MULTIPLICITY AND CONCENTRATION OF SOLUTIONS FOR CHOQUARD EQUATION VIA NEHARI METHOD AND PSEUDO-INDEX THEORY

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Abstract. This paper concerns the following nonlinear Choquard equation:

\[- \varepsilon^2 \Delta w + V(x)w = \varepsilon^{-\theta} W(x)(I_{\theta} * (W|w|^p))|w|^{p-2}w, \quad x \in \mathbb{R}^N, \quad (*)\]

where \( \varepsilon > 0, \ N > 2, \ I_{\theta} \) is the Riesz potential with order \( \theta \in (0, N) \), \( p \in \left[ 2, \frac{N+\theta}{N-2} \right) \), \( \min V > 0 \) and \( \inf W > 0 \). Under proper assumptions, we explore the existence, concentration, convergence and decay estimate of semiclassical solutions for (\(*\)). The multiplicity of solutions is established via pseudo-index theory. The existence of sign-changing solutions is achieved by minimizing the energy on Nehari nodal set.

1. Introduction and main results. We study the multiplicity of semiclassical solutions and the concentration phenomenon, convergence, decay estimate of ground-state solutions for a nonlinear Choquard equation. More precisely, we are interested in the following equation:

\[- \varepsilon^2 \Delta w + V(x)w = \varepsilon^{-\theta} W(x)(I_{\theta} * (W|w|^p))|w|^{p-2}w, \quad w \in H^1(\mathbb{R}^N), \quad (1.1)\]

where \( \varepsilon > 0, \ N > 2, \ \theta \in (0, N), \ p \in \left[ 2, \frac{N+\theta}{N-2} \right), \ V \) and \( W \) are bounded positive functions, and the Riesz potential \( I_{\theta} \) is defined as follows:

\[ I_{\theta}(x) := \frac{A_{\theta}}{|x|^{N-\theta}}, \quad x \in \mathbb{R}^N \setminus \{0\}, \quad \text{where} \quad A_{\theta} := \frac{\Gamma\left( \frac{N-\theta}{2} \right)}{2^{\theta} \pi^{\frac{N}{2}} \Gamma\left( \frac{\theta}{2} \right)}. \quad (1.2)\]

The Choquard equation once appeared in Fröhlich and Pekar’s model of polaron [24] and was afterwards introduced by Ph. Choquard in 1976 in the modelling of a one-component plasma [15]. It can be regarded as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics with nonrelativistic Newtonian gravity [4]. Moreover, it also associates with the Einstein-Klein-Gordon and Einstein-Dirac system [12]. Thereupon, Choquard equation has a rich Physical background.

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Solitary wave solution with the type
\[ \psi(x, t) = e^{-it/\varepsilon} w(x) \]
for the focusing time-dependent Hartree equation with \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^N,\)
\[ i\varepsilon \frac{\partial \psi}{\partial t} = -\varepsilon^2 \Delta \psi + (V(x) + 1)\psi - \varepsilon^{-\theta} W(x)(I_\theta * (W[|\psi|^p]))|\psi|^{p-2}\psi \]
corresponds to the solution of Eq.(1.1). In this sense, Choquard equation is known as stationary Hartree equation. This is one of the motivations for the study of Eq.(1.1).

As early as in 1977, E.H. Lieb [15] dealt with the following Choquard equation
\[ -\Delta u + au = 2u \left( \frac{1}{|x|} * |u|^2 \right), \quad x \in \mathbb{R}^3, \quad (1.3) \]
where \(a > 0,\) and he proved the existence and uniqueness (up to translations) of solutions by using symmetric decreasing rearrangement inequalities. Thereafter, P.L. Lions [17] also considered Eq.(1.3) and obtained the existence of infinitely many spherically symmetric solutions.

From then on, Choquard equation has attracted more and more attention from researchers. For instance, Ackermann [1] studied the following equation
\[ -\Delta u + V(x) u = (W * |u|^2) u, \quad x \in \mathbb{R}^3, \]
where \(V \in L^\infty(\mathbb{R}^3)\) was periodic separately in each coordinate direction with minimal period 1 and \(W \in L^1(\mathbb{R}^3) + L^r(\mathbb{R}^3)\) was nonnegative and even with \(r \geq 1,\) he first proved a nonlinear superposition principle for zeros of equivariant vector fields that were asymptotically additive in a well-defined sense and then used this result to achieve the existence of multibump solutions for the equation. Wei and Winter [27] studied
\[ -\varepsilon^2 \Delta u + V(x) u = \frac{1}{8\pi\varepsilon^2} \left( \frac{1}{|x|} * |u|^2 \right) u, \quad x \in \mathbb{R}^3, \]
where \(\varepsilon > 0,\) \(V \in C^2(\mathbb{R}^3)\) and \(\inf_{\mathbb{R}^3} V(x) > 0,\) they proved that for any given positive integer \(K,\) if \(P_1, P_2, \cdots, P_K \in \mathbb{R}^3\) were given nondegenerate critical points of \(V,\) then for \(\varepsilon\) sufficiently small, there existed a positive solution for the equation and this solution had exactly \(K\) local maximum points \(Q_i^\varepsilon (i = 1, 2, \cdots, K)\) with \(Q_i^\varepsilon \to P_i\) as \(\varepsilon \to 0.\) Ma and Zhao [18] settled the longstanding open problem concerning the classification of all positive solutions to Eq.(1.3) with \(a = 1\) and proved that all the positive solutions of this equation must be radially symmetric and monotone decreasing about some fixed point by the method of moving planes. Moroz and Schaftingen [20] proved the existence of groundstate solutions for Eq.(1.1) with \(\varepsilon = V(x) = W(x) = 1,\) \(p \in \left( \frac{N+\theta}{N}, \frac{N+\theta}{N-2} \right)\) and established the regularity, positivity, symmetry, monotony and decay asymptotics of groundstate solutions. Moroz and Schaftingen [21] studied Eq.(1.1) with \(W(x) = 1,\) they assumed the external potential \(V \in C\left( \mathbb{R}^N, [0, \infty) \right)\) had a local minimum and obtained the existence of a family of positive solutions concentrating to the local minimum of \(V\) under optimal assumptions on the decay of \(V\) and admissible range of \(p \geq 2\) by using variational methods and nonlocal penalization technique. Alves, Cassani, Tarsi and Yang [2] considered the equation
\[ -\varepsilon^2 \Delta u + V(x) u = \varepsilon^{-\theta} \left( \frac{1}{|x|^{2-\theta}} * F(u) \right) f(u), \quad x \in \mathbb{R}^2, \]
where $\varepsilon > 0$, $\theta \in (0,2)$, $V$ is continuous and $F$ is the primitive of $f$, they assumed that $f$ had a critical exponential growth in the sense of Trudinger-Moser and established the existence and concentration of positive groundstate solutions by variational methods. Bonheure, Cingolani and Schaftingen [6] studied the logarithmic Choquard equation

$$-\Delta u + au = \frac{1}{2\pi} \left( \ln \frac{1}{|x|} * |u|^2 \right) u, \quad x \in \mathbb{R}^2,$$

where $a > 0$, they derived the sharp asymptotic decay and nondegeneracy of the unique positive radially symmetric groundstate solution.

Recently, the discussion about sign-changing solutions for Choquard equations has appeared increasingly. Clapp and Salazar [8] studied

$$-\Delta u + V(x)u = \left( \frac{1}{|x|^{N-\theta}} * |u|^p \right) |u|^{p-2} u, \quad x \in \Omega,$$

where $\Omega$ is an exterior domain in $\mathbb{R}^N$, $N > 2$, $\theta \in (0,N)$, $p \in \left( \frac{N+\theta}{N-2}, \infty \right)$ and $V \in C(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} V > 0$, $\lim_{|x| \to \infty} V(x) > 0$, they established the existence of multiple sign-changing solutions with small energy in $H^1_0(\Omega)$ under symmetry assumptions on $\Omega$ and $V$. Ghimenti and Schaftingen [10] studied Eq.(1.1) with $\varepsilon = V(x) = W(x) = 1$ and constructed minimal energy odd solutions for $p \in \left( \frac{N+\theta}{N-2}, \frac{N+2}{N-2} \right)$ and minimal energy nodal solutions for $p \in \left( 2, \frac{N+\theta}{N-2} \right)$ by introducing a new minimax principle for least energy nodal solutions and developing new concentration-compactness lemmas for sign-changing Palais-Smale sequences, it’s interesting that the nonlinear Schrödinger equation as nonlocal counterpart of the Choquard equation does not have such solutions. Ruiz and Schaftingen [25] settled the open problem in [10] and proved that the least energy nodal solutions had an odd symmetry with respect to a hyperplane when $\theta$ was either close to 0 or close to $N$.

For more results about Choquard equations, see [3, 7, 11, 22, 26, 29] and the references therein. For local case, we mainly refer to [9].

Ding and Wei [9] considered the following Schrödinger equation

$$-\varepsilon^2 \Delta w + V(x)w = W(x)|w|^{p-2} w, \quad x \in \mathbb{R}^N,$$

where $\varepsilon > 0$, $p \in \left( 2, \frac{2N}{N-2} \right)$ and $V,W$ are continuous bounded positive functions, they studied the existence and concentration phenomena of semiclassical positive groundstate solutions. They also constructed the multiplicity of solutions including at least 1 pair of sign-changing solutions by pseudo-index theory and Nehari method.

Motivated by the works mentioned above, we intend to study the multiplicity and concentration of semiclassical solutions for the nonlinear Choquard equation (1.1). We shall use the ranges and interdependence of linear and nonlinear potentials to be the main assumptions. It is well known that the nonlocal convolution term with Riesz potential makes more difficulties in the process of discussion. We will prove that the convolution term and its derivative satisfy Brézis-Lieb type lemma. Finally we obtain the existence of multiple solutions, the least energy solutions and the least energy nodal solutions for Eq.(1.1). Moreover, we shall show the positive least energy solutions concentrate in a special set related to the minimum of linear potential and the maximum of nonlinear potential.

Now we state our assumptions and main results.
To describe our results, denote

\[ \tau := \min_{R_0} V, \quad V := \{ x \in \mathbb{R}^N : V(x) = \tau \}, \quad \tau_\infty := \liminf_{|x| \to \infty} V(x); \]

\[ k := \max_{R_0} W, \quad W := \{ x \in \mathbb{R}^N : W(x) = k \}, \quad k_\infty := \limsup_{|x| \to \infty} W(x); \]

\[ x_v \in V : \quad k_v := W(x_v) = \max_{V} W; \]

\[ x_w \in W : \quad \tau_w := V(x_w) = \min_{W} V. \]

(A1): Either (i) \( \tau < \tau_\infty \), and \( \exists R_0 > 0 \) such that \( W(x) \leq k_v, \forall |x| \geq R_0 \); or (ii) \( k > k_\infty \), and \( \exists R_0 > 0 \) such that \( V(x) \geq \tau_w, \forall |x| \geq R_0. \)

If (A1)-(i) holds, set \( \mathcal{A}_v := \{ x \in V : W(x) = k_v \} \cup \{ x \notin V : W(x) > k_v \}. \)

If (A1)-(ii) holds, set \( \mathcal{A}_w := \{ x \in W : V(x) = \tau_w \} \cup \{ x \notin W : V(x) < \tau_w \}. \)

In what follows, \( \mathcal{A} \) stands for \( \mathcal{A}_v \) in the case (A1)-(i), and \( \mathcal{A}_w \) in the case (A1)-(ii). Obviously, \( \mathcal{A} \) is bounded. Moreover, \( \mathcal{A} = V \cap W \) if \( V \cap W \neq \emptyset. \)

**Theorem 1.1.** Assume that (A0) holds and

\[ \tau < \tau_\infty, \quad k_v \geq k_\infty. \]  
(1.4)

Then for the maximal integer \( m \in \mathbb{N} \) with

\[ m < \left( \frac{\tau_\infty}{\tau} \right)^{\frac{\theta+2}{2(p-1)}} \cdot \left( \frac{k_v}{k_\infty} \right)^{\frac{2}{p+2}}, \]  
(1.5)

Eq.(1.1) possesses at least \( m \) pairs of solutions for small \( \varepsilon > 0 \). Furthermore, when \( m \geq 2 \) and \( p \in (2, \frac{N+\theta}{N-2}) \), among the solutions, at least one is positive, one is negative and two change sign.

**Theorem 1.2.** Assume that (A0) holds and

\[ k > k_\infty, \quad \tau_w \leq \tau_\infty. \]  
(1.6)

Then for the maximal integer \( m \in \mathbb{N} \) with

\[ m < \left( \frac{\tau_\infty}{\tau_w} \right)^{\frac{\theta+2}{2(p-1)}} \cdot \left( \frac{k}{k_\infty} \right)^{\frac{2}{p+2}}, \]  
(1.7)

all the conclusions of Theorem 1.1 keep true.

**Theorem 1.3.** Assume that (A0)-(A1) hold. Then for sufficiently small \( \varepsilon > 0 \), Eq.(1.1) has a positive least energy solution \( w_\varepsilon \). If additionally \( V, W \in C^1(\mathbb{R}^N, \mathbb{R}) \) and \( \nabla V, \nabla W \) are bounded, then \( w_\varepsilon \) satisfies that (i)(concentration) there exists a maximum point \( x_\varepsilon \) of \( w_\varepsilon \) with

\[ \lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, \mathcal{A}) = 0; \]

(ii)(decay estimate) for \( p \in (2, \frac{N+\theta}{N-2}) \), there exist \( C > 0 \) and \( R > 0 \) such that

\[ w_\varepsilon(x) \leq C \varepsilon^{\frac{N-2}{2}} |x - x_\varepsilon|^{\frac{1-N}{2}} \exp \left( \frac{-\sqrt{\varepsilon}}{4\varepsilon} |x - x_\varepsilon| \right), \quad \forall |x| \geq R; \]

(iii)(convergence) setting \( u_\varepsilon(x) := w_\varepsilon(\varepsilon x + x_\varepsilon) \), for any sequence \( x_\varepsilon \to x_0(\varepsilon \to 0) \),

\[ u_\varepsilon \to u \quad \text{in} \quad H^1(\mathbb{R}^N). \]
with $u$ being a least energy solution of
\[ -\Delta u + V(x_0)u = W^2(x_0)(I_\theta \ast u^p)u^{p-1}, \quad u > 0. \] (1.8)

If particularly $V \cap W \neq \emptyset$, then $\lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, V \cap W) = 0$ and up to a sequence, $u_\varepsilon \to u$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \to 0$ with $u$ being a least energy solution of
\[ -\Delta u + \tau u = k^2(I_\theta \ast u^p)u^{p-1}, \quad u > 0. \] (1.9)

The following lemma is useful to the proof of decay estimate of solutions.

**Lemma 1.4.** [20] Let $r \geq 0$ and $K \in C^1((r, \infty), \mathbb{R})$. If
\[ \lim_{s \to \infty} K(s) > 0 \]
and for some $t > 0$,
\[ \lim_{s \to \infty} K'(s)s^{1+t} = 0, \]
then there exists a nonnegative radial function $v : \mathbb{R}^N \setminus B_r(0) \to \mathbb{R}$ such that
\[ -\Delta v + K(|x|)v = 0, \quad x \in \mathbb{R}^N \setminus B_r(0) \]
and for some $r_0 \in (r, \infty)$,
\[ \lim_{|x| \to \infty} v(x)|x|^\frac{N-1}{2} \exp \left( \int_{r_0}^{|x|} \sqrt{K(s)}ds \right) = 1. \]

This paper is organized as follows: Section 2 offers some valuable information about the Riesz potential. Section 3 contains some preliminary results which are established by variational methods and play a key role in the arguments of main theorems. Section 4 and Section 5 contribute to the proofs of main results. In Section 4, we prove the multiplicity of solutions via Benci pseudo-index theory and show the existence of the least energy solutions and the least energy nodal solutions via Mountain Pass theorem and Location theorem, respectively. In Section 5, we discuss the convergence, concentration phenomenon and decay estimate of the positive least energy solutions.

For simplicity in the following arguments, we shall use different patterns of $C$ to denote various positive constants, and $o(1)$ to denote the quantities that tend to 0 as $n \to \infty$ or $j \to \infty$. Additionally, set
\[
\|u\| := \|u\|_{H^1(\mathbb{R}^N)}, \quad \|u\|_q := \|u\|_{L^q(\mathbb{R}^N)}, \quad (u,v)_1 := (u,v)_{H^1(\mathbb{R}^N)},
\]
\[
u^+ := \max\{0,u\}, \quad u^- := \min\{0,u\}, \quad \mathbb{R}_+ := (0, \infty).
\]

2. **Riesz potential.** In this section, we shall recall and prove some useful results about the Riesz potential.

