Second-order matter perturbations in a $\Lambda$CDM cosmology and non-Gaussianity

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Abstract
We obtain exact expressions for the effect of primordial non-Gaussianity on the matter density perturbation up to second order in a $\Lambda$CDM cosmology, fully accounting for the general relativistic corrections arising on scales comparable with the Hubble radius. We present our results both in the Poisson gauge and in the comoving and synchronous gauge, which are relevant for comparison to different cosmological observables.

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1. Introduction

In this paper we discuss how primordial non-Gaussianity (NG) in the cosmological perturbations is left imprinted in the large-scale structure (LSS) of the universe in a $\Lambda$CDM cosmology. We show how the information on the primordial non-Gaussianity, set on super-Hubble scales, flows into smaller scales through a complete general relativistic (GR) computation. Primordial NG thus leaves an observable imprint in the LSS. Another interesting finding is that, on sufficiently large scales, there is another additional source of non-Gaussianity which arises from GR corrections, the leading contributions of which are post-Newtonian terms as first pointed out in [1]. The importance of the signatures of primordial non-Gaussianity in the evolution of the matter density perturbations is due to the fact that future high-precision measurements of the statistics of the dark matter density will allow us to pin down the primordial non-Gaussianity, thus representing a tool complementary to studies of the cosmic microwave background (CMB) anisotropies. It is beyond the scope of this paper to go into the details of the various theoretical and observational methods related to this issue. This
paper will then serve as a basic guideline to capture the starting point expressions that relates primordial NG to the dark matter density and gravitational potentials, outlining some specific non-Gaussian signatures in the LSS that can be potentially interesting. We perform our calculations assuming a flat universe with pressure-less matter, i.e. cold dark matter (CDM) plus non-relativistic ordinary matter and a cosmological constant, hereafter ΛCDM cosmology. We present our results both in the Poisson gauge and in the comoving time-orthogonal gauge, which are relevant for comparison to observations.

The primordial NG considered in our analysis is set at primordial (inflationary) epochs on large (super-Hubble) scales. At later times cosmological perturbations re-enter the Hubble radius during the radiation or during the matter- and dark energy-dominated epochs. For scales re-entering the Hubble radius during the radiation dominated era one should include the radiation in the evolution equations, thus using a complete second-order matter transfer function also for those scales. A detailed treatment of it has been given in [2, 3]. The evolution of the dark matter perturbations up to second-order accounting for a radiation-dominated epoch, both analytically and numerically, is also investigated in [4, 5]. Here we will focus on large scales for which the effects arising during the radiation-dominated epoch can be neglected.

The plan of the paper is as follows. In section 2 we give the general form of the perturbed line element and we introduce the gauge-invariant curvature perturbation of uniform density hypersurface at first and second order, which is used to provide the inflationary initial conditions, including the effect of primordial non-Gaussianity. In section 3 we derive the second-order expression for the density perturbation in a ΛCDM cosmology, in the Poisson gauge, taking into full account both NG initial conditions and post-Newtonian corrections arising from the nonlinear evolution of perturbations according to the fully general relativistic treatment. Section 4 contains a similar calculation in the comoving-synchronous gauge. Section 5 contains our concluding remarks.

2. Metric perturbations and primordial non-Gaussianity

We consider a spatially flat universe filled with a cosmological constant Λ and a non-relativistic pressureless fluid of cold dark matter (CDM), whose energy–momentum tensor reads $T^{\mu\nu} = \rho u^\mu u_\nu$. Following the notations of [6], the perturbed line element around a spatially flat FRW background reads

$$ds^2 = a^2(\tau)\{- (1 + 2\phi) d\tau^2 + 2\dot{\phi} d\tau dx^i + [(1 - 2\psi) \delta_{ij} + \hat{\chi}_{ij}] dx^i dx^j\},$$  

where $a(\tau)$ is the scale factor as a function of the conformal time $\tau$. Here each perturbation quantity can be expanded into a first-order (linear) part and a second-order contribution, as for example, the gravitational potential $\phi = \phi^{(1)} + \phi^{(2)}/2$. Up to now we have not chosen any particular gauge. We can employ the standard split of the perturbations into the so-called scalar, vector and tensor parts, according to their transformation properties with respect to the three-dimensional space with metric $\delta_{ij}$, where scalar parts are related to a scalar potential, vector parts to transverse (divergence-free) vectors and tensor parts to transverse trace-free tensors. Thus, $\phi$ and $\psi$, the gravitational potentials, are scalar perturbations, and for instance, $\omega_i^{(r)} = \partial_i \omega^{(r)} + \omega_i^{(r)}$, where $\omega^{(r)}$ is the scalar part and $\omega_i^{(r)}$ is a transverse vector, i.e. $\nabla^i \omega_i^{(r)} = 0$ ($(r) = (1, 2)$ stand for the $r$th order of the perturbations). The symmetric traceless tensor $\hat{\chi}_{ij}$ generally contains a scalar, vector and tensor contribution, namely $\hat{\chi}_{ij} = D_{ij} \chi + \partial_i \chi_{j} + \partial_j \chi_{i} + \chi_{ij}$, where $D_{ij} \equiv \partial_i \partial_j - (1/3) \nabla^2 \delta_{ij}$, $\chi_i$ is a solenoidal
vector ($\partial^a x_i = 0$) and $\chi_{ij}$ represents a traceless and transverse (i.e. $\partial^a \chi_{ij} = 0$) tensor mode.

As for the matter component we split the mass density into a homogeneous $\rho(\tau)$ and a perturbed part as $\rho(x, \tau) = \rho(\tau)(1 + \delta^{(1)} + \delta^{(2)}/2)$ and we write the four velocity as $u^\mu = (\delta^{a} + v^a)/a$ with $u^a u_a = -1$ and $v^a = v^{(1)a} + v^{(2)a}/2$.

