The Painlevé property of $\mathbb{C}P^{N-1}$ sigma models

P P Goldstein$^1$ and A M Grundland,$^{2,3}$

$^1$ Theoretical Physics Division, National Centre for Nuclear Research, Hoza 69, 00-681 Warsaw, Poland
$^2$ Department of Mathematics and Computer Sciences, Université du Québec, Trois-Rivières. CP500 (QC) G9A 5H7, Canada
$^3$ Centre de Recherches Mathématiques. Université de Montréal. Montréal CP6128 (QC) H3C 3J7, Canada

E-mail: Piotr.Goldstein@fuw.edu.pl, grundlan@crm.umontreal.ca

Abstract. We test the $\mathbb{C}P^{N-1}$ sigma models for the Painlevé property. While the construction of finite action solutions ensures their meromorphicity, the general case requires testing. The test is performed for the equations in the homogeneous variables, with their first component normalised to one. No constraints are imposed on the dimensionality of the model or the values of the initial exponents. This makes the test nontrivial, as the number of equations and dependent variables are indefinite. A $\mathbb{C}P^{N-1}$ system proves to have a $(4N - 5)$-parameter family of solutions whose movable singularities are only poles, while the order of the investigated system is $4N - 4$. The remaining degree of freedom, connected with an extra negative resonance, may correspond to a branching movable essential singularity. An example of such a solution is provided.

PACS numbers: 02.30.Jr, 02.30.Ik, 02.30.Mv

AMS classification scheme numbers: 81T45, 35Q51, 35G50, 35A20

Key words: sigma models, integrability, singularity analysis, Painlevé property.

1. Introduction

Numerous physical applications of models with effective Lagrangians, in particular the $\mathbb{C}P^{N-1}$ sigma models [2, 3, 8, 16, 17, 20, 22], made these models an interesting subject to study [9, 11, 10]. The question of the integrability of the equations governing these models has found an apparently positive answer in the works of Din and Zakrzewski [11]. Moreover, the linear spectral problem is known for them, so (in principle) the initial problem may be solved by the inverse scattering method. However the above results only concern systems with finite action. On the other hand, if we are interested in the dynamics of the systems, we start from the corresponding Euler-Lagrange (EL) equations, which allow for a much larger class of solutions. A natural question arises, as to whether the equations remain integrable if we lift the assumption of finite action.
In the present paper we will discuss this question and provide a self-contained approach to the subject.

The first approach which we try when testing a system of equations for integrability is usually the Painlevé test in the form introduced in [1], or its generalisation to partial differential equations (PDEs) [21], with possible further refinements (as discussed in [5, 6, 19], which provide a comprehensive review of the method).

In our case, the Painlevé test entails extra difficulties due to the fact that the dimensionality of the \( \mathbb{C}P^{N-1} \) model and the number of equations are arbitrary. Nevertheless, the test can be carried out (Section 3).

In what follows, Section 2 contains a short summary of \( \mathbb{C}P^{N-1} \) models and possible methods of their description. We conclude that section by selecting the description (system of PDEs) suitable for the Painlevé test. In Section 3 we perform the test, obtaining a ‘nearly general’ local solution in the form of a Laurent series. By ‘nearly general’ we mean that our solution provides \( 4N - 5 \) first integrals out of the total of \( 4(N - 1) \) such integrals in the general solution, i.e. it yields one integral fewer than the order of the system. Section 4 contains a discussion of the missing first integrals. A counterexample, i.e. an example of the non-Painlevé behaviour, is given in the form of a solution which has an essential singular manifold with branching. The manifold depends on four parameters (although not on an arbitrary function), which means that the position of the singularity depends on the initial conditions.

2. \( \mathbb{C}P^{N-1} \) sigma models

Sigma models describe complex systems by a simple Lagrangian defined in terms of an effective field which lies in an appropriate space, while the complexity remains in the metric of the space.

\[
\mathcal{L} = \sum_{i,j=0}^{\infty} g_{ij} dz_i d\bar{z}_j, \tag{1}
\]

where \( z_i, \bar{z}_j \) represent the field variables in \( \mathbb{C}^N \), while \( g_{ij} \) is the metric tensor. A bar over a symbol denotes its complex conjugate (c.c.).

