On restricted edge-connectivity of half-transitive multigraphs *

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Abstract  Let $G = (V, E)$ be a multigraph (it has multiple edges, but no loops). We call $G$ maximally edge-connected if $\lambda(G) = \delta(G)$, and $G$ super edge-connected if every minimum edge-cut is a set of edges incident with some vertex. The restricted edge-connectivity $\lambda'(G)$ of $G$ is the minimum number of edges whose removal disconnects $G$ into non-trivial components. If $\lambda'(G)$ achieves the upper bound of restricted edge-connectivity, then $G$ is said to be $\lambda'$-optimal. A bipartite multigraph is said to be half-transitive if its automorphism group is transitive on the sets of its bipartition. In this paper, we will characterize maximally edge-connected half-transitive multigraphs, super edge-connected half-transitive multigraphs, and $\lambda'$-optimal half-transitive multigraphs.

Keywords: Multigraphs; Half-transitive multigraphs; Maximally edge-connected; Super edge-connected; Restricted edge-connectivity.

1 Introduction

A graph $G$ consists of vertex set $V$ and edge set $E$, where $E$ is a multiset of unordered pairs of (not necessarily distinct) vertices. A loop is an edge whose endpoints are the same vertex. An edge is multiple if there is another edge with the same endvertices; otherwise it is simple. The multiplicity of an edge $e$, denoted by $\mu(e)$, is the number of multiple edges sharing the same endvertices; the multiplicity of a graph $G$, denoted by $\mu(G)$, is the maximum multiplicity of its edges. A graph is a simple graph if it has no multiple edges or loops, a multigraph if it has multiple edges, but no loops, and a pseudograph if it contains both multiple edges and loops. The underlying graph of a multigraph $G$,
denoted by $U(G)$, is a simple graph obtained from $G$ by destroying all multiple edges. It is clear that $\mu(G) = 1$ if the graph $G$ is simple.

Let $G = (V, E)$ be a multigraph. Denote by $\lambda(G)$ the edge-connectivity of $G$. For $\lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of $G$, a multigraph $G$ with $\lambda(G) = \delta(G)$ is naturally said to be maximally edge-connected, or $\lambda$-optimal for simplicity. A multigraph $G$ is said to be vertex-transitive if for any two vertices $u$ and $v$ in $G$, there is an automorphism $\alpha$ of $G$ such that $v = \alpha(u)$, that is, $Aut(G)$ acts transitively on $V$. A bipartite multigraph $G$ with bipartition $V_1 \cup V_2$ is called half-transitive if $Aut(G)$ acts transitively both on $V_1$ and $V_2$. Mader [9] proved the following well-known result.

**Theorem 1.1.** [9] Every connected vertex-transitive simple graph $G$ is $\lambda$-optimal.

If $G$ is a vertex-transitive multigraph, then $G$ is not always maximally edge-connected. A simple example is the multigraph obtained from a 4-cycle $C_4$ by replacing each edge belongs to a pair of opposite edges in $C_4$ with $m$ ($m \geq 2$) multiple edges.

For half-transitive simple graphs, Liang and Meng [8] proved the following result:

**Theorem 1.2.** [8] Every connected half-transitive simple graph $G$ is $\lambda$-optimal.

The problem of exploring edge-connected properties stronger than the maximally edge-connectivity for simple graphs has been the theme of many research. The first candidate may be the so-called super edge-connectivity. We can generalize this definition to multigraphs. A multigraph $G$ is said to be super edge-connected, in short, super-$\lambda$, if each of its minimum edge-cut sets isolates a vertex, that is, every minimum edge-cut is a set of edges incident with a certain vertex in $G$. By the definitions, a super-$\lambda$ multigraph must be a $\lambda$-optimal multigraph. However, the converse is not true. For example, $K_m \times K_2$ is $\lambda$-optimal but not super-$\lambda$ since the set of edges between the two copies of $K_m$ is a minimum edge-cut which does not isolate any vertex.

The concept of super-$\lambda$ was originally introduced by Bauer et al. see [1], where combinatorial optimization problems in design of reliable probabilistic simple graphs were investigated. The following theorem is a nice result of Tindell [15], which characterized super edge-connected vertex-transitive simple graphs.

