Compact minimal surfaces in the Berger spheres

Francisco Torralbo

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Abstract In this article, we construct compact, arbitrary Euler characteristic, orientable and non-orientable minimal surfaces in the Berger spheres. Also, we show an interesting family of surfaces that are minimal in every Berger sphere, characterizing them by this property. Finally we construct, via the Daniel correspondence, new examples of constant mean curvature surfaces in \( \mathbb{S}^2 \times \mathbb{R} \), \( \mathbb{H}^2 \times \mathbb{R} \) and the Heisenberg group with many symmetries.

Keywords Surfaces · Minimal · Homogeneous 3-manifolds · Berger spheres

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1 Introduction

The study of minimal surfaces in space forms is a classical topic in differential geometry. It is well known that there is no compact minimal surface in the Euclidean 3-space or in the hyperbolic space. On the other hand, Lawson [9] showed that every compact surface except the projective plane can be minimally immersed in the 3-sphere. His construction is based on the following fact: if a minimal surface contains a geodesic arc, it can be continued as an analytic minimal surface by geodesic reflection (see Definition 1). He constructed compact minimal surfaces using this property and reflecting the Plateau solution over a suitable geodesic polygon. This method has been successfully applied to other spaces and other kind of surfaces, such as constant mean curvature ones (see, for example [6–8]).

Besides the space forms, the most regular 3-manifolds are the homogeneous Riemannian ones. Among them, the compact type examples, which are expected to admit compact minimal
surfaces as in the space form case, are the Berger spheres. Some of the simplest constant mean
curvature surfaces (including minimal ones), i.e., those invariant by a 1-parameter group of
isometries, have been studied by the author (cf. [12] and Sect. 3). In addition, some aspects
of the stability of constant mean curvature surfaces have been discussed (cf. [13]).

In this article, we deal with the following problem: the construction of compact minimal
surfaces in the Berger spheres. These 3-manifolds are homogeneous Riemannian with isom-
etry group of dimension 4, which are, roughly speaking, 3-spheres endowed with a family
of deformations of the round metric (see Sect. 2 for details). The main issue we are going to
face is the following: the reflection over a geodesic of the Berger sphere is not in general an
ambient isometry.

The first section is devoted to presenting in detail the family of Berger spheres and their
geodesics, and it shows when reflections across geodesics are given by ambient isometries
(cf. Lemma 1). The second section characterizes a new family of minimal examples as the
only ones that are minimal with respect to every Berger metric (see Proposition 1). Sections 4
and 5 develop the main result (cf. Theorems 1, 2 and Corollary 1) in the article:

Every compact surface except the projective plane can be minimally immersed in any
Berger sphere. Furthermore, in the oriented case, the immersion can be chosen without
self intersections.

Finally, in Sect. 6 we give some remarks about the sister immersions (cf. [2]) in $S^2 \times \mathbb{R}$,
$H^2 \times \mathbb{R}$, and the Heisenberg group associated to the minimal examples constructed in
Sect. 4.

2 Preliminaries

A Berger sphere is a usual 3-sphere $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ endowed with the metric

$$g(X, Y) = \frac{4}{\kappa} \left[ (X, Y) + \left( \frac{4\tau^2}{\kappa} - 1 \right) \langle X, V \rangle \langle Y, V \rangle \right]$$

where $\langle , \rangle$ stands for the usual metric on the sphere, $V(z, w) = J(z, w) = (iz, iw)$, for each
$(z, w) \in S^3$ and $\kappa, \tau$ are real numbers with $\kappa > 0$ and $\tau \neq 0$. From now on, we will
denote the Berger sphere $(S^3, g)$ as $S^3_b(\kappa, \tau)$. We note that if $\kappa = 4\tau^2$ then $S^3_b(\kappa, \tau)$ is,
up to homotheties, the round sphere. If $\kappa \neq 4\tau^2$ then, the group of isometries of $S^3_b(\kappa, \tau)$
is $\{A \in O(4) : AJ = \pm JA\}$, where $J$ is the matrix $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_1 \end{pmatrix}$ and $J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Iso$(S^3_b(\kappa, \tau))$ has two different connected components $U_+(2)$ and $U_-(2)$, given by

$$U_+(2) = \{A \in O(4) : AJ = JA\}, \quad U_-(2) = \{A \in O(4) : AJ = -JA\}$$

and each one is homeomorphic to the unitary group $U(2)$. $U_+(2)$, after identifying it with
the unitary group, acts in the usual way over a pair of complex numbers, whereas $U_-(2)$ acts
by conjugation, that is, given $A = \begin{pmatrix} z_1 & z_2 \\ w_1 & w_2 \end{pmatrix} \in U_-(2)$, then

$$A(z, w) = (z_1 \bar{z} + z_2 \bar{w}, w_1 \bar{z} + w_2 \bar{w})$$

Taking into account the above relation between the standard metric on the sphere and the
Berger metric, it is not difficult to get the next formula, that links the Levi–Civita connection
$\nabla$ of the round sphere to $\nabla^b$, the one associated to the Berger metric:
\[ \nabla^2_{X} Y = \nabla_X Y + \left( \frac{4\tau^2}{\kappa} - 1 \right) \left[ (Y, V) (JX) + (X, V) (JY) \right] \quad (2.1) \]

where \( J \) is the complex structure of \( \mathbb{C}^2 \), that is \( J(z, w) = (iz, iw) \), and \( ()^\top \) denotes the tangential component to the sphere.

The Hopf fibration \( \Pi : S^3_b(\kappa, \tau) \to S^2(\kappa) \), where \( S^2(\kappa) \) stands for the 2-sphere of radius \( 1/\sqrt{\kappa} \), defined by
\[
\Pi(z, w) = \frac{2}{\sqrt{\kappa}} \left( \bar{z}w, \frac{1}{2} (|z|^2 - |w|^2) \right),
\]
is a Riemannian submersion whose fibers are geodesics. The vertical unit Killing field is given by \( \xi = \frac{\kappa}{4\tau} V \).

