Generalized Rainbow Turán Numbers of Odd Cycles

József Balogh ∗ Michelle Delcourt † Emily Heath ‡ Lina Li §

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Abstract

Given graphs $F$ and $H$, the generalized rainbow Turán number $\text{ex}(n, F, \text{rainbow-}H)$ is the maximum number of copies of $F$ in an $n$-vertex graph with a proper edge-coloring that contains no rainbow copy of $H$. B. Janzer determined the order of magnitude of $\text{ex}(n, C_s, \text{rainbow-}C_t)$ for all $s \geq 4$ and $t \geq 3$, and a recent result of O. Janzer implied that $\text{ex}(n, C_3, \text{rainbow-}C_{2k}) = O(n^{1+1/k})$. We prove the corresponding upper bound for the remaining cases, showing that $\text{ex}(n, C_3, \text{rainbow-}C_{2k+1}) = O(n^{1+1/k})$. This matches the known lower bound for $k$ even and is conjectured to be tight for $k$ odd.

1 Introduction

The Turán number of a graph $H$ is the maximum number of edges in an $H$-free graph on $n$ vertices, denoted $\text{ex}(n, H)$. This has been generalized in many different ways. For example, the rainbow Turán number $\text{ex}^*(n, H)$, introduced in [1], is the maximum number of edges in a graph on $n$ vertices which can be properly edge-colored with no rainbow copy of $H$. Another natural variation is the generalized Turán number $\text{ex}(n, F, H)$, which is the maximum number of copies of a graph $F$ in an $n$-vertex graph that contains no copy of $H$, and was first studied systematically by Alon and Shikhelman [2]. Both of these problems have been extensively studied, see for example [3, 1] and [2, 4, 5, 6, 7, 8, 9, 10].

∗Department of Mathematics, University of Illinois Urbana-Champaign, Urbana, Illinois 61801, USA, and Moscow Institute of Physics and Technology, Russian Federation. E-mail: jobal@illinois.edu. Research supported by NSF RTG Grant DMS-1937241, NSF Grant DMS-1764123, Arnold O. Beckman Research Award (UIUC Campus Research Board RB 18132), the Langan Scholar Fund (UIUC), and the Simons Fellowship.

†Department of Mathematics, Ryerson University, Toronto, Ontario M5B 2K3, Canada. E-mail: mdelcourt@ryerson.ca. Research supported by NSERC under Discovery Grant No.2019-04269 and an AMS-Simons Travel Grant.

‡Department of Mathematics, Iowa State University, Ames, IA 50011, USA. Email: eheath@iastate.edu. Research supported by NSF RTG Grant DMS-1937241.

§Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada. Email: lina.li@uwaterloo.ca.
Gerbner, Mézéros, Methuku and Palmer [11] considered the following generalized problem which unites the two concepts above. Given two graphs \( F \) and \( H \), the \textit{generalized rainbow Turán number}, denoted by \( \text{ex}(n, F, \text{rainbow-}H) \), is the maximum number of copies of \( F \) in an \( n \)-vertex graph which can be properly edge-colored to avoid a rainbow copy of \( H \). Note that trivially we have \( \text{ex}(n, F, \text{rainbow-}H) \geq \text{ex}(n, F, H) \). The question for \( F = H \) has been studied for paths, trees, cycles and cliques, see [11, 12, 13].

Recently, B. Janzer [13] determined the order of magnitude of \( \text{ex}(n, C_s, \text{rainbow-}C_{2k}) \) for all cases except for \( s = 3 \). In the case \( s = 3 \), he gave the following bounds.

\textbf{Theorem 1.1} (Janzer [13]). If \( k \geq 2 \) is odd then \( \text{ex}(n, C_3, \text{rainbow-}C_{2k}) = \Omega(n^{1+1/k}) \), and if \( k \) is even then \( \text{ex}(n, C_3, \text{rainbow-}C_{2k+1}) = \Omega(n^{1+1/k}) \). Furthermore, for every integer \( k \geq 2 \), we have
\[
\text{ex}(n, C_3, \text{rainbow-}C_{2k}) = O(\text{ex}^*(n, C_{2k})),
\]
and
\[
\text{ex}(n, C_3, \text{rainbow-}C_{2k}) \geq \text{ex}(n, C_3, C_{2k}) = \Omega(\text{ex}(n, \{C_4, C_6, \ldots, C_{2k}\})),
\]
\[
\text{ex}(n, C_3, \text{rainbow-}C_{2k+1}) \geq \text{ex}(n, C_3, C_{2k+1}) = \Omega(\text{ex}(n, \{C_4, C_6, \ldots, C_{2k}\})).
\]

