Three-dimensional eccentric discs around Be stars

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ABSTRACT

One-armed oscillation modes in the circumstellar discs of Be stars may explain the cyclical variations in their emission lines. We show that a three-dimensional effect, involving vertical motion and neglected in previous treatments, profoundly influences the dynamics. Using a secular theory of eccentric discs that reduces the problem to a second-order differential equation, we show that confined prograde modes are obtained for all reasonable disc temperatures and stellar rotation rates. We confirm these results using a numerical analysis of the full set of linearized equations for three-dimensional isothermal discs including viscous terms that couple the horizontal motions at different altitudes. In order to make these modes grow, viscous damping must be overcome by an excitation mechanism such as viscous overstability.

Key words: accretion, accretion discs — circumstellar matter — hydrodynamics — stars: emission-line, Be

1 INTRODUCTION

Classical Be stars (Porter & Rivinius 2003) are rapidly rotating early-type stars that exhibit Balmer emission lines. It is widely agreed that these lines, which are generally double-peaked, originate in a relatively thin circumstellar disc that is in approximately Keplerian rotation (Okazaki 2007, and references therein). While the precise mechanism by which the disc is formed remains controversial, it is likely to resemble a viscous decretion disc that is expelled by the action of a torque at its inner boundary (Lee et al. 1991; Porter & Rivinius 2003).

Many Be stars show cyclical variations in their double-peaked emission lines over years or decades, with the red and blue peaks alternately becoming more prominent (Okazaki 1997, and references therein). An explanation of this phenomenon was given by Okazaki (1991), who proposed that a low-frequency, one-armed oscillation mode (Kato 1983) occurs in the disc. This is equivalent to saying that the disc becomes eccentric and the slow precession of its elliptical shape gives rise to the cyclical changes in the observed emission lines.

Eccentric discs, in which fluid elements, solid particles or stars follow elliptical orbits of variable eccentricity around a central mass, have further applications in systems as diverse as planetary rings, protoplanetary systems, close binary stars and galactic nuclei. A detailed understanding of the origin of eccentricity and the rates of precession in the circumstellar discs of Be stars would therefore be of general interest.

Okazaki (1991) originally considered a disc that orbits in a point-mass potential and obtained a sequence of retrograde modes in which the precession of the disc is in a direction opposite to its rotation. Retrograde precession is a natural consequence of the pressure forces in the disc, which cause a small departure from Keplerian rotation and allow the eccentricity to propagate in a wavelike manner. The global modes found by Okazaki (1991) are weighted towards the outer part of the disc and their periods become extremely long as the outer radius of the disc is increased to realistic values.

Papaloizou et al. (1992) and Savonije & Heemskerk (1993) considered the effect of the quadrupole gravitational potential associated with the rotational deformation of the star, which tends to cause a prograde precession of elliptical orbits. They showed that, when the quadrupole effect is taken into account, prograde modes can be obtained that are naturally confined in the inner part of the disc and are insensitive to the outer boundary condition. Subsequent observations confirmed that the precession is indeed prograde (Telting et al. 1994).

However, Okazaki (1997) found that confined prograde modes can be obtained only when the disc is sufficiently cool, so that the quadrupole effect dominates over the tendency of pressure to produce retrograde precession and extended modes. He concluded that a different mechanism is required in the hotter discs of early-type Be stars such as the prototype γ Cas, and proposed that radiative line forces could explain the required prograde precession. Unfortunately the modelling of radiative forces is subject to considerable uncertainty. Fürt & Harmanec (2006) found that the resulting
model has little predictive power owing to the sensitivity of the results to the parameters.

More recently, Papaloizou & Savonije (2006) investigated an alternative way to obtain prograde modes in hotter discs. This involves replacing the rigid inner boundary condition at the stellar surface with a free boundary condition, on the basis that a gap is formed between the star and the disc. Such a gap is not generally expected in the scenario of the viscous decretion disc but might be possible in alternative models.

