ON VECTOR-VALUED POINCARÉ SERIES OF WEIGHT 2

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Abstract. Given a pair $(\Gamma, \rho)$ of a Fuchsian group of the first kind, and a unitary representation $\rho$ of $\Gamma$ of arbitrary rank, the problem of construction of vector-valued Poincaré series of weight 2 is considered. When the genus of the group is zero, it is shown how an explicit basis for the space of these functions can be prescribed. Also, a connection is established between a complex structure on unitary character varieties for Fuchsian groups and the classical Eichler-Shimura isomorphism.

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1. INTRODUCTION

For a finite dimensional vector space $V$ over $\mathbb{C}$ with a Hermitian inner product and $\rho : \Gamma \to \text{Aut } V$ a unitary representation of a Fuchsian group $\Gamma$, an automorphic form of weight 2 with the representation $\rho$ is a holomorphic function $f : \mathbb{H} \to V$ satisfying

$$f(\gamma \tau) \gamma' \tau) = \rho(\gamma) f(\tau), \quad \forall \gamma \in \Gamma, \tau \in \mathbb{H}.\quad (1.1)$$

The case $V = \mathbb{C}$ is due to Petersson [Pet40, Pet48]. Whenever it is more convenient, we will consider the cases $V = \mathbb{C}^r$ or $V = \text{End } \mathbb{C}^r$ with their standard inner products.

The present investigation undertakes Lehner’s dictum [Leh64], which lists the 3 primary problems in the classical theory of automorphic forms as: (1) proving their existence, (2) constructing Poincaré series, and (3) finding a minimal spanning family among these. Our strategy is twofold, mixing aspects of the theory of holomorphic vector bundles over compact Riemann surfaces with the complex analysis on the upper half plane.
The geometric interpretation of cusp forms as global holomorphic sections of a vector bundle enables the computation of the dimension of the space they span by means of the Riemann-Roch theorem (a foundational result implicit in the work of André Weil [Wei38]), thus solving (1) (cf. [Hej83]). The main difficulty with (2) is that the standard Poincaré series fail to be absolutely convergent. This can be remedied following Hecke’s idea [Hec27], later extended by Petersson in the rank 1 case [Pet48], on the introduction of a convergence factor. The present approach is a simplification and extension to arbitrary rank, as it is possible to consider vector-valued Poincaré series of weight 2 as limits of certain absolutely convergent series within the Hilbert space of measurable automorphic forms of weight 2. In particular, the construction of the Poincaré series follows from the properties of the Cauchy-Riemann operators on Hilbert spaces. Regarding (3), at least in the genus 0 case, an approach based on a vector bundle analog of the uniformization theorem is possible, which is presented.

At a more substantial level, the author’s interest in the subject stems on its unifying relation with the deformation theory of complex structures and the theory of moduli (for this, refer to section 6), as in the work of Narasimhan-Mehta-Seshadri [NS65, MS80] on vector bundles over Riemann surfaces.

The contents of this work are organized as follows: Section 2 is devoted to introducing conventions; in section 3 we recall how the dimension of the space of cusp forms of weight 2 can be determined by means of the Riemann-Roch theorem on a suitable vector bundle; in section 4, Poincaré series are constructed by means of a limiting procedure in the Hilbert space of measurable automorphic forms, and moreover, it is shown how, with the aid of the Petersson inner product, a complete family can be determined; in section 5 it is shown how a basis for the space of cusp forms can be prescribed in the genus zero case. Finally, in section 6 it is shown how, in the concrete case of the representation $\text{Ad} \rho_C$ for $\rho$ irreducible, the classical Eichler-Shimura isomorphism manifests as the differential of the map from a moduli space of stable parabolic bundles with fixed weights and the character variety of unitary representations of a Fuchsian group.

2. Preliminary remarks

Let $\Gamma$ be a Fuchsian group (i.e. a discrete subgroup of $\text{PSL}(2, \mathbb{R})$) of the first kind.\footnote{In the classical literature, Fuchsian groups are also called principal-circle groups, or simply circle groups, and a Fuchsian group of the first kind is often called horocyclic. $H$-groups are also called zonal.} Among these we will be interested in the so-called $H$-groups ($\text{Grenzkreisgruppen}$). By definition, an $H$-group is a Fuchsian group of the first kind that is finitely generated and possesses parabolic elements. Examples of these model the fundamental groups of Riemann surfaces of finite type (compact surfaces with a finite number of points removed) but...
in general torsion is also allowed. It is a classical result [Hej83, Leh64] that \( \Gamma \) admits an explicit presentation in terms of \( 2g \) hyperbolic generators \( A_1, \ldots, A_g, B_1, \ldots, B_g, m \) elliptic generators \( S_1, \ldots, S_m \) and \( n \) parabolic generators \( T_1, \ldots, T_n \) with relations

\[
\prod_{i=1}^g [A_i, B_i] \cdot \prod_{j=1}^m S_j \cdot \prod_{k=1}^n T_k = 1,
\]

\[
S_1^{l_1} \cdots S_m^{l_m} = 1.
\]

The \textit{signature} of \( \Gamma \) is the collection of nonnegative integers \( (g; l_1, \ldots, l_m; n) \) with \( 2 \leq l_j < \infty, j = 1, \ldots, m \), subject to the inequality

\[
2 - 2g - \sum_{i=1}^m \left( 1 - \frac{1}{l_i} \right) - n < 0.
\]

The equivalence relation on Fuchsian groups is conjugation in \( \text{PSL}(2, \mathbb{R}) \).

The requirement \( n > 0 \) shall be emphasized, although most of the present results are still valid without this restriction.

