Joint Extraction of $m_s$ and $V_{us}$ From Hadronic $\tau$ Decays

Kim Maltman

Department of Mathematics and Statistics, York University,
4700 Keele St., Toronto, ON CANADA M3J 1P3

Carl E. Wolfe

Department of Physics and Astronomy, York University,
4700 Keele St., Toronto, ON CANADA M3J 1P3

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We study the simultaneous determination of $m_s$ and $V_{us}$ from flavor-breaking hadronic $\tau$ decay sum rules using weights designed to bring under better control problems associated with the slow convergence of the relevant $D = 2$ OPE series. Much improved stability and consistency is found as compared to the results of conventional analyses based on the “$(k,0)$ spectral weights”. Results for $m_s$ are in excellent agreement with those of recent strange scalar and strange pseudoscalar sum rule analyses, while those for $V_{us}$ agree within errors with the output from recent lattice-based $\Gamma[K_{\mu2}]/\Gamma[\pi\mu2]$ and $K_{\ell3}$-based analyses. Very significant error reductions are shown to be expected, especially for $V_{us}$, once the improved strange spectral data from the B factory experiments becomes available.

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I. BACKGROUND

Measurements of inclusive flavor $ij = ud,us$ vector (V) or axial vector (A) current induced hadronic $\tau$ decay distributions yield kinematically weighted linear combinations of the spectral functions, $\rho^{(J)}_{V/A;ij}$, of the spin $J = 0$ and 1 parts, $\Pi^{(J)}_{V/A;ij}$, of the relevant current-current correlators. Explicitly, with $R_{V/A;ij} \equiv \Gamma[\tau^- \rightarrow \nu_\tau \text{hadrons}_{V/A;ij}(\gamma)]/\Gamma[\tau^- \rightarrow \nu_\tau \bar{\nu}_\tau(\gamma)]$ [1],

$$R_{V/A;ij} = 12\pi^2|V_{ij}|^2 S_{EW} \int_{m_{\tau}^2}^{m_{\tau}^2} \frac{ds}{m_{\tau}^2} \left(1 - y_{\tau}\right)^2 \left[1 + 2y_{\tau}\right] \rho^{(0+1)}_{V/A;ij}(s) - 2y_{\tau}\rho^{(0)}_{V/A;ij}(s)\right)$$

where $y_{\tau} = s/m_{\tau}^2$, $V_{ij}$ is the flavor $ij$ CKM matrix element, $S_{EW} = 1.0201 \pm 0.0003$ is a short-distance electroweak correction, and the superscript $(0 + 1)$ denotes the sum of $J = 0$ and $J = 1$ contributions. Since the spectral function combinations in Eq. (1)

*CSSM, Univ. of Adelaide, Adelaide, SA 5005 AUSTRALIA
correspond to correlators with no kinematic singularities, each term on the RHS can be rewritten using the basic finite energy sum rule (FESR) relation,

\[
\int_{th}^{s_0} ds \, w(s) \rho(s) = \frac{-1}{2\pi i} \int_{|s|=s_0} ds \, w(s) \Pi(s) .
\]  

(2)

Analogous FESR’s, corresponding to spectral integrals, \( R_{ij}^{(k,m)} \), obtained by rescaling the kinematic weights in \( R_{ij} \) by \((1 - y_r)^k y_r^m \) before integration, are referred to as the “(k, m) spectral weight sum rules”. Similar spectral integrals and FESR’s can be constructed for \( s < m_s^2 \), for general non-spectral weights \( w(s) \), and for either of the correlator combinations \( \Pi^{(0+1)}_{ij} \) or \( s \Pi^{(0)}_{ij} \). We denote such spectral integrals generically by \( R_{ij}(s_0) \), and refer to the purely \( J = 0 \) contribution in “inclusive” FESR’s (those having both \( J = 0 + 1 \) and \( J = 0 \) contributions) as “longitudinal”, in what follows.

