SEVERAL ALMOST CRITICAL REGULARITY CONDITIONS BASED ON ONE COMPONENT OF THE SOLUTIONS FOR 3D N-S EQUATIONS

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Abstract. In this article, we establish several almost critical regularity conditions such that the weak solutions of the 3D Navier-Stokes equations become regular, based on one component of the solutions, say $u_3$ and $\partial_3 u_3$.

1. Introduction

In this paper, we consider sufficient conditions for the regularity of weak solutions of the Cauchy problem for the Navier-Stokes equations

$$\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= 0, \quad \text{in } \mathbb{R}^3 \times (0, T), \\
\nabla \cdot u &= 0, \quad \text{in } \mathbb{R}^3 \times (0, T), \\
u(t) u(x, 0) &= u_0, \quad \text{in } \mathbb{R}^3,
\end{align*} \tag{1.1}$$

where $u = (u_1, u_2, u_3) : \mathbb{R}^3 \times (0, T) \to \mathbb{R}^3$ is the velocity field, $p : \mathbb{R}^3 \times (0, T) \to \mathbb{R}$ is a scalar pressure, and $u_0$ is the initial velocity field, $\nu > 0$ is the viscosity. We set $\nabla_h = (\partial_{x_1}, \partial_{x_2})$ as the horizontal gradient operator and $\Delta_h = \partial^2_{x_1} + \partial^2_{x_2}$ as the horizontal Laplacian, and $\Delta$ and $\nabla$ are the usual Laplacian and the gradient operators, respectively. Here we use the classical notations

$$(u \cdot \nabla)v = \sum_{i=1}^{3} u_i \partial_{x_i} v_k, \quad (k = 1, 2, 3), \quad \nabla \cdot u = \sum_{i=1}^{3} \partial_{x_i} u_i,$$

and for sake of simplicity, we denote $\partial_{x_i}$ by $\partial_i$.

Let us recall the definition of Leray-Hopf weak solution. We set

$$\mathcal{V} = \{\phi : \text{ the 3D vector valued } C^\infty_0 \text{ functions and } \nabla \cdot \phi = 0\},$$

which will form the space of test functions. Let $H$ and $V$ be the closure spaces of $\mathcal{V}$ in $L^2$ under $L^2$-topology, and in $H^1$ under $H^1$-topology, respectively.

For $u_0 \in H$, the existence of weak solutions of (1.1) was established by Leray [20] and Hopf in [13], that is, $u$ satisfies the following properties:

(i) $u \in C_w([0, T); H) \cap L^2(0, T; V)$, and $\partial_t u \in L^1(0, T; V')$, where $V'$ is the dual space of $V$;
(ii) $u$ verifies (1.1) in the sense of distribution, i.e., for every test function $\phi \in C^\infty([0, T); V)$,
and for almost every \( t, t_0 \in (0, T) \), we have
\[
\int_{\mathbb{R}^3} u(x, t) \cdot \phi(x, t)dx - \int_{\mathbb{R}^3} u(x, t_0) \cdot \phi(x, t_0)dx
\]
\[
= \int_{t_0}^t \int_{\mathbb{R}^3} [u(x, t) \cdot (\phi_t(x, t) + \nu \Delta \phi(x, t))]dxds
\]
\[
+ \int_{t_0}^t \int_{\mathbb{R}^3} [(u(x, t) \cdot \nabla) \phi(x, t)] \cdot u(x, t)]dxds
\]

(iii) The energy inequality, i.e.,
\[
\|u(\cdot, t)\|^2_{L^2} + 2\nu \int_{t_0}^t \|\nabla u(\cdot, s)\|^2_{L^2}ds \leq \|u_0\|^2_{L^2},
\]
for every \( t \) and almost every \( t_0 \).

It is well known, if \( u_0 \in V \), a weak solution becomes strong solution of (1.1) on \((0, T)\) if, in addition, it satisfies
\[
u \in C([0, T); V) \cap L^2(0, T; H^2) \text{ and } \partial_t u \in L^2(0, T; H).
\]

We know the strong solution is regular (say, classical) and unique (see, for example, [29], [30]).

For the 2D case, just as the authors said in [4], the Navier-Stokes equations (1.1) have unique weak and strong solutions which exist globally in time. However, the global regularity of solutions for the 3D Navier-Stokes equations is a major and challenging problem, the weak solutions are known to exist globally in time, but the uniqueness, regularity, and continuous dependence on initial data for weak solutions are still open problems. Furthermore, strong solutions in the 3D case are known to exist for a short interval of time whose length depends on the initial data. Moreover, this strong solution is known to be unique and to depend continuously on the initial data (see, for example, [29], [30]).

There are many interesting sufficient conditions which guarantee that a given weak solution is smooth, and the first result is usually referred as Prodi-Serrin (PS) conditions (see [26] and [28]), i.e. if additional the weak solution \( u \) is in the class of
\[
u \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = 1, \quad s \in [3, \infty],
\]
then the weak solution becomes regular. As to \( s = 3 \), Escauriaza, Seregin and Šverák in [9] established the \( L^{\infty,3} \) regularity criterion which says that if a weak solution \( u \in L^{\infty}(0, T; L^3(\mathbb{R}^3)) \), then it is regular. It is well known that if \( (u, p) \) solves the Navier-Stokes equations, then so does \( (u_\lambda, p_\lambda) \) for all \( \lambda > 0 \), where \( u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t) \). The class of Serrin’s type is important from a viewpoint of scaling invariance, which implies that \( \|u_\lambda\|_{L^1L^s} = \|u\|_{L^1L^s} \) holds for all \( \lambda > 0 \) if and only if \( \frac{2}{t} + \frac{3}{s} = 1 \). The full regularity of weak solutions can also be proved under alternative assumptions on the gradient of the velocity \( \nabla u \). In 1995, Beirão da Veiga [11] established a Serrin’s type regularity criterion on the gradient of the velocity field, if
\[
\nabla u \in L^t(0, T; L^s(\mathbb{R}^3)), \quad \frac{2}{t} + \frac{3}{s} = 2, \quad s \in [\frac{3}{2}, \infty].
\]

It is shown that the additional conditions in terms of only one velocity component, say \( u_3 \), cannot satisfy the same PS conditions as above and have a gap remained (see, for example, [22], [35], [4], [36] and [14]). Similarly, when we provide sufficient conditions in terms of only one of the nine components of the gradient of velocity field (i.e., the velocity Jacobian matrix), the gap seems enlarged (see, for example, [3], [10], [11] and [36]). As to the results refer to \( \nabla u_3 \), one can find in [25], [37], [15] and [14]. The reason to lead to the gap is from the term \( u \cdot \nabla u \). Especially, when we give the conditions on \( u_3 \) in some Lebesgue space, the terms \( \partial_t u_j, i, j = 1, 2 \) are difficult to control. In order to make sure the sufficient conditions satisfy the PS indexes, authors may
consider the combined version of the regularity criterion (based on two or more components of velocity or gradient of velocity). For example, in [23], [16] and [2], authors investigated regularity criterion in terms of $\partial_3 u$. For other combined version of the critical regularity criterion, we refer to [23], [24], [33]. There are many other versions of regularity criteria on component of velocity, the component of gradient of velocity or the combined of the components. For example, see [5], [8], [13], [19], [21], [32], [27], [34].

