Improved homogenization estimates for high order elliptic systems

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In the whole space $\mathbb{R}^d$ ($d \geq 2$), we study homogenization of a divergence-form matrix elliptic operator $L_\varepsilon$ of an arbitrary even order $2m \geq 4$ with measurable $\varepsilon$-periodic coefficients, where $\varepsilon$ is a small parameter. We construct an approximation for the resolvent of $L_\varepsilon$ with the remainder term of order $\varepsilon^2$ in the operator $L^2$-norm. We impose no regularity conditions on the operator beyond ellipticity and boundedness of coefficients. We use two scale expansions with correctors regularized by the Steklov smoothing.

Keywords Homogenization; High order elliptic systems; Resolvent approximation; Steklov’s smoothing

1 Introduction and statement of the problem

1.1. We study homogenization of high order matrix elliptic operators with rapidly oscillating periodic coefficients. More exactly, we consider matrix elliptic operators $L_\varepsilon$ of arbitrary even order $2m \geq 4$ with measurable $\varepsilon$-periodic coefficients, where $\varepsilon$ is a small parameter. These operators act in the space of vector-valued functions $u : \mathbb{R}^d \to \mathbb{C}^n$ and are given formally by

$$(L_\varepsilon u)_j = (-1)^m \sum_{k=1}^{n} \sum_{|\alpha| = |\beta| = m} D^\alpha (A_{\alpha\beta}^j (x/\varepsilon) D^\beta u_k), \quad j = 1, \ldots, n.$$ (1.1)

Here, $\alpha = (\alpha_1, \ldots, \alpha_d)$ is the multiindex of length $|\alpha| = \alpha_1 + \ldots + \alpha_d$ with $\alpha_j \in \mathbb{Z}_{\geq 0}$; $D^\alpha$ denotes the multiderivative $D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d}$, $D_i = D_{x_i}$, $i = 1, \ldots, d$;

$$A = \{A_{\alpha\beta}^j(y)\}.$$ (1.2)

is an array of measurable 1-periodic coefficients defined on $\mathbb{R}^d$ with values in $\mathbb{C}^n$, indexed by integers $1 \leq j \leq n$, $1 \leq k \leq n$ and by multiindices $\alpha, \beta$ with $|\alpha| = |\beta| = m$.

Introducing $(n \times n)$-matrices $A_{\alpha\beta} = \{A_{\alpha\beta}^j\}_{j=1}^{n}$, we rewrite (1.1) more briefly as follows:

$$L_\varepsilon u = (-1)^m \sum_{|\alpha| = |\beta| = m} D^\alpha (A_{\alpha\beta}(x/\varepsilon) D^\beta u).$$ (1.3)

We further simplify the display of the operator $L_\varepsilon$. To this end we consider arrays $F = \{F_{j,\gamma}\}$ indexed by integers $j$ with $1 \leq j \leq n$ and by multiindices $\gamma$ with $|\gamma| = k$ for some $k$. If $\varphi : \mathbb{R}^d \to \mathbb{C}^n$ is a vector-valued function with weak derivatives of order up to $k$, then we view $\nabla^k \varphi$ as such an array, with

$$(\nabla^k \varphi)_{j,\gamma} = D^\gamma \varphi_j.$$
If $F$ and $G$ are such kind arrays of $L^2$ functions defined in $\mathbb{R}^d$ with values in $\mathbb{C}^n$, then the inner product of $F$ and $G$ in the space $L^2(\mathbb{R}^d)$ is given by

$$(F,G) = \sum_{j=1}^n \sum_{|\gamma|=k} \int_{\mathbb{R}^d} \overline{F_{j,\gamma}} G_{j,\gamma} \, dx,$$

and the corresponding $L^2$-norm is denoted by

$$\|F\| = (F,F)^{1/2}.$$ 

We let $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, denote the standard Lebesgue spaces with respect to Lebesgue measure. We use also the standard Sobolev space $H^m(\mathbb{R}^d) = \{u : \nabla^k u \in L^2(\mathbb{R}^d), 0 \leq k \leq m\}$, equipped with the norm

$$\|u\|_{H^m(\mathbb{R}^d)}^2 = \sum_{0 \leq k \leq m} \|\nabla^k u\|^2.$$ 

As known, the set $C_0^\infty(\mathbb{R}^d)$ of smooth compactly supported functions is dense in $H^m(\mathbb{R}^d)$ and the norm in this space can be equivalently introduced in a simpler way by

$$\|u\|_{H^m(\mathbb{R}^d)}^2 = \|\nabla^m u\|^2 + \|u\|^2.$$ 

If $F = \{F_{j,\alpha}\}$ is an array indexed by $\alpha$, $|\alpha| = m$, and integers $j$, $1 \leq j \leq n$, and $A$ is from (1.2), then $AF$ is the array given by

$$(AF)_{j,\alpha} = \sum_{k=1}^n \sum_{|\beta|=m} A_{\alpha\beta}^{jk} F_{k,\beta}.$$ 

Throughout this paper we assume that arrays of coefficients (1.2) satisfy the bound

$$\|A\|_{L^\infty(\mathbb{R}^d)} \leq \lambda_1$$ 

and the strict Gårding inequality

$$\text{Re} (\nabla^m \varphi, A \nabla^m \varphi) \geq \lambda_0 \|\nabla^m \varphi\|^2 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d)$$ 

(1.5)

for some positive constants $\lambda_1$ and $\lambda_0$.

Applying homothety to the integrals actually involved in (1.5), we obtain the similar inequality with an $\varepsilon$-periodic coefficients $A^\varepsilon$ and the same constant for all $\varepsilon \in (0,1]$, i.e.,

$$\text{Re} (\nabla^m \varphi, A^\varepsilon \nabla^m \varphi) \geq \lambda_0 \|\nabla^m \varphi\|^2 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d).$$

(1.6)

Here and in the rest of the paper, given a 1-periodic function $b(y)$, we denote by $b^\varepsilon$ or $(b)^{\varepsilon}$ the $\varepsilon$-periodic function of the variable $x$ obtained from $b(y)$ by substituting $y = x/\varepsilon$, i.e.,

$$b^\varepsilon(x) = b(x/\varepsilon).$$

For example, $A^\varepsilon = A(x/\varepsilon)$, $(N^k_{\alpha})^\varepsilon = N^k_{\alpha}(x/\varepsilon)$, $(D^\beta G_{\gamma\alpha})^\varepsilon = (D^\beta G_{\gamma\alpha}(y))|_{y=x/\varepsilon}$ and so on.

1.2. We consider the following problem for vector-valued functions:

$$u^\varepsilon \in H^m(\mathbb{R}^d), \quad L^\varepsilon u^\varepsilon + u^\varepsilon = f,$$

(1.7)
for an arbitrary right-hand side \( f \in H^{-m}(\mathbb{R}^d) \), where \( H^{-m}(\mathbb{R}^d) \) is dual to \( H^m(\mathbb{R}^d) \). By definition, a (weak) solution to Equation (1.7) satisfies the integral identity

\[
(\nabla^m \varphi, A^\varepsilon \nabla^m u^\varepsilon) + (\varphi, u^\varepsilon) = \langle f, \varphi \rangle, \quad \varphi \in H^m(\mathbb{R}^d),
\]

where \( \langle f, \varphi \rangle \) denotes the value of a functional \( f \in H^{-m}(\mathbb{R}^d) \) on an element \( \varphi \in H^m(\mathbb{R}^d) \). Thereby, according to (1.8) it is natural to present the operator from (1.1) and (1.3) as

\[
L_\varepsilon = \text{div}_m(A^\varepsilon \nabla^m),
\]

where \( \text{div}_m \) is adjoint to \( \nabla^m \). By property (1.6), the operator \( L_\varepsilon \) is uniformly coercive, i.e.,

\[
\text{Re}((L_\varepsilon + I)\varphi, \varphi) = \text{Re}(\nabla^m \varphi, A^\varepsilon \nabla^m \varphi) + (\varphi, \varphi) \geq c \|\varphi\|^2_{H^m(\mathbb{R}^d)} \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d),
\]

where \( c = \min(1, \lambda_0) \). By the Lax–Milgram theorem, there exists a unique solution \( u^\varepsilon \) to (1.7); it satisfies the \( \varepsilon \)-uniform bound

\[
\|u^\varepsilon\|_{H^m(\mathbb{R}^d)} \leq 1/c \|f\|_{H^{-m}(\mathbb{R}^d)},
\]

which means

\[
\|(L_\varepsilon + I)^{-1}\|_{H^{-m}(\mathbb{R}^d) \to H^m(\mathbb{R}^d)} \leq 1/c.
\]

The first qualitative results in homogenization of the operators (1.1) and the corresponding equations (1.7) were obtained in the scalar case long ago in 70s [1, 15]. Now, we are interested in operator type estimates for the homogenization error with respect to the small parameter \( \varepsilon \). These estimates can be formulated in terms of the resolvent \( (L_\varepsilon + I)^{-1} \) and its approximations in various operator norms. We continue the recent studies of [3–8], where the approaches proposed in [14] and [16] were applied in different situations concerning high order elliptic operators. Here, we construct approximations for resolvents of matrix-valued operators (1.1) with the remainder term of order \( O(\varepsilon^2) \) as \( \varepsilon \to 0 \) in the operator \( (L^2 \to L^2) \)-norm using for justification different technique in comparison with [3] where the scalar case was studied.

