FILTERED MODULES ON MOMENT GRAPHS AND PERIODIC PATTERNS

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Abstract. We define and study a certain category that naturally arises from the quotient of an affine moment graph by the action of a root lattice. We show that it contains enough projectives and that the standard multiplicities of indecomposable projectives are given by Lusztig's periodic polynomials, if the characteristic of the ground field is zero or big enough. Then we relate it to the combinatorial category defined by Andersen, Jantzen and Soergel (and hence to the theory of $G_1T$-modules).

1. Introduction

The main idea of the moment graph approach towards calculating multiplicities and other categorical data in representation theory is the following. First, one relates a deformed version of a category of modules with a Verma flag to a category of sheaves admitting a Verma flag on a moment graph. This usually is an exact equivalence, hence multiplicity conjectures for projectives can be transferred to a moment graph framework. The second step amounts to showing that the projective moment graph sheaves can be obtained by local hypercohomologies of equivariant parity sheaves on a flag variety. One then has two options to proceed. Either one proves a geometric decomposition theorem, from which it follows that parity sheaves are IC-complexes, and then uses the fact that in this case the multiplicities can be calculated using a Kazhdan–Lusztig type algorithm, or one proves the conjecture directly on the moment graph level. The latter was done for moment graphs associated with Coxeter systems and characteristic zero coefficients by Elias and Williamson [EW14] (in the equivalent setup of Soergel bimodules).

Lusztig's conjecture on the irreducible characters of reductive algebraic groups (cf. [Lus80a]) in positive characteristic $p$ can be translated, for $p$ at least the Coxeter number, into a conjecture about baby Verma multiplicities of projective $G_1T$-modules. More specifically, these multiplicities are conjectured to be given by the evaluation at 1 of Lusztig's periodic polynomials. Following the approach outlined above, a moment graph version for periodic polynomials is desirable. One candidate for this was given by Andersen, Jantzen and Soergel in [AJS94], which is used in their independence of $p$-result for Lusztig's conjecture. But their category is quite difficult to work with. In particular, in order to understand the indecomposable objects in this category one has to decompose the product of
translation functors applied to the unit object, which is, practically always, not feasible by direct calculation.

In this paper we propose a different category that seems to us much more conceptual. It is a category associated with the quotient of the affine moment graph, ordered by Lusztig’s generic order, by the action of the root lattice. It can be thought of as sheaves on the abstract moment graph quotient that are, in addition, endowed with a filtration by the set of alcoves. The resulting category carries a canonical exact structure, hence we can study projective objects therein.

Our two main results in this article are the following: First, we prove that in characteristic zero and in big enough characteristics, the multiplicities of the indecomposable projective objects are indeed given by Lusztig’s periodic polynomials. A decisive step for the proof is a functor from affine moment graph sheaves to our category, which allows us to use multiplicity results on affine moment graphs. Then we show that there is a functor from our category to the Andersen, Jantzen, Soergel category, and hence to $G_1T$-modules. In fact, Lusztig’s conjecture holds if the above multiplicity statement holds.

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3. Generalities on moment graphs

The following is a for the purposes of this article convenient definition of a moment graph.

3.1. Moment graphs. Let $X$ be a lattice, i.e. a free abelian group of finite rank.

Definition 3.1. A moment graph over $X$ is a graph given by a set of vertices $V$, a set of edges $E$, and a labelling $\alpha : E \to X/\{\pm 1\}$. We assume that the graph satisfies the following assumptions:

- Two vertices are connected by at most one edge.
- A vertex is not connected to itself.

Remark 3.2. In this paper it is convenient for us to assume that the labels on the graph are only determined up to a sign (cf. Definition 4.4). An arbitrary choice of signs will not change the structures that we are studying.

Let $G$ be a moment graph over the lattice $X$. Throughout the paper we fix a field $k$ of characteristic $\neq 2$ (the assumption on the characteristic will become important in Section 7.1). We denote by $X_k = X \otimes \mathbb{Z}k$ the $k$-vector space associated with $X$. Let $S = S(X_k)$ be the corresponding symmetric algebra. We consider $S$ as a $\mathbb{Z}$-graded algebra with $X_k \subset S$ being the homogeneous component of degree
2. Most of the modules we deal with in this paper are \( \mathbb{Z} \)-graded and we denote the graded components by \( M = \bigoplus_{n \in \mathbb{Z}} M_n \). Homomorphisms between graded modules are supposed to respect the grading. For \( l \in \mathbb{Z} \) we denote by \([l] \) the respective shift in the grading. It is defined by \( M[|n|] = M_{n+l} \). For any graded \( S \)-module \( N \) we denote by \( N^* = \bigoplus_{l \in \mathbb{Z}} \text{Hom}_S^l(N, S) \) with \( \text{Hom}_S^l(N, S) = \text{Hom}_S(N, S[|l|]) \) the graded dual of \( N \).

3.2. The structure algebra. Let \( \mathcal{X} \) be a subgraph of \( \mathcal{G} \).

**Definition 3.3.** The structure algebra of \( \mathcal{G} \) over \( \mathcal{X} \) is

\[
\mathcal{Z}(\mathcal{X}) = \left\{ (z_x) \in \prod_{x \in \mathcal{X}} S \mid z_x \equiv z_y \text{ mod } \alpha(E) \text{ for all edges } E: x \to y \text{ of } \mathcal{X} \right\}.
\]

The set \( \mathcal{Z}(\mathcal{X}) \) is an algebra by coordinatewise addition and multiplication. It contains a copy of \( S \) on the diagonal, and it is a unital, associative, graded \( S \)-algebra (note that the direct product in the definitions above should be taken componentwise, i.e. it should be a direct product in the category of graded \( S \)-modules). We denote by \( \mathcal{Z} = \mathcal{Z}(\mathcal{G}) \) the global structure algebra. For \( \mathcal{X}' \subset \mathcal{X} \subset V \) the projection \( \prod_{x \in \mathcal{X}} S \to \prod_{x \in \mathcal{X}'} S \) with kernel \( \prod_{x \in \mathcal{X} \setminus \mathcal{X}'} S \) induces a homomorphism \( \mathcal{Z}(\mathcal{X}) \to \mathcal{Z}(\mathcal{X}') \) of graded \( S \)-algebras.

**Remark 3.4.** Let us denote by \( C(\mathcal{X}) \) the set of connected components of the graph \( \mathcal{X} \). We then have a decomposition \( \mathcal{Z}(\mathcal{X}) = \prod_{\Omega \in C(\mathcal{X})} \mathcal{Z}(\Omega) \).

3.3. Localizations. Let \( L \) be a sublattice of \( X \). We say that \( L \) is saturated if the quotient \( X/L \) is torsion free, i.e. if the inclusion \( L \subset X \) splits. Let \( L \) be a saturated sublattice of \( X \).

**Definition 3.5.** We denote by \( \mathcal{G}^L \) the sub-moment graph of \( \mathcal{G} \) that we obtain by keeping all vertices and by deleting all edges that are labelled by elements that are not contained in \( L \).

For example, \( \mathcal{G}^\emptyset \) has the same vertex set as \( \mathcal{G} \), but it has no edges, so we can identify \( C(\mathcal{G}^\emptyset) \) with the set of vertices of \( \mathcal{G} \).

We set

\[
S_L := S[\alpha^{-1} \mid \alpha \in X, \alpha \neq 0, \alpha \notin L].
\]

As we invert homogeneous elements, this is a graded algebra as well. We have, in particular, \( S_X = S \) and \( S^\emptyset = S[\alpha^{-1} \mid \alpha \in X, \alpha \neq 0] \). For \( L \subset L' \) we have a canonical injective homomorphism \( S_{L'} \subset S_L \).

For an \( S \)-module \( M \) we set

\[
M_L := M \otimes_S S_L.
\]

If \( M \) is a graded module, then so is \( M_L \) in a natural way. But note that since we invert homogeneous elements in \( S \) of degree 2, we have an isomorphism \( M_L \cong M_L[2] \) if \( L \neq X \).
Lemma 3.6. Suppose that \( \mathcal{X} \) is a finite subgraph of \( \mathcal{G} \). Then the canonical inclusion \( \iota: \mathcal{Z}(\mathcal{X}) \to \mathcal{Z}(\mathcal{X}^L) \) is a bijection after localization, i.e. \( \iota^L: \mathcal{Z}(\mathcal{X})^L \to \mathcal{Z}(\mathcal{X}^L)^L \) is an isomorphism.

Proof. As \( \iota \) is injective and \( \mathcal{Z}(\mathcal{X})^L \) is torsion free, the map \( \iota^L \) is injective as well. We now show that it is surjective. Let \( \alpha_1, \ldots, \alpha_n \) be the labels of all edges of \( \mathcal{X} \) that are deleted in \( \mathcal{X}^L \). From the definition of \( \mathcal{Z} \) it the follows that \( \alpha_1 \cdots \alpha_n \mathcal{Z}(\mathcal{X}^L) \subset \mathcal{Z}(\mathcal{X}) \). This implies the surjectivity of \( \iota \). \( \square \)

3.4. Generic decompositions.

Definition 3.7. (1) For a vertex \( x \) of \( \mathcal{G} \) we define \( M^{0,x} \subset M^0 \) as the sub-\( \mathcal{Z} \)-module of all elements \( m \in M^0 \) on which \( (z_w) \in \mathcal{Z} \) acts as multiplication by \( z_x \in S \).

(2) We say that \( M \) is generically semisimple if \( M^0 = \bigoplus_{x \in \mathcal{V}} M^{0,x} \).

If \( \mathcal{G} \) is an infinite graph and \( M = \mathcal{Z} \) as a \( \mathcal{Z} \)-module it can occur that \( M^{0,x} = 0 \) for all \( x \), so \( M \) is not generically semisimple. On a finite graph this cannot happen, as we will show next.

Lemma 3.8. Suppose \( M \) is a \( \mathcal{Z} \)-module such that there exists a finite subgraph \( \mathcal{X} \) of \( \mathcal{G} \) and a \( \mathcal{Z}(\mathcal{X}) \)-action on \( M \) such that the \( \mathcal{Z} \)-action factors over the canonical homomorphism \( \mathcal{Z} \to \mathcal{Z}(\mathcal{X}) \). Then \( M \) is generically semisimple.

Proof. Note that \( M^0 \) is a \( \mathcal{Z}(\mathcal{X})^0 \)-module. By Lemma 3.6 we have \( \mathcal{Z}(\mathcal{X})^0 = \bigoplus_{x \in \mathcal{X}} S^0 \), which implies generic semisimplicity. \( \square \)

Definition 3.9. Let \( L \subset X \) be a saturated subset. We denote by \( \mathcal{Z}^L\)-mod\( ^f \) the full subcategory of the category of all \( \mathcal{Z}^L = \mathcal{Z}(\mathcal{G}^L) \)-modules that contains all objects \( M \) that have the following property:

- \( M \) is torsion free and finitely generated as an \( S^L \)-module.
- \( M \) is generically semisimple.

For saturated subsets \( L' \subset L \) of \( X \), localization \( M \mapsto M^L = M \otimes_S S^L \) yields a functor \( \mathcal{Z}^L\)-mod\( ^f \to \mathcal{Z}^{L'}\)-mod\( ^f \). We will write \( \mathcal{Z}\)-mod\( ^f \) instead of \( \mathcal{Z}^X\)-mod\( ^f \).

Lemma 3.10. Suppose that \( M \) is an object in \( \mathcal{Z}\)-mod\( ^f \). Let \( L \) be a saturated subset of \( X \). Then \( M^L \) carries a canonical action of \( \mathcal{Z}(\mathcal{G}^L) \). In particular, we have a canonical decomposition \( M^L = \bigoplus_{\Omega \subset C(\mathcal{G}^L)} M^{L,\Omega} \) of \( \mathcal{Z}(\mathcal{G}^L) \)-modules such that \( M^{L,\Omega} \subset \bigoplus_{x \in \Omega} M^{0,x} \).

We call the decomposition above the canonical decomposition of \( M^L \).

Proof. As \( M \) is supposed to be torsion free as an \( S \)-module, we have canonical inclusions \( M \subset M^L \subset M^0 = \bigoplus_{x \in \mathcal{X}} M^{0,x} \). Hence we can view each \( m \in M \) as an \( \mathcal{Y} \)-tuple \( m = (m_x) \) in such a way that \( z.m = (z_x m_x) \) for each \( z = (z_x) \in \mathcal{Z} \). The same formula defines an action of \( \mathcal{Z}^L \) on \( M^L \). \( \square \)
3.5. Interlude: A topology on partially ordered sets. Suppose that $\mathcal{X}$ is a set endowed with a partial order $\leq$. For an element $x$ of $\mathcal{X}$ we will use the short hand $\{\leq x\} = \{y \in \mathcal{X} \mid y \leq x\}$. The notations $\{\geq x\}$, $\{< x\}$, etc. have an analogous meaning. We will consider the following topology on $\mathcal{X}$.

**Definition 3.11.** A subset $\mathcal{A}$ of $\mathcal{X}$ is called **closed**, if $x \in \mathcal{A}$ and $y \leq x$ implies $y \in \mathcal{A}$. A subset $\mathcal{U}$ of $\mathcal{X}$ is called **open** if the complement $\mathcal{X} \setminus \mathcal{U}$ is closed.

**Remarks 3.12.**

- A subset $\mathcal{U}$ of $\mathcal{X}$ is open if and only if $x \in \mathcal{U}$ and $y \in \mathcal{X}$ with $x \leq y$ implies $y \in \mathcal{U}$.
- A subset $\mathcal{A}$ of $\mathcal{X}$ is closed if and only if $\mathcal{A} = \bigcup_{x \in \mathcal{A}} \{\leq x\}$.
- Arbitrary unions and arbitrary intersections of closed subsets are closed.
  The same holds for open subsets.

A subset $\mathcal{K}$ of $\mathcal{X}$ is locally closed if there are open subsets $\mathcal{U}_1 \subset \mathcal{U}_2$ with $\mathcal{K} = \mathcal{U}_2 \setminus \mathcal{U}_1$. This is the case if and only if there are closed subsets $\mathcal{A}_1 \subset \mathcal{A}_2$ with $\mathcal{K} = \mathcal{A}_2 \setminus \mathcal{A}_1$. Suppose that $\mathcal{K}$ is locally closed and choose $\mathcal{A}_1 \subset \mathcal{A}_2$ closed with $\mathcal{K} = \mathcal{A}_2 \setminus \mathcal{A}_1$. Set $\mathcal{K}_\leq := \bigcup_{x \in \mathcal{K}} \{\leq x\}$ and $\mathcal{K}_\lhd := \bigcup_{x \in \mathcal{K} \setminus \mathcal{K}_\leq} \{\leq x\}$. It is now easy to check that

- $\mathcal{K}_\lhd$ and $\mathcal{K}_\leq$ are closed.
- $\mathcal{K}_\leq \subset \mathcal{A}_2$ and $\mathcal{A}_2 = \mathcal{K}_\lhd \cup \mathcal{A}_1$.
- $\mathcal{K}_\lhd = \mathcal{K}_\lhd \cap \mathcal{A}_1$.
- $\mathcal{K} = \mathcal{K}_\lhd \setminus \mathcal{K}_\lhd$.

So $\mathcal{K}_\lhd$ and $\mathcal{K}_\leq$ are the minimal closed subsets such that $\mathcal{K} = \mathcal{K}_\lhd \setminus \mathcal{K}_\leq$.

3.6. **Filtrations on $\mathcal{Z}$-modules and Verma flags.** We now assume that we are given an additional piece of data on a moment graph, namely a partial order on the set of vertices.

**Definition 3.13.** An **ordered moment graph** is a moment graph with a partial order $\leq$ on its set of vertices such that the following holds:

- Two vertices that are connected by an edge are comparable.
- The partial order is generated by the pairs of vertices connected by an edge.

Suppose that $(\mathcal{G}, \leq)$ is an ordered moment graph and let $M$ be an object in $\mathcal{Z}\text{-mod}^f$. As $M$ is torsion free as an $S$-module, we consider it as a subspace in $M^0$.

**Definition 3.14.** Let $\mathcal{X} \subset \mathcal{V}$ be a subset with complement $\mathcal{Y} = \mathcal{V} \setminus \mathcal{X}$. We define $M_\mathcal{X} := M \cap \bigoplus_{x \in \mathcal{X}} M_0^{0,x}$ and $M_\mathcal{Y} := M / M_\mathcal{X}$.

Both $M_\mathcal{X}$ and $M_\mathcal{Y}$ are $\mathcal{Z}$-modules in a natural way. Now we define the notion of a $\leq$-Verma flag.
Definition 3.15. We say that $M \in \mathcal{Zmod}^{f}$ admits a $\triangleright$-Verma flag if for any $\triangleright$-open subset $\mathcal{J}$ of $\mathcal{V}$ the quotient $M^\mathcal{J}$ is graded free over $S$.

If $M$ admits a $\triangleright$-Verma flag and $\mathcal{J}$ is an open subset of $\mathcal{V}$ with closed complement $\mathcal{I}$, then $M_\mathcal{I}$ is a graded submodule of $M$, and $M_\mathcal{T}$ and $M^\mathcal{J}$ are contained in $\mathcal{Zmod}^{f}$ as well.

Definition 3.16. We denote by $C^{\triangleright}$ the full subcategory of $\mathcal{Zmod}^{f}$ that contains all objects that admit a $\triangleright$-Verma flag.

It is convenient for us to also give a localized version of the above definitions.

Definition 3.17. We say that $M \in \mathcal{ZLmod}^{f}$ admits a $\triangleright$-Verma flag if for any $\triangleright$-open subset $\mathcal{J}$ of $\mathcal{V}$ the quotient $M^\mathcal{J}$ is graded free over $S^L$. We denote by $C^{L,\triangleright}$ the corresponding full subcategory of $\mathcal{ZLmod}^{f}$.

3.7. $\mathcal{Z}$-modules and sheaves on $\mathcal{G}$. We also need the notion of a sheaf on the moment graph $\mathcal{G}$.

Definition 3.18. A $(k)$-sheaf $\mathcal{M}$ on $\mathcal{G}$ is given by the following data:

- a graded $S$-module $\mathcal{M}^x$ for each vertex $x$ of $\mathcal{G}$,
- a graded $S$-module $\mathcal{M}^E$ for each edge $E$ of $\mathcal{G}$ with $\alpha(E).\mathcal{M}^E = \{0\}$,
- a homomorphism $\rho_{x,E}: \mathcal{M}^x \to \mathcal{M}^E$ of graded $S$-modules whenever the vertex $x$ lies on the edge $E$.

A morphism $f: \mathcal{M} \to \mathcal{N}$ between the sheaves $\mathcal{M}$ and $\mathcal{N}$ on $\mathcal{G}$ is given by homomorphisms $f^x: \mathcal{M}^x \to \mathcal{N}^x$ and $f^E: \mathcal{M}^E \to \mathcal{N}^E$ of graded $S$-modules for all vertices $x$ and $E$. These data should satisfy the obvious compatibility conditions with respect to the $\rho$-maps.

We denote by $\mathcal{Gmod}^{f}$ the category of sheaves on $\mathcal{G}$ that have the property that $\mathcal{M}^x$ and $\mathcal{M}^E$ are finitely generated $S$-modules for each vertex $x$ and each edge $E$ and only finitely many $\mathcal{M}^x$ and $\mathcal{M}^E$ are non-zero. The morphisms in $\mathcal{Gmod}^{f}$ are morphisms of moment graph sheaves.

Definition 3.19. Let $\mathcal{X}$ be a subset of the vertices of $\mathcal{G}$. The space of sections of a sheaf $\mathcal{M}$ on $\mathcal{X}$ is

$$\Gamma(\mathcal{X}, \mathcal{M}) = \left\{ (m_x) \in \prod_{x \in \mathcal{X}} \mathcal{M}^x \left| \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \text{ for all edges } E: x \to y \text{ with } x, y \in \mathcal{X} \right. \right\} .$$

Note that $\Gamma(\mathcal{X}, \mathcal{M})$ carries a canonical structure of a graded $\mathcal{Z}(\mathcal{X})$-module. We write $\Gamma(\mathcal{V}, \mathcal{M})$ for the global sections $\Gamma(\mathcal{V}, \mathcal{M})$. For a subset $\mathcal{I}$ of $\mathcal{X}$ we denote by

$$\Gamma(\mathcal{X}, \mathcal{M})_\mathcal{I} = \left\{ (m_x) \in \Gamma(\mathcal{X}, \mathcal{M}) \left| m_x = 0 \text{ if } x \notin \mathcal{I} \right. \right\} .$$

the space of sections over $\mathcal{X}$ that are supported inside $\mathcal{I}$.
### 3.8. Moment graph localization of \(Z\)-modules.

In \cite{Nie08a} we defined a localization functor \(\mathcal{L}: \text{Z-mod} \to \text{G-mod}\) that is left adjoint to the global section functor \(\Gamma: \text{G-mod} \to \text{Z-mod}\).

**Proposition 3.20** (\cite{Nie08a}, Proposition 3.5 & 4.6). Suppose that \(M\) is an object in \(\mathcal{C}^\Delta\) and set \(\mathcal{M} = \mathcal{L}M\). Then the following holds:

1. The sheaf \(\mathcal{M}\) is \(\leq\)-flabby, i.e. for each \(\leq\)-open subset \(J\) of \(V\) the restriction map \(\Gamma(\mathcal{M}) \to \Gamma(J, \mathcal{M})\) is surjective.
2. The adjunction morphism \(M \to \Gamma(\mathcal{M})\) is an isomorphism.
3. For any closed subset \(I\) of \(V\) the above isomorphism identifies \(M_I\) with \(\Gamma(\mathcal{M})_I\).

Using the above proposition we can deduce the following properties for objects in \(\mathcal{C}^\Delta\).

**Proposition 3.21.** Let \(M\) be an object in \(\mathcal{C}^\Delta\). Then the following holds:

1. For closed subsets \(I_1\) and \(I_2\) of \(V\) we have
   - \(M_{I_1 \cup I_2} = M_{I_1} + M_{I_2}\),
   - \(M_{I_1 \cap I_2} = M_{I_1} \cap M_{I_2}\).
2. For any saturated subset \(L\) of \(X\) we have a canonical decomposition
   \[M^L = \bigoplus_{\Gamma \in C(\mathcal{G}^L)} M^{L, \Gamma}\]
   with \(\text{supp } M^{L, \Gamma} \subset \Gamma\).

*Proof.* Note that (1b) is a trivial consequence of the definition, and (2) is already proven in Lemma 3.10.

Now we prove (1a). Note that it is clear that \(M_{I_1} + M_{I_2} \subset M_{I_1 \cup I_2}\). We need to show that the converse inclusion holds. Set \(\mathcal{M} = \mathcal{L}M\). Then we can identify \(M\) with \(\Gamma(\mathcal{M})\) and for any closed subset \(I\) the subspace \(M_I\) is the space of sections of \(\mathcal{M}\) that are supported on \(I\). So let \(m = (m_x)\) be a section supported on \(I_1 \cup I_2\).

