Loop Equations for the d-dimensional n-Hermitian Matrix model

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Abstract

We derive the loop equations for the d-dimensional n-Hermitian matrix model. These are a consequence of the Schwinger-Dyson equations of the model. Moreover we show that in leading order of large $N$ the loop equations form a closed set. In particular we derive the loop equations for the $n = 2^k$ matrix model recently proposed to describe the coupling of Two-dimensional quantum gravity to conformal matter with $c > 1$. 
1 Introduction

The last three years have seen a rapid development in the knowledge relative to two-dimensional quantum gravity coupled to conformal matter with $c \leq 1$. This has been possible by the use of matrix model techniques and the large $N$ expansion, combined with the double scaling limit [1]. Very recently a new interpretation of the double scaling limit as a finite size rescaling has emerged [2][3] based on the renormalization group applied to large $N$ models [4][5].

Although the new techniques have provided a very deep understanding of two-dimensional quantum gravity coupled to $c \leq 1$ conformal matter, a barrier has emerged at $c = 1$, avoiding the continuation of the previous results beyond this point. The same barrier appears in the continuum treatment of non-critical strings. [6]

Various approaches to surpass this problem have been suggested. Among them we mention the use of reduced models to discuss the d-dimensional matrix model [7] which is the right model to discuss the d-dimensional string. Very recently we have obtained the loop equations for this model for arbitrary $N$ and show that they form a closed set in leading order of large $N$ [8]. For the moment they have not been solved but we see them as an important exact result that may permit to cross the barrier at $c = 1$.

Last year a $2^n$-Hermitian matrix model in zero dimensions have been proposed to describe a system with $c = n/2$ coupled to 2d gravity [9]. So a solution of this model is another way to cross the barrier.

Four years ago we obtained the loop equations for the zero dimensional two-matrix model [10], extending the solution of Metha for the partition function [11]. Last year we were able to solve these equations completely [12]. See also [13].

Motivated by these recent progress in the solution of the Two matrix model, in the present work we will derive the loop equations for the n-Hermitian matrix model in any space-time dimension $d$ and for arbitrary $N$ and show that they form a closed set in leading order of large $N$. They are a consequence of the Schwinger-Dyson equations of the model and the type of coupling among the different matrices. The couplings we consider include the $q$-state Potts matter fields coupled to Two Dimensional Gravity of [9] as a particular case.

This paper is organized as follows: In section 2 we review the Hidden BRST method [14] and use it to derive the loop equations in section 3.
section 4 we discuss in more detail the zero dimensional case. In particular, we compare our equations with the two matrix equations of [10]. In section 5 we present our conclusions and comment on future work.

2 Hidden BRST Symmetry and Schwinger-Dyson Equations

The usual method to derive Schwinger-Dyson equations (SDe) involves using the invariance of the path integral measure under field translations. Invariances of the action itself are not relevant for this derivation. In this section we review an alternative derivation which has been proposed some years ago [14]: By making use of a BRST symmetric extension of any action $S$, we can derive all SDe’s as BRST supersymmetric Ward identities of the new action.

To simplify the presentation, consider the path integral describing one bosonic field $\varphi(x)$:

$$Z = \int [d\varphi] e^{-S(\varphi)}. \quad (1)$$

The functional measure $[d\varphi]$ in the last equation is invariant under field translations:

$$\varphi(x) \rightarrow \varphi(x) + \epsilon(x). \quad (2)$$

The invariance of the functional measure implies the following statement:

$$\int [d\varphi] \frac{\delta [Fe^{-S(\varphi)}]}{\delta \varphi(x)} = 0 \quad (3)$$

From (3) we get

$$\langle \frac{\delta F}{\delta \varphi(x)} - F \frac{\delta S}{\delta \varphi(x)} \rangle = 0 \quad (4)$$

for any $F = F(\varphi)$. These are the SDe’s for the theory. Notice that, in general, $S$ is not invariant under (2) and that this lack of invariance plays no role in the derivation of the SDe’s We now introduce auxiliary fermionic variables $\psi(x), \bar{\psi}(x)$ and insert a trivial factor of unity

$$1 = \int [d\bar{\psi}][d\psi] e^{-\int dx \bar{\psi}(x)\psi(x)}$$
inside the partition function. The resulting action $\bar{S}$ is invariant under the following transformation:

$$\bar{S}[\varphi, \psi, \bar{\psi}] = S[\varphi] + \int dx \bar{\psi}(x)\psi(x)$$  \hspace{1cm} (5)

$$\delta \varphi(x) = \psi(x)$$  \hspace{1cm} (6)

$$\delta \psi(x) = 0$$  \hspace{1cm} (7)

$$\delta \bar{\psi}(x) = -\delta S/\delta \varphi(x)$$  \hspace{1cm} (8)

