LEVY LAPLACIAN ON MANIFOLD AND YANG-MILLS HEAT FLOW
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Abstract: A covariant definition of the Levy Laplacian on an infinite dimensional manifold is introduced. It is shown that a time-depended connection in a finite dimensional vector bundle is a solution of the Yang-Mills heat equations if and only if the associated flow of the parallel transports is a solution of the heat equation for the covariant Levy Laplacian on the infinite dimensional manifold.

2010 Mathematics Subject Classification: 70S15, 58J35
key words: Levy Laplacian, Yang-Mills equations, Yang-Mills heat equations, infinite dimensional manifold

1 Introduction

In our work, we relate two differential equations of the heat type: the quasi-linear Yang-Mills heat equations on a finite-dimensional manifold and the linear heat equation for the Levy Laplacian on an infinite-dimensional manifold. Namely, we generalize Accardi-Gibilisco-Volovich theorem on the equivalence of the Yang-Mills equations and the Laplace equation for the Levy Laplacian in the following way: we show that a time-depended connection in a finite-dimensional vector bundle is a solution of the Yang-Mills heat equations if and only if the associated flow of the parallel transports is a solution of the heat equation for the Levy Laplacian.

The Levy Laplacian is an infinite dimensional Laplacian which has not any finite dimensional analogs. It was introduced by Paul Levy on functions on $L^2(0,1)$ in the 1920s as follows. Let the second derivative of $f \in C^2(L^2(0,1), \mathbb{R})$ have the form

$$<f''(x)u,v> = \int_0^1 \int_0^1 K_V(x;t,s)u(t)v(s)dtds + \int_0^1 K_L(x;t)u(t)v(t)dt,$$  

(1)

where $K_V(x;\cdot,\cdot) \in L_2([0,1] \times [0,1])$ and $K_L(x;\cdot) \in L_\infty[0,1]$. (The kernels $K_V(x;\cdot,\cdot)$ and $K_L(x;\cdot)$ are called the Volterra kernel and the Levy kernel respectively.) Then the Levy Laplacian $\Delta_L$ acts on $f$ by the formula

$$\Delta_L f(x) = \int_0^1 K_L(x;t)dt.$$  

(2)

Another original definition of the Levy Laplacian by Paul Levy is the following. Let $\{e_n\}$ be an orthonormal basis in $L^2(0,1)$. Then the Levy Laplacian (generalized by the orthonormal basis $\{e_n\}$) acts on $f \in C^2(L^2(0,1), \mathbb{R})$ by the formula

$$\Delta^{\{e_n\}}_L f(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n <f''(x)e_k,e_k>.$$  

(3)

For some orthonormal bases (for example, for $e_n(t) = \sqrt{2} \sin \pi nt$) the definitions coincide on the domain of $\Delta_L$ (see [31] and also [18, 19, 27]).

The modern situation is the following. The term "Levy Laplacian" is used for various analogs and generalizations of the original Levy Laplacians $\Delta_L$ and $\Delta^{\{e_n\}}_L$. These Levy Laplacians act on functions (or generalized functions) over different infinite-dimensional...
spaces. One of these Levy Laplacians was introduced by Accardi, Gibilisco and Volovich in [3,4]. We will denote it by the symbol $\Delta^{AGV}_L$. The operator $\Delta^{AGV}_L$ was defined by analogy with (2). In [3,4], it was shown that a connection in a vector bundle over $\mathbb{R}^d$ is a solution of the Yang-Mills equations if and only if the parallel transport associated with the connection is a solution of the Laplace equation for the Laplacian $\Delta^{AGV}_L$. The definition of the Levy Laplacian $\Delta^{AGV}_L$ and the theorem on the relationship between the Levy Laplacian and the Yang-Mills equations was generalized for the case of manifold by Leandre and Volovich in [30]. Another definition of the Levy Laplacian on the infinite dimensional manifold was introduced by Accardi and Smolyanov in [9]. In this work, the Levy Laplacian was defined as the Cesaro mean of the second order directional derivatives by analogy with (3). The relationship of this Levy Laplacian and the Yang-Mills equations was studied in [40]. The relationship between the Yang-Mills equations and different Levy Laplacians was also studied in [41,42,43,44,45].

In the current paper, we introduce the definition of the Levy Laplacian on a manifold in terms of covariant derivatives. We define this operator as the composition of some infinite dimensional divergence and some nonstandard gradient. This covariant Levy Laplacian is an analog of operator (2). In the flat case, its definition coincides with the definition of $\Delta^{AGV}_L$. But in general case, its definition differs from the definition by Leandre and Volovich from [30]. Their definition was not in terms of covariant derivatives and was based on the triviality of the tangent bundle of the base infinite dimensional manifold. However, it seems that the covariant Levy Laplacian, the Levy Laplacian introduced by Leandre and Volovich and the Levy Laplacian introduced by Accardi and Smolyanov coincide on the domain of the first of them.

There are many papers devoted to the heat equations for the Levy Laplacians. In [18,19], some methods of infinite-dimensional analysis were used to study various differential equations with the Levy Laplacian including the heat equation. In [7,6,2,8], the Levy heat semigroup on the space generalized by the Fourier transforms of the measures on some infinite dimensional space was studied. The approach to the heat equation for the Levy Laplacian based on the white noise analysis was used in [35,28,5] (see also review [29]). Representations in the form of Feynman formulas for solutions of the heat equation for the Levy Laplacian on a manifold were obtained in [9]. Unfortunately, it seems that the Levy Laplacian $\Delta^{AGV}_L$, that is connected to the Yang-Mills equations, doesn’t coincide with the Laplacians that were used in the mentioned works, except [9] (see the discussion in [43]). Is it possible to transfer the technique of these works for the study of the Yang-Mills heat equations is an open question. Some possible ways for the application of the heat equation for the Levy Laplacian to study the Yang-Mills equations are discussed in [11].

The Yang-Mills heat flow is the gradient flow for the Yang-Mills action functional. It was introduced by Attiah and Bott in [10] and was studied by Donaldson in [15] (see also [16]). If the base manifold is 2-dimensional or 3-dimensional, than it is possible to construct a solution of the Yang-Mills equations by solving the Yang-Mills heat equations and letting time tend to infinity (see [37]). In the case of the structure group $U(1)$, the Yang-Mills heat equations are simply the heat equations for 1-forms and a solution of these equations tends to a harmonic 1-form as time tends to infinity (see [32]). In the dimension four, the blow-up does not occur for spherical symmetric solutions (see [23]). In the general case, the Yang-Mills heat equations have blow-up. The dependence of the heat equation for the Levy Laplacian on the dimension of the base manifold was never studied. It is interesting to study the behavior of solutions of this equation in the case then the Yang-Mills heat equation has a blow-up. The Yang-Mills equations and the Yang-Mills heat equations are also related in the following way. In [33,34], the proof of well-posedness of the Cauchy problem for the Yang-Mills equations on the Minkowski space based on the
application of the Yang-Mills heat flow was suggested. The approach to the Yang-Mills heat equations based on the (stochastic) parallel transport was used in [13]. But unlike ours, this approach was not based on the Levy Laplacian.

