A Note on Eye Movement

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Abstract

In a simplified fashion, the motion of the eyeball in its orbit consists of rotations around a fixed point. Therefore, this motion can be described in terms of the Euler’s angles of rigid body dynamics. However, there is a physiological constraint in the motion of the eye which reduces to two its degrees of freedom, so that one of Euler’s angles is not an independent variable. This paper reviews the basic features of the kinematics of the eye and the laws governing its motion.

Key words: Eye Movement, Rotations, Euler’s Angles, Listing’s Law, Donders’ Law.

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1 Introduction

The kinematics of the human eye has been extensively investigated in physics and medicine [1, 2, 3, 4, 5]. From the wealth of material available we have selected a few topics of more direct interest to physicists concerning with the description of rotational motion of the eye and its biological constraints.

In the simplified model of the eye, the organ is considered to be a spherical surface held by six muscles in a bony cavity known as orbit or eye socket. In the relaxed normal eye, the pressure of internal fluids maintains the radius of curvature of the eye at a constant value of approximately 12 mm. Since the actual size of the eyeball is in fact irrelevant to our analysis, as soon as it is constant, we assume the eyeball to be a spherical surface of unit radius.

We also disregard any translational motion of the eye with respect to the head, and consider only the rotary motions of the eyeball in the orbital socket. To a good approximation the center of rotation is fixed and coincides with the center of the eyeball $O$.

The basic rotations of the eye consist of its motions in looking from side to side along the horizontal meridian (a rotation around a vertical axis), and up and down (a rotation around a horizontal axis). These two motions are called cardinals.

There is a third possible motion called torsion, which is a rotation of the eye to compensate for an eventual tilting of the head while keeping the same direction of vision. Torsion also occurs when the pupil moves forcibly up or down, approaching the nose [7].

Instead of the cardinal angles just mentioned, we prefer to describe the motion of the eyeball in terms of Euler’s angles.

2 Euler’s Description of Motion

Suppose one holds the head upright and looks directly toward a distant target object horizontally straight in front. The line extending from $O$ to the fixation point is called the visual axis, and this position defines the primary position of the visual axis, to which all other will be referred (See [8] for some remarks on this). In changing the direction of the visual axis as the target object moves,
the eye rotates around the point $O$.

For the description of the rotational motion we set up a system of axis $OABC$ attached to the eyeball and moving with it. The position of the moving axes relative to a head-fixed system $OXYZ$ is determined by Euler’s angles. Assume initially the two systems coincide, and take the primary position of the visual axis to be the horizontal direction $OZ$ or $OC$ (The eye is looking to the reader), as shown in Fig. 1.

Starting from the primary position a general displacement of the eyeball can be effected by the following sequence of rotations (all of which performed in anti-clockwise direction). The eyeball is first turned around the axis $OZ$ through an angle $\psi$. This rotation brings the axes $OX$ and $OY$ to the positions $OX'$ and $OY'$. A second rotation around the axis $OY'$ through an angle $\theta$ moves $OZ$ to the position $OC$ and $OX'$ to the position $OA'$ both on the plane $OZX'$. A final rotation around $OC$ through an angle $\phi$ brings $OA'$ to $OA$ and $OY'$ to $OB$ both on the same plane $OA'Y'$.

In this representation, the torsion motion of the eyeball is simply the third rotation around the visual axis by an angle $\phi$.

3 Donders’ Law

In the movements of the eye there is a physiological constraint that limits the motion of the eyeball around the visual axis.

According to Donders’ law:

\textit{Every secondary position of the eyeball is associated with a definite and constant amount of torsion, no matter how the position be reached.}

This means that the eye needs only two parameters to be fully described, the value of the angle of torsion $\phi$ being fixed for given $\psi$ and $\theta$.

