Exactly and Quasi-Exactly Solvable two-mode Bosonic Hamiltonians

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Abstract

We develop a method to determine the eigenvalues and eigenfunctions of two-boson Hamiltonians include a wide class of quantum optical models. The quantum Hamiltonians have been transformed in the form of the one variable differential equation and the conditions for its solvability have been discussed. Applicability of the method is demonstrated on some simple physical systems.

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I. INTRODUCTION

For over ten years there has been a great deal of interest in quantum optical models which reveal new physical phenomena described by the Hamiltonians expressed as nonlinear functions of Lie algebra generators or boson operators \([1, 2, 3, 4, 5]\). Such systems have often been analyzed by using numerical methods, because the implementation of the Lie algebraic techniques to solve those problems is not very efficient and most of the other analytical techniques do not yield simple analytical expressions. They require tedious calculations \([6, 7, 8]\).

However, recently a new algebraic approach, essentially improving both analytical and numerical solution of the problems, has been suggested and developed for some nonlinear quantum optical systems \([9, 10, 11, 12]\). Most of such developments are mainly based on linear Lie algebras, but it is evident that there is no physical reason for symmetries to be only linear. Nonlinear Lie algebra techniques and their relations to the nonlinear quantum optical systems have been discussed \([13, 14, 15, 16]\). In both cases finite part of spectrum of the corresponding Hamiltonian can be exactly obtained in closed forms and these systems are known as quasi-exactly-solvable (QES) termed by Turbiner and Ushveridze \([17]\). Recently it has been proven that the single boson Hamiltonians also lead to a QES under some certain constraints \([18, 19]\).

The aim of this paper is to determine quasi-exact-solvability of the two-boson systems and discuss their possible applications in physics. As a particular case our model includes the solutions of the Karassiov-Klimov Hamiltonian and the Hamiltonian of the systems of photons and bosons expressed in a single mode form. These Hamiltonians are not only mathematically interesting but they have also potential interest in physics \([20, 21, 22]\).

The paper is organized as follows: In section 2, general form of the Hamiltonian has been constructed and its solution by using the invariance of the number operator have been discussed. This section includes some physical examples. In section 3 we present a transformation procedure in order to solve a wide range of the Hamiltonian. In section 4 the Hamiltonian is transformed in the form of the one variable differential equation and the conditions to obtain its eigenvalues and eigenfunctions are discussed. Finally, in section 5 we comment on the validity of our method and suggest the possible extensions of the problem.
II. TRANSFORMATION OF THE TWO-MODE BOSONIC SYSTEMS IN THE FORM OF SINGLE-VARIABLE DIFFERENTIAL EQUATION

Two mode bosonic Hamiltonians play an important role in nonlinear quantum optical systems. The Hamiltonians of such systems can be generalized as follows:

\[ H = \sum_{m_i} \alpha_{m_1,m_2,m_3,m_4} (a_1^+)^{m_1} (a_1)^{m_2} (a_2^+)^{m_3} (a_2)^{m_4} \]  

(1)

where \( \alpha_i \) is a constant and \( m_i \) determines order of the interaction. The boson creation, \( a_1 \), \( a_2 \), and annihilation \( a_1^+ \), \( a_2^+ \) operators obey the usual commutation relations

\[ [a_1, a_2] = [a_1, a_2^+] = [a_2, a_1^+] = [a_1^+, a_2] = 0, \quad [a_1, a_1^+] = [a_2, a_2^+] = 1. \]  

(2)

We assume that the conserved quantity of the physical system described by the Hamiltonian \( H \), correspond to the operator:

\[ K = s a_1^+ a_1 + p a_2^+ a_2. \]  

(3)