\( \Gamma \) admits a fundamental region \( \mathfrak{F} \) bounded by a finite number of geodesic arcs and vertices determined by the fixed points of \( \{S_i, T_j \} \). We will denote by \( \{e_1, \ldots, e_m \} \subset \mathbb{H} \) the set of elliptic fixed points \( (S_i(e_i) = e_i) \), and by \( \{p_1, \ldots, p_n \} \subset \mathbb{R} \cup \{\infty\} \) the set of cusps \( (T_i(p_i) = p_i) \) in the fundamental region \( \mathfrak{F} \). We follow Shimura’s convention [Shi71] on the construction of the compact quotient \( S = \Gamma \setminus \mathbb{H}^* \) with local parabolic (resp. elliptic) coordinates \( q_i = q \circ \sigma_i^{-1} \) (resp. \( \zeta_i = \zeta_i \circ \varphi_i^{-1} \)) where \( q = e^{2\pi \sqrt{-1} r}, \zeta : \mathbb{H} \to \mathbb{D} \) is the Cayley mapping, and \( \sigma_i, \varphi_j \in \text{PSL}(2, \mathbb{R}) \) satisfy

\[
(\sigma_i^{-1} T_i \sigma_i)(\tau) = \tau \pm 1.
\]

\[
\zeta \left( (\varphi_j^{-1} S_j \varphi_j)(\tau) \right) = e^{2\pi \sqrt{-1}/l_j} \cdot \zeta(\tau),
\]

The spectral decomposition of a unitary matrix \( M \in \text{U}(r) \) can be equivalently understood in terms of the data \( \{W, [U]\} \) consisting of a diagonal matrix of \textit{weights} \( W = \text{diag}(\alpha_1, \ldots, \alpha_r), 0 \leq \alpha_1 \leq \cdots \leq \alpha_r < 1 \) and a \textit{partial flag}

\[
[U] \in \text{U}(r)/\mathbb{Z} \left( e^{2\pi \sqrt{-1} W} \right),
\]

so that

\[
M = U e^{2\pi \sqrt{-1} W} U^{-1}.
\]

Thus, for a unitary representation \( \rho : \Gamma \to \text{U}(r) \), the collections of matrices \( \{\rho(T_i)\}_{i=1}^n, \{\rho(S_i)\}_{i=1}^m \) determine collections \( \{W_i, [U_i]\}_{i=1}^n, \{W_{n+i}, [U_{n+i}]\}_{i=1}^m \) of parabolic and elliptic data (cf. [MS80]). The diagonal elements \( \alpha_{ij} \) (resp. \( \alpha_{n+i,j} \)) are called \textit{parabolic} (resp. \textit{elliptic}) \textit{weights}. Notice that \( \alpha_{n+i,j} = n_{ij}/l_i \) for \( n_{ij} \in \mathbb{N} \).
Recall that an automorphic form of weight 2 is called regular if at each cusp $p_1, \ldots, p_n$,
\[
\lim_{\text{Im}(\tau) \to \infty} f(\sigma_i \tau)\sigma'_i(\tau) \quad \text{exists},
\]
and a cusp form if moreover
\[
\lim_{\text{Im}(\tau) \to \infty} f(\sigma_i \tau)\sigma'_i(\tau) = 0, \quad i = 1, \ldots, n.
\]
In other words, $\phi = f(\tau)d\tau$ is a vector-valued holomorphic differential on $\mathbb{H}$ satisfying $\gamma^*(\phi) = \rho(\gamma)\phi \forall \gamma \in \Gamma$, and with specific boundary behavior. For any choice of unitary matrix representatives $\{U_i\}_{i=1}^{n+m}$ of the parabolic and elliptic data, any regular $\rho$-automorphic form of weight 2 admits a $q$-series expansion near each cusp,
\[
(2.3) \quad f(\sigma_i \tau)\sigma'_i(\tau) = U_i q^{W_i} \left( \sum_{k=0}^{\infty} a_i(k)q^k \right),
\]
and at each elliptic fixed point we have the power series expansions,
\[
(2.4) \quad \frac{f(\varphi_i \tau)\varphi'_i(\tau)}{\zeta'^{-1}(\tau)\zeta'(\tau)} = U_{n+i} q^{W_{n+i}} \left( \sum_{k=0}^{\infty} b_i(k)q^k \right).
\]

Remark 1. The vanishing order of a cusp form differs from that of a regular form only when a given parabolic weight vanishes, $\alpha_{ij} = 0$, since then necessarily $a_i(0) = 0$.

Remark 2. When the representation is $\text{Ad} \rho \mathbb{C}^2$ and $f(\tau)$ is matrix-valued, the expansions at a cusp become
\[
(2.5) \quad f(\sigma_i \tau)\sigma'_i(\tau) = U_i q^{W_i} \left( \sum_{k=0}^{\infty} B_i(k)q^k \right) q^{-W_i}U_i^{-1}.
\]
f($\tau$) is regular if and only if each $B_i(0)$ is block-lower triangular (with respect to the weight multiplicities), and a cusp form if and only if each $B_i(0)$ is block-lower diagonal.

We will denote the spaces of regular automorphic forms and cusp forms of weight 2 by $\mathcal{M}_2(\Gamma, \rho)$ and $\mathcal{S}_2(\Gamma, \rho)$ respectively.

3. GEOMETRIC INTERPRETATION

We recall the construction of a vector bundle $E_\rho \to S$ with prescribed system of trivializations, explicitly depending on the choice of parabolic and elliptic weights of $\rho$ (cf. [NS65, MS80]). On $\mathbb{H}_0 = \mathbb{H} \setminus \Gamma$ we consider the local system $E_0 = \mathbb{H}_0 \times \mathbb{C}^* / \sim$ over $X = \Gamma \setminus \mathbb{H}_0$, where $(\tau, v) \sim (\gamma \tau, \rho(\gamma)v)$, for all $\gamma \in \Gamma$. To extend $\text{pr} : E_0 \to X$ to the elliptic fixed points and cusps, make
\[
\mathcal{U}_i = \langle T_i \rangle \setminus ((\sigma_i \cdot \mathbb{H}_0) \cup \{p_i\}), \quad \mathcal{U}_{n+i} = \langle S_i \rangle \setminus \mathbb{D}(e_i),
\]
where $\mathbb{D}(e_i)$ denotes the composition of $\rho$ and the adjoint representation of $U(r)$.
where \( \mathbb{H}_\delta = \{ \tau \in \mathbb{H} \mid \text{Im}\, \tau > \delta \} \), \( \epsilon = e^{-2\pi \delta} \) and \( \delta > 0 \) is sufficiently large so that the collection of open neighbourhoods \( \{ \mathcal{U}_i \}_{i=1}^{n+m} \) is pairwise disjoint in \( S \). For \( i = 1, \ldots, n \), \( j = 1, \ldots, m \), we define the functions

\[ 
\Phi_i : \text{pr}^{-1}(\mathcal{U}_i \setminus \{ p_i \}) \rightarrow \mathcal{U}_i \times \mathbb{C}^r, \quad \Psi_j : \text{pr}^{-1}(\mathcal{U}_{n+j} \setminus \{ e_j \}) \rightarrow \mathcal{U}_{n+j} \times \mathbb{C}^r,
\]

