Information content in uniformly discretized Gaussian noise: optimal compression rates
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Published in: International Journal of Modern Physics C, Vol.10, 687-716 (1999)

Abstract. We approach the theoretical problem of compressing a signal dominated by Gaussian noise. We present expressions for the compression ratio which can be reached, under the light of Shannon’s noiseless coding theorem, for a linearly quantized stochastic Gaussian signal (noise). The compression ratio decreases logarithmically with the amplitude of the frequency spectrum $P(f)$ of the noise. Entropy values and compression rates are shown to depend on the shape of this power spectrum, given different normalizations. The cases of white noise (w.n.), $f^n$ power-law noise —including $1/f$ noise—, (w.n.+$1/f$) noise, and piecewise (w.n.+1/$f^2$) noise are discussed, while quantitative behaviours and useful approximations are provided.

Keywords: Information Theory, Signal compression

1 Introduction

There are several motivations to consider the theoretical problem of compressing noise (or signals so stochastic that deserve this name). In some cases, the signal to be transmitted is intrinsically noisy (e.g. from scientific measurements) and needs to be compressed in a lossless way before any reduction process can be applied. One of the measured quantities which best exhibits this intrinsic randomness is the fluctuation of the cosmic microwave background (CMB) radiation. Considerable efforts have already been made in order to cope with the handling of such sort of data (see e.g. [1]-[3]). Like other signals from scientific instruments on-board space satellites, CMB-measurements produce high rates of noisy data that have to be sent to Earth via a more or less limited telemetry rate [4].

Electronic instruments (e.g. detectors, amplifiers) show characteristic low frequency instabilities ($1/f$ noise) to be added to white or thermal noise. When the signal measured with these instruments is weak, it can only be recovered from averaging many measurements. The averaging is possible only after a careful calibration of the low frequency instabilities, which in practice means that the whole (noisy) signal has to be transmitted (to Earth). This is an example that requires lossless compression of a signal dominated by noise. In the present work we would like to study, in a quantitative way, to what extent noise can be compressed.

This noise is usually treated as a Gaussian stochastic process with an arbitrary power spectrum (some relevant aspects of this type of processes have been considered in [5]-[6]). We shall assume that its values are discretized —quantized— in a uniform or linear way. Given the properties of a Gaussian distribution, it is
possible to find analytical approximations for its information content, and we will take advantage of them for obtaining the ideal —i.e., highest theoretically achievable— compression factor.

In the present work we make no reference to the error brought about by the discretization process itself. Yet, a few words on this subject are perhaps called for. A typical measure of the error caused is the *distortion* $D$ (many aspects of rate distortion theory are covered in [7]. The same philosophy has been applied to the minimum discrimination information (MDI) theory —see e.g. [8] and refs. therein). This magnitude is a sum—or integral— of the error between the continuous values of the initial random variable and the associated discrete ones, weighted by the probability distribution. According to the theory, for a given initial length of the random variable there is a minimal possible distortion. Then, $\sqrt{D}$ may be interpreted as the lowest ‘distortion noise’. Usually, this optimal $D$ falls as the length increases, but this implies to increase the entropy, thus setting a trade-off between compressibility and distortion.

The text is organized as follows. In Section 2 we present a basic introduction to the problem of data compression. In Section 3 we deal with the one-dimensional Gaussian case, which will be helpful for studying multidimensional Gaussian noise with possible correlations in Section 4. Information content and compression values are then discussed. In Section 5 our conclusions are presented. A number of calculations have been included in the appendices.

## 2 The basic data compression problem

Standard lossless data compression techniques are applied successfully only to data sets with some redundancy. This redundancy can be formally expressed using the entropy $H$. It is easy to show (see below) that it is not possible to compress a (uniformly) random distribution of measurements. If noise is discretized to a high resolution (as compared to its variance) the resulting distribution of numbers approaches a uniform distribution. This indicates that lossless compression might not be very efficient when the data is dominated by noise, but, as we shall see, the problem depends crucially on the digital resolution and the range of values to be stored.

Hypothetical data compression problems can be considered in the light of Shannon’s first theorem (see [9]). This theorem tells us that the Shannon entropy $H$ of a source is the lower bound to the average length of the code units or ‘words’ (In addition, we know that such a lower bound can be fairly well approached by means of some of the available methods for coding, such as Huffman’s, etc.). Then, the theoretical compression rate is defined as:

\[ c_{r, \text{opt}} \equiv \frac{\text{average length per code unit}}{\text{Shannon entropy per code unit}} \]

Of course, for this quotient to make sense, both quantities should be referred to the same type of code divisions (e.g. words, data values, blocks, packets, etc.) and must be written in the same length units (e.g. bits).

Thus, our problem entails the entropy of the stochastic process generating the noise under consideration. In our case, this noise will be the result of a Gaussian process with a specific power spectrum. Its outcome shall be represented by a random variable $\eta$, which can be assumed to be stationary in wide sense. The discrete set of $\eta(t)$-values for successive $t$ increases will be treated like the components of a multidimensional Gaussian variable.
with the power spectrum in question. Most of the time, we will deal with a bandwidth-limited spectrum, i.e., one where the frequencies are limited by an upper and a lower limit. Examining the associated Shannon entropy, we shall study the hypothetic chances of compressing the sort of data sequences generated by such processes. In particular, we will consider Gaussian white noise, Gaussian noise with correlation of the $1/f$-type, and Gaussian noise with a mixed correlation of the type white-noise + $1/f$-noise.

In general, the compression rate $c_r$ for finite sequences of symbols that have been encoded is usually defined as the quotient between the sequence lengths before and after the encoding process — $L_i$ and $L_f$, respectively — i.e., $c_r = \frac{L_i}{L_f}$. If $\{a_j\}$ and $\{\alpha_j\}$ ($j = 1, \ldots, N_s$) denote the initial and final — or encoded — sets of symbols, their average lengths are

$$
\mathcal{L}_i = \sum_{j=1}^{N_s} p_j L(a_j),
$$

$$
\mathcal{L}_f = \sum_{j=1}^{N_s} p_j L(\alpha_j),
$$

where $p_j$, $L(a_j)$ and $L(\alpha_j)$ give the probability of the $j$th symbol and its length in bits before and after encoding, respectively. When the sequences are long enough, the rate $c_r$ can be replaced with the quotient between the initial and final average lengths per symbol in the way $c_r \approx \frac{\mathcal{L}_i}{\mathcal{L}_f}$. We shall assume $L(a_j) = \mathcal{T}_i \forall_j$, i.e., that the initial data representation consists of symbols of the same length.

Shannon’s first theorem (also called noiseless coding theorem, see e.g. [10, 11]) provides theoretical lower (and upper) bounds to the final length per symbol in the way $H \leq \mathcal{L}_f$ ($\leq H + 1$), where $H$ is the Shannon entropy

$$
H = -\sum_j p_j \log_2(p_j). \tag{2.1}
$$

An efficient coding method will have to approach equality to the lower bound. For one dimension, the Huffman scheme is known to be reasonably close\(^1\)(see also the performance of other methods such as the Rice algorithm in [13]). Thus, the compression ratio will satisfy $c_r \approx \frac{\mathcal{T}_i}{\mathcal{L}_f} \approx \frac{\mathcal{T}_i}{H}$, being the equality the optimal case, given by

$$
c_{r, \text{opt}} \equiv \frac{\mathcal{T}_i}{H}. \tag{2.2}
$$

Let’s consider the case of an $N$-dimensional (vector) random variable. Since the probabilities must be now referred to a multivariate distribution, (2.1) is generalized to

$$
H_N = -\sum_{j_1, \ldots, j_N} p_{j_1, \ldots, j_N} \log_2(p_{j_1, \ldots, j_N}). \tag{2.3}
$$

We shall suppose that each of its components is a one-dimensional random variable of the same type. In addition, there might exist possible correlations among these components. There is a well-known inequality for any $N$-dimensional random variable $\tilde{\eta} = (\eta_1, \ldots, \eta_N)$ (Gaussian or not) relating the joint Shannon entropy $H_N$

\(^1\)To give an idea of this closeness, let’s quote a bound found in [12] calling $r \equiv \mathcal{T}_f - H$, and $p_{\text{max}} = \max(\{p_j\})$, then $r \leq p_{\text{max}} + \log_2\left(\frac{2 \log_2(e)}{e}\right) = p_{\text{max}} + 0.086$. 

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and the individual Shannon entropies of each component, \(H_1(\eta_j), j = 1, \ldots, N\), which reads

\[
H_N(\eta_1, \ldots, \eta_N) \leq H_1(\eta_1) + \ldots + H_1(\eta_N),
\]

(2.4)

or, equivalently,

\[
h(\eta_1, \ldots, \eta_N) \equiv \frac{H_N(\eta_1, \ldots, \eta_N)}{N} \leq \frac{1}{N} \sum_{j=1}^{N} H_1(\eta_j),
\]

(2.5)

where \(h\) denotes the joint Shannon entropy per component. Unlike \(H_N\), \(h\) does not grow extensively by merely increasing \(N\). When \(\eta_1, \ldots, \eta_N\) are all of the same type, (2.5) reduces to \(h \leq h_1\). Defining the initial length per component \(l_i\) as the analogue of \(L_i\) for each vector component, eq. (2.2) may be rewritten as

\[
c_{r, \text{opt}} \equiv \frac{l_i}{h}.
\]

(2.6)

It is essential to note that the equality in (2.5) is satisfied if and only if the \(N\) components \(\eta_1, \ldots, \eta_N\) are independent. Therefore, for independent variables of the same type, \(h = H_{N=1}\), and it is enough to study the \(N = 1\) case.

Observe that for a uniform distribution, where \(p_j = 1/N_s\), we have that \(h = l_i = \log_2(N_s)\); so, no compression is possible \((c_{r, \text{opt}} = 1)\).