Associated with the BRST-like symmetry, (6)–(8) is a set of Ward identities; for example, unbroken BRST implies:

$$\langle \delta [F(\varphi)\bar{\psi}(y)] \rangle = 0,$$

which is

$$\langle \int dx \frac{\delta F}{\delta \varphi(x)} \psi(x)\bar{\psi}(y) - F \frac{\delta S}{\delta \varphi} \rangle = 0.$$

Computing the average with respect to the fermionic variables we recover eq. (4). Notice that the BRST transformation we have just defined commutes with any symmetry the original action $S$ may have. In particular, if the original action is invariant under a $U(N)$ group, the BRST symmetry will commute with this group also and then its implications will be true order by order in the $1/N$ expansion.

In the next section we use the Schwinger-Dyson BRST symmetry to derive the loop equations for the $d$-dimensional $n$-Hermitian matrix model.

## 3 Loop Equations

In this section we will derive the loop equations for the model defined by the action:

$$S = \int d^dx \sum_{k,l} tr \left( \frac{1}{2} d_{ki} \partial_\mu M(x,k) \partial_\mu M(x,l) + V_l(M(x,l)) - \frac{1}{2} \sum_{i \neq j} c_{ij} tr M(x,i) M(x,j) \right)$$  \hspace{1cm} (9)

$M(x,l)$ is an $N \times N$ Hermitian matrix defined on the (euclidean) space-time point $x$, and $V_l$ is a local function of $M(x,l)$.  

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This action includes the systems studied in [9] (we follow their notation) if we take:

\[
d_{\tau\tau'} = \frac{1}{(2\sinh 2\beta)^n} \hat{\tau} \hat{\tau'} \exp \beta \tau \cdot \tau'
\]

\[
\tau \cdot \tau' = \sigma_1 \sigma'_1 + \sigma_2 \sigma'_2 + \ldots + \sigma_n \sigma'_n
\]

\[
\hat{\tau} = \sigma_1 \sigma_2 \ldots \sigma_n
\]

\[
c_{\tau\tau'} = d_{\tau\tau'}
\]

where \(\sigma = 1, 2, \ldots, q\), for the q-state Potts matter fields coupled to gravity.

The extended action and the BRST transformation are:

\[
S_{\text{ext}} = S + \int d^d x \sum_{l} \text{tr} \psi(x,l) \bar{\psi}(x,l)
\]

\[
\delta M(x,l) = \psi(x,l)
\]

\[
\delta \psi(x,l) = 0
\]

\[
\delta \bar{\psi}(x,l) = -\frac{\delta S}{\delta M(x,l)}
\]

Introduce the following representation of \(M(x,l)\):

\[
M(x,l) = \sum_{\alpha=1}^{N} m_{\alpha}(x,l) T^\alpha(x,l),
\]

where the projectors of the matrix \(M(x,l)\) satisfy:

\[
\sum_{\alpha=1}^{N} T^\alpha(x,l) = 1
\]

\[
\text{tr} T^\alpha(x,l) T^\beta(x,l) = \delta_{\alpha\beta}
\]

\[
T^\alpha(x,l) T^\beta(x,l) = \delta_{\alpha\beta} T^\alpha(x,l).
\]

If \(\delta\) denotes the BRST variation (see last section), we get the following BRST transformation for the projectors \(T^\alpha\) and eigenvalues \(m_{\alpha}\) (see the Appendix of [8]):

\[
\delta T^\alpha(x,l) = \sum_{\beta \neq \alpha} \frac{T^\alpha(x,l) \psi(x,l) T^\beta(x,l) + T^\beta(x,l) \psi(x,l) T^\alpha(x,l)}{m_{\alpha}(x,l) - m_{\beta}(x,l)}
\]

\[
\delta m_{\alpha}(x,l) = \text{tr} \psi(x,l) T^\alpha(x,l).
\]
Consider the following functional of $\lambda(x, l)$

$$K_{ij} = \int d^d x \sum_{l} \sum_{\alpha=1}^{N} \lambda_{\alpha}(x, l) T_{\alpha}(x, l)_{ij}. \quad (24)$$