(3.1) \( \Phi_i([\tau, v]) = ([\tau], q_i^{-W_i}U_i^{-1}v) \), \( \Psi_j([\tau, v]) = ([\tau], \zeta_j^{-W_{n+j}}U_{n+j}^{-1}v) \),

which depend on a choice of unitary matrix representatives \( \{ U_i \}_{i=1}^{n+m} \) of the parabolic and elliptic data and are readily seen to define biholomorphisms onto their images. The bundle \( E_\rho \) is defined as the identification

\[ E_\rho := E_0 \sqcup \Phi_i (\mathbb{D}_c \times \mathbb{C}^r) \sqcup \Psi_j (\mathbb{D}_c \times \mathbb{C}^r). \]

We can consider a sufficiently fine covering of \( S \) by extending \( \{ \mathcal{U}_i \}_{i=1}^{n+m} \) with a finite collection of domains \( \{ \mathcal{V}_j \} \) covering \( X \subset S \), thus obtaining

**Lemma 1.** For any choice of unitary matrix representatives \( \{ U_i \}_{i=1}^{n+m} \) of the parabolic and elliptic data, the vector bundle \( E_\rho \) is determined by the transition functions

\[
ge_{ij} = \begin{cases} 
\rho(\gamma_{ij}) & \text{on } \mathcal{V}_{ij}, \\
 q_i^{-W_i}U_i^{-1} & \text{on } \mathcal{U}_i \cap \mathcal{V}_j, \\
 \zeta_i^{-W_{n+i}}U_{n+i}^{-1} & \text{on } \mathcal{U}_{n+i} \cap \mathcal{V}_j.
\end{cases}
\]

(3.2)

**Proof.** This is a simple extension of the proof considered in [NS65], remark 6.2. It is clear that any two choices of unitary matrix representatives would yield equivalent systems of transition functions. \( \square \)

**Remark 3.** The bundle \( E_\rho \) satisfies (cf. [Wei38, MS80])

\[ c_1(E_\rho) = - \sum_{i=1}^{n+m} \sum_{j=1}^r \alpha_{ij}, \quad H^0(S, E_\rho) \cong (\mathbb{C}^r)^{\rho}. \]

When parabolic or elliptic generators are present, it is not longer true that \( E_\rho' = E_{\overline{\rho}} \), where \( \overline{\rho} = {\rho}^{-1} \) is the contragradient (or dual) representation.

**Lemma 2.** The parabolic and elliptic weights of \( \overline{\rho} \) are related to those of \( \rho \) as

\[
\alpha_{ij}' = \begin{cases} 
0 & \text{if } j \leq s_i, \\
1 - \alpha_{ir+s_i+1-j} & \text{if } j > s_i,
\end{cases}
\]

(3.3)

where \( s_i \) is the number of vanishing weights for each cusp or elliptic fixed point. The partial flags in the parabolic and elliptic data of \( \overline{\rho} \) satisfy \( U_i' = [\overline{U}_i \Pi_i] \), where \( \Pi_i \) is any permutation matrix satisfying

\[
\Pi_i^{-1}W_i \Pi_i = \text{diag}(\alpha_{i1}, \ldots, \alpha_{is_i}, \alpha_{ir}, \alpha_{ir-1} \ldots, \alpha_{is_i+1}).
\]

(3.4)

**Proof.** The proof is clear, since \( {\rho}^{-1}M^{-1} = \overline{M} \) whenever \( M \) is unitary. \( \square \)
Proposition 1. There is an isomorphism
\[\mathcal{M}_2(\Gamma, \rho) \cong H^0(S, E\rho \otimes K_S(D_p))\].

If \(\rho\) has nonvanishing elliptic weights, then
\[\mathcal{S}_2(\Gamma, \rho) \cong H^0(S, E\rho^\vee \otimes K_S(-D_e))\],
where \(D_e = \sum e_i, D_p = \sum p_j\), and \(E^\vee\) denotes the dual bundle of \(E\).

**Proof.** Let \(\phi = f(\tau)d\tau\). By the definition of the bundle \(E\rho\), it is only needed to analyze the behavior of \(\phi\) near the cusps and elliptic fixed points. Since
\[d\tau = \frac{1}{2\pi \sqrt{-1}} \frac{dq}{q}\],
when \(f(\tau)\) is regular, the \(q\)-series expansions \((2.3)\) imply that near each cusp \(p_i\), the local expressions \(q_i^{-W_i} U_i^{-1} \phi\) are meromorphic \(q_i\)-differentials with potential simple poles at each cusp. In turn, the local expansions \((2.4)\) and \((2.2)\) imply that near the elliptic fixed points,
\[\zeta_i^{-W_{n+i}} U_{n+i}^{-1} \phi = \left(\sum_{k=0}^{\infty} b_i(k) \zeta_k^{n+i}\right) d\zeta_i\].

According to lemma 1, these local differentials determine a section of the bundle \(E\rho^\vee \otimes K_S(-D_p)\) in the trivializations of the neighborhoods \(\{\mathcal{U}_i\}_{i=1}^{n+m}\) of the cusps and elliptic fixed points. This establishes \((3.5)\).

If \(f\) is a cusp form, the expansions \((2.3)\) satisfy the additional condition \((a_i(0))_j = 0\) whenever \(\alpha_{ij} = 0\). Thus, near each cusp \(p_i\) the local expression of the differential \(\phi\) can be equivalently rewritten as
\[\sigma_i^*(\phi) = U_i' q^{-W_i'} \left(\sum_{k=0}^{\infty} a_i'(k) q^k\right) dq\]
and similarly, since \(\rho\) has nonvanishing elliptic weights, near each elliptic fixed point \(e_i\)
\[\phi = V_i' \zeta_i^{-W_{n+i}} \left(\sum_{k=1}^{\infty} b_i'(k) \zeta_k^{n+i}\right) d\zeta_i\]
The transition functions of the bundles \(E_{\rho}^\vee, E_{\overline{\rho}}\) are related as \(g_{ij} = t_{ij}^{-1}\), and it follows from lemmas 1 and 2 that the cusp form condition is precisely the holomorphicity condition of sections of \(E_{\rho}^\vee \otimes K_S(-D_e)\) in the neighbourhoods \(\{\mathcal{U}_i\}_{i=1}^{n+m}\).

\[\square\]

**Corollary 1.** The dimension of the spaces of regular and cusp forms is
\[\dim \mathcal{M}_2(\Gamma, \rho) = \dim H^0(S, E_{\rho}^\vee(-D_p)) + r(g + n - 1) - \sum_{i,j} \alpha_{ij}\],
\[\dim \mathcal{S}_2(\Gamma, \rho) = \dim H^0(S, E_{\overline{\rho}}(D_e)) + r(g - m - 1) + \sum_{i,j} \alpha_{ij}'\].
Proof. A direct consequence of remark 3 and the Riemann-Roch theorem for vector bundles. □

