BRST OPERATOR FOR QUANTUM LIE ALGEBRAS: RELATION TO BAR COMPLEX

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Abstract

Quantum Lie algebras (an important class of quadratic algebras arising in the Woronowicz calculus on quantum groups) are generalizations of Lie (super) algebras. Many notions from the theory of Lie (super)algebras admit “quantum” generalizations. In particular, there is a BRST operator \( Q \) \((Q^2 = 0)\) which generates the differential in the Woronowicz theory and gives information about (co)homologies of quantum Lie algebras. In our previous papers a recurrence relation for the operator \( Q \) for quantum Lie algebras was given and solved. Here we consider the bar complex for q-Lie algebras and its subcomplex of q-antisymmetric chains. We establish a chain map (which is an isomorphism) of the standard complex for a q-Lie algebra to the subcomplex of the antisymmetric chains. The construction requires a set of nontrivial identities in the group algebra of the braid group. We discuss also a generalization of the standard complex to the case when a q-Lie algebra is equipped with a grading operator.

1 Introduction

The Woronowicz calculus [1] associates an algebra of exterior forms \( \Gamma \) and an enveloping algebra \( \mathcal{U} \) of the left invariant vector fields on \( \mathcal{A} \) to a Hopf algebra \( \mathcal{A} \). The algebra \( \mathcal{U} \) is called a quantum Lie algebra (q-Lie algebra for short). It is defined by relations \( \chi_i \chi_j - \sigma_{ij}^{km} \chi_k \chi_m = C_{ij}^k \chi_k \), where \( \{\chi_i\} \) is a set of
generators; ”structure constants” $\sigma^k_{ij}$ and $C^k_{ij}$ obey certain constraints, see Section 2. A q-Lie algebra is a non-homogeneous quadratic algebra. The general theory of quadratic algebras has been considered in a number of papers (see e.g. [2, 3, 4]). The case of q-Lie algebras is quite particular. An analog of the de Rham complex for $\mathcal{U}$ has been constructed in [1]. Mimicking the classical theory of Lie algebras, quantum analogs of the standard complex and BRST differential (with expected properties) have been introduced in [5, 6] (for a review of the general BRST-theory see [7]). In this paper we continue the investigation of q-Lie algebras along the lines of the theory of Lie algebras. Namely, (see e.g. [8]) we map the standard complex of the q-Lie algebra into the subspace of q-antisymmetric chains of the bar complex for $\mathcal{U}$. It turns out that the image of this map, just like in the case Lie algebras, is a subcomplex in the bar complex for $\mathcal{U}$. Moreover, the quantum BRST differential is the restriction of the bar differential $b$ to the subspace of the q-antisymmetric chains.

The paper is organized as follows. In Section 2 we recall the definition of q-Lie algebras. In Section 3 we introduce the exterior extensions of q-Lie algebra (an algebra of exterior forms $\Gamma$ and inner derivatives $\Gamma^*$) due to Woronowicz (our definition of the basis of differential forms is slightly different from the Woronowicz definition, see [5]) and develop the tools needed for the main constructions of Sections 4 and 5. Section 4 contains an explicit construction of the BRST operator of a q-Lie algebra. The presentation in the Sections 2-4 follows [12].

In Section 5 we discuss the bar complex $(C_n(\mathcal{U}), b)$ for a q-Lie algebra, its subcomplex of q-antisymmetric chains $(C_n(\mathcal{U}, \wedge L), b)$ and a chain map of the standard complex for q-Lie algebra to $(C_n(\mathcal{U}, \wedge L), b)$. Nontrivial identities in braid group algebra are used for proving the main Propositions of Sections 4 and 5.

Section 6 is a generalization of the constructions above to the situation when a q-Lie algebra is equipped with a grading operator. An example: a Lie super-algebra with grading given by parity. It is known that there are two choices of commutation relations between bosonic and fermionic ghosts. We explain this phenomenon in a general framework of a q-Lie algebra with a grading operator.

Section 7 summarizes the results of the paper.
2 Definitions and notation

The data defining a quantum Lie algebra with $N$ generators can be conveniently encoded into the following $(N + 1)^2 \times (N + 1)^2$ matrix [9]:

\[
R_{CD}^{AB} = \delta^i_A \delta^j_B \sigma_{ij}^{kl} \delta^C_k \delta^D_l + \delta^i_A \delta^j_B C_{ij}^k \delta^C_0 \delta^D_k + \delta^i_A \delta^j_B \delta^0_C \delta^D_0 + \delta^C_k \delta^D_0 \delta^0_A \delta^0_B
\]

\[
\equiv \delta^i_A \delta^j_B \sigma_{12} \delta^C_1 \delta^D_2 + \delta^i_A \delta^j_B C_{12}^{(2)} \delta^C_0 \delta^D_2 + \delta^i_A \delta^j_B \delta^0_C \delta^D_0 + \delta^C_0 \delta^D_0 \delta^0_A \delta^0_B .
\]

Here $1, 2, \ldots$ denote copies of an $N$-dimensional vector spaces $V_N$; $(1, 2, \ldots)$ are the corresponding outcoming and incoming vectors; small letters $i, j, k, l, \ldots = 1, 2, \ldots, N$ denote indices of the vector space $V_N$; capital letters $A, B, \ldots = 0, 1, \ldots, N$ denote indices of an $(N+1)$-dimensional space $V_{N+1}$ ($V_N$ is a subspace in $V_{N+1}$).

Eqs. (1) are equivalent to

\[
R^{ij}_{kl} = \delta^{ij}_{kl} , \quad R^{0j}_{kl} = C^j_{kl} , \quad R^{0A}_{B0} = R^{A0}_{0B} = \delta^A_B
\]

and the other components of $R$ vanish. We assume that the matrix $\sigma \in \text{End}(V_N \otimes V_N)$ has an eigenvalue 1.

Let $L_A = \{\chi_0, \chi_i\}$ ($i = 1, \ldots, N$) be a “quantum vector”:

\[
L_A = \delta_A^{(1)} \chi_1 + \delta_A^0 \chi_0 = \begin{pmatrix} \chi_0 \\ \chi_i \end{pmatrix}
\]

for the matrix $R$:

\[
R^{CD}_{AB} L_C L_D = L_A L_B .
\]

These relations are equivalent to

\[
[\chi_0, \chi_i] = 0 \quad \text{and} \quad (1 - \sigma_{12}) \chi_1 \chi_2) = C_{12}^{(1)} \chi_0 \chi_1 .
\]

