Monogamy of $\alpha$th Power Entanglement Measurement in Qubit Systems

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In this paper, we study the $\alpha$th power monogamy properties related to the entanglement measure in bipartite states. The monogamy relations related to the $\alpha$th power of negativity and the Convex-Roof Extended Negativity are obtained for $N$-qubit states. We also give a tighter bound of hierarchical monogamy inequality for the entanglement of formation. We find that the GHZ state and W state can be used to distinguish the $\alpha$th power the concurrence for $0 < \alpha < 2$. Furthermore, we compare concurrence with negativity in terms of monogamy property and investigate the difference between them.

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I. INTRODUCTION

Multiparticle entanglement is an important physical resource in quantum mechanics, which can be used in quantum computation, quantum communication and quantum cryptography. One of the most surprising phenomena for multipartite entanglement is the monogamy property, which may be as fundamental as the no-cloning theorem [1–4]. The monogamy property can be interpreted as the amount of entanglement between $\alpha$ subsystems $A, B$ and $C$ which may be entangled with each other [2], who showed that the squared concurrence $C^2$ follows this monogamy inequality. Osborne et al proved the squared concurrence follows a general monogamy inequality for $N$-qubit system [3]. Analogous to the Coffman-Kundu-Wootters (CKW) inequality, Ou et al proposed the monogamy inequality holds in terms of squared negativity $N^2$ [10]. Kim et al showed that the squared convex-roof extended negativity $\tilde{N}^2$ follows monogamy inequality [11]. Oliveira et al and Bai et al investigated entanglement of formation(EoF) and showed that the squared EoF $E^2$ follows the monogamy inequality [12–13]. A natural question is why those monogamy property above are squared entanglement measure? In fact, Zhu et al showed that the $\alpha$th power of concurrence $C^\alpha$ ($\alpha \geq 2$) and the $\alpha$th power of entanglement of formation $E^\alpha$ ($\alpha \geq \sqrt{2}$) follow the general monogamy inequalities [13].

In this paper, we study the monogamy relations related to $\alpha$th power of some entanglement measures. We show that the $\alpha$th power of negativity $\mathcal{N}^\alpha$ and the $\alpha$th power of convex-roof extended negativity (CREN) $\tilde{\mathcal{N}}^\alpha$ follows the hierarchical monogamy inequality for $\alpha \geq 2$ [15]. From the hierarchical monogamy inequality, the general monogamy inequalities related to $\mathcal{N}^\alpha$ and $\tilde{\mathcal{N}}^\alpha$ are obtained for $N$-qubit states. We find that the GHZ state and W state can be used to distinguish the $C^\alpha$ for $0 < \alpha < 2$, which situation was not clear in Zhu et al’s paper [14]. The hierarchical monogamy inequality for $E^\alpha$ is also discussed, which improved Bai et al’s result [13, 15].

This paper is organized as follows. In Sec. II we study the monogamy property of $\alpha$th power of negativity. In Sec. III we discuss the monogamy property of CREN. In Sec. IV we study the monogamy property of $\alpha$th power of EoF. In Sec. V we compare the monogamy property of concurrence with negativity. We summarize our results in Sec. VI.

II. MONOGAMY OF $\alpha$TH POWER OF NEGATIVITY

Given a bipartite state $\rho_{AB}$ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Negativity is defined as [9]:

$$\mathcal{N}(\rho_{AB}) = \frac{||\rho_{AB}^T|| - 1}{2},$$

(1)

where $\rho_{AB}^T$ is the partial transpose with respect to the subsystem $A$. $||X||$ denotes the trace norm of $X$, i.e. $||X|| = Tr\sqrt{XX^T}$. Negativity is a computable measure of entanglement, and which is a convex function of $\rho_{AB}$. $\mathcal{N}(\rho_{AB}) = 0$ if and only if $\rho_{AB}$ is separable for the $2 \otimes 2$ and $2 \otimes 3$ systems [21]. For the purposes of discussion, we use following definition of negativity:

$$\mathcal{N}(\rho_{AB}) = ||\rho_{AB}^T|| - 1.$$  

(2)

