ON THE KODAIRA DIMENSION OF $\overline{M}_{16}$

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ABSTRACT. We prove that the moduli space of curves of genus 16 is not of general type.

The problem of determining the nature of the moduli space $\overline{M}_g$ of stable curves of genus $g$ has long been one of the key questions in the field, motivating important developments in moduli theory. Severi [Sev] observed that $\overline{M}_g$ is unirational for $g \leq 10$, see [AC] for a modern presentation. Much later, in the celebrated series of papers [HM], [H], [EH], Harris, Mumford and Eisenbud showed that $\overline{M}_g$ is of general type for $g \geq 24$. Very recently, it has been showed in [FJP] that both $\overline{M}_{22}$ and $\overline{M}_{23}$ are of general type.

On the other hand, due to work of Sernesi [Ser], Chang-Ran [CR1], [CR2] and Verra [Ve] it is known that $\overline{M}_g$ is unirational also for $11 \leq g \leq 14$. Finally, Bruno and Verra [BV] proved that $\overline{M}_{15}$ is rationally connected. Our result is the following:

Theorem 1. The moduli space $\overline{M}_{16}$ of stable curves of genus 16 is not of general type.

A few comments are in order. The main result of [CR3] claims that $\overline{M}_{16}$ is uniruled. It has been however recently pointed out by Tseng [Ts] that the key calculation in [CR3] contains a fatal error, which genuinely reopens this problem (after 28 years!).

Before explaining our strategy of proving Theorem 1 recall the standard notation $\Delta_0, \ldots, \Delta_{\lfloor g/2 \rfloor}$ for the irreducible boundary divisors on $\overline{M}_g$, see [HM]. Here $\Delta_0$ denotes the closure in $\overline{M}_g$ of the locus of irreducible 1-nodal curves of arithmetic genus $g$. Our approach relies on the explicit uniruled parametrization of $\overline{M}_{15}$ found by Bruno and Verra [BV]. Their work establishes that through a general point of $\overline{M}_{15}$ there passes not only a rational curve, but in fact a rational surface. This extra degree of freedom, yields a uniruled parametrization of $\overline{M}_{15,2}$, therefore also a parametrization the boundary divisor $\Delta_0$ inside $\overline{M}_{16}$. We show the following:

Theorem 2. The boundary divisor $\Delta_0$ of $\overline{M}_{16}$ is uniruled and swept by a family of rational curves, whose general member $\Gamma \subseteq \Delta_0$ satisfies $\Gamma \cdot K_{\overline{M}_{16}} = 0$ and $\Gamma \cdot \Delta_0 > 0$.

Assuming Theorem 2 we easily conclude that $\overline{M}_{16}$ cannot be of general type.

Proof of Theorem 1. Note that in any effective representation of the canonical divisor

$$K_{\overline{M}_{16}} \equiv \alpha \cdot \Delta_0 + D,$$

where $\alpha \in \mathbb{Q}_{>0}$ and $D$ is an effective $\mathbb{Q}$-divisor on $\overline{M}_{16}$ not containing $\Delta_0$ in its support, we must have $\alpha = 0$. Indeed, we can choose the curve $\Gamma$ such that $\Gamma \nsubseteq D$, then we write

$$0 = \Gamma \cdot K_{\overline{M}_{16}} = \alpha \Gamma \cdot \Delta_0 + \Gamma \cdot D \geq \alpha \Gamma \cdot \Delta_0 \geq 0,$$

hence $\alpha = 0$. 

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Furthermore, since the singularities of $\overline{M}_g$ do not impose adjunction conditions [HM, Theorem 1], $\overline{M}_g$ is a variety of general type for a given $g \geq 4$ if and only the canonical class $K_{\overline{M}_g}$ is a big divisor class, that is, it can be written as
\[
K_{\overline{M}_g} = A + E,
\]
where $A$ is an ample $\mathbb{Q}$-divisor and $E$ is an effective $\mathbb{Q}$-divisor respectively. Assume that $K_{\overline{M}_{15}}$ can be written like in (1). It has already been observed that $\Delta_0 \not\subseteq \text{supp}(E)$, in particular $\Gamma \cdot E \geq 0$. Using Kleiman’s ampleness criterion, $\Gamma \cdot A > 0$, which yields the immediate contradiction $0 = \Gamma \cdot K_{\overline{M}_{15}} = \Gamma \cdot A + \Gamma \cdot E \geq \Gamma \cdot A > 0$. \hfill $\Box$