One can rescale the elements, $\chi_i \rightarrow \chi_0 \chi_i$. The algebra with rescaled generators $\chi_i$ ($i = 1, 2, \ldots, N$) subject to relations

\[
(1 - \sigma_{12}) \chi_1 \chi_2) = C_{12}^{(1)} \chi_1 ,
\]

is called a quantum Lie algebra if the matrix $R$ satisfies the Yang-Baxter equation,

\[
R^{CD}_{AB} R^{ES}_{MC} R^{FN}_{SD} = R^{CS}_{MA} R^{DN}_{SB} R^{EF}_{CD} \quad \text{or} \quad R_{23} R_{12} R_{23} = R_{12} R_{23} R_{12} .
\]
(here 1, 2 or 2, 3 denote copies of the vector spaces $V_{N+1}$ on which $R$-matrices act nontrivially). The Yang-Baxter equation for $R$ imposes the following conditions for the structure constants of the quantum Lie algebra (5):

\[
\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 , \tag{7}
\]

\[
C_1 \delta_3 C_1 = \sigma_2 C_1 \delta_3 C_1 + C_2 C_1 , \tag{8}
\]

\[
C_1 \delta_3 \sigma_1 = \sigma_2 \sigma_1 C_2 , \tag{9}
\]

\[
(\sigma_2 C_1 \delta_3 + C_2) \sigma_1 = \sigma_1 (\sigma_2 C_1 \delta_3 + C_2) . \tag{10}
\]

Here we have used a concise notation $\sigma_n := \sigma_{nn+1}$, $C_n := C_{n,n+1}$, $\delta_n := \delta^{(n-1)}$. 

3 Exterior extensions for quantum Lie algebra

To define the exterior extension of the quantum Lie algebra (5), one introduces quantum wedge algebras with generators $\gamma_i$ and $\Omega^i$ $(i = 1, \ldots N)$:

\[
\Omega^m \wedge \ldots \wedge \Omega^2 \wedge \Omega^1 = \Omega^m \otimes \ldots \otimes \Omega^2 \otimes \Omega^1 A_{1 \rightarrow n} , \tag{11}
\]

\[
\gamma_{1} \wedge \gamma_{2} \ldots \wedge \gamma_{n} = A_{1 \rightarrow n} \gamma_{1} \otimes \gamma_{2} \ldots \otimes \gamma_{n} . \tag{12}
\]

The cross-commutation relations are

\[
\gamma_{2} \Omega^2 = - \Omega^1 \sigma_{12}^{-1} \gamma_{1} + I_2 . \tag{13}
\]

In eqs. (11) and (12) we have used operators $A_{1 \rightarrow n}$, quantum analogues of the antisymmetric. They can be defined inductively (in fact they make sense already for the group algebra of the braid group $B_{M+1}$ with generators $\sigma_i$ $(i = 1, \ldots, M)$):

\[
A_{1 \rightarrow n} \equiv f_{1 \rightarrow n} A_{1 \rightarrow n-1} \quad \text{or} \quad A_{1 \rightarrow n} \equiv \overline{f}_{1 \rightarrow n} A_{2 \rightarrow n} , \tag{14}
\]

where $A_{1 \rightarrow 1} = 1$, $f_{k \rightarrow k} = \overline{f}_{k \rightarrow k} = 1$ and

\[
f_{k \rightarrow m} = 1 - f_{k \rightarrow m-1} \sigma_{m-1} \quad \text{or} \quad f_{k \rightarrow m} = f_{k+1 \rightarrow m} + (-1)^{m-k} \sigma_k \cdots \sigma_{m-1} ,
\]

\[
\overline{f}_{k \rightarrow m} = 1 - \overline{f}_{k+1 \rightarrow m} \sigma_k \quad \text{or} \quad \overline{f}_{k \rightarrow m} = \overline{f}_{k \rightarrow m-1} + (-1)^{m-k} \sigma_{m-1} \cdots \sigma_{k+1} \sigma_k . \tag{15}
\]
for \((k < m \leq n)\). Explicitly:
\[
f^{(\sigma)}_{k \rightarrow n} = f^{(\sigma)}_{k \rightarrow n} = 1 - \sigma_{n-1} + \sigma_{n-2} \sigma_{n-1} - \ldots + (-1)^{n-k} \sigma_k \sigma_{k+1} \ldots \sigma_{n-1} ,
\]
\[
\overline{f}^{(\sigma)}_{k \rightarrow n} = \overline{f}^{(\sigma)}_{k \rightarrow n} = 1 - \sigma_k + \sigma_{k+1} \sigma_k - \ldots + (-1)^{n-k} \sigma_{n-1} \ldots \sigma_{k+1} \sigma_k .
\]

If the sequence of operators \(A_{1 \rightarrow n}\) terminates at the step \(n = h + 1\) \((A_{1 \rightarrow n} = 0\) \(\forall n > h)\) for some matrix representation \(\rho\) of \(B_{M+1}\) then the number \(h\) is called the height of the matrix \(\rho(\sigma_{12})\).

The elements \(\gamma_i\) and \(\Omega^i\) are quantum analogues of ghost variables. The choice of the sign and the appearance of the braid matrix in (13) follows the conventional case of Lie (super)algebras. The commutation relations of \(\chi_i\) with \(\gamma_j\) and \(\Omega^k\) are
\[
\chi_2) \Omega^{(2)} = \Omega^{(1)} \left(\sigma_{12} \chi_1) + C_{12}^{(2)} \right), \quad \gamma_1) \chi_2) = \sigma_{12} \chi_1) \gamma_2) + C_{12}^{(1)} \gamma_1) .
\]

The algebra (5), (11) – (13) and (17) is the exterior extension of the quantum Lie algebra and (see [5]) this algebra gives rise to the Cartan differential calculus on quantum groups in the Woronowicz theory [1].

**Remark.** Let \(\hat{\sigma}_i\) \((i = 1, \ldots, M)\) be generators of the braid group \(B_{M+1}\):
\[
\hat{\sigma}_i \hat{\sigma}_{i+1} \hat{\sigma}_i = \hat{\sigma}_{i+1} \hat{\sigma}_i \hat{\sigma}_{i+1} , \quad [\hat{\sigma}_i, \hat{\sigma}_j] = 0 \text{ for } |i - j| > 1 .
\]

The important properties of the elements \(f^{(\hat{\sigma})}_{k \rightarrow n}\) and \(\overline{f}^{(\hat{\sigma})}_{k \rightarrow n}\) (16) of \(B_{M+1}\) are:
\[
f_{1 \rightarrow n} f_{1 \rightarrow n-1} \ldots f_{1 \rightarrow m} = x_n^{(n-m+1)} A_{m \rightarrow n} , \quad (n \geq m \geq 1) \tag{19}
\]
\[
\overline{f}_{1 \rightarrow n} \overline{f}_{2 \rightarrow n} \ldots \overline{f}_{m \rightarrow n} = x_n^{(n-m)} A_{1 \rightarrow m} , \quad (n \geq m \geq 1) \tag{20}
\]

where \(x_n^{(0)} = 1, x_n^{(1)} = f_{1 \rightarrow m}^{(\hat{\sigma})}, x_n^{(n)} = 1, x_n^{(n-1)} = \overline{f}_{1 \rightarrow n}^{(\hat{\sigma})}\) and \(x_{m+1}^{(2)}\) is given by
\[
x_{m+1}^{(2)} = f_{1 \rightarrow m}^{(\hat{\sigma})} + f_{1 \rightarrow m-1}^{(\hat{\sigma})} (\hat{\sigma}_m \hat{\sigma}_{m-1}) + f_{1 \rightarrow m-2}^{(\hat{\sigma})} (\hat{\sigma}_{m-1} \hat{\sigma}_{m-2}) (\hat{\sigma}_m \hat{\sigma}_{m-1})
\]
\[+ \ldots + f_{1 \rightarrow 2}^{(\hat{\sigma})} (\hat{\sigma}_3 \hat{\sigma}_2) \cdots (\hat{\sigma}_m \hat{\sigma}_{m-1}) + (\hat{\sigma}_2 \hat{\sigma}_1) \cdots (\hat{\sigma}_m \hat{\sigma}_{m-1}) .
\]