4. Poincaré series for weight 2

We will now provide a construction of the Poincaré series, and determine an infinite spanning set for the space of cusp forms. For this, it is enough to construct weak solutions to the Cauchy-Riemann equation on a suitable Hilbert space by means of a limiting procedure. For \( p, q = 0, 1 \), let \( \mathcal{E}^{p,q}_r(\Gamma, \rho) \) be the space of smooth, \( C^r \)-valued functions on \( \mathbb{H} \) satisfying

\[
  f(\gamma \tau) = \rho(\gamma) f(\tau) \quad \tau \in \mathbb{H}, \quad \gamma \in \Gamma,
\]

and which are compactly supported within a fundamental region \( \mathcal{F} \). For any \( f_1, f_2 \in \mathcal{E}^{p,q}_r(\Gamma, \rho) \), the Petersson inner product is defined as

\[
  \langle f_1, f_2 \rangle_P = 2^{p+q} \int \int \mathcal{F} \langle f_1, f_2 \rangle y^{2p+2q-2} dxdy
\]

where \( \langle f_1, f_2 \rangle = t \overline{f_1 f_2} \) (or tr\( (f_1 f_2^*) \) in the matrix-valued case) \(^3\) and \( \tau = x + \sqrt{-1}y \). A fundamental property of this inner product is its invariance under the action of \( \Gamma \). Let us denote by \( \mathcal{H}^{p,q}_r(\Gamma, \rho) \) the completion of \( \mathcal{E}^{p,q}_r(\Gamma, \rho) \) to a Hilbert space under the Petersson inner product. As a consequence of the \( q \)-series expansions at the cusps,

\[
  \mathcal{H}^{1,0}_r(\Gamma, \rho) \cap \mathfrak{M}_2(\Gamma, \rho) = S^2(\Gamma, \rho),
\]

so it is also possible to define

\[
  \mathfrak{S}_2(\Gamma, \rho) = \ker \overline{\partial},
\]

in terms of the Cauchy-Riemann operator \( \overline{\partial} : \mathcal{H}^{1,0}(\Gamma, \rho) \rightarrow \mathcal{H}^{1,1}(\Gamma, \rho) \). An admissible multi-index \( I = (i, j, l) \) is defined by the conditions \( 1 \leq i \leq n, \ 1 \leq j \leq r \) and \( l \geq 0 \), so that \( l > 0 \) if \( \alpha_{ij} = 0 \). Thus \( I \) is admissible if and only if \( \alpha_{ij} + l > 0 \). For any \( s > 0 \), let us consider the series

\[
  P^s_I(\tau) = \sum_{\gamma \in \Gamma_i \setminus \Gamma} (\sigma_i^{-1} \gamma)'(\tau) \text{Im}(\sigma_i^{-1} \gamma \tau)^s \left( q_i^l \circ \gamma \right) F_i(\gamma)e_j
\]

where \( \Gamma_i = \langle T_i \rangle \) and

\[
  F_i(\gamma) = \rho(\gamma)^{-1} U_i W_i \circ \gamma,
\]

which are independent of the choice of representatives in \( \Gamma_i \setminus \Gamma \), and satisfy (1.1) formally, together with the auxiliary series

\[
  Q^s_I(\tau) = \sum_{\gamma \in \Gamma_i \setminus \Gamma} |(\sigma_i^{-1} \gamma)'(\tau)|^2 \text{Im}(\sigma_i^{-1} \gamma \tau)^{s-1} \left( q_i^l \circ \gamma \right) F_i(\gamma)e_j
\]

which satisfy formally the functional equation

\[
  Q^s_I(\gamma \tau) |\gamma'(\tau)|^2 = \rho(\gamma) Q^s_I(\tau), \quad \gamma \in \Gamma, \ \tau \in \mathbb{H}.
\]

\(^3\)We emphasize the difference in notation between \( \| \cdot \| \) and \( \| \cdot \|_P \).
Lemma 3. For any admissible multi-index \( I = (i, j, l) \) and \( s > 0 \), the series \( P^s_I(\tau) \) and \( Q^s_I(\tau) \) are absolutely convergent for any \( \tau \in \mathbb{H} \) and uniformly convergent on compact sets. For any cusp \( p_k \),
\[
\{ \| \text{Im}(\sigma_k \tau) P^s_I(\sigma_k \tau) \|, \| \text{Im}(\sigma_k \tau)^2 Q^s_I(\sigma_k \tau) \| \} = O(y^{-s})
\]
as \( \text{Im}(\tau) \to \infty \).

Proof. Let \( \kappa_I = \alpha_{ij} + l \). Consider the series of norms of the summands in \eqref{eq:4.2}. Since for any \( \eta \in \text{PSL}(2, \mathbb{R}) \), \( \text{Im}(\eta(\tau)) = |\eta'(\tau)|\text{Im}(\tau) \), we obtain
\[
\frac{1}{\text{Im}(\tau)} \cdot \sum_{\gamma \in \Gamma \backslash \Gamma} e^{-2\pi \kappa_I \text{Im}(\sigma^{-1}_i \gamma \tau)} \text{Im}(\sigma^{-1}_i \gamma \tau)^{1+s}.
\]
Similarly, for the series of norms of the summands of \eqref{eq:4.3}, we get
\[
\frac{1}{\text{Im}(\tau)^s} \cdot \sum_{\gamma \in \Gamma \backslash \Gamma} e^{-2\pi \kappa_I \text{Im}(\sigma^{-1}_i \gamma \tau)} \text{Im}(\sigma^{-1}_i \gamma \tau)^{1+s}.
\]
The common series is majorized by the Eisenstein series
\[
E_i(\tau, 1 + s) = \sum_{\gamma \in \Gamma \backslash \Gamma} \text{Im}(\sigma^{-1}_i \gamma \tau)^{1+s},
\]
which is absolutely convergent for any \( s > 0 \) and \( \tau \in \mathbb{H} \), uniformly convergent on compact sets, and moreover, has the asymptotic behaviour at every cusp \( p_k \)
\[
E_i(\sigma_k \tau, s) - y^{1+s} = O(y^{-s}) \quad \text{as} \quad y \to \infty.
\]
\[\text{Kub73}\]. The presence of the exponential factors in the common series implies the claim, since \( \kappa_I > 0 \).

