Longitudinal and transverse structure functions in high-Reynolds-number turbulence

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Abstract. Using exact relations between velocity structure functions \cite{1–3} and neglecting pressure contributions in a first approximation, we obtain a closed system and derive simple order-dependent rescaling relationships between longitudinal and transverse structure functions. By means of numerical data with turbulent Reynolds numbers ranging from $\ReyL = 320$ to $\ReyL = 730$, we establish a clear correspondence between their respective scaling ranges, while confirming that their scaling exponents do differ. This difference does not seem to depend on the Reynolds number. Making use of the Mellin transform, we further map longitudinal to (rescaled) transverse probability density functions.

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1. Introduction

Intermittency is a ubiquitous feature of fluid turbulence: the scaling properties of flow quantities differ from Kolmogorov’s mean field theory [4]. For instance, in the inertial range of scales where flow properties are assumed to be independent of the details of energy injection and dissipation, the velocity increments do not have a monofractal structure. In homogeneous isotropic turbulence (HIT), two directions only matter in the computation of a velocity increment: the longitudinal one, $\Delta_r u$, taken along the separation and the transverse one, $\Delta_r v$, in which the difference of velocity components perpendicular to the separation are computed. Velocity structure functions are then defined as the ensemble average:

$$S_n(r) = \langle (\Delta_r u)^n (\Delta_r v)^m \rangle.$$  

For HIT, the von Kármán–Howarth relationship taken in the inertia range leads to Kolmogorov’s fourth/fifth law:

$$\langle \epsilon \rangle r^{-\frac{4}{5}}$$

where $\langle \epsilon \rangle$ is the mean energy dissipation rate per unit mass. While a monofractal inertial range behavior would then lead to $S_n(r) \propto r^{\zeta_n}$ with $\zeta_n \sim n/3$, intermittency means that $\zeta_n$ is a nonlinear (concave) function of $n$. Numerous works have been devoted to the study of the functional form of $\zeta_n$. We focus on a possible link between the longitudinal and transverse structure functions $S_{n,0}(r)$ and $S_{0,n}(r)$. There exist theoretical arguments that the longitudinal and the transverse show the same scaling [5]. However, both experimental data [6–9] and numerical simulations [10–13] show consistently different scaling exponents for longitudinal and transverse structure functions. Whether this difference can be attributed to a persistent small-scale anisotropy [14, 15] or to a finite Reynolds number effect [1] is an unsolved question to which we will come back below. Here, we note that in the case of the direct cascade in electron magnetohydrodynamics [16], it was demonstrated numerically that the difference vanishes with increasing numerical resolution and thus is a finite Reynolds number effect.

In this paper, we focus rather on the correspondence between scaling ranges of the longitudinal and transverse structure functions. Our approach is based on the observation that even though the (real space) velocity field of a turbulent flow coarse-grained at a scale $r$ is not smooth, the structure functions are smooth (differentiable) functions of $r$. We thus use the structure of the Navier–Stokes equation together with assumptions to derive constitutive relationships between $S_{n,0}(r)$ and $S_{0,n}(r)$. Specifically, we shall neglect the contributions from the pressure term. We start with exact scaling expressions derived by Hill [1], Hill and Boratav [2] and Yakhot [3]; we then obtain rescaling relationships between longitudinal and transverse structure functions. Our first finding is that, after rescaling, the longitudinal and transverse structure functions share the same inertial range, i.e. the same width in the extent of of scales where self-similarity is observed. This is important because the question of the location and span of the inertial range is often an issue in the analysis of turbulent data at (necessarily) finite Reynolds number. We stress, however, that the value of the longitudinal and transverse scaling exponents do differ. A second outcome of our simple ansatz is a direct mapping, using the Mellin transform, of the transverse and longitudinal probability density functions (PDFs). Differences that persist after the mapping are then due to the effect of the neglected terms, as pointed out in some previous attempts by Yakhot [3] and Gotoh and Nakano [17].

2. Rescaling relations between longitudinal and transverse structure functions

Our calculation traces back to the observation by Siefert and Peinke [18] that the Kármán equation (see Kármán and Howarth [19]) relating second-order longitudinal and transverse
structure functions can be interpreted as a Taylor expansion of a smooth function. To see this, we start with the Kármán equation

\[ S_{0,2}(r) = S_{2,0}(r) + \frac{r}{2} \frac{\partial}{\partial r} S_{2,0}(r), \]  

(1)

which is exact and contains no contribution from the pressure—it is a statement of incompressibility. Siefert and Peinke [18] observed that the structure function is a smooth function of \( r \) and that if the scale \( r \) is chosen in the inertial range, i.e. ‘small’ compared to the integral scale \( L \), equation (1) can be seen as a Taylor expansion:

\[ S_{0,2}(r) \approx S_{2,0}(r + \frac{r}{2}) = S_{2,0}\left(\frac{3}{2}r\right), \]  

(2)

where the function \( S_{2,0} \) is expanded about \( r \) for consistency with the exact relationship (1).