The Friedmann background equations are

$$3H^2 = a^2(8\pi G\rho(\tau) + \Lambda),$$

(2)

and

$$\rho'(\tau) = -3H\rho(\tau),$$

(3)

where a prime stands for differentiation with respect to conformal time, and $H = a'/a$. The matter density parameter is $\Omega_{\text{m}}(\tau) = 8\pi G a^2(\tau)\rho(\tau)/(3H^2(\tau))$.

Before recalling how one can parametrize the primordial non-Gaussianity, we need to provide a general definition for the amplitude of non-Gaussianity characterizing the matter density contrast beyond the usual second-order Newtonian contributions. It proves convenient to introduce an effective gravitational potential obeying the Poisson equation

$$-\nabla^2 \Phi = \frac{3}{2}\Omega_{\text{m}} H^2 \delta,$$

(4)

and at an initial epoch, deep in matter domination, we write

$$\Phi_{\text{in}} = \Phi^{(1)}_{\text{in}} + f_{\text{NL}}(\Phi^{(1)2}_{\text{in}} - (\Phi^{(1)}_{\text{in}})^2),$$

(5)

with the dimensionless nonlinearity parameter $f_{\text{NL}}$ setting the level of quadratic non-Gaussianity, and $\Phi \propto g(\tau), g(\tau)$ being the usual growth suppression factor (see section 3). Therefore, we will write the matter density contrast in Fourier space in terms of the linear density contrast by defining the kernel $K_\delta(k_1, k_2; \tau)$ depending on the wavevector of the perturbation modes as

$$\delta_k(\tau) = \delta^{(1)}_k(\tau) + \frac{1}{2}\delta^{(2)}_k(\tau)$$

$$= \delta^{(1)}_k(\tau) + \int \frac{d^3k_1d^3k_2}{(2\pi)^3} K_\delta(k_1, k_2; \tau) \delta^{(1)}(\tau) \delta^{(1)}(\tau) \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}).$$

(6)

We can write the kernel as

$$K_\delta(k_1, k_2; \tau) = \left[k^2 + 3\Omega_{\text{m}} H^2 f_{\text{NL}}(\mathbf{k}_1, \mathbf{k}_2, \tau) \frac{g_{\text{in}}}{g(\tau)} \frac{k^2}{k_1^2 k_2^2} \right] ,$$

(7)

where $k^2 = |\mathbf{k}_1 + \mathbf{k}_2|^2$ and $K_\delta(k_1, k_2; \tau)$ is the second-order Newtonian kernel.

2.1. Primordial non-Gaussianity and initial conditions

We conveniently fix the initial conditions at the time when the cosmological perturbations relevant for LSS are outside the horizon. A standard and convenient way to account for any initial primordial non-Gaussianity is to consider the curvature perturbation of uniform density hypersurfaces $\zeta = \zeta^{(1)} + \zeta^{(2)}/2 + \cdots$, where $\zeta^{(1)} = -\psi^{(1)} - \mathcal{H}\rho^{(1)}/\rho'$ and at second order [7, 12]

$$-\zeta^{(2)} = \bar{\psi}^{(2)} + \mathcal{H}\rho\bar{\psi}^{(2)}/\rho' - 2\mathcal{H}\rho\delta^{(1)}\rho/\rho' - 2\mathcal{H}\rho\bar{\psi}^{(1)} + 2\mathcal{H}\bar{\psi}^{(1)}$$

$$+ \left(\frac{\delta^{(1)}\rho}{\rho'}\right)^2 \left(\mathcal{H}\rho'' - \mathcal{H}' - 2\mathcal{H}^2\right),$$

(8)

In what follows, for our purposes we will neglect linear vector modes since they are not produced in standard mechanisms for the generation of cosmological perturbations (as inflation), and we also neglect tensor modes at linear order, since they give a negligible contribution to LSS formation.
where \( \dot{\psi} = \frac{1}{2} \psi' + \nabla^2 \chi / 6 \). This is a gauge-invariant quantity which remains constant on super-horizon scales after it has been generated during a primordial epoch (and possible isocurvature perturbations are no longer present). Therefore, \( \zeta(2) \) provides all the necessary information about the primordial level of non-Gaussianity. The conserved value of the curvature perturbation \( \zeta \) allows us to set the initial conditions for the metric and matter perturbations accounting for the primordial contributions. Different scenarios for the generation are characterized by different values of \( \zeta(2) \), while the post-inflationary nonlinear evolution due to gravity is common to all of them [7, 13–15]. For example, in standard single-field inflation \( \zeta(2) \) is generated during inflation and its value is \( \zeta(2) = 2(\zeta^{(1)}_1)^2 + O(\epsilon, \eta) \) [13, 16, 17], where \( \epsilon \) and \( \eta \) are the usual slow-roll parameters. Therefore, it turns out to be convenient to parametrize the primordial non-Gaussianity level in terms of the conserved curvature perturbation as in [20]

\[
\zeta(2) = 2a_{NL}(\zeta^{(1)})^2,
\]

where the parameter \( a_{NL} \) depends on the physics of a given scenario, and in full generality it can depend on scale and configuration. For example in the standard scenario \( a_{NL} \approx 1 \), while in the curvaton case (see e.g. [7]) \( a_{NL} = (3/4r) - r/2 \), where \( r \approx (\rho_\sigma/\rho)_D \) is the relative curvaton contribution to the total energy density at curvaton decay. In the minimal picture for the inhomogeneous reheating scenario, \( a_{NL} = 1/4 \). For other scenarios we refer the reader to [7–11]. One of the best techniques to detect or constrain the primordial large-scale non-Gaussianity is the analysis of the CMB anisotropies, for example by studying the CMB bispectrum [7, 18, 19]. The nonlinearity parameter \( f_{NL} \) as defined in equation (5) is defined also to make contact with the primordial non-Gaussianity entering in the CMB anisotropies. For large primordial non-Gaussianity, when \( |a_{NL}| \gg 1 \), \( f_{NL} \approx 5a_{NL}/3 \) (see [7, 20]).

