A SPECTRAL STUDY OF THE LINEARIZED BOLTZMANN OPERATOR IN $L^2$-SPACES WITH POLYNOMIAL AND GAUSSIAN WEIGHTS

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(Communicated by Yoshinori Morimoto)

Abstract. The spectrum structure of the linearized Boltzmann operator has been a subject of interest for over fifty years and has been inspected in the space $L^2(\mathbb{R}^d, \exp(|v|^2/4))$ by B. Nicolaenko [27] in the case of hard spheres, then generalized to hard and Maxwellian potentials by R. Ellis and M. Pinsky [13], and S. Ukai proved the existence of a spectral gap for large frequencies [33]. The aim of this paper is to extend to the spaces $L^2(\mathbb{R}^d, (1 + |v|)^k)$ the spectral studies from [13, 33]. More precisely, we look at the Fourier transform in the space variable of the inhomogeneous operator and consider the dual Fourier variable as a fixed parameter. We then perform a precise study of this operator for small frequencies (by seeing it as a perturbation of the homogeneous one) and also for large frequencies from spectral and semigroup point of views. Our approach is based on Kato’s perturbation theory for linear operators [22] as well as enlargement arguments from [25, 19].

1. Introduction.

1.1. The model. Consider a rarefied gas whose average number of particles located at position $x \in \Omega$, traveling at velocity $v \in \mathbb{R}^d$ at time $t \geq 0$ is given by $F(t,x,v)$, where $\Omega = \mathbb{T}^d$ or $\mathbb{R}^d$, and $d \geq 2$. Assume furthermore that the particles are uncorrelated, and that they undergo hard sphere collisions where the energy and momentum are conserved. Finally, we assume these binary collisions are the only interactions between particles. Under these conditions, this density satisfies the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F,F),$$  \hspace{1cm}  \text{(B)}

which is a transport equation whose source term takes into account the binary collisions between the particles. The operator $Q$ is called the Boltzmann operator or collision operator and is an integral bilinear operator defined as

$$Q(F,G)(v) := \int_{\mathbb{R}^d} \int_{S^{d-1}} |v - v_*| (F'G_*' - FG_*)dv_*d\sigma$$

where we used the standard notations $v$ and $v_*$ for the velocities of two particles before the collision,

2020 Mathematics Subject Classification. Primary: 34K08, 35Q20; Secondary: 34K27.

Key words and phrases. Spectral theory, pertubation theory, hydromynical limit, semigroup, Boltzmann, enlargement, polynomial weights.
– \( v' \) and \( v'_s \) for their velocities after the collision, given by

\[
v' := \frac{v + v_s}{2} + \frac{|v - v_s|}{2}\sigma, \quad v'_s := \frac{v + v_s}{2} - \frac{|v - v_s|}{2}\sigma,
\]

– \( F' := F(v') \), \( G'_s := G(v'_s) \) and \( G_s := G(v_s) \).

1.1.1. Equilibria. The (global) Maxwellian distributions, which write

\[
M_{\rho, \theta, u}(v) := \frac{\rho}{(2\pi\theta)^{d/2}} \exp\left(-\frac{|v-u|^2}{2\theta}\right)
\]

for some \( \rho, \theta > 0 \) and \( u \in \mathbb{R}^d \) can be shown to be equilibria of (B). We will denote in this paper the normal centered distribution \((\rho = 1, u = 0, \theta = 1)\) by \( M \).

1.1.2. Hydrodynamic limits. By choosing a system of reference values for length, time and velocity (see for example [17, 30]), we get a dimensionless version of the equation:

\[
\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon),
\]

where \( \varepsilon \) is the Knudsen number and corresponds to the mean free path, that is to say the average distance traveled by a particle between two collisions. Performing the linearization \( F^\varepsilon =: M + \varepsilon f^\varepsilon \), the equation rewrites in terms of \( f^\varepsilon \) as

\[
\varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} \mathcal{L} f^\varepsilon + Q(f^\varepsilon, f^\varepsilon), \quad (1)
\]

where \( \mathcal{L}h := Q(M, h) + Q(h, M) \). Letting \( \varepsilon \) go to zero, we expect to get the dynamics of a fluid as the amount of collisions will then go to infinity. This issue of unifying the mesoscopic and macroscopic points of view goes back to Hilbert, and several formal methods have been suggested by Hilbert [21], Chapman, Enskog and Grad [18]. These were made rigorous by C. Bardos, F. Golse and D. Levermore in [6, 7] by proving that if the (renormalized) solution of (1) and its first moments converge in some weak sense, then the limiting moments are (Leray) solutions to the incompressible Navier-Stokes-Fourier system.

To derive hydrodynamic equations from (1), one thus needs to solve the latter for any \( \varepsilon \ll 1 \) and prove the convergence of the solution when \( \varepsilon \) goes to zero. In [8], the authors rewrite (1) in integral form:

\[
f^\varepsilon(t) = S^\varepsilon\left(\frac{t}{\varepsilon^2}\right)f^\varepsilon(0) + \frac{1}{\varepsilon} \int_0^t S^\varepsilon\left(\frac{t-t'}{\varepsilon^2}\right)Q(f^\varepsilon(t'), f^\varepsilon(t'))dt', \quad (2)
\]

\[
S^\varepsilon(t) := \exp(t(\mathcal{L} - \varepsilon v \cdot \nabla_x)),
\]

construct smooth solutions in \( L^\infty_t H^s_x(\mathbb{R}^{1/2} (1 + |v|^s)) \) with \( s > d/2 \), and prove their convergence as \( \varepsilon \) goes to zero. In [15], the authors prove a converse result in the same functional space ; as long as a solution \( f^0 \) to the incompressible Navier-Stokes-Fourier system exists, a solution \( f^\varepsilon \) to (2) exists for \( \varepsilon \) small enough and \( f^\varepsilon \) converges to \( f^0 \). Both papers rely on the spectral study led by [13, 33] of the inhomogeneous linearized Boltzmann operator in \( L^2_t H^s_x(\mathbb{R}^{1/2} (1 + |v|^s)) \) which dictates the asymptotic of \( S^\varepsilon \) and \( \varepsilon^{-1} S^\varepsilon Q \). The theory of hydrodynamic limits for smooth solutions of the Boltzmann equation was partially extended to a larger class of Sobolev spaces with polynomial weights during the last decade: a Cauchy theory close to equilibrium was developed in [19] and their weak compactness with respect to \( \varepsilon \) was shown in [10]. The strong convergence could not be deduced as in [8, 15] since the spectral decomposition from [13] was not known to hold in the case of polynomial weights. This paper aims to provide such a generalization.
1.2. **Statement of the main results.** Let us define some notations used in the statement of Theorem 1.1 and 1.2. We denote \( L^2(m) \) the \( L^2 \) Hilbert space associated with the measure \( m^2(v)dv \). \( \mathcal{B}(X, Y) \) is the space of bounded linear operators from a Banach space \( X \) to another one \( Y \). For a linear operator \( \Lambda \), we denote by \( S_\Lambda \) and \( \Pi_{\Lambda, \lambda} \) is the spectral projector associated with an eigenvalue \( \lambda \in \Sigma_d(\Lambda) \), where \( \Sigma_d(\Lambda) \) is the discrete spectrum of \( \Lambda \). Finally, we write \( \Delta_\alpha := \{ \Re z > a \} \).

We will show that, similarly to the results in [13, 33], when considered as a closed operator in \( L^2(M^{-1/2}) \) or \( L^2(\langle v \rangle^k) \), the semigroup generated by \( \mathcal{L} - iv \cdot \xi \) has exponential decay in time (for large frequencies \( \xi \)), and splits (for small frequencies \( \xi \)) into a first part corresponding to its rightmost eigenvalues, and a remainder that decays exponentially in time. We also give some information on the regularity and asymptotics of the eigenvalues and eigenfunctions for \( |\xi| \ll 1 \).

**Theorem 1.1.** There exists \( k_* > 2 \) such that for any fixed \( k > k_* \), denoting the spaces \( E := L^2(M^{-1/2}) \), \( E(k) := L^2(\langle v \rangle^k) \), the operator \( \mathcal{L}_\xi := \mathcal{L} - iv \cdot \xi \) is closed in both spaces \( E(k) \) and \( E \) for any \( \xi \in \mathbb{R}^d \). Furthermore, the following holds:

1. **Spectral gaps and expansion of the eigenvalues.** For \( r > 0 \) small enough, there exist \( a, b > 0 \) such that, in both spaces,

   \[
   \Sigma(\mathcal{L}_\xi) \cap \Delta_{-a} = \{ \lambda_1(|\xi|), \ldots, \lambda_2(|\xi|) \} \subset \Sigma_d(\mathcal{L}_\xi), \quad |\xi| \leq r,
   \]

   \[
   \Sigma(\mathcal{L}_\xi) \cap \Delta_{-b} = \emptyset, \quad |\xi| \geq r,
   \]

   where the eigenvalues \( \lambda_j \) expand as an absolutely converging power series

   \[
   \lambda_j(|\xi|) = \sum_{n=1}^{\infty} \lambda_{j,n} |\xi|^n,
   \]

   with \( \lambda_{\pm 1}^{(1)} = \pm i \sqrt{1 + 2/d} \), \( \lambda_{j}^{(1)} = 0 \) for \( j = 0, 2 \), and \( \lambda_{j}^{(2)} < 0 \) for \( j = -1, \ldots, 2 \).

2. **Spectral decomposition and expansion of the projectors.** Denoting the set \( U := \{ \xi \in \mathbb{R}^d : 0 < |\xi| \leq r \} \), there exist projectors \( \mathcal{P}_j(\xi) \in \mathcal{B}(E(k), E) \) for any \( \xi \in U \) and \( j = -1, \ldots, 2 \), that expand as a power series

   \[
   \mathcal{P}_j(\xi) = \sum_{n=0}^{\infty} |\xi|^n \mathcal{P}_j^{(n)} \left( \frac{\xi}{|\xi|} \right), \quad \xi := \frac{\xi}{|\xi|},
   \]

   where the convergence is normal in \( \mathcal{B}(E(k), E) \), uniformly in \( \xi \in U \).

