Distributed Velocity-Constrained Consensus of Discrete-Time Multi-Agent Systems With Nonconvex Constraints, Switching Topologies, and Delays

Peng Lin, Wei Ren, and Huijun Gao

Abstract—In this technical note, a distributed velocity-constrained consensus problem is studied for discrete-time multi-agent systems, where each agent's velocity is constrained to lie in a nonconvex set. A distributed constrained control algorithm is proposed to enable all agents to converge to a common point using only local information. The gains of the algorithm for all agents need not to be the same or predesigned and can be adjusted by each agent itself based on its own and neighbors' information. It is shown that the algorithm is robust to arbitrarily bounded communication delays and arbitrarily switching communication graphs provided that the union of the graphs has directed spanning trees among each certain time interval. The analysis approach is based on multiple novel model transformations, proper control parameter selections, boundedness analysis of state-dependent stochastic matrices, exploitation of the convexity of stochastic matrices, and the joint connectivity of the communication graphs. Numerical examples are included to illustrate the theoretical results.

Index Terms—Constrained consensus, delays, multi-agent systems.

I. INTRODUCTION

In recent years, consensus problems in multi-agent systems have received a great deal of attention because of its important applications including formation control, attitude alignment of clusters of satellites, and flocking [1]–[15]. Most of the existing results concentrate on the ideal case where the state or input of each agent has no constraints. In some practical situations, the state or input of each agent is usually constrained to lie in a certain set, e.g., the saturation and dead zone of the velocity of physical vehicles.

Research on consensus problems with state or input constraints can be found in [6]–[13]. For example, article [6] introduced hyperbolic tangent functions to a consensus algorithm for continuous-time double-integrator multi-agent systems with a fixed undirected topology where the maximum amplitude of the control input of each agent is upper bounded. Also, from the view point of saturation control, articles [7]–[10] studied constrained control problems by a Lyapunov approach and showed that consensus can be achieved asymptotically or in finite time. However, in [6]–[10], it is assumed that each agent has continuous-time dynamics, the input constraint set of each agent is a hypercube and the communication graph is undirected. From the view point of projection control, article [11] proposed a projection algorithm for discrete-time multi-agent systems with switching topologies, where each agent is assumed to remain in a convex set. Founded on [11], article [12] took the communication delays into account and showed that the projection algorithm is robust to arbitrarily bounded communication delays, while article [13] studied the projection algorithm in a random environment and introduced a step size sequence for the consensus stability of the systems. However, in [11]–[13], it is assumed that the states of the agents are constrained in certain convex sets. When more general constraint sets are taken into account, the results in [11]–[13] cannot be directly applied due to the loss of the convexity of the constraint sets.

In this technical note, our objective is to solve the velocity-constrained consensus problem for discrete-time multi-agent systems with switching topologies and nonuniform communication delays. In contrast to [6]–[10], where the constraint set of each agent is a hypercube, here each agent’s velocity is constrained to lie in a nonconvex set. The communication graph considered is directed coupled with arbitrarily bounded communication delays and can be arbitrarily switching as long as the union of the graphs has directed spanning trees among each certain time interval. To solve the velocity-constrained consensus problem in this setting, a distributed control algorithm is proposed by applying a constrained control scheme using only local information. The gains of the algorithm for all agents need not to be the same or pre-designed and can be adjusted by each agent itself based on its own and neighbors’ information, which distinguishes it from the existing works on double-integrator consensus [3]–[5], where the feedback gains are uniform for all agents. Owing to the coexistence of the coupling of the position and velocity states and a velocity delay during the updating process of the position states, the nonlinearity caused by the nonconvex constraints would further lead to a stronger nonlinearity on the position states. Both nonlinearities are greatly different from those in [11]–[13] and the approaches there cannot be directly applied. Our analysis approach is to introduce multiple novel model transformations and select proper control parameters to transform the original system into an equivalent system whose system matrix is a state-dependent stochastic matrix. The state-dependent stochastic matrix has two features: one is that the nonzero entries are from an infinite set and the nonzero entries might not be uniformly lower bounded by a positive constant, and the other is that the stochastic matrix has zero diagonal entries. The coexistence of these two factors poses significant challenges. Most of the existing results on delay-related consensus require the number of possible nonzero entries to be finite (e.g., [2], [5], [12]) and existing

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1Throughout this technical note, when referring to a stochastic matrix, it means a row stochastic matrix.
Four examples of the constraint operator.

\[ \rho((V(N \parallel v) \cdot \max_{t \geq 0} v_{i} + v_{i} (kT) \cdot T) \cdot (v_{i}(k + 1)T) = x_{i}(kT) + v_{i}(kT)T \]

where \( x_{i}(kT) \in \mathbb{R}^{r} \) and \( v_{i}(kT) \in \mathbb{R}^{r} \) for some positive integer \( r \) are the position and velocity states of agent \( i \) and \( u_{i}(kT) \in \mathbb{R}^{r} \) is the control input. To simplify the notations, we replace all \( "(kT)" \) by \( "(k)" \). It is assumed that the initial conditions of \( x_{i}(k) \) and \( v_{i}(k) \) for all \( k \leq 0 \) and all \( i \) satisfy the dynamics of (1), and the velocity state of each agent \( v_{i}(k) \) is constrained to lie in a nonempty constraint set \( V_{i} \subseteq \mathbb{R}^{r} \) known only to agent \( i \).