3. Dark matter density perturbations at second order: Poisson gauge

The goal of this section is to compute the matter density contrast in the Poisson gauge [21], i.e. the generalization beyond linear order of the longitudinal gauge, by which a more direct comparison with the standard Newtonian approximation adopted in the interpretation of LSS observations and in N-body simulations in Eulerian coordinates is possible (see, however, [22] for a critical discussion of the potential problems connected to the use of the Poisson gauge on scales comparable with the Hubble radius). In the Poisson gauge one scalar degree of freedom is eliminated from the \( g_{00} \) component of the metric, and one scalar and two vector degrees of freedom are eliminated from \( g_{ij} \).

Let us briefly recall the results for the linear perturbations in the case of a non-vanishing cosmological \( \Lambda \) term. At linear order the traceless part of the \((i,j)\)-components of Einstein equations gives \( \phi^{(1)} = \psi^{(1)} = \psi \). Its trace gives the evolution equation for the linear scalar potential \( \psi \):

\[
\ddot{\psi} + 3H \dot{\psi} + a^2 \Delta \psi = 0.
\]

Selecting only the growing mode solution one can write

\[
\psi(x, \tau) = g(\tau) \psi_0(x),
\]

where \( \psi_0 \) is the peculiar gravitational potential linearly extrapolated to the present time \( (\tau_0) \) and \( g(\tau) = \dot{D}_\Lambda(\tau)/a(\tau) \) is the so-called growth-suppression factor, where \( \dot{D}_\Lambda(\tau) \) is the usual linear growing mode of density fluctuations in the Newtonian limit, i.e. the non-decaying solution of the differential equation

\[
D'' + \mathcal{H} D' - \frac{3}{2} \mathcal{H}^2 \Omega_{\Lambda} D = 0.
\]
We normalize the growth factor so that \( D_+(t_0) = a_0 = 1 \). The exact form of \( g \) can be found in \([23–25]\). In the \( \Lambda = 0 \) case \( g = 1 \). A very good approximation for \( g \) as a function of the redshift \( z \) is given in \([23, 24]\)

\[
g \propto \Omega_m \left[ \Omega_m^{2/7} - \Omega_\Lambda + (1 + \Omega_m/2)(1 + \Omega_\Lambda/70) \right]^{-1},
\]

with \( \Omega_m = \Omega_{om}(1 + z)^3/E^2(z), \Omega_\Lambda = \Omega_{\Lambda o}/E^2(z), E(z) \equiv (1 + z)\mathcal{H}(z)/\mathcal{H}_0 = [\Omega_{om}(1 + z)^3 + \Omega_{\Lambda o}]^{1/2} \) and \( \Omega_m, \Omega_\Lambda = 1 - \Omega_{om}, \Omega_{\Lambda o} \), the present-day density parameters of non-relativistic matter and cosmological constant, respectively. According to our normalization, \( g(z = 0) = 1 \). The energy and momentum constraints provide the density and velocity fluctuations in terms of \( \psi \) (see, for example, \([13]\) and \([26, 27]\) for the \( \Lambda \) case):

\[
\delta^{(1)} = \frac{1}{4\pi G a^2 \bar{\rho}} [\nabla^2 \psi - 3\mathcal{H}(\psi' + \mathcal{H}\psi)],
\]

\[
\nu_i^{(1)} = -\frac{1}{4\pi G a^2 \bar{\rho}} \partial_i(\psi' + \mathcal{H}\psi).
\]

In a similar way the expression of the second-order matter density contrast can be computed starting from the energy constraint given by the (0-0)-Einstein equation, once the evolution of the gravitational potentials is known. The latter has been already computed in detail in \([28]\) and in the following we summarize the main results.

The evolution equation for the second-order gravitational potential \( \psi^{(2)} \) is obtained from the trace of the \((i-j)\)-Einstein equations\(^5\)

\[
\psi^{(2)} + 3\mathcal{H}\psi^{(2)} + a^2\Lambda \psi^{(2)} = S(\tau),
\]

where \( S(\tau) \) is the source term

\[
S(\tau) = g^2 \Omega_m \mathcal{H}^2 \left[ \frac{(f - 1)^2}{\Omega_m} \psi_0^2 + 2\left( \frac{1}{\Omega_m} - \frac{3}{3} \right) a_0(x) \right. \\
\left. + g^2 \frac{4}{3} \left( \frac{f^2}{\Omega_m} + \frac{3}{2} \right) \nabla^{-2} \partial_i \partial_j (\partial_i \psi_0 \partial_j \psi_0) - (\partial_i \psi_0 \partial_j \psi_0) \right],
\]

where, for simplicity of notation, we have introduced

\[
a_0(x) = [\nabla^{-2} (\partial_i \psi_0 \partial_j \psi_0) - 3\nabla^{-4} \partial_i \partial_j (\partial_i \psi_0 \partial_j \psi_0)],
\]

and

\[
f(\Omega_m) = \frac{d \ln D_+}{d \ln a} = 1 + \frac{g'(\tau)}{Hg(\tau)},
\]

which can be written as a function of \( \Omega_m \) as \( f(\Omega_m) \approx \Omega_m(z)^{2/7} \) \([23, 24]\). In equation (17) \( \nabla^{-2} \) stands for the inverse of the Laplacian operator.

