Effective actions with fixed points

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Abstract

The specific form of the constant term in the asymptotic expansion of the heat-kernel on an axially-symmetric space with a codimension two fixed-point set of conical singularities is used to determine the conformal change of the effective action in four dimensions. Another derivation of the relevant coefficient is presented.

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1. Introduction and geometry

In previous work [1] we have used the transformation of the effective action, obtained by integrating the conformal anomaly in two dimensions, to relate the effective actions on regions of the two-sphere and plane. An extension to four dimensions is technically feasible. The relevant transformations when the manifolds have no boundary were given some time ago [2], and, when a boundary exists, in [3] (See also [4]). However, as a preliminary, it is necessary to extend the analysis to manifolds with conical, or vertex, singularities and this is the object of the present paper.

The general method depends upon knowing the constant term in the heat-kernel expansion, in this case the $C_2$ coefficient. When the manifold possesses a singular $O(2)$ fixed-point set of simple conical type, the extra term in $C_2$ has been determined by Fursaev [5]. In [6] the expression was rederived and its conformal behaviour discussed. A further analysis is presented in a later section.

We enlarge on the geometry. Let $\mathcal{N}$ be a totally geodesic submanifold of $\mathcal{M}$, in particular a fixed-point set of codimension two. The global symmetry group is $O(2)$, generated by the circular Killing vector $\partial/\partial \phi$, making $\mathcal{M}$ axially symmetric. Reperiodising the polar angle, $\phi$, from $2\pi$ to $\beta$ turns $\mathcal{M}$ into $\mathcal{M}_\beta$, a space with a simple conical singularity of ‘extent’ $\mathcal{N}$. Close to $\mathcal{N}$, $\mathcal{M}_\beta$ approximates to the product $C_\beta \times \mathcal{N}$ where $C_\beta$ is a cone of angle $\beta$.

As hyper-cylindrical coordinates of a point $P$ on $\mathcal{M}$ we take (i) the distance, $r$, from $P$ along the normal geodesic to its foot on $\mathcal{N}$, (ii) the coordinates, $y^a$, of this foot and (iii) the angle $\phi$, between the tangent to this geodesic, at $r = 0$, and some fiducial normal vector (parallely propagated along $\mathcal{N}$). $\phi$ is the rotation angle ‘about’ $\mathcal{N}$ and, with $r$, makes up the coordinates of the normal space. Close to $\mathcal{N}$, $r$ and $\phi$ are the usual plane polar coordinates.

The metric of $\mathcal{M}$ is generally $g_{\mu\nu}dx^\mu dx^\nu$ and in cylindrical coordinates is taken to be

$$ds^2 = dr^2 + f(r, y)d\phi^2 + g_{ab}(y, r)dy^ad^by^b$$

(1)

where $f(r, y)$ is an even function of $r$ and tends to $r^2$ as $r \to 0$. Also $g_{ab}(y, 0) = h_{ab}(y)$, the metric on $\mathcal{N}$. The region $r \leq b$ forms a tube, $U\mathcal{N}$, surrounding $\mathcal{N}$. If $g_{ab}(y, r) = h_{ab}(y)$, all the surfaces $r = \text{const.}, \phi = \text{const.}$ are totally geodesic.
2. Heat-kernel coefficients

The integrated heat-kernel expansion is written

\[ K_\beta(t) \sim \frac{1}{(4\pi t)^{d/2}} \sum_{n=0,1/2,\ldots}^\infty C_n t^n \]

where \( A_n \) is the usual volume integral, over \( \mathcal{M}_\beta \), of a local, scalar density involving the curvature of \( \mathcal{M} \). The \( F_n \) are the due to the conical singularity and are integrals over \( \mathcal{N} \). In particular, \( F_2 \) is given by

\[ F_2 = \int_{\mathcal{N}} f_2 h^{1/2} d^{d-2} y = \int_{\mathcal{N}} f_2 d\text{vol}_{\mathcal{N}}(y) \]

