Scalar potentials out of canonical quantum cosmology

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Using canonical quantization of a flat FRW cosmological model containing a real scalar field $\phi$ endowed with a scalar potential $V(\phi)$, we are able to obtain exact and semiclassical solutions of the so called Wheeler-DeWitt equation for a particular family of scalar potentials. Some features of the solutions and their classical limit are discussed.

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It is a common issue in Cosmology nowadays to make use of scalar fields $\phi$ as the responsible agents of some of the most intriguing aspects of our universe. Just to mention a few, we find scalar fields in inflaton (as seeds of the primordial perturbations for structure formation); in cold dark matter models of the formation of the actual cosmological structure, and also in dark energy models that intend to explain the current accelerated expansion of the universe.

The key feature for such a flexibility of scalar fields (spin-0 bosons) is the freedom one has to propose a scalar potential $V(\phi)$, which encodes in itself the (non gravitational) self-interactions among the scalar particles. The literature on scalar potentials is eminently vast, and most of the recent papers are aimed to explain the SnIa results that suggest the existence of dark energy. The hope is that the statistics of these vacua could resolve the smallness of the cosmological constant (the simplest candidate for dark energy).

Scalar fields also appear in the study of tachyon dynamics. For instance, in the unstable D-brane scenario, the scalar potential in the tachyon effective action around the minimum of the potential is of the form $V(\phi) = e^{-N^2(t)t}$. Currently, there has been a lot of interest in the study of tachyon driven cosmology.

On the other hand, scalar fields have also appeared within the so called (canonical) Quantum Cosmology (QC) formalism, which deals with a very early quantum epoch of the cosmos. Again, scalar fields act as matter sources, and then play an important role in determining the evolution of such an early universe.

QC means the quantization of minisuperspace models, in which the gravitational and matter variables have been reduced to a finite number of degrees of freedom. These models were extensively studied by means of Hamiltonian methods in the 1970s (for reviews see [21, 22]). It was first remarked by Kodama [28, 29], that solutions to the Wheeler-DeWitt equation (WDW) in the formulation of Arnowitt, Deser and Misner (ADM) are related to Ashtekar formalism (in the connection representation) by $\Psi_{ADM} = \Psi_{Ae^{\pm i\Phi}}$, where $\Phi$ is the homogeneous specialization of the generating functional of the canonical transformation from the ADM variables to Ashtekar’s. This function was calculated explicitly for the diagonal Bianchi type IX model by Kodama, who also found $\Psi_{A} = constant$ as solution. Since $\Phi$ is purely imaginary, for a certain factor ordering, one expects a solution of the form $\Psi = We^{\pm \phi}$.

Our aim in this paper is to determine which scalar potentials can arise as exact solutions to the WDW equation of QC, as well as which of them can be valid at the semiclassical level. For this we will use some of the ideas presented in the previous paragraphs to derive the WDW equation and find exact solutions to the quantum cosmological model. Though there are many solutions in principle, we will focus on those that may be relevant solutions for the early universe.

We begin by writing down the line element for a homogeneous and isotropic universe, the so called Friedmann-Robertson-Walker (FRW) metric, in the form

$$ds^2 = -N^2(t)dt^2 + a^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right],$$

where $a(t)$ is the scale factor, $N(t)$ is the lapse function, and $k$ is the curvature constant that takes the values 0, $+1$, $-1$, which correspond to a flat, a closed and an open universe, respectively. The effective action we are going to work on is

$$S_{tot} = S_g + S_{\phi} = \int dx^4 \sqrt{-g} \left[ R - 2\Lambda - \frac{\dot{\phi}^2}{2} + V(\phi) \right],$$

where $\phi$ is a scalar field endowed with a scalar potential $V(\phi)$, and $\Lambda$ is a cosmological constant [31]. The
Lagrangian for a flat FRW cosmological model is
\[
\mathcal{L} = \frac{1}{2} a \dot{\alpha}^2 - \frac{1}{2} \dot{\delta}^3 \Lambda N - \frac{a^3}{2} \dot{\phi}^2 + V(\phi) a^3 N, \tag{3}
\]
and then the canonical momenta are found to be
\[
P_a = \frac{\partial L}{\partial \dot{\alpha}} = \frac{a \dot{\alpha}}{N}, \quad \dot{a} = N P_a / a, \tag{4a}
\]
\[
P_\phi = \frac{\partial L}{\partial \dot{\phi}} = -\frac{a^3 \dot{\phi}}{N}, \quad \dot{\phi} = -\frac{N P_\phi}{a^3}. \tag{4b}
\]
We are now in position to write the corresponding Hamiltonian
\[
\mathcal{H} = \frac{1}{2} P_a^2 / a - \frac{1}{2} P_\phi^2 / a^3 - a^3 V(\phi, \Lambda), \tag{5}
\]
where we have written \( V(\phi, \Lambda) = 2V(\phi) - \Lambda \).