In this paper, we will get several almost critical regularity criterion based on only one velocity component $u_3$ and its partial derivative $\partial_3 u_3$. By using the anisotropic integrability properties on the spaces variable, we obtain a better result than previous ones. A crucial point is that we improve the inequality obtained in [3] to the anisotropic case (see Lemma 2.1 below for detail). In Theorem 1.1 below, we impose the assumption only on the $\partial_3 u_3$, we see that the indexes satisfy the “quasi-PS type” (the scaling indexes satisfy the strict inequality). In Theorem 1.4 we will give the quasi-PS type condition on $u_3$ to prove that regularity of $u$, and we see that the coupled condition on $\partial_3 u_3$ is scaling invariant.

Our main results can be stated in the following:

**Theorem 1.1.** Let $u$ be a Leray-Hopf weak solution to the 3D Navier-Stokes equations (1.1) with the initial value $u_0 \in V$. Suppose one of the following two items are satisfied.

(i) If $\partial_3 u_3$ satisfies

$$\sup_{0 \leq t \leq T} \left\| \partial_3 u_3(t) \right\|_{L^\alpha_{x_3}} \leq M, \text{ for some } M > 0,$$

where $\alpha$ and $\beta$ satisfy

$$1 \leq \alpha \leq \beta, \quad 2 < \beta \leq +\infty. \quad (1.3)$$

(ii) If $u_3$ and $\partial_3 u_3$ satisfy

$$u_3 \in L^\infty(0, T; L^3(\mathbb{R}^3)) \text{ and } \sup_{0 \leq t \leq T} \left\| \partial_3 u_3(t) \right\|_{L^\beta_{x_3}} \leq M, \text{ for some } M > 0,$$

where $\alpha$ and $\beta$ satisfy

$$\frac{1}{\alpha} + \frac{2}{\beta} < 2 \text{ and } 1 < \alpha \leq \beta, \quad \frac{3}{2} < \beta \leq 2. \quad (1.5)$$

Then $u$ is regular.

![Figure 1. Range of $(\alpha, \beta)$](image)

The domain "(1)" means the range of $(\alpha, \beta)$ in Theorem 1.1 (i). The domain "(2)" means the result of Theorem 1.1 (ii).
Remark 1.2. As we know that there is a large gap between the regularity criteria which have been obtained so far only on $\partial_3 u_3$ in Lebesgue space and the PS type condition:

$$\partial_3 u_3 \in L^1(0,T; L^s(\mathbb{R}^3)), \quad \frac{2}{s} + \frac{3}{s} = 2, \ s \in [\frac{3}{2}, \infty].$$

The purpose of (i) in Theorem 1.1 is to narrow this gap, and it shows that our criterion is of quasi-PS type. The range of the $(\alpha, \beta)$ is shown by the domain "(1)" in Figure 1. The condition (1.3) shows the different integrability on vertical and horizontal components. If we choose $\alpha = 1$ and $\beta$ tends to $2^+$, we see that the limit is a point of the line $1/\alpha + 2/\beta = 2$ (see Figure 1 for detail). When $\alpha = \beta > 2$, (1.3) becomes $\partial_3 u_3 \in L^\infty(0,T; L^\alpha(\mathbb{R}^3))$, $\alpha > 2$, and this result reduce to the endpoint version of regularity criterion of [3] (It can be obtained by using the method of [3] even though the authors did not mentioned). Moreover, we recall the endpoint version of regularity criterion $\partial_3 u_k \in L^\infty(0,T; L^3(\mathbb{R}^3))$ in Theorem 1.1 of [10]. We see that this result is also an improvement of the case of $j = k$.

Remark 1.3. Since the endpoint PS type condition on $u$ makes sure the weak solution regular (see [3]). The (ii) in Theorem 1.1 gives a depiction and comparison between the endpoint version of regularity criterion on $u_3$ and $u$. The range of the $(\alpha, \beta)$ is shown by the domain "(2)" in Figure 1. In case of $\frac{3}{2} \beta \leq 2$, we also see that the line $1/\alpha + 2/\beta = 2$ is the limit of the case of range of $(\alpha, \beta)$.

Theorem 1.4. Suppose that $u_0 \in V$, and $u$ is a Leray-Hopf weak solution to the 3D Navier-Stokes equations (1.1). Suppose that

$$1 < \alpha < +\infty, \ \max \left\{ \frac{11\alpha - 12}{3(\alpha - 1)}, 3 \right\} < s \leq \frac{11\alpha - 10}{3(\alpha - 1)},$$

and $u$ satisfies the following conditions

$$u_3 \in L^\infty(0,T; L^s(\mathbb{R}^3)),$$

and

$$\int_0^T \left\| \| \partial_3 u_3(\tau) \|_{L^s_3} \right\|_{L^\beta_{x_1,x_2}}^p \ d\tau \leq M, \ \text{for some } M > 0,$$

where

$$\beta = \frac{2\alpha}{(11\alpha - 10) - 3s(\alpha - 1)} \ \text{and } p = \frac{2\alpha}{3(\alpha - 1)(s - 3)}.$$

Then $u$ is regular.

Remark 1.5. We note that $\alpha, \beta$ and $p$ in Theorem 1.4 satisfy $1/\alpha + 2/\beta + 2/p = 2$. This means when we assume $s > 3$, we can give a scaling invariant condition on $\partial_3 u_3$.

In following theorem, we give the assumption on $u_3$ and $\partial_3 u_3$ with the time integrability.

Theorem 1.6. Suppose that $u_0 \in V$, and $u$ is a Leray-Hopf weak solution to the 3D Navier-Stokes equations (1.1). Suppose $u$ satisfies the following conditions

$$\int_0^T \left\| u_3(\tau) \right\|_{L^s}^q \ d\tau \leq M, \ \text{for some } M > 0,$$

and

$$\int_0^T \left\| \| \partial_3 u_3(\tau) \|_{L^s_3} \right\|_{L^\beta_{x_1,x_2}}^p \ d\tau \leq M, \ \text{for some } M > 0,$$

where $s$ and $q$, $\alpha$, $\beta$ and $p$ satisfy

$$\frac{3}{s} + \frac{2}{q} < 1; \ \frac{1}{\alpha} + \frac{2}{\beta} + \frac{2}{p} = 2,$$
and
\[
\frac{3}{2} < \beta < 2, \quad \frac{\beta}{2\beta - 2} < \alpha \leq \beta, \quad \frac{11\alpha\beta - 10\beta - 2\alpha}{3(\alpha - 1)\beta} \leq s \leq \infty,
\]
(1.13)

Then \( u \) is regular.

**Theorem 1.7.** Suppose that \( u_0 \in V \), and \( u \) is a Leray-Hopf weak solution to the 3D Navier-Stokes equations (1.11). Suppose \( u \) satisfies the following conditions
\[
\int_0^T \|u_3(\tau)\|_{L^q}^q \, d\tau \leq M, \text{ for some } M > 0,
\]
(1.14)
and
\[
\int_0^T \left\| \partial_3 u_3(\tau) \right\|_{L^q_{x_1,x_2}}^p \, d\tau \leq M, \text{ for some } M > 0,
\]
(1.15)
where \( s \) and \( q \), \( \alpha \), \( \beta \) and \( p \) satisfy
\[
\frac{3}{s} + \frac{2}{q} = 1; \quad \frac{1}{\alpha} + \frac{2}{\beta} + \frac{2}{p} < 2,
\]
(1.16)
and
\[
\frac{3}{2} \leq \beta \leq 2, \quad \frac{\beta}{2\beta - 2} < \alpha \leq \beta, \quad 3 \leq s \leq \frac{9\alpha\beta - 6\beta - 6\alpha}{(\alpha - 1)\beta},
\]
(1.17)

Then \( u \) is regular.

