Conformal weights of charged Rényi entropy twist operators for free scalar fields in arbitrary dimensions

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Abstract
I compute the conformal weights of the twist operators of free scalar fields for charged Rényi entropy in both odd and even dimensions. Explicit expressions can be found, in odd dimensions as a function of the chemical potential in the absence of a conical singularity and thence by images for all integer coverings. This method, developed some time ago, is equivalent, in results, to the replica technique. A review is given. The same method applies for even dimensions but a general form is more immediately available. For no chemical potential, the closed form in the covering order is written in an alternative way related to old trigonometric sums. Some derivatives are obtained. An analytical proof is given of a conjecture made by Bueno, Myers and Witczak-Krempa regarding the relation between the conformal weights and a corner coefficient (a universal quantity) in the Rényi entropy.

Keywords: charged Rényi entropy, twist operators, conformal weights

(Some figures may appear in colour only in the online journal)

1. Introduction and summary

The replica method is a popular technique in the calculation of entanglement and Rényi entropies which have proved to be valuable probes of aspects of condensed matter physics and quantum field theory. Relevant references are too many to list and I mention [1] and [2] as representative.

A conformal field-theoretic way of describing the ensuing cut structure involves the notion of ‘twist operators’, e.g. [3], and their conformal weights. These have some intriguing general properties which have been tested in specific situations, the simplest of which is free-
field theory. My rather restricted aim in the present work is to give some specific but explicit computations in this area which might prove useful for comparison or checking purposes. The quantity I will study particularly is the conformal weight of free scalars for charged Rényi entropy using a method that allows an arbitrary dimension, \( d \), to be treated easily. \( d = 3 \) is the only odd case treated in detail up to now. The technique I employ leads to a proof (section 4) of an interesting conjecture which relates the conformal weight to a corner coefficient arising in the expansion of Rényi entropy, [4].

In section 2, I set up some general formulae, transforming the problem to one in a Euclideanised cosmic string manifold (i.e. flat space with a single conical singularity). Thereby, in section 3, by putting together some previously derived results, I calculate, for all odd \( d \), the Rényi conformal weight, \( h_1 \), in the absence of the conical singularity but with a flux along the cone axis which can be interpreted as a Euclidean chemical potential. An existing image technique (equivalent to a replica relation) then allows the general Rényi weights, \( h_n \) (\( n \in \mathbb{N} \)), to be found as trigonometric sums. The zero chemical potential values agree with those that already exist for \( d = 3 \). Sections 7 and 8 give the corresponding even \( d \) calculation. Some derivatives are also computed. In section 10, the resulting polynomials in \( n \) are related to finite cosec sums whose evaluation (due to Jeffery in 1864) organise the polynomials in a more expressive way. Finally, an appendix contains a derivation of the image technique which was just written down in an earlier publication, [5].

2. The conformal weight

Since my object is a mere technical evaluation, for rapidity I present the basic equations as given in Belin et al [6], without derivations or much explanation.

The conformal weight, \( h_\mu \), of a spherical twist operator is derived in [6] using conformal transformation giving the result (I use their notation initially), in terms of the energy density, \( \mathcal{E}(T, \mu) \), on the \( d \)-dimensional hyperbolic cylinder

\[
 h_\mu(\mu) = \frac{2\pi n}{d - 1} (\mathcal{E}(T_0, \mu = 0) - \mathcal{E}(T_0/n, \mu)),
\]

where \( T_0 = 1/2\pi \) (I have set the radius \( R \) to unity), \( \mu \) is the chemical potential and \( n \) is the order of the replica covering.

One way of calculating \( \mathcal{E} \) is to make a further conformal transformation to a flat conical space i.e. to a Euclideanised cosmic string space. This yields the relation (in odd dimensions), [7],

\[
 \mathcal{E} = -r^d(d - 1) \langle T_{zz} \rangle,
\]

with the string metric written as

\[
 dz^2 = dr^2 + r^2d\phi^2 + d\mathbf{z} \cdot d\mathbf{z},
\]

the angular time point \( \phi \) being identified with \( \phi + 2\pi/q \). I now use \( q = 2\pi T \) and think of \( q \) as a real number.

The average on the right-hand side of (2) is with respect to free (complex) fields, \( \psi \), quasi-periodic under \( \phi \rightarrow \phi + 2\pi/q \) i.e. \( \psi(\phi + 2\pi/q, ...) = e^{2\pi i \delta} \psi(\phi, ...) \). As described many times before, [8–11], \( \delta \) could be regarded as a ‘magnetic flux’ through, or along, the

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1 As is, by now, standard, gauge equivalent are periodic fields with an appropriate vector potential in the equations of motion. See [8] for an extended discussion and other references.
‘axis’ of the cone which here is the co-dimensional 2 space coordinatised by $z$. Thermally, $T = q/2\pi$ would be the temperature and $\mu = \delta$ a (‘Euclidean’) chemical potential\textsuperscript{2}, [7, 12].

The (complex) scalar vacuum energy–momentum tensor has been computed with these boundary conditions awhile ago in [5] and, in order to avoid duplication, I just give the answer

$$\langle T_{zz} \rangle = \frac{1}{\pi} \frac{\Gamma(d/2)}{(4\pi r^2)^{d/2}} q \left( W_d(q, \delta) - \frac{d-2}{d-1} W_{d-2}(q, \delta) \right)$$

$$= \frac{1}{\pi} \frac{\Gamma(d/2)}{(4\pi r^2)^{d/2}} q \ Y(q, \delta)$$

$$= E(q, \delta),$$

(3)

which defines $Y(q, \delta)$, $E(q, \delta)$ and where $W_d$, on choice of a particular complex contour, becomes

$$W_d(q, \delta) = \int_0^\infty \frac{dr}{\cosh^d r^2 / 2} \frac{\cosh(q(\delta-1)r) \sin(\pi qr) - \cosh(q\delta r) \sin(\pi q(\delta-1))}{\cosh qr - \cos qr}.$$

(4)

Here $q \leq 1$ which corresponds to a conical angular excess. The values $q = 1/n$, $n \in \mathbb{N}$, give a multi-sheeted integral covering of the plane. Zero flux implies $\delta = 0$ or, equivalently, $\delta = 1$. Quantities for $\delta$ outside the range $0 \leq \delta \leq 1$ are computed by periodicity, $\delta + 1 \equiv \delta$.

