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for a Class of Simple Anosov Flows

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ABSTRACT. We consider perturbations of the Hamiltonian flow associated with the geodesic flow on a surface with constant negative curvature. We prove that, under a small perturbation, not necessarily of Hamiltonian character, the SRB measure associated with the flow exists and is analytic in the strength of the perturbation. An explicit example of “thermostatted” dissipative dynamics is considered.

1. Introduction

1.1. In recent time, much effort has been devoted to the analysis of hyperbolic systems, in part due to the Chaotic Hypothesis, introduced ten years ago in [GC], which states that a many particles systems in a nonequilibrium stationary state behaves as a uniformly hyperbolic dynamical system (Anosov or more generally Axiom A system), at least for the purpose of evaluating macroscopic observables. This hypothesis can be seen as a generalization of the ergodic hypothesis to nonequilibrium systems, at least for systems in a stationary state. Although it is very hard to prove uniform hyperbolicity for realistic model systems, ideas connected with the Chaotic Hypothesis have played an important role in analyzing the results of numerical or real experiments.

Several results have been obtained in this direction, among which the Gallavotti-Cohen Fluctuation Theorem (FT), a result concerning the large deviation functional of the phase space contraction rate (often identified with the entropy production rate), that extend the fluctuation-dissipation relation to systems in a nonequilibrium stationary state. The FT was proved rigorously in [G] for Anosov diffeomorphisms and then in [Ge] for Anosov flows. Furthermore several numerical tests have been conducted, using mathematical models of dissipative reversible systems and the chaotic hypothesis.

Most of the results quoted above are based on the existence of the Sinai-Ruelle-Bowen (SRB) measure. This existence was proved for a wide class of hyperbolic systems [BR],[S]. Unfortunately explicit expressions for the SRB measure are quite difficult to obtain and can be worked out only in particular cases, e.g. Anosov Coupled Lattice Map [BFG], while most of the models used in the simulations are based on continuous time.
dynamics (hyperbolic flows). We observe that, in order to obtain models for nonequilibrium systems, one can not consider the simplest examples of Anosov systems that, being volume preserving, are not dissipative.

In this paper we explicitly construct the SRB measure for a family of Anosov flows that includes dissipative cases. The flows considered are perturbations of the geodesic flow on a surface of constant negative curvature. Such a flow can be seen as a Hamiltonian flow restricted to the surface of unit energy. We will mainly consider perturbation arising by adding a force to the Hamiltonian equations of motion. If the chosen force is conservative (i.e. coming from a potential), the system remains Hamiltonian and volume preserving so that the stationary measure is not singular with respect to the volume measure. Otherwise, if the perturbation is non conservative, the system is expected to have an SRB measure singular with respect to the volume measure (dissipativity). Many of the models used in numerical works fall under this last category.

The paper is organized as follows. In section 2 we introduce the systems we consider and state the main results of the paper. Section 3, 4, 5 contain the proof of these results. A conclusive section gives comparison with known works and outlooks. Finally the Appendices contain some technical computations.

2. Model and main results

2.1. The geodesic flow. The complex upper half plane $\mathbb{C}_+ \overset{def}{=} \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$, endowed with the metric $g = y^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, is called the Lobachevskii plane. The isometries of this plane are given by the real, $2 \times 2$ matrices $h$ with $\det h > 0$ where, if $z \in \mathbb{C}_+$, the action of $h$ on $z$ is

$$zh = \frac{h_{11}z + h_{21}}{h_{12}z + h_{22}} \in \mathbb{C}_+,$$

for $h \overset{def}{=} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$.

Observe that $h$ and $h' = \lambda h$, for $\lambda \neq 0$, define the same transformation so that the isometries are naturally represented by the elements of $\text{PSL}(2, \mathbb{R})$.

A compact surface can be constructed from the Lobachevskii plane in the same way as the torus can be obtained from the plane $\mathbb{R}^2$. Given a Fuchsian subgroup $\Gamma \subset \text{PSL}(2, \mathbb{R})$ (see [P] for a precise definition), we can consider the equivalence relation generated by its action on $\mathbb{C}_+$,

$$z \sim z' \iff \exists \gamma \in \Gamma \mid z = z' \gamma.$$ 

The quotient set, indicated with $\Sigma = \mathbb{C}_+/\Gamma$, is the most general compact analytic surface with constant negative curvature. If $\Gamma$ is the smallest possible Fuchsian subgroup, we obtain a surface of genus two (2-torus).

We will consider as unperturbed dynamical system the flow generated by the Hamiltonian

$$H_0(x, y, px, py) = \frac{y^2}{2} (p_x^2 + p_y^2) \quad (2.1)$$

on the cotangent bundle $\mathcal{M} = T^*\Sigma$. For any given energy $\mathcal{E} > 0$, the surface

$$\mathcal{M}_\mathcal{E} = \{(x, y, px, py) \in \mathcal{M} : H_0(x, y, px, py) = \mathcal{E}\} ,$$

is a compact, invariant manifold. The geodesic flow on the surface $\Sigma$ can be identified in a natural way with the flow generated by (2.1) restricted to $\mathcal{M}_1$, see [GF].

To add a conservative force to such a system, we consider an analytic $\Gamma$–periodic function $\{V(z), z \in \mathbb{C}_+\}$, and the new Hamiltonian

$$H_\varepsilon = H_0 + \varepsilon V \quad (2.2)$$
which generates the equations of motion
\[\begin{align*}
\dot{x} &= y^2 p_x, \\
\dot{y} &= y^2 p_y, \\
\dot{p}_x &= -\frac{\partial V}{\partial x}, \\
\dot{p}_y &= -y(p_x^2 + p_y^2) - \epsilon \frac{\partial V}{\partial y}.
\end{align*}\] (2.3)

We can then add a non-conservative force to our system. To obtain well defined equations of motion it has to be covariant w.r.t. the transformations in \(\Gamma\). To obtain this we can consider the automorphic function of order one \(\phi\) defined by:
\[\phi(z\gamma) = \phi(z)j^2(z, \gamma), \quad \forall \gamma \in \Gamma,\]
where \(j(z, h) = h_{12}z + h_{22}\), see [F]. Setting
\[E_x = \frac{\phi(z) + \phi(z)}{2}, \quad E_y = \frac{\phi(z) - \phi(z)}{2i},\]
where \(\overline{z}\) is the complex conjugate, we obtain a force fields which is locally conservative, but is not the differential of a function. We can still define the potential difference between two points, \(z, z_0 \in \Sigma\),
\[U(z) - U(z_0) = \int_{z_0}^z dw \phi_1(w) + \int_{z_0}^{\overline{w}} d\overline{w} \overline{\phi_1(w)}\]
as a multivalued function.

The energy \(H_\epsilon\) computed along a motion which contains the force \((E_x, E_y)\) tends asymptotically to increases. In order to maintain it constant, we introduce a Gaussian thermostat, namely a momentum-dependent friction of the form \(\alpha(p) = p \cdot E/p^2\). Finally, the equations of motion for the perturbed flow on \(M_\epsilon \defeq \{(x, y, p_x, p_y) \in M : H_\epsilon(x, y, p_x, p_y) = \mathcal{E}\}\) are:
\[\begin{align*}
\dot{x} &= y^2 p_x, \\
\dot{y} &= y^2 p_y, \\
\dot{p}_x &= -\epsilon \frac{\partial V}{\partial x} + \epsilon' [E_x - \alpha(p)p_x], \\
\dot{p}_y &= -y(p_x^2 + p_y^2) - \epsilon \frac{\partial V}{\partial y} + \epsilon' [E_y - \alpha(p)p_y].
\end{align*}\] (2.4)

where \(\epsilon'\) is the strength of the non-conservative field: since only notational complications would arise from considering \(\epsilon \neq \epsilon'\), in the following, we will restrict ourselves to the case \(\epsilon = \epsilon'\). Under the dynamics eq. (2.4) \(H_\epsilon\) is an integral of the motion.