All of the treatments described so far are based on two-dimensional models of the disc that neglect aspects of its vertical structure and motion. At first sight, such an approach seems reasonable for studying eccentric modes in thin discs, where the motion might be assumed to be purely horizontal and independent of height. However, in presenting a three-dimensional, non-linear theory of eccentric discs, we have previously argued that three-dimensional effects are of considerable importance (Ogilvie 2001). In particular, the variation of the vertical gravitational acceleration around an elliptical orbit excites an oscillatory vertical motion in an eccentric disc that should not be neglected.

In this paper we show that, in fact, three-dimensional effects are essential to understanding the precession of eccentric discs around Be stars. They allow confined prograde modes to be obtained even when the stellar quadrupole moment is negligible and when the inner boundary is rigid. This property allows us to give a unified description of eccentric discs of Be stars of all stellar types without introducing uncertain radiative forces or modifying the inner boundary condition. While we do not deny that in some cases radiative forces might be important, or that the inner boundary condition might differ from a rigid one, we show that these innovations are unnecessary.

Most previous analyses have not discussed the processes that could cause eccentric modes to grow or decay, and which are therefore relevant to explaining the occurrence of eccentric discs around Be stars. Viscous overstability (Katz 1978; Latter & Ogilvie 2006) provides a possible explanation; Negueruela et al. (2001) have applied this idea to Be stars and estimated the associated growth rate. We defer to a future investigation a detailed analysis of the effects that could address the saturation of the growth mechanism and attempt to predict the observed amplitudes and precession rates of the eccentric modes, also remains to be undertaken.

The structure of this paper is as follows. In Section 2 we describe the basic state of the disc. We then discuss an approximate, secular theory of three-dimensional eccentric discs in Section 3, comparing it with the two-dimensional theories used by other authors. In Section 4 we solve the full linearized equations accurately using a spectral method, including some effects of viscosity, and compare the results with those of the secular theory. Conclusions are given in Section 5.

2 BASIC STATE OF THE DISC

We consider a basic state consisting of a steady, axisymmetric disc around a rotating star. Adopting cylindrical polar coordinates \((r, \phi, z)\), we write the gravitational potential of the star as

\[
\Phi = -GM(r^2 + z^2)^{-1/2} \left[ 1 + \frac{Q}{3} \frac{(r^2 - 2z^2)R^2}{(r^2 + z^2)^2} \right],
\]

where \(M\) and \(R\) are its mass and (equatorial) radius. This potential consists of monopole and quadrupole components, the latter being of dimensionless strength \(Q\) and arising from the rotational deformation of the star. (The parameter \(Q\) is related to the gravitational moment \(J_2\) used in planetary science by \(Q = 3J_2/2\).) For a star with uniform angular velocity \(\Omega_*,\) we have \(Q = k_2\Omega_3^2R^3/GM\), where \(k_2\) is the apsidal motion constant. We neglect higher-order multipole components as well as the self-gravitation of the disc.

In common with most other treatments, we assume that the stellar radiation maintains the disc at a constant temperature \(T_d\), slightly less than the effective temperature \(T_\star\) of the star. The isothermal sound speed \(c_s = (RT_d/\mu)^{1/2}\) is therefore constant, both in the equilibrium state and for the perturbations. We neglect any viscous or turbulent stresses and any resulting meridional motions in the disc.

The basic state then has velocity \(u = r\Omega(r)e_\phi\). The angular velocity \(\Omega\) is independent of \(z\) because the basic state is isothermal and therefore barotropic. The balance of forces requires \(r\Omega^2 e_\phi = \nabla(\Phi + h),\) where \(h = c_s^2 \ln \rho\) is the (pseudo-)enthalpy of an isothermal gas.

