ANALYTICAL AND NUMERICAL RESULTS ON THE
POSITIVITY OF STEADY STATE SOLUTIONS
OF A THIN FILM EQUATION

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Abstract. We consider an equation for a thin film of fluid on a rotating
cylinder and present several new analytical and numerical results on steady
state solutions. First, we provide an elementary proof that both weak and
classical steady states must be strictly positive so long as the speed of rotation
is nonzero. Next, we formulate an iterative spectral algorithm for computing
these steady states. Finally, we explore a non-existence inequality for steady
state solutions from the recent work of Chugunova, Pugh & Taranets.

1. Introduction. Thin liquid films appear in a wide range of natural and industrial
applications, such as coating processes, and are generally characterized by a small
ratio between the thickness of the fluid and the characteristic length scale in the
transverse direction. The physics of such processes are reviewed in Oron, Davis &
Bankoff,[14] and more recently in Craster & Matar, [11]. If one includes surface
tension in the model, the fluid is governed by a degenerate fourth order parabolic
equation. Such equations, studied by Bernis & Friedman, [7], Bertozzi & Pugh, [8],
and Beretta, Bertsch & dal Passo [6] still hold many challenges. See, for example,
[19, 20, 10] for progress on models which include convection, and Becker & Grün,[4],
for a thorough review of analytical progress.

There has been recent interest in equations governing the evolution of a thin,
viscous, film on the outer (or inner) surface of a cylinder rotating with constant
angular velocity, as in Figure 1. The dimensionless evolution is governed by

\[ u_t + (u^n (u_{\theta\theta} + u_\theta - \sin \theta) + \omega u_\theta) = 0. \]  

(1)

The solution, \( u = u(\theta, t) \), gives the evolution of the dimensionless thickness of the
fluid, parameterized by the angular variable, \( \theta \in \Omega \equiv [-\pi, \pi] \); \( u \) is periodic in \( \theta \). To
derive (1), one begins with the Navier-Stokes equations, together with a kinematic
boundary condition for the free surface. Subject to appropriate physical scalings and

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geometric symmetries, asymptotic analysis leads to equation (1), a dimensionless equation, and \( \omega \) is the dimensionless angular velocity. We refer the reader to, for instance, [16, 15] for a full derivation. For the nonlinearity, \( n = 1, 2, 3 \), depending on the boundary condition at the interface of the fluid and the cylinder. This can be generalized to non-integer values, but the focus of our work is on the integer cases. \( n = 3 \) is of particular interest and corresponds to a no-slip condition at the fluid-cylinder interface.

\[
\begin{align*}
R + u(\theta, t) \\
\end{align*}
\]

**Figure 1.** The grey region represents the thin film of liquid on the surface of the cylinder with dimensionless radius \( R \). The cylinder is rotating with constant, dimensionless, angular velocity \( \omega \). We assume symmetry in the \( z \) coordinate, so \( u \) is only a function of \( t \) and \( \theta \).

Though Buchard, Chugunova & Stephens, [9], gave a detailed examination of (1) when \( \omega = 0 \), our understanding of the case \( \omega \neq 0 \) remains incomplete. Some partial results on this case have made assumptions that solutions of (1) are strictly positive, such as in [18], where Pukhnachev proved, under a smallness assumption, positive solutions of (1) exist and are unique, using a contraction argument.

In this work, we further study non-negative steady state solutions, *time independent* solutions \( u(\theta) \) of (1), satisfying

\[
(u^n(u_{\theta\theta} + u_\theta - \sin \theta) + \omega u)_\theta = 0. 
\tag{2}
\]

This can be integrated once, to

\[
u^n(u_{\theta\theta} + u_\theta - \sin \theta) + \omega u = q, 
\tag{3}
\]

where \( q \), the constant of integration, is the dimensionless flux of the fluid.

Let us briefly explore the problem with some heuristic asymptotics in the \( n = 3 \) case. First, assume that \( 0 < q \ll \omega \), i.e.

\[
q = \varepsilon \omega, \quad 0 < \varepsilon \ll 1. 
\tag{4}
\]

Subject to this assumption, we may then perform a series expansion in \( \varepsilon \),

\[
u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \varepsilon^3 u^{(3)} + \ldots
\tag{5}
\]

Substituting this into

\[
u^3(u_{\theta\theta} + u_\theta - \sin \theta) + \omega u = \varepsilon \omega, 
\tag{6}
\]
we then match orders of $\varepsilon$. At order $\varepsilon^0$,
\[(u^{(0)})^3(\partial^3_\theta u^{(0)} + \partial_\theta u^{(0)} - \sin \theta) + \omega u^{(0)} = 0. \tag{7}\]
We shall assume the order $O(\varepsilon^0)$ term is trivial, $u^{(0)} = 0$; consequently, $u$ is $O(\varepsilon^1)$.

