Efficient Solution of a Class of Quantified Constraints with Quantifier Prefix Exists-Forall

Milan Hladík · Stefan Ratschan

Received: 14 December 2013 / Revised: 12 May 2014 / Accepted: 31 May 2014 / Published online: 29 July 2014
© Springer Basel 2014

Abstract In various applications the search for certificates for certain properties (e.g., stability of dynamical systems, program termination) can be formulated as a quantified constraint solving problem with quantifier prefix exists-forall. In this paper, we present an algorithm for solving a certain class of such problems based on interval techniques in combination with conservative linear programming approximation. In comparison with previous work, the method is more general—allowing general Boolean structure in the input constraint, and more efficient—using splitting heuristics that learn from the success of previous linear programming approximations.

Keywords Constraint solving · Decision procedures · Interval computation

Mathematics Subject Classification 65G20 · 65H20

1 Problem Description

We study the problem of finding \( x_1, \ldots, x_r \) such that

\[
\bigwedge_{i=1}^{n} \forall y_1, \ldots, y_s \in B_i \phi_i(x_1, \ldots, x_r, y_1, \ldots, y_s)
\]

where each \( B_i \) is a box (i.e., Cartesian product of closed intervals) in \( \mathbb{R}^s \) and each of the \( \phi_1, \ldots, \phi_n \) is a Boolean combination of inequalities where for each \( i \in \{1, \ldots, n\} \) only one of those inequalities contains the variables \( x_1, \ldots, x_r \) and this one inequality contains those variables only linearly. If no such \( x_1, \ldots, x_r \) exist, we want to detect this. Here is an illustrating example:

\[
\forall y_1 \in [0, 1], y_2 \in [-1, 1] \quad [y_1 \geq y_2 \lor x_1 \sin(y_1)y_2 + x_2y_1^2 \leq 0]
\]

\[
\forall y_1 \in [0, 1], y_2 \in [-1, 1] \quad [y_1 < y_2 \lor x_1 \cos(y_1)y_2 + x_2y_1^2 \leq 0].
\]
We also study the extension of this problem to the case where the conjunction may—in addition to constraints of the form

$$\forall y_1, \ldots, y_s \in B_i \phi_i(x_1, \ldots, x_r, y_1, \ldots, y_s)$$

also contain linear equalities in the variables $x_1, \ldots, x_r$ (the equalities can be viewed as a conjunction of inequalities, but this violates the condition that only one inequality contains $x_1, \ldots, x_r$).

In an earlier paper [22], we showed how to solve a special case, with restricted Boolean structure. The contributions of this paper are:

- The extension of the approach to arbitrary Boolean structure.
- The design of splitting heuristics that improves the performance of the algorithm by orders of magnitude.

Constraints of this type occur in various applications. Especially, they are useful in finding certificates for certain global system properties. For example, a Lyapunov function [15] represents a certificate for the stability of dynamical systems. In the case of global stability, such a function has to fulfill certain properties in the whole state space except for the original/equilibrium. After using an ansatz (often also called template) of the Lyapunov function as a polynomial with parametric coefficients, one can find the Lyapunov function by solving a universally quantified problem for those parameters. The fact that polynomials are linear in their coefficients corresponds to linearity of our variables $x_1, \ldots, x_r$. A similar situation occurs, for example in termination analysis of computer programs [4,18], or in the termination analysis of term-rewrite systems [7].

However, usually further work is usually necessary to apply the method studied in this paper to such problems: In the case of Lyapunov functions, one has to exempt one single point (the equilibrium) from the property which cannot be directly expressed by the boxes $B_1, \ldots, B_n$. In the case of termination analysis, such constraints have to be solved for the whole real space instead of boxes $B_1, \ldots, B_n$. In an earlier paper, we solved this problem for Lyapunov functions [10,22]. However, in other areas this is an open area for further research.

In the polynomial case, constraints over the domain of real numbers with quantifiers can always be solved due to Tarski’s classical result that the first-order theory of the real numbers allows quantifier elimination [28]. The area of computer algebra has developed impressive software packages for quantifier elimination [2,8]. However, those still have problems with scalability in the number of involved variables and, in general, they cannot solve non-polynomial problems. This was the motivation for several approaches to use interval based techniques for such problems, for special cases [1,11,14] and for arbitrary quantifier structure [21].

Alternative approaches for finding Lyapunov function certificates are based on techniques from real algebraic geometry [19,25]. However, those methods cannot solve general constraints of the form discussed in this paper (no general Boolean structure, no non-polynomial constraints).

In termination analysis of term-rewrite systems constraints with quantifier-prefix $\exists \forall$ are usually solved by first eliminating the universally quantified variables using conservative approximation [13,16], and then solving the remaining, existentially quantified problem. Again, this technique cannot solve general constraints of the form discussed in this paper (no general Boolean structure, no non-polynomial constraints).

The structure of the paper is as follows: In the next section, we will introduce the basic algorithm for solving the constraints. In Sect. 3 we will introduce splitting heuristics for the algorithm. In Sect. 4 we will extend the algorithm with equality constraints. In Sect. 5 we will prove termination of the algorithm for all non-degenerate cases. In Sect. 6 we discuss how, for a given box to split, choose the variable of that box to split. In Sect. 7 we will provide the results of computational experiments with the algorithm. And in Sect. 8 we will conclude the paper.

Throughout the paper, boldface variables denote objects that are intervals or contain intervals (e.g., interval vectors or matrices).

