The strong gravitational lensing for a gravitational source with an $f(R)$ global monopole

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Abstract

We investigate the gravitational lensing in strong field limit of a Schwarzschild black hole with a solid deficit angle owing to global monopole within the context of the $f(R)$ gravity theory. We show that the deflection angle and the strong field coefficients such as the minimum impact parameter, angular separation and the relative magnification are related not only to the monopole parameter but also to the $f(R)$ correction $\psi_0$. It is interesting that the tiny $f(R)$ parameter $\psi_0$ will make greater deviation on the angle and coefficients, offering a significant way to explore some possible distinct signatures of the Schwarzschild black hole with an $f(R)$ global monopole.

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I. Introduction

It is well known that the light deflection is one of the phenomena of gravitational lensing which is an important application of general relativity [1-7]. The massive source which causes the deflection is called gravitational lens. We can make use of the gravitational lensing to investigate the distant stars no matter they are bright or dim. In general, the deflection angle of photons passing close to a compact and massive source is expressed in integral forms, so it is difficult to discuss the detailed relations between the angle and the gravitational source. Alternatively we perform the calculation of the integral expressions in the limiting cases. Here we focus on the strong field approximation for the case that the light passes very close to a heavy compact body such as a black hole. Bozza put forward an analytical method to show that there exists a logarithmic divergence of the deflection angle in the proximity of the photon sphere [8-13]. The strong gravitational lensing was treated in a Schwarzschild black hole and a Reissner-Nordstrom black hole [8], a GMGHS charged black hole [14], a spinning black hole [15, 16], a braneworld black hole [17, 18], the deformed Horava-Lifshitz black hole [19], the black hole with global monopole [20], the black hole pierced by a cosmic string [21].

Several types of topological objects such as domain walls, cosmic strings and monopole may have been formed during the vacuum phase transition in the early Universe [22,23]. The basic idea is that these topological defects appeared due to breakdown of local or global gauge symmetries. A global monopole is a spherical symmetric topological defect formed in the phase transition of a system composed by a self-coupling triplet of scalar field whose original global O(3) symmetry is spontaneously broken to U(1). The properties of the metric outside a monopole are investigated in [24], which also show that the monopole exerts practically no gravitational force on nonrelativistic matter, but the space around it has a deficit solid angles and all light rays are deflected by the same angle. Recently $f(R)$ gravity is a type of modified gravity theory first proposed by Buchdahl [25]. As a possible alternative, $f(R)$ gravity theory has been introduced to explain the accelerated expansion of the universe instead of adding unknown forms of dark energy or dark matter [26-28]. The gravitational field of a global monopole in the modified gravity theory has been discussed [29]. Further the classical motion of a massive test particle in the spacetime of a global monopole in the context of $f(R)$ gravity theory is also studied [30].

Now we plan to probe the strong gravitational lensing on the massive source swallowing a global monopole governed by $f(R)$ theory. As mentioned above we have considered the gravitational lens equation for the massive global monopole in the strong field limit to exhibit the the relation between the deflection angle and the deficit solid angle subject to the monopole model parameters in the standard general relativity [20]. In the $f(R)$ theory the modified terms appear in the metric around the gravitational source with a global monopole and the modified metric is certainly different from the original one and is also not similar to the others. It is necessary to research on the influence from $f(R)$ theory on the deflection angle around the source. In section II we review the metric of massive global monopole in $f(R)$ gravity theory at first and present the expressions of the
strong gravitational lensing for the $f(R)$ global monopole. We perform the numerical estimation of gravitational lensing observables in the strong field limit for the global monopole with $f(R)$ influence in section III. We plot the dependence of the observational gravitational lensing parameters on the $f(R)$ correction. We discuss our results in the end.