As the Berger spheres are homogeneous Riemannian manifolds, in order to know all the geodesics it is enough to describe them at a fixed point, for example the tangential component to the sphere. A geodesic reflection across an horizontal or a vertical geodesic is an isometry. Notice that the horizontal and vertical geodesics are great circles. This terminology comes from the Hopf fibration, i.e., the vertical geodesics are the fibers of this Riemannian submersion. Notice that the horizontal and vertical geodesics are great circles. The ones starting at \((1, 0)\) are given by:
\[
\begin{align*}
    z_1(s) &= \exp \left( \frac{\kappa - 4\tau^2}{4\tau} \cos \theta \, s \right) \left[ \cos \left( \frac{\lambda \sqrt{\kappa}}{2} \, s \right) + i \frac{2\tau}{\sqrt{\kappa}} \frac{\cos \theta}{\lambda} \sin \left( \frac{\lambda \sqrt{\kappa}}{2} \, s \right) \right], \\
    z_2(s) &= \exp \left( \frac{i \varphi + \kappa - 4\tau^2}{4\tau} \cos \theta \, s \right) \left[ \sin \theta \frac{\lambda}{\lambda} \sin \left( \frac{\lambda \sqrt{\kappa}}{2} \, s \right) \right]
\end{align*}
\]
and \( \lambda^2 = \sin^2 \theta + \frac{4\tau^2}{\kappa} \cos^2 \theta \) (cf. [11]). We will say that a geodesic is horizontal, if its tangent vector is orthogonal to \( V \) and vertical, if its tangent vector is co-linear with \( V \). This terminology comes from the Hopf fibration, i.e., the vertical geodesics are the fibers of this Riemannian submersion. The ones starting at \((1, 0)\) are given by:
\[
\begin{align*}
    h_{\varphi}(s) &= \left( \cos \left( \frac{\sqrt{\kappa}}{2} \, s \right), \sin \left( \frac{\sqrt{\kappa}}{2} \, s \right)e^{i\varphi} \right), \\
    h'_{\varphi}(0) &= \frac{\sqrt{\kappa}}{2} \left( 0, e^{i\varphi} \right), \\
    v(s) &= \left( e^{i\frac{\sqrt{\kappa}}{4\tau} s}, 0 \right), \\
    v'(0) &= \frac{\kappa}{4\tau} (i, 0).
\end{align*}
\quad (2.2)
\]

**Remark 1** It is important to make the following observations:

1. Given two horizontal geodesics starting at \((1, 0)\), \( h_{\varphi_1} \) and \( h_{\varphi_2} \), the angle between them is \( \varphi_1 - \varphi_2 \).
2. The length of every vertical geodesic is \( 8\tau \pi / \kappa \), while the length of every horizontal geodesic is \( 4\pi / \sqrt{\kappa} \).

**Definition 1** A geodesic reflection across a geodesic \( \gamma \) of \( S^3_b(\kappa, \tau) \) is the map that sends a point \( p \) to its “opposite” point on a geodesic through \( p \) which meets \( \gamma \) orthogonally. More precisely, if \( \alpha \) is a geodesic that meets orthogonally \( \gamma \) at \( s = 0 \) and \( \alpha(s_0) = p \), then the geodesic reflection of \( p \) with respect to \( \gamma \) is the point \( \alpha(-s_0) \).

**Lemma 1** The geodesic reflection across an horizontal or a vertical geodesic is an isometry of \( S^3_b(\kappa, \tau) \).

**Proof** As the Berger spheres are homogeneous Riemannian manifolds, it is enough to check that the geodesic reflection across the vertical geodesic through \((1, 0)\) and all the horizontal geodesics through \((1, 0)\) are isometries.
In the first case, it is easy to see that such transformation is given by \( r(v(z, w)) = (z, -w) \), which is clearly an isometry of the Berger sphere.

In the second case, we fix first the horizontal geodesic given by \( h_0 \) [cf. (2.2)]. In this special case, it is not difficult to see that the geodesic reflection across \( h_0 \) is given by \( R_0(z, w) = (\overline{z}, \overline{w}) \), which is an isometry. Now, as the rotation of angle \( \theta \) around the vertical geodesic \( v \) through \((1, 0)\) is \( \rho(z, w) = (z, e^{i\theta}w) \), we can recover the geodesic reflection across any horizontal geodesic through \((1, 0)\) by conjugation with this rotation. Therefore, the geodesic reflection across the horizontal geodesic \( h_\phi \) [cf. (2.2)] is given by \( R_\phi(z, w) = (\overline{z}, e^{2i\phi}\overline{w}) \), which is an isometry. This finishes the proof. \( \square \)

3 New minimal surfaces in the Berger spheres

In [12], it is studied the simplest examples of constant mean curvature surfaces in the Berger spheres, namely the rotationally ones, that is, the constant mean curvature surfaces invariant by the 1-parameter group of isometries given by \( t \rightarrow (10e^{it}) \). Among them, the following minimal ones are included:

- the equator \( \{(z, w) \in S_3^b(\kappa, \tau) : \text{Im}(z) = 0\} \), which is the only minimal sphere, up to ambient isometries, in \( S_3^b(\kappa, \tau) \) (cf. Remark 5),
- the Clifford torus \( \{(z, w) \in S_3^b(\kappa, \tau) : |z|^2 = |w|^2 = \frac{1}{2}\} \),
- a 1-parameter family of examples, called unduloids because they look like euclidean unduloids. Some of them produce tori and, even more, for some Berger spheres with small bundle curvature \( \tau \) (with respect to a fixed \( \kappa \)) some of these tori are embedded (cf. [12, Remark 3.2]).

The first two examples are in fact minimal surfaces in every Berger sphere, that is, they are minimal surfaces in \( S_3^b(\kappa, \tau) \) for every \( \kappa > 0 \) and \( \tau \neq 0 \), so a natural question arises: Do more surfaces exist which are simultaneously minimal in every Berger sphere? The answer is yes, there are a new family of immersions that we will call helicoids (see next definition) which are minimal in every Berger sphere. This section is devoted to characterize this family of immersions as the unique one satisfying that property (see Proposition 1).

