WELL-POSEDNESS OF A PARAMETRICALLY FORCED NONLINEAR
SCHRÖDINGER EQUATION DRIVEN BY
TRANSLATION-INARIANT NOISE

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\textbf{Abstract.} We prove well-posedness in $H^\sigma(\mathbb{R})$ for any $\sigma \in [0, \infty)$ of a parametrically forced nonlinear Schrödinger equation (PFNLS) in one dimension driven by multiplicative Stratonovich noise which has spatially homogeneous statistics. The noise is white in time and correlated in space. We first construct local mild solutions via a fixed-point argument. We then formulate a blow-up criterion by showing that the equation has persistence of integrability and regularity as long as the $L^2(\mathbb{R})$-norm of the solution remains finite. Afterwards we derive a pathwise estimate on the $L^2(\mathbb{R})$-norm using a mild Itô formula. Our results also apply to the standard cubic NLS equation driven by multiplicative translation-invariant Stratonovich noise.

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1. INTRODUCTION

1.1. The parametrically forced nonlinear Schrödinger equation. Optic fibers that act as waveguides for electromagnetic signals form the basis for systems of fiber-optic communications, enabling long-distance communication at high bandwidth [1]. The behavior of a pulse propagating through an optic fiber is governed by the nonlinear Schrödinger (NLS) equation [2], which is an archetypical example of a nonlinear dispersive equation that is known to support solitary waves. The NLS equation has many further applications in physics, for instance in the description of Bose-Einstein condensates [3], deep-water waves [4], and plasma oscillations [5]. In these applications, the NLS equation describes the amplitude of a wave packet propagating through a nonlinear medium. We refer to [6] for a detailed treatment of the physical background.

In optic fibers, the nonlinear behavior arises due to a response of the refractive index of the fiber to an applied electric field known as the Kerr effect, leading to a cubic nonlinear term in the equation. Effective signal transmission in optic communication systems may be obstructed by the presence of linear loss in the fiber, weakening the signal as it propagates. Kutz et al. proposed a method of compensating loss using periodic phase-sensitive amplification [7], which has since become a popular approach for increasing feasible transmission lengths. The approach is modelled by the parametrically forced nonlinear Schrödinger (PFNLS) equation:

\[ dz = (i\Delta z - i\nu z - \epsilon(\gamma z - \mu z))dt + i\kappa|z|^2zdt \quad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+. \]  

(1.1)

Here, the complex-valued function \( z \) denotes the envelope of the electric field in an optic fiber, \( t \) is the distance along the fiber, and \( x \) denotes time in a translating frame that moves with the group velocity of light. The constants \( \gamma > 0 \) and \( \mu > 0 \) model linear loss in the fiber and phase-sensitive amplification, respectively. The constant \( \nu \in \mathbb{R} \) models a phase-advance of the signal-carrier, and the constant \( \kappa > 0 \) denotes the strength of the Kerr effect in the fiber. In this model, the local effect of the periodically spaced phase-sensitive amplifiers is averaged over the spacing length of the amplifiers. This description assumes that the amplifiers are closely spaced, which is valid for long propagation lengths [8]. In
particular, the model applies well to a re-circulating loop used for long-term storage of pulses in optical networks.

The literature on the NLS equation, in the absence of parametric forcing ($\nu = \epsilon = 0$), is vast. A concise overview can be found in [9]. One important feature that the NLS equation shares with other nonlinear dispersive equations, such as the Korteweg-de Vries equation, is the presence of conserved quantities [10]. Indeed, for initial data in $L^2(\mathbb{R})$ the NLS equation conserves the $L^2(\mathbb{R})$-norm

$$\mathcal{M}(z) = |z|^2_{L^2(\mathbb{R})},$$

and for initial data in $H^1(\mathbb{R})$, the NLS equation conserves the energy

$$E(z) = |z|^2_{H^1(\mathbb{R})} - \frac{\kappa}{2} |z|^4_{L^4(\mathbb{R})}. \quad (1.2)$$

These conservation laws are central to proving well-posedness results [9, Ch.4]. The presence of parametric forcing ($\epsilon, \gamma, \mu > 0$) in (1.1) breaks this structure, thereby complicating the analysis of (1.1).

1.2. A stochastic equation. In [8], Kath et al. also discuss two mechanisms that further inhibit signal transmission by introducing noise in the system, thereby transforming the description of pulse propagation into a stochastic partial differential equation. In this paper, we study the stochastic parametrically forced nonlinear Schrödinger equation (SPFNLS):

$$dz = (i\Delta z - i\nu z - \epsilon(\gamma z - \mu \overline{z}))dt + i\kappa |z|^2zdtd-i(z \circ dW) \quad \text{for} \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \quad (1.3)$$

The symbol $W$ denotes a real-valued Brownian motion in the Hilbert space $L^2(\mathbb{R}, \mathbb{R})$ that is white in the $t$-variable and correlated in the $x$-variable, and $\circ$ denotes the Stratonovich product.

The multiplicative noise term that we consider in (1.3) models phase noise induced by the coupling of light with the thermally excited acoustical modes of the fiber known as guided acoustic-wave Brillouin scattering (GAWBS) [8]. We use the Stratonovich product, as it is more realistic for physical applications. Indeed, in the absence of parametric forcing it allows for conservation of the $L^2(\mathbb{R})$-norm [11, Proposition 4.1]. The assumption on the correlation of the noise in the $x$-variable will be imposed on its covariance operator $\Phi\Phi^*$, and will be made precise later. It ensures that the resulting noise is sufficiently regular in the $x$-variable, which is required for the analysis. Physically, this means that the noise is correlated in time, which is a natural assumption in the context of GAWBS phase noise. The other noise effect proposed in [8] is due to quantum effects and results in an additive noise term. We focus in the present paper only on the multiplicative GAWBS phase noise.

The well-posedness of stochastic nonlinear Schrödinger equations such as (1.3) with $\nu = \epsilon = 0$ was first studied by de Bouard and Debussche in arbitrary spatial dimension $d$. In [11], the authors prove the existence of global mild solutions in $L^2(\mathbb{R})$ for an NLS equation with multiplicative noise in case the noise is sufficiently regular and the nonlinear term meets a sub-criticality condition. This condition is more restrictive than
in the deterministic case for dimensions $d \geq 4$. In the subsequent paper [12], the same authors also show well-posedness for initial data in $H^1(\mathbb{R})$. More recent works extend and generalize these existence results, for instance, to treat nonlinear noise terms [13], to analyze stochastic NLS equations on compact manifolds [14, 15] or to study blow-up solutions in case of a critical nonlinearity [16]. Barbu et al. [17] extend the results of de Bouard and Debussche to nonlinearities with exponents up to the same critical value as in the deterministic equation. A more recent work by Zhang [18] implies local smoothing and Strichartz estimates for a class of dispersive equations which contains the stochastic nonlinear Schrödinger equation. Their result does not imply ours, since only initial data in $L^2(\mathbb{R})$ are considered there.

In this paper, we establish the well-posedness of (1.3) in the Bessel space $H_\sigma(\mathbb{R})$ for any regularity $\sigma \in [0, \infty)$. We show that just like in the deterministic case, SPFNLS (and by extension, stochastic NLS) has persistence of regularity, meaning that the regularity of the solution is the same as the minimum of that of the noise and the initial data. This generalizes previous results on the one-dimensional cubic stochastic NLS, which generally restrict to the cases $\sigma = 0$ and $\sigma = 1$ in this setting. We expect our result to be optimal, since even for the deterministic NLS equation no global smoothing is available [10].

We are concerned with noise which has spatially and temporally homogeneous statistics, meaning that the correlation function does not depend on the time or the spatial position. Such noise is of particular interest for studying stability of solitary waves on the real line. To construct this noise, we convolve a space-time white noise with an $H_\sigma^\mathbb{R}$-function. An advantage of this approach is that it is easy to obtain an explicit expression for the spatial correlation function.

Due to the damping and parametric forcing, the $L^2(\mathbb{R})$-norm of the solution is no longer exactly conserved. Instead, we derive an integral equality which implies a pathwise a-priori estimate of the $L^2(\mathbb{R})$-norm and is sufficient to show well-posedness. The pathwise estimate also shows that the $L^2(\mathbb{R})$-norm of the solution decays exponentially if the damping parameter $\gamma$ is larger than the forcing parameter $\mu$. The estimate is derived by applying a mild Itô formula, which requires no extra regularity of the solution. This bypasses the need for cumbersome regularization procedures which are alluded to in previous works [11–13]. In our case, the mild Itô formula immediately simplifies to the regular Itô formula, which is a consequence of the fact that the $L^2(\mathbb{R})$-conservation of the NLS equation is shared by the linear Schrödinger equation. As far as we can tell this is not a general phenomenon, and there do not seem to be such cancellations when applying the same strategy to the energy functional $\mathcal{E}$ from (1.2).

1.3. Outline. This paper is organized as follows: in section 2, we formulate the setting and describe the main result, which is Theorem 2.8. Sections 3–5 contain the proof, which consists of three parts: local well-posedness, blow-up criteria, and a conservation law. In section 3 we show well-posedness of a truncated version of equation (1.3), and use this to construct a solution to (1.3) which is valid until a maximal stopping time $\tau^*$. In section
we formulate a blow-up criterion which shows that \( \tau^* \) can only occur before a terminal time \( T_0 \) if the \( L^2_x \)-norm of the solution diverges. Finally, we show in section 5 using a mild Itô formula that the \( L^2_x \)-norm of any solution can be controlled pathwise.

2. Setting & Main Result

2.1. Notation. For \( p \in [1, \infty] \) and \( K \in \{ \mathbb{R}, \mathbb{C} \} \) we denote by \( L^p(\mathbb{R}; K) \) the Lebesgue space of \( K \)-valued functions on the real line. In case \( K = \mathbb{C} \) we write \( L^p(\mathbb{R}) = L^p(\mathbb{R}; \mathbb{C}) \) or we use the shorthand notation \( L^p_x \). We write

\[
| \cdot |_{L^p_x} = \left( \int_{\mathbb{R}} | \cdot |^p dx \right)^{\frac{1}{p}}
\]

for its norm and in the case \( p = 2 \) we denote the inner product by

\[
\langle f, g \rangle_{L^2_x} = \int_{\mathbb{R}} f(x) \overline{g}(x) dx.
\]

For \( p \in [1, \infty] \), we write \( p' \) for its conjugate exponent, that is, the unique \( p' \in [1, \infty] \) such that \( \frac{1}{p} + \frac{1}{p'} = 1 \).

We denote the norm of general normed spaces \( X \) by \( | \cdot |_X \) and the inner product of general inner product spaces \( H \) by \( \langle \cdot, \cdot \rangle_H \). The space of bounded linear operators from a Banach space \( X \) to a Banach space \( Y \) is denoted by \( \mathcal{L}(X; Y) \) and the space of Hilbert-Schmidt operators between separable Hilbert spaces \( H \) and \( \tilde{H} \) as \( \mathcal{L}_2(H; \tilde{H}) \). If a mapping \( F \) between two Banach spaces \( X \) and \( Y \) is \( n \) times Fréchet differentiable at a point \( x_0 \in X \), then we denote its Fréchet derivative at \( x_0 \) in the directions \( h_1, \ldots, h_n \in X \) by \( dF(x_0)[h_1, \ldots, h_n] \).