   For \( j = 0, \pm 1 \), \( \mathcal{P}_j^{(0)} \left( \frac{\xi}{|\xi|} \right) \) is a projection onto \( \mathbb{C} \mathcal{E}_j^{(0)} \left( \frac{\xi}{|\xi|} \right) \), with

   \[
   e_0^{(0)} \left( \frac{\xi}{|\xi|} \right) = \left( 1 - \frac{1}{2} (|v|^2 - d) \right) M,
   \]

   \[
   e_{\pm 1}^{(0)} \left( \frac{\xi}{|\xi|} \right) = \left( 1 \mp \xi \cdot v + \frac{1}{d} (|v|^2 - d) \right) M,
   \]

   and \( \mathcal{P}_2^{(0)} \left( \frac{\xi}{|\xi|} \right) \) is a projection on \( \{ c \cdot vM : c \cdot \xi = 0 \} \), which is spanned by

   \[
   e_{2,\ell}^{(0)} \left( \frac{\xi}{|\xi|} \right) = C_{\ell} \left( \frac{\xi}{|\xi|} \right) \cdot vM, \quad \ell = 1, \ldots, d - 1,
   \]
where \((\vec{\xi}, C_1(\vec{\xi}), \ldots, C_{d-1}(\vec{\xi}))\) can be assumed to be any fixed orthonormal basis of \(\mathbb{R}^d\). Furthermore, they satisfy
\[
\mathcal{L}_\xi \mathcal{P}_j(\xi) = \lambda_j(\{|\xi|\}) \mathcal{P}_j(\xi),
\]
\[
\mathcal{P}_j(\xi) \mathcal{P}_\ell(\xi) = 0, \ j \neq \ell,
\]
\[
\sum_{j=-1}^{2} \mathcal{P}_j^{(0)}(\vec{\xi}) = \Pi_{L,0}, \ \vec{\xi} \in S^{d-1}.
\]

(3) - Expression of the projectors. For any \(\xi \in U\), any \(j = 0, \pm 1\), and any \(\ell = 1, \ldots, d-1\), there exist functions \(e_j(\xi), e_{2,\ell}(\xi) \in E\) and \(f_j(\xi), f_{2,\ell}(\xi) \in E(k)\) such that the projectors write
\[
\mathcal{P}_j(\xi) g = \langle g, f_j(\xi) \rangle_{E(k)} e_j(\xi), \ j = 0, \pm 1,
\]
\[
\mathcal{P}_2(\xi) g = \sum_{\ell=1}^{d-1} \langle g, f_{2,\ell}(\xi) \rangle_{E(k)} e_{2,\ell}(\xi),
\]
\[
\langle e_\alpha(\xi), f_\beta(\xi) \rangle_{E(k)} = \delta_\alpha, \beta,
\]

where \(\alpha\) and \(\beta\) are any indices among \(-1, 0, 1, (2, 1), \ldots, (2, d-1)\), and they have the following expansions:
\[
e_\alpha(\xi) = \sum_{n=0}^{\infty} \xi^n e_\alpha^{(n)}(\vec{\xi}),
\]
\[
f_\alpha(\xi) = \sum_{n=0}^{\infty} \xi^n f_\alpha^{(n)}(\vec{\xi}),
\]

where \(\alpha\) is any index among \(-1, 0, 1, (2, 1), \ldots, (2, d-1)\), and the convergence in normal, uniformly in \(\xi \in U\), in \(E\) and \(E(k)\) respectively.

Remark 1. A few precisions are to be made on these results.

1. In this theorem, \(e_\alpha : U \to E\), \(f_\alpha : U \to E(k)\) and \(\mathcal{P}_j : U \to \mathcal{B}(E(k), E)\) are measurable.
2. Using the fact that in the Hilbert space \(E\), \((\mathcal{L}_\xi)^* = \mathcal{L}_{-\xi}\), the relation \(\mathcal{O} \mathcal{L}_\xi \mathcal{O}^{-1} = \mathcal{L}_{\mathcal{O}^{-1} \xi}\), where \(\mathcal{O}\) is any real \(d \times d\) orthogonal matrix, and the first order expansion of the eigenvalues, one can show that for any \(\xi \in U\) and \(j = 0, 2,\)
\[
\lambda_j(\{|\xi|\}) = \lambda_{-1}(\{|\xi|\})\) and \(\lambda_j(\{|\xi|\}) \in \mathbb{R}.
\]

See the proof of Proposition 3.5 from [13].
3. Furthermore, one can deduce an expression of \(f_j(\xi)\) in terms of \(e_j(\xi)\) using the fact that, in \(E\), \(\mathcal{P}_2(\xi)^* = \mathcal{P}_2(-\xi)\), \(\mathcal{P}_j(\xi)^* = \mathcal{P}_{-j}(-\xi)\) for \(j = 0, \pm 1\), and using the relation
\[
\langle f, g \rangle_E = \langle f, g(\nu)^{-2k} M^{-1} \rangle_{E(k)}.
\]

Theorem 1.2. Under the same assumptions, denoting \(\mathcal{E} = E\) or \(E(k)\), there exists constants \(C > 0\) and \(\gamma \in (0, a)\) such that for any \(\xi \in \mathbb{R}^d\), \(\mathcal{L}_\xi\) generates on \(\mathcal{E}\)
a $C^0$-semigroup that splits as

$$S_{\mathcal{L}}(t) = \chi(\xi) \sum_{j=-1}^{2} e^{t\lambda_j(\xi)} \mathcal{P}_j(\xi) + \mathcal{V}(t,\xi), \ \xi \neq 0,$$

(18)

$$S_{\mathcal{L}}(t) = \sum_{j=-1}^{2} \mathcal{P}_j(0)(\tilde{\xi}) + \mathcal{V}(t,0) = \Pi_{\mathcal{L},0} + \mathcal{V}(t,0), \ \tilde{\xi} \in \mathbb{S}^{d-1},$$

(19)

where we have denoted $\chi$ the characteristic function of $\{|\xi| \leq r\}$, and the remainder $\mathcal{V}$ satisfies

$$\mathcal{P}_j(\xi) \mathcal{V}(t,\xi) = \mathcal{V}(t,\xi) \mathcal{P}_j(\xi) = 0,$$

(20)

$$\|\mathcal{V}(t,\xi)\|_{\mathcal{B}(\mathcal{E})} \leq Ce^{-\gamma t}.\tag{21}$$

In [3, 10, 19, 23, 25, 32, 31, 4], the authors consider the linearized flow $S_{\Lambda}$ of some form of the Boltzmann equation near a steady state and show it admits in various weighted Sobolev spaces a splitting

$$S_{\Lambda}(t)f = S_{\Lambda}(t)\Pi f + O(e^{-\sigma t}\|f - \Pi f\|),$$

(22)

where $\Pi$ is the projection on some finite dimensional $\Lambda$-stable subspace (in most cases, the null space of $\Lambda$). Such a decomposition was found using the enlargement theory introduced by C. Mouhot [25] then developed by M. Gualdani, S. Mischler and C. Mouhot [19] in various weighted Sobolev spaces, but in the spatially homogeneous case or on the torus, so that $\Lambda$ has a spectral gap. However, there is no spectral gap on the whole space, so one can study the linearized operator in Fourier space with fixed spatial frequency $\xi$ in order to construct a projector $\Pi$ satisfying (22). Furthermore, the splitting (22) allows to develop a Cauchy theory near an equilibrium, but in the context of asymptotic analysis, that is to say when $\Lambda$ and thus $\Sigma(\Lambda), \Pi, \sigma$ depend on a small parameter $\varepsilon$, one also needs to know their behavior as $\varepsilon$ goes to zero (see for instance [2, 8, 9, 10]). This is why we perform a precise study of the spectrum for $|\xi| \ll 1$, and the diffusive limit of the Boltzmann equation requires a second order expansion of the eigenvalues.

1.3. Method of proof and state of the art. Theorem 1.1 was initially proved in [13] in the space $L^2(M^{-1/2})$. The authors first proved that for some $\delta > 0$, the following equations are equivalent for $|\xi|, |\lambda| \leq \delta$:

$$(\mathcal{L} - iv \cdot \xi - \lambda) f = 0,$$

$$F(\lambda, \xi) f_0 = 0,$$

where $f_0$ is the projection of $f$ on $\mathcal{N}$, $f_1 := f - f_0$ is related to $f_0$ by $f_1 = G(\lambda, \xi) f_0$, and $F(\lambda, \xi) \in \mathcal{B}(\mathcal{N})$, $G(\lambda, \xi) \in \mathcal{B}(\mathcal{N}, \mathcal{N}^\perp)$ are smooth in $\xi$ and $\lambda$. They then proceed to solve $\det F(\lambda, \xi) = 0$ for $\lambda = \lambda(\xi)$ using the implicit function theorem, exhibit corresponding $f_0(\xi)$, construct the eigenfunctions and then the spectral projectors.

In their proof, the threshold $\delta$ was not found using constructive estimates, nor do they prove the existence of a spectral gap for $|\xi|$ bounded away from zero. However, their results hold for a general class of potentials, including hard and Maxwellian potentials with cut-off.

T. Yang and H. Yu [36] have a similar approach and still prove their results in $L^2(M^{-1/2})$, but they cover a broader class of kinetic equations. Furthermore, they prove the existence of a spectral gap for large $\xi$ using a key estimate from
and encounter the same difficulties as in this paper: they are able to provide constructive estimates for small and large frequencies, but need a non-constructive argument to deal with intermediate ones.

We also mention [12] and [35] who prove most of these results with similar approaches.

In this paper, we generalize the results from [13, 34] in spaces of the form $L^2(\langle v \rangle^k)$ using a new splitting of the homogeneous operator as well as an “enlargement theorem”, both from [19]. This splitting has the same properties in both Gaussian and polynomial spaces (dissipativity and relative boundedness, regularizing effect, see Lemma 2.2) which allows to treat both cases in a unified framework, and the aforementioned “enlargement theorem” guaranties that the spectral properties (structure of the spectrum and eigenspaces) do not depend on the specific choice of space, be it Gaussian or polynomial. We can therefore rely on previous studies of the Gaussian case when convenient.

As we deal with hard sphere case, the inhomogeneous operator in Fourier space can be seen as a relatively bounded perturbation of the homogeneous operator and thus be studied through classical (analytic) perturbation theory. In particular, all estimates are constructive, except for the exponential decay estimates for large frequencies.

Unlike [13] and [36] who compute the roots of the dispersion relations associated with the linear inhomogeneous Boltzmann equation, we prove that for small $\xi$, the zero eigenvalue (resp. the null space $\mathcal{N}$) of the homogeneous operator “splits” into several eigenvalues (resp. an invariant space $\mathcal{N}(\xi)$ isomorphic to $\mathcal{N}$). We then consider $(\mathcal{L}_\xi)_{|\mathcal{N}(\xi)} \in \mathcal{B}(\mathcal{N}(\xi))$ and straighten $\mathcal{N}(\xi)$ into $\mathcal{N}$ to get a new operator $\tilde{\mathcal{L}}(\xi) \in \mathcal{B}(\mathcal{N})$ conjugated to $(\mathcal{L}_\xi)_{|\mathcal{N}(\xi)}$ which we study using finite dimensional perturbation theory.

