On the singular hyperbolicity of star flows

Yi Shi  Shaobo Gan  Lan Wen*  

Abstract

We prove for a generic star vector field $X$ that if, for every chain recurrent class $C$ of $X$, all singularities in $C$ have the same index, then the chain recurrent set of $X$ is singular hyperbolic. We also prove that every Lyapunov stable chain recurrent class of a generic star vector field is singular hyperbolic. As a corollary, we prove that the chain recurrent set of a generic 4-dimensional star flow is singular hyperbolic.

1 Introduction

Let $M^d$ be a $d$-dimensional $C^\infty$ compact Riemannian manifold without boundary. Denote by $X^1(M^d)$ the space of $C^1$ vector fields on $M^d$, endowed with the $C^1$ topology. A vector field $X \in X^1(M^d)$ generates a $C^1$ flow $\phi_t = \phi_t^X$ on $M^d$, as well as the tangent flow $\Phi_t = d\phi_t$ on $TM^d$. Denote by $\text{Sing}(X)$ the set of singularities of $X$, and $\text{Per}(X)$ the set of periodic points of $X$. A singularity or a periodic orbit of $X$ are both called a critical orbit or a critical element of $X$.

A compact invariant set $\Lambda$ of $X$ is hyperbolic if there are two constants $C \geq 1, \lambda > 0$, and a continuous $\Phi_t$-invariant splitting

$$T_\Lambda M^d = E^s \oplus \langle X \rangle \oplus E^u$$

such that for every $x \in \Lambda$ and $t \geq 0$,

$$\|\Phi_t|_{E^s(x)}\| \leq Ce^{-\lambda t},$$

$$\|\Phi_{-t}|_{E^u(x)}\| \leq Ce^{-\lambda t}.$$  

Here $\langle X(x) \rangle$ denotes the space spanned by $X(x)$, which is 0-dimensional if $x$ is a singularity, or 1-dimensional if $x$ is regular. If $\Lambda$ consists of a critical element, denote the index of $\Lambda$ by $\text{Ind}(\Lambda) = \dim E^s$.

Let $\phi_t$ be the flow generated by a vector field $X$. For any $\varepsilon > 0, T > 0$, a finite sequence ${x_i}_{i=0}^n$ on $M$ is called $(\varepsilon, T)$-chain of $X$ if there are $t_i \geq T$ such that $d(\phi_{t_i}(x_i), x_{i+1}) < \varepsilon$ for any $0 \leq i \leq n - 1$. For $x, y \in M^d$, one says that $y$ is chain attainable from $x$ if there exists $T > 0$ such that for any $\varepsilon > 0$, there is an $(\varepsilon, T)$-chain $\{x_i\}_{i=0}^n$ with $x_0 = x$ and $x_n = y$. If $x$ is chain attainable from itself, then $x$ is called a chain recurrent point. The set of chain recurrent points is called chain recurrent set of $X$, denoted by $\text{CR}(X)$.

*2010 Mathematics Subject Classification: 37D30, 37D50
Key words and phrases: Singular hyperbolicity, star flow, Lyapunov stable class, shadowing
Chain attainability is a closed equivalence relation on CR(X). For each \( x \in \text{CR}(X) \), the equivalence class \( C(x) \) (which is compact) containing \( x \) is called the chain recurrent class of \( x \). A chain recurrent class is called trivial if it consists of a single critical element. Otherwise it is called nontrivial. Since every hyperbolic critical element \( c \) of \( X \) has a well-defined continuation \( c_Y \) for \( Y \) close to \( X \), the chain recurrent class \( C(c) \) also has a well-defined continuation \( C(c_Y, Y) \).

A compact invariant set \( \Lambda \) is called chain transitive if for every pair of points \( x, y \in \Lambda \), \( y \) is chain attainable from \( x \), where all chains are chosen in \( \Lambda \). Thus a chain recurrent class is just a maximal chain transitive set, and every chain transitive set is contained in a unique chain recurrent class.

A vector field \( X \in X^1(M^d) \) is called a star vector field or a star flow, if it satisfies the star condition, i.e., there exists a \( C^1 \) neighborhood \( U \) of \( X \) such that every critical element of every \( Y \in U \) is hyperbolic. The set of \( C^1 \) star vector field on \( M^d \) is denoted by \( X^*(M^d) \).

The notion of star system came up from the study of the famous stability conjecture. Recall that a classical theorem of Smale [28] (for diffeomorphisms) and Pugh-Shub [25] (for flows) states that Axiom A plus the no-cycle condition implies the \( \Omega \)-stability. Palis and Smale [23] conjectured that the converse also holds, which has been known as the \( \Omega \)-stability conjecture. In the study of the conjecture, Pliss, Liao and Mañé noticed an important condition called (by Liao) the star condition. As defined above, the star condition looks quite weak because, though involving perturbations, it concerns critical elements only, and the hyperbolicity considered is in an individual but not uniform way. Indeed, the \( \Omega \)-stability implies the star condition easily (Franks [7] and Liao [15]). Thus whether the star condition could give back Axiom A plus the no-cycle condition became a striking problem, raised by Liao [16] and Mañé [19]. An affirmative answer to the problem would, of course, contain the \( \Omega \)-stability conjecture. For diffeomorphisms, Aoki [1] and Hayashi [12] proved that the star condition indeed implies Axiom A plus the no-cycle condition. For flows, there are counterexamples if the flow has a singularity. For instance, the geometric Lorenz attractor [11], which has a singularity, is a star flow but fails to satisfy Axiom A. In fact, Liao [16] and Mañé [19] raised this problem for nonsingular star flows, and hence it was known as the nonsingular star flow problem. The problem was solved by Gan-Wen [9] proving that nonsingular star flows do satisfy Axiom A and the no-cycle condition.

These give rise to a new problem — to understand singular star flows, of which the geometric Lorenz attractor is one of the typical models. Note that, while being not structurally stable, the Lorenz attractor is quite robust under perturbations. Analytically, while being not hyperbolic, it exhibits quite some contractions and expansions. How to describe such a dynamics? Morales, Pacifico and Pujals [21] have given an appropriate notion about it, called singular hyperbolicity, which is of central importance to the subject. Their definition is for dimension 3, and the following higher dimensional version can be found in [32, 20].

**Definition 1.1.** (Positive singular hyperbolicity) Let \( \Lambda \) be a compact invariant set of \( X \in X^1(M^d) \). We say that \( \Lambda \) is positively singular hyperbolic of \( X \) if there are constants \( C \geq 1 \) and \( \lambda > 0 \), and a continuous invariant splitting
\[
T_\Lambda M = E^{ss} \oplus E^{cu}
\]

w.r.t. \( \Phi_t \) such that, for all \( x \in \Lambda \) and \( t \geq 0 \), the following three conditions are satisfied:
(1) $E^{ss}$ is $(C, \lambda)$-dominated by $E^{cu}$, i.e., $\|\Phi_t|_{E^{ss}(x)}\| \cdot \|\Phi_{-t}|_{E^{cu}(\phi_t(x))}\| \leq C e^{-\lambda t}$.

(2) $E^{ss}$ is uniformly contracting, i.e., $\|\Phi_t|_{E^{ss}(x)}\| \leq C e^{-\lambda t}$.

(3) $E^{cu}$ is sectionally expanding, i.e., for any 2 dimensional subspace $L \subset E^{cu}(x)$,

$$|\det (\Phi_t|_L)| \geq C^{-1} e^{\lambda t}.$$ 

We say that $\Lambda$ is negatively singular hyperbolic of $X$ if $\Lambda$ is positively singular hyperbolic of $-X$.

A union of finitely many positively singular hyperbolic sets is positively singular hyperbolic. Likewise for the negative case.

**Definition 1.2.** (Singular hyperbolicity) We say that $\Lambda$ is singular hyperbolic of $X$ if it is either positively singular hyperbolic of $X$, or negatively singular hyperbolic of $X$, or a disjoint union of a positively singular hyperbolic set of $X$ and a negatively singular hyperbolic set of $X$.

Using the notion of singular hyperbolicity, the following conjecture was formulated in [32]:

**Conjecture.** [32] For every star vector field $X \in X^*(M^d)$, the chain recurrent set $\text{CR}(X)$ is singular hyperbolic and consists of finitely many chain recurrent classes.

**Remark.** The conjecture is open even in 2-dimensional case.

In this paper we obtain some partial results to this conjecture. Let us say that a set $C$ has a homogeneous index for singularities if all the singularities in $C$ have the same index. Here are the main theorems of this paper.

**Theorem A.** There is a dense $G_\delta$ set $\mathcal{G}_A \subset X^*(M^d)$ such that, for every $X \in \mathcal{G}_A$, if a chain recurrent class $C$ of $X$ has a homogeneous index for singularities, then $C$ is positively or negatively singular hyperbolic.

**Remark.** The homogeneity requirement here looks restrictive. However, we will prove that, for generic star vector fields, any chain recurrent class can have at most two different indices for its singularities.

A direct consequence is the following

**Theorem B.** There is a dense $G_\delta$ set $\mathcal{G}_B \subset X^*(M^d)$ such that, for every $X \in \mathcal{G}_B$, if every chain recurrent class $C$ of $X$ has a homogeneous index for singularities, then the chain recurrent set $\text{CR}(X)$ is singular hyperbolic.

The next theorem states that, for generic star vector fields, if a chain recurrent class is Lyapunov stable, then it is singular hyperbolic.

**Theorem C.** There is a dense $G_\delta$ set $\mathcal{G}_C \subset X^*(M^d)$ such that, for every $X \in \mathcal{G}_C$, every Lyapunov stable chain recurrent class of $X$ is positively singular hyperbolic.

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1Theorem C is claimed in [2] under the assumption of the homogeneous property, i.e., the conclusion of our Theorem 5.7.
These theorems allow us to achieve the singular hyperbolicity of chain recurrent set in the 4 dimensional case.

**Theorem D.** There is a dense $G_δ$ set $G_D ⊂ \mathcal{X}^*(M^4)$ such that, for every $X ∈ G_D$, the chain recurrent set $\text{CR}(X)$ is singular hyperbolic.

We also obtain a description of ergodic measures of star flows, which could be thought as the counterpart of hyperbolic measures for diffeomorphisms. The following theorem is derived from a powerful shadowing lemma of Liao [17] and the estimation of size of invariant manifolds of Liao [18].

**Theorem E.** If $μ$ is an ergodic measure of a star flow, then $μ$ is a hyperbolic measure.

Theorem A, C and D are proved in Section 3 by admitting two technical theorems that will be proved in Section 4 and 5 respectively. A detailed version of Theorem E will be proved in Section 5 too.

**Acknowledgements.** We are very grateful for the invaluable suggestions of the anonymous referee. This work is partially supported by the Balzan research Project of J. Palis. YS is supported by Chinese Scholarship Council. SG is supported by 973 project 2011CB808002, NSFC 11025101 and 11231001. LW is supported by NSFC 11231001.

2 Preliminaries

2.1 Flows associated to a vector field

Given $X ∈ \mathcal{X}^1(M^d)$, $X$ generates a $C^1$ flow $ϕ_t : M^d → M^d$, and the tangent flow $Φ_t = dϕ_t : TM^d → TM^d$.

The usual linear Poincaré flow $ψ_t$ is defined as following. Denote the normal bundle of $X$ by $\mathcal{N} = \mathcal{N}^X = \bigcup_{x ∈ M^d \setminus \text{Sing}(X)} \mathcal{N}_x$, where $\mathcal{N}_x$ is the orthogonal complement of the flow direction $X(x)$, i.e.,

$$\mathcal{N}_x = \{v ∈ T_xM^d : v ⊥ X(x)\}.$$  

Given $v ∈ \mathcal{N}_x$, $x ∈ M^d \setminus \text{Sing}(X)$, $ψ_t(v)$ is the orthogonal projection of $Φ_t(v)$ on $\mathcal{N}_{ϕ_t(x)}$ along the flow direction, i.e.,

$$ψ_t(v) = Φ_t(v) - \frac{⟨Φ_t(v), X(ϕ_t(x))⟩}{∥X(ϕ_t(x))∥^2}X(ϕ_t(x)),$$

where $⟨·, ·⟩$ is the inner product on $T_xM$ given by the Riemannian metric.

We will need another flow $ψ^*_t : \mathcal{N} → \mathcal{N}$, which is called scaled linear Poincaré flow. Given $v ∈ \mathcal{N}_x$, $x ∈ M^d \setminus \text{Sing}(X)$,

$$ψ^*_t(v) = \frac{∥X(x)∥}{∥X(ϕ_t(x))∥}ψ_t(v) = \frac{ψ_t(v)}{∥Φ_t(ϕ_t(x))∥},$$

where $∥·∥$ is the norm on $T_xM$.
where \(\langle X(x)\rangle\) is the 1-dimensional subspace of \(T_xM^d\) spanned by the vector \(X(x) \in T_xM^d\). In a shadowing lemma of Liao (see Theorem 5.2), it is required some hyperbolicity with respect to this scaled linear Poincaré flow on the orbit arc.