with the integrand,

\[ f_2 = \frac{\pi}{B} (B^2 - 1) f_{2,1} + \frac{\pi}{360B} (B^4 - 1) f_{2,2} \]

where \( B = 2\pi/\beta \) and

\[ f_{2,1} = \left( \frac{1}{6} - \xi \right) R + \lambda_1 (\kappa \cdot \kappa) - 2\text{tr}(\kappa \cdot \kappa) + \lambda_2 \]

\[ f_{2,2} = \left( 2R_{\mu\nu\rho\sigma} n^\mu n^\rho (n^\nu \cdot n^\sigma) - R_{\mu\nu} n^\nu n^\mu - \frac{1}{2} \kappa \cdot \kappa \right) \]

\[ + \lambda_2 (\kappa \cdot \kappa - 2\text{tr}(\kappa \cdot \kappa)) \] (5)

The constant coefficients, \( \lambda_1 \) and \( \lambda_2 \), of the conformally covariant combination \( (\kappa \cdot \kappa - 2\text{tr}(\kappa \cdot \kappa)) \) are unknown. The \( n_i^\mu \) are the normals to \( \mathcal{N} \) and we may take \( n_i^\mu n_{i\mu} = \delta_{ij} \).

Although the extrinsic curvatures \( \kappa \) are zero for a fixed-point set, it is necessary to retain them when making general conformal transformations. A derivation of (5) is given in section 4.

3. Conformal transformations and the effective action

Under a Weyl rescaling, \( g_{\mu\nu} \rightarrow e^{-2\omega} g_{\mu\nu} \), \( \mathcal{M}_\beta \rightarrow \overline{\mathcal{M}}_\beta \) and \( \mathcal{N} \rightarrow \overline{\mathcal{N}} \). In order to preserve the topology, the transformation function \( \omega(r, \phi, y) \) must have period \( \beta \) in \( \phi \). In general, the O(2) symmetry is destroyed by the rescaling. \( \overline{\mathcal{N}} \) is a submanifold of \( \overline{\mathcal{M}}_\beta \), but not a totally geodesic one. It has nonzero extrinsic curvatures.
When evaluated on \( \mathcal{N} \), or equivalently on \( \overline{\mathcal{N}} \), \( \omega \) becomes independent of the coordinates of the normal space so that, at the singularity, \( \overline{\mathcal{M}}_{\beta} \to \overline{\mathcal{C}}_{\beta} \times \overline{\mathcal{N}} \) where \( \overline{\mathcal{C}}_{\beta} \) is a cone of angle \( \beta \) scaled by a factor depending on its position in \( \overline{\mathcal{N}} \). We could say that \( \overline{\mathcal{M}}_{\beta} \) has a ‘squashed’ conical singularity.

We turn now to an evaluation of the change in the renormalised effective action, \( W_R \), under a conformal transformation for conformal coupling, \( \xi = 1/6 \).

The technique used in the present paper is that explained in \([7, 3, 8, 9]\), involving the conformal transformation of the constant term in the heat-kernel expansion. The general formula is

\[
W_R[e^{-2\omega}g] - W_R[g] = \lim_{d \to d'} (4\pi)^{-d/2} \frac{C_{d'/2}^{(d)}[e^{-2\omega}g] - C_{d'/2}^{(d)}[g]}{d - d'},
\]

where \( d' \) is the dimension of \( \overline{\mathcal{M}}_{\beta} \) and \( d \) is an arbitrary dimension. We set \( d' = 4 \).

This method, in contrast to that of integrating the conformal anomaly, requires an application of finite conformal transformations in \( d \) dimensions. In the present instance there is little to choose between the two methods so far as effort goes.

The total coefficient \( C_2 \) contains the standard volume term \( A_2 \), which is dealt with in \([2]\). In the present paper we are interested only in the effect of \( F_2 \).