1.4. Outline of the paper. In Section 2, we show using results from [19] that there exist some threshold $k_* > 2$ such that in both spaces $L^2(M^{-1/2})$ and $L^2((v)^k)$ with $k > k_*$, $\mathcal{L}_\xi$ generates a strongly continuous semigroup, satisfies some rotation invariance property and the multiplication operator by $v$ is $\mathcal{L}_\xi$-bounded. Then, combining results from [19] and [35], we show the existence in both spaces of spectral gaps for small and large $\xi$: there exists $a, b > 0$ such that for large $\xi$, the spectrum does not meet $\Delta_{-b}$, and for small $\xi$, the part $\Sigma(\mathcal{L}_\xi) \cap \Delta_{-a}$ contains a finite amount of discrete eigenvalues enclosed by some fixed path $\Gamma$.

In Section 3, this path allows to transform the eigenvalue problem into an equivalent one on the finite dimensional null-space of $\mathcal{L}$ and in turn derive expansions for the eigenvalues and associated spectral projectors, thus proving Theorem 1.1.

In Section 4, we prove Theorem 1.2. The splitting comes from Theorem 1.1, and the decay estimate from Theorem B.2 whose assumptions are obtained using estimates from [35] combined with [19], and the continuity of the resolvent.

We recall in the appendix some results from spectral theory and semigroup theory.

1.5. Notations and definitions.

1.5.1. Function spaces. For any Borel function $m > 0$ and $p \in [1, \infty]$, we define the space $L^p(m)$ as the set of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f\|_{L^p(m)} := \|fm\|_{L^p} < \infty.$$
1.5.2. Operator theory. For some given Banach spaces $X$ and $Y$, we will denote the space of closed linear operators $\Lambda$ from their domain $\operatorname{D}(\Lambda)$ to $Y$ by $\mathcal{C}(X, Y)$. The space of bounded linear operators will be denoted $\mathcal{B}(X, Y)$. For any linear operator $\Lambda$, we denote its null space by $\operatorname{N}(\Lambda)$ and its range by $\operatorname{R}(\Lambda)$.

In particular, we write $\mathcal{C}(X) = \mathcal{C}(X, X)$ and $\mathcal{B}(X) = \mathcal{B}(X, X)$. We will also consider the resolvent set $\mathcal{P}(\Lambda)$ of $\Lambda$ which is defined to be the open set of all $z \in \mathbb{C}$ such that $\Lambda - z$ is bijective from $\operatorname{D}(\Lambda)$ onto $X$, and whose inverse is a bounded operator of $X$. The resolvent operator is an analytic function defined by

$$ R_{\Lambda}(z) := (z - \Lambda)^{-1}, $$

and cannot be continued analytically beyond this set. The complement of $\mathcal{P}(\Lambda)$ is called the spectrum of $\Lambda$ and is denoted $\Sigma(\Lambda) = \mathbb{C} - \mathcal{P}(\Lambda)$, which is therefore the set of all values $\lambda$ such that $\Lambda - \lambda$ is not boundedly invertible. When a spectral value $\lambda$ is isolated in the spectrum, or in other words when for some $\varepsilon > 0$ small enough

$$ \Sigma(\Lambda) \cap \{z \in \mathbb{C} | |z - \lambda| < \varepsilon\} = \{\lambda\}, $$

we may define the associated spectral projector

$$ \Pi_{\Lambda, \lambda} := \frac{1}{2\pi i} \int_{P} R_{\Lambda}(z) dz = \text{Res}(R_{\Lambda}; \lambda), $$

where $P$ is some closed path, encircling $\lambda$ and only $\lambda$ exactly once, and that does not meet the spectrum (a circle or any closed loop that can be continuously stretched within $\mathcal{P}(\Lambda)$ into a circle). It is well known that this operator is well defined and is a projector whose range satisfies the following inclusion

$$ \operatorname{N}(\Lambda - \lambda) \subset \operatorname{R}(\Pi_{\Lambda, \lambda}). $$

We call the left-hand side the geometric eigenspace and the right-hand side the algebraic eigenspace, and their dimensions are called respectively the geometric and algebraic multiplicities. When the algebraic multiplicity is finite, i.e. $\dim \operatorname{R}(\Pi_{\Lambda, \lambda}) < \infty$, $\operatorname{N}(\Lambda - \lambda) \neq \{0\}$ and the spectral value $\lambda$ is called a discrete eigenvalue, which we write $\lambda \in \Sigma_{d}(\Lambda)$.

We will also denote by $O(\mathbb{R}^{d})$ the set of $d \times d$ real orthogonal matrices, and denote the action of $O \in O(\mathbb{R}^{d})$ on any function $f$ defined on $\mathbb{R}^{d}$ by

$$ (Of)(v) := f(Ov). $$

In particular, if $\Phi$ is the multiplication operator by a function $\phi = \phi(v)$, then $O\Phi O^{-1}$ is the multiplication operator by $O\phi$.

1.5.3. Semigroup theory. For any $a \in \mathbb{R}$, we write $\Delta_{a} := \{\Re z > a\}$, and for any $C^{0}$-semigroup generator $\Lambda$, we write its semigroup $S_{\Lambda}(t)$.

2. General properties of the linearized operator. The linearized operator $\mathcal{L}$ has been extensively studied in the space $L^{2}(M^{-1/2})$ by Hilbert [21] and Grad [18], let us recall its main properties.

**Theorem 2.1.** Denote $E = L^{2}(M^{-1/2})$ and $L = \mathcal{L}_{|E}$. The operator $L$ is closed in $E$, self-adjoint, dissipative and densely defined. It splits as

$$ L = -\nu + K, $$

(23)
where $K$ is compact on $E$ and $\nu$ is a continuous function of $v \in \mathbb{R}^d$ defined by
\[ \nu(v) := \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} M_s |v - v_s| dv_s d\sigma \]
and satisfying for some $\nu_0, \nu_1 > 0$
\[ \nu_0(v) \leq \nu(v) \leq \nu_1(v). \]  \hfill (24)
There exists a spectral gap for some $a_0 \in (0, \nu_0)$:
\[ \Sigma(L) \cap \Delta_{-a_0} = \{ 0 \}, \]
where $\Delta_{-a_0} := \{ \text{Re} \ z > -a_0 \}$. The eigenvalue 0 is semi-simple and the null space of $L$, denoted $\mathcal{N}$, is spanned by the following basis, orthogonal in $E$:
\[ \{ \varphi_0(v) = M(v), \varphi_j(v) = v_j M(v), j = 1, \ldots, d, \varphi_{d+1}(v) = (|v|^2 - d) M(v) \}. \]

Finally, for any $O \in \mathcal{O} (\mathbb{R}^d)$, $OL = LO$.

**Remark 2.** The existence of this spectral gap has originally been proved using Weyl’s theorem. However, C. Baranger and C. Mouhot provided in [5, Theorem 1.1] an explicit estimate for $a_0$:
\[ a_0 \geq \frac{\pi}{48\sqrt{2e}}. \]

Using this decomposition, R. Ellis, M. Pinsky and S. Ukai [13, 33] proved in the space $L^2 (M^{-1/2})$ the theorems stated in Section 1.2. However, this decomposition does not have the same nice properties in the larger spaces of the form $L^2 ((v)^k)$.

M. Gualdani, S. Mischler and C. Mouhot introduced in [19] a new decomposition with similar properties which holds in both spaces and is presented in Lemma 2.2.

### 2.1. Closedness and decomposition of $\mathcal{L}_\xi$.

In this section, we present a decomposition of the linearized operator $\mathcal{L}_\xi = \mathcal{A} + \mathcal{B}_\xi$, where in both spaces $L^2 (M^{-1/2})$ and $L^2 ((v)^k)$, $\mathcal{A}$ boundedly maps its domain to $L^2 (M^{-1/2})$, $\mathcal{B}_\xi + \mathcal{A}$ is $m$-dissipative for some $a > 0$, and the multiplication operator by $v$ is $\mathcal{L}_\xi$-bounded.

The following lemma combines several results from [19] that were used to prove the existence of a spectral gap for $L - v \cdot \nabla_x$ in a large class of Sobolev spaces $W^{s,p}_x W^{\sigma,q}_v$. We focus instead on $\mathcal{L}_\xi$ in $L^2_v$ spaces, and also show the relative bound- edness of the multiplication operator by $v$.

**Lemma 2.2.** There exists some $k_* > 2$ such that for any $k > k_*$, the perturbed linearized Boltzmann operator splits as
\[ \mathcal{L}_\xi = \mathcal{B}_\xi + \mathcal{A} = \mathcal{B} - iv \cdot \xi + \mathcal{A}, \]
where, denoting $\mathcal{E} = L^2 (M^{-1/2})$ or $L^2 ((v)^k)$, the operator $\mathcal{A}_{|\mathcal{E}}$ is bounded from $\mathcal{E}$ to $L^2 (M^{-1/2})$, $(\mathcal{B}_\xi)_{|\mathcal{E}}$ and thus $(\mathcal{L}_\xi)_{|\mathcal{E}}$ are closed in $\mathcal{E}$ with the common dense domain $\{ f \in \mathcal{E} \mid vf \in \mathcal{E} \}$.

Furthermore, there exist $C > 0$ and $a_1 \in (0, \nu_0)$ such that for any $\xi \in \mathbb{R}^d$
\[ \mathcal{B}_\xi + a_1 \text{ is } m\text{-dissipative,} \]
\[ \|vf\|_{\mathcal{E}} \leq C (\|\mathcal{L}_\xi f\|_{\mathcal{E}} + \|f\|_{\mathcal{E}}), \ f \in D(\mathcal{L}_\xi). \]
Finally, for any $O \in \mathcal{O} (\mathbb{R}^d)$,
\[ O\mathcal{L}_\xi = \mathcal{L}_{O\xi} O. \]  \hfill (25)
Proof. In [19, section 4.3.3], the authors introduce a new splitting of the linearized operator \( L \), which allows to deal with polynomial weights:

\[
L = A_\delta + B_\delta = A_\delta + \left( -\nu + \mathcal{B}_\delta \right), \quad \delta \in (0,1),
\]

\[
A_\delta f(v) := \int_{\mathbb{R}^d \times \mathbb{R}^{d-1}} \Theta_\delta \left( M'_s f' + M' f'_s - M f_s \right) |v - v_s| dv_s d\sigma,
\]

\[
B_\delta f(v) := \int_{\mathbb{R}^d \times \mathbb{R}^{d-1}} \left( 1 - \Theta_\delta \right) \left( M'_s f' + M' f'_s - M f_s \right) |v - v_s| dv_s d\sigma,
\]

where \( \Theta_\delta = \Theta_\delta(v,v_s,\sigma) \) is some smooth function bounded by one on

\[
\{|v| \leq \delta^{-1}, 2\delta \leq |v - v_s| \leq \delta^{-1}, |\cos \theta| \leq 1 - 2\delta \},
\]

and supported in

\[
\{|v| \leq 2\delta^{-1}, \delta \leq |v - v_s| \leq 2\delta^{-1}, |\cos \theta| \leq 1 - \delta \},
\]

where \( \cos \theta := \sigma \cdot (v - v_s)/|v - v_s| \). According to [19, Lemma 4.12, (4.40)], \( \mathcal{B}_\delta \) satisfies the estimate

\[
\left\| \mathcal{B}_\delta f \right\|_{L^2((v)_{\nu^1/2})} \leq b_\delta(q - 1/2) \left\| f \right\|_{L^2((v)_{\nu^1/2})}, \quad q > 5/2,
\]

\[
b_\delta(q) \xrightarrow{\delta \to 0} b(q) := \frac{4}{\sqrt{(q+1)(q-2)}},
\]

\[
\left\| \mathcal{B}_\delta \right\|_{\mathcal{B}(L^2(M^{-1/2}))} \xrightarrow{\delta \to 0} 0.
\]