Proceeding with this assumption, the next few orders are:
\[
\begin{align*}
O(\varepsilon^1) &: \quad \omega u^{(1)} = \omega, \tag{8a} \\
O(\varepsilon^2) &: \quad \omega u^{(2)} = 0, \tag{8b} \\
O(\varepsilon^3) &: \quad - (u^{(1)})^3 \sin \theta + \omega u^{(3)} = 0. \tag{8c}
\end{align*}
\]
Consequently, the asymptotic approximation of the steady state is thus:
\[
u \sim \frac{q}{\omega} + \frac{1}{\omega} \left(\frac{q}{\omega}\right)^3 \sin \theta = \varepsilon + \frac{1}{\omega} \left(\frac{\varepsilon}{\omega}\right)^3 \sin \theta. \tag{9}\]
This corresponds to both the “strongly supercritical regime” of Benilov et al.,\cite{2, 5}, and the “small amplitude regime” of Pukhnachev,\cite{18}. A similar analysis applies to the case $n = 2$, for which an approximate steady state solution is
\[
u \sim \frac{q}{\omega} + \frac{1}{\omega} \left(\frac{q}{\omega}\right)^2 \sin \theta = \varepsilon + \frac{1}{\omega} \left(\frac{\varepsilon}{\omega}\right)^2 \sin \theta. \tag{10}\]
These asymptotic solutions are, to the first correction, positive.

In this work, we give an elementary proof that when $\omega \neq 0$ and $n \geq 2$, both weak and classical solutions must be strictly positive, without making any smallness assumptions; see Section 2. We emphasize that we are only interested in non-negative solutions. Indeed, one of the main challenges of fourth order parabolic equations, such as (1), is that they do not preserve sign; positivity is a nontrivial property.

For a more extended study of the steady states, it is helpful to explore the problem numerically. This can provide information on how, and if, $u$ and its derivatives develop singularities as $\omega \to 0$. For such a task, one needs a robust algorithm for computing solutions of (3). Motivated by the iteration argument of\cite{18}, we formulate and benchmark an iterative spectral algorithm in Section 3. This scheme is easy to implement and rapidly converges.

Our last result, appearing in Section 4, is motivated by Chugunova, Pugh, & Taranets,\cite{10}, who proved that for $n = 3$ there can be no positive steady state solutions when the rotation and flux parameters satisfy the inequality
\[
q > \left(\frac{2}{3}\right)^{3/2} \omega^{3/2}. \tag{11}\]
This was refinement of an earlier bound due to Pukhnachev,\cite{17}. We explore the ($\omega, q$) phase space and find evidence that this bound may be sharp. Furthermore, it appears that (11) is related to a transition between two different regimes of steady state solutions.

1.1. Notation and terminology. In what follows, we shall use the following conventions. $C^k_{\text{per}}(\Omega)$ is the set of continuous $2\pi$-periodic functions on $\Omega = [-\pi, \pi]$ with $k$ continuous derivatives. $H^k_{\text{per}}(\Omega)$ is the set of square integrable $2\pi$-periodic functions on $\Omega = [-\pi, \pi]$ with $k$ square integrable weak derivatives. For a function $\phi$ in one of these spaces, we shall denote its mean by $\bar{\phi}$. For brevity, we will also sometimes write
\[a \lesssim b\]
to indicate that there exists a constant $C > 0$ such that
\[ a \leq C b. \]

Finally, we say that $\theta_0$ is a **touchdown point** if $u(\theta_0) = 0$ and $\theta_0 \in \text{supp } u$. $u$ is said to have a **dry region** if $\text{supp } u \neq \Omega$; points $\theta_0$ in the dry region are not touchdown points.

2. **Positivity of steady states.** Here, we prove that both classical and weak steady state solutions of (1) are strictly positive provided $\omega \neq 0$. In [9], the authors proved that for $\omega = 0$, solutions need not be positive.

2.1. **Classical steady state solutions.** As noted, the case $n = 3$ corresponds to a no-slip condition at the surface of the cylinder. Recalling that $\omega$ is the non-dimensional angular velocity, physical intuition suggests that as long as $\omega \neq 0$, any steady state solution of (1) with $n = 3$ will coat the entire cylinder. The rotation “drags” the fluid with the cylinder, so that were there a dry patch, it could not be maintained; thus, it could not correspond to a steady state. The following elementary argument shows that this physical intuition is correct, and is in fact valid for $n \geq 2$.