The Riesz potential with order $\theta \in (0, N)$ of a function $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ is defined as
\[ (I_\theta \ast f)(x) := \int_{\mathbb{R}^N} \frac{A_\theta f(y)}{|x - y|^{N-\theta}}dy, \] (2.1)

where $A_\theta$ is the same as in (1.2). The integral in (2.1) converges in the classical Lebesgue sense for a.e. $x \in \mathbb{R}^N$ if and only if
\[ f \in L^1(\mathbb{R}^N, (1 + |x|)^{\theta-N}). \] (2.2)
Moreover, if (2.2) doesn’t hold, then (2.1) diverges everywhere in $\mathbb{R}^N$. 
The Riesz potential $I_\theta$ is well-defined as an operator in $L^q(\mathbb{R}^N)$ if and only if $q \in \left[1, \frac{N}{N-\theta}\right)$. Furthermore, if $q \in \left(1, \frac{N}{N-\theta}\right)$ and $r := \frac{Nq}{N-\theta}$, then

$$I_\theta : L^q(\mathbb{R}^N) \to L^r(\mathbb{R}^N)$$

is a bounded linear operator, which can be disclosed by the following Hardy-Littlewood-Sobolev inequality.

**Lemma 2.1.** [13] Let $\theta \in (0, N)$, $q \in \left(1, \frac{N}{\theta}\right)$. Then for any $f \in L^q(\mathbb{R}^N)$, $I_\theta * f \in L^{\frac{2Nq}{N-\theta q}}(\mathbb{R}^N)$ and

$$\left(\int_{\mathbb{R}^N} |I_\theta * f|^{\frac{Nq}{N-\theta q}} \, dx\right)^{\frac{1}{\frac{Nq}{N-\theta q}}} \leq C_{N, \theta, q} \left(\int_{\mathbb{R}^N} |f|^q \, dx\right)^{\frac{1}{q}}.$$

If $q = \frac{2N}{N+\theta}$ (see [16]), then the optimal constant in above inequality is

$$C_{N, \theta, q} = \frac{\Gamma\left(\frac{N-\theta}{2}\right)}{2^\theta \pi^\frac{N}{2} \Gamma\left(\frac{N+\theta}{2}\right)} \left(\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right)^{\frac{1}{N}}.$$

More generally, $I_\theta$ could be interpreted as the inverse of the fractional Laplacian operator $(-\Delta)^{\frac{\theta}{2}}$ (see [14]).

Applying Lemma 2.1 to the function $f = |u|^p \in L^{\frac{2Np}{N+\theta}}(\mathbb{R}^N)$ and by Hölder inequality, the following result is true.

**Lemma 2.2.** [23] Let $\theta \in (0, N)$. Then for any $u \in L^{\frac{2Np}{N+\theta}}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\theta * |u|^p)|u|^p \, dx \leq C_{N, \theta} \left(\int_{\mathbb{R}^N} |u|^{\frac{2Np}{N+\theta}} \, dx\right)^{\frac{N+\theta}{N}}.$$

If particularly $N > 2$, $p \in \left[\frac{N+\theta}{N}, \frac{N+\theta}{N-2}\right]$ and $u \in H^1(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} (I_\theta * |u|^p)|u|^p \, dx \leq C_{N, \theta, p} \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx\right)^p.$$

Indeed, $p \in \left[\frac{N+\theta}{N}, \frac{N+\theta}{N-2}\right]$ if and only if $\frac{2Np}{N+\theta} \in [2, 2^*]$, where $2^* := \frac{2N}{N-2}$.

It’s wonderful that the Brézis-Lieb type lemma holds for Riesz potential. To achieve the proof of this lemma, we need two essential results.

**Lemma 2.3.** Let $\Omega$ be a domain in $\mathbb{R}^N$ and $q, r \in (1, \infty)$, $\frac{1}{q} + \frac{1}{r} = 1$. If $u_n \to u$ in $L^q(\Omega)$ and $v_n \rightharpoonup v$ in $L^r(\Omega)$ as $n \to \infty$, then

$$\lim_{n \to \infty} \int_{\Omega} u_n v_n \, dx = \int_{\Omega} uv \, dx.$$

**Lemma 2.4.** Let $\Omega$ be a domain in $\mathbb{R}^N$, $r \in [1, \infty)$ and $\{u_n\}$ be a bounded sequence in $L^r(\Omega)$. If $u_n \rightharpoonup u$ a.e. in $\Omega$ as $n \to \infty$, then for any $q \in [1, r]$,

$$\lim_{n \to \infty} \int_{\Omega} \frac{1}{q} |u_n|^q - |u_n - u|^q - |u|^q \, dx = 0,$$

$$\lim_{n \to \infty} \int_{\Omega} \frac{1}{q} |u_n|^q - |(u_n - u)^\pm|^q - |u|^q \, dx = 0.$$

We would like to point out in advance that the constraint $p \in \left[2, \frac{N+\theta}{N-2}\right]$ is only needed for the second part in the following Brézis-Lieb type lemma, while the first part permits $p \in \left[\frac{N+\theta}{N}, \frac{N+\theta}{N-2}\right]$. 
Lemma 2.5. Let $N > 2$, $\theta \in (0,N)$, $p \in \left[2, \frac{N+\theta}{N-2}\right)$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$, then

(1) $\mathcal{D}(u_n) - \mathcal{D}(u_n - u) \to \mathcal{D}(u)$ as $n \to \infty$;

(2) $\mathcal{D}'(u_n) - \mathcal{D}'(u_n - u) \to \mathcal{D}'(u)$ in $H^{-1}(\mathbb{R}^N)$ as $n \to \infty$.

Proof. It follows from $p \in \left[2, \frac{N+\theta}{N-2}\right)$ that $\frac{2Np}{N+\theta} \in (2, 2^*)$. Thus $H^1(\mathbb{R}^N)$ is continuously embedded into $L^{\frac{2Np}{N+\theta}}(\mathbb{R}^N)$ and compactly embedded into $L^{\frac{2Np}{N+\theta}}_{loc}(\mathbb{R}^N)$.

(i) By Lemma 2.4, we get

\[ |u_n|^p - |u_n - u|^p \to |u|^p \quad \text{in} \quad L^{\frac{2Np}{N+\theta}}(\mathbb{R}^N) \quad \text{as} \quad n \to \infty. \tag{2.3} \]

In view of Lemma 2.1 and (2.3), we have

\[ I_\theta \ast (|u_n|^p - |u_n - u|^p) \to I_\theta \ast |u|^p \quad \text{in} \quad L^{\frac{2Np}{N+\theta}}(\mathbb{R}^N) \quad \text{as} \quad n \to \infty, \tag{2.4} \]

which, together with

\[ |u_n - u|^p \to 0 \quad \text{in} \quad L^{\frac{2Np}{N+\theta}}(\mathbb{R}^N) \quad \text{as} \quad n \to \infty \]

and Lemma 2.3, implies that

\[
\begin{align*}
\int_{\mathbb{R}^N} (I_\theta \ast |u_n|^p)|u_n|^p dx &- \int_{\mathbb{R}^N} (I_\theta \ast |u_n - u|^p)|u_n - u|^p dx \\
= &\int_{\mathbb{R}^N} (I_\theta \ast (|u_n|^p - |u_n - u|^p))(|u_n|^p - |u_n - u|^p) dx \\
&+ 2 \int_{\mathbb{R}^N} (I_\theta \ast (|u_n|^p - |u_n - u|^p))|u_n - u|^p dx \\
&\to \int_{\mathbb{R}^N} (I_\theta \ast |u|^p)|u|^p dx \quad \text{as} \quad n \to \infty.
\end{align*}
\]

(ii) For any $v \in H^1(\mathbb{R}^N)$,

\[
\begin{align*}
\int_{\mathbb{R}^N} (I_\theta \ast |u_n|^p)|u_n|^p - u_n|v dx &- \int_{\mathbb{R}^N} (I_\theta \ast |u_n - u|^p)|u_n - u|^p - u|v dx \\
= &\int_{\mathbb{R}^N} (I_\theta \ast (|u_n|^p - |u_n - u|^p)) (|u_n|^p - u_n - u)^p - u dx \\
&+ \int_{\mathbb{R}^N} (I_\theta \ast (|u_n|^p - |u_n - u|^p)) |u_n - u| dx \\
&+ \int_{\mathbb{R}^N} (I_\theta \ast |u_n - u|^p)(|u_n|^p - u_n - u)|v| dx \\
=: &D_1 + D_2 + D_3. \tag{2.5}
\end{align*}
\]

Firstly,

\[
\begin{align*}
|D_1 - \int_{\mathbb{R}^N} (I_\theta \ast |u|^p)|u|^p - u|v dx|
\leq &\int_{\mathbb{R}^N} |I_\theta \ast (|u_n|^p - |u_n - u|^p - |u|^p)\cdot (|u_n|^p - u_n - u)|v| dx \\
&+ \int_{\mathbb{R}^N} (I_\theta \ast |u|^p) \cdot (|u_n|^p - u_n - u)|v| dx.
\end{align*}
\]
Due to Hölder inequality and (2.4), we have
\[ D_{11} \leq |I_\theta \ast (|u_n|^p - |u_n - u|^p - |u|^p)| \frac{2N}{N+\theta} \cdot |u| \frac{2N}{N+\theta} \cdot |v| \frac{2N}{N+\theta} \]
\[ \leq o(1)\|v\|_1. \]  
(2.7)
By Hölder inequality, Minkowski inequality and Lemma 2.4, we have
\[ D_{12} \leq |I_\theta \ast |u|^p| \frac{2N}{N+\theta} \cdot \left( \int \left( |u_n|^{p-2}u_n - |u_n - u|^{p-2}(u_n - u) - |u|^{p-2}u \right)^p \frac{2N}{N+\theta} \right)^{N+\theta} 
\[ \leq |I_\theta \ast |u|^p| \frac{2N}{N+\theta} \cdot \left[ \left( \int \left( |u_n^+|^{p-1} - |u_n - u|^{p-1} - |u^+|^{p-1} \right)^p \frac{2N}{N+\theta} \right)^{N+\theta} 
\[ + \left( \int \left( |u_n^-|^{p-1} - |u_n - u|^{p-1} - |u^-|^{p-1} \right)^p \frac{2N}{N+\theta} \right)^{N+\theta} \right] \cdot |v| \frac{2N}{N+\theta} \leq o(1)\|v\|_1. \]
(2.8)
It follows from (2.6), (2.7) and (2.8) that
\[ D_1 \to \int (I_\theta \ast |u|^p)|u|^{p-2}u \, dx \quad \text{as } n \to \infty \text{ uniformly in } v \in H^1(\mathbb{R}^N). \]  
(2.9)
Secondly,
\[ D_2 = \int (I_\theta \ast (|u_n|^p - |u_n - u|^p - |u|^p)) |u_n - u|^{p-2}(u_n - u) \, dx 
\[ + \int (I_\theta \ast |u|^p)|u_n - u|^{p-2}(u_n - u) \, dx 
\[ := D_{21} + D_{22}. \]
\[ |D_{21}| \leq |I_\theta \ast (|u_n|^p - |u_n - u|^p - |u|^p)| \frac{2N}{N+\theta} \cdot |u_n - u| \frac{2N}{N+\theta} \cdot |v| \frac{2N}{N+\theta} \leq o(1)\|v\|_1. \]  
(2.11)
For any \( \delta > 0 \), there is \( R > 0 \) such that \( |I_\theta \ast |u|^p|_{L^{2N/(\theta+1)}(|x| > R)} < \delta \). Meanwhile, \( u_n \to u \) in \( L^{2N/(\theta+1)}(|x| \leq R) \) as \( n \to \infty \). Hence
\[ |D_{22}| \leq \int_{|x| > R} (I_\theta \ast |u|^p)|u_n - u|^{p-1}|v| \, dx + \int_{|x| \leq R} (I_\theta \ast |u|^p)|u_n - u|^{p-1}|v| \, dx \]
Similarly as (2.8), we get that
\[ \text{supp} \exists \delta > 0 \text{ such that} \]
\[ \text{for} \quad C := \left( \int R^N \frac{1}{|x|^{N-\theta}} \int \frac{1}{|x-y|^{N-\theta}} d\nu dy \right)^{\frac{2}{N-\theta}} \]
\[ \leq o(1) \|v\|_1. \]  

It follows from (2.10), (2.11) and (2.12) that
\[ D_2 \to 0 \quad \text{as} \quad n \to \infty \quad \text{uniformly in} \quad v \in H^1(\mathbb{R}^N). \]  

Thirdly,
\[ D_3 = \int R^N (I_\theta * |u_n - u|^p) \left( |u_n|^p - |u_n - u|^p - |u|^p - u \right) v dx \]
\[ + \int R^N (I_\theta * |u_n - u|^p) |u|^p v dx \]
\[ := D_{31} + D_{32}. \]  

Similarly as (2.8), we get that
\[ |D_{31}| \leq |I_\theta * |u_n - u|^p| \frac{2N}{N-\theta} \left( \left| |u_n|^p - |u_n - u|^p - |u|^p - u \right| \right) \frac{2N}{N-\theta} \]
\[ \leq o(1) \|v\|_1. \]  

For any \( \delta > 0 \), there is \( R_1 \) such that \( |u| \frac{2N}{N-\theta} (|x| > R_1) < \delta \). Fix \( R_1 > 0 \), then \( \exists R_2 > 0 \) large enough such that \( \frac{1}{(R_2 - R_1)^{2N}} < \delta \). Choose \( \eta \in C_0^\infty (\mathbb{R}^N) \) satisfying \( \text{supp} \eta \subset B_{R_2+1} \) and \( \eta \equiv 1 \) on \( B_{R_2} \) with \( |\nabla \eta| \leq 2 \). Noting the fact \( u_n \to u \) in \( L^{2N\theta/3} (|x| \leq R_2 + 1) \) as \( n \to \infty \) and Lemma 2.1, we get
\[ \int |x| \leq R_1 \left( |I_\theta * |u_n - u|^p| \frac{2N}{N-\theta} \right) dx \]
\[ = \int |x| > R_1 \left( \int_{|y| > R_2} A_\theta (|u_n - u|) \frac{1}{|x-y|^{N-\theta}} dy + \int_{|y| \leq R_2} A_\theta (|u_n - u|) \frac{1}{|x-y|^{N-\theta}} dy \right) \frac{2N}{N-\theta} dx \]
\[ \leq \frac{C_1}{(R_2 - R_1)^{2N}} + C_2 \left( \int_{\mathbb{R}^N} \left( \frac{A_\theta \eta (|u_n - u|) \frac{1}{|x-y|^{N-\theta}} dy \right) \frac{2N}{N-\theta} dx \right) \]
\[ \leq C_1 \delta + C_3 \left( \int_{\mathbb{R}^N} (\eta |u_n - u|^p) \frac{2N}{N-\theta} dx \right) \]
\[ \leq C \delta \]  

which implies that
\[ I_\theta * |u_n - u|^p \to 0 \quad \text{in} \quad L^{2N\theta/3} (|x| \leq R_1) \quad \text{as} \quad n \to \infty. \]  

Similarly as (2.12), we obtain that
\[ |D_{32}| \leq |I_\theta * |u_n - u|^p| \frac{2N}{N-\theta} \left( |u| \frac{2N}{N-\theta} (|x| > R_1) \right) \frac{2N}{N-\theta} \]
\[ + |I_\theta * |u_n - u|^p| \frac{2N}{N-\theta} (|x| \leq R_1) \frac{2N}{N-\theta} \]
\[ \leq o(1) \|v\|_1. \]
It follows from (2.14), (2.15) and (2.16) that
\[ D_3 \to 0 \quad \text{as } n \to \infty \text{ uniformly in } v \in H^1(\mathbb{R}^N). \] (2.17)

Uniting (2.5), (2.9), (2.13) and (2.17), we conclude that
\[
\int_{\mathbb{R}^N} (I_\theta * |u_n|^p)|u_n|^p u_n v dx - \int_{\mathbb{R}^N} (I_\theta * |u_n - u|^p)|u_n - u|^{p-2}(u_n - u)v dx
\to \int_{\mathbb{R}^N} (I_\theta * |u|^p)|u|^{p-2}u v dx \quad \text{as } n \to \infty \text{ uniformly in } v \in H^1(\mathbb{R}^N).
\] (2.18)

\[ \square \]

In terms of Lemma 2.5, the following result is absolutely true.