**Definition 2** Let \( c \in \mathbb{R} \) and consider \( \Phi_c : \mathbb{R}^2 \rightarrow S_3^b(\kappa, \tau) \) given by:

\[
\Phi_c(s, t) = \left( \cos(s)e^{ict}, \sin(s)e^{it} \right).
\]

Then, \( \Phi_c \) is a minimal immersion in any Berger sphere, that is, for every \( \kappa > 0 \) and \( \tau \neq 0 \). We will call these immersions helicoids since they are ruled by horizontal geodesics and they are invariant by the 1-parameter group of isometries of \( U(2) \) given by:

\[
\left( \begin{array}{cc}
e^{ict} & 0 \\ 0 & \ne^{it} \end{array} \right) : t \in \mathbb{R}
\]

**Remark 2**

1. If \( c = 0 \), we get the equator \( \{(z, w) \in \mathbb{C}^2 : \text{Im}(z) = 0\} \).
2. The immersions \( \Phi_c \) do not coincide with the family of minimal unduloids described in [12, Theorem 1.(iii)] because they are not invariant by the group of isometries considered there, except for the sphere where \( c = 0 \) (see previous item).
3. If \( c = m/n \) with \( m, n \in \mathbb{Z} \), we get the compact examples \( \tau_{m,n} \) constructed by Lawson [9, Sect. 7, p. 350]. It is interesting to remark that (cf. [9, Theorem 3] for more details):
To each pair of co-prime natural numbers \(m, n\), \(\tau_{m,n}\) is a compact surface with zero Euler characteristic.

- \(\tau_{m,n}\) is a Klein bottle if and only if \(m \cdot n\) is an even number.
- \(\tau_{1,1}\) is, up to an isometry, the Clifford torus and it is the only surface of this type that is embedded. More precisely,

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \Phi_1(s, t) = \frac{1}{\sqrt{2}} \left( e^{i(t+s)}, e^{i(t-s)} \right)
\]

Firstly, we are going to relate the mean curvature of an arbitrary surface in a Berger sphere to its mean curvature as a surface in the usual 3-sphere.

**Lemma 2** Let \(\Phi : \Sigma \to S^3\) be an immersion of a surface \(\Sigma\). Then, the mean curvatures \(H\) and \(H^b\) of \(\Phi\) in the round sphere \((S^3, \langle, \rangle)\) and the Berger sphere \(S^3_b(\kappa, \tau)\), respectively, are related by

\[
H^b = \frac{\sqrt{\kappa}}{2} \frac{1}{\sqrt{1 - \left(1 - \frac{\kappa}{4\tau^2}\right)v^2}} \left[ H + \frac{1}{2} \left(1 - \frac{\kappa}{4\tau^2}\right) \frac{\langle\nabla v, V\rangle}{v^2} \right]
\]

where \(v = \langle N, V \rangle\), \(N\) is a unit normal vector field of \(\Sigma\) in \((S^3, \langle, \rangle)\), and \(V(z, w) = (iz, iw)\).

**Proof** To begin with, if \(N\) and \(N^b\) are two unit normal vector field of \(\Phi\) in \(S^3\) and \(S^3_b(\kappa, \tau)\), respectively, then they are related by the equality:

\[
N^b = \frac{\sqrt{\kappa}}{2} \frac{1}{\sqrt{1 - \left(1 - \frac{\kappa}{4\tau^2}\right)v^2}} \left[ N - v \left(1 - \frac{\kappa}{4\tau^2}\right) V \right],
\]

where \(v = \langle N, V \rangle\).

Now, taking into account (2.1) and the above relation between the normal vector fields we get

\[
\frac{\kappa}{4f} g(\nabla_{e_i} N^b, e_j) = \left(4\tau^2 - 1\right) \langle e_i, V \rangle \langle e_j, JN \rangle
\]

\[
- \lambda_i \left(4\tau^2 - 1\right) \langle e_i, V \rangle \langle e_j, V \rangle
\]

\[
- \left(4\tau^2 - 1\right) \langle e_j, V \rangle e_i(v),
\]

where \(\{e_1, e_2\}\) is an orthonormal reference with respect to \(\langle, \rangle\) that diagonalizes the shape operator of \(\Phi : \Sigma \to (S^3, \langle, \rangle)\), \(\lambda_i\) are the principal curvatures, and \(f : \Sigma \to \mathbb{R}\) is the function given by \(f = \sqrt{\kappa} \frac{1}{2} \sqrt{1 - \left(1 - \frac{\kappa}{4\tau^2}\right)v^2}\). Finally, as \(-2H^b = \sum_{i,j=1}^2 g^{ij} g(\nabla_{e_i} N^b, e_j)\), a straightforward computation yields the result.

Thanks to the above result, if \(\Sigma\) is a minimal surface in the round sphere then it is minimal in some Berger sphere \(S^3_b(\kappa, \tau)\) (in fact, in every Berger sphere) if and only if \(\langle \nabla v, V \rangle = 0\). The equator \(\{(z, w) \in S^3 : \text{Im}(z) = 0\}\) and the Clifford torus \(\{(z, w) \in S^3 : |z|^2 = |w|^2 = \frac{1}{2}\}\) satisfy this condition so they are, as we already knew, minimal surfaces in every Berger sphere. Next proposition classifies all the surfaces that are simultaneously minimal in every Berger sphere.
Proposition 1 Let $\Phi : \Sigma \to \mathbb{S}^3$ be an immersion of a surface $\Sigma$. Suppose that $\Phi$ is a minimal immersion with respect to any Berger metric. Then, $\Phi$ is congruent to a piece of one of the helicoids $\Phi_c$ (cf. Definition 2).

Proof Let $\Sigma$ be a minimal surface in some Berger sphere (in fact, in every Berger sphere). Then, thanks to Lemma 2, it must be $(\nabla v, V) = 0$. Let $z = x + iy$ a conformal parameter for $\Phi : \Sigma \to (\mathbb{S}^3, (\cdot, \cdot))$ with conformal factor $e^{2u}$. Then, the compatibility equations (cf. [5, Theorem 2.3, p. 283]) of the immersion are given, for $\kappa = 4$, $\tau = 1$ (which is the case of the round sphere) and $H = 0$, by:

$$
p_{\bar{z}} = 0, \quad v_\bar{z} = iA - 2e^{-2u}\bar{A}p
$$

$$
A_{\bar{z}} = i\frac{e^{2u}}{2}v, \quad |A|^2 = \frac{e^{2u}}{4}(1 - v^2)
$$

(3.1)

where $A = (\Phi_z, V)$ and $p = (\Phi_{zz}, N)$. As the Hopf differential $\Theta = p(dz)^2$ is holomorphic, we can normalize it by $\lambda i$, with $\lambda \in \mathbb{R}$. Now, the condition $\langle \nabla v, V \rangle = 0$ becomes in

$$
0 = v_\bar{z}\bar{A} + v_{\bar{z}}A = -4\lambda e^{-2u}\text{Im}(A^2)
$$

If $\lambda = 0$, then $\Sigma$ is a totally umbilical minimal surface so it must be a totally geodesic sphere which corresponds to the case (1) in Remark 2.