Consider the open subset \(J = V \setminus (I_1 \cup I_2)\). We now define a section \(m_1\) of \(\mathcal{M}\) on \(J\) as follows. We set \(m_{1,x} = m_x\) if \(x \in I_1\), and \(m_{1,x} = 0\) otherwise. As \(m\) is a section and since \(I_1 \cap J\) and \(I_2 \cap J\) are not connected by any edge, we deduce that \(m_1\) is indeed a section. As \(\mathcal{M}\) is flabby, the restriction map \(\Gamma(\mathcal{M}) \to \Gamma(J, \mathcal{M})\) is surjective, and we can hence choose a preimage \(\tilde{m}_1\) of \(m_1\). Then \(\tilde{m}_1\) is supported inside \(I_1\) and coincides with \(m\) on \(I \cap J\). Hence \(m - \tilde{m}_1\) is supported on \(I_2\), and hence \(m \in M_{I_1} + M_{I_2}\). \(\square\)

### 3.9. An exact structure.

Let \(0 \to A \to B \to C \to 0\) be a sequence in \(\mathcal{C}^\Delta\).

**Definition 3.22.** We say that the above sequence is **exact** if for any open subset \(J\) the induced sequence

\[0 \to A^J \to B^J \to C^J \to 0\]

is an exact sequence of graded \(S\)-modules.
By Theorem 4.1 in [Fie08a] this indeed defines an exact structure in the sense of Quillen. In particular, we now have the notion of a projective object in \( C^\oplus \).

**Definition 3.23.** We say that an object \( P \) of \( C^\oplus \) is *projective* if the functor \( \text{Hom}_{C^\oplus}(P, \cdot) \) maps exact sequences in \( C^\oplus \) to short exact sequences of abelian groups.

**Remark 3.24.** In general, it is hard to establish the existence of projective objects in \( C^\oplus \) (and they don’t necessarily exist for general moment graphs). Using the theory of Braden–MacPherson sheaves one can show, however, that there are enough projectives once one replaces the freeness condition in Definition 3.15 by the more general reflexiveness condition (see [Fie08a, Section 4]). The moment graphs considered in this paper, however, carry additional structure that can be used to prove the existence of enough projectives in \( C^\oplus \).

### 4. Moment graphs with a group action

One of the main objects of study in this article are moment graphs that are acted upon by a group. Before we introduce these, we define and study a notion of an isomorphism of moment graphs.

#### 4.1. Moment graph isomorphisms

Morphisms of moment graphs were introduced in [Lan12]. We won’t need the most general notion of morphism. The following will suffice for the purposes of this article. Let \( G_1 \) and \( G_2 \) be moment graphs over the same lattice \( X \).

**Definition 4.1.** (1) A *twisted isomorphism* \( \sigma = (\sigma_V, \sigma_X) : G_1 \to G_2 \) between the moment graphs \( G_1 \) and \( G_2 \) is given by a bijection \( \sigma_V : V_1 \to V_2 \) between the sets of vertices of \( G_1 \) and \( G_2 \), and an automorphism \( \sigma_X : X \to X \) of the lattice \( X \), such that the following holds:

- The map \( \sigma_V \) induces an isomorphism of the underlying abstract graphs, i.e. the vertices \( x \) and \( y \) are connected by an edge in \( G_1 \) if and only if \( \sigma_V(x) \) and \( \sigma_V(y) \) are connected by an edge in \( G_2 \).
- Let \( x \) and \( y \) be vertices connected by an edge \( E : x \xrightarrow{\alpha} y \) in \( G_1 \) and let \( E' : \sigma_V(x) \xrightarrow{\alpha'} \sigma_V(y) \) be the image of \( E \) in \( G_2 \). Then \( \sigma_X(\alpha) = \pm \alpha' \).

(2) An *untwisted automorphism* is a twisted isomorphism \( \sigma : G_1 \to G_2 \) as above with \( \sigma_X = \text{id}_X \).

(3) If \( G_1 \) and \( G_2 \) are ordered moment graphs, then we call a (twisted) automorphism \( \sigma : G_1 \to G_2 \) *order preserving* (order reversing), if the bijection \( \sigma_V : V_1 \to V_2 \) preserves (reverses) the partial orders.

Now let \( G \) be a group. Here is our notion of a \( G \)-action:

**Definition 4.2.** A *\( G \)-moment graph* is a moment graph together with an action of \( G \) by untwisted moment graph automorphisms.
Note that we insist that $G$ acts by untwisted automorphisms, and that we do not care whether these automorphisms are order preserving or reversing or neither.

4.2. The induced automorphisms on the structure algebra. For any lattice automorphism $s: X \to X$ we denote by $S(s): S(X \otimes \mathbb{Z} k) \to S(X \otimes \mathbb{Z} k)$ the induced automorphism of the associated symmetric algebra over $k$. With a twisted automorphism $\sigma: \mathcal{G} \to \mathcal{G}$ we associate the homomorphism $\sigma_S: \prod_{x \in V} S \to \prod_{x \in V} S$ that is given by $\sigma_S(z_x) = z'_x$ with $z'_x = S(\sigma_X^{-1})(z_{\sigma_V(x)})$. In particular, if we denote by $e_y \in \prod_{x \in V} S$ the standard basis element supported on $y \in V$, then we have

$$\sigma_S(e_y) = e_{\sigma_V^{-1}(y)}.$$ 

Note that $\sigma_S$ is in general not $S$-linear, but $S$-twisted linear, i.e. we have $\sigma_S(f \cdot z) = S(\sigma_X^{-1})(f)\sigma_S(z)$ for $f \in S$ and $z \in \prod_{x \in V} S$.

Lemma 4.3. Let $\mathcal{X}$ be a subset of $\mathcal{V}$. Then $\sigma_S$ induces an $S$-twisted linear isomorphism

$$\mathcal{Z}(\sigma_V(\mathcal{X})) \sim \to \mathcal{Z}(\mathcal{X}).$$

In particular, we obtain an algebra automorphism $\sigma_Z: \mathcal{Z} \to \mathcal{Z}$.

Proof. From the definition it follows directly that $\sigma$ induces an $S$-twisted linear isomorphism $\prod_{x \in \sigma_V(\mathcal{X})} S \sim \to \prod_{x \in \mathcal{X}} S$. In order to prove the statement in the lemma it suffices to show that it maps $\mathcal{Z}(\sigma_V(\mathcal{X}))$ into $\mathcal{Z}(\mathcal{X})$.

So let $z = (z_x) \in \mathcal{Z}(\sigma_V(\mathcal{X}))$ and let $z' = (z'_x)$ be its image under $\sigma$. We have to check that for any edge $E: x \xrightarrow{\alpha} y$ with $x, y \in \mathcal{X}$ we have $z'_x \equiv z'_y \mod \alpha$. Indeed, we have

$$z'_x - z'_y = S(\sigma_X^{-1})(z_{\sigma_V(x)}) - S(\sigma_X^{-1})(z_{\sigma_V(y)})$$

$$= S(\sigma_X^{-1})(z_{\sigma_V(x)} - z_{\sigma_V(y)})$$

As $z_{\sigma_V(x)} \equiv z_{\sigma_V(y)} \mod \sigma_X(\alpha)$ by the definitions of $\mathcal{Z}$ and of a moment graph automorphism, we conclude that $S(\sigma_X^{-1})(z_{\sigma_V(x)} - z_{\sigma_V(y)})$ is divisible by $\alpha$ in $S$. □

4.3. Twisting of localizations. Let $\sigma: \mathcal{G} \to \mathcal{G}$ be a moment graph automorphism. Let $L \subset X$ be a saturated subset. Then $\sigma_X(L)$ is saturated as well, and the restriction of $\sigma$ to the subgraph $\mathcal{G}^L$ induces a twisted isomorphism

$$\sigma: \mathcal{G}^L \to \mathcal{G}^{\sigma_X(L)}.$$ 

In particular, we obtain a bijection $C(\mathcal{G}^L) \to C(\mathcal{G}^{\sigma(L)})$, $\Gamma \mapsto \sigma_V(\Gamma)$ between the sets of connected components.
4.4. Quotient graphs. Let us now fix a $G$-moment graph $\mathcal{G}$.

**Definition 4.4.** The quotient of $\mathcal{G}$ by the $G$-action is the moment graph $\mathcal{G} = G \backslash \mathcal{G}$ that is given by the following data:

1. Its set of vertices is $\overline{V} = G \backslash V$, the set of $G$-orbits in $V$.
2. Let $\Omega \neq \Omega'$ be two vertices and $\pm \alpha \in X / \langle \pm 1 \rangle$. Then $\Omega$ is connected to $\Omega'$ by a $\pm \alpha$-edge in $\mathcal{G}$ if and only if there are vertices $v \in \Omega$ and $v' \in \Omega'$ of $\mathcal{G}$ that are connected by a $\pm \alpha$-edge.

**Remarks 4.5.** Note that a vertex $v \in \Omega$ can be connected to several different $v' \in \Omega'$ by a $\pm \alpha$-edge (the moment graph $\tilde{\mathcal{G}}$ that we introduce in Section 6.8 is an example for this). Even if this is the case, we only have a single $\pm \alpha$-edge between $\Omega$ and $\Omega'$.

For a vertex $A \in V$ we denote by $\overline{A} = G.A \in \overline{V}$ its $G$-orbit. Each element $g$ of $G$ acts on $\mathcal{G}$ by a moment graph automorphism and as such induces an algebra automorphism on $\mathcal{Z}$ by Lemma 4.3. As the action is by untwisted automorphisms, the former are automorphisms of $\mathcal{Z}$ as an $S$-algebra. We let $\mathcal{Z}^G$ be the sub-$S$-algebra of $\mathcal{Z}$ that consists of all elements that are $g$-invariant for all $g \in G$. Let us denote by $\overline{\mathcal{Z}} = \mathcal{Z}(\overline{\mathcal{G}})$ the structure algebra (over $k$) of the quotient graph.

**Remark 4.6.** From the definitions it follows immediately that $\overline{\mathcal{Z}} = \mathcal{Z}^G$.

4.5. Filtrations on $\overline{\mathcal{Z}}$-modules. We will now endow modules over the structure algebra $\overline{\mathcal{Z}}$ with a filtration that is indexed by the partially ordered set $\mathcal{V}$. This filtration should replace the one we considered in Section 3.6. In contrast to the situation there the filtration that we consider here is not intrinsically defined by the module structure, but is an additional piece of datum.

We assume that the $G$-moment graph $\mathcal{G}$ carries in addition a partial order $\preceq$. Let $M$ be a $\overline{\mathcal{Z}}$-module.

**Definition 4.7.** A $\preceq$-filtration on $M$ is given by the data of sub-$\overline{\mathcal{Z}}$-modules $M_I$ of $M$ for each $\preceq$-closed subset $I$ of $\mathcal{V}$ such that the following conditions are satisfied. Let $I_1$ and $I_2$ be closed subsets of $\mathcal{V}$.

1. (P0) If $I_1 \subset I_2$, then $M_{I_1} \subset M_{I_2}$.
2. (P1) We have $M_{I_1 \cup I_2} = M_{I_1} + M_{I_2}$.
3. (P2) We have $M_{I_1 \cap I_2} = M_{I_1} \cap M_{I_2}$.

**Remark 4.8.**

1. Note that (P1) implies (P0), but it is convenient for us to state (P0) as a separate property.
2. Note that (P1) is equivalent to $M_I = \sum_{A \in I} M_{\preceq A}$ for all closed $I$. If (P0) holds, then (P1) is equivalent to $M_I = \sum M_{\preceq A}$ for all closed $I$, where the sum runs over all maximal elements $A$ in $I$.
3. Suppose that (P0) holds. For arbitrary closed $I_1$ and $I_2$ we then have a canonical homomorphism

$$M_{I_1} / M_{I_1 \cap I_2} \rightarrow M_{I_1 \cup I_2} / M_{I_2}.$$
This homomorphism is *injective* if and only if (P2) holds, and *surjective* if and only if (P1) holds.

For an open subset $\mathcal{J}$ of $\mathcal{V}$ with closed complement $\mathcal{I}$ we define

$$M^{\mathcal{J}} := M/M_{\mathcal{I}}.$$  

For open subsets $\mathcal{J}' \subset \mathcal{J}$ we then have a canonical quotient map $M^{\mathcal{J}} \to M^{\mathcal{J}'}$.

Let $\mathcal{K} \subset \mathcal{V}$ be a locally closed subset. Suppose that the data of submodules $M_{\mathcal{I}}$ satisfies property (P0). We then set

$$M[\mathcal{K}] = M_{\mathcal{I} \mathcal{K}}/M_{\mathcal{I} \mathcal{K}}.$$  

Note that a singleton $\{A\}$ is always locally closed. We write $M_{\{A\}}$ instead of $M[\{A\}]$.

**Definition 4.9.** Suppose that the data $M_{\mathcal{I}}$ satisfies (P0). We then define

$$\text{supp } M = \{ A \in \mathcal{A} \mid M_{\ll A} \neq M_{\gg A} \}.$$  

If $\mathcal{I}_1 \subset \mathcal{I}_2$ are closed subset such that $\mathcal{K} = \mathcal{I}_2 \setminus \mathcal{I}_1$, then we have inclusions $\mathcal{K}_{\mathcal{I}} \subset \mathcal{I}_1$ and $\mathcal{K}_{\ll} \subset \mathcal{I}_2$. From this we obtain a canonical homomorphism

$$M[\mathcal{K}] \to M_{\mathcal{I}_2}/M_{\mathcal{I}_1}.$$  

By Remark 4.8(3), this is an isomorphism if the filtration satisfies (P0-2).

4.6. **Dual filtrations.** Recall that we denote by $M^* = \text{Hom}_S(M, S)$ the graded dual of a graded $\mathcal{Z}$-module $M$. Suppose that $M$ carries a $\ll$-filtration, and suppose that for all pairs of closed subsets $\mathcal{I}' \subset \mathcal{I}$ the inclusion $M_{\mathcal{I}'} \subset M_{\mathcal{I}}$ splits in the category of graded $S$-modules. Then also the quotients $M^{\mathcal{J}} \to M^{\mathcal{J}'}$ split for open subsets $\mathcal{J}' \subset \mathcal{J}$. In particular, the graded dual of the quotient $M \to M^{\mathcal{J}}$ is a split injection $(M^{\mathcal{J}})^* \to M^*$.

Suppose that we are given an order reversing bijection $\sigma : \mathcal{V} \to \mathcal{V}$. For a $\ll$-closed subset $\mathcal{I}$ of $\mathcal{V}$ we then define

$$(M^*)_{\mathcal{I}} := (M^*_{\mathcal{I}'})^*.$$  

It is an easy exercise to check that (P0), (P1) and (P2) hold for this data as well, hence we obtain a $\ll$-filtration on $M^*$.

4.7. **$\ll$-Verma flags.** Suppose we are given a $\ll$-filtration on $M$. For an open subset $\mathcal{J}$ of $\mathcal{V}$ with closed complement $\mathcal{I}$ we set

$$M^{\mathcal{J}} := M/M_{\mathcal{I}}.$$  

**Definition 4.10.** Let $M$ be a $\mathcal{Z}$-module endowed with a $\ll$-filtration. We say that $M$ admits a $\ll$-Verma flag if for each open subset $\mathcal{J}$ of $\mathcal{A}$ the module $M^{\mathcal{J}}$ is a graded free $S$-module.
Let $\mathcal{I}' \subset \mathcal{I}$ be closed subsets of $\mathcal{V}$ with open complements $\mathcal{J}' = \mathcal{V} \setminus \mathcal{I}'$ and $\mathcal{J} = \mathcal{V} \setminus \mathcal{I}$. We now consider the inclusion $M_{\mathcal{I}'} \to M_{\mathcal{I}}$ and the corresponding quotient map $M^\mathcal{J} \to M^\mathcal{J}'$. From the short exact sequence (of $\mathcal{Z}$-modules)

$$0 \to M_{\mathcal{I}}/M_{\mathcal{I}'{}'} \to M/M_{\mathcal{I}'} \to M/M_{\mathcal{I}} \to 0$$

we obtain:

**Lemma 4.11.** The cokernel of $M_{\mathcal{I}'} \to M_{\mathcal{I}}$ is canonically isomorphic to the kernel of $M^{\mathcal{J}'} \to M^\mathcal{J}$. In particular, if the $\leq$-filtration satisfies (P1) and (P2) and if $A \in \mathcal{J}$ is a $\leq$-minimal element in an open subset $\mathcal{J}$, then we have a canonical isomorphism

$$M_{[A]} = \ker(M^\mathcal{J} \to M^{\mathcal{J} \setminus \{A\}}).$$

**Lemma 4.12.** Let $M$ be an object in $\mathcal{Z}$-mod endowed with a $\leq$-filtration that satisfies (P1) and (P2). Then $M$ admits a $\leq$-Verma flag if and only if $M_{[A]}$ is a graded free $S$-module for each $A \in \mathcal{V}$.

**Proof.** Let $\mathcal{J}$ be an open locally bounded set and suppose that $A$ is a minimal element in $\mathcal{J}$. Then we have an exact sequence of $\mathcal{Z}$-modules

$$0 \to M_{[A]} \to M^\mathcal{J} \to M^{\mathcal{J} \setminus \{A\}} \to 0$$

by Lemma 4.11. From this the claim follows easily by induction on the number of elements in the set $\mathcal{J} \cap \text{supp} M$. \(\square\)

4.8. **Localizations.** From now on we assume that $M$, endowed with a $\leq$-filtration, admits a Verma flag. Then for any pair $\mathcal{I}' \subset \mathcal{I}$ of closed subsets the inclusion $M_{\mathcal{I}'} \subset M_{\mathcal{I}}$ splits in the category of (graded) $S$-modules. Let $L$ be a saturated subset of $X$. Then we can consider the induced filtration on $M^L$. It is defined by setting

$$(M^L)^\mathcal{I} := (M^L)^L = M_{\mathcal{I}} \otimes_S S^L$$

for any closed subset $\mathcal{I}$ of $\mathcal{V}$. This clearly satisfies properties (P0-2). By what we observed above, we can reconstruct the filtration on $M$ from this induced filtration, i.e. we have $M_{\mathcal{I}} = M \cap (M^L)^\mathcal{I}$.

Let us now consider the decompositions of $G^L$ and $\overline{G}^L$ into connected components. As the group $G$ acts on $G$ by non-twisted graph automorphisms we get an induced action of $G$ on $G^L$. In particular, $G$ acts on the set $C(G^L)$ of connected components, and we obtain an identification

$$C(\overline{G}^L) = C(G^L)/G.$$ We denote the quotient map $\pi : C(G^L) \to C(\overline{G}^L)$.

By Lemma 3.10 we have a canonical decomposition $M^L = \bigoplus_{\mathcal{I} \in C(\overline{G}^L)} M^{L,\mathcal{I}}$. We will assume that this decomposition can be refined to a decomposition indexed by $C(G^L)$:
(P3) Let $\Gamma \in \mathcal{C}(\mathcal{G})$. Then there are sub-$\mathcal{Z}$-modules $M^{L,\Omega}$ of $M^{L,\Gamma}$ for each $\Omega \in \pi^{-1}(\Gamma)$, such that $\text{supp } M^{L,\Omega} \subset \Omega$ and $M^{L,\Gamma} = \bigoplus_{\Omega \in \pi^{-1}(\Gamma)} M^{L,\Omega}$.

Note that the above decomposition in (P3) will in general not be canonical. Note also that (P3) implies that $\text{supp } M^{L,\Gamma} \subset \bigcup_{\Omega \in \pi^{-1}(\Gamma)} \Omega$.

**Definition 4.13.** • Let $M$ be a $\mathcal{Z}$-module. The data of submodules $M^I \subset M$ for each $\preceq$-closed subset $I$ of $V$ is called an admissible filtration if it satisfies properties (P1), (P2) and (P3).

• We denote by $\mathcal{Z}$-mod$\preceq$ the category of $\mathcal{Z}$-modules that are finitely generated, reflexive and torsion free as $S$-modules and that are furthermore endowed with an admissible filtration. A morphism in $\mathcal{Z}$-mod$\preceq$ between the objects $M$ and $N$ is a $\mathcal{Z}$-module homomorphism $f: M \to N$ that respects the $\preceq$-filtration, i.e. that satisfies $f(M^I) \subset N^I$ for all closed subsets $I$ of $V$.

• We denote by $\mathcal{C}$-$\preceq$ the full subcategory of $\mathcal{Z}$-mod$\preceq$ that contains all objects that admit a $\preceq$-Verma flag.

4.9. **An exact structure.** Let $0 \to M \xrightarrow{f} N \xrightarrow{g} O \to 0$ be a sequence in $\mathcal{C}$-$\preceq$.

**Definition 4.14.** We say that the above sequence is exact, if for any open subset $J$ the induced sequence $0 \to M^J \xrightarrow{f^J} N^J \xrightarrow{g^J} O^J \to 0$ is an exact sequence of $S$-modules. In this case we call $g$ an admissible epimorphism and $f$ an admissible monomorphism.

Arguments similar to the arguments in the proof of [Fie08a, Theorem 4.1] show that this indeed gives an exact structure in the sense of Quillen.

**Lemma 4.15.** The sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} O \to 0$ is exact if and only if for any $A \in V$ the induced sequence $0 \to M_{[A]} \xrightarrow{f_{[A]}} N_{[A]} \xrightarrow{g_{[A]}} O_{[A]} \to 0$ is an exact sequence of $S$-modules.

**Proof.** Suppose the sequence $M \to N \to O$ is exact. Let $J \subset V$ be an open subset with minimal element $A \in J$, and set $J' = J \setminus \{A\}$. Then both sequences $0 \to M^J \to N^J \to O^J \to 0$ and $0 \to M^{J'} \to N^{J'} \to O^{J'} \to 0$ are exact, and hence also the kernel sequence $0 \to M_{[A]} \to N_{[A]} \to O_{[A]} \to 0$ is exact. (Note that we used Lemma 4.11.)

Now suppose that $M \to N \to O$ for any $A \in V$ is a sequence and that $0 \to M_{[A]} \to N_{[A]} \to O_{[A]} \to 0$ is exact for any $A$. Let $J$ be an open subset. Let $\mathcal{T}$ be the union of the supports of $M$, $N$ and $O$. Then $\mathcal{T}$ is finite, so we can prove
the above claim by induction on the number of elements in $\mathcal{T} \cap \mathcal{J}$. Denote by $\mathcal{U}$ the open subset generated by $\mathcal{T} \cap \mathcal{J}$, i.e.

$$\mathcal{U} = \bigcup_{C \in \mathcal{T} \cap \mathcal{J}} \{ B \in \mathcal{V} \mid C \subseteq B \}.$$ 

Then for $X = M, N, O$ we have $X^{\mathcal{J}} = X^{\mathcal{U}}$. Let $A \in \mathcal{U}$ be a minimal element. We then have a short exact sequence $0 \to X_A \to X^{\mathcal{U}} \to X^{\mathcal{U}\setminus A} \to 0$. As these sequences are functorial, the exactness of $0 \to M^{\mathcal{U}} \to N^{\mathcal{U}} \to O^{\mathcal{U}} \to 0$ follows from the exactness of $0 \to M^{\mathcal{U}\setminus A} \to N^{\mathcal{U}\setminus A} \to O^{\mathcal{U}\setminus A} \to 0$ and $0 \to M_A \to N_A \to O_A \to 0$. But note that the joint support of $M^{\mathcal{U}\setminus A}, N^{\mathcal{U}\setminus A}$ and $O^{\mathcal{U}\setminus A}$ is $\mathcal{T} \setminus \{ A \}$. Using the induction hypothesis we finish the proof. □

**Lemma 4.16.** Suppose that $M$ is an object in $\mathcal{C}^{\mathcal{J}}$. Let $\Omega \subset \mathcal{V}$ be a $G$-orbit and $\mathcal{I}$ a closed subset of $\mathcal{V}$. Then $M^{0,\Omega}_\mathcal{I}$ only depends on $\Omega \cap \mathcal{I}$.