The solution of equation (16) is then obtained using Green’s method with growing and decaying solutions of the homogeneous equation \( \psi_+(\tau) = g(\tau) \) and \( \psi_-(\tau) = h(\tau)/a^2(\tau) \), respectively. The second-order gravitational potentials then read \([28]\)

\[
\psi^{(2)}(\tau) = (B_1(\tau) - 2g(\tau)g_m - \frac{10}{3}(\Omega_{mNL} - 1)g(\tau)g_m) \psi_0^2 + (B_2(\tau) - \frac{4}{5}g(\tau)g_m) a_0(x) \\
+ B_3(\tau) \nabla^{-2} \partial_i \partial_j (\partial_i \psi_0 \partial_j \psi_0) + B_4(\tau) \partial_i \partial_j \psi_0 \partial_i \psi_0,
\]

\(^5\) The second-order perturbations of the Einstein tensor \( G^{ij}_m \) can be found for any gauge in appendix A of \([7, 16]\) and directly in the Poisson gauge, e.g. in \([2, 3]\). The perturbations of the energy–momentum tensor up to second order in the Poisson gauge have been computed in \([13]\) for a general perfect fluid (see \([7]\) for expressions in any gauge).
\[ \phi^{(2)}(\tau) = (B_1(\tau) + 4g^2(\tau) - 2g(\tau)g_m - \frac{10}{3}(\alpha_{NL} - 1)g(\tau)g_m)\psi_0^2 \\
+ \left[ B_2(\tau) + \frac{4}{3}g^2(\tau)\left( e(\tau) + \frac{1}{2} \right) - \frac{4}{3}g(\tau)g_m \right] \alpha_0(x) \\
+ B_3(\tau)\nabla^2 \partial_i \partial_j \left( \partial^i \varphi_0 \partial^j \varphi_0 \right) + B_4(\tau)\partial^i \varphi_0 \partial_i \varphi_0, \tag{21} \]

where we have introduced \( B_i(\tau) = \mathcal{H}_0^{-2} (f_0 + 3\Omega_{0m}/2)^{-1} \cdot \dot{B}_i(\tau) \) with the following definitions:

\[ \dot{B}_1(\tau) = \int_{\tau_n}^{\tau} d\tilde{\tau} \mathcal{H}^2(\tilde{\tau})(f(\tilde{\tau}) - 1)^2 C(\tau, \tilde{\tau}), \tag{22} \]

\[ \dot{B}_2(\tau) = 2 \int_{\tau_n}^{\tau} d\tilde{\tau} \mathcal{H}^2(\tilde{\tau})[2(f(\tilde{\tau}) - 1)^2 - 3 + 3\Omega_m(\tilde{\tau})]C(\tau, \tilde{\tau}), \tag{23} \]

\[ \dot{B}_3(\tau) = \frac{4}{3} \int_{\tau_n}^{\tau} d\tilde{\tau} \left( e(\tilde{\tau}) + \frac{3}{2} \right) C(\tau, \tilde{\tau}), \quad \dot{B}_4(\tau) = -\int_{\tau_n}^{\tau} d\tilde{\tau} C(\tau, \tilde{\tau}) \tag{24} \]

and

\[ C(\tau, \tilde{\tau}) = g^2(\tilde{\tau})a(\tilde{\tau}) \left[ g(\tau)\mathcal{H}(\tilde{\tau}) - g(\tilde{\tau})\alpha_0^2(\tilde{\tau})\mathcal{H}(\tau) \right], \tag{25} \]

with

\[ e(\Omega_m) \equiv \frac{f^2(\Omega_m)}{\Omega_m}. \tag{26} \]

The expression for \( \phi^{(2)} \) is obtained from the relation between \( \psi^{(2)} \) and \( \phi^{(2)} \):

\[ \nabla^2 \nabla^2 \psi^{(2)} = \nabla^2 \nabla^2 \phi^{(2)} - 4g^2 \nabla^2 \phi_0 \phi_0 - \frac{4}{3}g^2 \left( e + \frac{1}{2} \right) \left[ \nabla^2 (\partial^i \varphi_0 \partial^j \varphi_0) - 3\partial_i \partial_j (\partial^i \varphi_0 \partial^j \varphi_0) \right]. \tag{27} \]

which follows from the traceless part of the \((i,j)\)-component of Einstein equations [28]. Here \( \varphi_m = g_m \varphi_0 \) represents the initial condition taken at some time \( \tau_m \) deep in the matter-dominated era on super-horizon scales. Solutions (20) and (21) properly account for the non-Gaussian initial conditions parameterized by \( \alpha_{NL} - 1 \). These are obtained using the expression for \( \zeta^{(2)} \) during the matter-dominated epoch together with equation (27) and the second-order (0-0)-component of Einstein equations (both evaluated for a matter-dominated epoch), so that one can express \( \phi^{(2)} \) in terms of \( \zeta^{(2)} \) of equation (9), where \( \zeta^{(1)} = -3\varphi_m/3 \) (see [7, 13, 15, 20, 28]).