We now suppose that $\lambda \neq 0$. The last equation allows us to suppose that the real part of $A$ is zero. So, using (3.1), it must be

$$
A = i\frac{e^u}{2}\sqrt{1 - v^2}
$$

In that case, from the equation for $v_\bar{z}$ in (3.1), we get

$$
v_\bar{z} = -\frac{\sqrt{1 - v^2}}{2}(e^u + 2\lambda e^{-u}).
$$

From the above equation, we deduce that $v_y = 0$, i.e., $v = v(x)$. Using again the last equation, we get that $u = u(x)$ too.

It is well known that the minimal surfaces of the round sphere are controlled by the sinh-Gordon equation. In this case, as the conformal parameter depends only on one variable they must satisfy the equation

$$
u'' + (e^{2u} - 4\lambda^2 e^{-2u}) = 0,
$$

(3.2)

whose first integral is given by

$$(u')^2 + (e^{2u} - 4\lambda^2 e^{-2u}) = E
$$

(3.3)

The above equation has been intensively studied in the literature. Its solutions can be described in terms of Jacobi elliptic functions. In fact, it is possible to integrate explicitly the minimal immersions that they produce as follows:

$$
\Psi_{a,b} = \frac{1}{\sqrt{a^2 - b^2}} \left( \sqrt{e^{2u(x)} - b^2}e^{i(bF(x)+ay)}, \sqrt{a^2 - e^{2u(x)}}e^{i(aG(x)+by)} \right)
$$

(3.4)

where

$$
F(x) = \int_0^x \frac{P\left(e^{2u(t)}\right)}{e^{2u(t)} - b^2} dt, \quad G(x) = \int_0^x \frac{P\left(e^{2u(t)}\right)}{a^2 - e^{2u(t)}} dt,
$$

$$
P\left(e^{2u(t)}\right) = \left(a^2 - e^{2u(t)}\right)\left(e^{2u(t)} - b^2\right) - e^{2u(t)}u'(t)^2
$$

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and $u(x)$ is a solution of (3.2) for $\lambda^2 = a^2b^2/4$ and $E = a^2 + b^2$.

Now, among all these solutions, the only ones that satisfy the extra condition that $\langle \nabla \nu, V \rangle = 0$ are precisely, up to reparametrization, the family $\Phi_c$ and the proof finishes. $\square$

4 Compact minimal surfaces with arbitrary topology

In this section, we are going to construct new examples of compact minimal surfaces in the Berger spheres following a technique developed by Lawson in [9]. The main idea is to solve the Plateau problem over a nice geodesic polygon and to successively reflect the obtained minimal surface over the border geodesics to generate a compact minimal surface.

The key result that we are going to apply is the following reflection principle:

Given a minimal surface $\Sigma$ on a Berger sphere $\mathbb{S}^3(\kappa, \tau)$, if $\partial \Sigma$ contains a horizontal or a vertical geodesic arc $\gamma$ then $\Sigma$ can be continued as an analytic minimal surface across each non-trivial component of $\partial \Sigma \cap \gamma$ by geodesic reflection across $\gamma$.

More precisely, in the above situation we assume that $\Sigma$ is, at least, $C^2$ so we can extend it $C^2$ to the boundary $\gamma$ (cf. [4, Theorem 4, p. 40]). Hence, if we denote by $r$ the geodesic reflection across the border geodesic $\gamma$ then both $\Sigma$ and $r(\Sigma)$ are minimal surfaces because $r$ is an isometry (cf. Lemma 1). Even more, their tangent planes coincide along $\gamma$ because the map $r$ changes the co-normal vector field $\eta$ along $\gamma$ to $-\eta$. Therefore, as they are minimal surfaces, $\Sigma$ can be continued as an analytic minimal surface by $r(\Sigma)$.

4.1 Choosing the polygon

To help us to visualize the polygon, we consider the stereographic projection of the sphere from the point $(0, -1)$. The vertical geodesic $v_1$ from $(0, 1)$ becomes the $z$ axis and the one from $(1, 0)$, $v_2$, becomes the unit circle centered at the origin in the $xy$ plane. Choose $P_1, P_2$ two points in $v_2$ such that distance$(P_1, P_2) = (4\tau/\kappa)\theta$ and $Q_1, Q_2 \in v_1$ such that distance$(Q_1, Q_2) = (4\tau/\kappa)\phi$ (see Fig. 1).

We are going to consider the polygon $\Gamma = Q_1P_1Q_2P_2$, where its edges are the horizontal geodesics $f_1, f_2, h_1,$ and $h_2$ given by:

$$
\begin{align*}
  f_1(t) &= (\cos(t)e^{i\theta}, \sin(t)), & f_2(t) &= (\cos(t)e^{i\theta}, \sin(t)e^{i\phi}) & t \in [0, \pi/2] \\
  h_1(t) &= (\cos(t), \sin(t)), & h_2(t) &= (\cos(t), \sin(t)e^{i\phi}) & t \in [0, \pi/2]
\end{align*}
$$

Fig. 1 Polygon $\Gamma$ by stereographic projection from the point $(0, -1)$
Remark 3 It is not difficult to check that the group $G_\Gamma$ generated by geodesic reflections around the edges of $\Gamma$ is generated by the maps $H_1(z, w) = (\bar{z}, \bar{w})$, $H_2(z, w) = (\bar{z}, e^{2i\varphi}\bar{w})$, $F_1(z, w) \mapsto (e^{2i\theta}\bar{z}, \bar{w})$ and $F_2(z, w) = (e^{2i\varphi}\bar{z}, e^{2i\theta}\bar{w})$.

Moreover, two successive reflections around the vertices $P_1$ or $P_2$ (resp. $Q_2$ or $Q_2$) produce a rotation of angle $2\varphi$ (resp. $2\theta$) around the vertical geodesic $v_2$ (resp. $v_1$), i.e., the maps $(z, w) \mapsto (z, e^{2i\varphi}w)$ and $(z, w) \mapsto (e^{2i\theta}z, w)$ lie in the group $G_\Gamma$.

4.2 Solving the Plateau problem

In order to ensure the existence of a minimal surface with border $\Gamma$, we are going to use the following general result due to Meeks and Yau [10].

**Theorem** Let $W$ be a compact Riemannian 3-manifold with piecewise smooth mean convex boundary\(^1\). Let $\Gamma$ be a smooth collection of pairwise disjoint closed curves in $\partial W$, which bounds a compact orientable surface in $W$. Then, there exists an embedded orientable surface $\Sigma \subseteq W$ with $\partial \Sigma = \Gamma$ that minimizes area among all orientable surfaces with the same boundary.