**Lemma 2.6.** Let \( N > 2, \, \theta \in (0, N), \, p \in \left[ 2, \frac{N+\theta}{N-2} \right) \). If \( u_n \to u \) in \( H^1(\mathbb{R}^N) \) as \( n \to \infty \), then for any \( v \in H^1(\mathbb{R}^N) \),
\[ \langle D'(u_n), v \rangle \to \langle D'(u), v \rangle \quad \text{as } n \to \infty \]
where \( D(u) \) is defined as in Lemma 2.5.

**Proof.** According to (2.18), we just need to show that for any \( v \in H^1(\mathbb{R}^N) \),
\[ D_4 := \int_{\mathbb{R}^N} (I_\theta * |u_n - u|^p)|u_n - u|^{p-2}(u_n - u)v dx \to 0 \quad \text{as } n \to \infty. \]

For any \( \delta > 0 \), there is \( \tilde{R} > 0 \) such that \( |v| \leq \frac{2N}{Np} \langle |x| > \tilde{R} \rangle < \delta \). Moreover, \( u_n \to u \) in \( L^{\frac{2Np}{Np+\theta}}(|x| \leq \tilde{R}) \) as \( n \to \infty \). Therefore,
\[
|D_4| \leq |I_\theta * |u_n - u|^p| \cdot |u_n - u|^{p-1} \cdot |v| \leq \frac{2N}{Np} \cdot |u_n - u|^{p-1} \cdot |v| \leq \frac{2N}{Np} \cdot |v|.
\]
\[ = o(1). \]
\[ \square \]

3. **Preliminary results.** In this section, we present some results which are necessary for the arguments of our main results.

Let us first consider the following two Choquard equations for \( N > 2, \, \theta \in (0, N), \, p \in \left[ 2, \frac{N+\theta}{N-2} \right) \),
\[ -\Delta v + av = b^2(I_\theta * |v|^p)|v|^{p-2}v, \quad v \in H^1(\mathbb{R}^N), \]
where \( a > 0, \, b > 0, \) and
\[ -\Delta v + V^a(x)v = W^b_v(x)(I_\theta * (W^b_v(x))|v|^{p-2}v, \quad v \in H^1(\mathbb{R}^N), \]
where \( \tau \leq a \leq \tau_\alpha, \, k_\alpha \leq b \leq k \) and
\[ V^a(x) := \max\{a, V(x)\}, \quad V^\alpha(x) := V^\alpha(\varepsilon x), \]
\[ W^b_v(x) := \min\{b, W(x)\}, \quad W^b_v(x) := W^b(\varepsilon x). \]

**Definition 3.1.** (i) \( v \in H^1(\mathbb{R}^N) \) is a weak solution of Eq.(3.1) if for any \( \varphi \in H^1(\mathbb{R}^N) \),
\[
\int_{\mathbb{R}^N} (\nabla v \nabla \varphi + av \varphi) dx = \int_{\mathbb{R}^N} b^2(I_\theta * |v|^p)|v|^{p-2}v \varphi dx.
\]
(ii) $v_\varepsilon \in H^1(\mathbb{R}^N)$ is a weak solution of Eq. (3.2) if for any $\varphi \in H^1(\mathbb{R}^N)$,
\[
\int_{\mathbb{R}^N} (\nabla v_\varepsilon \nabla \varphi + V'_\varepsilon(x)v_\varepsilon \varphi) \, dx = \int_{\mathbb{R}^N} W_b^\varepsilon(x)(I_\theta * (W_b^\varepsilon |v_\varepsilon|^p))|v_\varepsilon|^{p-2}v_\varepsilon \varphi \, dx.
\]

Associated with Eq. (3.1) and Eq. (3.2) respectively, we define the energy functionals for each $v \in H^1(\mathbb{R}^N)$,
\[
J^{ab}(v) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + av^2) \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} b^2(I_\theta * |v|^p)|v|^p \, dx,
\]
\[
J^{\varepsilon ab}(v) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V'_\varepsilon(x)v^2) \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} W_b^\varepsilon(x)(I_\theta * (W_b^\varepsilon |v|^p))|v|^p \, dx;
\]
the Nehari manifolds
\[
N^{ab} := \{ v \in H^1(\mathbb{R}^N) \setminus \{0\} : (J^{ab})'(v), v \} = 0 \},
\]
\[
N^{\varepsilon ab} := \{ v \in H^1(\mathbb{R}^N) \setminus \{0\} : (J^{\varepsilon ab})'(v), v \} = 0 \};
\]
the least energies
\[
\bar{\vartheta}^{ab} := \inf_{N^{ab}} J^{ab}, \quad \bar{\vartheta}^{\varepsilon ab} := \inf_{N^{\varepsilon ab}} J^{\varepsilon ab};
\]
and the sets of least energy solutions
\[
R^{ab} := \{ v \in H^1(\mathbb{R}^N) : J^{ab}(v) = \bar{\vartheta}^{ab}, (J^{ab})'(v) = 0 \},
\]
\[
R^{\varepsilon ab} := \{ v_\varepsilon \in H^1(\mathbb{R}^N) : J^{\varepsilon ab}(v_\varepsilon) = \bar{\vartheta}^{\varepsilon ab}, (J^{\varepsilon ab})'(v_\varepsilon) = 0 \}.
\]

In particular, we set
\[
J^{\infty} := J^{\tau_\infty k_\infty}, \quad N^{\infty} := N^{\tau_\infty k_\infty}, \quad \bar{\vartheta}^{\infty} := \bar{\vartheta}^{\tau_\infty k_\infty};
\]
\[
J^{\varepsilon \infty} := J^{\tau_\infty k_\infty}, \quad N^{\varepsilon \infty} := N^{\tau_\infty k_\infty}, \quad \bar{\vartheta}^{\varepsilon \infty} := \bar{\vartheta}^{\tau_\infty k_\infty};
\]
\[
V^{\infty} := V^{\tau_\infty}, \quad W^{\infty} := W^{\tau_\infty}.
\]

Additionally, $J^{ab}, J^{\varepsilon ab} \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$. The critical points of $J^{ab}$ and $J^{\varepsilon ab}$ correspond to the weak solutions of Eq. (3.1) and Eq. (3.2), respectively.

3.1. The equation (3.1). In this subsection, we will construct some results for Eq. (3.1).

Lemma 3.2. There exist $\rho > 0$ and $\sigma > 0$ such that

(i) $J^{ab}(v) > \sigma$, $\forall||v||_1 = \rho$;

(ii) $\lim_{t \to \infty} J^{ab}(tv) = -\infty$ if $v \neq 0$.

The proof of Lemma 3.2 can be easily finished by applying Lemma 2.2.

Lemma 3.3.
\[
\bar{\vartheta}^{ab} = \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} J^{ab}(tv) = \inf_{\gamma \in \Gamma^{ab}} \max_{t \in [0,1]} J^{ab}(\gamma(t)) > 0,
\]
where
\[
\Gamma^{ab} := \{ \gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, J^{ab}(\gamma(1)) < 0 \}.
\]

Proof. For any $v \in H^1(\mathbb{R}^N) \setminus \{0\}$, there is a unique $t_v > 0$ such that $t_v v \in N^{ab}$ and $J^{ab}(t_v v) = \max_{t \geq 0} J^{ab}(tv)$. Thus
\[
\bar{\vartheta}^{ab} \geq \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} J^{ab}(tv).
\]  
(3.3)
For any \( v \in H^1(\mathbb{R}^N) \setminus \{0\} \), by Lemma 3.2-(ii), there is \( t_0 > 0 \) such that \( J^{ab}(t_0 v) < 0 \). Define \( \gamma_0(t) := t_0 v \) for \( t \in [0,1] \), then \( \gamma_0 \in \Gamma^{ab} \). Therefore,

\[
\max_{t \geq 0} J^{ab}(tv) \geq \max_{t \in [0,1]} J^{ab}(\gamma_0(t)) \geq \inf_{\gamma \in \Gamma^{ab}} \max_{t \in [0,1]} J^{ab}(\gamma(t)),
\]

which, together with Lemma 3.2-(i), implies that

\[
\inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} J^{ab}(tv) \geq \inf_{\gamma \in \Gamma^{ab}} \max_{t \in [0,1]} J^{ab}(\gamma(t)) > 0. \tag{3.4}
\]

For any \( \gamma \in \Gamma^{ab} \), it has to cross \( \mathcal{N}^{ab} \). Hence

\[
\inf_{\gamma \in \Gamma^{ab}} \max_{t \in [0,1]} J^{ab}(\gamma(t)) \geq \varrho^{ab}. \tag{3.5}
\]

Uniting (3.3), (3.4) and (3.5), we complete the proof. \( \square \)

**Lemma 3.4.** Let \( v \in H^1(\mathbb{R}^N) \setminus \{0\} \). Then

\[
\max_{t \geq 0} J^{ab}(tv) = \frac{p-1}{2p} \left( S^{ab}(v) \right)^{\frac{p}{p-1}},
\]

where

\[
S^{ab}(v) := \frac{\int_{\mathbb{R}^N} (|\nabla v|^2 + av^2) dx}{(\int_{\mathbb{R}^N} b^2(I_{\theta} * |v|^p)|v|^p dx)^{\frac{1}{p}}}. \]

**Lemma 3.5.** Both \( s^{ab} \) and \( \varrho^{ab} \) are attained in \( H^1(\mathbb{R}^N) \), where

\[
s^{ab} := \inf \{ S^{ab}(v) : v \in H^1(\mathbb{R}^N) \setminus \{0\} \} = \inf \left\{ S^{ab}(v) : \int_{\mathbb{R}^N} b^2(I_{\theta} * |v|^p)|v|^p dx = 1, v \in H^1(\mathbb{R}^N) \right\}
\]

with \( S^{ab}(v) \) defined as in Lemma 3.4.

**Proof.** The proof is similar to that of Theorem 1 in Moroz and Schaftingen [20]. For completeness, we give the details.

Firstly, we show that \( s^{ab} \) is attained in \( H^1(\mathbb{R}^N) \). Let \( \{v_n\} \subset H^1(\mathbb{R}^N) \) satisfy

\[
\int_{\mathbb{R}^N} b^2(I_{\theta} * |v_n|^p)|v_n|^p dx = 1, \quad \forall n \in \mathbb{N} \tag{3.6}
\]

and \( S^{ab}(v_n) \to s^{ab} \) as \( n \to \infty \). Then \( \{v_n\} \) is bounded in \( H^1(\mathbb{R}^N) \). If

\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} v_n^2 dx = 0,
\]

then Lions Lemma (see Lemma 1.21 in [28]) ensures that \( v_n \to 0 \) in \( L^{\frac{2N}{N+\theta}}(\mathbb{R}^N) \) as \( n \to \infty \), which contradicts with (3.6) by Lemma 2.2. Thus there exist \( \delta > 0 \) and \( y_n \in \mathbb{R}^N \) such that

\[
\int_{B_1(y_n)} v_n^2 dx > \delta. \tag{3.7}
\]

Set \( u_n := v_n(x + y_n) \), then \( \{u_n\} \) is also bounded in \( H^1(\mathbb{R}^N) \) and \( S^{ab}(u_n) = S^{ab}(v_n) \). Meanwhile, (3.6) deduces that

\[
\int_{\mathbb{R}^N} b^2(I_{\theta} * |u_n|^p)|u_n|^p dx = 1, \quad \forall n \in \mathbb{N}. \tag{3.8}
\]
Without loss of generality, we may assume that \( u_n \to u \) in \( H^1(\mathbb{R}^N) \) as \( n \to \infty \). Then \( u_n \to u \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \) as \( n \to \infty \), which, together with (3.7), implies \( u \neq 0 \). Noting that
\[
\int_{\mathbb{R}^N} (|\nabla u|^2 + au^2) \, dx
= \lim_{n \to \infty} \left( \int_{\mathbb{R}^N} (|\nabla u_n|^2 + au_n^2) \, dx - \int_{\mathbb{R}^N} (|\nabla (u_n - u)|^2 + a(u_n - u)^2) \, dx \right),
\]
we obtain
\[
s^{ab} \leq S^{ab}(u) = \lim_{n \to \infty} \left[ S^{ab}(u_n) \left( \frac{\int_{\mathbb{R}^N} b^2(I_{\partial*}|u_n|^p)|u_n|^p \, dx}{\int_{\mathbb{R}^N} b^2(I_{\partial}|u|^p)|u|^p \, dx} \right)^{\frac{1}{p}}
- S^{ab}(u_n - u) \left( \frac{\int_{\mathbb{R}^N} b^2(I_{\partial*}|u_n - u|^p)|u_n - u|^p \, dx}{\int_{\mathbb{R}^N} b^2(I_{\partial}|u|^p)|u|^p \, dx} \right)^{\frac{1}{p}} \right]
\leq s^{ab} \cdot \left( \int_{\mathbb{R}^N} b^2(I_{\partial*}|u|^p)|u|^p \, dx \right)^{-\frac{1}{p}} \cdot \limsup_{n \to \infty} \left[ \left( \frac{\int_{\mathbb{R}^N} b^2(I_{\partial*}|u_n|^p)|u_n|^p \, dx}{\int_{\mathbb{R}^N} b^2(I_{\partial}|u|^p)|u|^p \, dx} \right)^{\frac{1}{p}}
- \left( \frac{\int_{\mathbb{R}^N} b^2(I_{\partial*}|u_n - u|^p)|u_n - u|^p \, dx}{\int_{\mathbb{R}^N} b^2(I_{\partial}|u|^p)|u|^p \, dx} \right)^{\frac{1}{p}} \right].
\]
Due to (3.8) and Lemma 2.5-(i), we have for \( n \) sufficiently large,
\[
\int_{\mathbb{R}^N} b^2(I_{\partial*}|u_n - u|^p)|u_n - u|^p \, dx < 1. \tag{3.10}
\]
It follows from
\[
u_n \to u \quad \text{in} \quad L^\frac{2N}{N-2}(\mathbb{R}^N) \quad \text{as} \quad n \to \infty
\]
and Lemma 2.1 that
\[
I_{\partial^*}|u_n|^p \to I_{\partial^*}|u|^p \quad \text{in} \quad L^\frac{2N}{N-2}(\mathbb{R}^N) \quad \text{as} \quad n \to \infty.
\]
By Fatou’s Lemma and (3.8), we have
\[
0 < \int_{\mathbb{R}^N} b^2(I_{\partial^*}|u|^p)|u|^p \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} b^2(I_{\partial^*}|u_n|^p)|u_n|^p \, dx = 1.
\]
If \( \int_{\mathbb{R}^N} b^2(I_{\partial^*}|u|^p)|u|^p \, dx < 1 \), then it follows from (3.8), (3.9), (3.10), \( p \geq 2 \) and Lemma 2.5-(i) that
\[
s^{ab} \leq s^{ab} \cdot \limsup_{n \to \infty} \int_{\mathbb{R}^N} \frac{(I_{\partial^*}|u_n|^p)|u_n|^p - (I_{\partial^*}|u_n - u|^p)|u_n - u|^p)}{\int_{\mathbb{R}^N} b^2(I_{\partial}|u|^p)|u|^p \, dx} \, dx = s^{ab},
\]
which is a contradiction. Thus \( \int_{\mathbb{R}^N} b^2(I_{\partial^*}|u|^p)|u|^p \, dx = 1 \) and by (3.9),
\[
s^{ab} = s^{ab}(u).
\]
Next we will show \( \partial^{ab} \) is attained in \( H^1(\mathbb{R}^N) \). According to Lemma 3.3 and Lemma 3.4, we get that
\[
\partial^{ab} = \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{p-1}{2p} \left( s^{ab}(v) \right)^{\frac{p}{p-1}}.
\]
Noting that
\[
(S_{ab}(u))^{\frac{1}{\tau}} = \left( \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} S_{ab}(v) \right)^{\frac{1}{\tau}}
\]
\[
\leq \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} (S_{ab}(v))^{\frac{1}{\tau}} \leq (S_{ab}(u))^{\frac{1}{\tau}},
\]
we obtain
\[
\theta_{ab} = \frac{p-1}{2p} (S_{ab}(u))^{\frac{1}{\tau}}.
\]

Finally, set \( v := (s_{ab})^{\frac{1}{p-2}} u \), then \( S_{ab}(v) = S_{ab}(u) \), and \( v \neq 0 \) solves Eq.(3.1). Hence \( v \in N_{ab} \) and
\[
\theta_{ab} = \frac{p-1}{2p} (S_{ab}(v))^{\frac{1}{\tau}} = \frac{p-1}{2p} \int_{\mathbb{R}^N} (|\nabla v|^2 + av^2) \, dx = J_{ab}(v).
\]
That is \( v \in R_{ab} \).

In view of Theorem 3 in Moroz and Schaftingen [20], we have the following lemma.

