C*-ALGEBRAS ASSOCIATED TO TRANSFER OPERATORS FOR COUNTABLE-TO-ONE MAPS

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Abstract. Our initial data is a transfer operator $L$ for a continuous, countable-to-one map $\phi: \Delta \to X$ defined on an open subset of a locally compact Hausdorff space $X$. Then $L$ may be identified with a ‘potential’, i.e. a map $\varphi: \Delta \to X$ that need not be continuous unless $\phi$ is a local homeomorphism. We define the crossed product $C_0(X) \rtimes L$ as a universal $C^*$-algebra with explicit generators and relations, and give an explicit faithful representation of $C_0(X) \rtimes L$ under which it is generated by weighted composition operators. We explain its relationship with Exel-Royer’s crossed products, quiver $C^*$-algebras of Muhly and Tomforde, $C^*$-algebras associated to complex or self-similar dynamics by Kajiwara and Watatani, and groupoid $C^*$-algebras associated to Deaconu-Renault groupoids.

We describe spectra of core subalgebras of $C_0(X) \rtimes L$, prove uniqueness theorems for $C_0(X) \rtimes L$ and characterise simplicity of $C_0(X) \rtimes L$. We give efficient criteria for $C_0(X) \rtimes L$ to be purely infinite simple and in particular a Kirchberg algebra.

Introduction.

Since 1970’s transfer operators are indispensable tools in thermodynamical formalism and ergodic theory [Bow75], and even earlier such operators, named averaging operators, played an important role in the study of Banach spaces $C(X)$ of continuous functions on a compact space $X$, see [Pel68]. They are also crucial in the study of spectrum of weighted composition operators, see [ABL11], [BK21]. Transfer operators as a tool to construct $C^*$-algebras, were explicitly used for the first time by Exel in [Exe03] to present Cuntz-Krieger algebras as crossed products associated to topological Markov chains. Since then a number of generalisations and modifications of such crossed products were introduced, see for instance [ERo07], [ERE07], [Lar10], [BRV10], [Bro12]. Their general structure as Cuntz-Pimsner algebras is now quite well-understood, see [BR06], [Kwa17]. However, the detailed analysis of the associated $C^*$-algebras is usually limited to the case where the underlying mapping is a local homeomorphism on a compact Hausdorff space, see [EV06], [ERo07], [CS09], [BRV10], [Bro12]. The exceptions are $C^*$-algebras associated to rational maps [KW05] or maps whose inverse branches form a self-similar systems [KW06], [KW16]. All these $C^*$-algebras can be viewed as crossed products by transfer operators for finite-to-one maps admitting at most finite number irregular points. However, in many problems there is a natural need to study transfer operators for partial continuous maps that are countable-to-one. This concerns in particular infinite graph $C^*$-algebras [Rae05], [BRV10], [Bro12], [Kwa17] or thermodynamic formalism for countable Markov shifts, interest in which has been growing in recent years, see [Sar99], [EL99], [BEFR_1], [BEFR_2]. In the present paper we give a general, comprehensive account of the main structural results for crossed products by transfer operators for arbitrary partial continuous maps that are countable-to-one.

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More specifically we consider a continuous map \( \varphi : \Delta \to X \) defined on an open subset \( \Delta \) of a locally compact Hausdorff space \( X \). We assume that \( \varphi^{-1}(y) \) is countable for all \( y \in \Delta \). Then every bounded transfer operator for \( \varphi \) is a map \( L : C_0(\Delta) \to C_0(X) \) given by the formula
\[
L(a)(y) = \sum_{x \in \varphi^{-1}(y)} \varrho(x)a(x)
\]
where \( \varrho : \Delta \to [0, \infty) \) is a map that we call a potential. A potential \( \varrho \) is in general only upper semi-continuous and the main role in our analysis is played by the following two sets:
\[
\Delta_{\text{pos}} := \{ x \in \Delta : \varrho(x) > 0 \}, \quad \Delta_{\text{reg}} := \{ x \in \Delta_{\text{pos}} : \varrho \text{ is continuous at } x \}.
\]
So \( \Delta_{\text{reg}} \subseteq \Delta_{\text{pos}} \subseteq \Delta \subseteq X \). As we show \( \Delta_{\text{reg}} \) is an open subset of \( X \) and the restricted map \( \varphi : \Delta_{\text{reg}} \to X \) is a local homeomorphism. We define the crossed product \( C_0(X) \rtimes L \) as a universal \( C^* \)-algebra generated by the \( C^* \)-algebra \( C_0(X) \) and weighted operators \( at \), for \( a \in C_0(\Delta) \), subject to relations
\[
L(a) = tat^*, \ a \in C_0(\Delta), \quad a \sum_{i=1}^n u_i^K t^* u_i^K = a, \ a \in C_c(\Delta_{\text{reg}})
\]
where \( u_i^K \)'s is a suitably normalized partition of unity on \( K := \text{supp}(a) \), see [12] below. Apart from the case of covering maps on compact spaces treated in [EV06] this is the first general description of the crossed product \( C_0(X) \rtimes L \) in terms of explicit relations coming from \( L \). In other works the corresponding crossed product is usually defined and analyzed as the Cuntz-Pimsner algebra \( \mathcal{O}_{\mathcal{M}_L} \) associated to a \( C^* \)-correspondence \( \mathcal{M}_L \). We prove that \( C_0(X) \rtimes L \) is isomorphic to \( \mathcal{O}_{\mathcal{M}_L} \) (Theorem 3.8). We do not know whether in general \( C_0(X) \rtimes L \) can be naturally modelled by a topological quiver of Muhly and Tomforde [MT05]. One of our main structural result is the following the following version of (Cuntz-Krieger) uniqueness theorem (see Theorems 7.5, 7.7) that generalizes the corresponding results from [EV06], [ERo07], [CS09], [BRV10].

**Theorem A.** The following conditions are equivalent:

(i) Every representation of \( C_0(X) \rtimes L \) is faithful provided it is faithful on \( C_0(X) \).

(ii) The orbit representation of \( C_0(X) \rtimes L \) on \( \ell^2(\Delta) \) is faithful; this representation sends function in \( C_0(X) \) to operators of multiplication and the generator \( t \) to the weighted composition operator \( Th := \sqrt{\varrho} \circ \varphi \).

(iii) The map \( \varphi : \Delta_{\text{reg}} \to X \) is topologically free, that is the set of periodic points whose orbits are contained in \( \Delta_{\text{reg}} \) has empty interior.

If in addition \( \Delta_{\text{pos}} = \Delta_{\text{reg}} \) the above conditions are further equivalent to

(iv) \( C_0(X) \) is a maximal abelian \( C^* \)-subalgebra of \( C_0(X) \rtimes L \).

In general we characterise faithful representations of \( C_0(X) \rtimes L \) in terms of a canonical generalized expectation \( G \) for the inclusion \( C_0(X) \subseteq C_0(X) \rtimes L \) (Theorem 4.9). We construct \( G \) using a regular representation of \( C_0(X) \rtimes L \) on \( \ell^2(X \rtimes \mathbb{Z}) \). If \( \Delta_{\text{pos}} = \Delta_{\text{reg}} \), then \( G \) is a genuine conditional expectation and \( C_0(X) \rtimes L \) is naturally isomorphic to the \( C^* \)-algebra of the Renault-Deaconu groupoid for the partial local homeomorphism \( \varphi : \Delta_{\text{reg}} \to X \) (see Theorem 5.4). Then [iv] in Theorem A says that \( C_0(X) \) is a Cartan subalgebra of \( C_0(X) \rtimes L \) in the sense of Renault [Ren08]. We show by example that if \( \Delta_{\text{pos}} \neq \Delta_{\text{reg}} \), then topological freeness of \( \varphi : \Delta_{\text{reg}} \to X \) is not sufficient for maximal abelianess of \( C_0(X) \) in \( C_0(X) \rtimes L \).
We say that \( L \) is minimal if there are no non-trivial open subsets \( U \subseteq X \) such that \( \varphi(U \cap \Delta_{\text{pos}}) \subseteq U \) and \( \varphi^{-1}(U) \cap \Delta_{\text{reg}} \subseteq U \). As a corollary to Theorem \( \text{[A]} \) we get the following characterisation of simplicity (see Theorem \( \text{[7.11]} \)):

**Theorem B.** If \( \Delta_{\text{reg}} \) is infinite, then \( C_0(X) \rtimes L \) is simple if and only if \( L \) is minimal.

Inspired by notions of locally contractive groupoids \( \text{[Ana97]} \) and contractive topological graphs \( \text{[Kat04]} \) we define contractive transfer operators, see Definition \( \text{[8.1]} \). For such operators we get (see Theorem \( \text{[8.5]} \) and Corollary \( \text{[8.6]} \)):

**Theorem C.** If \( L \) is minimal and contractive, then \( C_0(X) \rtimes L \) is purely infinite and simple. If in addition \( X \) is second countable, then \( C_0(X) \rtimes L \) is a UCT-Kirchberg algebra (and so it is classifiable by its \( K \)-theory).

We illustrate the power of Theorem \( \text{[C]} \) by showing that it covers and unifies all purely infinite results in \( \text{[KW05]}, \text{[KW06]}, \text{[Ana97 Section 4]}, \text{[EHR 11]} \) (Examples \( \text{[8.7], [8.8], [8.9]} \)).

Another fundamental \( C^* \)-algebra associated to \( L \) is the fixed point algebra of the canonical circle gauge action on \( C_0(X) \rtimes L \). It is a direct limit \( A_\infty = \bigcup_{n=0}^\infty A_n \) of \( C^* \)-algebras

\[
A_n = \text{span}\{at^k \ell^{kb} : a, b \in C_0(\Delta_k), k = 0, \ldots, n\}
\]

where \( \Delta_k = \varphi^{-k}(\Delta) \) is the natural domain for \( \varphi^k \). The algebras \( A_n \) are interesting in their own right, see \( \text{[Kum83]} \), and the \( C^* \)-algebra \( A_\infty \) has important dynamical interpretations. For special self-similar maps \( A_\infty \) was studied in \( \text{[KW10]} \). When \( \Delta_{\text{reg}} = \Delta \), so that \( \varphi \) is a local homeomorphism, then \( A_\infty \) is a groupoid \( C^* \)-algebra of a generalized approximately proper equivalence relation on \( X \). This is a crucial tool in the study of Gibbs states via the Radon-Nikodym problem \( \text{[Ren05]}, \text{[BEFR2]} \). Also stable \( C^* \)-algebras for irreducible Smale spaces are naturally Morita equivalent to algebras of the form \( A_\infty \) (see Remark \( \text{[6.10]} \) below). Putting \( \Delta_{\text{pos},n} := \{x \in \Delta_n : \prod_{i=0}^{n-1} \varphi^i(x) \neq 0\} \) we describe the spectra of \( A_n \), \( n \in \mathbb{N} \), and \( A_\infty \), as follows (see Proposition \( \text{[6.2]} \) and Theorems \( \text{[6.6], [6.11]} \)):

**Theorem D.** For each \( n \in \mathbb{N} \) the algebra \( A_n \) is postliminary (Type I) and up to unitary equivalence all its irreducible representations are subrepresentations of the orbit representation on \( \ell^2(X) \). Namely, we have a bijection

\[
\hat{A}_n \cong \left( \bigcup_{k=0}^{n-1} \varphi^k(\Delta_{\text{pos},k}) \setminus \Delta_{\text{reg}} \right) \sqcup \varphi^n(\Delta_{\text{pos},n}),
\]

where a representation corresponding to \( y \in \varphi^k(\Delta_{\text{pos},k}) \) is the restriction of the orbit representation to the subspace \( \ell^2(\varphi^{-k}(\Delta)) \subseteq \ell^2(X) \). The Jacobson topology on \( \hat{A}_n \) in general is finer than the pushout topology on the right-hand side of \( \text{(1)} \). But the two topologies coincide for instance when the potential \( \varphi \) is continuous, and if in addition \( X \) is second countable, then the primitive ideal space of \( A_\infty \) is homeomorphic to the quasi-orbit space:

\[
\text{Prim}(A_\infty) \cong X/\sim
\]

where \( x \sim y \) iff \( \overline{O(x)} = \overline{O(y)} \) and the orbit of \( x \in X \) is \( O(x) := \bigcup_{k=0}^\infty \varphi^{-k}(\varphi^k(x)) \).

The paper is organized as follows. In Section \( \text{[1]} \) we discuss transfer operators for partial maps and the properties of the associated potential \( \varphi \). Covariant representations for transfer operators are introduced in Section \( \text{[2]} \). The crossed product \( C_0(X) \rtimes L \) and its relationship with previous constructions modeled by Cuntz-Pimsner algebras are discussed in Section \( \text{[3]} \).
In Section 4 based on the well known gauge-invariance uniqueness for Cuntz-Pimsner
algebras we prove faithfulness of the regular representation of $C_0(X) \rtimes L$, which in turn leads
us to a generalized expectation-invariance uniqueness theorem. We use the latter in Section 5
to prove that Reunaut-Deaconu groupoid $C^*$-algebras associated to a local homeomorphism
$\varphi$ is naturally isomorphic to $C_0(X) \rtimes L$. Section 6 is devoted to description of the spectrum of
algebras $A_n$ and $A_\infty$ and it contains the proof of Theorem D. Section 7 introduces topological
freeness for transfer operators and contains proof of Theorems A and B. Finally in Section 8
we give criteria for pure infiniteness for $C_0(X) \rtimes L$ (we prove Theorem C).

1. Transfer operators for partial maps and potentials

Throughout this paper $\varphi : \Delta \to X$ is a continuous map defined on an open subset $\Delta$ of a
locally compact space $X$. We refer to $(X, \varphi)$ as to a partial dynamical system. In addition
we will fix a bounded transfer operator for $(X, \varphi)$, which we will interpret as a potential for
the system $(X, \varphi)$. Namely, let us denote by $C_0(X)$ the $C^*$-algebra of continuous functions
on $X$ that vanish at infinity. We treat $C_0(\Delta)$ as an ideal in $C_0(X)$. By a transfer operator
for $(X, \varphi)$ we mean a positive linear map $L : C_0(\Delta) \to C_0(X)$ satisfying

\[ L((a \circ \varphi)b) = aL(b), \quad a \in C_0(X), b \in C_0(\Delta). \]

Remark 1.1. We could allow the transfer operator $L$ to attain values in the bounded continuous
functions $C_b(X)$, but then (2) forces $L$ to take values in $C_0(X)$ anyway. Indeed, if $b \in C_c(\Delta)$
is compactly supported with the support $K$ then taking $a \in C_c(X)$ such that $a|_{\varphi(K)} \equiv 1$
we get $L(b) = L((a \circ \varphi)b) = aL(b) \in C_c(X)$. Thus transfer operators map compactly supported functions
to compactly supported ones.

Transfer operator could be defined in purely $C^*$-algebraic terms as follows. Let $I$ be an ideal
in a $C^*$-algebra $A$ (by which we always mean a closed two-sided ideal). Let $\alpha : A \to M(I)$
be a non-degenerate $*$-homomorphism from $A$ to the multiplier $C^*$-algebra $M(I)$ of $I$. Such
maps are called partial endomorphisms of $A$ in [ERo07, Definition 1.1], [Kai03, Definition 3.12]. A (bounded) transfer operator for $\alpha$ is a positive linear map $L : I \to A$ satisfying

\[ L(\alpha(a)b) = aL(b), \quad a \in A, b \in I. \]

Positivity implies that $L$ is bounded and $*$-preserving. In addition the transfer equality (3)
implies that $L(I)$ is an ideal. Transfer operators introduced in [ERo07, Definition 1.2] are defined
on a not necessarily closed ideal in $I$, and thus in general they are unbounded.

Having the triple $(A, \alpha, L)$ as above and assuming that $A = C_0(X)$, we necessarily have
$I = C_0(\Delta)$, for an open set $\Delta \subseteq X$, and

\[ \alpha(a) = a(\varphi(x)), \quad x \in \Delta, \quad a \in A, \]

for a continuous map $\varphi : \Delta \to X$. Accordingly, $M(I) = C_b(\Delta)$ consists of continuous
bounded functions and $\alpha : C_0(X) \to C_b(\Delta)$. In particular, $\alpha : C_0(X) \to C_0(\Delta) \subseteq C_0(X)$
is an endomorphism of $C_0(X)$ if and only if the map $\varphi : \Delta \to X$ is proper, i.e. the preimage
of every compact set in $X$ is compact in $\Delta$. Furthermore, denoting by $\mathcal{M}(X)$ the space of
finite regular borel measures on $X$ equipped with the weak* topology, a transfer operator
$L : C_0(\Delta) \to C_0(X)$ for $\alpha$ is of the form

\[ L(a)(y) = \int_{\varphi^{-1}(y)} a(x)d\mu_y(x), \quad a \in C_0(\Delta), y \in X, \]
where $X \ni y \mapsto \mu_y \in \mathcal{M}(X)$ is a continuous map such that $\supp \mu_y \subseteq \varphi^{-1}(y)$ for every $y \in X$ and $\sup_{y \in X} \mu_y(X) = \|L\| < \infty$, cf., for instance, [ABL11], [Kwa12], [Kwa17]. If the preimages of $\varphi$ are countable, then this measure valued function can be replaced by a number valued function. We assume this throughout the paper.

**Standing assumption:**

$$|\varphi^{-1}(y)| \leq \aleph_0 \quad \text{for all } y \in X. \tag{5}$$

Under this assumption, the measures $\{\mu_y\}_{y \in X} \subseteq \mathcal{M}(X)$ appearing in (4) are discrete and putting $\varrho(x) := \mu_{\varphi(x)}(\{x\})$, $x \in \Delta$, we get that the corresponding transfer operator is given by

$$L(a)(y) = \sum_{x \in \varphi^{-1}(y)} \varrho(x)a(x). \tag{6}$$

We refer to the map $\varrho : X \to [0, \infty)$ as to the potential associated to $L$, and we put

$$\Delta_{\text{pos}} := \Delta \setminus \varrho^{-1}(0) = \{x \in \Delta : \varrho(x) > 0\}.$$  

Obviously, every map admits a zero transfer operator (so that $\Delta_{\text{pos}} = \emptyset$), but there is a large and important class of maps that admit a transfer operator with $\Delta_{\text{pos}} = \Delta$. This concerns essentially all local homeomorphisms, see Theorem 5.4 below, and all open finite-to-one maps on compact spaces. This last claim follows from a result of Pavlov and Troitsky [PT11, Theorem 1.1] – we thank Magnus Goffeng for pointing this to us:

**Theorem 1.2** (Pavlov, Troisky, [PT11]). Let $\varphi : \Delta \to X$ be a continuous surjection where $\Delta$ is a compact open subset of $X$. There exists a transfer operator $L : C(\Delta) \to C(X)$ with a strictly positive potential $\varrho : \Delta \to (0, +\infty)$ if and only if $\varphi$ is an open map with $\sup_{x \in X} |\varphi^{-1}(x)| < \infty$.

**Proof.** Under our assumptions the endomorphism $\alpha : C(X) \to C(\Delta)$, given by composition with $\varphi$, is a unital monomorphism – an inclusion. Conditional expectations $E$ for the inclusion $\alpha$ are in bijective correspondence with transfer operators $L$ for $\varphi$, given by $E = \alpha \circ L$. Thus the assertion follows from [PT11, Theorem 1.1] (in fact the ‘if part’ follows from the proof of [PT11, Theorem 4.3]). \hfill \square

We fix a transfer operator $L$ of the form (6). In general, $\varrho$ has the following properties.

**Proposition 1.3.** The potential $\varrho$ is upper semi-continuous, and so $\varrho$ is continuous at every point in $\varrho^{-1}(0)$. If $x_0 \in \Delta_{\text{pos}} = \Delta \setminus \varrho^{-1}(0)$, then the following are equivalent:

(i) $\varrho$ is continuous at $x_0$,

(ii) $\varphi$ is locally injective at $x_0$, i.e. there is open $U \subseteq \Delta$ with $x_0 \in U$ such that $\varphi|_U : U \to X$ injective,

(iii) $x_0$ is a local homeomorphism point for $\varphi$, i.e. there is open $U$ with $x_0 \in U$ such that $\varphi : U \to \varphi(U)$ is a homeomorphism and $\varphi(U)$ is open in $X$.

Moreover, $\varphi$ restricted to $\Delta_{\text{pos}}$ is an open map.

Before we get into the proof of Proposition 1.3, we first prove a couple of lemmas.

