Asymptotic entanglement and Lindblad dynamics: a perturbative approach

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Abstract

We consider an open bipartite quantum system with dissipative dynamics generated by $\hat{L}_\epsilon = \hat{L}_0 + \epsilon \hat{L}_1$, where $\hat{L}_0, \hat{L}_1$ are generators of Lindblad type and $0 < \epsilon \ll 1$. In order to study the entanglement of the stationary states of $\hat{L}_\epsilon$, we develop a perturbative approach and apply it to the physically significant case when $\hat{L}_0$ generates a reversible unitary dynamics, while $\hat{L}_1$ is a purely dissipative perturbation.

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1. Introduction

An open quantum system dynamics of Lindblad type \cite{1,2} is an effective description of the action of an environment $E$ on a quantum system $S$ weakly coupled to it. In general, the environment acts as a source of dissipation and noise; in spite of this, decoherence is not the only possible consequence. If suitably engineered, the coupling with the environment may also generate coherence and even entanglement \cite{3}, this possibility depending on the trade-off between dissipative effects and environment induced mixing \cite{4–10}.

Of particular interest is under what conditions the presence of an environment may induce convergence to asymptotic states with definite entanglement properties \cite{11–14}. In fact, controlling the coupling to the environment could then be used for preparing states with definite entanglement content \cite{15}. From this viewpoint, it is of great importance to know (1) the invariant states of a given Lindblad dynamics and (2) whether any initial state converges asymptotically to some stationary state. Apart from some older \cite{16–18} and more recent results \cite{19–21,23}, a full characterization of the asymptotic properties of open quantum systems and their asymptotic behavior is still to be achieved.

In the following, we will focus upon the following scenario: consider two finite-level systems $S_1$ and $S_2$, not directly interacting with each other, whose reversible, unitary dynamics
is generated by a Hamiltonian $H = H_1 + H_2$ via the generator $L_0[\rho] = -i[H, \rho]$. If weakly coupled to the same environment, on a long time scale, they undergo an open, dissipative dynamics generated by $L_\varepsilon = L_0 + \varepsilon L_1$, where $L_1$ is a generator of the Lindblad form and $\varepsilon$ measures the weakness of the coupling to the environment. In general, the addition of the perturbation term $\varepsilon L_1$ diminishes the number of invariant states with respect to those of $L_0$; however, in finite dimension, at least one invariant state will always survive and, by continuity, will be close to them: the issue is whether the remaining invariant states may be entangled or not.

Suppose the spectrum of the Hamiltonian $H = H_1 + H_2$ be non-degenerate; then $L_0[\rho_0] = 0$ only if $\rho$ is a separable state. Intuitively, if such states are well inside the closed convex subset of separable states, no dissipative perturbation $L_1$ could provide entangled states $\rho_\varepsilon$ such that $L_\varepsilon[\rho_\varepsilon] = 0$. Indeed, by continuity, such asymptotic states are perturbations of those of $L_0$, namely $\rho_\varepsilon = \rho_0 + \varepsilon \rho_1 + o(\varepsilon)$, and thus remain separable if $\rho_0$ is separable. In contrast, for separable stationary states $\rho_0$ on the boundary of the subset of separable states, it should be possible to construct entangled $\rho_\varepsilon$ by suitably engineered, small dissipative perturbations.

In the following, we give mathematical ground to these expectations by developing a systematic perturbation expansion of the states $\rho_\varepsilon$ that are invariant under generators of the form $L_\varepsilon = L_0 + \varepsilon L_1$ where $L_0$ and $L_1$ are generic Lindblad-type generators.

2. Perturbation theory

Let $S$ denote a $d$-level system with observables $X$ from the full matrix algebra $M_d(\mathbb{C})$ and states (density matrices) $\rho$ from the convex subset $S(S) \subset M_d(\mathbb{C})$ of positive matrices of trace 1. If $S$ is weakly coupled to its environment $E$, its time evolution is conveniently approximated by a Markovian Lindblad-type dynamics: $\rho \mapsto \rho_t = \gamma_t[\rho] = \exp(tL)[\rho]$ [1, 2] where $L$ is the generator of the semigroup of trace-preserving, completely positive maps $\gamma_t$, $\gamma_0 \circ \gamma_t = \gamma_t$. It incorporates in an effective manner the noise and dissipation due to the environment via a master equation of the form

$$\frac{\partial}{\partial t}\rho(t) = L[\rho(t)] = -i[H, \rho(t)] + D[\rho(t)],$$

where $M_d(\mathbb{C}) \ni H = H^\dagger$, while

$$D[\rho] = \sum_{\alpha} \left( h_\alpha \rho h_\alpha^\dagger - \frac{1}{2} \{ h_\alpha^\dagger h_\alpha, \rho \} \right).$$

where $h_\alpha, \sum_{\alpha} h_\alpha^\dagger h_\alpha \in M_d(\mathbb{C})$.

We shall denote by $S_\gamma \subset S$ the subset of stationary states of $\gamma_t$: they satisfy $L[\rho] = 0$ and form a convex subset of $K(\mathbb{L})$, the kernel of the generator.

The time evolution generated by (1) affects the states of the system, while its observables evolve according to the semigroup of dual maps $\gamma^T_t : X \mapsto X(t) = \gamma^T_t[X]$ generated by

$$\frac{\partial}{\partial t} X(t) = L^T_t[X(t)] = i[H, X(t)] + D^T_t[\rho(t)],$$

where

$$D^T_t[X] = \sum_{\alpha} \left( h_\alpha^\dagger X h_\alpha - \frac{1}{2} \{ h_\alpha^\dagger h_\alpha, X \} \right).$$

About the structure of the $\gamma_t$-invariant states, we have [16]
Proposition 1. The time average

\[ X \mapsto \mathbb{G}^T [X] = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathrm{d} \gamma_t^T [X] \]  

(5)
is a well-defined unital, completely positive map; its dual map \( \mathbb{G} : S \mapsto S \) defined by

\[ \text{Tr}(\mathbb{G}(\rho) X) = \text{Tr}(\rho \mathbb{G}^T [X]) \quad \forall X \in M_d(\mathbb{C}), \ \rho \in S \]  

(6)
is a completely positive, trace-preserving map which associates a state \( \rho \in S \) with a \( \gamma_t \)-invariant state: \( L \circ \mathbb{G}(\rho) = 0 \).