\( V_{us} \) and/or \( m_s \) are extracted using flavor-breaking differences, \( \delta R_{ij}^w(s_0) \), defined by

\[
\delta R_{ij}^w(s_0) = \left[ R_{ij}^w(s_0) / |V_{ij}|^2 \right] - \left[ R_{us}^w(s_0) / |V_{us}|^2 \right].
\]  

(3)

Since \( \delta R_{ij}^w(s_0) \) vanishes in the \( SU(3)_F \) limit, its OPE representation, \( \delta R_{OPE}^w(s_0) \), begins with a dimension \( D = 2 \) contribution, proportional to \( m_s^2 \). Experimental values for \( \delta R_{ij}^w(s_0) \) over a range of \( s_0 \) and \( w(s) \) allow \( m_s \) and/or \( V_{us} \) to be fitted, provided \( s_0 \) is large enough that insufficiently-well-known higher \( D \) OPE contributions are small \( [3, 4, 5, 6, 7, 8, 9] \). As stressed in Refs. \([7, 8, 9]\), the smallness of \( m_s \) (and hence of the flavor-breaking spectral integral differences) presents a challenge for the determination of \( m_s \), but a significant advantage for the determination of \( V_{us} \). Explicitly, one has, from Eq. \([3, 7]\),

\[
|V_{us}| = \sqrt{R_{us}^w(s_0) / \left( |R_{ud}^w(s_0)/|V_{ud}|^2| - \delta R_{OPE}^w(s_0) \right)} .
\]  

(4)

At scales \( \sim 2 - 3 \text{ GeV}^2 \), and for weights used in the literature, the dominant \( D = 2 \) term in \( \delta R_{OPE}^w(s_0) \) is much smaller than the separate \( ud, us D = 0 \) OPE contributions, and hence than the separate \( ud, us \) spectral integrals (for physical \( m_s \), typically at the few to several percent level). An uncertainty, \( \Delta (\delta R_{OPE}^w(s_0)) \), in \( \delta R_{OPE}^w(s_0) \) thus produces a fractional uncertainty in \( |V_{us}|, \sim \Delta (\delta R_{OPE}^w(s_0)) / 2 R_{us}^w(s_0), much smaller than that on \( \delta R_{OPE}^w(s_0) \) itself. Moderate precision for \( \delta R_{OPE}^w(s_0) \) thus suffices for high precision on \( |V_{us}| \), provided experimental errors can be brought under control.

In what follows, we perform a combined extraction of \( m_s \) and \( |V_{us}| \) based on existing spectral data. The \( ud \) data \([10, 11, 12]\) are already quite precise, but sizeable errors on the \( us \) data \([13, 14, 15]\) limit currently achievable precision. We thus focus on better controlling uncertainties on the theoretical (OPE) side of the analysis, especially those associated with slower-than-previously-anticipated convergence of the relevant \( D = 2 \) series \([16]\).
II. TECHNICAL ISSUES IN THE HADRONIC $\tau$ DECAY APPROACH

The first major stumbling block is the very bad behavior of the integrated longitudinal $D = 2$ OPE series. Even at the maximum scale $s_0 = m_\tau^2$ allowed by kinematics, the series shows no sign of converging [17]. Even worse, all truncation schemes employed in the literature, with no exceptions, badly violate constraints associated with spectral positivity [5]. A large part of the very strong unphysical $k$-dependence seen in results for $m_s$ from inclusive $(k, 0)$ spectral weight analyses was, in fact, associated with this spectral positivity violation problem [5]. Inclusive analyses employing the longitudinal OPE representation are thus untenable, and earlier results obtained using such analyses should be discarded in favor of those of more recent non-inclusive treatments [4, 7, 16].

Fortunately, the severe problems of the longitudinal $D = 2$ OPE representation are easily handled phenomenologically, for a combination of chiral and kinematic reasons. Apart from the $\pi$ and $K$ pole contributions, longitudinal spectral contributions vanish in the $SU(3)_F$ limit and are doubly-chirally suppressed away from it. Since the values of the longitudinal kinematic weight at the $K$ pole and in the region of the excited strange scalar and pseudoscalar (PS) resonances is essentially the same, this double chiral suppression is preserved in the ratio of non-pole to pole spectral integral contributions. The small residual non-pole $us$ PS and scalar contributions can, moreover, be well-constrained phenomenologically, the former via a sum rule analysis of the $us$ PS channel [4, 19], the latter via $K\pi$-scattering-data-based dispersive analyses [4, 18, 20] (the most reliable being the coupled-channel version discussed in Refs. [20], which incorporates short-distance QCD and chiral constraints). With the very accurately known $\pi$ and $K$ pole contributions, these results allow a reliable bin-by-bin subtraction of longitudinal contributions to the experimental distribution, and hence a direct determination of the $(0 + 1)$ spectral function, from which FESR’s not afflicted by the longitudinal $D = 2$ OPE problem can be constructed. Note that agreement between the values of $m_s$ obtained from scalar and PS sum rule analyses employing the resulting $us$ scalar and PS spectral “models” and those of recent $N_f = 2 + 1$ lattice simulations [22] rules out significantly larger non-pole longitudinal spectral strength. The residual non-pole part of the longitudinal subtraction is thus certainly small, and very well under control at the level required for $(0 + 1)$ FESR analyses. We therefore focus, in what follows, on sum rules involving the flavor-breaking combination $\Delta \Pi(s) \equiv \Pi_{V+A,uud}(s) - \Pi_{V+A,us}(s)$.