**Proposition 1** (Positivity of Classical Steady State Solutions). Let $u \in C^4_{\text{per}}(\Omega)$ be a classical steady-state solution of (1) with $n > 1$. If there is a point $\theta_0$ with $u(\theta_0) = 0$, then $\omega = 0$.

**Proof.** Assume that $u$ vanishes at $\theta_0$. Evaluating (3) at $\theta_0$, we have $q = 0$. Consequently,
\[ u^n(u_{\theta\theta\theta} + u_{\theta} - \sin \theta) + \omega u = 0. \tag{12} \]

On the support of $u$,
\[ u^{n-1}(u_{\theta\theta\theta} + u_{\theta} - \sin \theta) + \omega = 0. \tag{13} \]

Observe that we have the following bound for the first term in the above expression,
\[ |u^{n-1}(u_{\theta\theta\theta} + u_{\theta} - \sin \theta)| \leq u^{n-1}(\|u_{\theta\theta\theta}\|_{L^\infty} + \|u_{\theta}\|_{L^\infty} + 1). \]

Since $u$ is $C^4$, both $\|u_{\theta\theta\theta}\|_{L^\infty}$ and $\|u_{\theta}\|_{L^\infty}$ are finite. Therefore,
\[ \lim_{\theta \to \theta_0} u^{n-1}(u_{\theta\theta\theta} + u_{\theta} - \sin \theta) = 0 \]

This implies $\omega = 0$. 

Though a classical solution of (1) needs to be $C^4$, the above proof succeeds without change if $u$ is merely $C^2$ with bounded third derivative.

2.2. **Weak steady state solutions.** Positivity can also be proven under weaker assumptions on $u$. We say that a non-negative $u \in H^2_{\text{per}}(\Omega)$ is a **weak solution** of (2) if
\[ \int_{-\pi}^{\pi} (u^n \phi_{\theta})_\phi (u_{\theta\theta} + u - \sin(\theta)) - \omega u \phi_{\theta} d\theta = 0, \tag{14} \]
for all $\phi \in C^\infty_{\text{per}}(\Omega)$. We first establish some pointwise properties of a function, which need not be a solution to (14), in the neighborhood of a hypothetical point, $\theta_0$, at which $u(\theta_0) = 0$. 
Lemma 2.1. Let \( u \in H^2_{\text{per}}(\Omega) \) be non-negative. If there exists \( \theta_0 \) such that \( u(\theta_0) = 0 \), then

\[
\begin{align*}
  u'(\theta_0) &= 0 \\
  u(\theta) &= |u(\theta) - u(\theta_0)| \lesssim |\theta - \theta_0|^{3/2}, \\
  |u'(\theta)| &= |u'(\theta) - u'(\theta_0)| \lesssim |\theta - \theta_0|^{1/2}.
\end{align*}
\]

Proof. By virtue of a Sobolev embedding theorem in dimension one, \( H^2 \hookrightarrow C^{1,1/2} \) [1, 13]. Therefore, both \( u \) and \( u' \) are Hölder continuous with exponent 1/2;

\[
|u(\theta) - u(\theta_0)| \lesssim |\theta - \theta_0|^{1/2}
\]  

(16a)

\[
|u'(\theta) - u'(\theta_0)| \lesssim |\theta - \theta_0|^{1/2}.
\]  

(16b)

Now, either \( \theta_0 \) corresponds to a touchdown point, or it is in a dry region. If it is a touchdown point, then, because \( u \) is \( C^1 \) and non-negative, we must have that \( u'(\theta_0) = 0 \) too, otherwise there would be a zero crossing. If instead, \( \theta_0 \) corresponds to a dry region, then, again, we must have \( u'(\theta_0) = 0 \), since \( u \) is \( C^1 \) and there is a neighborhood of \( \theta_0 \) in which \( u \) is zero.

Refining (16), we can apply the mean value theorem and the Hölder continuity of \( u' \) to get

\[
|u(\theta) - u(\theta_0)| = |u'(\theta_*)||\theta - \theta_0| = |u'(\theta_*) - u'(\theta_0)||\theta - \theta_0| \lesssim |\theta - \theta_0|^{3/2},
\]

where \( \theta_* \) lies between \( \theta \) and \( \theta_0 \).

To prove the weak form of Proposition 1, we begin by deriving a weak analog of (3):

Lemma 2.2. If \( u \) is a weak solution, then there is a constant \( q \), depending only on \( u \), such that

\[
\int_{-\pi}^{\pi} (u^n)\delta(u\theta + u - \sin(\theta)) - \omega u\phi \, d\theta = q \int_{-\pi}^{\pi} \phi \, d\theta
\]  

(17)

for all \( \phi \in C^\infty_{\text{per}}(\Omega) \).

