Fuzzy W-closed sets

Talal Al-Hawary

Abstract: In this paper, we introduce the relatively new notion of fuzzy W-closed and fuzzy W-generalized closed sets. Several properties and connections to other well-known weak and strong fuzzy closed sets are discussed. Fuzzy W-generalized continuous and fuzzy W-generalized irresolute functions and their basic properties and relations to other fuzzy continuities are explored.

Subjects: Advanced Mathematics; Applied Mathematics; Foundations & Theorems

Keywords: Fuzzy W-open set; fuzzy W-closed set; fuzzy W-generalized closed; fuzzy W-generalized continuous function

AMS subject classifications: 54C08; 54H40

1. Introduction

Fuzzy topological spaces were introduced by Chakrabarty and Ahsanullah (1992) and Chang (1968). Let \((X, \mathcal{T})\) be a fuzzy topological space (simply, Fts). If \(\lambda\) is a fuzzy set (simply, F-set), then the closure of \(\lambda\) and the interior of \(\lambda\) will be denoted by \(\text{Cl}_\mathcal{T}(\lambda)\) and \(\text{Int}_\mathcal{T}(\lambda)\), respectively.

A F-set \(\lambda\) is called fuzzy open (simply, FO) if the image of every FO-set is FO. Clearly \(\lambda\) is a FC-set if and only if \(\lambda = \text{Cl}_\mathcal{T}(\lambda)\). A fuzzy set \(\lambda\) is called fuzzy generalized closed (simply, FGC) if the inverse image of every FC-set is FGC and \(\lambda\) is called fuzzy preopen (simply, FPO) if \(\lambda \subseteq \text{Int}_\mathcal{T}(\text{Cl}_\mathcal{T}(\lambda))\).

The collection of all FSO (resp., FSC and FGC) subsets of \(X\) will be denoted by \(\text{FSO}(X, \mathcal{T})\) (resp., \(\text{FSC}(X, \mathcal{T})\) and \(\text{FGC}(X, \mathcal{T})\)).

A fuzzy space is called an E.D. space if the closure of every FSO-set in it is FO. Equivalently, the interior of every FSC-set in it is SC. A fuzzy function \(f: (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})\) is called fuzzy continuous (simply, FCts) if the inverse image of every FC-set is FC.

The reader is referred to Al-Hawary (2008, 2017a, 2017b), Chakrabarty and Ahsanullah (1992), Chang (1968), Chaudhuri and Das (1993), Ekici (2007), Mahmoud and Fath (2004), Mursaleen et al. (2016), Nanda (1986), Pritha (2014) and Wong (1974).

PUBLIC INTEREST STATEMENT

We introduce the relatively new notion of fuzzy weak closed set and fuzzy weak generalized closed sets. Several properties and connections to other well-known weak and strong fuzzy closed sets are discussed. Fuzzy weak generalized continuous and fuzzy weak generalized irresolute functions and their basic properties and relations to other fuzzy continuities are explored.
In this paper, we introduce the relatively new notions of fuzzy $W$-closed sets, which is closely related to the class of fuzzy closed subsets. We investigate several characterizations of fuzzy $W$-open and fuzzy $W$-closed notions via the operations of interior and closure, for more on these notions for crisp sets see Al-Hawary (2004, 2007, 2013a, 2013b) and Al-Hawary and Al-Omari (2006, 2008, 2009). In Section 3, we introduce the notion of fuzzy $W$-generalized closed sets and study connections to other weak and strong forms of fuzzy generalized closed sets. Section 4 is devoted to introducing and studying fuzzy $W$-generalized continuous and fuzzy $W$-generalized irresolute functions and connections with other similar forms of fuzzy continuity.

2. Fuzzy $W$-closed sets

We begin this section by introducing the notions of fuzzy $W$-open and fuzzy $W$-closed subsets.

Definition 1 Let $\lambda$ be a fuzzy subset of a Fts space $(X, \mathfrak{I})$. The fuzzy $W$-interior of $\lambda$ is the union of all fuzzy open subsets of $X$ whose closures are contained in $\text{Cl}_W(\lambda)$, and is denoted by $\text{Int}_W(\lambda)$. $\lambda$ is called fuzzy $W$-open (simply, FWO) if $\lambda = \text{Int}_W(\lambda)$. The complement of a fuzzy FWO subset is called fuzzy fuzzy $W$-closed (simply, FWC). Alternatively, a fuzzy subset $\lambda$ of $X$ is fuzzy FWC if $\lambda = \text{Cl}_W(\lambda)$, where $\text{Cl}_W(\lambda) = \bigwedge_{\lambda \leq \lambda'} (\lambda' : \lambda' \leq \lambda, \lambda'$ is FC-set in $X$).