Very recently, O. Janzer [3] settled a well-known conjecture of Keevash, Mubayi, Sudakov and Verstraëte [1], proving that
\[
\text{ex}^*(n, C_{2k}) = \Theta(n^{1+1/k}). \tag{1}
\]
Together with Theorem 1.1, this yields \( \text{ex}(n, C_3, \text{rainbow-}C_{2k}) = O(n^{1+1/k}) \), which is tight for \( k \) odd.

In this paper, we study the remaining open case for cycles, proving an upper bound on \( \text{ex}(n, C_3, \text{rainbow-}C_{2k+1}) \) which matches the lower bound given by B. Janzer [13] for \( k \) even and which is expected to be sharp for \( k \) odd as well.

\textbf{Theorem 1.2}. For \( k \geq 2 \), we have
\[
\text{ex}(n, C_3, \text{rainbow-}C_{2k+1}) = O\left(n^{1+1/k}\right).
\]

In Section 2, we give a self-contained proof of the upper bound on \( \text{ex}(n, C_3, \text{rainbow-}C_5) \) which gives an explicit constant, although it is likely not best possible. In Section 3, we prove Theorem 1.2 by applying (1) to a subgraph in which every rainbow copy of \( C_{2k} \) extends to a rainbow copy of \( C_{2k+1} \) in the original graph. Throughout the paper, we use \( P_k \) to denote the path with \( k \) edges.

\section{No rainbow \( C_5 \)}

\textbf{Theorem 2.1}. We have \( \text{ex}(n, C_3, \text{rainbow-}C_5) \leq 32n^{3/2} \).
Proof. Let $G$ be an $n$-vertex graph with a proper edge-coloring $c$ containing no rainbow copy of $C_5$. First, we will show that $G$ contains at most $4|E(G)|$ triangles.

Fix a vertex $v \in V(G)$ and let $d$ denote the degree of $v$. We will count in two ways the pairs $(S, e)$ where $S \subseteq N(v)$ contains $\lceil d/2 \rceil$ neighbors of $v$ and $e \in E(G[S])$ satisfies $c(e) \neq c(vu)$ for every $u \in S$. There are $\binom{d}{\lceil d/2 \rceil}$ ways to choose the set $S$. Throw out any edge in $G[S]$ whose color appears on an edge incident with $v$ and $S$. Let $E'$ denote the set of remaining edges in $G[S]$. Now $E'$ must be rainbow $P_3$-free, otherwise we can find a rainbow copy of $C_5$ in $G$. Therefore, since Johnston, Palmer, and Sarkar [14] showed that $\text{ex}^*(n, P_3) = \frac{3}{2}n$, we have $|E'| \leq \frac{3}{2}\left\lceil \frac{d}{2} \right\rceil$. Thus, the number of triangles containing $v$ which are formed in this way, that is, the number of pairs of such a set $S$ and an edge from $E'$ of a different color, is at most $\frac{3}{2}\left\lceil \frac{d}{2} \right\rceil \binom{d}{\lceil d/2 \rceil}$.

On the other hand, we could instead first choose an edge $e$ in the neighborhood of $v$ to form our triangle, which can be done in $|E(G[N(v)])|$ ways, and then select an additional $\lceil d/2 \rceil - 2$ vertices from $N(v)$ to form the rest of $S$. However, we do not want the color of $e$ to appear on an edge incident to $v$. Since the edge coloring is proper, this only requires us to throw out at most one vertex from $N(v)$ since at most one edge incident to $v$ can have the same color as $e$. Thus, we can pick the remaining vertices of $S$ in at least $\binom{d-3}{\lceil d/2 \rceil - 2}$ ways. Therefore, we have

$$|E(G[N(v)])| \cdot \binom{d-3}{\lceil d/2 \rceil - 2} \leq \frac{3}{2}\left\lceil \frac{d}{2} \right\rceil \binom{d}{\lceil d/2 \rceil}.$$  

Thus, $|E(G[N(v)])| \leq 6d$, so the number of triangles in $G$ is at most $\frac{1}{3}\sum_{v \in V(G)}|E(G[N(v)])| \leq \frac{1}{3}\sum_{v \in V(G)}6d(v) = 4|E(G)|$.