Due to the different constraints of each agent’s driving forces in different directions, the velocities of the agents, e.g., quadrotors, might not lie in convex sets. Hence we make the following assumption for \( V_{i} \):

**Assumption 1:** Let \( V_{i} \subseteq \mathbb{R}^{r} \), \( i = 1, \cdots , n \), be nonempty bounded closed sets such that \( 0 \in V_{i} \), \( \max_{x \in V_{i}} ||S_{V_{i}}(x)|| = \bar{\rho}_{i} > 0 \) and \( \inf_{x \in V_{i}} ||S_{V_{i}}(x)|| = \underline{\rho}_{i} > 0 \) for all \( i \), where \( \bar{\rho}_{i} \) and \( \underline{\rho}_{i} \) are two positive constants, and \( S_{V_{i}}(\cdot) \) is a constraint operator such that \( S_{V_{i}}(0) = 0 \) and \( S_{V_{i}}(x) = \frac{1}{\underline{\rho}_{i}} \max_{0 \leq \alpha \leq 1} \{ [\beta \frac{\alpha}{1+\alpha} S_{V_{i}}(x)] \} \) for all \( x \neq 0 \).

The operator \( S_{V_{i}}(x) \) means to find the vector with the largest magnitude such that \( S_{V_{i}}(x) \) has the same direction as \( x \), \( ||S_{V_{i}}(x)|| \leq ||x|| \) and \( \lambda S_{V_{i}}(x) \in V_{i} \) for all \( 0 \leq \alpha \leq 1 \). (See Fig. 1 for illustrations.) It should be noted here that we do not impose any convexity assumption on each \( V_{i} \). The maximum \( \max_{x \in V_{i}} ||S_{V_{i}}(x)|| = \bar{\rho}_{i} \) means that the distance from any point in \( V_{i} \) to the origin is upper bounded. That is, the velocities of all agents cannot be arbitrarily large. The infimum \( \inf_{x \in V_{i}} ||S_{V_{i}}(x)|| = \underline{\rho}_{i} > 0 \) means that the distance from any point outside \( V_{i} \) to the origin is lower bounded by a positive constant. That is, each agent can move in any direction. Future work could be directed to the more general case where each agent might not be able to move toward certain directions.

Under the constraint that \( v_{i}(k) \in V_{i} \) for all \( i,k \), our objective is to design an algorithm for all agents to cooperatively reach a consensus on their position states at some vector, denoted by \( \bar{x} \in \mathbb{R}^{r} \), as \( k \to +\infty \), i.e., \( \lim_{k \to +\infty} x_{i}(k) = \bar{x} \) for all \( i \). From the dynamics of (1), the limit \( \lim_{k \to +\infty} x_{i}(k) = \bar{x} \) for all \( i \) means that \( \lim_{k \to +\infty} v_{i}(k) = 0 \) for all \( i \). That is, the velocity of each agent would actually converge to zero as \( k \to +\infty \).
IV. MAIN RESULTS

In this section, we study the velocity-constrained consensus problem for discrete-time multi-agent systems with switching topologies and communication delays. Motivated by the algorithms in [3]–[5] dealing with the case without constraints, we propose the control algorithm:

\[ u_i(k) = S_{\{v_i(k) - p_i(k)\}}(v_i(k))T + \pi_i(k) \]  

(2)

for all \( k \geq 0 \), where \( v_i(0) = S_{\{v_i(0)\}}(v_i(0)) \) is the feedback gain of agent \( i \), \( \pi_i(\cdot) = \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) (\pi_j - \pi_i(k)) \), \( \pi_i(k) \), and \( a_{ij}(k) \) denotes the edge weight of the edge \((j, i)\) (\( a_{ii}(k) > 0 \) for all \( j \in \mathcal{N}_i(k) \)). It is assumed that all \( \tau_j(\cdot) \) are upper bounded, i.e., \( \tau_j(\cdot) \leq M \) for some constant \( M > 0 \). When there are no constraints, the algorithm (2) would have the form of the algorithms introduced in [3]–[5]. If one agent receives multiple pieces of the state information from another agent at time \( k \), the latest piece would be used and all others dropped. Here, it is assumed that \( a_{ij}(k) \geq \mu_i \) for some positive number \( \mu_i \), when \( a_{ij}(k) > 0 \). The constraint operator is used to ensure the velocity of each agent to be lying in its corresponding constraint set, and the algorithm parameters of all agents, \( p_i(\cdot) \), need not remain the same and will be shown to be able to be adjusted by each agent itself based on the scaling factors of the constraint operator and the parameters of the previous time instant.

Remark 1: In [11]–[13], the projection operator is used to guarantee all agents with single-integrator discrete-time dynamics remain in their constraint sets. Different from [11]–[13], the system (1) takes the double-integrator form and there is a velocity delay inherent in the dynamics during the updating process of the position states. If the projection operator were used in (2), due to the coexistence of the nonconvexity of the constraint sets, the coupling of the position and velocity states, and the velocity delay, the nonlinearity caused by the projection operator would be hard to be measured or estimated and thus the system might become too complicated to analyze. Hence we do not adopt the projection operator in (2).

Define

\[ e_i(k) = \frac{||S_{\{v_i(k) - p_i(k)\}}(v_i(k))T + \pi_i(k)||}{||v_i(k) - p_i(k)\|} \]

(3)

for all \( k \geq 0 \). In particular, when \( v_i(k) = v_i(k)T + \pi_i(k) = 0 \), we define \( e_i(k) = 1 \). Clearly, \( 0 < e_i(k) \leq 1 \). Let \( b_i(k) = \frac{1 - e_i(k)}{\pi_i(k)} \). We make the following assumption.