If \( \gamma_t \) possesses a faithful invariant state, that is a full-rank density matrix \( \rho^* = \gamma_t[\rho^*] \), then the \( \gamma_t^T \)-invariant observables \( X = \gamma_t^T [X] \) form a subalgebra \( M_\rho \subset M_d(\mathbb{C}) \) and \( \mathbb{G}^T \) is a conditional expectation onto \( M_\rho \):

\[ \mathbb{G}^T [Y_1 XY_2] = Y_1 \mathbb{G}^T [X] Y_2 \quad \forall Y_{1,2} \in M_\rho, \ X \in M_d(\mathbb{C}). \]  

(7)

Controlling the structure of \( S_\gamma \) and, in particular, whether an initial state \( \rho \in S \) converges to \( \mathbb{G}(\rho) \) is a complicated matter with some partial clues [16–18]. Recently, such an issue has become again the subject of study [19–21, 23] also because of its increasing importance in quantum information [15]. Concerning \( \gamma_t \)-invariant states, the following result was obtained in [22] (see also [23]).

Proposition 2. Let \( L \) be the generator of a Lindblad-type dynamics \( \gamma_t \); one can always construct orthogonal stationary states \( \rho_j \) of \( \gamma_t \); \( L[\rho_j] = 0 \) and \( \rho_j \rho_k = 0 \) unless \( j = k \).

The following ones are instances of some possible scenarios.

Example 1. Let \( \mathbb{L}[\rho] = -i[H, \rho] \), where the Hamiltonian \( H = \sum_{j=1}^d E_j |j\rangle \langle j| \) has a non-degenerate spectrum; then, the stationary states \( \rho_j \) of proposition 2 are the orthogonal one-dimensional eigen-projectors \( |j\rangle \langle j| \). For later application, we need to extend the map \( \mathbb{G} \) of proposition 1 to the whole matrix algebra \( M_d(\mathbb{C}) \); in the present case it reads

\[ \mathbb{G}_d [X] = \sum_{j=1}^d |jX|j\rangle \langle j|. \]  

(8)

Clearly, because of the oscillatory behavior, there is no tendency to equilibrium: \( \gamma_t[\rho] \) does not converge to \( \mathbb{G}(\rho) \) when \( t \to +\infty \).

Example 2. Let \( H = 0 \) in (1) and \( h_a = |\psi\rangle \langle \alpha| \) in (2), where \( \{|\alpha\rangle\} \) is an orthonormal basis of \( \mathbb{C}^d \). Then,

\[ \mathbb{L}[\rho] = |\psi\rangle \langle \psi| - \rho \Rightarrow \gamma_t[\rho] = e^{-t} \rho + (1 - e^{-t}) |\psi\rangle \langle \psi|. \]  

(9)

Hence, \( \rho = |\psi\rangle \langle \psi| \) is the only stationary state and all others converge to it asymptotically: \( \gamma_t[\rho] \to |\psi\rangle \langle \psi| \).

The corresponding map \( \mathbb{G} \) is given by (on \( M_d(\mathbb{C}) \))

\[ \mathbb{G}_d [X] = \text{Tr}(X) |\psi\rangle \langle \psi|. \]  

(10)

Example 3. Let \( H = 0 \) in (1) and \( h_1 = |\psi\rangle \langle 1| \) in (2), where \( ||\psi|| = 1, \{|\alpha\rangle\}_{a=1}^d \) is an orthonormal basis and \( 0 < |\psi_1| = ||\langle 1|\psi\rangle|| < 1 \). Then,

\[ \mathbb{L}[\rho] = -(1|\rho|1) |\psi\rangle \langle \psi| - \frac{1}{2} |1\rangle \langle 1| \rho - \frac{1}{2} |\rho|11. \]  

(11)

Setting \( X_{\alpha\beta} = \langle \alpha|X|\beta\rangle \) for all \( X \in M_d(\mathbb{C}), \mu = 1 - |\psi_1|^2 > 0 \), \( \psi_a = \langle \alpha|\psi\rangle, \ \alpha \neq 1 \), the equations

\[ \dot{\rho}_{11} = -\mu \rho_{11}, \quad \dot{\rho}_{1a} = \psi_a \psi_\alpha^* \rho_{11} - \frac{1}{2} \rho_{1a}, \quad \dot{\rho}_{a\beta} = \psi_a \psi_\beta^* \rho_{11} \]

are obtained.
can easily be solved yielding
\[
\begin{align*}
\rho_{11}(t) &= e^{-\mu t} \rho_{11}, \\
\rho_{1a}(t) &= e^{-\mu t/2} \rho_{1a} + \psi_{1} \psi_{a}^{*} \rho_{11} \frac{e^{-\mu t/2} - e^{-\mu t}}{\mu - 1/2}, \\
\rho_{a\beta}(t) &= \rho_{a\beta} + \frac{1 - e^{-\mu t}}{\mu} \rho_{11} \psi_{a} \psi_{\beta}^{*},
\end{align*}
\]
where \(\alpha, \beta\) in the second expression are fixed by the initial conditions. Then,
\[
\mathbb{G}[X] = \sum_{\alpha, \beta \geq 2} \left( X_{\alpha\beta} + \frac{X_{11}}{\mu} \psi_{\alpha} \psi_{\beta}^{*} \right) |\alpha\rangle \langle \beta|.
\]
All states such that \(\rho = Q \rho Q\), where \(Q = \sum_{\alpha \geq 2} |\alpha\rangle \langle \alpha|\), are \(\gamma_{t}\)-invariant; also, \(\gamma_{t}[\rho] \longmapsto \mathbb{G}[\rho]\).

Despite the abstract characterizations of [21, 23], the convex subset of stationary states is difficult to control in practice; we shall thus concentrate on understanding how the invariant states of a semigroup \(\gamma_{t}^{(0)}\) are modified by a perturbation \(\mathbb{L}_{1}\) of its Lindblad generator \(\mathbb{L}_{0}\). Concretely, we will investigate the set \(S_{\epsilon}\) of stationary states of Lindblad-type dynamics \(\gamma_{t}^{(\epsilon)}\) generated by \(\mathbb{L}_{\epsilon} = \mathbb{L}_{0} + \epsilon \mathbb{L}_{1}\), \(0 < \epsilon \ll 1\). By switching on the perturbation, the dimension of \(S_{\epsilon}\) decreases, but, proposition 1 ensures the existence of at least one stationary state.