3 One-dimensional Gaussian variable

We will try to find this theoretical rate for a zero-mean Gaussian white noise \(\eta\) — whose probability density will be called \(f(\eta)\) — with variance equal to \(\sigma\), and whose values are discretized or ‘quantized’ to a given resolution. When discretizing, we gather results into intervals of some fixed width, which shall be denoted by \(\Delta \eta\). If this width is small enough, we may assume that all the values that have fallen into the same interval have, roughly, the same probability. Thus, to each interval we assign a ‘probability’ value as follows

\[
\eta \text{ in the interval around } \chi \left(\chi - \frac{\Delta \eta}{2}, \chi + \frac{\Delta \eta}{2}\right) \rightarrow \begin{cases} 
p(\chi) \Delta \eta & = \int_{\chi - \frac{\Delta \eta}{2}}^{\chi + \frac{\Delta \eta}{2}} d\zeta f(\zeta) \\
\approx f(\chi) \Delta \eta & = e^{-\frac{\chi^2}{2\sigma^2}} \frac{\Delta \eta}{\sqrt{2\pi\sigma^2}}
\end{cases}
\]

This will be done for each \(\eta^{(j)}\), with \(\eta^{(j)} = j \Delta \eta, j \in \mathbb{Z}\). Each interval will be called \(I^{(j)} = (\eta^{(j)} - \frac{\Delta \eta}{2}, \eta^{(j)} + \frac{\Delta \eta}{2})\).

In order to properly talk about probabilities, the set should be well normalized. Therefore, we write the probability that \(\eta\) takes a value in \(I^{(j)}\) as

\[
p_j \equiv p[\eta \in I^{(j)}] = \frac{p(\eta^{(j)})}{\sum_n p(\eta^{(n)})} = \frac{e^{-\frac{j^2(\Delta \eta)^2}{2\sigma^2}}}{Z},
\]

(3.1)
where

\[ Z = \sum_{n=-\infty}^{\infty} e^{-\frac{n^2(\Delta \eta)^2}{2\sigma^2}}. \] (3.2)

This \( Z \), introduced in order to fulfil the normalization condition, may be also regarded as the partition function of a system with energies \( \{E_n = \pi n^2\} \) at temperature \( T = \frac{2\pi \sigma^2}{(\Delta \eta)^2} \), with adequate new units for the Boltzmann constant.

In order to calculate the ideal compression rate, we need to find the Shannon entropy (2.1). Since \( N = 1 \), \( H = H_1 = h \), and the result (see subsec. A.1 in the appendix) is

\[ h = \log_2 \left[ \sqrt{2\pi e} \sigma \Delta \eta \right] + \mathcal{O} \left( \frac{2\pi \sigma^2}{(\Delta \eta)^2} e^{-\frac{2\pi^2}{3}\sigma^2}, \ldots \right) \] (3.3)

which depends on \( \sigma \) and \( \Delta \eta \) only through the dimensionless quotient

\[ \frac{\Delta \eta}{\sigma} \equiv \lambda \] (3.4)

Thus, the smaller \( \lambda \sim 1/\sqrt{T} \) (the higher the temperature) the larger the entropy \( h \). Compare this with the result of a naïve integration without discretization, which would be \( \log_2 \left( \sqrt{2\pi e} \sigma \right) \equiv h_{\text{cont}} \). In the \( \lambda \to 0 \) limit the exponentially small corrections vanish, but the logarithm of \( \lambda \) diverges. Thus,

\[ h \simeq h_{\text{cont}} - \log_2(\Delta \eta) \] (3.5)

(see explanation in refs. [14] or [15], or in app. 3, or our own comments below, after eq.(4.11)).

Let’s now write the initial mean length as \( \bar{l}_i = \log_2(N_s) \). This means that, using a suitable binary representation, \( N_s \) is the number of effectively distinct \( \eta \)-values that can be considered (although \( \bar{l}_i \) is an integer only when \( N_s \) is an exact power of 2, these variables will be treated as if they were real).

First, we can imagine a process in which the initial length per symbol \( \bar{l}_i \) has been fixed independently of \( \Delta \eta \) (this could be the case when we are worried about instabilities of the signal). Then, the optimal compression rate would just be the quotient

\[ c_{r,\text{opt}} \equiv \frac{\bar{l}_i}{h} \simeq \frac{\bar{l}_i}{\log_2(\sqrt{2\pi e}/\lambda)}. \] (3.6)

So that the larger we can make \( \lambda \), without loss of relevant information, the larger the compression. If the final sensibility \( S \) we need is obtained from some later average of \( M \) measurements of this noise \( \eta \), then we can make \( \lambda \simeq 1 \) as far as \( M \gtrsim (\sigma/S)^2 \). In this extreme case the compression can be as large as \( c_{r,\text{opt}} \simeq \bar{l}_i/2.047 \), e.g., \( c_{r,\text{opt}} = 7.8 \) for 16 bits symbols. Fig. 1 shows (as continuous lines) the entropy \( h \) and the compression \( c_{r,\text{opt}} \) as a function of \( \lambda \).

Another possibility is to work with \( \bar{l}_i \) as a function of \( R \) and \( \Delta \eta \). We suppose that the values of our random variable \( \eta \) span a range \( R \equiv \max(\eta) - \min(\eta) \). Assuming our discretization to be linear, it is clear that

\[ R = N_s \Delta \eta \] (3.7)
Figure 1: Shannon entropy per component $h$ (left) and associated optimal compression rate $c_{r, \text{opt}}$ (right) —by formulas (2.6), (3.6)— as functions of the discretization parameter $\lambda = \Delta \eta / \sigma$, for a fixed $\ell_i = 16$ bits. The three curves correspond to $n_p = 0$ with $P(\omega) = A = 1$ sec. (solid line), a combination of $n_p = 0$ and $n_p = -1$ of the form $P(\omega) = A_0 + \omega_0 / |\omega|$ with $A_0 = 1$, $\omega_0 = \omega_{\text{Max}} / 10$ (dashed line), and to $n_p = -1$ for $P(\omega) = A\omega_0 / |\omega|$ with the same value of $\omega_0$ (dotted line). No equal $\sigma_1-\sigma_p$-constraint has been imposed.

and, therefore,

$$\ell_i = \log_2 \left( \frac{R}{\Delta \eta} \right).$$

Formulae (2.6) and (3.3)-(3.8) enable us to put $c_{r, \text{opt}}$ as a function of either $\Delta \eta$ or $\ell_i$. Then,

$$c_{r, \text{opt}} \approx \frac{1}{\ell_i + \log_2 \left( \frac{\sqrt{2\pi e}}{2N_0} \right)} \cdot \frac{\ell_i}{\log_2 \left( \frac{\sqrt{2\pi e}}{2N_0} \right)}. \quad (3.10)$$

If we limit $R$ to a given number of $\sigma$’s—say $N_0$—around the origin, only the values in $(-N_0 \sigma, N_0 \sigma)$ will be taken into consideration. Thus, $R = 2N_0 \sigma$, and we can further write

$$c_{r, \text{opt}} \approx \frac{\ell_i}{\ell_i + \log_2 \left( \frac{\sqrt{2\pi e}}{2N_0} \right)} \cdot \frac{1}{\log_2 \left( \frac{\sqrt{2\pi e}}{2N_0} \right)}. \quad (3.10)$$

Note that $c_{r, \text{opt}}$ cannot be larger than one if $N_0 \leq N_{0, \text{crit}} \equiv \sqrt{\frac{2\pi e}{2}} \approx 2.0664$. This is interpreted as a critical size of the acceptable range. On the other hand, by taking larger and larger values of $N_0$ one could achieve arbitrarily high compression rates, but this would mean to collect sufficiently meaningful amounts of data very far from the mean. This could correspond to rare events which might not follow the Gaussian distribution.\footnote{Note that there is no contradiction here because even in the presence of non-Gaussian rare events, the bulk of the data might still be well described by a Gaussian so that our estimations could still yield a good approximation.}
Figure 2: Optimal compression rate $c_{r, \text{opt}}$—by formulas (3.8), (3.9)—as functions of the discretization interval $\lambda = \Delta \eta / \sigma$ (left) and as functions of the initial mean length in bits $\bar{l}_i$ (right). The three curves have the same parameters as in Fig. 1 but, this time, with variable $\bar{l}_i$ and $R = 2N_0\sigma$, $N_0 = 3$. Note that $c_{r, \text{opt}}$ has a divergence for small $\bar{l}_i$, which comes from the vanishing of $h$.

In general, a reasonable choice would be some $N_0$ moderately above $N_{0\text{crit}}$, but this depends critically on the subsequent data analysis we want to carry on with these data.

In Figure 2, we show $c_{r, \text{opt}}$ for the case of white noise (continuous line) with $R = 2N_0\sigma$, $N_0 = 3$. The main difference from Fig. 1 is that in the former $\bar{l}_i = 16$ bits while in Fig. 2, we choose $\bar{l}_i$ according to $\Delta \eta$ as in eq. (3.8) with $N_0 = 3$. Although the distance of three sigmas is already a long way from the mean, the compression rates found are rather small. In Fig. 2, we also show (right panel) $c_{r, \text{opt}}$ as a function of $\bar{l}_i$, showing how the compressibility increases as $\bar{l}_i$ gets small.

4 Multidimensional case: Gaussian stochastic processes

4.1 Uncorrelated Gaussian variables: white noise

Suppose now that we have $N$ uncorrelated Gaussian variables with different variances $\sigma_1, \ldots, \sigma_N$. Although we shall keep this general notation, we are only interested in processes where $\sigma_1 = \ldots = \sigma_N$, which is the case of a Gaussian stochastic process stationary in wide sense. As long as these $N$ variables are uncorrelated, we have to apply (2.4) as an equality, which, combined with (3.3) and equally quantizing in all dimensions, gives the joint entropy

$$H \equiv H_N = \log_2 \left[ \sqrt{\frac{2\pi e}{(\Delta \eta)^2}} \prod_{m=1}^{N} \sigma_m^2 \right] + O \left( N \frac{2\pi \sigma^2}{(\Delta \eta)^2} e^{-\frac{2\Delta \eta^2}{\sigma^2}} \ldots \right).$$  \hspace{1cm} (4.1)

Here we may interpret

$$\prod_{m=1}^{N} \sigma_m^2 = \det(C_0),$$  \hspace{1cm} (4.2)
\[ N \quad \lambda \equiv \frac{\Delta n}{\eta} \quad c_r^H \quad c_r^A \quad c_{r, \text{opt}} \]

|     | 1000 | 0.05 | 1.09 | 1.09 | 1.26 |
|-----|------|------|------|------|------|
| 10000 | 0.05 | 1.23 | 1.23 | 1.26 |
| 100000 | 0.05 | 1.25 | 1.25 | 1.26 |
|      | 0.2  | 1.81 | 1.83 | 1.83 |
|      | 0.4  | 2.34 | 2.36 | 2.37 |
|      | 0.6  | 2.81 | 2.85 | 2.85 |
|      | 0.8  | 3.27 | 3.31 | 3.32 |
|      | 1.0  | 3.67 | 3.69 | 3.80 |

Table 1: Comparison of optimal compression rates \( c_{r, \text{opt}} \) with actual rates from simulated Gaussian white noise compressed with implementations of the Huffman — \( c_r^H \) — and arithmetic — \( c_r^A \) — methods. These values are roughly a half of those in the solid-line curve of Fig. 1, as \( l_i \) is now equal to 8, instead of 16.

where \( C_0 \) is the diagonal matrix

\[ C_0 = \text{diag}(\sigma_1^2, \ldots, \sigma_N^2). \]  

It is useful to define an effective variance as:

\[ \sigma_0^2 \equiv \text{Det}^{1/N}(C_0), \]  

so that when \( \sigma_1 = \ldots = \sigma_N \) we will have \( \sigma_0 = \sigma_1 = \ldots = \sigma_N \), which also agrees with our definition of the 1-point sigma \( \sigma_{1p} \) below eq.(4.22).

