Gauge-boson propagator in out of equilibrium quantum-field system and the Boltzmann equation

A. NIÉGAWA

Department of Physics, Osaka City University
Sumiyoshi-ku, Osaka 558-8585, JAPAN

Abstract

We construct from first principles a perturbative framework for studying nonequilibrium quantum-field systems that include gauge bosons. The system of our concern is quasiuniform system near equilibrium or nonequilibrium quasistationary system. We employ the closed-time-path formalism and use the so-called gradient approximation. No further approximation is introduced. We construct a gauge-boson propagator, with which a well-defined perturbative framework is formulated. In the course of construction of the framework, we obtain the generalized Boltzmann equation (GBE) that describes the evolution of the number-density functions of gauge-bosonic quasiparticles. The framework allows us to compute the reaction rate for any process taking place in the system. Various processes, in turn, cause an evolution of the systems, which is described by the GBE.

1E-mail: niegawa@sci.osaka-cu.ac.jp
1 Introduction

Ultrarelativistic heavy-ion-collision experiments at the BNL Relativistic Heavy Ion Collider (RHIC) have begun and will soon start at the CERN Large Hadron Collider (LHC) in an anticipation of producing a quark-gluon plasma (QGP) [see, e.g., [1, 2]]. The QGP to be produced is an expanding nonequilibrium system. Studies of the QGP as such have just begun.

In previous papers, a perturbative framework has been formulated from first principles for dealing with out-of-equilibrium complex-scalar field system [3], $O(N)$ linear-sigma system [4], and the system that includes massless fermions [5]. In this paper, we take up the out-of-equilibrium quantum-field theories that includes gauge bosons. We deal with the gauge-boson and the FP-ghost sectors in Coulomb gauge and, following the same procedure as in [3, 4, 5], we construct the gauge-boson and FP-ghost propagators and, thereby, frame a perturbation theory. Only approximation we employ is the so-called gradient approximation (see below). We use the closed-time-path (CTP) formalism [6, 7, 8] of nonequilibrium statistical quantum-field theory. It turns out that the form of gauge-boson propagator is quite complicated.

Throughout this paper, we are interested in quasuniform systems near equilibrium or nonequilibrium quasistationary systems. Such systems are characterized by two different spacetime scales [7]: microscopic or quantum-field-theoretical and macroscopic or statistical. The first scale, the microscopic correlation scale, characterizes the reactions taking place in the system, while the second scale measures the relaxation of the system. For a weak coupling theory, in which we are interested in this paper, the former scale is much smaller than the latter scale [7]. In a derivative expansion of some quantity $F$ with respect to macroscopic spacetime coordinates $X$, we use the gradient approximation throughout:

$$F(X, ...) \simeq F(Y, ...) + (X - Y)^{\mu} \partial_{Y^{\mu}} F(Y, ...).$$ (1)

Let $\Delta(x, y)$ be a generic propagator. For the system of our concern, $\Delta(x, y)$, with $x - y$ fixed, does not change appreciably in $(x + y)/2$. We refer to the first term on the

\footnote{It should be noted, however, that, as the system approaches the critical point of the phase transition, the microscopic correlation scale diverges. Thus, the formalism developed in this paper applies to the systems away from the critical point.}
right-hand side (RHS) of Eq. (1) as the leading part (term) and to the second term as the gradient part (term). The self-energy part $\Pi(x, y)$ enjoys a similar property. Thus, as usual, we choose the relative coordinates $x - y$ as the microscopic coordinates while $X \equiv (x + y)/2$ as the macroscopic coordinates. A Fourier transform with respect to $x - y$ yields

$$
\Delta(x, y) = \int \frac{d^4 P}{(2\pi)^4} e^{-i P \cdot (x - y)} \Delta(X; P) \quad (X \equiv (x + y)/2)
$$

$$
(P^\mu = (p^0, p))
$$

together with a similar formula for $\Pi$. The above observation shows that $P^\mu$ in Eq. (3) can be regarded as the momentum of the quasiparticle participating in the microscopic reaction under consideration. We shall freely use $\Delta(x, y)$ or $\Delta(X; P)$ [$\Pi(x, y)$ or $\Pi(X; P)$], which we simply write $\Delta$ [$\Pi$] whenever obvious from the context.

The perturbative framework to be constructed accompanies the generalized Boltzmann equation (GBE) for the number density of quasiparticles. The framework allows us to compute any reaction rate by using the reaction rate formula [9]. Substituting the computed net production rates of quasiparticles into the GBE, one can determine the number densities as functions of macroscopic spacetime coordinates, which describes the evolution of the system.

For definiteness, we take up QCD. However, the procedure has little to do with QCD and then the framework to be constructed may be used for any theory that includes gauge boson(s) with almost no modification. The plan of the paper is as follows: In Sec. II, on the basis of CTP formalism, we construct the bare gauge-boson and FP-ghost propagators, with which the “bare-\(N\) scheme” may be constructed. In Sec. III, we set up the basis for formulating the “physical-\(N\) scheme.” In Sec. IV, we make up the self-energy-part resummed propagators. In Sec. V, we impose the condition that no large contribution appears in perturbative calculation, so that a “healthy” perturbative framework is constructed. It is shown that, on the energy-shell, the condition turns out to the generalized Boltzmann equation. In Sec. VI, we frame a concrete perturbative framework. Section VII is devoted to summary and outlook. Concrete derivation of various formula used in the text is made in Appendices.
2 Closed-time-path formalism

2.1 Preliminary

For definiteness, we take up QCD and adopt the Coulomb gauge. We deal only with the gluon and the FP-ghost sectors in an out of equilibrium QCD system. Singling out the free parts of the gluon and of the FP-ghosts, we write the Lagrangian (density) as

\[ \mathcal{L}(A^{(a)}(x), \eta^{(a)}(x), \bar{\eta}^{(a)}(x), ...) = -\frac{1}{2} A^{(a)\mu} D^{\mu\nu} A^{(a)\nu} + \bar{\eta}^{(a)} \nabla^2 \eta^{(a)} + \ldots, \]

\[ D^{\mu\nu} \equiv -g^{\mu\nu} \partial^2 + \partial^\mu \partial^\nu - \frac{1}{\alpha} \left[ (\partial^\mu - \partial^0 n^\mu)(\partial^\nu - \partial^0 n^\nu) \right]. \]  \hspace{2cm} (3)

Here \( n^\mu = (1, 0) \), \( A^{(a)\mu} \) ("\( a \)" the color index) is the gluon fields, and \( \eta^{(a)} \) and \( \bar{\eta}^{(a)} \) are the FP-ghost fields. The CTP formalism is formulated \[6, 7, 8\] by introducing an oriented closed-time path \( C = C_1 + C_2 \) in a complex-time plane, that goes from \(-\infty\) to \(+\infty\) \((C_1)\) and then returns from \(+\infty\) to \(-\infty\) \((C_2)\). The real time formalism is achieved by doubling every degree of freedom: For example, for gauge fields, \( A^{(a)\mu}(x_0, x) = A^{(a)\mu}(x_0, x) \) with \( x_0 \in C_1 \) and \( A^{(a)\mu}(x_0, x) = A^{(a)\mu}(x_0, x) \) with \( x_0 \in C_2 \). A field with suffix ‘\( i \)’ \((i = 1, 2)\) is called a type-\( i \) field. One can introduce a two-dimensional space, in which every field is a “vector” whose first (second) component is the type-1 (type-2) field. In equilibrium thermal field theory, this space is called the thermal space, so that we use this terminology in the following.

The classical contour action that governs the dynamics of nonequilibrium systems is written in the form

\[ \int_C dx_0 \int dx \mathcal{L}(A^{(a)\mu}(x), \eta^{(a)}(x), \bar{\eta}^{(a)}(x), ...) = \int_{-\infty}^{+\infty} dx_0 \int dx \hat{\mathcal{L}}(x), \]

\[ \hat{\mathcal{L}}(x) \equiv \mathcal{L}(A^{(a)\mu}_1(x), \eta^{(a)}_1(x), \bar{\eta}^{(a)}_1(x), ...) - \mathcal{L}(A^{(a)\mu}_2(x), \eta^{(a)}_2(x), \bar{\eta}^{(a)}_2(x), ...), \]  \hspace{2cm} (4)

where “\( \ldots \)” stands for quark fields. \( \hat{\mathcal{L}} \) here is sometimes called a hat-Lagrangian [cf. \[10\]]. Throughout this paper, we do not deal with initial correlations (see, e.g., [7]).

2.1.1 Gluon sector

Following standard procedure [7], the four kind of gluon propagators emerges:

\[ (\Delta^{(a)\mu\nu}_{11}(x, y)) \equiv -i \text{Tr} \left[ T \left\{ A^{(a)\mu}_1(x) A^{(a)\nu}_1(y) \right\} \rho \right], \]
\[
\begin{align*}
\Delta_{12}^{(ab)\mu\nu}(x, y) & \equiv -i \text{Tr} \left[ A_2^{(b)\nu}(y) A_1^{(a)\mu}(x) \rho \right], \\
\Delta_{21}^{(ab)\mu\nu}(x, y) & \equiv -i \text{Tr} \left[ A_1^{(a)\mu}(x) A_2^{(b)\nu}(y) \rho \right], \\
\Delta_{22}^{(ab)\mu\nu}(x, y) & \equiv -i \text{Tr} \left[ T \left\{ A_2^{(a)\mu}(x) A_2^{(b)\nu}(y) \right\} \rho \right],
\end{align*}
\]

(5)

where \( \rho \) is the density matrix, and \( T (\mathcal{T}) \) is the time-ordering (antitime-ordering) symbol. We restrict ourselves to the case where \( \rho \) is color singlet, so that \( \Delta_{ij}^{(ab)\mu\nu} = \delta_{ij} \Delta_{ij}^{\mu\nu} \). Throughout in the sequel, we drop the color index. At the end of calculation we set \( A_1^\mu = A_2^\mu \). Let us use bold-face letter \( \Delta_{ij}(x, y) \) for denoting \( 4 \times 4 \) matrix, whose \((\mu \nu)\)-component is \( \Delta_{ij}^{\mu\nu}(x, y) \). We further introduce a matrix propagator \( \hat{\Delta}(x, y) \), where the ‘caret’ denotes the \( 2 \times 2 \) matrix in thermal space, whose \((ij)\)-component is \( \Delta_{ij}(x, y) \). The matrix self-energy part \( \hat{\Pi}(x, y) \) is defined similarly.

