quarks and the polarization tensor for the gluons to one-loop order with the effective propagators. As is known from the exact integral equations for Green’s functions, we therefore neglect only three-point and higher correlations. Up to this level however, our approximation scheme is completely consistent.

As the parameters \( \gamma_F, \gamma_T \) and \( \gamma_E \) are due to interactions they are necessarily proportional to some positive power of the coupling constant \( g \) times the temperature \( T \). At temperatures high enough in order to ensure a small coupling constant the width parameters are necessarily small compared to the temperature. Therefore, we consider the limit of high temperature (for details see appendix B). As we want to provide a well-defined infrared limit with our parameterization of the spectral functions we determine \( \gamma_F, \gamma_T \) and \( \gamma_E \) precisely in this limit.

Let us start with the fermions. Our ansatz (30) for the propagator yields...
equation
\[ \gamma_F = \frac{1}{4} \text{Tr} \left[ \gamma_0 \Sigma^F(p_0, 0) \right]_{p_0 \to 0}. \] (31)

The expression for the one-loop self energy \( \Sigma^F(p, 0) \) is, according to diagrammatic rules for the bottom diagram in figure [1],
\[ \Sigma^F(p) = ig^2 \frac{N_c^2}{2} \int \frac{d^4k}{(2\pi)^4} S_F(p + k) (2S_T(k) + S_L(k) - S_E(k)) . \] (32)

For high temperatures and for positive \( p_0 \) we find, using the procedure outlined in appendix B:
\[ \frac{1}{4} \text{Tr} \left[ \gamma_0 \Sigma^F(p_0, 0) \right] = \frac{g^2 T^2 N_c^2 - 1}{16} \frac{1}{N_c} \frac{1}{(p_0 + \gamma_T + \gamma_F)} + \mathcal{O}(T) . \] (33)

For the electric and magnetic gluons the equations determining \( \gamma_E \) and \( \gamma_T \) are
\[ \xi \gamma_E^2 = \Pi_{00}(p_0, 0) \bigg|_{p_0 \to 0} \] (34)

and
\[ \gamma_T^2 = \frac{1}{2} P^T_{\mu\nu} \Pi_{\mu\nu}(p_0, 0) \bigg|_{p_0 \to 0} . \] (35)

The gluon polarization tensor has a rather complicated structure. We therefore show only the electric and transverse part which we need for (34) and (35); the following integrals correspond to the gluon tadpole, gluon/ghost loop and fermion loop diagrams depicted in figure [1]:
\[ \Pi_E(p_0, 0) = \Pi_{00}(p_0, 0) \]
\[ = -2g^2 N_c \int \frac{d^4k}{(2\pi)^4} S_T(k_0, k) \]
\[ + g^2 N_c \int \frac{d^4k}{(2\pi)^4} \left[ \xi^2 (2k_0 \gamma_E + \gamma_E^2) S_E(k_0, k) S_E(p_0 + k_0, k) \right. \]
\[ + 4(2k_0 \gamma_T + \gamma_T^2 + k^2) S_T(k_0, k) S_T(p_0 + k_0, k) \]
\[ - 2g^2 N_f \int \frac{d^4k}{(2\pi)^4} \left[ S_{F,0}(k, k) S_{F,0}(p_0 + k_0, k) - S_{F,i}(k, k) S_{F,i}(p_0 + k_0, k) \right] , \] (36)

\[ \Pi_T(p_0, 0) = \frac{1}{2} P^T_{\mu\nu} \Pi_{\mu\nu}(p_0, 0) \]
\[ = g^2 N_c \int \frac{d^4k}{(2\pi)^4} \left[ \xi \frac{1 + \cos^2(\theta)}{2} S_E(k_0, k) + \frac{3 - \cos^2(\theta)}{2} S_T(k_0, k) \right] \]
\[ + g^2 N_c \int \frac{d^4k}{(2\pi)^4} \left[ \xi (2\gamma_E k_0 + \gamma_E^2 - 4p_0 k_0 - \gamma_F) \frac{1 - \cos^2(\theta)}{2} \right] , \]
\[ \cdot S_E(k_0, k) S_E(p_0 + k_0, k) \\
+ (2\gamma_T(k_0 + p_0) + \gamma_T^2 - 2k_0p_0 - 3p_0^2) \frac{1 + \cos^2(\theta)}{2} \cdot S_E(k_0, k) S_T(p_0 + k_0, k) \\
+ 4k^2 \frac{1 - \cos^2(\theta)}{2} S_T(k_0, k) S_T(p_0 + k_0, k) \\
+ 2g^2 N_f \int \frac{d^4 k}{(2\pi)^4} \left[ S_{F,0}(k_0, k) S_{F,0}(p_0 + k_0, k) \\
+ S_{F,i}(k_0, k) p_i S_{F,j}(p_0 + k_0, k) p_j/p^2 \right]. \quad (37) \]

Here we have used the simple relations between \( S_E(p) \), \( S_L(p) \) and \( S_{gh}(p) \) due to (29).