The paper is organized as follows. In Sec. 2 we give preliminary information from the finite dimensional geometry about the Yang-Mills heat equation on a time depended connection in the finite dimensional vector bundle. In Sec. 3 we give preliminary information from the infinite dimensional geometry about the base Hilbert manifold of the $H^1$-curves. In Sec. 4 we introduce the $H^0$-gradient on the space of sections in the vector bundle over the base Hilbert manifold of the curves. We consider the parallel transport as a section in this vector bundle and find the value of the $H^0$-gradient on the parallel transport. In Sec. 5 we define the Levy Laplacian as the composition of the special infinite dimensional divergence and the $H^0$-gradient. We find the value of the Levy Laplacian on the parallel transport. In Sec. 6 we prove the theorem on the equivalence of the Yang-Mills heat equations and the heat equation for the Levy Laplacian.

2 Yang-Mills heat equations

Bellow, if $E_0$ is a vector bundle over a finite or infinite dimensional Hilbert manifold $M_0$, the symbol $C^\infty(M_0, E_0)$ denotes the space of smooth global sections in this bundle and the symbol $C^\infty(W_0, E_0)$ denotes the space of smooth local sections on an open set $W_0 \subset M_0$. In the infinite-dimensional case derivatives are understood in the Frechet sense.

Bellow, $M$ is a connected smooth compact $d$-dimensional Riemannian manifold or $\mathbb{R}^d$. Let $g$ be the Riemannian metric on $M$. We will raise and lower indices using the metric $g$ and we will sum over repeated indices. Let $E = E(\mathbb{C}^N, \pi, M, G)$ be a vector bundle over $M$ with the projection $\pi: E \to M$ and the structure group $G \subseteq SU(N)$. The fiber over $x \in M$ is $E_x = \pi^{-1}(x) \cong \mathbb{C}^N$. Let the Lie algebra of the structure group be $Lie(G) \subseteq su(N)$. Let $P$ be the principle bundle over $M$ associated with $E$ and $ad(P) = Lie(G) \times_G M$ be the adjoint bundle of $P$ (the fiber of $adP$ is isomorphic to $Lie(G)$). A connection in the vector bundle $E$ is a smooth section in $\Lambda^1 \otimes adP$. If $W_a$ is an open subset of $M$ and $\psi_a: \pi^{-1}(W_a) = W_a \times \mathbb{C}^N$ is a local trivialization of $E$, then in this local trivialization the connection $A$ is a smooth $su(N)$-valued 1-form $A^a(x) = A^a_\mu(x)dx^\mu = \psi_aA(x)\psi_a^{-1}$ on $W_a$. Let $\psi_a: \pi^{-1}(W_a) \cong W_a \times \mathbb{C}^N$ and $\psi_b: \pi^{-1}(W_b) \cong W_b \times \mathbb{C}^N$ be two local trivializations of $E$ and $\psi_{ab}: W_a \cap W_b \to G$ be the transition function. It means that $\psi_a \circ \psi_b^{-1}(x, \xi) = (x, \psi_{ab}(x)\xi)$ for all $(x, \xi) \in (W_a \cap W_b) \times \mathbb{C}^N$. Then for $x \in W_a \cap W_b$ the following holds

$$A^b(x) = \psi_{ab}^{-1}(x)A^a(x)\psi_{ab}(x) + \psi_{ab}^{-1}(x)d\psi_{ab}(x).$$

(4)

The connection defines the covariant derivative $\nabla$. If $\phi$ is a smooth section in $adP$, its covariant derivative has the form

$$\nabla_\mu \phi = \partial_\mu \phi + [A_\mu, \phi].$$

The curvature $F$ of the connection $A$ is a smooth section in $\Lambda^2 \otimes adP$. In the local trivialization, the curvature $F$ is the $su(N)$-valued 2-form $F^a(x) = \sum_\mu \theta^a_\mu(x)dx^\mu \wedge dx^\nu = \psi_aF(x)\psi_a^{-1}$, where $F^a_\mu = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + [A_\mu, A^a_\nu]$. For $x \in W_a \cap W_b$ the following holds

$$F^b(x) = \psi_{ab}^{-1}(x)F^a(x)\psi_{ab}(x).$$

(5)

The Yang-Mills action functional has the form

$$S_{YM}(A) = -\frac{1}{2} \int_M tr(F_\mu(x)F^{\mu\nu}(x))Vol(dx),$$

(6)
where $Vol$ is the volume form on the manifold $M$. The Euler-Lagrange equations for this action functional are

$$\nabla^\mu F_{\mu\nu} = 0. \quad (7)$$

Locally,

$$\nabla_x F_{\mu\nu} = \partial_x F_{\mu\nu} + [A_\lambda, F_{\mu\nu}] - F_{\mu\rho} \Gamma_\lambda^{\rho\nu} - F_{\nu\rho} \Gamma_\lambda^{\rho\mu},$$

where $\Gamma_\lambda^{\rho\mu}$ the Christoffel symbols of the Levi-Civita connection on $M$. Equations (7) are called the Yang-Mills equations.

The Yang-Mills heat equations are nonlinear parabolic differential equations on a time-depended connection $A(\cdot, \cdot) \in C^{1,\infty}([0, T] \times M, \Lambda^1 \otimes adP)$ (any partial derivative of $A(s, x)$ with respect to the $x$ variables is jointly $C^1$ on $[0, T] \times M$) of the form

$$\partial_s A_\mu(s, x) = \nabla^\mu F_{\mu\nu}(s, x). \quad (8)$$

For more information about these equations, in particular, for the initial value problem, the weak solutions, the blow-ups of solutions and the questions related to the gauge choice see [37, 13, 33, 34, 22].

## 3 Hilbert manifold of $H^1$-curves

For any sub-interval $I \subset [0, 1]$ the symbols $H^0(I, \mathbb{R}^d)$ and denote the spaces of $L_2$-functions and $H^1$-functions (absolutely continuous with finite energy) on $I$ with values in $\mathbb{R}^d$ respectively. Let

$$\|\gamma\|_0 = (\int_I (\dot{\gamma}(t), \gamma(t))_{\mathbb{R}^d} dt)^{\frac{1}{2}}$$

and

$$\|\gamma\|_1 = (\int_I (\dot{\gamma}(t), \gamma(t))_{\mathbb{R}^d} dt + \int_I (\ddot{\gamma}(t), \gamma(t))_{\mathbb{R}^d} dt)^{\frac{1}{2}}$$

be the Hilbert norms on the space $H^0(I, \mathbb{R}^d)$ and on the space $H^1(I, \mathbb{R}^d)$ respectively.