The basic object we will use to write the loop equations is:

$$u[\lambda] = e^{iK} \quad (25)$$

It fulfills the identity:

$$\frac{\delta u[\lambda]}{\delta \lambda_{\alpha}(x, l)} = \int_0^1 dt e^{itK} i T_{\alpha}(x, l) e^{i(1-t)K}, \quad (26)$$

which implies:

$$\frac{\delta tru[\lambda]}{\delta \lambda_{\alpha}(x, l)} = itr T_{\alpha}(x, l) u[\lambda] \quad (27)$$

The trace is taken with respect to the internal indices only. From $u[\lambda]$ we can compute all symmetric combinations of products of $T_{\alpha}$. For instance:

$$\frac{\delta^2 u[\lambda]}{\delta \lambda_{\beta_1}(x_1, l_1) \delta \lambda_{\beta_2}(x_2, l_2)} \bigg|_{\lambda=0} = -T_{\beta_2}(x_2, l_2) T_{\beta_1}(x_1, l_1) T_{\beta_2}(x_2, l_2) \frac{1}{2} \quad (28)$$

The Schwinger-Dyson equation is:

$$\langle tr[u[\lambda] \bar{\psi}(x, l)] \rangle = 0 \quad (29)$$

i.e.

$$\langle tr[\delta u[\lambda] \bar{\psi}(x, l) + tr[u[\lambda] \delta \bar{\psi}(x, l)] \rangle = 0. \quad (30)$$

Since

$$\delta \bar{\psi}(x, l) = \sum_k d_{lk} M(x, k) + \sum_{j \neq l} c_{lj} M(x, j) - \Delta_{l}(M) \quad (31)$$

where

$$\Delta_{l}(M) = V_{l}'(M(x, l)). \quad (32)$$

We get:

$$tru[\sum_k d_{lk} \Box M(x, k) - \Delta_{l}(M) + \sum_{j \neq l} c_{lj} M(x, j)] =$$

$$\sum_{\alpha} \sum_k (d_{lk}(m_{\alpha}(x, k) \Box x + 2 \partial_{\mu} m_{\alpha}(x, k) \partial_{\mu} + \Box x m_{\alpha}(x, k)) - \Delta_{\alpha}(x, k) \delta_{ik}) tru T_{\alpha}(x, k) +$$

$$\sum_{j \neq l} c_{lj} m_{\alpha}(x, j) tru T_{\alpha}(x, j)]. \quad (33)$$
We have that:
\[ \text{tru} T^\alpha(x, l) = -i \frac{\delta \text{tru}[\lambda]}{\delta \lambda_\alpha(x, l)}, \] (34)

and
\[ \delta u = \int_0^1 dt e^{it \sum_i \sum_{\alpha=1}^N \lambda_\alpha(x, l) T^\alpha(x, l)} \sum_{\alpha} \sum_{l} \lambda_\alpha(x, l) \sum_{\beta \neq \alpha} T^\alpha(x, l) \psi(x, l) T^\beta(x, l) \] (35)

\[ \frac{T^\alpha(x, l) \psi(x, l) T^\beta(x, l) + T^\beta(x, l) \psi(x, l) T^\alpha(x, l)}{m_\alpha(x, l) - m_\beta(x, l)}, \] (36)

We compute the fermionic average to get:
\[ \langle \delta u \bar{\psi}(x, l) \rangle = \langle \frac{i}{t} \sum_{\beta \neq \alpha} \int_0^1 \frac{\lambda_\alpha(x, l) - \lambda_\beta(x, l)}{m_\alpha(x, l) - m_\beta(x, l)} \text{tru}[t \lambda] T^\beta(x, l) \text{tru}[(1-t) \lambda] T^\alpha(x, l) \rangle. \] (37)

Since \( T^\alpha(x, l) \) appears only once in the trace, we can express it as a derivative of \( u \):
\[ \text{tru}[t \lambda] T^\beta(x, l) = \frac{-i}{t} \frac{\delta \text{tru}[t \lambda]}{\delta \lambda_\beta(x, l) (1-t) \lambda} \] (38)