**Theorem 1.3.** [15] A connected vertex-transitive simple graph $G$ which is neither a cycle nor a complete graph is super-$\lambda$ if and only if it contains no clique $K_k$ where $k$ is the degree of $G$.

For further study, Esfahanian and Hakimi [4] introduced the concept of restricted edge-connectivity for simple graphs. The concept of restricted edge-connectivity is one kind of conditional edge-connectivity proposed by Harary in [5], and has been successfully applied in the further study of tolerance and reliability of networks, see [2,3,7,11,12,18,20-22]. Let $F$ be a set of edges in $G$. Call $F$ a restricted edge-cut if $G - F$ is disconnected and contains no isolated vertices. The minimum cardinality over all restricted edge-cuts
is called *restricted edge-connectivity* of $G$, and denoted by $\lambda'(G)$. It is shown by Wang and Li [17] that the larger $\lambda'(X)$ is, the more reliable the network is. In [4], it is proved that if a connected simple graph $G$ of order $|V(G)| \geq 4$ is not a star $K_{1,n-1}$, then $\lambda'(G)$ is well-defined and $\lambda'(G) \leq \xi(G)$, where $\xi(G) = \min \{d(u) + d(v) - 2 : uv \in E(G)\}$ is the minimum edge degree of $G$. A simple graph $G$ with $\lambda'(G) = \xi(G)$ is called a $\lambda'$-optimal graph. It should be pointed out that if $\delta(G) \geq 3$, then a $\lambda'$-optimal simple graph must be super-$\lambda$. In fact, a graph $G$ is super-$\lambda$ if and only if $\lambda(G) < \lambda'(G)$, see [6]. Thus, the concepts of $\lambda$-optimal graphs, super-$\lambda$ graphs and $\lambda'$-optimal graphs describe reliable interconnection structures for graphs at different levels.

In [10], Meng studied the parameter $\lambda'$ for connected vertex-transitive simple graphs. The main result may be restate as follows:

**Theorem 1.4.** [10] Let $G$ be a $k$-regular connected vertex-transitive simple graph which is neither a cycle nor a complete graph. Then $G$ is not $\lambda'$-optimal if and only if it contains a $(k-1)$-regular subgraph $H$ satisfying $k \leq |V(H)| \leq 2k - 3$.

The authors in [13] proved the following result.

**Theorem 1.5.** [13] Let $G = (V_1 \cup V_2, E)$ be a connected half-transitive simple graph with $n = |V(G)| \geq 4$ and $G \not\cong K_{1,n-1}$. Then $G$ is $\lambda'$-optimal.

Since a graph $G$ is super-$\lambda$ if and only if $\lambda(G) < \lambda'(G)$, Theorem 1.5 implies the following corollary.

**Corollary 1.6.** The only connected half-transitive simple graphs which are not super-$\lambda$ are cycles $C_n (n \geq 4)$.

We can naturally generalize the concept of restricted edge-connectivity to multigraphs. The restricted edge-connectivity $\lambda'(G)$ of a multigraph $G$ is the minimum number of edges whose removal disconnects $G$ into non-trivial components. Similarly, define the minimum edge degree of $G$ as $\xi(G) = \min \{\xi(e) = d(u) + d(v) - 2\mu(e) : e = uv \in E(G)\}$, where $\xi(e) = d(u) + d(v) - 2\mu(e)$ is the edge degree of the edge $e = uv$ in $G$. By using a similar argument as in [4], we can prove that the restricted edge-connectivity of a connected multigraph $G$ is well-defined if $|V(G)| \geq 4$ and $U(G) \not\cong K_{1,n-1}$, but the inequality $\lambda'(G) \leq \xi(G)$ is not always correct. For example, the restricted edge-connectivity of the multigraph $G$ in Fig.1 is 6, but $\xi(G) = 4$.

![Fig.1](image-url)
In [14], we gave sufficient and necessary conditions for vertex-transitive multigraphs to be maximally edge-connected, super edge-connected and \( \lambda \)-optimal. In the following, we will study maximally edge-connected half-transitive multigraphs, super edge-connected half-transitive multigraphs, and \( \lambda' \)-optimal half-transitive multigraphs.