II. The deflection angle of a massive source with a $f(R)$ global monopole

The simplest model that gives rise to global monopole is described by the Lagrangian [24],

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - \frac{1}{4}\lambda(\phi^a \phi^a - \eta^2)^2$$

The triplet of field configuration showing a monopole is

$$\phi^a = \eta h(r) x^a,$$

Where $x^a x^a = r^2$. Here $\lambda$ and $\eta$ are parameters in this model. The model has a global $O(3)$ symmetry, which is spontaneously broken to $U(1)$. In order to couple this matter field to the gravitational field equation in the $f(R)$ theory and obtain their spherically symmetric solution, we adopt the line element as follow,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

In the $f(R)$ gravity theory, the action is given by,

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g}f(R) + S_m$$

where $f(R)$ is an analytical function of Ricci scalar $R$ and $\kappa = 8\pi G$. $G$ is the Newton constant. $g$ is the determinant of metric tensor. $S_m$ is the action associated with the matter fields. According to the metric formalism, the field equation leads,

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \Box F(R) = \kappa T_{\mu\nu}$$

where $F(R) = \frac{df(R)}{dr}$ and $T_{\mu\nu}$ is the minimally coupled energy-momentum tensor. Under the weak field approximation that assumes the components of metric tensor like $A(r) = 1 + a(r)$ and $B(r) = 1 + b(r)$ with $|a(r)|$ and $|b(r)|$ being smaller than unity, the field equation (5) was solved [29]. The metric is found finally,

$$A = B^{-1} = 1 - 8\pi G \eta^2 - \frac{2GM}{r} - \psi_0 r$$

Here the modification theory of gravity corresponds to a small correction on the general relativity like $F(R(r)) = 1 + \psi(r)$ with $\psi(r) \ll 1$. It can also be taken as the simplest analytical function of the radial coordinate $\psi(r) = \psi_0 r$. In this case the factor $\psi_0$ reflects the deviation of standard general relativity. Here the correction $\psi_0 r$ in the metric is linear, which is different from those in
the cases such as de Sitter spacetime and Reissner-Nordstrom metric etc.. It should be pointed that for a typical grand unified theory the parameter $\eta$ is of the order $10^{16}\text{GeV}$, which means $8\pi G\eta^2 \approx 10^{-5}$. The mass parameter is $M \sim M_{\text{core}}$ and $M_{\text{core}}$ is very small.

We choose that both the observer and the gravitational source lie in the equatorial plane with condition $\theta = \frac{\pi}{2}$. The whole trajectory of the photon is limited to the same plane. On the equatorial plane the metric reads,

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)d\varphi^2$$

(7)

where

$$C(r) = r^2$$

(8)

The functions $A(r)$ and $B(r)$ are in the form of Eq.(6). The deflection angle for the electromagnetic ray coming from a remote place can be expressed as a function of the closest approach,

$$\alpha(r_0) = I(r_0) - \pi$$

(9)

and

$$I(r_0) = 2 \int_{r_0}^{\infty} \frac{\sqrt{B(r)}}{\sqrt{C(r)}} \sqrt{\frac{C(r)A(r_0)}{C(r_0)A(r)}} - 1 \, dr$$

(10)

where $r_0$ is the minimum distance from the photon path to the source. It requires that the deflection angle turns to be infinite, meaning that the denominator of expression (10) is equal to the zero. To achieve this aim, we solve the equation $C'(r)C(r) = A'(r)A(r)$ to obtain the radius of the photon sphere. Certainly the closest approach distant $r_0$ must be larger than the radius of photon sphere or the photon will move around the gravitational source forever instead if escaping from the source. The radius of the photon sphere in the $f(R)$ global monopole metric is given by,

$$r_m = \frac{1 - 8\pi G\eta^2 - \sqrt{(1 - 8\pi G\eta^2) - 6GM\psi_0}}{\psi_0}$$

(11)

When we neglect the influence from $f(R)$ theory $\psi_0 = 0$, the photon sphere radius (11) will recover to that of metric by massive object involving global monopole within the frame of Einstein’s general relativity. The general formalism of deflection angle of strong field gravitational lensing worked by Bozza [8] can be written as,

$$\alpha(\theta) = -\tilde{a} \ln\left(\frac{\theta D_{\text{OL}}}{u_m} - 1\right) + \tilde{b} + O(u - u_m)$$

(12)

This issue is utilized to describe the propagation of electromagnetic ray close to the photon sphere. In order to know the coefficients in the expression (12), we expand the deflection angle (9) around the radius of photon $r_m$. First we introduce,
\[ z = 1 - \frac{r_0}{r} \]  