Assumption 2: Suppose that \( \tau_i(\cdot) \geq 0 \), \( b_i(0) \geq 0 \) for all \( k \geq 0 \) and all \( i \), and there exist a constant \( d_i > 0 \) such that \( \tau_i(k) > 4d_i \geq \frac{\pi_i(k)}{b_i(k)} \) for all \( i \) and all \( k \geq 0 \).

To illustrate, we show how to select \( p_i(\cdot) \) in a distributed manner to guarantee Assumption 2 in three steps:

1) Select \( p_i(0) \) such that \( 0 < p_i(0)T < 1 \). Then calculate \( b_i(0) \) according to the definition of \( b_i(k) \);

2) At each time \( k \), each agent assigns a proper weight to each nonzero \( a_{ij}(k) \) such that \( L(k)_{ij} \) is no larger than \( d_i \) for some constant \( d_i > \frac{\pi_i(0)}{b_i(k)} \),

3) At each time \( k \), based on \( p_i(k) \), select \( p_i(k+1) \) such that \( b_i(k+1) \geq b_i(k) \) and \( 0 < p_i(k+1)T < 1 \). Then calculate \( b_i(k+1) \) according to its definition.

Clearly, by selecting proper \( p_i(0) \) and nonzero \( a_{ij}(k) \), the first two steps can be easily realized. From the first step, we have that \( 0 < p_i(0)T < 1 \). Hence from the definition of \( e_i(k) \), we have that \( 0 < b_i(0)T < 1 \) and \( b_i(0) \geq p_i(0) \). Then there exists \( p_i(1) \) such that \( b_i(1)T = p_i(1)T < 1 \). That is, the third step is realized and we have that \( 0 < p_i(0)T \leq b_i(1)T \). By analogy, for all \( i \) and all \( k \), the third step can be realized and there exist \( p_i(k) \) such that \( 0 < p_i(k)T \leq b_i(k)T \leq p_i(k+1)T < 1 \). Moreover, from the second step, we have \( d_i < \frac{\pi_i(k)}{b_i(k)} \) for all \( i \) and all \( k \). That is, Assumption 2 is satisfied.

From the design rules above, it can be seen that the gains of the algorithm for all agents need not to be the same or predesigned, and they can be adjusted by each agent itself based on its own and neighbors' information.

Assumption 3: Suppose that there exist an infinite time sequence of \( k_1, k_2, \ldots \) and a positive integer \( \eta \) such that \( k_1 = 0, 0 < k_{m+1} - k_m \leq \eta \) for all \( n \) and the union of the graphs \( G(k_m), G(k_m + 1), \ldots, G(k_{m+1} - 1) \) has directed spanning trees.

Theorem 1: Under Assumptions 1–3, for the multi-agent system (1) with (2), all agents reach a consensus on their position states exponentially fast while their velocities remain in their corresponding constraint sets. Specifically,

1) there exist a vector \( \bar{x} \in R^r \) and two constants \( C > 0 \) and \( 0 < \mu \leq 1 \) such that \( ||x_i(k) - \bar{x}|| \leq C(1 - \mu^k) \) for all \( i \) and any \( k \geq 0 \);
2) \( \lim_{k \to \infty} v_i(k) = 0 \) and \( v_i(k) \in V_i \) for all \( i \).

Remark 2: In [6]–[10], it is assumed that the constraint set of each agent is a hypercube and the communication graph is undirected. It is hard to extend to consider more general nonconvex constraint sets and directed communication graphs, especially when communication delays are involved. In addition, in [11]–[13], it is assumed that the states of the agents are constrained in certain convex sets and the dynamics of the agents is in the form of single integrators. Their results cannot be directly applied here as well. The reasons mainly lie in three aspects. First, the agents in this technical note have two different states, position and velocity states, which are not independent but instead coupled in the form of integral. Unlike [11]–[13], the position states cannot be directly controlled. Second, the constraint sets are generally nonconvex and are on the velocities. The nonlinearity caused by the nonconvex constraints on velocities would further lead to a stronger nonlinearity on the position states. Both nonlinearities are different from and more complicated than that caused by convex constraint sets in [11]–[13]. Third, the constraint operator adopted in this technical note is different from the projection operator in [11]–[12]–[13]. The nonlinear dynamics of the two operators are different in nature.

For simplicity of expression, we only discuss the case of \( r = 1 \) in the proof of Theorem 1 and the case of \( r > 1 \) can be analyzed in the same way.

A. Multiple Model Transformations in the Proof of Theorem 1

To perform analysis on the closed-loop system (1) with (2), we first make multiple model transformations in three steps so as to use the property of nonnegative matrices to analyze the system stability for all \( k \geq 0 \).