Deduction of the gluon propagator \( \hat{\Delta} \) is carried out below in subsection B.

2.1.2 FP-ghost sector

The FP-ghost fields are unphysical fields, so that they are absent in the system under consideration. Then, the FP-ghost propagators are the same as in vacuum theory. This means that the \( 2 \times 2 \) matrix propagator that act on a thermal space is diagonal, whose diagonal elements are

\[
\begin{align*}
[\Delta_{yi}^{(ab)}(x, y)]_{11} & \equiv -i \langle 0 | T \left\{ \eta_i^{(a)}(x) \bar{\eta}_1^{(b)}(y) \right\} | 0 \rangle, \\
[\Delta_{yi}^{(ab)}(x, y)]_{22} & \equiv -i \langle 0 | \mathcal{T} \left\{ \eta_2^{(a)}(x) \bar{\eta}_2^{(b)}(y) \right\} | 0 \rangle.
\end{align*}
\]

In what follows, as in the gluon sector, we drop the color indices. Setting \( \eta_1 = \eta_2 \) and \( \bar{\eta}_1 = \bar{\eta}_2 \), the FP-ghost propagator may be written, in \( 2 \times 2 \) matrix notation, as

\[
\begin{align*}
\hat{\Delta}(x, y) & = \Delta_g(x, y) \hat{\tau}_3, \\
\Delta_g(x, y) & = -i \int \frac{d^4P}{(2\pi)^4} e^{-i P \cdot (x-y)} \frac{1}{P^2}.
\end{align*}
\]

(6)

As will be shown in Appendix A, the FP-ghost self-energy-part matrix \( \hat{\Pi}_g \) is also diagonal, and enjoys a property \( [\Pi_g]_{11} + [\Pi_g]_{22} = 0 \) with \( \text{Im}[\Pi_g(X; P)]_{11} = 0 \). Then we may write

\[
\hat{\Pi}_g(x, y) = \Pi_g(x, y) \hat{\tau}_3.
\]
A $\hat{\Pi}_g$-resummed FP-ghost propagator $\hat{G}_g$ ($= G_g \hat{\tau}_3$) is defined through a Schwinger-Dyson equation:

$$G_g = \Delta_g + \Delta_g \cdot \Pi_g \cdot G_g = \Delta_g + G_g \cdot \Pi_g \cdot \Delta_g.$$ 

Here we have used the short-hand notation $F \cdot G$, which is a function whose "$(x y)$-component" is

$$[F \cdot G](x, y) = \int d^4 z F(x, z)G(z, y).$$

Solving this to the gradient approximation, we obtain

$$G_g(X; P) = G^{(0)}_g(X; P) + G^{(1)}_g(X; P),$$
$$G^{(0)}_g(X; P) = -\frac{1}{p^2 + \Pi_g(X; P)},$$
$$G^{(1)}_g(X; P) = -\frac{i}{2} \frac{\partial \Pi_g(X; P)}{\partial \mu} \frac{\partial \Pi_g(X; P)}{\partial X_\mu} [G^{(0)}_g(X; P)]^3.$$

As will be observed in Sec. IV, to be consistent with the approximation we are taking, $G^{(1)}_g$ may be ignored, so that

$$\hat{G}_g(X; P) \simeq -\frac{\hat{\tau}_3}{p^2 + \Pi_g(X; P)}.$$

### 2.1.3 Vertex factors

The vertex factors may be read off from Eq. (4): There is no vertex that mixes type-1 fields with type-2 fields. Every vertex factor for interactions between the type-1 fields is the same as in vacuum theory, while the every vertex factor for the type-2 fields is of opposite sign to the corresponding vertex factor for the type-1 fields.

In the next subsection, we deduce the gluon propagator, Eq. (5).

### 2.2 Gluon propagator

We compute Eq. (5) with $A^\mu_1 = A^\mu_2 (\equiv A^\mu)$. For $A^\mu$, we use the plane-wave decomposition with helicity basis in vacuum theory. Straightforward but lengthy calculation yields

$$\Delta^{\mu\nu}(x, y) = \int \frac{d^4 P}{(2\pi)^4} e^{-iP \cdot (x-y)} G_{\mu\nu}(X; P) \quad (X \equiv (x + y)/2),$$

$$= \int \frac{d^4 P}{(2\pi)^4} e^{-iP \cdot (x-y)} G_{\mu\nu}(X; P) \quad (X \equiv (x + y)/2).$$
where

\[
\hat{\Delta}^{\mu \nu}(X; P) = \hat{\Delta}^{\mu \nu}_T(X; P) + \hat{\Delta}^{\mu \nu}_L(X; P) + \hat{\Delta}^{\mu \nu}_1(X; P) + \hat{\Delta}^{\mu \nu}_2(X; P) + \hat{\Delta}^{\mu \nu}_3(X; P),
\]

\[
\hat{\Delta}^{\mu \nu}_T(X; P) = -\mathcal{P}^{\mu \nu}_T(\hat{p}) \left[ \hat{\Delta}_{RA}(P) + [\Delta_R(P) - \Delta_A(P)] f(X; P) \hat{M}_+ \right],
\]

\[
\hat{\Delta}^{\mu \nu}_L(X; P) = \left[ n^{\mu \nu} - \frac{1}{p^2} P^{\mu \nu} \right] \frac{1}{p^2} \hat{\tau}_3,
\]

\[
\hat{\Delta}^{\mu \nu}_1(X; P) = g^{\mu \nu} g^{ij} \left[ \frac{i (1 + (\hat{p}^i)^2)}{2p^2 (1 - (\hat{p}^j)^2)} \left( \hat{p}^i \nabla^j_x - \hat{p}^j \nabla^i_x \right) 
- \frac{i \hat{p}^y}{p^2 (1 - (\hat{p}^y)^2)} \left( \delta^y \nabla^j_{\perp} - \delta^j \nabla^y_{\perp} \right) \right] [\Delta_R(P) - \Delta_A(P)] f(X; P) \hat{M}_+,
\]

\[
\hat{\Delta}^{\mu \nu}_2(X; P) = -i g^{\mu i} g^{\nu j} e^{ijk} p^k N_{\perp}(P) [\Delta_R(P) - \Delta_A(P)] \epsilon(p^0) \hat{M}_+,
\]

\[
\hat{\Delta}^{\mu \nu}_3(X; P) = g^{\mu \nu} g^{ij} \left\{ \left[ (1 - 2 \delta^{ij}) \delta^{ij} + \delta^{ij} \hat{p}^i \hat{p}^j - \frac{2 \hat{p}^i \hat{p}^j}{1 - (\hat{p}^y)^2} \right] \Re N_{LR}(P) 
- \frac{\hat{p}^y [\hat{p}^i (\hat{p} \times e^y)^j + \hat{p}^j (\hat{p} \times e^y)^i]}{1 - (\hat{p}^y)^2} \left( \delta^y \nabla^i_{\perp} - \delta^i \nabla^y_{\perp} \right) \epsilon(p^0) \Im N_{LR}(P) \right\} 
\times [\Delta_R(P) - \Delta_A(P)] \epsilon(p^0) \hat{M}_+,
\]

where the Greek letters, \( \mu \) and \( \nu \), take 0 - 3, while the Latin letters, \( i \) and \( j \), take 1, 2, and 3. To avoid possible confusion, we have written \((x, y, z)\) for \((1,2,3)\). In the above equations, \( e^y = (0, 1, 0) \), \( \nabla^i_x \equiv \partial / \partial X^i \), \( \epsilon^{123} = \epsilon^{xyz} = 1 \), and

\[
\mathcal{P}^{\mu \nu}_T(\hat{p}) = -g^{\mu i} g^{\nu j} (\delta^{ij} - \hat{p}^i \hat{p}^j) (\hat{p} \equiv p/p),
\]

\[
\hat{p} \equiv \hat{p} - (\hat{p} \cdot e^y) e^y,
\]

\[
\nabla^i_\perp \equiv \nabla^i_x - (\nabla_x \cdot \hat{p}) \hat{p}^i,
\]

\[
\hat{\Delta}_{RA}(P) = \begin{pmatrix} \Delta_R(P) & 0 \\ \Delta_R(P) - \Delta_A(P) & -\Delta_A(P) \end{pmatrix}, \quad \Delta_{R(A)}(P) = \frac{1}{P^2 \pm i \rho_0 0^+},
\]

\[
\hat{M}_\pm = \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix},
\]

\[
f(X; P) = \epsilon(p_0) N(X; P) - \theta(-p_0),
\]

\[
N(X; P) \equiv \frac{1}{2} \left[ N_{++}(X; |p^0|, \epsilon(p^0) \hat{p}) + N_{--}(X; |p^0|, \epsilon(p^0) \hat{p}) \right] 
\equiv \theta(p^0) n(X; |p^0|, \hat{p}) + \theta(-p^0) n(X; |p^0|, -\hat{p}),
\]

7
\[ N_-(X; P) \equiv \frac{1}{2} \left[ N_{++}(X; |p^0|, \epsilon(p^0)\hat{p}) - N_{--}(X; |p^0|, \epsilon(p^0)\hat{p}) \right] \]
\[ \simeq N_-(P), \] (19)
\[ N_{LR}(P) \equiv N_+(|p_0|, \epsilon(p^0)\hat{p}) \] (20)

with
\[ N_{\xi\zeta}(X; |p_0|, \hat{p}) \equiv \int d^3q e^{-iQ \cdot X} \text{Tr} \left[ a^\dagger_\xi(p - q/2) a_\zeta(p + q/2) \right] \quad (\xi, \zeta = +, -). \] (21)

Here \(|p_0| = p\), and \(Q = (q^0, q)\) with \(q_0 = q \cdot p/p\). \(a_+ (a_-)\) is an annihilation operator of a gluon with \(+ (-)\) helicity. \(a^\dagger_+ (a^\dagger_-)\) is a corresponding creation operator. In deriving Eqs. (19) - (21), we have assumed that \(\text{Tr}[a_\rho(p)a_\sigma(q)\rho] \simeq 0\). We have also assumed that \(N_-, \text{Eq. (19)}, \) and \(N_{LR}, \text{Eq. (20)}, \) are small quantities, so that \(\hat{\Delta}^{\mu\nu}_{L}\) and \(\hat{\Delta}^{\mu\nu}_{R}\) are of the order of gradient term \(\hat{\Delta}^{\mu\nu}_{G}\) that includes \(\nabla_X N\). More precisely \(N_\mu\) and \(N_{LR}(P)\) are of order of \((\nabla_X N)/p, \) and then, to the gradient approximation, \(X\)-dependences of \(N_-\) and of \(N_{LR}\) have been ignored.