Although it is possible to work generally, at least for even dimensions, a simplification occurs on setting $g = 1$, i.e., on removing the conical singularity. I give some of the details that were not exposed in [5]. Thus, from (4),

$$W_d(1, \delta) = 2 \sin \pi \delta \int_0^\infty dy \frac{\cosh y(2\delta - 1)}{\cosh y + 1}$$

$$= 2^d \cos \pi \delta \frac{\Gamma(d + 1 + \delta)\Gamma(d + 1 - \delta)}{\Gamma(d + 1)}$$

$$= \frac{2^d}{\Gamma(d + 1)} \frac{\sin(\pi(\delta - d))}{\Gamma(\delta - d)} \frac{\Gamma(\delta + d + 1)}{\Gamma(d + 1)},$$

(5)

which is an interpolation between odd and even dimensions. I have set $d = 2\delta + 1$ and $\delta = \delta - 1/2$ generally, and, in the special case of integer $d$, the ratio of Gamma functions in (5) factorises according to

$$\frac{\Gamma(x + d + 1)}{\Gamma(x + d)} = (-1)^{d/2} x^{(d+1)-1},$$

in terms of central factorials whose definition I repeat here for convenience, [13],

$$x^{(2\nu-1)} = x(x^2 - 1^2)(x^2 - 2^2) \ldots (x^2 - (\nu - 1)^2)$$

$$x^{(2\nu+1)-1} = \left(x^2 - \frac{1}{4}\right) \left(x^2 - \frac{9}{4}\right) \ldots \left(x^2 - \frac{(2\nu - 1)^2}{4}\right).$$

(6)

This dimensional continuation could be carried through into the energy density and the conformal weights, but I will not pursue this aspect.

\textsuperscript{2} The $\mu$ here equals $\mu/2\pi$ of [6]. See section 3. I will often call $\delta$ the phase.
Aiming towards the combination in (3),

\[ W_{d-2}(1, \delta) = 2^{d-2} \cos \pi \delta \frac{\Gamma(d + \delta) \Gamma(d - \delta)}{\Gamma(d - 1)} \]  

so that

\[ W_d(1, \delta) = 2^{d} \frac{(d + \delta)(d - \delta)}{d(d - 1)} W_{d-2}(1, \delta) \]  

whence the combination

\[ Y(1, \delta) = \frac{2^2 W_{d-2}(1, \delta)}{d(d - 1)} \left( \frac{d(d - 2)}{4} \right) \]

\[ = \frac{W_{d-2}(1, \delta)}{d(d - 1)} (1 - 4 \delta^2). \]  

Regarding this result, I note that \( W_d \) vanishes for zero flux \((\delta = 1/2)\) transcendentally for odd \(d\) because of the cosine factor. The conformal combination, (9), gives an extra algebraic zero.

The form, (9), is valid only for the range \(0 < \delta \leq 1\) (or \(-1/2 < \delta \leq 1/2\)) and must be extended beyond these by the periodicity coming from the basic definitions. So far \(d\), if integral, could be either even or odd.

From (6) or, equivalently, by iterating (8) down to either \(W_0\) (even \(d\)) or to \(W_1\) (odd \(d\)), the form of \(Y(1, \delta)\) can be found. Then, constructing \((T_{zz})\) from (3), one finds, for odd \(d\), the Plancherel form,

\[ r^d \langle T_{zz} \rangle = \frac{2^d}{\pi d(d - 1)} \frac{\Gamma(d/2)}{\Gamma(d + 1/2)} \frac{1}{(4\pi)^{d/2}} \left( \frac{1}{4} - \delta^2 \right) \left( (d - 1)^2 - \delta^2 \right) (1 - \delta^2) \delta \cot \pi \delta \]

\[ = \frac{1}{(4\pi)^{d/2} (2d + 1) \Gamma(d + 1/2)} \left( \frac{1}{4} - \delta^2 \right) \left( (d - 1)^2 - \delta^2 \right) (1 - \delta^2) \delta \cot \pi \delta, \]  

which is the answer written out in [5]. Even \(d\) will be treated later in sections 7 and 8.

The conformal weight follows from (1), and, now just for odd dimensions and \(n = 1\), is, on using (2),

\[ h^1(\mu) = \frac{1}{2(4\pi)^{d-1} (2d + 1) \Gamma(d + 1)} \left( \frac{1}{4} - \mu^2 \right) \left( (d - 1)^2 - \mu^2 \right) (1 - \mu^2) \mu' \cot \pi \mu', \]  

where \(\mu' = \mu - 1/2\). (Remember, in my conventions, \(\mu = 1\) corresponds to no chemical potential, i.e. \(\mu' = 1/2\).)