The mapping $\gamma : [0, 1] \to M$ is called $H^1$-curve if for any interval $I \subset [0, 1]$ and for any coordinate chart $(\phi_a, W_a)$ of the manifold $M$, such that $\gamma(I) \subset W_a$, it holds that $\phi_a \circ \gamma |_I \in H^1(I, \mathbb{R}^d)$. Let the symbol $\Omega$ denote the set of all $H^1$-curves in $M$. For any $x \in M$ let $\Omega_x = \{ \gamma \in \Omega : \gamma(0) = x \}$ and $\Omega_{x,x} = \{ \gamma \in \Omega : \gamma(1) = x \}$.

Fix $\gamma \in \Omega$. The mapping $X(\gamma; \cdot) : [0, 1] \to TM$ such that $X(\gamma; t) \in T_{\gamma(t)} M$ for any $t \in [0, 1]$ is a vector field along $\gamma$. We will also use the notation $X(\gamma)$ for $X(\gamma; \cdot)$. Let the symbol $H^\infty_0(TM)$ denote the Banach space of all $L^\infty$-fields along $\gamma$. The norm $\| \cdot \|_\infty$ on this space is defined by

$$\|X(\gamma)\|_\infty = \text{ess sup}_{t \in [0, 1]} (\sqrt{g(X(\gamma; t), X(\gamma; t)))}).$$

The symbol $H^0_0(TM)$ denotes the Hilbert space of all $H^0$-fields along $\gamma$. The scalar product on this space is defined by the formula

$$G_0(X(\gamma), Y(\gamma)) = \int_0^1 g(X(\gamma; t), Y(\gamma; t)) dt. \quad (9)$$

If $X(\gamma)$ is an absolutely continuous field along $\gamma \in \Omega$, its covariant derivative $\nabla X(\gamma)$ is the field along $\gamma$ defined by

$$\nabla X(\gamma; t) = \dot{X}(\gamma; t) + \Gamma(\gamma(t))(X(\gamma; t), \dot{\gamma}(t)), $$
where \((\Gamma(x)(X,Y))^\mu = \Gamma^\mu_{\lambda \nu}(x)X^\lambda Y^\nu\) in local coordinates. Let \(Q(\gamma;\cdot)\) denote the parallel transport generated by the Levi-Civita connection along the curve \(\gamma\). It is easy to show that

\[
\nabla X(\gamma; t) = Q(\gamma; t)\frac{d}{dt}(Q(\gamma; t)^{-1}X(\gamma; t)).
\]

The symbol \(H^1_\gamma(TM)\) denotes the Hilbert space of all \(H^1\)-fields along \(\gamma\). The scalar product on this space is defined by the formula

\[
G_1(X(\gamma), Y(\gamma)) = \int_0^1 g(X(\gamma; t), Y(\gamma; t))dt + \int_0^1 g(\nabla X(\gamma; t), \nabla Y(\gamma; t))dt. \tag{10}
\]

The set \(\Omega\) of all \(H^1\)-curves in \(M\) can be endowed with the structure of a Hilbert manifold in the following way (see \([17, 25, 26]\)). Let \(d(\cdot, \cdot)\) denote the distance on \(M\) generated by the metric \(g\). Let

\[
W(\gamma, \varepsilon) = \{\sigma \in \Omega : d(\gamma(t), \sigma(t)) < \varepsilon\text{ for all }t \in [0,1]\}.
\]

Let \(\tilde{W}(\gamma, \varepsilon) = \{X \in T_\gamma H^1([0,1], M) : \|X\|_\infty < \varepsilon\}\). Let \(\exp_\gamma\) denote the exponential mapping on the manifold \(M\) at the point \(x \in M\). For \(\gamma \in \Omega\) let the mapping

\[
\exp_\gamma : \tilde{W}(\gamma, \varepsilon) \to \Omega
\]

be defined by the formula

\[
\exp_\gamma(X)(t) = \exp_{\gamma(t)}(X(t)).
\]

It is known that \(\exp_\gamma\) is a bijection between \(W(\gamma, \varepsilon)\) and \(\tilde{W}(\gamma, \varepsilon)\). The structure of the Hilbert manifold on \(\Omega\) is defined by the atlas \((\exp_\gamma^{-1}, W(\gamma, \varepsilon))\). The set \(\Omega_x\) is a Hilbert submanifold of \(\Omega\) and the set \(\Omega_{x,x}\) is a Hilbert submanifold of \(\Omega_x\).

We consider two canonical vector bundles \(H^0\) and \(H^1\) over the Hilbert manifold \(\Omega\) (see \([25, 26]\)). The fiber of \(H^0\) over \(\gamma \in \Omega\) is the space \(H^0_\gamma(TM)\) and \(G_0(\cdot, \cdot)\) is a Riemannian metric on this bundle. The vector bundle \(H^1\) is the tangent bundle over the manifold \(\Omega\). Its fiber over \(\gamma \in \Omega\) is the space \(H^1_\gamma(TM)\) and \(G_1(\cdot, \cdot)\) is a Riemannian metric on this bundle. Let \(H^1_{0,0}\) denote the subbundle of \(H^1\) such that the fiber of \(H^1_{0,0}\) over \(\gamma \in \Omega\) is the space \(\{X \in H^1_\gamma(TM) : X(0) = X(1) = 0\}\).

A connection in a vector bundle over an infinite-dimensional manifold can be given by Christoffel symbols (see \([25, 26, 24]\)). Let \(M_0\) be a base Hilbert manifold modeled on a Hilbert space \(H_0\) and \(E_0\) be a Hilbert vector bundle over \(M_0\) with the fiber \(V_0\) and the projection \(\pi_0 : E_0 \to M_0\). If \(W_0\) is a coordinate chart on \(M_0\), then \(E_0\) has a local trivialization \(\pi_0^{-1}(W_0) \cong W_0 \times H_0\) and the tangent bundle \(TM_0\) over \(M_0\) has a local trivialization \(TW_0 \cong W_0 \times H_0\). Then the Christoffel symbols \(\Gamma^\gamma\) of the connection in \(E_0\) is a smooth function on \(W_0\) with values in the space of continuous bilinear functionals from \(V_0 \times H_0\) to \(V_0\). The Christoffel symbols are transformed under the coordinate transformations in the similar way as in the finite-dimensional case.

The Levi-Civita connection on the \(d\)-dimensional manifold \(M\) generates the canonical connection \(\nabla^H\) in the infinite-dimensional bundle \(H^0\). (We associate the connection and the covariant derivative generated by this connection). Let \(\sigma \in \Omega\). The Christoffel symbols \(\Gamma^\sigma\) of the connection \(\nabla^H\) in the coordinate chart \((\exp_\sigma^{-1}, W(\sigma, \varepsilon))\) are defined as follows.