Clearly the Hamiltonian conserves the number of particles in the system, when \( H \) commutes with the \( K \). Meanwhile, we have mention here, in general, conserved quantity of \( K \) type appears more general contexts than the number operator. Indeed a convenient quantum mechanical treatment of the \( m^{th} \) order interaction of the two-boson model exploits the \( SU(2) \) algebra structure of the problem, by expressing the Hamiltonian as a polynomial of the angular momentum operators whose degree depends on the degree of interaction. In this case the conserved quantity is the total angular momentum, \( J = j(j+1) \), which can be related to the number operator \( j = N/2 \), where \( N = a_1^+ a_1 + a_2^+ a_2 \). Therefore the conserved quantity \( J \) crucially depends on the conservation of the number of particles. The classical motion of the particle takes place in the space of angular momentum on the sphere of radius \( j(j+1) \). To have a physical insight in to \( K \), from other perspective, when the motion of the particle takes the place on an ellipsoid the conserved quantity \( K \) takes the form \( K \). The operator \( K \) and bosonic operators satisfy the commutation relations

\[ [K, a_1^+] = s a_1^+; \quad [K, a_1] = -s a_1; \quad [K, a_2^+] = p a_2^+; \quad [K, a_2] = -p a_2. \]  

(4)

If \( K \) and \( H \) commute, then they have the same eigenfunction. Therefore it is worth to seek the conditions for the commutation of \( H \) and \( K \). This can easily be done by using the
commutation relations (2) and (4) and we obtain the following relation
\[ [K, H] = \sum_{m_i} [(s(m_1 - m_2) + p(m_3 - m_4)] \alpha_{m_1 m_2 m_3 m_4} (a_1^+)^{m_1} (a_2^+)^{m_2} (a_3^+)^{m_3} (a_4^+)^{m_4}. \] (5)

It is obvious that the constant of motion \( K \) and \( H \) commute when the following set of equation is satisfied
\[ \sum_{m_i} [(s(m_1 - m_2) + p(m_3 - m_4)] = 0. \] (6)

The action of the operator \( K \) on the state \( |n_1, n_2\rangle \) is given by
\[ K |n_1, n_2\rangle = (s n_1 + p n_2) |n_1, n_2\rangle. \] (7)

In this section we seek the solution of the eigenvalue equation (7) in the Bargmann-Fock space. The usual realization of the bosonic operators in Hilbert space is given by
\[ a_1^+ = \frac{1}{\sqrt{2}} \left( -\frac{\partial}{\partial x_1} + x_1 \right); \quad a_1 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_1} + x_1 \right), \] (8a)
\[ a_2^+ = \frac{1}{\sqrt{2}} \left( -\frac{\partial}{\partial x_2} + x_2 \right); \quad a_2 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_2} + x_2 \right). \] (8b)

The operators can be transformed to the Bargmann-Fock space by introducing the operator
\[ \Gamma = \exp \left[ \frac{i}{8} (a_1^2 + a_1^{+2} + a_2^2 + a_2^{+2}) \right] \] (9)
and the similarity transformation
\[ b_1 = \Gamma a_1 \Gamma^{-1} = \frac{1}{\sqrt{2}} (a_1^+ - a_1), \quad b_1^+ = \Gamma a_1^+ \Gamma^{-1} = \frac{1}{\sqrt{2}} (a_1^+ + a_1) \]
\[ b_2 = \Gamma a_2 \Gamma^{-1} = \frac{1}{\sqrt{2}} (a_2^+ - a_2), \quad b_2^+ = \Gamma a_2^+ \Gamma^{-1} = \frac{1}{\sqrt{2}} (a_2^+ + a_2). \] (10)

The Bargmann-Fock space differential realizations of the operators take the form
\[ b_1 = \frac{\partial}{\partial x_1}; \quad b_1^+ = x_1; \quad b_1 = \frac{\partial}{\partial x_2}; \quad b_2^+ = x_2 \] (11)

Thus, in the Bargmann-Fock space the eigenvalue problem (7) leads to the following solution:
\[ \psi(x_1, x_2) = x_1^k \phi \left( x_2 x_1^{-\frac{1}{p}} \right). \] (12)

where \( k \) is given by
\[ k = n_1 + \frac{p}{s} n_2. \] (13)

The eigenfunction of the Hamiltonian can be obtained from the relation
\[ |n_1, n_2\rangle = \Gamma^{-1} \psi(x_1, x_2). \] (14)

In the following section the application of our procedure will be discussed explicitly.
A. Hamiltonian of the second Harmonic generation and generalization of the solvability conditions