The next lemma states the basic properties of star flows, proved in [15].

**Lemma 2.1.** ([15]) For any \(X \in \mathcal{X}^*(M^d)\), there is a \(C^1\) neighborhood \(\mathcal{U}\) and numbers \(\eta > 0\) and \(T > 0\) such that for any periodic orbit \(\gamma\) of \(Y \in \mathcal{U}\) with period \(\pi(\gamma) \geq T\), if \(\mathcal{N}_\gamma = N^s \oplus N^u\) is the hyperbolic splitting with respect to \(\psi^Y_t\) then

- For every \(x \in \gamma\) and \(t \geq T\), one has
  \[
  \|\psi^Y_t|_{N^s(x)}\| \leq e^{-\eta t},
  \]

- For every \(x \in \gamma\), then
  \[
  \prod_{i=0}^{[\pi(\gamma)/T] - 1} \|\psi^Y_t|_{N^u(\phi^Y_{i+1}(x))}\| \leq e^{-\eta \pi(\gamma)},
  \]

\[
\prod_{i=0}^{[\pi(\gamma)/T] - 1} m(\psi^Y_t|_{N^u(\phi^Y_{i+1}(x))}) \geq e^{\eta \pi(\gamma)}.
\]

Here \(m(A)\) is the mini-norm of \(A\), i.e., \(m(A) = \|A^{-1}\|^{-1}\).

Let \(E\) be a finitely dimensional vector space. Denote \(\wedge^2 E\) the second exterior power of \(E\). Given a linear isomorphism: \(A : E \rightarrow F\) between finitely dimensional vector spaces \(E\) and \(F\), denote \(\wedge^2 A : \wedge^2 E \rightarrow \wedge^2 F\) the linear isomorphism induced by \(A\). Now the second item of last theorem has the following consequence:

**Corollary 2.2.** For any \(X \in \mathcal{X}^*(M^d)\), there is a \(C^1\) neighborhood \(\mathcal{U}\) and numbers \(\eta > 0\) and \(T > 0\) such that for any periodic orbit \(\gamma\) of \(Y \in \mathcal{U}\) with period \(\pi(\gamma) \geq T\), if \(\mathcal{N}_\gamma = N^s \oplus N^u\) is the hyperbolic splitting with respect to \(\psi^Y_t\), \(E^{cs} = N^s \oplus \langle X \rangle\) and \(E^{cu} = N^u \oplus \langle X \rangle\) which are invariant subbundles of \(Y\), then we have for any \(x \in \gamma\),

\[
\prod_{i=0}^{[\pi(\gamma)/T] - 1} \|\wedge^2 \Phi^Y_t|_{E^{cs}(\phi^Y_{i+1}(x))}\| \leq e^{-\eta \pi(\gamma)},
\]

\[
\prod_{i=0}^{[\pi(\gamma)/T] - 1} m(\wedge^2 \Phi^Y_t|_{E^{cu}(\phi^Y_{i+1}(x))}) \geq e^{\eta \pi(\gamma)}.
\]

**Remark.** For simplicity, we will assume the constant \(T = 1\).

### 2.2 \(C^1\) connecting and generic results for flows

We need the following two versions of connecting lemmas.

**Lemma 2.3.** ([30]) For any vector field \(X \in \mathcal{X}^1(M^d)\) and any neighborhood \(\mathcal{U}\) of \(X\), for any point \(z \notin \text{Per}(X) \cup \text{Sing}(X)\), there exist \(L > 0, \rho > 1, \delta_0 > 0\) such that for any \(\delta \in (0, \delta_0]\), for any \(p\) and \(q\) in \(M \setminus \Delta, \Delta = \cup_{0 \leq t \leq L} \phi^X_t(B_\delta(z))\), if both the positive orbit of \(p\) and the negative orbit of \(q\) enter into \(B_{\delta/\rho}(z)\), then there is \(Y \in \mathcal{U}\) such that
• $q$ is on the positive orbit of $p$ with respect to the flow $\phi_t^Y$ generated by $Y$.

• $Y(x) = X(x)$ for any $x \in M \setminus \Delta$.

The connecting lemma of chains is also true for all the star flows, since all the critical elements of star flows are hyperbolic (see [3]).

Lemma 2.4. ([3]) Let $X \in X^*(M^d)$. For any $C^1$ neighborhood $U$ of $X$ and $x, y \in M^d$, if $y$ is chain attainable from $x$, then there exists $Y \in U$ and $t > 0$ such that $\phi_t^Y(x) = y$. Moreover, for every $k \geq 1$, let $\{x_{i,k}, t_{i,k}\}_{i=0}^{n_k^{k-1}}$ be a $(1/k, T)$-chain from $x$ to $y$ and denote by

$$\Lambda_k = \bigcup_{i=0}^{n_k^{k-1}} \phi_{[0,t_{i,k}]}(x_{i,k}).$$

Let $\Lambda$ be the upper Hausdorff limit of $\Lambda_k$, i.e., $\Lambda$ consists of points $z$ such that there exist $z_k \in \Lambda_k$ and $\lim_{k \to \infty} z_k = z$. Then for any neighborhood $U$ of $\Lambda$, there exists $Y \in U$ with $Y = X$ on $M \setminus U$ and $t > 0$ such that $\phi_t^Y(x) = y$. This strong version will be used in the proof of Lemma 4.2.

Remark. According to the proof of the above connecting lemma for chain ([3]), the conclusion can be strengthened as following: for any neighborhood $U$ of $\Lambda$, and for any finitely many (hyperbolic) critical elements $c_i, i = 1, 2, \ldots, j$, there exist a neighborhood $V_i$ of $c_i (i = 1, 2, \ldots, j)$ and $Y \in U$ with $Y = X$ on $(\cup_{i=1}^{j} V_i) \cup (M \setminus U)$ and $t > 0$ such that $\phi_t^Y(x) = y$. This strong version will be used in the proof of Lemma 4.2.

We need the following generic properties for star vector fields.

Lemma 2.5. There is a dense $G_\delta$ set $\mathcal{G} \subset X^*(M^d)$ such that for any $X \in \mathcal{G}$, one has

1. For every critical element $p$ of $X$, the chain recurrent class $C(p) = C(p_X, X)$ is continuous at $X$ in the Hausdorff topology.

2. If $p$ and $q$ are two different critical elements of $X$ with $C(p) = C(q)$, then there exists a $C^1$ neighborhood $U$ of $X$ such that for any $Y \in U$, one has $C(p_Y, Y) = C(q_Y, Y)$.

3. For any hyperbolic critical element $p$ of $X$, if $W^u(p) \subset C(p)$, then there is a $C^1$ neighborhood $U$ of $X$ such that for any $Y \in U$, $C(p_Y, Y)$ is Lyapunov stable.

4. For any nontrivial chain recurrent class $C$ of $X$, there exists a sequence of periodic orbits $Q_n$ such that $Q_n$ tends to $C$ in the Hausdorff topology.

Remark. Item 1, 2 and 3 is from [5] and item 4 is from [6].

3 Reducing the main theorems to two technical results

In this section we reduce the proofs of the main theorems to two technical theorems, Theorem 3.4 and 3.5. First we define the saddle value of a singularity, a crucial value for the analysis of singularities whose chain recurrent class is nontrivial.
Definition 3.1. Let $X \in \mathcal{X}^1(M^d)$ and $\sigma$ a hyperbolic singularity of $X$. Assume the Lyapunov exponents of $\Phi_t(\sigma)$ are
$$\lambda_1 \leq \cdots \leq \lambda_s < 0 < \lambda_{s+1} \leq \cdots \leq \lambda_d,$$
then the saddle value $sv(\sigma)$ of $\sigma$ is defined as
$$sv(\sigma) = \lambda_s + \lambda_{s+1}.$$

Definition 3.2. Let $X \in \mathcal{X}^1(M^d)$ and $\sigma$ a hyperbolic singularity of $X$. Assume that $\mathcal{C}(\sigma)$ is nontrivial and the Lyapunov exponents of $\Phi_t(\sigma)$ are
$$\lambda_1 \leq \cdots \leq \lambda_s < 0 < \lambda_{s+1} \leq \cdots \leq \lambda_d.$$
We say $\sigma$ is Lorenz-like, if the following conditions are satisfied:

- $sv(\sigma) \neq 0$.
- If $sv(\sigma) > 0$, then $\lambda_{s-1} < \lambda_s$, and $W^{ss}(\sigma) \cap \mathcal{C}(\sigma) = \{\sigma\}$. Here $W^{ss}(\sigma)$ is the invariant manifold corresponding to the bundle $E^{ss}_\sigma$ of the partially hyperbolic splitting $T_\sigma M = E^{ss}_\sigma \oplus E^{cu}_\sigma$, where $E^{ss}_\sigma$ is the invariant space corresponding to the Lyapunov exponents $\lambda_1, \lambda_2, \cdots, \lambda_{s-1}$ and $E^{cu}_\sigma$ corresponding to the Lyapunov exponents $\lambda_s, \lambda_{s+1}, \cdots, \lambda_d$.
- If $sv(\sigma) < 0$, then $\lambda_{s+1} < \lambda_{s+2}$, and $W^{uu}(\sigma) \cap \mathcal{C}(\sigma) = \{\sigma\}$. Here $W^{uu}(\sigma)$ is the invariant manifold corresponding to the bundle $E^{uu}_\sigma$ of the partially hyperbolic splitting $T_\sigma M = E^{cs}_\sigma \oplus E^{uu}_\sigma$, where $E^{cs}_\sigma$ is the invariant space corresponding to the Lyapunov exponents $\lambda_1, \lambda_2, \cdots, \lambda_{s+1}$ and $E^{uu}_\sigma$ corresponding to the Lyapunov exponents $\lambda_{s+2}, \lambda_{s+3}, \cdots, \lambda_d$.

Remark. If the singularity $\sigma$ is Lorenz-like, then the splitting (say, $T_\sigma M = E^{ss}_\sigma \oplus E^{cu}_\sigma$ in the case $sv(\sigma) > 0$) is a singular hyperbolic splitting over $\{\sigma\}$.

Although in the definition of Lorenz-like singularity (and singular hyperbolicity) it is allowed that $E^{uu}_\sigma$ is trivial (for $sv(\sigma) < 0$), i.e., $E^{uu}_\sigma = \{0\}$, we will show that for $C^1$ generic star vector field $X$, if $\mathcal{C}(\sigma)$ is nontrivial, then $E^{uu}_\sigma$ should be nontrivial (see Theorem 3.4 below). We need the important Main Theorem of Liao in [18] (see [31] for a generalization):

Theorem 3.3. ([18, Main Theorem]) Given $X \in \mathcal{X}^*(M)$, there exists a neighborhood $\mathcal{U}$ of $X$ such that
$$\sup_{Y \in \mathcal{U}} \# \{ P \subset M : P \text{ is a periodic sink of } Y \} < \infty.$$
by
\[ N(X) = \# \{ P \subset M : P \text{ is a periodic sink of } X \}. \]

According to Theorem 3.3, \( N(X) \) is well-defined. Since \( N(\cdot) \) is lower semi-continuous, there exists a dense \( G_\delta \) subset \( G_0 \subset X^*(M) \) such that \( N(\cdot) \) is continuous on \( G_0 \). Given \( X \in G_0 \), take a small neighborhood \( U \subset X^*(M) \) of \( X \) such that \( N(\cdot) \) is constant on \( U \).

We will prove that for any singularity \( \sigma \) of \( X \in G_0 \), if \( T_\sigma M \) is sectional contracting, then \( C(\sigma) \) is trivial. Otherwise, assume that \( C(\sigma) \) is nontrivial. Then according to \( C^1 \) connecting lemma (Lemma 2.4), there exists \( Y \in U \) such that \( Y \equiv X \) in a neighborhood of \( \sigma \), which implies that \( T_\sigma M \) is still sectional contracting for \( Y \), and \( Y \) has a homoclinic loop \( \Gamma \) associated to \( \sigma = \sigma_Y \). \( \Gamma \cup \{ \sigma \} \) is sectional contracting since the unique invariant measure is the atomic measure \( \delta_\sigma \) supported on \( \sigma \). It is easy to see that there is a sequence \( \{ Y_n \} \) tending \( Y \) and periodic orbit \( P_n \) of \( Y_n \) tending to \( \Gamma \cup \{ \sigma \} \) in the Hausdorff topology. Since the invariant measure supported on \( P_n \) converges to \( \delta_\sigma \), \( P_n \) is a sink of \( Y_n \) for \( n \) large enough and hence \( N(Y_n) \geq N(Y) + 1 \). This contradicts that \( N(\cdot) \) is constant on \( U \). \hfill \Box

From now on, we will only consider singularities which are neither sectional contracting nor sectional expanding.

