On quasi Steinberg characters of complex reflection groups

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Outline

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- Quasi $p$-Steinberg characters of symmetric groups
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- Murnaghan–Nakayama rule for the groups $G(r, 1, n)$
- Complex reflection groups and their representation theory
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**Notation:** $p$ is a prime; $G$ denotes a finite group.

**Definition ($p$-regular element of a group $G$)**

An element whose order is not divisible by $p$.

Let $p$ divide the order of $G$.

**Definition ($p$-Steinberg character)**

An irreducible character $\chi$ of $G$ such that $\chi(x) = \pm |C_G(x)|_p$, for every $p$-regular element $x$ in $G$, where $C_G(x)$ denotes the centralizer of $x$ in $G$.

**Definition (quasi $p$-Steinberg character)**

An irreducible character $\chi$ of $G$ is called a quasi $p$-Steinberg character if $\chi(g) \neq 0$ for all $p$-regular elements $g$ in $G$. 
Example

- All irreducible characters of a $p$-group are quasi $p$-Steinberg characters.
- Linear characters of a finite group are quasi $p$-Steinberg characters.

So, we concentrate on nonlinear characters when talking about quasi $p$-Steinberg characters.
Theorem (Paul and Singla, 2021)

For $n \geq 3$, let $\lambda$ be a partition of $n$ such that $\lambda \neq (n), (1^n)$ and $p$ be a prime. All triplets $(n, \lambda, p)$ such that $\chi_{\lambda}$ is a quasi $p$-Steinberg character of $S_n$ are given in Table below.

| $n$ | $\lambda$ | $p$ |
|-----|------------|-----|
| 3   | $(2, 1)$   | 2   |
| 4   | $(2, 2)$   | 2   |
| 4   | $(3, 1), (2, 1, 1)$ | 3   |
| 5   | $(4, 1), (2, 1, 1, 1)$ | 2   |
| 5   | $(3, 2), (2, 2, 1)$ | 5   |
| 6   | $(3, 2, 1)$ | 2   |
| 6   | $(4, 2), (2, 2, 1, 1)$ | 3   |
| 8   | $(5, 2, 1), (3, 2, 1, 1, 1)$ | 2   |
Murnaghan–Nakayama rule for \( S_n \)

**Definition (Skew hook/Rim hook/Ribbon)**

A skew diagram that is edgewise connected and contains no \( 2 \times 2 \) subset of boxes.

The height of a ribbon is equal to one less than the number of rows in the ribbon.

**Example**

![Example Diagram]

Its height is 3.
**Definition**

A ribbon tableau is a generalized tableau $T$ with positive integral entries such that the entries in the rows and columns of $T$ weakly increase, and all occurrences of a given entry lie in a single ribbon. The height of ribbon tableau $T$, denoted by $ht(T)$, is the sum of heights of all of its ribbons.

**Theorem (Murnaghan–Nakayama rule)**

For a partition $\lambda$ of $n$ and $\sigma \in S_n$, the character $\chi^\lambda$ is given by

$$
\chi^\lambda(\sigma) = \sum_{T} (-1)^{ht(T)},
$$

where the sum is over all ribbon tableaux $T$ of shape $\lambda$ and content given by the lengths of the cycles in $\sigma$. 
Example

Let $\lambda = (4, 2)$ and $\sigma = (1, 2, 3)(4, 5)$. Then, the ribbon tableaux are

\[
\begin{array}{cccc}
1 & 1 & 1 & 3 \\
2 & 2 & & \\
\end{array}, \quad 
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
1 & & 3 & \end{array}
\]

So, $\chi^\lambda(\sigma) = 0$. 
Representation theory of $G(r, 1, n)$

$$G(r, 1, n) : = \mathbb{Z}_r^n \rtimes S_n$$
$$= \{ (z_1, z_2, \ldots, z_n, \sigma) \mid z_i \in \mathbb{Z}_r \text{ for all } 1 \leq i \leq n, \sigma \in S_n \}.$$

Various ways to study representation theory of $G(r, 1, n)$

- Theory of symmetric functions (Specht’s Thesis/Macdonald’s book on “Symmetric functions and Hall polynomials”)
- Wigner–Mackey method of little groups
- The Okounkov–Vershik approach [Mishra and Srinivasan, 2016]

Theorem

The irreducible representations of $G(r, 1, n)$ are parametrized by $r$-partite partitions of $n$. 

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Murnaghan–Nakayama rule for $G(r,1,n)$

It was first proved by Stembridge in 1989. The version we state here is by Adin, Postnikov and Roichman in 2010.