The identities (19), (20) are equivalent to the factorization formula
\[
A_{1 \rightarrow n} = x_n^{(n-m)} A_{1 \rightarrow m} A_{m+1 \rightarrow n} \tag{22}
\]
The elements \( x_n^{(m)} \) can be defined inductively using the recurrent relation
\[
x_{n+1}^{(n-m+1)} = x_n^{(n-m)} - (-1)^{n-m} x_n^{(n-m+1)} \sigma_n \ldots \sigma_m
\]
and in particular we have
\[
x_{n+1}^{(2)} = (f_{n-n}^{(2)} + x_n^{(2)} \sigma_n \sigma_{n-1}) , \quad (n \geq 2) ,
\]
which gives (21). Then, we have \( x_{n+1}^{(3)} = x_n^{(2)} - x_n^{(3)} \sigma_n \sigma_{n-1} \sigma_{n-2} \) for \( (n \geq 3) \), etc.

Note that the elements \( x_n^{(m)} \) are alternating sums over the braid group elements which can be considered as quantum analogs of \((m, n - m)\) shuffles (the shuffles are obtained by projection \( \sigma_i \to -s_i \), where \( s_i \) are generators of symmetric group \( S_{M+1} \)). From this point of view the wedge products (11), (12) are related to the quantum shuffle products (about quantum shuffles and corresponding products see [10], [1]). The associativity of these products is provided by the identities: \( x_n^{(m-n)} x_{m-k}^{(m-k)} = x_n^{(m-k)} (T_{k} x_{n-k}^{(n-m)} T_{k}^{-1}) \) for \( k < m < n \), where \( T_{k} \sigma_{i} T_{k}^{-1} = \sigma_{i+k} \).

4 BRST operator for quantum Lie algebra

In the paper [5] we have given a recurrence which determines the BRST operator \( Q \) for the quantum Lie algebra:

**Proposition 1.** The BRST operator \( Q \) for the quantum Lie algebras (5), which satisfies the equation \( Q^2 = 0 \), should have the following form

\[
Q = \Omega_{i}^{\chi_{i}} + \sum_{r=1}^{h-1} Q_{(r)} ,
\]

where \( h \) is the height of the braid matrix \( \sigma_{12} \), the operators \( Q_{(r)} \) are given by

\[
Q_{(r)} = \Omega_{(r+1)}^{(r+1)} \otimes \Omega_{(r)}^{(r)} \otimes \ldots \otimes \Omega_{(1)}^{(1)} (AXA)_{[1\ldots r+1]}^{(1\ldots r)} (AXA)_{[1\ldots r+2]}^{(1\ldots r)} \ldots \otimes \gamma_{(r)}
\]

and \( (AXA)_{[1\ldots r+1]}^{(1\ldots r)} = A_{1\ldots r+1} (XA)_{[1\ldots r+2]}^{(1\ldots r)} = (AX)_{[1\ldots r+1]} \otimes A_{1\ldots r} \) are tensors which obey the following recurrent relations

\[
A_{1\ldots r+1} (XA)_{[1\ldots r+2]}^{(1\ldots r)} = A_{1\ldots r+1} ((-1)^{r} \sigma_{r} \sigma_{r-1} \ldots \sigma_{1} - 1) (XA)_{[2\ldots r+1]}^{(2\ldots r+1)}
\]
with the initial condition \( A_{12} X_{12}^{[1]} = -C_{12}^{[1]} \).

It should be stressed that the derivation of the recurrence (26) requires the consideration of the terms (in the equation \( Q^2 = 0 \)) linear in \( \chi_i \) only.

In [12] we presented the solution of eqn. (26) and defined coefficients \( (A X A)^{[1 \ldots r + 1]} \) explicitly. This was done with the help of an important identity in the group algebra of the braid group:

**Lemma 1.** Let \( \hat{\sigma}_i (i = 1, \ldots M) \) be generators of the braid group \( B_{M+1} \) (18). Then the following identity in the group algebra of \( B_{M+1} \) holds:

\[
Y_{1 \rightarrow r+1} \overline{f}_{1 \rightarrow r}^{(\hat{\sigma})} = \left( (-1)^{r+1} f_{1 \rightarrow r+1} \hat{\sigma}_r \cdots \hat{\sigma}_1 + \overline{f}_{1 \rightarrow r+1}^{(\hat{\sigma})} \right) Y_{2 \rightarrow r+1},
\]

where \( f_{1 \rightarrow r}^{(\hat{\sigma})} \) and \( \overline{f}_{1 \rightarrow r}^{(\hat{\sigma})} \) are defined in (16)

\[
Y_{k \rightarrow r+1} = (1 - \hat{\sigma}_r^2) (1 + \hat{\sigma}_{r-1} \hat{\sigma}_r^2) \cdots \left(1 - (-)^{r-k} \hat{\sigma}_k \hat{\sigma}_{k+1} \cdots \hat{\sigma}_r \hat{\sigma}_2 \right),
\]

\[
Y_{k \rightarrow k} = 1.
\]

**Proof.** With the help of eqs. (15), the identity (27) can be represented in the form

\[
Y_{1 \rightarrow r+1} \overline{f}_{1 \rightarrow r}^{(\hat{\sigma})} = F_{1 \rightarrow r+1} Y_{2 \rightarrow r+1},
\]

where

\[
F_{1 \rightarrow r+1} = \left( (-1)^r f_{1 \rightarrow r}^{(\hat{\sigma})} \hat{\sigma}_r^2 \cdots \hat{\sigma}_1 + \overline{f}_{1 \rightarrow r}^{(\hat{\sigma})} \right).
\]

One can prove the identity (29) by induction (see [12], [13]).

**Remark.** The operators \( Y_{k \rightarrow r+1} \) (28) have another form:

\[
Y_{k \rightarrow r+1} = (1 + \hat{\sigma}_r) (1 - \hat{\sigma}_{r-1} \hat{\sigma}_r) \cdots (1 + (-)^{r-k} \hat{\sigma}_k \hat{\sigma}_{k+1} \cdots \hat{\sigma}_r) f_{k \rightarrow r+1}^{(\hat{\sigma})}.
\]

This is nothing but the quantum version of the Zagier’s factorization identities [11] (which can be obtained from (31) by the projection \( \hat{\sigma}_i \rightarrow q s_i \), where \( s_i \) are generators of the symmetric group, \( s_i^2 = 1 \)). The proof of identities (31) is straightforward (see e.g. [12]).