Clearly $\text{Int}_W(\lambda) \leq \text{Int}_W(\lambda) \leq \text{Cl}_W(\lambda)$ and $\lambda \leq \text{Cl}_W(\lambda) \leq \text{Cl}_W(\lambda)$ and hence every FWC-set is a FWO-set, but the converses need not be true.

Example 1 Let $X = \{a, b, c\}$ and $\mathfrak{I} = \{0, 1, \chi_{\{a\}}, \chi_{\{b\}}, \chi_{\{a\}}\}$. Then $\chi_{\{a\}}$ is FC-set, but the converses need not be true.

Even the intersection of two FWO subsets needs not be FWO.

Example 2 Let $X = \{a, b, c, d\}$ and $\mathfrak{I} = \{0, 1, \chi_{\{b\}}, \chi_{\{c\}}, \chi_{\{b,c\}}, \chi_{\{a\}}, \chi_{\{a,c\}}\}$. Then $\chi_{\{a\}}$ and $\chi_{\{a,c\}}$ are FWO-sets, but $\chi_{\{a\}} \land \chi_{\{a,c\}} = \chi_{\{a\}}$ is not a FWO-set.

Next, we show that arbitrary unions of FWO subsets are FWO.

**Theorem 1** If $(X, \mathfrak{I})$ is a fuzzy space, then arbitrary union of FWO subsets are FWO.

If $\{\lambda_{\alpha} : \alpha \in \Delta\}$ is a collection of FWO subsets of $X$, then for every $\alpha \in \Delta$, $\text{Int}_W(\lambda) = \lambda_{\alpha}$. Hence

$$\text{Int}_W(\bigvee_{\alpha \in \Delta} \lambda_{\alpha}) = \bigvee_{\alpha \in \Delta} \text{Cl}_W(\mu) \leq \bigvee_{\alpha \in \Delta} \text{Cl}(\lambda_{\alpha})$$

$$= \bigvee_{\alpha \in \Delta} \text{Cl}_W(\lambda_{\alpha}) = \bigvee_{\alpha \in \Delta} \text{Int}_W(\lambda_{\alpha})$$

$$= \bigvee_{\alpha \in \Delta} \lambda_{\alpha}.$$

Hence $\bigvee_{\alpha \in \Delta} \lambda_{\alpha}$ is FWO.

**Corollary 1** Arbitrary intersection of FWC subsets are FWC, while finite unions of FWC subsets need not be FWO.

In classical topology, the interior of a set is a subset of the set itself. But this is not the case for FWO-sets. Next we show that $\lambda \leq \text{Int}_W(\lambda)$ and $\text{Int}_W(\lambda) \leq \lambda$ need not be true.

Example 3 Consider the space in Example 1. Then $\chi_{\{c\}} \not\leq \text{Int}_W(\chi_{\{c\}}) = 0$ and $\text{Int}_W(\chi_{\{a,b\}}) = 1 \not\leq \chi_{\{a,b\}}$.

**Corollary 2** If $\lambda$ is a fuzzy $\epsilon$-dense subset of $X (\text{Cl}_W(\lambda) = 1)$, then $\text{Int}_W(\lambda) = 1$. 

---

*AL-Hawary, Cogent Mathematics (2017), 4: 1343518
https://doi.org/10.1080/23311835.2017.1343518*
**Lemma 1** The intersection of a FC-set with a FeMC-set is FC.

Let $\lambda$ be a FC-set and $\mu$ be a FWC-set. For all $\gamma \leq Cl_W(\lambda \land \mu)$, then for every FO-set $\eta$ such that $\gamma \leq \eta$, $\eta \land (\lambda \land \mu) \neq 0$. Hence $\eta \land \lambda \neq 0$ and $Cl(\eta) \land Cl(\mu) \neq 0$. Thus $\gamma \leq Cl(\lambda) \land Cl_W(\mu) = \lambda \land \mu$. Therefore, $\lambda \land \mu$ is FC.