The weak derivative of a weakly differentiable function \( f \in L^p(\mathbb{R}) \) is denoted by \( \partial_x f \) and we write \( \Delta = \partial_x^2 \) for the Laplacian on the real line. For \( k \in \mathbb{N}_0 \) and \( p \in [1, \infty] \), we write \( W^{k,p}(\mathbb{R}) \) (or the shorthand \( W^{k,p}_x \)) for the Sobolev space of functions \( f \in L^p(\mathbb{R}) \) that are \( k \) times weakly differentiable with weak derivatives up to order \( k \) belonging to \( L^p(\mathbb{R}) \). We equip this space with the norm

\[
| \cdot |_{W^{k,p}_x} = \left( \sum_{\alpha=0}^{k} | \partial_x^\alpha \cdot |^{p}_{L^p_x} \right)^{\frac{1}{p}}.
\]

For \( r \in [1, \infty] \) we will also use the Lebesgue-Bochner spaces of the form \( L^r(I; X) \), where \( I \) is a real interval and \( X \) is a Banach space. These are the strongly Lebesgue-measurable functions \( f : I \to X \) such that \( t \mapsto \| f(t) \|_X \) is in \( L^r(I) \).

We will also make frequent use of fractional Bessel spaces. For \( \sigma \in [0, \infty) \) and \( p \in (1, \infty) \), the Bessel space \( H^{\sigma,p}(\mathbb{R}; K) \) consists of the functions \( f \in L^p(\mathbb{R}; K) \) for which the quantity

\[
|f|_{H^{\sigma,p}_x} = |(1 + \Delta)^{\frac{\sigma}{2}} f|_{L^p_x}
\]
is finite. Here, the fractional power of \((1 + \Delta)\) is defined using the Fourier multiplier with symbol \(\xi \mapsto (1 + |\xi|^2)^{\frac{\alpha}{2}}\). The space \(H^{\alpha,p}(\mathbb{R}; \mathbb{K})\) is a Banach space and we have continuous embeddings \(H^{\alpha_1,p}(\mathbb{R}; \mathbb{K}) \hookrightarrow W^{k,p}(\mathbb{R}; \mathbb{K}) \hookrightarrow H^{\alpha_2,p}(\mathbb{R}; \mathbb{K})\) if \(\alpha_1 \geq k \geq \alpha_2\) and \(k \in \mathbb{N}_0\). As a consequence, \(H^{k,p}(\mathbb{R}; \mathbb{K})\) is isomorphic to \(W^{k,p}(\mathbb{R}; \mathbb{K})\) for any \(k \in \mathbb{N}_0\). Proofs of these statements rely on the theory of singular integrals, and can for example be found in [19, Chapter 3]. We also note that \(H^{\sigma,2}(\mathbb{R}; \mathbb{K})\) is a Hilbert space with inner product \(\langle f, g \rangle_{H^{\sigma,2}(\mathbb{R}; \mathbb{K})} = \langle (1 + \Delta)^{\frac{\sigma}{2}} f, (1 + \Delta)^{\frac{\sigma}{2}} g \rangle_{L^2_x}\). As with \(L^p\) and \(W^{k,p}\), we will use the shorthand \(H^{\sigma,p}(\mathbb{R})\) or simply \(H^{\sigma,p}_x\) for \(H^{\sigma,p}(\mathbb{R}; \mathbb{R})\). We will omit the superscript \(p\) if \(p = 2\).

Lastly, we denote by \(\{S(t)\}_{t \in \mathbb{R}}\) the \(C_0\)-group on \(L^2_x\) generated by \(i\Delta : L^2(\mathbb{R}) \supset H^2(\mathbb{R}) \to L^2(\mathbb{R})\), which acts at \(t \in \mathbb{R}\) as the Fourier multiplier with symbol \(\xi \mapsto e^{-4\pi^2 |\xi|^2} [9, \text{Lemma 3.2.2}]\). Using Plancherel’s theorem, it can be seen that \(S(t)\) is unitary on \(L^2_x\). Since the Fourier multiplier of \(S(t)\) commutes with that of \((1 + \Delta)^{\frac{\sigma}{2}}\), it is immediate that \(S(t)\) is also a unitary group on \(H^\sigma_x\) for any \(\sigma\).

2.2. Stochastic set-up. Suppose we are given a stochastic basis, i.e. a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), where \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T_0]}\) is a complete and right-continuous filtration, and \(T_0 \in (0, \infty)\). When we speak of progressive measurability or predictability, it will always be with respect to \(\mathbb{F}\). Suppose that \(W(\cdot)\) is an \(L^2(\mathbb{R}; \mathbb{R})\)-cylindrical Brownian motion on \([0, T_0]\). In order to regularize \(W\), we introduce for any \(h \in H^\sigma(\mathbb{R}; \mathbb{R})\) the operator
\[
\Phi_h : f \mapsto h * f, \tag{2.1}
\]
which represents convolution by \(h\). We then write the stochastic PFNLS equation as
\[
dz = (i\Delta z - ivz - \epsilon(\gamma z - \mu z^3))dt - \frac{1}{2}zF_h dt + i\kappa|z|^2zdt - iz\Phi_h dW, \tag{2.2}
\]
for \(t \in [0, T_0]\), where \(z\) is a complex-valued process defined on \(\mathbb{R} \times [0, T_0]\). The formulation above should be interpreted in the Itô sense, and is equivalent to (1.3). The newly introduced \(F_h\) serves as a correction to the difference between the Stratonovich and Itô products, called the Itô correction or Itô drift, and is given by
\[
F_h(x) = \sum_{k \in \mathbb{N}} (\Phi_h e_k(x))^2 \quad \text{for } x \in \mathbb{R}, \tag{2.3}
\]
with \((e_k)_{k \in \mathbb{N}}\) an orthonormal basis of \(L^2(\mathbb{R}; \mathbb{R})\). We now collect some useful facts about \(F_h\) and \(\Phi_h\) which we prove in Appendix A.

**Proposition 2.1.** If \(h \in L^2(\mathbb{R}; \mathbb{R})\) and \(z \in L^2_x\), then the series in (2.3) is well-defined and we have the equalities
\[
F_h(x) = |h|_{L^2_x}^2, \tag{2.4}
\]
\[
|z\Phi_h|_{L^2(L^2(\mathbb{R}; \mathbb{R}); L^2_x)} = |z|_{L^2_x} |h|_{L^2_x}. \tag{2.5}
\]
If \(h \in H^\sigma(\mathbb{R}; \mathbb{R})\) and \(z \in H^\sigma_x\) for any \(\sigma \geq 0\), we have the estimate
\[
|z\Phi_h|_{L^2(L^2(\mathbb{R}; \mathbb{R}); H^\sigma_x)} \leq C_\sigma |z|_{H^\sigma_x} |h|_{H^\sigma_x}, \tag{2.6}
\]
for some constant \(C_\sigma\) depending only on \(\sigma\).
As $F_h(x)$ is a constant function by (2.4), we will henceforth simply write $F_h$. From the definition of $\Phi_h$, it is clear that this operator commutes with translation. This is also the case for $F_h$ by (2.4), and for $dW$ by definition. Thus, the noise terms do not break the temporal- and spatial translation symmetries inherent to (1.1).

Before we proceed with the mathematical analysis, we give a meaningful interpretation to our noise. Since $\dot{W}$ formally has a covariance operator equal to the identity, it can be seen using (2.1) that $\Phi_h\dot{W}$ formally satisfies the covariance relation

\[
E\left[\langle \Phi_h\dot{W}(t), \delta_x \rangle_{L^2_x}\langle \Phi_h\dot{W}(t'), \delta_{x'} \rangle_{L^2_x}\right] = \delta_0(t-t')(\tilde{h} * h)(x-x'),
\]

where $\delta_a$ denotes a Dirac mass at the point $x = a$, and $\tilde{h}$ denotes the reflection of $h$ around the origin. Therefore, $g := \tilde{h} * h$ can be interpreted as the spatial correlation function of our noise. Note that $g$ is an even function, so that the correlation only depends on $|x-x'|$.

Also, the variance at any point is given by $g(0) = |h|_{L^2_x}^2$, which means this quantity can be viewed as the strength of the noise.

2.3. Preliminary estimates. Let us collect several useful estimates for future reference. First, we recall that the $C_0$-group associated with the Schrödinger equation $\{S(t)\}_{t \in \mathbb{R}}$ satisfies a family of space-time smoothing estimates, originally proved by Strichartz [20]. The estimates apply to Bochner-Lebesgue-norms, in which the exponents are admissible pairs. These are defined as follows.

Definition 2.2 (Strichartz pairs). A pair $(r, p)$ of exponents is called Strichartz-admissible (or a Strichartz pair) if

\[
\frac{1}{2} = \frac{2}{r} + \frac{1}{p} \quad \text{and} \quad p \in [2, \infty).
\] (2.7)

Remark 2.3. Although the statements of Theorem 2.4 still hold for the pair $(2, \infty)$, we avoid this case because difficulties arise with the harmonic analysis of $H^{\sigma,p}_x$ and the theory of stochastic integration in $L^p$ if $p = \infty$.

In dimension one, the Strichartz estimates read as follows:

Theorem 2.4 (Strichartz estimates). Let $T \in (0, \infty)$ and $\sigma \in [0, \infty)$. Suppose that $(r, p)$ and $(\gamma, q)$ are Strichartz-admissible pairs.

1. (Fixed-time estimates) If $t \neq 0$, then $S(t)$ maps $H^{\sigma,p'}_x(\mathbb{R})$ continuously to $H^{\sigma,p}_x(\mathbb{R})$ and there exists a constant $C$ depending only on $p$ such that

\[
|S(t)z|_{H^{\sigma,p}_x} \leq C|t|^{-\left(\frac{1}{2} - \frac{1}{p}\right)}|z|_{H^{\sigma,p'}_x}
\]

for every $z \in H^{\sigma,p'}_x(\mathbb{R})$. 7
(2) (Homogeneous estimates) If \( z \in H^\sigma(\mathbb{R}) \), then the function \( t \mapsto S(t)z \) belongs to \( C([0, T]; H^\sigma(\mathbb{R})) \) \( \cap L^r(0, T; H^{\sigma, p}(\mathbb{R})) \). Furthermore, there exists a constant \( C \) depending only on \((r, p)\), such that

\[
|S(\cdot)z|_{L^r(0, T; H^{\sigma, p})} \leq C|z|_{H_\sigma^r}.
\]

(3) (Convolution estimates) If \( f \in L^r(0, T; H^{\sigma, p}(\mathbb{R})) \), then the function

\[
t \mapsto \Psi_f(t) = \int_0^t S(t - s)f(s)ds \quad \text{for } t \in [0, T],
\]

belongs to \( C([0, T]; H^\sigma(\mathbb{R})) \) \( \cap L^r(0, T; H^{\sigma, p}(\mathbb{R})) \). Furthermore, there exists a constant \( C \), depending only on \((r, p)\) and \((\gamma, q)\), such that

\[
|\Psi_f|_{L^r(0, T; H^{\sigma, p})} \leq C|f|_{L^r(0, T; H^{\sigma, p})},
\]

for every \( f \in L^r(0, T; H^{\sigma, p}(\mathbb{R})) \).