**Lemma 1.** Consider the generator \(\mathbb{L}_{\epsilon} = \mathbb{L}_{0} + \epsilon \mathbb{L}_{1}\), where both \(\mathbb{L}_{0,1}\) are generators in the Lindblad form and the semigroups \(\gamma_{t}^{(0)}\) and \(\gamma_{t}^{(\epsilon)}\) generated by \(\mathbb{L}_{0}\) and \(\mathbb{L}_{\epsilon}\) are \(\mathbb{L}_{0} + \epsilon \mathbb{L}_{1}\). Let \(n(0)\) be the number of \(\gamma_{t}^{(0)}\)-invariant orthogonal density matrices and \(n(\epsilon)\) that of \(\gamma_{t}^{(\epsilon)}\)-invariant orthogonal density matrices for \(0 < \epsilon \ll 1\); then \(n(0) \geq n(\epsilon)\).

**Proof.** From proposition 2, one can always choose density matrices such that \(\mathbb{L}_{\epsilon}[\rho_{j}(\epsilon)] = 0\) and \(\text{Tr}(\rho_{j}(\epsilon) \rho_{k}(\epsilon)) = 0\) for \(j \neq k\). In finite dimension, eigenvalues and eigen-projectors are continuous in \(\epsilon\); therefore, should \(\rho_{j}(\epsilon) \neq \rho_{k}(\epsilon)\) merge as \(\epsilon \to 0\), the continuity of the Hilbert–Schmidt scalar product would be violated.

Because of finite dimensionality, the solutions can always be expressed as converging series in powers of \(\epsilon\)
\[
\rho_{\epsilon} = \sum_{n \geq 0} \epsilon^{n} \rho_{n},
\]
where the operators \(\rho_{n}\) must solve the iterative procedure
\[
(\mathbb{L}_{0} + \epsilon \mathbb{L}_{1})[\rho(\epsilon)] = \mathbb{L}_{0}[\rho_{0}] + \sum_{n=1}^{\infty} \epsilon^{n} (\mathbb{L}_{0}[\rho_{n}] + \mathbb{L}_{1}[\rho_{n-1}]) = 0,
\]
whence
\[
\mathbb{L}_{0}[\rho_{0}] = 0, \quad \mathbb{L}_{0}[\rho_{n}] = -\mathbb{L}_{1}[\rho_{n-1}] \quad n \geq 1,
\]
where \(\rho_{0}\) is a stationary state of \(\gamma_{t}^{(0)}\). Also, since \(\text{Tr}(\rho(\epsilon)) = 1\), it follows that \(\text{Tr}(\rho_{n})\) must vanish at all orders. In the following, we discuss when \(\rho_{n} = -\mathbb{L}_{0}^{-1}[\mathbb{L}_{1}[\rho_{n-1}]]\) are acceptable solutions.

**Definition 1.** Let \(\mathbb{F} = \text{id} - \mathbb{G}\), where \(\mathbb{G}\) is as in proposition 1; since \(\mathbb{G}\) is trace-preserving, the image of \(\mathbb{M}_{d}(\mathbb{C})\) by \(\mathbb{F}\) consists of traceless matrices: \(\text{Tr}(\mathbb{F}[X]) = 0\) for all \(X \in \mathbb{M}_{d}(\mathbb{C})\).

**Lemma 2.** \(\mathbb{L}^{-1}\) can be defined as a map from \(\mathbb{F}[\mathbb{M}_{d}(\mathbb{C})]\) into itself.
Proof. Note that $\mathbb{G}$, as a time average, maps into the kernel of $L$ and leaves it invariant; thus, from $X = \mathbb{G}[X] + \mathbb{F}[X]$, it follows that $L^{-1}[X]$ is well defined on $M_d(\mathbb{C}) \ni X \neq 0$ only if $\mathbb{G}[X] = 0$. Then, $L^{-1}$ is constructed as a linear map from the range of $\mathbb{F}$ into itself such that $L \circ L^{-1} = L^{-1} \circ L = \text{id}$ on $\mathbb{F}(M_d(\mathbb{C}))$. This guarantees that $L^{-1}[0] = 0$; indeed, consider $Z = L^{-1}[X] - L^{-1}[Y]$, with $X = L[V] = Y = L[V + W]$, $W \neq 0$, $L[W] = 0$; then, $\mathbb{F}[Z] = \mathbb{F} \circ L^{-1}[X] - \mathbb{F} \circ L^{-1}[Y] = 0$. \hfill $\square$

Example 4. In the case of example 1, where $L[\rho] = -i[H, \rho]$ and $H$ is non-degenerate, $\mathbb{G}[X] = 0$ if and only if $\langle j|X|j \rangle = 0$ for all $j$; one thus obtains

$$L^{-1}[X] = \sum_{j \neq k} \langle j|X|k \rangle \frac{E_j - E_k}{X}.$$  

Example 5. In the case of example 2, $\mathbb{G}[X] = 0$ if and only if $\text{Tr}(X) = 0$; one can verify that, on traceless matrices,

$$L^{-1}[X] = -X.$$  

Example 6. Finally, in example 3, $\mathbb{G}[X] = 0$ if and only if $X$ is of the form

$$X = X_{11}|1\rangle\langle 1| + \sum_{a \geq 2} (X_{1a}|1\rangle\langle a| + X_{a1}|a\rangle\langle 1|) = \frac{X_{11}}{\mu} \sum_{a, \beta \geq 2} \psi_a \psi_\beta^* |\alpha\rangle\langle \beta|,$$

where the only free entries are $X_{1a}$ and $X_{a1}$, $\alpha \geq 1$. Then,

$$L^{-1}[X] = -\frac{1}{\mu} (X_{11}|1\rangle\langle 1| + \sum_{a, \beta \geq 2} X_{a\beta} |\alpha\rangle\langle \beta|)$$

$$- \sum_{a \geq 2} \frac{2}{\mu} \left( X_{aa} - \frac{2}{\mu} \psi_a \psi_a^* \right) |1\rangle\langle 1| + \left( X_{a1} - \frac{2}{\mu} \psi_a \psi_1^* \right) |\alpha\rangle\langle 1|. 

$$  

(17)

From the previous lemma, it follows that, in order to solve for $\rho_n$ in (15) by inverting $L_0$, one has to ensure that $\mathbb{G}_0[L_1[\rho_{n-1}]] = 0$ for all $n \geq 2$. The following lemma gives a sufficient condition for this to be true.

