ACOUSTIC WORMHOLES

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Abstract

Acoustic analogs of static, spherically symmetric massive traversable Lorentzian wormholes are constructed as a \textit{formal} extension of acoustic black holes. The method is straightforward but the idea is interesting in itself. The analysis leads to a new acoustic invariant for the massless counterpart of the Einstein-Rosen model of an elementary particle. It is shown that there is a marked, in a sense even counterintuitive, physical difference between the acoustic analogs of black holes and wormholes. The analogy allows us to also portray the nature of curvature singularity in the acoustic language. It is demonstrated that the light ray trajectories in an optical medium are the same as the sound trajectories in its acoustic analog. The implications of these analogies in the laboratory set up and in the different context of phantom energy accretion have been speculated.

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1 Introduction

Several years ago, Unruh \cite{1} proposed a novel idea as to how a sonic horizon in transonic flow could give out a thermal spectrum of sound waves mimicking Hawking’s general relativistic black hole evaporation. The radiation of such sound waves is now commonly known as Hawking-Unruh radiation \cite{2}. This basic acoustic analogy has been further explored in later years and under different physical circumstances by several authors \cite{3-11}. A very useful account and a detailed extension of some of these developments can be found in Visser \cite{4} wherein it is also demonstrated that the Hawking radiation is purely a kinematic effect quite independent of the machinery of Einstein’s field equations. This information is consistent with the fact that a detector with uniform acceleration $a$ in vacuum responds as though it were immersed in a thermal bath of temperature $T = \hbar a / 2\pi k c$, where $\kappa$ is Boltzmann’s constant, $\hbar$ is Planck’s constant and $c$ is the speed of light in vacuum \cite{2}. This is the same as Hawking temperature when one formally puts $a \equiv g$, where $g$ stands for the gravitational acceleration at the black hole surface. All in all, the past and present works surrounding the basic analogy have engendered the practical

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possibility of detecting Hawking-Unruh radiation in fluid (especially superfluid) models under appropriately simulated conditions [8-11]. Quite reasonably, there is a widespread current interest in this topic as these models open up an alternative window to look for many unknown effects of quantum gravity on black holes in the laboratory.

As things stand, the theoretical acoustic modelling can be achieved in two reciprocal ways: The first way is to start from the fluid equations for continuity, Euler equations and the equation of state. Linearize them and obtain the wave equation for sound waves. This process allows us to equivalently describe sound waves as propagating on a spacetime metric, called the acoustic metric. The equation of propagation can be cast into that of a massless minimally coupled scalar field [1,4]. Following this method, Visser [4] obtained the Painlevé-Gullstrand form of the Schwarzschild exterior metric up to a conformal factor and also the canonical acoustic metric. In that work, various types of acoustic regions are consistently defined leading to an improved formula for Hawking-Unruh radiation. The second way, though it is not the one usually followed, is to start from a given spacetime metric of general relativity and obtain an equivalent fluid description in terms of density, pressure, equation of state etc. This new method has been worked out in a recent paper by Visser and Weinfurtner [6] in the context of building the acoustical analog of the equatorial slice of a Kerr black hole. The analogy enables one to gain insight into Kerr spacetime itself as well as into the vortex-inspired experiments. All the above works have led to a systematic development of what might be generally called the physics of optical or acoustical black holes.

In this paper, a first hand idea of acoustic wormholes is advanced as a natural extension of acoustic black holes although it is known that the intrinsic topologies of black and wormholes are widely different. After briefly re-deriving the standard form of acoustic metric in Sec.2, we present, in Sec.3, a class of static, spherically symmetric, traversable Lorentzian wormholes of general relativity. In Sec.4, we adopt the second method to develop the acoustic analog of these wormholes including the zero mass limit and compare their characters with those of the acoustic analog of the Schwarzschild black hole. Geodesic equations are discussed in Sec.5. A summary together with some speculative remarks appear in Sec.6.

