Nonperturbative solution of the Nonconfining Schwinger Model with a generalized regularization

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Abstract
Nonconfining Schwinger model [1] is studied with a one parameter class of kinetic energy like regularization. It may be thought of as a generalization over the regularization considered in [3]. Phasespace structure has been determined in this new situation. The mass of the gauge boson acquires a generalized expression with the bare coupling constant and the parameters involved in the regularization. Deconfinement scenario has become transparent at the quark-antiquark potential level.
1 Introduction

QED in (1 + 1) dimension, e.g., Schwinger model [1] is a very interesting exactly solvable field theoretical model. It has been widely studied over the years by several authors in connection with the confinement aspect of quark [2, 3]. Here massless quarks interact with the Abelian gauge field. Gauge field acquires mass via a kind of dynamical symmetry breaking and the quarks disappear from the physical spectra. Recently, a new generalized version of this model has been proposed in [4]. In that paper we find that a one parameter class of regularization commonly used to study of chiral Schwinger model has been introduced in the vector Schwinger model. For a specific choice, i.e., for the vanishing value of the parameter the model reduces to the usual vector Schwinger model but for the other admissible value of this parameter the phaspace structure as well as the the physical spectra gets altered remarkably. This new regularization leads to a change in the confinement scenario of the quark too. In fact, the quarks gets liberated as it was happened in the Chiral Schwinger model [5].

In QED, a regularization gets involved when one calculates the effective action by integrating the fermions out. The ambiguity in the regularization has been exploited by different authors in different times and different interesting results have been extracted out [5, 6, 7, 8, 9]. The most remarkable one is the chiral Schwinger model as studied by Jackiw and Rajaraman [5]. They saved the long suffering from the nonunitarity problems of the chiral generalization of the Schwinger model as studied in [10] introducing the one parameter class of regularization. Regularization plays a crucial role in the confining aspect of the fermions too [4, 8]. In this connection, it is fair to admit that the relation between regularization and confinement is not yet so transparent.

In [11], the authors showed that the regularization of the Schwinger model allows some extra flexibility in the effective action of the theory and studied the model with a parameter dependent kinetic energy term for the gauge fields and got some interesting and acceptable result. We are interested here to investigate the Nonconfining Schwinger model with a one parameter class of kinetic energy term for the gauge field. We should mention here that regularization here also allows the same type of flexibility.

With this new modification the nature of the solution though does not change as it was happened in [11], but the physical mass and the bare cou-
pling constant acquires a generalized form along with the two regularization parameters. This generalization allows one to make the physical mass zero with an unusual limit of one of the parameter involved in the kinetic energy term of the gauge field. This limit makes the deconfinement scenario more transparent when it is investigated studying the quark-antiquark potential. We should admit here that though there were free chiral fermion in the spectrum of the Nonconfining Schwinger model [4], the deconfinement of fermions were not transparent at the quark-antiquark potential level. We therefore get motivated to study the deconfinement scenario in the present situation calculating the quark-antiquark potential.

In Sec.2, we have shown how the regularization of the Schwinger model allows the flexibility to modify the effective action with the introduction of two parameter dependent terms. One of these two is kinetic energy like term and the other one is conventional masslike term for the gauge fields. In fact, it involves a generalization of the gauge current. It is shown that the usual fermionic operator of the Schwinger model allows the current to be constructed in such a way that this generalized expression is obtained.

In Sec.3, the constraint analysis of the modified bosonized effective Lagrangian has been carried out and the phasespace structure has been determined. Here it is shown that the physical mass and the bare coupling constant acquires a generalized relation.

In Sec.4, Quark-antiquark potential has been calculated for this new situation. We have shown that the deconfinement scenario has become transparent at the quark-antiquark potential level.