Lemma 4. For any admissible multi-index \( I \) and \( s > 0 \), \( P^s_I(\tau) \in \mathcal{H}^{1,0}(\Gamma, \rho) \) and \( Q^s_I(\tau) \in \mathcal{H}^{1,1}(\Gamma, \rho) \). For any positive sequence \( s_n \to 0 \), the sequences \( P^{s_n}_I(\tau) \), \( Q^{s_n}_I(\tau) \) are Cauchy.

Proof. Consider the automorphic functions \( \| y P^s_I(\tau) \| \) and \( \| y^2 Q^s_I(\tau) \| \). It follows from the estimate \eqref{eq:4.4} in lemma 3 that for any cusp \( p_k \) and \( s > 0 \),
\[
\lim_{y \to \infty} \| \text{Im}(\sigma_k \tau) P^s_I(\sigma_k \tau) \| = \lim_{y \to \infty} \| \text{Im}(\sigma_k \tau)^2 Q^s_I(\sigma_k \tau) \| = 0.
\]
In particular, \( \| y P^s_I(\tau) \| \) and \( \| y^2 Q^s_I(\tau) \| \) are bounded in \( \mathfrak{K} \). Since \( \mathfrak{K} \) has finite hyperbolic area, this readily implies that
\[
\| P^s_I(\tau) \|_{p < \infty}, \quad \| Q^s_I(\tau) \|_{p < \infty}.
\]
Consider a positive sequence \( s_n \to 0 \). The absolute convergence and the asymptotics \eqref{eq:4.4} imply once again that
\[
\| y P^{s_n}_I(\tau) - y P^{s_m}_I(\tau) \| = |s_n - s_m| h(\tau),
\]
where \( h(\sigma_k \tau) = O(\ln(y) y^{-\min\{s_n, s_m\}}) \) as \( y \to \infty \) for every cusp \( p_k \), and in particular it is bounded within \( \mathfrak{K} \). A similar statement holds for the function \( \| y^2 (Q^{s_n}_I(\tau) - Q^{s_m}_I(\tau)) \| \), which implies that the sequence \( \{ P^{s_n}_I(\tau) \} \) (resp. \( \{ Q^{s_n}_I(\tau) \} \)) is Cauchy in \( \mathcal{H}^{1,0}(\Gamma, \rho) \) (resp. \( \mathcal{H}^{1,1}(\Gamma, \rho) \)).
**Definition 1.** Given an admissible multi-index $I = (i, j, l)$, the Poincaré series of weight 2 for the representation $\rho$ are defined as the Hilbert space limit

$$P_I(\tau) = 4\pi \kappa_I \cdot \lim_{s \to 0} P^s_I(\tau),$$

where $\kappa_I = \alpha_{ij} + l$.

**Corollary 2.** The functions $P_I(\tau)$ are holomorphic, that is, $P_I(\tau) \in \mathcal{S}_2(\Gamma, \rho)$.

**Proof.** This is a consequence of lemmas 3 and 4, since the functions $P^s_I(\tau)$ and $Q^s_I(\tau)$ were defined so that for any $s > 0$, the equation

$$\bar{\partial}P^s_I(\tau) = -\frac{s}{2\sqrt{-1}}Q^s_I(\tau)$$

is satisfied in the weak sense. In particular, $\bar{\partial}P^s_I(\tau) \to 0$ weakly as $s \to 0$ in $\mathcal{H}^{1,0}(\Gamma, \rho)$, which implies that $\bar{\partial}P_I(\tau) = 0$ weakly. It follows from Weyl's lemma [Kra72] that $P_I$ is indeed holomorphic, thus a cusp form.

The final result in this section shows that our construction overdetermines $\mathcal{S}_2(\Gamma, \rho)$, an essential property that is satisfied by the Poincaré series of weight $> 2$.

**Proposition 2.** Let $f(\tau)$ be a cusp form and $I = (i, j, l)$ a fixed admissible multi-index. Then

$$\langle f(\tau), P_I(\tau) \rangle_P = a_i(l)j.$$

**Proof.** The fundamental region $\mathfrak{F}$ can be chosen so that $\sigma^{-1}_i \cdot \mathfrak{F}$ lies in the semi-strip $0 \leq \text{Re}(\tau) \leq 1$ in $\mathbb{H}$, and the representatives of any given class $[\gamma]$ in $\Gamma_i \setminus \Gamma$ can be chosen so that $\sigma^{-1}_i \gamma \cdot \mathfrak{F}$ satisfies the same property. This is the basis of the classical “unfolding trick” for the expression of the inner product of the series (4.2) with any $f \in \mathcal{H}^{1,0}(\Gamma, \rho)$ (resp. $g \in \mathcal{H}^{1,1}(\Gamma, \rho)$) as the elementary integral

$$\langle P^s_I(\tau), f(\tau) \rangle_P = \int_0^\infty \int_0^1 \langle U_i q^W e_j, f(\sigma_i \tau) \sigma_i'(\tau) \rangle q^l y^s dxdy,$$

Consider the $q$-series expansions (2.3) for $f(\tau)$ at the cusp $p_i$. The inner product $\langle P^s_I(\tau), f(\tau) \rangle_P$ can thus be estimated as

$$\sum_{k=0}^\infty a_i(k)j \left( \int_0^1 e^{2\pi \sqrt{1-(l-k)^2}} dx \right) \cdot \left( \int_0^\infty e^{-2\pi (k+l+2\alpha_{ij})} y^s dy \right)$$

$$= \frac{\Gamma(1+s)}{(4\pi(l + \alpha_{ij}))^{1+s}} \cdot a_i(l)j.$$

The claim will follow if we let $s \to 0$. \hfill \Box

---

4This manipulation is genuine as a consequence of the complete additivity of the Lebesgue integral.
Let $\mathcal{I} = \{(i,j,l) : 1 \leq i \leq n, 1 \leq j \leq r, l \geq 0\}$ and \{\(f_1, \cdots, f_d\)} be an arbitrary basis for $\mathcal{S}_2(\Gamma, \rho)$. Consider the function
$$T : \mathcal{I} \to \mathbb{C}^d, \quad I \mapsto (w_{1I}, \cdots, w_{dI}) = w_I, \quad w_{kI} = \langle f_k, P_I \rangle$$
defined up to the left action of $\text{GL}(d, \mathbb{C})$. In particular, Petersson’s result on linear relations of cusp forms [Pet40] is still valid in the present context.

