Parallel Quantum Pebbling: Analyzing the Post-Quantum Security of iMHFs

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Abstract. The classical (parallel) black pebbling game is a useful abstraction which allows us to analyze the resources (space, space-time, cumulative space) necessary to evaluate a function $f$ with a static data-dependency graph $G$. Of particular interest in the field of cryptography are data-independent memory-hard functions $f_{G,H}$ which are defined by a directed acyclic graph (DAG) $G$ and a cryptographic hash function $H$. The pebbling complexity of the graph $G$ characterizes the amortized cost of evaluating $f_{G,H}$ multiple times as well as the total cost to run a brute-force preimage attack over a fixed domain $X$, i.e., given $y \in \{0, 1\}^*$ find $x \in X$ such that $f_{G,H}(x) = y$. While a classical attacker will need to evaluate the function $f_{G,H}$ at least $m = |X|$ times a quantum attacker running Grover’s algorithm only requires $\mathcal{O}(\sqrt{m})$ blackbox calls to a quantum circuit $C_{G,H}$ evaluating the function $f_{G,H}$. Thus, to analyze the cost of a quantum attack it is crucial to understand the space-time cost (equivalently width times depth) of the quantum circuit $C_{G,H}$. We first observe that a legal black pebbling strategy for the graph $G$ does not necessarily imply the existence of a quantum circuit with comparable complexity — in contrast to the classical setting where any efficient pebbling strategy for $G$ corresponds to an algorithm with comparable complexity evaluating $f_{G,H}$. Motivated by this observation we introduce a new (parallel) quantum pebbling game which captures additional restrictions imposed by the No-Deletion Theorem in Quantum Computing. We apply our new quantum pebbling game to analyze the quantum space-time complexity of several important graphs: the line graph, Argon2i-A, Argon2i-B, and DRSample. Specifically, (1) We provide a recursive quantum pebbling attack which shows that the space-time complexity of the line graph is at most $\mathcal{O}(N^{1+\epsilon})$ for any constant $\epsilon$. (2) We show that any $(\epsilon, d)$-reducible DAG has space-time complexity at most $\mathcal{O}(N\epsilon + dN^2d)$. In particular, this implies that the quantum space-time complexity of Argon2i-A and Argon2i-B are at most $\mathcal{O}(N^2 \log \log N/\sqrt{\log N})$ and $\mathcal{O}(N^2 / \sqrt{\log N})$, respectively. (3) We show that the quantum space-time complexity of DRSample is at most $\mathcal{O}(N^2 \log \log N/ \log N)$. It is an open question to construct a constant indegree graph with quantum space-time complexity $\Omega(N^2)$ or to give a general pebbling attack showing that no such graph exists.

Keywords: Parallel Quantum Pebbling · Argon2i · DRSample · Data-Independent Memory-Hard Function
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1 Introduction

The (parallel) black pebbling game [PH70, Coo73] is a powerful abstraction which can be used to analyze the resources (space, space-time, amortized space-time) necessary to evaluate any function $f_G$ with a static data-dependency graph $G$. In the black pebbling game we are given a directed acyclic graph (DAG) $G = (V, E)$ where nodes intuitively represent intermediate data values and edges represent dependencies between these values, e.g., if $z = x \times y$ then we would add directed edges from nodes $x$ and $y$ to node $z$ to indicate that $x$ and $y$ are required to compute $z$. However, while the parallel black pebbling game is a useful abstraction for classical computation it is not a suitable model for reversible computation as in quantum computation. In this paper, we introduce a parallel quantum pebbling game as an abstraction which can be used to analyze the resources required to build a reversible quantum circuit evaluating our function $f_G$. We use this new pebbling game to analyze the space-time cost of several important graphs (the line graph, Argon2i-A, Argon2i-B, DRSample) associated with prominent data-independent memory-hard functions (iMHFs) — used in cryptography to design egalitarian proof of work puzzles and to protect low-entropy secrets (e.g., passwords) against brute-force attacks.

Review: Parallel Black Pebbling. The classical parallel black pebbling game begins with no pebbles on the graph ($P_0 = \{\}$), and during each round of the pebbling game, we may only place a new pebble on a node $v$ if all of $v$’s parents were pebbled in the previous round. Intuitively, if the data value $X_v$ corresponding to node $v$ is computed as $X_v := H(X_u, X_{v-1})$ then $G$ would include directed edges $(u, v)$ and $(v-1, v)$ indicating that we cannot compute value $X_v$ (resp. place a pebble on node $v$) unless $X_u$ and $X_{v-1}$ are already available in memory (resp. we already have pebbles on nodes $u$ and $v - 1$). More formally, if $P_i \subseteq V$ denotes the set of pebbled nodes during round $i$, then we require that \text{parents}(P_{i+1} \setminus P_i, G) \subseteq P_i$ where \text{parents}(S, G) = \bigcup_{u \in S} \{u : (u, v) \in E\}. In the black pebbling game we are given a subset $T \subseteq V$ of target nodes (corresponding to output data values) and the goal of the black pebbling game is to eventually place a pebble on each node in $T$. A pebbling $P = (P_0, P_1, \ldots, P_t)$ is legal if $P_0 = \{\}$ and \text{parents}(P_{i+1} \setminus P_i, G) \subseteq P_i$ for each $i < t$. Intuitively, the requirement that \text{parents}(P_{i+1} \setminus P_i, G) \subseteq P_i$ enforces the natural constraint that we cannot compute a new data value before all dependent data values are available in memory. In the sequential pebbling game, we additionally require that $|P_{i+1} \setminus P_i| \leq 1$ so that only one new pebble can be placed on the graph in each round while the parallel pebbling game has no such restriction. Thus, a legal parallel (resp. sequential) pebbling of a data-dependency graph $G$ naturally corresponds to a parallel (resp. sequential) algorithm to compute $f_G$ and the number of pebbles $|P_i|$ on the graph in each round $i$ corresponds to memory usage during each round of computation.

The sequential black pebbling game has been used to analyze space complexity (e.g., see [HPV77, PTC76]) and to examine space-time tradeoffs, e.g., see [Cob66, Coo73, Pau75, PV76, Tom81]. In the field of cryptography, the parallel black pebbling game has been used to analyze the security of data-independent memory-hard functions (iMHFs). An iMHF $f_{G,H}$ is defined using a cryptographic hash function $H$ and a data-dependency graph $G$, e.g., [AS15, AB16, ABP17, BZ17]. The output of $f_{G,H}(x)$ is defined to be the label $X_N$ of the final sink node $N$ in $G$ where the label $X_1 = H(X)$ of the first (source) node is obtained by hashing the input and the label of each internal node $v$ is obtained by hashing the labels of all of $v$’s parents, e.g., if \text{parents}(v, G) = \{u, v - 1\}$ then we would set $X_v = H(X_u, x_{v-1})$. In many cryptographic applications (e.g., password hashing), we want to ensure that it is moderately expensive to evaluate $f_{G,H}$ to ensure that a
brute-force preimage attack (given $y$ find some $x$ such that $f_{G,H}(x) = y$) is prohibitively expensive even when the domain $X$ of inputs is smaller (e.g., low entropy passwords). When modeling the cryptographic hash function $H$ as a random oracle, one can prove that the cost to evaluate $f_{G,H}$ in the parallel random oracle model is exactly captured by the pebbling cost of $G$, e.g., see [AS15, AT17, ABP18]. Thus, we would like to pick a graph $G$ with high pebbling costs and/or understand the pebbling costs associated with candidate iMHFs. Prior work demonstrated that the amortized space-time complexity of prominent iMHF candidates, including Password Hashing Competition winner Argon2i, was lower than previously hoped [AB16, ABP17, AB17, BZ17]. On the positive side, recent work has shown how to use depth-robust graphs [EGS75] to construct iMHFs with (essentially) optimum amortized space-time complexity [ABP17, ABH17, BHK+19]. However, it is important to note that the classical black pebbling game does not include any rules constraining our ability to remove pebbles. We are allowed to remove pebbles from the graph at any point in time which corresponds to freeing memory and can be done to reduce the space usage. While the classical pebbling game allows us to discard pebbles at any point in time to free memory, this action is often not possible in a quantum circuit due to the No-Deletion Theorem [KPB00]. In this sense, the black pebbling game cannot be used to model reversible computation as in a quantum circuit and an efficient parallel black pebbling for a graph $G$ does not necessarily imply the existence of an quantum circuit $C_{G,H}$ with comparable cost.

**Review: Measuring Pebbling Costs.** There are several natural ways to measure the cost of a pebbling. The space cost of a pebbling $P = (P_0, \ldots, P_t)$ measures the maximum number of pebbles on the graph during any round, i.e., $\max_i |P_i|$ and the space complexity of a graph measures the minimum space cost over all legal pebblings of $G$. Similarly, the space-time cost of a pebbling $P = (P_0, \ldots, P_t)$ measures the product $t \times \max_i |P_i|$ and the cumulative pebbling cost is $\sum_i |P_i|$. Intuitively, space complexity measures the amount of memory (e.g., RAM) required for a computation and space-time cost measures the full cost of the computation by telling how long the memory will be locked up during computation. Cumulative pebbling costs gives the amortized space-time complexity of pebbling multiple copies of the graph $G$, i.e., when we are evaluating our function $f_G$ on multiple different inputs in parallel [AS15].

**(Quantum) Pre-Image Attacks.** Understanding the amortized space-time complexity of a graph $G$ is important to estimate the cost of a classical brute-force pre-image attack attack over a domain $X$ of size $m$. In particular, suppose we are given a target output $y$ (e.g., $y = f_{G,H}(x')$ for a secret input $x \in X$) and we wish to find some input $x' \in X$ such that $y = f_{G,H}(x')$. Classically, the space-time cost of a blackbox preimage attack would require us to evaluate the function $f_{G,H}$ on $\Omega(m)$ inputs. If the cumulative pebbling cost of $G$ is given by $\sum_i |P_i|$ then the total space-time cost of the preimage attack would scale proportionally to $m \sum_i |P_i|$, i.e., $m$ times the amortized space-time complexity. Thus, a more efficient black pebbling strategy for $G$ yields an lower cost pre-image attack.

In the context of quantum computing, Grover’s algorithm [Gro96] substantially reduces the cost of a brute-force preimage attack over a domain $X$ of size $m$. In particular, Grover’s algorithm only requires $O(\sqrt{m})$ blackbox queries to the function $f_{G,H}$ evaluating the function $f_{G,H}$ and this is optimal — any quantum algorithm using $f_{G,H}$ as a blackbox must make at least $\Omega(\sqrt{m})$ queries [BBBV97]. If we instantiate $f_{G,H}$ with a quantum circuit of width $w$ and depth $d$ then full Grover circuit would have width $W = O(w)$ and depth $D = d \times O(\sqrt{m})$. In particular, the total space-time (equivalently width-depth) cost of the attack would be $wd \times O(\sqrt{m})$. Thus, to analyze the cost of
a quantum preimage attack it is crucial to understand the space-time (or width-depth) cost of a quantum circuit \( C_{G,H} \) computing \( f_{G,H} \). Our goal will be to treat \( H \) as a black-box and use graph pebbling to characterize the space-time cost. A natural first attempt would be to use the classical black pebbling game to analyze the parallel pebbling cost of \( G \) as above. If this approach worked we could simply leverage prior (parallel) black pebbling analysis of prominent iMHF candidates, e.g., [AB16, ABP17, AB17, BZ17] to analyze the cost of a quantum pre-image attack. Unfortunately, this approach breaks down because a legal black pebbling strategy does not necessarily correspond to a valid quantum circuit \( C_{G,H} \) with comparable cost. Thus, we will require a different pebbling game to analyze the width-depth cost of the quantum circuit \( C_{G,H} \).

**Related Pebbling Games.** Prior work [Ben89, Krá01, MSR+] introduced a reversible pebbling game to capture restrictions imposed by the No-Deletion Theorem and analyze space-time tradeoffs in quantum computing. However, the pebbling game considered in these works is sequential and only allows for the addition/removal of one pebble in each round. As such the reversible pebbling game is also not suitable to analyze the space-time cost of a quantum circuit evaluating \( f_{G,H} \) since the circuit can evaluate \( H \) multiple times in parallel. There are several important subtleties that must be considered when extending the game to the parallel setting. More recently, [KSS21] introduced a new (sequential) pebbling game called the *spooky pebble game* to model measurement based deletion in quantum computation. Intuitively, measurement based deletion allows for the conversion of some qubits into (cheaper) classical bits which can later be used to restore the quantum state. The spooky pebble game only allows for sequential computation and the cost model ignores classical storage. Furthermore, a spooky quantum pebbling requires frequent intermediate measurements during computation which would present an enormous technical barrier to implement any spooky pebbling attack. By contrast, a pebbling attack in our parallel quantum pebbling game naturally corresponds to a quantum circuit which does not require any intermediate measurements and our cost model accounts for the total storage cost (classical + quantum). While [KSS21] introduced a spooky pebbling attack on the line graph, we note this spooky pebbling strategy does not yield an efficient quantum pebbling attack in our model as the classical storage is quite high, and the pebbling attack requires frequent intermediate measurements.

**Notation.** We use the notation \([N]\) (resp. \([a, b]\)) to denote the set \(\{1, \ldots, N\}\) (resp. \(\{a, a+1, \ldots, b\}\)) for a positive integer \(N\) (resp. \(a \leq b\)). The notation \(\overset{\to}{\in}\) denotes a uniformly random sampling, e.g., we say \(x \overset{\to}{\in} [N]\) when \(x\) is a uniformly sampled integer from 1 to \(N\). For simplicity, we let \(\log(\cdot)\) be a log base 2, i.e., \(\log x := \log_2 x\).

Let \(G = (V, E)\) be a directed acyclic graph (DAG) where we denote \(N\) to be the number of nodes in \(V = [N]\). Given a node \(v \in V\), we define \(\text{parents}(v, G)\) to be the immediate parents of node \(v\) in \(G\), and we extend this definition to a subset of nodes as well; for a set \(W \subseteq V\), we define \(\text{parents}(W, G) := \bigcup_{w \in W} \{ u : (u, w) \in E \}\). We let \(\text{ancestors}(v, G)\) be the set of all ancestors of \(v\) in \(G\), i.e., \(\text{ancestors}(v, G) := \bigcup_{i \geq 1} \text{parents}^i(v, G)\), where \(\text{parents}^1(v, G) = \text{parents}(v, G)\) and \(\text{parents}^i(v, G) = \text{parents}(\text{parents}^{i-1}(v, G), G)\). Similarly, for a set \(W \subseteq V\), we define \(\text{ancestors}(W, G) := \bigcup_{i \geq 1} \text{parents}^i(W, G)\), where \(\text{parents}^1(W, G) = \text{parents}(W, G)\) and \(\text{parents}^i(W, G) = \text{parents}(\text{parents}^{i-1}(W, G), G)\).

We denote the set of all sink nodes of \(G\) with \(\text{sinks}(G) := \{ v \in V : \#(v, u) \in E \}\) – note that \(\text{ancestors}(\text{sinks}(G), G) = V\). We define \(\text{depth}(v, G)\) to refer to the number of the longest directed path in \(G\) ending at node \(v\) and we define \(\text{depth}(G) = \max_{v \in V} \text{depth}(v, G)\) to refer to the number of nodes in the longest directed path in \(G\). Given a node \(v \in V\), we define \(\text{indeg}(v) := |\text{parents}(v, G)|\).
to denote the number of incoming edges into \( v \), and we also define \( \text{indeg}(G) := \max_{v \in V} \text{indeg}(v) \). Given a set \( S \subseteq V \) of nodes, we use \( G - S \) to refer to the subgraph of \( G \) obtained by deleting all the nodes in \( S \) and all edges that are incident to \( S \). We also use the notation \( S \leq k := S \cap [k] \) denotes the subset of \( S \) that only intersects with \([k]\).