Furthermore, using the Carleman representation of \( A_\delta \) (see [11]), this operator writes for some kernel \( k_\delta \in C_\infty(\mathbb{R}^d \times \mathbb{R}^d) \)

\[
A_\delta f(v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} k_\delta(v,v_s) f(v_s) dv_s.
\]

We consider in this proof \( k_\delta \) such that \( b(k - 1/2) = 1 \) and fix some \( k > k_s \) so that \( b(k - 1/2) < 1 \). We also consider \( \delta > 0 \) to be small enough so that

\[
a_1 := \nu_0 - \left\| \mathcal{B}_\delta \right\|_{\mathcal{B}(L^2(M^{-1/2}))} > 0, \quad \text{if } \mathcal{E} = L^2(M^{-1/2}),
\]

\[
a_1 := \nu_0 (1 - b_\delta(k - 1/2)) > 0, \quad \text{if } \mathcal{E} = L^2((v)^k).
\]

Step 1: Boundedness and closedness at \( \xi = 0 \). As \( A_\delta \) is an integral operator with a bounded and compactly supported kernel, it is clear that for any of the two spaces \( \mathcal{E} = L^2(M^{-1/2}), L^2((v)^k) \), this operator is bounded from \( \mathcal{E} \) to \( L^2(M^{-1/2}) \).

When \( \mathcal{E} = L^2(M^{-1/2}) \), \( B_\delta \) is the sum of a closed and a bounded operator, so it is closed and densely defined.

When \( \mathcal{E} = L^2((v)^k) \), note\(^1\) that \( \nu_0 > 1 \), which combined with (26) implies that \( \mathcal{B}_\delta \) is \( \nu \)-bounded, with \( \nu \)-bound equal to \( b_\delta(k) < 1 \). Hence \( B_\delta \) is closed on \( L^2((v)^k) \) by [22, Theorem IV-1.1].

In both cases, \( B_\delta \) and thus \( L \) are closed and defined on the dense domain

\[
D(B_\delta) = D(L) = \{ f \in \mathcal{E} \mid \nu f \in \mathcal{E} \}.
\]

Step 2: Dissipativity estimates. By the definition of \( a_1, B_\delta + a_1 \) is dissipative on \( L^2(M^{-1/2}) \).

\(^1\) see for instance [19, Remark 4.1]
In the polynomial space, we have from (26) that
\[ \left( \|B_{\delta} f \|_{L^2(v^k)} \right)^2 \leq b_0 (k-1/2) \| \nu^{1/2} f \|_{L^2(v^k)}^2 . \]
Thus, by the definition of a_1, we have
\[ \langle B_{\delta} f , f \rangle_{L^2(v^k)} \leq (1 - b_0 (k)) \| \nu^{1/2} f \|_{L^2(v^k)}^2 \leq -a_1 \| f \|_{L^2(v^k)} , \]
which yields the dissipativity of B_{\delta} + a_1 on L^2 (\langle v \rangle^k).

**Step 3: Relative bound and closedness of L_\xi and B_\xi.** First, let us show that \( \nu + iv \cdot \xi \) is \( L_\xi \)-bounded uniformly in \( \xi \in \mathbb{R}^d \):
\[
\| (\nu + iv \cdot \xi) f \|_\mathcal{E} \leq \| B_\xi f \|_\mathcal{E} + \| \mathcal{B}_\delta f \|_\mathcal{E} \\
\leq \| B_\xi f \|_\mathcal{E} + \beta \| \nu f \|_\mathcal{E} \\
\leq \| B_\xi f \|_\mathcal{E} + \beta \| (\nu + iv \cdot \xi) f \|_\mathcal{E} 
\]
where \( \beta = b_0 (k) \) for \( \mathcal{E} = L^2 (\langle v \rangle^k) \), and \( \beta = \| \mathcal{B}_\delta \|_{\mathcal{B}(L^2(M^{-1/2}))} \) for \( \mathcal{E} = L^2 (M^{-1/2}) \). In both cases, we assume \( \delta \) to be small enough so that \( \beta \in (0,1) \), which allows to write
\[
\| (\nu + iv \cdot \xi) f \|_\mathcal{E} \leq (1 - \beta)^{-1} \| B_\xi f \|_\mathcal{E} .
\]
We can now show the perturbation \( v \) is \( L_\xi \)-bounded:
\[
\| v f \|_\mathcal{E} \leq \nu_0^{-1} \| (\nu + iv \cdot \xi) f \|_\mathcal{E} \\
\leq \frac{1}{\nu_0 (1 - \beta)} \| B_\xi f \|_\mathcal{E} \\
\leq \frac{1}{\nu_0 (1 - \beta)} \left( \| L_\xi f \|_\mathcal{E} + \| A_\delta f \|_\mathcal{E} \right) .
\]
We thus have a control \( \| v f \|_\mathcal{E} \leq C \| (L_\xi f \|_\mathcal{E} + \| f \|_\mathcal{E}) \), where \( C = C(\mathcal{E}, \delta) \). Thanks to this uniform bound, we know (again, by \[22, IV-1.1\]) that for any \( \xi_0 \in \mathbb{R}^d \) and \( \xi \in \mathbb{R}^d \) satisfying \( |\xi - \xi_0| < 1/C \), \( L_\xi \) is closed if \( L_{\xi_0} \) is. Since \( L \) is closed, we deduce that \( L_\xi \) is closed for all \( \xi \in \mathbb{R}^d \). By the same reasoning and the second line of the previous sequence of estimates, we can show that \( B_\xi \) is closed for all \( \xi \in \mathbb{R}^d \).

Finally, \( B_\xi + a_1 \) is m-dissipative because \( B_\xi \) is boundedly invertible:
\[
B_\xi = \left( -1 + \mathcal{B}_\delta (\nu + iv \cdot \xi)^{-1} \right) (\nu + iv \cdot \xi) ,
\]
where \( \| \mathcal{B}_\delta (\nu + iv \cdot \xi)^{-1} \|_{\mathcal{B}(\mathcal{E})} \leq \beta \| \nu \|_{\mathcal{B}(\mathcal{E})} = \beta < 1. \) Lemma 2.2 is proved. \( \square \)

**Notation.** In the rest of this paper, we fix some \( k > k_* \) and denote the functional spaces
\[
E(k) := L^2 (\langle v \rangle^k) , \quad E := L^2 (M^{-1/2}) , \quad \mathcal{E} = E or E(k).
\]
When considering \( L_\xi \) on one of these spaces, we denote its resolvents
\[
\mathcal{R}(\lambda, \xi) := (\lambda - L_\xi)^{-1} , \\
\mathcal{R}(\lambda) := (\lambda - L)^{-1} = \mathcal{R}(\lambda, 0) .
\]
We also fix some \( a \in (0, \min\{a_0, a_1\}) \), where \( a_0 \) is that of Theorem 2.1 and \( a_1 \) of Lemma 2.2.
2.2. **Spectral gap properties of \( L_\xi \).** In this section, we show the existence of spectral gaps uniform in \( \xi \in \mathbb{R}^d \). More precisely, for small \( \xi \), \( \Delta_{-\mathbf{a}} := \{ \text{Re } z > -\mathbf{a} \} \) contains a finite amount of eigenvalues converging to zero, and lying in the interior of a fixed closed path \( \Gamma \). For large \( \xi \), the half plane \( \Delta_{-\mathbf{b}} = \{ \text{Re } z > -\mathbf{b} \} \) contains no spectral value, for some constant \( \mathbf{b} > 0 \).

After establishing some basic results on the resolvent \( \mathcal{R}(\lambda, \xi) \) (Proposition 1), we prove a spectral gap property (Proposition 2) using the decomposition from Theorem 2.1 for the case \( \mathcal{E} = E \) and an enlargement result from [19] to extend it to the case \( \mathcal{E} = E(k) \). The eigenvalues on the right-hand side of this gap are shown to be separated from the rest of the spectrum by a closed path \( \Gamma \) (Lemma 2). We conclude by proving the spectral gap property for large \( \xi \).

**Proposition 1.** Let \( \mathcal{E} \) be one of the spaces \( E \) or \( E(k) \). For any \( \lambda_0 \in \mathcal{P}(L_{\xi_0}) \), we have that \( v\mathcal{R}(\lambda_0, \xi_0) \in \mathcal{B}(\mathcal{E}) \) and the following expansion around \( (\lambda_0, \xi_0) \) holds:

\[
\mathcal{R}(\lambda, \xi) = \mathcal{R}(\lambda_0, \xi_0) \sum_{n=0}^{\infty} \left[ (\lambda_0 - \lambda)\mathcal{R}(\lambda_0, \xi_0) + iv \cdot (\xi_0 - \xi)\mathcal{R}(\lambda_0, \xi_0) \right]^n,
\]

whenever \( |\xi - \xi_0| \leq v\mathcal{R}(\lambda_0, \xi_0) \| \mathcal{E} \|_{\mathcal{B}(\mathcal{E})} \) and \( |\lambda - \lambda_0| \leq \| \mathcal{R}(\lambda_0, \xi_0) \|_{\mathcal{B}(\mathcal{E})} \). In particular, the resolvent is continuous on the following set, which is open:

\[
\{ (\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^d \mid \lambda \in \mathcal{P}(L_\xi) \}.
\]

**Proof.** Recall that for some \( C > 0 \), we have

\[
\|vf\|_{L_\xi} \leq C (\|\mathcal{L}_\xi f\|_{L_\xi} + \|f\|_{L_\xi}), \quad f \in D(L_\xi).
\]

We deduce that for any \( (\lambda_0, \xi_0) \in \mathbb{C} \times \mathbb{R}^d \) such that \( \lambda_0 \in \mathcal{P}(L_{\xi_0}) \)

\[
\|v\mathcal{R}(\lambda_0, \xi_0)\|_{L_\xi} \leq C (\|L_{\xi_0}\|_{\mathcal{B}(L_\xi)} \|f\|_{L_\xi} + \|\mathcal{R}(\lambda_0, \xi_0)\|_{L_\xi} \|f\|_{L_\xi}).
\]

Rewriting \( L_{\xi_0}\mathcal{R}(\lambda_0, \xi_0) = -1 + \lambda_0 \mathcal{R}(\lambda_0, \xi_0) \), we have for some constant \( C' = C''(\lambda_0, \xi_0) \) that

\[
\|v\mathcal{R}(\lambda_0, \xi_0)\|_{L_\xi} \leq C'' \|f\|_{L_\xi}.
\]

This means that \( v\mathcal{R}(\lambda_0, \xi_0) \in \mathcal{B}(L_\xi) \). Now, rewrite the resolvent as

\[
\mathcal{R}(\lambda, \xi) = (\lambda - L_\xi)^{-1} = (\lambda_0 - L_{\xi_0} - (\lambda_0 - \lambda) - iv \cdot (\xi_0 - \xi))^{-1}
\]

\[
= \mathcal{R}(\lambda_0, \xi_0) (1 - (\lambda_0 - \lambda)\mathcal{R}(\lambda_0, \xi_0) - i(\xi_0 - \xi) \cdot v\mathcal{R}(\lambda_0, \xi_0) )^{-1}
\]

whenever \( (\lambda, \xi) \) is close enough to \( (\lambda_0, \xi_0) \). In such a case, we have the Neumann expansion (27).