We observe that the structure of \(\hat{\Delta}^{\mu\nu}_{L}(P)\) in Eq. (11) is rather simple, which represents the propagator between “longitudinal gluons” as well as the gauge part. This is a reflection of the fact that, in the Coulomb gauge, only transverse modes are physical modes. Then, as in the FP-ghost propagator in Eq. (6), the \((11)\)-component of \(\hat{\Delta}^{\mu\nu}_{L}, \) \([\hat{\Delta}^{\mu\nu}_{L}(P)]_{11}\), is the same as in vacuum theory.

From Eq. (21), follows
\[ \theta(\pm p^0) P \cdot \partial_X N_{\xi\zeta}(X; |p^0|, \pm\hat{p}) = 0 \quad (|p_0| = p). \] (22)

Using this in Eqs. (17) and (18), we have
\[ P \cdot \partial_X f(X; P) = 0 \quad (|p_0| = p). \] (23)

As is obvious from the construction, or as can be directly shown from Eq. (23), we see that
\[ -\hat{\tau}_3 \mathcal{D}^{\mu\rho} \hat{\Delta}_{\rho\nu}(x, y) = -\hat{\Delta}^{\mu\rho}(x, y) \mathcal{D}_{\rho\nu} \hat{\tau}_3 = \delta^{\mu\nu} \delta^4(x - y), \] (24)
where \(\mathcal{D}^{\mu\rho}\) is as in Eq. (3). Two equations in Eq. (22) are “free Boltzmann equations.” One can construct a perturbation theory in a similar manner as in [3], where a complex-scalar field system is treated. We call the perturbation theory thus constructed the bare-\(N\) scheme, since \(N\) obeys the “free Boltzmann equation.” This scheme is equivalent [3] to the physical-\(N\) scheme, to which we now turn.
3 Construction of the physical-\(N\) scheme

We aim to construct a scheme in terms of the number density that is as close as possible to the physical number density. To this end, first of all, we abandon the “free Boltzmann equation” \((23)\). This means that \(f\) and \(A\) in the present (physical-\(N\)) scheme differs from the bare-\(N\) scheme counterparts. Specification of \(f\) is postponed until Sec. VI.

Now, in contrast to Eq. \((24)\), \(\hat{\Delta}\) is not an inverse of \(-\hat{\tau}_3 D_{\mu\nu}\). Straightforward calculation within the gradient approximation yields

\[
\hat{\tau}_3 D_{\mu\nu}(x, y) = -g_{\mu\nu}\delta^4(x - y) - i\hat{\tau}_3 \hat{M}_+ \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \times [\Delta_R(P) - \Delta_A(P)] \mathcal{P}_T^{\mu\nu}(\hat{p}) (P \cdot \partial_X f(X; P)) .
\]

In obtaining this, we have used \(P^2[\Delta_R(P^2) - \Delta_A(P^2)] \propto P^2\delta(P^2) = 0\). Our procedure of constructing a consistent scheme is as follows: We further modify \(A\) by adding a suitable \(\hat{\Delta}_{\text{add}}\) to \(\hat{\Delta}\). The conditions for \(\hat{\Delta}_{\text{add}}\) to be satisfied are

- \(\hat{\Delta}_{\text{add}}\) vanishes in the bare-\(N\) scheme.
- To the gradient approximation, the counterpart of Eq. \((24)\) turns out to

\[
\left(-\hat{\tau}_3 D_{\mu\nu} 1 - (\hat{L}_c)_{\mu\nu}\right) \cdot \left(\hat{\Delta} + \hat{\Delta}_{\text{add}}\right) = \delta^{\nu\mu} 1 , \quad (25)
\]

where use has been made of the short-hand notation \((7)\). ‘1’ is the matrix in spacetime whose \((x y)\)-component is \(\delta^4(x - y)\), and \(\hat{L}_c\) is some \((4 \times 4) \otimes (2 \times 2)\) matrix function.

It is a straightforward task to obtain the form of the required \(\hat{\Delta}_{\text{add}}\):

\[
\hat{\Delta}_{\text{add}}(X; P) = i\hat{M}_+ \mathcal{P}_T^{\mu\nu}(\hat{p}) \left[\Delta_R^2(P) + \Delta_A^2(P)\right] (P \cdot \partial_X f(X; P)) , \quad (26)
\]

from which we obtain for \(\hat{L}_c\) in Eq. \((25)\),

\[
\hat{L}_c^{\mu\nu}(x, y) = L_c^{\mu\nu} \hat{M}_- = 2i \hat{M}_- \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \mathcal{P}_T^{\mu\nu}(\hat{p}) (P \cdot \partial_X f(X; P)) ,
\]

with \(\hat{M}_-\) as in Eq. \((16)\). In obtaining Eq. \((25)\) with \(\hat{\Delta}_{\text{add}}\) as in Eq. \((26)\), we have used \((\hat{L}_c)_{\mu\nu} \cdot \hat{\Delta}_{\text{add}} \simeq (\hat{L}_c)_{\mu\nu} \cdot (\hat{\Delta} + \hat{\Delta}_{\text{add}})\), since the difference can be ignored to the
gradient approximation. In a similar manner, we find that $(\hat{\Delta}^{\mu\nu} + \hat{\Delta}_{\text{add}}^{\mu\nu}) \cdot (-\hat{\tau}_3 \vec{D}_{\mu\nu} - (\hat{L}_c)_{\mu\nu}) = \delta^{\mu\nu} 1$. Thus we have found that the $2 \times 2$ matrix propagator $(\hat{\Delta}^{\mu\nu} + \hat{\Delta}_{\text{add}}^{\mu\nu})$ is an inverse of $(-\hat{\tau}_3 \vec{D}_{\mu\nu} - (\hat{L}_c)_{\mu\nu})$, so that the free action is\[\]

$$-\frac{1}{2} \int d^4x d^4y \hat{A}_\mu(x) \left[ \hat{\tau}_3 \vec{D}_{\mu\nu} \delta^4(x - y) + L_c^{\mu\nu}(x, y) \hat{M}_- \right] \hat{A}_\nu(y),$$

$$\hat{A}^\mu = (\hat{A}_1^\mu, \hat{A}_2^\mu).$$

Since the term with $L_c(x, y)$ in Eq. (27) is absent in the original action, we should introduce the counter action to compensate it,

$$\mathcal{A}_c = \frac{1}{2} \int d^4x d^4y \hat{A}_\mu(x) L_c^{\mu\nu}(x, y) \hat{M}_- \hat{A}_\nu(y),$$

which yields a vertex

$$i L_c^{\mu\nu}(x, y) \hat{M}_- = -2\hat{M}_- \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x - y)} P^{\mu\nu}(\hat{p}) P \cdot \partial_x f(X; P)$$

$$\equiv i \hat{\Pi}^{(c)\mu\nu}(x, y).$$

For completeness, we construct a $\hat{L}_c$-resummed propagator. Since $\hat{M}_+ \hat{M}_- = \hat{M}_- \hat{M}_+ = 0$ and $\hat{\Pi}^{(c)\mu\nu} \propto \mathcal{P}^{\mu\nu}_T$, only piece that changes the form under the $\hat{L}_c$-resumination is $-\mathcal{P}^{\mu\nu}_T(\hat{p}) \hat{\Delta}_{RA}$ in $\hat{\Delta}_{\mu\nu}^T$, Eq. (10) with Eq. (16):

$$-\mathcal{P}^{\mu\nu}_T(\hat{p}) \hat{\Delta}_{\text{resum}}^{\mu\nu}(x, y) = -\mathcal{P}^{\mu\nu}_T(\hat{p}) \left\{ \hat{\Delta}_{RA}(x, y) - \sum_{n=1}^\infty \left[ \hat{\Delta}_{RA} \cdot \hat{\Pi}^{(c)} \cdot \hat{\Delta}_{RA} \right]^n (x, y) \right\}$$

$$= -\mathcal{P}^{\mu\nu}_T(\hat{p}) \left\{ \hat{\Delta}_{RA}(x, y) - \left[ \hat{\Delta}_{RA} \cdot \hat{\Pi}^{(c)} \cdot \hat{\Delta}_{RA} \right] (x, y) \right\}$$

$$\simeq -\mathcal{P}^{\mu\nu}_T(\hat{p}) \left\{ \hat{\Delta}_{RA}(x, y) + 2i \hat{M}_+ \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x - y)} \right.$$\[\]

$$\left. \times \Delta_R(P) \Delta_A(P) P \cdot \partial_x f(X; P) \right\}.$$

Here, use has been made of $\hat{\Delta}_{RA} \hat{M}_- \hat{\Delta}_{RA} \propto \hat{M}_+$. Then, substituting $\left( \hat{\Delta}_{\text{resum}}^{\mu\nu} \right)^{\mu\nu}$ for $\hat{\Delta}_{\text{resum}}^{\mu\nu}$, we obtain, with obvious notation,

$$\left( \hat{\Delta}_{\text{resum}}^{\mu\nu}(X; P) \right)^{\mu\nu} + \hat{\Delta}_{\text{add}}^{\mu\nu}(X; P)$$

$$= -\mathcal{P}^{\mu\nu}_T(\hat{p}) \left\{ \left[ \Delta_{RA}(X; P) + \hat{M}_+ [\Delta_R(P) - \Delta_A(P)] f(X; P) \right]$$

$$- i \hat{M}_+ \mathcal{P}^{\mu\nu}_T(\hat{p}) [\Delta_R(P) - \Delta_A(P)]^2 P \cdot \partial_x f(X; P) \right\}.$$
In closing this section, we emphasize that \( f(X; P) \) in the present scheme is an arbitrary function, provided that \( f(X_0^0, X; P) = -\theta(-p^0) + \epsilon(p^0)N(X_0^0, X; P) \), with \( X_0^0 \) the initial time, is a given initial data. We have introduced the counteraction \( A_c \), Eq. (28), so as to remain on the original theory. Thus, it cannot be overemphasized that the schemes with different \( f \)'s are mutually equivalent. If we choose the \( f = f_B \) that subjects to the “free Boltzmann equation” (23), the scheme reduces to the bare-N scheme in the last section. In Sec. VI, we shall choose \( f \), with which a well-defined perturbation theory is formulated. In any case, it is natural to choose \( f \), so that, when interactions are switched off, \( f \) turns out to \( f_B \).