The second problem concerns the slow convergence of the $(0 + 1)$ $D = 2$ OPE series. For scales $s_0 \sim 2 - 3$ GeV$^2$, $[\Delta \Pi]_{OPE}$ is dominated by its $D = 2$ contribution [16],

$$[\Delta \Pi(Q^2)]_{D=2}^{OPE} = \frac{3}{2\pi^2} \frac{\bar{m}_s}{Q^2} \left[ 1 + 2.333\bar{a} + 19.933\bar{a}^2 + 208.746\bar{a}^3 + (2378 \pm 200)\bar{a}^4 + \cdots \right],$$

(5)

where $\bar{a} = \alpha_s(Q^2)/\pi$ and $\bar{m}_s = m_s(Q^2)$, with $\alpha_s(Q^2)$ and $m_s(Q^2)$ the running coupling and strange quark mass in the $\overline{MS}$ scheme. The $O(\bar{a}^4)$ coefficient has been estimated using approaches previously successful in obtaining accurate predictions for the $O(\bar{a}^3)$ coefficient in Eq. (5) and $n_f$-dependent $O(\bar{a}^3 m_q^2)$ coefficients of the electromagnetic current correlator in advance of the explicit calculations of these values [23].
Since, with 4-loop running, independent high-scale determinations of $\alpha_s(M_Z)$ correspond to $a(m_t^2) \simeq 0.10 - 0.11$, Eq. \[24\] shows that, at the spacelike point on $|s| = s_0$, the convergence of the $(0 + 1) D = 2$ OPE series is marginal at best, even at the highest scales accessible in $\tau$ decay. While $|\alpha_s(Q^2)|$ decreases as one moves along the contour away from the spacelike point, allowing the convergence of the integrated series to be improved through judicious weight choices, this observation shows that, for weights not chosen specifically with this constraint in mind, one must expect to find very slow convergence. Through judicious weight choices, this observation shows that, for weights not chosen from the spacelike point, allowing the convergence of the integrated series to be improved.

In evaluating the integrated $(0 + 1) D = 2$ OPE contribution, the size of the last term kept, the level of residual scale dependence, and the difference between the direct correlator and Adler function evaluations, both truncated at the same order, have all been used as measures of the truncation uncertainty. The slow convergence of the integrated series, however, can make it hard to be sufficiently conservative. For example, the quadrature sum of the last term size plus residual scale dependence, used previously in the literature to estimate the $O(\tilde{a}^3)$ Adler function truncation uncertainty, yields a result $\sim 2.5$ times smaller than the actual difference between the $O(\tilde{a}^3)$-truncated Adler function and $O(\tilde{a}^4)$-truncated direct correlator results \[8\]. Since the growth of $\alpha_s$ with decreasing scale makes higher order terms relatively more important at lower scales, premature truncation of a slowly-converging series typically shows up as an unphysical $s_0$-dependence in extracted, nominally $s_0$-independent quantities. With polynomial weights, $w(y) = \sum_m c_my^m$, for which integrated $D = 2N + 2$ OPE contributions not suppressed by additional powers of $\alpha_s$ scale as $c_N/s_0^N$, such unphysical $s_0$-dependence can also result if unsuppressed higher $D$ contributions which might in principle be present are incorrectly assumed negligible and omitted from the analysis. Since, typically, not even rough estimates of $D > 6$ condensate combinations are available, such omission is most dangerous for $w(y)$ having large values of the coefficients $c_m$, with $m > 2$, where such $D = 2m + 2 > 6$ contributions will be enhanced. The $(2, 0)$, $(3, 0)$ and $(4, 0) J = 0 + 1$ spectral weights, $w^{(2, 0)}(y) = 1 - 2y - 2y^2 + 8y^3 - 7y^4 + 2y^5$, $w^{(3, 0)}(y) = 1 - 3y + 10y^3 - 15y^4 + 9y^5 - 2y^6$, and $w^{(4, 0)}(y) = 1 - 4y + 3y^2 + 10y^3 - 25y^4 + 24y^5 - 11y^6 + 2y^7$, are examples of weights having such large, potentially dangerous higher order coefficients.