We have the following result [12]:
Proposition 2. The explicit solution of the recurrent relations (26) is given by the formula

\[ (-1)^{r+1} (AXA)^{i_1 \ldots i_r}_{j_1 \ldots j_{r+1}} \]

\[ = \left( (1 - R_2) \left( 1 + R_{r-1} R_2 \right) \cdots \left( 1 + (-)^r R_1 \ldots R_{r-1} R_2 \right) \right)^{k_1 \ldots k_r \ 0}_{j_1 \ldots j_{r+1}} (A_{1 \ldots r})^{i_1 \ldots i_r}_{k_1 \ldots k_r}. \]

Here \( 1, \ldots, r-1, r \) label the copies of the vector space \( V_{N+1} \); \( R_2 := R_{r,r+1} \) and \( r, r+1 \) are the numbers of vector spaces \( V_{N+1} \) where the \( R \)-matrix (1) acts nontrivially; \( i_1, \ldots, j_1, \ldots, k_1, \ldots = 1, 2, \ldots N \) are the vector indices.

Proof. Using the definition of the antisymmetrizer (14) and the braid relation (7) we rewrite the right-hand side of (26) in the form

\[ A_{1 \ldots r+1} \left( (-1)^r \sigma_r \sigma_{r-1} \ldots \sigma_1 - 1 \right) (XA)^{2 \ldots r}_{|2 \ldots r+1}} \]

\[ = \left( (-1)^r f_{1 \ldots r+1} \sigma_r \sigma_{r-1} \ldots \sigma_1 - \mathcal{J}_{1 \ldots r+1} \right) A_{2 \ldots r+1} (XA)^{2 \ldots r}_{|2 \ldots r+1}} \]

Thus, the expression (32) is a solution of the equation (26) if the following identity holds

\[ (Y_{1 \ldots r+1})^{(1 \ldots 0 \ldots r+1)}_{1 \ldots r+1} \mathcal{J}_{1 \ldots r} = \left( (-1)^{r+1} f_{1 \ldots r+1} \sigma_r \ldots \sigma_1 + \mathcal{J}_{1 \ldots r+1} \right) (Y_{2 \ldots r+1})^{(2 \ldots 0 \ldots r)}_{|2 \ldots r+1}} \]

(33)

where

\[ Y_{k \ldots r+1} = (1 - R_2) \left( 1 + R_{r-1} R_2 \right) \cdots \left( 1 + (-)^r R_k R_{k+1} \ldots R_{r-1} R_2 \right). \]

As indicated by the structure of indices, the vector space \( V_{N+1} \) is a direct sum of \( V_N \) and a one-dimensional vector space \( V_0 \). Let \( P : V_{N+1} \to V_N \) and \( 1 - P : V_{N+1} \to V_0 \) be the corresponding projectors. Similarly, let \( \bar{P} \) and \( 1 - \bar{P} \) be the projectors associated to the dual space \( V_{N+1}^* \).

Define

\[ P_{1 \ldots r} := P_1 \ldots P_r, \quad P^{1 \ldots r \ 0} = \bar{P}_1 \ldots \bar{P}_r (1 - \bar{P})_{r+1}. \]

These operators split out the components which we need.
Eqn. (33) can be rewritten now in the form
\[ P_{1 \rightarrow r+1} Y_{1 \rightarrow r+1} f^{(R)}_{1 \rightarrow r+1} P_{1 \rightarrow 0} \]
\[ = P_{1 \rightarrow r+1} \left( (-1)^{r+1} f^{(R)}_{1 \rightarrow r+1} R_{r} \cdots R_{1} + \tilde{J}^{(R)}_{1 \rightarrow r+1} \right) Y_{2 \rightarrow r+1} P_{1 \rightarrow 0} \]
(34)
(where \( f^{(R)}_{1 \rightarrow r+1} \) and \( \tilde{J}^{(R)}_{1 \rightarrow r+1} \) are the \((N+1)\)-dimensional analogues of the operators (16)) in view of relations
\[ \tilde{J}^{(R)}_{1 \rightarrow r} P_{1 \rightarrow r} = P_{1 \rightarrow r} \tilde{J}^{(\sigma)}_{1 \rightarrow r} , \]
\[ Y_{2 \rightarrow r+1} P_{1 \rightarrow r} = P_{1 \rightarrow r+1} \left( P_{2 \rightarrow r+1} Y_{2 \rightarrow r+1} P_{2 \rightarrow 0} \right) \]
\[ \equiv P_{1 \rightarrow r+1} \left( Y_{2 \rightarrow r+1} \right)_{(2 \ldots r \ldots 0)}^{(2 \ldots r \ldots 1)} , \]
\[ P_{1 \rightarrow r+1} \left( (-1)^{r} f^{(R)}_{1 \rightarrow r+1} R_{r} \cdots R_{1} - \tilde{J}^{(R)}_{1 \rightarrow r+1} \right) P_{1 \rightarrow r+1} \]
\[ = \left( (-1)^{r} f^{(R)}_{1 \rightarrow r+1} \sigma_{r} \sigma_{r-1} \cdots \sigma_{1} - \tilde{J}^{(R)}_{1 \rightarrow r+1} \right) \]
which can be obtained directly using the explicit form of the \( R \)-matrix (2).

The equation (34) is fulfilled in view of the identity (27) which has been proven in Lemma 1. Indeed, if in the identity (27) we take the \( R \)-matrix representation for the braid group \( \rho_{R}(B_{M+1}) \): \( \rho_{R}(\hat{\sigma}_{k}) = R_{k} \) and act on this identity from the left and right by the projectors \( P_{1 \rightarrow r+1} \), \( P_{1 \rightarrow 0} \), then we deduce (34). This completes the proof of the Proposition 2.

In the next Section we need more information about the \( R \)-matrix representation of the braid group just considered. Introduce the Jucys-Murphy elements \( J_{r} \in \rho_{R}(B_{M+1}) \):
\[ J_{1} = 1 , \quad J_{r+1} = R_{r} J_{r} R_{r}. \] (35)
These elements form a commuting set in \( \rho_{R}(B_{M+1}) \): \([J_{r}, J_{m}] = 0 \) and satisfy
\[ R_{m} J_{r} = J_{r} R_{m}, \quad m < r . \] (36)

**Proposition 3.** The explicit form of the components
\[ Z_{r+1} := P_{1 \rightarrow r+1}(J_{r+1})P_{1 \rightarrow 0} \]
of Jucys-Murphy elements (35) in terms of the structure constants $C_n$ and
$\sigma_m$ is:
\[
Z_{r+1} = C_r + \sigma_r C_{r-1} \delta_{r+1} + \sigma_r \sigma_{r-1} C_{r-2} \delta_r \delta_{r+1} + \ldots
+ \sigma_r \sigma_{r-1} \ldots \sigma_2 C_1 \delta_3 \ldots \delta_{r+1} .
\]  
(37)
The elements $Z_r$ have the following properties
\[
\sigma_m Z_r = Z_r \sigma_m , \quad m \leq r - 2 ,
\]  
(38)
\[
Z_{r+1} = C_r + \sigma_r Z_r \delta_{r+1} .
\]  
(39)

Proof. Using the definition of Jucys-Murphy elements (35) and the representation (2) one can rewrite $P_{1 \to r+1}(J_{r+1})P^{1 \to 0}$ in the form
\[
= P_{1 \to r+1} (R_\Sigma \ldots (R_1)^2 \ldots R_\Sigma) P^{1 \to 0}
\]  
Then, again using (2) we deduce (37). To obtain the property (38) apply the operators $P_{1 \to r}$ and $P^{1 \to 0}$ to (36) from the left and from the right (for $m \leq r - 2$). The recurrent relation (39) is a direct consequence of the relations (37).