Consider $W$ the solid tetrahedron determined by $\Gamma$, $v_1$ and $v_2$ (see Fig. 1). Clearly, $W$ is a compact Riemannian 3-manifold whose border is a two-dimensional complex consisting of four two-dimensional compact simplices, which we will call faces, with interior angles less than or equal to $\pi$ if and only if $\varphi, \theta \leq \pi$. If we denote every face of $\partial W$ by $\blacktriangle ABC$, where $A$, $B$, and $C$ are the vertices, then we can parametrize them by the following equations:

\[
\Phi_1 : \left[0, \frac{\pi}{2}\right] \times [0, \varphi] \to \blacktriangle Q_1Q_2P_1 \subseteq S_b^3(\kappa, \tau), \quad \Phi_1(t, s) = \left(\cos(t), \sin(t)e^{i\kappa}\right),
\]
\[
\Phi_2 : \left[0, \frac{\pi}{2}\right] \times [0, \varphi] \to \blacktriangle Q_1Q_2P_2 \subseteq S_b^3(\kappa, \tau), \quad \Phi_2(t, s) = \left(\cos(t)e^{i\theta}, \sin(t)e^{i\kappa}\right),
\]
\[
\Phi_3 : \left[0, \frac{\pi}{2}\right] \times [0, \theta] \to \blacktriangle Q_1P_1P_2 \subseteq S_b^3(\kappa, \tau), \quad \Phi_3(t, s) = \left(\cos(t)e^{i\kappa}, \sin(t)\right),
\]
\[
\Phi_4 : \left[0, \frac{\pi}{2}\right] \times [0, \theta] \to \blacktriangle Q_2P_1P_2 \subseteq S_b^3(\kappa, \tau), \quad \Phi_4(t, s) = \left(\cos(t)e^{i\varphi}, \sin(t)e^{i\kappa}\right).
\]

Then, the only condition that we have to check in order to guarantee $W$ has a piecewise smooth mean convex boundary is that every face has non-negative mean curvature with respect to the inward pointing normal.

First of all, $\Phi_1$ is a minimal immersion because is a small piece of the minimal sphere $\{(z, w) \in S_b^3(\kappa, \tau) : \text{Im}(z) = 0\}$. On the other hand, $\Phi_2 = \rho_\theta \circ \Phi_1$, where $\rho_\theta(z, w) = (e^{i\theta}z, w)$ is the rotation around the vertical geodesic $v_1$ of angle $\theta$ so $\Phi_2$ is minimal too. Moreover $\Phi_3 = s \circ \Phi_1$, where $s(z, w) = (w, z)$ is an isometry of the Berger sphere, and $\Phi_4 = \rho_\varphi \circ \Phi_3$, where $\rho_\varphi(z, w) = (z, e^{i\varphi}w)$ is the rotation of angle $\varphi$ around the vertical geodesic $v_2$. Hence, both $\Phi_3$ and $\Phi_4$ are minimal immersions too. To sum up, $W$ has piecewise smooth mean convex boundary so the above theorem ensures us that there exists a minimal surface $\Sigma$ with border $\Gamma$ if and only if $\varphi, \theta \leq \pi$.

\(^1\) Suppose that $W$ is a compact Riemannian 3-manifold that embeds in the interior of another Riemannian 3-manifold. $W$ is said to have piecewise smooth mean convex boundary if $\partial W$ is a two-dimensional complex consisting of a finite number of smooth two-dimensional compact simplices with interior angles less than or equal to $\pi$, each one with non negative mean curvature with respect to the inward pointing normal.
4.3 Generating the surface

In the above situation, take $n, m \in \mathbb{N}$ and choose $\varphi = \pi/(n + 1)$ and $\theta = \pi/(m + 1)$. We denote by $\Sigma_{\Gamma}^{(m,n)}$ a minimal surface in $S^3_b(\kappa, \tau)$ with border $\Gamma$. Let $G_{\Gamma}$ the group of isometries generated by geodesic reflection around the edges of $\Gamma$. Then, thanks to the choice of $\varphi$ and $\theta$, $G_{\Gamma}$ is a discrete group so $\Sigma_{\Gamma}^{(m,n)} = \bigcup_{g \in G_{\Gamma}} g \left( \Sigma_{\Gamma}^{(m,n)} \right)$ is a compact minimal surface in $S^3_b(\kappa, \tau)$. Even more, if $H = \left\{ g \in G_{\Gamma} : g \left( \Sigma_{\Gamma}^{(m,n)} \right) = \Sigma_{\Gamma}^{(m,n)} \right\}$, then $\text{ord}(G)/\text{ord}(H) = 2(m + 1)(n + 1)$ because $2(n + 1)$ reflections at $v_2$ produce the identity and $(m + 1)$ reflections at $v_1$ of the resulting surface produce the whole surface $\Sigma_{\Gamma}^{(m,n)}$.

Moreover, if we denote by $K$ the Gauss curvature of $\Sigma_{\Gamma}^{(m,n)}$, then it is easy to check that, by the Gauss–Bonnet Theorem,

$$\int_{\Sigma_{\Gamma}^{(m,n)}} K = \frac{2\pi}{(m + 1)(n + 1)} (1 - mn)$$

and so

$$2\pi \chi(\Sigma_{\Gamma}^{(m,n)}) = \int_{\Sigma_{\Gamma}^{(m,n)}} K = \frac{\text{ord}(G)}{\text{ord}(H)} \int_{\Sigma_{\Gamma}^{(m,n)}} K = 4\pi (1 - mn),$$

where $\chi(\Sigma_{\Gamma}^{(m,n)})$ denotes the Euler characteristic. Finally, we get that the surface $\Sigma_{\Gamma}^{(m,n)}$ is oriented and its genus is $mn$. To sum up, we state the following result.

**Theorem 1** For every $g \geq 0$, there exists a compact embedded minimal surface of genus $g$ in $S^3_b(\kappa, \tau)$.

**Proof** By the construction method, we already know that there exists a compact minimal surface $\Sigma_{\Gamma}^{(m,n)}$ (possibly with singularities) in $S^3_b(\kappa, \tau)$ of genus $g = mn, m, n \in \mathbb{N}$.

