Abstract

We have previously found a new phase of cold nuclear matter based on a holographic gauge theory, where baryons are introduced as instanton gas in the probe D8/D8 branes. In our model, we could obtain the equation of state (EOS) of our nuclear matter by introducing fermi momentum. Then, here we apply this model to the neutron star and study its mass and radius by solving the Tolman-Oppenheimer-Volkoff (TOV) equations in terms of the EOS given here. We give some comments for our holographic model from a viewpoint of the other field theoretical approaches.
1 Introduction

In a holographic approach, the baryon is introduced as a soliton of vector mesons constructed in the probe $N_f$ flavor branes [1, 2, 3, 4]. The configuration of this soliton, which has a unit baryon number, is given by a finite-sized BPST instanton solution of the $SU(N_f)$ ($N_f \geq 2$) YM theory [3] in D8 branes. Here the size of the instanton is not arbitrary and determined to be as a finite size since the solution is embedded in the non-flat 4D partial space of the probe brane [2]. In this context, several approaches to study the nuclear system have been performed [6, 7, 8, 9, 10, 11, 12, 13].

Previously we have studied nuclear matter according to this holographic approach [14]. In our approach, the flavored Yang-Mills fields in the action of D8 branes are retained up to the quadratic term of the field strength. Its higher order terms are neglected here in the context of our dilute gas approximation of the instantons. The configuration of the flavored YM fields is obtained by supposing the dilute instanton gas form and by determining the size parameter of the instantons from the action principle [14]. After that, the chemical potential $\mu$ and the charge density $\bar{n}$ for the baryon are obtained in terms of the solution of the $U(1)$ gauge field, which is dual to the baryon number current operator. In this process, another important point is to find the embedding solution for the profile of D8/D8 brane. As for this profile, we use an antipodal solution [2], which is originally obtained in the case without any instantons. However, we could show that our antipodal solution, which is shown in [14], is also useful for any instanton (baryon) density $n_{\perp}$ [14].

In this setting, we could find an interesting phenomenon, a phase transition of the nuclear matter, which has been shown by $\bar{n}$-$\mu$ relation. Namely, $\bar{n}$ jumps from zero to a finite value $\bar{n}_c$ at a non-zero $\mu = \mu_c$. This corresponds to a transition from the vacuum to a non-trivial nuclear matter phase. It would be important to study the properties of this new phase of the nuclear matter and to find a possible candidate for this matter.

Our purpose is to exploit our analysis for $\bar{n}$-$\mu$ relation in order to identify the nuclear matter given here with a star. Since the baryon considered here is neutral, we could suppose such a star as a neutron star. It is an interesting problem to see that this proposal is reasonable or not by solving the TOV equations [15]. In solving the TOV equations however, we must know the relation of the energy density $\rho$ and pressure $p$ of the nuclear matter, namely EOS. In general, $\rho$ and $p$ are holographically obtained by using the bulk metric according to holographic renormalization [16, 17]. In the present case, however, the bulk metric is independent of the instanton gas since the flavor branes are treated as the probe.

An approach to solve the TOV equations has been performed by using a holographic EOS obtained in a context of holographic framework [18], however it leads to a large

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1 The charge density $\bar{n}$ is defined through the time component of $U(1)$ gauge field. While it is proportional to the baryon number $n$, we use $n$ in our present analysis.
radius star. Then, we propose an alternative way to get the relation of $\rho$ and $p$, EOS. We know the Luttinger theorem for interacting fermion system \[19\]. We apply this theorem to our nuclear matter. Then we acquire the other formula satisfied by the usual fermion system. One exception is the point that the mass of the fermion in our model is introduced as a function of the baryon number density. This density dependence of the fermion mass implies a non-trivial interaction among the fermion system given here, and this density dependent mass is determined from our holographic model. After that, we obtain a EOS of our holographic model, then the TOV equations\[15\] are solved.