\section{Preliminary}

Let \( G = (V, E) \) be a multigraph. For two disjoint non-empty subsets \( A \) and \( B \) of \( V \), let \( [A, B] = \{ e = uv \in E : u \in A \text{ and } v \in B \} \). For the sake of convenience, we write \( u \) for the single vertex set \( \{ u \} \). If \( A = V \setminus A \), then we write \( N(A) \) for \( [A, \overline{A}] \) and \( d(A) \) for \( |N(A)| \). Thus \( d(u) \) is just the degree of \( u \) in \( G \). Denote by \( G[A] \) the subgraph of \( G \) induced by \( A \).

An edge-cut \( F \) of \( G \) is called a \( \lambda \)-cut if \( |F| = \lambda(G) \). It is easy to see that for any \( \lambda \)-cut \( F \), \( G - F \) has exactly two components. If \( N(A) \) is a \( \lambda \)-cut of \( G \), then \( A \) is called a \( \lambda \)-fragment of \( G \). It is clear that if \( A \) is a \( \lambda \)-fragment of \( G \), then so is \( \overline{A} \). Let \( r(G) = \min\{|A| : A \text{ is a } \lambda \text{-fragment of } G\} \). Obviously, \( 1 \leq r(G) \leq \frac{1}{2} |V| \). A \( \lambda \)-fragment \( B \) is called a \( \lambda \)-atom of \( G \) if \( |B| = r(G) \). A \( \lambda \)-fragment \( C \) is called a strict \( \lambda \)-fragment if \( 2 \leq |C| \leq |V(G)| - 2 \). If \( G \) contains strict \( \lambda \)-fragments, then the ones with smallest cardinality are called \( \lambda \)-superatoms.

Similarly, we can give the definition of \( \lambda' \)-atom. A restricted edge-cut \( F \) of \( G \) is called a \( \lambda' \)-cut if \( |F| = \lambda'(G) \). For any \( \lambda' \)-cut \( F \), \( G - F \) has exactly two components. Let \( A \) be a proper subset of \( V \). If \( N(A) \) is a \( \lambda' \)-cut of \( G \), then \( A \) is called a \( \lambda' \)-fragment of \( G \). It is clear that if \( A \) is a \( \lambda' \)-fragment of \( G \), then so is \( \overline{A} \). Let \( r'(G) = \min\{|A| : A \text{ is a } \lambda' \text{-fragment of } G\} \). Obviously, \( 2 \leq r'(G) \leq \frac{1}{2} |V| \). A \( \lambda' \)-fragment \( B \) is called a \( \lambda' \)-atom of \( G \) if \( |B| = r'(G) \).

For a multigraph \( G \), the inequality \( \lambda(G) \leq \xi(G) \) is not always correct. But if \( G \) is a \( k \)-regular multigraph, we proved the following result.

\begin{lemma} \cite{14} \label{lem21}
Let \( G \) be a connected \( k \)-regular multigraph. Then \( \lambda'(G) \) is well-defined and \( \lambda'(G) \leq \xi(G) \) if \( |V(G)| \geq 4 \).
\end{lemma}

We call a bipartite multigraph \( G \) with bipartition \( V_1 \cup V_2 \) semi-regular if each vertex in \( V_1 \) has the same degree \( d_1 \) and each vertex in \( V_2 \) has the same degree \( d_2 \). For semi-regular bipartite multigraphs, a similar result can be obtained.

\begin{lemma} \label{lem22}
Let \( G \) be a connected semi-regular bipartite multigraph with bipartition \( V_1 \cup V_2 \). Then \( \lambda'(G) \) is well-defined and \( \lambda'(G) \leq \xi(G) \) if \( |V(G)| \geq 4 \) and \( U(G) \not\cong K_{1,n-1} \).
\end{lemma}

\textbf{Proof.} Assume each vertex in \( V_1 \) has degree \( d_1 \) and each vertex in \( V_2 \) has degree \( d_2 \). Assume, without loss of generality, that \( d_1 \leq d_2 \). Let \( e = uv \) be an edge such that \( \xi(e) = \xi(G) \), where \( u \in V_1 \) and \( v \in V_2 \). If \( G - \{u, v\} \) contains a non-trivial component, say \( C \), then \( N(V(C)) \) is a restricted edge-cut and \( |N(V(C))| = |N(\{u, v\})| = \xi(e) = \xi(G) \).
Thus assume that \( G - \{u, v\} \) only contains isolated vertices. If there is a vertex \( w \) other than \( v \) in \( V_2 \), then \( d_1 + d_2 \leq |N(V \setminus \{u, v\})| = |N(\{u, v\})| = \xi(e) = d_1 + d_2 - \mu(e) < d_1 + d_2 \) by \( |V(G)| \geq 4 \), a contradiction. Thus \( V_2 = \{v\} \) and \( U(G) \cong K_{1,n-1} \), also a contradiction. □