**Lemma 1.4.** Restriction of $\varphi$ to $\Delta_{\text{pos}}$ is an open map.

**Proof.** Every open set in $\Delta_{\text{pos}}$ is of the form $U \cap \Delta_{\text{pos}}$ where $U \subseteq \Delta$ is open in $\Delta$. Let $y_0 \in \varphi(U \cap \Delta_{\text{pos}})$ so that $y_0 = \varphi(x_0)$ for some $x_0 \in U \cap \Delta_{\text{pos}}$. Take any continuous function $0 \leq a \leq 1$ supported on $U$ and such that $a(x_0) = 1$. Then $\mu_{\varphi(x_0)}(a) \geq \varrho(x_0) > 0$ and...
\( \mu_y(a) = 0 \) for every \( y \notin \varphi(U) \). Since the map \( X \ni y \mapsto \mu_y(a) \) is continuous the set \( V := \{ y \in X : \mu_y(a) > 0 \} \) is open in \( X \). Clearly, \( y_0 = \varphi(x_0) \in V \subseteq \varphi(U \cap \Delta_{\text{pos}}) \). \( \square \)

**Lemma 1.5.** For any \( x_0 \in \Delta \) and \( \varepsilon > 0 \) there is a neighbourhood \( U_0 \) of \( x_0 \) such that for any open \( U \subseteq U_0 \), with \( x_0 \in U \), there is a neighbourhood \( V \) of \( \varphi(x_0) \) such that

\[
\sum_{x \in U \cap \varphi^{-1}(y)} \varrho(x) - \varrho(x_0) < \varepsilon \quad \text{for all } y \in V. \tag{7}
\]

**Proof.** Fix \( \varepsilon > 0 \). Since the measure \( \mu_{\varphi(x_0)} \) is regular there is a neighbourhood \( U_1 \) of \( x_0 \) such that \( \mu_{\varphi(x_0)}(U_1) < \mu_{\varphi(x_0)}(\{ x_0 \}) + \varepsilon \), which translates to

\[
\sum_{x \in U_1 \cap \varphi^{-1}(\varphi(x_0))} \varrho(x) < \varrho(x_0) + \varepsilon.
\]

Let \( U_0 \) be any neighbourhood of \( x_0 \) such that \( \overline{U_0} \subseteq U_1 \). Now for any neighbourhood \( U \subseteq U_0 \) of \( x_0 \) take two continuous functions such that

\[
0 \leq f_1, f_2 \leq 1, \quad f_1(x) = \begin{cases} 1, & x \in U \\ 0, & x \notin U_0 \end{cases}, \quad f_2(x) = \begin{cases} 1, & x = x_0 \\ 0, & x \notin U \end{cases}.
\]

Set \( V = \{ y : \mu_y(f_1) < \varrho(x_0) + \varepsilon \) and \( \mu_y(f_2) > \varrho(x_0) - \varepsilon \} \). Clearly, \( \varphi(x_0) \in V \) and for any \( y \in V \) we have \( \varrho(x_0) - \varepsilon < \mu_y(f_2) \leq \sum_{x \in U \cap \varphi^{-1}(y)} \varrho(x) = \mu_y(U) \leq \mu_x(f_1) < \varrho(x_0) + \varepsilon \). \( \square \)

**Corollary 1.6.** For any neighbourhood \( U \) of \( x_0 \in \Delta \) and any \( \varepsilon > 0 \) there exists a continuous function \( 0 \leq h \leq 1 \) supported on \( U \) such that \( h(x) \equiv 1 \) on an neighbourhood of \( x_0 \) and \( \varrho(x_0) \leq \max_{y \in X} \sum_{x \in \varphi^{-1}(y)} \varrho(x) h(x) < \varrho(x_0) + \varepsilon \).

**Proof.** We may assume that \( U \subseteq U_0 \) where \( U_0 \) is as in Lemma 1.5 and then find \( V \) corresponding to \( U \) in this lemma. Take any continuous function \( 0 \leq h \leq 1 \) supported on an open set contained with a boundary in \( U \cap \varphi^{-1}(V) \) and such that \( h(x) = 1 \) on an open neighbourhood of \( x_0 \). Then

\[
\varrho(x_0) \leq \sum_{x \in \varphi^{-1}(\varphi(x_0))} \varrho(x) h(x) \leq \max_{y \in X} \sum_{x \in \varphi^{-1}(y)} \varrho(x) h(x) = \| L(h) \|
\]

and \( \| L(h) \| = \max_{y \in V} \sum_{x \in \varphi^{-1}(y)} \varrho(x) h(x) \leq \max_{y \in V} \sum_{x \in U} \varrho(x) < \varrho(x_0) + \varepsilon \). \( \square \)

**Proof of Proposition 1.3.** Let \( x_0 \in X \) and \( \varepsilon > 0 \). Let \( U \) and \( V \) be open sets as in Lemma 1.5. Then \( U \cap \varphi^{-1}(V) \) is an open neighbourhood of \( x_0 \), and for any \( y \in U \cap \varphi^{-1}(V) \) we have \( \varrho(x) \leq \sum_{y \in U \cap \varphi^{-1}(\varphi(x))} \varrho(y) < \varrho(x_0) + \varepsilon \). Hence \( \varrho \) is upper continuous at \( x_0 \).

Now let \( x_0 \in \Delta \setminus \varphi^{-1}(0) \).

\( \text{(i) } \Rightarrow \text{(ii)} \) Suppose that \( \varrho \) is lower continuous at \( x_0 \). Then for any \( \varepsilon < \varrho(x_0)/3 \) there is a neighbourhood \( U \) of \( x_0 \) such that

\[
\varrho(x) > \varrho(x_0) - \varepsilon > 0 \quad \text{for all } x \in U. \tag{8}
\]

By Lemma 1.5 we may assume that there is an open neighbourhood \( V \) of \( \varphi(x_0) \) such that (7) holds. Then for \( W := U \cap \varphi^{-1}(V) \) we get

\[
\varrho(x_0) - \varepsilon < \mu_{\varphi(x)}(W) < \varrho(x_0) + \varepsilon \quad \text{for all } x \in W.
\]
We claim, that \( \varphi \) is injective on \( W \). Indeed, assume on the contrary that \( W \) contains two distinct points \( x_1, x_2 \) such that \( \varphi(x_1) = \varphi(x_2) \). Then by (8) we get
\[ \varrho(x_0) + \varepsilon > \mu_{\varphi(x_1)}(W) \geq \varrho(x_1) + \varrho(x_2) > 2(\varrho(x_0) - \varepsilon). \]
which contradicts \( \varepsilon < \varrho(x_0)/3 \).

(ii) \( \Rightarrow \) (iii) This follows from Lemma 1.4.

(iii) \( \Rightarrow \) (i) Suppose that \( x_0 \) is a local homeomorphism point, and let \( U \) be a neighbourhood of \( x_0 \) such that \( \varphi : U \to \varphi(U) \) is a homeomorphism. Let \( \varepsilon > 0 \). By Lemma 1.5 we may assume that there is a neighbourhood \( V \) of \( \varphi(x_0) \) such that (7) holds. But for any \( x \) in \( U \cap \varphi^{-1}(V) \) we have \( \varphi(x) \neq x \). Indeed, since
\[ \varphi \text{ is injective.} \]
and the inequality is equality when
\[ \varphi(x) = x. \]
Hence \( \varrho \) is continuous at \( x_0 \).

\[ \square \]

2. Covariant representations and regular points

Throughout the paper, we fix a transfer operator \( L : C_0(\Delta) \to C_0(X) \) of the form (6) where \( \varphi : \Delta \to X \) is a partial map and \( \varrho : \Delta \to [0, \infty) \) is the associated potential. We write \( A := C_0(X) \) and \( I := C_0(\Delta) \), and let \( \alpha : C_0(X) \to C_0(\Delta) \) be given by \( \alpha(a) = a \circ \varphi \).

**Definition 2.1.** A representation of the transfer operator \( L \) is a pair \((\pi, T)\) where \( \pi : A \to B(H) \) is a non-degenerate representation and \( T \in B(H) \) satisfies
\[ \pi(L(a)) = T^* \pi(a) T, \quad a \in I = C_0(\Delta). \]
We say that \((\pi, T)\) is faithful if \( \pi \) is faithful. We denote by
\[ C^*(\pi, T) := C^*(\pi(A) \cup \pi(I)T) \]
the \( C^* \)-algebra generated by \( \pi(A) \cup \pi(I)T \).

**Remark 2.2.** Without loss of generality, we could additionally assume in Definition 2.1 that \( TH \subseteq \pi(I)H \) (as composing \( T \) with the projection onto \( \pi(I)H \) does not affect (9) and the \( C^* \)-algebra \( C^*(\pi, T) \)). Assuming this we have \( ||T|| \leq ||L||^{\frac{1}{2}} \), with the equality when \( \pi \) is faithful. Indeed, since \( L : I \to A \) is positive, we have \( ||L|| = \lim_{\lambda} ||L(\mu_\lambda)|| \) for an approximate unit \( \{\mu_\lambda\} \) in \( I \), see for instance, Kwa17 Lemma 2.1. Hence
\[ ||T||^2 = ||T^*T|| = \lim_{\lambda} ||T^*\pi(\mu_\lambda)T|| = \lim_{\lambda} ||\pi(L(\mu_\lambda))|| \leq \lim_{\lambda} ||\mu_\lambda|| = ||L|| \]
and the inequality is equality when \( \pi \) is injective. However, in what follows, we will not assume that \( TH \subseteq \pi(I)H \), as we will be mainly concerned with operators of the form \( \pi(a)T \), for \( a \in I \), and then we always have \( ||\pi(a)T|| \leq ||a|| ||L||^{\frac{1}{2}} \).

**Remark 2.3.** The \( C^* \)-algebra \( C^*(\pi, T) \) is not affected if we replace \( \Delta \) by any open set \( U \) such that \( \Delta_{\text{pos}} \subseteq U \subseteq \Delta \), as then \( \pi(C_0(\Delta))T = \pi(C_0(U))T \). Indeed, \( ||\pi(a)T||^2 = ||\pi(L(a^*a))|| \) and \( ||L(a^*a)|| = \sup_{x \in X} \sum_{x \in \varphi^{-1}(y) \cap \Delta_{\text{pos}}} |a|^2(x) \varrho(x) \), so the norm of \( \pi(a)T \) depends only on values of \( a \) on \( \Delta_{\text{pos}} \). In particular, we may always assume that \( \Delta = \varphi^{-1}(\varphi(\Delta_{\text{pos}})) \), as the set \( \varphi^{-1}(\varphi(\Delta_{\text{pos}})) \) is open because the map \( \varphi : \Delta_{\text{pos}} \to X \) is open.

**Lemma 2.4.** Let \((\pi, T)\) be a representation of \( L \). We have the following commutation relations
\[ \pi(b)T\pi(a) = \pi(b\alpha(a))T, \quad a \in A, b \in I. \]
If in addition \( TH \subseteq \pi(I)H \) and \( \varphi \) is proper, then \( T\pi(a) = \pi(\alpha(a))T, a \in A \).
In view of Lemma 2.4, we have

Thus \( \text{span} I \) is a \( \pi \)-algebra, and so \( \pi(A) \cap \text{span}(I) TT^* \pi(I) \) is an ideal in \( \pi(A) \).

Proof. By Lemma 2.4, \( T \pi(a) \) and \( d := \pi(\alpha(a))T \) in the calculations above. Then all the terms \( c^*d, d^*d, c^*c, d^*c \) are equal to \( \pi(a^*) TT^* \pi(a) \). For instance, if \( \{\mu_\lambda\} \) is an approximate unit in \( I \), then

\[
\begin{align*}
  c^*d &= \pi(a^*) TT^* \pi(\alpha(a)) T = s-lim_\lambda \pi(a^*) TT^* \pi(\mu_\lambda) \pi(\alpha(a)) T \\
  &= s-lim_\lambda \pi(a^*) \pi(L(\mu_\lambda)\alpha(a)) = s-lim_\lambda \pi(a^*) \pi(L(\mu_\lambda)) \pi(a) = \pi(a^*) TT^* \pi(a).
\end{align*}
\]

Here \( s \)-lim stands for a limit in strong operator topology.

**Corollary 2.5.** If \((\pi, T)\) is a representation of \( L \), then

\[
\pi(I) TT^* \pi(I) = \text{span}\{\pi(a) TT^* \pi(b) : a, b \in I\}
\]

is a \( C^* \)-algebra, and so \( \pi(A) \cap \pi(I) TT^* \pi(I) \) is an ideal in \( \pi(A) \).

Proof. By Lemma 2.4, \( \pi(a) TT^* \pi(b) \cdot \pi(c) TT^* \pi(d) = \pi(a) TT^* \pi(\alpha(L(bc))d) \) for \( a, b, c, d \in I \). Thus \( \text{span}\{\pi(a) TT^* \pi(b) : a, b \in I\} \) is a \( * \)-algebra.

**Remark 2.6.** In view of Lemma 2.4 we have \( \pi(I) TT^* \pi(I) = \pi(I) T \pi(A) T^* \pi(I) \), and if \( TH \subseteq \pi(I) H \) and \( \varphi \) is proper, then \( \pi(I) TT^* \pi(I) = \pi(A) TT^* \pi(A) \).

The spectrum of the ideal in Corollary 2.5 is related to the set of regular points that we define as follows.

**Definition 2.7.** The set of regular points for \( \varphi \) is

\[
\Delta_{\text{reg}} := \{ x \in \Delta : \varphi(x) > 0 \text{ and } \varphi \text{ is continuous at } x \}.
\]

Clearly, \( \Delta_{\text{reg}} \) is an open set, and by Proposition 1.3 a point \( x \in \Delta \) is regular if and only if \( \varphi(x) > 0 \) and \( x \) is a local homeomorphism point for \( \varphi \).

**Remark 2.8.** We have a hierarchy of sets \( \Delta_{\text{reg}} \subseteq \Delta_{\text{pos}} \subseteq \Delta \) where \( \Delta_{\text{pos}} \) need not be open nor closed in \( X \). The map \( \varphi \) is open on \( \Delta_{\text{pos}} \) and in addition locally injective on \( \Delta_{\text{reg}} \).

**Proposition 2.9.** Let \((\pi, T)\) be a faithful representation of \( L \). Then

\[
\pi(A) \cap \pi(I) TT^* \pi(I) \subseteq \pi(C_0(\Delta_{\text{reg}})).
\]

Proof. To lighten the notation we will suppress \( \pi \) and we will write \( A \subseteq B(H) \). Let us fix \( a \in A \) such that \( a \notin C_0(\Delta_{\text{reg}}) \). That is, there is \( x_0 \in X \setminus \Delta_{\text{reg}} \neq 0 \) with \( a(x_0) \neq 0 \). We need to show that \( a \notin \text{span}(TT^*I) \) and to this end it suffices to show that for any \( a_i, b_i \in I, i = 1, \ldots, N \in \mathbb{N} \), we have

\[
\|a - \sum_{i=1}^{N} a_i TT^* b_i \| \geq |a(x_0)|.
\]

We first show a weaker inequality, which holds for an arbitrary \( y \in \Delta \) though,

\[
\|a - \sum_{i=1}^{N} a_i TT^* b_i \| \geq |a(y)| - \sqrt{\varphi(y)} \sum_{i=1}^{N} \|a_i\| \|b_i\|.
\]

Let $\varepsilon > 0$ and put $U := \{x \in \Delta : |a(x) - a(y)| < \varepsilon\}$. By Corollary 1.6 there is a continuous function $0 \leq h \leq 1$ supported on $U$ such that $h(x) = 1$ on an open neighbourhood of $y$ and

$$g(y) \leq \|L(h^2)\| < g(y) + \varepsilon.$$  

Thus for any $b \in C_0(X)$ one has

$$\|hb^*T\|^2 = \|T^*bh\|^2 = \|L(|b|^2h^2)\| \leq \|b\|^2\|L(h^2)\| \leq \|b\|^2(g(y) + \varepsilon).$$

Using this we get

$$\|a - \sum_{i=1}^N a_iTT^*b_i\| \geq \|h(a - \sum_{i=1}^N a_iTT^*b_i)h\| = \|ah^2 - \sum_{i=1}^N ha_iTT^*b_ih\|$$

$$\geq \|ah^2\| - \sum_{i=1}^N |ha_iT||T^*b_ih|$$

$$\geq \|h\|^2 - \sqrt{\|g(y)\| + \varepsilon}\sum_{i=1}^N |a_i||b_i|.$$ 

Passing with $\varepsilon$ to zero, we get (11). Now we consider two cases.

I). Suppose first that for each $\delta > 0$ every neighbourhood of $x_0$ contains a point $x$ with $g(x) < \delta$. Equivalently, there is a net $\{x_n\} \subseteq \Delta$ such that $x_n \to x_0$ and $g(x_n) \to 0$. Applying (11) to $y = x_n$ we have

$$\|a - \sum_{i=1}^N a_iTT^*b_i\| \geq |a(x_n)| - \sqrt{\|g(x_n)\| + \varepsilon}\sum_{i=1}^N |a_i||b_i|,$$

which by passing to the limit, gives (10).

II). Finally, suppose that there is $\delta > 0$ and an open neighbourhood $U$ of $x_0$ such that

$$\inf_{x \in U} g(x) \geq \delta > 0.$$ 

Clearly it is enough to consider the case when $a(x_0) \neq 0$. Let $\varepsilon > 0$. We may assume that $U \subseteq \{x \in \Delta : |a(x) - a(x_0)| < \varepsilon\}$. Also, since $x_0 \notin \Delta_{\text{reg}}$, $g$ is not continuous at $x_0$. Therefore, by Proposition 1.3 there exist two distinct points $x_1, x_2$ in $U$ such that $\varphi(x_1) = \varphi(x_2)$. Let $U_1, U_2 \subseteq U$ be two open disjoint sets with $x_1 \in U_1$ and $x_2 \in U_2$. By Corollary 1.6 for each $i = 1, 2$, there are continuous functions $0 \leq h_i \leq 1$ supported on $U_i$ such that $h_i(x_i) = 1$ and

$$g(x_i) \leq \|L(h_i)\| < g(x_i) + \varepsilon, \quad \text{and} \quad g(x_i) \leq \|L(h_i^2)\| < g(x_i) + \varepsilon.$$ 

Put $h := h_1 - \frac{\varphi(x_1)}{\varphi(x_2)}h_2$. Using that $h$ is supported on $U$ we get

$$\|ahT\| \geq \|hT||a(x_0)| - \varepsilon|.$$ 

On the other hand, $\|hT\|^2 = \|L(h^2)\| = \|L(h_1^2) + \left(\frac{\varphi(x_1)}{\varphi(x_2)}\right)^2 L(h_2^2)\| \geq \|L(h_1^2)\| \geq g(x_1) \geq \delta$. Moreover, for any $b \in C_0(X)$ we have

$$\|T^*bhT\| = \|L(hb)\| \leq \|L(h)\| \cdot \|b\| = \left\|L(h_1) - \frac{\varphi(x_1)}{\varphi(x_2)} L(h_2)\right\| \cdot \|b\|$$

$$\leq \left((g(x_1) + \varepsilon) - \frac{\varphi(x_1)}{\varphi(x_2)} g(x_2)\right) \cdot \|b\| = \varepsilon \cdot \|b\|. $$
Using all this, we get
\[
\|a - \sum_{i=1}^{N} a_i TT^* b_i\| \geq \frac{\| (a - \sum_{i=1}^{N} a_i TT^* b_i)(hT)\|}{\|hT\|}
\]
\[
\geq \frac{\|ahT\|}{\|hT\|} - \sum_{i=1}^{N} \frac{\|a_i T\| \cdot \|T^* b_i hT\|}{\|hT\|}
\]
\[
\geq (|a(x_0)| - \varepsilon) - \varepsilon \sum_{i=1}^{N} \frac{\|a_i T\| \cdot \|b_i\|}{\sqrt{\delta}}.
\]
Passing with \(\varepsilon\) to zero, we get (10). \(\square\)

**Definition 2.10.** We say that a representation \((\pi, T)\) of \(L\) is **covariant** if
\[
\pi(C_0(\Delta_{\text{reg}})) \subseteq \pi(I)TT^*\pi(I).
\]

Thus a faithful representation \((\pi, T)\) is covariant iff \(\pi(C_0(\Delta_{\text{reg}})) = \pi(I)TT^*\pi(I)\).

**Remark 2.11.** Every transfer operator admits a faithful covariant representation (see Example 2.14). If \(\Delta_{\text{reg}}\) is non-empty, then there are faithful representations that are not covariant. Indeed, if \((\pi, T)\) is any representation of \(L\) on a Hilbert space \(H\), then putting \(\tilde{H} := H \otimes \ell^2(\mathbb{N})\), \(\tilde{\pi} := \pi \otimes \text{id}\) and \(\tilde{T} := T \otimes U\) where \(U\) is the unilateral shift on \(\ell^2(\mathbb{N})\), we get a representation \((\tilde{\pi}, \tilde{T})\) of \(L\) with \(\tilde{\pi}(A) \cap \tilde{\pi}(I)TT^*\tilde{\pi}(I) = 0\), because \(UU^*\) is a non-trivial projection.

2.1. **Characterisations of covariant representations.** Let \(K\) be a compact subset of \(\Delta_{\text{reg}}\). Then we may find a finite cover \(\{U_i\}_{i=1}^{n}\) of \(K\) such that \(\bigcup_{i=1}^{n} U_i\) is contained in a compact subset of \(\Delta_{\text{reg}}\) and \(\varphi|_{U_i}\) is injective for every \(i = 1, \ldots, n\). Take a partition of unity \(\{v_i\}_{i=1}^{n} \subseteq C_0(\Delta_{\text{reg}})\) on \(K\) subordinated to \(\{U_i\}_{i=1}^{n}\). Then
\[
\begin{align*}
\tag{12} u_i^K := \sqrt{\frac{|v_i|}{\varphi}}, & \quad i = 1, \ldots, n,
\end{align*}
\]
are well defined functions in \(C_c(\Delta_{\text{reg}})\) because \(\varphi\) is bounded away from zero on \(\bigcup_{i=1}^{n} U_i\). We will use these functions to characterise covariant representations.