Before proceeding further note that in the expression for the second-order gravitational potentials of equation (20) and (21) we recognize two contributions. The term which dominates on small scales, \( \left[ B_3(\tau)\nabla^2 \partial_i \partial_j (\partial^i \varphi_0 \partial^j \varphi_0) + B_4(\tau)\partial^i \varphi_0 \partial_i \varphi_0 \right] \), which gives rise to the second-order Newtonian piece and is insensitive to any non-Gaussianity in the initial conditions. The remaining pieces in equations (20) and (21) correspond to contributions which tend to dominate on large scales with respect to those characterizing the Newtonian contribution, and whose origin is purely relativistic. In particular these are the pieces carrying the information on primordial non-Gaussianity. For a flat matter-dominated (Einstein–de Sitter) universe \( g(\tau) = 1 \) and \( B_1(\tau) = B_2(\tau) = 0 \), while \( B_3(\tau) \to (5/21)\tau^2 \) and \( B_4(\tau) \to -\tau^2/14 \), so that one recovers the expressions of [3] (see also [29]). In [27] second-order cosmological perturbations have been computed in the \( \Lambda \neq 0 \) case from the synchronous to the Poisson gauge, thus extending the analysis of [6], and the CMB temperature anisotropies induced by metric perturbations have been also considered by applying the expressions of [30] (see
also [28]). However, an important point to note is that both [6, 30] and [27] disregard any primordial nonlinear contribution from inflation.

The matter density contrast at second order can now be calculated from the (0-0)-Einstein equation

\[ 3\mathcal{H}\psi^{(2)} + 3\mathcal{H}^2 \phi^{(2)} - \nabla^2 \psi^{(2)} + 12\mathcal{H}\psi^{(1)}\phi^{(1)} - 2\nabla^2 (\psi^{(1)})^2 - 12\mathcal{H}^2 (\phi^{(1)})^2 - 12\mathcal{H}\phi^{(1)}\psi^{(1)} - 3(\psi^{(1)})^2 + \partial_i \psi^{(1)} \partial^i \psi^{(1)} - 4\psi^{(1)} \nabla^2 \psi^{(1)} = -3\mathcal{H}^2 \Omega_m \left( \frac{b^{(2)}}{2} + \nu^{(12)} \right). \]  

We thus arrive at the density contrast in the Poisson gauge

\[ \delta^{(2)} = \frac{1}{\Omega_m} \left[ (f - 1)^2 - \frac{2}{\mathcal{H}} \cdot \frac{A'(\tau)}{\mathcal{H}} - \frac{2}{\mathcal{H}} \cdot \frac{A(\tau) - 1}{\mathcal{H}} \right] \phi^{(2)} - \frac{2}{\mathcal{H}^2 \Omega_m g^2} \left[ \frac{B_2(\tau)}{\mathcal{H}} - \frac{4}{3} \mathcal{H} g^m + B_3(\tau) \right] \nabla^2 \phi^{(2)} + \cdots \]

\[ = \frac{2}{\Omega_m g^2} \left[ \frac{B_1(\tau)}{\mathcal{H}} \right] \nabla^2 \phi^{(2)} + \frac{2}{3\Omega_m g^2} \left[ 0 + \frac{4}{3} \mathcal{H} g^m - \frac{4}{3} \mathcal{H}^2 B_4(\tau) \right] \nabla^2 \phi^{(2)}, \]

where \( A(\tau) \equiv \left[ B_1(\tau) - 2gg_m - \frac{10}{3}(a_{NL} - 1)gg_m + 2g^2 \right] \nabla^2 \phi^{(2)} \) and \( \alpha(x, \tau) \) has the same expressions as \( \alpha_0(x) \) introduced in equation (18) with \( \phi \) in place of \( \phi_0 \). Note that we have explicitly verified that equation (29) in the limit of an Einstein–de Sitter universe recovers the expression for the matter density contrast obtained in [1].

Equation (29) is the main result of this section. It shows how the primordial NG, which is initially generated on large scales, is transferred to the density contrast on subhorizon scales. The expression for the density contrast is made of three contributions: a second-order Newtonian piece (the last two terms in equation (29), proportional to \( \tau^4 \) in an Einstein–de Sitter universe) which is insensitive to the nonlinearities in the initial conditions; a post-Newtonian (PN) piece (related to two gradient terms of the gravitational potential) which carries the most relevant information on primordial NG; the super-horizon terms (corresponding to the first two lines of equation (29)). Our findings show in a clear way that the information on the primordial NG set on super-Hubble scales flows into the PN terms, leaving an observable imprint in the LSS. Another interesting result which shows up in equation (29) is the presence in the post-Newtonian term of contributions different from the primordial NG which are due to weakly nonlinear corrections which switch on when the modes cross inside the Hubble radius and which can constitute an interesting additional source of non-Gaussianity since they can probe large-scale GR corrections.

3.1. The nonlinearity parameter \( f_{NL} \)

We can rewrite the matter density contrast in Fourier space in terms of the linear density contrast as in equation (6). We find

\[ K_n(N, k_1, k_2; \tau) = \frac{3}{4} \Omega_m \mathcal{H}^2 \left[ \frac{B_3(\tau)}{k_1^4 k_2^2} + \frac{B_4(\tau)}{k_1^2 k_2^4} \right], \]

where \( K_n(N, k_1, k_2; \tau) \) is the nonlinear contribution from inflation.

The results in [6, 27, 30] have initial conditions corresponding to our \( a_{NL} = 0 \).
and for the non-linearity parameter

\[
    f_{NL}^p(\mathbf{k}_1, \mathbf{k}_2; \tau) = \left[ \frac{5}{3}(a_{NL} - 1) + 1 - \frac{g}{g_{in}} - \frac{1}{2} B_1(\tau) \right] - \left( \frac{k_1^2 + k_2^2}{k^2} \right) \frac{g}{g_{in}} \\
    + \frac{(k_1 \cdot k_2)}{k^2} \left[ \frac{2}{3} \epsilon(\tau) + \frac{1}{2} \frac{g}{g_{in}} + \frac{2}{3} \frac{1}{2} \frac{B_2(\tau)}{g_{in}} + \frac{3}{2} \frac{H^2 B_2(\tau)}{g_{in}} \right] \\
    + \frac{(k \cdot k_1)(k \cdot k_2)}{k^4} \left[ \frac{3}{2} \hat{H}^2 (\tau) - 2 + \frac{3}{2} \frac{B_2(\tau)}{g_{in}} - \frac{(k \cdot k_1)(k \cdot k_2)}{k^2} \frac{k_1^2 + k_2^2}{k_1^2 k_2^2} \right] \\
    \times \frac{3}{2} \frac{H^2 f(\tau) B_2(\tau)}{g_{in}} - \frac{(k_1 \cdot k_2)}{k^2} \frac{k_1^2 + k_2^2}{k_1^2 k_2^2} \frac{3}{2} \frac{H^2 f(\tau)}{g_{in}} B_2(\tau). \tag{31}
\]