The resulting expressions for high temperature are:

\[ \Pi_E(p_0, 0) = -\frac{g^2 T^2 N_c}{6} + \frac{g^2 T^2 N_f \gamma_F}{3(2\gamma_T + p_0)} + \frac{g^2 T^2 N_c (6\gamma_T + p_0)}{6(2\gamma_T + p_0)} + \mathcal{O}(T), \quad (38) \]

\[ \Pi_T(p_0, 0) = p_0 \frac{g^2 T^2}{18} \left( \frac{N_f}{2\gamma_F + p_0} + \frac{2N_c}{2\gamma_T + p_0} \right) + \mathcal{O}(T). \quad (39) \]

Note, that quite similar results are obtained in a semiclassical approximation to the bremsstrahlung problem in hot and dense matter [21]. This indicates, that the high temperature limit we consider is closely related to a classical self-consistent calculation.

Obviously, \( \Pi_T(p_0, 0) \) vanishes for \( p_0 \to 0 \) suggesting \( \gamma_T = 0! \) On the other hand, it starts linear with \( p_0 \) supporting a term \( 2p_0\gamma_T \) as in our ansatz for the transverse gluon propagator. We interpret this behavior as an inconsistency of our ansatz for the transverse propagator requiring a modification.

The simplest modification of our ansatz such that \( S_T(p_0, 0)^{-1} \) vanishes for \( p_0 \to 0 \) can be achieved by a spectral function like (7) but now with \( \omega^2 = p^2 - \gamma_T^2 \). The modified transverse propagator then reads:

\[ S_T(p) = \frac{1}{|p_0|^2 + 2|p_0|\gamma_T + \mathbf{p}^2}. \quad (40) \]

It seems that this propagator still has an infrared problem. However, recall that the proper quantity to consider are correlations of the field strength tensor rather than the vector potentials. In particular, the time averaged screening of an external magnetic field is given by the momentum space correlation function

\[ \int dt \langle B_i^a(t, \mathbf{x}) B_j^a(0, \mathbf{y}) \rangle \approx \int d^3 \mathbf{p} \exp(i\mathbf{px}) \mathcal{P}_T^{ij} \mathcal{P}_{ij} S_T(0, \mathbf{p}) \propto \delta(\mathbf{x}). \quad (41) \]

Here, we have not displayed any factors which are irrelevant for the spatial behavior. Obviously, static magnetic fields are screened and, in particular, the screening length
Doing the previous calculations with the new spectral function for the transverse gluons yields the same results for the leading terms. A possible explanation for this feature is that at high temperature the kinetic terms are the dominant ones; these are unchanged by our modification. As the results are the same, our ansatz for the transverse gluons now is compatible with the infrared limit of the corresponding transverse polarization. The new equation determining $\gamma_T$ reads:

$$\gamma_T = \left( \frac{1}{4p_0^0} P^T_{\mu \nu} \Pi_{\mu \nu} (p_0, 0) \right)_{p_0 \to 0}. \quad (42)$$

Now we can solve the set of equations (31),(34) and (42) using the expressions (33),(38) and (39). The solutions with positive values for the spectral width parameters are

$$\gamma_E = gT \sqrt{\frac{N_f + 2N_c}{6\xi}}$$

$$\gamma_T = \frac{1}{16} \frac{N_c^2 - 1}{N_c} \frac{(gT)^2}{\gamma_F} - \gamma_F$$

$$\gamma_F = gT \frac{1}{12} \sqrt{11N_c - \frac{9}{N_c} - N_f - \sqrt{(N_f - 2N_c)^2 + 36(N_c^2 - 1)}}. \quad (43)$$

The solution for $\gamma_E$ seems to indicate a gauge dependence. However, only for $p_0 = 0$ the electric propagator $S_E(p)$ is gauge invariant. There it reads

$$S_E(0, p) = \frac{1}{p^2 + \xi \gamma_E^2}. \quad (44)$$

Obviously, $\xi \gamma_E^2$ is independent of $\xi$ and is just the famous “Debye”-screening mass for electric gluons \cite{4,22}.