**Lemma 3.6.** If \( u \in H^1(\mathbb{R}^N) \) is a least energy solution of Eq.(3.1), then \( u \in L^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N) \), \( u \) is either positive or negative, and \( u \) is radially symmetric up to translations.

**Lemma 3.7.** Let \( a_i > 0 \) and \( b_i > 0 \) for \( i = 1, 2 \).

(i) If \( \min\{a_2 - a_1, b_1 - b_2\} \geq 0 \), then \( \theta_{a_1b_1} \leq \theta_{a_2b_2} \);

(ii) If \( \min\{a_2 - a_1, b_1 - b_2\} \geq 0 \) and \( \max\{a_2 - a_1, b_1 - b_2\} > 0 \), then \( \theta_{a_1b_1} < \theta_{a_2b_2} \).

**Lemma 3.8.**
\[
\theta_{ab} = \left( \frac{a}{\tau_{\infty}} \right)^{\frac{a+2}{a-2}} \left( \frac{b}{\tau_{\infty}} \right)^{-\frac{N+2}{a-2}} \left( \frac{k_{\infty}}{b} \right)^{\frac{2}{\tau_{\infty}}} \theta_{\infty}.
\]
Moreover, (1.5) holds if and only if \( m\theta_{r_1b_1} < \theta_{\infty} \); (1.7) holds if and only if \( m\theta_{r_2b} < \theta_{\infty} \).

**Proof.** Setting \( u(x) := \left( \frac{\tau_{\infty}}{a} \right)^{\frac{a+2}{a-2}} \left( \frac{b}{\tau_{\infty}} \right)^{-\frac{N+2}{a-2}} v \sqrt{\frac{\tau_{\infty}}{a}} x \), then Eq.(3.1) is equivalent to
\[
-\Delta u + \tau_{\infty} u = k_{\infty}^2 (I_\theta * |u|^p)|u|^{p-2} u, \quad u \in H^1(\mathbb{R}^N).
\]
One can check that \( v \in N_{ab} \) if and only if \( u \in N_{\infty} \), and for any \( v \in N_{ab} \),
\[
J_{ab}(v) = \left( \frac{a}{\tau_{\infty}} \right)^{\frac{a+2}{a-2}} \left( \frac{b}{\tau_{\infty}} \right)^{-\frac{N+2}{a-2}} \left( \frac{k_{\infty}}{b} \right)^{\frac{2}{\tau_{\infty}}} J_{\infty}(u).
\]

3.2. The equation (3.2). In this subsection, we shall establish some results for Eq.(3.2).

**Lemma 3.9.** There exist \( \rho > 0 \) and \( \sigma > 0 \) both independent of \( \varepsilon, a, b \) and just dependent on \( N, \theta, p, \tau, k \) such that

(i) \( J_{ab}^\rho(v) > \sigma, \quad \forall \|v\|_1 = \rho \);

(ii) \( \lim_{t \to +\infty} J_{ab}^{\rho(t)}(tv) = -\infty \) if \( v \neq 0 \).
Lemma 3.10. 
\[ \vartheta^b = \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} J^b(t v) = \inf_{\gamma \in \Gamma^b} \max_{t \in [0,1]} J^b(\gamma(t)) > 0, \]
where 
\[ \Gamma^b := \{ \gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \ J^b(\gamma(1)) < 0 \}. \]

The proof of Lemma 3.10 is similar to that of Lemma 3.3 and so is omitted.

Lemma 3.11. If \( J^\infty \) possesses a (PS)\(_c \) sequence, then either \( c = 0 \) or \( c \geq \vartheta^\infty \).

Moreover, \( \vartheta^\infty \geq \vartheta^\infty \).

Proof. Let \( \{ v_n \} \subset H^1(\mathbb{R}^N) \) and \( J^\infty(v_n) \to c, \ (J^\infty)'(v_n) \to 0 \) in \( H^{-1}(\mathbb{R}^N) \) as \( n \to \infty \). Assume \( c \neq 0 \), we will prove \( c \geq \vartheta^\infty \).

Since \( \{ v_n \} \) is bounded in \( H^1(\mathbb{R}^N) \), we may assume \( v_n \to v \) in \( H^1(\mathbb{R}^N) \) as \( n \to \infty \) along a subsequence. Set \( z_n := v_n - v \). By the classical Brézis-Lieb Lemma, we have
\[ \begin{align*}
\int_{\mathbb{R}^N} (|\nabla v_n|^2 + V^\infty(x)v_n^2) \, dx \\
= \int_{\mathbb{R}^N} (|\nabla v|^2 + V^\infty(x)v^2) \, dx + \int_{\mathbb{R}^N} (|\nabla z_n|^2 + V^\infty(x)z_n^2) \, dx + o(1).
\end{align*} \] (3.11)

By slight amendment in the proof of Lemma 2.5, we get that
\[ \begin{align*}
\int_{\mathbb{R}^N} W^\infty(x)(I_\theta * (W^\infty|v_n|^p)) |v_n|^{p-2} v_n \, dx \\
= \int_{\mathbb{R}^N} W^\infty(x)(I_\theta * (W^\infty|v|^p)) |v|^{p-2} v \, dx \\
+ \int_{\mathbb{R}^N} W^\infty(x)(I_\theta * (W^\infty|z_n|^p)) |z_n|^{p-2} z_n \, dx + o(1),
\end{align*} \] (3.12)

and for any \( \varphi \in H^1(\mathbb{R}^N) \),
\[ \begin{align*}
\int_{\mathbb{R}^N} W^\infty(x)(I_\theta * (W^\infty|v_n|^p)) |v_n|^{p-2} v_n \varphi \, dx \\
= \int_{\mathbb{R}^N} W^\infty(x)(I_\theta * (W^\infty|v|^p)) |v|^{p-2} v \varphi \, dx \\
+ \int_{\mathbb{R}^N} W^\infty(x)(I_\theta * (W^\infty|z_n|^p)) |z_n|^{p-2} z_n \varphi \, dx + o(1)(||\varphi||_1). \] (3.13)

As the proof of Lemma 2.6, we get that for any \( \varphi \in H^1(\mathbb{R}^N) \),
\[ \begin{align*}
\int_{\mathbb{R}^N} W^\infty(x)(I_\theta * (W^\infty|v_n|^p)) |v_n|^{p-2} v_n \varphi \, dx \\
\to \int_{\mathbb{R}^N} W^\infty(x)(I_\theta * (W^\infty|v|^p)) |v|^{p-2} v \varphi \, dx \quad \text{as } n \to \infty,
\end{align*} \]
which ensures that \( (J^\infty)'(v) = 0 \). In virtue of (3.11), (3.12) and (3.13), we obtain that
\[ J^\infty(v_n) = J^\infty(v) + J^\infty(z_n) + o(1), \]
\[ (J^\infty)'(v_n) = (J^\infty)'(v) + (J^\infty)'(z_n) + o(1), \]
which imply that
\[ J^\infty(z_n) \to c - J^\infty(v) \quad \text{as } n \to \infty, \] (3.14)
\[ (J^\infty)'(z_n) \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N) \quad \text{as } n \to \infty. \] (3.15)
Similarly, if $\exists z_{n_k} = 0$, that is $v_{n_k} \equiv v$, then $J_\infty^\infty(v) = c \neq 0$ and $v \in \mathcal{N}_\infty^\infty$. Thus $c \geq \vartheta_\infty^\infty$.

If $z_n \neq 0$, $\forall n \in \mathbb{N}$, then $\exists t_n > 0$ such that $t_n z_n \in \mathcal{N}_\infty^\infty$. Hence
\[
J_\infty^\infty(t_n z_n) \geq \vartheta_\infty^\infty
\] (3.16)

and
\[
0 = \langle (J_\infty^\infty)'(t_n z_n), t_n z_n \rangle = t_n^2 \int_{\mathbb{R}^N} \left( |\nabla z_n|^2 + V_\infty^\infty(x) z_n^2 \right) dx \\
- t_n^{2p} \int_{\mathbb{R}^N} W_\infty^\infty(x) (I_\theta * (W_\infty^\infty|z_n|^p)) |z_n|^p dx.
\] (3.17)

Taking into account (3.15), we have
\[
o(1) = \langle (J_\infty^\infty)'(z_n), z_n \rangle = \int_{\mathbb{R}^N} \left( |\nabla z_n|^2 + V_\infty^\infty(x) z_n^2 \right) dx \\
- \int_{\mathbb{R}^N} W_\infty^\infty(x) (I_\theta * (W_\infty^\infty|z_n|^p)) |z_n|^p dx,
\]

which, together with (3.17), implies that
\[
(1 - t_n^{2p-2}) \int_{\mathbb{R}^N} W_\infty^\infty(x) (I_\theta * (W_\infty^\infty|z_n|^p)) |z_n|^p dx = o(1).
\] (3.18)

Additionally,
\[
\|z_n\|^2 \leq C \int_{\mathbb{R}^N} \left( |\nabla z_n|^2 + V_\infty^\infty(x) z_n^2 \right) dx \\
= C \int_{\mathbb{R}^N} W_\infty^\infty(x) (I_\theta * (W_\infty^\infty|z_n|^p)) |z_n|^p dx + o(1).
\]

If $\int_{\mathbb{R}^N} (I_\theta * |z_n|^p)|z_n|^p dx \to 0$ as $n \to \infty$, then $\|z_n\|_1 \to 0$ as $n \to \infty$. Thus $v_n \to v$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$ and $c = J_\infty^\infty(v) \geq \vartheta_\infty^\infty$. If $\int_{\mathbb{R}^N} (I_\theta * |z_n|^p)|z_n|^p dx \geq \delta > 0$, then $t_n \to 1$ as $n \to \infty$ by (3.18). Hence $J_\infty^\infty(t_n z_n) \to c - J_\infty^\infty(v)$ as $n \to \infty$ by (3.14), which implies that $c \geq J_\infty^\infty(v) + \vartheta_\infty^\infty \geq \vartheta_\infty^\infty$ by (3.16).

Finally, we show $\vartheta_\infty^\infty \geq \vartheta_\infty^\infty$. By definition,
\[
V_\infty^\infty(x) \geq \tau_0, \quad W_\infty^\infty(x) \leq k_0, \quad \forall x \in \mathbb{R}^N.
\]

Thus
\[
J_\infty^\infty(u) \geq J_\infty^\infty(u), \quad \forall u \in H^1(\mathbb{R}^N).
\]

In virtue of Lemma 3.3 and Lemma 3.10, $\vartheta_\infty^\infty$ and $\vartheta_\infty^\infty$ are Mountain Pass levels of $J_\infty^\infty$ and $J_\infty^\infty$, respectively. Therefore, $\vartheta_\infty^\infty \geq \vartheta_\infty^\infty$. \hfill \Box

**Remark 1.** Similarly, if $J_\infty^{ab}$ has a $(PS)_c$ sequence, then either $c = 0$ or $c \geq \vartheta_\infty^{ab}$.

**Lemma 3.12.** $J_\infty^{ab}$ satisfies $(PS)_c$ condition for all $c < \vartheta_\infty^\infty$.

**Proof.** Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ and $J_\infty^{ab}(v_n) \to c$, $(J_\infty^{ab})'(v_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$ as $n \to \infty$.

Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, we may assume $v_n \to v$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$ along a subsequence. Then $(J_\infty^{ab})'(v) = 0$ by Lemma 2.6.

Set $z_n := v_n - v$. Then $z_n \to 0$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$ and
\[
z_n \to 0 \quad \text{in } L_q^{loc}(\mathbb{R}^N) \quad \text{as } n \to \infty \quad \text{for } q \in [2, 2^*].
\] (3.19)

Due to the classical Brézis-Lieb Lemma and Lemma 2.5, we have
\[
J_\infty^{ab}(z_n) \to c - J_\infty^{ab}(v), \quad (J_\infty^{ab})'(z_n) \to 0 \quad \text{in } H^{-1}(\mathbb{R}^N) \quad \text{as } n \to \infty.
\] (3.20)
For any $\phi$ which, jointly with (3.19) and (3.20), implies that
\[
\lim_{|x| \to \infty} \left( V_\epsilon^\infty(x) - V_\epsilon^a(x) \right) = 0, \quad \lim_{|x| \to \infty} \left( W_\epsilon^\infty(x) - W_\epsilon^b(x) \right) = 0,
\]
which, together with (3.19) and (3.20), implies that
\[
\theta_c \leq \|\langle + k + x_1 \rangle \|_{H^-1} + \int_{|x| \leq R} \frac{k}{p} |\phi_\epsilon^a|^{2p} + \frac{2p}{N+1} d\phi_\epsilon^a \delta + C \left( |z_n|_{L^2(B_R)} + |z_n|_{L^{\frac{2Np}{N-p}}(B_R)} \right),
\]
which, jointly with (3.19) and (3.20), implies that
\[
\frac{J_\epsilon^\infty(z_n) - J_\epsilon^{ab}(z_n)}{n} \to c - J_\epsilon^{ab}(v) \quad \text{as} \quad n \to \infty.
\]
For any $\varphi \in H^1(\mathbb{R}^N)$, by Hölder inequality and Lemma 2.1, we have
\[
\left\| (J_\epsilon^\infty)'(z_n) - (J_\epsilon^{ab})'(z_n), \varphi \right\| \leq \frac{1}{2} \int_{|x| > R} |V_\epsilon^\infty(x) - V_\epsilon^a(x)| |\varphi| dx + \frac{1}{2} \int_{|x| \leq R} |V_\epsilon^\infty(x) - V_\epsilon^a(x)| |\varphi| dx
\]
which, together with (3.21) and (3.22), implies that
\[
\frac{J_\epsilon^\infty(z_n) - J_\epsilon^{ab}(z_n)}{n} \to c - J_\epsilon^{ab}(v) \quad \text{as} \quad n \to \infty.
\]
It follows from (3.21) and (3.22) that $\{z_n\}$ is a $(PS)_{\epsilon-J_\epsilon^{ab}(v)}$ sequence of $J_\epsilon^\infty$. By Lemma 3.11, either $c = J_\epsilon^{ab}(v)$ or $c \geq J_\epsilon^{ab}(v) + \theta_\epsilon^\infty$. But the latter contradicts with the assumption $c < \theta_\epsilon^\infty$. Thus $c = J_\epsilon^{ab}(v)$ and
\[
\frac{J_\epsilon^\infty(v_n) - J_\epsilon^{ab}(v_n)}{n} \to J_\epsilon^{ab}(v) \quad \text{as} \quad n \to \infty.
\]
Next we will prove \( v_n \to v \) in \( H^1(\mathbb{R}^N) \) as \( n \to \infty \). Since \( \langle (J_{\varepsilon}^{ab})'(v_n), v_n \rangle = o(1) \) and \( \langle (J_{\varepsilon}^{ab})'(v), v \rangle = 0 \), we have
\[
J_{\varepsilon}^{ab}(v_n) = \frac{p-1}{2p} \int_{\mathbb{R}^N} \left( |\nabla v_n|^2 + V_{\varepsilon}^n(x)v_n^2 \right) dx + o(1),
\]
\[
J_{\varepsilon}^{ab}(v) = \frac{p-1}{2p} \int_{\mathbb{R}^N} \left( |\nabla v|^2 + V_{\varepsilon}^a(x)v^2 \right) dx,
\]
which, together with (3.23), imply that \( \|v_n\|_1 \to \|v\|_1 \) as \( n \to \infty \). Therefore,
\[
v_n \to v \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{as} \quad n \to \infty.
\]
\[\square\]

**Lemma 3.13.** \( \limsup_{\varepsilon \to 0} \partial_{\varepsilon}^{ab} \leq \partial^{\alpha\beta} \), where \( \alpha = V^a(0) \), \( \beta = W^b(0) \). Moreover, if \( V(0) \leq a \), \( W(0) \geq b \), then \( \lim_{\varepsilon \to 0} \partial_{\varepsilon}^{ab} = \partial^{ab} \).