A plot of this function, suitably periodised, for \(d = 3\) was given, essentially, in [5] and is contained below \((n = 1)\) in figure 1, for present circumstances.
3. Use of images to give higher coverings

So far, to get explicit expressions, I have been restricted to working at \( q = 1 \), i.e. at \( n = 1 \). I now show how to extend the range to \( n \in \mathbb{N} \) using an image method described in [5] which gives a decomposition (or sum rule) for the energy density, \( \langle T_{zz} \rangle \) (as an example)\(^3\). From the connection (2) this translates to,

\[
\mathcal{E}(q/n, n\delta) = \frac{1}{n} \sum_{s=0}^{n-1} \mathcal{E}(q, \delta + s/n),
\]

where \( n \) (but not necessarily \( q \)) is integral. Knowing the result for any \( \delta \) is crucial for the applicability of the method.

The notation now is such that the first argument of \( \mathcal{E} \) is \( 2\pi \) times the inverse of the cone angle, and the second the phase change on circling this cone once. Thus if \( q = 1 \), the left-hand side refers to an \( n \)-fold covering of the plane with phase change \( n\delta \) and so the phase change on circling through just \( 2\pi \), i.e. around one leaf of the covering, is \( \delta \), which is the (Euclidean) chemical potential, \( \mu \), denoted \( \mu_E/2\pi \) in [6].

When \( q = 1 \), the summands are determined by the explicit expression in (10). Hence a closed form exists for the density for an \( n \)-fold covering of flat space. This gives a useful, and more elegant, alternative to a crude numerical treatment of (4) (which, however, applies for all \( q \leq 1 \)). In this case, expressed in terms of the conformal weight, (12) gives, in view of (1),

\[
h_n^{(o,e)}(\mu) = \sum_{s=0}^{n-1} h_1^{(o,e)}(\mu + s/n),
\]

valid for odd and even dimensions.

A graph of this for \( d = 3 \) appears in figure 1. Higher dimensions are easily computed. The meeting at \( \mu = 1/2 \) of the \( n = 1 \) and \( n = 2 \) curves follows from (12) on setting \( n = 2 \).

It is interesting to note that expressions similar to (12) hold for those spectral quantities which are determined linearly in terms of the Green function, for example the effective action, the heat-kernel and the local \( \zeta \)-function. (See the appendix.)

Exact numerical values of the \( h_n \) (for no chemical potential and \( d = 3 \)) were found in [4], by a different method, based on earlier expressions. My results for \( d = 3 \) computed using the

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\(^3\) The appendix contains more details.
above formulae agree with those exhibited in table 1 of this reference and will not be repeated. As further examples, in five-dimensions I find, for \( n = 5 \),

\[
\frac{3(645 - 83\sqrt{5})\sqrt{\sqrt{5} + 5}}{15632\sqrt{\pi}} \approx 8.3445842623 \times 10^{-4},
\]

and for \( n = 4 \),

\[
\frac{45\pi + 128}{40960\pi^2} \approx 6.663343844 \times 10^{-4}.
\]

Higher dimensions have similar forms which are quickly produced.

4. Proof of a conjecture concerning corner coefficients

A considerable amount of work has been done to find regularisation independent (universal) constants that appear during the calculation of entanglement and Rényi entropies. One such constant, \( \sigma_n \), is associated with the existence of a corner, or vertex, singularity in the entangling surface. The subscript \( n \in \mathbb{N} \) is the order of the replica covering used to compute the entropies. For the entanglement entropy the limit \( n \to 1 \) is required but the general case is relevant for Rényi entropies.

Casini and Huerta, \([1]\), gave an integral form for \( \sigma_n \) for free scalar fields in three-dimensions. Motivated by some relations between \( \sigma_n \) and the central charges conjectured and calculated by Bueno \textit{et al} \([14]\), Elvang and Hadjiantonis, \([15]\), recently have remarkably been able to reduce the rather complicated integral to a finite sum, which can be evaluated for a given \( n \).

Lately, Bueno \textit{et al} \([4]\), have used these results to make, and verify, the further conjecture that \( \sigma_n \) is related to the conformal weight, \( h_n \), of twist operators by

\[
\sigma_n = \frac{1}{\pi} \frac{h_n}{n-1},
\]

for any three-dimensional conformal field theory. For free fields, they checked this numerically for a large range of \( n \), using the finite sum for \( \sigma_n \) and an integral form for \( h_n \), derived earlier, which they also transformed into a (different) finite trigonometric sum for integral \( n \).

I begin the proof by writing down Elvang and Hadjiantonis’s expression for \( \sigma_n \),

\[
\sigma_n = \frac{1}{24\pi n^3(n-1)} \sum_{k=1}^{n-1} k(n-k)(n-2k)\tan\frac{k\pi}{k},
\]

which they obtained after considerable manipulation.

I show that the conjecture (14) is a simple, perhaps even trivial, consequence of (15) and the results above.

In the absence of a conical singularity \( (q = 1) \), the conformal weight in three-dimensions, is (see (11)),

\[
h_1(\delta) = \frac{1}{6} \left( \frac{1}{4} - \delta^2 \right) \delta \cot \pi \delta,
\]

where \( \delta = \delta - 1/2 \). \( \delta = 1 \) gives zero flux.
Next the image decomposition, (13), gives, for no chemical potential

$$h_n = \sum_{s=0}^{n-1} h_1 \left( \frac{s+1}{n} - \frac{1}{2} \right). \quad (17)$$

and all that remains is to substitute (16) into (17) which gives

$$h_n = -\frac{1}{6} \sum_{s=0}^{n-1} \left( \frac{1}{4} - \left( \frac{s+1}{n} - \frac{1}{2} \right)^2 \right) \frac{\pi (s+1)}{n} \tan \left( \frac{s+1}{n} \right)$$

$$= -\frac{1}{6} \sum_{k=1}^{n} \left( \frac{1}{4} - \left( \frac{k}{n} - \frac{1}{2} \right)^2 \right) \frac{\pi k}{n} \tan \left( \frac{k}{n} \right)$$

$$= \frac{1}{12n} \sum_{k=1}^{n-1} k(n-k)(n-2k) \tan \left( \frac{\pi k}{n} \right). \quad (18)$$

Comparing (15) and (18) produces an analytical proof of the conjecture, (14), of Bueno et al [4], as promised. A factor of 2 occurs because my analysis uses complex fields.