Because of Lemma 2.1 and Lemma 2.2, we call a regular multigraph (or a semi-regular bipartite multigraph) \( G \) \( \lambda \)-optimal if \( \lambda'(G) = \xi(G) \). Since each vertex-transitive multigraph is regular and each half-transitive multigraph is semi-regular, thus a vertex-transitive multigraph (or a half-transitive multigraph) \( G \) is \( \lambda \)-optimal if \( \lambda'(G) = \xi(G) \).

Recall that an imprimitive block for a permutation group \( \Phi \) on a set \( T \) is a proper, non-trivial subset \( A \) of \( T \) such that for every \( \varphi \in \Phi \) either \( \varphi(A) = A \) or \( \varphi(A) \cap A = \emptyset \). A subset \( A \) of \( V(G) \) is called an imprimitive block for \( G \) if it is an imprimitive block for the automorphism group \( Aut(G) \) on \( V(G) \). The following theorem shows the importance of imprimitive blocks:

**Theorem 2.3.** [16] Let \( G = (V, E) \) be a connected simple graph and \( A \) be an imprimitive block for \( G \). If \( G \) is vertex-transitive, then \( G[A] \) is also vertex-transitive.

By a similar argument as Theorem 2.3, we can obtain the following result for half-transitive multigraphs.

**Lemma 2.4.** Let \( G \) be a connected bipartite multigraph with bipartition \( V_1 \cup V_2 \). Assume \( A \) is an imprimitive block for \( G \) such that \( A \cap V_1 \neq \emptyset \) and \( A \cap V_2 \neq \emptyset \). If \( G \) is half-transitive, then \( G[A] \) is also half-transitive.

**Proof.** Since \( G \) is half-transitive, for any two vertices \( u, v \in A \cap V_i \) (\( i \in \{1, 2\} \)), there is \( \alpha \in Aut(G) \) such that \( \alpha(u) = v \). Because \( \alpha(A) \cap A \neq \emptyset \), we have \( \alpha(A) = A \) by \( A \) is an imprimitive block for \( G \). Thus the restriction of \( \alpha \) to \( A \) is an automorphism of \( G[A] \), which maps \( u \) to \( v \). It follows \( G[A] \) is a half-transitive multigraph.

### 3 Maximally edge-connected half-transitive multigraphs

In [9], Mader proved that any two distinct \( \lambda \)-atoms of a simple graph are disjoint. For multigraphs, this property still holds.

**Lemma 3.1.** Let \( G \) be a connected multigraph. Then any two distinct \( \lambda \)-atoms of \( G \) are disjoint.

**Proof.** Suppose to the contrary that there are two distinct \( \lambda \)-atoms \( A \) and \( B \) with \( A \cap B \neq \emptyset \). We have \( V(G) \setminus (A \cup B) \neq \emptyset \) by \( |A| \leq |V(G)|/2 \) and \( |B| \leq |V(G)|/2 \). Then \( N(A \cap B) \) and \( N(A \cup B) \) are edge-cuts of \( G \), thus \( d(A \cap B) = |N(A \cap B)| \geq \lambda(G) \) and \( d(A \cup B) = |N(A \cup B)| \geq \lambda(G) \). From the following well-known submodular inequality (see [16]),

\[
2\lambda(G) \leq d(A \cup B) + d(A \cap B) \leq d(A) + d(B) = 2\lambda(G),
\]
we conclude that both \(|d(A \cap B)| = \lambda(G)\) and \(|d(A \cup B)| = \lambda(G)\) hold. Thus \(A \cap B\) is a \(\lambda\)-fragment with \(|A \cap B| < |A|\), which contradicts to \(A\) is a \(\lambda\)-atom of \(G\). \(\Box\)

**Theorem 3.2.** Let \(G\) be a connected half-transitive multigraph with bipartition \(V_1 \cup V_2\). Assume each vertex in \(V_1\) has degree \(d_1\) and each vertex in \(V_2\) has degree \(d_2\). Then \(G\) is not maximally edge-connected if and only if there is a proper induced connected half-transitive multi-subgraph \(H\) of \(G\) such that

\[|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\} - 1,\]

where \(A_1 = V_1 \cap V(H), A_2 = V_2 \cap V(H), d'_1\) is the degree of each vertex of \(A_1\) in \(H\) and \(d'_2\) is the degree of each vertex of \(A_2\) in \(H\).