2 Basic Algorithm

We use the following algorithm (which generalizes an algorithm [22] that solves constraints of a more specific form arising in the analysis of ordinary differential equations):
1. For each $i \in \{1, \ldots, n\}$, substitute the intervals of $B_i$ corresponding to $y_1, \ldots, y_s$ into $\phi_i$, and evaluate using interval arithmetic. As a result, all the inequalities that do not contain $x_1, \ldots, x_r$ are simplified to an inequality of the form $I \leq 0$, where $I$ is an interval, and every inequality (one for each $i \in \{1, \ldots, n\}$) that does contain $x_1, \ldots, x_r$ to an inequality of the form $p_1 x_1 + \ldots + p_r x_r \leq q$.

where the $p_1, \ldots, p_r$, $q$ are intervals.

2. Replace any inequality of the form $I \leq 0$ with a (not necessarily strictly) negative upper bound of $I$ by the Boolean constant $T$ (for “true”).

3. Replace any inequality of the form $I \leq 0$ with a (strictly) positive lower bound of $I$ by the Boolean constant $\mathcal{F}$ (for “false”).

4. Simplify the constraint further using basic reasoning with the Boolean constants $T$ and $\mathcal{F}$ (e.g., simplify $T \land \phi$ to $T$, $\mathcal{F} \lor \phi$ to $\phi$).

5. If the resulting constraint is a Boolean constant, we are done (if the result is the Boolean constant $T$, no solution exists).

6. If the constraint is an interval linear system of inequalities $P x \leq q$ (i.e., all disjunctions $\lor$ in the formula have been removed by the simplifications in Step 4), we reduce the interval linear system to a linear system $A z \leq b$ using the method of Rohn and Kreslová [12, 24], and solve this system using linear programming. If the result is not yet an interval linear system, we continue with the next step.

7. If the previous step resulted in a solvable linear program, we have a solution to the original problem. If no, we choose an $i \in \{1, \ldots, n\}$, split the box $B_i$ into pieces $B^1, B^2$, replace the original constraint by

$$\bigwedge_{j \in \{1, \ldots, i-1, i+1, \ldots, n\}} \left( \forall y_1, \ldots, y_s \in B_j \right) \left[ \phi_j(x_1, \ldots, x_r, y_1, \ldots, y_s) \right] \land$$

$$\left( \forall y_1, \ldots, y_s \in B^1 \right) \left[ \phi_1(x_1, \ldots, x_r, y_1, \ldots, y_s) \right] \land$$

$$\left( \forall y_1, \ldots, y_s \in B^2 \right) \left[ \phi_1(x_1, \ldots, x_r, y_1, \ldots, y_s) \right]$$

and iterate from Step 1 of the algorithm.

Note that this algorithm only splits boxes with bounds pertaining to the variables $y_1, \ldots, y_s$ but not wrt. variables $x_1, \ldots, x_r$. This is the main advantage of such an algorithm over a naive algorithm that substitutes sample points for the free variables $x_1, \ldots, x_n$. Completeness can be preserved due to completeness of the Rohn/Kreslová algorithm which we now describe in more detail:

The basic idea is, to replace each variable $x_i$ by two non-negative variables $x_i^1$ and $x_i^2$, and then to rewrite each term $[p_i, \overline{p}_i] x_i$ of the interval system of inequalities to $[p_i, \overline{p}_i] (x_i^1 - x_i^2)$ which is equal to $[p_i, \overline{p}_i] x_i^1 - [p_i, \overline{p}_i] x_i^2$. Based on the fact that the inequalities should hold for all elements of the intervals, we can now exploit the fact that the $x_i^1$ and $x_i^2$ are non-negative. Hence, we can now replace the interval coefficients of $x_i^1$ with their upper endpoint, and the interval coefficients of $x_i^2$ with their lower endpoint, resulting in $\overline{p_i} x_i^1 - p_i x_i^2$. The result is an linear system of inequalities of the form $\overline{P} x^1 - P x^2 \leq q$, $x_1 \geq 0$, $x_2 \geq 0$.

3 An Informed Splitting Strategy

The major building block of the algorithm that we left open is the splitting strategy: Which box to choose for splitting in Step 7 and along which variable to split it. In this section we will develop such a strategy. We will first describe the basic idea for the linear system $A z \leq b$ created in Step 6 of the algorithm (Sect. 3.1), then we will study how to take into account the fact that the linear system was created from an interval linear system $P x \leq q$ by the algorithm of Rohn and Kreslová (Sect. 3.2), then we will study how to ensure convergence of the strategy (Sect 3.3), take into account interval evaluation from Step 1 of the main algorithm (Sect. 3.4), and summarize the result into a sub-algorithm of our main algorithm (Sect. 3.5).
3.1 Basic Idea

Our goal is to have a strategy that is

- complete: if the problem has a solution, we will eventually find it
- efficient: the algorithm converges to a solution as fast as possible.

It is not too difficult to ensure completeness of the algorithm: Just ensure that the width of all boxes goes to zero [21,22]. However, the result can be highly inefficient: each split increases the size of the constraint to solve, slowing down the algorithm. Hence it is essential to concentrate on splits that bring the constraint closer to solvability.

Since splitting heuristics for classical interval branch-and-bound (or branch-and-prune) algorithms are well-studied [6,9,20] we assume that in Step 6 the algorithm already arrived at an interval linear system of inequalities.

We will try to come up with splits that bring the next linear program closer to solvability. For achieving this, we need some measure of what it means for an infeasible system of inequalities $A z \leq b$ (as created by Step 6 of the algorithm) to be close to solvability, which will lead us to a method for determining how it can be brought closer to solvability by splitting.

The overall approach is to

1. use the minimum of the residual $\max_{i \in \{1,...,n\}} (Az - b)_i$, that is
   $$\min_z \max_{i \in \{1,...,n\}} (Az - b)_i$$
   as a measure of closeness to feasibility (here the index $i$ denotes the $i$-th entry of the vector $Az - b$),
2. to compute the corresponding minimizer, and then to
3. use those splits that promise to improve the residual for this minimizer the most.