A similar conclusion can be reached for fermion fields using the fermion image relation outlined in the appendix.

5. Higher dimensions

It is easy to find the higher dimensional weights, $h_n$, but the corresponding vertex constants, $\sigma_n$, are not yet available.

A conjecture, similar to (14), has been made by Bueno and Myers, [16]. For example, in five-dimensions

$$\sigma_n^{(5)} = \frac{16}{9} \frac{h_n}{n-1},$$

which, using the results above, yields

$$\sigma_n^{(5)} = \frac{1}{360\pi(n-1)n^2} \sum_{k=1}^{n-1} k(n-k)(n-2k)(n+2k)(3n-2k) \tan \left( \frac{\pi k}{n} \right),$$

as my prediction for the corner contribution in five-dimensions. Other dimensions are easily calculated.

6. Derivatives in the odd case

Although (11) is an explicit expression for the conformal weight, $h_1(\mu)$, it is helpful to evaluate a few lower derivatives with respect to $\mu$ at $\mu = 1 \equiv 0$ in order to compare, if possible, with other results.

I define $h^{(\mu)}_{0,0} = \partial_\mu h^{(\mu)}_{0,0}(\mu)|_{\mu=1}$ to agree with the notation of [6]. Because of the double zero at $\mu' = 1/2$, the first derivative $h^{(\mu)}_{0,0}$ vanishes, in agreement with the remarks in [6]. This would not be so in the non-conformal case. The second derivative also follows easily as the specific function of dimension,

$$h^{(\mu)}_{0,2} = \frac{\Gamma(\hat{D} + 1/2)\Gamma(\hat{D} - 1/2)}{(4\pi)^{(d-3)/2}}.$$

I could not find anything in [6] with which to compare this quantity directly.
7. Even dimensions

I now turn to the easier case of even dimensions and I will, initially, again restrict to the \( q = 1 \) case i.e. no conical singularity, only an Aharonov–Bohm flux. The relevant expression is given in (5). The general formulae are the same but now

\[
W_0 = \sin \pi \delta \Gamma(\delta)\Gamma(1 - \delta) = \pi
\]
and the expressions are entirely algebraic. In place of (10) one has, [5],

\[
r^d \left< T_{zz} \right> = \frac{1}{(4\pi)^\delta (2\delta + 1)\Gamma(\delta + 1)} \left( \frac{1}{4} - \delta^2 \right) \left( \frac{1}{4} - \delta^2 \right) \cdots ((\delta - 1)^2 - \delta^2) \right].
\]

(19)

where, in the particular case of \( \delta = 1/2 \), the product in square brackets is empty and there is only one zero at \( \delta = 1/2 \).

Correspondingly, the conformal weight is

\[
h_1^{(c)}(\mu) = \frac{(4\pi)^{1-\delta}}{2(2\delta + 1)\Gamma(\delta + 1)} \left( \frac{1}{4} - \mu^2 \right) \left( \frac{1}{4} - \mu^2 \right) \cdots ((\mu - 1)^2 - \mu^2) \right].
\]

(20)

When \( \delta = 1/2 \), because of the periodicity in \( \mu \), the first derivative with respect to \( \mu \) suffers a discontinuity at \( \mu = 0, 1 \) as shown in figure 2.

The image method used previously to extend \( q = 1 \) to \( q = 1/n \) applies here as well but, for even dimensions, it is also possible to give a closed expression for any \( q \), i.e. any cone angle, [5]. This is because the complex contour can be chosen to encircle the origin and the result of a standard residue calculation is of polynomial form. Although the analysis is available in the cited references, in order to be self-contained I present some essentials in section 9. Here I point out that the polynomials are given explicitly in [5], which contains a list of conformal conical energy densities with numerical coefficients. For example, the \( d = 4 \) conformal weight as a function of the chemical potential and covering order is

\[
h_n^{(4)} = \frac{1}{3n^3} \left( \frac{n^4 - 1}{45} + \frac{2}{3} \mu^2 (\mu - 1)^2 \right).
\]

\[\text{Figure 2. } d = 2. \text{ Conformal weight for coverings, } n.\]

\[\text{Figure 2. } d = 2. \text{ Conformal weight for coverings, } n.\]

\[\text{4 This is the same as the discontinuity in the second virial coefficient for anyons, } \delta \text{ corresponding to the statistics determining parameter there, [17].}\]
A systematic way of finding such polynomials is given in section 9. Furthermore, section 10 has another way of organising the polynomials by relating them to trigonometric sums.

Plots are given for \( d = 2 \) and \( d = 4 \) and a few coverings in figures 2 and 3.

8. Singularities. Phase transitions

The explicit expressions, (11) and (20), show that derivatives of the plane weights \( h_{\mu}^{(\alpha,\epsilon)}(\mu) \) have discontinuities at \( \mu = 0, 1 \) as a consequence of the imposed periodicity. The first derivative with respect to \( \mu \), or \( \mu' \), for \( d = 2 \), the third for \( d = 3 \) and 4 the fifth for \( d = 5 \) and 6, and so on, as briefly noted in [5].

The image expression (12) is periodic in \( \mu \) with period \( 1/n \) and shows that the singular points for the weights of the \( n \)-fold cover are at \( \mu = s/n \), \( s = 1 \) to \( s = n \). (This is most obvious in figure 2.)