We denote with \( P_{G,T} \) and \( P_{G,T}^\parallel \) the set of all legal sequential and parallel classical pebblings of \( G \) with target set \( T \), respectively. In the case where \( T = \text{sinks}(G) \), we simply write \( P_G \) and \( P_G^\parallel \), respectively.

### 1.1 Our Results

We introduce the parallel quantum pebbling game as a tool to analyze the (amortized) space-time cost of a quantum circuit evaluating a function \( f \) with a static data-dependency graph \( G \). As we discuss there are several key subtleties that arise when extending the sequential reversible pebbling game to the parallel setting. We argue that any parallel quantum pebbling \( P = (P_0, \ldots, P_t) \) of the graph \( G \) corresponds to a quantum circuit \( C_P \) evaluating \( f \) with comparable costs, e.g., the depth of the quantum circuit \( C_P \) corresponds to the number of pebbling rounds \( t \) and the width of the circuit corresponds to the space complexity of the pebbling i.e., \( \max_i |P_i| \).

As an application, we use the parallel quantum pebbling game to analyze the space-time cost of several important password hashing functions \( f_{G,H} \) including PBKDF2, BCRYPT, Argon2i, and DRSample.

#### Quantum Pebbling Attacks on Line Graphs

We first focus on analyzing the quantum pebbling cost of a line graph \( L_N \) with \( N \) nodes \( \{1, \ldots, N\} \) and edges \((i, i+1)\) for each \( 1 \leq i < N \). Classically, there is a trivial black pebbling strategy for the line graph with simply walks a single pebble from node 1 to node \( N \) over \( N \) pebbling rounds, i.e., in each round \( i \) we place a new pebble on node \( i \) and then delete the pebble on node \( i-1 \). This pebbling strategy is clearly optimal as the maximum space usage is just 1 and the space-time cost is just \( N \times 1 = N \). However, this simple pebbling strategy is no longer legal in the quantum pebbling game and it is a bit tricky just to find a quantum pebbling strategy whose space-time cost is significantly lower than \( O(N^2) \) — the space-time cost of the naive pebbling strategy which avoids removing pebbles. We give a recursive quantum pebbling attack with space-time complexity \( O(N^{1+\epsilon}) \) for any constant \( \epsilon > 0 \).

Because the space-time complexity of the line graph \( G = L_N \) is so low, it is a poor choice for an iMHF \( f_{G,H} \) or for password hashing \[BHZ18\]. However, the line graph \( L_N \) naturally corresponds to widely deployed password hashing algorithms like BCRYPT \[PM99\] and PBKDF2 \[Kal00\] which use hash iteration to increase costs where the parameter \( N \) controls the number of hash iterations. Thus, to understand the cost of a (quantum) brute-force password cracking attack it is useful to analyze the (quantum) pebbling cost of \( L_N \).

#### Quantum Pebbling Attack for Depth-Reducible DAGs

We give a generic quantum pebbling attack on any \((e, d)\)-reducible DAG \( G \) with space-time cost \( O(Ne + dN2^d) \) which corresponds to a meaningful attack whenever \( e = o(N) \) and \( d2^d = o(N) \). A DAG \( G \) is said to be \((e, d)\)-reducible if there is a subset \( S \subseteq V \) of at most \( e \) nodes such that any length \( d \) path \( P \) in \( G \) contains at least one node in \( S \). As we show this leads to meaningful quantum pebbling attacks on Argon2i, the winner of the Password Hashing Competition. Specifically, we demonstrate how to construct depth-reducing sets for Argon2i-A (an older version of Argon2i) and Argon2i-B (the current version of Argon2i).
with $e = o(N)$ and $d2^d = o(N)$. This leads to quantum pebbling attacks with space-time complexity $O(N^2 \log \log N/\sqrt{\log N})$ and $O(N^2/\sqrt{\log N})$ against Argon2i-A and Argon2i-B, respectively.

In the classical pebbling setting, Alwen and Blocki [AB16] previously gave a generic pebbling attack on $(e, d)$-reducible DAGs with amortized space-time cost $O(Ne + N^2d/e)$. However, this pebbling attack is not legal in the quantum setting, and without amortization the space-time cost is still $N^2$ — the average number of pebbles on the graph per round is just $e + Nd/e$ but at the peak the pebbling strategy still requires $\Omega(N)$ pebbles. In our pebbling strategy, the maximum space usage is $O(e + d2^d)$.

Quantum Pebbling Attack against DRSample. Finally, we use the quantum pebbling game to analyze DRSample [ABH17] — a proposal to update the edge distribution in Argon2i with a depth-robust graph. With high probability, a randomly sampled DRSample DAG $G$ will not be $(e, d)$-reducible for parameters $e, d$ as large as $e = \Omega(N/\log N)$. Thus, the generic quantum pebbling attack on $(e, d)$-reducible graphs does not apply. We give an alternate pebbling strategy by partitioning the nodes of $G$ into $\lceil N/b \rceil$ consecutive blocks of size $b$ and converting a quantum pebbling of the line graph $L[\lceil N/b \rceil]$ into a legal quantum pebbling of $G$. The quantum pebbling strategy will be cost effective as long as we have an efficient pebbling strategy for $L[\lceil N/b \rceil]$ and the graph $G$ does not contain too many “long” edges $(u, v)$ with $|v - u| \geq b$ — we show that DRSample does not contain too many long edges when $b = N/\log^2 N$. Combined with our quantum pebbling strategies for the line graph, this leads to an attack on DRSample with space-time cost at most $O(N^2 \log \log N/\log N)$.

We give an efficient quantum pebbling algorithm which transforms a legal quantum pebbling $P' = (P_1', \ldots, P_v')$ of the line graph $L[\lceil N/b \rceil]$ into a legal quantum pebbling $P = (P_1, \ldots, P_t)$ of a DAG $G = (V, E)$. Intuitively, the pebbling transformation works as follows: (1) partition the nodes in $G$ into $\lceil N/b \rceil$ blocks $B_1, \ldots, B[\lceil N/b \rceil]$ each containing $b$ consecutive nodes. (2) Whenever we place a new pebble on a node $v \in L[\lceil N/b \rceil]$ in the pebbling $P'$, we pebble all of the nodes in block $B_v$ over the next $b$ rounds in the pebbling $P$ for $G$. (3) Whenever we remove a pebble from a node $v \in L[\lceil N/b \rceil]$ in the pebbling $P'$, we remove all of the nodes in block $B_v$ over the next $b$ rounds with the possible exception of nodes $u \in B_v$ which have a long outgoing edge $(u, w)$ with $|w - u| \geq b$.

The new pebbling requires $t = O(bt')$ rounds and if the pebbling $P'$ for $L[\lceil N/b \rceil]$ uses at most $s' = \max_j |P'_j|$ pebbles then the pebbling $P$ requires space at most $bs' + (#\text{skip})$ where $#\text{skip}$ is upper bounded by the number of long edges $(u, v) \in E$ with $|v - u| \geq b$. Thus, the total space-time complexity will be $O(b^2s't' + N#\text{skip})$ and we will be able to obtain an efficient quantum pebbling attack as long as $b = o(N)$ and $(#\text{skip}) = o(N)$.

We observe that with high probability, a randomly sampled DRSample graph $G = (V, E)$ has the property that there are not too many directed edges $(u, v)$ with length $|v - u| > b = N/\log^2 N$. In particular, we will have $(#\text{skip}) = O(N \log \log N/\log N)$ with high probability. We can use our previous quantum pebbling attacks on the line graph to obtain a quantum pebbling of $L[\lceil \log^2 N \rceil]$ using time $t' = O(\log^2 N)$ and space $s' = o(\log N)$. Combined with our pebbling transformation, this leads to a quantum pebbling attack on DRSample with space-time cost at most $O(N^2 \log \log N/\log N)$.

1.2 Technical Overview

Defining the Parallel Quantum Pebbling Game. We begin by defining and motivating the quantum pebbling game. We want to ensure that any legal (parallel) quantum strategy for $G$
corresponds to a quantum circuit \( C_{G,H} \) evaluating \( f_{G,H} \) that could be used as part of a preimage attack using Grover’s algorithm.

We first consider the parallel quantum random oracle model \([BDF+11]\) where the random oracle is a function \( H : \{0,1\}^{\leq 2\lambda} \rightarrow \{0,1\}^\lambda \). In the parallel quantum random oracle model we are given access to a quantum oracle maps basis states of the form \( |x_1, y_1, \ldots, x_k, y_k, z\rangle \) to the new state \( |x_1, y_1 \oplus H(x_1), \ldots, x_k, y_k \oplus H(x_k), z\rangle \). Here, \( x_1, \ldots, x_k \) denote the queries, \( y_1, \ldots, y_k \) denote the output registers and \( z \) denotes any auxiliary data. Notice that if \( y_i = 0^\lambda \) then the \( i^{th} \) output register will just be \( H(x_i) \) after the query is submitted.

Now consider the function \( f(x) = H^N(x) \) where \( H^1(x) = H(x) \) and \( H^{i+1}(x) = H(H^i(x)) \). The data-dependency graph for \( f \) is simply the line graph \( G = L_N \). In our quantum pebbling game, we want to ensure that each pebbling transition corresponds to a legal state transition in the quantum random oracle model. If \( N = 5 \), then the pebbling configuration \( P_i = \{2, 3, 4\} \) intuitively corresponds to a quantum state containing the labels \( X_2 = H^2(x), X_3 = H^3(x) \) and \( X_4 = H^4(x) \). From this state, we could use \( X_4 \) and an input register and submit the query \(|X_4, 0^\lambda\rangle\) to the random oracle to obtain \( X_5 = H(X_4) \) from the resulting state \(|X_4, H(X_4)\rangle\). Similarly, while we cannot simply delete \( X_3 \) we could uncompute this value by using \( X_3 \) as an output register and submitting the random oracle query \(|X_2, X_3\rangle\) to obtain the new state \(|X_2, H(X_2) \oplus X_3\rangle = |X_2, 0^\lambda\rangle\) in which the label \( X_3 \) has been removed. However, without the label \( X_1 \) there is no way to uncompute \( X_2 \) without first recomputing \( X_1 \).

The above example suggests that we extend the parallel pebbling game by adding the rule that \( \text{parents}(P_i \setminus P_{i+1}, G) \subseteq P_i \), i.e., a pebble can only be deleted if all of its parents were pebbled at the end of the previous pebbling round. While this rule is necessary, it is not yet sufficient to prevent impossible quantum state transitions. In particular, the rule would not rule out the pebbling transition from \( P_i = \{1, 2, \ldots, i\} \) to the new configuration \( P_{i+1} = \{\} \) where all labels have been removed from memory. This pebbling transition would correspond to a quantum transition from a state in which labels \( X_1, \ldots, X_i \) are stored in memory to a new state where all of these labels have been uncomputed after just one (parallel) query to the random oracle. Because quantum computation is reversible this would also imply that we could directly transition from the original state (no labels computed) to a state in which all of the labels \( X_1, \ldots, X_i \) are available after just one (parallel) query to the quantum random oracle. However, it is known that computing \( X_i = H^i(x) \) requires at least \( i \) rounds of computation even in the parallel quantum random oracle model e.g., see \([BLZ21]\). Thus, the pebbling transition from \( P_i = \{1, 2, \ldots, i\} \) to \( P_{i+1} = \{\} \) must be disallowed by our quantum pebbling rules as the corresponding quantum state transition is impossible.

We address this last issue by adding another pebbling rule: if \( v \in \text{parents}(P_i \setminus P_{i-1}, G) \cup \text{parents}(P_{i-1} \setminus P_i, G) \), then \( v \in P_i \). Intuitively, the rule ensures that if the label \( X_v \) appeared in an input register to either compute or uncompute some other data-label then we cannot also uncompute \( X_v \) in this round, i.e., we must keep a pebble at node \( v \).

Finally, if we plan to use Grover’s algorithm in a preimage attack we need a quantum circuit \( C_{G,H} \) which computes \( f_{G,H} \) with no side effects, i.e., maps the basis state \(|x, y, z\rangle\) to the new state \(|x, y \oplus f_{G,H}(x), z\rangle\). In particular, this means that intermediate data values that are not part of the final output will need to be uncomputed. Thus, we add the requirement that the final quantum pebbling state is \( P_i = \text{sink}(G) \), i.e., each of the sink nodes of \( G \) have pebbles and all other pebbled have been removed. By contrast, in the classical black pebbling game there is no requirement that pebbles on other nodes must be removed.
In the sequential quantum pebbling game we add the requirement that we can only add or remove one pebble in each round. Formally, we require that \(|(P_i \cup P_{i-1}) \setminus (P_i \cap P_{i-1})| \leq 1\). We denote with \(P^{*}_{G,T}\) and \(P^{*}_{G,T}\) the set of all legal sequential and parallel quantum pebblings of \(G\) with target set \(T\), respectively. Note that we have \(P^{*}_{G,T} \subseteq P^{*}_{G,T}\) since any sequential quantum pebbling is still a legal parallel quantum pebbling. We will mostly be interested in the case where \(T = \text{sinks}(G)\) in which case we simply write \(P^{*}_{G}\) and \(P^{*}_{G}\), respectively.

We make several observations about the quantum pebbling game. First, any legal quantum pebbling of a DAG \(G\) is also a legal (classical) parallel black pebbling of \(G\) since we only added additional pebbling restrictions, i.e., \(P^{*}_{G} \subseteq P_{G}\) and \(P^{*}_{G}\) \(\subseteq P_{G}\). Thus, any lower bounds on the classical parallel pebbling cost of \(G\) will immediately carry over to the quantum setting. However, upper bounds will not necessarily carry over since classical pebbling attacks may not be legal in the quantum pebbling game. Second, we observe that the following sequential quantum pebbling strategy works for any DAG \(G = (V = [N], E)\). In the first \(N\) rounds, pebble all nodes in topological order without deleting any pebbles. In the next \(N - 1\) rounds remove pebbles from all nodes (excluding \(\text{sinks}(G)\)) in reverse topological order. More formally, assuming that \(1, \ldots, N\) is a topological order and that node \(N\) is the only sink node we have \(P_i = [i]\) for each \(i \leq N\) and \(P_{N+j} = [N] \setminus [N - j, N - 1]\) for each \(j \leq N - 1\). The pebbling requires \(N\) pebbles and finishes in \(t = 2N - 1\) rounds so the space-time cost is \(2N^2 - N\). We refer to the above sequential strategy as the naïve quantum pebbling for a graph \(G\).

**Quantum Pebbling Attack on the Line Graph.** We give a quantum pebbling attack on the line graph \(L_N\) with space-time cost \(O(N^{1+\epsilon})\) for any constant \(\epsilon > 0\). To provide intuition, we first describe a simpler quantum pebbling attack with space-time cost \(O(N\sqrt{N})\). The basic idea is to start by pebbling the first \(k = \sqrt{N}\) nodes and then remove pebbles from nodes \(\{1, \ldots, k - 1\}\) leaving only a pebble on node \(k\). Formally, we set \(P_i = [i]\) for each \(i \leq k\) and we set \(P_{k+j} = [k] \setminus [k - j, k - 1]\) for \(j \leq k - 1\). In the next \(2k - 1\) rounds, we will pebble nodes \([k + 1, 2k]\) and then remove pebbles from nodes \([k + 1, 2k - 1]\) leaving pebbles only on the nodes \(k\) and \(2k\). Repeating this process, we will reach a pebbling configuration with pebbles only on nodes \(\{jk : j \leq N/k\}\) after at most \(2N\) pebbling rounds. Finally, we can reverse the above process to remove pebbles on all nodes but \(N\), e.g., we can begin the final removal process by (re)pebbling all of the nodes \(N - 2k + 1, \ldots, N - k - 1\) and then removing the pebble on node \(N - k\). In Appendix B.1, we give an example of this quantum pebbling strategy on the line graph \(L_9\) with \(N = 9\) and \(k = \sqrt{N} = 3\) — see Figure 3. The above pebbling strategy requires at most \(4N\) pebbling rounds and uses at most \(N/k + k = 2\sqrt{N}\) pebbles for a total space-time cost of \(8N\sqrt{N}\).

