TWISTED ARITHMETIC SIEGEL WEIL FORMULA
ON $X_0(N)$

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Abstract. In this paper, we study twisted arithmetic divisors on
the modular curve $X_0(N)$ with $N$ square-free. For each pair $(\Delta, r)$
where $\Delta > 0$ and $\Delta \equiv r^2 \mod 4N$, we constructed a twisted
arithmetic theta function $\tilde{\phi}_{\Delta,r}(\tau)$ which is a generating function
of arithmetic twisted Heegner divisors. We prove the modularity
of $\tilde{\phi}_{\Delta,r}(\tau)$, along the way, we also identify the arithmetic pairing
$\langle \tilde{\phi}_{\Delta,r}(\tau), \tilde{\omega}_N \rangle$ with special value of some Eisenstein series, where
$\tilde{\omega}_N$ is a normalized metric Hodge line bundle.

1. INTRODUCTION

Eisenstein series plays an important role in the arithmetic geometry
and number theory. Kudla conjectured that the derivative of Eisen-
stein series is closely related to the arithmetic intersection number on
Shimura varieties, commonly called arithmetic Siegel-Weil formula. In
[KRY1], Kudla, Rapoport and Yang gave such a formula on a division
Shimura curve. Kudla and Yang worked out a result on the modular
curve $X_0(1)$ [Ya]. The associated Eisenstein series is Zagier’s famous
Eisenstein series [HZ] of weight $3/2$. In [BF2], Bruinier and Funke
gave another proof of the main formula in [Ya] via theta lifting. We
extended the arithmetic Siegel-Weil formula to modular curve $X_0(N)$
with $N$ square free in [DY].

This paper is a sequel to [DY]. We replace the Heegner divisors
by twisted Heegner divisors, which were first studied by Bruinier and
Ono in [BO]. Interestingly, the derivative part of the Eisenstein series
disappear in such a case.

Let $N > 0$ be a positive integer, and let

$$(1.1) \quad V = \{ w = (w_1, w_2, w_3) \in M_2(\mathbb{Q}) : \text{tr}(w) = 0 \},$$

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with the quadratic form \( Q(w) = N \det w = -Nw_2w_3 - Nw_1^2 \). Let

\[
L = \left\{ w = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in M_2(\mathbb{Z}) \mid a, b, c \in \mathbb{Z} \right\} \subset V,
\]

and let \( L^\sharp \) be its dual lattice. Then \( \text{SL}_2 \cong \text{Spin}(V) \) acts on \( V \) by conjugation, i.e., \( g \cdot w = gwg^{-1} \), and \( \Gamma_0(N) \) acts on \( L^\sharp / L \) trivially. Let \( \mathbb{D} \) be the associated Hermitian domain of positive lines in \( V_\mathbb{R} \), then it is isomorphic to upper half plane \( \mathbb{H} \) via (2.2) which preserves the action of \( \text{SL}_2 \). We identify \( X_0(N) \) with compactification of \( \Gamma_0(N) \setminus \mathbb{D} \).

For each \( \mu \in \frac{L^\sharp}{L} \), let \( L_{\mu} = \mu + L \) and \( \left[ n \right] = \left\{ w \in L_{\mu} : Q(w) = n \right\} \).

Let \( \Delta \in \mathbb{Z}_{>0} \) be a positive fundamental discriminant which is a square modulo \( 4N \). Let \( L^\Delta = \Delta L \) with quadratic form \( Q^\Delta(x) = \frac{Q(x)}{\Delta} \), then its dual lattice \( L^\Delta,\sharp = L^\sharp \). Associated to \( \Delta \) is a generalized ‘genus character’ \( \chi_\Delta : \frac{L^\sharp}{L^\Delta} \to \{ \pm 1 \} \), defined by Gross, Kohnen, and Zagier in \([\text{GKZ}]\), which can be rephrased as a map (see Section 2.2 for detail). Fix a class \( r \mod (2N) \) with \( \Delta = r^2 \mod (4N) \), we have then a twisted Heegner divisor for any \( \mu \in \frac{L}{L} \) and a positive rational number \( n \in \mathbb{Q}(\mu) + \mathbb{Z} \).

\[
(1.3) \quad Z_{\Delta,r}(n, \mu) := \sum_{w \in \Gamma_0(N) \setminus L_{\mu}[\Delta \mathbb{N}]} \chi_\Delta(w) Z(w) \in \text{Div}(X_0(N))_{\mathbb{Q}},
\]

which is defined over \( \mathbb{Q}(\sqrt{\Delta}) \). Here \( Z(w) = \mathbb{R}w \) is the point on \( X_0(N) \) given by the positive line \( \mathbb{R}w \).

We will construct twisted Kudla’s Green function \( \Xi_{\Delta,r}(n, \mu, v) \) for \( Z_{\Delta,r}(n, \mu) \) in Section 4. All these functions are smooth at cusps, which are different from the Green functions \( \Xi(n, \mu, v) \) in \([\text{DY}]\). These functions are well-defined and smooth when \( n \leq 0 \).

Now assume that \( N \) is square free. Let \( X_0(N) \) be the canonical integral model over \( \mathbb{Z} \) of \( X_0(N) \) as defined in \([\text{KM}]\) (see Section 5.1). Then we could define twisted arithmetic divisors in arithmetic Chow group \( \widehat{\text{CH}}^1 (X_0(N)) \) by

\[
(1.4) \quad Z_{\Delta,r}(n, \mu, v) = \begin{cases} (Z_{\Delta,r}(n, \mu), \Xi_{\Delta,r}(n, \mu, v)) & \text{if } n > 0, \\ (0, \Xi_{\Delta,r}(n, \mu, v)) & \text{otherwise}, \end{cases}
\]

where \( Z_{\Delta,r}(n, \mu) \) is the Zariski closure of \( Z_{\Delta,r}(n, \mu) \) in \( X_0(N) \). Now we could define the twisted arithmetic theta function as follows.
Definition 1.1.

(1.5) \[ \hat{\phi}_{\Delta,r}(\tau) = \sum_{n \equiv Q(\mu)(\mod Z)} q_n^r e^{i \tau n} e^\mu \in \widehat{CH}_R^1(\mathcal{X}_0(N)) \otimes \mathbb{C}[L^2/L] [[q,q^{-1}]], \]

where \( q_\tau = e^{2\pi i \tau} \).

Let \( \Gamma' \) be the metaplectic cover of \( \text{SL}_2(\mathbb{Z}) \) which acts on \( \mathbb{C}[L^2/L] \) via the Weil representation \( \rho_L \) (see (2.3)) and let \( \{ e_\mu : \mu \in L^2/L \} \) be the standard basis of \( \mathbb{C}[L^2/L] \). Then

\[ E_L(\tau,s) = \sum_{\gamma' \in \Gamma' \backslash \Gamma} (v^{s-1} e_\mu_{\rho_L,\gamma'}) |_{3/2,\rho_L,\gamma'} \]

is a vector valued Eisenstein series of weight \( 3/2 \), where the Petersson slash operator is defined on functions \( f : \mathbb{H} \rightarrow \mathbb{C}[L^2/L] \) by

\[ (f |_{3/2,\rho_L,\gamma'})(\tau) = \phi(\tau)^{-3} \rho_L^{-1}(\gamma') f(\gamma\tau), \]

and \( \gamma' = (\gamma,\phi) \in \Gamma' \). Define the normalized Eisenstein series [DY equation (1.5)]

(1.6) \[ \mathcal{E}_L(\tau,s) = -\frac{s}{4} \pi^{-s-1} \Gamma(s) \zeta^{(N)}(2s) N^{\frac{1}{2}+\frac{3}{2}s} E_L(\tau,s) \]

where

\[ \zeta^{(N)}(s) = \zeta(s) \prod_{p|N} (1 - p^{-s}). \]

Let \( \mathbb{H}_N \) be the metrized Hodge bundle on \( \mathcal{X}_0(N) \) with certain normalized Petersson metric (see ([Ku2]) and (1.11)). The main result of this paper is as follows.

**Theorem 1.2.** Let \( \Delta > 1 \) be a fundamental discriminant, then

(1.7) \[ \deg \hat{\phi}_{\Delta,r}(\tau) = 0 \]

and

(1.8) \[ \langle \hat{\phi}_{\Delta,r}(\tau), \mathbb{H}_N \rangle = \frac{1}{\varphi(N)} \log(u_\Delta) h(\Delta) \mathcal{E}_L(\tau,1), \]

where \( u_\Delta > 1 \) is the fundamental unit of quadratic field \( \mathbb{Q}(\sqrt{\Delta}) \), \( h(\Delta) \) is its class number and \( \varphi \) is the Euler function.

It is interesting to compare it with the main result in [DY], which we state it here for convenience.
Theorem 1.3. [DY, Theorem 1.3] When \( \Delta = 1 \),
\[
\deg \hat{\varphi}_{\Delta,r}(\tau) = \frac{2}{\varphi(N)} \mathcal{E}_L(\tau, 1),
\]
and
\[
\langle \hat{\varphi}_{\Delta,r}(\tau), \hat{\omega}_N \rangle = \frac{1}{\varphi(N)} \left( \mathcal{E}'_L(\tau, 1) - \sum_{p|N} \frac{p}{p-1} \mathcal{E}_L(\tau, 1) \log p \right).
\]

Here is the basic idea in the proof of Theorem 1.2. For any \( \Gamma_0(N) \) invariant function \( f \), we define the twisted theta lift by
\[
(1.9) \quad I_{\Delta,r}(\tau, f) = \int_{\mathcal{X}_0(N)} f(z) \Theta_{\Delta,r}(\tau, z),
\]
where \( \Theta_{\Delta,r}(\tau, z) \) is the twisted Kudla-Millson theta kernel defined by (2.16), following [AE, Section 4].

Define the normalized Eisenstein series of weight 0 as in [DY] by
\[
\mathcal{E}(N, z, s) = \frac{N^{2s}}{2\pi} \Gamma(s) \zeta(N)(2s) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\Im(\gamma z))^s.
\]

We prove the following theorem in Section 3.

Theorem 1.4. When \( \Delta \) is a fundamental discriminant,
\[
(1.10) \quad I_{\Delta,r}(\tau, \mathcal{E}(N, z, s)) = \Delta^{\frac{s}{2}} \Lambda(\varepsilon_\Delta, s) \mathcal{E}_L(\tau, s),
\]
where \( \Lambda(\varepsilon_\Delta, s) = L(\varepsilon_\Delta, s) \Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}} \) is the completed \( L \)-series associated to the character \( \varepsilon_\Delta(n) = (\Delta n) \).