2.2. Canonical coordinates. A simpler representation of the unperturbed dynamics was introduced in [CEG]. We consider the canonical transformation from \(M\setminus\{H_0 = 0\}\) to \(G \defeq \text{GL}(2, \mathbb{R})/\Gamma\)
\[\begin{pmatrix} p_x & p_y \\ x & y \end{pmatrix} \mapsto \begin{pmatrix} p_1 & q_2 \\ -p_2 & q_1 \end{pmatrix} \defeq g,\]
defined by
\[\begin{align*}
p_x + ip_y &= \frac{i}{2} \det^2(g)j^2(i, g^{-1}) \\
x + iy &= ig^{-1}.
\end{align*}\] (2.5)

This transforms the equations of motion (2.3) into those generated by the new Hamiltonian (with slight abuse of notation, we still call \(H_\epsilon\) and \(V\) the Hamiltonian and the potential as functions of the matrix \(g\))
\[H_\epsilon(g) = \frac{\det^2(g)}{8} + \epsilon V(g).\] (2.6)
Clearly $H_\varepsilon$ is an analytic function of $g$. Introducing the matrices
\begin{equation}
\sigma^0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \tag{2.7}
\end{equation}
the Hamilton equation derived from (2.6) reads
\begin{equation}
\dot{g} = -\frac{\det(g)}{4}g\sigma^3 + \varepsilon\sigma^1 \frac{\partial V}{\partial g}(g)\sigma^y, \tag{2.8}
\end{equation}
for $\sigma^x \equiv (\sigma^+ + \sigma^-)$ and $\sigma^y \equiv (\sigma^+ - \sigma^-)$. The non-conservative equations of motion (2.4) reads
\begin{equation}
\dot{g} = -\frac{\det(g)}{4}g\sigma^3 + \varepsilon\sigma^1 \frac{\partial V}{\partial g}(g)\sigma^y - \varepsilon c(g)g\sigma^y \equiv \frac{\det(g)}{4}[ -g\sigma^3 + \varepsilon F(g) ], \tag{2.9}
\end{equation}
where the function $c(g)$ is:
\begin{equation}
c(g) = \frac{1}{2\det^2(g)} \left[ \frac{\phi(ig^{-1})}{j^2(i, g^{-1})} + \frac{\phi(ig^{-1})}{j^2(-i, g^{-1})} \right].
\end{equation}
This is a explicit example of a non-conservative system. Moreover it is possible to prove that the systems (2.9) generically have a positive average space contraction rate, see [BGM].

2.3. Remark. Our techniques can be extended to a more general case. Given a Hamiltonian
\begin{equation}
H_\varepsilon(g) \equiv H_0(g) + \varepsilon V(g) \tag{2.10}
\end{equation}
like in eq.(2.6) we can consider any analytic vector field $\nu_\varepsilon$ on $\mathcal{M}$, $\varepsilon$-close to the Hamiltonian vector field generated by $H_0$ and tangent to the level surfaces of $H$. Clearly the flow generated by such a vector field preserve $H_\varepsilon$ and the following results hold in this more general situation.

2.4. The conjugation. Let $\Phi_t : G_\varepsilon \rightarrow G_\varepsilon$ and $\Phi_t^\varepsilon : G_\varepsilon^\varepsilon \rightarrow G_\varepsilon^\varepsilon$ be the flows generated by the Hamiltonian $H_0$ and by the dissipative system in eq.(2.9), respectively. As a first step we want to prove that these two flows can be conjugated by a change of coordinate. Differently from the case of Anosov diffeomorphisms, [GBG], this is not enough to map $\Phi_t$ into $\Phi_t^\varepsilon$, but a local rescaling of time is also required. The details are given in the following theorem. To state it we need some notations:
\begin{align*}
G_\varepsilon &= \{ g \in G \mid H_0(g) = \varepsilon \}, & G_{>\varepsilon} &= \{ g \in G \mid H_0(g) > \varepsilon \}, \\
G_\varepsilon^\varepsilon &= \{ g \in G \mid H_\varepsilon(g) = \varepsilon \}, & G_{>\varepsilon}^\varepsilon &= \{ g \in G \mid H_\varepsilon(g) > \varepsilon \}.
\end{align*}

Theorem 1. Conjugation. Given $\varepsilon > 0$, there exists an $\tilde{\varepsilon} > 0$ such that, for any $\varepsilon : |\varepsilon| \leq \tilde{\varepsilon}$ there are functions $h_\varepsilon : G_{>\varepsilon} \rightarrow G_{>\varepsilon}^\varepsilon$, and $\tau_\varepsilon : G_{>\varepsilon} \rightarrow \mathbb{R}$, Hölder continuous in $g$ and analytic in $\varepsilon$, such that
\begin{equation}
h_\varepsilon \circ \Phi_t = \Phi_t^\varepsilon \circ h_\varepsilon, \quad \text{for} \quad T_{\varepsilon} \equiv \int_0^t ds (\tau_\varepsilon \circ \Phi_s). \tag{2.11}
\end{equation}
Furthermore, $H_0 \equiv H_\varepsilon \circ h_\varepsilon$, so that $h_\varepsilon(G_\varepsilon) = G_\varepsilon^\varepsilon$.

The proof, given in section 3, is based on the hyperbolicity of the unperturbed flow, which is discussed in the next section.
The function \( h_\varepsilon \) is the space conjugation, while \( \tau_\varepsilon \) is the time conjugation. Even if \( h_\varepsilon \) conjugate the flow from \( G_\varepsilon \) to \( G_\varepsilon^\varepsilon \), the existence of a conjugation from whole the \( G \) to itself cannot be uniform in \( \varepsilon \). Indeed, fixed \( \varepsilon \), if \( E < \varepsilon \sup_g V(g) \) the topology of \( G_\varepsilon^\varepsilon \) is different from that of \( G_\varepsilon \), and no conjugation is possible.

### 2.5. Hyperbolicity

If the tangent space \( T_g G_\varepsilon^\varepsilon \) can be splitted into three continuous, \( \Phi^\varepsilon \)-covariant, one-dimensional, linear subspaces:

\[
T_g G_\varepsilon^\varepsilon = E_g^+ \oplus E_g^- \oplus E_g^3
\]

where \( E_g^3 \) is parallel to the flow and there exists constants \( c, \lambda > 0 \) such that

\[
\| (T_g \Phi^\varepsilon_t) w \| \leq c e^{-\lambda t}\| w \| \quad \text{for} \quad w \in E_g^-, \ t \geq 0
\]

\[
\| (T_g \Phi^\varepsilon_t) w \| \leq c e^{\lambda t}\| w \| \quad \text{for} \quad w \in E_g^+, \ t \leq 0
\]

then the flow \( \Phi^\varepsilon \) is hyperbolic on \( G_\varepsilon^\varepsilon \). Moreover \( E_g^+ \), \( E_g^- \), and \( E_g^3 \) are called the unstable, stable and neutral subspace, respectively.

The unperturbed flow, \( \Phi \), is hyperbolic on \( G_\varepsilon \), for every \( \varepsilon > 0 \). The solution of (2.8) is explicitly given by:

\[
\Phi_t(g) \overset{\text{def}}{=} e^{-\left(\det(g)/4\right)t} \mod \Gamma.
\]

and it is clear that \( E_g^\alpha \) is generated by \( g \sigma^\alpha \), for \( \alpha = \pm, 3 \) and \( \lambda = \sqrt{2E} \).

The four curves

\[
\Phi^\alpha_\varepsilon(g) \overset{\text{def}}{=} e^{-\varepsilon \sigma^\alpha} \mod \Gamma \quad \text{for} \quad \alpha = 3, 0, \pm
\]

are the integral manifold of the vector fields \( w^\alpha(g) = -(2\varepsilon) \sigma^\alpha \), for \( a = 0, \pm, 3 \). We remark that \( \Phi_t \equiv \Phi^3_t \det(g)/4 \)

and that \( \Phi^0 \) is orthogonal to \( G_\varepsilon \).