The WDW equation for this model is achieved by replacing \( P_\alpha \) by \(-i \partial_\alpha \) in Eq. (3) with \( q^\mu = (a, \phi) \). We now perform the change of variable \( a = e^{\alpha} \); hence the total Hamiltonian can be written as
\[
\mathcal{H} = e^{-3\alpha} / 2 \left[ -\frac{\partial^2}{\partial \alpha^2} + \frac{\partial}{\partial \alpha} + \frac{\partial^2}{\partial \phi^2} - e^{6\alpha} V(\phi, \Lambda) \right] = 0. \tag{6}
\]

Following the suggestion by Hartle and Hawking we do a semi-general factor ordering on \( e^{-3\alpha} \) and \( P_\alpha \), and then
\[
- e^{-(3-q)\alpha} \partial_\alpha e^{-q \alpha} \partial_\alpha = -e^{-3\alpha} \partial_\alpha^2 + q e^{-3\alpha} \partial_\alpha, \tag{7}
\]
where \( q \) is any real constant. Under this factor ordering the WDW reads
\[
\Box \Psi + Q \frac{\partial \Psi}{\partial \alpha} - e^{6\alpha} V(\phi, \Lambda) \Psi = 0, \tag{8}
\]
where \( Q = q + 1, \Psi \) is called the wave function of the universe, and \( \Box \equiv -\partial^2_\alpha + \partial^2_\phi \) is the two dimensional d’Alambertian operator.

Before going further, we would like to mention here the so called semiclassical limit of the WDW equation. This is achieved by taking \( \Psi = e^{-S} \), and imposing the usual WKB conditions on \( S \)
\[
\left( \frac{\partial S}{\partial \alpha} \right)^2 \gg \left| \frac{\partial^2 S}{\partial \alpha^2} \right|, \quad \left( \frac{\partial S}{\partial \phi} \right)^2 \gg \left| \frac{\partial^2 S}{\partial \phi^2} \right|. \tag{9}
\]
Hence, the WDW equation, under a particular factor ordering \( (Q = 0) \), becomes what is called the Einstein-Hamilton-Jacobi (EHJ) equation,
\[
(\Box S)^2 - U = 0. \tag{10}
\]

This equation is also obtained if we introduce the following transformation on the canonical momenta \( P_\alpha \to \frac{\partial S}{\partial \phi} \) in equation (8). In consequence, \( \mathcal{H} = 0 \) is equivalent to the known Friedmann equation, and Eqs. (10) provide the classical solutions of the Einstein-Klein-Gordon equations.

Eq. (8) resembles the equation of a damped massive wave equation, where the mass term is provided by the scalar potential. Thus, it is interesting to note that, for a free scalar field \( V(\phi) = 0 \) and \( Q = 0 \), the wave function of the universe consists of two wave solutions traveling along the directions \( \alpha \pm \phi \), which are indeed the classical trajectories on the \( \{ \alpha, \phi \} \) plane.

Taking the following ansatz for the wave function
\[
\Psi(\alpha, \phi) = W(\alpha, \phi)e^{-S(\alpha, \phi)} \tag{11}
\]
where \( S(\alpha, \phi) \) is now termed as the superpotential function, Eq. (8) can be splitted into the EHJ equation (10) for \( S \), and into the following equations
\[
W \left( \Box S + Q \frac{\partial S}{\partial \alpha} \right) + 2 \nabla W \cdot \nabla S = 0 \tag{12a}, \\
\Box W + Q \frac{\partial W}{\partial \alpha} = 0, \tag{12b}
\]
with \( \nabla W \cdot \nabla S \equiv - (\partial_\alpha W) (\partial_\alpha S) + (\partial_\phi W) (\partial_\phi S) \), \( (\nabla)^2 \equiv - (\partial_\alpha^2 + (\partial_\phi^2) \), and \( U(\alpha, \Lambda) = e^{6\alpha} V(\phi, \Lambda) \).

In order to find solutions of the WDW equation, we will choose to solve Eqs. (10) and (12a), whose solutions will have to comply with Eq. (12b), which will be our constraint equation.

