THE LOGARITHMIC ENTROPY FORMULA FOR THE LINEAR HEAT EQUATION ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper we introduce a new logarithmic entropy functional for the linear heat equation on complete Riemannian manifolds and prove that it is monotone decreasing on complete Riemannian manifolds with nonnegative Ricci curvature. Our results are simpler version, without Ricci flow, of R.-G. Ye’s recent result (arXiv: math.DG/0708.2008). As an application, we apply the monotonicity of the logarithmic entropy functional of heat kernels to characterize Euclidean space.

1. Introduction

Given a compact n-dimensional Riemannian manifold \((M, g_0)\) without boundary, the Ricci flow is the following evolution equation

\[
\frac{\partial g}{\partial t} = -2\text{Ric}
\]

with the initial condition \(g(x, 0) = g_0(x)\), where \(\text{Ric}\) denotes the Ricci tensor of the metric \(g(x, t)\). The Ricci flow equation was introduced by R. Hamilton to approach the geometrization conjecture in [1]. Recently, studying various entropy functionals along the Ricci flow is a very powerful tool for understanding of Riemannian manifolds. A nice example is that G. Perelman [2] introduced the following shrinking entropy functional

\[
\mathcal{W}(g(t), f(t), \tau) := \int_M \left[ \tau \left( R + |\nabla f|^2 \right) + f - n \right] (4\pi\tau)^{-n/2} e^{-f} d\mu,
\]

where \(\tau > 0\) and \(d\tau/dt = -1\), \(R\) and \(d\mu\) denote the scalar curvature and the volume form of the metric of \(M\), respectively. He proved that this entropy functional is nondecreasing along the Ricci flow coupled to a backward heat-type equation. More precisely, if \(g(t)\) is a solution to the Ricci flow (1.1) and the coupled function \(f(x, t)\) satisfies the evolution equation

\[
\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau},
\]

then Perelman proved that

\[
\frac{\partial \mathcal{W}}{\partial t} = 2\tau \int_M \left| \text{Ric} + \nabla^2 f - \frac{g}{2\tau} \right|^2 (4\pi\tau)^{-n/2} e^{-f} d\mu.
\]

The monotonicity of this entropy can be used to prove that shrinking breathers must be shrinking gradient Ricci solitons. More importantly, the monotonicity

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property is also fundamental in proving Hamilton’s little loop conjecture or what Perelman calls the no local collapsing theorem (see [2] or [3]).

Another example is that in [4], M. Feldman, T. Ilmanen and L. Ni tweaked some signs for Perelman’s shrinking entropy \( W \), and constructed a new entropy \( W^+ \) corresponding to expanding Ricci solitons, i.e.,

\[
W^+(g(t), f_+(t), \sigma) := \int_M \left[ \sigma \left( R + |\nabla f_+|^2 \right) - f_+ + n \right] (4\pi\sigma)^{-n/2} e^{-f_+} d\mu,
\]

where \( \sigma > 0 \) and \( d\sigma/dt = 1 \). They showed that this expanding entropy \( W^+ \) is also monotone nondecreasing on closed Riemannian manifolds under the Ricci flow \( (1.1) \) coupled to the backward heat-type equation

\[
\frac{\partial f_+}{\partial t} = -\Delta f_+ + |\nabla f_+|^2 - R - \frac{n}{2\sigma}
\]

and constant precisely on expanding Ricci solitons.

Besides the above two entropies, R.-G. Ye in [5] also introduced a new logarithmic entropy functional

\[
\mathcal{Y}_a(g(t), u(t), t) := -\int_M u^2 \log u^2 d\mu + \frac{n}{2} \log \left[ \int_M \left( |\nabla u|^2 + \frac{R}{4} u^2 \right) d\mu + a \right] + 4at
\]

on an \( n \)-dimensional closed Riemannian manifold under the Ricci flow \( (1.1) \), where \( a \) is some constant. Here the coupled function \( u(t) \) satisfies

\[
\frac{\partial u}{\partial t} = -\Delta u - \frac{|\nabla u|^2}{u} + \frac{R}{2} u
\]

such that \( \int_M u^2 d\mu = 1 \). In other words, \( u^2(t) \) solves to the conjugate heat-type equation

\[
\frac{\partial u^2}{\partial t} = -\Delta u^2 + Ru^2.
\]

satisfying \( \int_M u^2 d\mu = 1 \). Under some suitable assumption, Ye can use the monotonicity of Perelman’s entropy to derive the monotonicity of this logarithmic entropy functional along the Ricci flow \( (1.1) \) coupled to the heat-type equation \( (1.4) \). If we let \( u = (4\pi\tau)^{-n/4} e^{-f/2} \), then Perelman’s entropy functional can be rewritten as

\[
W(g(t), u(t), \tau) = -\int_M u^2 \log u^2 d\mu + 4\tau \left[ \int_M \left( |\nabla u|^2 + \frac{R}{4} u^2 \right) d\mu + a \right] - 4a\tau - \frac{n}{2} \log(4\pi\tau) - n
\]

for an arbitrary constant \( a \), and heat-type equation \( (1.2) \) becomes equation \( (1.4) \). We would like to point out that Ye’s entropy \( (1.3) \) is often called logarithmic entropy because of the appearance of an additional logarithmic operation compared with Perelman’s entropy functional \( (1.5) \).