The transformations needed are

\[
R_{\mu\nu} n^\nu n^\mu \to e^{2\omega}(R_{\mu\nu} n^\nu n^\mu + (d - 2)\omega_{\mu\nu} n^\mu n^\nu + 2\Delta_2 \omega + 2(d - 2)\Delta_1 \omega)
\]

\[
R_{\mu\nu\rho\sigma} (n^\mu, n^\rho)(n^\nu, n^\sigma) \to e^{2\omega}(R_{\mu\nu\rho\sigma} (n^\mu, n^\rho)(n^\nu, n^\sigma) + 2\omega_{\mu\nu} n^\mu n^\nu + 2\Delta_1 \omega),
\]

where \( \omega_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu} \), and

\[
\kappa_{ab} \to e^{-\omega}(\kappa_{ab} + h_{ab} n^\mu \omega_{\mu}),
\]

\[
\kappa \to e^{\omega}(\kappa + (d - 2)n^\mu \omega_{\mu})
\]

for codimension 2. Also, for reference, \( h^{1/2} \to e^{(2-d)\omega} h^{1/2} \).

The change in \( h^{1/2} f_{2,2} \), for example, is

\[
(4 - d) \left( \omega_{f_{2,2}} + 2\omega n^\mu \omega_{\mu} + \omega_{\mu\nu} n^\mu n^\nu + (1 + d/2)\omega_{\mu\nu} n^\mu n^\nu + \kappa n^\mu \omega_{\mu} + \lambda_2 (d - 2)(\omega_{\mu\nu} n^\mu n^\nu + 2\kappa n^\mu \omega_{\mu}) \right) + O((4 - d)^2) - 2\tilde{\Delta}_2 \omega.
\]

all multiplied by \( h^{1/2} \). \( \tilde{\Delta}_2 \) is the Laplacian intrinsic to \( \mathcal{N} \).
It is then easy to show that the change in the integral (3) is proportional to
\((d-4)\) so that, from (6), our final result can be written

\[
W_R[e^{-2\omega}g] - W_R[g] = \int_{\mathcal{N}} \Delta w \, d\text{vol}_\mathcal{N}(y)
\]

with

\[
\Delta w = -\omega f_2 + \frac{2\pi}{B} \sum_{k=1}^{2} (B^k - 1) \Delta w_k,
\]

where, after setting \(d = 4\) and \(\kappa = 0\),

\[
\Delta w_1 = -\frac{\lambda_1}{8\pi^2} \omega_{\mu|\nu} n^\mu n^\nu
\]

and

\[
\Delta w_2 = -\frac{1}{16\pi^2} \left(2\omega_{\mu|\nu} + \omega_{\mu\nu} n^\mu n^\nu + (2\lambda_2 + 3) \omega_{\mu|\nu} n^\mu n^\nu \right).
\]

The constants \(\lambda_1\) and \(\lambda_2\) remain undetermined. The derivation of the corresponding terms in the presence of a conventional boundary is somewhat complicated, involving either a direct solution of the differential equations or the cleaner, but still longish, functorial properties, [10].

Note that, even if \(\mathcal{N}\) were an O(2) fixed-point subspace of \(\mathcal{M}_\beta\), there would still be a contribution from the \(-\kappa_\kappa/2\) term in (5). In this case \(\omega_{\mu|\nu} n^\mu\) is zero removing dependence on \(\lambda_1\) and \(\lambda_2\) in the final answer.

4. The heat-kernel expansion

For completeness we present a derivation of (4) with (5), similar to that in [5]. The basic idea is that, close to \(\mathcal{N}\), the heat-kernel on \(\mathcal{M}_\beta\) is approximated by that obtained by a process of *reperiodisation* from the heat-kernel on \(\mathcal{M}\). This has been described and used in earlier work [11–13] where the Green functions in some specific curved spaces, e.g. de Sitter and Schwarzschild, were considered.