Step 1: From the definition of the constraint operator \( S_{\{v_i(\cdot)\}}(\cdot) \) and \( S_{\{v_i(\cdot)\}}(\cdot) \) have the same direction for any nonzero \( x \). From (3), we have

\[ S_{\{v_i(k) - p_i(k)\}}(v_i(k)T + \pi_i(k)) = e_i(k)[v_i(k) - p_i(k)\] \[ v_i(k)T + \pi_i(k)] + e_i(k)\pi_i(k). \]

Recall that \( 0 < e_i(k) \leq 1 \). It follows from the definition of \( b_i(k) \) that \( 0 < p_i(k)T < 1 \). \( b_i(k)T \leq 1 \), \( b_i(k)T < 1 \) and \( b_i(k) \geq p_i(k) \). Because \( 1 - b_i(k)T = e_i(k)(1 - p_i(k)T) \), it follows that

\[ S_{\{v_i(k) - p_i(k)\}}(v_i(k)T + \pi_i(k)) = e_i(k)[v_i(k) - b_i(k)v_i(k)T + e_i(k)\pi_i(k). \]

Note that here \( \bar{x} \) is the consensus vector for all agents’ position states.
Let $\phi(k) = [x'_1(k), v'_1(k), \cdots, x'_n(k), v'_n(k)]^T$ and $E(k) = \text{diag}\{e_1(k), \cdots, e_n(k)\}$. Then the system (1) with (2) can be written as

$$
\phi(k + 1) = \{\tilde{A}(k) - [E(k)L_0(k) \otimes I_2] \tilde{B}\} \phi(k) + \sum_{m=0}^{M} [E(k)\Phi_m(k) \otimes I_2] \tilde{B} \phi(k - m),
$$

(4)

where $\tilde{A}(k) = \text{diag}\{\tilde{A}_1(k), \cdots, \tilde{A}_n(k)\}$ with $\tilde{A}_i(k) = \begin{bmatrix} 1 \quad -b_i(k) \end{bmatrix}^T$, $\tilde{B} = I_2 \otimes \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\tilde{L}(k) = \text{diag}(L(k))$, and the $i$th entry of $\Phi_m(k)$ ($m = 0, 1, \cdots, M$) is either zero or equal to the weight of the edge $(j, i)$ if $\tau_{ij} = m$.

Step 2): To partly decouple the integral relationship of the position and velocity of each agent, we introduce another variable substitution. Let $Q(k) = \text{diag}\{Q_1(k), \cdots, Q_n(k)\}$ with $Q_i(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}_i \xi_i(k) = Q(k)\phi(k), A(k) = \text{diag}\{A_1(k), \cdots, A_n(k)\}$ with $A_i(k) = \begin{bmatrix} 1 & -b_i(k) \end{bmatrix}^T$, $B(k) = \text{diag}\{B_1(k), \cdots, B_n(k)\}$ with $B_i(k) = \begin{bmatrix} 0 & 0 \\ v_i(k) \end{bmatrix}$ and $F = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}$. Clearly, $A(k) = Q(k)\tilde{A}(k)Q^{-1}(k)$.

It follows that

$$
\xi(k + 1) = Q(k + 1)\phi(k + 1) = Q(k + 1)Q(k)^{-1}Q(k)\tilde{A}(k) - [E(k)L_0(k) \otimes I_2] \tilde{B}Q(k)^{-1}Q(k)\phi(k) + Q(k + 1)Q(k)^{-1}Q(k) \times \sum_{m=0}^{M} [E(k)\Phi_m(k) \otimes I_2] \tilde{B}Q(k)^{-1}(k)Q(k)\phi(k - m).
$$

(5)

Here, it should be emphasized that

$$
Q^{-1}(k)B(k)[E(k)\Phi_m(k) \otimes F]\xi(k - m) = Q^{-1}(k)B(k)[E(k)\Phi_m(k) \otimes F]Q(k - m)\phi(k - m) = Q^{-1}(k)B(k)[E(k)\Phi_m(k) \otimes F]Q(k)\phi(k - m)
$$

for all $k > 0$ and all $0 \leq m \leq M$, because the entries of the even columns of $B(k)[E(k)\Phi_m(k) \otimes F]$ are all zero and the odd entries of $\xi(k - m), \phi(k - m)$ and $Q(k)\phi(k - m)$ are all equal correspondingly.

Step 3): To use the property of nonnegative matrices for the analysis of the system, we introduce an augmented system. Similar to [5], let $Z(k) = [\xi^T(k), \xi^T(k - 1), \cdots, \xi^T(k - M)]^T$ and $\Psi(k)$ be a matrix composed of $(M + 1)^2 2n \times 2n$ square blocks such that $\Psi(k)_{i,j} = A(k) - [E(k)L_0(k) - E(k)\Phi_0(k) \otimes I_2] \tilde{B} [B(k), \Psi(k)]_{i,j} = E(k)\Phi_0(k) \otimes I_2 \tilde{B} [B(k), \Psi(k)]_{i,j} = I_{2n}$ for all $2 \leq i \leq M + 1$ and all other blocks are zero matrices, where $[\Psi(k)]_{i,j}$ denotes the $(i, j)$th block. It follows that

$$
Z(k + 1) = \text{diag}\{Q(k + 1)Q^{-1}(k), I_{2nM}\} \Psi(k)Z(k).
$$

(6)

Remark 3: Since all introduced transformation matrices are non-singular, the system (1) with (2) is equivalent to the system (6) without information loss.