2 Acoustic metric from fluid equations

For completeness, let us first survey in brief how the acoustic metric is developed. Consider an irrotational, nonrelativistic, inviscid, barotropic fluid so that the equations are

\[ \nabla \times \mathbf{v} = 0 \Rightarrow \mathbf{v} = \nabla \Psi \] (1)

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \] (2)

\[ \rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p \] (3)

\[ p = p(\rho) \] (4)
in which all relevant terms have their usual meanings. Now linearize the equations around a background exact solution set \([p_0(t, \vec{x}), \rho_0(t, \vec{x}), \Psi_0(t, \vec{x})]\) such that
\[
p = p_0 + \delta p_0 + 0(\delta p_0)^2, \quad \rho = \rho_0 + \delta \rho_0 + 0(\delta \rho_0)^2, \quad \Psi = \Psi_0 + \delta \Psi_0 + 0(\delta \Psi_0)^2 \tag{5}
\]
where \(\delta\) denotes a small perturbation to the relevant quantities. Then the perturbations satisfy the equation [1]
\[
-\partial_t \left[ \frac{\partial p}{\partial p} \rho_0 (\partial_t \Psi_1 + \vec{\nabla}_0 \cdot \vec{\nabla} \Psi_1) \right] + \vec{\nabla} \cdot \left[ \rho_0 \vec{\nabla} \Psi_1 - \frac{\partial p}{\partial p} \rho_0 \vec{\nabla}_0 \left( \partial_t \Psi_1 + \vec{\nabla}_0 \cdot \vec{\nabla} \Psi_1 \right) \right] = 0. \tag{6}
\]
This equation can be neatly rewritten as the minimally coupled wave equation [4]
\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu \nu} \frac{\partial \Psi_1}{\partial x^\nu} \right) = 0 \tag{7}
\]
where \(g_{\mu \nu}\) is the acoustic metric, \(\Psi_1 \equiv \delta \Psi_0\), \(g = \det |g_{\mu \nu}|\), and
\[
ds_{\text{acoustic}}^2 = g_{\mu \nu} dx^\mu dx^\nu = \rho_0 c_s^2 \left[ -c_s^2 dt^2 + (dx^i - \omega_0^i dt) (dx^j - \omega_0^j dt) \right] \delta_{ij} \tag{8}
\]
in which \(i, j = 1, 2, 3\) and the speed of sound is given by \(c_s^2 = \frac{dp}{d\rho}\). This is geometrization of acoustics. Our strategy in this paper is to follow the reverse route, that is, we cast a given metric of general relativity into the form (8) and find out the fluid variables for an analogous acoustic configuration. Before closing this section, we note that in a physical situation where \(\vec{\nabla}_0 = 0\), and \(\rho_0/c_s = \) constant, we have an exact Minkowski spacetime where the role of signal speed is played by that of sound waves. This phenomenon calls for some sort of sonic special relativity where the observers are equipped with only sound waves, that is, a world where the observers can only “hear” but not “see”. This scenario has been explicitly recognized and contemplated upon by Visser [4]. It is of some historical interest to note that such a sonic relativity was in fact conceived decades ago where the subsonic and supersonic Lorentz-like transformations contained \(c_s\) as the invariant speed [12]. At that time, it was advanced not as a formal theory but rather as a technical tool for solving various acoustical problems in linearized sub- and supersonic aerodynamics including the Earth’s magnetospheric bow shock geometry. Recently, these concepts have received additional support, though not yet a confirmation, in the current context of classical acoustical black holes that fundamentally issue forth from fluid equations ab initio. However, it is quite likely that quantum effects could destroy the inherent Lorentz invariance at the microscopic level [13].

3 Wormhole solutions of general relativity

Let us consider the field equations of the Einstein minimally coupled scalar field theory which are given by (We adopt units \(8\pi G = c = \hbar = 1\), and signature convention \(-, +, +, +\)):
\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = T_{\mu \nu} \tag{9}
\]
\[ T_{\mu\nu} = \alpha \left[ \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \Phi \Phi^\sigma \right] \] (10)

\[ \Phi_{,\mu} = 0 \] (11)

where \( \alpha \) is an arbitrary constant that can have any sign, \( \Phi \) is the scalar field, \( R_{\mu\nu} \) is the Ricci tensor and the semicolon denotes covariant derivatives with respect to the metric \( g_{\mu\nu} \).