The nonlinearity parameter \( f_{NL} \) is defined via equation (4) and in this way it generalizes the standard definition of \([31]\) inferred from the Newtonian gravitational potential. In order to obtain the expressions in equations (30) and (31) we have performed an expansion in \((H/k_1)^2 \ll 1 \) up to terms \((H/k_1)^4 \) starting from equation (29). In equation (31) the primordial NG is clearly evident in the piece proportional to \((a_{NL} - 1) \). The remaining terms are due to the horizon scale post-Newtonian corrections we commented about in equation (29). The nonlinearity (NG) induced by these terms show a specific and non-trivial shape dependence that can help in detecting them. Of course their relative importance increases with the scale, and in fact it has been shown in \([32]\) that, through the large-scale halo bias techniques investigated in \([33, 34]\), these GR corrections are potentially detectable.

It is also worth noting that some of the terms entering in the nonlinearity parameter vanish in the limit of a vanishing cosmological constant (e.g. \( B_1(\tau) \) and \( B_2(\tau) \) go to zero in this limit).

A final comment on equation (6). In this expression, when the various modes are well inside the horizon, one can take the usual configuration for the linear density perturbations with \( \delta_k^{(1)}(\tau) \propto 2k^2 T(k)/3 \Omega_m H_0^2 D_s(\tau) \), where \( T(k) \) is the usual linear matter transfer function. In this way one partially accounts for the effects of the transition from a radiation- to a matter-dominated epoch. In fact a full computation would require to study up to second order the evolution of the density perturbations also during the radiation-dominated era. In this way one can recover a full matter transfer function up to second order. Details about such a computation can be found in \([3–5]\). For example, \([5]\) shows that the corrections from the full matter transfer function, when accounting for a matching at second order to the radiation epoch, give a relative correction with respect to the Newtonian kernel that is of order \( a_{eq} \sim 10^{-4} \) and that such a correction is equivalent to the effect of a primordial NG of \( f_{NL} \sim 4 \). That these small-scale corrections are small is easy to understand. By looking at equation (28) one realizes that accounting for the radiation epoch at second order the matter density perturbations from, say approximately the matter-radiation equality epoch onward, will get a correction which scales like \( \delta_k^{(2)}(eq_{eq}) \tau^2 \), which rescales the matching initial conditions to the radiation epoch. However, we expect such a term to be negligible w.r.t. the Newtonian part which scales like \( (\delta_k^{(1)}(\tau))^2 \propto \tau^4 \), and also w.r.t. a sizable primordial NG.

4. Dark matter density perturbations at second order: comoving-synchronous gauge

Let us now see how the second-order matter perturbations and gravitational potentials are obtained in the comoving-synchronous gauge.

We make use of the formalism developed in \([6, 35–37]\) which the reader is referred to for more details. The synchronous, time-orthogonal gauge is defined by setting \( g_{00} = -a^2(\tau) \) and \( g_{0i} = 0 \), so that the line element takes the form

\[
    ds^2 = a^2(\tau)[-d\tau^2 + \gamma_{ij}(\mathbf{x}, \tau) dx^i dx^j]. \tag{32}
\]
For our fluid containing irrotational, pressure-less matter plus \( \Lambda \), this also implies that the fluid four-velocity field is given by \( u^\mu = (1/a, 0, 0, 0) \), so that \( x \) represents comoving ‘Lagrangian’ coordinates for the fluid element (indeed, the possibility of making the synchronous, time-orthogonal gauge choice and comoving gauge choice simultaneously is a peculiarity of fluids with vanishing spatial pressure gradients, i.e. vanishing acceleration, which holds at any time, i.e. also beyond the linear regime). A very efficient way to write down Einstein and continuity equations is to introduce the peculiar velocity-gradient tensor \([37]\)

\[
\dot{\varphi}_{ij} = u^j_i - \frac{a'}{a} \delta_{ij} = \frac{1}{2} \gamma_{ij},
\]

where we have subtracted the isotropic Hubble flow. Here, semicolons denote covariant differentiation. From the continuity equation \( T^\mu_\nu = 0 \), we infer the exact solution for the density contrast \( \delta = \delta \rho / \rho \) \([6, 37]\):

\[
\delta(x, \tau) = (1 + \delta_0(x))\psi(x, \tau) / \psi_0(x) - 1/2 - 1,
\]

where \( \psi = \det_{ij} \). The subscript ‘0’ denotes the value of quantities evaluated at the present time.

From equation (34) it is evident that in the comoving-synchronous gauge the only independent degree of freedom is the spatial metric tensor \( \gamma_{ij} \). The energy constraint reads

\[
\dot{\varphi}_{ij} + 4H\varphi_{ij} + R = 6H^2\Omega_m \delta,
\]

where \( R_{ij} \) is the Ricci tensor associated with the spatial metric \( \gamma_{ij} \) with scalar curvature \( \mathcal{R} = R_{ij} \). The momentum constraint reads \( \dot{\varphi}_{kl} = \varphi_{ij} \), where bars stand for covariant differentiation in the three-space with metric \( \gamma_{ij} \). Finally, one can use the Raychaudhuri equation

\[
\dot{\varphi}_{ij} + 2H\varphi_{ij} + \varphi\dot{\varphi}_{ij} + \frac{1}{2} \varphi \dot{\varphi}_{kl} - \varphi^2 \delta_{ij} + R_{ij} - \frac{1}{2} \mathcal{R} \delta_{ij} = 0.
\]

Note that these equations are exact and describe the fully nonlinear evolution of cosmological perturbations (up to the time of caustic formation). In order to show how the primordial non-Gaussianities appear in in the matter density contrast, we then perform a perturbative expansion up to second order in the fluctuations of the metric.