In [18], evidence from experimental data Taylor-based Reynolds numbers between 180 and 550 was presented to support this view. The success of the approach introduced by Siefert and Peinke [18] motivated us to extend their re-interpretation of differential relations to structure functions of higher orders, making use of the exact relationships derived by Hill and Boratav [2], Hill [1] and Yakhot [3]. Hill derived these relations directly by inventing a clever matrix algorithm which allowed him to efficiently simplify the derivation and calculations. Yakhot [3], on the other hand, derived an equation for the characteristic function \( Z = \langle \lambda \cdot \Delta u_r \rangle \), where \( \Delta u_r \) denotes a velocity increment over the distance \( r \). Structure–function relations can then be obtained by differentiating the characteristic function \( Z \).

As an illustrative example, consider the relation for even-order mixed structure functions derived by Yakhot [3]:

\[ \frac{\partial S_{2n,0}}{\partial r} + \frac{2}{r} S_{2n,0} + \frac{2(2n - 1)}{r} S_{2n-2,2} = C_p + C_f. \]  

(3)

The term \( C_p \) contains contributions from the (unknown) pressure field and is the reason why the system cannot be closed. The term \( C_f \) contains the contributions from the large-scale forcing and can safely be ignored in the inertial range as proposed by Kurien and Sreenivasan [20]. These authors also analyzed and compared these relations to measurements in atmospheric turbulence at a Taylor-based Reynolds number of about 10 700. One of their findings was that for even-order structure functions the pressure contributions can be an order of magnitude smaller than the terms directly related to the structure functions. A detailed numerical study on the role of the pressure term has been realized by Gotoh and Nakano [17]. In order to obtain closed expressions, we shall hereafter neglect the pressure contributions. Although this assumption is quite crude, we are already able to capture the dominant features of the relationship between longitudinal and transverse structure functions, such as amplitudes and common inertial range. Pressure (or energy injection) contributions will then appear as departures from predictions of this closed system of equations.

In order to demonstrate the procedure, we start with formulae for the fourth-order structure functions and neglect the contributions from the pressure and the large-scale forcing (see also equations (11) and (13) in [20])

\[ 3S_{2,2}(r) \approx S_{4,0}(r) + \frac{r}{2} \frac{\partial}{\partial r} S_{4,0}(r) \approx S_{4,0}\left(\frac{3}{2}r\right), \]  

(4)

\[ \frac{1}{3} S_{0,4}(r) \approx S_{2,2}(r) + \frac{r}{4} \frac{\partial}{\partial r} S_{2,2}(r) \approx S_{2,2}\left(\frac{5}{4}r\right), \]  

(5)
which can be combined into
\[ S_{4,0} \left( \frac{3}{2} \frac{\partial}{\partial r} r \right) \approx S_{0,4}(r). \] (6)

For the sixth-order structure functions we obtain similarly (see equations (12), (15) and (14) in [20])
\[ 5S_{4,2}(r) \approx S_{6,0}(r) + \frac{r}{2} \frac{\partial}{\partial r} S_{6,0}(r) \approx S_{6,0} \left( \frac{3}{2} r \right), \]
\[ S_{2,4}(r) \approx S_{4,2}(r) + \frac{r}{4} \frac{\partial}{\partial r} S_{4,2}(r) \approx S_{4,2} \left( \frac{5}{4} r \right), \]
\[ \frac{1}{5} S_{0,6}(r) \approx S_{2,4}(r) + \frac{r}{6} \frac{\partial}{\partial r} S_{2,4}(r) \approx S_{2,4} \left( \frac{7}{6} r \right). \]

Again, combining these equations results in the simple relation
\[ S_{6,0} \left( \frac{3}{2} \frac{7}{6} \frac{\partial}{\partial r} r \right) \approx S_{0,6}(r). \] (7)

In general, the rescaling for even-order structure functions reads
\[ S_{n,0} \left( \frac{3}{2} \frac{7}{6} \frac{\partial}{\partial r} r \right) = S_{n,0} \left( \frac{\Gamma(n+2)}{2^n \Gamma^2(n/2+1)} r \right) \approx S_{0,n}(r). \] (8)

In order to demonstrate that the Taylor expansion is valid also for higher-order structure functions, we look at the differential relation for the fourth-order structure function obtained from equations (4) and (5)
\[ S_{04}(r) = S_{40}(r) + \frac{7}{8} \frac{\partial S_{40}(r)}{\partial r} r + \frac{1}{8} \frac{\partial^2 S_{40}(r)}{\partial r^2} r^2. \] (9)

In figure 1, we compare the longitudinal (black), transverse (blue), rescaled longitudinal (red) and the one using the differential relation (green). The difference between the rescaled longitudinal and the one using the differential relation is negligible.