For any maximally entangled state in two-qubit system, this definition of negativity is equal to 1.
For a bipartite pure state $|\psi_{AB}\rangle$, the concurrence is defined as:

$$C(|\psi_{AB}\rangle) = \sqrt{2[1-\text{Tr}(\rho_A^2)]} = 2\sqrt{\det \rho_A}, \quad (3)$$

where $\rho_A$ is the reduced density matrix of subsystem A. For a mixed state $\rho_{AB}$, the concurrence can be defined as:

$$C(\rho_{AB}) = \min_i p_i C(|\psi_{i,AB}\rangle), \quad (4)$$

where the minimum is taken over all possible pure state decompositions $\{p_i, |\psi_{i,AB}\rangle\}$ of $\rho_{AB}$.

The next lemma builds a relationship between negativity and concurrence in a $2 \otimes m \otimes n$ system ($m \geq 2, n \geq 2$).

**Lemma 1.** For a pure state $|\psi_{ABC}\rangle$ in a $2 \otimes m \otimes n$ system ($m \geq 2, n \geq 2$), the negativity of bipartition $A|BC$ is equal to its concurrence: $N_{A|BC} = C_{A|BC}$, where $N_{A|BC} = N(|\psi_{ABC}\rangle)$ and $C_{A|BC} = C(|\psi_{ABC}\rangle)$.

**Proof:** Based on the Schmidt decomposition, we can write the bipartition $A|BC$ as $|\psi_{A|BC}\rangle = i \sqrt{\lambda_i} |\phi_{i,BC}\rangle$, where $\lambda_i$ are Schmidt coefficients and $\sum_i \lambda_i = 1$. $\{|\phi_{i,BC}\rangle\}$ are orthogonal basis for system $A$ and system $BC$ respectively. The density operator $\rho_{ABC} = \sum_i \lambda_i |\psi_{i,BC}\rangle \langle \psi_{i,BC}|$, the partial transpose of $\rho_{ABC}$ with respect to system $A$ is given by: $\rho_{ABC}^T_A = \sum_i \lambda_i |\psi_{i,BC}\rangle \langle \phi_{i,BC}| \otimes |\phi_{i,BC}\rangle \langle \phi_{i,BC}|$. The negativity of $|\psi_{A|BC}\rangle$ is:

$$N_{A|BC} = \|\rho_{ABC}^T_A\| - 1
= \|\sum_i \lambda_i |\phi_{i,B}\rangle \langle \phi_{i,B}| \otimes |\phi_{i,C}\rangle \langle \phi_{i,C}|\| - 1
= \|\sum_i \lambda_i |\phi_{j,B}\rangle \langle \phi_{j,B}| \otimes |\phi_{i,C}\rangle \langle \phi_{i,C}|\| - 1
= \|\sum_j \lambda_j |\phi_{j,B}\rangle \langle \phi_{j,B}| \otimes \sum_i \lambda_i |\phi_{i,C}\rangle \langle \phi_{i,C}|\| - 1
= \|R \otimes R^l\| - 1
= \|R\|^2 - 1
= (\sum_i \lambda_i)^2 - 1
= 2\sqrt{\lambda_0 \lambda_1}
= 2\sqrt{\det \rho_A}
= C_{A|BC},$$

where $R = \sum_j \lambda_j |\phi_{j,B}\rangle \langle \phi_{j,B}|$, and we have used the property of trace norm: $\|A \otimes B\| = \|A\| \otimes \|B\|$. $\square$

Now we will study the monogamy property of $\alpha$th power of negativity $N^\alpha$.

**Theorem 1.** For a pure state $|\psi_{A|BC}\rangle$ in a $2 \otimes 2 \otimes 2^{N-2}$ system, the $\alpha$th power of negativity satisfies the monogamy inequality:

$$N^\alpha_{A|BC} \geq N^\alpha_{AB} + N^\alpha_{AC}, \quad (5)$$

for $\alpha \geq 2$, and satisfy the polygamy inequality:

$$N^\alpha_{A|BC} < N^\alpha_{AB} + N^\alpha_{AC}, \quad (6)$$

for $\alpha \leq 0$.

