ON THE C*-ALGEBRA GENERATED BY TOEPLITZ OPERATORS AND FOURIER MULTIPLIERS ON THE HARDY SPACE OF A LOCALLY COMPACT GROUP

UĞUR GÜL

Dedicated to Prof. Aydın Aytuna on the occasion of his 65th birthday

ABSTRACT. Let $G$ be a locally compact abelian Hausdorff topological group which is non-compact and whose Pontryagin dual $\Gamma$ is partially ordered. Let $\Gamma^+ \subseteq \Gamma$ be the semigroup of positive elements in $\Gamma$. The Hardy space $H^2(G)$ is the closed subspace of $L^2(G)$ consisting of functions whose Fourier transforms are supported on $\Gamma^+$. In this paper we consider the C*-algebra $C^*(T(G) \cup F(C(\Gamma^+)))$ generated by Toeplitz operators with continuous symbols on $G$ which vanish at infinity and Fourier multipliers with symbols which are continuous on one point compactification of $\Gamma^+$ on the Hilbert-Hardy space $H^2(G)$. We characterize the character space of this C*-algebra using a theorem of Power.

INTRODUCTION

For a locally compact abelian Hausdorff topological group $G$ whose Pontryagin dual $\Gamma$ is partially ordered, one can define the positive elements of $\Gamma$ as $\Gamma^+ = \{ \gamma \in \Gamma : \gamma \geq e \}$ where $e$ is the identity of the group $G$ and the Hardy space $H^2(G)$ as

$$H^2(G) = \{ f \in L^2(G) : \hat{f}(\gamma) = 0 \quad \forall \gamma \notin \Gamma^+ \}$$

where $\hat{f}$ is the Fourier transform of $f$. It is not difficult to see that $H^2(G)$ is a closed subspace of $L^2(G)$ and since $L^2(G)$ is a Hilbert space there is a unique orthogonal projection $P : L^2(G) \to H^2(G)$ onto $H^2(G)$.

This definition of the Hardy space $H^2(G)$ is motivated by Riesz theorem in the classical cases when $G = \mathbb{T}$ i.e when $G$ is the unit circle, which characterizes the Hardy class functions among $f \in L^2(\mathbb{T})$ as the space of functions whose negative Fourier coefficients vanish and by the Paley-Wiener theorem when $G = \mathbb{R}$, the real line since the group Fourier transform is the Fourier series when $G = \mathbb{T}$ and coincides with the Euclidean Fourier transform when $G = \mathbb{R}$.

One can extend the theory of Toeplitz operators to this setting by defining a Toeplitz operator with symbol $\phi \in L^\infty(G)$ as $T_\phi = PM_\phi$ where $M_\phi$ is the multiplication by $\phi$ and $P$ is the orthogonal projection of $L^2$ onto $H^2$. Such a definition was first considered by Coburn and Douglas in [2]. However the Toeplitz operators considered in [2] were more general since no partial order was assumed on the dual $\Gamma$ whereas the Hardy space was defined as the space of functions whose

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2000 Mathematics Subject Classification. 47B35.

Key words and phrases. C*-algebras, Toeplitz Operators, Hardy space of a locally compact group.

Date: 21/02/2014.
Fourier transforms are supported on a fixed sub-semigroup $\Gamma_0$ of $\Gamma$. The definition of Hardy space of groups whose duals are partially ordered and their Toeplitz operators were introduced and studied by Murphy in [7] and [8]. However in these papers [7] and [8], Murphy studies the case where $G$ is compact. In this paper we will study the case where $G$ is not compact. One very important assumption that we will make is that $\Gamma^+$ separates the points of $G$, i.e. for any $t_1, t_2 \in G$ satisfying $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$.

The Toeplitz C*-algebra of a locally compact group is defined as

$$T(G) = C^* \{ T_\phi : \phi \in C_0(G) \} \cup \{ I \}$$

where $C_0(G)$ is the space of continuous functions vanishing at infinity and $I$ is the identity operator. In the study of this Toeplitz C*-algebra, the most important notions are the commutator ideal $\text{com}(G) = I^*(\{ T_\phi T_\psi - T_\psi T_\phi : \phi, \psi \in C_0(G) \})$, the semi-commutator ideal $\text{scom}(G) = I^*(\{ T_\phi - T_\psi : \phi, \psi \in C_0(G) \})$ and the symbol map $\Sigma : C(\hat{G}) \to T(G)/\text{com}(G)$, $\Sigma(\phi) = [T_\phi]$ where $\hat{G}$ is the one point compactification of $G$ and $[T_\phi]$ denotes the equivalence class of $T_\phi$ modulo $\text{com}(G)$. It is not difficult to see that $\text{com}(G) \subseteq \text{scom}(G)$. We start by proving the following important result whose proof is adapted from [6]:

**Lemma 1.** Let $G$ be a locally compact abelian Hausdorff topological group whose Pontryagin dual $\Gamma$ is partially ordered and let $\Gamma^+$ be the semigroup of positive elements of $\Gamma$. Suppose that $\Gamma^+$ separates the points of $G$ i.e. for any $t_1, t_2 \in G$ with $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$. Let $\text{com}(G)$ and $\text{scom}(G)$ be the commutator and the semi-commutator ideal of the Toeplitz C*-algebra $T(G)$ respectively. Then

$$\text{com}(G) = \text{scom}(G)$$

It is shown in [2] and [7] that $\Sigma : C(\hat{G}) \to T(G)/\text{com}(G)$ is an isometry but is not a homomorphism since it may not preserve the multiplication. However $\Sigma : C(\hat{G}) \to T(G)/\text{scom}(G)$ is a homomorphism and combining this fact with Lemma 1 above we deduce that the symbol map $\Sigma : C(\hat{G}) \to T(G)/\text{com}(G)$ is an isometric isomorphism which means that

$$M(T(G)) = \hat{G}$$

where $M(A)$ is the character space of a C*-algebra $A$.

We introduce another class of operators acting on $H^2(G)$ which are called “Fourier multipliers”. These operators in the classical case $G = \mathbb{R}$ were introduced in [4]. The space of Fourier multipliers is defined as

$$F(C(\Gamma^+)) = \{ D_\theta = F^{-1} M_\theta F |_{H^2(G)} : \theta \in C(\Gamma^+) \}$$

where $F : L^2(G) \to L^2(\Gamma)$ is the Fourier transform. By Plancherel theorem it is not difficult to see that the image $F(H^2(G))$ of $H^2(G)$ under the Fourier transform is equal to $L^2(\Gamma^+)$. Again it is not difficult to see that $F(C(\Gamma^+))$ is isometrically isomorphic to $C(\Gamma^+)$. This means that

$$M(F(C(\Gamma^+))) = \Gamma^+$$

Lastly we consider the C*-algebra generated by $T(G)$ and $F(C(\Gamma^+))$ which we denote by $\Psi(C_0(G), C(\Gamma^+))$ i.e.

$$\Psi(C_0(G), C(\Gamma^+)) = C^*(T(G) \cup F(C(\Gamma^+)))$$
Using a Theorem of Power [9], [10] which characterizes the character space of the C*-algebra generated by two C*-algebras as a certain subset of the cartesian product of character spaces of these two C*-algebras, we prove following theorem:

**Main Theorem.** Let $G$ be a non-compact, locally compact abelian Hausdorff topological group whose Pontryagin dual $\Gamma$ is partially ordered and let $\Gamma^+$ be the semi-group of positive elements of $\Gamma$. Suppose that $\Gamma^+$ separates the points of $G$ i.e. for any $t_1, t_2 \in G$ with $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$. Let

$$\Psi(C_0(G), C(\Gamma^+)) = C^*(T(G) \cup F(C(\Gamma^+)))$$

be the C*-algebra generated by Toeplitz operators and Fourier multipliers on $H^2(G)$. Then for the character space $M(\Psi)$ of $\Psi(C_0(G), C(\Gamma^+))$ we have

$$M(\Psi) \cong (\hat{G} \times \{\infty\}) \cup (\{\infty\} \times \Gamma^+).$$

1. Preliminaries

In this section we fix the notation that we will use throughout and recall some preliminary facts that will be used in the sequel.

Let $S$ be a compact Hausdorff topological space. The space of all complex valued continuous functions on $S$ will be denoted by $C(S)$. For any $f \in C(S)$, $\|f\|_\infty$ will denote the sup-norm of $f$, i.e.

$$\|f\|_\infty = \sup\{|f(s)| : s \in S\}.$$  

If $S$ is a locally compact Hausdorff topological space, $C_0(S)$ will denote the space of continuous functions $f$ which vanish at infinity i.e. for any $\varepsilon > 0$ there is a compact subset $K \subset S$ such that $|f(x)| < \varepsilon$ for all $x \notin K$. For a Banach space $X$, $K(X)$ will denote the space of all compact operators on $X$ and $B(X)$ will denote the space of all bounded linear operators on $X$. The real line will be denoted by $\mathbb{R}$, the complex plane will be denoted by $\mathbb{C}$ and the unit circle group will be denoted by $\mathbb{T}$. The one point compactification of a locally compact Hausdorff topological space $S$ will be denoted by $\hat{S}$. For any subset $S \subset B(H)$, where $H$ is a Hilbert space, the C*-algebra generated by $S$ will be denoted by $C^*(S)$ and for any subset $S \subset A$ where $A$ is a C*-algebra, the closed two-sided ideal generated by $S$ will be denoted by $I^*(S)$.

For any $\phi \in L^\infty(G)$ where $G$ is a Borel space (a topological space with a regular measure on it), $M_\phi$ will be the multiplication operator on $L^2(G)$ defined as

$$M_\phi(f)(t) = \phi(t)f(t).$$

For convenience, we remind the reader of the rudiments of theory of Banach algebras, some basic abstract harmonic analysis and Toeplitz operators.

