Abstract—In this paper, we generalize the fundamental relation between the derivative of the mutual information and the minimum mean squared error (MMSE) to multiuser setups. We prove that the derivative of the mutual information with respect to the signal to noise ratio (SNR) is equal to the MMSE plus a covariance induced due to the interference, quantified by a term with respect to the cross correlation of the multiuser input estimates, the channels and the precoding matrices. We also derive new relations for the gradient of the conditional and nonconditional mutual information with respect to the MMSE. Capitalizing on the new fundamental relations, we derive closed form expressions of the mutual information for the multiuser channels, particularly the two user multiple access Gaussian channel driven by binary phase shift keying (BPSK) to illustrate and shed light on methods to derive similar expressions for higher level constellations. We capitalize on the new unveiled relation to derive the multiuser MMSE and mutual information in the low-SNR regime.

Index Terms—Estimation Theory; Gradient of conditional mutual information; Gradient of non-conditional mutual information; Gradient of joint mutual information; Information Theory; Interference; MAC; MMSE; Multiuser I-MMSE; Mutual Information.

I. INTRODUCTION
Connections between information theory and estimation theory dates back to the work of Duncan, in [1] who showed that for the continuous-time additive white Gaussian noise (AWGN) channel, the filtering minimum mean squared error (causal estimation) is twice the input output mutual information for any underlying signal distribution. Recently, Guo, Shamai, and Verdu have illuminated intimate connections between information theory and estimation theory in a seminal paper, [2]. In particular, Guo et al. have shown that in the classical problem of information transmission through the conventional AWGN channel unveil that the derivative of the mutual information with respect to the SNR is equal to the smoothing minimum mean squared error (noncausal estimation); a relationship that holds for scalar, vector, discrete-time and continuous-time channels regardless of the input statistics. The relevance of these recent connections comes from the fact that mutual information and MMSE are two canonical operational measures in information theory and estimation theory: mutual information measures the reliable information transmission rate between the input and the output of a system for a specific signaling scheme, while MMSE measures the minimum mean squared error in estimating the input given the output. Later Palomar and Verdu generalized this relation to linear vector Gaussian channels [3], [4]. The mutual information was also represented as an integral of a certain measure of the estimation error in Poisson channels [5], [6]. There have been extensions of these results to the case of mismatched input distributions in the scalar Gaussian channel in [7] and [8]. Most recently, Ghanem in [9], [10], derived the gradient of the mutual information with respect to arbitrary parameters for the multiple access Gaussian channels, a relation that extends the relation for the case of mutually interfering inputs in linear vector Gaussian channels to the case of multiple non-mutually interfering inputs and with mutual interference, a starting point to the results in this work. The implications of a framework involving key quantities in information theory and estimation theory are countless both from the theoretical [11], [12] and the more practical perspective, [13], [14], [15], [16], [17], [18], [19]. The intimate connection between information measures an estimation measures allow few explicit closed form expressions of the mutual information for binary inputs to be derived, particularly ones for BPSK and QPSK over the single input single output (SISO) channel, [2], [20], [21]. Therefore, it is of particular importance to address connections between information theory and estimation theory for the multiuser case in order to understand the communication framework under such inputs and try to provide explicit forms when multiple accessing and interfering inputs coexist.

In this paper, we first revisit the connections between the mutual information and the MMSE for the multiuser setup, see also [10], [16]. Therefore,
the fundamental relation between the derivative of the mutual information and the MMSE, known as I-MMSE identity, and defined for point to point channels with any noise or input distributions in [2] is not anymore suitable for the multiuser case. Therefore, we generalize the I-MMSE relation to the multiuser case. Moreover, we generalize the relations for linear vector Gaussian channels in [3] to multiuser channels where we extend these relations to the per-user gradient of the mutual information with respect to the MMSE, channels and precoders (power allocation) matrices of the user and the interferers. Then, we derive new closed form expressions for the mutual information for single user and multiuser scalar Gaussian channels driven by BPSK inputs. Further, we analyze the MAC Gaussian channel model at the asymptotic regime of low SNR and capitalize on the new unveiled connections between the mutual information and the MMSE to derive the low SNR expansion of the mutual information in a multiuser setup.

Throughout the paper, the following notation is employed, boldface uppercase letters denote matrices, lowercase letters denote scalars. The superscript, $(.)^{-1}$, $(.)^T$, $(.)^*$, and $(.)^\dagger$ denote the inverse, transpose, conjugate, and conjugate transpose operations. The $(\nabla)$ denotes the gradient of a scalar function with respect to a variable. The $E[.]$ denotes the expectation operator. The $||.||$ and $Tr \{ \}$ denote the Euclidean norm, and the trace of a matrix, respectively.

The rest of the paper is organized as follows; section II introduces the system model. Section III introduces the new fundamental relations between the mutual information and the MMSE. Section IV introduces the new closed form expression of the mutual information. Section V introduces analysis at the asymptotic regime of low SNR.

II. System Model

Consider the deterministic complex-valued vector channel,

$$\mathbf{y} = \sqrt{snr} \mathbf{H}_1 \mathbf{P}_1 \mathbf{x}_1 + \sqrt{snr} \mathbf{H}_2 \mathbf{P}_2 \mathbf{x}_2 + \mathbf{n}, \quad (1)$$

where the $n_r \times 1$ dimensional vector $\mathbf{y}$ and the $n_t \times 1$ dimensional vectors $\mathbf{x}_1$, $\mathbf{x}_2$ represent, respectively, the received vector and the independent zero-mean unit-variance transmitted information vectors from each user input to the MAC channel. The distributions of both inputs are not fixed, not necessarily Gaussian nor identical. The $n_r \times n_t$ complex-valued matrices $\mathbf{H}_1$, $\mathbf{H}_2$ correspond to the deterministic channel gains for both input channels (known to both encoder and decoder) and $\mathbf{n} \sim CN(0, I)$ is the $n_r \times 1$ dimensional complex Gaussian noise with independent zero-mean unit-variance components. The optimization of the mutual information is carried out over all $n_t \times n_t$ precoding matrices $\mathbf{P}_1$, $\mathbf{P}_2$ that do not increase the transmitted power.

III. New Fundamental Relations between the Mutual Information and the MMSE

The first contribution is given in the following theorem, which provides a generalization of the I-MMSE identity to the multiuser case and traverses back to the same identity of the single user case.

**Theorem 1:** The relation between the derivative of the mutual information with respect to the snr and the non-linear MMSE for a multiuser Gaussian channel satisfies:

$$\frac{dI(snr)}{dsn} = \text{mmse}(snr) + \psi(snr) \quad (2)$$

Where,

$$\text{mmse}(snr) = Tr \{ \mathbf{H}_1 \mathbf{P}_1 \mathbf{E}_1 (\mathbf{H}_1 \mathbf{P}_1)^\dagger \} + Tr \{ \mathbf{H}_2 \mathbf{P}_2 \mathbf{E}_2 (\mathbf{H}_2 \mathbf{P}_2)^\dagger \}, \quad (3)$$

$$\psi(snr) = -Tr \{ \mathbf{H}_1 \mathbf{P}_1 \mathbf{E}_y [\mathbf{x}_1 \mathbf{y} \mathbf{E}_{x_2 \mathbf{y}} [\mathbf{x}_2 \mathbf{y}][\mathbf{H}_2 \mathbf{P}_2]^\dagger] \}$$

$$- Tr \{ \mathbf{H}_2 \mathbf{P}_2 \mathbf{E}_y [\mathbf{x}_2 \mathbf{y} \mathbf{E}_{x_1 \mathbf{y}} [\mathbf{x}_1 \mathbf{y}][\mathbf{H}_1 \mathbf{P}_1]^\dagger] \},$$