The embeddedness follows from the fact that the fundamental piece $\Sigma_{\Gamma}^{(m,n)}$ fits in the solid tetrahedron determined by the border $\Gamma$ and this tetrahedron does not intersect itself as we go on reflecting it by the isometries of the group $G_{\Gamma}$.

Finally, we prove that no singularity appears. We know that $\Sigma_{\Gamma}^{(m,n)}$ is regular up to the border $\Gamma$ and at the interior of the edges of $\Gamma$ the surface is smooth too (cf. [4, Theorem 4, p. 40]) so we only have to check the regularity at the vertices $P_1, P_2, Q_1, Q_2$, and their reflections. But this is a consequence of a more general removable singularity result (cf. [1, Proposition 1]). In order to apply that result, we only recall that the surface $\Sigma_{\Gamma}^{(m,n)}$ is locally embedded around the vertex (it is, in fact, embedded). \hfill $\Box$

**Remark 4**

1. We can produce the other type of compact minimal surfaces by just changing the geodesic polygon over the tetrahedron $\partial W$. More precisely, take the geodesic polygon $\tilde{\Gamma}$ which joins the points $P_1 Q_1 Q_2 P_2$ in Fig. 1 with angles $\varphi = \pi/(n + 1)$ and $\theta = \pi/(m + 1)$. Then, by the previous process, we can produce a compact minimal surface that, in this case, always has Euler characteristic zero. In fact, the resulting surface can be explicitly described by the immersion $\Phi_{m/n}$ given in Definition 2.

2. Let $r \in \mathbb{N}$ and $T_r$ the group generated by the map

$$(z, w) \rightarrow \left( e^{\frac{2\pi i}{r}} z, e^{\frac{2\pi i}{r}} w \right).$$
Notice that \( T \subset \text{Iso}(S^3_b(\kappa, \tau)) \) for every \( \kappa \) and \( \tau \). We will denote the quotient \( S^3_b(\kappa, \tau)/T_r \) with the induced metric by \( L'_b(\kappa, \tau) \). Then \( L'_b(\kappa, \tau) \) is a homogeneous Riemannian\(^2\) 3-manifold with isometry group of dimension 4.

For each \( r \in \mathbb{N} \) such that \( r \) divides \( n + 1 \) and \( m + 1 \), the surface \( \Sigma^{(m, n)} \) is invariant by the group \( T_r \) and so it is induced to the quotient \( L'_b(\kappa, \tau) \) as a compact minimal surface with genus \( 1 - (1 - mn)/r \).

### 5 Non-orientable compact minimal surfaces

Now we are going to construct, following the same procedure as in the previous section, compact non-orientable minimal surfaces. We start with the points \( P_1 \), \( P_1 \), \( Q_1 \), and \( Q_2 \) as in Sect. 4, and we define a polygon \( \Gamma' = P_1 Q_2 P_2 (-P_2) \), where its edges are the horizontal geodesics \( f_1 \), \( f_2 \), \( h_2 \) and the vertical one \( v_2 \), parametrized by (see Fig. 2):

\[
\begin{align*}
    f_1(t) &= (\cos(2t) e^{i\theta}, \sin(2t)), \quad f_2(t) = (\cos(t) e^{i\theta}, \sin(t) e^{i\phi}) \quad t \in [0, \pi/2] \\
    h_2(t) &= (\cos(t), \sin(t) e^{i\phi}) \quad t \in [0, \pi/2] \\
    v_2(t) &= (e^{i\theta}, 0) \quad t \in [-\theta, 0]
\end{align*}
\]

Consider \( W \) the region delimited by the minimal spheres \( \{(z, w) \in S^3 : \text{Im}(w) = 0\} \), \( \{(z, w) \in S^3 : \text{Im}(e^{-i\theta} z) = 0\} \) and \( \{(z, w) \in S^3 : \text{Im}(e^{i\phi} w) = 0\} \). Then, the polygon \( \Gamma' \) is in the border of \( W \) and \( W \) has piecewise smooth mean convex boundary if and only if the angles \( \varphi, \theta \leq \pi \).

Take \( n, m \in \mathbb{N} \) with \( n \) odd and choose \( \varphi = \pi/(n + 1) \) and \( \theta = \pi/(m + 1) \). Therefore, there exists an embedded minimal surface \( \Lambda^{(m, n)}_{\Gamma'} \) with border \( \Gamma' \). Let \( G_{\Gamma'} \) the group of isometries generated by geodesic reflections around the edges of \( \Gamma' \). Then, thanks to the choice of \( \varphi \) and \( \theta \), \( G_{\Gamma'} \) is a discrete group so \( \Lambda^{(m, n)} = \bigcup_{g \in G_{\Gamma'}} g(\Lambda^{(m, n)}_{\Gamma'}) \) is a compact minimal surface (possibly with singularities) in \( S^3_b(\kappa, \tau) \).

Let \( \Lambda \) be the set constructed by reflecting the surface \( \Lambda^{(m, n)}_{\Gamma'} \) \( 2(m + 1) \) times at \( Q_2 \), and then reflecting the resulting configuration \( (n + 1) \) times at \( P_2 \). We claim that \( \Lambda = \Lambda^{(m, n)} \).

It is sufficient to prove that \( \Lambda \) is invariant under the reflection around the edges of \( \Gamma' \). First, it is clear by the construction of \( \Lambda \) that it is invariant under the reflection under the edges

\[^2\text{The isometry } \left( \begin{array}{cc} a & -b \\ b & a \end{array} \right) \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \text{ sends the point } (\alpha, \beta) \text{ to } (a, b) \text{ and it is induced to the quotient } L'_b(\kappa, \tau) \text{ because it commutes with } T_r.\]
$Q_2 P_2$, $Q_2 P_1$, and $Q_1 P_2$. Furthermore, as $n$ is odd, the reflection around $P_1(-P_2)$ coincides with reflecting $(n+1)/2$ times around $P_2$ so the claim is proved.

By the previous reasoning, we also know that if $H = \{ g \in G_{\Gamma'} : g \left( \Lambda_{\Gamma'}^{(m,n)} \right) = \Lambda_{\Gamma'}^{(m,n)} \}$, then $\mathrm{ord}(G)/\mathrm{ord}(H) = 2(m+1)(n+1)$.