If \( \alpha \) is negative, or \( \Phi \) is imaginary, then the stress tensor \( T_{\mu\nu} \) has an overall negative sign so that it represents exotic matter necessary for threading wormholes [14]. We consider a solution of the form

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{-2 \phi(r)} dt^2 + e^{-2 \psi(r)} \left[ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right] \] (12)

\[ \phi(r) = -\beta \ln \left( \frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \right), \quad \psi(r) = (\beta - 1) \ln \left( 1 - \frac{m}{2r} \right) - (\beta + 1) \ln \left( 1 + \frac{m}{2r} \right) \] (13)

\[ \Phi(r) = \left[ \frac{2(1 - \beta^2)}{\alpha} \right]^{\frac{1}{2}} \ln \left[ \frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \right] \] (14)

where the two undetermined constants \( m \) and \( \beta \) are related to the source strengths of the gravitational and scalar parts of the configuration. The solution was first put in that form by Buchdahl [15] and it describes all weak field tests of gravity just as accurately as does the Schwarzschild metric. The conserved mass of the configuration is given by

\[ M = m \beta. \] (15)

Once the scalar component is set to zero ( \( \beta = 1 \Rightarrow \Phi = 0 \) ), the solutions (13), (14) reduce to the Schwarzschild black hole in accordance with Wheeler’s “no scalar hair” conjecture. Physically, this indicates the possibility that the scalar field be radiated away during collapse so that the end result is a Schwarzschild black hole [16]. The class of solutions (13), (14) describes asymptotically flat traversable wormholes with finite tidal forces [17]. The throat appears at the coordinate radii

\[ r_0^\pm = \frac{m}{2} \left[ \beta \pm \sqrt{\beta^2 - 1} \right], \] (16)

that connects two asymptotically flat regions, one at \( r = \infty \) and the other appearing at \( r = 0 \). Here we take only the positive sign \( (r_0^+) \). To find the second asymptotic region, note that, under the transformation \( r = \frac{m^2}{\xi} \), the metric \( ds^2(r) \rightarrow (-1)^{2\beta} ds^2(\xi) \). Thus, in order to ensure the form invariance of the metric including the signature, we must have \( \beta \in N \). It is obvious that the metric becomes flat also at \( \xi = \infty \), that is, at \( r = 0 \). The \( r- \) and the \( \xi- \) coordinates meet at \( r = \xi = m/2 \). In other words, either \( r \) or \( \xi \) can be taken as a global coordinate patch. The requirement that the throat radii be real implies that \( \beta^2 > 1 \). Under this wormhole condition, consider the two mutually exclusive possibilities for \( \Phi \): It could be either real or imaginary. The reality of \( \Phi \) [more precisely, the reality
of the scalar charge \( M_S = -m \left[ (1 - \beta^2)/\alpha \right]^{1/2} \) obtained from \( \Phi \simeq M_S/r \) demands that \( \alpha < 0 \). Alternatively, one could have \( \alpha > 0 \) and allow for an imaginary \( \Phi \). The latter choice presents no pathology or inconsistency in the wormhole physics, as is recently shown in Ref.[18]. In both cases, however, we have a negative sign before the stress tensor (10) and consequently almost all energy conditions are violated. In what follows, we need to concentrate on the metric part only. Under this setting, we now proceed to find the acoustic model of the wormhole.

4 Acoustic wormholes

With fluid, it is *not* possible to set up the geometry having the *topological appearance* of a wormhole that is opening at two ends to asymptotically flat spacetimes. What one can do is to examine how the fluid quantities behave at the corresponding coordinate values \((r = \infty, r = r_0^+, r = 0)\) in the wormhole geometry. The situation is quite similar to the acoustic analogue of Schwarzschild vacuum which is modelled by real (even flowing) fluid filling all space. It is not possible to find also the topology (e.g., the double cone in the Kruskal-Szekeres extension) in the fluid field. With this limitation in mind, consider the acoustic metric (8) and choose \( \nabla \Phi_0 = 0 \). Then formally replace the vacuum speed of light \( c \) in (12) [which we have set to unity] by the asymptotic speed of sound \( c_\infty \) corresponding to a linear medium, equate the metric (12) with the metric (8), that is, set \( \tilde{g}_{\mu \nu} = g_{\mu \nu} \).

Immediately one obtains the density and sound speed profiles respectively as

\[
\rho_0(r) = \rho_\infty \left( 1 - \frac{m^2}{4r^2} \right) \quad (17)
\]

\[
c_s(r) = c_\infty \left( \frac{1 - \frac{m^2}{2r}}{1 + \frac{m^2}{2r}} \right)^{\frac{2\beta}{1 - m^2/2r}} \left( 1 - \frac{m^2}{4r^2} \right)^{-1}. \quad (18)
\]

It must be emphasized again that we are looking for an acoustical analog of the wormhole. Consequently, the fluid density \( \rho_0 \) is *not* the exotic energy density of the scalar field \( \Phi \) which is actually negative for \( \beta^2 > 1 \) and is given by

\[
\rho(\Phi) = -\frac{256m^2r^2(\beta^2 - 1)(1 - \frac{m}{2r})^{2\beta}(1 + \frac{m}{2r})^{-2\beta}}{(m^2 - 4r^2)^4}. \quad (19)
\]