**Proof.** Assume, without loss of generality, that \(d_1 \leq d_2\). If \(G\) is not maximally edge-connected, then \(\lambda(G) \leq d_1 - 1\). Let \(A\) be a \(\lambda\)-atom of \(G\) and \(H = G[A]\). By Lemma 3.1, we know \(A\) is an imprimitive block for \(G\). Thus \(H\) is a connected half-transitive multigraph by Lemma 2.4. Assume each vertex in \(A \cap V_1\) has degree \(d'_1\) in \(H\) and each vertex in \(A \cap V_2\) has degree \(d'_2\) in \(H\). Then \(|A \cap V_1|(d_1 - d'_1) + |A \cap V_2|(d_2 - d'_2) = d(A) = \lambda(G) \leq d_1 - 1\).

Now we prove the sufficiency. Assume \(G\) contains a proper induced connected half-transitive multi-subgraph \(H\) such that \(|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\} - 1\), then \(\lambda(G) \leq d(V(H)) = |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\} - 1\), that is, \(G\) is not maximally edge-connected. \(\Box\)

## 4 Super edge-connected half-transitive multigraphs

In [10], Tindell studied the intersection property of \(\lambda\)-superatoms of vertex-transitive simple graphs. For half-transitive multigraphs, we have the following lemma.

**Lemma 4.1.** Let \(G\) be a connected half-transitive multigraph with bipartition \(V_1 \cup V_2\). Assume \(G\) is not super edge-connected, \(A\) and \(B\) are two distinct \(\lambda\)-superatoms. If \(|A| = |B| \geq 3\), then \(A \cap B = \emptyset\).

**Proof.** Assume each vertex in \(V_1\) has degree \(d_1\) and each vertex in \(V_2\) has degree \(d_2\). Without loss of generality, assume that \(d_1 \leq d_2\). If \(A \cap B \neq \emptyset\), then by a similar argument as the proof of Lemma 3.1, we can conclude that \(|d(A \cap B)| = |d(A \cup B)| = \lambda(G)|. We claim that \(|A \cap B| = 1\). Otherwise, if \(|A \cap B| \geq 2\), then \(|V(G) \setminus (A \cup B)| \geq |A \cap B| \geq 2\). Since \(G[A], G[V \setminus A], G[B]\) and \(G[V \setminus B]\) are connected, we have \(G[A \cup B]\) and \(G[V \setminus (A \cap B)]\) are connected. If \(G[A \cap B]\) is not connected, then we have \(d(A \cap B) \geq 2\lambda(G)\), a contradiction. If \(G[A \cap B]\) is connected, then \(A \cap B\) is a strict \(\lambda\)-fragment with \(|A \cap B| < |A|\), which contradicts to \(A\) is a \(\lambda\)-superatom. Hence \(|A \cap B| = 1\).

Let \(C = V(G) \setminus B\). Then \(|A \cap C| = |A \setminus (A \cap B)| \geq 2\), and \(A, V(G) \setminus A, C\) and \(V(G) \setminus C\) are all strict \(\lambda\)-fragments. By a similar argument as above we can deduce that \(A \cap C\) is a strict \(\lambda\)-fragment with \(|A \cap C| < |A|\), which is impossible. \(\Box\)
\textbf{Theorem 4.2.} Let $G$ be a connected half-transitive multigraph with bipartition $V_1 \cup V_2$. Assume each vertex in $V_1$ has degree $d_1$, each vertex in $V_2$ has degree $d_2$ and $|V(G)| \geq 2 \min\{d_1, d_2\} + 2$. Then $G$ is not super edge-connected if and only if there is a proper induced connected half-transitive multi-subgraph $H$ of $G$ such that

$$|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\},$$

where $A_1 = V_1 \cap V(H)$, $A_2 = V_2 \cap V(H)$, $d'_1$ is the degree of each vertex of $A_1$ in $H$ and $d'_2$ is the degree of each vertex of $A_2$ in $H$.