For any \(t \in [0,1]\) we consider the normal coordinate chart on \(M\) at the point \(\sigma(t)\) and the Christoffel symbols \(\Gamma^\sigma_{\gamma(t)}\) of the Levi-Civita connection on \(M\) in this coordinate chart. If \(\gamma \in W(\sigma, \varepsilon), X \in H^1_{\gamma(0)}(TM)\) and \(Y \in H^1_{\gamma(t)}(TM)\), then \((\Gamma^\gamma_{\gamma(t)}(X,Y))(t) \in T_{\gamma(t)}M\) for almost all \(t\) is defined by

\[
(\Gamma^\gamma_{\gamma(t)}(X,Y))(t) = \Gamma^\sigma_{\gamma(t)}(\gamma(t))(X(t), Y(t)). \tag{11}
\]
in the normal coordinate chart on $M$ at the point $\sigma(t)$. In \cite{25,26}, it is proved that Cristoffel symbols $\left(11\right)$ correctly define the connection in the vector bundle $\mathcal{H}^0$. Let $X \in C^\infty(W(\sigma, \varepsilon), \mathcal{H}^0)$ and $Y \in C^\infty(W(\sigma, \varepsilon), \mathcal{H}^1)$. Then in the normal coordinate chart on $M$ at the point $\sigma(t)$ we have the following expression for the covariant derivative

$$\nabla^\mathcal{H}_Y X(\gamma; t) = d_Y X(\gamma; t) + \Gamma_{\sigma(t)}(\gamma(t))(X(t), Y(t)). \quad \left(12\right)$$

**Example 1.** Let the section $\psi$ in $\mathcal{H}^0$ be defined by $\psi(\gamma; t) = \dot{\gamma}(t)$. It holds that $\nabla^\mathcal{H}_Y \psi(\gamma; t) = d_Y \psi(\gamma; t)$ (see \cite{23,26}).

**Remark 1.** The connection $\nabla^\mathcal{H}$ is Riemannian. It means that for any smooth local sections $Y, Z$ in $\mathcal{H}^0$ and any smooth local section $X$ in $\mathcal{H}^1$ the following holds

$$d_X G_0(Y, Z) = G_0(\nabla^\mathcal{H}_X Y, Z) + G_0(Y, \nabla^\mathcal{H}_X Z). \quad \left(13\right)$$

### 4 First derivative and $H^0$-gradient of parallel transport

Let $\mathcal{E}$ be the vector bundle over $\Omega$ that its fiber over $\gamma \in \Omega$ is the space $L(E_{\gamma(0)}, E_{\gamma(1)})$ of all linear mappings from $E_{\gamma(0)}$ to $E_{\gamma(1)}$. The parallel transport $U_{1,0}$ generated by the connection $A$ in $\mathcal{E}$ can be considered as a section in $\mathcal{E}$ defined in the following way. Let $\psi_a : \pi^{-1}(W_a) \cong W_a \times \mathbb{C}^N$ be a local trivialization of the vector bundle $E$ and let $A^a$ be a local 1-form of the connection $A$ on the open set $W_a \subset M$. For $\gamma \in \Omega$ such that $\gamma([0, 1]) \subset W_a$ we can consider the system of differential equations

$$\begin{cases} \frac{d}{dt} U^a_{t,s}(\gamma) = -A^a(\gamma(t)) \dot{\gamma}^\mu(t) U^a_{t,s}(\gamma) \\ \frac{d}{ds} U^a_{t,s}(\gamma) = U^a_{t,s}(\gamma) A^a(\gamma(s)) \dot{\gamma}^\mu(s) \\ U^a_{t,t}|_{t=s} = Id. \end{cases} \quad \left(14\right)$$

Then $U_{1,0}(\gamma) = \psi_a^{-1} U^a_{1,0}(\gamma) \psi_a$ is the parallel transport along $\gamma$ generated by the connection $A$. If $\gamma([s, t]) \subset W_a \cap W_b$ and $A^a$ and $A^b$ are the local 1-forms of the connection $A$ on the open sets $W_a$ and $W_b$ respectively, then equality \left(14\right) implies that

$$U^a_{t,s}(\gamma) = \psi_{ab}(\gamma(t)) U^a_{t,s}(\gamma) \psi_{ba}(\gamma(s)). \quad \left(15\right)$$

For arbitrary $\gamma \in \Omega$ let consider the family of local trivializations $\psi_{ai} : \pi^{-1}(W_{ai}) \cong W_{ai} \times \mathbb{C}^N$ of the vector bundle $E$ and the partition $c = t_1 \leq t_2 \leq \ldots \leq t_n = d$ such that $\gamma([t_i, t_{i+1}]) \subset W_{ai}$. Let

$$U_{d,c}^{a_0, a_1}(\gamma) = U_{t_n, t_{n-1}}^{a_0, a_1}(\gamma) \psi_{a_0 a_{n-1}}(\gamma(t_{n-1})) \ldots U_{t_3, t_2}^{a_2, a_1}(\gamma) \psi_{a_2 a_1}(\gamma(t_2)) U_{t_2, t_1}^{a_1}(\gamma). \quad \left(16\right)$$

Then $U_{d,c}(\gamma) = \psi_{a_0}^{-1} U_{d,c}^{a_0, a_1}(\gamma) \psi_{a_1}$ and $U_{1,0}(\gamma)$ is a parallel transport along $\gamma$. By \left(15\right), the definition of the parallel transport does not depend on the choice of the partition and the choice of the family of trivializations. In \cite{17}, it is proved that the mapping $\Omega \ni \gamma \rightarrow U_{1,0}(\gamma)$ is a smooth section in the vector bundle $\mathcal{E}$. The parallel transport does not depend on the choice of parametrization of the curve $\gamma$ and $U_{t,s}(\gamma)$ coincide with the parallel transport along the restriction of $\gamma$ on the interval $[s, t]$. Also, the parallel transport satisfies the multiplicative property:

$$U_{t,s}(\gamma) U_{s,r}(\gamma) = U_{t,r}(\gamma) \quad \text{for} \quad r \leq s \leq t. \quad \left(17\right)$$