Take the residue of both sides, we obtain the first identity in Theorem 1.2. For a modular form \( f \) of weight \( k \), we define its renormalized Petersson metric as
\[
(1.11) \quad \| f(z) \| = |f(z)(4\pi e^{-C y})^{\frac{k}{2}}|,
\]
where \( C = \frac{\log 4\pi + \gamma}{2} \).

Combining the above theorem with the Kronecker limit formula for \( \Gamma_0(N) \) [DY, Theorem 1.5], we obtain the following result.

Theorem 1.5.
\[
(1.12) \quad -\frac{1}{12} I_{\Delta,r}(\tau, \log \| \Delta_N \|) = \begin{cases} \\
\mathcal{E}'_L(\tau, 1) \quad & \text{if } \Delta = 1, \\
\log(u_\Delta) h(\Delta) \mathcal{E}_L(\tau, 1) \quad & \text{if } \Delta > 1.
\end{cases}
\]

From above two theorems, we can reduce the proof of the second identity in Theorem 1.2 to the comparison between the Fourier coefficients of \( I_{\Delta,r}(\tau, \log \| \Delta_N \|) \) and intersection number \( \langle \hat{Z}_{\Delta,r}(n, \mu, v), \hat{\omega}_N \rangle \). We will do it in the Section 5.

Finally, we obtain the following modularity result.
Theorem 1.6. \( \hat{\phi}_{\Delta,r}(\tau) \) is a vector valued modular form for \( \Gamma' \) of weight \( \frac{k}{2} \), valued in \( \mathbb{C}[L^{r}/L] \otimes \hat{\mathcal{H}}_{r}^{1}(X_0(N)) \).

The case \( \Delta = 1 \) is proved in [DY]. In [BO, Section 6], Bruinier and Ono proved that

\[
A_{\Delta,r}(\tau) = \sum_{\mu \in L^{r}/L} \sum_{n > 0} Z_{\Delta,r}(n, \mu) q_{r}^{n} e_{\mu}
\]

is a cusp form valued in \( S_{\frac{k}{2},\rho_{L}} \otimes J(Q(\sqrt{\Delta})) \), where \( J \) is the Jacobian of \( X_{0}(N) \). Notice \( A_{\Delta,r}(\tau) \) is the generic component of \( \hat{\phi}_{\Delta,r}(\tau) \). So modularity of \( \hat{\phi}_{\Delta,r}(\tau) \) (Theorem 1.6) is an integral version of their result.

This paper is organized as follows. We will introduce some notations and introduce the twisted Kudla-Millson theta function in Section 2. We will introduce the twisted theta lifting and use it to prove the Theorem 1.4 and 1.5 in Section 3. In Section 4, we define twisted Kudla’s Green functions, and show these functions are smooth at cusps. In Section 5, we will define the arithmetic theta functions and prove the main result Theorem 1.2. In the last section, we obtain the modularity Theorem 1.6.

Acknowledgments. Add later.

Part 1. Theta lifting and Kronecker limit formula

2. Basic set-up and theta lifting

Let \( V \) be the quadratic space and let \( L \) be the even integral lattice defined in the introduction. Then \( \text{SL}_{2} \cong \text{Spin}(V) \) acts on \( V \) by conjugation, i.e., \( g.w = gwg^{-1} \). Notice that \( \Gamma_{0}(N) \) acts on \( L^{r}/L \) trivially. Let \( \mathbb{D} \) be the Hermitian domain of positive real lines in \( V_{\mathbb{R}} \):

\[
\mathbb{D} = \{ z \subset V_{\mathbb{R}}; \dim z = 1 and ( , ) |_{z} > 0 \}.
\]

The isomorphism between \( \mathbb{H} \) and \( \mathbb{D} \) is given by the map

\[
(2.1) \quad z = x + iy \mapsto w(z) = \frac{1}{\sqrt{Ny}} \begin{pmatrix} -x & z \bar{z} \\ -1 & x \end{pmatrix}.
\]

The inverse is

\[
(2.2) \quad w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto z(w) = \frac{2aN + \sqrt{D}}{2cN}, \quad D = -4NQ(w).
\]

This isomorphism is \( \text{SL}_{2}(\mathbb{R}) \)-action compatible and induces an isomorphism between \( Y_{0}(N) = \Gamma_{0}(N) \backslash \mathbb{H} \) and \( \Gamma_{0}(N) \backslash \mathbb{D} \). Let \( X_{0}(N) \) be the usual compactification of \( Y_{0}(N) \).
2.2. Twisted Heegner divisors. Let $\text{Mp}_{2,\mathbb{R}}$ be the metaplectic double cover of $\text{SL}_2(\mathbb{R})$, which can be realized as pairs $(g, \phi(g, \tau))$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, $\phi(g, \tau)$ is a holomorphic function of $\tau \in \mathbb{H}$ such that $\phi(g, \tau)^2 = j(g, \tau) = ct + d$. Let $\Gamma'$ be the preimage of $\text{SL}_2(\mathbb{Z})$ in $\text{Mp}_{2,\mathbb{R}}$ with two generators

$$S = \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \quad T = \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right).$$

We denote the standard basis of $\mathbb{C}[L^2/L]$ by $\{e_\mu = L_\mu : \mu \in L^1/L\}$. Then there is a Weil representation $\rho_L$ of $\Gamma'$ on $\mathbb{C}[L^2/L]$ given by \textcolor{blue}{[Boc]}

$$\rho_L(T)e_\mu = e(Q(\mu))e_\mu,$$

$$\rho_L(S)e_\mu = \frac{e(\frac{1}{8})}{\sqrt{|L^2/L|}} \sum_{\mu' \in L^1/L} e(-\mu, \mu')e_{\mu'}.$$ 

This Weil representation $\rho_L$ is closed related to the Weil representation $\omega$ of $\text{Mp}_2,\mathbb{A}$ on $S(V_\omega)$ (see \textcolor{blue}{[BHY]}).

The slash operator with weight $\frac{3}{2}$ is defined on functions $f : \mathbb{H} \to \mathbb{C}[L^2/L]$ given by

$$f|_{\frac{3}{2}, \rho_L} \gamma'/(\tau) = \phi(\tau)^{-3} \rho_L^{-1}(\gamma')f(\gamma\tau),$$

where $\gamma' = (\gamma, \phi) \in \Gamma'$.

2.1. The Weil representation. Let $\text{Mp}_{2,\mathbb{R}}$ be the metaplectic double cover of $\text{SL}_2(\mathbb{R})$, which can be realized as pairs $(g, \phi(g, \tau))$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, $\phi(g, \tau)$ is a holomorphic function of $\tau \in \mathbb{H}$ such that $\phi(g, \tau)^2 = j(g, \tau) = ct + d$. Let $\Gamma'$ be the preimage of $\text{SL}_2(\mathbb{Z})$ in $\text{Mp}_{2,\mathbb{R}}$ with two generators

$$S = \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \quad T = \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right).$$

We denote the standard basis of $\mathbb{C}[L^2/L]$ by $\{e_\mu = L_\mu : \mu \in L^1/L\}$. Then there is a Weil representation $\rho_L$ of $\Gamma'$ on $\mathbb{C}[L^2/L]$ given by \textcolor{blue}{[Boc]}

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This Weil representation $\rho_L$ is closed related to the Weil representation $\omega$ of $\text{Mp}_2,\mathbb{A}$ on $S(V_\omega)$ (see \textcolor{blue}{[BHY]}).

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$$f|_{\frac{3}{2}, \rho_L} \gamma'/(\tau) = \phi(\tau)^{-3} \rho_L^{-1}(\gamma')f(\gamma\tau),$$

where $\gamma' = (\gamma, \phi) \in \Gamma'$.

2.2. Twisted Heegner divisors. Let $\Delta \in \mathbb{Z}_{>0}$ be a fundamental discriminant which is a square modulo $4N$, and let $L^\Delta = \Lambda L$ with renormalized quadratic form $Q_\Delta(w) = \frac{Q(w)}{\Delta}$. Then it is easy to check $L^\Delta \sharp = L^\sharp$. Let $\Gamma_\Lambda$ be the subgroup of $\Gamma_0(N)$ which acts on $L^\Delta \sharp/L^\Delta$ trivially. It is not hard to check that the map

$$\chi_\Lambda \left( \begin{pmatrix} \frac{a}{2N} & -b \sqrt{N} \\ c & \frac{a}{2N} \end{pmatrix} \right) = \begin{cases} \left( \frac{\Delta}{n} \right), & \text{if } \Delta \mid b^2 - 4Na \text{ and } \frac{b^2 - 4Na}{\Delta} \text{ is a square modulo } 4N \text{ and } (a, b, c, \Delta) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

gives a well-defined map

$$\chi_\Lambda : L^\Delta \sharp/L^\Delta \to \{ \pm 1 \}.$$ 

Here $n$ is any integer prime to $\Delta$ represented by one of the quadratic form $[N_1a, b, N_2c]$ with $N_1N_2 = N$ and $N_1, N_2 > 0$, and $[a, b, Nc] = ax^2 + bxy + Ncy^2$ is the integral binary quadratic form corresponding to $w = \left( \frac{x}{c}, \frac{y}{N} \right)$. Indeed, $\chi_\Lambda(w) = \chi_\Lambda([a, b, Nc])$ is the generalized genus character defined in \textcolor{blue}{[GKZ, Section 1]} (see also \textcolor{blue}{[BO, Section ...
4]). We leave it to readers to check that $\chi(w + L^\Delta) = \chi(w)$ and so $\chi_{\Delta}$ induces a map on $L^{\Delta,\sharp}/L^\Delta$. It is known [GKZ] that the map is invariant under the action of $\Gamma_0(N)$ and the action of all Atkin-Lehner involutions, i.e.,

$$\chi_{\Delta}(\gamma w \gamma^{-1}) = \chi_{\Delta}(w) \text{ and } \chi_{\Delta}(W_M w W_M^{-1}) = \chi_{\Delta}(w),$$

where $\gamma \in \Gamma_0(N)$ and $W_M$ is the Atkin-Lehner involution with $M \mid N$.