Proof. First, observe that for any \( \phi \in C^\infty_{\text{per}}(\Omega) \), \( \phi - \bar{\phi} \) has mean zero and therefore has an antiderivative in \( C^\infty_{\text{per}}(\Omega) \). Given a test function \( \phi \), let \( \varphi \) solve \( \varphi_\theta = \phi - \bar{\phi} \). Plugging in \( \varphi \) in (14) in place of \( \phi \), we have

\[
\int_{-\pi}^{\pi} (u^n)\delta(u\theta + u - \sin(\theta)) - \omega u\phi \, d\theta = \bar{\phi} \int_{-\pi}^{\pi} (u^n)\delta(u\theta + u - \sin(\theta)) - \omega u \, d\theta.
\]  

(18)

We may now take

\[
q = 2\pi \int_{-\pi}^{\pi} (u^n)\delta(u\theta + u - \sin(\theta)) - \omega u \, d\theta.
\]

We next derive a weak analog of equation (12) in the event that \( u \) has a touchdown or a dry region.

Lemma 2.3. For \( n > 1 \), let \( u \) be a weak solution satisfying (17) for all \( \phi \in C^\infty_{\text{per}}(\Omega) \). If \( u(\theta_0) = 0 \) for some \( \theta_0 \), then \( q = 0 \).
Proof. Given \( \varepsilon > 0 \), let \( \phi \in C_{\text{per}}^\infty(\Omega) \) be a compactly supported test function satisfying:

\[
0 \leq \phi \leq 1, \quad \text{supp } \phi \subseteq (-1, +1), \quad \int_{-\pi}^{\pi} \phi \, d\theta = 1.
\]  

(19)

Let

\[
\phi^\varepsilon(\theta) \equiv \varepsilon^{-1} \phi(\varepsilon^{-1}(\theta - \theta_0))
\]  

(20)

Then \( \text{supp } \phi^\varepsilon \subseteq I_\varepsilon \equiv [\theta_0 - \varepsilon, \theta_0 + \varepsilon] \). Substituting \( \phi^\varepsilon \) into (17), we compute

\[
|q| \leq |\omega| \int_{I_\varepsilon} u \phi^\varepsilon d\theta + \int_{I_\varepsilon} |n u^{n-1} u_\theta \phi^\varepsilon + u^n \phi^n_\theta| \left| u_{\theta\theta} + u - \sin \theta \right| d\theta
\]

\[
\leq |\omega| ||u||_{L^\infty(I_\varepsilon)} + ||n u^{n-1} u_\theta \phi^\varepsilon + u^n \phi^n_\theta||_{L^2(I_\varepsilon)} ||u_{\theta\theta} + u - \sin \theta||_{L^2(I_\varepsilon)}
\]  

(21)

\[
\leq |\omega| ||u||_{L^\infty(I_\varepsilon)} + C\sqrt{2\varepsilon} \left( ||n u^{n-1} u_\theta \phi^\varepsilon||_{L^\infty(I_\varepsilon)} + ||u^n \phi^n_\theta||_{L^\infty(I_\varepsilon)} \right)
\]  

The constant \( C \) depends on the \( L^2 \) norms of \( u \) and its second derivative, which are both finite by assumption.

Using estimates (15), we have that within \( I_\varepsilon \),

\[
u^n \phi^n_\theta \leq \varepsilon^{2\nu-2}
\]  

(22a)

\[
u u^{n-1} u_\theta \phi^\varepsilon \leq \varepsilon^{3\nu-2}
\]  

(22b)

Substituting this pointwise analysis into (21),

\[
|q| \lesssim \varepsilon^{3\nu-6} + \varepsilon^{(3\nu-3)/2}. 
\]  

(23)

Since this holds for all \( \varepsilon > 0 \), we conclude that \( q = 0 \) for \( n > 1 \).

Using Lemmas 2.2 and 2.3, we now know that a weak solution with a touchdown or a dry region satisfies

\[
\int_{-\pi}^{\pi} (\phi u^n) \phi(u_{\theta\theta} + u - \sin(\theta)) - \omega u \phi \, d\theta = 0.
\]  

(24)

This is precisely the weak form of equation (12).

We now proceed to our main result in the weak case, under the additional restriction that \( n \geq 2 \):

**Proposition 2** (Positivity of Weak Solutions). Let \( u \) be a weak solution with \( n \geq 2 \). If \( S = \{ \theta : u(\theta) > 0 \} \neq \Omega \), then \( \omega = 0 \).