Thus, the entropy per component is conveniently written as in the previous case (3.3):

\[ h = \log_2 \left( \sqrt{2\pi e} \frac{\sigma_0}{\Delta \eta} \right) + O \left( \frac{2\pi \sigma^2}{(\Delta \eta)^2} e^{-\frac{\sigma^2}{(\Delta \eta)^2}} \right) \equiv h_0(\sigma_0) \]  

where the 0-subscript means that this is the uncorrelated case. As we shall see below, to deal with correlations will just mean the replacement of \( C_0 \) with a new correlation matrix — say \( C \) in (4.4).

We have done simulations of Gaussian noise with \( \sigma_1 = \ldots = \sigma_N \) and the data have been represented with a fixed \( l_i = 8 \) bits. The \( N \)-dimensional variable is then compressed by the Huffman and arithmetic methods, and the compression rate \( c_r^H \) is found as the quotient between the sizes of the initial and the compressed files. This actual compression rate is then compared to the optimal one, i.e., to \( c_{r, \text{opt}} = \frac{7}{h} \). The results are presented in Table 1. The agreement is better as \( N \) increases. The explanation is that, in practice, the compressed files take up some further space for storing the conversion tables between both symbol sets. Obviously, since the number of different symbols is fixed — \( 2^l_i = 256 \)— the relative contribution caused by the size of these tables decreases as \( N \) grows.
4.2 Gaussian variables with correlation: coloured noise

Now, suppose that we have an \( N \)-dimensional variable \( \vec{\eta} = (\eta_1, \ldots, \eta_N) \) whose components are correlated according to the entries of some covariance matrix \( C \). By mathematical definition, \( C(\eta_j, \eta_k) = \langle (\eta_j - \overline{\eta_j})(\eta_k - \overline{\eta_k}) \rangle \), where \( \langle \ldots \rangle \) denotes statistical average, and \( \overline{\eta} \equiv \langle \eta \rangle \). In the case of zero-mean variables, it reduces to

\[
C(\eta_j, \eta_k) = \langle \eta_j \eta_k^* \rangle \equiv C_{jk} \tag{4.6}
\]

(for the changes to be made when the mean is not zero, see sec. 3.4). In practice, a discretization or shot noise fluctuation could be added and the theoretical correlation would be changed to \( C_{jk} = \langle \eta_j \eta_k^* \rangle + \frac{1}{\langle N \rangle} \). In general, this is of little interest as it just amounts to a constant increase of the power spectrum. The values of \( \vec{\eta} \) can correspond to continuous random variable \( \eta = \eta(t) \) sampled in \( N \) time intervals (\( \vec{\eta} = \eta(\vec{t}) \)). For a wide sense stationary stochastic process we have that \( C_{jk} = C_{j-k} \) can only be a function of \( j-k \), i.e., the covariance matrix is a Toeplitz matrix.

A sequence of a Gaussian stochastic process has a joint probability density given by

\[
f(\eta) \propto e^{-\frac{1}{2} \vec{\eta}^T C^{-1} \vec{\eta}}.
\]

In the absence of correlations \( C \) is just the \( C_0 \) of \((4.3)\) and therefore \( C^{-1} = \text{diag}(1/\sigma_1^2, \ldots, 1/\sigma_N^2) \), but now we expect the presence of nonvanishing off-diagonal coefficients. We may assume that all the \( \vec{\eta} \) components are real. Each dimension will be discretized in the same way as for the one-dimensional case. Therefore, we will consider the joint probabilities

\[
p_{j_1, \ldots, j_N} = p[\eta_1 \in I_{j_1}, \ldots, \eta_N \in I_{j_N}] = \frac{1}{Z} e^{-\frac{(\Delta \eta)^2}{2} (j_1, \ldots, j_N)^T C^{-1} (j_1, \ldots, j_N)}, \tag{4.7}
\]

where the normalizing quantity \( Z \) is given by

\[
Z = \sum_{n_1, \ldots, n_N} e^{-\frac{(\Delta \eta)^2}{2} (n_1, \ldots, n_N)^T C^{-1} (n_1, \ldots, n_N)}. \tag{4.8}
\]

The ensuing Shannon entropy (see subsec. \( A.2 \) in the appendix) is

\[
H \equiv H_N = \log_2 \left[ \sqrt{\det \left( \frac{2\pi e}{(\Delta \eta)^2} C \right)} \right] + O \left( N \frac{2\pi \sigma^2}{(\Delta \eta)^2} e^{-\frac{2\pi^2 \sigma^2}{(\Delta \eta)^2}} \right), \tag{4.9}
\]

where this \( \sigma \) is given in section \( A.2 \). Note that the next-to-leading terms are, again, exponentially small, and their typical size can be adequately expressed as a function of a dimensionless parameter \( \frac{\Delta \eta}{\sigma} \equiv \lambda \). Before going on, some comments are in order. The previous relation can be rewritten in the form

\[
H = \log_2 \left[ \sqrt{\det(2\pi e C)} \right] - N \log_2(\Delta \eta) + \text{exponentially small part} \tag{4.10}
\]

The first term on the r.h.s. is just the result of having calculated \( H \) after replacing the multiple sum in \((4.8)\) with a multiple integral. Therefore, we shall call it \( H_{\text{cont}} \). Further, in the continuum limit, \( \lambda \to 0 \) and the
exponential corrections should vanish. This leads to

$$H = H_{\text{cont}} - N \log_2(\Delta \eta),$$

(4.11)

When an entropy associated to a discretization of width $\Delta \eta$ is compared with its continuous version, we realize that we gain $N$ times the ‘information’ leaked by mistaking a single element of unit length for an interval of size $\Delta \eta$, which is $N \left[ - \log_2(\Delta \eta) + \log_2(1) \right] = -N \log_2(\Delta \eta)$. In terms of entropy per component, (4.11) becomes $h = h_{\text{cont}} - \log_2(\Delta \eta)$, which generalizes (3.3), as now $h_{\text{cont}}$ has the same expression as in (3.5) but changing $\sigma$ by $\sigma_e$. Furthermore, there is a critical $\Delta \eta$-value for which the whole $h$ vanishes. When this happens, the discretization is so coarse that the little resolution kept is not enough to store any effective information at all.

Another convenient way of writing the entropy per component is

$$h = \log_2 \left[ \sqrt{2\pi e \sigma_e \Delta \eta} \right] + O \left( \frac{2\pi \sigma_e^2}{(\Delta \eta)^2} e^{-\frac{2\pi\sigma_e^2}{(\Delta \eta)^2}}, \ldots \right)$$

(4.12)

were we have now that the effective variance is:

$$\sigma_e^2 \equiv \text{Det}^{1/N}(C),$$

(4.13)

These expressions generalize to correlated variables the result in eq. (4.5) for $h_0$ by just replacing $C_0$ with $C$ and $\sigma_0$ with $\sigma_e$. Thus, for a general covariance matrix $C$ we only need to find $\sigma_e$ above to obtain the corresponding entropy.

### 4.2.1 Calculation of Det(C)

The next task is the calculation of the determinant of $C$. Going to Fourier space —see sec. B, subsec. B.2— one obtains the relation

$$\text{Det}(C) = \left( \frac{\Delta \omega}{2\pi \Delta t} \right)^N \text{Det} \left( \hat{C} \right),$$

(4.14)

where $\hat{C}$ is the Fourier-space representation of $C$, $\Delta t$ is the Fourier time sampling interval, $\Delta \omega$ the associated frequency interval and, taking into account that $N$ samples are considered, $\Delta \omega = \frac{2\pi}{N\Delta t}$. In order to find concrete results, some sort of hypothesis on $\hat{C}$ has to be made. Here we consider stationary (or homogeneous) processes, for which the the covariance matrix is a Toeplitz matrix, and therefore $\hat{C}$ is diagonal —see subsec. B.3— so that $\langle \hat{\eta}(\omega) \hat{\eta}^*(\omega') \rangle = P(\omega) \delta_{\text{Dirac}}(\omega - \omega')$, whose discrete version yields:

$$\hat{C}_{jk} = P(\omega_j) \frac{\delta_{jk}}{\Delta \omega},$$

(4.15)

i.e., $\hat{C}$ is a diagonal matrix. In all these cases, the problem boils down to the properties of the $P(\omega)$ function. If we denote by $P$ the diagonal matrix:

$$P \equiv \text{diag}(P(\omega_{-N/2}), \ldots, P(\omega_{N/2})).$$

(4.16)

we can write the effective rms correlation $\sigma_e$ that appears in (4.12) by:

$$\sigma_e^2 \equiv \text{Det}^{1/N}(C) = \left( \frac{1}{2\pi \Delta t} \text{Det}^{1/N}(P) \right)^{1/N} \left[ \prod_j P(\omega_j) \right].$$

(4.17)
The white noise case corresponds to the constant power spectra \( P(w) = A \) and the matrix \( P \) is proportional to the identity. In this case,

\[
\sigma_e^2 = \sigma_0^2 = \frac{A}{2\pi \Delta t}, \tag{4.18}
\]

showing that the larger the sampling interval \( \Delta t \) the smaller the variance, as expected.