\textbf{Proof.} Assume, without loss of generality, that $d_1 \leq d_2$. If $G$ is not super edge-connected, then $G$ contains $\lambda$-superatoms. Let $A$ be a $\lambda$-superatom of $G$ and $H = G[A]$. If $|A| = 2$, then $H$ is isomorphic to a multigraph which contains two vertices and $t$ edges between these two vertices. Thus $H$ is an induced $t$-regular connected half-transitive multi-subgraph of $G$. Therefore $|A \cap V_1|(d_1 - t) + |A \cap V_2|(d_2 - t) = d(A) = \lambda(G) \leq d_1$.

In the following, we assume that $|A| \geq 3$.

By Lemma 4.1, we know $A$ is an imprimitive block for $G$. Thus $H$ is a connected half-transitive multigraph by Lemma 2.4. Assume each vertex in $A \cap V_1$ has degree $d'_1$ in $H$ and each vertex in $A \cap V_2$ has degree $d'_2$ in $H$. Thus $|A \cap V_1|(d_1 - d'_1) + |A \cap V_2|(d_2 - d'_2) = d(A) = \lambda(G) \leq d_1$.

Now we prove the sufficiency. Assume $G$ contains a proper induced connected half-transitive multi-subgraph $H$ such that $|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\}$, then $d(V(H)) = |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\}$. If $G - V(H)$ contains no isolated vertices, then $V(H)$ is a strict $\lambda$-fragment. Thus $G$ is not super edge-connected. Assume $G - V(H)$ contains an isolated vertex $w$, then $N(w) = N(V(H))$. Since $|A_1| \leq \min\{d_1, d_2\}$ and $|A_2| \leq \min\{d_1, d_2\}$ by $|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\}$, we see that $G$ is not connected by $|V(G)| \geq 2 \min\{d_1, d_2\} + 2$, a contradiction. $\square$

5 \textbf{$\lambda'$-optimal half-transitive multigraphs}

In [19], the authors proved the following fundamental result for studying the restricted edge-connectivity of simple graphs.

\textbf{Theorem 5.1.} [19] Let $G = (V, E)$ be a connected simple graph with at least four vertices and $G \not\cong K_{1,n-1}$. If $G$ is not $\lambda'$-optimal, then any two distinct $\lambda'$-atoms of $G$ are disjoint.

For multigraphs, we cannot obtain a similar result as in Theorem 5.1. But for half-transitive multigraphs, the similar result holds.

\textbf{Lemma 5.2.} Let $G$ be a connected multigraph with $\delta(G) \geq 2\mu(G)$. If $G$ contains a $\lambda'$-atom $A$ with $|A| \geq 3$, then each vertex in $A$ has at least two neighbors in $A$. 

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**Proof.** By contradiction, assume there is a vertex $u \in A$ such that $u$ contains only one neighbor in $A$. Let $v$ be the only neighbor of $u$ in $A$. Set $A' = A\{u\}$. Then both $G[A']$ and $G[A\bar{A}]$ are connected. We have $|A'| \geq 2$ by $|A| \geq 3$. Clearly, $|\bar{A}| = |A| + 1 \geq 4$. Thus $[A', \bar{A}]$ is a restricted edge-cut. Since $\delta(G) \geq 2\mu(G)$, we have

$$\lambda'(G) \leq ||A', \bar{A}|| = ||A, \bar{A}|| + \mu(uv) - (d(u) - \mu(uv)) \leq ||A, \bar{A}|| = \lambda'(G).$$

It follows that $A'$ is a $\lambda'$-fragment with $|A'| < |A|$, which contradicts to $A$ is a $\lambda'$-atom. $\square$

**Lemma 5.3.** Let $G$ be a connected half-transitive multigraph with bipartition $V_1 \cup V_2$ and $\delta(G) \geq 2\mu(G)$. Assume $G$ is not $\lambda'$-optimal, $A$ and $B$ are two distinct $\lambda'$-atoms. Then $|A| = |B| \geq 3$ and $A \cap B = \emptyset$.

**Proof.** Assume each vertex in $V_1$ has degree $d_1$ and each vertex in $V_2$ has degree $d_2$. Without loss of generality, assume that $d_1 \leq d_2$.