4 Resummation of the self-energy part

4.1 Preliminary

Throughout in the sequel, we restrict to the strict Coulomb gauge \( \alpha = 0 \) [cf. Eq. (3)]. The bare propagator consists of six pieces, \( \hat{\Delta}_T + \hat{\Delta}_L + \sum_{i=1}^{3} \hat{\Delta}_i + \hat{\Delta}_{\text{add}} \), Eqs. (4) and (20). \( \hat{\Delta}_T + \hat{\Delta}_L \) is the leading part and \( \hat{\Delta}_1 \) and \( \hat{\Delta}_{\text{add}} \) are the gradient parts, which represents variation in the macroscopic spacetime coordinates \( X_\mu \), through first-order derivative \( \partial_{X_\mu} f(X; P) \). We have assumed that \( \hat{\Delta}_2 \) and \( \hat{\Delta}_3 \) are of the same order of magnitude as the gradient parts. Interactions among the fields give rise to reactions taking place in a system, which, in turn, causes a nontrivial change in the number density of quasiparticles. Thus, the self-energy part \( \hat{\Pi} \) ties with the gradient parts. More precisely, \( \hat{\Pi} \) is of the same order of magnitude as \( \hat{\Delta}_{\text{add}} (\hat{\Delta}_T + \hat{\Delta}_L)^{-2} \) and \( \hat{\Delta}_i (\hat{\Delta}_T + \hat{\Delta}_L)^{-2} (i = 1 - 3) \). Hence, in computing \( \hat{\Pi} \) in the approximation under consideration, it is sufficient to keep the leading part (i.e., the part with no \( X_\mu \)-derivative). Then \( \hat{\Pi}(X; P) \) may be decomposed as

\[
\hat{\Pi}^{\mu\nu}(X; P) = \mathcal{P}_T^{\mu\nu}(\hat{p})\hat{\Pi}_T(X; P) + n^\mu n^\nu \hat{\Pi}_L(X; P) + \frac{p_0}{p} (n^\mu \hat{p}_\nu + \hat{p}_\nu n^\mu) \hat{\Pi}_C(X; P) + \hat{p}_\mu \hat{p}_\nu \hat{\Pi}_D(X; P). 
\]

Here and in the following, we use such a notation as \( v^\mu \) for denoting a four-vector whose 0th component is zero: \( v^\mu \equiv (0, v) \). Then, \( \hat{p}^\mu = (0, \hat{p}) \).

Within the gradient approximation, it is sufficient to perform a \( \hat{\Pi} \)-resummation for the leading part, \( \hat{\Delta}_T + \hat{\Delta}_L \). This is because the corrections to other pieces due to
the resummation are of higher order. Thus, for the gradient parts of $\hat{\Delta}$ as well as for $\hat{\Delta}_{\text{add}}$, one can use the formulae in the bare-$N$ scheme in Sec. II [cf. above argument after Eq. (30)]. In particular, for $f$ in the gradient parts, one can use $f (= f_B)$ as in Eq. (23) in the bare-$N$ scheme. For the purpose of performing resummation, as usual, it is convenient to introduce the “standard form” \[10, 3, 4, 5\]

\[
\hat{B}_L \cdot \hat{\Delta}_{\text{diag}} \cdot \hat{B}_R (x, y) = \int d^4z_1 \int d^4z_2 \hat{B}_L(x, z_1) \hat{\Delta}_{\text{diag}}(z_1, z_2) \hat{B}_R(z_2, y),
\]

\[
\hat{\Delta}_{\text{diag}} = \text{diag} (\hat{\Delta}^\mu_{\rho}, -\hat{\Delta}^\mu_{\nu}),
\]

\[
\Delta^\mu_{R(A)}(P) = -\tilde{g}^{\mu\nu}(P)\Delta_{R(A)}(P),
\]

\[
\hat{B}_L = \begin{pmatrix} 1 & f \\ 1 + f & 1 \end{pmatrix}, \quad \hat{B}_R = \begin{pmatrix} 1 + f & f \\ 1 & 1 \end{pmatrix},
\]

\[
\tilde{g}^{\mu\nu}(P) = g^{\mu\nu} + \frac{1}{p^2} \left(1 + \frac{P^2}{p^2}\right) P^\mu P^\nu - \frac{p}{p^2} (P^\mu n^\nu + P^\nu n^\mu)
\]

\[
= \mathcal{D}^{\rho\mu}_{\nu, P} - \frac{p^2}{p^2} n^\mu n^\nu + \frac{P^2}{p^4} P^\mu P^\nu,
\]

where $f (= f(x, y))$ is the inverse Fourier transform of $f(X; P)$. It is to be noted that $-\Delta^\mu_{R(A)}(x - y)$ is an inverse of $\mathcal{D}^{\rho\mu}_{\nu, P}$, Eq. (3):

\[
\mathcal{D}^{\rho\mu}_{\nu, P} \hat{\Delta}^\nu_{R(A)} = \Delta^\mu_{R(A)} \mathcal{D}^{\nu\rho}_{\mu, P} = -\tilde{g}^{\mu\nu}.
\]

Computing Eq. (32) to the gradient approximation, we obtain

\[
\hat{\Delta}_{\text{base}}^\mu_{\nu}(x, y) = \hat{\Delta}_T^\mu_{\nu}(x, y) + \hat{\Delta}_L^\mu_{\nu}(x, y) + \hat{\Delta}_{\text{add}}^\mu_{\nu}(x, y)
\]

\[
= \left[ \hat{B}_L \cdot \hat{\Delta}_{\text{diag}} \cdot \hat{B}_R \right] (x, y) + M_+ \Delta^\mu_{K}(x, y),
\]

\[
\Delta^\mu_{K}(X; P) = -\frac{i}{p^2} \left[ \frac{P}{p^2} (i \mu \nabla^\mu + i \rho \nabla^\rho) f + 2 \frac{n^\mu n^\nu}{p^2} P \cdot \nabla f \right],
\]

where $P/P^2$ is the principal part of $1/(P^2 \pm i0^+)$. As mentioned above, within the gradient approximation, it is sufficient to take $\hat{\Delta}_T + \hat{\Delta}_L$ as a “resummed part.” It is obvious from the above observation that one can freely include gradient part(s) into the “resummed part.” We include the gradient part $\hat{\Delta}_{\text{add}}$ and take $\hat{\Delta}_{\text{base}}$ as the “resummed part.”

It is to be noted that, from Eqs. (14), (17), (19), and (23), follows

\[
\sum_{i, j=1}^{2} (-)^{i+j} (\Delta_{\text{base}})_{ij} = 0.
\]

This relation holds for $\hat{\Delta} + \hat{\Delta}_{\text{add}}$ (in place of $\hat{\Delta}_{\text{base}}$).
4.2 Self-energy-part resummed propagator

As will be shown in Appendix A [cf. Eq. (A.7)],

\[ \sum_{i,j=1}^{2} \Pi_{ij} = 0 \]  

(38)

holds. A \( \hat{\Pi} \)-resummed propagator \( \hat{G} \) obeys the Schwinger-Dyson equation:

\[ \hat{G} = \hat{\Delta}_{\text{base}} - \hat{\Delta}_{\text{base}} \cdot \hat{\Pi} \cdot \hat{G} = \hat{\Delta}_{\text{base}} - \hat{\Delta}_{\text{base}} \cdot \hat{\Pi} \cdot \hat{G} \]  

(39)

It is clear that, for solving this equation, it is convenient to introduce the “\( \hat{B} \)-transformed” quantities

\[ \hat{G} \equiv \hat{B}_{L}^{-1} \cdot \hat{G} \cdot \hat{B}_{R}^{-1}, \]  

(40)

\[ \hat{\Delta}_{\text{base}} \equiv \hat{B}_{L}^{-1} \cdot \hat{\Delta}_{\text{base}} \cdot \hat{B}_{R}^{-1} = \begin{pmatrix} \Delta_{R} & \Delta_{K} \\ 0 & -\Delta_{A} \end{pmatrix}, \]  

(41)

\[ \hat{\Pi} \equiv \hat{B}_{R} \cdot \hat{\Pi} \cdot \hat{B}_{L} = \begin{pmatrix} \Pi_{R} & \Pi_{K} \\ 0 & -\Pi_{A} \end{pmatrix}. \]  

Here \( \Delta_{K} \) is as in Eq. (36), and \( \Pi_{R(A)} \) and \( \Pi_{K} \) are, in respective order, the inverse Fourier transforms of

\[ \Pi_{R(A)}(X;P) = \Pi_{11} + \Pi_{12} \]  

(42)

\[ \Pi_{K}(X;P) = [1 + f(X;P)]\Pi_{12}(X;P) - f(X;P)\Pi_{21}(X;P) \]

\[ -\frac{i}{2} \{ \Sigma_{R} + \Sigma_{A}, f \}_{\text{P.B.}}. \]  

(43)

In Eq. (43), \( \{ ..., ..., \}_{\text{P.B.}} \) is defined as

\[ \{ A, B \}_{\text{P.B.}} \equiv \frac{\partial A(X;P)}{\partial X^{\mu}} \frac{\partial B(X;P)}{\partial P_{\mu}} - \frac{\partial A(X;P)}{\partial P_{\mu}} \frac{\partial B(X;P)}{\partial X^{\mu}}. \]  

(44)

In obtaining Eq. (43) [Eqs. (12) and (13)], use has been made of Eq. (37) [Eq. (38)]. Although the last term in Eq. (43) may be dropped to the approximation under consideration, we have kept it.

