World sheets of spinning particles

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The classical spinning particles are considered such that quantization of classical model leads to an irreducible massive representation of the Poincaré group. The class of gauge equivalent classical particle world lines is shown to form a $((d+1)/2)$-dimensional world sheet in $d$-dimensional Minkowski space, irrespectively to any specifics of the classical model. For massive spinning particles in $d = 3, 4$, the world sheets are shown to be circular cylinders. The radius of cylinder is fixed by representation. In higher dimensions, particle’s world sheet turns out to be a toroidal cylinder $\mathbb{R} \times T^D$, $D = [(d-1)/2]$. Proceeding from the fact that the world lines of irreducible classical spinning particles are cylindrical curves, while all the lines are gauge equivalent on the same world sheet, we suggest a method to deduce the classical equations of motion for particles and also to find their gauge symmetries. In $d = 3$ Minkowski space, the spinning particle path is defined by a single fourth-order differential equation having two zero-order gauge symmetries. The equation defines particle’s path in Minkowski space, and it does not involve auxiliary variables. A special case is also considered of cylindric null curves, which are defined by a different system of equations. It is shown that the cylindric null curves also correspond to irreducible massive spinning particles. For the higher-derivative equation of motion of the irreducible massive spinning particle, we deduce the equivalent second-order formulation involving an auxiliary variable. The second-order formulation agrees with a previously known spinning particle model.

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I. INTRODUCTION

The history of classical spinning particle models has developed over nine decades starting from the work by Frenkel\textsuperscript{1}. For a brief summary of the first seventy years and corresponding bibliography we refer to review\textsuperscript{2}. In the introduction, we discuss some generalities about irreducible classical spinning particle models, and then explain the idea of describing the particle dynamics by the world sheets of certain type.

The common feature shared by all the classical spinning particle models is that their degrees of freedom include, besides particle’s position in space, also the position in some “internal space.” In geometric terms, the configuration space $\mathcal{M}$ of spinning particle is a fiber bundle over Minkowski space, with coordinates on fibers describing configurations of spinning degrees of freedom. $\mathbb{Z}_2$-grading can be assigned to the fibers, so the spinning degrees of freedom can be described by Grassmann-odd or -even coordinates. Be the fibers even or odd, they can describe both integer and half-integer spins. Once the Poincaré-invariant action functional is found for the trajectories on $\mathcal{M}$, the particle model is defined at classical level, and it can be quantized.

Once the classical model has Poincaré symmetry, the space of quantum states should carry the Poincaré group representation which can be reducible, or irreducible. The Kirillov-Konstant-Souriau method\textsuperscript{3–5} tells us that if the quantum mechanical system realizes irreducible representation, the classical limit is the dynamical system corresponding to the coadjoint orbit of the group. The classical action functional of the system is defined by symplectic form on the co-orbit. In this way, we know that the physical phase space of the classical irreducible massive spinning particle is a coadjoint orbit of the Poincaré group corresponding to nonvanishing mass and spin. The latter fact means\textsuperscript{6} that for any irreducible classical spinning particle model, irrespectively to any specifics of fibers over Minkowski space chosen to describe spinning degrees of freedom, all the on-shell gauge invariant observables have to be functions of the conserved momentum $p$, and angular momentum $J$. The admissible values of $p$ and $J$ are restricted only by the spin-shell and mass-shell constraints.

Given the above-mentioned understanding of irreducibility at classical level, the spinning particle models are typically constructed in three main stages. First, the particle’s configuration space $\mathcal{M}$ is chosen to be a fiber bundle over Minkowski space. This bundle is given by the Cartesian product $\mathcal{M} = \mathbb{R}^{1,d-1} \times S$ of Minkowski space...

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to some manifold $S$ (the typical fiber). The manifold $S$ is supposed to be a homogeneous space of the Poincaré group. In the old-fashioned terminology, the models are categorized by types of coordinates introduced in the internal space $S$. In this way, the vectorial, tensorial, spinorial, twistorial, etc. types of spinning particle models are distinguished\textsuperscript{2}. At the second stage, all the Poincaré invariants are identified of the trajectories on $M$ (usually without higher derivatives), and the most general ansatz is constructed for the action functional. At the third stage, the action is specified in the way that makes the model irreducible. The latter means that all the on-shell gauge invariants are functions of Noether’s momentum $p$ and angular momentum $J$. These conserved quantities must satisfy the mass-shell and spin-shell conditions, being free from any other constraint.

In works\textsuperscript{22,23}, the configuration space of the spinning particle has been suggested to be the product of Minkowski space and 2-sphere, $M_6 = \mathbb{R}^{1,3} \times S^2$. These models can be viewed as minimal in the sense that $S^2$ is a manifold of the minimal possible dimension where the Poincaré group acts transitively. In the paper\textsuperscript{22}, two Poincaré invariants were found of the trajectories on $M_6$, while in paper\textsuperscript{23} one more Poincaré invariant was found, being of Wess-Zumino type. Once all the invariants of trajectories on $M_6$ are included in the action, the model becomes universal. Depending on the constant parameters in the Lagrangian, the model is able to describe the irreducible massive and massless spinning particle, and irreducible particle with continuous helicity. The model without third invariant\textsuperscript{22} describes the irreducible massive spinning particle, though it admits inclusion of interaction only with the constant curvature gravity\textsuperscript{22} and constant electromagnetic field. The minimal models of irreducible spinning particle models are also known for higher dimensions\textsuperscript{16–18} and references therein. The most frequent common feature of the reformulations is that some linear space $\mathcal{L}$ of higher dimension is considered as the phase space of spinning particle instead of $T^*M_6$, while some extra constraints are imposed on linear coordinates. Most typically, the constraints are quadratic in coordinates. They reduce $\mathcal{L}$ to $T^*M_6$. For example, the “bilocal” irreducible spinning particle model was recently developed in Ref.\textsuperscript{15} with the phase space being a squared cotangent bundle to Minkowski space, $\mathcal{L} = T^*\mathbb{R}^{1,3} \times T^*\mathbb{R}^{1,3}$. The quadratic constraints are imposed on $\mathcal{L}$ reducing the dynamics to sub-bundle $T^*M_6 \subset \mathcal{L}$, and the model turns out equivalent to that proposed in\textsuperscript{22}. The minimal models of irreducible spinning particle are also known for higher dimensions\textsuperscript{16–18} and in $d = 9–15$. In the latter case, the configuration space is $M_4 = \mathbb{R}^{1,2} \times S^3$.

A somewhat special class of spinning particle models are higher-derivative theories. No internal space is engaged, while the spinning degrees of freedom come from extra derivatives involved in the equations of motion in Minkowski space. The first model with a Lagrangian involving the arc length and curvature of particle’s world line was suggested by Pisarski\textsuperscript{21}. Later, the particle models with higher-derivative invariants included in the Lagrangian were studied from various viewpoints. We mention here only some of these works which particularly address the issue of irreducibility of the models. Once the arc length and curvature are both included linearly in the action, the model is reducible with the certain Regge trajectory connecting mass and spin\textsuperscript{22}. It was also noticed that the linear in curvature Lagrangian has an extra gauge symmetry of W-type and describes the irreducible massless spinning particle\textsuperscript{23–25}. These conclusions extend to a more general action and to higher dimensions: linear in curvature actions describe irreducible massless spinning particles, while any other combination of the arc length, and/or higher-derivative invariants results in reducible representation\textsuperscript{25–27}. It is also relevant to mention the model with the Lagrangian, being the curvature, with the trajectories restricted to the class of null curves\textsuperscript{28,29}. It corresponds to a reducible representation involving massive states\textsuperscript{28,29}. The attractive feature of higher-derivative spinning particle models is that they describe the dynamics of point particles in terms of lines in Minkowski space, without a recourse to any internal space. The irreducible models are known in terms of Minkowski space trajectories only for massless particles, while all the known higher-derivative models for massive particles are reducible. In this article, we show that the irreducible massive spinning particle dynamics can be described by higher-derivative equations for its paths in Minkowski space, without introducing any auxiliary configuration space to accommodate the spinning degrees of freedom.

Let us announce the key observation we proceed from: once the massive spinning particle is irreducible, the equivalence class of classical particle paths in $d$-dimensional Minkowski space forms a $[(d + 1)/2]$-dimensional surface. By equivalent paths we understand the trajectories which are connected by gauge transformations. In other words, the trajectories of irreducible massive spinning particle are world sheets rather than world lines. The space of all world sheets in Minkowski space turns out to be isomorphic to the Poincaré group co-orbit for the corresponding representation. The geometry of the spinning particle world sheets turns out very simple. In $d = 3$ and $d = 4$, the world sheets are $2d$ cylinders of fixed radius with timelike axis. The cylinder axis is directed along the particle’s momentum, while the position of the axis in space is determined by the particle’s angular momentum. The radius of cylinder is fixed by spin. In higher dimensions, the particle world sheets are the toroidal cylinders, $\mathbb{R} \times T^{(d−1)/2}$, where $T^{(d−1)/2}$ is $[(d−1)/2]$-torus. The radii of the torus are fixed by the eigenvalues of the Casimir

\textsuperscript{1} In $d = 3$, the more general Lagrangian, being a linear combination of the curvature and torsion, has been considered in\textsuperscript{30}. 
operators corresponding to an irreducible representation. There is a subtle issue related to the fact that different world sheets can intersect with each other, so some special world lines can belong to several world sheets. We will term the world lines atypical paths if they belong to different world sheets. Once the world line belongs to a unique world sheet, it is termed a typical path. Given the typical path, it defines the world sheet, while the atypical one does not. As we shall see, all the typical paths are gauge equivalent on the same world sheet, while the atypical paths do not mix up by gauge transformations with any other line. As the physical phase space of the irreducible particle, being understood as the Poincaré group co-orbit, is isomorphic to the variety of the particle world sheets, we consider only typical world lines as equivalence class of observable paths. The atypical lines are excluded, because they do not represent a unique world sheet.