Another remarkable feature of these results is the decrease of $\gamma_F$ with $N_f$. Indeed, for

$$N_c > \frac{N_c^2 - 1}{13} \quad (45)$$

Table 1: Self-consistent spectral width parameters for high temperature QCD

|          | $N_c = 2$ |          | $N_c = 3$ |
|----------|-----------|----------|-----------|
|          | $N_f = 1$ | $N_f = 2$ | $N_f = 3$ | $N_f = 1$ | $N_f = 2$ | $N_f = 3$ |
| $\sqrt{\xi} \gamma_E/(gT)$ | 0.913  | 1.080  | 1.225  | 1.080  | 1.155  | 1.225  |
| $\gamma_T/(gT)$ | 0.273  | 0.390  | 0.380  | 0.315  | 0.344  | 0.380  |
| $\gamma_F/(gT)$ | 0.199  | 0.168  | 0.260  | 0.28   | 0.271  | 0.260  |
Table 2: Transverse plasma frequencies of high temperature QCD

we get a complex $\gamma_F$. This is not a very serious problem for the values of $N_f$ and $N_c$ that are realized in nature; for $N_c = 3$ the limit is $N_f < 12$. However, this behavior has to be understood from a purely theoretical point of view as it signals some strange and unexpected behavior. A possible explanation is that for large $N_f$ the one-loop diagrams are not the appropriate ones to describe the leading behavior. In this case, one would have to consider a large-$N_f$ expansion instead of a loop expansion.

In table 2, we provide a listing of $\gamma_E$, $\gamma_T$ and $\gamma_F$ from (43) in units of $gT$ for several values of $N_c$ and $N_f$.

For completeness we also mention the results for pure Yang-Mills theory:

\begin{align*}
\gamma_E & = gT \sqrt{\frac{N_c}{3 \xi}} \\
\gamma_T & = gT \sqrt{\frac{N_c}{6}}.
\end{align*}

(46)

One may also determine the plasma frequency of the system. We concentrate on the transverse oscillations, because only these are gauge invariant. They are solutions of the equation

\[(i\omega_T)^2 = \Pi_T(i\omega_T, 0).\]

(47)

There is always one solution with $\omega_T = 0$ because $\Pi_T(p_0, 0)$ is proportional to $p_0$. Numerical values for the other solutions are listed in table 2. A remarkable feature is the strong damping of these modes. Their imaginary part is about half of the real part. Moreover, there is also an overdamped mode. The ratio between the
2 in contrast to the standard value $\sqrt{3}$ [3, 22]. The explanation for this discrepancy may be the inclusion of an imaginary part of $\omega_T$ which is not negligible.

4 The partition function of hot QED

Essentially the same procedure can be applied to QED. Let us briefly recapitulate the most important steps. The partition function is now

$$Z_{QED} = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\bar{c} \mathcal{D}c \exp(-S_{QED})$$  (48)

with the action

$$S_{QED} = \int d^4x \left[ \bar{\psi}_I(x) \gamma_\mu (\partial_\mu - ieA_\mu(x)) \psi_I(x) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} 
+ \frac{1}{2} (\Lambda_\mu A_\mu(x))^2 + \Lambda_\mu \bar{c}(x) \partial_\mu c(x) \right],$$  (49)

where $l$ is again a “flavor” index. The electromagnetic field strength tensor is

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a,$$  (50)

and the gauge is the same as for QCD.

We now define a modified unperturbed action $\tilde{S}_0$:

$$S_{QED} = \tilde{S}_0 + \tilde{S}_{\text{int}}$$

$$= S_0 + S_\gamma + S_{\text{int}} - S_\gamma$$

$$= \tilde{S}_0 + \tilde{S}_{\text{int}}$$

$$\tilde{S}_0 = S_0 + S_\gamma.$$  (51)

Repeating the same procedure as for QCD means that we also introduce a spectral width for the ghosts. One might ask, whether this is not a contradiction to the fact that the ghosts are free fields that decouple from the physical fields. However, this question is irrelevant in our method, since we are dealing with generalized free fields having a continuous mass spectrum. Hence, we may choose any spectral width parameter for unobservable fields [3].