The Bruno-Verra parametrization of $\overline{M}_{15}$. The parametrization of $\Delta_0$ and the proof of Theorem 1 uses several important results from [BV], which we now recall. We denote by $\mathcal{H}_{15,9}$ the Hurwitz space parametrizing degree 9 covers $C \to \mathbb{P}^1$, where $C$ is a smooth curve of genus 15. Then $\mathcal{H}_{15,9}$ is birational to the parameter space $\mathcal{G}_{15,9}^1$ classifying pairs $(C, A)$, where $[C] \in \overline{M}_{15}$ and $A \in W^1_9(C)$ is a pencil. By residuation, $\mathcal{G}_{15,9}^1$ is isomorphic to the parameter space $\mathcal{G}_{15,19}^6$ classifying pairs $(C, L)$, where $C$ is a smooth curve of genus 15 and $L \in W^3_{19}(C)$. In particular, $\mathcal{G}_{15,19}^6$ is irreducible. Note that the general fibre of the forgetful map $\mathcal{G}_{15,19}^6 \to \overline{M}_{15}$ is 1-dimensional.

We pick a general element $[C, L] \in \mathcal{G}_{15,19}^6$, in particular $L$ is very ample and $h^0(C, L) = 7$. We set $A := \omega_C \otimes L^7 \in W^1_9(C)$. We may assume that $A$ is base point free and the pencil $|A|$ has simple ramification. We consider the multiplication map \[
\phi_L : \text{Sym}^2 H^0(C, L) \to H^0(C, L^2).
\]
Since $C$ is Petri general, $h^1(C, L^2) = 0$, therefore $h^0(C, L^2) = 2 \cdot 19 + 1 - 15 = 24$. Furthermore, via a degeneration argument it is shown in [BV] Theorem 3.11, that for a general choice of $(C, L)$, the map $\phi_L$ is surjective, hence $h^0(\mathbb{P}^6, \mathcal{I}_C|_{\mathbb{P}^6}(2)) = \dim(\text{Ker}(\phi_L)) = 4$, that is, the degree 19 curve $C \subseteq \mathbb{P}^6$ lies on precisely 4 independent quadrics. We let
\[
S := \text{Bs}|\mathcal{I}_C|_{\mathbb{P}^6}(2)\vert
\]
be the base locus of the system of quadrics containing $C$. It is further established in [BV] Theorem 3.11 that under our generality assumptions, $S$ is a smooth surface. From the adjunction formula it follows that $\omega_S = \mathcal{O}_S(1)$, that is, $S$ is a canonical surface. We write down the exact sequence
\[
0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0.
\]
From the adjunction formula $\mathcal{O}_C(C) \cong \omega_C \otimes \omega_{S/C}^\vee = \omega_C \otimes L^7 = A \in W^3_9(C)$. Since $S$ is a regular surface, by taking cohomology in (3), we obtain
\[
h^0(S, \mathcal{O}_S(C)) = h^0(S, \mathcal{O}_S) + h^0(C, A) = 3.
\]
Observe also from the sequence (3) that the linear system $|\mathcal{O}_S(C)|$ is base point free, for $|\mathcal{O}_C(C)| = |A|$ is so. This concludes our recap of results from [BV], amounting to the fact that through a general point $[C] \in \overline{M}_{15}$ there passes a rational surface, obtained as the image of the moduli map \[
\mathbb{P}^2 \cong |\mathcal{O}_S(C)| \dashrightarrow \overline{M}_{15}.
\]
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The uniruledness of the boundary divisor $\Delta_0$ in $\overline{M}_{16}$.

We now lift the construction discussed above from $\overline{M}_{15}$ to the moduli space $\overline{M}_{15,2}$ of 2-pointed stable curves of genus 15. We start with a general curve $C$ of genus 15 and consider the correspondence

$$\Sigma := \left\{ (A, x + y) \in W_9^1(C) \times C_2 : H^0(C, A(-x - y)) \neq 0 \right\},$$

endowed with the projections $\pi_1: \Sigma \to W_9^1(C)$ and $\pi_2: \Sigma \to C_2$ respectively. Here $C_2$ is the second symmetric product of $C$. It follows that $\Sigma$ is an irreducible surface and that $\pi_2$ is generically finite. Indeed, for a general point $2x \in C_2$, we can invoke for instance [EH, Theorem 1.1] to conclude that $\pi_2^{-1}(2x)$ is finite. The fibre $\pi_1^{-1}(A)$ is irreducible whenever $A$ has simple ramification.