**Proof.** Let \( \bar{V}_\varepsilon(x) := V^{\varepsilon}_\varepsilon(x) - \alpha \) and \( \bar{W}_\varepsilon(x) := \beta - W^{\varepsilon}_\varepsilon(x) \). Then
\[
\bar{V}_\varepsilon(x) \to 0, \quad \bar{W}_\varepsilon(x) \to 0, \quad \text{a.e. in} \ \mathbb{R}^N \quad \text{as} \ \varepsilon \to 0. \quad (3.24)
\]
Meanwhile,
\[
J_{\varepsilon}^{ab}(v) = J^{\alpha\beta}(v) + \frac{1}{2} \int_{\mathbb{R}^N} \bar{V}_\varepsilon(x)v^2 dx + \frac{\beta}{p} \int_{\mathbb{R}^N} \bar{W}_\varepsilon(x)(I_\theta * |v|^p)|v|^p dx
\]
\[
- \frac{1}{2p} \int_{\mathbb{R}^N} \bar{W}_\varepsilon(x)(I_\theta * (\bar{W}_\varepsilon |v|^p))|v|^p dx. \quad (3.25)
\]
By Lemma 3.5, \( \exists \varepsilon \in \mathcal{R}^{\alpha\beta} \), that is \( J^{\alpha\beta}(\varepsilon) = \rho^{\alpha\beta} \) with \( \varepsilon \in \mathcal{N}^{\alpha\beta} \). Let \( t_\varepsilon > 0 \) such that \( t_\varepsilon e \in \mathcal{N}^{ab}_\varepsilon \), then
\[
\max_{t \geq 0} J_{\varepsilon}^{ab}(te) = J_{\varepsilon}^{ab}(t_\varepsilon e) \geq \partial_{\varepsilon}^{ab}. \quad (3.26)
\]
Since \( J_{\varepsilon}^{ab}(t_\varepsilon e) \to -\infty \) as \( t \to +\infty \), there exists \( T_0 > 0 \) such that
\[
J_{\varepsilon}^{ab}(t_\varepsilon e) < 0, \quad \forall t > T_0. \quad (3.27)
\]
By (3.26) and (3.27), we get \( t_\varepsilon \leq T_0 \). Without loss of generality, we can assume \( t_\varepsilon \to t_0 \) as \( \varepsilon \to 0 \). Noting (3.24), (3.25) and (3.26), if follows from Lebesgue dominated convergence theorem that
\[
\partial_{\varepsilon}^{ab} \leq J_{\varepsilon}^{ab}(t_\varepsilon e) + \frac{\beta t_\varepsilon^2}{2} \int_{\mathbb{R}^N} \bar{V}_\varepsilon(x)e^2 dx + \frac{\beta t_\varepsilon^2}{p} \int_{\mathbb{R}^N} \bar{W}_\varepsilon(x)(I_\theta * |e|^p)|e|^p dx
\]
\[
- \frac{t_\varepsilon^2}{2p} \int_{\mathbb{R}^N} \bar{W}_\varepsilon(x)(I_\theta * (\bar{W}_\varepsilon |e|^p))|e|^p dx \to J^{\alpha\beta}(t_0 e) \leq J^{\alpha\beta}(e) = \rho^{\alpha\beta} \quad \text{as} \ \varepsilon \to 0.
\]
Therefore, \( \limsup_{\varepsilon \to 0} \partial_{\varepsilon}^{ab} \leq \partial^{\alpha\beta} \).

Next, if \( V(0) \leq a, \ W(0) \geq b \), then \( \alpha = a, \ \beta = b \). Hence
\[
\bar{V}_\varepsilon(x) \geq 0, \quad \bar{W}_\varepsilon(x) \geq 0, \quad \forall x \in \mathbb{R}^N,
\]
which, together with (3.25) and \( \beta \geq \bar{W}_\varepsilon(x) \) for any \( x \in \mathbb{R}^N \), imply that
\[
J_{\varepsilon}^{ab}(v) \geq J^{\alpha\beta}(v), \quad \forall v \in H^1(\mathbb{R}^N).
\]
Thus, as Mountain Pass levels, \( \vartheta^a \geq \vartheta^\alpha \). In virtue of
\[
\vartheta^\alpha \leq \liminf_{\varepsilon \to 0} \vartheta^a \leq \limsup_{\varepsilon \to 0} \vartheta^a \leq \vartheta^\alpha,
\]
we obtain that \( \lim_{\varepsilon \to 0} \vartheta^a = \vartheta^\alpha = \vartheta^a \).

\begin{lemma}
Let \( \tau \leq a \leq \tau_\infty \), \( k_\infty \leq b \leq k \). If
\[
\left( \frac{V^a(0)}{\tau_\infty} \right)^{\frac{s+2}{(N-2)(s-1)}} < \frac{W^b(0)}{k_\infty},
\]
then there exists \( \varepsilon^a > 0 \) such that for all \( \varepsilon \leq \varepsilon^a \), \( \vartheta^a \) is attained at some \( \varepsilon^a \). 
\end{lemma}

\begin{proof}
Noting Lemma 3.8 and (3.28), we have \( \vartheta^\alpha \geq \vartheta^\infty \), where \( \alpha = V^a(0) \) and \( \beta = W^b(0) \). By Lemma 3.13 and Lemma 3.11, \( \exists \varepsilon^a > 0 \) such that
\[
\vartheta^a < \vartheta^\infty \leq \vartheta^a, \quad \forall \varepsilon \leq \varepsilon^a.
\]
By Lemma 3.12, \( J^a \) satisfies (PS)\( _{a+} \) condition for all \( \varepsilon \leq \varepsilon^a \), which, together with Lemma 3.9 and Lemma 3.10, implies that \( \vartheta^a \) is attained at \( \varepsilon^a \). Since \( J^a(v) = J^a(|v|) \) for any \( v \in H^1(\mathbb{R}^N) \), we may assume \( v^a \geq 0 \). By bootstrap method and elliptic regularity theory, \( v^a \in C^2(\mathbb{R}^N) \). By strong maximum principle, \( v^a > 0 \).
\end{proof}

\subsection{Preparation for sign-changing solutions.}

In order to prove the existence of sign-changing solutions, we define the Nehari nodal set
\[\mathcal{N}^a := \{ v \in H^1(\mathbb{R}^N) : v^+ \neq 0, \langle (J^a)'(v), v^\pm \rangle = 0 \}\]
and the least energy nodal value
\[\zeta^a := \inf_{\mathcal{N}^a} J^a.\]
Indeed, \( \mathcal{N}^a \cap \mathcal{M}^a \neq \emptyset \) and \( \zeta^a \geq \vartheta^a > 0 \).

\begin{lemma}
For any \( v \in H^1(\mathbb{R}^N) \) with \( v^\pm \neq 0 \), there exists a unique pair of \( (s_v, t_v) \in \mathbb{R}_+ \times \mathbb{R}_+ \) such that
\[
s_v v^+ + t_v v^- \in \mathcal{M}^a, \quad J^a(s_v v^+ + t_v v^-) = \max_{s,t \geq 0} J^a(s v^+ + t v^-).
\]
\end{lemma}

\begin{proof}
For any \( v \in H^1(\mathbb{R}^N) \) with \( v^\pm \neq 0 \), consider the mapping
\[g(s, t) := \langle (J^a)'(sv^+ + tv^-), sv^+ \rangle, \langle (J^a)'(sv^+ + tv^-), tv^- \rangle\]
for \( (s, t) \in \mathbb{R}_+ \times \mathbb{R}_+ \). Since
\[
\langle (J^a)'(sv^+ + tv^-), sv^+ \rangle = s^2 \int_{\mathbb{R}^N} (|\nabla v^+|^2 + V^a(x)|v^+|^2) dx
\]
and
\[
\langle (J^a)'(sv^+ + tv^-), tv^- \rangle = t^2 \int_{\mathbb{R}^N} (|\nabla v^-|^2 + V^a(x)|v^-|^2) dx
\]
we obtain the following inequalities:
\[
\begin{align*}
&- s^p \int_{\mathbb{R}^N} W^b(x)(I_\theta + (W^b|v^-|^p))|v^+|^p dx, \\
&- s^p \int_{\mathbb{R}^N} W^b(x)(I_\theta + (W^b|v^+|^p))|v^+|^p dx, \\
&\text{and} \\
&\langle (J^a)'(sv^+ + tv^-), tv^- \rangle = t^2 \int_{\mathbb{R}^N} (|\nabla v^-|^2 + V^a(x)|v^-|^2) dx
\end{align*}
\]
and
\[
\begin{align*}
&- t^p \int_{\mathbb{R}^N} W^b(x)(I_\theta + (W^b|v^-|^p))|v^-|^p dx, \\
&- t^p \int_{\mathbb{R}^N} W^b(x)(I_\theta + (W^b|v^+|^p))|v^+|^p dx, \\
&\text{and} \\
&\langle (J^a)'(sv^+ + tv^-), tv^- \rangle = t^2 \int_{\mathbb{R}^N} (|\nabla v^-|^2 + V^a(x)|v^-|^2) dx
\end{align*}
\]
then there exist $R > r > 0$ such that for any $s, t \in [r, R]$,
\[
\langle J_e^{ab}((rv^+ + tv^+), rv^+) \rangle > 0, \quad \langle J_e^{ab}((rv^+ + tv^-), rv^-) \rangle > 0, \quad \langle J_e^{ab}((sv^+ + rv^+), rv^-) \rangle < 0, \quad \langle J_e^{ab}((sv^+ + rv^-), rv^-) \rangle < 0.
\]
By Miranda’s theorem [19], there exist $(s_v, t_v) \in (r, R) \times (r, R)$ such that
\[
\langle J_e^{ab}((sv^+ + t_vv^-), sv^+ + t_vv^-) \rangle = 0 = \langle J_e^{ab}((sv^+ + t_vv^-), sv^+ + t_vv^-) \rangle.
\]
That is $s_vv^+ + t_vv^- \in M_e^{ab}$.

Noting $p \geq 2$ and $J_e^{ab}(s_vv^+ + t_vv^-)$ is strictly concave for $(s, t) \in \mathbb{R}^+ \times \mathbb{R}^+$, it follows from
\[
\frac{\partial J_e^{ab}((sv^+ + t_vv^-), sv^+ + t_vv^-)}{\partial s} \bigg|_{(s, t)} = \frac{1}{ps_v} \langle J_e^{ab}((sv^+ + t_vv^-), sv^+ + t_vv^-) \rangle = 0,
\]
\[
\frac{\partial J_e^{ab}((sv^+ + t_vv^-), sv^+ + t_vv^-)}{\partial t} \bigg|_{(s, t)} = \frac{1}{pt_v} \langle J_e^{ab}((sv^+ + t_vv^-), sv^+ + t_vv^-) \rangle = 0
\]
that $(s_v^+, t_v^+)$ is the unique maximum point of $J_e^{ab}(s_vv^+ + t_vv^-)$ for $(s, t) \in \mathbb{R}^+ \times \mathbb{R}^+$. Hence $J_e^{ab}(sv^+ + t_vv^-) = \max_{s, t \geq 0} J_e^{ab}(sv^+ + t_vv^-) = \max_{s, t \geq 0} J_e^{ab}(sv^+ + t_vv^-) = \max_{s, t \geq 0} J_e^{ab}(sv^+ + tv^-)$.

Let $\mathbb{B}^2$ denote the closed unit disc in the plane $\mathbb{R}^2$ and deg be the classical topological degree of Brower. As similar as the proof of Proposition 3.2 in Ghimenti and Schaftingen [10], we have the following minimax principle.

**Lemma 3.16.** If $p \in \left(2, \frac{N+2}{N-2}\right)$, then for any $\delta > 0$,
\[
\zeta_e^{ab} = \inf_{\gamma \in \Gamma} \sup_{\mathbb{B}^2} J_e^{ab} \circ \gamma,
\]
where
\[
\Gamma := \left\{ \gamma \in C(\mathbb{B}^2, \mathbb{H}^1_0(\mathbb{R}^N)) : \xi \left(\gamma(\partial \mathbb{B}^2)\right) \neq 0, \deg(\xi \circ \gamma) = 1 \right\}
\]
and $J_e^{ab} \circ \gamma \frac{\xi^2}{\|\xi\|^2} \leq \left(\zeta_e^{ab} \frac{1}{\|\xi\|^2} - (\xi_e^{ab} \frac{1}{\|\xi\|^2} + \delta) \right)$ on $\partial \mathbb{B}^2$.

with the mapping $\xi = (\xi_+, \xi_-) \in C(\mathbb{H}^1(\mathbb{R}^N), \mathbb{R}^2)$ defined by
\[
\xi_\pm(u) := \left\{ \begin{array}{ll}
\int_{\mathbb{R}^N} W_e^\pm(x) \left(1 + W_e^\pm(|u|^p)\right) |u|^p dx - 1 & \text{if } u \neq 0, \\
-1 & \text{if } u = 0.
\end{array} \right.
\]
Moreover, $\forall \gamma \in \Gamma$, $M_e^{ab} \cap \gamma(\mathbb{B}^2) \neq \emptyset$.

**Remark 2.** The continuity of $\xi$ on the subset of constant-sign functions in $H^1(\mathbb{R}^N)$ is ensured by the Hölder inequality, the Hardy-Littlewood-Sobolev inequality (Lemma 2.1), the Minkowski inequality, the classical Sobolev inequality and the assumption $p > 2$.

4. **Proofs of the multiplicity of solutions.** In this section, we will prove the existence of multiple solutions, the least energy solutions and the least energy nodal solutions for Eq.(1.1).

Setting $v(x) := w(x \epsilon)$, the Eq.(1.1) is equivalent to
\[
- \Delta v + V(\epsilon x)v = W(\epsilon x) \left(\int_\theta (W(\epsilon y) |v(y)|^p) |v|^{p-2} v dy \right) v \in H^1(\mathbb{R}^N).
\]
If $v_\epsilon(x)$ is a solution of Eq.(4.1), then $w_\epsilon(x) = v_\epsilon(\frac{x}{\epsilon})$ is a solution of Eq.(1.1).
Noting $V(\varepsilon x) = V^\varepsilon(x)$, $W(\varepsilon x) = W^\varepsilon(x)$, we find that Eq. (4.1) is a particular form of Eq. (3.2). For simplicity, we denote

$$J_\varepsilon := J^{\varepsilon r_k}, \quad N_\varepsilon := N^{\varepsilon r_k}, \quad \vartheta_\varepsilon := \vartheta^{\varepsilon r_k}, \quad R_\varepsilon := R^{\varepsilon r_k},$$

$$V_\varepsilon := V^{\varepsilon^r}, \quad M_\varepsilon := M^{\varepsilon r_k}, \quad \zeta_\varepsilon := \zeta^{\varepsilon r_k}, \quad W_\varepsilon := W^{\varepsilon^r}.$$

**Proof of Theorem 1.1.** Without loss of generality, we may assume $x_v = 0$. Then $V(0) = \tau$, $W(0) = k_v$. By (1.4) and (1.5), we get $m \geq 1$.

**Step 1.** We shall construct a $m$-dimensional subspace $E_{r_m}$ of $H^1(\mathbb{R}^N)$ such that

$$\sup_{v \in E_{r_m}} J_\varepsilon(v) < \vartheta^\infty, \quad \forall \varepsilon \geq r_m, \quad \forall \varepsilon \leq \varepsilon_m,$$

where $r_m$ and $\varepsilon_m$ are existing constants depending on $m$.

Choose $a = \tau$, $b = k_v$ in Eq. (3.1), by Lemma 3.5 and Lemma 3.6, there exists $v \in \mathcal{R}^{\kappa_v}$ and $v(x) = v(|x|) > 0$. Let $r > 0$, $\chi_r \in C_0^\infty(\mathbb{R}_+)$ satisfy $\chi_r(s) = 1$ for $s \leq r$ and $\chi_r(s) = 0$ for $s \geq r + 1$ with $|\chi'_r(s)| \leq 2$. Set

$$v_r(x) := \chi_r(|x|)v(x), \quad x \in \mathbb{R}^N.$$

It follows from

$$\|v_r - v\|_1^2 = \int_{\mathbb{R}^N} \left( |\nabla((\chi_r(|x|) - 1)v)|^2 + ((\chi_r(|x|) - 1)v)^2 \right) dx$$

$$\leq C \int_{|x| > r} (|\nabla v|^2 + v^2) dx$$

$$\to 0 \text{ as } r \to \infty$$

that

$$v_r \to v \text{ in } H^1(\mathbb{R}^N) \text{ as } r \to \infty,$$

$$v_r \to v \text{ in } L^{\frac{2N}{N+\rho}}(\mathbb{R}^N) \text{ as } r \to \infty,$$

$$|v_r|^p \to |v|^p \text{ in } L^{\frac{2N}{N+\rho}}(\mathbb{R}^N) \text{ as } r \to \infty,$$

$$I_{\theta} * |v_r|^p \to I_{\theta} * |v|^p \text{ in } L^{\frac{2N}{N+\rho}}(\mathbb{R}^N) \text{ as } r \to \infty,$$

$$\int_{\mathbb{R}^N} (I_{\theta} * |v_r|^p)|v_r|^p dx \to \int_{\mathbb{R}^N} (I_{\theta} * |v|^p)|v|^p dx \text{ as } r \to \infty.$$
as \( \varepsilon \to 0 \) uniformly on any bounded set of \( x \). Hence (4.2) and (4.3) deduce that

\[
\max_{t \geq 0} J_\varepsilon(t v_r) = \frac{p-1}{2p} \left( \frac{\left( \int_{\mathbb{R}^N} \left( (\nabla v_r^2 + V_\varepsilon(x) v_r^2 \right) dx \right)^p}{\int_{\mathbb{R}^N} W_\varepsilon(x) (I_\theta + (W_\varepsilon|v_r|)) |v_r|^p dx} \right)^{\frac{1}{p-1}}
\]

\[
= \max_{t \geq 0} J^{r_k v} (t v_r)
\]

Define

\[
\phi_{rj}(x) := v_r \left( x_1 - 2j(r+1), x_2, \ldots, x_N \right)
\]

for \( j = 0, 1, \ldots, m - 1 \), and set

\( E_{r_m} := \text{span} \{ \phi_{rj}(x) : j = 0, 1, \ldots, m - 1 \} \).