The following theorem comes from the “enlargement/factorization theory” initiated by C. Mouhot [25] and developed in [19, 24], which constitutes a breakthrough as it allows to derive spectral information on a linear operator \( \Lambda \) in a large space from information in a smaller one, on which \( \Lambda \) is usually well understood. Such results allowed to generalize a quantitative exponential rate of convergence to equilibrium for the Boltzmann equation in various settings (homogeneous or inhomogeneous, with or without angular cuttof, with weights ranging from Gaussian to polynomial) in [1, 3, 10, 19, 20, 23, 25, 32, 31].

**Proposition 2.** Denote \( L_\xi = (L_\xi|_{\mathcal{E}} \) and \( L_\xi = (L_\xi|_{E(k)}) \).
1. For any $\xi \in \mathbb{R}^d$, the following set consists of a finite amount of discrete eigenvalues:

$$D(\xi) := \Sigma(L_\xi) \cap \overline{\Delta_{-a}} = \Sigma(L_\xi) \cap \Delta_{-a}.$$  

2. For any $\lambda \in D(\xi)$, we have

$$\left(\Pi_{L_\xi, \lambda}\right)_{|E} = \Pi_{L_\xi, \lambda},$$

$$\mathcal{R}\left(\Pi_{L_\xi, \lambda}\right) = \mathcal{R}\left(\Pi_{L_\xi, \lambda}\right),$$

$$N(L_\xi - \lambda) = N(L_\xi - \lambda),$$

and the following factorization formula holds on $\Delta_{-a} - D(\xi)$:

$$\mathcal{R}_{L_\xi} = \mathcal{R}_{\mathcal{B}_\xi} + \mathcal{R}_{L_\xi} A \mathcal{R}_{\mathcal{B}_\xi},$$  

(29)

3. For any $r > 0$, there exists a $T = T(r) > 0$ such that

$$D(\xi) \subset [-a, 0] + i[-T, T], \ |\xi| \leq r.$$  

(30)

**Remark 3.** The previous proposition means that $\mathcal{P}(L_\xi) \cap \Delta_{-a}$, $\Pi_{L_\xi, \lambda}$, $\mathcal{R}\left(\Pi_{L_\xi, \lambda}\right)$ and $N(L_\xi - \lambda)$ can be considered without ambiguity on the space we are working with (the spectral projectors can be restricted to $E$ or extended to $\mathcal{E}(k)$ by density).

**Proof of Proposition 2.** This is a direct application of [19, Theorem 2.1] whose assumptions are met by Lemma 2.2, except for the fact that $\Sigma(L_\xi) \cap \Delta_{-a}$ is made up of a finite amount of discrete eigenvalues, which is proven below.

For any $\lambda \in \Delta_{-\nu_0}$ such that $\|K\mathcal{R}_{\nu + iv, \xi}(\lambda)\|_{\mathcal{B}(\mathcal{E})} < 1$, with $K$ from Theorem 2.1, the resolvent can be factored as

$$\mathcal{R}_{L_\xi}(\lambda) = \mathcal{R}_{\nu + iv, \xi}(\lambda) \left(1 - K\mathcal{R}_{\nu + iv, \xi}(\lambda)\right)^{-1}.$$  

(31)

The following lemma from S. Ukai [33, Lemma 4.4.1] allows to get such estimates for $K\mathcal{R}_{\nu + iv, \xi}$.

**Lemma 2.3.** For any $\delta > 0$, we have

$$\sup_{\sigma \geq -\nu_0 + \delta} \left\|K\mathcal{R}_{\nu + iv, \xi}(\sigma + i\tau)\right\|_{\mathcal{B}(\mathcal{E})} \rightarrow 0, |\xi| \rightarrow 0.$$

(32)

Therefore, by estimate (32), for any $r > 0$, there exists $T = T(r) > 0$ such that

$$\Delta_{-a} \cap \{|\Im z| \geq T\} \subset \mathcal{P}(L_\xi)$$  

(33)

whenever $|\xi| \leq r$. Furthermore, as $L$ is a non-positive self-adjoint operator according to Theorem 2.1, and $iv \cdot \xi$ is skew-symmetric, for any $|\xi| \leq r$,

$$\Sigma(L_\xi) \subset (-\infty, 0] + i[-T, T].$$  

(34)

Combining (33) and (34), we have for any $|\xi| \leq r$

$$\Sigma(L_\xi) \cap \Delta_{-a} \subset [-a, 0] + i[-T, T].$$

However, as $L_\xi$ is the sum of a compact operator and the multiplication operator by $\nu + iv \cdot \xi$, whose range does not meet $\Delta_{-a}$, [22, Theorem IV-5.26] tells us that $\Sigma(L_\xi) \cap \Delta_{-a} \subset \Sigma_d(L_\xi)$.

In conclusion, $\Sigma(L_\xi) \cap \Delta_{-a}$ is a compact discrete set, thus finite, which yields the conclusion.

**Lemma 2.4.** For any $r_\star > 0$, there exists $b_\star \in (0, \nu_0)$ such that

$$\Sigma(L_\xi) \cap \Delta_{-b_\star} = \emptyset, \ |\xi| \geq r_\star.$$
Proof. Once again, we can consider the case \( E = E \). By (32), for some large enough \( R_s > 0 \), we have that \( \Sigma(L_\xi) \cap \Delta_{-a} = \emptyset \) whenever \( |\xi| \geq R_s \). Consider now \( r_s > 0 \) and the sets

\[
X := \{ (\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^d \mid \lambda \in \Sigma(L_\xi), \lambda \in \Delta_{-a}, r_s \leq |\xi| \leq R_s \},
\]

\[
Y := \{ z : \Re z \geq 0 \} \times \{ \xi : r_s \leq |\xi| \leq R_s \}.
\]

If we can show that \( X \) is compact and does not meet \( Y \), then we shall have the conclusion with \( b_s := \text{dist}(X, Y) > 0 \).

**Step 1:** Compactness of \( X \). This set is closed by Proposition 1. Arguing as in the proof of Proposition 2, there exists a \( T > 0 \) such that, for any \( |\xi| \leq R_s \),

\[
\Delta_{-a} \cap \{|\Im z| \geq T\} \subset \text{P}(L_\xi).
\]

Thus \( X \) is compact because it is closed and contained in the bounded set

\[
[-a, 0] + i[-T, T] \times \{ r_s \leq |\xi| \leq R_s \}.
\]

**Step 2:** \( X \) does not meet \( Y \). We know that \( X \) is made up of pairs \((\lambda, \xi)\) such that \( \lambda \in \Sigma_a(L_\xi) \) by Proposition 2, let us now show that for any \((\lambda, \xi) \in X\), we have \( \Re \lambda \leq 0 \). As \( L_\xi \) is dissipative, it is enough to show that it has no eigenvalue in \( i\mathbb{R} \) for \( \xi \neq 0 \). Let us argue by contradiction and consider an eigenvalue \( i\tau \in i\mathbb{R} \) and an associated (non-zero) eigenfunction decomposed \( f = f_0 + f_1 \in \mathcal{N} \oplus \mathcal{N}^\perp \).

Suppose \( f_1 \neq 0 \). As \( \Re \langle L_\xi f, f \rangle \leq -a_0 \| f_1 \|^2_{L_\xi} < 0 \) by the coercivity of \( L \) on \( \mathcal{N}^\perp \), and \( \Re \langle L_\xi f, f \rangle = \Re \langle i\tau \| f \|_{L_\xi}^2 \rangle = 0 \) because \( f \) is an eigenfunction, we get a contradiction, therefore \( f \in \mathcal{N} \). But this would mean that \( L_\xi f = -i\nu \cdot \xi f = i\tau f \), which is impossible as \( f \neq 0 \) and \( \xi \neq 0 \).

3. The eigen problem for small \( \xi \). We show in this section that for small \( \xi \), the eigenvalue 0 of the unperturbed operator \( L \) splits into several semi-simple eigenvalues \( \lambda_{-1}(\xi), \ldots, \lambda_2(\xi) \) of the perturbed operator \( L_\xi \). We also show that these eigenvalues, corresponding spectral projectors and eigenfunctions have Taylor expansions in \( |\xi| \) near \( |\xi| = 0 \).

We rely mostly on perturbation theory and draw inspiration from Kato’s reduction process [22, Section II-2.3]: considering \( \mathcal{P}(\xi) = \sum \Pi_{L_\xi, \lambda} \), where the sum is taken over all eigenvalues \( \lambda \) in \( \Sigma(L_\xi) \cap \Delta_{-a} \), and \( \mathcal{U}(\xi) \) an isomorphism of \( \mathcal{E} \) mapping \( \mathcal{N} \) onto \( \text{R}(\mathcal{P}(\xi)) \), we have

\[
\Sigma(L_\xi) \cap \Delta_{-a} = \Sigma \left( (L_\xi)_{|\text{R}(\mathcal{P}(\xi))} \right) \Sigma \left( \mathcal{U}(\xi)^{-1} L_\xi \mathcal{U}(\xi)_{|\mathcal{N}} \right).
\]

The eigenproblem is thus reduced to the one involving the operator \( \mathcal{U}(\xi)^{-1} L_\xi \mathcal{U}(\xi) \) on the finite dimensional space \( \mathcal{N} \).

In the following, we present Taylor expansions of \( \mathcal{P}(\xi) \) (Lemma 3.1) and \( \mathcal{U}(\xi) \) (Lemma 3.2). We then define the auxiliary operator

\[
\tilde{L}(\xi) := \frac{1}{|\xi|} \mathcal{U}^{-1}(\xi) L_\xi \mathcal{U}(\xi)_{|\mathcal{N}} \in \mathcal{B}(\mathcal{N}),
\]

show we can assume \( \xi = (r, 0, \ldots, 0) \), and give a Taylor expansion (Lemmas 3.3 and 3.4) so that we may use Kato’s theory to solve our eigenproblem.