B. Consensus Stability Analysis in the Proof of Theorem 1

Let $\Gamma(k, s) = \prod_{i=1}^{k} \text{diag}\{Q(i + 1)\Psi(i), I_{2nM}\} \Psi(i)$ be the transition matrix of the system (6). The relation between $Z(k + 1)$ and $Z(s)$ for all $k \geq s \geq 0$ can be described by

$$
Z(k + 1) = \Gamma(k, s)Z(s).
$$

(7)

In the following, we will perform consensus analysis on the system (7). Below are some lemmas prepared for the main theorem. Specifically, Lemma 1 studies the stochasticity of the matrices $\text{diag}\{Q(k + 1)Q^{-1}(k), I_{2nM}\} \Psi(k)$ and $\Gamma(k, s)$, and the lower boundedness of the scaling factor $e_i(k)$ and an auxiliary matrix that will be used for the proof of Lemma 2. Lemma 2 proves that there exists at least one column among the first 2n columns of $\Gamma(k, s)$ such that each entry is larger than some positive constant when $k - s$ is sufficiently large. Lemma 3 proves that all columns of the transition matrix $\Gamma(k, s)$ exponentially tend to the same as $k \to +\infty$.

Lemma 1: Under Assumptions 1 and 2,

1) $\text{diag}\{Q(k + 1)Q^{-1}(k), I_{2nM}\} \Psi(k)$ and $\Gamma(k, s)$ are stochastic matrices for any $k \geq s \geq 0$;
2) For all $i, e_i(k) \geq \frac{1}{2nT\tau_{ij}(k)}$;
3) Let $\Theta(k) = \text{diag}\{Q(k + 1)Q^{-1}(k), I_{2nM}\} \Psi(k)$. Each nonzero entry of $\Theta(k)$ is uniformly lower bounded by some positive constant.

Proof:

1) Let $k \geq s \geq 0$. Note that

$$
Q_i(k + 1)Q_i^{-1}(k) = \begin{bmatrix} 1 - \frac{b_i(k)}{v_i(k + 1)} & 0 \\ \frac{b_i(k)}{v_i(k + 1)} & \frac{b_i(k)}{v_i(k + 1)} \end{bmatrix}.
$$

(8)

Clearly, each row sum of the matrices $Q(k + 1)Q^{-1}(k)$ is 1. Calculating $\sum_{i=1}^{M+1} [\Psi(k)]_{i,1} \Phi_i(k)$, we have $\sum_{i=1}^{M+1} [\Psi(k)]_{i,1} = \Phi_1(k)1 = 1$. That is, each row sum of the first 2n row of $\Psi(k)$ is 1. Observing the form of $\Psi(k)$, each of its row sums is 1.

Recall that when $0 < p_i(k)T < 1, b_i(k) \geq p_i(k)$ and $0 < b_i(k)T < 1$. Under Assumption 2, we have for all $i$ and all $k$

$$
\frac{1}{T} > b_i(k + 1) \geq p_i(k + 1) \geq b_i(k) \geq p_i(k)
$$

(9)

and

$$
p_i^2(k) > 4d_i.
$$

(10)

Thus, $\frac{b_i(k)}{v_i(k + 1)} \leq 1, 1 - \frac{b_i(k)}{v_i(k + 1)} > \frac{1}{T}$ and $\frac{b_i^2(k)}{v_i(k + 1)} > 4d_i$ for all $i$ and all $k$. It follows that all entries of $Q(k + 1)Q^{-1}(k)$ and $\Psi(k)$ are nonnegative. Thus, the matrices $Q(k + 1)Q^{-1}(k)$, $\Psi(k)$, $Q(k + 1)Q^{-1}(k)\Psi(k)$ and $\Gamma(k, s)$ for any $k \geq s \geq 0$ are stochastic matrices.

2) Since $Z(k) = \Gamma(k - 1, 0)Z(0)$, each $Z_i(k)$ is a convex combination of $Z_i(0)$ for all $i$ and thus $\|Z_i(k)\| \leq \max_j\{\|Z_j(0)\|\}$ for all $i$. It follows from the definitions of $x_i(k)$, $v_i(k)$, $Z_i(k)$ and $\xi_i(k)$ that

$$
\|\pi_i(k)\| \leq 2nT[L(k)]_{i,i} \max_{j}\{\|Z_j(0)\|\}
$$

and

$$
\|v_i(k) - p_i(k)v_i(k)T\| \leq \|\pi_i(k)\| = \frac{b_i(k)}{v_i(k)}(Z_1(k) - Z_2(k))\|.
$$

Note that $b_i(k) < \frac{1}{T}$ and $\|L(k)]_{i,i}\| \leq d_i$ under Assumption 2. It follows that

$$
\|v_i(k) - p_i(k)v_i(k)T + \pi_i(k)\| \leq \left(\frac{1}{T} + 2nTd_i\right)\max_j\{\|Z_j(0)\|\}
$$
When $v_i(k) - p_i(k)v_i(k)T + \pi_i(k) = S v_i(k) - p_i(k)v_i(k)T + \pi_i(k), e_i(k) = 1$. When $v_i(k) - p_i(k)v_i(k)T + \pi_i(k) \neq S v_i(k) - p_i(k)v_i(k)T + \pi_i(k)$, under Assumption 1, 
\[ \|S v_i(k) - p_i(k)v_i(k)T + \pi_i(k)\| \geq \rho. \]
Thus,
\[ e_i(k) = \frac{\|S v_i(k) - p_i(k)v_i(k)T + \pi_i(k)\|}{\|v_i(k) - p_i(k)v_i(k)T + \pi_i(k)\|} \geq \left(\frac{\rho}{\|\cdot\|}\right)^{\frac{1}{\mu+1}}. \]