In view of the above discussion, $s_0$-stability tests are essential components of any FESR determination of $V_{us}$ and/or $m_s$. The existence of a stability window in $s_0$ for extracted quantities or, if not a stability window, then a window within which the observed instability is safely smaller than the estimated theoretical uncertainty, is crucial to establishing the reliability of the theoretical error estimate.
III. THE SPECTRAL AND NON-SPECTRAL WEIGHT ANALYSES

The OPE and spectral integral inputs used in our analysis are outlined below. The fact that we restrict our attention to FESR’s involving the V+A spectral combination, to weights satisfying \( w(s = s_0) = 0 \), and to scales \( s_0 > 2 \text{ GeV}^2 \), all serve to strongly suppress possible residual OPE breakdown effects [6, 26, 27].

Integrals of the leading \( D = 2 \) contribution to \( [\Delta \Pi(Q^2)]_{\text{OPE}} \), given in Eq. (5), are evaluated using two versions of the \( O(\bar{a}^4) \)-truncated CIPT prescription [29], one involving the RG-improved \( D = 2 \) correlator contribution \( [\Delta \Pi]_{\text{D=2}}^{\text{OPE}} \), the other the equivalent Adler function contribution, \( [\Delta \Pi]_{\text{D=2}}^{\text{OPE}} = -Q^2 d[\Pi(Q^2)]_{\text{D=2}}^{\text{OPE}} / dQ^2 \). The difference between the two expressions, which would agree to all orders, but differ here by terms of \( O(\bar{a}^5) \) and higher, is used as one measure of the truncation uncertainty. For \( \bar{a} \) and \( \bar{m}_s \) we employ exact solutions corresponding to the 4-loop-truncated \( \beta \) and \( \gamma \) functions [30], with initial condition \( \alpha_s(m^2) = 0 \). The remaining initial condition, \( m_s(2 \text{ GeV}) \), is either taken as input or obtained as part of the fit.

Expressions for the \( D = 4 \) and \( D = 6 \) contributions are given in Ref. [1]. The dominant \( D = 4 \) term, proportional to \( \langle m_s \bar{s}s \rangle \), is evaluated using ChPT quark mass ratios [28], GMOR for the light quark condensate, and \( r_c \equiv \langle m_\ell \ell \rangle / \langle m_s \bar{s}s \rangle = 0.8 \pm 0.2 \). \( D = 6 \) contributions are estimated using the vacuum saturation approximation, and assigned an uncertainty of \( \pm 500\% \). Contributions with \( D > 6 \) are assumed negligible, \( s_0 \)-stability studies being employed to test the self-consistency of this assumption.

For the spectral integrals we work with the ALEPH \( ud \) [11] and \( us \) [13] data, for which both data and covariance matrices are publicly available. A small global renormalization of the \( ud \) data is performed to reflect minor changes in the \( e, \mu \) and total strange branching fractions since the original ALEPH publication. Following the prescription of Ref. [31], we also perform mode-by-mode rescalings to reflect current (PDG06 [25]) values of the branching fractions of the various strange decay modes [32]. Errors on the \( K \) and \( \pi \) pole contributions are reduced by using the more precise expectations based on \( \Gamma[\pi\mu_2] \) and \( \Gamma[K\mu_2] \). With current data, \( ud \) and \( us \) spectral integral errors are at the \( \sim 0.5\% \) and \( \sim 3 - 4\% \) levels, respectively, for the weights to be discussed below. BABAR and BELLE will drastically reduce the size of the \( us \) errors in the near future.