**Notation.** We will denote the spectral projector on the null space of \( L \) by

\[
P := \Pi_{L, 0}.
\]
Lemma 3.1. There exists $r > 0$ such that, for $|\xi| \leq r$, the projector
\[
P(\xi) := \sum_{\lambda \in D} \Pi_{\mathcal{L}_\xi, \lambda},
\]
where $D := \Sigma(\mathcal{L}_\xi) \cap \Delta_{-a}$, expands in $\mathcal{B}(\mathcal{E})$:
\[
P(\xi) = P + \sum_{n=1}^{\infty} |\xi|^n P(n) \left( \xi \right), \quad \xi := \xi/|\xi|,
\]
where the convergence is normal in $\mathcal{B}(\mathcal{E})$, uniformly in $|\xi| \leq r$, and
\[
P(1) \left( \xi \right) = iPv \cdot \xi S + iSv \cdot \xi P \in \mathcal{B}(\mathcal{E}),
\]
where $S$ is the reduced resolvent of $\mathcal{L}$ at $\lambda = 0$ (see (45) for the definition). Furthermore, $P(\xi)|_E$ and $R P(\xi))$ do not depend on the choice of space $E = E$ or $E = E(k)$.

Proof. Consider the closed curve
\[
\Gamma := \partial([a,0] + i[-T,T]),
\]
where $T$ is that of Lemma 2 for $r = 1$. Choose some $r \in (0,1)$ such that
\[
\sup_{\lambda \in \Gamma} \left|v \mathcal{R}(\lambda)\right|_{\mathcal{B}(\mathcal{E})} < \frac{1}{r},
\]
where the left-hand side is finite as $\Sigma(\mathcal{L}) \cap \Delta_{-a} = \{0\}$ in virtue of Theorem 2.1 and Proposition 2. The following series thus converges normally in $\mathcal{B}(\mathcal{E})$, uniformly in $(\lambda,\xi) \in \Gamma \times \{\xi : |\xi| \leq r\}$:
\[
\mathcal{R}(\lambda,\xi) = \mathcal{R}(\lambda) \sum_{n=0}^{\infty} |\xi|^n \left(-iv \cdot \xi \mathcal{R}(\lambda)\right)^n.
\]
In particular, for any $|\xi| \leq r$, $\Gamma$ does not meet $\Sigma(\mathcal{L}_\xi)$ and encloses the eigenvalues in $\Sigma(\mathcal{L}_\xi) \cap \Delta_{-a}$. By integrating this series along $\lambda \in \Gamma$, we get the expansion of $P(\xi)$. The expression of the coefficients comes from the residue Theorem and the fact that $\lambda = 0$ is a semi-simple eigenvalue of $\mathcal{R}(\lambda)$, combined with the expansion (44). The last point of the lemma comes from point 2. of Proposition 2. \qed

Lemma 3.2. There exists $r > 0$ and a family of invertible maps $U(\xi) \in \mathcal{B}(\mathcal{E})$ for any $|\xi| \leq r$ such that $U(\xi)$ maps $N = R\left(P\right)$ onto $R\left(P(\xi)\right)$, and $U(\xi)|_E$ does not depend on the choice $E = E$ or $E = E(k)$. Furthermore, they follow the expansion
\[
U(\xi) = \text{Id} + \sum_{n=1}^{\infty} |\xi|^n U^{(n)} \left( \xi \right),
\]
where the convergence is normal in $\mathcal{B}(\mathcal{E})$, uniformly in $|\xi| \leq r$, and
\[
U^{(1)} \left( \xi \right) = iPv \cdot \xi S - iSv \cdot \xi P \in \mathcal{B}(\mathcal{E}),
\]
where $S$ is the reduced resolvent of $\mathcal{L}$ at $\lambda = 0$ (see (45) for the definition).

Proof. Kato’s process [22, Section I-4.6] shows that whenever two bounded projectors $P$ and $Q$ are such that $\|P - Q\|_{\mathcal{B}(\mathcal{E})} < 1$, we can define an invertible map $U$ satisfying the relation $UP^{-1} = Q$ by
\[
U = U'(1 - R)^{-1/2} = (1 - R)^{-1/2} U'.
\]
where we have noted
\[
R = (P - Q)^2,
\]
\[
(1 - R)^{-1/2} = \sum_{n=0}^{\infty} \left(\frac{-1/2}{n}\right)(-R)^n,
\]
\[
U' = QP + (1 - Q)(1 - P).
\]

By assuming \( r > 0 \) to be small enough so that \( ||P(\xi) - P||_{\mathcal{B}(E)} < 1 \) whenever \( |\xi| \leq r \), we define this way \( U(\xi) = U \) with \( P = P \) and \( Q = P(\xi) \). The existence of the expansion comes from the expansion of \( P(\xi) \), and the coefficients can be computed from the latter, using the fact that \( PS = SP = 0 \).

The fact that \( U(\xi)|_{E} \) does not depend on the choice of \( E \) comes from the last point of Lemma 3.1.

**Lemma 3.3.** The reduced operator defined by
\[
\tilde{\mathcal{L}}(\xi) := \frac{1}{|\xi|}U(\xi)^{-1}\mathcal{L}_\xi U(\xi)|_{\mathcal{N}} \in \mathcal{B}(\mathcal{N})
\]
does not depend on the initial choice of space \( \mathcal{E} = E, E(k) \), and has the expansion
\[
\tilde{\mathcal{L}}(\xi) = -iP\tilde{\xi} \cdot v + |\xi|P\tilde{\xi} \cdot vS\tilde{\xi} \cdot v + \sum_{n=2}^{\infty} |\xi|^n \tilde{\mathcal{L}}(n)(\tilde{\xi}),
\]
where the convergence is normal uniformly in \( |\xi| \leq r \).

Furthermore, for any \( |\xi| \leq r \), its spectrum is related to the one of \( \mathcal{L}_\xi \) by
\[
\Sigma(\mathcal{L}_\xi) \cap \Delta_{-a} = \Sigma\left(|\xi|\tilde{\mathcal{L}}(\xi)\right).
\]

**Proof.** By Lemmas 3.1 and 3.2, the operator \( \tilde{\mathcal{L}}(\xi) \) is well defined for \( \xi \neq 0 \), maps \( \mathcal{N} \) onto itself, and does not depend on the choice of space \( \mathcal{E} \). As \( \tilde{\mathcal{L}}(\xi) \) has a Taylor expansion around \( \xi = 0 \) in \( |\xi| \), we just need to check that the same is true for \( \frac{1}{|\xi|}\mathcal{L}_\xi \).

For the same reason as in Proposition 3.1, the series
\[
\mathcal{L}_\xi \mathcal{P}(\xi) = \frac{1}{2i\pi} \int_\Gamma z\mathcal{R}(z, \xi)dz = \sum_{n=0}^{\infty} \frac{|\xi|^n}{2i\pi} \int_\Gamma z\mathcal{R}(z) \left(-iv \cdot \tilde{\xi}\mathcal{R}(z)\right)^n dz,
\]
converges normally in \( \mathcal{B}(E) \) for \( |\xi| \leq r \) small enough. Using the residue Theorem, the first terms are

- for \( |\xi|^0 : 0 \), because 0 is a semi-simple eigenvalue of \( \mathcal{L} \), and thus a simple pole of \( \mathcal{R}(z) \),
- for \( |\xi|^1 : -iPv \cdot \tilde{\xi}P \),
- for \( |\xi|^2 : P(v \cdot \tilde{\xi})^2 S + S(v \cdot \tilde{\xi}P)^2 \).

We get (38) by combining this expansion with (37).

Finally, as \( \Gamma \) circles the eigenvalues in \( \Sigma(\mathcal{L}_\xi) \cap \Delta_{-a} \) and \( \mathcal{P}(\xi) \) is the spectral projector associated with them, we have \( \Sigma\left(|\xi|\tilde{\mathcal{L}}(\xi)\right) = \Sigma(\mathcal{L}_\xi) \cap \Delta_{-a} \) according to [22, Theorem III-6.17], and (39) holds as \( \tilde{U}(\xi) \) is an isomorphism mapping \( \mathcal{N} \) onto \( R(P(\xi)) \).
Before we prove Theorem 1.1, we need the following lemma that allows to assume $\xi$ to be of the form $(r,0,\ldots,0)$ where $r \in [0,\mathbf{r}]$, and to deal with the fact that we do not know whether or not the eigenvalues $\lambda_0$ and $\lambda_2$ of this theorem are distinct.

Lemma 3.4. For $0 < |\xi| \leq \mathbf{r}$ and any $O \in O(\mathbb{R}^d)$ such that $O\bar{\xi} = (1,0,\ldots,0)$,

\[ O\bar{\xi} = \bar{\xi}(|\xi|,0,\ldots,0)O. \tag{40} \]

Furthermore, there exist $\bar{\lambda}_2(r) \in \mathbb{C}$ and a $3 \times 3$ matrix $A(r)$ such that the operator $\bar{\xi}(r) := \bar{\xi}(r,0,\ldots,0)$ writes in the basis $\{\varphi_2, \ldots, \varphi_d, \varphi_0, \varphi_1, \varphi_{d+1}\}$

\[ \bar{\xi}(r) = \begin{pmatrix} \bar{\lambda}_2(r)Id_{d-1} & O_{(d-1)\times 3} \\ O_{3\times (d-1)} & A(r) \end{pmatrix}. \tag{41} \]

Proof. Recall that whenever $O \in O(\mathbb{R}^d)$ is such that $O\bar{\xi} = (1,0,\ldots,0)$, we have the relation $O\xi = \bar{\xi}(\xi,0,\ldots,0)O$. As $R(z,\xi)$, $P(\xi)$, $U(\xi)$ and $L(\xi)$ are constructed from $L(\xi)$, (40) holds.

Step 1: Block decomposition. Let $j \in \{2, \ldots, d\}$ and $k \neq j$. Consider the orthogonal symmetry $O : v_j \leftrightarrow -v_j$. Noting that $O\bar{\xi}(r) = \bar{\xi}(r)O$ and $O\varphi_j = -\varphi_j$, we have

\[ \left\langle \bar{\xi}(r)\varphi_j, \varphi_k \right\rangle_\xi = \left\langle O\bar{\xi}(r)\varphi_j, O\varphi_k \right\rangle_\xi \\
= \left\langle \bar{\xi}(r)O\varphi_j, O\varphi_k \right\rangle_\xi \\
= -\left\langle \bar{\xi}(r)\varphi_j, \varphi_k \right\rangle_\xi. \]

Therefore, $\left\langle \bar{\xi}(r)\varphi_j, \varphi_k \right\rangle_\xi = 0$, and similarly $\left\langle \bar{\xi}(r)\varphi_j, \varphi_j \right\rangle_\xi = 0$. We conclude that $\bar{\xi}(r)$ has the matrix representation

\[ \begin{pmatrix} B(r) & O_{(d-1)\times 3} \\ O_{3\times (d-1)} & A(r) \end{pmatrix}, \]

where $B(r)$ is some diagonal $(d-1) \times (d-1)$ matrix.