**Proof:** See Appendix A.

The per-user MMSE is given respectively as follows:

$$\mathbf{E}_1 = \mathbf{E}_y [(\mathbf{x}_1 - \hat{\mathbf{x}}_1)(\mathbf{x}_1 - \hat{\mathbf{x}}_1)^\dagger] \quad (4)$$

$$\mathbf{E}_2 = \mathbf{E}_y [(\mathbf{x}_2 - \hat{\mathbf{x}}_2)(\mathbf{x}_2 - \hat{\mathbf{x}}_2)^\dagger]. \quad (5)$$

The input estimates of each user input is given respectively as follows:

$$\hat{\mathbf{x}}_1 = \mathbf{E}_{x_1 \mathbf{y}} [\mathbf{x}_1 \mathbf{y}] = \sum_{x_1, x_2} x_1 p_{y|x_1, x_2}(y|x_1, x_2) p_{x_1}(x_1) p_{x_2}(x_2) \frac{p_y(y)}{p_y(y)} \quad (6)$$

$$\hat{\mathbf{x}}_2 = \mathbf{E}_{x_2 \mathbf{y}} [\mathbf{x}_2 \mathbf{y}] = \sum_{x_1, x_2} x_2 p_{y|x_1, x_2}(y|x_1, x_2) p_{x_1}(x_1) p_{x_2}(x_2) \frac{p_y(y)}{p_y(y)} \quad (7)$$
The conditional probability distribution of the Gaussian noise is defined as:
\[ p_{y|x_1, x_2}(y|x_1, x_2) = \frac{1}{\pi \sigma^2} e^{-\|y - \sqrt{\sigma^2}H_1 P_1 x_1 - \sqrt{\sigma^2}H_2 P_2 x_2\|^2} \] (8)

The probability density function for the received vector \( y \) is defined as:
\[ p_y(y) = \sum_{x_1, x_2} p_{y|x_1, x_2}(y|x_1, x_2) p_{x_1}(x_1) p_{x_2}(x_2). \] (9)

Henceforth, the system MMSE with respect to the SNR is given by:
\[ \text{mmse}(\text{snr}) = \mathbb{E}_y \left[ \| H_1 P_1 (x_1 - \mathbb{E}_{x_1|y} x_1|y) \|^2 \right] \\
+ \mathbb{E}_y \left[ \| H_2 P_2 (x_2 - \mathbb{E}_{x_2|y} x_2|y) \|^2 \right], \] (10)
\[ = \text{Tr} \left\{ H_1 P_1 E_1 (H_1 P_1)^\dagger \right\} + \text{Tr} \left\{ H_2 P_2 E_2 (H_2 P_2)^\dagger \right\} \] (11)
as given in Theorem [I]

Note that the term \( \text{mmse}(\text{snr}) \) is due to the users MMSEs, particularly, \( \text{mmse}(\text{snr}) = \text{mmse}_1(\text{snr}) + \text{mmse}_2(\text{snr}) \) and \( \psi(\text{snr}) \) are covariance terms that appear due to the covariance of the interferers. Those terms are with respect to the channels, precoders, and non-linear estimates of the user inputs. When the covariance terms vanish to zero, the mutual information with respect to the SNR will be equal to the mmse with respect to the SNR, this applies to the single user and point to point communications. Therefore, the result of Theorem [I] is a generalization of previous result and boils down to the result of Guo et al, [2] under certain conditions which are: (i) when the cross correlation between the inputs estimates equals zero (ii) when interference can be neglected (i.e. interference is very weak, very strong, or aligned) and (iii) under the single user setup.

Such generalized fundamental relation between the change in the mutliuser mutual information and the SNR is of particular relevance. Firstly, such result allows us to understand the behavior of per-user rates with respect to the interference due to the mutual interference and the interference due to other users behaviour in terms of power levels and channel strengths. In addition, the result allows us to be able to quantify the losses incurred due to the interference in terms of bits. Therefore, when the term \( \psi(\text{snr}) \) equals zero. The derivative of the mutual information with respect to the SNR equals the total \( \text{mmse}(\text{snr}) \):
\[ \frac{dI(\text{snr})}{d\text{snr}} = \text{mmse}(\text{snr}), \] (12)
which matches the result by Guo et. al in [2].

A. The Conditional and Non-Conditional I-MMSE and a Remark on Interference Channels

The implication of the derived relation on the interference channel is of particular relevance. It is worth to note that we can capitalize on the new fundamental relation to extend the derivative with respect to the SNR to the conditional mutual information. To make this more clear, we capitalize on the chain rule of the mutual information which states the following:
\[ I(x_1, x_2; y) = I(x_1; y) + I(x_2; y|x_1) \] (13)

Therefore, through this observation we can conclude the following theorem.

**Theorem 2**: The relation between the derivative of the conditional and the non-conditional mutual information and their corresponding minimum mean squared error satisfies, respectively:
\[ \frac{dI(x_2; y|x_1)}{d\text{snr}} = \text{mmse}_2(\text{snr}) + \psi(\text{snr}) \] (14)
\[ \frac{dI(x_1; y)}{d\text{snr}} = \text{mmse}_1(\gamma \text{snr}) \] (15)

**Proof**: Taking the derivative of both sides of (13), and subtracting the derivative of \( I(x_1; y) \) which is equal to \( \text{mmse}_1(\gamma \text{snr}) \), \( \gamma \) is a scaling factor, due to the fact that \( x_1 \) is decoded first considering the other users’ input \( x_2 \) as noise. Therefore, Theorem [2] has been proved.

Note that the derivative of the conditional mutual information as well as the non-conditional mutual information can be scaled due to different SNRs in a two-user interference channel. However, the scaling is straightforward to apply. For further details, refer to [21], [16].

The following theorems in addition to Theorem [I] generalizes the connections between information theory and estimation theory to the multiuser case.

**Theorem 3**: The relation between the gradient of the mutual information with respect to the channel
and the non-linear MMSE for a two user-MAC channel with arbitrary inputs (1) satisfies:
\[
\nabla H_1 I(x_1, x_2; y) = H_1 P_1 E_1 P_1^\dagger - H_2 P_2 E[x_2 x_1^\dagger] P_1^\dagger
\]
(16)
\[
\nabla H_2 I(x_1, x_2; y) = H_2 P_2 E_2 P_2^\dagger - H_1 P_1 E[x_1 x_2^\dagger] P_2^\dagger
\]
(17)

**Proof:** The detailed proof has been provided in Appendix B. [10]

**Theorem 4:** The relation between the gradient of the mutual information with respect to the precoding matrix and the non-linear MMSE for a two-user MAC channel with arbitrary inputs (1) satisfies:
\[
\nabla P_1 I(x_1, x_2; y) = H_1^\dagger H_1 P_1 E_1 - H_1^\dagger H_2 P_2 E[x_2 x_1^\dagger]
\]
(18)
\[
\nabla P_2 I(x_1, x_2; y) = H_2^\dagger H_2 P_2 E_2 - H_2^\dagger H_1 P_1 E[x_1 x_2^\dagger]
\]
(19)

**Proof:** The detailed proof has been provided in Appendix C. [10]

Theorems 3 and 4 provides intuitions about the change of the mutual information with respect to the changes in the channel or the precoding (power allocation). There is a straightforward connection between both changes when each gradient is scaled with respect to the changing auxiliary parameter, we can write this connection as follows:
\[
\nabla P_1 I(x_1, x_2; y) P_1^\dagger = H_1^\dagger H_1 I(x_1, x_2; y)
\]
(20)
\[
\nabla P_2 I(x_1, x_2; y) P_2^\dagger = H_2^\dagger H_2 I(x_1, x_2; y)
\]
(21)

Note that we can derive the gradient of the mutual information with respect to any arbitrary parameter following similar steps of the proof of the previous two theorems. Note also that the derived relations in (16), (17), (18), and (19) reduce to the relation between the gradient of the mutual information and the non-linear MMSE derived for the linear vector Gaussian channels [3] if the cross correlation between the input estimates is zero.

