On the prime geodesic theorem for hyperbolic 3-manifolds

Muharem Avdispahić

University of Sarajevo, Department of Mathematics, Zmaja od Bosne 33-35, 71000 Sarajevo, Bosnia and Herzegovina

Correspondence
Muharem Avdispahić, University of Sarajevo, Department of Mathematics, Zmaja od Bosne 33-35, 71000 Sarajevo, Bosnia and Herzegovina.
Email: mavdispa@pmf.unsa.ba

Abstract
Through the Selberg zeta approach, we reduce the exponent in the error term of the prime geodesic theorem for cocompact Kleinian groups or Bianchi groups from Sarnak’s $5/3$ to $3/2$. At the cost of excluding a set of finite logarithmic measure, the bound is further improved to $13/9$.

KEYWORDS
hyperbolic 3-manifolds, prime geodesic theorem, Selberg zeta function

MSC (2010)
11M36, 57M50, 58J50

1 | INTRODUCTION

Let $\mathbb{H}^3$ denote the 3-dimensional hyperbolic space and let $\Gamma$ be a cofinite subgroup of $PSL(2, \mathbb{C})$. The quotient $\Gamma \backslash \mathbb{H}^3$ is a 3-dimensional hyperbolic manifold of finite volume. The prime geodesic theorem in this setting says that the number $\pi_\Gamma(x)$ of prime geodesics $P$ with the length $l(P) \leq \log x$ equals

$$\pi_\Gamma(x) = li(x^2) + \sum_{n=1}^{M} li(x^{s_n}) + E(x), \quad (1.1)$$

where $s_1, \ldots, s_M$ are the real zeros of the Selberg zeta function $Z_\Gamma$ lying in the interval $(1, 2)$ and $E(x)$ is the error term.

For groups of the form $\Gamma = \Gamma_D = PSL(2, \mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-D})$ of class number one, Sarnak [18] proved

$$\pi_\Gamma(x) = li(x^2) + O\left(x^{5/3 + \varepsilon}\right). \quad (1.2)$$

In the particular case of $\Gamma = PSL(2, \mathbb{Z}[i])$, Koyama [12] obtained

$$\pi_\Gamma(x) = li(x^2) + O\left(x^{11/14 + \varepsilon}\right)$$

under the mean-Lindelöf hypothesis.

While Bianchi groups are noncompact, the fact that the contribution of the continuous spectra is dominated by the contribution of the discrete spectra enables one to derive (1.2) for these groups as well. The additional ingredient for achieving (1.2) is the knowledge of the lower bound for the first eigenvalue of the Laplace–Beltrami operator in the respective setting.

Using the explicit formula for the integrated Chebyshev functions of an appropriate order, we shall decrease the exponent in the error term $E(x)$. 

Theorem 1.1. Let $\Gamma \subset PSL(2, \mathbb{C})$ be a cocompact group or a noncompact cofinite group that satisfies the condition

$$\sum_{\gamma_n > 0} \frac{x^{\beta_n - 1}}{\gamma_n^2} = O\left(\frac{1}{1 + (\log x)^3}\right) \quad (x \to \infty),$$

(1.3)

where $\beta_n + i\gamma_n$ are poles of the scattering determinant. Then,

$$\pi_\Gamma(x) = li(x^2) + \sum_{n=1}^M li(x^{s_n}) + O\left(\frac{x^3}{\log x}\right) \quad (x \to \infty).$$

Corollary. If $\Gamma$ is a Bianchi group, then

$$\pi_\Gamma(x) = li(x^2) + O\left(\frac{x^3}{\log x}\right) \quad (x \to \infty).$$

The bound in Theorem 1.1 and Corollary is the 3-dimensional analogue of Randol's $O\left(\frac{x^3}{\log x}\right)$ in the prime geodesic theorem for Riemann surfaces [17]. If $\Gamma \subset PSL(2, \mathbb{R})$ is a cocompact Fuchsian group (or, for that matter, a noncompact cofinite group satisfying an analogue of (1.3) [10, p. 477]), it is possible to reduce the exponent in Randol's estimate to $\frac{7}{10}$ outside a set of finite logarithmic measure [1]. Under the generalized Lindelöf hypothesis, one can reach $\frac{5}{8}$ in the case of $PSL(2, \mathbb{Z})$, i.e., come half a way between $\frac{3}{4}$ and the expected exponent $\frac{1}{2}$, outside a set of finite logarithmic measure [2]. For a simple proof of the bound $\frac{2}{3}$ in the latter case without Lindelöf hypothesis, see [3].

