 Unsourced Multiple Access With Random User Activity

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Abstract—To account for the massive uncoordinated random access scenario, which is relevant for the Internet of Things, Polyanskiy et al. (2017) proposed a novel formulation of the multiple-access problem, commonly referred to as unsourced multiple access, where all users employ a common codebook and the receiver decodes up to a permutation of the messages. In this paper, we extend this seminal work to the case where the number of active users is random and unknown a priori. We define a random-access code accounting for both misdetection (MD) and false alarm (FA), and derive a random-coding achievability bound for the Gaussian multiple access channel. Our bound captures the fundamental trade-off between MD and FA probabilities. It suggests that the lack of knowledge of the number of active users entails a small penalty in energy efficiency when the target MD and FA probabilities are high. However, as the target MD and FA probabilities decrease, the energy efficiency penalty becomes more significant. For example, in a typical IoT scenario with frames-length 19200 complex channel uses and 25–300 active users in average, the required energy per bit to achieve both MD and FA probabilities below $10^{-3}$, predicted by our bound, is only 0.5–0.7 dB higher than that predicted by the bound in Polyanskiy et al. (2017) for a known number of active users. This gap increases to 3–4 dB when the target MD probability and/or FA probability is below $10^{-3}$. Taking both MD and FA into account, we use our bound to benchmark the energy efficiency of slotted-ALOHA with multi-packet reception, of a decoder that simply treats interference as noise, and of some recently proposed unsourced multiple access schemes. Numerical results suggest that, when the target MD and FA probabilities are high, it is effective to estimate the number of active users, then treat this estimate as the true value, and use a coding scheme that performs well for the case of known number of active users. However, this approach becomes energy inefficient when the requirements on MD and FA probabilities are stringent.

Index Terms—Multiple-access channels, unsourced multiple access, random-coding bound, misdetection, false alarm.

I. INTRODUCTION

The Internet of Things (IoT) enables a variety of applications, such as autonomous driving, smart homes, and smart cities, by providing wireless access to a massive number of devices. A significant fraction of IoT devices is battery limited and transmits short packets in a sporadic and uncoordinated manner [2], [3]. This calls for a new theoretical framework that helps to understand the fundamental limits on the energy efficiency achievable in massive uncoordinated access and provide guidelines for system design. To this end, Polyanskiy [4] proposed a novel formulation for the multiple-access problem, commonly referred to as unsourced multiple access that relies on three key assumptions: i) all users employ a common codebook and the decoder only aims to return a list of messages; ii) the error event is defined per user and the error probability is averaged over the users; iii) each user sends a fixed amount of information bits within a finite-length frame. This formulation provides a unified framework within which traditional as well as modern random access protocols [5] give achievability results. In [4], an upper bound on the minimum energy per bit achievable on the Gaussian multiple access channel (MAC) was derived. Modern random access schemes exhibit a large gap to this bound, and devising coding schemes approaching the bound is an active area of research [6], [7], [8], [9], [10], [11], [12]. The framework proposed in [4] has recently been extended to the quasi-static fading channel [13], multiple-antenna channel [14], [15], and a setup with common alarm messages [16].

The achievability bound in [4] is established for the scenario in which the number of active users is fixed and known to the receiver. This assumption has also been considered in most extensions of [4]. In practice, however, since IoT devices access the channel at random times and in a grant-free manner, the number of active users varies over time, and hence, it is typically unknown to the receiver. Therefore, the bound in [4] may provide an overoptimistic benchmark on the performance of random-access schemes operating in the practically relevant scenario in which the number of active
users is not known to the receiver. In this scenario, the decoder needs to determine the size of the list of transmitted messages. Choosing a list size smaller than the number of active users will result in misdetections (MDs), i.e., transmitted messages that are not included in the decoded list, whereas choosing it larger than the number of active users will result in false alarms (FAs), i.e., decoded messages that have not been transmitted. Furthermore, additional MDs and FAs may occur in the decoding process. There is a trade-off between MD and FA probabilities: a decoder that always outputs the whole codebook will never misdetect, but has FA probability close to one; similarly, a decoder that always outputs an empty set will never raise an FA but will always misdetect.

Characterizing the MD–FA trade-off is a fundamental engineering challenge. In the IoT, an MD can cause the system to be unaware of an important event reported by the devices, whereas an FA can trigger an unnecessary reaction that interrupts the system operation. Depending on the application, the MD probability can be more critical than the FA probability or vice versa. For example, in security inspection, where the cost of an MD can be extremely high (such as not detecting a threat) while the cost of an FA is relatively low (simply a further inspection), the MD probability should be favored. In healthcare, alarm fatigue, i.e., desensitization of clinicians due to high exposure to alarms, has been recognized as a serious issue. If MDs committed by health monitors do not cause vital consequences, the FA probability should be kept low to avoid alarm fatigue [17], [18].

The MD–FA trade-off was not addressed in [4]. Achievability bounds based on variable-length codes and feedback have been proposed for the general random-access channel [19], and in particular for the Gaussian MAC [20], with unknown number of active users. However, the authors considered the joint-user error event instead of the per-user error event, and thus, MD and FA were not explicitly considered. The per-user error probability achieved with this random-coding scheme was considered in [21, Sec. V-E], but the MD–FA trade-off was not addressed. To summarize, a random-coding bound accounting for both MD and FA, which can serve as a benchmark for unsourced multiple access with random user activity, is missing.

Most of the practical coding schemes that have been proposed so far for common-codebook massive random access require knowledge of the number of active users. Modern variations of the ALOHA protocol, such as irregular repetition slotted ALOHA (IRSA) [22], can also operate when the number of active users is unknown. However, most of the results available for these schemes pertain to the packet loss rate, which accounts only for MD. The successive interference cancellation coding scheme proposed in [7] is also analyzed in terms of MD only. Note that minimizing the MD probability alone can entail a high FA probability. In [23], a tensor-based communication scheme was proposed, and both MD and FA probabilities were reported in the performance evaluation. Another scheme for which both MD and FA probabilities have been reported was recently proposed in [24] for the quasi-static fading MAC and for the case in which the receiver has a large number of antennas. However, in [23] and [24], the MD and FA probabilities are not reported separately but rather through their sum.

**Contributions:** In this work, we extend Polyanskiy’s bound to the case where the number of active users is random and unknown. Our contributions are summarized as follows. We first extend the definition of a random-access code provided in [4] to account for both MD and FA probabilities. We then derive a random-coding achievability bound for the Gaussian MAC. Unlike [4], we do not assume that the receiver knows the number of active users. To circumvent this issue, we let the decoder seek the best list size within a predetermined interval around an estimated value of the number of active users. Our decoding metric to determine the list of transmitted messages, which is based on nearest neighbor decoding, is similar to the one used in [16]. However, different from [16], we limit the decoded list size to belong to a finite-size set to avoid noise overfitting, especially in the low energy-per-bit regime. We use our random-coding bound to characterize both MD and FA in slotted ALOHA with multi-packet reception (SA-MPR). Finally, we derive a random-coding bound for a scheme that simply treats interference as noise, referred to as treating-interference-as-noise (TIN), in which the number of active users is unknown.

To gain engineering insights on the role of the knowledge of the number of active users, we compare our bound with the bound in [4, Th. 1]. We also use our bound to benchmark the energy efficiency of SA-MPR and TIN, which do not require a priori knowledge of the number of active users. Finally, we consider the scheme based on sparse regression codes (SPARCs) proposed in [8] and its enhancement in [10], which were both derived for the case of known number of active users. We adapt these schemes to the case of unknown number of active users by performing an energy-based estimation of the number of active users and by letting the decoder treat the estimate as the true value. Numerical results pertaining to a scenario with 300 active users in average and frame length 19200 complex channel uses show that to achieve both MD and FA probabilities below $10^{-3}$, the required energy per bit predicted by our achievability bound is only 0.65 dB higher than that predicted by the bound for a known number of active users [4, Th. 1]. For the same setting, the required energy per bit predicted by our bound is 9 dB, 4.1 dB, and 3.6 dB lower than that of the SA-MPR bound, SPARC [8], and enhanced SPARC [10], respectively. The gap between the performance of enhanced SPARC with known number of active users and with unknown number of active users is small. This suggests that, for mild requirements on $P_{MD}$ and $P_{FA}$, it is sufficient to adapt existing coding schemes that perform well in the case of known number of active users by simply adding an active-user estimation step and then treating the estimated number of active users as the true value. On the contrary, when we consider MD probability and/or FA probability below $10^{-3}$, the gap between our bound and the bound for a known number of active users in [4] is much larger: around 3–4 dB. For these stringent requirements, it turns out that it is energy inefficient to simply treat the estimated number of active users as the true value due to the errors that occur in the estimation step.
To summarize, our results suggest that for the Gaussian MAC, the lack of knowledge of the number of active users entails a small loss in terms of energy efficiency if the target MD and FA probabilities are high, as typically considered in the literature. In this case, it is effective to first estimate the number of active users, then treat this estimate as the true value, and use a coding scheme that performs well for the case where the number of active users is known. However, for more stringent targets, the loss due to the lack of knowledge of the number of active users might be large. It remains unclear whether this large gap is fundamental or pertains more stringent targets, the loss due to the lack of knowledge of the number of active users, then treat this estimate as the true value, and use a coding scheme that performs well for the number of active users, which is assumed to be known. We therefore need to account for both MD and FA probabilities. We next rigorously define these two quantities, as well as the notion of a random-access code.

Definition 1 (Random-Access Code for the Gaussian MAC): Consider the $K_n$-user Gaussian MAC with $K_n \sim P_{K_n}$. An $(M, n, \epsilon_{MD}, \epsilon_{FA})$ random-access code for this channel, where $M$ is the size of the codebook, $n$ is the codeword length, and $\epsilon_{MD}, \epsilon_{FA} \in (0, 1)$, consists of:

- A random variable $U$ defined on a set $\mathcal{U}$ that is revealed to both the transmitters and the receiver before the start of the transmission.
- An encoding function $f : \mathcal{U} \times [M] \rightarrow \mathbb{C}^n$ that produces the transmitted codeword $x_i = f(U, w_i)$, satisfying the power constraint, of user $i$ for a given message $w_i$ uniformly distributed over $[M]$.
- A decoding function $g : \mathcal{U} \times \mathbb{C}^n \rightarrow \mathcal{P}([M])$ that provides an estimate $\widehat{W} = \{\tilde{w}_1, \ldots, \tilde{w}_n\}$ of the list of transmitted messages. Let $\widehat{W} = \{\tilde{w}_1, \ldots, \tilde{w}_n\}$ denote the set of distinct elements of $W = \{w_1, \ldots, w_K\}$. We assume that the decoding function satisfies the following constraints on the MD and FA probabilities:

$$P_{MD} = \mathbb{E} \left[ \frac{1}{|W|} \sum_{i=1}^{n} \mathbb{P} \left( \tilde{w}_i \notin \widehat{W} \right) \right] \leq \epsilon_{MD} \quad (2)$$

$$P_{FA} = \mathbb{E} \left[ \frac{1}{|\tilde{W}|} \sum_{i=1}^{|\tilde{W}|} \mathbb{P} \left( \tilde{w}_i \notin \widehat{W} \right) \right] \leq \epsilon_{FA} \quad (3)$$

The expectations in (2) and (3) are with respect to the random user activity, and we use the convention $0/0 = 0$ to circumvent the case $|W| = 0$ or $|\tilde{W}| = 0$.

Remark 1: According to Definition 1, the receiver aims to produce the list of distinct transmitted messages—there are neither MD nor FA if $\tilde{W} = W$. Therefore, the event that two users transmit the same message does not result in an error if the message is included in the list of decoded messages.

In the definition of random-access code proposed in [4, Def. 1], the decoder outputs a list of messages of size equal to the number of active users, which is assumed to be known. In such a setup, an MD implies an FA, and vice versa. Hence, MD and FA events either occur simultaneously or do not occur simultaneously. In our setup, the number of decoded messages

$$\bf{y} = \sum_{i=1}^{K_n} x_i + \bf{z}, \quad (1)$$

where $\bf{z} \sim \mathcal{CN}(\bf{0}, \bf{I}_n)$ is the Gaussian noise, which we assume being independent of $\{x_i\}_{i=1}^{K_n}$. We also assume that the transmitted signals satisfy the power constraint $\|x_i\|^2 \leq nP, \forall i \in [K_n]$, almost surely. We further assume that the receiver does not know $K_n a priori$, but can choose to estimate it. As in [4], our model differs from the classical MAC in that the total number of users is not limited, all users employ the same codebook, and the receiver decodes up to a permutation of the messages. However, as opposed to [4], where the number of active users is assumed to be fixed and known, we assume that $K_n$ is random and unknown. We therefore need to account for both MD and FA probabilities. We next rigorously define these two quantities, as well as the notion of a random-access code.\footnote{Our definition of a random-access code can be extended straightforwardly to more general MACs with permutation-invariant channel law, as considered in [4, Def. 1].}

II. RANDOM-ACCESS CHANNEL

We consider a MAC in which a random number $K_n$ of users transmit their messages to a receiver over $n$ uses of a stationary memoryless additive white Gaussian noise channel. Here, $K_n$ follows a distribution with probability mass function (PMF) $P_{K_n}$. Let $x_i \in \mathbb{C}^n$ be the signal transmitted by user $i$ over $n$ channel uses. The corresponding channel output is given by

$$\bf{y} = \sum_{i=1}^{K_n} x_i + \bf{z}, \quad (1)$$
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|---|---|---|
| | | \[ |W| \] can be different from the number of distinct transmitted messages \(|W|\). This motivates the definition of MD and FA probabilities provided in (2) and (3), respectively.

**Remark 2:** In the next section, we shall use a random-coding argument to obtain achievability bounds. We will construct a codebook ensemble for which (2) and (3) holds on average. Specifically, the MD and FA probabilities, averaged over the codebook ensemble, are upper-bounded by \(\epsilon_{\text{MD}}\) and \(\epsilon_{\text{FA}}\), respectively. Unfortunately, this does not imply that there exists a single code in this ensemble that achieves both (2) and (3). In other words, the fact that a random code satisfies both (2) and (3) on average does not imply the existence of a deterministic code that satisfies these constraints. The introduction of the random variable \(U\) in Definition 1 allows us to circumvent this issue by enabling randomized coding strategies. Specifically, proceeding as in [25, Th. 19], one can show that there exists a randomized coding strategy that achieves both (2) and (3) and involves time-sharing among at most three deterministic codes (i.e., \(|U| \leq 3\) in this ensemble. In fact, following the improvement in the size of the common randomness reported in [21, Th. 8], one can show that \(|U| \leq 2\) suffices.