The models prove to be rich in interesting properties provided that the metric depends on the fields, i.e. the model is nonlinear. Even simple nonlinear cases, like the \( \mathbb{C}P^{N-1} \) models have many applications, from two dimensional gravity to biological membranes [4, 12, 16]. In these models the independent variables \( \xi^1, \xi^2 \) are from the Riemann sphere or a 2D Minkowski space, \( z \in S^N \), while the differential in (1) is expressed in terms of the \( z \)-dependent covariant derivatives \( D_\mu \) by

\[
D_\mu z = \partial_\mu z - (z^\dagger \cdot \partial_\mu z)z, \quad \partial_\mu = \partial_{\xi^\mu}, \quad \mu = 1, 2. \tag{2}
\]

producing a Lagrangian density of the form

\[
\mathcal{L} = \frac{1}{4} (D_\mu z)^\dagger \cdot (D_\mu z), \tag{3}
\]
Painlevé property of $\mathbb{C}P^{N-1}$ sigma models

where the convention of summation over repeating Greek indices is assumed, $z$ are complex unit vectors in $\mathbb{C}^N$, a dagger denotes the Hermitian conjugate, and $\partial$ and $\bar{\partial}$ are the derivatives with respect to $\xi = \xi^1 + i\xi^2$ and $\bar{\xi} = \xi^1 - i\xi^2$ respectively. The normalisation of $z$ requires that

$$z^\dagger \cdot z = 1, \quad z = (z_0, ..., z_{N-1}). \quad (4)$$

The EL equations corresponding to the Lagrangian (3)

$$D_\mu D_\mu z + (D_\mu z)^\dagger \cdot (D_\mu z) z = 0, \quad (5)$$

are simple, but they are not suitable for testing the Painlevé property: due to the normalisation (4), a pole of $z$ has to correspond to a zero of $z^\dagger$, at least for real $\xi^\mu$. For the same reason, we do not analyse even simpler equations satisfied by the rank-1 projectors $P = z \otimes z^\dagger$, namely

$$[\partial \bar{\partial} P, P] = 0, \quad (6)$$

The necessary freedom is achieved if we use the homogeneous unnormalised field variables $f$, such that

$$z = f/(f^\dagger \cdot f)^{1/2}, \quad \mathbb{C} \ni \xi \mapsto f(\xi, \bar{\xi}) = (f_0(\xi, \bar{\xi}), ..., f_{N-1}(\xi, \bar{\xi})) \in \mathbb{C}^N \setminus \{0\}, \quad (7)$$

whose dynamics is governed by the unconstrained EL equations

$$\left( \mathbb{I} - \frac{f \otimes f^\dagger}{f^\dagger \cdot f} \right) \left[ \partial \bar{\partial} f - \frac{1}{f^\dagger \cdot f} \left( (f^\dagger \cdot \bar{\partial} f) \partial f + (f^\dagger \cdot f) \bar{\partial} f \right) \right] = 0. \quad (8)$$

The way in which these vector functions are constructed makes them elements of a Grassmannian space $\text{Gr}(1, \mathbb{C}^N)$ [22] and suggests that equations (8) are invariant under multiplication of $f$ by any scalar function (which may easily be checked by direct calculation). This property leaves too much freedom for the shape of possible singularities. However if we normalise the homogeneous variables in such a way that the first component $f_0$ is equal to 1, we eventually obtain a system of equation suitable for the Kovalevsky-Gambier analysis, commonly known as the Painlevé test. The equations in terms of the affine variables $w = (w_1, ..., w_{N-1})$, such that

$$w_i = f_i/f_0, \quad i = 1, ..., N-1 \quad (\text{generically } f_0 \neq 0). \quad (9)$$