See [9, Theorem 3.2.5, p. 35] or [20, 21] for a proof.

We will also need a stochastic version of the convolution estimate (2.9). For this we use the following theorem, proved in a more abstract setting by Brzeźniak and Millet [22]. Since the proof is straightforward, we include it in Appendix B for the sake of completeness.

**Theorem 2.5** (Stochastic Strichartz estimates). Let \( T \in (0, \infty) \), \( \sigma \in [0, \infty) \), and let \((r, p)\) satisfy (2.7). Let \( H \) be a Hilbert space and let \( W \) be an \( H \)-cylindrical Wiener process. If \( u \in L^2(\Omega, L^2(0, T; \mathcal{L}_2(H; H_x^\sigma))) \) is \( \mathbb{F} \)-predictable, then the stochastic convolution

\[
t \mapsto \Upsilon_u(t) = \int_0^t S(t - s)u(s)dW(s) \quad \text{for } t \in [0, T]
\]

takes values in \( L^r(0, T; H_x^{\sigma, p}) \), \( \mathbb{P} \)-almost surely. Furthermore, there exists a constant \( C \) depending only on \((r, p)\), such that

\[
\mathbb{E}\left[|\Upsilon_u|_{L^r(0, T; H_x^{\sigma, p})}^2\right] \leq C^2 \mathbb{E}\left[|u|_{L^2(0, T; \mathcal{L}_2(H; H_x^\sigma))}^2\right].
\]

(2.10)

In the case \( r = \infty, p = 2 \), the stochastic convolution is in fact almost surely \( H_x^{\sigma} \)-continuous, and we can also take the \( C([0, T]; H_x^{\sigma}) \)-norm on the left-hand side of (2.10).

2.4. Main result. Before we formulate our main result, we introduce our notion of a solution.

**Definition 2.6.** Let \( h \in L^2(\mathbb{R}; \mathbb{R}) \), and let \( z_0 \) be \( F_0 \)-measurable and \( L^2_{x} \)-valued. A predictable process \( z \) taking values in \( C([0, T_0]; L^2_{x}) \cap L^1(0, T_0; L^2_{x}) \) is a mild solution of (2.2)
if the mild-solution formula
\[ z(t) = S(t)z_0 - \int_0^t S(t-s)(ivz(s) + \epsilon(\gamma z(s) - \mu \overline{z}(s)) + \frac{1}{2} F_h z(s))\,ds \]
\[ + i\kappa \int_0^t S(t-s)(|z(s)|^2z(s))\,ds - i \int_0^t S(t-s)(z(s)\Phi_h dW(s)) \]
is satisfied for all \( t \in [0, T_0] \), \( \mathbb{P} \)-almost surely.

**Remark 2.7.** By the Strichartz estimate (2.9) with \((\gamma, q) = (4, \infty)\) and \( \sigma = 0 \) (which holds by Remark 2.3), the condition \( z \in L^4(0, T_0; L_x^2) \) ensures that the integral in (2.11) containing the cubic term is well-defined.

We can now formulate our main existence and uniqueness result.

**Theorem 2.8** (Global well-posedness). Let \( \sigma \in [0, \infty) \), \( h \in H^\sigma(\mathbb{R}; \mathbb{R}) \), and let \( z_0 \) be \( F_0 \)-measurable and \( H_x^\sigma \)-valued. Then (2.2) has a unique mild solution \( z \), which additionally, \( \mathbb{P} \)-almost surely, takes values in \( C([0, T_0]; H_x^\sigma) \cap L^r(0, T_0; H_x^{\sigma,p}) \) for any \((r, p)\) satisfying (2.2). Furthermore, we have the pathwise a-priori estimate
\[ |z(t)|_{L_x^2} \leq e^{-c(\gamma - \mu)t} |z(0)|_{L_x^2}, \]
for any \( t \in [0, T_0] \), \( \mathbb{P} \)-almost surely.

**Remark 2.9.** By complex interpolation, \( C([0, T]; H_x^0) \cap L^r(0, T; H_x^{\sigma,p}) \) embeds contractively into \( L^q(0, T; H_x^{\sigma,q}) \) for any Strichartz pair \((\gamma, q)\) which satisfies \( q \leq p \). Thus, it suffices to prove Theorem 2.8 for pairs \((r, p)\) where \( p \) is large.

3. **Local well-posedness**

Following the approach of de Bouard and Debussche in [11,12], we first establish well-posedness of a version of (2.2) in which the nonlinear term \( z \mapsto |z|^2z \) is truncated. The truncation provides control of the nonlinearity, which is otherwise not Lipschitz continuous.

We now fix \( \sigma, h, \) and \( z_0 \) which are as in Theorem 2.8, as well as a pair \((r, p)\) which satisfies (2.7) and \( p \geq 8 \). All of these are to be used throughout the entire proof. By Remark 2.9, the extra condition on \( p \) is not restrictive. We will additionally assume that \( z_0 \in L^2(\Omega; H_x^\sigma) \), and only lift this condition at the very end.

For any \( T \in [0, T_0] \), we introduce the Banach spaces
\[ X_T := C([0, T]; H_x^\sigma) \cap L^r(0, T; H_x^{\sigma,p}), \]
\[ Y_T := C([0, T]; H_x^\sigma) \cap L^{\frac{2r}{r-p}}(0, T; L_x^8). \]

We will formulate a contraction argument in \( L^2(\Omega; X_T) \), with a cutoff function based on the \( Y_T \)-norm. Since the pairs \((r, p)\) and \((\infty, 2)\) both satisfy (2.7), we can freely replace the mixed Lebesgue-Bochner norms on the left-hand side of (2.8), (2.9), and (2.10) by
the $X_T$-norm, and will do so throughout. By Remark 2.9 and the embedding $H^p \rightarrow L^p$, we see that $X_T$ embeds into $Y_T$ with a $T$-independent constant, and we will use this fact without further mention.

Let furthermore $\theta \in C_0^\infty(\mathbb{R})$ fulfill $\text{supp } \theta \subseteq (-2, 2)$, $\theta(x) = 1$ for $x \in [-1, 1]$ and $\theta(x) \in [0, 1]$ for $x \in \mathbb{R}$. Furthermore, fix some $R \geq 1$ and define $\theta_R(x) := \theta(x/R)$.

The truncated mild equation then takes the form

$$ z(t) = S(t)z_0 - \int_0^t S(t-s)(i\nu z(s) + \varepsilon(\gamma z(s) - \mu \overline{z}(s))) + \frac{1}{T} F_h z(s) \, ds \quad (3.2) $$

$$ + i\kappa \int_0^t S(t-s)(\theta_R(|z(s)|_Y)|z(s)|^2 z(s)) \, ds - i \int_0^t S(t-s)z(s)\Phi_h dW(s) $$

for each $t \in [0, T_0]$, $\mathbb{P}$-almost surely.

The following proposition asserts that equation (3.2) has a unique solution in $L^2(\Omega; X_{T_0})$.

**Proposition 3.1** (Global well-posedness of truncated equation). Equation (3.2) has a unique mild solution $z$ in the sense of Definition 2.6, which additionally satisfies $z \in L^2(\Omega; X_{T_0})$.

For the proof of Proposition 3.1, we take inspiration from the fixed point argument that was applied to the stochastic NLS equation with initial data in $L^2$ in [11, Proposition 3.1]. The use of Theorem 2.5, which was unknown at the time, significantly simplifies the proof.

Before we give the proof, we derive two estimates which will be frequently used.

**Lemma 3.2.** There exists a constant $C < \infty$ such that the estimates

$$ |fgh|_{L^2_T(0,T; H^\frac{s}{2})} \leq C T^\frac{1}{2} |f|_{H^\frac{s}{2}} |g|_{Y_T} |h|_{Y_T}, $$

$$ ||z||^2_{L^2_T(0,T; H^\frac{s}{2})} \leq C T^\frac{1}{2} |z|_{C([0,T]; H^\frac{s}{2})} ||z||^2_{L^2_T(0,T; X_T)}, $$

hold for all $f$, $g$, $h$, $z \in Y_T$ and all $T \in (0, \infty)$.

**Proof.** Since $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$, it follows from the Kato-Ponce inequality (see for instance [23, Theorem 1.4]) that

$$ |fgh|_{H^\frac{s}{2}} \leq C (|f|_{H^\frac{s}{2}} |gh|_{L^2_T} + |f|_{L^2_T} |gh|_{H^\frac{s}{2}}). $$

Applying Hölder’s inequality and the Kato-Ponce inequality once more using $\frac{1}{4} = \frac{1}{8} + \frac{1}{8}$ and $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$ respectively gives

$$ |fgh|_{H^\frac{s}{2}} \leq C (|f|_{H^\frac{s}{2}} |g|_{L^2_T} |h|_{L^2_T} + |f|_{L^2_T} |g|_{H^\frac{s}{2}} |h|_{L^2_T} + |f|_{L^2_T} |g|_{L^2_T} |h|_{H^\frac{s}{2}}). $$

By applying Hölder’s inequality several times (in time), we also get

$$ |uvw|_{L^2_T(0,T)} \leq T^\frac{1}{2} |u|_{C([0,T])} |v|_{L^\frac{16}{5}_T(0,T)} |w|_{L^\frac{16}{5}_T(0,T)} $$

(3.6)
for any $u \in C([0,T])$ and $v, w \in L^\infty(0,T)$. By combining (3.5) and (3.6) we get
\[
|fh|_{L^2(0,T;H^s_x)} \leq C_T \left( |f|_{C([0,T];H^s_x)} |g|_{L^\infty(0,T;L^2_x)} |h|_{L^\infty(0,T;L^2_x)} \right)
\]
\[
+ |f|_{L^\infty(0,T;L^2_x)} |g|_{C([0,T];H^s_x)} |h|_{L^\infty(0,T;L^2_x)}
\]
\[
+ |f|_{L^\infty(0,T;L^2_x)} |g|_{L^\infty(0,T;L^2_x)} |h|_{C([0,T];H^s_x)}.
\]

The estimate (3.3) now follows from (3.1b), and (3.4) follows by taking $f = z$, $g = z$, $h = \tau$.