**Definition 3.5.** Let \( X \in X^*(M^d) \) and \( \sigma \in \text{Sing}(X) \) such that \( C(\sigma) \) is nontrivial. Then the periodic index \( \text{Ind}_p(\sigma) \) of \( \sigma \) is defined as

\[ \text{Ind}_p(\sigma) = \begin{cases} s, & \text{if } \text{sv}(\sigma) < 0, \\ s - 1, & \text{if } \text{sv}(\sigma) > 0. \end{cases} \]

For a periodic orbit \( P \) of \( X \), we define \( \text{Ind}_p(P) = \text{Ind}(P) \).

**Remark.** The notion of periodic index of singularity is to describe the index of periodic orbits derived from the perturbation of homoclinic loop associated to the corresponding singularity. Our definition does not concern the case that the saddle value of singularity is zero, which could not occur if we admit the generic assumptions. However, we will prove in Lemma 4.2 that for every \( X \in X^*(M^d) \) and \( \sigma \in \text{Sing}(X) \), if \( C(\sigma) \) is nontrivial, then \( \text{sv}(\sigma) \neq 0 \). This result justifies our definition.

The next theorem studies the singularities of a nontrivial chain recurrent class for a generic star flow. We show that these singularities are all Lorenz-like, that is, the tangent space of the singularity admits a partially hyperbolic splitting, and the strong stable/unstable manifold intersects the chain recurrent class only at the singularity. The proof will be given in Section 4.

**Theorem 3.6.** For any \( X \in X^*(M^d) \) and \( \sigma \in \text{Sing}(X) \), if the chain recurrent class \( C(\sigma) \) is nontrivial, then any singularity \( \rho \in C(\sigma) \) is Lorenz-like. Moreover, there is a dense \( G_\delta \) subset \( G_1 \subset X^*(M^d) \) and if we further assume that \( X \in G_1 \), then \( \text{Ind}_p(\rho) = \text{Ind}_p(\sigma) \).

**Remark.** From this theorem and the definition of periodic index of singularity, it follows that, for a generic star vector field \( X \) and any nontrivial chain recurrent class \( C(\sigma) \) of \( X \), if \( \rho \in C(\sigma) \cap \text{Sing}(X) \), then the index of \( \rho \) can only be \( \text{Ind}_p(\sigma) + 1 \) if \( \text{sv}(\rho) > 0 \), or \( \text{Ind}_p(\sigma) \) if \( \text{sv}(\rho) < 0 \).
The next theorem states that if the singularities of a chain recurrent class are all Lorenz-like and have the same index, then the chain recurrent class is singular hyperbolic. The proof will be given in Section 5.

**Theorem 3.7.** There is a dense $G_δ$ subset $\mathcal{G}_2 \subset X^*(M^d)$ such that for any $X \in \mathcal{G}_2$ and $\sigma \in \text{Sing}(X)$, if $C(\sigma)$ is nontrivial and for any singularity $\rho \in C(\sigma)$, $\text{Ind}(\rho) = \text{Ind}(\sigma)$, then $C(\sigma)$ is positively or negatively singular hyperbolic.

**Remark.** Notice that Theorem 3.6 talks about the periodic index of singularities, while Theorem 3.7 talks about the index (not periodic index) of singularities.

Now we give the proofs of Theorem A, C and D by assuming Theorem 3.6 and 3.7. A detailed version of Theorem E (Theorem 5.6) will be proved in section 5.

**Proof of Theorem A.** Let $\mathcal{G}_A = \mathcal{G}_2$, which is a dense $G_δ$ subset of $X^*(M^d)$. Let $X \in \mathcal{G}_A$, and $C$ be a chain recurrent class of $X$. If $C \cap \text{Sing}(X) = \emptyset$, then we apply [9] to conclude that $C$ is a hyperbolic set, which is of course singular hyperbolic. Now, assume that there exists some singularity $\sigma \in C$. If $C = \{\sigma\}$, from star condition, $C$ is hyperbolic and hence singular hyperbolic. If $C$ is nontrivial, Theorem 3.7 tells us that $C$ is positively or negatively singular hyperbolic. This proves Theorem A.

**Proof of Theorem C.** We let $\mathcal{G}_C = \mathcal{G}_0 \cap \mathcal{G}_1 \cap \mathcal{G}_2$. Consider any $X \in \mathcal{G}_C$ and any Lyapunov stable chain recurrent class $C$ of $X$. If $C \cap \text{Sing}(X) = \emptyset$, then [9] guarantees that $C$ is a hyperbolic attractor. So we only need to consider the case when $C$ contains some singularity. Since $C$ is Lyapunov stable, we must have $W^u(\sigma) \subset C$ for any $\sigma \in C \cap \text{Sing}(X)$.

**Claim.** For any $\sigma \in C \cap \text{Sing}(X)$, we have that $sv(\sigma) > 0$.

**Proof of Claim:** Otherwise, assume that there exists $\sigma \in C \cap \text{Sing}(X)$, $sv(\sigma) < 0$. By Theorem 3.6, $\sigma$ is Lorenz-like, i.e., there exists a negatively singular hyperbolic splitting $T_\sigma M = E^{cs}_\sigma \oplus E^{uu}_\sigma$. According to Theorem 3.4, $T_\sigma M$ is not sectional contracting. So, $E^{uu}_\sigma$ is nontrivial. Hence, $W^u(\sigma) \setminus C \neq \emptyset$, which contradicts $W^u(\sigma) \subset C$.

Now the singularities in $C$ have the same index. Applying Theorem 3.7, we conclude that $C$ is positively singular hyperbolic. This proves Theorem C.

Combining these results, we could show that the singular hyperbolicity of chain recurrent set for generic star flows in dimension 4.

**Proof of Theorem D.** We assume $\text{dim}(M) = 4$ and $\mathcal{G}_D = \mathcal{G}_0 \cap \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}$ which is a dense $G_δ$ subset of $X^*(M^4)$, where $\mathcal{G}$ is the dense $G_δ$ set in Lemma 2.5. As in the proofs of the above theorems, for any $X \in \mathcal{G}_D$, we only need to consider a nontrivial chain recurrent class $C$ of $X$ such that there exists $\sigma \in C \cap \text{Sing}(X)$.

If there exists some singularity $\rho \in C$ such that $\text{Ind}(\rho) = 3$, then $\text{dim}(E^u(\rho)) = 1$ and $W^u(\rho)$ has two separatrices. Since we assume $X \in \mathcal{G}$, $C(\rho_X, X) = C$ depends continuously on $X$ and hence is robustly nontrivial.

**Claim.** $W^u(\rho) \subset C$ and, consequently, $C$ is Lyapunov stable.
Proof of Claim: In fact, suppose on the contrary that one separatrix \( \text{Orb}(x_1) \) of \( W^u(\rho) \) is not contained in \( C \). By the upper semi-continuity of chain recurrent class, we know this holds robustly. The non-triviality of \( C(\rho) \) implies the other separatrix \( \text{Orb}(x_2) \) of \( W^u(\rho) \) is contained in \( C \). Using the connecting lemma for chains, you can perturb \( \text{Orb}(x_2) \) to be the homoclinic orbit associated to \( \rho \). Then applying the \( \lambda \)-lemma, an arbitrarily small perturbation could make the positive orbit of \( x_2 \) arbitrarily close to \( x_1 \), which is no longer contained in \( C(\rho) \). Combining all these perturbations together, we get a vector field \( Y \) arbitrarily \( C^1 \) close to \( X \), such that

\[
W^u(\rho_Y) \cap C(\rho_Y, Y) = \{\rho_Y\},
\]

contradicting the fact that \( C(\rho_X, X) \) is robustly nontrivial. \qed

From the claim and Theorem C, \( C \) is positively singular hyperbolic.

If there are some singularity \( \rho \in C \) such that \( \text{Ind}(\rho) = 1 \), we just need to consider \( -X \). Then following the analysis above directly, \( C \) is Lyapunov stable for \( -X \), which is negatively singular hyperbolic for \( X \). So we can reduce to the case that all the singularities contained in \( C \) have the same index 2, which allows us to applying Theorem A. As a result, \( C \) is singular hyperbolic.

Now we have proved that every chain recurrent class of \( X \) is singular hyperbolic. And hence, \( \text{CR}(X) \) is singular hyperbolic. This proves Theorem D. \qed

4 Analysis of singularities

In this section, we will analyze the singularities contained in a nontrivial chain recurrent class for some \( X \in \mathcal{X}^*(M^d) \). Our main technique is the extended linear Poincaré flow introduced in [14], which has been proved to be a useful tool in the analysis of non-isolated singularities (e.g., see [32, 10, 2]).

First we state a lemma on the estimation of index of periodic orbits which accumulate on singularities and their homoclinic orbits. Then we use the dominated splitting of the extended linear Poincaré flow to achieve the properties of Lyapunov exponents of singularities. Especially, we will conclude that all the singularities whose chain recurrent class are nontrivial are Lorenz-like.

Lemma 4.1. Let \( X \in \mathcal{X}^*(M^d) \), \( \sigma \in \text{Sing}(X) \) and \( \Gamma = \text{Orb}(x) \) be a homoclinic orbit associated to \( \sigma \). Assume that there exists a sequence of star vector fields \( \{X_n\} \) converging to \( X \) in the \( C^1 \) topology and periodic orbit \( P_n \) of \( X_n \) with index \( l \) such that \( \{P_n\} \) converges to \( \Gamma \cup \{\sigma\} \) in the Hausdorff topology. Then there exist two subspaces \( E, F \subset T_\sigma M \) such that

1. \( E \) is \((l + 1)\)-dimensional and sectional contracting:

\[
\frac{1}{k} \sum_{i=0}^{k-1} \log \| \wedge^2 \Phi_{X}^i|_{\Phi_{X}(E)} \| \leq -\eta, \quad k = 1, 2, \ldots
\]

2. \( F \) is \((d - l)\)-dimensional and sectional expanding:

\[
\frac{1}{k} \sum_{i=0}^{k-1} \log m(\wedge^2 \Phi_{X}^i|_{\Phi_{X}(F)}) \geq \eta, \quad k = 1, 2, \ldots
\]
Here the constant $\eta$ comes from corollary 2.2.

Moreover, we have the following estimation of the index of periodic orbits:

\[ \text{Ind}(\sigma) - 1 \leq l = \text{Ind}(P_n) \leq \text{Ind}(\sigma). \]

Proof. Let the hyperbolic splitting of $P_n$ be

\[ T_{P_n}M = E^s(P_n) \oplus (X_n(P_n)) \oplus E^u(P_n). \]

Consider the $X_n$-invariant subspace

\[ E_n = E^s(P_n) \oplus (X_n(P_n)) \]

on $P_n$. Since $P_n$ tends to the homoclinic loop associated to $\sigma$, their periods must tend to infinity as $n \to \infty$. For $n$ large enough, you can apply Corollary 2.2 to get the following estimations

\[ [\pi(x_n)]^{-1} \prod_{i=0}^{[\pi(x_n)]-1} \| \wedge^2 \Phi^X_1 |_{E_n(\phi^X_{x_n(x_n)})} \| \leq e^{-\eta \pi(x_n)} \]

for any $x_n \in P_n = \text{Orb}(x_n)$. Then for any $\epsilon > 0$, Pliss Lemma (24) gives some point $p_n \in P_n$ satisfying

\[ \frac{1}{k} \sum_{i=0}^{k-1} \log \| \wedge^2 \Phi^X_1 |_{\phi^X_n(E_n(p_n))} \| \leq -\eta + \epsilon, \quad k = 1, 2, \cdots \]

Assume $p_n$ tends to $y \in \Gamma \cup \{\sigma\}$. Taking some subsequence if necessary, one can assume $E_n(p_n) \to E(y)$, then we have

\[ \frac{1}{k} \sum_{i=0}^{k-1} \log \| \wedge^2 \Phi^X_1 |_{\phi^X_n(E(y))} \| \leq -\eta + \epsilon, \quad k = 1, 2, \cdots \]

Now the Pliss Lemma (24) allows us to find $n_j \to \infty$ such that

\[ \frac{1}{k} \sum_{i=0}^{k-1} \log \| \wedge^2 \Phi^X_1 |_{\phi^X_{i+n_j}(E(y))} \| \leq -\eta + 2\epsilon, \quad k = 1, 2, \cdots \]

Since $\phi_{n_j}(y)$ tends to $\sigma$ as $j \to \infty$, we derive a subspace $E \subset T_\sigma M$ with $\text{dim}E = \text{dim}E_n(p_n) = l + 1$ and

\[ \frac{1}{k} \sum_{i=0}^{k-1} \log \| \wedge^2 \Phi^X_1 |_{\phi^X(E)} \| \leq -\eta + 2\epsilon, \quad k = 1, 2, \cdots \]

So, $E$ is sectional contracting under $\Phi^X_t$. Notice that we can choose the constant $\epsilon$ arbitrarily small, this give us the proof of first item.