A. The \((k, 0)\) spectral weight analyses

As noted above, slow convergence of the integrated \( D = 2 \) OPE series is both expected, and observed [8, 16], for the \((k, 0)\) spectral weights. Further evidence that the OPE sides of the \((k, 0)\) sum rules are not under good control is provided by Figure 1. The figure shows the 1\( \sigma \) contours for joint fits of \( m_s \) and \( |V_{us}| \) to the \( s_0 = m_s^2 \) experimental spectral integrals for various pairs of \((k, 0)\) spectral weights. It is clear that no sensible common fit region exists, and hence that a reliable joint fit for \( m_s \) and \( |V_{us}| \) cannot be obtained using the \((k, 0)\) spectral weights considered previously in the literature.

Further consideration is warranted for the \((0, 0)\) analysis, which has been proposed in the literature as a particularly favorable one for the determination of \( |V_{us}| \) [7].
Indeed, with $m_s$ from other sources as input, the $s_0 = m_s^2$ version of this analysis, would, if reliable, allow an improved determination of $|V_{us}|$ with only improved $us$ branching fractions as input, a feature special to this weight and this $s_0$ value. Unfortunately, on the OPE side, the $D = 2$ truncation uncertainty appears much larger than previously anticipated [8], as can be seen from (i) the rather poor agreement between the truncated correlator and Adler function evaluations; (ii) the strong deterioration in this situation with increasing truncation order; and (iii) the very significant $s_0$-instability in the output values of $|V_{us}|$, at least for the PDG04 value of $m_s$ used as input in Ref. [8]. In Figure 2 we demonstrate that this $s_0$-instability is not an artifact of the particular $m_s$ employed in Ref. [8]. The figure shows the OPE and spectral integrals as a function of $s_0$ for the $(0,0)$ FESR, for a range of different fixed input $m_s$. The value of $|V_{us}|$ needed for this comparison was obtained by matching the OPE and spectral integral versions of $\delta R^{(0,0)}(m_s^2)$. Note that very strong correlations exist amongst the OPE integrals for different $s_0$, and, similarly, amongst the spectral integrals for different $s_0$. Strongly discrepant $s_0$-dependences for the OPE and spectral integrals, as seen in the figure for the entire range of $m_s$ considered, thus rule out the possibility that an acceptable $s_0$-stability in $|V_{us}|$ might be obtained from the $(0,0)$ analysis, in its current form, for any input value of $m_s$. Even restricting our attention to the rather narrow range of $s_0$ values within 0.4 GeV$^2$ of $m_s^2$, the level of $s_0$-instability in $|V_{us}|$ is $>0.0020$, more than a factor of 2 larger than previous estimates of the total theoretical uncertainty. In Ref. [8], in order to obtain a more conservative estimate of the truncation uncertainty, we employed twice the sum, in quadrature, of the last included term and the integrated correlator-Adler function difference. The resulting estimated truncation uncertainty (total theoretical uncertainty) in $|V_{us}|$ was then $\pm 0.0020 (\pm 0.0022)$ for $s_0 = m_s^2$, compatible at least with the observed $s_0$-instability in the limited $s_0$ region noted above. While this estimate is considerably more conservative than those employed previously in the literature, we argue that, given the observed level of $s_0$-instability, less cautious assessments can not be justified. Since the $D = 2$ truncation uncertainty is unlikely to be reduced, we conclude that, unfortunately, theoretical uncertainties preclude a $(0,0)$ spectral weight determination of $|V_{us}|$ at the sub-$\pm 0.0020$ level. Fortunately, alternate weights with significantly improved $D = 2$ convergence behavior exist, allowing one to take advantage of the general approach proposed in Refs. [7].

B. The non-spectral-weight analyses

In Ref. [4], three non-spectral polynomial weights, $w_{10}(y)$, $\bar{w}_{10}(y)$, and $w_{20}(y)$, designed specifically to produce improved integrated $D = 2$ convergence, were constructed. The significantly improved convergence is displayed explicitly in Ref. [4], and Table 1 of Ref. [8]. The weights were also designed to (i) keep higher order coefficients which might enhance $D > 6$ contributions small, and, (ii) strongly suppress spectral integral contributions from the region above $s \sim 1$ GeV$^2$, where $us$ errors are large. Here we consider also a fourth weight, $w_{8}(y)$, with lower degree, but less strong suppression of the high-$s$ part of the $us$ spectrum. While the latter feature leads to larger $us$ spectral
integral errors with present data, the weight will be useful for future analyses based on data with smaller us spectral errors. Having an additional non-spectral weight also allows us to investigate more fully the issue of the mutual consistency of analyses associated with different improved-convergence non-spectral weight choices.

