QUANTISATION OF KIPPENHAHN’S CONJECTURE

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Abstract. Semidefinite programming optimises a linear objective function over a spectrahedron, and is one of the major advances of mathematical optimisation. Spectrahedra are described by linear pencils, which are linear matrix polynomials with hermitian matrix coefficients. Our focus will be on dimension-free linear pencils where the variables are permitted to be hermitian matrices. A major question on linear pencils, and matrix theory in general, is Kippenhahn’s Conjecture. The conjecture states that given a linear pencil $xH + yK$ if the hermitian matrices $H$ and $K$ generate the full matrix algebra, then the pencil must have at least one simple eigenvalue for some $x$ and $y$. The conjecture is known to be false, via a single counterexample due to Laffey. A dimension-free version of the conjecture, known as the Quantum Kippenhahn Theorem, has recently been proven true non-constructively. We present a novel family of counterexamples to Kippenhahn’s Conjecture, and use this family to construct concrete examples of the Quantum Kippenhahn Theorem.

1. Introduction

Semidefinite programming is one of the major advances in mathematical optimisation of the last thirty years. It is widely used in quantitative science, in such fields as control theory, computational finance, signal processing, and fluid dynamics, among many other applications [21] [8]. It involves the optimisation of a linear objective function of several variables with respect to constraints represented by linear matrix inequalities on those variables. In many cases, after translation in the space of variables, a clean way of representing such constraints is with a linear pencil and an associated linear matrix inequality:

Definition 1 (Linear pencil and linear matrix inequality). A linear pencil is an expression of the form

$$L(x_1, \ldots, x_n) = \sum_{i=1}^{n} A_i x_i$$

where the $A_i$ are hermitian matrices, the $x_i \in \mathbb{R}$ are variables. Linear pencils whose coefficients are hermitian matrices (not necessarily diagonal) give rise to linear matrix inequalities (LMIs). A linear matrix inequality (LMI) is an expression of the form

$$I + L(x_1, \ldots, x_n) \succeq 0,$$

where $L$ is a linear pencil and the relation $\succeq$ stands for ‘positive semi-definite’, meaning non-negative eigenvalues.

Linear pencils are ubiquitous in matrix theory and numerical analysis (e.g. the generalized eigenvalue problem), and they frequently appear in (real) algebraic geometry (cf. 

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The use of hermitian matrices (as opposed to arbitrary matrices) in the definition ensures convexity, and a full set of real eigenvalues simplify consideration of such inequalities. Inequality (2) defines a feasible region called a spectrahedron \([1]\). Spectrahedra are always convex and semialgebraic.

The boundary of a spectrahedron associated to a linear pencil \(L\) satisfies \(\text{Det}(I + L) = 0\). This boundary will be generically smooth, but may contain singularities such as sharp corners. The determinant of \(L\) vanishes at each of the singularities. A linear function on the space of variables in which the spectrahedron lies will have a global direction of greatest increase and will have flat level sets. This means that the optimisation of a linear function over the spectrahedron will often be found at one of the singularities. To see this, consider a generically smooth convex set, with some sharp extreme points, approaching a flat plane. In many orientations, it will collide first with one of its extreme points. It is therefore of interest to be able to identify singularities algebraically. However, simply looking for vanishing points of the gradient is not always sufficient, as the following example shows.

**Example 1** (Double circle).

\[
I + L = \begin{pmatrix}
1 - x & y \\
y & 1 + x \\
1 - y & x \\
x & 1 + y
\end{pmatrix} \succeq 0.
\]

This spectrahedron is a disc. Evaluating the determinant of the LMI will give us \(\text{Det}(I + L) = (1 - x^2 - y^2)^2\), and setting this to zero gives us the boundary. However the presence of the square in the determinant means that the gradient vanishes at every point on the boundary, falsely giving the impression of singularities where there are none. However, \(L\) is composed of two block diagonal submatrices, each of which determine the same feasible region. By recognizing this and selecting only one of these sub-matrices to define the LMI, we can define the same feasible region and correctly determine that there are no singularities on the boundary. In this instance, we were saved from purely algebraic singularities by \(L\) being block diagonal, but will this always be true?

Our work was motivated by the 1951 conjecture of Rudolf Kippenhahn \([6]\):

**Conjecture 1** (Kippenhahn \([6]\)). Let and \(H, K\) be hermitian \(2n \times 2n\) matrices, and let \(f = \text{det}(xH + yK + I) \in \mathbb{R}[x, y]\). Let \(A\) be the algebra generated by \(H\) and \(K\). If there exists \(k \in \mathbb{N}, k \geq 2\) and a \(g \in \mathbb{C}[x, y]\) such that \(f = g^k\) then there is some unitary matrix \(U\) such that \(U^* (xH + yK)U\) is block diagonal, and thus \(A \neq M_n(\mathbb{C})\).

Kippenhahn’s conjecture can also be illustrated geometrically. Given a linear pencil \(L = I + xH + yK\) as in Conjecture \([1]\) its determinant \(f = \text{det} L\) gives rise to the affine scheme \(\text{Spec} \mathbb{C}[x, y]/(f)\). If \(f = g^k\) with \(k \geq 2\) then this scheme is obviously nonreduced - see for example Chapter 5, Section 3.4 of \([16]\). This conjecture is now known to be false. Our objective is to extend the understanding of the counterexamples to this conjecture. Kippenhahn orginally gave a more general form of this conjecture where \(f\) is permitted to be a product of more than one irreducible polynomial, and the matrices need not be of even order. Kippenhahn’s conjecture linked the multiplicity of eigenvalues of a certain type of matrix polynomial to the algebra generated by the coefficients of that polynomial. In his paper \([6]\), Kippenhahn proved that his conjecture holds for \(n \leq 2\). Shapiro extended the validity range of the conjecture in a series of 1982 papers. In the first of these papers \([17]\) she demonstrated that if \(f\) has a linear factor of multiplicity greater than \(n/3\), then
the conjecture holds. This proves the more general form of the conjecture conjecture for the case of $3 \times 3$ matrices. The second paper [18] shows that if $f = g^{n/2}$ where $g$ is quadratic, then the conjecture holds. This, combined with [17], proves the conjecture for matrices of order 4 and 5. The final paper [19] showed that the conjecture holds if $f$ is a power of a cubic factor. This is sufficient to prove the conjecture for order 6 matrices, that is for $n = 3$ in the form we are interested in, where $f$ is a power of an irreducible polynomial, but not Kippenhahn’s orginal more general form. Buckley recently gave a proof of the same result as a corollary to a more general result about Weierstrass cubics [2].

In his 1983 paper [10], Laffey disproved the Conjecture for $n = 4$ with a single counterexample. In 1998 Li, Spitkovsky, and Shukla disproved Kippenhahn’s more general form of the conjecture for $n = 3$ by constructing a family of counterexamples (the LSS counterexample) of the form $f = \det(I + xH + yK) = g^2h$, where both $g$ and $h$ are quadratics [13].

In [11] we introduced a theorem which allows for the construction of a class of one-parameter families of counterexamples of Kippenhahn’s conjecture, for $n \geq 4$. Here we give a simplified version of that theorem:

**Theorem 1** (Simplified constructibility theorem [11]). Let $H$ and $K$ be $2n \times 2n$ hermitian matrices over $\mathbb{C}$. Then if, in the basis in which $K$ is diagonal, with any equal diagonal entries being consecutive,

1. $K$ has eigenvalues of multiplicity at most 2,
2. the $2 \times 2$ blocks of $H$ lying across the top row are all invertible,
3. there exist distinct $2 \times 2$ blocks $H_{1i}$ and $H_{1j}$ lying in the top row of $H$ so that $H_{1i}H_{1i}^*$ and $H_{1j}H_{1j}^*$ do not commute,

then the algebra generated by $H$ and $K$ is $M_{2n}(\mathbb{C})$.

1.1. **Main results.** By giving criteria for a pair of matrices $H$ and $K$ to generate the full matrix algebra, this theorem provides an avenue for constructing counterexamples to Kippenhahn’s Conjecture. The criteria are quite broad, and allow for various families of counterexamples. We will use Theorem [11] in Section 2 to introduce a novel counterexample family with different properties from the family introduced in [11]. Make the following definitions:

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The key properties of these $2 \times 2$ matrices are as follows:

$$\alpha^2 = \beta^2 = -U^2 = I_2,$$
$$\alpha U + U \alpha = 0 = \beta U + U \beta,$$
$$\alpha \beta - \beta \alpha = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$
Now let $n \geq 4$ be an integer, and define $q = \frac{(n-2)(n+2)}{2} + 1$. From the above matrices, build the following $2n \times 2n$ skew symmetric matrices:

$$A_{(b)} = \begin{pmatrix}
qU & \alpha & \alpha & \alpha & \alpha & \cdots & \alpha \\
-\alpha & (q+1)U & ba + \beta & ba & 0 & \cdots & 0 \\
-\alpha & -ba - \beta & (q+2)U & 0 & \cdots & 0 \\
-\alpha & -ba & 0 & (q+3)U & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-\alpha & -ba & 0 & 0 & \cdots & (q+n-1)U
\end{pmatrix}, \quad (3)$$

$$B = \begin{pmatrix}
U \\
U \\
\ddots \\
U
\end{pmatrix}, \quad (4)$$

where $b \in \mathbb{R}, b \neq 0$. We then define

$$H_{(b)} = A_{(b)}^2, \quad K = A_{(b)}B + BA_{(b)}, \quad (5)$$

giving rise to the linear pencil

$$L_{(b)}(x, y) = xH_{(b)} + yK, \quad (6)$$

where $x$ and $y$ are real and we have indicated dependence on the parameter $b$ using a subscript. Note that $B^2 = -I$, and consider the expression

$$xA_{(b)} + yB.$$ 

Since this is an even-order skew symmetric matrix for all values of $x$ and $y$, its eigenvalues will come in imaginary conjugate pairs. Squaring the expression we have

$$(xA_{(b)} + yB)^2 = x^2(A_{(b)})^2 + xy(A_{(b)}B + BA_{(b)}) + yB^2$$

$$= x^2H_{(b)} + xyK - y^2I,$$

which will have eigenvalues which are the squares of the eigenvalues of $A_{(b)} + yB$. Since $A_{(b)} + yB$ has imaginary conjugate pairs of eigenvalues, $x^2H_{(b)} + xyK - y^2I$ must have real negative eigenvalues of even multiplicity. Since adding a multiple of the identity only shifts eigenvalues by a constant factor, $x^2H_{(b)} + xyK$ must also have real negative eigenvalues of even multiplicity, and with a shift of variables we can say the same about $xH_{(b)} + yK$. This claim is rigorously proven as Lemma 1 in Section 2. Care needs to be taken for the case where $y = 0$. Evaluating $K$, we have

$$K = \begin{pmatrix}
-2qI_2 & -2(q+1)I_2 & \cdots & -2(q+n-1)I_2 \\
\end{pmatrix}. \quad (7)$$

This structure emerges from the properties of $\alpha$ and $\beta$ and the way in which they cancel out in the anti-commutator of $A_{(b)}$ and $B$. Notice how the non-zero $2 \times 2$ blocks of $K$ are simply the identity times the coefficients of the diagonal blocks of $A_{(b)}$, with a factor of $-2$. This ties in with the first condition of Theorem 1. In Section 2, we will show that $H_{(b)}$ and $K$ as specified satisfy the requirements of Theorem 1 and therefore span the full matrix algebra and violate Kippenhahn’s conjecture. We summarise these results into the following theorem:
Theorem 2 (New Counterexample Family). Let \( b \neq 0 \) be a real number, and let \( n \geq 4 \) be an integer. Then let \( 2n \times 2n \) skew-symmetric matrices \( A(b) \) and \( B \) be as defined in Equations (3) and (4). Let \( H(b) = (A(b))^2 \) and \( K = A(b)B + BA(b) \). Then the linear pencil \( L = xH(b) + yK \), where \( x, y \in \mathbb{R} \), has the following properties:

1. All of the eigenvalues of \( L(b) \) have even multiplicity,
2. \( H(b) \) and \( K \) generate the full matrix algebra \( M_{2n}(\mathbb{C}) \), for all \( b \neq 0 \) and for all \( x, y \in \mathbb{R} \). Therefore \( H(b) \) and \( K \) violate Kippenhahn’s Conjecture.

The first key property to note is that the diagonal entries of \( A(b) \) are all different multiples of \( U \). This ensures that \( K \) has eigenvalues of multiplicity no more than two. The coefficients based on the constant \( q \) have a double purpose, first purpose being to establish the proper structure of \( K \), and second purpose to control the eigenvalues of \( H(b) \) and \( K \) in a specific way. The precise meaning of this statement will be expanded upon in the latter part of this introduction, where we discuss quantisation.

The second key property of this family of counterexamples is presence of non-commuting blocks. In both cases, \( A(b) \), from which \( H(b) \) is built, is filled almost entirely off the diagonal with \( \alpha \), except for \( \beta \) which is placed at a specific location. To get an impression of what is happening here, consider the upper-left \( 8 \times 8 \) block of \( A(b) \):

\[
\begin{pmatrix}
qU & \alpha & \alpha & \alpha \\
-\alpha & (q + 1)U & b\alpha + \beta & b\alpha \\
-\alpha & -b\alpha - \beta & (q + 2)U & 0 \\
-\alpha & -b\alpha & 0 & (q + 3)U
\end{pmatrix}
\]

When \( A(b) \) is squared to give \( H \), observe what happens when the second row of blocks of \( A(b) \) is multiplied by the third column and fourth column respectively. In the first instance, the \( b\alpha \) term in the top row lines up with a zero and vanishes, and in the second instance it is an \( b\alpha + \beta \) term which vanishes. Simplify by using \( \alpha^2 = I_2 \), and we are left with non-commuting blocks in the second row of \( H(b) \). After symmetric permutation of rows and columns, all of the conditions for Theorem 1 are met.

1.2. Quantisation. Consider a coupled linear system, such as two electronic circuits linked by a feedback circuit. The behaviour of the coupled system can be studied independently of the dimension of the matrices describing each distinct part. Once dissipation or any sort of dampening is included, an inequality enters the picture. Such a situation can often be modelled by an LMI with the real variables replaced with non-commutative variables. Most often the non-commutative variables of interest are hermitian matrices, due to their mathematical richness and physical relevance. The term free analysis is used for the study of such LMIs; it refers to ‘dimension-free’. Free analysis is widely used in operator algebra, mathematical physics, and quantum information theory. See Chapter 8 of [1] for a survey of this material. Further information about free LMIs may be found in [3]. The process of replacing commutative variables with non-commutative variables is referred to as quantisation.