As known in classical homogenization, the homogenized operator \( \hat{L} \) corresponding to (1.11) is of the same class (1.4) and (1.5) as the original operator \( L_\varepsilon \), but much simpler. In a similar display as in (1.3), we write it in the form

\[
\hat{L} = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha \hat{A}_{\alpha\beta} D^\beta, \tag{1.12}
\]

where the constant \((n \times n)\)-matrices \( \hat{A}_{\alpha\beta} \) are defined with help of auxiliary periodic problems on the unit cube (see (2.1)). The homogenized problem will be

\[
u \in H^m(\mathbb{R}^d), \quad \hat{L}u + u = f. \tag{1.13}
\]

Similarly as in [15], where the scalar case was considered, the \( G \)-convergence of \( L_\varepsilon \) to \( \hat{L} \) can be proved. This result implies, in particular, the strong resolvent convergence of \( L_\varepsilon \) to \( \hat{L} \) in the space \( L^2(\mathbb{R}^d) \), which, in terms of the solutions to (1.7) and (1.13), means the limit relation

\[
\lim_{\varepsilon \to 0} \|u^\varepsilon - u\|_{L^2(\mathbb{R}^d)} = 0 \tag{1.14}
\]

for any right-hand side function \( f \in L^2(\mathbb{R}^d) \). Recently the stronger operator convergence of \( L_\varepsilon \) to \( \hat{L} \) was established in [3], namely, the uniform resolvent convergence in the operator \( L^2(\mathbb{R}^d) \)-norm with the following convergence rate estimate

\[
\|(L_\varepsilon + I)^{-1} - (\hat{L} + I)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C\varepsilon, \tag{1.15}
\]
where the constant $C$ depends on the dimension $d$, the order $2m$ of the operator $L_\varepsilon$ and the numbers $\lambda_0$, $\lambda_1$ from (1.4) and (1.5) (the dependence of constants on $d$ and $2m$ will not be mentioned further). This result is formulated and proved in [3] for the scalar case, but the proof admits extension to vector problems. The main result in [3] is even stronger; it concerns approximations of the resolvent $(L_\varepsilon + I)^{-1}$ in the operator $(L^2(\mathbb{R}^d) \to H^m(\mathbb{R}^d))$-norm by the sum $(\hat{L} + I)^{-1} + \varepsilon^m K_\varepsilon$ of the resolvent of the homogenized operator $\hat{L}$ and the correcting operator. Furthermore,

$$\| (L_\varepsilon + I)^{-1} - (\hat{L} + I)^{-1} - \varepsilon^m K_\varepsilon \|_{L^2(\mathbb{R}^d) \to H^m(\mathbb{R}^d)} \leq C\varepsilon,$$  

(1.16)

where the constant $C$ is of the same type as in (1.15). In the case of the matrix operator $L_\varepsilon$, the operator $K_\varepsilon$ in (1.16) can be determined by the following relations:

$$K_\varepsilon f(x) = \sum_{k=1}^n \sum_{|\gamma|=m} N^k_\gamma(x/\varepsilon) S^\varepsilon D^\gamma u_k(x), \quad u(x) = (\hat{L} + I)^{-1} f(x),$$  

(1.17)

where the vector-valued functions $N^k_\gamma(y)$, indexed by the integer $k$, $1 \leq k \leq n$, and the multiindex $\gamma$, $|\gamma|=m$, are solutions to auxiliary problems (2.2) on the periodicity cell $Y = [-1/2, 1/2]^d$; $S^\varepsilon$ is the Steklov smoothing operator defined by

$$(S^\varepsilon \varphi)(x) = \int_{[-1/2, 1/2]^d} \varphi(x - \varepsilon \omega) \, d\omega$$  

(1.18)

whenever $\varphi \in L^1_{loc}(\mathbb{R}^d)$. Note that

$$\| \varepsilon^m K_\varepsilon \|_{L^2(\mathbb{R}^d) \to H^m(\mathbb{R}^d)} \leq C, \quad \| K_\varepsilon \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C.$$  

(1.19)

Then the estimate (1.15) is readily obtained from (1.16) if we first weaken the operator norm, passing from $(L^2(\mathbb{R}^d) \to H^m(\mathbb{R}^d))$-norm to $(L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d))$-norm, and then transfer the term $\varepsilon^m K_\varepsilon$ to the remainder, due to the second inequality in (1.19).

In selfadjoint case, the similar results as (1.15) and (1.16) were proved in [13] and [2], respectively. The authors used spectral approach based on the Floquet–Bloch transform tightly linked with periodic setting. Our approach admits extension to operators with coefficients not necessarily pure periodic but, e.g., locally periodic (see [10] for the case $m = 1$).

1.3. Now we are aimed to obtain an $\varepsilon^2$-order approximation of the resolvent $(L_\varepsilon + I)^{-1}$ in the operator $(L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d))$-norm, and it will be the sum of the zeroth approximation $(\hat{L} + I)^{-1}$ and some correcting term of order $\varepsilon$, namely,

$$(L_\varepsilon + I)^{-1} = (\hat{L} + I)^{-1} + \varepsilon K_1 + O(\varepsilon^2).$$  

(1.20)

This result is exactly formulated in Theorem 5.1 and proved in Section 5. To prove (1.20) we rely substantially on (1.16). In Section 4, we reproduce the proof of (1.16) in the case of matrix-valued operators since many arguments of it are used in derivation of (1.20). As it is seen from (1.17), to regularize the corrector we use the Steklov smoothing operator, which plays the key role in our method; its properties are presented in Section 3. Section 2 is devoted to auxiliary periodic problems.

In what follows, we systematically refer to the differentiation formula for the product

$$D^\alpha(wv) = \sum_{\gamma \leq \alpha} c_{\alpha, \gamma} D^\gamma w D^{\alpha-\gamma} v = (D^\alpha w)v + \sum_{\gamma < \alpha} c_{\alpha, \gamma} D^\gamma w D^{\alpha-\gamma} v$$  

(1.21)

for suitably differentiable functions $v$ and $w$ with some constants $c_{\alpha, \gamma}$, where $c_{\alpha, 0} = c_{\alpha, \alpha} = 1$. The sum in (1.21) is taken over all multiindices $\gamma$ such that $\gamma \leq \alpha$ or $\gamma < \alpha$. We assume that $\gamma \leq \alpha$ if $\gamma_i \leq \alpha_i$ for all $1 \leq i \leq d$ and $\gamma < \alpha$ if, in addition, for at least one index $i$ we have the strict inequality $\gamma_i < \alpha_i$.  

4
2 Periodic problems

In this section, we introduce auxiliary periodic problems on the the unit cube \( Y = [-1/2, 1/2]^d \). The inner product and the norm in the space \( L^2(Y) \) is denoted by \( (\cdot, \cdot)_Y \) and \( \| \cdot \|_Y \).

2.1. On the set of smooth 1-periodic vector-functions \( u \in C_{\text{per}}^\infty(Y, \mathbb{C}^n) \) with zero mean

\[
(\langle u \rangle) = \int_Y u(y) \, dy,
\]

we introduce the norm \( \| \nabla^m u \|^{1/2}_Y \) and denote by \( \mathcal{W} \) the completion of this set in this norm. It is known (see Lemma 3.1 in [3]) that the inequality (1.5) for functions in \( C_0^\infty(\mathbb{R}^d) \) yields a similar inequality for smooth periodic functions

\[
\text{Re}(\nabla^m \varphi, A \nabla^m \varphi)_Y \geq \lambda_0 \| \nabla^m \varphi \|_Y^2 \quad \forall \varphi \in C_{\text{per}}^\infty(Y, \mathbb{C}^n) \quad (2.1)
\]

which can be extended by closure to the entire space \( u \in \mathcal{W} \). Due to (2.1), the operator

\[
L = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha(A_{\alpha\beta}(y)D^\beta)
\]

acting from \( \mathcal{W} \) into its dual \( \mathcal{W}' \) is coercive. Given \( n \)-dimensional vectors \( e^k = \{ \delta_{jk} \}_j, 1 \leq k \leq n \), where \( \delta_{jk} \) is the Kronecker delta, we consider the problem on the cell of periodicity

\[
N^k_\gamma \in \mathcal{W}, \quad \sum_{|\alpha|=|\beta|=m} D^\alpha(A_{\alpha\beta}(y)D^\beta N^k_\gamma(y)) = - \sum_{|\alpha|=m} D^\alpha(A_{\alpha\gamma}(y)e^k), \quad (2.2)
\]

for any multiindex \( \gamma \) with \( |\gamma|=m \) and integer \( k, 1 \leq k \leq n \). We can briefly rewrite it as

\[
LN^k_\gamma = F^k_\gamma \quad (N^k_\gamma \in \mathcal{W}),
\]

where \( F^k_\gamma \) is the functional on \( \mathcal{W} \). Therefore, the Lax–Milgram theorem guarantees the unique solvability of (2.2) with the estimate for the solution

\[
\|N^k_\gamma\|_\mathcal{W} \leq c, \quad c = \text{const}(\lambda_0, \lambda_1). \quad (2.3)
\]

Solutions to the problems of type (2.2) can be understood in the sense of the integral identity over the cell \( Y \) for test periodic functions or in the sense of distributions on \( \mathbb{R}^d \). This double point of view applies to relations of solenoidal type, for example, (2.8), and will be used in our analysis.