Lemma 3. Given $\mathbb{L}_c = L_0 + \varepsilon L_1$, if $L_0[\rho_0] = 0$ for a unique state $\rho_0$ and $L_1[\rho_0] \neq 0$, then $\mathbb{L}_c[\rho_0] = 0$ for a unique $\rho_c$ given by

$$\rho_c = \sum_{n=0}^{\infty} (-\varepsilon)^n (\mathbb{L}_0^{-1} \circ \mathbb{L}_c)^n[\rho_0] = \frac{1}{1 + \varepsilon \mathbb{L}_0^{-1} \circ \mathbb{L}_c}[\rho_0].$$  

(19)

Proof. As $\mathbb{G}_0$ maps into the kernel of $L_0$ and is trace-preserving, from the hypothesis of the lemma it follows that $\mathbb{G}_0 \circ L_1[\rho_0] = \lambda \rho_0$. Then, $\lambda = \text{Tr}(\mathbb{G}_0 \circ L_1[\rho_0]) = \text{Tr}(L_1[\rho_0]) = 0$ implies $\mathbb{G}_0 \circ L_1[\rho_0] = 0$ so that $L_0$ can be inverted on $L_1[\rho_0]$ and one can solve the first recursive relation in (15). As $L_0^{-1}$ maps into $\mathbb{F}_0[M_d(\mathbb{C})]$ where $F_0 = 1 - \mathbb{G}_0$ and $\mathbb{G}_0$ is the trace-preserving map in (5) corresponding to $L_0$, then $\text{Tr}(\rho_1) = 0$. Iterating this argument yields the result. \hfill $\square$

Example 7. For $L_0$ as in example 2, there is only one invariant state so that lemma 3 applies. Furthermore, since $L_0^{-1}[X] = -X$ (19) yields $\rho_c = (1 - \varepsilon L_1)^{-1}[\rho]$. In such a case of a unique invariant state under $L_0$, we can make some preliminary considerations about the entanglement of the unique state invariant under $L_0 + \varepsilon L_1$. Consider a separable pure state $\rho = \mathcal{P} \otimes Q \in M_d(\mathbb{C})$, where $\mathcal{P} = |\phi\rangle\langle \phi|$ and $Q = |\chi\rangle\langle \chi|$; then, suitable non-local
perturbations, \( \mathbb{L}_1 \) may entangle it. Indeed, by partial transposition [3], \( \rho_i \mapsto \rho_i^{\perp} \), operated on the second party with respect to an orthonormal basis starting with \( |\chi\rangle \), one obtains

\[
\rho_i^{\perp} = P \otimes Q + \varepsilon (\mathbb{L}_1 [P \otimes Q])^\dagger + o(\varepsilon).
\]

By projecting with \( \Pi_\perp \) onto a subspace orthogonal to \( P \otimes Q \), it follows that

\[
\text{Tr}(\rho_i^{\perp} \Pi_\perp) = \varepsilon \text{Tr}(\mathbb{L}_1 \Pi_\perp [P \otimes Q])^\dagger + o(\varepsilon).
\]

If \( \mathbb{L}_1 [\rho] = -i[H_1 \otimes 1 + 1 \otimes H_2 + H_{12}, \rho] \), where \( H_{12} \) is a non-local coupling of the two sub-systems, then the quantity

\[
\text{Tr}(\rho_i^{\perp} \Pi_\perp) \simeq -\varepsilon \text{Tr}(\mathbb{L}_1 (H_{12} + P \otimes Q))^\dagger
\]

can be made negative by suitably choosing \( H_{12} \); then, one violates the positivity of partial transposition at order \( \varepsilon \) and \( \rho_i \) is entangled at that order.

Entanglement can also be obtained via a purely dissipative time evolution as the one generated by \( \mathbb{L}_1 \) as in (11); indeed, choosing \( |1\rangle \langle 1| = P \otimes Q \) yields

\[
\text{Tr}(\rho_i^{\perp} \Pi_\perp) \simeq \varepsilon \text{Tr}(\mathbb{L}_1 (|\psi\rangle \langle \psi|)^\dagger),
\]

which can become negative by a suitable choice of entangled \( |\psi\rangle \) and \( \Pi_\perp \).

The possibility of generating entanglement in the above two cases comes from the fact that the zeroth-order state \( \rho_0 \) is on the border of the closed subset of separable states and can thus be moved into the open complementary subset of entangled states by suitable terms of order \( \varepsilon \).

2.1. \( \dim(\ker(\mathbb{L}_0)) \geq 2 \)

If, as in examples 1 and 3, the kernel of \( \mathbb{L}_0 \) contains more than one stationary state, still one may seek a \( \rho_0 \) such that \( \mathbb{L}_0 [\rho_0] = 0 \) and

\[
\mathbb{G}_0 \circ \mathbb{L}_1 [\rho_0] = \mathbb{L}_1 [\rho_0] = 0, \quad \text{where} \quad \mathbb{L}_1 := \mathbb{G}_0 \circ \mathbb{L}_1 \circ \mathbb{G}_0,
\]

so that the first-order correction can be obtained as

\[
\rho_1 = -\mathbb{L}_0^{-1} \circ \mathbb{L}_1 [\rho_0].
\]

In order to continue the iteration in (14) and obtain

\[
\rho_2 = -\mathbb{L}_0^{-1} \circ \mathbb{L}_1 [\rho_1],
\]

again by inverting \( \mathbb{L}_0 \), one has first to ensure that

\[
\mathbb{G}_0 \circ \mathbb{L}_1 [\rho_1] = -\mathbb{G}_0 \circ \mathbb{L}_1 \circ \mathbb{L}_0^{-1} \circ \mathbb{L}_1 [\rho_0] = 0,
\]

and, analogously, for the higher order contributions to (13).