**Proof:** When $\alpha \geq 2$, by using Lemma 1, we obtain $N^\alpha_{A|BC} = C^\alpha_{A|BC}$. Combine with the result from Re. [14] and the last inequality is due to for any mixed state in a $2 \otimes d$ ($2 \leq d$) quantum system, concurrence is an upper bound of negativity, i.e. $N_{AB} \leq C_{AB}$. When $\alpha \leq 0$, without loss of generality, assuming $N_{AB} \geq N_{AC} > 0$, we have:

$$N^\alpha_{A|BC} \leq (N^\alpha_{AB} + N^\alpha_{AC})^{\frac{1}{2}} = N^\alpha_{AB} (1 + \frac{N^\alpha_{AC}}{N^\alpha_{AB}})^{\frac{1}{2}} < N^\alpha_{AB} [1 + (\frac{N^\alpha_{AC}}{N^\alpha_{AB}})^{\frac{1}{2}}] = N^\alpha_{AB} + N^\alpha_{AC},$$

where we used the property for the second inequality: $(1 + x)^{\frac{1}{2}} < 1 + \frac{x}{2} (x > 0, i \leq 0)$. If $N_{AB} = 0$ or $N_{AC} = 0$, the inequality $N^\alpha_{A|BC} < N^\alpha_{AB} + N^\alpha_{AC}$ obviously holds. $\square$

If we consider any $N$-qubit pure state $|\psi_{A_1A_2...A_N}\rangle$ in $k$-partite cases with $k = \{3, 4, \ldots, N\}$. From Theorem 1, a set of hierarchical monogamy inequalities of $N^\alpha$ holds:

$$N^\alpha_{A_1A_2...A_N} \geq \sum_{i=2}^{k-1} N^\alpha_{A_1A_i} + N^\alpha_{A_1|A_k...A_N}, \quad (10)$$

for $\alpha \geq 2$, and a set of hierarchical polygamy inequalities of $N^\alpha$ holds:

$$N^\alpha_{A_1A_2...A_N} < \sum_{i=2}^{k-1} N^\alpha_{A_1A_i} + N^\alpha_{A_1|A_k...A_N}, \quad (11)$$

for $\alpha \leq 0$.

These set of hierarchical relations can be used to detect the multipartite entanglement in these $k$-partite cases [15]. We can also obtain the following result:

**Corollary 1.** For any $N$-qubit pure state $|\psi_{A_1A_2...A_N}\rangle$ the general monogamous inequality hold:

$$N^\alpha_{A_1A_2...A_N} \geq N^\alpha_{A_1A_2} + \ldots + N^\alpha_{A_1A_N}, \quad (12)$$

for $\alpha \geq 2$, and the general polygamous inequalities hold:

$$N^\alpha_{A_1A_2...A_N} < N^\alpha_{A_1A_2} + \ldots + N^\alpha_{A_1A_N}, \quad (13)$$

for $\alpha \leq 0$.

We can see the result of $\alpha = 2$ from Re. [10] is a special case of our monogamy inequality Eq. (12).
III. MONOGAMY OF $\alpha$TH POWER CONVEX-ROOF EXTENDED NEGATIVITY

Given a bipartite state $\rho_{AB}$ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. CREN is defined as the convex roof extended of negativity on pure states $^{[13]}$:

$$\tilde{\mathcal{N}}(\rho_{AB}) = \min \sum_i p_i \mathcal{N}(|\psi^i_{AB}\rangle),$$

where the minimum is taken over all possible pure state decompositions $\{p_i, |\psi^i_{AB}\rangle\}$ of $\rho_{AB}$. Obviously, the CREN of a pure state is equal to its Negativity. CREN gives a perfect discrimination of PPT bound entangled states and separable states in any bipartite quantum systems $^{[22][23]}$. We have following result for CREN:

**Theorem 2**. For a mixed state $\rho_{ABC}$ in a $2 \otimes 2 \otimes 2^{N-2}$ system, the following monogamy inequality holds:

$$\tilde{\mathcal{N}}_{A|BC}^\alpha \geq \tilde{\mathcal{N}}_{AB}^\alpha + \tilde{\mathcal{N}}_{AC}^\alpha,$$

for $\alpha \geq 2$, and following polygamy inequality holds:

$$\tilde{\mathcal{N}}_{A|BC}^\alpha < \tilde{\mathcal{N}}_{AB}^\alpha + \tilde{\mathcal{N}}_{AC}^\alpha,$$

for $\alpha \leq 0$.