The potential and enthalpy in the midplane are \(\Phi_m(r) = \Phi(r, 0) \approx -[GM/r][1 + (Q/3)(R/r)^2]\) and \(h_m(r) = \Phi(r, 0),\) and the radial force balance implies

\[
\Omega^2 = \frac{GM}{r^3} \left[ 1 + Q \left( \frac{R}{r} \right)^2 \right] + \frac{1}{r} \partial_r h_m.
\]

The associated epicyclic frequency \(\kappa(r)\) is given by

\[
\kappa^2 = \frac{1}{r^2} \partial_r (r^4 \Omega^2) = \frac{GM}{r^3} \left[ 1 - Q \left( \frac{R}{r} \right)^2 \right] + \frac{1}{r^3} \partial_r (r^3 \partial_r h_m).
\]

We also refer to the Keplerian angular velocity \(\Omega_K(r)\) defined by

\[
\Omega_K^2 = \frac{GM}{r^3}.
\]

For a thin disc we may expand the potential about the midplane to obtain \(\Phi \approx \Phi_m + \frac{1}{2}\Omega^2 z^2,\) where the vertical frequency \(\Omega_z(r)\) is given by

\[
\Omega_z^2 = \frac{GM}{r^3} \left[ 1 + 3Q \left( \frac{R}{r} \right)^2 \right].
\]

The vertical force balance then implies \(h = h_m - \frac{1}{2}\Omega_z^2 z^2\) and so \(\rho = \rho_m \exp[-(z^2/2H^2)],\) where \(\rho_m(r)\) is the midplane density and \(H(r) = c_s/\Omega_z\) is the scaleheight. The surface density is \(\Sigma(r) = (2\pi)^{1/2} \rho_m H.\)

Since the basic state is steady and axisymmetric, wave modes may be considered in which the dependence on azimuth and time is of the form \(\exp(\pm\phi - \omega t),\) where \(m\) is the azimuthal wavenumber and \(\omega\) is the wave frequency. In the case \(m = 1\) that is of interest here, \(\omega\) is also the angular pattern speed of the mode in an inertial frame of reference. This can be identified with the precession rate of the eccentric disc, which is positive if the precession is prograde (i.e. in the same direction as the rotation of the disc).
3 SECULAR TREATMENT

3.1 Comparison of two- and three-dimensional theories

An approximate theory can be derived in which the disc is assumed to be nearly Keplerian and the frequency $\omega$ of the wave is much less than the orbital frequency $\Omega$ of the disc. This type of approximation, which is related to the secular theory of celestial mechanics, has been discussed in the two-dimensional linear case by Tremaine (2001), Papaloizou (2002) and Goodchild & Ogilvie (2006), and in the three-dimensional non-linear case by Ogilvie (2001). The derivation of this theory in the case of a three-dimensional isothermal disc around a Be star is presented in Appendix A which draws on the analysis of Section 4 below. The two-dimensional approximation is also discussed there.

In the secular theory the radial velocity $u(r)$ satisfies the equation

$$\omega - f)u = \frac{1}{2r^2} \partial_r \left[ \frac{c^2 \Sigma}{\Omega} \partial_r \left( \frac{\Sigma u}{r \Omega} \right) \right],$$

where $f(r)$ is given by

$$f = \frac{\Omega^2 - \kappa^2}{2\Omega}$$

in a two-dimensional disc, but

$$f = \frac{\Omega^2 - \kappa^2}{2\Omega} + \frac{9\epsilon^2}{4r^2\Omega}$$

in a three-dimensional disc. With the dependence $e^{i \phi - \kappa t}$ assumed above, the radial velocity is related to the complex eccentricity $E(r) = e e^{i \phi}$, where $e$ is the eccentricity and $\phi(r)$ is the longitude of periastron measured in a frame of reference that rotates with the angular pattern speed $\omega$ of the mode, by $u^* = i r \Omega E$ (Ogilvie 2001). It can be seen from equation (6) that $f$ is a local contribution to the global precession rate $\omega$ of the eccentric mode. Indeed, an integral expression for the mode frequency in the case of rigid boundary conditions $\Sigma(u = 0 \text{ at } r = r_{\text{in}} \text{ and } r = r_{\text{out}})$ is