We can also express the entropy as a difference from the entropy \( h_0 \) of a white noise spectrum of amplitude \( P = A \) by:

\[
h = h_0 + \frac{1}{N} \log_2 \left[ \sqrt{\det \left( \frac{1}{A} P \right)} \right]. \tag{4.19}
\]

In general, given two power spectra \( P_1 \) and \( P_2 \) with effective correlations \( \sigma_{e1} \) and \( \sigma_{e2} \), the entropy differences are given by:

\[
h_2 - h_1 = \log_2 \left[ \frac{\sigma_{e2}}{\sigma_{e1}} \right] = \log_2 \left[ \frac{\det^{1/2N}(P_2)}{\det^{1/2N}(P_1)} \right] = \frac{1}{N} \log_2 \left[ \sqrt{\det \left( P_2 P_1^{-1} \right)} \right]. \tag{4.20}
\]

Entropy comparison for equal-\( \sigma_{1p} \) processes. From the expression above (4.17) it is clear that \( \sigma_e^2 \) is linearly proportional to the amplitude of the power spectrum \( P(w) \), so that \( h \) will depend (logarithmically) on the normalization of \( P(w) \). It is interesting to compare the entropy for different shapes of \( P(w) \) which have been normalized in the same way. Here we will consider the case where we normalize \( P(w) \) so that \( \vec{\eta} \) has the same 1-point variance. We will see that this is equivalent to fix the traces of the \( P \) matrix (4.16).

First, using eq.(B.13) and the properties of the trace, we get

\[
\text{Det}(C) = \left[ \frac{\text{Tr}(C)}{\text{Tr}(\hat{C})} \right]^N \text{Det}(\hat{C}). \tag{4.21}
\]

Now, bearing in mind the usual definition of the 1-point variance: \( \sigma^2 \), which reads \( \sigma_{1p}^2 \equiv C(\eta(t), \eta(t)) \), let’s introduce

\[
\sigma_{1p}^2 (C) \equiv \frac{1}{N} \text{Tr}(C) = \frac{1}{2\pi \Delta t} \frac{1}{N} \text{Tr}(P). \tag{4.22}
\]

For the case of uncorrelated variables (white noise) with equal sigma: \( \sigma_1 = \ldots = \sigma_N \equiv \sigma_0 \), we have that \( \sigma_{1p} = \sigma_e = \sigma_0 \) in eq.(4.17). In general \( \sigma_{1p} \neq \sigma_e \) when there are correlations.

Using this definition, we have from (4.21):

\[
\sigma_e^2 \equiv \det^{1/N}(C) = \sigma_{1p}^2 \frac{\det^{1/N}(P)}{\text{Tr}(P)/N} \tag{4.23}
\]

Inserting this result into (4.12), we can write:

\[
h = h_{1p} + \frac{1}{2N} \log_2 \left[ \frac{\det(P)}{[\text{Tr}(P)/N]^N} \right], \tag{4.24}
\]

where

\[
h_{1p} = h_0(\sigma_{1p}) = \log_2 \left[ \sqrt{2\pi e} \frac{\sigma_{1p}^2}{\Delta \eta} \right] + \text{exponentially small part}. \tag{4.25}
\]
These new formulae are adequate for comparing processes with the same value of $\sigma_1^2$ and different $P$’s (i.e., different power spectra). The 1-point entropy $h_1p$ denotes the entropy per component of a white noise with a variance $\sigma_1^2 = \ldots = \sigma_N^2 = \sigma_1^2$, as in this case $P \propto I$, causing the second term on the r.h.s. of (4.24) to vanish.

For any square and positive semidefinite matrix $M$, the inequality $\frac{1}{N} \text{Tr}(M) \geq \text{Det}^{1/N}(M)$ holds. Both $C$ and $P$ satisfy these conditions. Therefore $\sigma_\varepsilon^2 \leq \sigma_1^2$ and $h \leq h_1p$. The equality is achieved when $P \propto I$, i.e., only for the white noise itself. In any other case, a Gaussian process with the same $\sigma_1p$ has smaller effective variance and lower entropy than the corresponding white noise. This is easy to understand from (2.4) or (2.5).

**Asymptotic expressions.** When the exact form of $\text{Det}(P)$ is not easy to obtain, we can resort to the following procedure. We may assume that $P(-\omega) = P(\omega)$ and that the mode with $\omega_0 = 0$ has to be removed, as often happens (this mode is related to the correlation at $t \to \infty$ and, if one requires that the system be ergodic, it should vanish). Then,

$$\log_2[\text{Det}(P)] = \sum_{j=-N/2}^{N/2} \log_2[P(\omega_j)] = 2 \sum_{j=1}^{N/2} \log_2[P(\omega_j)],$$

and an application of the Euler-Maclaurin summation formula (see e.g. [17]), leads us to the approximation

$$\sum_{j=1}^{N/2} \log_2[P(\omega_j)] = \frac{1}{\Delta \omega} \int_{\omega_{\min}}^{\omega_{\max}} d\omega \log_2[P(\omega)] + \frac{1}{2} \left( \log_2[P(\omega_{\max})] + \log_2[P(\omega_{\min})] \right) + \text{higher order terms in } \Delta \omega.$$  

(4.26)

The same method can be applied to the calculation of $\text{Tr}(P)$, in (4.24), i.e.,

$$2 \sum_{j=1}^{N/2} P(\omega_j) = 2 \left[ \frac{1}{\Delta \omega} \int_{\omega_{\min}}^{\omega_{\max}} d\omega P(\omega) + \frac{1}{2} (P(\omega_{\max}) + P(\omega_{\min})) + \text{higher order terms in } \Delta \omega \right].$$

(4.27)

**Filters.** Quite often, stochastic processes go through what is called a filter. Formally, filters can be pictured as multiplicative changes in the power spectrum. Therefore, everything happens as if we had a new power spectrum function, say $P'$, coming from the replacement

$$P(\omega) \rightarrow P'(\omega) = P(\omega)\phi(\omega),$$

where the $\phi$ function is the frequency response of the filter itself. Let $h'$ denote the new entropy per component. It is immediate that the change caused by the introduction of $\phi$ will be given by

$$h' = h + h_\phi,$$

$$h_\phi = \frac{1}{N} \log_2[\sqrt{\text{Det}(\Phi)}], \quad \Phi = \text{diag}(\phi(\omega_{-N/2}), \ldots, \phi(\omega_{N/2})), $$

(4.29)

where $h$ denotes the entropy per component for the same process when no filter is present.

**4.2.2 Simple power-law power spectrum**

Here, we will consider a power spectrum of the type

$$P(\omega) = A \left( \frac{\omega}{\omega_0} \right)^{n_p},$$

(4.30)
where $A$ is a constant that sets the overall amplitude and $w_0$ some characteristic scale that sets the time units. Taking into account the discrete $\omega$-values (B.11) we evaluate

$$\det \left( \frac{1}{A^P} \right) = \prod_j P'(\omega_j) = \left( \frac{\Delta \omega}{\omega_0} \right)^N n_p \left[ \left( \frac{N}{2} \right)! \right]^{2n_p}$$

(4.31)

(where the zero mode $j = 0$ has been omitted). Making use of Stirling’s approximation for large $N/2$, and using the frequency relations (3.11), we find:

$$\sigma_e^2 \equiv \det 1/N(C) \simeq \frac{A}{2\pi \Delta t} \left( \frac{\pi}{e w_0 \Delta t} \right)^n \sigma_0^2 \left( \frac{\omega_{\text{Max}}}{e w_0} \right)^n,$$

where $\sigma_0$ corresponds to the white noise case ($n_p = 0$). If we normalize the spectrum at $w_0 = \omega_{\text{Max}}$ then for $n_p < 0$ we have that $h > h_0$ and the optimal compression rate has to decrease, while for $n_p > 0$ we have $h < h_0$. Some special values are given in table 2, and are also illustrated by Fig. 1. However, this comparison depends on the normalization and involves noises with different values of $\sigma_{1p}$, as we have only changed the value of $n_p$ without doing anything to maintain the initial $\sigma_{1p}$. In this case, by eq. (4.22),

$$\sigma_{1p}^2 = \frac{1}{2\pi \Delta t} \frac{1}{N} \sum_j P'(\omega_j) = \frac{A}{\pi N \Delta t} \left( \frac{\Delta \omega}{w_0} \right)^n S_n (\frac{N}{2}), \quad S_n (\frac{N}{2}) \equiv \sum_{j=1}^{N/2} j^{n_p}. \quad (4.33)$$

Making use of (4.24), we are led to

$$h = h_1 + \frac{1}{N} \log_2 \left[ \left( \frac{N}{2} \right)! \right] - \frac{1}{2} \log_2 \left[ \sum_{j=1}^{N/2} j^{n_p} \right]$$

$$= h_1 + \frac{1}{2} \log_2 \left( \frac{N}{2} \right) - \frac{1}{2} \log_2 (e) - \frac{1}{2} \log_2 \left[ \sum_{j=1}^{N/2} j^{n_p} \right] + O \left( \frac{\log_2(N)}{N} \right), \quad (4.34)$$

where the Stirling approximation has been applied. When $n_p > -1$, we apply the Euler-Maclaurin summation formula (4.28) and obtain

$$\sigma_{1p}^2 = \frac{1}{2\pi \Delta t} \frac{1}{n_p + 1} \left( \frac{\pi}{w_0 \Delta t} \right)^n \left[ 1 + O \left( \frac{1}{N} \right) \right], \quad \text{for } n_p > -1. \quad (4.35)$$

For the $n_p = -1$ case may be more straightforwardly estimated by using

$$S_{-1} \left( \frac{N}{2} \right) = \Psi \left( \frac{N}{2} + 1 \right) + \gamma = \ln \left( \frac{N}{2} \right) + \gamma + O \left( \frac{1}{N} \right), \quad (4.36)$$

where $\gamma$ is Euler’s constant: $\gamma \simeq 0.57721 \ldots$. So, $\sigma_{1p}^2$ becomes

$$\sigma_{1p}^2 = \frac{(Aw_0)}{2\pi \Delta t} \left[ \ln \left( \frac{N}{2} \right) + \gamma + O \left( \frac{1}{N} \right) \right] \quad \text{for } n_p = -1. \quad (4.37)$$

Then, by the previous formulas and by (4.24),

$$h = \begin{cases} 
  h_1 - \frac{n_p}{2} \log_2(e) + \frac{1}{2} \log_2(n_p + 1) + O \left( \frac{\log_2(N)}{N} \right), & \text{for } n_p > -1, \\
  h_1 + \frac{1}{2} \log_2(e) - \frac{1}{2} \log_2 \left[ \ln \left( \frac{N}{2} \right) + \gamma \right] + O \left( \frac{\log_2(N)}{N} \right), & \text{for } n_p = -1, 
\end{cases} \quad (4.38)$$
where $h_{1p}$, given by (4.23), is the entropy per component of a white noise with the $\sigma_0 = \sigma_{1p}$. Note that, although it seems that $h$ diverges with $N$ for $n_p = -1$, this is an artifact of this type of comparison with a fixed $\sigma_{1p}$. Although $\sigma_{1p}^2$ diverges logarithmically with $N$, the information content does not, as $\sigma_e^2$ in eq.(4.32) is finite:

$$\sigma_e^2 = \frac{(Aw_0)}{2\pi^2} e. \quad \text{for } n_p = -1 \quad (4.39)$$

Some examples are illustrated by the 5th column of Table 2 and Fig. 3.