2 Kinetic energy like regularization

Schwinger model is defined by the lagrangian density

\[ \mathcal{L}_F = \bar{\psi}(i\gamma^\mu - eA^\mu)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \]

where the Lorentz indices takes the value 0 and 1 in (1 + 1) dimensional spacetime and the other notations are all standard. The coupling constant \( e \) has the dimension of mass. In (1 + 1) dimension, the most general ansatz for \( A_\mu \) is
\[ A_\nu = -\frac{\sqrt{\pi}}{e}(\tilde{\partial}_\nu \sigma + \partial_\nu \tilde{\eta}), \]  

(2)

where \( \tilde{\partial}_\mu = \epsilon_{\mu\nu} \partial^\nu \) with \( \epsilon_{01} = 1 \). \( \sigma \) and \( \eta \) are two scalar fields. It is constructive to define a pseudo scalar field \( \eta \) defined by the equation \( \tilde{\partial} \eta = \partial \tilde{\eta} \). From the above definitions one finds

\[ F_{\mu\nu} = -\frac{\sqrt{\pi}}{e}(\partial_\mu \tilde{\partial}_\nu - \tilde{\partial}_\mu \partial_\nu)\sigma = \frac{\sqrt{\pi}}{e} \epsilon_{\mu\nu} \Box \sigma. \]  

(3)

The equation of motion to be solved here is

\[ [i\partial / + \sqrt{\pi} \tilde{\gamma}_\mu (\sigma + \eta)]\psi = 0. \]  

(4)

Equation (4) has the solution

\[ \psi(x) = e^{\sqrt{\pi} \gamma_5 (\sigma(x) + \eta(x))} : \psi(0)(x), \]  

(5)

where the notations ‘:’ indicates the normal ordering with respect to the Fock space operator \( \sigma \) and \( \eta \) and \( \psi(0) \) is the zero mass free Dirac field defined by

\[ i\partial /\psi(0)(x) = 0. \]  

(6)

Here the gauge current has been calculated using point splitting regularization. In this paper, we have made a generalization over the conventional construction [1]. Current is regularized in the following way

\[ J^{\text{reg}}_\mu = [\bar{\psi}(x + \epsilon) \gamma_\mu : e^{\epsilon e \int_{x^-}^{x^+} dx^\nu [a A_\nu(x) - 2\alpha \partial^\nu F_{\mu\nu}]} : \psi(x) - V.E.V], \]  

(7)

where \( a \) and \( \alpha \) are two arbitrary parameters. The term \( V.E.V. \) stands for vacuum expectation value. In our generalization two new parameters are involved. One of these two is connected with conventional masslike regularization and the other one is connected with the kinetic energy like regularization. This generalization though does not maintain gauge invariance like the usual vector Schwinger model it maintains the Lorentz invariance which is the basic need of a theory to be physically sensible. Here we should admit that a gauge invariant regularization some times scores over the gauge non-invariant regularization because it corresponds to the increased symmetry of the theory but it has to be remembered that the increased symmetry is not
the symmetry of the physical states its mere the symmetry of the effective action which has to be broken by gauge fixing. Therefore, gauge non-invariant regularization should be equally acceptable. Moreover, the present modification is describing a generic situation.

The calculation of $J_{\mu}^{\text{reg}}$ is straightforward. Using $A_\mu$ and $\psi$ and keeping the terms up to first order we obtain

\[
J_{\mu}^{\text{reg}} = J_{\mu}^f - i\sqrt{\pi} \lim_{\epsilon \to 0} <0|\bar{\psi}(0)(x + \epsilon)\gamma_\mu [a(\gamma_5 \epsilon^\nu \partial_\nu + \epsilon^\nu \tilde{\partial}_\nu)\sigma + \eta)](\sigma + \eta) \rangle 
+ 2\alpha \epsilon^\nu \tilde{\partial}_\nu [\sigma(\sigma + \eta)]|\psi(0)|0>,
\]  
(8)

After a little simplification we have

\[
J_{\mu}^{\text{reg}} = J_{\mu}^f - \frac{1}{\sqrt{\pi}} \left[ a \frac{\epsilon_\mu \epsilon_\nu - \tilde{\epsilon}_\mu \tilde{\epsilon}_\nu}{\epsilon_2} \tilde{\partial}_\nu(\sigma + \eta) + 2\alpha \frac{\epsilon_\mu \epsilon_\nu}{\epsilon^2} \tilde{\partial}_\nu [\sigma(\sigma + \eta)] \right],
\]  
(9)

where

\[
J_{\mu}^f := \bar{\psi}(0)(x)\gamma_\mu \psi(0)(x).
\]  
(10)