In Section 8 we investigate the application of quantisation to the Kippenhahn conjecture. Applied to a linear pencil such as in the Kippenhahn conjecture, a linear pencil of the form \( L = xH + yK \) gives rise to

\[
L(X, Y) = X \otimes H + Y \otimes K,
\]
where $X$ and $Y$ are hermitian matrices. In this dimension-free setting Kippenhahn’s conjecture has been proven to hold true as Corollary 5.6 of [7]:

**Theorem 3** (Quantum Kippenhan Theorem). If $A_1, ..., A_g$ generate $M_d(\mathbb{C})$ as an $\mathbb{C}$-algebra, then there exist $n \in \mathbb{N}$ and $X_1, ..., X_g \in M_n(\mathbb{C})$ such that

$$\dim \ker \left( I_{nd} + \sum_{i=1}^{g} X_i \otimes A_i \right) = 1.$$ 

The implication of this result is that given any counterexample $H$ and $K$ to Kippenhahn’s conjecture in the commutative setting, some hermitian matrices $X$ and $Y$ exist in the free setting for which $H$ and $K$ are no longer a counterexample, that is, $X \otimes H + Y \otimes K$ does not have a square characteristic polynomial. The proof of the theorem is not constructive in the sense that it proves that $X$ and $Y$ must exist such that the Kippenhahn conjecture holds true, but does not identify such $X$ and $Y$. We have been able to improve on this situation. For all of the known counterexamples to Kippenhahn’s Conjecture, including our own, we have explicitly identified $2 \times 2$ hermitian matrices $X$ and $Y$ which successfully quantise the counterexamples and restore Kippenhahn’s conjecture in a dimension-free setting.

**Theorem 4** (Counterexample Quantisation). Let $H(b)$ and $K$ be as in Theorem 2, to any order $n$, and let

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1/q^2 \\ 1/q^2 & 1 \end{pmatrix},$$

where $q = \frac{(n+2)(n-2)}{2} + 1$. Then $X \otimes H(b) + Y \otimes K$ has at least 6 simple eigenvalues at almost all parameter values $b$ and thus tightens the Quantum Kippenhahn Theorem.

Theorem 4 is the main result of Section 3 and will be proven there. We finish this paper in Sections 4 and 5 with a short discussion of the quantisation of Laffey’s counterexample and Li, Spitkovsky, and Shukla’s counterexample to the more general form of the conjecture. With these results, the theorems of Klep and Volčič in [7] are given concrete expression.

2. A NOVEL COUNTEREXAMPLE TO KIPPENHAHN’S CONJECTURE

In this section we will construct a novel family of counterexamples to Kippenhahn’s Conjecture as described in Theorem 2. This family will be similar in structure to that in constructed in [11], but will have different values along the diagonal, and a slightly different off-diagonal structure. Theorem 1 allows a significant amount of flexibility in the structure of the counterexamples it generates, and we have chosen the following structure so as to produce nice eigenvalue behaviour. This specifics of this will become clear in Section 3. First we will state and prove the double eigenvalue property referred to in the Introduction:

**Lemma 1** (Double eigenvalue lemma). Let $A$ and $B$ be skew-hermitian matrices of order $2n$, and let $B^2 = -I$. Define $H = A^2$ and $K = AB + BA$. Then, for all $x, y \in \mathbb{R}$, $xH + yK$ has eigenvalues only of even multiplicity.

**Proof.** Consider first $Ax + By$. Take some $x_0$ and $y_0$ in $\mathbb{R}$, where $x_0 \neq 0$. The eigenvalues of the skew-hermitian matrix $Ax_0 + By_0$ will be purely imaginary and will exist in conjugate pairs. Note that a given pair may appear more than once. Denote such pairs by $\pm i \lambda_k$, where $\lambda_k$ is real and $k$ ranges from 1 to $n$. 

Then the eigenvalues of \((Ax_0 + By_0)^2\) will be \(-\lambda_k^2\), obviously coming in pairs. Since the same pair of eigenvalues of \(Ax_0 + By_0\) may occur several times, we cannot say for sure that each \(-\lambda_k^2\) has multiplicity 2, but we can be sure that it has even multiplicity. Let \(v_k\) be an eigenvector of \(Ax_0 + By_0\) with eigenvalue \(i\lambda_k\), and \(w_k\) be an eigenvector of \(Ax_0 + By_0\) with eigenvalue \(-i\lambda_k\). Since \(v_k\) and \(w_k\) belong to different eigenspaces of \(Ax_0 + By_0\), the subspace span\(\{v, w\}\) which they generate is two-dimensional.

Then \((Ax_0 + By_0)^2v_k = -\lambda_k^2v_k\) and \((Ax_0 + By_0)^2w_k = -\lambda_k^2w_k\). But \(A^2 = H\), \(AB + BA = K\), and \(B^2 = -I_{2n}\), so we have

\[
(Ax_0 + By_0)^2 = Hx_0^2 + Kx_0y_0 - I_{2n}y_0^2, \\
(Hx_0^2 + Kx_0y_0)v_k = (y_0 - \lambda_k^2)v_k.
\]

Dividing through by \(x_0 \neq 0\) we have that

\[
(Hx_0 + Ky_0)v_k = \frac{1}{x_0}(y_0 - \lambda_k^2)v_k.
\]

Therefore, \(v_k\) is an eigenvector of \(Hx_0 + Ky_0\) with eigenvector \(\frac{1}{x_0}(y_0 - \lambda_k^2)\). Repeating this process for \(w_k\), we see that \(w_k\) is also an eigenvector of \(Hx_0 + Ky_0\) with eigenvector \(\frac{1}{x_0}(y_0 - \lambda_k^2)\). Therefore \(v_k\) and \(w_k\) span a two-dimensional eigenspace of \(Hx_0 + Ky_0\).

For the case where \(x_0 = 0\), we simply have \(Ky_0\), which clearly has paired eigenvalues because of the diagonal structure of \(K\). Therefore, for every \(x_0\) and \(y_0\) in \(\mathbb{R}\) \(Hx_0 + Ky_0\) has eigenvalues all of even multiplicity. \(\square\)

2.1. Definition of the counterexample family. Take an integer \(n \geq 4\), real number \(b \neq 0\) and let \(q = 1 + \frac{(n-2)(n+2)}{2}\). As before, set

\[
\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Now define \(2n \times 2n\) matrices \(A(b)\) as in Equation \((\text{3})\) and \(B = I_n \otimes U\).

The significance of the choice of \(q = 1 + \frac{(n-2)(n+2)}{2}\) will become clear when we set bounds on the eigenvalues of the system in Section \((\text{3})\), as will the reason for isolating \(q\) into a variable of its own rather than simply expressing the diagonal entries in terms of \(n\). Define symmetric matrices

\[
H(b) = (A(b))^2, K = A(b)B + BA(b).
\]

Lemma \((\text{1})\) allows us to conclude that \(xH(b) + yK\) has eigenvalues of all even multiplicities, and its characteristic polynomial is of the form \(f = q^2\). So we have met the first condition for Kippenhahn’s conjecture.

2.2. The matrix algebra generated. We will show that each of the three conditions of Theorem \((\text{1})\) applies to \(H(b)\) and \(K\).

(1) Recall \(K\) from Equation \((\text{7})\). Clearly the first condition for Theorem \((\text{1})\) is satisfied.

(2) Let us evaluate the \(2 \times 2\) blocks of the top row and the second row of \(H(b)\). First the top row:

\[
\begin{pmatrix} -q^2 - n & 0 \\ 0 & -q^2 - n \end{pmatrix}, \begin{pmatrix} -2b & -2 \\ 0 & -2b \end{pmatrix}, \begin{pmatrix} b & -1 \\ -3 & b \end{pmatrix}, \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -n - 1 \\ -n - 1 & 0 \end{pmatrix}.
\]
Each of these blocks is invertible for all permitted values of $b$, that is $b \neq 0$, except for $b = \pm 3, \pm \sqrt{3}$. Now consider the second row:

$$
\begin{pmatrix}
-2b & 0 \\
-2 & -2b
\end{pmatrix}, \begin{pmatrix}
-2b^2 - q^2 - 2q - 3 & 0 \\
0 & -2b^2 - q^2 - 2q - 3
\end{pmatrix}, \begin{pmatrix}
0 & -b \\
-b & -2
\end{pmatrix},
$$

$$
\begin{pmatrix}
-1 & -2b \\
-2b & -1
\end{pmatrix}, \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}, \ldots, \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}.
$$

Each of these blocks is invertible for all permitted values of $b$, except for $b = \pm \frac{1}{2}$.

Therefore for any allowed value of $b$, at least one of these rows has all invertible blocks. By simultaneous symmetric permutation of $H_{(b)}$ and $K$, we can place the invertible row at the top. Such simultaneous symmetric permutation will not affect the algebra generated by $H_{(b)}$ and $K$, nor will it affect Condition 1 on $K$. Therefore $H_{(b)}$ satisfies Condition 2 in general.

(3) Take the third and fourth blocks of the first row of $H_{(b)}$:

$$H_{13} = \begin{pmatrix}
 b & -1 \\
-3 & b
\end{pmatrix}, H_{14} = \begin{pmatrix}
 b & -3 \\
-3 & b
\end{pmatrix}.$$

Then the commutator of $H_{13}H_{13}^T$ and $H_{14}H_{14}^T$ is \( \begin{pmatrix} 0 & 48b \\ -48b & 0 \end{pmatrix} \) which must always be non-zero. The commutator of the products of the third and fourth blocks of the second row with their own transposes likewise evaluates to \( \begin{pmatrix} 0 & -16b \\ 16b & 0 \end{pmatrix} \), which must also be non-zero. So no matter which row we have placed at the top in step 2, Condition 3 is satisfied.

We can therefore conclude that $H_{(b)}$ and $K$ violate the Kippenhahn Conjecture.

3. Quantisation of the New Counterexample

As mentioned in the Introduction, quantisation of a linear pencil is the process of replacing the commutative variables $x_i$ with non-commutative variables $X_i$, typically hermitian matrices, to give an expression of the form $L(X) = \sum X_i \otimes A_i$.

Such quantised pencils arise particularly in coupled linear systems.

Quantising a linear pencil such as in the Kippenhahn conjecture, we have $L(X, Y) = X \otimes H + Y \otimes K$, where $X$ and $Y$ are hermitian matrices. In this setting Kippenhahn’s conjecture has been proven to hold true in an arbitrary number of variables as Corollary 5.6 of [7]:

**Theorem 5** (Quantum Kippenhahn Theorem). If $A_1, \ldots, A_g$ generate $M_d(F)$ as a $F$-algebra, then there exist $n \in \mathbb{N}$ and $X_1, \ldots, X_g \in M_n(F)$ such that

$$\operatorname{Dim Ker} \left( I_{nd} + \sum_{i=1}^{g} X_i \otimes A_i \right) = 1.$$

This Quantum Kippenhahn Theorem holds true for some hermitian $X$ and $Y$, but the proof does not identify such $X$ and $Y$. We have been able to improve on this situation. For the counterexample introduced in Section [2] we have explicitly identified $2 \times 2$ hermitian matrices $X$ and $Y$ which successfully quantise the counterexample and restore
Kippenhahn’s conjecture in a dimension-free setting. Our approach will be to note that the entries of \( L(b) \) vary smoothly with \( b \), and set \( b \) to zero. We will then use Gershgorin’s Disc Theorem and the Fundamental Symmetric Polynomials to show the desired result for \( b = 0 \). We will then use results from perturbation theory \([5]\) to extend this result to almost all values of \( b \).

To begin with, let us reconsider \( A(b) \) as stated in Equation (3). Notice how each element of \( A(b) \) is a polynomial function at most linear in \( b \), and recall that \( q \) is fixed depending on \( n \). Now set \( b = 0 \). It will be recalled that \( b = 0 \) is specifically excluded from the valid range of the counterexample, but we may still consider it as a way to inspect the behaviour of the eigenvalues of \( L(b) = X \otimes H(b) + Y \otimes K \). Recall that \( K \) does not have any dependence on \( b \).

With \( b \) set to zero, we have

\[
A(0) = \begin{pmatrix}
qU & \alpha & \alpha & \alpha & \cdots & \alpha \\
-\alpha & (q+1)U & \beta & 0 & & \\
-\alpha & -\beta & (q+2)U & 0 & & \\
-\alpha & 0 & 0 & (q+3)U & & \\
\vdots & & & & & \\
-\alpha & & & & & (q+n-1)U
\end{pmatrix}.
\]

Notice how each \( 2 \times 2 \) block is either purely diagonal (the \( \alpha \) blocks) or purely off-diagonal (the \( U \) and \( \beta \) blocks). Therefore, we can perform a symmetric permutation of the rows and columns of \( A(0) \), and likewise of \( B \), separating \( A(0) \) into two \( n \times n \) diagonal and off-diagonal blocks. Remember that \( A(0) \) is of dimension \( 2n \times 2n \). This permutation splits the elements of the \( \alpha \) blocks across the diagonal blocks and the elements of \( U \) and \( \beta \) across the off-diagonal blocks. This is simply a change of basis order, and by an abuse of notation we may continue to refer to the altered matrix as \( A(0) \):

\[
\begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
-1 & & & & \\
-1 & & & & \\
\vdots & & & & \\
-1 & & & & \\
q & 0 & 0 & \cdots & 0 \\
0 & (q+1) & 1 & & \\
0 & -1 & (q+2) & & \\
\vdots & & & & \\
0 & & & & (q+n-1)
\end{pmatrix}
\begin{pmatrix}
-q & 0 & 0 & \cdots & 0 \\
0 & -(q+1) & 1 & & \\
0 & -1 & -(q+2) & & \\
\vdots & & & & \\
0 & & & & -(q+n-1)
\end{pmatrix}.
\]

Now we can symmetrically swap the 1st and \((n+1)^{th}\) row, and likewise the 1st and \((n+1)^{th}\) column, to get

\[
\begin{pmatrix}
-q & 1 & 1 & \cdots & 1 \\
1 & (q+1) & 1 & & \\
1 & -1 & (q+2) & & \\
\vdots & & & & \\
1 & & & & (q+n-1)
\end{pmatrix}
\begin{pmatrix}
q & -1 & -1 & \cdots & -1 \\
-1 & -(q+1) & 1 & & \\
-1 & -1 & -(q+2) & & \\
\vdots & & & & \\
-1 & & & & -(q+n-1)
\end{pmatrix}.
\]
It is important to note that, with these symmetric permutations, the eigenvalues of $H_{(0)}$ and $K$ and any linear pencils thereof, as well as the algebra generated by $H_{(0)}$ and $K$, are not affected. Therefore we can freely adjust $H_{(0)}$ and $K$ to suit our purposes, so long as the permutations are symmetric and applied equally to both $H_{(0)}$ and $K$, or $A_{(0)}$ and $B$.