Figure 3: Entropy and optimal compression rate for different power spectra with the same $\sigma_{1p}$, but (unlike in Fig 2) keeping $l_i = 16$ bits fixed and $\omega_0 = \omega_{\text{min}}$. The present set of cases is: $n_p = 0$ (solid line), $n_p = -1$ (dashed line) and $n_p = +1$ (dotted line).

Fig. 4 shows the entropy $h$ as a function of the spectral index $n_p$ given by the above formulas. As can be seen, $h$ has a maximum at $n_p = 0$, as expected.

### 4.2.3 ‘$f^0 + 1/f$’ spectrum

In practice, realistic power spectra include often combinations of several powers. This new example corresponds to a power spectrum including two terms: one with $n_p = 0$ (white noise) and another with $n_p = -1$ (usually called $1/f$ noise), which we write as

$$P(\omega) \equiv A \left(1 + \frac{\omega_k}{|\omega|}\right) = A \left(1 + \frac{f_k}{|f|}\right), \quad (4.40)$$

where $f$ stands for frequency $w \equiv 2\pi f$, and $f_k$ for the so called knee frequency, where both contributions are equal. We shall assume that $w$ has been discretized as in the previous cases. Because a direct evaluation of $\text{Det}(P)$ would not be so easy now, we shall apply the above commented approximation based on the Euler–Maclaurin summation formula. After performing the integration (4.27) for the $P(\omega)$ of eq. (4.40) one gets

$$\sigma_e^2 = \frac{A}{2\pi \Delta t} \left(1 + \frac{\omega_k}{\omega_{\text{Max}}}\right) \left[\frac{\omega_{\text{Max}} + \omega_k}{\omega_{\text{min}} + \omega_k}\right]^{\omega_k/\omega_{\text{Max}}} \quad (4.41)$$
The corresponding entropy is just given by eq. (4.12). When $\omega_k << \omega_{\text{Max}}$ we recover the white noise case eq. (4.18), while in the case $\omega_k >> \omega_{\text{Max}}$ the $1/f$ noise dominates and we recover eq. (4.39), as expected. We observe that a combined power spectrum (4.40) with reasonably small $A$ is effectively equivalent to one of the type $P(\omega) = A \left( \frac{|\omega|}{\omega_0} \right)^{n_p}$ with an intermediate $n_p$ between 0 and $-1$. An illustration of the values of $h$ and optimal compression for this case is shown in Fig. 4 and also in Fig. 3 as dashed line.

Typically we will have that $\omega_{\text{min}} << \omega_{\text{Max}}$ and also $\omega_{\text{min}} << \omega_k$. In this case the only relevant parameter is $r \equiv \omega_k/\omega_{\text{Max}}$:

$$\sigma^2_e = \frac{A}{2\pi \Delta t} \left( 1 + r \right)^{1+r} \frac{1}{r^r} = \sigma_0^2 \left( 1 + r \right)^{1+r} \frac{1}{r^r},$$

(4.42)

where $r = 0$ reproduces the white noise case and large $r$ reproduces the $1/f$ case ($n_p = -1$) with arbitrarily large normalization. For $r = 1$ we have that the effective variance of the signal is four times as large as the white noise part $\sigma^2_e = 4\sigma_0^2$, so that the entropy will be one unit larger with the combined spectrum than with the white noise alone. Other values for $h$ and $c_r$ as a function of $r$ are shown in Fig. 4. In this case $\lambda = \Delta \eta/\sigma_0 = 1$ so that $h_0 \approx 2.047$ and $c_{r, \text{opt}} \approx 3.91$ ($\bar{l}_t = 8$ bits) which agrees with the values at $r = 0$.

Another way to compare the two cases is to use an equal $\sigma_{1p}$ comparison with a white noise. In this case:

$$\sigma_{1p}^2 = \frac{1}{N} \text{Tr}(C) = \frac{A}{2\pi \Delta t} \left[ 1 + \frac{\omega_k}{\omega_{\text{Max}}} S_{-1} \left( \frac{N}{2} \right) \right],$$

(4.43)

and, using (4.24),

$$h = h_{1p} - \log_2 \left[ \sqrt{\frac{\omega_{\text{Max}} + \omega_k}{\omega_{\text{Max}} + \bar{l}_t}} \right] + \frac{\omega_k}{\omega_{\text{Max}} \omega_{\text{Min}}} \log_2 \left[ \sqrt{\frac{\omega_{\text{Max}} + \omega_k}{\omega_{\text{Min}} + \omega_k}} \right] + O \left( \frac{\log_2(N)}{N} \right),$$

(4.44)
Figure 5: Entropy $h$ and optimal compression $c_r$, opt ($\bar{t}_i = 8$ bits) as a function of $r = \omega_k/\omega_{\text{Max}}$ for a ‘$f^0 + 1/f$’ noise. We have chosen $\lambda = \Delta \eta/\sigma_0 = 1$ and symbols of $\bar{t}_i = 8$ bits.

where $h_{1p}$ stands for the entropy per component of a Gaussian white noise with the same 1-point variance $\sigma_{1p}^2$. An example of this type of noise is shown in the 4th column of Table 2 and Fig.4. Of course, the $\omega_{\text{Max}}/\omega_k \rightarrow 0$ limit of this expression yields the $n_p = -1$ case of (4.38) (see also the 5th column of Table 2).

| $\lambda = \Delta \eta/\sigma_0$ | $h_0 = h_{1p}$ | $h$ |
|---|---|---|
| $n_p = 0$ | $f^0 + 1/f$ | $n_p = -1$ |
| $\sigma_0$ | $\sigma_0$ | $\sigma_{1p} = \sigma_0$ | $\sigma_{1p} = \sigma_0$ |
| 0.05 | 6.37 | 7.37 | 5.89 | 5.71 |
| 0.25 | 4.05 | 5.05 | 3.57 | 3.39 |
| 0.50 | 3.05 | 4.05 | 2.57 | 2.39 |
| 1.00 | 2.05 | 3.05 | 1.57 | 1.39 |

Table 2. Shannon entropy per component $h$ for large $N$, and several values of $\lambda = \Delta \eta/\sigma_0$. The purely white-noise case $h_0$ for a given $\sigma_0$ and $\lambda$ are listed in column 2. Columns 3 and 4 gives the results for a combination $P(\omega) = A(1 + \omega_k/|\omega|)$, with $\omega_k = \omega_{\text{Max}}$ ($r = 1$) when the white noise part is fixed to the same $\sigma_0$ (column 3) and when the 1-point sigma is fixed to $\sigma_{1p} = \sigma_0$ (column 4). In column 5 we have listed the values for a correlation of the $n_p = -1$ type $P(\omega) = A(w_0/|\omega|)$ and $\sigma_{1p} = \sigma_0$. In the last two cases $N = 1000$.

We can see there how $h < h_{1p}$ when we compare spectra normalized to have the same $\sigma_{1p}^2$, while $h > h_{1p}$ when we just add a term $(1/f)$ to the (constant) white noise power spectrum. The interpretation is simple, as shown in eq. (4.12) the entropy is given by the effective correlation. On the one hand, adding power always increases $\sigma_e$ (see eq. (4.17)), and therefore $h$. But, on the other hand, $\sigma_e^2 \leq \sigma_{1p}^2$ so that, when $\sigma_{1p}$ is fixed, any power spectrum gives smaller $h$ than the white noise and, as we said above, this can be easily understood in
Figure 6: Comparison between purely white noise (solid line) and two processes of the type $P(f) \propto 1 + f_k/|f|$ with $f_k = 10$ Hz (dashed line) and 100 Hz (dotted line), for $\tilde{f}_t = 16$ bits, and without imposing the equal-$\sigma_{1p}$ constraint. In both cases $h > h_0$, while in the analogous example of Fig. 3 it happened just the opposite, in the light of inequality (2.3). This change of behaviour can be seen comparing Figs. 1 and 2 with Fig. 6.

4.2.4 Examples of piecewise-mixed spectra

1. Here we study the piecewise-defined spectrum:

$$P(\omega) = \begin{cases} 
A, & \text{for } \omega \leq \omega_L, \\
A \frac{\omega_L}{\omega}, & \text{for } \omega_L < \omega \leq \omega_H, \\
A \frac{\omega_L^2 \omega_H}{\omega^2}, & \text{for } \omega_H < \omega \leq \omega_{\text{Max}}.
\end{cases}$$

(4.45)

The result of applying (4.19) and making asymptotic approximations for large values of $\frac{\omega_L}{\Delta \omega}$, $\frac{\omega_H}{\Delta \omega}$, and $\frac{\omega_{\text{Max}}}{\Delta \omega}$ is

$$h = h_0 + \left(1 - \frac{\omega_L + \omega_H}{2\omega_{\text{Max}}}ight) \log_2(e) - \frac{1}{2} \log_2 \left(\frac{\omega_{\text{Max}}^2}{\omega_L \omega_H}\right) + \text{higher order terms.}$$

(4.46)

2. Another case which can be of interest is:

$$P(\omega) = \begin{cases} 
A', & \text{for } \omega \leq \omega_L, \\
A + \frac{B}{|\omega|}, & \text{for } \omega_L < \omega \leq \omega_{\text{Max}}.
\end{cases}$$

(4.47)

Taking now as reference the case in which $B = 0$ and $A' = A$, we may write

$$h = h(A' = A, B = 0) + \log_2 \left[\sqrt{1 + \frac{B}{A\omega_{\text{Max}}}} - \frac{\omega_L}{\omega_{\text{Max}}} \log_2 \left(\sqrt{1 + \frac{B}{A\omega_L}}\right)\right]$$

$$+ \frac{B}{\omega_{\text{Max}} A} \log_2 \left[\sqrt{A\omega_{\text{Max}} + B} + \frac{\omega_L}{\omega_{\text{Max}} A} \log_2 \left(\sqrt{\frac{A'}{A}}\right)\right] + \text{higher order terms.}$$

(4.48)
5 Conclusions

We have studied the Shannon entropy $h$ of a Gaussian discrete noise $\eta_i$ characterized by its power spectrum $P$. It amounts to $h \simeq \log_2 \left( \sqrt{2\pi e} \sigma_c / \Delta\eta \right)$, where $\sigma_c = \sigma_c(P)$ is given by eq.(4.17) and $\Delta\eta$ is the discretization width. The finite-$N$ corrections to this formula are exponentially small (eqs.(A.6) and (A.14) in Appendix A).