**Corollary 3.** Let \{\(P_I(\tau)\)} be a collection of Poincaré series of weight 2, \{\(\lambda_I\)\} $\subset \mathbb{C}$. The relation
$$\sum \lambda_I P_I(\tau) = 0$$
is satisfied if and only if
$$\sum \lambda_I w_I = 0.$$

5. **The genus zero case**

We are interested in constructing an explicit basis for the space $\mathcal{S}_2(\Gamma, \rho)$ when $g = 0$. This condition will be assumed henceforth throughout this section. It will also be assumed that $\rho$ is irreducible and all parabolic and elliptic weights are different from 0. The latter condition implies that every regular form is a cusp form, so $\mathfrak{M}_2(\Gamma, \rho) = \mathcal{S}_2(\Gamma, \rho)$.

The Birkhoff-Grothendieck theorem captures the fundamental difference between the cases $g = 0$ and $g > 0$, since for the former, it states that every vector bundle over $\mathbb{P}^1$ is holomorphically equivalent to a direct sum of line bundles of the form $\mathcal{O}(a)$,

$$E \cong \bigoplus_{j=1}^r \mathcal{O}(a_j), \quad a_1 \leq \cdots \leq a_r.$$  

In particular, the space $H^0(\mathbb{P}^1, E)$ can be identified with the space of vector-valued polynomials of degree at most $a_j$ in the $j$th-entry (or 0 if $a_j < 0$). Thus,

$$\dim H^0(\mathbb{P}^1, E) = \sum_{j=1}^r \max\{a_j + 1, 0\}.$$

Now, in the case when $E = E_\rho$ is constructed from a unitary representation of $\Gamma$, we have the following results [MT14] regarding the previous bundle isomorphism and the splitting type. Let $N = \text{diag}(a_1, \ldots, a_r)$.

**Lemma 5.** If $\rho$ is irreducible and has non-vanishing parabolic and elliptic weights, then

$$E_\rho^\vee = E_\rho(D_e + D_p),$$

and the Birkhoff-Grothendieck splitting coefficients of $E_\rho$ satisfy

$$-(n + m) < a_j < 0, \quad j = 1, \cdots, r.$$
Lemma 6. Let $\rho : \Gamma \to U(r)$ be an admissible representation for a fixed set of weights $W$. For every choice of representatives $\{U_i\}_{i=1}^{n+m}$ there exists a holomorphic function $Y : \mathbb{H} \to GL(r, \mathbb{C})$ satisfying

$$(5.1) \quad Y(\gamma \tau) = Y(\tau) \rho(\gamma)^{-1}, \quad \forall \gamma \in \Gamma, \tau \in \mathbb{H},$$

with $\zeta$-series expansions at each elliptic fixed point

$$(5.2) \quad Y(\varphi_i \tau) = \left( \sum_{k=0}^{\infty} C_{n+i}(k) \zeta^{1,k} \right) \zeta^{-1} W_{n+i-1} \gamma \zeta \tau,$$

$i = 1, \ldots, m$,

and having the $q$-series expansions

$$(5.3) \quad Y(\sigma_i \tau) = \left( \sum_{k=0}^{\infty} C_i(k) q^k \right) q^{-W_i} \zeta \tau,$$

$i = 1, \ldots, n - 1$,

$$(5.4) \quad Y(\tau) = q^{-N} \left( \sum_{k=0}^{\infty} C_n(k) q^k \right) q^{-W_n} \zeta,$$

where the constant terms $C_i(0) \in GL(r, \mathbb{C})$ for $i = 1, \ldots, n + m$. When $\rho$ is irreducible, the set $Y(\rho)$ of all functions $Y$ with these properties is a torsor for the projectivization of the automorphism group $Aut E_\rho$ of the bundle $E_\rho$.

Let $E_\rho \cong \bigoplus_{j=1}^{r} \mathcal{O}(a_j)$ be realized by a matrix-valued function $Y$ as in lemma 6. It follows from proposition 1 that

$$\mathcal{S}_2(\Gamma, \rho) \cong H^0 \left( \mathbb{P}^1, \bigoplus_{j=1}^{r} \mathcal{O}(a_j) \right),$$

where $b_j = -a_j - m - 2$. Let us consider a **hauptmodul** $J : \mathbb{H}^* \to \mathbb{P}^1$ (i.e. an automorphic function for $\Gamma$ that is univalent on $\mathfrak{H}$ and regular at each cusp), normalized so that $J(p_n) = \infty$. Then each column $v_k$ of $t \gamma J(G_e \circ J) \gamma^t$, where $G_e(z) = \prod_{j=1}^{m} (z - J(e_j))$, corresponds to a meromorphic section of $\mathcal{O}(b_k)$. We should distinguish the columns $\{v_1, \ldots, v_t\}$ ($t \leq r$) for which $b_k \geq 0$, since in this case, the functions $(g_k \circ J) v_k$, where $g_k(z)$ is a polynomial of degree at most $b_k$, are cusp forms. In particular, the functions $\{v_1, \ldots, v_t\}$ are cusp forms, and for every fixed cusp $\{p_i\}$, $i \neq n$, $\dim(\text{Span}(a_1(0), \ldots, a_n(0))) = t$ (as a consequence of proposition 1, the univalence of $J$ on $\mathfrak{H}$ (cf. [MT14]), and lemma 6, since $\det C_i(0) \neq 0$). Therefore at least $t$ of the $r$ Poincaré series $\{P_I\}$, $I = (i, j, 0)$ would be nonzero. Since these Poincaré series are obviously linearly independent,

$$t \leq \dim \mathcal{S}_2(\Gamma, \rho).$$

Similarly, the following rough upper bound follows from lemma 5

$$(5.5) \quad \dim \mathcal{S}_2(\Gamma, \rho) = \sum_{j=1}^{t} (b_j + 1) < t(n - 1),$$

It is thus required that $n \geq 3$, a necessary condition when $m = 0$. 


Theorem 1. Let $\mathcal{J}$ be the collection of multi-indices $I = (i,j,0)$, with $|\mathcal{J}| = \dim S_2(\Gamma, \rho)$, determined by lexicographic order, up to a permutation of the cusps. The Poincaré series $\{P_I\}_{I \in \mathcal{J}}$ form a basis for $S_2(\Gamma, \rho)$.

Proof. Consider the lexicographic subcollection $\mathcal{J}$ of the set of indices of the form $(i,j,0)$ with $|\mathcal{J}| = \dim S_2(\Gamma, \rho)$, and assume that $V = \text{Span}\{P_I\}_{I \in \mathcal{J}}$ is a proper subset of $S_2(\Gamma, \rho)$, so that $V^\perp$ is nontrivial.

The case $r = 1$ is straightforward, since any nonzero $f \in V^\perp$ would correspond to a holomorphic section with $b+1$ zeros by proposition 2, but $c_1(\mathcal{O}(b)) = b$, a contradiction.