Let $A$ be a Banach algebra. Then its character space $M(A)$ is defined as

$$M(A) = \{x \in A^* : x(ab) = x(a)x(b) \quad \forall a, b \in A\}$$

where $A^*$ is the dual space of $A$. If $A$ has identity then $M(A)$ is a compact Hausdorff topological space with the weak* topology. When $A$ is commutative $M(A)$ is called the maximal ideal space of $A$. For a commutative Banach algebra $A$ the Gelfand transform $\Gamma : A \rightarrow C(M(A))$ is defined as

$$\Gamma(a)(x) = x(a).$$

If $A$ is a commutative C*-algebra with identity, then $\Gamma$ is an isometric *-isomorphism between $A$ and $C(M(A))$. If $A$ is a C*-algebra and $I$ is a two-sided closed ideal of $A$, then
then the quotient algebra $A/I$ is also a $C^\ast$-algebra (see [5]). For a Banach algebra $A$, we denote by $\text{com}(A)$ the closed ideal in $A$ generated by the commutators $\{a_1a_2 - a_2a_1 : a_1, a_2 \in A\}$. It is an algebraic fact that the quotient algebra $A/\text{com}(A)$ is a commutative Banach algebra. The reader can find detailed information about Banach and $C^\ast$-algebras in [11] and [5] related to what we have reviewed so far.

On a locally compact abelian Hausdorff topological group $G$ there is a unique (up to multiplication by a constant) translation invariant measure $\lambda$ on $G$ i.e. for any Borel subset $E \subset G$ and for any $x \in G$,

$$\lambda(xE) = \lambda(E)$$

where $xE = \{xy : y \in E\}$ is the translate of $E$ by $x$. This measure is called the Haar measure of $G$. Let $L^1(G)$ be the space of integrable functions with respect to this measure. Then $L^1(G)$ becomes a commutative Banach algebra with multiplication as the convolution defined as

$$(f \ast g)(t) = \int_G f(ts^{-1})g(s)d\lambda(s)$$

The Pontryagin dual $\Gamma$ of $G$ is defined to be the set of all continuous homomorphisms from $G$ to the circle group $T$:

$$\Gamma = \{ \gamma : G \to T : \gamma(st) = \gamma(s)\gamma(t) \text{ and } \gamma \text{ is continuous} \}$$

It is a well known fact that $\Gamma$ is in one to one correspondence with the maximal ideal space $M(L^1(G))$ of $L^1(G)$ via the Fourier transform:

$$<\gamma, f> = \hat{f}(\gamma) = \int_G \gamma(t)f(t)d\lambda(t)$$

When $\Gamma$ is topologized by the weak* topology coming from $M(L^1(G))$, $\Gamma$ becomes a locally compact abelian Hausdorff topological group with point-wise multiplication as the group operation:

$$(\gamma_1\gamma_2)(t) = \gamma_1(t)\gamma_2(t)$$

Let $\tilde{\lambda}$ be a fixed Haar measure on $\Gamma$. Plancherel theorem asserts that the Fourier transform $\mathcal{F}$ is an isometric isomorphism of $L^2(G)$ onto $L^2(\Gamma)$:

$$\mathcal{F}(f)(\gamma) = \hat{f}(\gamma) = \int_G \gamma(t)f(t)d\lambda(t)$$

with inverse $\mathcal{F}^{-1}$ defined as

$$\mathcal{F}^{-1}(f)(t) = \tilde{f}(t) = \int_\Gamma \gamma(t)f(\gamma)d\tilde{\lambda}(\gamma)$$

Here we note that $\tilde{\lambda}$ is normalized so that the above formula for the inverse Fourier transform holds. For detailed information on abstract harmonic analysis consult [12].

A partially ordered group $\Gamma$ is a group with partial order $\geq$ on it satisfying $\gamma_1 \geq \gamma_2$ implies $\gamma_1 \gamma \geq \gamma_2 \gamma \forall \gamma \in \Gamma$. This definition of the ordered group was given in [8]. Let $\Gamma^+ = \{ \gamma \in \Gamma : \gamma \geq e \}$ be the semi-group of positive elements of $\Gamma$ where $e$ is the unit of the group $\Gamma$. Let $G$ be a locally compact abelian Hausdorff topological group and let $\Gamma$ be the Pontryagin dual of $G$. Then the Hardy space $H^2(G)$ is defined as

$$H^2(G) = \{ f \in L^2(G) : \hat{f}(\gamma) = 0 \ \forall \gamma \notin \Gamma^+ \}$$
The Hardy space $H^2(G)$ is a closed subspace of $L^2(G)$ and since $L^2(G)$ is a Hilbert space, there is a unique orthogonal projection $P : L^2(G) \to H^2(G)$. For any $\phi \in L^\infty(G)$ the Toeplitz operator $T_\phi : H^2(G) \to H^2(G)$ is defined as

$$T_\phi = PM_\phi$$

Toeplitz operators satisfy the following algebraic properties:

- $T_{c\phi + \psi} = cT_\phi + T_\psi \quad \forall c \in \mathbb{C}, \quad \forall \phi, \psi \in C(\hat{G})$
- $T_\phi^* = T_\hat{\phi} \quad \forall \phi \in C(\hat{G})$

The proofs of these properties are the same as in the classical case where $G = \mathbb{T}$ (or $G = \mathbb{R}$) and can be found in [3].