Now Eq. (39) is transformed into

\[ \hat{G} = \hat{\Delta}_{\text{base}} - \hat{\Delta}_{\text{base}} \cdot \hat{\Pi} \cdot \hat{G} = \hat{\Delta}_{\text{base}} - \hat{\Delta}_{\text{base}} \cdot \hat{G} \cdot \hat{\Pi} \cdot \hat{\Delta}_{\text{base}}, \]  

13
which can easily be solved to give

\[
\hat{\mathbf{G}} = \begin{pmatrix} \mathbf{G}_R & \mathbf{G}_K \\ 0 & -\mathbf{G}_A \end{pmatrix},
\]

\[
\mathbf{G}_{R(A)} = \left[ \Delta_{R(A)} + \Pi_{R(A)} \right]^{-1},
\]

\[
\mathbf{G}_K = \mathbf{G}_R \cdot \left[ \Delta_R^{-1} \cdot \Delta_K \cdot \Delta_A^{-1} + \Pi_K \right] \cdot \mathbf{G}_A.
\]

Substituting this back into Eq. (40), we obtain, after Fourier transformation,

\[
\hat{\mathbf{G}}(X; P) \approx \hat{\mathbf{G}}_{RA}(X; P) + \hat{M}_+ \left[ \mathbf{G}_R(X; P) - \mathbf{G}_A(X; P) \right] f(X; P)
\]

\[
+ \hat{M}_+ \left[ \mathbf{G}_K(X; P) - \frac{i}{2} \{ \mathbf{G}_R + \mathbf{G}_A, f \}_{P.B.} \right],
\]

where

\[
\hat{\mathbf{G}}_{RA}(X; P) \equiv \begin{pmatrix} \mathbf{G}_R(X; P) & 0 \\ \mathbf{G}_R(X; P) - \mathbf{G}_A(X; P) & -\mathbf{G}_A(X; P) \end{pmatrix}.
\]

Here we note that \( \hat{\Pi} \) consists of two pieces,

\[
\hat{\Pi} = \hat{\Pi}^{\text{loop}} + \hat{\Pi}^{(c)},
\]

where \( \hat{\Pi}^{\text{loop}} \) is the contribution from loop diagrams and \( \hat{\Pi}^{(c)} \) is as in Eq. (29), which has come from the counter action \( \mathcal{A}_c \). It should be remarked that some \( \hat{\Pi}^{\text{loop}} \) contains internal vertex(es) \( \hat{\Pi}^{(c)}. \) [cf. the argument at Sec VI below.] From the above definitions of \( \Pi_{R(A)}, \Pi_K, \) and \( \hat{\Pi}^{(c)}, \) we have

\[
\Pi^{(c)}_R = \Pi^{(c)}_A = 0,
\]

\[
\Pi^{(c)}_{K\mu\nu}(X; P) = -2i\mathcal{P}^{\mu\nu}(\hat{p}) P \cdot \partial_X f(X; P).
\]

From Eq. (43), we obtain, after some manipulation,

\[
\mathbf{G}_{R(A)}(X; P) \simeq \mathbf{G}^{(0)}_{R(A)}(X; P) + \mathbf{G}^{(1)}_{R(A)}(X; P),
\]

\[
\mathbf{G}^{(0)}_{R(A)}(X; P) = -\frac{\mathcal{P}^{\mu\nu}(\hat{p})}{P^2 - \Pi_{R(A)}^T(X; P)} + \frac{n^\mu n^\nu}{p^2 + \Pi_{R(A)}^L(X; P)},
\]

\[
\mathbf{G}^{(1)}_{R(A)}(X; P) = \frac{i}{2p^2 P^2 - \Pi_{R(A)}^T} \left[ \frac{1}{P^2 - \Pi_{R(A)}^T} \left( n^\mu \nabla_{\mu} - n^\nu \nabla_{\nu} \right) \Pi_{R(A)}^T \right.
\]

\[
- \frac{p_0}{p^2 + \Pi_{R(A)}^L} \left( n^\mu \nabla_{\mu} - n^\nu \nabla_{\nu} \right) \Pi_{R(A)}^C \right].
\]
Although $G_{R(A)}^{(1)}$ can be ignored to the gradient approximation, we have displayed it. Straightforward calculation of the last term in Eq. (47) using the definitions (45), (46), and (44) yields (cf. Appendix B)

$$G_K(X; P) - \frac{i}{2} \{G_R + G_A, f\}_{P.B.} = G_K^{(1)} + G_K^{(2)},$$

where

$$G_K^{(1)\mu\nu} = -i \left[ 2p \cdot \partial_X f + \left\{ \Re \Pi_{R}^{T}, f \right\}_{P.B.} + i(\Pi_{K}^{T})^{\text{loop}} \right] \frac{P_{T}^{\mu\nu}}{(P^2 - \Pi_{R}^{T})(P^2 - \Pi_{A}^{T})},$$

$$G_K^{(2)} = \frac{P^0}{P^2} \left[ \frac{n^\nu \nabla_{\perp}^\mu f}{(P^2 - \Pi_{R}^{T})(P^2 + \Pi_{A}^{T})} - \frac{n^\mu \nabla_{\perp}^\nu f}{(P^2 - \Pi_{A}^{T})(P^2 + \Pi_{K}^{T})} \right] + \frac{i}{2P} \left( \frac{1}{P^2 - \Pi_{K}^{T}} - \text{c.c.} \right) \left( p^\mu \nabla_{\perp}^\nu f - p^\nu \nabla_{\perp}^\mu f \right),$$

$$+ i \Re \left[ \frac{P_{T}^{\mu\nu}}{(P^2 - \Pi_{R}^{T})^2} \left( 2P \cdot \partial_X f + \left\{ \Pi_{R}^{T}, f \right\}_{P.B.} \right) + \frac{n^\mu n^\nu}{(P^2 + \Pi_{K}^{T})^2} \left( 2P \cdot \nabla_X f + \left\{ \Pi_{R}^{T}, f \right\}_{P.B.} \right) \right],$$

where

$$(\Pi_{K}^{T(L)})^{\text{loop}} \equiv [1 + f] \left( \Pi_{12}^{T(L)} \right)^{\text{loop}} - f \left( \Pi_{21}^{T(L)} \right)^{\text{loop}}.$$ (55)

In obtaining Eqs. (53) and (54), use has been made of $(\Pi_{A}^{T(L)})^* = \Pi_{R}^{T(L)}$, which is proved in Appendix A.

### 5 Generalized Boltzmann equation

#### 5.1 Energy-shell and physical number densities

For later use, referring to Eq. (51), we define the energy-shell for the transverse mode by

$$\Re \left[ P^2 - \Re \Pi_{R}^{T}(X; P) \right]_{p^\nu = \pm \omega_T(X; \pm p)} = 0.$$ (56)

It is well known [2] that, in equilibrium quark-gluon plasma, “longitudinal” mode called plasmon appears for soft $p (= O(gT))$ [g the QCD coupling constant]. The
energy-shell of such modes is defined by \[\text{cf. Eq. (51)}\]

\[
\text{Re} \left[ p^2 + \text{Re}\Pi^L_R(X; P) \right]_{p^0 = \pm \omega_L(X; \mp P)} = 0.
\]  (57)

Useful formulae that hold on the energy-shell are displayed in Appendix C.

We restrict to the \(p^0 > 0\) part. The \(p^0 < 0\) part yields the same result, since a gluon and a corresponding antigluon is the same. In some of the formulae in this subsection, we shall drop the argument \(X\).

For obtaining expressions for the number densities of the transverse and “longitudinal” modes, we compute the statistical average of the operators,

\[
\mathcal{N}_T(x) = -i A^{(+)}_j(x) \frac{\partial}{\partial x^0} A^{(-)}_j(x),
\]

\[
\mathcal{N}_L(x) = i A^{(+)}_0(x) \frac{\partial}{\partial x^0} A^{(-)}_0(x).
\]

Here the superscript “(+)” [“(−)" stands for the positive [negative] frequency part. Statistical averages of \(\mathcal{N}_T\) and of \(\mathcal{N}_L\) yield, in respective order,

\[
\text{Tr} [\mathcal{N}_T(x) \rho] = i \int \frac{d^4 P}{(2\pi)^4} \frac{\theta(p_0)p_0}{P^2 - \Pi^L_R} \left[ G^{ij}_{12}(X; P) + G^{ij}_{21}(X; P) \right],
\]

\[
\text{Tr} [\mathcal{N}_L(x) \rho] = -i \int \frac{d^4 P}{(2\pi)^4} \frac{\theta(p_0)p_0}{P^2 + \Pi^L_R} \left[ G^{00}_{12}(X; P) + G^{00}_{21}(X; P) \right].
\]  (58)

It is to be noted that “\(x\)” here is a macroscopic spacetime coordinates [cf. Sec. I].

We first compute the leading contributions to Eq. (58). Substituting the leading parts of \(G_{21}\) and of \(G_{12}\) (cf. Eq. (47)) and using Eq. (51), we obtain

\[
\text{Tr} [\mathcal{N}_T(x) \rho] = 2i \int \frac{d^4 P}{(2\pi)^4} \frac{\theta(p_0)p_0}{P^2 - \Pi^L_R} \left[ \frac{1}{P^2 - \Pi^L_R} - \text{c.c.} \right] [2f(X; P) + 1],
\]

\[
\text{Tr} [\mathcal{N}_L(x) \rho] = -i \int \frac{d^4 P}{(2\pi)^4} \frac{\theta(p_0)p_0}{P^2 + \Pi^L_R} \left[ \frac{1}{P^2 + \Pi^L_R} - \text{c.c.} \right] [2f(X; P) + 1].
\]

The narrow-width approximation\[\text{Im}\Pi^L_R \to -\epsilon(p_0)^0\] yields

\[
\text{Tr} [\mathcal{N}_T(x) \rho] \simeq 2 \int \frac{d^3 p}{(2\pi)^3} Z_T(\omega_T(p), p) n(\omega_T(p), p) + \ldots, \quad (59)
\]

\[
\text{Tr} [\mathcal{N}_L(x) \rho] \simeq \int \frac{d^3 p}{(2\pi)^3} Z_L(\omega_L(p), p) n(\omega_L(p), p) + \ldots. \quad (60)
\]

\[\text{In the case of equilibrium system, the narrow-width approximation is a good approximation for hard modes but not for soft modes.}\]
Here \( n \) is as in Eq. (18), and ‘...’ stands for the contribution from \( 2f + 1 \geq \epsilon(p^0) \) [ cf. Eq. (17)], which is the vacuum-theory contribution corrected by the medium effect. Z’s in Eqs. (52) and (60) are the wave-function renormalization factors, Eqs. (C.1) and (C.4) in Appendix C. If there are several modes, summation should be taken over all modes. The factor ‘2’ on the RHS of Eq. (59) is a reflection of the two independent polarizations.