$$\omega = \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{\Sigma r |u|^2}{\Omega} \, dr = \int_{r_{\text{in}}}^{r_{\text{out}}} f \frac{\Sigma r |u|^2}{\Omega} \, dr$$

$$- \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{c^2 r^3}{2\Omega} \left( \partial_r \left( \frac{\Sigma u}{r \Omega} \right) \right)^2 \, dr,$$

which follows from equation (6) after multiplication by $\Sigma u/\Omega$ and an integration by parts. The first term on the right-hand side shows the contribution to the precession rate from $f$ in the form of an integral weighted by the structure of the mode, while the second term shows a retrograde contribution associated with pressure. (Note that the pressure also contributes indirectly through its effect on $f$.)

In a two-dimensional disc the integral over $u$ is given by equation (10) and corresponds to the expected pressure for the local apsidal precession rate. Note that, in this case, $f \approx \Omega - \kappa$ if, as assumed, the disc is nearly Keplerian ($|\Omega^2 - \kappa^2| \ll \Omega^3$). There is, in fact, more than one version of the two-dimensional theory. Savonije & Heemskerk (1993) work throughout with vertically averaged equations and find

$$f = \frac{Q}{r^5 \Omega} \frac{\Omega}{\Omega^2} - \frac{c^2}{2r^2\Omega} \partial_r (r^2 \partial_r \ln \Sigma).$$

Okazaki (1997) calculates $\Omega$ and $\kappa$ using a three-dimensional equilibrium disc but then applies vertically averaged equations for the perturbations. He therefore uses

$$f = \frac{Q}{r^5 \Omega} \frac{\Omega}{\Omega^2} - \frac{c^2}{2r^2\Omega} \partial_r (r^2 \partial_r \ln \rho_m) + \frac{9\epsilon^2}{4r^2\Omega}$$

The last term represents an additional local contribution to the prograde precession of the disc that arises from the three-dimensional dynamics including the vertical motion; it is discussed further in Section 3.2 below. For the parameters relevant to Be stars, this term is never negligible (it corresponds to a precession period of order 1 yr at the stellar surface) and declines much more slowly with radius than the quadrupole term.

We refer to the three theories described above as ‘2DS’ (equation (10) above; Savonije & Heemskerk 1993; Papaloizou & Savonije 2004), ‘2DO’ (equation (11) above; Okazaki 1994, 1997) and ‘3D’ (equation (12)). Note, however, that the secular approximations were usually not employed in the cited papers. The 3D secular theory is based on similar assumptions and approximations to the non-linear analysis of Ogilvie (2001). In particular, some viscous or other stress is required to couple different layers in the disc so that they tend to adopt the same eccentricity. In the absence of such stresses the eccentricity may vary significantly with $z$ and the results can differ. This complication is discussed in Section 4 where the full linearized equations are solved.

3.2 Application to power-law discs

It is consistent with the spirit of the secular approximation to neglect the differences between $\Omega$, $\kappa$, $\Omega_\xi$ and $\Omega_K$ except where essential (i.e. in the quantity $\Omega^2 - \kappa^2$). Accordingly, we may take $\Omega \propto r^{-3/2}$ and $H \propto r^{1/2}$. For a surface density profile $\Sigma \propto r^{-\sigma}$ the density in the midplane varies as $\rho_m \propto r^{\sigma+3/2}$. We introduce the dimensionless radial coordinate

$$x = \frac{r}{R}$$

and the small parameter

$$\epsilon = c_m \left( \frac{R}{GM} \right)^{1/2},$$

which is a measure of the angular semithickness $H/r$ of the disc at the stellar surface. The three theories then give

2DS: $\frac{f}{\Omega} = Q x^2 - \frac{3}{2} \epsilon^2 x$,

2DO: $\frac{f}{\Omega} = Q x^2 + \frac{1}{2} \left( \sigma + \frac{3}{2} \right) \epsilon^2 x$,

3D: $\frac{f}{\Omega} = Q x^2 + \frac{1}{2} \left( \sigma + 6 \right) \epsilon^2 x$.  

