Quantum gravity effect on the Hawking radiation of charged rotating BTZ black hole

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Abstract

In this study, the quantum gravity effect on the tunnelling radiation of charged massive particles with spin-0, spin-1/2 and spin-1 from 2+1 dimensional charged rotating Banados-Teitelboim-Zanelli (BTZ) black hole is looked into by using the Hamilton-Jacobi approach. For this, we calculate the modified Hawking temperature of the black hole by using the modified Klein-Gordon, the Dirac and the vector boson equations based on the Generalized Uncertainty Principle (GUP), and we notice that the modified Hawking temperature of the black hole depends not only on the black hole properties, but also on the angular (orbital+spin) momentum, energy, charge and mass of the tunnelling particle. Furthermore, we discuss the stability of the black hole in the context of the modified heat capacity, and observe that it undergoes a second-type phase transition in the presence of the quantum gravity effect, but a first-type transition in the absence of the quantum gravity effect.

Keywords: BTZ black hole, Quantum gravity, tunnelling, Hawking radiation

1. Introduction

It is one of the most important problems of modern physics to construct a self-consistent quantum gravity theory by merging quantum mechanics with general relativity. With such a theory, we expect to clarify the fundamental physical problems in gravity, e.g. the origin of the universe and the final stage of a black hole. At present, there are several theories as candidate such as the string theory and the loop quantum gravity theory which exhibit some features already expected from a self-consistent quantum gravity theory. The common feature of these theories is that they all point out the existence of a minimum observable length in the order of the Planck scale. The existence of such a minimal length leads to the generalized Heisenberg uncertainty principle (GUP), and it causes in some modifications on the quantum mechanical relations. Together with the modifications, the intrinsic properties of a particle as an extended object begin to emerge in the quantum gravity effects.

With the well-known studies of Bekenstein, Bardeen, Carter and Hawking, a black hole has been considered as a thermodynamical system. From these studies, in particular, in the Hawking’s studies that consider quantum mechanical methods in a
curved spacetime it was proved that the thermal radiation of a black hole, known as Hawking radiation, stem from the quantum vacuum fluctuation near the black hole horizon. Since then, to get this radiation it has been put forward various alternative approaches. For instance, the Hamilton-Jacobi approach based on the tunnelling process of an elementary particle throughout the classically forbidden trajectory from inside to outside of a black hole horizon is one of the effective ways to derive the radiation [25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. In all the studies realized by the context of the standard Heisenberg uncertainty principle, it has been seen that the Hawking radiation of a black hole depends on only the properties of black hole. However, in the studies performed in the GUP context, it has been come out that the Hawking radiation is depended on both the properties of the black hole and the tunneling particle [47, 48, 11, 12, 13, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 14]. At that case, it can be determined that what kind of particle tunnels from a black hole [15, 16]. With this motivation, in the study, we will investigate the Hawking radiation of the 2 + 1-dimensional charged rotating Banados-Teitelboim-Zanelli (hereafter CR-BTZ) black hole by using the quantum tunnelling process of the charged massive scalar, Dirac and vector boson particles under the GUP effects. From the derived results for the scalar particle, we will also discuss the stability of the black hole.

The structure of this paper is as follows: In the context of the GUP, we find the modified Hawking temperature of the black hole by using the tunnelling process of the charged massive scalar particle in section-2, charged massive Dirac particle in section-3 and charged massive vector particle in section-4. In section-5, using the modified Hawking temperature of the charged massive scalar particle, we calculate the modified heat capacity of the black hole. Additionally, the local stability of the black hole is discussed before the conclusion.

2. Tunneling of the charged massive scalar particle from the CR-BTZ black hole

The BTZ black hole is an exact solution of the (2 + 1)-dimensional Einstein-Maxwell gravity theory and its charged and rotating case includes more rich mathematical and physical structure than its non-charged and non-rotating cases. For this reason, the CR-BTZ black hole satisfies a perfect spacetime background to emerge the GUP effects. In order to emerge these effects, we start by writing its explicit form of the spacetime background as follows [62];

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2\left[d\phi + N^\theta(r)dt\right]^2,$$

where the $f(r)$ and $N^\theta(r)$ are

$$f(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4l^2} - \frac{Q^2}{2} \ln(r),$$

$$N^\theta(r) = -\frac{J}{2r^2},$$

where $l$ is radius related to the cosmological constant as $l^2 = -1/\Lambda$, and, $M$, $Q$ and $J$ are the mass, electric charge and angular momentum of the black hole, respectively. For convenience, we carry out the dragging coordinate transformation $d\phi = d\phi + N^\theta dt$ by means of the Killing vectors, $(\partial_t)$ and $(\partial_\phi)$. Then, the metric takes the following form

$$ds^2 = f(r)dt^2 - \frac{1}{2f(r)}dr^2 - r^2d\phi^2.$$
The mass and the angular velocity of the outer horizon of the black hole are given as follows:

\[ M = \frac{r_+^2}{f} + \frac{J}{4r_+^2} - \frac{Q^2}{2} \ln(r_+), \quad (3) \]

\[ \Omega_+ = \frac{J}{2r_+}, \quad (4) \]

respectively, where \( r_+ \) is the radius of the outer horizon.