Sequence of ribbons

A sequence of ribbons $b = (b_1, b_2, \ldots, b_t)$ corresponding to an $r$-partite Young diagram $\lambda = (\lambda_0, \ldots, \lambda_{r-1})$ is obtained from a sequence of $r$-partite Young diagrams

$$\emptyset = \lambda^{(0)} \subseteq \ldots \subseteq \lambda^{(t)} = \lambda$$

by defining $b_i := \lambda^{(i)} \setminus \lambda^{(i-1)}$ for $1 \leq i \leq t$ such that each $b_i$ has $r - 1$ empty components and the nonempty component is a ribbon.
An \emph{\(r\)-partite ribbon tableau} \(T\) of shape \(\lambda\) is obtained by filling the boxes in the nonempty component of the ribbon \(b_i\) with entry \(i\) for each \(1 \leq i \leq t\).

\begin{itemize}
  \item \textbf{\(i\)-th index, \(i\)-th length and \(i\)-th height of \(T\)}
  \begin{align*}
    f_T(i) &:= \text{index in } \lambda^{(i)} \text{ of the nonempty component in the } r\text{-tuple } b_i; \\
    l_T(i) &:= \text{number of boxes in the nonempty component in } b_i; \\
    ht_T(i) &:= \text{one less than the number of rows in the nonempty component in } b_i.
  \end{align*}
\end{itemize}
Murnaghan–Nakayama rule for $G(r, 1, n)$

\[
\chi^\lambda(\pi) = \sum_{T \in RT_c(\lambda)} \prod_{i=1}^{t} (-1)^{ht_T(i)} \omega^{f_T(i) \cdot z(c_i)}
\]

where

\[
\pi = (z_1, z_2, \ldots, z_n, \sigma),
\]

cycle decomposition of $\sigma$ is given by $c = (c_1, c_2, \ldots, c_t)$,

for $1 \leq i \leq t$, $l(c_i) = \text{length of the cycle } c_i$, $z(c_i) = \text{color of the cycle } c_i$,

$\omega$ is a primitive $r$-th root of unity,

$RT_c(\lambda)$ is the set of $r$-partite ribbon tableaux $T$ of shape $\lambda$ such that $l_T(i) = l(c_i)$ for all $1 \leq i \leq t$. 
Complex reflection groups $G(r, q, n)$

**Definition**

For a positive integer $q$ which divides $r$, we define a subgroup $G(r, q, n)$ of $G(r, 1, n)$ as follows:

$$G(r, q, n) := \{(z_1, z_2, \ldots, z_n, \sigma) \in G(r, 1, n) \mid \sum_{i=1}^{n} z_i \equiv 0 \pmod{q} \}.$$

By Shephard–Todd classification, the family $G(r, q, n)$ is the only infinite family of finite irreducible complex reflection groups.
**Special subfamilies in the family** $G(r, q, n)$

- Cyclic group of order $r$, $\mathbb{Z}/r\mathbb{Z} = G(r, 1, 1)$;
- Dihedral group of order $2r$, $D_{2r} = G(r, r, 2)$;
- Symmetric group $S_n = G(1, 1, n)$;
- Weyl group of type $B_n$ is $G(2, 1, n)$;
- Weyl group of type $D_n$ is $G(2, 2, n)$.

**Notation:** $m = \frac{r}{q}$

**Representation theory of** $G(r, q, n)$

The irreducible $G(r, q, n)$-modules are parametrized by the ordered pairs $(\tilde{\lambda}, \delta)$, where $\tilde{\lambda}$ is an $(m, q)$-necklace with total $n$ boxes and $\delta \in C_{\lambda}$, the stabilizer subgroup for the necklace $\tilde{\lambda}$. 
Main Results
Theorem (M., Paul and Singla)

Given a partition $\lambda$ of $n$, define $\hat{\lambda}^j = (\lambda_0, \lambda_1, \ldots, \lambda_j, \ldots, \lambda_{r-1})$, where $\lambda_j = \lambda$ for some $0 \leq j \leq r - 1$, and $\lambda_k = \emptyset$ for $k \neq j$. Then, $\chi^{\hat{\lambda}^j}$ is a quasi $p$-Steinberg character of $G(r, 1, n)$ if and only if $\chi^{\lambda}$ is a quasi $p$-Steinberg character of $S_n$.