**Remark 4.17.** In more explicit terms this means the following. For a closed subset $\mathcal{I}$ we denote by $\mathcal{I}' = \mathcal{I} \cap \Omega$ the smallest closed subset that contains $\mathcal{I} \cap \Omega$. Then $\mathcal{I}' \subset \mathcal{I}$ and the claim is that the inclusion $M^{0,\Omega}_{\mathcal{I}'} \subset M^{0,\Omega}_\mathcal{I}$ is a bijection.

**Proof.** Let $\mathcal{I}'$ be defined as above, and denote by $M[\mathcal{K}]$ the quotient of the inclusion $M_{\mathcal{I}'} \subset M_{\mathcal{I}}$. By assumption, $\Omega \cap \mathcal{K} = \emptyset$, so the admissibility of the filtration implies that $(M[\mathcal{K}])^{0,\Omega} = 0$. Hence $M^{0,\Omega}_{\mathcal{I}'} = M^{0,\Omega}_\mathcal{I}$. □

### 4.10. Admissible filtrations and subtopologies

Now suppose that we are given a subtopology on $\mathcal{V}$ (i.e. a subset $\mathcal{J}$ of the set of closed subsets in $\mathcal{V}$). Suppose that the subtopology $\mathcal{J}$ has the following property:

- For each $G$-orbit $\Omega$ in $\mathcal{V}$, the induced family $\{ \mathcal{I} \cap \Omega \}_{\mathcal{I} \in \mathcal{J}}$ contains each closed subset of $\Omega$.

**Lemma 4.18.** If $\mathcal{J}$ satisfies the above property, then the following holds. Let $M$ and $N$ be objects in $\mathcal{C}^{\mathcal{J}}$ and let $f: M \to N$ be a homomorphism of $\mathcal{C}$-modules. If $f(M_{\mathcal{I}}) \subset N_{\mathcal{I}}$ for all $\mathcal{I} \in \mathcal{J}$, then $f$ respects the $\leq$-filtrations.

(In particular, an admissible $\leq$-filtration on $M$ is already determined by the data of submodules $M_{\mathcal{I}}$ for $\mathcal{I} \in \mathcal{J}$.)

**Proof.** Let $f: M \to N$ be as above. As $M$ and $N$ admit Verma flags, we can recover the filtrations on both modules from the induced generic filtrations. So it suffices to show that $f^0: M^0 \to N^0$ maps $M^{0,\Omega}_{\mathcal{I}}$ into $N^{0,\Omega}_{\mathcal{I}}$ for each $\Omega$ and each closed subset $\mathcal{I}$. By our assumption on the subtopology we can find $\mathcal{I}' \in \mathcal{J}$ with $\mathcal{I}' \cap \Omega = \mathcal{I} \cap \Omega$. By Lemma 4.16 we have $M^{0,\Omega}_\mathcal{I} = M^{0,\Omega}_{\mathcal{I}'}$ and $N^{0,\Omega}_\mathcal{I} = N^{0,\Omega}_{\mathcal{I}'}$. But the assumption in the statement implies that $f^0(M^{0,\Omega}_{\mathcal{I}'}) \subset N^{0,\Omega}_{\mathcal{I}'}$. □
4.11. **The Krull-Schmidt theorem.** Let $M$ be an object in $\mathbb{Z}\text{-mod}^{\leq}$. By definition, $M$ is a finitely generated module over $S$. Hence all its homogeneous components are finite dimensional $k$-vector spaces and any degree zero endomorphism of $M$ splits into a direct sum of endomorphisms of finite dimensional vector spaces.

Let $f : M \to M$ be a morphism in $\mathbb{Z}\text{-mod}^{\leq}$. We define

$$\ker f^\infty := \{m \in M \mid f^n(m) = 0 \text{ for some } n \gg 0\} = \bigcup_{n \in \mathbb{N}} \ker f^n,$$

$$\text{im } f^\infty := \bigcap_{n \in \mathbb{N}} \text{im } f^n.$$

Clearly, $\ker f^\infty$ and $\text{im } f^\infty$ are sub-$\mathbb{Z}$-modules of $M$. Now $f$ preserves any homogeneous component of $M$. As these are finite dimensional vector spaces, we can use the Fitting-decomposition for finite dimensional vector spaces to obtain a direct sum decomposition

$$M = \ker f^\infty \oplus \text{im } f^\infty$$

in the category of $\mathbb{Z}$-modules. We now show that this direct sum decomposition is compatible with the $\leq$-filtration of $M$. By this we mean that for each $A \in \mathring{\mathcal{V}}$ we have

$$M_{\leq A} = (\ker f^\infty \cap M_{\leq A}) \oplus (\text{im } f^\infty \cap M_{\leq A}).$$

It is clear that the right hand side is contained in the left hand side, so we only have to show the reverse inclusion.

For this we consider the restriction of $f$ to $M_{\leq A}$. This yields an endomorphism $f_{\leq A} : M_{\leq A} \to M_{\leq A}$. The same arguments as before give us a decomposition

$$M_{\leq A} = \ker(f_{\leq A})^\infty \oplus \text{im } (f_{\leq A})^\infty.$$

Clearly, $\ker(f_{\leq A})^\infty = \ker f^\infty \cap M_{\leq A}$ and $\text{im } (f_{\leq A})^\infty \subset \text{im } f^\infty \cap M_{\leq A}$. Hence

$$M_{\leq A} = \ker(f_{\leq A})^\infty \oplus \text{im } (f_{\leq A})^\infty \subset (\ker f^\infty \cap M_{\leq A}) \oplus (\text{im } f^\infty \cap M_{\leq A}),$$

which is the reverse inclusion that we wanted to show. Hence for any endomorphism $f$ of $M$ in $\mathbb{Z}\text{-mod}^{\leq}$ we obtain a Fitting-decomposition

$$M = \ker f^\infty \oplus \text{im } f^\infty$$

in $\mathbb{Z}\text{-mod}^{\leq}$. The usual arguments now yield the Krull-Schmidt theorem.

**Theorem 4.19.**

1. Each $M \in \mathbb{Z}\text{-mod}^{\leq}$ admits an up to isomorphism and rearrangement unique decomposition into indecomposables.

2. Each endomorphism of an indecomposable object $M$ in $\mathbb{Z}\text{-mod}^{\leq}$ is either an automorphism or nilpotent. In particular, $\operatorname{End}_{\mathbb{Z}\text{-mod}^{\leq}}(M)$ is a local ring.
Proof. We first prove the second statement. So let $M$ be an indecomposable object in $\mathcal{Z}\text{-mod}^\otimes$ and $f: M \to M$ an endomorphism that is not an automorphism. As $f$ decomposes into a direct sum of endomorphisms of finite dimensional vector spaces, we deduce that $\ker f \neq 0$. Hence $\ker f^\infty \neq 0$. As $M$ is indecomposable, $M = \ker f^\infty$. As $M$ is finitely generated $M = \ker f^n$ for some $n \gg 0$, so $f$ is nilpotent.

Standard arguments can now be used to show that $\text{End}_{\mathcal{Z}\text{-mod}^\otimes}(M)$ is a local ring, and from this part (1) of the theorem can be deduced, again using standard arguments. □

5. Twisted automorphisms of $G$-moment graphs

Suppose that $\mathcal{G}$ is a $G$-moment graph and $\sigma: \mathcal{G} \to \mathcal{G}$ is a twisted automorphism. We say that $\sigma$ normalizes the $G$-action if for any $x \in \mathcal{V}$ we have $\sigma_\mathcal{V}(x) \in G.\sigma_\mathcal{V}(x)$.

Lemma 5.1. Suppose that $\sigma: \mathcal{G} \to \mathcal{G}$ normalizes the $G$-action on $\mathcal{G}$. Then $\sigma$ induces a twisted automorphism $\overline{\sigma}: \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ on the quotient graph. In particular, we obtain an algebra automorphism $\overline{\sigma}: \overline{\mathcal{Z}} \to \overline{\mathcal{Z}}$.

Proof. The assumption on $\sigma$ implies that $\sigma$ induces a bijection on the vertex set of $\overline{\mathcal{G}}$, and together with the lattice automorphism we obviously obtain a moment graph automorphism. □

5.1. Twisted $\overline{\mathcal{Z}}$-modules. Let $M$ be a $\overline{\mathcal{Z}}$-module (with $\rho^M: \overline{\mathcal{Z}} \to \text{End}_\mathcal{Z}(M)$ as the structural morphism). We denote by $M^\overline{\sigma}$ the $\overline{\mathcal{Z}}$-module that we obtain by twisting the $\overline{\mathcal{Z}}$-structure with $\overline{\sigma}$, i.e. we have $M = M^\overline{\sigma}$ as an abelian group, and $\rho^{M^\overline{\sigma}} = \rho \circ \overline{\sigma}: \overline{\mathcal{Z}} \to \text{End}_\mathcal{Z}(M^\overline{\sigma})$ as the structural map.

Let $L \subset X$ be a saturated subset. As $\sigma_X: X \to X$ is a lattice automorphism, the subset $\sigma_X(L)$ is again saturated in $X$. Moreover, $\sigma$ induces an isomorphism $\sigma^L: \mathcal{G}^L \to \mathcal{G}^{\sigma_X(L)}$ of subgraphs of $\mathcal{G}$. In particular, we obtain a bijection $C(\mathcal{G}^L) \to C(\mathcal{G}^{\sigma_X(L)}), \Gamma \mapsto \sigma_\mathcal{V}(\Gamma)$ between the sets of connected components.

Lemma 5.2. We have a canonical identification

$$(M^L)^{\overline{\sigma}} = (M^\overline{\sigma})^{\sigma_X(L)}.$$ 

Proof. The statement follows from the fact that $\alpha \in X$ acts on $M^{\overline{\sigma}}$ as multiplication by $\sigma_X^{-1}(\alpha)$. □

5.2. Twisting functors on $\overline{\mathcal{C}}^{\leq}$. Now suppose that $\sigma: \mathcal{G} \to \mathcal{G}$ normalizes the $G$-action and is, in addition, order preserving. Let $M$ be an object in $\overline{\mathcal{Z}}\text{-mod}^{\leq}$. We now define a $\leq$-filtration on $M^{\overline{\sigma}}$ as follows. Let $I \subset \mathcal{V}$ be a $\leq$-closed subset. Then $\sigma_\mathcal{V}^{-1}(I)$ is closed as well and we set

$$(M^{\overline{\sigma}})_I := (M_{\sigma_\mathcal{V}^{-1}(I)})^{\overline{\sigma}}.$$
For the open complement $J$ of $I$ we then have $(M^\mathfrak{I})^J = (M^{\sigma^{-1}(J)})^\mathfrak{I}$. It is clear that this data satisfies the properties (P0-2).

**Lemma 5.3.** We have $\text{supp } M^\mathfrak{I} = \sigma_{\mathcal{V}}(\text{supp } M)$.

**Proof.** We have $A \in \text{supp } M^\mathfrak{I}$ if and only if $(M^\mathfrak{I})_{\sigma A} \neq (M^\mathfrak{I})_{\leq A}$. By definition this means $M_{\sigma^{-1}(\leq A)} \neq M_{\sigma^{-1}(\leq A)}$. The latter is the case if and only if $\sigma^{-1}(A) \in \text{supp } M$. \qed

**Lemma 5.4.**

1. The filtration defined above is an admissible $\preceq$-filtration, so we obtain a functor $\overline{\mathcal{T}}_\sigma: \mathcal{Z}\text{-mod}^\preceq \to \mathcal{Z}\text{-mod}^\preceq$.

2. The functor $\overline{\mathcal{T}}_\sigma$ preserves the subcategory $\mathcal{C}^\preceq$ and its restriction is exact.

**Proof.** We start with the proof of part (1). We have to check property (P3). Let $L \subset X$ be a saturated subset. Then $L' = \sigma^{-1}(L)$ is saturated again. Let $\Gamma$ be a connected component of $\mathcal{G}^L$. Then $\Gamma' = \sigma^{-1}(\Gamma)$ is a connected component of $\mathcal{G}^{L'}$. As the filtration on $M$ satisfies property (P3) there is a decomposition

$$M_{L',\Omega'} = \bigoplus_{\Omega' \in \pi^{-1}(\Gamma')} M_{L',\Omega'}$$

with $\text{supp } M_{L',\Omega'} \subset \Omega'$. Twisting this decomposition yields

$$(M^\mathfrak{I})^L = (M_{L'})^\mathfrak{I} = \bigoplus_{\Omega' \in \pi^{-1}(\Gamma')} (M_{L',\Omega'})^\mathfrak{I}.$$ 

By Lemma 5.3 we have $\text{supp } (M_{L',\Omega'})^\mathfrak{I} = \sigma_{\mathcal{V}}(\text{supp } M_{L',\Omega'}) \subset \sigma_{\mathcal{V}}(\Omega')$. As $\sigma_{\mathcal{V}}(\Omega')$ runs through the elements of $\pi^{-1}(\Gamma)$ as $\Omega'$ runs through $\pi'^{-1}(\Gamma')$, this is what we wanted to show.

Now we prove part (2). Suppose that $M$ is an object in $\mathcal{C}^\preceq$. If $J \subset \mathcal{V}$ is an open subset, then $\sigma^{-1}(J)$ is open as well. As $\mathfrak{I}$ induces an automorphism of $\mathfrak{S}$, we deduce that $(M^{\sigma^{-1}(J)})^\mathfrak{I}$ is a graded free $\mathfrak{S}$-module. This coincides with $(M^\mathfrak{I})^J$. So $M^\mathfrak{I}$ is contained in $\mathcal{C}^\preceq$ as well. Similar arguments show that the functor $M \mapsto M^\mathfrak{I}$ is exact on $\mathcal{C}^\preceq$. \qed

5.3. **Duality functors on $\mathcal{C}^\preceq$.** Suppose now that $\sigma: \mathcal{G} \to \mathcal{G}$ normalizes the $G$-action and is, in addition, order reversing and an involution (i.e., an automorphism with $\sigma^2 = \text{id}_\mathcal{G}$). Let $M \in \mathcal{Z}\text{-mod}^\preceq$. We define the object $\overline{\mathcal{D}}_\sigma M \in \mathcal{Z}\text{-mod}^\preceq$ as follows. As a $\mathfrak{S}$-module, we set

$$\overline{\mathcal{D}}_\sigma M := \text{Hom}_{\mathfrak{S}}(M, \mathfrak{S})^\mathfrak{I}.$$ 

Suppose that $I \subset \mathcal{V}$ is a $\preceq$-closed subset with open complement $J$. Then $\sigma^{-1}(I)$ is open with closed complement $\sigma^{-1}(J)$. We define the inclusion $(\overline{\mathcal{D}}_\sigma M)_I \to \overline{\mathcal{D}}_\sigma M$ as the dual of the quotient $M \to M^{\sigma^{-1}(I)}$. In particular, we have

$$(\overline{\mathcal{D}}_\sigma M)_I = (M^{\sigma^{-1}(I)})^\ast.$$
This is the space of graded $S$-homomorphisms from $M$ to $S$ that vanish on $M_{\sigma^{-1}(J)}$. By what we observed in Section 4.6 this data satisfies the properties (P0-2).

**Lemma 5.5.** We have $\text{supp} \overline{D}_\sigma M = \sigma_{\mathcal{V}}(\text{supp} M)$.

**Proof.** Easy exercise. □

**Lemma 5.6.**

1. The above defines an admissible $\leq$-filtration.
2. If $M$ is an object in $\mathcal{C}^\leq$, then so is $\overline{D}_\sigma M$ and the resulting functor $\overline{D}_\sigma : \mathcal{C}^\leq \rightarrow \mathcal{C}^\leq$ is exact.

**Proof.** The proof of property (1) follows the same arguments as the corresponding proof in Lemma 5.4. We only have to replace Lemma 5.3 by Lemma 5.5.

Now let us prove part (2). Let $J$ be an open subset of $\mathcal{V}$ with closed complement $I$. By the definition of the filtration we have $(\overline{D}_\sigma M)_J = (M_{\sigma^{-1}(J)})^*$. There is a short exact sequence

$$0 \rightarrow M_{\sigma^{-1}(J)} \rightarrow M \rightarrow M_{\sigma^{-1}(I)} \rightarrow 0.$$ 

As $M$ and $M_{\sigma^{-1}(I)}$ are supposed to be graded free $S$-modules, so is $M_{\sigma^{-1}(J)}$. Hence its dual is graded free as well. The exactness statement is proven similarly. □

6. The affine moment graph

From now on we fix a finite irreducible root system $R$ in a real finite dimensional vector space $V$. For any $\alpha \in R$ we denote by $\alpha^\vee \in V^* = \text{Hom}_R(V, \mathbb{R})$ the corresponding coroot. We denote by $R^\vee = \{\alpha^\vee | \alpha \in R\}$ the coroot system in $V^*$. We also fix a system $R^+ \subset R$ of positive roots and denote by $\Delta \subset R^+$ the corresponding set of simple roots. The positive coroots are then given by $R^{\vee,+} = \{\alpha^\vee \in R^\vee | \alpha \in R^+\}$, and the simple coroots by $\Delta^\vee = \{\alpha^\vee \in R^{+,\vee} | \alpha \in \Delta\}$. We denote by $X \subset V$ the weight lattice for the above data, i.e.

$$X = \{\lambda \in V | \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha^\vee \in R^\vee\}.$$ 

Here, and in the following, we denote by $\langle \cdot, \cdot \rangle : V \times V^* \rightarrow \mathbb{R}$ the natural pairing. We have $\mathbb{Z}R \subset X$.

**6.1. Alcove geometry.** For any $\alpha^\vee \in R^\vee$ and $n \in \mathbb{Z}$ we define

$$H_{\alpha^\vee,n} := \{\mu \in V | \langle \mu, \alpha^\vee \rangle = n\},$$

$$H_{\alpha^\vee,n}^+ := \{\mu \in V | \langle \mu, \alpha^\vee \rangle > n\},$$

$$H_{\alpha^\vee,n}^- := \{\mu \in V | \langle \mu, \alpha^\vee \rangle < n\}.$$ 

We call the $H_{\alpha^\vee,n}$ reflection hyperplanes. We clearly have $H_{\alpha^\vee,n}^{(\pm)} = H_{-\alpha^\vee,-n}^{(\mp)}$. 

Definition 6.1. • The connected components of \( V \setminus \bigcup_{\alpha^\vee \in R^+, n \in \mathbb{Z}} H_{\alpha^\vee, n} \) are called \textit{alcoves}. We denote by \( \mathcal{A} \) the set of alcoves.

• The connected components of \( V \setminus \bigcup_{\alpha^\vee \in \Delta^+, n \in \mathbb{Z}} H_{\alpha^\vee, n} \) are called \textit{boxes}. We denote by \( \mathcal{B} \) the set of boxes.

Let \( A \in \mathcal{A} \) be an alcove. Then there is a unique set \( \{n_{\alpha^\vee}\}_{\alpha^\vee \in R^+, +} \) of integers such that

\[
A = \{ \mu \in V \mid n_{\alpha^\vee} < \langle \mu, \alpha^\vee \rangle < n_{\alpha^\vee} + 1 \text{ for all } \alpha^\vee \in R^+, + \}.
\]

The lower closure of \( A \) is then defined as

\[
\underline{A} := \{ \mu \in V \mid n_{\alpha^\vee} \leq \langle \mu, \alpha^\vee \rangle < n_{\alpha^\vee} + 1 \text{ for all } \alpha^\vee \in R^+, + \},
\]

and the upper closure as

\[
\overline{A} := \{ \mu \in V \mid n_{\alpha^\vee} < \langle \mu, \alpha^\vee \rangle \leq n_{\alpha^\vee} + 1 \text{ for all } \alpha^\vee \in R^+, + \}.
\]

Let \( \lambda \) be an element in \( X \). The statements in the following definitions are clear.

Definition 6.2. • There is a unique alcove \( A_\lambda^+ \) that contains \( \lambda \) in its lower closure, and a unique alcove \( A_\lambda^- \) that contains \( \lambda \) in its upper closure.

• We say that \( A \in \mathcal{A} \) is a \textit{special} alcove if \( A = A_{\lambda}^- \) for some \( \lambda \in X \).

• We call \( A_0^+ \) the \textit{fundamental alcove}.

• There is a unique box \( \Pi_{\lambda} \) that contains \( A_{\lambda}^- \).

6.2. \textbf{Translations}. For \( \lambda \in X \) we denote by \( t_\lambda : V \to V \) the affine translation \( \mu \mapsto \lambda + \mu \). We clearly have

\[
t_\lambda(H_{\alpha^\vee, n}) = H_{\alpha^\vee, n + \langle \lambda, \alpha^\vee \rangle}, \quad t_\lambda(H_{\alpha^\vee, n}^\pm) = H_{\alpha^\vee, n + \langle \lambda, \alpha^\vee \rangle}^\pm.
\]

Hence \( t_\lambda \) induces bijections

\[
t_{\lambda, \mathcal{A}} : \mathcal{A} \to \mathcal{A}, \quad t_{\lambda, \mathcal{B}} : \mathcal{B} \to \mathcal{B}
\]

and we clearly have \( t_{\lambda, \mathcal{A}}(A_\mu^\pm) = A_{\lambda + \mu}^\pm \) and \( t_{\lambda, \mathcal{B}}(\Pi_{\mu}) = \Pi_{\lambda + \mu} \).

6.3. \textbf{The (extended) affine Weyl group}. We denote by

\[
s_{\alpha^\vee, n} : V \to V
\]

the affine reflection with fixed point hyperplane \( H_{\alpha^\vee, n} \). Note that each reflection \( s_{\alpha^\vee, n} \) stabilizes the weight lattice \( X \) and the root lattice \( \mathbb{Z} \Delta \). The affine Weyl group is the subgroup \( \hat{W} \subset \text{Aff}(V) \) generated by the affine reflections \( s_{\alpha^\vee, n} \) for \( \alpha^\vee \in R^\vee \) and \( n \in \mathbb{Z} \). We denote by \( \hat{S} \) the reflections at hyperplanes that have a codimension 1 intersection (in \( V \)) with the closure of the fundamental alcove \( A_0^+ \).

Then \( (\hat{W}, \hat{S}) \) is a Coxeter system.

