1 Introduction

Quantum systems in finite dimensional Hilbert spaces proved to be a good laboratory for investigations of connections between quantum and classical characteristics of chaotic and integrable dynamics. In particular statistical properties of quantum spectra are well established signatures of integrability on the classical level \cite{Haa00}.

Recently we investigated a general procedure of attaining the classical limit for quantum systems with Hamiltonians defined as polynomials in generators of finite-dimensional compact Lie algebras \cite{SK06}. We generalized to an arbitrary algebra $\mathfrak{g}$ the procedure proposed in \cite{GK98} and \cite{GHK00}, where a particular example of the $\mathfrak{su}_3$ algebra was used to illustrate connection between spectral statistics and properties of the limiting classical Hamiltonian system. In the above cited papers we discuss at length the physical relevance of model Hamiltonians defined in terms of generators of an arbitrary $\mathfrak{g}$, as well as the meaning of (various) classical limits of them, so here we only mention that they describe e.g., collections of many (say $N$) identical multilevel (or spin-like) systems. The classical limit is attained when $N$ goes to infinity and is performed by increasing to infinity the dimension of the irreducible representation in the space of which the Hamiltonian of the system acts. In cases when the rank of the Lie algebra is greater than one, no unique way of attaining the classical limit via a sequence of irreducible representation with increasing dimensionality can be, a priori, singled out. This fact has some interesting consequences discussed thoroughly in the above cited papers (like e.g., different limiting statistics of the eigenvalues of the same system, but for different paths to the classical limit).

In the present paper we want to study more closely spectral statistics of Hamiltonians defined in terms of elements of $\mathfrak{su}_3$ along sequences of irreducible representations mimicking the transition to the classical limit. Formally, the classical limit, is often treated as ‘going with the value of the Planck constant to zero’. Such a statement taken literally is meaningless - we should specify with respect to value of which physical quantity the Planck constant becomes
negligible, so that the classical description of the system is justified. In the above mentioned situation of $N M$-level systems and a ‘nonlinear’ Hamiltonian in the form of a polynomial in the generators of $\mathfrak{su}_n$, the classical limit was obtained for $N \to \infty$, and the ‘effective Planck constant’ (the quantity which approached zero in the classical limit) was scaled appropriately with the dimensions of representations. Such a scaling was motivated by purely physical considerations. In the present paper we show that for an arbitrary $\mathfrak{g}$ there is a unique proper scaling relevant for the classical limit. This is one of the main result of the paper justifying our approach to study the outlined class of models in order to understand the transition to classical limit in well controllable models.

In the following we will concentrate on Hamiltonians taking the form of second order polynomial in elements of $\mathfrak{su}_3$, inspired by the so called Lipkin Hamiltonian of a three-shell nuclear model, often invoked in various investigation of quantum-chaotic phenomena [LS90a, LS90b, WIC98]. We will be able to thoroughly investigate limiting spectral properties of the linear part of the Hamiltonian in all possible cases of attaining the classical limit and give some remarks on the properties of the full nonlinear Hamiltonian, which will be an object of our further studies in a forthcoming paper.

Let us briefly sketch the contents of the remaining sections. In the following one we give the definition of the nearest neighbour distribution of eigenvalues of a hermitian matrix - the main object characterizing a quantum system investigated in the paper. Section 3 is concerned with rays of irreducible representations and an abstract notion of hermitian operator. In Section 4 a limit theorem for simple operators, i.e., Lie algebra elements is given. Under some assumptions on the starting representation these operators will have Dirac statistics as limiting nearest neighbour distribution. A proof of this is given by estimating the number of different eigenvalues of the represented operator along the ray through the starting representation. As a corollary we obtain Dirac statistics for certain monomials in $\mathcal{U}(\mathfrak{g})$.

If we consider operators which are not given as monomials there is need for a suitable choice of rescaling. The already announced main result of Section 5 is that under all possible ways of rescaling only the one used in [GHK00] and [SK06] gives interesting and physically meaningful results. For this we give physical arguments as well as mathematical facts.

In the last section we deal with an explicit ‘nonlinear’ example. We consider the full Lipkin-Hamiltonian. By giving explicit formulas for the action of operators in arbitrary representation we are able to complete our statements from Section 4 concerning the nearest neighbour statistics for the linear part for arbitrary choice of the sequence of representations and make some remarks about the structure of the matrix of the full Lipkin Hamiltonian in arbitrary representation.

2 Nearest neighbour distributions

The statistics we deal with are the nearest neighbour distributions which are defined as follows

**Definition 1.** Let $A$ be an $N \times N$ hermitian matrix with eigenvalues, counted with multiplicity, $x_1 \leq \cdots \leq x_N$. 
The **nearest neighbour distribution** of \( A \) is the Borel measure on \( \mathbb{R} \) given by

\[
\mu_A = \frac{1}{N} \sum_{i=1}^{N-1} \delta_{x_N-x_1}(x_{j+1}-x_j)
\]

if \( x_1 \neq x_N \), and

\[
\mu_A = \frac{N-1}{N} \delta_0
\]

if \( x_1 = \ldots = x_N \).