Fix a $r \mod 2N$ class with $r^2 \equiv \Delta \mod 4N$. For any $\mu \in L^\sharp/L$ and a positive rational number $n \in Q(\mu) + Z$, we define the twisted Heegner divisor by

$$Z_{\Delta, r}(n, \mu) := \sum_{w \in \Gamma_0(N) \setminus L_{\mu}[n\Delta]} \chi_{\Delta}(w) Z(w) \in \text{Div}(X_0(N))_Q,$$

which is defined over $Q(\sqrt{\Delta})$. Notice that we count each point $Z(w) = \mathbb{R} w$ with multiplicity $\frac{2}{|\Gamma_w|}$ in the orbifold $X_0(N)$, where $\Gamma_w$ is the stabilizer of $w$ in $\Gamma_0(N)$. So our definition is the same as that in [AE, Section 5] and [BO, Section 5].

Now define

$$Z_{\Delta}(n, \delta) := \sum_{w \in \Gamma_{\Delta} \setminus L_{\delta}^{\Delta}[n]} Z(w) \in \text{Div}(X_{\Gamma_{\Delta}}).$$

Recall the natural map

$$\pi_{\Gamma_{\Delta}} : X_{\Gamma_{\Delta}} \longrightarrow X_0(N)$$

is a covering map with the degree $[\Gamma_{\Delta} : \Gamma_{\Delta}]$. Here for any congruence group, $\hat{\Gamma} = \Gamma/(\Gamma \cap \{ \pm 1 \})$.

**Lemma 2.1.** Let $n \equiv Q(\mu) \ (\mod Z)$ be a positive number, then

$$\sum_{\delta \in L^{\Delta,\sharp}/L^\Delta, \delta \equiv r_{\mu}(L)} \chi_{\Delta}(\delta) Z_{\Delta}(n, \delta) = \pi_{\Gamma_{\Delta}}^*(Z_{\Delta, r}(n, \mu)), \tag{2.9}$$

where $\pi_{\Gamma_{\Delta}}^*$ is the pullback

$$\pi_{\Gamma_{\Delta}}^* : Z^1(X_0(N)) \longrightarrow Z^1(X_{\Gamma_{\Delta}}).$$

**Proof.** Write $\Gamma = \Gamma_0(N)$, and notice $L^\sharp = L^{\Delta,\sharp}$. For $w \in \Gamma \setminus L_{\mu}[\Delta n]$, $\pi_{\Gamma_{\Delta}}^{-1}(Z(w)) = \{ Z(w_1), ..., Z(w_g) \}$, then $\pi_{\Gamma_{\Delta}}^*(Z(w)) = Z(w_1) + ... + Z(w_g)$.
Thus one has
\[
\pi^*_{\Gamma,\Delta}(Z_{\Delta,r}(n,\mu)) = \sum_{w \in \Gamma \setminus \mathbb{L}_\mu[\Delta \eta]} \chi_\Delta(w) \pi^*_{\Gamma,\Delta}(Z(w))
\]
\[
= \sum_{\delta \in L^1 / L^\Delta, \delta \equiv r \mu(L)} \chi_\Delta(\delta) \sum_{w \in \Gamma \setminus \mathbb{L}_{\delta}[\eta]} Z(w)
\]
\[
(2.10)
\]
\[
= \sum_{\delta \in L^1 / L^\Delta, \delta \equiv r \mu(L)} \chi_\Delta(\delta) Z_\Delta(n,\delta)
\]
Then we obtain the result. □

2.3. Twisted Kudla-Millson theta function. Following Kudla and Millson ([KM], [BF2, Section 3]), for \(z = x + iy \in \mathbb{H}\), there is a decomposition
\[
V_\mathbb{R} = \mathbb{R} w(z) \oplus \mathbb{R} w(z)^{\perp}, \quad w = w_1 + w_{1\perp}.
\]
Define \(R(w, z)_\Delta = -(w_{1\perp}, w_{1\perp})_\Delta\), and the majorant
\[
(w, w)_\Delta = (w_1, w_3)_{\Delta} + R(w, z)_\Delta,
\]
where \((,)_\Delta = (,)_\mathbb{A}\) is the bilinear form associated to the quadratic form \(Q_\Delta\). One has
\[
(2.11) \quad R(w, z)_\Delta = \frac{1}{2} (w, w(z))_\Delta^2 - (w, w)_\Delta.
\]
For \(w = (\begin{smallmatrix} w_1 \\
 w_2 \\
 w_3 \\
- w_1 \end{smallmatrix}) \in V_\mathbb{R}\),
\[
(2.12) \quad (w, w(z))_\Delta = -\sqrt{\frac{N}{y \sqrt{\Delta}}} (w_3 z \bar{z} - w_1 (z + \bar{z}) - w_2).
\]
Let
\[
\varphi^0_\Delta(w, z) = \left( (w, w(z))_\Delta^2 - \frac{1}{2\pi} \right) e^{-2\pi R(w, z)_\Delta} \mu(z)
\]
\[
(2.13) \quad \text{and } \varphi_\Delta(w, \tau, z) = e(Q_\Delta(w, \tau)) \varphi^0_\Delta(\sqrt{\omega} w, z),
\]
be the differential forms on \(V_\mathbb{R}\) valued in \(\Omega^{1,1}(\mathbb{D})\), where \(\mu(z) = \frac{dx \, dy}{y^2}\).

For any \(\delta \in L^\Delta / L^\Delta\), define
\[
(2.14) \quad \Theta_\delta(\tau, z) = \sum_{w \in L^\Delta_\delta} \varphi_\Delta(w, \tau, z),
\]
where \(L^\Delta_\delta = L^\Delta + \delta\). Then
\[
(2.15) \quad \Theta_{L^\Delta}(\tau, z) = \sum_{\delta \in L^1 / L^\Delta} \Theta_\delta(\tau, z) e_\delta
\]
is a vector valued Kudla-Millson theta function, which is a nonholomorphic modular form of weight $3/2$ of $(\Gamma', \rho_L \Delta)$ with respect to the variable $\tau$ with values in $\Omega^{1,1}(X_{\Gamma_0}, \Delta)$, where $X_{\Gamma_0}$ is the modular curve $\Gamma_0 \backslash \mathbb{H}$.

Fix a class $r \mod 2N$ with $r^2 \equiv \Delta \mod 4N$. Following the Bruinier and Ono’s work [BO], Alfes and Ehlen constructed a $\mathbb{C}[L^2/L]$-valued theta function [AE, Section 4]

$$\Theta_{\Delta, r}(\tau, z) := \sum_{\mu} \Theta_{\Delta, r, \mu}(\tau, z) e_{\mu},$$

where

$$\Theta_{\Delta, r, \mu}(\tau, z) = \sum_{\substack{\delta \in L^2/L, \delta \equiv r \mu \mod 1 \in \mathbb{Z},\mathbb{Z} \cap L^2/L, \Delta \equiv Q(\delta) \mod 1}} \chi_{\Delta}(\delta) \Theta_{\delta}(\tau, z).$$

This twisted theta function has a good transformation properties just like the classical Kudla-Millson theta functions.

**Proposition 2.2.** [AE, Proposition 4.1] The theta function $\Theta_{\Delta, r}(\tau, z)$ is a non-holomorphic $\mathbb{C}[L^2/L]$-valued modular form of weight $3/2$ for the representation $\rho_L$. Furthermore, it is a non-holomorphic automorphic form of weight $0$ for $\Gamma_0(N)$ in the variable $z \in \mathbb{D}$.

## 3. Twisted theta lift

Following Alfes and Ehlen [AE], we consider the twisted theta lifting: for any $\Gamma_0(N)$-invariant function $f(z)$, the lifting is given by

$$I_{\Delta, r}(\tau, f) := \int_{X_0(N)} f(z) \Theta_{\Delta, r}(\tau, z) = \sum_{\mu \in L^2/L} \int_{X_0(N)} f(z) \Theta_{\Delta, r, \mu}(\tau, z) e_{\mu},$$

if the integral is convergent.

In this section, we consider the lift of Eisenstein series $E(N, z, s)$ and Petersson norm $\log \|\Delta_N\|$. From [BF2, Proposition 4.1], one knows the theta function is $O(e^{-Cy^2})$ around cusps, as $y \to \infty$ for some constant $C > 0$. Then these two lifts are convergent.

We first recall a result of Alfes and Ehlen.

**Proposition 3.1.** [Al, Proposition 3.1], [Eh] Let $K = \mathbb{Z}$ with the quadratic form $Q(x) = -N x^2 (K^2/K \cong L^2/L)$, then

$$\Theta_{\Delta, r}(\tau, z) = -y N^{3/2} \sum_{n=1}^{\infty} \gamma_{\tau}^{1/2} \sum_{\gamma \in \Gamma_0 \backslash \Gamma'} \left[ \exp(-\pi y^2 N n^2 v_{\Delta}) v^{-3/2} \sum_{\lambda \in K^2} e(\Delta Q(\lambda) \tau - 2N \lambda n x) e_{r \lambda} \right] \gamma dxdy.$$
Now we are ready to prove Theorem 1.4 which we restate here for convenience. We follow the idea in the proof of [AE, Theorem 6.1], where the case $N = 1$ is considered.

**Theorem 3.2.** When $\Delta$ is a fundamental discriminant, 
\begin{equation}
I_{\Delta,r}(\tau, E(N, z, s)) = \Delta^{\frac{3}{2}} \Lambda(\varepsilon_{\Delta}, s) E_L(\tau, s),
\end{equation}
where $\Lambda(\varepsilon_{\Delta}, s) = L(\varepsilon_{\Delta}, s) \Gamma(\frac{s}{2}) \pi^{-\frac{s}{2}}$ is the completed $L$-series associated to the character $\varepsilon_{\Delta}$.