In minimal models of irreducible spinning particles in various dimensions, it has been previously noticed\textsuperscript{15,16,19} that particle’s paths in Minkowski space always lie on a surface with the symmetry axis being the conserved momentum. This is interpreted as the Zitterbewegung phenomenon. A somewhat similar phenomenon occurs in some massless spinning particle models. In particular, the paths of chiral fermions in $d = 4$ are noticed to lie on 3-planes.\textsuperscript{21,22} What we state here goes beyond these observations in several respects. The first is that the very fact of irreducibility of massive spinning particle constrains particle’s paths in Minkowski space to be cylindric lines. This fact does not depend on any specifics of the particle model, like the choice of internal configuration space. Second, every causal typical path on a specific cylinder is gauge equivalent to any other causal line on the same cylinder. Third, the definition of the particle trajectory as a cylindric line in Minkowski space has differential consequences. These relations between the derivatives of the path can be viewed as equations of motion for the particle, and also as an alternative definition of cylindric lines. These higher-derivative equations for the paths in Minkowski space are not Lagrangian. In $d = 3$, these equations turn out to be equivalent to Lagrangian equations of lower order constructed for the minimal model of spinning particle.\textsuperscript{19} For higher dimensions, similar equivalence is expected, though it is a more technically complex issue which we do not address in this article.

II. IRREDUCIBILITY OF REPRESENTATION AND PARTICLE WORLD SHEET IN MINKOWSKI SPACE

Consider quantum mechanics of particle in Minkowski space.\textsuperscript{3} The action of the Poincaré group in the space of physical states is generated by the operators of momentum $\hat{p}_a$ and angular momentum $\hat{J}_{ab}$. The representation is implied to be unitary of the Poincaré group, so the operators $\hat{p}_a, \hat{J}_{ab}$ have to be Hermitian. Once the particle is a point object in Minkowski space, the space of states is supposed to admit representation by functions of the particle position $x$ and maybe some other variables of a spinning sector. The momentum is supposed to generate translations in Minkowski space, so

$$\hat{p}_a x^b = i \delta^b_a. \quad (1)$$

The angular momentum is supposed to generate Lorentz transformations of Minkowski space, so it can be represented as a sum of orbital momentum and spin momentum,

$$\hat{J}_{ab} = x_a \hat{p}_b - x_b \hat{p}_a + \hat{S}_{ab}. \quad (2)$$

Once Lorentz transformations of Minkowski space coordinates are generated by the orbital momentum $x_a \hat{p}_b - x_b \hat{p}_a$, the action of the spin generator on $x$ should be trivial,

$$\hat{S}_{ab} f(x) = 0, \quad \forall f \in C^\infty (R^{1,d-1}). \quad (3)$$

The latter fact also means that $[\hat{p}, \hat{S}]=0$. Rel. (2) can be considered as a definition for the operator of spin $\hat{S}$ in terms of momentum $\hat{p}$, angular momentum $\hat{J}$ and position $x$ of the point particle in Minkowski space,

$$\hat{S} = \hat{J} - x \wedge \hat{p}. \quad (4)$$

Once the representation is irreducible of the Poincaré group, any linear operator acting on the space of states, and commuting with $\hat{p}$ and $\hat{J}$, should be a multiple of unit. This applies not only to the elements of the universal enveloping algebra of the Poincaré group, but also to any polynomial in the Casimir operators constructed from $\hat{S}$. The spin momentum operators $\hat{S}$ generate the Lorentz group representation which should be irreducible in its own turn, to avoid reducibility of the Poincaré group representation. This means that for a nondegenerate representation

\textsuperscript{2} If zero component of momentum is positive, the path is considered causal once $\dot{x}^0 > 0$.

\textsuperscript{3} We use a mostly positive signature of the Minkowski metric throughout the paper.
of the Poincaré group generated by the operators (1) and (2), the Casimir operators $C_k(\hat{p}, \hat{J})$, $k = 0, 1, \ldots, [(d-1)/2]$, of the Poincaré group and the Casimir operators of the Lorentz group $C^L_l(S)$, $l = 0, 1, \ldots, [d/2] - 1$ should be a multiple of unit
\[ C_0 \equiv \hat{p}^2 = -m^2, \quad C_k(\hat{p}, \hat{J}) = s_k, \quad k = 1, \ldots, [(d-1)/2]; \] (5)
\[ C^L_l \equiv \hat{S}^2 = \text{sign}(\varrho)\hat{p}^2, \quad C^L_l(S) = \varrho_l, \quad l = 1, \ldots, [d/2] - 1, \] (6)
where $m, \varrho, s_k, \varrho_l$ are real numbers being the eigenvalues of the Casimir operators. Irreducibility of the representation further means that any operator commuting to all the generators $\hat{p}, \hat{J}$ is spanned by the Casimir operators (5), (6).

For example, in $d = 3$, we have three operators commuting with the generators (1), (2): the operator of squared spin $\hat{S}^2$, and two Casimir operators of the Poincaré group, mass $\hat{p}^2$ and spin $W = \hat{p} \cdot \hat{J}$. In $d = 4$ we have four commuting operators: the operator of squared spin $\hat{S}^2$, contraction of spin and its dual $\frac{1}{2}\epsilon_{abcd}\hat{S}^{ab}\hat{S}^{cd}$, and two Casimir operators of the Poincaré group – mass $\hat{p}^2$ and operator of squared Pauli-Lubanski vector $W^2, W_a = \frac{1}{2}\epsilon_{abcd}J^{bc}\hat{p}^d$. In $d$-dimensional Minkowski space, the number of commuting operators is $d$.

The above-mentioned obvious facts have consequences for the classical limit of the quantum theory where the Poincaré group representation is irreducible while the Lorentz generators have the structure (2). Once a quantum observable is a multiple of unit, the corresponding classical quantity has to be constrained to a constant. This means, in the classical limit of the irreducible quantum theory, the constraints are imposed on $p, J, x$,
\[ p^2 + m^2 = 0, \quad C_k(p, J) = s_k, \quad k = 1, \ldots, [(d-1)/2]; \] (8)
\[ (S(x, p, J))^2 - \text{sign}(\varrho)\hat{p}^2 = 0, \quad C^L_l(S(x, p, J)) = \varrho_l, \quad l = 1, \ldots, [d/2] - 1, \] (9)
where $S(x, p, J) = J - x \wedge p$. These constraints on $x, p, J$ are immediate classical counterparts of the quantum irreducibility conditions (5), (6). Irreducibility of the Poincaré group representation in quantum theory also assumes relation (7) to hold, hence no other independent constraints, besides (5), (6), can arise at the classical level between position, momentum and angular momentum. Notice that the ordering ambiguities for the operators $x, p, J$ cannot affect relations (5), (6) because in the classical limit the distinctions disappear between different symbols of operators.

Constraints (5) fix the level of the classical Casimir functions of the Poincaré group, restricting the admissible values of classical conserved quantities of the particle – momentum $p$ and angular momentum $J$. These constraints mean that $p, J$ are reduced to the co-orbit of the Poincaré group. Once the Poincaré group representation is irreducible at the quantum level, any classical path $x(\tau)$ of the particle in Minkowski space has to satisfy algebraic equations (9) involving arbitrary constants of motion $p, J$ subject to constraints (5) and the fixed constants $m, s, \varrho$. The algebraic equations (9) define a surface in Minkowski space. As we explain below in this section, in any dimension, these surfaces have a timelike axis of symmetry defined by the particle momentum. The same axis further means that any operator commuting to all the generators $\hat{p}, \hat{J}$ is spanned by the Casimir operators (5), (6).

Once the representation (1), (2) is irreducible, it should completely define the quantum dynamics. Therefore, the classical dynamics have to be completely defined by the classical limit of irreducibility conditions. In particular, this means that the classical particle paths in Minkowski space are defined by the fact that they satisfy the algebraic equations (9) with constant parameters $p, J$ constrained by (5). Given the constants of motion – momentum and angular momentum – no requirement can be imposed on the classical path with $\dot{x}^0 > 0$ other than to lie on the surface defined by equations (9). That is why, we refer to the surfaces (9) as world sheets of the spinning particle.

This understanding of irreducible spinning particle classical dynamics has consequences. In particular, any two trajectories with $\dot{x}^0 > 0$ have to be considered as equivalent once they lie on the same world sheet (i.e., on the surface with the same specific values of $p$ and $J$). This understanding may seem quite different from the usual formulation of classical theory where the trajectories are defined as solutions of equations of motion, being an ODE system. The equivalence between different ODE solutions is usually understood as a consequence of gauge symmetry of the EoMs, while the constants of motion are the integrals of the ODE system. In the next section, we demonstrate that the usual form of classical dynamics can be deduced for the spinning particle from the algebraic equations of world sheets (5), (6).

Now, let us discuss the geometry of the world sheets. Given momentum and angular momentum subject to classical irreducibility relations (5), particle’s world sheet is a level surface of the Lorentz group Casimir functions (1) in any dimension. At first, consider the simplest case $d = 3$. In $d = 3$, Eqs. (5) read
\[ p^2 + m^2 = 0, \quad (p, J) = ms. \] (10)
The Lorentz group has a single Casimir function in $d = 3$. The level surface is defined as

$$S^2 = \text{sign}(\rho)\rho^2, \quad \rho = \text{const}. \quad (11)$$

If $\rho < 0$, the vector of spin momentum $S^a = \frac{1}{2} \epsilon^{abc} S_{bc}$ belongs to a two-sheet hyperboloid; for $\rho > 0$, $S$ is in the one-sheet hyperboloid. For $\rho = 0$, the vector of spin lies on the cone.