In order to have a notion of limit of these distribution we make use of the following distance between measures which is common in statistics

**Definition 2.** Let \( \mu, \nu \) be Borel measures on \( \mathbb{R}_{\geq 0} \) of finite mass. The **Kolmogorov-Smirnov** distance of \( \mu \) and \( \nu \) is given by

\[
d_{KS}(\mu, \nu) := \sup_{x \in \mathbb{R}_{\geq 0}} \left| \int_0^x d\mu - \int_0^x d\nu \right|.
\]

We will use the notion of convergence induced by this distance exclusively in text. Since weak convergence of measure is implied by the pointwise convergence of distribution functions the convergence in the sense of Kolmogorov-Smirnov also implies weak convergence.

### 3 Setting

We fix a semi-simple, compact Lie group \( K \) and its complexification \( G \) whose Lie algebras are denoted by \( \mathfrak{k} \) and \( \mathfrak{g} \). We denote the universal enveloping algebra of \( \mathfrak{g} \) by \( \mathcal{U}(\mathfrak{g}) \).

By convention each considered representation of \( K \) will be irreducible and unitary and therefore finite-dimensional. To have a simple notation we use the same letter for the representations of \( K \) and \( G \) and the induced Lie algebra representation. Moreover, this letter will be used for the induced representation on \( \mathcal{U}(\mathfrak{g}) \). If \( \rho : K \rightarrow \mathcal{U}(V) \) is such an irreducible, unitary representation, \( \rho : \mathcal{U}(\mathfrak{g}) \rightarrow \text{End}(V) \) is thus the induced representation on the universal enveloping algebra from the original \( \rho \).

There are numerous possibilities of sequences of irreducible representations of \( K \), but using the results from [SK06] a physical meaningful limit is provided by **rays** of irreducible representations.

**Definition 3.** Let \( \rho : K \rightarrow \mathcal{U}(V) \) be an irreducible representation of highest weight \( \lambda \).

A **ray** through \( \rho \) is a sequence of irreducible representations \( (\rho_k : \mathcal{U}(\mathfrak{g}) \rightarrow \text{End}(V_k))_{k \in \mathbb{N}} \), such that \( \rho_k \) has the highest weight \( k \cdot \lambda \).

We would like to emphasize that the irreducible representations of highest weight \( k \cdot \lambda \) are exactly the leading irreducible representations in the \( k \)-fold tensor product of the representation \( \rho \).

Since we would like to treat the operators to be discussed as Hamiltonians of some physical systems, we need a representation independent notion of hermitian operators.
Lemma 4. There exists a unique $\mathbb{R}$-linear map $\dagger : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ satisfying

- $\xi = -\xi^\dagger$ for $\xi \in \mathfrak{k}$
- $(c\xi_1 \cdots \xi_p)^\dagger = \overline{c}\xi_p^\dagger \cdots \xi_1^\dagger$ for $\xi_j \in \mathcal{U}(\mathfrak{g})$, $c \in \mathbb{C}$

and for all $\xi \in \mathcal{U}(\mathfrak{g})$ with $\xi^\dagger = \xi$

\begin{equation}
\rho(\xi) \text{ is hermitian, for each irreducible, unitary representation } \rho : K \to \text{End}(V).}
\end{equation}

Proof. First, one defines $\dagger$ on the full tensor algebra $\mathcal{T}(\mathfrak{g})$ by the above requirements and checks by a direct calculation that is passes to the quotient.

The statement about the image $\rho(\xi)$ follows by another calculation from the defining properties and the fact that $\rho : \mathcal{U}(\mathfrak{g}) \to \text{End}(V)$ is an algebra homomorphism.

Thus, we may propose the following definition.

Definition 5. An operator $\xi \in \mathcal{U}(\mathfrak{g})$ is called abstract hermitian operator, if $\xi = \xi^\dagger$. The real subspace of abstract hermitian operators in $\mathcal{U}(\mathfrak{g})$ is denoted by $\mathcal{H}$.

Remark 6. Note, that $\xi$ may not be an abstract hermitian operator although $\rho(\xi)$ is hermitian, e.g., every $\xi$ in the kernel of $\rho$ is represented as hermitian matrix.

4 Simple Operators

In this section the spectral statistics of simple operators, i.e., hermitian generators of a semi-simple compact Lie group, are considered along sequences of irreducible representations. The main result of this section is the following

Theorem 7. If $G$ is simple, rank $G \geq 2$ and $\rho$ is an irreducible representation of highest weight $\lambda$, such that $\lambda$ is in the interior of the Weyl chamber, then for all $\xi \in i\mathfrak{k}$

\begin{equation}
\lim_{k \to \infty} \mu_{\rho_k(\xi)} = \delta_0
\end{equation}

The strategy for the proof is as follows. First, $\xi$ is contained in the complexified Lie algebra $\mathfrak{k}^\mathbb{C}$ of some fixed maximal torus $T$. We compare the dimension of the representation space $V_k$ with the number of different weights of the representation, since the weights evaluated at $\xi$ are the eigenvalues of $\rho_k(\xi)$. The quotient of these numbers is shown to converge to zero, hence in the limit we will recover the Dirac measure for the nearest neighbor distribution of $\rho_k(\xi)$.