For the second item, we only need to consider $-X$.

Now for the estimation of the index of $P_n$, if we assume $\text{Ind}(\sigma) < l = \text{Ind}(P_n)$, then

\[ \text{dim}(E \cap E^u(\sigma)) \geq \text{dim}E + \text{dim}E^u(\sigma) - d \geq l + 1 + d - (l - 1) - d = 2. \]

However, since $E$ is sectional contracting and $E^u(\sigma)$ is sectional expanding, this is absurd. So $l = \text{Ind}(P_n) \leq \text{Ind}(\sigma)$. For the other side of the inequality, we only need to consider $-X$, and the same argument as above will show that $l = \text{Ind}(P_n) \geq \text{Ind}(\sigma) - 1$. This finishes the proof of the lemma.
Remark. From this lemma and its proof, one can see

• If some periodic orbit is sufficiently close to a homoclinic loop associated to some singularity \(\sigma\) of a star flow, then the index of the periodic orbit could only be \(\text{Ind}(\sigma) - 1\) or \(\text{Ind}(\sigma)\).

• In this lemma, we do not need to assume that the star flow is generic.

Let us recall some basic definitions in [14]. Denote by

\[
G^1 = G^1(M^d) = \{ L : L \text{ is a 1-dimensional subspace of } T_xM^d, x \in M^d \}
\]

the Grassmannian manifold of \(M^d\). Given \(X \in \mathcal{X}^1(M^d)\), the tangent flow \(\Phi_t\) induces a flow

\[
\Phi_t : G^1 \to G^1 \\
L \mapsto \Phi_t(L)
\]
on \(G^1\).

Let \(\beta : G^1 \to M^d\) and \(\xi : TM^d \to M^d\) be the corresponding bundle projections. It naturally induces a (pullback) bundle

\[
\beta^*(TM^d) = \{ (L, v) \in G^1 \times TM^d : \beta(L) = \xi(v) \}.
\]

Then \(\beta^*(TM^d)\) is a \(d\)-dimensional vector bundle over \(G^1\) with the bundle projection

\[
\iota : \beta^*(TM^d) \to G^1 \\
\iota(L, v) = L.
\]

Then we could lift the tangent flow \(\Phi_t\) to \(\beta^*(TM^d)\), which is called extended tangent flow, (still) denoted by

\[
\Phi_t : \beta^*(TM^d) \to \beta^*(TM^d) \\
\Phi_t(L, v) = (\Phi_t(L), \Phi_t(v)).
\]

Let

\[
\mathcal{P} = \{ (L, v) \in \beta^*(TM^d) : v \in L \}.
\]

This is a 1-dimensional subbundle of \(\beta^*(TM^d)\) over \(G^1\), which is invariant under any extended tangent flow. Similarly, we could define the normal bundle of \(\mathcal{P}\) as follows

\[
\mathcal{N} = \mathcal{P}^\perp = \{ (L, v) \in \beta^*(TM^d) : v \perp L \}.
\]

Then \(\mathcal{N}\) is a \((d - 1)\)-dimensional subbundle of \(\beta^*(TM^d)\) over \(G^1\). Now for every \(X \in \mathcal{X}^1(M^d)\), we could define the extended Poincaré flow of \(X\)

\[
\psi_t = \psi_t^X : \mathcal{N} \to \mathcal{N}
\]
to be

\[
\psi_t(L, v) = \pi(\Phi_t(L, v)), \quad \forall (L, v) \in \mathcal{N},
\]

where \(\pi\) is the orthogonal projection from \(\beta^*(TM^d)\) to \(\mathcal{N}\) along \(\mathcal{P}\).

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For a compact invariant set $\Lambda$ of $X \in \mathcal{X}^1(M^d)$, we denote

$$B(\Lambda) = \{ L \in G^1 : \beta(L) \in \Lambda, \exists X_n \to X, p_n \in \text{Per}(X_n), \text{Orb}(p_n, X_n) \hookrightarrow \Lambda, \text{such that } \langle X_n(p_n) \rangle \to L \}.$$ 

$$B^i(\Lambda) = \{ L \in G^1 : \beta(L) \in \Lambda, \exists X_n \to X, p_n \in \text{Per}(X_n), \text{Ind}(p_n) = j, \text{Orb}(p_n, X_n) \hookrightarrow \Lambda, \text{such that } \langle X_n(p_n) \rangle \to L \}.$$ 

Here $\text{Orb}(p_n, X_n) \hookrightarrow \Lambda$ means that the Hausdorff upper limit of $\text{Orb}(p_n, X_n)$ is contained in $\Lambda$.

**Lemma 4.2.** Let $X \in \mathcal{X}^*(M^d)$ and $\sigma \in \text{Sing}(X)$. Assume that the Lyapunov exponents of $\Phi_t(\sigma)$ are

$$\lambda_1 \leq \cdots \leq \lambda_s < 0 < \lambda_{s+1} \leq \cdots \leq \lambda_d.$$ 

If $C(\sigma)$ is nontrivial, then

1. either $\lambda_{s-1} \neq \lambda_s$ or $\lambda_{s+1} \neq \lambda_{s+2}$.
2. if $\lambda_{s-1} = \lambda_s$, then $\lambda_s + \lambda_{s+1} < 0$.
3. if $\lambda_{s+1} = \lambda_{s+2}$, then $\lambda_s + \lambda_{s+1} > 0$.
4. if $\lambda_{s-1} \neq \lambda_s$ and $\lambda_{s+1} \neq \lambda_{s+2}$, then $\lambda_s + \lambda_{s+1} \neq 0$.

**Proof.** Fix $\sigma \in \text{Sing}(X)$ such that $C(\sigma)$ is nontrivial and denote $s = \text{Ind}(\sigma)$. By changing the Riemannian metric, we can assume that $E^s(\sigma) \perp E^u(\sigma)$. Since $C(\sigma)$ is nontrivial, there exist $x \in C(\sigma) \cap \text{W}^u(\sigma) \setminus \{ \sigma \}$ and $y \in C(\sigma) \cap \text{W}^s(\sigma) \setminus \{ \sigma \}$. For any small $C^1$ neighborhood $\mathcal{U}$ of $X$, according to Lemma 2.4 and its remark, there exists a neighborhood $\mathcal{V}$ of $\sigma$, and $Y \in \mathcal{U}$ such that $Y = X$ on $V$ and $y = \phi^t_Y(x)$ for some $t > 0$. By considering $\phi_N^t(y)$ and $\phi_{-N}(x)$ for $N > 0$ large enough, we may assume that $x, y \in V$, which implies $\sigma_Y = \sigma$ exhibits a homoclinic orbit $\Gamma = \text{Orb}(z)$. Note that $X$ and $Y$ exhibit the same Lyapunov exponents at the singularity $\sigma_Y = \sigma$.

Choose two sequences of regular points $x_n \to x$ and $y_n \to y$, such that $\phi^t_{x_n}(x_n) = y_n$. Connecting $x_n$ to $x$ and $y_n$ to $y$, we derive a sequence of vector fields $Y_n \to Y$ and $x_n \in \text{Per}(Y_n)$ such that $\text{Orb}(x_n)$ converge to $\Gamma \cup \{ \sigma \}$.

Considering the compact $Y$-invariant set $\Lambda = \Gamma \cup \{ \sigma \}$, from Lemma 4.1 we know that

$$s - 1 \leq \lim_{n \to \infty} \text{Ind}(x_n) \leq s$$

which also implies that either $\beta(B^{s-1}(\Lambda)) = \Lambda$ or $\beta(B^s(\Lambda)) = \Lambda$. Assume the first case holds. Then the linear Poincaré flows $\psi^Y_t$ of all these periodic orbits admit the uniform dominated splitting

$$\frac{\|\psi^Y_t|_{N^s(x)}\|}{m(\psi^Y_t|_{N^u(x)})} \leq e^{-2\eta t};$$
for some constant \( \eta > 0 \) and \( \forall x \in \text{Orb}(x_n), \forall t \geq 1 \). Since the constant \( \eta \) is uniform for any \( n \) and the extended linear Poincaré flow is a continuous linear flow on a continuous bundle, by taking limits in this framework, we get a dominated splitting

\[ N_\Delta = \mathcal{E} \oplus \mathcal{F} \]

over \( \Delta \) with \( \dim \mathcal{E} = s - 1 \), \( \dim \mathcal{F} = d - s \). Here \( \Delta \subset \mathbb{G}^1 \) is the set of limit points of \( \{ (Y_n(x)) : x \in P_n \} \) and contained in \( B^{s-1}(\Lambda) \). Since \( P_n \) converges to the homoclinic loop, then we can choose \( p_n \in \text{Orb}(x_n) \) such that

\[ \lim_{n \to \infty} \langle X(p_n) \rangle \subset E^u(\sigma). \]

This implies that \( \Delta^u(\sigma) = \{ L \in \Delta : L \subset E^u(\sigma) \} \), which is a nonempty and compact invariant set under \( \Phi^Y_t = \Phi^X_t \) when restricted on \( \mathbb{G}^1(\sigma) = \{ L \in \mathbb{G}^1 : \beta(L) = \sigma \} \). If we restrict the extended linear Poincaré flow on \( N_{\Delta^u(\sigma)} \), it will also admit the dominated splitting with the same constant \( 2\eta \). Since we have assumed \( E^s(\sigma) \perp E^u(\sigma) \), we have

\[ E^s(\sigma) \subset N_{\Delta^u(\sigma)} \]

and

\[ \psi_t^Y|_{E^s(\sigma) \cap N_{\Delta^u(\sigma)}} = \Phi^Y_t|_{E^s(\sigma)}. \]

Since \( \dim N^s = s - 1 \) and \( \dim E^s(\sigma) = s \), \( E^s(\sigma) \) admits a dominated splitting w.r.t. the tangent flow \( \Phi^Y_t \) with the same constant \( \eta \), i.e.,

\[ E^s(\sigma) = E^{ss}(\sigma) \oplus E^c(\sigma) \]

is a \( \Phi^Y_t \)-invariant splitting, where \( \dim E^{ss}(\sigma) = s - 1 \) and \( \dim E^c(\sigma) = 1 \). Moreover, it satisfies

\[ \frac{\| \Phi^Y_t|_{E^{ss}(\sigma)} \|}{m(\Phi^Y_t|_{E^c(\sigma)})} \leq e^{-2\eta t}. \]

This implies that the Lyapunov exponents of \( \sigma_Y = \sigma \) satisfy \( \lambda_{s-1} \leq \lambda_s - 2\eta \). Since \( Y = X \) on a small neighborhood of \( \sigma \), the same inequality holds for \( X \).

If we assume \( \beta(B^s(\Lambda)) = \Lambda \), then the same analysis shows that \( \lambda_{s+1} \leq \lambda_{s+2} - 2\eta \). This proves the first item of this lemma.

For the rest three items, we need

Claim. \( \quad \bullet \) If \( \beta(B^s(\Lambda)) = \Lambda \), then \( \lambda_s + \lambda_{s+1} \leq -\eta \).

\( \quad \bullet \) If \( \beta(B^{s-1}(\Lambda)) = \Lambda \), then \( \lambda_s + \lambda_{s+1} \geq \eta \).