The affine Weyl group permutes the set of alcoves \( \mathcal{A} \), and \( \mathcal{A} \) is a principal homogeneous set for this action. That means that the map \( \hat{W} \to \mathcal{A}, w \mapsto A_w := \)
$w(A^+_0)$ is a bijection. Via this bijection, the natural right action of $\hat{W}$ on itself yields a right action of $\hat{W}$ on $\mathcal{A}$ (i.e. $A_w = A_{xw}$).

The finite Weyl group is the subgroup $\mathcal{W}$ of $\hat{W}$ that is generated by the linear reflections, i.e. by the $s_{\alpha^\vee,0}$ with $\alpha^\vee \in R^\vee$. For an element $w$ in the finite Weyl group $\mathcal{W}$ we have

$$w(H_{\alpha^\vee,n}) = H_{w(\alpha^\vee),n},$$

$$w(H_{\alpha^\vee,n}^\pm) = H_{w(\alpha^\vee),n}^\pm,$$

for $\alpha^\vee \in R^{\vee,+}$ and $n \in \mathbb{Z}$.

The finite Weyl group normalizes the subgroup $X$ in Aff$(V)$ (embedded as the group of translations $\{t_\lambda\}_{\lambda \in X}$), as we have

$$w \circ t_\lambda \circ w^{-1} = t_{w(\lambda)}$$

for $w \in \hat{W}$ and $\lambda \in X$. The root lattice $\mathbb{Z}R \subset X$ is contained in $\hat{W}$, as

$$s_{\alpha^\vee,1}s_{\alpha^\vee,0} = t_\alpha$$

for each $\alpha^\vee \in R^\vee$. So $\mathbb{Z}R$ is a normal subgroup and we have

$$\hat{W} = \mathcal{W} \ltimes \mathbb{Z}R.$$ The extended affine Weyl group is the semidirect product $\hat{W}^{ext} = \mathcal{W} \ltimes X$ inside Aff$(V)$. Hence it contains the affine Weyl group. As both $X$ and $\mathcal{W}$ stabilize the set of reflection hyperplanes, the elements of the extended affine Weyl group naturally act on the set of alcoves and the set of walls.

Let $\lambda \in X$.

**Definition 6.3.**

- We define $\mathcal{W}_\lambda$ as the stabilizer of $\lambda$ in $\hat{W}$.
- We define $\mathcal{S}_\lambda \subset \mathcal{W}_\lambda$ as the set of reflections at hyperplanes that contain a wall of $A^+_\lambda$. Then $(\mathcal{W}_\lambda, \mathcal{S}_\lambda)$ is a finite Coxeter system. We denote by $w_\lambda$ the element in $\mathcal{W}_\lambda$ of maximal (Coxeter-)length. We then have $w_\lambda(A^+_\lambda) = A^-_\lambda$.

The stabilizer $\mathcal{W} = \mathcal{W}_0$ of $0 \in V$ is the finite Weyl group. We denote by $\mathcal{A}^\circ = \mathcal{W}(A^+_0) = \mathcal{W}(A^-_0)$ the set of alcoves that contain 0 in their closure.

**6.4. Partial orders on $\mathcal{A}$.** As before we consider the bijection $\hat{W} \overset{\sim}{\rightarrow} \mathcal{A}$ that is given by the map $w \mapsto A_w = w(A^+_0)$. We denote the inverse map by $A \mapsto w_A$. We denote by $\leq$ the partial order that we obtain from the Bruhat order on the Coxeter system $(\hat{W}, \mathcal{S})$ by transport of structure along the above map.

We denote by $\leq$ the partial order on $\mathcal{A}$ that is generated by the relations $A \leq s_{\alpha^\vee,n}(A)$ if $A \subset H_{\alpha^\vee,n}$ (and hence $s_{\alpha^\vee,n}(A) \in H_{\alpha^\vee,n}^+$) for a positive coroot $\alpha^\vee$ and $n \in \mathbb{Z}$.

**Remark 6.4.** Note that the orders $\leq$ and $\leq$ coincide in the fundamental chamber, and on the subset $\mathcal{A}^\circ$ they are opposite.
Note that the translations $t_{\lambda, A} : A \to A$ preserve the order $\preceq$, i.e. we have $A \preceq B$ if and only if $t_\gamma(A) \preceq t_\gamma(B)$ for all $\gamma \in X$.

For the following result, see \cite{Lus80b}.

**Lemma 6.5.** Let $A$ be an alcove and $s \in \hat{S}$ and assume that $A \preceq As$.

1. The set $\{A, As\}$ is an interval, i.e. for any $B \in \mathcal{A}$ with $A \preceq B \preceq As$ we have $B = A$ or $B = As$.
2. For $B \in \mathcal{A}$ we have:
   (a) If $B \preceq As$, then $Bs \preceq As$.
   (b) If $A \preceq B$, then $A \preceq Bs$.

**6.5. Length functions.** For any $\lambda \in X$ let $\ell_\lambda : A \to \mathbb{Z}$ be the length function relative to $\lambda$ (cf. \cite{Lus80b}). For an alcove $A$, $\ell_\lambda(A)$ is obtained as follows. Fix a path $\gamma$ in the real vector space $V$ from the special alcove $A_{-\lambda}$ to $A$ (nothing will depend on this choice), then $\ell_\lambda(A) = -(m_+ - m_-)$, where $m_+$, respectively $m_-$, is the number of hyperplanes that $\gamma$ crosses from negative to positive side, respectively positive to negative side. Clearly, $\ell_\lambda(A^+_\lambda) = -l(w_\lambda)$ (here $l$ is the length function of the Coxeter system $(W_\lambda, S_\lambda)$), and if $s$ is an affine simple reflection such that $A \prec As$, then $\ell_\lambda(As) = \ell_\lambda(A) - 1$.

Note that here we use a convention that is opposite to the one used in \cite{Lus80b} or \cite{Soe97}.

**6.6. The map $\alpha \uparrow : \mathcal{A} \to \mathcal{A}$.** Let $\alpha \in \mathbb{R}^+$ and $A \in \mathcal{A}$ an arbitrary alcove. Then there is a unique $n \in \mathbb{Z}$ such that

$$A \subset H_{\alpha, n}^- \cap H_{\alpha, n-1}^+.$$  

We define $\alpha \uparrow A = s_{\alpha, n}(A)$. Then $A \prec \alpha \uparrow A$, and $\alpha \uparrow : \mathcal{A} \to \mathcal{A}$ is a bijection. We denote the inverse map by $\alpha \downarrow : \mathcal{A} \to \mathcal{A}$.

**6.7. A linearization.** Let $\hat{X}^\vee = X^\vee \oplus \mathbb{Z}$ be the affine coweight lattice. We will now define an action of the extended affine Weyl group on $\hat{X}^\vee$. First, we define the action of an element in the finite Weyl group $w \in W$ by

$$w.(v, n) = (w(v), n).$$

Secondly, we define the action of an element $\lambda$ in the weight lattice $X$ by

$$\lambda.(v, n) = (v, n + (\lambda, v)).$$

We then have

$$w \circ \lambda \circ w^{-1}(v, n) = (v, n + (\lambda, w^{-1}v))$$

$$= (v, n + w(\lambda, v))$$

$$= w(\lambda),(v, n),$$

so the above actions combine to give a linear action of the extended affine Weyl group on $\hat{X}^\vee$. 

6.8. The affine moment graph $\hat{G}$ and the periodic moment graph $\tilde{G}$. Now we define the two moment graphs that we are interested in in this paper.

**Definition 6.6.** The affine moment graph $\hat{G}$ is defined as follows.
- Its set of vertices is $A$.
- The vertices $A$ and $B$ are connected by a single edge if and only if there is $\alpha \in R^+$ and $n \in \mathbb{Z}$ with $B = s_{\alpha, n}(A)$.
- The edge $A - s_{\alpha, n}(A)$ is labeled by $\pm(\alpha, n) \in \hat{X}^\vee/(\pm 1)$.

Note that we would like to consider $\hat{G}$ as a $\mathbb{Z}R$-moment graph by considering the action of the root lattice on the set $A$. But, as we will see, these are genuinely twisted automorphisms, so we first have to reduce the labels on $\hat{G}$ using the lattice homomorphism $\hat{X}^\vee \rightarrow X^\vee$, $(v, n) \mapsto (v, n) := v$. This lattice homomorphism induces a $k$-algebra homomorphism $\hat{S} \rightarrow S$ that we also denote by $f \mapsto f$. From now on we will consider $S$ as an $\hat{S}$-algebra via this homomorphism. For convenience, we abbreviate $\delta^\vee = (0, 1) \in \hat{X}^\vee$, so the kernel of the homomorphism $\cdot$ is $\mathbb{Z}\delta^\vee$. For any $\hat{S}$-module $M$ we define

$$M_{\delta^\vee = 0} := M \otimes_{\hat{S}} S.$$ 

The following is the definition of the moment graph that we obtain from $\hat{G}$ by “setting $\delta^\vee = 0$.”

**Definition 6.7.** The moment graph $\tilde{G}$ is the following moment graph over the lattice $X^\vee$:
- Its set of vertices is the set of alcoves $A$. This set is partially ordered by the generic Bruhat order.
- The vertices $A, B \in A$ are connected by an edge if there is $\alpha \in R^+$ and $n \in \mathbb{Z}$ with $B = s_{\alpha, n}A$.
- The edge $A - s_{\alpha, n}A$ is labeled by $\pm\alpha \in X^\vee/(\pm 1)$.

We denote by $\hat{Z} = \mathbb{Z}(\hat{G})$ and by $\tilde{Z} = \mathbb{Z}(\tilde{G})$ the structure algebras of $\hat{G}$ and $\tilde{G}$ over the field $k$. Note that $\hat{Z}$ is a $\hat{S}$-algebra, while $\tilde{Z}$ is an $S$-algebra.

**Remark 6.8.** In the following we will consider $\hat{G}$ and $\tilde{G}$ as ordered moment graphs via the partial order which is opposite to the generic one $\succeq$, so that, for any $A \in A$, the sets $\{\succ A\}$ and $\{\succeq A\}$ are closed in the induced topology. Observe that, according to this choice, a maximal element of a $\succeq$-closed subset is in fact a minimal element with respect to the generic Bruhat order. In the case of $\hat{G}$, we will at one point also use the Bruhat order $\preceq$.

6.9. Properties of $\preceq$. We will need the following simple result.

**Lemma 6.9.** Let $s \in \hat{S}$.

1. Let $\mathcal{I}$ be a $\succeq$-closed subset of $A$. Then $\mathcal{I} \cup Is$ is a $\succeq$-closed subset of $A$ as well.
(2) Let $A$ be an alcove with $A \succ As$ and let $\mathcal{I}$ be a $\succeq$-closed subset that contains $A$ as a $\succeq$-minimal element. Then $As$ is $\succeq$-minimal in $(\mathcal{I} \cup \mathcal{I}s) \setminus \{As\}$.

Proof. Let us prove the first statement. Let $C \in \mathcal{I}s$ and suppose that $D \succeq C$. We need to show that $D \in \mathcal{I} \cup \mathcal{I}s$. If $C \succ C$s, then $D \succeq Cs$. From $Cs \in \mathcal{I}$ we deduce $D \in \mathcal{I}$. If $Cs \succ C$ then $Ds \succeq C$ by part (2b) of Lemma 6.5. Now we can replace $D$ with $Ds$ if necessary and assume that $D \succ Ds$. Then $D \succ Cs$ by part (2b) of Lemma 6.5. Hence $D \in \mathcal{I}$.

Now we prove the second statement. Suppose that $B \in \mathcal{I} \cup \mathcal{I}s$ and $As \succ B$. Then $A \succ B$ and part (2a) of Lemma 6.5 implies $A \succ Bs$. But either $B$ or $Bs$ is contained in $\mathcal{I}$, which contradicts the minimality of $A$ in $\mathcal{I}$. Finally, suppose there is $B \in \mathcal{I} \cup \mathcal{I}s$ such that $A \succ B$. We can assume $Bs \prec B$, so $As \succeq B$ by part (2a) of Lemma 6.5, so $B = As$. \hfill \qed

6.10. Twisted automorphisms of $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}}$. In Section 6.3 we showed that the extended affine Weyl group naturally acts on the set of alcoves $\mathcal{A}$. In Section 6.7 we showed that it also acts on $\hat{X}^\vee$ by lattice automorphisms. For $w \in \hat{\mathcal{W}}_{ext}$ let us denote the corresponding maps by $\sigma_{w,\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ and $\sigma_{w,\hat{X}^\vee} : \hat{X}^\vee \to \hat{X}^\vee$.

Lemma 6.10. (1) For each $w \in \hat{\mathcal{W}}_{ext}$ the pair $\sigma_w = (\sigma_{w,\mathcal{A}}, \sigma_{w,\hat{X}^\vee})$ defines a twisted automorphism of $\hat{\mathcal{G}}$.
(2) If $w = t_\lambda$ is an affine translation, then $\sigma_{t_\lambda} : \hat{\mathcal{G}} \to \hat{\mathcal{G}}$ preserves the order $\succeq$.
(3) For each $\lambda \in \hat{X}$, the twisted automorphism $\sigma_{w_\lambda}$ reverses the order $\succeq$.

Hence, we obtain for any $\lambda \in \hat{X}$ a twisting functor $T_{t_\lambda}$ and a duality functor $D_{w_\lambda}$ on $\hat{\mathcal{C}}^\succeq$. In order to simplify and harmonize the notation, we will write $\hat{T}_{t_\lambda}$ and $\hat{D}_{w_\lambda}$ for these functors in the following. Now the twisted automorphisms that we considered above descend to twisted automorphisms on $\hat{G}$:

Lemma 6.11. (1) For each $w \in \hat{\mathcal{W}}_{ext}$ the twisted automorphism $\sigma_w : \hat{\mathcal{G}} \to \hat{\mathcal{G}}$ descends to a twisted automorphism on $\hat{\mathcal{G}}$.
(2) If $\lambda \in \hat{X}$ and $w = t_\lambda$ is the respective affine translation, then the corresponding automorphism $\sigma_{w_\lambda} : \hat{\mathcal{G}} \to \hat{\mathcal{G}}$ is untwisted.

Proof. Part (1) follows from the fact that for each $w \in \hat{\mathcal{W}}_{ext}$ we have $w(\delta^\vee) = \delta^\vee$ (this is an immediate consequence of the formulas in Section 6.7). For part (2), note that $t_\lambda(\alpha^\vee + n\delta^\vee) = \alpha^\vee + (n + \langle \lambda, \alpha^\vee \rangle)\delta^\vee$, so the induced linear map $X^\vee \to X^\vee$ is the identity. \hfill \qed

In particular, we have the root lattice $\mathbb{Z}R$ acting on $\hat{\mathcal{G}}$ by untwisted automorphisms. From now on we consider $\hat{\mathcal{G}}$ as a $\mathbb{Z}R$-moment graph. We denote the quotient graph $\bar{\mathcal{G}} = \mathbb{Z}R \backslash \hat{\mathcal{G}}$, and we denote its structure algebra by $\bar{Z}$. We denote by $\bar{\mathcal{C}}^\succeq$ the category of objects with a Verma flag in $\mathbb{Z}R$-mod$^\succeq$. 
6.11. The structure algebras of $\hat{G}$, $\tilde{G}$ and $\overline{G}$. Before we can compare the structure algebras $\hat{Z}$, $\tilde{Z}$ and $\overline{Z}$ we need to introduce some homomorphisms.

We denote the projection $\hat{X}^\vee \to X^\vee$ with kernel $\mathbb{Z}\delta^\vee$ by $v \mapsto \tau$. It induces a surjective algebra homomorphism $\hat{S} \to S$ that we also denote by $f \mapsto \overline{f}$. The product homomorphism $\prod_{A \in \mathcal{A}} \hat{S} \to \prod_{A \in \mathcal{A}} S$ (the product of the projections $\hat{S} \to S$) induces, clearly, a homomorphism
\[
\pi: \hat{Z} \to \tilde{Z}.
\]
of algebras.

We denote by $\tau: \hat{S} \to \hat{Z}$ the inclusion of the constant subalgebra $\hat{S}$ on the diagonal, i.e. $\tau(f) = (z_A)$ with $z_A = f$ for all $w \in \hat{W}$. There is also a twisted embedding $\eta$ of $\hat{S}$ into $\tilde{Z}$. For $\lambda \in \hat{X}^\vee$ we define $\eta(\lambda) = (\eta(\lambda)_A)_{A \in \mathcal{A}}$ by $\eta(\lambda)_A = w_A(\lambda)$.

It is easy to check that $\eta(\lambda) \in \tilde{Z}$. So we obtain a $\mathbb{Z}$-linear homomorphism $X^\vee \to \tilde{Z}$ that induces a homomorphism $\eta: \hat{S} \to \tilde{Z}$ of $k$-algebras. We will consider the tensor product
\[
\hat{S} \otimes_k \hat{S} \xrightarrow{\tau \otimes \eta} \tilde{Z}.
\]

Let $\tilde{G}^0 \subset \tilde{G}$ be the full submoment graph with vertices $\mathcal{A}^0$. Then $\tilde{G}^0$ can be identified with the finite moment graph $\mathcal{G}$ associated with the root system $R$. We denote by $\tilde{Z}^0$ its structure algebra over the lattice $X^\vee$.

Here is our main result on the structure algebras.

**Lemma 6.12.** (1) The kernel of the homomorphism $\pi: \hat{Z} \to \tilde{Z}$ is $\tilde{Z}\delta^\vee$.

(2) If $\operatorname{ch} k$ is good for $R^\vee$, then the subalgebra $\overline{Z} = \tilde{Z}^{Z R}$ of $\tilde{Z}$ is contained in the image of $\pi$.

(3) If $\operatorname{ch} k$ is good for $R^\vee$, then the restriction induces an isomorphism $\tilde{Z}^{Z R} \to \tilde{Z}^0$.

**Proof.** We prove part (1). Let $(z_B) \in \tilde{Z}$ be in the kernel. Then $z_B$ is divisible by $\delta^\vee$ in $\hat{S}$ for each $B \in \mathcal{A}$. Define $z'_B := (\delta^\vee)^{-1}z_B \in \hat{S}$. We now show that $z' = (z'_B) \in \prod_{B \in \mathcal{A}} \hat{S}$ is contained in $\tilde{Z}$ as well. So let $B \dashv C$ be an edge of $\tilde{Z}$. Then $z_B - z_C$ is divisible by the label of the edge, i.e. by $\alpha^\vee + n \delta^\vee$ for some $\alpha^\vee \in R^\vee$ and $n \in \mathbb{Z}$. As $\alpha^\vee + n \delta^\vee$ and $\delta^\vee$ are coprime elements in $\hat{S}$, also $z'_B - z'_C$ is divisible by $\alpha^\vee + n \delta^\vee$. So $z' \in \tilde{Z}$.

Now we prove parts (2) and (3). First, observe that the image of the composition
\[
\hat{S} \otimes \hat{S} \xrightarrow{\tau \otimes \eta} \tilde{Z} \xrightarrow{\pi} \tilde{Z}
\]
is contained in $\tilde{Z}^{Z R}$. Indeed, the image of $f \otimes g$ is the element $z = (z_A)$ with $z_A = w_A(f)g$. For $A \in \mathcal{A}$ and $\lambda \in Z R$ we have
\[
z_{t_A}(z) = w_{t_A}(f)g = \overline{t_A}w_A(f)g = w_A(f)g = z_A.
\]
Hence the image of the above map is contained in the $\mathbb{Z}R$-invariant subalgebra of $\tilde{Z}$.

Now note that the restriction homomorphism $\tilde{Z}^R \to \tilde{Z}$ is injective, as $\mathcal{A}^o$ is a set of representatives of the $\mathbb{Z}R$-orbits in $\mathcal{A}$. But by [Dem73], the composition $\tilde{S} \otimes \tilde{S} \to \tilde{Z}^R \to \tilde{Z}$ is surjective. Hence $\tilde{S} \otimes \tilde{S} \to \tilde{Z}^R$ is surjective as well.

6.12. A sufficient condition for projectivity.

**Proposition 6.13.** Let $L$ be a saturated subset of $X$. Suppose that $\overline{P}$ is an object in $\overline{C}^{L,\geq}$ that satisfies the following properties:

- As a $\overline{Z}^L$-module, $\overline{P}$ is projective.
- The support of $\overline{P}$ contains a unique minimal element $A$, and for each vertex $\Omega$ of $\overline{G}$ we either have $\supp \overline{P} \cap \pi^{-1}(\Omega) = \emptyset$ or $\supp \overline{P} \cap \pi^{-1}(\Omega) = \{ B \}$, where $B$ is the minimal alcove in $\{ \geq A \} \cap \pi^{-1}(\Omega)$.

Then $\overline{P}$ is projective in $\overline{C}^{L,\geq}$.

**Proof.** The projectivity of $\overline{P}$ follows from this statement: We have, functorially in $N \in \overline{C}^{L,\geq}$,

$$\text{Hom}_{\overline{C}^{L,\geq}}(\overline{P}, N) = \text{Hom}_{\overline{Z}^L}(\overline{P}, N_{\geq A}) = \text{Hom}_{\overline{Z}^L}(\overline{P}, N_{\geq A}).$$

Let us prove these identities. As the support of $\overline{P}$ is contained in $\{ \geq A \}$, the first identity is part of the definition of the category $\overline{C}^{L,\geq}$. So we need to show that the inclusion $\text{Hom}_{\overline{C}^{L,\geq}}(\overline{P}, N_{\geq A}) \subset \text{Hom}_{\overline{Z}^L}(\overline{P}, N_{\geq A})$ is a bijection. In other words, we have to show that each $\overline{Z}^L$-linear homomorphism $f: \overline{P} \to N_{\geq A}$ respects the $\geq$-filtration. But this is an easy consequence of the mimality condition that we supposed on the support of $\overline{P}$. \hfill \square

6.13. A restriction functor $\Upsilon : \overline{C}^{\geq} \to \overline{C}^{\geq}$. We now construct a functor $\Upsilon$ from $\hat{Z}^{\geq}$ to $\overline{Z}^{\geq}$ and we show that it restricts to a functor from $\overline{C}^{\geq}$ to $\overline{C}^{\geq}$.

For any $\hat{S}$-module $M$ we set $M_{\delta^v = 0} = M \otimes_{\overline{S}} \overline{S}$. Let $M$ be an object in $\hat{Z}^{\geq}$. Then $M_{\delta^v = 0}$ is a $\hat{Z}_{\delta^v = 0}$-module. By Lemma 6.12 this $S$-algebra is the image of the restriction functor $\hat{Z} \to \overline{Z}$ and it contains the subalgebra $\overline{Z} = \overline{Z}^R$. As a $\overline{Z}$-module we define $\Upsilon M$ to be $\text{Res}_{\overline{Z}_{\delta^v = 0}}^\overline{Z} M_{\delta^v = 0}$. Now note that $M$ carries a canonical $\leq$-filtration, and this induces a filtration on $M_{\delta^v = 0}$ (i.e. $(M_{\delta^v = 0})_{\overline{T}} := (M_{\overline{T}}) \otimes_{\overline{S}} \overline{S}$). If $M$ admits a $\geq$-Verma flag, then Proposition 3.21 says that this filtration satisfies the properties (P1), (P2) and (P3), hence we obtain an object $\Upsilon(M)$ in $\overline{Z}^{\geq}$. As we can identify the quotient $\Upsilon(M)_{\overline{T}}$ with $\text{Res}_{\overline{S}}^\overline{Z} M_{\overline{T}}$, the object $\Upsilon(M)$ admits a $\geq$ Verma flag and we obtain an exact (!) functor

$$\Upsilon : \overline{C}^{\geq} \to \overline{C}^{\geq}.$$
7. Translation Functors

In this section we define translation functors on module categories associated with the moment graphs $\mathcal{G}$ and $\mathcal{Y}$.