3) From (9) and (10), we have $b_i(k) \geq \rho(0) - \frac{\rho(0)T}{\mu(0)} > T$ and $b_i^2(k) \geq \frac{\rho^2(0)}{\mu(0)} > 4d_i$. Hence, $p_i(0)T \leq \frac{\rho(0)T}{\mu(0)}$ and $\frac{\rho(0)T}{\mu(0)} \geq \frac{\rho(0)T}{\mu(0)} - \frac{\rho(0)T}{\mu(0)} > 0$. Note that $a_i(k) \geq \mu_i$. It follows that each nonzero entry of diag$(Q(k+1)Q^{-1})$ is stochastic. Thus, (k+1)Q^{-1})$. Statistical approach with its zero entries in the matrix $\theta(s, k)$ might not be uniformly lower bounded by a positive constant. The role of such nonzero entries might have no difference with the zero entries in the matrix $\theta(s, k)$, which might make all columns of $\theta(s, k)$ not converge to a common vector. For the convergence analysis of the stochastic matrices, most of the existing results on delay-related consensus require the number of possible nonzero entries to be finite (e.g., [2], [5], [12]) and existing approaches based on the results in [16] require the stochastic matrices to have positive diagonal entries and their nonzero entries to be uniformly lower bounded by a positive constant. Though the results of [14] allow for an infinite number of edge weights and zero diagonal entries, the union of the communication graphs among each certain time interval is assumed to be strongly connected and each nonzero entry of the stochastic matrices is assumed to be uniformly lower bounded by a positive constant. As a result, existing results cannot be directly applied to deal with the problem studied in this technical note. In addition, in our algorithm, it is not required that the feedback gains for all agents be the same, which distinguishes our results from the works on double-integrator consensus [3]–[5], where the feedback gains must be uniform among all agents.

Lemma 2: Under Assumptions 1–3, there exists a positive integer $h \in \{1, \ldots, 2n\}$ and a number $0 < \mu \leq 1$ such that $\Gamma(k) - \mu \Gamma(k)$ is a nonnegative matrix. So, to prove this lemma, we only need to prove that there exist two positive numbers $h \in \{1, \ldots, 2n\}$ and $0 < \mu \leq 1$ such that $\Gamma(k) - \mu \Gamma(k)$ is a nonnegative matrix. Therefore, it follows that $\max_k \{\Gamma(k, s) \}_{h i} - \mu \Gamma(k, s)_{h i} \leq C_0(1 - \tilde{\mu})^{-2}$ for all $k \geq s$, where $C_0 = (1 - \tilde{\mu})^{-2}$ and $\tilde{\mu} \geq 4n(M + 1)$.

Proof: Define $\Gamma(k, s) = \prod_{i=s}^{k} \Theta(i)$, where $\Theta(i)$ has been defined in Lemma 1. Obviously, $\Gamma(k, s) - \Gamma(k, s)$ is a nonnegative matrix. Let $\tilde{\Gamma}(k, s)$ be the directed graph whose edge weight matrix is $\Gamma(k, s)$. By calculations, for all $k \in \{1, \ldots, n\}$, all $i \in \{1, \ldots, M\}$, $i = \{1, \ldots, 2n\}$, $\tilde{\Gamma}(k, k)_{j2j-1} = \Theta(k)_{j2j-1} > 0$, $\tilde{\Gamma}(k, k)_{j2j-1} = \Theta(k)_{j2j-1} > 0$, and $\tilde{\Gamma}(k, k)_{j2j} = \Theta(k)_{j2j} > 0$, and hence nodes $j$ and $2j - 1$ are strongly connected and there is an edge from node $j$ to node $2j - 1$ in graph $\tilde{\Gamma}(k, k)$. Also, note that $\tilde{\Gamma}(k, k)_{j2j} = \Theta(k)_{j2j} > 0$ for all $j \in \{1, \ldots, 2n\}$ and all $k$ and hence $\tilde{\Gamma}(k, s)_{j2j} > 0$, i.e., each node has an edge to itself in the graph $\tilde{\Gamma}(k, s)$, for all $j \in \{1, \ldots, 2n\}$ and all $k \geq s \geq 0$. Since the union of $\tilde{\Gamma}(k, 0), \ldots, \tilde{\Gamma}(k, k+1)$ has at least a directed spanning tree except Assumption 3, the graph $\tilde{\Gamma}(k, 0)$ has a directed spanning tree with a root node $i \in \{1, \ldots, 2n\}$, such that the graph $\tilde{\Gamma}(k, 0)$ has a directed spanning tree with a root node $i \in \{1, \ldots, 2n\}$ for any positive integer $c$.