For computing the minimum of the residual, we reformulate

$$\min_z \max_{i \in \{1,...,n\}} (Az - b)_i$$

as the constrained optimization problem

$$\min_{z,\rho} \rho \text{ subject to } \rho = \max_{i \in \{1,...,n\}} (Az - b)_i$$

which is

$$\min_{z,\rho} \rho \text{ subject to } \rho \geq (Az - b)_1, \ldots, \rho \geq (Az - b)_n,$$

from where we arrive at the linear program

$$\min_{z,\rho} \rho \text{ subject to } Az - b \leq [1,\ldots,1]^T \rho.$$

Let $z^*, \rho^*$ be the resulting minimizer. If the residual $\rho^* \leq 0$ then we know that the system $Az \leq b$ is solvable. If not, then the constraint violation vector $Az^* - b$ provides information on how much the individual constraints contribute to non-solvability.

We try to decrease the constraint violation of the row of $A$ for which $Az^* - b$ is maximal. Denote this row by $i$. The constraint corresponding to this row is of the form $a_1 z_1^* + \ldots + a_{2r} z_{2r}^* \leq b$ where each coefficient $a_j, j \in \{1,...,2r\}$ results from an endpoint of some interval in the interval system $Px \leq q$. Now we want to choose a $j \in \{1,...,2r\}$ such that splitting will aim at changing the coefficient $a_j$ as much as possible. We assume that the change that we can expect for coefficient $a_j$ if using such a split, is given by some real number $\delta_j$. Under this assumption, the inequality will change to

$$a_1 z_1^* + \ldots + a_{j-1} z_{j-1}^* + (a_j + \delta_j) z_j^* + a_{j+1} z_{j+1}^* + \ldots + a_{2r} z_{2r}^* \leq b$$

\footnote{With the exception of degenerate cases, see Sect. 5.}
which is
\[ a_1 z_1^* + \ldots + a_{2r} z_{2r}^* \leq b - \delta_j z_j^*, \]
resulting in an improvement \( -\delta_j z_j^* \).

Hence we can expect the maximal improvement of the residual by choosing \( j \) as
\[
\arg \max_{j \in \{1, \ldots, 2r\}} -\delta_j z_j^*.
\]

For analyzing how \( \delta_j \) should look like, we have to analyze the precise form of the system \( Az - b \) which we will do in the next sub-section.

3.2 Exploiting Structure

Now observe that the linear program that we used in the previous sub-section is not arbitrary, but is the result of the
Rohn/Kreslová transformation of an interval system of linear inequalities of the form
\[
\min \rho \quad \text{subject to} \quad P x_1 - P x_2 - b \leq [1, \ldots, 1]^T \rho, \quad x_1 \geq 0, \quad x_2 \geq 0.
\]

Observe that the entries of the underlying interval linear system of inequalities \( P x \leq q \) are created by interval evaluation (Step 1 of the main algorithm). Assuming that splitting shrinks large entries of the interval matrix \( P \) more than small intervals, the change \( \delta_j \) that we can expect for \( a_j \) from splitting is proportional to the width \( w(p_{v(j)}) \) of the corresponding interval \( p_{v(j)} \) in the \( i \)-th row of \( (p_1, \ldots, p_n) \) of \( P \). However, since splitting results in a sub-interval \( p'_{v(j)} \subseteq p_{v(j)} \), the expected change for lower bounds of intervals is positive, and for upper bounds of intervals is negative.

Analyzing the left-hand side \( P x_1 - P x_2 - b \) of the linear program resulting from the Rohn/Kreslová transformation, we observe that the \( x_1 \) have coefficient \( P \), that is, the sign of upper bounds is positive, and the \( x_2 \) have coefficient \( -P \) that is, the sign of lower bounds is negative. Combining this with the fact that lower bounds will be increased and upper bounds be decreased by splitting, the expected change \( \delta_j = -w(p_{v(j)}) \), resulting in
\[
\arg \max_{j \in \{1, \ldots, 2r\}} -\delta_j z_j = \arg \max_{j \in \{1, \ldots, 2r\}} w(p_{v(j)}) z_j^*.
\]

Now observe furthermore, that the coefficients of the linear program come in pairs that refer to the two bounds the same intervals, and hence also their width is the same. So, instead of
\[
\arg \max_{j \in \{1, \ldots, 2r\}} w(p_{v(j)}) z_j^*
\]
we can directly refer to the interval matrix:
\[
\arg \max_{j \in \{1, \ldots, r\}} \{ w(p_j) \max[x_1^j, x_2^j] \}
\]
where \( x_1^j \) and \( x_2^j \) refer to the individual entries of the vectors of variables as introduced by the Rohn/Kreslová transformation.

We also note the following:

**Lemma 1** Let \( \overline{P} \) and \( P \) be real matrices in \( \mathbb{R}^{n \times r} \) such that for every \( i \in \{1, \ldots, n\}, j \in \{1, \ldots, r\}, P_{i,j} < \overline{P}_{i,j}. \)
Let \( b \) be a real vector in \( \mathbb{R}^n \). Then for every solution \( x_1, x_2 \) of the linear program
\[
\min \rho \quad \text{subject to} \quad \overline{P} x_1 - P x_2 - b \leq [1, \ldots, 1]^T \rho, \quad x_1 \geq 0, \quad x_2 \geq 0
\]
for every \( j \in \{1, \ldots, r\} \), either \( x_1^j \) or \( x_2^j \) is zero.
Proof Let $P^c$ be the center of the interval matrix $P$, that is the matrix that contains the midpoint of the corresponding intervals of $P$. Let $P^\Delta$ be the matrix that contains for every entry the width of the corresponding interval of $P$. Then the above linear program is equivalent to

$$\min \rho, \ P^c(x^1 - x^2) + P^\Delta(x^1 + x^2) - b \leq [1, \ldots, 1]^T \rho, \ x^1 \geq 0, x^2 \geq 0$$