At first glance, it seems like increasing the parameter \(k > \sqrt{N}\) would only increase the total pebbling cost. However, our second observation is that we can apply our pebbling strategy recursively to pebble each interval. For example, in the first phase we began by pebbling all of the nodes in the interval \([1, k]\) and then removing pebbles from nodes \([1, k - 1]\). Any quantum pebbling of the line graph \(L_k\) will accomplish the same outcome allowing us to apply our quantum pebbling attack recursively, e.g., if \(k = N^{2/3}\) then we could pebble node \(k\) and also remove pebbles from nodes \(1, \ldots, k - 1\) in time \(4k\) using space \(2\sqrt{k}\). Thus, a single level of recursion reduces the overall space usage to \(N/k + 2\sqrt{k}\) whilst also doubling the number of pebbling rounds to \(8N\) — setting \(k = N^{2/3}\) the total space time cost is reduced to \(O(N^{4/3})\). Additional levels of recursion allows us to further reduce the quantum pebbling cost to \(O(N^{1+\epsilon})\) for any constant \(\epsilon\). In particular, if we let \(T(m, \ell)\) be the minimum amount of time to pebble node \(m\) and remove pebbles from \(1, \ldots, m - 1\)
using only space $\ell$ we can inductively argue that $T(k^c, ck) \leq 2^{c-1}k^c$ for any integer $c \geq 2$. Thus, for $N = k^\ell$ the total space-time cost is at most $2^{c-1}ck^{c+1} = 2^{c-1}cN^{1+1/c}$ where the parameter $c \geq 2$ can be tuned to minimize the costs.

**Generic Quantum Pebbling Attack on Depth-Reducible Graphs.** We give a generic quantum pebbling attack on any $(e, d)$-reducible DAG $G = (V = [N], E)$ with maximum indegree 2. The space-time cost of our quantum pebbling attack is at most $O\left(Ne + Nd^2\right)$. Thus, the attack will be superior to the naïve quantum pebbling strategy as long as $e = o(N)$ and $d^2 = o(N)$. We begin with a depth-reducing set $S \subseteq V$ of size $|S| \leq e$. Our quantum pebbling strategy will never remove pebbles from the set $S$ until after all of the sink nodes in $G$ are pebbled and we are ready to remove pebbles from the remaining nodes. On each round $i \leq N$ we will place a new pebble on node $\{i\}$. To ensure that this step is legal, we consider subgraph formed by all of node $i$’s ancestors in $G - S$. Since $G - S$ does not contain a directed path of length $d$ and each node has at most 2 parents there are at most $2^d$ ancestors of node $i$ in $G - S$. Once again applying the observation that the depth of $G - S$ is at most $d$ we can start to pebble $i$’s ancestors in round $i - d - 1$ to ensure that $i$’s immediate parents are pebbled by round $i - 1$. After we place a pebble on node $i$ we can remove pebbles from $i$’s ancestors in $G - S$ over the next $d$ rounds. Since we only keep pebbles on the set $S$ and the ancestors of up to $2d$ nodes in $G - S$, the maximum space usage of this quantum pebbling strategy will be $O\left(e + d2^d\right)$.

We apply the generic attack to Argon2i-A and Argon2i-B. In particular, we apply ideas from [AB17, BZ17] to show that Argon2i-A (resp. Argon2i-B) graphs are $(e, d)$-reducible with $e = O\left(N \log \log N/\sqrt{\log N}\right)$ and $d = \log N/\log \log N$ (resp. $e = O\left(N/\sqrt{\log N}\right)$ and $d = (\log N)/2$). This leads to quantum pebbling attacks with cost $O\left(N^2 \log \log N/\sqrt{\log N}\right)$ and $O\left(N^2/\sqrt{\log N}\right)$ for Argon2i-A and Argon2i-B, respectively. An intriguing open question is whether or not these are the best quantum pebbling attacks for Argon2i-A and Argon2i-B?

**Quantum Pebbling Attack on DRSample.** We provide a general quantum pebbling attack on any DAG $G$ with the property that $G$ contains few skip nodes (defined below). Intuitively, given a DAG $G = (V, E)$ with $|V| = N$ and a parameter $b \geq 1$, we can imagine partitioning the nodes of $V$ into consecutive blocks $B_1 = \{v_1, \ldots, v_b\}, B_2 = \{v_{b+1}, \ldots, v_{2b}\}, \ldots, B_{\lceil N/b\rceil} = \{v_{\lfloor N/b\rfloor - 1}, v_{\lfloor N/b\rfloor}, \ldots, v_N\}$ such that we have $\lceil N/b \rceil$ blocks in total and each block contains exactly $b$ nodes (with the possible exception of the last block if $N/b$ is not an integer). We call a node $u$ in block $B_i$ a skip node if $G$ contains a directed edge $(u, v)$ from $u$ to some node $v \in B_j$ with $j > i + 1$ and we call the edge $(u, v)$ a skip edge, i.e., the edge $(u, v)$ skips over the block $B_{i+1}$ entirely.

We first observe that if the graph $G$ contained no skip edges then it would be trivial to transform a (parallel) quantum pebbling $P'$ of the line graph $L_{\lceil N/b \rceil} = (V', E')$ with space-time cost $\Pi_{st}^\dagger(P')$ into a (parallel) quantum pebbling $P$ of $G$ with space-time cost $O\left(b^2\Pi_{st}^\dagger(P')\right)$. In particular, placing a pebbling on node $v' \in V'$ of the line graph corresponds to $b$ rounds in which we pebble all nodes in block $B_{i'}$. Thus, the pebbling time increases by a factor of $O\left(b\right)$ and the total space usage also increases by a factor $b$. Unfortunately, this strategy may result in an illegal quantum pebbling when $G$ contains skip edges. However, we can modify the above strategy to avoid removing pebbles on skip nodes which intuitively increases our space usage by $s$ — the total number of skip nodes in the graph $G$. The procedure $P = \text{Trans}(G, P', b)$ is formally described in Algorithm 1 in Appendix D, and an example for the quantum pebbling strategy can be found in in Figure 6 in Appendix B. As long as $s$ is sufficiently small, we obtain an efficient parallel quantum pebbling attack on $G$. In particular, given a quantum pebbling $P'$ of the line graph $L_{\lceil N/b \rceil} = (V', E')$ with space-time
cost $Π^{\ast \parallel}_{st}(P')$ we can find a quantum pebbling $P$ of $G$ with space-time cost $O\left(sN + b^2Π^{\ast \parallel}_{st}(P')\right)$. Combining this observation with our efficient quantum pebbling attacks on the line graph we can see that the space-time costs will be at most $O\left(sN + b^2(N/b)^{1+\epsilon}\right)$ for any constant $\epsilon > 0$. For graphs like DRSample [ABH17], we can show that (whp) the number of skip nodes is at most $s = O\left(\frac{N \log \log N}{\log N}\right)$ when we set the block size $b = O\left(\frac{N}{\log^2 N}\right)$ leading to a quantum pebbling attack with space-time cost $O\left(\frac{N^2 \log \log N}{\log N}\right)$.

### 2 Quantum Pebbling Games

The biggest difference between the classical and quantum pebbling games occurs when removing pebbles from a pebbling configuration. In a classical setting, we can always delete any pebbles in any point in time when they are no longer needed. On the other hand, in a quantum setting, this is not feasible by no-cloning theorem. Since we can only free a pebble by querying a random oracle at the same input, we can observe that a pebble can be deleted only if we know all of its parents, i.e., all of its parents were previously pebbled. The following definition captures this property:

**Definition 1 (Quantum Parallel/Sequential Graph Pebbling).** Let $G = (V,E)$ be a DAG and let $T \subseteq V$ be a target set of nodes to be pebbled. A quantum pebbling configuration (of $G$) at round $i$ is a subset $P_i \subseteq V$ and a legal parallel quantum pebbling of $T$ is a sequence $P = (P_0, \ldots, P_t)$ of quantum pebbling configurations of $G$ where $P_0 = \emptyset$ and which satisfies conditions (1), (2), (3), and (4) below. A sequential quantum pebbling additionally needs condition (5).

1. The pebbling should end with pebbles only on the target nodes, i.e., $P_t = T$.
2. A pebble can be added only if all of its parents were pebbled at the end of the previous pebbling round, i.e., $\forall i \in \{t\} : x \in (P_i \setminus P_{i-1}) \Rightarrow \text{parents}(x, G) \subseteq P_{i-1}$.
3. (Quantum No-Deletion Property) A pebble can be deleted only if all of its parents were pebbled at the end of the previous pebbling round, i.e., $\forall i \in \{t\} : x \in (P_{i-1} \setminus P_i) \Rightarrow \text{parents}(x, G) \subseteq P_{i-1}$.
4. (Quantum Reversibility) If a pebble was required to generate new pebbles (or remove pebbles), then we must keep the corresponding pebble around, i.e., $\forall i \in \{t\} : x \in \text{parents}(P_i \setminus P_{i-1}, G) \cup \text{parents}(P_{i-1} \setminus P_i, G) \Rightarrow x \in P_i$.
5. (Sequential pebbling only) At most one pebble is added or removed in each round, i.e., $\forall i \in \{t\} : |(P_i \cup P_{i-1}) \setminus (P_i \cap P_{i-1})| \leq 1$.

We denote with $P^\parallel_{G,T}$ and $P^{\ast \parallel}_{G,T}$ the set of all legal sequential and parallel quantum pebblings of $G$ with target set $T$, respectively. Note that we have $P^{\ast \parallel}_{G,T} \subseteq P^{\ast \parallel}_{G,T}$. We will mostly be interested in the case where $T = \text{sinks}(G)$ in which case we simply write $P^\parallel_G$ and $P^{\ast \parallel}_G$, respectively.

We further define a relaxed version of a (parallel/sequential) quantum pebbling, where we allow extra pebbles in addition to the pebbles on the target nodes at termination of pebbling steps, i.e., we let $P = (P_0, \ldots, P_t)$ be a legal relaxed (parallel/sequential) quantum pebbling if the condition (1) above is relaxed to (1') such as:

1'. The pebbling should end with pebbles on the target nodes, i.e., $P_t \supseteq T$.

We denote with $\overline{P}^\parallel_{G,T}$ and $\overline{P}^{\ast \parallel}_{G,T}$ the set of all legal relaxed sequential and parallel quantum pebblings of $G$ with target set $T$, respectively. Similarly, in case where $T = \text{sinks}(G)$, we simply write $\overline{P}^\parallel_G$ and $\overline{P}^{\ast \parallel}_G$, respectively.
Remark 1. The reason we require condition (1), i.e., the pebbling must end with pebbles exactly on the target nodes, is as follows. Often times, graphs are used to encode certain functions so that the sink nodes correspond to the output labels. Suppose that we implement it for Grover’s search, in which case we would want to wipe out memory, i.e., given a function $f$, it requires a quantum circuit computing $f$. It should produce the output with no side effects. In particular, a quantum circuit that computes $f$ should send $|x, y, A⟩$ to $|x, f(x) \oplus y, A⟩$ such that an input register $x$ and an auxiliary state $A$ are untouched. A quantum circuit that computes $f$ with side-effects e.g., maps $|x, y, A⟩$ to $|x, f(x) \oplus y, A'⟩$ would not be compatible with Grover’s algorithm. We also require condition (4) to make sure that it is not possible to remove multiple pebbles at once. For example, if $P_i = \{1, \ldots, i\}$ in a line graph and if we do not have the condition (4), then $P_{i+1} = \{\}$ does not violate the quantum pebbling rules, however, it violates the reversibility of quantum circuits because we cannot place pebbles on $\{1, \ldots, i\}$ at once from the empty set. Furthermore, the condition (4) also prevents the situation that a pebble is used as an input and an output register simultaneously, e.g., $P_i = \{1, 2, 3\}$ and $P_{i+1} = \{1, 2, 4\}$ in a line graph do not violate the quantum pebbling rules if we do not have the condition (4), which is indeed illegal since no shallow quantum circuit can delete a pebble from node 3 and place a pebble on node 4 at the same time.

As we discussed in Remark 1, relaxed quantum pebbling is not suitable for Grover’s search. Thus, we remark that except for the specific circumstances we will consider the regular quantum (parallel/sequential) pebblings. We also remark that if we do not specify whether the (classical/quantum) pebblings are parallel or sequential, then we always refer to parallel (classical/quantum) pebblings.

Definition 2 (Quantum Pebbling Complexity). Given a DAG $G = (V, E)$, we essentially use the same definitions for the quantum pebbling complexity as defined in the previous literature [AS15, ABP17, ABP18]. That is, the standard notion of time, space, space-time and cumulative pebbling complexity (cc) of a quantum pebbling $P = \{P_0, \ldots, P_t\} \in \mathcal{P}_G^{\star\parallel}$ are also defined to be:

- (time complexity) $\Pi^\star_{t,\parallel}(P) = t$,
- (space complexity) $\Pi^\star_{s,\parallel}(P) = \max_{i \in [t]} |P_i|$,  
- (space-time complexity) $\Pi^\star_{st,\parallel}(P) = \Pi^\star_{t,\parallel}(P) \cdot \Pi^\star_{s,\parallel}(P)$, and
- (cumulative pebbling complexity) $\Pi^\star_{cc,\parallel}(P) = \sum_{i \in [t]} |P_i|$.

For $\alpha \in \{s, t, st, cc\}$ and a target set $T \subseteq V$, the parallel quantum pebbling complexities of $G$ are defined as

$$\Pi^\star_{\alpha,\parallel}(G, T) = \min_{P \in \mathcal{P}_G^{\star\parallel}} \Pi^\star_{\alpha,\parallel}(P).$$

When $T = \text{sinks}(G)$ we simplify notation and write $\Pi^\star_{\alpha,\parallel}(G)$.

We define the time, space, space-time and cumulative pebbling complexity of a sequential quantum pebbling $P = \{P_0, \ldots, P_t\} \in \mathcal{P}_G^\star$ in a similar manner: $\Pi^\star_t(P) = t, \Pi^\star_s(P) = \max_{i \in [t]} |P_i|$, $\Pi^\star_{st}(P) = \Pi^\star_t(P) \cdot \Pi^\star_s(P)$, and $\Pi^\star_{cc}(P) = \sum_{i \in [t]} |P_i|$. Similarly, for $\alpha \in \{s, t, st, cc\}$ and a target set $T \subseteq V$, the sequential quantum pebbling complexities of $G$ are defined as $\Pi^\star_{\alpha}(G, T) = \min_{P \in \mathcal{P}_G^\star} \Pi^\star_{\alpha}(P)$. When $T = \text{sinks}(G)$ we simplify notation as well and write $\Pi^\star_{\alpha}(G)$.

When compared to the definition of a classical pebbling, we can observe that a quantum pebbling has more restrictions, i.e., it only allows us to have pebbles exactly on the target nodes at the
end of the pebbling steps, and it further requires quantum no-deletion property and quantum reversibility. This implies that any legal quantum pebblings are also legal classical pebblings, i.e., $P_G,T \subseteq P^*_G(T)$ (resp. $P_G,T \subseteq P^*_G(T)$). This implies that for any graph $G$, target set $T$ and cost metric $\alpha \in \{s, t, st, cc\}$, we have $\Pi^\parallel_{\alpha}(G, T) \leq \Pi^\parallel_{s}(G, T)$ (resp. $\Pi^\parallel_{\alpha}(G, T) \leq \Pi^\parallel_{\alpha}(G, T)$) for a DAG $G = (V, E)$ and a target set $T \subseteq V$, where $\Pi^\parallel_{\alpha}(G, T)$ (resp. $\Pi^\parallel_{\alpha}(G, T)$) denotes the parallel (resp. sequential) classical pebbling complexities which are defined essentially the same as in Definition 2 with a classical pebbling $P = \{P_0, \ldots, P_t\} \in P^\parallel_G$ (resp. $P_G$). This means that any lower bound on the classical pebbling complexity of a graph $G$ immediately carries over to the quantum setting and and upper bound (attack) on the quantum pebbling cost immediately carries over to the setting classical pebbling.