We can look at $A_{(0)}$ as being composed of four blocks

\[
A_{(0)} = \begin{pmatrix}
0 & A_1 \\
-A_1^T & 0
\end{pmatrix},
\]

and therefore we will have

\[
H_{(0)} = \begin{pmatrix}
-A_1 A_1^T & 0 \\
0 & -A_1^T A_1
\end{pmatrix} = \begin{pmatrix}
H_1 & 0 \\
0 & H_2
\end{pmatrix}.
\]

Likewise,

\[
K = \begin{pmatrix}
K_1 & 0 \\
0 & K_1
\end{pmatrix},
\]

\[
K_1 = \begin{pmatrix}
-2q & 0 & 0 & 0 \\
0 & -2(q+1) & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & -2(q+n-1)
\end{pmatrix}.
\]

This is the critical difference between the counterexample family introduce here as compared with the counterexample family of [11]. This counterexample family becomes block diagonal when the parameter $b$ is set to the forbidden value of zero, whereas the previous counterexample family of [11] did not. Notice how $A_1$ is almost symmetric. It differs from its transpose only in the $(2,3)$ and $(3,2)$ entries. Let us see what effect this has on $H_1$ and $H_2$. Let

\[
A_1 = \begin{pmatrix}
A_2 & P \\
P^T & Q
\end{pmatrix},
\]

where $Q$ is diagonal,

\[
A_2 = \begin{pmatrix}
q & -1 & -1 \\
-1 & -(q+1) & 1 \\
-1 & -1 & -(q+2)
\end{pmatrix},
\]

\[
P = \begin{pmatrix}
-1 & -1 \\
0 & \ldots & 0 \\
0 & 0
\end{pmatrix}.
\]

Let us now evaluate the two diagonal blocks of $H_{(0)}$:

\[
H_1 = -\begin{pmatrix}
A_2 A_2^T + PP^T & A_2 P + PQ \\
PP^T A_2^T + QQ^T & Q^2
\end{pmatrix},
\]

\[
H_2 = -\begin{pmatrix}
A_2^T A_2 + PP^T & A_2^T P + PQ \\
PP^T A_2 + QQ^T & Q^2
\end{pmatrix}.
\]

Observe that

\[
A_2 P = A_2^T P = \begin{pmatrix}
-q & -q \\
0 & \ldots & 0 \\
0 & 0
\end{pmatrix}.
\]
Therefore $H_1$ and $H_2$ differ only in the top left $3 \times 3$ block. We can evaluate them explicitly as

$$H_1 = \begin{pmatrix}
-q^2 - n + 1 & 0 & -3 & -3 & \ldots & -n + 1 \\
0 & -q^2 - 2q - 3 & 0 & -1 & -1 \\
-3 & 0 & -q^2 - 4q - 6 & -1 & -1 \\
-3 & -1 & -1 & -q^2 - 6q - 10 & -1 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-n + 1 & -1 & -1 & -1 & \ldots & -(q + n - 1)^2 - 1
\end{pmatrix},$$

$$H_2 = \begin{pmatrix}
-q^2 - n + 1 & -2 & -1 & -3 & \ldots & -n + 1 \\
-2 & -q^2 - 2q - 3 & -2 & -1 & -1 \\
-1 & -2 & -q^2 - 4q - 6 & -1 & -1 \\
-3 & -1 & -1 & -q^2 - 6q - 10 & -1 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-n + 1 & -1 & -1 & -1 & \ldots & -(q + n - 1)^2 - 1
\end{pmatrix}. $$

Notice how the $q$ terms cancel out everywhere except along the diagonal, and additionally how there is a difference of about $-2q$ between consecutive diagonal entries in both $H_1$ and $H_2$. We are going to make this observation precise, and use the diagonal entries to get the eigenvalues under control, using the following famous theorem of Gershgorin.

**Theorem 6** (Gershgorin’s Disc Theorem [4]). Let $M = (m_{ij}) \in M_n(\mathbb{C})$, and let

$$R_i(M) = \sum_{j \neq i} |m_{ij}|,$$

where $1 \leq i \leq n$, denote the deleted absolute row sums of $M$. Then all the eigenvalues of $M$ are located in the union of $n$ discs

$$\bigcup_{i=1}^n \{ z \in \mathbb{C} : |z - m_{ii}| \leq R_i(M) \}.$$

Furthermore, if a union of $k$ of these $n$ discs forms a connected region that is disjoint from all the remaining $n - k$ discs, then there are precisely $k$ eigenvalues of $M$ in this region. If all of the Gershgorin discs of $M$ are disjoint, then $M$ has only simple eigenvalues.

Since every matrix is similar to its transpose, the theorem may also be stated in terms of the deleted absolute column sums. For a non-hermitian matrix this set of column discs may provide a more accurate estimate. With hermitian matrices the two sets of sums are equal.

**Lemma 2.** $H_1$ and $H_2$ have only simple eigenvalues and are similar for all $n \geq 4$.

**Proof.** Let us examine the spacing of the diagonal entries of $H_1$ and $H_2$. Denote these diagonal entries by $D(t)$. The first three of these are

$$D(1) = -q^2 - n + 1, D(2) = -q^2 - 2q - 3, D(3) = -q^2 - 4q - 6,$$

and the remainder are of the form

$$D(t) = -(q + t - 1)^2 - 1,$$

for $4 \leq t \leq n$. Recall the definition of $q$:

$$q = 1 + \frac{(n - 2)(n + 2)}{2}.$$
We can evaluate
\[ D(1) - D(2) = -n + 2q + 4 = n^2 - n + 2 > 0. \]
Therefore we have \( D(1) > D(2). \) Now compare the remaining diagonal entries:
\[ D(2) - D(3) = 2q + 3 = n^2 + 1 > 0, \]
\[ D(3) - D(4) = 2q + 4 = n^2 + 2 > 0, \]
\[ D(t) - D(t + 1) = 2(q + t) - 1 = n^2 - 5 + t > 0, \]
where \( 4 \leq t \leq n - 1. \)
So the diagonal entries of \( H_1 \) and \( H_2 \) form a descending sequence
\[ D(1) > D(2) > \ldots > D(n), \]
and the difference between entries always increases. The smallest difference is \( D(1) - D(2), \) regardless of \( n. \)

Now let us examine the row sums of \( H_1 \) and \( H_2, \) as in Gershgorin’s theorem. It is seen that
\[ R_1(H_1) = R_1(H_2) = \sum_{i=1}^{n-1} i = \frac{1}{2}n(n - 1). \]
Likewise,
\[ R_2(H_1) = n - 3, \quad R_2(H_2) = n + 1, \]
\[ R_3(H_1) = R_3(H_2) = n, \quad R_t(H_1) = R_t(H_2) = n + t - 3, \quad 4 \leq t \leq n. \]
Remember that the row sums are absolute values. The largest of the row sums \( R(2), \ldots, R(n) \) is the last one, \( R(n) = 2n - 2. \) Let us compare this to the first row sum \( R(1): \)
\[ R(1) - R(n) = \frac{1}{2}n^2 - \frac{1}{2}n - 2n + 3 = \frac{1}{2}(n - 4)(n - 1) + 1 > 0 \quad \forall n \geq 4. \]
Therefore the radius of the first Gershgorin disc of \( H_1 \) and \( H_2, \) centered on \( D(1), \) is the largest of all the Gershgorin discs. Furthermore,
\[ D(1) - D(2) < D(t) - D(t + 1) \quad \forall 2 \leq t \leq n - 1. \]
This can be easily verified for \( n \geq 4, \) the allowed range of \( n, \) by referring to the separations of the \( \{D(t)\} \) shown above. We can see that
\[ \frac{1}{2}(D(1) - D(2)) = \frac{1}{2}(n^2 - n + 2) > \frac{1}{2}n(n - 1) = R(1). \]
Therefore the largest of the Gershgorin discs of both \( H_1 \) and \( H_2 \) has a radius less than half of the smallest distance between two consecutive diagonal entries, when treated as points on the real line. Therefore all Gershgorin discs of \( H_1 \) and \( H_2, \) considered as separate matrices, are disjoint. Theorem [6] tells us that \( H_1 \) and \( H_2 \) both therefore have only simple eigenvalues. However, \( H_{(b)} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \) is the square of a skew-symmetric matrix \( A_{(b)}. \)
Since skew-symmetric matrices have purely imaginary eigenvalues in complex-conjugate pairs, \( H_{(b)} \) must be negative definite with eigenvalues appearing with even multiplicity. But because \( H_{(b)} \) is block diagonal, its eigenvalues are simply the union of the eigenvalues of \( H_1 \) and \( H_2. \) We have just shown that \( H_1 \) and \( H_2 \) have only simple eigenvalues (of multiplicity 1). Therefore if \( H_1 \) and \( H_2 \) had eigenvalues \( \lambda \) and \( \rho \) respectively, with \( \lambda \neq \rho, \) then both \( \lambda \) and \( \rho \) would be simple eigenvalues of \( H_{(b)}. \) Since this is impossible, we must
conclude that $H_1$ and $H_2$ must have exactly the same eigenvalues, all simple. Thus $H_1$ and $H_2$ are similar.

□

Proposition 7. $H_1$ and $H_2$ are not trivially similar. That is, the similarity matrix $S$ relating $H_1$ and $H_2$ is not of the form

$$S = \left( \begin{array}{c} \tilde{S} \\ I_{n-3} \end{array} \right),$$

where $\tilde{S}$ is a similarity matrix relating the first principal $3 \times 3$ blocks of $H_1$ and $H_2$.

Proof. If two matrices are similar, they must have the same determinant. Compare the determinants of the upper-left $3 \times 3$ blocks of $H_1$ and $H_2$:

$$\text{Det} \begin{pmatrix} -q^2 - n + 1 & 0 & -3 \\ 0 & -q^2 - 2q - 3 & 0 \\ -3 & 0 & -q^2 - 4q - 6 \end{pmatrix} = -nq^4 - 6nq^3 - 17nq^2 - 24nq - 18n - q^6 - 6q^5 - 16q^4 - 18q^3 + 8q^2 + 42q + 45,$$

$$\text{Det} \begin{pmatrix} -q^2 - n + 1 & -2 & -1 \\ -2 & -q^2 - 2q - 3 & -2 \\ -1 & -2 & -q^2 - 4q - 6 \end{pmatrix} = -nq^4 - 6nq^3 - 17nq^2 - 24nq - 14n - q^6 - 6q^5 - 16q^4 - 18q^3 + 8q^2 + 42q + 33.$$

Taking the difference, we have $\text{Det}(H_1) - \text{Det}(H_2) = -4(n - 3)$, which is non-zero for all $n \geq 4$. Therefore the first principal $3 \times 3$ blocks of $H_1$ and $H_2$ are not similar and so no trivial similarity matrix $S = \text{diag}(\tilde{S}, I_{2n-3})$ can be constructed for $H_1$ and $H_2$. □

Before proceeding we introduce some notation and give a few preliminary lemmas.

Definition 2 (Submatrix notation). Given a square matrix $M$ of order $n$ and an indexing set $I \subseteq \{1, 2, ..., n\}$, we may denote the principal submatrix consisting of the rows and columns indexed by $I$ as $M(I)$.

Definition 3 (Anchor point). Given a submatrix $M(I)$ we may refer to the elements of $I$ and the anchor points of $M(I)$.

Example 2. Let

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 0 \end{pmatrix},$$

then

$$M\{1,3\} = \begin{pmatrix} 1 & 3 \\ 3 & 8 \end{pmatrix},$$

and has anchor points 1 and 3.

Definition 4 (Elementary symmetric polynomial of eigenvalues). Let $M$ be an $n \times n$ hermitian matrix, and let $m_1, ..., m_n$ be its eigenvalues. Let $k \leq n$ be a natural number, and define $K$ as the set of all indexing sets of natural numbers of the form $\{i_1, i_2, ..., i_k\}$ where $i_1 < i_2 < ... < i_k$. Then

$$e_k(M) = \sum_{I \in K} \prod_{i \in I} m_i$$

is the $k^{th}$ symmetric polynomial of eigenvalues of $M$. 

Example 3. Suppose $M$ has eigenvalues $m_1, m_2, m_3$. Then

$$e_2(M) = m_1 m_2 + m_1 m_3 + m_2 m_3.$$ 

Lemma 3. Take a hermitian matrix $M$ with eigenvalues $m_1, \ldots, m_n$. Let $k \leq n$ be a natural number, and define $K$ as the set of all indexing sets of natural numbers of the form $\{i_1, i_2, \ldots, i_k\}$ where $i_1 < i_2 < \ldots < i_k$. Then

$$e_k(M) = \sum_{I \in K} \det M\{I\}.$$ 

Proof. See Theorem 1.2.12 in [4], page 42. □

Lemma 4. The difference in determinant between the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \begin{pmatrix} a & e \\ c & f \end{pmatrix}$$

is $ef - bc$. Likewise, the difference in determinant between the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \begin{pmatrix} e & b \\ c & f \end{pmatrix}$$

is $ad - ef$.

Proof. Straightforward. □

Lemma 5. Let

$$P_1 = \begin{pmatrix} a & 0 & -3 \\ 0 & b & 0 \\ -3 & 0 & c \end{pmatrix}$$

and

$$P_2 = \begin{pmatrix} a & -2 & -1 \\ -2 & b & -2 \\ -1 & -2 & c \end{pmatrix}.$$ 

Then $e_2(P_1) = e_2(P_2)$.

Proof. Apply Lemma 4 to each of the $2 \times 2$ submatrices of $P_1$ and $P_2$. The differences are

$$\det(P_1\{1,2\}) - \det(P_2\{1,2\}) = 4,$$

$$\det(P_1\{1,3\}) - \det(P_2\{1,3\}) = -8,$$

$$\det(P_1\{2,3\}) - \det(P_2\{2,3\}) = 4,$$

the sum of which is zero. □

Now, let us define

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & q^{-2} \\ q^{-2} & 1 \end{pmatrix}.$$ 

We will now set $b = 0$ and examine the eigenvalues of

$$L(0) = X \otimes H(0) + Y \otimes K = \begin{pmatrix} H(0) & q^{-2}K \\ q^{-2}K & K \end{pmatrix}.$$ 

We can split $L(0)$ into two $2n \times 2n$ diagonal blocks using symmetric permutation:

$$L(0) = \begin{pmatrix} H_1 & q^{-2}K_1 \\ q^{-2}K_2 & H_1 \\ 0 & q^{-2}K_1 \\ 0 & q^{-2}K_2 \end{pmatrix} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}.$$
Here we are denoting the diagonal blocks of \( L_{(0)} \) by \( L_1 \) and \( L_2 \).