III. RANDOM-CODING BOUND

**A. Proposed Random-Coding Bound for \(K_a\) Random and Unknown**

We first review the random-coding bound in [4, Th. 1]. Let \(W = \{w_1, \ldots, w_{K_a}\}\) be the set of transmitted messages. Each of the active users picks a codeword \(c_{w_i}, i \in [K_a]\), from a common codebook containing \(M\) codewords \(c_1, \ldots, c_M\) drawn independently from the distribution \(CN(0, P^2 I_n)\) for a fixed \(P^2 < P\). To convey message \(w_i\), the corresponding active user transmits \(c_{w_i}\) provided that \(\|c_{w_i}\|^2 \leq np\). Otherwise, it transmits the all-zero codeword. That is, \(x_i = c_{w_i} I\{\|c_{w_i}\|^2 \leq np\}\). The receiver employs a minimum distance decoder, in which the list of decoded messages is obtained as \(\hat{W} = \arg \min_{W' \subseteq [M], |W'| = K_a} \|c(W') - y\|^2\). The error analysis involves manipulations of unions of the pairwise error events via a change of measure and the application of the Chernoff bound combined with Gallager’s \(\rho\)-trick [26, p. 136]. An alternative bound is also obtained by writing the pairwise error event as an inequality involving information densities, and by applying the tail bound on the information density given in [27, Cor. 17.1].

**Remark 3:** Using independent and identically distributed (i.i.d.) Gaussian codewords allows for a tractable analysis that leads to explicit bounds for fixed \(n\) and \(P\). An alternative approach consists in using codebooks with codewords uniformly distributed on the power sphere, i.e., the so-called spherical/shell codebooks. For the conventional Gaussian MAC [20], spherical codebooks were shown to achieve a better second-order (dispersion) term in the asymptotic expansion of the achievable rate region as \(n \rightarrow \infty\) than i.i.d. Gaussian codebooks. For the UMA setting, however, it is unclear if spherical codebooks achieve a better second-order term. Furthermore, with spherical codebooks, it appears challenging to obtain explicit closed-form bounds for fixed \(n\) and \(P\).

In the following, we derive a random-coding bound for the case in which \(K_a\) is random and unknown to the receiver. Specifically, we consider a random-coding scheme with the same encoder as in [4].\(^2\) The new challenge in our setting with respect to [4] is that the receiver does not know \(K_a\), and thus cannot use this number to set the decoded list size. To overcome this challenge, we let the receiver estimate \(K_a\) from \(y\), and then decide the best list size within an interval around the initial estimate of \(K_a\). Specifically, given \(K_a'\), the receiver produces a list of decoded messages as

\[
\hat{W} = \arg \min_{W' \subseteq [M], |W'| = K_a'} \|c(W') - y\|^2, \tag{5}
\]

where \(m(y, K)\) is a suitably chosen metric, and \(K_a'\) and \(K_a\) are suitably chosen lower and upper limits on \(K_a'\), respectively. The metric \(m(y, K)\) and the parameters \(K_a'\) and \(K_a\) can, for example, be chosen based on prior knowledge (if available) on the distribution of \(K_a\). Then, given \(K_a'\), the receiver produces a list of decoded messages as

\[
K_a' = \arg \max_{K_a \in \mathcal{K}} m(y, K) \tag{4}
\]

where \(K_a' = \max\{K_a', K_a' - r\} \) and \(\mathcal{K}_a = \min\{K_a, K_a' + r\}\), with \(r\) being a nonnegative integer. We refer to \(r\) as the decoding radius. Note that if \(r = 0\), the receiver outputs \(K_a'\) codewords, i.e., it treats the estimate \(K_a'\) as the true value. An error analysis of this random-coding scheme conducted along similar lines as in [4] leads to the following result.

**Theorem 1 (Random-Coding Bound, \(K_a\) Random and Unknown):** Fix \(P^2 < P\), \(r\), \(K_a\), and \(K_a (K_a \leq K_a)\). For the \(K_a\)-user Gaussian MAC with \(K_a \sim P_{K_a}\), there exists an \((M, n, \epsilon_{\text{MD}}, \epsilon_{\text{FA}})\) random-access code satisfying the power constraint \(P\) for which

\[
\epsilon_{\text{MD}} = \sum_{K_a = \max\{K_a, 1\}}^{K_a} \left( \sum_{K_a = \max\{K_a, 1\}}^{K_a} \sum_{t = 1}^{K_a} \frac{t + (K_a - \mathcal{K}_a)^+}{K_a} \right) + \hat{p}, \tag{6}
\]

\[
\epsilon_{\text{FA}} = \sum_{K_a = K_a}^{K_a} \left( \sum_{K_a = K_a}^{K_a} \sum_{t = t}^{K_a} \frac{t + (K_a - K_a)^+}{K_a} \right) + \hat{p}, \tag{7}
\]

\(^2\)Strictly speaking, our encoding function takes not only the message as input (as in [4]) but also the common randomness variable \(U\). However, for brevity, we omit \(U\) in the encoding and decoding functions. For details on how this common randomness is incorporated in the encoding and decoding functions, we refer the readers to [21] and [25].
where

\[
\tilde{p} = 2 - \sum_{K_a = K_t}^{K_a} P_{K_a}(K_a) - \mathbb{E}_{K_a} \left[ \frac{M!}{M^{K_a}(M-K_a)!} \right] \\
+ \mathbb{E}_{[K_a]} \left[ \Gamma(n,nP/P') \right]^{-1} \left[ \Gamma(n) \right],
\]

(8)

\[
p_t = \sum_{t \in T_t} p_{t,t'},
\]

(9)

\[
p_{t,t'} = e^{-nE(t,t')},
\]

(10)

\[
E(t,t') = \max_{\rho, \rho_1 \in [0,1]} -\rho_1 t'R_1 - \rho_1 R_2 + E_0(\rho, \rho_1),
\]

(11)

\[
E_0(\rho, \rho_1) = \max_{\lambda} \rho \lambda \ln(1 - \rho_1 P_2b),
\]

(12)

\[
a = \rho \lambda - \frac{\mu}{1 + \rho't'},
\]

(13)

\[
b = \rho \lambda - \frac{\mu}{1 + \rho't'},
\]

(14)

\[
\mu = \frac{1 + \rho't'}{\rho},
\]

(15)

\[
P_2 = 1 + ((K_a - K_a')^+) + (K_a' - K_a)^+) P',
\]

(16)

\[
R_1 = \frac{1}{\ln(1 + |W_a|)} \ln \left( \frac{M - \max_{t'} \{K_a, K_a'\}}{t'} \right),
\]

(17)

\[
R_2 = \frac{1}{\ln(1 + |W_a|)} \ln \left( \min_{t'} \{K_a, K_a'\} \right),
\]

(18)

\[
q_t = \inf_{\gamma} \left( \mathbb{P}[l_t \leq \gamma] + \mathbb{E}_{t \in T_t} \exp(n(t'R_1 + R_2) - \gamma) \right),
\]

(19)

\[
q_{t,t'} = \inf_{\gamma} \left( \mathbb{P}[l_t \leq \gamma] + \exp(n(t'R_1 + R_2) - \gamma) \right),
\]

(20)

\[
T = \{0 : \min_{t'} \{K_a, K_a', M - K_a - (K_a - K_a')^+\} \}
\]

(21)

\[
T_t = \left\{ ((K_a - K_a')^+) - (K_a' - K_a)^+ + \max\{K_a', 1\} \right\},
\]

(22)

\[
\mathbb{P}[K_a = K_a'] = \sum_{t'} \mathbb{P} \left[ m(y_0, K_a') > m(y_0, K_a) \right],
\]

(23)

\[
\xi(K_a, K_a') = \min_{K_a \in [K_a,K_a'] : K_a \neq K_a'} \mathbb{P} \left[ m(y_0, K_a') > m(y_0, K_a) \right].
\]

(24)

\[
\xi(K_a, K_a') = \min_{K_a \in [K_a,K_a'] : K_a \neq K_a'} \mathbb{P} \left[ m(y_0, K_a') > m(y_0, K_a) \right].
\]

(25)

\[
in \{25\}, \ y_0 \sim CN(0, (1 + K_a P') I_n). \]

The random variable \( l_t \) in (19) and (20) is defined as

\[
l_t = \min_{w_{MD} \in [K_a, K_a']^+ \cup \cdots K_a]} u_t(c(W_{FA}) + c(W_{MD})),
\]

(26)

where \( W_{FA} = [K_a + 1 : K_a'], W_{MD} = [K_a - K_a^+] \cup W_{MD} \), and the information density is defined as

\[
\mu(c(W_{MD}) : y \mid c(W \setminus W_{MD}))
\]

\[
en \ln(1 + (t + (K_a - K_a')^+) + \nu)
\]

\[
+ \left( \|y - c(W \setminus W_{MD})\|^2 + (t + (K_a - K_a')^+) \right)
\]

\[
- \|y - c(W_{MD}) - c(W \setminus W_{MD})\|^2. \]

(27)

Proof: The proof follows the footsteps of [4] with some new ingredients to overcome the challenges posed by the random and unknown number of active users. First, for the proposed two-step decoder, the quality of both the estimation and decoding steps needs to be analyzed. Second, due to the possible mismatch between the numbers of transmitted and decoded messages, the sets of MDs and FAs need to be carefully handled. See Appendix A for details.

Some remarks are in order.

i) Let the sets of misdetected messages and falsely alarmed messages be denoted by

\[
W_{MD} = \hat{W} \setminus \hat{W},
\]

(28)

\[
W_{FA} = \tilde{W} \setminus \hat{W},
\]

(29)

respectively. Let \( K_a \rightarrow K_a' \) denote the event that the estimation step outputs \( K_a' \) when \( K_a \) users are active. Given \( K_a \rightarrow K_a' \), note that the set of possible decoded list sizes \([K_a' : K_a']\) might not contain the number \( K_a \) of transmitted messages. If \( K_a' < K_a \), the decoder commits at least \( K_a - K_a' \) MDs; if \( K_a' > K_a \), the decoder commits at least \( K_a' - K_a \) FAs. Furthermore, there can be additional MDs and FAs occurring during the decoding process. Accordingly, we further break down the sets \( W_{MD} \) and \( W_{FA} \) as follows. We set \( W_{MD} = W_{MD} \cup W_{MD} ' \), where \( W_{MD} ' \) denotes the list of \((K_a - K_a')^+\) initial MDs due to insufficient decoded list size, and \( W_{MD} \) the additional MDs that occur during decoding. Similarly, we set \( W_{FA} = W_{FA} \cup W_{FA} ' \), where \( W_{FA} ' \) denotes the list of \((K_a' - K_a)^+\) initial FAs due to excessive decoded list size, and \( W_{FA} \) the additional FAs. This is explained in details in Appendix A.

ii) The bounds in (6) and (7) are obtained by writing the MD and FA probabilities as \( P_{MD} = \mathbb{E} \left[ \frac{|W_{MD}|}{|W|} \right] \) and \( P_{FA} = \mathbb{E} \left[ \frac{|W_{FA}|}{|W|} \right] \). Similar to [4], to facilitate the bounding, we make a change of measure over which these expectations are taken at a cost of adding a constant bounded by \( \tilde{p} \) given in (8). We then expand these expectations over possible values of the size of the sets \( W, \bar{W}, W_{MD} \), and \( W_{FA} \). The terms \( \mu(c(K_a - K_a')^+) \) and \( \kappa_a - \kappa - (K_a - K_a')^+ + \kappa_a' + (K_a' - K_a)^+ \) are realizations of \( \frac{|W_{MD}|}{|W|} \) and \( \frac{|W_{FA}|}{|W|} \), respectively, given that \( K_a \rightarrow K_a' \), \( |W_{MD}| = t \), and \( |W_{FA}| = t' \). Furthermore, the probabilities \( \mathbb{P} \left[ K_a \rightarrow K_a' \mid |W_{MD}| = t \right] \) and \( \mathbb{P} \left[ K_a \rightarrow K_a' \mid |W_{MD}| = t \right] \) associated with these realizations are upper-bounded by \( P_{K_a}(K_a) \min_{\{p_t, q_t, \xi(K_a, K_a')\}} \) and \( P_{K_a}(K_a) \min_{\{p_t, q_t, \xi(K_a, K_a')\}} \), respectively. Here, \( \xi(K_a, K_a') \) is an upper bound on \( \mathbb{P}[K_a \rightarrow K_a'] \), whereas \( \min_{\{p_t, q_t\}} \) and \( \min_{\{p_t, q_t\}} \) are upper bounds.
on \( \mathbb{P}[|W_{\text{MD}}| = t] \) and \( \mathbb{P}[|W_{\text{MD}}| = t', |W_{\text{FA}}| = \ell'] \), respectively.

iii) The bounds \( p_{t,t'} \) and \( q_{t,t'} \) on \( \mathbb{P}[|W_{\text{MD}}| = t, |W_{\text{FA}}| = \ell'] \) are obtained following the error-exponent-based and dependence-testing-based approaches, respectively. Similar approaches are used to obtain the bounds \( p_0 \) and \( q_0 \), respectively, on \( \mathbb{P}[|W_{\text{MD}}| = t] \). Both approaches have been used in [4] and can improve the bounds in different regimes. Numerical experiments suggest that \( p_{t,t'} \) dominates for medium and high values of \( t, t' \) and \( K_a \), while \( q_{t,t'} \) dominates when \( t, t' \), and \( K_a \) are small. Computing \( q_t \) and \( q_{t,t'} \) is more cumbersome than \( p_t \) and \( p_{t,t'} \). Thus, one can limit the evaluation of these terms to small \( t, t' \), and \( K_a \) to reduce the complexity. This only loosens the bound slightly.

iv) The parameters \( K_i \) and \( K_a \) are introduced to facilitate the numerical evaluation of the bounds \( \epsilon_{\text{MD}} \) and \( \epsilon_{\text{FA}} \). In particular, we use them to avoid infinite sums over \( K_a \) and \( K_i \), since the domain of \( K_a \) can be unbounded. It is often convenient to set \( K_i \) to be the largest value and \( K_a \) the smallest value for which \( \sum_{K_a = K_i} P_{a}(K_a, \ell_K) \) exceeds a predetermined threshold.

v) The term \( 1 - \mathbb{E}_{K_a} \left[ \frac{M!}{Mn_a(M-K_a)!} \right] \) in \( \bar{p} \) (see (8)) can be upper-bounded by \( \mathbb{E}_{K_a} \left[ \frac{M!}{Mn_a(M-K_a)!} \right] \) as in [4].

vi) The term \( E_{\ell_K=1} \) in (17) can be upper-bounded by \( \frac{1}{\ln \max (K_a, K_i^\prime) } \ln \nu \), which allows for a numerically stable computation when \( \max (K_a, K_i^\prime) \) is large.

vii) The optimal \( \lambda \) in (12) is given by the largest real root of the cubic function \( c_1 x^3 + c_2 x^2 + c_3 x + c_4 \) with

\[
\begin{align*}
c_1 &= -\rho \lambda_1 (\lambda_1 + 1) t^P P_2 P_3^2, \\
c_2 &= \rho \lambda_1 t^P P_3^2 - \rho \lambda_1 (\lambda_1 + 1) t^P P_2 P_3 \\
c_3 &= (\lambda_1 + 1) t^P P_2 P_3 \\
c_4 &= (\lambda_1 + 1) t^P P_2 P_3
\end{align*}
\]

where \( P_2 \) is given by (16) and \( P_3 = (t' + \rho t) P' \).