This natural reduction of the degrees of freedom is connected with the fact that when a target object is viewed with the head in a certain position relative to it, the image of the object must be formed on the same region of the retina whenever the gaze is directed to the same point of the object, no matter what the movements the eye might have made between two gazes of the same
It is easy to see that after the rotations by the angles $\psi$ and $\theta$, the eyeball at the position $C$ will have turned by an amount $\psi$ relative to the object viewed in comparison with its relative position at the primary position $Z$. In order that the gaze affect the same elements of the retina the third rotation around $OC$ by an amount $\phi$ must restore the initial relative position of the eyeball and the object. This is accomplished by performing a rotation $\phi = -\psi$ around $OC$.

4 Listing’s Law

Listing explained how the position of the eye could be described by only two independent quantities. According to Listing’s law for the motion of the eyeball [11]:

Any movement of the eye from the primary position to any other secondary position is equivalent to a single rotation around an axis perpendicular both to $OZ$ and $OC$ through an angle $ZOC$.

In Fig. 1 this axis of rotation is the axis $OY'$, making an angle $\psi$ (the meridional angle) with the vertical plane in the primary position, and the angle of rotation is $\theta$ (also called the eccentricity). This law does not specifies how the eye actually moves from the primary position to any secondary position. It is not even necessary that the motion be effected in this way, but the result must be the same whatever the path the gaze may have taken to reach that final position [12] (Compare with Euler’s theorem: The general displacement of a rigid body with one point fixed is a rotation around some axis).

In these terms, Donders’ law appears as a statement that the position of the eye as a whole is uniquely determined by the direction of the visual axis.

The constraint on the movement of the eye expressed by Listing’s law can not be of mechanical nature only, but may also have a neurological component, since it has been observed that the eyes do not obey Listing’s law during sleep [12].


5 Torsion and False Torsion

In the motion of the eye away from the primary position in the manner prescribed by Listing’s law, which is by a rotation around $OY'$ through an angle $\theta$, the vertical meridian plane of the eye in the primary position $OYZ$ traces out the surface of a cone of vertex $O$ and semi-angle $\psi$ formed by the axes $OY$ and $OY''$ and bisected by the axis of rotation $OY'$, as shown in Fig. 2. In its rotation the axis $OY$ moves to its new position $Oy$. The axes $OY$, $OY''$ and $Oy$ meet the unit sphere at the points $P, Q, R$ respectively. The straight line $PQ$ cuts the axis $OY'$ at $T$ defining the lengths $PT = \sin \psi$ and $OT = \cos \psi$.

This motion is associated with a definite inclination (also called a tilt or torsion) of the vertical meridian plane in its secondary position $OyC$ with respect to the vertical plane $OYC$ passing through $OY$ and the visual axis in the secondary position [13]. The inclination depends on the meridional and eccentricity angles in the following way. In Fig. 2, the vertical plane $OYC$ is normal to the plane $OTR$, and cuts the line $TR$ at $S$. The angle of inclination $\rho$ is the angle between $ORS$ in the plane $ORT$. From the triangle $OTR$ we have [7]

$$
\tan(\psi - \rho) = \frac{ST}{OT},
$$

where $ST = PT \cos \theta$ and $OT = \cos \psi$. Thus $\tan(\psi - \rho) = \tan \psi \cos \theta$. Solving for $\tan \rho$ we get

(Check it [7] and compare with [10, 13])

$$
\tan \rho = \frac{\tan \psi (1 - \cos \theta)}{1 + \tan^2 \psi \cos \theta}.
$$

In our use of Euler’ angles, the torsional rotation around $OC$ by $-\psi$ was intended to bring the eyeball to the position it would assume under Listing’s law. Since the final orientation of the eye depends on the way rotations are carried out, had we chosen another set of axis to describe the motion of the eye we would have to perform a torsion motion of different magnitude. Because of this, the torsional motion required to be performed in order to bring the eyeball to the position specified by Listing’s law when using any other system of axis is often called a false torsion [10].

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Figure 1: Rotary Motions of the Human Eye
Figure 2: The Tilt of the Vertical