The above mentioned entropy functionals are all considered with the metric evolved by the Ricci flow. Below we recall an entropy functional proposed by L. Ni [6] on a closed manifold with a fixed metric. Let \( (M, g) \) be an \( n \)-dimensional closed Riemannian manifold with a fixed metric. Ni [6] (see also chapter 16 in [3])
considered the linear heat equation

\[ (\frac{\partial}{\partial t} - \Delta) \tilde{u} = 0 \]

and introduced the following entropy

\[ W(f, \tau) := \int_M (\tau |\nabla f|^2 + f - n) (4\pi \tau)^{-n/2} e^{-f} d\mu, \]

where \((f, \tau)\) satisfies

\[ \tilde{u} = \frac{e^{-f}}{(4\pi \tau)^{n/2}} \text{ and } \int_M e^{-f} (4\pi \tau)^{n/2} d\mu = 1 \]

with \(\tau > 0\). By direct computation, Ni obtained the following result.

**Theorem A.** (L. Ni [6]) Let \((M, g)\) be an \(n\)-dimensional closed Riemannian manifold. Assume that \(\tilde{u}\) is a positive solution to the heat equation (1.6) with

\[ \int_M \tilde{u} d\mu = 1. \]

Let the smooth function \(f\) be defined as \(\tilde{u} = (4\pi \tau)^{-n/2} e^{-f}\) and \(\tau = \tau(t)\) with \(d\tau/dt = 1\). Then

\[ \frac{dW}{dt} = -\int_M 2\tau \left( |\nabla^2 f - g/2\tau|^2 + \text{Ric}(\nabla f, \nabla f) \right) \tilde{u} d\mu. \]

In particular, if \(M\) has nonnegative Ricci curvature, then \(W(f, \tau)\) is monotone decreasing along the heat equation (1.6).

We remark that if \(\tilde{u} = H\) is the positive fundamental solution of the heat equation (1.6), then Ni’s entropy formula on closed manifolds can be generalized to complete noncompact manifolds with nonnegative Ricci curvature (see Lemma 4.1 below).

In this paper, motivated by the work of Ye [5], we will introduce a new logarithmic entropy functional for the linear heat equation (1.6) on a complete (possibly noncompact) manifold under the static metric. This entropy functional is very similar to the appearance of the logarithmic entropy functional along the Ricci flow introduced by Ye [5]. Following similar arguments to that of Ye [5], we employ the property of Ni’s entropy functional and derive the monotonicity of our logarithmic entropy functional for the heat equation (1.6) as long as the Ricci curvature of the manifold is nonnegative. As an application, on the noncompact case, we apply the monotonicity of our logarithmic entropy functional of heat kernels to characterize the Euclidean space. The main results of this paper are Theorem 2.3 in Section 2, Theorem 4.4 in Section 4 and Theorem 5.1 in Section 5.

The rest part of this paper is organized as follows. In Section 2, we first give some definitions of logarithmic entropies and then we state Theorem 2.3 which may be a natural generalization of Ye’s logarithmic entropy. After that, in Section 8 we give a detailed proof of Theorem 2.3. The proof mainly follows the arguments of Ye’s logarithmic entropy functional along Ricci flow [5]. The most difference is that our proof here makes use of the monotonicity of Ni’s entropy formula. In Section 4, we generalize Theorem 2.3 to the complete noncompact setting (see Theorem 4.4). In Section 5, we apply Theorem 4.4 to give a characterization of Euclidean space. In Section 6, we generalize Theorem 2.3 to the case of the weighted heat equation.
2. Monotonicity of the logarithmic entropy

Now we start to give several definitions, which are very similar to Ye’s definitions in the Ricci flow case.

**Definition 2.1.** Let $(M, g)$ be an $n$-dimensional closed Riemannian manifold. For any function $u \in W^{1,2}(M)$ satisfying $\int_M u^2 d\mu = 1$, we first define the logarithmic entropy formula as follows:

$$\mathcal{Y}_0(u) := -\int_M u^2 \log u^2 d\mu + \frac{n}{2} \log \left( \int_M |\nabla u|^2 d\mu \right),$$

where function $u$ satisfies $\int_M |\nabla u|^2 d\mu > 0$.

Then we define the logarithmic entropy formula with remainder $a$ ($a > -\lambda$) as follows:

$$\mathcal{Y}_a(u) := -\int_M u^2 \log u^2 d\mu + \frac{n}{2} \log \left( \int_M |\nabla u|^2 d\mu + a \right),$$

where $u \in W^{1,2}(M)$ and $\int_M u^2 d\mu = 1$. Here $\lambda$ denotes the first nonzero eigenvalue of the Laplace operator.

In general the above constant $a$ in $\mathcal{Y}_a(u)$ is not arbitrary. We add the restricted condition: $a > -\lambda$ to guarantee that $\int_M |\nabla u|^2 d\mu + a > 0$.

In the end, we will give the adjusted logarithmic entropy formula including another parameter $t$.

**Definition 2.2.** Let $u$ and $a$ be the same as the above definition. We define

$$\mathcal{Y}_a(u, t) := -\int_M u^2 \log u^2 d\mu + \frac{n}{2} \log \left( \int_M |\nabla u|^2 d\mu + a \right) - 4at.$$

The form of this entropy is similar to (1.3), with the most difference is that here $\mathcal{Y}_a(u, t)$ is defined under the static metric.