Such discontinuities in thermal physics usually signal phase transitions. A similar conclusion in two-dimensions is reached by Belin et al [6], by a different method. For a free massless fermion, they determine the \( n \) positions of the transitions to be at \( \mu = (2k - 1)/2n \) to \( \mu = (2k + 1)/2n \) with \( k \) an integer.

9. Even \( d \) again. Derivatives

By continuing the polar angle into the complex plane, Sommerfeld used a contour method to find the Green function, \( G \), or an equivalent quantity like the diffusion kernel, on a cone of arbitrary angle. The method amounts to the re-periodisation of the Green function without a conical singularity, \( G_0 \). Accordingly, a contour manipulation enables \( G_0 \) to be subtracted from \( G \) from the start. This can be thought of as a renormalisation. One can also allow for a flux along the cone axis by appropriate choice of \( G_0 \).

For even dimensions (to which the following analysis is restricted) the relevant massless \( G_0 \) possesses only poles and the contour can be deformed into a closed one around a point which can be taken to be the origin for a particular, inessential, choice of angular coordinate.

Computation of field theory quantities from \( G \), such as vacuum averages, proceeds by action on this contour integral. A summary of this approach is given in [18].
The method works if a conical part can be displayed in the metric and if $G_0$ is known. Such is the case for flat space, of course, and gives the cosmic string expressions as used in this paper, and in [5].

The vacuum averages of interest here then devolve upon the computation, in even dimensions, of the quantity, see (3),

$$W_d(q, \delta) = \frac{1}{4} \oint_C d\alpha \frac{\cos (q\tilde{\alpha})}{\sin^d(\alpha/2) \sin(q\alpha/2)},$$

where $C$ is a small clockwise loop around $\alpha = 0$. A closed form for this can be found in terms of Bernoulli polynomials, as detailed in [5], and below. I set, thermally, $\delta = \delta - 1/2 = \mu - 1/2 = \mu'$. The needed power series expansion of the integrand in (21) can be given in terms of higher Bernoulli polynomials, $B_n$. Equivalent to these are the, here more convenient, higher Nörlund $D$-polynomials defined, generally, by the central translation

$$D_{\nu}(x|\omega) = 2^\nu B_{\nu}(x + (\sum \omega_i)/2|\omega),$$

where $\omega$ is an $n$-vector of components $\omega_i$. (Refer to Nörlund, [19], for any unexplained notation). In the case here, $n = d + 1$ and $\omega = (q, 1_d)$.

The standard expansions then yield

$$\frac{\cos (q\mu' \alpha)}{\sin^d(\alpha/2) \sin(q\alpha/2)} = \frac{2^{d+1}}{q} \frac{\sin((\alpha/2) + \frac{\alpha}{2})}{\sin(\alpha/2)} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu)!} \left(\frac{\alpha}{2}\right)^{2\nu} D_{2\nu+1}(q\mu'|q, 1_d).$$

Hence, by residues, for even $d$, the exact answer is

$$W_d(q, \mu) = \frac{(-1)^{d/2}}{d!} \pi D_d^{(d+1)}(q\mu'|q, 1_d).$$

Although this can be taken as a general closed form, I will, equivalently, concentrate on the derivatives with respect to $\mu'$ and $q$ at $\mu' = 1/2$ and $q = 1$. This will be sufficient to give the derivatives of $h_n = h_{1/q}$ from (1) to (3). References [3] and [6] give general properties of these derivatives so a specific evaluation is useful. Because this is a rather particular calculation, it is relegated to an appendix.

### 10. Trigonometric sums

In this final section, I derive an alternative organisation of the polynomials in $n$ which occur in the conformal weights in even dimensions, $d = 2g$.

A different evaluation of the contour integral (21) yields the well known twisted trigonometric image sum, which occurs in many places, (see [20]),

$$W_d(q, \delta) = -\frac{\pi}{q} \sum_{l=1}^{q-1} \cos(2\pi\delta l) \cot \left(\frac{\pi l}{q}\right),$$

where the flux, $\delta = r/q$, is rational. I deal mostly with the case of no flux, $\delta = 1$, i.e. no chemical potential.

Although, to begin with, $q$ is integral, after the sum has been performed to give a rational function of $q$ one can set $q = 1/n$, now with, e.g. $n$ integral, [21]. This can be justified by an appeal to Carlson’s theorem, [22], which says, roughly, that a function which vanishes on the integers, and has some convergence properties as $n \to \infty$, is identically zero.
The earliest evaluation of (25) known to me is by Jeffery, [23], and I will utilise his results, as expounded in [21]. They are obtained by straightforward algebra.

The answer is the explicit function of $q$,

$$W_2(q, 1) = \frac{\pi^{2g-1}}{q \Gamma(2g)} \sum_{i=0}^{g-1} \frac{\Gamma(2i + 2)}{2^{2i}} A_i^g (1 - q^{2i+2}) \frac{\zeta(2i + 2)}{\pi^{2i+2}}, \quad (26)$$

where the $A_i^g$ are constants related to the differentials of nothing, or, equivalently, to central factorial numbers. They have a combinatorial significance or they can be evaluated by recursion. They vanish if $i \geq g$ and have been tabulated since 1909. (See also [24].)