**B. Gradient of the Conditional and Non-Conditional Mutual Information**

Theorem 4 shows how much rate is lost due to the other user, this is due to the fact that some terms in the gradient of the mutual information preclude the effect of the mutual interference of the main links. Therefore, we can account for such quantified rate loss via optimal power allocation and optimal precoding. Then, in order to be able to understand the achieved rates of each user in a MAC channel, we capitalize on the chain rule of the mutual information to derive the conditional mutual information as follows,
\[
I(x_1, x_2; y) = I(x_2; y) + I(x_1; y|x_2)
\]
(22)
and,
\[
I(x_1, x_2; y) = I(x_1; y) + I(x_2; y|x_1)
\]
(23)

Where the joint mutual information term is defined as follows:
\[
I(x_1, x_2; y) = E \left[ \frac{p_{y|x_1, x_2}(y|x_1, x_2)}{\sum_{x_1', x_2'} p_{y|x_1', x_2'}(y|x_1', x_2') p_{x_1}(x_1') p_{x_2}(x_2')} \right]
\]
(24)

Where the signal \( x_2 \) is considered as noise. The conditional mutual information is defined as follows:
\[
I(x_1; y|x_2) = E \left[ \frac{p_{y|x_2}(y|x_2)}{\sum_{x_2'} p_{y|x_2'}(y|x_2') p_{x_2}(x_2')} \right]
\]
(25)

Where the signal \( x_2 \) is considered as noise. The conditional mutual information is defined as follows:
\[
I(x_1; y|x_2) = E \left[ \frac{p_{y|x_1, x_2}(y|x_1, x_2)}{\sum_{x_1'} p_{y|x_1', x_2}(y|x_1', x_2) p_{x_1}(x_1') p_{x_2}(x_2')} \right]
\]
(27)
\[
I(x_2; y|x_1) = E \left[ \frac{p_{y|x_1, x_2}(y|x_1, x_2)}{\sum_{x_2'} p_{y|x_1, x_2'}(y|x_1, x_2') p_{x_1}(x_1) p_{x_2}(x_2')} \right]
\]
(28)

Clearly, we know that \( I(x_2; y) \) is the mutual information when user 2 is decoded first considering user 1 signal as noise. Therefore, we can write it as follows,
\[
I(x_2; y) = E \left[ \log \frac{p_{y|x_2}(y|x_2)}{p_y(y)} \right]
\]
(29)

\[
p_{y|x_2}(y|x_2) = \frac{1}{\pi \sigma^2} e^{-(y - \sqrt{\alpha} x_2)(P_2^H H_2 P_1 + 1)^{-1} (y - \sqrt{\alpha} x_2)}
\]
(30)

\[
p_y(y) = \sum_{x_2'} p_{y|x_2'}(y|x_2') p_{x_2'}(x_2')
\]
(31)

Based on such definition, we conclude the following theorem which provides a new fundamental relation between the gradient of the mutual information with
respect to the precoder of the other user given that the other user will be secondly decoded.

**Theorem 5:** The gradient of the mutual information with respect to the precoder, for a scaled user power when the other user input is considered as noise is as follows,

\[
\nabla_{P_1} I(x_2; y) = H_2 P_2 E_0 P_1^2 H_1^\dagger H_1 P_1 (P_1^* H_1^\dagger H_1 P_1 + I) \tag{32}
\]

\[
\nabla_{P_2} I(x_1; y) = H_1 P_1 E_0 P_1^2 H_1^\dagger H_2 P_2 (P_2^* H_2^\dagger H_2 P_2 + I) \tag{33}
\]

**Proof:** The proof follows similar steps of the proof of Theorem [4].

When each user is transmitting over a single channel, the new relation in Theorem [5] will be more clearly understood in terms of the effect of the interference plus noise power scaling on the gradient, see [16]. In other words, Theorem [5] is of particular relevance to understand how the rate changes and can be adapted based on the changes of the channel and power of the interference.

**Corollary 1:** The gradient of the conditional mutual information will be as follows,

\[
\nabla_{P_1} I(x_1; x_2; y) = \nabla_{P_1} I(x_1, x_2; y) - \nabla_{P_1} I(x_2; y) \tag{34}
\]

\[
\nabla_{P_2} I(x_2; x_1; y) = \nabla_{P_2} I(x_1, x_2; y) - \nabla_{P_2} I(x_1; y) \tag{35}
\]

**Proof:** The proof of the corollary follows from the chain rule of mutual information and the gradient of the mutual information derived for the sum rate and non-conditional rates.

### IV. Multiuser MMSE and Mutual Information Closed Forms

The only known explicit closed forms of the MMSE and the mutual information are for BPSK inputs, [2], and QPSK [20] for the SISO channel. In the SISO case, the relation between the mutual information and the MMSE for the signle user setup allows the derivation of this form. For a MAC channel as an example of multiuser channels, we will first consider a unit power, unit channel coefficient for simplicity, and we will then capitalize on the new unveiled generalization of the relation between the mutual information and the two user MMSE with the covariance. Therefore, we derive new explicit closed form expressions of the MMSE and the mutual information for each user in the two user Gaussian MAC driven by BPSK. To derive a closed form expression of the conditional and non-conditional mutual information for each user under the MAC, we capitalize again on the chain rule of the mutual information stated in [23]. The first user which will be decoded first given that the other user is noise, therefore the MMSE and the mutual information of user 1 will be respectively, given by the following theorems.

**Theorem 6:** The non-conditional \( \text{mmse}_1'(\text{snr}) \) of the user 1 decoded first and scaled with the other user noise is given by:

\[
\text{mmse}_1'(\text{snr}) = 1 - \frac{1}{4\sqrt{\pi}} \int_{y \in \mathbb{R}} \tanh \left( \frac{\sqrt{\text{snr}}}{2} y \right) e^{-\left(y - \frac{\text{snr}}{2}\right)^2} dy \tag{36}
\]

**Proof:** See Appendix B, part I.

**Theorem 7:** The non-conditional mutual information \( I_1'(\text{snr}) \) of the user 1 decoded first and scaled with the other user noise is given by:

\[
I_1'(\text{snr}) = \frac{\text{snr}}{4} - \frac{1}{4\sqrt{\pi}} \int_{y \in \mathbb{R}} \log \cosh \left( \frac{\sqrt{\text{snr}}}{2} y \right) e^{-\left(y - \frac{\text{snr}}{2}\right)^2} dy \tag{37}
\]

**Proof:** See Appendix B, part II.

However, the conditional MMSE and conditional mutual information of user 2 that will be decoded next under the MAC given that the first user is decoded first are given on the following theorems.