**Proposition 2.12.** Let \((\pi, T)\) be a representation of \(L\). The following are equivalent:

(i) \((\pi, T)\) is covariant.

(ii) for every \(a \in C_c(\Delta_{\text{reg}})\) supported on a set where \(\varphi\) is injective we have \(\pi(a)TT^*\pi(u) = \pi(a)\) for some \(u \in C_0(X)\).

(iii) for every \(a \in C_c(\Delta_{\text{reg}})\) supported on a set where \(\varphi\) is injective we have \(\pi(a)TH = \pi(a)H\).

(iv) For every \(a \in C_c(\Delta_{\text{reg}})\) supported on \(K\) we have we have
\[
\pi(a)\sum_{i=1}^{n} \pi(u_i^K)TT^*\pi(u_i^K) = \pi(a).
\]

(v) for every \(x_0 \in \Delta_{\text{reg}}\) and \(\varepsilon > 0\) there is a neighbourhood \(U\) of \(x_0\) such that for every \(a, b \in C_0(U)\) with \(\|a\|, \|b\| \leq 1\) we have
\[
\|\pi(a)TT^*\pi(b) - \varphi(x_0)\pi(ab)\| < \varepsilon.
\]
The above conditions hold whenever \( \pi(I)TH = H \) (which is equivalent to \( \pi(A)TH = H \) when \( \varphi \) is proper).

**Proof.** Clearly, [i] implies [iii]. Since \( C_c(\Delta_{\text{reg}}) \) is dense in \( C_0(\Delta_{\text{reg}}) \) and every element in \( C_c(\Delta_{\text{reg}}) \) is a finite sum of functions supported on sets where \( \varphi \) is injective, we see that [ii] also implies [i]. For converse implications, let \( a \in C_c(\Delta_{\text{reg}}) \) have support \( K \) such that \( \varphi|_K \) is injective and let \( \varphi \in C_c(\Delta_{\text{reg}}) \) be such that \( u|_K = (\varphi|_K)^{-1} \). For every \( b \in C_0(\Delta) \) and \( x \in \Delta_{\text{reg}} \) we have \( a(x)\alpha(L(ub))(x) = a(x)\sum_{t \in \varphi^{-1}(\varphi(x))} \theta(t)u(t)b(t) = a(x)b(x) \). Hence

\[
\left( \pi(a)TT^*\pi(u) \right)\pi(b)T = \pi(a\alpha(L(ub)))T = \pi(a)\pi(b)T.
\]

Thus \( \pi(a)TT^*\pi(u) = \pi(a) \) whenever \( \pi(a) \) is determined by its action on \( \pi(I)TH \). Both [i] and [iii] imply this. Indeed, if [i] holds then \( \pi(a) \in \pi(I)TT^*\pi(I) \), and if we assume that \( \pi(a) \) is injective and \( \pi(a) \) and [iii] imply this. Indeed, if [i] holds then \( \pi(a) \in \pi(I)TT^*\pi(I) \), and if we assume [iii] we get

\[
\pi(a)H = \pi(a)TH = \pi(a)T\pi(A)H = \pi(a)\pi(\alpha(A))TH = \pi(a)\pi(I)TH.
\]

Hence [i] and [iii] are equivalent and they follow from the condition \( \pi(I)TH = H \). If \( \varphi \) is proper, then \( \pi(I)TH = \pi(A\alpha(A))TH = \pi(A)T\pi(\alpha(A))H = \pi(A)TH \).

Since \( C_c(\Delta_{\text{reg}}) \) is dense in \( C_0(\Delta_{\text{reg}}) \), [iv] readily implies [i]. Conversely, if we assume [i] then for every \( a \in C_c(\Delta_{\text{reg}}) \) the operator \( \pi(a) \in \pi(I)TT^*\pi(I) \) is determined by its action on \( \pi(I)TH \). Moreover, for every \( b \in I \) we have \( a\sum_{i=1}^n u_i^K \alpha(L(u_i^K b)) = ab \). Thus

\[
\left( \pi(a)\sum_{i=1}^n \pi(u_i^K)TT^*\pi(u_i^K) \right)\pi(b)T = \pi\left( a\sum_{i=1}^n u_i^K \alpha(L(u_i^K b)) \right)T = \pi(a)\pi(b)T.
\]

This implies [iv]. Hence [i] \( \Rightarrow \) [iv].

[iv] \( \Rightarrow \) [v]. Let \( U \) be a neighbourhood of \( x_0 \in \Delta_{\text{reg}} \) such that \( \varphi|_U \) is injective and \( U \subseteq \{ x \in \Delta_{\text{reg}} : |\varphi(x) - \varphi(x_0)| < \varepsilon \} \). Take any \( a,b \in C_0(U) \) with \( ||a||, ||b|| \leq 1 \). Note that \( qa \in C_0(U) \) and \( ||qab - q(x_0)ab|| < \varepsilon \). The argument in the proof that [iii] implies [i] shows that \( \pi(a)TT^*\pi(b) = \pi(qab) \). Hence [v] holds.

[v] \( \Rightarrow \) [iii]. Let \( a \in C_c(\Delta_{\text{reg}}) \) have support \( K \) such that \( \varphi|_K \) is injective. Without loss of generality we may assume that \( ||a|| \leq 1 \). By [v] and compactness of \( K \) there is a partition of unity \( \{ u_i \}_{i=1}^n \) on \( K \) subordinate to an open cover \( \{ U_i \}_{i=1}^n \) of \( K \) such that for every \( i = 1,\ldots,n \) there is a point \( x_i \in U_i \) such that for every \( b \in C_0(U_i) \), \( ||b|| \leq 1 \), we have \( ||\pi(u_i a)TT^*\pi(b) - \varphi(x_i)\pi(ab)|| < \varphi(x_i)/2 \). Clearly, \( a \) satisfies [iii] if each \( u_i a \) satisfies [iii]. Hence we may assume that \( a \in C_0(U) \) where \( \varphi|_U \) is injective and there is \( x_0 \in U \) such that

\[
||\pi(a)TT^*\pi(b) - \varphi(x_0)\pi(ab)|| < \varphi(x_0)/2,
\]

for any \( b \in C_0(U) \), \( ||b|| \leq 1 \). Now let \( \{ \mu_\lambda \} \) be an approximate unit in \( C_0(U) \) and let \( P := \text{s-lim} \pi(\mu_\lambda) \) be the projection given by the strong limit. Then we have

\[
||P(TT^*P - \varphi(x_0)P)|| \leq \varphi(x_0)/2.
\]

Thus \( ||\varphi(x_0)P TT^*P - P|| \leq 1/2 < 1 \) and therefore the operator \( 1/\varphi(x_0)P TT^*P : PH \rightarrow PH \) is invertible. In particular, \( P_U TH = P_U H \) and this gives \( \pi(a)TH = \pi(a)P_U TH = \pi(a)P_U H = \pi(a)H \). \( \square \)
Remark 2.13. Assume $X = \Delta_{\text{reg}}$. Equivalently, $\varphi : X \to X$ is a local homeomorphism and $\varphi > 0$ is strictly positive. Then condition (iv) in Proposition 2.12 reduces to

$$\sum_{i=1}^{n} \pi(u_i^X)TT^* \pi(u_i^X) = 1,$$

which is the condition identified by Exel and Vershik in [EV06]. Also conditions in Proposition 2.12 are equivalent to the condition $\pi(A)TH = H$, which is called axiom (A3) in [BK21].

Example 2.14 (Orbit representation). There is a natural faithful covariant representation $(\pi_o, T_o)$ of $L$ on the Hilbert space $\ell^2(X)$. We will call it the orbit representation. Namely we define a faithful representation $\pi_o : C_0(X) \to B(\ell^2(X))$ by

$$(\pi_o(a)h)(x) := a(x)h(x), \quad a \in C_0(X), h \in \ell^2(X).$$

Let $\{1_x\}_{x \in X}$ be the standard orthonormal basis of $\ell^2(X)$. Since $\sum_{x \in \varphi^{-1}(y)} \varphi(x) \leq ||L||$, $y \in X$, there is $T_o \in B(\ell^2(X))$ such that $T_o 1_y := \sum_{x \in \varphi^{-1}(y)} \sqrt{\varphi(x)} 1_x$, $y \in X$. Its adjoint is given by $T_o^* 1_x = \sqrt{\varphi(x)} 1_{\varphi(x)}$, for $x \in \Delta$, and $T_o^* 1_x = 0$ for $x \not\in \Delta$. Equivalently,

$$(T_o h)(x) = \begin{cases} \sqrt{\varphi(x)} h(\varphi(x)), & x \in \Delta \\ 0, & x \not\in \Delta \end{cases}, \quad (T_o^* h)(y) = \sum_{x \in \varphi^{-1}(y)} \sqrt{\varphi(x)} h(x),$$

for $h \in \ell^2(X)$. Clearly $(\pi_o, T_o)$ is a faithful representation of $L$. To see that it is covariant we show condition (iii) in Proposition 2.12. Let $a \in C_c(\Delta_{\text{reg}})$ be supported on a set $K$ such that $\varphi|_K$ is injective and let $u \in C_c(\Delta_{\text{reg}})$ be such that $u|_K = (\varphi|_K)^{-1}$. Then

$$(\pi_o(a)T_o T_o^* \pi_o(u)h)(x) = a(x) \sqrt{\varphi(x)} \left( \sum_{t \in \varphi^{-1}(\varphi(x))} \sqrt{\varphi(t)} u(t) h(t) \right) = a(x) h(x) = (\pi_o(a)h)(x).$$

Hence $\pi_o(a) = \pi_o(a)T_o T_o^* \pi_o(u)$.

3. The crossed product

Recall that $L : C_0(\Delta) \to C_0(X)$ is a transfer operator for a partial map $\varphi : \Delta \to X$, $A = C_0(X)$, $I = C_0(\Delta)$, and $\alpha : A \to M(I)$ is given by composition with $\varphi$. Let us consider a universal $*$-algebra $\mathcal{A}(L)$ generated by $C_c(X)$ (viewed as a $*$-algebra) and an element $t$ subject to relations

$$(14) \quad L(a) = t a t^*, \quad a t b = a \alpha(b)t, \quad \text{for all } a \in C_c(\Delta), b \in C_c(X),$$

and for every compact $K \subseteq \Delta_{\text{reg}}$ and $a \in C_c(\Delta_{\text{reg}})$ supported on $K$

$$(15) \quad a = \sum_{i=1}^{n} u_i^K t t^* u_i^K = a$$

where $u_i^K$’s are given by (12). Note that these relations are satisfied by operators coming from covariant representations of $L$, see Lemma 2.3 and Proposition 2.12.

Definition 3.1. The algebraic crossed-product $C_c(X) \rtimes_{\text{alg}} L$ is the $*$-subalgebra of $\mathcal{A}(L)$ generated by $C_c(X)$ and $C_c(\Delta)t$.

To describe the structure of $C_c(X) \rtimes_{\text{alg}} L$ we need to iterate partial transfer operators. Let

$$\Delta_n := \varphi^{-n}(X), \quad I_n := C_0(\Delta_n), \quad n \in \mathbb{N}.$$
We put $\Delta_0 := X$ and $I_0 := A = C_0(X)$. So $\Delta_n$ is a natural domain for the partial map $\varphi^n$; the composition $\underbrace{\varphi \circ \cdots \circ \varphi}_{n \text{ times}}$ makes sense on $\Delta_n$. Define $\alpha^n : A \to M(I_n)$ to be the partial endomorphism of $A$ given by composition with $\varphi^n : \Delta_n \to X$. Having the map $\varphi : \Delta \to [0, \infty)$, that defines $L$ via (16), for each $n \in \mathbb{N}$ we define $\varrho_n : \Delta_n \to [0, \infty)$ by

$$\varrho_n(x) := \prod_{i=0}^{n-1} \varphi^i(x), \quad x \in \Delta_n.$$  

We also put $\varrho_0 \equiv 1$. Then the formula

$$L^n(a)(y) := \sum_{x \in \varphi^{-n}(y)} \varrho_n(x)a(x), \quad a \in C_0(\Delta_n), y \in X,$$

defines a transfer operator $L^n : I_n \to A$ for $\alpha^n : A \to M(I_n)$. To describe the maps $L^n$ and $\alpha^n$ more algebraically, note that (19)

$$I^0_n := \operatorname{span}\{a_1\alpha(a_2\alpha(\ldots a_n)) : a_1, \ldots, a_n \in C_c(\Delta)\}$$

is a dense *-subalgebra of $I_n$ (we put $I^0_0 := C_c(X)$). Thus $L^n$ and $\alpha^n$ are determined by the following formulas, for $a_1, \ldots, a_n \in C_c(\Delta)$ and $a \in C_c(X)$:

$$L^n(a_1\alpha(a_2\alpha(\ldots a_n))) = L(L(\ldots L(L(a_1)a_2)a_3\ldots)a_n)$$

(17)

$$a_1\alpha(a_2\alpha(\ldots a_n)) \cdot \alpha^n(a) = a_1\alpha(a_2\alpha(\ldots a_n\alpha(a))$$

(18)

**Lemma 3.2.** The algebraic crossed-product is the following linear span

$$C_c(X) \rtimes_{\operatorname{alg}} L = \operatorname{span}\{at^n t^m b : a \in I^0_n, b \in I^0_m, n, m \in \mathbb{N}_0\}.$$  

Moreover, for all $n, m, k, l \in \mathbb{N}_0$ and $a \in I^0_n, b \in I^0_m, c \in I^0_k, d \in I^0_l$, we have

$$(at^n t^m b) \cdot (ct^k t^l d) = \begin{cases} at^n t^{m-k+l} t^{l+1} (b)(c)d & m \geq k, \\
\alpha^n(a)(b)(c) t^{k-m+n} t^l d & m < k. \end{cases}$$

(20)

**Proof.** Using (17) and (18) we get that (14) generalizes to

$$L^n(a) = t^n at^n, \quad at^n b = a\alpha^n(b)t^n, \quad \text{for all } a \in I^0_n, b \in A, n \in \mathbb{N}.$$  

For instance, for $n = 2$, and $a_1, a_2 \in I$, we have

$$L^2(a_1\alpha(a_2)) = L(L(a_1)a_2) = t^2(t^* a_1 a_2) t = t^* a_1\alpha(a_2) t^2,$$

$$t^2 a_1\alpha(a_2) = a_1 a_2 \alpha(b) \cdot t = a_1\alpha(a_2)(b) \cdot t^2 \alpha^2(b) t^2.$$

Using (21) one readily gets (20). In turn (20) implies that the self-adjoint linear space $\operatorname{span}\{at^n t^m b : a \in I^0_n, b \in I^0_m, n, m \in \mathbb{N}_0\}$ is closed under multiplication. Hence it is a *-algebra, and clearly it is generated by $I^0_0 \cup I^0_1$. This proves (19). \hfill \Box

By universality every covariant representation $(\pi, T)$ of $L$ induces (uniquely) a representation $\pi \rtimes T$ of the *-algebra $C_c(X) \rtimes_{\operatorname{alg}} L$ where $\pi \rtimes T(a) = \pi(a), a \in C_c(X)$, and $\pi \rtimes T(at) = aT$ for $a \in C_c(\Delta)$. Namely, $\pi \rtimes T(\sum_{i=1}^n a_i t^n t^m b_i) = \sum_{i=1}^n a_i T^n t^m b_i$ for $a_i \in I^0_{n_i}, b_i \in I^0_{m_i}, i = 1, \ldots, n$. We put

$$\|x\|_{\max} := \sup\{\|\pi \rtimes T(x)\| : (\pi, T) \text{ is a covariant representation for } L\}.$$
It is easily verified that $\| \cdot \|_{\text{max}}$ is a $C^*$-seminorm (a submultiplicative seminorm satisfying the $C^*$-equality). It is finite because $\| \sum_{i=1}^n a_i t^{m_i} b_i \|_{\text{max}} \leq \sum_{i=1}^n \| a_i \| \| b_i \| (\| L^{m_i} \| \| L^{m_i} \|)^{1/2}$, cf. Remark 2.2. Restriction $\| \cdot \|_{\text{max}}$ to $C_c(X)$ coincides with the unique $C^*$-norm on $A$, because there exists a faithful covariant representation, see Example 2.14. In other words, the (self-adjoint and two-sided) ideal
\[ \mathcal{N} := \{ x \in C_c(X) \times_{\text{alg}} L : \| x \|_{\text{max}} = 0 \} \]
intersects $C_c(X)$ trivially.

**Definition 3.3.** The crossed product of $A$ by the transfer operator $L$ is the $C^*$-algebra $A \rtimes L$ obtained by the Hausdorff completion of $C_c(X) \times_{\text{alg}} L$ in $\| \cdot \|_{\text{max}}$:
\[ A \rtimes L = \overline{C_c(X) \times_{\text{alg}} L}^{\| \cdot \|_{\text{max}}}. \]

**Remark 3.4.** Since $C_c(X) \cap \mathcal{N} = \{ 0 \}$, we may and we will treat $C_c(X)$ as a $*$-subalgebra of $A \rtimes L$. The closure of $C_c(X)$ in $A \rtimes L$ will be identified with $A$. We will also abuse the notation and write $at^n, a \in I_n^l$, for their images in $A \rtimes L$. In fact we extend this notation to any $a \in I_n = C_0(\Delta_n)$ by writing $at^n$ for the limit in $A \rtimes L$ of a sequence $a_n t^n$ where $\{ a_n \}_{n=1}^\infty \subseteq I_n$ converges uniformly to $a$. So by Lemma 3.2 we have
\[ A \rtimes L = \overline{\text{span}\{ at^n t^m b : a \in I_n, b \in I_m, n, m \in \mathbb{N}_0 \}}. \]

**Proposition 3.5.** Assume that $A \rtimes L \subseteq B(H)$ is represented in a faithful and non-degenerate way on a Hilbert space $H$. The crossed product $A \rtimes L$ is the universal $C^*$-algebra for covariant representations of $L$:

(i) $A \rtimes L$ contains $A$ as a $C^*$-subalgebra, and is generated by $A$ and $It$ for $t \in B(H)$ such that $L(a) = t^* at, a \in I$ and $C_0(\Delta_{\text{reg}}) \subseteq \text{Tel}$. 

(ii) Every covariant representation $(\pi, T)$ of $L$ induces a representation $\pi \times T$ of $A \rtimes L$ where $\pi \times T(a) = \pi(a), a \in A$, and $\pi \times T(\pi(a)) = \pi(\pi(a))T, a \in I$.

Every $C^*$-algebra possessing properties [i], [ii] is isomorphic to $A \rtimes L$ by an isomorphism which is identity on $A$.

**Proof.** [i] and [ii] follow by construction. To see the last part, assume that $C = C^*(A \cup I)$ is a $C^*$-algebra, represented on a Hilbert space $K$, that satisfy analogues of [i], [ii]. Then [ii] for $A \rtimes L$ and $C$ give $*$-epimorphisms $\Psi : A \rtimes L \rightarrow C$ and $\Phi : C \rightarrow A \rtimes L$ which clearly are inverse to each other.

**Remark 3.6.** Proposition 3.5 shows that $A \rtimes L$ depends only on $L : I \rightarrow A$, or equivalently on $q : \Delta \rightarrow X$ (it depends only on $q$ up to continuous factors, see Corollary 4.6 below).

### 3.1. Cuntz-Pimsner picture and other constructions

Let $L : I \rightarrow A$ be a transfer operator for the partial endomorphism $\alpha : A \rightarrow M(I)$. The $C^*$-correspondence $M_L$ associated to $L$, cf. [Exe03, LRo07], is a Hausdorff completion of the $A$-bimodule $I$ where $a \cdot \xi \cdot b = \alpha \xi \cdot b$, for $\xi \in I, a, b \in A$, in the $A$-valued pre-inner product given by $\langle \xi, \eta \rangle_A := L(\xi^* \eta), \xi, \eta \in I$. A representation of $M_L$ is a pair $(\pi, \psi)$ where $\pi : A \rightarrow B(H)$ is a non-degenerate representation and $\psi : M_L \rightarrow B(H)$ is a (necessarily linear) map such that $\pi(a) \psi(\xi) \pi(b) = \psi(a \xi \cdot b)$ and $\psi(\xi) \psi(\eta) = \pi(\xi, \eta)_A$ for $a, b \in A, \xi, \eta \in M_L$.

**Lemma 3.7.** Every representation $(\pi, \psi)$ of $M_L$ comes from a representation $(\pi, T)$ of $L$ in the sense that $\psi(q(\xi)) = \pi(\xi) T, \xi \in I$, where $q : I \rightarrow M_L$ is the canonical quotient map. This gives a bijective correspondence between representations $(\pi, \psi)$ of $M_L$ and representations $(\pi, T)$ of $L$ satisfying $TH \subseteq \overline{\pi(I)H}$. 