We get the following loop equation:
\[ \langle \sum_{\beta \neq \alpha} \int_0^1 dt \lambda_\alpha(x, l) - \lambda_\beta(x, l) \frac{1}{m_\alpha(x, l) - m_\beta(x, l)} t(1-t) \delta \lambda_\beta(x, l) \delta \lambda_\alpha(x, l) \rangle = \]
\[ -\langle \sum_{\alpha} \sum_{j \neq l} c_{ij} m_\alpha(x, j) \frac{\delta \text{tru}[\lambda]}{\delta \lambda_\alpha(x, j)} \rangle + \]
\[ \sum_k d_{lk} (m_\alpha(x, k) \Box_x + 2 \partial_\mu m_\alpha(x, k) \partial_\mu + \Box_x m_\alpha(x, k) - \Delta_\alpha(x, k) \delta_{lk}) \frac{\delta \text{tru}[\lambda]}{\delta \lambda_\alpha(x, k)} \rangle, \] (39)

which is valid to any order in the \( 1/N \) expansion.

We can simplify the equation if we restrict ourselves to the leading order in \( 1/N \). Then we can apply Witten’s factorization property \( \text{Witten} \) to the loop equations to prove that:
\[ F(M) \sim \langle F(M) \rangle + O(N^{-a}), \quad a > 0 \]
for any $U(N)$-invariant $F$. In particular, we obtain that:

$$m_\alpha(x, l) \sim \langle m_\alpha(x, l) \rangle = \langle m_\alpha(0, l) \rangle.$$ 

That is, in leading order of large $N$ the loop equation becomes a closed system for $u[\lambda]$: 

$$
\sum_{\beta \neq \alpha} \int_0^1 dt \frac{\lambda_\alpha(x, l) - \lambda_\beta(x, l)}{m_\alpha(l) - m_\beta(l)} \frac{1}{t(1 - t)} \frac{\delta tru(t\lambda)}{\delta \lambda_\beta(x, l)} \frac{\delta tru[(1 - t)\lambda]}{\delta \lambda_\alpha(x, l)} = 
$$

$$- \sum_\alpha \left( \sum_{j \neq l} c_{ij} m_\alpha(j) \frac{\delta tru[\lambda]}{\delta \lambda_\alpha(x, j)} + \sum_k d_{ik} (m_\alpha(k) \Box_x - \Delta_\alpha(k) \delta_{ik}) \frac{\delta tru[\lambda]}{\delta \lambda_\alpha(x, k)} \right), \quad (39)
$$

We could derive this closed system because we chose to write the Schwinger-Dyson (loop) equations in terms of $T^\alpha(x, l)$ rather than $M(x, l)$. Also by using the BRST Schwinger-Dyson transformation we could avoid this difficult change of variables in the path integral and the (highly) non-trivial measure in the $(m_\alpha, T^\alpha)$ basis.

The loop equations must be solved subject to the following boundary conditions:

$$
\frac{tru}{\delta \lambda_\alpha(x, l)} \bigg|_{\lambda=0} = N 
$$

$$
\frac{\delta tru}{\delta \lambda_\alpha(x, l)} = i 
$$

$$
\frac{\delta^2 tru}{\delta \lambda_\alpha(x, l) \delta \lambda_\beta(x, l)} \bigg|_{\lambda=0} = -\delta_{\alpha\beta} \quad (40)
$$

$u = \text{constant (}\lambda \text{ independent})$ is a particular solution of the loop equations, but it does not satisfy the boundary conditions. We must look for a non-trivial solution.

We get an equivalent formulation of the loop equations noticing that only the derivative of $tru$ appears in the loop equations. So identify:

$$
v[\lambda]_\alpha(x, l) = \frac{\delta tru}{\delta \lambda_\alpha(x, l)}. \quad (41)
$$
So that the loop equation is equivalent to the following system of equations:

\[
\sum_{\beta \neq \alpha} \int_0^1 \frac{dt}{t(1-t)} v[\lambda]_{\beta}(x,l)v[(1-t)\lambda]_{\alpha}(x,l) = - \sum_{\alpha} \left( \sum_{j \neq l} c_{ij} m_{\alpha}(j) v[\lambda]_{\alpha}(x,j) + \sum_k d_{lk}(m_{\alpha}(k) \Box_x - \Delta_{\alpha}(k) \delta_{lk}) v[\lambda]_{\alpha}(x,k) \right) + \sum_{k} d_{lk}(m_{\alpha}(k) \Box_x - \Delta_{\alpha}(k) \delta_{lk}) v[\lambda]_{\alpha}(x,k) \]