**Proof.** One has by Proposition 3.1,
\begin{align*}
\Theta_{\Delta,r}(\tau, z) &= -\frac{y N^{3/2}}{v^{3/2} \Delta} \sum_{n=1}^{\infty} \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma'} n^2 \left( \frac{\Delta}{n} \right) \exp\left( -\frac{\pi y^2 N n^2 |c\tau + d|^2}{v \Delta} \right) v^{-3/2} |c\tau + d|^3 \\
&\times (c\tau + d)^{-3/2} \sum_{\lambda \in K^2} e(\Delta Q(\lambda) \tau - 2 N \lambda_n x) \rho_K^{-1}(\gamma) e_{r,\lambda} dx dy \\
&= -\frac{y N^{3/2}}{v^{3/2} \Delta} \sum_{n=1}^{\infty} \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma'} n^2 \left( \frac{\Delta}{n} \right) \exp\left( -\frac{\pi y^2 N n^2 |c\tau + d|^2}{v \Delta} \right) \\
&\times (c\tau + d)^{3/2} \sum_{\lambda \in K^2} e(\Delta Q(\lambda) \tau - 2 N \lambda_n x) \rho_K^{-1}(\gamma) e_{r,\lambda} dx dy.
\end{align*}

Here for every coprime pair $(c, d)$, there is unique $\gamma = \left( \begin{smallmatrix} c \\ d \end{smallmatrix} \right) \in \Gamma_{\infty} \setminus \Gamma'$, and we could identify them.

Unfolding the integral, for $\Re(s) > 1$, we have
\begin{align*}
I_{\Delta,r}(\tau, E(N, z, s)) &= \int_{\Gamma_{\infty} \setminus \mathcal{H}} \Theta_{\Delta,r}(\tau, z) y^s \\
&= -\frac{N^{3/2}}{v^{3/2} \Delta} \sum_{n=1}^{\infty} n^2 \left( \frac{\Delta}{n} \right) \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma'} (c\tau + d)^{3/2} \int_{0}^{\infty} e\left( -\frac{\pi y^2 N n^2 |c\tau + d|^2}{\Delta} \right) y^{s+1} dy \\
&\times \rho_K^{-1}(\gamma) \int_{0}^{1} \sum_{\lambda \in K^2} e(\Delta Q(\lambda) \tau - 2 N \lambda_n x) e_{r,\lambda} dx.
\end{align*}

Notice that
\begin{align*}
\int_{0}^{1} \sum_{\lambda \in K^2} e(\Delta Q(\lambda) \tau - 2 N \lambda_n x) e_{r,\lambda} dx &= e_{\mu_0}.
\end{align*}

Then one has
\[
\int_{\Gamma_\infty \setminus \mathbb{H}} \Theta_L(\tau, z) y^s \\
= -\frac{N^{\frac{s}{2}}}{2v^{\frac{s}{2}}\Delta} \sum_{n=1}^{\infty} n^2 \left( \frac{\Delta}{n} \right) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma'} \frac{v^{\frac{s+2}{2}}(c\tau + d)^{3/2}\Gamma\left(\frac{s}{2} + 1\right)}{\pi^{s+2} |c\tau + d|^{s+2} N^{\frac{s+2}{2}n^{s+2}}} \rho_K^{-1}(\gamma) e_{\mu_0} \\
= -\frac{1}{2} N^{-\frac{s}{2}} \Delta^{\frac{s}{2}} L(\varepsilon, s) \Gamma\left(\frac{s}{2} + 1\right) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma'} \frac{v^{\frac{s+1}{2}}(c\tau + d)^{3/2}}{\pi^{s+1} n^{s+2}} |c\tau + d|^{s+2} \rho_K^{-1}(\gamma) e_{\mu_0} \\
= -N^{-\frac{s}{2}} \Delta^{\frac{s}{2}} \frac{s}{4\pi} \Delta(\varepsilon, s) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma'} \left( v^{\frac{s-1}{2}} e_{\mu_0} \right) |3/2, \rho_L, \gamma|.
\]

Notice in the last equality we identify \( \rho_L \) with \( \rho_K \) because the isomorphism \( K^\sharp/K \cong L^\sharp/L \) keep the quadratic form. For the normalized Eisenstein series, we obtain
\[
I_{\Delta,r}(\tau, \mathcal{E}(N, z, s)) = \Delta^{\frac{s}{2}} \mathcal{E}_L(\tau, s).
\]

□

Taking residue of both sides of the equation (3.2) at \( s = 1 \), we have the following result.

**Corollary 3.3.**

\[\tag{3.3} I_{\Delta,r}(\tau, 1) = \begin{cases} \frac{2}{\varphi(N)} \mathcal{E}_L(\tau, 1) & \text{if } \Delta = 1, \\
0 & \text{if } \Delta > 1. \end{cases}\]

Recall the modular form of weight \( k = 12\varphi(N) \) for \( \Gamma_0(N) \) defined in [DY, (1.6)]:
\[\tag{3.4} \Delta_N(z) = \prod_{\ell | N} \Delta(tz)^{a(t)}\]
with
\[a(t) = \sum_{r | t} \mu\left(\frac{t}{r}\right) \mu\left(\frac{N}{r}\right) \varphi\left(\frac{N}{r}\right) \varphi\left(\frac{N}{r}\right),\]
where \( \mu(n) \) is the the Möbius function and \( \varphi(N) \) is the Euler function.

For a modular form \( f \) of weight \( k \) and level \( N \), we define its normalized Petersson metric as
\[\tag{3.5} ||f(z)|| = |f(z)(4\pi e^{-C}y)^{\frac{k}{2}}|\]
where \( C = \frac{\log 4\pi + \gamma}{2} \) with Euler constant \( \gamma \).
Theorem 3.4.

\[ (3.6) \quad -\frac{1}{12} I_{\Delta,r}(\tau, \log \|\Delta_N\|) = \begin{cases} \mathcal{E}_L(\tau, 1) & \text{if } \Delta = 1, \\ \log(u_\Delta)h(\Delta)\mathcal{E}_L(\tau, 1) & \text{if } \Delta > 1, \end{cases} \]

where \( u_\Delta > 1 \) is the fundamental unit and \( h(\Delta) \) is the class number of real quadratic field with discriminant \( \Delta \).

Proof. We proved the case \( \Delta = 1 \) in [DY, Theorem 1.6]. Now we assume \( \Delta > 1 \). From the Kronecker limit formula for \( \Gamma_0(N) \)[DY, Theorem 1.5]

\[ \lim_{s \to 1} \left( \mathcal{E}(N, z, s) - \varphi(N)\zeta^*(2s-1) \right) = -\frac{1}{12} \log \left( y^{6\varphi(N)} | \Delta_N(z) | \right), \]

one has

\[ -\frac{1}{12} I_{\Delta,r}(\tau, \log \|\Delta_N(z)\| y^{6\varphi(N)} |) = \lim_{s \to 1} \left( I_{\Delta,r}(\tau, \mathcal{E}(N, z, s)) - I_{\Delta,r}(\tau, \varphi(N)\zeta^*(2s-1)) \right). \]

Here \( \zeta^*(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})\zeta(s) \). From the Theorem 1.4 and Corollary 3.3, we obtain

\[ -\frac{1}{12} I_{\Delta,r}(\tau, \log \|\Delta_N(z)\|) = \sqrt{\Delta}\Lambda(\varepsilon, 1)\mathcal{E}_L(\tau, 1) = \log(u_\Delta)h(\Delta)\mathcal{E}_L(\tau, 1). \]

\[ \Box \]

4. Twisted Kudla’s Green function

Following Kudla’s methods in [Ku1], we construct twisted Kudla’s Green function for \( Z_{\Delta,r}(n, \mu) \) in this section.

For \( r > 0 \) and \( s \in \mathbb{R} \), let

\[ (4.1) \quad \beta_s(r) = \int_1^{\infty} e^{-rt} t^{-s} dt, \]

and

\[ (4.2) \quad \xi_\Delta(w, z) = \beta_1(2\pi R(w, z)\Delta). \]

Definition 4.1 (Twisted Kudla’s Green functions). For \( n \in Q(\mu) + \mathbb{Z} \), define

\[ (4.3) \quad \Xi_{\Delta,r}(n, \mu, v)(z) = \sum_{\delta \in L^*/L, \delta \equiv \mu L} \chi(\delta)\Xi_L(\delta)\Xi_{\Delta,r}(n, \delta, v)(z), \]
where $\Xi_{L,\ell}(n,\delta,v)(z)$ is the Kudla’s Green function associated to the lattice $L^\Delta$ with quadratic form $Q_\Delta$ given by

$$\Xi_{L,\ell}(n,\delta,v)(z) = \sum_{0 \neq w \in L^\Delta [n]} \xi_\Delta(\sqrt{v}w, z).$$

So one has

$$\Xi_{\Delta, r}(n, \mu, v)(z) = \sum_{0 \neq w \in L_{r,\ell} [\Delta]} \chi_\Delta(w) \xi_\Delta(\sqrt{v}w, z),$$

which is clearly invariant under $\Gamma_0(N)$. Recall from the Lemma 2.1

$$\sum_{\delta \in L^\ell / L^\Delta, \delta \equiv \mu(L)} \chi_\Delta(\delta) Z_\Delta(n, \delta) = \pi_{\ell,\mu}^*(Z_{\Delta, r}(n, \mu)).$$

Since $\Xi_{L,\ell}(n, \delta, v)$ is a Green function for $Z_\Delta(n, \delta)$ on $X_{\ell,\ell} = \Gamma_\ell \backslash \mathbb{H}$ by [Kn], we have thus the following lemma.

**Lemma 4.2.** When $n > 0$, $\Xi_{\Delta, r}(n, \mu, v)(z)$ is a Green function for $Z_{\Delta, r}(n, \mu)$ on $Y_0(N) = \Gamma_0(N) \backslash \mathbb{H}$, and satisfies the following current equation,

$$dd^c[\Xi_{\Delta, r}(n, \mu, v)(z)] + \delta_{\Delta, r}(n, \mu) = [\omega_{\Delta, r}(n, \mu, v)],$$

where $\omega_{\Delta, r}(n, \mu, v)$ is the differential form

$$\omega_{\Delta, r}(n, \mu, v) = \sum_{w \in L_{\mu}[\Delta]} \chi_\Delta(w) \varphi^0_\Delta(w, z).$$

When $n \leq 0$, $\Xi_{\Delta, r}(n, \mu, v)(z)$ is smooth on $Y_0(N)$.

Now we consider behaviors of these Green functions at cusps. We recall some definitions given by Funke and Bruinier, see [BF1] and [Fu]. Let $\text{Iso}(V)$ be the set of isotropic non-zero vectors $\ell = Qw$ of $V$, with $0 \neq w \in V$ and $Q(w) = 0$. Given $\ell = \text{span}\{ (a,b) \} \in \text{Iso}(V)$, let $P_\ell = \frac{\ell}{\ell}$ be the associated cusp, which depends only on the equivalence class of isotropic line $\ell$. Two isotropic lines give the same cusp in $Y_0(N)$ if and only if there is $\gamma \in \Gamma_0(N)$ such that $\gamma \ell_1 = \ell_2$.