**Corollary 2** If $\lambda$ is a FSC subset of an E.D. fuzzy space, then $Cl(\lambda) = Cl\{\lambda\}$.

We only need to show $Cl_W(\lambda, \lambda) \leq Cl(\lambda)$ when $\lambda$ is a FSC-set. For all $\mu \leq Cl_W(\lambda)$ and all $\eta$ FO-set such that $\mu \neq \eta$, we have $Cl(\eta) \land Cl(\lambda) \neq 0$. As $X$ is E.D., $Cl(\lambda) \land Cl(\lambda) = 0$. Hence $\exists \gamma \leq Cl(\eta)$ and $\gamma \leq Cl(\lambda)$ which is FO. Hence $\gamma \land Cl(\lambda) \neq 0$ and as $\lambda$ is FSC, $\eta \land \lambda \neq 0$. Therefore $Cl_W(\lambda, \lambda) \leq Cl(\lambda)$.

We remark that $X$ being an E.D. fuzzy space is necessary in Lemma 2.

**Example 4** Consider the space in 2. Then $Cl_W(\lambda) = \lambda$.

**Corollary 3** In an E.D. fuzzy space, a $\lambda$-set is $\lambda$-set if and only if it is FWO-set.

### 3. Fuzzy $W$-generalized closed sets

In this section, we introduce the notion of fuzzy $W$-generalized closed set. Moreover, several interesting properties and constructions of these fuzzy subsets are discussed.

**Definition 2** A fuzzy subset $\lambda$ of a space $X$ is called fuzzy $W$-generalized closed (simply, FGC) if whenever $\mu$ is a FO subset such that $\lambda \leq \mu$, we have $Cl_W(\lambda) \leq Cl(\lambda) = 0$. Hence $\lambda$ is fuzzy $W$-generalized-open (simply, FWO) if $1 - \lambda$ is FGC.

**Theorem 2** A fuzzy subset $\lambda$ of $(X, \mathcal{H})$ is FGC if and only if $\mu \leq \lambda$ whenever $\mu \leq \lambda$ and $\mu$ is FC-set in $(X, \mathcal{H})$.

Let $\lambda$ be a FWO-set and $\mu$ be a FC subset such that $\mu \leq \lambda$. Then $1 - \lambda \leq 1 - \mu$. As $1 - \lambda$ is FGC and as $1 - \mu$ is FO, $Cl_W(1 - \lambda) \leq 1 - \mu$. So $\mu \leq 1 - Cl_W(1 - \lambda) = Cl_W(\lambda)$.

Conversely if $1 - \lambda \leq \mu$ then the FC-set $1 - \mu \leq \lambda$. Thus $1 - \mu \leq Cl_W(1 - \lambda)$ and so $Cl_W(1 - \lambda) \leq \mu$.

Next we show the class of FGC-sets is properly placed between the classes of FC- and FWO-sets. Moreover, the class FGC-sets is properly placed between the classes of FC-sets and FWO-sets. A FC-set is trivially FGC and clearly every FGC-set is FC and every FWO-set is FGC as $Cl(\lambda) \leq Cl_W(\lambda)$ for every $\lambda$-set in space $X$. In Example 1, $\lambda = \chi_{[a,c]}$ is a FC-set that is not FWO. In Example 2, $\lambda = \chi_{[a,b,d]}$ is FGC, but it is not FeGC as $Cl_W(\chi_{[a,c]} \cup \chi_{[b,d]}) = 1$.

The following is an immediate result from Lemma 2:

**Theorem 3** If $\lambda$ is a FSC subset of a fuzzy E.D. space $X$, the following are equivalent:

1. $\lambda$ is a FWO-set;
2. $\lambda$ is a FGC-set.
Its clear that if $\lambda \leq \mu$, then $\text{Int}_W(\lambda) \leq \text{Int}_W(\mu)$ and $\text{Cl}_W(\lambda) \leq \text{Cl}_W(\mu)$.

**Lemma 3** If $\lambda$ and $\mu$ are $F$ subsets of a space $X$, then $\text{Cl}(\lambda \vee \mu) = \text{Cl}(\lambda) \vee \text{Cl}(\mu)$ and $\text{Cl}(\lambda \wedge \mu) \leq \text{Cl}(\lambda) \wedge \text{Cl}(\mu)$.

Since $\lambda$ and $\mu$ are $F$-subsets of $\lambda \vee \mu, \text{Cl}_W(\lambda \vee \mu), \text{Cl}_W(\lambda \wedge \mu)$.

**Corollary 5** Finite unions of $FWGC$-sets are $FWGC$.

While the finite intersections of $FWGC$-sets need not be $FWGC$.

