Control of Stirling engine. Simplified, compressible model

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Abstract. A one-dimensional free boundary problem on a motion of a heavy piston in a tube filled with viscous gas is considered. The system of governing equations and boundary conditions is derived. The obtained system of differential equations can be regarded as a mathematical model of an exterior combustion engine. The existence of a weak solution to this model is proved. The problem of maximization of the total work of the engine is considered.

1. Problem formulation
The paper deals with a free boundary problem which describes the motion of a heavy piston in a tube filled with viscous gas. It is assumed that the viscous gas occupies the interval $(0, 1)$. The piston is modeled by a material point $a \in (0, 1)$. Its motion is described by a function $x = a(t)$, $t \in (0, T)$. We also assume that the dynamics of the piston is governed by the equation

$$m \ddot{a}(t) + G(\dot{a}(t), a(t)) = F,$$

where $F$ is the hydrodynamical force acting on the piston, $G$ is a given function. This means that the piston is connected with some mechanical device which dynamics is described by the function $G$. The state of the gas is completely described by the density $\rho(x, t)$, the velocity $u(x, t)$, and the temperature $\vartheta(x, t)$. Among the other thermodynamic quantities there are the internal energy $e$, the pressure $p$, and the entropy $s$. The three quantities are assumed to be functions of state variable $\rho$ and $\vartheta$. Typically, the pressure formula admits the representation

$$p = p_e(\rho) + p_\vartheta(\vartheta) \vartheta.$$

The internal energy is defined by the relation

$$\frac{\partial e}{\partial \rho} = \frac{p_e(\rho)}{\rho^2},$$

which yields the representation

$$e = P_e(\rho) + Q(\vartheta), \quad P_e(\rho) = \int_1^\rho z^{-2} p_e(z) dz,$$
where \( Q \) is an arbitrary function of \( \theta \). The entropy \( s(\varrho, \theta) \) is defined by the relations
\[
s = \int_1^\varrho \frac{Q(z)}{z} dz - P_\theta(\varrho), \quad P_\theta(\varrho) = \int_1^\varrho \frac{p_\theta(z)}{z^2} dz.
\]

In the simple case of barotropic gas, the pressure is a function of the density, \( p = p(\varrho) \), and the entropy \( s \) equals zero. This eliminates the temperature from the equations.

In this framework, the problem can be formulated as follows. Denote by \( \Omega \) the rectangular \( (0, 1) \times (0, T) \) and set
\[
\Omega^- = \{(x, t) : 0 < x < a(t), \quad 0 < t < T\}, \quad \Omega^+ = \{(x, t) : a(t) < x < 1, \quad 0 < t < T\}
\]

The problem is to find the velocity, the density, the temperature, and the trajectory of the piston satisfying the following equations and boundary conditions
\[
\frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho u)}{\partial x} = 0 \quad \text{in} \quad \Omega^\pm, \tag{2}
\]
\[
\frac{\partial (\varrho u)}{\partial t} + \frac{\partial (\varrho u^2)}{\partial x} + \frac{\partial (p - \nu u_x)}{\partial x} = 0 \quad \text{in} \quad \Omega^\pm, \tag{3}
\]
\[
\frac{\partial}{\partial t} \left( \varrho \left( \frac{u^2}{2} + e \right) \right) + \frac{\partial}{\partial x} \left( u \left( \frac{\varrho^2}{2} + p e + p \right) \right) - \frac{\partial}{\partial x} (\kappa(\theta) \frac{\partial \theta}{\partial x}) - \nu \frac{\partial}{\partial x} (u u_x) = 0 \quad \text{in} \quad \Omega^\pm, \tag{4}
\]
\[
u \dot{a}(t), t = \dot{a}(t) \quad \text{in} \quad (0, T), \tag{5}
\]
\[
m \dot{a}(t) + G(\dot{a}(t), a(t)) = (T^+(a(t), t) - T^-(a(t), t)) \quad \text{in} \quad (0, T), \tag{6}
\]
\[
\left( \kappa \frac{\partial \theta}{\partial x} \right)^+(a(t), t) - \left( \kappa \frac{\partial \theta}{\partial x} \right)^-(a(t), t) = (T^-(t) - T^+(t)) u(a(t), t) \quad \text{in} \quad (0, T), \tag{7}
\]
\[
\theta^+(a(t), t) = \theta^-(a(t), t). \tag{8}
\]

Here \( \kappa(\theta) \) is a given positive function, mostly a constant. The stress is defined by the equality
\[
T = \nu u_x - p.
\]

We also use the notation
\[
T^\pm(t) := \lim_{x \to a(t) \pm 0} T(x, t), \quad \vartheta^\pm(t) := \lim_{x \to a(t) \pm 0} \vartheta(x, t). \tag{9}
\]