(42)

\[
\frac{\delta v[\lambda]_{\alpha}(x,l)}{\delta \lambda_{\beta}(y,j)} = \frac{\delta v[\lambda]_{\beta}(y,j)}{\delta \lambda_{\alpha}(x,l)}
\]

(43)

\[
v[\lambda]_{\alpha}(x,l) \big|_{\lambda=0} = i
\]

(44)

\[
\frac{\delta v[\lambda]_{\alpha}(x,l)}{\delta \lambda_{\beta}(x,l)} \big|_{\lambda=0} = -\delta_{\alpha\beta}
\]

(45)

Notice that:

\[
tr M(x_1, l_1)M(x_2, l_2)\ldots M(x_n, l_n) = \sum_{\alpha_1 \ldots \alpha_n} m_{\alpha_1} \ldots m_{\alpha_n} (l_n) tr T^{\alpha_1}(x_1, l_1) \ldots T^{\alpha_n}(x_n, l_n)
\]

\[
= \frac{1}{N} \sum_{\alpha_1 \ldots \alpha_n} m_{\alpha_1} \ldots m_{\alpha_n} [tr T^{\alpha_1}(x_1, l_1) \ldots T^{\alpha_n}(x_n, l_n) + \text{all permutations of indices } \alpha] (46)
\]

Then it is enough to know \(tr u[\lambda]\) to get all correlations of the matrix \(M(x, l)\) that are symmetric under permutations of the space-time points \((x_i, l_i)\).

A continuum version of the large-\(N\) loop equation is obtained by the usual manipulations. That is we introduce the functions \(\lambda(z, x, l)\) and \(m(z, l)\) defined for \(0 \leq z \leq 1\) and the density of eigenvalues \(\rho_t\), and make the following identifications:

\[
\lambda_{\alpha}(x, l) = \lambda(\frac{\alpha}{N}, x, l)
\]

(47)

\[
m_{\alpha}(l) = \sqrt{N}m(\frac{\alpha}{N}, l)
\]

(48)

\[
\rho_t(m_t) = \frac{dx_t}{dm_t}
\]

(49)

\[
\int dm_t \rho_t(m_t) = 1.
\]

(50)
The large-$N$ loop equation becomes:

\[
P \int dm_1 dm_2 \rho_l(m_1) \rho_l(m_2) \int_0^1 dt \frac{\lambda(m_1, y, l) - \lambda(m_2, y, l)}{m_1 - m_2} \frac{1}{t(1-t)} \times \frac{\delta tru[t\lambda]}{\delta \lambda(m_2, y, l)} \frac{\delta tru[(1-t)\lambda]}{\delta \lambda(m_1, y, l)} = - \sum_{j \neq l} c_{ij} \int dm \rho_j(m) m \frac{tru[\lambda]}{\delta \lambda(m, y, j)} \\
- \int dm \rho_l(m) (m \square_y - \Delta_t(m)) \frac{tru[\lambda]}{\delta \lambda(m, y, l)}; \quad (51)
\]

\(P\) stands for the principal value of the integral over \(m_i\).

4 The Zero dimensional case

In this section we discuss the zero dimensional case. The equations are:

\[
\sum_{\beta \neq \alpha} \int_0^1 dt \frac{\lambda_\alpha(l) - \lambda_\beta(l)}{m_\alpha(l) - m_\beta(l)} \frac{1}{t(1-t)} \frac{\delta tru[t\lambda]}{\delta \lambda_\beta(l)} \frac{\delta tru[(1-t)\lambda]}{\delta \lambda_\alpha(l)} = \\
- \sum_{\alpha} \left[ \sum_{j \neq l} c_{ij} m_\alpha(j) \frac{\delta tru[\lambda]}{\delta \lambda_\alpha(j)} - \Delta_\alpha(l) \frac{\delta tru[\lambda]}{\delta \lambda_\alpha(l)} \right]. \quad (52)
\]

We want to check that we reproduce the equations of [10] for \(n = 2\). In this case we have:

\[
u[\lambda] = 1 + i \sum_{\alpha,l} \lambda_\alpha(l) + \frac{\lambda^2}{2} \left\{ \sum_{\alpha} (\lambda_\alpha(1)^2 + \lambda_\alpha(2)^2) + 2 \sum_{\alpha,\beta} \lambda_\alpha(1) \lambda_\beta(2) tr(T_\alpha(1) T_\alpha(2)) \right\} + \frac{\lambda^3}{6} \left\{ \sum_{\alpha} (\lambda_\alpha(1)^3 + \lambda_\alpha(2)^3) + 3 \sum_{\alpha,\beta} (\lambda_\alpha(1)^2 \lambda_\beta(2) + \lambda_\alpha(1) \lambda_\beta(2)^2) tr(T_\alpha(1) T_\beta(2)) \right\} + o(\lambda^4) \quad (53)
\]

Inserting this into the loop equations we readily get the system of [10]. In the same way we can get the equations for the higher correlators of \(T_\alpha(1), T_\beta(2)\).