Calling \( \lambda^\pm(g) = \pm \det(g)/2 = \pm \sqrt{2H_0(g)} \) and \( \lambda^3 \equiv 0 \) the Lyapunov exponents of \( \Phi_t \), and using that the commutation relation among the matrices \( \{\sigma^i\}_{i=0,3,\pm} \) are

\[
[\sigma^3, \sigma^+] = 2\sigma^+ \quad [\sigma^3, \sigma^-] = -2\sigma^- \quad [\sigma^+, \sigma^-] = \sigma^3,
\]

we obtain that:

\[
\Phi_t \circ \Phi^\alpha_\varepsilon = \Phi^\alpha_\varepsilon \exp(t\lambda^\alpha(g)) \circ \Phi_t.
\]

### Theorem 2. Hyperbolicity

For any energy \( \varepsilon > 0 \), there exists \( \bar{\varepsilon} > 0 \) such that, for any \( \varepsilon : |\varepsilon| \leq \bar{\varepsilon} \)

the flow \( \Phi^\varepsilon \) on \( G_\varepsilon^\varepsilon \) is hyperbolic for every \( \varepsilon' > \varepsilon \). In particular, there exist vector fields \( \{w^\alpha_\varepsilon\}_{\alpha=0,\pm} \)

and functions \( \{\lambda^\alpha_\varepsilon\}_{\alpha=0,\pm} \)

on \( G_\varepsilon^\varepsilon \) such that

\[
T \Phi^\varepsilon_t w^\alpha_\varepsilon = \exp \left\{ \int_0^t ds \left( \lambda^\alpha_\varepsilon \circ \Phi_t^s \right) \right\} (w^\alpha_\varepsilon \circ \Phi_t^s), \quad \text{for} \quad \alpha = 0, \pm.
\]

Furthermore, \( \{w^\alpha_\varepsilon \circ h_\varepsilon\}_{\alpha=0,\pm} \) and \( \{\lambda^\alpha_\varepsilon \circ h_\varepsilon\}_{\alpha=0,\pm} \), are analytic in \( \varepsilon \), and Hölder continuous in \( g \).

Notwithstanding we called the conjugation a change of variables, since it is not differentiable -but only Hölder-continuous- this theorem is not a direct consequence of theorem 1. The fact that \( \{\lambda^\alpha_\varepsilon \circ h_\varepsilon\}_{\alpha=0,\pm} \), rather than \( \{\lambda^\alpha_\varepsilon\}_{\alpha=0,\pm} \), are analytic in \( \varepsilon \), will be important for the construction of the SRB measure.

### 2.6. SRB distribution

For any energy \( \varepsilon \) we can define the SRB measure on \( G_\varepsilon^\varepsilon \):

\[
\mu_\varepsilon^\varepsilon(\mathcal{O}) \overset{\text{def}}{=} \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{d\tau}{T} \left( \mathcal{O} \circ \Phi^\varepsilon_\tau \right)(g),
\]

5
provided that such a limit exists and is constant Lebesgue-almost everywhere in \( g \) for every continuous function \( \mathcal{O} \). Such a measure exists, is unique and ergodic, if the dynamical system is Anosov and topologically mixing, i.e., it is hyperbolic in the whole \( \mathcal{G}_E \) and the stable and the unstable manifold are dense \( \mathcal{G}_E \).

The flow \( \Phi \) is Anosov since it is also Hamiltonian, it is easy to prove that its SRB measure is the Lebesgue measure. Regarding \( \Phi^\varepsilon \), uniform hyperbolicity was established in Theorem 2, while topological mixing is a direct consequence of the existence of the conjugation.

**Theorem 3. Analyticity of the SRB measure.** Given \( E > 0 \), there exists \( \bar{E} > 0 \), such that, for any \( \varepsilon : |\varepsilon| < \bar{E} \) the SRB measure \( \mu^\varepsilon_E \), is analytic in \( \varepsilon \) for every \( E' > E \); i.e., for any analytic \( \mathcal{O} : G \to \mathbb{R} \), the mean value \( \mu^\varepsilon_E(\mathcal{O}) \) is analytic in \( \varepsilon \).

This is our main result. The proof will consist in an explicit construction of the SRB measure.

To summarize, for any energy \( E > 0 \), and \( \varepsilon \) small enough, we have constructed an hyperbolic structure and the corresponding SRB measure on each one of the leaves \( \{\mathcal{G}_E^\varepsilon\}_{\varepsilon > E} \) in \( \mathcal{G}_E^\varepsilon \). The set \( \{\mu^\varepsilon_E\}_{\varepsilon > E} \) is an invariant measure on \( \mathcal{G}_E^\varepsilon \).

### 3. Proof of Theorem 1.

#### 3.1. Directional derivatives.**

For any smooth \( f \) on \( G \) we define the directional derivative along the curves \( \{\Phi^\alpha\}_{\alpha=0,\pm,3} \), as:

\[
(\mathcal{L}_\alpha f)(g) \overset{def}{=} \frac{d(f \circ \Phi^\alpha)}{d\xi}(\xi = 0)(g).
\]

These derivatives satisfy the relation \((\mathcal{L}_\alpha w^3) - (\mathcal{L}_3 w^\alpha) = \lambda^\alpha w^\alpha \). Since the stable, unstable and neutral directions are tangent to \( \mathcal{G}_E \), whereas \( w^0 \) is transversal to it, we have

\[
(\mathcal{L}_\alpha H_0)(g) = 0 \quad \text{for} \quad \alpha = 3, \pm \quad g \in \mathcal{G}_E
\]

\[
(\mathcal{L}_0 H_0)(g) \neq 0 \quad \text{for} \quad g \in \mathcal{G}_E.
\]

Given \( \gamma < 1 \) and a function \( f \) on \( G \), we also define the directional Hölder derivative along \( \{\Phi^\alpha\}_{\alpha=0,\pm,3} \) as

\[
(\mathcal{L}_\alpha^\gamma f)(g) \overset{def}{=} \sup_{\varepsilon < |\varepsilon| \leq 1} \frac{|f(\circ \Phi^\varepsilon g) - f(g)|}{|\varepsilon|}\gamma,
\]

if the supremum is finite.

#### 3.2. Construction of the Conjugation.

In order to find a solution of (2.11), let us differentiate it w.r.t. \( t \), for \( t = 0 \):

\[
(\mathcal{L}_3 h_\varepsilon)(g) = \left(\frac{\det h_\varepsilon}{\det g}\right)\tau_\varepsilon g \left[w^3 \circ h_\varepsilon + \varepsilon F \circ h_\varepsilon\right](g).
\]

We will look for a solution \( h_\varepsilon \) and \( \tau_\varepsilon \) of the form

\[
h_\varepsilon(g) = g + \sum_{\alpha=0,\pm,3} \delta h_\varepsilon^\alpha(g) w^\alpha(g) = \sum_{\alpha=0,\pm,3} \left[\delta_0,\alpha + \delta h_\varepsilon^\alpha(g)\right]w^\alpha(g)
\]

\[
\tau_\varepsilon = 1 + \delta \tau_\varepsilon
\]

where \( \delta_{\alpha,\beta} \) is the Kronecker symbol. Projecting along the directions \( \{w^\alpha(g)\}_{\alpha=0,\pm,3} \) and using the identity following (2.1), yields (see the Appendix for the details):

\[
(\mathcal{L}_3 \delta h_\varepsilon^\alpha)(g) - \lambda^\alpha \delta h_\varepsilon^\alpha(g) = \varepsilon F^\alpha(g) + R^\alpha_\varepsilon \left(\delta h_\varepsilon^0, \delta h_\varepsilon^3, \delta h_\varepsilon^+, \delta h_\varepsilon^-, \delta \tau_\varepsilon\right)
\]

\[
+ \delta_{\alpha,3}(\delta \tau_\varepsilon(g) - 2\delta h_\varepsilon^0(g))
\]
In the r.h.s. of (3.6), \( \{ F^\alpha : \mathcal{G} \to \mathbb{R} \}_{\alpha = 0, \pm, 3} \) are analytic functions of \( g \), depending neither on \( \delta h_\varepsilon \), nor on \( \delta \tau_\varepsilon \); while \( \{ R^\alpha_\ominus : \mathbb{R}^5 \to \mathbb{R} \}_{\alpha = 0, \pm, 3} \) are analytic functions of the form

\[
R^\alpha_\ominus (f_1, f_2, f_3, f_4, f_5) = \varepsilon \sum_{i=1}^5 C^R_{\alpha,i} f_i + O(f^2)
\]

(3.7)

for suitable constants \( \{ C^R_{\alpha,i} \}_{i = 1, \ldots, 5} \), with \( O(f^2) \) of order 0 in \( \varepsilon \). The last term in (3.6) is linear in \( \delta h_\varepsilon^0 \), and in \( \delta \tau_\varepsilon \), but we singled it out because it is order 0 in \( \varepsilon \).