Let $\ell_\infty = \text{span}\{ (0,1) \} \in \text{Iso}(V)$ and then $P_\infty = \infty$ be the associated cusp. In general, for an isotropic line $\ell$, there exists $\sigma_\ell \in \text{SL}_2(\mathbb{Z})$ such that $\ell_\infty = \sigma_\ell \ell$. Then

$$\sigma_\ell \Gamma_0 \sigma_\ell^{-1} = \{ \pm \left( \begin{array}{cc} 1 & m \kappa_\ell \\ 0 & 1 \end{array} \right) , m \in \mathbb{Z} \},$$

where $\Gamma_\ell \subseteq \Gamma_0(N)$ is the stabilizer of $\ell$ and $\kappa_\ell > 0$ is the classical width of the associated cusp $P_\ell$. On the other hand, there is another positive number $\beta_\ell > 0$, depending on $L$ and the cusp $P_\ell$, such that $\left( \begin{array}{cc} 0 & \beta_\ell \\ 0 & 0 \end{array} \right)$ is a primitive element in $\ell_\infty \cap \sigma_\ell L$. The Funke constant $\varepsilon_\ell = \frac{\kappa_\ell}{\beta_\ell}$ at cusp $P_\ell$.
is defined in [Fu Section 3], which is called width by Funke. We will simply denote $\kappa = \kappa_\infty$.

Let $\delta_{w,\ell}$ be the number of isotropic lines $\ell_w \in \text{Iso}(V)$ which is perpendicular to $w$ and belongs to the same cusp as $\ell$. We often drop the index $\ell$ when $\ell = \ell_\infty$. In such a case, $w$ is orthogonal to two isotropic lines $\ell_w = \text{span}\{X\}$ and $\tilde{\ell}_w = \text{span}\{\tilde{X}\}$ such that $(w, X, \tilde{X})$ is a positively oriented basis of $V$. We denote $w \sim \ell_w$. Notice that $\ell_w = \ell_{-w}$.

When $-4Nn$ is a square, Funke defined in [Fu Section 3]

\[(4.8) \quad L_{\mu,\ell}[n] = \{ w \in L_{\mu}[n] ; w \sim \ell \}, \]

Here we use the different notation.

**Lemma 4.3.** Let $-4Nn$ be a square, then

\[(4.9) \quad \sum_{w \in \Gamma_0(N) \backslash L_{\mu}[n]} \delta_w = | \Gamma_{\ell_\infty} \setminus L_{\mu,\ell_\infty}[n] | + | \Gamma_{\ell_\infty} \setminus L_{-\mu,\ell_\infty}[n] | . \]

Moreover, if $2\mu \notin L$ and $L_{\mu}[n] \neq \phi$, then one of the numbers $| \Gamma_{\ell_\infty} \setminus L_{\mu,\ell_\infty}[n] |$, $| \Gamma_{\ell_\infty} \setminus L_{-\mu,\ell_\infty}[n] |$ is $\sqrt{-4Nn}$, the other is 0, so the sum is $\sqrt{-4Nn}$.

**Proof.** For any $w \in \Gamma_0(N) \backslash L_{\mu}[n]$ with $\delta_w \neq 0$, assume $w^\perp = \text{Span}\{\ell_\infty, \ell\}$.

From the definition of the $\delta_w$, this implies $w \sim \ell_\infty$ or $-w \sim \ell_\infty$.

If $\delta_w = 1$, then $w \in L_{\mu,\ell_\infty}[n]$ or $-w \in L_{-\mu,\ell_\infty}[n]$.

If $\delta_w = 2$, there exists $\gamma \in \Gamma_0(N)$ such that $\ell = \gamma \ell_\infty$. $w \in L_{\mu,\ell_\infty}[n]$ ($-w \in L_{-\mu,\ell_\infty}[n]$) implies that $-\gamma w \in L_{-\mu,\ell_\infty}[n]$ ($\gamma w \in L_{\mu,\ell_\infty}[n]$) respectively.

There is a map

\[(4.10) \quad f : \Gamma_{\ell_\infty} \setminus L_{\mu,\ell_\infty}[n] \cup \Gamma_{\ell_\infty} \setminus L_{-\mu,\ell_\infty}[n] \longrightarrow S \subseteq \Gamma_0(N) \backslash L_{\mu}[n], \]

given by if $w \in \Gamma_{\ell_\infty} \setminus L_{\mu,\ell_\infty}[n]$, $f(w) = w$; if $w \in \Gamma_{\ell_\infty} \setminus L_{-\mu,\ell_\infty}[n]$, $f(w) = -w$. Here $S$ is the subset $\{ w \in \Gamma_0(N) \backslash L_{\mu}[n] \mid \delta_w \neq 0 \}$.

Restriction on each subset $\Gamma_{\ell_\infty} \setminus L_{\mu,\ell_\infty}[n]$ or $\Gamma_{\ell_\infty} \setminus L_{-\mu,\ell_\infty}[n]$, the map is injective. If $w, w' \in \Gamma_{\ell_\infty} \setminus L_{\mu,\ell_\infty}[n]$ and $f(w) = f(w')$, then there exists $\sigma \in \Gamma_0(N)$ such that $\sigma w = w'$. It's need to keep orientation, so $\sigma \in \Gamma_{\ell_\infty}$, and then $w = w' \in \Gamma_{\ell_\infty} \setminus L_{\mu,\ell_\infty}[n]$.

For $w \in S$, when $\delta_w = 1$, there is only one preimage $w \in \Gamma_{\ell_\infty} \setminus L_{\mu,\ell_\infty}[n]$ or $-w \in \Gamma_{\ell_\infty} \setminus L_{-\mu,\ell_\infty}[n]$; when $\delta_w = 2$, there are exactly two preimages $(w, -\gamma w)$ or $(\gamma w, -w)$ depends on $w \in L_{\mu,\ell_\infty}[n]$ or not.

From above discussions, counting both sides of $f$, one obtains

\[(4.11) \quad \sum_{w \in \Gamma_0(N) \backslash L_{\mu}[n]} \delta_w = | \Gamma_{\ell_\infty} \setminus L_{\mu,\ell_\infty}[n] | + | \Gamma_{\ell_\infty} \setminus L_{-\mu,\ell_\infty}[n] | . \]
For any $\mu$, it’s known that $|\Gamma_{\ell,\infty} \setminus L_{\mu,\ell,\infty}[n]| = \sqrt{-4Nn}$ or $0$ ([Fu, Section 3]). From [DY, Lemma 6.2], if $2\mu \not\in L$

$$\sum_{w \in \Gamma_0(N) \setminus L_\mu[n]} \delta_w = \sqrt{-4Nn} \text{ or } 0,$$

This proves the lemma.

**Lemma 4.4.**

\[(4.12) \quad \sum_{w \in \Gamma_{\ell,\infty} \setminus L_{\mu,\ell,\infty}[n_\Delta]} \chi_{\Delta}(w) = 0.\]

**Proof.** The representatives for $\Gamma_{\ell,\infty} \setminus L_{\mu,\ell,\infty}[n_\Delta]$ is given by

$$\left\{ \left( \frac{k - j\beta_{\ell,\infty}}{-k} \right), j = 0, \ldots, 2kN - 1 \right\},$$

with $D = 4N^2k^2$ and $\beta_{\ell,\infty} = \frac{1}{N}$.

If $k$ is not of the form $\frac{k'}{2N}$ for some $k' \in \mathbb{Z}_{>0}$, then $\chi_{\Delta}(w) = 0$ by the definition of $\chi_{\Delta}$. So we could assume $k = \frac{k'}{2N}$. Then one has

\[(4.13) \quad \sum_{w \in \Gamma_{\ell,\infty} \setminus L_{\mu,\ell,\infty}[n_\Delta]} \chi_{\Delta}(w) = \sum_{j=0}^{2kN-1} \left( \frac{\Delta}{j} \right) = \sum_{j=0}^{\Delta k'-1} \left( \frac{\Delta}{j} \right) = 0.\]

The main purpose of this section is to prove the following result.

**Theorem 4.5.** Let the notations be as above, and $n \in Q(\mu) + \mathbb{Z}$, then $\Xi_{\Delta,r}(n,\mu,v)(z)$ is smooth at the cusps.

Moreover, let $0 \neq \ell \in \text{Iso}(V)$ be an isotropic vector and $P_\ell$ be the associated cusp. Around the cusp,

$$\lim_{q_\ell \to 0} \Xi_{\Delta,r}(n,\mu,v)(z) = 0,$$

where $q_\ell$ is a local parameter at the cusp $P_\ell$.

**Proof.** Let $D = -4N\Delta n$ and we split the proof into three cases: $D$ is not a square, $D > 0$ is a square and $D = 0$.

**Case 1:** We first assume that $D$ is not a square. This case follows directly from the [DY, Theorem 5.1].

**Case 2:** Next we assume that $D > 0$ is a square. We first work on $X_{\Gamma,\Delta}$ as $\Xi_{\ell,\Delta}(n,\delta,v)$ is defined over $X_{\Gamma,\Delta}$. Let $P_\ell$ be the cusp associated to an isotropic line $\ell \in \text{Iso}(V)$. We have, near the cusp $P_\ell$, by [DY, Theorem 5.1] and (4.3)

\[(4.14) \quad \Xi_{\Delta,r}(n,\mu,v)(z) = -g_{\Delta,r}(n,\mu,v,P_\ell) \log |q_\ell|^2 - 2\psi_{\Delta}(n,\mu,v;q_\ell),\]
where \( \psi_\Delta(n, \mu, v; q_\ell) \) is a smooth function of \( q_\ell \) (as two real variables \( q_\ell \) and \( \bar{q}_\ell \)) and
\[
\lim_{q_\ell \to 0} \psi_\Delta(n, \mu, v; q_\ell) = 0.
\]

Here
\[
g_{\Delta, r}(n, \mu, v, P_\ell) = \frac{1}{8\pi \sqrt{-nv}} \beta_{3/2}(-4\pi nv) \alpha_{\Delta, r}(n, \mu, P_\ell),
\]
\[
\alpha_{\Delta, r}(n, \mu, P_\ell) = \sum_{w \in \Gamma_0(N) \setminus L_{r\mu}[n\Delta]} \chi_\Delta(w) \delta_{w, \ell}
\]
and \( 0 \leq \delta_{w, \ell} \leq 2 \) is the number of isotropic lines \( \ell_w \in \text{Iso}(V) \) which is perpendicular to \( w \) and belongs to the same cusp as \( \ell \). Now all terms in (4.14) can be descended to modular curve \( X_0(N) \).