We are interested in the integrated, diagonal \(K_\beta\)

\[
K_\beta(t) = \int_{\mathcal{M}_\beta} K_\beta(y,r;y,r,t)
\]

and its asymptotic expansion (2).
Because $\mathcal{M}_\beta$ is identical to $\mathcal{M}$ off $\mathcal{N}$, the local heat-kernel expansions of $K_\beta$ and $K_{2\pi}$ will be the same in $\mathcal{M}_\beta - \mathcal{N}$. It is therefore sufficient for the asymptotic expansion to write, thickening out $\mathcal{N}$ by setting $\mathcal{M}_\beta = U\mathcal{N} \cup U\overline{\mathcal{N}}$,

$$K_\beta(t) = \int_{U\mathcal{N}} K_\beta(y, r; y, r, t) + \int_{U\overline{\mathcal{N}}} K_{2\pi}(y, r; y, r, t)$$  \hspace{1cm} (10)

valid up to terms exponentially small as $t \to 0$.

Because of the O(2) symmetry, the Laplacian heat-kernel $K_{2\pi}(y, r, \phi; y', r', \phi', t)$ depends on the polar angles through the difference $\phi - \phi'$ only, and the approximation in the narrow tube $U\mathcal{N}$ is conveniently written as a contour integral,

$$K_{\beta,\delta}(\phi - \phi', t) \approx \int_A K_{2\pi}(\alpha, t) P(\alpha - \phi + \phi'; \beta, \delta) d\alpha$$  \hspace{1cm} (11)

where $P(\alpha; \beta, \delta)$ is the reperiodising function,

$$P(\alpha; \beta, \delta) = \frac{e^{\pi i \alpha (2\delta - 1)/\beta}}{2i \beta \sin (\pi \alpha/\beta)}.$$  \hspace{1cm} (12)

As is our wont, a phase change $\delta$, $0 < \delta \leq 1$, has been included. This will not be made use of here but allows one to discuss fluxes running along the singularity. The contour $A$ has two parts. In the upper half-plane it runs from $(\pi - \epsilon) + i\infty$ to $(-\pi + \epsilon) + i\infty$ and in the lower half-plane from $(-\pi + \epsilon) - i\infty$ to $(\pi - \epsilon) - i\infty$.

It is helpful to exhibit the arguments now,

$$K_{\beta,\delta}(y, r; y', r', t) \approx \frac{1}{2i} \int_A K_{2\pi}(y, r; y', r', \alpha, t) e^{\pi i \alpha (2\delta - 1)/\beta} \cot \left( \frac{\pi(\alpha - \phi + \phi')}{\beta} \right) d\alpha.$$  \hspace{1cm} (13)

The ‘complex point’ $r'_\alpha$ has polar coordinates $(r', \alpha + \phi)$.

We also give the expression for $\delta = 1$, i.e. no phase change,

$$K_\beta(y, r; y', r', t) \approx \frac{1}{2i} \int_A K_{2\pi}(y, r; y', r', \alpha, t) \cot \left( \frac{\pi(\alpha - \phi + \phi')}{\beta} \right) d\alpha$$  \hspace{1cm} (14)

where the symmetry of the contour under reversal of $\alpha$ has been used and the fact that $K_{2\pi}(\phi, t) = K_{2\pi}(-\phi, t)$ by orientation arguments. (The metric (1) is unchanged under reversal of $\phi$. If the cone were spinning, these arguments would have to be revised.)

To isolate the effect of the singularities, the contour $A$ is deformed to a small loop around the origin plus two infinite ‘vertical’ curves, labelled $A'$, which, to
avoid problems, are taken to skirt the origin. The small loop evaluates by the pole at $\alpha = 0$ to $K_{2\pi}$ and so one has the split
\[ K_\beta \approx K_{2\pi} + K'_\beta \] (15)
where $K'_\beta$ is the effect of the singularity and is given by a formula like (14) but now over the $A'$ contour.