We therefore consider the generic form
\[
\frac{f}{\Omega} = Qx^{-2} + sx^2 x, \tag{18}
\]
where \( s \) is a constant. It is convenient to work with dimensionless, rescaled values of the mode frequency and quadrupole strength, \( \tilde{\omega} \) and \( \tilde{Q} \), defined by
\[
\omega = \epsilon^2 \left( \frac{GM}{R^3} \right)^{1/2} \tilde{\omega},
\]
\[
Q = \epsilon^2 \tilde{Q}. \tag{19}
\]
Equation (18) then becomes
\[
\left[ \tilde{\omega} - \left( \tilde{Q} x^{-7/2} + sx^{-1/2} \right) \right] u = \frac{1}{2x^2} \partial_x \left[ x^{\sigma+3} \partial_x (x^{-\sigma+1/2} u) \right], \tag{20}
\]
which is an eigenvalue problem of Sturm–Liouville form, when considered together with appropriate boundary conditions. Note that the parameter \( \epsilon \) drops out of the equation under the above rescalings. For prograde modes (\( \tilde{\omega} > 0 \)), one solution of this equation decays exponentially as \( x \to \infty \), while the other solution grows exponentially. We therefore consider the equation in the domain \( 1 < x < \infty \), requiring solutions to satisfy a rigid boundary condition \( u = 0 \) at the stellar surface \( x = 1 \) and to decay as \( x \to \infty \).

Reasonable values of \( \tilde{Q} \) for Be stars can be estimated following Firt & Harmanec (2006). Their Table 4 is based on stellar models of intermediate main-sequence age. There is considerable uncertainty in the applicability of these models because of the rapid rotation of Be stars. Nevertheless, using these data, we estimate that \( \tilde{Q} \) ranges from approximately 8 at the lower end (\( M = 2.51 \, M_\odot \)) to approximately 12 at the upper end (\( M = 15.85 \, M_\odot \)). These values assume, somewhat arbitrarily, a disc temperature of \( T_d = (2/3) T_e \) and a stellar rotation rate that is 95% of the critical value. Similarly, for the stellar models of \( \gamma \) Cas and 59 Cyg described in Tables 2 and 3 of Firt & Harmanec (2006), we estimate \( \tilde{Q} \approx 9 \) and 13, respectively. It is clear that significantly larger values of \( \tilde{Q} \) cannot be obtained by further increasing the stellar rotation rate, and in fact smaller values may be more appropriate. The values of \( \epsilon \) for all these models are in the range 0.021–0.026, again assuming that \( T_d = (2/3) T_e \).

### 3.3 Solutions in the absence of a quadrupole

Consider first the case without a quadrupole term, \( \tilde{Q} = 0 \).

The solution that decays as \( x \to \infty \) is then
\[
u = 2[(\sigma + 2)^2 - 8s]^{1/2}, \quad z = 4(2\tilde{\omega})^{1/2} x^{1/4}, \tag{23}
\]
with
\[
u = 2[(\sigma + 2)^2 - 8s]^{1/2}, \quad z = 4(2\tilde{\omega})^{1/2} x^{1/4}, \tag{23}
\]
where \( K_\nu \) is the modified Bessel function of the second kind of order \( \nu \) (Abramowitz & Stegun 1965). Note that \( z \) is real and positive for the prograde modes of interest. This solution has no zeros in \( z > 0 \) if \( \nu \) is real, but does so if \( \nu \) is imaginary.