In order to investigate the quantum gravity effects on the tunneling process of the scalar particles from the black hole, by using the GUP relations, the standard Klein-Gordon equation is modified as

\[ \left\{ (-i\hbar)^2 \partial_t^2 - M_0^2 - i\hbar q(\partial_t A^0) - 2i\hbar qA^0 \partial_t + q^2 A_\mu A^\mu \right\} \left[ \Phi - 2\alpha (-i\hbar^2 \partial_t \partial^t + M_0^2) \Phi \right] + (i\hbar)^2 \partial_t \partial^t \Phi = 0, \quad (5) \]

where \( \Phi, M_0 \) and \( A_\mu \) are the modified wave function, mass of the scalar particle and the electromagnetic potential vector, respectively. In addition to this, \( \alpha = \alpha_0/M_p^2 \) with the \( M_p^2 \) and \( \alpha_0 \) are the Planck mass and dimensionless parameter, respectively \([6, 7]\). For the non-vanishing component of the electromagnetic potential, \( A_0 \), and \( \alpha^2 = 0 \), the explicit form of the modified Klein-Gordon equation can be simplified by the following way:

\[ \frac{\hbar^2}{f} \frac{\partial^2 \Phi}{\partial t^2} + \frac{\hbar^2}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} + 2\alpha \hbar^2 \frac{\partial}{\partial t} \left( \partial_t \partial^t \Phi \right) + \frac{2\alpha \hbar^2}{r^2} \frac{1}{r} \frac{\partial^2 \Phi}{\partial \phi^2} \]

\[ -\hbar \frac{\partial \Phi}{\partial t} \left( \frac{q^2 A_0^2}{f} - M_0^2 \right) \left( 1 - 2\alpha M_0^2 \right) \Phi + 2i\alpha \hbar A_0 \frac{\partial \Phi}{\partial t} \left( 1 - 2\alpha M_0^2 \right) \frac{\partial \Phi}{\partial \phi} = 0. \quad (7) \]

Then, the modified Klein-Gordon equation in the CR-BTZ black hole background and the electromagnetic potential, \( A_0 = -Q \ln(\xi) \), is written as

After that, if the modified wave function of the scalar particle, \( \Phi(t, r, \phi) \), is defined as

\[ \Phi(t, r, \phi) = A \exp \left( \frac{i}{\hbar} S(t, r, \phi) \right), \quad (8) \]

where \( A \) is a constant and \( S(t, r, \phi) \) is the classical action function. Substituting the wave function expression into the Eq. (7) and subsequently neglecting the higher order terms of \( \hbar \), we get the modified Hamilton-Jacobi equation as follows:

\[ \left( \frac{\partial S}{\partial t} \right)^2 - f^2 \left( \frac{\partial S}{\partial r} \right)^2 - \frac{f}{r^2} \left( \frac{\partial S}{\partial \phi} \right)^2 = \left( q^2 A_0^2 - M_0^2 \right) - \alpha \frac{2f}{r^2} \left( \frac{\partial S}{\partial \phi} \right)^2 \]

\[ + 2\alpha \hbar A_0 \left( 1 - 2\alpha M_0^2 \right) \left( \frac{\partial S}{\partial t} \right) = \alpha \left[ 2M_0^2 \left( q^2 A_0^2 - M_0^2 \right) + 2f^3 \left( \frac{\partial S}{\partial r} \right)^2 \right] / 3. \]
To solve this equation, by using separation of variable method, the \( S(t, r, \phi) \) can be separated as
\[
S(t, r, \phi) = -(E - j\Omega_r) t + j\phi + W(r) + C
\]
in which \( C \) is a complex constant, \( \Omega_r \) is the angular velocity of the outer horizon, and \( E, j \) and \( W(r) = W_0(r) + \beta W_1(r) \) are energy, angular momentum and radial trajectory of the particle, respectively [52, 15, 16]. After some calculations, the radial trajectory \( W_\pm(r) \) are obtained as
\[
W_\pm(r) = \pm \int \frac{\sqrt{(E - j\Omega_r - qA_0)^2 - f \left( M_0^2 + f^2 / r^2 \right)}}{f} [1 + a\Theta] \, dr,
\]
where \( W_+ (r) \) and \( W_- (r) \) correspond to the outgoing and incoming particle trajectories, respectively, and the abbreviation \( \Theta \) is
\[
\Theta = \frac{f^2 \left( M_0^2 - j^2 / r^2 \right) + qA_0 (M_0^2 - j^2 / r^2) [2(E - j\Omega_r) - qA_0]}{f \left( (E - j\Omega_r - qA_0)^2 - f \left( M_0^2 + j^2 / r^2 \right) \right)} - \frac{f \left( (E - j\Omega_r)^2 - f \left( M_0^2 + j^2 / r^2 \right) \right)}{f}.
\]
As \( f(r) \approx (r - r_+) f'(r_+) \) near the outer horizon, the \( W_\pm(r) \) are computed as
\[
W_\pm(r_+) = \pm \text{Im} \left[ \frac{E - j\Omega_r + qQ \ln(\frac{r}{r_+})}{H} \right] [1 + a\Sigma],
\]
with the abbreviations \( \Sigma \) and \( H \) are
\[
\Sigma = \frac{4 \text{Im} qQ H \ln(\frac{r}{r_+}) [2(E - j\Omega_r) + qQ \ln(\frac{r}{r_+}) + 3H(E - j\Omega_r)^2 r_+^2 \left( M_0^2 + j^2 / r_+^2 \right)]}{2H r_+^2 \left( E - j\Omega_r + qQ \ln(\frac{r}{r_+}) \right)^2},
\]
\[
H = \frac{2r_+^2 - \frac{J^2}{2r_+^2} - \frac{Q^2}{2r_+}}{2r_+},
\]
respectively. On the other hand, the tunneling probabilities of particles crossing the outer horizon are given by
\[
P_{\text{out}} = \exp \left[ \frac{-2}{\hbar} \text{Im} W_+(r_+) \right],
\]
\[
P_{\text{in}} = \exp \left[ \frac{-2}{\hbar} \text{Im} W_-(r_+) \right].
\]
Hence, the tunneling probability of the particle is