Proof. If \((\pi, T)\) is a representation of \(L\), then \(\psi(q(\xi)) := \pi(\xi)T, \xi \in I,\) is well defined because \(\|\pi(\xi)T\|^2 = \|\pi(L(\xi^*\xi))\| \leq L(\xi^*\xi) = \|q(\xi)\|\). and clearly, \((\pi, \psi)\) is a representation of \(M_L\). Let \((\pi, \psi)\) be a representation of \(M_L\) and let \(\mu_\lambda\) be an approximate unit in \(I = C_0(\Delta)\). We claim that the net of operators \(T_\lambda := \psi(q(\mu_\lambda))\) is strongly Cauchy. Indeed, let \(h \in H\) and \(\lambda \leq \lambda',\) in the directed set \(\Lambda\). Then
\[
\|\left(T_\lambda - T_{\lambda'}\right)h\|^2 = \langle h, L(\mu_\lambda - \mu_{\lambda'})^2h \rangle \leq \langle h, L(\mu_\lambda - \mu_{\lambda'})h \rangle.
\]
Since the net \(\{L(\mu_\lambda)\}_{\lambda \in \Lambda}\) is strongly convergent the last expression tends to zero. Hence \(T := s\lim_{\lambda \in \Lambda} T_\lambda\) defines a bounded operator. For every \(a \in C_0(\Delta)\) we have
\[
T*aT = s\lim_{\lambda \in \Lambda} T_\lambda^n a T_\lambda = \lim_{\lambda \in \Lambda} L(\mu_\lambda a \mu_\lambda) = L(a).
\]
Thus \((\pi, T)\) is a representation of \(L\) satisfying \(TH \subseteq \pi(I)\overline{H}\).

Theorem 3.8. The crossed product \(A \rtimes_L\) is naturally isomorphic with Katsura’s Cuntz-Pimsner algebra \(O_{M_L}\). In particular, \(A \rtimes L\) is always nuclear, and satisfies the Universal Coefficient Theorem (UCT) if \(A\) is separable (equivalently \(X\) is second countable).

Proof. By [Kat04], Propositions 3.3 and 4.9 there is the largest ideal \(J_{M_L}\) in \(A\) such that for every faithful representation \((\pi, \psi)\) of \(M_L\) we have
\[
\{a \in A : \pi(a) \in \overline{\psi(M_L)\psi(M_L^*)}\} \subseteq J_{M_L}.
\]

The faithful representation \((\pi, \psi)\) of \(M_L\) is called covariant if the above inclusion is an equality. Hence by Lemma 3.7 and Propositions 2.9 (and Example 2.14) we have \(J_{M_L} = C_0(\Delta_{\text{reg}})\) and we have a bijective correspondence between covariant representations \((\pi, \psi)\) of \(M_L\) and covariant representations \((\pi, T)\) of \(L\) satisfying \(TH \subseteq \pi(I)\overline{H}\). By definition \(O_{M_L}\) is generated by the range of a universal covariant representation of \(M_L\). By Proposition 3.5 and Remark 2.2 \(A \rtimes L\) is generated by a universal covariant representations \((\pi, T)\) of \(L\) satisfying \(TH \subseteq \pi(I)\overline{H}\). This gives a natural isomorphism \(A \rtimes L \cong O_{M_L}\), cf. the last part of Proposition 3.5.

Since \(A\) is commutative (and hence nuclear), \(A \rtimes L \cong O_{M_L}\) is nuclear by [Kat04], Corollary 7.4. If \(A\) is separable, then satisfies the UCT by [Kat04], Proposition 8.8.

Remark 3.9. We have seen in the above proof that Katsura’s ideal \(J_{M_L}\) for \(M_L\) is \(C_0(\Delta_{\text{reg}})\). It also follows form the proof of Lemma 3.7 that the \(C^*\)-correspondence \(M_L\) is naturally isomorphic to \(\overline{\pi(I)H}\) with operations coming from the \(C^*\)-algebra \(A \rtimes L\).

Corollary 3.10. The crossed product \(A \rtimes L\) is naturally isomorphic to the crossed product \(O(A, \alpha, L)\) by the partial endomorphism \(\alpha\) defined in [ERo07].

Proof. The crossed product \(O(A, \alpha, L)\) in [ERo07] is defined to be \(O_{M_L}\).

Corollary 3.11. If \(\Delta = X\) and \(q > 0\) on a dense subset of \(X\), then \(A \rtimes L\) is naturally isomorphic to the Exel’s crossed product \(A \rtimes_{\alpha, L} N\) [Exe03], generalised to the non-unital case in [BRV10].

Proof. The assumptions mean that \(\alpha : A \to A\) is non-degenerate and \(L\) is faithful. Thus the assertion follows from [Kwa17], Proposition 4.9.

We naturally associate to \(L\) a topological correspondence in the sense of [BHM21], Definition 2.1, see also [CKO19], Subection 9.3. The underlying topological directed graph \((E^0, E^1, s, r)\) is the graph of \(\varphi:\)
\[
E^0 := X, \quad E^1 := \Delta, \quad r(x) := x, \quad s(x) := \varphi(x).
\]
It is equipped with the continuous family of measures \( \mu = \{ \mu_y \}_{y \in X} \) along fibers of \( \varphi \) given by \( \mu_y(a) := L(a)(y), a \in C_c(X) \). Note that we only have \( \text{supp} \mu_y \subseteq s^{-1}(y), y \in X \). Thus the topological correspondence \( Q := (X, \Delta, id, \varphi, \mu) \) is a topological quiver in the sense of [MT05] iff \( \text{supp} \mu_y = s^{-1}(y), y \in X \) iff \( \Delta = \Delta_{\text{pos}} \) (note that we use a convention where \( s \) and \( r \) play the opposite role in [MT05]).

**Lemma 3.12.** The \( C^* \)-correspondence \( M_Q \) associated to the topological correspondence \( Q = (X, \Delta, id, \varphi, \mu) \) in [BHM21, Definition 2.4], cf. [MT05] 3.1, coincides \( M_L \).

**Proof.** This follows immediately from the constructions (definitions). \( \square \)

**Corollary 3.13.** If \( \Delta = \Delta_{\text{pos}} \), so that \( Q = (X, \Delta, id, \varphi, \mu) \) is a topological quiver, then the crossed product \( A \rtimes L \) is naturally isomorphic to the quiver \( C^* \)-algebra associated to \( Q \) by Muhy and Tomforde in [MT05].

**Proof.** By definition the quiver \( C^* \)-algebra is the Cuntz-Pimsner algebra of \( M_Q \), which by Lemma 3.12 is equal to \( M_L \). Hence the assertion follows from Theorem 3.8. \( \square \)

**Remark 3.14.** Note that if \( \Delta_{\text{pos}} \) is locally compact, then \( Q_{\text{pos}} := (X, \Delta_{\text{pos}}, id, \varphi, \mu) \) is a topological quiver. Moreover, if \( \Delta_{\text{pos}} \subseteq \Delta \) is open, we may apply Corollary 3.13 to the restricted map \( \varphi : \Delta_{\text{pos}} \to X \), to conclude that \( A \rtimes L \) is the quiver algebra associated to \( Q_{\text{pos}} \).

If \( \Delta_{\text{pos}} \) is closed in \( \Delta \) and \( \Delta \) is normal, one may show that the \( C^* \)-correspondences \( M_L \) and \( M_{Q_{\text{pos}}} \) are isomorphic and hence \( A \rtimes L \) is again the quiver algebra of \( Q_{\text{pos}} \). We do not know whether \( A \rtimes L \) has a natural topological quiver model in general.

**Example 3.15** (Maps on Riemann surfaces). Let \( \varphi : \Delta \to X \) be a non-constant holomorphic map defined on an open connected subset \( \Delta \) of a Riemann surface \( X \) (so that \( \Delta \) is a Riemann surface as well). Let \( x \in \Delta \). By branching lemma, \( \varphi \) locally at \( x \) looks like \( z \to z^d \), and then \( m(x) := d \in \mathbb{N} \) is the degree of \( \varphi \) at \( x \). In particular, \( \varphi^{-1}(y) \) is a discrete set of \( \Delta \), for every \( y \in X \). Assume that \( \varphi \) is proper. Then it is surjective and the number \( d := \sum_{x \in \varphi^{-1}(y)} m(x) \), called the degree of \( \varphi \), does not depend on \( y \in X \) and is finite. In particular,

\[
L(a)(y) := \sum_{x \in \varphi^{-1}(y)} m(x)a(x), \quad a \in C_0(\Delta)
\]

defines a transfer operator for \( \varphi \), and \( \| L \| = d \). If \( \Delta = X = \hat{\mathbb{C}} = \overline{\mathbb{C}} \cup \{ \infty \} \) is the Riemann sphere, then \( \varphi \) is a rational function \( R : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) and the crossed product \( C(\hat{\mathbb{C}}) \rtimes L \) is isomorphic to the \( C^* \)-algebra \( \mathcal{O}_R(\hat{\mathbb{C}}) \) associated to \( R \) in [KW05] (which is \( \mathcal{O}_{ML} \) by definition). If \( R \) is of degree at least two and has an exceptional point, then \( R \) is conjugated either to a polynomial or a map \( z \to z^d \) for some \( d \in \mathbb{Z} \setminus \{0\} \). [Bea91, Theorem 4.1.2]. The rational map \( R \) (and the transfer operator \( L \)) restricts to the Julia set \( J_R \) and Fatou set \( F_R \) and the crossed products \( C(J_R) \rtimes L \) and \( C_0(F_R) \rtimes L \) to the \( C^* \)-algebras studied in [KW05].

**Example 3.16** (Branched coverings with finite system of branches). Let \( \varphi : \Delta \to X \) be a continuous surjective partial map such that \( \varphi^{-1} \) has a finite system of branches, i.e. there is a finite collection of partial maps \( \{ \gamma_i \}_{i=1}^N \) such that each \( \gamma_i : X \to \Delta \) is continuous injective and \( \varphi^{-1}(y) = \{ \gamma_i(y) : i = 1, ..., N \} \). Then \( g(x) := |\{ i : x \in \gamma_i(X) \}| \) defines a potential for \( \varphi \) as clearly

\[
L(a)(y) = \sum_{i=1}^N a(\gamma_i(y)) = \sum_{x \in \varphi^{-1}(y)} g(x)a(x), \quad a \in C_0(\Delta),
\]
defines a transfer operator $L : C_0(\Delta) \to C_0(X)$ for $\varphi$. If $X = \Delta$ is compact and $\gamma_i$ is a proper contraction, for all $i$, the crossed product $A \times L$ is naturally isomorphic to the the $C^*$-algebra associated to the self-similar set $X$ in [FW06] (it is defined there as $O_{M_\lambda}$). The model example is the tent map $\varphi : [0, 1] \to [0, 1]$ where $\varphi(x) = 1 - |1 - 2x|$ and $L(a)(y) = a(y) + a(1 - y)$.

If a map has infinitely many branches one may define a transfer operator by using a scaling function that will make the sums converge:

**Example 3.17.** Let $X = [0, 1]$, $\Delta = (0, 1]$ and $\varphi(x) = \sin \frac{x}{2}$. On may define a transfer operator for $\varphi$ by the formula $L(a)(y) = 2^{[y=\pm 1]} \sum_{x \in \varphi^{-1}(y)} e^{-1/x} a(x)$.

4. INVARIANCE UNIQUENESS THEOREMS AND THE REGULAR REPRESENTATION

An important consequence of universality of $A \times L$ is that it is equipped with a circle gauge action $\gamma : \mathbb{T} \to \text{Aut}(A \times L)$. Namely, for each $\lambda \in \mathbb{T}$ the pair $(\text{id}_A, \lambda t)$ may be treated as covariant representation of $L$. Hence by Proposition 3.1(ii) there is an $*$-epimorphism $\gamma_\lambda : A \times L \to A \times L$ such that

$\gamma_\lambda|_A = \text{id}_A$, and $\gamma_\lambda(at) = \lambda at$, $a \in I$.

Moreover, we clearly have $\gamma_1 = \text{id}_{A \times L}$ and $\gamma_{\lambda_1} \circ \gamma_{\lambda_2} = \gamma_{\lambda_1 \lambda_2}$ for $\lambda_1, \lambda_2 \in \mathbb{T}$. Thus $\gamma : \mathbb{T} \to \text{Aut}(A \times L)$ is a group homomorphism. Its fixed points form a $C^*$-algebra $A_\infty := \{x \in A \times L : \gamma_\lambda(x) = x \text{ for all } \lambda \in \mathbb{T}\}$. We call $A_\infty$ the core $C^*$-subalgebra of $A \times L$. It is well known, see, for instance [Rae05 Proposition 3.2], that the formula $E(x) := \int_{\mathbb{T}} \gamma_\lambda(x) d\lambda$ defines a faithful conditional expectation onto $A_\infty$. That is, $E$ is norm one projection onto $A_\infty$, which is necessarily a completely positive $A_\infty$-bimodule map, see [Tak02, III, Theorem 3.4, IV, Corollary 3.4]. Faithfulness here means that $E(a^*a) = 0$ implies $a = 0$ for all $a \in A \times L$.

**Proposition 4.1.** We have

$A_\infty = \text{span}\{at^n t^m b : a, b \in I_n, n \in \mathbb{N}_0\}$,

and the conditional expectation $E : A \times L \to A_\infty$ is the unique contractive projection onto $A_\infty$ such that $E(at^n t^m b) = 0$ for $n \neq m$ ($a \in I_n$, $b \in I_m$).

**Proof.** We have $E(at^n t^m b) = at^n t^m b \int_{\mathbb{T}} \lambda^{n-m} d\lambda$ which is zero when $n \neq m$ and $at^n t^m b$ when $n = m$. This determines $E$ and implies that $A_\infty = \text{span}\{at^n t^m b : a, b \in I_n, n \in \mathbb{N}_0\}$. □

The gauge-invariance uniqueness for Cuntz-Pimsner algebras implies the following

**Theorem 4.2.** Let $(\pi, T)$ be a faithful covariant representation of $L$ and let $C^*_{\pi}(\pi, T)$ be the $C^*$-algebra generated by $\pi(A) \cup (\pi(I)T)$. Then $\pi \times T$ is faithful on the core subalgebra $A_\infty$ of $A \times L$, and the following conditions are equivalent:

(i) $\pi \times T$ is an isomorphism, i.e. $A \times L \cong C^*_{\pi}(\pi, T)$;

(ii) $C^*_{\pi}(\pi, T)$ is equipped with a circle gauge-action, i.e. a group homomorphism $\gamma : \mathbb{T} \to \text{Aut}(C^*_\pi(\pi, T))$ where $\gamma_{\pi(A)} = \text{id}_{\pi(A)}$ and $\gamma_{\pi(\pi(a)T)} = z \pi(a)T$, for $z \in \mathbb{T}$, $a \in I$;

(iii) There is a conditional expectation from $C^*_{\pi}(\pi, T)$ onto $(\pi \times T)(A_\infty) \subseteq C^*_{\pi}(\pi, T)$ that annihilates all the operators of the form $\pi(\pi(a)T^n m \pi(b))$ with $n \neq m$, $a \in I_m$, $b \in I_n$.

**Proof.** Faithfulness of $\pi \times T$ on $A_\infty$ follows from Theorem 3.8 and [Kat04, Theorem 6.4]. Implications [(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)] are obvious, cf. Proposition 4.1. Assume (iii) and denote by $E_\pi$ the conditional expectation from $C^*(\pi, T)$ onto $(\pi \times T)(A_\infty)$. Then $E_\pi \circ \pi \times T = \pi \times T \circ E_\pi$, and this composite map is faithful because $E$ is faithful and $\pi \times T$ is faithful on the range of $E$. This implies that $\pi \times T$ is faithful on the whole of $A \times L$. □
Corollary 4.3. For each \( n \in \mathbb{N} \), \( A \times L^n \) is naturally isomorphic to the \( C^* \)-subalgebra of \( A \times L \) generated by \( A \cup I_n t^n \).

Proof. By (21), we see that \( (id, t^n) \) is a faithful representation of \( L^n \) into \( A \times L \). We use Proposition 2.12(ii) to show that the representation \( (id, t^n) \) is covariant. To this end note that the set of regular points for \( \tilde{\varphi}_n \) is \( \Delta_{reg}^n = \{ x \in \Delta_n : x, \varphi(x), ..., \varphi^{n-1}(x) \in \Delta_{reg} \} \). Let \( a \in C_c(\Delta_{reg}^n) \) with support \( K_0 \) contained in an open set \( U \subseteq \Delta_{reg}^n \) where \( \varphi^n|_U \) is injective.

Put \( a_0 := a \) and for each \( k = 1, ..., n - 1 \) let \( a_k \in C_0(\varphi^k(U)) \) be such that \( a_k|_{\varphi^k(K)} \equiv 1 \), so that then we have \( a = \prod_{k=0}^{n-1} a_k(a_k) \). For each \( k = 0, ..., n - 1 \), the map \( \varphi|_{\varphi^k(U)} \) is injective.

Hence by Proposition 2.12(ii) there is \( a_k \in C_0(\Delta_n) \) such that \( a_k = a_k t^n u_k \). In particular, since \( a_0 = a \in C_0(\Delta_n) \), we may assume that \( u_0 \in C_0(\Delta_n) \). Then \( u := \prod_{k=0}^{n-1} a_k(u_k) \) is well defined and using (21) we get

\[
\text{id} t^n u = a_0 (\text{id} t\ldots a_{n-1} t^* u_{n-1} \ldots t^* u_1 t^*) u_0 = \prod_{k=0}^{n-1} a_k(a_k) = a.
\]

Hence \( (id, t^n) \) is a covariant representation of \( L^n \). It is equipped with a circle gauge-action. Indeed, if \( \gamma : \mathbb{T} \to \text{Aut}(A \times L) \) is the gauge-action \( \gamma^n \) on \( A \times L \), then the desired gauge action on \( \text{span}\{a_0 t^n b : a \in I_n, b \in I_n, m \in \mathbb{N}_0\} \) can be defined by the formula \( \gamma^n(b) := \gamma_0(b) \), for any \( \lambda \in \mathbb{T} \) and \( z \in \mathbb{T} \) such that \( z^n = \lambda \). Hence we have the natural isomorphism \( A \times L^n \cong \text{span}\{a_0 t^n b : a \in I_n, b \in I_n, m \in \mathbb{N}_0\} \) by Theorem 4.2.

\[\square\]

4.1. Regular representation and generalised expectations. The orbit representation \( (\pi_o, T_o) \) defined in Example 2.14 in general does give a faithful representation of \( A \times L \). Tensoring it with the regular representation \( \lambda \) of \( \mathbb{Z} \) guarantees that:

Definition 4.4. The regular representation of the transfer operator \( L \) is the pair \( (\tilde{\pi}, \tilde{T}) \) where \( \tilde{\pi} : C_0(X) \to B(H) \) and \( \tilde{T} \in B(H) \) act on \( H := \ell^2(X) \otimes \ell^2(\mathbb{Z}) \cong \ell^2(X \times \mathbb{Z}) \) by \( \tilde{\pi} = \pi_o \otimes id_{\ell^2(\mathbb{Z})} \) and \( \tilde{T} = T_o \otimes \lambda \). Thus using the standard orthonormal basis \( \{1_{x,n}\}_{x \in X, n \in \mathbb{Z}} \) of \( H \) we have

\[
\tilde{\pi}(a) 1_{x,n} = a(x) 1_{x,n}, \quad \tilde{T} 1_{y,n} = \sum_{x \in \varphi^{-1}(y)} \sqrt{\varphi(x)} 1_{x,n+1}.
\]

Proposition 4.5. The regular representation \( (\tilde{\pi}, \tilde{T}) \) is a faithful covariant representation of \( L \) that extends to a faithful representation \( \tilde{\pi} \times \tilde{T} \) of \( A \times L \), so \( A \times L \cong C^*(\tilde{\pi}, \tilde{T}) \).