Effecting the split (15), and combining with (10), one finds
\[ K_\beta(t) \sim \frac{\beta}{2\pi} K_2\pi(t) + \frac{1}{2\beta i} \int_{A'} \int_{U_N} K_{2\pi}(y, r; y, r_\alpha, t) \cot \left( \frac{\pi \alpha}{\beta} \right) d\alpha \] (16)
where $K_{2\pi}(t)$ is the integrated kernel on the smooth manifold $\mathcal{M}$ and has the standard asymptotic expansion. It will not be considered further. The $\beta/2\pi$ is a volume factor that reflects the $O(2)$ symmetry. The second term is the effect of the singularity. We denote it by $K'_\beta(t)$. The point $(y, r_\alpha)$ is $(y, r)$ rotated through $\alpha$ about $N$.

As explained in our earlier works, the contour $A'$ can be replaced by a small clockwise loop around the origin, and we will imagine this to have been done.

Donnelly [14] has elucidated the asymptotic expansion of
\[ K_{2\pi}(\phi, t) = \int_{\mathcal{M}} K_{2\pi}(y, r; y, r_\phi, t) = \frac{2\pi}{\beta} \int_{A'} K_{2\pi}(y, r; y, r_\phi, t) \] (17)
and his results can be substituted directly into (16) for, as we see, the complex activity takes place in the normal space, the point $y$ of $\mathcal{N}$ being a spectator.

It is clear from the classic results of Minakshisundaram and Pleijel that, up to exponentially small terms, the integral in (17) gets its value from the fixed-point set $r = r_\alpha$ i.e. from $\mathcal{N}$, and so, following Donnelly, for $\phi \neq 0$,
\[ K_{2\pi}(\phi, t) \sim \frac{1}{(4\pi t)^{(d-2)/2}} \sum_{n=0}^{\infty} t^n \int_{\mathcal{N}} b_n(\phi, y) d\text{vol}_{\mathcal{N}}(y). \] (18)
Substitution into (16), after setting $\phi \to \alpha$, gives
\[ K'_\beta(t) \sim \frac{1}{(4\pi t)^{(d-2)/2}} \sum_{n=0}^{\infty} t^n \int_{\mathcal{N}} \int_{A'} b_n(\alpha, y) \cot \left( \frac{\pi \alpha}{\beta} \right) d\alpha d\text{vol}_{\mathcal{N}}(y) \] (19)
and we now concentrate on the contour integral part of this equation
\[ b_n(y) = \frac{1}{4\pi i} \int_{A'} b_n(\alpha, y) \cot \left( \frac{\pi \alpha}{\beta} \right) d\alpha. \] (20)
If $S(\phi)$ is the linear O(2) (actually SO(2)) action on the normal fibre, the general form of the coefficients is, [14],

$$b_n(\phi, y) = \frac{1}{|\det (1 - S)|} b'_n(\phi, y)$$  \hspace{1cm} (21)

where $b'_n(\phi, y)$ is an O($d - 2$)$\times$O(2) invariant polynomial in the components of $T \equiv (1 - S)^{-1}$, the curvature of $M$ and its covariant derivatives.

We indicate the origin of (18) and (21), [14,15]. The local Minakshisundaram-Pleijel parametrix expansion is

$$K_{2\pi}(x; x', t) \sim e^{-\Omega(x,x')/2t} \sum_{n=0}^{\infty} a_n(x, x') t^n$$  \hspace{1cm} (22)

where $\Omega(x, x')$ is half the square of the geodesic distance between $x$ and $x'$. This is substituted into (17) and the integral divided into one over $N$, with coordinates $y$, and one over the normal fibre, with coordinates $r$. Because of the exponential cutoff the integral is restricted to the tubular neighbourhood $U_N$. Transforming the fibre coordinates to normal coordinates, $x^i$, based at $y$, $2\Omega$ becomes $\tilde{\Omega}(1 - S)(1 - S)x$. (This is actually a diagonal form.) The remainder of the integrand, including the volume form and the $a_n$ coefficients, is expanded about the point $(y, 0)$ and the integrals over the $x^i$ extended to $\pm \infty$, again up to exponentially small errors. Standard Gaussian integrals, familiar from perturbation theory, then yield (21). The nontrivial O(2) tensor dependence of $b'_n$ originates in the expansion of the volume factor. We remark that Donnelly’s general expression shows that there is always the contribution $a_n(y, y)$ to $b'_n(\phi, y)$.