For any closed Riemannian manifold with a fixed metric, let $\lambda$ denote the first nonzero eigenvalue of the Laplace operator. Let $u = u(x, t)$ be a smooth positive solution of the following equation

$$\frac{\partial u}{\partial t} = \Delta u + \frac{|\nabla u|^2}{u}$$

such that the normalization condition

$$\int_M u^2 d\mu = 1$$

holds for all $t$. In fact, if we let $\tilde{u} = u^2$, then $\tilde{u}$ satisfies heat equation (1.6) with the restraint condition

$$\int_M \tilde{u} d\mu = 1.$$

Now we state the monotonicity of the adjusted logarithmic entropy formula on closed manifolds as follows.

**Theorem 2.3.** Let $M$ be an $n$-dimensional closed Riemannian manifold with the nonnegative Ricci curvature. Then the adjusted logarithmic entropy $\mathcal{Y}_a(u, t)$ ($a > -\lambda$) is monotonic with respect to the static metric.
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−λ) is monotone decreasing along the heat-type equation (2.1) (namely, $u^2$ solves the heat equation (1.6)) with $\int_M u^2d\mu = 1$. More precisely,

$$\frac{d}{dt}H_a(u(t), t) \geq \frac{n}{4\omega} \int_M \left[ -2 \frac{\nabla^2 u}{u} + 2 \frac{\nabla u \otimes \nabla u}{u^2} - \frac{4\omega}{n} g \left( \frac{\nabla u}{u}, \frac{\nabla u}{u} \right) \right] u^2 d\mu$$

$$= \frac{n}{4\omega} \int_M \left( f_{ij} - \frac{4\omega}{n} g_{ij} \right)^2 + 4R \bar{f}_{ij} \bar{f}_{ij} \frac{e^{-\bar{f}}}{(4\pi t)^{n/2}} d\mu,$$

where $u = (4\pi t)^{-n/2} e^{-\bar{f}/2}$ and $\omega := \omega(u(x, t), a) = \int_M |\nabla u|^2 d\mu + a > 0$.

We need to emphasize that in a general setting, the function $\bar{f}$ here may be different from the function $f$ employed in the proof of Theorem 2.3 given below. This function $\bar{f}$ is used for the purpose of simplifying the expressions in the above formulas.

**Remark 2.4.** We can generalize this result to the case of noncompact manifolds, which will be discussed in Section 4.

**Remark 2.5.** We can also generalize Theorem 2.3 to the case of the weighted heat equation, which is treated in Section 6. For the concepts of the weighted heat equation, the reader can refer to [7, 8] or [9].

### 3. Proof of Theorem 2.3

Now we introduce Ni’s entropy formula

$$W(f, \tau) = \int_M (\tau |\nabla f|^2 + f - n) \frac{e^{-f}}{(4\pi \tau)^{n/2}} d\mu,$$

for the heat equation (1.6) with $\int_M \tilde{u} d\mu = 1$. Here $\tilde{u} = u^2$ satisfying

$$u = (4\pi \tau)^{-n/2} e^{-f/2},$$

i.e.,

$$f = -\log u^2 - \frac{n}{2} \log \tau - \frac{n}{2} \log(4\pi).$$

So we can rewrite Ni’s entropy as follows:

$$W(f, \tau) = -\int_M u^2 \log u^2 d\mu + (4\tau) \cdot \left( \int_M |\nabla u|^2 d\mu + a \right) - \frac{n}{2} \log(4\tau)$$

$$- 4a \tau - \frac{n}{2} \log \pi - n,$$

where $a$ is an arbitrary constant.

At first, we have the following useful fact.

**Lemma 3.1.** Assume $a > -\lambda$. Let $u \in W^{1,2}(M)$ with

$$\int_M u^2 d\mu = 1.$$

Then the minimum of the function

$$h(s) = s \left( \int_M |\nabla u|^2 d\mu + a \right) - \frac{n}{2} \log s$$

is obtained at $s = \left( \int_M |\nabla u|^2 d\mu + a \right)^{1/n}$. 

$$\frac{d}{dt}H_a(u(t), t) \geq \frac{n}{4\omega} \int_M \left[ -2 \frac{\nabla^2 u}{u} + 2 \frac{\nabla u \otimes \nabla u}{u^2} - \frac{4\omega}{n} g \left( \frac{\nabla u}{u}, \frac{\nabla u}{u} \right) \right] u^2 d\mu$$

$$= \frac{n}{4\omega} \int_M \left( f_{ij} - \frac{4\omega}{n} g_{ij} \right)^2 + 4R \bar{f}_{ij} \bar{f}_{ij} \frac{e^{-\bar{f}}}{(4\pi t)^{n/2}} d\mu,$$
for $s > 0$ is given by

$$\min h = \frac{n}{2} \log \left( \int_M |\nabla u|^2 d\mu + a \right) + \frac{n}{2} \left( 1 - \log \frac{n}{2} \right)$$

and is achieved at the unique minimum point

$$s = \frac{n}{2} \left( \int_M |\nabla u|^2 d\mu + a \right)^{-1}.$$

**Proof.** The proof can proceed essentially along the same line as in [5]. For completeness, we still present the proof in detail. Let