**Lemma 6.** \( L_1 \) and \( L_2 \) each have only simple eigenvalues.

**Proof.** We will deal with \( L_1 \) and \( L_2 \) one at a time.

The case of \( L_1 \): We will use Gershgorin’s Disc Theorem. First examine the discs due to \( H_1 \). Here are the diagonal entries again:

\[
D(1) = -q^2 - n + 1, \quad D(2) = -q^2 - 2q - 3, \\
D(3) = -q^2 - 4q - 6, \quad D(t) = -(q + t - 1)^2 - 1 \quad 4 \leq t \leq n.
\]

As in the proof of Lemma 2, the diagonal entries are in descending order and the smallest gap is between \( D(1) \) and \( D(2) \):

\[
D(1) - D(2) = n^2 - n + 2.
\]

The row sums are much as in the proof of Lemma 2, but they each pick up a single term from the presence of \( q^{-2}K \):

\[
R_1(L_1) = \frac{1}{2}n(n - 1) + \frac{2}{q}, \quad R_2(L_1) = n - 3 + \frac{2q + 1}{q^2}, \\
R_3(L_1) = n + \frac{2q + 2}{q^2}, \quad R_t(L_1) = n + t - 3 + \frac{2q + t - 1}{q^2}, \quad 4 \leq t \leq n.
\]

In the proof of Lemma 2 we established that of the row sums \( \{R_1(L_1), \ldots, R_n(L_1)\} \) it is \( R_n(L_1) \) that is largest. This is still true, since each of the extra terms \( \{2/q, (2q + 1)/q^2, \ldots, (2q + t - 1)/q^2, \ldots\} \) is larger than the previous one. Therefore let us compare \( R_1(L_1) \) and \( R_n(L_1) \):

\[
R_1(L_1) - R_n(L_1) = \frac{1}{2}n(n - 1) + \frac{2}{q} - 2n + 3 - \frac{2q + n - 1}{q^2} \\
= (n - 1) \left( \frac{n - 4}{2} - \frac{1}{q^2} \right) + 1.
\]

This expression increases monotonically with \( n \). At \( n = 4 \), it evaluates to

\[
R_1(L_1) - R_n(L_1) = 3 \left( -\frac{1}{17^2} \right) + 1 = \frac{286}{289} > 0.
\]

This difference can only increase as \( n \) grows larger, so we can say that for all \( n \geq 4 \), \( R_1(L_1) \) is the largest row sum. Now let us see how the Gershgorin disc associated with \( R_1(L_1) \) fits into the space between \( D(1) \) and \( D(2) \):

\[
R_1(L_1) = \frac{1}{2}n(n - 1) + \frac{2}{q}, \\
\frac{1}{2}(D(1) - D(2)) = \frac{1}{2}n(n - 1) + 1.
\]

But since \( q > 2 \) for all \( n \geq 4 \), we can see that \( R_1(L_1) < \frac{1}{2}(D(1) - D(2)) \) for all \( n \geq 4 \). We have already shown that every subsequent row sum is smaller than \( R_1(L_1) \) and that every pair of diagonal entries is farther apart. This means that none of the Gershgorin discs can make it halfway across the space between each pair of diagonal entries, and so all of the Gershgorin discs due to \( H_1 \) and \( q^{-2}K_1 \) are disjoint.

Now consider the lower right block of \( L_1 \) equal to \( K_1 \). The diagonal entries are now:

\[
D(1 + n) = -2q, \quad D(2 + n) = -2(q + 1).
\]
\[ D(3 + n) = -2(q + 2) \quad D(t + n) = -2(q + t - 1) \quad 4 \leq t \leq n. \]

The space between any two diagonal entries is exactly 2. The row sums come from the off-diagonal $q^{-2}K$ block:

\[
\begin{align*}
R_{n+1}(L_1) &= \frac{2}{q} & R_{n+2}(L_1) &= \frac{2(q + 1)}{q^2} \\
R_{n+3}(L_1) &= \frac{2(q + 2)}{q^2} & R_{n+t}(L_1) &= \frac{2(q + t - 1)}{q^2} \quad 4 + n \leq t \leq 2n.
\end{align*}
\]

The largest of these row sums is $R_{2n}(L_1)$:

\[
R_{2n}(L_1) = \frac{2(q + n - 1)}{q^2} < \frac{4((n - 2)(n + 2) + n + 1)}{(n - 2)^2(n + 2)^2} = \frac{4(n - 1)}{(n - 2)^2(n + 2)}.
\]

Substituting $n = 4$ into this expression, we evaluate it to be $R_{2n}(L_1) = 1/2$. The denominator of this expression is a cubic which increases monotonically in the range $n \geq 4$. Since the denominator is a cubic, once it is greater than the numerator, which is linear, and increasing monotonically, it will always be greater than the numerator. Therefore $R_{2n}(L_1) < 1$ for all $n \geq 4$, and so none of the Gershgorin discs centred on the diagonal entries $D(1 + n), ..., D(2n)$ can intersect with any of the other Gershgorin discs centred on those diagonal entries.

We have separated the Gershgorin disc into two groups, centered around $D(1), ..., D(2)$ and $D(1 + n), ..., D(2n)$, and shown that these groups are disjoint amongst themselves. It remains to show that the Gershgorin discs centred around $D(2n)$ and $D(1)$ do not overlap, since these are the elements of each group which approach each other the closest. The separation of these is

\[ D(2n) - D(1) = -2(q + n - 1) + q^2 + n - 1 = q^2 - 2q - (n + 1), \]

and the sum of their respective row sums is

\[
R_1(L_1) + R_{2n}(L_1) = \frac{1}{2} n(n - 1) + \frac{2}{q} + \frac{2(q + n - 1)}{q^2} = \frac{q^2 n(n + 1) + 4q + 2n - 2}{q^2}.
\]

The difference $D(2n) - D(1)$ is easily shown to be dominated by $q^2$ and to be positive for $n = 4$. Since $q$ grows as $n$ grows larger, we see that the combined radius is of the Gershgorin discs centred around $D(2n)$ and $D(1)$ is not large enough to cover the distance between them, for all $n \geq 4$.

Therefore all of the Gershgorin discs of $L_1$ are disjoint, and so $L_1$ has only simple eigenvalues.

The case for $L_2$: This is very much the same procedure as before, except there is a difference in the first three row sums $R_1(L_2), ..., R_3(L_2)$ due to the difference between $H_1$ and $H_2$. This difference does not affect the outcome, and the calculations are almost exactly the same so we will omit them here. \qed

We have shown that $L_1$ and $L_2$, the two diagonal components of $L_{(0)}$, have only simple eigenvalues. The eigenvalues of $L$ are the union of the eigenvalues of $L_1$ and $L_2$. Therefore, any repeated eigenvalues of $L$ must be eigenvalues which are shared between $L_1$ and $L_2$. The following theorem shows that there must be at least some eigenvalues which are not shared between $L_1$ and $L_2$, and which are therefore simple eigenvalues of $L_{(0)}$ as a whole.

**Theorem 8.** $L_{(0)}$ has at least six simple eigenvalues.
**Proof.** Recall that $L_{(0)} = \text{diag}(L_1, L_2)$. Denote the eigenvalues of $L_1$ by

$$
\lambda_1, \ldots, \lambda_{2n},
$$

and the eigenvalues of $L_2$ by

$$
\rho_1, \ldots, \rho_{2n}.
$$

If the eigenvalues were exactly the same, then all symmetric polynomials involving the eigenvalues would also be the same. We will compare the elementary symmetric polynomials of $L_1$ and $L_2$.

First symmetric polynomial of eigenvalues: this is simply the trace. $L_1$ and $L_2$ share the same trace, since $H_1$ differs from $H_2$ only in off-diagonal elements, and so we can conclude that

$$
e_1(L_1) = \sum_{i=1}^{2n} \lambda_i = \sum_{i=1}^{2n} \rho_i = e_1(L_2).
$$

Second symmetric polynomial of eigenvalues: the second symmetric polynomial of the eigenvalues of a matrix $M$ is equal to the sums of symmetric $2 \times 2$ minors of $M$:

$$
e_2(M) = \sum_{i<j} m(i, j),
$$

where $m(i, j)$ is the minor formed by deleting all rows and columns except numbers $i$ and $j$. Let us therefore compare such minors of $L_1$ and $L_2$. Keep in mind that $L_1$ and $L_2$ are the same everywhere except for the $3 \times 3$ block in the upper left:

$$
L_1 = \begin{pmatrix}
-q^2 - n + 1 & 0 & -3 & \cdots \\
0 & -q^2 - 2q - 3 & 0 & \\
-3 & 0 & -q^2 - 4q - 6 & \\
\vdots & \vdots & \ddots & \\
\end{pmatrix},
$$

$$
L_2 = \begin{pmatrix}
-q^2 - n + 1 & -2 & -1 & \cdots \\
-2 & -q^2 - 2q - 3 & -2 & \\
-1 & -2 & -q^2 - 4q - 6 & \\
\vdots & \vdots & \ddots & \\
\end{pmatrix}.
$$

Let us now examine the possible values of $l_1(i, j)$ and $l_2(i, j)$, the $2 \times 2$ minors of $L_1$ and $L_2$. Since $L_1$ and $L_2$ differ only in the first three rows and columns,

$$
l_1(i, j) = l_2(i, j) \quad \text{if } i, j > 3.
$$

Therefore when considering the difference between $e_2(L_1)$ and $e_2(L_2)$ we need to consider only those minors for which $i \leq 3$.

Consider now $i < 3, j \geq 3$. There is now one anchor point in the first three diagonal entries of $L_1$, and one anchor point elsewhere. Deleting all other rows and columns for the construction of the minor will invariably delete the off-diagonal entries of the first $3 \times 3$ block. Therefore $l_1(i, j) = l_2(i, j)$ for $i \leq 3, j > 3$.

We need therefore only consider those minors which are obtained from the top-left $3 \times 3$ block. We now have

$$
e_2(L_1) = \det \begin{pmatrix}
-q^2 - n + 1 & 0 & -3 \\
0 & -q^2 - 2q - 3 & 0 \\
\end{pmatrix} + \det \begin{pmatrix}
-q^2 - n + 1 & -3 \\
-3 & -q^2 - 4q - 6 & \\
\end{pmatrix}
$$

$$
+ \det \begin{pmatrix}
-q^2 - 4q - 6 & 0 \\
0 & -q^2 - 2q - 3 & \\
\end{pmatrix} + \text{ (Shared terms)},
$$

$$
e_2(L_2) = \det \begin{pmatrix}
-q^2 - n + 1 & -2 & -1 \\
-2 & -q^2 - 2q - 3 & \\
-1 & -1 & -q^2 - 4q - 6 & \\
\end{pmatrix}.
$$
where we have grouped the terms arising from minors anchored at diagonal entries outside the first three into a ‘shared terms’ group. Upon evaluating these three determinants, we will get terms in $q$, and three constant terms. In both cases these constant terms sum to $-9$, and so we have

$$e_2(L_1) = -9 + \text{(Shared terms)} = e_2(L_2).$$

Therefore the second symmetric polynomial of eigenvalues is equal for both $L_1$ and $L_2$.

Third symmetric polynomial of eigenvalues: we do not require this one for our purposes, so for brevity it will be skipped. Using similar reasoning to the case of the second symmetric polynomial, it can be shown that the third symmetric polynomial of eigenvalues is indeed the same for both $L_1$ and $L_2$.

Fourth symmetric polynomial of eigenvalues: now we are choosing a sequence of indices $i < j < k < m$ and using these to construct the symmetric $4 \times 4$ minors of $L_1$ and $L_2$. Many of these minors will be shared between the two matrices, and we can avoid these when considering the difference in the overall sums. The possible choices of $i, j, k, m$ fall into categories, which we will list and consider in turn. First, split $L_1$ and $L_2$ into sectors, as follows:

$$L_t = \begin{pmatrix} P & Q \\ Q & R \end{pmatrix}.$$

Here $P$ represents the first $3 \times 3$ block of $H_1$ and $H_2$, which is the only point of difference between $L_1$ and $L_2$. To save space, use the following notation for the diagonal entries of $P$:

$$P_1 = -q^2 - n + 1, \quad P_2 = -q^2 - 2q - 3, \quad P_3 = -q^2 - 4q - 6.$$

$Q$ represents the rest of $H_1$ and $H_2$. $R$ represents the rest of $L_1$ and $L_2$, consisting of $K$ in the bottom right corner, and the two off-diagonal copies of $q^{-2}K$. We will now look through the possibilities for constructing $4 \times 4$ minors, and identify those which have the potential to be different between $L_1$ and $L_2$.

Case 1: $i, j, k, m$ are all drawn from $Q$ and $R$.

Since $L_1$ and $L_2$ are identical in these sectors, minors which result from the choices of $i, j, k, m$ from these two sectors alone can be disregarded as ‘shared terms’ when comparing $e_4(L_1)$ and $e_4(L_2)$. The conclusion here is that at least one of the anchor points must be drawn from $P$, a fact we will refer to again.

Case 2: $i, j, k, m$ are all drawn from $P$ and $Q$.

Since $P$ and $Q$ together make up $H_1$ or $H_2$, choosing $i, j, k, m$ from these sectors gives a sum of minors that is equal to $e_4(H_1)$ or $e_4(H_2)$. But we already know that $H_1$ and $H_2$ have the same eigenvalues, and so $e_4(H_1) = e_4(H_2)$. Therefore minors which result from the choices of $i, j, k, m$ from these two sectors alone can be disregarded as shared terms. Therefore at least one of the anchor points must be drawn from $R$.

Case 3: $i$ drawn from $P$, and $j, k, m$ are drawn from $Q$ and $R$.

In this case none of the off-diagonal entries in $P$ are drawn into the sub-matrix. Since these are the only point of difference between $L_1$ and $L_2$, the resulting minors are shared between $e_4(L_1)$ and $e_4(L_2)$. Therefore at least two anchor points must come from $P$.