Proof. The proof is by contradiction. Assuming that \( S \neq \Omega \), let \( \theta_0 \) be a point on the boundary of \( S \). Given \( \varepsilon > 0 \) sufficiently small, take \( y \in \Omega \) such that the interval \( I_{\varepsilon/2} \equiv [y - \frac{\varepsilon}{2}, y + \frac{\varepsilon}{2}] \) is contained in \( S \) and \( |\theta_0 - y| = \varepsilon \). Now, let \( \psi \) be a compactly supported test function as in (19), and let

\[
\psi^\varepsilon(\theta) = 2\varepsilon^{-1} \psi(2\varepsilon^{-1}(\theta - y)).
\]

This function then has support in \( I_{\varepsilon/2} \) as in Figure 2. Finally, define \( \phi^\varepsilon = \psi^\varepsilon / u^n \), which is as smooth as \( u \), by the conditions on \( \varepsilon \) and \( y \).

Plugging \( \phi^\varepsilon \) into (24), we see:

\[
|\omega| \int_{I_{\varepsilon/2}} u^{1-n} \phi^\varepsilon \, d\theta \leq \int_{I_{\varepsilon/2}} |\psi^\varepsilon_\theta| \left| u_{\theta\theta} + u - \sin \theta \right| d\theta
\]

\[
\leq ||\psi^\varepsilon_\theta||_{L^2(I_{\varepsilon/2})} \left| u_{\theta\theta} + u - \sin \theta \right|_{L^2(I_{\varepsilon/2})}.
\]  

(25)
Since \( u \lesssim \varepsilon^{\frac{3}{2}} \) on the interval and \( n > 1 \), we can bound the integral on the left from below:

\[
\int_{I_{\varepsilon/2}} u^{1-n} \psi \, d\theta \gtrsim \varepsilon^{\frac{3}{2}(1-n)}.
\]  
(26)

By the construction of \( \psi \), \( \| \psi \|_{L^2(I_{\varepsilon/2})} \lesssim \varepsilon^{-3/2} \). Therefore,

\[
|\omega| \lesssim \varepsilon^{\frac{3}{2}(n-2)} \|u_{\theta\theta} + u - \sin \theta\|_{L^2(I_{\varepsilon/2})}
\]  
(27)

For \( n > 2 \), we immediately observe that since \( \varepsilon > 0 \) is arbitrary, \( \omega = 0 \). For \( n = 2 \), we know that since \( (u_{\theta\theta})^2 \) and \( u^2 \) are measurable,

\[
\|u_{\theta\theta} + u - \sin \theta\|_{L^2(I_{\varepsilon/2})} \to 0
\]
as \( \varepsilon \to 0 \), which implies \( \omega = 0 \) in this case.

\[\text{Figure 2. In the proof of Proposition 2, we choose a smooth, compactly supported function } \psi \text{ with support contained in the support of } u, \text{ but close to a zero of } u.\]

3. **A computational examination of steady state solutions.** While we have given a simple proof that when \( \omega \neq 0 \) and \( n \geq 2 \) steady state solutions must be positive, how their profiles change with \( \omega \) requires computation. In this section, we develop an iterative spectral scheme, inspired by the fixed point analysis of [18], to compute the solutions in a small amplitude regime.

3.1. **Description of algorithm.** In what follows, we focus on the case \( n = 3 \), although it can be easily adapted to other values of \( n \). We seek to rewrite (3) in the form

\[
u = L^{-1} f(u) + g(\theta)
\]  

where \( L \) is a linear operator, \( f \) contains the nonlinear terms and \( g \) is a driving term. This motivates the iteration scheme

\[
u^{(j+1)} = L^{-1} f(u^{(j)}) + g(\theta).
\]
To begin, we divide (3) by \( u^3 \), to obtain
\[
(\partial_\theta^3 + \partial_\theta) u = \sin \theta + qu^{-3} - \omega u^{-2}.
\]
As \( \partial_\theta^3 + \partial_\theta \) has a non trivial kernel, the lefthand side cannot be inverted without careful projection. Pukhnachev, [18], resolved this problem by adding and subtracting appropriate terms to the equation. First, we introduce the variable \( v \),
\[
v \equiv u - \frac{q}{\omega}.
\]
Then
\[
Lv \equiv (\partial_\theta^3 + \partial_\theta + \frac{\omega^4}{q^4}) v = \sin \theta - \frac{\omega v}{(v + \frac{q}{\omega})^3} + \frac{\omega^4}{q^4} v
\]
\[
= \sin \theta + \frac{\omega^5}{q^3} v^2 \frac{3q^2 + 3q \omega v + \omega^4 v^2}{(q + \omega v)^3} \equiv \sin \theta + F(v).
\]
The right hand side of (29) contains a driving term, \( \sin \theta \), and a nonlinear term of order \( O(v^2) \). Rearranging (29),
\[
v = \frac{q^3}{\omega^4} \sin \theta + L^{-1} F(v)
\]
We now have our iteration scheme
\[
v^{(j+1)} = \frac{q^3}{\omega^4} \sin \theta + L^{-1} F(v^{(j)}).
\]
In the Fourier domain, this corresponds to
\[
\hat{v}_k^{(j+1)} = \frac{q^3}{\omega^4} \frac{\delta_{k,1} - \delta_{k,-1}}{2i} + \frac{\hat{F}_k^{(j)}}{-ik^3 + ik + \frac{\omega^4}{q^4}}, \quad F^{(j)} = F(v^{(j)})
\]
Plots of solutions computed using this algorithm appear in Figures 3 and 4.