In particular, Donnelly calculates for any codimension (we include the $\xi R$ coupling)

$$b'_0(\phi, y) = 1$$

$$b'_1(\phi, y) = \left( \frac{1}{6} - \xi \right) R + \frac{1}{6} R^l_i + \frac{1}{3} R_{ijkl} T^{ijl} T^{k} - \frac{1}{3} R_{ijkl} T^{ik} T^{jl} - R_{ij} T^{ik} T^{jl}$$  \hspace{1cm} (23)

where

$$R_{ijkl} = R_{\mu\nu\rho\sigma} n^\mu_i n^\nu_j n^\rho_k n^\sigma_l$$ and $$R_{ij} = R_{\mu\nu\rho\sigma} h^{\rho\sigma} n^\mu_i n^\nu_j.$$  \hspace{1cm} (24)

$h_{\mu\nu} = g_{\mu\nu} - n_\mu.n_\nu$ is equivalent to the metric on $N$, therefore

$$R_{ij} = R_{\mu\nu} n^\mu_i n^\nu_j - R_{\mu\nu\rho\sigma} n^\rho.n^\sigma n^\mu_i n^\nu_j$$

and

$$R^l_i = R_{\mu\nu} n^\mu_i n^\nu - R_{\mu\nu\rho\sigma} (n^\rho.n^\sigma)(n^\mu.n^\nu).$$
For codimension two the calculation is simplified by noting that
\[ R_{ijkl} = \frac{1}{2} R^{(2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \quad \text{where} \quad R^{(2)} = R_{\mu\nu\rho\sigma}(n^\mu . n^\rho)(n^\nu . n^\sigma) \]
so
\[ R_{ij} = R_{\mu\nu} n_i^\mu n_j^\nu - \frac{1}{2} R^{(2)}\delta_{ij}. \]

Substituting into (23) gives
\[
 b'_1(\phi, y) = \left(1 - \xi\right)R + \frac{1}{6} R^i - R^{(2)} \text{tr} T^2 - \frac{1}{6} R^{(2)}(\text{tr} T)^2 + R_{\mu\nu} n_i^\mu n_j^\nu (T^2)^{ij}. \tag{25}
\]

With respect to normal coordinates,
\[
 S = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}; \quad T = \frac{1}{2(1 - \cos \phi)} \begin{pmatrix} 1 - \cos \phi & \sin \phi \\ -\sin \phi & 1 - \cos \phi \end{pmatrix}
\]
and
\[
 T^2 = -\frac{1}{2(1 - \cos \phi)} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}.
\]
In terms of indices
\[
 T_{ij} = \frac{1}{2} \left( \delta_{ij} + \frac{\sin \phi}{1 - \cos \phi} \epsilon_{ij} \right) = \frac{1}{2} \left( \delta_{ij} + \cot(\phi/2)\epsilon_{ij} \right). \tag{26}
\]

The characteristic equation is \( T^2 - T = -1 \det T = -\tilde{T}T. \)

One sees from the symmetry of the matrices that, in (25), \((T^2)^{ij}\) can be replaced by \(\delta^{ij}\) \text{tr} \(T^2\)/2. Furthermore \(\text{tr} T = 1\) giving,
\[
 b'_1(\phi, y) = \frac{1}{6} \left( (1 - 6\xi)R + (1 + 6\text{tr} T^2) \left( R_{\mu\nu} n^\mu . n^\nu - 2R^{(2)} \right) \right). \tag{27}
\]