To get the energy density, or conformal weight, according to (3), the combination $Y(q, 1)$ is required

$$Y(q, 1) = W_2(q, 1) - \frac{2g - 2}{2g - 1} W_{2g-2}(q, 1)$$

$$= \frac{\pi^{2g-1}}{\Gamma(2g)} \frac{1}{q} \sum_{i=0}^{g-1} \frac{\Gamma(2i + 2)}{2^{2i}} B_i^g (1 - q^{2i+2}) \frac{\zeta(2i + 2)}{\pi^{2i+2}}, \quad (27)$$

using $A_i^{g-1} = 0$ and where the $B_i^g$ are the easily found constants

$$B_i^g = A_i^g - (g - 1)^2 (A_i^{g-1} - \delta_{i,g-1}).$$

Note that $B_0^g = 0$ which means that there is no term proportional to $q$ in $Y(q, 1)$. This is a consequence of conformal coupling, [5], in the algebraic sense that, in the combination, (27), the coefficient of the second term depends on the specific conformal value of the coefficient, usually denoted by $\xi$, of the scalar curvature term in the propagation equation.

Putting things together and setting $q = 1/n$, the conformal weight is

$$h_n = \frac{2^{4g-2}}{(4\pi)^{g-1}} \frac{\Gamma(g)}{\Gamma(2g)} \sum_{i=1}^{g-1} \frac{\Gamma(2i + 2)}{2^{2i}} B_i^g (n - n^{-2i+1}) \frac{\zeta(2i + 2)}{\pi^{2i+2}}, \quad (28)$$

which is of the same form as equation (B.16) in [3] after a change of summation variable from $i$ to $j$ where $j = g - 1 - i$. The constants are different because the methods are different. The expressions are positive for $n > 1$.

In [21], I also computed (following Jeffery) the ‘fermionic’ sum, obtained by setting $\delta = (q - 1)/2q$, i.e. $\mu = (q - 1)/2$, in (25). This evaluated to a sum similar to (27) except that the Riemann $\zeta$-function was replaced by a Dirichlet $\eta$-function. I note that a related sum occurs in the spin-half calculations of [3].

11. Comments and conclusion

Explicit expressions for the conformal weights of free scalar twist operators have been given, and some plotted against chemical potential, for both odd and even dimensions for all covering integers. For odd dimensions this last is possible by virtue of an image relation involving the fluxes along the conical axis.

In even dimensions, polynomials valid for all cone angles can be found. Further, relating the required energy densities (and, therefore the conformal weights) to a known
trigonometric sum gives a more convenient form to the polynomials, at least for no chemical potential.

The derivatives of the conformal weights with respect to the chemical potential have discontinuities indicating phase transitions.

As an alternative to conformally transforming to a hyperbolic $d$-cylinder, the standard transformation to a conically deformed $d$-sphere can be applied. This has the advantage of leading to a compact space. The method has been detailed by Belin et al [6] in the case of $S^3$. Their technique involves finding the eigenvalues and their degeneracies, then constructing the logdet by direct summation with ad hoc zeta function regularisations.

In the language of the orbifolded sphere I have employed before, the sphere is divided into $2q$ equal lunes of angle $\pi/q$. If $q$ is integral, the lunes properly tile the sphere but one can extend $q$ into the reals and treat lunes of arbitrary angle, even bigger than $2\pi$. The spectrum on a periodic double lune, of angle $2\pi/q$, is given as the union of the Dirichlet and Neumann spectra on the single lune. This is best seen by using the description where the Aharonov–Bohm flux appears in the field equations. The ensuing analysis will be presented at another time.

The proof of the conjecture of [4] in section 4 shows the utility, in some circumstances, of conformally transforming to the flat conical space rather than to the hyperbolic cylinder.

The considerable simplification effected by Elvang and Hadjiantonis of the integral to a sum which has another interpretation would suggest that there is a way of obtaining this form more directly.

**Appendix A. A replica equivalent**

For completeness I enlarge on the analysis leading to the decomposition (12). This is only an elaboration of the relation given in [5] which relies on an earlier discussion of quantum field theory on a cone, [9], itself based on a more general image construction, [25], on multiply connected manifolds.

I start with an expression, stated for the heat kernel, $K(x, x')$, but valid for any quantity given linearly in terms of it (such as the Green function, $G$, energy–momentum density, etc), on a manifold for which there is a metric containing a conical (two-dimensional) part. Examples are the cosmic string, Rindler, de Sitter, Schwarzschild spaces and spheres, in the Euclidean case. It is not necessary to know $K$ explicitly to discuss the general formula. The polar angle of the cone is denoted by $\phi$. The remaining coordinates will not be displayed and I write the heat-kernel as $K = K(\phi, \phi') = K(\phi - \phi')$ and set $\phi' = 0$. The metric is assumed SO(2) invariant about the cone axis, which is the SO(2) fixed-point set. I keep the distance from the cone axis constant so the analysis reduces to that on the circle.

For a cone, $\phi$ has the range $0 \leq \phi \leq \beta$, with end points identified, and then $K = K_\beta$. The polar angle can be unrolled by letting $\beta \to \infty$ and one has the homotopy class relation, [9],

$$K_\beta(\phi; \delta) = \sum_{m=-\infty}^{\infty} e^{2\pi im \delta} K_\infty(\phi + m\beta; 0),$$

for the heat-kernel on a $\beta$-cone for a complex field that picks up a phase $\exp(2\pi i \delta)$ on circling the cone once ($\phi \to \phi + \beta$), as allowed by general theory on a multiply connected space. $\delta$ can be referred to as a flux, and everything has to be periodic in it.
Therefore (the limits could be $s = j$ to $s = n - 1 + j$),
\[
\frac{1}{n} \sum_{s=0}^{n-1} K_j(\phi; \delta + s/n) = \frac{1}{n} \sum_{m=\infty}^{n-1} e^{2\pi im(\delta + s/n)} K_\infty(\phi + m\beta; 0) \\
= \sum_{m=\infty}^{n-1} e^{2\pi im \delta} K_\infty(\phi + m\beta; 0) \bigg|_{m=-l n}^{l \in \mathbb{Z}} \\
= \sum_{l=-\infty}^{\infty} e^{2\pi il \delta} K_\infty(\phi + l n\beta; 0) \bigg|_{m=-l n}^{l \in \mathbb{Z}} \\
= K_{n\beta}(\phi; n\delta),
\] (30)
which is the required decomposition, as given in\textsuperscript{5} [5].