The first thing to notice is that $\sigma_c$ changes linearly with the amplitude of $P$, so that the entropy increases logarithmically with $P$. For a given normalization, how does the entropy depend on the shape of the power spectrum? We can compare the entropy of two types of noise using the entropy difference $\Delta h = h - h_0$. In cases with power-law spectra $P(\omega) \propto \left( \frac{|\omega|}{\omega_0} \right)^{n_p}$, $\Delta h$ can be quite sensitive to the choice of $\omega_0$, whose variations may even cause a reversal of the sign of $\Delta h$. This type of change is due to the already commented logarithmic dependence of $h$ on the amplitude of $P$. If we fix the (1-point) variance of the noise, we have seen that the maximum entropy (minimum compression) is the one given by white noise (or constant $P$), as expected. For $P(\omega) \propto \omega^{n_p}$ spectra with fixed one-point variance, we have that the larger $|n_p|$ the smaller the entropy for $n_p > -1$ (eg eq. (4.38) and Fig.2). Notice that when $\Delta\eta > \sqrt{2\pi e} \sigma_c$ we have $h < 0$ indicating that the data have been discretized with such a low resolution that there is no information left.

We have defined the optimal compression rate as the ratio of the initial average length per code unit $\bar{t}_i$ over the Shannon entropy $h$ per component: $c_r, \text{ opt} \equiv \frac{\bar{t}_i}{h}$. For a linearly discretized data set with $\bar{t}_i = N_{\text{bits}} = \log_2(N_s)$ bits the optimal compression rate depends on the discretization width $\Delta\eta$ through a simple relation:

$$c_r, \text{ opt} \equiv \frac{\bar{t}_i}{h} \simeq \frac{N_{\text{bits}}}{\log_2 \left( \sqrt{2\pi e} \sigma_c / \Delta\eta \right)},$$

(5.1)

The choice of $\Delta\eta$ is in principle arbitrary and depends on what we want to do in the data processing of the signal (noise). The final compression factors will depend only on the ratio of these two quantities $\lambda \equiv \frac{\Delta\eta}{\sigma}$ and the number of bits $N_{\text{bits}}$ chosen to represent the data. Another way of writing this results is: $c_r, \text{ opt} \simeq \frac{\log_2(R) - \log_2(\Delta\eta)}{h_{\text{cont}} - \log_2(\Delta\eta)}$, where $R$ is the range of the random variable and $h_{\text{cont}}$ is a constant depending on the type of process, which may be interpreted as the Shannon entropy per component in the continuum limit. In mathematical terms, $h_{\text{cont}}$ involves the determinant of the correlation matrix. If the initial length $\bar{t}_i$ is held fixed, independently of $\Delta\eta$, the relation is just $c_r, \text{ opt}(\Delta\eta) \simeq \frac{\bar{t}_i}{h_{\text{cont}} - \log_2(\Delta\eta)}$.

The purely white noise case ($n_p = 0$) offers rather slight hopes, for moderate ranges $R$. If we choose $R = (-N_0 \sigma, N_0 \sigma)$ with $N_0 = 3$, and $\lambda = \Delta\eta / \sigma = 0.25$ the compression rate is of $c_r, \text{ opt} = 1.13$—only marginally above one—and, yet, this happens at the expense of losing resolution to the extent that only four distinct values are observed within each interval of width $\sigma$. Less resolution than that may be too little for many applications. One could wonder what happens, in the opposite case, when resolution is kept at any cost. For a binning of $2^8$ distinct intervals within the same range, $\lambda$ has to take on such a value that the compression rate is a meagre 1.07. Such a thinly spaced binning means that the white noise is seen very much like a uniformly distributed one, and has a similar uncompressibility.

On the other hand, for fixed $\sigma_{1p}^2$ a negative spectral index lowers the effective information and helps compression. Moreover, the optimal compression rate increases as the sampling time interval decreases. As we
see in Fig. 2, when $\Delta \eta = 0.25$ the compression rate for $n_p = -1$ with the same $\sigma_{1p}$ as for the white noise is $\sim 1.4$. Moreover, the difference between $n_p = -1$ and $n_p = 0$ increases as the discretization parameter $\lambda = \Delta \eta/\sigma$ grows. However, one cannot think of arbitrarily raising its value, as such a thing would imply a widening of the discretization error, and an even greater loss in resolution for the values of our variables.

A combination of both types has also been studied by taking a ‘mixed’ power spectrum with $n_p = 0$ plus $1/f$ (i.e. $n_p = -1$) terms. If the coefficient of the $n_p = 0$ part is low enough, the behaviour shown is intermediate between purely $n_p = 0$ and purely $n_p = -1$, and can be interpreted as if it just had an effective $n_p$ between both values. When $P(\omega) \propto \left(A_0 + \frac{\omega_0}{\omega}\right)$, if $A_0$ is set to 1, $h$ is not too sensitive to increases in $\omega_0$ much above the knee frequency. On the contrary, if $\omega_0$ is kept constant, variations in $A_0$ may easily change the sign of $\Delta h$. As a common feature to all possible situations, one observes an increase in compressibility as the measured data involve more and more correlation, i.e. larger dominance of their spectral $n^p$-parts with $n_p \neq 0$ (see Fig. 4).

Imagine a situation of a data set that consists of a slowly varying signal (to be stored in $\tilde{l}_i$ bits) plus large amplitude noise that dominates over the signal on large frequencies. The signal is to be recovered by averaging the noise after transmission (and therefore compression) and a careful calibration of instabilities in the noise. This is a common situation for scientific measurements on-board satellites collecting data with low signal-to-noise ratio. In this case the noise component can be kept with a low resolution and one can choose $\Delta \eta \simeq \sigma_e$ which gives $h \simeq 2.05$ indicating that all information is contained effectively in two bits. Then, high compression rates $c_{r, \text{opt}} \simeq \tilde{l}_i/2$ could be obtained: e.g. $c_{r, \text{opt}} \simeq 8$ for $\tilde{l}_i \simeq 16$ bits. To achieve such a high compression values in practice, an efficient coding method has to be used. For one dimension, the Huffman and arithmetic schemes are known to be reasonably close to the optimal value. When data (symbols) are correlated in a manifest way, as the general case considered here, other methods have to be used in combination. One of the simplest methods that take into account correlations is run-length encoding, where the signal is converted to a stream of integers that indicate how many consecutive symbols are equal (see [19]). This would be quite efficient in the situation we have just mentioned.

The data discretization or ‘quantization’ process causes a distortion error. This issue has not been considered in the present paper, as we have kept it outside of the scope of this study (i.e., we have started from a data set already quantized in a given way). Nevertheless, the results in ref. [20] (Chap. 13) for a univariate Gaussian source indicate that the ‘best expected’ average error for a representation of a given length $\tilde{l}_i$ decreases as $\tilde{l}_i$ increases. This confirms the intuitive idea that a random variable like $\eta$ is better described as $\tilde{l}_i$ grows. However, when this happens the entropy grows too, and the compression chances are reduced.

**A Appendix: discrete calculations**
A.1 One-dimensional case

First, we rewrite the $Z$ of (3.2) as

$$Z = \sum_{n=-\infty}^{\infty} e^{-\frac{n^2\lambda^2}{2}} = \theta \left( \frac{\lambda^2}{2\pi}; 0 \right),$$  \hspace{1cm} (A.1)

where

$$\lambda \equiv \frac{\Delta \eta}{\sigma}$$  \hspace{1cm} (A.2)

is the size of the discretization interval in units of $\sigma$, and

$$\theta(\beta; m) \equiv \sum_{n=-\infty}^{\infty} n^{2m} e^{-\pi \beta n^2}$$

is a notation for the sort of Jacobi elliptic theta functions appearing in this calculation.

Note that the discretization has enabled us to deal with a discrete probability set —(3.1)— thus avoiding the well-known difficulties associated with $H$ for continuous probability distributions. In our own case (calling $H \equiv H_1$ all through this subsection),

$$H = -\frac{1}{\ln(2)} \left\{ -\frac{\lambda^2}{2} \theta \left( \frac{\lambda^2}{2\pi}; 1 \right) - \ln \left[ \theta \left( \frac{\lambda^2}{2\pi}; 0 \right) \right] \right\}. $$

For $m = 1$, we just observe that

$$\theta(\beta; 1) = -\frac{1}{\pi} \frac{d}{d\beta} \theta(\beta; 0).$$

Using this, we arrive at

$$H = -\frac{1}{\ln(2)} \left\{ \beta \frac{d}{d\beta} \ln[\theta(\beta; 0)] - \ln[\theta(\beta; 0)] \right\}, \hspace{1cm} \text{with} \hspace{0.5cm} \beta = \frac{\lambda^2}{2\pi} = \frac{1}{T}. $$ \hspace{1cm} (A.3)

By (A.1), this can also be written as

$$H = \frac{1}{\ln(2)} \frac{d}{dT} [T \ln(Z)]. $$ \hspace{1cm} (A.4)

Up to the trivial change of units —or, equivalently, a conventional modification of the Boltzmann constant— $H$ is the thermodynamical entropy $S$ of a one-particle system at temperature $T$ with partition function $Z$. In the situation we are studying, this $Z$ is $Z(T) = \theta \left( \frac{1}{T}; 0 \right)$ as given by eq. (3.2). However, the validity of eq. (A.4) is quite general: in fact, for any system with probabilities of the form

$$p_J = \frac{e^{-E_J/T}}{Z}, \hspace{1cm} \text{where} \hspace{0.5cm} Z = \sum_I e^{-E_I/T}$$

(where $I, J$ can be single indices or multiple indices), one may check that, after applying the definition (2.1) or (2.3), eq. (A.4) holds. Therefore, we might as well have started our calculation of $H$ from eq. (A.4) itself (and we will do so for the $N$-dimensional case). Analogously, $-T \ln(Z)$ plays the role of the Helmholtz free energy $F$, satisfying the relation $S = -\frac{dF}{dT}$. 