We start by giving an estimation the dimension of an irreducible representation in terms of the highest weight.

Lemma 8. Let $\rho$ be an irreducible representation with the highest weight $\lambda$ and $\lambda = \sum_j \lambda_j f_j$ the decomposition of $\lambda$ into the fundamental weights $f_j$.

Then the dimension of $\rho_\lambda$ is bounded from below by

\[ \dim \rho_\lambda \geq \prod_{\alpha \in \Pi^+, \langle \lambda, \alpha \rangle > 0} \frac{\langle \lambda, \alpha \rangle}{\langle \delta, \alpha \rangle}, \]
where \( \Pi^+ \) denotes the set of positive roots and \( \delta = \frac{1}{2} \sum_{\alpha \in \Pi^+} \alpha \).

Moreover, the number \( n_\lambda \) of possible weights of \( \rho_\lambda \) is bounded from above as follows

\[
n_\lambda \leq \text{ord}(W) \cdot \prod_j (\lambda_j + 1).
\]  

(7)

**Proof.** The Weyl’s dimension formula \([WG99]\) reads

\[
\dim \rho_\lambda = \prod_{\alpha \in \Pi^+} \frac{\langle \delta + \lambda, \alpha \rangle}{\langle \delta, \alpha \rangle} = \prod_{\alpha \in \Pi^+} \left( 1 + \frac{\langle \lambda, \alpha \rangle}{\langle \delta, \alpha \rangle} \right).
\]

(8)

Now, \( \langle \lambda, \alpha \rangle \geq 0 \) and \( \langle \delta, \alpha \rangle > 0 \) for all positive roots \( \alpha \). Thus, the first inequality is clear.

Starting from \( \lambda \) we get all other weights by subtracting multiples of the roots. The lattice of roots is a sublattice of the lattice of weights, so we can reach every weight by subtracting multiples of the fundamental weights \( f_j \).

There are at most \( \prod_j (\lambda_j + 1) \) of such possible substractions which give positive weights and every weight is in the \( W \)-orbit of a positive weight, which has at most \( |W| \) elements. This proves the second inequality.

Now, we give a proof of Theorem \[7\]

**Proof of Theorem \[7\]** Let \( Q \) be the set

\[
Q := \{ \alpha \in \Pi^+ : \langle \lambda, \alpha \rangle > 0 \}
\]

(9)

and \( q := \text{card } Q \). We claim that \( Q = \Pi^+ \). Indeed, since \( \lambda \) is in the interior of the Weyl chamber, we have

\[
\langle \lambda, \alpha_j \rangle > 0
\]

(10)

for all simple roots \( \alpha_j \). But the positive roots are just positive linear combinations of simple roots, thus \( Q = \Pi^+ \).

There exist exactly \( r := \text{rank } G \) simple roots and \( q > r \) because \( G \) is assumed to be simple and \( r \geq 2 \). If \( q = r \) were true, the root system could be decomposed into 1-dimensional pieces, which would contradict the assumptions.

Combining the above two estimates, \[9\] and \[7\], we obtain

\[
\frac{\# \text{differ. eigenvalues of } \rho_k(\xi)}{\dim \rho_k} \leq \frac{\text{ord}(W) \prod_j (k\lambda_j + 1)}{\prod_{\alpha \in Q} \frac{\langle k\lambda, \alpha \rangle}{\langle \delta, \alpha \rangle}} < \text{const}(\lambda)k^{r-q},
\]

(11)

since the number of different eigenvalues is less then the number of weights of the representation.

The right-hand side of \[11\] converges to zero if \( k \) goes to \( \infty \) which finishes the proof.

**Remark 9.** Theorem \[7\] is still valid for \( G \) semi-simple provided that \( q > \text{rank } G \).
5 Rescaling

If we consider operators $\xi \in H$ which are not ‘homogeneous’, e.g., $\xi = \eta_1 + \eta_2\eta_3$, there is a question of the right scaling of the operators when the dimensionality of irreducible representations increases. It will in general happen that the distance between the maximal and minimal eigenvalues of $\eta_2\eta_3$ along a ray of representations grow much faster than the distance for $\eta_1$. Thus, as will be shown, the lower degree terms vanish in the limit. It is therefore necessary to rescale the operators in dependence of the parameter $k$.

Definition 10. Fix a basis $\xi_1, \ldots, \xi_n$ of $g$ and identify $\mathcal{U}(g)$ with $\mathbb{C}[\xi_1, \ldots, \xi_n]$. Let $\rho : K \to U(V)$ be an irreducible representation and $(\rho_k)_{k \in \mathbb{N}}$ be the ray through $\rho$.

A rescaling map is a family of maps $(r_k : \mathcal{U}(g) \to \mathcal{U}(g))_{k \in \mathbb{N}}$ given as

$$r_k(\xi_j) = \frac{1}{s_k} \xi_j$$

(12)
on the generators and extended multiplicatively, where each $s_k \in \mathbb{N}^*$ and $\lim_{k \to \infty} s_k = \infty$.