**Example 8.** Consider the case where \( \mathbb{L}_0 [\rho] = -i[H_{0,1}, \rho] \) with \( H_0 \) non-degenerate. With \( H_0 \langle E_j^0 | E_j^0 \rangle = E_j^0 \langle E_j^0 | E_j^0 \rangle \) and using (8), one obtains

\[
\mathbb{L}_1 [\rho] = -i \sum_{i,j=1}^{d} |E_i^0 \rangle \langle E_i^0 | [H_i, |E_i^0 \rangle \langle E_i^0 | E_j^0 ]] |E_j^0 \rangle \langle E_j^0 | E_j^0 | = 0
\]

for all \( \rho \). Then, with \( \rho_0 = \sum_{k=0}^{d} p_k |E_k^0 \rangle \langle E_k^0 | \), using (16), one computes

\[
\rho_1 = -\mathbb{L}_0^{-1} \circ \mathbb{L}_1 [\rho_0] = - \sum_{j \neq k} \frac{p_k - p_j}{E_j^0} [H_i, |E_j^0 \rangle \langle E_j^0 | E_j^0 ]] |E_j^0 \rangle \langle E_j^0 | E_j^0 |. 
\]
From non-degenerate perturbation theory, the perturbation of $|E_\ell^0\rangle$ to first order in $\varepsilon$ is the eigenvector $|\psi^{(\ell)}_\varepsilon\rangle$ of $H_\varepsilon = H_0 + \varepsilon H_1$ given by

$$|\psi^{(\ell)}_\varepsilon\rangle = |E_\ell^0\rangle + \varepsilon \sum_{j \neq \ell} \frac{\langle E_\ell^0 | H_1 | E_j^0 \rangle}{E_j^0 - E_\ell^0} |E_j^0\rangle.$$

Thus one sees that, to order $\varepsilon$, $|\psi^{(\ell)}_\varepsilon\rangle|\psi^{(\ell)}_\varepsilon\rangle$ reproduces $\rho_1$ with $\rho_0 = |E_\ell^0\rangle\langle E_\ell^0|$. Furthermore, $G_0 \circ L_1[\rho_1] = 1\sum_{j \neq k} \frac{P_j - P_k}{E_j^0 - E_k^0} \frac{\langle E_k^0 | H_1 | E_j^0 \rangle^2}{E_j^0 - E_k^0} \left( |E_\ell^0\rangle\langle E_\ell^0| - |E_j^0\rangle\langle E_j^0| \right) = 0.$

Thus, $\rho_2 = (L_0^{-1} \circ L_1)^2[\rho_0]$ and so on with higher orders.

Unlike in the previous example, it may happen that (22) is not satisfied by the chosen $\rho_0$.

**Example 9.** Consider $L_0$ as in example 1 and $L_1$ as in example 2: the solution to

$$L_0[\rho] = 0 \quad \text{and to} \quad \tilde{L}_1[\rho] = \sum_{i,j=1}^d \rho_{ij} (|i\rangle\langle i|) |j\rangle\langle j|) = 0$$

must have the form $\rho_0 = \sum_{j=1}^d |\psi(j)\rangle^2 |j\rangle\langle j|$. Then, a natural candidate for the first order perturbation contribution $\rho_1$ is, using $L_0^{-1}$ as in example 4,

$$\rho_1 = -L_0^{-1} \circ L_1[\rho_0] = -L_0^{-1}[\rho] |\psi| - \rho_0 = -i \sum_{j \neq k} \frac{\psi(j)\psi^*(k)}{E_j - E_k} |j\rangle\langle k|.$$  \hspace{1cm} (23)

However, with this choice, it turns out that

$$G_0 \circ L_1[\rho_1] = G_0[|\psi\rangle\langle \psi| - \rho_1] = \sum_{j=1}^d |\psi(j)\rangle^2 |j\rangle\langle j| \neq 0.$$

A possible strategy to overcome the problem exposed by the previous example is as follows: in lemma 2, $L_0^{-1}$ is defined as a map from the range of $F_0$ into itself. Thus, given a first-order perturbation contribution $\rho_1 = -L_0^{-1} \circ L_1[\rho_0]$, one can always add to it (see the proof of the lemma) $\sigma_1 \in M_d(\mathbb{C})$ such that $L_0[\sigma_1] = 0$ whence $L_0[\rho_1 + \sigma_1] = L_0[\rho_1] = -L_1[\rho_0]$. One can thus try to find an appropriate $\gamma^{(\ell)}_\varepsilon$-invariant matrix $\sigma_1$ such that

$$G_0 \circ L_1[\rho_1] + G_0 \circ L_1[\sigma_1] = G_0 \circ L_1[\rho_0] + \tilde{L}_1[\sigma_1] = 0,$$

where we have used that $G_0[\sigma_1] = \sigma_1$. Thus, if such $\sigma_1$ can be found it is of the form

$$\sigma_1 = -\tilde{L}_1^{-1} \circ G_0 \circ L_1[\rho_1] = \tilde{L}_1^{-1} \circ G_0 \circ L_1 \circ L_0^{-1} \circ L_1[\rho_0].$$  \hspace{1cm} (24)

where the inverse $\tilde{L}_1^{-1}$ of $\tilde{L}_1$ is defined as in lemma 2 and thus $\text{Tr}(\sigma_1) = 0$. Then, one would obtain the second-order perturbation contribution $\rho_2 = -L_0^{-1} \circ L_1[\rho_1 + \sigma_1].$

**Remark 1.** Of course, the existence of $\sigma_1$ is equivalent to the invertibility of $\tilde{L}_1$. In general, $\tilde{L}_1$ is not a generator of the Lindblad form; namely, it does not generate a semigroup of completely positive maps on the set of all matrices, even if $L_1$ does. However, in the following section, we shall consider the setting of example 9 and prove that $\tilde{L}_1$ generate a positivity preserving semigroup on the density matrices commuting with $H$. [7]
3. Dissipative perturbation of a unitary evolution

In the following, we restrict to a less general situation than the ones addressed in the previous section; namely, we will stick to purely dissipative perturbations $L_1 = D$ as in (2) of a generator $L_0[\rho] = -\text{i}[H, \rho]$ as in example 1:

$$L_\varepsilon[\rho] = -\text{i}[H, \rho] + \varepsilon D[\rho], \quad D[\rho] = \sum_a \left( h_a \rho h_a^\dagger - \frac{1}{2} \left[ h_a^\dagger h_a, \rho \right] \right). \quad (25)$$

