HOFER-ZEHNDER CAPACITY AND A HAMILTONIAN CIRCLE
ACTION WITH NONCONTRACTIBLE ORBITS

KEI IRIE

ABSTRACT. Let \((M,\omega)\) be an aspherical symplectic manifold, which is closed or convex. Let \(U\) be an open set in \(M\), which admits a circle action generated by an autonomous Hamiltonian \(H \in C^\infty(U)\), such that each orbit of the circle action is not contractible in \(M\). Under these assumptions, we prove that the Hofer-Zehnder capacity of \(U\) is bounded by the Hofer norm of \(H\). The proof uses a variant of the energy-capacity inequality, which is proved by the theory of action selectors.

1. INTRODUCTION

1.1. Hofer-Zehnder capacity. First we fix some notations. We set \(S^1 := \mathbb{R}/\mathbb{Z}\). For any topological space \(X\), we set \(\pi_1(X) := C^0(S^1, X) / \sim\), where \(\gamma \sim \gamma'\) means that \(\gamma\) and \(\gamma'\) are homotopic. For each \(\gamma \in C^0(S^1, X)\), \(\bar{\gamma} \in C^0(S^1, X)\) is defined as \(\bar{\gamma}(t) := \gamma(-t)\). Since \(\gamma \sim \gamma' \implies \bar{\gamma} \sim \bar{\gamma}'\), one can define \(\bar{\alpha} \in \pi_1'(X)\) for any \(\alpha \in \pi_1'(X)\). When \(X\) is path connected, \(c_X\) denotes the element in \(\pi_1'(X)\) which consists of contractible loops on \(X\).

We introduce a refinement of the Hofer-Zehnder capacity, taking into account free homotopy classes of periodic orbits. Let \((M,\omega)\) be a symplectic manifold. We always assume that \(\partial M = \emptyset\). For \(H \in C^\infty(M)\), its Hamiltonian vector field \(X_H \in \mathfrak{X}(M)\) is defined by the equation \(\omega(X_H, \cdot) = -dH(\cdot)\). For any \(S \subset \pi_1'(M)\), \(\mathcal{H}^S_{HZ}(M,\omega)\) denotes the set of \(H \in C^\infty_0(M)\) which satisfies the following properties:

1. \(H \leq 0\) and \(\{H = 0\} \neq \emptyset\).
2. There exists a nonempty open set \(U \subset M\) such that \(H|_U \equiv \min H\).
3. Any nonconstant periodic orbit \(\gamma\) of \(X_H\) satisfying \(|\gamma| \in S\) has period \(> 1\).

Then we define
\[
c^S_{HZ}(M,\omega) := \sup\{- \min H | H \in \mathcal{H}^S_{HZ}(M,\omega)\}.
\]
Following properties are immediate from the definition:

- For any \(S, S' \subset \pi_1'(M)\), \(S \subset S' \implies c^S_{HZ}(M,\omega) \geq c^{S'}_{HZ}(M,\omega)\).
- For any nonempty open set \(U\) in \(M\), let \(i_U^M : U \to M\) denote the inclusion map, and let \((i_U^M)_* : \pi_1'(U) \to \pi_1'(M)\) denote the induced map. Then, for any \(S \subset \pi_1'(M)\),
  \[
c^S_{HZ}((i_U^M)_*^{-1}(S)) (U,\omega) \leq c^S_{HZ}(M,\omega).
\]
- Abbreviating \(c_{HZ}(M,\omega) := c^\pi_1(M)(M,\omega)\), \(c_{HZ}(M,\omega) \leq c^S_{HZ}(M,\omega)\) for any \(S \subset \pi_1'(M)\). Moreover, \(c_{HZ}(U,\omega) \leq c_{HZ}(M,\omega)\) for any nonempty open set \(U\) in \(M\).

Date: December 23, 2011.
2010 Mathematics Subject Classification. 53D40, 70H12.
$c_{HZ}(M, \omega)$ defined as above coincides with the original Hofer-Zehnder capacity \((3, 4)\).

1.2. Main result. First we fix some terminologies. Let \((M, \omega)\) be a symplectic manifold such that \(\partial M = \emptyset\).