One can check that \( (\phi_{ri}, \phi_{rj}) = 0 \) if \( i \neq j \). Thus \( \dim E_{r_m} = m \). Similarly as (4.4), for all \( j = 1, 2, \ldots, m - 1 \),

\[
\max_{t \geq 0} J_\varepsilon(t \phi_{rj}) = \frac{p-1}{2p} \left( \frac{\left( \int_{\mathbb{R}^N} \left( (\nabla \phi_{rj}^2 + V_\varepsilon(x) \phi_{rj}^2 \right) dx \right)^p}{\int_{\mathbb{R}^N} W_\varepsilon(x) (I_\theta + (W_\varepsilon|\phi_{rj}|)) |\phi_{rj}|^p dx} \right)^{\frac{1}{p-1}}
\]

\[
= \max_{t \geq 0} J^{r_k v} (t \phi_{rj}) = \max_{t \geq 0} J^{r_k v} (t v_r)
\]

\( \to \vartheta^{r_k v} \) as \( \varepsilon \to 0, r \to \infty \), respectively.

Consequently, \( \forall \delta > 0, \exists r_\delta > 0, \exists \varepsilon_\delta > 0 \) such that

\(
\max_{t \geq 0} J_\varepsilon(t \phi_{rj}) \leq \vartheta^{r_k v} + \delta, \quad \forall r \geq r_\delta, \forall \varepsilon \leq \varepsilon_\delta, \quad j = 0, 1, \ldots, m - 1. \tag{4.5}
\)

For any \( v \in E_{r_m} \), we may assume \( v = t_0 \phi_{r0} + t_1 \phi_{r1} + \cdots + t_{m-1} \phi_{r(m-1)} \), where \( t_j \in \mathbb{R} \) for \( j = 0, 1, \ldots, m - 1 \). Then

\[
(I_\theta + (W_\varepsilon|v|)) |v|^p \geq \sum_{j=0}^{m-1} (I_\theta + (W_\varepsilon|t_j \phi_{rj}|)) |t_j \phi_{rj}|^p.
\]

Hence by (4.5),

\[
J_\varepsilon(v) \leq J_\varepsilon(t_0 \phi_{r0}) + J_\varepsilon(t_1 \phi_{r1}) + \cdots + J_\varepsilon(t_{m-1} \phi_{r(m-1)})
\]

\[
\leq m(\vartheta^{r_k v} + \delta), \quad \forall r \geq r_\delta, \forall \varepsilon \leq \varepsilon_\delta,
\]

which implies that

\[
\sup_{v \in E_{r_m}} J_\varepsilon(v) \leq m(\vartheta^{r_k v} + \delta), \quad \forall r \geq r_\delta, \forall \varepsilon \leq \varepsilon_\delta.
\]

Noting Lemma 3.8, we can choose \( 0 < \delta < \vartheta^{r_k v} - \vartheta^{r_k v} \), then \( \exists r_m > 0, \exists \varepsilon_m > 0 \) such that

\[
\sup_{v \in E_{r_m}} J_\varepsilon(v) < \vartheta^{r_k v}, \quad \forall r \geq r_m, \forall \varepsilon \leq \varepsilon_m. \tag{4.6}
\]
Step 2. We will define constants $c_1, c_2, \ldots, c_m$ and verify that they are critical values of $J_\varepsilon$.

Consider the symmetric group $\mathbb{Z}_2 = \{id, -id\}$ and denote
$$\Sigma := \{A \subset H^1(\mathbb{R}^N) : A \text{ is closed and } A = -A\}.$$  

For any $A \in \Sigma$, the Krasnoselski genus of $A$ is defined by
$$\text{gen}(A) := \inf \{n : \exists g \in C(A, \mathbb{R}^n \setminus \{0\}) \text{ and } g \text{ is odd }\}.$$  

Set
$$\mathcal{H} := \{h \in C(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)) : h \text{ is an odd homeomorphism}\}$$

and for any $A \in \Sigma$, define
$$i(A) := \min_{h \in \mathcal{H}} \text{gen}(h(A) \cap \partial B_\rho),$$

where $\rho > 0$ is a constant defined in Lemma 3.9. Thus $i(A)$ is a version of Benci pseudo-index of $A$. Let
$$c_j := \inf_{i(A) \geq j} \sup_{v \in A} J_\varepsilon(v), \quad j = 1, 2, \ldots, m.$$  

Clearly, $c_1 \leq c_2 \leq \cdots \leq c_m$. Next we will show $c_1 \geq \sigma$ and $c_m \leq \sup_{v \in E_{rm}} J_\varepsilon(v)$, where $\sigma > 0$ was defined in Lemma 3.9.

For any $A \in \Sigma$ and $i(A) \geq 1$, we have $\text{gen}(A \cap \partial B_\rho) \geq 1$, which implies $A \cap \partial B_\rho \neq \emptyset$. It follows from Lemma 3.9-(i) that $\sup_{v \in A} J_\varepsilon(v) > \sigma$. Hence $c_1 \geq \sigma$.

Taking into account that Krasnoselski genus satisfies dimension property (see Benci [5]), we get
$$\text{gen}(h(E_{rm}) \cap \partial B_\rho) = \dim E_{rm} = m, \quad \forall h \in \mathcal{H},$$

which implies $i(E_{rm}) = m$. Thus $c_m \leq \sup_{v \in E_{rm}} J_\varepsilon(v)$.

Noting (4.6) and Lemma 3.11, we get that for any $r \geq r_m$, $\varepsilon \leq \varepsilon_m$,
$$\sigma \leq c_1 \leq c_2 \leq \cdots \leq c_m \leq \sup_{v \in E_{rm}} J_\varepsilon(v) < \vartheta_\infty \leq \vartheta_\varepsilon.$$  

Now we prove $c_j (1 \leq j \leq m)$ are critical values of $J_\varepsilon$ by applying the Theorem 1.4 in Benci [5]. Set
$$c_0 := \sigma, \quad c_\infty := \sup_{v \in E_{rm}} J_\varepsilon(v),$$

$$(J_\varepsilon)^c := \{v \in H^1(\mathbb{R}^N) : J_\varepsilon(v) \leq c\},$$

$$K_c := \{v \in H^1(\mathbb{R}^N) : J_\varepsilon(v) = c, \ (J_\varepsilon)'(v) = 0\}.$$  

Since $J_\varepsilon$ is an even functional, then
$$(J_\varepsilon)^c \in \Sigma, \quad K_c \in \Sigma, \quad \forall c \in [c_0, c_\infty].$$  

By (4.7) and Lemma 3.12, $J_\varepsilon$ satisfies $(PS)_c$ condition for any $c \in [c_0, c_\infty]$, which implies that
$$K_c \text{ is compact in } H^1(\mathbb{R}^N), \quad \forall c \in [c_0, c_\infty].$$  

For any $c \in [c_0, c_\infty]$, $d > 0$ and $(K_c)_d := \{v \in H^1(\mathbb{R}^N) : \text{dist}(v, K_c) < d\}$, choose $\delta = \frac{d}{4}$, we will show that there exists $\varepsilon > 0$ such that
$$\| (J_\varepsilon)'(v) \| \geq \frac{8\varepsilon}{\delta}, \quad \forall v \in J_\varepsilon^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \setminus (K_c)_\frac{d}{4}.$$  

(4.10)
Assume by contradiction that (4.10) is false, then \( \exists v_n \notin (K_c)_{\frac{\delta}{4}} \) such that

\[
\frac{2}{n} \leq J_\varepsilon(v_n) \leq c + \frac{2}{n}, \quad \| (J_\varepsilon)'(v_n) \| < \frac{8}{\delta n}.
\]

Letting \( n \to \infty \), we have

\[
J_\varepsilon(v_n) \to c, \quad (J_\varepsilon)'(v_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N).
\]

That is \( \{v_n\} \) is a \((PS)_c\) sequence of \( J_\varepsilon \). Without loss of generality, we may assume

\[
v_n \to v_0 \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{as} \quad n \to \infty.
\]

Thus \( J_\varepsilon(v_0) = c \), \( (J_\varepsilon)'(v_0) = 0 \), namely, \( v_0 \in K_c \), which contradicts with \( v_n \notin (K_c)_{\frac{\delta}{4}} \).

Choose \( S := H^1(\mathbb{R}^N) \setminus (K_c)_d \), then \( S_{2\delta} = H^1(\mathbb{R}^N) \setminus (K_c)_{\frac{\delta}{4}} \) and (4.10) imply that

\[
\| (J_\varepsilon)'(v) \| \geq \frac{8\varepsilon}{\delta}, \quad \forall v \in J_\varepsilon^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}.
\]

By deformation lemma, \( \exists \tilde{\eta} \in C([0,1] \times H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)) \) such that \( \tilde{\eta}(t, \cdot) \) is an odd homeomorphism of \( H^1(\mathbb{R}^N) \) for any \( t \in [0,1] \) and \( \tilde{\eta}(1, (J_\varepsilon)^{c+\varepsilon}) \cap S) \subset (J_\varepsilon)^{c-\varepsilon} \).

Set \( \eta := \tilde{\eta}(1, \cdot) \), then \( \eta \) is an odd homeomorphism of \( H^1(\mathbb{R}^N) \) and

\[
\eta((J_\varepsilon)^{c+\varepsilon}) \cap (K_c)_d \subset (J_\varepsilon)^{c-\varepsilon}. \tag{4.11}
\]

For any \( A \in \Sigma \) and \( A \subset (J_\varepsilon)^{c_0} \), it follows from Lemma 3.9-(i) that \( A \cap \partial B_\rho = \emptyset \). Thus \( \text{gen}(A \cap \partial B_\rho) = 0 \) and

\[
i(A) = \min_{h \in \mathcal{H}} \text{gen}(h(A) \cap \partial B_\rho) = 0. \tag{4.12}
\]

Choose \( \tilde{A} = E_{rm} \), then

\[
\tilde{A} \subset (J_\varepsilon)^{c_\infty} \quad \text{and} \quad i(\tilde{A}) = i(E_{rm}) = m \geq 1. \tag{4.13}
\]

Uniting (4.8), (4.9), (4.11), (4.12) and (4.13), we obtain that \( c_j (1 \leq j \leq m) \) are critical values of \( J_\varepsilon \), and \( \text{gen}(K_c) \geq r + 1 \) if \( c := c_k = c_{k+1} = \cdots = c_{k+r} \) with \( k \geq 1 \) and \( k + r \leq m \). Since \( J_\varepsilon \) is even, we conclude that \( J_\varepsilon \) has at least \( m \) pairs of critical points which are also solutions of Eq.(4.1).

**Step 3.** We will prove Eq.(4.1) has at least one positive and one negative least energy solutions.

Noting (1.5) and \( m \geq 1 \), we get that

\[
\left( \frac{\varepsilon}{r_\infty} \right)^{\frac{q+2}{4} - \frac{(N-2)(p-1)}{4} + \frac{1}{p}} \frac{\varepsilon}{k_\infty} < \frac{1}{k_\infty}, \quad \text{which, together with} \quad V(0) = \tau, \quad W(0) = k_v, \quad \text{implies that}
\]

\[
\left( \frac{\tau^r(0)}{r_\infty} \right)^{\frac{q+2}{4} - \frac{(N-2)(p-1)}{4} + \frac{1}{p}} < \frac{W^k(0)}{k_\infty}.
\]

By Lemma 3.14, there exists \( \varepsilon^{\tau k} > 0 \) such that \( \partial \varepsilon \) is attained at \( v_\varepsilon > 0 \) for all \( \varepsilon \leq \varepsilon^{\tau k} \). Therefore, \( v_\varepsilon \) and \( -v_\varepsilon \) are positive and negative least energy solutions of Eq.(4.1), respectively.

**Step 4.** We will prove Eq.(4.1) has at least one pair of sign-changing solutions when \( m \geq 2 \) and \( 2 < p < \frac{N+\theta}{N-2} \).
Firstly, let $e \in \mathcal{R}^{rk_v}$ with $e > 0$. That is, $J^{rk_v} (e) = \vartheta^{rk_v}$ and $(J^{rk_v})'(e) = 0$. Set $\chi_r \in C_0^\infty (\mathbb{R}_+)$ satisfy $\chi_r(s) = 1$ for $s \leq r$ and $\chi_r(s) = 0$ for $s \geq r + 1$ with $|\chi'_r(s)| \leq 2$. Choose $r > 0$ and $x_r \in \mathcal{R}^N$ with $|x_r|$ large and
\[
\text{dist} (B_{r+1}(0), B_{r+1}(x_r)) > 0.
\]
Define for $x \in \mathbb{R}^N$,
\[
e^+_r (x) := \chi_r(|x|)e(x) \geq 0, \quad e^-_r (x) := -\chi_r(|x - x_r|)e(x) \leq 0.
\]
Then $\text{supp} e^+_r \cap \text{supp} e^-_r = \emptyset$ and
\[
e^+_r \rightarrow e, \quad e^-_r \rightarrow -e \quad \text{in } H^1 (\mathbb{R}^N) \quad \text{as } r \rightarrow \infty. \quad (4.14)
\]
Hence by Lemma 3.15, there exists $(s_r, t_r) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that
\[
s_r e^+_r + t_r e^-_r \in \mathcal{M}_r. \quad (4.15)
\]
Secondly, since $\langle (J_e)'(s_r e^+_r + t_r e^-_r), s_r e^+_r + t_r e^-_r \rangle = \langle (J_e)'(s_r e^+_r + t_r e^-_r), t_r e^-_r \rangle = 0$, we have
\[
\int_{\mathbb{R}^N} \left( |\nabla e^+_r|^2 + V_e (x) |e^+_r|^2 \right) \, dx
\]
\[
= s_r^{p-2} t_p \int_{\mathbb{R}^N} W_e (x) (I_\theta * (W_e |e^-_r|^p)) |e^+_r|^p \, dx \quad (4.16)
\]
and
\[
\int_{\mathbb{R}^N} \left( |\nabla e^-_r|^2 + V_e (x) |e^-_r|^2 \right) \, dx
\]
\[
= t_r^{p-2} s_p \int_{\mathbb{R}^N} W_e (x) (I_\theta * (W_e |e^+_r|^p)) |e^-_r|^p \, dx \quad (4.17)
\]
which deduce that $s_r$ and $t_r$ are bounded for $r \in (0, \infty)$. Without loss of generality, we may assume that
\[
s_r \rightarrow s_0, \quad t_r \rightarrow t_0 \quad \text{as } r \rightarrow \infty.
\]
Letting $\varepsilon \rightarrow 0$ in (4.16) and (4.17), we get that
\[
\int_{\mathbb{R}^N} \left( |\nabla e^+_r|^2 + \tau |e^+_r|^2 \right) \, dx
\]
\[
= s_r^{p-2} t_p \int_{\mathbb{R}^N} k^2_{\varepsilon} (I_\theta * |e^-_r|^p) |e^+_r|^p \, dx \quad (4.18)
\]
and
\[
\int_{\mathbb{R}^N} \left( |\nabla e^-_r|^2 + \tau |e^-_r|^2 \right) \, dx
\]
\[
= t_r^{p-2} s_p \int_{\mathbb{R}^N} k^2_{\varepsilon} (I_\theta * |e^+_r|^p) |e^-_r|^p \, dx \quad (4.19)
\]
Setting $r \rightarrow \infty$ in (4.18) and (4.19), by (4.14) we obtain that
\[
\int_{\mathbb{R}^N} (|\nabla e|^2 + \tau e^2) \, dx = \left( s_0^{p-2} t_0^p + s_0^{2p-2} \right) \int_{\mathbb{R}^N} k^2_{\varepsilon} (I_\theta * e^p) e^p \, dx
\]
and
\[
\int_{\mathbb{R}^N} (|\nabla e|^2 + \tau e^2) \, dx = \left( t_0^{p-2} s_0^p + t_0^{2p-2} \right) \int_{\mathbb{R}^N} k^2_{\varepsilon} (I_\theta * e^p) e^p \, dx.
Since \( \int_{\mathbb{R}^N} (|\nabla v|^2 + \tau e^2) \, dx = \int_{\mathbb{R}^N} k_v^2 (I_0 * e^p)e^p \, dx \), we have \( s_0 = t_0 = 2^{-\frac{1}{p}}. \)

Thirdly, by (4.15), we have

\[
\zeta_\varepsilon \leq J_\varepsilon(s_\varepsilon e^\tau_\varepsilon + t_\varepsilon e^-\tau_\varepsilon)
= \frac{p-1}{2p} \int_{\mathbb{R}^N} \left( (|\nabla (s_\varepsilon e_\tau^\tau_\varepsilon + t_\varepsilon e^-\tau_\varepsilon)|^2 + V_\varepsilon(x)(s_\varepsilon e_\tau^\tau_\varepsilon + t_\varepsilon e^-\tau_\varepsilon)^2) \right) \, dx
= \frac{p-1}{2p} \int_{\mathbb{R}^N} \left( (|\nabla (s_\varepsilon e_\tau^\tau_\varepsilon + t_\varepsilon e^-\tau_\varepsilon)|^2 + \tau(s_\varepsilon e_\tau^\tau_\varepsilon + t_\varepsilon e^-\tau_\varepsilon)^2) \right) \, dx \quad (as \ \varepsilon \to 0)
\]

\[
\to \frac{p-1}{2p} \left( s_0^2 + t_0^2 \right) \int_{\mathbb{R}^N} (|\nabla e|^2 + \tau e^2) \, dx \quad (as \ \varepsilon \to \infty)
= (s_0^2 + t_0^2) \vartheta^{\tau k_v}.
< 2\theta^{\tau k_v}
\]

Noting \( m \geq 2 \), due to Lemma 3.8 and Lemma 3.11, we get

\[
2\theta^{\tau k_v} < \vartheta^{\infty} \leq \vartheta^{\infty}_\varepsilon.
\]

In view of (4.20) and (4.21), we conclude that \( \zeta_\varepsilon < \vartheta^{\infty}_\varepsilon \) for \( \varepsilon \) small enough, which implies that \( J_\varepsilon \) satisfies (PS)\(_\varepsilon\) condition for \( \varepsilon \) small enough.