\[ \Gamma = e^{-\frac{\hbar}{\pi} \text{Im} S} = \frac{P_{\text{out}}}{P_{\text{in}}} = e^{-\frac{E_{\text{total}}}{T_{\text{KG}} H}}, \]  

(13)

where \( E_{\text{total}} \) is total energy of the scalar particle, and \( T_{\text{KG}} \) is the modified Hawking temperature of the outer horizon for the scalar particle \([63, 64, 65, 66]\). Then, the tunneling probability of the charged massive scalar particle for the black hole is written as

\[ \Gamma = \exp \left( -\frac{4\pi [E - j\Omega + qQ \ln(r^+)]}{\hbar H} \right) \]  

(14)

and thus the modified Hawking temperature, \( T_{\text{KG}} \), is obtained as follows

\[ T_{\text{KG}} = T_H \frac{1}{1 + a\Sigma}. \]  

(15)

where \( T_H \) is the standard Hawking temperature of the black hole and its explicit expression is

\[ T_H = \frac{\hbar}{4\pi} \left( \frac{2r_+}{\ell^2} - \frac{J^2}{2r_+^2} - \frac{Q^2}{2r_+} \right). \]  

(16)

Furthermore, neglecting the higher order \( \alpha \) terms (since \( \alpha \ll 1 \)), we find the modified Hawking temperature of the black hole as follows;

\[ T_{\text{KG}} \approx T_H [1 - a\Sigma]. \]  

(17)

This result indicates that the modified Hawking temperature of the charged massive scalar particle is lower than the standard Hawking temperature. Moreover, it shows that the modified Hawking temperature depends on not only the black hole but also the tunneling particle properties.

3. Tunneling of the charged massive Dirac particle from the CR-BTZ black hole

In order to make a discussion on a tunneling Dirac particle from the black hole under the GUP effect, by means of the GUP relations, the standard Dirac equation \([67]\) is be modified as follows;

\[ -\sigma^i(x)\partial_i \Psi = \left( \sigma^i(x)\partial_i - \sigma^j(x)\Gamma_\mu - \frac{m_0}{\hbar} \sigma^j(x)\frac{\partial^i}{\hbar} A_\mu \right) \left( 1 + \alpha h^2 \partial_j \partial^j - am_0^2 \right) \Psi, \]  

(18)

and its explicit form is writing as

\[ \sigma^0(x)\partial_0 \Psi + \sigma^i(x) \left( 1 - am_0^2 \right) \partial_i \Psi + i\alpha h^2 \sigma^j(x)\partial_i \left( \partial_j \partial^j - \frac{m_0}{\hbar} \right) \Psi - \frac{m_0}{\hbar} \left( 1 + \alpha h^2 \partial_j \partial^j - am_0^2 \right) \Psi - \sigma^j(x)\frac{\partial^i}{\hbar} A_\mu \left( 1 + \alpha h^2 \partial_j \partial^j - am_0^2 \right) \Psi = 0, \]  

(19)

where \( \Psi \) is the modified Dirac spinor, \( m_0 \) is mass of the Dirac particle, \( \sigma^j(x) \) are the spacetime dependent Dirac matrices, and \( \Gamma_\mu \) are spin affine connection for spin-1/2 particle \([67]\). Using
Eq. (2), the spinorial affine connections are derived in terms of Pauli matrices by the following way:

$$\Gamma_0 = -\frac{i}{4} f(r) \sigma^3 \sigma^1, \quad \Gamma_1 = 0, \quad \Gamma_2 = \frac{1}{2} \sqrt{f(r)} \sigma^1 \sigma^2. \quad (20)$$

To proceed the tunneling probability of a charged massive Dirac particle from the black hole, we use the following ansatz for the modified wave function;

$$\Psi(x) = \exp\left(\frac{i}{\hbar} \mathcal{S}(t, r, \phi) \right) \begin{pmatrix} A(t, r, \phi) \\ B(t, r, \phi) \end{pmatrix}, \quad (21)$$

where the $A(t, r, \phi)$ and $B(t, r, \phi)$ are the functions of spacetime coordinates. Inserting the Eqs. (20) and (21) into the Eq. (19), then, we obtain the following coupled equations for the leading order in $\hbar$ and $\alpha$:

$$A \left[ \frac{1}{\sqrt{f}} \left( \frac{\partial \mathcal{S}}{\partial t} \right) + m_0 \left( 1 - \alpha m_0^2 \right) + \frac{am_0}{r} \left( \frac{\partial \mathcal{S}}{\partial \phi} \right)^2 + am_0 f \left( \frac{\partial \mathcal{S}}{\partial r} \right)^2 \right] + A \left[ \frac{qA_0}{\sqrt{f}} + \frac{aqA_0}{\sqrt{f}} \left( \frac{\partial \mathcal{S}}{\partial t} \right)^2 + \frac{aqA_0}{\sqrt{f}} \left( \frac{\partial \mathcal{S}}{\partial \phi} \right)^2 - \frac{aqA_0 m_0^2}{\sqrt{f}} \right] + B \left[ i \sqrt{f} \left( 1 - \alpha m_0^2 \right) \left( \frac{\partial \mathcal{S}}{\partial t} \right)^2 + \frac{1 - \alpha m_0^2}{r} \left( \frac{\partial \mathcal{S}}{\partial \phi} \right)^2 + i \alpha f^{3/2} \left( \frac{\partial \mathcal{S}}{\partial r} \right)^2 \right] + B \left[ \frac{\alpha}{r} \sqrt{f} \left( \frac{\partial \mathcal{S}}{\partial t} \right)^2 + \frac{\alpha f}{r} \left( \frac{\partial \mathcal{S}}{\partial \phi} \right)^2 + \frac{\alpha}{r} \left( \frac{\partial \mathcal{S}}{\partial \phi} \right)^2 \right] = 0, \quad (22)$$

These coupled equations own non-trivial solutions for the $A(t, r, \phi)$ and $B(t, r, \phi)$ when the determinant of the coefficient matrix is vanished. Neglecting the terms that contain higher order of the $\alpha$ parameter, it gives us to the modified Hamilton-Jacobi equation for the charged massive Dirac particle:

$$\frac{1}{f} \left( \frac{\partial \mathcal{S}}{\partial t} \right)^2 + \frac{2qA_0}{f} \left( \frac{\partial \mathcal{S}}{\partial t} \right)^2 + \frac{q^2 A_0^2}{f} - f \left( \frac{\partial \mathcal{S}}{\partial \phi} \right)^2 - \frac{1}{r^2} \left( \frac{\partial \mathcal{S}}{\partial \phi} \right)^2 - m_0^2 + \alpha \left[ 2m_0^2 - \frac{4f}{r^2} \left( \frac{\partial \mathcal{S}}{\partial \phi} \right)^2 + \frac{2f}{r^2} \left( \frac{\partial \mathcal{S}}{\partial \phi} \right)^4 - 2f \left( \frac{\partial \mathcal{S}}{\partial \phi} \right)^4 \right]$$
\[
+\alpha \left[ 2q^2\Lambda_0^2 \left( \frac{\partial S}{\partial r} \right)^2 + \frac{2q\Lambda_0}{r^2 f} \left( \frac{\partial S}{\partial \Omega} \right) \right] + \frac{2q\Lambda_0}{r^2 f} \left( \frac{\partial S}{\partial \phi} \right) + \frac{2q^2\Lambda_0^2}{r^2 f} \left( \frac{\partial S}{\partial \Omega} \right)^2 \right] = 0. \tag{23}
\]

Using the separation of variable method, the \( S(t, r, \phi) \) can be written as \( S(t, r, \phi) = -(E - j\Omega_+)t + j\phi + K(r) + C \) and then the radial trajectory \( K_\pm(r) \) are found as

\[
K_\pm(r) = \pm \int \frac{\sqrt{(E - j\Omega_+ - q\Lambda_0)^2 - f(r) \left[ (m_0^2 - \beta^2 f^2) \right]}}{f(r)} \left[ 1 + \alpha \chi \right] dr, \tag{24}
\]

where \( K_+(r) \) and \( K_-(r) \) correspond to the outgoing and incoming particle trajectories, respectively, and the abbreviation \( \chi \) is

\[
\chi = \frac{(E - j\Omega_+)(E - j\Omega_+ - q\Lambda_0) \left[ 2m_0^2 f(r) - (E - j\Omega_+ - q\Lambda_0) \right]}{f(r) \left[ (E - j\Omega_+ - q\Lambda_0)^2 - f(r) \left( m_0^2 + j^2 / r^2 \right) \right]}. \tag{25}
\]

Then, by integrating the radial equation, \( K_\pm(r) \) are obtained as

\[
K_\pm(r_\pm) = \pm i \hbar \frac{\sqrt{(E - j\Omega_+ + qQ \ln(\frac{\mp r_{\pm}}{\mp r_\pm}))}}{H} \left[ 1 + \alpha \Pi \right],
\]

where the \( \Pi \) is

\[
\Pi = \frac{3H(E - j\Omega_+)m_0^2 r_+^2}{2r_+^2 H(E - j\Omega_+ + qQ \ln(\frac{\mp r_{\pm}}{\mp r_\pm})) - \left[ (E - j\Omega_+)(j^2 H + 4q^2 Q^2 r_+ \ln(\frac{\mp r_{\pm}}{\mp r_\pm})) + 4(E - j\Omega_+)^2 qQ r_+ \right]}.
\]

Accordingly, by using the Eqs. (12) and (13) for the Dirac particle, the modified Hawking temperature of the charged massive Dirac particle in terms of the standard Hawking temperature, \( T^D_H \), is obtained as

\[
T^D_H = \frac{\hbar H}{4\pi} \frac{1}{\left[ 1 + \alpha \Pi \right]}, \tag{26}
\]

or

\[
T^D_H \approx T_H \left[ 1 - \alpha \Pi \right]. \tag{27}
\]

This result shows that the modified Hawking temperature of the charged massive Dirac particle is lower than the standard Hawking temperature and depends on both the black hole and the particle properties.