As a consequence of the bundle splittings, every cusp form $f(\tau)$ can be uniquely expressed as a linear combination

$$f(\tau) = \sum_{k=1}^{t} (g_k \circ J)v_k. \quad (5.6)$$

Pick any nonzero $f \in V^\perp$, and consider its splitting $(5.6)$. Also, express $\dim S_2(\Gamma, \rho) = c + d$, $c, d \in \mathbb{Z}$, $0 \leq d < t$. By hypothesis, $(P_I(\tau), f(\tau))_P = 0$ for all $I = (i,j,0)$, $1 \leq i \leq c$, $1 \leq j \leq t$. This implies that for each $1 \leq k \leq t$, the polynomial $g_k(z)$ would have a zero at each $J(p_i)$, $i = 1,\ldots,c$, since otherwise $a_k(0) = \lambda \cdot t C_i(0)_k$ (the $k$th-column of the matrix $t C_i(0)$) in the $q$-series expansion $(2.3)$ of $(g_k \circ J)v_k$ at $p_i$, with $\lambda \neq 0$, and thus there would be a nontrivial linear relation among several columns of at least one constant matrix $C_i(0)$, $1 \leq i \leq c$, a contradiction. Therefore, the total number of zeros of the polynomials $g_k$ would be bounded below by $ct$, but

$$ct > \sum_{k=1}^{t} b_k = \sum_{k=1}^{t} c_1(\mathcal{O}(b_k)),$$

a contradiction. This concludes the proof.

6. Character varieties and the Eichler-Shimura isomorphism

Fixing a Fuchsian group of the first kind $\Gamma$ together with a set of parabolic and elliptic weights $\mathcal{W} = \{W_i\}_{i=1}^{n+m}$, leads to the construction of the associated unitary character variety

$$\mathcal{X}_w = \text{Hom}(\Gamma, U(r))^0 / U(r),$$

where the superscript 0 stands for the admissible irreducible unitary representations, i.e., the irreducible representations whose parabolic and elliptic generators have prescribed weights $\mathcal{W}$. When nonempty, $\mathcal{X}_w$ is a real-analytic manifold of dimension $2r^2(g-1) + 2 + \sum_{i=1}^{n+m} \dim \mathcal{F}_r$, where

$$\mathcal{F}_r = U(r)/Z\left(e^{2\pi \sqrt{-1} W_i}\right),$$

5The restriction $i \neq n$ can be removed with the action of $\text{PSL}(2, \mathbb{C})$ on the choice of $J$.

6In the case $g = 0$, necessary and sufficient conditions under which such a choice admits a non-empty moduli space are described in [Bel01, Bis02].
are the partial flag manifolds associated with each weight matrix. The character variety can be constructed in terms of the manifold $\mathcal{P} \subset \psi^{-1}(I)$ of regular points for the map

$$
\psi : U(r)^{2g} \times \prod_{i=1}^{n+m} \mathcal{F}_i \to SU(r),
$$

$$(M_1, N_1, \ldots, M_g, N_g, [U_1], \ldots, [U_{n+m}])$$

$$\mapsto \prod_{i=1}^{g} (M_i N_i M_i^{-1} N_i^{-1}) \prod_{i=1}^{n+m} U_i e^{2\pi \sqrt{-1} W_i U_i^{-1}}$$

modulo the free and proper action of $\text{PSU}(r)$ corresponding to conjugation of representations, since it can be verified that a point in $\psi^{-1}(I)$ is regular if and only if its associated representation is irreducible, and the latter condition is preserved under conjugation (cf. [NS64, MS80]). The tangent space at a point $[\rho]$ of such quotient is in correspondence with the first parabolic cohomology group $H^1_P(\Gamma, \text{Ad} \rho)$ (see [Shi71] for details on this group).

An Eichler integral of weight 0 with the representation $\text{Ad} \rho_C$ is a holomorphic function $\mathcal{E} : \mathbb{H} \to \text{End} \mathbb{C}^r$ satisfying

$$\mathcal{E}(\gamma \tau) = \rho(\gamma) \mathcal{E}(\tau) \rho(\gamma)^{-1} + z(\gamma), \quad \gamma \in \Gamma, \tau \in \mathbb{H},$$

where $z : \Gamma \to \text{End} \mathbb{C}^r$ satisfies the 1-cocycle condition

$$z(\gamma_1 \gamma_2) = z(\gamma_1) + \rho(\gamma_1) z(\gamma_2) \rho(\gamma_1)^{-1}, \quad \gamma_1, \gamma_2 \in \Gamma.$$  

If $\mathcal{E}$ is an Eichler integral of weight 0, then $\mathcal{E}'$ is a cusp form of weight 2. Parabolic cohomology is related to cusp forms by means of the Eichler-Shimura isomorphism [Shi71]

$$\mathcal{E}_2(\Gamma, \text{Ad} \rho_C) \cong H^1_P(\Gamma, \text{Ad} \rho),$$

given explicitly by

$$\mathcal{L}_{ES}(f) = [\text{Re}(z_f)] = [(z_f - z_f^*)/2]$$

where $z_f$ is any 1-cocycle associated to an antiderivative of $f$. The Eichler-Shimura isomorphism is an isomorphism of real vector spaces, thus turning the character variety into an almost complex manifold. According to Mehta-Seshadri [MS80], the moduli space $\mathcal{M}_w$ of stable parabolic bundles with given parabolic weights $\mathcal{W}^7$ is isomorphic (as a set) to $\mathcal{K}_w$, and the infinitesimal deformations of a given stable parabolic bundle $^8 \mathcal{E}_s \cong (E_\rho)_s$ are modeled by

$$H^1(\text{Par End} \mathcal{E}) \cong H^1(E_{\text{Ad} \rho_C}),$$

---

7From the point of view of the uniformization theory, elliptic generators can also be considered in the correspondence between stable parabolic bundles and irreducible unitary representations after projection in a suitable branched covering of $\mathbb{H}$, as in this case the parabolic data would only have an additional finite order restriction.