\textbf{Lemma 3.3.} Define the operators
\[
(T_0 z_0)(t) := S(t)z_0,
\]
\[
(T_1 z)(t) := -\int_0^t S(t-s)W(z(s) + \epsilon(\gamma z(s) - \mu \tau(s))) + \frac{1}{2} F_h z(s) ds,
\]
\[
(T_2 z)(t) := i\kappa \int_0^t S(t-s)(\theta_R(|z|)z|z|^2z) ds,
\]
\[
(T_3(z, h))(t) := -i \int_0^t S(t-s)z(s)\Phi_h dW(s).
\]

Then there exists a constant $C$ such that the following inequalities hold for all $z_0$, $z$, $z'$, $h$, and $T \in [0, T_0]$:  
\[
|T_0 z_0|_{X_T} \leq C |z_0|_{H^s_x},
\]
\[
|T_1 z|_{X_T} \leq C T |z|_{C([0,T];H^s_x)},
\]
\[
|T_2 z - T_2 z'|_{X_T} \leq C T^2 \|z - z'|_{Y_T},
\]
\[
|T_3(z, h)|_{L^2(\Omega;X_T)} \leq C T^2 \|h|_{H^s_x} \|z|_{L^2(\Omega;C([0,T];H^s_x))}.
\]

\textbf{Proof.} Inequality (3.8a) follows directly from (2.8) and (3.8b) follows from (2.9).

To obtain (3.8d) we use Theorem 2.5 and (2.6) to find
\[
|T_3(z, h)|_{L^2(\Omega;X_T)} = \int_0^T S(t-s)z(s)\Phi_h dW(s)|_{L^2(\Omega;X_T)}
\]
\[
\leq C \int_0^T |z|_{H^s_x} |\Phi_h|_{L^2(\Omega;L^2(0,T;L^2_x;H^s_x))} |dW(s)|_{L^2(\Omega;L^2_x;H^s_x))}
\]
\[
\leq C \|h|_{H^s_x} \|z|_{L^2(\Omega;L^2_x;H^s_x)}
\]
\[
\leq C \|h|_{H^s_x} \|z|_{L^2(\Omega;X_T)}.
\]

Finally we derive (3.8c). First, we use the convolution Strichartz estimate (2.9) with $(\gamma, q) = (8, 4)$ to obtain
\[
|T_2 z - T_2 z'|_{X_T} \leq C \theta_R(|z|) z z \overline{z} - \theta_R(|z'|) z' z' \overline{z}' \leq C I.
\]
Now given \( z, z' \), we define
\[ t_1 := \sup \{ t \in [0, T) : |z'|_{Y_t} \geq 2R \}, \]  
\[ t_2 := \sup \{ t \in [0, T) : |z|_{Y_t} \geq 2R \}. \] (3.9a) (3.9b)

Without loss of generality we assume \( t_1 \leq t_2 \). By splitting up the Lebesgue integral over the intervals \((0, t_1), (t_1, t_2)\) and \((t_2, T)\) we can estimate
\[ I \leq |\theta_R(|z|_{Y_t}) z z \mathbb{E} - \theta_R(|z'|_{Y_t}) z' z' \mathbb{E}|_{L^2(0, t; H^s_x \mathbb{E})} \]
\[ + |\theta_R(|z'|_{Y_t}) (z - z') z \mathbb{E}|_{L^2(0, t; H^s_x \mathbb{E})} \]
\[ + |\theta_R(|z'|_{Y_t}) z' (z - z') \mathbb{E}|_{L^2(0, t; H^s_x \mathbb{E})} \]
\[ + |\theta_R(|z'|_{Y_t}) z' (z - z')|_{L^2(0, t; H^s_x \mathbb{E})} \]
\[ =: I_1 + I_2 + I_3. \]

We split \( I_1 \) up into four terms:
\[ I_1 \leq |(\theta_R(|z|_{Y_t}) - \theta_R(|z'|_{Y_t})) z z \mathbb{E}|_{L^2(0, t; H^s_x \mathbb{E})} \]
\[ + |\theta_R(|z'|_{Y_t}) (z - z') z \mathbb{E}|_{L^2(0, t; H^s_x \mathbb{E})} \]
\[ + |\theta_R(|z'|_{Y_t}) z' (z - z') \mathbb{E}|_{L^2(0, t; H^s_x \mathbb{E})} \]
\[ + |\theta_R(|z'|_{Y_t}) z' (z - z')|_{L^2(0, t; H^s_x \mathbb{E})} \]
\[ =: J_1 + J_2 + J_3 + J_4. \]

Notice that we have enlarged the domain of integration in the expression for \( J_1 \) from \((0, t_1)\) to \((0, t_2)\). This is done so that we can re-use the estimate later.

To estimate \( J_1 \) we first notice that
\[ |\theta_R(|z|_{Y_s}) - \theta_R(|z'|_{Y_s})| \leq C_R |z|_{Y_s} - |z'|_{Y_s} \leq C_R |z - z'|_{Y_s} \leq C_R |z - z'|_{Y_T} \quad (3.10) \]
for any \( s \in [0, T] \), since \( \theta_R \) is Lipschitz and the \( Y_t \)-norm is nondecreasing in \( t \). Therefore, we can estimate
\[ J_1 \leq C_R |z - z'|_{Y_T} |z z \mathbb{E}|_{L^2(0, t; H^s_x \mathbb{E})} \leq CC_R T^{\frac{1}{2}} |z - z'|_{Y_T} |z|_{Y_t}^3 \]
\[ \leq CC_R T^{\frac{1}{2}} |z - z'|_{Y_T} |z|_{Y_t}^3 \quad (3.9b) \]
(3.11)

To estimate \( J_2 \), observe that \( |\theta_R| \leq 1 \) so that
\[ J_2 \leq |(z - z') z \mathbb{E}|_{L^2(0, t; H^s_x \mathbb{E})} \leq C t_1^{\frac{1}{2}} |z - z'|_{Y_{t_1}} |z|_{Y_{t_1}}^2 \quad (3.9b) \]
(3.9c)
since we have assumed \( t_1 \leq t_2 \). The terms \( J_3 \) and \( J_4 \) are estimated similarly.
To get an appropriate estimate for $I_2$, we observe that here we may rewrite
\begin{align*}
\theta_R(|z|_{Y_L})z(s)z(s)\overline{z}(s) - \theta_R(|z'|_{Y_L})z'(s)\overline{z}'(s) & \overset{(3.9a)}{=} \theta_R(|z|_{Y_L})z(s)z(s)\overline{z}(s) \\
\overset{(3.9a)}{=} (\theta_R(|z|_{Y_L}) - \theta_R(|z'|_{Y_L}))z(s)z(s)\overline{z}(s),
\end{align*}

since $\theta_R(|z'|_{Y_L}) = 0$ for any $s \in [t, T]$. Therefore, taking the $L^2(t, t_2; H^{a, d}_x)$-norm gives
\[ I_2 \leq |(\theta_R(|z|_{Y_L}) - \theta_R(|z'|_{Y_L}))zz\overline{z}|_{L^2(t, t_2; H^{a, d}_x)} \leq J_1, \]

at which point we may re-use (3.11).

Finally, we notice that the integrand in $I_3$ is zero because of the cutoff functions and (3.9).

\[\square\]

**Proof of Proposition 3.1.** Define the operator
\[ \mathcal{T}(z, z_0, h)(t) := \left(\mathcal{T}_0z_0 + \mathcal{T}_1z + \mathcal{T}_2z + \mathcal{T}_3(z, h)\right)(t). \]

From (3.8) and (bi-)linearity of $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_3$ it is immediate that $\mathcal{T}(\cdot, z_0, h)$ maps $L^2(\Omega, X_T)$ into itself and that the estimate
\[ |\mathcal{T}(z, z_0, h) - \mathcal{T}(z', z_0, h)|_{L^2(\Omega, X_T)} \leq C(T + T^d + T^d|h|_{H^a_2})|z - z'|_{L^2(\Omega, X_T)} \]
holds for any $T \in [0, T]$. This immediately implies
\[ |\mathcal{T}(z, z_0, h) - \mathcal{T}(z', z_0, h)|_{L^2(\Omega, X_T)} \leq \frac{1}{2}|z - z'|_{L^2(\Omega, X_T)} \]
for $T$ sufficiently small depending only on $h$. Thus, by the contraction-mapping principle, $\mathcal{T}(\cdot, z_0, h)$ has a unique fixed point on this space. Since the fixed points of $\mathcal{T}(\cdot, z_0, h)$ coincide exactly with the mild solutions to (3.2), this shows existence and uniqueness on $[0, T]$.

To get a solution on $[0, T_0]$, we notice that $T$ was chosen independently of $z_0$. Thus, since $u(T) \in L^2(\Omega; H^a_2)$, it is possible to restart the solution at time $T$ with initial value $u(T)$ to get a solution on $[T, 2T]$. Repeating this and patching together the solutions, we obtain a solution on $[0, T_0]$.

\[\square\]

Let us denote by $z_R$ the unique solution of the truncated equation (3.2) with radius $R$ given by Proposition 3.1. We define for $R \geq 1$ the stopping times
\[ \tau_R := \sup\{t \in [0, T_0] : |z_R|_{Y_L} \leq R\}, \]

which is the first time the norm $|z_R|_{Y_L}$ reaches size $R$, and before this time no truncation takes place. Two solutions $z$ and $z_R$ should therefore coincide on $[0, \min\{\tau_R, \tau_R'\}]$. This is stated in the following lemma.

**Lemma 3.4.** Let $R, R' \geq 1$. Then for each $t \in [0, \min\{\tau_R, \tau_R'\}]$ we have $\mathbb{P}$-almost surely
\[ z_R(t) = z_R'(t). \]
Proof. To obtain a contraction estimate for some deterministic time \( T \), we first extend the solutions \( z_R \) and \( z_{R'} \) beyond the stopping time \( \tau := \min\{\tau_R, \tau_{R'}\} \) by setting

\[
\tilde{z}_R(t) := \begin{cases} 
  z_R(t) & \text{if } t \in [0, \tau), \\
  y_R(t) & \text{if } t \in [\tau, T_0],
\end{cases}
\]

where \( y_R \) is the unique solution to the linear equation

\[
y_R(t) = S(t - \tau)z_R(\tau) - \int_{\tau}^{t} S(t - s)(i\nu y_R(s) + \epsilon(\gamma y_R(s) - \mu \overline{y_R}(s))) + \frac{1}{2} F_h y_R(s) \, ds
\]

\[
- i \int_{\tau}^{t} S(t - s)y_R(s) \Phi_h dW(s),
\]

that corresponds to full truncation \( (R = 0) \) in (3.2). We extend \( \tilde{z}_{R'} \) in a similar manner and write

\[
\tilde{z}_{R'}(t) - \tilde{z}_R(t) = (T_1(\tilde{z}_{R'} - \tilde{z}_R))(t) + (T_3(\tilde{z}_{R'} - \tilde{z}_R, h)(t) + I(t),
\]

where \( T_1 \) and \( T_3 \) are as in (3.7) and

\[
I(t) := i\kappa \int_{0}^{t} S(t - s)(|\tilde{z}_{R'}(s)|^2 \tilde{z}_{R'}(s) - |\tilde{z}_R(s)|^2 \tilde{z}_R(s)) \, ds.
\]

Similarly to how we estimated \( J_2, J_3 \) and \( J_4 \) in the proof of Lemma 3.3, we can estimate

\[
|I|_{X_T} \leq C||\tilde{z}_{R'}|^2 \tilde{z}_{R'} - |\tilde{z}_R|^2 \tilde{z}_R|_{L^2(0, T; H_{\ast}^{\gamma})} \leq \max\{R, R'\}^2 C' T^\gamma |\tilde{z}_{R'} - \tilde{z}_R|_{Y_T}
\]

for any \( T \in [0, T_0] \). Combining this with the previous estimates (3.8b) and (3.8d), we find

\[
|\tilde{z}_{R'} - \tilde{z}_R|_{L^r(0, T; X_T)} \leq C(T + T^\gamma)|\tilde{z}_{R'} - \tilde{z}_R|_{L^r(0, T; X_T)},
\]

for some constant \( C \) depending only on \( R, R' \) and \( h \).