In order to test our results, we use numerical data from pseudo-spectral simulations of incompressible Navier–Stokes turbulence, as described in [21]—the LaTu code. A statistically stationary flow is maintained by keeping constant the Fourier modes in the lowest two shells. All the results are averaged over several large-eddy turnover times (over two in the case of \( \mathfrak{R}_h = 730 \)). Parameters of these high-Reynolds-number simulations are given in table 1.

Figure 2 shows the application of the rescaling formula (8) to the 2nd, 4th, 6th and 10th-order structure functions obtained from a Navier–Stokes simulation with 2048\(^3\) grid points and parameters as described in table 1. We choose this dataset because it contains ten large-eddy turnover times and thus provides reliable statistics for high-order structure functions. In each subfigure of figure 2, the unscaled structure functions are shown in the lower part (solid lines) and the structure functions rescaled according to equation (8) are shown on top (dashed lines). Note that the original structure functions are shifted for clarity. The effect of the rescaling transformation (8) is twofold: first, the amplitudes from the dissipative scales up to the integral scales are now very similar. In addition, the range of scales over which a power-law behavior is observed is now identical for the two SFS, although the scaling exponents differ slightly. This is evidenced in the inset of each subfigure in figure 2 where the logarithmic derivatives of the structure–functions with respect to scale have been plotted, and the vertical lines mark
Figure 1. Comparison of structure functions: longitudinal (black), transverse (blue), rescaled longitudinal (red) and the one using the differential relation (green).

Table 1. Parameters of the numerical simulations. $\Re_\lambda = \sqrt{15 u_{\text{rms}} L / \nu}$: the Taylor–Reynolds number; $u_{\text{rms}}$, root-mean-square velocity; $\epsilon_k$, the mean kinetic energy dissipation rate; $\nu$, the kinematic viscosity; $dx$, the grid-spacing; $\eta = (\nu^3 / \epsilon_k)^{1/4}$, the Kolmogorov dissipation length scale; $\tau_\eta = (\nu / \epsilon_k)^{1/2}$, the Kolmogorov time scale; $L = (2/3E)^{3/2} / \epsilon_k$, the integral scale; $T_L = L / u_{\text{rms}}$, large-eddy turnover time; $N^3$, the number of collocation points.

| $\Re_\lambda$ | $u_{\text{rms}}$ | $\epsilon_k$ | $\nu$ | $dx$ | $\eta$ | $\tau_\eta$ | $L$ | $T_L$ | $N^3$ |
|---------------|----------------|-------------|------|-----|-------|---------|-----|-----|------|
| 730           | $0.192$       | $3.8 \times 10^{-3}$ | $1.0 \times 10^{-5}$ | $1.53 \times 10^{-3}$ | $7.2 \times 10^{-4}$ | 0.05 | 1.85 | 9.6 | 4096$^3$ |
| 460           | $0.189$       | $3.6 \times 10^{-3}$ | $2.5 \times 10^{-5}$ | $3.07 \times 10^{-3}$ | $1.45 \times 10^{-3}$ | 0.083 | 1.85 | 9.9 | 2048$^3$ |
| 320           | $0.187$       | $3.5 \times 10^{-3}$ | $0.5 \times 10^{-4}$ | $6.14 \times 10^{-3}$ | $2.45 \times 10^{-3}$ | 0.12 | 1.85 | 10 | 1024$^3$ |

the scaling interval. Note that both effects could not be achieved by an order-independent fixed rescaling factor of 3/2.

For this data set the transverse increments are more intermittent than the rescaled ones. In order to address the question of whether this difference in scaling might depend on Reynolds number, we show in figure 3 the logarithmic derivative of the eighth-order structure function for three different simulations. With increasing Reynolds number the inertial range increases but the scaling exponent (the value of the plateau) remains the same. We measure a value of approximately 2 for the transverse functions and 2.21 for the longitudinal ones. For comparison we included data from randomized snapshots originating from the simulation with $\Re_\lambda = 730$. In detail, for each Fourier mode we randomly change the phase while preserving its amplitude and incompressibility of the flow. This preserves the energy spectrum but destroys the structure of
the flow (energy cascade, coherent structures, etc). The randomized longitudinal and transverse structure–functions exhibit the same scaling exponent, now close to the trivial 8/3 value.

A general assumption is that the possible remaining large-scale anisotropy in the small scales is expected to decrease with the Reynolds number. We remark that Biferale et al [14] showed that the differences in high-order exponents remain even, if measured in the purely isotropic sector. That the curves in figure 3 fall on top of each other within the inertial range of scales is an indication that the observed differences in the scaling exponents are not due to large-scale anisotropies. This indicates that the differences observed above have to be attributed to the specific small-scale structures of the flow.