The system of equations (2)-(8) should be supplemented with the boundary conditions. Choose functions \( U^\pm : (0, T) \to \mathbb{R}, \Theta^\pm : (0, T) \to \mathbb{R}^+, \) and \( R^\pm : (0, T) \to \mathbb{R}^+ \). Denote by \( \Sigma_{in}^\pm \subset [0, T] \) the inlets and by \( \Sigma_{out}^\pm \subset [0, T] \) the outlets defined by \( U \),
\[
\Sigma_{in}^- = \{ t \in (0, T) : U(0, t) > 0 \},
\]
\[
\Sigma_{in}^+ = \{ t \in (0, T) : U(1, t) < 0 \},
\]
\[
\Sigma_{out}^\pm = (0, T) \setminus \Sigma_{in}^\pm.
\]

We set the boundary conditions
\[
u(0, t) = U^-(t), \quad u(1, t) = U^+(t) \quad \text{for} \quad t \in (0, T), \tag{11}
\]
\[
\varrho(0, t) = R^-(t) \quad \text{on} \quad \Sigma_{in}^-,
\]
\[
\varrho(1, t) = R^+(t) \quad \text{on} \quad \Sigma_{in}^+,
\]
\[
\theta(0, t) = \Theta^-(t), \quad \theta(1, t) = \Theta^+(t) \quad \text{for} \quad t \in (0, T). \tag{14}
\]
At the initial moment \( t = 0 \) the velocity and the density distributions are prescribed

\[
\begin{align*}
u(x,0) &= u_0(x), & \varrho(x,0) &= R_0(x), & \theta(0,x) &= \Theta_0(x) & \text{for } x \in (0,1).
\end{align*}
\] (15)

The initial position of the piston is given

\[ a(0) = a_0. \] (16)

The equations, boundary, and initial conditions (2)–(16) form a closed system of equations for the velocity, the density, the temperature, and the trajectory of the piston. This system can be regarded as a mathematical model of an exterior combustion engine often called the Stirling engine.

An intensive treatment of one-dimensional compressible Navier-Stokes equations starts with pioneering book by Antontsev, Kazhikhov, and Monakhov [1] on the global theory for one-dimensional homogeneous problems. Although the theory of strong solutions is satisfactory in what concerns the specific boundary conditions, many issues of global behavior of solutions for large data in general boundary conditions are far from being understood.

A global theory of weak solutions was developed by P.L. Lions. We refer the reader to the books by Lions [3], Feireisl [2], and Novotný & Straškraba [4] for the state of the art in the domain. The theory of weak solutions to inhomogeneous boundary value problems for compressible Navier-Stokes equations was developed in the book by Plotnikov & Sokolowski [5]. The problem on the motion of the piston was studied in the paper by Shelukhin [6] under the assumptions that \( \mathcal{U}_\pm = 0 \) and \( \mathcal{G} = 0 \).

In this paper we are focused on the case when the thermodynamical process is adiabatic and the dynamics of the piston is conservative. This means that

\[
p = c\varrho^\gamma, \quad \gamma > 1, \quad G(\dot{a},a) = V'(a), \quad (17)
\]

where \( V : (0,1) \to \mathbb{R} \) is a mechanical potential. In this case equations (2)–(16) read

\[
\begin{align*}
\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}(\varrho u) &= 0 \quad \text{in} \quad \Omega^\pm, \\
\frac{\partial (\varrho u)}{\partial t} + \frac{\partial}{\partial x}(\varrho u^2) + \frac{\partial}{\partial x}(p - \nu u_x) &= 0 \quad \text{in} \quad \Omega^\pm, \\
u(a(t),t) &= \dot{a}(t) \quad \text{in} \quad (0,T), \\
m\ddot{a}(t) + V'(a(t)) &= (\mathbb{T}^+(a(t),t) - \mathbb{T}^-(a(t),t)) \quad \text{in} \quad (0,T).
\end{align*}
\] (18) (19) (20) (21)

\[
\begin{align*}
u(0,t) &= U^-(t), & u(1,t) &= U^+(t) & \text{for} \quad t \in (0,T), \\
\varrho(0,t) &= R^-(t) & \text{on} \quad \Sigma^-_{\operatorname{in}}, \\
\varrho(1,t) &= R^+(t) & \text{on} \quad \Sigma^+_{\operatorname{in}}, \\
\end{align*}
\] (22) (23) (24)

\[
\begin{align*}
u(x,0) &= u_0(x), & \varrho(x,0) &= R_0(x) > 0, & \text{for} \quad x \in (0,1), & a(0) &= a_0. \\
\end{align*}
\] (25)

Furthermore, we assume that the potential \( V \) and the given data satisfy the following conditions

**H.1** \( V : (0,1) \to \mathbb{R}^+ \) is a smooth positive function such that

\[
\lim_{a \to 0^+} V(a) = \lim_{a \to 1-0} V(a) = \infty.
\] (26)
The functions $U^\pm$ belong to the class $C^2[0,T]$. Every set $\Sigma^\pm_\text{in}$ consists of finite number of open intervals.