Proof of Claim: We just prove the first item, then for the second one we only need to consider \(-Y\). Recall the definition of \( \beta(B^s(\Lambda)) = \Lambda \), which means the homoclinic loop \( \Lambda \) is the Hausdorff limit of periodic orbits \( \text{Orb}(x_n) \) of \( Y_n \) with index \( s \). Applying Lemma 4.1 we know that there exists an \((s + 1)\)-dimensional subspace \( E \subset T_y M \), such that

\[ \frac{1}{k} \sum_{i=0}^{k-1} \log \| \lambda^2 \Phi^Y_t |_{\Phi^Y(E)} \| \leq -\eta, \ k = 1, 2, \ldots \]

On the other hand, \( \beta(B^s(\Lambda)) = \Lambda \) implies that \( \lambda_{s+1} < \lambda_{s+2} \). Denote by \( E^{cs} \) the direct sum of the generalized eigenspaces associated to \( \lambda_i, i = 1, 2, \ldots, s + 1 \), which is
an \((s + 1)\)-dimensional \(\Phi^Y\)-invariant subspace of \(T_\sigma M\). Then the dominated splitting on \(T_\sigma M\) implies \(E^{cs}\) must admit the estimation above, i.e.,

\[
\frac{1}{k} \sum_{i=0}^{k-1} \log \| \wedge^2 \Phi^Y_i \|_{\Phi^Y_\sigma(E^{cs})} \leq -\eta, \quad k = 1, 2, \ldots
\]

However, if we assume \(\lambda_s + \lambda_{s+1} > -\eta\), we can pick a pair of eigenvectors \(u\) and \(v\) associated to \(\lambda_s\) and \(\lambda_{s+1}\) respectively. So we have the following equalities

\[
\| \Phi^Y_t(u) \| = e^{\lambda_s t} \| u \|, \quad \forall t > 0 ,
\]

\[
\| \Phi^Y_t(v) \| = e^{\lambda_{s+1} t} \| v \|, \quad \forall t > 0 .
\]

Since we have assumed \(E^s(\sigma) \perp E^u(\sigma)\), which implies \(u \perp v\), so we have

\[
\frac{1}{k} \sum_{i=0}^{k-1} \log \| \wedge^2 \Phi^Y_i \|_{\Phi^Y_\sigma(E^{cs})} \geq \lambda_s + \lambda_{s+1} > -\eta, \quad k = 1, 2, \ldots
\]

This is a contradiction. So we must have \(\lambda_s + \lambda_{s+1} \leq -\eta\). This finishes the proof of the claim.

Now we prove item 2 of this lemma. If \(\lambda_{s-1} = \lambda_s\), then by the analysis above, the homoclinic loop \(\Lambda = \Gamma \cup \{\sigma\}\) could only be accumulated by periodic orbits of index \(s\). This proves \(\beta(B^s(\Lambda)) = \Lambda\). So we can apply the first item of the claim to show that \(\lambda_s + \lambda_{s+1} \leq -\eta\).

Item 3 is just item 2 of \(-X\).

Item 4 could be proved in the same way. In this case, we have two possibilities. Either \(\beta(B^s(\Lambda)) = \Lambda\) or \(\beta(B^{s-1}(\Lambda)) = \Lambda\). Corresponding to these two cases, the claim guarantee that we have either \(\lambda_s + \lambda_{s+1} \leq -\eta\) or \(\lambda_s + \lambda_{s+1} \geq \eta\). This finishes the proof of this lemma.

Remark. In the proof of this lemma, you can see that \(|\lambda_s + \lambda_{s+1}| \geq \eta\). Moreover, either \(\lambda_{s-1}\) and \(\lambda_s\), or \(\lambda_{s+1}\) and \(\lambda_{s+2}\) should admit a uniform gap which is \(2\eta\).

Corollary 4.3. For any \(X \in \mathcal{X}^*(M^d)\) and any \(\sigma \in \text{Sing}(X)\), if \(C(\sigma)\) is nontrivial, then

\[
sv(\sigma) \neq 0.
\]

Lemma 4.4. Let \(X \in \mathcal{X}^*(M^d)\) and \(\sigma \in \text{Sing}(X)\). Let \(\Gamma = \text{Orb}(x)\) be a homoclinic orbit associated to \(\sigma\). Assume there exists a sequence of vector fields \(\{X_n\}\) converging to \(X\) in the \(C^1\) topology and a sequence of periodic orbits \(P_n\) of \(X_n\) such that \(P_n\) converges to \(\Gamma \cup \{\sigma\}\) in the Hausdorff topology. Then we have

\[
\lim_{n \to \infty} \text{Ind}(P_n) = \text{Ind}_p(\sigma),
\]

i.e., for \(n\) large enough, \(\text{Ind}(P_n) = \text{Ind}_p(\sigma)\).
Proof. We have proved that the saddle value of $\sigma$ is not equal to zero. Without loss of generality we assume $\text{sv}(\sigma) > 0$, otherwise we consider $-X$. Then the periodic index $\text{Ind}_p(\sigma) = s - 1$, where $s = \text{Ind}(\sigma)$. Moreover, the Lyapunov exponents of $\sigma$ satisfies $\lambda_{s-1} < \lambda_s$, which determines a dominated splitting of $T_\sigma M$:

$$T_\sigma M = E^{ss}(\sigma) \oplus E^c(\sigma) \oplus E^u(\sigma).$$

Here $E^c(\sigma)$ is the eigenspace associated to $\lambda_s$, and the saddle value $\text{sv}(\sigma) > 0$ insures that the invariant subspace $E^c(\sigma) \oplus E^u(\sigma)$ is sectional expanding.

Since Lemma 4.1 has guaranteed that $\text{Ind}(P_n) \geq s - 1$ for $n$ large enough, so we only need to show that $\text{Ind}(P_n) > s - 1$ leads to a contradiction. If $\text{Ind}(P_n) > s - 1$, also according to Lemma 4.1, $T_\sigma M$ contains a sectional contracting subspace $E$ of dimension $s + 1$.

Then we have

$$\dim(E \cap (E^c(\sigma) \oplus E^u(\sigma))) \geq \dim E + \dim(E^c(\sigma) \oplus E^u(\sigma)) - d \geq s + 1 + d - s + 1 - d = 2.$$  

However, we notice that $E$ is sectional contracting and $E^c(\sigma) \oplus E^u(\sigma)$ is sectional expanding. This is absurd. So we have proved $\text{Ind}(P_n) \leq s - 1$ for $n$ large enough. This finishes the proof of the lemma. \hfill \Box

Remark. This lemma asserts that when a periodic orbit is close enough to a homoclinic loop associated to some singularity, then its index has to be equal to the periodic index of the singularity. When we consider another kind of critical elements, periodic orbits, this also holds. Precisely, if the periodic orbit $Q_n$ tends to a homoclinic orbit $\Gamma = \{\text{Orb}(x)\}$ associated to some periodic orbit $P$, then we must have $\text{Ind}(Q_n) = \text{Ind}(P)$ for $n$ large enough. The reason here is that $\Gamma \cup P$ is a hyperbolic set since $\Gamma$ should be a transverse homoclinic loop (see [9]).

Lemma 4.5. Let $X \in \mathcal{X}^*(M^d)$ be a $C^1$ generic vector field and $\sigma \in \text{Sing}(X)$. Then for every critical element $c$ in $C(\sigma)$,

$$\text{Ind}_p(c) = \text{Ind}_p(\sigma).$$

Proof. Here we take a $C^1$ generic $X \in \mathcal{X}^*(M^d)$ satisfying item 2 of Lemma 4.5, i.e., if $p$ and $q$ are two different critical elements of $X$ with $C(p) = C(q)$, then there exists a $C^1$ neighborhood $\mathcal{U}$ of $X$ such that for any $Y \in \mathcal{U}$, one has $C(py, Y) = C(qY, Y)$. Assume that there exists a critical element $c$ contained in $C(\sigma)$ such that

$$\text{Ind}_p(c) \neq \text{Ind}_p(\sigma).$$

Fix a $C^1$ neighborhood $\mathcal{U} \subset \mathcal{X}^*(M^d)$ as above and all our perturbations will be contained in $\mathcal{U}$. We will show that some perturbation $Z \in \mathcal{U}$ has a periodic orbit with zero Lyapunov exponent, which is a contradiction. First, we need the following sublemma.

Sublemma 4.6. There exists $Y \in \mathcal{U}$ arbitrarily $C^1$ close to $X$ such that there is a heteroclinic cycle associated to $\sigma_Y$ and $c_Y$, i.e., there exist two regular points $x$ and $y$ such that
• $\text{Orb}(x, Y) \subseteq W^s(\sigma_Y) \cap W^u(c_Y)$.
• $\text{Orb}(y, Y) \subseteq W^u(\sigma_Y) \cap W^s(c_Y)$.

Proof. If $c$ is not a singularity with index $s = \text{Ind}(\sigma)$, then either

$$\dim W^s(\sigma) + \dim W^u(c) \geq d + 1,$$

or

$$\dim W^u(\sigma) + \dim W^s(c) \geq d + 1.$$ 

Without loss of generality we assume that the first case holds. Then we can choose $x_s \in W^s(\sigma) \cap C(\sigma)$ and $x_u \in W^u(c) \cap C(\sigma)$ and apply the connecting lemma for chains (Lemma 2.4) to create a heteroclinic orbit

$$x \in W^s(\sigma_{X_1}) \cap W^u(c_{X_1})$$

for some $X_1 \in U$. Moreover, since $\dim W^s(\sigma) + \dim W^u(c) \geq d + 1$, one can assume this intersection is transverse after an arbitrary small $C^1$ perturbation when necessary. Since we still have $C(\sigma_{X_1}, X_1) = C(c_{X_1}, X_1)$ which is nontrivial, we could choose

$$y_u \in W^u(\sigma_{X_1}) \cap C(\sigma_{X_1}) \quad \text{and} \quad y_s \in W^s(c_{X_1}) \cap C(\sigma_{X_1}).$$

Moreover, we may assume that $X_1$ satisfies item 4 of Lemma 2.5 so that you can apply the connecting lemma of Wen-Xia (Lemma 2.3) to get some

$$y \in W^u(\sigma_Y) \cap W^s(c_Y)$$

for some $Y \in U$ and $Y = X$ on $M \setminus \text{Orb}(x)$ (see the proof Theorem C in [8] for details). This finishes the proof of the claim in the case that $c$ is not a singularity with index $s = \text{Ind}(\sigma)$.

Now we assume that $c$ is a singularity with the same index of $\sigma$. The difficulty here is that we could not achieve a transverse heteroclinic orbit which will allow us to "connect twice". So we will need more assumptions on the vector field after the first connecting.

First, we choose $x_s \in W^s(\sigma) \cap C(\sigma)$ and $x_u \in W^u(c) \cap C(\sigma)$ and applying the connecting lemma for chains to create a heteroclinic orbit

$$\Gamma = \text{Orb}(x) \subseteq W^s(\sigma_{X_1}) \cap W^u(c_{X_1}).$$

Then we consider $\overline{W^u(\sigma_{X_1}, X_1)}$, the closure of the unstable manifold of $\sigma_{X_1}$, which is lower semi-continuous with respect to $X_1$. Denote

$$D_\Gamma = \{ S \in U : S|_{\{\sigma_{X_1}\} \cup \{ c_{X_1} \}} = X_1|_{\{\sigma_{X_1}\} \cup \{ c_{X_1} \}} \}$$

the set of all vector fields that coincide with $X_1$ on $\{\sigma_{X_1}\} \cup \Gamma \cup \{ c_{X_1} \}$. Then $D_\Gamma$ is a closed subset of $A^1(M^d)$, which is also a Baire set. This fact allows us to choose $X_2 \in D_\Gamma$ arbitrarily $C^1$ close to $X_1$, which is a continuous point of $\overline{W^u(\sigma_{X_2}, X_2)}$ in $D_\Gamma$.

Claim.

$$c_{X_2} \in \overline{W^u(\sigma_{X_2}, X_2)}.$$

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Proof of Claim: Otherwise, there exists an open neighborhood $V$ of $\overline{W^u(\sigma_{X_2}, X_2)} \cap C(\sigma_{X_2})$, such that $c_{X_2} \in M^d \setminus V$. We choose some $y \in W^u_{loc}(\sigma_{X_2}, X_2) \cap C(\sigma_{X_2}) \cap V$. Then $c_{X_2}$ is chain attainable from $y$, i.e., there exists a sequence of chain $\{y^n_i, t^n_i\}_{i=1}^{l_n}$, $\forall t^n_i > T$, $n = 1, 2, 3 \dotsc$ (for some $T > 0$) which satisfy

$$d(\phi^n_{t_i}(y^n_i), y^n_{i+1}) < \frac{1}{n}, \quad y^n_1 = y \quad \text{and} \quad d(\phi^X_{t_i}(y^n_i), c_{X_2}) < \frac{1}{n},$$

for all $1 \leq i \leq l_n - 1$ and $n > 0$. Denote by $w_n$ the point at which, for the first time, the chain $\{y^n_i, t^n_i\}_{i=1}^{l_n}$ does not belong to $V$. Then $\{w_n\}$ will converge to some point $w \in \partial V \cap C(\sigma_{X_2})$, which does not belong to $\overline{W^u(\sigma_{X_2}, X_2)}$. Moreover, we assert that $w$ does not belong to $\Gamma$, otherwise the chain before $w_n$ will accumulate to $c_{X_2}$ first, which contradicts the fact that $w_n$ is the first point that escapes from $V$. For the same reason, the Hausdorff limit of these chains from $y$ to $w_n$ is far from $\Gamma$. We will use the connecting lemma for chains here. One has

- There exists chains with arbitrarily small jumps from $y$ to $w$.
- All these chains and their Hausdorff limit do not intersects $\overline{\Gamma}$.