Although we follow the approach in [4], the novel aspects of our proof are that we provide a combined analysis of both the estimation and decoding steps. Furthermore, we single out the initial MDs and FAs that can not be avoided and carefully count the additional MDs and FAs that can occur during the decoding process. The separate treatment of the initial and additional MDs and FAs is crucial for our bound.

In the following theorem, we provide closed-form expressions for \( \xi(K_a, K_i') \) for two different estimators of \( K_a \).

**Theorem 2 (Closed-Form Expressions for \( \xi(K_a, K_i') \)):** For the maximum likelihood (ML) estimation of \( K_a \), i.e., \( m(y; K) = \ln p_{y|K}(y|K) \), \( \xi(K_a, K_i') \) is given by

\[
\xi(K_a, K_i') = \min_{K_a' \neq K_i} \{ K < K_a' \} \frac{\Gamma(n, \xi(K_a, K_i'))}{\Gamma(n)} + \{ K > K_a' \} \frac{\Gamma(n, \xi(K_a, K_i'))}{\Gamma(n)},
\]

where

\[
\xi(K_a, K_i') = n \ln \left( \frac{1 + K P' + K_i' P'_t}{1 + K P' - K_i' P'_t} \right) \left( \frac{1}{1 + K P'} \right)^{1-\epsilon}.
\]

For an energy-based estimation \( \xi(K_a, K_i') \) of \( K_a \), i.e., \( m(y; K) = \frac{1}{||y||^2 - n(1 + K P')} \), \( \xi(K_a, K_i') \) is given by (34) with

\[
\xi(K_a, K_i') = \frac{n}{1 + K a^P} \left( 1 - \frac{K}{2} P' \right).
\]

**Proof:** See Appendix B.

As mentioned in Remark i, since our decoder outputs a list of size within the interval \( [K_a' : K_i'] \), it commits initial MDs or FAs when the true \( K_a \) falls outside of this interval. In Section V-A, we shall discuss why we choose to restrict the decoded list size to this interval. Due to the initial MDs and FAs, the MD and FA probabilities do not vanish even when \( P \to \infty \) and all other system parameters are kept fixed. In other words, the bounds \( \epsilon_{\text{MD}} \) and \( \epsilon_{\text{FA}} \) in Theorem 1 exhibits error floors when \( P \) is large. To characterize this effect, we put forth the following asymptotic lower bounds on \( \epsilon_{\text{MD}} \) and \( \epsilon_{\text{FA}} \), which are obtained by assuming that no additional MD or FA occurs on top of the initial MDs or FAs.

**Corollary 1 (Asymptotic Lower Bounds on \( \epsilon_{\text{MD}} \) and \( \epsilon_{\text{FA}} \)):** With ML or energy-based estimation of \( K_a, \epsilon_{\text{MD}} \) and \( \epsilon_{\text{FA}} \) given in (6) and (7), respectively, satisfy

\[
\lim_{P \to \infty} \epsilon_{\text{MD}} \geq \epsilon_{\text{MD}},
\]

\[
\lim_{P \to \infty} \epsilon_{\text{FA}} \geq \epsilon_{\text{FA}}.
\]
if the probability of having additional MDs and FAs vanishes, i.e., \( \min[p_{t}, q_{t}] \to 0 \) for \( t \neq 0 \) and \( \min[p_{t}, q_{t}] \to 0 \) for \( (t,t') \neq (0,0) \) as \( P \to \infty \). With \( \rho = \rho_1 = 1 \), the optimal \( \lambda \) in (12) is given by \( \lambda = 1/(2P_2) \). Thus, by replacing the maximization over \( \rho \) and \( \rho_1 \) in (11) with \( \rho = \rho_1 = 1 \), we obtain that \( E(t,t') \geq -t'R_2 - R_2 + \ln (1 + (t+ t')P_2) \).

It follows that

\[
\tilde{p}_{t,t'} \leq \left( M - \max\left\{ K_a, \frac{K_2'}{t'} \right\} \right) \left( \min\left\{ K_a, K'_2 \right\} \right) \left( 1 + \frac{(t + t')P_2}{4P_2} \right)^{-n} \cdot \left( 1 + \frac{(t + t')P_2}{4P_2} \right)^{-n}.
\]

(41)

If \( K_a \in [K'_2, \frac{K_2}{t'}] \), i.e., \( P_2 = 1 \), the right-hand side of (41) vanishes as \( P' \to \infty \). Otherwise, the right-hand side of (41) converges to

\[
\tilde{p}_{t,t'} = \left( M - \max\left\{ K_a, \frac{K'_2}{t'} \right\} \right) \left( \min\left\{ K_a, K'_2 \right\} \right)^{-n} \cdot \left( 1 + \frac{(t + t')P_2}{4(P_2 + (K_2' - K_2)^+)} \right)^{-n} \cdot \left( 1 + \frac{(t + t')P_2}{4(P_2 + (K_2' - K_2)^+)} \right)^{-n}.
\]

(42)

(43)

Observe that \( \tilde{p}_{t,t'} \) is small if \( n \) is relatively large compared to \( \ln M \) and \( \ln K_a \), which is true for relevant values of \( n, M \) and \( K_a \) in the IoT. Specifically, in typical IoT scenarios, \( \log_2 M \) and \( K_a \) are in the order of \( 10^2 \) to \( 10^3 \), while \( K_a/n \) is from \( 10^{-4} \) to \( 10^{-3} \)—see [4] and [28, Rem. 3].

For example, with \( (M,n) = (2^{(100)}, 15000) \) and \( K_a \leq 300 \) as considered in [4] and many follow-up works, assume that \( (K_2 - K_2^+ - (K_2^+ - K_2)^+) \leq 20 \) then \( \tilde{p}_{t,t'} < 10^{-128} \) for every \( t \leq 300 \) and \( t' \leq 300 \). As a consequence, \( p_{t,t'} \) and \( p_t \) are very small. We conclude that \( \lim_{P \to \infty} \epsilon_{\text{MD}} \) and \( \lim_{P \to \infty} \epsilon_{\text{FA}} \) approach closely \( \tilde{\epsilon}_{\text{MD}} \) and \( \tilde{\epsilon}_{\text{FA}} \), respectively. In other words, \( \tilde{\epsilon}_{\text{MD}} \) and \( \tilde{\epsilon}_{\text{FA}} \) essentially characterize the error floors of \( \epsilon_{\text{MD}} \) and \( \epsilon_{\text{FA}} \), respectively, as \( P \to \infty \). We shall further validate this argument through numerical experiments in Section IV.

The choice of the decoding radius \( r \) turns out to be crucial. It can be optimized according to the target MD and FA probabilities. On the one hand, a large decoding radius results in a reduction of the initial MDs and FAs, and thus a reduction of the error floors. Indeed, it is easy to verify that both \( \tilde{\epsilon}_{\text{MD}} \) in (38) and \( \tilde{\epsilon}_{\text{FA}} \) in (40) decrease with \( r \). As \( r \to \infty \), both \( \tilde{\epsilon}_{\text{MD}} \) and \( \tilde{\epsilon}_{\text{FA}} \) vanish. On the other hand, a large decoding radius leads to overfitting, especially when \( P \) is small. Specifically, as we shall clarify in Section V-A, when the noise dominates the signal and the interference, increasing \( r \) seems to increase the chance that the decoder (5) returns a list containing codewords whose sum is closer in Euclidean distance to the noise than to the sum of the transmitted codewords. When \( P \) is sufficiently small, numerical experiments indicate that it is optimal to set \( r = 0 \), which introduces a bias that helps overcome overfitting. In this case, we obtain the following achievability bound.

Corollary 2 (Zero Decoding Radius): Fix \( P' < P, K_{t}, \) and \( K_a \) \((K_t \leq K_a)\). There exists an \((M, n, \epsilon_{\text{MD}}, \epsilon_{\text{FA}})\) random-access code satisfying the power constraint \( P \) for which

\[
\epsilon_{\text{MD}} = \sum_{K_a=\max\{K_t, 1\}}^{K_a} \left( P_{K_a}(K_a) \sum_{K_a=K_t}^{K_a} \psi + (K_a - K_a')^+ \right) \min\{p_t, q_t, \xi(K_a, K'_a)\} + \tilde{p},
\]

(44)

\[
\epsilon_{\text{FA}} = \sum_{K_a=K_t}^{K_a} \left( P_{K_a}(K_a) \sum_{K_a=K_t}^{K_a} \psi + (K_a - K_a')^+ \right) \min\{p_t, q_t, \xi(K_a, K'_a)\} + \tilde{p},
\]

(45)

where \( \psi = \min\{K'_a, K_a, M - \max\{K_a, K'_a\}\} \). and \( \tilde{p} \) are given by (8), (10), and (20), respectively, with \( r = 0 \).

If the number of active users is fixed to \( K_a \), by letting \( K'_a = K_a \), one obtains from Corollary 2 a trivial generalization of [4, Th. 1] to the complex case.

Our random-coding scheme belongs to a family of random-coding schemes that first estimate the number of active users and then exploit this estimate to choose the number of returned messages. The next theorem gives an ensemble converse bound on the MD and FA probabilities, i.e., a converse bound on the MD and FA probabilities averaged over a random codebook ensemble, for schemes of this type.

Theorem 3 (An Ensemble Converse Bound for a Family of Coding Schemes): Fix \( K_{t} \) and \( K_a \) \((K_t \leq K_a)\). Consider a decoding function that first estimates \( K_a \) by \( K'_a \) as in (4) and then returns a list of at least \( K'_a \) and at most \( K''_a \) messages, where \( K'_a \) and \( K'_a \) are functions of \( K_a \). Consider a random-access code consisting of \( M \) length-\( n \) codewords drawn independently from a distribution \( P_a \) and the aforementioned decoding function. It holds that

\[
\mathbb{E}[P_{\text{MD}}] \geq \Sigma_{\text{MD}}
\]

\[
= \mathbb{E}_{K_a} \left[ \frac{M!}{M!} \right] \sum_{K_a=\max\{K_t, 1\}}^{K_a} \left( P_{K_a}(K_a) \right) \cdot \sum_{K'_a=K_t}^{K_a} \left( K_a - K'_a \right)^+ \cdot \mathbb{P}[K_a \to K'_a],
\]

(46)

\[
\mathbb{E}[P_{\text{FA}}] \geq \Sigma_{\text{FA}}
\]

\[
= \mathbb{E}_{K_a} \left[ \frac{M!}{M!} \right] \sum_{K_a=\max\{K_t, 1\}}^{K_a} \left( P_{K_a}(K_a) \right) \cdot \sum_{K'_a=K_t}^{K_a} \left( K_a - K'_a \right)^+ \cdot \mathbb{P}[K_a \to K'_a],
\]

(47)

\[\text{Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.} \]
Here, the expectations on the left-hand sides of (46) and (47) are with respect to the codeword distribution $P_x$, and $\mathbb{P}[K_a \to K_a']$ is the probability that the estimation step outputs $K_a'$ when $K_a$ users are active. That is, $\mathbb{P}[K_a \to K_a'] = \mathbb{P}[K_a = K_a' | \hat{K}_a = K_a]$, where the distribution of the received signal $y$ is the one induced by the codeword distribution $P_x$ and the Gaussian noise via (1).

**Proof:** See Appendix D.

**Remark 5:** A practical approach to adapt the coding schemes proposed for the case of known $K_a$ to the setting where $K_a$ is unknown is to first estimate $K_a$ and then treat this estimate as the true $K_a$ in the decoding process. In this way, the number of decoded messages is equal to the estimate of $K_a$. For this approach, our random-coding bound with zero decoding radius, provided in Corollary 2, gives an achievability bound, whereas Theorem 3 with $\mathcal{K}_a' = \mathcal{K}_a$ and with $P_x$ being the i.i.d. Gaussian codeword distribution with average power $P$ gives an ensemble converse bound. A general converse bound on the MD and FA probabilities seems difficult to obtain, as we discuss in Section V-B.

**B. Application to Slotted ALOHA With Multi-Packet Reception**

Our random-coding bound can also be applied to SA-MPR to investigate the resulting MD–FA trade-off. Consider an SA-MPR scheme where a length-$n$ frame is divided into $L$ slots, and where each user chooses randomly a slot in which to transmit. For $K_a$ active users, the number of users transmitting in a slot follows a binomial distribution with parameters $(K_a, 1/L)$. The PMF of the number of active users in a slot, $K_{SA}$, is given by

$$P_{K_{SA}}(K_{SA}) = \sum_{K_a=0}^{K_{SA}} P_{K_a}(K_a) \binom{K_a}{K_{SA}} (1 - 1/L)^{K_a - K_{SA}}.$$  \hspace{1cm} (48)

Existing analyses of slotted ALOHA usually assume that the decoder can detect perfectly if no user, one user, or more than one user have transmitted in a slot. Furthermore, it is usually assumed that a collision-free slot leads to successful message decoding. However, in the presence of noise, single-user decoding in a collision-free slot may fail. Furthermore, the more the slots, the shorter the slot length over which a user transmits its signal. In this case, the single-user code becomes shorter and thus less resilient to noise. To account for both detection and decoding errors, in Corollary 3 below we apply the decoder introduced in (5) in a slot-by-slot manner, and obtain a random-coding bound similar to Theorem 1.

**Corollary 3 (Random-Coding Bound for SA-MPR):** For the Gaussian MAC with the number of active users distributed according to $P_{K_a}$ and frame length $n$, an SA-MPR scheme with $L$ slots can achieve the MD and FA probabilities given in (6) and (7), respectively, with codeword size $M$, codeword length $n/L$, power constraint $PL$, and per-slot number of active users distributed according to $P_{K_{SA}}$ defined in (48) (i.e., $P_{K_a}$ is replaced by $P_{K_{SA}}$ in (6) and (7)).

