RELATIVISTIC CORRECTIONS TO THE SUNYAEV-ZELDOVICH EFFECT

ANTHONY CHALLINOR 1 AND ANTHONY LASENBY 2

Mullard Radio Astronomy Observatory, Cavendish Laboratory, Madingley Road, Cambridge, CB3 0HE, United Kingdom

Received 1997 August 11; accepted 1998 January 5

ABSTRACT

We present an extension of the Kompaneets equation which allows relativistic effects to be included to any desired order. Using this, we are able to obtain simple analytic forms for the spectral changes due to the Sunyaev-Zeldovich effect in hot clusters, correct to first and second order in the expansion parameter $\theta_e = k_B T_e / m c^2$. These analytic forms agree with previous numerical calculations of the effect based upon the multiple scattering formalism and are expected to be very accurate over all regions of the cosmic microwave background spectrum for $k_B T_e$ up to $\sim 10$ keV. Our results confirm previous conclusions that the result of including relativistic corrections in the Sunyaev-Zeldovich effect is a small reduction in the amplitude of the effect over the majority of the spectrum: specifically, we find $\Delta T / T = - 2 y (1 - 17/100 \theta_e + 123/400 \theta_e^2)$ (correct to second order) in the Rayleigh-Jeans region, where $y$ is the usual Comptonization parameter. For a typical cluster temperature of 8 keV, this amounts to a correction downward to the value of the Hubble constant derived using combined X-ray and Rayleigh-Jeans Sunyaev-Zeldovich information by about 5%.

Subject headings: cosmic microwave background — galaxies: clusters: general — radiative transfer — relativity — scattering

1. INTRODUCTION

Nonrelativistic treatments of the Sunyaev-Zeldovich effect usually employ the Kompaneets equation (Kompaneets 1957) to determine the distortion of the cosmic microwave background (CMB) spectrum. The Kompaneets equation does not, however, include relativistic effects, which may be important for hot clusters in which $k_B T_e \gtrsim 10$ keV. For this reason, and because of the low optical depth of typical clusters, relativistic treatments of the Sunyaev-Zeldovich effect usually employ a multiple scattering description of the Comptonization process (Wright 1979; Fabbri 1981; Taylor & Wright 1989; Loeb, McKee, & Lahav 1991; Rephaeli 1995). Including relativistic effects in this procedure gives a complicated expression for the spectral distortion, which is best handled by numerical techniques (see, for example, Rephaeli 1995).

In this paper, we show how the Kompaneets equation may be extended to include relativistic effects in a self-consistent manner, allowing the Sunyaev-Zeldovich effect in hot clusters to be described on the basis of a Kompaneets-type equation. The extension of the Kompaneets equation can be carried out to arbitrary orders in relativistic effects, although we shall only consider the lowest order corrections here. The resulting equation conserves photons at every order, and we demonstrate that it is consistent with previous numerical calculations of the effect based upon the multiple scattering formalism and are expected to be very accurate over all regions of the cosmic microwave background spectrum for $k_B T_e$ up to $\sim 10$ keV. Our results confirm previous conclusions that the result of including relativistic corrections in the Sunyaev-Zeldovich effect is a small reduction in the amplitude of the effect over the majority of the spectrum: specifically, we find $\Delta T / T = - 2 y (1 - 17/100 \theta_e + 123/400 \theta_e^2)$ (correct to second order) in the Rayleigh-Jeans region, where $y$ is the usual Comptonization parameter. For a typical cluster temperature of 8 keV, this amounts to a correction downward to the value of the Hubble constant derived using combined X-ray and Rayleigh-Jeans Sunyaev-Zeldovich information by about 5%.

We employ natural units, $c = \hbar = 1$, except in the discussion of the Sunyaev-Zeldovich effect in § 4.

2. EXTENDING THE KOMPANEETS EQUATION

In this paper we shall not consider effects due to the peculiar motion of the cluster (such effects give rise to a kinematic correction to the Sunyaev-Zeldovich effect). For a comoving cluster, the CMB photon distribution function is isotropic and may be denoted $n(\omega)$, where $\omega$ is the photon energy. The electrons are assumed to be in thermal equilibrium at temperature $T_e$ and are described by an isotropic distribution function, $f(E)$, where $E$ is the photon energy. The Boltzmann equation describing the evolution of $n(\omega)$ may be written as (Buchler & Yeh 1976)

$$\frac{\partial n(\omega)}{\partial t} = -2 \int \frac{d^3 p}{(2\pi)^3} d^3 p' \sum_{\omega'} W(\omega) [n(\omega')[1 + n(\omega')]] f(E) \times$$

$$- n(\omega') [1 + n(\omega')] f(E') \, , \quad (1)$$

where $W$ is the invariant transition amplitude for Compton scattering of a photon of 4-momentum, $k'$, by an electron (of charge $e$ and mass $m$) with 4-momentum, $p''$, to a photon momentum, $k''$, and an electron momentum, $p''$.