**Remark 1.8.** Very recently, J. Y. Chemin and P. Zhang in [6] considered the sufficient additional condition in homogeneity Sobolev spaces rather than Lebesgue spaces, and got the regularity criterion involving only one component of velocity
\[
u_3 \in L^7(0, T; \dot{H}^\sigma(\mathbb{R}^3)), \quad \gamma \in ]4, 6[.
\]
(1.18)
The derivative on \( u_3 \) is order of \( \sigma = \frac{1}{2} + \frac{2}{\gamma} \in ]\frac{5}{6}, 1[ \), and the embedding \( \dot{H}^{\frac{1}{2} + \frac{2}{\gamma}}(\mathbb{R}^3) \hookrightarrow L^\gamma(\mathbb{R}^3) \) implies the range of \( \eta = \frac{3\gamma}{2} - 2 \) is \( ]\frac{9}{7}, 6[ \). We find there are some interesting inspirations between Theorem 1.6 or Theorem 1.7 and (1.18). In Theorem 1.6 we see that the condition on \( \partial_3 u_3 \) is critical, that is scaling invariant, the range of \( p \in ]4, \infty[ \) is larger than the one of \( \gamma \in ]4, 6[ \) in (1.18), the condition (1.10) is weaker than that in (1.18) because of \( L^\gamma(0, T; \dot{H}^\sigma(\mathbb{R}^3)) \hookrightarrow L^\gamma(0, T; L^n(\mathbb{R}^3)) \). We see that the indexes of \( u_3 \) in (1.10) is of “quasi-PS type”, and the range of \( (s, q) \) is larger than \((\gamma, \eta)\) in (1.18), and in particular, the range of \( q \) can be enlarged to \([3, \infty[ \), since for any \( \epsilon > 0 \), there exist \( \alpha, \beta \) satisfying (1.13) such that \( 0 < \frac{11\alpha\beta - 10\beta - 2\alpha}{3(\alpha - 1)\beta} - 3 < \epsilon \). While Theorem 1.7 gives another depiction on the regularity criterion on \( u_3 \), in which we assume the indexes on \( u_3 \) is critical and that on \( \partial_3 u_3 \) is of quasi-PS type and range of the indexes are correspondingly expanded. Moreover, Theorem 1.6 and 1.7 also generalize the results of [32]. We see that the author in [32] considered the special case of the Theorem 1.6 and 1.7 with \( \alpha = \beta \).

For the convenience, we recall the following version of the three-dimensional Sobolev and Ladyzhenskaya inequalities in the whole space \( \mathbb{R}^3 \) (see, for example, [7], [12], [17]). There exists a positive constant \( C \) such that
\[
\|u\|_r \leq C\|u\|_{L^2(\mathbb{R}^3)}^{\frac{s}{2r}} \|\partial_1 u\|_{L^2(\mathbb{R}^3)}^{\frac{s}{2r}} \|\partial_2 u\|_{L^2(\mathbb{R}^3)}^{\frac{s}{2r}} \|\partial_3 u\|_{L^2(\mathbb{R}^3)}^{\frac{s}{2r}} \leq C\|u\|_{L^2(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)},
\]
(1.19)
for every \( u \in H^1(\mathbb{R}^3) \) and every \( r \in ]2, 6[ \), where \( C \) is a constant depending only on \( r \).
2. Proof of Main Results

In this section, under the assumptions of the Theorem 1.1, Theorem 1.4, Theorem 1.6 or Theorem 1.7 in Section 1 respectively, we prove our main results. First of all, by using the energy inequality, for Leray-Hopf weak solutions, we have (see, for example, [29, 30] for detail)

\[ \|u(\cdot, t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(\cdot, s)\|_{L^2}^2 ds \leq K_1, \quad (2.1) \]

for all \(0 < t < T\), where \(K_1 = \|u_0\|_{L^2}^2\).

It is well known that there exists a unique strong solution \(u\) local in time if \(u_0 \in V\). In addition, this strong solution \(u \in C((0, T^*); V) \cap L^2(0, T^*; H^2(\mathbb{R}^3))\) is the only weak solution with the initial datum \(u_0\), where \(0 < T^* \leq T\), is the maximal interval of existence of the unique strong solution. If \(T^* \geq T\), then there is nothing to prove. If, on the other hand, \(T^* < T\), then our strategy is to show that the \(H^1\) norm of this strong solution is bounded uniformly in time over the interval \((0, T^*)\), provided additional conditions in Theorem 1.1, Theorem 1.4, Theorem 1.6 or Theorem 1.7 in Section 1 are valid. As a result the interval \((0, T^*)\) cannot be a maximal interval of existence, and consequently \(T^* \geq T\), which concludes our proof.

In order to prove the \(H^1\) norm of the strong solution \(u\) is bounded on interval \((0, T^*)\), combing with the energy equality \((2.1)\), it is sufficient to prove

\[ \|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \leq C, \quad \forall \ t \in (0, T^*), \quad (2.2) \]

where the constant \(C\) depends on \(T, K_1\). Before we prove the main theorem, we show the following lemma.

**Lemma 2.1.** Let us assume that

\[ 1 \leq \alpha, \beta, s, a, t \leq \infty, \quad 2 < r \leq \infty, \quad \text{and} \quad 0 \leq \theta \leq 1, \quad (2.3) \]

where \(\alpha, \beta, s, r\) and \(\theta\) satisfy

\[ \frac{1}{a} + \frac{1}{t} = \frac{\beta - 1}{\beta}, \quad (2.4) \]

and

\[ \frac{1}{(r - 1)\alpha} + \frac{\theta}{\alpha} = \frac{1 - \theta}{s(\alpha - 1)}, \quad (2.5) \]

then we have the following estimates

\[ \left| \int_{\mathbb{R}^3} \phi f g d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 \right| \leq C \left\| \partial_3 \phi \right\|_{L^2} \left\| \frac{\partial f}{\partial_1} \right\|_{L^2}^{\frac{1}{2}} \left\| \partial_3 \phi \right\|_{L^2} \left\| \frac{\partial g}{\partial_2} \right\|_{L^2}^{\frac{1}{2}} \left\| \phi \right\|_{L^2} \left\| \frac{\partial f}{\partial_1} \right\|_{L^2}^{\frac{1}{2}} \left\| g \right\|_{L^2} \left( \frac{1 - \theta}{s(\alpha - 1)} \right)^{\frac{r - 2}{2}} \times \left\| f \right\|_{L^2} \left\| \partial_1 f \right\|_{L^2} \left\| \partial_2 f \right\|_{L^2} \left( \frac{1 - \theta}{s(\alpha - 1)} \right)^{\frac{r - 2}{2}} \right\|_{L^2}, \quad (2.6) \]
Proof. Without loss of generality, we assume that the functions $\phi, f, g \in C^\infty_0(\mathbb{R}^3)$. By using of Gagliardo-Nirenberg and Hölder’s inequalities, we have

\[
\left| \int_{\mathbb{R}^3} \phi f\,dx_1\,dx_2\,dx_3 \right| \\
\leq C \int_{\mathbb{R}^3} \left[ \max_{x_3} |\phi| \left( \int_{\mathbb{R}} |f|^2\,dx_3 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} |g|^2\,dx_3 \right)^{\frac{1}{2}} \right] \,dx_1\,dx_2
\]

\[
\leq C \left[ \int_{\mathbb{R}^2} \left( \max_{x_3} |\phi| \right)^r \,dx_1\,dx_2 \right]^{\frac{1}{r}} \left[ \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} |f|^2\,dx_3 \right)^{\frac{r-2}{r}} \,dx_1\,dx_2 \right]^{\frac{2}{r} - \frac{2}{r}} \times \left[ \int_{\mathbb{R}^3} |g|^2\,dx_1\,dx_2\,dx_3 \right]^{\frac{1}{2}}
\]