### 4. Tunneling of the charged massive vector boson from the CR-BTZ black hole

Given the GUP relations, the standard massive vector boson equation given in [68] can be modified as follows;

\[
-\beta^\lambda(x) \partial_\lambda \overline{\Psi} = \left( i \beta^\mu(x) \partial_\mu - \beta^\mu(x) \frac{\partial_\mu}{\hbar} A_\mu - \frac{\hbar_0}{q^2} - \beta^\mu(x) \frac{q}{\hbar} A_\mu \right) \left( 1 + \alpha \hbar^2 \partial_\lambda \partial^\lambda - \alpha \mu_\lambda \right) \overline{\Psi}
\]
spectively. The $\beta$ spin-1 particle are given as
\[ i\beta^s(x)\partial_0\Psi + i\beta^s(x)(1 - \alpha\mu_0^2)\partial_r\Psi + i\alpha\hbar^2\beta^s(x)\partial_0(\partial_r\Psi) - \frac{\mu_0}{\hbar}(1 + \alpha\hbar^2\partial_r\partial_r - \alpha\mu_0^2)\Psi \\
- i\beta^s(x)\Sigma_\mu(1 + \alpha\hbar^2\partial_r\partial_r - \alpha\mu_0^2)\Psi - \beta^s(x)\frac{g_0}{\hbar}\Lambda_\mu(1 + \alpha\hbar^2\partial_r\partial_r - \alpha\mu_0^2)\Psi = 0, \quad (28) \]
where the $\Psi$ and $\mu_0$ are the modified wave function and mass of the vector boson particle, respectively. The $\beta^s(x)$ and $\Sigma_\mu$ are the Kemmer matrices and spin connection coefficients for the spin-1 particle are given as
\[ \beta^s(x) = \overline{\beta^s(x)} \otimes I + I \otimes \overline{\beta^s(x)}, \]
\[ \Sigma_\mu(x) = \Gamma_\mu(x) \otimes I + I \otimes \Gamma_\mu(x), \quad (29) \]
respectively. To calculate the quantum gravity effect on the Hawking temperature of the black hole, we use the following ansatz for the modified wave function \[ \Psi(x) = \exp\left( iS(t, r, \phi) \right) \begin{pmatrix} A(t, r, \phi) \\ B(t, r, \phi) \\ B(t, r, \phi) \\ D(t, r, \phi) \end{pmatrix}. \quad (30) \]
Then, by using Eqs. \(20\), \(29\) and \(30\), the modified massive vector boson equation is reduced to the three coupled differential equations after neglecting the terms with $h$:
The Eq. (31) has nontrivial solutions for the coefficients \( A(t, r, \phi) \), \( B(t, r, \phi) \) and \( D(t, r, \phi) \) under the condition that the coefficients matrix determinant is 0. Hence, the modified Hamilton-Jacobi equation for the charged massive vector boson particle is derived as follows:

\[
\begin{align*}
B & \left[ -\sqrt{f} \frac{\partial S}{\partial r} - i \frac{1}{r} \frac{\partial S}{\partial \phi} - \alpha \sqrt{f} \left( \frac{\partial S}{\partial r} \right)^2 + \alpha \mu_0 \sqrt{f} \frac{\partial S}{\partial r} - \frac{1}{r} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] \\
& \quad + B \left[ -i \alpha f \left( \frac{\partial S}{\partial r} \right)^2 + \alpha \mu_0 \sqrt{f} \frac{\partial S}{\partial r} - \alpha f \sqrt{f} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] \\
& \quad + D \left[ i \frac{\mu_0}{2} \frac{1 - \alpha \mu_0^2}{f} + i \frac{\alpha \mu_0 f}{2 \sqrt{f}} \left( \frac{\partial S}{\partial r} \right)^2 + i \alpha \mu_0 f \sqrt{f} \left( \frac{\partial S}{\partial r} \right)^2 - i \frac{\mu_0}{2} \sqrt{f} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] \\
& \quad + D \left[ i \frac{\alpha q \mu_0^2}{\sqrt{f}} - i \frac{\alpha q A_0}{r^2} \left( \frac{\partial S}{\partial r} \right)^2 - i \alpha q A_0 \sqrt{f} \left( \frac{\partial S}{\partial r} \right)^2 - i \sqrt{f} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] = 0
\end{align*}
\]