read

$$\begin{align*}
1 + \sum_{l=1}^{N-1} \bar{w}_l w_l & \partial \bar{\partial} w_i - \sum_{l=1}^{N-1} (\bar{w}_l \partial \bar{w}_l \partial w_i + \bar{w}_l \partial w_i \bar{\partial} w_i) = 0, \quad (10a) \\
1 + \sum_{l=1}^{N-1} w_l \bar{w}_l & \tilde{\partial} \tilde{\partial} w_i - \sum_{l=1}^{N-1} (w_l \partial \bar{w}_l \partial \bar{w}_i + w_l \partial \bar{w}_i \bar{\partial} \bar{w}_i) = 0, \quad (10b)
\end{align*}$$

where the complex conjugates of (10a) have been written separately as (10b) because the complex conjugation will no longer link the variables $w_i$ with $\bar{w}_i$ when we extend the independent variables analytically to the double complex plane $\mathbb{C}^2$ (as it is done in the Painlevé test). Therefore, in what follows, we put quotation marks in 'complex
Painlevé property of $\mathbb{C}P^{N-1}$ sigma models

conjugation’ when we write about the symmetry which turns unbarred quantities into the barred ones and vice versa.

Equations (10a, 10b) will be the subject of further analysis. They constitute a system of $2(N - 1)$ second-order PDEs, which requires $4(N - 1)$ first integrals to build the general solution.

3. The Painlevé test

To perform the test, we look for the solution of the system (10a, 10b), extended to the double complex plane $(\xi, \bar{\xi}) \in \mathbb{C}^2$, in the form of a Laurent series about a movable noncharacteristic singularity manifold

$$\Phi(\xi, \bar{\xi}) = \bar{\xi} - \varphi(\xi) \quad \text{(Kruskal’s simplification),}$$

(11)

where the function $\varphi$ defining the singularity manifold is a holomorphic function of $\xi$, while the coefficients of the expansion are analytic in their arguments $(\xi, \bar{\xi})$.

The condition of being noncharacteristic excludes the surfaces $\xi = 0$ and $\bar{\xi} = 0$, which in turn eliminates locally holomorphic and locally antiholomorphic functions $w$, $\bar{w}$, including the solutions of Din and Zakrzewski [11, 9]. On the other hand, the selection of non-characteristic singularity manifolds makes possible both the Kruskal simplification ([15]) and the assumption $\varphi'(\xi) \neq 0$.

In the series below, we adopt the notation in which a superscript for $\Phi$ is simply an exponent, while a superscript for a dependent variable, e.g. $w_i^n$ denotes the $n$-th order coefficient in the Laurent expansion of $w_i$. Additionally, it is convenient to extend the notation to negative $n$, assuming

$$w_i^n = 0 \quad \text{whenever } n < 0, \text{ for all } i = 1, \ldots, N - 1.$$  

(12)

We do not limit the number of dependent variables $w_i$ and allow a priori the possibility that the initial exponents at each $w_i$ may be different. Thus the Laurent expansion has the form

$$w_i = \sum_{n=0}^{\infty} w_i^n(\xi) \Phi^{n-\alpha_i},$$

(13a)

$$\bar{w}_i = \sum_{n=0}^{\infty} \bar{w}_i^n(\xi) \Phi^{n-\beta_i},$$

(13b)

where for all $i$ we have $w_i^0 \neq 0$ and $\bar{w}_i^0 \neq 0$ (otherwise we would start from higher-order terms). We also assume that $\alpha_i > 0$ and $\beta_i > 0$ for all $i$.

Let us substitute (13) into our equations (10a, 10b). As these equations are of 3rd degree, the resulting equations contain quadruple sums (a sum over the components of $w$ and products of 3 sums of the Laurent series). We first rearrange the latter sums $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \rightarrow \sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{p=0}^{n}$, where $k = m + n + p$. Then we shift the dummy index $k$ in such a way that all terms indexed with the same $k$ become proportional to the same powers of $\Phi$ (the range of $k$ remains unchanged thanks to our convention (12)). Next we require that the coefficients of the same powers in $\Phi$ vanish.
Painlevé property of $\mathbb{C}P^{N-1}$ sigma models