We define coefficient matrices \( \hat{A}_{\alpha\beta}, |\alpha|=|\beta|=m \), of the homogenized operator \( \hat{L} \) in (1.12) by the following relations:

\[
\hat{A}_{\alpha\beta} e^k = \langle A_{\alpha\beta}(\cdot)e^k + \sum_{|\gamma|=m} A_{\alpha\gamma}(\cdot)D^\gamma N^k_\beta(\cdot) \rangle, \quad 1 \leq k \leq n. \quad (2.4)
\]

Setting

\[
e_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \neq \beta, \end{cases}
\]

we have

\[
\hat{A}_{\alpha\beta} e^k = \langle \sum_{|\gamma|=m} A_{\alpha\gamma}(\cdot)(e_{\gamma\beta} e^k + D^\gamma N^k_\beta(\cdot)) \rangle, \quad 1 \leq k \leq n. \quad (2.5)
\]
There arises the array of the homogenized coefficients \( \hat{A} = \{ \hat{A}^{jk}_{\alpha \beta} \} \), indexed by integers \( 1 \leq j \leq n, \ 1 \leq k \leq n \) and by multiindices \( \alpha, \beta \) with \( |\alpha| = |\beta| = m \). This array inherits the properties (1.4) and (1.5) (see Lemma 3.2 in \([3]\))], which ensures for the solution of (1.13) the elliptic estimate

\[
\|u\|_{H^{2m}(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)}. \tag{2.6}
\]

Setting

\[
g^{k}_{\alpha \beta}(y) = \sum_{|\gamma|=m} A^{\alpha \gamma}_{\gamma \beta}(y)(e^{\gamma \beta}c^k + D^\gamma N^{\beta}_{\gamma}(y)) - \hat{A}^{\alpha \beta}c^k \tag{2.7}
\]

for all admissible indices \( \alpha, \beta, \) and \( k \), we obtain the relations

\[
\langle g^{k}_{\alpha \beta} \rangle = 0 \ \forall \alpha, \beta, k \quad \text{and} \quad \sum_{|\alpha|=m} D^\alpha g^{k}_{\alpha \beta} = 0 \ \forall \beta, k, \tag{2.8}
\]

which allow us to use the following assertion proved in \([3]\) (see also \([17]\)).

**Lemma 2.1** Assume that \( \{g_{\alpha}\}_{|\alpha|=m} \in L^2_{\text{per}}(Y)^{\hat{m}} \) (\( \hat{m} \) is the number of multiindices of length \( m \)) and

\[
\langle g_{\alpha} \rangle = 0 \ \forall \alpha, \quad \sum_{|\alpha|=m} D^\alpha g_{\alpha} = 0. \tag{2.9}
\]

Then there exists a matrix \( \{G_{\gamma \alpha}\}_{|\alpha|=|\gamma|=m} \) from \( H^m_{\text{per}}(Y)^{\hat{m} \times \hat{m}} \) such that

\[
G_{\gamma \alpha} = -G_{\gamma \alpha}, \quad \sum_{|\gamma|=m} D^\gamma G_{\gamma \alpha} = g_{\alpha}, \tag{2.10}
\]

\[
\|G_{\gamma \alpha}\|_{H^{m}(Y)} \leq c \sum_{|\alpha|=m} \|g_{\alpha}\|_{L^2(Y)}, \quad c = \text{const}(d, m). \tag{2.11}
\]

For any fixed admissible indices \( \beta \) and \( k \), the vector \( \{g^{k}_{\alpha \beta}\}_{|\alpha|=m} \), with components from (2.7), satisfies the assumptions of Lemma 2.1 due to (2.8). Consequently, there is a matrix \( \{G^{k}_{\gamma \alpha \beta}\}_{\gamma, \alpha, \beta} \), \( |\alpha|=|\gamma|=m \), from \( H^m_{\text{per}}(Y)^{\hat{m} \times \hat{m}} \) such that identities of the form (2.10) hold componentwise, i.e.,

\[
G^{k}_{\alpha \beta \gamma} = -G^{k}_{\gamma \alpha \beta}, \quad g^{k}_{\alpha \beta} = \sum_{|\gamma|=m} D^\gamma G^{k}_{\gamma \alpha \beta}, \tag{2.12}
\]

and \( G^{k}_{\gamma \alpha \beta} \) satisfies \( H^m \)-estimate of type (2.11).

**2.2.** As a corollary of Lemma 2.1 we have

**Lemma 2.2** Let the periodic vector \( \{g_{\alpha}(y)\} \) and matrix \( \{G_{\gamma \alpha}(y)\} \) be taken from Lemma 2.1. Then

\[
g_{\alpha}(x/\varepsilon)\Phi(x) = \sum_{|\gamma|=m} D^\gamma (\varepsilon^m G_{\gamma \alpha} \Phi) - \sum_{|\gamma|=m} \sum_{\mu<\gamma} \varepsilon^{m-|\mu|} c_{\gamma, \mu} (D^\mu G_{\gamma \alpha})^\varepsilon D^{\gamma-\mu} \Phi \tag{2.13}
\]

for any \( \Phi \in C^\infty_0(\mathbb{R}^d) \) and all indices \( \alpha, |\alpha|=m \), where the constants \( c_{\gamma, \mu} \) are from (1.21).

Furthermore, for the vector

\[
\{M_{\alpha}\}_{|\alpha|=m}, \quad M_{\alpha} = \sum_{|\gamma|=m} D^\gamma (G_{\gamma \alpha} \Phi), \tag{2.14}
\]

we have

\[
\sum_{|\alpha|=m} D^\alpha M_{\alpha} = 0 \quad (\text{in the sense of distributions on } \mathbb{R}^d). \tag{2.15}
\]
Lemma 3.2

If \( \frac{1}{|\gamma|} \) for any derivative \( D^\alpha \),

\[ g_\alpha(y) = \sum_{|\gamma|=m} D^\gamma G_{\gamma\alpha}(y), \quad g_\alpha(x/\varepsilon)\Phi(x) = \sum_{|\gamma|=m} D^\gamma (\varepsilon^m G_{\gamma\alpha}(x/\varepsilon))\Phi(x), \]

and (2.13) follows from the product rule (1.21). The property (2.15) implies the identity

\[ (\varphi; \sum_{|\alpha|=m} D^\alpha M_\alpha) = 0 \quad \forall \varphi \in C^\infty_0(\mathbb{R}^d), \]

which is valid since

\[ (\varphi; \sum_{|\alpha|=m} D^\alpha M_\alpha) = \sum_{|\gamma|=|\alpha|=m} (\varphi; D^\alpha D^\gamma (G_{\gamma\alpha}^\varepsilon \Phi)) = \sum_{|\gamma|=|\alpha|=m} (D^\gamma D^\alpha \varphi, G_{\gamma\alpha}^\varepsilon \Phi), \]

where the last sum is equal to zero due to the skew-symmetry of the matrix \( G_{\gamma\alpha} \) (see (2.10)_1).

\[ \square \]

3 Properties of smoothing

In the present paper, we prove error estimates of homogenization by the method, coming from [14] and [16] (see also the overview [17]), where it is proposed to overcome the difficulties, caused by the lack of regularity in data, with help of \( \varepsilon \)-smoothing operators, for example, Steklov’s smoothing operator \( S^\varepsilon \) defined in (1.13). Here are some well known properties of \( S^\varepsilon \):

\[ \| S^\varepsilon \varphi \| \leq \| \varphi \| \quad \forall \varphi \in L^2(\mathbb{R}^d), \quad (3.1) \]

\[ \| S^\varepsilon \varphi - \varphi \| \leq (\sqrt{d}/2)\varepsilon \| \nabla \varphi \| \quad \forall \varphi \in H^1(\mathbb{R}^d), \quad (3.2) \]

where \( \| \cdot \| \) denotes the \( L^2(\mathbb{R}^d) \)-norm. We mention also the evident property \( S^\varepsilon (D^\alpha \varphi) = D^\alpha (S^\varepsilon \varphi) \) for any derivative \( D^\alpha \), which is exploited systematically in the sequel.

Along with (3.1) and (3.2), we use properties of the operator \( S^\varepsilon \) given in the following two assertions; they were firstly highlighted and proved in [16].