**C. Random-Coding Bound for a Scheme That Treats Interference as Noise**

We next present a random-coding union bound with parameter $s$ (RCUs) for TIN to account for both MD and FA when the number of active users is unknown. This is an extension of the dependence testing (DT) bound for TIN with a known number of active users proposed in [4, Sec. III]. We consider a random-coding scheme with the same encoder as in Theorem 1 and a TIN decoder. Specifically, the receiver first estimates the number of active users $K_a$ using the metric $m(y, K)$, as in the decoder analyzed in Theorem 1. Then, it outputs a list composed of the $K_a'$ codewords that are closest to the received signal $y$ in Euclidean distance, i.e.,

$$\hat{W} = \arg \min_{W = \{w'_1, \ldots, w'_l\} \subset [M]} \sum_{i=1}^{K_a'} \|y - c_{w'_i}\|^2.$$  \hspace{1cm} (49)

Operationally, the receiver decodes the message of each user by treating the signals of all other users as noise. An error analysis of this coding scheme gives the following result.

**Theorem 4 (Random-Coding Bound for TIN):** Fix $P' < P$ and nonnegative integers $K_e \leq K_u$. For the $K_u$-user Gaussian MAC with $K_a \sim P_{K_a}$, there exists an $(M, n, \epsilon_{MD}, \epsilon_{FA})$ random-access code satisfying the power constraint $P$ for which, for every $s > 0$,

$$\epsilon_{MD} = \mathbb{P}_{K_a = \max\{K_e, 1\}} \mathbb{P}_{K_a = K_e} \left( (K_a - K_a')^+ \xi(K_a, K_a') + \min\{K_a, K_a'\} \min \{\eta(s), \xi(K_a, K_a')\} \right) + \bar{p},$$

$$\epsilon_{FA} = \sum_{K_a = K_e}^{K_u} \sum_{K_a' = K_e}^{K_u} \mathbb{P}_{K_a}(K_a) \left( (K_a' - K_a)^+ \xi(K_a, K_a') + \min\{K_a, K_a'\} \min \{\eta(s), \xi(K_a, K_a')\} \right) + \bar{p},$$

where $\bar{p}$ is given in (8), $\xi(K_a, K_a')$ is given in (25), and

$$\eta(s) = \mathbb{P}_{s} \left[ \frac{1}{u} \sum_{i=1}^{n} ts(x_i; y_i) \leq \ln \left( \frac{M - K_u}{u} \right) \right].$$  \hspace{1cm} (52)

Here, $u$ is uniformly distributed on $[0, 1]$ and $[x_1 \ldots x_n]^T \sim \mathcal{CN}(0, \hat{P}I_n)$ with $\hat{P} = P'/\left(1 + (K_a - 1)P\right)$. Furthermore, given $x_i = x_i$, we have that $y_i \sim \mathcal{CN}(x_i, 1)$. Finally, $ts(\cdot; \cdot)$ is the generalized information density, given by

$$ts(x; y) = -s|y - x|^2 + \frac{|y|^2}{1 + s\hat{P}} + \ln(1 + s\hat{P}).$$  \hspace{1cm} (53)

**Proof:** See Appendix E.

When $K_u$ is fixed to $K_a$ and $K_a' = K_a$, by setting $K_e = K_a = K_u$ and $s = 1$, we obtain from Theorem 4 that

$$\epsilon_{MD} = \epsilon_{FA} = \eta(1) + \frac{M!}{MK_u(M - K_u)!} + \frac{\Gamma(n, nP/P')}{\Gamma(n)}.$$  \hspace{1cm} (54)

This bound is similar to the one presented in [4, Sec. III]. To simplify the computation of $\eta(s)$, we can use the normal approximation given in [29, Eq. (7)]. Specifically, the term
η(s) can be expanded as

$$
\eta(s) = Q\left(\frac{nC - \ln(M - K_a)}{\sqrt{nV}}\right) + O\left(\frac{1}{\sqrt{n}}\right)
$$

where $$I_s = E[I_s(x_i; y_i)]$$, $$V_s = E[I_s(x_i; y_i) - I_s]^2$$, and $$Q(\cdot)$$ is the Gaussian Q function. Note that, by choosing $$s = 1$$, we obtain the normal approximation for the AWGN channel with Gaussian input, which is capacity achieving but not optimal from a dispersion perspective [30]. It then follows that

$$
\eta(1) = Q\left(\frac{nC - \ln(M - K_a)}{\sqrt{nV}}\right) + O\left(\frac{1}{\sqrt{n}}\right),
$$

where $$C = I_1 = \ln(1 + \frac{1}{1 + (K_a - 1)^{\frac{1}{m'}}})$$ and $$V = V_1 = \frac{2P}{1 + (K_a - 1)^{\frac{1}{m'}}}$$.

### IV. NUMERICAL EXPERIMENTS

We numerically evaluate the proposed random-coding bound and compare it with the random-coding bound in [4] and with the performance of some unsourced multiple access schemes, namely, SA-MPR, TIN, and the schemes in [8] and [10]. We assume that $$K_a \sim \text{Pois}(E[K_a])$$ and the distribution of $$K_a$$ is known by the receiver. Poisson processes are commonly used to model message arrivals in packet-data networks and result in accurate models in many realistic IoT scenarios [31]. We assume an information payload per user of $$k = \log_2 M = 128$$ bits and consider a transmission duration of $$n = 19200$$ complex channel uses (i.e., 38400 real degrees of freedom). As performance metric, we consider the average energy per bit $$E_b/N_0 = nP/k$$ needed to operate at a given target MD and FA probabilities.

In Fig. 1, we compare the random-coding bound provided in Theorem 1 with that in [4, Th. 1] in terms of the required $$E_b/N_0$$ so that neither $$P_{MD}$$ nor $$P_{FA}$$ exceed $$10^{-1}$$. For our bound, we consider the ML estimation of $$K_a$$ and set the decoding radius $$r = 0$$, i.e., $$K_a = K_a' = K_a''$$. Numerical experiments indicate that this choice of decoding radius is optimal for the targeted MD and FA probabilities.\(^8\) We choose $$K_a$$ to be the largest value and $$K_a$$ the smallest value for which $$\mathbb{P}[K_a \notin [K_a, K_a]] < 10^{-9}$$.

Due to computational complexity, the terms $$q_t$$ and $$q_{t,t'}$$ are evaluated for $$t = 1$$ and $$K_a \leq 50$$ only. For the bound in [4, Th. 1], we average over the Poisson distribution of $$K_a$$. This corresponds to the scenario in which $$K_a$$ is random but known to the receiver. As can be seen, similar to the bound in [4, Th. 1], our bound exhibits two behaviors as a function of the average number of active users: one where the required $$E_b/N_0$$ is almost constant and finite-blocklength effect dominates, and one where it grows with the average number of active users and multi-user interference dominates. To achieve $$\max\{P_{MD}, P_{FA}\} \leq 10^{-1}$$, the extra required $$E_b/N_0$$ due to the lack of knowledge of $$K_a$$ is about 0.5–0.7 dB. This small gap suggests that for the mild target $$\max\{P_{MD}, P_{FA}\} \leq 10^{-1}$$, lack of knowledge of the number of active users entails a small penalty in terms of energy efficiency.

\(^8\) We discuss the trade-offs resulting from the choice of the decoding radius later on in this section.

In Fig. 1, we also show the performance of the SA-MPR bound given in Corollary 3 where we optimize $$L$$ and the decoding radius for each $$E[K_a]$$. We further consider the possibility to encode $$\lceil\log_2 L\rceil$$ extra bits for each user in the slot index, and assume perfect decoding of these bits. We refer to this scheme as SA-MPR with slot-index coding. Furthermore, we plot the RCU bounds on the performance of TIN given in Theorem 4 optimized over $$s$$, and its normal approximation evaluated with $$s = 1$$. We also evaluate the performance of two state-of-the-art practical schemes, namely:

- the SPARC scheme proposed in [8], which employs a concatenated coding structure, consisting of an inner approximate message passing (AMP) decoder followed by an outer tree decoder;
- an enhancement, proposed in [10], of the SPARC scheme in [8], where belief propagation between the inner AMP decoder and the outer tree decoder is introduced.

---

**Fig. 1.** The required $$E_b/N_0$$ to achieve $$\max\{P_{MD}, P_{FA}\} \leq 10^{-1}$$ as a function of $$E[K_a]$$ for $$k = 128$$ bits, $$n = 19200$$ channel uses, and $$K_a \sim \text{Pois}(E[K_a])$$. Solid lines represent schemes/bounds with $$K_a$$ unknown; dashed lines represent schemes/bounds with $$K_a$$ known.
Note that the SPARC and enhanced SPARC schemes were proposed for the Gaussian MAC with known number of active users. To adapt these schemes to the case of unknown $K_a$, we follow the approach discussed in Remark 5: we first employ an energy-based estimation of $K_a$, and then treat this estimate as the true $K_a$ in the decoding process. As shown in Fig. 1, TIN (light blue) and SA-MPR (pink), even with slot-index coding, become energy inefficient as $\mathbb{E}[K_a]$ increases. The enhanced SPARC scheme (orange) achieves the closest performance to our bound for $\mathbb{E}[K_a] \geq 100$ and outperforms the original SPARC scheme by about 0.5 dB for large $\mathbb{E}[K_a]$. The performance of enhanced SPARC with known $K_a$ is only slightly better than that with unknown $K_a$. This suggests that, for the considered mild requirements on $P_{MD}$ and $P_{FA}$, it is effective to adapt existing coding schemes that perform well for known $K_a$ by simply adding a $K_a$-estimation step, and then treating the estimate of $K_a$ as the true value.

In Fig. 2, we plot the bounds on $P_{MD}$ and $P_{FA}$ in Theorem 1 as a function of $E_b/N_0$ for $\mathbb{E}[K_a] \in \{50, 200\}$, and different decoding radius $r$. We also show the bound in [4, Th. 1] for $K_a$ known. The MD and FA probabilities drop at a certain $E_b/N_0$, and then saturate to error floors. The existence of a waterfall regime and an error-floor regime is more evident in the case $\mathbb{E}[K_a] = 200$. The error floors are due to the initial MDs and FAs, and are characterized by $\epsilon_{MD}$ and $\epsilon_{FA}$ in Corollary 1. The ensemble converse in Theorem 3 computed for the i.i.d. Gaussian ensemble with average power $P$ for the considered decoding radii is also depicted. We observe that the converses for the MD and FA probabilities are similar, and follow closely our achievability bounds after the waterfall. This explains the waterfall effect: the additional MDs and FAs occurring in the decoding process dominate when the $E_b/N_0$ is small, while the initial MDs and FAs dominate in the error-floor regime. We further observe that setting $r = 0$ leads to $\epsilon_{MD} = \epsilon_{FA}$ and turns out optimal in the low $E_b/N_0$ regime, where noise overfitting is the bottleneck. This is in agreement with the results reported in Fig. 1. An increase of the decoding radius results in a performance improvement in the moderate and high $E_b/N_0$ regime, where setting $r = 0$ yields high error floors. Our numerical results indicate that zero decoding radius should be used when the target MD and FA probabilities are higher than about $8 \times 10^{-3}$ for $\mathbb{E}[K_a] = 50$ and $6.5 \times 10^{-3}$ for $\mathbb{E}[K_a] = 200$. Otherwise, if the target MD and FA probabilities are low, one should increase the decoding radius to lower the error floors and thus meet the requirements in the waterfall regime. Remarkably, as illustrated in Fig. 2, there is a sharp difference between the case $r = 0$ and the case $r > 0$ in terms of the $E_b/N_0$ value at which the waterfall region starts. Moreover, this $E_b/N_0$ value appears to coincide for all positive $r$.

When asymmetric requirements on $P_{MD}$ and $P_{FA}$ are considered, one can adjust the decoding interval $[K_a^r : K_a^u]$ in (5) to achieve different MD-FA trade-offs. Specifically, one can set $K_a^r = \max\{K_a, K_a^u - r\}$ and $K_a^u = \min\{K_a, K_a^u + r\}$ where $(r_l, r_u)$ is a pair of lower and upper decoding radii. The random-coding bound in Theorem 1 can be extended accordingly. In Fig. 3, we consider this extension and show the required $E_b/N_0$ to achieve different MD-FA trade-offs. Specifically, we can set $K_a^r = \max\{K_a, K_a^u - r\}$ and $K_a^u = \min\{K_a, K_a^u + r\}$ where $(r_l, r_u)$ is a pair of lower and upper decoding radii. The random-coding bound in Theorem 1 can be extended accordingly. In Fig. 3, we consider this extension and show the required $E_b/N_0$ to achieve different MD-FA trade-offs. Specifically, one can set $K_a^r = \max\{K_a, K_a^u - r\}$ and $K_a^u = \min\{K_a, K_a^u + r\}$ where $(r_l, r_u)$ is a pair of lower and upper decoding radii. The random-coding bound in Theorem 1 can be extended accordingly. In Fig. 3, we consider this extension and show the required $E_b/N_0$ to achieve different MD-FA trade-offs.
the same for the two requirements with $P_{FA} \leq 10^{-3}$, and it is slightly lower for the requirement with $P_{FA} \leq 10^{-1}$. We also plot the bound for $K_a$ known [4, Th. 1]. In contrast to the scenario in Fig. 1, where the gap between our bound and the bound for $K_a$ known was small, the gap is now significantly larger. Specifically, our bound suggests that one needs additional 3–4 dB in $E_b/N_0$ when $P_{MD}$ and/or $P_{FA}$ are required to be lower than $10^{-3}$ when our random-coding scheme is employed. This gap tends to be smaller as $\mathbb{E}[K_a]$ increases. It remains unclear whether this gap is fundamental. The slope of the curves for the bound in Theorem 1 suggests that the dominating factor for $\mathbb{E}[K_a]$ ≤ 300 is still the finite-blocklength effect, and the multi-user interference will kick in for higher $\mathbb{E}[K_a]$. In Fig. 3, we also show the required $E_b/N_0$ for the enhanced SPARC scheme with $K_a$ known/unknown. We further plot a lower bound on this $E_b/N_0$ value resulting from evaluating the right-hand sides of (46) and (47) for the distribution on the received signal $y$ induced by the SPARC codebook and for $r = 0.8$. Recall that the lower bound is obtained by counting the MDs and FAs that occur in the estimation step only, while assuming that no additional MD or FA occurs in the decoding step. Recall also that we adapt the enhanced SPARC scheme to the case of unknown $K_a$ by simply treating the estimate of $K_a$ as the true $K_a$. The performance of the enhanced SPARC scheme with $K_a$ unknown is very close to the lower bound and drastically worse than enhanced SPARC with $K_a$ known. This confirms that the estimation step is indeed the bottleneck. Furthermore, the lower bound for enhanced SPARC exhibits a large gap to the achievability bound in Theorem 1. For example, the gap is about 7 dB for only 25 active users in average. This large gap suggests that this approach, which simply uses the estimate of $K_a$ to set the decoded list size and relies on existing coding schemes proposed for $K_a$ known, becomes energy inefficient for stringent requirements on $P_{MD}$ and/or $P_{FA}$. This calls for more sophisticated methods to overcome the bottleneck of estimating $K_a$ and handle effectively the uncertainty about the number of active users.