Let $\emptyset \neq \tilde{\Gamma}(k, 0) \subseteq \{1, \ldots, 2n\}$ be a set such that each of its element is a root node of $\tilde{\Gamma}(k, 0)$, and let $(\tilde{\Gamma}(k, 0), 1)$ be the maximum value of the number of directed paths from any node of $\tilde{\Gamma}(k, 0)$ to any other node without going through the same node $\emptyset \neq \tilde{\Gamma}(k, 0)$ for any positive integer $c$. Note that $\tilde{\Gamma}(k, 0)_{j2} \geq 0$ for all $j \in \{1, \ldots, 2n\}$. When $\tilde{\Gamma}(k, 0)_{j2} \neq 0$ or $\emptyset \neq \tilde{\Gamma}(k, 0)$, then from graph theory, $\emptyset \neq \tilde{\Gamma}(k, 0)_{j2} \geq 0$. When $\emptyset \neq \tilde{\Gamma}(k, 0)_{j2} \neq 0$ or $\emptyset \neq \tilde{\Gamma}(k, 0)$, then from graph theory, $\emptyset \neq \tilde{\Gamma}(k, 0)_{j2} \geq 0$. When $\emptyset \neq \tilde{\Gamma}(k, 0)_{j2} \neq 0$ or $\emptyset \neq \tilde{\Gamma}(k, 0)$, then from graph theory, $\emptyset \neq \tilde{\Gamma}(k, 0)_{j2} \geq 0$. When $\emptyset \neq \tilde{\Gamma}(k, 0)_{j2} \neq 0$ or $\emptyset \neq \tilde{\Gamma}(k, 0)$, then from graph theory, $\emptyset \neq \tilde{\Gamma}(k, 0)_{j2} \geq 0$.
Take $\epsilon < \frac{\rho(\theta_j - \sigma_j)}{2(2 \rho)}$ and $k_m - 1 > T_\theta$. If $[\Gamma(k_m - 1, s)]_{ij} \leq \frac{\rho_j + \sigma_j}{2}$, we have

$$
[\Gamma(k_m + 1 - 1, s)]_{ij}
$$

$$
= \sum_{l=1}^{2n(M+1)} [\Gamma(k_m + 1 - 1, k_m)]_{ij} [\Gamma(k_m - 1, s)]_{ij} + (\hat{\mu})
$$

$$
+ [\Gamma(k_m + 1 - 1, k_m)]_{ij} (\theta_j + \epsilon) + \frac{\hat{\mu}(\theta_j + \sigma_j)}{2}
$$

$$
\leq (1 - \hat{\mu})(\theta_j + \epsilon) + \frac{\hat{\mu}(\theta_j + \sigma_j)}{2}
$$

$$
< \theta_j - \epsilon
$$

for all $j$, where the first inequality has used (11) and the second inequality has used the fact that $\sum_{j=1}^{2n(M+1)} [\Gamma(k_m + 1 - 1, k_m)]_{ij} = 1$. This contradicts with (11). Similarly, if $[\Gamma(k_m - 1, s)]_{ij} > \frac{\rho_j + \sigma_j}{2}$, we have

$$
[\Gamma(k_m + 1 - 1, s)]_{ij}
$$

$$
= \sum_{l=1}^{2n(M+1)} [\Gamma(k_m + 1 - 1, k_m)]_{ij} [\Gamma(k_m - 1, s)]_{ij} + (\hat{\mu})
$$

$$
+ [\Gamma(k_m + 1 - 1, k_m)]_{ij} (\theta_j + \epsilon) + \frac{\hat{\mu}(\theta_j + \sigma_j)}{2}
$$

$$
\geq (1 - \hat{\mu})(\sigma_j - \epsilon) + \frac{\hat{\mu}(\theta_j + \sigma_j)}{2}
$$

$$
> \sigma_j + \epsilon,
$$

for all $j$, where the first inequality has used (12). This also yields a contradiction. Therefore, there exists a constant $0 \leq \rho_i(s) \leq 1$ for each $i$ such that $\lim_{h \to \infty} [\Gamma(k, s)]_{hi} = \rho_i(s)$ for all $h$. Moreover, from the stochasticity of $\Gamma(k, s)$, we have $\sum_{i=1}^{2n(M+1)} \rho_i(s) = 1$.

2) From Lemma 2 again, there exists a positive integer $q \in \{1, \cdots, 2n\}$ and a positive number $0 < \mu \leq 1$ such that $[\Gamma(k_h - 1, k_{(h-1)\bar{a}})]_{ij} \geq \hat{\mu}$ for all $i$ and all positive integers $h > 0$. Note that

$$
[\Gamma(k_h - 1, s)]_{ij}
$$

$$
= \sum_{l=1}^{2n(M+1)} [\Gamma(k_h - 1, k_{(h-1)\bar{a}})]_{ij} [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij}
$$

$$
+ [\Gamma(k_h - 1, k_{(h-1)\bar{a}})]_{ij} [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij}
$$

$$
\geq (1 - \hat{\mu})(\theta_j + \epsilon) + \frac{\hat{\mu}(\theta_j + \sigma_j)}{2}
$$

(13)

From (13), we have

$$
\max_i [\Gamma(k_h - 1, s)]_{ij} \leq (1 - \hat{\mu}) \max_i [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij}
$$

$$
+ \hat{\mu} [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij}
$$

and

$$
\min_i [\Gamma(k_h - 1, s)]_{ij} \geq (1 - \hat{\mu}) \min_i [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij}
$$

$$
+ \hat{\mu} [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij}
$$

Hence,

$$
\max_i [\Gamma(k_h - 1, s)]_{ij} - \min_i [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij} \leq (1 - \hat{\mu})
$$

$$
\times (\max_i [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij} - \min_i [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij}).
$$

Thus,

$$
\max_i [\Gamma(k_h - 1, s)]_{ij} - \min_i [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij} \leq (1 - \hat{\mu})^{h-1}
$$

$$
\times (\max_i [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij} - \min_i [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij}).
$$