1 A.D.Challinor@mrao.cam.ac.uk.
2 A.N.Lazenby@mrao.cam.ac.uk.
Substituting this form for \( f(E) \) into equation (1) and expanding the term in brackets in the integrand in powers of \( \Delta x \), where

\[
\Delta x \equiv \frac{\omega'}{k_B T_e} - \frac{\omega}{k_B T_e},
\]


gives a Fokker-Planck expansion:

\[
\frac{\partial n(x)}{\partial t} = 2 \left[ \frac{\partial n}{\partial x} + n(1 + n) \right] I_1
+ 2 \left[ \frac{\partial^2 n}{\partial x^2} + 2(1 + n) \frac{\partial n}{\partial x} + n(1 + n) \right] I_2
+ 2 \left[ \frac{\partial^3 n}{\partial x^3} + 3(1 + n) \frac{\partial^2 n}{\partial x^2} + 3(1 + n) \frac{\partial n}{\partial x} + n(1 + n) \right] I_3
+ 2 \left[ \frac{\partial^4 n}{\partial x^4} + 4(1 + n) \frac{\partial^3 n}{\partial x^3} + 6(1 + n) \frac{\partial^2 n}{\partial x^2} \right]
+ 4(1 + n) \frac{\partial n}{\partial x} + n(1 + n) \right] I_4 + \cdots ,
\]

where

\[
I_n = \frac{1}{n!} \int \frac{d^3 p}{(2\pi)^3} d^3 p' d^3 k' W(f(E)(\Delta x))^n ,
\]

which does not depend on \( n(\omega) \).

The calculation of the \( I_n \) may be performed by expanding the integrand in powers of \( p/m \) and \( \omega/m \). The factor \( f(E) \) is handled by the expansion

\[
f(E) \approx e^{(u - m)k_B T_e} e^{-\left[ 1 + \frac{1}{2} \theta_e u^2 + \frac{1}{4} \theta_e^2 u^4 (u - 4) + \cdots \right]} ,
\]

and the chemical potential, \( \mu \), may be eliminated by introducing the electron number density, \( N_e \), which evaluates to

\[
N_e = \frac{m \sqrt{2 \pi}}{\pi^2} \Gamma \left( \frac{3}{2} \right) k_B T_e^{3/2} e^{(\mu - m)k_B T_e}
\]

\[
\times \left[ 1 + \frac{15}{8} \theta_e + \frac{105}{128} \theta_e^2 + O(\theta_e^3) \right] .
\]

Note that equation (12) is an asymptotic expansion about \( \theta_e = 0 \) of the result

\[
N_e = \frac{m^3}{\pi^3} \theta_e K_2 \left( \frac{1}{\theta_e} \right) e^{m/k_B T_e} ,
\]

where \( K_2(x) \) is a modified Bessel function, which suggests that our series expansions of the \( I_n \) will only be asymptotic series. The calculations of the \( I_n \) are ideally suited to symbolic computer algebra packages (we use Maple). Expressing the results in terms of \( \theta_e \), \( x \), and the Thomson cross section, \( \sigma_T \), we find

\[
I_1 = \frac{1}{2} \sigma_T N_e \theta_e x [(4 - x) + \theta_e (10 - \frac{472}{635} x + \frac{710}{276} x^2)] + O(\theta_e^2)
\]

\[
I_2 = \frac{1}{2} \sigma_T N_e \theta_e x^2 [1 + \theta_e (\frac{472}{635} - \frac{649}{276} x + \frac{710}{276} x^2)] + O(\theta_e^3)
\]

\[
I_3 = \frac{1}{2} \sigma_T N_e \theta_e x^3 [\frac{7}{8} (6 - x) \theta_e] + O(\theta_e^4)
\]

\[
I_4 = \frac{1}{2} \sigma_T N_e \theta_e x^4 (7 \theta_e) + O(\theta_e^5) .
\]

For \( n > 4 \), \( I_n \) is third-order or higher in \( \theta_e \). For CMB photons passing through a cluster at redshift \( z \), we have \( x \simeq 6.2 \times 10^{-4} (1 + z) / k_B T_e \), where \( x \) is the average of \( x \) over a Planck distribution, and \( k_B T_e \) is expressed in electron volts. The electron temperature is typically \( \lesssim 10 \) keV, so that \( x \ll 1 \).