3. Implications for longitudinal and transverse probability density functions

Since the rescaling property has the effect to make the longitudinal and transverse structure functions fall nearly on top of each other, we want to understand the effect of the rescaling transformation on the PDFs. In this subsection, we try to map longitudinal PDFs to transverse ones using the rescaling property expressed through equation (8). The rescaling transformations
Figure 3. Logarithmic derivative of longitudinal and transverse structure functions for three different Reynolds numbers and in the case of a randomized velocity field.

were derived for even-order structure functions. Thus, in the following, we disregard skewness effects and consider only the symmetric part of the PDFs. We first approximate the numerically obtained longitudinal PDFs with a log-normal distribution using the expression given in [22]:

$$P_L(\Delta u, r) = \frac{1}{\Delta u \sqrt{\ln r}} \int_{-\infty}^{\infty} e^{-x^2} \exp \left[ -\left( \frac{\ln \frac{\Delta u}{r^{\frac{1}{2}}}}{4b \ln r} \right)^2 \right] dx,$$

for which a fit is obtained with the values $a = 0.383$ and $b = 0.0166$ [22]. In figure 4, the numerically obtained PDF and the fit $P_L(\Delta u, r)$ are shown for several spatial scales. We apply the inverse Mellin transform

$$P_L(\Delta u, r) = \frac{1}{\Delta u} \int_{-\infty}^{\infty} dn S(n, r)(\Delta u)^{-n}$$

with $S(n, r) = A(n)r^{\xi(n)}$, and we follow the procedure in [22] that fixes the amplitude by going to the Gaussian limit for large spatial differences:

$$A(n) = (n - 1)!! = \frac{2^{n/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} x^n \, dx.$$  

Now a mapping from the longitudinal PDFs to the transverse PDFs is obtained by inserting the rescaling relation (8) into the expression for the structure–functions:

$$P_T(\Delta u, r) = \frac{1}{\Delta u} \int_{-\infty}^{\infty} dn A(n)(C(n)r^{\xi(n)}(\Delta u)^{-n},$$

where $C(n) = \frac{n(n+2)}{2!r^{\xi(n)/2+1}}$ as in equation (8).

Since the ansatz $S(n, r) = A(n)r^{\xi(n)}$ does not contain a cutoff at integral and dissipation scales, this cutoff is inserted into the rescaling function $C(n)$. In both regions outside the inertial range, a smooth behavior is expected. On scales close to the integral range, a Gaussian
behavior for both longitudinal and transverse increments is expected and thus no rescaling is necessary. This justifies choosing $C(n)$ to be constant for $n \leq 0$. The cutoff at the dissipation scale is achieved by choosing $C(n)$ to be constant for $n > 6$. The precise value of the chosen $n$ is dependent on the actual Reynolds number and the effect of choosing a different bound allows for a widening of the transformed PDF. This reflects the fact that the Reynolds number has a similar effect on the width of the PDF. Evaluating the integral (11) using a saddle point approximation, we obtain a mapping from the log-normal fit of the longitudinal PDF $P_L(\Delta u, r)$ to a new PDF $P_T(\Delta u, r)$ which is compared with the numerically obtained data in figure 4 for increments ranging from the near-dissipation range to integral scales. One can observe that the agreement is especially remarkable in the inertial range ($r = 106\eta$ and $r = 212\eta$). We do not expect perfect agreement for all scales since in that case there would be no room for differences between longitudinal and transverse structure–functions. Thus the discrepancy for $r = 21\eta$ and

Figure 4. Longitudinal PDF (lower points) and fit with log-normal distribution (green line); transverse PDF (upper points) and mapped distribution (red line). The PDFs for different increments ranging from the near-dissipation range to integral scales are shown. ($\eta_L = 460$, $\eta$ denotes the dissipation and $L$ is the integral scale.)
\[ r = 42 \eta \] just represents the missing contributions of the pressure term. Therefore, this method of mapping the PDFs is also a promising candidate for applications such as PDF modeling of turbulent flows (see [23] and references therein).

4. Conclusions and outlook

In this paper, we have suggested a new way of analyzing experimental and numerical data for longitudinal and transverse structure functions in Eulerian data of a turbulent velocity field. This procedure yields a mapping between the longitudinal and transverse scales, which provides a consistent reference point for the identification of the inertial ranges of scales of turbulent flows. In addition, the derived scale correspondence allows for a direct mapping of the full probability density of transverse and longitudinal structure functions. This may be of much practical interest as the distributions carry more complete information than a subset of their moments. The gap of longitudinal and transverse structure–function exponents seems to depend not on Reynolds number but on the small-scale structure of the flow. The proposed mapping may help clarify the role played by the pressure terms. Future work will be devoted to the analysis of other turbulent systems such as magnetohydrodynamics, where the addition of the magnetic pressure term yields an interesting comparison.

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