Thus, we have learned that \( n \) is the number density of gluonic quasiparticle. Undoing the narrow-width approximation yields further corrections to the number densities.

Let us turn to analyze the contributions from the other parts of \( \hat{G} \) in Eq. (17). Inspection of Eq. (58) with Eqs. (17) - (54) shows that all but \( G^{(1)}_K \), Eq. (53), yield well-defined corrections to the physical number densities due to the medium effect. \( G^{(1)}_K \) contains

\[
\frac{1}{[P^2 - \Pi^T_R]} \left[ P^2 - (\Pi^T_R)^* \right]
\]

and

\[
\frac{1}{[p^2 + \Pi^L_R]} \left[ p^2 + (\Pi^L_R)^* \right].
\]

In the narrow-width approximation \( \text{Im} \Pi^T_R \to -\epsilon(p_0)0^+ \) [\( \text{Im} \Pi^L_R \to -\epsilon(p_0)0^+ \)], Eq. (51) [Eq. (52)] develops pinch singularity at \( p_0 = \omega_T(p) \) \( [p_0 = \omega_L(p)] \) in a complex \( p^0 \)-plane. Then the contributions of \( G^{(1)}_K \) to Eq. (58) diverge in this approximation. In practice, \( \text{Im} \Pi^T_R (\propto g^2) \) is a small quantity (at least for hard modes), so that the contribution, although not divergent, is large. This invalidates the perturbative scheme and a sort of “renormalization” is necessary for the number densities \[3, 4, 5\]. This observation leads us to introduce the condition \( G^{(1)}_K = 0 \) on the energy-shells:

\[
\begin{align*}
2P \cdot \partial_X f + \left\{ \text{Re} \Pi^T_R, f \right\}_{\text{P.B.}} + i(\Pi^T_R)^\text{loop} \bigg|_{p_0 = \pm \omega_T(X; \pm p)} &= 0, \\
2p \cdot \nabla_X f + \left\{ \text{Re} \Pi^L_R, f \right\}_{\text{P.B.}} + i(\Pi^L_R)^\text{loop} \bigg|_{p_0 = \pm \omega_L(X; \pm p)} &= 0,
\end{align*}
\]

These serve as determining equations for so far arbitrary \( f \). (See below, for more details.) Now the above-mentioned large contributions, which turn out to the diverging contributions (due to the pinch singularities) in the narrow-width approximation, do not appear. Thus, the contributions from \( G^{(1)}_K \) to Eq. (58) also yields well-defined corrections to the number densities.
### 5.2 Generalized Boltzmann equation

We are now in a position to disclose the physical meaning of Eqs. (63) and (64). With the help of the formulae in Appendix C, Eq. (63) with \( p_0 \to \omega_T(X; p) \) turns out to

\[
(Z_T(X; P))^{-1} \left[ \frac{\partial}{\partial X^0} + v_T(X; p) \cdot \frac{\partial}{\partial X} \right] n(X; P)
+ \frac{1}{2 \omega_T(X; p)} \frac{\partial \text{Re} \Pi^T_K(X; P)}{\partial X^\mu} \frac{\partial n(X; P)}{\partial P_\mu}
= -\frac{i}{2 \omega_T(X; p)} \left( \Pi^T_K \right)^{\text{loop}} (X; P) .
\]  

(65)

Here \( v_T (\equiv \partial \omega_T(X; p)/\partial p) \) is the group velocity of the mode [cf. Eq. (C.2)]. As will be shown in Appendix D, the RHS of Eq. (65) is related to the net production rate, \( \Gamma^T_{\text{net } p} \), of the mode \( p^0 = \omega_T(X; p) \). Using Eqs. (D.1) and (C.3), we obtain

\[
\left[ \frac{\partial}{\partial X^0} + v_T(X; p) \cdot \frac{\partial}{\partial X} \right] n(X; P)
+ \frac{\partial \omega_T(X; p)}{\partial X^\mu} \frac{\partial n(X; P)}{\partial P_\mu} = \Gamma^T_{\text{net } p}(X; p) .
\]

This can further be rewritten in the form,

\[
\left( \frac{d}{dX^0} + v_T(X; p) \cdot \frac{d}{dX} \right) n(X; \omega_T(X; p), \hat{p})
- \frac{\partial \omega_T(X; p)}{\partial X} \frac{dn}{d\hat{p}} = \Gamma^T_{\text{net } p}(X; p) .
\]

(66)

Similarly, Eq. (64) with \( p_0 = \omega_L(X; p) \) yields

\[
\left( \frac{d}{dX^0} + v_L(X; p) \cdot \frac{d}{dX} \right) n(X; \omega_L(X; p), \hat{p})
- \frac{\partial \omega_L(X; p)}{\partial X} \frac{dn}{d\hat{p}} = \Gamma^L_{\text{net } p}(X; p) .
\]

(67)

\( n \) here is essentially (the main part of) the relativistic Wigner function, and Eqs. (66) and (67) are the generalized relativistic Boltzmann equation for gluonic quasiparticles [cf. (11)].

For the sake of definiteness, we are taking QCD and have dealt with the gluon sector. The quark sector has already been dealt with in [5]. As far as the gluon sector is concerned to the gradient approximation, one can use the “free quark”
distribution function that subjects to the “free Boltzmann equation.” This can be seen as follows: In the gluon sector, the quark distribution function enters through the quark propagators involved in the gluon self-energy part $\hat{\Pi}$. As observed at the beginning of Sec. IV, $\hat{\Pi}$ is related to the gradient part of the gluon propagator. Thus, in calculating $\hat{\Pi}$, one can use the 0th-order expression (of the derivative expansion) for the propagators involved in $\hat{\Pi}$.

6 Perturbation theory

As has been discussed in the preceding section, the propagator in the physical-$N$ scheme is free from the pinch singular term (in the narrow-width approximation) and then the perturbative calculation of some quantity yields “healthy” perturbative corrections. For constructing a concrete perturbative scheme, one more step is necessary.

To extend the conditions (63) and (64) to off the energy-shells, we divide $f$ into two pieces [cf. Eq. (17)]

$$f(X; P) = \epsilon(p^0)N(X; P) - \theta(-p^0) = \tilde{f}(X; P) + f_0(X; P) = [\epsilon(p^0)\tilde{N}(X; P) - \theta(-p^0)] + \epsilon(p^0)N_0(X; P).$$

(68)

$f_0$ (and then also $\tilde{f}$) is defined as follows: Let $R_i(X; p)$ ($i = 1, 2, ...$) be a region in a $p^0$-plane that includes $i$th energy-shell. We choose $R_i(X; p)$, such that, for $i \neq j$, $R_i \cap R_j = \emptyset$. On each energy-shell, $N_0(X; P) = N(X; P)$, and, in whole $p^0$-region but $R_i(X; p)$ ($i = 1, 2, ...$), $N_0(X; P)$ vanishes. $\partial N_0(X; P)/\partial X$ and $\partial N_0(X; P)/\partial P$ exist and $N_0(X; P)$ obeys

$$2P \cdot \partial_X N_0 + \{\text{Re} \Pi^T_R, N_0\}_{\text{P.B.}} = -i \left( \Pi^T_K \right)_{\text{loop}},$$

$$2p \cdot \nabla_X N_0 + \{\text{Re} \Pi^L_R, N_0\}_{\text{P.B.}} = -i \left( \Pi^L_K \right)_{\text{loop}}.$$  

(69)

Then, $G^{(1)\mu\nu}_K$ in Eq. (53) turns out to

$$G^{(1)\mu\nu}_K = -i \left[ 2P \cdot \partial_X \tilde{f} + \{\text{Re} \Pi^T_R, \tilde{f}\}_{\text{P.B.}} + i(\Pi^T_K)^{\text{loop}} \right] \frac{\mathcal{P}^{\mu\nu}}{(P^2 - \Pi^T_R)(P^2 - \Pi^T_A)} \frac{n^\mu n^\nu}{(p^2 + \Pi^L_R)(p^2 + \Pi^L_A)}.$$  

(70)
It is obvious from the above construction that this $G^{(1)\mu\nu}_K$ does not possess pinch singularities in narrow-width approximation, and thus “healthy” perturbation theory is established.

It is worth mentioning that there is arbitrariness in the choice of the regions $R_i(X; p) (i = 1, 2, ... )$. Furthermore, the choice of the functional forms of $\tilde{f}$ and of $f_0$ is also arbitrary, provided that

$$ f(X^0 = X^0_{in}, X; P) + f_0(X^0 = X^0_{in}; X; P) = f(X^0 = X^0_{in}; X; P), $$

where $f(X^0 = X^0_{in}; X; P)$ is the initial data with $X^0_{in}$ the initial time. As has been discussed at the end of Sec. III, these arbitrariness are not the matter.

To summarize, to the gradient approximation, the (resummed) propagator $\hat{G}^{\mu\nu}$ of the theory is

$$ \hat{G}^{\mu\nu} = \hat{G}^{\mu\nu} + \hat{\Delta}^{\mu\nu}_1 + \hat{\Delta}^{\mu\nu}_2 + \hat{\Delta}^{\mu\nu}_3. \quad (71) $$

Here $\hat{\Delta}^{\mu\nu}_1 - \hat{\Delta}^{\mu\nu}_3$ are as in Eqs. (12) - (14) and $\hat{G}^{\mu\nu}$ is as in Eq. (17) with Eqs. (18) - (53), provided that $G^{(1)\mu\nu}_K$ is given by Eq. (70). $f$ consists of two pieces as in Eq. (68). $f_0 (= \epsilon(p^0)N_0)$ subjects to Eq. (69), which is to be solved under a given initial data. It is to be noted that $\hat{\Delta}^{\mu\nu}_1 - \hat{\Delta}^{\mu\nu}_3$ are the gradient parts, so that, if one wants, for $f$ in $\hat{\Delta}^{\mu\nu}_1 - \hat{\Delta}^{\mu\nu}_3$, one can substitute the solution to the “free Boltzmann equation,” Eq. (23).