7.1. Translation Functors on $\mathcal{G}^<$ and $\mathcal{G}^>$. We now fix an element $s \in \mathcal{S}$ and consider its right action on the set $\mathcal{A}$. We call a subset $\mathcal{X}$ of $\mathcal{A}$ s-invariant if $\mathcal{X}s = \mathcal{X}$. Suppose $\mathcal{X}$ is such a subset. We then consider the $\mathcal{S}$-linear homomorphism $\gamma_s: \prod_{A \in \mathcal{X}} \mathcal{S} \rightarrow \prod_{A \in \mathcal{X}} \mathcal{S}$ that is given by $\gamma_s((z_A)) = (z'_A)$, where $z'_A = z_{As}$. Clearly, $\gamma_s^2 = id$. It is easy to check that $\gamma_s$ stabilizes the sub-$\mathcal{S}$-algebra $\hat{Z}(\mathcal{X})$ in $\bigoplus_{A \in \mathcal{X}} \mathcal{S}$, hence it defines and $\hat{\mathcal{S}}$-algebra automorphism on $\hat{Z}(\mathcal{X})$. We denote by $\hat{Z}(\mathcal{X})^s \subset \hat{Z}$ the sub-$\mathcal{S}$-algebra that contains all $\gamma_s$-invariant elements, i.e.

$$\hat{Z}(\mathcal{X})^s = \left\{(z_A) \in \hat{Z}(\mathcal{X}) \mid z_A = z_{As} \text{ for all } A \in \mathcal{X}\right\},$$

and by $\hat{Z}(\mathcal{X})^{-s}$ the sub-$\hat{Z}(\mathcal{X})^s$-module of s-antiinvariant elements, i.e.

$$\hat{Z}(\mathcal{X})^{-s} = \left\{(z_A) \in \hat{Z}(\mathcal{X}) \mid z_A = -z_{As} \text{ for all } A \in \mathcal{X}\right\}.$$

Let $\alpha^\vee \in \hat{\mathcal{X}}^\vee$ be the affine positive coroot associated with the simple reflection $s$. We denote by $c^s$ the image of $\eta(\alpha^\vee)$ under the homomorphism $\hat{Z} \rightarrow \hat{Z}(\mathcal{X})$. Clearly, $c^s \in \hat{Z}(\mathcal{X})^{-s}$.

Lemma 7.1. We have $\hat{Z}(\mathcal{X}) = \hat{Z}(\mathcal{X})^s \oplus \hat{Z}(\mathcal{X})^{-s}$, and $\hat{Z}(\mathcal{X})^{-s}$ is a graded free $\hat{Z}(\mathcal{X})^s$-module with generator $c^s$.

Let $M$ be an object in $\hat{Z}\text{-mod}^f$. Let $\mathcal{X} \subset \mathcal{A}$ be an s-invariant subset and suppose that supp $M \subset \mathcal{X}$. Then there is a unique $\hat{Z}(\mathcal{X})$-action on $M$ such that the $\hat{Z}$-action factors over the homomorphism $\hat{Z} \rightarrow \hat{Z}(\mathcal{X})$. As this latter map is compatible with the s-action on both algebras, we obtain a homomorphism

$$\hat{Z} \otimes \hat{Z}. M \rightarrow \hat{Z}(\mathcal{X}) \otimes \hat{Z}(\mathcal{X})^s. M.$$

Lemma 7.2. The above homomorphism is an isomorphism. In particular, there is an action of $\hat{Z}(\mathcal{X})$ on $\hat{Z} \otimes \hat{Z}. M$ such that the $\hat{Z}$-action factors over the homomorphism $\hat{Z} \rightarrow \hat{Z}(\mathcal{X})$.

Proof. By Lemma 7.1, both sides are identified (as S-modules) with $1 \otimes M \oplus c^s \otimes M$. □

Definition 7.3. The translation functor $\hat{\vartheta}_s: \hat{Z}\text{-mod}^f \rightarrow \hat{Z}\text{-mod}^f$ is given by $M \mapsto \hat{Z} \otimes \hat{Z}. \cdot [1]$.

As $\hat{Z}$ is free as a $\hat{Z^s}$-module, the tensor product functor $\hat{Z} \otimes \hat{Z}. \cdot$ preserves the property of being torsion free as an $\hat{S}$-module. Moreover, by Lemma 7.2 the
support of $\vartheta_s M$ is contained in the finite set $\text{supp } M \cup (\text{supp } M)_s$, so we indeed defined a functor from $\hat{Z} \text{-mod}^f$ to $\hat{Z} \text{-mod}^f$.

In Section 5 in [Fie08b] we showed that the translation functors preserve the subcategory $\hat{\mathcal{C}}^\leq$ of $\hat{Z} \text{-mod}^f$ of modules admitting a $\leq$-Verma flag. Furthermore, we showed that the resulting functors $\hat{\vartheta}_s: \hat{\mathcal{C}}^\leq \to \hat{\mathcal{C}}^\leq$ are exact. The arguments used in [Fie08b] can also be applied to the partial order $\succeq$ and yield the same results (cf. [Fie11]). Let us state this in a theorem.

**Theorem 7.4.** Let $s \in \hat{S}$.

1. The translation functor $\hat{\vartheta}_s$ preserves the subcategories $\hat{\mathcal{C}}^\leq$ and $\hat{\mathcal{C}}^\succeq$.
2. The resulting functors $\hat{\vartheta}_s: \hat{\mathcal{C}}^\leq \to \hat{\mathcal{C}}^\leq$ and $\hat{\vartheta}_s: \hat{\mathcal{C}}^\succeq \to \hat{\mathcal{C}}^\succeq$ are exact and $\text{self-adjoint}$.

### 7.2. Translation functors on $\hat{\mathcal{C}}^\succeq$

The aim of this section is to prove that the translation functors on $\hat{\mathcal{C}}^\succeq$ descend with respect to the functor $\Upsilon: \hat{\mathcal{C}}^\succeq \to \mathcal{C}^\succeq$. So the main result of this section is the following.

**Theorem 7.5.** For each $s \in \hat{S}$ there is a functor $\vartheta_s: \mathcal{C}^\succeq \to \mathcal{C}^\succeq$ such that the diagram

$$
\begin{array}{ccc}
\hat{\mathcal{C}}^\succeq & \xrightarrow{\Upsilon} & \mathcal{C}^\succeq \\
\hat{\vartheta}_s \downarrow & & \downarrow \vartheta_s \\
\hat{\mathcal{C}}^\succeq & \xrightarrow{\Upsilon} & \mathcal{C}^\succeq 
\end{array}
$$

naturally commutes.

The first step in the proof of the above result is the definition of $\vartheta_s$ on the level of $\mathbb{Z}$-modules. Note that the map $\mathcal{A} \to \mathcal{A}$, $A \mapsto As$ associated with $s \in \hat{S}$ commutes with the left $ZR$-action on $\mathcal{A}$, so it defines a map $\mathcal{A} \to \hat{\mathcal{A}}$, $\Omega \mapsto \Omega s$ on the $ZR$-orbits. For an $s$-invariant subset $\mathcal{X}$ of $\mathcal{A}$ we define the subalgebras $\overline{Z}(\mathcal{X})^s$ and $\overline{Z}(\mathcal{X})^{-s}$ of invariants and antiinvariants in $\overline{Z}(\mathcal{X})$ as before:

$$
\overline{Z}(\mathcal{X})^s := \{(z_\Omega) \in \bigoplus_{\Omega \in \mathcal{X}} S \mid z_\Omega = z_\Omega s \text{ for all } \Omega \in \mathcal{X}\},
$$

$$
\overline{Z}(\mathcal{X})^{-s} := \{(z_\Omega) \in \bigoplus_{\Omega \in \mathcal{X}} S \mid z_\Omega = -z_\Omega s \text{ for all } \Omega \in \mathcal{X}\}.
$$

The analogue of Lemma 7.1 in the quotient case is the following.

**Lemma 7.6.** Suppose that $\mathcal{X}$ is an $s$-invariant subset of $\overline{\mathcal{A}}$. Then we have a decomposition $\overline{Z}(\mathcal{X}) = \overline{Z}(\mathcal{X})^s \oplus \overline{Z}(\mathcal{X})^{-s}$, and $\overline{Z}(\mathcal{X})^{-s}$ is a free $\overline{Z}(\mathcal{X})^s$-module of graded rank $v^2$ (i.e. $\overline{Z}(\mathcal{X})^{-s} \cong \overline{Z}(\mathcal{X})^s[-2]$).

Let $M$ be an object in $\overline{Z} \text{-mod}^f$. We will now consider the $\overline{Z}$-module

$$
\vartheta_s(M) := \overline{Z} \otimes_{\overline{Z}^s} M[1].
$$
It follows from Lemma 7.6 that $\overline{\vartheta}(M)$ is again torsion free as an $S$-module.

Suppose that $\mathcal{A}$ is an $s$-invariant subset of $\overline{\mathcal{A}}$ that contains $\overline{\text{supp}}M$. Then there is a unique action of $\overline{\mathcal{Z}}(\mathcal{A})$ on $M$ such that $\overline{\mathcal{Z}}$ acts via the homomorphism $\overline{\mathcal{Z}} \to \overline{\mathcal{Z}}(\mathcal{A})$. Clearly, the $s$-invariants $\overline{\mathcal{Z}}^s$ act via the homomorphism $\overline{\mathcal{Z}}^s \to \overline{\mathcal{Z}}(\mathcal{A})^s$, and we obtain a canonical homomorphism $\overline{\mathcal{Z}} \otimes \overline{\mathcal{Z}}^s M \to \overline{\mathcal{Z}}(\mathcal{A}) \otimes \overline{\mathcal{Z}}(\mathcal{A})^s M$. By Lemma 7.6, $\overline{\mathcal{Z}}$ is a free $\overline{\mathcal{Z}}^s$-module of graded rank $1 + v^2$, and $\overline{\mathcal{Z}}(\mathcal{A})$ is a free $\overline{\mathcal{Z}}(\mathcal{A})^s$-module of graded rank $1 + v^2$. From this we immediately obtain the next result.

**Lemma 7.7.** Let $\mathcal{A}$ be an $s$-invariant subset of $\overline{\mathcal{A}}$ that contains $\overline{\text{supp}}M$. Then the natural homomorphism $\overline{\mathcal{Z}} \otimes \overline{\mathcal{Z}}^s M \to \overline{\mathcal{Z}}(\mathcal{A}) \otimes \overline{\mathcal{Z}}(\mathcal{A})^s M$ is an isomorphism. In particular we have $\overline{\text{supp}}\overline{\vartheta}sM \subset \overline{\text{supp}}M \cup (\overline{\text{supp}}M)^s$, and hence $\overline{\vartheta}sM$ is again an object in $\overline{\mathcal{Z}}$-mod$^f$. We now need to define a filtration.

### 7.3. An admissible $\succeq$-filtration on $\overline{\mathcal{Z}} \otimes \overline{\mathcal{Z}}^s M$.

Suppose that $M$ admits a $\succeq$-Verma flag. In this case we can endow $\overline{\vartheta}sM$ with a $\succeq$-filtration in the following way. For a closed subset $\mathcal{I}$ of $\mathcal{A}$ and a $\mathbb{Z}\mathcal{R}$-orbit $\Omega \in \overline{\mathcal{A}}$ we first define $\mathcal{I}_\Omega \subset \mathcal{A}$ as the closure of $(\mathcal{I} \cap \Omega) \cup (\mathcal{I} \cap \Omega)^s$ (this is the smallest $s$-invariant closed subset containing $\mathcal{I} \cap \Omega$). We consider $(\overline{\vartheta}_s(M)_{\mathcal{I}_\Omega})^{0,\Omega}$ as a subset in $(\overline{\vartheta}_s M)^{0,\Omega}$ and define

$$(\overline{\vartheta}_s M)_{\mathcal{I}} := \overline{\vartheta}_s M \cap \bigoplus_{\Omega \in \mathcal{A}} (\overline{\vartheta}_s(M)_{\mathcal{I}_\Omega})^{0,\Omega}.$$ 

Note that $\mathcal{I}'_\Omega \subset \mathcal{I}_\Omega$ if $\mathcal{I}' \subset \mathcal{I}$ and hence $(\overline{\vartheta}_s M)_{\mathcal{I}'} \subset (\overline{\vartheta}_s M)_{\mathcal{I}}$, so our definition satisfies property (P0). Also $(\mathcal{I}_1 \cap \mathcal{I}_2)_\Omega = (\mathcal{I}_1)_\Omega \cap (\mathcal{I}_2)_\Omega$, from which we deduce property (P2). We will now show that this filtration also satisfies properties (P1) and (P3), and that $\overline{\vartheta}_s M$ admits a $\succeq$-Verma flag. We begin with the following. Let $L \subset X$ be a saturated subset, and let $\Gamma$ be a connected component of $\overline{\mathcal{G}}^L$. Then $\Gamma s$ is a connected component of $\overline{\mathcal{G}}^L$ as well.

**Lemma 7.8.** Suppose that $\Gamma \neq \Gamma s$. If $M \in \overline{\mathcal{Z}}$-mod$^f$. Then $(\overline{\vartheta} M)^L_{\mathcal{I}} = M^L_{\mathcal{I}} \oplus M^L_{\mathcal{I},\Gamma s}$.

**Proof.** Note that $\mathcal{Z}(\Gamma) \oplus \mathcal{Z}(\Gamma s)$ is an $s$-stable direct summand of $\mathcal{Z}(\overline{\mathcal{G}}^L)$, and the projection of the subalgebra of $s$-invariants on either summand is an isomorphism.
We have
\[
\overline{\vartheta}_s M)^{L, \Gamma} = \mathcal{Z}(\Gamma)(\overline{\vartheta}_s M^L)
\]
\[
= \mathcal{Z}(\Gamma)(\mathcal{Z} \otimes_{\mathcal{Z}s} M^L)
\]
\[
= \mathcal{Z}(\Gamma)(\mathcal{Z}(\mathcal{G}^\Gamma) \otimes_{\mathcal{Z}(\mathcal{G}^\Gamma)s} M^L)
\]
\[
= \mathcal{Z}(\Gamma)(\mathcal{Z}(\Gamma) \oplus \mathcal{Z}(\Gamma s)) \otimes (\mathcal{Z}(\Gamma) \oplus \mathcal{Z}(\Gamma s))^s (M^{L, \Gamma} \oplus M^{L, \Gamma s})
\]
\[
= M^{L, \Gamma} \oplus M^{L, \Gamma s}.
\]

Lemma 7.9. If $\mathcal{I}$ is $s$-invariant, then we have $(\overline{\vartheta}_s M)_{\mathcal{I}} = \overline{\vartheta}_s (M_{\mathcal{I}})$.

Proof. Note that if $\mathcal{I}$ is $s$-invariant we have $\mathcal{I} \cap \Omega \subset \mathcal{I}$ and $\mathcal{I} \cap \Omega s \subset \mathcal{I}$, so $\mathcal{I}_{\Omega} \subset \mathcal{I}$. Hence $(\overline{\vartheta}_s M_{\mathcal{I}_{\Omega}})^{0, \Omega} \subset (\overline{\vartheta}_s M_{\mathcal{I}})^{0, \Omega}$. As $\overline{\vartheta}_s(M_{\mathcal{I}}) = \overline{\vartheta}_s M \cap \bigoplus_{\Omega}(\overline{\vartheta}_s M_{\mathcal{I}})^{0, \Omega}$ it suffices to show that $(\overline{\vartheta}_s M_{\mathcal{I}})^{0, \Omega} = (\overline{\vartheta}_s M_{\mathcal{I}_{\Omega}})^{0, \Omega}$. By Lemma 7.8 we have
\[
(\overline{\vartheta}_s M_{\mathcal{I}})^{0, \Omega} = M_{\mathcal{I}}^{0, \Omega} \oplus M_{\mathcal{I}}^{0, \Omega s}
\]
and we have an analogous equation for $\overline{\vartheta}_s M_{\mathcal{I}_{\Omega}}$. By Lemma 7.8, $M_{\mathcal{I}}^{0, \Omega} \oplus M_{\mathcal{I}}^{0, \Omega s}$ only depends on $\mathcal{I} \cap \Omega$ and $\mathcal{I} \cap \Omega s$. From $\mathcal{I} \cap \Omega = \mathcal{I}_{\Omega} \cap \Omega$ and $\mathcal{I} \cap \Omega s = \mathcal{I}_{\Omega} \cap \Omega s$ we deduce the claim. □

Lemma 7.10. We have $\text{supp} \overline{\vartheta}_s M \subset \text{supp} M \cup (\text{supp} M)s$.

Proof. For notational convenience let us set $Z = \overline{\vartheta}_s M$. Suppose that $A \in \mathcal{A}$ is such that $A \succ As$. Then $\{\succeq As\}$ and $\{\succeq As\} \setminus \{A, As\}$ are $s$-invariant closed subsets. Consider the following inclusions
\[
Z_{\{\succeq As\} \setminus \{A, As\}} \subset Z_{\{\succeq As\} \setminus \{As\}} \subset Z_{\{\succeq As\}},
\]
which by Lemma 7.9 we can identify with
\[
\overline{\vartheta}_s(M_{\{\succeq As\} \setminus \{A, As\}}) \subset \overline{\vartheta}_s(M_{\{\succeq As\} \setminus \{As\}}) \subset \overline{\vartheta}_s(M_{\{\succeq As\}}).
\]
Suppose that $A \notin \text{supp} M$ and $As \notin \text{supp} M$. We have to show that $A \notin \text{supp} Z$ and $As \notin \text{supp} Z$. But we have $M_{\{\succeq As\} \setminus \{A, As\}} = M_{\{\succeq As\}}$, hence the above inclusions are bijections and, in particular, $Z_{\{As\}} = 0$, so $As \notin \text{supp} Z$. By property (P2) the canonical homomorphism
\[
Z_{\{\succeq A\}}/Z_{\{\succ A\}} \rightarrow Z_{\{\succeq As\} \setminus \{As\}}/Z_{\{\succeq As\} \setminus \{A, As\}}
\]
is injective. From the bijections above we deduce that the space on the right hand side is 0, hence so is the space on the left hand side. Hence $A \notin \text{supp} Z$. □

Lemma 7.11. The filtration defined above on $\overline{\vartheta}_s M$ satisfies property (P3).
Proof. Let $L$ be a saturated subset of $X$ and let $\Gamma$ be a connected component of $\mathcal{G}^L$. As $M$ satisfies property $(P3)$ there are sub-$\mathcal{Z}$-modules $M^{L,\Omega}$ for each $\Omega \in \Gamma$ such that $\text{supp} M^{L,\Omega} \subset \Omega$ and $M^{L,\Omega} = \bigoplus_{\Omega \in \Gamma} M^{L,\Omega}$. Let us fix some $\Omega$ in $\Gamma$. By Lemma 7.10 we have $\text{supp} \vartheta_s M^{L,\Omega} \subset \Omega \cup \Omega_s$. Clearly $\Omega_s$ is also a connected component of $\mathcal{G}^L$. In the case $\Omega_s = \Omega$ we hence have $\text{supp} \vartheta_s M^{L,\Omega} \subset \Omega$. In the case $\Omega_s \neq \Omega$ the images $\overline{\Omega}$ and $\overline{\Omega} \Omega_s$ are distinct connected components of $\mathcal{G}^L$. Then we have a decomposition
\[ \mathcal{Z}(\overline{\Omega} \cup \overline{\Omega}) = \mathcal{Z}(\overline{\Omega} \cup \overline{\Omega}) \oplus \mathcal{Z}(\overline{\Omega}) \]
and each direct summand is a free $\mathcal{Z}(\overline{\Omega} \cup \overline{\Omega})^s$-module (as right multiplication by $s$ is a moment graph isomorphism between $\Omega$ and $\Omega_s$). Using Lemma 7.7 we deduce
\[ \vartheta_s(M^{L,\Omega}) = \mathcal{Z}(\overline{\Omega} \cup \overline{\Omega}) \otimes_{\mathcal{Z}(\overline{\Omega} \cup \overline{\Omega})^s} M^{L,\Omega} = \mathcal{Z}(\overline{\Omega}) \otimes_{\mathcal{Z}(\overline{\Omega} \cup \overline{\Omega})^s} M^{L,\Omega} \]
and the first module is supported in $\Omega_s$, while the second is supported in $\Omega$. Hence $\vartheta_s(M^{L,\Omega})$ splits in any case according to composition of $\mathcal{G}^L$ into connected components. Hence $(P3)$ holds for $\vartheta_s M$. \hfill \Box

It remains to prove property $(P1)$ and that $\vartheta_s M$ admits a $\succeq$-Verma flag. It is now convenient to formalize $(P1)$ in dependence on a closed subset $\mathcal{I}$ of $\mathcal{A}$:

$(P1)_\mathcal{I}$ We have $(\vartheta_s M)_\mathcal{I} = \sum_A (\vartheta_s M)_{\succeq A}$, where the sum is taken over all (minimal) elements $A$ of $\mathcal{I}$.

We will now prove the following statements:

(S1) If $\mathcal{I}$ is an $s$-invariant closed subset, then $(P1)_\mathcal{I}$ holds.
(S2) Suppose that $\mathcal{I}$ is $s$-invariant and that $(P1)_{\mathcal{I}'}$ holds for all closed subsets $\mathcal{I}'$ of $\mathcal{I}$. Then $(\vartheta_s M)_\mathcal{I}$ admits a Verma flag.
(S3) Let $\mathcal{I}$ be a closed subset, and let $C$ be a minimal element in $\mathcal{I}$ with $C \succ C_s$. Suppose that $(P1)_{\mathcal{I} \setminus \{C\}}$ and $(P1)_{\succ C_s}$ hold. Then $(P1)_\mathcal{I}$ holds.
(S4) Let $A$ be an alcove with $A \succ A_s$. Assume that $(P1)_\mathcal{I}$ holds for each closed subset $\mathcal{I}$ of $\{\succ A_s\} \setminus \{A\}$. Then $(P1)_{\succ A_s}$ holds.

Proposition 7.12. The above statements imply that the $\succeq$-filtration on $\vartheta_s M$ is admissible and that $\vartheta_s M$ admits a Verma flag.