Since \( N \) is square-free, the Atkin-Lehner involutions act on the cusps of \( X_0(N) \) transitively. It’s known that \( \chi_\Delta \) is invariant under the Atkin-Lehner involutions, so
\[
\alpha_{\Delta, r}(n, \mu, P_\ell) = \sum_{w \in \Gamma_0(N) \setminus L_{r\mu}[n\Delta]} \chi_\Delta(w) \delta_{w, \ell}
= \sum_{w \in \Gamma_0(N) \setminus L_{r\sigma_\ell \mu}[n\Delta]} \chi_\Delta(w) \delta_{w, \ell}\infty
= \alpha_{\Delta, r}(n, \sigma_\ell \mu, P_\ell\infty).
\]

Here \( \sigma_\ell \in \text{SL}_2(\mathbb{Z}) \) is an Atkin-Lehner operator such that \( \sigma_\ell \ell = \ell_{\infty} \) and \( L_{r\sigma_\ell \mu} = L + r\sigma_\ell \mu \). Therefore, it suffices to show \( \alpha_{\Delta, r}(n, \mu, P_{\ell\infty}) = 0 \) for any \( \mu \).

Recall that \( \delta_w = \delta_{w, \ell_{\infty}} \) is the number of isotropic lines \( \ell \in \text{Iso}(V) \) which is perpendicular to \( w \) and belongs to the cusp \( P_{\infty} \). For \( w \in L_{r\mu}[n\Delta], \delta_w = 0 \) unless \( w \in L_{r\mu, \ell_{\infty}[n\Delta]} \) or \( w \in L_{-r\mu, \ell_{\infty}[n\Delta]} \).

When \( 2r\mu \notin L, \delta_w = 1 \) (see [DY, Lemma 6.2]). From the Lemma 4.3, one has
\[
(4.15) \quad \alpha_{\Delta, r}(n, \mu, P_{\ell\infty}) = \sum_{w \in \Gamma_{\ell\infty} \setminus L_{r\mu, \ell_{\infty}[n\Delta]} \chi_\Delta(w) + \sum_{w \in \Gamma_{\infty} \setminus L_{-r\mu, \ell_{\infty}[n\Delta]} \chi_\Delta(w).
\]

According to Lemma 4.4, one has
\[
(4.16) \quad \alpha_{\Delta, r}(n, \mu, P_{\ell\infty}) = 0.
\]
When $2r\mu \in L$, from the proof of Lemma 4.3

$$\alpha_{\Delta, r}(n, \mu, P_{\ell\infty}) = \sum_{\delta_w=2, w \in \Gamma_{\ell\infty} \setminus L_{r\mu, \ell\infty} [n\Delta]} 2\chi_{\Delta}(w) + \sum_{\delta_w=1, w \in \Gamma_{\ell\infty} \setminus L_{r\mu, \ell\infty} [n\Delta]} \chi_{\Delta}(w) + \sum_{\delta_w=1, -w \in \Gamma_{\ell\infty} \setminus L_{r\mu, \ell\infty} [n\Delta]} \chi_{\Delta}(-w).$$

Since

$$\chi_{\Delta}(-w) = sgn(\Delta)\chi_{\Delta}(w) = \chi_{\Delta}(w),$$

one has

$$\alpha_{\Delta, r}(n, \mu, P_{\ell\infty}) = \sum_{w \in \Gamma_{\ell\infty} \setminus L_{r\mu, \ell\infty} [n\Delta]} 2\chi_{\Delta}(w). \quad (4.17)$$

So from equation (4.17) and Lemma 4.4, one obtains that

$$\alpha_{\Delta, r}(n, \mu, P_{\ell\infty}) = 0. \quad (4.18)$$

Then we complete the whole case $D > 0$.

**Case 3:** We finally assume $D = 0$. When $n = 0$, we just need to consider the case $r\mu = 0$. Notice that if $r\mu \neq 0$, $L_{r\mu}[0]$ is empty. From the [DY] Theorem 5.1, around cusp $P_{\ell}$

$$\Xi_{\Delta, r}(0, \mu, v) = -\sum_{\delta \in L/L^\Delta, \Delta \equiv 0(L), \ell \cap L^\Delta \neq \emptyset} \chi_{\Delta}(\delta) \left( \frac{\varepsilon_{\ell}}{2\pi \sqrt{vN}} \left( \log |q_{\ell}|^2 + 2 \log(-\log |q_{\ell}|^2) \right) \right)$$

$$-2 \sum_{\delta \in L/L^\Delta, \Delta \equiv 0(L), \ell \cap L^\Delta \neq \emptyset} \chi_{\Delta}(\delta) \psi_{\ell}(0, \delta, v; q_{\ell}). \quad (4.19)$$

Here $\varepsilon_{\ell}$ is the Funke constant of $\ell$ and $\psi_{\ell}(0, \delta, v; q_{\ell})$ is smooth functions of $q_{\ell}$, and

$$\lim_{q_{\ell} \to 0} \psi_{\ell}(0, \delta, v; q_{\ell}) = \begin{cases} a_{\ell} & \text{if } \delta \in L^\Delta \\ b_{\ell} & \text{if } \delta \notin L^\Delta \end{cases} \quad (4.20)$$

for some constant $a_{\ell}$ and $b_{\ell}$. When $\delta \in L^\Delta$, $\chi_{\Delta}(\delta) = 0$, so

$$\lim_{q_{\ell} \to 0} \sum_{\delta \in L/L^\Delta, \Delta \equiv 0(L), \ell \cap L^\Delta \neq \emptyset} \chi_{\Delta}(\delta) \psi_{\ell}(0, \delta, v; q_{\ell}) = \sum_{\delta \in L/L^\Delta, \Delta \equiv 0(L), \ell \cap L^\Delta \neq \emptyset} \chi_{\Delta}(\delta) b_{\ell}. \quad (4.21)$$
Combing it with equation (4.19), it suffices to consider
\[
\sum_{\delta \in L^\Delta/L^\Delta, \delta \equiv 0(L), \ell \cap L^\Delta \neq \phi} \chi(\delta).
\]

We assume that \(\ell \cap L = \mathbb{Z}\lambda_{\ell}\), where \(\lambda_{\ell}\) is the primitive element. So the representatives for all \(\delta \in L^\Delta/L^\Delta\) with \(\delta \equiv r\mu \equiv 0(L)\) such that \(\ell \cap L^\Delta \neq \phi\) are given by
\[
\{m\lambda_{\ell}, m = 0, 1, \ldots, \Delta - 1\}.
\]
One has
\[
\sum_{\delta \in L^\Delta/L^\Delta, \delta \equiv 0(L), \ell \cap L^\Delta \neq \phi} \chi(\delta) = \sum_{m=0}^{\Delta-1} \chi(m\lambda_{\ell}) = 0.
\]

From the equation (4.19) and (4.21), we know \(\Xi_{\Delta,r}(0, \mu, v)\) is smooth around all cusps \(P_{\ell}\) and goes to zero when \(q_{\ell} \to 0\). This finishes the proof of the theorem. \(\square\)

Let
\[
Z_{\Delta,r}(n, \mu) = \begin{cases} Z_{\Delta,r}(n, \mu) & \text{if } n > 0, \\ 0 & \text{if otherwise}. \end{cases}
\]

**Corollary 4.6.** Let notations and assumption be as in Theorem 4.5, then \(\Xi_{\Delta,r}(n, \mu, v)\) is a Green function for \(Z_{\Delta,r}(n, \mu)\) on \(X_0(N)\) in the usual Gillet-Soulé sense, i.e.,
\[
\text{dd}^c[\Xi_{\Delta,r}(n, \mu, v)] + \delta Z_{\Delta,r}(n, \mu) = [\omega_{\Delta,r}(n, \mu, v)].
\]

Bruinier and Ono constructed automorphic Green function for divisor \(Z_{\Delta,r}(f)\) in the work [BO], where \(f\) is the weight 1/2 harmonic weak Maass form. When take some special \(f\), one could get the Green function \(\Phi_{\Delta,r}(n, \mu)\) for divisor \(Z_{\Delta,r}(n, \mu)\). Similarly as recent work of Ehlen and Sankaran [ES], we could ask a question if
\[
\sum_{\mu} \sum_{n} \Phi_{\Delta,r}(n, \mu)q^n e_{\mu} - \sum_{\mu} \sum_{n} \Xi_{\Delta,r}(n, \mu, v)q^n e_{\mu}
\]
is a modular form. They didn’t prove case associated to modular curve in that paper.

5. **Twisted arithmetic theta function**

In this section, we assume that \(N\) is square free.

Following [KM], let \(\mathcal{X}_0(N) (\mathcal{X}_0'(N))\) be the moduli stack over \(\mathbb{Z}\) of cyclic isogenies of degree \(N\) of elliptic curves (generalized elliptic curves) \(\pi : E \to E'\), such that \(\ker \pi\) meets every irreducible component
of each geometric fiber. The stack $\mathcal{X}_0(N)$ is regular and proper flat over $\mathbb{Z}$ and $\mathcal{X}_0(N)(\mathbb{C}) = X_0(N)$. It is a DM-stack. It is regular over $\mathbb{Z}$ and smooth over $\mathbb{Z}[\frac{1}{N}]$.

When $p | N$, the special fiber $\mathcal{X}_0(N) \pmod{p}$ has two irreducible components $\mathcal{X}_p^\infty$ and $\mathcal{X}_p^0$. Let $\mathcal{X}_p^\infty(\mathcal{X}_p^0)$ be the component which contains the cusp $\mathcal{P}_\infty \pmod{p}(\mathcal{P}_0 \pmod{p})$. Here $\mathcal{P}_\infty$ and $\mathcal{P}_0$ are Zariski closure of cusp infinity and zero. Let $\mathcal{Z}_{\Delta,(n,\mu)}$ be the Zariski closure of $\mathcal{Z}_{\Delta,r}(n,\mu)$.