$$\omega = \int_M |\nabla u|^2 d\mu + a.$$

Then we have

$$h(s) = \omega s - \frac{n}{2} \log s.$$

Since $a > -\lambda$, we have $\omega > 0$. So we know $h(s) \to \infty$ as $s \to \infty$, and $h(s) \to \infty$ as $s \to 0$. Therefore the function $h$ achieves its minimum at the unique minimum point $s = \frac{\omega}{2}$, since

$$h'(s) = \omega - \frac{n}{2s}.$$

Hence the minimum of $h$ is

$$h \left( \frac{n}{2 \omega} \right) = \frac{n}{2} - \frac{n}{2} \log \left( \frac{n}{2 \omega} \right)$$

$$= \frac{n}{2} \log \omega + \frac{n}{2} \left( 1 - \log \frac{n}{2} \right).$$

This completes the proof of the lemma. \hfill \square

Using this lemma we have a lower bound of Ni’s entropy when $a > -\lambda$.

**Lemma 3.2.** Assume $a > -\lambda$. Then for each $\tau > 0$, we have

$$W(f, \tau) \geq - \int_M u^2 \log u^2 d\mu + \frac{n}{2} \log \left( \int_M |\nabla u|^2 d\mu + a \right) - 4a\tau + b(n),$$

where

$$b(n) := - \frac{n}{2} \log \pi - \frac{n}{2} \left( 1 + \log \frac{n}{2} \right).$$

Moreover, we have

$$W \left( f, \frac{n}{8\omega(u, a)} \right) = - \int_M u^2 \log u^2 d\mu + \frac{n}{2} \log \left( \int_M |\nabla u|^2 d\mu + a \right) - \frac{na}{2\omega(u, a)} + b(n),$$

where

$$\omega(u, a) := \int_M |\nabla u|^2 d\mu + a.$$

**Proof.** This conclusion follows from the above lemma and (3.2) immediately. \hfill \square

**Remark 3.3.** Lemma 3.2 is still true for complete noncompact Riemannian manifolds as long as $W(f, \tau)$ is finite.

Now we can finish the proof of Theorem 2.3.
Proof of Theorem 2.3. Let \( u(x,t) \) be a smooth positive solution of equation (2.1). Let \( t_1 \leq t_2 \) and for \( t \in [t_1, t_2] \), we define for a given \( \sigma > 0 \)

\[
\tau = \tau(t) = t - t_1 + \sigma.
\]

Assume that \( f(x,t) \) is defined by (3.1), i.e.,

\[
u = (4\pi\tau)^{-n/4}e^{-f/2}.
\]

Obviously, \( f(x,t) \) solves the following equation

\[
\frac{\partial f}{\partial t} = \Delta f - |\nabla f|^2 - \frac{n}{2\tau}
\]

According to Ni’s entropy monotonicity formula along the heat equation, we have for \( f = f(x,t) \) and \( \tau = \tau(t) \) (\( d\tau/\partial t = 1 \))

\[
\frac{dW}{dt} = -\int_M 2\tau \left( |f_{ij} - \frac{g_{ij}}{2\tau}|^2 + R_{ij} f_i f_j \right) \tilde{u} \, d\mu
\]

on time interval \([t_1, t_2]\). Therefore

\[
W(f(t_2), t_2 - t_1 + \sigma) - W(f(t_1), \sigma) = -2 \int_{t_1}^{t_2} \int_M \tau \left( |f_{ij} - \frac{g_{ij}}{2\tau}|^2 + R_{ij} f_i f_j \right) \tilde{u} \, d\mu \, dt.
\]

If we choose

\[
\sigma = \frac{n}{8\omega(u(x,t_1), a)},
\]

then the above equality becomes

\[
W(f(t_2), t_2 - t_1 + \sigma) + 2 \int_{t_1}^{t_2} \int_M \tau \left( |f_{ij} - \frac{g_{ij}}{2\tau}|^2 + R_{ij} f_i f_j \right) \tilde{u} \, d\mu \, dt
\]

\[
= W(f(t_1), \sigma) - \int_M u^2 \log u^2 \, d\mu \bigg|_{t_1}^{t_2} + \frac{n}{2} \log \left( \int_M |\nabla u|^2 \, d\mu + a \right) \bigg|_{t_1}^{t_2} - 4a\sigma + b(n),
\]

where we used (3.3). On the other hand, by Lemma 3.2, we notice that

\[
W(f(t_2), t_2 - t_1 + \sigma) \geq - \int_M u^2 \log u^2 \, d\mu \bigg|_{t_1}^{t_2} + \frac{n}{2} \log \left( \int_M |\nabla u|^2 \, d\mu + a \right) \bigg|_{t_1}^{t_2} - 4a(t_2 - t_1 + \sigma) + b(n).
\]

Combining this with (3.4) yields

\[
- \int_M u^2 \log u^2 \, d\mu \bigg|_{t_1}^{t_2} + \frac{n}{2} \log \left( \int_M |\nabla u|^2 \, d\mu + a \right) \bigg|_{t_1}^{t_2} - 4a\sigma + b(n)
\]

\[
\geq 2 \int_{t_1}^{t_2} \int_M \tau \left( |f_{ij} - \frac{g_{ij}}{2\tau}|^2 + R_{ij} f_i f_j \right) \tilde{u} \, d\mu \, dt - \int_M u^2 \log u^2 \, d\mu \bigg|_{t_1}^{t_2}
\]

\[
+ \frac{n}{2} \log \left( \int_M |\nabla u|^2 \, d\mu + a \right) \bigg|_{t_1}^{t_2} - 4a(t_2 - t_1 + \sigma) + b(n).
\]