The spatial metric tensor can be expanded as

\[
\gamma_{ij} = (1 - 2\psi^{(1)} - \chi^{(2)})\delta_{ij} + \frac{1}{2} \chi^{(2)}_{ij},
\]

where \( \chi^{(1)}_{ij} \) and \( \chi^{(2)}_{ij} \) are traceless tensors and include scalar, vector and tensor (gravitational waves) perturbations. As usual we split the density contrast into a linear and a second-order part as \( \delta(x, \tau) = \delta^{(1)}(x, \tau) + \delta^{(2)}(x, \tau) \). At linear order the growing-mode solutions in the comoving-synchronous gauge are given by

\[
\psi^{(1)}(x, \tau) = \frac{5}{3} \varphi_m(x) + \frac{2}{9H^2(\tau)\Omega_m(\tau)} \nabla^2 \varphi(x, \tau),
\]

\[
\chi^{(1)}_{ij}(x, \tau) = D_{ij} \chi^{(1)}(x, \tau), \quad \chi^{(1)}(x, \tau) = -\frac{4}{3H^2(\tau)\Omega_m(\tau)} \varphi(x, \tau),
\]

where \( \varphi(x, \tau) \) is the growing-mode scalar potential defined in equation (11). The linear density contrast \( \delta^{(1)} \) in this gauge is related to \( \varphi \) via the usual Poisson equation, namely

\[
\nabla^2 \varphi(x, \tau) = \frac{3}{2} H^2(\tau)\Omega_m(\tau) \delta^{(1)}(x, \tau).
\]
In writing $X_{ij}^{(1)}$ we have eliminated the residual gauge ambiguity of the synchronous
gauge as in [6].

We have also assumed that linear vector modes are absent, since they are
not produced in standard mechanisms for the generation of cosmological perturbations (as
inflation). We have also neglected linear tensor modes, since they play a negligible role in
LSS formation.

By perturbing equation (34) up to second order, we get
\[
\delta^{(2)} = \delta^{(2)}_0 + 3(\psi^{(2)} - \psi^{(2)}_0) + \frac{1}{2} \left( D_{s}^{2} - 1 \right) \left[ \frac{1}{2} (\nabla^2 \chi^{(1)}_0)^2 + \frac{1}{3} \partial^i \partial^j \chi^{(1)}_0 \partial_i \partial_j \chi^{(1)}_0 \right] \nonumber
- (D_s - 1) \left( 2\psi^{(1)}_0 + \frac{1}{3} \nabla^2 \chi^{(1)}_0 \right) \nabla^2 \chi^{(1)}_0 .
\]

To compute the metric perturbation $\psi^{(2)}$, we can use the evolution equation, equation (37). To this aim it proves convenient to write
\[
\psi^{(2)} = \psi^{(2)}_0 - \frac{1}{3} \delta^{(2)}_0 + \frac{1}{6} \partial^i \partial^j \chi^{(1)}_0 \partial_i \partial_j \chi^{(1)}_0 - \frac{1}{2} \left( 2\psi^{(1)}_0 + \frac{1}{3} \nabla^2 \chi^{(1)}_0 \right) \nabla^2 \chi^{(1)}_0 + \frac{1}{12} \left( \nabla^2 \chi^{(1)}_0 \right)^2 + \xi .
\]

where we have introduced the variable $\xi$ which is determined by the equation
\[
\xi'' + \mathcal{H} \xi' - \frac{3}{2} \mathcal{H}^2 \Omega_m \xi = \frac{1}{4} \nabla^2 \chi^{(1)}_0 \left[ \frac{1}{2} (\nabla^2 \chi^{(1)}_0)^2 - \left( 1 + \frac{2}{3\epsilon(\Omega_m)} \right) \partial^i \partial^j \chi^{(1)}_0 \partial_i \partial_j \chi^{(1)}_0 \right] .
\]

An approximate solution of this equation can be obtained by making use of the usual
approximation (see e.g. [38, 39]) $e(\Omega_m) \approx 1$, which holds true for reasonable values of $\Omega_m$.

Under these circumstances we can write
\[
\xi(\tau) \approx A D_s(\tau) + \frac{D_s^2(\tau)}{14} \left[ \frac{1}{2} (\nabla^2 \chi^{(1)}_0)^2 - \frac{5}{3} \partial^i \partial^j \chi^{(1)}_0 \partial_i \partial_j \chi^{(1)}_0 \right] ,
\]