5 Relation to bar chain complex

We recall first some standard material. For every associative algebra $\mathcal{U}$ one forms the bar complex $(C_n(\mathcal{U}), b)$:

a) $C_n(\mathcal{U})$ is the space $\underbrace{\mathcal{U} \otimes \ldots \otimes \mathcal{U}}_n$;

b) for
\[
T = a_1 \otimes a_2 \otimes \ldots \otimes a_n \in C_n(\mathcal{U}) ,
\]  
the action of the boundary operator $b : C_{n+1}(\mathcal{U}) \to C_n(\mathcal{U})$ is given by
\[
b(a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}) = \sum_{i=1}^n (-1)^{n-i} (a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) .
\]  
(41)
Using the associativity of $\mathcal{U}$ one can check directly that $b^2 (a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}) = 0$. Moreover, for a unital algebra $\mathcal{U}$, the complex $(C_n(\mathcal{U}), b)$ is acyclic since one can define a homotopy operator $\delta: C_n(\mathcal{U}) \rightarrow C_{n+1}(\mathcal{U})$

$$\delta(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = (a_1 \otimes a_2 \otimes \ldots \otimes a_n \otimes 1)$$

which satisfies $\delta b + b \delta = id$.

Let $\mathcal{U}$ be the q-Lie algebra with generators $\chi_i$ and defining relations (5). Consider the chains $T$ (as in (40)) with all elements $a_k$ belonging to the set $\{\chi_j\}$ of the generators of the q-Lie algebra and define their q-antisymmetric linear combination which is given by the contraction with the numerical tensor $A_{1\rightarrow n} \in End(V^\otimes n)$ (14):

$$\chi_1 \wedge \chi_2 \wedge \ldots \wedge \chi_n := A_{1\rightarrow n} \{\chi_1 \otimes \chi_2 \otimes \ldots \otimes \chi_n\} \in C_n(\mathcal{U}) . \quad (42)$$

In fact we shall need the following chains:

$$a \otimes \{\chi_1 \wedge \chi_2 \wedge \ldots \wedge \chi_n\} \in C_{n+1}(\mathcal{U}) \quad (43)$$

with arbitrary $a \in \mathcal{U}$. Denote the subspace of such chains by $C_{n+1}(\mathcal{U}, \wedge L) \in \mathcal{U} \otimes L \wedge \ldots \wedge L$, where $L$ is a linear span of $\{\chi_1\}$.

Now we formulate one of the main results of this paper.

**Proposition 4.** The boundary operator $b$ (41) preserves the q-antisymmetry,

$$b: C_{n+1}(\mathcal{U}, \wedge L) \longrightarrow C_n(\mathcal{U}, \wedge L) . \quad (44)$$

In other words, we have a complex $(C_n(\mathcal{U}, \wedge L), b)$ which is a subcomplex of the bar complex $(C_n(\mathcal{U}), b)$.

**Proof.** Acting by the boundary operator (41) on the q-antisymmetric chains (43), we deduce

$$b \{a \otimes \chi_1 \wedge \chi_2 \wedge \ldots \wedge \chi_{n+1}\} = (-1)^n A_{1\rightarrow n+1} \{a \chi_1 \otimes \chi_2 \otimes \ldots \otimes \chi_{n+1}\}$$

$$+ a \otimes A_{1\rightarrow n+1} \left[ \sum_{k=1}^{n} (-1)^{n-k} \{\chi_1 \otimes \ldots \otimes \chi_k \chi_{k+1} \otimes \ldots \otimes \chi_{n+1}\} \right] \quad (45)$$
\[
= (-1)^n \mathcal{F}_{1\rightarrow n+1} \{ a \chi_1 \otimes \chi_2 \wedge \ldots \wedge \chi_{n+1} \}
\]
\[
+ a \otimes A_{1\rightarrow n+1} \sum_{k=1}^{n} (-1)^{n-k} t^{(k)}_{k,k+1} \{ \chi_1 \otimes \ldots \otimes \chi_k \otimes \chi_{k+2} \otimes \ldots \otimes \chi_{n+1} \},
\]
where structure constants \( t^{(k)}_{j;k} \) satisfy
\[
C^{(k)}_{j;k;k+1} = (1 - \sigma_k) t^{(k)}_{k;k+1} ;
\]
tensors \( \mathcal{F}_{1\rightarrow n+1} \) are introduced in (14) – (16) and we have used the defining relations (5) and the properties of "antisymmetrizers"
\[
A_{1\rightarrow n+1} \chi(k) \chi_{k+1}) = A_{1\rightarrow n+1} t^{(k)}_{k;k+1} \chi(k).
\]
Thus, to prove (44) we need to represent the second term in (46) in the form
\[
a \otimes W^{(1\ldots n)}_{1\ldots n+1} \{ \chi_1 \wedge \ldots \wedge \chi_n \} = a \otimes \sum_{k=1}^{n} (-1)^{n-k} t^{(k)}_{k,k+1} \delta_{k+2\rightarrow n+1} \chi_1 \otimes \ldots \otimes \chi_n \}
\]
Therefore it is sufficient to prove:

**Lemma 2.** There exist tensors \( W_{n+1} := W^{(1\ldots n)}_{1\ldots n+1} \) such that
\[
A_{1\rightarrow n+1} \left( \sum_{k=1}^{n} (-1)^{n-k} t^{(k)}_{k,k+1} \delta_{k+2\rightarrow n+1} \right) = W_{n+1} A_{1\rightarrow n}
\]
where \( \delta_{n+1\rightarrow n} := 1, \delta_{k+2\rightarrow n+1} := \delta_{k+2} \cdots \delta_{n+1} \) is a shift operator \( i \rightarrow i + 1 \) for \( k < i \leq n \). The explicit form of the tensors \( W_n \) (in terms of the quantum Lie algebra structure constants \( \sigma^{(j)}_{ki} \) and \( C^{(j)}_{ik} \)) is:
\[
W_{n+1} = C_n - (1 - \sigma_n) C_{n-1} \delta_n + (1 - \sigma_{n-1} + \sigma_n \sigma_{n-1}) C_{n-2} \delta_n \delta_{n+1} - \\
\ldots + (-1)^{n+1} [1 - \sigma_2 + \sigma_3 \sigma_2 - \ldots + (-1)^{n+1} \sigma_2 \cdots \sigma_2] C_1 \delta_3 \cdots \delta_{n+1}
\]
\[
= Z_n + Z_{n-1} \delta_n + Z_{n-2} \delta_n \delta_{n+1} + \ldots + (-1)^{n+1} Z_2 \delta_3 \cdots \delta_{n+1},
\]
where the elements \( Z_k \) are defined in (37), (39) (we fix \( Z_2 = C_1 = W_2 \)).