The compatibility conditions for Eqs. (10) and (11) impose some restrictions on the parameters $\rho$ and $s$. Using the second relation (10), we get the spin constraint,

$$\langle p, S \rangle = ms. \quad (12)$$

Applying the obvious identity

$$[p, S]^2 \equiv \langle p, S \rangle^2 - p^2 S^2 \geq 0, \quad (13)$$

which is valid for any vector $S$ and timelike vector $p$, we see that the spin constraints (11) and (12) are consistent only if

$$s^2 + \text{sign}(\rho)\rho^2 \geq 0. \quad (14)$$

The admissible configurations of spin momentum $S^a$ are defined by the intersection of the quadric (11) and the plane (12). In general, it is a circle $S_1$, whose radius is given by the square root of the lhs of inequality (14),

$$r = \sqrt{s^2 + \text{sign}(\rho)\rho^2}. \quad (15)$$

We skip this degenerate case where the spin does not have independent degree of freedom.

Substituting the spin vector $S$ as a function of the conserved quantities $p$ and $J$ from relation (11) into (11), we get the equation of a four-parameter variety of surfaces in 3d Minkowski space,

$$\left( J - [x, p] \right)^2 = \text{sign}(\rho)\rho^2. \quad (16)$$

The surfaces are parametrized by two conserved vectors $p$ and $J$ subject to two constraints (10). In the other words, every element of the variety of 2d-surfaces (16) in Minkowski space corresponds to the point of the Poincaré group co-orbit.

Let us see that the quadric surfaces defined by equation (16) are cylinders. It is convenient to introduce instead of $p$ and $J$ two other constant vectors, $n$ and $y$,

$$y = \frac{1}{m^2} [p, J], \quad n = \frac{p}{m} \Leftrightarrow J = m[y, n] - sn, \quad p = mn. \quad (17)$$

Whenever $p$ and $J$ are constrained by relations (10), the vectors $n, y$ are orthogonal to each other and $n$ is normalized,

$$n^2 = -1, \quad (n, y) = 0. \quad (18)$$

In terms of normalized constant vectors $n, y$ the equation (16) reads

$$(x - y)^2 + (n, x)^2 = r^2, \quad r = \frac{1}{m} \sqrt{s^2 + \text{sign}(\rho)\rho^2}. \quad (19)$$

It is the equation of a circular cylinder of radius $r$ in 3d Minkowski space. The timelike unit vector $n$ is directed along the axis of cylinder. The vector $y$, being orthogonal to $n$, connects the origin of reference system with the axis of cylinder. Once $n$ is timelike, $y$ is spacelike. By Eq. (19), the vector $n = p/m$ defines the direction of the cylinder axis, while the vector $y$ specifies the position of the axis in space. A cylinder of fixed radius with any position of the timelike axis is admissible. In this way, $n$ and $y$ parameterize the variety of all possible cylinders with timelike axis and fixed radius.

The case of $s = \rho = 0$ obviously corresponds to a spinless point particle, so the paths cannot be anything but the straight lines. With nonvanishing spin, $s \neq 0$, the particle paths should be cylindric lines. The latter fact,

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4 In the special case of $\rho = -s$, the cylinder collapses to a straight line. We do not address this degenerate case in this paper. Also notice that in many of the minimal spinning particle models in $d = 4$, the spin $S_{ab}$ squares to zero, so $\rho = 0$. The radius of cylinder in this case is just $s/m$.\]
as we see in the next sections, is sufficient to completely define the classical dynamics, including the equations of motion for the particle in \( d = 3 \).

A similar picture can be seen in four dimensions. Eqs. (8) restrict admissible values of the conserved quantities – momentum and angular momentum – to the Poincaré group co-orbit. There are two Casimir functions of the Lorentz group. Hence two equations (9) are imposed on the particle position in Minkowski space, given any concrete values of \( p \) and \( J \). One of the equations for \( x \) is a quadric, and another one defines a plane. The intersection of the quadric and the plane is a 2\( d \)-cylinder whose timelike axis is defined by particle’s momentum. The radius of cylinder vanishes in the spinless limit, and the surface degenerates into a timelike straight line directed along the momentum. Somewhat special is the case when the plane is tangent to the quadrics, not transverse. We do not elaborate on this degenerate case here, though it seems important.

In \( d > 4 \), the Lorentz group has \([d/2]\) Casimir functions. The number and order of Eqs. (9) is growing with dimension, so the direct study of the system becomes more cumbersome than in lower dimensions. Instead of direct study of the equations, we use the reasons of symmetry to establish the structure of particle’s world sheets. At first, consider Eqs. (9) in the rest reference system of the particle. Since the equations are covariant, the conclusions can be extended to a general system by a Poincaré transformation. In the rest system, the momentum and angular momentum read

\[
p = (m, 0, \ldots, 0), \quad J = (J^{ij}), \quad i, j = 1, 2, \ldots, d - 1.
\]

The condition that the Lorentz boost \( J_{00} \) vanishes is not a restriction. If \( J_{00} \neq 0 \), the boost can be absorbed by the coordinate transformation

\[
x^a \mapsto x^a + \frac{1}{m^2} J^{0a}.
\]

Once \( p \) and \( J \) are fixed, the group of symmetries of the particle world sheet equations (9) is the stabilizer subgroup of the Poincaré group with respect to \( p \) and \( J \). It includes the translations along the time axis and spatial rotations that preserve \( J \),

\[
x^0 \mapsto x^0 + \varepsilon, \quad x^i \mapsto \Lambda^i_j x^j.
\]

Here, \( \varepsilon \) is some constant, and \( \Lambda \) is the element of the stabilizer subgroup \( H_J \subset SO(d - 1) \) with respect to \( J \). The parameters of transformation (22) are \( \varepsilon \) and \( \Lambda \). In general, the transformations (22) are independent, and their number coincides with the dimension of the surface defined by equations (9). Therefore, the world sheet is the orbit of the stabilizer subgroup along the fixed axis is a straight line. The stabilizer subgroup \( H_J \) is isomorphic to the maximal torus in \( SO(d - 1) \) passing through \( J \), and it is Abelian. Its orbit is a torus. The most general \([(d + 1)/2]\)-dimensional surface, being invariant w.r.t. both transformations, is a toroidal cylinder

\[
\Sigma = \mathbb{R} \times T^D, \quad T^D = S^1 \times \ldots \times S^1, \quad D = [(d - 1)/2].
\]

The levels of the Poincaré and Lorentz group Casimir functions (5), (6) determine the radii of the circles \( r_k = r_k(m, s_k, \varrho_l) \). For some special combinations of the parameters \( m, s_k, \varrho_l \), one or more circles can collapse to points. The examples of degenerate cases of this sort have been noticed above in \( d = 3 \) and \( d = 4 \).

As we have observed in this section, any path of irreducible spinning particle should lie on a (toroidal) cylinder whose position in the space is defined by particle’s momentum and angular momentum, while the radii are fixed by the representation. We have also seen that irreducibility of the Poincaré group representation does not impose any other restriction on the world line of the particle. The latter fact means that any two causal typical world lines have to be considered equivalent once they belong to the same world sheet.

### III. EQUATIONS OF MOTION FOR WORLD LINES ON THE WORLD SHEETS

In this section, we demonstrate that the definition of particle’s paths as causal cylindric lines has consequences. First, it defines gauge symmetries of the system. Second, it defines the equations of motion imposed on particle’s
trajectories. And third, it defines all the independent conserved quantities $p$ and $J$ as functions of trajectory and its derivatives. So, the usual formulations of the classical theory based on equations of motion are deduced from the fact that equivalent paths form world sheets.

To explain the method we use for deriving the EoMs from the algebraic equations of particle’s world sheet, we discuss at first a more general problem of the same type we have to solve here. Consider a manifold $M$, $\dim M = d$ and $m$-parameter set of $n$ smooth functions $C_{\alpha}(x, y) \in C^\infty(M)$, $\alpha = 1, \ldots, n$, with $x$ being local coordinates on $M$, and $y_A, A = 1, \ldots, m$, are the parameters. Consider $m$-parameter variety of surfaces $\Sigma \subset M$. The element $\Sigma_y$ of the variety is defined as the zero locus of functions $C_{\alpha}(x, y)$ with certain parameters $y$.

$$\Sigma_y = \{x \in M | C_{\alpha}(x, y) = 0, \ \alpha = 1, \ldots, n\}, \quad \dim \Sigma_y = d - n. \quad (24)$$

A simple example of this setup can be the 3-parameter variety of spheres of fixed radius $r$ in $3d$ Euclidean space. The equation reads $(x - y)^2 - r^2 = 0$. The components of the radius-vector $y$ of the sphere center are considered as the parameters.

The line $x(\tau)$ is said to be of $\Sigma$-type if it lies on some representative of family of surfaces $\Sigma_y$, i.e. $\exists y : x(\tau) \subset \Sigma_y$. Given the set of surfaces $\Sigma_y$, the problem is to define the class of $\Sigma$-type lines by an ODE system. For example, the question can be to find the ODE such that the general solution is the class of spherical lines. The problem of interest for us is to describe the class of cylindrical lines.