In Fig. 4, we plot the bounds $\epsilon_{MD}$ and $\epsilon_{FA}$ in Theorem 1 as a function of $E_b/N_0$ for $\mathbb{E}[K_a] \in \{50, 200\}$ for the decoding radii considered in Fig. 3. Solid lines represent $\epsilon_{MD}$, while dashed lines represent $\epsilon_{FA}$. We observe again that the waterfall region of either $\epsilon_{MD}$ or $\epsilon_{FA}$ starts at a similar $E_b/N_0$ value for various values of $(r_e, r_u)$ different from $(0, 0)$. We also observe that, after the waterfall, our achievability bounds approach closely the ensemble converse in Theorem 3. The $E_b/N_0$ value that satisfies the requirements on both $P_{MD}$ and $P_{FA}$ is dictated by the $E_b/N_0$ value that satisfy the requirement on $P_{FA}$. This explains why the $E_b/N_0$ values according to Theorem 1, shown in Fig. 3, is similar for the two requirements with $P_{FA} \leq 10^{-3}$.

V. DISCUSSION

In this section, we provide some additional remarks on our choice of using a two-step decoder in our random-coding achievability bound, and the challenges involved in obtaining a general converse bound.

A. The Two-Step Decoder

An alternative to the proposed two-step decoder, which first estimates the number of active users and then the list...
of messages, is a joint decoder that estimates both at the same time. Such a joint decoder operates according to the following rule
\[
\hat{\mathcal{W}} = \arg \min_{W' \subseteq [M]: K_e \leq |W'| \leq K_u} \|c(W') - y\|^2
\]  
(55)

where the limits \(K_e\) and \(K_u\) are chosen based on prior knowledge on the distribution of \(K_a\). Note that this decoder is a special case of our two-step decoder when i) the \(K_a\)-estimation step (4) is skipped, and ii) \(K_e' = K_e\) and \(K_u' = K_u\) in the message-decoding step (5). Thus, a random-coding bound for the joint decoder (55) follows directly from Theorem 1. This bound is stated in the following corollary.

**Corollary 4 (Random-Coding Bound for the Joint Decoder):** Fix \(P' < P, K_e,\) and \(K_u (K_e \leq K_u)\). For the \(K_a\)-user Gaussian MAC with \(K_a \sim P_{K_a}\), there exists an \((M, n, \epsilon_{MD}, \epsilon_{FA})\) random-access code satisfying the power constraint \(P\) for which
\[
\epsilon_{MD} = \sum_{K_a = \max\{K_e, 1\}}^K P_{K_a}(K_a) \sum_{t = 0}^{K_a} \frac{t}{K_a} \min\{p_t, q_t\} + \hat{\rho},
\]
\[
\epsilon_{FA} = \sum_{K_a = K_e}^K P_{K_a}(K_a) \sum_{t = 0}^{K_a} \sum_{t' \in T_t} \frac{t'}{K_a - t + t'} \min\{p_{t,t'}, q_{t,t'}\} + \hat{\rho},
\]
(56)
\(57\)

where \(\hat{\rho}, p_t, p_{t,t'}, q_t,\) and \(q_{t,t'}\) are given by (8), (9), (10), (19), and (20), respectively, with \(K_e' = K_e\) and \(K_u' = K_u\).

Unfortunately, the bound in Corollary 4 results in a low energy efficiency. For example, for a similar setting as in Fig. 1, i.e., \(k = 128\) bits, \(n = 19200\) channel uses, and \(K_a \sim \text{Pois}(E[K_a])\), to achieve \(\max\{P_{MD}, P_{FA}\} \leq 10^{-4}\), the required \(E_b/N_0\) for the joint decoder is between 4.5 dB and 5 dB for \(E[K_a] \in [25 : 300]\). To understand the drawback of this bound, we shall now inspect the terms \(p_t\) and \(p_{t,t'}\). Recall that 
\[
p_t = \sum_{t' \in \mathcal{T}_t} p_{t,t'} \quad \text{and} \quad p_{t,t'} = e^{-\rho \rho'}(t, t') \text{with the error exponent } E(t, t') = -\rho\rho' R_1 - \rho R_2 + E_0(p, p_1).
\]
Here, \(E_0(p, p_1)\) stems from the Chernoff bound on the probability of the pairwise error event \(\|c(W') - y\|^2 \leq \|c(W) - y\|^2\), while \(R_1\) and \(R_2\) stem from a tightened union bound over all possible sets of \(t'\) falsely alarmed messages and \(t\) misdetected messages, respectively. These terms scale differently with \(t'\), making it nontrivial to understand how \(p_{t,t'}\) varies with \(t'\). Specifically, for fixed \(t, P',\) and \(M,\) the term \(-\rho R_2\) is a constant, \(E_0(p, p_1)\) increases logarithmically with \(t'\), and \(-\rho \rho' R_1\) decreases linearly with \(t'\).

In Fig. 5 and Fig. 6, we plot the values of \(p_{t,t'}\) and \(p_t\) for \(k = 128\) bits, \(n = 19200\) channel uses, and \(K_a = E[K_a] = 50\). We set \(K_e = (E[K_a] - r)^+\) and \(K_u = E[K_a] + r\) for a chosen nonnegative integer \(r\). We first consider \(r = r_0\), where \(r_0\) is the largest value of \(r\) such that \(P[K_a \notin [K_e : K_u]] < 10^{-3}\). In Fig. 5, we consider \(E_b/N_0 = 2\) dB, which yields a small \(P'\). As shown in Fig. 5(a), \(p_{t,t'}\) increases with \(t'\) for a fixed \(t\). This is because for a small \(P'\), the term \(E_0(p, p_1)\) increases slowly with \(t'\), and the term \(-\rho \rho' R_1\) dominates for all values of \(t'\), driving the error exponent \(E(t, t')\) towards 0. When \(r = r_0\), the interval \([K_e : K_u] = \text{large}\). Therefore, both \(T_t = (\max\{K_e, 1\} - K_a + t)^+ : (K_u - K_a + t)\) and \(T_{t'} = [K_e - K_u + t)^+ : (K_u - K_u + t)\) contain large values of \(t'\) for which \(p_{t,t'}\) is close to 1. As a consequence, \(\epsilon_{FA}\) is large due to the sum over \(t' \in T_t\) in (57). Furthermore, \(p_{t,t'} = \sum_{t' \in T_t} p_{t,t'}\) is also large, as seen in Fig. 5(b), leading to a large \(\epsilon_{MD}\). In Fig. 6, we consider \(E_b/N_0 = 4.6\) dB, i.e., a higher \(P'\). As shown in Fig. 6(a), for a fixed \(t,\) the term \(p_{t,t'}\) first decreases and then increases with \(t'\). This is because when \(P'\) is sufficiently high, \(E_0(p, p_1)\) dominates for small \(t'\) but \(-\rho \rho' R_1\) eventually dominates as \(t'\) grows. However, \(p_{t,t'}\) increases rather slowly with \(t'\) and remains small for all \(t' \in T_t\). It follows that \(p_t = \sum_{t' \in T_t} p_{t,t'}\) is small, as seen in Fig. 6(b).
Fig. 5. The bounds $p_{t,t'}$ in (10) and $p_t$ in (9) applied to the joint decoder (55) for $k = 128$ bits, $n = 19200$ channel uses, $K_a = E[K_a] = 50$, and $E_b/N_0 = 2$ dB. Here, $K_{\ell} = (E[K_a] - r)^+ + K_u = E[K_a] + r$; $r_0$ is the largest value of $r$ such that $P[K_a \notin [K_{\ell} : K_u]] < 10^{-9}$ with $K_a \sim \text{Pois}(50)$.

Fig. 6. The bounds $p_{t,t'}$ in (10) and $p_t$ in (9) for the setting in Fig. 5 but with $E_b/N_0 = 4.6$ dB.

(a) The bound $p_{t,t'}$ on the probability of having $t$ misdetections and $t'$ false alarms for a given $t$ and $t' \in T_t = [(K_t - K_a + t)^+ : (K_u - K_a + t)]$.

(b) The bound $p_t = \sum_{t' \in T_t} p_{t,t'}$ on the probability of having $t$ misdetections.

The fact that $p_{t,t'}$ increases quickly with $t'$ at low $E_b/N_0$ seems to indicate that the joint decoder tends to commit many false alarms in this regime. While it remains unclear if this interpretation is correct, we provide a possible explanation as follows. At low $E_b/N_0$, i.e., when the noise dominates, the joint decoder often returns a list (of typically big size) of wrong codewords whose sum is closer to $y$ than to the sum of the transmitted codewords. Although it is unlikely that $\|c(W') - y\|^2 \leq \|c(W) - y\|^2$ for a given set $W'$ of large size, the probability that this is true for at least one of $M$ possible sets is still significant since $M$ is large. In essence, by searching over a large set of codewords, the decoder ends up approximating the additive noise component in the received signal. We refer to this effect as “noise overfitting”.

Indeed, this rapid increase may simply be due to the looseness of Gallager’s $\rho$-trick. To mitigate noise overfitting, we simply reduce the feasible set over which the minimization in (55) is performed. This is similar to using an inductive bias to restrict the hypothesis class in order to overcome overfitting in statistical learning [32, Sec. 2.3]. Specifically, we reduce $r$. Indeed, this allows us to avoid large values of $t'$ in $T_t$ and $\mathcal{T}_t$, and thus reduce $p_t$. For the scenario in Fig. 5, setting $r = 1$ results in $T_t = \mathcal{T}_t = [(t - 1)^+ : (t + 1)]$. These intervals do not contain large values of $t'$ for which $p_{t,t'}$ is close to 1, as shown in Fig. 5(a). It follows that $p_t$ is drastically reduced, as shown in Fig. 5(b). Furthermore, to adapt the inductive bias to the received signal, we choose the feasible set based on the estimate $K'_a$ of $K_a$. Specifically, we replace the feasible set in (55) with the set $\{W' \subset [M] : K'_a \leq |W'| \leq K'^+_a\}$ where $[K'_a : K'^+_a]$ is an interval around $K'_a$. We control the size of this interval via the decoding radius $r$. It turns out that, to satisfy mild requirements on the MD and FA probabilities, setting
$K_a = K_a' = K_a''$, i.e., $r = 0$, leads to high energy efficiency, whereas for more stringent requirements, the interval $[K_a' : K_a'']$ should be progressively enlarged.

An alternative method to overcome noise overfitting is to introduce a regularization term in (55) that penalizes a choice of $|W|$ far from $E[K_a]$. We have tried this method but did not obtain a better bound than the one provided by the two-step decoder.

In short, although our two-step decoder might be suboptimal, it effectively mitigates noise overfitting and achieves the highest energy efficiency among the approaches that we have considered.

**B. A General Converse**

A general converse bound on the MD and FA probabilities appears difficult to obtain. A possible approach is to assume that $K_a$ is known to the receiver. However, even for this case, a tight converse bound is not available in the literature. In [4], only a conjectured converse bound was provided. A converse bound on the required power to achieve a target MD probability $\epsilon_{MD}$ for fixed and known $K_a$ was reported in [33]. This converse is based on two different approaches. In the first approach, one casts a UMA code as a single-user code with list decoding, and applies the result on minimum energy to send $k$ bits through the Gaussian channel [34]. Here, the list size is $K_a$. In the second approach, one computes the rate-distortion function between two binary vectors that indicate the transmitted and decoded messages. The average distortion between these vectors is bounded by $2K_a\epsilon_{MD}$. Then, this rate-distortion function is upper-bounded by the sum-capacity $n \log(1 + K_aP)$ of the Gaussian MAC. The bound obtained via the first approach dominates when $K_a$ is small whereas the one obtained via the second approach dominates when $K_a$ is large. However, this second bound holds only for $\epsilon_{MD} \leq \frac{1}{K_a}$. For $\epsilon_{MD} > \frac{1}{K_a}$, which typically holds in massive IoT applications, one needs to rely on the bound obtained from the first approach, which exhibits a large gap from the achievability bound for large $K_a$.

The converse in [33] does not generalize naturally to the case of unknown $K_a$, where both the MD and FA probabilities need to be considered, for the following two reasons. First, in the list-decoding-based approach, the list size $K_a$ is not fixed and known, and the error event associated with single-user list decoding accounts for MD only. Second, in the rate-distortion-based approach, it is nontrivial to express/bound the average distortion between the aforementioned binary vectors in terms of the target MD and FA probabilities.

**VI. Conclusion**

To account for the random user activity in the IoT, we proposed a formulation for unsecured multiple access where both the identity and the number of active users are unknown. We derived a random-coding bound for the Gaussian MAC that reveals a trade-off between misdetection and false alarm. Our bound provides an estimate of the penalty in terms of energy efficiency due to the lack of knowledge of the number of active users, and serves as a benchmark to assess the performance of practical schemes. Numerical results show that for the Gaussian MAC, if the target misdetection and false-alarm probabilities are sufficiently high, e.g., $10^{-1}$, the lack of knowledge of the number of active users entails a small loss. In this case, it is effective to adapt a coding scheme that performs well for the case of known number of active users, by first estimating the number of active users and then treating this estimate as the true value in the decoding process. However, for stringent target misdetection and false-alarm probabilities, e.g., $10^{-3}$, numerical results suggest that the loss due to the lack of knowledge of the number of active users might be significant. It remains unclear if this is loss is fundamental or pertains to the considered random-coding scheme only. For stringent requirements, adapting existing coding schemes proposed for known number of active users by simply treating the estimate of the number of active users as perfect is energy inefficient even for a small number of active users. Therefore, more sophisticated methods are needed to handle effectively the uncertainty about the number of active users.

**Appendix A**

**Proof of Theorem 1**

The following well-known results will be used in the proof.

**Lemma 1** (Change of Measure [35, Lemma 4]): Let $p$ and $q$ be two probability measures. Consider a random variable $x$ supported on $\mathcal{H}$ and a function $f : \mathcal{H} \to [0, 1]$. It holds that

$$E_p[f(x)] \leq E_q[f(x)] + d_{TV}(p, q)$$

where $d_{TV}(p, q)$ denotes the total variation distance between $p$ and $q$.

**Lemma 2** (Chernoff bound [36, Th. 6.2.7]): For a random variable $x$ with moment-generating function $E[e^{px}]$ defined for all $|t| \leq b$, it holds for all $\lambda \in [0, b]$ that

$$P[x \leq x] \leq e^{\lambda x} E[e^{-\lambda x}]$$

**Lemma 3** (Gallager’s $\rho$-trick [26, p. 136]): It holds that $P[\cup_1 A_i] \leq (\sum_1 P[A_i])^\rho$ for every $\rho \in [0, 1]$.