Such Gallagherian approach to prime geodesic theorems leads to the following result in our 3-dimensional setting.

Theorem 1.2. Let $\Gamma \subset PSL(2, \mathbb{C})$ be a cocompact group or a noncompact congruence group for some imaginary quadratic number field. Then there exists a set $E$ of finite logarithmic measure such that

$$\pi_\Gamma(x) = li(x^2) + \sum_{n=1}^M li(x^{s_n}) + O\left(\frac{x^{12}}{(\log x)^7} (\log \log x)^{\frac{7}{6}}\right),$$

as $x \to \infty$, $x \notin E$.

2 | PRELIMINARIES

We use the upper half-space model

$$\mathbb{H}^3 = \{(x, y, r) \in \mathbb{R}^3 \mid r > 0\} = \{(z, r) \mid z \in \mathbb{C}, r > 0\} = \{z + rj \mid r > 0\}$$

with the hyperbolic metric $ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}$ and volume form $dv = \frac{dx dy dr}{r^3}$. The Laplace–Beltrami operator is defined by

$$\Delta = -r^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2} \right) + r \frac{\partial}{\partial r}.$$ 

The group $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm I\}$ is the group of orientation preserving isometries of $\mathbb{H}^3$. It acts on $\mathbb{H}^3$ transitively by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (v) = \frac{(az + b)(cz + d) + a\bar{c}r^2 + rj}{|cz + d|^2 + |c|^2r^2},$$

where we put $v = z + rj$.

Discrete subgroups of $PSL(2, \mathbb{C})$ are known as Kleinian groups. We shall consider cocompact $\Gamma \subset PSL(2, \mathbb{C})$ as well as a certain class of noncompact cofinite $\Gamma$ in which cases the quotient space $\Gamma \backslash \mathbb{H}^3$ is a compact respectively finite volume hyperbolic 3-manifold.
If the trace $tr(P)$ of $P \in \Gamma \setminus \{I\}$ is real, $P$ is called hyperbolic, parabolic or elliptic depending whether $|tr(P)|$ is larger, equal or less than 2. In all other cases, $P$ is loxodromic.

Every hyperbolic or loxodromic $P \in \Gamma$ is conjugate in $PSL(2, \mathbb{C})$ to a unique element

$$\begin{pmatrix} a(P) & 0 \\ 0 & a(P)^{-1} \end{pmatrix}; \quad |a(P)| > 1.$$\hspace{1cm} (2.1)

The norm of $P$ is defined by $N(P) = |a(P)|^2$. For $P$ there exist exactly one primitive hyperbolic or loxodromic $P_0 \in \Gamma$ and exactly one $n \in \mathbb{N}$ such that $P = P_0^n$.

Based on a correspondence between conjugate classes of $\Gamma$ and free homotopy classes of closed continuous paths on $\Gamma \setminus \mathbb{H}^3$ (see, e.g., [5] for necessary details), we are interested in the number $\pi_\Gamma(x)$ of primitive hyperbolic or loxodromic conjugate classes $P_0$ with the norm $N(P_0) \leq x$, i.e., we are interested in $\pi_\Gamma(x) = \sum_{N(P_0) \leq x} 1$. This resembles the situation with the problem of distribution of prime numbers $\pi(x) = \sum_{p \leq x} 1$ that led Riemann to introduce his famous zeta function.

The Selberg zeta function $Z_\Gamma$ is defined by

$$Z_\Gamma(s) = \prod_{P_0} \prod_{k,l} \left( 1 - a(P_0)^{-2k} \overline{a(P_0)^{-2l}} N(P_0)^{-s} \right), \quad \text{Re}s > 2,$$

where the first product is over all primitive hyperbolic or loxodromic conjugacy classes of $\Gamma$ and the second product is over all pairs of nonnegative integers such that $k \equiv l \pmod{m(P_0)}$, $m(P)$ denoting the order of the torsion of the centralizer of $P$.