\[
\leq C \left[ \int_{\mathbb{R}^3} |\phi|^{r-1} |\partial_3\phi| \,dx_1\,dx_2\,dx_3 \right]^{\frac{1}{r}} \left[ \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} |f|^{\frac{2r}{r-2}}\,dx_3 \right)^{\frac{r-2}{r}} \,dx_1\,dx_2 \right]^{\frac{2}{r} - \frac{2}{r}} \left\| g \right\|_{L^2}
\]  

\[
\leq C \left( \left\| \partial_3\phi \right\|_{L^2_{x_3\cdot x_2}}^\frac{1}{r} \right) \left( \left\| |\phi|^{r-1} \partial_3\phi \right\|_{L^2_{x_3\cdot x_2}} \left\| f \right\|_{L^2_{x_3\cdot x_2}} \left\| \partial_1f \right\|_{L^2_{x_3\cdot x_2}} \left\| \partial_2f \right\|_{L^2_{x_3\cdot x_2}} \left\| g \right\|_{L^2}
\]

\[
\leq C \left( \left\| \partial_3\phi \right\|_{L^2_{x_3\cdot x_2}}^\frac{1}{r} \right) \left( \left\| |\phi|^{r-1} \partial_3\phi \right\|_{L^2_{x_3\cdot x_2}} \left\| f \right\|_{L^2_{x_3\cdot x_2}} \left\| \partial_1f \right\|_{L^2_{x_3\cdot x_2}} \left\| \partial_2f \right\|_{L^2_{x_3\cdot x_2}} \left\| g \right\|_{L^2}
\]

in above inequalities, we used (2.3) and (2.4). The proof is completed. \( \square \)

Proof of Theorem 1.1. We first prove (i). For the $\alpha, \beta$, we set $s = 2$,

\[
r = \frac{\beta(3\alpha - 2)}{\alpha(\beta + 1) - \beta},
\]

and

\[
\theta = \frac{\beta - \alpha}{2\alpha \beta - \alpha - \beta},
\]

then such $s, r$ and $\theta$ satisfy (2.4). We select that

\[
a = \frac{\alpha \beta + \alpha - \beta}{\alpha \beta - \beta}, \quad t = \frac{\beta(\alpha \beta + \alpha - \beta)}{\beta - \alpha},
\]

then the selected $a$ and $t$ satisfy (2.5). Because of

\[
r - 2 = \frac{\alpha(\beta - 2)}{\alpha \beta + \alpha - \beta},
\]

it is easy to check that (2.3) is also satisfied by (1.4) and (2.10). Furthermore, we see that

\[
(1 - \theta)(r - 1)a = s = 2, \quad \theta(r - 1)t = \beta.
\]
Therefore, taking the inner product of the equation (1.1) with $-\Delta_h u$ in $L^2$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h \nabla u\|_{L^2}^2 = \int_{\mathbb{R}^3} [(u \cdot \nabla) u] \Delta_h u dx$$

$$= \sum_{i,j=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \Delta_h u_j dx$$

$$= \sum_{i,j=1}^2 \int_{\mathbb{R}^3} u_i \partial_i u_j \Delta_h u_j dx + \sum_{i=1}^2 \int_{\mathbb{R}^3} u_i \partial_i u_3 \Delta_h u_3 dx$$

$$+ \sum_{j=1}^2 \int_{\mathbb{R}^3} u_3 \partial_3 u_j \Delta_h u_j dx + \int_{\mathbb{R}^3} u_3 \partial_3 u_3 \Delta_h u_3 dx$$

$$= J_1(t) + J_2(t) + J_3(t) + J_4(t).$$

By integrating by parts a few times and using the incompressibility condition, we get $J_1(t), J_2(t)$ as follows

$$J_1(t) = \frac{1}{2} \sum_{i,j=1}^2 \int_{\mathbb{R}^3} \partial_i u_j \partial_i u_j \partial_3 u_3 dx - \int_{\mathbb{R}^3} \partial_1 u_1 \partial_2 u_2 \partial_3 u_3 dx - \int_{\mathbb{R}^3} \partial_1 u_2 \partial_2 u_1 \partial_3 u_3 dx$$

$$J_2(t) = -\sum_{i,k=1}^2 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_3 \partial_3 u_3 dx - \sum_{i,k=1}^2 \int_{\mathbb{R}^3} u_i \partial_i u_3 \partial_3 u_3 dx$$

$$- \sum_{i,k=1}^2 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_3 \partial_3 u_3 dx + \frac{1}{2} \sum_{i,k=1}^2 \int_{\mathbb{R}^3} \partial_1 u_i \partial_1 u_3 \partial_3 u_3 dx$$

$$J_3(t) = -\sum_{i,k=1}^2 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_3 \partial_3 u_3 dx - \frac{1}{2} \sum_{k=1}^2 \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_3 \partial_3 u_3 dx.$$

From $J_1(t), J_2(t), J_3(t), J_4(t)$ it follows that

$$\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h \nabla u\|_{L^2}^2 \leq C \int_{\mathbb{R}^3} |u_3||\nabla u||\nabla_h u| dx$$

(2.13)

Applying Lemma 2.1 with the parameters $r, \theta, a, t$ in (2.8), (2.9) and (2.10) respectively, we have

$$\frac{1}{2} \frac{d}{dt} \|\nabla_h u\|_{L^2}^2 + \nu \|\nabla_h \nabla u\|_{L^2}^2 \leq C \int_{\mathbb{R}^3} |u_3||\nabla u||\nabla_h u| dx$$

(2.14)
Integrating (2.14) in time, applying Young’s inequality and the energy inequality (2.1), we get
\[
\|\nabla h(u(t))\|_{L^2}^2 + 2\nu \|\nabla h \nabla u\|_{L^2}^2 \\
\leq \|\nabla h_0\|_{L^2}^2 + C \int_0^t \left( \|\partial_3 u_3\|_{L^2} \right)^{\frac{1+a(r-1)}{2}} \left( \|u_3\|_{L^2} \right)^{\frac{r}{2}} \|\nabla u\|_{L^2}^\frac{r}{2} \|\nabla \nabla u\|_{L^2} \|\nabla \nabla u\|_{L^2} d\tau \\
\leq \|\nabla h_0\|_{L^2}^2 + C \int_0^t \left( \|\partial_3 u_3\|_{L^2} \right)^{\frac{1+a(r-1)}{2}} \left( \|u_3\|_{L^2} \right)^{\frac{r}{2}} \|\nabla u\|_{L^2}^\frac{r}{2} d\tau + \nu \int_0^t \|\nabla h \nabla u\|_{L^2}^2 d\tau.
\] (2.15)

Absorbing the last term in (2.15), and using (1.3) and (2.1), we have
\[
\|\nabla h(u(t))\|_{L^2}^2 + \nu \|\nabla h \nabla u\|_{L^2}^2 \leq C,
\] (2.16)
where the constant $C$ depends only on $M, K_1$. Next, we also use $-\Delta u$ as test function, and get
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2
\]
\[= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_k u_j \ dx
\]
\[= \sum_{j=1}^3 \int_{\mathbb{R}^3} u_3 \partial_3 u_j \Delta u_j \ dx + \sum_{i=1}^3 \sum_{j=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \Delta u_j \ dx + \sum_{j=1}^3 \int_{\mathbb{R}^3} u_3 \partial_3 u_j \partial_3 u_j \ dx
\]
\[= L_1(t) + L_2(t) + L_3(t)
\]