Step 2: The diagonal block. Consider the orthogonal symmetry $O : v_j \leftrightarrow v_j+1$ where $j \in \{2, \ldots, d-1\}$. Noting that $O\bar{\xi}(r) = \bar{\xi}(r)O$, we have

\[ \left\langle \bar{\xi}(r)\varphi_j, \varphi_j \right\rangle_\xi = \left\langle O\bar{\xi}(r)\varphi_j, O\varphi_j \right\rangle_\xi \\
= \left\langle \bar{\xi}(r)O\varphi_j, O\varphi_j \right\rangle_\xi \\
= \left\langle \bar{\xi}(r)\varphi_{j+1}, \varphi_{j+1} \right\rangle_\xi. \]

We then conclude to (41) by induction on $j$. \hfill \Box

Proof of Theorem 1.1. The first steps of the proof rely on the following formulas which come from (38) and (41)

\[ \bar{\xi}(r) = \begin{pmatrix} \bar{\lambda}_2(r)Id_2 & O_3 \\ O_2 \end{pmatrix} \\
= -iPv_1 + rv_1v_1sv_1 + o(r). \]
**Step 1: The multiple eigenvalue.** The operator $\tilde{L}(r)$ has an obvious $(d-1)$-dimensional eigenvalue $\tilde{\lambda}_2(r)$. The corresponding eigenvectors are $\varphi_2, \ldots, \varphi_d$, and as they are normalized for the inner product of $E$, we have

$$
\tilde{\lambda}_2(r) = -i(v_1 \varphi_2, \varphi_2)_E + r(Sv_1 \varphi_2, v_1 \varphi_2)_E + o(r),
$$

because $v_1 \varphi_2 M^{-1}(v)$ is odd in $v_1$. The first order derivative $(Sv_1 \varphi_2, v_1 \varphi_2)_E$ is negative because $v_1 \varphi_2 \notin \mathcal{N}$ and $(Sf, f)_E \leq -a \|f\|_E^2$ for any $f \in \mathcal{N}^\perp$.

**Step 2: The simple eigenvalues.** We are now going to investigate the eigenvalues of $\tilde{L}(r)$ on the subspace $X := \{\varphi_0, \varphi_1, \varphi_{d+1}\}$, that is to say, we are going to study $A(r)$. We have that

$$
A(r) = -iPv_1 + rPv_1Sv_1 + o(r) \text{ on } X.
$$

The matrix representation of $A(0)$ is

$$
A(0) = -i \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2/d} \\ 0 & \sqrt{2/d} & 1 \end{pmatrix}.
$$

One can show that $A(0)$ is diagonalizable with the following eigenvalues and corresponding eigenvectors

$$
\begin{align*}
\psi_{-1} &= \left( 1 + v_1 + \frac{1}{d} (|v|^2 - d) \right) M, \\
\psi_0 &= \left( 1 - \frac{1}{2} (|v|^2 - d) \right) M, \\
\psi_1 &= \left( 1 - v_1 + \frac{1}{d} (|v|^2 - d) \right) M.
\end{align*}
$$

By [22, Theorem II-5.4], $A(r)$ is diagonalizable with three distinct simple eigenvalues $\tilde{\lambda}_j(r) = ij \sqrt{1 + 2/d} + \beta_j r + o(r)$ for $r$ small enough, where

$$
\beta_j = (Sv_1 \psi_j, v_1 \psi_j)_E < 0,
$$

because $v_1 \psi_j \notin \mathcal{N}$.

Denoting $\lambda_j(|\xi|) := |\xi| \tilde{\lambda}_j(|\xi|)$, we have (5), and (3) using the relation (39). The spectral gap property (4) is just Lemma 2.4. Point (1) is proved.

**Step 3: The spectral decomposition.** We have the decomposition

$$
\sum_{j=-1}^{2} \tilde{P}_j(r) = \text{Id}_{\mathcal{N}}, \quad \tilde{P}_j(r)\tilde{P}_k(r) = \delta_{j,k} \tilde{P}_j(r),
$$

$$
\tilde{L}(r) = \sum_{j=-1}^{2} \tilde{\lambda}_j(r) \tilde{P}_j(r),
$$

where $\tilde{P}_j(r)$ is the one-dimensional spectral projector of $A(r)$ associated with $\tilde{\lambda}_j(r)$ and extended by 0 on $\text{Span} (\varphi_2, \ldots, \varphi_d)$, and $\tilde{P}_2(r)$ is the projection on $\text{Span} (\varphi_2, \ldots, \varphi_d)$ parallel to $X$. 
By (40), we go back to the general case of $\xi$ not necessarily of the form $(r, 0, \ldots, 0)$, using $O \in \mathcal{O}(\mathbb{R}^d)$ such that $O\xi = (1, 0, \ldots, 0)$:

$$L_\xi P(\xi) = |\xi| U(\xi)O\overline{L}(|\xi|)O^{-1}U(\xi)^{-1}$$

$$= \sum_{j=1}^2 \lambda_j(|\xi|)P_j(\xi),$$

where we have defined $P_j(\xi) := U(\xi)O\overline{P}_j(|\xi|)O^{-1}U(\xi)^{-1}$. By Lemma 3.2, $U(\xi)$ has a Taylor expansion in $\mathcal{B}(E)$ and $U(\xi)^{-1}$ has one in $\mathcal{B}(E(k))$, therefore this projector has the expansion (6) in $\mathcal{B}(E(k), E)$, and $P_j^{(0)}(\xi) = O\overline{P}_j^{(0)}O^{-1}$. We have thus proved (10)-(11), and (12) comes from the definition of $P(\xi)$ in the case $|\xi| = 0$.

**Step 5: Range of the projectors for $|\xi| = 0$.** For $j = 0, \pm 1$, $P_j^{(0)}(\xi)$ is a projection onto the subspace spanned by $e_j^{(0)}(\xi)$, where

$$e_0^{(0)}(\xi) := O^{-1}\psi_0 = \left(1 - \frac{1}{2} (|v|^2 - d)\right) M,$$

$$e_{\pm 1}^{(0)}(\xi) := O^{-1}\psi_{\pm 1} = \left(\pm \bar{v} \cdot v + \frac{1}{d} (|v|^2 - d)\right) M,$$

and $P_2^{(0)}(\xi)$ is a projection on the subspace

$$\text{Span} \left\{e_2^{(0)}(\xi), \ldots, e_d^{(0)}(\xi)\right\} = \left\{c \cdot v M \mid c \cdot \bar{\xi} = 0\right\},$$

$$e_j^{(0)}(\xi) := O^{-1}\varphi_j = C_j(\xi) \cdot v M, j = 2, \ldots, d,$$

where $(\xi, C_2(\xi), \ldots, C_d(\xi))$ is an arbitrary orthonormal basis of $\mathbb{R}^d$. Point (2) is proved.

**Step 6: Expression of the projectors.** Consider $\{e_j(\xi)\}_{j=2}^d$ the family obtained by the Gram-Schmidt orthogonalization of $\{P_j(\xi)e_j^{(0)}(\xi)\}_{j=2}^d$ for the inner product of $E(k)$, and denote $e_j(\xi) := P_j(\xi)e_j^{(0)}(\xi)$ for $j = 0, \pm 1$. Note that by (6), the function $P_j(\xi)e_j^{(0)}(\xi)$ follows itself an expansion of the form (16) with the same $e_j^{(0)}(\xi)$, and since this family is orthogonal for $|\xi| = 0$:

$$\forall 2 \leq j < k \leq d, \quad \langle e_j^{(0)}(\xi), e_k^{(0)}(\xi) \rangle_{E(k)} = \langle O^{-1}\varphi_j, O^{-1}\varphi_k \rangle_{E(k)} = \langle \varphi_j, \varphi_k \rangle_{E(k)} = 0,$$

and the orthogonalization process is analytic, the $e_j(\xi)$ satisfy (16).

Define the functions $f_j(\xi) := P_j(\xi)^*e_j(\xi)$ where the adjoint is considered for the inner product of $E(k)$. They have the expansion (17) by (6). They satisfy the biorthogonality relation (15) by (11) when $j$ or $k = 0, \pm 1$, and by the orthogonalization when $j, k \in \{2, \ldots, d\}$.

Point (3) is proved. \hfill \Box
Remark 4. The coefficients $C_j \left( \tilde{\xi} \right)$ in Theorem 1.1 can therefore be assumed to be measurable, but not continuous as it is a non-vanishing tangent vector field on the sphere $\mathbb{S}^{d-1}$, by the hairy ball theorem.

4. Exponential decay of the semigroup.

Proof of Theorem 1.2. The proof will use the following factorization in $\mathcal{B}(E)$ that comes from the combination of (29) and (31):

$$R_{\mathcal{L}_\xi}(z) = R_{\mathcal{B}_\xi}(z) + R_{-(\nu + iv \cdot \xi)}(z) (1 - K R_{-(\nu + iv \cdot \xi)}(z))^{-1} A R_{\mathcal{B}_\xi}(z)$$

which holds whenever $\|K R_{-(\nu + iv \cdot \xi)}(z)\|_{\mathcal{B}(E)} < 1$.

Step 1: Global estimates. Note that as $\mathcal{L}_\xi - \omega$ is dissipative for $\omega = -a + \|A\|_{\mathcal{B}(E)}$ according to Lemma 2.2 and the fact that $iv \cdot \xi$ is skew-symmetric, we have

$$\forall \xi \in \mathbb{R}^d, \|S_{\mathcal{L}_\xi}(t)\|_{\mathcal{B}(E)} \leq e^{\omega t}.$$

Furthermore, (32) means that for some $T > 0$, $\|K R_{-(\nu + iv \cdot \xi)}(z)\|_{\mathcal{B}(E)} \leq \frac{1}{2}$ holds if $|\Im z| \geq T$ and $\Re z \geq -a$. The factorization (42) combined with the dissipativity of $B_\xi + a_1$ from Lemma 2.2 yields the following bound for $z \in \Delta_{-a} \cap \{ |\Im z| \geq T \}$ and $\xi \in \mathbb{R}^d$:

$$\|R_{\mathcal{L}_\xi}(z)\|_{\mathcal{B}(E)} \leq \|R_{\mathcal{B}_\xi}(z)\|_{\mathcal{B}(E)} + 2 \|R_{-(\nu + iv \cdot \xi)}(z)\|_{\mathcal{B}(E)} \|A R_{\mathcal{B}_\xi}(z)\|_{\mathcal{B}(E)} \leq (a_1 - a)^{-1} + \frac{2\|A\|_{\mathcal{B}(E)}}{(\nu_0 - a)(a_1 - a)} \leq M,$$

for some $M > 0$. The dissipativity of $\mathcal{L}_\xi - \omega$ tells us that, taking $T$ large enough, we may assume

$$\forall \xi \in \mathbb{R}^d, z \in U, \|R_{\mathcal{L}_\xi}(z)\|_{\mathcal{B}(E)} \leq M, \quad (43)$$

where $U := \Delta_{-a} \cap \{ |z + a| \geq T \}$.