3.3. Implicit solution. We can implicitly solve (3.6). For every continuous \( f : \mathcal{G} \to \mathbb{R} \), it is possible to invert the operators \( \{ L_3 - \lambda^3 \}_{\beta = \pm} \):

\[
(L_3 - \lambda^3)^{-1} f = \int_0^{\sgn(\lambda^3)\infty} dt \; e^{(L_3 - \lambda^3)t} f = \int_0^{\sgn(\lambda^3)\infty} dt \; e^{-\lambda^3t}(f \circ \Phi_t), \quad \beta = \pm
\]

(3.8)

where the exponentially decaying factor guarantees convergence.

The implicit solutions for the stable and the unstable components of the conjugation are then:

\[
\delta h^\beta_\varepsilon = \int_0^{\sgn(\beta)\infty} dt \; e^{-\lambda^3t}(R^\beta_\ominus \circ \Phi_t) + \varepsilon \int_0^{\sgn(\beta)\infty} dt \; e^{-\lambda^3t}(F^\beta_\ominus \circ \Phi_t), \quad \beta = \pm,
\]

(3.9)

for \( R^\beta_\ominus \circ \Phi_t \equiv R^\alpha_\ominus (\{ \delta h^\alpha_\varepsilon \circ \Phi_t \}_{\alpha = 0, \pm, 3}, \delta \tau_\varepsilon, \Phi_t \) \). The equation for \( \delta h^3_\varepsilon \) cannot be solved in the same way since \( \lambda^3 \equiv 0 \). Nonetheless, we can choose \( \tau_\varepsilon \) so that the r.h.s. member of (3.6), for \( \alpha = 3 \), is identically zero:

\[
\delta \tau_\varepsilon = 2 \delta h^0_\varepsilon - \varepsilon F^3 - R^3_\ominus (\delta h^0_\varepsilon, 0, \delta h^+_\varepsilon, \delta h^-_\varepsilon, \delta \tau_\varepsilon).
\]

(3.10)

Since also \( \lambda^0 \equiv 0 \), a similar problem occurs for the equation corresponding to \( \delta h^0_\varepsilon \); in this case, it is possible to obtain an equation for \( \delta h^0_\varepsilon \) using that \( H_\varepsilon \circ h_\varepsilon = H_0 \). Considering the transversality condition (3.2) and the implicit equations for the level surfaces, one can solve (3.6) in terms of \( \delta h^0_\varepsilon \) only, obtaining:

\[
\delta h^0_\varepsilon = - \frac{1}{\mathcal{L}_0 H_0} \left[ H_0 \circ h_\varepsilon - H_0 - \sum_{\alpha} (\mathcal{L}_\alpha H_0) \cdot \delta h^\alpha_\varepsilon + \varepsilon V \circ h_\varepsilon \right]
\]

\[
def \equiv - \varepsilon \frac{V}{\mathcal{L}_0 H_0} - O(\delta h^0_\varepsilon, \delta h^3_\varepsilon, \delta h^+_\varepsilon, \delta h^-_\varepsilon, \delta \tau_\varepsilon),
\]

(3.11)

where \( O \) can be written as in (3.7), for certain other constants \( \{ C^C_{\alpha,i} \}_{i = 1, \ldots, 5} \). The fact that \( u^0 \) is orthogonal to the level surfaces of \( H_0 \), (see (3.2)) guarantees that this expression is well defined for any \( g \in \mathcal{G}_\varepsilon \) and \( \varepsilon \) small enough.

3.4. Existence of the conjugation. Observe that the equations (3.9), (3.10) and (3.11) can be naturally seen as defining a function \( f \equiv \{ f^\alpha : \mathcal{G} \to \mathbb{R}^4 \}_{\alpha = 0, \pm, 3} \) for \( f^0 = \delta h^0_\varepsilon, f^\pm = \delta h^\pm_\varepsilon \) and \( f^3 = \delta \tau_\varepsilon \). We will look for a solution of the above equations in the Banach space, \( \mathcal{B} \), defined by the norm \( \| f \|_\gamma \equiv \max_{\alpha} \| f^\alpha \|_\gamma \), with

\[
\| f^\alpha \|_\gamma \equiv \| f^\alpha \| + \sum_{\beta = \pm} \| \mathcal{L}_\beta f^\alpha \| + \sum_{\beta = 3, 0} \| \mathcal{L}_\beta f^\alpha \|,
\]

where \( \| u \|_\gamma \equiv \sup_{g \in \mathcal{G}} | u(g) | \) for \( u : \mathcal{G} \rightarrow \mathbb{R} \).

The equation for the conjugation is given in terms of the operator
(Lf)α \overset{\text{def}}{=} \begin{cases} (L_3 - \lambda^\alpha)f^\alpha & \text{if } \alpha = \pm 1 \\ f^\alpha & \text{if } \alpha = 0, 3 \end{cases},

and the function

\[ S_\varepsilon^\alpha(f) \overset{\text{def}}{=} \begin{cases} \varepsilon \mathcal{F}^\alpha + \mathcal{R}_\varepsilon^\alpha(f^0, 0, f^+, f^-, f^3) & \text{for } \alpha = \pm 1 \\ -\varepsilon(L_0H_0)^{-1} \cdot V + \mathcal{O}(f^0, 0, f^+, f^-, f^3) & \text{for } \alpha = 0 \\ -\varepsilon [(L_0H_0)^{-1} \cdot 2V + \mathcal{F}^3] - \left(2\mathcal{O} + \mathcal{R}^3_\varepsilon\right)(f^0, 0, f^+, f^-, f^3) & \text{for } \alpha = 3 \end{cases} \tag{3.12} \]

**Lemma 1.** There exists \( \bar{\varepsilon} > 0 \) such that, for any \( \varepsilon : |\varepsilon| \leq \bar{\varepsilon} \), the equation

\[ Lf = S_\varepsilon(f) \tag{3.13} \]

has unique solution in the ball of \( \mathcal{B} \) with radius \( |\varepsilon|C \), for a suitable \( C \). Such a solution is analytic in \( \varepsilon \).

**Proof.** We first bound the norm of \( L^{-1} \). From (2.17) it follows that

\[ \sup_{|\zeta| > 0} \frac{|f \circ \Phi \circ \Phi^\alpha - f \circ \Phi_t|}{|\zeta|^\gamma} = e^{\gamma t\lambda^\alpha} \sup_{\zeta > 0} \frac{|(f \circ \Phi^\alpha \exp\{t\lambda^\alpha\} - f) \circ \Phi_t|}{|\zeta|^\gamma \exp\{\gamma t\lambda^\alpha\}} \leq e^{\gamma t\lambda^\alpha} \left( \|L_3^\alpha f\| + 2\|f\| \right) \]

from this, it is easy to get the bound \( \|L^{-1}\|_\gamma \leq 5/\lambda_+ (1 - \gamma) \).