Proof. Note that we already checked $(P2)$ and $(P3)$. Hence we only have to show that $(P1)_\mathcal{I}$ holds for all closed subsets $\mathcal{I}$ of $\mathcal{A}$ by $(S2)$. We prove this by induction. If $\mathcal{I} \cap \text{supp} \vartheta_s M = \emptyset$, then $(\vartheta_s M)_\mathcal{I} = 0$ and $(P1)_\mathcal{I}$ holds trivially. So suppose that $\mathcal{I}$ is minimal such that $(P1)_\mathcal{I}$ fails. By $(S1)$, $\mathcal{I}$ cannot be $s$-invariant. So there must be a minimal element $C$ in $\mathcal{I}$ with $C \succ C_s$. By minimality of $\mathcal{I}$ we deduce from $(S3)$ that $(P1)_{\succ C_s}$ fails. Hence there must be also a maximal element $A$ in $\mathcal{A}$ such that $A \succ A_s$ and such that $(P1)_{\succ A_s}$ does not hold. We now claim that the maximality of $A$ implies that $(P1)_\mathcal{I}$ holds for each closed subset
of \{\nless As\} \setminus \{A\} (once we proved this, \(P_1\nless As\) follows from (S4) and we have a contradiction). We again show this by induction. So let \(I\) be a minimal closed subset of \(\{\nless As\} \setminus \{A\}\) such that \(P_1\nless As\) fails. If all minimal elements \(C\) in \(I\) are such that \(Cs \nless C\), then \(I\) is \(s\)-invariant, which contradicts (S1). Otherwise there is some minimal \(C \in I\) with \(C \nless Cs\). But this contradicts (S3) in view of the minimality of \(I\) (note that, by part (2a) of Lemma 6.5, \(C \nless A\)).

We prove the statements (S1-4) step by step.

**Lemma 7.13.** If \(I\) is an \(s\)-invariant closed subset, then \(P_1\nless As\) holds.

**Proof.** Let \(I\) be \(s\)-invariant. For each minimal \(A \in I\) we then have \(As \nless A\), so \(\{\succeq A\}\) is \(s\)-invariant as well. Using Lemma 7.9 we deduce

\[
(\overline{\vartheta}_s M)_I = \overline{\vartheta}_s(M_I) = \overline{\vartheta}_s(\sum_A M_{\succeq A}) = \sum_A \overline{\vartheta}_s M_{\succeq A} = \sum_A (\overline{\vartheta}_s M)_{\succeq A},
\]

(where the sums above are over all minimal \(A\) in \(I\)).

**Lemma 7.14.** Suppose that \(I\) is \(s\)-invariant and that \(P_1\nless As\) holds for all closed subsets \(I'\) of \(I\). Then \((\overline{\vartheta}_s M)_I\) admits a Verma flag.

**Proof.** For convenience let us abbreviate \(Z = \overline{\vartheta}_s M\). By Lemma 4.12 we have to prove that the subquotients \(Z_{\{C\}}\) with \(C \in I\) are graded free \(S\)-modules. So let us fix \(C \in I\). Suppose that \(I' \subset I\) is closed with \(C \in I'\) minimal. As \(P_1\) and \(P_2\) are satisfied, we have a canonical isomorphism \(Z_{\{C\}} \cong Z_{I'}/Z_{I' \setminus \{C\}}\). As \(I\) is supposed to be \(s\)-invariant, we have \(Cs \in I\) as well. We will now prove that both \(Z_{\{C\}}\) and \(Z_{\{Cs\}}\) are graded free. Let us set \(I' = \{\succeq Cs\}\). We have a short exact sequence

\[0 \to Z_{I' \setminus \{C\}}/Z_{I' \setminus \{C, Cs\}} \to Z_{I'}/Z_{I' \setminus \{C, Cs\}} \to Z_{I'}/Z_{I' \setminus \{C\}} \to 0.\]

By what we have observed above, the submodule can be identified with \(Z_{\{Cs\}}\), and the quotient with \(Z_{\{C\}}\). As \(\mathcal{Z}\)-modules they are supported on \(\overline{Cs}\) and \(\overline{C}\), resp. We now study the object in the middle. As both \(I'\) and \(I' \setminus \{C, Cs\}\) are \(s\)-invariant, Lemma 7.13 and the exactness of \(\overline{\vartheta}_s(\cdot)\) on the level of \(\mathcal{Z}\)-modules imply

\[
Z_{I'}/Z_{I' \setminus \{C, Cs\}} = (\overline{\vartheta}_s M)_{I'}/(\overline{\vartheta}_s M)_{I' \setminus \{C, Cs\}} = \overline{\vartheta}_s(M_{I'}/M_{I' \setminus \{C, Cs\}}) = \overline{\vartheta}_s(M_{\{C, Cs\}}).
\]
As $M_{[C,C_s]}$ is supported (as a $\overline{Z}$-module) on $\{C,C_s\}$, Lemma 7.14 implies that

$$\overline{\vartheta}_s(M_{[C,C_s]}) = \overline{Z}(\{C,C_s\}) \otimes_{\overline{Z}(\{C,C_s\})} M_{[C,C_s]}$$

$$= \{(z_{\tau}, z_{\rho}) \in S \otimes_2 | z_{\tau} \equiv z_{\rho} \mod \alpha^\vee\} \otimes S M_{[C,C_s]},$$

where $\alpha^\vee$ is the label connecting the edge between $\overline{C}$ and $\overline{C_s}$. The subspace supported on $\overline{C_s}$ is $S(0, \alpha^\vee) \otimes S M_{[C,C_s]} \cong M_{[C,C_s]}[-2]$, while the quotient is $\overline{Z}(\overline{C}, \overline{C_s})/S(0, \alpha^\vee) \otimes S M_{[C,C_s]} \cong S \otimes S M_{[C,C_s]} \cong M_{[C,C_s]}$.

While proving the previous lemma, we have shown the following property of translation functors.

**Lemma 7.15.** Suppose that the filtration on $\overline{\vartheta}_s M$ is admissible. Then we have natural isomorphism for all alcove $A$ with $A \succ As$

$$(\overline{\vartheta}_s M)_{[A]} \cong M_{[A,As]}[-1] \text{ and } (\overline{\vartheta}_s M)_{[As]} \cong M_{[A,As]}[1].$$

**Lemma 7.16.** Let $I$ be a closed subset, and let $C$ be a minimal element in $I$ with $C \succ C_s$. Suppose that $(P1)_{I\{C\}}$ and $(P1)_{C_s}$ hold. Then $(P1)_I$ holds.

**Proof.** Again we abbreviate $Z = \overline{\vartheta}_s M$. We will now show that the canonical homomorphism

$$(*) \quad Z_{\succ C} \to Z_I/Z_{I\setminus C}$$

is surjective. If this is true, then $Z_I$ is generated by the submodules $Z_{\succ C}$ and $Z_{I\setminus \{C\}}$. As, by assumption, $Z_{I\setminus \{C\}}$ is generated by the union of the $Z_{\succ D}$ with $D \in I \setminus \{C\}$, $Z_I$ is generated by $Z_{\succ D}$ with $D \in I$, so $(P1)_I$ holds.

Now set $\tilde{I} = I \cup I_s$. This is a closed subset of $\mathcal{A}$ with minimal element $C_s$. Set $I' = \tilde{I} \setminus C_s$. This has $C$ as a minimal element. As $I \subset I'$ we have a canonical homomorphism $Z_I \to Z_{I'}$. In order to prove the surjectivity of the map $(*)$ it suffices to prove the surjectivity of the composition $Z_{\succ C} \to Z_I/Z_{I\setminus C} \to Z_I/Z_{I'\setminus C}$ as, by property (P2), the rightmost map is injective. Now consider the following diagram with two exact columns

$\begin{array}{ccc}
Z_{\succ C}/Z_{\succ C} & \xrightarrow{f} & Z_{\succ C}/Z_{\{C\}\setminus C} & \xrightarrow{g} & Z_{I'}/Z_{I'\setminus C} \\
\uparrow & & \uparrow & & \uparrow \\
Z_{\succ C}/Z_{\{C\}\setminus C} & \xrightarrow{h} & Z_{\tilde{I}}/Z_{\tilde{I}\setminus \{C,C_s\}} \\
\uparrow & & \uparrow \\
Z_{\succ C}/Z_{\succ C} & \xrightarrow{i} & Z_{\tilde{I}}/Z_{\tilde{I}'}. 
\end{array}$

Again all horizontal maps are injective by (P2). We want to show that $g \circ f$ is surjective. Now we consider the middle row, i.e. the homomorphism $h$. Note that all the indexing sets there are $s$-invariant. By Lemma 7.9 both modules can be identified with $\overline{\vartheta}_s(M_{[C,C_s]})$ and hence $h$ is an isomorphism. This immediately
implies that \( i \) is surjective. Hence \( h \) and \( i \) are isomorphisms, hence so is \( g \). So we only have to show that \( f \) is surjective. But this is implied by \((P1)_{>C_s}\).

\[\square\]

**Lemma 7.17.** Let \( A \) be an alcove with \( A \succ As \). Assume that \((P1)_I\) holds for each closed subset \( I \) of \{\(\succ As\)\} \(\setminus\{A\}\). Then \((P1)_{>As}\) holds.

**Proof.** Again we set \( Z = \overline{\theta}_s M \). In a first step we show that \( Z_{\succ As} \) is a graded free \( S \)-module. As \( \{\succ As\} \) is \( s \)-invariant, we already know that \( Z_{\succ As} \) is graded free as an \( S \)-module. Now we have a look at the short exact sequence

\[
0 \rightarrow Z_{\{\succ As\}\setminus\{A, As\}} \rightarrow Z_{\succ As} \rightarrow Z_{\{\succ As\}/Z_{\{\succ As\}\setminus\{A, As\}}} \rightarrow 0.
\]

As \( \{\succ As\} \setminus\{A, As\} \) is \( s \)-invariant, Lemma 7.14 implies that the module on the left is graded free. We can identify the module on the right with \( \overline{\theta}_s(M_{\succ As}/M_{\{\succ As\}\setminus\{A, As\}}) = \overline{\theta}_s(M_{[A, As]}) \), hence it is graded free as well. As we observed in the proof of Lemma 7.14, the submodule \( (\overline{\theta}_s(M_{[A, As]}))_{\succ A} \) is graded free. But its preimage in \( Z_{\succ As} \) is the submodule \( Z_{\succ As} \), so this latter module is graded free as well.

As a second step we prove that the quotient \( Z_{\succ As}/Z_{\succ A} \) is a graded free \( S \)-module. By assumption, the quotient \( Z_{\{\succ As\}\setminus\{A, As\}}/Z_{\succ A} \) is graded free. As we have a short exact sequence

\[
0 \rightarrow Z_{\{\succ As\}\setminus\{A, As\}}/Z_{\succ A} \rightarrow Z_{\succ As}/Z_{\succ A} \rightarrow (Z_{\{\succ As\}}/Z_{\{\succ As\}\setminus\{A, As\}})_{\succ A} \rightarrow 0,
\]

the module in the middle is graded free.

Note that in order to show \((P1)_{>As}\) we need to show that the homomorphism \( Z_{\succ A} \rightarrow Z_{\succ As}/Z_{\succ As\setminus A} \) is surjective. Set \( \mathcal{X} = \{\succ As\} \setminus\{\succ A\} \). Hence \( \{\succ As\} \setminus\{\succ A\} = A \cup \mathcal{X} \). Also note that for each \( C \in \mathcal{X} \) we have \( C \succ C \). In particular, the \( ZR \)-orbit if \( A \) does not meet \( \mathcal{X} \). We will now consider the quotient \( Z_{\succ As}/Z_{\succ A} \). Its support is \( A \cup \mathcal{X} \). By what we observed above we have a canonical decomposition \( (Z_{\succ As}/Z_{\succ A})^0 = R \oplus Q \), where \( R \) is supported on \( A \) and \( Q \) is supported on \( \mathcal{X} \). We will now show that this decomposition induced a decomposition on \( Z_{\succ As}/Z_{\succ A} \). As this module is a graded free \( S \)-module it suffices to show that we have an induced decomposition on each localization \( (Z_{\succ As}/Z_{\succ A})^\alpha \) for \( \alpha \in X \). But for each such \( \alpha \), the connected component in \( \overline{G}^\alpha \) containing \( A \) does intersect \( \mathcal{X} \) trivially. So property \((P3)\) (which we have already proven) ensures the decomposition.

Hence we have a direct sum decomposition \( Z_{\succ As}/Z_{\succ A} = R \oplus Q \) with \( \text{supp } R \subset \{A\} \) and \( \text{supp } Q \subset \mathcal{X} \). By definition, the image of \( Z_{\succ A} \rightarrow Z_{\succ As}/Z_{\succ A} \) is \( R \). Now consider the short exact sequence

\[
0 \rightarrow Z_{\succ As\setminus A}/Z_{\succ A} \rightarrow Z_{\succ As}/Z_{\succ A} \rightarrow Z_{\succ As}/Z_{\succ As\setminus A} \rightarrow 0.
\]

The space on the right hand side is supported on \( \{A\} \) and by what we have shown above, the composition \( Z_{\succ A} \rightarrow Z_{\succ As}/Z_{\succ A} \rightarrow Z_{\succ As}/Z_{\succ As\setminus A} \) is an isomorphism. Hence property \((P1)_{>As}\) holds.

\[\square\]

Now we can prove Theorem 7.3.
Proof. Note that the inclusion $\mathcal{Z} = \tilde{\mathcal{Z}}^{\mathbb{Z}R} \subset \pi(\tilde{\mathcal{Z}})$ of Lemma 6.12 is compatible with the $s$-action, i.e. it restricts to an inclusion of the $s$-invariants $\mathcal{Z}^{s} \subset \pi(\tilde{\mathcal{Z}}^{s})$. Hence we obtain a natural homomorphism $\mathcal{Z} \otimes_{\mathbb{Z}} M_{s^{v}=0} \to (\tilde{\mathcal{Z}} \otimes_{\tilde{\mathcal{Z}}} M)_{s^{v}=0}$. As $\mathcal{Z} = \mathcal{Z}^{s} \oplus \mathcal{Z}^{s}[-2]$ as a $\mathcal{Z}^{s}$-module and, analogously, $\tilde{\mathcal{Z}} = \tilde{\mathcal{Z}}^{s} \oplus \tilde{\mathcal{Z}}^{s}[-2]$ as a $\tilde{\mathcal{Z}}^{s}$-module, the above homomorphism must be an isomorphism. We obtain an identification of $\mathcal{X} \circ \Upsilon(M)$ with $\Upsilon \circ \mathcal{X}^{s} (M)$ as $\mathcal{Z}$-modules that is functorial in $M$. We have to show that this also identifies the $\succ$-filtrations. By Lemma 7.18 it suffices to show that $\mathcal{X} \circ \Upsilon(M)_{\mathcal{I}}$ is identified with $\Upsilon \circ \mathcal{X}^{s} (M)_{\mathcal{I}}$ for $s$-invariant closed subsets $\mathcal{I}$. But for those $\mathcal{I}$ we have $\mathcal{X} \circ \Upsilon(M)_{\mathcal{I}} = \Upsilon \circ \mathcal{X}^{s} (M)_{\mathcal{I}}$ and $\Upsilon \circ \mathcal{X}^{s} (M)_{\mathcal{I}} = \Upsilon \circ \mathcal{X}^{s} (M)_{\mathcal{I}}$. The functoriality of the identification now finishes the proof.

7.4. Adjunction. Let $M$ be an object in $\mathcal{Z}\text{-mod}^{f}$. We now consider the $\mathcal{Z}$-modules $\mathcal{Z} \otimes_{\mathcal{X}} M$ and $\text{Hom}_{\mathcal{X}}(\mathcal{Z}, M)$. On the first, $\mathcal{Z}$ acts by multiplication on the first factor, and on the second by multiplying the argument, i.e. $z.f(m) = f(zm)$.

Lemma 7.18. The functors $\text{Hom}_{\mathcal{X}}(\mathcal{Z}, \cdot): \mathcal{Z}\text{-mod}^{f} \to \mathcal{Z}\text{-mod}^{f}$ and $\mathcal{Z} \otimes_{\mathcal{X}} \cdot [2]: \mathcal{Z}\text{-mod}^{f} \to \mathcal{Z}\text{-mod}^{f}$ are isomorphic.

Proof. Note that by Lemma 7.6 the structure algebra $\mathcal{Z}$ is a free $\mathcal{X}$-module of graded rank $1 + v^{2}$. Let $x$ and $y$ be homogenous basis elements of degree 0 and 2. Let $x^{s}, y^{s} \in \text{Hom}_{\mathcal{X}}(\mathcal{Z}, \mathcal{Z})$ be the dual basis. These elements are of degree 0 and $-2$, resp. The homomorphism

$$(\mathcal{Z} \otimes_{\mathcal{X}} M)[2] \to \text{Hom}_{\mathcal{X}}(\mathcal{Z}, M)$$

$$x \otimes m \mapsto (z \mapsto y^{s}(z)m)$$

$$y \otimes m \mapsto (z \mapsto x^{s}(z)m)$$

is a graded homomorphism of $\mathcal{Z}$-modules with inverse

$$\text{Hom}_{\mathcal{X}}(\mathcal{Z}, M) \to (\mathcal{Z} \otimes_{\mathcal{X}} M)[2]$$

$$f \mapsto x \otimes f(y) + y \otimes f(x).$$

Proposition 7.19. The functor $\mathcal{X}: \mathcal{C}^{\mathbb{Z}} \to \mathcal{C}^{\mathbb{Z}}$ is self-adjoint, i.e. there is an isomorphism

$$\text{Hom}_{\mathcal{C}^{\mathbb{Z}}}(\mathcal{X}(\cdot), \cdot) \cong \text{Hom}_{\mathcal{C}^{\mathbb{Z}}}(\cdot, \mathcal{X}(\cdot)).$$

Proof. In Lemma 7.18 we identified the functors $\mathcal{Z} \otimes_{\mathcal{X}} \cdot [2]$ and $\text{Hom}_{\mathcal{X}}(\mathcal{Z}, \cdot)$ on the level of $\mathcal{Z}$-modules. Hence we only have to check that the usual functorial identification

$$\phi: \text{Hom}_{\mathcal{Z}}(\mathcal{Z} \otimes_{\mathcal{X}} M, N) \cong \text{Hom}_{\mathcal{Z}}(M, \text{Hom}_{\mathcal{X}}(\mathcal{Z}, N)) = \text{Hom}_{\mathcal{Z}}(M, \mathcal{Z} \otimes_{\mathcal{X}} N[2])$$

$$f \mapsto (m \mapsto (z \mapsto f(z \otimes m)))$$

is an isomorphism.
sends homomorphisms respecting the $\succeq$-filtrations to morphisms that do so as well. By Lemma 4.18, it suffices to check this on homomorphisms respecting only the submodules associated with $s$-invariant closed subsets (for which we can use Lemma 7.9). Let $I$ be such a subset. By the functoriality of $\phi$, a homomorphism $f: \mathcal{Z} \otimes_{\mathcal{S}} M \to N$ maps $(\vartheta_s M)_I = \vartheta_s(M_I)$ into $N_I$ if and only if $\phi(f)$ maps $M_I$ into $\vartheta_s(N_I) = (\vartheta_s N)_I$. □

**Proposition 7.20.** The functor $\vartheta_s$ is exact.

*Proof.* Let $M \to N \to O$ be an exact sequence in $\mathcal{C}^{\succeq}$. Note that $\vartheta_s$ preserves Verma flags, so by Lemma 4.15 it is enough to check that for any $A \in \mathcal{A}$ the sequence

$$0 \to (\vartheta_s M)_{[A]} \to (\vartheta_s N)_{[A]} \to (\vartheta_s O)_{[A]} \to 0$$

is exact. By 7.15 this is equivalent, up to a simultaneous shift, to the exactness of the sequence

$$0 \to M_{[A, As]} \to N_{[A, As]} \to O_{[A, As]} \to 0.$$ 

The exactness of this sequence is an easy consequence of the exactness of the original one. □

From Propositions 7.19 and 7.20 we deduce the next result.

**Corollary 7.21.** Suppose that $P \in \mathcal{C}^{\succeq}$ is projective. Then $\vartheta_s(P) \in \mathcal{C}^{\succeq}$ is projective as well.

### 8. Projective objects

In this section we study the projective objects in $\hat{\mathcal{C}}^\prec$, $\mathcal{C}^\prec$ and $\mathcal{C}^\succeq$. The category $\mathcal{C}^\succeq$ does not contain any projective object.

We first introduce the *Verma-type objects* in our categories.

#### 8.1. Standard objects in $\hat{\mathcal{C}}^\prec$, $\mathcal{C}^\prec$ and $\mathcal{C}^\succeq$

For any alcove $A$ let us define “standard objects” as follows. We denote by $\hat{\mathcal{V}}(A)$ the $\hat{\mathcal{S}}$-module that is graded free of graded rank one as an $\hat{\mathcal{S}}$-module, and on which $(z_B) \in \hat{\mathcal{Z}}$ acts as multiplication by $z_A$. It is then clear that $\hat{\mathcal{V}}(A)$ admits a $\succeq$ and a $\preceq$-Verma flag, and that $\text{supp} \, \hat{\mathcal{V}}(A) = \{A\}$.

For the quotient situation we set $\nabla(A) := \Upsilon(\hat{\mathcal{V}}(A))$. Note that this is graded free of graded rank 1 as an $S$-module, and $(z_B) \in \mathcal{Z}$ acts on it as multiplication by $z_{\mathcal{I}}$. The filtration is such that $\nabla(A)_I = \{0\}$ if $A \not\in I$, and $\nabla(A)_I = \hat{\mathcal{V}}(A)$ if $A \in I$. Clearly, $\nabla(A) \in \mathcal{C}^\prec$ and $\text{supp} \, \nabla(A) = \{A\}$.
8.2. Projective objects in \( \hat{\mathcal{C}}^\infty \). Here we quote one of the main results about the projective objects in the category \( \hat{\mathcal{C}}^\infty \).

**Theorem 8.1.**

1. For any \( A \in \mathcal{A} \) there is an up to isomorphism unique indecomposable projective object \( \hat{\mathcal{P}}(A) \) in \( \hat{\mathcal{C}}^\infty \) that admits an admissible epimorphism onto \( \hat{\mathcal{V}}(A)[l(w_A)] \).
2. We have \( \text{supp} \hat{\mathcal{P}}(A) = \{ B \in \mathcal{A} | B \leq A \} \).
3. Let \( \hat{\mathcal{P}} \) be an arbitrary projective object in \( \hat{\mathcal{C}}^\infty \). Then there are \( A_1, \ldots, A_n \in \mathcal{A} \) and \( l_1, \ldots, l_n \in \mathbb{Z} \) with
   \[
   \hat{\mathcal{P}} \cong \hat{\mathcal{P}}(A_1)[l_1] \oplus \cdots \oplus \hat{\mathcal{P}}(A_n)[l_n].
   \]

The multiset of pairs \( (A_1, l_1), \ldots, (A_n, l_n) \) is uniquely determined by \( \hat{\mathcal{P}} \).

