Effects of a minimal length on the thermal properties of a Dirac oscillator

Abdelmalek Boumali
Laboratoire de Physique Appliquée et Théorique,
Université de Tébessa, 12000, W. Tébessa, Algeria.

Lyazid Chetouani
Département de Physique, Université de Constantine, W. Constantine, 25000, Algeria.

Hassan Hassanabadi
Department of Physics, University of Shahrood, Shahrood Iran.

(Dated: December 15, 2015)

Abstract

The effect of the minimal length on the thermal properties of a Dirac oscillator is considered. The canonical partition function is well determined by using the method based on the Epstein Zeta function. Through this function, all thermodynamics properties, such as the free energy, the total energy, the entropy, and the specific heat, have been determined. Moreover, this study leads to a minimal length in the interval $10^{-16} < \Delta x < 10^{-14} \, \text{m}$ with the following physically acceptable condition $\beta > \beta_0 = \frac{1}{m_0 c}$. We show that this condition is obtained directly through the properties of the Epstein Zeta function, and the minimal length $\Delta x$ coincide with the reduced Compton wavelength $\lambda = \frac{h}{m_0 c}$.

∗boumali.abdelmalek@gmail.com
†lyazidchetouani@gmail.com
‡h.hasanabadi@shahroodut.ac.ir
I. INTRODUCTION

The Dirac relativistic oscillator is an important potential both for theory and application. It was for the first time studied by Ito et al. They considered a Dirac equation in which the momentum $\vec{p}$ is replaced by $\vec{p} - im\beta \omega \vec{r}$, with $\vec{r}$ being the position vector, $m$ the mass of particle, and $\omega$ the frequency of the oscillator. The interest in the problem was revived by Moshinsky and Szczepaniak, who gave it the name of Dirac oscillator (DO) because, in the non-relativistic limit, it becomes a harmonic oscillator with a very strong spin-orbit coupling term. Physically, it can be shown that the (DO) interaction is a physical system, which can be interpreted as the interaction of the anomalous magnetic moment with a linear electric field. The electromagnetic potential associated with the DO has been found by Benitez et al. The Dirac oscillator has attracted a lot of interest both because it provides one of the examples of the Dirac’s equation exact solvability and because of its numerous physical applications (see and references therein). Recently, Franco-Villafane et al. exposed the proposal of the first experimental microwave realization of the one-dimensional DO. Quimbay et al. show that the Dirac oscillator can describe a naturally occurring physical system. Specifically, the case of a two-dimensional Dirac oscillator can be used to describe the dynamics of the charge carriers in graphene, and hence its electronic properties. This idea has been also proved in the calculations of the thermal properties of graphene using Zeta function.

The unification between the general theory of relativity and the quantum mechanics is one of the most important problems in theoretical physics. This unification predicts the existence of a minimal measurable length on the order of the Planck length. All approaches of quantum gravity show the idea that near the Planck scale, the standard Heisenberg uncertainty principle should be reformulated. The minimal length uncertainty relation has appeared in the context of the string theory, where it is a consequence of the fact that the string cannot probe distances smaller than the string scale $h \sqrt{\beta}$, where $\beta$ is a small positive parameter called the deformation parameter. This minimal length can be introduced as an additional uncertainty in position measurement, so that the usual canonical commutation relation between position and momentum operators becomes $[\hat{x}, \hat{p}] = i\hbar (1 + \beta p^2)$. This commutation relation leads to the standard Heisenberg uncertainty relation $\Delta \hat{x} \Delta \hat{p} \geq i\hbar (1 + \beta (\Delta p)^2)$, which clearly implies the existence of a non-zero minimal length $\Delta x_{\text{min}} = \hbar \sqrt{\beta}$. This mod-
The principal aim of this paper is to study the effect of the presence of a nonzero minimal length on the thermal properties of the Dirac oscillator in one and two dimensions. For this, we use the formalism based on the Epstein Zeta function to calculate the canonical partition function in both cases. We expect that the introduction of a minimal length have important consequences on these properties.