If $\Gamma$ is cocompact, the functional equation for $Z_\Gamma$ reads (see [5, Cor. 4.4 on p. 209])

$$Z_\Gamma(2-s) = \exp\left(-\frac{\text{vol}(\Gamma \setminus \mathbb{H}^3)}{3\pi}(s-1)^3 + E(s-1)\right) Z_\Gamma(s),$$

where $E = \sum_{\rho \in \rho(R)} \frac{\log N(P_\rho)}{m(R)|tr(\rho)|^2 - 4}$, the sum being taken over all elliptic conjugacy classes of $\Gamma$.

In general case, one has (see [8, Theorem 4.4])

$$Z_\Gamma(2-s) = Z_\Gamma(s) \left( \frac{\Gamma(2-s)}{\Gamma(s)} \right)^{4\kappa h_\Gamma} \left[ \varphi(1-s) \right]^{4\kappa} \prod_{k=1}^{l} \left( \frac{s-1-q_k}{1-s-q_k} \right)^{4\kappa b_k} \cdot \exp \left[ \int_0^{s-1} 4\pi \kappa \text{vol}(\Gamma \setminus \mathbb{H}^3) t^2 dt + \kappa_1 (s-1) \right],$$

where $h_\Gamma$ is the number of cusps, $\kappa$ and $\kappa_1$ are certain constants, $q_k$ ($1 \leq k \leq l$) are the poles of the scattering determinant $\varphi$ in $(0,1]$ with order $b_k$.

(The standard symbol $\Gamma(s)$ for Euler's gamma function appearing in the last equation should not cause any confusion with the notation for the group.)

The relationship between zeros of $Z_\Gamma$ and the discrete spectrum of the Laplace–Beltrami operator on $L^2(\Gamma \setminus \mathbb{H}^3)$ is given by $s(2-s) = \lambda$. The latter operator being essentially self-adjoint, one has that $\lambda_n \nearrow +\infty$ ($n \to +\infty$). So, there are finitely many zeros $s_0, \ldots, s_M$ lying in $(1,2)$ ($\bar{s}_0, \ldots, \bar{s}_M$ in $[0,1]$). All others $s_n = 1 \pm 2t_n$, $t_n > 0$, corresponding to discrete eigenvalues $\lambda_n > 0$, are on the critical line $\text{Re}(s) = 1$.

This serves as a ground for the expectation that $E(x) = O(x^{1+\varepsilon})$ in (1.1). However, the density of zeros of $Z_\Gamma$ has prevented all the efforts in establishing the analogue of von Koch's theorem [11, p. 84] for Riemann surfaces or higher dimensional manifolds.

For groups considered in this paper, discrete eigenvalues are distributed according to

$$N_\Gamma(T) = \# \{ \lambda_n : \lambda_n \leq 1 + T^2 \} \sim \frac{\text{vol}(\Gamma \setminus \mathbb{H}^3)}{6\pi^2} T^3.$$\hspace{1cm} (2.1)
Namely, (2.1) is the Weyl law for cocompact $\Gamma$. If $\Gamma$ is noncompact cofinite and satisfies (1.3), then \[ \int_{-T}^{T} \frac{q'}{q} (1 + it) \, dt = o(T^2) \], what in combination with the extended Weyl law [5, Thm. 5.4. on p. 307]

\[ N_\Gamma(T) - \frac{1}{4\pi} \int_{-T}^{T} \frac{q'}{q} (1 + it) \, dt \sim \frac{\nu \left( \Gamma \backslash \mathbb{H}^{3} \right)}{6\pi^{2}} T^{3} \]

yields (2.1).

An analogue of the classical von Mangoldt function is given by

\[ \Lambda_\Gamma(P) = \frac{N(P) \log N(P_0)}{m(R) |a(P) - a(P)^{-1}|^2}, \]

where $P_0$ is a primitive element associated to $P$.

Similar to the Riemann zeta case, a convenient tool to study the distribution of prime geodesics is provided by the Chebyshev function

\[ \psi_{0,\Gamma}(x) = \sum_{N(P) \leq x} \Lambda_\Gamma(P) \]

and its integrated versions $\psi_{1,\Gamma}(x) = \int_0^x \psi_{0,\Gamma}(t) \, dt, \psi_{n,\Gamma}(x) = \int_0^x \psi_{n-1,\Gamma}(t) \, dt$.

Explicit formulas with an error term for appropriately integrated Chebyshev function are the starting ground in our proofs.