The Toeplitz C*-algebra $\mathcal{T}(G)$ is defined to be the C*-algebra generated by continuous symbols on $G$:

$$\mathcal{T}(G) = C^*\{T_\phi : \phi \in C_0(G)\} \cup \{I\}$$

where $I$ is the identity operator and $C_0(G)$ is the space of continuous functions which vanish at infinity:

$$C_0(G) = \{f : G \to \mathbb{C} : f \text{ is continuous and } \forall \epsilon > 0 \quad \exists K \subset \subset G \mid |f(t)| < \epsilon \quad \forall t \not\in K\}$$

where $K \subset \subset G$ denotes a compact subset of $G$. Actually one has

$$\mathcal{T}(G) = C^*\{T_\phi : \phi \in C(\hat{G})\}$$

where $\hat{G}$ is the one-point compactification of $G$. In the case where $G$ is compact one has $G = \hat{G}$ and the most prototypical concrete example of this case is $G = \mathbb{T}$. This case was analyzed by Coburn in [4]. The famous result of Coburn asserts that for any $T \in \mathcal{T}(\mathbb{T})$ there are unique $K \in K(H^2(\mathbb{T}))$ and $\phi \in C(\mathbb{T})$ such that $T = T_\phi + K$. Hence the quotient algebra $\mathcal{T}(\mathbb{T})/K(H^2(\mathbb{T}))$ modulo the compact operators is isometrically isomorphic to $C(\mathbb{T})$. The two sided closed *-ideal $\text{com}(G)$ generated by the commutators is called the commutator ideal of $\mathcal{T}(G)$:

$$\text{com}(G) = I^*\{[T_\phi, T_\psi - T_\psi, T_\phi] : \phi, \psi \in C(\hat{G})\}$$

and the semi-commutator ideal $\text{scom}(G)$ is defined as

$$\text{scom}(G) = I^*\{[T_\phi, T_\psi - T_\psi, T_\phi] : \phi, \psi \in C(\hat{G})\}$$

The symbol map $\Sigma : C(\hat{G}) \to \mathcal{T}(G)/\text{com}(G)$ is defined as

$$\Sigma(\phi) = [T_\phi]$$

where $[,]$ denotes the equivalence class modulo $\text{com}(G)$. In [2] and [8] it is shown that $\Sigma$ is an isometry. The symbol map $\Sigma$ also preserves the * operation however is not a homomorphism i.e does not preserve multiplication. But if $\text{com}(G) = \text{scom}(G)$ then it is an isometric isomorphism. We will show under certain conditions that $\text{com}(G) = \text{scom}(G)$.

We introduce another class of operators which we call the “Fourier multipliers”. This class of operators in the case $G = \mathbb{R}$ was introduced in [4] and proved to be useful in calculating the essential spectra of a class of composition operators. The Fourier multiplier $D_\theta : H^2(G) \to H^2(G)$ with symbol $\theta \in C(\mathbb{T}^+)$ is defined as

$$D_\theta(f)(t) = (\mathcal{F}^{-1}M_\theta\mathcal{F}(f))(t)$$
The most prototypical example of a Fourier multiplier is a convolution operator with kernel \( k \in L^1(G) \):

\[
(T_k f)(t) = \int_G k(ts^{-1}) f(s) d\lambda(s)
\]

It is not difficult to see that actually \( T_k = D_k \) where \( \hat{k} \) denotes the Fourier transform of \( k \). The set of all Fourier multipliers \( F(C(\Gamma^+)) \) defined as

\[
F(C(\Gamma^+)) = \{ D_{\theta} : \theta \in C(\Gamma^+) \}
\]

is a commutative C*-algebra since the map \( D : C(\Gamma^+) \to F(C(\Gamma^+)) \) defined as \( D(\theta) = D_\theta \) is an isometric *-isomorphism.

Lastly we consider the C*-algebra generated by Toeplitz operators with continuous symbols and continuous Fourier multipliers. The main result of this paper is a characterization of the character space \( M(\Psi) \) of \( \Psi(C_0(G), C(\Gamma)) \). We know that

\[
M(F(C(\Gamma^+))) \cong \Gamma^+,
\]

under certain conditions we have \( \text{com}(G) = \text{com}(G) \) and this implies that

\[
M(T(G)) \cong \hat{G}.
\]

We will use the following theorem due to Power [9, 10] in identifying the character space of \( \Psi(C_0(G), C(\Gamma)) \):