**Theorem 8:** The conditional \( \text{mmse}_2'(\text{snr}) \) of user 2 decoded second given that user 1 in the MAC channel is decoded first with BPSK inputs is given by:

\[
\text{mmse}_2'(\text{snr}) = 1 - \frac{1}{\sqrt{2\pi}} \int_{y \in \mathbb{R}} \tanh \left( \frac{\sqrt{\text{snr}y}}{2} \right) e^{-\left(y - \frac{\text{snr}}{2}\right)^2} dy \tag{38}
\]

**Proof:** The proof follows similar steps as in Appendix B, Part I, and follows the formula in [2].

**Theorem 9:** The conditional mutual information \( I_2'(\text{snr}) \) of user 2 decoded second given that user 1 in the MAC channel is decoded first with BPSK inputs is given by:

\[
I_2'(\text{snr}) = \frac{\text{snr}}{\sqrt{2\pi}} \int_{y \in \mathbb{R}} \log \cosh \left( \frac{\sqrt{\text{snr}y}}{2} \right) e^{-\left(y - \frac{\text{snr}}{2}\right)^2} dy \tag{39}
\]

**Proof:** The proof follows similar steps as in Appendix B, Part II, and follows the formula in [2].

Notice that if both users are time sharing or decoded jointly, at such point, a maximum sum rate is achievable, therefore, each users rate will follow the one
in Theorem 9. Such case is similar to two parallel channels of each user, therefore, the sum rate is the sum of each individual rate. However, from Theorems 7 to 8, its straightforward to conclude the following corollaries that defines the total MMSE and mutual information of a two user MAC driven by BPSK inputs.

**Corollary 2:** The total $\text{mmse}(\text{snr})$ of two users MAC channel with BPSK inputs is given by:

$$
\text{mmse}(\text{snr}) = \text{mmse}'_{1}(\text{snr}) + \text{mmse}'_{2}(\text{snr}) - \psi(\text{snr})
$$

(40)

Where,

$$
\text{mmse}'_{1}(\text{snr}) = \text{mmse}_{1}(\text{snr})
$$

(41)

and,

$$
\text{mmse}'_{2}(\text{snr}) = \text{mmse}_{2}(\text{snr}) + \psi(\text{snr})
$$

(42)

**Proof:** See Appendix B, part II.

We shall now capitalize on the unveiled connection between the mutual information and the MMSE plus the covariance of the input estimates. In the case when both user inputs are decoded jointly, such covariance terms can be easily shown to be equal, i.e., $\psi(\text{snr}) = 0$, and so they cancel each other over all permutations of the inputs. Therefore, the joint mutual information is just the sum of the rates of both inputs or the integral of the MMSE of both users. However, a general form of the joint mutual information that clarifies the new fundamental relation when $\psi(\text{snr}) \neq 0$ is given by the following corollary.

**Corollary 3:** The total $I(\text{snr})$ of two users MAC channel with BPSK inputs is given by:

$$
I(\text{snr}) = I'_{1}(\text{snr}) + I'_{2}(\text{snr})
$$

(43)

$I'_{1}(\text{snr})$ corresponds to the mutual information of user 1 given the other user is considered as noise and $I'_{2}(\text{snr})$ is the mutual information of user 2 given that user 1 is decoded first.

**Proof:** See Appendix B, part II.

Figure 1 illustrates the mutual information per user in a MAC and the sum rates under equivalent powers and compared to the case of two users over SISO parallel channels. Its quite clear now, why the mutual information for a MAC Gaussian channel approaches 1.5 bits/sec/Hz when both inputs have similar power, incurring 0.5 bits/sec/Hz loss, as previously explained in [10], and why it doesn’t approach the one of parallel Gaussian channels unless unbalanced power allocation takes place - the so called mercury waterfilling-. Therefore, when both users can be decoded jointly and not successively, the rate loss collapse and the achieved rate is the sum over each single user mutual information, which approaches 2 bits/sec/Hz for BPSK at high SNRs. However, when successive decoding takes place, an unfair rate allocation takes place, were the user decoded first will pay the price from its achievable rates. This can be also well explained in terms of the MMSE, where the MMSE of the user decoded first have a scaled SNR, with a scaling factor less than one. This will let this scaled MMSE not to decay to zero, however, it saturates at high SNR to a point above the zero, at 0.5 for this example.

![Figure 1](image-url)  
*Figure 1. The two-user per MAC rates and sum rates with BPSK inputs.*

V. **MULTIUSER I-MMSE IN THE LOW-SNR REGIME**

We now consider the two-user MAC Gaussian channel with arbitrary input distributions in the regime of low-snr. Consider a zero-mean uncorrelated complex inputs, with $\mathbb{E}[x_{1}x_{1}^{*}] = I$, $\mathbb{E}[x_{2}x_{2}^{*}] = I$, $\mathbb{E}[x_{1}x_{1}^{T}] = 0$, and $\mathbb{E}[x_{2}x_{2}^{T}] = 0$. We consider the low-snr expansion to the MMSE of equation (10). Note that it can be easily deduced that the Taylor expansion of the non-linear MMSE in (10) will lead to the first order Taylor expansion of the linear MMSE for the Gaussian inputs setup. Thus,
the low-snr expansion of the MMSE matrix can be expressed as:

$$E = I - (H_1 P_1)^\dagger H_1 P_{1,snr} - (H_2 P_2)^\dagger H_2 P_{2,snr} + O(snr^2),$$  \hspace{1cm} (44)$$

with $E = E_1 + E_2$. Consequently,

$$mmse(snr) = Tr \left\{ H_1 P_1 E_1 (H_1 P_1)^\dagger \right\} + Tr \left\{ H_2 P_2 E_2 (H_2 P_2)^\dagger \right\}$$

$$= Tr \left\{ H_1 P_1 (H_1 P_1)^\dagger \right\} + Tr \left\{ H_2 P_2 (H_2 P_2)^\dagger \right\}$$

$$- Tr \left\{ (H_1 P_1 (H_1 P_1)^\dagger)^2 \right\} .snr$$

$$- Tr \left\{ (H_2 P_2 (H_2 P_2)^\dagger)^2 \right\} .snr + O(snr^2).$$  \hspace{1cm} (46)$$

Note that due to our new result of Theorem 1, we cannot apply immediately the fundamental relationship between mutual information and MMSE in [3], [2]. Therefore, applying our new result, the low-snr Taylor expansion of the mutual information is given in the following theorem.

**Theorem 10:** The low-snr Taylor expansion of the mutual information of the two user MAC is given by:

$$I(snr) = Tr \left\{ H_1 P_1 (H_1 P_1)^\dagger \right\} .snr + Tr \left\{ H_2 P_2 (H_2 P_2)^\dagger \right\}$$

$$- Tr \left\{ (H_1 P_1 (H_1 P_1)^\dagger)^2 \right\} .snr^2 - Tr \left\{ (H_2 P_2 (H_2 P_2)^\dagger)^2 \right\} .snr^2 + O(snr^3).$$  \hspace{1cm} (47)$$

**Proof:** See Appendix C

The wideband slope — which indicates how fast the capacity is achieved in terms of required bandwidth — is inversely proportional to the second order terms of the mutual information in the low-snr Taylor expansion (47). Therefore, this term is a key low-power performance measure since the bandwidth required to sustain a given rate with a given low power, i.e., minimal energy per bit, is inversely proportional to this term [22]. Further, it's clear that the 5th and 6th term in (47) cancels each other proving that the mutual information is insensitive to the input distribution at the low SNR. However, at medium to high SNR, those terms play a fundamental role in the rate losses encountered.

**VI. CONCLUSIONS**

We provide a generalization of the fundamental relation between the mutual information and the MMSE to a new fundamental relation which applies to multiuser channel setups. Further, we proved our generalized framework deriving the relation for the joint mutual information, conditional and non-conditional mutual information. We capitalize on our unveiled generalized relation to find explicit closed forms of the mutual information of multiuser channels driven by BPSK inputs, and to derive the multiuser I-MMSE at the regime of low SNR. Besides its contribution to the quantification of data rates obtained from different levels of constellations; in a multiuser setup, the new fundamental relation will have high impact on future designs of transmission schemes that are interference-aware, due to the awareness of the covariance introduced due to the interference. It will also have impact on statistical signal processing applications that are based on classification of mixtures of data in a measurement system.