In other words we put the elements in $\mathcal{U}(g)$ in normal ordering with respect to the chosen basis and substitute $\frac{1}{s_k} \xi_j$ for each $\xi_j$. Thus, rescaling is induced by the linear automorphism of $g$ given by multiplication with $s_k^{-1}$. Although this automorphism is independent of the chosen basis, the resulting extension to a vector space automorphism of $\mathcal{U}(g)$ is not, since it depends on the identification of $\mathcal{U}(g)$ with $\mathbb{C}[X_1, \ldots, X_n]$. More precisely, it depends on the ordering of basis vectors. Two different scaling procedures can be termed ‘natural’

1. Rescaling by the dimension: $s_k := \dim \rho_k$
2. Rescaling by the parameter: $s_k := k$ for $k > 0$ and $s_0 := 1$.

Our main goal in this section is to show that only the latter will give physically interesting results.

5.1 A Physical Argument

As indicated in the introduction and explained in details in [GK98, GHK00, SK06], we want to treat the limiting procedure described in the previous paragraphs as corresponding to taking the classical limit of the physical system in question. Using the momentum map we associate to an irreducible representation $\rho : K \to U(V)$ with highest weight $\lambda$ the coadjoint orbit $K.\lambda$ in $\mathfrak{k}^*$. The classical limit is the performed exactly by increasing $k$ to infinity along the ray through $\rho$ the corresponding sequence of coadjoint orbits. In this way $k \to \infty$ should correspond to $\hbar \to 0$.

We will show that $k$ can be identified with $\hbar^{-1}$ by invoking a semiclassical argument to express the volume the phase space in terms of the Planck constants. We will show that the rescaling by inverse parameter is the only natural choice, since a first order operator should be measured in the natural scale, i.e., in terms of $\hbar$, and their rescaling should be

$$r_k(\xi_j) = \hbar \xi_j = \frac{1}{k} (\xi_j).$$

(13)

Operators of higher degree should then be rescaled accordingly, which implies rescaling by the parameter. To this end let us formulate
Proposition 11. Let \( \rho \) be an irreducible representation with highest weight \( \lambda \) in the interior of the Weyl chamber. Then

\[
k = \frac{1}{\hbar}.
\]

Proof. For the proof of (14) recall that the volume of the phase space should be proportional to

\[
\frac{1}{\hbar^f},
\]

where \( f \) denotes the number of degrees of freedom in the classical phase space, i.e. the dimension of the coadjoint orbit through the highest weight. By a standard fact of the theory of compact Lie groups \([\text{Kir68}]\) this number is equal to the number of positive roots, i.e.,

\[
f = \#\Pi^+.
\]

But the volume of the phase space \( K.\lambda \) can be expressed in terms of the dimension of an irreducible representation. For this let \( \delta \) denote half of the sum of positive roots and set \( \lambda' = \lambda - \delta \). It is proved in \([\text{Kir68}]\) that

\[
\text{vol}(K.\lambda) = \dim \rho_{\lambda'}.
\]

Using Weyl’s dimension formula it follows that

\[
\text{vol}(K.\lambda) = \prod_{\alpha \in \Pi^+} \frac{\langle \lambda' + \delta, \alpha \rangle}{\langle \delta, \alpha \rangle} = \prod_{\alpha \in \Pi^+} \frac{\langle \lambda, \alpha \rangle}{\langle \delta, \alpha \rangle}.
\]

Therefore the volume of \( K.(k\lambda) \) is proportional to \( k^f \). Comparing with the proportionality factor in (15) implies

\[
k = \frac{1}{\hbar}
\]

and completes the proof.

5.2 A Mathematical Argument

Although the reader may be already convinced by the above reasoning, we would like to give a more mathematical argument, which applies also to representations on the border of the Weyl chamber.

We state the following lemma:

Lemma 12. Let \( \xi_H = \sum I a_I \xi_I \in \mathcal{U}(\mathfrak{g}) \) be given with \( \xi^I_H = \xi_I \) and consider the ray \( (\rho_k : K \to \mathcal{U}(V_k))_{k \in \mathbb{N}^+} \) through an irreducible representation \( \rho : K \to \mathcal{U}(V) \) of highest weight \( \lambda \).

Then

\[
\|\rho_k(\xi_H)\|_{\text{End}(V_k)} \leq \sum I |a_I| c(\lambda)^{|I|} \cdot k^{|I|}
\]