**Example 6** Consider the space $X = \{a, b, c, d, e\}$ and $\mathfrak{F} = \{0, 1, \mathcal{X}_{\lambda a}, \mathcal{X}_{\lambda b}, \mathcal{X}_{\lambda c}\}$. Then $\lambda = \mathcal{X}_{\lambda a}$ and $\mu = \mathcal{X}_{\lambda c}$ are $F\text{FC}_G$ -sets as the only suppers fuzzy set of them is 1, but $\lambda \wedge \mu = \mathcal{X}_{\lambda c}$ is not a $F\text{FC}_G$-set.

**Theorem 4** The intersection of a $FWGC$-set with a $FWC$-set is $FWGC$.

**Proof** Let $\lambda$ be a $FWGC$-set and $\mu$ be a $FWC$-set. Let $\gamma$ be a $FO$-set such that $\lambda \wedge \mu \leq \gamma$. Then $\lambda \leq \gamma \vee (1 - \mu)$. Since $1 - \mu$ is $FWO$, by Corollary 3, $\gamma \vee (1 - \mu)$ is $FO$ and since $\lambda$ is $FWGC$, $\text{Cl}_W(\lambda \wedge \mu) \leq \text{Cl}_W(\lambda) \wedge \text{Cl}_W(\mu) = \text{Cl}_W(\lambda) \wedge \mu \leq (\gamma \vee (1 - \mu)) \wedge \mu = \gamma \wedge \mu \leq \gamma$.

4. **Fuzzy W-g-continuous and Fuzzy W-g-irresolute functions**

**Definition 3** A fuzzy function $f : (X, \mathfrak{F}) \to (Y, \mathfrak{F}')$ is called

1. fuzzy $W$-$g$-continuous (simply, $FWGC$-sets) if $f^{-1}(\lambda)$ is a $FWGC$-set in $(X, \mathfrak{F})$ for every $FC$-set $\lambda$ of $(Y, \mathfrak{F}')$,
2. fuzzy $W$-$g$-irresolute (simply, $FWGI$) if $f^{-1}(\lambda)$ is a $FWGC$ -set in $(X, \mathfrak{F})$ for every $FWGC$-set $\lambda$ of $(Y, \mathfrak{F}')$.

**Lemma 4** Let $f : (X, \mathfrak{F}) \to (Y, \mathfrak{F}')$ be $FWGC$. Then $f$ is $FGTS$.

Follows from the fact that every $FWGC$-set is $FGT$. The converse of the preceding Lemma needs not be true.

**Example 7** Consider the space $(X, \mathfrak{F})$ in Example 5 and the identity function $f : (X, \mathfrak{F}) \to (X, \mathfrak{F}')$ where $\mathfrak{F}' = \{0, 1, \mathcal{X}_{\lambda a}\}$. Since $f^{-1}(\mathcal{X}_{\lambda a}) = \mathcal{X}_{\lambda a} \neq \text{Cl}(\mathcal{X}_{\lambda a})$, $f$ is not $F\text{GTS}$, but $f$ is $F\text{CT}$ and hence $F\text{GTS}$.

Even the composition of $FWGC$ functions needs not be $FWGC$.

**Example 8** Let $f$ be the function in Example 6. Let $\mathfrak{F}' = \{0, 1, \mathcal{X}_{\lambda a}\}$. Let $g : (X, \mathfrak{F}) \to (X, \mathfrak{F}')$ be the identity function. It is easily observed that $g$ is also $F\text{GTS}$ as the only super set of $\mathcal{X}_{\lambda c}$ is 1. But the composition function $g \circ f$ is not $F\text{GTS}$ as $\mathcal{X}_{\lambda c}$ is a $FC$-set in $(X, \mathfrak{F}')$ and it is not a $F\text{GTS}$-set in $(X, \mathfrak{F})$.

We end this section by giving a necessary condition for a $FWGC$ function to be $FWGI$.

**Theorem 5** If $f : (X, \mathfrak{F}) \to (Y, \mathfrak{F}')$ is a bijective, $FO$- and $FWGC$-function, then $f$ is $FWGI$.