Let $j \in \{1, \ldots, r\}$ be arbitrary, but fixed, and assume that both $x^1_j$ and $x^2_j$ are non-zero. Then, we can replace $x^1_j$ by $x^1_j - \varepsilon$ and $x^2_j$ by $x^2_j + \varepsilon$, where $\varepsilon > 0$. As a result, the value of the first term $P^c(x^1 - x^2)$ stays unchanged, while the value of the second term $P^\Delta(x^1 + x^2)$ has decreased. Hence, we can decrease the minimum of the linear program, which is a contradiction to the assumption that the original values $x^1_j$ or $x^2_j$ were a solution of the linear program. \hfill \square

3.3 Ensuring Convergence

The basic idea, as described in the previous section, does not result in a converging method. We will demonstrate this on a concrete example, taking into account the precise form of how the system of inequalities is created by the method of Rohn and Kreslová. For this, assume the interval inequality

$$[-1, 3]x_1 + [-3, 1]x_2 \leq -2$$

and the corresponding inequality

$$3x^1_1 + x^2_2 + x^1_2 + 3x^2_1 \leq -2, \ x^1_1 \geq 0, x^2_2 \geq 0, x^1_2 \geq 0, x^2_1 \geq 0.$$  

The resulting linear program

$$\min x^1_1, x^2_1, x^1_2, x^2_2, \rho \quad \text{subject to} \ 3x^1_1 + x^2_2 + x^1_2 + 3x^2_1 + 2 \leq [1, \ldots, 1]^T \rho, \ x^1_1 \geq 0, x^2_2 \geq 0, x^1_2 \geq 0, x^2_1 \geq 0$$

has the solution $\rho = 2, \ x^1_1 = 0, x^2_2 = 0, x^1_2 = 0, x^2_1 = 0$ which corresponds to the values $x_1 = 0, x_2 = 0$ of the original interval inequality. Evaluating our heuristics, we get

$$\arg \max_{j \in \{1, 2\}} \ w(p_j) \ \max\{x^1_j, x^2_j\} = \arg \max_{j \in \{1, 2\}} \ 0 = 0$$

Hence our heuristics already compute the value 0 for each coefficient, suggesting that no shrinking of interval coefficients is necessary anymore (Theorem 1 in Sect. 5 will provide a more general characterization of such behavior). Still, we have not yet found a solution of $[-1, 3]x + [-3, 1]y \leq -2$, and the residual value $\rho > 0$ correctly indicates this. Moreover, a shrinking of the first interval, for example, resulting in

$$[2, 3]x + [-3, 1]y \leq -2$$

leads to a solvable system.

Analyzing the problem, we see that for the solution of $[2, 3]x + [-3, 1]y \leq -2, \ x \neq 0$! So the original heuristics was misleading, since it mistakenly assumed $x = 0$. In other words, while the minimizer $z^*$ of the linear program $\min_{z, \rho} \ A z - b \leq [1, \ldots, 1]^T \rho$ gives some orientation on which coefficients to shrink, it need not necessarily be a solution of the original input constraint, and hence may be misleading.

To fix the problem, we assume that the minimizer $x^1, x^2$ to the linear program only approximates the final solution of the input constraint that we are looking for. For each $j \in \{1, \ldots, r\}$, the final solution might instead be located in an interval $[x^1_j - x^2_j - \varepsilon, x^1_j - x^2_j + \varepsilon]$ around the corresponding solution $x^1_j - x^2_j$ of the interval system of linear inequalities.

We will now analyze the corresponding changed value of the term $\max\{x^1_j, x^2_j\}$ used in the computation of the heuristic value

$$\arg \max_{j \in \{1, \ldots, r\}} \ w(p_j) \ \max\{x^1_j, x^2_j\}.$$
In the case where $x_j^1 - x_j^2 \geq 0$, by Lemma 1, $x_j^2 = 0$. Hence the original value of the term $\max\{x_j^1, x_j^2\}$ is $x_j^1$, and the corresponding changed value is $x_j^1 + \varepsilon$. In the case where $x_j^1 - x_j^2 < 0$, the original value is $x_j^2$, and the changed value is $x_j^2 + \varepsilon$. Putting those cases together, the changed value is $\max\{x_j^1, x_j^2\} + \varepsilon$.

Hence the corresponding heuristic value can be up to

$$\arg \max_{j\in\{1,\ldots,r\}} \left[w(p_j)[\max\{x_j^1, x_j^2\} + \varepsilon]\right]$$

which we will use, for a user-provided constant $\varepsilon > 0$.

We will see later (Sect. 5), that even if the constant $\varepsilon$ does not correctly estimate the difference between the solution of the current linear program and a solution of the original constraint $\phi$, for constraints that have a non-degenerate solution, the resulting method always converges to such a solution, if $\varepsilon > 0$.

In general, we will use heuristics of the form

$$\arg \max_{j\in\{1,\ldots,r\}} h(p_j, x_j^1, x_j^2)$$

and show convergence under certain conditions of this function $h$.

3.4 Interval Evaluation

Up to now, we know which row(s) of $P$ to split, that is, for which $i \in \{1, \ldots, n\}$ to split the box $B_i$. We also know, which bound of which interval in that row of $P$ we want to decrease, but we still do not know which coordinate of $B_i$ result in the biggest decrease of that bound. For determining this, observe that each entry of the interval matrix $P$ results from interval evaluation of a certain expression on the box $B_i$. Hence we need to infer, for a given arithmetical expression and an interval for each variable in that expression, which split of an interval results in the biggest decrease of the given resulting (lower or upper) bound of interval evaluation. There are many possible choices for this. Hence our approach will be parametric in the concrete method used. We will assume a function splitheur such that for a given arithmetical expression $t$, box $B$, and sign $s \in \{-, +\}$, splitheur($t, B, s$)

- returns a variable of $B$ to split for improving the lower/upper bound (depending on $s \in \{-, +\}$) of the interval evaluation of $t$ on $B$, and for which
- repeated splitting according to this function converges, that is, for the sequence $B^1, B^2, \ldots$ created by splitting according to this function, for every $\varepsilon > 0$ there is a $k$ such that for all $i \geq k$ the width of $t(B^i)$ is smaller than $\varepsilon$.