**Lemma 3.1** If \( \varphi \in L^2(\mathbb{R}^d), \ b \in L^2_{per}(Y), \) and \( b^\varepsilon(x) = b(x/\varepsilon), \) then \( b^\varepsilon S^\varepsilon \varphi \in L^2(\mathbb{R}^d) \) and

\[ \| b^\varepsilon S^\varepsilon \varphi \| \leq \|b^\varepsilon\|^2/\|\varphi\|. \quad (3.3) \]

**Lemma 3.2** If \( b \in L^2_{per}(Y), \ b(0) = 0, \ b^\varepsilon(x) = b(x/\varepsilon), \varphi \in L^2(\mathbb{R}^d), \) and \( \psi \in H^1(\mathbb{R}^d), \) then

\[ \| (b^\varepsilon S^\varepsilon \varphi, \psi) \| \leq C\varepsilon \|b^\varepsilon\|^2/\|\varphi\| \| \nabla \psi \|, \quad C = const(d). \quad (3.4) \]

The above estimates (3.2) and (3.3) can be sharpened under higher regularity conditions. For example,

\[ \| S^\varepsilon \varphi - \varphi \| \leq C\varepsilon^2 \| \nabla^2 \varphi \| \quad \forall \varphi \in H^2(\mathbb{R}^d), \quad C = const(d), \quad (3.5) \]

whence, by duality,

\[ \| S^\varepsilon \varphi - \varphi \|_{H^{-2}(\mathbb{R}^d)} \leq C\varepsilon^2 \|\varphi\|_{L^2(\mathbb{R}^d)} \quad \forall \varphi \in L^2(\mathbb{R}^d), \quad C = const(d), \]

thereby, for \( m \geq 2 \)

\[ \| S^\varepsilon \varphi - \varphi \|_{H^{-m}(\mathbb{R}^d)} \leq C\varepsilon^2 \|\varphi\|_{L^2(\mathbb{R}^d)} \quad \forall \varphi \in L^2(\mathbb{R}^d), \quad C = const(d). \quad (3.6) \]

The \( L^2 \)-form in (3.4) has a larger smallness order in the following situation.
Lemma 3.3 If $b \in L^2_{\text{per}}(Y)$, $\langle b \rangle = 0$, $b^\varepsilon(x) = b(x/\varepsilon)$, and $\varphi, \psi \in H^1(\mathbb{R}^d)$, then
\[
|((b^\varepsilon S^\varepsilon \varphi, \psi)| \leq C \varepsilon^2 (|b|^2)^{1/2} \|\nabla \varphi\| \|\nabla \psi\|, \quad C = \text{const}(d). \tag{3.7}
\]

We extend Lemma 3.3 and Lemma 3.2 as follows.

Lemma 3.4 If $\alpha, \beta \in L^2_{\text{per}}(Y)$, $(\alpha, \beta)_{Y} = 0$, $\alpha^\varepsilon(x) = \alpha(x/\varepsilon)$, $\beta^\varepsilon(x) = \beta(x/\varepsilon)$, $\varphi, \psi \in H^1(\mathbb{R}^d)$, then
\[
|((\alpha^\varepsilon S^\varepsilon \varphi, \beta^\varepsilon S^\varepsilon \psi)| \leq C \varepsilon^2 (|\alpha|^2)^{1/2} (|\beta|^2)^{1/2} \|\nabla \varphi\| \|\nabla \psi\|, \quad C = \text{const}(d). \tag{3.8}
\]

Lemma 3.5 If $\alpha, \beta \in L^2_{\text{per}}(Y)$, $\alpha^\varepsilon(x) = \alpha(x/\varepsilon)$, $\beta^\varepsilon(x) = \beta(x/\varepsilon)$, $\varphi \in L^2(\mathbb{R}^d)$, and $\psi \in H^1(\mathbb{R}^d)$, then
\[
|((\alpha^\varepsilon S^\varepsilon \varphi, \beta^\varepsilon S^\varepsilon \psi) - (\alpha, \beta)_Y(\varphi, \psi)| \leq C \varepsilon (|\alpha|^2)^{1/2} (|\beta|^2)^{1/2} \|\nabla \varphi\| \|\nabla \psi\|, \quad C = \text{const}(d). \tag{3.9}
\]

In the above assertions, $(\alpha, \beta)_Y$ denotes the inner product in $L^2_{\text{per}}(Y)$.

The proof of (3.5), (3.7)–(3.9) can be found in [9], [11].

4 Preliminaries

4.1. To approximate the solution $u^\varepsilon$ of (1.7), we consider the function
\[
\tilde{u}^\varepsilon(x) = u(x) + \varepsilon^m \sum_{k=1}^n \sum_{|\gamma| = m} N^k_\gamma(x/\varepsilon) D^\gamma u_k(x), \tag{4.1}
\]
composed of the solutions to the problems (1.13) and (2.2), and try to prove the estimate
\[
\|u^\varepsilon - \tilde{u}^\varepsilon\|_{H^m(\mathbb{R}^d)} \leq c \varepsilon \|f\|_{L^2(\mathbb{R}^d)}, \quad c = \text{const}(\lambda_0, \lambda_1). \tag{4.2}
\]

We first suppose that $f \in C^\infty_0(\mathbb{R}^d)$ and, thus, the vector-function $u$ in (4.1) is infinitely differentiable and, together with its derivatives, is decreasing at infinity sufficiently rapidly, so that $\tilde{u}^\varepsilon \in H^m(\mathbb{R}^d)$ and the discrepancy of the function $\tilde{u}^\varepsilon$ in (1.7), that is $(L_\varepsilon + I)\tilde{u}^\varepsilon - f$, can be calculated. Namely,
\[
(L_\varepsilon + I)\tilde{u}^\varepsilon - f = (L_\varepsilon + I)\tilde{u}^\varepsilon - (\hat{L} + I)u = L_\varepsilon \tilde{u}^\varepsilon - \hat{L}u + (\tilde{u}^\varepsilon - u),
\]
and by (1.3) and (1.12)
\[
(L_\varepsilon + I)\tilde{u}^\varepsilon - f = (-1)^m \sum_{|\alpha| = m} D^\alpha (\Gamma_{\alpha}(\tilde{u}^\varepsilon, L_\varepsilon) - \Gamma_{\alpha}(u, \hat{L})) + (\tilde{u}^\varepsilon - u), \tag{4.3}
\]
where we introduce the generalized gradients
\[
\Gamma_{\alpha}(\tilde{u}^\varepsilon, L_\varepsilon) = \sum_{|\beta| = m} A_{\alpha, \beta} D^\beta \tilde{u}^\varepsilon, \quad \Gamma_{\alpha}(u, \hat{L}) = \sum_{|\beta| = m} \hat{A}_{\alpha, \beta} D^\beta u \tag{4.4}
\]
for all multiindices $\alpha$, $|\alpha| = m$. By the rule (1.21),
\[
D^\beta (\varepsilon^m (N^k_\gamma)^\varepsilon D^\gamma u_k) = (D^\beta N^k_\gamma)^\varepsilon D^\gamma u_k + \sum_{\mu < \beta} \varepsilon^{m-|\mu|} c_{\beta, \mu} (D^\mu N^k_\gamma)^\varepsilon D^{\beta+\gamma-\mu} u_k.
\]

Thus, from (4.4) and (4.1) we get
\[
\Gamma_{\alpha}(\tilde{u}^\varepsilon, L_\varepsilon) = \sum_{|\beta| = m} A_{\alpha, \beta}^\varepsilon D^\beta (u + \varepsilon^m \sum_{k=1}^n \sum_{|\gamma| = m} (N^k_\gamma)^\varepsilon D^\gamma u_k).
\]
\[
= \sum_{|\beta|=m} (A^\varepsilon_{\alpha\beta} D^\beta u + \sum_{k=1}^n \sum_{|\gamma|=m} A^\varepsilon_{\alpha\gamma} (D^\gamma N^k_{\beta})^\varepsilon D^\beta u_k) + \sum_{k=1}^n \sum_{|\beta|=m} \varepsilon^{-|\mu|} A^\varepsilon_{\alpha\beta \gamma\mu} (D^\mu N^k_{\beta})^\varepsilon D^{\beta+\gamma-\mu} u_k.
\]

Using (2.5) and (2.7), we transform the first sum in the above representation of \( \Gamma_a(\tilde{u}^\varepsilon, L_\varepsilon) \):

\[
\sum_{|\beta|=m} (A^\varepsilon_{\alpha\beta} D^\beta u + \sum_{k=1}^n \sum_{|\gamma|=m} A^\varepsilon_{\alpha\gamma} (D^\gamma N^k_{\beta})^\varepsilon D^\beta u_k) = \sum_{k=1}^n \sum_{|\beta|=m} A^\varepsilon_{\alpha\gamma} (e^\gamma N^k_{\beta})^\varepsilon D^\beta u_k
\]

and write this representation in the form

\[
\Gamma_a(\tilde{u}^\varepsilon, L_\varepsilon) = \Gamma_a(u, \tilde{L}) + \sum_{k=1}^n \sum_{|\beta|=m} (g^k_{\alpha\beta})^\varepsilon D^\beta u_k, \tag{4.4}
\]

and write this representation in the form

\[
\Gamma_a(\tilde{u}^\varepsilon, L_\varepsilon) = \Gamma_a(u, \tilde{L}) + \sum_{k=1}^n \sum_{|\beta|=m} (g^k_{\alpha\beta})^\varepsilon D^\beta u_k + \sum_{k=1}^n \sum_{|\beta|=m} \varepsilon^{-|\mu|} A^\varepsilon_{\alpha\beta \gamma\mu} (D^\mu N^k_{\beta})^\varepsilon D^{\beta+\gamma-\mu} u_k. \tag{4.5}
\]