We choose \( C \geq \|L^{-1}\| \cdot \max\{1, 4\|\mathcal{F}\|_\gamma, 4\|(L_0H_0)^{-1}V\|_\gamma\} \). From (3.7), there exists a \( \gamma, \varepsilon \)-independent constant \( C_0 > 1 \) such that, for any \( f, \tilde{f} \) in the ball \( \mathcal{B}_\varepsilon \overset{\text{def}}{=} \{ f \in \mathcal{B} : \|f\|_\gamma \leq |\varepsilon|C \}, \)

\[ \|\mathcal{O}(f) - \mathcal{O}(\tilde{f})\|_\gamma, \|\mathcal{R}_\varepsilon^\alpha(f) - \mathcal{R}_\varepsilon^\alpha(\tilde{f})\|_\gamma \leq |\varepsilon|C_0 \|f - \tilde{f}\|_\gamma. \tag{3.14} \]

Indeed, it is possible to write \( \mathcal{O}(f) - \mathcal{O}(\tilde{f}) = \sum_{j=1}^5 \int_0^1 \partial_j \partial_t \mathcal{O} \circ (tf + (1-t)\tilde{f}) \) and similarly for \( \mathcal{R}_\varepsilon^\alpha \); furthermore, the H"older derivative of a product of functions is bounded by the product of the H"older derivatives of each functions. From (3.14) it follows

\[ \|S_\varepsilon^\alpha(f) - S_\varepsilon^\alpha(\tilde{f})\|_\gamma \leq |\varepsilon|3C_0 \|f - \tilde{f}\|_\gamma. \tag{3.15} \]

By the choice of \( C \) and using (3.15) for \( \tilde{f} \equiv 0 \), we have that, choosing \( \bar{\varepsilon} = \lambda_+ \bar{\varepsilon} L^{-1} S_\varepsilon \) sends \( \mathcal{B}_\varepsilon \) into itself. Moreover (3.15) implies that the application \( L^{-1} S_\varepsilon \) is a contraction in \( \mathcal{B}_\varepsilon \), by the previous choice, \( \bar{\varepsilon} < \lambda_+ \bar{\varepsilon} L^{-1} S_\varepsilon \). Since \( \mathcal{F} \) and \( V \) are analytic, the solution of (3.13) is unique in \( \mathcal{B}_\varepsilon \) and is the limit of a sequence of functions which are analytic in \( \{ \varepsilon \in \mathbb{C} : |\varepsilon| \leq \bar{\varepsilon} \} \); by Vitali theorem the solution is also analytic. \( \blacksquare \)

This Lemma concludes the proof of Theorem 1.

### 4. Proof of Theorem 2.

**4.1. Unstable Direction.** The second step toward the construction of an analytic SRB measure consist in constructing the perturbed unstable direction \( w_\varepsilon^+(g) \) and the associated Lyapunov exponent \( \lambda_\varepsilon^+(g) \). These quantities are both defined in (2.18).
As expected from the general theory[A], the unstable direction of the perturbed system \( w^+_\varepsilon \) is \textit{generically} not analytic in \( \varepsilon \). To construct the SRB measure we need unstable direction computed in the conjugated point \( h_\varepsilon \), which we will see to be analytic in \( \varepsilon \).

Calling \( v^+_\varepsilon \overset{\text{def}}{=} w^+_\varepsilon \circ h_\varepsilon \) and \( L_\varepsilon \overset{\text{def}}{=} \lambda^+ \circ h_\varepsilon \overset{\text{def}}{=} \lambda^+ + \delta L_\varepsilon \), we will compute (2.18) for time \( t \) replaced by \( T^\varepsilon_\varepsilon \) and position \( h_\varepsilon(g) \) rather than \( g \). Using also (2.11), we obtain:

\[
(T_{h_\varepsilon} \Phi^\varepsilon_{T^\varepsilon_\varepsilon}) v^+_\varepsilon = e^\int_0^t ds \left( \tau_s \circ \Phi_\varepsilon \right) (L^+_\varepsilon \circ \Phi_\varepsilon) \left( (v^+_\varepsilon \circ \Phi_t) \right). 
\]

\( \text{(4.1)} \)

\[ 4.2. \text{Construction of the Unstable Direction.} \] Proceeding as in the previous section, taking the time derivative of both sides of equation (4.1) at \( t = 0 \) we obtain:

\[
(T_{h_\varepsilon} \Phi^\varepsilon_0) v^+_\varepsilon - \frac{1}{\tau_\varepsilon} \frac{\det(g)}{4} (L_3 v^+_\varepsilon) = L_\varepsilon \cdot v^+_\varepsilon
\]

\( \text{(4.2)} \)

We now write \( v^+_\varepsilon \) as \( v^+_\varepsilon = w^+ + \sum_{a=0,3,-} \delta V^a \varphi^a \). Projecting along the direction \( w^+ \), calling \( F^\varepsilon = L + F \) and defining \( F^\varepsilon \) such that \( F = \sum_{a=0,3,-} F^a \varphi^a \) and \( F^\varepsilon \) such that \( F^\varepsilon = \sum_{a=0,3,-} F^a \varphi^a \), after some lengthy but straightforward algebra, reported in the Appendix, we get

\[
\delta L_\varepsilon = \frac{\det(g)}{4} \left[ \varepsilon F^{\varepsilon,+} (g) - \delta \tau_\varepsilon (g) - P^\varepsilon \left( \delta V^0_\varepsilon, \delta V^3_\varepsilon, \delta V^3_\varepsilon, \delta L_\varepsilon \right) \right]
\]

while, projecting along the other directions, we get

\[
\left[ L_3 - (\lambda^a - \lambda^+) \right] \delta V^a_\varepsilon (g) = \varepsilon F^a_\varepsilon (g) - \delta \tau_\varepsilon (g) + P^a_\varepsilon \left( \delta V^0_\varepsilon, \delta V^3_\varepsilon, \delta V^3_\varepsilon, \delta L_\varepsilon \right)
\]

where \( \{P^a_\varepsilon\}_{a=0,3,-} \) can be written as in (3.7). In order to solve (4.3) and (4.4), as for (3.6), we first need to replace the term \( 2 \delta \tau_\varepsilon \delta V^3_\varepsilon \) in the r.h.s. of (4.4), with the expression obtained by implicitly solving the equation for \( \alpha = 0 \):

\[
\delta V^a_\varepsilon (g) = \int_{-\infty}^0 ds \ e^{s \lambda^+} \left[ \varepsilon F^{\varepsilon,+} + P^\varepsilon \right] \circ \Phi_s ,
\]

for \( P^\varepsilon \circ \Phi_s = P^\varepsilon \left( \{ \delta V^a_\varepsilon \circ \Phi_s \}_{a=0,3,-}, \delta L_\varepsilon \circ \Phi_s \right) \). Substituting into (4.4), we get

\[
\left[ L_3 - (\lambda^a - \lambda^+) \right] \delta V^a_\varepsilon = \varepsilon \tilde{F}^a_\varepsilon + P^a_\varepsilon \left( \delta V^0_\varepsilon, \delta V^3_\varepsilon, \delta V^3_\varepsilon, \delta L_\varepsilon \right)
\]

(4.5)

for suitable \( \{\tilde{F}^a_\varepsilon\}_{a=0,3,-} \), which depend neither on \( \{\delta V^a_\varepsilon\}_{a=0,3,-} \), nor on \( \delta L_\varepsilon \), is linear in \( \varepsilon \) and Hölder continuous in \( g \). Moreover, \( \{P^a_\varepsilon\}_{a=0,3,-} \) are analytic in their arguments, and can be written as in (3.7), for suitable constants \( \{\tilde{C}_{j,a}\}_{j=1,\ldots,4} \), \( a=0,3,- \).

\[ 4.3. \text{Existence of the perturbed unstable direction.} \] Calling \( f^0 = \delta V^0_\varepsilon \), \( f^3 = \delta V^3_\varepsilon \), \( f^- = \delta V^-_\varepsilon \) and \( f^+ = \delta L^+_\varepsilon \), we can look for a solution of the (4.3) and (4.4) in the Banach space \( \mathcal{B} \) introduced in section 3.4. Again we introduce the operator:

\[
(Mf)^{\alpha}_{\varepsilon} \overset{\text{def}}{=} \begin{cases} f^a & \text{if } \alpha = + \\ L_3 - (\lambda^a - \lambda^+) & \text{if } \alpha = -, 0, 3 \end{cases}
\]

and the function

\[
T^\varepsilon_\varepsilon (f) \overset{\text{def}}{=} \begin{cases} \varepsilon \frac{\det(g)}{4} F^{\varepsilon,+} - \delta \tau_\varepsilon \frac{\det(g)}{4} + P^{\varepsilon,+} (f^0, f^3, f^-, f^+) & \text{for } \alpha = + \\ \varepsilon \tilde{F}^{\varepsilon,+} + P^\varepsilon (f^0, f^3, f^-, f^+) & \text{for } \alpha = -, 0, 3 \end{cases}
\]
and we prove the following lemma.