The problem is threefold: (i) to find the ODE system such that the set of all its solutions coincides with the class of $\Sigma$-lines; (ii) to define the constant parameters $y$ as integrals of the ODE system; (iii) to find all the gauge transformations such that they map a line on $\Sigma_y$ to any other line on the surface with the same parameters $y$. For example, consider $\Sigma$ being the class of spheres $S^2$ of fixed radius $r$ in $3d$ Euclidean space. For the class of spherical lines, the answers are well known to the first two questions. The ODE of spherical lines (see, e.g., \[33\]) reads

$$r^2 = \frac{1}{\kappa^2} + \frac{(\omega')^2}{\omega^2 \kappa^2}, \quad (25)$$

where $\kappa$ is the curvature of the line, $\omega$ is the torsion, and a prime denotes the derivative by the natural parameter on the line. The parameter $y$, being the radius vector of the center of sphere, is identified as the integral of motion for the equation of spherical lines \[25\]. It reads

$$y = x + \frac{1}{\kappa^2} x'' + \frac{\omega'}{\kappa^2 \omega} [x', x''], \quad (26)$$

The answer to the third question seems unknown in the literature, though it is quite simple. The infinitesimal gauge symmetry transformations of the equation of spherical lines \[25\] read

$$\delta_1 x = [x', [x', x'']] \epsilon_1, \quad \delta_2 x = [x', [x'', x''']] \epsilon_2, \quad (27)$$

where the infinitesimal transformation parameters $\epsilon_1, \epsilon_2$ are arbitrary functions of parameter $\tau$ on the curve.

Once the world lines of spinning particles are cylindrical, they comprise the class of interest in this paper. This class is much less studied in geometry, though there are some works on cylindrical lines in $d = 3$ Euclidean space \[34,35\]. In Minkowski space, the issue of general cylindric lines has not been studied yet, to the best of our knowledge.

Let us explain the general idea of deducing the ODE system for the lines of $\Sigma$-type, given the system of algebraic equations which defines the family of surfaces $\Sigma$,

$$C_{\alpha}(x, y) = 0, \quad \alpha = 1, \ldots, n. \quad (28)$$

Once the line $x(\tau)$ lies on $\Sigma$, Eqs. \[28\] should remain valid along the line. This has differential consequences: the functions $C_{\alpha}(x(\tau), y)$ have to conserve along the line $x(\tau) \subset M$,

$$\dot{C}_{\alpha}(x, y) \equiv \dot{x}^\alpha \frac{\partial C_{\alpha}(x, y)}{\partial x^\alpha} = 0, \quad \alpha = 1, \ldots, n. \quad (29)$$

The same is true for the second and higher derivatives,

$$\frac{d^k}{d\tau^k} C_{\alpha}(x, y) = 0, \quad k = 1, 2, \ldots. \quad (30)$$

Under appropriate regularity conditions imposed on $C_{\alpha}(x, y)$, all the parameters $y$ can be expressed from Rel. \[30\] as functions of $x, \dot{x}, \ddot{x}, \ldots$,

$$y_A = \bar{y}_A(x, \dot{x}, \ddot{x}, \ldots), \quad A = 1, 2, \ldots, m. \quad (31)$$
Substituting the functions $\tilde{y}_A(x, \dot{x}, \ddot{x}, \ldots)$ instead of all the parameters $y_A$ in the original algebraic equations (28), we arrive at the ODE system

$$C_a(x, \tilde{y}(x, \dot{x}, \ddot{x}, \ldots)) = 0. \quad (32)$$

The functions $\tilde{y}_A(x, \dot{x}, \ddot{x}, \ldots)$ are by construction the integrals of motion for this system, so the solutions lie on the surface $\Sigma_y$ (24). In this way, the ODE system is constructed for the lines of $\Sigma$-type.

Consider the issue of gauge symmetry for the ODE system (32). The quantities $\tilde{y}$ conserve on solutions, while the conserved quantities must be gauge invariant. This means that the gauge symmetry can connect the solutions which correspond to the same parameters $y$. It further means that the gauge transformation should be a symmetry transformation of the surface $\Sigma_y$ with every fixed set of parameters $y$. This amounts to saying that the gauge symmetry is generated by the vector fields tangential to $\Sigma_y$. Let us choose the basis of the vector field $R_i = R_i^a(x, y)\partial_a$, $i = 1, \ldots, d - n$ in the tangent bundle of $\Sigma$, i.e.,

$$R_i^a(x, y) \frac{\partial C_a(x, y)}{\partial x^a} \bigg|_{\Sigma_y} = 0, \quad TS_y = \text{span}\{R_i\}. \quad (33)$$

Then, the infinitesimal gauge transformation reads

$$\delta x^a = R_i^a(x, \tilde{y}(x, \dot{x}, \ddot{x}, \ldots))\varepsilon^i(\tau), \quad (34)$$

where the gauge parameters $\varepsilon^i(\tau)$ are arbitrary functions of $\tau$. By construction, this transformation leaves the equations (32) invariant.

Applying this methodology, one can deduce, in principle, the equations for the lines of various $\Sigma$-types. For spherical lines this method works very easily leading to the equations of spherical lines (29), their integrals of motion (26), and gauge symmetries (27). For general cylindric lines it also works, in principle, while Eqs. (30) are more complex algebraic relations in the case of cylinder, so it is much more problematic to explicitly solve them for the parameters $y$ (31). In fact, a similar program has been implemented for cylindric lines in $\mathbb{R}^3$. The differential equation for the lines has been obtained in an implicit form. In (32), another implicit form of the cylindric line equations has been deduced employing an alternative ad hoc method in 3d Euclidean space. To the best of our knowledge, the problem of cylindric lines has never been solved in Minkowski space, even in $d = 3$. In Minkowski space, however, the class of null curves is admissible on the cylinders with timelike axis. This does not have an analogue in the Euclidean case. For this class of curves, the method of deducing ODE’s works very well along the lines described above. At the first glance, it may seem strange to see the massive particles propagating by the lightlike world lines. In fact, this is not so strange because the coordinates are not gauge invariant quantities for these particles, while the gauge invariants can have usual properties being typical for massive particles, including timelike momentum. The higher-derivative models with null curves were previously known\cite{29,24} such that describe reducible massive representation. In the next subsection, we deduce the equations for cylindric null curves. These classical equations of motion correspond to irreducible massive representation. In Subsection III.B, we deduce the higher-order equations for timelike cylindrical world lines, though they are obtained in an implicit form. In Subsection III.C, we introduce an auxiliary variable, that allows us to obtain the second-order equations for timelike cylindrical lines in an explicit form. These equations agree with previously known equations\cite{22,28} describing the classical irreducible massive spinning particle.

### A. Lightlike world lines on a cylinder with a timelike axis

Let $x(\sigma)$ be a null curve parameterized by the pseudoarc length such that $(x''', x''') = 1$, while $x'$ squares to zero. Hereafter in this section a prime denotes derivative by $\sigma$.

The cylinder of radius $r$ with time like axis is defined by the equation

$$C(x, y, n) \equiv (x - y)^2 + (n, x)^2 - r^2 = 0, \quad (35)$$

where vector parameters $n$ and $y$ satisfy (18). The vectors $n$ and $y$ parameterize all the possible positions of the cylinder in space, given the fixed radius $r$. Once the cylinder is a world sheet of massive spinning particle, the parameters $n, y$ are connected with particle’s momentum and angular momentum by relations (17). With two vector parameters subject to two scalar constraints, the variety of cylinders is a 4-parameter set of surfaces in $d = 3$ Minkowski space. To express these parameters in terms of derivatives of the line, one has to consider differential
consequences of the algebraic equation (35) up to fourth order. Equations (30) for the cylinder are specified as
\begin{align*}
(x', x + n(n, x) - y) &= 0, \\
(x'', x + n(n, x) - y) + (n, x')^2 &= 0, \\
(x''', x + n(n, x) - y) + 3(n, x'')(n, x') &= 0, \\
(x''''', x + n(n, x) - y) + 4(n, x''')(n, x') + 3(n, x'')^2 - 1 &= 0.
\end{align*}

(36)

Introduce the difference vector
\[ d = x + n(n, x) - y \iff y = x + n(n, x) - d. \]

(37)

The vectors \( d \) and \( n \) are subject to conditions
\[ (d, n) = 0, \quad n^2 = -1. \]

(38)

The vector \( d \) is normal to the cylinder axis, and it connects the axis with the current point on the surface. In terms of \( d \), the equation of cylinder (35) reads
\[ d^2 = r^2. \]

(39)

With the difference vector \( d \), the differential consequences (36) are reformulated as
\begin{align*}
(x', d) &= 0, \\
(x'', d) + (n, x')^2 &= 0, \\
(x''', d) + 3(n, x'')(n, x') &= 0, \\
(x''''', d) + 4(n, x''')(n, x') + 3(n, x'')^2 - 1 &= 0.
\end{align*}

(40)

It is convenient to (40) w.r.t. \( n, d \) by decomposing all the vector quantities in the Frenet-Serret moving frame. The moving frame has to be adapted to the description of the null curves. It is given by the TNB triad normalized as
\[ (N, N) = (T, B) = 1, \quad (T, T) = (B, B) = (T, N) = (B, N) = 0. \]

(41)

The Frenet-Serret formulas have the form
\begin{align*}
x' &= T, \\
T' &= N, \\
N' &= -\kappa T - B, \\
B' &= \kappa N,
\end{align*}

where the curvature \( \kappa \) reads
\[ \kappa = \frac{1}{2}(x''', x'''). \]

(43)

Decompose the derivatives of \( x \) up to the fourth order in the TNB frame,
\begin{align*}
x' &= T, \\
x'' &= N, \\
x''' &= -\kappa T - B, \\
x'''' &= -\kappa' T - 2\kappa N.
\end{align*}

(44)

For the vectors \( d \) and \( n \) we chose the ansatz
\[ d = \gamma(\beta T - N), \quad n = -\frac{\alpha^2 \beta^2 + 1}{2\alpha} T + \alpha \beta N + \alpha B, \]

(45)

which automatically satisfies three equations: (38) and the first equation in (40). The system (38), (40) has no special solutions with \( \alpha = 0 \), so this ansatz (45) is most general.