By integrating by parts a few times and using the incompressibility condition, we get $L_1(t), L_2(t), L_3(t)$ as follows
\[
L_1(t) = - \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_j u_3 \partial_3 u_j \partial_k u_j \ dx - \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} u_3 \partial_3 u_j \partial_k u_j \ dx
\]
\[= - \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_j u_3 \partial_3 u_j \partial_k u_j \ dx + \frac{1}{2} \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_3 u_3 \partial_k u_j \partial_k u_j \ dx,
\]
\[
L_2(t) = - \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_j u_3 \partial_3 u_j \partial_k u_j \ dx - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_k u_j \ dx
\]
\[= - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_i u_3 \partial_i u_j \partial_k u_j \ dx + \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_i u_3 \partial_k u_j \partial_k u_j \ dx,
\]
\[
L_3(t) = - \sum_{j=1}^3 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 u_j \partial_3 u_j \ dx = \frac{1}{2} \sum_{j=1}^3 \int_{\mathbb{R}^3} (\partial_i u_1 + \partial_2 u_2) \partial_3 u_j \partial_3 u_j \ dx.
\]

Therefore, by (1.19) and Hölder’s inequalities, for every $i (i = 1, 2, 3)$ we have
\[
|L_i(t)| \leq C \int_{\mathbb{R}^3} |\nabla h u| |\nabla u|^2 \ dx
\]
\[\leq C \|\nabla h u\|_{L^2} \|\nabla u\|_{L^4}^2
\]
\[\leq C \|\nabla h u\|_{L^2} \|\nabla u\|_{L^2}^\frac{1}{2} \|\nabla h \nabla u\|_{L^2} \|\Delta u\|_{L^2} \|\nabla u\|_{L^2}^\frac{1}{2},
\] (2.17)

and hence we have
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 \leq C \|\nabla h u\|_{L^2} \|\nabla u\|_{L^2}^\frac{1}{2} \|\nabla h \nabla u\|_{L^2} \|\Delta u\|_{L^2} \|\nabla u\|_{L^2}^\frac{1}{2}.
\] (2.18)
Integrating (2.18), applying Hölder’s and Young’s inequalities and combining (2.10) and (2.1), we obtain
\[
\|\nabla u\|_{L^2}^2 + 2\nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \\
\leq \|\nabla u(0)\|_{L^2}^2 + \left(\sup_{0 \leq s \leq t} \|\nabla h u\|_{L^2}\right) \left(\int_0^t \|\nabla u\|_{L^2}^2 d\tau\right)^\frac{1}{2} \\
\times \left(\int_0^t \|\nabla h \nabla u\|_{L^2}^2 d\tau\right)^\frac{1}{4} \left(\int_0^t \|\Delta u\|_{L^2}^2 d\tau\right)^\frac{3}{4},
\]
(2.19)
Absorbing the last term on the right hand side of (2.19), it immediately implies that (2.22). We complete the proof of Theorem 1.1 (i).

Next, we prove (ii). Similar to the proof of Theorem 1.1 (i), we will apply Lemma 2.1 get the desired result. Therefore, for the $\alpha, \beta$ in (1.5) and (1.6), we set $s = 3$,
\[
r = \frac{\beta(4\alpha - 3)}{\alpha(\beta + 1) - \beta},
\]
(2.20)
and
\[
\theta = \frac{\beta - \alpha}{3\alpha \beta - \alpha - 2\beta},
\]
(2.21)
then such $s, r$ and $\theta$ satisfy (2.23). We select that
\[
a = \frac{\alpha \beta + \alpha - \beta}{\alpha \beta - \beta}, \quad t = \frac{\beta(\alpha \beta + \alpha - \beta)}{\beta - \alpha},
\]
(2.22)
then the selected $a$ and $t$ satisfy (2.24). Note that
\[
r - 2 = \frac{2\alpha \beta - \beta - 2\alpha}{\alpha \beta + \alpha - \beta},
\]
by (1.6), we see that (2.22) and above equality imply (2.23) holds, and furthermore,
\[(1 - \theta)(r - 1)a = s = 3, \quad \theta(r - 1)t = \beta.
\]
Therefore, taking the inner product of the equation (1.1) with $-\Delta h u$ in $L^2$, applying Lemma 2.1 with the parameters in (2.20) - (2.22), similar to the proof of Theorem 1.1 we have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla h u\|_{L^2}^2 + \nu \|\nabla h \nabla u\|_{L^2}^2 \\
\leq C \int_{\mathbb{R}^3} |u_{33}| |\nabla u| |\nabla h \nabla u| dx \\
\leq C \left\|\partial_3 u_{33}\right\|_{X^3_x}^{\frac{1}{\theta}} \left\|\partial_3 u_{33}\right\|_{L^\infty_{x_1,x_2}} \left\|\partial_3 u_{33}\right\|_{X^3_{x_1,x_2}} \left\|\partial_3 u_{33}\right\|_{L^\infty_{x_1,x_2}} \left\|\partial_3 u_{33}\right\|_{L^\infty_{x_1,x_2}} \left\|\nabla u\right\|_{L^2}^{\frac{r-2}{r}} \left\|\nabla \nabla u\right\|_{L^2}^{\frac{1}{2}} \left\|\frac{1}{2} \partial_3 \nabla u\right\|_{L^2} \left\|\frac{1}{2} \partial_3 \nabla u\right\|_{L^2} \left\|\nabla h \nabla u\right\|_{L^2} \\
\leq C \left\|\partial_3 u_{33}\right\|_{X^3_x}^{\frac{1}{\theta}} \left\|\partial_3 u_{33}\right\|_{L^\infty_{x_1,x_2}} \left\|\partial_3 u_{33}\right\|_{L^\infty_{x_1,x_2}} \left\|\partial_3 u_{33}\right\|_{L^\infty_{x_1,x_2}} \left\|\nabla u\right\|_{L^2}^{\frac{r-2}{r}} \left\|\nabla \nabla u\right\|_{L^2}^{\frac{1}{2}} \left\|\frac{1}{2} \partial_3 \nabla u\right\|_{L^2} \left\|\frac{1}{2} \partial_3 \nabla u\right\|_{L^2} \left\|\nabla h \nabla u\right\|_{L^2} \\
\leq C \left\|\partial_3 u_{33}\right\|_{X^3_x}^{\frac{1}{\theta}} \left\|\partial_3 u_{33}\right\|_{L^\infty_{x_1,x_2}} \left\|\partial_3 u_{33}\right\|_{L^\infty_{x_1,x_2}} \left\|\partial_3 u_{33}\right\|_{L^\infty_{x_1,x_2}} \left\|\nabla u\right\|_{L^2}^{\frac{r-2}{r}} \left\|\nabla \nabla u\right\|_{L^2}^{\frac{1}{2}} \left\|\frac{1}{2} \partial_3 \nabla u\right\|_{L^2} \left\|\frac{1}{2} \partial_3 \nabla u\right\|_{L^2} \left\|\nabla h \nabla u\right\|_{L^2}^{r+2},
\]
(2.23)
Integrating \((2.23)\) in time, applying Young's inequality and the assumption \((1.5)\), we get

\[
\begin{align*}
\|\nabla_h u(t)\|_{L^2}^2 + 2\nu \|\nabla_h \nabla u\|_{L^2}^2 \\
\leq \|\nabla_h u_0\|_{L^2}^2 + C \int_0^t \left\| \Omega_{\beta} u_3 \right\|_{L^3_{x_2}}^{\frac{1+\theta(r-1)}{1-\theta(r-1)}} \|u_3\|_{L^3_{x_2}}^{\frac{(1-\theta)(r-1)}{1+\theta(r-1)}} \|\nabla u\|_{L^2_{x_2}}^{\frac{r-2}{r}} \|\nabla_h \nabla u\|_{L^2_{x_2}}^{\frac{r+2}{r}} \, d\tau
\end{align*}
\]

\[(2.24)\]