**Power’s Theorem.** Let \( C_1, C_2 \) be C*-subalgebras of \( B(H) \) with identity, where \( H \) is a separable Hilbert space, such that \( M(C_i) \neq \emptyset \), where \( M(C_i) \) is the space of multiplicative linear functionals of \( C_i \), \( i = 1, 2 \) and let \( C \) be the C*-algebra that they generate. Then for the commutative C*-algebra \( \hat{C} = \text{com}(C) \) we have \( M(\hat{C}) = P(C_1, C_2) \subset M(C_1) \times M(C_2) \), where \( P(C_1, C_2) \) is defined to be the set of points \( (x_1, x_2) \in M(C_1) \times M(C_2) \) satisfying the condition:

Given \( 0 \leq a_1 \leq 1, 0 \leq a_2 \leq 1, a_1 \in C_1, a_2 \in C_2 \),

\[
x_i(a_i) = 1 \quad \text{with} \quad i = 1, 2 \quad \Rightarrow \quad \|a_1a_2\| = 1.
\]

The proof of this theorem can be found in [9]. Power’s theorem will give the character space \( M(\Psi) \) of \( \Psi(C_0(G), C(\Gamma)) \) as a certain subset of the cartesian product \( \hat{G} \times \Gamma^+ \).

### 2. The Character Space of \( \Psi(C_0(G), C(\Gamma)) \)

In this section we will concentrate on the C*-algebra \( \Psi(C_0(G), C(\Gamma)) \). But before that we will identify the character space \( M(T(G)) \) of \( T(G) \) under certain conditions. The condition that we will pose on \( G \) is that \( \Gamma^+ \) separate the points of \( G \) i.e. for any \( t_1, t_2 \in G \) with \( t_1 \neq t_2 \) there is \( \gamma \in \Gamma^+ \) such that \( \gamma(t_1) \neq \gamma(t_2) \). Under this condition we show that \( \text{com}(G) = \text{com}(G) \) and this implies that \( M(T(G)) \cong \hat{G} \). Hence we begin by proving the following lemma whose proof is adapted from the proof of Theorem 2.2 of [6]:
Proposition 2. Let $G$ be a locally compact abelian Hausdorff topological group whose Pontryagin dual $\Gamma$ is partially ordered and let $\Gamma^+$ be the semigroup of positive elements of $\Gamma$. Suppose that $\Gamma^+$ separates the points of $G$ i.e. for any $t_1, t_2 \in G$ with $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$. Let $\text{com}(G)$ and $\text{scom}(G)$ be the commutator and the semi-commutator ideal of the Toeplitz C*-algebra $\mathcal{T}(G)$ respectively. Then

\[ \text{com}(G) = \text{scom}(G) \]

Proof. It is trivial that $\text{com}(G) \subseteq \text{scom}(G)$ hence we need to show that $\text{scom}(G) \subseteq \text{com}(G)$:

Let $B = \{ \phi \in C_0(G) : T_\phi T_\psi - T_{\phi \psi} \in \text{com}(G) \}$ then $B$ is a self-adjoint subalgebra of $C_0(G)$: Let $\psi \in B$ then since $T_\phi T_\psi - T_{\phi \psi}$ we have

\[ (T_\phi T_\psi - T_{\phi \psi})^* = (T_{\phi \psi} - T_\phi T_\psi)^* \]

and hence $T_\phi T_\psi - T_{\phi \psi} \in \text{com}(G)$ and $T_{\phi \psi} - T_\phi T_\psi \in \text{com}(G)$.

Since $\psi \in B$ and $\text{com}(G)$ is an ideal we have $T_\psi (T_{\psi_1 \psi_2} - T_{\psi_1} T_{\psi_2}) \in \text{com}(G)$, $(T_{\phi_1 \psi_1} T_{\psi_2} - T_{\phi_1} T_{\psi_1})T_{\psi_2} \in \text{com}(G)$ and hence if $T_\phi T_{\psi_1 \psi_2} - T_{\phi \psi_1 \psi_2} \in \text{com}(G)$ for any $\phi \in C_0(G)$.

We have the following corollary of proposition 2:

Corollary 3. Let $G$ be a locally compact abelian Hausdorff topological group whose Pontryagin dual $\Gamma$ is partially ordered and let $\Gamma^+$ be the semigroup of positive elements of $\Gamma$. Suppose that $\Gamma^+$ separates the points of $G$ i.e. for any $t_1, t_2 \in G$ with $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$. Let $\mathcal{T}(G)$ be the Toeplitz C*-algebra with symbols in $C(\hat{G})$ acting on $H^2(G)$. Then we have

\[ M(\mathcal{T}(G)) \cong \hat{G} \]

Proof. The symbol map $\Sigma : C(\hat{G}) \to \mathcal{T}(G)/\text{com}(G)$, $\Sigma(\phi) = [T_\phi]$ is an isometry that preserves the *-operation and $\Sigma : C(\hat{G}) \to \mathcal{T}(G)/\text{scom}(G)$ is multiplicative.
Since $\text{com}(G) = s\text{com}(G)$, $\Sigma$ is an isometric isomorphism. Since characters kill the commutators we have
\[
M(T(G)) = M(T(G)/\text{com}(G)) \cong M(C(G)) = \hat{G}
\]

We will need the following small observation in proving our main theorem:

**Lemma 4.** Let $G$ be a locally compact, non-compact, abelian Hausdorff topological group and let $K_1, K_2 \subset G$ be two non-empty compact subsets of $G$. Then there is $t_0 \in G$ such that $K_1 \cap (t_0 K_2) = \emptyset$ where $t_0 K_2 = \{t_0 t : t \in K_2\}$.