On substituting (45) into (40), we get three non-trivial equations
\[ \gamma - \alpha^2 = 0, \quad \beta(\gamma - 3\alpha^2) = 0, \quad 2\kappa \gamma - 4\kappa \alpha^2 + 5\alpha^2 \beta^2 + 1 = 0. \]

(46)

The solution to this system of equations reads
\[ \alpha = (2\kappa)^{-\frac{1}{2}}, \quad \beta = 0, \quad \gamma = (2\kappa)^{-1}. \]

(47)

In such a way we arrive at the following equations for the cylindrical null curves:
\[ 2\kappa \equiv (x''', x''') = \frac{1}{r}, \quad (x', x') = 0. \]

(48)
These equations have only one gauge symmetry, being reparametrization invariance:

\[ \delta_x x = x' \varepsilon. \]  

(49)

The general recipe of deducing gauge symmetry \(^{53}\) for the lines on surface implies a pair of gauge transformations for the general lines on a cylinder. The null curves are not general lines because they are subject to an extra differential equation, \(p^2 = 0\). The extra equation reduces the gauge symmetry.

Relations \(^{53}\) can be interpreted as equations of null helices. Every helix is a cylindrical line, while not every cylindrical line is a helix. In \(d = 3\), any cylindrical null curve is a helix, however, as we see.

In higher dimensions, the null helices have been studied in details in ref.\(^ {28,29}\). As is seen from the classification of the paper\(^ {36}\), the generic classes of null helices with independent fixed curvatures are defined by more parameters than the dimension of nondegenerate Poincaré group co-orbit. So general null helices cannot describe an irreducible massive spinning particle in \(d > 4\), unlike \(d = 3, 4\). Some null helices in \(d > 4\) are not even toroidal cylindrical lines. Because of that, we expect that only some special types of null helices with certain combinations of curvatures are destined to describe the dynamics of irreducible massive spinning particles in the dimension greater than four.

Let us find the cylinder parameters \(n, y\) as integrals of motion of equations \(^{48}\). Upon the identification \(d = -rx''\), the parameters read,

\[ n = r^\frac{3}{2}x''' + r^{-\frac{1}{2}}x', \quad y = x + rx'''(x, x') + x'''(x, x') + x'(x, x'') + r^{-1}x'(x, x') + rx'''. \]  

(50)

Let us check that the equations of motion \(^{48}\) describe the irreducible massive spinning particle indeed. The momentum \(p\) and angular momentum \(J\) are connected to the cylinder parameters \(n, y\) by relations \(^{17}\). Substituting \(^{17}\) into \(^{17}\) we get \(p, J\):

\[ p = m(r^\frac{1}{2}x''' + r^{-\frac{1}{2}}x'), \quad J = [x + rx''', p] - \frac{s}{m}p. \]  

(51)

These quantities conserve on shell by construction. The gauge invariance of \(p, J\) is also obvious. Given the radius-to-spin relation \(^{19}\), \(p\) and \(J\) satisfy the Casimir constraints \(^{10}\) in \(d = 3\) with mass \(m\) and spin \(s\). So, the gauge invariant integrals of motion define the co-orbit of the Poincaré group, which is four dimensional. To prove the irreducibility of the system, we have to check that the physical degree of freedom (DoF) number is four, so there are no other degrees of freedom besides the ones on the co-orbit. The DoF number can be counted in various ways. Once the equations of motion \(^{48}\) are non-Lagrangian, we count the DoF by using the formula \(^{8}\) from Ref.\(^ {32}\). This counting method does not appeal to Lagrangian or Hamiltonian formalism. So, the DoF number is counted for any ODE system by the formula:

\[ n = \sum_{k=0} k(n_k - r_k - l_k), \]  

(52)

where \(n_k\) is the number of the equations of the order \(k\) for every \(k\), the \(r_k\) is the number of gauge symmetries of order \(k\), \(l_k\) is the number of gauge identities of the order \(k\). In the case at hand, these numbers read \(n_3 = 1\) (constant curvature equation), \(n_1 = 1\) (null-curve condition), \(r_0 = 1\) (reparametrization gauge symmetry). Substituting all the ingredients into \(^{52}\) we get \(n = 4\), as it should be for a massive spinning particle in \(d = 3\).

Notice that in work\(^ {28,29}\) the higher-derivative Lagrangian models are proposed for describing massive spinning particles by null curves in \(d = 3, 4\). The Lagrangian is a curvature of the lightlike line in Minkowski space \(^8\) while no auxiliary internal space is introduced for spinning degrees of freedom. The equations of motion in these models describe 10 DoF in \(d = 4\) and 6 in \(d = 3\), so the theories correspond to reducible representations. Noether’s momentum conserves in these models and it has a non-vanishing square. These reducible representations include massive modes, even though the world lines are lightlike. As noticed in\(^ {28,29}\), the dynamics can be selected of the irreducible theory by adding the mass-shell constraint by hand to the variational equations of motion. In the view of these previously known facts, it does not seem strange that the higher-derivative equations for the null curves in Minkowski space can describe the classical dynamics that corresponds to irreducible massive representation. The equations of motion \(^{48}\) have been deduced above proceeding from the fact that all the paths of irreducible spinning particle in \(d = 3\) have to be cylindrical lines. As we have seen in this subsection, the restriction of the class of cylindrical lines to the cylindrical null curves does not break irreducibility.

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6 This action can be interpreted as the pseudoarc length, once the velocity squares to zero.
B. Timelike world lines on the cylinder with timelike axis

In the beginning of Section III, the general algorithm was proposed for deducing the differential equations of motion for the world lines from the algebraic equations of the world sheet. In the previous subsection, we have explicitly implemented the algorithm and deduced the equations of motion with the restriction that the admissible world lines are null curves. In this subsection, we apply the same algorithm to deduce the equations for the time like world lines.

The problem has a subtlety that some cylindric lines do not uniquely define the cylinder they lie on. In a different wording, the line can belong to the intersection of two or more cylinders of the same radius. We consider these lines as atypical. Depending on the relative position of the two cylinders, the intersection can be either a closed path or a straight line. In Minkowski space, none of the closed lines can be a causal path, so these curves are unrelated to the classical trajectories of spinning particles. We systematically ignore them. The straight lines constitute a special set of atypical cylindrical curves which have a smaller gauge symmetry compared to the typical lines on the same world sheet. Below in this section (see relation (76)), we demonstrate that each straight line defines its own class of gauge equivalent atypical trajectories. These trajectories do not define the world sheet they lie on. As the physical phase space is isomorphic to the space of world sheets, the atypical world lines do not represent any of the equivalent physical evolutions of the particle. For this reason, they are excluded from the class of physical trajectories.

Let $x(\tau)$ be a timelike curve parameterized by the natural parameter $\tau$, so we have the normalization condition for the velocity and its consequences

$$ (x', x') = -1, \quad (x', x'') = 0, \quad (x', x''') + (x'', x''') = 0. \tag{53} $$

Throughout this section, prime denotes derivative by $\tau$.

Differentiating the cylinder equation (35) four times and accounting for identities (53), we arrive at the relations connecting the constants $n, y$ with the derivatives of the cylindric line

$$ (x', x + n(n, x) - y) = 0, $$

$$ (x'', x + n(n, x) - y) + (n, x') = 0, $$

$$ (x''', x + n(n, x) - y) + 3(n, x')(n, x') = 0, $$

$$ (x''''', x + n(n, x) - y) + 4(n, x')(n, x') + 3(n, x'')^2 - (x'')^2 = 0. \tag{54} $$

The problem is to solve these relations w.r.t. the constant parameters $n, y$ expressing them all in terms of derivatives of the timelike world line $x(\tau)$. We proceed to the solution of these equations by reformulating them in terms of the constant vector $n$ and the difference vector $d \tag{57}$.

$$ (x', d) = 0, $$

$$ (x'', d) + (n, x') = 0, $$

$$ (x''', d) + 3(n, x')(n, x') = 0, $$

$$ (x''''', d) + 4(n, x')(n, x') + 3(n, x'')^2 - (x'')^2 = 0. \tag{55} $$

where the vectors $d$ and $n$ are orthogonal to each other, and $n$ is normalized; see $\tag{58}$. In terms of $n, d$, the equation of cylinder $\tag{35}$ takes the form $\tag{39}$.

Now, we seek to express $d$ and $n$ from $\tag{55}$ in terms of derivatives of the line. The vector $y$ is defined by Rel. $\tag{58}$. Once these vectors are expressed in terms of derivatives of $x$, the vector-valued functions

$$ n = n(x', x'', x''''), \quad y = y(x', x'', x'''), \tag{56} $$

define the integrals of motion. They conserve whenever the curve lies on the cylinder with fixed radius $r$. The values of integrals of motion $n$ and $y$ parametrise all the possible positions of the cylinder in space. Once the cylinder is a world sheet of massive spinning particle, the parameters $n, y$ are connected with particle’s momentum and angular momentum by relations $\tag{14}$. The ODE describing the timelike curves that lie on the cylinder is obtained by substituting $\tag{56}$ into the cylinder equation $\tag{35}$, or equivalently, $d(x', x'', x''')$ into $\tag{59}$. The equations for $d, n \tag{55}$ is a system of polynomial equations, much like Eqs. $\tag{40}$ in the case of null curves. The system $\tag{55}$ for the integrals of timelike curves is more complex, however, comparing to $\tag{40}$. It has a solution, which we can find only in an implicit form.