The Eq. (31) has nontrivial solutions for the coefficients \( A(t, r, \phi) \), \( B(t, r, \phi) \) and \( D(t, r, \phi) \) under the condition that the coefficients matrix determinant is 0. Hence, the modified Hamilton-Jacobi equation for the charged massive vector boson particle is derived as follows:

\[
\begin{align*}
& \frac{1}{f} \left( \frac{\partial S}{\partial t} \right)^2 + \frac{2q A_0}{f} \left( \frac{\partial S}{\partial r} \right)^2 + q^2 A_0^2 \frac{1}{f} - \frac{1}{r^2} \left( \frac{\partial S}{\partial \phi} \right)^2 - f \left( \frac{\partial S}{\partial r} \right)^2 - \frac{\mu_0^2}{4} \\
& \quad + \alpha \left[ \frac{9 \mu_0^2 f}{4} \left( \frac{\partial S}{\partial r} \right)^2 + 9 \frac{\partial S}{\partial \phi} \left( \frac{\partial S}{\partial \phi} \right)^2 + \frac{3 \mu_0^2}{4} - 3 \frac{1}{r^2} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] \\
& \quad + \alpha \left[ \frac{6 f}{r^2} \left( \frac{\partial S}{\partial r} \right)^2 + \frac{9 \mu_0^2}{4} \left( \frac{\partial S}{\partial \phi} \right)^2 + \frac{3 \mu_0^2}{4} - 3 \frac{1}{r^2} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] \\
& \quad + \alpha \left[ \frac{4q A_0}{r^2} \left( \frac{\partial S}{\partial r} \right)^2 + \frac{36 q A_0^2}{f} \left( \frac{\partial S}{\partial r} \right)^2 - \frac{3 q^2 A_0^2}{f} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] \\
& \quad + \alpha \left[ \frac{4q A_0}{r^2} \left( \frac{\partial S}{\partial r} \right)^2 + \frac{3 q^2 A_0^2}{f} \left( \frac{\partial S}{\partial \phi} \right)^2 - \frac{3 q^2 A_0^2}{f} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] = 0
\end{align*}
\]

(32)

Thanks to the separation of variable of the \( S(t, r, \phi) \), the radial trajectory \( K_s(r) \) are

\[
K_s(r) = \pm \int \sqrt{(E - j \Omega_+ - q A_0)^2 - f(\frac{\partial S}{\partial \phi} + \frac{\partial S}{\partial r})} \frac{1 + \alpha \Gamma}{f} dr
\]

(33)

where \( \Gamma \) is

\[
\Gamma = \frac{(E - j \Omega_+)(E - j \Omega_+ - q A_0)[5f\mu_0^2 - 4(E - j \Omega_+ - q A_0)^2]}{4f(E - j \Omega_+ - q A_0)^2 - f(\frac{\partial S}{\partial \phi} + \frac{\partial S}{\partial r})}
\]

Using a contour that has a semicircle around the pole at the outer horizon, the \( K_s(r) \) integrals in Eq (33) are calculated as

\[
K_s(r_s) = \pm \ln \frac{(E - j \Omega_+ + q Q \ln(\frac{\phi}{\pi}))}{H} [1 + \alpha \Delta],
\]

(34)

where \( \Delta \) is

\[
\Delta = \frac{(E - j \Omega_+) \left[ 9r_s^2 \mu_0^2 H - 4(4q Q(E - j \Omega_+)r_s + j^2 H + 4r_s q^2 Q^2 \ln(\frac{\phi}{\pi})) \right]}{8r_s^2 H (E - j \Omega_+ + q Q \ln(\frac{\phi}{\pi}))}.
\]
Hence, using the Eqs.(12) and (13) for the vector boson particle, the modified Hawking temperature becomes

\[ T_{\text{VB}}^\prime = \frac{\hbar}{4\pi} \left[ 1 + \alpha \Delta \right]^{-1}, \]

or

\[ T_{\text{VB}}^\prime = T_H \left[ 1 - \alpha \Delta \right] \]

with the standard Hawking temperature, \( T_H \), in Eq.(16). This result shows that the modified Hawking temperature of the charged massive vector boson particle is lower than the standard temperature and different from that of the modified Hawking temperatures of the scalar and Dirac particle.

5. Stability analysis

The local stability of a black hole can be analyzed by the heat capacity [69]. If the heat capacity is positive, then the black hole is locally stable or else it is unstable [70, 71, 72]. For this reason, to discuss the stability of the CR-BTZ black hole, we firstly calculate its modified heat capacity.

The heat capacity at constant charge, \( Q \), and constant angular momentum, \( J \), of the black hole can be expressed by using the following relation

\[ C_{Q,J} = \left( \frac{\partial M}{\partial T_H} \right)_{Q,J} \] (36)

where \( M \) and \( T_H \) are the mass and the standard Hawking temperature of the black hole, respectively. Making a comparison to the quantum gravity effects with respect to the standard ones, we derive the modified heat capacity of the black hole from the modified Hawking temperature of the charged massive scalar particle. Also, it is important to note that the heat capacity derived from the modified Hawking temperature is going to be different to each particle. Using Eqs.(4) and (15), the modified heat capacity of the black hole \( (C_{Q,J}) \) becomes

\[ C_{Q,J} = \frac{2\pi}{r_+ \hbar} \left[ E - j\Omega_s + qQ \ln(\frac{\pi}{\alpha}) \right] \left( \frac{\mathcal{A} - \mathcal{B}}{\mathcal{A} - \mathcal{B}} \right)^2 \] (37)

where the constant \( X, Y, \mathcal{A} \) and \( \mathcal{B} \) are functions of the \( E, J, j, l, q, Q, M_0, r_s \) and \( \alpha \) (see Appendix).