The same statement applies also to pressure. Eqs.(7) and (11) imply that, formally, \( \Psi_1 \equiv \Phi \) but the underlying physics is entirely different: \( \Psi_1 \) represents a fluid perturbation propagating with local speed \( c_s \) on top of a spacetime \( \tilde{g}_{\mu \nu} \) while the dilaton \( \Phi \) mediates a medium range force similar to the electromagnetic force that propagates with the speed of light \( c \). For a more elaborate discussion on this point, see Ref.[19]. Returning to the fluid description, we can rewrite the Euler equation (3) for \( p_0 = p_0(r) \) by explicitly displaying the force required to hold the fluid configuration in place against the pressure gradient:

\[
\vec{f} = \rho_0 \left( \vec{v}_0, \nabla \right) \vec{v}_0 + c_s^2 \partial_r \rho_0 \hat{r}. \quad (20)
\]

It is now possible to find the pressure profile by integrating the equation \( \vec{v}_0 = 0 \):
\[
f_r = \frac{dp_0}{dr} = c_s^2 \frac{d\rho_0}{dr}
\]

which gives, under the physical condition that \( p_\infty \to 0 \),

\[
p_0(r) = 8\rho_\infty c_\infty^2 \left[ -1 + (2r - m)^{4\beta - 1}(2r + m)^{-4\beta + 1}(m^2 + 4r^2 + 16mr\beta) \right] / 16(16\beta^2 - 1) \tag{22}
\]

The barotropic equation of state can be found by eliminating \( r \) from Eqs. (17) and (22) as

\[
p_0(\rho_0) = \frac{\rho_\infty c_\infty^2}{2(16\beta^2 - 1)} \left[ -1 + \left( \frac{1 - \Omega}{\Omega} \right)^{4\beta - 1} \left( \frac{1 + \Omega}{\Omega} \right)^{4\beta + 1} \left( 1 + \frac{8\beta}{\Omega} + \frac{1}{\Omega^2} \right) \right] \tag{23}
\]

where \( \Omega \equiv \left( 1 - \frac{\rho_0}{\rho_\infty} \right)^{\frac{1}{2}} \). The corresponding results for the Schwarzschild black hole can be obtained by setting \( \beta = 1 \) for which there is a horizon at \( r = r_H = m/2 \). It is of interest to note that \( p_0(r) \) does not involve the scalar field parameter \( \beta \) at all, and it is of the same form as in the Schwarzschild case [6]. At the throat of the wormhole, the pressure is given by

\[
p_0(r_0^-) = \frac{\rho_\infty c_\infty^2}{2(16\beta^2 - 1)} \left[ -1 + 10\beta \Pi (-1 + \Pi)^{4\beta - 1} (1 + \Pi)^{-4\beta + 1} \right] \tag{24}
\]

where \( \Pi \equiv \beta + \sqrt{\beta^2 - 1} \). The equations (17), (18), (22)-(24) represent the exact acoustic model for the wormhole for \( \beta > 1 \). From the expression (22), it follows that the pressure drops from zero at infinity to negative values as one proceeds to the throat. This phenomenon is similar to that occurring in acoustic black holes. By varying \( \beta \), the negative value can be brought as near to zero as we please. For instance, as \( \beta \to \infty \), we have \( p_0(r_0^-) \to 0^- \). This implies that the fluid pressure increases to zero as we enlarge the throat \( (r_0^+) \) of the wormhole. The limit \( \beta \to \infty \) also implies that the mass of the wormhole \( M \) be large. This property of pressure enhancement proportional to the enlargement of mass is a generic property (it actually occurs for any value of \( \beta > 1 \)) of the class of wormholes under consideration here. However, with regard to the overall energy condition, normalizing \( \rho_\infty = c_\infty = 1 \), we can see that \( p_0(r) + p_0(r) > 0 \) for \( r \in [r_0^+, \infty) \) and \( p_0(r) + p_0(r) \to 1 \) rapidly as \( r \to \infty \). Also, \( p_0(r_0^-) + p_0(r_0^-) > 0 \) for \( \beta > 1 \) and \( p_0(r_0^-) + p_0(r_0^+) \to 1 \) very fast as \( \beta \to \infty \), that is, there is no violation of energy condition either at the throat or elsewhere whatever be the size of the wormhole. In contrast, for the acoustic analog of the Schwarzschild black hole, \( p_0(r_H) + p_0(r_H) < 0 \) implying that the null energy condition is violated at the horizon. Roughly at \( r = 0.51m \), \( \rho_0 + p_0 = 0 \) and for \( r > 0.51m \), we have \( \rho_0 + p_0 > 0 \). Thus, there is an extremely thin layer of exotic acoustic material of the order of thickness \( 0.01m \) around the horizon. This energy violating behavior of the acoustic analogue of Schwarzschild black holes is very different from that of traversable wormholes. A separate detailed examination of other variants of energy conditions would be worthwhile. We had, in the above assumed, for simplicity, \( \nabla \rho_0 = 0 \), but we could as well introduce any velocity profile \( v_0 = v_0(r) \) in which case only
the pressure profile \( p_0 = p_0(r) \) would be different from and perhaps more complicated than Eq.(22).