Absorbing the last term in \((2.24)\), and using \((1.5)\) and the energy inequality \((2.1)\), we have

\[
\|\nabla_h u(t)\|_{L^2}^2 + \nu \|\nabla_h \nabla u\|_{L^2}^2 \leq C,
\]

where the constant \(C\) depends only on \(M, K_1\). For the rest, we can give the same process as in Theorem \((1.1)\) (i) to prove \((2.2)\), and then to complete the proof of this theorem. \(\Box\)

**Proof of Theorem \((1.4)\).** In view of the condition

\[
1 < \alpha < +\infty, \quad \max \left\{ \frac{11\alpha - 12}{3(\alpha - 1)}, 3 \right\} < s \leq \frac{11\alpha - 10}{3(\alpha - 1)},
\]

we give the parameters \(\beta, \theta, a, t, r\) as follows, and we will check one by one that all of them satisfy the assumptions in Lemma \((2.1)\).

\[
\beta = \frac{2\alpha}{(11\alpha - 10) - 3s(\alpha - 1)},
\]

\[(2.26)\]

\[
r = \frac{2s(\alpha - 1) + 2\alpha}{(13\alpha - 12) - 3s(\alpha - 1)},
\]

\[(2.27)\]

\[
\theta = \frac{3s(\alpha - 1) - 11\alpha + 12}{5s(\alpha - 1) - 11\alpha + 12},
\]

\[(2.28)\]

\[
a = \frac{13\alpha - 12 - 3s(\alpha - 1)}{2(\alpha - 1)},
\]

\[(2.29)\]

and

\[
t = \frac{2\alpha[(13\alpha - 12) - 3s(\alpha - 1)]}{[(11\alpha - 10) - 3s(\alpha - 1)][3s(\alpha - 1) - 11\alpha + 12]}.
\]

\[(2.30)\]

By \((1.7)\), we have \(\beta > 1\). In fact, \(s \leq \frac{11\alpha - 10}{3(\alpha - 1)}\) (if \(s = \frac{11\alpha - 10}{3(\alpha - 1)}\), then \(\beta = \infty\)) implies that

\[
s > \frac{9\alpha - 10}{3(\alpha - 1)} \iff \beta > 1,
\]

and then also by \((1.7)\), we see that

\[
s > \frac{11\alpha - 12}{3(\alpha - 1)} \implies s > \frac{9\alpha - 10}{3(\alpha - 1)}.
\]

For \(r\), because of

\[
s \leq \frac{11\alpha - 10}{3(\alpha - 1)} \implies s \leq \frac{13\alpha - 12}{3(\alpha - 1)},
\]

\[(2.31)\]

by \((2.21)\) and \((2.31)\), we have

\[
r - 2 = \frac{8(s - 3)(\alpha - 1)}{(13\alpha - 12) - 3s(\alpha - 1)} > 0 \iff \left\{ \begin{array}{l} s > 3 \\ \alpha > 1. \end{array} \right.
\]

\[(2.32)\]

By \((1.8)\) and \((2.25)\), it is obviously that \(0 \leq \theta < 1\). As to \(a\), we see that

\[
s \leq \frac{11\alpha - 10}{3(\alpha - 1)} \implies a \geq 1,
\]

\[(2.33)\]
and moreover, we have

\[
t - 1 = \frac{9(\alpha - 1)^2 s^2 - 6(12\alpha - 11)(\alpha - 1)s + (21\alpha - 20)(7\alpha - 6)}{[(11\alpha - 10) - 3s(\alpha - 1)] [3s(\alpha - 1) - 11\alpha + 12]} > 0.
\]

Besides, one can check that

\[
\begin{aligned}
\frac{1}{a} + \frac{1}{t} &= \frac{\beta - 1}{\beta}, \\
\frac{1 - \theta}{s} + \frac{\theta(1 - \alpha)}{\alpha} &= \frac{\alpha - 1}{\alpha(r - 1)}. 
\end{aligned}
\]

(2.33)

Therefore, all the conditions of Lemma 2.1 are satisfied. Similar to Theorem 1.1, we begin with (2.14) and by using the parameters defined in (2.20)-(2.30) and Lemma 2.1, we get

\[
\frac{1}{2} \int \frac{d}{dt} \left\| \nabla h u \right\|_{L^2}^2 + \nu \left\| \nabla_h \nabla u \right\|_{L^2}^2 \\
\leq C \int |u_3| \left\| \nabla h \nabla u \right\|_{L^2}^2 dx \\
\leq C \left\| \partial_3 u_3 \right\|_{L^\alpha_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\beta_{x_1, x_2}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_1, x_2}} \\
\times \left\| \nabla u \right\|_{L^2}^{r-2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \nabla_h \nabla u \right\|_{L^2} \\
\leq C \left\| \partial_3 u_3 \right\|_{L^\alpha_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\beta_{x_1, x_2}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_1, x_2}} \\
\times \left\| \nabla u \right\|_{L^2}^{r-2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \nabla_h \nabla u \right\|_{L^2} \\
\leq C \left\| \partial_3 u_3 \right\|_{L^\alpha_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\beta_{x_1, x_2}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_1, x_2}} \\
\times \left\| \nabla u \right\|_{L^2}^{r-2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \nabla_h \nabla u \right\|_{L^2}^{r+2}.
\]

Integrating (2.33) in time, applying Young’s inequality and the assumption (1.8), we get

\[
\left\| \nabla_h u(t) \right\|_{L^2}^2 + 2\nu \left\| \nabla_h \nabla u \right\|_{L^2}^2 \\
\leq \left\| \nabla_h u_0 \right\|_{L^2}^2 + C \int_0^t \left\| \partial_3 u_3 \right\|_{L^\alpha_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\beta_{x_1, x_2}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_1, x_2}} \\
\times \left\| \nabla u \right\|_{L^2}^{r-2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \nabla_h \nabla u \right\|_{L^2} \\
\leq \left\| \nabla_h u_0 \right\|_{L^2}^2 + C \int_0^t \left\| \partial_3 u_3 \right\|_{L^\alpha_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\beta_{x_1, x_2}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_1, x_2}} \\
\times \left\| \nabla u \right\|_{L^2}^{r-2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \nabla_h \nabla u \right\|_{L^2} \\
\leq \left\| \nabla_h u_0 \right\|_{L^2}^2 + C \int_0^t \left\| \partial_3 u_3 \right\|_{L^\alpha_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\beta_{x_1, x_2}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_1, x_2}} \\
\times \left\| \nabla u \right\|_{L^2}^{r-2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \nabla_h \nabla u \right\|_{L^2}^{r-2} \left\| \nabla_h \nabla u \right\|_{L^2}^{r+2} d\tau.
\]

(2.35)

Absorbing the last term in (2.35), we have

\[
\left\| \nabla_h u(t) \right\|_{L^2}^2 + \nu \left\| \nabla_h \nabla u \right\|_{L^2}^2 \\
\leq \left\| \nabla_h u_0 \right\|_{L^2}^2 + C \int_0^t \left\| \partial_3 u_3 \right\|_{L^\alpha_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\beta_{x_1, x_2}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_1, x_2}} \\
\times \left\| \nabla u \right\|_{L^2}^{r-2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \nabla_h \nabla u \right\|_{L^2} \\
\leq \left\| \nabla_h u_0 \right\|_{L^2}^2 + C \int_0^t \left\| \partial_3 u_3 \right\|_{L^\alpha_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\beta_{x_1, x_2}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_3}} \left\| \partial_3 \partial_3 u_3 \right\|_{L^\lambda_{x_1, x_2}} \\
\times \left\| \nabla u \right\|_{L^2}^{r-2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \partial_2 \nabla u \right\|_{L^2} \left\| \nabla_h \nabla u \right\|_{L^2}^2 d\tau.
\]