where the \( c_j(\lambda) \) is a constant depending only on \( \lambda \) and \( \| \cdot \|_{\text{End}(V_k)} \) denotes the operator norm on \( \text{End}(V_k) \).
Proof. We use the explicit construction of irreducible representations by Borel-Weil. For this let 
\( r := \operatorname{rank} G \) and 
\[
S_j = (s_1^{(j)}, \ldots, s_{d(j)}^{(j)}), \quad j = 1, \ldots, r
\]  
(21)
denote a basis of the \( j \)-th fundamental representation. These are holomorphic sections in a holomorphic line bundle 
\[ L_j \to G/B \]  
(22) 
where \( B \) is a Borel subgroup of \( G \) and \( L = G \times \chi_j \mathbb{C} \), such that \( \chi_j : B \to \mathbb{C} \) is the exponentiated character of the fundamental weight \( \lambda_j \). The irreducible representation with highest weight \( \lambda \) is then given by the action on sections of the line bundle 
\[ L = L_1^{\otimes \lambda_1} \otimes \cdots \otimes L_r^{\otimes \lambda_r} \to G/B. \]  
(23) 
By the theorem of Borel-Weil the tensors of the form 
\[
S_I^1 \otimes \cdots \otimes S_I^r,
\]  
(24) 
with \( I_1, \ldots, I_r \) multiindices of degree \( |I_j| = \lambda_j \) constitute a generating system of the space of sections.

Without loss of generality we may assume that \( \xi_1 \) is represented by a diagonal hermitian matrix in every fundamental representation, whose spectral norms a bounded by a constant \( c_{\xi_1} \). Since the operator norm is equal to the spectral norm, we wish to give an estimate for the maximal absolute value of an eigenvalue of \( \xi_1 \) in \( \rho_\lambda \).

But on the generating system of vectors given by (24) the action is on each factor separately, so we have 
\[
\| \rho_k(\xi_1) \| \leq c_{\xi_1} (\lambda_1 + \ldots + \lambda_r) =: c_{\xi_1} |\lambda|.
\]  
(25) 
For this recall, that \( \xi_j \) acts as differential operator of degree 1, therefore we get the \( \lambda_j \)'s as scalars and not in the exponent.

Clearly, the same argument can be carried out for \( \xi_2, \ldots, \xi_n \). Thus, we have the following estimate 
\[
\| \rho_k(\xi_j) \| \leq c_k (\lambda_1 + \ldots + \lambda_r) = c_k |\lambda|
\]  
(26) 
for all \( j = 1, \ldots, n \) with the constant \( c := \max_j c_{\xi_j} \).

Now, consider \( \gamma = \sum_I a_I X^I \). Then 
\[
\| \rho_k(\gamma) \|_{\text{End}(V_k)} \leq \sum_I |a_I| \| \rho_k(\xi_1) \|_{\text{End}(V_k)}^{I_1} \cdots \| \rho_k(\xi_n) \|_{\text{End}(V_k)}^{I_n}.
\]  
(27) 
Using the estimates given by (26) and Lemma 8 we see that 
\[
\| \rho_k(\gamma) \|_{\text{End}(V_k)} \leq \sum_I |a_I| k^{|I|} \cdot (c |\lambda|)^{|I|},
\]  
(28) 
which completes the proof. \( \square \)

Remark 13. If \( \xi \in \mathfrak{g}, \xi^\dagger = \xi \) and \( \rho_1(\xi) \neq 0 \), then the asymptotic growth of the norm is given by 
\[
\| \rho_1(\xi) \| \sim c \cdot k,
\]  
(29) 
where \( c \) is a constant, which depends on \( \lambda \) and \( \| \rho_1(\xi) \| \). This can be seen by a short calculation for the action of \( \xi \) on the tensors in (21).
Recall that for a highest weight \( \lambda \) the set \( Q \) is defined as \( Q = \{ \alpha \in \Pi_+ : \langle \alpha, \lambda \rangle \} \) and \( q = \# Q \). We use the Lemma and the remark to prove the following theorem

**Theorem 14.** Consider the ray \( (\rho_m : K \to U(V_m))_{m \in \mathbb{N}} \) through an irreducible representation \( \rho : K \to U(V) \) of highest weight \( \lambda \) and assume \( q > 2 \).

Let \( \xi_H = \xi + \sum_{|I|>1} a_I \Xi^I \in \mathcal{U}(\mathfrak{g}) \) be given, such that \( \xi^\dagger_H = \xi_H \) and \( \xi \in \mathfrak{g} \) with \( \|\rho_1(\xi)\| \neq 0 \). Then the nearest neighbor distributions of rescaled \( \xi_H \) along the ray agree with the limit of the nearest neighbor distribution of \( \xi \), if the rescaling factor \( s_K \) grows at least like \( k^{1+\epsilon} \).

**Proof.** We claim that \( \left\| \frac{1}{s_k} \rho(\Xi^I) \right\|_{\text{End}(V_k)} \leq \tilde{c} \frac{k^{|I|}}{s_k^{(1+\epsilon)|I|}} \),

where \( \tilde{c} \) is a constant. On the other hand by the remark above \( \rho_k(\frac{1}{s_k} \xi) \) has norm bound from below by some constant. Thus, \( \|\rho_k(r_k(\xi)) - \rho(r_k(\xi_H))\|_{\text{End}(V_k)} \to 0 \) as \( k \to \infty \).

Analogously, one would like to treat the case of “underscaling”, i.e., if \( s_k \) grows slower than \( k^{1-\epsilon} \). In that case we would expect the limit of the nearest neighbour distribution to depend only on the top homogeneous part of \( \xi_H \). While this is the case in all known examples, we can give no proof here. The problem being that a lower bound for the growth of the norm in the top homogeneous part is needed. If the norms of the top homogeneous part behave asymptotically like in (28) then with an analogous argument like in the proof above one can show convergence to the nearest neighbour distribution of the top homogeneous part.