**VII. APPENDIX A: PROOF OF THEOREM 1**

The conditional probability density for the two-user MAC can be written as follows:

$$p_{y|x_1,x_2}^{-snr}(y|x_1, x_2) = \frac{1}{\pi r} e^{-\|y-\sqrt{snr} H_1 x_1 - \sqrt{snr} H_2 x_2\|^2}$$

Thus, the corresponding mutual information is:

$$I(x_1, x_2; y) = E\left[ \log \left( \frac{p_{y|x_1,x_2}(y|x_1, x_2)}{p_y(y)} \right) \right]$$

$$I(x_1, x_2; y) = -n_r \log(\pi e) - E \left[ \log(p_y(y)) \right]$$

Then, the derivative of the mutual information with respect to the SNR is as follows:

$$\frac{dI(x_1, x_2; y)}{dnr} = - \frac{\partial}{\partial snr} \int p_y(y) \log (p_y(y)) dy$$

$$= - \int (p_y(y) \frac{1}{p_y(y)} + \log(p_y(y))) \frac{\partial p_y(y)}{\partial snr} dy$$

$$= - \int (1 + \log(p_y(y))) \frac{\partial p_y(y)}{\partial snr} dy$$
Where the probability density function of the received vector $y$ is given by:

$$p_y(y) = \sum_{x_1,x_2} p_{y|x_1,x_2}(y|x_1,x_2)p_{x_1,x_2}(x_1,x_2) \quad (55)$$

$$= \mathbb{E}_{x_1,x_2} [p_{y|x_1,x_2}(y|x_1,x_2)] \quad (56)$$

The derivative of the conditional output with respect to the SNR can be written as:

$$\frac{\partial p_{y|x_1,x_2}(y|x_1,x_2)}{\partial \text{SNR}} =$$

$$= - \frac{1}{\text{SNR}} \left( \mathbb{E}_{y|x_1,x_2} \right) \frac{\partial}{\partial \text{SNR}} \left( \left( y - \sqrt{\text{SNR}} \mathbb{E}_{x_1,x_2}(H_1 P_1 x_1 - \sqrt{\text{SNR}} H_2 P_2 x_2) \right) \right) \times$$

$$\left( y - \sqrt{\text{SNR}} H_1 P_1 x_1 - \sqrt{\text{SNR}} H_2 P_2 x_2 \right)$$

$$= - \frac{1}{\text{SNR}} \left( \mathbb{E}_{y|x_1,x_2} \right) \frac{\partial}{\partial \text{SNR}} \left( \left( H_1 P_1 x_1 \right) - (H_2 P_2 x_2) \right) \times$$

$$\mathbb{E}_{y|x_1,x_2} \left( \frac{\partial y}{\partial \text{SNR}} \right)$$

$$= - \frac{1}{\text{SNR}} \left( \mathbb{E}_{y|x_1,x_2} \right) \times$$

$$\left( H_1 P_1 x_1 \right) - (H_2 P_2 x_2) \right) \times$$

$$\mathbb{E}_{y|x_1,x_2} \left( \frac{\partial y}{\partial \text{SNR}} \right)$$

Therefore, we have:

$$\mathbb{E}_{x_1,x_2} \left[ \frac{\partial p_{y|x_1,x_2}(y|x_1,x_2)}{\partial \text{SNR}} \right] =$$

$$\mathbb{E}_{x_1,x_2} \left[ - \frac{1}{\text{SNR}} \left( \mathbb{E}_{y|x_1,x_2} \right) \times \right.$$

$$\left. \left( H_1 P_1 x_1 \right) - (H_2 P_2 x_2) \right) \times$$

$$\mathbb{E}_{y|x_1,x_2} \left( \frac{\partial y}{\partial \text{SNR}} \right) \right]$$

$$= - \frac{1}{\text{SNR}} \left( \mathbb{E}_{y|x_1,x_2} \right) \times$$

$$\left( H_1 P_1 x_1 \right) - (H_2 P_2 x_2) \right) \times$$

$$\mathbb{E}_{y|x_1,x_2} \left( \frac{\partial y}{\partial \text{SNR}} \right)$$

Substitute (59) into (54), we get:

$$\frac{dI(x_1,x_2; y)}{dsnr} = \frac{1}{\text{SNR}} \left( \mathbb{E}_{y|x_1,x_2} \right) \times$$

$$\left( H_1 P_1 x_1 \right) - (H_2 P_2 x_2) \right) \times$$

$$\mathbb{E}_{y|x_1,x_2} \left( \frac{\partial y}{\partial \text{SNR}} \right)$$

Using integration by parts applied to the real and imaginary parts of $y$ we have:

$$\int \left( 1 + \log (p_y(y)) \right) \frac{\partial y}{\partial \text{SNR}} \frac{p_{x_1,x_2}(y|x_1,x_2)}{dy} =$$

$$\int \left( 1 + \log (p_y(y)) \right) \frac{p_{x_1,x_2}(y|x_1,x_2)}{dy} \left( \frac{\partial y}{\partial \text{SNR}} \right)$$

The first term in (62) goes to zero as $\|y\| \rightarrow \infty$.

Therefore,

$$\frac{dI(x_1,x_2; y)}{dsnr} = \frac{1}{\text{SNR}} \mathbb{E}_{x_1,x_2} \left[ - \frac{1}{\sqrt{\text{SNR}}} \int \nabla_y p_{y}(y) \, dy \right] \times$$

$$\left( (H_1 P_1 x_1 + H_2 P_2 x_2) \right) \times$$

$$\mathbb{E}_{y|x_1,x_2} \left( \frac{\partial y}{\partial \text{SNR}} \right)$$

$$= \frac{1}{\sqrt{\text{SNR}}} \mathbb{E}_{x_1,x_2} \left[ - \frac{1}{\sqrt{\text{SNR}}} \int \nabla_y p_{y}(y) \, dy \right] \times$$

$$\left( (H_1 P_1 x_1 + H_2 P_2 x_2) \right) \times$$

$$\mathbb{E}_{y|x_1,x_2} \left( \frac{\partial y}{\partial \text{SNR}} \right)$$

$$= \frac{1}{\sqrt{\text{SNR}}} \mathbb{E}_{x_1,x_2} \left[ - \frac{1}{\sqrt{\text{SNR}}} \int \nabla_y p_{y}(y) \, dy \right] \times$$

$$\left( (H_1 P_1 x_1 + H_2 P_2 x_2) \right) \times$$

$$\mathbb{E}_{y|x_1,x_2} \left( \frac{\partial y}{\partial \text{SNR}} \right)$$

Substituting (66) into (65) we get:

$$\frac{dI(x_1,x_2; y)}{dsnr} = \frac{1}{\sqrt{\text{SNR}}} \mathbb{E}_{x_1,x_2} \left[ - \frac{1}{\sqrt{\text{SNR}}} \int \nabla_y p_{y}(y) \, dy \right] \times$$

$$\left( (H_1 P_1 x_1 + H_2 P_2 x_2) \right) \times$$

$$\mathbb{E}_{y|x_1,x_2} \left( \frac{\partial y}{\partial \text{SNR}} \right)$$

Therefore,

$$\frac{dI(x_1,x_2; y)}{dsnr} = \frac{1}{\sqrt{\text{SNR}}} \mathbb{E}_{x_1,x_2} \left[ - \frac{1}{\sqrt{\text{SNR}}} \int \nabla_y p_{y}(y) \, dy \right] \times$$

$$\left( (H_1 P_1 x_1 + H_2 P_2 x_2) \right) \times$$

$$\mathbb{E}_{y|x_1,x_2} \left( \frac{\partial y}{\partial \text{SNR}} \right)$$

Substituting (66) into (65) we get:

$$\frac{dI(x_1,x_2; y)}{dsnr} = \frac{1}{\sqrt{\text{SNR}}} \mathbb{E}_{x_1,x_2} \left[ - \frac{1}{\sqrt{\text{SNR}}} \int \nabla_y p_{y}(y) \, dy \right] \times$$

$$\left( (H_1 P_1 x_1 + H_2 P_2 x_2) \right) \times$$

$$\mathbb{E}_{y|x_1,x_2} \left( \frac{\partial y}{\partial \text{SNR}} \right)$$

Therefore, the derivative of mutual information with respect to the SNR and the per user mmse and input estimates (or covariances) is as follows:

$$\frac{dI(x_1,x_2; y)}{dsnr} = \text{mmse}1(snr) + \text{mmse}2(snr)$$

$$- \left\{ \text{Tr} \left[ H_1 P_1 E_y \left[ x_1 x_2 \right] \times (H_1 P_2)^T \right] \right\}$$
Therefore, we can write the derivative of the derivative of the mutual information with respect to the\( snr \) as follows:
\[
\frac{dI(snr)}{dsnr} = mmse(snr) + \psi(snr)
\] (67)