In the context of quantum preimage attacks, parallel space-time costs are arguably the most relevant metric. In particular, the depth of the full Grover circuit scales with the number of queries to our quantum circuit $C_{G,H}$ for $f_{G,H}$ multiplied by the number of pebbling rounds for $G$. Similarly, the width of the full Grover circuit will essentially be given by the space usage of our pebbling. Thus, the space-time of Grover’s algorithm will scale directly with $\Pi^\parallel_{s}(P)$. The cumulative pebbling complexity would only be relevant in settings where we are running multiple instances of Grover’s algorithm in parallel and can amortize space usage over multiple inputs. Thus, while cumulative pebbling complexity is still interesting to study, we primarily focus on space-time costs in this work.

3 Quantum Pebbling Attacks and Applications on iMHFs

3.1 Warmup: Quantum Pebbling Attack on a Line Graph

We first consider two widely deployed hash functions, PBKDF2 \cite{Kal00} and BCRYPT \cite{PM99}, as motivating examples for analyzing a line graph. Basically, they are constructed by hash iterations so can be modeled as a line graph when simplified. Hence, pebbling analysis of a line graph tells us about the costs of PBKDF2 and BCRYPT. Although there there has been some effort to replace such password-hash functions with memory-hard functions such as Argon2 or SCRYPT \cite{BHZ18}, PBKDF2 and BCRYPT are still commonly used by a number of organizations. Thus, it is still important to understand the costs of an offline brute-force attack on passwords protected by functions like PBKDF2 and BCRYPT. In fact, NIST recommends to use memory-hard functions for password hashing \cite{GNP+17} but they still allow PBKDF2 and BCRYPT when used with long enough hash iterations. Hence, there is still value to analyze the quantum resistance of these functions. Our quantum pebbling attack on DRSample relies on efficient pebbling strategies for line graphs as a subroutine providing further motivation to understand the quantum pebbling costs of a line graph.

**Quantum Pebbling Strategy.** We define $L_N$ to denote a line graph with $N$ nodes, i.e., $L_N := (V = [N], E)$ where $E = \{(i, i + 1) : i \in [N - 1]\}$. We can easily observe that $\Pi^\parallel_{s}(L_N) = \mathcal{O}(N^2)$ since a naïve quantum pebbling strategy will have $\Pi^\parallel_{s}(L_N) = 2N$ and $\Pi^\parallel_{s}(L_N) = N$ when we pebble all the nodes from 1 to $N$ and delete pebbles from $N - 1$ to 1, which is possible since parents$(i, L_N) = \{i - 1\}$ was pebbled in previous rounds for all $i = 2, \ldots, N - 1$. We can reduce the space-time cost to $\mathcal{O}(N\sqrt{N})$ by the following strategy:
Appendix B.1

One could.

Figure 3 can be found in Theorem 1. We give an example of a quantum pebbling on a line graph with depth-reducing sets with $d(S)$ there exists a subset $S \subseteq V$ into $\Pi$. Theorem 1. We generalize this observation and reduce the space-time cost to $O(N \sqrt{N})$. Since the time cost is bounded by $\Pi_{st}^\star (P) \leq 2N$, we observe that $\Pi_{st}^\star (P) \leq 2N (k + \lceil N/k \rceil)$. Since $k + \lceil N/k \rceil$ becomes minimum when $k = \lceil N/k \rceil$ by the equality condition of the AM-GM inequality, we choose $k \approx \sqrt{N}$ and we can conclude that $\Pi_{st}^\star (P) = O(N \sqrt{N})$.

In Appendix B.1, we give an example of a quantum pebbling on a line graph $L_n$. As we discussed above, we divide the graph into chunks of size $k = \sqrt{N} = 3$ as shown in Figure 3. One could generalize this observation and reduce the space-time cost to $O(N^{1+\epsilon})$ for any constant $\epsilon > 0$ by applying this strategy recursively, i.e., we can inductively upper bound the space-time parallel quantum pebbling cost for a line graph $L_{nc} = (V, E)$ with $|V| = k^c$ for an integer $c > 1$ by dividing $V$ into $k$ chunks of size $k^{c-1}$, where we already know the upper bound of the space-time cost of a line graph of size $k^{c-1}$ by the inductive hypothesis. The full proof of Theorem 1 can be found in Appendix C.

Theorem 1. $\Pi_{st}^\star (L_n) = O(N^{1+\epsilon})$ for any constant $\epsilon > 0$.

3.2 Quantum Pebbling Attacks on $(e, d)$-reducible DAGs

In this section, we introduce another type of quantum pebbling attack on $(e, d)$-reducible DAGs with depth-reducing sets with $d$ very small. Recall that a DAG $G = (V, E)$ is $(e, d)$-reducible if there exists a subset $S \subseteq V$ with $|S| \leq e$ such that the subgraph $G - S$ does not contain a path of length $d$. Here, we call such subset $S$ a depth-reducing set. In this paper, we only consider DAGs with constant indegree, and especially the current state-of-the-art constructions of iMHFs have indegree 2. Therefore, we will assume that $\text{indeg}(G) = 2$ for the DAGs that we consider.

Since the graph has indegree 2, if we find a depth-reducing set $S$ such that $G - S$ has depth $d$, then we observe that $|\text{ancestors}(v, G - S)| \leq 2^d$ for any node $v$ in $G - S$. If $d$ is small, i.e., $d \ll \log N$, then $2^d \ll N$ and we can expect that the space-time cost for pebbling such $(e, d)$-reducible DAG becomes $o(N^2)$. More precisely, we start with giving a regular pebbling strategy (without quantum restrictions) for such DAGs.
Classical Black Pebbling Strategy. We begin by giving a classical pebbling strategy with small space-time complexity. Note that prior pebbling strategies focused exclusively on minimizing cumulative pebbling cost, but the pebbling attacks of Alwen and Blocki [AB16] for $(e, d)$-reducible graphs still have the space-time cost $\omega(N^2)$. \footnote{When it comes to cumulative pebbling cost, if $G$ is $(e, d)$-reducible then Alwen and Blocki [AB16] showed that $cc(G) \leq \min_{g \geq d} \left( eN + gN \cdot \text{deg}(G) + \frac{N^2}{g} \right) = o(N^2)$.}

We first introduce the following helpful notation. For nodes $x$ and $y$ in a DAG $G = (V, E)$, let $\text{LongestPath}_G(x, y)$ denote the number of nodes in the longest path from $x$ to $y$ in $G$. Then for a node $w \in V$, a depth-reducing set $S \subseteq V$, and a positive integer $i \in \mathbb{Z}_{>0}$, we first define a set $A_{w,S,i}$ which consists of the nodes $v$ where the longest directed path from $v$ to $w$ in $G - S_{\leq w-1}$ has length $i$, i.e., it contains exactly $i$ nodes.

$$A_{w,S,i} := \left\{ v : \text{LongestPath}_{G-S_{\leq w-1}}(v, w) = i \right\}.$$  

It is trivial by definition that for any $v \in V$, $A_{v,S,1} = \{v\}$.

Let $G = (V = [N], E)$ be an $(e, d)$-reducible DAG. We observe that $\text{depth}(G_{\leq k} - S_{\leq k}) \leq d$ is still true for any $k \leq N$. At round $k$, we have always ensure that we have pebbles on the set $S_{\leq k}$ and on $\{k\}$ itself. Further, at round $k$, we can look $d$ steps into the future so that at round $k + d$ we can pebble node $k + d$ without delay. Hence, we start to repebble ancestors($k + d, G - S$) in this round and because $\text{depth}(G_{\leq k} - S_{\leq k}) \leq d$ we are guaranteed to finish within $d$ rounds — just in time to pebble node $k + d$. Taken together, in round $k$, we have pebbles on $\{k\}$, $S_{\leq k}$, and ancestors($k + i, G - S$) for all $i \leq d$. More precisely, for $v \in V$, let $P_v = S_{\leq v} \cup \bigcup_{j=1}^{d} \bigcup_{i=j}^{d} A_{v-1+j,S,i}$. Since each ancestor graph has size at most $2^d$ and there are at most $d$ of them, we observe that the total number of pebbles in each round is at most $1 + |S_{\leq k}| + \sum_{i=1}^{d} |\text{ancestors}(k + i, G - S)| \leq 1 + e + d2^d$. Hence, we have that $\Pi_{st}^\parallel(G) \leq N(1 + e + d2^d)$.

Quantum Pebbling Strategy. While the above strategy works in the classical setting it will need to be tweaked to obtain a legal quantum pebbling. In particular, after node $k + d$ is pebbled we cannot immediately remove pebbles from all nodes in ancestors($k + d, G - S$) because this would violate our quantum reversibility property. Instead, we can reverse the process and unpebble nodes in ancestors($k + d, G - S$) over the next $G - S$ rounds — with the possible exception of nodes $v \in \text{ancestors}(k + d, G - S)$ which are part of ancestors($k + d + j, G - S$) and are still required for some future node $k + d + j$. Thus, if a DAG $G$ is $(e, d)$-reducible we can establish the following result.

**Theorem 2.** Let $G = (V = [N], E)$ be an $(e, d)$-reducible DAG. Then $\Pi_{st}^\parallel(G) = O(Ne + Nd2^d)$.

We will give the proof of Theorem 2 later in the subsection. To prove Theorem 2, we first would need to give a legal quantum pebbling for an $(e, d)$-reducible DAG $G$. Lemma 1 provides the desired quantum pebbling for $G$.

**Lemma 1.** Let $G = (V = [N], E)$ be an $(e, d)$-reducible DAG and let $S \subseteq V$ be a depth-reducing set. Define

$$B_v := \bigcup_{j=1}^{d+1} \bigcup_{i=j}^{d+1} (A_{v+1-j,S,i} \cup A_{v-1+j,S,i}),$$

where $\text{LongestPath}_{G}(v, w) = i$. When it comes to cumulative pebbling cost, if $G$ is $(e, d)$-reducible then Alwen and Blocki [AB16] showed that $cc(G) \leq \min_{g \geq d} \left( eN + gN \cdot \text{deg}(G) + \frac{N^2}{g} \right) = o(N^2)$. \footnote{When it comes to cumulative pebbling cost, if $G$ is $(e, d)$-reducible then Alwen and Blocki [AB16] showed that $cc(G) \leq \min_{g \geq d} \left( eN + gN \cdot \text{deg}(G) + \frac{N^2}{g} \right) = o(N^2)$.}
for \( v \in V \). Then \( P = (P_0, P_1, \ldots, P_{2N}) \), where each pebbling configuration is defined by

- \( P_0 = \emptyset \),
- for \( v \in [N] \), \( P_v := S_v \cup B_v \), and
- for \( N < v \leq 2N \), \( P_v := P_{2N-v} \cup \{N\} \),

is a legal parallel quantum pebbling for \( G \).

Before proving Lemma 1, we observe the following key claim. The proof of Claim 1 can be found in Appendix C.

**Claim 1.** For \( v \in [N] \), \( \text{parents}(P_v \setminus P_{v-1}, G) \cup \text{parents}(P_{v-1} \setminus P_v, G) \subseteq P_{v-1} \cap P_v \).

**Proof of Lemma 1:** We want to show that it satisfies conditions in Definition 1.

**Condition (1):** \( P_{2N} = \{N\} \).
- It is clear that \( P_{2N} = P_0 \cup \{N\} = \{N\} \) which is the only target node of the pebbling game.

**Condition (2):** \( \forall v \in [2N] : x \in (P_v \setminus P_{v-1}) \Rightarrow \text{parents}(x, G) \subseteq P_{v-1} \).
- If \( v \in [N] \), by Claim 1, we have \( \text{parents}(P_v \setminus P_{v-1}) \subseteq P_{v-1} \cap P_v \subseteq P_{v-1} \).
- If \( N < v \leq 2N \), we have \( P_v \setminus P_{v-1} = (P_{2N-v} \cup \{N\}) \setminus (P_{2N-v+1} \cup \{N\}) = P_{2N-v} \setminus P_{2N-v+1} \).
  Let \( w = 2N - v + 1 \), then we have that \( w \in [N] \) and \( P_v \setminus P_{v-1} = P_w \setminus P_{w-1} \). Now we want to show that \( \text{parents}(P_{w-1} \setminus P_w, G) \subseteq P_{v-1} \cap P_v \), which also holds by Claim 1.

**Condition (3):** \( \forall v \in [2N] : x \in (P_{v-1} \setminus P_v) \Rightarrow \text{parents}(x, G) \subseteq P_{v-1} \).
- If \( v \in [N] \), by Claim 1, we have \( \text{parents}(P_{v-1} \setminus P_v) \subseteq P_{v-1} \cap P_v \subseteq P_{v-1} \).
- If \( N < v \leq 2N \), we have \( P_{v-1} \setminus P_v = (P_{2N-v} \cup \{N\}) \setminus (P_{2N-v+1} \cup \{N\}) = P_{2N-v} \setminus P_{2N-v+1} \).
  Then similarly, letting \( w = 2N - v + 1 \), we have that \( w \in [N] \) and \( P_{v-1} \setminus P_v = P_w \setminus P_{w-1} \). Now we want to show that \( \text{parents}(P_w \setminus P_{w-1}, G) \subseteq P_{v-1} = P_w \cup \{N\} \), which also holds by Claim 1.

**Condition (4):** \( \forall v \in [2N] : x \in \text{parents}(P_v \setminus P_{v-1}, G) \cup \text{parents}(P_{v-1} \setminus P_v, G) \Rightarrow x \in P_v \).
- If \( v \in [N] \), this is clear from Claim 1 since \( \text{parents}(P_v \setminus P_{v-1}, G) \cup \text{parents}(P_{v-1} \setminus P_v, G) \subseteq P_{v-1} \cap P_v \subseteq P_v \).
- If \( N < v \leq 2N \), by similar argument from above, by letting \( w = 2N - v + 1 \), we have that \( w \in [N] \) and \( \text{parents}(P_v \setminus P_{v-1}, G) \cup \text{parents}(P_{v-1} \setminus P_v, G) = \text{parents}(P_w \setminus P_{w-1}, G) \subseteq P_{w-1} \cap P_w \subseteq P_{w-1} \subseteq P_{w-1} \cup \{N\} = P_w \).

Taken together, we can conclude that for any \( v \in [2N] \), \( P_v \) is a legal quantum pebbling configuration for \( G \). \qed

Now we are ready to prove Theorem 2.

**Proof of Theorem 2:** Let \( P = (P_0, P_1, \ldots, P_{2N}) \) as defined in Lemma 1, in which we showed that it is a legal quantum pebbling. Clearly, \( \Pi_t^s \| (P) = 2N \). Further, we observe that \( \Pi_s^* \| (P) \leq \frac{2N}{s} \). Therefore, the time complexity of \( \Pi_t^s \| (P) \) is \( 2N \) and the space complexity of \( \Pi_s^* \| (P) \) is \( \frac{2N}{s} \).
max_{v \in V} \{|S_{\leq v}| + |B_v| + 1\}. Since we assume that \text{indeg}(G) = 2, we have
\[ |B_v| = \left| \bigcup_{j=1}^{d+1} \bigcup_{i=j}^{d+1} (A_{v+1-j,S,i} \cup A_{v-1+j,S,i}) \right| \leq \sum_{j=1}^{d+1} \sum_{i=j}^{d+1} |A_{v+1-j,S,i}| + |A_{v-1+j,S,i}| \leq \sum_{j=1}^{d+1} \sum_{i=j}^{d+1} 2^{i+1} = 8d^2 + 2.