Hence, for \( T \) sufficiently small, both sides of (3.14) are zero, and we have \( \tilde{z}_{R'} = \tilde{z}_R \) on \( [0, T] \), \( \mathbb{P} \)-almost surely. Since the choice of \( T \) did not depend on \( z_0 \), we can repeat this procedure on \( [T, 2T] \) and so on to get equality of \( \tilde{z}_{R'} \) and \( \tilde{z}_R \) on \( [0, T_0] \), \( \mathbb{P} \)-almost surely. \( \mathbb{P} \)-almost sure equality of \( z_{R'} \) and \( z_R \) on \( [0, \tau] \) then follows by definition.

Using the stopping times \( \tau_R \) introduced in (3.12) we now define

\[
\tau^* := \sup_{R \geq 1} \tau_R.
\]

Let us construct a maximal solution \( z \) by setting \( z(t) := z_R(t) \) on \( [0, \tau_R] \) for each \( R \geq 1 \). By Lemma 3.4, this process is well-defined on \( [0, \tau^*] \). We combine our findings about \( z \) so far in the following proposition.

Proposition 3.5 (Local well-posedness of SPFNLS). The following statements hold \( \mathbb{P} \)-almost surely:

1. \( z \in X_t \) for every \( t \in [0, \tau^*) \),
2. \( z \) satisfies (2.11) for all \( t \in [0, \tau^*) \),
3. \( \tau^* < T_0 \) implies \( \lim_{t \uparrow \tau^*} z(t)|_{Y_t} = \infty \).
Remark 3.6. Since \( \tau_R = \sup \{ t \in [0, T_0] : |z|_{Y_t} \leq R \} \) and the \( Y_t \)-norm is non-decreasing with \( t \) we see that \( \tau_R \) is \( \mathbb{P} \)-almost surely non-decreasing with \( R \).

4. Blowup

In this section we show that the \( Y_t \)-norm of a solution cannot blow up unless the \( C([0, t]; L_x^2) \)-norm also blows up. This can be viewed as a consequence of the fact that the one-dimensional cubic NLS is mass-subcritical.

**Proposition 4.1** (Blow-up criterion). The implication

\[
\sup_{t \in [0, \tau^*)} |z|_{C([0, t]; L_x^2)} < \infty \implies \sup_{t \in [0, \tau^*)} |z|_{Y_t} < \infty
\]

holds \( \mathbb{P} \)-almost surely.

Proposition 4.1 is an immediate consequence of the following two lemmata and (3.1b).

**Lemma 4.2** (Persistence of integrability). The implication

\[
\sup_{t \in [0, \tau^*)} |z|_{C([0, t]; L_x^2)} < \infty \implies \sup_{t \in [0, \tau^*)} |z|_{L_x^\infty(0, t; L_x^p)} < \infty
\]

holds \( \mathbb{P} \)-almost surely.

**Lemma 4.3** (Persistence of regularity). The implication

\[
\sup_{t \in [0, \tau^*)} |z|_{L_x^\infty(0, t; L_x^p)} < \infty \implies \sup_{t \in [0, \tau^*)} |z|_{C([0, t]; H_x^s)} < \infty
\]

holds \( \mathbb{P} \)-almost surely.

Lemmata 4.2 and 4.3 share many similarities, but are still different enough to warrant separate proofs.

**Proof of Lemma 4.2.** We fix some \( M \geq 1 \) and define the stopping time

\[
\tau := \sup \{ t \in [0, \tau^*) : |z|_{C([0, t]; L_x^2)} \leq M \}, \tag{4.1a}
\]

as well as a recursive sequence of stopping times according to \( \tau_0 = 0 \) and

\[
\tau_{N+1} := \sup \{ t \in [\tau_N, \tau] : |z|_{L_x^\infty(\tau_N, t; L_x^p)} \leq 2KM \}, \quad N \in \mathbb{N}_0, \tag{4.1b}
\]

where \( K \) is the constant from (2.8). Additionally, we define the event

\( A := \{ \omega \in \Omega : \tau_N < \tau, \forall N \in \mathbb{N}_0 \} \) and claim that \( \mathbb{P}(A) = 0 \).

To see this, we start the solution from time \( \tau_N \) and get the \( \mathbb{P} \)-almost sure equality

\[
\begin{align*}
z(t) &= S(t - \tau_N)z(\tau_N) - \int_{\tau_N}^{t} S(t - s)(ivz(s) + \epsilon(\gamma z(s) - \mu z(s)) + \frac{i}{2}z(s)F_h)ds \\
&\quad + ik \int_{\tau_N}^{t} S(t - s)(|z(s)|^2z(s))ds - i \int_{\tau_N}^{t} S(t - s)z(s)\Phi_h dW(s) \\
&=: T_0 + T_1 + T_2 + T_3 \tag{4.2}
\end{align*}
\]
for every \( t \in [\tau_N, \tau^*]. \) Since the estimates from Lemma 3.3 are invariant under time translation and the pair \((\frac{16}{M}, 8)\) satisfies (2.7) we see that

\[
|T_0|_{L^{\frac{16}{M}}(\tau_N, \tau_{N+1}; L^8_x)} \overset{(3.8a)}{\leq} K|z(\tau_N)|_{L^2_x}^{(4.1)} \leq KM, \\
|T_1|_{L^{\frac{16}{M}}(\tau_N, \tau_{N+1}; L^8_x)} \overset{(3.8b)}{\leq} C(\tau_{N+1} - \tau_N)|z|_{C([\tau_N, \tau_{N+1}]; L^2_x)}^{(4.1)} \leq C(\tau_{N+1} - \tau_N)M.
\]

(4.3) (4.4)

To estimate \( T_2 \) we use Lemma 3.2. The pairs \((\frac{16}{M}, 8)\) and \((8, 4)\) both satisfy (2.7), which implies

\[
|T_2|_{L^{\frac{16}{M}}(\tau_N, \tau_{N+1}; L^8_x)} \overset{(2.9)}{\leq} C||z||^2_{L^8_x(\tau_N, \tau_{N+1}; L^8_x)} \overset{(3.4)}{\leq} C'(\tau_{N+1} - \tau_N)^\frac{3}{2}|z|_{C([\tau_N, \tau_{N+1}]; L^2_x)}^2 |z|^2_{L^{\frac{16}{M}}(\tau_N, \tau_{N+1}; L^8_x)} \overset{(4.1)}{\leq} C'M(2KM)^2(\tau_{N+1} - \tau_N)^\frac{3}{2}.
\]

(4.5)

Taking the \( L^{\frac{16}{M}}(\tau_N, \tau_{N+1}; L^8_x)\)-norm of (4.2) and using the triangle inequality along with (4.3)-(4.5) gives

\[
|z|_{L^{\frac{16}{M}}(\tau_N, \tau_{N+1}; L^8_x)} \leq KM + C(\tau_{N+1} - \tau_N)M + 4C'K^2M^3(\tau_{N+1} - \tau_N)^\frac{3}{2} + |\int_{\tau_N}^{t} S(t-s)z(s)\Phi_\omega dW(s)|_{L^{\frac{16}{M}}(\tau_N, \tau_{N+1}; L^8_x)}.
\]

(4.6)

From (4.1b) it is clear that we must have the equality \( |z|_{L^{\frac{16}{M}}(\tau_N, \tau_{N+1}; L^8_x)} = 2KM \) for every \( N \) if \( \omega \in A. \) On the other hand, since \( \tau_N \) is increasing with \( N \) and bounded by \( T_0, \) the second and third term on the right-hand side of (4.6) converge to zero as \( N \to \infty. \)

Combining these facts, we see that \( P(A) \) is bounded by the probability that the events

\[
A_N := \left\{ \omega \in \Omega : \left| \int_{\tau_N}^{t} S(t-s)z(s)\Phi_\omega dW(s)|_{L^{\frac{16}{M}}(\tau_N, \tau_{N+1}; L^8_x)} \geq \frac{KM}{2} \right\}
\]

occur for infinitely many \( N. \) However, using Markov’s inequality and Theorem 2.5, we can estimate

\[
\frac{1}{4} K^2 M^2 P(A_N) \leq \mathbb{E} \left[ |\int_{\tau_N}^{t} S(\cdot - s)z(s)\Phi_\omega dW(s)|_{L^{\frac{16}{M}}(\tau_N, \tau_{N+1}; L^8_x)}^2 \right] \leq \mathbb{E} \left[ |\int_{0}^{t} S(\cdot - s)z(s)1_{[\tau_N, \tau_{N+1}]}(s)\Phi_\omega dW(s)|_{L^{\frac{16}{M}}(0, T_0; L^8_x)}^2 \right] \overset{(2.10)}{\leq} C^2\mathbb{E} \left[ |1_{[\tau_N, \tau_{N+1}]}z\Phi_\omega|_{L^2_x(0, T_0; L^2_x)}^2 \right] \leq C^2 C_\sigma |h||_L^2 \mathbb{E} \left[ |z|_{L^2_x(\tau_N, \tau_{N+1}; L^8_x)}^2 \right].
\]

(2.6)
Since
$$\sum_{N=0}^{\infty} \mathbb{E} \left[ |z|^2_{L^2(\tau_N, \tau_{N+1}; L^2_x)} \right] = \mathbb{E} \left[ \sum_{N=0}^{\infty} |z|^2_{L^2(\tau_N, \tau_{N+1}; L^2_x)} \right] \leq \mathbb{E} \left[ |z|^2_{L^2(0, \tau; L^2_x)} \right] \leq M^2 T_0 < \infty$$
by Fubini’s theorem, we see that the probabilities $\mathbb{P}(A_N)$ are summable. Thus, $\mathbb{P}(A) = 0$ by the Borel-Cantelli lemma. By definition of $A$, this implies $\sup_{t \in [0, \tau]} |z|_{L^2(0, t; L^2_x)} < \infty$, $\mathbb{P}$-almost surely. Recalling that $M$ was arbitrary, we finish the proof by choosing $M$ larger than $\sup_{t \in [0, \tau^*]} |z|_{C([0, t]; L^2_x)}$ (if this quantity is finite) so that $\tau = \tau^*$ by (4.1a).