Case 4: $i, j$ from $P$, and $k, m$ from $R$. 

$P$ and $Q$ together make up $H_1$ or $H_2$, choosing $i, j, k, m$ from these sectors gives a sum of minors that is equal to $e_4(H_1)$ or $e_4(H_2)$. But we already know that $H_1$ and $H_2$ have the same eigenvalues, and so $e_4(H_1) = e_4(H_2)$. Therefore minors which result from the choices of $i, j, k, m$ from these two sectors alone can be disregarded as shared terms. Therefore at least one of the anchor points must be drawn from $R$. 

The factor of $-9$ is obtained from the constant terms by considering the difference in the overall sums.
Each choice of \( k \) and \( m \), giving anchor points \( K_k \) and \( K_m \) from \( R \) will result in \( 2 \times 2 \) minors drawn from the following submatrix:

\[
\begin{pmatrix}
  P_1 & (0 \text{ or } -2) & (-3 \text{ or } -1) & r_k \\
  (0 \text{ or } -2) & P_2 & (0 \text{ or } -2) & r_m \\
  (-3 \text{ or } -1) & (0 \text{ or } -2) & P_3 & K_k \\
  r_1 & r_2 & K_m & M
\end{pmatrix},
\]

where \( r_1 \) and \( r_2 \) represent the first three rows of the columns in corresponding to \( k \) and \( m \).

There is limited freedom in the entries of \( r_k \) and \( r_m \). There can be at most one non-zero entry in each one, equal to \( q^{-2}K_k \) or \( q^{-2}K_m \) respectively, if \( k \) or \( m \leq 3 + n \), and the any nonzero entry in \( r_m \) must be lower in the column than a corresponding non-zero entry in \( r_k \). This means that it is possible, depending on the choice of \( k \) and \( m \) for there to be a non-zero entry in \( r_k \) and only zero entries in \( r_m \), but not vice-versa. All of this follows because the \( R \) sector is made up entirely of multiples of the matrix \( K_1 \). Examine each of the cases in turn. Firstly, the case for all zeros:

\[
\begin{pmatrix}
  P_1 & (0 \text{ or } -2) & (-3 \text{ or } -1) & 0 \\
  (0 \text{ or } -2) & P_2 & (0 \text{ or } -2) & 0 \\
  (-3 \text{ or } -1) & (0 \text{ or } -2) & P_3 & K_k \\
  0 & 0 & 0 & K_m
\end{pmatrix}.
\]

Since this matrix is block diagonal, extracting the \( 4 \times 4 \) minors as required will lead to

\[K_kK_m(2 \times 2 \text{ sum of minors obtained from } P)\].

But we already know from the analysis of \( e_2 \) that these minors will be the same for \( L_1 \) and \( L_2 \), and so the \( 4 \times 4 \) minors are the same between \( L_1 \) and \( L_2 \) in this case.

Now the case for non-zero entries in both columns. Without loss of generality, we can assume the following arrangement:

\[
\begin{pmatrix}
  P_1 & (0 \text{ or } -2) & (-3 \text{ or } -1) & q^{-2}K_k \\
  (0 \text{ or } -2) & P_2 & (0 \text{ or } -2) & 0 \\
  (-3 \text{ or } -1) & (0 \text{ or } -2) & P_3 & q^{-2}K_m \\
  q^{-2}K_k & 0 & q^{-2}K_m & K_m
\end{pmatrix}.
\]

There are three choices of \( 4 \times 4 \) minors in keeping with our requirements, and their sum is:

\[
\begin{vmatrix}
  P_1 & (0 \text{ or } -2) & q^{-2}K_k & 0 \\
  (0 \text{ or } -2) & P_2 & 0 & q^{-2}K_m \\
  q^{-2}K_k & 0 & K_k & K_m \\
  0 & q^{-2}K_m & 0 & q^{-2}K_m
\end{vmatrix} + \begin{vmatrix}
  (0 \text{ or } -2) & P_2 & 0 & q^{-2}K_m \\
  (0 \text{ or } -2) & P_3 & 0 & 0 \\
  0 & q^{-2}K_m & 0 & K_k \\
  q^{-2}K_k & 0 & K_k & K_m
\end{vmatrix} + \begin{vmatrix}
  P_1 & (0 \text{ or } -2) & (0 \text{ or } -2) & 0 \\
  q^{-2}K_k & 0 & q^{-2}K_m & 0 \\
  0 & 0 & 0 & 0 \\
  (0 \text{ or } -2) & P_3 & 0 & 0
\end{vmatrix}.
\]

With a symmetric permutation, we can make the off-diagonal blocks of each the second minor symmetric, and the cost of reversing the order of the \( K_k \) and \( K_m \). This will not change the value of the minor. Now, we will have a situation where all of the off-diagonal blocks commute with the bottom right block, and so we can use the following identity of block matrices

\[
\begin{vmatrix}
  M & N \\
  E & F
\end{vmatrix} = |MF - NE|,
\]
for $E$ and $F$ commuting. The product of all of the off diagonal blocks will be a diagonal matrix, and so we will end up with an expression of the form

$$
\begin{pmatrix}
\text{terms} & K_k(0 \text{ or } -2) \\
K_m(0 \text{ or } -2) & \text{terms}
\end{pmatrix}
\begin{pmatrix}
\text{terms} & K_m(0 \text{ or } -2) \\
K_k(0 \text{ or } -2) & \text{terms}
\end{pmatrix}

+ \begin{pmatrix}
\text{terms} & K_k(-3 \text{ or } -1) \\
K_m(-3 \text{ or } -1) & \text{terms}
\end{pmatrix}.
$$

The only room for a difference to emerge in this expression between $L_1$ and $L_2$ is in the off diagonal terms. For $L_1$ these become

$$-9K_kK_m$$

and for $L_2$ these become

$$K_kK_m(-4 - 4 - 1) = -9K_kK_m,$$

and so the terms are the same for both $L_1$ and $L_2$. This is a common feature of what we are doing here. Many expressions reduce down to the same set of $2 \times 2$ minors coming from $P$, with the same useful equality property emerging from the off-diagonals. The remaining two possibilities for non-zero entries in both columns of $R$ work out in exactly the same way, and will not be repeated.

We are left with the possibility of a non-zero entry in the first column of $R$ and zeros in the second. Without loss of generality:

$$
\begin{pmatrix}
P_1 & (0 \text{ or } -2) & (-3 \text{ or } -1) & q^{-2}K_k & 0 \\
(0 \text{ or } -2) & P_2 & (0 \text{ or } -2) & 0 & 0 \\
(-3 \text{ or } -1) & 0 & P_3 & 0 & 0 \\
q^{-2}K_k & 0 & 0 & K_k & K_m
\end{pmatrix}.
$$

Same procedure as in the previous step:

$$
\begin{pmatrix}
P_1 & (0 \text{ or } -2) & q^{-2}K_k & 0 \\
(0 \text{ or } -2) & P_2 & 0 & 0 \\
q^{-2}K_k & 0 & K_k & K_m \\
(-3 \text{ or } -1) & P_3 & 0 & 0 \\
q^{-2}K_k & 0 & K_k & K_m
\end{pmatrix}

+ \begin{pmatrix}
\end{pmatrix}.
$$

again leading to an expression of the form

$$
\begin{pmatrix}
\text{terms} & K_k(0 \text{ or } -2) \\
K_m(0 \text{ or } -2) & \text{terms}
\end{pmatrix}
\begin{pmatrix}
\text{terms} & K_m(0 \text{ or } -2) \\
K_k(0 \text{ or } -2) & \text{terms}
\end{pmatrix}

+ \begin{pmatrix}
\text{terms} & K_k(-3 \text{ or } -1) \\
K_m(-3 \text{ or } -1) & \text{terms}
\end{pmatrix}.
$$

The same conclusion follows, and we see that there can be no difference between the minors of $L_1$ and $L_2$ constructed in this fashion. Therefore we can conclude that either the anchor points are of the form $i, j \in P$, $k \in Q$, $m \in R$, or of the form $i, j, k \in P$, $m \in R$.

Case 5: $i, j$ from $P$, $k$ from $P$ or $Q$, and $m$ from $R$.

Since $m$ is now definitely from $R$, we must have that $m > n$. To simplify the remaining steps, we will split the indexing of anchor points into indices $1, \ldots, n$ for $i, j, k \in P$ and $Q$, and separate indices $1, \ldots, n$ for $m$ in $R$. This amounts to redefining $m$ by $m \leftarrow m - n$. We
will proceed by fixing \( m \) and evaluating the possible minors from all other valid choices of \( i, j \) and \( k \). We will go through every \( m \) from 1 to \( n \) (using the new indexing system) and collect the sum of the differences. First consider the case where \( m > 3 \). Consider the minor formed by \( i, j, k \in P \):

\[
\begin{vmatrix}
-q^2 - n + 1 & 0 \text{ or } -2 & (-3 \text{ or } -1) \\
(0 \text{ or } -2) & -q^2 - 2q - 3 & 0 \text{ or } -2 \\
(-3 \text{ or } -1) & 0 \text{ or } -2 & -q^2 - 4q - 6
\end{vmatrix}
= 0.
\]

Since \( m > 3 \), there are only zeros in the final row and column. Remembering that \( q = (1/2)(n - 2)(n + 2) + 1 \), we can directly evaluate this term, and we find that the difference due to this term is

\[
\Delta(1, 2, 3, m) = -4(n - 3)(-2(q + m - 1)) = 4(n + 2)(n - 2) + 8m.
\]

Now suppose that we are taking \( k \in Q \). We shall fix \( k \), and evaluate each of \( \Delta(1, 2, k, m) \), \( \Delta(1, 3, k, m) \), and \( \Delta(2, 3, k, m) \), and sum over \( k \). Each minor will be of the form

\[
\begin{vmatrix}
2 \times 2 \text{ submatrices} & (-k + 1 \text{ or } -1) & 0 \\
\text{of } P & -1 & 0 \\
(-k + 1 \text{ or } -1) & -1 & -(q + k - 1)^2 & c \\
0 & 0 & c & -2(q + m - 1)
\end{vmatrix}
\]

where the top left block represents the three \( 2 \times 2 \) submatrices of \( P \), and \( c \) can either be 0 if \( m - k \neq 0 \), or \( -2q^{-2}(q + m - 1) \) if \( m - k = 0 \). This non-zero entry occurs when the row (column) originating from \( k \) lines up with a non-zero entry of the top right block of \( L_1 \) (or \( L_2 \)). Since \( m > 3 \), such lining up cannot occur from any of the rows (columns) due to \( i \) or \( j \), and so the remaining two entries in the fourth row and column must be zero. For clarity, let us express the differences as follows:

\[
\Delta(1, 2, k, m) = \begin{vmatrix}
p_1 & 0 & 1 - k & 0 \\
0 & p_2 & -1 & 0 \\
1 - k & -1 & t & c \\
0 & 0 & c & s
\end{vmatrix} - \begin{vmatrix}
p_1 - 2 & 1 - k & 0 \\
-2 & p_2 & -1 & 0 \\
1 - k & -1 & t & c \\
0 & 0 & c & s
\end{vmatrix},
\]

\[
\Delta(1, 3, k, m) = \begin{vmatrix}
p_1 - 3 & -1 & 1 - k & 0 \\
-3 & p_3 & -1 & 0 \\
1 - k & -1 & t & c \\
0 & 0 & c & s
\end{vmatrix} - \begin{vmatrix}
p_1 - 1 & 1 - k & 0 \\
-1 & p_3 & -1 & 0 \\
1 - k & -1 & t & c \\
0 & 0 & c & s
\end{vmatrix},
\]

\[
\Delta(2, 3, k, m) = \begin{vmatrix}
p_2 & 0 & -1 & 0 \\
0 & p_2 & -1 & 0 \\
-1 & -1 & t & c \\
0 & 0 & c & s
\end{vmatrix} - \begin{vmatrix}
p_2 - 2 & -1 & 0 \\
-2 & p_2 & -1 & 0 \\
-1 & -1 & t & c \\
0 & 0 & c & s
\end{vmatrix},
\]

where \( s = -2(q + m - 1) \), \( t = -(q + k - 1)^2 \), and \( p_1, p_2, p_3 \) are the diagonal entries of \( P \). In the Laplace expansion of these determinants, only those terms which involve the entries at the (1,2) and (2,1) positions will differ between these two expansions. All others will be shared and will cancel out. Therefore by using Lemma 4 and subtracting matching terms in the Laplacian expansions we can evaluate

\[
\Delta(1, 2, k, m) = -4 \left( c^2 - s(k + t - 1) \right)
\]

\[
\Delta(1, 3, k, m) = 4 \left( 2c^2 - s(k + 2t - 1) \right)
\]

\[
\Delta(2, 3, k, m) = 4 \left( -c^2 + st + s \right),
\]
which sums to $4s = -8(q + m - 1)$. The contribution to $\Delta(i, j, k, m)$ for $i, j \in P$, $k \in Q$, and $m \in R, m > 3$ for a fixed $m$ and fixed $k$ is $-8(q + m - 1)$. Summing this over $k \in Q$, we simply multiply by $(n - 3)$ to get

$$\sum_{k>3} \Delta(i, j, k, m) = -8(n - 3)(q + m - 1) = -4(n + 2)(n - 2) - 8m.$$ 

But this is equal to the negative of $\Delta(1, 2, 3, m)$ which we obtained earlier, cancelling out. Keep in mind that this sum is for a fixed $m$. So we have

$$\sum_{m>3} \Delta(i, j, k, m) = 0.$$ 

Now we shall turn our attention to the remaining case, which is $m \leq 3$. Again fix $k$, and consider $m = 1$. First, consider $\Delta(1, 2, 3, 1)$:

$$\Delta(1, 2, 3, 1) = \begin{vmatrix} 
-q^2 - n + 1 & 0 & -3 & -2q^{-1} \\
0 & -q^2 - 2q - 3 & 0 & 0 \\
-3 & 0 & -q^2 - 4q - 6 & 0 \\
-q^2 - n + 1 & -2 & -q^2 - 2q - 3 & -2q^{-1} \\
-2 & -q^2 - 2q - 3 & -2 & 0 \\
-1 & -2 & -q^2 - 4q - 6 & 0 \\
-2q^{-1} & 0 & 0 & -2q 
\end{vmatrix},$$

which, repeatedly using Lemma 4 and expanding only those terms which do not cancel out, gives us

$$\Delta(1, 2, 3, 1) = \frac{8((n - 3)q^3 - 2)}{q^2}.$$ (8)