where $A$ is an integration constant which can be determined by the energy constraint at
the initial time. Note that it is precisely the sub-leading, post-Newtonian term proportional
to the linear growing mode $D_s$ which brings all the relevant information about primordial
and GR-induced non-Gaussianity. It is also important to stress that the time dependence
of this term, which comes from the homogeneous solution of the above equation, is
exact, while the above approximation only affects the fastest growing Newtonian terms,
i.e. those proportional to $D_s^2$. Using this procedure and providing the initial data (formally
at $\tau \to 0$) in terms of the gauge-invariant curvature perturbation, which in this gauge reads
$\chi^{(1)}_{m} = -\psi^{(1)}_m - (1/6) \nabla^2 \chi^{(1)}_{m} = -5\psi^{(1)}_m/3$ and $\chi^{(2)}_{m} = -\psi^{(2)}_m - (1/6) \nabla^2 \chi^{(2)}_{m} = 50\psi^{(2)}_m/9$, we
finally obtain
\[
\delta^{(2)}(\tau) = \frac{100}{9\mathcal{H}^2} \left[ f(\Omega_{lm}) + \frac{3}{2} \Omega_m \right]^{-1} \left\{ D_s(\tau) \left[ \frac{3}{4} - a_{NL} \right] (\nabla \psi_m)^2 + (2 - a_{NL}) \psi_m \nabla^2 \psi_m \right\} \nonumber
+ \frac{D_s^2(\tau)}{14\mathcal{H}^2} \left[ f(\Omega_{lm}) + \frac{3}{2} \Omega_m \right]^{-1} \left\{ 5(\nabla \psi_m)^2 + 2a_{NL} \psi_m \nabla^2 \psi_m \right\} ,
\]

where we made use of the following property:
\[
\mathcal{H} D_s + \frac{1}{2} \mathcal{H}^2 \Omega_m D_s = \text{const},
\]

whose validity can be easily proven on the basis of equation (12) and of the Friedmann
equation $\mathcal{H}' = -\mathcal{H}^2 + (3/2) \mathcal{H}^2 \Omega_m = 0$. One may note that even the Newtonian part (i.e. the one
proportional to $D_s^2$) of the second-order matter perturbations differs from the Poisson gauge

\footnote{More in general, at any order $n$ in perturbation theory the scalar potentials $\psi^{(n)}$ and $\chi^{(n)}$ can be shifted by arbitrary
constant amounts $\delta \psi^{(n)}_0$ and $\delta \chi^{(n)}_0$, only provided $\delta \psi^{(n)}_0 + (1/6) \nabla^2 \delta \chi^{(n)}_0 = 0$.}
expression. This is a well-known feature that is to be ascribed to the different meaning of the mass density when moving from Eulerian to Lagrangian coordinates, which only appears at second and higher orders (see, e.g., [40]).

As we did in the Poisson-gauge case, we can re-express this result in terms of a suitable potential $\Phi$ in order to introduce a nonlinearity parameter $f_{\text{NL}}$ with the usual meaning. We obtain (see also [32])

$$f_C^{\text{NL}}(k_1, k_2) = \frac{5}{3} \left[ (a_{\text{NL}} - 1) - 1 + \frac{5}{2} \frac{k_1 \cdot k_2}{k^2} \right].$$

Note that the nonlinearity parameter is different w.r.t. the one in equation (31), not only because they have been computed in two different gauges, but also because in the comoving synchronous gauge the post-Newtonian term giving rise to $f_{\text{NL}}^{\text{S}}$ can be easily written in an exact form where the growing mode $D_\tau(\tau)$ is factored out, while this is not the case for the Poisson gauge. It is also important to stress here that the expression for $f_{\text{NL}}$ obtained in this gauge is the one to be used to evaluate the effect of NG on the Lagrangian bias of dark matter halos, as recently stressed in [41]. In [32] it was shown that small effects in the above equation which come purely from the general relativistic evolution, i.e. the term which survives in the limit $a_{\text{NL}} = 1$, are potentially detectable for some planned LSS surveys.

### 5. Concluding remarks

In this paper we have investigated the effect of primordial and GR-induced non-Gaussianities on the second-order matter density perturbation in a $\Lambda$CDM cosmology. The calculation has been performed in two popular gauges, the Poisson gauge and the comoving time-orthogonal one, which are useful for comparison with observations, depending on the particular quantity under study. For instance, in evaluating the effect of NG on the mass function and Lagrangian bias of dark matter halos, the comoving gauge expression is more appropriate, while the Poisson gauge formulas are more suitable for gravitational lensing studies. The strongest present limits on $f_{\text{NL}}$ come from the analysis of the angular bispectrum of WMAP temperature anisotropy data. Indeed, Komatsu et al [42], analyzing the 7-years WMAP data obtain the 95% limits $-10 < f_{\text{NL}}^{\text{local}} < 74$, and $-214 < f_{\text{NL}}^{\text{equilateral}} < 266$. The analysis of Planck data both in temperature and E-mode polarization is expected to improve the accuracy by almost an order of magnitude. A complementary and very powerful information on the amplitude and shape of primordial NG will come from the study of the galaxy clustering (e.g. [43]) and other LSS datasets, such as weak gravitational lensing (e.g. Refs [44, 45]) and redshifted 21 cm background anisotropy (e.g. [46, 47]). Primordial NG in LSS data can be searched for by various techniques: abundance of massive and/or high-redshift objects [48–54], abundance of voids [55], higher order statistics such as bispectrum and trispectrum (see, e.g. [56]), large-scale clustering of halos, thought as rare high peaks of the dark matter distribution [33, 34, 51, 57–59] and their cross-correlation with the CMB via the integrated Sachs–Wolfe effect [60], small-scale NG corrections to the matter power-spectrum [59, 61]. For instance, Slosar et al [62], exploiting the scale dependence of the NG correction to the halo linear bias $\Delta b_{\text{NG}} \propto f_{\text{NL}}^{\text{local}} k^{-2}$, obtained the 95% confidence range $-29 < f_{\text{NL}}^{\text{local}} < 70$, using the two-point function of a combination of datasets. The prospects for sensibly narrowing these limits with the advent of galaxy surveys sampling regions comparable to the Hubble volume are analyzed in [63]. Indeed, as shown in [32], there are very promising prospects to observe NG signatures down to the limits of the order unity GR corrections discussed in this paper. This largely motivates theoretical efforts to obtain accurate predictions of these effects.
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