**Proof.** Since $G$ is non-compact and locally compact there is a one-point compactification $\hat{G}$ of $G$. Hence there is a point at infinity $\infty \in \hat{G}$ such that $\infty \notin G$. Assume that the lemma does not hold i.e. there are two non-empty compact subsets $K_1, K_2 \subset G$ such that $K_1 \cap (t K_2) \neq \emptyset$ for all $t \in G$. Now take a net $\{t_\alpha\} \subset G$ such that $\lim_{\alpha} t_\alpha = \infty$. Since $\forall \alpha \in I$ we have $K_1 \cap (t_\alpha K_2) \neq \emptyset$, there are $x_\alpha \in K_1$ and $y_\alpha \in K_2$ such that $x_\alpha = t_\alpha y_\alpha$. Since $K_1$ and $K_2$ are compact there are $x_0 \in K_1$, $y_0 \in K_2$ and sub-nets $x_{\alpha_1} \in K_1$, $y_{\alpha_2} \in K_2$ such that $\lim x_{\alpha_1} = x_0$ and $\lim y_{\alpha_2} = y_0$. One can further find a common sub-net index set $I_0 \subset I$ such that $\lim_{\beta} x_\beta = x_0$ and $\lim_{\beta} y_\beta = y_0$. Since $x_\beta = t_\beta y_\beta$, $\lim_{\beta} t_\beta y_\beta = \infty$ and multiplication is continuous this implies that
\[
x_0 = \lim_{\beta} x_\beta = \lim_{\beta} t_\beta y_\beta = \infty
\]
but this contradicts to the fact that $x_0 \in K_1$. This contradiction proves the lemma.

Now we will show the following lemma which will shorten the proof of our main theorem. The proof of the following lemma is adapted from [13]:

**Lemma 5.** Let $G$ be a locally compact abelian Hausdorff topological group with Pontryagin dual $\Gamma$. Let $\phi \in C_0(G)$ and $\theta \in C_0(\Gamma)$ each have compact supports. Then $D_\phi M_\theta$ is a compact operator on $L^2(G)$ where $D_\theta = F^{-1} M_\theta F$.

**Proof.** Let $K_1 \subset G$ and $K_2 \subset \Gamma$ be compact supports of $\phi$ and $\theta$ respectively. Then for any $f \in L^2(G)$ we have
\[
(D_\phi M_\theta f)(t) = \int_{K_2} \gamma(t) \theta(\gamma) \left( \int_{K_1} \gamma(\tau) \phi(\tau) f(\tau \gamma) d\lambda(\tau) \right) d\hat{\lambda}(\gamma)
\]
\[
= \int_{K_1} \phi(\tau) \int_{K_2} \gamma(t \tau^{-1}) \theta(\gamma) d\hat{\lambda}(\gamma) f(\tau \gamma) d\lambda(\tau)
\]
where
\[
k(t, \tau) = \phi(\tau) \int_{K_2} \gamma(t \tau^{-1}) \theta(\gamma) d\hat{\lambda}(\gamma)
\]
Now consider
\[
\int_G \int_G |k(t, \tau)|^2 d\lambda(t) d\lambda(\tau) = \int_G \int_G |\phi(\tau) \int_{K_2} \gamma(t \tau^{-1}) \theta(\gamma) d\hat{\lambda}(\gamma)|^2 d\lambda(t) d\lambda(\tau)
\]
\[
\leq \| \phi \|^2 \int_{K_1} \int_G |\theta(t \tau^{-1})|^2 d\lambda(t) d\lambda(\tau) = \| \phi \|^2 \int_{K_1} \| \theta \|^2 d\lambda(\tau)
\]
\[
= \| \phi \|^2 \int_{K_1} \| \theta \|^2 d\lambda(\tau) = \| \phi \|^2 \| \theta \|^2 \lambda(K_1) < \infty
\]
This implies that $D_\theta M_\phi$ is Hilbert-Schmidt and hence compact. □

Now we are ready to prove our main theorem as follows:

**Main Theorem.** Let $G$ be a non-compact, locally compact abelian Hausdorff topological group whose Pontryagin dual $\Gamma$ is partially ordered and let $\Gamma^+$ be the semigroup of positive elements of $\Gamma$. Suppose that $\Gamma^+$ separates the points of $G$ i.e. for any $t_1, t_2 \in G$ with $t_1 \neq t_2$ there is $\gamma \in \Gamma^+$ such that $\gamma(t_1) \neq \gamma(t_2)$. Let

$$\Psi(C_0(G), C(\Gamma^+)) = C^*(T(G) \cup F(C(\Gamma^+)))$$

be the $C^*$-algebra generated by Toeplitz operators and Fourier multipliers on $H^2(G)$. Then for the character space $M(\Psi)$ of $\Psi(C_0(G), C(\Gamma^+))$ we have

$$M(\Psi) \cong (\hat{G} \times \{\infty\}) \cup (\{\infty\} \times \hat{\Gamma}^+)$$