Likewise, by direct calculation,

$$\Delta(1, 2, 3, 2) = \frac{8(q + 1)((n - 3)q^4 + 4q + 4)}{q^4},$$ (9)

$$\Delta(1, 2, 3, 3) = \frac{8(q + 2)((n - 3)q^4 - 2q - 4)}{q^4}. $$ (10)

Summing these, we have that

$$\sum_{m=1}^{3} \Delta(1, 2, 3, m) = 8 \left( 3n(q + 1) - \frac{4}{q^4} \right).$$ (11)

Now let us examine the case for a fixed $k > 3$ and a fixed $m < 3$. Start with $m = 1$, and use Lemma 4 to evaluate the differences:

$$\Delta(1, 2, k, 1) = \begin{vmatrix} 
p_1 & 0 & 1 - k & c \\
p_2 & -1 & 0 & 0 \\
1 - k & -1 & t & 0 \\
c & 0 & 0 & s 
\end{vmatrix} - \begin{vmatrix} 
p_1 & -2 & 1 - k & c \\
-2 & p_2 & -1 & 0 \\
1 - k & -1 & t & 0 \\
c & 0 & 0 & s 
\end{vmatrix} = 4s(-1 + k + t),$$

$$\Delta(1, 3, k, 1) = \begin{vmatrix} 
p_1 & -3 & 1 - k & c \\
-3 & p_3 & -1 & 0 \\
1 - k & -1 & t & 0 \\
c & 0 & 0 & s 
\end{vmatrix} - \begin{vmatrix} 
p_1 & -1 & 1 - k & c \\
-1 & p_3 & -1 & 0 \\
1 - k & -1 & t & 0 \\
c & 0 & 0 & s 
\end{vmatrix} = -4s(-1 + k + 2t),$$
\[ \Delta(2, 3, k, 1) = \begin{vmatrix} p_2 & 0 & -1 & 0 \\ 0 & p_3 & -1 & 0 \\ -1 & -1 & t & 0 \\ 0 & 0 & 0 & s \end{vmatrix} - \begin{vmatrix} p_2 & -2 & -1 & 0 \\ -2 & p_3 & -1 & 0 \\ -1 & -1 & t & 0 \\ 0 & 0 & 0 & s \end{vmatrix} = 4s(1 + t), \]

where \( s = -2q, c = -2/q, t = -(q + k - 1)^2 \), and \( p_1, p_2, p_3 \) are the diagonal entries of \( P \). The sum of these differences is

\[ \Delta(i, j, k, 1) = 4s = -8q \] (12)

for fixed \( k > 3 \), and hence

\[ \sum_{k>3} \Delta(i, j, k, 1) = -(n - 3)8q \] (13)

Similar calculations give us

\[ \sum_{k>3} \Delta(i, j, k, 2) = -(n - 3)8(q + 1), \] (14)

\[ \sum_{k>3} \Delta(i, j, k, 3) = -(n - 3)8(q + 2). \] (15)

for \( k > 3 \). Now, adding each of these to Equations (8), (9), (10), we find that

\[ \sum \Delta(i, j, k, m) = \frac{32(q + 1)^2}{q^4} - \frac{16(q + 2)^2}{q^4} - \frac{16}{q^2} = -\frac{32}{q^4} \neq 0 \]

We have shown that the difference of the fourth symmetric polynomial of eigenvalues of \( L_1 \) and \( L_2 \) is non-zero, and so therefore \( L_1 \) and \( L_2 \) must have different eigenvalues. Therefore, \( L(0) \) has at least some simple eigenvalues. We will now place lower bounds on how many eigenvalues must differ between \( L_1 \) and \( L_2 \), and therefore a lower bound on how many simple eigenvalues \( L(0) \) must have.

Suppose that the eigenvalues of \( L_1 \) and \( L_2 \) are the same except for one eigenvalue. Denote the eigenvalues of \( L_1 \) by

\[ \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \]

and the eigenvalues of \( L_2 \) by

\[ \{\lambda_1 + t, \lambda_2, \ldots, \lambda_n\}, \]

where \( t \neq 0 \). But then, the trace of \( L_1 \) cannot equal the trace of \( L_2 \), which we know it must since \( e_1(L_1) = e_1(L_2) \). Therefore \( L_1 \) and \( L_2 \) must differ in at least two eigenvalues, such that the sum of eigenvalues is the same for both:

\[ L_1 : \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \]

\[ L_2 : \{\lambda_1 + t, \lambda_2 - t, \lambda_3, \ldots, \lambda_n\}, \]
where \( t \neq 0 \). Now let us evaluate \( e_2(L_1) \) and \( e_2(L_2) \):

\[
e_2(L_1) = \lambda_1 \lambda_2 + (\lambda_1 + \lambda_2) \sum_{i=3}^{n} \lambda_i + \text{shared terms}
\]

\[
e_2(L_2) = (\lambda_1 + t)(\lambda_2 - t) + (\lambda_1 + t) \sum_{i=3}^{n} \lambda_i + (\lambda_2 - t) \sum_{i=3}^{n} \lambda_i + \text{shared terms}
\]

\[
= (\lambda_1 + t)(\lambda_2 - t) + (\lambda_1 + \lambda_2) \sum_{i=3}^{n} \lambda_i + \text{shared terms}
\]

We know that \( e_2(L_1) = e_2(L_2) \), and so

\[
\lambda_1 \lambda_2 = (\lambda_1 + t)(\lambda_2 - t) = \lambda_1 \lambda_2 + t(\lambda_2 - \lambda_1) - t^2,
\]

leading to

\[
t(\lambda_2 - \lambda_1 - t) = 0.
\]

Since \( t \neq 0 \), we have that \( t = \lambda_2 - \lambda_1 \). But for this value of \( t \), we have that \( \lambda_1 + t = \lambda_2 \), and \( \lambda_2 + t = \lambda_1 \), which would mean that the eigenvalues of \( L_2 \) are the same as the eigenvalues of \( L_1 \), which is a contradiction. Therefore \( L_1 \) and \( L_2 \) must differ in at least three eigenvalues. There may be more eigenvalues not in common, but three is a lower bound. Therefore, since the eigenvalues of \( L_0 \) are the union of the eigenvalues of \( L_1 \) and \( L_2 \), we can finally conclude that \( L_0 \) has at least six simple eigenvalues.

Having shown the desired condition of some simple eigenvalues at a specific value of the parameter \( b \) for the Kippenhahn counterexample \( L_{(b)} \), we now wish to extend this condition to cover as many values of \( b \) as possible. We will use results from perturbation theory found in Chapter 2 of [5] and Chapter 5 of [9].

**Theorem 9.** \( L_{(b)} \) has some simple eigenvalues at almost all \( b \).

**Proof.** Consider the characteristic polynomial of \( L_{(b)} \):

\[
\text{Det}(L_{(b)} - \lambda I) = 0.
\]

As stated on page 63 of [5], this is an algebraic equation in \( \lambda \) of degree \( 2n \), the order of \( L_{(b)} \), with coefficients which are holomorphic in \( b \). It follows (see page 119 of [9], or page 64 of [5]) that the roots of (16) are branches of analytic functions of \( b \) with only algebraic singularities. This implies that the number of distinct eigenvalues of \( L_{(b)} \) is a constant independent of \( b \) except at some exceptional points where the analytic functions representing each root of (16) meet. Since these eigenvalue functions are analytic and susceptible to a power series expansion, there can only be a finite number of such crossing points in any compact interval of \( \mathbb{R} \), and only a countable number overall. At an exceptional point, the number of distinct eigenvalues can only decrease, never increase.

We already know that \( L_{(0)} \) has at least six simple eigenvalues, and according to the above reasoning these eigenvalues can only collide at a countable number of values of \( b \). Therefore \( L_{(b)} \) has some simple eigenvalues at almost every \( b \in \mathbb{R} \), that is at all except a measure zero set of values of \( b \).

This result establishes Theorem 4 from the Introduction. We have shown that the second of our counterexamples can be quantised for all orders for almost all values of the parameter \( b \). This is an improvement on the situation with the first counterexample, for which we were only able to show this result for orders 8 and 10. The tradeoff is that we have not shown that every eigenvalue is simple for almost all \( b \), but only a subset of the
eigenvalues. Regardless, this constitutes a quantisation of the Kippenhahn conjecture at all orders.

4. QUANTISATION OF LI, SPITKOVSKY AND SHUKLA’S COUNTEREXAMPLE

In this section we will give a general description of the process of quantisating the counterexample of Li, Spitkovsky, and Shukla (the LSS counterexample) to the more general form of Kippenhahn’s Conjecture. First we will establish a preliminary lemma.

Lemma 7. Let \( M = \begin{pmatrix} P & Q \\ Q & 0 \end{pmatrix} \) be a matrix over a field \( F \), where \( P \) and \( Q \) are symmetric and invertible. Then for any eigenvalue \( \lambda \) of \( M \) of multiplicity \( m \), the auxiliary matrix \( \tilde{M}(\lambda) \) of \( M \), defined as

\[ \tilde{M}(\lambda) = P + \frac{1}{\lambda} Q^2 \]

has \( \lambda \) as an eigenvalue.

If \( m > 1 \), then \( \lambda \) must also have multiplicity greater than 1 with respect to \( \tilde{M}(\lambda) \).

Proof. Let \( \begin{pmatrix} u \\ v \end{pmatrix} \) be an eigenvector of \( M \), with eigenvalue \( \lambda \):

\[ M \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Pu + Qv \\ Qu \end{pmatrix} = \left( \begin{array}{c} \lambda u \\ \lambda v \end{array} \right). \]

Since \( \det(M) = \det(Q)^2 \neq 0 \), we know that \( M \) is invertible and \( \lambda \neq 0 \). Comparing components, we see that

\[ v = \frac{1}{\lambda} Qu. \]

We then have

\[ \tilde{M}(\lambda) u = Pu + \frac{1}{\lambda} Q^2 u = \lambda u. \]

So \( u \) is an eigenvector of the auxiliary matrix \( \tilde{M}(\lambda) \) with eigenvalue \( \lambda \). Note that \( u \neq 0 \) because \( u \) is an eigenvector. Suppose now that \( \lambda \) appeared with multiplicity greater than one as an eigenvalue of \( M \). Then there must be some \( \lambda \)-eigenvector \( \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \neq k \begin{pmatrix} u \\ v \end{pmatrix} \) for any \( k \in F \). By the above reasoning, both \( u \) and \( u_2 \) are eigenvectors of \( \tilde{M}(\lambda) \) with eigenvalue \( \lambda \). Suppose that \( u_2 = ku \). Then \( v_2 = (k/\lambda) Qu \), as above, and so we would have \( \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = k \begin{pmatrix} u \\ v \end{pmatrix} \), which is not possible. Therefore we must conclude that \( u_2 \) is not a multiple of \( u \). Therefore \( \lambda \) must be an eigenvalue of \( \tilde{M}(\lambda) \) of multiplicity at least two, thus proving the result. \( \square \)

The auxiliary matrix (17) will be of use in this section, and in Section 5 where we quantise Laffey’s seminal counterexample [10].

In their paper [13], Li, Spitkovsky, and Shukla described the following \( 6 \times 6 \) counterexample to the weak form of Kippenhahn’s conjecture, where the characteristic polynomial need not be square. Take a matrix \( A \) defined by

\[
A = \begin{pmatrix}
0 & x & 0 & cy & 0 & 0 \\
0 & 0 & y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -cx & 0 & \sqrt{1-c^2}\xi & 0 \\
0 & 0 & 0 & 0 & 0 & \eta \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]
where \( x, y, \xi, \eta, c > 0 \), and \( x^2 + y^2 = \xi^2 + \eta^2 = 1, c < 1/2 \). Then define \( H = \Lambda + \Lambda^T \) and \( K \) via \( 2A = H + iK \):

\[
H = \begin{pmatrix}
0 & x & 0 & cy & 0 & 0 \\
x & 0 & y & 0 & 0 & 0 \\
y & 0 & 0 & -cx & 0 & 0 \\
cy & -cx & 0 & \sqrt{1 - c^2\xi} & 0 & \eta \\
0 & 0 & \sqrt{1 - c^2\xi} & 0 & \eta & 0 \\
0 & 0 & 0 & \eta & 0 & 0
\end{pmatrix},
\]

\[
K = \begin{pmatrix}
0 & -ix & 0 & -icy & 0 & 0 \\
ix & 0 & -iy & 0 & 0 & 0 \\
y & 0 & -icx & 0 & 0 & 0 \\
icy & 0 & icx & 0 & -i\sqrt{1 - c^2\xi} & 0 \\
0 & 0 & i\sqrt{1 - c^2\xi} & 0 & -i\eta & 0 \\
0 & 0 & 0 & i\eta & 0 & 0
\end{pmatrix}.
\]

We will now quantise this counterexample. Define hermitian \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \), and consider \( L(s) = sX \otimes H + Y \otimes K \), where \( s \in \mathbb{R}, s > 1 \). We find that

\[
L(s) = \begin{pmatrix}
0 & M(s) \\
M(s)^T & 0
\end{pmatrix}, \quad \text{where } M(s) = \\
\begin{pmatrix}
0 & (s+1)x & 0 & c(s+1)y & 0 & 0 \\
(s-1)x & 0 & (s+1)y & 0 & 0 & 0 \\
0 & (s-1)y & 0 & -c(s-1)x & 0 & 0 \\
(c(s-1)y & 0 & -c(s+1)x & 0 & \sqrt{1 - c^2\xi(s-1)} & 0 \\
0 & 0 & 0 & \sqrt{1 - c^2\xi(s+1)} & 0 & \eta(s-1) \\
0 & 0 & 0 & 0 & \eta(s-1) & 0
\end{pmatrix}.
\]

The eigenvalues of \( L(s) \) will be plus/minus the eigenvalues of \( M(s)M(s)^T \), so by showing that values exist of \( s \) for which \( M(s)M(s)^T \) has simple eigenvalues, we will show the same for \( L(s) \). Firstly, \( \text{Det}(M(s)) = -c^2\eta^2(s-1)^3(s+1)^3(s^2 + y^2)^2 \), which for the allowed values of \( c, \eta \), and \( s \) is non-zero.

Our approach is to evaluate the discriminant of \( M(s) \) for \( s = 2, 3, 4 \), and show that for any values of the parameters of the counterexample, at least one of these values of \( s \) gives a non-zero discriminant. Therefore there is always some \( sX \) and \( Y \) for which \( M \) and thus \( L \) have a non-zero discriminant, and therefore only simple eigenvalues. The process involves lengthy calculation of characteristic polynomials and the discriminants thereof, performed using Mathematica, and a Gröbner basis calculation, performed using Macaulay2. The code for these calculations is included in Section 7.