We note that any change of variables of the form \( v = u + \alpha \) as in (28) would yield an operator with trivial kernel, allowing us to proceed as above. However, for our specific choice of \( \alpha \), the denominator appearing in the definition of \( F \) is simply \( \omega u \). By the results in section 2, we expect that this will be uniformly positive, so the above operator should be relatively well-behaved.

3.2. Performance of the spectral iterative method. When our algorithm successfully converges, it does so quite rapidly, as shown in Figure 5, where we plot the error between iterates \( v^{(j)} \) and the final iterate. We see that the error is proportional to \( e^{-\alpha N} \), although it is clear that the rate, \( \alpha \), varies with \( q \) and \( \omega \). For other values, related to the threshold (11), our solution diverges. While this will be further discussed in Section 4, we believe this to be closely related to the transition between the “small amplitude regime”, and a fundamentally nonlinear regime; see (9).

The accuracy of our approximation is quite good, even using a modest number of grid points. Figure 6, shows how quickly the maximum and minimum of the numerical solution converge to one with \( 10^5 \) grid points as the number of grid points increases. Again, solutions with values of \( q \) and \( \omega \) closer to (11) converge less rapidly; the solution in black is much closer to this threshold than the other three.
3.3. **Mass constraints and non-uniqueness.** Given values of $q$ and $\omega$, our algorithm successfully converges to non-trivial solutions. However, for a given value of $q$ and $\omega$, there may be more than one solution of (3). To study this phenomenon, it is helpful to introduce the mass of a solution

$$\mathcal{M} \equiv \int_{-\pi}^{\pi} u(\theta)d\theta. \quad (33)$$

Benilov, Benilov & Kopteva, [5], fixed $\omega$ and numerically explored the relationship between the flux and the mass as they varied. They found non-uniqueness of the solutions, in the sense that for a certain range of $\mathcal{M}$, there were multiple solutions with different values of $q$. Alternatively, for a certain range of $q$, there were multiple solutions with different values of $\mathcal{M}$. See Figure 14 of their work for a phase diagram, and see our Figures 9 and 10 for similar diagrams.

This non-uniqueness affects our algorithm. For a given $(\omega, q)$, the solution that the method converges to may not be the solution we desire, as the following example
shows. Using AUTO, [12], if we seek a solution with $M = 1$ and $\omega = .09$, we will obtain the solution pictured in Figure 7 (a). This solution has flux parameter $q = 0.0114039$.

Suppose we now fix $\omega = .09$, and continue the solution by reducing the mass value $M$, while permitting the flux parameter to vary as necessary. As seen Figure 7 (b), there will be a second solution, with the same $q$, but slightly smaller mass. For $q = 0.01140392$, the second solution is at $M = 0.912782$. Thus, for a given $(\omega, q)$, there may be multiple solutions, each with a different mass– which solution does the iterative spectral algorithm converge to? As Figure 7 (a) shows, it converges to the solution with the smaller mass. The agreement between the AUTO solution and our method’s solution is quite good; after the AUTO solution is splined onto the regular grid of the spectral solution, the pointwise error is $O(10^{-7})$. 

Figure 5. $L^\infty$ error between approximate solutions and the final iterate, at 512 grid points, computed to a tolerance of $10^{-10}$.

(a) The error in the maximum of the solution. (b) The error in the minimum of the solution.

Figure 6. Convergence of the iterative spectral method, as measured by an error in the minimum and maximum, as the grid size is increased.
The inability of our algorithm to recover the $M = 1$ solution can be traced to the development of a linear instability in the iteration scheme. Let $v^{(j)} = v + p^{(j)}$, where $v$ is a solution, and $p^{(j)}$ is a perturbation at iterate $j$. If we linearize (31) about $v$, we have

$$p^{(j+1)} = L^{-1} \left[ \frac{\omega^5 6q^3 + 6q^2 \omega v + 4q \omega v^2 + \omega^3 v^3}{q^3} (q + \omega v)^4 \right] \equiv L_v p^{(j)}.$$  

If the spectrum of $L_v$ lies strictly inside the unit circle, the perturbation will vanish, and the solution $v$ will be stable with respect to the iteration. However, if there are points in the spectrum of this operator which lie outside the unit circle, the solution will be unstable to the iteration, and we would expect our algorithm to diverge.