Proof. Using that \( (\pi_o, T_o) \) is a faithful covariant representation of \( L \), one readily concludes that \( (\tilde{\pi}, \tilde{T}) \) is also a faithful covariant representation of \( L \). By Theorem 4.2 to prove that \( \tilde{\pi} \times \tilde{T} \) is faithful it suffices to show that \( C^*(\tilde{\pi}, \tilde{T}) \) has the appropriate gauge action. To this end, for each \( z \in \mathbb{T} \) we define a unitary operator \( U_z \in B(\ell^2(X \times \mathbb{Z})) \) by the formula \( U_z 1_{x,n} := z^n 1_{x,n}, \quad x \in X, \quad n \in \mathbb{Z} \). Putting \( \gamma_z(b) := U_z b U_z^* \), \( b \in C^*(\tilde{\pi}, \tilde{T}) \), we get \( \gamma_z|_{\varphi(A)} = id_{\varphi(A)} \) and \( \gamma_z(\pi(a) \tilde{T}) = z \pi(a) \tilde{T} \) for \( z \in \mathbb{T} \) and \( a \in I \). Hence \( \gamma : \mathbb{T} \to \text{Aut}(C^*(\tilde{\pi}, \tilde{T})) \) is the desired homomorphism. \(\square\)

Corollary 4.6. Let \( L \) and \( L' \) be transfer operators for a fixed partial map \( \varphi : \Delta \to X \) and let \( \varphi, \varphi' : \Delta \to [0, +\infty) \) be the corresponding potentials. Assume that there is a continuous strictly positive map \( \omega : \Delta \to (0, \infty) \) such that \( \varphi' = \varphi \omega \). Then

\[
C^*(\tilde{\pi}, \tilde{T}) = C^*(\tilde{\pi}, \tilde{T}'),
\]

where \( (\tilde{\pi}, \tilde{T}) \) and \( (\tilde{\pi}, \tilde{T}') \) are regular representations of \( L \) and \( L' \) respectively. Thus \( C_0(X) \times L \) and \( C_0(X) \times L' \) are naturally isomorphic.
Proof. For any $a \in C_c(\Delta)$ we have $a\omega^{1/2}, a\omega^{-1/2} \in C_c(\Delta)$, $\tilde{\pi}(a)\tilde{T}' = \tilde{\pi}(a\omega^{1/2})\tilde{T}$ and $\tilde{\pi}(a)\tilde{T} = \tilde{\pi}(a\omega^{-1/2})\tilde{T}'$. Hence $\tilde{\pi}(C_c(\Delta))\tilde{T}' = \tilde{\pi}(C_c(\Delta))\tilde{T}$ which implies $\tilde{\pi}(C_0(\Delta))\tilde{T}' = \tilde{\pi}(C_0(\Delta))\tilde{T}$ and this gives the assertion. \hfill \Box

Using the regular representation we prove existence of a canonical faithful completely positive map from $C_0(X) \rtimes L$ to the $C^*$-algebra $B(X)$ of all bounded Borel complex valued maps on $X$. We denote by $\delta_{i,j}$ the Kronecker symbol.

**Lemma 4.7.** There is a faithfully completely positive map $G : C_0(X) \rtimes L \to B(X)$ such that

$$G(at^{k}t^{l}b) = \delta_{k,l} \cdot ab_0$$

for all $a \in I_k, b \in I_l$ and $k,l \in \mathbb{N}_0$. In particular, $G$ is a (genuine) conditional expectation from $C_0(X) \rtimes L$ onto $C_0(X)$ iff $\varrho : \Delta \to [0, +\infty)$ is continuous.

Proof. In view of Proposition 4.5 we may identify $\varphi$ for the generalised expectation. Let $\pi$ be a faithful covariant representation of $\varphi$. If $b \in F(0) = 0$, which by faithfulness of $\varphi$ implies that $b = 0$. Hence $\pi \times T$ is faithful. \hfill \Box

**Remark 4.8.** The above map $G$ is an identity on $A = C_0(X) \subseteq B(X)$. Therefore $G$ is a generalised expectation for the $C^*$-inclusion $A \subseteq A \rtimes L$ in the sense of [KM21] Definition 3.1.

**Theorem 4.9.** Let $(\pi, T)$ be a faithful covariant representation of $L$. Then $\pi \times T : A \rtimes L \to C^*(\pi, T)$ is faithful if and only if there is a bounded linear map $F : C^*(\pi, T) \to B(X)$ such that $F(\pi(a)T^kT'^l\pi(b)) = \delta_{k,l} \cdot ab_0$, for all $a \in I_k, b \in I_l, k, l \in \mathbb{N}_0$.

Proof. If $\pi \times T$ is faithful, then $F$ exists by Lemma 4.7. Conversely, if $F$ exists, then for any $b \in A \rtimes L$ with $\pi \times T(b) = 0$, we have $G(b^*b) = F(\pi \times T(b^*b)) = F(\pi \times T(b)^* \pi \times T(b)) = F(0) = 0$, which by faithfulness of $G$ implies that $b = 0$. Hence $\pi \times T$ is faithful. \hfill \Box

5. Local homeomorphisms and the groupoid model

In this section we assume that $\varphi : \Delta \to X$ is a local homeomorphism. We show that the groupoid $C^*$-algebra associated in [Ren00] to $(X, \varphi)$ is naturally isomorphic to the crossed product of $C_0(X)$ by a transfer operator. We first discuss existence of a transfer operator for $(X, \varphi)$. 
Lemma 5.1 (cf. [ER07, Lemma 2.1]). Let \( \varphi : \Delta \to X \) be a local homeomorphism. For any continuous function \( \varphi : \Delta \to [0, +\infty) \) with \( \sup_{y \in X} \sum_{x \in \varphi^{-1}(y)} \varphi(x) a(x) < \infty \), the formula
\[
L(a)(y) = \sum_{x \in \varphi^{-1}(y)} \varphi(x) a(x)
\]
defines a transfer operator \( L : C_0(\Delta) \to C_0(X) \) for \( \varphi \). Moreover every transfer operator for \( \varphi \) is of the above form (even if we drop our standing assumption (5)).

Proof. For each \( a \in C_0(\Delta) \) and \( y \in Y \) we have \( |L(a)(y)| \leq \sum_{x \in \varphi^{-1}(y)} |\varphi(x)| a(x) \leq \|a\| \cdot M \) where \( M := \sup_{y \in X} \sum_{x \in \varphi^{-1}(y)} \varphi(x) \). Hence \( L \) is a well defined bounded linear operator from \( C_0(\Delta) \) to the space of bounded functions on \( X \). Clearly, \( L \) is positive and satisfies the transfer identity (2). Thus it suffices to show that \( L(a) \) is continuous on \( X \) for any \( a \in C_c(\Delta) \), see Remark 1.1(1). Let \( K \) be the compact support of \( a \) and take any \( y \in X \). If \( y \notin \varphi(K) \), then \( L(a)(y) = 0 \) and as \( X \setminus \varphi(K) \) is open, \( L(a) \) is continuous at \( y \). Assume then that \( y \in \varphi(K) \). Since \( \varphi \) is a local homeomorphism, \( \varphi^{-1}(y) \cap K \) is finite, and we may find pairwise disjoint, non-empty open sets \( \{U_i\}_{i=1}^n \) covering \( \varphi^{-1}(y) \cap K \) and such that \( \varphi|_{U_i} \) is injective for any \( i = 1, \ldots, n \). By [ER07, Lemma 2.1] we may find open \( V \subseteq \bigcap_{i=1}^n \varphi(U_i) \) containing \( y \) and such that \( \varphi^{-1}(V) \cap (K \setminus \bigcup_{i=1}^n U_i) = \emptyset \). So \( \varphi^{-1}(V) \cap K \subseteq \bigcup_{i=1}^n U_i \). Using this we see that \( L(a)|_V = \sum_{i=1}^n (\varphi \circ \varphi^{-1}|_{U_i} \cdot (a \circ \varphi|_{U_i}^{-1}) \). Since the latter sum is finite and involves only continuous functions, \( L(a)|_V \) is continuous. This finishes the proof of the first part.

For the second part note that since \( \varphi \) is a local homeomorphism, for every \( y \in X \), \( \varphi^{-1}(y) \) is discrete. Hence the measures in (4) have to be discrete, here we do not need our standing assumption (5). Thus every transfer operator \( L \) for \( \varphi \) is of the form (5) and the associated potential \( \varphi \) is continuous by Proposition 1.3. In addition, \( \sup_{y \in X} \sum_{x \in \varphi^{-1}(y)} \varphi(x) = \|L\| < \infty \).

The important question is whether we can find a strictly positive continuous \( \varphi : \Delta \to (0, +\infty) \) with \( \sup_{y \in X} \sum_{x \in \varphi^{-1}(y)} \varphi(x) < \infty \). Note that a necessary condition for this is our standing assumption (5). We answer this question in the affirmative in two important cases.

Example 5.2. If \( \varphi : \Delta \to X \) is a proper local homeomorphism, then for each \( y \in X \) the preimage \( \varphi^{-1}(y) \) is finite and in fact the map \( X \ni y \mapsto |\varphi^{-1}(y)| \in \mathbb{N}_0 \) is continuous (locally constant), see [BRV10, Lemma 2.2] where it is assumed that \( \Delta = \varphi(\Delta) = X \) but the proof works in our setting. Thus putting \( \varphi(x) := |\varphi^{-1}(\varphi(x))|^{-1}, x \in \Delta \), we get a continuous strictly positive function \( \varphi > 0 \) such that \( \sum_{x \in \varphi^{-1}(y)} \varphi(x) = 1 \) for every \( y \in \varphi(\Delta) \). The corresponding transfer operator is given by the formula
\[
(22) \quad L(a)(y) = \frac{1}{|\varphi^{-1}(y)|} \sum_{x \in \varphi^{-1}(y)} a(x), \quad a \in C_0(\Delta).
\]
If \( \varphi \) is not proper, (22) fails to define a transfer operator even if we assume \( \sup_{y \in X} |\varphi^{-1}(y)| < \infty \) (the function \( L(a)(y) \) may be discontinuous).

Example 5.3. Let \( \varphi : \Delta \to X \) be any local homeomorphism, but assume that there is a partition of unity \( \{f_n\}_{n=1}^\infty \) subordinated to a countable cover \( \{U_n\}_{n=1}^\infty \) of \( \Delta \) such that \( \varphi|_{U_n} \) is injective. Such a partition exists if \( \Delta \) is second countable or more generally if \( \Delta \) is \( \sigma \)-compact. Then
\[
\varphi(x) = \sum_{n=1}^\infty \frac{1}{|\varphi|_n f_n(x), \quad x \in \Delta,
\]
defines a continuous strictly positive function \( \varphi : \Delta \to (0, +\infty) \) such that for every \( y \in X \) we have \( \sum_{x \in \varphi^{-1}(y)} \varphi(x) \leq 1 \). Thus \( \varphi \) yields a transfer operator.
For an introduction to the theory of étale, locally compact, Hausdorff groupoids we recommend [Sim20]. For any such a groupoid $G$ the groupoid $C^*$-algebra $C^*_r(G)$ is the maximal $C^*$-completion of the $*$-algebra $C_c(G)$ with operations:

$$(f * g)(\gamma) = \sum_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1)g(\gamma_2) \quad \text{and} \quad f^*(\gamma) = \overline{f(\gamma^{-1})},$$

where $\gamma, \gamma_1, \gamma_2 \in G$, $f, g \in C_c(G)$. Then the embedding $C_c(G) \subseteq C_0(G)$ extends to a contractive embedding $C^*_r(G) \subseteq C_0(G)$, so that we may view elements of $C^*_r(G)$ as functions on $G$ and the formulas for algebraic operations remain valid. Also, identifying $X$ with $G^0 := \{(x, 0, x) : x \in X\}$, $C_0(X) \subseteq C^*_r(G)$ is a non-degenerate $C^*$-subalgebra and there is a conditional expectation $F$ from $C^*_r(G)$ onto $C_0(X)$ given by restriction of functions. This conditional expectation $F$ is faithful if $G$ is amenable.

The transformation groupoid or Renault-Deaconu groupoid associated to $(X, \varphi)$ is an étale, amenable, locally compact, Hausdorff groupoid, see [Ren00], where

$$G := \{(x, n - m, y) : n, m \in \mathbb{N}_0, x \in \Delta_n, y \in \Delta_m, \varphi^n(x) = \varphi^m(y)\},$$

the groupoid structure is given by $(x, n, y)(y, m, z) := (x, n + m, z)$, $(x, n, y)^{-1} := (y, -n, x)$, and the topology is defined by the basic open sets $\{(x, n - m, y) : (x, y) \in U \times V, \varphi^n(x) = \varphi^m(y)\}$ where $U \subseteq \Delta_n$, $V \subseteq \Delta_m$ are open sets such that $\varphi^n|_U$ and $\varphi^m|_V$ are injective. For full local homeomorphisms on compact spaces the isomorphism in the following theorem is well known, see [EV06], [ERe07], [BK21].

**Theorem 5.4.** Assume $\varphi : \Delta \to X$ is a local homeomorphism and let $g : \Delta \to (0, +\infty)$ be any strictly positive continuous map with $\sup_{y \in X} \sum_{x \in \varphi^{-1}(y)} g(x) < \infty$ (such a map always exists when $\varphi$ is proper or $\Delta$ is $\sigma$-compact). Then $L(a)_y = \sum_{x \in \varphi^{-1}(y)} g(x)a(x)$ is a well defined transfer operator $L : C_0(\Delta) \to C_0(X)$ for $\varphi$, and we have an isomorphism

$$C^*_r(G) \cong C_0(X) \rtimes L$$

where $G$ is the Renault-Deaconu groupoid associated to $\varphi$. This isomorphism is determined by the formula

$$\Phi(a_n \otimes b_m) := a_n g_n^{-\frac{1}{2}} t^m g_m^{-\frac{1}{2}} b_m, \quad a_n \in C_c(\Delta_n), b_m \in C_c(\Delta_m), n, m \in \mathbb{N}_0,$$

where $(a_n \otimes b_m)(x, k, y) = \delta_{k, n - m} \cdot a_n(x)b_m(y)$.

**Proof.** Let us assume that $C^*_r(G) \subseteq B(H)$ is represented in a faithful and non-degenerate way on some Hilbert space $H$. Let $\{\mu_{\lambda}\}_{\lambda \in \Lambda} \subseteq C_c(\Delta)$ be an approximate unit in $I$ and consider the net of functions $\{T_{\lambda}\}_{\lambda \in \Lambda} \subseteq C_c(G)$ given by $T_{\lambda}(x, 1, \varphi(x)) = \mu_{\lambda}(x)g(x)^{\frac{1}{2}}$ and $T_{\lambda}(x, n, y) = 0$ if $(n, y) \neq (1, \varphi(x))$. We claim that $\{T_{\lambda}\}_{\lambda \in \Lambda}$ is strongly Cauchy. Indeed, let $a \in A$, $h \in H$ and $\lambda \leq \lambda'$, in the directed set $\Lambda$. We have $T_{\lambda}^* a T_{\lambda'} = L(\mu_{\lambda} a_{\lambda'})$ in the $*$-algebra $C_c(G)$. Thus

$$\| (T_{\lambda} - T_{\lambda'}) ah \|^2 = \langle h, L(\alpha(a^*) (\mu_{\lambda'} - \mu_{\lambda})^2 \alpha(a)) h \rangle \leq \langle h, a^* L(\mu_{\lambda'} - \mu_{\lambda}) ah \rangle = \langle ah, L(\mu_{\lambda'} - \mu_{\lambda}) ah \rangle.$$

Since the net $\{L(\mu_{\lambda})\}_{\lambda \in \Lambda}$ is strongly convergent the last expression tends to zero. Hence $T := \lim_{\lambda \in \Lambda} T_{\lambda}$ defines a bounded operator. For every $a \in C_0(\Delta)$ we have

$$T^* \alpha T = \lim_{\lambda \in \Lambda} T_{\lambda}^* \alpha T_{\lambda} = \lim_{\lambda \in \Lambda} L(\mu_{\lambda} a_{\lambda}) = L(a).$$
If $a$ is supported on a set $K$ such that $\varphi|_K$ is injective, then taking $u \in C_c(\Delta_{reg})$ such that $u|_K = (q|_K)^{-1}$ we get $aTT^*u = s\lim_{\lambda \in A} aT_\lambda u = \lim_{\lambda \in A} \mu_\lambda a = a$. Hence $(\mathrm{id}, T)$ is a covariant representation of $L$ by Proposition 2.12. Thus we have a $*$-homomorphism $\mathrm{id} \times T : C_0(X) \rtimes L \to B(H)$. It takes values in $C^*(G)$ because if $a \in C_c(\Delta)$, then $aT \in C_c(G)$ where $aT(x, k, y) = \delta_{(k, y), (1, \varphi(x))} \cdot a(x)\varrho(x)^{\frac{1}{2}}$. More generally, one readily checks that for $a_n \in C_c(\Delta_n)$, $b \in C_c(\Delta_m)$ we have $a_n g_n^{-\frac{1}{2}}T^m T^* g_m^{-\frac{1}{2}} b_m = a_n \otimes b_m \in C_c(G)$. Since functions $a_n \otimes b_m$ span $C_c(G)$ we conclude that $\mathrm{id} \times T : C_0(X) \rtimes L \to C^*(G)$ is a surjective $*$-homomorphism that intertwines the conditional expectations $G : C_0(X) \times L \to C_0(X)$ and $F : C^*(G) \to C_0(X)$. Hence $\mathrm{id} \times T$ is an isomorphism by Corollary 11.9. Its inverse is as described in the assertion.

**Example 5.5** (Deaconu-Muhly $C^*$-algebras associated with branched coverings). We consider a slightly more general situation than in [DM01] and by a branched self-covering we mean a continuous open and surjective map $\varphi : X \to X$ of a locally compact, $\sigma$-compact space, for which there is a closed set $S \subseteq X$ such that $\varphi|_{X \setminus S}$ is a local homeomorphism. The $C^*$-algebra $DM(X, \varphi)$ associated to $\sigma$ in [DM01] is by definition the $C^*$-algebra of the Renault-Deaconu groupoid associated to the partial local homeomorphism $\varphi : X \setminus S \to X$. Thus by Theorem 5.4 we have

$$DM(X, \varphi) \cong C_0(X) \rtimes L,$$

where $L : C_0(X) \to C_0(X)$ is any transfer operator for $\varphi : X \to X$ given by continuous $\varrho : X \to [0, +\infty)$ with $S = \varrho^{-1}(0)$. If in addition, $S$ has empty interior, then $C_0(X) \rtimes L$ is Exel’s crossed product.

**Example 5.6** (Graph $C^*$-algebras). Let $E = (E^0, E^1, r, s)$ be a countable directed graph ($r, s : E^1 \to E^0$ are range and source maps). The boundary space $\partial E = E^\infty \cup E_s^\infty \cup E_m^\infty$ of $E$, cf. [Web13], [Bro12, Subsection 4.1] or [Kwa17], as a set consist of all infinite paths and of finite paths that start in sources or in infinite emitters. It is a locally compact Hausdorff space with topology generated by cylinder sets and their complements. The one-sided topological Markov shift associated to $E$ is the map $\sigma : \partial E \setminus E^0 \to \partial E$ defined, for $\mu = \mu_1\mu_2\ldots \in \partial E \setminus E^0$, by the formulas

$$\sigma(\mu) := \mu_2\mu_3\ldots \text{ if } \mu \notin E^1, \quad \text{and} \quad \sigma(\mu) := s(\mu_1) \text{ if } \mu = \mu_1 E^1.$$

This is a countable-to-one local homeomorphism. So we have a partial endomorphism $\alpha : C_0(\partial E) \to M(C_0(\partial E \setminus E^0))$. One may always find strictly positive numbers $\lambda = \{\lambda_e\}_{e \in E^1}$, such that the formula

$$L(a)(\mu) = \sum_{e \in E^1, \epsilon \in \partial E} \lambda_e a(\epsilon \mu)$$

defines a bounded map $L : C_0(\partial E \setminus E^0) \to C_0(\partial E)$ ([Kwa17, Proposition 5.4] characterises when this happens), and then $L$ is a transfer operator for $\sigma$. By [Kwa17, Theorem 5.6] we then also have

$$C^*(E) \cong C_0(\partial E) \rtimes L,$$

where $C^*(E)$ is the graph $C^*$-algebra - the universal $C^*$-algebra generated by partial isometries $\{s_e : e \in E^1\}$ and mutually orthogonal projections $\{p_v : v \in E^1\}$ such that $s_e^*s_e = p_s(e), s_es_e^* \leq p_{r(e)}$ and $p_v = \sum_{r(e)=v} s_es_e^*$ whenever the sum is finite.

**Example 5.7** (Exel-Laca $C^*$-algebras). Let $I$ be any set and let $A = \{A(i, j)_{i,j \in I}\}$ be a $\{0, 1\}$-matrix over $I$ with no identically zero rows. The Exel-Laca algebra $O_A$ is the universal
\[ C^*\text{-algebra generated by partial isometries } \{s_i : i \in I\} \text{ with commuting initial projections and mutually orthogonal range projections satisfying } s_i^*s_is_j^*s_j = A(i, j)s_j^*s_j^* \text{ and} \]

\[ \prod_{i \in E} s_i^*s_i \prod_{j \in F} (1 - s_j^*s_j) = \sum_{k \in I} \prod_{i \in E} A(i, k) \prod_{j \in F} (1 - A(j, k))s_k^*s_k^* \]

whenever \( E, F \subseteq I \) are finite sets such that \( \prod_{i \in E} A(i, k) \prod_{j \in F} (1 - A(j, k)) \) is non zero only for finitely many \( k \in I \). For any word \( \alpha = \alpha_1...\alpha_n \) in \( I \) admissible by \( A \) we put \( s_\alpha = s_{\alpha_1}...s_{\alpha_n} \). Then

\[ D_A := \overline{\text{span}} \{ s_\alpha \left( \prod_{i \in E} s_i^*s_i \right) s_\alpha^* : E \subseteq I \text{ is a finite set}, \alpha \text{ is a finite word} \} \]

is a commutative \( C^*\)-subalgebra of \( O_A \). The spectrum \( X \) of this algebra is a second countable totally disconnected space described in \[14.99\], as a certain subset of \( \{0, 1\}^F \) where \( F \) is a free group generated by \( I \). It is also described in \[Ren00\] as a spectrum of a certain Boolean algebra that model a Markov shift. In particular, there is a naturally associated partial local homeomorphism \( \varphi : \Delta \to X \) defined on an open dense subset \( \Delta \subseteq X \). The space of infinite admissible words

\[ X_A := \{ \omega \in I^\mathbb{N} : A(\omega_n, \omega_{n+1}) = 1 \text{ for } n \in \mathbb{N} \} \]

embeds naturally into \( X \) (and is dense in \( X \) when \( A \) is irreducible), in a way that

\[ \varphi(\omega_1\omega_2...) = \omega_2\omega_3..., \quad \text{for } \omega \in X_A \cap \Delta. \]

Moreover, by \[Ren00\] Proposition 4.8 \( O_A \) is isomorphic to the \( C^*\)-algebra \( C^*(G) \) of the Renault-Deaconu groupoid associated to \( \varphi \). Thus by Theorem 5.23 \( O_A \) is isomorphic to the crossed product \( C_0(X) \rtimes L \) for a certain transfer operator \( L \) for \( \varphi \). Such an isomorphism is described in \[ERo07\] Proposition 2.13 for an unbounded transfer operator, and one can make the operator bounded by choosing appropriate potential \( \rho \). For instance, as in Example 5.6 it suffices to choose positive numbers \( \lambda = \{\lambda_i\}_{i \in I} \), such that the formula

\[ T_\lambda := \sum_{i \in I} \sqrt{\lambda_i} s_i \]

converges strictly in \( O_A \), cf. \[Kwa17\] Proposition 5.4. Then \( L(a) := T_\lambda(a)T_\lambda^* \) defines a bounded transfer operator for \( \varphi \) and \( O_A \cong C_0(X) \rtimes L \).