By Lemma 2.4, there is $X_3$ which is arbitrarily $C^1$ close to $X_2$, such that

- $w \in \overline{W^u(\sigma_{X_3}, X_3)}$.
- The perturbation region does not intersect $\overline{\Gamma}$, which implies $X_3 \in \mathcal{D}_\Gamma$.

This fact shows that we could enlarge $\overline{W^u(\sigma_{X_2}, X_2)}$ to $w$ by an arbitrarily small $C^1$ perturbation in $\mathcal{D}_\Gamma$, which contradicts that $X_2$ is a continuous point of $\overline{W^u(\sigma_{X_2}, X_2)}$ in $\mathcal{D}_\Gamma$. This finishes the proof of the claim.

Thus $c_{X_2} \in \overline{W^u(\sigma_{X_2}, X_2)}$, which implies that $c_{X_2}$ could be accumulated by some regular orbits contained in $\overline{W^u(\sigma_{X_2}, X_2)}$. So there exists some point $z$ such that

$$z \in \overline{W^u(\sigma_{X_2}, X_2)} \cap W^s_{loc}(c_{X_2}, x_2).$$

One assumes that every $\varepsilon$-perturbation of $X_2$ is still in $\mathcal{U}$ for some $\varepsilon > 0$. With the help of $C^1$-connecting lemma of Wen-Xia (Lemma 2.3), for $\varepsilon > 0$, there are $L > 0$ and two neighborhoods $\tilde{W}_z \subset W_z$ of $z$ such that if one denotes $W_{L,z} = \cup_{0 \leq t \leq L} \phi^X_t(W_z)$, one has

- $W_{L,z}$ is disjoint from $\overline{\Gamma}$.
- The positive orbit of some $y \in W^u_{loc}(\sigma_{X_2}, X_2)$ intersects $\tilde{W}_z$.

By Lemma 2.3, there is $Y$ $\varepsilon$-close to $X_2$ such that

- $Y$ has a heteroclinic orbit: $\text{Orb}(y, Y) \subseteq W^u(\sigma_Y) \cap W^s(c_Y)$.
- $\Gamma = \text{Orb}(x, Y) \subseteq W^s(\sigma_Y) \cap W^u(c_Y)$ is still a heteroclinic orbit.

This finishes the proof of the sublemma. \qed
Now we continue to prove Lemma 4.5. For simplicity, we will assume that $Y$ is $C^1$ linearizable around $\sigma_Y$ and $c_Y$, and exhibits the heteroclinic cycle

$$\Gamma_{0,0} = \Gamma_Y = \{\sigma_Y\} \cup \{c_Y\} \cup \text{Orb}(x, Y) \cup \text{Orb}(y, Y),$$

where $\text{Orb}(x, Y) \subseteq W^s(\sigma_Y) \cap W^u(c_Y)$ and $\text{Orb}(y, Y) \subseteq W^u(\sigma_Y) \cap W^s(c_Y)$.

In two disjoint linearizable neighborhoods of $\sigma_Y$ and $c_Y$, choose two pairs of points $\{x, y\}$ and $\{u, v\}$ such that

- $x \in W^s_{loc}(\sigma_Y) \cap \text{Orb}(x)$ and $y \in W^u_{loc}(\sigma_Y) \cap \text{Orb}(y),$
- $u \in W^u_{loc}(c_Y) \cap \text{Orb}(x)$ and $v \in W^s_{loc}(c_Y) \cap \text{Orb}(y).$

Then we can choose two pairs of continuous segments $\{x_r, y_r\}$, $0 \leq r \leq 1$ and $\{x_t, y_t\}$, $0 \leq t \leq 1$ such that

- $\phi_{t^0}^Y(x_r, 0) = x_r$ and $\phi_{t^0}^Y(y_r, 0) = y_r$;
- $\phi_{t^0}^Y(y_r, 0) = x_r$ and $\phi_{t^0}^Y(x_r, 0) = y_r$.

Connecting $x_r$ to $x_r$, $y_r$ to $y_r$, $x_t$ to $x_t$, and $y_t$ to $y_t$ continuously, we get a continuous family of star vector fields $\{Y_r, t : 0 \leq r, t \leq 1\} \subseteq U \subset X^*(M^d)$ with two parameters $r$ and $t$ such that

- $\lim_{r \to 0} Y_r = Y$.
- $Y_{0, t}$ exhibits a homoclinic orbit associated to $\sigma_Y$, denoted by $\Gamma_{0, t}$ for $0 \leq t \leq 1$.
- $Y_{r, 0}$ exhibits a homoclinic orbit associated to $c_Y$, denoted by $\Gamma_{r, 0}$ for $0 \leq r \leq 1$.
- $Y_{r, t}$ exhibits a periodic orbit $\Gamma_{r, t}$ satisfying

$$\lim_{r \to 0} \Gamma_{r, t} = \Gamma_{0, t} \quad \text{and} \quad \lim_{t \to 0} \Gamma_{r, t} = \Gamma_{r, 0}.$$  

We fix some $r_0 > 0$ and let $t \to 0$, for $t = t_0$ small enough, Lemma 4.4 insures that

$$\text{Ind}(\Gamma_{r_0, t_0}) = \text{Ind}_p(c_Y).$$

Then letting $\Gamma_{r, t_0} \to \Gamma_{0, t_0}$ as $r \to 0$, and applying Lemma 4.4 again, we know there is some $r_1 < r_0$ such that

$$\text{Ind}(\Gamma_{r_1, t_0}) = \text{Ind}_p(\sigma_Y) \neq \text{Ind}(\Gamma_{r_0, t_0}).$$

Since the family of vector fields $\{Y_r, t_0 : r_1 \leq r \leq r_0\}$ is continuous on the parameters $r$ in the $C^1$ topology, the Lyapunov exponents of $\Gamma_{r, t_0}$ is also continuous on $r$. This implies that there must be some $r_2$ with $r_1 < r_2 < r_0$, such that $\Gamma_{r_2, t_0}$ is a nonhyperbolic periodic orbit, contradicting $Y_{r_2, t_0} \in U \subset X^*(M^d)$. This finishes the proof of the lemma. \hfill \Box

**Lemma 4.7.** Let $X \in X^*(M^d)$ and $\sigma$ be a singularity of $X$ such that $C(\sigma)$ is nontrivial. Then for every singularity $\rho$ in $C(\sigma)$, we have

- if $\text{sv}(\rho) > 0$, then $W^{ss}(\rho) \cap C(\sigma) = \{\rho\}$. 

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Proof of Theorem 3.6. We take the dense singularities contained in a singular chain recurrent class have the same index, then this

Theorem 3.6 follows from Corollary 4.3, Lemma 4.5 and 4.7 directly. This ends the proof here. Assume sv(ρ) > 0 (if sv(ρ) < 0 we consider −X). Then from Lemma 4.2 we know that there exists a dominated splitting

\[ T_ρM = E^{ss}(ρ) \oplus E^c(ρ) \oplus E^u(ρ), \]

which can be assumed to be mutually orthogonal. Suppose on the contrary that \( W^{ss}(ρ) \cap C(σ) \neq \{ρ\} \). Then applying the connecting lemma of chains, there exists some star vector field \( Y \) arbitrarily \( C^1 \) close to \( X \) exhibiting a strong homoclinic connection

\[ Γ \subset W^{ss}(ρ_Y, Y) \cap W^u(ρ_Y, Y). \]

Moreover, we can assume \( Y \) is linearizable around \( ρ_Y \). Then using the perturbation around the singularities to generate periodic orbits accumulating the homoclinic loop, we get a sequence of vector fields \( \{Y_n\} \) and \( p_n \in \text{Per}(Y_n) \) satisfying \( p_n \to ρ \) and

\[ \langle Y_n(p_n) \rangle \hookrightarrow E^{ss}(ρ_Y) \oplus E^u(ρ_Y) \setminus (E^{ss}(ρ_Y) \cup E^u(ρ_Y)). \]

Since we have \( \text{Ind}(p_n) = \text{Ind}_ρ(ρ_Y) = \text{Ind}(ρ_Y) - 1 = s - 1 \), we can choose some nonzero \( v \) such that \( L = \langle v \rangle \in B^{−1}(C(σ_Y)) \) and \( v \in E^{ss}(ρ_Y) \oplus E^u(ρ_Y) \). Let \( v = v^{ss} + v^u \), where \( v^{ss} \in E^{ss}(ρ_Y) \) and \( v^u \in E^u(ρ_Y) \). Without loss of generality, we can assume that \( |v^{ss}| = |v^u| \). Let \( w = v^{ss} - v^u \), then \( v \perp w \). So, \( (L, w) \in N_L \). Denote \( (Lt, w_t) = ψ^Y_t(L, w) \).

Since \( E^{ss}(ρ_Y) \) is contracting and \( E^u(ρ_Y) \) is expanding, we have

1. \( L_t \hookrightarrow E^u(ρ_Y) \) and \( \langle w_t \rangle \hookrightarrow E^{ss}(ρ_Y) \), as \( t \to +∞ \).
2. \( L_t \hookrightarrow E^{ss}(ρ_Y) \) and \( \langle w_t \rangle \hookrightarrow E^u(ρ_Y) \), as \( t \to −∞ \).

There exists a dominated splitting \( N_{B^{s−1}(C(σ_Y)) \cap T_ρM} = E \oplus F \) with index \( s - 1 \), since \( L \) is the limit of flow directions of periodic orbits. Now we consider two cases:

Case 1: \( (L, w) \in E_L \). In this case, consider \( t \to −∞ \). There exists \( t_n \to −∞ \) such that \( Lt_n \to L' \in E^{ss}(ρ_Y) \). According to the continuity of \( E_L \), we know that \( (Lt_n, w_{t_n}) \in E_L \cap E_{L'} \). However we know that \( \langle w_{t_n} \rangle \hookrightarrow E^u(ρ_Y) = F_{L'} \). This is a contradiction.

Case 2: \( (L, w) \notin E_L \). In this case, consider \( t \to +∞ \). There exists \( t_n \to +∞ \) such that \( L_{t_n} \to L' \in E^u(ρ_Y) \). Since \( E < F \), we have \( (L_{t_n}, w_{t_n}) \hookrightarrow F_{L'} \). However we know that \( \langle w_{t_n} \rangle \hookrightarrow E^{ss}(ρ_Y) = E_{L'} \). This is also a contradiction.

This finishes the proof of Lemma 4.7.

We end this section by summarizing these results to deduce Theorem 3.6.

Proof of Theorem 3.6. We take the dense \( G_δ \) subset \( G_1 \) satisfying Lemma 4.5. Then Theorem 3.6 follows from Corollary 4.3, Lemma 4.5 and 4.7 directly. This ends the proof of Theorem 3.6.

5 Singular hyperbolicity of singular chain recurrent classes

In this section, we will give a proof of Theorem 3.7, which states that if all the singularities contained in a singular chain recurrent class have the same index, then this
chain recurrent class must be singular hyperbolic. During the proof, we will obtain a nice
description for the ergodic measures of star flows (Theorem 5.6). The main techniques
we will use are Liao’s Shadowing Lemma (Theorem 5.2) and his estimation of the size of
invariant manifolds (Theorem 5.4).

First, we define quasi-hyperbolic arcs for the scaled linear Poincaré flow (see Section
2 for definition).

**Definition 5.1.** Given \( X \in \mathcal{X}^1(M^d) \) and \( x \not\in \text{Sing}(X) \), the orbit arc \( \phi_{[0,T]}(x) \) is called \((\eta,T_0)^*\) quasi hyperbolic with respect to a direct sum splitting \( \mathcal{N}_x = E(x) \oplus F(x) \) and the
scaled linear Poincaré flow \( \psi^*_T \) if there exists \( \eta > 0 \) and a partition
\[ 0 = t_0 < t_1 < \cdots < t_l = T, \text{ where } t_{i+1} - t_i \in [T_0,2T_0] \]
such that for \( k = 0,1,\cdots,l-1 \), we have
\[
\prod_{i=0}^{k-1} \| \psi^*_{t_{i+1} - t_i} |_{E(x)} \| \leq e^{-\eta t_k},
\]
\[
\prod_{i=k}^{l-1} m(\psi^*_{t_{i+1} - t_i} |_{F(x)}) \geq e^{\eta (t_l - t_k)},
\]
\[
\frac{\| \psi^*_{t_{k+1} - t_k} |_{E(x)} \|}{m(\psi^*_{t_{k+1} - t_k} |_{F(x)})} \leq e^{-\eta (t_{k+1} - t_k)}.
\]

**Remark.** This definition is similar to the usual quasi hyperbolic orbit arc for linear
Poincaré flow. The only difference is that we consider the scaled linear Poincaré flow
instead of the usual linear Poincaré flow.