Our special choice for the ghost propagator is obtained by enforcing the BRST identities (29). This ensures the (gauge dependent) choice $S_{L2}(p) = 0$, and therefore is motivated by simplicity of computation.
The equations for $\gamma_F, \gamma_E$ and $\gamma_T$ are the same as in the QCD-case, namely (31), (34) and (42). The fermion self energy now is:

$$\Sigma^F(p) = i e^2 \int \frac{d^4k}{(2\pi)^4} S_F(p+k)(2S_T(k) + S_L(k) - S_E(k)) \ .$$  (52)

For high temperature and for positive $p_0$ we find

$$\frac{1}{4} \mathrm{Tr} \left[ \Gamma_0 \Sigma^F(p_0, 0) \right] = \frac{e^2 T^2}{8} \frac{1}{(p_0 + \gamma_T + \gamma_F)} + \mathcal{O}(T) \ .$$  (53)

The electric and transverse component of the photon polarization tensor read:

$$\Pi_E(p_0, 0) = \Pi_{00}(p_0, 0) = -4e^2 N_f \int \frac{d^4k}{(2\pi)^4} \left[ S_{F,0}(k, 0)S_{F,0}(p_0 + k_0, k) - S_{F,i}(k, 0)S_{F,i}(p_0 + k_0, k) \right] ,$$  (54)

$$\Pi_T(p_0, 0) = \frac{1}{2} P^T_{\mu\nu} \Pi_{\mu\nu}(p_0, 0) = 4e^2 N_f \int \frac{d^4k}{(2\pi)^4} \left[ S_{F,0}(k, 0)S_{F,0}(p_0 + k_0, k) + S_{F,i}(k, 0)p_iS_{F,j}(p_0 + k_0, k)p_j/p^2 \right] .$$  (55)

The resulting expressions for high temperature are:

$$\Pi_E(p_0, 0) = \frac{2e^2 T^2 \gamma_F N_f}{3(2\gamma_F + p_0)} + \mathcal{O}(T) \ ,$$  (56)

$$\Pi_T(p_0, 0) = \frac{e^2 T^2 N_f p_0}{9(2\gamma_F + p_0)} + \mathcal{O}(T) \ .$$  (57)

Solving the self-consistency equations for the spectral width parameters in QED then yields

$$\gamma_E = eT \sqrt{\frac{N_f}{3\zeta}}$$

$$\gamma_T = \frac{(eT)^2 N_f}{36 \gamma_F}$$

$$\gamma_F = \frac{eT}{2} \sqrt{\frac{1}{2} - \frac{N_f}{9}} .$$  (58)

Again we find that for $N_f > 4$ the fermionic spectral width $\gamma_F$ becomes imaginary.
Finally, we also give the result for the transverse plasmon mode:

$$\omega_T = eT \frac{1}{2} \sqrt{\frac{5}{9} N_f - \frac{1}{2} - 1 \gamma_F}$$  \hspace{1cm} (59)$$

For standard electrodynamics with $N_f = 1$ this yields

$$\omega_T = eT \frac{1}{2 \sqrt{18}} (1 - i \sqrt{7}) \approx eT (0.118 - 0.312i)$$  \hspace{1cm} (60)$$

This is a strongly damped oscillation. Here, the ratio between electric screening mass $\sqrt{\xi \gamma_E}$ and the real part of $\omega_T$ deviates even more strongly from $\sqrt{3}$ than for QCD. However, since the plasma oscillation is also stronger damped than in the QCD case, this only supports our previously stated conclusion that one may not neglect the imaginary part for the determination of the plasmon mode.

5 Summary and conclusions

In the present paper, we have investigated the self-consistent determination of finite temperature spectral broadening for QCD and QED in the infrared limit, i.e., for zero “particle” momenta. For the “electric” part of the gauge boson propagator, we reproduce the well-known Debye screening mass, whereas for the fermions as well as for the transverse gauge bosons we acquire new results: In contrast to the method of hard thermal loops [2] we obtain, that all spectral width parameters $\gamma_F$, $\gamma_E$, $\gamma_T$ are of the order $gT$, respective $eT$, in the high temperature limit.

Based on results for massive fermions [7] we believe that in some regimes the constant spectral width obtained here might be a good approximation to the full solution of the problem. However, we have formulated a consistent recipe for the perturbative expansion of QCD and QED in terms of generalized free fields [19], whose continuous mass spectrum is described by our effective propagators (eqn. (30), transverse boson propagator replaced by (40)).

The skeleton expansion using these propagators may serve to calculate the full QCD and QED Green’s functions [3, 8]. Indeed, if the estimate (1) holds for any infrared regulator, i.e. if $M$ may be replaced by a typical $\gamma$, our results suggest that now a perturbative calculation of high temperature QCD or QED is possible; the expansion parameter in this case would be $g$ instead of $g^2$.