Though we shall not attempt to prove properties of the spectrum of $\tilde{L}$, we can discretize our problem and compute eigenvalues of the induced matrix. Computing the eigenvalues of a discrete approximation of $L_v$ at the two AUTO solutions, we find that the matrix corresponding to the $M = 1$ solution does, in fact, have an eigenvalue outside the unit circle, $\lambda_{\text{unstable}} = 1.017059$; see Figure 8. The discretized operator is defined as

$$\tilde{L}_v = \left(D^3 + D + \frac{\omega^4}{q^4} I\right)^{-1} \text{diag} \{F'(v)\}.$$  

with the differentiation matrix $D$, a dense Toeplitz matrix for band limited interpolants, as in [21]. The value of the unstable eigenvalue converges rapidly as we increase the number of grid points, as shown in Table 1.

An interesting question to ask is if this instability is related to an instability in (1) for the steady state solutions. Linearizing (1) about a steady state, a perturbation
Figure 8. The eigenvalues of the discretized $\tilde{L}_v$ for the two solutions computed using AUTO with $\omega = 0.09$ and $q = 0.0114039$. Note that for the $M = 1$ solution, there is an eigenvalue outside the unit circle at $\lambda = 1.017059$. Computed with 2048 grid points.

Table 1. Values of the unstable eigenvalue of $\tilde{L}_v$ for the $\omega = 0.09$, $q = 0.0114039$ and $M = 1$ solution as a function of the number of grid points.

| No. of Grid Points | $\lambda_{\text{unstable}}$ |
|--------------------|-----------------------------|
| 16                 | 1.01711308844               |
| 64                 | 1.01705921834               |
| 256                | 1.01705919446               |
| 2048               | 1.01705919124               |

$p$ evolves according to

$$\partial_t p = \partial_\theta \left\{ u^3 \left[ p \left( \frac{\omega^4}{q^3} - 3(qu - u) - \omega u^3 - \omega u^3 - (\partial_\theta^3 p + \partial_\theta p + \frac{\omega^4}{q^3} p) \right) \right] \right\}$$

$$= \partial_\theta \left\{ u^3 \left[ F'(v)p - Lp \right] \right\}$$

$$= \partial_\theta \left\{ u^3 L \left[ L^{-1} F'(v)p - p \right] \right\},$$

where $L, F,$ and $v$ are defined above. Noting the definition of $L_v$ in (34), (36) may be written as

$$\partial_\lambda p = \partial_\theta \left\{ u^3 (L_v - I)p \right\} \equiv H_u p.$$

Unfortunately, it is unclear how the spectrum of $L_v$, an integro-differential operator related to a third order operator, connects to the spectrum of $H_u$, a fourth order operator.

Finally, we note that the $(M, q)$ phase space of solutions with fixed $\omega$ can be more tortuous than that shown in Figure 7 (b). Indeed, as we send $\omega \to 0$, a singular limit, there may be not just multiple solutions of different mass and the same flux, but also multiple solutions of different flux and the same mass. Such
The \((M,q)\) phase space of solutions with rotation parameter \(\omega = 5 \times 10^{-6}\). Note that there can be as many as six solutions of different mass and the same flux, and as many as three solutions of different flux and the same mass. Computed using AUTO.

The \((M,\omega)\) phase space of solutions with flux parameter \(q = 5 \times 10^{-9}\). As was the case in Figure 9, there can be multiple solutions with different rotation speeds and the same mass, and multiple solutions with the same rotation speeds and different mass. Computed using AUTO.

Examples appear in Figures 9 and 10. These cases are more extensively explored in [5] and in Badali et al, [3].

4. **Non-existence results.** As mentioned in the introduction, in [10], the authors prove that solutions to (3) do not exist for all \((\omega,q)\) pairs. Indeed, they show there are no solutions when (11) holds. Interestingly, this relation is stated entirely in terms of \(q\) and \(\omega\), and makes no consideration of the mass, \(M\).

We now explore, numerically, (11). For fixed mass, we compute the associated solution at a variety of \(\omega\) values, determining \(q\). The results are appear in Figure 11. As \((\omega,q)\) cannot uniquely determine \(u\), we rely on AUTO and its continuation algorithms, rather than our own iterative scheme.
Examining the figure, we remark on the two features found in each fixed mass curve. Far to the right of the non-existence line, the $\omega q$ relationship appears to be linear. Indeed, for moderate mass values, the relationship appears to be of the form (4). It is in this “small amplitude” regime that our iterative spectral algorithm succeeds.