It follows that

\[
\mathcal{V}_a(u(t_1), t_1) \geq \mathcal{V}_a(u(t_2), t_2) + 2 \int_{t_1}^{t_2} \int_M \tau \left( |f_{ij} - \frac{g_{ij}}{2\tau}|^2 + R_{ij} f_i f_j \right) \tilde{u} \, d\mu \, dt.
\]
Therefore, we obtain
\[-\frac{dY_a(u(t),t)}{dt} \geq 2\sigma \int_M \left( |f_{ij} - g_{ij}|^2 + R_{ij} f_i f_j \right) \frac{e^{-f}}{(4\pi\sigma)^{n/2}} d\mu \]
\[= \frac{n}{4\omega} \int_M \left[ -2 \frac{\nabla^2 u}{u} + 2 \frac{\nabla u \otimes \nabla u}{u^2} - \frac{4\sigma}{n} g \right]^2 + 4Ric \left( \frac{\nabla u}{u}, \frac{\nabla u}{u} \right) u^2 d\mu.\]

Hence the desired theorem follows.  

4. ON THE COMPLETE NONCOMPACT CASE

When \((M^n, g)\) is complete and noncompact, all of our above discussions hold in Section 3 as long as the integrations by parts make sense and all the integrals involved are finite. In an analogous way as the above argument, we can show that the monotonicity of the adjusted logarithmic entropy \(Y_a(u, t)\) for the heat kernel, still holds for any complete Riemannian manifold with nonnegative Ricci curvature.

First, we recall the following entropy formula on complete noncompact manifolds (see Corollary 16.17 and Theorem 16.27 in [3])

**Lemma 4.1.** Let \((M, g)\) be an \(n\)-dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature. For \(\tilde{u} = u^2 = (4\pi\tau)^{-\frac{n}{2}} e^{-f} = H\), the heat kernel of heat equation (1.6) satisfying \(\int_M \tilde{u} d\mu = 1\) and \(\tau = \tau(t)\) with \(d\tau/dt = 1\), Ni’s entropy quantity \(W(f, \tau)\) is finite. Moreover,

\[
\frac{dW}{dt} \leq - \int_M 2\tau \left| \nabla^2 f - \frac{g}{2\tau} \right|^2 u^2 d\mu.
\]

The above lemma has been carefully proved in [3], which involves estimates for the heat kernel and its first and second derivatives. On the other hand, we have the following known facts (see Corollaries 16.15 and 16.16 in [3]), which are useful for our discussion on the noncompact case.

**Lemma 4.2.** Let \((M, g)\) be an \(n\)-dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature. For \(\tilde{u} = u^2 = (4\pi\tau)^{-\frac{n}{2}} e^{-f} = H\), the heat kernel of heat equation (1.6) satisfying \(\int_M \tilde{u} d\mu = 1\) and \(\tau = \tau(t)\) with \(d\tau/dt = 1\), there exists a constant \(C(n) < \infty\) such that

\[
\int_M f(x, y, \tau) H(x, y, \tau) d\mu \leq C(n)
\]

and

\[
(4.1) \quad \int_M |\nabla f|^2 H d\mu < \infty
\]

for any \(x \in M\) and \(\tau > 0\).

Note that estimate (4.1) follows by Corollary 16.16 in [3] and Li-Yau’s heat kernel upper bounds (Corollary 3.1 in [10]). By Lemma 4.2, we easily have the following proposition.

**Proposition 4.3.** Let \((M, g)\) be an \(n\)-dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature. For \(\tilde{u} = u^2 = (4\pi\tau)^{-\frac{n}{2}} e^{-f} = H\), the heat kernel of heat equation (1.6) satisfying \(\int_M \tilde{u} d\mu = 1\) and \(\tau = \tau(t)\) with \(d\tau/dt = 1\), the adjusted logarithmic entropy quantity \(Y_a(u, t)\) in Definition 2.2 (for any \(x \in M\), \(t \in \mathbb{R}\) and \(a > -\lambda\)) is finite.
Proof. Since
\[-\log H = f + \frac{n}{2} \log (4\pi \tau)\]
then we have
\[(4.2) \quad -\int_M u^2 \log u^2 d\mu = \int_M fH d\mu + \frac{n}{2} \log (4\pi \tau)\]
and
\[(4.3) \quad \int_M |\nabla u|^2 d\mu = \frac{1}{4} \int_M |\nabla f|^2 H d\mu.\]
We also notice that
\[\mathcal{Y}_a(u, t) = -\int_M u^2 \log u^2 d\mu + \frac{n}{2} \log \left( \int_M |\nabla u|^2 d\mu + a \right) - 4at.\]
Hence our conclusion easily follows by (4.2), (4.3) and Lemma 4.2. □

From Proposition 4.3, we know that the entropy \(\mathcal{Y}_a(u, t)\) is well-defined for \(u^2 = H\), being the heat kernel, on complete noncompact Riemannian manifolds with nonnegative Ricci curvature. Using Lemma 4.1, we can apply the same trick as in the proof of Theorem 2.3 and obtain the following monotonicity of the adjusted logarithmic entropy for the heat kernel on complete (possibly noncompact) Riemannian manifolds.