The outline of this paper is as follows. In the next section, the instanton gas model is set up and how to obtain the EOS is explained. In the section 3, the holographic EOS is given from our analysis, then it is used in the next section to solve the TOV equations. We give some speculation on our result, which give a rather small mass and radius for the neutron star, and an estimation based on the two phase components is given. In the final section, summary and discussions are given.

2 Instanton gas and EOS

2.1 Instanton gas

Here baryon is introduced as an instanton soliton found in the $N_f$ D8 branes, which are embedded as probe in the D4 stacked bulk \[2\] in the type IIA string theory. We set as $N_f = 2$, for simplicity, then the $U(2) = U(1) \times SU(2)$ gauge fields in the D8 brane action are given as follows

$$A_b = A_b(z) \delta_b^0,$$

$$Q_i = \rho_i \epsilon_{ijk} x^k,$$

$$Q_i = \rho_i \tau^i,$$

where $A_b ((f_1)_{ij}, (f_1)_{iz})$ denotes the $U(1)$ gauge field ($SU(2)$ gauge field strength). Further, $\rho_i (a^m)$ denotes the instanton size (position), $\epsilon_{123} = 1$, $i, j = 1, 2, 3$ and $m = 1, \ldots, 4$, where $x^4 = z$. Then, $Q$ is extended into the following multi-instanton form,

$$Q = \sum_{i=1}^{N_I} Q_i,$$

$$Q_i = \frac{2\rho_i^2}{(x^m - a_i^m)^2 + \rho_i^2},$$

where $N_I$ denotes the instanton number. By neglecting the higher order terms of $Q$ and the overlapping of instantons, the $U(1)$ gauge field and the instanton size are determined as a solution of the brane action with the Chern-Simons term. The chemical potential of the baryon is obtained by using the result for $E_z = \partial_z A_0$ as \[14\]

$$\mu = \mu_c + \int_0^\infty dz E_z,$$
where $\mu$ and $\mu_c$ are defined as

$$
\mu = A_0(\infty), \quad \mu_c = A_0(0).
$$

(2.6)

After that, we can obtain the relation between the instanton density, $n = N_I/V_3$ where $V_3$ is the three dimensional volume, and the chemical potential $\mu$ of the baryon. From this result, we can derive the equation of the nuclear matter given here. The details of our model are given in [14].

2.2 Density dependent mass and EOS

As mentioned in the introduction, we apply the Luttinger theorem for interacting fermion system [19] to our nuclear matter. Then we may write the baryon number density $n$ as

$$
n = \frac{g S_2}{3(2\pi)^3} \frac{k_F^3}{k_F} = \frac{8\pi}{3(2\pi)^3} k_F^3
$$

(2.7)

by introducing the fermi momentum $k_F$ for our baryon. $g (= 2)$ is a number of spin degrees of freedom of the baryon, $S_2$ is a volume of two-dimensional surface of 3-sphere. In order to exploit this relation and our result $\mu = \mu(n)$, we introduce an effective baryon mass $m(n)$, which depends on the density $n$. This effective mass is interpreted as a reflection of the complicated interaction among the nuclear system given here. A similar concept is seen in the ordinary field theory approach [20]. Then it is possible to express $\mu$ as

$$
\mu = \mu(n) = \sqrt{k_F^2 + m^2(n)}.
$$

(2.8)

In this context, by extending the formula for the free fermion gas, we get the energy density $\rho$ and pressure $p$ of the nuclear matter are obtained as,

$$
\rho = \rho_c + \frac{8\pi}{3(2\pi)^3} \int_{k_c}^{k_F} dk \frac{k^2}{\sqrt{k^2 + m^2(n)}},
$$

(2.9)

$$
p = p_c + \frac{8\pi}{3(2\pi)^3} \int_{k_c}^{k_F} dk \frac{k^4}{\sqrt{k^2 + m^2(n)}},
$$