**Lemma 4**: Let $x \sim \mathcal{CN}(\mu, \sigma^2 I_n)$. It holds that

$$E[e^{-\gamma \|x\|^2}] = (1 + \gamma \sigma^2)^{-n} \exp \left( -\gamma \|\mu\|^2 \right) \frac{1}{1 + \gamma \sigma^2}, \forall \gamma > \frac{1}{\sigma^2}.$$  

**Proof:** We write $E[e^{-\gamma \|x\|^2}] = E\left[\exp \left( -\frac{\gamma}{2} \|\frac{x}{\sigma}\|^2 \right)\right]$. Notice that $\|\frac{x}{\sigma}\|^2$ follows the noncentral chi-square distribution with $2n$ degrees of freedom and noncentrality parameter $\frac{2\|\mu\|^2}{\sigma^2}$. We obtain (60) by using that the moment-generating function of this distribution is known in closed form.

We present next an error analysis of the random-coding scheme introduced in Section III. Denote by $\mathcal{W}_{MD}$ the set

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10In [33, Slide 31], the step $E[S \log S] \geq t \log t$ holds for $t \in (0, 1/e]$ only. Here, $S$ is the number of mistyped messages and $t = K_a\epsilon_{MD}$. Since $\epsilon_{MD}$ is the target MD probability, we have that $E[S] \leq t$. Jensen’s inequality implies that $E[S \log S] \geq E[S] \log E[S]$, but $E[S \log E[S]] \geq t \log t$ only if $t \in (0, 1/e]$. Therefore, the second bound in the converse reported in [33] holds only for $\epsilon_{MD} \leq \frac{1}{K_a}$. 

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of misdetected messages, i.e., \( W_{\text{MD}} = \tilde{W} \setminus \tilde{\tilde{W}} \), and by \( W_{\text{FA}} \) the set of falsely alarmed messages, i.e., \( W_{\text{FA}} = \tilde{W} \setminus \tilde{W} \). The MD and FA probabilities, defined respectively in (2) and (3), can be expressed as the average fraction of misdetected and falsely alarmed messages as

\[
P_{\text{MD}} = \mathbb{E}\left[ \frac{|W_{\text{MD}}|}{|W|} \right], \quad (61)
\]

\[
P_{\text{FA}} = \mathbb{E}\left[ \frac{|W_{\text{FA}}|}{|W|} \right]. \quad (62)
\]

### A. Change of Measure

Since \( \frac{|W_{\text{MD}}|}{|W|} \) and \( \frac{|W_{\text{FA}}|}{|W|} \) are nonnegative random variables that are upper-bounded by one, we can apply Lemma 1. Specifically, we replace the measure over which the expectation is taken by the one under which: i) there are at least \( K_\ell \) and at most \( K_u \geq K_a \) active users, i.e., \( K_\ell \leq K_a \leq K_u \); ii) the active users transmit distinct messages, i.e., \( |W| = K_a \) and \( \tilde{w}_1, \ldots, \tilde{w}_{K_a} \) are sampled uniformly without replacement from \( [M] \); iii) \( x_i = c_{w_i}, \forall i \), instead of \( x_i = c_{w_i}, \forall i \in \{\|c_{w_i}\|^2 \leq nP\} \).

It then follows from [37, Eq. (41)] that the total variation between the original measure and the new one is upper-bounded by \( \mathbb{P}\{K_a \notin [K_\ell : K_u]\} + \mathbb{P}\{|\tilde{W}| < K_a\} + \mathbb{P}\{U\} \), where \( U = \{\|c_{w_{i}}\|^2 \leq nP, \forall i \in [K_a]\} \) and \( \tilde{U} \) denotes the complement of \( U \). We compute these probabilities as follows:

- To compute the first probability, we simply use that

\[
\mathbb{P}\{K_a \notin [K_\ell : K_u]\} = 1 - \sum_{K_a=K_\ell}^{K_u} P(K_a). \quad (63)
\]

- To evaluate \( \mathbb{P}\{|\tilde{W}| < K_a\} \), consider a given \( K_a = K_a \). Since \( w_1, \ldots, w_{K_a} \) are drawn uniformly and independently from \( [M] \), there are \( M^{K_a} \) possible \( K_a \)-tuples. Among them, \( (M!/(M-K_a)!M^{K_a}) \) tuples have nonduplicate elements. Therefore, \( \mathbb{P}\{|\tilde{W}| = K_a\} = \frac{M!}{(M-K_a)!M^{K_a}} \). As a consequence,

\[
\mathbb{P}\{|\tilde{W}| < K_a\} = 1 - \mathbb{P}\{|\tilde{W}| = K_a\} \quad (64)
\]

- The probability \( \mathbb{P}\{U\} \) can be finally evaluated as

\[
\mathbb{P}\{U\} = \mathbb{E}_{K_a} \left[ \sum_{i=1}^{K_a} \mathbb{P}\{\|c_{w_i}\|^2 > nP\} \right] \quad (65)
\]

\[
\leq \mathbb{E}_{K_a} \sum_{i=1}^{K_a} \frac{\mathbb{P}\{\|c_{w_i}\|^2 > nP\}}{\Gamma(n/nP/P')} \quad (66)
\]

\[
= \mathbb{E}\left[ K_a \frac{\Gamma(n, nP/P')}{\Gamma(n)} \right] \quad (67)
\]

where (66) follows from the union bound and (67) holds since \( \|c_{w_i}\|^2 \) follows the Gamma distribution with shape \( n \) and scale \( P' \).

From the above calculations, we deduce that the total variation between the two measures is upper-bounded by \( \bar{p} \) defined in (8). Hence, applying Lemma 1 to the random quantities \( \frac{|W_{\text{MD}}|}{|W|} \) and \( \frac{|W_{\text{FA}}|}{|W|} \), we consider implicitly the new measure from now on at a cost of adding \( \bar{p} \) to the original expectations.

It remains to bound the MD and FA probabilities given in (61) and (62), respectively, under the new measure. For the sake of clarity, in Appendix A-B, we shall prove a bound on \( P_{\text{MD}} \) and \( P_{\text{FA}} \) for a special case where i) \( K_a \) and \( K_a' \) are fixed and \( r = 0 \), i.e., there are always \( K_a \) users transmitting and the decoder always outputs a list of size \( K_a' \); ii) \( K_a' \leq \min\{K_a, M - K_a\} \). Later, in Appendix A-C, we shall show how to extend the proof to the general case where \( K_a \) and \( K_a' \) are random and \( r \geq 0 \).

### B. A Special Case

In the aforementioned special case, (6) and (7) become

\[
e_{\text{MD}} = \sum_{t=0}^{K_a' - K_a} \frac{t + K_a - K_a'}{K_a} \min\{p_{t,t}, q_{t,t}\} + \bar{p}, \quad (68)
\]

\[
e_{\text{FA}} = \sum_{t=0}^{K_a' - K_a} \frac{t}{K_a} \min\{p_{t,t}, q_{t,t}\} + \bar{p}, \quad (69)
\]

where \( p_{t,t} \) and \( q_{t,t} \) will be given shortly. The task is to show that \( e_{\text{MD}} \) and \( e_{\text{FA}} \) are indeed upper bounds of \( P_{\text{MD}} \) and \( P_{\text{FA}} \), respectively, in this special case.

Since the decoded list size \( K_a' \) is smaller than the number of transmitted messages \( K_a \), \( K_a - K_a' \) messages are initially misdetected, and there can be \( t \in [0 : K_a'] \) additional MDs occurring during the decoding process. Exploiting symmetry, we assume without loss of generality (w.l.o.g.) that \( \tilde{W} = K_a \) and that the list of messages that are initially misdetected due to insufficient decoded list size is \( W_{\text{MD}} = [K_a - K_a'] \). Furthermore, let \( W_{\text{AMD}} = W_{\text{MD}} \cup W_{\text{MD}} \) denote the set of \( t \) additional MDs. Note that \( W_{\text{AMD}} \) is a generic subset of \( [K_a - K_a' + 1 : K_a] \). Note also that \( t \) is the number of FAs, i.e., \( |W_{\text{FA}}| = t \). The relation between these sets of messages is depicted in Fig. 7.

Using the above definitions, the set of transmitted messages can be expressed as

\[
W = \tilde{\tilde{W}} = W_{\text{MD}} \cup W_{\text{AMD}} \cup (\tilde{W} \setminus W_{\text{MD}}),
\]

and the received signal is

\[
y = c(W_{\text{MD}}) + c(W_{\text{AMD}}) + c(\tilde{W} \setminus W_{\text{MD}}) + z,
\]

Fig. 7. A diagram depicting the relation between the sets of messages defined for the special case in Appendix A-B.
Since the messages in $W_{\text{IMD}}$ are always misdetected, the best approximation of $W$ that the decoder can produce is $W_{\text{AMD}} \cup (\hat{W} \setminus W_{\text{MD}})$. However, under the considered error event $\hat{W} \to W$, the messages in $W_{\text{AMD}}$ are misdetected by the ones in $W_{\text{FA}}$, and thus the actual decoded list is $W_{\text{FA}} \cup (\hat{W} \setminus W_{\text{MD}})$. Therefore, $\hat{W} \to W$ implies that $\|y - c(W_{\text{AMD}}) - c(W_{\text{FA}})\| < \|y - c(W_{\text{AMD}}) - c(W_{\text{MD}})\|^2$, which is equivalent to
\[
\|c(W_{\text{IMD}}) + c(W_{\text{AMD}}) - c(W_{\text{FA}}) + z\|^2 < \|c(W_{\text{IMD}}) + c(W_{\text{AMD}}) + z\|^2.
\]

(70)

Let $F(W_{\text{IMD}}, W_{\text{AMD}}, W_{\text{FA}})$ denote the set of $(W_{\text{IMD}}, W_{\text{AMD}}, W_{\text{FA}})$ such that (70) holds.

We now compute the expectations in (61) and (62). Recall that, under assumptions just stated, we have $|W_{\text{MD}}| = t + K_\lambda - K'_{\lambda'}$, $|W_{\text{FA}}| = |W_{\text{AMD}}| = t$, and $|\hat{W}| = K'_{\lambda'}$. It follows from (61) and (62) that, after the change of measure in Appendix A-A, $P_{\text{MD}}$ and $P_{\text{FA}}$ can be bounded as
\[
P_{\text{MD}} \leq \sum_{t=0}^{K'_{\lambda'}} \frac{t + K_\lambda - K'_{\lambda'}}{K_\lambda} \mathbb{P}[|W_{\text{AMD}}| = t] + \bar{p},
\]
\[
P_{\text{FA}} \leq \sum_{t=0}^{K'_{\lambda'}} \frac{t}{K_\lambda} \mathbb{P}[|W_{\text{AMD}}| = t] + \bar{p}.
\]

(71)
(72)

Next, we proceed to bound $\mathbb{P}[|W_{\text{AMD}}| = t]$ following two approaches. The first approach is based on error exponent analyses, resulting in the term $p_{t,t}$ in (68). The second approach is a variation of the DT bound [38, Th. 17], resulting in $q_{t,t}$ in (68).

1) The Error-Exponent-Based Approach: By writing the event $|W_{\text{AMD}}| = t$ as a union of the pairwise error events $F(W_{\text{IMD}}, W_{\text{AMD}}, W_{\text{FA}})$, we have that
\[
\mathbb{P}[|W_{\text{AMD}}| = t] = \mathbb{P}\left[ \bigcup_{|W_{\text{AMD}}| = t} F(W_{\text{IMD}}, W_{\text{AMD}}, W_{\text{FA}}) \right].
\]

(73)

Next, given $c(W_{\text{IMD}})$, $c(W_{\text{AMD}})$, and $z$, it holds for every $\lambda > -1$ that
\[
\mathbb{P}[F(W_{\text{IMD}}, W_{\text{AMD}}, W_{\text{FA}})] \
\leq e^\lambda \|c(W_{\text{IMD}}) + z\|^2 \cdot \mathbb{E}_{c(W_{\text{FA}})}[e^{-\lambda \|c(W_{\text{IMD}}) + c(W_{\text{AMD}}) - c(W_{\text{FA}}) + z\|^2}]
\leq e^\lambda \|c(W_{\text{IMD}}) + z\|^2 (1 + \lambda t P')^{-n} \cdot \exp\left( -\lambda \|c(W_{\text{IMD}}) + c(W_{\text{AMD}}) + z\|^2 \right),
\]

(74)
(75)

where (74) follows from the Chernoff bound in Lemma 2, and (75) follows by computing the expectation in (74) using Lemma 4. Next, we apply Gallager’s $\rho$-trick in Lemma 3 and conclude that, given $c(W_{\text{IMD}})$, $c(W_{\text{AMD}})$, and $z$, it holds for every $\rho \in [0, 1]$ that
\[
\mathbb{P}\left[ \bigcup_{|W_{\text{FA}}| = |K_{\lambda'} + M|} F(W_{\text{IMD}}, W_{\text{AMD}}, W_{\text{FA}}) \right] \leq \left( M - K_\lambda \right)^\rho \left( 1 + \lambda t P' \right)^{-n} \cdot \exp\left( -\lambda \|c(W_{\text{IMD}}) + z\|^2 \right),
\]

(76)

Taking the expectation over $c(W_{\text{AMD}})$ using Lemma 4, we obtain for given $c(W_{\text{IMD}})$ and $z$ that
\[
\mathbb{P}\left[ \bigcup_{|W_{\text{FA}}| = |K_{\lambda'} + M|} F(W_{\text{IMD}}, W_{\text{AMD}}, W_{\text{FA}}) \right] \leq \left( M - K_\lambda \right)^\rho \left( 1 + \lambda t P' \right)^{-n} \cdot \mathbb{E}[\exp(\rho |c(W_{\text{IMD}}) + z\|^2 - n \rho a_0)]
\leq \left( M - K_\lambda \right)^\rho \left( 1 + \lambda t P' \right)^{-n} e^{-n \rho a_0} (1 - \rho_1 P_2 b_0)^{-n},
\]

(77)
(78)
(79)

where $a_0$ and $b_0$ are obtained by taking $t' = t$ in (13) and (14), respectively. Now applying Gallager’s $\rho$-trick again, we obtain that, for every $\rho_1 \in [0, 1]$,
\[
\mathbb{P}\left[ \bigcup_{|W_{\text{FA}}| = |K_{\lambda'} + M|} F(W_{\text{IMD}}, W_{\text{AMD}}, W_{\text{FA}}) \right] \leq \left( M - K_\lambda \right)^\rho \left( 1 + \lambda t P' \right)^{-n} \cdot \mathbb{E}[\exp(\rho_1 |c(W_{\text{IMD}}) + z\|^2 - n \rho_1 a_0)]
\leq \left( M - K_\lambda \right)^\rho_1 \left( 1 + \lambda t P' \right)^{-n} e^{-n \rho_1 a_0} (1 - \rho_1 P_2 b_0)^{-n},
\]