### 5.3 A remark on the commutativity

Let us briefly discuss rescaling from the semiclassical viewpoint, by pointing that the postulated rescaling ensures the commutativity of the dynamical variables in the semiclassical limit. For simple operators let’s consider \( r_k(\xi_1 \xi_2) = \frac{1}{s_k} \xi_1 \xi_2 = \frac{1}{s_k} \xi_2 \xi_1 + \frac{1}{s_k} [\xi_1, \xi_2] \). As the commutator is a linear combination of \( \xi_j \), the respective term will vanish as \( k \) goes to infinity. A similar reasoning can be applied to arbitrary homogenous polynomials in \( \xi_j \). So, in this narrow sense the amount of commutativity is increased, although the operators themselves are non-commutative. We like to think of this as a sign of the semiclassical nature of the limit.

### 6 The Lipkin-Hamiltonian

In this section we will make some remarks about more complicated operators on \( SU_3 \), i.e. simple “polynomials” in the universal enveloping algebra of \( \mathfrak{sl}(3, \mathbb{C}) \).

Recall that the fundamental irreducible representations \( \rho_k \) of the group \( SU_n \) are given as the natural representations of \( SU_n \) on the vector spaces

\[
\bigwedge_{i=1}^k \mathbb{C}^n \text{ for } k = 1, \ldots, n - 1.
\]
By the theorem of Peter and Weyl every irreducible representation is realized as a subrepresentation of the natural $SU_n$ representation on $L^2(SU_n)$. It is known, what the generating functions for them are: let $g \in SU_n$ be the matrix
\[
\begin{pmatrix}
g_{11} & \ldots & g_{1n} \\
\vdots & \ddots & \vdots \\
g_{n1} & \ldots & g_{nn}
\end{pmatrix},
\]
then the vector space of the irreducible representation with highest weight $\lambda = \sum_{k=1}^{n-1} \lambda_k f_k$, where $f_k$ denotes the highest weight of $\rho_k$ is given as the linear span of the $k$-homogeneous polynomials in the minors of $g$ of the appropriate degree taken from the first columns only. This means, we take the homogeneous polynomials of degree $\lambda_1$ in the $1 \times 1$-minors of the first column, the homogeneous polynomials of degree $\lambda_2$ in the $2 \times 2$-minors of the first two columns and so forth, and then take every possible multiplicative combination of them.

For $SU_3$ we have to consider only the $1 \times 1$-minors
\[
x_1 := g_{11}, x_2 := g_{12}, x_3 := g_{13}
\]
and the $2 \times 2$-minors
\[
y_1 := g_{21}g_{32} - g_{31}g_{22}, y_2 := g_{31}g_{12} - g_{11}g_{32}, y_3 := g_{11}g_{22} - g_{21}g_{12}.
\]
A generating set for the representation subspace of $L^2(SU_3)$ corresponding to the highest weight $\lambda = (\lambda_1, \lambda_2)$ is then generated by the polynomials in the $x_i$ and $y_j$ which are homogeneous of degree $\lambda_1$ in the $x_i$ and homogeneous of degree $\lambda_2$ in the $y_j$. Unfortunately, these are not always linearly independent due to the relation
\[
x_1y_1 + x_2y_2 + x_3y_3 = 0.
\]

It can be shown, that this is the only relation among the variables, i.e., the irreducible representations are given as the bi-graded subrepresentations of the representation on the $\mathbb{C}$-algebra
\[
R := \mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3]/(x_1y_1 + x_2y_2 + x_3y_3).
\]
The necessary details for calculating these relations for arbitrary $SU_n$ can be found in [MS05] and [DRS74]. In the first book there is actually a nice interpretation in terms of Gröbner bases.

It was shown in the last section that elements from $\mathfrak{su}_3$ will have spectral statistics converging to the Dirac measure in the interior of the Weyl chamber. We want to study a little more complicated object, called the Lipkin-Hamiltonian in nuclear physics (cf. [GHK00]). It is an abstractly hermitian operator in $\mathcal{T}(\mathfrak{sl}_3(\mathbb{C}))$ given by
\[
\xi_{\text{Lipkin}} = aT_3 + b \sum_{i \neq j} S_{ij}^2,
\]
where $T_3 = diag(1, 0, -1)$ and $S_{ij}$ are the matrices with 1 at position $(i, j)$ and 0 elsewhere and $a, b$ denote some real constants. For $b = 0$, we already know that the spectral statistics in the interior of the Weyl chamber converge to zero. Before we go any further, we will first calculate, how these operators act.
We start with the example of $S_{12}$. It acts on a vector $v$ by $\frac{d}{dt}\big|_{t=0} \exp(S_{12}t).v$. Since we are acting on functions, the action on the argument is given by multiplication with the inverse.