**Lemma 4.** The map $\widehat{D} = G_0 \circ D \circ G_0,$ with $G_0$ given by (8), generates a positive, trace-preserving map on the $\gamma_1^{(0)}$-invariant states $\rho$ that commute with $H.$

**Proof.** If $S_0 \ni \rho = \sum_{j=1}^d \rho_{jj} |j\rangle\langle j|$, then,

$$\widehat{D}[\rho] = \sum_{i,j=1}^d \rho_{ii} (j| \mathcal{D}(i) \langle i|) |j\rangle\langle j| = \sum_{i,j=1}^d \rho_{ii} \sum_a (|j \langle h_a |i\rangle|^2 - \delta_{ij} |j \langle h_a |j\rangle|^2).$$

Then,

$$\dot{\rho}_{jj} = \sum_{i=1}^d \rho_{ii} \sum_a (|j \langle h_a |i\rangle|^2 - \delta_{ij} |j \langle h_a |j\rangle|^2) \geq -h \rho_{jj}, \quad (27)$$

where $h = \left\| \sum_a h_a^\dagger h_a \right\|.$ Therefore, the eigenvalues of any $\rho \in S_0$ remain positive while evolving with $\exp(t\widehat{D}).$ \hfill \Box

**Example 10.** Consider $\widehat{L}_1$ as in example 9; on density matrices $S_0 \ni \rho = \sum_{j=1}^d \rho_{jj} |j\rangle\langle j|$ that commute with $H$, one finds that $\exp(t\widehat{D})[\rho]$ converges to the unique invariant state $\rho_0 = \sum_{j=1}^d |\psi(j)\rangle^2 |j\rangle\langle j|.$ Indeed, (27) yields

$$\dot{\rho}_{jj} = |\psi(j)|^2 \rho_{jj} - \rho_{jj}, \quad \rho_{jj}(t) = |\psi(j)|^2 (1 - e^{-t}) + \rho_{jj} e^{-t}.$$ 

As already observed, even if one knows the structure of the invariant states of $L_\varepsilon$, it remains to be proved that one actually has asymptotic convergence to them. As showed in lemma 1, even a very weak perturbation $\varepsilon D$ of $L_0$ in general decreases the dimension of the kernel of $L_\varepsilon$; this is why adding a suitable engineered dissipative perturbation can be used to drive a system into a certain stationary state which may even be chosen to be pure [15]. However, one must check that all other eigenvalues of $L_\varepsilon$ get a negative real part. This cannot hold in general [23]: purely imaginary eigenvalues can remain, but only if the Lindblad dynamics becomes trivial when reduced to the subspace which supports the stationary states. The following lemma provides sufficient conditions for this not to be the case.

**Lemma 5.** Assume that, given the generator (25), no projection $P \neq 1$ can satisfy (1) $[P, H] = [P, h_a] = 0$ and (2) $P h_a P = c_a P$ for all $a$. Then, the non-zero eigenvalues of $L_\varepsilon$ have a strictly negative real part.

**Proof.** The matrix units $|j\rangle\langle k|$, $j \neq k$, are such that

$$L_0[|j\rangle\langle k|] = -\text{i}(E_j - E_k)|j\rangle\langle k|.$$ 

Consider $L_\varepsilon[\rho_{jk}(\varepsilon)] = \lambda_{jk}(\varepsilon) \rho_{jk}(\varepsilon)$ and the expansion of eigen-matrices $\rho_{jk}(\varepsilon) = \rho_{jk}^{(0)} + \varepsilon \rho_{jk}^{(1)}$ and eigenvalues $\lambda_{jk}(\varepsilon) = -\text{i}(E_j - E_k) + \varepsilon \eta_{jk}$ to first order in $\varepsilon$. Inserted into the eigenvalue equation, this yields

$$L_0[\rho_{jk}^{(0)}] + \varepsilon \left( L_1[\rho_{jk}^{(0)}] + L_0[\rho_{jk}^{(1)}] \right) \simeq -\text{i}(E_j - E_k) \rho_{jk}^{(0)} + \varepsilon (\eta_{jk} \rho_{jk}^{(0)} - \text{i}(E_j - E_k) \rho_{jk}^{(1)}),$$

where

$$\eta_{jk} = \text{i}(E_j - E_k) + \varepsilon (\eta_{jk}^{(1)} + \varepsilon \eta_{jk}^{(2)} + \ldots).$$

This implies that $\eta_{jk}$ and $\rho_{jk}^{(0)}$ w.r.t. class $\rho_{jk}^{(0)}$: a strictly negative real part.
whence
\[-i[H, \rho^{(0)}_{jk}] = -i(E_j - E_k)\rho^{(0)}_{jk}, \quad -i[H, \rho^{(1)}_{jk}] + \mathbb{L}_1[\rho^{(0)}_{jk}] = \eta_{jk}\rho^{(0)}_{jk} - i(E_j - E_k)\rho^{(1)}_{jk}.\]

Setting $\rho^{(0)}_{jk} = |j\rangle\langle k|$, one obtains
\[\eta_{jk} = \langle j|\mathbb{L}_1[|j\rangle\langle k|]|k\rangle = \sum_\alpha \left( \langle j|h_\alpha\rangle\langle k|h^\dagger_\alpha|k\rangle - \frac{1}{2}\left( \langle j|h^\dagger_\alpha h_\alpha|j\rangle + \langle k|h^\dagger_\alpha h_\alpha|k\rangle \right) \right),\]
and when either $i \neq j$ or $k \neq \ell$,
\[\langle i|\rho^{(1)}_{jk}|\ell\rangle = \frac{\langle i|\mathbb{L}_1[|j\rangle\langle k|]|\ell\rangle}{i(E_i - E_\ell - E_j + E_k)}.
\]