Proof of the easier part

Assuming $\chi^{\hat{\lambda}^j}$ to be a quasi $p$-Steinberg character of $G(r, 1, n)$, it follows that $\chi^{\lambda}$ is a quasi $p$-Steinberg character of $S_n$ by the following identity:

$$\chi^{\lambda}(\sigma) = \chi^{\hat{\lambda}^j}((0, \ldots, 0, \sigma)).$$
Theorem (M., Paul and Singla)

For an $r$-partite partition $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{r-1})$ of $n$, the irreducible character $\chi^\lambda$ is a quasi $p$-Steinberg character of $G(r, 1, n)$ in exactly the following cases:

**General case**

(i) $\lambda_j \vdash n$ for some $j$ and $\lambda_k = \emptyset$ for all $j \neq i$, and

(ii) $\chi^\lambda_j$ is a quasi $p$-Steinberg character of $S_n$.

**Additional cases for $n < 5$:**

(a) For $n = 2$, the character $\chi^\lambda$ is a quasi $2$-Steinberg character when $\lambda_j = (1)$ for some $j$, $\lambda_k = (1)$ for some $k \neq j$, and $\lambda_l = \emptyset$ for all $l \notin \{j, k\}$.

(b) For $n = 3$, the character $\chi^\lambda$ is a quasi $3$-Steinberg character when $\lambda_j \vdash 2$ for some $j$, $\lambda_k = (1)$ for some $k \neq j$ and $\lambda_l = \emptyset$ for all $l \notin \{j, k\}$.

(c) For $n = 4$, the character $\chi^\lambda$ is a quasi $2$-Steinberg character when $\lambda_j \vdash 3$ for some $j$, $\lambda_k = (1)$ for $k \neq j$, and $\lambda_l = \emptyset$ for all $l \notin \{j, k\}$.
Sketch of the proof

$p \nmid n$: Consider the element $\alpha = (0, \ldots, 0, (1, 2, \ldots, n))$ whose type is $((n), \emptyset, \ldots, \emptyset)$. Now $\chi^\lambda(\alpha) \neq 0$ implies that $\lambda_j \vdash n$ for some $j$, $\lambda_k = \emptyset$ for $k \neq j$, and

$$\chi^\lambda(\alpha) = \chi^{\lambda_j}((1, 2, \ldots, n)).$$

Thus, $\chi^{\lambda_j}$ is a quasi $p$-Steinberg character of $S_n$.

Why are there additional cases for $n < 5$?

$p \mid n$: $p \nmid n - 1$. One of the subcases is $\lambda_j \vdash n - 1$ for some $j$, $\lambda_k = (1)$ for some $k \neq j$ and $\lambda_l = \emptyset$ for all $l \notin \{j, k\}$.

When $n \geq 5$, we have the following observations:

- Either $\alpha_2 = (0, \ldots, 0, (1, 2, \ldots, n - 2)(n - 1, n))$ or $\alpha_3 = (0, \ldots, 0, (1, 2, \ldots, n - 3)(n - 2, n - 1, n))$ is $p$-regular;

- Also, $\chi^\lambda(\alpha_2) = \chi^\lambda(\alpha_3) = 0$. 
Notation: \((\chi^\lambda)^*\) denotes an irreducible character of \(G(r, q, n)\) which appears in \(\text{Res}^{G(r,1,n)}_{G(r,q,n)} \chi^\lambda\). Note that \((\chi^\lambda)^*\) may not be unique.

Theorem (M., Paul and Singla)

The irreducible character \((\chi^\lambda)^*\) is a quasi \(p\)-Steinberg character of \(G(r, q, n)\) in exactly the following cases:

General case

\(\chi^\lambda\) is a quasi \(p\)-Steinberg character of \(G(r, 1, n)\). In this case,

\[(\chi^\lambda)^* = \text{Res}^{G(r,1,n)}_{G(r,q,n)} \chi^\lambda.\]
Additional cases:

(a) For $n = 3$, $p = 2$, the three two-dimensional characters $(\chi^\lambda)^*$ in $\text{Res}^{G(r,1,3)}_{G(r,q,3)} \chi^\lambda$ are quasi $2$-Steinberg characters. This case arises if and only if $r$ and $q$ are multiples of $3$, and $k = j + \frac{r}{3}$, $l = j + \frac{2r}{3}$ for $0 \leq j \leq \frac{r}{3} - 1$.