### 3 PROOF OF THEOREM 1.1

Let $\Gamma$ be a cofinite group satisfying the condition (1.3), Hejhal's technique [9, pp. 85–86 and pp. 103–110], when transferred from the Riemann surfaces case to the present setting, gives the explicit formula for $\psi_{3,\Gamma}$ in the following form

\[ \psi_{3,\Gamma}(x) = A_0 x^3 + A_1 x^3 \log x + A_2 x^2 + A_3 x + A_4 \]

\[ + \sum_{n=0}^{M} x_{s_n}^3 + \sum_{n=0}^{M} x_{s_n + 3}^3 + \sum_{n=0}^{M} x_{s_n + 1}^3 (s_n + 2) (s_n + 3) \]

\[ + \sum_{0 \leq s_n \leq T} x_{s_n}^3 (s_n + 1) (s_n + 2) (s_n + 3) + \sum_{0 \leq s_n \leq T} x_{s_n + 1}^3 (s_n + 2) (s_n + 3) \]

\[ + O \left( \frac{x^3}{T} \right) + \sum_{s_n \geq 0} x_{\rho_n}^3 (\rho_n + 1) (\rho_n + 2) (\rho_n + 3) + \sum_{\tilde{\rho}_n \geq 0} x_{\tilde{\rho}_n}^3 (\tilde{\rho}_n + 1) (\tilde{\rho}_n + 2) (\tilde{\rho}_n + 3) \]

(cf. the calculations yielding to [15, Thm. 5.4]. These are not affected by the fact that $\Omega_\pm$ estimates in [15] seem to remain unproved).

Here $s_n = 1 + it_n, \tilde{s}_n = 1 - it_n$ are the zeros of $Z_\Gamma$ coming from the discrete spectrum and $\rho_n = \beta_n + i\gamma_n, \tilde{\rho}_n = \beta_n - i\gamma_n$ are those from the continuous spectrum.

If $\Gamma$ is a cocompact group, then the last two sums in the explicit formula above are obviously void.

The asymptotics of a nonnegative nondecreasing function $\psi_{0,\Gamma}$ can be easily derived from the asymptotics of $\psi_{3,\Gamma}$. One introduces the functions

\[ \Delta_{\pm}^+ f(x) = \int_{x}^{x+h} \int_{t}^{t+h} f''''(u) \, du \, dv \, dt = f(x + 3h) - 3f(x + 2h) + 3f(x + h) - f(x) \]

and

\[ \Delta_{\pm}^- f(x) = \int_{x-h}^{x} \int_{t-h}^{t} f''''(u) \, du \, dv \, dt. \]
By the mean value theorem, we get
\[ h^{-3} \Delta^+_3 \sum_{n=0}^{M} \frac{x^{s_n + 3}}{s_n (s_n + 1)(s_n + 2)(s_n + 3)} = \sum_{n=0}^{M} \frac{x^{s_n}}{s_n} + O(h^2). \]

Further, for the zeros on the critical line, we have
\[ h^{-3} \Delta^+_3 \sum_{|s_n| \leq T} \frac{x^{s_n + 3}}{s_n (s_n + 1)(s_n + 2)(s_n + 3)} = O\left( x \sum_{|s_n| \leq T} \frac{1}{s_n} \right) = O(xT^2). \]

Now,
\[ h^{-3} \Delta^+_3 \sum_{\nu_n \geq 0} \frac{x^{\rho_n + 3}}{\rho_n (\rho_n + 1)(\rho_n + 2)(\rho_n + 3)} = O\left( \frac{1}{h} \sum_{\nu_n \geq 0} \frac{x^{\rho_n + 1}}{|\rho_n||\rho_n + 1|} \right) = O\left( \frac{x^2}{h} \right) \]
by (1.3).

Thus,
\[ \psi_{\omega, \Gamma} (x) \leq h^{-3} \Delta^+_3 \psi_{3, \Gamma} (x) = \sum_{n=0}^{M} \frac{x^{s_n}}{s_n} + O(h^2) + O(xT^2) + O\left( \frac{x^4}{h^3 T} \right) + O\left( \frac{x^2}{h} \right). \]

The optimal choice for the first three \(O\)-summands on the right-hand side is \( h \sim x^{\frac{3}{2}}, T \sim x^{\frac{1}{2}} \). The fourth term \( O\left( \frac{x^2}{h} \right) \) is dominated by the obtained bound \( O\left( \frac{x^{\frac{1}{2}}}{h} \right) \).