Fourthly, choose \( 0 < \delta < \vartheta^{\infty}_\varepsilon \) in Lemma 3.16, then for any \( \gamma \in \Gamma \), \( J_\varepsilon \circ \gamma < \zeta_\varepsilon - \tilde{\delta} \) on \( \vartheta^\mathbb{R}^2 \) for some \( \tilde{\delta} > 0 \). Thus \( \gamma(\vartheta^\mathbb{R}^2) \cap J_\varepsilon^{-1}([\zeta_\varepsilon - \tilde{\delta}, \zeta_\varepsilon + \tilde{\delta}]) = \emptyset \) for any \( \gamma \in \Gamma \).

By means of Location theorem (see Theorem 2.20 in Willem [28]), there exists \( v_\varepsilon \in H^1(\mathbb{R}^N) \) such that

\[
\text{dist}(v_\varepsilon, M_\varepsilon) \to 0, \ J_\varepsilon(v_\varepsilon) \to \zeta_\varepsilon, \ (J_\varepsilon)'(v_\varepsilon) \to 0 \text{ in } H^{-1}(\mathbb{R}^N) \text{ as } n \to \infty.
\]

Finally, for \( \varepsilon \) small enough, going if necessary to a subsequence,

\[
v_\varepsilon \to v^\varepsilon \text{ in } H^1(\mathbb{R}^N) \text{ as } n \to \infty,
\]

which, together with (4.22), implies that

\[
v^\varepsilon \in M_\varepsilon, \ J_\varepsilon(v^\varepsilon) = \zeta_\varepsilon, \ (J_\varepsilon)'(v^\varepsilon) = 0.
\]

We conclude that \( v^\varepsilon \) and \(-v^\varepsilon\) are a pair of sign-changing solutions for Eq. (4.1).

This completes the proof.

\[ \square \]

**Proof of Theorem 1.2.** We can assume without loss of generality that \( x_w = 0 \). Then \( V(0) = \tau_w, \ W(0) = k \). By (1.6) and (1.7), we get \( m \geq 1 \). Taking \( a = \tau_w, \ b = k \) in Eq. (3.1), there exists \( v \in \mathcal{R}^{\tau_w k} \). The following arguments are similar to the proof of Theorem 1.1, so the details are omitted.

---

5. **Proofs of convergence, concentration and decay estimate of solutions.**

In this section, we shall prove the convergence, concentration and decay estimate of the positive least energy solutions for Eq. (1.1). Namely we give the proof of Theorem 1.3.

**Proof of Theorem 1.3.** We shall deal with the case (A1)-(i), the other case can be handled similarly. Without loss of generality, we assume \( x_v = 0 \). Then \( V(0) = \tau, \ W(0) = k_v \). Obviously, (A1)-(i) implies that (1.4) holds. It follows from Theorem 1.1 that Eq.(1.1) has a positive least energy solution \( w_\varepsilon(x) \) and Eq.(4.1) has a positive least energy solution \( v_\varepsilon(x) = w_\varepsilon(\varepsilon x) \).

**Step 1.** We shall prove the convergence of \( v_\varepsilon \) as \( \varepsilon \to 0 \) up to a sequence after translations.
Let $\varepsilon_j \to 0 (j \to \infty)$, $v_j := v_{\varepsilon_j} \in \mathcal{R}_{\varepsilon_j}$ with $v_j > 0$. Since
\[
\vartheta_{\varepsilon_j} = J_{\varepsilon_j}(v_j) = \frac{p - 1}{2p} \int_{\mathbb{R}^N} \left( |\nabla v_j|^2 + V_{\varepsilon_j}(x)v_j^2 \right) dx \geq C\|v_j\|_1^2,
\]
it follows from Lemma 3.13 that \{\{v_j\}\} is bounded in $H^1(\mathbb{R}^N)$. By $v_j \in \mathcal{N}_{\varepsilon_j}$ and Lemma 2.2, we have
\[
\|v_j\|_1^2 \leq C \int_{\mathbb{R}^N} (I_0 * v_j^p)v_j^p dx \leq \bar{C}\|v_j\|_1^{2p}. \tag{5.1}
\]
Suppose
\[
\lim_{j \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} v_j^2 dx = 0,
\]
then by Lions Lemma (see Lemma 1.21 in [28]), Lemma 2.1 and Lemma 2.3, we have
\[
v_j \to 0 \quad \text{in } L^{2\frac{N}{N-p}}(\mathbb{R}^N) \quad \text{as } j \to \infty, \quad \int_{\mathbb{R}^N} (I_0 * v_j^p)v_j^p dx \to 0 \quad \text{as } j \to \infty,
\]
which contradicts with (5.1). Therefore, there exist $\delta > 0$ and $y_j' \in \mathbb{R}^N$ such that
\[
\int_{B_1(y_j')} v_j^2 dx \geq \delta. \tag{5.2}
\]
Set $u_j(x) := v_j(x + y_j')$, $\hat{V}_{\varepsilon_j}(x) := V_{\varepsilon_j}(x + y_j')$, $\hat{W}_{\varepsilon_j}(x) := W_{\varepsilon_j}(x + y_j')$. Then $u_j$ solves
\[
- \Delta u_j + \hat{V}_{\varepsilon_j}(x)u_j = \hat{W}_{\varepsilon_j}(x)(I_0 * (\hat{W}_{\varepsilon_j} u_j^p))u_j^{p-2}, \quad u_j > 0 \tag{5.3}
\]
with least energy
\[
\dot{\vartheta}_{\varepsilon_j} = J_{\varepsilon_j}(u_j) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u_j|^2 + \hat{V}_{\varepsilon_j}(x)u_j^2 \right) dx - \frac{1}{2p} \int_{\mathbb{R}^N} \hat{W}_{\varepsilon_j}(x)(I_0 * (\hat{W}_{\varepsilon_j} u_j^p))u_j^p dx \tag{5.4}
\]
Moreover, $(I_0 * (\hat{W}_{\varepsilon_j} u_j^p))(x + y_j') = (I_0 * (\hat{W}_{\varepsilon_j} u_j^p))(x)$ for any $x \in \mathbb{R}^N$, which ensures that
\[
\dot{\vartheta}_{\varepsilon_j} = \dot{J}_{\varepsilon_j}(u_j) = J_{\varepsilon_j}(v_j) = \vartheta_{\varepsilon_j}. \tag{5.5}
\]
In view of the boundness of \{u_j\}, we can assume without loss of generality that
\[
u_j \to u \quad \text{in } H^1(\mathbb{R}^N) \quad \text{as } j \to \infty, \tag{5.6}
\]
\[
u_j \to u \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^N) \quad \text{as } j \to \infty \quad \text{for } q \in [2, 2^*), \tag{5.7}
\]
which, together with (5.2), implies $u \neq 0$.

Since $V$ and $W$ are bounded, going if necessary to a subsequence, we assume
\[
V_{\varepsilon_j}(y_j') \to V_0 \quad \text{and} \quad W_{\varepsilon_j}(y_j') \to W_0 \quad \text{as } j \to \infty. \tag{5.8}
\]
By the boundness of $\nabla V : |\nabla V(x)| \leq M$, $\forall x \in \mathbb{R}^N$, we get that for any given $r > 0$,
\[
|\hat{V}_{\varepsilon_j}(x) - V_{\varepsilon_j}(y_j')| = \left| \int_0^1 \nabla V(\varepsilon_j y'_j + t\varepsilon_j x)\varepsilon_j x dt \right| \leq \varepsilon_j M r, \quad \forall x \in B_r(0).
\]
Thus $\hat{V}_{\varepsilon_j}(x) \to V_0$ as $j \to \infty$ uniformly on any bounded set of $x$. Similarly, 
$\hat{W}_{\varepsilon_j}(x) \to W_0$ as $j \to \infty$ uniformly on any bounded set of $x$. As the proof of 
Lemma 3.13, we have

$$\limsup_{j \to \infty} \hat{V}_{\varepsilon_j} \leq \partial^{V_0W_0}. \quad (5.9)$$

By (5.3), (5.6), (5.7), Lemma 2.1 and Lemma 2.3, we obtain that for any $\varphi \in C^\infty_0(\mathbb{R}^N)$,

$$0 = \lim_{j \to \infty} \int_{\mathbb{R}^N} \left[ \nabla u_j \nabla \varphi + \hat{V}_{\varepsilon_j}(x)u_j \varphi - \hat{W}_{\varepsilon_j}(x)(I_{\theta} \ast (\hat{W}_{\varepsilon_j} u_j^p))u_j^{p-1} \varphi \right] dx$$

$$= \int_{\mathbb{R}^N} \left[ \nabla u \nabla \varphi + V_0 u \varphi - W_0(I_{\theta} \ast (W_0 u^p))u^{p-1} \varphi \right] dx,$$

which deduces that $u$ solves

$$-\Delta u + V_0 u = W_0^2(I_{\theta} \ast u^p)u^{p-1}, \quad u > 0 \quad (5.10)$$

with energy

$$J^{V_0W_0}(u) : = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + V_0 u^2 \right) dx - \frac{1}{2p} \int_{\mathbb{R}^N} W_0^2(I_{\theta} \ast u^p)u^p dx$$

$$= \frac{p-1}{2p} \int_{\mathbb{R}^N} W_0^2(I_{\theta} \ast u^p)u^p dx \geq \partial^{V_0W_0}. \quad (5.11)$$

According to Fatou’s Lemma,

$$\int_{\mathbb{R}^N} W_0^2(I_{\theta} \ast u^p)u^p dx \leq \liminf_{j \to \infty} \int_{\mathbb{R}^N} \hat{W}_{\varepsilon_j}(x)(I_{\theta} \ast (\hat{W}_{\varepsilon_j} u_j^p))u_j^p dx. \quad (5.12)$$

Uniting (5.4), (5.9), (5.11) and (5.12), we obtain

$$\partial^{V_0W_0} \leq J^{V_0W_0}(u) \leq \liminf_{j \to \infty} \hat{J}_{\varepsilon_j}(u_j) \leq \limsup_{j \to \infty} \hat{J}_{\varepsilon_j} \leq \partial^{V_0W_0}. \quad (5.13)$$

Therefore,

$$\lim_{j \to \infty} \hat{J}_{\varepsilon_j} = \partial^{V_0W_0} = J^{V_0W_0}(u) . \quad (5.13)$$

Set $\eta \in C^\infty_0(\mathbb{R}^N)$ satisfy $\text{supp}\eta \subset B_2$ and $\eta \equiv 1$ on $B_1$ with $|\nabla \eta| \leq 2$. Define

$$\tilde{u}_j(x) := \eta \left( \frac{x}{j} \right) u(x), \quad x \in \mathbb{R}^N,$$

$$z_j(x) := u_j(x) - \tilde{u}_j(x), \quad x \in \mathbb{R}^N.$$

Then

$$\tilde{u}_j \to u \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{as} \quad j \to \infty,$$

$$\tilde{u}_j \to u \quad \text{in} \quad L^q(\mathbb{R}^N) \quad \text{as} \quad j \to \infty \quad \text{for} \quad q \in [2, 2^*],$$

$$\tilde{u}_j \to u \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{as} \quad j \to \infty$$

and

$$z_j \to 0 \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{as} \quad j \to \infty,$$

$$z_j \to 0 \quad \text{in} \quad L^q_{\text{loc}}(\mathbb{R}^N) \quad \text{as} \quad j \to \infty \quad \text{for} \quad q \in [2, 2^*],$$

$$z_j \to 0 \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{as} \quad j \to \infty.$$
Next we will show \( \tilde{J}_{\epsilon_j}(z_j) \to 0 \) and \( \langle \tilde{J}_{\epsilon_j}'(z_j), z_j \rangle \to 0 \) as \( j \to \infty \), where

\[
\tilde{J}_{\epsilon_j}(z_j) := \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla z_j|^2 + \widehat{V}_{\epsilon_j}(x) |z_j|^2 \right) dx - \frac{1}{2p} \int_{\mathbb{R}^N} \hat{W}_{\epsilon_j}(x) \left( I_\theta \ast (\hat{W}_{\epsilon_j}|z_j|^p) \right) |z_j|^p dx.
\]

Firstly, we claim

\[
\|z_j\|_1^2 = \|u_j\|_1^2 - \|\tilde{u}_j\|_1^2 + o(1). \tag{5.14}
\]

Indeed, \( \|u_j\|_1^2 - \|\tilde{u}_j\|_1^2 - \|z_j\|_1^2 = 2(u_j, \tilde{u}_j)_1 - 2\|\tilde{u}_j\|_1^2 \), where \( \|\tilde{u}_j\|_1 \to \|u\|_1 \) and

\[
|(u_j, \tilde{u}_j)_1 - (u, u)_1| \leq |(u_j, \tilde{u}_j - u)_1| + |(u_j - u, u)_1| \to 0 \quad \text{as} \quad j \to \infty.
\]

Secondly, we claim

\[
\int_{\mathbb{R}^N} \widehat{V}_{\epsilon_j}(x) |z_j|^2 dx = \int_{\mathbb{R}^N} \widehat{V}_{\epsilon_j}(x) |u_j|^2 dx - \int_{\mathbb{R}^N} \widehat{V}_{\epsilon_j}(x) |\tilde{u}_j|^2 dx + o(1). \tag{5.15}
\]

Indeed, \( \{z_j\} \) is bounded in \( H^1(\mathbb{R}^N) \) and in \( L^q(\mathbb{R}^N) \) for any \( q \in [2, 2^*) \). Moreover, \( \forall \delta > 0, \exists \epsilon(\delta) > 0 \) such that

\[
|z_j|^q - |z_j|^q = ||z_j + \tilde{u}_j|^q - |z_j|^q| \leq \delta|z_j|^q + c(\delta)|\tilde{u}_j|^q
\]

and

\[
f_j := (|u|^q - |\tilde{u}|^q - |\tilde{u}|^q - \delta|z|^q) \leq (1 + c(\delta))|\tilde{u}|^q \leq (1 + c(\delta))|u|^q.
\]

By Lebesgue dominated convergence theorem, \( \int_{\mathbb{R}^N} f_j dx \to 0 \) as \( j \to \infty \). Thus

\[
\int_{\mathbb{R}^N} |z_j|^q - |z_j|^q - |\tilde{u}_j|^q| dx \leq \int_{\mathbb{R}^N} f_j dx + \delta \int_{\mathbb{R}^N} |z_j|^q dx = o(1). \tag{5.16}
\]