We tabulate for a better view the profiles of fluid quantities both for wormhole and black hole (with \( \rho_\infty = c_\infty = 1 \)):

| \( \beta > 1 \) | \( \beta = 1 \) |
|---|---|
| \( r = \infty \) | \( r = r_{th} \) |
| \( \rho_0 \) | \( \rho_0(r_0^+) \) |
| \( c_s \) | \( c_s(r_0^+) \) |
| \( p_0 \) | \( p_0(r_0^+) - \frac{1}{16\beta^2-1} \) |
| 1 | \( \infty \) | 1 | 0 | \( \infty \) |
| 1 | 0 | 0 |

It is only the overall profile, as evidenced above, that distinguishes between black and wormholes. Of particular interest is the zero mass case \( (M = m\beta = 0) \). This case actually illustrates the massless counterpart of what was originally conceived by Einstein and Rosen in their two-sheeted elementary particle model [20] with the so called “bridge” (throat) representing the positive mass particle. This original concept has later been extended by Wheeler [21] and discussed well enough in the literature [18, 21-23]. To arrive at this massless case, we can set \( \beta = 0, m \neq 0 \) in Eqs.(17),(18) and (22). (The other alternative, \( \beta \neq 0, m = 0 \) is trivial). We choose \( m = -im \) where \( m > 0 \), so that \( r_0^+ = m/2 \). Then the relevant fluid profiles are

\[
\rho_0(r) = \rho_\infty \left(1 + \frac{m^2}{4r^2}\right), c_s(r) = c_\infty \left(1 + \frac{m^2}{4r^2}\right)^{-1}, p_0(r) = \rho_\infty c_\infty^2 \left(1 + \frac{4r^2}{m^2}\right)^{-1}.
\]

We immediately find that

\[
\rho_0 c_s = \rho_\infty c_\infty = \text{constant}
\]

which illustrates a new invariant of the particle model in its acoustical analog. Such an invariant is unavailable in the massive case. Moreover, at \( r = r_0^+ = m/2 \), we can see that \( p_0(r_0^+) = \rho_\infty c_\infty^2 \) and \( \rho_0(r_0^+) = 2\rho_\infty \). Actually, there is a sharp increase in pressure at the throat from the asymptotic zero value implying a condensation of fluid at the “bridge” which seems to tally well with the particle model. This feature is in contradistinction to the pressure decrease from the asymptotic zero value to any negative value at the throat or to \( p_0(r_H) = -\rho_\infty c_\infty^2/30 \) at the horizon \( (r_H = m/2) \) in the acoustic analogs of a massive wormhole or Schwarzschild black hole respectively. Also, \( \rho_0(r) + \rho_0(r) > 0 \) for \( r \in [m/2, \infty) \) showing that it is possible to model the Einstein-Rosen particle with ordinary, as opposed to exotic, matter satisfying a simple equation of state \( p_0(\rho_0) = \rho_\infty c_\infty^2/(1 - \rho_\infty/\rho_0) \).