The Hamiltonian (1) includes various physical Hamiltonians, second and third harmonic generation effective Hamiltonians. This gives us an opportunity to test our approach because those Hamiltonians have been studied in literature. Consider the following Hamiltonian:

$$ H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \kappa (a_1^+)^2 a_2 + \kappa a_2^+ (a_1)^2 \quad (15) $$

which can be related to the Hamiltonian (1), when the parameters

$$ \alpha_{m_1 m_2, m_3, m_4} = 0 \quad (16) $$

except that

$$ \alpha_{1,0,0,0} = \omega_1, \alpha_{0,0,1,1} = \omega_2, \alpha_{2,0,0,1} = \kappa, \alpha_{0,2,1,0} = \kappa \quad (17) $$

The condition (16) is satisfied when $p = 2s$. The constants $\omega_1$ and $\omega_2$ are the angular frequencies of two independent harmonic oscillators characterized by annihilation and creation operators, $\kappa$ and $\kappa$ are the coupling coefficients that determine the strength of the interaction of the oscillators. In the following we use the Bargmann-Fock realization, where creation and annihilation operators $a_i^+$ and $a_i$ are replaced by $b_i^+$ and $b_i$, respectively. The eigenfunction (12) is of the form

$$ \psi(x_1, x_2) = (x_1)^k \phi(z) \quad (18) $$

where $z = x_2 x_1^{-1/2}$. The solution of this system describes a quantum mechanical state of $H$ provided that $\phi(z)$ belongs to the Bargmann-Fock space. The eigenvalue equation of the second harmonic generation Hamiltonian can be written as

$$ H \psi(x_1, x_2) = E \psi(x_1, x_2) \quad (19) $$

Insertion of (18) into (19) yields the following differential equation

$$ \left[ 4\kappa z^2 \frac{d^2}{dz^2} + (\kappa + z (\omega_2 - 2\omega_1 + 2\kappa z(3 - 2k))) \frac{d}{dz} + \omega_2 + k\omega_1 + \kappa z(k(k - 1) - E) \right] \phi(z) = 0. \quad (20) $$
According to Turbiner\[17\] the Hamiltonian (20) is quasi-exactly solvable. The eigenvalue equation (20) can be obtained as follows. Let us assume that the function \(\phi(z)\) is a polynomial in \(z\) with the coefficients being functions of energy:

\[
\phi(z) = \sum_{m=0}^{\infty} P_m(E) z^m. \tag{21}
\]

When inserted \(\phi(z)\) in (20) then we obtain the following three-term recurrence relation:

\[
\kappa (k-2m)(k-2m-1)P_{m+1}(E) + \left(\omega_2 + \kappa \omega_1 + m(\omega_2 - 2\omega_1) - E\right)P_m(E) + \kappa m P_{m-1}(E) = 0. \tag{22}
\]

The function \(P_m(E)\) terminates when \(m > k/2\), and therefore the roots of the \(P_m(E)\) belong to the spectrum of the Hamiltonian. For the corresponding problem, Beckers, Brihaye and Debergh\[13\] have obtained three different Hamiltonians in the framework of nonlinear algebra. Two of the Hamiltonians are PT-symmetric and they related to the sextic harmonic oscillator. The third Hamiltonian is exactly the same one in (20) which was obtained in a different method. The Hamiltonian (20) can also be transformed in the form of the Schrödinger equation by changing variable \(z = -1/\kappa y^2\) and defining the wavefunction

\[
\phi(z) = e^{-\int W(y) dy} \psi(y) \tag{23}
\]

where

\[
W(y) = \frac{k}{y} + \frac{(\omega_2 - 2\omega_1)y}{4} - \frac{\kappa \omega_1 y^3}{4} \tag{24}
\]

Substituting (23) in (20) we obtain the following equation:

\[
H = -\frac{1}{2} \frac{d^2}{dy^2} + \frac{1}{4} \left((2k + 5)\omega_2 - 2\omega_1\right) y^2 - \frac{1}{8} \kappa \omega_2 (\omega_2 - 2\omega_1)y^4 + \frac{1}{16} \kappa \omega_1^2 y^6. \tag{25}
\]