It is, of course, a very interesting problem to see if the methods of [12] can be extended to the n-matrix chain.
5 Discussion and Open Problems

We have been able to derive the loop equations for the d-dimensional n-matrix model. They are a consequence of the Schwinger-Dyson equations satisfied by the Green functions of the model. Due to the factorization property of U(N)-invariant operators in large N the loop equations form a closed set that may be the starting point for a non-perturbative description of the system. In particular our result applies to the q-state Potts matter fields coupled to two dimensional gravity that have been previously proposed to describe the non-critical string with \( c > 1 \).

There are several interesting open problems. It is known that the loop equations for \( d = 0 \) models can be realized as Virasoro\(^1\) and \( W_3 \)\(^2\) constraints on the free energy of the system. We wonder whether this is true for the present case too. In addition the loop equations could be used as the starting point for a different approximation to the Physics of the model, perhaps on the lines suggested by the solution of the Two matrix model in \(^1\).

Acknowledgements

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References

[1] For a review of matrix models, see P. Ginsparg and G. Moore, Yale and Los Alamos preprint YCTP-P23-92, LA-UR-92-3479(1993) hep-th/9304011; P. Di Francesco, P. Ginsparg and J. Zinn-Justin, "2D Gravity and Random Matrices", hep-th/9306153.

[2] P.H. Damgaard and U.H. Heller, "On spin and matrix models in the complex plane", CERN-TH-6956-93, hep-lat/9307016.

[3] S. Hikami, " Finite N analysis of matrix models for n-Ising spin on a random surface", hep-th/9309003.

[4] E. Brezin and J. Zinn-Justin, Phys. Lett. B288(1992)54.
[5] J. Alfaro and P.H. Damgaard, Phys. Lett. B289(1992)342.

[6] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. A3(1988)819.

[7] L. Alvarez-Gaumé, J. Barbón and C. Crnković, Nucl. Phys. B394(1993)383.

[8] J. Alfaro, ”Loop Equations for the d-dimensional One Hermitian Matrix model”, CERN-TH-6966/93, [hep-th 9308051], to be published in Phys. Lett. B.

[9] E. Brezin and S. Hikami, Phys. Lett. B283(1992)203.

[10] J. Alfaro and J.C. Retamal, Phys. Lett. B222(1989)429.

J. Alfaro, “Hidden BRST and large N”, Lecture at the Cargèse Workshop on Probabilistic Methods in Quantum Field Theory and Quantum Gravity, eds. P.H. Damgaard, H. Huffel and A. Rosemblum (Plenum Press, New York, 1990), p. 279;

id., “Hidden BRST Symmetry Schwinger-Dyson Equations and Large-N”, Lecture at the XIV Johns Hopkins Workshop on Nonperturbative methods in Low Dimensional Quantum Field Theories, eds. G. Domokos, Z. Horvath and S. Kovesi-Domokos (World Scientific, Singapore, 1991), p. 481.

[11] C. Itzykson and J.B. Zuber, J. Math. Phys. 21(1980)411; M.L. Mehta, Commun. Math. Phys. 79(1981)327.

[12] J. Alfaro, Phys. Rev. D47(1993)4714.

[13] M. Staudacher, Phys. Lett. B305(1993)332.

[14] J. Alfaro and P.H. Damgaard, Phys. Lett. 222B(1989)425 and Ann. Phys. 202(1990)398.

[15] E. Witten, in Recent Developments in Gauge Theories, Proceedings of the NATO Advanced Study Institute, Cargese, 1979, eds. G. ’t Hooft et al. (Plenum, New York, 1980). In the context of stochastic quantization this point has been discussed in J. Alfaro, Phys. Lett. 148B(1984)157. See also id., Prog. Theor. Phys. Suppl. 111(1993)401.
[16] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B348(1991)435; M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. A6(1991)1385.

[17] E. Gava and K.S. Narain, Phys. Lett. B263(1991)213.