As an aside, it is of interest to see what the acoustic analog looks like at the naked singularity that occurs at \( r_{NS} = m/2 \) for \( \beta < 1 \). (Note that, for \( \beta > 1 \), one has from Eq.(19), \( \rho(\Phi) < 0 \), and the geometry inevitably is that of a wormhole so that the minimum distance to the center that the travellers can get to is \( r_0^+ > r_{NS} \). That is, the singular surface is practically inaccessible to them.) We see from the Eqs.(17), (18) and (22) that

(i) \( \rho_0(r_{NS}) = 0, \rho_0(r_{NS}) = -\frac{\rho_\infty c_\infty^2}{2(16\beta^2-1)}, c_s(r_{NS}) = 0 \) for \( \frac{1}{2} < \beta < 1 \)
(ii) \( \rho_0(r_{NS}) = 0, \rho_0(r_{NS}) = -\frac{\rho_\infty c_\infty^2}{6}, c_s(r_{NS}) = \frac{1}{4} \) for \( \beta = \frac{1}{2} \)
(iii) \( \rho_0(r_{NS}) = 0, c_s(r_{NS}) = \infty \) for \( \beta < \frac{1}{2} \)

whereas \( p_0(r_{NS}) \) blows up only at \( \beta = \frac{1}{4} \) while remaining finite elsewhere. We notice
that the behavior of fluid parameters closely resembles those at the black hole horizon except in the minor deviation in numerical values. Only case (iii) throws up really singular behavior. These illustrate a new (acoustic) interpretation of a curvature singularity of general relativity. Whether it is possible to acoustically model this singularity including the black/wormholes in the laboratory is still anybody’s guess.

We close this section by giving another particularly simple class of solutions to Eqs. (9)-(11) given by \( \alpha = -2 \):

\[
\phi(r) = \frac{M}{r}, \quad \psi(r) = -\frac{M}{r}, \quad \Phi = -\frac{M}{r}
\]

where \( M \) is the tensor mass as is revealed by the expansion of the metric which coincides up to second order with the Robertson expansion [24] of a centrally symmetric gravitational field. The solution was proposed by Yilmaz [25] several decades ago, but it can also be obtained, like the first example, by means of a conformal rescaling of the vacuum Brans-Dicke equations. This is a singularity free solution representing a class of traversable wormholes [26] with the throat occurring at \( r_{th} = M > 0 \). The energy density is \( \rho(\Phi) = \frac{-M^2}{r^4} < 0 \) so that the weak energy condition is violated. The acoustic parameters are given by

\[
\rho_0(r) = \rho_\infty, \quad p_0(r) = 0, \quad c_s(r) = c_\infty e^{-\frac{2M}{r}}
\]

This static configuration resembles a dust-like fluid. The massless limit is trivial as \( M = 0 \) in Eqs.(27) leads only to flat spacetime.

5 Geodesic equations

To what extent the formal acoustic analogue developed above makes real sense, except satisfying some curiosity, is not clear. However, there is always a basic but natural question, no matter whether we are concerned with black or wormholes. We want to know if the “medium” description of gravitational interaction can be used also to encompass other effects like, e.g., geodesic motions of general relativity. On a hindsight, it appears that the acoustic medium formulated in Eqs.(17), (18) [and consequently, in Eq.(22)] could be good enough for this purpose. To describe the situation, we would digress a little, but the arguments might still be worthwhile.

Several years ago, it was shown, in a theoretically exact formulation, that light motion perceives gravity as an optical refractive medium [27]. In fact, the medium notion arose as early as in the twenties of the last century when Eddington [28] first conceived it in a somewhat approximate manner followed by the development of what is now widely known as Gordon’s optical metric [29]. Only recently, these ideas have been revived and applied to the theoretical investigation of acoustical/optical black holes [30]. Along these lines of thought, and following an input from Evans and Rosenquist [31], it has been shown that, in the spherically symmetric case, the geodesic motion of light can be cast in an exact Newtonian “\( F = ma \)” form [27] given by

\[
\frac{d^2\tau}{dA^2} = \nabla \left( \frac{n^2}{2} \right)
\]
in which, formally, \( n(r) = \frac{c_{\infty}}{c_s} \), and \( A \) is the Evans-Rosenquist action defined by \( dA = dt/n^2 \). Note, importantly, that it is the same relation between \( c_{\infty} \) and \( c_s \) as in Eq.(18) that determines \( n \) in Eq.(29). That is, both optical and acoustical descriptions share the same refractive index, although the two media are governed by different sets of constitutive equations. This suggests, \textit{prima facie}, that the light ray trajectories in the optical medium could be the same as the sound ray trajectories in the analogous acoustic medium.