20
A ‘finely’ or thinly spaced discretization means that \( \lambda \) should be small. However, the above expression of \( \theta(\beta; m) \) as a series is obviously inadequate when \( \beta = \frac{\lambda^2}{2\pi} \ll 1 \). Such a difficulty will be overcome by recalling the remarkable theta function identity (see e.g. ref. [16])

\[
\theta(\beta; 0) = \frac{1}{\sqrt{\beta}} \theta \left( \frac{1}{\beta}; 0 \right).
\] (A.5)

Applying now this identity to (A.3) or (A.4), expanding each part for small \( \lambda \) and differentiating, one finds

\[
H = \frac{1}{\ln(2)} \left[ \frac{1}{2} + \ln \left( \frac{\sqrt{2\pi}}{\lambda} \right) + 2 \left( 1 - \frac{2\pi^2}{\lambda^2} \right) e^{-\pi^2/\lambda^2} - 2e^{-4\pi^2/\lambda^2} + \frac{8}{3} e^{-6\pi^2/\lambda^2} + \mathcal{O} \left( \frac{2\pi}{\lambda^2} e^{-8\pi^2/\lambda^2} \right) \right]
\] (A.6)

which, regarded as an expansion, is quickly convergent for \( 0 < \lambda \ll \sqrt{2\pi} \). (One should notice that, actually, the two expressions have a generous overlap around \( \beta \simeq 1 \) where both converge and any of them can be consistently used).

There is no explicit dependence on \( \sigma \), as the only relevant variable is the relative discretization size \( \lambda \). Even for moderately large values of \( \lambda \), the next-to-leading part of \( H \) is very small: e.g. for \( \lambda = 1 \) we have \( e^{-\pi \beta} = 2\pi e^{-4\pi^2/\lambda^2} \simeq 1.7 \cdot 10^{-8} \), \( e^{-2\pi \beta} = 2\pi^2 e^{-4\pi^2/\lambda^2} \simeq 4.5 \cdot 10^{-17} \); at \( \lambda = 1/2 \) these two quantities become \( 1.3 \cdot 10^{-33} \) and \( 6.6 \cdot 10^{-68} \), respectively. Neglecting such terms, we easily obtain a good approximate formula, which may be reexpressed as

\[
H = \log_2 \left( \sqrt{\frac{2\pi e}{\lambda^2}} \right) + \mathcal{O} \left( \frac{2\pi}{\lambda^2} e^{-2\pi^2/\lambda^2} \ldots \right),
\] (A.7)

with \( \lambda \) given by (A.2). This yields eq. (3.3)

### A.2 N-dimensional case

After looking at the \( Z \) of eq. (4.8), let’s introduce, for convenience, the new notations

\[
\sigma \equiv \min(\{\sigma_1, \ldots, \sigma_N\}), \quad \chi^{-1} \equiv \sigma^2 C^{-1}, \quad \lambda \equiv \frac{\Delta \eta}{\sigma},
\] (A.8)

which enable us to write

\[
Z = \sum_{n_1, \ldots, n_N = -\infty}^{\infty} e^{-\frac{\lambda^2}{2} (n_1, \ldots, n_N)^T \chi^{-1} (n_1, \ldots, n_N)}.
\] (A.9)

In terms of the multidimensional Jacobi theta function

\[
\theta_N(\beta|M) \equiv \sum_{n_1, \ldots, n_N = -\infty}^{\infty} e^{-\pi \beta \sum_{i,j=1}^{N} M_{ij} n_i n_j},
\] (A.10)

we can put

\[
Z = \theta_N \left( \beta|\chi^{-1} \right), \quad \beta \equiv \frac{\chi^2}{2\pi} = \frac{1}{T}.
\] (A.11)

By (A.4), the joint Shannon entropy (now \( H \equiv H_N \)) becomes

\[
H = \frac{d}{dT} \left[ T \log_2(Z) \right] = \log_2 \left[ \theta_N (\beta|\chi^{-1}) \right] - \beta \frac{d}{d\beta} \log_2 \left[ \theta_N (\beta|\chi^{-1}) \right].
\] (A.12)
We are interested in approximations for small $\beta$, but the present expressions are inadequate for this situation. The way out is to take advantage of a Jacobi identity for multidimensional theta functions, namely,

$$ \theta_N (\beta | M) = \frac{1}{\sqrt{|\text{Det}(M)|^{1/2} \beta^{N/2}}} \theta_N \left( \frac{1}{\beta} M^* \right), \quad (A.13) $$

which, unlike the initial expression, may be expanded for small $\beta$. Doing so (and noting that $C$ has to be real when viewed in configuration space),

$$ H = \frac{N}{\ln(2)} \left[ \frac{1}{2} + \ln \left( \frac{1}{\sqrt{\beta}} \right) + \frac{1}{2N} \ln \text{Det}(\chi) + \mathcal{O} \left( \frac{1}{\beta} e^{-\frac{2}{\beta} \min(C/\sigma^2)} \right) \right] $$

$$ = \log_2 \left[ \sqrt{\text{Det} \left( \frac{2\pi e}{(\Delta\eta)^2} C \right)} \right] + \mathcal{O} \left( N \frac{2\pi \sigma^2}{(\Delta\eta)^2} e^{-\frac{2}{\beta} \min(C)} \right), \quad (A.14) $$

where $\min(C)$ means the minimum over the (positive) eigenvalues of the correlation matrix $C$, and where the relations (A.8) and the definition of $\beta$ in (A.11) have been used. More terms of this expansion can be obtained explicitly by using eq. (A.6). Notice, however, that each order of (A.6) gives rise here, in principle, to $N$ different orders (the first $N$ of them corresponding to the sequence of eigenvalues of $C$, increasing in magnitude). The bottom line gives us (4.9).

**B Appendix: useful Fourier-space results**

**B.1 Discrete Fourier transforms**

The continuous transforms taken as reference are

$$ \hat{\eta}(k) = \int dx \ e^{-ikx} \eta(x), \quad \{ \} \quad \text{(B.1)} $$

$$ \eta(x) = \int dk \ 2\pi e^{ikx} \hat{\eta}(k), \quad \{ \} $$

where $k$ and $x$ are a pair of conjugate variables. Discretizing them,

$$ k_n = n\Delta k, \quad \{ \} \quad \text{(B.2)} $$

$$ x_n = n\Delta x, \quad \{ \} $$

and calling

$$ \hat{\eta}_n \equiv \hat{\eta}(k_n), \quad \{ \} \quad \text{(B.3)} $$

$$ \eta_n \equiv \eta(x_n), \quad \{ \} $$

we construct discrete transforms which, in the continuum limit, reproduce (B.1):

$$ \hat{\eta}_n = \Delta x \sum_m e^{-ik_n x_m} \eta_m, \quad \{ \} $$

$$ \eta_n = \frac{\Delta k}{2\pi} \sum_m e^{ik_n x_m} \hat{\eta}_m. \quad (B.4) $$

Taking into account (B.2) and the correct relation between sampling intervals, i.e. $\Delta k = \frac{2\pi}{N\Delta x}$, one realizes that

$$ k_n x_m = k_m x_n = \frac{2\pi}{N} mn. \quad (B.5) $$
Therefore, we can write

\[
\begin{align*}
\hat{\eta}_n &= \Delta x (W \hat{\eta})_n, \\
\eta_n &= \frac{\Delta k}{2\pi} (W^* \hat{\eta})_n.
\end{align*}
\] (B.6)

where \( W \) is the symmetric matrix with coefficients \( W_{mn} = e^{i \frac{2\pi}{N} mn} \). After renaming

\[
x \rightarrow t, \\
k \rightarrow \omega,
\]

this yields the expressions (B.8).

### B.2 Fourier-space relation involving \( \text{Det}(C) \)

For convenience, we prefer to handle the Fourier-space representation of \( C \)—which we shall denote by \( \hat{C} \)—rather than \( C \) itself (we will see that \( \hat{C} \) is simpler). A vector \( \vec{\eta} \) and its discrete Fourier transform \( \hat{\eta} \) are related by expressions of the type

\[
\begin{align*}
\hat{\eta} &= \Delta t W \vec{\eta}, \\
\vec{\eta} &= \frac{\Delta \omega}{2\pi} W^* \hat{\eta},
\end{align*}
\] (B.8)

where \( W \) indicates a matrix whose coefficients are given by \( W_{mn} = e^{i \frac{2\pi}{N} mn} \) (see subsec. [B.1]). \( \Delta t \) is a \( t \)-interval which now has to be interpreted as the time lapse between two successive Fourier ‘samplings’. If we imagine that \( \eta_j = \eta(t_j) \), then \( t_j - t_{j-1} = \Delta t, \forall j \). \( \Delta \omega \) is the corresponding interval in ‘angular frequency’ or conjugate space. Taking into account the usual relation between the sampling interval and the associated angular frequency (or conjugate momentum) range that can be correctly sampled in conjugate space, one has the following relation between \( \Delta t, \Delta \omega \) and \( N \):

\[
\Delta \omega = 2\pi \frac{1}{N\Delta t},
\] (B.9)

The discrete values of \( \omega \) are

\[
\omega_j = j \Delta \omega, \ j = -N/2, \ldots, N/2.
\] (B.10)

Let \( \omega_{\text{min}} \) and \( \omega_{\text{Max}} \) denote the minimum and maximum nonzero absolute values of \( \omega \). Then,

\[
\begin{align*}
\omega_{\text{min}} &\equiv \omega_1 = \frac{\Delta \omega}{N} = \frac{2\pi}{N\Delta t} = 2\pi f_{\text{min}}, \\
\omega_{\text{Max}} &\equiv \omega_{N/2} = \frac{N}{2} \Delta \omega = \frac{\pi}{\Delta t} = 2\pi f_{\text{Max}}.
\end{align*}
\] (B.11)

We have here introduced frequencies — \( f \)’s — in the way \( \omega = 2\pi f \), as usual.