This paper is organized as follows: in sec. II, we propose a method based on Epstein Zeta function to calculate the canonical partition function of the Dirac oscillator in one and two dimensions. Sec. III is devoted to present the different results concerning the thermodynamics quantities of this oscillator. Finally, sec. VI will be a conclusion.
II. ZETA THERMAL PARTITION FUNCTION OF A DIRAC OSCILLATOR IN ONE AND TWO DIMENSIONS

A. Framework theory

The two-dimensional zeta Epstein function \( Z \) is defined for \( \Re s > 1 \), by

\[
Z(s) = \sum_{n,m=-\infty}^{\infty} \frac{1}{(am^2 + bmn + cn^2)^s},
\]

(1)

where \( a, b, c \) are real numbers with \( a > 0 \) and \( D = b^2 - 4ac \). By defining that \( D = 4ac - b^2 > 0 \), then the following quantity

\[
Q(m,n) = am^2 + bmn + cn^2,
\]

(2)

is a positive-definite binary quadratic form of discriminant \( D \). In this case, we have

\[
Z(s) = \sum_{n,m=-\infty}^{\infty} \frac{1}{Q(m,n)^s}.
\]

(3)

Now, by using the following substitutions

\[
x = \frac{b}{2a}, \quad y = \frac{\sqrt{D}}{2a}, \quad \tau = x + iy,
\]

(4)

Eq. (3)

\[
Z(s) = \sum_{n,m=-\infty}^{\infty} \frac{1}{a^s |m + n\tau|^{2s}}.
\]

(5)

By following the procedure used in [31], the final form of two-dimensional Epstein zeta function is

\[
Z(s) = 2a^{-s}\zeta(2s) + 2a^{-s}y^{1-2s}\sqrt{\pi}\frac{\zeta(2s - 1)\Gamma(s - \frac{1}{2})}{\Gamma(s)} + \frac{2a^{-s}y^{1-2s}\pi^s}{\Gamma(s)}H(s),
\]

(6)

with [31]

\[
H(s) = 4\sum_{k=1}^{\infty} \sigma_{1-2s}(k) k^{s-\frac{1}{2}} \cos (2k\pi x) K_{s-\frac{1}{2}}(2k\pi y),
\]

(7)

where \( \sigma_{\nu}(k) \) denotes the sum of the \( \nu \)-th powers of the divisors of \( k \), that is,

\[
\sigma_{\nu}(k) = \sum_{d|k} d^{\nu} = \sum_{d|k} \left( \frac{k}{d} \right)^{\nu}.
\]

(8)
B. The zeta thermal function

We start with the following eigenvalues of a one-dimensional Dirac oscillator in the presence of minimal length $\beta^{[21]}$

$$\epsilon_n = m_0c^2\sqrt{1 + 2\frac{\hbar\omega}{m_0c^2}n + \beta\frac{\hbar^2\omega^2}{c^2}n^2}. \quad (9)$$

With the substitutions

$$b = 2r, a = r^2\frac{\beta}{\beta_0}, \quad \left(r = \frac{\hbar\omega}{m_0c^2}, \beta_0 = \frac{1}{m_0^2c^2}\right), \quad (10)$$

we get

$$\epsilon_n = m_0c^2\sqrt{an^2 + bn + 1}. \quad (11)$$

In what follow, we choose $r = 1$. Given the energy spectrum, we can define the partition function via

$$Z_1 = \sum_n e^{-\tilde{\beta}\epsilon_n}, \quad (12)$$

where $\tilde{\beta} = \frac{1}{k_BT}$ with $k_B$ is the Boltzmann constant. In our case, $Z$ reads

$$Z_1 = \sum_n e^{-\frac{1}{\tau}\sqrt{an^2 + bn + 1}}. \quad (13)$$

with $\tau = \frac{k_BT}{m_0c^2}$. Now, we put that

$$\chi = \frac{1}{\tau}\sqrt{an^2 + bn + 1}, \quad (14)$$

and by using the following relation $[35]$

$$e^{-\chi} = \frac{1}{2\pi i} \int_C ds \chi^{-s} \Gamma(s), \quad (15)$$

the sum is transformed into

$$\sum_n e^{-\frac{1}{\tau}\sqrt{an^2 + bn + 1}} = \frac{1}{2\pi i} \int_C ds \left(\frac{1}{\tau}\right)^{-s} \sum_n \{an^2 + bn + 1\}^{-\frac{s}{2}} \Gamma(s) = \frac{1}{2\pi i} \int_C ds \left(\frac{1}{\tau}\right)^{-s} \mathcal{Z}(s) \Gamma(s), \quad (16)$$

and $\Gamma(s)$ and $\mathcal{Z}(s)$ are respectively the Euler and one-dimensional Epstein zeta function $[31]$, where

$$\mathcal{Z}(s) = \sum_n \frac{1}{Q(1,n)^s}. \quad (17)$$
with
\[ Q(1, n) = an^2 + bn + 1. \] (18)