The virtues of a finite temperature perturbation theory using such propagators with a nontrivial spectral function lie at hand:

1. It takes into account the Narnhofer-Thirring theorem described in the intro-
2. It takes into account that the basis of a perturbative expansion has to be chosen according to the symmetry of the problem: The symmetry group of a temperature state is not the Poincaré group with its stable particle representations, but rather possesses only representations with continuous mass spectrum [3, 4].

3. It is free of unphysical infrared divergences in every order, because products of mass-shell \( \delta \)-functions cannot appear.

Finally, our results for the transverse plasmon modes indicate a rather strong damping in contrast to the frequently used assumption that damping of these collective modes is suppressed by a factor of \( g \) or \( e \). Due to the large imaginary part of these plasmon frequencies the relation \( m_{el} = \sqrt{3} \text{Re}(\omega_T) \) no longer holds.

acknowledgement

The authors wish to thank B. Friman and J. Knoll for fruitful discussions.

A Notation

We use a euclidean metric \( g_{\mu\nu} = \delta_{\mu\nu} \). Space-time integrals are denoted by

\[
\int d^4 x = \int d^3 x \int_0^\beta d\tau , \ \beta = 1/T .
\]

The euclidean \( \gamma \)-matrices are Hermitean and obey the anticommutation relations

\[
\{ \gamma_\mu, \gamma_\nu \} = 2\delta_{\mu\nu} .
\]

The gluon fields \( A_\mu = A_\mu^a T^a \) are local elements of the Lie algebra SU(N). As such \( A_\mu \) is traceless and Hermitean. The commutation relations of the generators \( T^a \) are

\[
[T^a, T^b] = if^{abc} T^c .
\]

They are normalized such that

\[
\text{Tr} \left[ T^a T^b \right] = \frac{1}{2} \delta^{ab}
\]

and their completeness relation reads

\[
T^a_{\alpha\beta} T^a_{\rho\sigma} = \frac{1}{2} \left( \delta_{\alpha\sigma} \delta_{\rho\beta} - \frac{1}{N} \delta_{\alpha\beta} \delta_{\rho\sigma} \right) .
\]
This implies
\[ T^\alpha_\alpha T^\beta_\beta = \frac{N^2 - 1}{2N} \delta^\alpha_\beta \quad . \tag{66} \]

A useful relation for the structure constants reads
\[ f^{abc} f^{bdc} = N \delta^{ad} \quad . \tag{67} \]

Fourier transforms of a field \( \phi(x) \) are defined as follows:
\[
\phi(x) = \phi(\tau, x) = \int \frac{d^4 p}{(2\pi)^4} \phi(p) e^{i p x} \\
= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\beta} \sum_n \phi(\nu_n^\pm, p) e^{i \nu_n^\pm \tau + i p x} . \tag{68} \]

Here, \( \nu_n^\pm \) are the Matsubara frequencies for bosons (fermions) depending on the character of the field \( \phi \), and defined in (5).

**B High temperature limit**

Sums over Matsubara frequencies are transformed into contour integrals using the well-known identity:
\[
\sum_n f(\nu_n) = \int_{-\infty}^{\infty} \frac{dz}{2\pi} f(z) \pm \int_{-\infty}^{\infty} \frac{dz}{4\pi} \frac{f(z) + f(-z)}{\exp(-i\beta z) \mp 1} . \tag{69} \]

The upper sign is for bosonic \( \nu_n \), the lower sign for fermionic \( \nu_n \).

The remaining one-loop integrals have the following typical form (see ref. [8] for details):
\[
\int_{-\infty}^{\infty} dE \frac{1}{\exp \left( \frac{E}{T} \right) \pm 1} \int_0^{\infty} dk \frac{k^2}{A_1(E, k)} \int_{-\infty}^{\infty} dE' \frac{A_2(E', k)}{E - E' + p_0} . \tag{70} \]

The \( E' \) integration as well as the \( k \)-integration are carried out analytically, as contour integrals in the complex plane of the corresponding integration variable.