We proceed to solving $\tag{55}$ as follows. We consider the overdetermined system consisting of cylinder equation $\tag{39}$, its differential consequences $\tag{55}$ and constraints $\tag{58}$. This system includes seven equations for two vector unknowns $d$ and $n$, which have six components, given the constraints between them $\tag{58}$. At first, we solve the system consisting of six simplest equations: $\tag{35}, \tag{39}$, and three from $\tag{55}$. At this stage, the integrals of motion
are found. Then, we substitute the solution to the last equation (55). The consistency condition for the system gives the ODE for the cylindrical curves. Below, we elaborate on these manipulations.

It is convenient to rewrite equations (59), (58), (55) by decomposing all the vector quantities in the Frenet-Serret moving frame. The moving frame has to be adapted to the description of timelike curves. It is given by the TNB triad normalized as

\[(T, T) = -1, \quad (N, N) = (B, B) = 1, \quad (T, N) = (T, B) = (N, B) = 0.\]  

The Frenet-Serret differentiation formulas have the form

\[x' = T, \quad T' = \kappa N, \quad N' = \kappa T + \omega B, \quad B' = -\omega N,\]  

where the curvature \(\kappa\) and torsion \(\omega\) read

\[\kappa = (x'', x''')^{\frac{1}{2}}, \quad \omega = \frac{(x', x'', x''')}{(x'', x''')}\]  

Decompose the derivatives of \(x\) up to fourth order in the TNB frame,

\[x' = T, \quad x'' = \kappa N, \quad x''' = \kappa' N + \kappa(\kappa T + \omega B), \quad x'''' = 3\kappa\kappa'T + (\kappa'' + \kappa^3 - \kappa\omega^2) N + (2\kappa' \omega + \kappa\omega') B.\]  

Introduce the ansatz for expansion of vectors \(d\) and \(n\) in the TNB frame

\[d = r(\alpha_1 N + \alpha_2 B), \quad n = \beta_1 T - \beta_2 (\alpha_2 N - \alpha_1 B),\]  

where the unknown quantities \(\alpha_1, \alpha_2, \beta_1, \beta_2\) are subject to the conditions

\[\alpha_1^2 + \alpha_2^2 - 1 = 0, \quad \beta_1^2 - \beta_2^2 - 1 = 0.\]  

By construction, this ansatz automatically resolves relations (58), (59) imposed on \(d\) and \(n\), and the first equation (55). The nontrivial equations are

\[
\begin{align*}
\kappa r\alpha_1 + \beta_2^2 = 0, \quad 3\kappa\beta_1\beta_2\alpha_2 + \kappa' r\alpha_1 + \kappa\omega r\alpha_2 = 0, \\
(\kappa'' + \kappa^3 - \kappa\omega^2)\alpha_1 + (2\kappa' \omega + \kappa\omega')\alpha_2) r + 4(\kappa' \alpha_2 - \kappa\omega\alpha_1)\beta_1\beta_2 - \kappa^2(3\alpha_1^2\beta_2^2 - 7\beta_2^2 - 3) = 0.
\end{align*}
\]

The solution to (62) and (63) reads

\[\alpha_1 = -\frac{1}{r\kappa} \frac{1}{\sqrt{z^2 - 1}}, \quad \alpha_2 = \frac{\kappa'}{\kappa^3} \frac{1}{3z + r\omega(z^2 - 1)}, \quad \beta_1 = \pm \frac{z}{\sqrt{z^2 - 1}}, \quad \beta_2 = \pm \frac{1}{\sqrt{z^2 - 1}},\]  

with \(z\) being a root of the algebraic equation

\[P_1(z) = r^4\kappa^4 z^8 \omega^2 + 6r^3\kappa^4 z^7 \omega - r^2 \kappa^4 (4r^2 \omega^2 - 9) z^6 + 18r^3\kappa^4 z^5 \omega - r^2 (18\kappa^4 - 6r^2 \kappa^4 \omega^2 + \kappa^2 + \omega^2 \kappa^2) z^4 + 6r^2 \kappa^2 \omega (3r^2 \kappa^2 - 1) z^3 - (9\kappa^2 + 4r^4 \kappa^4 \omega^2 - 2r^2 \kappa^2 \omega^2 - 9r^2 \kappa^2 - 2r^2 \kappa^2 r^2) z^2 - 6r^2 \kappa^2 \omega (r^2 \kappa^2 - 1) z - r^2 (-\kappa^2 + \kappa^2 \omega^2 + r^2 \kappa^2 \omega^2) = 0.\]  

In terms of the variable \(z\) defined by Eq. (66), the vectors \(d\) and \(n\) read

\[d = -\frac{1}{r\kappa} \frac{1}{\sqrt{z^2 - 1}} \frac{1}{\sqrt{z^2 - 1}} [x', x'']; \quad n = \frac{z}{\sqrt{z^2 - 1}} x' - \frac{x'}{\kappa^3} \frac{1}{3z + r\omega(z^2 - 1)} \frac{1}{\sqrt{z^2 - 1}} (x', x'').\]  

Given \(d\) and \(n\), the integral of motion \(y\) can be found by formula (57).
Leaving only the leading contributions in the curvature and torsion, we find \( \alpha \) as the curvature and torsion. It is convenient to express \( n \) in an implicit way, because \( z \) is not expressed from Eq. (69) explicitly. Eqs. (63), (64) can be approximately solved if the curvature, torsion and their derivatives are considered small in comparison to the radius of the cylinder, i.e.,

\[
rx \sim r^2x' \sim r^3x'' \sim r\omega \sim r^2\omega' \ll 1.
\]

(69)

In this approximation, \( \alpha_2 = \beta_1 = 1 \), while \( \alpha_1, \beta_2 \) become the small parameters of the same order of magnitude as the curvature and torsion. It is convenient to express \( \alpha_1, \beta_2 \) from the second relation (63) and relation (64). Leaving only the leading contributions in the curvature and torsion, we find

\[
\alpha_1 = \frac{1}{r} \left( 2rxx'\omega + 3rx'^2\omega' + 9x^3 \right) \quad \beta_2 = r \left( x'x'\omega - 3x^2x' - x' (2x'\omega + x\omega') \right) \frac{1}{4(x')^2 - 3xx''}.
\]

(69)

Under assumption (69), both these quantities are small, so the approximation \( \alpha_1, \beta_2 \ll 1 \) is consistent. The vectors \( d \) and \( n \) in this approximation read

\[
n = x' + r \frac{x''x\omega - 3x^2x' - x' (2x'\omega + x\omega')}{4(x')^2 - 3xx''} x'', \quad d = 2rxx'\omega + 3rx'^2\omega' + 9x^3 \frac{1}{4(x')^2 - 3xx''} x'' + r[x', x''].
\]

(70)

These formulas replace (67), (68) in the class of cylindrical curves whose curvature and torsion subject to condition (69). It is seen that \( n \) remains regular once the curvature tends to zero \( \kappa, \omega \to 0 \), while \( d \) becomes singular. The singularity is quite natural because the straight line does not determine a unique cylinder it belongs to, unlike the typical cylindrical curve.

Substituting the solution (65) to (64), we get another polynomial equation:

\[
P_2(z) = 3x^3r^3\omega z^8 + 9x^2r^2z^7 - r^3(-x^2\omega^3 - 2x^2\omega - x'\omega + 6x^4\omega + xx''\omega)z^6
\]

\[
-r^2(-7x^3\omega^2 - 4x'^2 + 3xx'' + 9x^4)z^5 - 3r(r^2x'\omega' - 4x'^2\omega - r^2xx''\omega + r^2x'\omega^3 + 2r^2x^2\omega)z^4
\]

\[
+ r^2(6xx'' - 14x^2\omega^2 - 8x^2 - 9x^4)z^3 + 3r(2r^2x^2\omega - 5x^2\omega + 2r^2x'\omega + r^2x'\omega' + r^2x'^2\omega^3 - r^2xx''\omega)z^2
\]

\[
+ (-9r^2 + 4x'^2r^2 + 9r^2x^4 + 7r^2x^2\omega - 3r^2xx''z) - r(r^2x^3\omega^3 + 2r^2x^2\omega + 3r^2x'\omega - r^2xx''\omega + r^2x'\omega')
\]

\[
- 3x^2\omega) = 0.
\]

(71)

For a cylindrical curve the l.h.s. of (69) and (71) must vanish simultaneously.

To keep the contact with the earlier results on the ODE for cylindric lines in Euclidean space, we note that the auxiliary variable \( z \) is introduced in the way to get the simplest possible expressions for \( d \) and \( n \). This simplification comes with a price of more cumbersome equations (66) and (71) compared to the Euclidean analogues from Ref. 34. In particular, both the polynomials (66) and (71) have eighth order, while in Ref. 34 the analogues have the orders eight and six. This difference is insignificant because the orders can always be reduced by Euclid’s algorithm.