Right now, we use this function for only the coefficient chosen by our heuristics. It might also make sense to try it on all coefficients, and choose the best one.

3.5 Algorithm

The resulting algorithm is called from the main algorithm in Step 7 in the case where in Step 6 we arrived at an (unsolvable) interval linear system. The algorithm has the following form (where, for an arithmetical expression $t$ and a box $B$, eval($t, B$) denotes interval evaluation of $t$ on $B$):

**Input:** expressions $t_{i,j}, i \in \{1, \ldots, n\}, j \in \{1, \ldots, r\}$ s.t. $t_{i,j}$ is the coefficient of $x_j$ in $\phi_i$,
boxes $B_i$, $i \in \{1, \ldots, n\}$

**Output:** $i \in \{1, \ldots, n\}$, $k \in \{1, \ldots, s\}$ suggesting to split box $B_i$ at its $k$-th coordinate

1: let $P$ be the $(n \times r)$-interval matrix s.t. $P_{i,j} = \text{eval}(t_{i,j}, B_i), i \in \{1, \ldots, n\}, j \in \{1, \ldots, r\}$
2: $(x^1, x^2) \leftarrow \arg \min_{\rho} \bar{P}x^1 - \underline{P}x^2 - b \leq [1, \ldots, 1]^T \rho, x^1 \geq 0, x^2 \geq 0$
3: $d \leftarrow \bar{P}x^1 - \underline{P}x^2 - b$ // residual
4: $i \leftarrow \arg \max_{i \in \{1, \ldots, n\}} d_i$ // box $B_i$ to split
5: $j \leftarrow \arg \max_{j \in \{1, \ldots, r\}} h(P_{i,j}, x_j^1, x_j^2)$ // coefficient to improve
6: return $i$, splitheur($t_{i,j}, B_i, \text{sgn}(x^1 - x^2)_j$)
We will call this version of the algorithm the split-worst version. We will also consider an alternative version that, instead of splitting only the box corresponding to the maximal constraint violation (as computed in Line 4), splits all boxes with positive constraint violation. We will call that version of the algorithm, the split-all version.

As already discussed above, if for the solution $\rho$ computed in Line 2, $\rho \leq 0$, then we know that the interval linear system $Px \leq q$ from Line 6 of the main algorithm is solvable. Moreover, since $\rho \leq 0$ is equivalent to the linear system of Rohn/Krevlavá being solvable, this computes the same information as Line 6 of the main algorithm and hence no solving has to be done there. In other words, instead of solving the Rohn/Krevlavá linear system of equations, we solve the linear program

$$\min \rho, \overline{P}x^1 - \underline{P}x^2 - b \leq [1, \ldots, 1]^T \rho, x^1 \geq 0, x^2 \geq 0$$

and use it both for determining the overall solution of the algorithm and heuristics for splitting.

4 Equality Constraints

Now we analyze the extended problem that—in addition to constraints of the form $\forall y_1, \ldots, y_s \in B, \phi_i(x_1, \ldots, x_r, y_1, \ldots, y_s)$—also contain linear equalities over the variables $x_1, \ldots, x_r$. Viewing each equality as a conjunction of two inequalities one sees that in that case, the two inequalities force the optimum $\rho^*$ of

$$\min \rho, \overline{P}x^1 - \underline{P}x^2 - b \leq [1, \ldots, 1]^T \rho, x^1 \geq 0, x^2 \geq 0$$

to be zero. So in this case, the heuristics in the form described above are not useful. In order to handle equalities better, we do not view such equalities as two inequalities, but we handle them directly. That is we solve the linear program

$$\min \rho, \overline{P}x^1 - \underline{P}x^2 - b \leq [1, \ldots, 1]^T \rho, C(x^1 - x^2) = d, x^1 \geq 0, x^2 \geq 0$$

where $Cx = d$ is the linear system of equations containing all the linear equalities.

5 Convergence

Our main algorithm consists of a loop that continues until a solution has been found. In this section we will answer the question: Will the loop terminate for all input constraints? Again we will assume that in Step 6 the algorithm already arrived at an interval linear system of inequalities, since convergence of basic interval branch-and-bound (or branch-and-prune) algorithms is not difficult to show (e.g., it follows as a special case of Theorem 6 in [21]).

Observe that the only place where the algorithm approximates, is the interval evaluation in Step 1 of the main algorithm. In the whole section, for the formal proofs, we assume that the resulting linear programs are solved precisely, using rational number arithmetic. Still, in practice, it suffices to solve them approximately, for example, based on floating-point arithmetic.

In the following we will denote by $\hat{P}x \leq \hat{q}$ the system of linear interval inequalities that would result from the input constraint if interval evaluation would be non-overapproximating. In a similar way, we will denote by $\hat{A}z \leq \hat{b}$ the system of linear inequalities corresponding to $\hat{P}x \leq \hat{q}$.

Definition 1 We call $z$ a robust solution of a system of inequalities and equalities $Az \leq b \land Cz = d$ iff $Az \leq b \land Cz = d$. We call a constraint $\phi$ robust if the corresponding system $Az \leq \hat{b} \land Cz = d$ has a robust solution.

In other words, a robust solution of a system $Az \leq b \land Cz = d$ is an interior point of $Az \leq b$ that satisfies the equalities $Cz = d$.