Therefore,
\[
\text{Re}(\eta_{jk}) = \sum_\alpha \left( \text{Re}\left( \langle j|h_\alpha\rangle\langle k|h^\dagger_\alpha|k\rangle \right) - \frac{1}{2}\left( \langle j|h^\dagger_\alpha h_\alpha|j\rangle + \langle k|h^\dagger_\alpha h_\alpha|k\rangle \right) \right)
\]
\[
\leq -\frac{1}{2} \sum_\alpha |\langle j|h_\alpha\rangle| - |\langle k|h^\dagger_\alpha|k\rangle|^2.
\]

The above inequality is strict unless $\langle j|h^\dagger_\alpha h_\alpha|j\rangle = |\langle j|h_\alpha\rangle|^2$ and same for $\langle k|h^\dagger_\alpha h_\alpha|k\rangle$. Then, $\text{Re}(\eta_{jk}) = 0$ would imply that, for all $\alpha$,
\[h_\alpha = \langle j|h_\alpha\rangle(|j\rangle\langle j| + |k\rangle\langle k|) + \sum_{i: i \neq j, k} \langle i|h_\alpha\rangle|\ell\rangle\langle \ell|,
\]
whence, contrary to the assumptions, $h_\alpha$ reduces to a scalar multiple of $P = |j\rangle\langle j| + |k\rangle\langle k|$ on the subspace projected out by $P$.

Concerning the perturbation of the eigenvalue 0, choose $\rho_0$ such that $\mathbb{L}_0[\rho] = 0$, but $\mathbb{D}[\rho] \neq 0$; then, to first order in $\varepsilon$,
\[(\mathbb{L}_0 + \varepsilon\mathbb{D})[\rho + \varepsilon\rho_1 + o(\varepsilon)] = (\varepsilon\alpha_1 + o(\varepsilon))[(\rho + \varepsilon\rho_1 + o(\varepsilon))] \implies \mathbb{L}_0[\rho_1] + \mathbb{D}[\rho] = \alpha_1\rho.
\]

By splitting $\mathbb{D}[\rho] = G_0 \circ \mathbb{D}[\rho] + F_0 \circ \mathbb{D}[\rho]$, where $F_0 = \text{id} - G_0$, one obtains the solutions $\rho_1 = -\mathbb{L}_0^{-1} \circ F_0 \circ \mathbb{D}[\rho]$ and $\mathbb{D}[\rho] = \alpha \rho$. Since $\mathbb{D}$ generates a trace-preserving positive map on the $\gamma^{(0)}_t$-invariant states, the eigenvalue $\alpha$ must be negative. \(\square\)

**Example 11.** Consider $D$ as in example 2, where $h_\alpha = |\psi\rangle\langle \alpha|$. Then, $[P, |\psi\rangle\langle \alpha|] = 0$ for all $1 \leq \alpha \leq d$ yields $P = 1$.

### 3.1. Entanglement production

In this section we study whether an appropriate, purely dissipative Lindblad dynamics can create entanglement even when it is a weak perturbation of a non-entangling unitary dynamics. The fact that a Lindblad dynamics that does not include a unitary part is able to create entanglement is shown in [11] in a very concrete example, and the fact that a unitary evolution can be added if the invariant state is an eigenstate of the unitary evolution is the result in [15]. Here we concentrate on the assumption that this is exactly not the case; instead, we tackle the situation where the states invariant under the unitary time evolution are all separable. First, we observe that

**Lemma 6.** Let the generator $\mathbb{L}_0$ be given by a Hamiltonian of the form $H = H_1 \otimes 1 + 1 \otimes H_2$, where $H_1$ has eigenvalues $E_{1,k}$ and $H_2$ has eigenvalues $E_{2,l}$ where $E_{1,k} \neq E_{2,l} \quad \forall k, l$. Then, all $\gamma^{(0)}_t$-invariant states are separable. If the solutions of $\mathbb{D}[ho] = 0$, where $\mathbb{D}$ is as in (26), are
not on the border of the set of separable states, then there exists \( \epsilon_0 \) such that for all \( \epsilon \leq \epsilon_0 \) the invariant state is unique and separable.

**Proof.** Let \( \epsilon_0 \) be smaller than the radius in which the perturbation expansion of \( \rho(\epsilon) \) such that \( \mathbb{L}_n[\rho(\epsilon)] = 0 \) converges. Because of proposition 2, if all solutions of \( \hat{D} [\rho] = 0 \) are invertible, then there can exist only one solution; as a consequence (see the proof of lemma 3), there can be only one \( \rho(\epsilon) \) within its convergence radius. Further, in a sufficiently small neighborhood of a state not on the border of the convex set of separable states, all states are separable. \( \square \)

The following result will instead provide instances of a contrary behavior, more along the lines of example 7, showing the possibility of creating entanglement by weak dissipative perturbations.

**Proposition 3.** Consider the generator (25) with a non-entangling Hamiltonian as in lemma 6 and a dissipative perturbation \( \hat{D} \) such that \( \hat{D} [\rho] = 0 \) has only one solution. Then, there is a unique state in the kernel of \( \mathbb{L}_n \), given by the perturbation expansion \( \rho(\epsilon) = \sum_n \epsilon^n \rho_n \) where

\[
\rho_n = (-i)^n ((id - \hat{D}^{-1} \circ \mathbb{G}_0 \circ D) \circ \mathbb{L}_n^{-1} \circ D)^n [\rho_0].
\]

**Proof.** Given a zeroth-order approximation \( \rho_0 \) such that \( \mathbb{L}_0[\rho_0] = 0 \) and \( \hat{D} [\rho_0] = 0 \), we put ourselves in the most general situation where \( \mathbb{G}_0 \circ D \circ \mathbb{L}_0^{-1} \circ D[\rho_0] \neq 0 \). We then add to \( \rho_1 = -\mathbb{L}_0^{-1} \circ D[\rho_0] \) a matrix \( \sigma_1 \) such that \( \mathbb{L}_0[\sigma_1] = 0 \); the new matrix \( \tilde{\rho}_1 = \rho_1 + \sigma_1 \) still solves \( \mathbb{L}_0[\tilde{\rho}_1] = -\hat{D} [\rho_0] \). Since we want to solve \( \mathbb{L}_0[\rho_2] = -\hat{D} [\tilde{\rho}_1] \) by inverting \( \mathbb{L}_0 \), we seek \( \sigma_1 \) such that (see (24))

\[
\mathbb{G}_0 \circ D[\rho_1] + \hat{D} [\sigma_1] = 0.
\]