Substituting these expansions for the \( I_n \) into the series given in equation (7), we develop an expansion of \( \partial n/\partial t \) in \( \theta_e \). In this paper, we shall only be concerned with the lowest order relativistic corrections and start by retaining all terms up to \( O(\theta_e^3) \). To include all such terms consistently, it is necessary to retain only the first four terms in the series equation (7). A lengthy calculation gives the result

\[
\frac{\partial n(x)}{\partial t} = \sigma_T N_e \theta_e \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^2 j(x) \right] ,
\]

where the current \( j(x) \) is given by

\[
j(x) = x^2 \left[ \frac{\partial n}{\partial x} + n(1 + n) \right] + \theta_e \left[ \frac{5}{2} \frac{\partial n}{\partial x} + n(1 + n) \right]
+ \frac{21}{5} x \frac{\partial}{\partial x} \left[ \frac{\partial n}{\partial x} + n(1 + n) \right] + \frac{7}{10} x^2 \left[ \frac{\partial^2 n}{\partial x^2} + 2 \frac{\partial n}{\partial x} \right]
\times (1 + 2n) + \frac{\partial n}{\partial x} \left( 1 - 2 \frac{\partial n}{\partial x} \right) \right] + O(\theta_e^3) .
\]
equation (Kompaneets 1957), although it would seem natural to retain only terms in $I_1$ and $I_2$, would be incorrect, since it would not obey photon conservation. To be specific, if only $I_1$ and $I_2$ are retained, then photon conservation requires that the current be given by

$$j(x) = 2 \frac{I_2}{\sigma_T n_e \theta_e} \left[ \frac{\partial n}{\partial x} + n(1 + n) \right],$$

and consistency with the expansion of the Boltzmann equation (eq. [7]) requires that

$$I_1 = \frac{\partial I_2}{\partial x} + 2 \frac{I_2}{x} - I_2 .$$

Calculating $I_2$ to $O(\theta_e^2)$ in the limit of small $x$ gives the current $j(x)$ (in the limit of small $x$), where

$$j(x) = x^3 \left[ \frac{\partial n}{\partial x} + n(1 + n) \right] \left( 1 + \frac{47}{2} \theta_e \right),$$

which amounts to a relativistic correction to the cross section appearing in the usual Kompaneets equation. This result is incorrect, since the approach is not self-consistent to the order that the answer is quoted. To see this, one need only calculate $I_1$, using equation (18) and the expression for $I_2$ from equation (14). The expression for $I_1$ obtained by this procedure only agrees with the direct calculation (eq. [14]) to $O(\theta_e^2)$, not to $O(\theta_e^4)$. One obtains a better approximation to the true current (eq. [16]) by calculating $I_1$ directly and then using equation (18) to calculate an effective value of $I_2$, which may then be used to deduce the current. This procedure reproduces the first term in the $O(\theta_e)$ correction in $j(x)$, in the limit of small $x$, and is similar to that employed by Fabbri (1981) in obtaining his equation (11). However, it is only the full expression equation (16) that is consistent with the original Boltzmann equation (eq. [1]) to $O(\theta_e^2)$.

### 3. RATE OF ENERGY TRANSFER

As a check on the consistency of equation (16) with existing results in the literature, we calculate the energy transfer rate between the electrons and a Planckian distribution of photons at temperature $T_e$.

Multiplying the continuity equation (eq. [15]) by $x^3$ and integrating, we find

$$\frac{\partial}{\partial t} \int_0^\infty x^3 n(x) dx = - \sigma_T n_e \theta_e \int_0^\infty x^2 j(x) dx .$$

The left-hand side of equation (20) is proportional to the rate at which the photons are gaining energy per unit volume, denoted $dE_\omega/dt$. Substituting a Planckian distribution for $n(\omega)$ in equation (16) and integrating gives the result

$$\frac{dE_\omega}{dt} = 4E_\omega \sigma_T n_e \theta_e \left[ 1 + \frac{5}{2} \theta_e - 21 \frac{\zeta(6)}{\zeta(4)} \theta_e + O(\theta_e^2) \right],$$

where

$$\theta_e \equiv \frac{k_B T_e}{m},$$

and $\zeta(x)$ is the Riemann zeta function.