Lemma 2 If $z$ is a robust solution of $Az \leq b \land Cz = d$, then there is an $\varepsilon > 0$ s.t. that for all $A'$, and $b'$ differing from $A$ and $b$ not more than $\varepsilon$ for each entry, $A'z \leq b' \land Cz = d$. 
Proof} Since $z$ is a robust solution, $Az - b$ is a vector of negative numbers. From this we can compute an upper bound on the allowed changes of $A$ and $b$. □

It is not difficult to ensure convergence of the algorithm:

**Lemma 3** Assume that the splitting strategy ensures that the width of every bounding box $B_i$ goes to zero. Then the algorithm will terminate for robust inputs.

**Proof** Assume an arbitrary iteration of the algorithm. As above, denote by $\hat{P}_x \leq \hat{q}$ the interval system of inequalities that the algorithm would compute if using the precise range instead of over-approximating interval evaluation in Step 1. Denote by $\hat{A}z \leq \hat{b}$ the corresponding system of linear inequalities the algorithm computes from $\hat{P}_x \leq \hat{q}$ in Step 6. Assuming, in addition, a system of linear equalities $Cz = d$, let $\hat{z}$ be the robust solution of $\hat{A}z \leq \hat{b} \land Cz = d$, so $\hat{A}\hat{z} < \hat{b}$. Due to the fact that the algorithm does not compute the precise range in Step 1, but over-approximates it using interval evaluation, the algorithm will compute with an interval matrix $P \supseteq \hat{P}$. The over-approximation error goes to zero due to convergence of interval arithmetic. Hence $\hat{A}$ will be approximated increasingly well. So, due to Lemma 2, $\hat{A}z \leq \hat{b} \land Cz = d$ will eventually hold and the algorithm terminates. □

However, our heuristics do not necessarily ensure that the width of every bounding box goes to zero: Even if it would ensure that every bounding box is split infinitely often, it might still happen that the width of some bounding box does not go to zero, because a certain coordinate of the box is not split infinitely often. This might not even be necessary for termination, because this coordinate might correspond to a variable that does not occur in an coefficient term.

**Theorem 1** Consider the split-all version of the algorithm with heuristics of the form

$$\max_{j \in \{1, \ldots, n\}} h(p_j, x_j^1, x_j^2)$$

where

(a) $\lim_{w(p) \to 0} h(p, x) = 0$

(b) $w(p) > 0$ implies $h(p, x) > 0$.

Then we have: If the input constraint has a robust solution, then the algorithm terminates.

**Proof** We consider the split-all version of the algorithm and assume that the algorithm does not terminate. Then it creates an infinite sequence of unsolvable interval linear programs and corresponding linear programs (see Line 6 of the main algorithm). In each iteration all $B_i$ with positive constraint violation are split. Hence, all those bounding boxes are split infinitely often. In each iteration, in Line 5 of the algorithm from Sect. 3.5, a coefficient $j$ for improvement is chosen. If all coefficients are chosen infinitely often, then due to convergence of splitheur the width of all coefficients goes to zero, which implies that the constraint will eventually have non-positive constraint violation and the corresponding bounding box would not be chosen for splitting, which is a contradiction.

Hence, non-termination implies that at least one of the coefficients is not chosen infinitely often. Let us analyze the state of the algorithm where all coefficients that are chosen finitely often will not be chosen any more. All other coefficients are chosen infinitely often, which means that due to convergence of splitheur, their interval width goes to zero. Hence, due to Assumption (a), their $h$-value goes to zero. Moreover, due to Assumption (b) the coefficients that are not split any more, have positive $h$-value. This implies that Line 5 eventually chooses one of them, a contradiction. So the algorithm terminates. □

Clearly, the heuristics

$$\arg \max_{j \in \{1, \ldots, r\}} \left[ w(p_j) [\max \{x_j^1, x_j^2\} + \epsilon] \right],$$

with $\epsilon > 0$, as developed in Sect. 3, fulfill the assumptions of the theorem, and hence the algorithm converges.
6 Variable Splitting Heuristics

In this section we discuss, how the function splitheur\((t, B, s)\), that we introduced in Sect. 3.4, can be implemented.

A widely used technique (e.g., in global optimization \([6]\)) for this is to use derivatives of the arithmetical expression. An alternative would be Corollary 2.1.2 in Neumaier’s book \([17]\). However, that would need the computation of interval over-approximation of derivatives, or interval Lipschitz constants, respectively. Moreover, those techniques are only a priori estimates of the decrease that might fail to give exact information. In order to arrive at more precise information, we use the observation that we already have a fixed set of usually small expressions that we want to analyze. Hence, interval evaluation of those expressions will usually take negligible time compared to the rest of the algorithm. Hence we explicitly try all possible splits and compare their effect on the width of the result of interval evaluation \([5]\):

\[
\text{splitheur}(t, B, s) = \arg \max_{i \in \{1, \ldots, |B|\}} \min_k \{|b^k(t(B)) - b^k(t(\text{split}^k(B, i))))|\}
\]

where \(b^k\) takes the upper-/lower bound respectively of the argument interval according to \(s\), and \(\text{split}^k(B, i)\) denotes the \(k\)-th box resulting from splitting the box \(B\) at variable \(i\) (usually \(k \in \{1, 2\})

However, the method as described up to now does not ensure convergence of the method. The reason is the following: Assume a term \(r\) in \(n\) variables. Assume intervals \(I_1, \ldots, I_n\) on which we evaluate \(t\). Let \(I_j^-\) be the lower and \(I_j^+\) be the upper half of \(I_j\). Assume a procedure that replaces that interval \(I_j\) by its lower or upper half, for which this results in the biggest decrease of interval evaluation of \(t\). Repeated application of this procedure does not result in the width of interval evaluation going to zero. For example, for the term \(x^2 + y\) with \(x \in [-1, 1], y \in [-0, 2]\), splitting \([-1, 1]\) does not result in any improvement at all. However, it is necessary for global convergence.