Setting that
\[ x = \frac{b}{2}, \quad y = \sqrt{\frac{D}{2}}, \text{ with } D = 4a - b^2 > 0, \] (19)
we find the restriction on the deformation parameter \( \beta \)
\[ \beta > \beta_0 = \frac{1}{m_0^2 c^2}, \] (20)
and consequently, (17) is transformed into
\[ Z(s) = 2a^{-\frac{s}{2}} \zeta(s) + \frac{2a^{-\frac{s}{2}} y^{1-s}\sqrt{\pi}}{\Gamma\left(\frac{s}{2}\right)} \zeta(s - 1) \Gamma\left(\frac{s}{2} - \frac{1}{2}\right) + \frac{2a^{-\frac{s}{2}} y^{\frac{s}{2} - \frac{\pi}{2}}}{\Gamma\left(\frac{s}{2}\right)} H\left(\frac{s}{2}\right). \] (21)

Now, the final partition function is
\[ Z_{1D} = \frac{1}{2\pi i} \int_C ds \left(\frac{1}{\tau}\right)^{-s} Z(s) \Gamma(s), \] (22)
or
\[ Z_{1D} = \frac{1}{2\pi i} \int_C ds \left(\frac{1}{\tau}\right)^{-s} 2a^{-\frac{s}{2}} \zeta(s) \Gamma(s) + \frac{1}{2\pi i} \int_C ds \left(\frac{1}{\tau}\right)^{-s} 2a^{-\frac{s}{2}} y^{1-s}\sqrt{\pi} \zeta(s - 1) \Gamma\left(\frac{s}{2} - \frac{1}{2}\right) \Gamma(s). \]
\[ + \frac{1}{2\pi i} \int_C ds \left(\frac{1}{\tau}\right)^{-s} 2a^{-\frac{s}{2}} y^{\frac{s}{2} - \frac{\pi}{2}} \Gamma\left(\frac{s}{2}\right) H\left(\frac{s}{2}\right) \Gamma(s) \] (23)

The first integral has two poles in \( s = 0 \) and \( s = 1 \), the second has three poles in \( s = 0, s = 1 \) and \( s = 2 \), and finally the third has a pole at \( s = 0 \). By Applying the residues theorem, we get
\[ Z_{1D} = 2\zeta(0) + \frac{2}{\sqrt{a}} \{ \zeta(1) + \zeta(0) \} \tau + \frac{2\pi}{ay} \tau^2. \] (24)

The last integral goes to the zero because of the following relations
\[ \frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}} \right\}, \] (25)
with where \( \gamma \) is Euler ‘s constant given by
\[ \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log(n) \right), \] (26)
Elizalde \[31, 32\] also mentioned that this formula is very useful and its practical application quite simple: in fact, the two first terms are just nice, while the last one is quickly convergent and thus absolutely harmless in practice.
Thus, the final partition function for the one-dimensional Dirac oscillator becomes

$$Z_{1D}(\tau, \alpha) = \frac{2\pi}{\alpha \sqrt{\alpha - 1}} \tau^2 + \frac{1}{\sqrt{\alpha}} \tau - 1,$$

(27)

with $\alpha = \frac{\beta}{\beta_0}$, so $a = \alpha$ and $y = \sqrt{\alpha - 1}$. We note here that the case of a two-dimensional can be treated in the same way as that used in one dimension: starting with the following form of the spectrum of energy (see [29])

$$\bar{\epsilon}_n = m_0 c^2 \sqrt{1 + 4 \frac{\hbar \omega}{m_0 c^2} n + 4 \beta \frac{\hbar^2 \omega^2}{c^2} n^2},$$

(28)

and by the same procedure as described above, a final wanted partition function of a two-dimensional Dirac oscillator is

$$Z_{2D}(\tau, \alpha) = \frac{\pi}{4\alpha \sqrt{\alpha - 1}} \tau^2 + \frac{1}{2\sqrt{\alpha}} \tau - 1.$$

(29)

Finally, all thermal properties for both cases can be obtained by using the following relations

$$\mathcal{F} \equiv \frac{F}{mc^2} = -\tau \ln(Z),$$

$$U \equiv \frac{U}{mc^2} = \tau^2 \frac{\partial \ln(Z)}{\partial \tau},$$

$$S \equiv \frac{S}{k_B} = \ln(Z) + \tau \frac{\partial \ln(Z)}{\partial \tau},$$

$$C \equiv \frac{C}{k_B} = 2\tau \frac{\partial \ln(Z)}{\partial \tau} + \tau^2 \frac{\partial^2 \ln(Z)}{\partial \tau^2}.$$

(30)

(31)