- \((M, \omega)\) is called aspherical when \(\omega|_{\pi_2(M)} = 0\).
- \((M, \omega)\) is called convex when there exists an increasing sequence \(M_1 \subset M_2 \subset \cdots\) of compact codimension 0 submanifolds of \(M\), which satisfies \(\bigcup_i M_i = M\) and the following property for each \(i \geq 1\): there exists a vector field \(X_i\) defined on some neighborhood of \(\partial M_i\) in \(M_i\), which points strictly outwards on \(\partial M_i\) and \(L_{X_i} \omega = \omega\).
- A circle action \(S^1 \circ M\) is called Hamiltonian action generated by \(H \in C^\infty(M)\), when there holds \(\frac{d}{dt}(t \cdot x)|_{t=0} = X_H(x)\) for any \(x \in M\).

The main result of this note is the following:

**Theorem 1.1.** Let \((M, \omega)\) be a connected aspherical symplectic manifold, which is closed or convex. Let \(U\) be an open set in \(M\), which admits a Hamiltonian circle action generated by \(H \in C^\infty(U)\). Suppose that for any \(x \in U\), \(\gamma^x : S^1 \rightarrow M; t \mapsto t \cdot x\) is not contractible in \(M\), and \([\gamma^x]\) \(\in \pi_1(M)\) does not depend on \(x \in U\). Then, setting \(\alpha := [\gamma^x] \in \pi_1(M)\),

\[
\|c_{HZ}^{\alpha}(U, \omega)\| \leq \sup_{x \in U} H - \inf H.
\]

**Remark 1.2.** In \([7]\), L. Macarini gives a similar upper bound of \(c_{HZ}(U, \omega)\), provided that \(U\) is a connected open set in a geometrically bounded symplectic manifold, and \(U\) admits a free Hamiltonian circle action, which satisfies an additional condition on “the order of the action”. For precise statement, see Theorem 1.1 in \([7]\).

Theorem 1.1 is proved in section 2. First we give the following application:

**Corollary 1.3.** Let \(N\) be a compact connected Riemannian manifold, \(\omega_N\) be the standard symplectic form on \(T^*N\), and \(DT^*N := \{(q, p) \in T^*N \mid |p| < 1\}\). Suppose that \(N\) admits a circle action (which may not preserve the metric), such that for any \(x \in N\), \(\gamma^x : S^1 \rightarrow N; t \mapsto t \cdot x\) is not contractible. Then, setting \(\alpha := [\gamma^x] \in \pi_1(N)\),

\[
\|c_{HZ}^{\alpha}(DT^*N, \omega_N)\| = \leq 2 \sup_{x \in N} \text{length}(\gamma^x).
\]

**Proof.** Let \(Z\) be a vector field on \(N\), which generates the given circle action, i.e. \(Z_x := \frac{d}{dt}(t \cdot x)|_{t=0} \in N\). Up to reparametrization of the action, we may assume that \(\sup_{x \in N} |Z_x| \leq \sup_{x \in N} \text{length}(\gamma^x) =: L\). The circle action on \(N\) naturally extends to a Hamiltonian circle action on \(T^*N\) generated by \(H \in C^\infty(T^*N)\), where \(H(q, p) := p(Z_q)\). Since \(DT^*N \subset H^{-1}((-L, L))\), we get

\[
\|c_{HZ}^{\alpha}(DT^*N, \omega_N)\| \leq \|c_{HZ}^{\alpha}(H^{-1}((-L, L)), \omega_N)\| \leq 2L,
\]

where the second inequality follows from Theorem 1.1.

**Remark 1.4.** In \([3]\), the author proved \(c_{HZ}(DT^*N, \omega_N) < \infty\) under same assumption as Corollary 1.3, based on the correspondence between the pair-of-pants product in Floer homology of cotangent bundles and the loop product on homology of loop spaces.
As a specific case of Corollary 1.3 we recover the following result of M. Jiang [6]:

**Corollary 1.5.** Let $N$ be a flat torus: $N := \mathbb{R}/a_1\mathbb{Z} \times \cdots \times \mathbb{R}/a_n\mathbb{Z}$, where $n \geq 1$ and $0 < a_1 \leq \cdots \leq a_n$. Then, $c_{HZ}(DT^*N, \omega_N) \leq 2a_1$.

2. **Proof**

2.1. **Action selector.** To prove Theorem 1.1 we use the notion of action selectors. Let $(M, \omega)$ be an aspherical symplectic manifold, and $\mathcal{H}(M) := C_0^\infty(M \times [0, 1])$.