Therefore, Theorem [1] has been proved as a generalization of the one by Guo, Shamai, Verdu in [2] to the multiuser case.

VIII. APPENDIX B: MULTIUSER MMSE(\( snr \)) AND I(\( snr \)) FOR BPSK INPUTS

A. Part I

Consider the simplified case for a channel model given by:
\[
y = \sqrt{snr} x_1 + \sqrt{snr} x_2 + n,
\] (68)

The total MMSE is given as:
\[
mmse(snr) = mmse_1(snr) + mmse_2(snr)
\] (69)

Therefore, we can write the non-linear MMSE matrix for each user respectively as:
\[
E_1 = E[(x_1 - E[x_1|y])(x_1 - E[x_1|y])^\dagger] = E[x_1x_1^\dagger] - E[E[x_1|y]E[x_1|y]],
\] (70)
\[
E_2 = E[(x_2 - E[x_2|y])(x_2 - E[x_2|y])^\dagger] = E[x_2x_2^\dagger] - E[E[x_2|y]E[x_2|y]],
\] (71)
with,
\[
E[x_1|y] = \frac{\sum x_1x_2 x_1p_{y|x_1,x_2}(y|x_1,x_2)p_{x_1}(x_1)p_{x_2}(x_2)}{p_y(y)}
\] (72)
\[
E[x_2|y] = \frac{\sum x_1x_2 x_2p_{y|x_1,x_2}(y|x_1,x_2)p_{x_2}(x_2)}{p_y(y)}
\] (74)

Its of particular importance to notice that the conditioning over \( x_1 \) in \( p_{y|x_1,x_2}(y|x_1,x_2) \) inside \( E[x_2|y] \) is just for clarity. However, if \( x_1 \) is decoded first, then when decoding (estimating) \( x_2 \) next, we remove \( x_1 \). Therefore, \( E[x_2|y] \) is in fact equals to \( E[x_2|y-x_1] \), this will make \( p_{y|x_1,x_2}(y|x_1,x_2) \) equals to \( p_{y|x_2}(y|x_2) \) and \( p_{x_1}(x_1) \) will be absent according from the equation.

For the two user MAC driven by BPSK inputs, the values of \( x_1 = \{1, -1\} \) and \( x_2 = \{1, -1\} \). The non-linear estimates in (73) and (75) consider that both user inputs are decoded jointly. However, we are interested in successive decoding of the users inputs. Therefore, the non-linear estimate of user 1 decoded first in the MAC considering user 2 as noise that scales the SNR of user 1, and so we can write it with respect to all possible permutations of the possible inputs of user 1 as follows:
\[
E[x_1|y] = \frac{\sum x_1 x_1p_{y|x_1,x_2}(y|x_1,x_2)p_{x_1}(x_1)p_{x_2}(x_2)}{\sum x_1 x_1 p_y(x_1)p_{x_2}(x_2)}
\] (76)
\[
E[x_1|y] = \frac{e^{-\frac{(y-y_1)^2}{4}} - e^{-\frac{(y+y_1)^2}{4}}}{e^{-\frac{(y-y_1)^2}{4}} + e^{-\frac{(y+y_1)^2}{4}}}
\] (77)

However,
\[
E[E[x_1|y]E(x_1|y)]' = \int \left( \frac{\sum x_1 x_1p_{y|x_1,x_2}(y|x_1,x_2)p_{x_1}(x_1)p_{x_2}(x_2)}{p_y(y)} \right)^2 p_y(y) dy
\] (78)
\[
E[E[x_1|y]E(x_1|y)]' = \frac{1}{8\pi} \int \frac{e^{-\frac{(y-y_1)^2}{4}} - e^{-\frac{(y+y_1)^2}{4}}}{e^{-\frac{(y-y_1)^2}{4}} + e^{-\frac{(y+y_1)^2}{4}}} (e^{-\frac{(y-y_1)^2}{4}} - e^{-\frac{(y+y_1)^2}{4}}) dy
\] (79)

Digging into the depth of the right hand side of equation (79), we have:
\[
(y - \sqrt{snr})^2 = y^2 - 2\sqrt{snr}y + snr
\] (80)
\[
(y + \sqrt{snr})^2 = y^2 + 2\sqrt{snr}y + snr
\] (81)

Thus,
\[
e^{-\frac{(y-y_1)^2}{4}} - e^{-\frac{(y+y_1)^2}{4}} = e^{-\frac{y_1^2}{2}} - e^{-\frac{y_1^2}{2}}
\] (82)
\[
e^{-\frac{(y-y_1)^2}{4}} + e^{-\frac{(y+y_1)^2}{4}} = e^{-\frac{y_1^2}{2}} + e^{-\frac{y_1^2}{2}}
\] (83)
\[
= tanh \left( \frac{\sqrt{snr}}{2} \mathcal{R}(y) \right)
\] (83)
It follows that:

\[ E \left[ \mathbf{x}_1 | \mathbf{y} \right] = \frac{1}{8\pi} \int \tanh \left( \frac{\sqrt{\text{snr}} y}{2} \right) \left( e^{-\frac{(y - \sqrt{\text{snr}} x)^2}{4}} - e^{-\frac{(y + \sqrt{\text{snr}} x)^2}{4}} \right) dy \]  
(84)

Therefore,

\[ E \left[ \mathbf{x}_1 | \mathbf{y} \right] = \frac{1}{8\pi} \int_{y \in \mathbb{C}} \tanh \left( \frac{\sqrt{\text{snr}} y}{2} \right) e^{-\frac{(y - \sqrt{\text{snr}} x)^2}{4}} \] 
- \frac{1}{8\pi} \int_{y \in \mathbb{C}} \tanh \left( \frac{\sqrt{\text{snr}} y}{2} \right) e^{-\frac{(y + \sqrt{\text{snr}} x)^2}{4}} dy \]  
(85)

However, it is known that:

\[ \tanh(-x) = -\tanh(x) \]  
(86)

and the expectation remains the same if \( y \sim \mathcal{N}(\sqrt{\text{snr}}, 1) \) replaced by \( y \sim \mathcal{N}(-\sqrt{\text{snr}}, 1) \), due to symmetry, therefore, we have:

\[ E \left[ \mathbf{x}_1 | \mathbf{y} \right] = \frac{1}{4\pi} \int_{y \in \mathbb{C}} \tanh \left( \frac{\sqrt{\text{snr}} y}{2} \right) e^{-\frac{(y - \sqrt{\text{snr}} x)^2}{4}} dy \]  
(87)

Therefore, due to marginalization of the complex domain into the real domain, substituting \( E[\mathbf{x}_1 | \mathbf{x}_1^*] = 1 \) into (70) the scaled MMSE of user 1 over a MAC channel with BPSK inputs is given by:

\[ \text{mmse}'_{1}(\text{snr}) = 1 - \frac{1}{4\sqrt{\pi}} \int_{y \in \mathbb{R}} \tanh \left( \frac{\sqrt{\text{snr}} y}{2} \right) \times \] 
\[ e^{-\frac{(y - \sqrt{\text{snr}} x)^2}{4}} dy \]  
(88)

Therefore, Theorem 6 has been proved.