Here we have used the identity

\[
<0|\bar{\psi}_\alpha(0)(x + \epsilon)\psi_\beta(0)(x)|0> = -i \frac{(\epsilon^\mu \gamma_\mu)_{\alpha\beta}}{2\pi \epsilon^2}
\]  
(11)

Taking the symmetric limit, i.e., averaging over the point splitting direction $\epsilon$ we obtain the final expression of $J_{\mu}^{\text{reg}}$:

\[
J_{\mu}^{\text{reg}} = -\frac{1}{\sqrt{\pi}} \tilde{\partial}_\mu(\phi + a(\sigma + \eta) + \alpha \Box \sigma),
\]  
(12)

Where $\phi$ is the potential of the free fermionic current defined by

\[
J_{\mu}^f(x) := \bar{\psi}(0)(x)\gamma_\mu \psi(0)(x) :,
\]  
(13)

with $\phi$ satisfying the bosonic equivalent of the free fermion $\psi(0)$. We, therefore, have

\[
J_{\mu}^{\text{reg}}(x) = -\frac{1}{\sqrt{\pi}} \tilde{\partial}_\mu \phi + \frac{e}{\sqrt{\pi}} a A_\mu - \alpha \frac{\epsilon^2}{\sqrt{\pi}} \partial^\nu F_{\nu\mu}.
\]  
(14)
3 Determination of phasespace structure from the bosonized version of the theory

The effective bosonized lagrangian density which gives the current given in equation (14) is

\[ \mathcal{L}_B = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - g \tilde{\alpha} \partial_\mu \phi A^\mu + \frac{1}{2} a g^2 A_\mu A^\mu - \frac{\tilde{\alpha}}{4} F_{\mu \nu} F^{\mu \nu}. \]  

(15)

\( g \) and \( \tilde{\alpha} \) are defined by \( g = \frac{\rho}{\sqrt{\pi}} \) and \( \tilde{\alpha} = \frac{\rho^2}{4 \pi} \) for later convenience. The usual kinetic energy term for the gauge field is absorbed within the parameter \( \alpha \).

Let us now proceed with the constraint analysis of the system. To determine the physical phasespace structure it is necessary to calculate the momenta corresponding to the field \( A_0 \), \( A_1 \), and \( \phi \). From the standard definition of the momentum we obtain

\[ \pi_0 = 0, \]  

(16)

\[ \pi_1 = F_{01}, \]  

(17)

\[ \pi_\phi = \dot{\phi} - g A_1, \]  

(18)

where \( \pi_0, \pi_1 \) and \( \pi_\phi \) are the momenta corresponding to the field \( A_0 \), \( A_1 \) and \( \phi \). Using the equations (16), (17), and (refM3), the Hamiltonian density are calculated:

\[ \mathcal{H} = \frac{1}{2} (\pi_\phi + g A_1)^2 + \frac{1}{2 \tilde{\alpha}} \pi_1^2 + \frac{1}{2} \phi'^2 + \pi_1 A_0' - g A_0 \phi' - \frac{1}{2} a g^2 (A_0^2 - A_1^2). \]  

(19)

\( \omega = \pi_0 = 0 \), is the familiar primary constraint of the theory. The preservation of the constraint \( \omega \) requires \( [\omega(x), H(x)] = 0 \), which leads to the Gauss' law as a second class constraint:

\[ \ddot{\omega} = \pi_1' + g \phi' + a g^2 A_1 = 0. \]  

(20)

Treating (20) as strong condition one can eliminate \( A_0 \) and obtain the reduced Hamiltonian density as follows.