For any $H \in \mathcal{H}(M)$ and $t \in [0, 1]$, $H_t \in C_0^\infty(M)$ is defined as $H_t(x) := H(t, x)$. For any $H \in \mathcal{H}(M)$, $(\varphi_t^H)_{0 \leq t \leq 1}$ is the flow generated by $(X_{H_t})_{0 \leq t \leq 1}$. i.e. $\varphi_t^H : M \to M$ is defined as

$$\varphi_0^H = \text{id}_M, \quad \frac{d}{dt}\varphi_t^H = X_H(\varphi_t^H)(0 \leq t \leq 1).$$

For any $H, K \in \mathcal{H}(M)$, we define $\bar{H}, H \ast K \in \mathcal{H}(M)$ as

$$\bar{H}(t, x) := -H(t, \varphi_t^H(x)), \quad H \ast K(t, x) := H(t, x) + K(t, (\varphi_t^H)^{-1}(x)).$$

It is easy to verify the following properties:

- $\varphi_t^H = (\varphi_t^H)^{-1}$, $\varphi_t^{H \ast K} = \varphi_t^H \circ \varphi_t^K$ for any $0 \leq t \leq 1$.
- $(\mathcal{H}(M), \ast)$ is a group. The unit element is $0$, and the inverse of $H$ is $\bar{H}$.

For any $H \in \mathcal{H}(M)$ and $x \in \text{Fix}(\varphi_1^H)$, we define $\gamma_H : S^1 \to M$ as $\gamma_H(t) := \varphi_t^H(x)$. We define $\mathcal{P}(H) := \{\gamma_H \mid x \in \text{Fix}(\varphi_1^H)\}$. $\mathcal{P}^0(H)$ denotes the set of $\gamma \in \mathcal{P}(H)$ which is contractible in $M$. Setting $D := \{z \in \mathbb{C} \mid |z| \leq 1\}$, for any contractible $\gamma : S^1 \to M$, we take $\tilde{\gamma} : D \to M$ so that $\tilde{\gamma}(e^{2\pi i t}) = \gamma(t)$ and define

$$\mathcal{A}_H(\gamma) := \int_D \omega - \int_{S^1} H_t(\gamma(t))dt.$$ 

It is well-defined since we have assumed that $(M, \omega)$ is aspherical. Then we define

$$\Sigma^c(H) := \{\mathcal{A}_H(\gamma) \mid \gamma \in \mathcal{P}^0(H)\}.$$

It is well-known that $\Sigma^c(H)$ is a nowhere dense subset in $\mathbb{R}$ (see Proposition 3.7 in [8]). Finally, for any $H \in \mathcal{H}(M)$, we set

$$E_-(H) := -\int_0^1 \min H_t dt, \quad E_+(H) := \int_0^1 \max H_t dt, \quad \|H\| := E_-(H) + E_+(H).$$

**Definition 2.1.** Let $(M, \omega)$ be a connected aspherical symplectic manifold. An action selector for $(M, \omega)$ is a map $\sigma : \mathcal{H}(M) \to \mathbb{R}$ which satisfies the following axioms:

- (AS1) $\sigma(H) \in \Sigma^c(H)$ for any $H \in \mathcal{H}(M)$.
- (AS2) For any $H \in \mathcal{H}_H^{c_{HZ}}(M, \omega)$, $\sigma(H) = -\min H$.
- (AS3) $\sigma(H) \leq E_-(H)$ for any $H \in \mathcal{H}(M)$.
- (AS4) $\sigma$ is continuous with respect to the $C^0$-topology of $\mathcal{H}(M)$.
- (AS5) $\sigma(H \ast K) \leq \sigma(H) + E_-(K)$ for any $H, K \in \mathcal{H}(M)$.

**Remark 2.2.** The above set of axioms for action selectors follows that in [2], although our sign conventions are different from [2]. Moreover, our notion of Hofer-Zehnder admissible Hamiltonians is wider than that in [2].
Our proof of Theorem 1.1 is based on the following result:

**Theorem 2.3** ([3], [1]). Let $(M, \omega)$ be a connected aspherical symplectic manifold.

1. When $M$ is closed, there exists an action selector for $(M, \omega)$.
2. When $M$ is convex, there exists an action selector for $(M, \omega)$.

(1) was proved by M. Schwarz in [3], based on the Piunikhin-Salamon-Schwarz isomorphism. (2) was proved by U. Frauenfelder and F. Schlenk in [1], based on [3] and Floer theory for convex symplectic manifolds ([1]).

### 2.2. A variant of the energy-capacity inequality.

First we prove the following result, which can be considered to be a variant of the energy-capacity inequality:

**Theorem 2.4.** Let $(M, \omega)$ be a connected aspherical symplectic manifold, which is closed or convex. Let $U$ be an open set in $M$ and $H \in \mathcal{H}(M)$ such that:

1. $\varphi^1_H|_U = \text{id}_U$.
2. For any $x \in U$, $\gamma^1_H : S^1 \to M; t \mapsto \varphi^1_H(x)$ is not contractible. Moreover, $[\gamma^1_H] \in \pi_1(M)$ does not depend on $x \in U$.