(2.36)

where we note that \(\frac{2(1+\theta(r-1))}{r-2} = \frac{\alpha}{2(r-3)(\alpha - 1)}\). Next, we apply the estimates on \(\left\| \nabla u(t) \right\|_{L^2}^2\). In view of (2.18), and integrating it in time, applying Hölder’s and Young’s inequalities and combining
Proof of Theorem 1.6. Since $\alpha, \beta$ and $s$ satisfy (1.12) and (1.13), for any arbitrary small positive constant $\epsilon$ satisfying $0 < \epsilon < \min \left\{ \frac{4}{10}, \frac{8 \beta - 12}{11 \beta - 12} \right\}$, we can choose $\alpha$ such that

$$\frac{\beta}{2 \beta - 2} < \alpha \leq \frac{(4 - 10 \epsilon)^{\beta}}{(8 - 11 \epsilon) \beta + 2 \epsilon - 8},$$

where we used $\beta < 2$, and then we choose

$$s = \frac{\alpha \beta - 2 \beta + 2 \alpha}{(1 - \epsilon)(\alpha - 1) \beta}.$$
From (2.41), it is easy to check that
\[ s \geq \frac{11\alpha\beta - 10\beta - 2\alpha}{3(\alpha - 1)\beta} > 3, \]
where we use the fact that \( \frac{\beta}{2\beta - 2} < \alpha \Rightarrow \frac{11\alpha\beta - 10\beta - 2\alpha}{3(\alpha - 1)\beta} > 3. \) Next, we set
\[ r = \frac{(s\alpha + \alpha - s)\beta}{\alpha\beta + \alpha - \beta}, \]
(2.43)
\[ \theta = \frac{\beta - \alpha}{s\alpha\beta - s\beta - \alpha + \beta}, \]
(2.44)
\[ a = \frac{\alpha\beta + \alpha - \beta}{(\alpha - 1)\beta}, \]
(2.45)
\[ t = \frac{(\alpha\beta + \alpha - \beta)\beta}{\beta - \alpha}. \]
(2.46)

From (2.43), we have
\[ r - 2 = \frac{(s - 1)\alpha\beta - (s - 2)\beta - 2\alpha}{\alpha\beta + \alpha - \beta}, \]
and by (2.42), we have \( 3\alpha\beta s - 11\alpha\beta - 3\beta s + 10\beta + 2\alpha \geq 0 \) and \( 2\alpha\beta - \beta - 2\alpha > 0. \) Therefore, one has
\[ 3\alpha\beta s - 11\alpha\beta - 3\beta s + 10\beta + 2\alpha > 0 \iff 3[(s - 1)\alpha\beta - (s - 2)\beta - 2\alpha] \geq 4(2\alpha\beta - \beta - 2\alpha) > 0, \]
and finally we get \( r > 2 \) and \( (s - 1)\alpha\beta - (s - 2)\beta - 2\alpha > 0. \) Since
\[ (s - 1)\alpha\beta > (s - 2)\beta + 2\alpha \iff s\alpha\beta - s\beta - \alpha + \beta > \alpha\beta + \alpha - \beta = (\alpha - 1)\beta + \alpha > 0, \]
it is easy for us to get \( 0 \leq \theta < 1. \) Moreover, \( a \geq 1 \) and \( t \geq 1 \) are obviously. All parameters selected above satisfy the conditions of Lemma 2.1 and similar to Theorem 1.4 (see (2.33) and (2.34)), one has
\[
\begin{align*}
\|\nabla_h u(t)\|^2_{L^2} + \nu \|\nabla_h \nabla u\|^2_{L^2} & \leq \|\nabla_h u_0\|^2_{L^2} + C \int_0^t \left( \left\|\partial_3 u_3\right\|_{L^3_{x_3}} \right)^{1+\theta(r-1)} \left\| u_3 \right\|_{L^r_{x_3}} \left\| \nabla_h u \right\|_{L^2} \left\| \nabla_h \nabla u \right\|_{L^2} d\tau \\
& \leq \|\nabla_h u_0\|^2_{L^2} + C \int_0^t \left( \left\|\partial_3 u_3\right\|_{L^3_{x_3}} \right)^{1+\theta(r-1)} \left\| u_3 \right\|_{L^r_{x_3}} \left\| \nabla_h \nabla u \right\|_{L^2} d\tau \\
& = \|\nabla_h u_0\|^2_{L^2} + C \int_0^t \left( \left\|\partial_3 u_3\right\|_{L^3_{x_3}} \right)^{1+\theta(r-1)} \left\| u_3 \right\|_{L^r_{x_3}} \left\| \nabla_h \nabla u \right\|_{L^2} d\tau.
\end{align*}
\]
(2.47)

Next, we apply the estimates on \( \|\nabla u(t)\|^2_{L^2}. \) In view of (2.18), and integrating it in time, applying Hölder’s and Young’s inequalities and combining (2.47) and (2.41), we obtain
\[
\begin{align*}
\|\nabla u\|^2_{L^2} + 2\nu \int_0^t \|\Delta u\|^2_{L^2} d\tau & \leq \|\nabla u_0\|^2_{L^2} + \left( \sup_{0 \leq s \leq t} \|\nabla_h u\|^2_{L^2} \right) \left( \int_0^t \|\nabla u\|^2_{L^2} d\tau \right)^t \\
& \times \left( \int_0^t \|\nabla_h \nabla u\|^2_{L^2} d\tau \right)^{t(1-\beta)/2\alpha} \left( \int_0^t \|\Delta u\|^2_{L^2} d\tau \right)^{t/4} \\
& \leq C \left( \int_0^t \left\|\partial_3 u_3\right\|_{L^3_{x_3}} \right)^{1+\theta(r-1)} \left\| u_3 \right\|_{L^r_{x_3}} \left( \int_0^t \|\Delta u\|^2_{L^2} d\tau \right)^{t/4} \\
& \times \left( \int_0^t \|\Delta u\|^2_{L^2} d\tau \right)^{t/4} + \|\nabla_h u_0\|^2_{L^2} \left( \int_0^t \|\Delta u\|^2_{L^2} d\tau \right)^{t/4} + \|\nabla u_0\|^2_{L^2}.
\end{align*}
\]
(2.48)
By Hölder’s and Young’s inequalities, one has
\[
\|\nabla u\|_{L^2}^2 + 2\nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \leq C \left( \int_0^t \left\| \partial_3 u_3 \right\|_{L^2_{x_3}}^{2(\alpha-1)\beta \gamma - (s-2)\beta - 2\alpha} \left\| u_3 \right\|_{L^2_{x_1', x_2}}^{2(\alpha-1)\beta \gamma - (s-2)\beta - 2\alpha} \left\| \nabla u \right\|_{L^2_{x}}^2 d\tau \right)^{\frac{1}{2}}
\]
\[
+ C \left( 1 + \|\nabla u_0\|_{L^2}^\frac{8}{3} \right) + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau
\]  
(2.49)