(b) For $n = 4$, $p = 3$, the two three-dimensional characters $(\chi^\lambda)^*$ in $\text{Res}^{G(r,1,4)}_{G(r,q,4)} \chi^\lambda$ are quasi $3$-Steinberg characters. This case arises if and only if $r$ and $q$ are both even, and $k = j + \frac{r}{2}$ for $0 \leq j \leq \frac{r}{2} - 1$. 
Sketch of the proof

Case 1: $p 
mid n$.

Subcase (1a): $p 
mid n - 1$.

The element $\alpha_1 = (0, \ldots, 0, (1, 2, \ldots, n - 1))$ is a $p$-regular element of $G(r, q, n)$. So, $(\chi^\lambda)^*(\alpha_1) \neq 0$. This implies that $\chi^\lambda(\alpha_1) \neq 0$. Then, $\lambda$ can be of one of the two forms:

(i) either $\lambda_j \nmid n$ for some $j$, $\lambda_k = \emptyset$ for all $k \neq j$:

$$(\chi^\lambda)^* = \text{Res}_{G(r,q,n)}^{G(r,1,n)} \chi^\lambda.$$  

Also, $\lambda = \hat{\lambda}_j$. $\chi^\hat{\lambda}_j$ is a quasi $p$-Steinberg character of $G(r, 1, n)$,

or
(ii) $\lambda_j \vdash n - 1$ for some $j$, $\lambda_k = (1)$ for some $k \neq j$, $\lambda_l = \emptyset$ for all $l \notin \{j, k\}$: the corresponding irreducible character $(\chi^\lambda)^*$ is not a quasi $p$-Steinberg character of $G(r, q, n)$ because of the following observations when $n \geq 3$:

- $(\chi^\lambda)^* = \text{Res}_{G(r, q, n)}^{G(r, 1, n)} \chi^\lambda$;
- The element $\alpha = (0, \ldots, 0, (1, 2, \ldots, n))$ is $p$-regular;
- $\chi^\lambda(\alpha) = 0$.

For $n = 2$, $(\chi^\lambda)^*$ is not a quasi $p$-Steinberg character if $\chi^\lambda$ does not decompose as a representation of $G(r, q, n)$.
Subcase (1b): $p | n − 1$. Then, $p | n − 2$ and

$\alpha_2 = (0, \ldots, 0, (1, 2, \ldots, n − 2))$ is $p$-regular. Then, $\chi^\lambda(\alpha_2) \neq 0$. Then one of the following is true:

(i) $\lambda_j \vdash n$ for some $j$, $\lambda_k = \emptyset$ for all $k \neq j$;

(ii) $\lambda_j \vdash n − 1$ for some $j$, $\lambda_k = (1)$ for some $k \neq j$, $\lambda_l = \emptyset$ for all $l \notin \{j, k\}$;

(iii) $\lambda_j \vdash n − 2$ for some $j$, $\lambda_k \vdash 2$ for some $k \neq j$, $\lambda_l = \emptyset$ for all $l \notin \{j, k\}$;

(iv) $\lambda_j \vdash n − 2$ for some $j$, $\lambda_k = (1)$ for some $k \neq j$, $\lambda_l = (1)$ for some $l \notin \{j, k\}$, and $\lambda_u = \emptyset$ for all $u \notin \{j, k, l\}$;
For \( n \geq 5 \), when \( \lambda \) is of one of the forms (ii)-(iv), \( (\chi^\lambda)^* \) is not a quasi \( p \)-Steinberg character of \( G(r, q, n) \). And, if it is of form (i), then \( \hat{\chi}^j \) is a quasi \( p \)-Steinberg character of \( G(r, 1, n) \).

Here, \( n \neq 2 \) as \( p \mid n - 1 \).

What happens when \( n = 3, p = 2 \) or \( n = 4, p = 3 \)?

\( n = 3, p = 2 \). The only important form is \( \lambda_j = (1) \) for some \( j \), \( \lambda_k = (1) \) for some \( k \neq j \), \( \lambda_l = (1) \) for some \( l \notin \{j, k\} \), and \( \lambda_u = \emptyset \) for all \( u \notin \{j, k, l\} \).

Also, \( \chi^\lambda \) decomposes into three two-dimensional irreducible characters of \( G(r, q, n) \) if and only if \( r \) and \( q \) are multiples of 3, and \( k = j + \frac{r}{3}, l = j + 2\frac{r}{3} \) for \( 0 \leq j \leq \frac{r}{3} - 1 \). And, in such a case, all these three two-dimensional irreducible characters of \( G(r, q, n) \) are quasi 2-Steinberg characters.

Case 2: \( p \mid n \) is studied using similar types of arguments.
Thank you