Then, setting $\alpha := [\gamma^1_H] \in \pi_1(M)$, $c^{(\text{H})}_{\mathbb{R}Z}((\epsilon_M, \alpha))(U, \omega) \leq \|H\|$. 

The proof is similar to the proof of the energy-capacity inequality in [2] (section 2.1 in [2]). In the following, $\sigma : \mathcal{H}(M) \to \mathbb{R}$ denotes an action selector for $(M, \omega)$, which exists due to Theorem 2.3.

Suppose that $U, H, \alpha$ are as in Theorem 2.4. We have to show $- \min K \leq \|H\|$ for any $K \in \mathcal{H}_{\mathbb{R}Z}^{(\epsilon_M, \alpha)}(M)$, $\text{supp} K \subset U$. Since $K \in \mathcal{H}_{\mathbb{R}Z}^{(\epsilon_M)}(M)$, $\sigma(K) = - \min K$ by (AS2). Hence it is enough to show $\sigma(K) \leq \|H\|$.

First notice the following lemma:

**Lemma 2.5.** For any $\chi \in C^\infty([0, 1])$ satisfying $\int_0^1 \chi(t) \, dt = 1$, we set $K_\chi \in \mathcal{H}(M)$ by $K_\chi(x, t) := K(x)\chi(t)$. Then, $\sigma(K_\chi) = \sigma(K)$.

**Proof.** For $0 \leq s \leq 1$, set $\chi_s := s\chi + (1 - s)$. Then, it is easy to verify that $\Sigma^0(K_{\chi_s}) = \Sigma^0(K)$ for any $s$. By (AS1), $\sigma(K_{\chi_s}) \in \Sigma^0(K)$ for any $0 \leq s \leq 1$. On the other hand, $\sigma(K_{\chi_s})$ depends continuously on $s$ by (AS4). Since $\Sigma^0(K)$ is nowhere dense, $[0, 1] \to \mathbb{R}; s \mapsto \sigma(K_{\chi_s})$ is a constant function. Hence $\sigma(K_\chi) = \sigma(K)$. \qed

**Remark 2.6.** The above lemma is same as Lemma 2.2 in [2]. We have included the proof for the convenience of the reader.

Take $\chi \in C^\infty([0, 1])$ so that $\int_0^1 \chi(t) \, dt = 1$ and $\text{supp} \chi \subset (1/2, 1)$. By Lemma 2.5, it is enough to show $\sigma(K_{\chi}) \leq \|H\|$. After reparametrizing in $t$, we may assume that $H_t \equiv 0$ for $1/2 \leq t \leq 1$. Then

$$aK_\chi * H(t, x) = \begin{cases} H(t, x) & (0 \leq t \leq 1/2) \\ aK_\chi(t, x) & (1/2 \leq t \leq 1) \end{cases},$$

where $aK_\chi$ is defined

4
We claim that $\Sigma^o(aK_{\chi} * H) \subset \Sigma^o(H)$ for any $0 \leq a \leq 1$. Let $x \in \text{Fix}(\varphi^1_{aK_{\chi} * H})$ such that $\gamma^x_{aK_{\chi} * H}$ is contractible in $M$. We distinguish two cases:

- Suppose $x \in U$. Since $\varphi^1_H|_U = \text{id}_U$, $x \in \text{Fix}(\varphi^1_{aK_{\chi}})$. Since $0 \leq a \leq 1$ and $K \in \mathcal{K}^{(\ell)}(M)$, $[\gamma^x_{aK_{\chi}}] \neq \alpha$. On the other hand, $\gamma^x_{aK_{\chi} * H}$ is a concatenation of $\gamma^x_H$ and $\gamma^x_{aK_{\chi}}$, and $[\gamma^x_H] = \alpha$. Hence $\gamma^x_{aK_{\chi} * H}$ is not contractible: it contradicts our assumption.

- Suppose $x \notin U$. Since $\varphi^1_{aK_{\chi}}|_{M \setminus U} = \text{id}_{M \setminus U}$, $x \in \text{Fix}(\varphi^1_{aK_{\chi}})$. Using $\text{supp}K \subset U$, it is easy to verify that $\gamma^x_{aK_{\chi} * H} = \gamma^x_H$, $aK_{\chi} * H(t, \gamma^x_{aK_{\chi} * H}(t)) = H(t, \gamma^x_H(t))$. Hence $\mathcal{A}_{aK_{\chi} * H}(\gamma^x_{aK_{\chi} * H}) = \mathcal{A}_H(\gamma^x_H) \in \Sigma^o(H)$.