B. Part II

Due to the relation between the MMSE and the mutual information for SISO channels, the mutual information for user 1 decoded first and with a \( 2\sigma^2 \) scaled \( \text{snr} \) is given by:

\[ I_{1}(\text{snr}) = \frac{\text{snr}}{4} - \frac{1}{4\sqrt{\pi}} \int_{y \in \mathbb{R}} \log \cosh \left( \frac{\sqrt{\text{snr}} y}{2} \right) e^{-\frac{(y - \sqrt{\text{snr}} x)^2}{4}} dy \]  
(89)

Where \( \sigma^2 = 2 \) is the sum of the noise and interference power variance on both sides of the axis. Following similar steps to the ones above, user 2 will be decoded next given (conditioned) on the knowledge of user 1 who is decoded first, therefore, the non-linear estimate of user 2 message removing user 1 message is:

\[ E[\mathbf{x}_2 | \mathbf{y}] = \frac{\sum \mathbf{x}_1 \mathbf{p}_y|\mathbf{x}_1 \mathbf{x}_2 \mathbf{p}_x(\mathbf{x}_2)}{\sum \mathbf{x}_2 \mathbf{p}_y|\mathbf{x}_2 \mathbf{x}_2 \mathbf{p}_x(\mathbf{x}_2)} \]  
(90)

Doing the same steps as before, and capitalizing on the new unveiled relation, the MMSE of user 2 will be given by,

\[ \text{mmse}_2(\text{snr}) = 1 - \frac{1}{\sqrt{2\pi}} \int_{y \in \mathbb{R}} \tanh \left( \sqrt{\text{snr}} y \right) e^{-\frac{(y - \sqrt{\text{snr}} x)^2}{4}} dy \]  
(91)

and the mutual information for user 2 decoded next is given by:

\[ I_{2}(\text{snr}) = \frac{\text{snr}}{4} - \frac{1}{4\sqrt{\pi}} \int_{y \in \mathbb{R}} \log \cosh \left( \sqrt{\text{snr}} y \right) e^{-\frac{(y - \sqrt{\text{snr}} x)^2}{4}} dy \]  
(92)

Notice that due to the new fundamental relation between \( \text{mmse}(\text{snr}) \) and the mutual information, we can observe the effect of the covariance terms \( \psi(\text{snr}) \), given as;

\[ \psi(\text{snr}) = \mathbb{E}_y [\mathbb{E}_{x_1 | y} [\mathbf{x}_1 | \mathbf{y}]] \mathbb{E}_{x_2 | y} [\mathbf{x}_2 | \mathbf{y}] - \mathbb{E}_y [\mathbb{E}_{x_2 | y} [\mathbf{x}_2 | \mathbf{y}]] \mathbb{E}_{x_1 | y} [\mathbf{x}_1 | \mathbf{y}] \]  
(93)

Following similar steps to the ones before, we can see that the covariance term will have a negative value which explains the loss in the mutual information in \( I_{1}(\text{snr}) \). Therefore, the covariances of such setup are given as:

\[ \mathbb{E}_y [\mathbb{E}_{x_1 | y} [\mathbf{x}_1 | \mathbf{y}]] \mathbb{E}_{x_2 | y} [\mathbf{x}_2 | \mathbf{y}] - \mathbb{E}_y [\mathbb{E}_{x_2 | y} [\mathbf{x}_2 | \mathbf{y}]] \mathbb{E}_{x_1 | y} [\mathbf{x}_1 | \mathbf{y}] \]  
(94)

and,

\[ \mathbb{E}_y [\mathbb{E}_{x_2 | y} [\mathbf{x}_2 | \mathbf{y}]] \mathbb{E}_{x_1 | y} [\mathbf{x}_1 | \mathbf{y}] \]  
(95)

Both covariance terms are not equal, which can be explained by \( p_y(y) \) that is different in the integration based on who is decoded first.

The new fundamental relation between the mutual information and the MMSE plus covariance states that,

\[ dI(\text{snr}) = \frac{\text{mmse}_1(\text{snr}) + \text{mmse}_2 + \psi(\text{snr})}{d\text{snr}} \]  
(96)

However, we derive the mutual information \( I_{1}(\text{snr}) \) based on the following:

\[ \frac{dI_{1}(\text{snr})}{d\text{snr}} = \text{mmse}'_{1}(\text{snr}) \]  
(97)
and the mutual information $I'_2(snr)$ was derived based on the following:

$$\frac{dI'_2(snr)}{dsnr} = mmse_2(snr)$$

(98)

It follows from (96) that,

$$\frac{dI_2(snr)}{dsnr} = mmse_2(snr) + \psi(snr)$$

(99)

Therefore,

$$\frac{dI_2(snr)}{dsnr} - \frac{dI'_2(snr)}{dsnr} = \psi(snr)$$

(100)

This means that:

$$mmse(snr) = mmse_1(snr) + mmse_2(snr)$$

= mmse'_1(snr) + mmse_2(snr)

= mmse'_1(snr) + mmse'_2(snr) - \psi(snr)$$

(101)

and,

$$I(snr) = I'_1(snr) + I'_2(snr)$$

(102)

Moreover, when both inputs are decoded jointly, this covariances $\psi(snr)$ collapse to zero due to equality and so the mutual information will be the sum of the integral of both users MMSEs $mmse_1(snr) + mmse_2(snr)$. Therefore, Theorem 7 has been proved with the following corollaries.

**IX. APPENDIX C: PROOF OF THEOREM 10**

First we will find the low-snr expansion of the MMSE matrix $E_1 + E_2$ for user 1 and user 2 in (44) as $snr \rightarrow 0$. Therefore, we will first derive the low-snr expansion of the conditional probability exponent given as:

$$|y - \sqrt{snr}H_1P_1x_1 - \sqrt{snr}H_2P_2x_2|^2$$

$$= (y - \sqrt{snr}H_1P_1x_1 - \sqrt{snr}H_2P_2x_2)\dagger \times$$

$$= \psi(snr)\left(\psi(snr) \right) + \psi(snr)\left(\psi(snr) \right) - \psi(snr)\left(\psi(snr) \right)$$

$$= \psi(snr)\left(\psi(snr) \right) + \psi(snr)\left(\psi(snr) \right) - \psi(snr)\left(\psi(snr) \right)$$

(101)

Hence,

$$p_{y|x_1,x_2}(y|x_1,x_2) =$$

$$\frac{1}{\pi^{nr}} \exp\left(\frac{-\|y - \sqrt{snr}H_1P_1x_1 - \sqrt{snr}H_2P_2x_2\|^2}{\pi^{nr}}\right)$$

$$= \frac{1}{\pi^{nr}} \exp\left(-\|y\|^2\right) \times$$

$$\left(1 + 2\sqrt{snr}R(y^\dagger H_1P_1x_1) + 2\sqrt{snr}R(y^\dagger H_2P_2x_2) + \mathcal{O}(snr)\right)$$

(104)