The opposite direction, \( \psi_{\omega, \Gamma} (x) \geq h^{-3} \Delta^+_3 \psi_{3, \Gamma} (x) \), is treated analogously and yields the same result.

Hence,
\[ \psi_{\omega, \Gamma} (x) = \frac{x^2}{2} + \sum_{n=1}^{M} \frac{x^{s_n}}{s_n} + O\left( \frac{x^{\frac{3}{2}}}{h} \right) (x \to \infty). \]

As well-known, the latter relation implies
\[ \pi_{\Gamma} (x) = li(x^2) + \sum_{n=1}^{M} li(x^{s_n}) + O\left( \frac{x^{\frac{3}{2}}}{\log x} \right) (x \to \infty). \]

Proof of Corollary. The groups \( PSL (2, O_K) \) satisfy the condition (1.3). Luo, Rudnick and Sarnak [14] proved that the smallest eigenvalue \( \lambda_1 \) is bounded from below by \( \frac{171}{196} \). It was further increased to \( \frac{160}{169} \) by Koyama [13]. This implies \( \sum_{n=1}^{M} \frac{x^{s_n}}{s_n} = O\left( x^{\frac{16}{11}} \right) \) what is obviously less than \( x^{\frac{3}{2}} \). Thus, the prime geodesic theorem takes the form
\[ \pi_{\Gamma} (x) = li(x^2) + O\left( \frac{x^{\frac{3}{2}}}{\log x} \right) (x \to \infty). \]

\[ \square \]

4 \ | PROOF OF THEOREM 1.2

To reduce the exponent in the error term of the prime geodesic theorem, we need a better control of the growth of \( \sum_{|s_n| \leq T} \frac{x^{s_n}}{s_n} \). One Gallagerian way to progress in this direction is related to exclusion of a set of finite logarithmic measure, as initially demonstrated in the classical Riemann zeta setting [7].
In contrast to the situation of the previous Section, the appropriate starting point here is the explicit formula for \( \psi_{2, \Gamma} \) with an error term

\[
\psi_{2, \Gamma}(x) = A_0 x^2 + A_1 x^2 \log x + A_2 x + A_3 + \sum_{n=0}^{M} \frac{x^{s_n+2}}{s_n (s_n+1) (s_n+2)} + \sum_{n=0}^{M} \frac{x^{\widetilde{s}_n+2}}{\widetilde{s}_n (\widetilde{s}_n+1) (\widetilde{s}_n+2)} + \sum_{0 \leq \tau_n \leq T} \frac{x^{\rho_n+2}}{\rho_n (\rho_n+1) (\rho_n+2)} + \sum_{\gamma_n \geq 0} \frac{x^{\tilde{\rho}_n+2}}{\tilde{\rho}_n (\tilde{\rho}_n+1) (\tilde{\rho}_n+2)} + O\left(\frac{x^5}{T}\right).
\]

Let \( n = \lfloor \log x \rfloor \) and take \( \varepsilon > 0 \) arbitrarily small. For \( 0 < Y < T \), denote by \( E_n \) the set

\[
\left\{ x \in [e^n, e^{n+1}]: \left| \sum_{|t_n| \leq T} x^{s_n} / (s_n+1) (s_n+2) \right| > x^\alpha (\log x)^\beta (\log \log x)^{\beta + 2\varepsilon} \right\}.
\]

(5)

Then

\[
\mu^x E_n = \int_{E_n} \frac{dx}{x} \leq \int_{e^n}^{e^{n+1}} \left| \sum_{Y < |t_n| \leq T} x^{s_n} / (s_n+1) (s_n+2) \right|^2 \frac{dx}{x^{1+2\alpha} (\log x)^{2\beta} (\log \log x)^{2\beta + 4\varepsilon}}
\]

\[
\ll \frac{e^{n(6-2\alpha)}}{n^2 (\log n)^{2\beta + 4\varepsilon}} \int_{e^n}^{e^{n+1}} \left| \sum_{Y < |t_n| \leq T} x^{s_n} / (s_n+1) (s_n+2) \right|^2 \frac{dx}{x}.
\]

(4.1)

By the Gallagher lemma [6], the last integral is dominated by

\[
\int_{-\infty}^{+\infty} \left| \sum_{Y < |t_n| < t_n+1} 1 / s_n^3 \right|^2 dt.
\]