Applying Lemma 2.2 to the term \((g^k_{\alpha\beta})^\varepsilon D^\beta u_k\) (note that the vector \(\{g^k_{\alpha\beta}\}_a\), with \(\beta\) and \(k\) fixed, satisfies the assumptions of Lemma 2.2) we obtain

\[
(g^k_{\alpha\beta})^\varepsilon D^\beta u_k = \sum_{|\gamma|=m} D^\gamma (\varepsilon^m (G^k_{\gamma\alpha\beta})^\varepsilon D^\beta u_k) - \sum_{|\gamma|=m} \varepsilon^{-|\mu|} c_{\gamma\mu} (D^\mu G^k_{\gamma\alpha\beta})^\varepsilon D^{\gamma-\mu} D^\beta u_k. \tag{4.6}
\]

Setting

\[
M^k_{\alpha\beta} := \sum_{|\gamma|=m} D^\gamma ((G^k_{\gamma\alpha\beta})^\varepsilon D^\beta u_k),
\]

we get the vector \(\{M^k_{\alpha\beta}\}_{|\alpha|=m}\) with the property (2.15). Therefore,

\[
\sum_{|\alpha|=m} D^\alpha \sum_{k=1}^n (g^k_{\alpha\beta})^\varepsilon D^\beta u_k = - \sum_{k=1}^n \sum_{|\alpha|=|\beta|=m} D^\alpha \varepsilon^{-|\mu|} A^\varepsilon_{\alpha\beta \gamma\mu} (D^\mu G^k_{\gamma\alpha\beta})^\varepsilon D^{\beta+\gamma-\mu} u_k. \tag{4.7}
\]

From (4.5) and (4.7) it follows that

\[
\sum_{|\alpha|=m} D^\alpha (\Gamma_a(\tilde{u}^\varepsilon, L_\varepsilon) - \Gamma_a(u, \tilde{L})) = \sum_{k=1}^n \sum_{|\alpha|=|\beta|=m} D^\alpha \varepsilon^{-|\mu|} A^\varepsilon_{\alpha\beta \gamma\mu} (D^\mu N^k_{\beta})^\varepsilon D^{\beta+\gamma-\mu} u_k.
\]
Thus, the discrepancy of the function \( \hat{u}^\varepsilon \) with Equation (1.7) is represented by the sum
\[
(A_\varepsilon + I)\hat{u}^\varepsilon - f = \sum_j \varepsilon^{n_j} b_j(x/\varepsilon)\Phi_j(x) + \sum_{|\alpha|=m} D^\alpha \sum_j \varepsilon^{n_j} \tilde{b}_j(x/\varepsilon)\hat{\Phi}_j(x), \quad n_j \geq 1. \tag{4.9}
\]
Here, all \( \varepsilon \)-periodic functions \( b_j(x/\varepsilon) \) and \( \tilde{b}_j(x/\varepsilon) \) are formed of 1-periodic functions from the list
\[
A_{\alpha\beta}^k, N_k^\gamma, D^\mu N_k^\gamma, C_{\gamma\alpha\beta}^k, D^\mu G_{\gamma\alpha\beta}^k, \tag{4.10}
\]
including the coefficients from (1.12), solutions to the cell problems (2.2) together with their derivatives of order up to \( m \), and components of the matrix potentials from (2.12) together with their derivatives of order up to \( m \). The functions \( \Phi_j \) and \( \hat{\Phi}_j \) in (4.9) coincide with the components \( u_{k_1} \), \( 1 \leq k \leq n \), of the function \( u \), or their derivatives \( D^\mu u_k \) of order up to \( 2m \).

Since \( (L_\varepsilon + I)\hat{u}^\varepsilon = f \), from (4.9) we find \( (L_\varepsilon + I)(\hat{u}^\varepsilon - u^\varepsilon) = O(\varepsilon) \), whence, by the energy estimate of the type (1.11), we derive
\[
\|\hat{u}^\varepsilon - u^\varepsilon\|_{H^m(\mathbb{R}^d)} = O(\varepsilon), \tag{4.11}
\]
which is not the same as the desired estimate (1.12). If we specify the majorant on the right-hand side (1.11), we cannot guarantee that it will have the form as in (1.12). We remind also that the above computations and the final estimate make sense only under strong regularity condition on \( f \). We will show further how to overcome these difficulties.

4.2. Now we take an approximation for the solution of (1.7) in the form
\[
\hat{u}^\varepsilon(x) = w^\varepsilon(x) + \varepsilon^m U_m^\varepsilon(x) \tag{4.12}
\]
with
\[
U_m^\varepsilon(x) = \sum_{k=1}^{n} \sum_{|\gamma|=m} N_k^\gamma(x/\varepsilon)D^\gamma w_k^\varepsilon(x), \tag{4.13}
\]
\[
w^\varepsilon(x) = S^\varepsilon u(x), \tag{4.14}
\]
where \( S^\varepsilon \) is Steklov’s smoothing operator defined in (1.18); \( u(x) \) and \( N_k^\gamma(y) \), for all multiindices \( \gamma, |\gamma| = m \), and integers \( k, 1 \leq k \leq n \), are solutions to the problems (1.13) and (2.2) respectively.

**Lemma 4.1** Assume that the function \( f \) in (1.7) belongs to \( L^2(\mathbb{R}^d) \). Then the function defined in (4.12)–(4.14) approximates the solution to the problem (1.7) with the estimate (4.3).

Under conditions of Lemma 4.1, the function (4.12) belongs to the space \( H^m(\mathbb{R}^d) \) because each term in the corrector (4.13), together with its derivatives of order up to \( m \), belongs to the space \( L^2(\mathbb{R}^d) \). For example, the differentiation of order \( m \) of the corrector (4.13) yields the products
\[
\varepsilon^{|\mu|} (D^\mu N_k^\gamma)^\varepsilon D^{\gamma+\alpha-\mu} w_k^\varepsilon, \quad 0 \leq |\mu| \leq m, \quad |\alpha| = |\gamma| = m, \tag{4.15}
\]
and we handle the terms of type (4.15) by straightforward applying of Lemma 3.1. To this end, note that \( w^\varepsilon = S^\varepsilon u \) and \( u \in H^{2m}(\mathbb{R}^d) \) with estimate (2.6), besides, \( D^\mu N_k^\gamma \in L^2_{\text{per}}(Y) \) and estimate (2.3) holds. Therefore,
\[
\|(D^\mu N_k^\gamma)^\varepsilon D^{\gamma+\alpha-\mu} w_k^\varepsilon\| \leq \|D^\mu N_k^\gamma\|_Y \|D^{\gamma+\alpha-\mu} u_k\| \leq C \|f\|. \tag{4.16}
\]

The discrepancy of the function (4.12) with equation (1.7) can be represented as
\[
L_\varepsilon \hat{u}^\varepsilon + \hat{u}^\varepsilon - f = (L_\varepsilon + I)\hat{u}^\varepsilon - (\hat{A} + I)w^\varepsilon + (S^\varepsilon f - f) = (L_\varepsilon \hat{u}^\varepsilon - \hat{L}w^\varepsilon) + (\hat{u}^\varepsilon - w^\varepsilon) + (S^\varepsilon f - f), \tag{4.17}
\]
where we used the equality \((\hat{L} + I)w^\varepsilon = S^\varepsilon f\) obtained by applying the operator \(S^\varepsilon\) to both sides of (1.13) and taking into account that \(S^\varepsilon u = w^\varepsilon\).

Comparing (4.12) with (4.1) and (4.17) with (4.3), we see that \(\tilde{u}^\varepsilon\) is related to \(u^\varepsilon\) in (4.12) in the same way as \(\tilde{u}^\varepsilon\) is related to \(u\) in (1.1) but the structure of representations (4.17) and (4.3) is slightly different. The calculations in Subsection 4.1 made for the function \(\tilde{u}^\varepsilon\) defined in (4.1) can be repeated for \(\tilde{u}^\varepsilon\) defined in (4.12). All the expressions and passages have meaning and are justified with help of Lemma 3.1 similarly as it is done in (4.16). Thus, a counterpart of (4.9) will be

\[
(L_\varepsilon + I)\tilde{u}^\varepsilon - (\hat{L} + I)w^\varepsilon = \sum_j \varepsilon^{n_j} b_j(x/\varepsilon) \Phi_j(x) + \sum_{|\alpha|=m} D^\alpha \sum_j \varepsilon^{n_j} \tilde{b}_j(x/\varepsilon) \tilde{\Phi}_j(x) =: F_\varepsilon, \tag{4.18}
\]

where \(n_j \geq 1\), \(b_j\) and \(\tilde{b}_j\) are formed of the functions (4.10), \(\Phi_j\) and \(\tilde{\Phi}_j\) coincide with the function \(w^\varepsilon = S^\varepsilon u\) or its derivatives of order up to \(2m\). By Lemma 4.1 the right-hand side of (4.18), denoted by \(F_\varepsilon\), admits an estimate with the required majorant

\[
\|F_\varepsilon\|_{H^{-m}(\mathbb{R}^d)} \leq C\varepsilon\|f\|_{L^2(\mathbb{R}^d)}. \tag{4.19}
\]

It remains to estimate the last term on the right-hand side of (4.17). Due to (3.6),

\[
\|S^\varepsilon f - f\|_{H^{-m}(\mathbb{R}^d)} \leq C\varepsilon^2\|f\|_{L^2(\mathbb{R}^d)}. \tag{4.20}
\]

Now we are ready to obtain (4.2). Indeed,

\[
L_\varepsilon \tilde{u}^\varepsilon + u^\varepsilon - f = L_\varepsilon \tilde{u}^\varepsilon + \tilde{u}^\varepsilon - (L_\varepsilon \tilde{u}^\varepsilon + u^\varepsilon) = (L_\varepsilon + I)(\tilde{u}^\varepsilon - u^\varepsilon),
\]

whence, in view of (4.17) and (4.18), \((L_\varepsilon + I)(\tilde{u}^\varepsilon - u^\varepsilon) = F_\varepsilon + (S^\varepsilon f - f)\). It remains to apply the energy estimate for this equation and take into account (4.19) and (4.20). Lemma 4.1 is proved.