When $\Delta = 1$, $\mathcal{Z}_{\Delta,r}(n,\mu)$ has a moduli interpretation, one could check details in [DY, Section 6].

We define arithmetic divisor in $\widehat{\text{CH}}_R^1(\mathcal{X}_0(N))$ by

(5.1) \[ \widehat{\mathcal{Z}}_{\Delta,r}(n,\mu,v) = (\mathcal{Z}_{\Delta,r}(n,\mu), \Xi_{\Delta,r}(n,\mu,v)). \]

The twisted arithmetic theta function ($q = e(\tau)$) is defined to be

(5.2) \[ \widehat{\phi}_{\Delta,r}(\tau) = \sum_{n \equiv Q(\mu)(mod \, \mathbb{Z})} \widehat{\mathcal{Z}}_{\Delta,r}(n,\mu,v) q^n e_\mu \in \widehat{\text{CH}}_R^1(\mathcal{X}_0(N)) \otimes \mathbb{C}[L^1/L][[q,q^{-1}]]. \]

5.1. **Arithmetic intersection.** The Gillet-Soulé intersection theory (see [So]) has been extended to arithmetic divisors with log-log singularities and equivalently metrized bundles with log singularities ([BKK], [Kü1], [Kü2]). We will use Kühn’s settings in [Kü1] as in [DY] and we refer to [Kü1] for details.

Let $S=$\{cusps\} and define $\widehat{\text{Pic}}_R(\mathcal{X}_0(N), S)$ be the group of metrized line bundles with log-log singularity along $S$ with $\mathbb{R}$-coefficients.

For an arithmetic divisor $\widehat{\mathcal{Z}} = (\mathcal{Z}, g)$ with log-log-singularity along $S$, $g$ is a smooth function on $X_0(N) \setminus \{\mathcal{Z}(\mathbb{C}) \cup S\}$, and satisfying the following conditions:

\[ dd^c g + \delta_{\mathcal{Z}(\mathbb{C})} = [\omega], \]

\[ g(t_j) = -2\alpha_j \log(-\log(|t_j|^2)) - 2\beta_j \log |t_j| - 2\psi_j(t_j) \quad \text{near } S_j, \]

for some smooth function $\psi_j$ and some $(1,1)$-form $\omega$ which is smooth away from $S$. Here $t_j$ is local parameter.

We could view the metrized line bundle $\widehat{\mathcal{L}} = (\mathcal{L}, || ||)$ as an arithmetic divisor $\widehat{\mathcal{Z}} = (\text{div}(s), -\log ||s||^2)$ with canonical section $s$.

Around cusp $S_j$,

\[ ||s(t_j)|| = (-\log |t_j|^2)^{\alpha_j} |t_j|^{|\text{ord}_{S_j}(s)|} \varphi(t_j). \]

Then one obtains that

(5.3) \[ \alpha_j(g) = \alpha_j(s), \quad \beta_j(g) = \text{ord}_{S_j}(s), \quad \psi_j(t_j) = \log \varphi(t_j). \]
Define $\widehat{CH}_1^R(\mathcal{X}_0(N), S)$ be the quotient of the $\mathbb{R}$-linear combination of the arithmetic divisors of $\mathcal{X}$ with log-log growth along $S$ by $\mathbb{R}$-linear combinations of the principal arithmetic divisors with log-log growth along $S$. One has $\widehat{Pic}_R(\mathcal{X}_0(N), S) \cong \widehat{CH}_1^R(\mathcal{X}_0(N), S)$.

From [Kü1, Proposition 1.4], there is an extended height paring [DY, Proposition 4.1]

$$\widehat{CH}_1^R(\mathcal{X}_0(N), S) \times \widehat{CH}_1^R(\mathcal{X}_0(N), S) \to \mathbb{R},$$

such that if $Z_1$ and $Z_2$ are divisors intersect properly, then

$$\langle (Z_1, g_1), (Z_2, g_2) \rangle = (Z_1 \cdot Z_2)_{\text{fin}} + \frac{1}{2} g_1 \ast g_2.$$

The degree map is given by

$$(5.4) \quad \text{deg} : \widehat{CH}_1^R(\mathcal{X}_0(N), S) \to \mathbb{R}, \quad \text{deg}(Z, g) = \int_{\mathcal{X}} \omega = \langle (Z, g), (0, 2) \rangle.$$

It is $\text{deg} Z$ when $g$ is a Green function without log-log singularity.

Define $\widehat{CH}_1^R(\mathcal{X}_0(N)) := \widehat{CH}_1^R(\mathcal{X}_0(N), \text{empty})$, which is the usual arithmetic Gillet-Soulé Chow group.

5.2. The metrized Hodge bundle. Let $\omega_N$ be the Hodge bundle on $\mathcal{X}_0(N)$ (see [KM]). Then there is an isomorphism $\omega_N^2 \cong \Omega_{\mathcal{X}_0(N)/\mathbb{Z}}(-S)$, which is canonically isomorphic to the line bundle of modular forms of weight 2 for $\Gamma_0(N)$.

The normalized Petersson metric of line bundle of modular form $\mathcal{M}_k(\Gamma_0(N))$, which is given by

$$\|f(z)\| = |f(z)(4\pi e^{-C}y)^k|,$$

where $k$ is the weight, $C = \frac{\log 4\pi + \gamma}{2}$ and $\gamma$ is Euler constant. We denote this metric line bundle by $\hat{\mathcal{M}}_k(\Gamma_0(N))$. This metrized induces a metric on $\omega_N$, and we denote this metrized line bundle by $\hat{\omega}_N$. We will identify $\hat{\omega}_N^k \cong \hat{\mathcal{M}}_k(\Gamma_0(N))$, under which $\hat{\omega}_N$ becomes the class of arithmetic divisor

$$(5.5) \quad \frac{1}{k} \hat{\text{Div}}(\Delta_N) = \frac{1}{k}(\text{Div} \Delta_N, -\log \|\Delta_N(z)\|^2).$$

Here the modular form $\Delta_N(z)$ of weight $k = 12\varphi(N)$ is given by (3.4).

We need the following two lemmas in [DY Section 6].

Lemma 5.1.

$$(5.6) \quad \text{Div} \Delta_N = \frac{r k}{12} P_\infty - k \sum_{p|N} \frac{p}{p-1} \chi_p^0,$$
where
\[ r = N \prod_{p | N} (1 + p^{-1}) = [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]. \]

Here \( P_\infty \) is the Zariski closure of cusp \( \infty \) and \( X_0^0 \) is a vertical component of \( X_0(N) \).

**Lemma 5.2.** Let \( q_z = e(z) \) be a local parameter of \( X_0(N) \) at the cusp \( P_\infty \). The metrized line bundle \( \hat{\omega}^k = \hat{\mathcal{M}}_k(N) \) has log singularity along cusps with all \( \alpha \)-index \( \alpha_P = \frac{k}{2} \) at every cusp \( P \). At the cusp \( P_\infty \), one has
\[ \|\Delta_N(z)\| = (-\log |q_z|^2)^{\frac{k}{2}} |q_z|^k \varphi(q_z), \]
with
\[ \varphi(q_z) = e^{-\frac{kC}{2}} \prod_{n=1}^\infty |(1 - q_z)^{24C(n)}|. \]

Indeed, \( \hat{\text{Div}}(\Delta_N) = (\text{Div}(\Delta_N), -\log \|\Delta_N(z)\|^2) \) is an arithmetic divisor with log-log singularity at cusps.

**5.3. Main results.** Firstly, we prove the following proposition which is an analogue of [DY, Proposition 6.7].

**Proposition 5.3.** Let \( \Delta > 1 \), for every prime \( p | N \), one has
\[ \langle \hat{\varphi}_{\Delta,r}(\tau), X_p^0 \rangle = \langle \hat{\varphi}_{\Delta,r}(\tau), X_p^\infty \rangle = 0. \]

**Proof.** Since \( \chi_\Delta(-w) = \chi_\Delta(w) \), similarly to work [DY], we obtain
\[ w_N^*(\hat{\varphi}_{\Delta,r}(\tau)) = \hat{\varphi}_{\Delta,r}(\tau). \]
It is known that \( w_N^* X_p^0 = X_p^\infty \) with \( w_N = (0_N, -1) \). \( w_N \) is an isomorphism, then one has
\[ \langle \hat{\varphi}_{\Delta,r}(\tau), X_p^0 \rangle = \langle \hat{\varphi}_{\Delta,r}(\tau), X_p^\infty \rangle \]
\[ = \frac{1}{2} \langle \hat{\varphi}_{\Delta,r}(\tau), X_p \rangle \]
\[ = \frac{1}{2} \hat{\text{I}}_{\Delta,r}(\tau, 1) \log p = 0. \]

Notice that the principal arithmetic divisor \( \hat{\text{Div}}(p) = (X_p, -\log p^2) \), so \( X_p = (0, \log p^2) \) in arithmetic Chow group . \( \square \)

**Theorem 5.4.** Let the notations be as above, then
\[ \langle \hat{\varphi}_{\Delta,r}(\tau), \hat{\omega}_N \rangle = \frac{1}{\varphi(N)} \log(u_\Delta) h(\Delta) E_L(\tau, 1). \]

**Proof.** We denote
\[ (5.7) \quad \hat{\Delta}_N = \left( \frac{rk}{12} P_\infty, -\log \|\Delta_N(z)\|^2 \right), \]
so
\[
\hat{\text{Div}}(\Delta_N) = \hat{\Delta}_N - k \sum_{p \mid N} \frac{p}{p-1} x_p^0.
\]

From the Proposition 5.3, one has
\[
\langle \hat{\phi}_{\Delta,r}(\tau), \hat{\text{Div}}(\Delta_N) \rangle = \langle \hat{\phi}_{\Delta,r}(\tau), \hat{\Delta}_N \rangle.
\]

Now it’s only need to prove
\[
\langle \hat{\phi}_{\Delta,r}(\tau), \hat{\Delta}_N \rangle = 12 \log(u_{\Delta}) h(\Delta) E_L(\tau,1),
\]
which amounts to check their Fourier coefficients term by term.