In Figure 3 we display the matches between the OPE and spectral integral differences for $w_{20}$ and $w_{10}$, for a range of input $m_s(2\text{ GeV})$. The spectral integral differences are computed as described above for the analogous $(0, 0)$ spectral weight case, shown in Figure 2. The $\hat{w}_{10}$ and $w_8$ matches are intermediate in quality, and hence not shown separately. The contrast with the $(0, 0)$ spectral weight case is immediately evident.

In Figure 4 we display the joint fit, $s_0 = m_T^2$, $1\sigma$ contours, analogous to those of Figure 1 for various pairs of the non-spectral weights. Also shown are the $1\sigma$ contours for two 3-fold and the full 4-fold fits. A good common fit region for $m_s$ and $|V_{us}|$ obviously exists, in sharp contrast to the situation for the $(k, 0)$ spectral weights.

We first consider results for $|V_{us}|$ obtained, as in previous studies, using single weight analyses with $s_0 = m_T^2$ and external $m_s$ input. With the average of strange scalar and PS sum rule and $N_f = 2 + 1$ lattice results, $m_s(2\text{ GeV}) = 94 \pm 6$ MeV, advocated in the last of Refs. [20], one finds $|V_{us}| = 0.2209 \pm 0.0029_{\text{exp}} \pm 0.0017_{\text{th}}$ for $w_{20}$, $0.2210 \pm 0.0030_{\text{exp}} \pm 0.0010_{\text{th}}$ for $\hat{w}_{10}$, $0.2206 \pm 0.0032_{\text{exp}} \pm 0.0007_{\text{th}}$ for $w_{10}$, and $0.2218 \pm 0.0037_{\text{exp}} \pm 0.0009_{\text{th}}$ for $w_8$, with large experimental errors, dominated by those on the us distribution. The results are consistent, but cannot be directly averaged due to strong correlations.

A cautious approach, given the intrinsically slow convergence of the $(0 + 1)$ $D = 2$ series, is to ignore external information on $m_s$, perform a combined fit for $m_s$ and $|V_{us}|$, and verify that the results for $m_s$ are consistent with what is known from other sources. The results for the 3-fold, $s_0 = m_T^2$, $w_{20}$, $\hat{w}_{10}$, $w_{10}$ fit, including theoretical errors on the same footing as experimental ones in the minimization process, are $m_s(2\text{ GeV}) = 89 \pm 26$ MeV and $|V_{us}| = 0.2202 \pm 0.0046$. The analogous full 4-fold non-spectral weight fit yields $m_s(2\text{ GeV}) = 96 \pm 31$ MeV and $|V_{us}| = 0.2208 \pm 0.0052$. The larger errors in the second case result from the less strong suppression of the high-s us data region by $w_8$.

Since the combined joint fit results for $m_s$ are in excellent agreement with the external average noted above, it makes sense now to perform a combined $s_0 = m_T^2$ fit for $|V_{us}|$ with $m_s(2\text{ GeV}) = 94 \pm 6$ MeV as input. The lower-error 3-fold fit (without $w_8$) yields

$$|V_{us}| = 0.2209 \pm 0.0031,$$  \hspace{1cm} (6)

which we take as our main result. Note that, had the fit been performed without including the uncertainty on the central value for $m_s$, the output central value for $|V_{us}|$ would have been shifted up to 0.2221. The result of Eq. (6), though lower in central value, is compatible, within mutual errors, with both the recent $K^{\ell\ell}$ determination, $|V_{us}| = 0.2249 \pm 0.0019$ [33] (based on the conventional Leutwyler-Roos estimate for $f_+(0)$) and the $\Gamma[K_{\mu2}]/\Gamma[\pi\mu2]$ determination, $|V_{us}| = 0.2223^{+0.0026}_{-0.0035}$ (based on the updated 2006 MILC evaluation of $f_K/f_\pi$ [22]). We comment further below on the status/reliability of the current hadronic-\tau-decay-based analysis.
IV. DISCUSSION, CONCLUSIONS AND PROSPECTS