If $|A| = 2$, then $\lambda'(G) = d(A) = d_1 + d_2 - 2\mu(uv) \geq \xi(G)$ (where $A = \{u, v\}$), which contradicts that $G$ is not $\lambda'$-optimal. Thus $|A| \geq 3$.

Suppose to the contrary that $A \cap B \neq \emptyset$. Set $C = A \cap B$, $A_1 = A \cap \bar{B}$, $B_1 = B \cap \bar{A}$ and $D = \bar{A} \cap \bar{B} = \bar{A} \cup B$. In the following, we will derive a contradiction by a series of claims.

Clearly, one of the following two inequalities must holds:

$$||[A_1, C]) \leq ||[C, B_1]] + ||[C, D]],$$

$$||[B_1, C]| \leq ||[C, A_1]] + ||[C, D]].$$

In the following, we always assume, without loss of generality, that inequality (1) holds.

**Claim 1.** $A_1$ satisfies one of the following two conditions: (i) $A_1 = \{v_{21}\}(v_{21} \in V_2)$ and $d_1 > 2\mu(G)$, or (ii) $A_1 = \{v_{11}, \cdots, v_{1m}\}(v_{1i} \in V_1$ for $1 \leq i \leq m)$ and $d_2 > (m - 1)d_1 + 2\mu(G)$.

It follows from inequality (1) that

$$d(A_1) = ||[A_1, D]] + ||[A_1, C]] + ||[A_1, B_1]] \leq d(A) = \lambda'(X).$$

Assume $G[A_1]$ has a component $\bar{G}$ with $|V(\bar{G})| \geq 2$. Set $F = V(\bar{G})$. Since $G[B]$ and $G[\bar{A}]$ are both connected, and $B \cap \bar{A} \neq \emptyset$, we see that $G[\bar{A}]$ is connected. Furthermore, since $G$ is connected, every component of $G[A_1]$ is joined to $G[\bar{A}]$, and thus $G[F]$ is connected. So $[F, \bar{F}]$ is a restricted edge-cut with $|d(F)| \leq \lambda'(G)$. Because $A$ is a $\lambda'$-atom and $F$ is a proper subset of $A$, we obtain $d(F) > d(A) = \lambda'(G)$, it is a contradiction. Thus, each component in $G[A_1]$ is an isolated vertex. By $d(A_1) \leq \lambda'(G) < d_1 + d_2 - 2\mu(G)$, we can derive that $A_1$ satisfies one of the following two conditions: (i) $A_1 = \{v_{21}\}(v_{21} \in V_2)$ and $d_1 > 2\mu(G)$, or (ii) $A_1 = \{v_{11}, \cdots, v_{1m}\}(v_{1i} \in V_1$ for $1 \leq i \leq m)$ and $d_2 > (m - 1)d_1 + 2\mu(G)$.

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Claim 2. $C \not\subseteq V_1$ and $C \not\subseteq V_2$.

By contradiction. Suppose $C \subseteq V_1$. Then $G[C]$ is an independent set. Since we have assumed that $|[A_1, C]| \leq |[C, B_1]| + |[C, D]|$, there exists a vertex $v$ in $C$ such that

$$|[v, A_1]| \leq |[v, D]| + |[v, B_1]|. \quad (3)$$

Set $F = A \setminus \{v\}$, then

$$d(F) = d(A) - |[v, D]| - |[v, B_1]| + |[v, A_1]| \leq d(A) = \lambda'(X).$$

Since $G[A]$ is connected and $C$ is an independent set, we have $|[v, A_1]| \geq 1$. It follows from inequality (3) that $|[v, A_1]| \leq 1$. So, $G[F]$ is connected. We claim that each component in $G[F]$ has at least 2 vertices. In fact, if there is an isolated vertex $u$ in $G[F]$, then $u$ is the only vertex adjacent to $u$ in $G[A]$, which contradicts to Lemma 5.2. Now, similarly as in the proof of Claim 1, a contradiction arises, since $F$ contains a smaller $\lambda'$-fragment than $A$. $C \not\subseteq V_2$ can be proved similarly.

Claim 3. $d(D) < \lambda'(G)$ and $D$ is an independent set contained in $V_1$.