6. Spectra of the core subalgebras

We now proceed to the analysis of the internal structure the core subalgebra \( A_\infty \) of \( A \rtimes L \). The fundamental fact is that \( A_\infty = \bigcup_{n \in \mathbb{N}_0} A_n \) is a direct limit of algebras that can be further decomposed into ‘liminary pieces’. Namely, for each \( n \in \mathbb{N}_0 \) we put

\[ K_n := \overline{\text{span}} \{ s_i^*s_is_i^*s_i^* : i \in I, a, b \in I, k, m \in \mathbb{N}_0 \} \]

Proposition 6.1. For each \( n \in \mathbb{N}_0 \), \( A_n \) and \( K_n \) are \( C^*\)-algebras of \( A_\infty \). Moreover, \( K_nK_m = K_{m+n} \) for \( n \leq m \) and \( A_n \cap K_{n+1} = K_n \cap K_{n+1} = K_n \cap \overline{\text{span}} \{ s_i^*s_is_i^*s_i^* : i \in I \} \).

Proof. By Remark 3.9 \( J_{ML} = C_0(\Delta_{\text{reg}}) \) and we have an isomorphism of \( C^*\)-correspondences \( M_L \cong T \). By Corollary 1.3 this implies that for any \( n \in \mathbb{N} \) we have \( M_{L^n} \cong T_n \), and so also \( M_{L^{n+1}} \cong M_{L^n} \), see \[16\], \[17\] and \[18\]. In particular, \( K_n \) is naturally isomorphic with compact operators on \( M_{L^n} \), and the assertion follows from the corresponding facts for Cuntz-Pimsner algebras, see, for instance, \[Kat04\], Lemma 5.4, Propositions 5.9, 5.11].
Recall that $I_n = C_0(\Delta_n)$, where $\Delta_n$ is the domain of $\varphi^n$, and $I_0 = A = C_0(X)$. We put

$$\Delta_{\text{pos,n}} := \Delta_n \setminus \varrho_n^{-1}(0) = \{ x \in \Delta_n : \varrho_n(x) > 0 \}, \quad n \in \mathbb{N}_0,$$

which is the natural domain for the $n$-th iterate of the partial map $\varphi|_{\Delta_{\text{pos,n}}}$ where $\Delta_{\text{pos}} := \{ x \in \Delta : \varrho(x) > 0 \}$. Using the transfer identity, we see that the closure of $L^n(I_n)$ is an ideal in $A$. Its spectrum is

$$\overline{I^n(I_n)} = \{ y \in X : \varphi^{-n}(y) \setminus \varrho_n^{-1}(0) \neq \emptyset \} = \varphi^n(\Delta_{\text{pos,n}}),$$

and in particular, this set is open in $X$. For any positive function $\rho : \Omega \to (0, \infty)$ we denote by $\ell^2(\Omega, \rho)$ the weighted $\ell^2$-space consisting of those functions $f : \Omega \to \mathbb{C}$ for which $\|f\|_2 := (\sum_{x \in \Omega} |f(x)|^2 \rho(x))^{1/2} < \infty$. This is a Hilbert space unitarily isomorphic to $\ell^2(\Omega)$ via the map $\ell^2(\Omega, \rho) \ni 1_x \mapsto \sqrt{\rho(x)}1_x \in \ell^2(\Omega)$.

**Proposition 6.2.** For each $n \in \mathbb{N}$ the algebra $K_n$ is finitary (in fact it has a continuous trace) and up to unitary equivalence all its irreducible representations are subrepresentations of the orbit representation. Moreover, we have a homeomorphism

$$\overline{K_n} \cong \varphi^n(\Delta_{\text{pos,n}})$$

under which the representation $\pi^n_y$ of $K_n$ corresponding to $y \in \varphi^n(\Delta_{\text{pos,n}})$ acts on $H^n_y := \ell^2(\varphi^{-n}(y) \setminus \varrho_n^{-1}(0), \varrho_n)$ and is defined by

$$\pi^n_y(at^n b)h = a \cdot \left( \sum_{x \in \varphi^{-n}(y)} \varrho_n(x)b(x)h(x) \right),$$

for $a, b \in I_n$ and $h \in H^n_y$. The map $U \mapsto I_n t^n C_0(U) t^n I_n$ is a bijection between open subsets of $\varphi^n(\Delta_{\text{pos,n}})$ and ideals in $K_n$.

**Proof.** As in the proof of Proposition 6.1 we may identify $M_{L^n}$ with $\overline{I^n t^n}$ and then $K_n$ is identified with the algebra of compact operators on $M_{L^n}$. Hence $\overline{I^n t^n}$ is a Morita-Rieffel equivalence bimodule between $K_n = \overline{I_n t^n t^n I_n}$ and $\overline{t^n I_n t^n} = L^n(I_n) = C_0(\varphi^n(\Delta_{\text{pos,n}}))$. Such an equivalence preserves spectra and a number of other properties, see [RW98]. In particular $K_n$ has a continuous trace, because $C_0(\varphi^n(\Delta_{\text{pos,n}}))$ has it. The ideal in $K_n$ corresponding via the equivalence $M_{L^n} = \overline{I_n t^n}$ to an ideal $C_0(U)$ in $C_0(\varphi^n(\Delta_{\text{pos,n}}))$ is $(M_{L^n}, C_0(U) M_{L^n}) = \overline{I_n t^n C_0(U) t^n I_n}$. This correspondence extends to a homeomorphism $\overline{K_n} \cong \varphi^n(\Delta_{\text{pos,n}})$ where the representation $\pi_y$ of $K_n$ corresponding to $y \in \varphi^n(\Delta_{\text{pos,n}})$ acts on $H_y := M_{L^n} \otimes_{\text{ev}_y} \mathbb{C}$ which by construction is the Hausdorff completion of the algebraic tensor product $C_0(\Delta_n) \otimes \mathbb{C}$ in seminorm coming from the sesqui-linear form determined by

$$\langle c_1 \otimes \lambda_1, c_2 \otimes \lambda_2 \rangle = \overline{c_1} L(c_1^* c_2)(y)\lambda_2 = \sum_{x \in \varphi^{-n}(y)} \varrho_n(x)\overline{\lambda_1} c_1(x)\lambda_2 c_2(x).$$

Then $\pi_y$ is determined by the formula $\pi_y(at^n b)[c \otimes \lambda] = [a L^n(bc) \otimes \lambda]$, for $a, b, c \in C_0(\Delta_n)$, $\lambda \in \mathbb{C}$. Using this one readily sees that the map $[c \otimes 1] \mapsto c|_{\varphi^{-n}(y) \setminus \varrho_n^{-1}(0)}$ determines a unitary

$$H_y \cong H^n_y = \ell^2(\varphi^{-n}(y) \setminus \varrho_n^{-1}(0), \varrho_n)$$

that intertwines $\pi_y$ and $\pi^n_y$. Moreover, the subspace $G_y := \ell^2(\varphi^{-n}(y) \setminus \varrho_n^{-1}(0))$ of $\ell^2(X)$ is invariant under the action of $\pi_o \times T_o(K_n)$, because for $a \in A$ and $x \in \varphi^{-n}(y) \setminus \varrho_n^{-1}(0)$ we have

$$\pi_o(a)1_x = a(x)1_x, \quad T_o^n 1_x = \sum_{x' \in \varphi^{-n}(y) \setminus \varrho_n^{-1}(0)} \sqrt{\varrho_n(x')}\varrho_n(x)1_{x'}. $$
Thus we have a subrepresentation $\sigma_y : K_n \to B(G_y)$ of $\pi_0 \times T_0|K_n$ where $\sigma_y(at^*b) = \pi_0(a)T_0^\circ \pi_0(b)|G_y$. The canonical isomorphism $G_y \cong H_y^n$ is a unitary equivalence between $\sigma_y$ and $\pi_y^n$.

**Remark 6.3.** If $\Delta = X$ is compact, then $t \in A \times L$ and the unique extension of $\pi_y^n$ to $A + K_n$ is defined by the formulas $\pi_y^n(a)h = a \cdot h$, $\pi_y^n(t^nt^m)h = \left(\sum_{x \in \varphi^{-1}(V)} \theta_n(x)h(x)\right) \cdot 1$, for $a \in A$ and $h \in H_y^n$. Thus $\pi_y^n(a)$ is a multiplication operator and $\pi_y^n(t^nt^m)$ is a rank one operator whose range consists of constant functions.

Having a continuous map $f : U \to Y$ defined on an open subset $U$ of a topological space $X$, we may attach $X$ to $Y$ along $f$ to get the space $X \cup_f Y := (X \sqcup Y)/(x \sim f(x)$ for all $x \in U$) equipped with the quotient topology. This is the pushout of $f : U \to Y$ and the inclusion map $U \subseteq X$. We may always identify $X \cup_f Y$ with the disjoint union $X \cup_f Y := (X \setminus U) \sqcup Y$ where the second summand $Y$ is open in $X \cup_f Y$ and if the map $f$ is open, then the open sets in $X \cup_f Y$ can be identified with pairs of open sets $V \subseteq X$, $W \subseteq Y$ satisfying $\varphi^{-1}(W) = V \cap U$ (then the corresponding open set in $X \cup_f Y$ is $V \setminus U \cup W$). We use this construction to describe the spectrum of the $C^*$-algebras $A_n$, as $A_{n+1} = A_n + K_n$ may be viewed as a pushout of $A_n$ and $K_{n+1}$ via the $C^*$-algebra $A_n \cap K_{n+1}$.

**Lemma 6.4.** For each $n \in \mathbb{N}$, we have continuous bijection form $K_n \xy K_{n+1}$ onto the pushout of $\varphi^n(\Delta_{\text{pos},n})$ and $\varphi^{n+1}(\Delta_{\text{pos},n+1})$ along the partial homeomorphism $\varphi : \varphi^n(\Delta_{\text{pos},n}) \cap \Delta_{\text{reg}} \to \varphi^{n+1}(\Delta_{\text{pos},n+1})$. We have a continuous bijection

$$
K_n \xy K_{n+1} \cong \varphi^n(\Delta_{\text{pos},n}) \setminus \Delta_{\text{reg}} \cup \varphi^{n+1}(\Delta_{\text{pos},n+1}),
$$

where the topology on the right hand side consists of sets $U_n \setminus \Delta_{\text{reg}} \cup U_{n+1}$ where $U_n \subseteq \varphi^n(\Delta_{\text{pos},n})$, $U_{n+1} \subseteq \varphi^{n+1}(\Delta_{\text{pos},n+1})$ are open and $\varphi^{-1}(U_{n+1}) = U_n \cap \Delta_{\text{reg}}$.

**Proof.** Since $K_{n+1}$ is an ideal in $K_n + K_{n+1}$ we may identify $\tilde{K}_{n+1} \cong \varphi^{n+1}(\Delta_{\text{pos},n+1})$ with an open subset of $K_n + K_{n+1}$. Its complement is naturally identified with the spectrum of the quotient $K_n/(K_{n+1} \cap K_n) \cong (K_n + K_{n+1})/K_{n+1}$. By Proposition 6.11 $K_n \cap K_{n+1} = \tilde{I}_n t^n C_0(\Delta_{\text{reg}}) t^m \tilde{I}_n$ is an ideal in $K_n$. Hence using the homeomorphisms from Proposition 6.2 we may identify $\tilde{K}_{n+1}$ with $\varphi^{n+1}(\Delta_{\text{pos},n+1})$ and $K_n \cap K_{n+1}$ with $\varphi^n(\Delta_{\text{pos},n}) \cap \Delta_{\text{reg}}$. Accordingly, we get the bijection (23), which restricts to homeomorphisms $\varphi^{n+1}(\Delta_{\text{pos},n+1})$ and $K_n \cap K_{n+1} \subseteq \varphi^n(\Delta_{\text{pos},n}) \setminus \Delta_{\text{reg}}$. Any representation $\pi$ that is in $K_n + \tilde{K}_{n+1} \subseteq \tilde{K}_{n+1}$ is a representation of $K_n$ that vanishes on $K_{n+1}$. Every ideal in $K_n$ is of the form $\tilde{I}_n t^n C_0(\tilde{U}) t^m \tilde{I}_n$.

Hence the bijection (23) becomes continuous if $\tilde{L}_n(I_n) \setminus \Delta_{\text{reg}} \cup L^{n+1}(I_{n+1})$ is equipped with the pushout topology.

The pushout topology on the right hand side of (23) is always $T_0$, and the continuous bijection (23) might be a homeomorphism even when this topology is non-Hausdorff, see [KW16] and Example 6.8 below. However, in general the pushout topology is weaker than the topology of the spectrum $K_n + \tilde{K}_{n+1}$, and a general description of the topology of the latter requires more than just the pushout data.
Example 6.5. Let us consider $A_1 = A + K_1 = \hat{K}_0 + K_1$ associated to the transfer operator $L(a)(y) = a(y/2)$ for the tent map $\varphi : [0,1] \to [0,1]$, $\varphi(x) = 1 - |1 - 2x|$. Then $\varphi = 1_{[0,1/2]}$, $\Delta_{\text{pos}} = [0,1/2]$ and $\Delta_{\text{reg}} = [0,1/2]$.

So as sets we have

$$\hat{A}_1 \cong X \setminus \Delta_{\text{reg}} \cup \varphi(\Delta_{\text{pos}}) = [1/2,1] \cup [0,1].$$

The pushout topology on the right hand side is the usual one with the only exception that neighbourhoods of 1/2 in the first summand contain sets of the form $[1/2,1/2 + \varepsilon) \cup (1-\varepsilon,1)$. So in particular the pushout topology is not Hausdorff in this case (it is $T_0$ though). The topology on $\hat{A}_1$ is larger and in fact $\hat{A}_1$ is homeomorphic to the direct union of two closed intervals. Indeed, the operator $tt^*$ in the regular representation becomes the multiplication operator by the characteristic function $1_{[0,1/2]}$. So $A_1$ is generated by $A = C[0,1]$ and $tt^* = 1_{[0,1/2]}$ and $A_1 = \text{Att}^* \oplus A(1-tt^*) = C[0,1/2] \oplus C[1/2,1]$. The extra open set in $\hat{A}_1$ (not seen by the pushout topology) comes from the ideal generated by the element $1 - tt^*$ which is neither in $K_0 = A$ nor in $K_1$. Thus the precise description of $\hat{A}_1$ seem to require some additional algebraic data that is difficult to pin down.

Theorem 6.6. Let $\varphi : \Delta \to [0,\infty)$ be a potential associated to a transfer operator $L : C_0(\Delta) \to C_0(X)$ for $\varphi : \Delta \to X$. For each $n \in \mathbb{N}$ the algebra $A_n$ is postliminary and we have a natural bijection

$$\hat{A}_n \cong \left( \bigcup_{k=0}^{n-1} \varphi^k(\Delta_{\text{pos}},k) \setminus \Delta_{\text{reg}} \right) \sqcup \varphi^n(\Delta_{\text{pos}},n).$$

More specifically, for every irreducible representation $\pi$ of $A_n$ there is a maximal $k \leq n$ with $\pi(K_k) \neq 0$ and a unique $y \in \varphi^k(\Delta_{\text{pos}},k)$ ( $y \in \varphi^k(\Delta_{\text{pos}},k) \setminus \Delta_{\text{reg}}$ if $k < n$) such that $\pi \cong \pi^k_y$ where $\pi^k_y$ is a representation of $A_n$ on $l^2(\varphi^{-k}(y) \setminus \varphi^{-1}(0),g_k)$ determined by

$$\pi^k_y(at^it^ib)h = a \cdot \left( \sum_{x \in \varphi^{-i}(y)} \varrho_i(x)b(x)h(x) \right), \quad a,b \in I, \ i = 1, \ldots, k,$$

and $\pi^k_y(K_i) = 0$ for all $k < i \leq n$. If we equip the right hand side of (23) with the topology that consists of sets $\left( \bigcup_{k=0}^{n-1} U_k \setminus \Delta_{\text{reg}} \right) \sqcup U_n$ where $U_k$ is an open subset of $\varphi^k(\Delta_{\text{pos}},k)$, for $k = 0, \ldots, n$, and $U_k \cap \Delta_{\text{reg}} = \varphi^{-1}(U_k)$ for $k < n$, then (24) is continuous and its inverse is continuous when restricted to each direct summand.

Proof. We prove this by induction. The assertion holds for $n = 1$ by Lemma 6.4. Assume that for certain $n$ we have $\hat{A}_n \cong \left( \bigcup_{k=0}^{n-1} \varphi^k(\Delta_{\text{pos}},k) \setminus \Delta_{\text{reg}} \right) \sqcup \varphi^n(\Delta_{\text{pos}},n)$ as in the assertion. Here $\left( \bigcup_{k=0}^{n-1} \varphi^k(\Delta_{\text{pos}},k) \setminus \Delta_{\text{reg}} \right)$ corresponds to the closed set $\hat{A}_n \setminus \hat{K}_n$ and $\hat{K}_n \cong \varphi^n(\Delta_{\text{pos}},n)$ is the homeomorphism from Proposition 6.2.

By Proposition 6.1 $K_n + K_{n+1}$ is an ideal in $A_{n+1}$. The corresponding open subset of $\hat{A}_{n+1}$ is $\hat{K}_n + \hat{K}_{n+1} \cong L^n(I_n) \setminus \Delta_{\text{reg}} \sqcup \Delta_{\text{pos}} + L^{n+1}(I_{n+1})$ as described in Lemma 6.3. Its complement $\hat{A}_{n+1} \setminus \hat{K}_n + \hat{K}_{n+1} \cong \hat{A}_n \setminus \hat{K}_n \cong \left( \bigcup_{k=0}^{n-1} L^k(I_k) \setminus \Delta_{\text{reg}} \right)$. Since $A_n \cap K_{n+1} = K_n \cap K_{n+1}$, see Proposition 6.1 we conclude that the topology on $\hat{A}_{n+1}$ is described as in the assertion. The ranges of all representations in $\hat{A}_n$ contain compact operators. Hence $A_n$ is postliminary. \qed
Remark 6.7. If $\pi$ is an irreducible representation of $A_n$, there is a ‘dynamical procedure’ of determining $y$ and $k$ for which $\pi \cong \pi^k_y$. Namely, the set $Z := \{x \in X : a(x) \neq 0 \text{ implies } \pi(a) \neq 0 \text{ for all } a \in A\}$ is closed and there is $k \leq n$ such that $\varphi^k(Z)$ is a singleton. If there is a minimal $k < n$ such that $\varphi^k(Z) = \{y\} \notin \Delta_{\text{reg}}$, then $\pi \cong \pi^k_y$. Otherwise $\pi \cong \pi_y$ where $\varphi^n(Z) = \{y\}$.

Example 6.5 shows that the continuous bijection (24) in general fails to be a homeomorphism. Obviously, it is a homeomorphism when the pushout topology on the right hand side of (24) is Hausdorff, and less obviously, when $\varrho$ is continuous, see Theorem 6.11 below. This may also happen in a non-continuous and non-Hausdorff case:

Example 6.8. Let us consider the standard transfer operator $L(a)(y) = \frac{1}{2}[a(\frac{x}{2}) + a(1 - \frac{x}{2})]$ for the tent map $\varphi : [0, 1] \to [0, 1], \varphi(x) = 1 - |1-2x|$. Then $\varrho = \frac{1}{2}1_{X\setminus\{\frac{1}{2}\}} + 1_{\{\frac{1}{2}\}}, \Delta_{\text{pos}} = X = [0, 1]$ and $\Delta_{\text{reg}} = X \setminus \{\frac{1}{2}\} = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. Accordingly,

$$\hat{A}_n \cong \bigcup_{k=0}^{n-1} \pi^{k}_{1/2} \cup \{\pi^n_x : x \in [0, 1]\}$$

where the pushout topology on the right hand side can be described as follows: $\{\pi^n_x : x \in [0, 1]\}$ is an open set homeomorphic to $[0, 1]$ and each $\pi^k_{1/2}$ has a basis of neighbourhoods of the form $\{\pi^k_{1/2}\} \cup \{\pi^n_x : x \in (0, \varepsilon)\}$, if $k < n - 1$, and $\{\pi^{n-1}_{1/2}\} \cup \{\pi^n_x : x \in (1 - \varepsilon, 1)\}$, if $k = n - 1$ (so $\pi^k_{1/2}, k < n - 1$, cannot be separated from $\pi^n_0$ and $\pi^{n-1}_{1/2}$ cannot be separated from $\pi^n_1$).