**Proof**: We only prove the first monogamy inequality, the proof of second inequality is similar to the proof of Theorem 1. Assuming a mixed state $\rho_{ABC}$ in a $2 \otimes 2 \otimes 2^{N-2}$ system, by using the Lemma 1, the definition of CREN and concurrence, we have:

$$\tilde{\mathcal{N}}_{A|BC} = \min \sum_i p_i \mathcal{N}(|\psi^i_{A|BC}\rangle) = \min \sum_i p_i \mathcal{C}(|\psi^i_{A|BC}\rangle) = \mathcal{C}_{A|BC}.$$

Thus we have:

$$\tilde{\mathcal{N}}_{A|BC}^\alpha = \mathcal{C}_{A|BC}^\alpha \geq \mathcal{C}_{AB}^\alpha + \mathcal{C}_{AC}^\alpha \geq \tilde{\mathcal{N}}_{AB}^\alpha + \tilde{\mathcal{N}}_{AC}^\alpha,$$

for $\alpha \geq 2$, where the second inequality is due to for any mixed state in a $2 \otimes d$ ($2 \leq d$) quantum system, concurrence is an upper bound of negativity.

From Theorem 2, a set of hierarchical monogamy inequalities of $\tilde{\mathcal{N}}^\alpha$ holds for any $N$-qubit mixed state $\rho_{A_1A_2:...A_N}$ in $k$-partite cases with $k = \{3, 4, \ldots, N\}$:

$$\tilde{\mathcal{N}}_{A_1|A_2:...A_N}^\alpha \geq \sum_{i=2}^{k-1} \tilde{\mathcal{N}}_{A_1|A_i}^\alpha + \tilde{\mathcal{N}}_{A_1|A_k:...A_N}^\alpha,$$

for $\alpha \geq 2$, and a set of hierarchical polygamy inequalities of $\tilde{\mathcal{N}}^\alpha$ holds:

$$\tilde{\mathcal{N}}_{A_1|A_2:...A_N}^\alpha \leq \sum_{i=2}^{k-1} \tilde{\mathcal{N}}_{A_1|A_i}^\alpha + \tilde{\mathcal{N}}_{A_1|A_k:...A_N}^\alpha,$$

for $\alpha \leq 0$.

We also have the following corollary:

**Corollary 2**. For a mixed state $\rho_{A_1A_2:...A_N}$ in a $N$-qubit system, the $\alpha$th power of CREN satisfies:

$$\tilde{\mathcal{N}}_{A_1|A_2:...A_N}^\alpha = \tilde{\mathcal{N}}_{A_1|A_2}^\alpha + \cdots + \tilde{\mathcal{N}}_{A_1|A_N}^\alpha,$$

for $\alpha \geq 2$ and

$$\tilde{\mathcal{N}}_{A_1|A_2:...A_N}^\alpha < \tilde{\mathcal{N}}_{A_1|A_2}^\alpha + \cdots + \tilde{\mathcal{N}}_{A_1|A_N}^\alpha,$$

for $\alpha \leq 0$.

We can see the result of $\alpha = 2$ from Ref. $^{[11]}$ is a special case of our monogamy inequality Eq. $^{[24]}$.