We now fix a general element $[C, x, y] \in \overline{M}_{15,2}$. Then there exist finitely many pencils $A \in W_9^1(C)$ containing both points $x$ and $y$ in the same fibre. Furthermore, each of these pencils $A$ can be assumed to be base point free with simple ramification and general enough such that $L := \omega_C \otimes A^r \in W_9^6(C)$ is very ample and in the embedding

$$\varphi_L: C \hookrightarrow \mathbb{P}^6$$

the curve $C$ lies on precisely 4 independent quadrics intersecting in a smooth canonical surface $S$ defined by (2).

Proposition 3. With the notation above, if $h^0(C, A(-x - y)) = 1$, then $\dim |I_{\{x,y\}}(C)| = 1$.

Proof. It follows from the commutativity of the following diagram, keeping in mind that $h^0(S, \mathcal{O}_S(C)) = 3$ and that the first column is injective.

$$
\begin{array}{ccc}
0 & \longrightarrow & H^0(S, I_{\{x,y\}}(C)) \\
\downarrow & & \downarrow \text{res} \\
0 & \longrightarrow & H^0(C, A(-x - y))
\end{array}
\begin{array}{ccc}
& & \longrightarrow H^0(S, \mathcal{O}_S(C)) \\
& & \downarrow \cong \\
& & H^0(\mathcal{O}_{\{x,y\}}(C))
\end{array}
$$

$\square$

We now introduce the moduli map of the pencil introduced in Proposition 3.

(4) $m: \mathbb{P} = |I_{\{x,y\}}(C)| \to \overline{M}_{15,2},$

where the marked points of the pencil are the base points $x$ and $y$ respectively. Composing $m$ with the clutching map $\overline{M}_{15,2} \to \Delta_0 \subseteq \overline{M}_{16}$, we obtain a pencil $\xi: \mathbb{P} \to \Delta_0$.

We set

(5) $R := m^*(\mathbb{P}) \subseteq \overline{M}_{15,2}$ and $\Gamma := \xi^*(\mathbb{P}) \subseteq \overline{M}_{16}$.

Proposition 4. Every curve inside the pencil $\Gamma \subseteq \overline{M}_{16}$ corresponds to a nodal curve which does not belong to any of the boundary divisors $\Delta_1, \ldots, \Delta_8$.

Proof. Keeping the notation above, for a generic choice of $(A, x + y) \in \Sigma$, the pencil

$$P := |I_{\{x,y\}}(C)|$$

corresponds to a generic line inside $|\mathcal{O}_S(C)|$. As already pointed out, $|\mathcal{O}_S(C)|$ is base point free on the surface $S$ defined by (2), giving rise to a regular map of degree 9

(6) $f: S \to \mathbb{P}^2 = |\mathcal{O}_S(C)|^\vee.$
It suffices to show that the inverse image $P$ under $f$ of a general pencil of lines in $P^2$ consists only of integral curves with at most one node. This is achieved in several steps.

**(i)** We rule out the possibility that $P$ contains a reducible 1-nodal curve $C' = F + M$, where $F$ and $M$ are integral smooth curves on $S$ such that $F \cdot M = 1$. Assume this is the case. By the Hodge Index Theorem, we have $F^2 \cdot M^2 < 1$. Assume $F^2 = 0$. For a general element $C' \in P$, we have $F \cdot C' = 1$. Moreover a general $C'$ intersects $F$ at one point $q$ such that $O_F(C') \cong O_F(q)$. If $h^0(F, O_F(q)) = 1$, then it is easily seen that $q$ is a base point of $|O_S(C)|$, therefore $q$ is a base point of $|A| = |O_C(C)|$. This contradicts the generality of the genus 15 curve $C$, which has gonality 9. If, on the other hand, $h^0(F, O_F(q)) \geq 2$, then $F$ is a smooth rational curve. Since $F^2 = 0$, it follows $F \cdot K_S = -2$, contradicting the very ampleness of $K_S$. The same argument works if $M^2 = 0$, hence we may assume $F^2 \cdot M^2 \neq 0$.

Let $F^2 \leq -2$ then $F \cdot C' \leq -1$ and $F$ is a fixed component of $|O_S(C)|$, a contradiction again. Finally, assume $F^2 = -1$ and then $F \cdot C' = 0$. Choose a general point $z \in F \setminus M$. Since $|O_S(C)|$ is base point free, it follows that $|T_{\{z\}}(C)|$ is a pencil. Each curve of this pencil must contain $F$, thus that $|M| = |C - F|$ is a pencil and $h^0(F, O_F(M)) = 2$. Hence $F$ is an exceptional line on the minimal surface $S$, a contradiction. Summarizing, $P$ can only contain any curves of compact type.