\[ \mathcal{H}_r = \frac{1}{2} (\pi_\phi + g A_1)^2 + \frac{1}{2 a g^2} (\pi_1' + g \phi')^2 + \frac{1}{2} \left( \frac{\pi_1^2}{\tilde{\alpha}} + \phi'^2 \right) + \frac{1}{2} a g^2 A_1^2. \]  

(21)
It is straightforward to show that the Dirac brackets of the remaining variable remain canonical. Using the canonical Dirac brackets the following first order equations of motion are found out from the Hamiltonian density (21)

\[
\dot{A}_1 = \frac{1}{\alpha} \pi_1 - \frac{1}{ag^2} (\pi_1'' + g\phi''), \quad (22)
\]

\[
\dot{\phi} = \pi_\phi + gA_1, \quad (23)
\]

\[
\pi_\phi = \frac{a + 1}{a} \phi'' + \frac{1}{ag} \pi_1'', \quad (24)
\]

\[
\dot{\pi}_1 = -g\pi_\phi - (a + 1)g^2 A_1. \quad (25)
\]

A little algebra converts the above first order equation into the following second order equations:

\[
(\Box + (1 + a)\frac{g^2}{\alpha})\pi_1 = 0, \quad (26)
\]

\[
\Box[\pi_1 + g(1 + a)\phi] = 0. \quad (27)
\]

Equation (26) describes a massive scalar field with mass \( m = \sqrt{\frac{1+a}{\alpha}} g \) and equation (27) describes a massless scalar field. Therefore, the structure of the spectra remains unchanged. However, the mass term of the massive boson gets altered and it has a crucial link with the regularization parameter \( a \) and \( \alpha \). In the usual vector Schwinger model mass term remains constant but it acquires a generalized form with the bare coupling constant \( g \) in the present situation :

\[
m^2 = (\frac{a + 1}{\alpha})g^2. \quad (28)
\]

The mass here gets scaled down by a factor \( \sqrt{\frac{a+1}{\alpha}} \). The identical situation were happened in the vector meson model (12), where mass of the physical particle gets altered because of the variation of regularization like this situation. This is interesting to note that one can make this mass zero with an unusual limit \( \tilde{\alpha} \to \infty \). The parameter \( a \) can not do so because we have seen in [4] that \( a \) has a restriction \( a \geq 0 \) for the theory to be physically sensible. Not only that but also, this unusual limit makes the deconfining phenomena
more transparent in the quark-antiquark potential level. We will consider this fact in our discussion in the next Section.

4 Calculation of quark-antiquark potential

In this Section, we calculate the quark-antiquark potential following the text and study the effect of the above mentioned unusual limit on the confinement aspect of quark. From the study of the previous Section we can conclude that the above theory can be reduced to a theory involving two free fields \(\pi_1\) and \(\pi_1 + e\phi\). These two fields are not properly normalized. One can show that the properly normalized fields are

\[
\sigma = \frac{1}{\sqrt{1 + a}}\pi_1, \tag{29}
\]

\[
\rho = \frac{1}{\sqrt{a(a + 1)}}[\pi_1 + g(a + 1)\phi]. \tag{30}
\]

The original fields \(\phi\) and \(A_\mu\) can be expressed in terms of the free fields by

\[
\phi = \frac{1}{\sqrt{a + 1}}(\sqrt{a}\rho - \sigma), \tag{31}
\]

\[
A_\mu = \frac{1}{g}\epsilon_{\mu\nu}\partial^\nu \frac{1}{\sqrt{a(a + 1)}}(\sqrt{a}\sigma + \rho). \tag{32}
\]

The gauge current for this model is found out to be

\[
J_\mu = -\epsilon_{\mu
\nu}\partial^\nu \phi + agA_\mu. \tag{33}
\]

Substituting (11) and (12) in the equation (33) we find

\[
J_\mu = -\sqrt{a + 1}\epsilon_{\mu\nu}\partial^\nu \sigma. \tag{34}
\]