Finally, if $K$ denotes the Gaussian curvature of $\Lambda_{\Gamma'}^{(m,n)}$, then, by the Gauss-Bonnet theorem,

$$\int_{\Lambda_{\Gamma'}^{(m,n)}} K = \frac{\pi}{(n+1)(m+1)} (1 - mn)$$

and so

$$2\pi \chi \left( \Lambda_{\Gamma'}^{(m,n)} \right) = \int_{\Lambda_{\Gamma'}^{(m,n)}} K = \frac{\mathrm{ord}(G)}{\mathrm{ord}(H)} \int_{\Lambda_{\Gamma'}^{(m,n)}} K = 2\pi (1 - mn).$$

That is, $\chi \left( \Lambda_{\Gamma'}^{(m,n)} \right) = 1 - mn$. It is clear that, if $1 - mn$ is odd, the resulting surface is non-orientable. Moreover, if $1 - mn$ is even, the surface is also non-orientable. To prove this, we will follow the same argument Lawson used: we consider a non-contractible, closed curve on the piece of the surface $\Lambda_{\Gamma'}^{(m,n)} \cup R \left( \Lambda_{\Gamma'}^{(m,n)} \right) \cup U$ where $R$ is the reflection around $P_1(-P_2)$ and $U$ is a small neighborhood of $P_2$ on the surface $\Lambda^{(m,n)}$. Such a curve is orientation reversing.

**Theorem 2** For every pair of natural numbers $n$ and $m$, with $n$ odd, there exists a non-orientable, compact minimal surface $\Lambda^{(m,n)}$ with Euler characteristic $1 - mn$.

**Proof** The previous construction ensures that there exists a compact (possibly with singularities) non-orientable minimal surface $\Lambda^{(m,n)}$ with Euler characteristic $1 - mn$. Now, if we look locally at a vertex, the surface only goes once around it as we reflect the small piece $\Lambda_{\Gamma'}^{(m,n)}$ with respect to the edges of $\Gamma'$, so $\Lambda^{(m,n)}$ is locally embedded around the vertex. Hence, by the same argument as in the proof of Theorem 1, we know that the surface $\Lambda^{(m,n)}$ is smooth. Finally, no branching points appear at the vertices because, as we notice before, the surface is locally embedded around them.

As a corollary from Theorem 1 and Theorem 2, we get the following:

**Corollary 1** Every compact surface but the projective plane can be minimally immersed into any Berger sphere $\mathbb{S}^3_B(k, \tau)$. Even more, for orientable surfaces the immersions can be chosen without self-intersections.

**Remark 5** The statement of the Corollary 1 does not extend to the case of the projective plane since the only minimal sphere is, up to congruences, the equator \{$(z, w) : \mathrm{Im}(z) = 0$\} which is embedded. This last assertion follows from [3, Theorem 2.3.1], where the authors show that every constant mean curvature sphere in any homogeneous Riemannian manifold with isometry group of dimension four (in particular, in the Berger spheres) must be rotationally invariant and [12, Theorem 1], where it is showed that the minimal rotationally sphere is, up to congruences, the equator \{$(z, w) : \mathrm{Im}(z) = 0$\}.

### 6 Some remarks about the Daniel sister surfaces corresponding to the constructed examples

An important tool in the description of constant mean curvature surfaces in space forms is the classical Lawson correspondence. It establishes an isometric one-to-one local correspondence...
between constant mean curvature surfaces in different space forms that allows to pass, for instance, from minimal surfaces in $\mathbb{S}^3$ to constant mean curvature 1 surfaces in $\mathbb{R}^3$. Lawson used it to construct double periodic constant mean curvature surfaces in $\mathbb{R}^3$ from the compact minimal ones that he constructed in $\mathbb{S}^3$ (cf. [9, Sect. 14]).

In 2007, Daniel [2, Theorem 5.2] generalized the Lawson correspondence to the context of homogeneous Riemannian spaces, establishing relations between constant mean curvature immersions in different simply connected homogeneous Riemannian spaces with isometry group of dimension 4, and he called these immersions as sister immersions.

Although the Daniel correspondence is a general tool, we will restrict ourselves to the isometric correspondence between minimal surfaces in the Berger spheres and constant mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{S}^2$ and $\mathbb{H}^2$ stand for the 2-sphere and the hyperbolic plane of constant curvature 1 and $-1$, respectively, and the Heisenberg group $\text{Nil}_3$. We can state it as follows:

**Proposition 2** Let $\Sigma^{(m,n)}$ a compact minimal surface of $\mathbb{S}^3_b(\kappa, \tau)$ given in Theorem 1 and $\tilde{\Sigma}^{(m,n)}$ its universal cover. Let $G \subseteq \text{Iso}(\mathbb{S}^3_b(\kappa, \tau))$ the group of congruences of $\Sigma^{(m,n)}$ and $\tilde{G}$ the extension of $G$ to isometries of $\tilde{\Sigma}^{(m,n)}$ by the deck transformations of the covering. Then, each element of $\tilde{G}$ extends to a congruence of the sister immersion.

**Proof** Let $\phi : \tilde{\Sigma}^{(m,n)} \rightarrow E$, where $E$ is some homogeneous manifold with isometry group of dimension 4, the sister immersion of the minimal immersion of $\tilde{\Sigma}^{(m,n)}$ in $\mathbb{S}^3_b(\kappa, \tau)$. Take a fundamental piece $\Sigma_\Gamma$. Suppose that $\Sigma_\Gamma$ and $\Sigma_\Gamma^n$ are two domains in the universal cover bijectively mapped into $\Sigma$ and $g(\Sigma_\Gamma^{(m,n)})$, respectively, with $g \in G$. Let $\tilde{g}$ the lift of $g$ to the universal cover.