(2.10)

where $\rho_c$ and $p_c$ denote the critical density and pressure respectively. Here notice that the lower bound of the momentum, $k_0$, is introduced as

$$
n_c = \frac{8\pi}{3(2\pi)^3} k_c^3
$$

(2.11)

since the nuclear matter considered here is defined for $n \geq n_c$. This is our proposal to obtain the energy momentum tenser of the nuclear matter based on the holographic approach. Then we obtain EOS, the relation between $\rho$ and $p$, at any density to solve the TOV.
3 Numerical Data of $\mu(n)$ and $m(n)$ and EOS

Now, we reconstruct the information of density dependence of an effective nucleon mass $m(n)$ in nuclear matter phase from our ($\mu, n$) data\(^2\) by using the eqns. (2.7) and (2.8).

From Fig.2 of the previous paper [14], we get the empirical relation

$$\mu^2 \sim an^\alpha$$  \hspace{1cm} (3.1)

with $a = 1500, \alpha = 1.0$ in natural unit (mass unit is set to GeV). This is shown in Fig.1 left. As $a \gg 1$, we can estimate $\mu^2 \gg k_F^2$ and $\mu(n) \sim m(n)$ from the eqns. (2.7) and (2.8) which is shown in Fig.1 right. We obtain $m(n)$ and see the relation

$$m^2(n) \sim \mu^2(n) \sim an^\alpha = a\left(\frac{8\pi}{3(2\pi)^3}k_F^3\right)^\alpha$$  \hspace{1cm} (3.2)

The value of the fermi momentum and the mass $m_0$ at phase transition point are

$$k_c \equiv \left(\frac{3(2\pi)^3}{8\pi}n_c\right)^{1/3} \sim 0.55, \quad m_0^2 \equiv m^2(n_c) \sim 4.99.$$  

Next, we calculate the fermi momentum dependence of the energy density and the pressure from eqns. (2.9) and (2.10).

As $k \leq k_F << m(n)$, the energy function $\sqrt{k^2 + m^2(n)}$ can be approximated as

$$\sqrt{k^2 + m^2(n)} \sim m(n)$$  \hspace{1cm} (3.3)

in the integral.

\(^2\)Data points are limited in the range of $\langle r \rangle >$ nucleon size where $\langle r \rangle$ is the separation among nucleons. The dilute gas approximation is available.
Then we can calculate the integrals in eqns. (2.9) and (2.10),

\[ \rho (k_F) = \rho_c + b_1 \left( k_F^{3\alpha/2+3} - k_c^3 k_F^{3\alpha/2} \right), \quad (3.4) \]

\[ p (k_F) = p_c + b_2 \left( k_F^{5-3\alpha/2} - k_c^5 k_F^{-3\alpha/2} \right), \quad (3.5) \]

where \( \rho_c \equiv \rho (k_c) \), \( p_c \equiv p (k_c) \) and

\[ b_1 = \sqrt{\frac{a \left( \frac{1}{3\pi^2} \right)^{\alpha}}{3\pi^2}}, \quad b_2 = \frac{1}{15\pi^2 \sqrt{a \left( \frac{1}{3\pi^2} \right)^{\alpha}}}. \]

From eqns. (3.4) and (3.5), we obtain the relation as

\[ \rho (p) = \rho_c + b_1 \left( \frac{1}{b_2} (p - p_c) + k_c^5 k_F^{-3\alpha/2} \right)^{3\alpha/2+3} - k_c^3 k_F^{3\alpha/2} \right]. \quad (3.6) \]

Parameters \( a(= 1500) \) and \( \alpha(= 1.0) \) lead to \( b_1 = 0.239 \) and \( b_2 = 0.952 \times 10^{-3} \). Fig 2 shows the relationship between \( \rho \) and \( p \).