\textit{Proof of Lemma 4.3.} We fix some $M \geq 1$ and define the stopping times

\begin{align}
\tau &:= \sup \left\{ t \in [0, \tau^*], |z|_{L^2(0, t; L^2_x)} \leq M \right\}, \\
\tau_k &:= \sup \left\{ t \in [0, \tau], |z|_{C([0, t]; H^s_x)} \leq 2^k \right\}, \quad k \in \mathbb{N}_0,
\end{align}

(4.7a) (4.7b)
as well as the event $A := \{ \omega \in \Omega : \tau_k < \tau, \forall k \in \mathbb{N}_0 \}$. We claim that $\mathbb{P}(A) = 0$.

To see this, we start the mild solution at $\tau_k$ to get

\begin{align}
z(t) &= S(t - \tau_k)z(\tau_k) - \int_{\tau_k}^{t} S(t - s)(i\nu z(s) + \epsilon(\gamma z(s) - \mu \overline{z}(s)) + \frac{1}{2} z(s) F_h) ds \\
&\quad + i\kappa \int_{\tau_k}^{t} S(t - s)( |z(s)|^2 z(s)) ds - i \int_{\tau_k}^{t} S(t - s) z(s) \Phi_h dW(s) \\
&=: T_0 + T_1 + T_2 + T_3
\end{align}

(4.8)
for every $t \in [\tau_k, \tau^*)$, $\mathbb{P}$-almost surely. Since $S(t)$ is unitary and using the definition of $\tau_k$ it follows that

\begin{align}
|T_0|_{C([\tau_k, \tau_{k+1}]; H^s_x)} &= |z(\tau_k)|_{H^s_x} \leq 2^k,
\end{align}

(4.9)
\begin{align}
|T_1|_{C([\tau_k, \tau_{k+1}]; H^s_x)} &\leq |z|_{L^1(\tau_k, \tau_{k+1}; H^s_x)} \leq 2^{k+1} (\tau_{k+1} - \tau_k).
\end{align}

(4.10)

To estimate $T_2$ we use Theorem 2.4 and Lemma 3.2 again to find

\begin{align}
|T_2|_{C([\tau_k, \tau_{k+1}]; H^s_x)} &\leq C |z|^2_{L^2(\tau_k, \tau_{k+1}; H^s_x)} \leq C' (\tau_{k+1} - \tau_k) \frac{1}{2} |z|_{C([\tau_k, \tau_{k+1}]; H^s_x)} |z|^2_{L^2(\tau_k, \tau_{k+1}; H^s_x)} \\
&\leq C' M^2 2^{k+1} (\tau_{k+1} - \tau_k)^{\frac{1}{2}}.
\end{align}

(4.11)

From (4.7b), along with continuity of the $C([0, t]; H^s_x)$-norm in $t$ it is clear that

$|z|_{C([\tau_k, \tau_{k+1}]; H^s_x)} = 2^{k+1}$ if $\tau_k < \tau$, which holds for all $k$ if $\omega \in A$. Taking the $C([\tau_k, \tau_{k+1}]; H^s_x)$-norm of (4.8), using the triangle inequality along with (4.9)-(4.11) and dividing by $2^k$ then
leaves to
\[ 2 \leq 1 + 2(\tau_{k+1} - \tau_k) + 2C' M^2 (\tau_{k+1} - \tau_k) \frac{1}{2} \]
\[ + 2^{-k} \left| \int_{\tau_k}^t S(t-s) z(s) \Phi_h dW(s) \right|_{C([\tau_k, \tau_{k+1}] ; H_x^s)} \]  

for every \( k \in \mathbb{N}_0 \) if \( \omega \in \Omega \). Since \( \tau_k \) is nondecreasing and bounded by \( T_0 \), the second and third term on the right-hand side of (4.12) converge to zero as \( k \to \infty \). This implies that \( \mathbb{P}(A_k) \) is bounded by the probability that the events

\[ A_k := \left\{ \omega \in \Omega : \left| \int_{\tau_k}^t S(t-s) z(s) \Phi_h dW(s) \right|_{C([\tau_k, \tau_{k+1}] ; H_x^s)} \leq 2^{k-1} \right\} \]

occur for infinitely many \( k \). However, using Markov’s inequality and Theorem 2.4 we can estimate

\[ \mathbb{P}(A_k) \leq 2^{-2k+2} \mathbb{E} \left[ \sup_{t \in [\tau_k, \tau_{k+1}]} \left| \int_{\tau_k}^t S(t-s) z(s) \Phi_h dW(s) \right|_{H_x^s}^2 \right] \]
\[ \leq 2^{-2k+2} \mathbb{E} \left[ \sup_{t \in [0, T_0]} \left| \int_{0}^t S(t-s) z(s) 1_{[\tau_k, \tau_{k+1}]}(s) \Phi_h dW(s) \right|_{H_x^s}^2 \right] \]
\[ \leq 2^{-2k+2} C^2 \mathbb{E} \left[ |z\Phi_h|^2_{L^2(\tau_k, \tau_{k+1} ; L^2(\mathbb{R} ; H_x^s))} \right] \]
\[ \leq 2^{-2k+2} C^2 \mathbb{E} \left[ |\tau_{k+1} - \tau_k|^2 \right] \]
\[ \leq 16 C^2 \mathbb{E} \left[ \tau_{k+1} - \tau_k \right]. \]

Furthermore, by Fubini’s theorem we have

\[ \sum_k \mathbb{E} \left[ \tau_{k+1} - \tau_k \right] = \mathbb{E} \left[ \sum_k (\tau_{k+1} - \tau_k) \right] \leq T_0. \]

Therefore, the probabilities \( \mathbb{P}(A_k) \) are summable so that \( \mathbb{P}(A) = 0 \) by the Borel-Cantelli lemma. By definition of \( A \), this implies \( \sup_{t \in [0, T_\ast]} |z|_{C([0, t] ; H_x^s)} < \infty \), \( \mathbb{P} \)-almost surely. We finish the proof by recalling that \( M \) was arbitrary, and choosing \( M \) larger than \( \sup_{t \in [0, T_\ast]} |z|_{L^2_x(0, t; L^s_x)} \) (which is possible if this quantity is finite) gives \( \tau = \tau^\ast \) by (4.7a). \( \square \)

5. Conservation

Having formulated a blow-up criterion in terms of the \( L^2_x \)-norm, we now show that this norm can be controlled pathwise. This will yield global well-posedness of (2.11) in combination with Proposition 4.1.

**Proposition 5.1.** The inequality
\[ |z(t)|_{L^2_x} \leq e^{-c(\gamma-\mu)t} |z(0)|_{L^2_x} \]  
holds, \( \mathbb{P} \)-almost surely, for every \( t \in [0, \tau^\ast) \).
Proof. We fix some $R$ and show that the claim holds for all $t \in [0, \tau_R]$, where $\tau_R$ is as in (3.12). Letting $R \to \infty$ then completes the proof by definition of $\tau^*$ (3.15).

To show the claim, we apply to $z_R$ the mild Itô formula proved by Da Prato, Jentzen and Röckner [24, Theorem 1] with the functional

$$M(z) := \frac{1}{2}|z|^2_{L^2_z},$$

which has first and second Fréchet derivatives given by

$$dM(z)[h_1] = \text{Re} \langle h_1, z \rangle_{L^2_z}, \quad d^2M(z)[h_1, h_2] = \text{Re} \langle h_1, h_2 \rangle_{L^2_z}.$$ 

Since $S(t)$ is unitary on $L^2_z$, the equalities

$$M(S(t)z) = \frac{1}{2}|S(t)z|^2_{L^2_z} = \frac{1}{2}|z|^2_{L^2_z},$$

$$dM(S(t)z)[S(t)h_1] = \text{Re} \langle S(t)h_1, S(t)z \rangle_{L^2_z} = \text{Re} \langle h_1, z \rangle_{L^2_z},$$

$$d^2M(S(t)z)[S(t)h_1, S(t)h_2] = \text{Re} \langle S(t)h_1, S(t)h_2 \rangle_{L^2_z} = \text{Re} \langle h_1, h_2 \rangle_{L^2_z}$$

hold for every $t \in \mathbb{R}$ and $z, h_1, h_2 \in L^2_z$, and thus the mild Itô formula coincides exactly with the regular Itô formula, except without the term $i\Delta$. Since additionally $z_R(t) = z(t)$ for all $t \in [0, \tau_R]$ by definition, this gives the $\mathbb{P}$-almost sure equality

$$M(z(t)) = M(z(0)) + \text{Re} \int_0^t \langle -i\nu z(s) + ik|z(s)|^2 z(s), z(s) \rangle_{L^2_z} ds$$  \hspace{1cm} (5.2a)

$$- \text{Re} \int_0^t \langle \gamma z(s) - \mu z(s), z(s) \rangle_{L^2_z} ds$$  \hspace{1cm} (5.2b)

$$- \frac{1}{2} \text{Re} \int_0^t \langle z(s)F_h, z(s) \rangle_{L^2_z} ds$$  \hspace{1cm} (5.2c)

$$- \text{Re} \int_0^t \langle iz(s)\Phi_h dW(s), z(s) \rangle_{L^2_z}$$  \hspace{1cm} (5.2d)

$$+ \frac{1}{2} \text{Re} \int_0^t \langle z(s)\Phi_h^2 |z(s)|^2_{L^2_z} \rangle_{L^2_z} ds.$$  \hspace{1cm} (5.2e)

for all $t \in [0, \tau_R]$. From the fact that $\langle uv, w \rangle_{L^2_z} = \langle v, \overline{w} \rangle_{L^2_z}$, we see that

$$\langle -i\nu z(s) + ik|z(s)|^2 z(s), z(s) \rangle_{L^2_z} = -i\nu|z(s)|^2_{L^2_z} + ik|z(s)|^2_{L^2_z}.$$

Taking the real part shows that the second term on the right-hand side of (5.2a) vanishes. Similarly, we can rewrite

$$\langle iz(s)\Phi_h dW(s), z(s) \rangle_{L^2_z} = i \langle \Phi_h dW(s), |z(s)|^2 \rangle_{L^2_z}.$$

Since $W(s)$ and $h$ are both real-valued, the inner product on the right-hand side always results in a real scalar. Thus, (5.2d) also vanishes. Finally, from Proposition 2.1 we see that

$$|z(s)|^2_{L^2_z} \leq |z(s)|^2_{L^2_z} |h^2_{L^2_z} \leq |z(s)|^2_{L^2_z} = \langle z(s)F_h, z(s) \rangle_{L^2_z}.$$
Therefore, (5.2c) and (5.2e) cancel exactly. Combining all this, we see that (5.2) simplifies to

\[ M(z(t)) = M(z(0)) - \Re \int_0^t \langle \epsilon (\gamma z(s) - \mu \overline{z}(s)), z(s) \rangle ds \]

\[ = M(z(0)) - \epsilon \int_0^t |\gamma z(s)|_{L^2_x}^2 - \mu \Re \langle \overline{z}(s), z(s) \rangle_{L^2_x} ds. \]

Applying the Cauchy-Schwarz inequality allows us to deduce the inequality

\[ |z(t)|_{L^2_x}^2 \leq |z(0)|_{L^2_x}^2 - 2\epsilon \int_0^t (\gamma - \mu) |z(s)|_{L^2_x}^2 ds, \]

which implies (5.1) after using Grönwall’s lemma and taking square roots. □

Proof of Theorem 2.8. By combining Proposition 4.1 with (5.1) it is immediate that \( \tau^* = T_0, \mathbb{P}\)-almost surely, and the solution \( z \) is global. This completes the proof of Theorem 2.8 if \( z_0 \in L^2(\Omega; H^s_x) \), so it only remains to lift this assumption.