Under the assumption that $\alpha_i > 0$, $\beta_i > 0$ for all $i = 0, ..., N - 1$, there is no balance of terms with exponents of different form in the lowest order. Therefore, the initial exponents are obtained from the equations satisfied by the coefficients of the lowest-order terms $k = 0$ rather than those satisfied by the exponents of those terms. In the lowest-order, we obtain for $i = 1, ..., N - 1$

\[
\sum_{l=1}^{N-1} \bar{w}_l^0 w_l^0 (2\alpha_l - \alpha_i - 1) = 0
\]

(14a)

\[
\sum_{l=1}^{N-1} w_l^0 \bar{w}_l^0 (2\beta_l - \beta_i - 1) = 0,
\]

(14b)

where the prime denotes the derivative with respect to $\xi$ (we have omitted the $\xi$-dependence of $\varphi$ and all the $w$'s).

Equations (14), divided by $w_i^0 \varphi' \alpha_i$ or $\bar{w}_i^0 \varphi' \beta_i$, constitute two separate systems of linear equations: one for $\alpha_1, ..., \alpha_{N-1}$ and a similar one for $\beta_1, ..., \beta_{N-1}$. It is evident that

\[
\alpha_1 = ..., = \alpha_{N-1} = \beta_1 = ..., = \beta_{N-1} = 1
\]

(15)

solves both systems (14). A question arises as to whether this is the only solution with all positive $\alpha_i$ and $\beta_i$. We will show that this is indeed the case. The proof below is performed for the system (14a). The proof for (14b) is identical.

Proof. Let our movable singularity manifold intersects the plane $\bar{\xi} = \xi^*$ (the plane of real $\xi^1$ and $\xi^2$), where the asterisk denotes the actual complex conjugation. Consider the matrix of coefficients, $B_{ij} = 2\bar{w}_i^0 w_i^0 - \delta_{ij} \sum_{l=1}^{N-1} \bar{w}_l^0 w_l^0$. For each $j = 1, ..., N - 1$, all elements of the $j$-th column are identical, with the exception of the diagonal element. If we add all the columns to the first one and then subtract the first row of the resulting matrix from each of the other rows, we obtain a triangular matrix, with zeros everywhere except for the first row and the main diagonal. The diagonal has $\sum_{l=1}^{N-1} \bar{w}_l^0 w_l^0$ in the first row and minus this sum in all other rows. Hence the determinant of the system can be calculated explicitly

\[
\det B = - \left( \sum_{l=1}^{N-1} \bar{w}_l^0 w_l^0 \right)^{N-1}.
\]

(16)

On the complex plane $\bar{\xi} = \xi^*$ the barred $w^0$'s are indeed the complex conjugates of their unbarred counterparts. Hence all the components of the sum in (16) are positive on this plane, as we have assumed that $w_l^0$ and $\bar{w}_l^0$ are nonzero for all $l$. Continuity of these coefficients ensures that they remain positive in some neighbourhood of this plane. Thus the determinant (16) is nonzero (negative) and the solution (15) is unique in some domain. But the initial exponents have to be independent of $(\xi, \bar{\xi})$, hence the solution (15) is unique in the neighbourhood of the whole singularity manifold.

□
Putting all $\alpha_i$ and $\beta_i$ equal to 1, as in (15), we obtain the recurrence relations at $k > 0$. For each $k = 1, 2, \ldots$ they have the form of a system of $2(N-1)$ linear algebraic equations with the unknowns $w_1^k, \ldots, w_{N-1}^k$ and $\bar{w}_1^k, \ldots, \bar{w}_{N-1}^k$

$$(k-1)k\varphi'\left(\sum_{l=1}^{N-1}w_i^0\bar{w}_i^0w_i^k + 2k\sum_{l=1}^{N-1}\bar{w}_i^0w_i^0\bar{w}_i^k\right)$$

$$= -\varphi'\sum_{l=1}^{N-1}\sum_{n=1}^{k-n}\left[\sum_{p=0}^{n-1}(n-1)(n-2p)w_i^0\bar{w}_i^{k-n-p}w_i^p + 2nw_i^0\bar{w}_i^{k-n}w_i^n\right]$$