Let us start with Eq. (10), which is an equation for the superpotential function only. If \( S(\alpha, \phi) = (1/\mu)e^{3\gamma} g(\phi) \), then Eq. (10) becomes an ordinary differential equation for \( g(\phi) \) in terms of the scalar potential as
\[
\left( \frac{dg}{d\phi} \right)^2 - 9g^2 = \mu^2 V(\phi, \Lambda). \tag{13}
\]

This last equation has several exact solutions, which can be generated in the following way. Let us consider that \( V(\phi, \Lambda) = g^2 F(g) \), where \( F(g) \) is an arbitrary function of its argument. Thus, Eq. (13) can be written in quadratures as
\[
\Delta \phi = \int \frac{d \ln g}{\sqrt{9 + \mu^2 F(g)}}. \tag{14}
\]

From Eq. (14) we can solve for \( g \) as a function of \( \phi \), and then find the corresponding scalar potential that leads to an exact solution of the EHJ. Some solutions for the scalar potential are shown in Table I.

Next, we assume that \( W = e^{[u(\alpha) + v(\phi)]} \) in Eq. (12a); after a bit of algebra, we obtain
\[
W = \exp \left\{ \frac{k}{2} \left[ \frac{\alpha}{3} + \int \frac{d\phi}{\partial_\phi (\ln g)} \right] + \frac{Q}{2} - \frac{\mu^2}{4} \int \frac{d[V(\phi)]}{(\partial_\phi g)^2} \right\}, \tag{15}
\]
where \( k \) is an arbitrary constant. One only needs to verify under which conditions solutions in Eqs. (13) and (15) comply with the constraint equation (12b), which takes
the following form

$$\partial_\phi^2 \Psi + \left( \partial_\phi \Psi \right)^2 - \frac{k^2 - 9Q^2}{36} = 0,$$
\[(16a)\]

$$\partial_\phi \Psi = \frac{k}{2} \frac{1}{\partial_\phi (\ln g)} - \frac{\mu^2}{4} \frac{\partial_\phi \left[ V(\phi) \right]}{\left( \partial_\phi g \right)^2}.$$  
\[(16b)\]

It is clear that the constraint equation is not easy to satisfy; but it can be done approximately as we shall show below.

For completeness, the classical solutions of the EHJ equation arising from Eq. \((14)\) are given by

$$\alpha = \frac{3}{3} + \int \frac{d\phi}{\partial_\phi (\ln g)} = \text{const.},$$
\[(17a)\]

$$-\mu \int \frac{d\phi}{\partial_\phi g} = \Delta t.$$  
\[(17b)\]

We will now focus our attention on the two simplest cases shown in Table 1. For \(F(g) = V_0\), for which the scalar potential is of the exponential form, the exact solution of the WDW equation reads

$$\Psi = \exp \left[ \frac{k}{2} \left( \frac{\alpha}{3} - \frac{\phi}{B} \right) + \frac{Q}{2} \alpha + \frac{\mu^2 V_0}{2B} \phi - \frac{1}{\mu} e^{-3\alpha - 2\beta} \right] \Psi_0,$$
\[(18a)\]

$$k = -3 \left[ 3 + B\sqrt{1 + Q^2/(\mu^2 V_0)} \right],$$
\[(18b)\]

where the last equation is the solution to the constraint \((12)\).

One needs a not increasing wave function in order to recover classical solutions at late times. Therefore, the wave function should be well behaved for \(|\alpha| \to \infty\) for fixed \(\phi\). These two conditions are accomplished if \(k > 0\); which in turn implies that \(\mu^2 V_0 > 0\). Such a wave function is shown in Fig. 1 where we see that it is different from zero only in a certain region of the \(\alpha, \phi\) plane. Moreover, the wave function has a very well defined boundary line, and this would tell us that the region for which \(\phi < (3/B)\alpha\) would not be allowed for the evolution of the universe.

It is interesting to note that the scalar potential of the form \(V \sim e^{-2B\phi}\), where \(B\) is an arbitrary parameter, is one of the most studied in the literature, for basic literature see \([1, 2, 3, 4, 12, 13, 22, 34]\) and references therein. Such an exponential potential, in a scalar field dominated universe, is inflationary if \(2B < \sqrt{7}\).
FIG. 2: Semiclassical wave function for a sinh scalar potential, which is the second solution shown in Table I. Here, $k = 0.5$, $\mu = 1$ and $Q = 0$. Note that the wave function is limited by two well-definite boundary lines; it vanishes in the region in which $S > 1$. However, there is a divergence in the line $\phi = 0$, in which solution (19) does not comply with the constraint equation (12b), see text for details.

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