Taken together, \( \Pi_{st}^\bullet(P) = \Pi_{i}^\bullet(P) \Pi_{v}^\bullet(P) \leq 2N(e + 8d^2 + 3) = \mathcal{O}(Ne + Nd^2) \). Hence, we can conclude that \( \Pi_{st}^\bullet(G) = \min_{P \in \mathcal{P}_{G,\{N\}}} \Pi_{st}^\bullet(P) = \mathcal{O}(Ne + Nd^2). \quad \square \)

**Analysis of Argon2i.** There are a number of variants for the Argon2i graphs. We will focus on Argon2i-A [BCS16] and Argon2i-B\(^2\) [BDKJ16] here. Recall that Argon2i-A is a graph \( G = (V = [N], E) \), where \( E = \{(i, i+1) : i \in [N-1]\} \cup \{(r(i), i)\} \), where \( r(i) \) is a random value that is picked uniformly at random from \([i-2]\). Argon2i-B has the same structure, expect that \( r(i) \) is not picked uniformly at random but has a distribution as follows:
\[
\Pr [r(i) = j] = \Pr_{x \in [N]} \left( 1 - \frac{x^2}{N^2} \right) \in (j-1, j].
\]

**Lemma 2.** Let \( G_{\text{Arg-A}} = (V_A = [N], E_A) \) and \( G_{\text{Arg-B}} = (V_B = [N], E_B) \) be randomly sampled graphs according to the Argon2i-A and Argon2i-B edge distributions, respectively. Then with high probability, the following holds:

1. \( G_{\text{Arg-A}} \) is \((e_1, d_1)\)-reducible for \( e_1 = \frac{N}{d} + \frac{N \ln N}{\lambda} \) and \( d_1 = d' \lambda \), for any \( 0 < \lambda < N \) and \( 0 < d' < \frac{N}{\lambda} \).
2. \( G_{\text{Arg-B}} \) is \((e_2, d_2)\)-reducible for \( e_2 = \frac{N}{d} + \frac{N \ln N}{2\lambda} \) and \( d_2 = d' \lambda \), for any \( 0 < \lambda < N \) and \( 0 < d' < \frac{N}{\lambda} \).

Alwen and Blocki [AB16, AB17] established similar bounds to Lemma 2, but focused on parameter settings where the depth \( d \) is large. By contrast, we will need to pick a depth-reducing set with a smaller depth parameter \( d \ll \log N \) to minimize the \( d^2 \) cost term in our pebbling attack. The full proof of Lemma 2 can be found in Appendix C. Here, we only give a brief intuition of the proof. To reduce the depth of a graph, we follow the approach of [AB16, AB17] and divide \( N \) nodes into \( \lambda \) layers of size \( N/\lambda \) and then reduce the depth of each layer to \( d' \) so that the final depth becomes \( d = d' \lambda \). To do so, we delete all nodes with parents in the same layer, and then delete one out of \( d' \) nodes in each layer. And then we count the number of nodes to be deleted in both steps for each graph.

Applying the result from Lemma 2 to Theorem 2, we have the following space-time cost of quantum pebbling for Argon2i-A and Argon2i-B. Intuitively, we obtain Corollary 1 by setting \( \lambda = \sqrt{\log N} \) and \( d' = \lambda/\ln \lambda \approx 2\sqrt{\log N}/\log \log N \) (resp. \( \lambda = \sqrt[3]{\log^2 N} \) and \( d' = \sqrt[3]{\log N}/2 \)) in Lemma 2 for Argon2i-A (resp. Argon2i-B). The full proof of Corollary 1 is in Appendix C.

\(^2\) We will follow the naming convention following [AB17] throughout the paper.
Corollary 1. Let $G_{\text{Arg-A}} = (V_A = |N|, E_A)$ and $G_{\text{Arg-B}} = (V_B = |N|, E_B)$ be randomly sampled graphs according to the Argon2i-A and Argon2i-B edge distributions, respectively. Then with high probability, $\Pi_{st}^{\#}(G_{\text{Arg-A}}) = O\left(\frac{N^2 \log \log N}{\sqrt{\log N}}\right)$, and $\Pi_{st}^{\#}(G_{\text{Arg-B}}) = O\left(\frac{N^2}{\sqrt{\log N}}\right)$.

Remark 2. Our quantum pebbling attacks on Argon2i-A and Argon2i-B have space-time cost $\Pi_{st}^{\#}(G_{\text{Arg-A}}) = O\left(\frac{N^2 \log \log N}{\sqrt{\log N}}\right)$, and $\Pi_{st}^{\#}(G_{\text{Arg-B}}) = O\left(\frac{N^2}{\sqrt{\log N}}\right)$ respectively. In the classical setting it was known that $\Pi_{cc}^{\#}(G_{\text{Arg-A}}) = \tilde{O}(N^{1.708})$ and $\Pi_{cc}^{\#}(G_{\text{Arg-B}}) = \tilde{O}(N^{1.768})$ [ABP17, BZ17].

While these pebbling attacks achieve more impressive reductions in cost, we stress that the attacks are (1) non-quantum and (2) the space-time complexity of these classical pebbling attacks is still $\Omega(N^2)$ since there will be a few pebbling rounds with $\Omega(N)$ pebbles on the graph. We remark that since any quantum pebbling is a legal classical pebbling that it immediately follows that $\Pi_{st}^{\#}(G_{\text{Arg-A}}) = O\left(\frac{N^2 \log \log N}{\sqrt{\log N}}\right)$, and $\Pi_{st}^{\#}(G_{\text{Arg-B}}) = O\left(\frac{N^2}{\sqrt{\log N}}\right)$. The best known classical lower bounds for Argon2i-A and Argon2i-B are $\Pi_{cc}^{\#}(G_{\text{Arg-A}}) = \Omega(N^{1.66})$ and $\Pi_{cc}^{\#}(G_{\text{Arg-B}}) = \tilde{\Omega}(N^{1.75})$ which immediately implies that $\Pi_{st}^{\#}(G_{\text{Arg-A}}) = \Omega(N^{1.66})$, and $\Pi_{st}^{\#}(G_{\text{Arg-B}}) = \tilde{\Omega}(N^{1.75})$. Thus, there remains a gap between the best upper/lower bounds for $\Pi_{st}^{\#}(G_{\text{Arg-A}})$ and $\Pi_{st}^{\#}(G_{\text{Arg-B}})$.

3.3 Quantum Pebbling Attacks using an Induced Line Graph

In this section, we give another general strategy to pebble directed acyclic graphs by “reducing” the DAG $G$ to a line graph, as shown in Figure 1. Intuitively, given a DAG $G = (V, E)$ with $|V| = N$ and an integer parameter $b \geq 1$, we can partition $V$ into consecutive blocks $B_1, \ldots, B_{\lfloor N/b \rfloor}$ such that each block contains exactly $b$ nodes, while for the last block we can have less than $b$ nodes if $N/b$ is not an integer.

![Figure 1: A line graph $L_{\lfloor N/b \rfloor}$ induced from a DAG $G$. Note that each block in an original graph corresponds to a node in the corresponding line graph, e.g., a block $B_i$ in $G$ that consists of five nodes corresponds to the node $v'_i$ in $L_{\lfloor N/b \rfloor}$.](image)

Notation. Now we consider a quantum pebbling $P'$ of the line graph $L_{\lfloor N/b \rfloor} = (V' = \lfloor N/b \rfloor, E')$. Intuitively, each node in $L_{\lfloor N/b \rfloor}$ corresponds to each block in $G$. To transform $P'$ into a pebling $P$ of $G$, it will be useful to introduce some notation. Given a node $v' \in V'$ and the pebbling $P'$ of $L_{\lfloor N/b \rfloor}$, we define $\text{LastDelete}(P', v') := \max\{i : v' \in P'_i\}$ to denote the unique index $i$ such that node
in the block but not the entire block. Hence, we need additional pebbles from all nodes except for the last node in the block.

For the final time, i.e., in round \( r \), delete pebbles from the block for the last time, i.e., after round \( r \). To overcome this barrier, when we convert \( v' \in P' \), but \( v' \notin P'_j \) for all rounds \( j > i \), i.e., the pebble on node \( v' \) was removed for the final time in round \( i + 1 \). Similarly, it will be convenient to define LastAdd(\( P' \)) := \max \{ i : \lceil N/b \rceil \notin P'_{i-1} \} to be the unique round where a pebble was placed on the last node \( v = \lceil N/b \rceil \) for the final time (Note: it is possible that a legal pebbling \( P' \) places/removes a pebble on node \( v = \lceil N/b \rceil \) several times).

We make a couple of basic observations. First, we note that if \( u' < v' \) then \( \text{LastDelete}(P', u') > \text{LastDelete}(P', v') \) since we need node \( v' - 1 \) on the graph to remove a pebble from node \( v' \). Similarly, we note that for any node \( v' < \lceil N/b \rceil \) that \( \text{LastDelete}(P', v') > \text{LastAdd}(P') \) since we need node \( \lceil N/b \rceil - 1 \) to be pebbled before we can place a pebble on the final node. Given our graph \( G = (V, E) \), a parameter \( b \), and a partition \( B_1, \ldots, B_{\lceil N/b \rceil} \) of \( V \) into consecutive blocks of size \( b \), we define Skip(\( B_i, G \)), for each \( i \), to be the set of all skip nodes in block \( B_i \), i.e., the set of nodes with an outgoing edge that skips over block \( B_{i+1} \):

\[
\text{Skip}(B_i, G) := \{ v \in B_i : \exists j > i + 1 \text{ such that } v \in \text{parents}(B_j, G) \}.
\]

We further define NumSkip(\( G, b \)) as the total number of skip nodes in \( G = (V, E) \) after partitioning the set of nodes \( V \) into consecutive blocks of size \( b \), i.e.,

\[
\text{NumSkip}(G, b) := \sum_{i=1}^{\lceil N/b \rceil} |\text{Skip}(B_i, G)|,
\]

where \( B_i \)’s are defined as before.

**Pebbling Attempt 1.** Our first approach to convert \( P' \in P^*_{L_{\lceil N/b \rceil}} \) to a legal quantum pebbling \( P \) of \( G \) is as follows. Since each node in \( L_{\lceil N/b \rceil} \) corresponds to a block (of size at most \( b \)) in \( G \), we can transform placing a pebble on a node in \( L_{\lceil N/b \rceil} \) to pebbling all nodes in the corresponding block in \( G \) in at most \( b \) steps. Similarly, we can convert removing a pebble on a node in \( L_{\lceil N/b \rceil} \) to removing pebbles from all nodes in the corresponding block in \( G \) in at most \( b \) steps. It gives us \( \Pi_s^* \| (P) \leq b \Pi_s^* \| (P') \) since each node is transformed to a block of size at most \( b \), and \( \Pi_t^* \| (P) \leq b \Pi_t^* \| (P') \) since one pebbling/removing step in \( L_{\lceil N/b \rceil} \) is transformed to at most \( b \) pebbling/removing steps in \( G \).

However, this transformation does not yield a legal quantum pebbling of \( G \) due to the skip nodes. In particular, given a quantum pebbling configuration \( P_k' = \{ v' \} \) of \( L_{\lceil N/b \rceil} \), it is legal to proceed as \( P_{k+1}' = \{ v', v' + 1 \} \). However, when converting it to a quantum pebbling of \( G \), one would need to place pebbles on block \( B_{v'+1} \) while only having pebbles on block \( B_v \). This could be illegal if there is a node \( v \in V \) such that \( v \in B_i \) for \( i < v' \) and \( v \in \text{parents}(B_{v'+1}, G) \), i.e., \( v \) is a skip node in \( B_i \), because \( v \) must be previously pebbled to place pebbles on block \( B_{v'+1} \).

**Quantum Pebbling Strategy.** To overcome this barrier, when we convert \( P' \in P^*_{L_{\lceil N/b \rceil}} \) to a legal quantum pebbling \( P \) of \( G \), we define a transformation \( P = \text{Trans}(G, P', b) \) which convert placing/removing a pebble on/from a node \( v' \) in \( L_{\lceil N/b \rceil} \) to placing/removing pebbles on/from all nodes in the corresponding block \( B_{v'} \) in \( G \) in at most \( b \) steps as our first attempt, but when we remove pebbles from \( B_{v'} \) in \( G \), we keep skip nodes for the block in the transformation until we delete pebbles from the block for the last time, i.e., after round \( \text{LastDelete}(P', v') \), since these skip nodes will no longer needed to pebble nodes in other blocks in the future.

Furthermore, for the last block (in \( G \)), when a pebble is placed on the last node (in \( L_{\lceil N/b \rceil} \)) for the final time, i.e., in round \( \text{LastAdd}(P') \), we indeed want to only pebble the last node (sink node) in the block but not the entire block. Hence, we need additional (at most \( b - 1 \)) steps to remove pebbles from all nodes except for the last node in the block.
We can argue the legality of the converted pebbling of $G$ because pebbling steps in each block is legal and keeping skip nodes during the transformation does not affect the legality of pebbling. Intuitively, whenever we pebble a new node $v$ in $L_{\lfloor N/b \rfloor}$ the node $v-1$ must have been pebbled in the previous round. Thus, in $G$ we will have pebbled on all nodes in the block $B_{v-1}$. Now for every node $w \in B_v$ and every edge of the form $(u, w)$ we either have (1) $u \in B_{v-1}$, (2) $u \in B_v$ or (3) $u \in B_j$ with $j < v-1$. In the third case $u$ is a skip node and will already be pebbled allowing us to legally place a pebble on node $w$. Similarly, in the first case we are guaranteed that $u$ is already pebbled before we begin pebbling nodes in block $B_v$ since every node in $B_{v-1}$ is pebbled and in the second case $u$ will be (re)pebbled before node $w$. A similar argument shows that all deletions are legal as well. The full proof of Lemma 3 can be found in Appendix C.

Lemma 3. Let $G = (V = [N], E)$ and $b \in [N]$ be a parameter. If $P' \in \mathcal{P}_{L_{\lfloor N/b \rfloor}}^*$, then $P = \text{Trans}(G, P', b) \in \mathcal{P}_{G}^*$.

The entire procedure $\text{Trans}(G, P', b)$ is formally described in Algorithm 1 in Appendix D, and an example for the quantum pebbling strategy can be found in Figure 6 in Appendix B. Now we observe the following theorem describing the space-time cost of the converted pebbling in terms of the cost of the reduced pebbling of the line graph.