We are now in a position to calculate the quark anti-quark potential. One can expect that the potential of any quark-antiquark pair \((q, \bar{q})\) would exhibit the behavior of the characteristic of screened charge because of the intrinsic Higgs mechanism induced by vacuum polarization. To do that one needs to compare the Hamiltonian in the presence of the external \((q, \bar{q})\) source with
that in the absence of that external source. In the presence of the test charge
\(-Q\) at \(-\frac{L}{2}\) and \(+Q\) at \(+\frac{L}{2}\) the charge density will be reduced to

\[ J_0 = \frac{1}{g}Q[\delta(x + \frac{L}{2}) - \delta(x - \frac{L}{2}) + \sqrt{1 + a\partial_1\sigma} = \sqrt{1 + a\partial_1(\sigma - \Theta)}, \]  

(35)

where

\[ \Theta = \frac{g}{\sqrt{1 + a}}Q[\theta(x + \frac{L}{2})\theta(\frac{L}{2} - x)]. \]  

(36)

The external charge interacts with the massive field only and change the mass of the field. Therefore, the presence of the external charges changes the Hamiltonian of the massive part the theory only. This change in the Hamiltonian is found as follows.

\[ H(\Theta) = \int dx[\pi_x^2 + (\partial_1\sigma)^2 + \frac{1 + a}{\tilde{\alpha}}g^2(\sigma - \Theta)^2]. \]  

(37)

The potential which is the difference between the ground state energy in presence of the external charge with the same quantity in the absence of the external charge is calculated as follows. The ground state energy in presence of the external source is

\[ E(\Theta) = <\omega_Q[\Theta]|H[\Theta]|\omega_Q[\Theta] >. \]  

(38)

Using the completeness condition on the states of \(\sigma\) we have

\[ E(\Theta) = \int d\sigma d\sigma' <\omega_Q[\Theta]|\sigma > <\sigma|H[\Theta]|\sigma' > <\sigma'|\omega_Q[\Theta]. \]  

(39)

and

\[ <\sigma|H[\Theta]|\sigma' > = \delta(\sigma - \sigma') \int dx[-\frac{\partial^2}{\partial_1\sigma(x)^2} + \sigma(x)D\sigma'(x) + \frac{g^2}{\tilde{\alpha}}(1 + a)\Theta^2 \]  

\[ - \frac{2g^2}{\tilde{\alpha}}(1 + a)\sigma\Theta], \]  

(40)

where \(D = \frac{g^2}{\alpha}(1 + a) - \partial_1^2\). After a little algebra the potential comes out to be

\[ E[\Theta] - E[0] = \frac{1}{2} \int [\frac{g^2}{\tilde{\alpha}}(1 + a)\Theta^2 - \frac{g^4}{\alpha_2(1 + a)^2}\Theta D^{-1}\Theta]. \]  

(41)
The operator $D^{-1}$ has the following matrix element.

$$<x|D^{-1}|y> = <x|\left(\frac{g^2}{\tilde{\alpha}}(1+a) - \partial^2\right)|y>$$

$$= \frac{1}{2g}\sqrt{\frac{1+a}{\tilde{\alpha}}}e^{-\sqrt{\frac{1+a}{\tilde{\alpha}}}|x-y|}. \quad (42)$$

Substituting (42) in (41) the quark-antiquark potential $V(L)$ is obtained:

$$V(L) = E[\Theta] - E[0]$$

$$= \frac{Q^2}{2g}\sqrt{\frac{1+a}{\tilde{\alpha}}}[1 - e^{-\sqrt{\frac{1+a}{\tilde{\alpha}}}gL}] \quad (43)$$

$V(L)$ is the quark-antiquark potential for the Nonconfining Schwinger model with the new regularization considered for investigation in this paper. The above potential comes out to be

$$V_{NCSM}(L) = \frac{Q^2}{2g}\sqrt{1+a}[1 - e^{-gL}], \quad (44)$$

for Nonconfining Schwinger model, and

$$V_{VSM}(L) = \frac{Q^2}{2g}[1 - e^{-gL}], \quad (45)$$

for usual vector Schwinger model.