Let \( z_0 \in \mathcal{M}(\Omega, \mathcal{F}_0, \mathbb{P}; H^s_x) \) be given. Consider the measurable mapping

\[ \Omega \ni \omega \mapsto e^{-|z_0(\omega)|_{H^s_x}} \in [0, 1], \]

and define the probability measure \( \tilde{\mathbb{P}} \) as

\[ \tilde{\mathbb{P}}(A) = (\mathbb{E} e^{-|z_0|_{H^s_x}})^{-1} \int_A e^{-|z_0(\omega)|_{H^s_x}} d\mathbb{P}(\omega), \quad A \in \mathcal{F}. \]

Then, \( \tilde{\mathbb{P}} \) and \( \mathbb{P} \) are absolutely continuous with respect to each other and we claim that \( z_0 \in L^2(\Omega, \tilde{\mathbb{P}}; H^s_x) \). To see this, note that by elementary calculus we have

\[ |z_0|_{H^s_x}^2 e^{-|z_0|_{H^s_x}} \leq \sup_{x \in \mathbb{R}^+} x^2 e^{-x} = 4e^{-2}, \]

so that

\[ |z_0|_{L^2_x(\Omega, \tilde{\mathbb{P}}; H^s_x)}^2 = \int_\Omega |z_0|_{H^s_x}^2 d\tilde{\mathbb{P}} = (\mathbb{E} e^{-|z_0|_{H^s_x}})^{-1} \int_\Omega |z_0|_{H^s_x}^2 e^{-|z_0|_{H^s_x}} d\mathbb{P} \]

\[ \leq (\mathbb{E} e^{-|z_0|_{H^s_x}})^{-1} \int_\Omega 4e^{-2} d\mathbb{P} < \infty \]

where \( (\mathbb{E} e^{-|z_0|_{H^s_x}})^{-1} \) is finite since

\[ \mathbb{P}(e^{-|z_0|_{H^s_x}} = 0) = \mathbb{P}(|z_0|_{H^s_x} = \infty) = 0, \]

as \( z_0 \) is an \( H^s_x \)-valued random variable. It follows that there exists a unique mild solution \( z \) which \( \tilde{\mathbb{P}} \)-almost surely satisfies (2.11) for \( t \in [0, T_0] \), where the stochastic integral is computed under the law of \( \tilde{\mathbb{P}} \). Since \( \tilde{\mathbb{P}} \) is absolutely continuous with respect to \( \mathbb{P} \), it follows from [25, Theorem 2.14] that the stochastic integral computed under the law of \( \tilde{\mathbb{P}} \) is \( \tilde{\mathbb{P}} \)-indistinguishable from its version computed under the law of \( \mathbb{P} \), that is, the stochastic integrals computed under the laws of \( \mathbb{P} \) and \( \tilde{\mathbb{P}} \) are equal for all \( t \in [0, T_0] \), \( \tilde{\mathbb{P}} \)-almost surely.
It follows that the equality (2.11) holds also $\mathbb{P}$-almost surely. We conclude that $z$ is a mild solution of (1.3).

6. Concluding Remarks

The well-posedness of the stochastic parametrically forced nonlinear Schrödinger equation provides a foundation for further research on the effect of noise on the PFNLS equation. One obvious topic of interest is the stability of solitary wave solutions, in particular for the application to optic fibers, where signals are encoded in travelling waves. A stable solitary wave is expected to maintain its shape when it is subject to small perturbations, which is important for long attainable transmission lengths and the feasibility of long-time storage in the application of fiber-loops. In the deterministic case, this question of stability was addressed by Kapitula and Sandstede [26], where it was shown that perturbations from a primary solitary wave decay exponentially over time.

In case that $\mu > \gamma$, i.e., enough amplification is supplied, equation (1.1) admits two solitary standing wave solutions $\zeta_j$ of the form

$$\zeta_j(x) = e^{i\theta_j} \sqrt{\beta_j} \text{sech}(\sqrt{\beta_j} x),$$  \hspace{1cm} (6.1)

for $j = 1, 2$, where $\theta_1, \theta_2 \in [0, 2\pi)$ are the two solutions to $\cos(2\theta_j) = \gamma / \mu$, and $\beta_j = \omega + \epsilon \mu \sin(2\theta_j)$. As equation (1.1) is translation invariant, shifting the solitary waves by an arbitrary constant $a \in \mathbb{R}$ produces two families of solutions. The solitary wave for which $\sin(2\theta_j) > 0$ is the stable solitary wave investigated in [26], and the other solitary wave (for which $\sin(2\theta_j) < 0$) was shown to be unstable [27]. We denote the stable solitary wave by $\zeta$ and write $\theta$ and $\beta$ for the constants $\theta_j$ and $\beta_j$ associated to $\zeta$. The stability analysis in [26] relies on computing the spectrum of the linear operator associated with the linearization of (1.1) around the solitary wave. It is convenient to consider a phase-shifted frame, where $\phi = e^{-\theta_i} z$ solves

$$d\phi = (i\Delta \phi - i\nu \phi - \epsilon(\gamma \phi - \mu e^{-2\theta_i} \bar{\phi}))dt + i\kappa|\phi|^2 \phi dt$$  \hspace{1cm} (6.2)

if and only if $z$ solves the PFNLS equation (1.1). The phase-shifted solitary wave $e^{-\theta_i} \zeta$ is then a stationary solution to (6.2). Linearizing around $e^{-\theta_i} \zeta$ in (6.2) gives rise to the linearization operator $\mathcal{L} : L^2_x \supset H^2_x \to L^2_x$ given by

$$\mathcal{L}v = i\Delta v - i\nu v - \epsilon(\gamma v - \mu e^{-2\theta_i} \bar{v}) + i\kappa(2|\zeta|^2 v + e^{-2\theta_i} \zeta^2 \bar{v}),$$

in the sense that a perturbation $v = \phi - e^{-\theta_i} \zeta$, where $\phi$ solves (6.2), satisfies $\partial_t v = \mathcal{L}v + i\kappa R(v)$, where $R(v)$ is the nonlinear remainder

$$R(v) = e^{\theta_i} \zeta v^2 + 2e^{-\theta_i} \zeta |v|^2 + |v|^2 v.$$  

It is known that the spectrum of the linearization $\sigma(\mathcal{L})$ is located strictly to the left of the imaginary axis, except for a simple eigenvalue at zero [26, 28]. This eigenvalue arises due to the translation invariance of (1.1). Exponential stability of the semigroup
\( \{e^{\xi t}\} \) on the remaining modes then follows via an additional resolvent estimate and the Gearhart-Prüss theorem [29].

With these tools available from the deterministic theory, a natural progression of this work is to analyze the stability of the primary solitary wave in the presence of noise. This topic of stochastic stability has already been explored for (travelling) waves in various settings, for instance by Krüger & Stannat [30] in the setting of stochastic reaction-diffusion equations, and by the first author and co-authors for the stochastic FitzHugh-Nagumo equations [31]. Future work may investigate the stability of the solitary wave \( \zeta \) in the stochastic parametrically forced NLS equation. Also, stability analysis of solitons could be carried out for other stochastic dispersive equations which have been shown to be well-posed, such as the Korteweg-de Vries equation [32] or the Gross-Pitaevskii equation [33].

**Appendix A. Hilbert-Schmidt operators**

**Proof of Proposition 2.1.** Let \( h \in L^2(\mathbb{R}; \mathbb{R}) \) and let \( e_k \) be any orthonormal basis of \( L^2(\mathbb{R}; \mathbb{R}) \). Since \( \Phi_h f = h * f \), we see using Parseval’s identity that

\[
\sum_{k \in \mathbb{N}} (\Phi_h e_k(x))^2 = \sum_{k \in \mathbb{N}} \langle h(-x), e_k \rangle_{L^2}^2 = |h(-x)|_{L^2}^2 = |h|_{L^2}^2.
\]

Since \( |h|_{L^2}^2 \) is independent of \( x \) and the choice of basis \( e_k \), this shows the claims made until equation (2.4). Using Parseval’s identity and Fubini’s theorem, we can also compute

\[
|z\Phi_h|_{L^2(L^2(\mathbb{R}; \mathbb{R}); L^2)}^2 = \sum_{k \in \mathbb{N}} |z\Phi_h e_k|_{L^2}^2 = \sum_{k \in \mathbb{N}} \int_{\mathbb{R}} |z(x)|^2 \langle h(-x), e_k \rangle_{L^2}^2 dx
\]

\[
= \int_{\mathbb{R}} |z(x)|^2 \sum_{k \in \mathbb{N}} \langle h(-x), e_k \rangle_{L^2}^2 dx = \int_{\mathbb{R}} |z(x)|^2 |h(-x)|_{L^2}^2 dx
\]

\[
= |z|_{L^2}^2 |h|_{L^2}^2,
\]

which shows (2.5). Next we show by induction that (2.6) holds when \( \sigma = 2n \) for some nonnegative integer \( n \). The base case is implied by (2.5), so we now assume that the statement holds for some \( n \). By elementary computations, we find

\[
(1 + \Delta)(z\Phi_h f) = (1 + \Delta)(z(h * f))
\]

\[
= z(h * f) + \Delta z(h * f) + 2 \partial_x z(\partial_x h * f) + z(\Delta h * f)
\]

\[
= z\Phi_h f + \Delta z\Phi_h f + 2 \partial_x z(\Phi_{\partial_x h} f) + z(\Phi_{\Delta h} f)
\]

Combining this with the triangle inequality and the induction hypothesis gives

\[
|z\Phi_h|_{L^2(L^2(\mathbb{R}; \mathbb{R}); H^{n+2})} = |(1 + \Delta)(z\Phi_h)|_{L^2(L^2(\mathbb{R}; \mathbb{R}); H^{n+2})}
\]

\[
= |z\Phi_h f + \Delta z\Phi_h f + 2 \partial_x z(\Phi_{\partial_x h} f) + z(\Phi_{\Delta h} f)|_{L^2(L^2(\mathbb{R}; \mathbb{R}); H^{n+2})}
\]

\[
\leq C(|z|_{H^{n+2}}|h|_{H^n} + |\Delta z|_{H^{n+2}}|h|_{H^n} + 2 |\partial_x z|_{H^{n+2}}|\partial_x h|_{H^n} + |z|_{H^{n+2}}|\Delta h|_{H^n})
\]

\[
\leq C'(|z|_{H^{n+2}}|h|_{H^n})^2.
\]
To show the statement for general $\sigma$, let $n$ be an integer such that $2n \geq \sigma$, and let $\theta \in [0, 1]$ be such that $\sigma = 2n\theta$. We have already shown that the bilinear map 

$$T : H^{2k}_x \oplus H^{2k}(\mathbb{R}; \mathbb{R}) \to \mathcal{L}_2(\mathcal{L}^2(\mathbb{R}; \mathbb{R}), H^{2k}_x),$$

$$(z, h) \mapsto z \cdot \Phi_h$$

is bounded for any nonnegative integer $k$. By complex interpolation (using the notation $[\cdot, \cdot]_\theta$ for the intermediate space) between the case $k = 0$ and $k = n$, it follows that $T$ is also a bounded bilinear map from 

$$[L^2_x, H^{2k}_x]_\theta \oplus [L^2(\mathbb{R}; \mathbb{R}), H^{2k}(\mathbb{R}; \mathbb{R})]_\theta = H^{2\sigma}_x \oplus H^\sigma(\mathbb{R}; \mathbb{R})$$

to 

$$[\mathcal{L}_2(L^2(\mathbb{R}; \mathbb{R}), L^2_x), \mathcal{L}_2(L^2(\mathbb{R}; \mathbb{R}), H^{2n}_x)]_\theta = \mathcal{L}_2(L^2(\mathbb{R}; \mathbb{R}), H^{\sigma}_x).$$

(A.1)

For interpolation of bilinear functions we have used [34, 10.1], and the isomorphism (A.1) is shown for $\gamma$-radonifying operators (which generalize Hilbert-Schmidt operators) in [35, Theorem 9.1.25].