**(ii)** The essential step in our argument involves proving that $P$ contains no curves with singularities worse than nodes. Precisely, we show that $|O_S(C)|$ contains only finitely many non-nodal curves. Note first that the branch curve $B \subseteq P^2$ of $f$ is reduced, else we contradict the assumption that the pencil $A \in W_0^1(C)$ on $C$ has simple ramification. We distinguish two cases, depending on whether the map $f : S \to P^2$ given by $[5]$ is finite or not. Assume first $f$ is finite. We introduce the discriminant curve

$$J := \left\{ C' \in |O_S(C)| : C' \text{ is singular} \right\}.$$ 

The dual curve $B^\vee$ is contained in $J$. Since $B$ is reduced, the general tangent line to $B$ is tangent at exactly one point $p \in B$ and with multiplicity 2. A standard local calculation shows that $f^* (T_p B) \in |O_S(C)|$ is a one-nodal curve, singular at exactly one point $z \in f^{-1}(p)$. The complement $J \setminus B^\vee$ is the (possibly empty) union of (some of) the pencils $P_b$, where $b \in B_{\text{sing}}$ and $P_b$ is defined as the pull-back by $f$ of the pencil of lines in $P^2$ through $b$. In view of the numerical situation at hand (that is, $C^2 = 9$), the geometric possibilities for a pencil $P_b \subset J$ are quite constrained. Since $f$ is finite, the pencil $P_b$ has no fixed component. Let $Z := \text{Bs}(P_b)$. Then a general $C' \in P_b$ is integral and smooth along $C' \setminus Z$. Moreover, each $C' \in P_b$ is singular at a given point $z \in Z$ and a general such $C'$ has multiplicity $m \geq 2$ at $z$. Necessarily, the differential $df_z : T_z(S) \to T_p(P^2)$ is zero. Since $m^2 \leq C^2 = (C')^2 = 9$, we find $m \leq 3$. We discuss the possible cases. Let

$$\sigma : S' \to S$$

be the blow-up of $S$ at $z$ and denote by $E \subseteq S'$ the exceptional divisor. The pencil $|O_{S'}(\sigma^* C - mE)|$ is the strict transform of $P_b$. Observe that the restriction map

$$r : H^0(S', O_{S'}(\sigma^* C - mE)) \to H^0(E, O_E(m))$$
is not zero, hence \( \text{Im}(\nu) \) defines a linear series \( p_b \) on \( E \cong \mathbb{P}^1 \). Either \( p_b \) is a pencil or a constant divisor of degree \( m \in \{2, 3\} \).

If \( m = 3 \), then \( \text{supp}(Z) = \{z\} \). This point is an ordinary triple point for any curve \( C' \in P_b \). If \( m = 2 \), then either each \( C' \in P_b \) has a node, or else, each \( C' \in P_b \) has a cusp at \( z \). Indeed, if \( p_b \) is a pencil on \( E \), then each \( C' \in P_b \) is nodal at \( z \). If \( p_b = \{u_1 + u_2\} \) consists of a fixed divisor, then \( P_b \) contains a unique curve \( C_z \) having multiplicity at least 3 at \( z \). If \( u_1 \neq u_2 \), all other curves \( C' \in P_b \setminus \{C_z\} \) are nodal at \( z \), whereas if \( u_1 = u_2 \), then all such \( C' \) are cuspidal at \( z \). Summarizing, since \( C_9 = 9 \), the general curve \( C' \in P_b \) either has a unique triple point, or at most two singular points of multiplicity 2. All these cases can be ruled out by a parameter count, that ultimately contradicts the generality of the pair \( (C, A) \in \mathcal{H}_{15, 9} \) we started with. For instance, assuming each \( C' \in P_b \) is cuspidal at \( z \), passing to the normalization \( \nu: \tilde{C} \to C' \), setting \( z := \nu^{-1}(z) \), we obtain that \( A := \nu^*(\mathcal{O}_{C'}(C')) \in W_9^1(\tilde{C}) \) verifies \( h^0(\tilde{C}, A(-4z)) \geq 1 \). This implies that the pair \([C', \omega_S \otimes \mathcal{O}_{C'}]\) lies in a codimension two subvariety of the Hilbert scheme of degree 19 curves of genus 15 in \( \mathbb{P}^6 \), thus contradicting the general choice of \( (C, A) \in \mathcal{H}_{15, 9} \). The remaining cases can be dealt with similarly.