We may assume that every edge of $G$ is in a triangle, otherwise we could delete an edge without decreasing the number of triangles. Now assume towards a contradiction that $|E(G)| \geq 8n^{3/2}$.

One-by-one, delete vertices of degree less than $4\sqrt{n}$ in $G$. Note that not all vertices are deleted, since otherwise $|E(G)| < 4n^{3/2}$. Denote by $G'$ the remaining induced subgraph with minimum degree $\delta(G') \geq 4\sqrt{n}$, and let $n' = |V(G')|$. Fix an arbitrary vertex $v \in V(G')$.

A cherry is a path of length 2. We will form an auxiliary graph $F$ with vertex set $V(F) = N_{G'}(v)$ which contains an edge $uw$ if and only if there are at least seven cherries of the form $uxw$ in $G'$. We will show that $F$ must contain a vertex of degree at least 3 and use this vertex to find a rainbow copy of $C_5$ in $G$.

Let $S \subseteq N_{G'}(v)$ be a set of $\lceil \sqrt{n} \rceil$ vertices. We will count the cherries in $G'$ with endpoints in $S$. Since each vertex $x$ in $V(G')$ is the center vertex in exactly $\binom{d_3(x)}{2}$ cherries of this

\[d_S(x) = |N_G(x) \cap S|.\]
Figure 1: If there is a vertex of degree at least 3 in $F$, then $G$ contains a rainbow copy of $C_5$. Edges in $F$ are dashed while edges in $G$ are solid.

In this form, we can count the desired cherries as follows:

$$\sum_{x \in V(G')} \left( \frac{d_S(x)}{2} \right) \geq n' \left( \frac{1}{n'} \sum_{x \in S} d_S(x) \right) = n' \left( \frac{1}{n'} \sum_{s \in S} d_{G'}(s) \right) \geq \frac{n' \cdot \delta(G') |S|}{2} \geq \frac{(\delta(G') |S|)^2}{4n'} \geq \frac{16n|S|^2}{4n'} > 7 \left( \frac{|S|}{2} \right),$$

where we use convexity and the fact that $\delta(G') \geq 4\sqrt{n}$. Thus, there must be some pair of vertices in $S$ which are the endpoints of at least seven cherries in $G'$, and hence, these vertices are adjacent in $F$. Since $S$ was an arbitrary set of $\lceil \sqrt{n} \rceil$ vertices in $N_{G'}(v) = V(F)$, we have shown that $\alpha(F) \leq \sqrt{n}$, where $\alpha(F)$ denotes the independence number of $F$. This gives $\Delta(F) \geq |F|/\alpha(F) - 1 \geq 4\sqrt{n}/\sqrt{n} - 1 \geq 3$, so there is a vertex $u$ in $F$ of degree at least 3. Let $x, y, z$ be neighbors of $u$ in $F$. By assumption, the edge $uv$ is in at least one triangle in $G$, so there is a vertex $w \in V(G)$, possibly in $\{x, y, z\}$, which forms a triangle with $uv$.

Let $c(vw) = 1$ and $c(uw) = 2$. Then since $c$ is a proper coloring, at least one of $vx, vy,$ and $vz$ is colored with a new color, say $c(vx) = 3$. Since $ux$ is an edge in $F$, there are at least seven cherries in $G'$ with endpoints $u$ and $x$, and hence, at least one with new colors 4 and 5 which avoids $v$ and $w$. Thus, there is a rainbow copy of $C_5$ in $G$, and we reach a contradiction. Therefore, $G$ must contain at most $8n^{3/2}$ edges, and hence, at most $32n^{3/2}$ triangles, as desired.

3 No rainbow $C_{2k+1}$

Proof of Theorem 1.2. Let $G$ be an $n$-vertex graph with a proper edge-coloring $f : E(G) \to C$ containing no rainbow copy of $C_{2k+1}$. We may assume each edge in $G$ is in at least one triangle.