The proof of the next theorem could be found in [16] (see [10] for more explanations).

**Theorem 5.2.** ([16]) Given \( X \in \mathcal{X}^1(M^d) \), a compact set \( \Lambda \subset M^d \setminus \text{Sing}(X) \), and \( \eta > 0, T_0 > 0 \), for any \( \varepsilon > 0 \) there exists \( \delta > 0 \), such that for any \((\eta,T_0)^*\) quasi hyperbolic orbit arc \( \phi_{[0,T]}(x) \) with respect to some direct sum splitting \( \mathcal{N}_x = E(x) \oplus F(x) \) and the
scaled linear Poincaré flow \( \psi^*_T \) which satisfies \( x, \phi_T(x) \in \Lambda \) and \( d(E(x),\psi_T(E(x))) \leq \delta \),
there exists a point \( p \in M^d \) and a \( C^1 \) strictly increasing function \( \theta : [0,T] \to \mathbb{R} \) such that

- \( \theta(0) = 0 \) and \( 1 - \varepsilon < \theta'(t) < 1 + \varepsilon \);
- \( p \) is a periodic point with \( \phi_{\theta(t)}(p) = p \);
- \( d(\phi_t(x),\phi_{\theta(t)}(p)) \leq \varepsilon |X(\phi_t(x))|, \text{ } t \in [0,T] \).

**Remark.** In this theorem, the compactness of \( \Lambda \) guarantees the two ends of the quasi
hyperbolic string to be uniformly far from the singularities. But we do not require the
compact set \( \Lambda \) to be invariant. Some part of the quasi hyperbolic string can be very close
to singularities. If the ends of the string are close to singularity, the conclusion may not
hold.
The second theorem of Liao we need is the significant estimation for the size of invariant manifolds. Such kind of theorems are well-known in the case of diffeomorphism and non-singular flow (e.g., see [27]). If there is a singularity, however, it would be very subtle and difficult to estimate the size of invariant manifolds when the regular orbits approximate the singularity. As before, we first introduce the definition of \((\eta, T, E)^*\) contracting orbit arcs.

**Definition 5.3.** Let \(X \in \mathcal{X}(M^d)\), \(\Lambda\) a compact invariant set of \(X\), and \(E \subset N_{\Lambda \setminus \text{Sing}(X)}\) an invariant bundle of the linear Poincaré flow \(\psi_t\). For \(\eta > 0\) and \(T > 0\), \(x \in \Lambda \setminus \text{Sing}(X)\) is called \((\eta, T, E)^*\) contracting if for any \(n \in \mathbb{N}\),

\[
\prod_{i=0}^{n-1} \| \psi_T^i|_{E(\phi_T(x))} \| \leq e^{-n\eta}.
\]

Similarly, \(x \in \Lambda \setminus \text{Sing}(X)\) is called \((\eta, T, E)^*\) expanding if it is \((\eta, T, E)^*\) contracting for \(-X\).

**Theorem 5.4.** ([18]) Let \(X \in \mathcal{X}(M^d)\) and \(\Lambda\) a compact invariant set of \(X\). Given \(\eta > 0, T > 0\), assume that \(N_{\Lambda \setminus \text{Sing}(X)} = E \oplus F\) is an \((\eta, T)\)-dominated splitting with respect to the linear Poincaré flow. Then, for any \(\varepsilon > 0\), there is \(\delta > 0\) such that if \(x\) is \((\eta, T, E)^*\) contracting, then there is a \(C^1\) map \(\kappa : E_x(\delta|X(x)|) \rightarrow N_x\) such that

- \(d_{C^1}(\kappa, \text{id}) < \varepsilon\).
- \(\kappa(0) = 0\).
- \(W_{\delta|X(x)|}^{cs}(x) \subset W^*(\text{Orb}(x))\), where \(W_{\delta|X(x)|}^{cs}(x) = \exp_x(\text{Image}(\kappa))\).

Here \(E_x(r) = \{v \in E_x : |v| \leq r\}\).

**Remark.** Compared with the cases of diffeomorphisms and non-singular flows, we can see that this theorem is quite reasonable. In those two cases, if we have a uniform contraction for the derivatives in the future, we can achieve a uniform size of stable manifolds. But here, because of the interference of singularities, we could only expect the size of stable manifolds to be proportional to the flow speed. This could also be thought as some kind uniform size of invariant manifolds.

For the proof of Theorem 3.7, we still need the Ergodic Closing Lemma of Mañé. We call a point \(x \in M - \text{Sing}(X)\) is strongly closable for \(X\), if for any \(C^1\) neighborhood \(U\) of \(X\), and any \(\delta > 0\), there exists \(Y \in U\), \(y \in M\), and \(\tau > 0\) such that the following items are satisfied:

- \(\phi_{\tau}^Y(y) = y\).
- \(d(\phi_t^X(x), \phi_t^Y(y)) < \delta\), for any \(0 \leq t \leq \tau\).

The set of strongly closable points of \(X\) will be denoted by \(\Sigma(X)\). The following flow version of the Ergodic Closing Lemma can be found in [29].

**Theorem 5.5.** ([29]) For any \(X \in \mathcal{X}(M)\), \(\mu(\text{Sing}(X) \cup \Sigma(X)) = 1\) for every \(T > 0\) and every \(\phi_T^X\)-invariant Borel probability measure \(\mu\).
Now with the help of these theorems, we can give a description for ergodic measures of star flows. The next theorem is a detailed version of Theorem E.

Given a $C^1$ vector field $X$, an ergodic measure $\mu$ of $X$ is called hyperbolic if $\mu$ has at most one zero Lyapunov exponent, whose invariant subspace is spanned by $X$.

**Theorem 5.6.** Let $X \in \mathcal{X}^*(M^d)$. Then any ergodic measure $\mu$ of $X$ is hyperbolic. Moreover, if $\mu$ is not the atomic measure on any singularity, then

$$\text{supp}(\mu) \cap H(P) \neq \emptyset,$$

where $P$ is a periodic orbit with the index of $\mu$, i.e., the stable dimension of $P$ and $\mu$ coincide.

**Proof.** Since $\mu$ is ergodic for the time-$t$ map $\phi_t$ except at most countable many $t$ ([26]), we can choose $T$ large enough so that

- $\mu$ is ergodic for the time-$T$ map $\phi_T$,
- $\exists \eta > 0$ such that the constants $T$ and $\eta$ satisfy the conclusion of Lemma 2.1.

If $\mu$ supports on some critical element, then from the definition of star flows, it should be hyperbolic. So for the rest of the proof, we will assume that $\mu$ does not support on any critical element. We will first use the ergodic closing lemma to show $\mu$ is hyperbolic; then apply the argument of Katok and Liao’s shadowing lemma (Theorem 5.2) to prove the existence of the accumulation of periodic orbits; and finally, the estimation of the size of stable and unstable manifolds (Theorem 5.4) will guarantee these periodic orbits are homoclinic related.

Applying Theorem 5.6 there exists some point $x \in B(\mu) \cap \text{supp}(\mu) \cap \Sigma(X)$ and $X_n \in \mathcal{X}^1(M^d)$, $x_n \in M^d$, $\tau_n > 0$ such that

- $\phi^X_{\tau_n}(x_n) = x_n$, where $\tau_n$ is the minimal period of $x_n$;
- $d(\phi^X_t(x), \phi^X_{\tau_n}(x_n)) < 1/n$, for any $0 < t < \tau_n$;
- $\|X_n - X\|_{C^1} < 1/n$.

Here $B(\mu)$ is the set of generic points of $\mu$. Recall that $x$ is a generic point of $\mu$ if for any continuous function $\xi : M^d \to \mathbb{R}$,

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi(\phi^T_i(x)) = \int \xi(y) d\mu(y).$$

Since $\mu$ does not support on any critical element, we know that $\tau_n \to \infty$ as $n \to \infty$, and the ergodic measure $\mu_n$ supported on the periodic orbit of $x_n$ will converge to $\mu$ in the sense of weak topology. From Lemma 2.1 we know that for any $x \in \text{Orb}(x_n)$, $m \in \mathbb{N}$,

$$\prod_{i=0}^{[m\tau_n/T] - 1} \|\psi^{-1}_{T^n} |_{N^s(\phi^X_{\tau_n})}\| \leq e^{-m\eta\tau_n},$$

$$\prod_{i=0}^{[m\tau_n/T] - 1} m(\psi^{-1}_{T^n} |_{N^u(\phi^X_{\tau_n})}) \geq e^{m\eta\tau_n}.$$
These inequalities imply
\[
\int \log \| \psi_T^X_n \|_{N^s(x)} \, d\mu_n(x) \leq -\eta,
\]
\[
\int \log m(\psi_T^X_n) \, d\mu_n(x) \geq \eta.
\]
We may assume that the index of Orb\((x_n)\) is the same, then item 1 of Lemma 2.1 gives a dominated splitting on the limit: \(N^s \oplus N^u\). By considering the extended linear Poincaré flow \(\psi_T^X(L, v)\), since \(\psi\) is continuous in \(T, X, L, v\) (see Lemma 3.1 in [14]), we get that
\[
\int \log \| \psi_T^X \|_{N^s(x)} \, d\mu(x) \leq -\eta,
\]
\[
\int \log m(\psi_T^X) \, d\mu(x) \geq \eta.
\]
This proves that \(\mu\) is hyperbolic for \(X\).

Since \(\mu\) does not support on any critical element,
\[
\int \log \| \Phi_T^{\langle X(x) \rangle} \| \, d\mu(x) = 0.
\]
We get that
\[
\int \log \| \psi_T^* \|_{N^s(x)} \, d\mu(x) \leq -\eta,
\]
\[
\int \log m(\psi_T^*) \, d\mu(x) \geq \eta,
\]
equivalently,
\[
\int \log \| \psi_T^* \|_{N^u(x)} \, d\mu(x) \leq -\eta.
\]
By Birkhoff Ergodic Theorem, we know that for \(\mu\)-almost every \(z \in M\), we have
\[
\int \log \| \psi_T^* \|_{N^s(x)} \, d\mu(x) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \log \| \psi_T^* \|_{N^s(x)} \leq -\eta,
\]
\[
\int \log \| \psi_T^* \|_{N^u(x)} \, d\mu(x) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \log \| \psi_T^* \|_{N^u(x)} \leq -\eta.
\]

Following Katok’s argument [13], for every \(K > 0\), let \(\Lambda_K\) be the set of points \(x \in \text{supp}(\mu) \cap B(\mu)\) such that for each \(k > 0\) one has
\[
\prod_{i=0}^{k-1} \| \psi_T^* \|_{N^s(x)} \leq Ke^{-k\eta},
\]
\[
\prod_{i=0}^{k-1} \| \psi_T^* \|_{N^u(x)} \leq Ke^{-k\eta}.
\]
Then \(\mu(\Lambda_K) \to 1\) as \(K \to \infty\). So, for \(K\) large enough, \(\mu(\Lambda_K) > 0\). Since \(\mu\) could not support on any critical element and is ergodic, we have \(\mu(\text{Sing}(X)) = 0\). So for some \(\delta > 0\), \(\Delta_K = \Lambda_K \setminus B(\text{Sing}(X), \delta)\) has positive measure, where \(B(\text{Sing}(X), \delta)\) is the \(\delta\)-neighborhood of \(\text{Sing}(X)\) in \(M\). Note that \(\Delta_K\) is a closed set. According to Poincaré
recurrence theorem, this implies that for every $z \in \text{supp } \mu|_{\Delta_K}$, one can find orbit arcs $\phi_{[0,m_nT]}(x_n)$ such that $x_n, \phi_{m_nT}(x_n)$ belong to $\Delta_K$, the distances $d(x_n,z), d(z, \phi_{m_nT}(x_n))$ are arbitrarily small and the non-invariant atomic measure

$$\mu_n = \frac{1}{m_n} \sum_{i=0}^{m_n-1} \delta_{\phi_i(x_n)}$$

is arbitrarily close to $\mu$. In particular, for each $0 \leq k \leq m_n$ we have

$$\prod_{i=0}^{k-1} \|\psi^*_T|_{N^s(\phi^s_T(x_n))}\| \leq Ke^{-k\eta}, \quad \text{and} \quad \prod_{i=0}^{k-1} \|\psi^*_T|_{N^u(\phi^u_T(x_n))}\| \leq Ke^{-k\eta}.$$

Since here the end points of the quasi-hyperbolic orbit arc are uniformly $\delta-$away from the singularities of $X$, we can apply the shadowing lemma of Liao (Theorem 5.2): there exists a sequence of periodic points $p_n$ converge to $z$, such that the atomic measure supported on $\text{Orb}(p_n, X)$ converge to $\mu$. Moreover, the property of shadowing original $(\eta,T)^*$ quasi hyperbolic orbit arcs guarantees that some $q_n \in \text{Orb}(p_n)$ is $(\eta/2, T, N^s)^*$ contracting and $(\eta/2, T, N^u)^*$ expanding: for $n$ large enough (to eliminate the constant $K$) and every $k \in \mathbb{N}$,

$$\prod_{i=0}^{k-1} \|\psi^*_T|_{N^s(\phi^s_T(q_n))}\| \leq e^{-k\eta/2},$$

$$\prod_{i=0}^{k-1} \|\psi^*_T|_{N^u(\phi^u_T(q_n))}\| \leq e^{-k\eta/2}.$$  

Then Theorem 5.4 shows that $q_n$ will have a uniform size of local stable and unstable manifolds, which guarantees that for $n$ large enough, periodic orbits $\text{Orb}(p_n) = \text{Orb}(q_n)$ are mutually homoclinic related, and hence $z \in H(p_n)$. This finishes the proof of the theorem. \(\square\)

Remark. 1. Theorem E is the first conclusion of this theorem.