Theorem 3. Given a DAG $G = (V, E)$ with $|V| = N$ nodes, a reduced line graph $L_{\lfloor N/b \rfloor} = (V', E')$ with $|V'| = \lfloor N/b \rfloor$ nodes (where $b$ is a positive integer), and a legal quantum pebbling $P' \in \mathcal{P}_{L_{\lfloor N/b \rfloor}}^*$, there exists a legal quantum pebbling $P = \text{Trans}(G, P', b) \in \mathcal{P}_{G}^*$ such that

$$\Pi_{st}^* (P) \leq 2b^2 \Pi_{st}^* (P') + 2b \Pi_{t}^* (P') \cdot \text{NumSkip}(G, b).$$

Proof. Consider the algorithm $P = \text{Trans}(G, P', b)$ as shown in Algorithm 1 in Appendix D. We argue that the quantum pebbling $P$ is legal in Appendix C and focus here on analyzing the cost of the pebbling $P$. First, we consider the time cost of $P$. Notice that in each round $P_j'$ in $P'$ (of the line graph $L_{\lfloor N/b \rfloor}$), we have two cases: if $j \neq \text{LastAdd}(P', \lfloor N/b \rfloor)$, we need $b$ rounds to place/remove pebbles in the corresponding blocks in $G$; otherwise, i.e., $j = \text{LastAdd}(P', \lfloor N/b \rfloor)$, we need $b + N - (\lfloor N/b \rfloor - 1) - 1 \leq 2b$ rounds to place/remove pebbles in the corresponding blocks in $G$. Hence, we have

$$\Pi_{t}^* (P) \leq b \left( \Pi_{t}^* (P') - 1 \right) + 2b = b \left( \Pi_{t}^* (P') + 1 \right).$$

When it comes to the space cost of the pebbling $P$, we need space for the pebbling $P'$ multiplied by the block size since each node in $P'$ has a 1-1 correspondence between each block of size $b$ in $G$. Furthermore, we additionally needs space for the skip nodes as they should not be removed to make the pebbling $P = \text{Trans}(G, P', b)$ legal. That is, we have

$$\Pi_{s}^* (P) \leq b \cdot \Pi_{s}^* (P') + \text{NumSkip}(G, b).$$

Combining these inequalities together, we can conclude that

$$\Pi_{st}^* (P) = \Pi_{s}^* (P) \cdot \Pi_{t}^* (P) \leq \left( b \cdot \Pi_{s}^* (P') + \text{NumSkip}(G, b) \right) \cdot b \left( \Pi_{t}^* (P') + 1 \right) = 2b^2 \Pi_{st}^* (P') + 2b \Pi_{t}^* (P') \cdot \text{NumSkip}(G, b).$$
Analysis on DRSample. DRSample [ABH17] is the first practical construction of an iMHF which modified the edge distribution of Argon2i. Consider a DAG $G = (V = [N], E)$. Intuitively, similar to Argon2i, each node $v \in V \setminus \{1\}$ has at most two parents, i.e., there is a directed edge $(v - 1, v) \in E$ and a directed edge from a random predecessor $r(v)$. While Argon2i-A picks $r(v)$ uniformly at random from $[v - 2]$, DRSample picks $r(v)$ according to the following random process: (1) We randomly select a bucket index $i \leq \log v$, (2) We randomly sample $r(v)$ from the bucket $B_i(v) = \{ u : 2^{i-1} < v - u \leq 2^i \}$. We observe the following lemma which (whp) upper bounds the number of skip nodes when we sample $G$ according to this distribution.

**Lemma 4.** Let $G_{DRS} = (V_{DRS} = [N], E_{DRS})$ be a randomly sampled graph according to the DRSample edge distribution. Then with high probability, we have $\text{NumSkip}(G_{DRS, \left\lceil \frac{N}{\log^2 N} \right\rceil}) = \mathcal{O}\left(\frac{N \log \log N}{\log N}\right)$.

The full proof of Lemma 4 can be found in Appendix C. Here, we only give a brief intuition. To count the number of skip nodes, we need to find edges with length longer than $b$ so that the edge skips over a block. There are at most $\log v - \log b$ (out of $\log v$) buckets which potentially could result in a skip node i.e., any edge $(r(v), v)$ with length $v - r(v) \leq b$ cannot produce a new skip node. The probability that the edge $(r(v), v)$ is longer than $b$ is at most $1 - \log b/\log v \leq 1 - \log b/\log N = \log(N/b)/\log N$. Thus, the expected number of skip nodes in DRSample is at most $N \log(N/b)/\log N$ and standard concentration bounds imply that the number of skip nodes will be upper bounded by $\mathcal{O}(N \log(N/b)/\log N)$ with high probability. Setting $b = \lceil N/\log^2 N \rceil$ we can conclude that the expected number of skip nodes in DRSample is at most $\mathcal{O}(N \log \log N/\log N)$ with high probability. Applying the result from Lemma 4 to Theorem 3, we have the following space-time cost of quantum pebbling for DRSample.

**Corollary 2.** Let $G_{DRS} = (V_{DRS} = [N], E_{DRS})$ be a randomly sampled graph according to the DRSample edge distribution. Then with high probability, $\Pi_{str,\|}(G_{DRS}) = \mathcal{O}\left(\frac{N^2 \log \log N}{\log N}\right)$.

**Proof.** Given $G_{DRS}$, we can consider a reduced line graph $L_{[\log^2 N]} = (V', E')$ with $|V'| = \lceil \log^2 N \rceil$. Then by Theorem 3, we have

$$\Pi_{str,\|}(G_{DRS}) \leq 2 \left(\frac{N}{\log^2 N}\right)^2 \Pi_{str,\|}(L_{[\log^2 N]}) + \frac{2N}{\log^2 N} \cdot \Pi_{str,\|}(L_{[\log^2 N]}) \cdot \text{NumSkip}(G_{DRS, \left\lceil \frac{N}{\log^2 N} \right\rceil}).$$

By Theorem 1, we have $\Pi_{str,\|}(L_{[\log^2 N]}) = \mathcal{O}\left(\log^{2(1+\epsilon)} N\right)$ and $\Pi_{str,\|}(L_{[\log^2 N]}) = \mathcal{O}\left(\log^2 N\right)$ for any constant $\epsilon > 0$. By Lemma 4, we have that for any $0 < \epsilon < \frac{1}{2}$,

$$\Pi_{str,\|}(G_{DRS}) \leq 2 \left(\frac{N}{\log^2 N}\right)^2 \mathcal{O}\left(\log^{2(1+\epsilon)} N\right) + \frac{2N}{\log^2 N} \cdot \mathcal{O}(\log^2 N) \cdot \mathcal{O}\left(\frac{N \log \log N}{\log N}\right)$$

$$= \mathcal{O}\left(\frac{N^2 \log \log N}{\log^2 N} + \frac{N^2 \log \log N}{\log N}\right) = \mathcal{O}\left(\frac{N^2 \log \log N}{\log N}\right). \quad \square$$

4 Conclusion and Open Questions

We introduced a new (parallel) quantum pebbling game and applied it to analyze the quantum space-time complexity of a line graph, Argon2i-A, Argon2i-B, and DRSample. In particular, we
showed that the quantum space-time cost of pebbling a line graph of size $N$ is $O(N^{1+\epsilon})$ for any constant $\epsilon > 0$ using a recursive pebbling strategy. We also showed that there is a quantum pebbling strategy for an $(e, d)$-reducible indegree-2 DAG $G$ of size $N$ with the space-time cost $O((Ne + Nd^2)/e)$, which becomes meaningful whenever $e = o(N)$ and $d^2 = o(N)$. We applied this attack to Argon2i-A and Argon2i-B to yield quantum pebbling attacks with space-time cost $O(N^2 \log \log N / \sqrt{\log N})$ and $O(N^2 / \sqrt{\log N})$ for Argon2i-A and Argon2i-B, respectively. Finally, we introduced a general quantum pebbling attack on a DAG $G$ of size $N$ by reducing the graph to a line graph $L_{[N/b]}$, and given a legal quantum pebbling $P'$ of the line graph with space-time cost $H_{st}^*(P')$, we provided a legal quantum pebbling $P$ of $G$ with space-time cost $O\left(sN + b^2H_{st}^*(P')\right)$, where $s$ denotes the number of skip nodes in $G$. Tuning the parameter $b = O(N/\log^2 N)$ the skip number for DRSample is $O\left(\frac{N \log \log N}{\log N}\right)$ leading to a quantum pebbling attack with space-time cost $O\left(N^2 \log \log N / \log N\right)$.

We have several interesting open questions. [BHK+19] proposed a new iMHF candidate called DRS+BRG (DRSample plus Bit-Reversal Graph) by overlaying a bit-reversal graph [LT82, FLW14] on top of DRSample, which provides the best resistance to known classical pebbling attacks. It is still an open question for computing the quantum space-time complexity of DRS+BRG, i.e., whether the quantum space-time cost is below $O(N^2)$. We remark that there is no small depth-reducing set for DRS+BRG and the extra bit-reversal edges indicate that the number of skip nodes will be large as well, which imply that neither attack in our work would be applicable on DRS+BRG. It is also remained open if there is any DAG with constant indegree having quantum space-time cost $\Omega(N^2)$, or there are any attacks that rule out this possibility. We are also interested if there are more efficient quantum pebbling attacks for Argon2i-A/B, or there are stronger lower bounds for these DAGs. Finally, while the space-time cost is the most relevant metric for quantum preimage attacks, it is still worthwhile to study the quantum cumulative pebbling cost (cc) of iMHF candidates, i.e., Argon2i-A/B, DRSample, DRS+BRG, etc.

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A Empirical Analysis of Pebbling Attacks on a Line Graph

In this section, we present an empirical analysis of pebbling attacks on a line graph $L_N$. In particular, we implemented our recursive pebbling attack and ran it on line graphs $L_N$ with $N = 2^{10}, 2^{11}, \ldots, 2^{24}$. In each instantiation of an attack, we fixed different recursion parameter $k = 0, 1, \ldots, 10$ and measured the space-time cost of the quantum pebbling attack.

Figure 2 (left) shows the minimum ST-cost for each parameter choice of $N$ (the size of the graph) and $k$ (the level of recursion), along with the optimal value of $k$, i.e., one that produces the minimum ST-cost. We see that the optimal value of $k$ is $k = 3$ for $N = 2^{10}, \ldots, 2^{17}$, and $k = 4$ for $N = 2^{18}, \ldots, 2^{24}$. The plot measures the space-time cost, space cost, and time cost, and we can see that the space-time cost essentially scales linearly in log scale. Another observation is that as the optimal level of recursion jumps from $k = 3$ to $k = 4$ (i.e., when $N = 2^{18}$), the space cost decreases notably, however, the increased time cost compensates it so that the space-time cost still scales linearly in log scale.

Figure 2 (right) explores the impact of the recursion parameter on a quantum pebbling quality for the line graphs of various size. For readability, we only contained the result for $N = 2^{12}, 2^{16}, 2^{20}$, and $2^{24}$. As we discussed before, the optimal value for the level of recursion lies either on $k = 3$ or $k = 4$ depending on the size of the graph, i.e., too many level of recursion indeed increases the space-time cost.

B Quantum Pebbling Strategy Examples

B.1 Example for a Line Graph

Consider a line graph $G$ with $N = 9$ and we want to follow the quantum pebbling strategy above using chunks. By the observation above, we choose $k \approx \sqrt{N} = \sqrt{9}$ as an optimal parameter here. Hence, we divide the graph into chunks of size $k = 3$, which we illustrate this by drawing dashed lines as shown in Figure 3. Now we proceed as follows:
We start by placing a pebble from node 1; we have $P_1 = \{1\}$, $P_2 = \{1, 2\}$, and $P_3 = \{1, 2, 3\}$. Now we reach the end of the first chunk, so we simultaneously start placing a pebble on the next chunk (node 4) and deleting a pebble from the current chunk except for the last node of the chunk (start deleting node 2, which is possible since $\text{parents}(2, G) = \{1\}$ was already pebbled. See condition (3) of Definition 1).

- Note that we do not delete (indeed cannot delete) node 3 because we need to place a pebble on node 4, and at a later point in time we will want to delete a pebble from node 4 to save space cost, which is only possible when we have a pebble on node 3.

- We will repeat the same adding/deleting procedure when we reach node 6 and 9 which are the last node in the second and the third chunk.

- After we reach the target node 9, we would need to delete intermediate nodes with pebbles. To do so, we repeat the same strategy though work backwards instead. That is, we start deleting pebbles in the current chunk while start adding pebbles in the previous chunk at the same time. (In $P_{10}$ in Figure 3, we delete a pebble from node 8 and add a pebble on node 4 at the same time. The reason we start to add a pebble on node 4 is that we can only delete a pebble from node 6 when its parents were all previously pebbled, which is node 5 in this example.)

- Note that we do not need extra space cost for this deleting procedure because we only add pebbles while we delete the same number of pebbles in each round. We can observe that until the 9th round, the space cost is 5, and we do not need more than 5 pebbles until the last round where we only have a pebble on the target node 9.
Hence, in this example, the time cost $II^\star_t(G) = 17 \leq 2N$ and the space cost $II^\star_s = 5 \leq 2\sqrt{N}$. Therefore, we observe that $II^\star_t(G) = 17 \cdot 5 = 85 \leq 4N\sqrt{N} = O\left(N\sqrt{N}\right)$. We remark that the time cost could be cut by half when we do not require to clear all the pebbles on the intermediate nodes, though in either case the space-time complexity has the same order.

### B.2 Example on an $(e, d)$-Reducible Graph

In this example, we give a DAG $G = (\{V = [N], E\})$ with $N = 16$, and $E = \{(i, i + 1) : i \in [15]\} \cup \{((i - 1)4 + 1, (i - 1)4 + 3), ((i - 1)4 + 1, (i - 1)4 + 4), ((i - 1)4 + 1, (i - 1)4 + 5), ((i - 1)4 + 1, (i - 1)4 + 6) : i \in [3]\} \cup \{(13, 15), (13, 16)\}$, as shown in Figure 4. We observe that $G$ is $(4, 3)$-reducible.

Fig. 4: An $(e, d)$-reducible DAG $G$ of $N = 4N'$ nodes, with $e = N'$ and $d = 3$ (we set $N' = 4$ in the figure above). Note that with depth-reducing set $S = \{1, 5, 9, 13\}$, we have an original DAG $G$ (top) and the induced subgraph $G - S$ (bottom).

Recall that $P = (P_0, P_1, \ldots, P_{2N})$ such that $P_0 = \emptyset$, for $v \in [N], P_v := S_{\leq v} \cup B_v$, and for $N < v \leq 2N, P_v := P_{2N - v} \cup \{N\}$ is a legal quantum pebbling for $G$, as shown in Lemma 1, where $B_v := \bigcup_{j=1}^{d+1} \bigcup_{i=j}^{d+1} (A_{v+1 - j, S, i} \cup A_{v-1 + j, S, i})$, with the definition $A_{w, S, i} := \{v : \text{LongestPath}_{G - S_{w-1}}(v, w) = i\}$. For example, when we compute $P_8$ for the graph above, it is described as

$$P_8 = S_{\leq 8} \cup B_8$$

$$= \{1, 5\} \cup \bigcup_{j=1}^{4} \bigcup_{i=j}^{4} (A_{9-j, S, i} \cup A_{7+j, S, i})$$

$$= \{1, 5\} \cup (A_{8, S, 1} \cup A_{8, S, 2} \cup A_{8, S, 3} \cup A_{8, S, 4}) \cup (A_{7, S, 2} \cup A_{7, S, 3} \cup A_{7, S, 4} \cup A_{9, S, 2} \cup A_{9, S, 3} \cup A_{9, S, 4})$$

$$\quad \cup (A_{6, S, 3} \cup A_{6, S, 4} \cup A_{10, S, 3} \cup A_{10, S, 4}) \cup (A_{5, S, 4} \cup A_{11, S, 4})$$

$$= \{1, 5\} \cup \{6, 7\} \cup \{6, 7, 8\} \cup \{\} \cup \{2\}$$

$$= \{1, 2, 5, 6, 7, 8\}.$$

Then entire pebbling process is illustrated in Figure 5. Note that for our example, $II^\star_t(P) = 32 = 2N$ and $II^\star_s(P) = 9$, which leads to $II^\star_t(P) = 32 \cdot 9 = 288$. While this is not a significant improvement on the naive pebbling strategy for small $N = 16$, the space-time costs scale with $O(N)$ for the graphs defined above.
B.3 Example of a Quantum Pebbling Using an Induced Line Graph

In this example, we give a DAG $G = (V, E)$ with the edge distribution as illustrated in Figure 6. As we discussed in Section 3.3, we reduce our DAG $G$ to a line graph $L_6$ by choosing the block size $b = 3$. Given an efficient quantum pebbling $P'$ of $L_6$ as shown in Figure 7, we apply $\text{Trans}(G, P', b = 3)$ to produce a legal quantum pebbling of $G$. Note that we have $\text{LastAdd}(P', 6) = 6$, hence, in quantum pebbling rounds of $G$ that corresponds to $P'_6$, we pebble all nodes in $B_6$ and delete pebbles from the block in reverse topological order except for the last node as shown in
Algorithm 1 in Appendix D, which takes $b + N - (\lceil N/b \rceil - 1)b - 1 = 3 + 18 - (6 - 1)3 - 1 = 5$ steps to complete. We also note that pebbles colored in red are skip nodes, which will be kept until the corresponding block is deleted for the last time, i.e., we keep a skip node $v \in B_i$ until we reach rounds that correspond to $P'_j$ (of $L_6$) with $j = \text{LastDelete}(P', i)$.