For the remaining \( E \)-integration, we use the fact that the integrand is an odd function of \( E \) depending only on the scale \( \gamma \),
\[
f(E) = \gamma \tilde{f}\left( \frac{E}{\gamma} \right) , \quad f(E) = -f(-E) . \tag{71} \]

multiplied by a temperature distribution function. For simplicity we discuss the case where \( f(E) \) has energy-dimension 1.
The $E$-integration may be split up into two parts,

$$
\int_0^\infty dE \frac{f(E)}{\exp \left( \frac{E}{T} \right)} \pm 1 = \gamma T \int_0^\infty dx \frac{\tilde{f}(x \frac{T}{\gamma})}{\exp(x) \pm 1} + \gamma^2 \int_0^1 dx \frac{\tilde{f}(x)}{\exp \left( x \frac{\gamma}{T} \right) \pm 1}.
$$

(72)

In the first part, we may use an approximation for $\tilde{f}$ at large arguments,

$$
\tilde{f} \left( x \frac{\gamma}{T} \right) \approx \frac{xT}{\gamma} + \mathcal{O} \left( \frac{\gamma}{xT} \right).
$$

(73)

The contribution of the leading term to the complete expression is of order $T^2$. The remainder is at most of order $\gamma T$.

In the second piece of the integrals, we expand the distribution function for small $x$. The resulting integrals are infrared finite because $\tilde{f}(x)$ is finite at $x \to 0$. As it is an odd function of its argument it vanishes at least proportional to $x$ for small $x$. The leading term arises for bosons where the distribution function behaves as

$$
\frac{1}{\exp \left( x \frac{\gamma}{T} \right) - 1} \approx \frac{T}{\gamma x} + \mathcal{O}(1).
$$

(74)

This yields a contribution of order $\gamma T$.

References

[1] J.I.Kapusta, *Finite temperature field theory*,
(Cambridge University Press, New York 1989)

[2] E.Braaten and R.D.Pisarski, Nucl.Phys. B337 (1990) 569

[3] A.Linde, Phys.Lett. B96 (1980) 289;
   D.J.Gross, R.D.Pisarski and L.G.Yaffe, Rev.Mod.Phys. 53 (1981) 43

[4] E.Braaten, *Solution to the Perturbative Infrared Catastrophe in Hot Gauge Theories*
   Northwestern University preprint NUHEP-TH-94-24 (1994), [hep-ph/9409434](https://arxiv.org/abs/9409434)

[5] H.J.Borchers, R.N.Sen, Commun.Math.Phys.21 (1975) 101

[6] N.P.Landsman,
   Phys.Rev.Lett. 60 (1988) 1990 and Ann.Phys. 186 (1988) 141
[7] P.A. Henning, R. Sollacher and H. Weigert, 
_Fermion damping rate in a hot medium_, 
GSI-Preprint 94-56 (1994), subm. to Phys. Rev. D (hep-ph/9409280)

[8] P.A. Henning, 
_TUNFD for quantum fields with continuous mass spectrum_, 
Habilitation thesis (Darmstadt 1993), Phys. Rep. in press

[9] R. Kubo, J. Phys. Soc. Japan **12** (1957) 570; 
C. Martin and J. Schwinger, Phys. Rev. **115** (1959) 1342

[10] N.P. Landsman and Ch. G. van Weert, Phys. Rep. **145** (1987) 141

[11] H. Narnhofer, M. Requardt and W. Thirring, 
Commun. Math. Phys. **92** (1983) 247

[12] V.V. Lebedev and A.V. Smilga, Ann. Phys. **202** (1990) 229

[13] R. Baier and R. Kobes, _Damping Rate of a Fast Fermion in Hot QED_ 
University of Bielefeld preprint BI-TP 94/14 (1994), hep-ph 9403335

[14] T. Matsubara, Progr. Theoret. Phys. **14** (1955) 351

[15] G. Baym and N. D. Mermin, J. Math. Phys. **2** (1961) 232

[16] A. L. Fetter and J. D. Walecka, _Quantum Theory of Many-Particle Systems_ 
(McGraw-Hill, New York 1971)

[17] A. O. Caldeira and A. J. Leggett, Physica **121A** (1983) 587

[18] S. Nadkarni, Phys. Rev. **D 27** (1983) 917

[19] A. L. Licht, Ann. Phys. **34** (1965) 161

[20] C. Becchi, A. Rouet and R. Stora, Phys. Lett. **B 52** (1974) 344; 
I. V. Tyutin, Lebedev Report No. FIAN 39 (1975)

[21] J. Knoll and D. N. Voskresensky, 
_Bremsstrahlung in dense matter_ 
GSI-preprint in preparation.

[22] D. Pines, _Elementary Excitations in Solids_ 
(Benjamin, New York 1964)