To match a rigid boundary condition at \( x = 1 \) we therefore require \( \nu^2 < 0 \). The 2DS theory gives \( 8s = 4\sigma \), so \( \nu^2 = 4(\sigma^2 + 4) \) and modes are never confined. The 2DO theory gives \( 8s = 4\sigma + 6 \), so \( \nu^2 = 4(\sigma^2 - 2) \) and modes may be confined for \( \sigma^2 < 2 \). The 3D theory gives \( 8s = 4\sigma + 24 \), so \( \nu^2 = 4(\sigma^2 - 20) \) and modes may be confined for \( \sigma^2 < 20 \).

### 3.4 Critical quadrupole strength

The integral expression for the dimensionless frequency eigenvalue of a mode that satisfies a rigid boundary condition at the stellar surface and decays as \( r \to \infty \) is (cf.
equation.[4]  
\[
\tilde{\omega} \int_1^\infty x^{-\sigma+5/2} u^2 \, dx = \int_1^\infty (\tilde{Q} x^{-\sigma-1} + sx^{-\sigma+2}) u^2 \, dx \\
- \frac{1}{2} \int_1^\infty x^{\sigma+3} \left[ \partial_x (x^{-\sigma+1/2} u) \right]^2 \, dx,
\]
where we take \( u \) to be real. Following [Papaloizou & Savonije 2006], we note that this expression has the usual variational property associated with self-adjoint eigenvalue problems. In this case, a confined prograde mode exists if and only the right-hand side of this equation can be made positive by a trial function \( u(x) \) satisfying the appropriate boundary conditions. This is clearly possible if either \( \tilde{Q} \) or \( s \) is large enough. What is the minimum value of \( \tilde{Q} \) (for a given value of \( s \)) such that the right-hand side can just be made to vanish for a non-trivial \( u \)? This happens when  
\[
\tilde{Q} \int_1^\infty x^{-\sigma-1} u^2 \, dx = \frac{1}{2} \int_1^\infty x^{\sigma+3} \left[ \partial_x (x^{-\sigma+1/2} u) \right]^2 \, dx \\
- \int_1^\infty sx^{-\sigma+2} u^2 \, dx.
\]
The minimum value of \( \tilde{Q} \) for which this equation can be satisfied is given by the corresponding Euler–Lagrange equation, which is just equation (21) with \( \tilde{\omega} \) set to zero. The solutions are  
\[
u = \frac{1}{3} \left( \frac{\sigma + 2}{\sigma - 1} \right)^{1/2}, \quad w = \frac{\tilde{Q}^{1/2}}{3} \left( \frac{x}{\tilde{Q}} \right)^{-3/2}, \]
with
\[
\mu = \frac{\nu}{6} = \frac{\sigma + 2}{3 \sigma - 1}, \quad w = \left( \frac{\tilde{Q}}{3} \right)^{3/2}.
\]

If \( \mu^2 < 0 \) we have seen that confined modes can be found, in principle, even if \( \tilde{Q} = 0 \). Consider then the case \( \mu^2 > 0 \). The solution that decays as \( x \to \infty \) is \( J_\mu(w) \). The critical condition for a rigid inner boundary condition is therefore \( \tilde{Q} = 9w_\mu^4/8 \), where \( w_\mu \) is the first zero of \( J_\mu(w) \) in \( w > 0 \). (This is an increasing function of \( \mu \) and therefore a decreasing function of \( s \).)

For example, if \( \sigma = 2 \), we require \( \tilde{Q} > 10.8 \) for confinement in the 2DO theory, or \( \tilde{Q} > 15.9 \) in the 2DS theory. These values are not very sensitive to \( \sigma \); [Papaloizou & Savonije 2006] quote \( \tilde{Q} > 17.3 \) for \( \sigma = 5/2 \) and a rigid inner boundary condition, which agrees with this analysis.