The modified heat capacity vanishes for \( X = Y \) and \( \mathcal{A} = \mathcal{B} \), but this point doesn’t indicate a first-type phase transition because the modified heat capacity is always positive in this case. On the other hand, the modified heat capacity diverges at point \( \mathcal{A} = \mathcal{B} \) for \( X \neq Y \). In that case, the signature of the modified heat capacity changes according to relation between \( \mathcal{A} \) and \( \mathcal{B} \). Then, we can mention that the black hole has a second-type phase transition in the presence of the quantum gravity effect. To elucidate the stability properties of the black hole, if we plot the \( C_{Q,J} \) heat capacity for the special values of the constants \( E, J, j, l, q, Q, M_0 \) and \( \alpha \) since its expression is very complicated, then we can see that the black hole undergoes only a second-type phase transition in the presence of the quantum gravity effect (solid line in Fig.-1).
On the other hand, in the absence of the GUP effect, i.e. \( \alpha = 0 \), the modified heat capacity of the black hole reduced to the standard one [73]:

\[
C_{Q,J} = \frac{4\pi r_+}{\hbar} \left( \frac{4r_+^4 - J^2 \ell^2 - Q^2 \ell^2 r_+^2}{4r_+^4 + 3J^2 \ell^2 + Q^2 \ell^2 r_+^2} \right).
\]

(38)

Then, the standard heat capacity vanishes at the point \( 4r_+^4 = J^2 \ell^2 + Q^2 \ell^2 r_+^2 \). The black hole unstable in the region \( 0 < r_+ < 0.8 \) but stable in the region \( 0.8 < r_+ \), as is depicted in Figure-1. Therefore, we can say that the black hole undergoes the first type of phase transition in order to become stable in the absence of the GUP effect (dashed line in Fig.-1).

However, in the presence of the quantum gravity effect, we see that there are two regions, \( 0 < r_+ < 0.51 \) and \( 0.55 < r_+ < 0.84 \), in which the black hole is unstable and that the black hole is stable in the rest two regions that are \( 0.51 < r_+ < 0.55 \) and \( 0.84 < r_+ \). On the other hand, it is important to emphasize that the black hole is stable in the region, \( 0.51 < r_+ < 0.55 \), due to quantum gravity effect while it is unstable in this region in the absence of quantum gravity effect.

Also, in the presence of the quantum gravity effect, if the angular momentum of the black
hole is negative, i.e. $J < 0$, the black hole becomes unstable in the region $0.51 < r_+ < 0.55$ (solid line in Figure-2), but, in the absence of the quantum gravity effect, the stability properties of the black hole doesn’t change, i.e, the black hole is unstable in the region $0 < r_+ < 0.84$ and stable in the region $0.84 < r_+$ (dashed line in Figure-2). As is delineated in Fig.-1 and 2, under the GUP effects, it has been seen that the black hole at $r_+ = 0.84$ undergoes second-type phase transition to become stable instead of $r_+ = 0.80$. Therefore, we can say that the quantum gravity effect grows the outer horizon $r_+$.

6. Concluding remarks

In this study, we have investigated the quantum gravity effect on the thermodynamical properties of the CR-BTZ black hole in the context of the quantum mechanical tunneling process of the charged massive particles such as spin-0 scalar particle, spin-1/2 Dirac particle and spin-1 vector boson particle. Our calculations show that both the modified Hawking temperature of the CR-BTZ black hole depend not only on the black hole properties, but also on the properties of the tunneling particle. Also, it is worth to note that the modified Hawking temperatures calculated via tunneling of the particles are different from each other. Using the modified Hawking temperature of the charged massive scalar particle, we analyse the local stability of the black hole and show that the modified heat capacity of the black hole depends on both the properties of the black hole and the tunneling particle, as well. From the modified heat capacity of the scalar particle, we find that the black hole undergoes only a second-type phase transition in the presence of the quantum gravity effect (see Figure-1) although it possesses a first-type transition in the absence of quantum gravity effect. Also, in the absence of the quantum gravity effect, i.e. $\alpha=0$, both the modified heat capacity and Hawking temperature of the black hole can be reduced to the standard one. Also, from Fig.-1 and 2, we see that the quantum gravity effect grows the outer horizon $r_+$, i.e., it gives more deep insight about the black hole.

Finally, we can say that, thanks to the quantum gravity effect, the nature of the tunneling particles play an important role in understanding the evolution of a black hole such that it can shed some light on the final stage of black holes.

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Appendix

The constants in Eq.(37) are given as follows:

\[
X = 3a\tilde{E}r_+^2J^2 \left[ J^2M_0^2 + \frac{2}{3\alpha} (r_+^2Q^2 + J^2) + \frac{J^2}{r_+^2} + Q^2 (J^2 + M_0^2r_+^2) + \frac{4}{3}\tilde{E}QQr_+^2 \right] \\
+ 2r_+^2\tilde{E}Q \ln\left( \frac{r_+}{r_0} \right) \left[ J^2 + \frac{r_+^2Q}{J^2} (Q + \alpha\tilde{E}Q) + 2\alphaJ^2 \left( \frac{1}{r_+^2} + Q^2 \right) \right] \\
+ QQ \ln\left( \frac{r_+}{r_0} \right) \left[ \frac{1}{J^2} \left( 1 + 2\alpha J^2 \left( \frac{1}{r_+^2} + Q^2 \right) \right) + r_+^2Q^2 \right]
\]
\[ Y = 4 r^2 j Q \ln \left( \frac{r^2}{Q^2} \right) \left( 2r^2 + 8 \alpha j^2 \right) + 2q Q \ln \left( \frac{r^2}{Q^2} \right) \left( r^2 + 2 \alpha j^2 \right) \] 