These results give the explicit dependence on the angle \(\phi\). If \(\phi\) is replaced by the complex angle \(\alpha\), the expressions can be substituted into the contour integral (20). Noting that \(|\det(1 - S(\phi))| = 2(1 - \cos \phi) \to 2(1 - \cos \alpha)\) and \(\text{tr} T^2 = -1 + 1/(1 - \cos \phi)\), from (20), (21) and (27) we encounter the polynomials, [16],
\[
 P_k(\beta, \delta) = \frac{1}{\beta i} \int_{A'} \frac{1}{(1 - \cos \alpha)^k} \frac{\cos(\pi \alpha(2\delta - 1)/\beta)}{\sin(\pi \alpha/\beta)} d\alpha. \tag{28}
\]
A routine residue calculation gives

\[
P_1(\beta, \delta) = \frac{1}{3}(B^2 - 1) - 2B^2\sigma,
\]

\[
P_2(\beta, \delta) = \frac{1}{90}(B^2 - 1)(B^2 + 11) - \frac{1}{3}B^2\sigma(B^2\sigma + 2)
\]

\[
P_3(\beta, \delta) = \frac{1}{3780}(B^2 - 1)(2B^4 + 23B^2 + 191),
\]

\[
- \frac{1}{90}B^2\sigma(B^4\sigma(2\sigma + 1) - 15B^2\sigma - 24),
\]

\[
P_4(\beta, \delta) = \frac{1}{85050}(B^2 - 1)(B^2 + 11)(3B^4 + 10B^2 + 227),
\]

\[
- \frac{1}{5670}B^2\sigma(B^6(3\sigma^2 + 4\sigma + 2) + 28B^4\sigma(2\sigma + 1) + 294B^2\sigma + 432),
\]

where \( \sigma = \delta(1 - \delta) \). In this paper \( \sigma = 0 \).

Combining terms in (27) we obtain (4) and (5) for \( \xi = 0 \), as promised.

5. The general coefficient

A typical term in the general coefficient \( b'_n(\alpha, y) \) has the form

\[
M_{ij...kl}T^{ij}...T^{kl}
\]

where \( M_{..} \) is an appropriate combination of the curvature, \( R_{\mu\nu\rho\sigma} \), its covariant derivatives and the normal vectors \( n^\mu_i \). Using (26), a parity argument, or the symmetry of the \( \alpha \)-integral, shows that there can only be an even number of \( \epsilon \)-symbols so (30) reduces to a series of contractions of \( M \). From (26) it follows that each pair of \( \epsilon \)-symbols will increase the order of the polynomials, \( P_k \), by one.

From dimensions, the maximum number of pairs in \( b'_n \) equals \( n \), an example being \( (R_{ijkl}T^{ij}T^{kl})^n \), and so the general form of (20) is

\[
b_{n-1}(y) = \frac{2\pi}{B} \sum_{k=1}^{n} P_k(B)G_{nk}
\]

where the \( G_{nk} \) are integrals over \( \mathcal{N} \) of a local, scalar expression constructed from the curvature of \( \mathcal{M} \), its covariant derivatives and the normals \( n^\mu_i \). Form (31) was given by Fursaev [5]. It allows one to set up a functorial method on the lines of Branson and Gilkey [10] for the determination of the coefficients.
6. Comments

Donnelly’s expression can also be applied to the special case where $\beta$ is an integral part of $2\pi$, $\beta = 2\pi/q$, $q \in \mathbb{Z}$. This would involve a preimage summation which can be effected to yield the expected answer, as mentioned by Fursaev [5].

A natural extension is to higher codimensions. However a simple process similar to that of periodisation does not appear to exist. If we think of codimension two as corresponding to a dihedral angle (with sides identified) then codimension three corresponds to a trihedral corner, and the heat-kernel for such a domain is unknown except for special cases.

Regarding the conformal transformation, in order to apply the result (7) with confidence, it would be helpful to have an independent check. This would entail finding two conformally related spaces, with conformally related singular fixed-point sets, on which one could determine the effective action, or at least that part due to the singularities. A possibility is the Einstein universe with a cosmic string [17]. This is conformal to $\mathbb{R}^4$ with a string.

Another possibility is to check the heat-kernel coefficients (5) themselves. This can be done in any dimension since the coefficients are universal. However it necessitates finding a tractable space with a deformed conical submanifold *i.e.* one with nonzero extrinsic curvatures.

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