This topology coincides with the standard topology of $\hat{A}_n$, as using the regular representation one can see that $A_n$ is naturally isomorphic the $C^*$-subalgebra of $C([0, 1], M_{2^n}(\mathbb{C}))$ consisting continuous matrix valued functions $a$ satisfying

$$a(1) \in M_{2n-1}(\mathbb{C}) \oplus M_{2n-1}(\mathbb{C}), \quad a(0) \in M_{2n-1+1}(\mathbb{C}) \oplus M_{2n-2}(\mathbb{C}) \oplus ... \oplus M_2(\mathbb{C}) \oplus \mathbb{C}.$$ 

In particular, representations $\pi^n_1$ and $\pi^{n-1}_{1/2}$ are of dimension $2^{n-1} = |\varphi^{-n}(1)|$, and $\pi^n_0$ is of dimension $2^{n-1} + 1 = |\varphi^{-n}(0)|$. This example is covered by the main result of \[KW16\].

The algebra $A_\infty$ as a rule is not postliminary (the example of Glimm algebras shows that $A_\infty$ will usually be antiliminary, cf. \[Ped79\] Theorem 6.5.7]). Accordingly, one can not hope to describe the spectrum $\hat{A}_\infty$ completely in a reasonable way. However, the inductive limit of spaces $\hat{A}_n$ will give a dense subset of $A_\infty$, and when the continuous maps (24) are open one can use them to describe the primitive ideal space $\text{Prim}(A_\infty)$ of $A_\infty$. We show how to do it in the case when $\varrho$ is continuous.

6.1. The case of a continuous potential. When $\varrho$ is continuous then $A_\infty$ has a natural étale groupoid model. The groupoid in question is a motivating example in the theory of approximately proper equivalence relations \[Ren05\], which recently has been generalized to cover partial local homeomorphisms \[BEFR2\]. Namely, assume that $\varphi : \Delta \to X$ is a local homeomorphism. For each $n \in \mathbb{N}$ consider the equivalence relation

$$R_n := \{(x, y) \in \Delta_n \times \Delta_n : \varphi^n(x) = \varphi^n(y)\}$$

as an étale groupoid with the product topology inherited from $\Delta_n \times \Delta_n$. Then we get a generalized approximately proper (in short GAP) equivalence relation on $X \times X$

$$R := \bigcup_{n=0}^{\infty} R_n,$$
equipped with the inductive limit topology, i.e. $U \subseteq R$ is open iff $U \cap R_n$ is open for every $n \in \mathbb{N}$, see [BEFR20] Proposition 5.6]. Similarly, each finite union $\bigcup_{k=0}^{n} R_k$ becomes an étale groupoid. All these groupoids are amenable (see the proof of [Ren00] Proposition 2.4)).

**Proposition 6.9.** Assume $\varphi : \Delta \to X$ is a local homeomorphism and let $\varphi : \Delta \to (0, +\infty)$ be any strictly positive continuous map with $\sup_{y \in X} \sum_{x \in \varphi^{-1}(y)} \varphi(x) < \infty$. Equivalently, fix a transfer operator $L : C_0(\Delta) \to C_0(\Delta)$ with continuous potential $\varphi > 0$. We have a natural isomorphisms

\[ K_n \cong C^*(R_n), \quad A_n \cong C^*(\bigcup_{k=0}^{n} R_k), \quad n \in \mathbb{N}, \quad A_\infty \cong C^*(R), \]

where $K_n := \overline{\text{int}}(t^n t^{1-n} I_n)$, $A_n := K_0 + \cdots + K_n$, $A_\infty = \bigcup_{n=0}^{\infty} A_n$ are core subalgebras of $A \rtimes L$, and $R$ is the gap relation associated to $\varphi$.

**Proof.** We view $R$ and $\bigcup_{k=0}^{n} R_k$ as open subgroupoids of $\mathcal{G}$ with the same unit space $X$. Then the inclusions $C_c(\bigcup_{k=0}^{n} R_k) \subseteq C_c(R) \subseteq C_c(\mathcal{G})$ extend to $*$-homomorphisms $C^*(\bigcup_{k=0}^{n} R_k) \to C^*(R) \to C^*(\mathcal{G})$ that intertwine the canonical faithful conditional expectations onto $C_0(X)$ (the groupoids in question are amenable). Hence the aforementioned $*$-homomorphisms are faithful and we may write $C^*(\bigcup_{k=0}^{n} R_k) \subseteq C^*(R) \subseteq C^*(\mathcal{G})$. Now it is immediate that the isomorphism from Theorem 5.4 restricts to isomorphisms $C^*(\bigcup_{k=0}^{n} R_k) \cong A_n, C^*(R) \cong A_\infty$.

The groupoid $R_n$ can be viewed as the restriction of $\bigcup_{k=0}^{n} R_k$ to an open invariant subset $\Delta_n \subseteq X$. Hence $C^*(R_n)$ can be identified with an ideal in $C^*(\bigcup_{k=0}^{n} R_k)$ generated by $C_0(\Delta)$, see [Sim20] Proposition 4.3.2]. Restriction of the isomorphism $C^*(\bigcup_{k=0}^{n} R_k) \cong A_n$ gives $C^*(R_n) \cong K_n$.

**Remark 6.10.** By [Wie14], all irreducible Smale spaces $(\tilde{X}, \tilde{\varphi})$ with totally disconnected stable sets are inverse limits for certain finite-to-one continuous surjections $\varphi : X \to X$ satisfying Wieler’s axioms. By Proposition 6.9 and [DGMW18] Theorem 5.6, if $\varphi$ is an open Wieler map, then the stable algebra $S$ and the stable Ruelle algebra $R_s$ of the Smale space $(\tilde{X}, \tilde{\varphi})$, cf. [PS99], are Morita-equivalent to the algebras $A_\infty$ and $A \rtimes L$, respectively.

The groupoids $R_n, \bigcup_{k=0}^{n} R_k, R$ are not only amenable but also principle (all the isotropy groups are trivial). In particular, inclusions $C_0(X) \subseteq C^*(\bigcup_{k=0}^{n} R_k) \cong A_n, C^*(R) \cong A_\infty$ are $C^*$-diagonals in the sense of Kumjian [Kum86], and the ideals in $A_n$ and $A_\infty$ correspond to open invariant sets in $\bigcup_{k=0}^{n} R_k$ and $R$, respectively (see [Sim20] Theorem 4.3.3 or [BL20] Corollary 3.12]). Also the primitive ideal spaces can be identified with quasi-orbit spaces, see [BL20] Corollary 3.19 or [KM20] Theorem 7.17. This allows us to improve Theorem 6.10 in the case when $\varphi$ is continuous, as follows.

**Theorem 6.11.** If $\varphi : \Delta \to [0, +\infty)$ is continuous, then the continuous bijection in [21] is a homeomorphism, and we have natural homeomorphisms

\[ \tilde{A}_n \cong \text{Prim}(A_n) \cong X/\sim_n, \]

where $x \sim_n y$ iff there is $k \leq n$ such that $x, y \in \Delta_{\text{pos}, k}$ and $\varphi^k(x) = \varphi^k(y)$. If in addition $X$ is second countable, we have a homeomorphism

\[ \text{Prim}(A_\infty) \cong X/\sim \]

where $x \sim y$ iff $\overline{\mathcal{O}_R(x)} = \overline{\mathcal{O}_R(y)}$ and $\mathcal{O}_R(x) := \bigcup_{k=0, x \in \Delta_{\text{pos}, k}} \varphi^{-k}(\varphi^k(x))$ is the orbit of $x \in X$. 
Proof. Since $g$ is continuous we may assume that $\Delta$ is equal to $\Delta_{\text{pos}} = \Delta_{\text{reg}}$, so that $g > 0$ and we may apply isomorphisms from Proposition 6.9. The orbits of $x$ for the groupoid $R_{[0,n]} := \bigcup_{k=0}^n R_k$ are given by $\mathcal{O}_n(x) = \bigcup_{k=0}^n, x \in \Delta_{\text{pos},k} \varphi^{-k}(\varphi^k(x))$. We have a bijection
$$\left( \bigcup_{k=0}^{n-1} \varphi^k(\Delta_{\text{pos},k}) \setminus \Delta_{\text{reg}} \right) \cup \varphi^n(\Delta_{\text{pos},n}) \xrightarrow{\sim} X/\sim_n$$
that sends a point $y$ in $k$-th summand to $\varphi^{-k}(y)$, which is an $R_{[0,n]}$-orbit. Using this bijection one checks that open $R_{[0,n]}$-invariant subsets of $X$ correspond to open sets in the pushout topology of the right-hand side of (24). Since ideals in $A_n \cong C^*_\omega(\bigcup_{k=0}^n R_k)$ correspond bijectively to open $R_{[0,n]}$-invariant sets, this gives the first part of the assertion (we have $\hat{A}_n \cong \text{Prim}(A_n)$ because $A_n$ is postliminary, in fact Type $I_0$).

The second part follows because $\mathcal{O}_R(x)$ is the orbit of $x$ under the groupoid $R$ and the primitive ideal space $\text{Prim}(A_{\infty}) \cong \text{Prim}(C^*(R))$ is homeomorphic to the quasi-orbit space for $R$ by [BL20, Corollary 3.19].

7. Topological free transfer operators and simplicity

A full map on locally compact Hausdorff space is called topologically free if the set of its periodic points has empty interior. We will introduce topological freeness for a partial map $\varphi : \Delta \to X$ by reducing it to the case of a full map. Namely, we will restrict $\varphi$ to its essential domain $\Delta_{\infty} := \bigcap_{n=1}^\infty \Delta_n \cap \varphi^n(\Delta_n)$, $\Delta_n = \varphi^{-n}(\Delta)$, $n \in \mathbb{N}$, which gives a full map $\varphi : \Delta_{\infty} \to \Delta_{\infty}$, see [KL20, Definition 3.1]. As a starting step of defining topological freeness for transfer operators we analyse this notion for open partial maps.

Definition 7.1. A partial continuous open map $\varphi : \Delta \to X$ is topologically free if the set of periodic points for $\varphi : \Delta_{\infty} \to \Delta_{\infty}$ has empty interior in $\Delta_{\infty}$.

Lemma 7.2. Suppose that $\varphi : \Delta \to X$ is a partial continuous open map of $X$. The following conditions are equivalent:

(i) $\varphi$ is topologically free;

(ii) for every $n \in \mathbb{N}$ the set $\{x \in \Delta_n : \varphi^n(x) = x\}$ has empty interior in $X$;

(iii) for every $k, l \in \mathbb{N}_0$ with $l < k$, $\{x \in \Delta_k : \varphi^k(x) = \varphi^l(x)\}$ has empty interior in $X$.

Proof. Note that $\{x \in \Delta_n : \varphi^n(x) = x\} \subseteq \Delta_{\infty}$ is closed in $\Delta_{\infty}$, because $\Delta_{\infty}$ is Hausdorff and $\varphi$ is continuous. Moreover, $\Delta_{\infty}$ is a Baire space, as it is a $G_\delta$ subset of the locally compact Hausdorff space $X$. Thus $\{x \in \Delta_{\infty} : \exists n \in \mathbb{N} \varphi^n(x) = x\} = \bigcap_{n \in \mathbb{N}} \{x \in \Delta_n : \varphi^n(x) = x\}$ has empty interior if and only if each of the intersected sets has empty interior. This proves

(i)$\Rightarrow$(ii).

The implication (ii)$\Rightarrow$(iii) is immediate. For the converse assume that there is a non-empty open set $U \subseteq \{x \in \Delta_k : \varphi^k(x) = \varphi^l(x)\}$ where $l < k$. Then $V := \varphi^l(U)$ is non-empty open set contained in $\{x \in \Delta_n : \varphi^n(x) = x\}$ where $n := k - l \in \mathbb{N}$. \qed

If we consider a transfer operator $L$ associated to a partial map $\varphi : \Delta \to X$, then the natural ‘domain of openness’ for $\varphi$ is $\Delta_{\text{pos}}$, see Proposition 1.3. The next lemma shows that in the definition of topological freeness we can equally-well use the smaller set $\Delta_{\text{reg}}$.

Lemma 7.3. If $L$ is a transfer operator for a map $\varphi : \Delta \to X$, then $\varphi : \Delta_{\text{reg}} \to X$ is topologically free if and only if $\varphi : \Delta_{\text{pos}} \to X$ is topologically free.
**Proof.** Since $\Delta_{\text{reg}} \subseteq \Delta_{\text{pos}}$, topological freeness of $\varphi : \Delta_{\text{pos}} \to X$ implies topological freeness of $\varphi : \Delta_{\text{reg}} \to X$. Assume now that $\varphi : \Delta_{\text{pos}} \to X$ is not topologically free. So there is a non-empty open set $U \subseteq \Delta_{\text{pos},n} = \Delta_n \setminus \varrho_n^{-1}(0)$, for some $n \in \mathbb{N}$, such that for each $x \in U$ we have $x = \varphi^n(x)$. Then $\varphi$ is injective on each of the sets $U$, $\varphi(U)$, ..., $\varphi^{n-1}(U)$ and since they are contained in $\Delta_{\text{pos}} = \Delta \setminus \varrho^{-1}(0)$ it follows from Proposition 7.3 that they are in fact contained in $\Delta_{\text{reg}}$. Therefore $\varphi : \Delta_{\text{reg}} \to X$ is not topologically free. \qed

The foregoing observations naturally lead to the following definition, which agrees with the version of topological freeness suggested in [CKO19, Example 9.14].

**Definition 7.4.** We say that the transfer operator $L$ is topologically free if the restricted open map $\varphi : \Delta_{\text{reg}} \to X$ is topologically free.

**Theorem 7.5.** Let $L$ be a transfer operator for a partial map $\varphi : \Delta \to X$. The following conditions are equivalent:

(i) Every faithful covariant representation $(\pi, T)$ of $L$ extends to a faithful representation $\pi \rtimes T$ of the crossed product $A \rtimes L$.

(ii) $A$ detects ideals in $A \rtimes L$, i.e. $A \cap N \neq \{0\}$ for any non-zero ideal $N$ in $A \rtimes L$.

(iii) The orbit representation $(\pi_o, T_o)$ introduced in Example 2.14 extends to a faithful representation $\pi_o \rtimes T_o$ of the crossed product $A \rtimes L$.

(iv) The map $\varphi : \Delta_{\text{reg}} \to X$ is topologically free.

**Proof.** The equivalence $[\text{(i)} \Leftrightarrow \text{(iii)}]$ is straightforward and implication $[\text{(i)} \Rightarrow \text{(iii)}]$ is trivial. To prove $[\text{(iii)} \Rightarrow \text{(iv)}]$ assume $\varphi : \Delta_{\text{reg}} \to X$ is not topologically free. Then there is a non-empty open set $U \subseteq \Delta_n$, such that $\varphi^n|_U = \text{id}|_U$ and $\varphi^k(U) \subseteq \Delta_{\text{reg}}$ for $k = 0, \ldots, n-1$. In particular, $\varrho_n$ is continuous and non-zero at every point in $U$. Thus for any non-zero $a \in C_c(U) \subseteq A$ we have $a\sqrt{\varrho_n} \in A$. Using the regular representation one readily calculates that $at^n - a\sqrt{\varrho_n}$ is a non-zero element of $A \rtimes L \cong C^*(\pi(A) \cup \pi(I)T)$, but $(\pi_o \rtimes T_o)(at^n - a\sqrt{\varrho_n}) = 0$. Hence $\pi_o \rtimes T_o$ is not faithful.

Implication $[\text{(iv)} \Rightarrow \text{(i)}]$ follows from [CKO19, Example 9.14] as by Lemma 7.3 condition described there is equivalent to topological freeness as we define it. \qed

Topological freeness for homeomorphisms appeared already in the work of Zeller-Meier, see [ZM68, Proposition 4.14], who used it to characterise when $C_0(X)$ is maximal abelian in the associated crossed product. This result was generalised to crossed products by local homeomorphisms by Carlsen and Silvestrov in [CS09]. However, it seems that there is no obvious generalisation of this fact if we allow irregular points (discontinuity points of $\varrho$). More specifically, let $A'$ denote the commutant of $A = C_0(X)$ in $A \rtimes L$. So $A$ is maximal abelian in $A \rtimes L$ iff $A = A'$. Using the generalised expectation $G$ introduced in Lemma 4.7 we clearly have

$$C_0(X) \subseteq \{ b \in A \rtimes L : G(b) = b \} \subseteq C_0(X)'.$$

It turns out that when $\varrho$ is discontinuous already the first inclusion might be proper.

**Example 7.6.** For the tent map $\varphi : [0,1] \to [0,1]$ where $\varphi(x) = 1 - |1 - 2x|$ and $\varrho = 1_{[0,\frac{1}{2}]}$, we have the transfer operator $L(a)(y) = a(y^2)$. Using the regular representation one readily calculates that

$$G(t^n t^m) = t^n t^m = 1_{[0,\frac{1}{2}]}.$$
Accordingly, $C_0(X) \subseteq \{ b \in A \times L : G(b) = b \}$ as the latter contains some functions that are discontinuous at points $\frac{1}{2n}$, $n \in \mathbb{N}$. Hence $C_0(X) \neq C_0(X)'$ even though the map $\varphi$ is topologically free.

For the sake of completeness we will generalise the main result of [CS00] (to partial, not necessarily surjective maps on locally compact spaces, and arbitrary continuous $\varphi$). The proof is based on Renault’s characterisation of Cartan subalgebras [Ren08], see also [KM20] 7.2.

**Theorem 7.7.** Suppose that the transfer operator $L$ is given by a continuous $\varphi$. Then the equivalent conditions in Theorem 7.5 are further equivalent to each of the following:

(i) The $C^*$-algebra $C_0(X)$ is maximal abelian in $C_0(X) \rtimes L$.
(ii) $C_0(X)$ is a Cartan subalgebra of $C_0(X) \rtimes L$, in the sense of [Ren08].
(iii) The partial map $\varphi : \Delta_{\text{reg}} = \Delta_{\text{pos}} \to X$ is topologically free.

**Proof.** The transfer operator $L : I \to A$ for $\varphi : \Delta \to X$ restricts to the transfer operator $L_{\text{reg}} : C_0(\Delta_{\text{reg}}) \to A$ for the partial homeomorphism $\varphi : \Delta_{\text{reg}} \to X$. Since we assume $\varphi$ is continuous we get $\Delta_{\text{reg}} = \Delta \setminus \varphi^{-1}(0)$ and the crossed products $A \rtimes L$ and $A \rtimes L_{\text{reg}}$ are naturally isomorphic (their regular representations coincide, see Proposition 4.5). Thus by Theorem 5.4 we may identify $A \rtimes L$ with the groupoid $C^*$-algebra $C^*(G)$ of the Renault-Deaconu groupoid $G$ associated to $\varphi : \Delta_{\text{reg}} \to X$. By the work of Renault’s [Ren08], $C_0(X)$ is maximal abelian in $C^*(G)$ if and only if $C_0(X)$ is a Cartan subalgebra of $G$ if and only if the groupoid $G$ is effective (Renault considered second countable groupoids but his theory works without this assumption, see [KM20] 7.2) By [Ren08] Corollary 3.3 and [Ren00] Proposition 2.3 the groupoid $G$ is effective if and only if the map $\varphi : \Delta_{\text{reg}} \to X$ is topologically free (local homeomorphisms satisfying (iii) in Lemma 7.2 are called essentially free in [Ren00]). This proves the desired equivalence. $\square$

### 7.1. Simplicity

The following definition and lemma are compatible with [ER07] Definition 3.1 and Proposition 3.2] where partial local homeomorphisms are considered.

**Definition 7.8.** Let $U \subseteq X$. We say $U$ is positively invariant if $\varphi(U \cap \Delta_{\text{pos}}) \subseteq U$, $U$ is negatively invariant if $\varphi^{-1}(U) \cap \Delta_{\text{reg}} \subseteq U$, and $U$ is invariant if it is both positively and negatively invariant. We say that $L$ is minimal if there are no non-trivial open invariant sets.