(80)
(81)

where the last equality follows by computing the expectation in (80) jointly over $c(W_{\text{IMD}})$ and $z$ using Lemma 4, and by setting $P_2 = 1 + (K_\lambda - K'_{\lambda'}) P'$. Finally, substituting the result into (73), we obtain
\[
\mathbb{P}[|W_{\text{AMD}}| = t] \leq K_{\lambda'}/t^{\rho_1} \left( M - K_\lambda \right)^\rho_1 \left( 1 + \lambda t P' \right)^{-n} \cdot \exp\left( -\lambda \|c(W_{\text{IMD}}) + z\|^2 \right),
\]

(82)
(83)

2) The DT-Based Approach: Next, we present an alternative bound on $\mathbb{P}[|W_{\text{AMD}}| = t]$. Consider the channel law $P_{y}(c(W_{\text{AMD}}), c(W_{\text{MD}}))$ with input $c(W_{\text{IMD}})$ and output $y$ where $|W_{\text{AMD}}| = t$. The corresponding information density
messages are returned.\(\)−

Note that (70) is equivalent to

\[
\gamma_t(c(W_{\text{MD}}); y | c(W \setminus W_{\text{MD}})) = \min_{\gamma \in W_{\text{MD}}} \gamma_t(c(W_{\text{MD}}); y | c(W \setminus W_{\text{MD}})).
\]

For a fixed arbitrary \(\gamma\), it follows that

\[
\mathbb{P}[|W_{\text{MD}}| = t] = \mathbb{P}[I_t \leq \gamma] + \mathbb{P}[I_t > \gamma]
\]

(85)

\[
\leq \mathbb{P}[I_t \leq \gamma] + \mathbb{P}[I_t > \gamma]
\]

(86)

\[
= \mathbb{P}[I_t \leq \gamma]
\]

(87)

\[
= \mathbb{P}[I_t \leq \gamma] + \mathbb{P}[I_t > \gamma]
\]

(88)

Here, (87) follows by writing explicitly the event \(|W_{\text{MD}}| = t\), and (88) by relaxing the inequality inside the second probability. Using that \(\mathbb{P}[i(x; y) > \gamma] \leq e^{-\gamma}, \forall x \in \mathcal{X}\) [27, Cor. 17.1], we obtain

\[
\mathbb{P}[\gamma_t(c(W_{\text{FA}}); y | c(W \setminus W_{\text{MD}})) > \gamma] \leq e^{-\gamma}.
\]

(89)

Then, by applying the union bound and taking the infimum over \(\gamma\), we conclude that

\[
\mathbb{P}[|W_{\text{MD}}| = t] \leq \inf \{ \mathbb{P}[I_t \leq \gamma] + (K_a - K'_a) e^{-\gamma} \}
\]

(90)

\[
q_t, t.
\]

(91)

This concludes the DT-based approach.

It follows from (83) and (91) that \(\mathbb{P}[|W_{\text{MD}}| = t] \leq \min\{p_t, q_t\}\). Introducing this bound into (71) and (72), we obtain that the MD and FA probabilities, averaged over the Gaussian codebook ensemble, are upper-bounded by \(\epsilon_{\text{MD}}\) and \(\epsilon_{\text{FA}}\) given in (68) and (69), respectively.

\[
C. \text{The General Case}
\]

We now explain how the result in the special case considered in the previous subsection can be extended to the general case where \(K_a\) and \(K'_a\) are random and \(r \geq 0\). For random \(K_a\) and \(K'_a\), one has to take into account all the possible combinations of the number of transmitted messages and decoded messages when computing the expectations in (61) and (62).

Consider the event that \(K_a\) users are active and the estimation of \(K_a\) results in \(K'_a\), which we denote by \(K_a \rightarrow K'_a\). As in the special case, we assume w.l.o.g. that \(\mathcal{W} = [K_a]\). Furthermore, exploiting symmetry, we let \(W_{\text{MD}} = [(K_a - K'_a) + 1 : (K_a - K'_a)]\) be the list of initial MDs due to insufficient decoded list size, and \(W_{\text{MD}} = W_{\text{MD}} \setminus W_{\text{MD}}\) denote the list of \(t\) additional MDs occurring during the decoding process. Note also that, if \(K'_a > K_a\), the decoder always outputs more than \(K_a\) messages. Hence, at least \(K'_a - K_a\) decoded messages are falsely alarmed.

Exploiting symmetry, we let w.l.o.g. \(W_{\text{FA}} = [K_a + 1 : K'_a]\) the list of initial FA\(s\) due to excessive decoded list size, and \(W_{\text{FA}} = W_{\text{FA}} \setminus W_{\text{FA}}\) denote the list of \(t\) additional FAs occurring during the decoding process. In Fig. 8, we depict the relation between these sets of messages. Under these assumptions, \(W_{\text{MD}}\) and \(W_{\text{FA}}\) are generic subsets of \([K_a - K'_a] + 1 : K_a\) and \(\max[K_a, K'_a] + 1 : M\), respectively.

Note that in the special case considered in Appendix A-B, \(t\) can take value from 0 to \(K'_a\) while \(t' = t\). In the general case, instead:

- The possible values of \(t\) belong to the set \(T\) defined in (21). This is because the number of MDs, given by \(t + (K_a - K'_a)\), is upper-bounded by the total number \(K_a\) of transmitted messages, and by \(M - K'_a\) (since at least \(K'_a\) messages are returned).

- Given \(t\), the integer \(t'\) takes value in \(T_t\) defined in (23) because: i) the decoded list size, given by \(K_a - t - (K_a - K'_a)\), must be in \([K'_a : K_a]\); ii) the number of FAs, given by \(t' + (K'_a - K_a)\), is upper-bounded by the number \(M - K_a\) of messages that are not transmitted, and by the maximal number \(K'_a\) of decoded messages.

- If the decoded list size is further required to be strictly positive, then \(t'\) takes value in \(T_t\) defined in (22).
Using the above definitions, the best approximation of $\hat{W}$ that the decoder can produce is $W_{\text{IFA}} \cup W_{\text{MID}} \cup (W \setminus W_{\text{MD}})$, while the actual decoded list, under $\hat{W} \rightarrow W$, is $W_{\text{IFA}} \cup W_{\text{MD}} \cup (W \setminus W_{\text{MD}})$. Therefore, $\hat{W} \rightarrow W$ implies that $\|y - c(W_{\text{IFA}}) - c(W_{\text{MD}}) - c(W \setminus W_{\text{MD}})\| < \|y - c(W_{\text{IFA}}) - c(W_{\text{MD}}) - c(W \setminus W_{\text{MD}})\|^2$, which is equivalent to
\[
\|c(W_{\text{MD}}) + c(W_{\text{IFA}}) - c(W_{\text{MD}}) + z\|^2 < \|c(W_{\text{MD}}) - c(W_{\text{IFA}}) + z\|^2.
\]
(92)

Let $F(W_{\text{MD}}, W_{\text{MID}})$ denote the set of $(W_{\text{MD}}, W_{\text{MID}}, W_{\text{IFA}})$ such that (92) holds.

We now compute the expectations in $P_{\text{MD}}$ given by (61) and $P_{\text{FA}}$ given by (62). Given $|W_{\text{MD}}| = t$ and $|W_{\text{IFA}}| = t'$, we have that $|W_{\text{MD}}| = t + (K_a - K_a^+) + |W_{\text{FA}}| = t + (K_a' - K_a)^+ + |W_{\text{MD}}|$, and $|\hat{W}| = K_a - t - (K_a - K_a^+) + t' + (K_a' - K_a)^+$. It follows from (61) and (62) that, (after the change of measure in Appendix A-A, $P_{\text{MD}}$ and $P_{\text{FA}}$ can be bounded as
\[
P_{\text{MD}} \leq \sum_{K_a = \max(K_i), K_a' = K_i} K_a \sum_{K_a = \max(K_i), K_a' = K_i} P_{K_a}(K_a) \sum_{K_a = \max(K_i), K_a' = K_i} K_a \sum_{t' \in T} \frac{t + (K_a - K_a^+ + |W_{\text{MD}}|)}{K_a}|W_{\text{MD}}| = t, K_a \rightarrow K_a') + \tilde{p},
\]
(93)
\[
P_{\text{FA}} \leq \sum_{K_a = K_i} K_a \sum_{K_a = K_i} P_{K_a}(K_a) \sum_{K_a = K_i} K_a \sum_{t' \in T} \frac{t + (K_a' - K_a)^+ + t' + (K_a' - K_a)^+}{K_a-t - (K_a - K_a^+) + t' + (K_a' - K_a)^+} \cdot \mathbb{P}(|W_{\text{MD}}| = t, |W_{\text{IFA}}| = t', K_a \rightarrow K_a') + \tilde{p}.
\]
(94)

Next, we proceed to bound the joint probability $\mathbb{P}(|W_{\text{MD}}| = t, K_a \rightarrow K_a')$ in (93) and the joint probability $\mathbb{P}(|W_{\text{MD}}| = t, |W_{\text{IFA}}| = t', K_a \rightarrow K_a')$ in (94). Let $A(K_a, K_a') = \{m(y, K_a') > m(y, K), \forall K \neq K_a'\}$. Since the event $K_a \rightarrow K_a'$ implies that $|\hat{W}| \in [K_a' : K_a^+]$ and that $A(K_a, K_a')$ occurs, we have
\[
\mathbb{P}(|W_{\text{MD}}| = t, K_a \rightarrow K_a') \leq \mathbb{P}(|W_{\text{MD}}| = t, |\hat{W}| \in [K_a' : K_a^+], A(K_a, K_a'))
\]
(95)
\[
\leq \min \left\{ \mathbb{P}(|W_{\text{MD}}| = t, |\hat{W}| \in [K_a' : K_a^+], A(K_a, K_a')) \right\},
\]
(96)
where (96) follows from the fact that the joint probability is upper-bounded by each of the individual probabilities. Similarly, it follows that
\[
\mathbb{P}(|W_{\text{MD}}| = t, |W_{\text{IFA}}| = t', K_a \rightarrow K_a') \leq \min \left\{ \mathbb{P}(|W_{\text{MD}}| = t, |W_{\text{IFA}}| = t', |\hat{W}| \in [K_a' : K_a^+], A(K_a, K_a')) \right\}.
\]
(97)

We next present the bounds on the probabilities $\mathbb{P}[A(K_a, K_a')]$, $\mathbb{P}[|W_{\text{MD}}| = t, |\hat{W}| \in [K_a' : K_a^+]$, and $\mathbb{P}[|W_{\text{MD}}| = t, |W_{\text{IFA}}| = t', |\hat{W}| \in [K_a' : K_a^+]

1) Bound on $\mathbb{P}[A(K_a, K_a')]$: We have
\[
\mathbb{P}[A(K_a, K_a')] = \mathbb{P}[m(y, K_a') > m(y, K), \forall K \neq K_a']
\]
(98)
\[
\leq \min_{K \in [K_a : K_a']} \mathbb{P}[m(y, K_a') > m(y, K)].
\]
(99)

Note that under the new measure, $y \sim \mathcal{CN}(0, (1 + K_a P') I_n)$. We follow two approaches to bound $\mathbb{P}[|W_{\text{MD}}| = t, |\hat{W}| \in [K_a' : K_a^+]$. In the first approach, we write the event $\{|W_{\text{MD}}| = t, |\hat{W}| \in [K_a' : K_a^+]$ as a union of the pairwise events and obtain
\[
\mathbb{P}[|W_{\text{MD}}| = t, |\hat{W}| \in [K_a' : K_a^+] = \mathbb{P} \left[ \bigcup_{t' \in T} \bigcup_{K_a = K_i} K_a \sum_{t' \in T_{\text{MD}}} K_a-t - (K_a - K_a^+) + t' + (K_a' - K_a)^+ \right.
\]
(100)

Then, by applying the Chernoff bound, Gallager’s $\rho$-trick, and Lemma 4 following similar steps as in Appendix A-B, we obtain
\[
\mathbb{P}[|W_{\text{MD}}| = t, |\hat{W}| \in [K_a' : K_a^+] \leq p_t
\]
(102)
with $p_t$ given by (9). In the second approach, we consider the channel law $P_{\text{F}}(c(W_{\text{MD}}); y | c(W \setminus W_{\text{MD}}))$ with input $c(W_{\text{MD}})$ and output $y$ where $|W_{\text{MD}}| = t$. The corresponding information density $u_t(c(W_{\text{MD}}); y | c(W \setminus W_{\text{MD}}))$ is defined in (27). Note that (92) is equivalent to
\[
\frac{u_t(c(W_{\text{MD}}); y | c(W \setminus W_{\text{MD}}))}{u_t(c(W_{\text{MID}}); y | c(W \setminus W_{\text{MD}}))}.
\]
(103)

Then, by proceeding as in Appendix A-B2, we obtain
\[
\mathbb{P}[|W_{\text{MD}}| = t, |\hat{W}| \in [K_a' : K_a^+] \leq q_t
\]
(104)
with $q_t$ given by (19).

3) Bound on $\mathbb{P}[|W_{\text{MD}}| = t, |W_{\text{IFA}}| = t', |\hat{W}| \in [K_a' : K_a^+]$: First, we have that
\[
\mathbb{P}[|W_{\text{MD}}| = t, |W_{\text{IFA}}| = t', |\hat{W}| \in [K_a' : K_a^+] = \mathbb{P} \left[ \bigcup_{t' \in T} \bigcup_{K_a = K_i} K_a \sum_{t' \in T_{\text{MD}}} K_a-t - (K_a - K_a^+) + t' + (K_a' - K_a)^+ \right.
\]
(105)
Notice that the probability \( \mathbb{P}[|W_{AMD}| = t, |W_{IFA}| = t', |W| \in [K'_a : K''_a]] \) differs from the probability \( \mathbb{P}[|W_{AMD}| = t, |W_{IFA}| = t', |W| \in [K'_a : K''_a]] \) in (101) only in that the union over \( t' \in \mathcal{T}_t \) is absent. By applying the Chernoff bound, Gallager's p-trick, and Lemma 4 following similar steps as in Appendix A-B1, we conclude that

\[
\mathbb{P}[|W_{AMD}| = t, |W_{IFA}| = t', |W| \in [K'_a : K''_a]] \leq p_{t,t'}
\]

with \( p_{t,t'} \) given by (10). Alternatively, bounding \( \mathbb{P}[|W_{AMD}| = t, |W_{IFA}| = t', |W| \in [K'_a : K''_a]] \) as in Appendix A-B2, we obtain

\[
\mathbb{P}[|W_{AMD}| = t, |W_{IFA}| = t', |W| \in [K'_a : K''_a]] \leq q_{t,t'}
\]

with \( q_{t,t'} \) given by (20).