Let $g_{\mathcal{E}}$ and $g_{\mathcal{H}^0 \otimes \mathcal{E}}$ denote the natural Riemannian metrics in the bundle $\mathcal{E}$ and $\mathcal{H}^0 \otimes \mathcal{E}$ respectively. If $X, Y$ are local sections in $\mathcal{E}$ and $\Phi, \Psi$ are local sections in $\mathcal{H}^0$, then

$$g_{\mathcal{E}}(\Phi, \Psi) = -tr(\Phi \Psi),$$

$$g_{\mathcal{E} \otimes \mathcal{H}^0}(X \otimes \Phi, Y \otimes \Psi) = G_0(X, Y) g_{\mathcal{E}}(\Phi, \Psi).$$
Definition 1. The domain \( \text{dom grad}_H \) of the \( H^0 \)-gradient consists of all \( \varphi \in C^\infty(\Omega, \mathcal{E}) \) such that there exists \( J_\varphi \in C^\infty(\Omega, \mathcal{E} \otimes H^0) \), that the following equality holds

\[
g_{\mathcal{E} \otimes H^0}(J_\varphi(\gamma), X(\gamma) \otimes \Phi(\gamma)) = g_{\mathcal{E}}(d_X \varphi(\gamma), \Phi(\gamma))
\]

for any \( \gamma \in \Omega \), any local smooth section \( X \) in \( H_{0,0}^1 \) and any local smooth section \( \Phi \) in the bundle \( \mathcal{E} \). The \( H^0 \)-gradient is a linear mapping \( \text{grad}_H \colon \text{dom grad}_H \to C^\infty(\mathcal{E} \otimes H^0) \) defined by the formula

\[
\text{grad}_H \varphi = J_\varphi.
\]

Remark 2. Any connection \( B \) in \( E \) generates the connection in \( \mathcal{E} \) such that (see [30])

\[
\nabla^B_Y \Psi(\gamma) = d_Y \Psi(\gamma) + B_\mu(\gamma(1))Y^\mu(\gamma; 1)\Psi(\gamma) - \Psi(\gamma)B_\mu(\gamma(0))Y^\mu(\gamma; 0).
\]

If \( Y \) is a section in \( H_{0,0}^1 \), then \( \nabla^B_Y \Psi(\gamma) = d_Y \Psi(\gamma) \). So the definition of the \( H^0 \)-gradient is covariant.

Example 2. Let \( f \in C^\infty(M, \mathbb{R}) \). Let \( L_f \colon \Omega \to \mathbb{R} \) be defined by

\[
L_f(\gamma) = \int_0^1 f(\gamma(t))dt.
\]

Then

\[
\text{grad}_H L_f(\gamma; t) = \nabla f(\gamma(t)),
\]

where \( \nabla \) is the gradient on the manifold \( M \).

The following lemma is Duhamel’s Principle (see [17]).

Lemma 1. Let \( V \) be a finite dimensional inner product space. For any \( Z \in L_2([c, d], \text{End}(V)) \) there exists a unique \( P(Z) \in H^1([c, d], \text{End}(V)) \) such that \( \frac{d}{dt}P(Z; t) = -Z(t)P(Z; t) \) for almost all \( t \) and \( P(Z; c) = \text{Id} \). The mapping \( L_2([c, d], \text{End}(V)) \to Z \mapsto P(Z; d) \) is \( C^\infty \)-smooth and \( P(Z; t) \in \text{Aut}(V) \) for all \( t \in [c, d] \). Furthermore, the first derivative of \( P \) has the form

\[
< DP(Z; d), \delta Z > = -P(Z; d) \int_c^d P^{-1}(Z; t)\delta Z(t)P(Z; t)dt.
\]  \hspace{1cm} (18)

Proof. For clarity, we present the idea of the proof. For the complete proof see [17]. Consider the function \( R(t) = P^{-1}(Z_2; t)P(Z_1; t) \). Then

\[
\frac{d}{dt}R(t) = \frac{d}{dt}P^{-1}(Z_2; t)P(Z_1; t) + P^{-1}(Z_2; t)\frac{d}{dt}P(Z_1; t) = \]

\[
= P^{-1}(Z_2; t)(Z_2(t) - Z_1(t))P(Z_1; t) \hspace{1cm} (19)
\]

and

\[
P(Z_2; d)(R(1) - R(0)) = P(Z_2; d)(P^{-1}(Z_2; d)P(Z_1; d) - \text{Id}) = P(Z_1; d) - P(Z_2; d). \hspace{1cm} (20)
\]

Together (19) and (20) imply the formula

\[
P(Z_1; d) - P(Z_2; d) = -P(Z_2; d) \int_c^d P^{-1}(Z_2; t)(Z_2(t) - Z_1(t))P(Z_1; t)dt.
\]

The statement of the proposition can be deduced from this formula. \( \square \)
**Proposition 1.** The first derivative of the parallel transport has the form

\[
d_X U_{d,c}(\gamma) = - \int_c^d U_{d,t}(\gamma) F_{\mu \nu}(\gamma(t)) X^\mu(\gamma; t) \dot{\gamma}^\nu(t) U_{t,c}(\gamma) dt - \\
- A_\mu(\gamma(d)) A_\mu(\gamma(c)) X^\mu(\gamma; c). \quad (21)
\]

*Proof.* Consider the partition \( c = t_1 < t_2 < \ldots < t_n-1 < t_n = d \) and the family of local trivializations \( \psi_{a_i} : \pi^{-1}(W_{a_i}) \cong W_{a_i} \times \mathbb{C}^N \) of the vector bundle \( E \) such that \( \gamma([t_i, t_{i+1}]) \subset W_{a_i} \). Lemma 1 implies

\[
d_X U_{a_i}^{a_i}(\gamma) = \int_{t_i}^{t_{i+1}} U_{t_{i+1}, t_i}^{a_i}(\gamma)(-\partial_\gamma A_{\mu}^a(\gamma(t)) X^\mu(\gamma; t) \dot{\gamma}^\nu(t) - A_\mu(\gamma(t)) X^\mu(\gamma; t)) U_{t_i}^{a_i}(\gamma) dt.
\]

Integrating by parts, we have

\[
d_X U_{a_i}^{a_i}(\gamma) = - \int_{t_i}^{t_{i+1}} U_{t_{i+1}, t_i}^{a_i}(\gamma) F_{\mu \nu}(\gamma(t)) X^\mu(\gamma; t) \dot{\gamma}^\nu(t) U_{t_i}^{a_i}(\gamma) dt - \\
- A_\mu(\gamma(t_{i+1})) X^\mu(\gamma; t_{i+1}) U_{t_{i+1}, t_i}^{a_i}(\gamma) + U_{t_i}^{a_i}(\gamma) A_{\mu}^a(\gamma(t_i)) X^\mu(\gamma; t_i). \quad (22)
\]

Also we have

\[
d_X \psi_{a_i+1 a_i}(\gamma(t_i)) = \partial_\gamma \psi_{a_i+1 a_i}(\gamma(t_i)) X^\mu(\gamma; t_i).
\]