**Proof.** We will use Power’s Theorem. In the setup of Power’s theorem $C_1 = T(G)$ and $C_2 = F(C(\Gamma^+))$. By corollary 3 we have $M(C_1) = \hat{G}$ and we have $M(C_2) = \hat{\Gamma}^+$. So we need to determine $(t, \gamma) \in \hat{G} \times \hat{\Gamma}^+$ satisfying for $0 \leq \phi, \theta \leq 1$, $\phi(t) = \theta(\gamma) = 1$ implies $\| T_{\theta} D_{\phi} \| = 1.$

Let $(t, \gamma) \in G \times \hat{\Gamma}^+$. Let $\phi \in C(\hat{G})$, $\theta \in C(\hat{\Gamma}^+)$ such that $0 \leq \phi, \theta \leq 1$ and $\phi(t) = \theta(\gamma) = 1$. Let us also assume that $\theta$ and $\phi$ have compact supports. Let $\theta \in C(\hat{G})$ such that $0 \leq \theta \leq 1$, $\theta$ has compact support and $\theta |_{C(\hat{\Gamma}^+)} = \theta$. Since

$$\| T_{\theta} D_{\phi} \| \leq \| M_\phi D_\theta \|_{L^2(G)}$$

it suffices to show that $\| M_\phi D_\theta \|_{L^2(G)} < 1$. We will also assume that $\phi(s) < 1 \forall s \in G - \{t\}$. Since $(M_\phi D_\theta)^* = D_\theta^* M_\phi$ and $D_\theta^* M_\phi$ is compact by Lemma 5, $M_\phi D_\theta$ is also compact. Hence $M_\phi D_\theta (M_\phi D_\theta)^* = M_\phi D_\theta^2 M_\phi$ is a compact self-adjoint operator on $L^2(G)$ and this implies that $\| M_\phi D_\theta^2 M_\phi \| = \mu$ where $\mu$ is the largest eigenvalue of $M_\phi D_\theta^2 M_\phi$. Let $f \in L^2(G)$ be the corresponding eigenvector such that $\| f \| = 1$, then we have

$$\mu = \| \mu f \| = \| (M_\phi D_\theta^2 M_\phi f) \| \leq \| D_\theta^2 M_\phi f \| \leq 1$$

since $\phi(s) < 1 \forall s \in G - \{t\}$. This implies that $\| D_\theta M_\phi \| = \| M_\phi D_\theta^2 M_\phi \| < 1$. This means that $(t, \gamma) \notin M(\Psi) \forall (t, \gamma) \in G \times \hat{\Gamma}^+$. So if $(t, \gamma) \in M(\Psi)$ then either $t = \infty$ or $\gamma = \infty$.

Now let $t \in \hat{G}$ and $\gamma = \infty$. Let $\phi \in C(\hat{G})$ and $\theta \in C(\hat{\Gamma}^+)$ such that $0 \leq \phi, \theta \leq 1$ and $\phi(t) = \theta(\infty) = 1$. Observe that $P = D_{\chi_{\Gamma^+}}$ where $\chi_{\Gamma^+}$ is the characteristic function of $\Gamma^+$. So we have $D_\theta T_{\phi} = D_\theta D_{\chi_{\Gamma^+}} M_\phi = D_{\chi_{\Gamma^+}} D_\theta M_\phi = D_\theta M_\phi$. Since $\mathcal{F}$ is unitary we have

$$\| D_\theta M_\phi \|_{H^2(G)} = \| \mathcal{F} D_\theta M_\phi \mathcal{F}^{-1} \|_{L^2(\Gamma^+)} = \| M_\phi \mathcal{F} M_\phi \mathcal{F}^{-1} \|_{L^2(\Gamma^+)}$$

Since $\theta(\infty) = 1$ we have $\forall \epsilon > 0$, $\exists \gamma_0 \in \Gamma^+$ such that $1 - \epsilon \leq \theta(\gamma) \leq 1 \forall \gamma \geq \gamma_0$. Consider the operator $S_{\gamma_0} : L^2(\Gamma^+) \to L^2(\Gamma^+)$ defined as $$(S_{\gamma_0} f)(\gamma) = f(\gamma_0^{-1} \gamma),$$

then $S_{\gamma_0}$ is an isometry. Observe that

$$\int_{\Gamma^+} \gamma_0(t) u(t) f(u) d\lambda(u) = \gamma_0(t) \tilde{f}(t) = (M_{\gamma_0} \tilde{f})(t)$$

for $\tilde{f}$ the completion of $f$ in the $L^2(\Gamma^+)$ norm.
Hence we have $S_{t_0} = FM_{t_0}F^{-1}$ which implies that