(4.2)

To proceed further, we need a more precise form of the Weyl law. If \( \Gamma \) is cocompact, then \( N_{\Gamma}(T) = \frac{\text{vol}(\Gamma \backslash \mathbb{H}^3)}{6\pi^2} T^3 + O\left(\frac{T^2}{\log T}\right) \) [4] (see also [5, Theorem 5.6. and remarks on pp. 228–289]). However, if \( \Gamma \) is a noncompact congruence subgroup for some imaginary quadratic number field, we still have \( N_{\Gamma}(T) = \frac{\text{vol}(\Gamma \backslash \mathbb{H}^3)}{6\pi^2} T^3 + O(T^2) \) according to [16]. Thus, \( N_{\Gamma}(T+1) - N_{\Gamma}(T) = O(T^2) \) in both cases. This yields

\[
\left| \sum_{Y < |t_n| < t_n+1} 1 / s_n^3 \right|^2 = O\left(\frac{1}{t_n^2}\right).
\]

Hence, the integral (4.2) is bounded by \( O\left(\frac{1}{Y}\right) \). Looking back at (4.1), we obtain

\[
\mu^x E_n \ll \frac{e^{n(6-2\alpha)}}{Y n^{2\beta} (\log n)^{2\beta + 4\varepsilon}}.
\]
Putting \( Y = e^{(6-2\alpha)n^{1-\beta}} (\log n)^{1-2\beta+\frac{\varepsilon}{2}} \), we get

\[
\mu^* E_n \ll \frac{1}{n (\log n)^{1+\varepsilon}}.
\]

Therefore, the set \( E = \bigcup E_n \) has a finite logarithmic measure.

Following the same line of argumentation as in Section 3, we derive

\[
\psi_{0,\Gamma}(x) \leq h^{-2} \Delta^+_2 \psi_{2,\Gamma}(x)
\]

\[
= \sum_{n=0}^M \frac{x^n}{s_n} + O \left( \frac{x^2}{h^2} \right) + \frac{1}{h^2} \left| \Delta^+_2 \sum_{|s_n| \leq \gamma} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} \right| + O \left( \frac{x^5}{h^2 T} \right) + \frac{1}{h} O \left( \sum_{|\rho_n| \leq \gamma} \frac{x^{\rho_n+1}}{|\rho_n|} \right),
\]

where

\[
\Delta^+_2 f(x) = f(x \pm 2h) - 2 f(x \pm h) + f(x) .
\]

Now, splitting \( \Delta^+_2 \sum_{|s_n| \leq \gamma} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} \) into

\[
\Delta^+_2 \sum_{|s_n| \leq \gamma} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \Delta^+_2 \sum_{|s_n| \leq \gamma} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)},
\]

we arrive at

\[
\psi_{0,\Gamma}(x) - \sum_{n=0}^M \frac{x^n}{s_n} \ll h^2 + \sum_{|s_n| \leq \gamma} \frac{x^{s_n+2}}{s_n(s_n+1)(s_n+2)} + \frac{x^5}{h^2 T} + \frac{x^2}{h^2} + \frac{x^2}{h}
\]

\[
\ll h^2 + xY^2 + \frac{x^5}{h^2 T} + \frac{x^2}{h} + \frac{x^2}{h}
\]

(4.3)

on the complement of \( E \). Recall that \( Y \sim x^{6-2\alpha} (\log x)^{1-2\beta} (\log \log x)^{1-2\beta+\frac{\varepsilon}{2}} \). Optimizing the first three summands on the right-hand side of (4.3), we get

\[
x^\alpha (\log x)^\beta (\log \log x)^{\beta+\varepsilon} = x^{13-4\alpha} (\log x)^{2-4\beta} (\log \log x)^{2-4\beta+\varepsilon}.
\]

This yields \( \alpha = \frac{26}{9} \) and \( \beta = \frac{4}{9} \). We get

\[
\psi_{0,\Gamma}(x) \leq \sum_{n=0}^M \frac{x^n}{s_n} + O \left( \frac{x^3}{\gamma} (\log x)^{\frac{2}{\gamma}} (\log \log x)^{\frac{\gamma+\varepsilon}{\gamma}} \right) + O \left( \frac{x^5}{h^2 T} \right) + O \left( \frac{x^2}{h} \right).
\]

It is easily checked that \( \frac{x^2}{h} \) is dominated by \( O \left( \frac{x^{15}}{h^{15}} \right) \). Finally, \( T > Y \) can be chosen so that \( O \left( \frac{x^5}{h^2 T} \right) \) does not affect the bound.