**Proof** Let $\lambda$ be a $FWGC$ subset of $Y$ and let $f^{-1}(\lambda) \leq \eta$, where $\eta \in \mathfrak{F}$. Clearly, $\lambda \leq f(\eta)$. Since $f(\eta) \in \mathfrak{F}'$ and since $\lambda$ is $FWGC$, $\text{Cl}_W(\lambda) \leq f(\eta)$ and thus $f^{-1}(\text{Cl}_W(\lambda)) \leq \eta$. Since $f$ is $FWGC$ functions and since $\text{Cl}_W(\lambda)$ is $FC$ in $Y$, $f^{-1}(\text{Cl}_W(\lambda))$ is $FWGC$. $f^{-1}(\text{Cl}_W(\lambda)) \leq \text{Cl}_W(f^{-1}(\text{Cl}_W(\lambda))) = f^{-1}(\text{Cl}_W(\lambda)) \leq \eta$. Therefore, $f^{-1}(\lambda)$ is $FWGC$ and hence, $f$ is $FWGI$. $\blacksquare$
5. Conclusion

Several characterizations of fuzzy $W$-open and fuzzy $W$-closed notions via the operations of interior and closure are explored and the notion of fuzzy $W$-generalized closed sets is studied. Finally, fuzzy $W$-generalized continuous and fuzzy $W$-generalized irresolute functions are discussed and connections with other similar forms of fuzzy continuity are established.

Funding
The authors received no direct funding for this research.

Author details
Talal AL-Hawary
E-mail: talalhawary@yahoo.com
ORCID ID: http://orcid.org/0000-0002-5167-5265
1 Department of Mathematics, Yarmouk University, Irbid, Jordan.

Citation information
Cite this article as: Fuzzy $W$-closed sets, Talal AL-Hawary, Cogent Mathematics (2017), 4: 1343518.

References
Al-Hawary, T. (2008). Fuzzy $\omega_0$-open sets. Bulletin of the Korean Mathematical Society, 45, 749–755.
Al-Hawary, T. (2013). $p$-closed sets. Acta Universitatis Apulensis, 35, 29–36.
Al-Hawary, T. (2013a). Fuzzy M-open sets. Theory and Applications of Mathematics & Computer Science, 7, 72–77.
Al-Hawary, T. (2017a). Fuzzy L-closed sets. Theory and Applications of Mathematics & Computer Science, 7, 72–77.
Al-Hawary, T. A. (2004). $\omega$-generalized closed sets. International Journal of Applied Mathematics, 16, 341–353.
Al-Hawary, T. A. (2007). Continuous mappings via regular open sets. Mutah Lil-Buhuth Wad-Dirasat, 2, 15–26.
Al-Hawary, T. A. (2013). $c$-open sets. Acta Scientiarum-Technology, 35, 111–115.
Al-Hawary, T. A., & Al-Ornari, A. (2006). Decompositions of continuity. Turkish Journal of Mathematics, 30, 187–195.
Al-Hawary, T. A., & Al-Ornari, A. (2008). Generalized b-closed sets. Mutah Lil-Buhuth Wad-Dirasat, 5, 27–39.
Al-Hawary, T. A., & Al-Ornari, A. (2009). $\theta$-generalized regular closed sets. Mutah Lil-Buhuth Wad-Dirasat, 2, 21–29.
Chakraborty, M. K., & Ashanullah, T. M. (1997). Fuzzy topology on fuzzy sets and tolerance topology. Fuzzy Sets and Systems, 45, 103–108.
Chang, C. L. (1968). Fuzzy topological spaces. Journal of Mathematical Analysis and Applications, 24, 182–190.
Chaudhuri, A. K., & Das, P. (1993). Some results on fuzzy topology on fuzzy sets. Fuzzy Sets and Systems, 56, 331–336.
Ekici, E. (2007). On the forms of continuity for fuzzy functions. Annals of University of Craiova Mathematics and Computer Science, 34, 58–65.
Mahmoud, F. S., & Foth, M. A. (2004). A generalization of continuity. Fuzzy Sets and Systems, 145, 27–39.
Mursaleen, M., Srivastava, H. M., & Sharma, S. K. (2015). Generalized statistically convergent sequences of fuzzy numbers. Journal of Intelligent & Fuzzy Systems, 30, 1511–1518.
Nandh, S. (1996). On Fuzzy topological spaces. Fuzzy Sets and Systems, 19, 193–197.
Pritha, M. (2014). Fuzzy generalized closed sets in fuzzy topological spaces. International Journal of Fuzzy Mathematics and Systems, 4, 299–304.
Wong, C. K. (1974). Fuzzy points and local properties of fuzzy topology. Journal of Mathematical Analysis and Applications, 46, 316–328.