Fig. 6: A parallel quantum pebbling $P = \{P_1, \ldots, P_{35}\}$ of a DAG $G$ using an induced line graph $L_6$. The pebbling for $L_G$, which is $P' = \{P'_1, \ldots, P'_{11}\}$, is shown in Figure 7, using the pebbling strategy from Section 3.1. Pebbles colored in red are skip pebbles that cannot be removed until we remove the block of pebbles for the last time, i.e., for each block $B_i$, we keep pebbles on the skip nodes until we reach $P'_j$ with $j = \text{LastDelete}(P', i)$.
Fig. 7: A quantum pebbling for a line graph with 6 nodes. Note that we mark a pebble on node $i$ in round $j$ with “D” if $j = \text{LastDelete}(P', i)$, and with “A” if $j = \text{LastAdd}(P', i)$.

C Missing Proofs

Reminder of Theorem 1. $\Pi_{st}^{\bullet,\|}(L_N) = O(N^{1+\epsilon})$ for any constant $\epsilon > 0$.

Proof of Theorem 1: Let $G' = (V', E')$ be a line graph with $V = [k^c]$ and $E' = \{(i, i+1) : i \in [k^c - 1]\}$. Then from Lemma 5 below, we have that $\Pi_{st}^{\bullet,\|}(G') \leq c2^{c-1}k^{c+1}$. Substituting $N = k^c$, we get the space-time cost $\Pi_{st}^{\bullet,\|}(G) \leq c2^{c-1}N^{1+1/c}$.

Since $c$ was arbitrary, for any constant $\epsilon = 1/c > 0$ we have $\Pi_{st}^{\bullet,\|}(G) \leq \frac{2^{1-1}}{\epsilon}N^{1+\epsilon} = O(N^{1+\epsilon})$. □

Lemma 5. Let $c > 1$ be an integer and $G = (V, E)$ be a line graph where $V = [k^c]$ and $E = \{(i, i+1) : i \in [k^c - 1]\}$ for some constant $k$. Then $\Pi_{st}^{\bullet,\|}(G) \leq c2^{c-1}k^{c+1}$.

Proof. Let $T(m, \ell)$ be the minimum amount of time to pebble node $m$ and remove pebbles from $1, \ldots, m-1$ using only space $\ell$. Now we claim that $T(k^c, ck) \leq 2^{c-1}k^c$ for any integer $c \geq 2$. We will use induction to prove this claim.

- If $c = 2$, then our goal is to pebble node $k^2$ and remove pebbles from $1, \ldots, k^2 - 1$ using only space $2k$. Using the same trick from before – that we divide into chunks of size $k$ and leave a pebble on every $k^{th}$ node – the space cost is $(k^2/k) + k = 2k$ and the time cost will be at most $k^2 + k^2 = 2k^2$, where the first $k^2$ denotes the time to move forward to place a pebble on node $k^2$, and the second $k^2$ denotes the time to work backwards to remove the pebbles from other nodes. Hence, we have that $T(k^2, 2k) \leq 2k^2$.

- Suppose that we have $T(k^c, ck) \leq 2^{c-1}k^c$ and we want to pebble node $k^{c+1}$ using the same strategy recursively, e.g., divide into chunks of size $k^c$ and leave a pebble on every $k^{th}$ node, and recursively do the same strategy for each chunk. This strategy will require additional space $k = k^{c+1}/k^c$ while we already have a strategy to pebble each chunk with space $ck$, so the total
space usage will be \(k + ck = (c+1)k\). The time cost will be \(2k \cdot T(k^c, ck)\) since we have \(k\) chunks and a factor of 2 due to move forward and backwards as illustrated in the base case. Hence, we have that \(T(k^{c+1}, (c+1)k) = 2k \cdot T(k^c, ck) \leq 2k \cdot 2^{c-1}k^c = 2^c k^{c+1}\), which proves the claim. Hence, with this recursive strategy we have that \(\Pi_\ast^t(G) = \Pi_\ast^d(G) \cdot \Pi_\ast^t(G) \leq (ck) \cdot (2^{c-1}k^c) = 2^{c-1}k^{c+1}\).

**Reminder of Claim 1.** For \(v \in [N]\), \(\text{parents}(P_v \setminus P_{v-1}, G) \cup \text{parents}(P_{v-1} \setminus P_v, G) \subseteq P_{v-1} \cap P_v\).

**Proof of Claim 1:** We observe that for \(v \in [N]\), \(P_v \setminus P_{v-1} \subseteq \bigcup_{i=0}^{d} A_{v+i,S,i+1}\), and \(P_{v-1} \setminus P_v \subseteq A_{v,S,2} \cup \left(\bigcup_{i=1}^{d+1} A_{v-i,S,i}\right)\). Then by Claim 2 below, we have

\[
\text{parents}(P_v \setminus P_{v-1}, G) \setminus S \subseteq \bigcup_{i=0}^{d} \text{parents}(A_{v+i,S,i+1}, G)
\]

\[
\subseteq \bigcup_{i=0}^{d} A_{v+i,S,i+2}
\]

\[
= \left(\bigcup_{i=0}^{d-1} A_{v+i,S,i+2}\right) \cup A_{v-1+d,S,d+2}
\]

\[
= \bigcup_{i=0}^{d-1} A_{v+i,S,i+2} \subseteq P_{v-1} \cap P_v,
\]

and

\[
\text{parents}(P_{v-1} \setminus P_v, G) \setminus S \subseteq \text{parents}(A_{v,S,2}, G) \cup \left(\bigcup_{i=1}^{d+1} \text{parents}(A_{v-i,S,i}, G)\right)
\]

\[
\subseteq A_{v,S,3} \cup \left(\bigcup_{i=1}^{d} A_{v-i,S,i+1}\right)
\]

\[
= A_{v,S,3} \cup \left(\bigcup_{i=1}^{d} A_{v-i,S,i+1}\right) \cup A_{v-d,S,d+2}
\]

\[
= A_{v,S,3} \cup \left(\bigcup_{i=1}^{d} A_{v-i,S,i+1}\right) \subseteq P_{v-1} \cap P_v,
\]

where we have \(A_{v-1+d,S,d+2} = A_{w-d,S,d+2} = \emptyset\) by the \((e, d)\)-reducibility. Taken together, we have \(\text{parents}(P_v \setminus P_{v-1}, G) \cup \text{parents}(P_{v-1} \setminus P_v, G) \subseteq S \subseteq P_{v-1} \cap (P_{v-1} \cap P_v) = P_{v-1} \cap P_v\). 

**Claim 2.** \(\text{parents}(A_{w,S,i}, G) \setminus S \subseteq A_{w,S,i+1}\).

**Proof.** If \(x \in A_{w,S,i}\) then by definition we have \(\text{LongestPath}_{G-S_{x-1}}(x, w) = i\). For any \(x' \in \text{parents}(x, G) \setminus S\), we observe that \(\text{LongestPath}_{G-S_{x-1}}(x', w) = 1 + \text{LongestPath}_{G-S_{x-1}}(x, w) = i + 1\), which completes the proof. 

\[29\]
Reminder of Lemma 2. Let $G_{\text{Arg-A}} = (V_A = [N], E_A)$ and $G_{\text{Arg-B}} = (V_B = [N], E_B)$ be randomly sampled graphs according to the Argon2i-A and Argon2i-B edge distributions, respectively. Then with high probability, the following holds:

(1) $G_{\text{Arg-A}}$ is $(e_1, d_1)$-reducible for $e_1 = \frac{N}{d'} + \frac{N \ln \lambda}{\lambda}$ and $d_1 = d' \lambda$, for any $0 < \lambda < N$ and $0 < d' < \frac{N}{\lambda}$.
(2) $G_{\text{Arg-B}}$ is $(e_2, d_2)$-reducible for $e_2 = \frac{N}{d'} + \frac{2N}{\sqrt{\lambda}}$ and $d_2 = d' \lambda$, for any $0 < \lambda < N$ and $0 < d' < \frac{N}{\lambda}$.

Proof of Lemma 2: We divide $N$ nodes into $\lambda$ layers of size $N/\lambda$ and reduce the depth of each layer to $d'$ so that the final depth becomes $d_1 = d_2 = d' \lambda$ for both Argon2i-A and Argon2i-B. To do so, we (a) delete all nodes with parents in the same layer, and (b) delete one out of $d'$ nodes in each layer. Let $\text{Delete}_i$ be the event that a node $v$ in $i^{\text{th}}$ layer is deleted in step (a), i.e., $r(v)$ remains in the same layer.

(1) For $G_{\text{Arg-A}}$, since all the layers have the same number of nodes and $r(v)$ is picked uniformly at random from $[v-2]$, we observe that $\Pr[\text{Delete}_i] \leq \frac{1}{i}$. It is clear that we delete $N/d'$ nodes in step (b). Hence,

\[ e_1 = \frac{N}{d'} + (\# \text{ nodes deleted in step (a)}) = \frac{N}{d'} + \sum_{i=1}^{\lambda} \Pr[\text{Delete}_i] \cdot \frac{N}{\lambda} \approx \frac{N}{d'} + \frac{N \ln \lambda}{\lambda}. \]

(2) For $G_{\text{Arg-B}}$, since we have $i \left(1 - \frac{x^2}{N^2}\right) \in (j-1, j]$ if and only if $N \sqrt{1 - \frac{i}{t}} \leq x < N \sqrt{1 - \frac{i-1}{t}}$, we have that $\Pr[r(i) = j] = \sqrt{1 - \frac{i-1}{t}} - \sqrt{1 - \frac{i}{t}}$. Similarly, we have $\Pr[a < r(i) < b] = \Pr_{x \in [N]} \left[i \left(1 - \frac{x^2}{N^2}\right) \in (a, b-1)\right] = \sqrt{1 - \frac{a}{t}} - \sqrt{1 - \frac{b-1}{t}}$. Thus,

\[
\Pr[\text{Delete}_i] = \Pr \left[ \frac{(i-1)N}{\lambda} < r(v) < v \right] = \sqrt{1 - \frac{(i-1)N}{v}} - \sqrt{1 - \frac{v-1}{v}} = \sqrt{1 - \frac{(i-1)N}{\lambda v}} - \sqrt{1 - \frac{1}{v}} \leq \sqrt{1 - \frac{i-1}{i}} - \sqrt{\frac{\lambda}{iN}} = \sqrt{\frac{1}{i}} - \sqrt{\frac{\lambda}{iN}},
\]
where the last inequality holds since \( \sqrt{1 - \frac{(i-1)N}{\lambda v}} - \sqrt{\frac{1}{v}} \) is an increasing function of \( v \) and the largest possible \( v \) is \( iN/\lambda \) since it should lie in the \( i \)th layer. Hence,

\[
e_2 = \frac{N}{d'} + \sum_{i=1}^{\lambda} \Pr[\text{Delete}_i] \cdot \frac{N}{\lambda}
\]

\[
\leq \frac{N}{d'} + \left( \frac{N}{\lambda} - \sqrt{\frac{N}{\lambda}} \right) \sum_{i=1}^{\lambda} \sqrt{\frac{1}{i}}
\]

\[
\leq \frac{N}{d'} + \left( \frac{N}{\lambda} - \sqrt{\frac{N}{\lambda}} \right) \left( \int_{1}^{\lambda} \frac{dx}{\sqrt{x}} + 1 \right)
\]

\[
= \frac{N}{d'} + \left( \frac{N}{\lambda} - \sqrt{\frac{N}{\lambda}} \right) (2\sqrt{\lambda} - 1) \leq \frac{N}{d'} + \frac{2N}{\sqrt{\lambda}}.
\]

\[\square\]

**Reminder of Corollary 1.** Let \( G_{\text{Arg-A}} = (V_A = [N], E_A) \) and \( G_{\text{Arg-B}} = (V_B = [N], E_B) \) be randomly sampled graphs according to the Argon2i-A and Argon2i-B edge distributions, respectively. Then with high probability, \( \Pi_{\text{st}}^{\text{st}}(G_{\text{Arg-A}}) = O\left( \frac{N^2 \log \log N}{\sqrt{\log N}} \right) \), and \( \Pi_{\text{st}}^{\text{st}}(G_{\text{Arg-B}}) = O\left( \frac{N^2}{\sqrt{\log N}} \right) \).

**Proof of Corollary 1:** From Theorem 2 and Lemma 2, we have

\[
\Pi_{\text{st}}^{\text{st}}(G_{\text{Arg-A}}) \leq O\left( N + Ne + Nd^2d' \right) \simeq O\left( N + \frac{N^2}{d'} + \frac{N^2 \ln \lambda}{\lambda} + \lambda d' 2\lambda d' N \right).
\]

To make the upper bound optimal, we want to make the upper bound as small as possible. Hence, we want to find \( d' \) and \( \lambda \) such that \( \frac{N^2}{d'} \simeq \frac{N^2 \ln \lambda}{\lambda} \simeq \lambda d' 2\lambda d' N \) as much as possible. Hence, \( d' = \frac{\lambda}{\ln \lambda} \) and \( \lambda \) should satisfy \( \frac{\lambda^3}{(\ln \lambda)^2} 2\lambda^2 \ln \lambda \simeq \lambda d' 2\lambda d' N \). Setting \( \lambda = \sqrt{\log N} \), we have \( d' = \frac{\lambda}{\ln \lambda} = \frac{2\sqrt{\log N}}{\ln \log N} \) and \( d = d' \lambda = \frac{2\log N}{\ln \log N} \). Thus,

\[
\Pi_{\text{st}}^{\text{st}}(G_{\text{Arg-A}}) \leq O\left( N + \frac{2N^2 \ln \log N}{2\sqrt{\log N}} + \frac{2N \log N}{\ln \log N} \right)
\]

\[
= O\left( N + \frac{2N^2 \ln \log N}{2\sqrt{\log N}} + \frac{2N^2 \ln \log N}{\ln \log N} \right)
\]

\[
= O\left( \frac{N^2 \log \log N}{\sqrt{\log N}} \right),
\]

since \( \ln x = (\ln 2)(\log x) \) for any \( x > 0 \).

For Argon2i-B, we have

\[
\Pi_{\text{st}}^{\text{st}}(G_{\text{Arg-B}}) \leq O\left( N + Ne + Nd^2d' \right) \simeq O\left( N + \frac{N^2}{d'} + \frac{2N^2}{\sqrt{\lambda}} + \lambda d' 2\lambda d' N \right).
\]

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Similarly, to make the upper bound optimal, we want to make \( \frac{N^2}{\lambda} \approx \frac{2N^2}{\sqrt{\lambda}} \approx \lambda d' 2^{\lambda d'} \) as much as possible. Hence, we have \( d' \approx \sqrt{\frac{N}{2}} \) and plugging in \( \lambda = \sqrt{\log^2 N} \) and \( d' = \sqrt{\log N}/2 \), we have

\[
\Pi_{st}^{\alpha, \beta}(G_{\text{Arg-B}}) \leq O\left( N + \frac{4N^2}{\sqrt{\log N}} + \frac{N\sqrt{N\log N}}{2} \right)
= O\left( \frac{N^2}{\sqrt{\log N}} \right).
\]

\[\square\]

**Reminder of Lemma 3.** Let \( G = (V = [N], E) \) and \( b \in [N] \) be a parameter. If \( P' \in \mathcal{P}_{L([N/b]), b}^{\alpha, \beta} \), then \( P = \text{Trans}(G, P', b) \in \mathcal{P}_{\mathcal{G}}^{\alpha, \beta} \).