This result may be compared with a direct evaluation of the energy transfer, obtained by multiplying the transition rate $W$ by the energy transfer $\omega' - \omega$ and integrating over all collisions:

$$\frac{dE_\omega}{dt} = 4 \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} \times d^3 p d^3 p' W(\omega)[1 + n(\omega')] (\omega' - \omega) .$$

The integral may be evaluated by a consistent expansion of the integrand. A lengthy calculation gives the result in equation (21), which also agrees with the noncovariant calculation of Woodward (1970) (who gives higher order corrections also). Note that if we had considered only $I_1$ and $I_2$, then using $j(x)$ given by equation (19), we would have obtained (in the limit of small $\theta_e$)

$$\frac{dE_\omega}{dt} = 4E_\omega \sigma_T n_e \theta_e - 21 \frac{\zeta(6)}{\zeta(4)} \theta_e + O(\theta_e^2) ,$$

which grossly overestimates the importance of the relativistic corrections.

### 4. THE SUNYAEV-ZELDOVICH EFFECT

In this section we apply the generalized Kompaneets equation (to first-order in relativistic corrections) to the calculation of the Sunyaev-Zeldovich effect in optically thin clusters. We consider higher order effects in the next section.

Following the standard assumptions, we assume that the optical depth is sufficiently small that the spectral distortions are small. In this limit, we may solve equation (15) iteratively. The lowest order solution is obtained by substituting the initial photon distribution, $n_0(x)$, into the current (eq. [16]). The integral over time is then trivial and may be replaced by an integral along the line of sight through the cluster, giving

$$\Delta n(x) = \frac{y}{x^3} \frac{\partial}{\partial x} \left[ x^2 j(x) \right],$$

where $j(x)$ is evaluated with $n_0(x)$, and

$$y \equiv \sigma_T \int N_e \theta_e \, dl ,$$

where the integral is taken along the line of sight through the cluster.

For the CMB, we take the initial (undistorted) photon distribution to be Planckian with temperature $T_0$:

$$n_0(x) = \frac{1}{e^{xT} - 1} ,$$

where $x \equiv T_e/T_0$ is the (large) ratio of electron temperature to the CMB temperature. Evaluating equation (25) in the limit of large $x$, we find the following fractional distortion:

$$\frac{\Delta n(X)}{n(X)} = \frac{y X e^x}{e^x - 1} \left( X \coth \left( \frac{1}{2} X \right) \right) + \frac{7}{10} X^3 \coth^3 \left( \frac{1}{2} X \right) \times \left[ X \coth \left( \frac{1}{2} X \right) - 3 \right] ,$$

(28)
correct to first-order in relativistic effects, where

\[ X \equiv \frac{\hbar \omega}{k_B T_0}. \]  

(29)

The first two terms in large parentheses in equation (28) give the usual nonrelativistic Sunyaev-Zeldovich expression, while the terms proportional to \( \theta_e \) are the lowest order relativistic correction. Equation (28) agrees with the result in Stebbins (1998). In the Rayleigh-Jeans limit (small \( X \)), we find

\[ \frac{\Delta n(X)}{n(X)} \approx -2y \left( 1 - \frac{17}{10} \theta_e + O(\theta_e^2) \right). \]  

(30)

In Figure 1 we plot the change in spectral intensity, \( \theta_e \Delta I/y \), as a function of \( X \), where

\[ \Delta I = \frac{X^3}{e^x - 1} \frac{\Delta n}{n}. \]  

(31)

Also plotted in Figure 1 are the nonrelativistic predictions made with the standard Kompaneets equation. The curves in Figure 1 are for \( k_B T_0 = 5, 10, \) and \( 15 \) keV, which are the same as the parameters used by Rephaeli (1995) in his Figure 1. His calculations, which were based on the multiple scattering formalism (Wright 1979) and required a numerical analysis, give results in excellent agreement with ours, which only require the use of the simple expression given as equation (28). This suggests that there is no problem in principle with applying the Boltzmann equation to the problem of Comptonization in clusters, even though the optical depth may be very small. Similar conclusions were reached by Fabbrri (1981), but his demonstration was restricted to low-temperature clusters in which relativistic effects are not important.

It is clear from Figure 1 that for \( X \lesssim 8 \), the relativistic corrections lead to a reduction in the magnitude of the intensity change, compared to the nonrelativistic prediction. This in turn leads to a reduction in the inferred value of the Hubble constant determined by the Sunyaev-Zeldovich route.