Then, we get two different sister immersions of $\tilde{\Sigma}_\Gamma$, the initial $\phi$ and $\psi = \phi \circ \tilde{g}$, of the same surface $\tilde{\Sigma}_\Gamma$. By uniqueness, there exists an isometry $F : E \rightarrow E$ such that $\psi = F \circ \phi$. We claim that $F$ is a symmetry of the whole immersion $\phi : \Sigma^{(m,n)} \rightarrow E$. In fact, $\phi (\Sigma^{(m,n)})$ and

$\square$
\[(F \circ \phi) \left( \Sigma (m,n) \right) \text{ are two constant mean curvature surfaces in } E \text{ such that they share a open piece, } \phi(\bar{\Sigma}_F), \text{ so they coincide, that is } F, \text{ is a congruence of the immersion } \phi. \quad \square \]

In the product space case, as we are going to see, it is possible to precise which kind of symmetry it is produced in the sister immersion. From now on, we will denote \( \mathbb{S}^2 \times \mathbb{R} \) and \( \mathbb{H}^2 \times \mathbb{R} \) as \( M^2(\epsilon) \times \mathbb{R} \) for \( \epsilon = 1 \) and \( \epsilon = -1 \), respectively. Based on the Daniel correspondence, the shape operator \( S^* \) of the constant mean curvature immersion in \( M^2(\epsilon) \times \mathbb{R} \) is related to the shape operator \( S \) of the minimal immersion in the Berger sphere as follows:

\[
S^* = -JS + \tau \text{Id}, \quad \text{where } J = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right). \tag{6.1}
\]

Besides, if we write the vertical Killing field as \( \xi = T + \nu N \), where \( N \) is a unit normal to the surface \( \Sigma \) in the Berger sphere, then \( (0, 1) \in T(M^2(\epsilon) \times \mathbb{R}) \equiv TM^2(\epsilon) \times \mathbb{R} \) is given by

\[
(0, 1) = -JT + \nu N^* \tag{6.2}
\]

where \( N^* \) is unit normal to \( \Sigma \) in \( M^2(\epsilon) \times \mathbb{R} \).

**Proposition 3** Let \( \Sigma \) be a simply connected surface. Suppose that there exists an isometric minimal immersion of \( \Sigma \) in \( \mathbb{S}^3_3(\kappa, \tau) \). Then, there is a correspondence in the sister immersions between the following symmetries of the surface \( \Sigma \):

(a) a reflection around a horizontal geodesic of \( \mathbb{S}^3_3(\kappa, \tau) \) and

(b) a reflection with respect a vertical plane of \( M^2(\epsilon) \times \mathbb{R} \).

**Remark 6**

(1) The relation (6.1) reminds the analogous one between constant mean curvature one surfaces in \( \mathbb{R}^3 \), considered as conjugate cousins of minimal surfaces in \( \mathbb{S}^3 \). In fact, this relation is the key to prove the lemma.

(2) Unfortunately, the relation between the shape operators in the Daniel correspondence between minimal immersions in the Berger spheres and constant mean curvature immersions in the Heisenberg group is not as good as in the product spaces. This behavior does not allow us to describe which kind of symmetries has the sister immersion in the Heisenberg group.

**Proof** We are going to see that (a)⇒(b) and the same argument goes backward. We follow the above notation, and we consider \( \gamma = (\beta, h) \subseteq \Sigma \subseteq M^2(\epsilon) \times \mathbb{R} \) parametrized by arc length and \( M^2(\epsilon) \times \mathbb{R} \subseteq \mathbb{R} \times \mathbb{R}^3 \) (or \( \mathbb{R}^3_1 \times \mathbb{R} \) if \( \epsilon = -1 \)) isometrically immersed with unit normal along \( \gamma \) given by \( (\beta, 0) \). Then, we can view \( J\gamma' \), where \( J \) is the rotation of \( \pi/2 \) in the tangent plane, as a curve in \( \mathbb{R}^4 \) (or \( \mathbb{R}^4_1 \)). We claim that \( J\gamma' \) is constant. Firstly we get that

\[
\langle (J\gamma'), \gamma' \rangle = -\langle J\gamma', \nabla_{\gamma'} \gamma' \rangle = 0, \quad \text{as } \gamma \text{ is a geodesic of } \Sigma
\]

\[
\langle (J\gamma'), J\gamma' \rangle = 0, \quad \text{as } J\gamma' \text{ has length } 1
\]

\[
\langle (J\gamma'), N^* \rangle = -\langle J\gamma', dN^*(\gamma') \rangle = (J\gamma', S^*\gamma') = (J\gamma', -S\gamma' + \tau\gamma') = -\langle \gamma', S\gamma' \rangle = 0, \quad \text{as } \gamma \text{ is an asymptote line of } \Sigma
\]

where we take into account the relation (6.1). Thus, \( (J\gamma')' \) is proportional to \( (\beta, 0) \) as it does not have components in the reference \( \{\gamma', J\gamma', N^*\} \) of \( T_\gamma M^2(\epsilon) \times \mathbb{R} \). But, as \( \gamma \) is
horizontal as a curve of the Berger sphere, we know that \( \{ \gamma', \xi \} = \{ \gamma', T \} = 0 \) and so it must be \( \gamma' = JT/\sqrt{1 - v^2} \). Therefore, using (6.2),

\[
\langle \gamma', (0, 1) \rangle = \frac{1}{\sqrt{1 - v^2}} \langle JT, -JT + vN^s \rangle = -\sqrt{1 - v^2},
\]

and so \((0, 1) = -\sqrt{1 - v^2} \gamma' + vN^s \). The last equation implies that \( \{ J \gamma', (0, 1) \} = 0 \). Hence

\[
\{ (J \gamma')', (\beta, 0) \} = -\{ J \gamma', (\beta', 0) \} = h' \{ J \gamma', (0, 1) \} = 0,
\]

where we have used that \( 0 = \{ J \gamma', \gamma' \} = \{ J \gamma', (\beta', 0) \} + \{ J \gamma', h'(0, 1) \} \), and the claim is proved.

Finally, \( J \gamma' = (v, 0) \in \mathbb{R}^3 \times \mathbb{R} \). On the one hand,

\[
\langle \gamma, (v, 0) \rangle' = \langle \gamma', (v, 0) \rangle = \langle \gamma', J \gamma' \rangle = 0,
\]

so \( \langle \gamma, v \rangle \) is constant. On the other hand,

\[
\langle \gamma, (v, 0) \rangle = \langle \beta, v \rangle = 0
\]
as \( \beta \) is normal to \( M^2(\epsilon) \) and \( v \) is tangent. All these relations say that \( \gamma \subseteq P \), where \( P \) is the vertical plane defined by \( P = \{(p, t) \in M^2(\epsilon) \times \mathbb{R} : \langle p, v \rangle = 0 \} \). Moreover, since the tangent plane of \( \Sigma \) along \( \gamma \) is spanned by \( \{ \gamma', J \gamma' = (v, 0) \} \), the surface \( \Sigma \) is orthogonal to the vertical plane \( P \).

Finally, as the reflexion with respect to \( P \) is an isometry of \( M^2(\epsilon) \times \mathbb{R} \) and the constant mean curvature surface \( \Sigma \) is orthogonal to \( P \), we can extend it by this isometry. \( \square \)

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