**Appendix B. Stochastic Strichartz estimates**

To prove Theorem 2.5 we distinguish between the cases $p = 2$ and $p > 2$.

**Case $p > 2$.** For every $s \in [0, T]$, we define the operator

$$\phi(s) : H^\sigma_x \to L^r(0, T; H^\sigma_x),$$

$$h \mapsto 1_{[s, T]}(\cdot)S(\cdot - s)h,$$

and notice that (2.8) implies

$$\sup_{s \in [0, T]} |\phi(s)|_{\mathcal{L}(H^\sigma_x; L^r(0, T; H^\sigma_x))} =: L < \infty. \quad (B.1)$$

With this definition of $\phi$, we can rewrite the stochastic convolution as

$$\Upsilon_u = \int_0^T \phi(s)u(s)dW(s),$$

where the integrand on the right-hand side is interpreted as an $\mathcal{L}(H; L^r(0, T; H^\sigma_x))$-valued process.

Since $p \in (2, \infty)$, the space $L^p_x$ has the martingale type-$2$ property [36, Proposition 3.5.30]. Using the lifting operator $(1 + \Delta)^{\sigma/2}$, this property immediately extends to $H^\sigma_x$. Using (2.7) we see that $r \geq 4$ so that the space $L^r(0, T; H^\sigma_x)$ has this property as well. Therefore, we can use the Itô estimate (see for example [37, Theorem 4.6]) to find

$$\mathbb{E} \left[ |\Upsilon_u|_{L^r(0, T; H^\sigma_x)}^2 \right] \leq C^2 \mathbb{E} \left[ |\phi u|_{L^2(0, T; \gamma(H; L^r(0, T; H^\sigma_x)))}^2 \right] \leq C^2 L^2 \mathbb{E} \left[ |u|_{L^2(0, T; \mathcal{L}_2(H; H^\sigma_x))}^2 \right],$$

where we have used the ideal property of $\gamma$-radonifying operators in the final step. \hfill \Box
Case $p = 2$. Since $(r, p)$ satisfies (2.7) we have $r = \infty$. Using the fact that $S(t)$ is unitary on $H_\sigma^*$ together with the Burkholder-Davis-Gundy inequality [38, Theorem 4.4] we find

$$E \left[ \sup_{t \in [0,T]} \left| \int_0^t S(t-s)u(s) dW(s) \right|^2_{H_\sigma^*} \right] = E \left[ \sup_{t \in [0,T]} \left| \int_0^t S(-s)u(s) dW(s) \right|^2_{H_\sigma^*} \right]$$

$$\leq C^2 E \left[ |S(-\cdot)u(\cdot)|_{L^2(0,T;L^2(H;H^*_\sigma))}^2 \right] = C^2 E \left[ |u|^2_{L^2(0,T;L^2(H;H^*_\sigma))} \right]$$

□

REFERENCES

[1] Govind P. Agrawal. Fiber-optic communication systems, volume 222. John Wiley & Sons, 2012.

[2] Govind P. Agrawal. Nonlinear fiber optics. In Nonlinear Science at the Dawn of the 21st Century, pages 195–211. Springer, 2000.

[3] Jared C. Bronski, Lincoln D. Carr, Bernard Deconinck, and J. Nathan Kutz. Bose-Einstein condensates in standing waves: The cubic nonlinear Schrödinger equation with a periodic potential. Phys. Rev. Lett., 86(8):1402, 2001.

[4] Nikolay Vitanov, Amin Chabchoub, and Norbert Hoffmann. Deep-water waves: On the nonlinear Schrödinger equation and its solutions. J. Theor. Appl. Mech., 43, 01 2013.

[5] Padma Kant Shukla and Bengt Eriasson. Nonlinear interactions between electromagnetic waves and electron plasma oscillations in quantum plasmas. Phys. Rev. Lett., 99(9):096401, 2007.

[6] Catherine Sulem and Pierre-Louis Sulem. The nonlinear Schrödinger equation: self-focusing and wave collapse, volume 139. Springer Science & Business Media, 2007.

[7] Nathan Kutz, Cheryl Hile, William Kath, Ruo-Ding Li, and Prem Kumar. Pulse propagation in nonlinear optical fiber lines that employ phase-sensitive parametric amplifiers. J. Opt. Soc. Am. B, 11(10):2112–2123, 1994.

[8] William Kath, Antonio Mecozzi, Prem Kumar, and Christopher Goedde. Long-term storage of a soliton bit stream using phase-sensitive amplification: effects of soliton–soliton interactions and quantum noise. Opt. Commun., 157(1-6):310–326, 1998.

[9] Thierry Cazenave. An introduction to nonlinear Schrödinger equations, volume 22. Universidade federal do Rio de Janeiro, Centro de ciências matemáticas e da natureza, Instituto de matemática, 1989.

[10] Terence Tao. Nonlinear dispersive equations: local and global analysis. Number 106. American Mathematical Soc., 2006.

[11] Anne de Bouard and Arnaud Debussche. A stochastic nonlinear Schrödinger equation with multiplicative noise. Comm. Math. Phys., 205(1):161–181, 1999.

[12] Anne de Bouard and Arnaud Debussche. The stochastic nonlinear Schrödinger equation in $H^1$. Stoch. Anal. Appl., 21(1):97–126, 2003.

[13] Fabian Hornung. The nonlinear stochastic Schrödinger equation via stochastic Strichartz estimates. J. Evol. Equ., 18(3):1085–1114, 2018.

[14] Zdzisław Brzeźniak, Fabian Hornung, and Lutz Weis. Martingale solutions for the stochastic nonlinear Schrödinger equation in the energy space. Probab. Theory Relat. Fields., 174(3):1273–1338, 2019.

[15] Zdzisław Brzeźniak, Fabian Hornung, and Lutz Weis. Uniqueness of martingale solutions for the stochastic nonlinear Schrödinger equation on 3D compact manifolds, 2022.

[16] Michael Röckner, Yiming Su, and Deng Zhang. Multi-bubble Bourgain-Wang solutions to nonlinear Schrödinger equation, 2021.
[17] Viorel Barbu, Michael Röckner, and Deng Zhang. Stochastic nonlinear Schrödinger equations. *Nonlinear Anal. Theory Methods Appl.*, 136:168–194, 2016.

[18] Deng Zhang. Strichartz and local smoothing estimates for stochastic dispersive equations with linear multiplicative noise. *SIAM Journal on Mathematical Analysis*, 54(6):5981–6017, 2022.

[19] Elias M. Stein. *Singular Integrals and Differentiability Properties of Functions (PMS-30)*. Princeton University Press, 1970.

[20] Robert Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.

[21] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Am. J. Math.*, 120(5):955–980, 1998.

[22] Z. Brzeźniak and A. Millet. On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact riemannian manifold. *Potential Analysis*, 41(2):269–315, Aug 2014.

[23] Archil Gulisashvili and Mark A. Kon. Exact smoothing properties of Schrödinger semigroups. *American Journal of Mathematics*, 118(6):1215–1248, 1996.

[24] Giuseppe Da Prato, Arnulf Jentzen, and Michael Roeckner. A mild Itô formula for SPDEs. *Transactions of the American Mathematical Society*, 372, 2019.

[25] Philip E. Protter. Stochastic differential equations. In *Stochastic integration and differential equations*, pages 249–361, Springer, 2005.

[26] Todd Kapitula and Björn Sandstede. Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations. *Phys. D*, 124(1-3):58–103, 1998.

[27] J. Nathan Kutz and William L. Kath. Stability of pulses in nonlinear optical fibers using phase-sensitive amplifiers. *SIAM J. Appl. Math.*, 56(2):611–626, 1996.

[28] J. C. Alexander, M. G. Grillakis, C. K. R. T. Jones, and Björn Sandstede. Stability of pulses on optical fibers with phase-sensitive amplifiers. *Z. Angew. Math. Phys.*, 48(2):175–192, 1997.

[29] Jan Prüss. On the spectrum of $C_0$-semigroups. *Trans. Am. Math. Soc.*, 284(2):847–857, 1984.

[30] Jennifer Krüger and Wilhelm Stannat. A multiscale-analysis of stochastic bistable reaction–diffusion equations. *Nonlinear Anal.*, 162:197–223, 2017.

[31] Katharina Eichinger, Manuel V. Gnann, and Christian Kuehn. Multiscale analysis for traveling-pulse solutions to the stochastic FitzHugh-Nagumo equations. *To appear in Ann. Appl. Probab.*, 2021.

[32] A. de Bouard, A. Debussche, and Y. Tsutsumi. White noise driven Korteweg–de Vries equation. *Journal of Functional Analysis*, 169(2):532–558, 1999.

[33] Anne de Bouard and Reika Fukuizumi. Stochastic fluctuations in the Gross–Pitaevskii equation. *Nonlinearity*, 20(12):2823, nov 2007.

[34] A. Calderón. Intermediate spaces and interpolation, the complex method. *Studia Mathematica*, 24(2):113–190, 1964.

[35] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis. *Analysis in Banach Spaces: Volume II: Probabilistic Methods and Operator Theory*, volume 67. Springer, 2018.

[36] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis. *Analysis in Banach Spaces: Volume I: Martingales and Littlewood-Paley Theory*, volume 63. Springer, 2016.

[37] Jan van Neerven, Mark Veraar, and Lutz Weis. Stochastic integration in Banach spaces – a survey. In Robert C. Dalang, Marco Dozzi, Franco Flandoli, and Francesco Russo, editors, *Stochastic Analysis: A Series of Lectures*, pages 297–332, Basel, 2015. Springer Basel.

[38] Jan van Neerven, Mark C. Veraar, and Lutz Weis. Stochastic integration in UMD Banach spaces. *Ann. Probab.*, 35(4):1438–1478, 2007.