As \( 1/b_2 \ll 1 \), this equation is approximated by the following EOS

\[ p - p_c = \frac{b_2}{b_1} (\rho - \rho_c)^\gamma. \quad (3.7) \]

The index \( \gamma (\approx \frac{5-3\alpha/2}{3\alpha/2+3} = 0.78 < 1) \) is abnormal. This is in contrast to normal stars(\( \gamma = 4/3 \)) or the ideal fermion gas(\( \gamma = 5/3 \)). The origin of \( \alpha = 1 \) is the effective nucleon mass \( m(n) \). As the nucleon density increases, the effective nucleon mass and nucleon size increase in our holographic model [14].
Fig. 2: The relationship between $\rho$ and $p$. It is approximated as $p - p_c = \text{const.} \times (\rho - \rho_c)\gamma$ with $\gamma = 0.78$.

4 Application to Neutron Star and solution of TOV Equations

Since we have EOS as eq. (3.7), we can reduce unknown function in TOV equations [15] and can solve them. The TOV equations are

$$\frac{dp(r)}{dr} = \frac{G(m(r) + 4\pi r^3 p(r)/c^2)}{c^2 r^2} \left( \frac{c^2 \rho(r) + p(r)}{1 - 2Gm(r)/c^2 r} \right),$$

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r),$$

(4.1)

where $G$ is the Newton’s constant, $p(r)$ and $m(r)$ are the pressure and the mass respectively. We introduce new parameters $r_0, M_0, p_0$ as follows,

$$\frac{c^2}{G} = \frac{M_0}{r_0},$$

$r_0 = 1 \text{ km},$

$M_0 = 1.35 \times 10^{30} \text{ kg} (= 0.68 \times M_\odot (\text{solarmass})),$

$p_0 = \frac{M_0}{r_0^3} c^2 (= 1.22 \times 10^{38} \text{ N/m}^2).$

The TOV equations can expressed by the dimensionless values $x = r/r_0, Y(x) = m/M_0, Z(x) = p/p_0$,

$$\frac{dZ(x)}{dx} x^2 (1 - 2Y(x)/x) = (Y(x) + 4\pi x^2 Z(x)) \left( \frac{dY(x)}{dx} / 4\pi x^2 + Z(x) \right).$$

(4.2)
When we combine the EOS and the TOV equations, we can estimate the radius and mass of neutron star depending on parameters of $\rho_c$ and $p_c$.

(1) the case of $p_c = 0$.

This case is interesting and neutron star is composed of the nuclear matter in new phase. Calculated radius and mass depend on the parameter $\rho_c$ and are shown in Fig. 3.

If we can select $\rho_c \approx 0.0002$ GeV$^4$, we obtain the neutron star of $R \approx 20$ km and $M \approx M_s$.

When $\rho_c > 0.0005$ GeV$^4$, the radius and mass are quite small.

As $\rho_c \to 0$, $R, M \to \infty$ and the neutron star becomes unstable[18].

the meaning of $\rho_c$

We have found that the resultant mass and radius of neutron star depend on the critical energy density $\rho_c$. What is the natural value of $\rho_c$? A naive expectation is that $\rho_c \approx m_Bn_{cr}$. In this case, $\rho_c$ is approximately given as 0.005 GeV$^4$ and then both the mass and radius get quite small.

The following consideration may be useful to understand our results. Suppose that a neutron star is composed by $A$ neutrons, where mass and mutual distance are given as $m_B$ and $d_0$. Then the neutron star mass $M_{NS}$ and radius $R_{NS}$ will be

$$M_{NS} \approx m_B A, \quad R_{NS} \approx d_0 A^{1/3}. \quad (4.3)$$

If we supposed that the neutron star is the star which has the smallest radius not to be the black hole, the radius should agree with the Schwarzschild radius $R = 2GM$. By plugging (4.3) into this condition, one obtains the following results:

$$M_{NS} \approx \left( \frac{1}{2G} \right)^{3/2} m_B^{-1/2} d_0^{3/2}, \quad R_{NS} \approx \left( \frac{1}{2G} \right)^{1/2} m_B^{-1/2} d_0^{3/2}. \quad (4.4)$$
Note that we can obtain the moderate values of the neutron star mass $M_{NS}$ and radius $R_{NS}$ by using the familiar values of the parameters $m_B, G$ and $d_0$. This implies that above very simple analysis gives good estimation for the neutron star. In our case, combined with the relation $\frac{4}{3}\pi d_0^3 n_{cr} \approx 1$, one can estimate a numerical value of $d_0$. Since we have $n_{cr}$ larger than the value of the normal nuclear density $0.16\text{fm}^{-3}$ in [14], we get the values of $M_{NS}$ and $R_{NS}$ which are smaller than previous one. However the suppression from this effect is not as strong as it explains our result. It is remained as an open problem.