IV. MONOGAMY OF $\alpha$TH POWER OF ENTANGLEMENT OF FORMATION

Given a bipartite state $\rho_{AB}$ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the entanglement of formation (EoF) is defined as $^{[18][19]}$:

$$E(\rho_{AB}) = \min \sum_i p_i E(|\psi^i_{AB}\rangle),$$

where $E(|\psi^i_{AB}\rangle) = -Tr\rho_A \log_2 \rho_A = -Tr\rho_B \log_2 \rho_B$ is the von Neumann entropy, the minimum is taken over all possible pure state decompositions $\{p_i, |\psi^i_{AB}\rangle\}$ of $\rho_{AB}$. In Ref. $^{[20]}$, Wootters derived an analytical formula for a two-qubit mixed state $\rho_{AB}$:

$$E(\rho_{AB}) = h(1 + \sqrt{1 - C_{AB}^2})^\alpha,$$

where $h(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$ is the binary entropy and $C_{AB}$ is the concurrence of $\rho_{AB}$ which is given by Eq. $^{[3]}$ and Eq. $^{[14]}$. Bai et al have proven a set of hierarchical monogamy inequalities holds for the squared EoF in a $2 \otimes 2 \otimes 2^{N-2}$ system $^{[12]}$:

$$E_{A_1|A_2:...A_N}^2 \geq \sum_{i=2}^{k-1} E_{A_1|A_i}^2 + E_{A_1|A_k:...A_N}^2,$$

for $\alpha \geq 2$, and the following polygamy inequality holds:

$$E_{A_1|A_2:...A_N}^2 \leq E_{A_1|A_2}^2 + E_{A_1|A_3:...A_N}^2,$$

for $\alpha \leq 0$.

**Proof**: Let’s consider a tripartite pure state $|\phi_{ABC}\rangle$ in a $2 \otimes 2 \otimes 2^{N-2}$ system. Based on the Schmidt decomposition, the $2^{N-2}$-dimensional qubit $\mathbf{C}$ can be viewed as
an effect four-dimensional qubit $\mathbb{F}$. Therefore, we can consider the monogamy relationship in a $2 \otimes 2 \otimes 4$ system:

$$E^\alpha_{A|BC} = E^\alpha(C^2_{A|BC})$$

$$\geq E^\alpha(C^2_{AB} + C^2_{AC})$$

$$\geq E^\alpha(C^2_{AB}) + E^\alpha(C^2_{AC})$$

$$= E^\alpha(\rho_{AB}) + E^\alpha(\rho_{AC}),$$

(28)

where the first inequality is due to $E(C^2)$ is a monotonic increasing function and $C^2_{A|BC} \geq C^2_{AB} + C^2_{AC}$ holds, the second inequality is due to the fact $\mathbb{F}^\alpha(C_t^2 + C_s^2) \geq E^\alpha(C_t^2) + E^\alpha(C_s^2)$ for all $\alpha \geq \sqrt{2}$, the last equality is due to a mixed state $\rho_{AC}$ in a $2 \otimes d$ system, $E(\rho_{AC}) = E(C^2(\rho_{AC}))$. Thus, we complete our discussion on pure state.

Consider a mixed state $\rho_{ABC}$ in a $2 \otimes 2 \otimes 2^{N-2}$ system. We use an optimal convex decomposition $\{p_i, |\phi^i_{ABC}\rangle\}$:

$$E(\rho_{A|BC}) = \sum_i p_i E(\langle \phi^i_{ABC} | \phi^i_{ABC} \rangle),$$

(29)

we can derive

$$E(\rho_{A|BC}) = \sum_i p_i E(\langle \phi^i_{ABC} | \phi^i_{ABC} \rangle)$$

$$= \sum_i p_i E[\mathbb{F}^2(\langle \phi^i_{ABC} | \phi^i_{ABC} \rangle)]$$

$$\geq E[\sum_i p_i \mathbb{F}^2(\langle \phi^i_{ABC} | \phi^i_{ABC} \rangle)]$$

$$\geq E[\mathbb{F}^2(\rho_{A|BC})]$$

$$\geq \sqrt{E^\alpha(C^2_{AB}) + E^\alpha(C^2_{AC})}$$

$$= E^\alpha(\rho_{AB}) + E^\alpha(\rho_{AC}),$$

(30)

where the first equality is the definition of mixed state, we have used that $E(C^2)$ is a convex function in the first inequality, the second inequality can be derived by Cauchy-Schwarz inequality: $(\sum_i x_i^2)(\sum_i y_i^2) \geq (\sum_i x_i y_i)^2$, with $x_i = \sqrt{p_i} x_i^2 = \sqrt{p_i} \mathbb{F}^2(\langle \phi^i_{ABC} | \phi^i_{ABC} \rangle)$. Thus proving the monogamy inequality. On the other hand, it is easy to check the polygamy inequality for $\alpha \leq 0$. □