We have shown that it is possible to construct non-spectral weights which improve the convergence behavior of the integrated $D = 2$ OPE series and allow a self-consistent joint fit for $m_s$ and $|V_{us}|$. While the results reported above for $|V_{us}|$ are compatible within errors with those from other sources, one should bear in mind that the errors on the present hadronic $\tau$ decay data are sufficiently large that non-trivial shifts in the central value are likely when the B-factory data becomes available. To see why this is the case, note that the branching fractions for observed strange decay modes in the current data go down only to the $\sim 3 \times 10^{-4}$ level. Missing modes with lower branching fractions could have a non-trivial impact on the extracted value of $|V_{us}|$. Missing a strange mode with branching fraction $1 \times 10^{-4}$, for example, would lower $|V_{us}|$ by $\sim 0.0004$. At the desired level of precision, one thus needs to detect all strange modes with branching fractions at the few-to-several-$10^{-5}$ level. Such determinations are certainly feasible at the B-factory experiments where preliminary results for branching fractions at this level (with errors at the few-to-several-$10^{-6}$ level) have already been reported [34].

Regarding the goal of improved future precision, one should note that the $\sim 1/2\%$ uncertainty on the $ud$ spectral integrals, though negligible in comparison to the $us$ errors at present, contributes $\sim 0.0005$ to the uncertainty on $|V_{us}|$, assuming the current assessment of the non-strange spectral errors is correct. One should also bear in mind here the disagreement between the isovector vector spectral contributions implied by isospin-breaking electroproduction data and those measured directly in $\tau$ decay experiments [35]. For example, were the electroproduction results for $\pi\pi$ (whose implied $\tau \rightarrow \pi\pi\nu_\tau$ branching fraction is $4.5\sigma$ below that measured directly in $\tau$ decay) to be correct, the value of $|V_{us}|$ obtained from the $\tau$ decay analysis above would be raised by $\sim 0.0018$. Even the slightly lower central value of the preliminary BELLE $\tau \rightarrow \pi\pi\nu_\tau$ measurement (which is compatible, within its somewhat larger systematic error, with the earlier $\tau$ world average value) would produce an increase of $\sim 0.0006$ in $|V_{us}|$. Resolving the discrepancy between the $\pi\pi$ electroproduction and $\tau$ decay results is thus important not just for clarifying expectations for the hadronic contribution to $(g - 2)_\mu$ in the Standard Model, but also for finalizing future $\tau$ decay determinations of $|V_{us}|$.

We finally comment that, re-running the analysis with the central values of the $us$ spectral distribution unchanged but with errors reduced by a factor of 5 (covariances by a factor of 25), and assuming $m_s(2 \text{ GeV})$ to be known to $\pm 5 \text{ MeV}$, a combined total fit error below 0.0010 is achieved. Such an improvement in the $us$ spectral distribution errors should be well within reach of the final B-factory analyses [36]. Further improvement on the theoretical component of the total error is almost certainly also possible through the use of new weights with significantly less strong suppression of the high-$s$ region. Such weights, however, will become useful only once the errors in the $us$ spectrum above the $K^*$ have been significantly reduced. The necessity of self-consistency and stability checks also means that an interaction between theorists and experimentalists will certainly be required for successful development of such weights.
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FIG. 1: $m_s, |V_{us}|$ joint fits using $s_0 = m_T^2$ and various pairs of $(k,0)$ spectral weights

\[ V_{us} \cdot m_s \text{ One-Sigma Contours} \]

FIG. 2: $(0,0)$ spectral weight OPE and spectral integrals vs. $s_0 \text{ (GeV}^2\text{)}$ for various input $m_s(2 \text{ GeV})$ and $|V_{us}|$ obtained by matching the resulting OPE and spectral integrals at $s_0 = m_T^2$
FIG. 3: $w_{20}$ and $w_{10}$ OPE and spectral integrals (GeV$^2$) vs. $s_0$ for various input $m_s$ (2 GeV) and $|V_{us}|$ obtained by matching the resulting OPE and spectral integrals at $s_0 = m_s^2$.

FIG. 4: 2-, 3- and 4-fold, $s_0 = m_s^2$, joint $m_s$, $|V_{us}|$ non-spectral weight fits