$$+ \sum_{l=1}^{N-1}\sum_{n=1}^{k-n}\left[\sum_{p=0}^{n-1}\bar{w}_i^{k-n-p-1}((n-p)w_i^p(w_i^n)' + (n-1)(w_i^p)'w_i^n)\right]$$

$$+ (k-3)(k-4)\varphi'w_i^{k-2} - (k-4)(w_i^{k-3})'$$

and a similar set of $N-1$ equations for the ‘complex conjugates’ $\bar{w}_1^k, \ldots, \bar{w}_{N-1}^k$. Note that the unknowns $\bar{w}_1^k, \ldots, \bar{w}_{N-1}^k$ are absent from the left-hand sides (lhs) of (17) and similarly, the unknowns $w_1^k, \ldots, w_{N-1}^k$ are absent from the lhs of the conjugate system (although the systems remain coupled with each other through the right-hand sides (rhs)). This absence means that the matrix of coefficients of the complete linear system is a direct sum of two square matrices and its determinant is a product of their determinants. The Fuchs indices or resonances (we use the second name to avoid misunderstandings in our multi-index notation) are calculated from the requirement that the determinant vanish.

The first matrix has the elements

$$A_{ij}^k = \varphi'k\left[(k-1)\delta_{ij}\left(\sum_{l=1}^{N-1}w_i^0\bar{w}_i^0\right) - 2w_i^0\bar{w}_j^0\right].$$

The second component of the direct sum is its ‘complex conjugate’. The instances in which the determinant of their direct sum vanishes are listed in Table 1.

| Resonance | Multiplicity | Linear dependence of the rows |
|-----------|--------------|------------------------------|
| $k = 0$   | $2(N-1)$     | Each row identically vanishes. |
| $k = 1$   | $2(N-2)$     | Row no. $i$, where $i = 2, \ldots, N-1$, is equal to the 1st row multiplied by $w_i^0/w_i^0$; the second component of the direct sum is the ‘c.c.’ of the first one. |
| $k = -1$  | 2            | If each row $i$, where $i, i = 1, \ldots, N-1$ is multiplied by $w_i^0/w_i^0$ and the products are added together, the result is zero. The second component of the direct sum is the ‘c.c.’ of the first one. |

Altogether we have $4N - 4$ zeros. This number is equal to the total order of the system of PDEs. Hence there are no more resonances.

We now test the compatibility of the resonances by checking whether the rhs of the equations (17) have the same linear dependence between rows as their lhs.

For $k = 0$, all terms on the rhs contain $w$ with a negative superscript, which according to our convention (12) means that they are equal zero. Hence the whole rhs...
Painlevé property of $\mathbb{C}P^{N-1}$ sigma models

is equal to zero, as it should be. Consequently, it leaves room for $2(N-1)$ arbitrary functions of $\xi$ (first integrals).

For $k = 1$, the rhs of the $i$-th equation, $i = 1, \ldots, N-1$, reduces to $w_i^0 \sum_{l=1}^{N-1} \bar{w}_l^0 (w_l^0)'$, i.e. all rows are proportional. For instance, we may take the first row and write each of the rows no. $i = 2, \ldots, N-1$ as equal to the first row multiplied by $w_i^0/w_1^0$. This satisfies the linear dependence condition of Table 1. This leaves room for another $2(N-2)$ arbitrary functions.

The above verification of compatibility cannot be performed for negative zeros. One of the two zeros $k = -1$ corresponds to the arbitrariness of $\varphi(\xi)$. The compatibility of the other zero at $k = -1$ remains unknown.

The verified zeros allow us to introduce a total of $4N-6$ arbitrary functions of $\xi$. These are $w_i^0$ and $\bar{w}_i^0$ for $i = 1, \ldots, N-1$ and $w_i^1$ and $\bar{w}_i^1$ for $i = 2, \ldots, N-1$. Together with the arbitrary singularity manifold $\varphi$ (corresponding to one of the two zeros $k = -1$), they constitute a set of $2N-5$ first integrals. There remains the second zero $k = -1$, which is the cause of the missing $(4N-4)$-th first integral. This problem will be addressed in the next section.