The functions $R^\pm \in L^\infty(0,T)$ are nonnegative. The function $R_0 \in L^\infty(0,1)$ is strictly positive. Moreover,

$$u_0 \in L^\infty(0,1), \quad 0 < a_0 < 1.$$ 

Condition (26) prevents the destruction of the mechanical system.

### 2. Weak solution. Existence theorem.

The triplet

$$(\rho, u) : \Omega \rightarrow \mathbb{R}^+ \times \mathbb{R}, \quad a : (0, T) \rightarrow (0, 1)$$

is a weak solution to problem (18)–(25) if the following conditions are satisfied:

- The state variables satisfy

  $$\rho, \rho_u, \rho u^2 \in L^\infty(0,T;L^1(\Omega)), \quad u \in L^2(0,T;W^{1,2}(0,1)), \quad a \in W^{1,2}(0,1) \quad (27)$$

  $$u(0,t) = U^-(t), \quad u(1,t) = U^+(t) \quad \text{on} \ (0,T). \quad (28)$$

- The integral identity

  $$\int_\Omega \rho u \zeta_t + \rho u^2 \zeta_x + (p - \nu u_x) \zeta_x dx dt + \int_0^1 \rho_0(x) u_0(x) \zeta(x,0) dx$$

  $$+ m u_0(a_0) \zeta(a_0) + \int_0^T (m \dot{a}(t)^2 \partial_x \zeta(a(t),t) - V'(a(t)) \zeta(a(t),t)) dt = 0 \quad (29)$$

holds for all functions $\zeta \in C^1(\Omega)$ such that

$$\zeta(x,T) = 0, \quad \zeta(0,t) = \zeta(1,t) = 0.$$ 

- The integral identity

  $$\int_\Omega (\rho \partial_t \psi + \rho u \cdot \nabla \psi) dx dt = \int_0^T \psi R^+ U^+ dt - \int_0^T \psi R^- U^- dt - \int_0^1 R_0 \psi(\cdot,0) dx \quad (30)$$

holds for all functions $\psi \in C^1(\Omega)$ satisfying

$$\psi = 0 \quad \text{in a neighborhood of} \ \Sigma^\pm_\text{out} \cup \{(0,1) \times \{T\}\},$$

which means that the support of the test function $\psi$ meets the boundary of the rectangular $\Omega$ at the inlet $\Sigma^\pm_\text{in}$ and the bottom $(0,1) \times \{0\}$.

The first main result of the study is the following

**Theorem 1** Assume that Conditions (H.1)–(H.3) are fulfilled. Then problem (18)–(25) has a weak solution which satisfies the estimates

$$\|u\|_{L^2(0,T;W^{1,2}(0,1))} + (\rho \|u\|_{L^\infty(0,T;L^1(0,1))} + \|p\|_{L^\infty(0,T;L^1(0,1))} \leq c,$$

$$\sup_{t \in (0,T)} \dot{a}(t)^2 + V(a(t)) \leq c, \quad (31)$$

where the constant $c$ depends only on $T$ as well as the boundary and initial data.

The proof is based on the methods developed in [5]. Following [5] we use the multistep regularization. The weak solutions is obtained as a limit of a sequence of approximate solutions to regularized equations. The main difficulty is the derivation of estimate (31).
3. The optimization problem.
Recall that the power of the hydrodynamical forces acting on the piston is equal to $(\mathbb{T}^+(a(t), t) - \mathbb{T}^-(a(t), t)) \dot{a}(t)$. Assume that the functions $U^\pm$ and $u_0, R_0$ are fixed and satisfy conditions (H.1)–(H.3). In this case the total work $W$ of hydrodynamical forces acting on the piston becomes the function of $R^\pm$,

$$ W(R^\pm) = \int_0^T (\mathbb{T}^+(a(t), t) - \mathbb{T}^-(a(t), t)) \dot{a}(t) dt $$

(32)

In view of Theorem 1 the functional $W$ is well defined on the set of bounded nonnegative functions. Fix an arbitrary $N > 0$ and consider the set of admissible data

$$ \mathfrak{M} = \{ R^\pm \in L^\infty(0, T) : \| R^\pm \|_{L^\infty(0, T)} \leq N \}.$$

The natural question is to find the boundary data which maximize the total work of the engine. This problem can be formulated as follows: to find an admissible couple $R^\pm \ast \in \mathfrak{M}$ such that

$$ W(R^\pm \ast) = \sup_{R^\pm \in \mathfrak{M}} W(R^\pm), \quad (33)$$

where the cost function is given by (32). The following theorem on solvability of variational problem (33) is the second main result of the study.

**Theorem 1** Assume that conditions (H.1)–(H.3) are fulfilled. Then problem (33) has at least one solution $R^\pm \ast \in \mathfrak{M}$.

The proof is based on Theorem 1. The key observation is that the functional $W$ is star weakly continuous on $\mathfrak{M}$.

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