Absorbing the last term on the right hand side of (2.49), and thanks to the energy inequality (2.4), we get
\[
\|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \leq C \left( 1 + \|\nabla u_0\|_{L^2}^\frac{8}{3} \right) + C \left( \int_0^t \left\| \nabla u_3 \right\|_{L^2_{x_3}}^{3(s-1)\alpha\beta - (s-2)\beta - 2\alpha} \left\| u_3 \right\|_{L^2_{x_1', x_2}}^{3(s-1)\alpha\beta - (s-2)\beta - 2\alpha} \left\| \nabla u \right\|_{L^2_{x}}^2 d\tau \right) .
\]  
(2.50)

Now, we set the pair of conjugate indexes as follows
\[
h = \frac{3(s-1)\alpha\beta - (s-2)\beta - 2\alpha}{4(2\alpha - \beta - 2\alpha)},
\]
and
\[
h' = \frac{3(s-1)\alpha\beta - (s-2)\beta - 2\alpha}{3\alpha\beta s - 11\alpha\beta - 3\beta s + 10\beta + 2\alpha},
\]
where we note that (1.13) implies \( h \geq 1 \) and then \( h' > 1 \). Therefore, by Young’s inequality and (2.50), it follows that
\[
\|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \leq C \left( 1 + \|\nabla u_0\|_{L^2}^\frac{8}{3} \right) + C \left( \int_0^t \left\| \nabla u_3 \right\|_{L^2_{x_3}}^{2\alpha\beta - 2\alpha} \left\| u_3 \right\|_{L^2_{x_1', x_2}}^{3\alpha\beta s - 11\alpha\beta - 3\beta s + 10\beta + 2\alpha} \left\| \nabla u \right\|_{L^2_{x}}^2 d\tau \right) .
\]  
(2.51)

Applying (2.42), we have
\[
3 + \frac{3\alpha\beta s - 11\alpha\beta - 3\beta s + 10\beta + 2\alpha}{4(\alpha - 1)\beta s} = \frac{\alpha\beta - 2\beta + 3\alpha\beta s - 3\beta s + 2\alpha}{4(\alpha - 1)\beta s}
\]
\[
= 1 - \frac{\epsilon}{4}.
\]  
(2.52)

Therefore, since \( \epsilon \) is arbitrary, by Gronwall’s inequality and (1.10), (1.13), (2.51) implies that
\[
\|\nabla u\|_{L^2}^2 + \nu \int_0^t \|\Delta u\|_{L^2}^2 d\tau \leq C \left( 1 + \|\nabla u_0\|_{L^2}^\frac{8}{3} \right) (1 + M) e^{CM},
\]  
(2.53)

for all \( t \in (0, T^*) \). Therefore, the \( H^1 \) norm of the strong solution \( u \) is bounded on the maximal interval of existence \( (0, T^*) \). This completes the proof of Theorem 1.6.

\[ \Box \]

Proof of Theorem 1.7. Since \( \alpha, \beta \) and \( s \) satisfy (1.10) and (1.17), for any arbitrary small positive constant \( \epsilon \) satisfying \( 0 < \epsilon < \min \left\{ \frac{\alpha}{8}, \frac{2\beta - 2}{\beta} \right\} \), we can choose \( \alpha \) such that
\[
\frac{4\beta}{(8 - \epsilon)\beta - 8} \leq \alpha \leq \frac{\beta}{(2 - \epsilon)\beta - 2}.
\]  
(2.54)
and then we choose
\[ s = \frac{(1 + \epsilon)\alpha\beta - 2\beta + 2\alpha}{(\alpha - 1)\beta}. \] (2.55)

From (2.55), it is easy to check that
\[ 3 \leq s \leq \frac{9\alpha\beta - 6\beta - 6\alpha}{(\alpha - 1)\beta}. \]

Next, we set \( r, \theta, a, t \) as in (2.48), (2.49), (2.50), and (2.51) respectively. From (2.48), we have
\[ r - 2 = \frac{(s - 1)\alpha\beta - (s - 2)\beta - 2\alpha}{\alpha\beta + \alpha - \beta}, \]
and by (1.17), we have \( 2\alpha\beta - \beta - 2\alpha > 0 \). Therefore, one has
\[ 2\alpha\beta - \beta - 2\alpha \geq 0 \iff \frac{\alpha\beta - 2\beta + 2\alpha}{(\alpha - 1)\beta} < 3 \leq s \iff (s - 1)\alpha\beta - (s - 2)\beta - 2\alpha > 0, \]
and finally we get \( r > 2 \). Since
\[ (s - 1)\alpha\beta > (s - 2)\beta + 2\alpha \iff s\alpha\beta - \beta - \alpha + \beta > \alpha\beta + \alpha - \beta = (\alpha - 1)\beta + \alpha > 0, \]
it is easy for us to get \( 0 < \theta < 1 \). Moreover, \( a \geq 1 \) and \( t \geq 1 \) are obviously. All parameters selected above satisfy the conditions of Lemma 2.1. Similar to the proof of Theorem 1.6, we have (2.50), and we restate below
\[ \|\nabla u\|_{L^2_t}^2 + \nu \int_0^t \|\Delta u\|_{L^2_t}^2 \, d\tau \leq C \left( 1 + \|\nabla u_0\|_{L^2}^2 \right) \]
\[ + C \left( \int_0^t \left( \|\partial_3 u_3\|_{L^2_{t,x}} \right)^{\frac{\alpha}{2} - \frac{2\beta}{\alpha(\alpha + 2)}} \|u_3\|_{L^2} \right) \]
\[ \left( \|\nabla u\|_{L^2_t}^2 \right) \] (2.56)

Now, we set the pair of conjugate index as follows
\[ h = \frac{3((s - 1)\alpha\beta - (s - 2)\beta - 2\alpha}{4\beta(\alpha - 1)(s - 3),} \]
and
\[ h' = \frac{3((s - 1)\alpha\beta - (s - 2)\beta - 2\alpha)}{9\alpha\beta - \alpha\beta s + \beta s - 6\beta - 6\alpha}, \]
where we note that (1.17) implies \( h \geq 1 \) and then \( h' > 1 \). Therefore, by Young’s inequality, (2.56) implies that
\[ \|\nabla u\|_{L^2_t}^2 + \nu \int_0^t \|\Delta u\|_{L^2_t}^2 \, d\tau \leq C \left( 1 + \|\nabla u_0\|_{L^2}^2 \right) \]
\[ + C \left( \int_0^t \left( \|\partial_3 u_3\|_{L^2_{t,x}} \right)^{\frac{\alpha}{2} - \frac{2\beta}{\alpha(\alpha + 2)}} + \|u_3\|_{L^2}^2 \right) \|\nabla u\|_{L^2_t}^2 \, d\tau \] (2.57)

Applying (2.55), we have
\[ \frac{1}{\alpha} + \frac{2}{\beta} + \frac{9\alpha\beta - \alpha\beta s + \beta s - 6\beta - 6\alpha}{4\alpha\beta} \]
\[ = \frac{9\alpha\beta + \beta s - 2\beta + 2\alpha - \alpha\beta s}{4\alpha\beta} \]
\[ = 2 - \frac{\epsilon}{4}. \] (2.58)

Therefore, since \( \epsilon \) is arbitrary, by Gronwall’s inequality and (1.14)-(1.17), (2.57) implies that
\[ \|\nabla u\|_{L^2_t}^2 + \nu \int_0^t \|\Delta u\|_{L^2_t}^2 \, d\tau \leq C \left( 1 + \|\nabla u_0\|_{L^2}^2 \right) (1 + M)e^{CM}, \] (2.59)
for all \( t \in (0, T^*) \). Therefore, the \( H^1 \) norm of the strong solution \( u \) is bounded on the maximal interval of existence \( (0, T^*) \). This completes the proof of Theorem 1.7. \( \square \)
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