4. The question of the double resonance at $k = -1$

The negative resonances, except for a single $k = -1$ resonance, correspond to essential singular points. In the $\mathbb{C}P^{N-1}$ model, they are connected with the coupling between $w$ and $\bar{w}$ (if not for the coupling we would have two separate systems, each possessing a single resonance $k = -1$). A singularity connected with the phase may indeed be essential. A question arises: does the essential singularity introduce multivaluedness in the solution or not.

The authors tried the perturbative Painlevé analysis of [7] for the $\mathbb{C}P^1$ model. Up to the third order in the perturbation of the Laurent series (13) all the resonances are compatible. However the order at which an incompatibility may occur is difficult to predict. Being unable to prove the Painlevé or non-Painlevé property by any systematic method, we limit ourselves to a counterexample.

An example of a solution (an envelope solitary wave) which has branching at a point dependent on the initial conditions has been derived by Lie group analysis and the corresponding symmetry reduction of the $\mathbb{C}P^1$ model in [13, 14]. A typical solution of the kind reads

$$w(\xi, \bar{\xi}) = R \exp[i(\xi/a - f)], \quad \bar{w}(\xi, \bar{\xi}) = R \exp[-i(\xi/a - f)],$$

where

$$R = \pm \sqrt{(p-1) \cosh g + p + 1 \over (p-1) \cosh g - p - 1},$$

$$f = \arctan \left( p + 1 \over 2 \sqrt{-p} \tanh g \right) + \left( p + 2 \sqrt{-p} - 1 \right) \chi - 2 \sqrt{-p} \chi_0 \over 2(p-1) + d,$$

and

$$g = (p+1)(\chi - \chi_0) \over 2(p-1),$$

where

$$\chi = {\xi \over a} - {\bar{\xi} \over b}. \quad (19)$$
To ensure that \( w \) and \( \bar{w} \) are complex conjugates of each other when the remaining quantities are real, it is usually assumed that \( p < -1 \), however the solution is valid for any \( p \).

This solution (as well as several other solutions in the form of elliptic functions) is associated with multileaf surfaces [13, 14]. It is singular for \( \chi - \chi_0 = (k+1/2)i\pi, \ k \in \mathbb{Z} \). For these values of \( \chi \), the argument of arctan in (19) becomes infinite, which results in branching (i.e. multivaluedness of the arctan function). These singularities do not lie on characteristics (\( \xi = \text{const} \) and \( \bar{\xi} = \text{const} \)), which makes them proper for the analysis. The position of the singularities depends on four parameters: \( p, \ a, \ b, \ \chi_0 \), and thus also on the initial conditions, which contradicts the usual understanding of the Painlevé property. However the authors are aware that a more constructive answer to the question of compatibility at the negative resonance would be provided by a non-Painlevé solution with its position dependent on an arbitrary function rather than a few parameters. We do not have such a solution.

The action integral for the example (19) is not finite, hence it is compatible with the theorem of Din and Zakrzewski [11, 9, 22]. Neither does it contradict the classical result of [1], because it cannot be obtained as a solution of a Gelfand-Levitan-Marchenko equation [18] with a finite integral kernel.

The \( \mathbb{C}P^1 \) model is a limit case of \( \mathbb{C}P^{N-1} \) models, where all but one affine coordinates (and all but one ‘complex conjugates’) tend to zero. Thus the absence of the Painlevé property in the \( \mathbb{C}P^1 \) infers its absence for all \( \mathbb{C}P^{N-1} \) models.

Conclusion

We have shown that the equations governing the behaviour of \( \mathbb{C}P^{N-1} \) models, without the constraint of finite action, may have solutions with movable singularities in the form of pole manifolds. The order of the poles is one for all dependent variables (the calculation based on the usual assumption of the Painlevé test, i.e. negative initial exponents, eliminates poles of other orders). For the \( \mathbb{C}P^{N-1} \) model equations, the Laurent series about a pole manifold is consistent at all \( 4N-5 \) nonnegative resonances. This way, it provides a family of solutions with \( 4N-5 \) parameter functions (first integrals) within the domain of convergence of the series. However, branching may still occur at essential singular points. We have provided an example of a solution which is multivalued in the neighbourhood of a sequence of non-characteristic movable singular manifolds. Their position depends on the initial conditions through four parameters. It would be desirable to find a deformation of such solutions turning them into solutions depending on an arbitrary function.