By Theorem 1.5, it is suffice to prove
\[
\langle \hat{Z}_{\Delta,r}(n,\mu,v), \hat{\Delta}_N \rangle = -\int_{X_0(N)} \log \|\Delta_N(z)\| \omega_{\Delta,r}(n,\mu,v).
\]

From the intersection formula [Küll Proposition 1.4], we get
\[
\langle Z_{\Delta,r}(n,\mu,v), \hat{\Delta}_N \rangle = (Z_{\Delta,r}(n,\mu,v), \frac{r k}{12} P_{\infty})_{\text{fin}} + \frac{1}{2} g_1 * g_2.
\]

Around the cusp \(P_{\infty}\), from Theorem 4.5, we have
\[
\lim_{q \to 0} \Xi_{\Delta,r}(n,\mu,v)(z) = 0,
\]
then index \(\alpha_{1,j}\) and \(\psi_{1,j}(0)\) in equation (5.3) associated to \(\hat{Z}_{\Delta,r}(n,\mu,v)\) are equal to zero (see details in [DY Proposition 4.1]). The star product here is
\[
g_1 * g_2 = \Xi_{\Delta,r}(n,\mu,v)(z) * (-\log \|\Delta_N\|^2)
\]
\[
= -2 \int_{X_0(N)} \log \|\Delta_N\| \omega_{\Delta,r}(n,\mu,v).
\]

It’s easy to find that \(Z_{\Delta,r}(n,\mu,v)\) and \(P_{\infty}\) have no intersection. Then we obtain
\[
\langle \hat{Z}(n,\mu,v), \hat{\Delta}_N \rangle = -\int_{X_0(N)} \log \|\Delta_N\| \omega_{\Delta,r}(n,\mu,v).
\]
This finishes the proof.

\[\square\]

6. Modularity of the arithmetic theta function

In this section, we will prove the modularity of \(\hat{\phi}_{\Delta,r}(\tau)\). We will follow the methods in [KRY2 Chapter 4] and [DY Section 8]. To simplify the notation, we denote in this section \(X = X_0(N)\) and \(X = X_0(N)\), and let \(S\) be the set of cusps of \(X\). Let \(g_{GS}\) be a Gillet-Soulé Green function for the divisor \(\text{Div} \Delta_N\) (without log-log singularity), and
let $\hat{\Delta}_{GS} = (\text{Div} \Delta_N, g_{GS}) \in \widehat{CH}_1^1(\mathcal{X})$, and $f_N = g_{GS} + \log \|\Delta_N\|^2$. One has

$$\hat{\Delta}_{GS} = \widehat{\text{Div}}(\Delta_N) + a(f_N),$$

where $a(f_N) = (0, f_N) \in \widehat{CH}_1^1(\mathcal{X}, S)$.

Let $A(X)$ be the space of smooth functions $f$ on $X$ which are conjugation invariant ($\text{Frob}_\infty$-invariant), and let $A^0(X)$ be the subspace of functions $f \in A(X)$ with

$$\int_X f \mu_{GS} = 0,$$

where $\mu_{GS} = c_1(\hat{\Delta}_{GS})$.

For each $p|N$, let $Y_p = X_\infty - pX_p$, and $Y_p^\vee = \frac{1}{(Y_p, Y_p)} Y_p$. Finally let $\widetilde{MW}$ be the orthogonal complement of $\mathbb{R}\hat{\Delta}_{GS} + \sum_{p|N} \mathbb{R}Y_p^\vee + \mathbb{R}\kappa(1) + a(A^0(X))$ in $\widehat{CH}_1^1(\mathcal{X})$.

Recall the result [DY] Proposition 8.3 as follows. Let

$$\text{MW} = J_0(N) \otimes \mathbb{R},$$

where $J_0(N)$ is the jacobian, then there is an isomorphism

$$(6.1) \quad \widetilde{MW} \cong MW, \quad \widehat{Z} = (Z, g_Z) \mapsto Z,$$

where $Z$ is the generic fiber of $\mathcal{Z}$.

**Proposition 6.1.** ([KRY2] Propositions 4.1.2, 4.1.4, [DY] Proposition 8.2] One has

$$\widehat{CH}_1^1(\mathcal{X}) = \widetilde{MW} \oplus (\mathbb{R}\hat{\Delta}_{GS} + \sum_{p|N} \mathbb{R}Y_p^\vee + \mathbb{R}\kappa(1)) \oplus a(A^0(X)).$$

More precisely, every $\widehat{Z} = (Z, g_Z)$ decomposes into

$$\widehat{Z} = \widehat{Z}_{MW} + \frac{\deg \widehat{Z}}{\deg \hat{\Delta}_{GS}} \hat{\Delta}_{GS} + \sum_{p|N} \langle \widehat{Z}, Y_p \rangle Y_p^\vee + 2\kappa(\widehat{Z})a(1) + a(f_{\widehat{Z}})$$

for some $f_{\widehat{Z}} \in A^0(X)$, where

$$\kappa(\widehat{Z}) \deg \hat{\Delta}_{GS} = \langle \widehat{Z}, \hat{\Delta}_{GS} \rangle - \frac{\deg \widehat{Z}}{\deg \hat{\Delta}_{GS}} \langle \hat{\Delta}_{GS}, \hat{\Delta}_{GS} \rangle.$$

Finally, let $\Delta_z$ be the Laplacian operator with respect to $\mu_{GS}$. Then the space $A^0(X)$ has an orthonormal basis $\{f_j\}$ with

$$\Delta_z f_j + \lambda_j f_j = 0, \quad \langle f_i, f_j \rangle = \delta_{ij}, \quad \text{and} \quad 0 < \lambda_1 < \lambda_2 < \cdots,$$
where the inner product is given by
\[ \langle f, g \rangle = \int_{X_0(N)} f \bar{g} \mu_{GS}. \]
In particular, every \( f \in A^0(X) \) has the decomposition
\[ f(z) = \sum \langle f, f_j \rangle f_j. \] (6.2)
Recall ([KRY2, (4.1.36)]) that
\[ d_z d_c z f = \Delta_z(f) \mu_{GS}. \] (6.3)

Now we could prove the following modularity result.

**Theorem 6.2.** Let the notation be as above. Then
\[ \hat{\phi}_{\Delta,r}(\tau) = \tilde{\phi}_{MW}(\tau) + a(\phi_{SM}) + \phi_1(\tau) a(1) \]
where \( \tilde{\phi}_{MW}(\tau) \) is a modular form of \( \Gamma' \) of weight \( 3/2 \) and representation \( \rho_L \) valued in finite dimension vector space \( MW \otimes \mathbb{C}[L^z/L] \), \( \phi_{SM} \) and \( \phi_1(\tau) \) are modular forms of \( \Gamma' \) of weight \( 3/2 \) and representation \( \rho_L \) valued in \( A^0(X_0(N)) \otimes \mathbb{C}[L^z/L] \).

**Proof.** Since \( \deg(Z_{\Delta,r}(n,\mu)) = 0 \), one knows that the generic part \( A_{\Delta,r}(\tau) \in J_0(N) \otimes \mathbb{C}[L^z/L][[q]]. \) Here
\[ A_{\Delta,r}(\tau) = \sum_{n>0,\mu} Z_{\Delta,r}(n,\mu) q^n e_{\mu} \] (6.4)
is modular by Bruinier and Ono’s result [BO, Section 6]. Let \( \tilde{\phi}_{MW} \) be the image of \( A_{\Delta,r}(\tau) \) under the isomorphism (6.1), and it’s also modular.

From decomposition result Proposition 6.1 one has
\[ \hat{\phi}_{\Delta,r}(\tau) = \tilde{\phi}_{MW} + a(\phi_{SM}) + \phi_1(\tau) a(1), \] (6.5)
where \( \tilde{\phi}_{MW} = (A_{\Delta,r}(\tau), g_{MW}) \) and \( g_{MW} \) is the harmonic Green function and \( \phi_1(\tau) = \frac{2}{\deg(\Delta_{GS})} (\hat{\phi}_{\Delta,r}, \hat{\Delta}_{GS}) \) which is modular from the later Lemma 6.3. Then one has
\[ \phi_{SM} = \Xi_{\Delta,r}(\tau, z) - g_{MW} - \phi_1(\tau), \]
where \( \Xi_{\Delta,r}(\tau, z) = \sum_\mu \sum_n \Xi_{\Delta,r}(n, \mu, v) q^n e_{\mu}. \) In the above decomposition, the part in \( \mathbb{R} \hat{\Delta}_{GS} + \sum_{p|N} \mathbb{R} \gamma_p \) is zero.

Finally, one has by (6.2),
\[ \phi_{SM}(\tau, z) = \sum_j \langle \phi_{SM}, f_j \rangle f_j. \] (6.6)
Simple calculation gives
\[
\langle \phi_{SM}, f_j \rangle = -1 \lambda_j \int_{X_0(N)} \phi_{SM}(\tau, z) \Delta_z(f_j) \mu_{GS}
\]
\[
= -1 \lambda_j \int_{X_0(N)} \phi_{SM}(\tau, z) d_z \bar{d}_z f_j
\]
\[
= -1 \lambda_j \int_{X_0(N)} d_z \bar{d}_z \phi_{SM}(\tau, z) f_j.
\]
So
\[
-\lambda_j \langle \phi_{SM}, f_j \rangle = \int_{X_0(N)} d_z \bar{d}_z (\Xi_{\Delta,r}(\tau, z) - g_{MW}) f_j = \int_{X_0(N)} \Theta_{\Delta,r}(\tau, z) f_j
\]
is modular, since \( \Theta_{\Delta,r}(\tau, z) \) is modular. From the spectral decomposition (6.6), \( \phi_{SM} \) is modular.

□

From the proof of above theorem, we have the following result.

**Lemma 6.3.** \( \langle \hat{\phi}_{\Delta,r}, \hat{\Delta}_{GS} \rangle \) is a vector valued modular form of \( \Gamma' \) valued in \( \mathbb{C}[L^2/L] \) of weight 3/2 and representation \( \rho_L \).

**Proof.** From the intersection formula [Kü1, Proposition 1.4], one has
\[
\langle \hat{\phi}_{\Delta,r}, \hat{\Delta}_{GS} \rangle = \frac{1}{2} \int_X g_{GS} d\bar{d} \Xi_{\Delta,r}(\tau, z) = \frac{1}{2} \int_X g_{GS} d\bar{d} (\phi_{SM} + g_{MW})
\]
Since \( \phi_{SM} \) and \( g_{MW} \) are modular from above theorem, we know \( \langle \hat{\phi}_{\Delta,r}, \hat{\Delta}_{GS} \rangle \) is modular.

□

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