4.3. In operator terms, the estimate (4.12) implies (4.16).

Relying on calculations of Section 4.1, we can specify the right-hand side of (4.17) as follows.

**Lemma 4.2** 
(i) Assume that the function \(\tilde{u}^\varepsilon = w^\varepsilon + \varepsilon^m U_m^\varepsilon\) is defined in (4.12)–(4.14). Then

\[
(L_\varepsilon + I)\tilde{u}^\varepsilon - f = (-1)^m \sum_{|\alpha|=m} D^\alpha r_\varepsilon^\alpha + r_\varepsilon^0 + (S^\varepsilon f - f) =: F^\varepsilon, \tag{4.21}
\]

where

\[
\begin{align*}
& r_\varepsilon^0 = \varepsilon^m U_m^\varepsilon, \tag{4.13} \\
& r_\varepsilon^\alpha = \sum_{k=1}^n \sum_{|\beta|=m} (g_{\alpha\beta}^k)^\varepsilon D^\beta w_k^\varepsilon + \varepsilon \sum_{k=1}^n \sum_{|\beta|=m} \sum_{\gamma<\mu, |\gamma|=m-1} A_{\alpha\gamma}^\varepsilon c_{\gamma,\mu} (D^\mu N_{\beta}^k)^\varepsilon D^{\beta + \gamma - \mu} w_k^\varepsilon + w_{\alpha}^\varepsilon. \tag{4.23}
\end{align*}
\]

and \(w_{\alpha}^\varepsilon\) combines all the terms with a factor \(\varepsilon^j\), \(j \geq 2\), which come from the expansion of type (4.9). The above-mentioned functions \(w^\varepsilon\), \(N_{\alpha}^k\), and \(g_{\alpha\beta}^k\) are defined in (4.14) (2.2), (2.7) respectively.

(ii) The right-hand side function \(F^\varepsilon\) in (4.21) satisfies the estimate

\[
\|F^\varepsilon\|_{H^{-m}(L^2(\mathbb{R}^d))} \leq C\varepsilon\|f\|_{L^2(\mathbb{R}^d)}, \quad C = \text{const}(\lambda_0, \lambda_1); \tag{4.24}
\]

and, for one of its component \(w_{\alpha}^\varepsilon\) (see (4.23)), we have the sharper estimate

\[
\|w_{\alpha}^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon^2\|f\|_{L^2(\mathbb{R}^d)}, \quad C = \text{const}(\lambda_0, \lambda_1). \tag{4.25}
\]
In display (4.23), we do not use representations of type (4.6) and (4.7), which were given to justify the estimate (4.24); on the other hand, as for the expansion of type (4.5), we split in it the sum $\sum_{\mu<\gamma}$ into two parts with $|\mu|=m-1$ and $|\mu|<m-1$, and the latter one forms the term $w_\alpha^*\gamma$ of order $\varepsilon^2$.

5 Improved $L^2$-approximations

In this section, we prove our main result formulated in Theorem 5.1 concerning improved $L^2$-approximations of the resolvent $(L_\varepsilon + I)^{-1}$. As a preliminary, we introduce all necessary homogenization attributes for an adjoint operator which participate in our further calculations.

5.1. Let $A^*$ denote the adjoint array of $A$ from (1.2), i.e., $A^* = \{A_{\alpha\beta}^{*jk}(y)\}$ with $A_{\alpha\beta}^{*jk}(y) = A_{\beta\alpha}^{jk}(y)$. For $h \in L^2(\mathbb{R}^d)$, let $v^\varepsilon$ be the weak solution to

$$v^\varepsilon \in H^m(\mathbb{R}^d), \quad L_\varepsilon^* v^\varepsilon + v^\varepsilon = h,$$

where $L_\varepsilon^* = \text{div}_m((A^*)^\varepsilon \nabla^m)$ is the adjoint of $L_\varepsilon$ (see (1.9)), and let $v$ be the weak solution to

$$v \in H^m(\mathbb{R}^d), \quad \tilde{L}_* v + v = h,$$

where $\tilde{L}_* = \text{div}_m(\hat{A}_*^\varepsilon \nabla^m)$ and $\hat{A}_*$ is adjoint of $\hat{A}$. It is known that (5.2) is the homogenized problem for (5.1) since passing to the adjoint operator and homogenization are commutable for $L_\varepsilon$, i.e.,

$$(A^*)^{\text{hom}} = (A^{\text{hom}})^*, \quad \text{where} \quad A^{\text{hom}} = \hat{A}. \tag{5.3}$$

Inspite of (5.3), we need to introduce cell problems

$$N_{\gamma k}^* \in W, \quad \sum_{|\alpha|=|\beta|=m} D^\alpha (A_{\alpha\beta}^* (y) D^\beta N_{\gamma k}^* (y)) = - \sum_{|\alpha|=m} D^\alpha (A_{\alpha\gamma}^* (y) e^k), \tag{5.4}$$

for any multiindex $\gamma$ with $|\gamma|=m$ and integer $k$, $1 \leq k \leq n$. Solutions to (5.4) generate formally the homogenized coefficients for the equation (5.2) similarly as in (2.5) and likewise vectors $g_{\alpha\beta}^{sk}$ similarly as in (2.7). Thus,

$$A_{\alpha\beta}^{\text{hom}} e^k = \langle \sum_{|\gamma|=m} A_{\alpha\gamma}^* (\cdot) (e_{\gamma\beta} e^k + D^\gamma N_{\gamma k}^* (\cdot)) \rangle \tag{5.5}$$

and

$$g_{\alpha\beta}^{sk} (y) = \sum_{|\gamma|=m} A_{\alpha\gamma}^* (y) (e_{\gamma\beta} e^k + D^\gamma N_{\gamma k}^* (y)) - A_{\alpha\beta}^{\text{hom}} e^k \tag{5.6}$$

for all admissible indices $\alpha$, $\beta$, and $k$, which imply the relations

$$\langle g_{\alpha\beta}^{sk} \rangle = 0 \quad \forall \alpha, \beta, k, \quad \sum_{|\alpha|=m} D^\alpha g_{\alpha\beta}^{sk} = 0 \quad \forall \beta, k. \tag{5.7}$$

Similarly as in (1.12)-(1.14), we take an approximation for the solution of (5.1) in the form

$$\tilde{v}^\varepsilon (x) = z^\varepsilon (x) + \varepsilon^m V_m^{\varepsilon} (x), \quad V_m^{\varepsilon} (x) = \sum_{k=1}^{n} \sum_{|\gamma|=m} N_{\gamma k}^* (x/\varepsilon) D^\gamma z_k^\varepsilon (x), \quad z^\varepsilon (x) = S^\varepsilon v (x), \tag{5.8}$$

where
where \(v\) is the solution to (5.2); and, as in Lemma 4.1, we claim the estimate
\[
\|v^\varepsilon - \tilde{v}^\varepsilon\|_{H^m(\mathbb{R}^d)} \leq c\varepsilon \|h\|_{L^2(\mathbb{R}^d)},
\]
(5.9)
In what follows, we often refer to the elliptic estimate relating to (5.2)
\[
\|v\|_{H^{2m}(\mathbb{R}^d)} \leq c\|h\|_{L^2(\mathbb{R}^d)},
\]
(5.10)
and also to the estimate
\[
\|\tilde{v}^\varepsilon\|_{H^m(\mathbb{R}^d)} \leq c\|h\|_{L^2(\mathbb{R}^d)};
\]
(5.11)
the latter is obtained in a similar way as (4.16). Here and hereafter, \(c = \text{const}(\lambda_0, \lambda_1)\).