We define some functions as follow,

\[ R(z, r_0) = \frac{2\sqrt{C(r_0)A(r)B(r)}}{C'(r)} \frac{r_0}{(1 - z)^2} \]  

\[ f(z, r_0) = \frac{1}{\sqrt{A(r_0) - \frac{C(r_0)}{C(r)}}} \]  

\[ f_0(z, r_0) = \frac{1}{\sqrt{\alpha z + \beta z^2}} \]  

\[ I_D(r_0) = \int_0^1 R(0, r_m)f_0(z, r_0)dz \]  

\[ I_R(r_0) = \int_0^1 [R(z, r_0)f(z, r_0) - R(0, r_m)f_0(z, r_0)]dz \]

where

\[ \alpha = \frac{r_0}{C_0}(C_0A_0 - C_0A'_0) \]  

\[ \beta = \frac{r_0}{2C_0^3}[2(A'_0C_0 - A_0C'_0)(r_0C_0C'_0 - C_0^2) - r_0(A''_0C_0 - A_0C''_0)C_0^2] \]

with \( A_0 = A(r_0), C_0 = C(r_0) \). In Eq. (12) \( D_{OL} \) means the distance between observer and gravitational lens. By conservation of the angular momentum around the source, the closest approach distance is related to the impact parameter \( u \) defined as \( u = \sqrt{C(r_0)/A(r_0)} \). Further the minimum impact parameter corresponding to \( r_0 = r_m \) is certainly \( u_m = u \mid_{r_0=r_m} \). The strong field limit coefficients \( \bar{a} \) and \( \bar{b} \) can be expressed as,

\[ \bar{a} = \frac{R(0, r_m)}{2\sqrt{\beta_m}} \]  

\[ \bar{b} = -\pi + I_R(r_m) + \bar{a} \ln \frac{2\beta_m}{A(r_m)} \]

where

\[ \beta_m = \beta \mid_{r_0=r_m} \]  

In the context of \( f(R) \) gravity theory we substitute the Barriola-Vilenkin-monopole type metric (7) into the derivation above to show that,

\[ R(z, r_0) = 2 \]  

(24)
and

\[ f(z, r_0) = \left[ -\frac{2GM}{r_0} z^3 + \left( \frac{6GM}{r_0} - l_0 \right) z^2 + \left( 2l_0 - \frac{6GM}{r_0} - \psi_0 r_0 \right) z \right]^{-\frac{1}{2}} \]  

(25)

with \( l_0 = 1 - 8\pi G\eta^2 \). It is clear that the function \( R(z, r_0) \) is regular and the function \( f(z, r_0) \) diverges as \( z \to 0 \). We expand the argument of the square root in \( f(z, r_0) \) to the second order in \( z \),

\[ f_0(z, r_0) = \frac{1}{\sqrt{\alpha z + \beta z^2}} \]  

(26)

with

\[ \alpha = 2l_0 - \frac{6GM}{r_0} - \psi_0 r_0 \]  

(27)

\[ \beta = \frac{6GM}{r_0} - l_0 \]  

(28)

According to the definition above and Eq. (21) and (22), we derive the coefficients \( \bar{a}, \bar{b} \), and minimum impact parameter \( u_m \) of the deflection angle under the deviation of Einstein’s general relativity,

\[ \bar{a} = \frac{1}{(l_0^2 - 6GM\psi_0)^{\frac{1}{4}}} \]  

(29)

\[ \bar{b} = -\pi + \frac{1}{h^{\frac{1}{2}}} \ln \left( \frac{3(\sqrt{3}l_0 - 2\sqrt{h} + \sqrt{h})^4}{8\sqrt{h}(2\sqrt{h} - l_0)} \right) \]  

(30)

\[ u_m = \frac{l_0 - \sqrt{h}}{\psi_0 \sqrt{2\sqrt{h} - l_0}} \]  