Based on the discussion above, we show that for a mixed state $\rho_{A_1A_2...A_N}$ in a $2 \otimes 2 \otimes 2^{N-2}$ system, a set of hierarchical monogamy inequalities holds for the $\alpha$th power of $\text{EoF}$ in $k$-partite case with $k = \{3, 4, \ldots, N\}$:

$$E^\alpha_{A_1|A_2...A_N} \geq \sum_{i=2}^{k-1} E^\alpha_{A_1A_i} + E^\alpha_{A_1|A_2...A_N},$$

(31)

for $\alpha \geq \sqrt{2}$, which can be an improvement of Bai et al’s work. And a set of hierarchical polygamy inequalities holds:

$$E^\alpha_{A_1|A_2...A_N} \leq \sum_{i=2}^{k-1} E^\alpha_{A_1A_i} + E^\alpha_{A_1|A_2...A_N},$$

(32)

for $\alpha \leq 0$. When $k = N$, the general monogamy inequality hold:

$$E^\alpha_{A_1A_2...A_N} \geq E^\alpha_{A_1A_2} + \cdots + E^\alpha_{A_1A_N},$$

(33)

for $\alpha \geq \sqrt{2}$, the specific case have been revealed in Ref. [14]. We also have the general polygamy inequality:

$$E^\alpha_{A_1A_2...A_N} \leq E^\alpha_{A_1A_2} + \cdots + E^\alpha_{A_1A_N},$$

(34)

for $\alpha \leq 0$.

\section{MONOGAMY OF $\alpha$TH POWER CONCURRENCE VS MONOGAMY OF $\alpha$TH POWER NEGATIVITY}

Based on the monogamy inequality of concurrence $\mathbb{F}$, Ref. [14] considered the general monogamy inequalities of $\alpha$th power concurrence in an $N$-qubit mixed state $\rho_{A_1A_2...A_N}$, and claimed the following inequalities holds:

$$C^\alpha_{A_1|A_2...A_N} \geq C^\alpha_{A_1A_2} + \cdots + C^\alpha_{A_1A_N},$$

(35)

for $\alpha \geq 2$. While the polygamy inequalities holds:

$$C^\alpha_{A_1|A_2...A_N} \leq C^\alpha_{A_1A_2} + \cdots + C^\alpha_{A_1A_N},$$

(36)

for all $\alpha \leq 0$. It’s not clear for $0 < \alpha < 2$. In this section, we will discuss the monogamy property of $\alpha$th power of concurrence and $\alpha$th power of negativity, for $0 < \alpha < 2$. For convenience, we define the “residual tangle” of $\alpha$th power of concurrence as:

$$\tau^\alpha(\psi_{A_1A_2...A_N}) = C^\alpha_{A_1A_2...A_N} - C^\alpha_{A_1A_2} - \cdots - C^\alpha_{A_1A_N},$$

(37)

and define the “residual tangle” of $\alpha$th power of concurrence as:

$$\tau^\alpha(\psi_{A_1A_2...A_N}) = N^\alpha_{A_1A_2...A_N} - N^\alpha_{A_1A_2} - \cdots - N^\alpha_{A_1A_N},$$

(38)

Interestingly, We find that the $N$-qubit GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes N + |1\rangle \otimes N),$$

(39)

and $N$-qubit W state

$$|W\rangle = \frac{1}{\sqrt{N}}(|00\cdots 01\rangle + |00\cdots 10\rangle + \cdots + |10\cdots 00\rangle),$$