In Figure 2 we plot the fractional change in the Rayleigh-Jeans brightness temperature, \( \Delta T_{RJ}/T_0 \) (divided by \( y/\theta_e \)), for the same \( \theta_e \) as in Figure 1. The relativistic corrections to the change in the Rayleigh-Jeans brightness temperature are significant even at low frequency, unlike the corrections to the intensity, where relativistic corrections are small in the Rayleigh-Jeans part of the spectrum.

5. HIGHER-ORDER EFFECTS

We have found that for \( k_B T_e \gtrsim 10 \) keV, the second-order relativistic effects make a significant contribution to the spectral distortion, while third-order effects are only significant for \( k_B T_e \gtrsim 15 \) keV.

These calculations require a straightforward extension of the method of §2 to include terms at \( O(\theta^2_e) \) (for second-order relativistic effects). For the calculation to \( O(\theta^2_e) \), it is necessary to retain the first six terms of equation (7) and to calculate \( I_1 \) through \( I_6 \) to \( O(\theta^2_e) \). The first iteration of equation (15) for \( T_c \gg T_0 \) gives the following next order (in \( \theta_e \)) correction to \( \Delta n/n \):

\[
\frac{\Delta n(X)}{n(X)} \left|_{2} \right. = \theta^2_e \frac{y X e^x}{e^x - 1} \left\{ -\frac{15}{2} + \frac{1023}{8} X \coth \left( \frac{1}{2} X \right) \\
- \frac{868}{5} X^2 \coth^2 \left( \frac{1}{2} X \right) + \frac{329}{5} X^3 \coth^3 \left( \frac{1}{2} X \right) \\
- \frac{44}{5} X^4 \coth^4 \left( \frac{1}{2} X \right) + \frac{11}{30} X^5 \coth^5 \left( \frac{1}{2} X \right) \\
+ \frac{X^2}{30 \sinh^2 \left( 1/2X \right)} \left[ -2604 + 3948X \coth \left( \frac{1}{2} X \right) \\
- 1452X^2 \coth^2 \left( \frac{1}{2} X \right) + 143X^3 \coth^3 \left( \frac{1}{2} X \right) \\
+ \frac{X^4}{60 \sinh^4 \left( 1/2X \right)} \left[ -528 + 187X \coth \left( \frac{1}{2} X \right) \right] \right\}. \]  

(33)
In the Rayleigh-Jeans limit, we find
\[
\frac{\Delta n(x)}{n(x)} \approx -2y \left[ 1 - \frac{17}{10} \theta_e + \frac{123}{40} \theta_e^2 + O(\theta_e^3) \right].
\] (34)

In Figure 3 we compare the spectrum of \(\Delta I\) calculated with equation (28) to the spectrum with the correction (eq. [33]) included, for \(k_B T_e = 5, 10,\) and 15 keV (\(\theta_e \approx 0.01, 0.02,\) and 0.03, respectively). In each case, the second-order relativistic effects are not significant in the Rayleigh-Jeans part of the spectrum. This is to be expected from inspection of equation (34), where the \(\theta_e^2\) term is clearly insignificant for the values of \(\theta_e\) considered. For \(k_B T_e = 5\) keV, the second-order effects are insignificant over the entire spectrum. However, for \(k_B T_e \gtrsim 10\) keV, the second-order effects make a significant contribution to the relativistic correction to the Kompaneets-based prediction outside the Rayleigh-Jeans region. We have verified that the third-order corrections are negligible over the entire spectrum for \(k_B T_e \approx 10\) keV. This is confirmed by a comparison of the curves in Figure 3 with the points that are the results of a direct Monte Carlo evaluation of the Boltzmann collision integral with \(n(\alpha)\) given by the Planck distribution, \(n_0(x)\) (Gull & Garrett 1998). The second-order effects should be included in the analysis of high-frequency data for hot clusters. The magnitude of the second-order correction to the Sunyaev-Zeldovich result for the rather mild values of \(\theta_e\), considered here is symptomatic of the asymptotic nature of the series expansion of \(\partial n/\partial \tau\) in \(\theta_e\). However, for the majority of clusters considered in Sunyaev-Zeldovich analyses, the inclusion of the first two relativistic corrections should be sufficient, particularly for experiments working in the Rayleigh-Jeans region of the spectrum.