### 5. Quantisation of Laffey’s Counterexample

Here is the counterexample given by Laffey in [10]:

\[
H = \begin{pmatrix}
-122 & 0 & 12 & 18 & -30 & 18 & 26 & 10 \\
0 & -122 & -6 & -12 & -16 & -28 & 20 & -16 \\
12 & -6 & -218 & 0 & 44 & 8 & 24 & 12 \\
18 & -12 & 0 & -218 & -2 & -34 & -10 & 22 \\
-30 & -16 & 44 & -2 & -216 & 0 & -12 & -8 \\
18 & -28 & 8 & -34 & 0 & -216 & -8 & 36 \\
26 & 20 & 24 & -10 & -12 & -8 & -120 & 0 \\
10 & -16 & 12 & 22 & -8 & 36 & 0 & -120
\end{pmatrix}.
\]
In particular, $\lambda$ must have only simple eigenvalues. Each of the elements of $S$ must conclude that $\lambda$ is a constant in the context of the auxiliary matrix, each of the these polynomials is a fixed constant. By Lemma 7, we know that $\lambda$ is an eigenvalue of $\tilde{L}(\lambda)$, and our aim is to determine the multiplicity of $\lambda$ with respect to $\tilde{L}(\lambda)$. We can take the commutator of $\tilde{L}(\lambda)$ with a skew symmetric matrix $S = \{s_{ij}\}$. This commutator has 36 distinct elements, a sample of which is listed here:

1. $-4\lambda(6s_{1,3} + 9s_{1,4} - 15s_{1,5} + 9s_{1,6} + 13s_{1,7} + 5s_{1,8})$,
2. $2(8\lambda s_{1,2} - 22\lambda s_{1,4} + \lambda s_{1,4} + 47\lambda s_{1,5} + 6\lambda s_{1,7} + 4\lambda s_{1,8} + 6\lambda s_{3,5} + 9\lambda s_{4,5} - 9\lambda s_{5,6} - 13\lambda s_{5,7} - 5\lambda s_{5,8} - 24s_{1,5})$,
3. $2\lambda(9s_{1,2} + 48s_{2,4} + 8s_{2,5} + 17s_{2,6} + 5s_{2,7} - 11s_{2,8} - 3s_{3,4} + 8s_{4,5} + 14s_{4,6} - 10s_{4,7} + 8s_{4,8})$,
4. $2(9\lambda s_{1,2} - 4\lambda s_{2,3} + 17\lambda s_{2,4} + 47\lambda s_{2,6} + 4\lambda s_{2,7} - 18\lambda s_{2,8} - 3\lambda s_{3,6} - 6\lambda s_{4,6} - 8\lambda s_{5,6} - 10\lambda s_{5,7} + 8\lambda s_{6,8} - 24s_{2,6})$,
5. $4\lambda(3s_{1,2} + 24s_{2,3} - 11s_{2,5} - 2s_{2,6} - 6s_{2,7} - 3s_{3,4} + 4s_{3,5} + 7s_{3,6} - 5s_{3,7} + 4s_{3,8})$,
6. $2(5\lambda s_{1,4} + 9\lambda s_{1,8} - 8\lambda s_{2,4} - 6\lambda s_{2,8} + 6\lambda s_{3,4} + 4\lambda s_{4,5} - 18\lambda s_{4,6} - 49\lambda s_{4,8} - 8\lambda s_{5,8} - 17\lambda s_{6,8} - 5\lambda s_{7,8} - 24s_{4,8})$.

Since $\lambda$ is a constant in the context of the auxiliary matrix, each of the these polynomials is linear in the $s_{ij}$, and so taking a Gröbner basis using Mathematica is immediately possible. This leads to the following Gröbner basis:

$$\lambda s_{7,8}, \lambda s_{6,8}, \lambda s_{6,7}, \lambda s_{5,8}, \lambda s_{5,7}, \lambda s_{5,6}, s_{4,8}, s_{4,7}, s_{4,6}, s_{4,5}, s_{3,8}, s_{3,7}, s_{3,6}, s_{3,5},$$

$$\lambda s_{3,4}, s_{2,8}, s_{2,7}, s_{2,6}, s_{2,5}, \lambda s_{2,4}, \lambda s_{2,3}, s_{1,8}, s_{1,7}, s_{1,6}, s_{1,5}, \lambda s_{1,4}, \lambda s_{1,3}, \lambda s_{1,2}.$$
6. Final remarks on quantisation

While Theorem 3 fixes the disproven Kippenhahn conjecture, the obtained quantities and thus the bounds on the size \( n \) of \( X \) and \( Y \) needed are exponential in the size \( d \) of \( A \) and \( B \). These bounds are likely far from optimal, a view suggested by the small size of the \( X \) and \( Y \) which we have found for the Kippenhahn conjecture counterexamples. We believe that \( n = d \) should suffice. At the same time it would be interesting to find matrices \( A \) and \( B \) which generate \( M_d(\mathbb{C}) \), such that for generic \( 2 \times 2 \) hermitian \( X \) and \( Y \), \( \text{Det}(I + A \otimes X + B \otimes Y) \) is a square. We conjecture that the smallest such \( A \) and \( B \) have to be of size at least 10.

7. Appendix: Computer algebra system code

Here is the code which was used to perform the calculations of Sections 4 and 5.

7.1. Code for quantising Li, Spitkovsky, and Shukla’s counterexample. Mathematica and Macaulay2 were used for this set of calculations. This section first presents the Mathematica code, which provides input for Macaulay2, and then the Macaulay2 code.

(*Generates a skew symmetric matrix of chosen size *)

skewSymbol[order_] :=
  Module[{row = {}, matrix = {}, columnCount = {}, j = 0, k = 0,
    increment = 1, symIncrement = 0, incrementRow},
    Print["\n\n"];
    While[increment < order,
      (* Produces a strictly upper triangular matrix *)
      row = Table[Subscript[s, increment, k], {k, increment + 1, order}];
      PrependTo[row, 0];
      AppendTo[matrix, row];
      symIncrement = order;
      increment ++;
    ];
    (* Adds a final zero row *)
    AppendTo[matrix, {0}];
    (* Scans up the columns of the new upper triangular matrix, and fills in the lower triangle. *)
    While[symIncrement > 1,
      columnCount = Reverse[Range[symIncrement - 1]];
      Scan[PrependTo[
        matrix[[symIncrement]], -matrix[[#]][[symIncrement - # + 1]]] &,
        columnCount];
      symIncrement --;
    ];
(*A check that the output really is symmetric, then return *)
If[Transpose[matrix] == -matrix, Return[matrix];
    Print["Internal error, aborting"];
    matrix = IdentityMatrix[order]; Return[matrix];
];

(* Shorthand for the matrix commutator *)
comm[matrix1_, matrix2_] := Module[{},
    Return[Simplify[matrix1 . matrix2 - matrix2 . matrix1]];
];

A = ( {
    {0, x, 0, c y, 0, 0},
    {0, 0, y, 0, 0, 0},
    {0, 0, 0, 0, 0, 0},
    {0, 0, -c x, 0, \[Xi] Sqrt[1 - c^2], 0},
    {0, 0, 0, 0, \[Eta]},
    {0, 0, 0, 0, 0, 0}
} );

H = A + Transpose[A];
K = Simplify[-I (2 A - H)];
Print["Here we have A, where x, y, \[Xi], \[Eta], c > 0, and 
         x^2 + y^2 = \[Xi]^2 + \[Eta]^2 = 1, c < 1/2"];
Print[MatrixForm[A]]; Print["Define H = A + A^T"];
Print[MatrixForm[H]]; Print["And K via 2A = H + i K"];
Print[MatrixForm[K]]; 

Print["Now define hermitian X"];
X = ( {
    {0, 1},
    {1, 0}
} );
Print[MatrixForm[X]]; Print["and Y"];

Y = ( {
    {0, 0},
    {0, 1}
} );
{0, I},
{-I, 0}
};
Print[MatrixForm[Y]];

Print[ "and L = X\[CircleTimes]H + Y\[CircleTimes]K, which
is composed of complex conjugate blocks M and M^*, where M
is given by"];
Print["Eigenvalues of L will be +/- sqrt[eigenvalues of M M^t"];

M = L[[1 ;; 6, 7 ;; 12]]; 
Print[M // mf];
Print["The determinant of M is shown below, and is non-zero
for our purposes:"]; Print[Det[M] // Factor];

Print["We will evaluate M M^t for various values of s."];

Print["\n"];
Print["Now try s = 2. M M^t ="];
M2 = (M.Transpose[M]) /. {s -> 2} // sf;
Print[M2 // mf];

Print["Rearrange:"]; 
M2 = Transpose[M2[{{1, 3, 5, 2, 4, 6}}][{{1, 3, 5, 2, 4, 6}}]]; Print[M2 // mf];

Print["Its characteristic polynomial, c2:"]; c2 = CharacteristicPolynomial[M2, t] // Factor;
Print[c2];

Print["Now we take the discriminant d2 of c2, and simplify
with what we know about x, y, [Eta], and [Xi]. We will
arrange the factors of p2, q2, and r2 of d2 in a column,
and deal with each in turn:"];

d2 = Discriminant[c2, t] // sf // Factor;

d2 = sf[d2 /. {x^k_ :> Sqrt[1 - y^2]^k, [Eta]^l_ ->
Sqrt[1 - [Xi]^2]^l}] // Factor;

bads2 = Drop[Map[First, FactorList[d2]], 6];
Print["p2:"];
p2 = bads2[[1]];  
Print[p2];  
Print["q2:"];
q2 = bads2[[2]];  
Print[bads2[[2]]];  
Print["r2:"];
r2 = bads2[[3]];  
Print[bads2[[3]]];

Print["\n"];

Print["Let's look for zeros of q2 in the domain of interest. First we will change variables in q2 as 
\text{c}^k \text{RuleDelayed} \text{c}^{(k/2)}, \text{y}^l \text{RuleDelayed} \text{a}^{(l/2)}, \text{\Xi}^m \text{RuleDelayed} \text{b}^{(m/2)}"]; 
q2 = q2 /. {c^ k_ :> c^( k /2) , y^ l_ :> a^(l /2) , \Xi^ m_ :> b^( m /2)};
Print[q2];

Print["Now use the Reduce command to look for zeros of this reduced q2 in the domain of interest:"];
Print[Reduce[{q2 <= 0, a > 0, b > 0, c > 0, 1 > a, 1 > b, 1 > c}, {a, b, c}]];  
Print["No such zeros exist"];

Print["\n"];

Print["Now we shall look for zeros of r2 in the domain of interest. First we will change variables in r2 as 
\text{c}^k \text{RuleDelayed} \text{c}^{(k/2)}, \text{y}^l \text{RuleDelayed} \text{a}^{(l/2)}, \text{\Xi}^m \text{RuleDelayed} \text{b}^{(m/2)}"]; 
r2 = r2 /. {c^ k_ :> c^(k/2), y^ l_ :> a^(l/2), \Xi^ m_ :> b^(m/2)};
Print[r2];

Print["Now use the Reduce command to look for zeros of this reduced r2 in the domain of interest:"];
Print[Reduce[\{r2 <= 0, a > 0, b > 0, c > 0, 1 > a, 1 > b, 1 > c\}, \{a, b, c\}]];  
Print["No such zeros exist"];

Print["\n"];  
Print["The first factor, p2, will be dealt with at the end."];

Print["\n"];  
Print["Now try s = 3. M M^t ="];  
M3 = (M.Transpose[M]) /. \{s -> 3\} // sf;  
Print[M3 // mf];

Print["Rearrange:"];  
M3 = Transpose[M3[\{1, 3, 5, 2, 4, 6\}]]\{1, 3, 5, 2, 4, 6\};

Print[M3 // mf];

Print["Its characteristic polynomial, c3:"];  
c3 = CharacteristicPolynomial[M3, t] // Factor;

Print[c3];

Print["Now we take the disrimnant d3 of c3, and simplify with what we know about x, y, \[Eta], and \[Xi]. We will arrange the factors of p3, q3, and r3 of d3 in a column, and deal with each in turn:"];  
d3 = Discriminant[c3, t] // sf // Factor;

d3 = sf[d3 /. \{x^k_ :> Sqrt[1 - y^2]^k, \[Eta]^l_ -> Sqrt[1 - \[Xi]^2]^l\_} // Factor;

bads3 = Drop[Map[First, FactorList[d3]], 6];

Print["p3:"];  
p3 = bads3[[1]];  
Print[p3];

Print["q3:"];  
q3 = bads3[[2]];  
Print[q3];

Print["r3:"];  
r3 = bads3[[3]];

Print[r3];
Print["\n"];

Print["Let's look for zeros of q3 in the domain of interest. First we will change variables in q3 as c^k \[RuleDelayed\]c^(k/2), y^l \[RuleDelayed\]a^(l/2), \[Xi]^m \[RuleDelayed\]b^(m/2)"];

q3 = q3 /. {c^k_ :> c^(k/2), y^l_ :> a^(l/2), \[Xi]^m_ :> b^(m/2)};
Print[q3];

Print["Now use the Reduce command to look for zeros of this reduced q3 in the domain of interest:"];
Print[Reduce[{q3 <= 0, a > 0, b > 0, c > 0, 1 > a, 1 > b, 1 > c}, {a, b, c}]];
Print["No such zeros exist"];
Print["\n"];

Print["Now we shall look for zeros of r3 in the domain of interest. First we will change variables in r3 as c^k \[RuleDelayed\]c^(k/2), y^l \[RuleDelayed\]a^(l/2), \[Xi]^m \[RuleDelayed\]b^(m/2)"];

r3 = r3 /. {c^k_ :> c^(k/2), y^l_ :> a^(l/2), \[Xi]^m_ :> b^(m/2)};
Print[r3];

Print["Now use the Reduce command to look for zeros of this reduced r3 in the domain of interest:"];
Print[Reduce[{r3 <= 0, a > 0, b > 0, c > 0, 1 > a, 1 > b, 1 > c}, {a, b, c}]];
Print["No such zeros exist"];
Print["\n"];
Print["The first factor, p3, will be dealt with at the end ."];

Print["\n"];

Print["Now try s = 4. M M^t ="];
M4 = (M.Transpose[M]) /. {s -> 4} // sf;
Print[M4 // mf];