(31)

where \( h = l_0^2 - 6GM\psi_0 \). In the strong field limit the deflection angle of massive source with deficit angle involving the correction from modified gravity approach is,

\[ \alpha(\theta) = -\frac{1}{(l_0^2 - 6GM\psi_0)^{\frac{1}{4}}} \ln \left( \frac{\theta D_{OL}}{\frac{\eta}{\psi_0} \sqrt{\frac{3}{2\sqrt{h} - l_0}}} \right) - 1 \]  

\[ + \frac{1}{h^{\frac{1}{2}}} \ln \left( \frac{3(\sqrt{3}l_0 - 2\sqrt{h} + \sqrt{h})^4}{8\sqrt{h}(2\sqrt{h} - l_0)} \right) - \pi \]  

(32)

Our results including the strong field limit coefficients \( \bar{a}, \bar{b} \), the deflection angle and the minimum impact parameter \( u_m \) of the angle contain the factor \( \psi_0 \) which stands for the influence of \( f(R) \) theory. If we exclude the \( f(R) \)-correction as \( \psi_0 = 0 \), both of the coefficients \( \bar{a} \) and \( \bar{b} \), parameter \( u_m \) and the deflection angle \( \alpha(\theta) \) will recover to be those of the Schwarzschild black hole with a global monopole controlled by Einstein’s general relativity [20]. When the factor \( \psi_0 \) appear in our expressions, certainly these coefficients, parameter and angle will change. The strong field gravitational lensing can help us to confirm whether the Einstein’s general relativity need to be revised.
We relate the position and the magnification to the strong field limit coefficients for the sake of comparing our results with the observable evidence. The position of the source and the images are related through the lens equation derived in [9] given by,

$$\beta = \theta - \frac{D_{LS}}{D_{OL}} \triangle \alpha_n$$  \hspace{1cm} (33)

where $D_{LS}$ represents the distance between the lens and the source. Here $D_{OS} = D_{OL} + D_{LS}$. $\beta$ denotes the angular separation between the source and the lens. $\theta$ is the angular separation between the lens and the image. The offset of the deflection angle is expressed as $\triangle \alpha_n = \alpha(\theta) - 2n\pi$ by subtracting all the times run around the source by photons. Due to $u_m \ll D_{OL}$ the position of the $n$-th relativistic image can be approximated as,

$$\theta_n = \theta^0_n + \frac{u_m e_n (\beta - \theta^0_n) D_{OS}}{\bar{a}D_{LS}D_{OL}}$$  \hspace{1cm} (34)

where

$$e_n = e^{\frac{\beta - 2n\pi}{a}}$$  \hspace{1cm} (35)

while the second term in the right-hand side of Eq. (34) is much smaller than $\theta^0_n$ and we introduce $\theta^0_n$ as $\alpha(\theta^0_n) = 2n\pi$. The magnification of $n$-th relativistic image is the inverse of the Jacobian evaluated at the position of the image and is obtained as,

$$\mu_n = \frac{1}{(\frac{\partial}{\partial \theta})^{\theta^0_n}_{\theta_n}}$$

$$= \frac{u_m e_n (1 + e_n) D_{OS}}{\bar{a}\beta D_{LS}D_{OL}}$$  \hspace{1cm} (36)

In the limit $n \to \infty$ the asymptotic position of approached by a set of images $\theta_\infty$ relates to the minimum impact parameter as,

$$u_m = D_{OL} \theta_\infty$$  \hspace{1cm} (37)

As an observable the angular separation between the first image and the others is defined as,

$$s = \theta_1 - \theta_\infty$$  \hspace{1cm} (38)

where $\theta_1$ represents the outermost image in the situation that the outermost one is thought as a single image and all the remaining ones are packed together at $\theta_\infty$. As another observable the ratio of the flux from the first image and the flux of all the other images is,

$$r = \frac{\mu_1}{\sum_{n=2}^{\infty} \mu_n}$$  \hspace{1cm} (39)

According to $e^{\frac{2\pi}{a}} \gg 1$ and $e^{\frac{\beta}{b}} \sim 1$, these observables can be written in terms of the deflection angle parameters $\bar{a}$ and $\bar{b}$ as,
$$s = \theta_\infty e^{\frac{\bar{b} - \bar{a}}{\bar{a}}} \quad (40)$$

$$r = e^{\frac{2\bar{a}}{\bar{a}}} \quad (41)$$

It is significant that the strong field limit coefficients such as $\bar{a}$, $\bar{b}$ and $u_m$ are directly connected to the observables like $r$ and $s$. It is then possible for us to probe whether the original general relativity need to be generalized in virtue of the strong field gravitational lensing for a Schwarzschild black hole with a global monopole.