**Theorem 4.4.** Let \((M, g)\) be an \(n\)-dimensional complete (possibly noncompact) Riemannian manifold with nonnegative Ricci curvature. The adjusted logarithmic entropy \(\mathcal{Y}_a(u, t)\) (\(a > -\lambda\)) with \(\tilde{u} = u^2 = H\), the heat kernel of heat equation (1.6) satisfying \(\int_M u^2 d\mu = 1\), is monotone decreasing. More precisely,
\[(4.4) \quad \frac{d}{dt} \mathcal{Y}_a(u(t), t) \geq \frac{n}{4\omega} \int_M \left[ -2 \frac{\nabla^2 u}{u} + 2 \frac{\nabla u \otimes \nabla u}{u^2} - \frac{4\omega}{n} g \right]^2 u^2 d\mu\]
where \(u = (4\pi t)^{-n/4} e^{-\tilde{f}/2}\) and \(\omega := \omega(u(x, t), a) = \int_M |\nabla u|^2 d\mu + a > 0\).

**Remark 4.5.** In general, \(\tilde{f}\) may be different from the function \(f\) in Lemma 4.1. Here function \(\tilde{f}\) is used for the purpose of simplifying the expressions in the above formulas.

## 5. Application

In this section, we will use the monotonicity of the entropy functional \(\mathcal{Y}_a(u(t), t)\) of the heat kernel obtained in Section 4 to characterize Euclidean space.

**Theorem 5.1.** Let \((M^n, g)\) be a complete Riemannian manifold with nonnegative Ricci curvature and constant \(a > -\lambda\), where \(\lambda\) is the first nonzero eigenvalue of Laplace operator. If \(\mathcal{Y}_a(u(t_1), t_1) = \mathcal{Y}_a(u(t_2), t_2)\) for some \(t_1 < t_2\), with \(u^2\) being the heat kernel, then \((M^n, g)\) is isometric to Euclidean space.

**Proof.** Since \(\mathcal{Y}_a(u(t_1), t_1) = \mathcal{Y}_a(u(t_2), t_2)\) for some \(t_1 < t_2\), using Theorem 4.4, the monotonicity formula (4.4) (see also (3.5)) implies that
\[\tilde{f}_{ij} - \frac{4\omega}{n} g_{ij} \equiv 0 \iff f_{ij} - \frac{g_{ij}}{2(t - t_1 + \sigma)} \equiv 0\]
on \((t_1, t_2)\), where
\[
\sigma = \frac{n}{8\omega(u(x, t_1), a)} \quad \text{and} \quad \omega(u(x, t_1), a) = \int_M |\nabla u|^2 d\mu + a > 0.
\]

In other words,
\[
f_{ij} - \frac{g_{ij}^2}{2t} \equiv 0
\]
for all \(t \in (\sigma, t_2 - t_1 + \sigma)\). That is, for any such \(t\), \(\varphi := 4tf\) satisfies

\[
\nabla_i \nabla_j \varphi = 2g_{ij}
\]

From this, we see that \(\varphi\) attains its minimum at some point \(O \in M\). We claim that
\[
\varphi(x) - \varphi(O) = d^2(x, O).
\]

To prove this claim, let \(x \in M\) be any point and consider a unit speed minimal geodesic \(\gamma : [0, d(x, O)] \rightarrow M\) joining \(O\) to \(x\). We have
\[
\frac{d^2}{ds^2} \varphi(\gamma(s)) = \nabla_i \nabla_j \varphi \dot{\gamma}^i \dot{\gamma}^j = 2.
\]

Since \(\nabla \varphi(O) = \tilde{O}\), we have \(\varphi(\gamma(s)) = \varphi(O) + d^2(O, \gamma(s))\) for all \([0, d(x, O)]\) and hence the claim follows.

Therefore taking the trace of (5.1) yields
\[
\Delta(d^2) \equiv 2n.
\]

This, together with the assumption that Ricci curvature is nonnegative, implies \((M^n, g)\) is isometric to Euclidean space. \(\square\)

Remark 5.2. The observation in proof of Theorem 5.1 that \(f_{ij} - \frac{g_{ij}^2}{2t} \equiv 0\) for all \(t \in (\sigma, t_2 - t_1 + \sigma)\) implies that \((M, g)\) is isometric to the standard Euclidean space has been made by different authors. For example, see Corollary 1.3 in [6], Proposition 2 in [11], Theorem 3 in [12] or Theorem 7.1 in [13].

6. Further Remarks

While we only considered the linear heat equation in the previous sections, the arguments in Sections 2 and 3 are in fact valid for the weighted heat equation as well.