Print["Rearrange:"];
M4 = Transpose[M4][[{1, 3, 5, 2, 4, 6}]]][[{1, 3, 5, 2, 4, 6}]];
Print[M4 // mf];

Print["Its characteristic polynomial, c4:"];
c4 = CharacteristicPolynomial[M4, t] // Factor;
Print[c4];

Print["Now we take the discriminant d4 of c4, and simplify with what we know about x, y, \[Eta], and \[Xi]. We will arrange the factors of p4, q4, and r4 of d4 in a column, and deal with each in turn:" ];

d4 = Discriminant[c4, t] // sf // Factor;

d4 = sf[d4 /. {x^k_ -> Sqrt[1 - y^2]^k, \[Eta]^ l_ -> Sqrt[1 - \[Xi]^2]^l}] // Factor;

bads4 = Drop[Map[First, FactorList[d4]], 6];
Print["p4:"];
p4 = bads4[[1]];
Print[p4];
Print["q4:"];
q4 = bads4[[2]];
Print[q4];
Print["r4:"];
r4 = bads4[[3]];
Print[r4];

Print["\n"];
Print["Let's look for zeros of $q_4$ in the domain of interest. First we will change variables in $q_4$ as $c^k \cdot c^{(k/2)}$, $a^l \cdot a^{(l/2)}$, $b^m \cdot b^{(m/2)}$;"

$q_4 = q_4 /. \{c^k_\rightarrow c^{(k/2)}, a^l_\rightarrow a^{(l/2)}, b^m_\rightarrow b^{(m/2)}\};$

Print[q4];

Print["Now use the Reduce command to look for zeros of this reduced $q_4$ in the domain of interest:"];

Print[Reduce[{q4 <= 0, a > 0, b > 0, c > 0, 1 > a, 1 > b, 1 > c}, \{a, b, c\}]];
Print["No such zeros exist"]; Print["\n"];

Print["Now we shall look for zeros of $r_4$ in the domain of interest. First we will change variables in $r_4$ as $c^k \cdot c^{(k/2)}$, $a^l \cdot a^{(l/2)}$, $b^m \cdot b^{(m/2)}$;"

$r_4 = r_4 /. \{c^k_\rightarrow c^{(k/2)}, a^l_\rightarrow a^{(l/2)}, b^m_\rightarrow b^{(m/2)}\};$

Print[r4];

Print["Now use the Reduce command to look for zeros of this reduced $r_4$ in the domain of interest:"];

Print[Reduce[{r4 <= 0, a > 0, b > 0, c > 0, 1 > a, 1 > b, 1 > c}, \{a, b, c\}]];
Print["No such zeros exist"]; Print["\n"];

Print["The first factor, $p_4$, will be dealt with next."];

Print["\n"];
Print["Now collect p2, p3, and p4, and show via a Groebner basis that, no matter what the values of parameters of the family of counterexamples, one of the values of s = 2, 3, or 4 will ensure that one of p2, p3 or p4 cannot be equal to zero. This will mean by an appropriate choice of s we can ensure that the discriminant of the characteristic polynomial of M cannot be zero, and so M and thus L cannot have any repeated eigenvalues. First we will perform the same change of variables \([Xi] \rightarrow a, y \rightarrow b\) as with q and r, and prepare p2, p3, and p4 for entrance into Macaulay2."];

Print["\n"];

p2 = InputForm[p2 /. \{\(Xi\) \rightarrow a, y \rightarrow b\}];
p3 = InputForm[p3 /. \{\(Xi\) \rightarrow a, y \rightarrow b\}];
p4 = InputForm[p4 /. \{\(Xi\) \rightarrow a, y \rightarrow b\}];

Print["\n"];

Print["The follow code is then entered into Macaulay2:"];
Print["\n"];

Print["R = QQ[a, b, c];"];
Print["p2 = " p2];
Print["p3 = " p3];
Print["p4 = " p4];
Print["I = ideal (p2, p3, p4);"];
Print["r = radical I;"];
Print["gens gb r"];
Print["\n"];

7.1.2. Macaulay2 code. This code is the final output of the Mathematica code as presented in the previous subsection.

The output is a Groebner basis for p2, p3, and p4. All of the terms in this basis must vanish if p2, p3 and p4 are all to vanish.

\(R = QQ[a, b, c];\)
\[ p_2 = -9 + 54a^2 - 117a^4 + 108a^6 - 36a^8 + 54b^2 - 206a^2b^2 + 212a^4b^2 - 4a^6b^2 - 56a^8b^2 - 117a^4 + 212a^2b^4 + 320a^4b^4 - 680a^6b^4 + 256a^8b^4 + 108b^6 - 4a^2b^6 - 680a^4b^6 + 512a^6b^6 - 36b^8 - 56a^2b^8 + 256a^4b^8 + 18c^2 - 90a^2c^2 + 162a^4c^2 - 126a^6c^2 - 36a^8c^2 - 108b^2c^2 + 404a^2c^2 - 528a^4b^2c^2 + 276a^6b^2c^2 - 44a^8b^2c^2 + 2^2c^2 + 234a^4b^4c^2 - 526a^2b^4c^2 + 12a^4b^4c^2 - 536a^6b^4c^2 - 256a^8b^4c^2 - 216b^6c^2 + 100a^2b^6c^2 - 948a^4b^6c^2 - 768a^6b^6c^2 + 72b^8c^2 + 112a^2b^8c^2 - 512a^4b^8c^2 - 9c^4 + 36a^2c^4 - 54a^4c^4 + 36a^6c^4 - 9a^8c^4 + 54b^2c^4 - 198a^2b^2c^4 + 270a^4b^2c^4 - 162a^6b^2c^4 + 36a^8b^2c^4 - 1^2c^4 - 117b^4c^4 + 314a^2b^4c^4 - 213a^4b^4c^4 - 48a^6b^4c^4 + 64a^8b^4c^4 + 108b^6c^4 + 96a^2b^6c^4 - 6^4c^4 - 268a^4b^6c^4 + 256a^6b^6c^4 - 36b^8c^4 - 56a^2b^8c^4 + 256a^4b^8c^4 \]

\[ p_3 = -4 + 24a^2 - 52a^4 + 48a^6 - 16a^8 + 24b^2 - 111a^2b^2 - 172a^4b^2 - 99a^6b^2 + 14a^8b^2 - 52a^4 + 172a^2b^4 - 130a^4b^4 - 30a^6b^4 + 36a^8b^4 + 48b^6 - 99a^2b^6 - 30a^4b^6 + 72a^6b^6 - 16b^8 + 14a^2b^8 + 36a^4b^8 + 8c^2 - 40a^2c^2 + 72a^4c^2 - 56a^6c^2 + 16a^8c^2 - 48b^2c^2 + 199a^2b^2c^2 - 293a^4b^2c^2 + 181a^6b^2c^2 - 39a^8b^2c^2 + 104b^4c^2 - 331a^2b^4c^2 + 297a^4b^4c^2 - 34a^6b^4c^2 - 2 - 36a^8b^4c^2 - 96b^6c^2 + 200a^2b^6c^2 + 13a^4b^6c^2 - 108a^6b^6c^2 + 32b^8c^2 - 28a^2b^8c^2 - 72a^4b^8c^2 - 4c^4 + 16a^2c^4 - 24a^4c^4 + 16a^6c^4 - 4a^8c^4 + 2b^2c^4 - 220a^2b^2c^4 - 88a^4b^2c^4 + 120a^4b^4c^4 - 72a^6b^4c^4 + 16a^8b^4c^4 - 52a^4b^4c^4 + 159a^2b^4c^4 - 153a^4b^4c^4 + 37a^6b^4c^4 + 9a^8b^4c^4 + 48b^6c^4 + 101a^2b^6c^4 + 17a^4b^6c^4 + 36a^2b^6c^4 + 16a^4b^6c^4 + 14a^2b^8c^4 + 36a^4b^8c^4 - 8c^4 \]
\[ p_4 = -225 + 1350a^2 - 2925a^4 + 2700a^6 - 900a^8 + 1350b^2 - 6494a^2b^2 + 10676a^4b^2 - 10816a^6b^2 + 11816a^8b^2 + 900b^4 - 6494a^2b^4 + 10676a^4b^4 - 10816a^6b^4 + 1816a^8b^4 + 1024c^2 - 2250a^2c^2 + 4050a^4c^2 - 3150a^6c^2 + 900b^2c^2 + 11444a^2b^2c^2 - 17232a^4b^2c^2 + 5850b^4c^2 - 19870a^2b^4c^2 + 20460a^4b^4c^2 - 5444a^6b^4c^2 - 900b^6c^2 + 1800a^2c^4 - 2250a^4c^4 + 1350a^6c^4 - 225b^2c^4 - 1350a^2c^6 - 3b^2c^6 + 3a^2c + 3b^2c + c^3 - c, \]

\[ I = \text{ideal (} p_2 , p_3 , p_4 \text{)}; \]

\[ r = \text{radical } I; \]

\[ \text{gens gb } r \]

This code then outputs the following Gröbner basis for the ideal generated by \( p_1, p_2, \) and \( p_3 \):

\[ o7 = | a^3bc - abc , 2ab^3c^3 - 2ab^3c^4 + abc , \]

\[ a^4c^3+3a^2b^2c^3+2b^4c^3-2a^4c^2-2a^2b^2c^2-2b^4c^2-2a^2c^3-3b^2c^3+3a^2c^3+3b^2c^3+3a^2c + 3b^2c^2+c^3-c , \]

\[ 4a^5b^2+4a^3b^4+a^5c^2-2ab^4c^2+2a^5-8a^3b^2-2ab^4-2a^3c^2+3a^3+3ab^2+ac^2-a , \]

\[ 4b^5c^3+2a^2b^2c^3-4b^5c^3-2a^2b^2c^3-2b^4c^3-2a^2c^3-3b^2c^3-3a^2bc+6b^3c^2+2bc^3-2bc , \]

\[ 4a^2b^4c^2-4a^2b^2c^2-2a^2b^4-4a^4c^2-2a^2b^2c^2-2b^4c^2+4a^8+8a^2b^2+2a^2c^2+3b^2c^2-2a^8-3b^2c^2+1 | \]

The first term of of this Gröbner basis is \( a^3bc - abc \). This can only be zero if \( a = 0 \) or ±1 or \( b \) or \( c = 0 \), but none of these conditions are permitted. Therefore, at least one of \( p_2, p_3 \) or \( p_4 \) must be non-zero for any possible configuration of the variables. The discriminant of \( M \) and thus \( L \) is therefore shown to be non-zero for at least one of \( s = 2, 3, \) or 4.
Therefore no matter what values of the counterexample parameters are chosen, we can select non-commutative variables $sX$ and $Y$ such that $L$ from the LSS counterexample has all simple eigenvalues.

### 7.2. Code for quantising Laffey’s counterexample.
These calculations were done entirely in Mathematica. The same Mathematica modules `skewSymbol` and `comm` as used in the previous section are also used in this code, but have been omitted for brevity.

(* Laffey’s H and K*)

```
H = ( { 
  { -122, 0, 12, 18, -30, 18, 26, 10},
  {0, -122, -6, -12, -16, -28, 20, -16},
  {12, -6, -218, 0, 44, 8, 24, 12},
  {18, -12, 0, -218, -2, -34, -10, 22},
  {-30, -16, 44, -2, -216, 0, -12, -8},
  {18, -28, 8, -34, 0, -216, -8, 36},
  {26, 20, 24, -10, -12, -8, -120, 0},
  {10, -16, 12, 22, -8, 36, 0, -120} } ) ;
K = ( { 
  { -4, 0, 0, 0, 0, 0, 0, 0},
  {0, -4, 0, 0, 0, 0, 0, 0},
  {0, 0, 4, 0, 0, 0, 0, 0},
  {0, 0, 0, 4, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0} } ) ;
```

(*Non-commutative X and Y*)

```
X = ( { 
  {1, 0},
  {0, 0} 
} ) ;
Y = ( { 
  {0, 1},
  {1, 0} 
} ) ;
```

(*The auxiliary matrix $L_1$ associated with $X$ and $Y$ for a given eigenvalue $\[\Lambda\]$ of $L = X \,[\text{CircleTimes}] \ H + Y \,[\text{CircleTimes}] \ K \ast$*)

```
L1 = \[\Lambda\] H + K.K;
```

`Print[MatrixForm[L1]];`
(* A general skew symmetric matrix of the same size as \( L_1 \) *)

\[ S_1 = \text{skewSymbol}[\text{Length}[L_1]]; \]

(*The commutator of \( L_1 \) and \( S_1 \) which must vanish if \( \lambda \) 
is to have multiplicity greater than one *)

\[ \text{CommLS1} = \text{comm}[L_1, S_1]; \]

Print[MatrixForm[S1]];
Print[MatrixForm[CommLS1]];

(*Extract the elements of the commutator, which are linear in
 the elements of \( L_1 \) and \( S_1 \) since \( \lambda \) is a constant
 with respect to the auxiliary matrix, and remove
duplicates and zeros. *)

\[ \text{eqs1} = \text{Flatten}[\text{UpperTriangularize}[\text{CommLS1}]]; \]
\[ \text{eqs1} = \text{DeleteCases}[\text{eqs1}, 0]; \]

(*Extract the elements of \( S_1 \) as variables, and display the
 total number of such variables *)

\[ \text{vars1} = \text{DeleteCases}[\text{Flatten}[S_1], 0]; \]
\[ \text{vars1} = \text{Union}[\text{vars1} /. -x_\_ \rightarrow x]; \]
\Print[\text{Length}[\text{vars1}]];

(*Both the elements of the commutator and the variables
 displayed as columns *)

Print[Column[\text{eqs1}]];
Print["\n"]; Print[Column[\text{vars1}]];

(*If the commutator vanishes, then the Groebner basis of the
 elements of the commutator, treated as linear polynomials,
 must also vanish. *)

\[ \text{gb1} = \text{GroebnerBasis}[\text{eqs1}, \text{vars1}]; \]
(*We see that the Groebner basis is precisely the elements of the skew-symmetric matrix \( S_1 \), up to a factor of \( \lambda \) which we know must be non-zero and is constant in this context. We confirm that all elements of \( S_1 \) and included by comparing the sizes of the set of variables and the Groebner basis, *)

Print[gb1];
Print[Length[gb1]]; Print[Length[gb1] - Length[vars1]];  

(*We have shown that the commutator of \( L_1 \) and \( S_1 \) can only vanish if every element of \( S_1 \) also vanishes, and so \( \lambda \) cannot have multiplicity more than one. *)

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