As $\omega$ continues to decrease, the scaling changes to accommodate (11), and appears to follow a curve like $q \propto \omega^\beta$, for a value of $\beta > 3/2$. For all masses, the deviation from (11) is less than an order of magnitude. Thus, there is not an obvious small parameter from which to formulate a series expansion. We also remark that there are many intersections between the curves of constant mass; the lack of uniqueness of a solution for a given $(\omega, q)$ is quite common. As $\omega$ and $q$ approach this highly nonlinear regime, our algorithm becomes limited. As discussed in the preceding section unstable eigenvalues can appear, precluding convergence to solutions with a desired mass, and the rate of convergence slows. Interestingly, if we extend the axes in Figure 11 further southwest, so as to include the parameters appearing in 9 and 10, the plot is essentially unchanged. While the curves are closer together
than the region corresponding to Figure 7, there is nothing in Figure 11 suggesting multiple solutions with the same mass.

Turning to Figure 12, we see the tendency of the solutions to form singularities, vertical asymptotes, as \( \omega \) tends towards zero for fixed \( q \). These were computed using our spectral iterative algorithm; the asymptotes occurred where the algorithm failed to converge in a reasonable number of iterations. As these figures show, it is possible to get quite close to the theoretical threshold of (11). This suggests that the constraint may be sharp, and this warrants further investigation. Even if the constant, \((2/3)^{2/3}\), is not correct, the exponent \(3/2\) appears to be sharp.

![Graphs showing thickness of solutions with grid size of \(2^{10}\) as \(\omega\) varied.](image)

(a) \( q = 1 \times 10^{-3}\). The mass of the final solution was 0.3951

(b) \( q = 1 \times 10^{-4}\). The mass of the final solution was 0.1612

**Figure 12.** Thickness of solutions with a grid size of \(2^{10}\) as \(\omega\) is varied. The dashed lines denote the observed threshold at which our spectral iterative algorithm fails and the theoretical threshold, (11) beyond which no solutions should exist. Computed using the spectral iterative method.

5. **Discussion.** In this work, we have given a simple proof of the positivity of steady state solutions of a thin film equation with rotation, subject to modest assumptions, provided \( n \geq 2 \). It may be possible to employ additional regularity properties of the steady state solution to lower this threshold; we only relied on Sobolev embeddings in our result.

We have also formulated an easy to implement algorithm for computing the steady state solutions for a given \((\omega,q)\) pair. However, in cases where multiple solutions are known to exist, an instability can develop about one of the solutions, precluding convergence to other solutions. We conjecture that the preference is related to the proximity to the non-existence curve, where the parameters have left the small amplitude regime. Finding a way to stabilize the spectral algorithm in this regime remains an open problem.

It may appear that our spectral algorithm is of limited use, applicable only in a small amplitude regime. But it should prove helpful in the computational benchmarking of other algorithms, and for precomputing steady states for time dependent simulations. Within the parameter regime where it applies, its advantage is its simplicity. Indeed, we found it to be more manageable than AUTO, which required us
to first perform continuation from a known analytic solution. We resolved this by beginning with the problem
\[ u^3(u_{\theta\theta} + u_\theta + \alpha \sin \theta) + \omega u = q, \tag{38} \]
which has the constant solution \( u = q/\omega \) at \( \alpha = 0 \) for any \( q \) and \( \omega \neq 0 \). The particular \( q \) and \( \omega \) are selected so as not to violate (11), and we then perform continuation in \( \alpha \) up till \( \alpha = 1 \). From that point, we can explore the \((q, \omega, M)\) phase space of solutions.

Furthermore, the spectral algorithm may be thought of as a numerical implementation of the fixed point analysis in [18]. Under this interpretation, the failure of the iteration algorithm to compute the \( M = 1 \) solution due to the linear instability (Figure 8) suggests that this fixed point analysis cannot be used to obtain the \( M = 1 \) solution. However, there is abundant numerical evidence for its existence. Thus, an outstanding problem is to develop an analysis of the steady states outside of this small amplitude regime.

Finally, our computations suggest further exploration of the non-existence curve is warranted. Figures 11 and 12 suggest that the nonexistence relationship may be sharp. Furthermore, the phase diagram indicates that this inequality separates two physical regimes. In the first, where \( \omega \) and \( q \) are nearly linearly related, the solution is nearly constant, and can be approximated by (9). In order to accommodate the nonexistence constraint, the phase diagram bends, and we enter a regime where the parameters are nonlinearly related, and the steady state solutions cease to be well described by unimodal bumps.

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