5.2. Possessing the \(H^m\)-estimate (4.2) of order \(\varepsilon\) for the function \(\tilde{v}^\varepsilon\) defined in (4.12), we seek for approximations of the solution \(u^\varepsilon\) in \(L^2\)-norm with accuracy of order \(\varepsilon^2\). To this end, we study the \(L^2\)-form
\[
(h, \tilde{u}^\varepsilon - u^\varepsilon), \quad h \in L^2(\mathbb{R}^d).
\]
Representing \(h\) in terms of the solution to (5.1), namely, as \(h = (L^\varepsilon + I)v^\varepsilon\), we get
\[
(h, \tilde{u}^\varepsilon - u^\varepsilon) = (v^\varepsilon, (L^\varepsilon + I)(\tilde{u}^\varepsilon - u^\varepsilon)) \overset{(5.8)}{=} (v^\varepsilon, (L^\varepsilon + I)\tilde{u}^\varepsilon - f) \overset{(4.21)}{=} (v^\varepsilon, F^\varepsilon) = (v^\varepsilon - \tilde{v}^\varepsilon, F^\varepsilon) + (\tilde{v}^\varepsilon, F^\varepsilon),
\]
with \(\tilde{v}^\varepsilon\) defined in (5.8). Thanks to (5.9) and (4.21), we have \((v^\varepsilon - \tilde{v}^\varepsilon, F^\varepsilon) \simeq 0\). Here and hereafter, the sign \(\simeq\) denotes an equality modulo terms \(T\) estimated as follows:
\[
|T| \leq c\varepsilon^2 \|f\|_{L^2(\mathbb{R}^d)} \|h\|_{L^2(\mathbb{R}^d)}.
\]
(5.12)
We call such kind terms \(T\) inessential. Thus
\[
(h, \tilde{u}^\varepsilon - u^\varepsilon) \simeq (\tilde{v}^\varepsilon, F^\varepsilon) \overset{(4.21)}{=} (\tilde{v}^\varepsilon, (-1)^m \sum |\alpha|=m D^\alpha r^\alpha_{\varepsilon} + r^0_{\varepsilon} + (S^\varepsilon f - f)) \overset{(5.8)}{=} \sum |\alpha|=m (D^\alpha \tilde{v}^\varepsilon, r^\alpha_{\varepsilon}) + (\tilde{v}^\varepsilon, r^0_{\varepsilon}) + (\tilde{v}^\varepsilon, (S^\varepsilon f - f)) =: I_1 + I_2 + I_3.
\]
(5.13)
Here
\[
I_3 := (\tilde{v}^\varepsilon, S^\varepsilon f - f) \leq \|\tilde{v}^\varepsilon\|_{H^m(\mathbb{R}^d)} \|S^\varepsilon f - f\|_{H^{-m}(\mathbb{R}^d)} \leq c\varepsilon^2 \|h\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)},
\]
(5.11) \(3.6\)
and
\[
I_2 := (\tilde{v}^\varepsilon, r^0_{\varepsilon}) \leq \|\tilde{v}^\varepsilon\|_{L^2(\mathbb{R}^d)} \|r^0_{\varepsilon}\|_{L^2(\mathbb{R}^d)} \overset{(5.11)}{\leq} c\varepsilon^2 \|h\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}
\]
(5.11) \(4.22\)
because \(m \geq 2\) and \(\|r^0_{\varepsilon}\|_{L^2(\mathbb{R}^d)} \leq c\varepsilon^m \|f\|_{L^2(\mathbb{R}^d)}\), which is proved by applying Lemma 3.1 and estimates (2.3) and (2.6). Therefore, in the representation (5.13), the terms \(I_2\) and \(I_3\) are inessential and it remains to estimate the sum
\[
I_1 := \sum |\alpha|=m (D^\alpha \tilde{v}^\varepsilon, r^\alpha_{\varepsilon}) \overset{(4.23)}{=} \sum |\alpha|=m \sum k=1 |\beta|=m (g^k_{\alpha\beta})^\varepsilon \varepsilon D^\beta w^\varepsilon_k + \varepsilon \sum |\alpha|=m \sum k=1 |\beta|=m \sum \mu<\gamma, \mu|=m-1 A_{\alpha\gamma} c_{\gamma,\mu} (D^\mu N^k_{\beta})^\varepsilon D^\beta+\gamma-\mu w^\varepsilon_k =: I_{11} + I_{12}.
\]
(5.14)
In view of the estimate (4.25), we dropped here inessential terms with \(w^\varepsilon_k\) coming from (4.23).
By the rule (1.21),

\[
D^\alpha \tilde{v}^\varepsilon \overset{[5.8]}{=} D^\alpha (z^\varepsilon + \varepsilon^m \sum_{j=1}^{n} \sum_{|\gamma|=m} (N^s_\gamma)^\varepsilon D^\gamma z_j^\varepsilon)
\]

\[
= \sum_{j=1}^{n} \sum_{|\gamma|=m} (e^j e_{\alpha \gamma} + (D^\alpha N^s_\gamma)^\varepsilon) D^\gamma z_j^\varepsilon + \sum_{j=1}^{n} \sum_{|\alpha|=|\gamma|=m} \sum_{\mu<\alpha} \varepsilon^{m-|\mu|} c_{\alpha,\mu} (D^\mu N^s_\gamma)^\varepsilon D^{\gamma+\alpha-\mu} z_j^\varepsilon.
\]

Consequently,

\[
I_{11} := \sum_{|\alpha|=m} \sum_{|\beta|=m} (D^\alpha \tilde{v}^\varepsilon, \sum_{k=1}^{n} \sum_{|\gamma|=m} (g^k_{\alpha \beta})^\varepsilon D^\beta w^\varepsilon_k)
\]

\[
\overset{[5.15]}{\simeq} \sum_{j,k=1}^{n} \sum_{j=1}^{n} \sum_{\gamma=m} \sum_{\mu<\alpha,|\mu|=m-1} c_{\alpha,\mu} ((D^\mu N^s_\gamma)^\varepsilon D^{\gamma+\alpha-\mu} z_j^\varepsilon, (g^k_{\alpha \beta})^\varepsilon D^\beta w^\varepsilon_k)
\]

Being inessential, terms with a factor \(\varepsilon^s, s \geq 2\), (or with \(|\mu| < m - 1\), which come from the last sum in (5.15), are dropped here; the necessary estimate (5.12) is obtained for them by Lemma 3.4. Likewise, the entire first sum in the representation of \(I_{11}\) is inessential due to Lemma 3.4 and the properties (2.3) of \(g^k_{\alpha \beta}\). Note that here \(w^\varepsilon = S^\varepsilon u\) and \(z^\varepsilon = S^\varepsilon v\) and the solutions \(u, v\) to the homogenized equations are regular enough to apply Lemma 3.4 (see (2.6) and (5.10)). Thus, we get

\[
I_{11} \simeq \varepsilon \sum_{j,k=1}^{n} \sum_{\gamma=m} \sum_{\mu<\alpha,|\mu|=m-1} c_{\alpha,\mu} ((D^\mu N^s_\gamma)^\varepsilon D^{\gamma+\alpha-\mu} z_j^\varepsilon, (g^k_{\alpha \beta})^\varepsilon D^\beta w^\varepsilon_k),
\]

which, by Lemma 3.3, yields (we recall again that \(w^\varepsilon = S^\varepsilon u\) and \(z^\varepsilon = S^\varepsilon v\))

\[
I_{11} \simeq \varepsilon \sum_{j,k=1}^{n} \sum_{\gamma=m} \sum_{\mu<\alpha,|\mu|=m-1} c_{\alpha,\mu} (D^\mu N^s_\gamma, g^k_{\alpha \beta}) (D^{\gamma+\alpha-\mu} v_j, D^\beta u_k).
\]

Similarly, we transform the remaining part in (5.14); namely, the sum

\[
I_{12} := \varepsilon \sum_{|\alpha|=m} \sum_{k=1}^{n} \sum_{|\gamma|=m} \sum_{\mu<\gamma,|\mu|=m-1} A^\varepsilon_{\alpha \gamma} c_{\gamma,\mu} (D^\mu N^k_\beta)^\varepsilon D^{\beta+\gamma-\mu} w^\varepsilon_k
\]

\[
= \varepsilon \sum_{j,k=1}^{n} \sum_{\gamma=m} \sum_{\mu<\gamma,|\mu|=m-1} c_{\gamma,\mu} (D^\mu N^k_\beta)^\varepsilon D^{\beta+\gamma-\mu} w^\varepsilon_k
\]

\[
= \varepsilon \sum_{k=1}^{n} \sum_{|\gamma|=m} (\Gamma_\gamma (\tilde{v}^\varepsilon, L^*_k), \sum_{\mu<\gamma,|\mu|=m-1} c_{\gamma,\mu} (D^\mu N^k_\beta)^\varepsilon D^{\beta+\gamma-\mu} w^\varepsilon_k),
\]

where we introduced the generalized gradient \(\Gamma_\gamma (\tilde{v}^\varepsilon, L^*_k) = \sum_{|\alpha|=m} (A^\varepsilon)^\varepsilon_{\alpha \gamma} D^\alpha \tilde{v}^\varepsilon\) in a similar way as (4.4)_1. The counterpart of (4.5) relating to the operator \(L^*_\varepsilon\) is valid; it implies