Since $\max_i [\Gamma(k_h - 1, s)]_{ij} - \min_i [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij} \leq 1$ from the stochasticity and nonnegativity of the matrix $\Gamma(k_h, s)$,

$$
\max_i \{[\Gamma(k, s)]_{ij}\} \leq \max_i \{[\Gamma(k - 1, s)]_{ij}\}
$$

$$
\text{and}
$$

$$
\min_i \{[\Gamma(k - 1, s)]_{ij}\} \leq \min_i \{[\Gamma(k, s)]_{ij}\}
$$

for all $k > s$, we have $\max_i [\Gamma(k, s)]_{ij} - \min_i [\Gamma(k, s)]_{ij} \leq \max_i [\Gamma(k_h - 1, s)]_{ij} - \min_i [\Gamma(k_{(h-1)\bar{a}} - 1, s)]_{ij} \leq (1 - \hat{\mu})^{h-1}$ for all $k \leq k \leq k_{(h-1)\bar{a}} - 1$. Since $\hat{\mu} > 1$ for all $k > 0$, we have $\max_i [\Gamma(k, s)]_{ij} - \rho_i(s) \leq (1 - \hat{\mu})^{h-1} \leq C_0(1 - \hat{\mu})^{\theta_j}$ for all $k > s$, where $C_0 = (1 - \hat{\mu})^{-2}$.

Remark 5: From Lemma 3, we can see that each row of the product of stochastic matrices exponentially converge to a certain vector when the union of the edges whose weights are lower bounded by a certain positive constant among each certain time interval has a spanning tree, even when the stochastic matrices have zero diagonal entries and some of their nonzero entries are arbitrarily close to zero.

Proof of Theorem 1: Under Assumptions 1–3, from Lemmas 1 and 3, $\Gamma(k, s)$ is a stochastic matrix for any $k \geq s$, and there are constants $0 \leq \rho_i(s) \leq 1$ and $0 < \hat{\mu} \leq 1$ for all $i \in \{1, \cdots, 2n(M + 1)\}$ such that $\sum_{i=1}^{2n(M+1)} \rho_i(s) = 1$ and

$$
\max_i \{[\Gamma(k, s)]_{ij} - \rho_i(s)\} \leq C_0(1 - \hat{\mu})^{\theta_j}
$$

for all $k \geq 0$ and $s \geq 0$, where $C_0 = (1 - \hat{\mu})^{-2}$ and $\theta_j \geq 2n(M + 1)$. Since the initial conditions of $x_i(k)$ and $v_i(k)$ for all $k \leq 0$ satisfy the dynamics of (1), from (7) and the definitions of $Z(k)$ and $\xi(k)$, the solution of $Z(k)$ exists for any given $k \geq 0$. Let $s = M$ and $x = \sum_{j=1}^{2n(M+1)} \rho_j(s) Z_j(s)$. Thus, $\|Z_i(k, s) - \overline{x}\| = \|\sum_{j=1}^{2n(M+1)} \rho_j(s) Z_j(s)\| \leq C_0(1 - \hat{\mu})^{\theta_j}$ for all $k \geq 0$. Since $0 < \mu \leq 1$, it follows that there exist two constants $C > 0$ and $0 < \mu \leq 1$ such that $\|x_i(k, s) - \overline{x}\| \leq C(1 - \mu)^k$ for any $k \geq 0$ and all $i$.

V. A Numerical Example

Consider a multi-agent system consisting of 4 agents in a plane. The velocity of each agent $v_i$ is constrained to lie in a nonempty non-convex set $V_i = \{x | \|x\| \leq 1\} \cup \{x | -0.5 \leq \{1, 0\}^T x \leq 0.5, 0 \leq [0, 1]^2 x \leq 1.5\}$ for all $i$. At each time, only one edge of the graph shown in Fig. 2 is available to transmit the information and the switching sequence of the edges is $(1, 2), (2, 3), (3, 4), (4, 1), (1, 2), \cdots$. The weight of each edge is 0.5. The sample time of the system is $T = 0.2$ s. The initial conditions of all agents are taken as $x_i(0) = x_i(0)$ and $v_i(k) = 0$ for all $k < 0$. The delay for the edge (1, 2) is 7 s, for the edge (2, 3) is 2T s and for the edges (3, 4) and (4, 1) is 3T s. According to the design rule of $p_i(r)$ that satisfies Assumption 2, the parameters of the
control algorithms (2) are taken as \( p_i(0) = 1.5 \) and \( p_i(k) = b_i(k - 1) \) for all \( i \) and all \( k \geq 1 \). Clearly, Assumptions 1–3 are all satisfied. Fig. 3 shows the simulation results of the multi-agent system (1) with (2). It is clear that all agents eventually reach a consensus while their velocities remain their constraint sets, which is consistent with Theorem 1.

VI. CONCLUSION

In this technical note, a distributed velocity-constrained consensus problem has been studied for discrete-time multi-agent systems. Each agent’s velocity is constrained to lie in a nonconvex set. The communication graph considered is directed coupled with arbitrarily bounded communication delays and can be arbitrarily switching under the condition that the union of the graphs has directed spanning trees among each certain time interval. A distributed control algorithm has been proposed by applying a constrained control scheme using only local information. To analyze the velocity-constrained consensus problem, we have first introduced multiple novel model transformations and selected proper control parameters to transform the original system into an equivalent system with stochastic matrices. Due to the existence of communication delays and constraints, the stochastic matrices are state-dependent and have zero diagonal entries, and their nonzero entries might not be uniformly lower bounded by a positive constant. To overcome these coexisting challenges, with the help of an auxiliary matrix, we have proved that the transition matrix of the equivalent system has at least one column with all positive entries over a certain time interval and used the convexity of a stochastic matrix to show that all rows of the transition matrix tend to the same exponentially.

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