(2) the case of $p_c \neq 0$.

In this case, the outer core is needed to hold the pressure balance with the inner core of nuclear matter and neutron star has the structure of two phases.

If we assume the outer core is an ideal Fermi gas, $p$ and $\rho$ are expressed by

$$\rho = \frac{8\pi}{(2\pi)^3} \int_0^{k_F} dk \: k^2 \sqrt{k^2 + m^2},$$

$$p = \frac{8\pi}{3(2\pi)^3} \int_0^{k_F} dk \: k^4 \sqrt{k^2 + m^2},$$

with nucleon mass ($m = 0.94 \text{ GeV}$).

Then, the approximate equation of state is given as

$$p = p_c (\rho/\rho_c) ^\gamma$$

with $\gamma = 5/3$.

We can estimate the radius and mass of neutron star with two phase structure. The calculated radius and mass are shown in Fig[1].

When $p_c \geq 10^{-5} \text{ GeV}^4 (\rho_c \geq 0.0005 \text{ GeV}^4)$, the radius and mass of the inner core is quite small and the outer core dominates. In case of $p_c \leq 10^{-5} \text{ GeV}^4 (\rho_c \leq 0.0005 \text{ GeV}^4)$, inner core of the nuclear matter phase begins to dominate. Over the wide range of $p_c$, we obtain $R \simeq 10 \sim 20 \text{ km}$ and $m \simeq M_s$. 

5 Summary and Discussions

In this paper, based on the previous study of a holographic nuclear matter, we tried to construct the equation of states (EOS) to obtain the mass-radius relation of neutron stars. The relation between the chemical potential ($\mu$) and the nucleon number density ($n$) leads to the following EOS

$$p - p_c \propto (\rho - \rho_c)^\gamma$$

with $\gamma = 0.78$, which is less than 1 and quite different from the ordinary ones. As the nucleon number density increases, the effective nucleon mass and nucleon size also increase in our holographic model. This is the origin of our EOS.

When we apply the nuclear matter to the neutron star, interesting results about the mass and radius of neutron star are obtained. In the case of $p_c = 0$ and $\rho_c = 0$, the neutron star is unstable. While in our case, there is a phase boundary with $\rho_c \neq 0$. Due to the property, the neutron star can be stable.

When $p_c \neq 0$, two phase structure is realized. Over the wide range of $p_c$, we obtain the normal neutron star.

In our setup, baryons have been introduced as dilute instanton gas and the weak attractive force which is needed to form the stable nuclear matter is realized via the Dirac-Born-Infeld action in the curved background. However, the repulsive interaction which has been seen in the two baryon system [4] is absent. We have neglected the overlap among instantons in the equation (2.4). Because of the absence of the repulsive force, the mass and radius of neutron stars we obtained here get much smaller than usual. In the language of four dimensional field theories, people introduce one pion exchange as an attractive force between baryons, while omega meson exchange plays a role of a repulsive core [20].
There are some other aspects of the current work. One of them is the magnetic field in neutron star. To this end, it is necessary to extend our model so as to incorporate the electromagnetic fields. This point will be asked in the future.