\[
\Gamma_\gamma (\tilde{v}^\varepsilon, L^*_\varepsilon) = \Gamma_\gamma (z^\varepsilon, (L^*)^{hom}) + \sum_{j=1}^{n} \sum_{|\alpha|=m} (g^s_{\alpha \gamma})^\varepsilon D^\alpha z_j^\varepsilon + O(\varepsilon)
\]
with $\Gamma(\varepsilon, (L^\star)^\hom) = \sum_{|\alpha|=m} (A^\star)^\hom_{\alpha} D^\alpha \varepsilon$, where $O(\varepsilon)$ denotes the counterpart of the last sum in (4.5) relating to $L^\star_\varepsilon$, which yields inessential terms after substituting (5.17) in the representation of $I_{12}$. The generalized gradient $\Gamma(\varepsilon, (L^\star)^\hom)$ yields likewise inessential terms after substituting (5.17) in the representation of $I_{12}$ due to Lemma 4.2 (note that $\langle D^\mu N^k_\beta \rangle = 0$). Finally, the representation of $I_{12}$ is simplified as follows:

$$I_{12} \simeq \varepsilon \sum_{j,k=1}^n \sum_{|\alpha|=|\beta|=|\gamma|=m} \sum_{\mu<\alpha,|\mu|=m-1} c_{\gamma,\mu}((U^\star)^\gamma_{\alpha j} D^\alpha \varepsilon, (D^\mu N^k_\beta)^\gamma (D^\beta+\gamma-\mu) u^j_k),$$

whereof, by applying Lemma 3.5 as in the case of (5.16), we obtain

$$I_{12} \simeq \varepsilon \sum_{j,k=1}^n \sum_{|\alpha|=|\beta|=|\gamma|=m} \sum_{\mu<\alpha,|\mu|=m-1} c_{\gamma,\mu}((U^\star)^\gamma_{\alpha j} D^\alpha \varepsilon, (D^\mu N^k_\beta)^\gamma (D^\beta+\gamma-\mu) u^j_k),$$

or, integrating by parts (note that $|\gamma-\mu| = 1$),

$$I_{12} \simeq -\varepsilon \sum_{j,k=1}^n \sum_{|\alpha|=|\beta|=|\gamma|=m} \sum_{\mu<\alpha,|\mu|=m-1} c_{\alpha,\mu}((g^\gamma_{\alpha j}, D^\mu N^k_\beta)^\gamma (D^\alpha+\gamma-\mu) v_j, D^\beta u_k).$$

To make $I_{12}$ more coinciding with $I_{11}$ we replace in it indices $\alpha$ with $\gamma$ and vice versa; thereby,

$$I_{12} \simeq -\varepsilon \sum_{j,k=1}^n \sum_{|\alpha|=|\beta|=|\gamma|=m} \sum_{\mu<\alpha,|\mu|=m-1} c_{\alpha,\mu}((g^\gamma_{\alpha j}, D^\mu N^k_\beta)^\gamma (D^\alpha+\gamma-\mu) v_j, D^\beta u_k).$$

(5.18)

Combining (5.13), (5.14), (5.16), (5.18), and estimates for the terms $I_2, I_3$, we conclude that

$$(h, \tilde{u}^\varepsilon - u^\varepsilon) \simeq \varepsilon \sum_{j,k=1}^n \sum_{|\alpha|=|\beta|=|\gamma|=m} \sum_{\mu<\alpha,|\mu|=m-1} b^j_k^{\alpha\beta\gamma\mu}(D^\alpha+\gamma-\mu) v_j, D^\beta u_k)$$

(5.19)

with coefficients

$$b^j_k^{\alpha\beta\gamma\mu} = c_{\alpha,\mu}(D^\mu N^k_\gamma g^\alpha_{\beta j})^\gamma - c_{\alpha,\mu}(g^\gamma_{\alpha j}, D^\mu N^k_\beta)^\gamma.$$

(5.20)

Now we simplify the form $(h, \tilde{u}^\varepsilon - u^\varepsilon)$ itself, arguing in a standard way:

$$(h, \tilde{u}^\varepsilon - u^\varepsilon) \overset{(4.12)}{=} (h, \varepsilon^m U^\varepsilon_m) \overset{(4.13)}{=} (h, \varepsilon^m U^\varepsilon_m) \overset{(4.13,\ 5.5)}{\simeq} (h, w^\varepsilon - u^\varepsilon) \overset{(5.21)}{\simeq} (h, w^\varepsilon - u^\varepsilon),$$

where at the last step we recall that $w^\varepsilon = S^\varepsilon u$. From (5.19) and (5.21) we deduce

$$(h, u - u^\varepsilon) \simeq \varepsilon(-1)^{m+1} \sum_{j,k=1}^n \sum_{|\alpha|=|\beta|=|\gamma|=m} \sum_{\mu<\alpha,|\mu|=m-1} b^j_k^{\alpha\beta\gamma\mu}(v_j, D^\alpha+\gamma-\mu) u_k),$$

(5.22)

taking into account that $|\alpha + \gamma - \mu| = m+1$. Let $M$ denote the matrix differential operator acting in the space of vector-valued functions $\varphi: \mathbb{R}^d \to \mathbb{C}^n$ according the formula

$$M \varphi = (-1)^m \sum_{|\alpha|=|\beta|=|\gamma|=m} \sum_{\mu<\alpha,|\mu|=m-1} B^\alpha\beta\gamma\mu D^\alpha+\beta+\gamma-\mu \varphi, \quad B^\alpha\beta\gamma\mu = \{b^j_k^{\alpha\beta\gamma\mu}\}_{j,k=1}^n.$$
Finally, we have
\[(h, (\hat{L} + I)^{-1} f + \varepsilon \mathcal{K}_1 f - (L_\varepsilon + I)^{-1} f) \simeq 0\]
with
\[\mathcal{K}_1 = (\hat{L} + I)^{-1} M (\hat{L} + I)^{-1},\]
where \(M\) is the identity matrix.

wherefrom, recalling the definition of the symbol \(\simeq\) (see the fragment with (5.12)), we get
\[\|(\hat{L} + I)^{-1} f + \varepsilon \mathcal{K}_1 f - (L_\varepsilon + I)^{-1} f\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon^2 \|f\|_{L^2(\mathbb{R}^d)},\]
which implies the asymptotic (1.20). Thus we established the following

**Theorem 5.1** Let \(\mathcal{K}_1\) denote the operator defined by (5.24), (5.25), and (5.20). Then the estimate
\[\|(\hat{L} + I)^{-1} + \varepsilon \mathcal{K}_1 - (L_\varepsilon + I)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C\varepsilon^2,\]
holds. Here \(\hat{L}\) is the homogenized operator defined in (1.12). In definition (5.20), we use the vector-functions \(N_\beta^x, g^x_{\alpha\beta}, N^x_\gamma, g^x_{\alpha\gamma}\), and constants \(c_{\alpha,\mu}\), defined in (2.2), (2.7), (5.4), (5.6), and (1.17), respectively.

In communication [12], the similar result concerning \(\varepsilon^2\) order resolvent approximations in \(L^2\)-operator norm is formulated for selfadjoint matrix fourth order operators; moreover, \(\varepsilon^3\) order resolvent approximations is also given in [12].

### 5.3.
Now we consider scalar operators (1.1) (i.e., \(n = 1\)) satisfying conditions (1.4), (1.5) and suppose additionally that \(A = \{A_{\alpha\beta}(y)\}\) is an array of real-valued and symmetric coefficients, i.e., \(A_{\alpha\beta} = A_{\beta\alpha}\). In this case, \(L_\varepsilon = L^*_\varepsilon\), thereby, homogenization attributes of the adjoint problem are the same as for the original one. In particular, \(N_\beta = N^*_\beta, g_{\alpha\beta} = g^*_{\alpha\beta}\); moreover, these functions are real-valued. Therefore, \(I_{11}\) and \(I_{12}\) acquire the simpler form than in (5.16) and (5.18); namely,
\[I_{11} \simeq \varepsilon \sum_{\alpha,\beta,\gamma, \mu < \alpha} c_{\alpha,\mu}(D^\mu N_\gamma, g_{\alpha\beta} Y(D^{\gamma + \alpha - \mu} v, D^\beta u),\]
\[I_{12} \simeq -\varepsilon \sum_{\alpha,\beta,\gamma, \mu < \alpha} c_{\alpha,\mu}(D^\mu N_\beta, g_{\alpha\gamma} Y(D^{\gamma + \alpha - \mu} v, D^\beta u),\]
with summation over admissible multiindices. Replacing \(\gamma\) with \(\beta\) and vice versa, we get
\[I_{12} \simeq -\varepsilon \sum_{\alpha,\beta,\gamma, \mu < \alpha} c_{\alpha,\mu}(D^\mu N_\gamma, g_{\alpha\beta} Y(D^{\beta + \alpha - \mu} v, D^\gamma u),\]
where \((D^{\beta + \alpha - \mu} v, D^\gamma u) = (D^{\gamma + \alpha - \mu} v, D^\beta u);\) thus, \(I_{11} + I_{12} \simeq 0\) and, by arguments in Subsection 5.2, \((L_\varepsilon + I)^{-1} = (\hat{L} + I)^{-1} + O(\varepsilon^2)\) in operator \(L^2(\mathbb{R}^d)\)-norm, which is known already from [5], [6].
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