5.1. The Crossover Frequency

The accurate determination of the crossover frequency, \(X_0\) (where the thermal component of the spectral distortion vanishes), is essential for reliable subtraction of the kinematic contribution to the Sunyaev-Zeldovich effect (Rephaeli 1995). In Figure 4 we plot the crossover frequency as a function of \(k_B T_e\). For \(k_B T_e \lesssim 20\) keV, we find that \(X_0\) is well approximated by the linear relation
\[
X_0 \approx 3.83(1 + \theta_e)\) (35)

For comparison, Rephaeli (1995) found \(X_0\) to be approximated by \(X_0 \approx 3.83(1 + \theta_e)\) in the interval \(k_B T_e = 1-50\) keV, while Fabbri (1981) found \(X_0 \approx 3.83(1 + 1.1\theta_e)\) for \(k_B T_e \leq 150\) keV. It is clear that our calculation favors Fabbri's expression. For \(k_B T_e \gtrsim 20\) keV, \(X_0\) calculated with the first three relativistic corrections departs from the linear prediction (eq. [35]). However, we do not regard this as indicative of a breakdown of the linear approximation, since it is clear from Figure 3 that the inclusion of higher order terms may have a significant effect on the value of the crossover frequency.

6. CONCLUSION

We have shown how the Kompaneets equation may be generalized to include relativistic effects in a self-consistent manner. The resulting equation guarantees photon conservation at each order and is in agreement with direct calculations of the energy transfer rate between the plasma and a Planckian distribution of photons.

We have applied this formalism to the calculation of the Sunyaev-Zeldovich effect in optically thin clusters. We presented simple analytic expressions for the first two relativistic corrections to the usual Kompaneets-based expression for the spectral distortion, which are in excellent agreement with the numerical calculations of Rephaeli (1995) for electron temperatures \(\lesssim 10\) keV. This provides further evidence that the low optical depth of clusters does not forbid the application of the Boltzmann equation to the calculation of the Sunyaev-Zeldovich effect (Fabbri 1981). The asymptotic nature of the expansion of \(\partial n/\partial \tau\) in \(\theta_e\) requires the inclusion of higher order corrections to calculate the effect in hotter clusters in the Wien region of the spectrum. While the calculation of higher order corrections is not problematic, the bad convergence properties of the series means that ultimately one must resort to a numerical
calculation of the collision integral (Corman 1970) (or employ the multiple scattering formalism [Wright 1979]) to calculate the effect in very hot clusters.

Our calculations support the conclusions reached in Rephaeli & Yankovitch (1997): including relativistic effects leads to a small decrease in the value of the Hubble constant, inferred from combined X-ray and Sunyaev-Zeldovich information. For a cluster temperature of \( T \approx 8 \) keV, the reduction in \( H_0 \) due to relativistic effects is \( \approx 5\% \) for measurements made in the Rayleigh-Jeans region.

We would like to express our gratitude to N. Itoh, Y. Kohyama, and S. Nozawa, whose preprint (Itoh, Kohyama, & Nozawa 1998) stimulated this research and to S. Gull and A. Garrett for allowing us to quote the (unpublished) results of a Monte Carlo simulation.

REFERENCES

Berestetskii, V. B., Lifshitz, E. M., & Pitaevskii, L. P. 1982, Quantum Electrodynamics: Landau and Lifshitz Course of Theoretical Physics (2d ed; Oxford: Pergamon Press)

Buchler, J. R., & Yesh, W. R. 1976, ApJ, 210, 440

Corman, E. G. 1970, Phys. Rev. D, 1, 2734

Fabbri, R. 1981, ApSS, 77, 529

Gull, S. F., & Garrett, A. 1998, in preparation

Itoh, N., Kohyama, Y., & Nozawa, S. 1998, ApJ, submitted

Kompaneets, A. S. 1957, Soviet Phys. JETP, 4, 730

Lasenby, A. N., & Jones, M. E. 1997, in The Extragalactic Distance Scale, ed. M. Livio, M. Donahue, & N. Panagia (Cambridge: Cambridge Univ. Press), 76

Loeb, A., McKee, C. F., & Lahav, O. 1991, ApJ, 374, 44

Rephaeli, Y. 1995, ApJ, 445, 33

Rephaeli, Y., & Yankovitch, D. 1997, ApJ, 481, L55

Saunders, R. 1996, in Microwave Background Anisotropies: Proc. 16th Moriond Meeting, ed. F. R. Bouchet (Gif-sur-Yvette: Editions Frontières), 377

Stebbins, A. 1998, ApJ, submitted

Taylor, G. B., & Wright, E. L. 1989, ApJ, 339, 619

Woodward, P. 1970, Phys. Rev. D, 1, 2731

Wright, E. L. 1979, ApJ, 232, 348