**Proof.** To prove (49) we use the identities (7) – (10) for the structure constants and the relation (47) between \( C^{(j)}_{ik} \) and \( t^{(j)}_{ik} \).
Eq. (49) is obviously fulfilled for \(n = 1\). Assume that (49) is correct for 
\(n = m - 1\). We have to prove (49) for \(n = m\). For the left hand side of (49) we obtain:
\[
A_{1 \rightarrow m+1} \left( \sum_{k=1}^{m} (-1)^{m-k} t_{k,k+1}^{(k)} \delta_{k+2} \cdots \delta_{m+1} \right)
\]
\[
= A_{1 \rightarrow m+1} \left( \sum_{k=1}^{m-1} (-1)^{m-k} t_{k,k+1}^{(k)} \delta_{k+2} \cdots \delta_{m} \right) \delta_{m+1} + t_{m,m+1}^{(m)}
\]
\[
= \left( -f_{m+1} W_{m} \delta_{m+1} + f_{m+1} f_{m} t_{m,m+1}^{(m)} \right) A_{1 \rightarrow m-1}
\]
\[
= \left[ -f_{m+1} W_{m} \delta_{m+1} + x_{m+1} C_{m} \right] A_{1 \rightarrow m-1}.
\]

In the second equality we used (14), (49); in the third equality we used (19) with 
\(n - m = 2\). Here and below, to simplify notation, we put 
\(f_{m} := f_{1 \rightarrow m}\) and \(x_{m} := x_{m}^{(2)}\).

Comparing with the right-hand side of (49), we obtain the equation
\[
-f_{m+1} W_{m} \delta_{m+1} + x_{m+1} C_{m} = W_{m+1} f_{m},
\]
which is an identity for \(m = 2\) in view of (9) and (10). We prove (51) by induction. For this we note (formula (50)) that one has the inductive relation
\[
W_{n+1} = Z_{n+1} - W_{n} \delta_{n+1}.
\]

Substitute to the left-hand side of (51) the recurrence relations (15), (23), (52) and take into account relations (9), (39), (38) and the base of induction:
\[
-W_{m} \delta_{m+1} + f_{m} \sigma_{m} (Z_{m} - W_{m-1} \delta_{m}) \delta_{m+1} + (f_{m} + x_{m} \sigma_{m} \sigma_{m-1}) C_{m}
\]
\[
= -W_{m} \delta_{m+1} + f_{m} (\sigma_{m} Z_{m} \delta_{m+1} + C_{m}) + W_{m} f_{m-1} \delta_{m+1} \sigma_{m-1}
\]
\[
= (Z_{m+1} - W_{m} \delta_{m+1}) - f_{m-1} \sigma_{m-1} Z_{m+1} + W_{m} \delta_{m+1} f_{m-1} \sigma_{m-1}
\]
\[
= (Z_{m+1} - W_{m} \delta_{m+1})(1 - f_{m-1} \sigma_{m-1}) = W_{m+1} f_{m}.
\]

Thus, we proved (51) and, therefore, (49). This finishes the proof of the Proposition 4.

The identity (49) guarantees that the q-antisymmetric chains (43) form the subcomplex in the bar complex for q-Lie algebra \(\mathcal{U}\) (5). The identity
\[
(b)^{2} \{a \otimes \chi_{1} \wedge \chi_{2} \wedge \ldots \wedge \chi_{n+1} \} = 0
\]
for the boundary operator (45) directly follows from the identity \( b^2 = 0 \) for the bar differential \( b \) (41).

In analogy with the theory of usual Lie algebras (see e.g. [8]), we relate the complex \((C_n(\mathcal{U}, \wedge \mathcal{L}), b)\) with the standard complex for the quantum Lie algebra. Namely, we relate the chains

\[
C_n = a \otimes A_{1 \rightarrow n}(\chi_1 \otimes \cdots \otimes \chi_n) \in C_n(\mathcal{U}, \wedge \mathcal{L})
\]

with wedge polynomials in \( \gamma \)'s with coefficients in \( \mathcal{U} \)

\[
M_n = a \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n, \quad a \in \mathcal{U},
\]

by means of the one-to-one map \( i: C_n \leftrightarrow M_n \),

\[
i(a \{ \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n \}) = a \otimes A_{1 \rightarrow n} \{ \chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_n \}
\]

The BRST operator \( Q \) (24), (25), (32) acts on the expressions (53) from the right and, according to (13), the variables \( \Omega \)'s in \( Q \) should be considered as "right \( q \)-derivatives" over \( \gamma \)'s (in particular \( \forall a \in \mathcal{U} \) the right action of \( \Omega^i \) on \( a \) gives zero).

**Proposition 5.**

1. \( Q^2 = 0 \);

2. The map \( i \) (54) is the chain map, \( b \circ i = i \circ Q \).

**Proof.** Consider only first two terms in the expression (24):

\[
Q = \Omega^{(1} \chi_{1)} - \Omega^{(2} \wedge \Omega^{(1} t^{(1)}_{12)} \gamma_{1)} + \ldots .
\]

Using (13), (17), (37) and (39) one can prove by induction the following relations:

\[
\gamma_1 \wedge \cdots \wedge \gamma_n \Omega^{(n} = (-1)^n \Omega^{(0} \sigma_0^{-1} \cdots \sigma_{n-1}^{-1} \gamma_0) \wedge \cdots \wedge \gamma_{n-1})
\]

\[
+(-1)^{n-1} \mathcal{F}^{(\sigma)}_{1 \rightarrow n} \sigma_1^{-1} \cdots \sigma_{n-1}^{-1} \gamma_1) \wedge \cdots \wedge \gamma_{n-1}) ,
\]

\[
\gamma_1 \wedge \cdots \wedge \gamma_{n-1} \chi_n = \sigma_{n-1} \cdots \sigma_1 \chi_1 \gamma_2 \wedge \cdots \wedge \gamma_n + \mathcal{Z}_n \gamma_1 \wedge \cdots \wedge \gamma_{n-1} .
\]

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Thus, using (20) and (56), the right action of the first two terms (55) of \( Q \) on (53) gives

\[
Q \left( a^{\gamma_1} \land \ldots \land ^{\gamma_n} \right) = (-1)^{n-1} T^{(\sigma)}_{1-n} \left( a^{\chi_1^{\gamma_2} \land \ldots \land ^{\gamma_n}} \right) + \\
+(-1)^{n-1} T^{(\sigma)}_{1-n} \sigma_1^{-1} \ldots \sigma_{n-1}^{-1} Z_n \left( a^{\gamma_1} \land \ldots \land ^{\gamma_{n-1}} \right) - \\
-x_n^{(n-2)} (\sigma_1^{-1} \ldots \sigma_{n-1}^{-1}) (\sigma_1^{-1} \ldots \sigma_{n-2}^{-1}) C_{n-1} \left( a^{\gamma_1} \land \ldots \land ^{\gamma_{n-1}} \right) + \ldots . 
\]

According to the Proposition 1 the first two terms in (55), together with the condition

\[
Q^2 \left| \text{terms linear in } \chi = 0 , 
\]

define BRST operator \( Q \) uniquely. So we need only to check that the formula (57) (for \( n = 1, 2 \)) is compatible with \( b \circ i = i \circ Q \) and eqs. (46), (48), (54). It can be done directly. Therefore \( Q^2 = 0 \) and \( i \) is a chain map. 