The last equation is the Hamiltonian of the sextic Harmonic oscillator. Consequently the procedure given here can be applied to obtain eigenfunction and eigenvalues of the various physical Hamiltonians. The validity of the procedure depends on the choice of the \(\alpha_{m_1,m_2,m_3,m_4}\). One can easily be obtain various physical Hamiltonians by appropriate choice.
of $\alpha_{m_1 m_2, m_3, m_4}$ and by considering the conditions given in (11). For instance one can easily obtain $n^{th}$ harmonic generation Hamiltonian by setting:

$$\alpha_{1,1,0,0} = \omega_1, \alpha_{0,0,1,1} = \omega_2, \alpha_{n,0,0,1} = \kappa, \alpha_{0,n,1,0} = \overline{\kappa}$$

and the remaining values of the $\alpha_{m_1 m_2, m_3, m_4} = 0$. Then the Hamiltonian (11) takes the form

$$H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \kappa (a_1^+)^n a_2 + \overline{\kappa} a_2^+ (a_1)^n.$$  (27)

In the following section we continue to explore the solution of the Hamiltonian (11), in the Bargmann-Fock space by developing a similarity transformation procedure.

### III. TRANSFORMATION OF THE BOSONIC OPERATORS

In this section we present two transformation procedure to obtain the conditions of the solvability of the Hamiltonian (11). The following transformations allow us to study a wide range of physical systems. Let us introduce the following similarity transformation induced by the operator

$$S = (a_2^+)^{\eta a_1} a_1$$  (28)

where $\eta$ is a constant and it can be determined by considering the transformation of the number operator (3). The operator $S$ acts on the state $|n_1, n_2\rangle$ as follows,

$$S |n_1, n_2\rangle = (a_2^+)^{\eta n_1} |n_1, n_2\rangle = \sqrt{\frac{n_2!}{(n_2 + \eta n_1)!}} |n_1, n_2 + \eta n_1\rangle.$$  (29)

Since $a_1$ and $a_2$ commute, the transformation of $a_1$ and $a_1^+$ under $S$ can be obtained by writing $a_2^+ = e^b$, with $[a_1, b] = [a_1^+, b] = 0$,

$$S a_1^+ S^{-1} = e^{\eta b a_1^+} a_1^+ e^{-\eta b a_1^+} a_1 = a_1^+(a_2^+)^{\eta}$$

$$S a_1 S^{-1} = e^{\eta b a_1^+} a_1 e^{-\eta b a_1^+} a_1 = a_1(a_2^+)^{-\eta}$$

and transformation of $a_2$ and $a_2^+$ as follows

$$S a_2^+ S^{-1} = (a_2^+)^{\eta a_1} a_2^+ (a_2^+)^{-\eta a_1} a_1 = a_2^+$$

$$S a_2 S^{-1} = (a_2^+)^{\eta a_1} a_2 (a_2^+)^{-\eta a_1} a_1 = a_2 - \eta a_1^+ a_1 (a_2^+)^{-1}.$$  (31)
In a similar manner, we prepare the other similarity transformation which is useful to study QES of the Hamiltonian (1), by introducing the following operator:

\[ T = a_2^{\alpha a_1^+} a_1 \]  (32)

The operator \( T \) acts on the two-boson state as

\[ T |n_1, n_2\rangle = a_2^{\alpha n_1} |n_1, n_2\rangle = \sqrt{\frac{n_2!}{(n_2 - \alpha n_1)!}} |n_1, n_2 - \alpha n_1\rangle. \]  (33)

Since \( a_1 \) and \( a_2 \) commute, the transformation of \( a_1 \) and \( a_1^+ \) under \( S \) can be obtained by letting \( a_2 = e^c \) with \([a_1, c] = [a_1^+, c] = 0\),

\[ Ta_1^+T^{-1} = e^{\alpha a_1^+ a_1} = a_1^+ (a_2)^\alpha \]

\[ Ta_1T^{-1} = e^{-\alpha a_1^+ a_1} = a_1 (a_2)^{-\alpha} \]  (34)

The transformation of \( a_2 \) and \( a_2^+ \) is as follows:

\[ Ta_2^+T^{-1} = a_2^{\alpha a_1^+} a_2^+ a_1^+ a_1 = a_2^+ + \alpha a_1^+ a_1 a_2^{-1} \]

\[ Ta_2T^{-1} = a_2^{\alpha a_1^+} a_2 a_2^+ a_1^+ = a_2 \]  (35)

These two transformations lead to the conditions of the quasi-exact-solvability of the Hamiltonian (1).