The equation (29) can be further extended to massive particle motion so that there is now a combined geodesic equation [32]

\[
\frac{d^2 \vec{r}}{dA^2} = \nabla \left( \frac{N^2}{2} \right), \quad N = n \left[ 1 - \frac{m_0^2 e^{-2\phi}}{H^2} \right]^{1/2}
\] (30)

where \( m_0 \) is the rest mass of the test particle and \( H \) is the total conserved energy that can be normalized to unity. The new refractive index \( N \) for massive particles can be used to exactly describe all the gravitational and cosmological kinematic effects. This reformulation of geometry allows us to view known classical effects from an altogether different angle. To cite an example, the anti-de Sitter geometry can be described as Maxwell’s “fish-eye” lens [33] giving rise to closed time-like curves. Moreover, Eq.(30) provides an easy way to introduce quantum concepts in a semiclassical manner in terms of matter de Broglie waves [32]. The method underlying the above approach has subsequently been extended to the rotating Kerr solution by Alsing [34]. He assumes the form of the metric on the equatorial plane (\( \theta = \pi/2 \)) to be of the general form

\[
ds^2 = -h(\vec{r})[dt - g_i dx^i]^2 + \xi^{-2}(\vec{r})\delta_{CD}dx^C dx^D
\] (31)

where \( dl^2 = \delta_{CD}dx^C dx^D \) is the spatial metric on the flat two-plane. (To first order in \((m/r)\), the approximate Kerr metric can always be written in that form with \( g_\phi \approx -2ma/r \), where \( a \) is the angular momentum per unit mass.) It has then been shown that the exact geodesic equations on the equatorial slice read (no simplifying assumptions of weak gravity or low velocity)[34]

\[
\frac{d^2 \vec{r}}{dA^2} = \nabla \left( \frac{n^2 v^2}{2} \right) + \frac{d\vec{r}}{dA} \times \text{curl} \vec{g}
\] (32)

where \( n(\vec{r}) = \xi^{-1} h^{-1/2} \), \( v(\vec{r}) = n^{-1}(\vec{r}) \left[ 1 - \frac{h(\vec{r})}{H^2} \right]^{1/2} \), \( \vec{g} \equiv (g_i) = -\frac{q}{m_0} \). Evidently, Eq.(32) has the form of a gravitational Lorentz-like force equation with a charge to mass ratio \((q/m_0)\) provided we make the formal identifications

\[
A \equiv t, \quad A' \equiv (\varphi, \vec{A}), \quad \varphi = \left( -\frac{n^2 v^2}{2} \right) \left( \frac{q}{m_0} \right)^{-1}, \quad \vec{A} = \vec{g} \left( \frac{q}{m_0} \right)^{-1}
\] (33)

\[
\vec{E} = -\frac{\partial \vec{A}}{\partial t'} - \nabla \varphi, \quad \vec{B} = \nabla \times \vec{A}.
\] (34)

Complementarily, the vacuum Einstein field equations themselves can be formally split into gravi-electric and the gravi-magnetic parts satisfying an appropriately defined set of Maxwell’s equations [35]. Returning now to the Eqs.(30) and (32), we mention that these
are useful also as an alternative tool for dealing with other problems such as the quantum interference of thermal neutrons in a curved spacetime [36]. As a further implication of the medium idea, an analog of the historical Fizeau effect can be envisaged in a gravity field [37]. These possibilities indicate that the alternative approach captures the kinematic essence of curved spacetime geometry in a reasonably satisfactory manner. Such analogies also provide curious theoretical insights both into the real acoustic medium and into the gravitational field as a result of wisdom borrowed from one regime and employed in the other. An experimental verification of the gravitational Fizeau effect, although we do not know at the moment exactly how it could be technically performed, will stand for a new test of gravity in addition to providing indirect support to the vortex-type experiments being pursued [8,38].