$$
\exp(S_{12}t)^{-1} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\
 g_{21} & g_{22} & g_{23} \\
 g_{31} & g_{32} & g_{33}\end{pmatrix} = \begin{pmatrix} g_{11} - tg_{21} & g_{12} - tg_{22} & g_{13} - tg_{23} \\
 g_{21} & g_{22} & g_{23} \\
 g_{31} & g_{32} & g_{33}\end{pmatrix}.
$$

\hspace{1cm} (38)

Thus, we get the following substitution rules for the arguments $x_i$ and $y_i$:

$$
x_1 = g_{11} \mapsto g_{11} - tg_{21} = x_1 - tx_2 \\
x_2 = g_{21} \mapsto x_2 \\
x_3 = g_{31} \mapsto x_3 \\
y_1 = g_{21}g_{32} - g_{31}g_{22} \mapsto y_1 \\
y_2 = g_{31}g_{12} - g_{11}g_{32} \mapsto g_{31}(g_{12} - tg_{12}) - (g_{11} - tg_{21})g_{32} = y_2 + ty_1 \\
y_3 = g_{21}g_{32} - g_{31}g_{22} \mapsto (g_{11} - tg_{21})g_{22} - g_{21}(g_{12} - tg_{22}) = y_3
$$

\hspace{1cm} (39)

Applying this to the basis elements $x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^{b_1}y_2^{b_2}y_3^{b_3}$ the action is given by

$$
\frac{d}{dt}\big|_{t=0} \exp(S_{12}t).x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^{b_1}y_2^{b_2}y_3^{b_3} = -a_1x_1^{a_1-1}x_2^{a_2+1}x_3^{a_3}y_1^{b_1}y_2^{b_2}y_3^{b_3} + b_2x_1^{a_1}x_2^{a_2}x_3^{b_3}y_1^{b_1+1}y_2^{-1}y_3^{b_3}
$$

\hspace{1cm} (40)

Thus, the $S_{12}$ action is given by the action of the differential operator

$$
-x_2 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_2}
$$

\hspace{1cm} (41)

on the bi-homogeneous polynomials. In fact, analogously we can calculate that the action of $S_{ij}$ is given by

$$
-x_j \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_j}.
$$

\hspace{1cm} (42)

The actions of $T_3$ corresponds to the differential operator

$$
-x_1 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_3} + y_1 \frac{\partial}{\partial y_1} - y_3 \frac{\partial}{\partial y_3}.
$$

\hspace{1cm} (43)

Remark 15. The reader should be aware of the fact that in the chosen basis operators which are abstract hermitian, e.g., $S_{12} + S_{21}$, are not represented as hermitian matrices.

We begin our discussion by showing that $T_3$ has the Dirac measure as limiting spectral statistic in any case. Recall, that we proved this only for the interior of the Weyl chamber in Section [4].

Lemma 16. Let $\lambda$ be the highest weight of a non-trivial irreducible representation of $SU_3$ and assume that $\lambda$ lies on the border of the Weyl chamber of $SU_3$, i.e. $\lambda = (\lambda_1,0)$ or $\lambda = (0,\lambda_2)$ and $\lambda \neq (0,0)$.

Then the spectral statistics of the operator $T = \text{diag}(1,0,-1) \in \mathfrak{sl}_3(\mathbb{C})$ along the irreducible representations $\rho_*, m\lambda$ converge to the Dirac measure for $m \to \infty$. 

11
Proof. For this we consider only the case $\lambda_2 = 0$, the other case being analogous.

The element $T$ operates on a polynomial $x_1^{a_1}x_2^{a_2}x_3^{\lambda_1-a_1-a_2}$ by

$$T.x_1^{a_1}x_2^{a_2}x_3^{\lambda_1-a_1-a_2} = (-a_1 + \lambda_1 - a_1 - a_2)x_1^{a_1}x_2^{a_2}x_3^{\lambda_1-a_1-a_2}. \quad (44)$$

So the eigenvalues of $T$ are given by $-2a_1 - a_2 + \lambda_1$ with $0 \leq a_1, a_2$ and $a_1 + a_2 \leq \lambda_1$. We see for example that the zero eigenvalue has increasing multiplicity if $\lambda_1$ increases, because

$$0 = 2 \cdot 0 + \lambda_1 - \lambda_1 = 2 \cdot 1 + (\lambda_1 - 2) - \lambda_1 = 2 \cdot 2 + (\lambda_1 - 4) - \lambda_1 = \ldots \quad (45)$$

Let us count the number of different eigenvalues now. This number is the number the fibers of the map $(a_1, a_2) \mapsto 2a_1 + a_2$, which is $2\lambda_1 - 1$. To see this, just note that $(a_1, a_2)$ is in the same fiber as $(a_1 + 1, a_2 - 2)$. So every fiber can be represented by an element of the form $(0, a_2)$ or $(a_1, \lambda_1)$ and these are exactly $2\lambda_1 - 1$ elements.