Both in (44) and (45), the potential tends to constant value for large separation of the test charges, i.e., for the situation when $L$ approaches towards infinity. This fact has good agreement with usual vector Schwinger model since it is known that quarks gets confined during the process but the result was not so transparent in the analysis of the usual Nonconfining Schwinger model. It is little confusing too. There were free massless boson the spectrum and in $(1+1)$ dimension, boson can be interpreted in terms of fermion. So at that stage without calculating the quark-antiquark potential it was natural to conclude that the fermions are not confined. However we find no signature of the presence of fermion when we study the quark-antiquark potential carefully because only the vanishing value of the potential at large $L$ suggests deconfinement. Nonconfining Schwinger model with this new regularization can rectify the above confusion because if we take the limit $\tilde{\alpha} \to \infty$ the potential $V(L)$ approaches to zero. Certainly, it is the signature of free quark.
5 Concluding remarks

Schwinger model was studied in [4] with a one parameter class of masslike regularization. In this paper, we study this model with a more general regularization and determine its phasespace structure. The model with this generalized regularization remains exactly solvable. The most interesting and crucial property namely the deconfinement aspect of quark has been investigated calculating the quark-antiquark potential. To modify the model we point split the current which was defined as the product of the two fermionic operator. Schwinger gave a prescription to insert an exponential line integral of the gauge field. But this was not the only choice. To be more specific, it was one of the many possible choices. Following [11], we also inset an extra factor in the exponential involving field strength. Of course, the introduction of complicated term may make the theory complicated and spoil the exactly solvable nature. However, the regularization chosen here keeps the solvable nature intact and makes the physical property interesting. The equations of motion obtained in this new situation is different from the usual vector Schwinger model but it is identical in nature to the Nonconfining Schwinger model excepting the difference in the mass term of the massive boson. Unlike the Schwinger model, mass here acquires a generalized expression with the bare coupling constant and the parameter involved in the regularization. Certainly, the equation can be converted into free fields like the usual Vector Schwinger model.

In the usual vector Schwinger model we find only one massive boson with a constant mass proportional to the square of the bare coupling constant. However, in the Nonconfining Schwinger model and in the present situation spectra are different in nature. We find a massless boson along with the massive boson. Mass in both cases acquires a generalized expression where the regularization parameters are involved. In the Nonconfining Schwinger model, only one regularization parameter is involved whereas in the present situation there are two independent regularization parameter. Here the generalization is such that one can make the mass term zero with an unusual limit and this unusual limit can rectify the confusion of deconfinement scenario of the Nonconfining Schwinger model at the quark-antiquark level. In this context, we should mention that though there were free massless boson in the spectrum of the Nonconfining Schwinger model the signature of deconfinement was absent in the quark-antiquark potential level.
Question may be raised: which regularization is better gauge invariant or non-invariant? Obviously, there is no specific answer. We should admit that some times gauge invariance scores over the gauge non-invariance because of the increased symmetry of the theory. However, it has to be remember that the symmetry increased here is not at all a symmetry of the states its rather a symmetry of the effective action which has to be broken by gauge fixing in order to extract out the real physical contents. So both the regularizations make sense and should be acceptable in the theoretical laboratory. Concerning confinement, another question may be raised: which regularization is better confining or deconfining? This question also has no specific answer. Certainly, we should make some comments on this issue. Confinement and deconfinement is really a crucial question. Hitherto, there is no such model from which one can have clear explanation regarding confinement and deconfinement scenario of quark. Inspite of the absence of the signature of finding free quarks from the experiment experimentalist of recent time have started to believe that QGP phase exists. So a practical benefit may follow from this work that this new version may be useful to study the QGP phase just as the usual vector Schwinger model has been used to study the confinement. In this sense, both the model should be accepted in the theoretical laboratory.

The question of confinement and deconfinement discussed in this paper is limited in (1 + 1) dimension. It would certainly be interesting what happens to this question in (3 + 1) dimension since confinement and deconfinement are of real physical interest. Last but not the least this type of investigation may throw some light in the fact that whether there is any connection to the confinement and regularization. More qualitative investigations are needed indeed in this issue.

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