Furthermore, by the form of its coefficients and by eq. (B.8), it is clear that the \( W \) matrix satisfies

\[
W^T = W, \\
W^{-1} = \frac{\Delta t \Delta \omega}{2\pi} W^*.
\] (B.12)
and, consequently, $W^{-1} = \frac{\Delta t \Delta \omega}{2\pi} W^{T*}$. In other words, up to a multiplicative scalar constant, $W$ is a unitary operator. Taking now formula (4.6), we apply (B.8) and (B.12) to write the $C$ matrix in terms of Fourier-space objects, and quickly obtain

$$C = \left(\frac{\Delta \omega}{2\pi \Delta t}\right) W^{-1} \hat{C} W,$$

where $\hat{C}$ is the above mentioned Fourier-space representation of $C$, i.e., it is the matrix whose coefficients read

$$\hat{C}_{jk} = \langle \hat{\eta}_j \hat{\eta}_k^* \rangle.$$

Formula (B.13) is telling us that

$$\text{Det}(C) = \left(\frac{\Delta \omega}{2\pi \Delta t}\right)^N \text{Det}(\hat{C}),$$

independently of $W$.

### B.3 Power Spectrum

Recall first the definition (4.6) of the covariance matrix: $C_{jk} = \langle \eta_j \eta_k^* \rangle$, where $\langle \ldots \rangle$ denotes statistical average over realizations of the stochastic process $\eta$. For a stationary stochastic process we have that $C_{jk} = C_{j-k}$ can only be a function of $j - k$, e.g., the covariance matrix is Toeplitz matrix. It is a simple exercise to show that in this case the covariance matrix in Fourier space $\hat{C}$ is always diagonal:

$$\hat{C}_{jk} = \langle \hat{\eta}_j \hat{\eta}_k^* \rangle \propto \delta_{jk}$$

The power spectrum is then defined as:

$$\hat{C}_{jk} \equiv P(\omega_j) \frac{\delta_{jk}}{\Delta \omega},$$

in analogy with the continuous definition:

$$\langle \hat{\eta}(\omega) \hat{\eta}^*(\omega') \rangle = P(\omega) \delta_{\text{Dirac}}(\omega - \omega').$$

### B.4 Nonzero mean

In the practical handling of data, it is sometimes necessary to introduce offsets, with the consequence that a variable which had initially a zero mean may lose such a property. Assuming that $\langle \eta \rangle = 0$, let’s suppose that an offset $a \in \mathbb{R}$ is added to $\eta$. Thus, for the new variable $\eta' \equiv \eta + a$ one has $\langle \eta' \rangle = a$. From the definition (4.6), we find that the covariance matrix of $\eta'$ is just

$$C' = C + a^2 I,$$

where $C$ is the covariance matrix of $\eta$, and $I$ is the identity matrix. Relating now covariance and power spectrum by eqs. (B.13) and (B.15) —or (B.17)—, one realizes that the new power spectrum is simply

$$P' = P + 2\pi \Delta t a^2 I,$$

i.e., the previous one shifted by a constant, which corresponds to the entropy change coming from the knowledge of $\langle \eta' \rangle = a$. Proceeding in this way, it is possible to use the same computational methods as for the zero-mean case, with the only difference that $P$ has to be modified according to eq. (B.20).
C Appendix: The continuous random variable case

As it is well-known, Shannon’s entropy was firstly designed to deal with discrete random variables

\[ H(\eta) \equiv - \sum_j p(\eta_j) \log_2[p(\eta_j)], \]  

(C.1)

where the index runs through all possible different countable values of the r.v.. The problem with the continuous r.v. is that different \( \eta_j \)'s do not form a partition. To define \( H(\eta) \) we form first the discrete r.v. \( \eta_\Delta \) obtained by rounding off \( \eta \)

\[ \eta_\Delta \equiv n \Delta \eta \quad \text{if} \quad n \Delta \eta - \Delta \eta < x \leq n \Delta \eta. \]  

(C.2)

Clearly,

\[ P(\eta_\Delta = n \Delta \eta) = P(n \Delta \eta - \Delta \eta < \eta \leq n \Delta \eta) = \int_{n \Delta \eta - \Delta \eta}^{n \Delta \eta} d\eta f(\eta) = \Delta \eta \bar{f}(n \Delta \eta), \]  

(C.3)

where \( \bar{f}(n \Delta \eta) \) is a number between the maximum and minimum of \( f(\eta) \) in the interval \((n \Delta \eta - \Delta \eta, n \Delta \eta)\). Applying Shannon’s definition, we have:

\[ H(\eta_\Delta) = - \sum_{n=-\infty}^{\infty} \Delta \eta \bar{f}(n \Delta \eta) \log_2[\Delta \eta \bar{f}(n \Delta \eta)] \]  

(C.4)

and, since

\[ \sum_{n=-\infty}^{\infty} \Delta \eta \bar{f}(n \Delta \eta) = \int_{-\infty}^{\infty} d\eta f(\eta) = 1, \]  

(C.5)

we conclude that

\[ H(\eta_\Delta) = - \log_2(\Delta \eta) - \sum_{n=-\infty}^{\infty} \Delta \eta \bar{f}(n \Delta \eta) \log_2[\bar{f}(n \Delta \eta)]. \]  

(C.6)

As \( \Delta \eta \to 0 \), the r.v. \( \eta_\Delta \) tends to \( \eta \); however, its entropy \( H(\eta_\Delta) \) tends to \( \infty \) because \( -\log_2(\Delta \eta) \to \infty \). This is why we define the entropy \( H(\eta) \) of \( \eta \) not as the limit of \( H(\eta_\Delta) \) but as the limit of the sum \( H(\eta_\Delta) + \log_2(\Delta \eta) \) when \( \Delta \eta \to 0 \), i.e.:

\[ H(\eta_\Delta) + \log_2(\Delta \eta) \to \int_{-\infty}^{\infty} d\eta f(\eta) \log_2[f(\eta)] \quad \text{as} \quad \Delta \eta \to 0. \]  

(C.7)

So, the definition of ‘entropy’ for a continuous variable is:

\[ H(\eta) = \int_{-\infty}^{\infty} d\eta f(\eta) \log_2[f(\eta)], \]  

(C.8)

where the integration extends only over the region where \( f(\eta) \neq 0 \), as we have \( f(\eta) \log_2[f(\eta)] = 0 \) if \( f(\eta) = 0 \). This ‘entropy’ is more usually called differential entropy in the literature and its definition can also be extended to multivariate probability distributions. It is easy to see then that the above limit translates into:

\[ H(\eta_\Delta) + \log_2(\Delta \eta)^N \to \int_{-\infty}^{\infty} d\eta f(\eta) \log_2[f(\eta)] \quad \text{as} \quad \Delta \eta \to 0 \]  

(C.9)
when the $N$-dimensional space of $\vec{\eta}$ is latticed with $\Delta \eta$-boxes. So, we could approximate $H(\vec{\eta}\Delta) \simeq -\log_2(\Delta \eta)^N + H(\vec{\eta})$. In our case we have defined the compression ratio as $c_{r,\text{opt}} \equiv \frac{\text{average length}}{h}$, where $h \equiv \frac{H(\vec{\eta}\Delta)}{N}$. If we look just back to the approximate:

$$h \approx -\log_2(\Delta \eta) + H(\vec{\eta})/N.$$  \hfill (C.10)

The last summand in the previous expression is the average uncertainty per sample in a block of $N$ consecutive samples. The limit $N \to \infty$ of it is what is known as differential entropy rate:

$$\overline{h}(\vec{\eta}) = \lim_{N \to \infty} \frac{H(\eta_1, \ldots, \eta_N)}{N} \quad \hfill (C.11)$$

So, if we imagine that we have a stochastic process infinitely long and $\vec{\eta}$ is a vector r.v. whose dimension tends to infinity (i.e. $\eta_j = \eta(t_j)$ and we take samples for a long time or just many samples) we could then approximate:

$$h \simeq -\log_2(\Delta \eta) + \overline{h}(\vec{\eta}). \quad \hfill (C.12)$$

Regarding $\overline{h}(\vec{\eta})$ as the ‘continuous part’ of the entropy per component —i.e. $h_{\text{cont}}$—, this relation amounts to eq. (3.5).

### C.1 Entropy in the continuous case

For the one-dimensional Gaussian distribution in eq.(3.1) it is straightforward to show that:

$$\overline{h} \equiv h_{\text{cont}} = \log_2 \left[ \sqrt{2\pi e} \sigma \right],$$ \hfill (C.13)

in agreement with eq.(3.3) in the limit of small $\Delta \eta$, as expected from the comments in the previous section. For the case of $N$-dimensional Gaussian noise with correlations, we can use the fact that $\overline{h}(\vec{\eta})$ is well-known (see eg. [15]) for a Gaussian stochastic process with power spectrum $P(\omega)$:

$$\overline{h}(\vec{\eta}) = \log_2[\sqrt{2\pi e}] + \frac{1}{4\pi} \int_{-\pi}^{\pi} d\omega \log_2[\tilde{P}(\omega)]. \quad \hfill (C.14)$$

where $\tilde{P}(\omega)$ refers to the discrete stochastic process derived from the continuous one $P(\omega)$ by the relation

$$\tilde{P}(\omega) = \frac{1}{\Delta t} \sum_{m=-\infty}^{\infty} P\left(\omega + \frac{2\pi m}{\Delta t}\right), \quad -\pi \leq \omega \leq \pi,$$ \hfill (C.15)

where $\Delta t$ is the sampling interval that discretizes the process. For power spectra with a bandwidth limitation this reduces to (see [18]):

$$\tilde{P}(\omega) = \frac{1}{\Delta t} P\left(\frac{1}{\Delta t} \omega\right) \quad \hfill (C.16)$$

where $\tilde{P}(\omega)$ refers to the process $\eta_n = \eta(t = n\Delta t)$. In this case we can do a simple change of variables $\omega' = \omega/\Delta t$ in eq.(C.14) to find:

$$\frac{1}{4\pi} \int_{-\pi}^{\pi} d\omega \log_2[\tilde{P}(\omega)] = -2 \log_2 \Delta t + \frac{\Delta t}{2\pi} \int_{0}^{\pi/\Delta t} d\omega' \log_2[P(\omega')]. \quad \hfill (C.17)$$
where we have used the parity of $P(\omega)$ and the fact that the range in eq.(C.14) is symmetric. Recalling that $\omega_{\text{Max}} = \pi/\Delta t$ and we are using $\omega_{\text{min}} \simeq 0$ we see can that this calculation is equivalent to the Euler-Maclaurin summation formula eq.(4.27), so that the continuous calculation of the entropy given by eq.(C.14) and eq.(C.12) yields identical results to those of the discrete calculation eq.(4.12) in the limit of large $N$.

Acknowledgements

We would like to thank Pablo Fosalba for stimulating discussions. This work has been supported by CSIC, and by DGES (MEC), project PB96-0925.

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