One way of solving this problem is, to take the time since the last split into account. For example, we could use

\[
\text{splitheur}(t, B, s) = \arg \max_{i \in \{1, \ldots, |B|\}} \left[ c(i) + \min_k \{|b^k(t(B)) - b^k(t(\text{split}^k(B, i))))|\} \right]
\]

where \(c(i)\) is a function that increases with the time of the last split of variable \(i\). If this function goes to infinity with the time of the last split, then every variable will be split eventually, ensuring convergence. The result is some compromise between round-robin-splitting (which ensures convergence) and aggressive local improvement. In order to make this heuristics independent of the size of \(B\) (which decreases during the algorithm) it makes sense to use some scaling with \(w(t(B))\) in the function \(c(i)\).

7 Computational Experiments

We did experiments on examples for computing Lyapunov-like functions \([22]\), with the heuristic function

\[
\arg \max_{j \in \{1, \ldots, r\}} \left[ w(p_j) [\max\{x^1_j, x^2_j\} + \varepsilon] \right]
\]

and \(\varepsilon = 0.001\).

For the resulting examples we have \(\phi_1 = \ldots = \phi_r\) with different bounding boxes for each branch \(i \in \{1, \ldots, r\}\). The bounding boxes and the inequality constraints of the examples are as follows:

**Example A:**

\[
B_1 = [0.8, 1.2] \times [0.3, 0.49], \quad B_2 = [0.8, 1.2] \times [0.51, 0.7], \quad B_3 = [1.01, 1.2] \times [0.49, 0.51], \quad B_4 = [0.8, 0.99] \times [0.49, 0.51]
\]

where \(\phi\) is of the form

\[
x_1(2y_1^3y_2 - 2y_1^2 + y_1) + x_2(y_1^2y_2 - y_1 + 0.5) + x_3(y_1^2y_2 - y_1^2y_2 - y_1y_2 + 0.5y_1 + 0.5y_2) \]
\[
+ x_4(0.5 - y_1y_2^2) + x_5((-2)y_1^2y_2^2 + y_2) \leq -0.0001
\]

**Example B:**

\[
B_1 = [-0.8, 0.8] \times [-0.8, -0.1], \quad B_2 = [-0.8, 0.8] \times [0.1, 0.8], \quad B_3 = [-0.8, -0.1] \times [-0.1, 0.1], \quad B_4 = [0.1, 0.8], [-0.1, 0.1]
\]
The extension of the algorithm to a more general class of constraints, for example to ensure applicability in invariant computation [23,26,27].

Table 1 Results of experiments

|       | Round-robin | Split-worst | Split-all |
|-------|-------------|-------------|-----------|
|       | Splits      | Time        | Splits    | Time        |
| A     | 48          | 0.260       | 21        | ε           | 5          | ε         |
| B     | 545         | 8.777       | 112       | 0.648       | 6          | ε         |
| C     | 20          | 0.128       | 20        | 0.128       | 5          | ε         |
| D     | 719         | 70.176      | 719       | 70.176      | 10         | 2.472     |
| A'    | 0           | ε           | 0         | ε           | 0          | ε         |
| B'    | 719         | 19.233      | 82        | 0.300       | 9          | ε         |
| C'    | 244         | 6.472       | 4         | ε           | 3          | ε         |
| D'    | 0           | ε           | 0         | ε           | 0          | ε         |

where $\phi$ is of the form

$$x_1(-2y_1^2 + 2y_1y_2) + x_2(0.2y_1y_2 - 4y_2^2 - 2y_1^3y_2 - 0.2y_1^3y_2) \leq -0.0001$$

Example C:

$B_1 = [-0.4, 0.4] \times [-0.4, -0.1], B_2 = [-0.4, 0.4] \times [0.1, 0.4],$

$B_3 = [-0.4, -0.1] \times [-0.1, 0.1], B_4 = [0.1, 0.4] \times [-0.1, 0.1]$  

where $\phi$ is of the form

$$x_1(-16y_1^6 + 24y_1^5 - 8y_1^4) + x_2(-12y_1^5 + 18y_1^4 - 6y_1^3)
+x_3(-8y_1^3 + 12y_1^3 - 4y_1^2) + x_4(4y_2^2) \leq -0.000001$$

Example D:

$B_1 = [-0.2, 0.2] \times [-0.2, 0.2] \times [-0.2, -0.1], B_2 = [-0.2, 0.2] \times [-0.2, 0.2] \times [0.1, 0.2],$

$B_3 = [-0.2, 0.2] \times [-0.2, -0.1] \times [-0.1, 0.1] B_4 = [-0.2, 0.2] \times [0.1, 0.2] \times [-0.1, 0.1],$

$B_5 = [-0.2, -0.1] \times [-0.1, 0.1] \times [-0.1, 0.1], B_6 = [0.1, 0.2] \times [-0.1, 0.1] \times [-0.1, 0.1]$  

where $\phi$ is of the form

$$x_1(-2y_1y_2) + x_2(-2y_2y_3) + x_3(-2y_3^2 - 2y_1y_3 + 2y_1^3y_3) + x_4(-y_1^2 - y_1y_3)
+x_5(y_1^2 - 2y_1y_2 - y_1y_3 - y_2y_3 + y_1^4) + x_6(-2y_2y_3 - y_1y_2 - y_2y_3 + y_1^3y_2) \leq -0.0001$$

In all four cases, we normalized the first coefficient $a$ to 1. To create versions with equality constraints we used the pre-processing method described in Section 4.3. of [22]. We will denote the result by $A^{'}, B^{'}, C^{'}, D^{'},$ and $D_1$. The results of the experiments can be seen in Table 1. Here, round-robin refer to the classical round-robin splitting heuristics where variables are split one after the other, and that we used in earlier work [22]. Empty entries correspond to cases where the algorithm did not terminate within 10 min, and $\varepsilon$ corresponds to cases where the algorithm terminates in less than 0.1 s. The experiments were done based on an implementation in the programming language Objective Caml using the LP solver Glpk, on a Linux operating system and a 64-bit 2.80GHz processor.