**Proof.** Part (1) is proven in Theorem 6.1 of [Fie08b]. There it is also shown that \( \text{supp} \hat{\mathcal{P}}(A) \subset \{ B \in \mathcal{A} | B \leq A \} \). Equality follows from Corollary 6.5 in loc.cit. together with the construction of the Braden–MacPherson sheaf in Section 6.1 in loc.cit. This establishes part (2). Part (3) can now be established with standard argument (cf. the proof of Theorem 8.5 in this paper). \( \square \)

Recall that we denote by \( \mathcal{A}^0 \) the set of all alcoves that contain 0 in their closure. Let us consider \( \hat{\mathcal{Z}}(\mathcal{A}^0) \), the structure algebra of \( \hat{\mathcal{G}} \) over the set \( \mathcal{A}^0 \).

**Proposition 8.2.** Suppose that char \( k \neq 3 \) is \( R \) is of type \( G_2 \). Then we have \( \text{supp} \hat{\mathcal{P}}(A_0^-) = \mathcal{A}^0 \) and \( \hat{\mathcal{P}}(A_0^-) \) is a free \( \hat{\mathcal{Z}}(\mathcal{A}^0) \)-module of rank 1.

**Proof.** Note that \( \{ B \in \mathcal{A} | B \leq A_0^- \} = \{ w(A_0^+) | w \in \hat{\mathcal{W}}, w \leq w_0 \} = \mathcal{W}(A_0^+) = \mathcal{A}^0 \). By [Fie08a] the indecomposable objects in \( \hat{\mathcal{C}}^\infty \) occur as the space of global sections of the Braden–MacPherson sheaf on the \( \leq \)-closed submoment graph of \( \hat{\mathcal{G}} \). In the case of \( \hat{\mathcal{P}}(A_0^-) \), this is the finite Bruhat graph associated to the root system \( R \). In this case the Braden–MacPherson sheaf coincides with the structure sheaf, so its space of global sections coincides with the structure algebra. Hence \( \hat{\mathcal{P}}(A_0^-) = \hat{\mathcal{Z}}(\mathcal{A}^0)[l(w_0)] \). \( \square \)

8.3. Kazhdan–Lusztig polynomials. We denote by \( \mathcal{L} = \mathbb{Z}[v, v^{-1}] \) the ring of integral Laurent polynomials in the variable \( v \). The affine Hecke algebra \( \hat{\mathcal{H}} \) is the free \( \mathcal{L} \)-module with basis \( \{ T_x \}_{x \in \hat{\mathcal{W}}} \) and the unique \( \mathcal{L} \)-bilinear multiplication that satisfies

\[
T_x \cdot T_y = T_{xy}, \text{ if } l(xy) = l(x) + l(y),
\]

\[
T_s^2 = v^{-2}T_e + (v^{-2} - 1)T_s, \text{ for } s \in \hat{\mathcal{S}}.
\]

Then \( T_e \) is the identity element in \( \hat{\mathcal{H}} \) and \( T_x \) is invertible in \( \hat{\mathcal{H}} \) for all \( x \in \hat{\mathcal{W}} \). Moreover, \( \hat{\mathcal{H}} \) carries a \( \mathbb{Z} \)-linear involution \( H \mapsto \overline{H} \) that is determined by \( \overline{v} = v^{-1} \) and \( \overline{T_x} = T_x^{-1} \). An element \( H \in \hat{\mathcal{H}} \) is called self-dual if \( \overline{H} = H \). Following [Soc97], we renormalize the standard basis of \( \hat{\mathcal{H}} \) and set \( H_x := v^{l(x)}T_x \) for \( x \in \hat{\mathcal{W}} \).
Theorem 8.3 ([KL79] Soe97, Theorem 2.1]). For any \( x \in \hat{\mathcal{W}} \) there exists a unique self-dual element \( H_x \) in \( H_x + \sum_y v^\mathbb{Z}[v] H_y \subset \hat{H} \).

For example, for \( s \in \hat{S} \) we have \( H_s = H_s + vH_e \). We define, for \( x, y \in \hat{\mathcal{W}} \), the polynomial \( h_{y,x} \in \mathbb{Z}[v] \) by

\[
H_x = \sum_y h_{y,x} H_y.
\]

8.4. Graded multiplicities for \( \hat{\mathcal{C}}^\leq \). Suppose \( M \) is a graded free \( S \)-module of finite rank. Then there is a well defined multiset \( \{ l_1, \ldots, l_n \} \) of integers such that \( M \cong \bigoplus_{i=1}^n S[l_i] \). We define the graded rank of \( M \) as \( \text{rk} M = \sum_{i=1}^n v^{-l_i} \in \mathcal{L} \).

Now let \( \hat{M} \) be an object in \( \hat{\mathcal{C}}^\leq \). Then \( \hat{M}_{[A, \leq]} \) is a graded free \( S \)-module of finite rank for each \( A \in \mathcal{A} \). We define

\[
h_{\leq}(\hat{M}) := \sum_{w \in \hat{\mathcal{W}}} v^{l(w)} \text{rk} \hat{M}_{[w, \leq]} H_w.
\]

The following is a truly deep result.

Theorem 8.4. Suppose that \( \text{char} k = 0 \). For each alcove \( A \in \mathcal{A} \) we have \( h_{\leq}(\mathcal{P}(A)) = H_{w_A} \).

Proof. By [Fie08a] there is an exact equivalence between \( \hat{\mathcal{C}}^\leq \otimes S \hat{S} \), where \( \hat{S} \) is the completion of \( S \) at the maximal ideal \( \hat{X}^\vee \cdot S \), and the category of deformed modules admitting a Verma flag inside the principal block of category \( \mathcal{O} \) over the affine Kac–Moody algebra associated to the root system \( R \). Via this equivalence and the BGG-reciprocity, the above statement is equivalent to the affine Kazhdan–Lusztig conjecture at a negative level, that was proven by Kashiwara and Tanisaki in [KT95]. \( \square \)

8.5. Projective objects in \( \mathcal{C}^\geq \). Here is the analogue of Theorem 8.1 for the category \( \mathcal{C}^\geq \). Recall that we denote by \( l \) the Bruhat length of \( w_0 \), the longest element in the finite Weyl group.

Theorem 8.5. (1) Let \( \lambda \in X \) and \( A \subset \Pi_\lambda \). Then there is an up to isomorphism unique indecomposable projective object \( \mathcal{P}(A) \) in \( \mathcal{C}^\geq \) that admits an admissible epimorphism onto \( \nabla(A)[l + \ell_\lambda(A)] \).

(2) For each \( A \in \mathcal{A} \) we have \( \text{supp} \mathcal{P}(A) \subset \{ \geq A \} \) and \( A \in \text{supp} \mathcal{P}(A) \).

(3) Let \( \mathcal{P} \) be an arbitrary projective object in \( \mathcal{C}^\leq \). Then there are \( A_1, \ldots, A_n \in \mathcal{A} \) and \( l_1, \ldots, l_n \in \mathbb{Z} \) with

\[
\mathcal{P} \cong \mathcal{P}(A_1)[l_1] \oplus \cdots \oplus \mathcal{P}(A_n)[l_n].
\]

The multiset of pairs \( (A_1, l_1), \ldots, (A_n, l_n) \) is uniquely determined by \( \mathcal{P} \).

We need some preparation before we can prove the theorem above.
Lemma 8.6. Suppose that char $k \neq 3$ is $R$ is of type $G_2$. Let $A$ be an alcove in the antifundamental box $\Pi_0$. Then the object $\Upsilon(\hat{P}(A))$ is projective in $\mathcal{C}^-$ and admits an admissible epimorphism $\Upsilon(\hat{P}(A)) \to \nabla(A)[l(w_A)]$.

Proof. We prove the statement by induction on $\ell_0(A)$. If $\ell_0(A) = 0$, then $A = A_0$. As a $\mathbb{Z}$-module $\Upsilon(\hat{P}(A_0^-))$ is free of rank 1 by Proposition 8.2. Moreover, its support is $\mathcal{A}^\circ$, which has a unique maximal element $A_0^-$. By Proposition 6.13 $\Upsilon(\hat{P}(A_0^-))$ is hence projective. The existence of an admissible epimorphism $\hat{P}(A_0^-) \to \hat{\nabla}(A_0^-)[l(w_A^-)]$ implies, by the exactness of $\Upsilon$, the existence of an admissible epimorphism $\Upsilon(\hat{P}(A_0^-)) \to \Upsilon(\hat{\nabla}(A_0^-)[l(w_A^-)]) \cong \nabla(A_0^-)[l(w_A^-)]$.

Now let $A$ be an arbitrary alcove in $\Pi_0$ with $\ell_0(A) > 0$ and choose $s \in \hat{S}$ with $\ell_0(As) < \ell_0(A)$ (and hence $As \succ A$). By Theorem 7.5 we have $\Upsilon(\hat{\vartheta}_s\hat{P}(As)) \cong \hat{\vartheta}_s\Upsilon(\hat{P}(As))$. Corollary 7.21 and our induction assumption imply that $\Upsilon(\hat{\vartheta}_s\hat{P}(As))$ is projective in $\mathcal{C}^-$.

By the exactness and self-adjointness of $\hat{\vartheta}_s$ we deduce that $\hat{\vartheta}_s\hat{P}(As)$ is projective in $\mathcal{C}^-$. As it admits an admissible epimorphism onto $\hat{\nabla}(A)[l(w_A)], \hat{\nabla}(A)$ must occur as one of its direct summands. Hence $\Upsilon(\hat{P}(A))$ is a direct summand of $\Upsilon(\hat{\vartheta}_s\hat{P}(As))$, hence projective in $\mathcal{C}^-$. As $A$ is also a $\succ$-minimal element in the support of $\Upsilon(\hat{P}(A))$, there exists an admissible epimorphism $\Upsilon(\hat{P}(A)) \to \nabla(A)[l(w_A)]$. □

Now we can prove the first part of Theorem 8.5.

Proposition 8.7. For each $A \in \Pi_\lambda$ there is an indecomposable projective object $\hat{P}(A)$ in $\mathcal{C}^-$ that admits an admissible epimorphism onto $\nabla(A)[l + \ell_\lambda(A)]$. Moreover, we have $\text{supp} \hat{P}(A) \subset \{ \succ A \}$ and $A \in \text{supp} \hat{P}(A)$.

Proof. If $A \in \Pi_0$, then $\Upsilon(\hat{P}(A))$ is projective and admits an admissible epimorphism onto $\nabla(A)[l(w_A)] = \nabla(A)[l + \ell_0(A)]$ by Lemma 8.6. There must be an indecomposable direct summand of $\Upsilon(\hat{P}(A))$ that admits such an epimorphism as well. This is our $\hat{P}(A)$. From this construction we also deduce $\text{supp} \hat{P}(A) \subset \{ \succ A \}$.

Suppose now that $A \in \Pi_{\lambda}$ is an arbitrary alcove. Then $A - \lambda \in \Pi_0$. As $T_\lambda$ is an exact equivalence of $\mathcal{C}^-$ we obtain that $T_\lambda \hat{P}(A - \lambda)$ is indecomposable and projective and admits an admissible epimorphism onto $\nabla(A)[l + \ell_0(A - \lambda)] = \nabla(A)[l + \ell_\lambda(A)]$. □

In order to finish the proof of Theorem 8.5 we need to show the following statement (note that the uniqueness statement in part (1) of loc.cit. easily follows from the next claim).

Proposition 8.8. Let $\hat{P}$ be an arbitrary projective object in $\mathcal{C}^-$. Then there are $A_1, \ldots, A_n \in \mathcal{A}$ and $l_1, \ldots, l_n \in \mathbb{Z}$ with $\hat{P} \cong \hat{P}(A_1)[l_1] \oplus \cdots \oplus \hat{P}(A_n)[l_n]$. 
Moreover, the multiset of pairs \((A_1, l_1), \ldots, (A_n, l_n)\) is uniquely determined by \(\overline{P}\).

Proof. Suppose that \(A\) is a \(\succeq\)-minimal alcove in the support of \(\overline{P}\). Then there is an admissible epimorphism \(\overline{P} \to \overline{V}(A)[n]\) for some \(n \in \mathbb{Z}\). The projectivity of \(\overline{P}\) and of \(\overline{P}(A)\) implies that there are homomorphisms \(f : \overline{P} \to \overline{P}(A)[n]\) and \(f' : \overline{P}(A)[n] \to \overline{P}\) such that the diagrams commute (the downward homomorphisms are fixed admissible epimorphisms). The composition \(f' \circ f\) is an endomorphism of \(\overline{P}(A)[n - l(w_A)]\) that cannot be nilpotent, hence is an automorphism. As \(\overline{P}(A)\) is indecomposable, the homomorphism \(f\) must split. Hence we can prove the existence of the claimed decomposition by induction on the number of elements in \(\text{supp}\ \overline{P}\). The uniqueness statement follows from the fact that \(\text{supp} \overline{P}(A) \subset \{\succeq A\}\) and \(A \in \text{supp} \overline{P}(A)\) for all \(A \in \mathfrak{A}\). \(\square\)

We continue with exhibiting some properties of the projectives in \(\overline{C}^\prec\).

8.6. A localization of projectives. Let \(L\) be a saturated subset of \(X^\prec\).

Lemma 8.9. (1) If \(\alpha^\prec \not\in L\) for all \(\alpha^\prec \in R^\prec\), then \(\overline{V}(A)^L\) is projective in \(\overline{C}^L{\succeq}\) for all \(A \in \mathfrak{A}\).

(2) Suppose that \(L = Q\alpha^\prec \cap X\). Then \(\overline{P}(A)^L\) fits into a short exact sequence

\[
0 \to \overline{V}(\alpha \uparrow A)^L \to \overline{P}(A)^L \to \overline{V}(A)^L \to 0
\]

for all \(A \in \mathfrak{A}\).

Proof. In the situation of part (1) we have \(\mathcal{Z}(\overline{G}^L) = \bigoplus_{x \in \overline{\mathfrak{V}}} S^L\). Then \(\overline{V}(A)\) satisfies all properties of Proposition 6.13 hence is indecomposable and projective.

So let us prove part (2). By assumption, \(\overline{G}^L\) splits into a disjoint union of subgraphs of the form \(\Omega \xrightarrow{\alpha^\prec} \Omega'\). Let \(\Omega = \overline{A}\) be the orbit containing \(A\), and consider the structure algebra

\[
\mathcal{Z}(\Omega \xrightarrow{\alpha^\prec} \Omega') = \{(z_\Omega, z_{\Omega'}) \in S^{\oplus 2} | z_\Omega \equiv z_{\Omega'} \mod \alpha^\prec\}.
\]

Note that this is free over \(S\) with basis \((1, 1)\) and \((0, \alpha^\prec)\). Now let us define \(\overline{P} \in \overline{Z}^L{\succeq}\) as follows. As a \(\overline{Z}\)-module we set \(\overline{P} = \mathcal{Z}(\Omega \xrightarrow{\alpha^\prec} \Omega')^L\). The
Lemma 8.10. Let $L$ be a saturated subset of $X^\vee$. If $P$ is projective in $\mathcal{C}^\vee$, then $P^L$ is projective in $\mathcal{C}^{L,\vee}$.

Proof. As the functors $\overline{T}_\mu$ for $\mu \in X$ and $\overline{v}_s$ preserve projectivity in $\mathcal{C}^{L,\vee}$, and since each indecomposable projective in $\mathcal{C}^{L,\vee}$ is a direct summand of an object that we obtain by repeatedly applying $\overline{T}_\mu$’s and $\overline{v}_s$’s to $P(A_0^-)$, we only have to check that $P(A_0^-)^L$ is projective in $\mathcal{C}^{L,\vee}$. But this is an easy consequence of Proposition 6.13 and of the properties of $P(A_0^-)$ that we exhibited in the proof of Lemma 8.6. □

Lemma 8.11. Let $A$ be an alcove in the antifundamental box $\Pi_0$, let $w \in \mathcal{W}$ and let $\alpha \in R$. Then $\alpha \uparrow w(A) \subseteq A$ if and only if $\alpha \downarrow w(A) \not\subseteq A$.

Proof. Since $A$ is an alcove in the antifundamental box $\Pi_0$, the set $\{\leq A\}$ is $\mathcal{W}$-invariant (with respect to the left-action). Hence $\alpha \uparrow w(A) \subseteq A$ if and only if $w^{-1}(\alpha \uparrow w(A)) = w^{-1}(\alpha) \uparrow A \subseteq A$. This is the case if and only if $w^{-1}(\alpha) \downarrow A \not\subseteq A$, which in turn is the case if any only if $w(w^{-1}(\alpha) \downarrow A) = \alpha \downarrow w(A) \not\subseteq A$. □

Lemma 8.12. Let $A$ be an alcove in $\Pi_0$. Then $\mathcal{W}(A) \subset \text{supp} \overline{P}(A)$.

Proof. Since $\overline{P}(A)$ is a direct summand of $\mathcal{Y}(\hat{P}(A))$ we have $\text{supp} \overline{P}(A) \subset \{\leq A\}$. We prove by induction on the (Coxeter-)length of $w \in \mathcal{W}$ that $w(A) \in \text{supp} \overline{P}(A)$.

The case $w = e$ is clear. So suppose that $w(A) \in \text{supp} \overline{P}(A)$ and let $s$ be a simple finite reflection with $ws(A) \leq w(A)$. Set $\alpha := w(\alpha_s)$. Then $ws(A) = \alpha \uparrow w(A)$.

By Lemma 8.11 we have $\alpha \downarrow w(A) \not\subseteq A$, hence $\alpha \downarrow w(A) \not\subseteq \text{supp} \overline{P}(A)$.

Now let $L = \mathbb{Q}(\alpha) \cap X$. By Lemma 8.10 the object $\overline{P}(A)^L$ is projective and by Lemma 8.9 it splits into a direct sum of non-split extension of $\overline{V}(B)$ and $\overline{V}(\alpha \uparrow B)$, for various $B \in \mathcal{A}$. As we have $w(A) \in \text{supp} \overline{P}(A)$, but $\alpha \downarrow w(A) \not\subseteq \text{supp} \overline{P}(A)$, we deduce that $\alpha \uparrow w(A) = ws(A) \in \text{supp} \overline{P}(A)$. □

Lemma 8.13. Suppose that char $k \neq 3$ is $R$ is of type $G_2$. Let $A$ be an alcove in $\Pi_\lambda$. Then we have an admissible monomorphism

$$\nabla(w_\lambda(A))[-l - l_\lambda(A)] \to \overline{P}(A).$$
Proof. Using the twist functors we can reduce the claim to the case that \( A \) is contained in the antifundamental box \( \Pi_0 \). Let \( s_1, \ldots, s_n \in \tilde{S} \) be such that \( A = A_0 s_1 \cdots s_n \) and suppose that \( n \) is minimal. Then \( \overline{\mathbf{P}}(A) \) is a direct summand of \( \overline{v}_{s_n} \cdots \overline{v}_{s_1} \overline{\mathbf{P}}(A_0^-) \). By Proposition 8.2 we have that \( w_0(A_0^-) = A_0^+ \) is a \( \succeq \)-maximal element in \( \text{supp} \overline{\mathbf{P}}(A_0^-) \) and \( \overline{\mathbf{P}}(A_0^-)[A_0] \cong S[-l] \). The minimality of \( n \) implies that \( A \) is a \( \succeq \)-minimal element on \( \text{supp} \overline{v}_{s_n} \cdots \overline{v}_{s_1} \overline{\mathbf{P}}(A_0^-) \) and \( (\overline{v}_{s_n} \cdots \overline{v}_{s_1} \overline{\mathbf{P}}(A_0^-))[A] = S[-l - \ell_0(A)] \). From Lemma 8.12 we deduce that \( w_0(A) \in \text{supp} \overline{\mathbf{P}}(A) \), hence \( w_0(A) \) must be a \( \succeq \)-maximal element in \( \text{supp} \overline{\mathbf{P}}(A) \) and \( \overline{\mathbf{P}}(A)[w_0(A)] \cong S[-l - \ell_0(A)] \). This implies the statement.

Now we can prove a duality result on the projectives.

**Theorem 8.14.** Let \( \lambda, \mu \in \mathcal{X} \) and suppose that \( A \) is contained in \( \Pi_\lambda \). Then

\[
\mathbb{D}_\mu \overline{\mathbf{P}}(A) \cong \overline{\mathbf{P}}(w_\mu \circ w_\lambda(A)).
\]

**Proof.** Note that \( \mathbb{D}_\mu \) is an exact autoequivalence of \( \mathcal{C}^- \). So \( \mathbb{D}_\mu \overline{\mathbf{P}}(A) \) is an indecomposable projective object in \( \mathcal{C}^- \). From Lemma 8.13 we can deduce that there is an admissible epimorphism

\[
\mathbb{D}_\mu \overline{\mathbf{P}}(A) \to \mathbf{V}(w_\mu \circ w_\lambda(A))[l + \ell_\lambda(A)].
\]

As \( \ell_\lambda(A) = \ell_{w_\mu(\lambda)}(w_\mu \circ w_\lambda(A)) \) we deduce \( \mathbb{D}_\mu \overline{\mathbf{P}}(A) \cong \overline{\mathbf{P}}(w_\mu \circ w_\lambda(A)) \) from the uniqueness statement in Theorem 8.5.

\( \square \)

9. Periodic Patterns

9.1. The periodic polynomials. We denote by

\[
\mathcal{P} = \bigoplus_{A \in \mathcal{A}} \mathcal{L}A
\]

the free \( \mathcal{L} \)-algebra generated by the set of alcoves \( \mathcal{A} \). It carries an action of the affine Hecke algebra \( \widehat{H} \) on the right, and this action is determined by the formula

\[
A \cdot H_{s} = \begin{cases} 
A s + vA, & \text{if } A \succ A_s, \\
A s + v^{-1}A, & \text{if } A \prec A_s.
\end{cases}
\]

Note that in [Soc97] the opposite convention for the partial order is used. For \( \lambda \in \mathcal{X} \) the bijection \( t_\lambda : \mathcal{A} \to \mathcal{A} \) induces an \( \mathcal{L} \)-linear automorphism \( \mathcal{P} \to \mathcal{P} \) that we denote by \( P \mapsto \lambda + P \) as well. This automorphism is \( \widehat{H} \)-linear.

We call a \( \mathbb{Z} \)-linear map \( f : \mathcal{P} \to \mathcal{P} \) \( \widehat{H} \)-skew linear, if \( f(P \cdot H) = f(P) \cdot \overline{H} \) for all \( P \in \mathcal{P} \) and \( H \in \widehat{H} \). We for each \( \lambda \in \mathcal{X} \) we define \( E_\lambda \in \mathcal{P} \) by

\[
E_\lambda = \lambda + \sum_{x \in \mathcal{W}} v^{l(x)} x(A_0^-),
\]

and we denote by \( \mathcal{P}^\circ \subset \mathcal{P} \) the \( \widehat{H} \)-submodule generated by all \( E_\lambda \).
Theorem 9.1 ([Lus80b], [Soe97]).

1. On \( \mathcal{P}^o \) there exists a unique \( \tilde{\mathcal{H}} \)-skew linear involution \( P \mapsto \overline{P} \) with \( \overline{E}_\lambda = E_\lambda \) for all \( \lambda \in X \).