Then

\[
d_X U_{a_i+1,a_i}^{a_i+1,a_i}(\gamma) = d_X (U_{a_i+1,a_i}^{a_i+1,a_i}(\gamma) \psi_{a_i+1 a_i}(\gamma(t_i)) U_{t_i}^{a_i}(\gamma)) = \\
= -\psi_{a_i+1}(\gamma) \int_{t_i}^{t_{i+2}} U_{t_{i+2}, t_i}^{a_i+1,a_i}(\gamma) F_{\mu \nu}(\gamma(t)) X^\mu(\gamma; t) \dot{\gamma}^\nu(t) U_{t_i}^{a_i}(\gamma) dt - \\
- A_\mu(\gamma(t_{i+2}))) X^\mu(\gamma; t_{i+2}) U_{t_{i+2}, t_i}^{a_i+1,a_i}(\gamma) + U_{t_i}^{a_i}(\gamma) A_{\mu}^a(\gamma(t_{i+1})) X^\mu(\gamma; t_{i+1}) + \\
+ U_{t_{i+1}, t_i}^{a_i+1,a_i}(\gamma) A_{\mu}^a(\gamma(t_i)) \psi_{a_i+1 a_i}(\gamma(t_i)) - \psi_{a_i+1 a_i}(\gamma(t_i)) A_{\mu}^a(\gamma(t_i)) + \\
+ \partial_\gamma \psi_{a_i+1 a_i}(\gamma(t_i)) X^\mu(\gamma, t_i) U_{t_{i+1}, t_i}^{a_i+1,a_i}(\gamma). \quad (23)
\]

Due to (11), the last summand in (23) is equal to zero. So Leibniz’s rule for (16) implies

\[
d_X U_{1,0}^{a_0 a_1}(\gamma) = -\psi_{a_0}(\gamma) \int_0^1 U_{1,t}(\gamma) F_{\mu \nu}(\gamma(t)) X^\mu(\gamma; t) \dot{\gamma}^\nu(t) U_{t,0}(\gamma) dt - \\
- A_\mu(\gamma(1)) X^\mu(\gamma; 1) U_{1, 0}^{a_0 a_1}(\gamma) + U_{1, 0}^{a_0 a_1}(\gamma) A_{\mu}^a(\gamma(0)) X^\mu(\gamma; 0)
\]

and, therefore, the statement of the proposition. \( \square \)

The following proposition is a direct corollary of Proposition 1.

**Proposition 2.** The following holds

\[
\text{grad}_{H_0} U_{1,0}(\gamma; t)^\mu = -U_{1,t}(\gamma) F_{\mu \nu}(\gamma(t)) \dot{\gamma}^\nu(t) U_{t,0}(\gamma).
\]

**Remark 3.** The first derivative of the parallel transport is well-known in the literature (see, for example, [21, 17]). The non-commutative Stokes formula is based on formula (21) for \( X(0) = X(1) \) (see [17] and also Remark 2.10 in [21]).
5 Covariant Levy divergence and Levy Laplacian

Let $\otimes^2 T^* M$ and $\wedge^2 T^* M$ be the bundles of the symmetric and antisymmetric tensors of type $(0,2)$ over $M$ respectively. Let $R_1$ be the vector bundle over $\Omega$ which fiber over $\gamma \in \Omega$ is the space of all $H^0$-sections in $\otimes^2 T^* M$ along $\gamma$. Let $R_2$ be the vector bundle over $\Omega$ which fiber over $\gamma \in \Omega$ is the space of all $H^1$-sections in $\wedge^2 T^* M$ along $\gamma$.

Let $C^\infty_{AGV}(\Omega, H^1_{0,0} \otimes H^0_{0,0} \otimes E)$ denote the space of all sections $K$ in $H^1_{0,0} \otimes H^1_{0,0} \otimes E$ that have the form

$$K(\gamma) < X, Y > = \int_0^1 \int_0^1 K^L(\gamma; s, t) < X(\gamma; t), Y(\gamma; t) > ds dt +$$

$$+ \int_0^1 K^S(\gamma; t) < \nabla X(\gamma; t), Y(\gamma; t) > dt +$$

$$+ \frac{1}{2} \int_0^1 K^S(\gamma; t) < \nabla Y(\gamma; t), X(\gamma; t) > dt,$$

where $K^L \in C^\infty(\Omega, R_1 \otimes E)$, $K^S \in C^\infty(\Omega, R_2 \otimes E)$ and $K^V \in C^\infty(\Omega, H^0 \otimes H^0 \otimes E)$.

Remark 4. Tensors of the type $[23]$ were in fact considered by Accardi, Gibilisco and Volovich in $[3, 4]$. The kernel $K^L$ is called the Volterra kernel, $K^L$ is called the Levy kernel and $K^S$ is called the singular kernel. By analogy with $[4, 30]$, it can be proved that these kernels are uniquely defined.

Definition 2. The domain $\text{dom} \text{div}_L$ of the (covariant) Levy divergence consists of all $\psi \in C^\infty(H^0 \otimes E)$ such that there exists $K^L \in C^\infty_{AGV}(\Omega, H^1_{0,0} \otimes H^1_{0,0} \otimes E)$ that the following holds

$$g_{H^0 \otimes E}(\nabla^H \psi(\gamma), Y(\gamma) \otimes \Phi(\gamma)) = g_{E}(K^L(\gamma) < X(\gamma), Y(\gamma) >, \Phi(\gamma))$$

for any $\gamma \in \Omega$, for any local sections $X, Y$ in $H^1_{0,0}$ and for any local section $\Phi$ in $E$. The Levy divergence is a linear mapping $\text{div}_L : \text{dom} \text{div}_L \rightarrow C^\infty(E)$ defined by the formula

$$\text{div}_L \psi(\gamma) = \int_0^1 K^L_{\psi \mu \nu}(\gamma; t) g^{\mu \nu}(\gamma(t)) dt,$$

where $K^L_\psi$ is the Levy kernel of the $K^L$.

Remark 5. The notion of the Levy divergence was in fact introduced in $[12]$. See $[12, 14, 15]$ for more information about the connection of this divergence with the Yang-Mills fields.

Definition 3. The value of the Levy Laplacian $\Delta_L$ on $\varphi \in C^\infty(\Omega, E)$ is defined by

$$\Delta_L \varphi = \text{div}_L(\text{grad}_{H^0} \varphi).$$

Example 3. Let $L_f : \Omega \rightarrow \mathbb{R}$ be defined as in Example 2. Then

$$\Delta_L L_f(\gamma) = \int_0^1 \Delta_{(M,g)} f(\gamma(t)) dt,$$

where $\Delta_{(M,g)}$ is the Laplace-Beltrami operator on the manifold $M$.