III. Numerical estimation of observables in the strong field limit in the context of $f(R)$ theory

We should estimate the numerical values of the coefficients and observables of gravitational lensing in the strong field limit for the massive source involving a global monopole within the frame of $f(R)$ theory to show the correction to the standard general relativity. We calculate these coefficients and observables according to equations above and show their dependence on the factor $\psi_0$ standing for the deviation of Einstein’s gravity in the figures. From Fig. 1 we plot the several curves showing that the increasing global monopole model parameter leads the minimum impact parameter $u_m$ greater for some different value of $\psi_0$. These curves are similar. Although the factor $\psi_0$ is tiny as mentioned above, the difference among the curves is manifest when the global monopole parameter $\eta$ is large enough. In Fig. 2 we choose $\frac{\theta D_{\text{OL}}}{GM} = 1$ and show the relation between the deflection angle and the monopole model parameter $\eta$ for various value of correction factor $\psi_0$. The influence from the correction is evident although the shapes of these curves resemble each other. With different value of factor $\psi_0$, it is shown that with the increase of model parameter $\eta$, all of the angular separations become larger while all of the relative magnifications decrease in the Fig. 3 and Fig. 4. The curves of the angular separation subject to the factor $\psi_0$ with different value are similar. The curves of relative magnification with respect to the different $\psi_0$ are also similar. The difference from the factor $\psi_0$ is manifest in the Figure 3 and 4. The strong gravitational lensing for massive source with a global monopole is efficient to study the $f(R)$ gravity theory because the deviation owing to the correction to the Einstein’s general relativity is obvious although the correction is very small.

IV. Summary

Here we investigate gravitational lensing in the strong field limit for the Schwarzschild black hole spacetime with a solid deficit angle owing to a global monopole in the context of $f(R)$ gravity theory. The linear term which is proportional to the radial coordinate represent the influence from the extended theory of gravity and is added to the background metric. The linear correction is completely different from the de Sitter case. The curves of the dependence of the minimum impact
parameter $u_m$, the relative flux $r$ and the angular separation $s$ on the monopole model parameter $\eta$ due to the correct factor $\psi_0$ with different value respectively are similar themselves. The difference among the curve shapes are manifest although the factor $\psi_0$ is tiny. It means that any slight modification embodied in the metric will exhibit this revisal appearing in the gravitational lensing in an amplification way. The effect from the correction to the original gravity theory is evident, which provides us a way to confirm whether the Einstein’s general relativity needs to be generalized.

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Figure 1: The solid, dotted and dashed curves of the dependence of the minimum impact parameter $u_m$ in the unit of $GM$ for a strong field limit in the Schwarzschild black hole with an $f(R)$ global monopole on the model parameter $G\eta^2$ for correct factor $\psi_0 = 0.02, 0.01, 0$ respectively.
Figure 2: The solid, dot and dashed curves correspond to the dependence of the deflection angle $\alpha$ in the Schwarzschild black hole with an $f(R)$ global monopole on model parameter $G\eta^2$ for correct factor $\psi_0 = 0.02, 0.01, 0$ respectively.
Figure 3: The solid, dot and dashed curves correspond to the dependence of the angular separation $s$ in the Schwarzschild black hole with an $f(R)$ global monopole on model parameter $G\eta^2$ in unit of $\theta_\infty$ for correct factor $\psi_0 = 0.02, 0.01, 0$ respectively, the asymptotic position of approached by a set of images.
Figure 4: The solid, dot and dashed curves correspond to the dependence of the relative flux $r$ in the Schwarzschild black hole with an $f(R)$ global monopole on model parameter $G\eta^2$ for correct factor $\psi_0 = 0.02, 0.01, 0$ respectively.