(iii) Assume now \( f \) is not finite and that for \( p \in B_{\text{reg}} \) as before, \( f^{-1}(p) \) decomposes as \( F \cup Z \), where \( F \) is of pure dimension one and \( Z \) is zero-dimensional. Moreover, \( \text{supp}(F) \cap \text{supp}(Z) = \emptyset \) and \( F \) is reduced. The pencil \( |\mathcal{O}_F(C - F)| \) is base point free. Then a general \( C' \in |\mathbb{I}_{f^{-1}(p)}(C)| \) is a reducible nodal curve that is smooth along \( \text{supp}(Z) \) and contains \( F \) as a component. Writing \( C' = F + M \), then \( M \) is a smooth integral integral. As we have seen, \( F \cdot M > 1 \), hence \( C' \) cannot be of compact type, which completes our proof.

Before stating our next result, recall that one sets \( \delta_i := [\Delta_i] \in CH^1(\overline{\mathcal{M}}_g) \) for \( 0 \leq i \leq \lfloor g/2 \rfloor \). We denote as usual by \( \lambda \in CH^1(\overline{\mathcal{M}}_g) \) the Hodge class. Recall also the formula [HM] for the canonical class of \( \overline{\mathcal{M}}_g \):

\[
K_{\overline{\mathcal{M}}_g} \equiv 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_1^{g/2} \in CH^1(\overline{\mathcal{M}}_g).
\]

**Proposition 5.** The rational curve \( \Gamma \) is a sweeping pencil for the boundary divisor \( \Delta_0 \). Its intersection numbers with the standard generators of \( CH^1(\overline{\mathcal{M}}_{16}) \) are as follows:

\[
\Gamma \cdot \lambda = 22, \quad \Gamma \cdot \delta_0 = 143, \quad \Gamma \cdot \delta_j = 0 \quad \text{for} \quad j = 2, \ldots, 8.
\]

**Proof.** First we construct a fibration whose moduli map is precisely the rational curve \( m: \mathbb{P}^1 \to \overline{\mathcal{M}}_{15, 2} \) considered in [3]. We consider the curve \( C \subseteq S \) and observe that since \( \mathcal{O}_C(C) \cong A \in W_9^1(C) \), we have that \( C^2 = 9 \), that is, the pencil \( |\mathbb{I}_{x,y}(C)| \) has precisely 9 base points, namely \( x, y \), as well as the 7 further points lying in the same fibre of the pencil \( |A| \) as \( x \) and \( y \). We consider the blow-up surface \( \epsilon: \tilde{S} = \text{Bl}_9(S) \to S \) at these 9 points. It comes equipped with a fibration

\[
\pi: \tilde{S} \to \mathbb{P}^1,
\]
as well as with two sections \( E_x, E_y \subseteq \tilde{S} \) corresponding to the exceptional divisors at \( x \) and \( y \) respectively.

In order to compute the intersection numbers of \( R = m(\mathbb{P}) \) with the tautological classes on \( \overline{\mathcal{M}}_{15, 2} \), we use for instance [Tan]. The subscript indicates the moduli space.
on which the intersection number is computed.

\[(R \cdot \lambda)_{\overline{M}_{15,2}} = \chi(S, O_S) + g - 1 = h^2(S, O_S) + g = h^3(S, O_S) + 15 = 22.\]

Here we have used \(H^1(S, O_S) = H^1(S, O_S) = 0\), as well as the fact that \(S\) is a canonical surface, hence \(\omega_S = O_S(1)\), therefore \(h^2(S, O_S) = h^2(S, O_S) = 7\). Furthermore, recalling that all curves in the fibres of \(m\) are irreducible, we find via [Tan] that

\[(R \cdot \delta_0)_{\overline{M}_{15,2}} = c_2(S) + 4(g - 1) = c_2(S) + 56.\]

From the Euler formula, \(c_2(S) = 12 \chi(S, O_S) - K_S^2\). We have already computed that \(\chi(S, O_S) = 8\), whereas \(K_S^2 = K_S^2 - 9 = \text{deg}(S) - 9 = 7\), for \(S\) an intersection of 4 quadrics. Thus \(c_2(S) = 12 \cdot 8 - 7 = 89\), leading to \((R \cdot \delta_0)_{\overline{M}_{15,2}} = 89 + 4 \cdot 14 = 145\).