**Lemma 2.** There exists $\bar{\varepsilon} > 0$ such that, for any $\varepsilon : |\varepsilon| \leq \bar{\varepsilon}$, the equation

$$Mf = T_\varepsilon(f)$$

(4.7)

has unique solution in the ball of $\mathcal{B}$ of radius $\varepsilon C$, for a suitable $C$. Such a solution is analytic in $\varepsilon$.

**Proof.** It follows from arguments similar to those used in the proof of Lemma 1. □

Clearly the perturbed stable direction and Lyapunov exponent can be constructed in the very same way.

5. Proof of Theorem 3.

5.1. Markov Partition. It is worthwhile to remark that for topologically mixing Anosov flows the foliations $E^+$ and $E^-$ are not jointly integrable and therefore it is not possible to find a surface which contains a finite piece of the stable and unstable manifold of a given point (see [Pl]). This is why the following construction of the Markov partition, [B1] and [R], is slightly different from a naive generalization of the construction of a Markov partition for diffeomorphisms.

We first consider the unperturbed flow $\Phi$. Fixed $\delta > 0$, we define the local weak-stable and weak-unstable manifolds passing through $g$ as

$$W^3_{\delta} \pm(g) \overset{\text{def}}{=} \left\{ (\Phi_t \circ \Phi_{\pm}^\pm)(g) : |\zeta|, |t| < \delta \right\};$$

which are clearly $C^\omega$ manifolds. Let $\mathcal{D}$ be any closed $C^\omega$ disk of dimension 2 in $G_\varepsilon$, transverse in each point to the flow $\Phi$. Given two close points on $\mathcal{D}$, $g, g'$ with $d(g, g') \leq \alpha_1$, for $\alpha_1$ small enough,

$$(g,g')_\mathcal{D} \overset{\text{def}}{=} W^3_{\delta} - (g) \cap W^3_{\delta} + (g') \cap \mathcal{D}$$

(5.1)

consists of a single point. We will say that $T$ is a rectangle on $\mathcal{D}$ if $(g,g')_\mathcal{D} \in T$ for any $g, g' \in T$.

The manifolds $W^3_T(g) = \{(g',g')_\mathcal{D} : g' \in T\}$ and $W^3_T(g) = \{(g',g')_\mathcal{D} : g' \in T\}$ are the projection of the stable and of the unstable manifolds through $g$ on the rectangle $T$, which can be seen as:

$$T = \langle W^+_T(g), W^-_T(g) \rangle .$$

Given a family of closed rectangles $\{T_1, \ldots, T_N\}$ on disks $\{D_1, \ldots, D_N\}$ such that $T_i \subset \text{int} D_i$ and $T_i = \overline{\text{int} T_i}$, we will call it a proper family of rectangles if there exists $\alpha > 0$ such that $\mathcal{G}_\varepsilon = \bigcup_{j=1}^N \bigcup_{t \in [0, \alpha]} \Phi_{-t}(T_j)$; and for any $i \neq j$ at least one of the sets $D_i \cap \bigcup_{t \in [0, \alpha]} \Phi_t(D_j)$ and $D_j \cap \bigcup_{t \in [0, \alpha]} \Phi_t(D_i)$ is empty.

Let $\Pi = \bigcup_{j=1}^N T_j$ and define the ceiling function, $\theta : \Pi \to \mathbb{R}_+$, as the smallest strictly positive time required for $\Phi_t(g)$ to cross $\Pi$; and the Poincaré map, $\mathcal{H} : \Pi \to \Pi$, as $\mathcal{H}(g) = \Phi_{\theta(g)}(g)$.

Finally, the proper family of rectangles, $\{T_1, \ldots, T_N\}$, is called Markov partition if it satisfies following condition: for any $g \in T_i$ such that $\mathcal{H}^{\pm 1} (g) \in T_j$ one has that $\mathcal{H}^{\pm 1} (W^\mp (g)) \subset T_j$. In particular, it is possible to show that the flow $\Phi$ admits a Markov partition of the rectangles $\{T_1, \ldots, T_N\}$ on disks $\{D_1, \ldots, D_N\}$.

5.2. Symbolic dynamics

Let $A$ be the incidence matrix associated with $\mathcal{H}$, i.e.

$$A_{i,j} = \begin{cases} 1 & \text{if } \text{int} T_i \cap \mathcal{H}(\text{int} T_j) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

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Since the dynamics is mixing, there exists an integer \( k \) such that the matrix \( A^k \) has only non-zero entries. We introduce the space of sequences
\[
\Sigma_A \overset{\text{def}}{=} \left\{ \sigma \in \{1, \ldots, N\}^\mathbb{Z} : A_{\sigma_i, \sigma_{i+1}} = 1, i \in \mathbb{Z} \right\},
\]
the shift map, \( \rho : \Sigma_A \rightarrow \Sigma_A \), such that \( (\rho \sigma)_j = \sigma_{j+1} \) and the coding map, \( X : \Sigma_A \rightarrow \Pi \), such that \( X(\sigma) = \bigcap_{i=-\infty}^{+\infty} \mathcal{H}^{-i}(\text{int} T_{\sigma_i}) \). We remark that \( \mathcal{H} \circ X = X \circ \rho \); furthermore, endowing the space \( \Sigma_A \) with the distance \( d(\sigma, \sigma') = e^{-\nu(\mathcal{A} \circ \nu)} \), for \( \nu(\sigma, \sigma') = \max \{ n \in \mathbb{N} : \sigma_i = \sigma_i' \forall i : |i| \leq n \} \), the map \( \rho \) is continuous, and \( X \) is Hölder continuous.

Finally, the coding is inherited by all \( g \in \mathcal{G}_E \): after calling
\[
Y = \{ (\sigma, t) \in \Sigma_A \times \mathbb{R}_+ : 0 \leq t \leq (\theta \circ X)(\sigma) \}
\]
and identifying \( (\sigma, (\theta \circ X)(\sigma)) \) with \( (\rho \sigma, 0) \), let \( q : Y \rightarrow \mathcal{G}_E \) be the one-to-one map defined by \( q(\sigma, t) = (\Phi_t \circ X)(\sigma) \); then
\[
(\Phi_t \circ q)(\sigma, s) = q(\rho^k \sigma, t') \tag{5.3}
\]
for the unique \( k \) such that \( t' = t + s - \sum_{j=0}^{k-1} (\theta \circ X \circ \rho^j)(\sigma) \) satisfies \( 0 \leq t' < (\theta \circ X \circ \rho^k)(g) \).

5.3. **SRB measure.** Given a Hölder continuous \( f : \Sigma_a \rightarrow \mathbb{R} \), we can associate to it the equilibrium state with potential \( f \), i.e. an \( \rho \)-invariant, Gibbs measure \( \nu_f \) on \( \Sigma_A \), defined by the formal Hamiltonian
\[
H(\sigma) \overset{\text{def}}{=} \sum_{j=-\infty}^{+\infty} f(\rho^j \sigma), \tag{5.4}
\]
see [B2] for proofs and details.