By Claim 2, $|C| \geq 2$. We claim that $d(C) > \lambda'(G)$. In fact, if $G[C]$ contains a component of order at least 2, then similar to the proof of Claim 1, we can show that $[C, C] \subseteq C$ contains a restricted edge-cut, and thus $d(C) > \lambda'(G)$. Otherwise, we assume that each component in $G[C]$ is an isolated vertex. Since not all vertices in $C$ are from the same bipartition, there must be at least one vertex in $V_2$. From $|C| \geq 2$, we have $d(C) \geq d_2 + d_1 > \xi(G) \geq \lambda'(X)$. Thus, we have that $d(C) > \lambda'(G)$.

From the well-known submodular inequality (see [16]), we have

$$d(C) + d(D) \leq d(A) + d(B) = 2\lambda'(G). \quad (4)$$

By (4) and $d(C) > \lambda'(G)$, we obtain $d(D) < \lambda'(G)$. Applying a similar argument as above, we can show that $D$ is an independent set contained in $V_1$.

Let $s = |D|$. Then $s \geq 2$ and

$$d(D) = sd_1. \quad (5)$$

Denote by $e_1$ the number of edges in $G[\overline{C}]$. Clearly,

$$d(C) = d(\overline{C}) = \sum_{v \in \overline{C}} d(v) - 2e_1. \quad (6)$$

Since $G[\overline{B}]$ is connected and $D$ is an independent set contained in $V_1$, Claim 1 (ii) can not hold, Thus, Claim 1 (i) is true. Since $G$ is a bipartite multigraph, we have

$$e_1 \leq 2s\mu(G). \quad (7)$$

Combining this with (4), (5) and (6), we see that

$$2d_1 + 2d_2 - 4\mu(G) - sd_1 > 2\lambda'(G) - d(D) \geq d(C) \geq sd_1 + 2d_2 - 4s\mu(G).$$

This implies $d_1 < 2\mu(G)$, contradicting to the assumption that $d_1 \geq 2\mu(G)$. $\square$
Theorem 5.4. Let $G$ be a connected half-transitive multigraph with bipartition $V_1 \cup V_2$ and $\delta(G) \geq 2\mu(G)$. Assume each vertex in $V_1$ has degree $d_1$, each vertex in $V_2$ has degree $d_2$, $|V_1| \geq \xi(G)$ and $|V_2| \geq \xi(G)$. Then $G$ is not $\lambda'$-optimal if and only if there is a proper induced connected half-transitive multi-subgraph $H$ of $G$ such that

$$|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \xi(G) - 1,$$

where $A_1 = V_1 \cap V(H)$, $A_2 = V_2 \cap V(H)$, $d'_1$ is the degree of each vertex of $A_1$ in $H$ and $d'_2$ is the degree of each vertex of $A_2$ in $H$.

Proof. Assume, without loss of generality, that $d_1 \leq d_2$. If $G$ is not $\lambda'$-optimal, then $G$ contains $\lambda'$-atoms. Let $A$ be a $\lambda'$-atom of $G$ and $H = G[A]$. By Lemma 5.3, we have $|A| \geq 3$ and $A$ is an imprimitive block for $G$. Thus $H$ is a connected half-transitive multigraph by Lemma 2.4. Assume each vertex in $A \cap V_1$ has degree $d'_1$ in $H$ and each vertex in $A \cap V_2$ has degree $d'_2$ in $H$. Thus $|A \cap V_1|(d_1 - d'_1) + |A \cap V_2|(d_2 - d'_2) = d(A) = \lambda'(G) \leq \xi(G) - 1$.

Now we prove the sufficiency. Assume $G$ contains a proper induced connected half-transitive multi-subgraph $H$ such that $|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \xi(G) - 1$, then $d(V(H)) = |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \xi(G) - 1, |A_1| \leq \xi(G) - 1$ and $|A_2| \leq \xi(G) - 1$. If $G - V(H)$ contains a non-trivial component, say $B$, then $[B, \overline{B}]$ is a restricted edge-cut and $d(B) \leq d(V(H)) \leq \xi(G) - 1$. Thus $G$ is not $\lambda'$-optimal. Now we assume that each component of $G - V(H)$ is an isolated vertex, then $|N(V(G) \setminus V(H))| \geq d_1 + d_2 > \xi(G)$ by $|V_1| \geq \xi(G)$ and $|V_2| \geq \xi(G)$. On the other hand, $|N(V(G) \setminus V(H))| = |N(V(H))| \leq \xi(G) - 1$, it is a contradiction. \(\square\)

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