8The subscript $^*$ indicates that the vector bundle $\mathcal{E}$ is endowed with a parabolic structure.
which also plays the role of the tangent space at $E_*$ in the moduli space. In virtue of the Petersson inner product and Serre duality, theorem 1, and that $\text{Ad} \rho_\mathcal{C}$ is self-dual, i.e., equivalent to its contragradient representation, we obtain

\begin{equation}
H^1(\text{Par End } E) \cong \mathfrak{S}_2(\Gamma, \text{Ad} \rho_\mathcal{C})^*.
\end{equation}

Recall that it is possible to introduce complex coordinates analogous to the Bers coordinates on Teichmüller spaces. Namely, given $\nu \in \mathfrak{S}_2(\Gamma, \text{Ad} \rho_\mathcal{C})^*$ and $\epsilon \in \mathbb{C}$ sufficiently small, the differential equation

\[ f^{-1}f_\epsilon = \epsilon \nu \]

admits a solution $f^{\epsilon \nu} : \mathbb{H} \to \text{GL}(r, \mathbb{C})$ satisfying $f^{\epsilon \nu}(\gamma \tau) = \rho^{\epsilon \nu}(\gamma) f^{\epsilon \nu}(\tau) \rho(\gamma)^{-1}$ for all $\gamma \in \Gamma$, with $\rho^{\epsilon \nu}$ unitary and irreducible (see [TZ08] for details and normalizations). Let us call $\mathcal{F}$ the map

\[ \mathcal{F} : \mathcal{M}_w \to \mathcal{K}_w, \]

given by sending an equivalence class of stable parabolic bundles $[E_*]$ to the equivalence class of irreducible unitary representations $[\rho]$ so that $[E_*] = [(E\rho)_*]$.

**Theorem 2.** With respect to the complex coordinates given by the choice of an arbitrary basis in $\mathfrak{S}_2(\Gamma, \text{Ad} \rho_\mathcal{C})^*$, the differential of $\mathcal{F}$ at the point $[(E\rho)_*]$ is given by

\begin{equation}
(6.2) \quad d\mathcal{F}_{[(E\rho)_*]}(\nu) = -2\mathcal{L}_{ES}(\nu^*).
\end{equation}

**Proof.** Since the complex line $\text{Span}_\mathbb{C}\{\nu\} \subset \mathfrak{S}_2(\Gamma, \text{Ad} \rho_\mathcal{C})^*$ corresponds to the real plane $\text{Span}_\mathbb{R}\{\nu, \sqrt{-1}\nu\}$, it is enough to determine the image of $\nu, \sqrt{-1}\nu$, as tangent vectors at $[(E\rho)_*]$, under the differential $d\mathcal{F}$. Let $\epsilon = x + \sqrt{-1}y$ be sufficiently small, and $[\rho^{\epsilon \nu}]$ the associated irreducible unitary representation in $\mathcal{K}_w$. Then

\[ d\mathcal{F}_{[(E\rho)_*]}(\nu) = (\rho_x|_{\epsilon=0}) \rho^{-1}, \quad d\mathcal{F}_{[(E\rho)_*]}(\sqrt{-1}\nu) = (\rho_y|_{\epsilon=0}) \rho^{-1}, \]

as unitary parabolic cocycles. Let $\mathcal{E}_- = -\frac{\partial f^{\epsilon \nu}}{\partial \epsilon}|_{\epsilon=0}$ and $\dot{\rho}_- = \frac{\partial \rho^{\epsilon \nu}}{\partial \epsilon}|_{\epsilon=0}$.

Then $\mathcal{E}_-$ is an Eichler integral of weight 0 with the representation $\text{Ad} \rho_\mathcal{C}$, satisfying $\mathcal{E}_-^* = \nu^*$ (see [MT14] for details on the proof) and, in particular, its $\text{End} \mathcal{C}_\epsilon$-parabolic 1-cocycle is $z_{\nu^*} = -\dot{\rho}_- \rho^{-1}$. It readily follows from the chain rule that

\[ (\rho_x|_{\epsilon=0}) \rho^{-1} = z_{\nu^*}^* - z_{\nu^*} = -2\mathcal{L}_{ES}(\nu^*), \]

and

\[ (\rho_y|_{\epsilon=0}) \rho^{-1} = \sqrt{-1}(z_{\nu^*} + z_{\nu^*}^*) = -2\mathcal{L}_{ES}(-\sqrt{-1}\nu^*). \]

\[ \square \]
Remark 4. Theorem 2 shows indirectly that the almost complex structure on $\mathcal{H}_w$ induced by the Eichler-Shimura isomorphism (in the same spirit of [NS64]) would be integrable, thus determining a complex structure on the character variety.

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References

[Bel01] Prakash Belkale, Local systems on $\mathbb{P}^1 - S$ for $S$ a finite set, Compos. Math. 129 (2001), 67–86.
[Bis02] Indranil Biswas, A criterion for the existence of a flat connection on a parabolic vector bundle, Adv. geom. 2 (2002), 231–241.
[Hec27] Erich Hecke, Theorie der Eisensteinischen Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik, Abh. Math. Sem. Univ. Hamburg 5 (1927), 199–224.
[Hej83] Dennis A. Hejhal, The Selberg trace formula for $\text{PSL}(2, \mathbb{R})$ vol. 2., Lecture Notes in Mathematics 1001. Springer-Verlag, Berlin (1983).
[Kra72] Irwin Kra, Automorphic forms and Kleinian groups, W. A. Benjamin, Inc. (1972).
[Kub73] Tomio Kubota, Elementary theory of Eisenstein series, John Wiley & sons (1973).
[Leh64] Joseph Lehner, Discontinuous groups and automorphic functions, Mathematical Surveys, No. VIII American Mathematical Society, Providence, R.I. (1964).
[MS80] V. B. Mehta and C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures, Math. Ann. 248 (1980), 205–239.
[MT14] C. Meneses and L. Takhtajan, Singular connections, WZNW action, and moduli of parabolic bundles on the sphere, preprint (2014).
[NS64] M. S. Narasimhan and C. S. Seshadri, Holomorphic vector bundles on a compact riemann surface, Math. Ann. 155 (1964), 69–80.
[NS65] , Stable and unitary vector bundles on a compact riemann surface, Ann. of Math. (2) 82 (1965), 540–567.
[Pet40] Hans Petersson, über eine Metrisierung der automorphen Formen und die Theorie der Poincaré’schen Reihen, Math. Ann. 117 (1940), 453–537.
[Pet48] , Automorphe Formen als metrische Invarianten. I, II, Math. Nachr. 1 (1948), 158–212, 218–257.
[Shi71] Goro Shimura, Introduction to the arithmetic theory of automorphic forms, Princeton University Press (1971).
[TZ08] Leon Takhtajan and Peter Zograf, The first Chern form on moduli of parabolic bundles, Math. Ann. 341 (2008), no. 1, 113–135.
[Wei38] André Weil, Généralisation des fonctions abéliennes, J. Math. Pures Appl. 17 (1938), 47–87.