Now, let \( \Lambda^+_t(g) \) be the Jacobian of the linear map \( T\Phi_t : E^+_g \rightarrow E^+_\Phi_t(g) \); and let
\[
\lambda^+_t(g) \overset{\text{def}}{=} - \frac{d \ln \Lambda^+_t(g)}{dt} \bigg|_{t=0}
\]
which exists and is analytic in \( g \). Finally we define the potential \( \tilde{f}^+ \) as
\[
\tilde{f}^+_t(g) \overset{\text{def}}{=} \int_0^t \theta(g) \, ds \left( \lambda^+_s \circ \Phi_s \right)(g). \tag{5.5}
\]
Given a continuous function \( \mathcal{O} \) on \( \mathcal{G}_E \) we can the SRB measure \( \mu_E \) for \( \Phi \) is given by
\[
\mu_E(\mathcal{O}) = \nu_{\tilde{f}^+ \circ X}(\mathcal{O} \circ X)
\]
where
\[
\tilde{O}(g) \overset{\text{def}}{=} \int_0^t \theta(g) \, ds \left( \mathcal{O} \circ \Phi_s \right)(g),
\]
see [BR] (theorem 5.1). Since \( \Phi \) is a Hamiltonian flow, \( \mu \) is the Lebesgue measure.

For the perturbed, non-Hamiltonian flow, \( \Phi_t \), the SRB measure is generally not absolutely continuous w.r.t. the Lebesgue measure. Contrary to the naive expectation, the rectangles \( T^t_j \overset{\text{def}}{=} \mathcal{H}(T_j) \) do not yield a Markov Partition, since they are not portions of smooth disks.
We first observe that the disk $D_i$, $i = 1, \ldots, N$, can be seen as the intersection a smooth disk $D_i$ of dimension 3 in $G$ with the energy surface $G\varepsilon$. In this way we can define the disks $D_i^\varepsilon = D_i \cup G\varepsilon$. Let now $\Delta_i$ be the open neighborhood of $D_i$ defined by $\Delta_i = \cup_{r: r < \delta} \Phi_i^r(D_i)$. On $\bigcup_{i=1}^{N} \Delta_i$ we can define, for $\varepsilon$ small enough, the maps $s_\varepsilon(g)$ as the solution of $\Phi_{s_\varepsilon(g)}^\varepsilon(g) \in D_i$ and $q_\varepsilon(g) = \Phi_{s_\varepsilon(g)}^\varepsilon(g)$.

We can define the map $p_\varepsilon: \bigcup_{i=1}^{N} D_i \rightarrow \bigcup_{i=1}^{N} D_i^\varepsilon$ as

$$p_\varepsilon(g) = q_\varepsilon \circ h_\varepsilon(g)$$

which is clearly analytic in $\varepsilon$ and Hölder continuous in $g$. It is easy to see that the sets $T_i^\varepsilon = p_\varepsilon(T_i)$ form a Markov partition for $\Phi^\varepsilon$ on $G\varepsilon$. We can define the perturbed ceiling function $\theta_\varepsilon : \Pi^\varepsilon = \bigcup_{j=1}^{N} T_j^\varepsilon \rightarrow \mathbb{R}^+$ and the perturbed Poincaré map $\mathcal{H}_\varepsilon : \Pi^\varepsilon \rightarrow \Pi^\varepsilon$, as $\mathcal{H}_\varepsilon(g) = \Phi_{s_\varepsilon(g)}^\varepsilon(g)$. Clearly $p_\varepsilon$ conjugates $\mathcal{H}$ with $\mathcal{H}_\varepsilon$. Finally, the coding map for the perturbed flow $\Phi_\varepsilon$, $X_\varepsilon : \Sigma_A \rightarrow \Pi^\varepsilon$ is given by $X_\varepsilon = p_\varepsilon \circ X$.

Given a Hölder continuous function $\mathcal{O}$, its average w.r.t. $\mu_\varepsilon^\varepsilon$ is given by

$$\mu_\varepsilon^\varepsilon(\mathcal{O}) = \nu_{f_\varepsilon^+ \circ X_\varepsilon}(\tilde{\mathcal{O}}_\varepsilon \circ X_\varepsilon)$$

(5.5)

where $f_\varepsilon^+, \tilde{\mathcal{O}}_\varepsilon : \Pi^\varepsilon \rightarrow \mathbb{R}$ are defined as before:

$$f_\varepsilon^+(g) = \int_0^{\theta_\varepsilon(g)} ds \left( \lambda_\varepsilon^+ + \Phi_\varepsilon^\varepsilon(g) \right), \quad \tilde{\mathcal{O}}_\varepsilon(g) = \int_0^{\theta_\varepsilon(g)} ds \left( \mathcal{O} \circ \Phi_\varepsilon^\varepsilon(g) \right).$$

We observe that $(\Phi_\varepsilon^\varepsilon \circ p_\varepsilon)(g) = (\Phi_{s_\varepsilon + (s_\varepsilon \circ h_\varepsilon)}^\varepsilon(g) \circ h_\varepsilon)(g)$. Calling $\tilde{\theta}_\varepsilon : \bigcup_{j=1}^{N} T_j^\varepsilon \rightarrow \mathbb{R}^+$ the ceiling function for the Hölder continuous manifold $\bigcup_{j=1}^{N} T_j^\varepsilon$, we also have $(\theta_\varepsilon \circ p_\varepsilon)(g) = (\tilde{\theta}_\varepsilon \circ h_\varepsilon)(g) + (s_\varepsilon \circ h_\varepsilon \circ \mathcal{H})(g) - (s_\varepsilon \circ h_\varepsilon)(g)$. Therefore

$$(f_\varepsilon^+ \circ p_\varepsilon)(g) = \int_0^{(\theta_\varepsilon \circ p_\varepsilon)(g)} ds \left( \lambda_\varepsilon^+ + \Phi_\varepsilon^\varepsilon \circ p_\varepsilon(g) \right)$$

$$= \int_0^{(\tilde{\theta}_\varepsilon \circ h_\varepsilon)(g) + (s_\varepsilon \circ h_\varepsilon \circ \mathcal{H})(g)} ds \left( \lambda_\varepsilon^+ + \Phi_\varepsilon^\varepsilon \circ h_\varepsilon(g) \right)$$

$$= \int_0^{(\tilde{\theta}_\varepsilon \circ h_\varepsilon)(g)} ds \left( \lambda_\varepsilon^+ + \Phi_\varepsilon^\varepsilon \circ h_\varepsilon(g) \right) + (\tilde{\mathcal{F}}_\varepsilon^+ \circ \mathcal{H})(g) - \tilde{\mathcal{F}}_\varepsilon^+(g)$$

for a suitable, Hölder continuous function $\tilde{\mathcal{F}}_\varepsilon^+ : \Pi \rightarrow \mathbb{R}$. It is well known that, due to its cocycle structure, the term $(\tilde{\mathcal{F}}_\varepsilon^+ \circ \mathcal{H})(g) - \tilde{\mathcal{F}}_\varepsilon^+(g)$ in the last line of (5.6) can be neglected. In the remaining integral we perform the change of integration variable from $s$ to $s' : s = T_j^\varepsilon(g)$ and we use the identities $(\theta_\varepsilon \circ h_\varepsilon)(g) = T_\varepsilon^\varepsilon(g)$ and $(\Phi_{T_\varepsilon^\varepsilon(g)} \circ h_\varepsilon)(g) = (h_\varepsilon \circ \Phi_\varepsilon^\varepsilon)(g)$ to get

$$\int_0^{(\tilde{\theta}_\varepsilon \circ h_\varepsilon)(g)} ds \left( \lambda_\varepsilon^+ + \Phi_\varepsilon^\varepsilon \circ h_\varepsilon(g) \right) = \int_0^{T_\varepsilon^\varepsilon(g)} ds \left[ \tau_\varepsilon \left( \lambda_\varepsilon^+ \circ h_\varepsilon \circ \Phi_\varepsilon^\varepsilon(g) \right) \right].$$

The last expression is clearly analytic in $\varepsilon$ due to the analyticity of $\tau_\varepsilon$ and of $\lambda_\varepsilon^+ \circ h_\varepsilon \equiv L_\varepsilon^+$. Observe that this integral is the potential we would have obtained considering directly the set $T_\varepsilon$ as a Markov partition.