The same procedure applies to \( \psi_{0,\Gamma}(x) \geq h^{-2} \Delta^- \psi_{2,\Gamma}(x) \). Hence,

\[
\psi_{0,\Gamma}(x) = \sum_{n=0}^M \frac{x^n}{s_n} + O \left( \frac{x^{13}}{\gamma} (\log x)^{\frac{2}{\gamma}} (\log \log x)^{\frac{\gamma+\varepsilon}{\gamma}} \right) \quad (x \rightarrow \infty, x \notin E),
\]

what gives us

\[
\pi_{\Gamma}(x) = li(x^2) + \sum_{n=1}^M li(x^n) + O \left( \frac{x^{13}}{\gamma} (\log x)^{\frac{2}{\gamma}} (\log \log x)^{\frac{\gamma+\varepsilon}{\gamma}} \right),
\]

as \( x \rightarrow \infty, x \notin E \).
REFERENCES

[1] M. Avdispahić, Prime geodesic theorem of Gallagher type, arXiv:1701.02115.
[2] M. Avdispahić, Prime geodesic theorem for the modular surface, arXiv:1702.01699v2.
[3] M. Avdispahić, Gallagherian PGT on \( \text{PSL}(2, \mathbb{Z}) \), Funct. Approx. Comment. Math., http://doi.org/10.7169/facm.1686.
[4] J. J. Duistermaat, J. A. C. Kolk, and V. S. Varadarajan, Spectra of compact locally symmetric manifolds of negative curvature, Invent. Math. 52 (1979), no. 1, 27–93.
[5] J. Elstrodt, F. Grunewald, and J. Mennicke, Groups acting on hyperbolic space, Springer-Verlag, Berlin, Heidelberg, 1998.
[6] P. X. Gallagher, A large sieve density estimate near \( \sigma = 1 \), Invent. Math. 11 (1970), 329–339.
[7] P. X. Gallagher, Some consequences of the Riemann hypothesis, Acta Arith. 37 (1980), 339–343.
[8] R. Gangolli and G. Warner, Zeta functions of Selberg's type for some non-compact quotients of symmetric spaces of rank one, Nagoya Math. J. 78 (1980), 1–44.
[9] D. A. Hejhal, The Selberg trace formula for \( \text{PSL}(2, \mathbb{R}) \), vol. I, Lecture Notes in Math., vol. 548, Springer, Berlin, 1976.
[10] D. A. Hejhal, The Selberg trace formula for \( \text{PSL}(2, \mathbb{R}) \), vol. II, Lecture Notes in Math., vol. 1001, Springer, Berlin, 1983.
[11] A. E. Ingham, The distribution of prime numbers, Cambridge University Press, 1932.
[12] S. Koyama, Prime geodesic theorem for the Picard manifold under the mean-Lindelöf hypothesis, Forum Math. 13 (2001), 781–793.
[13] S. Koyama, The first eigenvalue problem and tensor products of zeta functions, Proc. Japan Acad. Ser. A 80 (2004), no. 5, 35–39.
[14] W. Luo, Z. Rudnick, and P. Sarnak, On Selberg's eigenvalue conjecture, Geom. Funct. Anal. 5 (1995), 387–401.
[15] M. Nakasuji, Prime geodesic theorem for hyperbolic 3-manifolds: general cofinite cases, Forum Math. 16 (2004), 317–363.
[16] M. R. Palm, Explicit \( GL(2) \) trace formulas and uniform, mixed Weyl laws, PhD thesis, Georg-August-Universität Göttingen, 2012.
[17] B. Randol, On the asymptotic distribution of closed geodesics on compact Riemann surfaces, Trans. Amer. Math. Soc. 233 (1977), 241–247.
[18] P. Sarnak, The arithmetic and geometry of some hyperbolic three-manifolds, Acta Math. 151 (1983), 253–295.

How to cite this article: Avdispahić M. On the prime geodesic theorem for hyperbolic 3-manifolds. Mathematische Nachrichten. 2018;291:2160–2167. https://doi.org/10.1002/mana.201700190