The right-hand side is periodic in $\delta$ with period $1/n$, as is the left-hand side, which follows on shifting the summation limits and bringing them back using the (usual) periodicity of $K_j(\ast; \delta) = K_j(\ast; \delta + 1)$. This means that $\delta = 1/n$ is the same as $\delta = 1$ and equivalent to zero chemical potential.

If, for example, $\beta = 2\pi$, the equation expresses the projection from a (covering) rolled up circle of circumference $2\pi n$ down to one of circumference $2\pi$ (of the same radius). For no flux through the big circle, (30) shows that quantities on the integer covering conical manifold can be expressed as a sum of quantities on a non-conical manifold ($\beta = 2\pi$), but one with a non-zero flux through an axis, as mentioned in [5]. Because of the projection, (30) can be referred to as a (pre-) image sum, see [25].

For no flux on the right-hand side of (30) I need, in my conventions, $n\delta = 1$, so that the fluxes on the left-hand side, are $(s + 1)/n$ and I can write (30) as
\[
K_{2n}(\phi) = \frac{1}{n} \sum_{k=0}^{n-1} K_{2k}(\phi; k/n),
\] (31)
which is, essentially, the result that leads to (13) as used in section 4. Some subtractions might be necessary to obtain finite quantities.

I emphasise that this decomposition is valid for any manifold that contains an SO(2) invariant cone.

The same relation holds, at least formally, for the effective action (equivalently free energy, partition function) and could therefore be regarded as a cheap derivation of the replica equation
\[
\log Z[n] = \sum_{k=0}^{n-1} \log Z_{k/n},
\]
e.g. [26] equation (5), without any diagonalisation of coupled ‘twist fields’.

The disappearance of the factor $1/n$ is a volume effect as the quantities in (12), and similar relations, refer to densities.

The extension to fermionic fields can be effected by putting $\delta \rightarrow \delta \pm 1/2$ in (29) to allow for the extra minus sign on circling the $\beta$-cone once. This results in the relation
\[
K_{n\beta}^f(\phi; n(\delta + 1/2) + 1/2) = \frac{1}{n} \sum_{s=0}^{n-1} K_{^f\delta}(\phi; \delta + s/n),
\]
The second argument is the flux through the cone.

\textsuperscript{5} From the corresponding relation for Bernoulli polynomials it could be referred to as a multiplication or a transformation formula.
Assuming no flux on the left-hand side gives
\[
\delta = \delta_0 = -\frac{n - 1}{2n}
\]
so
\[
K_{n,0}^f(\phi; 1) = \frac{1}{n} \sum_{s=0}^{n-1} K_s^f(\phi; \delta_0 + s/n).
\]
(32)

I refer to this as the zero chemical potential case. In particular
\[
K_{2n}^f(\phi; 1) = \frac{1}{n} \sum_{s=0}^{n-1} K_{2s}^f(\phi; \delta_0 + s/n).
\]
(33)
or, after translating the summation variable from \(s\) to \(k = s - (n-1)/2\),
\[
K_{2n}^f(\phi; 1) = \frac{1}{n} \sum_{k=-(n-1)/2}^{(n-1)/2} K_{2k}^f(\phi; k/n).
\]
(34)
Equations (33) and (34) replace (31). If \(n = 1\) then \(s = 0, k = 0\) and the flux argument of \(K_{2\pi}\)
is 0, equivalent to 1, which is correct by periodicity.

The partition function equations
\[
\log Z_f[n] = \sum_{s=0}^{n-1} \log Z_{s/n+\delta_0} K_s^f(\phi; k/n).
\]
then follow.

For non-zero chemical potential, \(\mu\), one sets
\[
\delta = \mu + \delta_0
\]
and (32) is replaced by
\[
K_{n,0}^f(\phi; \mu) = \frac{1}{n} \sum_{s=0}^{n-1} K_s^f(\phi; \mu + \delta_0 + s/n)
\]
\[
= \frac{1}{n} \sum_{k=-(n-1)/2}^{(n-1)/2} K_{2k}^f(\phi; \mu + k/n)
\]
(35)
which are periodic in \(\mu\) with period 1/\(n\). Hence zero chemical potential can be achieved by choosing any of \(\mu = m/n, m \in N\).

**Appendix B. Derivatives for even \(d\)**

In this appendix I pursue the question of the derivatives of the conformal weights raised in section 9.

I prefer the derivatives with respect to \(q\) rather than \(n, =1/q\), as \(q W_0(q, \mu')\) is a bipolynomial in \(q\) and \(\mu'\) and most derivatives become zero after a while. The derivatives with respect to \(n\) can easily be reconstituted.
I define the derivatives

\[
W_{ab}(d, q, \mu') \equiv \frac{1}{a! b!} \partial_a^\mu \partial_b^\mu W_d(q, \mu)
\]

\[
W_{ab}(d) \equiv W_{ab}(d, q, \mu')|_{\mu' = 1/2, q = 1}.
\] (36)

If one wishes to display the polynomial content of \((36)\) explicitly then there is the expansion, \([19] \text{ p 162}\)

\[
D_{d}^{(d+1)}(q \mu'|q, 1_d) = \sum_{s=0}^{d} \binom{d}{s} (2q \mu')^s D_{d-s}^{(d+1)} [q, 1_d] ,
\] (37)

in terms of higher Nörlund D-numbers, \(D_{\nu}^{(n)}[\ast]\), which are zero if \(\nu\) is odd so the sum in \((37)\) is over even integers because, here, \(d\) is even.