Theorem 1. The following holds

$$\Delta_L U_{1,0}(\gamma) = - \int_0^1 U_{1,0}(\gamma) \nabla^\mu F_{\mu \nu}(\gamma(t)) \gamma^\nu(t) U_{1,0}(\gamma(t)) dt.$$
Proof. Bellow, we denote $\text{grad}_{\mu(t)} U_{1,0}(\gamma; t)$ by $J(\gamma; t)$. At first we find the covariant derivative of the $J$. In local coordinates, we have the following expression for the directional derivative $dy J$:

$$
dy J^\mu(\gamma; t) = -dy U_{1,t}(\gamma) F^\mu_\nu(\gamma(t)) \gamma^\nu(t) U_{t,0}(\gamma) - U_{1,t}(\gamma) F^\mu_\nu(\gamma(t)) \dot{\gamma}^\nu(t) dy U_{t,0}(\gamma) - U_{1,t}(\gamma) \partial_t F^\mu_\nu(\gamma(t)) Y^\lambda(\gamma; t) \dot{\gamma}^\nu(t) U_{t,0}(\gamma) - U_{1,t}(\gamma) F^\mu_\nu(\gamma(t)) \dot{Y}^\nu(\gamma; t) U_{t,0}(\gamma).
$$

Let $Y(\gamma; 0) = Y(\gamma; 1) = 0$. Using formulas (11), (21), we obtain that

$$
\nabla^\mu_{\nu} J(\gamma; t) = -U_{1,t}(\gamma) F^\mu_\nu(\gamma(t)) \dot{\gamma}^\nu(t) (\int_0^t U_{t,s}(\gamma) F_{\lambda k}(\gamma(s)) Y^\lambda(\gamma; s) \dot{\gamma}^k(s) U_{s,0}(\gamma) ds) - (\int_t^1 U_{1,s}(\gamma) F_{\lambda k}(\gamma(s)) Y^\lambda(\gamma; s) \dot{\gamma}^k(s) U_{s,t}(\gamma) ds) F^\mu_\nu(\gamma(t)) \dot{\gamma}^\nu(t) U_{t,0}(\gamma) - U_{1,t}(\gamma) \nabla_\lambda F^\mu_\nu(\gamma(t)) Y^\lambda(\gamma; t) \dot{\gamma}^\nu(t) U_{t,0}(\gamma) - U_{1,t}(\gamma) F^\mu_\nu(\gamma(t)) \nabla Y^\nu(\gamma; t) U_{t,0}(\gamma).
$$

If also $X(\gamma; 0) = X(\gamma; 1) = 0$, the equality

$$
\int_0^1 U_{1,t}(\gamma) \nabla_\mu F_{\nu \lambda}(\gamma(t)) X^\mu(\gamma; t) Y^\nu(\gamma; t) \dot{\gamma}^\lambda(t) U_{t,0}(\gamma) dt + \int_0^1 U_{1,t}(\gamma) F_{\mu \nu}(\gamma(t)) X^\mu(\gamma; t) \nabla Y^\nu(\gamma; t) U_{t,0}(\gamma) =
$$

$$
= \frac{1}{2} \int_0^1 U_{1,t}(\gamma)(\nabla_\mu F_{\nu \lambda}(\gamma(t)) \dot{\gamma}^\lambda(t) + \nabla_\nu F_{\mu \lambda}(\gamma(t)) \dot{\gamma}^\lambda(t)) X^\mu(\gamma; t) Y^\nu(\gamma; t) U_{t,0}(\gamma) dt + \frac{1}{2} \int_0^1 U_{1,t}(\gamma) F_{\mu \nu}(\gamma(t))(X^\mu(\gamma; t) \nabla Y^\nu(\gamma; t) + Y^\mu(\gamma; t) \nabla X^\nu(\gamma; t)) U_{t,0}(\gamma) dt
$$

(29)

can be obtained by integrating by parts, using the Bianchi identities

$$
\nabla_\mu F_{\nu \lambda} + \nabla_\nu F_{\mu \lambda} + \nabla_\lambda F_{\mu \nu} = 0
$$

and renaming of indices. Formulas (28) and (29) together imply that $J$ belongs to the domain of the Levy divergence. The Volterra kernel $K^V_{J\mu}(\gamma; t, s)$ of $K^V_J$ has the form

$$
K^V_{J\mu}(\gamma; t, s) = \begin{cases} U_{1,t}(\gamma) F_{\mu \lambda}(\gamma(t)) \dot{\gamma}^\lambda(t) U_{t,s}(\gamma) F_{\nu \kappa}(\gamma(s)) \dot{\gamma}^\kappa(s) U_{s,0}(\gamma), & \text{if } t \geq s \\ U_{1,s}(\gamma) F_{\nu \kappa}(\gamma(s)) \dot{\gamma}^\kappa(s) U_{s,t}(\gamma) F_{\mu \lambda}(\gamma(t)) \dot{\gamma}^\lambda(t) U_{t,0}(\gamma), & \text{if } t < s,
\end{cases}
$$

(30)

the Levy kernel $K^L_{J\mu}(\gamma; t)$ has the form

$$
K^L_{J\mu}(\gamma; t) = \frac{1}{2} U_{1,t}(\gamma)(-\nabla_\mu F_{\nu \lambda}(\gamma(t)) \dot{\gamma}^\lambda(t) - \nabla_\nu F_{\mu \lambda}(\gamma(t)) \dot{\gamma}^\lambda(t)) U_{t,0}(\gamma),
$$

and the singular kernel has the form

$$
K^S_{J\mu}(\gamma; t) = U_{1,t}(\gamma) F_{\mu \nu}(\gamma(t)) U_{t,0}(\gamma).
$$

It means that

$$
\Delta_L U_{1,0}(\gamma) = div_L J(\gamma) = - \int_0^1 U_{1,t}(\gamma) \nabla_\mu F_{\mu \nu}(\gamma(t)) \dot{\gamma}^\nu(t) U_{t,0}(\gamma) dt.
$$
Remark 6. As it was mentioned in the introduction, the first Levy Laplacian on the infinite dimensional manifold was introduced in [30]. This Laplacian acts on a space of sections in a vector bundle over \( \Omega \times \). The definition of these operator is based on the triviality of the tangent bundle over \( \Omega \). In the case \( M = \mathbb{R}^d \), both this Levy Laplacian and the covariant Levy Laplacian (26) coincide with the Levy Laplacian introduced in [3]. The Levy Laplacian as the Cesaro mean of the second order directional derivatives was defined on a space of sections in a vector bundle over \( \Omega \times \) in [9]. The values of Levy Laplacians introduced in [30, 9] on the parallel transport coincide with (27) (see [30, 40]). Definitions 1 and 6 of the \( H_0 \)-gradient and the Levy Laplacian can be transferred to the infinite dimension bundles over \( \Omega \). In this case, we conjecture that all three Levy Laplacians coincide on the domain of the covariant Levy Laplacian (26).

Remark 7. Laplacians on abstract infinite-dimensional Hilbert manifolds were considered in the literature (see [14]). It is interesting is it possible to define the Levy Laplacian on the abstract Hilbert manifold and to study the heat equation for this operator. It seems that the definition of the Levy Laplacian as the Cesaro mean of the second order directional derivatives (see [9]) can be useful for this purpose.

In [9], the Feynman approximations were obtained for the solutions of the heat equation for the Levy Laplacian. It is interesting whether it is possible to develop a related approach of the quasi-Feynman approximations to this equation (see [38]).