In order to make a clear statement of our result for the weighted heat equation, we need to recall some basic facts about the \(m\)-dimensional Bakry-Émery Ricci curvature (please see [13], [15], [16] and [7] for more details). Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold, and \(h\) be a \(C^2\) function. We define a symmetric diffusion operator
\[
L := \Delta - \nabla h \cdot \nabla,
\]
which is the infinitesimal generator of the Dirichlet form
\[
\mathcal{E}(\varphi_1, \varphi_2) = \int_M (\nabla \varphi_1, \nabla \varphi_2) d\nu, \quad \forall \varphi_1, \varphi_2 \in C_0^\infty(M),
\]
where \(\nu\) is an invariant measure of \(L\) given by \(d\nu = e^{-h}d\mu\). It is well-known that \(L\) is self-adjoint with respect to the weighted measure \(d\nu\). Given a smooth metric measure space \((M, g, e^{-h}d\mu)\), the \(\infty\)-dimensional Bakry-Émery Ricci curvature by

\[
\text{Ric}(L) := \text{Ric} + \text{Hess}(h),
\]
where \( \text{Hess} \) denotes the Hessian of the metric \( g \). We also define the \( m \)-dimensional Bakry-Émery Ricci curvature of the diffusion operator \( L \) as follows:

\[
Ric_{m,n}(L) := Ric(L) - \frac{\nabla h \otimes \nabla h}{m-n},
\]

where \( m := \text{dim}_{BE}(L) \geq n \) is called the Bakry-Émery dimension of \( L \), which is a constant and is not necessarily to be an integer. In general the number \( m \) is not equal to the manifold dimension \( n \), unless the operator \( L \) is the Laplace operator. If \( m = \infty \), then \( Ric_{m,n}(L) = Ric(L) \).

A remarkable feature of \( Ric_{m,n}(L) \) is that Laplacian comparison theorems hold for \( Ric_{m,n}(L) \) in the metric measure space \((M^m, g, e^{-h}d\mu)\) that look like the case of Ricci tensor in a \( m \)-dimensional manifold [7] (see also [9], [17] and [18]). When \( h \) is constant, \( Ric(L) \) and \( Ric_{m,n}(L) \) both recover the ordinary Ricci curvature.

Given a smooth metric measure space \((M, g, e^{-h}d\mu)\), we consider the weighted heat equation

\[
\partial_t \tilde{u} - L \tilde{u} = 0.
\]

In June 2006, X.-D. Li (see also [19]) introduced the following entropy formula

\[
W(f, \tau) := \int_M (\tau |\nabla f|^2 + f - m) (4\pi\tau)^{-m/2} e^{-f} d\nu,
\]

where \( m \) is a finite constant satisfying \( m \geq n \), and \((f, \tau)\) satisfies

\[
\tilde{u} = \frac{e^{-f}}{(4\pi\tau)^{m/2}} \quad \text{and} \quad \int_M \frac{e^{-f}}{(4\pi\tau)^{m/2}} d\nu = 1
\]

with \( \tau > 0 \). By the direct calculation, he obtained the following result.

**Theorem B.** (X.-D. Li [19]) Let \( M \) be an \( n \)-dimensional closed Riemannian manifold. Assume that \( \tilde{u} \) is a positive solution to the weighted heat equation (6.1) with \( \int_M \tilde{u} d\nu = 1 \). Let \( f \) be defined by

\[
\tilde{u} = (4\pi\tau)^{-m/2} e^{-f}
\]

and \( \tau = \tau(t) \) satisfies \( d\tau/dt = 1 \). Then

\[
\frac{dW}{dt} = -\int_M 2\tau \left[ |\nabla^2 f - \frac{g}{2\tau}|^2 + Ric_{m,n}(L)(\nabla f, \nabla f) \right] \tilde{u} d\nu
- \int_M \left( \sqrt{\frac{2\tau}{m-n}} \nabla h \cdot \nabla f + \sqrt{\frac{m-n}{2\tau}} \right)^2 \tilde{u} d\nu.
\]

In particular, if \( Ric_{m,n}(L) \geq 0 \), then \( W(f, \tau) \) is monotone decreasing along the weighted heat equation (6.1) with \( \int_M \tilde{u} d\nu = 1 \).

**Remark 6.1.** In fact, Li [19] proved that the above result holds on complete (non-compact) manifolds when \( \tilde{u} = H \), being the heat kernel of the weighted heat equation (6.1). The above Theorem B also appeared in [8].

**Proof.** For the convenience of the reader, we give a sketch of the proof of this theorem. If we let

\[
\phi = -\log \tilde{u} \quad \text{and} \quad \omega = 2L\phi - |\nabla \phi|^2,
\]

then

\[
\phi_t = L\phi - |\nabla \phi|^2.
\]
Now using the Bochner formula, we can derive that
\[
\left( \frac{\partial}{\partial t} - L \right) \omega = -2|\nabla^2 \phi|^2 - 2Ric(L)(\nabla \phi, \nabla \phi) - 2(\nabla \omega, \nabla \phi).
\]
If we set \( U := \tau(2Lf - |\nabla f|^2) + f - m = \tau \omega + f - m \), then
\[
\left( \frac{\partial}{\partial t} - L \right) U = -2\tau Ric_m,n(L)(\nabla f, \nabla f) - 2\tau \left| \nabla^2 f - \frac{g}{2\tau} \right|^2
\]
\[\quad - \left( \sqrt{\frac{2\tau}{m-n}} \nabla h \cdot \nabla f + \sqrt{\frac{m-n}{2\tau}} \right)^2 - 2(\nabla U, \nabla f).\]

\[\quad - \tilde{\omega} \left( \sqrt{\frac{2\tau}{m-n}} \nabla h \cdot \nabla f + \sqrt{\frac{m-n}{2\tau}} \right)^2.\]