The Painlevé analysis is nontrivial for these models due to the indefinite number of equations and dependent variables.
Painlevé property of $\mathbb{C}P^{N-1}$ sigma models

Acknowledgments

AMG’s work was supported by a research grant from NSERC of Canada. P.P.G. wishes to thank the Centre de Recherches Mathématiques (Université de Montréal) for the NSERC financial support provided for his visit to the CRM.

References

[1] Ablowitz M J, Ramani A and Segur H (1980) A connection between nonlinear evolution equations and ordinary differential equations of P-type J. Math. Phys. 21 715–721 and 1006–1015.
[2] Babelon O (2007) A short introduction to Classical and Quantum Integrable Systems, Univ. Paris 6.
[3] Babelon O, Bernard D and Talon M (2003) Introduction to Classical Integrable Systems, Cambridge University Press.
[4] Carrol R and Konopelchenko B (1996) Generalized Weierstrass–Enneper inducing conformal immersions and gravity Int. J. Mod. Phys. A 11 1183–1216.
[5] Conte R (1999) The Painlevé approach to nonlinear ordinary differential equations, chapter 3, 77–180 in The Painlevé Property One Century Later, ed. R. Conte, New York, Springer Verlag.
[6] Conte R and Musette M (2008) The Painlevé Handbook, Dordrecht, Springer.
[7] Conte R, Fordy A. P and Pickering A (1993), A perturbative Painlevé approach to nonlinear differential equations, Physica D 69 33–58.
[8] Davydov A (1999) Solitons in Molecular Systems, New York Kluver.
[9] Din A M (1984) The Riemann-Hilbert problem and finite-action CPN?1 model solutions, Nucl Phys B 233,269–288.
[10] Din A.M., Horvath Z. and Zakrzewski W.J. (1984) The Riemann–Hilbert problem and finite action $\mathbb{C}P^{N-1}$ solutions, Nucl. Phys. B 233 269.
[11] Din A M, Zakrzewski W J, (1980) General classical solutions in the $\mathbb{C}P^{N-1}$ model Nucl Phys B 174 397–406.
[12] Gross D G, Piran T and Weinberg S (1992) Two-dimensional Quantum Gravity and Random Surfaces, Singapore: World Scientific.
[13] Grundland A M and Šnobl (2006) Description of surfaces associated with $\mathbb{C}P^{N-1}$ sigma models on Minkowski space J Geom Phys 56 512–531.
[14] Grundland A M and Šnobl (2006) Surfaces Associated with Sigma Models Stud Appl Math 117 335-351.
[15] Jimbo M, Kruskal M D and Miwa T (1982) Painlevé test for the self-dual Yang-Mills equation, Phys. Lett. A 92 59–60.
[16] Landolfi G (2003) On the Canham-Helfrich membrane model J Phys A: Math Theor 36 4699 (16pp).
[17] Manton N.and Sutcliffe P (2004) Topological Solitons, Cambridge University Press
[18] Marchenko V A (2011) Sturm-Liouville operators and Applications AMS
[19] Musette M (1999) The Painlevé analysis for nonlinear partial differential equations, chapter 8, 517–562 in The Painlevé Property One Century Later, ed. R. Conte, New York, Springer Verlag.
[20] Rajaraman R (2002) $CP_N$ Solitons in quantum Hall systems, Europhys J B 28 157-162.
[21] Weiss J, Tabor M and Carnevale G (1983) The Painlevé property for partial differential equations, J. Math. Phys. 24 522–526.
[22] Zakrzewski W J (1989) Low Dimensional Sigma Models, ch. 4 and 8–11, Bristol, Adam Hilger.