8 Conclusion

We have shown how to efficiently solve a class of quantified constraints. Computational experiments show that the corresponding splitting heuristics result in efficiency improvements by orders of magnitude.

Possibilities for further research include:

- The application of the algorithm to areas such as termination analysis [4,16,18].
- The extension of the algorithm to a more general class of constraints, for example to ensure applicability in invariant computation [23,26,27].
Acknowledgments Stefan Ratschan’s work was supported by the Czech Science Foundation (GA ČR) grant number P202/12/J060 with institutional support RVO:67985807. Milan Hladík’s work was supported by the Czech Science foundation (GA ČR) grant number P402-13-10660S.

References

1. Benhamou, F., Goualard, F.: Universally quantified interval constraints. In: Proceedings of the Sixth International Conference on Principles and Practice of Constraint Programming (CP’2000), Number 1894 in LNCS, Springer Verlag, Singapore (2000)
2. Brown, C.W.: QEPCAD B: a system for computing with semi-algebraic sets via cylindrical algebraic decomposition. SIGSAM Bull. 38(1), 23–24 (2004)
3. Caviness, B.F., Johnson J.R.: Quantifier Elimination and Cylindrical Algebraic Decomposition. Springer, Wien (1998)
4. Cousot, P.: Proving program invariance and termination by parametric abstraction, Lagrangian relaxation and semidefinite programming. In: Cousot, R. (ed.) VMCAI’05, Number 3385 in LNCS, pp. 1–24, Springer, Berlin (2005)
5. Csendes, T., Klatte, R., Ratz, D.: A posteriori direction selection rules for interval optimization methods. Central Eur. J. Oper. Res. (2000)
6. Csendes, T., Ratz, D.: Subdivision direction selection in interval methods for global optimization. SIAM J. Numer. Anal. 34(3), 922–938 (1997)
7. Dershowitz, N.: Termination of rewriting. J. Symbol. Comput. 3, 69–116 (1987)
8. Dolzmann, A., Sturm, T.: Redlog: computer algebra meets computer logic. SIGSAM Bull. 31(2), 2–9 (1997)
9. Douillard, T., Jermann, C.: Splitting heuristics for disjunctive numerical constraints. In: Proceedings of the 2008 ACM symposium on Applied computing, pp. 140–144. ACM, New York (2008)
10. Giesl Peter, A., Hafstein, S.F.: Revised CPA method to compute Lyapunov functions for nonlinear systems. J. Math. Anal. Appl. 410(1), 292–306 (2014)
11. Goldsztejn, A., Michel, C., Rueher, M.: Efficient handling of universally quantified inequalities. Constraints 14(1), 117–135 (2009)
12. Hladík, M.: Weak and strong solvability of interval linear systems of equations and inequalities. Linear Algebra Appl. 438(11), 4156–4165 (2013)
13. Hong, H., Jakuš, D.: Testing positiveness of polynomials. J. Autom. Reason. 21(1), 23–38 (1998)
14. Jaulin, L., Walter, É.: Guaranteed tuning, with application to robust control and motion planning. Automatica 32(8), 1217–1221 (1996)
15. Khalil, H.K.: Nonlinear Systems. 3rd edn. Prentice Hall, New York (2002)
16. Lucas, S.: Practical use of polynomials over the reals in proofs of termination. In: PPDP ’07: Proceedings of the 9th ACM SIGPLAN international conference on Principles and practice of declarative programming, pp 39–50, ACM, New York (2007)
17. Neumaier, A.: Interval Methods for Systems of Equations. Cambridge Univ. Press, Cambridge (1990)
18. Podelski, A., Rybalchenko, A.: A complete method for the synthesis of linear ranking functions. In VMCAI: Verification, Model Checking, and Abstract Interpretation, vol. 2937 of LNCS. Springer, Berlin (2004)
19. Prajna, S., Papachristoudoulou, A., Seiler, P., Parrilo, P.A.: SOSTOOLS and its control applications. In: Positive Polynomials in Control, pp. 273–292. Springer Verlag, Berlin (2005)
20. Ratschan, S.: Search heuristics for box decomposition methods. J. Glob. Optim. 24(1), 51–60 (2002)
21. Ratschan, S.: Efficient solving of quantified inequality constraints over the real numbers. ACM Trans. Comput. Logic 7(4), 723–748 (2006)
22. Ratschan, S., She, Z.: Providing a basin of attraction to a target region of polynomial systems by computation of Lyapunov-like functions. SIAM J. Contr. Optim. 48(7), 4377–4394 (2010)
23. Rodriguez-Carbonell, E., Kapur, D.: Automatic generation of polynomial loop invariants: algebraic foundations. In: Proceedings International Symposium on Symbolic and Algebraic Computation, ISSAC-2004, (2004)
24. Rohn, J., Kreslová, I.: Linear interval inequalities. Linear Multilinear Algebra 38, 79–82 (1994)
25. Sankaranarayanan, S., Chen, X., Abrahám, E.: Lyapunov function synthesis using Handelman representations. In: The 9th IFAC Symposium on Nonlinear Control Systems, pp. 576–581, (2013)
26. Sankaranarayanan, S., Sipma, H.B., Manna, Z.: Constructing invariants for hybrid systems. Formal Methods Syst. Des. 32(1), 25–55 (2008)
27. Sturm, T., Tiwari, A.: Verification and synthesis using real quantifier elimination. In: Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation, pp. 329–336. ACM, New York (2011)
28. Tarski, A.: A Decision Method for Elementary Algebra and Geometry. Univ. of California Press, Berkeley, 1951. Also in [3]