Once two of the polynomial equations (66) and (71) share a common root, the resultant of polynomials \( P_1(z) \) and \( P_2(z) \) must vanish. The resultant can be computed as the determinant of the Sylvester matrix. Given two polynomials, respectively of degree \( m \) and \( n \),

\[
P_1(z) = a_0 + a_1z + a_2z^2 + \ldots + a_mz^m, \quad P_2(z) = b_0 + b_1z + b_2z^2 + \ldots + b_nz^n,
\]

(72)

the associated Sylvester matrix \( (S_{ij}) \) is the \((n + m) \times (n + m)\) matrix defined as follows:

\[
S_{ij} = \begin{cases} 
    a_{m+i-j}, & i = 1, \ldots, n, \\
    b_{i-j}, & j = 1, \ldots, n+m,
    \\
    0, & \text{otherwise.}
\end{cases}
\]

(73)

The pair of polynomial equations (66), (71) defines the cylindric timelike line. These two equations are equivalent to a single equation stating that the determinant of the Sylvester matrix (73) vanishes for the polynomials \( P_1(z) \) and \( P_2(z) \) (66), (71).

\[
\det (S_{ij}(r, x', x''), \omega, \omega, \omega')) = 0.
\]

(74)

---

7 Given two univariate polynomials \( P_1(z) \) and \( P_2(z) \) as in (72), suppose their roots are \( u_1, \ldots, u_m \) and \( w_1, \ldots, w_n \), respectively. The resultant reads

\[
\text{Res}(P_1, P_2) = a_0^m b_0^n \prod_{i=1}^{m} \prod_{j=1}^{n} (u_i - w_j).
\]

The resultant vanishes if and only if the polynomials have a common root.
The coefficients of the polynomials, and hence the entries of the Sylvester matrix, are the functions of the curvature \( \kappa \), torsion \( \omega \), their derivatives \( \kappa', \omega', \omega'' \), and radius \( r \). This means, Eq. (74) is the fourth-order ODE for the particle path \( x(\tau) \). It is the ODE which defines the timelike world lines of the particle.

The l.h.s. of equation (74) is a quite certain quantity being the resultant of polynomials \((66)\) and \((71)\) constructed by the rule \((65)\). The explicit form of the resultant as a function of curvature, torsion and their derivatives is too long expression to print out, and it is not very informative. It seems a matter of principle, however, to establish that the timelike paths of irreducible massive spinning particle in \( d = 3 \) Minkowski space are defined by a single fourth-order ODE for \( x(\tau) \), without a recourse to any internal space for the spinning degrees of freedom.

By construction, Eq. (74) should have two gauge symmetries. Applying the general recipe \((34)\) for constructing the gauge symmetry to the equation of the cylinder \((39)\), we find the gauge symmetry transformations, infinitesimal gauge transformation parameters being arbitrary functions of \( \tau \).

Here, the vectors \( n, d \) are expressed in terms of \( x \) and its derivatives by formulas \((67)\), \((68)\), and \( \varepsilon_1, \varepsilon_2 \) are the infinitesimal gauge transformation parameters being arbitrary functions of \( \tau \). In the limit of small curvature and torsion \((69)\), \((70)\), the formulas \((75)\) take the form

\[
\delta_{\varepsilon_1} x = n \varepsilon_1, \quad \delta_{\varepsilon_2} x = [n, d] \varepsilon_2.
\]

where \( x' \) and \( x'' \) are the velocity and acceleration vectors in the natural parametrization of the curve. For the straight lines \( x'' = 0 \), the second gauge transformation \((75)\) becomes singular. The remaining gauge symmetry is reparametrization, so each straight line constitutes its own class of gauge equivalent atypical curves. By this reason, the straight lines are unconnected by gauge transformations with the typical cylindrical lines.

The physical DoF number can be counted for the model \((74)\) by formula \((52)\). In the case at hand, we have \( n_4 = 1 \) (the equation \((74)\) of motion involves the fourth-order derivatives), \( r_0 = 2 \) (two gauge symmetry transformations \((75)\) have the zero order). Substituting all the ingredients into \((52)\), we get \( n = 4 \), as it should be for the massive spinning particle in \( d = 3 \).

The momentum \( p \) and angular momentum \( J \) are connected to the cylinder parameters \( n, y \) by relations \((17)\). Substituting \((67)\), \((68)\) into \((17)\) we get, \( p, J \):

\[
p = m \left( \frac{z}{\sqrt{z^2 - 1}} x' - \frac{1}{\sqrt{z^2 - 1}} \frac{1}{\kappa' 3 z + \omega(z^2 - 1)} \frac{1}{\sqrt{z^2 - 1}} x'' - \frac{1}{r \kappa' (z^2 - 1) \sqrt{z^2 - 1}} [x', x''] \right);
\]

\[
J = [x - d, p] - \frac{s}{m} p,
\]

with the vector \( d \) given by \((67)\), and \( z \) being the common root of \((66)\) and \((71)\). The quantities \((74)\), \((76)\) conserve on shell by construction. Given the radius-to-spin relation \((19)\), \( p \) and \( J \) satisfy constraints \((10)\) in \( d = 3 \) with mass \( m \) and spin \( s \). So, the gauge invariant integrals of motions define the co-orbit of the Poincaré group, which is four dimensional. For the above reasons, the single fourth-order equation \((74)\) describes an irreducible massive spinning particle moving along a timelike path.

C. Explicit second-order equations for the timelike world lines of spinning particle

In this section, we derive the second-order equations defining the timelike lines lying on the cylinder with the timelike axis. The main idea is to introduce an appropriate auxiliary angle-type variable such that can absorb certain combinations of derivatives of \( x(\tau) \). A similar idea was implemented in the work\(^{35}\) for deducing an explicit form of ODE system for cylindric lines in \( d = 3 \) Euclidean space. We introduce the angle variable in a different way compared to\(^{35}\) that seems us better suited the Minkowski space specifics and leading to a simpler ODE system. Besides that, our second order system is variational, while the action functional has a simple geometric meaning. Furthermore, the same action was previously known in the minimal spinning particle model suggested in ref.\(^{19}\). In this model, the particle configuration space is chosen to be \( M_4 = \mathbb{R}^{1,2} \times S^1 \). The factor \( S^1 \), however, has been considered just as a fiber over Minkowski space unrelated to the fact that the particle paths are cylindrical lines in Minkowski space. Now, we see that the structure of the configuration space of the minimal spinning particle model in \( d = 3 \) and model Lagrangian follow from the fact that the particle evolves by the cylindric world sheet in Minkowski space.

In the previous subsection, we have got the fourth-order ODE \((74)\) defining the cylindric lines \( x(\tau) \). Now, we are going to diminish the order of the ODE for cylindric lines by making use of auxiliary angular variable absorbing
In terms of the auxiliary angle, the spin vector $S$ involving the difference vector $d$ (37) in the ODE formulation. By construction, $d$ is normal to the cylinder axis, and it connects the axis with the current particle position on the surface. Once the radius of cylinder is fixed, the difference vector satisfies the following constraints,

$$ (n, d) = 0, \quad d^2 = r^2. \quad (79) $$

Given the point on the cylinder, the vectors $n$ and $[n, d]$ span the tangent space to the cylinder. If the curve $x(\tau)$ lies on the cylinder, its tangent vector $\dot{x} = \frac{dx}{d\tau}$ is tangential to the cylinder. And vice versa, if the velocity is tangential to the cylinder all over the line, the path belongs to the cylinder. Since the cylinder tangent space is spanned by the vectors $n$ and $[n, d]$, any cylindrical line is defined by the first-order equations,

$$ \dot{x} = e_1 n + e_2 [n, d], \quad (80) $$

with $\tau$ being some parameter on the curve $x(\tau)$, not necessarily natural. The expansion coefficients $e_1$ and $e_2$ can be arbitrary functions of $\tau$. These coefficients, $e_1$ and $e_2$, can be understood as einbeins. In Ref. 38, it has been shown that each einbein induces a gauge transformation of the ODE system. In the case at hand, the generators of transformations are the coefficients at einbeins in the ODE system (80). In this way, one can see that the first-order equations (80) automatically have gauge symmetries (75).

Eqs. (80) describe particle’s path in Minkowski space in terms of the conserved unit vector $n$ and difference vector $d$. The latter is subject to constraints (79). Now, we solve the constraints in a parametric form. The solution depends on the constants $s$ and $\varrho$ (81), which define the level of the corresponding classical Casimir function. Below, we mostly consider the case when the spin vector is spacelike or null, i.e. $\varrho \geq 0$. It turns out to be the model with the configuration space $M_4 = \mathbb{R}^{1,2} \times S^1$ considered in Ref. 19. The case of the timelike spin can be treated in a similar way.

The world line is supposed to be timelike and casual, i.e.

$$ \dot{x}^2 < 0, \quad \dot{x}^0 > 0. \quad (81) $$

Provided that $\varrho \geq 0$, the constraints (83) admit a solution w.r.t. $D$ in terms of the auxiliary null vector $b$:

$$ d = \frac{s[b, n] - \varrho(b + n(n, b))}{m(n, b)}, \quad b^2 = 0. \quad (82) $$

The auxiliary angle variable $\varphi$ is introduced as a parametrization of $b$, because any null vector is defined (up to inessential overall factor) by a single angle,

$$ b(\varphi) = (1, -\sin \varphi, \cos \varphi). \quad (83) $$

In terms of the auxiliary angle, the spin vector $S$ reads

$$ S = \frac{s b - \varrho [b, n]}{(b, n)}, \quad S^2 = \varrho^2 \geq 0. \quad (84) $$

The radius of cylinder is given by $r = \frac{1}{2} \sqrt{s^2 + \varrho^2}$.