**IV. SOLVABILITY OF THE HAMILTONIAN**

In this section we discuss the solvability of the Hamiltonian (1). The conserved quantity \( K \) of the physical system describes the states of the corresponding Hamiltonian. The transformation of \( K \) under the operator \( S \) is given by

\[ K' = SKS^{-1} = (s - p\eta) a_1^+ a_1 + pa_2^+ a_2 \]  (36)

The Hamiltonian (1) is characterized by the total number of \( a_1 \) and \( a_2 \) bosons when the conserved quantity of the physical system \( K \) commutes with the whole Hamiltonian. In the transformed case it is only the number of \( a_2 \) bosons that characterize the system under the
condition $\eta = s/p$. According to (36) the representation is characterized by a fixed number $a_2^+a_2 = k$. Therefore in the transformed form the Hamiltonian can be expressed as one boson operator $a_1$, when the condition (11) is taken into consideration. The transformed form of the Hamiltonian (11) can be written as:

$$\tilde{H} = SHS^{-1} = \sum_{m_1} \alpha_{m_1,m_2,m_4}(a_1^+)^{m_1}(a_1)^{m_2}(k - \frac{s}{p}a_1^+a_1)^{m_4}$$

(37)

The difference between (11) and (37) is that while in the first the total number of $a_1$ and $a_2$ bosons characterize the system, in the later it is only the number of $a_2$ bosons that characterize the system. Therefore the representation is characterized by a fixed number $k$ and in (37), the Hamiltonian is expressed in terms of one boson operator $a_1$. The transformed Hamiltonian $\tilde{H}$, in the Bargmann-Fock space, which plays an important role in the quasi-exact solution of the equation (11). It can be transformed in the form of the one dimensional differential equations in the Bargmann-Fock space when the boson operators are realized as

$$a_1 = \frac{d}{dx}, \quad a_1^+ = x.$$  

(38)

The basis function of the primed generators of the system is the degree of polynomial of order $k$,

$$P_n(x) = (x^0, x^1, \cdots, x^k).$$

(39)

Action of (37) on the (39), in the Bargmann-Fock space can be written as:

$$\tilde{H}P_n(x) = \sum P_n(E)x^n$$

(40)

where $E$ is the eigenfunction of the Hamiltonian and the polynomial $P(E)$ can be written as

$$\sum_{m_1} \alpha_{m_1,m_2,m_4}(k - \frac{s}{p}n)^{m_4} \frac{n!}{(n - m_2)!}P_{n+m_1-m_2}(E) - EP_n(E) = 0$$

(41)

The wavefunction is itself the generating function of the energy polynomials. The eigenvalues are then produced by the roots of such polynomials. If the $E_j$ is a root of the polynomial $P_{j+1}(E)$, the series (41) terminates at $j > k\frac{p}{s}$ and $E_j$ belongs to the spectrum of the corresponding Hamiltonian. The eigenvalues are then obtained by finding the roots of such polynomials.
The constant of motion $K$ characterize the system can be transformed, by the operator $T$ is given by

$$K' = TKT^{-1} = (s + p\eta)a_1^+a_1 + pa_2^+a_2. \quad (42)$$

The Hamiltonian (1) can be characterized, in the transformed case, $\eta = -s/p$. Thus according to (42) the representation is characterized by a fixed number $a_2^+a_2 = k$. Therefore the transformed Hamiltonian includes one boson operator $a_1$, when the condition (6) is taken into consideration, and it can be written as:

$$H' = THT^{-1} = \sum_{m_i} \alpha_{m_1,m_2,m_3}(a_1^+)^{m_1}(a_1)^{m_2}(k + \frac{s(m_1 - m_2)}{pm_3} + \frac{s}{p}a_1^+a_1)^{m_2} \quad (43)$$

The Hamiltonian can be expressed as one dimensional differential equation in the Bargmann-Fock space. In order to solve the Hamiltonian $H'$ we can follow the procedure given (38) through (41). In this case the polynomial function is terminated when $k$ is constrained to

$$k > -\frac{s(m_1 - m_2)}{pm_3} - \frac{s}{p}n \quad (44)$$

Consequently we have obtained two classes of Hamiltonians whose spectrum can be obtained (quasi)exactly.