Determination of $f$ or $N$ proceeds order by order in perturbation theory to get $f = f_0 + f_1 + f_2 + ...$. Provided that perturbative computation is completed up to $(n - 1)$th order, one can proceed to $n$th order calculation. $f$ or $N$ enters through $\hat{\Delta}^{\mu\nu}_{base}$, Eq. (55), and through $\hat{\Pi}^{(c)\mu\nu} (\propto P \cdot \partial f)$, Eq. (29). For $f$ in $\hat{\Delta}^{\mu\nu}_{base}$ and in $\hat{\Pi}^{(c)}$ in Eq. (69), we substitute $\sum_{j=0}^n f_j$, while for $f$ in $\hat{\Pi}^{(c)}$ involved in some $\hat{\Pi}^{loop}$, Eq. (49), we use appropriate $\sum_{i=0}^j f_i \ (j \leq n - 1)$. Then, proceeding as above, one can determine $f_n$, and thereby any physical quantity may be computed. Here we recall that we are taking the gradient approximation. Then, in proceeding to higher orders, one should check whether or not the consequences of the calculation of the order under consideration is consistent with the gradient approximation. One suspects that, in general, only first few orders meet this criterion.

As to the vertex, as has been mentioned at Sec. II A, the matrix vertex $\hat{V}$ is diagonal and its (2 2)-component is of opposite sign relative to the (1 1)-component.
It is to be noted that the two-point vertex $iL_{\epsilon}^{\mu\nu}(x, y)\hat{M}_-$ in Eq. (29) has already been built into $\hat{G}^{\mu\nu}$ (cf. Eqs. (49) and (50)) and is absent in the perturbative framework using $\hat{G}^{\mu\nu}$ in Eq. (71).

7 Summary and outlook

In this paper we have dealt with out-of-equilibrium perturbation theory for gauge bosons and FP-ghosts in Coulomb gauge. The gauge-boson and FP-ghost propagators are constructed from first principles. Only approximation we have employed is the so-called gradient approximation, so that the perturbative framework applies to the quasiumiform systems near equilibrium or the nonequilibrium quasistationary systems. The framework allows us to compute any reaction rates.

There comes out naturally the generalized Boltzmann equation (GBE) that describes the spacetime evolution of the number densities of quasiparticles, through which the evolution of the system is described.

The GBE for a quark-gluon plasma (nonequilibrium QCD) is “directly” derived in [12] [cf., also, [13]]. The “derivation” of the GBE in this paper is quite different from that in [12]. What we have shown here is that the requirement of the absence of large contributions from the perturbative framework leads to the GBE. This means that the quasiparticles thus defined are the well-defined modes in the medium. Conversely, if we start with defining the quasiparticles such that their number density functions subject to the GBE, then, on the basis of them, well-defined perturbation theory may be constructed.

Acknowledgemente

This work was supported in part by a Grant-in-Aide for Scientific Research ((C)(2) (No. 12640287)) of the Ministry of Education, Science, Sports and Culture of Japan.
Appendix A: Properties of the self-energy parts

A.1: Gluon self-energy part

Let us start with studying the properties of the propagator. From Eqs. (9) - (20) and Eq. (26), we obtain

\[
\begin{align*}
[i\Delta^{\mu\nu}_{11}(X;P)]^* & = i\Delta^{\nu\mu}_{22}(X;P) = i\Delta^{\nu\mu}_{22}(X;-P), \\
[i\Delta^{\mu\nu}_{12(21)}(X;P)]^* & = i\Delta^{\nu\mu}_{12(21)}(X;P) = \Delta^{\nu\mu}_{21(12)}(X;-P),
\end{align*}
\]

where use has been made of \( f(X;P) = -\theta(-p_0) + \epsilon(p_0)n(X;\epsilon(p_0)p) \).

Now, we observe the properties of vertices. As has been mentioned in Sec. IIA and Sec. VI, the vertex matrix is diagonal and its (2 2)-component is of opposite sign relative to the (1 1)-component, \( \hat{V} = \text{diag}(v^{(1)}, v^{(2)}) = \text{diag}(v, -v) \). For QCD, let \( \hat{V}_3 = (v_3(P), -v_3(P)) \) be a 3-gluon-vertex factor and \( \hat{V}_4 = (v_4, -v_4) \) be a 4-gluon-vertex factor. Noting that \( v_3(P) \) is real and linear in \( P \), and \( v_4 \), being independent of \( P \), is pure imaginary, we see that

\[
\begin{align*}
(v_3^{(1)}(P))^* & = v_3(P) = -v_3(-P) = v_3^{(2)}(-P), \\
(v_4^{(1)})^* & = -v_4 = v_4^{(2)}
\end{align*}
\]

(A.1)

holds. There is one more (two-point) vertex (29), which satisfies

\[
\begin{align*}
[iL^\mu\nu_c(X;P)\hat{M}_{-}]^* & = iL^\mu\nu_c(X;-P)\hat{M}_{-}.
\end{align*}
\]

(A.2)

For quark \( \bar{f} \) and FP-ghost, one can write down the similar relations, which we do not display here.

From the diagrammatic analysis of \( \Pi(X;P) \) using the above relations, one can straightforwardly obtain the relations,

\[
\begin{align*}
[\Pi^{\mu\nu}_{11}(X;P)]^* & = -\Pi^{\mu\nu}_{22}(X;-P) = -\Pi^{\mu\nu}_{22}(X;P), \\
[\Pi^{\mu\nu}_{12(21)}(X;P)]^* & = -\Pi^{\mu\nu}_{21(12)}(X;-P) = -\Pi^{\mu\nu}_{21(12)}(X;P).
\end{align*}
\]

Using the decomposition (31) for the leading parts of \( \Pi(X;P) \), we obtain

\[
\begin{align*}
[(\Pi_S(X;P))_{11}]^* & = -\Pi_S(X;P), \\
[(\Pi_S(X;P))_{12(21)}]^* & = -\Pi_S(X;P),
\end{align*}
\]

\( (S = T, L, C, D) \).
The last equation shows that \((\Pi_5(X;P))_{12(21)}\) are pure imaginary.

Let us turn to analyze \(\hat{\Pi}\) in configuration space:

\[
\hat{\Pi}(x, y) = \int \prod_{i=1,2,...} [dz_i du_i dv_i] 
\times \hat{\Pi}(x,y; z_1, z_2, ...; (u_1, v_1), (u_2, v_2), ...), \quad (A.4)
\]

where \(x\) and \(y\) are the spacetime points to which the external legs are attached, \(z_i (i = 1, 2, \ldots)\) are the internal vertex-points, and \((u_i, v_i) [i = 1, 2, \ldots]\) are the sets of internal vertex-points of \(iL_c(u_i, v_i)\hat{M}_-\). From Eq. (3), which leads to Eqs. (8) - (20), and Eq. (26), we see that

\[
\Delta_{1i}(x, y) = \Delta_{2i}(x, y) \quad \text{for} \quad x^0 > y^0 \quad (i = 1, 2). \quad (A.5)
\]

Here we use the “largest-time technique” [14]. Using the relation (A.5) and Eq. (A.1) (in configuration space) together with their quark and FP-ghost counterparts, one can show that, in Eq. (A.4), when \(z^0_i\) of \((z^0_1, z^0_2, \ldots)\) is the largest time, then \(\hat{\Pi}(x, y) = 0\). As can be shown by using Eq. (A.2) in configuration space, this is also the case for the largest \(u^0_i\) or for the largest \(v^0_i\) of \([(u^0_1, v^0_1), (u^0_2, v^0_2), \ldots]\). Then, in Eq. (A.4), the largest time is \(x^0\) or \(y^0\).

Similar argument as above shows that

\[
\Pi_{21}(x, y) = -\Pi_{11}(x, y), \quad \Pi_{22}(x, y) = -\Pi_{12}(x, y) \quad \text{for} \quad x^0 > y^0,
\]

\[
\Pi_{11}(x, y) = -\Pi_{12}(x, y), \quad \Pi_{21}(x, y) = -\Pi_{22}(x, y) \quad \text{for} \quad x^0 < y^0. \quad (A.6)
\]

From the first (second) relation, we get \(\Pi_A(x, y) \equiv \Pi_{11} + \Pi_{21} = 0\) for \(x^0 > y^0\), and \(\Pi_R(x, y) \equiv \Pi_{11} + \Pi_{12} = 0\) for \(x^0 < y^0\), as they should be. From Eq. (A.7), follows

\[
\sum_{i,j=1}^{2} \Pi_{ij}(x, y) = 0. \quad (A.7)
\]

Using the decomposition (31), after Fourier transformation, we obtain

\[
\sum_{i,j=1}^{2} (\Pi_5(X;P))_{ij} = 0 \quad (S = T, L, C, D). \quad (A.8)
\]

The relation (A.3) leads to

\[
[\Pi_A^S(X;P)]^* = \Pi_R^S(X;P) \quad (S = T, L, C, D). \quad (A.9)
\]
A.2: FP-ghost self-energy part

In every diagram representing the FP-ghost self-energy part $\hat{\Pi}_g$, two external FP-ghost lines are connected with each other. Then, the fact that the bare FP-ghost propagator is diagonal [cf. Eq. (6)] tells us that the matrix self-energy part $\hat{\Pi}_g$ is also diagonal, $\hat{\Pi}_g = \text{diag}(\Pi_g^{(1)}, \Pi_g^{(2)})$.