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References

[1] T. Sakai and S. Sugimoto, “Low energy hadron physics in holographic QCD,” Prog. Theor. Phys. 113, 843 (2005) [hep-th/0412141].
[2] H. Hata, T. Sakai, S. Sugimoto and S. Yamato, “Baryons from instantons in holographic QCD,” Prog. Theor. Phys. 117, 1157 (2007) [hep-th/0701280].
[3] K. Hashimoto, T. Sakai and S. Sugimoto, “Holographic Baryons: Static Properties and Form Factors from Gauge/String Duality,” Prog. Theor. Phys. 120, 1093 (2008) [arXiv:0806.3122 [hep-th]].
[4] K. Hashimoto, T. Sakai and S. Sugimoto, “Nuclear Force from String Theory,” Prog. Theor. Phys. 122, 427 (2009) [arXiv:0901.4449 [hep-th]].
[5] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Yu. S. Tyupkin, “Pseudoparticle Solutions of the Yang-Mills Equations,” Phys. Lett. B 59, 85 (1975).
[6] K. -Y. Kim, S. -J. Sin and I. Zahed, “Dense hadronic matter in holographic QCD,” [hep-th/0608046].
[7] O. Bergman, G. Lifschytz and M. Lippert, “Holographic Nuclear Physics,” JHEP 0711, 056 (2007) [arXiv:0708.0326 [hep-th]].
[8] K. -Y. Kim, S. -J. Sin and I. Zahed, “Dense holographic QCD in the Wigner-Seitz approximation,” JHEP 0809, 001 (2008) [arXiv:0712.1582 [hep-th]].
[9] M. Rozali, H. -H. Shieh, M. Van Raamsdonk and J. Wu, “Cold Nuclear Matter In Holographic QCD,” JHEP 0801, 053 (2008) [arXiv:0708.1322 [hep-th]].
[10] W. -y. Chuang, S. -H. Dai, S. Kawamoto, F. -L. Lin and C. -P. Yeh, “Dynamical Instability of Holographic QCD at Finite Density,” Phys. Rev. D 83, 106003 (2011) [arXiv:1004.0162 [hep-th]].
[11] V. Kaplunovsky, D. Melnikov and J. Sonnenschein, “Baryonic Popcorn,” JHEP 1211, 047 (2012) arXiv:1201.1331 [hep-th].

[12] S. Seki and S. -J. Sin, “Chiral Condensate in Holographic QCD with Baryon Density,” JHEP 1208, 009 (2012) arXiv:1206.5897 [hep-th].

[13] J. de Boer, B. D. Chowdhury, M. P. Heller and J. Jankowski, “Towards a holographic realization of the Quarkyonic phase,” Phys. Rev. D 87, 066009 (2013) arXiv:1209.5915 [hep-th].

[14] Kazuo Ghoroku, Kouki Kubo, Motoi Tachibana, Tomoki Taminato, Fumihiko Toyoda, “Holographic cold nuclear matter as dilute instanton gas” , Phys. Rev. D 87, 066006(2013), arXiv:1211.2499 [hep-th].

[15] J. R. Oppenheimer and G. M. Volkoff, “On Massive neutron cores,” Phys. Rev. 55, 374 (1939); R.C. Tolman, “Static solutions of Einstein’s field equations for spheres of fluid,” Phys. Rev. 55, 364 (1939).

[16] K. Skenderis, “Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence,” Class. Quant. Grav. 19, 5849 (2002) hep-th/0209067.

[17] S. de Haro, S. N. Solodukhin and K. Skenderis, “Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence,” Commun. Math. Phys. 217, 595 (2001) hep-th/0002230.

[18] Y. Kim, C. -H. Lee, I. J. Shin and M. -B. Wan, “Holographic equations of state and astrophysical compact objects,” JHEP 1110, 111 (2011) arXiv:1108.6139 [hep-ph].

[19] J. M. Luttinger, “Fermi Surface and Some Simple Equilibrium Properties of a System of Interacting Fermions,” Phys. Rev. 119, 1153 (1960).

[20] A. Schmitt, “Dense matter in compact stars: A pedagogical introduction,” Lect. Notes Phys. 811, 1 (2010) arXiv:1001.3294 [astro-ph.SR].