\[ A = r^2 \tilde{E}^2 \left[ 16 r^2 \left( 2r^2 + \tilde{J}^2 + 9 \alpha j^2 r^2 \right) + \alpha Q^4 \tilde{t} \left( 3j^2 + 4 \alpha j^2 r^2 + 6qQ \alpha \tilde{E}^{-1} + \frac{4q^2 J^2}{Q^2} \right) \right] \]

\[ + 6 \alpha \tilde{E}^2 J^2 \tilde{M}_0^2 r_+^2 \left[ 4 \tilde{E} + \frac{8Q^2 J^2}{3 \tilde{E} \tilde{M}_0^2} \right] r_+^4 + q Q \left( \frac{\tilde{J}^2 \tilde{I}^2 + \tilde{J}^2}{\tilde{E} \tilde{M}_0^2} + 2 \tilde{E} Q^2 r_+^4 \right) \]

\[ + 48 \alpha \tilde{E}^2 \tilde{M}_0^2 \ln \left( \frac{r^2}{Q^2} \right) \left[ J^2 r^2 + \alpha \tilde{J}^2 \left( 12 \alpha r^2 + \frac{Q^4 \tilde{t}^2}{4 \alpha r^4} \right) \right] + 4 \alpha \left( J^2 r^2 + 2 \alpha r^2 \right) + 2J^2 \tilde{M}_0^2 + \frac{Q^4 \tilde{t}^2}{r_+^2} \frac{11}{12} Q^2 + \frac{q^2 J^2}{3} J^2 \right] \]

\[ + 48 \alpha \tilde{E}^2 \tilde{M}_0^2 r_+^{10} \left( \tilde{E} + 2qQ \right) \]

\[ B = 16 \tilde{E}^2 Q^2 \tilde{E}_0^2 r_+^2 \left( 1 + 11 \alpha q^{-1} Q^2 + \frac{3}{8} \tilde{E}^{-1} Q^2 + \alpha r^2 \right) \left[ q Q + \tilde{E} \left( 6 + \frac{Q^2 \tilde{t}^2}{2r_+^2} + \frac{3J^2 \tilde{I}^2}{2 \alpha q r_+^4} \right) Q + \frac{3J^2 \tilde{I}^2}{4Qr_+^2} \right] \]

\[ + 4 \alpha q Q \tilde{J}^2 \tilde{E} \ln \left( \frac{r^2}{Q^2} \right) \left[ M_0^2 r_+^2 \left( 9 \tilde{E} \tilde{M}_0^2 + 3 \tilde{E} \tilde{M}_0^2 Q^{-1} + 4 \tilde{E} q Q r_+^4 \right) + 4 \tilde{E} q Q r_+^4 + 11 \left( J^2 \tilde{I}^2 + 8 \tilde{E}^{-1} \tilde{I}^2 r_+^4 \right) \right] \]

\[ + 6 \alpha \tilde{E}^2 Q^2 \tilde{E}_0^2 \left( \frac{1}{2} + 3 \left( \frac{\tilde{E}}{qQ} + \frac{J^2}{q^2 r_+^4} \right) \right) + 4 \alpha J^2 \tilde{E} \tilde{M}_0^2 r_+^4 \left[ 6 \tilde{E} r_+^2 + \tilde{Q} \tilde{t}^2 \left( 16 \tilde{E} r_+^2 + \tilde{Q} \tilde{t}^2 + \frac{1}{2} \tilde{Q}^2 \right) + J^2 \tilde{I}^2 \frac{Q^2 \tilde{t}^2 + 8 \tilde{E}^{-1} \tilde{I}^2 \tilde{t}^2}{2} \right] \]

\[ + 4 \alpha q \tilde{Q} \tilde{t}^2 \left( \frac{1}{2} + \frac{3}{4} \alpha Q \tilde{t}^2 \right) + Q^2 \tilde{I}^2 \left( 16 \tilde{E} \tilde{M}_0^2 + 3 \tilde{M}_0^2 + 6 \right) r_+^4 + 8 \tilde{E} Q^{-2} r_+^2 + 4 \alpha q \tilde{E} q Q \tilde{t}^{-1} \]

\[ + 2 \alpha q \tilde{J}^2 \tilde{E} \tilde{M}_0^2 r_+^4 \left( \frac{5}{4} \tilde{E} \tilde{M}_0^2 + \tilde{Q} \tilde{t}^2 \right) + \frac{5}{4} \tilde{E} \tilde{J}^2 \tilde{Q} \tilde{t}^{-1} + 12 \alpha \tilde{J}^2 \tilde{Q} \tilde{t}^{-1} + \frac{5}{4} \tilde{E} \tilde{J}^2 \tilde{Q} \tilde{t}^{-1} \]

\[ + 16 \alpha q^2 \tilde{Q} \tilde{t}^2 \tilde{E} \ln \left( \frac{r^2}{Q^2} \right) \left[ \tilde{J}^2 \tilde{I}^2 \left( 6 \alpha r^2 + \frac{3}{4} \tilde{J}^2 \tilde{I}^2 \right) + \left( J^2 Q \tilde{J}^2 + 5 \tilde{E} \tilde{J}^2 \tilde{Q} \tilde{t}^{-1} \right) r_+^2 \right] \]

with \( \tilde{E} = E - j \Omega \).