Choose \( q = 2 \) in (5.16), we find (5.15) holds. Meanwhile, (5.16) with \( q = 2 \) and (5.14) imply that

\[
|\nabla z_j|^2 = |\nabla u_j|^2 - |\nabla \tilde{u}_j|^2 + o(1).
\]

Thirdly, we claim

\[
\int_{\mathbb{R}^N} \hat{W}_{\epsilon_j}(x) \left( I_\theta \ast (\hat{W}_{\epsilon_j}|z_j|^p) \right) |z_j|^p dx
\]

\[
= \int_{\mathbb{R}^N} \hat{W}_{\epsilon_j}(x) \left( I_\theta \ast (\hat{W}_{\epsilon_j}|u_j|^p) \right) |u_j|^p dx - \int_{\mathbb{R}^N} \hat{W}_{\epsilon_j}(x) \left( I_\theta \ast (\hat{W}_{\epsilon_j}|\tilde{u}_j|^p) \right) |\tilde{u}_j|^p dx + o(1). \tag{5.18}
\]

Indeed, taking into account that \( \{z_j\} \) is bounded in \( L^{2N/p}(\mathbb{R}^N) \), it follows from (5.16) with \( q = p \) that

\[
|u_j|^p - |\tilde{u}_j|^p - |z_j|^p \to 0 \quad \text{in} \quad L^{2N/p}(\mathbb{R}^N) \quad \text{as} \quad j \to \infty
\]

and

\[
I_\theta \ast (|u_j|^p - |\tilde{u}_j|^p - |z_j|^p) \to 0 \quad \text{in} \quad L^{2N/p}(\mathbb{R}^N) \quad \text{as} \quad j \to \infty,
\]

which, together with

\[
|z_j|^p \to 0 \quad \text{in} \quad L^{2N/p}(\mathbb{R}^N) \quad \text{as} \quad j \to \infty,
\]

\[
\tilde{u}_j^p \to u^p \quad \text{in} \quad L^{2N/p}(\mathbb{R}^N) \quad \text{as} \quad j \to \infty,
\]

\[
I_\theta \ast \tilde{u}_j^p \to I_\theta \ast u^p \quad \text{in} \quad L^{2N/p}(\mathbb{R}^N) \quad \text{as} \quad j \to \infty
\]
Moreover, obtain Lemma 2.3, imply that
\[
\int_{\mathbb{R}^N} \hat{W}_{\varepsilon_j}(x) \left[ (I_\theta \ast (\hat{W}_{\varepsilon_j} | u_j|^p)) |u_j|^p \\
- (I_\theta \ast (\hat{W}_{\varepsilon_j} | \tilde{u}_j|^p)) |\tilde{u}_j|^p - (I_\theta \ast (\hat{W}_{\varepsilon_j} | z_j|^p)) |z_j|^p \right] dx
\]
= \int_{\mathbb{R}^N} \hat{W}_{\varepsilon_j}(x) \left( I_\theta \ast \left( \hat{W}_{\varepsilon_j} (|u_j|^p - |\tilde{u}_j|^p - |z_j|^p) \right) \right) \left( |u_j|^p - |\tilde{u}_j|^p - |z_j|^p \right) dx
+ 2 \int_{\mathbb{R}^N} \hat{W}_{\varepsilon_j}(x) \left( I_\theta \ast \left( \hat{W}_{\varepsilon_j} (|u_j|^p - |\tilde{u}_j|^p - |z_j|^p) \right) \right) |\tilde{u}_j|^p dx
+ 2 \int_{\mathbb{R}^N} \hat{W}_{\varepsilon_j}(x) \left( I_\theta \ast (\hat{W}_{\varepsilon_j} |\tilde{u}_j|^p) \right) |z_j|^p dx
\rightarrow 0 \quad \text{as} \quad j \rightarrow \infty.

Fourthly, it follows from Lebesgue dominated convergence theorem that
\[
\int_{\mathbb{R}^N} \hat{V}_{\varepsilon_j}(x) \hat{u}_j^2 \, dx = \int_{\mathbb{R}^N} V_0 u^2 \, dx + o(1), \quad (5.19)
\]
\[
\int_{\mathbb{R}^N} \hat{W}_{\varepsilon_j}(x) \left( I_\theta \ast (\hat{W}_{\varepsilon_j} | u_j|^p) \right) dx = \int_{\mathbb{R}^N} W_0^2 (I_\theta \ast (u|^p)) u^p \, dx + o(1). \quad (5.20)
\]
Moreover,
\[
|\nabla \hat{u}_j|^2 = |\nabla u|^2 + o(1). \quad (5.21)
\]
Uniting (5.15), (5.17), (5.18), (5.19), (5.20), (5.21), (5.13), (5.3) and (5.10), we obtain
\[
\hat{J}_{\varepsilon_j} (z_j) = \hat{J}_{\varepsilon_j} (u_j) - \hat{J}_{\varepsilon_j} (\tilde{u}_j) + o(1)
= \hat{J}_{\varepsilon_j} - J^{V_0 W_0} (u) + o(1)
= o(1) \quad (5.22)
\]
and
\[
\left\langle (\hat{J}_{\varepsilon_j})' (z_j), z_j \right\rangle = \left\langle (\hat{J}_{\varepsilon_j})' (u_j), u_j \right\rangle - \left\langle (\hat{J}_{\varepsilon_j})' (\tilde{u}_j), \tilde{u}_j \right\rangle + o(1)
= \left\langle (\hat{J}_{\varepsilon_j})' (u_j), u_j \right\rangle - \left\langle (J^{V_0 W_0})' (u), u \right\rangle + o(1)
= o(1). \quad (5.23)
\]
Finally, by (5.22) and (5.23), we achieve that
\[
o(1) = \hat{J}_{\varepsilon_j} (z_j) - \frac{1}{2p} \left\langle (\hat{J}_{\varepsilon_j})' (z_j), z_j \right\rangle
= \frac{p - 1}{2p} \int_{\mathbb{R}^N} \left( |\nabla z_j|^2 + \hat{V}_{\varepsilon_j} (x) |z_j|^2 \right) \, dx
\geq C |z_j|^2,
\]
which implies $z_j \rightarrow 0$ in $H^1(\mathbb{R}^N)$ as $j \rightarrow \infty$. Thus
\[
|u_j - u|_1 \leq |z_j|_1 + |\tilde{u}_j - u|_1 \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty.
\]
That is
\[
u_j \rightarrow u \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{as} \quad j \rightarrow \infty.
\]

**Step 2.** We claim $u_j (x) \rightarrow 0$ as $|x| \rightarrow \infty$ uniformly in $j \in \mathbb{N}$. 
If the thesis is false, jointly with sub-solution estimate, then \( \exists \delta > 0, x_n \in \mathbb{R}^N \) with \( |x_n| \to \infty, j_n \in \mathbb{N} \) and \( C > 0 \) independent of \( n \) such that
\[
\delta \leq u_{j_n}(x_n) \leq C \left( \int_{B_1(x_n)} (u_{j_n})^2 \, dx \right)^{\frac{1}{2}}.
\]
Noting \( u_{j_n} \to u \) in \( L^2(\mathbb{R}^N) \) as \( n \to \infty \) and by Minkowski inequality, one has
\[
\delta \leq C \left( \int_{\mathbb{R}^N} (u_{j_n} - u)^2 \, dx \right)^{\frac{1}{2}} + C \left( \int_{B_1(x_n)} u^2 \, dx \right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n \to \infty.
\]
That is a contradiction.

**Step 3.** We claim \( \{ \xi_j \gamma_j \} \) is bounded in \( \mathbb{R}^N \).

If the thesis is not true, then there exists \( |\xi_j \gamma_j| \to \infty \) as \( j \to \infty \) along a subsequence. Hence
\[
V_0 \geq \tau_0 > \tau \quad \text{and} \quad W_0 \leq k_0 \leq k_v,
\]
which, together with Lemma 3.7, imply that \( \vartheta^{V_0 W_0} > \vartheta^{r k_v} \). However, by (5.5), (5.13) and Lemma 3.13,
\[
\vartheta^{V_0 W_0} = \lim_{j \to \infty} \vartheta_{\varepsilon_j} = \lim_{j \to \infty} \vartheta_{\varepsilon_j} \leq \limsup_{j \to \infty} \vartheta_{\varepsilon_j} \leq \vartheta^{r k_v}.
\]
That is a contradiction.

Therefore, going if necessary to a subsequence, we may assume
\[
\varepsilon_j \gamma_j \to x_0 \quad \text{as} \quad j \to \infty.
\]
Noting (5.8), we have
\[
V_0 = V(x_0) \quad \text{and} \quad W_0 = W(x_0).
\]
By (5.10), we find that \( u \) is a least energy solution of Eq. (1.8).

**Step 4.** We claim \( \{ \varepsilon \gamma_{\varepsilon_j} \} \) is bounded, where \( \gamma_{\varepsilon_j} \in \mathbb{R}^N \) is a maximum point of \( v_{\varepsilon_j} \).

Assume by contradiction that there is \( \varepsilon_j \to 0 \) with \( |\varepsilon_j \gamma_j| \to \infty \), where \( \gamma_j := \gamma_{\varepsilon_j} \) is a maximum point of \( v_j := \varepsilon_{\varepsilon_j} \). Repeating Step 1, 2, 3, one can get that \( \exists \gamma_j \in \mathbb{R}^N \) such that
\[
\gamma_j = \varepsilon_j \gamma_j \to u \neq 0 \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{as} \quad j \to \infty,
\]
\[
u_j(x) \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{uniformly in} \quad j \in \mathbb{N},
\]
\[
\{ \varepsilon \gamma_j \} \text{ is bounded in } \mathbb{R}^N.
\]
Thus \( |\varepsilon_j \gamma_j - \varepsilon_j \gamma_j' - |\varepsilon_j \gamma_j| \to \infty \text{ as } j \to \infty \), which implies that \( |\gamma_j - \gamma_j'| \to \infty \text{ as } j \to \infty \). Then \( \max_{\mathbb{R}^N} v_j = \gamma_j(y_j) = u_j(y_j - \gamma_j') \to 0 \text{ as } j \to \infty \). Hence \( \max_{\mathbb{R}^N} u_j \to 0 \text{ as } j \to \infty \). Noting \( u_j > 0 \), one has \( u_j(x) \to 0 \text{ as } j \to \infty \text{ uniformly in } x \in \mathbb{R}^N \), which contradicts with (5.26).

**Step 5.** We claim \( \lim_{\varepsilon \to 0} \text{dist}(\varepsilon \gamma_{\varepsilon_j}, A_u) = 0 \).

By Step 4, there exists \( \varepsilon_j \to 0 \) with
\[
\varepsilon_j \gamma_j \to y_0 \quad \text{as} \quad j \to \infty,
\]
where \( y_j := \gamma_{\varepsilon_j} \) is a maximum point of \( v_j := \varepsilon_{\varepsilon_j} \). It is sufficient to check that \( y_0 \in A_u \).
By Step 1 and Step 3, there exists \( y'_j \in \mathbb{R}^N \) satisfying \( u_j(x) = v_j(x + y'_j) \) and (5.24). By Step 2, we can assume \( u_j(x'_j) = \max_{\mathbb{R}^N} u_j \) and \( \{x'_j\}_j \) is bounded in \( \mathbb{R}^N \). Thus \( y_j = x'_j + y'_j \) and \( \varepsilon_j y_j = \varepsilon_j x'_j \to 0 \) as \( j \to \infty \), which, together with (5.24), (5.25) and (5.27), imply that

\[
y_0 = x_0, \quad V(y_0) = V_0, \quad W(y_0) = W_0.
\]  

(5.28)

Assume indirectly that \( y_0 \notin \mathcal{A}_v \), then either \( V(y_0) = \tau \), \( W(y_0) < k_v \) or \( V(y_0) > \tau \), \( W(y_0) \leq k_v \). By Lemma 3.7,

\[
y^{V(y_0)W(y_0)} > \vartheta_{k_v}.
\]  

(5.29)

Uniting (5.5), (5.13), (5.28), (5.29) and Lemma 3.13, we obtain

\[
\lim_{j \to \infty} \vartheta_{\varepsilon_j} = \lim_{j \to \infty} \varepsilon_j \vartheta_{\varepsilon_j} = \varepsilon\vartheta_{\varepsilon_0} = \varepsilon V_0W_0 = \varepsilon^{V(y_0)W(y_0)} > \vartheta_{\varepsilon_0} \geq \limsup_{j \to \infty} \vartheta_{\varepsilon_j}.
\]  

That is a contradiction.

If particularly \( \mathcal{V} \cap \mathcal{W} \neq \emptyset \), then \( x_0 \in \mathcal{A}_v = \mathcal{V} \cap \mathcal{W} \). Hence

\[
\lim_{\varepsilon \to 0} \text{dist}(\varepsilon y_\varepsilon, \mathcal{V} \cap \mathcal{W}) = 0 \quad \text{and} \quad V(x_0) = \tau, \quad W(x_0) = k,
\]  

which, jointly with Eq.(1.8), deduce that \( u \) is a least energy solution of Eq.(1.9).

**Step 6.** For \( \rho \in \left(2, \frac{N+\theta}{N-2}\right) \), we claim there exist \( C > 0 \) and \( \tilde{R} > 0 \) such that for all small \( \varepsilon > 0 \),

\[
u_{\varepsilon}(x) \leq C|x|^{1-N} \exp\left(-\frac{\sqrt{\tau}}{2}|x|\right), \quad \forall|x| \geq \tilde{R}.
\]

We verify its correctness for any sequence. By Step 2, we have

\[
\lim_{|x| \to \infty} \tilde{W}_{\varepsilon_j}(x)(I_\theta * (\tilde{W}_{\varepsilon_j} u^{\rho}_j))(x)(u_j(x))^{p-2} = 0
\]

uniformly in \( j \in \mathbb{N} \), which implies that there exists \( \tilde{R} > 0 \) such that

\[
\tilde{W}_{\varepsilon_j}(x)(I_\theta * (\tilde{W}_{\varepsilon_j} u^{\rho}_j))(x)(u_j(x))^{p-2} \leq \frac{3}{4}\tau, \quad \forall|x| \geq \tilde{R}, \quad \forall j \in \mathbb{N}.
\]  

(5.30)

Hence, by (5.3) and (5.30), we have

\[
-\Delta u_j + \frac{\tau}{4} u_j \leq 0, \quad \forall|x| \geq \tilde{R}, \quad \forall j \in \mathbb{N}.
\]

By Lemma 1.4 with \( K = \frac{\tau}{4} \), there exists \( v \in C^2(\mathbb{R}^N \setminus B_R(0), \mathbb{R}) \) such that

\[
\begin{cases}
-\Delta v + \frac{\tau}{4} v = 0 & |x| \geq \tilde{R}, \\
v(x) \geq u_j(x) & |x| = \tilde{R}, \quad j \in \mathbb{N}, \\
0 \leq v(x) \leq C|x|^{1-N} \exp\left(-\frac{\sqrt{\tau}}{2}|x|\right) & |x| \geq \tilde{R},
\end{cases}
\]

where \( C > 0 \) depends on \( \tau \). By comparison principle, we obtain that

\[
u_j(x) \leq C|x|^{1-N} \exp\left(-\frac{\sqrt{\tau}}{2}|x|\right), \quad \forall|x| \geq \tilde{R}, \quad j \in \mathbb{N}.
\]

**Step 7.** Set \( x_\varepsilon = \varepsilon y_\varepsilon \). Then \( \nu_{\varepsilon}(x_\varepsilon) = v_{\varepsilon}(y_\varepsilon) \). By Step 4, \( x_\varepsilon \) is a maximum point of \( \nu_{\varepsilon} \) and \( \{x_\varepsilon\}_\varepsilon \) is bounded in \( \mathbb{R}^N \). By Step 5, \( \lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, \mathcal{A}_v) = 0 \). By Step 1 and Step 2, \( \nu_{\varepsilon}(x) = v_{\varepsilon}(x + y'_\varepsilon) = \nu_{\varepsilon}(x + x_\varepsilon - \varepsilon y'_\varepsilon) \), where \( x'_\varepsilon = y_\varepsilon - y'_\varepsilon \) is a maximum point of \( \nu_{\varepsilon} \) with \( \varepsilon x'_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). By Step 6,

\[
\nu_{\varepsilon}(x) \leq C\varepsilon^{\frac{N-2}{4\varepsilon}} |x - x_\varepsilon|^{1-N} \exp\left(-\frac{\sqrt{\tau}}{4\varepsilon} |x - x_\varepsilon|\right), \quad \forall|x| \geq R,
\]
where } \bar{R} := \tilde{R} + \sup_{x \in \mathbb{R}^2} |x_z| .

The proof is completed.

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