Hence we have verified $\Sigma^o(aK_{\chi} * H) \subset \Sigma^o(H)$ for $0 \leq a \leq 1$. By (AS4), $\sigma(aK_{\chi} * H)$ depends continuously on $a$. Since $\Sigma^o(H)$ is nowhere dense, $[0, 1] \to \mathbb{R}$, $a \mapsto \sigma(aK_{\chi} * H)$ is a constant function. Hence $\sigma(H) = \sigma(K_{\chi} * H)$. Finally, we obtain $\sigma(K_{\chi}) \leq \|H\|$ by

$$\sigma(K_{\chi}) = \sigma(K_{\chi} * H * \tilde{H}) \leq \sigma(K_{\chi} * H) + E_-(\tilde{H}) = \sigma(H) + E_+(H) \leq E_-(H) + E_+(H) = \|H\|.$$ 

\hfill $\Box$

2.3. Proof of Theorem 1.1. Finally we prove Theorem 1.1.

For any $S \subset \pi^1(U)$, it is easy to verify that $c^S_{\mathcal{H}}(U, \omega) = \sup_{V} \{ c^S_{\mathcal{H}}(V, \omega), \text{ where } V \text{ runs over all open sets of } U \text{ with compact closures. Hence it is enough to show}
$$c^{(i\gamma)^{-1}(c_{\mathcal{M}, \phi})}_{\mathcal{H}}(V, \omega) \leq \sup H - \inf H$$

for any open $V \subset U$ with a compact closure. Since $H \in C^\infty(U)$ generates the given circle action on $U$, $V' := \bigcup_{0 \leq t \leq 1} \varphi^1_{tH}(V)$ is invariant under the circle action, and it is again an open set of $U$ with compact closure. Then there exists $\rho \in C^\infty_0(U)$ such that $0 \leq \rho \leq 1$ and $\rho|_{V'} \equiv 1$. Then $\rho H \in C^\infty_0(U)$ extends to $M$, and $\varphi^1_{\rho H}|_V = \varphi^1_{\rho H}|_{V'}$ for any $0 \leq t \leq 1$. By adding constant to $H \in C^\infty(U)$ if necessary, we may assume that $\inf H \leq 0 \leq \sup H$. Then $\inf \rho H \geq \inf H$, $\sup \rho H \leq \sup H$, hence $\|\rho H\| \leq \|H\|$. Finally we get
$$c^{(i\gamma)^{-1}(c_{\mathcal{M}, \phi})}_{\mathcal{H}}(V, \omega) \leq \|\rho H\| \leq \|H\|.$$

The first inequality follows from Theorem 2.2 (applied to $V$ and $\rho H$). \hfill $\Box$

Acknowledgement. The author would like to appreciate Professor Kenji Fukaya for his encourgement. The author is supported by Grant-in-Aid for JSPS fellows.

References

[1] U. Frauenfelder, F. Schlenk, *Hamiltonian dynamics on convex symplectic manifolds*, Israel J. Math. 159, 1–56, 2007.

[2] U. Frauenfelder, V. Ginzburg, F. Schlenk, *Energy capacity inequalities via an action selector*, Geometry, Spectral Theory, and Dynamics, Contemp. Math. vol. 387, AMS, 129–152, 2005.

[3] H. Hofer, E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser, Basel, 1994.

[4] H. Hofer, E. Zehnder, *A new capacity for symplectic manifolds*, Analysis et cetera. Academic Press, 405–428, 1990.
[5] K. Irie, *Hofer-Zehnder capacity of unit disk cotangent bundles and the loop product*, arXiv:1110.2244, 2011.

[6] M. Jiang, *Hofer-Zehnder symplectic capacity for two dimensional manifolds*, Proc. Roy. Soc. Edinburgh Sect. A 123, 945–950, 1993.

[7] L. Macarini, *Hofer-Zehnder capacity and Hamiltonian circle actions*, Commun. Contemp. Math. 6, 913–945, 2004.

[8] M. Schwarz, *On the action spectrum for closed symplectically aspherical manifolds*, Pacific J. Math. 193, 419–461, 2000.

[9] C. Viterbo, *Functors and computations in Floer homology with applications I*, Geom. Funct. Anal. 9, 985–1033, 1999.

**Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan**

*E-mail address: iriek@math.kyoto-u.ac.jp*