As in Section 2, we begin by giving a bound on the number of triangles in $G$ in terms of the number of edges in $G$. Fix a vertex $v \in V(G)$ and let $d$ denote the degree of $v$. Pick a set $S \subset N(v)$ containing $\lceil d/2 \rceil$ neighbors of $v$, and throw out any edge in $G[S]$ which is colored using some color which appears on an edge from $v$ to $S$. 

\[ \sum_{x \in V(G')} \left( \frac{d_S(x)}{2} \right) \geq n' \left( \frac{1}{n'} \sum_{x \in S} d_S(x) \right) = n' \left( \frac{1}{n'} \sum_{s \in S} d_{G'}(s) \right) \geq \frac{n' \cdot \delta(G') |S|}{2} \geq \frac{(\delta(G') |S|)^2}{4n'} \geq \frac{16n|S|^2}{4n'} > 7 \left( \frac{|S|}{2} \right), \]
Then \( G[S] \) cannot contain a rainbow copy of \( P_{2k-1} \). A result of Ergemlidze, Györi, and Methuku [15] showed that \( \text{ex}^*(n, P_{k+1}) < \left( \frac{9k}{7} + 2 \right) n \). Therefore, we obtain

\[
|E(G[S])| \leq \text{ex}^* \left( \left\lceil \frac{d}{2} \right\rceil, P_{2k-1} \right) \leq \frac{18k - 4}{7} \cdot \left\lceil \frac{d}{2} \right\rceil.
\]

By counting the triangles containing \( v \) and two adjacent vertices in \( S \) in two ways, we get

\[
|E(G[N(v)])| \cdot \left( \frac{d - 3}{\lfloor d/2 \rfloor - 2} \right) \leq \frac{18k - 4}{7} \cdot \left\lceil \frac{d}{2} \right\rceil \cdot \left( \frac{d}{\lfloor d/2 \rfloor} \right).
\]

Thus, \( |E(G[N(v)])| \leq \frac{72k - 16}{21} d \), and the number of triangles in \( G \) is at most

\[
\frac{1}{3} \sum_{v \in V(G)} |E(G[N(v)])| \leq \frac{72k - 16}{21} \cdot \frac{144k - 32}{21} |E(G)|.
\]

We will show that \( |E(G)| = O(n^{1+1/k}) \).

Assume towards a contradiction that \( G \) has more edges. For each edge \( uv \in E(G) \), arbitrarily fix a vertex \( w = w(uv) \) such that \( u, v, \) and \( w \) form a triangle in \( G \). We will find a subgraph of \( G \) in which any rainbow copy of \( C_{2k} \) can be extended to a rainbow copy of \( C_{2k+1} \) in \( G \). To this end, randomly select a partition of \( V(G) \) into parts \( A \) of size \( \lceil n/2 \rceil \) and \( B \) of size \( \lfloor n/2 \rfloor \). Similarly, take a random partition of \( C \) into parts \( X \) and \( Y \) of sizes \( \lceil |E|/2 \rceil \) and \( \lfloor |E|/2 \rfloor \), respectively.

Let \( F \) be the subgraph with vertex set \( B \) which contains an edge \( uv \in E(G[B]) \) if and only if the vertex \( w = w(uv) \) is in \( A \) and \( f(w) \in X \) while \( f(uw), f(vw) \in Y \). Then the expected number of edges in \( F \) is least \( |E(G)|/64 \), so we can fix partitions \( (A, B) \) and \( (X, Y) \) such that the corresponding graph \( F \) has at least this many edges.

Note that \( F \) inherits the proper edge-coloring \( f \) from \( G \), so we can apply (1) to this subgraph. Since \( \text{ex}^*(n, C_{2k}) = O(n^{1+1/k}) \), there must be a rainbow copy of \( C_{2k} \) in \( F \). This cycle contains only vertices in \( B \), so we can replace an arbitrary edge \( uv \) in the cycle by a pair of edges \( uw \) and \( vw \) with \( w \in A \) to create a copy of \( C_{2k+1} \) in \( G \). Furthermore, the cycle in \( F \) is colored only with colors from \( X \), while the new edges have colors from \( Y \), so we have found a rainbow copy of \( C_{2k+1} \). But this is a contradiction, since \( G \) contains no rainbow copies of \( C_{2k+1} \). Thus, \( |E(G)| = O(n^{1+1/k}) \), which implies that the number of triangles in \( G \) is \( O(n^{1+1/k}) \), as desired.■

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