**Lemma 7.9.** Let $U$ be an open subset of $X$ and put $J := C_0(U)$ and recall that $I = C_0(\Delta)$.

(i) $U$ is positively invariant iff $L(J \cap I) \subseteq J$;
(ii) $U$ is negatively invariant iff $L^{-1}(J) \cap C_0(\Delta_{\text{reg}}) \subseteq J$ iff $L^{-1}(J) \cap C_0(\Delta_{\text{reg}}) \subseteq J$;

**Proof.** (i) If $U$ is positively invariant, then for any $y \notin U$ we have $\varphi^{-1}(y) \cap \Delta_{\text{pos}} = \emptyset$, which implies $L(a)(y) = 0$ for any $a \in J \cap I$. Thus $L(J \cap I) \subseteq J$. If $U$ is not positively invariant, then there is $x \in U \setminus \varphi^{-1}(0)$ such that $\varphi(x) \notin U$. Taking any positive $a \in I \cap J$ with $a(x) \neq 0$, we get $L(a)(\varphi(x)) = \sum_{t \in \varphi^{-1}(\varphi(x))} g(t)a(t) > 0$ which shows that $L(J \cap I) \not\subseteq J$.

(ii) This follows from the equalities $C_0(\varphi^{-1}(U) \cap \Delta_{\text{reg}}) = L^{-1}(J) \cap C_0(\Delta_{\text{reg}})$ and $C_0(\varphi^{-1}(U) \cap \Delta_{\text{reg}}) = L^{-1}(J) \cap C_0(\Delta_{\text{reg}})$, which are readily verified. $\square$

**Example 7.10** (Directed graphs with one circuit). Suppose that $\varphi : X \to X$ is a directed graph with one circuit, i.e. $X$ is a countable discrete set and there is a point $x \in X$ such that for any $y \in X$ we have $\varphi^n(y) = x$ for some $n \geq 1$, cf. [BJSS17]. In particular, $x$ is a periodic point and $\varphi$ is a local homeomorphism. Take any $\varphi : X \to (0, +\infty)$ with $\sup_{y \in X} \sum_{x \in \varphi^{-1}(y)} g(x) < \infty$, so that it defines a transfer operator $L$ for $\varphi$. Then $X = \Delta_{\text{pos}} = \Delta_{\text{reg}}$ and the transfer operator $L$ is minimal but $\varphi$ is not topologically free. Hence, $A \rtimes L$ is not simple by Theorem 7.5.
Theorem 7.11. The following conditions are equivalent:

(i) the crossed product \( A \rtimes L \) is simple;
(ii) \( L \) is minimal and \( \varphi : \Delta_{\text{reg}} \to X \) is topologically free;
(iii) \( L \) is minimal and \( \varphi : \Delta_{\text{reg}} \to X \) is not a directed graph with one circuit.

Proof. We first show that (i) \(\Rightarrow\) (ii). If \( \varphi : \Delta_{\text{reg}} \to X \) is not topologically free, then \( A \rtimes L \) is not simple by Theorem 7.5. So let us assume that \( \varphi : \Delta_{\text{reg}} \to X \) is topologically free. Then for any non-zero ideal \( N \) in \( A \rtimes L \) the ideal \( J := A \cap N \) in \( A \) is non-zero. Hence \( J = C_0(U) \) for some non-empty open set \( U \). Note that \( L(J \cap I) = t^*INI t \subseteq K \), so \( L(J \cap I) \subseteq J \). Also for any \( a \in L^{-1}(J) \cap C_c(\Delta_{\text{reg}}) \), using (15), we obtain

\[
a = \sum_{i,j=1}^{n} u_i^K t^* u_i^K a u_j^K t^* u_j^K = \sum_{i,j=1}^{n} u_i^K tL(u_i^K a u_j^K) t^* u_j^K
\]

\[
= \sum_{i,j=1}^{n} u_i^K tL(\alpha(u_i^K \circ \varphi|_{U_i}^{-1})a \alpha(u_j^K \circ \varphi|_{U_j}^{-1})) t^* u_j^K
\]

\[
= \sum_{i,j=1}^{n} u_i^K t((u_i^K \circ \varphi|_{U_i}^{-1})L(a)(u_j^K \circ \varphi|_{U_j}^{-1}) t^* u_j^K \in IT J t^* T \subseteq N,
\]

so \( a \in J \). Hence \( U \) is an invariant set by Lemma 7.9. If \( U \not= X \) then \( N \not= A \rtimes L \) and \( A \rtimes L \) is not simple. Conversely, if \( N \not= A \rtimes L \) then \( U \not= X \) because otherwise \( N \) would contain \( A \) and \( A \rtimes L = A(A \rtimes L) \) would be \( N \).

Implication (i) \(\Rightarrow\) (iii) is clear and to prove the converse assume that \( L \) is minimal but \( \varphi : \Delta_{\text{reg}} \to X \) is not topologically free. Then there is a non-empty open set \( U \subseteq \Delta_{\text{reg},n} \) for some \( n \geq 1 \) such that \( \varphi^n|_U = \text{id} \) and \( U, \varphi(U), ..., \varphi^{n-1}(U) \) are pairwise disjoint. For any disjoint open sets \( V_1, V_2 \subseteq U \) the sets \( U_i := \bigcup_{k \in \mathbb{N}, k=0, ..., n} \varphi^{-m}(\varphi^k(V_i)) \) are disjoint open and invarient. Thus minimality of \( L \) forces \( U = \{x\} \) to be a singleton and \( \varphi : \Delta_{\text{reg}} \to X \) to be a directed graph with one circuit. \( \square \)

Remark 7.12. If \( \Delta = \Delta_{\text{reg}} \), then \( E := (X, \Delta, \text{id}, \varphi) \) is a topological graph in the sense of Katsura [Kat04] and Theorem 7.11 could be deduced from [Kat06, Theorem 8.12]. In this regular case, one could also get simplicity criteria for \( A \rtimes L \) using the Renault-Deaconu groupoid model, see [KM21] and references therein.

8. Locally contractive transfer operators and pure infiniteness

The following definition is inspired by [Kat08, Definition 2.7].

Definition 8.1. We say that an open set \( V \subseteq X \) is contracting if there are pairwise disjoint, non-empty open sets \( U_k \subseteq \Delta_{\text{reg},n_k} \cap V \) for \( k = 1, ..., m, n_k \geq 1 \), such that

\[
V \not\subseteq \bigcup_{k=1}^{m} U_k \quad \text{and} \quad \nabla \subseteq \bigcup_{k=1}^{m} \varphi^{n_k}(U_k).
\]

We say that \( L \) is contracting if \( \Delta = \Delta_{\text{pos}} \) and there is \( x_0 \in \Delta \) such that every neighbourhood of \( x_0 \) contains a contracting open set and \( \bigcup_{n=0}^{\infty} \varphi^{-n}(x_0) = X \).

If \( \Delta = \Delta_{\text{reg}} \), then \( L \) is contractive iff the topological graph \( E := (X, \Delta, \text{id}, \varphi) \) is contractive in the sense of [Kat08], and we could use [Kat08, Theorem A] to show that \( A \rtimes L \) is purely
Proposition 8.4. Assume that \( \varphi : \Delta_{\text{reg}} \to X \) is topologically free, \( \Delta = \Delta_{\text{pos}} \) and there is \( x_0 \in \Delta \) such that \( \bigcap_{n=0}^{\infty} \varphi^{-n}(x_0) \cap \Delta_{\text{reg},n} = X \). For any non-zero positive \( b \in A \times L \) there is \( d \in A \times L \) and \( a \in A \) which is 1 on some neighbourhood of \( x_0 \) and \( \|d^*bd - a\| < 1/2 \).

Proof. Put \( \varepsilon := \|E(b)\|/5 \) where \( E \) is the conditional expectation onto the core \( A_{\infty} \). Choose a positive \( b_0 \in \text{span}\{a^n t^m c : a \in I_n, c \in I_m, n, m \in \mathbb{N}_0\} \) such that \( \|b - b_0\| < \varepsilon \). Since \( \Delta = \Delta_{\text{pos}} \), in view of Lemma 7.13 we see that \( \varphi : \Delta \to X \) is topologically free, and this implies that the topological quiver \( Q = (X, \Delta, \text{id}, \varphi, \mu) \) satisfies condition (L). By Corollary 3.13 the \( C^* \)-algebra associated to \( Q \) is isomorphic to \( A \times L \). Hence we may apply [MT05, Proposition 6.14] to conclude that there is \( d_0 \in A \times L \) and \( a_0 \in C_c(X)^+ \) such that \( \|d_0\| \leq 1 \), \( \|a_0\| = \|E(b_0)\| \) and \( \|d_0^*bd_0 - a_0\| < \varepsilon \). Since \( \|a_0\| = \|E(b_0)\| > \|E(b)\| - \varepsilon = 4\varepsilon \), the open set

\[ U := \{x \in X : a_0(x) > 4\varepsilon\} \]

is non-empty. As \( \bigcup_{n=0}^{\infty} \varphi^{-n}(x_0) \cap \Delta_{\text{reg},n} \) is dense in \( X \) there is \( n \in \mathbb{N} \) and an open subset \( U_0 \subseteq U \cap \Delta_{\text{reg},n} \) such that \( \varphi^n|_{U_0} \) is a local homeomorphism onto an open neighbourhood \( V_0 \) of \( x_0 \). Take \( c_0 \in C_c(V_0)^+ \), \( \|c_0\| \leq 1 \), such that \( c_0 \) is 1 on an open neighbourhood \( V \subseteq V_0 \) of \( x_0 \). Then \( c := (a_0 d_0)^{-1} \cdot c_0 \circ (\varphi^n|_{U_0})^{-1} \in C_c(U_0) \) is such that \( a := L^n(c a c) \) is 1 on \( V \) and \( \|ct^n\|^2 = \|L^n(x_0)\| \leq \max_{x \in U_0} a_0(x) < (4\varepsilon)^{-1} \). Thus putting \( d := d_0 c t^n \) we get

\[ \|d^*bd - a\| = \|t^*a d_0 c t^*d x_0 - x_0^*a c t x_0\| \leq \|d_0^*bd_0 - a_0\| \cdot \|ct^n\|^2 \leq \|d_0^*bd_0 - a_0\| \cdot \|(c t^n)\|^2 < (\varepsilon + \varepsilon)(4\varepsilon)^{-1} < 1/2. \]
Theorem 8.5. If $L$ is minimal and contractive, then $A \times L$ is purely infinite and simple.

Proof. By Proposition 8.3 $A \times L$ is simple and contains an infinite projection $p$. Hence it suffices to show that for each non-zero positive $b_0 \in A \times L$, the hereditary $C^*$-subalgebra $b_0(A \times L)b_0$ generated by $b_0$ contains a projection equivalent to the infinite projection $p$. To this end, note that by Theorem 7.11 $\varphi : \Delta_{\text{reg}} \to X$ is topologically free. Let $x_0 \in \Delta$ be such that $\bigcup_{n=0}^{\infty} \varphi |_{\Delta_{\text{reg}}}^{-1}(x_0) = X$ and every neighbourhood of $x_0$ contains a non-empty contracting open set. By Lemma 8.4 there is $d \in A \times L$ and $a \in A$ which is 1 on a neighbourhood $V$ of $x_0$ and $\|d^*b_0d - a\| < 1/2$. We may take $V$ to be a precompact open contracting set. By Lemma 8.2 there are non-zero $b, c \in A \times L$ such that

$$b^*bb = b, \quad b^*bc = c, \quad b^*c = 0 \quad \text{and} \quad ab = b.$$  

Since $A \times L$ is simple and $c^*c \neq 0$ there are $b_1, \ldots, b_l \in A \times L$ such that $p = \sum_{k=1}^l b_k^*c^*cb_k$, see [Kat02, Lemma 4.1]. Set $e := \sum_{k=1}^l b_k^*c^*b^*c = \sum_{k=1}^l b_k^*c^*cb_k = p$.

In particular, $\|e\| = 1$. Using all these we get

$$\|e^*d^*b_0de - p\| = \|e^*(d^*b_0de - a)e\| < 1/2.$$

Let $f$ be the characteristic function of the interval $(\frac{1}{2}, \frac{3}{4})$. Then $p_0 := f(e^*d^*b_0de)$ is a well defined projection with $\|p_0 - e^*d^*b_0de\| < 1/2$, cf. [RLL00, Lemma 2.2.4]. Hence $\|p_0 - p\| < 1$ and therefore $p_0$ and $p$ are equivalent, cf. [RLL00, Proposition 2.2.5]. Then $q := f(\sqrt{b_0de}e^*d^*\sqrt{b_0})$ is a projection in $b_0(A \times L)b_0$ which is equivalent to $p_0$ and hence to the infinite projection $p$. \hfill \Box

Corollary 8.6. If $X$ is second countable and $L$ is minimal and contractive, then $A \times L$ is a Kirchberg algebra, i.e. a simple, separable, nuclear, purely infinite $C^*$-algebra satisfying the UCT.

Proof. Combine Theorems 8.5 and 8.8. \hfill \Box

Example 8.7 ($C^*$-algebras of rational maps). Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree at least two, and let $X = \Delta = J_R$ the Julia set for $R$. The transfer operator $L : C(J_R) \to C(J_R)$ for $\varphi : J_R \to J_R$, considered in Example 3.15 is minimal and contractive. Indeed, $\Delta_{\text{pos}} = J_R$ is uncountable and $\Delta_{\text{pos}} \setminus \Delta_{\text{reg}}$ is finite by [Bea91, Corollary 2.7.2, Theorem 4.2.4]. Moreover, for any open $V \subseteq J_R$ there is $n \in \mathbb{N}$ such that $R^n(V) = J_R$, by [Bea91, Theorem 4.2.5], and $\bigcup_{n=0}^{\infty} R^{-n}(z) = J_R$ for every $z \in J_R$, by [Bea91, Theorem 4.2.7]. So one may find $z_0 \in J_R$ whose inverse orbit $\bigcup_{n=0}^{\infty} R^{-n}(z)$ does not contain any critical point, and any open neighbourhood of $z_0$ contains a contractive open set. Hence $C(J_R) \times L$ is simple and purely infinite. This recovers [KW05, Theorem 3.8] as a special case of Theorem 8.5. Note that the Fatou set $F_R = \hat{\mathbb{C}} \setminus J_R$ is open and invariant for $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Thus $C(\hat{\mathbb{C}}) \times L$ is simple if and only if $\hat{\mathbb{C}} \setminus J_R$.

As a by product we also recover the main result of [Ham16]. Namely, let $\mu^L$ be the Lyubich measure. It is a $\varphi$-invariant regular probability measure whose support is $J_R$. Denoting by $T_\varphi$ the composition operator on $L_2(\mu^L)$ and identifying $C(J_R)$ with operators of multiplication on $L_2(\mu^L)$ we get $L(a) = d \cdot T_\varphi a T_\varphi$ where $d$ is degree of $R$, see [Lyu83, Lemma, p. 366]. Thus
using Proposition 2.12 one gets that \((id, d^{1/2}T_\varphi)\) is covariant representation of \(L : C(J_R) \to C(J_R)\) and therefore the simple C*-algebra \(C(J_R) \rtimes L\) is isomorphic to the C*-subalgebra of \(B(L_2(\mu^L))\) generated by \(C(J_R)\) and the composition operator \(T_\varphi\).

**Example 8.8** (Branched expansive coverings). Consider the transfer operator from Example 3.16 where \(\varphi : X \to X\) is a continuous map on a compact metric space \(X\) whose inverse has a finite number of continuous branches \(\gamma = \{\gamma_i\}^{N}_{i=1}\) that are proper contractions and \(X\) is self-similar for \(\gamma\). In other words, \(X\) is covered by compact sets \(\Delta_i, \ i = 1, ..., N,\) such that \(\varphi : \Delta_i \to X\) is an expansive homeomorphism \((\gamma_i = \varphi|_{\Delta_i}^1)\). As in [KW06] we assume the open set condition for \(\gamma\), which in terms of \(\varphi\) says that there is a non-empty open set \(V \subseteq X\), such that \(\varphi^{-1}(V) \subseteq V\) and \(\varphi^{-1}(V) \cap \Delta_i \cap \Delta_j = \emptyset\) for \(i \neq j\). Then \(\varphi^{-1}(V)\) is necessarily an open dense set in \(X\) not intersecting the set of branching points \(B = \bigcup_{i \neq j}\{x \in \Delta_i \cap \Delta_j\}\) and we have \(\Delta_{\text{reg}} = X \setminus B\), cf. [KW06] Proposition 2.6]. Using this one infers that each of the sets \(\varphi^n(B)\) has empty interior. Thus \(X \setminus \bigcup_{n=0}^\infty \varphi^n(B)\) is dense in \(X\) by Baire theorem. For any \(x \in X \setminus \bigcup_{n=0}^\infty \varphi^n(B)\) its negative orbit \(\bigcup_{n=0}^\infty \varphi^{-n}(x)\) lies entirely in \(\Delta_{\text{reg}}\). Every negative orbit is dense in \(X\). Indeed, \(A := \bigcup_{n=0}^\infty \varphi^{-n}(x)\) is a closed set with \(\varphi^{-1}(A) \subseteq A\), which implies \(A = X\) by the uniqueness of the self-similar set \(X\), see [Huc81]. Using expansiveness of \(\varphi\) we conclude that every neighbourhood of \(x_0 \in X \setminus \bigcup_{n=0}^\infty \varphi^n(B)\) contains a non-empty contracting open set. Minimality is clear. Hence \(C(X) \rtimes L\) is a unital Kirchberg algebra by Corollary 5.1. This recovers [KW06] Theorem 3.8] when the systems of contractive maps form inverse branches of a continuous map.

Recall that the Hutchinson measure \(\mu^H\) is the unique regular probability measure such that \(\mu^H(A) = 1/N \sum_{i=1}^N \mu^H(\gamma_i(A))\) for all Borel \(A \subseteq X\). It support is \(X\) and so we may identify \(C(X)\) with operators of multiplication on \(L_2(\mu^H)\). If \(\mu^H(B) = 0\) (which is automatic when \(X \subseteq \mathbb{R}^d\) and \(\gamma_i\)'s are similitudes, see [Sch94]), then the composition operator \(T_\varphi\) is an isometry on \(L_2(\mu^L)\) satisfying \(L(a) = 1/N \cdot T_\varphi^a T_\varphi\), see [Ham19]. Thus using Proposition 2.12 one sees that \((id, N^{-1/2}T_\varphi)\) is a covariant representation of \(L : C(J_R) \to C(J_R)\) and therefore \(C(X) \rtimes L\) is isomorphic to the C*-subalgebra of \(B(L_2(\mu^H))\) generated by \(C(X)\) and the composition operator \(T_\varphi\). This recovers the main result of [Ham19].

**Example 8.9** (Expanding local homeomorphisms). Assume \(\varphi : X \to X\) is an open continuous expanding map on a compact metric space, cf. [Ana97], [BK21] and references therein. Then any continuous \(g : X \to (0, \infty)\) defines a transfer operator \(L\) for \(\varphi\) and \(C(X) \rtimes L \cong C^*(G)\), by Theorem 4.4. By [BK21] Lemma 7.4], \(\varphi\) is topologically free if and only if \(X\) has no isolated periodic points. Clearly, \(L\) is minimal if and only if \(\varphi\) is \emph{minimal}, i.e. there is no non-trivial open set \(U\) with \(\varphi^{-1}(U) = U\). Thus assuming \(X\) is infinite, by Theorem 7.11 we get that

\[
C(X) \rtimes L \text{ is simple if and only if } \varphi \text{ is minimal.}
\]

Assume now that there are no wandering points in \(X\), or equivalently that periodic points are dense in \(X\). Then by spectral decomposition, cf. [BK21] Theorem 2.5], \(\varphi\) is minimal iff \(\varphi\) is topologically transitive iff for every non-empty open \(U \subseteq X\) there is \(N \in \mathbb{N}\) such that \(\bigcup_{k=1}^N \varphi^k(U) = X\). Thus if \(\varphi\) is minimal, then every negative orbit \(\bigcup_{n=0}^\infty \varphi^{-n}(x)\) is dense in \(X\) and every non-trivial open subset \(V \subseteq X\) is contracting, so in particular \(L\) is contracting. Hence by Corollary 5.1, we get

\[
C(X) \rtimes L \text{ is a Kirchberg algebra, if } \varphi \text{ is minimal and has no wandering points.}
\]

This last statement improves [Ana97] Proposition 4.2] (in the minimal case) and implies [EHR11] Proposition 4.2]. If there are no wandering points, then by [BK21] Proposition 3.8]
there exists a \( \varphi \)-invariant Borel probability measure \( \mu \) with support \( X \) such that identifying \( C(X) \) with operators of multiplication on \( L_2(\mu) \) we have \( L(1)^{-1}L(a) = T_\varphi aT_\varphi \), for \( a \in C(X) \), where \( T_\varphi \in B(L_2(\mu)) \) is the composition operator with \( \varphi \). Also \( (id, L(1)^{-1/2}T_\varphi) \) is a covariant representation of \( L \). Thus is if \( X \) has no isolated periodic points, then \( C(X) \times L \) is isomorphic to the \( C^* \)-subalgebra of \( B(L_2(\mu)) \) generated by \( C(X) \) and the composition operator \( T_\varphi \). If in addition \( \varphi \) is minimal (topologically transitive) and \( \ln \varphi \) is Hölder continuous, then the above measure \( \mu \) is unique and it is the Gibbs measure for \( \varphi \) and \( \ln \varphi \).

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