It now follows from (96), (100), (102), and (104) that

\[
\mathbb{P}[|W_{AMD}| = t, K_a \to K'_a] \leq \min \{p_{t}, q_{t}, \xi(K_a, K'_a)\}.
\]

From (97), (100), (106), and (107), we obtain that

\[
\mathbb{P}[|W_{AMD}| = t, |W_{IFA}| = t', K_a \to K'_a] \leq \min \{p_{t,t'}, q_{t,t'}, \xi(K_a, K'_a)\}.
\]

Substituting these bounds on \( \mathbb{P}[|W_{AMD}| = t, K_a \to K'_a] \) and \( \mathbb{P}[|W_{AMD}| = t, |W_{IFA}| = t', K_a \to K'_a] \) into (93) and (94), we deduce that the MD and FA probabilities, averaged over the Gaussian codebook ensemble, are upper-bounded by \( \epsilon_{MD} \) and \( \epsilon_{FA} \) given in (6) and (7), respectively. Finally, by proceeding as in [25, Th. 19], one can show that there exists a randomized coding strategy that achieves (6) and (7) and involves time-sharing among at most three deterministic codes, as explained in Remark 2.

\section*{Appendix B}

\section*{Proof of Theorem 2}

Let \( y_0 \sim \mathcal{CN}(0, (1 + K_a P') I_a) \). The probability density function of \( y_0 \) is given by

\[
p_{y_0}(y_0) = \frac{1}{\pi^\frac{n}{2}(1+K_a P')} \exp\left(-\frac{\|y_0\|^2}{1+K_a P'}\right).
\]

Therefore, with ML estimation of \( K_a \), we have that

\[
\log p_{y_0}(y_0) = -\frac{\|y_0\|^2}{1+K_a P'} - n \ln(1+K_a P') - n \ln \pi.
\]

Consequently, the event \( m(y_0, K_f) > m(y_0, K) \) can be written as

\[
\frac{\|y_0\|^2}{1+K_a P'} + n \ln(1+K_a P') < \frac{\|y_0\|^2}{1+K'_a P'} + n \ln(1+K_a P'),
\]

or equivalently,

\[
\|y_0\|^2 \left(\frac{1}{1+K_a P'} - \frac{1}{1+K'_a P'}\right) < n \ln \left(\frac{1+K_a P'}{1+K_a P'}\right).
\]

Using the fact that \( \|y_0\|^2 \) follows a Gamma distribution with shape \( n \) and scale \( 1 + K_a P' \), we deduce that \( \xi(K_a, K'_a) \) is given by (34) with \( \zeta(K_a, K'_a) \) given by (35).

For energy-based estimation, i.e., \( m(y, K) = -\|y\|^2 - n(1+K_a P') \), we can show after some manipulations that the event \( m(y_0, K'_f) > m(y_0, K) \) is equivalent to

\[
\begin{cases}
\|y_0\|^2 > n \left(1 + \frac{K_a + K'_a}{2} P'\right), & \text{if } K_a < K'_a, \\
\|y_0\|^2 < n \left(1 + \frac{K_a + K'_a}{2} P'\right), & \text{if } K_a > K'_a.
\end{cases}
\]

Thus, using that \( \|y_0\|^2 \) is Gamma distributed, we deduce that \( \xi(K_a, K'_a) \) is given by (34) with \( \zeta(K_a, K'_a) \) given by (36).

\section*{Appendix C}

\section*{Proof of Corollary 1}

We evaluate the bounds \( \epsilon_{MD} \) and \( \epsilon_{FA} \) given in (6) and (7), respectively, in the limit \( P \to \infty \). First, the optimal value of \( P' \) minimizing these bounds must grow with \( P \) since otherwise \( \tilde{p} \) will be large. Therefore, as \( P \to \infty \), we can assume without loss of optimality that \( P' \to \infty \). Next, when \( t = t' = 0 \), we can verify that \( a = b = 0 \), thus \( E_0(\rho, p_1) = 0 \) and \( E(0, 0) = 0 \), achieved with \( \rho = p_1 = 0 \). Therefore, \( p_0 = p_0 = 0 = e^{-n \rho} = 1 \). We can also verify that \( q_0 \) and \( q_0, 0 \) both converge to 1 as \( P' \to \infty \). When \( P' \to \infty \), \( \xi(K_a, K'_a) \) given in Theorem 2 converges to the right-hand side of (34) with \( \xi(K_a, K'_a) = n \ln \left( \frac{K_a}{K'_a} \right) - \frac{1}{2} \left( \frac{1}{K_a} - \frac{1}{K'_a} \right)^{-1} 
\)

for ML estimation of \( K_a \) and \( \xi(K_a, K'_a) = n \ln \left( \frac{K_a + K'_a}{2K'_a} \right) 
\)

for energy-based estimation of \( K_a \). Furthermore, the last term in \( \tilde{p} \) given by (8) vanishes and thus \( \tilde{p} \to \tilde{p} \). Finally, the lower bounds \( \epsilon_{MD} \) and \( \epsilon_{FA} \) follows by substituting the asymptotic values of \( p_0, q_0, 0, 0, 0, \xi(K_a, K'_a) \), and \( \tilde{p} \) computed above into \( \epsilon_{MD} \) and \( \epsilon_{FA} \), and by setting \( q_{t,t'} \) to zero for \( t \neq 0 \), and setting \( q_{t,t'} \) to zero for \( (t, t') \neq (0, 0) \).

\section*{Appendix D}

\section*{Proof of Theorem 3}

The MD and FA probabilities are computed as

\[
P_{MD} = \mathbb{E}_r \left[ \frac{|W_{MD}|}{|W|} \right] \quad \text{and} \quad P_{FA} = \mathbb{E}_r \left[ \frac{|W_{FA}|}{|W|} \right],
\]

respectively. As derived in (63) in Appendix A-A, the probability that at least two active users choose the same message to transmit is given by

\[
\mathbb{P}[|W| = K_a] = E_{K_a} \left[ \frac{M!}{M^a (M-K_a)!} \right].
\]

We have that

\[
P_{MD} = \mathbb{P}[|W| = K_a] \mathbb{E}_r \left[ \frac{|W_{MD}|}{|W|} \right] \mathbb{E}_r \left[ \frac{|W|}{|W|} = K_a \right] + \mathbb{P}[|W| < K_a] \mathbb{E}_r \left[ \frac{|W_{MD}|}{|W|} \right] \mathbb{E}_r \left[ \frac{|W|}{|W|} < K_a \right] \quad (112)
\]

\[
\geq E_{K_a} \left[ \frac{M!}{M^a (M-K_a)!} \right] \mathbb{E}_r \left[ \frac{|W_{MD}|}{|W|} \right] \mathbb{E}_r \left[ \frac{|W|}{|W|} = K_a \right] \quad (113)
\]

Similarly, it follows that

\[
P_{FA} \geq E_{K_a} \left[ \frac{M!}{M^a (M-K_a)!} \right] \mathbb{E}_r \left[ \frac{|W_{FA}|}{|W|} \right] \mathbb{E}_r \left[ \frac{|W|}{|W|} = K_a \right]. \quad (114)
\]

Denote the event that the estimation step outputs \( K'_a \) when \( K_a \) users are active, which we denote by \( K_a \to K'_a \). Under
this event and the condition in the right-hand side of (113) and (114), we have that $|\mathcal{W}| = K_a$ and $K'_a \leq |\mathcal{W}| \leq K'_a$. The second expectations in the right-hand side of (113) and (114) can be expanded as

$$E\left[\frac{|\mathcal{W}_{MD}|}{|\mathcal{W}|} | \hat{\mathcal{W}} = K_a\right] = \sum_{K_a \in D, \ K_a > 0} \left( P_{K_a}(K_a) \sum_{K'_a = K_a}^K \frac{|\mathcal{W} \setminus \hat{\mathcal{W}}|}{K_a} \mathbb{P}[K_a \rightarrow K_a] \right),$$

(115)

$$E\left[|\mathcal{W}_{FA}| | \hat{\mathcal{W}} = K_a\right] \geq \sum_{K_a \in D} \left( P_{K_a}(K_a) \sum_{K'_a = K_a}^K \frac{|\mathcal{W} \setminus \hat{\mathcal{W}}|}{K_a'} \mathbb{P}[K_a \rightarrow K_a'] \right),$$

(116)

where $D$ is the domain of $K_a$. Furthermore, it is straightforward that

$$|\hat{\mathcal{W}} \setminus \mathcal{W}| \geq (|\hat{\mathcal{W}}| - |\mathcal{W}|) \geq (K_a - K'_a)^+, \quad (117)$$

$$|\mathcal{W} \setminus \hat{\mathcal{W}}| \geq (|\mathcal{W}| - |\hat{\mathcal{W}}|) \geq (K'_a - K_a)^+. \quad (118)$$

The first inequality in (117) becomes equality if $\hat{\mathcal{W}} \subseteq \mathcal{W}$, i.e., all decoded messages have been transmitted. The first inequality in (118) becomes equality if $\mathcal{W} \subseteq \hat{\mathcal{W}}$, i.e., all transmitted messages are decoded. In short, the first inequalities in (117) and (118) become equalities if the MDs and FAs are caused by the mismatch between $K_a$ and $K'_a$ only. The second inequalities in (117) and (118) are due to $K'_a \leq |\mathcal{W}| \leq K'_a$. Next, by substituting (117) into (115) and (118) into (116), by ignoring the terms for $K_a \notin \{K_a : K_a\}$, and by averaging over $P_a$, we obtain the converse bounds (46) and (47).

**APPENDIX E**

**PROOF OF THEOREM 4**

We first apply the same change of measure as in Appendix A-A at a cost of adding the term $\tilde{p}$. Then $P_{MD}$ and $P_{FA}$ can be bounded as

$$P_{MD} \leq \sum_{K_a = \max\{K_e, 1\}}^K \sum_{K'_a = K_e}^K \sum_{j=1}^{K_a} P_{K_a}(K_a) \mathbb{P}\left[\tilde{w}_j \notin \hat{\mathcal{W}}, K_a \rightarrow K'_a\right] + \tilde{p},$$

(119)

$$P_{FA} \leq \sum_{K_a = \max\{K_e, 1\}}^K \sum_{K'_a = K_e}^K \sum_{j=1}^{K_a} P_{K_a}(K_a) \min\left\{\mathbb{P}\left[\tilde{w}_j \notin \hat{\mathcal{W}}\right], \mathbb{P}[K_a \rightarrow K'_a]\right\} + \tilde{p},$$

(120)

where $\mathbb{P}[w_j \notin \hat{\mathcal{W}}]$ implies that at least one codeword that was not transmitted is closer to $y$ than $c_{w_j}$. Therefore, for a given codebook $\{c_1, \ldots, c_M\}$, $\mathbb{P}[w_j \notin \hat{\mathcal{W}}]$ can be upper-bounded as

$$\mathbb{P}[w_j \notin \hat{\mathcal{W}}] \leq \mathbb{P}\left[\exists i \in [M] \setminus \mathcal{W}: \|y - c_i\|^2 \leq \|y - c_{w_j}\|^2\right]$$

(125)

$$= \mathbb{P}\left[\bigcup_{i \in [M] \setminus \mathcal{W}} \{\|y - c_i\|^2 \leq \|y - c_{w_j}\|^2\}\right].$$

(126)

By applying the union bound on (126), using the fact that the codewords are i.i.d., we conclude that

$$\mathbb{P}[w_j \notin \hat{\mathcal{W}}] \leq \mathbb{E}_{x,c} \min\{1, (M - K_a)\mathbb{P}[\|y - \tilde{c}\|^2 \leq \|y - c_{w_j}\|^2 | y, c]\},$$

(127)

where $\{y, c, \tilde{c}\}$ has the same joint distribution as $\{y, c_{w_j}, c_i\}$, $i \in [M] \setminus \mathcal{W}$. Next, by applying the Chernoff bound and
proceeding as in [29, App. A], we obtain the following RCUs bound [38, Th. 16] for every $s > 0$: \[
\mathbb{P}\left[ w_j \notin \hat{W} \right] \leq \mathbb{E}\left[ \min \left\{ 1, \exp\left( \ln(M - K_a) - \sum_{i=1}^{n} \eta_s(x_i; y_i) \right) \right\} \right]
\]
with $\eta_s(x; y)$ given by (53) and $\{x_i, y_i\}$ defined in Theorem 4. Finally, by observing that, for every positive random variable $\nu$, it holds that $\mathbb{E}\left[ \min\{1, \nu\} \right] = \mathbb{P}[\nu \geq u]$ where $u$ is uniformly distributed on $[0, 1]$, we obtain that the right-hand side of (128) is given by $\eta_s$ defined in (52).

2) Bound on $\mathbb{P}\left[ \hat{w}_j \notin \hat{W} \right]$ for $j \in \{\min\{K_a, K'_a\}\}$: The event $\hat{w}_j \notin \hat{W} = \{w_1, \ldots, w_{K_a}\}$ implies that $\hat{w}_j$ is closer to $y$ than at least one transmitted codeword. In this case, we assume w.l.o.g. that $\|y - c_{w_j}\|^2 \leq \|y - c_{w_1}\|^2$. It follows that
\[
\mathbb{P}\left[ \hat{w}_j \notin \hat{W} \right] \leq \mathbb{P}\left[ \bigcup_{\hat{w}_j \in \hat{W}} \left\{ \|y - c_{\hat{w}_j}\|^2 \leq \|y - c_{w_1}\|^2 \right\} \right].
\]
(129)

Next, by applying the union bound, then applying the Chernoff bound and proceeding as in [29, App. A], we deduce that $\mathbb{P}\left[ \hat{w}_j \notin \hat{W} \right]$ is upper-bounded by the right-hand side of (128), which can be expressed as $\eta_s$ defined in (52).

Furthermore, the probability $\mathbb{P}[K_a \rightarrow K'_a]$ can be upper-bounded by $\xi(K_a, K'_a)$ as in Appendix A-C1. By substituting the bounds on $\mathbb{P}[w_j \notin \hat{W}]$, $\mathbb{P}[\hat{w}_j \notin \hat{W}]$, and $\mathbb{P}[K_a \rightarrow K'_a]$ into (123) and (124), we deduce that $P_{MD}$ and $P_{FA}$, averaged over the Gaussian codebook ensemble, are upper-bounded by $\varepsilon_{MD}$ and $\varepsilon_{FA}$ given in (50) and (51), respectively. Finally, by proceeding as in [25, Th. 19], one can show that there exists a randomized coding strategy that achieves (6) and (7) simultaneously, and involves time-sharing among at most three deterministic codes.

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