2. From the proof, we can see that points in $\Lambda_K$ close to $z$ also belong to $H(P)$. So, $\mu(H(P)) > 0$. Since $\mu$ is ergodic, $\mu(H(P)) = 1$, i.e., $\mu$ is supported on $H(P)$. Especially,

$$\text{supp}(\mu) \subseteq \text{Per}(X).$$

3. According to Theorem 5.6, if $\mu$ is a nontrivial ergodic measure of a star vector field, then the measurable entropy of $\mu$ is positive.

Applying the description of invariant measures of star flows, we prove the following homogeneous property for generic star flows.

**Theorem 5.7.** For a $C^1$ generic star vector field $X$ and any chain recurrent class $C$ of $X$, there exists a neighborhood $U$ of $C$ such that all the critical elements contained in $U$ have the same periodic index with the critical elements contained in $C$.  

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Proof. For the case where \( C \) does not contain any singularities, we refer to [9] which showed that the homogeneous property holds for the nonsingular chain recurrent class of any star vector fields. So we will focus on the case where \( C = C(\sigma) \) is nontrivial and the vector field \( X \) satisfies the generic properties which will guarantee the conclusion of Lemma 4.5.

Now we assume that there exists a sequence of periodic orbits \( \{P_n\} \) whose Hausdorff limit is contained in \( C(\sigma) \), and

\[
\text{Ind}_p(P_n) = \text{Ind}(P_n) = k \neq \text{Ind}_p(\sigma).
\]

Without loss of generality, we may assume that \( k > \text{Ind}_p(\sigma) \).

The invariant probability measure \( \mu_n \) supported on \( P_n \) will converge to an invariant measure \( \tilde{\mu} \) whose support is contained in \( C(\sigma) \). Denote by

\[
\xi(x) = \inf_{E \subset T_x M, \dim E = k+1} \sup_{L \subset E, \dim L = 2} \log |\det (\Phi_T|_L)|.
\]

It is easily seen that \( \xi : M \to \mathbb{R} \) is continuous. Since

\[
\int \xi(x) d\mu_n \leq -\eta,
\]

we have

\[
\int \xi(x) d\tilde{\mu} \leq -\eta.
\]

Then, the Ergodic Decomposition Theorem allows us to find an ergodic invariant measure \( \mu \) supported on \( C(\sigma) \) which also satisfies the above estimation

\[
\int \xi(x) d\mu \leq -\eta.
\]

Obviously, \( \mu \) could not support on any singularity in \( C(\sigma) \). Theorem 5.6 tells us that \( \mu \) is hyperbolic with index \( \geq k \) and supp(\( \mu \)) \( \cap H(q) \neq \emptyset \) for some periodic point \( q \) with index \( \geq k \). By the definition of chain recurrent class and homoclinic class, we know that \( q \in C(\sigma) \). However, this is impossible because \( \text{Ind}_p(q) \geq k > \text{Ind}_p(\sigma) \) which contradicts to the conclusion of Lemma 4.5. This finishes the proof of the theorem.

Now we can finish the proof of Theorem 3.7 with the help of the description of ergodic measures and the homogeneous property of star vector fields.

Proof of Theorem 3.7. We take the dense \( G_\delta \) subset \( G_2 \subseteq G_1 \subseteq X^*(M^d) \) whose elements also satisfy the generic properties stated in Theorem 5.7 and the fourth item of Lemma 2.3. For any \( X \in G_2 \) and a nontrivial chain recurrent class \( C(\sigma) \) where \( \sigma \in \text{Sing}(X) \), from Lemma 2.3 we know that there exists a sequence of periodic orbits \( \{Q_n\} \) converge to \( C(\sigma) \) in the Hausdorff topology. Without loss of generality, we may assume that \( \text{sv}(\sigma) > 0 \). By Theorem 3.6 and the conclusion of Lemma 1.5 for any \( \rho \in \text{Sing}(X) \cap C(\sigma) \), \( \text{sv}(\rho) > 0 \). Moreover, the homogeneous property and \( W^{s\ast}(\rho) \cap C(\sigma) = \{\rho\} \) (from Theorem 3.6) for any \( \rho \in C(\sigma) \cap \text{Sing}(X) \) guarantees that

\[
\text{Ind}(Q_n) = \text{Ind}_p(\rho) = \dim E^{s\ast}(\rho), \quad \forall \rho \in C(\sigma) \cap \text{Sing}(X).
\]
This implies $\beta(B^k(C(\sigma))) = C(\sigma)$ (where $k = \dim E^{ss}(\rho)$) and it has a continuous splitting of the extended tangent flow over the compactification of $C(\sigma)$:

$$\beta^*(T_{C(\sigma)}M^d) |_{B^k(C(\sigma))} = N^s \oplus P \oplus N^u.$$ 

Recall that $P$ is the limit of flow line, which is $\Phi_t$-invariant. $N^{s/u}$ are contained in the normal bundle, which is invariant by the extended linear Poincaré flow $\psi_t$, and $E^{cs/cu} = N^{s/u} \oplus P$ is $\Phi_t$-invariant. Changing the metric if necessary, we can assume that $E^{ss}(\rho) \perp E^{cu}(\rho)$ for any singularity $\rho$. Since $W^{ss}(\rho) \cap C(\sigma) = \{\rho\}$, we know that $P |_{B^k(\{\rho\})} \subseteq B^k(\{\rho\}) \times E^{cu}(\rho)$. Consequently, the domination of the extended linear Poincaré flow $N^s \prec N^u$ ensures that

$$N^s |_{B^k(\{\rho\})} = B^k(\{\rho\}) \times E^{ss}(\rho), \quad \forall \rho \in C(\sigma) \cap \text{Sing}(X).$$

**Claim.** There exists a mixed dominated splitting $(N^s, \psi_t) \prec (P, \Phi_t)$ on $B^k(C(\sigma))$, i.e., there exists $T > 0$ such that

$$\frac{\|\psi_T^*|_{N^s}\|}{m(\Phi_T|_P)} \leq \frac{1}{2}.$$

**Proof of Claim:** The claim is equivalent to say that the scaled linear Poincaré flow $\psi_t^*$ restricted on $N^s$ is uniformly contracting. If it is not uniformly contracting, then there exists an ergodic invariant measure $\mu$ whose support is contained in $C(\sigma)$ such that

$$\int \log \|\psi_T^*|_{N^s(x)}\| \, d\mu(x) \geq 0.$$

It is easy to see that the push-forward measure $\beta_\ast(\mu)$ on $M$ can not to be the atomic measure at singularity since $P |_{B^k(\{\rho\})} \subseteq B^k(\{\rho\}) \times E^{cu}(\rho)$ and $N^s |_{B^k(\{\rho\})} = B^k(\{\rho\}) \times E^{ss}(\rho)$. So, the above inequality is also satisfied for the measure $\beta_\ast(\mu)$ on $M$. Moreover, the inequality implies that the dimension of invariant subspace associated to negative Lyapunov exponents of (the hyperbolic measure) $\beta_\ast(\mu)$ is less than $k$. Theorem 5.6 tells us that $\text{supp}(\beta_\ast(\mu)) \cap H(P) \neq \emptyset$ for some periodic orbit $P$ with index less than $k$. This contradicts to the homogeneous property stated in Lemma 4.5. This finishes the proof of the claim. 

Since $P |_{B^k(\{\rho\})} \subseteq B^k(\{\rho\}) \times E^{cu}(\rho)$, a similar proof as the above claim shows that $N^s$ is uniformly contracting with respect to $\psi_t$.

The rest part of the proof is to show that $\Phi_t$ admits a partially hyperbolic splitting over $T_{C(\sigma)}M$. This is almost exactly the same as the proof Theorem A in [14], and we just sketch the proof for the convenience of reader. By Lemma 2.1 and the claim we have

$$(N^s, \psi_t) \prec (N^u, \psi_t) \quad \text{and} \quad (N^s, \psi_t) \prec (P, \Phi_t).$$

According to Lemma 5.5 of [14] (see also [14], Lemma 4.4) the above dominations imply that we have the mixing dominated splitting $(N^s, \psi_t) \prec_{T_0} (E^{cu}, \Phi_t)$ for some $T_0 > 0$. So the linear bundle map

$$\Phi_{T_0} : \beta^*(TM) |_{B^k(C(\sigma))} \rightarrow \beta^*(TM) |_{B^k(C(\sigma))},$$

can be expressed as

$$\Phi_{T_0} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} : N^s \oplus E^{cu} \rightarrow N^s \oplus E^{cu},$$

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where $A = \psi_{T_0} |_{N^s}$, $D = \Phi_{T_0} |_{E^{cu}}$. Moreover the mixed domination $(N^s, \psi_t) \prec_{T_0} (E^{cu}, \Phi_t)$ implies that

$$\frac{\|A\|}{m(D)} \leq \frac{1}{2}.$$ 

Then the calculation in [14, Lemma 5.6] tells us there exists a $\Phi_{T_0}$-invariant subbundle, denoted by $E^{ss}$. This give a continuous $\Phi_{T_0}$-invariant splitting

$$\beta^*(TM) \mid_{B^k(C(\sigma))} = E^{ss} \oplus E^{cu}.$$ 

The compactness of $B^k(C(\sigma))$ and continuity of this invariant splitting guarantees there exists some constant $C > 0$, such that

$$\|\Phi_{T_0} \mid_{E^{ss}}\| \leq C \|\psi_{T_0} \mid_{N^s}\|.$$ 

Finally, for any $t > 0$, $E^{ss} \oplus E^{cu}$ is a $\Phi_t$-invariant dominated splitting by the uniqueness of dominated splitting. And this splitting induces a $\Phi_t$-invariant dominated splitting $E^{ss} \oplus E^{cu}$ on $T_{C(\sigma)}M$, and $E^{ss}$ is uniformly contracting with respect to $\Phi_t$ since $N^s$ is uniformly contracting with respect to $\psi_t$.

Now we have proved that $T_{C(\sigma)}M = E^{ss} \oplus E^{cu}$ is a $\Phi_t$-invariant partially hyperbolic splitting. For the singular hyperbolicity, we only need to show that $\Phi_t \mid_{E^{cu}}$ is sectional expanding. This is exactly the same as the proof of the claim. If it is not, then we can find an ergodic measure on $C(\sigma)$ such that its dimension of stable bundle is larger than $k = \text{Ind}_f(\sigma)$. The fact that the saddle values of all the singularities contained in $C(\sigma)$ are larger than 0 excludes the possibility that this measure is an atomic measure at any singularity. Then Theorem 5.6 allows us to find a periodic orbit contained in $C(\sigma)$ whose index is larger than $k$. This contradicts the homogeneous property of $X$, and finishes the proof of this theorem. 

\[\square\]

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Yi Shi <polarbearsy@gmail.com>: School of Mathematical Sciences, Peking University, Beijing 100871, China & Institut de Mathématiques de Bourgogne, Université de Bourgogne, Dijon 21000, France

Shaobo Gan <gansb@pku.edu.cn>: School of Mathematical Sciences, Peking University, Beijing 100871, China

Lan Wen <lwen@math.pku.edu.cn>: School of Mathematical Sciences, Peking University, Beijing 100871, China