**Proof of Lemma 3:** We want to show that it satisfies conditions in **Definition 1**.

**Condition (1):** \( P_{tb+N-([N/b]-1)b-1} = \{N\} \).

- It is clear by construction because we remove all nodes except for the target node \( N \).

**Condition (2):** \( \forall j \in [tb + N - ([N/b] - 1)b - 1] : v \in (P_j \setminus P_{j-1}) \Rightarrow \text{parents}(v, G) \subseteq P_{j-1} \).

- We first observe that whenever we pebble a new node \( w \) in \( L_{[N/b]} \), the node \( w - 1 \) must have been pebbled in the previous round.
- Suppose that \( v \in B_w \) for some \( w \in [N/b] \). For every edge of the form \((u, v)\), we have the following possibilities:
  (a) If \( u \in B_w \), \( u \) must be (re)pebbled before node \( v \) since both \( u \) and \( v \) corresponds to placing the node \( w \) in \( L_{[N/b]} \). Hence, \( u \in P_{j-1} \).
  (b) If \( u \in B_{w-1} \), we are guaranteed that \( u \) is already pebbled before we begin pebbling nodes in block \( B_w \) since every node in \( B_{w-1} \) is pebbled. Hence, \( u \in P_{j-1} \).
  (c) If \( u \in B_j \) with \( j < w - 1 \), then \( u \) is a skip node and will already be pebbled before placing a pebble on \( v \). Hence, \( u \in P_{j-1} \).
- Taken together, we have \( \text{parents}(v, G) \subseteq P_{j-1} \).

**Condition (3):** \( \forall j \in [tb + N - ([N/b] - 1)b - 1] : v \in (P_{j-1} \setminus P_{j}) \Rightarrow \text{parents}(v, G) \subseteq P_{j-1} \).

- We first observe that whenever we remove a pebble from \( w \) in \( L_{[N/b]} \), the node \( w - 1 \) must have been pebbled in the previous round.
- Suppose that \( v \in B_w \) for some \( w \in [N/b] \). For every edge of the form \((u, v)\), we have the following possibilities:
  (a) If \( u \in B_w \), a pebble on \( u \) is not yet removed in the previous round because we remove pebbles in \( B_w \) in a reverse topological order. Hence, \( u \in P_{j-1} \).
  (b) If \( u \in B_{w-1} \), we are guaranteed that \( u \) is already pebbled before we begin removing nodes in block \( B_w \) since every node in \( B_{w-1} \) is pebbled. Hence, \( u \in P_{j-1} \).
  (c) If \( u \in B_j \) with \( j < w - 1 \), then \( u \) is a skip node and will already be pebbled before removing a pebble from \( v \). Hence, \( u \in P_{j-1} \).
- Taken together, we have \( \text{parents}(v, G) \subseteq P_{j-1} \).

**Condition (4):** \( \forall j \in [tb + N - ([N/b] - 1)b - 1] : v \in \text{parents}(P_j \setminus P_{j-1}, G) \cup \text{parents}(P_{j-1} \setminus P_{j}, G) \), then \( v \in P_{j} \).
Lemma 4.

- If $v \in \text{parents}(P_j \setminus P_{j-1}, G)$, then there exists some $v' \in P_j \setminus P_{j-1}$ and some $w \in \ceil{N/b}$ such that $(v, v') \in E$ and $v' \in B_w$. Now we have the following possibilities:
  (a) If $v \in B_w$, then $v$ must be (re)pebbled before node $v'$ and keep pebbled since both $u$ and $v$ corresponds to placing the node $w$ in $L_{\ceil{N/b}}$. Hence, $v \in P_j$.
  (b) If $v \in B_{w-1}$, we are guaranteed that $v$ is already pebbled when we begin pebbling nodes in block $B_w$ since every node in $B_{w-1}$ is pebbled. Hence, $v \in P_j$.
  (c) If $v \in B_j$ with $j < w - 1$, then $v$ is a skip node and will already be pebbled and keep pebbled when placing a pebble on $v'$. Hence, $v \in P_j$.
- If $v \in \text{parents}(P_{j-1} \setminus P_j, G)$, then there exists some $v'' \in P_{j-1} \setminus P_j$ and some $w' \in \ceil{N/b}$ such that $(v, v'') \in E$ and $v'' \in B_{w'}$. Now we have the following possibilities:
  (a) If $v \in B_{w'}$, a pebble on $v$ is not yet removed in $P_j$ because we remove pebbles in $B_{w'}$ in a reverse topological order. Hence, $v \in P_j$.
  (b) If $v \in B_{w'-1}$, we are guaranteed that $v$ is already pebbled when we begin removing nodes in block $B_{w'}$ since every node in $B_{w'-1}$ is pebbled. Hence, $v \in P_j$.
  (c) If $v \in B_j$ with $j < w' - 1$, then $v$ is a skip node and will already be pebbled and keep pebbled when removing a pebble from $v'$. Hence, $v \in P_j$.

Taken together, we can conclude that if $P' \in \mathcal{P}_{L_{\ceil{N/b}}}$, then $P = \text{Trans}(G, P', b) \in \mathcal{P}_{G}^\oplus$.

**Reminder of Lemma 4.** Let $G_{\text{DRS}} = (V_{\text{DRS}} = [N], E_{\text{DRS}})$ be a randomly sampled graph according to the DRSample edge distribution. Then with high probability, we have $\text{NumSkip} \left( G_{\text{DRS}}, \ceil{N / \log^2 N} \right) = \mathcal{O} \left( \frac{N \log \log N}{\log N} \right)$.

**Proof of Lemma 4:** For each $v \in V_{\text{DRS}}$, let $Y_v$ be an indicator random variable for the event that $v - r(v) > b$. Then we observe that $\text{NumSkip}(G_{\text{DRS}}, b) \leq \sum_{v \in V_{\text{DRS}}} Y_v$, since $\text{NumSkip}(G_{\text{DRS}}, b)$ is upper bounded by the number of edges that skip over a block. Since there are at most $\log v$ buckets for $r(v)$ and $\log b$ buckets with $v - r(v) \leq b$, we have $\Pr[v - r(v) > b] \leq 1 - \frac{\log b}{\log v} \leq 1 - \frac{\log b}{\log N} = \frac{\log(N/b)}{\log N}$. Hence, by linearity of expectation it follows that

$$\mathbb{E}[\text{NumSkip}(G_{\text{DRS}}, b)] \leq \sum_{v \in V_{\text{DRS}}} \mathbb{E}[Y_v] = \sum_{v \in V_{\text{DRS}}} \Pr[v - r(v) > b] \leq \sum_{v \in V_{\text{DRS}}} \frac{\log(N/b)}{\log N} = \frac{N \log(N/b)}{\log N}.$$  

As the expected value is the sum of independent random variables, we can use Chernoff bounds with $\mu = \frac{N \log(N/b)}{\log N} \geq \sum_{v \in V_{\text{DRS}}} \mathbb{E}[Y_v]$ to show that for any constant $\delta > 0$, we have

$$\Pr[\text{NumSkip}(G_{\text{DRS}}, b) > (1 + \delta)\mu] < \exp \left( -\frac{\delta^2 N \log(N/b)}{3 \log N} \right).$$

Hence, with high probability, we have $\text{NumSkip}(G_{\text{DRS}}, b) = \mathcal{O} \left( \frac{N \log(N/b)}{\log N} \right)$. Setting $b = \frac{N}{\log^2 N}$, we get the desired result. \qed
D Quantum Pebbling Strategy using an Induced Line Graph

Algorithm 1: The Procedure \( \text{Trans}(G, P', b) \).

**Input:** A constant-indegree DAG \( G = (V = [N], E) \), a parameter \( b \) (size of the block), and a legal quantum pebbling \( P' = \{ P'_0, P'_1, \ldots, P'_t \} \in \mathcal{P}^*_G \) for an induced line graph \( L_{[N/b]} \).

**Output:** A legal quantum pebbling \( P \in \mathcal{P}^*_G \) of \( G \).

1. Partition \( V = [N] \) into \( B_1, \ldots, B_{[N/b]} \) where \( B_i = \{(i-1)b + 1, (i-1)b + 2, \ldots, ib\} \) for \( i \in [\lfloor N/b \rfloor - 1] \) and \( B_{[N/b]} = \{(\lfloor N/b \rfloor - 1)b + 1, (\lfloor N/b \rfloor - 1)b + 2, \ldots, N\} \).

2. Initialize \( P_{0,b}^i = \emptyset \) and \( P_{j,k}^i = \emptyset \) for each \( i \in [\lfloor N/b \rfloor] \), \( j \in [t] \), and \( k \in [f(j)] \), where \( f(j) = b + N - (\lfloor \frac{N}{b} \rfloor - 1)b - 1 \) if \( j = \text{LastAdd}(P', [N/b]) \), and \( f(j) = b \) elsewhere.

3. for \( i = 1, \ldots, [N/b] - 1 \) \( \text{// for each block } B_i, \ldots, B_{[N/b]-1} \text{ except for the last one} \)

   for \( j = 1, \ldots, t \) \( \text{// for each round in } P' \)

   if \( j \neq \text{LastAdd}(P', [N/b]) \) then

   \[
   \{ P_{j,1}^i, \ldots, P_{j,b}^i \} \leftarrow \text{BlockPebble}(B_i, b, S_i, P', P_{j-1,f(j)}^i, i, j).
   \]

   else \( \text{i.e., } j = \text{LastAdd}(P', [N/b]) \)

   \[
   \{ P_{j,1}^i, \ldots, P_{j,b}^i \} \leftarrow \text{BlockPebble}(B_i, b, S_i, P', P_{j-1,f(j)}^i, i, j).
   \]

   Maintain pebbles for the extra \( N - (\lfloor \frac{N}{b} \rfloor - 1)b - 1 \leq b - 1 \) steps, i.e.,

   \[
   P_{j,b}^i = P_{j,b+1}^i = \ldots = P_{j,b+N-(\lfloor \frac{N}{b} \rfloor-1)b-1}^i.
   \]

4. for \( j = 1, \ldots, t \) \( \text{// for the last block } B_{[N/b]} \text{ and for each round in } P' \)

   if \( j \neq \text{LastAdd}(P', [N/b]) \) then

   \[
   \{ P_{j,1}^{\lfloor \frac{N}{b} \rfloor}, \ldots, P_{j,b}^{\lfloor \frac{N}{b} \rfloor} \} \leftarrow \text{LastBlockPebble}(N, b, P', P_{j-1,f(j)}^{\lfloor \frac{N}{b} \rfloor}, j).
   \]

   else \( \text{i.e., } j = \text{LastAdd}(P', [N/b]) \)

   \[
   \{ P_{j,1}^{\lfloor \frac{N}{b} \rfloor}, \ldots, P_{j,b}^{\lfloor \frac{N}{b} \rfloor} \} \leftarrow \text{LastBlockPebble}(N, b, P', P_{j-1,f(j)}^{\lfloor \frac{N}{b} \rfloor}, j).
   \]

   Delete pebbles from the block in a reverse topological order, except for the sink node, with \( N - (\lfloor N/b \rfloor - 1)b - 1 \) steps, i.e., \( P_{b+k}^{\lfloor \frac{N}{b} \rfloor} = P_{b+k-1}^{\lfloor \frac{N}{b} \rfloor} \setminus \{ N - k \} \) for \( k = 1, \ldots, N - (\lfloor N/b \rfloor - 1)b - 1 \).

5. for \( k = 1, \ldots, f(j) \) do

   \[
   P_{j,k} = \bigcup_{i=1}^{[N/b]} P_{j,k}^i.
   \]

   if \( j \leq \text{LastAdd}(P', [N/b]) \) then \( \text{// Ordering the pebbling configurations} \)

   \[
   P_{j-1,b+k} \leftarrow P_{j,k}
   \]

   else

   \[
   P_{N-(\lfloor N/b \rfloor-1)b-1+(j-1)b+k} \leftarrow P_{j,k}
   \]

6. return \( P = \{ P_1, \ldots, P_{b+N-(\lfloor N/b \rfloor-1)b-1} \} \).
**Algorithm 2: The Subfunction BlockPebble(B, b, S, P′, P₀, i, j).**

**Input:** A set of nodes B, a parameter b (size of the set), a set of skip pebbles S, a legal quantum pebbling \( P' = \{P'_0, P'_1, \ldots, P'_t\} \), a pebbling configuration \( P_0 \) on B, and parameters i and j.

**Output:** A legal relaxed quantum pebbling \( P = \{P_1, \ldots, P_b\} \) of the set B.

1. Assert \(|B| = b\).
2. **if** \( i \in P'_j \setminus P'_{j-1} \) **then**
   3. Place pebbles in the block B with b steps, i.e., \( P_1 = P_0 \cup \{(i-1)b+1\} \), and \( P_k = P_{k-1} \cup \{(i-1)b+k\} \) for \( k = 2, \ldots, b \).
4. **else if** \( i \in P'_{j-1} \setminus P'_j \) **then**
5. **if** \( j-1 = \text{LastDelete}(P', i) \) **then**
   6. Delete pebbles from the block B in a reverse topological order with b steps, i.e., \( P_1 = P_0 \setminus \{ib\} \), and \( P_k = P_{k-1} \setminus \{ib-(k-1)\} \) for \( k = 2, \ldots, b \).
7. **else**
   8. Delete pebbles from the block B except for the skip nodes, i.e., \( P_1 = P_0 \setminus (\{ib\} \setminus S) \), and \( P_k = P_{k-1} \setminus (\{ib-(k-1)\} \setminus S) \) for \( k = 2, \ldots, b \).
9. **else**
10. Maintain pebbles in the block B for b steps, i.e., \( P_0 = P_1 = \cdots = P_b \).
11. **return** \( P = \{P_1, \ldots, P_b\} \).

---

**Algorithm 3: The Subfunction LastBlockPebble(N, b, P′, P₀, j).**

**Input:** A parameter N, b, a legal quantum pebbling \( P' = \{P'_0, P'_1, \ldots, P'_t\} \), a pebbling configuration \( P_0 \) of the last block, and a parameter j.

**Output:** A legal relaxed quantum pebbling \( P = \{P_1, \ldots, P_b\} \) of the last block.

1. **if** \( \lfloor N/b \rfloor \in P'_j \setminus P'_{j-1} \) **then**
2. Place pebbles in the block with \( N - (\lfloor N/b \rfloor - 1)b \) steps, and maintain the status for the next \( b - N + (\lfloor N/b \rfloor - 1)b \) steps, i.e., \( P_1 = P_0 \cup \{(\lfloor N/b \rfloor - 1)b+1\} \), \( P_k = P_{k-1} \cup \{(\lfloor N/b \rfloor - 1)b+k\} \) for \( k = 2, \ldots, N - (\lfloor N/b \rfloor - 1)b \), and \( P_{N - (\lfloor N/b \rfloor - 1)b} = P_{N - (\lfloor N/b \rfloor - 1)b+1} = \cdots = P_b \).
3. **else if** \( \lfloor N/b \rfloor \in P'_{j-1} \setminus P'_j \) **then**
4. Delete pebbles from the block in a reverse topological order with \( N - (\lfloor N/b \rfloor - 1)b \) steps, and maintain the status for the next \( b - N + (\lfloor N/b \rfloor - 1)b \) steps, i.e., \( P_1 = P_0 \setminus \{N\} \), \( P_k = P_{k-1} \setminus \{N-(k-1)\} \) for \( k = 2, \ldots, N - (\lfloor N/b \rfloor - 1)b \), and \( P_{N - (\lfloor N/b \rfloor - 1)b} = P_{N - (\lfloor N/b \rfloor - 1)b+1} = \cdots = P_b \).
5. **else**
6. Maintain pebbles in the block for b steps, i.e., \( P_0 = P_1 = \cdots = P_b \).
7. **return** \( P = \{P_1, \ldots, P_b\} \).