Having said all the above, we must underline a mathematical limitation of Eqs.(30) and (32): In order to integrate these equations, an arbitrary prescription of medium refractive index alone does not suffice; one needs to additionally specify the metric components $g_{00}$ and/or $g_{0i}$. For some classes of metrics, however, it might be possible to re-express them solely in terms of the medium parameter $n$. Fortunately, in the Schwarzschild problem which is of most practical importance, it is possible to write $e^{-2\phi(r)} = g_{00}(m/2r) \equiv g_{00}(n)$, after solving a cubic equation in $(m/2r)$ coming from $n(r) = (1 + m/2r)^{3}/(1 - m/2r)$. We can interpret $m$ as the total mass of the spherically symmetric fluid or the gravitating mass, as the case may be. One could then expect the geodesic equations (30) and (32) to be generally valid for any arbitrary functional prescription of $n$ for massless particles, and thereby $N(n)$ for massive test particles, without the need of further specifications as to whether this $n$ or $N$ refer to gravity field or real fluid medium. For instance [32], the Klein-Gordon equation for spinless particles, under the WKB approximation, leads directly to the eikonal equation $(\nabla S_{0})^{2} - \vec{p}^{2} = 0$, where $|\vec{p}| = HN$, and $S_{0}(\vec{r}, t) = S_{0}(\vec{r}, t) - Ht$ is the eikonal or phase. In the acoustic regime, from the kinematic wave front and the ray tube analyses [39], the ray geometry satisfies the same eikonal equation $(\nabla S_{0})^{2} - \frac{1}{c_{s}^{2}} = 0$, where we have defined $c_{s} = c_{\infty}/N$ in harmony with the phase speed $c = c_{\infty}/n$ for massless test particles. These arguments seem to imply that sound rays would bend in the acoustic medium $n = n(r)$ just as much as light rays would do in the Schwarzschild gravity. This concludes our discussion of geodesics.

6 Summary and remarks

Let us summarize our results. The present work is quite straightforward, but new, with possibly more to follow in future: (1) We have developed acoustic analogs of static traversable wormhole geometries of general relativity in which the stress tensor is provided by the minimally coupled scalar field. This, in turn, implies that we have essentially modelled the wormhole exotic material and geometry together by means of usual fluid variables. (Rotating wormholes [40] can likewise be acoustically modelled following the developments in Ref.[6]). The wormhole analogs have been found to correspond to energy condition satisfying fluid or ordinary matter. This result stands in direct contrast to the fact that a very thin shell of exotic fluid is required to wrap up the Schwarzschild black hole horizon in its acoustic analog. The distinction appears somewhat counterintuitive.
since, in the ordinary description, it is the wormhole, not the black hole, that contains energy violating exotic matter. (2) The acoustic analog of massless wormholes describes the singularity free Einstein-Rosen bridge model of elementary particles in a very interesting way. In this special case, we have found a new acoustic invariant that distinguishes the particle model from the massive analogs. (3) The nature of curvature singularity in the massive case has been brought forth in terms of the acoustic language. We have seen that the acoustic behaviors at the naked singularity are not too different from those at the horizon surface except in case (iii). Therefore, acoustically speaking, the occurrence of a naked singularity is just as viable as that of a horizon. This wisdom from acoustics could have implications for Penrose’s cosmic censorship conjecture in geometric general relativity. Finally, (4) we have demonstrated that a gravitational optical medium shares the same refractive index with the corresponding acoustic model in the simplest case of spherical symmetry. It is argued that, in the eikonal approximation, the equations of ray trajectories are exactly the same both in a gravity field and in the acoustic medium so long as both are described by the same $n = n(r)$. Therefore, the amount of bending of the rays in two situations should be the same.

To what extent the above analog features of traversable wormholes can be implemented in a laboratory set up is a moot question. In this context, we know that the Hawking-Unruh radiation occurs in a fluid in transonic motion while, in our model, the background fluid is assumed to be motionless. In this situation, the only way to detect a horizon or a throat is to track the phonon trajectories deep down into suitably simulated acoustic media. Then the measurement of Fresnel transmission ($T$) and reflection ($R$) coefficients [41], which depend only on the refractive index, could unambiguously reveal the signatures of those distinguished surfaces present in the medium. For the horizon, $R = 1$ and $T = 0$, that is, total internal reflection, while for the throat, $R$ and $T$ have non-zero values depending on $\beta$, satisfying $R + T = 1$.

As an investigation in a different direction, it might be worthwhile to review the problem of accretion of phantom energy ($\rho + p < 0$) into black holes ($\beta = 1$) [42] via the mathematical machinery of standard hydrodynamics. (Note however that a black hole’s mass can evolve also due to other reasons, for instance, by a changing Brans-Dicke scalar at the horizon, see Ref.[43].) The phantom accretion can be viewed, within the present framework, as a problem of superposition of two fluids: one is the analog acoustic fluid and the other is the phantom fluid, with the accretion ending when “equilibrium” is reached. In a similar fashion, accretion to wormholes ($\beta > 1$) [44] can be treated using the present acoustic model. It will be of interest to see to what extent the results from the standard geometric approach would agree with those from the analog fluid approach. Work is underway. Notably, a recent and very interesting work by Nojiri and Odintsov [45] shows that the phantom stage is transient, and so is the accretion to black holes with the consequence that the masses do not vanish.

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