III. NUMERICAL RESULTS AND DISCUSSIONS

Before presenting our results concerning the thermal quantities of one and two-dimensional Dirac oscillator, two remarks can be made: (i) in Table. I we show some values of $\beta_0$ together with the minimal length $\Delta x$ for some fermionic particles: . This parameter

| Symbol          | Mass $(\text{MeV}/c^2)$ | $\beta_0 = \frac{1}{m^2c^2} \left( \frac{x^2}{kg^2m^2} \right)$ | $\Delta x \approx h\sqrt{\beta_0} \text{ (m)}$ |
|-----------------|-------------------------|-------------------------------------------------|-----------------------------------------------|
| electron/positron | $e^-/e^+$             | 0.511                                           | $3.971566887 \times 10^{30}$                  |
| proton/anti-proton | $p/\bar{p}$          | 938.272                                         | $3.971566887 \times 10^{30}$                  |
| muon            | $\mu^-/\mu^+$        | 105.7                                           | $3.143705046 \times 10^{38}$                  |
| tauon           | $\tau^-/\tau^+$      | 1777                                            | $1.105704217 \times 10^{36}$                  |

Table I: Some values of both $\beta_0$ and minimal length $\Delta x = \sqrt{\beta_0}$.  

has been determinate through the properties of Epstein Zeta function, and this restriction
leads to the minimal length $\Delta x \simeq \hbar \sqrt{\beta_0}$. According to Table. I the minimal length lies in the interval $10^{-16} < \Delta x < 10^{-14} \text{m}$. In addition, we can see that $\Delta x \simeq \hbar \sqrt{\beta_0} = \frac{\hbar}{m_0 c} = \bar{\lambda}$, where $\bar{\lambda}$ is the reduced Compton wavelength: so the minimal length has the same order as the reduced Compton wavelength. (ii) in Figure. I we study, the effect of the presence of minimal length on the spectrum of energy. In this context, the reduced spectrum of energy as a function of the quantum number $n$ for different values of $\alpha$ are depicted in fig. I. This figure reveals that the effect deformation parameter $\beta$ on the energy spectrum is significant.

![Figure 1: Energy spectrum $\frac{\epsilon}{m_0 c^2}$ versus quantum number $n$ for different values of $\alpha = \frac{\beta}{\beta_0}$.](image)

Moreover, we note that we have only restrict ourselves to stationary states of positive energy. The reason for this is twofold [36]: (i) the Dirac oscillator possesses an exact Foldy–Wouthuysen transformation (FWT): so, the positive- and negative-energy solutions never mix. (ii) The solutions with infinite degeneracy do not correspond to physical states since there is not Lorentz finite representation for them. Thus, according to these arguments, we can assume that only particles with positive energy are available in order to determine the thermodynamic properties of our oscillator in question.

Now, we are ready the present our numerical results on the thermal properties of a Dirac oscillator in one and two dimensions: in Fig. 2 we show all thermal properties of the one dimensional Dirac oscillator for different values of $\alpha$. According to this figure, we can confirm that the parameter $\beta$ plays a significant role on these properties, and the effect of this parameter is very important on the thermodynamic properties. In particularity, the
curves of the reduced specific heat, for different values of $\beta$, tend to the an asymptotic limit at $2$, and they separated in the range of the reduce temperature $\tau$ between $0$ and $\sim 10$.

Figure 2: Thermal properties of a one dimensional Dirac oscillator for different values of $\alpha = \frac{\beta}{\beta_0}$.

For the case of a two-dimensional oscillator, and according to the Eqs. (9) and (28), we conclude that the method of determining the canonical partition function will be the same in both cases. As a consequence, all thermal properties can be found by the same manner as in the one-dimensional case. These properties are depicted in Fig. 3.
In this work, we have study the influence of the minimal length on the thermal properties of a Dirac oscillator in one and two dimensions. The statistical quantities of both cases were investigated by employing the Zeta Epstein function method. All this properties such as the free energy, the total energy, the entropy, and the specific heat, show the important effect of the presence of minimal length on the thermodynamics properties of a Dirac oscillator. Moreover, the formalism based on the properties of Zeta Epstein function allows us to calculate the values of minimal length $\Delta x = \hbar \sqrt{\beta}$ for some fermionic particles as shown in Table. These values coincide well with the reduced Compton wavelength $\bar{\lambda}$.

**IV. CONCLUSION**

In this work, we have study the influence of the minimal length on the thermal properties of a Dirac oscillator in one and two dimensions. The statistical quantities of both cases were investigated by employing the Zeta Epstein function method. All this properties such as the free energy, the total energy, the entropy, and the specific heat, show the important effect of the presence of minimal length on the thermodynamics properties of a Dirac oscillator. Moreover, the formalism based on the properties of Zeta Epstein function allows us to calculate the values of minimal length $\Delta x = \hbar \sqrt{\beta}$ for some fermionic particles as shown in Table. These values coincide well with the reduced Compton wavelength $\bar{\lambda}$.
ACKNOWLEDGMENTS

The authors wish to thank Prof Emilio Elizalde for his helpful comments and discussions about the Epstein Zeta Function.

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