The fact that the map \( i \) (54) is the chain map demonstrates the relation between the standard complex for the q-Lie algebra (with our BRST operator) and the subcomplex \( (C_n(U, \land L), b) \) of the bar complex \( (C_n(U), b) \).

**Remark.** One can show that

\[
W_{n+1} = P_{1-n+1} (f^{(R)}_{1-n+1} - 1) R_1 \ldots R_n P^{1-n,0} .
\]

The relation (51) can be obtained from the identity

\[
f^{(R)}_{1-m+1} (f^{(R)}_{1-m} R_1 \ldots R_{m-1}) R_m = (f^{(R)}_{1-m+1} R_1 \ldots R_m) f^{(R)}_{1-m}
\]

by action of \( P^{1-m,0} \) and \( P_{1-m+1} \) from the right and left. The identity (59) is equivalent to

\[
f^{(R)}_{1-m+1} f^{(R)}_{1-m} = f^{(R)}_{1-m+1} f^{(R)}_{2-m+1}
\]

which is nothing but the associativity condition

\[
T^{(1)}_{m+1} T^{(m-1)}_m = T^{(m+1)}_m T^{(1)}_{m+1} T^{-1}_1
\]

considered in the Remark in Section 3. To prove all these statements we need the following relations

\[
(f^{(R)}_{1-n+1} - 1) R_1 \ldots R_n P^{1-n,0} \\
= P^{1-n+1} W_{n+1} + \sum_{k=1}^{n} (-1)^k P^{1-k} (1 - P^{k+1}) P^{k+2-n+1} \delta_{k+2} \ldots \delta_{n+1}
\]
and
\[ f_{\frac{m}{m}}^{(R)} P^{1-m} = P^{1-m} f_{\frac{m}{m}}^{(\sigma)}. \]

### 6 A generalization

#### 6.1
We consider the similarity transformation of the \( R \)-matrix
\[ R_{12} \to R'_{12} = D_1 D_2 R_{12} D_1^{-1} D_2^{-1} \quad (60) \]
which corresponds to the linear transformation of the generators \( L_B \to D_B^C L_C \). The new matrix \( (60) \) satisfies Yang-Baxter equation \((6)\) for all numerical matrices \( D \). If the matrix \( D \) has the special form
\[ D^0_0 = 1, \quad D^0_i = D^i_0 = 0, \quad D^i_j = D^j_i, \]
then the new \( R \) matrix \((60)\) has the same form \((1)\) with transformed structure constants
\[ \sigma_{12} \to \sigma'_{12} = D_1 D_2 \sigma_{12} D_1^{-1} D_2^{-1}, \quad C_{|12|}^{(2)} \to C'_{|12|}^{(2)} = D_1 D_2 C_{|12|}^{(2)} D_1^{-1} D_2^{-1}. \quad (61) \]
This transformation leaves invariant the linear space (denoted by \( \mathcal{L} \)) spanned by the \( N \) elements \( \chi_i \). The q-Lie algebra is quadratic for any choice of the basis in \( \mathcal{L} \) but the structure constants in \((5)\) change according to \((61)\).

The interesting special case is when the matrix \( R \) is conserved under transformation \((60)\) \( R_{12} = R'_{12} \). Then the \( N \times N \) matrix \( D^i_j \) is such that
\[ D_1 D_2 \sigma_{12} = \sigma_{12} D_1 D_2, \quad D_1 D_2 C_{|12|}^{(2)} = C_{|12|}^{(2)} D_2. \quad (62) \]

**Lemma.** In this case the \( R \)-matrix \( D_1 D_2 D_1^{-1} D_2^{-1} \) satisfies Yang-Baxter equation.

The proof is straightforward. Now one can consider a quadratic algebra with defining relations
\[ (D_1 R_{12} D_1^{-1}) \ L'_1 L'_2 = L'_{12} \]
In the components, for the quantum vector \( L'_A = \{ \xi_0, \xi_i \} \), we obtain
\[ \xi_i \xi_0 = D^i_0 \xi_0 \xi_j \quad \text{and} \quad (1 - D_1 \sigma_{12} D_1^{-1}) \xi_{12} = D_1 C_{12}^{(1)} \xi_0 \xi_1. \quad (63) \]
This algebra is related to the q-Lie algebra (5) by a transformation

\[ \chi_i = \xi_0^{-1}\xi_i. \quad (64) \]

6.2. There is a generalization of the exterior algebra (11)–(13) and (17) arising from the considerations just above. One can introduce a grading operator \( g \) with commutation relations

\[ g \gamma_1 = D_1 \gamma_1 g, \quad g \Omega^{(1)} = \Omega^{(1)} D_1^{-1} g, \quad g \chi_1 = D_1 \chi_1 g, \quad (65) \]

where \( D^i_j \) is the matrix which satisfies (62). Note that the grading operator \( g \) can be included into play if we consider the quadratic algebra (63) and take \( g = \xi_0^{-1} \).

Relations (65) are consistent with relations (5), (11)–(13) and (17). Then the Woronowicz differential algebra (11)–(13), (17) can be rewritten for the new generators

\[ \tilde{\Omega}^{(1)} = \Omega^{(1)} g^{-1}, \quad \tilde{\gamma}_1 = g \gamma_1 \]

in the form:

\[ \tilde{\gamma}_{(2)} \tilde{\Omega}^{(2)} = -\tilde{\Omega}^{(1)} D_1^{-1} \sigma_{12}^{-1} D_1 \tilde{\gamma}_{(1)} + I_2, \quad (66) \]

\[ \tilde{\Omega}^{(1)} \wedge \tilde{\Omega}^{(2)} \wedge \ldots \wedge \tilde{\Omega}^{(n)} = \tilde{\Omega}^{(1)} \otimes \tilde{\Omega}^{(2)} \otimes \ldots \otimes \tilde{\Omega}^{(n)} A^{(D)}_{1 \to n}, \quad (67) \]

\[ \tilde{\gamma}_{(1)} \wedge \tilde{\gamma}_{(2)} \ldots \wedge \tilde{\gamma}_{(n)} = A^{(D)}_{1 \to n} \tilde{\gamma}_{(1)} \otimes \tilde{\gamma}_{(2)} \ldots \otimes \tilde{\gamma}_{(n)}, \quad (68) \]

\[ \chi_{(2)} \tilde{\Omega}^{(2)} = \tilde{\Omega}^{(1)} (\sigma_{12} D_1 \chi_1 + C^{(2)}_{12}), \quad \tilde{\gamma}_{(1)} \chi_{(2)} = \sigma_{12} D_1 \chi_1 \tilde{\gamma}_{(2)} + C^{(1)}_{12} \tilde{\gamma}_{(1)}, \quad (69) \]

where \( A^{(D)}_{1 \to n} = D_1^{-1} \ldots D_{n-1}^{-1} A_{1 \to n} D_1 \cdots D_{n-1} \).