$$S_{t_0}(FM_\phi F^{-1}) = (FM_\phi F^{-1})S_{t_0}.$$  

Now let $f \in L^2(\Gamma^+)$ such that $\| (FM_\phi F^{-1})f \|_2 > 1 - \epsilon$ and $\| f \|_2 = 1$ then for $g = FM_\phi F^{-1}f$ we have

$$\| M_\phi S_{t_0}g \|_2 \geq (1 - \epsilon)^2$$

since $S_{t_0}g$ is supported on $\{ \gamma : \gamma \geq \gamma_0 \}$, $\theta(\gamma) \geq 1 - \epsilon \forall \gamma \geq \gamma_0$ and $\| S_{t_0}g \|_2 \geq 1 - \epsilon$.  

Since $S_{t_0}g = (FM_\phi F^{-1})S_{t_0}f$ we have

$$\| (M_\phi FM_\phi F^{-1})(S_{t_0}f) \|_2 \geq (1 - \epsilon)^2.$$  

Since $S_{t_0}$ is an isometry we have $\| S_{t_0}f \|_2 = 1$ and this implies that

$$\| M_\phi FM_\phi F^{-1} \| \geq (1 - \epsilon)^2$$

$\forall \epsilon > 0$. Therefore we have $\| M_\phi FM_\phi F^{-1} \| = \| D_\phi T_\phi \| = 1$. Hence $(t, \infty) \in M(\Psi)$ $\forall t \in G$.

Now let $\gamma \in \Gamma^+$ and $t = \infty$. Let $\phi \in C(\hat{G})$ and $\theta \in C(\hat{\Gamma}^+)$ such that $0 \leq \phi, \theta \leq 1$ and $\phi(\infty) = \theta(\gamma) = 1$. Since $\phi(\infty) = 1$, for any $\epsilon > 0$ there is a compact subset $K_1 \subset G$ such that $1 - \epsilon \leq \phi(t) \leq 1 \forall t \notin K_1$. Let $\hat{\theta} = \chi_{\Gamma^+} \theta$. Then we have

$$D_\phi T_\phi = D_\theta D_{\chi_{\Gamma^+}} M_\phi = D_{\chi_{\Gamma^+} \phi} M_\phi = D_\theta M_\phi.$$  

Let $\epsilon > 0$ be given. Let $g \in H^2(G)$ so that $\| g \|_2 = 1$ and $\| D_\theta g \|_2 \geq 1 - \epsilon$. Let $K_2 \subset G$ be a compact subset of $G$ so that

$$(\int_{K_2} | g(t) |^2 \ d\lambda(t))^{\frac{1}{2}} \geq 1 - \epsilon.$$  

By Lemma 4 we have $t_0 \in G$ such that $K_1 \cap (t_0 K_2) = \emptyset$. Let $(S_{t_0}g)(t) = g(tt_0^{-1})$ then

$$(\int_{t_0 K_2} | S_{t_0}g(t) |^2 \ d\lambda(t))^{\frac{1}{2}} = (\int_{K_2} | g(t) |^2 \ d\lambda(t))^{\frac{1}{2}} \geq 1 - \epsilon.$$  

and this implies that

$$\| S_{t_0}g - M_\phi S_{t_0}g \|_2 \leq 2\epsilon.$$  

We observe that $S_{t_0} = F^{-1} M_{t_0} F$ where $\hat{t_0} : \Gamma \to \mathbb{C}$ defined as $\hat{t_0}(\gamma) = \gamma(t_0)$. This implies that $S_{t_0}$ is unitary and we have $D_{\theta} S_{t_0} = S_{t_0} D_{\theta}$. Since $\| D_{\theta} \| = 1$ we have

$$\| D_{\theta} S_{t_0}g - D_{\theta} M_\phi S_{t_0}g \|_2 \leq 2\epsilon.$$  

Since $S_{t_0}$ is unitary for $f = S_{t_0}g$ we have $\| f \|_2 = 1$ and $D_{\theta} S_{t_0} = S_{t_0} D_{\theta}$ together with $\| D_{\theta} g \|_2 \geq 1 - \epsilon$ implies that

$$\| D_{\theta} M_\phi f \|_2 \geq 1 - 3\epsilon.$$  

Since $\epsilon > 0$ is arbitrary we have $\| D_{\theta} M_\phi \| = \| D_{\theta} T_\phi \| = 1$. Therefore we have $(\infty, \gamma) \in M(\Psi)$, $\forall \gamma \in \Gamma^+$. Our theorem is thus proven. \hfill $\square$

3. Acknowledgements

The author wishes to express his sincere thanks to Prof. Riza Ertürk of Hacettepe University for useful discussions on Lemma 4.
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ÜGÜR GÜL,
HACETTEPE UNIVERSITY, DEPARTMENT OF MATHEMATICS, 06800, BEYTEPE, ANKARA, TURKEY
E-mail address: gulugur@gmail.com