Let us now rewrite Eqs. (80) in terms of the unconstrained angular variable $\varphi$. Multiplying both sides of (80) by $n$ and $[n, d]$, we express the einbeins

$$ e_1 = -(n, \dot{x}), \quad e_2 = \frac{1}{r^2}([n, d], \dot{x}). \quad (85) $$

Accounting for Eqs. (87) and (82), we get

$$ e_1 = \frac{(b, \dot{x})}{(b, n)} - \frac{s}{m} \frac{\dot{\varphi}}{(b, n)}, \quad e_2 = \frac{1}{r^2}([n, d], \dot{d}) = \frac{\dot{\varphi}}{(b, n)}. \quad (86) $$

Substituting the vector $b$ (82) and the einbeins (83) into (80), we arrive at the first-order equations for the cylindrical curves in the form that involves the auxiliary angular variable $\varphi$

$$ \dot{x} = \frac{(b, \dot{x})}{(b, n)} n + \frac{s}{m} \frac{b}{(b, n)^2} \dot{\varphi} - \frac{\varrho \varrho}{m} \frac{b}{(b, n)^2} \dot{\varphi}. \quad (87) $$
The cylinder parameters \( n, y \) are explicitly defined as integrals of motion for the equations of motion (87).

\[
n = \frac{\dot{x}}{\sqrt{-\dot{x}^2 \left( 1 - \frac{2s}{m} \frac{\dot{\varphi}}{m (b, \dot{x})^2} - \frac{\dot{\varphi}^2}{m^2 (b, \dot{x})^2} \right)}} - \frac{\dot{b}}{m} \frac{\dot{\varphi}}{m (b, \dot{x})^2} + \frac{b}{m} \frac{s}{m (b, \dot{x})^2} + \frac{\dot{\varphi}^2}{m^2 (b, \dot{x})^2} + \frac{\dot{b} \dot{\varphi}^2}{m^2 (b, \dot{x})^2},
\]

(88)

\[
y = \frac{\dot{x}}{\sqrt{-\dot{x}^2 \left( 1 - \frac{2s}{m} \frac{\dot{\varphi}}{m (b, \dot{x})^2} - \frac{\dot{\varphi}^2}{m^2 (b, \dot{x})^2} \right)}} + \frac{\dot{y}}{m} \frac{s}{m (b, \dot{x})^2} + \frac{\dot{\varphi}}{m} \frac{\dot{b}}{m} \frac{s}{m (b, \dot{x})^2} + \frac{\dot{b} \dot{\varphi}}{m^2 (b, \dot{x})^2} + \frac{\dot{b}}{m} \frac{\dot{\varphi}}{m (b, \dot{x})^2} + \frac{\dot{b} \dot{\varphi}^2}{m^2 (b, \dot{x})^2},
\]

(89)

Given the relations between the cylinder parameters \( n, y \) and particle momentum \( p \), and angular momentum \( J \) (17), the relations above define the conserved quantities of the particle in terms of particle’s path in the Minkowski space \( x(\tau) \), and auxiliary angle variable \( \varphi(\tau) \). Substituting (88) and (89) into (17), we get particle’s conserved quantities

\[
p = mn,
\]

\[
J = [x, p] - \frac{\dot{b}}{m (b, \dot{x})^2} + \frac{b}{m} \frac{s}{m (b, \dot{x})^2} + \frac{\dot{b}}{m} \frac{\dot{\varphi}}{m (b, \dot{x})^2} + \frac{\dot{b} \dot{\varphi}}{m^2 (b, \dot{x})^2} + \frac{\dot{b} \dot{\varphi}^2}{m^2 (b, \dot{x})^2},
\]

(90)

Consider the equations for the cylindrical lines (87). After excluding the cylinder parameters \( n, y \) by making use of relations (88), (89) and their consequences (90), we arrive at the explicit second-order equations for the particle paths \( x(\tau) \) and the auxiliary angular variable \( \varphi(\tau) \),

\[
\dot{p} \equiv \frac{d}{d\tau} \left( \frac{\dot{x}}{\sqrt{-\dot{x}^2 \left( 1 - \frac{2s}{m} \frac{\dot{\varphi}}{m (b, \dot{x})^2} - \frac{\dot{\varphi}^2}{m^2 (b, \dot{x})^2} \right)}} - \frac{\dot{b}}{m} \frac{\dot{\varphi}}{m (b, \dot{x})^2} + \frac{b}{m} \frac{s}{m (b, \dot{x})^2} + \frac{\dot{\varphi}^2}{m^2 (b, \dot{x})^2} + \frac{\dot{b} \dot{\varphi}^2}{m^2 (b, \dot{x})^2},
\]

\[
\dot{J} \equiv \frac{d}{d\tau} \left( [x, p] + s \frac{b}{m (b, \dot{x})^2} \right) = 0.
\]

(91)

There are Noether identities between equations (91). One identity is a consequence of relation (37), and one more follows from normalization condition \( p^2 + m^2 = 0 \) identically satisfied by the momentum (90). The identities read

\[
(p, \dot{p}) = 0, \quad \dot{J} + s \frac{m}{p^2} = 0.
\]

(92)

Given the identities, the second-order system for the cylindrical lines involves two independent equations for four variables: three space-time coordinates \( x \) and one auxiliary angular variable \( \varphi \). In particular, it is sufficient to require the normalized vector \( p \) to conserve, while the other equations will follow from this one. The gauge symmetries for equations (91) read

\[
\delta_{\varepsilon_1} x = p \varepsilon_1, \quad \delta_{\varepsilon_1} \varphi = 0; \quad \delta_{\varepsilon_2} x = 0, \quad \delta_{\varepsilon_2} \varphi = \varepsilon_2,
\]

(93)

with \( \varepsilon_1, \varepsilon_2 \) being the infinitesimal gauge parameters. The first gauge transformation is a shift of the world line along the axis of cylinder. The second gauge symmetry acts by rotations in the plane orthogonal to the axis. As a particular consequence of the transformation law (92), the angular variable is a pure gauge. Given the identities and gauge symmetries, the degree of freedom number for the system (91) can be counted by formula (52). It equals to four, as it should be for the irreducible spinning particle in \( d = 3 \).
By linear combining, Eqs. (91) can be brought into the Lagrangian form with the action functional

\[ S[x(\tau), \varphi(\tau)] = \int \left( -m \sqrt{-\dot{x}^2} \left( 1 - \frac{2s}{m} \frac{\dot{\varphi}}{\dot{x}} - \frac{\dot{\varphi}^2}{m^2 (b, \dot{x})^2} \right) - \delta \left( \frac{\partial b \dot{\varphi}}{b \dot{x}} \right) \right) d\tau. \] (94)

This action has been suggested in Ref.\textsuperscript{12} to describe the irreducible spinning particle in \( d = 3 \) by the configuration space \( M = \mathbb{R}^{1,2} \times S^1 \), where the factor \( S^1 \) is the configuration space of spin. From the viewpoint of the Lagrangian formalism, the integrals of motion \( p, J \) (90) are just Noether’s conserved quantities associated to the Poincaré symmetry of the action.

Let us mention about the geometric interpretation of the action (94). Consider the extremal value of the action

\[ S[x(\tau), \varphi(\tau)] \bigg|_{\frac{dx}{d\tau} = \frac{d\varphi}{d\tau} = 0} = -m \int (n, \dot{x}) d\tau. \] (95)

Obviously, the on-shell value of particle’s action is the arclength of the cylindrical line projection on the axis of particle’s world sheet. This provides the interpretation for the particle action from the viewpoint of geometry of path as such, without appealing to any fiber bundle over Minkowski space.

IV. CONCLUDING REMARKS

Let us briefly summarize what we have observed in this paper about dynamics of classical massive spinning particles. Once the Poincaré group representation is irreducible and nondegenerate for quantum spinning particle in Minkowski space, the classical evolutions of the particle are constrained to the world sheets. In \( d = 3,4 \) the world sheets are 2d cylinders. In \( d > 4 \) dimensions, the world sheets are toroidal cylinders \( \mathbb{R} \times T^D \), with the torus dimension \( D = [(d - 1)/2] \). The radii of the cylinders are fixed by representation. Positions of the world sheets in Minkowski space are defined by particle’s conserved momenta and angular momenta subject to the conditions constraining them to a nondegenerate co-orbit of the Poincaré group. So, the space of particle’s world sheets is isomorphic to the co-orbit of the Poincaré group. All the causal world lines on the same world sheet are gauge equivalent to each other. The particle paths, being understood as cylindrical lines in Minkowski space, can be defined by an ODE system. We demonstrate a general scheme of deducing such a system. The latter ODE system can be understood as classical equations of motion for the irreducible massive spinning particle. The equations for the cylindrical lightlike line are explicitly deduced in \( d = 3 \), and they turn out to be equations of null helices. Even though the lines are lightlike, the equations describe classical dynamics of irreducible massive spinning particle. The timelike cylindrical world lines in \( d = 3 \) are shown to be defined by a single fourth-order equation with two zero-order gauge symmetries. This higher-derivative equation of motion is deduced for the irreducible massive spinning particle in an implicit form. By introducing an auxiliary angle variable, being a pure gauge degree of freedom, this equation is shown to reduce to an equivalent explicit second-order system. The latter equations have been previously known as EoMs of the minimal model of the \( d = 3 \) irreducible spinning particle\textsuperscript{12}.

Overall, we see that the classical dynamics of irreducible spinning particles in various dimensions are completely defined by their world sheets in Minkowski space. The usual form of classical dynamics, being based on EoMs, is deduced from the world-sheet formulation. It is also seen that the EoMs of irreducible massive spinning particle can be formulated in terms of paths in Minkowski space, without recourse to any internal configuration space attributed to spinning degrees of freedom.

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