But the dimension of the representation on the space on homogeneous polynomials of degree $\lambda_1$ is $\frac{1}{2} \lambda_1(\lambda_1 - 1)$. Thus the quotient of the number of eigenvalues and the dimension is

$$\frac{2\lambda_1 - 1}{\frac{1}{2} \lambda_1(\lambda_1 - 1)} = \frac{4}{\lambda_1 - 1} - \frac{2}{\lambda_1(\lambda_1 - 1)} \quad (46)$$

and converges to zero if $\lambda_1 \to \infty$. Thus, we have proved the lemma.

Our basic input for the Lipkin Hamiltonian is the explicit action of the $S_{ij}^2$. These act by sparse matrices as we are going to discuss now.

Analogously, the action of other $S_{ij}$ can be calculated. We give the action in a tabular form here, where we just write $[\ldots] := [a_1, a_2, a_3, b_1, b_2, b_3]$ instead of the rather chumsy $x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^{b_1}y_2^{b_2}y_3^{b_3}$.

$$S_{12} : [\ldots] \mapsto -a_1[a_1 - 1, a_2 + 1, a_3, b_1, b_2, b_3] + b_2[a_1, a_2, a_3, b_1 + 1, b_2 - 1, b_3]$$

$$S_{13} : [\ldots] \mapsto -a_1[a_1 - 1, a_2, a_3 + 1, b_1, b_2, b_3] + b_3[a_1, a_2, a_3, b_1 + 1, b_2, b_3 - 1]$$

$$S_{21} : [\ldots] \mapsto -a_2[a_1 + 1, a_2 - 1, a_3, b_1, b_2, b_3] + b_1[a_1, a_2, a_3, b_1 - 1, b_2 + 1, b_3]$$

$$S_{23} : [\ldots] \mapsto -a_2[a_1, a_2 - 1, a_3 + 1, b_1, b_2, b_3] + b_3[a_1, a_2, a_3, b_1 + 1, b_2, b_3 - 1]$$

$$S_{31} : [\ldots] \mapsto -a_3[a_1 + 1, a_2, a_3 - 1, b_1, b_2, b_3] + b_1[a_1, a_2, a_3, b_1 - 1, b_2, b_3 + 1]$$

$$S_{32} : [\ldots] \mapsto -a_3[a_1, a_2 + 1, a_3 - 1, b_1, b_2, b_3] + b_2[a_1, a_2, a_3, b_1 - 1, b_2 + 1, b_3] \quad (47)$$

Applying every $S_{ij}$ twice we get

$$S_{12}^2 : [\ldots] \mapsto a_1[a_1 - 1] [a_1 - 2, a_2 + 2, a_3, b_1, b_2, b_3]$$

$$+ b_2(b_2 - 1) [a_1, a_2, a_3, b_1 + 2, b_2 - 2, b_3]$$

$$- 2a_1 b_2 [a_1 - 1, a_2 + 1, a_3, b_1 + 1, b_2 - 1, b_3]$$

$$S_{13}^2 : [\ldots] \mapsto a_1[a_1 - 1] [a_1 - 2, a_2, a_3 + 2, b_1, b_2, b_3]$$

$$+ b_3(b_3 - 1) [a_1, a_2, a_3, b_1 + 2, b_2, b_3 - 2]$$

$$- 2a_1 b_3 [a_1 - 1, a_2, a_3 + 1, b_1 + 1, b_2, b_3 - 1] \quad (48)$$

$$S_{21}^2 : [\ldots] \mapsto a_2[a_2 - 1] [a_1 + 2, a_2 - 2, a_3, b_1, b_2, b_3]$$

$$+ b_1(b_1 - 1) [a_1, a_2, a_3, b_1 - 2, b_2 + 2, b_3]$$

$$- 2a_2 b_1 [a_1 + 1, a_2 - 1, a_3, b_1 - 1, b_2 + 1, b_3]$$

$$;$$
From this we can read of the matrix representation of the $S_{ij}^2$ on the vector space of bi-graded polynomials of bidegree $(\lambda_1, \lambda_2)$ in $\mathbb{R}$, where we choose the representatives $[a_1, a_2, a_3, b_1, b_2, b_3]$ with $a_3b_3 = 0$, i.e. we replace every $x_3y_3$ by $-x_1y_1 - x_2y_2$. We see immediately that $S_{12}^2$ and $S_{21}^2$ do not increase $a_3$ or $b_3$, so the matrix representation of $S_{12}^2$ and $S_{21}^2$ has at most 3 entries not equal to zero in each column. The other $S_{ij}$ have two summands which may increase either $a_3$ or $b_3$, so they have at most 5 non-zero entries in each column. Thus, the sum $\sum_{i \neq j} S_{ij}^2$ has at most $2 \cdot 3 + 4 \cdot 5 = 26$ non-zero entries in each column, regardless of the bi-degree $(\lambda_1, \lambda_2)$. Each of these entries has absolute value bounded by $\max \{\lambda_1, \lambda_2\}^2$.

So, it is possible to compute the Lipkin Hamiltonian numerically and get graphical representations as in the following figure:

![Graphical representation of the Lipkin Hamiltonian]

Here, the nearest neighbor statistics are drawn for the Lipkin Hamiltonian in the irreducible representation corresponding to $\lambda_1 = \lambda_2 = 8$, which is 728 dimensional.

## 7 Acknowledgments

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