(40)

can be used to distinguish the monogamous property of $\tau^\alpha(\psi_{A_1A_2...A_N})$ for $0 < \alpha < 2$. In other words, $N$-qubit GHZ state is monogamous for the $\alpha$th power concurrence and $N$-qubit W state is polygamous for the $\alpha$th power concurrence, where $0 < \alpha < 2$. For $N$-qubit GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes N + |1\rangle \otimes N),$$

(41)

the concurrence $C_{A_1A_2...A_N} = 1$, $C_{A_1A_k} = 0, k = \{2, 3, \ldots, N\}$. Thus, the “residual tangle” $\tau^\alpha(\langle GHZ \rangle) = \ldots$
1 > 0, N-qubit GHZ state is monogamous for the $\alpha$th power concurrence. For N-qubit W state
\[ |W\rangle = \frac{1}{\sqrt{N}}(|00\cdots01\rangle + |00\cdots10\rangle + \cdots + |10\cdots00\rangle), \]
the concurrence $C_{A_1A_2\cdots A_N} = \frac{\sqrt{N}}{\sqrt{N-1}}, C_{A_i A_k} = \frac{2}{\sqrt{N}}, k = \{2, 3, \ldots, N\}$. Thus, the "residual tangle" $\tau^n(|W\rangle) = \left(\frac{2}{\sqrt{N}}\right)^{N}[\langle(N-1)\frac{2}{\sqrt{N}} - (N-1)] < 0$ for all $0 < \alpha < 2$. N-qubit W state is polygamous for the $\alpha$th power concurrence.

For the "residual tangle" $\tau^n(|\psi_{A_1A_2\cdots A_N}\rangle)$. The negativity of N-qubit GHZ state $N_{A_1A_2\cdots A_N} = 1, N_{A_i A_k} = 0, k = \{2, 3, \ldots, N\}$. Thus, $\tau^n(|GHZ\rangle) = 1 > 0$, it is coincide with $\tau^C(|GHZ\rangle)$. The situation is different when we consider $\tau^n(|\psi_{A_1A_2\cdots A_N}\rangle)$ for N-qubit W state. One obtain that $N_{A_1A_2\cdots A_N} = \frac{1}{\sqrt{N}}\sqrt{N-1}$ and $N_{A_i A_k} = \frac{1}{\sqrt{N}}\sqrt{2(N-2)^2 + 4 - 2(N-2)\sqrt{(N-2)^2 + 4}}, k = \{2, 3, \ldots, N\}$. It is easy to check that $\tau^n(|W\rangle) = \frac{1}{\sqrt{N}}[2^\alpha(N-1)\frac{2}{\sqrt{N}} - (N-1)[2(N-2)^2 + 4 - 2(N-2)\sqrt{(N-2)^2 + 4}]\alpha]$. $\tau^n(|W\rangle)$ can be positive and negative, as showed in Fig:1, we have plotted $\tau^n(|W\rangle)$ as the function of $\alpha$ for $0 < \alpha < 2$, and consider $N = 3$, $N = 4$ and $N = 5$ respectively. We find $\tau^n(|W\rangle)$ is not always negative, which is different than the case of $\tau^C(|W\rangle)$.

VI. CONCLUSION

In this paper, We studied the monogamy property of $\alpha$th power of entanglement measure in bipartite states. In particular, we investigated the monogamy properties of negativity and CREN in detail. We showed that the $\alpha$th power of negativity, CREN are monogamous for $\alpha \geq 2$ and polygamous for $\alpha \leq 0$. We improved the hierarchical monogamy inequality for the $\alpha$th power of EoF, and show that the $\alpha$th power of EoF is hierarchical monogamous for $\alpha \geq \sqrt{2}$. Finally, we discussed the monogamy property of $\alpha$th power of concurrence. We found the N-qubit GHZ state and N-qubit W state can be used to distinguish the $\alpha$th power the concurrence for $0 < \alpha < 2$ in qubit system. We compared concurrence with negativity in terms of monogamy property and showed the difference between them.

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