Hence
\[
\left( \frac{\partial}{\partial t} - L \right) (U \tilde{u}) = -2\tau \tilde{\omega}Ric_m,n(L)(\nabla f, \nabla f) - 2\tau \tilde{\omega} \left| \nabla^2 f - \frac{g}{2\tau} \right|^2
\]
\[\quad - \tilde{\omega} \left( \sqrt{\frac{2\tau}{m-n}} \nabla h \cdot \nabla f + \sqrt{\frac{m-n}{2\tau}} \right)^2.\]

Note that
\[
W(f, \tau) = \int_M (U \tilde{u})d\nu \quad \text{and} \quad \int_M |\nabla f|^2 \tilde{u}d\nu = \int_M Lf \tilde{u}d\nu.
\]
Therefore the result follows by integrating the equality (6.3) with respect to the measure \( d\nu \).

Following the above definitions of Section 2, we give the corresponding adjusted logarithmic entropy formula for the diffusion operator \( L \).

**Definition 6.2.** Let \((M, g, e^{-h}d\mu)\) be a metric measure space, where \( M \) is an \( n \)-dimensional closed Riemannian manifold. Let \( \kappa \) denote the first nonzero eigenvalue of the diffusion operator \( L \). For any function \( u \in W^{1,2}(M) \) satisfying \( \int_M u^2d\nu = 1 \), if \( a > -\kappa \), then we define a new logarithmic entropy with respect to this metric measure space
\[
H_a(u, t) := -\int_M u^2 \log u^2d\nu + \frac{m}{2} \log \left( \int_M |\nabla u|^2d\nu + a \right) - 4at.
\]

Parallel to Lemma 3.2, we have the following property.

**Lemma 6.3.** Assume that \( a > -\kappa \). Then for each \( \tau > 0 \), the entropy \( W \) defined by (6.2) satisfies
\[
W(f, \tau) \geq -\int_M u^2 \log u^2d\nu + \frac{m}{2} \log \left( \int_M |\nabla u|^2d\nu + a \right) - 4at + c(m),
\]
where
\[
c(m) := \frac{m}{2} \log \pi - \frac{m}{2} \left( 1 + \log \frac{m}{2} \right).
\]

Moreover,
\[
W\left( f, \frac{m}{2\omega(u, a)} \right) = -\int_M u^2 \log u^2d\nu + \frac{m}{2} \log \left( \int_M |\nabla u|^2d\nu + a \right) - \frac{ma}{2\omega(u, a)} + c(m),
\]
where \( \omega(u, a) := \int_M |\nabla u|^2d\nu + a \).
For any metric measure space \((M, g, e^{-h}d\mu}\), let \(u = u(x, t)\) be a smooth positive solution of the following equation

\[
\frac{\partial u}{\partial t} = Lu + \frac{|\nabla u|^2}{u}
\]

such that the normalization condition \(\int_M u^2 d\nu = 1\) holds for all \(t\). Using Lemma [6.3] and the arguments of proving Theorem 2.3, we can prove the following theorem.

**Theorem 6.4.** Let \((M, g, e^{-h}d\mu)\) be a metric measure space, where \(M\) is an \(n\)-dimensional closed Riemannian manifold. If the \(m\)-dimensional Bakry-Émery Ricci curvature is nonnegative and \(a > -\kappa\), then the adjusted logarithmic entropy \(H^a(u, t)\) is monotone decreasing along the heat-type equation (6.4) with \(\int_M u^2 d\nu = 1\). More precisely,

\[
-\frac{dH^a}{dt} \geq \frac{m}{4\omega} \int_M \left[ -\frac{2}{u} \nabla^2 u + \frac{2}{u^2} \nabla u \otimes \nabla u - \frac{4\omega}{m} g \right]^2 + 4\text{Ric}_{m,n}(L) \left( \frac{\nabla u}{u}, \frac{\nabla u}{u} \right) u^2 d\nu
\]

\[
+ \int_M \left( \sqrt{\frac{m}{\omega(m-n)}} \frac{\nabla h \cdot \nabla u}{u} - \sqrt{\frac{4\omega(m-n)}{m}} \right)^2 u^2 d\nu
\]

\[
= \frac{n}{4\omega} \int_M \left( f_{ij} - \frac{4\omega}{m} g_{ij} \right)^2 + \text{Ric}_{m,n}(L) (\nabla \tilde{f}, \nabla \tilde{f}) \frac{e^{-f}}{(4\pi t)^{m/2}} d\nu
\]

\[
+ \int_M \left( \frac{m}{4\omega(m-n)} \frac{\nabla h \cdot \nabla \tilde{f}}{\nabla \tilde{f}} + \sqrt{\frac{4\omega(m-n)}{m}} \right)^2 \frac{e^{-\tilde{f}}}{(4\pi t)^{m/2}} d\nu,
\]

where \(u = (4\pi t)^{-m/4} e^{-\tilde{f}/2}\) and \(\omega := \omega(u(x, t), a) = \int_M |\nabla u|^2 d\nu + a > 0\).

**Proof.** The proof is very similar to that of Theorem 2.3. Hence we leave this proof to the interested reader. Notice that here we should apply Theorem B instead of Theorem A and Lemma 6.3 to prove this theorem. \(\square\)

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