Due to the fact that the Levy Laplacian can be defined as the averaging of finite-dimensional Laplacians, it would be interesting to investigate whether it is possible to obtain the heat semigroup for the Levy Laplacian by averaging of the semigroups for these finite-dimensional operators (for the method of the averaging of semigroups see [36, 39]).

6 Heat equation for Levy Laplacian and Yang-Mills heat equations

In this section, \( A(\cdot, \cdot) \in C^{1,\infty}([0,T] \times M, \Phi^1 \otimes \text{ad}E) \) and \( U_{1,0}(s, \gamma) \) is the parallel transport generated by the connection \( A(s, \cdot) \) along the curve \( \gamma \in \Omega \).

Proposition 3. For any \( \gamma \in \Omega \) the following holds

\[
\partial_s U_{1,0}(s, \gamma) = - \int_0^1 U_{1,t}(s, \gamma) \partial_s A_\mu(s, \gamma(t)) \dot{\gamma}^\mu(t) U_{t,0}(s, \gamma) dt. \tag{31}
\]

Proof. Consider the partition \( 0 = t_1 < t_2 < \ldots < t_{n-1} < t_n = 1 \) and the family of local trivializations \( \psi_{a_i} : \pi^{-1}(W_{a_i}) \cong W_{a_i} \times \mathbb{C}^N \) of the vector bundle \( E \) such that \( \gamma([t_i, t_{i+1}]) \subset W_{a_i} \). Due to the fact that the time-depended connection belongs to the class \( C^{1,\infty} \), the mapping

\[
[0,1] \ni s \mapsto A_{\mu i}^a(s, \gamma(t)) \in L_2([t_i, t_{i+1}], \text{Lie}(G))
\]

is differentiable for any \( i \in \{1, \ldots, n\} \). Lemma 1 implies that

\[
\partial_s U_{i+1,i}(s, \gamma) = \int_{t_i}^{t_{i+1}} U_{t_{i+1},t_i}(s, \gamma)(-\partial_s A_{\nu i}^a(s, \gamma(t)) \dot{\gamma}^\nu(t)) U_{t,t_i}^a(s, \gamma) dt.
\]

Then Leibniz’s rule for (16) implies the statement of the proposition. \( \square \)

Theorem 2. The following two assertions are equivalent:

1) the flow of connections \( [0,T] \ni s \mapsto A(s, \cdot) \) is a solution of the Yang-Mills heat equations (8):
2) the flow of parallel transports \([0,T] \ni s \mapsto U_{1,0}(s,\cdot)\) is a solution of the heat equation for the Levy Laplacian:

\[
\partial_t U_{1,0}(s,\gamma) = \Delta_L U_{1,0}(s,\gamma).
\]  

(32)

**Proof.** Let the flow of the parallel transports be a solution of the Yang-Mills heat equations. Fix any curve \(\gamma \in C^1([0,1],M)\). Let the curve \(\gamma^r \in \Omega\) be defined by

\[
\gamma^r(t) = \begin{cases} 
  \gamma(t), & \text{if } t \leq r, \\
  \gamma(r), & \text{if } t > r.
\end{cases}
\]

Let us introduce the function \(R \in C^1([0,1], L(E_{\gamma(0)}, E_{\gamma(1)}))\) by the formula:

\[
R(r) = U_{1,r}(s,\gamma)(\partial_s U_{1,0}(s,\gamma^r) - \Delta_L U_{1,0}(s,\gamma^r)).
\]  

Due to the invariance with respect to the reparametrization of the parallel transport and due to the multiplicative property (17), we have

\[
R(r) = \int_0^r U_{1,t}(s,\gamma)(\partial_s A_{\nu}(s,\gamma(t))\gamma^\nu(t) - \nabla_\mu F^\mu_{\nu}(s,\gamma(t))\dot{\gamma}^\nu(t))U_{1,0}(s,\gamma)dt.
\]  

(34)

If \(U_{1,0}(\cdot,\cdot)\) is a solution of (22) then \(\partial_s U_{1,0}(s,\gamma_r) - \Delta_L U_{1,0}(s,\gamma_r) = 0\) and, therefore, \(R(r) \equiv 0\). Differentiating (34), we obtain

\[
\frac{d}{dr}R(r) = U_{1,r}(s,\gamma)(\partial_s A_{\nu}(s,\gamma(r))\gamma^\nu(r) - \nabla_\mu F^\mu_{\nu}(s,\gamma(r))\dot{\gamma}^\nu(r))U_{1,0}(s,\gamma) \equiv 0.
\]

It means that

\[
\partial_s A_{\nu}(s,\gamma(r))\gamma^\nu(r) - \nabla_\mu F^\mu_{\nu}(s,\gamma(r))\dot{\gamma}^\nu(r) = 0
\]

for all \(\gamma \in C^1([0,1],M)\) and for all \(r \in [0,1]\). So \(A(s,\cdot)\) is the Yang-Mills heat flow. The other side of the theorem is trivial. \(\square\)

**Remark 8.** If the connection \(A\) is time-independent, Theorem 2 becomes the Accardi-Gibilisco-Volovich theorem on the equivalence of the Yang-Mills equations and the Laplace equation for the Levy Laplacian.

**Remark 9.** Let \(f(s,\cdot)\) be a solution of the heat equation on the manifold \(M\):

\[
\partial_s f(s,\cdot) = \Delta_M g f(s,\cdot).
\]

Let the family of functionals \(L_{f(s,\cdot)}\) on \(\Omega\) be defined as in Examples 2 and 3. Then \(L_{f(s,\cdot)}\) is a solution of the heat equation for the Levy Laplacian:

\[
\partial_s L_{f(s,\cdot)} = \Delta_L L_{f(s,\cdot)}.
\]

**Remark 10.** The definitions of the \(H^0\)-gradient and the Levy Laplacian can be transferred to the infinite-dimensional bundle over \(\Omega_{x,x}\). In this case, these definitions have the simplest form. We don’t know whether Accardi-Gibilisco-Volovich theorem holds in this case: is it true that if \(\Delta_L U_{1,0}(\gamma) = 0\) for any \(\gamma \in \Omega_{x,x}\), then the connection associated with this parallel transport \(U_{1,0}\) is a solution of the Yang-Mills equations. Our proof of Theorem 2 essentially uses the fact that the endpoints of the curves from the base manifold are not fixed.

In this context, the following result is interesting. In [17], it is shown that if an operator-valued function on \(\Omega_{x,x}\) has some properties of the parallel transport (smoothness, group property, invariance with respect to reparametrization), then it is truly the parallel transport generated by some connection in \(E\). For the generalization of this result for a groupoid see [20].
7 Acknowledgments

The author would like to express his deep gratitude to L. Accardi, O. G. Smolyanov and I. V. Volovich for helpful discussions.

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