Step 2: Small $\xi$. From Theorem 1.1, up to a reduction of $r$, for some $\delta > 0$,

$$\Sigma(\mathcal{L}_\xi) \cap \Delta_{-a + 2\delta} = \Sigma(\mathcal{L}_\xi) \cap \Delta_{-a} = \{ \lambda_{-1}(|\xi|), \ldots, \lambda_{2}(|\xi|) \}$$

whenever $|\xi| \leq r$. In particular, $-a + \delta + i\mathbb{R} \subset P(\mathcal{L}_\xi)$ for any $|\xi| \leq r$, and by the continuity of $R_{\mathcal{L}_\xi}(z)$ in $(z, \xi)$ combined with (43), we have for some $K_0^{(-)} > 0$,

$$\forall |\xi| \leq r, \sup_{-a + \delta + i\mathbb{R}} \|R_{\mathcal{L}_\xi}\|_{\mathcal{B}(E)} \leq K_0^{(-)}.$$

Denote the following invariant subspaces and restriction by

$$N(\xi) := R(P(\xi)), N(\xi)_{\perp} := R(1 - P(\xi)), \mathcal{L}_\xi^\perp := (\mathcal{L}_\xi\mid_{\bar{N}(\xi)_{\perp}}).$$

By [22, Theorem III-6.17], $\Sigma(\mathcal{L}_\xi^\perp) \cap \Delta_{-a} = \emptyset$, and the semigroup and the resolvent associated with $\mathcal{L}_\xi$ split along the direct sum $E = N(\xi) \oplus N(\xi)_{\perp}$ as

$$S_{\mathcal{L}_\xi}(t) f = \sum_{j=-1}^{2} e^{\lambda_j(|\xi|) t} \mathcal{P}_j(\xi) f + S_{\mathcal{L}_\xi^\perp}(t) f_{\perp},$$

$$R_{\mathcal{L}_\xi}(z) = \sum_{j=-1}^{2} \frac{\mathcal{P}_j(\xi)}{\lambda_j(|\xi|) - z} f + R_{\mathcal{L}_\xi^\perp}(z) f_{\perp},$$

where $\lambda_j(|\xi|)$ are the eigenvalues of $P_j(\xi)$.
where \( f^\perp = f - \mathcal{P}(\xi)f \). Using the fact that \( \mathcal{R}_{L^+_{\xi}} \) is holomorphic on \( \Delta_{-a} \) and the maximum modulus principle, we deduce from the relation between \( \mathcal{R}_{L^+_{\xi}} \) and \( \mathcal{R}_{L^+_{\xi}}^{+} \), and the previous estimates, the bound

\[
\forall z \in \Delta_{-a+\delta}, \quad \|R_{L^+_{\xi}}(z)\|_{\mathcal{B}(\mathcal{N}(\xi)^{+})} \leq \max\{M, R_0^{(+)} + 4/\delta\} =: K^{(-)},
\]

which is uniform in \( |\xi| \leq r \).

We have shown that for any fixed \( |\xi| \leq r \), the operator \( L^+_{\xi} \) satisfies the assumptions of Theorem B.2 with 
\( X = \mathcal{N}(\xi)^{+}, \alpha = \omega, C_\alpha = 1, \beta = -a + \delta, K_\beta = K^{(-)} \).

We thus have the bound

\[
\forall |\xi| \leq r, \quad \|S_{L^+_{\xi}}(t)\|_{\mathcal{B}(\mathcal{N}(\xi)^{+})} \leq C^{(-)}e^{-a+\delta}t,
\]

for some \( C^{(-)} > 0 \). For \( |\xi| \leq r \), we define \( \mathcal{V}(t, \xi) \) to be \( S_{L^+_{\xi}}(t) \) extended by 0 on \( \mathcal{N}(\xi) \) (note that it does not change its growth estimate).

**Step 3: Large \( \xi \).** By Lemma 2.4, for some \( b \in (0, a) \), we have

\[
\forall |\xi| \geq r, \quad \Sigma(L^+_{\xi}) \cap \Delta_{-b} = \emptyset.
\]

By (32), we may assume that for some large enough \( R > r \),

\[
\forall |\xi| \geq R, \quad \sup_{\Delta_{-b}} \|R_{L^+_{\xi}}\|_{\mathcal{B}(\mathcal{E})} \leq M
\]

also holds. Again, the continuity of \( R_{L^+_{\xi}} \) implies the existence of a bound \( K_0^{(+)} > 0 \) on \( -b + i\mathbb{R} \) uniform in \( |\xi| \geq r \), and by a similar argument as in Step 2, we prove

\[
\forall |\xi| \geq r, \quad \|R_{L^+_{\xi}}(\xi)\|_{\mathcal{B}(\mathcal{E})} \leq K^{(+)}
\]

for some \( K^{(+)} > 0 \). We invoke once again Theorem B.2 with \( X = \mathcal{E}, \alpha = \omega, C_\alpha = 1, \beta = -b, K_\beta = K^{(+)} \) to obtain

\[
\forall |\xi| \geq r, \quad \|S_{L^+_{\xi}}(t)\|_{\mathcal{B}(\mathcal{E})} \leq C^{(+)}e^{-bt}
\]

for some \( C^{(+)} > 0 \). For \( |\xi| \geq r \), we define \( \mathcal{V}(t, \xi) \) to be \( S_{L^+_{\xi}}(t) \). We finally get the conclusion with \( \gamma = \min\{-a + \delta, b\} \) and \( C = \max\{C^{(-)}, C^{(+)}\} \).

**Acknowledgments.** The author would like to thank Isabelle Gallagher and Isabelle Tristani for their guidance and constant support. Many thanks are owed to the anonymous reviewers for their careful reading and for pointing out several works related to this study.

**Appendix A. Spectral theory.** Consider a Banach space \( X \) and \( \Lambda \in \mathcal{C}(X) \). If \( \lambda \in \Sigma_d(\Lambda) \), then \( \lambda \) is a finite order pole of the resolvent, which can be expanded as

\[
R_\lambda(\lambda + h) = \sum_{k=1}^{m} \frac{D^k}{h^{k+1}} + \frac{1}{h} \Pi_{\lambda, \lambda} - \sum_{n=0}^{\infty} h^n S^{n+1}.
\]  

The operator \( D \in \mathcal{B}(X) \) is called the eigennilpotent and satisfies

\[
D^m = 0, \\
D\Pi_{\lambda, \lambda} = \Pi_{\lambda, \lambda} D = D, \\
\lambda\Pi_{\lambda, \lambda} = \Pi_{\lambda, \lambda} \lambda = \lambda\Pi_{\lambda, \lambda} + D.
\]
The operator $S \in \mathcal{B}(X)$ is called the reduced resolvent and satisfies

$$
Sf = \begin{cases} 
0, & f \in R(\Pi_{A,\lambda}), \\
-(\lambda - \Lambda)^{-1} f, & f \in R(1 - \Pi_{A,\lambda}),
\end{cases}
$$ (45)

$$
SA \subset \Lambda S = 1 - \Pi_{A,\lambda}.
$$

The eigenvalue $\lambda$ is said to be semi-simple when both eigenspaces are equal, or equivalently when the eigen-nilpotent is zero (which is the same as saying the eigenvalue is a pole of order 1).

When two closed simple paths $\Gamma_1$ and $\Gamma_2$ with values in the resolvent set of $\Lambda$ are such that $\Gamma_1$ lies in the exterior of $\Gamma_2$, we have

$$
\int_{\Gamma_1} R_{\Lambda}(z)dz \int_{\Gamma_2} R_{\Lambda}(z)dz = 0.
$$

For a detailed presentation of these results, see [22, Section III-6.5].

**Appendix B. Semigroup theory.** The famous Hille-Yosida Theorem ((1) $\Leftrightarrow$ (2) below, see for example [29, Chapter 1, Theorem 3.1]) and Lummer-Phillips Theorem ((1) $\Leftrightarrow$ (3) below, [29, Chapter 1, Theorem 4.3]) give necessary and sufficient conditions for a closed and densely defined operator to be a $C^0$-semigroup generator.

**Theorem B.1** (Hille-Yosida-Lummer-Phillips). Let $\Lambda$ be a closed and densely defined operator on a Banach space $X$, the following conditions are equivalent for any $C > 0$ and $\omega \in \mathbb{R}$:

1. $\Lambda$ generates a $C^0$-semigroup satisfying $\|S_{\Lambda}(t)\|_{\mathcal{B}(X)} \leq C e^{\omega t}$,
2. $\Sigma(\Lambda) \cap \Delta_{\omega} = \emptyset$ and $\|R_{\Lambda}(z)\|_{\mathcal{B}(X)} \leq \frac{C}{|\text{Re} z - \omega|}$ for $z \in \Delta_{\omega}$,
3. $C \| (\Lambda - z)f \|_X \geq (z - \omega)\|f\|_X$ for $f \in D(\Lambda)$, $z > \omega$, and $P(\Lambda) \cap \Delta_{\omega} \neq \emptyset$.

Note that when $X$ is a Hilbert space, an m-dissipative operator, that is to say an operator $\Lambda$ such that

$$
\text{Re} \langle \Lambda f, f \rangle_X \leq 0, \ P(\Lambda) \cap \Delta_0 \neq \emptyset,
$$
satisfies the equivalent conditions of Theorem B.1 with $C = 1$ and $\omega = 0$.

Furthermore, still in a Hilbert setting, the growth estimate is directly linked to the size of the half-plane on which the resolvent is bounded: we give here a version of [14, V-Theorem 1.11] in which we specify the dependency of the constant in the growth estimate.

**Theorem B.2** (Gearhart-Prüss-Greine). Consider $\Lambda$ a $C^0$-semigroup generator on a Hilbert space $X$, satisfying $\|S_{\Lambda}(t)\|_{\mathcal{B}(X)} \leq C_\alpha e^{\alpha t}$, and whose resolvent is defined and uniformly bounded on $\Delta_\beta$ by $K_\beta$. The semigroup satisfies $\|S_{\Lambda}(t)\|_{\mathcal{B}(X)} \leq C_\beta e^{\beta t}$ for some constructive constant $C_\beta > 0$ depending on $K_\beta, C_\alpha, \alpha$ and $\beta$.

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Received October 2020; revised April 2021.

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