This generalized algebra embeds into the bar complex; the embedding is similar to the one in the particular case \( D = 1 \) and we don’t give details.

6.3. The quantum Lie algebras defined by eqs (5), (7)–(10) generalize the usual Lie (super)algebras. Indeed, in the non-deformed case, the braid matrix \( \sigma_{ij} = (-1)^{(m)(k)} \delta_j^m \delta_i^k \) is a super-permutation matrix (here \( m = 0, 1 \) is the parity of a generator \( \chi_m \)). Eqn. (7) is fulfilled (and we have additionally \( \sigma^2 = 1 \)). Eqn. (8) coincides with the Jacobi identity for Lie (super)-algebras. Eqn. (9) is then equivalent to the \( Z_2 \)-homogeneity condition \( C_{jk}^i = 0 \) for \( i \neq (j) + (k) \). Eqn. (10) follows from (9) in this case.
In this sense the exterior algebras (11) – (13), (66) – (68) generalize the Heisenberg algebras of the fermionic (and bosonic) ghosts and anti-ghosts. The grading matrix \( D \) now is the matrix \( D^m_n = (-1)^{i(n)} \delta^m_n \) which defines the well known automorphism of superalgebras, and we have

\[
(D_1 \sigma_{12} D_1^{-1})_{ij}^{mk} = -(-1)^{(m+1)(k+1)} \delta^m_j \delta^k_i.
\]

7 Conclusion

We have investigated the special class of quadratic algebras \( U \), the quantum Lie algebras (5). We have considered the exterior extension of \( U \) by the ghost algebra (11), (12), (13), (17) and constructed the BRST operator \( Q \). Using the BRST operator (24), (25), (32) one can build analogues of the standard and de Rham complexes for quantum Lie algebras. As we have shown in this paper, there is a map \( i \) from the standard complex to the subcomplex in the bar complex of \( U \). The map \( i \) induces (from the bar differential \( b \)) a differential on the standard complex. We compared this induced differential with the operator \( Q \). The operator \( Q \) is defined uniquely by two properties: the initial terms (55) and the condition (58). We verified that the induced differential satisfies these two properties. Therefore it coincides with \( Q \). Moreover it follows that \( Q^2 = 0 \). For the construction of the map \( i \) we need to check a number of complicated identities on the structure constants \( \sigma_{ij}^{kl} \) and \( C^{ij}_k \). The similar identities have appeared in the inductive definition of the BRST operator for q-Lie algebras. An elegant proof of some of these identities is based on combining the constants \( \sigma_{ij}^{kl} \) and \( C^{ij}_k \) into a bigger matrix \( R_{AB}^{CD} \) which realizes an \( R \)-matrix representation of the braid group \( B_\bullet \). Then we use the properties of the Jucys - Murphy elements in \( B_\bullet \), the quantum shuffle product and several important identities in the braid group algebra.

We presented a generalization of the constructions above to the situation when a q-Lie algebra is equipped with a grading operator.

Some results from Section 5 can be considered as an explicit realization of certain facts from the paper [3] for the special choice of the inhomogeneous quadratic algebras.

Note that the particular example of our construction of the BRST operator for the case of the quantum Lie algebra \( U_q(gl(N)) \) has been already considered in detail in our paper [14]. In this case we have constructed also
the quantum analogues of the anti-BRST operator and the quantum Laplace operator.

Also, we hope that our construction of the BRST operator for the quantum Lie algebras will be useful for constructions of BRST operators for any quadratic algebras (even for infinite dimensional algebras, such as $W_3$) and for producing the quantum $W$-algebras with the help of the quantum affine algebras (see [15] and references therein) by means of the quantum analogue of the Hamiltonian reduction procedure via the quantum BRST technique.

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References

[1] S. L. Woronowicz, “Differential calculus on compact matrix pseudogroups (quantum groups)”, Comm. Math. Phys. 122 (1989) 125.

[2] B. L. Feigin and B. L. Tsygan, ”Cyclic homology of algebras with quadratic relations, universal enveloping algebras and group algebras”, In “K-Theory, Arithmetic and Geometry”, ed. Yu. I. Manin (1987) LNM 1289, p. 210;

P. Feng and B. L. Tsygan, ”Hochschild and cyclic homology of quantum groups”, Comm. Math. Phys. 140 (1991) 481.

[3] A. Braverman and D. Gaitsgory, ”The Poincare-Birkhoff-Witt theorem for quadratic algebras of Koszul type”, J. of Algebra 181 (1996) 315; hep-th/9411113.

[4] A. Polishchuk and L. Positselski, ”Quadratic algebras”, Preprint (1996).
[5] A. P. Isaev and O. Ogievetsky, "BRST operator for quantum Lie algebras and differential calculus on quantum groups", *Theor. Math. Phys.* **129** No.2 (2001) 1558; math.QA/0106206.

[6] C. Burdik, A. P. Isaev and O. Ogievetsky, "Standard complex for quantum Lie algebras", *Phys. Atomic Nuclei*, **64** No. 12 (2001) 2101; math.QA/0010060.

[7] M. Henneaux and C. Teitelboim, "Quantization of gauge systems", Univ. Press, Princeton (1992).

[8] H. Cartan and S. Eilenberg, "Homological algebra", Princeton University Press (1956).

[9] D. Bernard, "A remark on quasi-triangular quantum Lie algebra", *Phys. Lett. B* **260** (1991) 389.

[10] M. Rosso, "Quantum groups and quantum shuffles", *Invent. Math.* **133** (1998) 133.

[11] D. B. Zagier, "Realizability of a model in infinite statistics", *Comm. Math. Phys.* **147** (1992) 199.

[12] A. P. Isaev and O. Ogievetsky, "BRST operator for quantum Lie algebras: explicit formula", *Int. J. Mod. Phys.* A Vol. 19, Supplement (2004) 240.

[13] M. Graña, "On Nichols algebras of low dimension", *Contemp. Math.* **267** (2000) 111.

[14] A. P. Isaev and O. Ogievetsky, "BRST and anti-BRST operators for quantum linear algebra $U_q(gl(N))$", *Nucl. Phys.* B (Proc. Suppl.) **102** (2001) 306.

[15] A. Sevostyanov, "Drinfeld-Sokolov reduction for quantum groups and deformations of $W$-algebras", Selecta Math., New Series **8** (2002) 637; math.QA/0107215.