In a similar manner to the above subsection, we obtain [cf. Eqs (A.8) and (A.9)]

$$\Pi_g^{(1)}(X; P) + \Pi_g^{(2)}(X; P) = 0,$$

$$\left[\Pi_g^{(1)}(X; P)\right]^* = -\Pi_g^{(2)}(X; P).$$

Then, we see that

$$\hat{\Pi}_g(X; P) = \hat{\tau}_3 \Pi_g(X; P), \quad \text{Im}\Pi_g(X; P) = 0.$$

Appendix B: On the derivation of $G_K$ in Eqs. (53) and (54)

Here we deal with a part of $G_K$ in Eq. (46),

$$\left[ G_R \cdot \Delta_R^{-1} \cdot \Delta_K \cdot \Delta_A^{-1} \cdot G_A \right] (x, y) \simeq \int \frac{d^4P}{(2\pi)^4} e^{-P \cdot (x-y)} G'_K(X; P),$$

$$G'_K(X; P) = G_R(X; P) \Delta_R^{-1}(P) \Delta_K(X; P) \Delta_A^{-1}(P) G_A(X; P), \quad \text{(B.1)}$$

which is valid within the approximation under consideration. We are adopting the strict Coulomb gauge $\alpha = 0$. Since $(\Delta_R^{-1}(P))^{\mu\nu} = (\Delta_A^{-1}(P))^{\mu\nu} = -D^{\mu\nu}(P)$ [cf. Eq. (34)] contains a term $-g^{\mu i} g^{\nu j} p^i p^j / \alpha (\alpha \to 0)$, care should be taken for computing Eq. (B.1). We need $O(\alpha)$ contribution to $G_{R(A)}$. For the present purpose, it is sufficient to obtain the leading-order expression, which is obtained from Eq. (45):

$$G_{R(A)}^{\mu\nu}(X; P) \simeq G^{(0)\mu\nu}_{R(A)}(X; P) + \alpha G^{\prime \mu\nu}_{R(A)}(X; P),$$

$$G^{\prime \mu\nu}_{R(A)}(X; P) = \frac{1}{P^2} \left[ \frac{P_0}{P^2} \left( \frac{p^2 + \Pi_R^{C}}{p^2 + \Pi_R^{T}} \right)^2 + \frac{P_0}{p} \left( \hat{\rho}_\mu \hat{\rho}_\nu + n^\mu \hat{\rho}^\nu \right) \left( \frac{p^2 + \Pi_R^{C}}{p^2 + \Pi_R^{T}} \right) + \hat{\rho}^\mu \hat{\rho}^\nu \right].$$

24
where $G^{(0)\mu\nu}_{R(A)}$ is as in Eq. (51) and $\tilde{p}^\mu \equiv (0, \tilde{p})$ as in Eq. (31).

We observe that

$$G^{(0)\mu\nu}_{R} p^\nu \equiv p^\nu G^{(0)\nu\mu}_{R(A)} = 0,$$

$$p^\rho \frac{\partial}{\partial p^\rho} G'_{R} p^\nu = \frac{1}{p^2} \left[ p^\nu + \frac{p^2 + \Pi^C_{R(A)}}{p^2 + \Pi^L_{R(A)}} p_0 n^\nu \right].$$

Substituting these and Eqs. (51) and (31) into Eq. (B.1), we obtain, after some manipulations,

$$G'_K(X; P) = -\frac{i}{p^2} \left[ \frac{2p^2 n^\mu n^\nu}{(p^2 + \Pi^L_\perp)(p^2 + \Pi^L_A)} \mathbf{p} \cdot \nabla_X f + \frac{p^\mu \nabla^\mu_\perp f}{P^2 - \Pi^T_R} \right. \left. \nabla_X f + \frac{p^0 \nabla^\mu_\perp f}{P^2 - \Pi^T_A} + p_0 \left( \frac{\Pi^C_{R} n^\mu \nabla^\mu_\perp}{p^2 + \Pi^L_\perp(p^2 - \Pi^T_\perp)} + \frac{\Pi^C_{A} n^\nu \nabla^\nu_\perp}{p^2 + \Pi^L_A(p^2 - \Pi^T_A)} \right) \right].$$

### Appendix C: On the energy shell

Here we display some formulae, which hold on the energy-shells of quasiparticles [cf. Eqs. (56) and (57) with Eq. (51)]. In most formulae in this Appendix, the argument $X$ is dropped. We restrict to the positive $p_0$ part.

**Transverse modes:**

We first define the wave-function renormalization factors through taking derivative of Eq. (56) with respect to $p^0$:

$$\left( Z_T(\omega_T(p), p) \right)^{-1} = 1 - \frac{1}{\omega_T} \frac{\partial}{\partial p^0} \text{Re} \Pi^T_R(p^0, p) \bigg|_{p^0 = \omega_T(p)}. \quad (C.1)$$

The group velocities of the modes are obtained from the definition (56):

$$\mathbf{v}_T(p) \equiv \left( \frac{d\omega_T(p)}{dp} \right) \frac{Z_T(\omega_T(p), p)}{\omega_T(p)} \times \left( p + \frac{1}{2} \frac{\partial}{\partial p} \text{Re} \Pi^T_R(p^0, p) \bigg|_{p^0 = \omega_T(p)} \right). \quad (C.2)$$

Differentiation of Eq. (56) with respect to $X$ leads to

$$\frac{\partial \omega_T(X; p)}{\partial X} = Z_T(\omega_T(p), p) \frac{\partial}{\partial X} \text{Re} \Pi^T_R(X; \omega_T(X; p), p) \left( \frac{2\omega_T}{\partial X} \right). \quad (C.3)$$
“Longitudinal” modes:

From Eq. (57), we obtain

\[
(Z_L(\omega_L(p), p))^{-1} = \frac{1}{2\omega_L} \frac{\partial \text{Re} \Pi^L_R(p^0, p)}{\partial p^0} \bigg|_{p^0 = \omega_L(p)},
\]

(C.4)

\[
v_L(p) \equiv \frac{d\omega_L(p)}{dp} = -\frac{Z_L(\omega_L(p), p)}{\omega_L(p)} \left[ p + \frac{1}{2} \frac{\partial \text{Re} \Pi^L_R(\omega_L(p), p)}{\partial p} \bigg|_{p^0 = \omega_L(p)} \right].
\]

Appendix D: Net production rates

From Eqs. (51), (56) and (57), we see that the projection operators onto \( p^0 = \omega_T \) mode is \( P^\mu_\nu_T(\hat{p}) \) \((\mu,\nu)\). Then, the production (decay) rates of the transverse modes and of the “longitudinal” modes are written as \[7, 3, 4, 5\]

\[
\Gamma_T^{p(d)}(p) = \frac{i}{2\omega_T} Z_T(\omega_T(p), p) \frac{1}{2} \Pi_T^{\mu\nu} \mu\nu \left( P_T(\hat{p}) \right)_{\mu\nu}
\]

and

\[
\Gamma_L^{p(d)}(p) = \frac{i}{2\omega_L} Z_L(\omega_L(p), p) \Pi_L^{\mu\nu} \mu\nu \left( P_L(\hat{p}) \right)_{\mu\nu}
\]

respectively. Here \( Z \)'s are the wave-function renormalization factor, Eqs. (C.1) and (C.4). Thus, the net production rate is

\[
\Gamma_{net}^{p(L)}(p) = (1 + n(\omega_T(L)(p), \hat{p})) \Gamma_T^{p(L)}(X; p) - n(\omega_T(L)(p), \hat{p}) \Gamma_d^{p(L)}(p)
\]

\[
= \frac{i}{2\omega_T} Z_T(L)(\omega_T(L)(p), p) \left( P_T^{\mu}(\omega_T(L)(X; p), p) \right)_{\mu\nu} \left( P_L^{\nu}(\omega_T(L)(X; p), p) \right)_{\nu\mu}
\]

(D.1)

where \( \left( P^{T(L)}_K \right)^{\text{loop}} \) is as in Eq. (55).
References

[1] J. D. Bjorken, *Acta Phys. Polon. B* **28**, 2773 (1997); See also, e.g., K. Rajagopal, in Quark-Gluon Plasma 2, edited by R. C. Hwa (World Scientific, Singapore, 1995), p.484.

[2] M. Le Bellac, “Thermal Field Theory,” Cambridge University Press, Cambridge, England, 1996.

[3] A. Niégawa, Prog. Theor. Phys. **102**, 1 (1999).

[4] A. Niégawa, Phys. Rev. D **62**, 125004 (2000).

[5] A. Niégawa, [hep-ph/0101216](http://arxiv.org/abs/hep-ph/0101216).

[6] J. Schwinger, J. Math. Phys. **2**, 407 (1961); L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) (Sov. Phys. JETP **20**, 1018 (1965)); R. A. Craig, J. Math. Phys. **9**, 605 (1968); J. Rammer and H. Smith, Rev. Mod. Phys. **58**, 323 (1986).

[7] K.-C. Chou, Z.-B. Su, B.-L. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985).

[8] N. P. Landsman and Ch. G. van Weert, Phys. Rep. **145**, 141 (1987).

[9] A. Niégawa, K. Okano, and H. Ozaki, Phys. Rev. D **61**, 056004 (2000). For the case of equilibrium systems, see, A. Niégawa, *ibid*. **57**, 1379 (1998).

[10] H. Umezawa, “Advanced Field Theory — Micro, Macro, and Thermal Physics.” AIP, New York, 1993; Y. Yamanaka, H. Umezawa, K. Nakamura, and T. Arimitsu, Int. J. Mod. Phys. **A9**, 1153 (1994); Y. Yamanaka and K. Nakamura, Mod. Phys. Lett. **A9**, 2879 (1994); H. Chu and H. Umezawa, Int. J. Mod. Phys. **A9**, 1703 and 2363 (1994).

[11] P. Zhuang and U. Heinz, Phys. Rev. D **57**, 6525 (1998); S. Ochs and U. Heinz, Ann. Phys. (N.Y.) **266**, 351 (1998).

[12] J.-P. Blaizot and E. Iancu, Nucl. Phys. **B557**, 183 (1999).

[13] S. Mrówczyński, Phys. Part. Nucl. **30**, 419 (1999) [hep-ph/9805435](http://arxiv.org/abs/hep-ph/9805435), and earlier works quoted therein.
[14] R. Kobes, Phys. Rev. D 42, 562 (1990).