Since we assumed \( \hat{D} \) to have only one state in its kernel, it cannot vanish on \( \mathbb{G}_0 \circ D[\rho_1] \); the latter is a traceless matrix and both its normalized positive and negative parts would be states in the kernel of \( \hat{D} \) (see the proof of lemma 3). Therefore,

\[
\tilde{\rho}_1 = (id - \hat{D}^{-1} \circ \mathbb{G}_0 \circ D)[\rho_1] = -(id - \hat{D}^{-1} \circ \mathbb{G}_0 \circ D) \circ \mathbb{L}_0^{-1} \circ D[\rho_0].
\]

Iterating this construction, one obtains the contributions to the perturbation expansion as in (28). \( \square \)

**Remark 2.** In general, according to lemma 2, in order to solve equation (29) for a generic \( \mathbb{L}_1 \) in the place of \( D \), one has to consider the map \( \hat{G}_1 \) that remains associated with it by the time average (5), and check whether \( \hat{G}_1 \circ \mathbb{G}_0 \circ D[\rho_1] = 0 \).

**Example 12.** The map \( \hat{D} \) in example 10 has only one invariant state; according to (5) its inverse is given by \( \hat{D}^{-1} [X] = -X \) on \( X \) such that \( \text{Tr}(X) = 0 \). Therefore, by means of (29) and (23), equation (28) with \( n = 1 \) and \( \rho_0 = \sum_{i=1}^{d} \rho_i |i\rangle \langle i| \) yields

\[
\rho_1 = -(id - \mathbb{G}_0 \circ D) \circ \mathbb{L}_0^{-1} \circ D[\rho_0] = \sum_{i=1}^{d} |\psi(i)\rangle^2 |i\rangle \langle i| - i\epsilon \sum_{j \neq k} \frac{\psi(j)\psi^*(k)}{E_j - E_k} |j\rangle \langle k|
\]

\[
\rho(\epsilon) = (1 + \epsilon) \sum_{i=1}^{d} |\psi(i)\rangle^2 |i\rangle \langle i| - i\epsilon \sum_{j \neq k} \frac{\psi(j)\psi^*(k)}{E_j - E_k} |j\rangle \langle k| + o(\epsilon).
\]

Consider the bipartite setting of lemma 6 and set \( 1 \leq \alpha, \beta \leq a, a^2 = d \),

\[ |j\rangle = |\alpha\beta\rangle = |\alpha\rangle \otimes |\beta\rangle \quad \text{where} \quad H|\alpha\beta\rangle = E_{\alpha\beta}|\alpha\beta\rangle, \quad E_{\alpha\beta} = E_{1,\alpha} + E_{2,\beta}. \]
By transposing the first party with respect to the orthonormal basis \{\ket{\alpha}\}_{a=1}^{a}$, as in example 7, one obtains

$$\rho^T(\varepsilon) = (1 + \varepsilon) \sum_{a, \beta=1}^{a} |\psi_{a\beta}|^2 \ket{\alpha} \bra{\alpha} \otimes \ket{\beta} \bra{\beta} - i\varepsilon \sum_{(a, \beta) \neq (\gamma, \delta)} \frac{\psi_{a\beta} \psi_{\gamma\delta}^*}{E_{a\beta} - E_{\gamma\delta}} \ket{\gamma} \bra{\alpha} \otimes \ket{\beta} \bra{\delta} + o(\varepsilon).$$

Suppose \(\psi_{a_1\beta_1} = \psi_{a_2\beta_2} = 0\) for \(a_1 \neq a_2\) and \(\beta_1 \neq \beta_2\); then, choosing an entangled state \(\ket{\phi}\) supported only in the subspace spanned by \(\ket{\alpha_2\beta_1}\) and \(\ket{\alpha_1\beta_2}\), one calculates

$$\langle \phi | \rho^T(\varepsilon) | \phi \rangle = \varepsilon \frac{\text{Im}(\phi_{a_1\beta_1} \psi_{a_2\beta_1} \phi_{a_2\beta_2}^* \psi_{a_1\beta_2}^*)}{E_{1\alpha_1} + E_{2\beta_2} - (E_{1\alpha_2} + E_{2\beta_1})} + o(\varepsilon).$$

This expectation can always be made negative and thus, by applying the partial transposition criterion, \(\rho(\varepsilon)\) results entangled to order \(\varepsilon\). Note that the assumptions on the coefficients ensure that the projection of \(\ket{\psi}\) onto the subspace spanned by \(\ket{\alpha_2\beta_1}\) and \(\ket{\alpha_1\beta_2}\) is entangled; furthermore, example 11 ensures that \(\rho(\varepsilon)\) is an asymptotic state for the given time evolution.

4. Summary

A practical control of the invariant states of quantum dynamical semigroups generated by Lindblad-type generators is still partial, while this knowledge would very much be needed due to the fact that quantum information protocols might use suitably engineered open quantum time evolutions to achieve entanglement asymptotically.

In this paper, we have considered an asymptotic approach to this problem and studied the fate of the separable invariant states of two non-interacting quantum systems when their unitary time evolution is weakly perturbed by a Lindblad-type dissipative contribution \(\varepsilon \mathbb{L}_1\) to the generator \(\mathbb{L}_0\).

The investigation has been conducted by developing the first theoretical steps of a perturbative approach to the stationary states \(\rho(\varepsilon)\) of the perturbed generator \(\mathbb{L}_\varepsilon = \mathbb{L}_0 + \varepsilon \mathbb{L}_1\) and the preliminary results have been applied to show how small suitable dissipative corrections to the unitary dynamics, not entangling by itself, may indeed lead to entangled asymptotic invariant states, \(\mathbb{L}_\varepsilon [\rho(\varepsilon)] = 0\).

The main theoretical tool used to obtain these results has been the practical handling of the first steps of an iterative procedure that provides the contributions to the perturbative expansion of \(\rho(\varepsilon)\); the examples we have presented show that in some cases the iterative procedure can be performed. From a theoretical point of view, it remains to be understood whether the construction of the perturbative contributions to \(\rho(\varepsilon)\) can always be achieved by an appropriate choice of the zeroth-order seed as in section 2.1 or whether the summation breaks at a finite order because of the lack of analyticity.

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