V. CONCLUSION

In this paper we have prepared a general method to obtain the solution of two boson Hamiltonian. By using either solution of number operator or similarity transformation, we have been able to provide a QES of the various physical Hamiltonians. Furthermore, it has been given that two boson Hamiltonian can be reduced to single variable differential equation in the Bargmann-Fock space. It is also important to mention here that the methods given here can be used to solve higher order differential equations.

The method given here, can easily be extended to solve the Hamiltonians that include multi-boson or fermion-boson systems. This extension leads to the solution of the various physical problems, such as Pauli equation, Rabi Hamiltonian, Jaynes-Cummings Hamiltonian.

[1] J. H. Eberly, N. B. Narozhny and J. J. Sanchez-Mondragon Phys. Rev. Lett. 44 1329 (1980).
[2] V. P. Karassiov J. Phys. A 27 153 (1994).
[3] V. P. Karassiov and A. B. Klimov Phys. Lett. A 189 43 (1994).
[4] J. Delgado, A. Luis, L. L. Sánchez-Soto and A. B. Klimov J. Opt. B: Quantum Semiclass. Opt. 2 33 5 (2000).
[5] A. B. Klimov, J. L. Romero, J. Delgado and L. L. Sánchez-Soto J. Opt. B: Quantum Semiclass. Opt. 5 34 6 (2003).
[6] V. P. Karassiov Phys. Lett. A 238 19 (1998).
[7] A. Bandilla, G. Drobny and I. Jex Phys. Rev. A 53 507 (1996).
[8] B. Jurco J. Math. Phys. 30 1289 (1989).
[9] A. B. Klimov and L. L. Sanchez-Soto Pyhs. Rev. A 61 063802 (2000).
[10] V. P. Karassiov, A. A. Gusey and S. I. Vinitsky hep-lat/0105152 (2001).
[11] G. Alvarez and R. F. Alvarez-Estrada J. Phys. A: Math. Gen. 34 10045 (2001)
[12] G. Alvarez and R. F. Alvarez-Estrada J. Phys. A: Math. Gen. 28 5767 (1995).
[13] J. Beckers, Y. Brihaye and N. Debergh J. Phys. A: Math. Gen. 32 2791 (1999).
[14] T. Tjin Int. J. Mod. Phys. A7 6175 (1992).
[15] B. Abdesselam, J. Beckers, A. Chakrabarti and N. Debergh J. Phys. A 29 3075 (1996).
[16] V.Sunilkumar, B. A. Bambah, R. Jagannathan, P. K. Panigrahi and V. Srinivasan J. Opt. B: Quantum Semiclass. Opt. 2 126 (2000).
[17] A. V. Turbiner and A. G. Ushveridze Phys. Lett. A 126 181 (1987).
[18] S. N. Dolya and O. B. Zaslavskii J. Phys. A: Math. Gen. 33 L369 (2000).
[19] S. N. Dolya and O. B. Zaslavskii J. Phys. A: Math. Gen. 34 5955 (2001).
[20] J. Perina Quantum Statistics of Linear and Nonlinear Optical Phenomena (Dordrecht: Kluwer, 1991) chap. 10
[21] J. Bajer and A. Miranowicz J. Opt. B: Quantum Semiclass. Opt. 2 L10 (2000)
[22] Qu Fa, Wei Bao-Hua, K. W. Yu and Lui Cui-Hong J. Phys. Condens. Mat. 8 2957 (1996)
[23] Y. Alhassid, F. Gürsey and F. Iachello F Ann. Phys.(N. Y.) 148 346 (1983)
[24] B. G. Wybourne Classical Groups for Physicist (John Wiley & Sons New York, 1974)
[25] Y. Alhassid and R. D. Levine Phys. Rev. A 18 89 (1978)
[26] Y. Alhassid and S. E. Koonin Phys. Rev. C 23 1590 (1981)