A. Derivation of the Multiuser MMSE at the Low SNR

The first term of the MMSE matrix of user 1 $E_1$ is $\mathbb{E}[x_1x_1^\dagger] = I$. However, to find the second term of $E_1$, $\mathbb{E}[x_1y]$ defined as:

$$\mathbb{E}[x_1|y] = \frac{\sum_{x_1,x_2} x_1 p_{y|x_1,x_2}(y|x_1,x_2) p_{x_1}(x_1) p_{x_2}(x_2)}{p_y(y)}$$

(105)

We need to substitute (104) into (105) as follows:

$$\mathbb{E}_y[\mathbb{E}_{x_1|y}[x_1] | \mathbb{E}_{x_1|y}[x_1]^\dagger]] =$$

$$\int_{y \in C^{nsn}} \frac{1}{\pi^{nr}} \exp\left(-\|y\|^2\right) p_{x_1}(x_1) p_{x_2}(x_2) \times$$

$$\sum_{x_1,x_2} x_1 \left(1 + \sum_{i=1}^2 2\sqrt{snr}R(y^\dagger H_iP_ix_i) + \mathcal{O}(snr)\right) \times$$

$$\sum_{x_1,x_2} x_2 \left(1 + \sum_{i=1}^2 2\sqrt{snr}R(y^\dagger H_iP_ix_i) + \mathcal{O}(snr)\right)$$

$$\left(\sum_{x_1,x_2} x_1 \left(1 + \sum_{i=1}^2 2\sqrt{snr}R(y^\dagger H_iP_ix_i) + \mathcal{O}(snr)\right)\right)^\dagger dy$$
Recall that \( \mathbb{E}[x_1] = \sum_{x_1} x_1 p_{x_1}(x_1) = 0 \), \( \mathbb{E}_{x_1,x_2}[x_1 x_1^T] = 0 \), \( \mathbb{E}_{x_1,x_2}[x_2 x_2^T] = 0 \), \( \mathbb{E}_{x_1,x_2}[x_1 x_1^T] = I \), and \( \mathbb{E}_{x_1,x_2}[x_2 x_2^T] = I \). Therefore, the numerator of (105) is given by,

\[
\exp(-|y|^2) \times \left( \sum_{x_1,x_2} x_1 (1 + 2 \sqrt{\text{snr}} (y^H H_1 P_1 x_1)) p_{x_1}(x_1)p_{x_2}(x_2) + 2 \sqrt{\text{snr}} (y^H H_2 P_2 x_2) + O(\text{snr}) \right) p_{x_1}(x_1)p_{x_2}(x_2) 
\]

It follows that:

\[
\mathbb{E}_y[\mathbb{E}_{x_1|y}[x_1|y] (\mathbb{E}_{x_1|y}[x_1|y])^\dagger] = (H_1 P_1)^\dagger H_1 P_1 \text{snr} + O(\text{snr}^2) \quad (108)
\]

Consequently, the low-snr expansion of the MMSE matrix of user 1 \( E_1 \) is given as follows:

\[
E_1 = I - (HP)^\dagger H_1 P_1 \text{snr} + O(\text{snr}^2) \quad (109)
\]

Similarly, the low-snr expansion of the MMSE matrix of user 2 \( E_2 \) is given as follows:

\[
E_2 = I - (HP)^\dagger H_2 P_2 \text{snr} + O(\text{snr}^2) \quad (110)
\]

Therefore, we can express the low-snr expansion of the total MMSE in terms of the snr as follows:

\[
\text{MMSE}(\text{snr}) = Tr \left\{ H_1 P_1 E_1 (H_1 P_1)^\dagger \right\} + Tr \left\{ H_2 P_2 E_2 (H_2 P_2)^\dagger \right\} 
\]

\[
= Tr \left\{ H_1 P_1 \left( I - (H_1 P_1)^\dagger H_1 P_1 \text{snr} + O(\text{snr}^2) \right) (H_1 P_1)^\dagger \right\} + Tr \left\{ H_2 P_2 \left( I - (H_2 P_2)^\dagger H_2 P_2 \text{snr} + O(\text{snr}^2) \right) (H_2 P_2)^\dagger \right\} 
\]

\[
= Tr \left\{ H_1 P_1 (H_1 P_1)^\dagger \right\} - Tr \left\{ \left( H_1 P_1 (H_1 P_1)^\dagger \right)^2 \right\} \text{snr} 
\]

\[
+ Tr \left\{ H_2 P_2 (H_2 P_2)^\dagger \right\} - Tr \left\{ \left( H_2 P_2 (H_2 P_2)^\dagger \right)^2 \right\} \text{snr} 
\]

\[
+ O(\text{snr}^2)
\]

B. Derivation of the Multiuser Mutual Information at the Low SNR

We shall now capitalize on the unveiled generalization of the fundamental relation between the mutual information and the MMSE plus covariance. Therefore, using similar steps to derive the low-snr expansion of the covariance given by,

\[
\psi(\text{snr}) = Tr \left\{ H_1 P_1 E_y[\mathbb{E}_{x_1|y}[x_1|y] \mathbb{E}_{x_2|y}[x_2|y]] (H_2 P_2)^\dagger \right\} - Tr \left\{ H_2 P_2 E_y[\mathbb{E}_{x_2|y}[x_2|y] \mathbb{E}_{x_1|y}[x_1|y]] (H_1 P_1)^\dagger \right\} 
\]

Substituting the low-snr expansion of \( \mathbb{E}_{x_1|y}[x_1|y] \) and \( \mathbb{E}_{x_2|y}[x_2|y] \) into the covariance, the low-snr expansion of covariance \( \psi(\text{snr}) \) as \( \text{snr} \to 0 \) is given by:

\[
\psi(\text{snr}) = Tr \left\{ H_1 P_1 (H_1 P_1)^\dagger H_2 P_2 (H_2 P_2)^\dagger \right\} \text{snr} 
\]

\[
- Tr \left\{ H_2 P_2 (H_2 P_2)^\dagger H_1 P_1 (H_1 P_1)^\dagger \right\} \text{snr}
\]
Its clear that at the low-snr the covariance $\psi(snr) = 0$ and so the main part that affects the mutual information will be the MMSE. Therefore, capitalizing on the fundamental relation which states that,

$$\frac{dI(snr)}{dsnr} = mmse(snr) + \psi(snr)$$

(111)

The mutliuser mutual information at the low snr regime is the integral of both sides of (111), and so its given by:

$$I(snr) = Tr \left\{ H_1P_1(H_1P_1)\dagger \right\} snr$$

$$+ Tr \left\{ H_2P_2(H_2P_2)\dagger \right\} snr$$

$$- Tr \left\{ (H_1P_1(H_1P_1)\dagger)^2 \right\} snr^2$$

$$- Tr \left\{ (H_2P_2(H_2P_2)\dagger)^2 \right\} snr^2 +$$

$$+ Tr \left\{ H_1P_1(H_1P_1)\dagger H_2P_2(H_2P_2)\dagger \right\} snr^2$$

$$- Tr \left\{ H_2P_2(H_2P_2)\dagger H_1P_1(H_1P_1)\dagger \right\} snr^2 + \mathcal{O}(snr^3)$$

(112)

This simplifies to the integral of the total MMSE at the low-snr, since the second order term of the snr sums to zero, thus,

$$I(snr) = Tr \left\{ H_1P_1(H_1P_1)\dagger \right\} snr$$

$$+ Tr \left\{ H_2P_2(H_2P_2)\dagger \right\} snr$$

$$- Tr \left\{ (H_1P_1(H_1P_1)\dagger)^2 \right\} snr^2$$

$$- Tr \left\{ (H_2P_2(H_2P_2)\dagger)^2 \right\} snr^2$$

$$+ \mathcal{O}(snr^3)$$

(113)

Therefore, Theorem [10] has been proved.

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