2. For each \( A \in \mathcal{A} \) there exists a unique element \( \overline{P}_A \in \mathcal{P}^o \) with \( \overline{P}_A = P_A \) and \( P_A \in A + \sum_B v\mathbb{Z}[v]B \).

We define \( p_{A,B} \in \mathbb{Z}[v] \) by \( \overline{P}_A = \sum_B p_{B,A}B \) and call these polynomials, following Soergel, the periodic polynomials.

9.2. Lusztig’s algorithm. One proves the existence of the elements \( \overline{P}_A \) by induction on the number \( \ell_\lambda(A) \), where \( \lambda \) is such that \( A \in \Pi_\lambda \). If \( \ell_\lambda(A) = 0 \), then \( A = A_\lambda^- \) and the element \( \overline{P}_A = E_\lambda \) satisfies all requirements. If \( \ell_\lambda(A) > 0 \), then there exists \( s \in \hat{S} \) with \( As \in \Pi_\lambda \) and \( \ell_\lambda(As) < \ell_\lambda(A) \). We assume that \( \overline{P}_{As} \) exists and consider the element \( \overline{P}_{As} \cdot \overline{H}_s \). It is self-dual and it is contained in \( A + \sum_B \mathbb{Z}[v]B \). So let us define the polynomials \( a_B \in \mathbb{Z}[v] \) by

\[
\overline{P}_{As} \cdot \overline{H}_s = \sum_B a_B B.
\]

Suppose that \( B \neq A \) is such that \( a_B(0) \neq 0 \). One shows that this implies \( \ell_\mu(B) < \ell_\lambda(A) \), where \( \mu \) is such that \( B \in \Pi_\mu \). Hence we can assume the existence of \( \overline{P}_B \). Now the element \( \overline{P}_{As} \cdot \overline{H}_s - \sum_{B \neq A} a_B(0) \overline{P}_B \) satisfies all our requirements.

9.3. Characters of objects in \( \overline{C}^\infty \). Let \( M \) be an object in \( \overline{C}^\infty \). Then each subquotient \( M_{[A]} \) is a graded free \( S \)-module of finite rank. Let \( \lambda \in X \). With \( M \) we now associate the “\( \lambda \)-character”

\[
h_\lambda(M) := \sum_{A \in \mathcal{A}} (v^{l_\lambda(A)}rk M_{[A]}) A \in \mathcal{P}.
\]

Note that \( h_\lambda(M) \) and \( h_\mu(M) \) differ only by multiplication by a power of \( v \).

Lemma 9.2. Let \( \lambda \in X \), \( s \in \hat{S} \) and let \( M \) be an object in \( \overline{C}^\infty \). Then we have \( h_\lambda(\overline{\vartheta}_sM) = h_\lambda(M) \cdot \overline{H}_s \).

Proof. From Lemma 7.13 and the equation \( rk(M_{[A],As}) = rk(M_{[A]}) + rk(M_{[As]}) \) we deduce

\[
\overline{rk}(\overline{\vartheta}_sM)_{[A]} = \begin{cases} v^{-1}(rk M_{[A]} + rk M_{[As]}), & \text{if } As \succ A, \\ v(rk M_{[A]} + rk M_{[As]}), & \text{if } A \succ As. \end{cases}
\]

From this we deduce

\[
v^{-1}h_\lambda(\overline{\vartheta}_sM) = \sum_{A \in \mathcal{A}} v^{l_\lambda(A)} \overline{rk}(\overline{\vartheta}_sM)_{[A]} A
\]

\[
= \sum_{A \prec As} v^{l_\lambda(A)-1}(rk M_{[A]} + rk M_{[As]})A + \sum_{A \succ As} v^{l_\lambda(A)+1}(rk M_{[A]} + rk M_{[As]})A.
\]
On the other hand,

\[ v^{-l} h_\lambda(M) \cdot H_s = \left( \sum_{A \in A^s} v^{\ell_A(A)} \text{rk} M[A] A \right) \cdot H_s \]

\[ = \sum_{A \prec As} v^{\ell_A(As)} \text{rk} M[As] A + \sum_{A \succ As} v^{\ell_A(As)} \text{rk} M[As] A \]

\[ + \sum_{A \prec As} v^{\ell_A(As)} \text{rk} M[As] A + \sum_{A \succ As} v^{\ell_A(As)} \text{rk} M[As] A \]

\[ = \sum_{A \prec As} v^{\ell_A(As)} \text{rk} M[As] A + \sum_{A \succ As} v^{\ell_A(As)} \text{rk} M[As] A \]

\[ = v^{-l} h_\lambda(\vec{S})M. \]

\[ \square \]

**Lemma 9.3.** Suppose that \( \text{char } k \neq 3 \) is \( R \) is of type \( G_2 \). We have \( h_\lambda(P(A^\lambda_\emptyset)) = E_\lambda \).

**Proof.** We start the proof with a remark. Note that for any \( \mu \in X \) we have

\[ h_{\lambda+\mu}(P(\mu + A)) = \mu + h_\lambda(P(A)) \]

as \( P(\mu + A) \cong T_\mu(P(A)) \). Moreover, we have

\[ P_{\mu + A} = \mu + P(A). \]

Hence, if we prove, then we can immediately deduce it for the alcove \( t_\mu(A) \) for each \( \mu \in X \) and therefore it is enough to prove the statement for \( \lambda = 0 \). From Lemma 8.6 we deduce that \( \hat{P}(A^\emptyset_\emptyset) \) is a direct summand of \( \Upsilon(\hat{P}(A^\emptyset_\emptyset)) \). But the latter is a free \( \mathbb{Z} \)-module of rank 1 by Proposition 8.2 and Lemma 6.12 hence it is indecomposable. On the support of \( \hat{P}(A^\emptyset_\emptyset) \) (i.e. on \( \mathcal{O}^0 \)), the partial orders \( \succeq \) and \( \preceq \) are opposite, so we can identify \( \hat{P}(A^\emptyset_\emptyset)[B,\succeq] \) with \( \hat{P}(A^\emptyset_\emptyset)[B,\preceq] \otimes_S S \) as graded \( S \)-modules.

Now \( \hat{P}(A^\emptyset_\emptyset) \) is the space of global sections of the Braden–MacPherson sheaf on the full sub-moment graph of \( \hat{G} \) with vertices \( \{ B \in \mathcal{O} \mid B \leq A^\emptyset_\emptyset \} \). This moment graph coincides with the moment graph associated with the finite root system.
Hence
\[ \text{rk} \hat{P}(A_0^-)_{\leq B, \succeq} = \text{rk} \hat{P}(A_0^-)_{\leq B, \preceq} = \begin{cases} v^{-l+2l(x)}, & \text{if } B = x(A_0^-), x \in \mathcal{W}, \\ 0, & \text{otherwise}. \end{cases} \]

Note that \( \ell_0(x(A_0^-)) = -l(x) \). We obtain
\[ h_0(\hat{P}(A_0^-)) = \sum_{x \in \mathcal{W}} v^{l(x)} x(A_0^-) \]
\[ = \sum_{x \in \mathcal{W}} v^{l(x)} x(A_0^-) \]
\[ = \sum_{x \in \mathcal{W}} v^{l(x)} x(A_0^-) \]
\[ = E_0. \]

The following is the main result of this paper.

**Theorem 9.4.** Suppose that \( \text{ch} k = 0 \). Let \( \lambda \in X \) and let \( A \) be an alcove in \( \Pi_\lambda \). Then
\[ h_\lambda(\hat{P}(A)) = P_A. \]

**Proof.** We prove the statement by induction on \( \ell_\lambda(A) \). If \( \ell_\lambda(A) = 0 \), i.e. \( A = A^- \), the claim is already proven in Lemma 9.3 (even without the assumption on the characteristic). So now let \( A \neq A^- \) be an alcove in \( \Pi_\lambda \) and assume that the claim holds for any alcove \( B \) with \( \ell_\mu(B) < \ell_\lambda(A) \), where \( \mu \) is such that \( B \in \Pi_\mu \).

By what we have observed in the proof of Lemma 9.3 we can assume that \( A \) is an alcove in \( \Pi_0 \). Then \( \hat{P}(A) \) is a direct summand of \( \Upsilon(\hat{P}(A)) \) by Lemma 8.6.

There is \( s \in \hat{S} \) with \( As \succ A \) and \( As \in \Pi_0 \). By the inductive hypothesis we have
\[ h_0(\hat{P}(As)) = P_{As} \in As + \sum_{B \succ As} v\mathbb{Z}[v]B. \]
From Lemma 9.2 we deduce
\[ h_0(\vartheta_s \hat{P}(As)) = P_{As} \cdot H_s \in A + \sum_{B \succ A} \mathbb{Z}[v]B. \]

Let us define the polynomials \( a_B \in \mathbb{Z}[v] \) for \( B \succ A \) by
\[ h_0(\vartheta_+ \hat{P}(As)) = P_A + \sum_{B \succ A} a_B B. \]
We then have
\[ P_A = h_0(\vartheta_+ \hat{P}(As)) - \sum_{B \succ A} a_B(0) P_B \]
by Lusztig’s algorithm. Let us define the numbers \( n_{B,r} \) for \( B \succ A \) and \( r \in \mathbb{Z} \) by
\[ \vartheta_+ \hat{P}(As) = \hat{P}(A) \oplus \bigoplus_{B \succ A, r \in \mathbb{Z}} n_{B,r} \cdot \hat{P}(B)[r]. \]

In order to prove \( h_0(\hat{P}(A)) = P_A \) it is enough to show the following:
• If $n_{B,r} \neq 0$, then $r = 0$.
• For each $B \succ A$ we have $n_{B,0} = a_B(0)$.

From (\*) we deduce

$$\text{rk } \overline{\mathcal{P}}(\mathcal{A})|_B \in v^{-l_0(B)}\mathbb{Z}[v].$$

Now suppose that $\mathcal{P}(B)[r]$ occurs as a direct summand of $\overline{\mathcal{P}}(\mathcal{A})$ for some $B \succ A$. We will now determine $r$. Suppose that $B$ is contained in $\Pi_\mu$. Then

$$\text{rk } \mathcal{P}(B)[r] = v^{-r-l_\mu(B)}$$
and we deduce $-r - l - \ell_\mu(B) \geq -l - \ell_0(B)$ or

$$r \leq \ell_0(B) - \ell_\mu(B)$$

The number on the right hand side is independent of $B$ and equals $\ell_0(A^-)$. By Theorem 8.14, $\overline{\mathcal{P}}(\mathcal{A})$ is $\overline{\mathcal{D}}_0$-selfdual. Hence $\overline{\mathcal{P}}(\mathcal{A})$ is $\overline{\mathcal{D}}_0$-selfdual, so $\overline{\mathcal{D}}_0(\mathcal{P}(B)[r])$ occurs as a direct summand of $\overline{\mathcal{P}}(\mathcal{A})$ as well. Again by Theorem 8.14

$$\overline{\mathcal{D}}_0(\mathcal{P}(B)[r]) \cong \mathcal{P}(w_0 \circ w_\mu(B))[−r].$$

The same arguments as above yield (note that $w_0 \circ w_\mu(B) \in \Pi_{w_0(\mu)}$)

$$-r \leq \ell_0(w_0 \circ w_\mu(B)) - \ell_{w_0(\mu)}(w_0 \circ w_\mu(B)) = \ell_0(A^-_{w_0(\mu)}) = -\ell_0(A^-_\mu).$$

From both inequalities we deduce $r = \ell_0(A^-_\mu)$. Then

$$\text{rk } \mathcal{P}(B)[r] = v^{-l_\mu(B) - \ell_0(A^-_\mu)} = v^{-l - \ell_0(B)},$$

hence $n_{B,r} \leq a_B(0)$. Now the right hand side is $\neq 0$ only if $B \in \Pi_0$, i.e. $\mu = 0$. Hence $r = 0$. Moreover, we have shown that for all $B \succ A$ we have

$$n_{B,0} \leq a_B(0).$$

In particular, each coefficient of $h_0(\mathcal{P}(\mathcal{A}))$ is bigger or equal than the corresponding coefficient of $\mathcal{P}_A$. So it suffices to show $\text{rk } \mathcal{P}(\mathcal{A})|_{B, \geq} \leq p_{B,A}(1)$, where by $\text{rk } M \in \mathbb{N}$ we denote the ungraded rank of a free $S$-module $M$.

We now use the fact that $\mathcal{P}(\mathcal{A})$ is a direct summand of $\Upsilon(\mathcal{P}(\mathcal{A}))$. In particular, for the ungraded ranks we have

$$\text{rk } \mathcal{P}(\mathcal{A})|_{B, \geq} \leq \text{rk } \mathcal{P}(\mathcal{A})|_{B, \leq}.$$

By Theorem 8.4, $\text{rk } \mathcal{P}(\mathcal{A})|_{B, \leq} = h_{w_B,w_A}(1)$. By [Lus80b, Corollary 5.3], $p_{B,A}(1) = h_{w_B,w_A}(1)$. Hence $\text{rk } \mathcal{P}(\mathcal{A})|_{B, \geq} \leq p_{B,A}(1)$. ☐

10. An application

In the book [AJS94] Andersen, Jantzen and Soergel established one of the major steps in the proof of Lusztig’s conjecture on the characters of a reductive group $G$ over a field of positive characteristics. They defined a combinatorial category $\mathcal{K}$ and a full subcategory $\mathcal{M}$ that is generated from a unit object by repeatedly applying various translation functors $\mathcal{T}^* : \mathcal{K} \to \mathcal{K}$. They then defined a fully faithful functor from a deformed category of $G_1 T$-modules ($G_1$ being the
kernel of the Frobenius homomorphism on $G$ and $T$ a maximal torus of $G$) to $\mathcal{K}$ and showed that the subcategory containing certain projective $G_1T$-modules is equivalent to $\mathcal{M}$, and that baby Verma multiplicities of those projectives can be calculated inside $\mathcal{M}$. As the character problem for rational irreducible $G$-modules can be translated into a multiplicity problem for projective $G_1T$-modules, one can transfer Lusztig’s conjecture into the “combinatorial” framework of the category $\mathcal{K}$. Now, the category $\mathcal{K}$ can be defined over almost any ring, and this made it possible to compare Lusztig’s modular conjecture with the analogous, characteristic 0 conjecture on quantum group representations. The latter is a theorem by work of Kazhdan and Lusztig.

In [Fie11] a functor $\Psi$ from the full subcategory of projective objects in $\hat{\mathcal{C}}^\leq$ (which is denoted by $\mathcal{H}_k$ in loc.cit.) to $\mathcal{K}$ was defined and it was shown that the subcategory $\mathcal{H}^\circ$ of projectives associated to alcoves in the antifundamental chamber is mapped into $\mathcal{M}$. Together with the work of Andersen, Jantzen and Soergel, this gave an alternative proof of Lusztig’s modular conjecture. Moreover, the localization results in [FW14] yield a functor from parity sheaves on affine flag manifolds that are associated to affine Weyl group elements that are maximal in their $W$-orbit to $\mathcal{M}$, hence giving a direct link between the geometry of affine flag manifolds and modular representation theory. (For the experts: this link is independent of the geometric Satake correspondence).

In the following we show that there is a functor $\Theta: \mathcal{C}^\leq \to \mathcal{K}$ and that there is a factorization of $\Psi: \mathcal{H}^\circ \to \mathcal{K}$ as $\Psi = \Theta \circ \Upsilon$. From this we deduce that the projective objects in $\mathcal{C}^\leq$ are mapped, by $\Theta$, to $\mathcal{M}$ and hence to projective $G_1T$-modules. In particular, a detailed understanding of $\mathcal{C}^\leq$ over a field of positive characteristic might help us understand the degeneration of projective $G_1T$-modules at exceptional primes.

10.1. The category $\mathcal{K}$. The “combinatorial category” of Andersen, Jantzen and Soergel is the following.

**Definition 10.1** ([AJS94], cf. [Fie11]). Let $\mathcal{K}$ be the category that consists of objects $M = \{M(A)\}_{A \in \mathcal{A}}, \{M(A, \beta)\}_{A \in \mathcal{A}, \beta \in \mathbb{R}^+}$, where

1. $M(A)$ is an $S^\theta$-module for each $A \in \mathcal{A}$ and
2. for $A \in \mathcal{A}$ and $\beta \in \mathbb{R}^+$, $M(A, \beta)$ is an $S^\beta$-submodule of $M(A) \oplus M(\beta \uparrow A)$.

A morphism $f: M \to N$ in $\mathcal{K}$ is given by a collection $(f_A)_{A \in \mathcal{A}}$ of homomorphisms $f_A: M(A) \to N(A)$ of $S^\theta$-modules, such that for all $A \in \mathcal{A}$ and $\beta \in \mathbb{R}^+$, $f_A \oplus f_{\beta \uparrow A}$ maps $M(A, \beta)$ into $N(A, \beta)$.

We are going to use the following shift functors in $\mathcal{K}$.
**Definition 10.2.** Let \( \lambda \in X \). The functor \( T^K_\lambda : K \to K \) is defined as follows. Let \( M \) be an object in \( K \). We set
\[
T^K_\lambda(M)(A) = M(t_{\lambda, A})(\lambda + A),
\]
\[
T^K_\lambda(M)(\beta, A) = M(\beta, t_{\lambda, A})(\beta + A).
\]
The action on the spaces of morphisms is the obvious one.

Note that \( \lambda + (\beta \uparrow A) = \beta \uparrow (\lambda + A) \), so the above is well-defined.

10.2. **Special objects in** \( K \). Andersen, Jantzen and Soergel defined in [AJS94] a certain object \( P_0 \) in \( K \) and a translation functor \( T'^s : K \to K \) for any \( s \in \hat{S} \). The category of special objects \( \mathcal{M}' \) in is then the full subcategory of \( K \) that contains all direct sums of direct summands of objects that are obtained by applying various translation functors to \( P_0 \).

In Section 5.3 of [Fie11] we gave a different definition of the translation functors \( T^s \) that lead to a different, but equivalent subcategory \( \mathcal{M} \) of special objects inside \( K \). In Section 5.5 of loc. cit. we defined a functor from the category \( \hat{\mathcal{H}} \) (denoted \( \mathcal{H} \) in [Fie11]) to \( K \) and showed (Theorem 5.4 in loc.cit.) that its image is contained in \( \mathcal{M} \). It is convenient for us to slightly enlarge the category \( \mathcal{M} \) by applying the shift functors \( T^K_\lambda \) to its objects.

**Definition 10.3.** We denote by \( \mathcal{M}' \) the full subcategory of \( K \) that contains all direct sums of objects of the form \( T^K_\lambda M \), where \( M \) is an object in \( \mathcal{M} \) and \( \lambda \in X \).

10.3. **The functor** \( \Psi : \hat{\mathcal{H}}^{\leq} \to K \). In Section 5.5 of [Fie11] we constructed the functor \( \Psi \) as follows. Let \( M \) be an object in \( \hat{\mathcal{H}}^{\leq} \). We then set
\[
\Psi(M)(A) = M_{\delta^\vee = 0}^B,
\]
\[
\Psi(M)(\beta, A) = \im(((M_{\leq A}^\beta)_{\delta^\vee = 0}^\beta \to M_{\delta^\vee = 0}^A \oplus M_{\delta^\vee = 0}^{0, \beta \uparrow A}),
\]
where the image above refers to the composition
\[
(M_{\leq A}^\beta)_{\delta^\vee = 0}^\beta \subset \bigoplus_{B \in \mathcal{A}} M_{\delta^\vee = 0}^{B, \beta \uparrow A} \oplus \frac{pr}{\|_{\|}} M_{\delta^\vee = 0}^{0, A} \oplus M_{\delta^\vee = 0}^{0, \beta \uparrow A}.
\]

10.4. **The functor** \( \Theta : \overline{C}^{\leq} \to K \). Let \( \overline{M} \) be an object in \( \overline{C}^{\leq} \). We define an object \( \Theta(\overline{M}) \) in \( K \) as follows. Let \( A \) be an alcove and suppose that \( \Omega \) is its \( \mathbb{Z}R \)-orbit. We define
\[
\Theta(\overline{M})(A) := \overline{M}_{\Omega}^{[A]}.
\]
For \( A \in \mathcal{A} \) and \( \beta \in R^+ \) we set
\[
\Theta(\overline{M})(A, \beta) := \overline{M}_{\Omega}^{[A, \beta \uparrow A]}.
\]
Note that this is a torsion free \( S^\beta \)-module, and hence it is naturally contained in
\[
(\overline{M}_{\Omega}^{[A, \alpha \uparrow A]}) \otimes_{\mathbb{Z}R} S^\theta = \overline{M}_{\Omega}^{[A]} \oplus \overline{M}_{\Omega}^{[A, \alpha \uparrow A]}.
\]
This clearly defines a functor \( \Theta : \overline{C}^{\leq} \to K \) and we have \( T^K_\lambda \circ \Theta = \Theta \circ T^\lambda \).
Proposition 10.4.  
(1) Let $\overline{P}$ be an object in $\mathcal{C}^<$ that is a direct summand of an object contained in $\Upsilon(\hat{\mathcal{H}}^\circ)$. Then $\Theta(\overline{P})$ is contained in $\mathcal{M}$.

(2) Suppose that $\overline{P}$ is a projective object in $\mathcal{C}^>$. Then $\Theta(\overline{P})$ is contained in $\mathcal{M}'$.

Proof. For each indecomposable projective object $\overline{P}$ there is $\lambda \in X$ such that $T^C_\lambda \overline{P}$ is a direct summand of an object in the image of $\Upsilon(\hat{\mathcal{H}}^\circ)$ (cf. Lemma 8.6). Since $T^C_\lambda \circ \Theta = \Theta \circ T^C_\lambda$, statement (2) follows from statement (1).

Now let $\hat{P}$ be an object in $\hat{\mathcal{H}}^\circ$. We have to show that there is a functorial isomorphism between $\Psi(\hat{P})$ and $\Theta \circ \Upsilon(\hat{P})$. We have
\[
\Theta \circ \Upsilon(\hat{P})(A) = (\Upsilon \hat{P})^{0,\Omega}_{[A]}
\]
\[
= \left( \bigoplus_{\beta \in \Omega} \hat{P}^{0,\beta B}_{\delta^\vee = 0}[A] \right)
\]
\[
= \hat{P}^{0,A}_{\delta^\vee = 0}
\]
\[
= \Psi(\hat{P})(A)
\]
and
\[
\Theta \circ \Upsilon(\hat{P})(\beta, A) = (\Upsilon \hat{P})^{\beta,\Omega,\beta^\vee \Omega}_{[A,\beta^\vee A]}
\]
\[
= \text{im} \left( (\Upsilon \hat{P})^\beta_{\geq A} \rightarrow (\Upsilon \hat{P})^{0,\beta A} \oplus (\Upsilon \hat{P})^{0,\beta^\vee A} \right)
\]
\[
= \text{im} \left( (\hat{P}_{\delta^\vee = 0})^\beta_{\geq A} \rightarrow \hat{P}^{0,A}_{\delta^\vee = 0} \oplus \hat{P}^{0,\beta^\vee A}_{\delta^\vee = 0} \right)
\]
\[
= \Psi(\hat{P})(\beta, A)
\]

\[\square\]

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