Furthermore, the \(D_{\nu}[\ast]\) are themselves polynomials in \(q\), with

\[
q^{t-1} D_{d-t}^{(d+1)} [q, 1_d] = \sum_{t=0}^{d-x} \binom{d-x}{t} q^{t+1} D_{d-1}^{(1)} [1] D_{d-t-1}^{(d)} [1_d].
\] (38)

For shortness, the quantities with all parameters equal to 1 (historically the most studied case) are denoted by

\[
D_{d}^{(d)} \equiv D_{d}^{(d)} [1_d],
\]

and are sometimes referred to as the (higher) Nörlund D-numbers. In particular one sets \(D_{\nu} \equiv D_{\nu}^{(1)}\). These are the \(d\)-analogues of the more familiar Bernoulli numbers and can be computed in similar ways. Then setting \(q = 1\) in \((38)\) gives a recursion for the higher numbers. There are other means of finding them however.

Simply as a check at this point, I confirm that the previously found product form, e.g. \((19)\), at \(q = 1\), results from the method of this section. From \((24)\), it is sufficient to give the polynomials, \(D_{d}^{(d+1)}(x) \equiv D_{d}^{(d+1)}[x 1_{d+1}]\). Here \(d\) is even.

The result follows from the standard descending factorial form for the \(B_{\nu-1}^{(n)}(x)\) given by Nörlund, [19], equation \((4)\) p 186 as a simple consequence of recursion. The transition to \(D, \) \((22)\), corresponds to a central translation of \(B, \) and gives the (even) central factorial

\[
D_{d}^{(d+1)}(x) = 2^d \left( x^2 - \frac{1}{4} \right) \left( x^2 - \frac{9}{4} \right) \ldots \left( x^2 - \frac{(d - 1)^2}{4} \right)
\] (39)

which, when used in \((24)\), allows expression \((19)\) to be regained.\(^6\)

The series expressions are easily programmed and I give two examples for the derivatives, \(h_{a, b}^{(n)}\), of the conformal weights obtained by combining \((36)\), \((1)\), \((2)\) and \((3)\). I display them without the factor of \(1/(4\pi)^{d/2}\) appearing in \((3)\) and also without the factor of \(\pi\) in \((1)\) in order to give simple fractions. For \(d = 2, \)\(^6\)

\(^6\) The coefficients of the expansion \((39)\) in powers of \(x^2\) are defined to be the central factorial numbers, or the central differentials of nothing. Then \((37)\) produces a relation between these quantities and the higher Nörlund D-numbers. All this is more or less standard but not widely used. Consult Nörlund’s classic text, [27], for some relevant expansions. The expression \((39)\) also yields a combinatorial form for these D-numbers as the product of the first \(d/2\) odd numbers taken a certain number of times.
And for \( d = 4 \),

\[
\begin{array}{c|cccc|c|cccc}
 b \backslash a & 0 & 1 & 2 & 3 & 4 \\
 0 & 0 & -4 & -4 & 0 & \\
 1 & -4/3 & -4 & 0 & 0 \\
 2 & 2/3 & 0 & 0 & 0 \\
 3 & -2/3 & 0 & 0 & 0 \\
\end{array}
\]

Note that, unless \( a = 0 \), the derivatives terminate beyond \( d \) or \( d - 1 \) because of the polynomial structure.

Other dimensions can easily be computed. Ideally one would like to have the dependence on dimension displayed explicitly but this seems more difficult.

### B.1. Computation of the D-numbers

As a brief aside, I turn to the purely mathematical topic of the computation of the Nörlund D-numbers, a well known occupation. Equation (38) involves the even variety, \( D_{2n-2l}^{(2n)} \).

Probably the most useful expression is the one which comes from the relation with the central differentials of nothing (or the central factorial numbers). This is.

\[
D_{2m-2l}^{(2n)} = 2^{2m-2l} \left( \frac{2n - 2l)!}{2l(2n - 1)!} \right) \left( \frac{D^2_{2l} \prod_{i=0}^{n-1} (x^2 - i^2)}{D^2_{2l-1} \prod_{i=1}^{n-1} (x^2 - i^2)} \right) \bigg|_{x=0}
\]

where \( D_s = d/dx \). The second expression in (40) also results from an application of the relation between the \( D^{(n)} \) and the generalised Bernoulli polynomials written in terms of products, [19] p 193. It is less convenient than the top line which has a neater combinatorial form, the limit of the term in brackets being equal to the product of the squares of the first \( n - 1 \) integers, taken a certain number of times. Precisely

\[
\frac{D^{2l} \left( \frac{g^{(2n)}}{(2l)!} \right)}{(2l)!} = \text{co}_{2l} \prod_{i=0}^{n-1} \left( x^2 - i^2 \right) = \text{co}_{2l-2} \prod_{i=1}^{n-1} \left( x^2 - i^2 \right)
\]

\[
= (-1)^l \sum_{i_1 < \ldots < i_l} i_1^2 i_2^2 \ldots i_l^2.
\]

This can be taken as an explicit computational formula for \( D_{2n-2l}^{(2n)} \), if \( \nu < n \),

\[
D_{2n-2l}^{(2n)} = (-1)^l 2^{2n-2l} \left( \frac{2n - 1}{2l - 1} \right)^{n-1} \sum_{i_1 < \ldots < i_{n-l}} i_1^2 i_2^2 \ldots i_{n-l}^2,
\]

which avoids recursion and Bernoulli numbers.

See also Mansour and Dowker, [28].
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