If we denote by \(\psi_x, \psi_y \in CH^1(\overline{M}_{15,2})\) the cotangent classes corresponding to the marked points labelled by \(x\) and \(y\) respectively, we compute furthermore

\[R \cdot \psi_x = -E^2_x = 1 \quad \text{and} \quad R \cdot \psi_y = -E^2_y = 1.\]

We now pass to the pencil \(\xi: P^1 \to \overline{M}_{16}\) obtained from \(m\) by identifying pointwise the disjoint sections \(E_x\) and \(E_y\) on the surface \(\tilde{S}\). First, using (8) we observe that

\[\Gamma \cdot \lambda = \xi(P) \cdot \lambda = (R \cdot \lambda)_{\overline{M}_{15,2}} = 22.\]

Furthermore, using Proposition [4] we conclude that \(\Gamma \cdot \delta_i = 0\) for \(i = 1, \ldots, 8\). Finally, invoking for instance [CR3, page 271], we find that

\[\Gamma \cdot \delta_0 = (R \cdot \delta_0)_{\overline{M}_{15,2}} = (R \cdot \psi_x)_{\overline{M}_{15,2}} = (R \cdot \psi_y)_{\overline{M}_{15,2}} = 145 - 2 = 143.\]

\[\square\]

**Proof of Theorem 2.** Since the image of \(m\) passes through a general point of \(\overline{M}_{15,2}\), the rational curve \(\Gamma \subseteq \overline{M}_{16}\) constructed in Proposition 5 is a sweeping curve for the boundary divisor \(\Delta_0\). Using the expression (7) for the canonical divisor of \(\overline{M}_{16}\), we compute

\[\Gamma \cdot K_{\overline{M}_{16}} = 13 \Gamma \cdot \lambda - 2 \Gamma \cdot \delta_0 = 13 \cdot 122 - 2 \cdot 143 = 0.\]

Also \(\Gamma \cdot \Delta_0 = 143 > 0\), which finishes the proof. \[\square\]

**The slope of \(\overline{M}_{16}\).**

The slope of an effective divisor \(D\) on the moduli space \(\overline{M}_g\) not containing any boundary divisor \(\Delta_i\) in its support is defined as the quantity \(s(D) := \frac{a}{\min_{i \geq 0} b_i}\), where \([D] = a\lambda - \sum_{i=0}^{8} b_i \delta_i \in CH^1(\overline{M}_g)\), with \(a, b_i \geq 0\). Then the slope \(s(\overline{M}_g)\) of the moduli space \(\overline{M}_g\) is defined as the infimum of the slopes \(s(D)\) over such effective divisors \(D\).

**Corollary 6.** We have that \(s(\overline{M}_{16}) \geq \frac{13}{2}\).

**Proof.** For any effective divisor \(D\) on \(\overline{M}_{16}\) containing no boundary divisor in its support, we may assume that the curve \(\Gamma\) constructed in Proposition 5 does not lie inside \(D\), hence \(\Gamma \cdot D \geq 0\). Writing \([D] = a\lambda - \sum_{i=0}^{8} b_i \delta_i\), using Theorem 2 we obtain \(\frac{a}{b_0} \geq \frac{13}{2}\). Furthermore, using [FP] Theorem 1.4, we conclude that for this divisor \(D\) we have \(b_i \geq b_0\) for \(i = 1, \ldots, 8\), that is, \(s(D) = \frac{a}{b_0} \geq \frac{13}{2}\). \[\square\]
Final remarks: Our results establish that $\overline{M}_{16}$ is not of general type. Showing that the Kodaira dimension of $\overline{M}_{16}$ is non-negative amounts to constructing an effective divisor $D$ on $\overline{M}_{16}$ having slope $s(D) \leq s(K_{\overline{M}_{16}}) = \frac{13}{2}$. Currently the known effective divisor on $\overline{M}_{16}$ of smallest slope is the closure in $\overline{M}_{16}$ of the Koszul divisor $Z_{16}$ consisting of curves $C$ having a linear system $L \in W^7_1(C)$ such that the image curve $\varphi_L: C \to \mathbb{P}^6$ is ideal-theoretically not cut out by quadrics. It is shown in [F1, Theorem 1.1] that $Z_{16}$ is an effective divisor on $\overline{M}_{16}$ and $s(Z_{16}) = \frac{407}{61} = 6.705...$. In a related direction, it is shown in [F2] that the canonical class of the space of admissible covers $\overline{M}_{16}$ is effective.

Note that one has a generically finite cover $\overline{M}_{16,9} \to \overline{M}_{16}$.

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