To conclude the proof it is enough to observe that $\mathcal{O} \circ X_\varepsilon$ is clearly analytic in $\varepsilon$ since $\theta_\varepsilon \circ p_\varepsilon(g)$ is. This implies that $\nu_{f_\varepsilon^+ \circ X_\varepsilon}(\tilde{\mathcal{O}}_\varepsilon \circ X_\varepsilon)$ is the average, w.r.t. a Gibbs state defined by potentials analytically depending on $\varepsilon$, of a function analytically depending on $\varepsilon$. The theorem follows from standard results on Gibbs states, see [GBG].
6. Conclusion and outlook.

The geodesic motion on a surface with constant negative curvature is the simplest example of continuous time Anosov system. The structural stability of these systems, namely the existence of the conjugation between two close flows, was first proved in [A]. Later on, in [KKPW] and in [LMM] (in particular in appendix A) very general results on the regularity of $h_\varepsilon$ in $\varepsilon$ were proved using the contracting mapping theorem or implicit function theorem, a point of view introduced by Moser, [Mo] and Mather [Ma]. In all above papers only the case $\mathcal{G}_\varepsilon = \mathcal{G}$ has been considered.

Our technique is more in the spirit of [BKL] (see also [BFG] and [GBG]). While [KKPW] discusses the regularity of the topological entropy of the system, and consequently of the “equilibrium states” associated to a generic Hölder continuous “potential”, we study the analyticity of a special equilibrium state, the SRB measure. In order to do it, we also have to construct and to prove analyticity in $\varepsilon$ of the contraction rate of the unstable phase space.

Finally it would be very interesting to study a lattice of coupled Anosov flows like it was done for Anosov diffeomorphisms. In this case, already coupling two flows results in a very difficult problem. To obtain such a coupling is enough to consider the Hamiltonian flow generated by the Hamiltonian $H_\varepsilon(g_1, g_2) = H_0(g_1) + H_0(g_2) + \varepsilon V(g_1, g_2)$ for a suitable potential $V$ analytic and $\Gamma$ periodic in $g_1$ and $g_2$. The main difficulty here is that, for $\varepsilon = 0$, one has that $H_0(g_i)$, $i = 1, 2$, are two independent conserved quantities, while for $\varepsilon \neq 0$ they are no more conserved. This implies that the coupled system cannot be uniformly hyperbolic and most of the techniques used in this paper do not apply directly. Several works have addressed the problem of the SRB measure for non uniformly hyperbolic systems, see e.g. [HP]. We hope to come back on this problem in the future.

Appendix A1. Explicit computations.

A1.1. Explanation of (3.6). Taking the time derivative in $t = 0$, the l.h.s. of the first equation in (2.11) gives

$$\frac{\det(g)}{4} \left[ w^3(g) + \sum_{\alpha=0, \pm 3} \delta h_\varepsilon^\alpha(g) (\mathcal{L}_3 w^\alpha)(g) + \sum_{\alpha=0, \pm 3} (\mathcal{L}_3 \delta h_\varepsilon^\alpha)(g) w^\alpha(g) \right]$$

Therefore (3.6) follows from the identity $(w^3 \circ h_\varepsilon)(g) = w^3 + \sum_{\alpha=0, \pm 3} \delta h_\varepsilon^\alpha(g) (\mathcal{L}_\alpha w^3)(g)$ and from (3.5), which gives

$$\left( \frac{\det \circ h_\varepsilon(g)}{\det(g)} \right) = 1 - 2\delta h_\varepsilon^0(g) + (\delta h_\varepsilon^0)^2(g) - (\delta h_\varepsilon^3)^2(g) - \delta h_\varepsilon^+(g) \delta h_\varepsilon^-(g).$$

A1.2. Explanation of (4.3) and (4.4). Using the decomposition for $v_\varepsilon$ after (4.2), (4.2) reads:

$$\begin{align*}
(\mathcal{L}_+ \Phi_0^\varepsilon)(g) + \sum_{\alpha=0,3,-} (\mathcal{L}_a \Phi_0^\varepsilon)(g) \delta V^\alpha(g) &- \frac{1}{\tau_\varepsilon(g)} \frac{\det(g)}{4} (\mathcal{L}_3 w^+)(g) \\
- \frac{1}{\tau_\varepsilon(g)} \frac{\det(g)}{4} \sum_{\alpha=0,3,-} \delta V^\alpha(g) (\mathcal{L}_3 w^\alpha)(g) &- \frac{1}{\tau_\varepsilon(g)} \frac{\det(g)}{4} \sum_{\alpha=0,3,-} w^\alpha(g) (\mathcal{L}_3 \delta V^\alpha)(g) \\
&= L_\varepsilon(g) w^+(g) + L_\varepsilon(g) \sum_{\alpha=0,3,-} \delta V^\alpha(g) w^\alpha(g) + (T_{h_\varepsilon(g)} \Phi_0^\varepsilon - T_g \Phi_0^\varepsilon) v_\varepsilon(g) .
\end{align*}$$

(4.1)
From (2.9) we get:
\[
\frac{\text{det}(g)}{4} (L_+ w^3) + \frac{\text{det}(g)}{4} \sum_{a=0,3,-} (L_a w^3) \delta V^a + \frac{\text{det}(g)}{2} w^3 \delta V^0
\]
\[= - \frac{1}{\tau_\varepsilon} \frac{\text{det}(g)}{4} (L_3 w^+) - \frac{1}{\tau_\varepsilon} \frac{\text{det}(g)}{4} \sum_{a=0,3,-} \delta V^a (L_3 w^a) - \frac{1}{\tau_\varepsilon} \frac{\text{det}(g)}{4} \sum_{a=0,3,-} w^a (L_3 \delta V^a)
\]
\[= L_\varepsilon \cdot w^+ + L_\varepsilon \sum_{a=0,3,-} \delta V^a w^a
\]
\[- \frac{\varepsilon \text{det}(g)}{4} (L_+ F) - \frac{\varepsilon \text{det}(g)}{4} \sum_{a=0,3,-} (L_a F) \delta V^a - \frac{\varepsilon \text{det}(g)}{2} F \delta V^0 + (\delta V_0 F_\varepsilon^* - T_\varepsilon \Phi_\varepsilon^* \delta V_0) v_\varepsilon(g)
\]
Using the identity following (3.1) and the decomposition $L_\varepsilon = \lambda^+ + \delta L_\varepsilon$ we obtain
\[
\delta \tau_\varepsilon (L_+ w^+) - \sum_{a=0,3,-} (L_3 \delta V^a - (\lambda^a - \lambda^+) \delta V^a) w^a + 2 w^3 \delta V^0
\]
\[= 4 \frac{\text{det}}{\text{det}} \delta L_\varepsilon \cdot w^+ - \varepsilon (L_+ F) + \mathcal{P}_\varepsilon (\delta V_0^0, \delta V_3^0, \delta V_3^-, \delta L_\varepsilon) .
\]
Projecting along the direction $w^+$, calling $\mathcal{F}^{\alpha(+)_{\varepsilon}} F_\alpha F$ and defining $\mathcal{F}^{\alpha}, P^{\alpha}$ such that $F = \sum_{a=0,3,\pm} \mathcal{F}^{\alpha} w^a$ and similarly for $P^{\alpha}$; finally defining $\mathcal{F}^{\alpha,\beta}$ such that $\mathcal{F}^{\alpha,\beta} = \sum_{a=0,3,\pm} \mathcal{F}^{\alpha,\beta} w^a$, we get (4.3) and (4.4).

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