### 3.5 Schrödinger analogy

A useful description of confined modes uses an analogy with bound states in quantum mechanics. If \( y = x^{1/4} \) and \( u = x^{\sigma/2-13/8} \psi(y) \), then \( \psi(y) \) satisfies the Schrödinger equation
\[ - \frac{d^2 \psi}{dy^2} + [V(y) - E]\psi = 0 \quad (28) \]
with an effective potential
\[ V(y) = \frac{1}{4} (16\sigma^2 + 64\sigma + 63 - 128s)y^{-2} - 32\tilde{\omega}y^{14} \quad (29) \]
and an effective energy eigenvalue \( E = -32\tilde{\omega} \). The coefficient of \( 1/y^2 \) in \( V \) is \((16\sigma^2 + 63)\) for the 2DS theory, \((16\sigma^2 - 33)\) for 2DO and \((16\sigma^2 - 321)\) for 3D. It is therefore likely to be negative only in the 3D theory. Note that the disc occupies the region \( y > 1 \). A bound state of negative energy, equivalent to a confined prograde mode, can be obtained if there is a sufficiently deep and wide potential well. The \( Q \) term tends to create a deep well close to the stellar surface. In the 2D theories it competes with the \( y^{-2} \) term, which contributes a repulsive potential. In the 3D theory, however, the \( y^{-2} \) term creates a much wider well, allowing broader modes with slower precession rates, even if \( \tilde{Q} = 0 \). These potentials are plotted in Fig. 2 for the case \( \sigma = 2 \) and \( \tilde{Q} = 10 \). For these parameters the 3D potential is deep and wide enough to support a bound state in the case of a rigid inner boundary condition, while the others are not.

### 3.6 Precession rates and mode shapes

We now compute the eigenvalues \( \tilde{\omega} \) of equation (21) by a shooting method, adopting a rigid outer boundary condition at the stellar surface and seeking prograde confined modes that decay as \( x \to \infty \). (In practice this is done by imposing a rigid outer boundary condition and verifying that the eigenvalue is completely insensitive to the value of the outer radius provided it is sufficiently large.) The results are shown in the left panel of Fig. 3. Here we confirm that confined prograde modes exist in the 2D theories only for sufficiently
large values of $\tilde{Q}$, as described in Section [34]. In contrast, the 3D theory allows such modes to be obtained for any value of $\tilde{Q}$. Furthermore, the precession rates are much larger in the 3D theory. Since the realistic values of $\tilde{Q}$ are probably in the vicinity of 10 or smaller, it is clear that the three-dimensional effects are of essential importance in the case of a rigid inner boundary condition.

In the right panel of Fig. 3 we show comparable results for the free inner boundary condition considered by Papaloizou & Savonije (2006), meaning that the Lagrangian pressure perturbation is zero at $r = R$. In the secular theory this is equivalent to $\partial_r E = 0$, or $\nu + 2r \partial_r u = 0$. As described by Papaloizou & Savonije (2006), a free inner boundary condition allows prograde modes to be found for smaller values of $\tilde{Q}$ in the 2DS theory. It is still true, however, that the three-dimensional effects have an important effect on the results. Since the eigenfunctions obtained with a free inner boundary condition are generally peaked at the inner radius, there may also be a conflict between the nonlinear development of such an eccentric mode and the existence of a stellar surface, unless there is a wide gap between the star and the disc.

To convert these eigenvalues into physical units, we again make use of the stellar models in Firt & Harmanec (2006). We then find that the precession period is

$$P = \frac{C_P}{\omega} \text{ yr},$$

(30)

where $C_P$ ranges from approximately 1.6 at the lower end to approximately 2.3 at the upper end (or 2.4 for $\gamma$ Cas and 1.8 for 59 Cyg). This conversion factor is independent of assumptions regarding the stellar rotation rate, except inversely as the rotation affects the equatorial radius, but $C_P$ is inversely proportional to the assumed disc temperature.

The shapes of the confined modes in the 3D theory are illustrated in Fig. 4. For larger values of $\tilde{Q}$, the mode is increasingly confined in the inner part of the disc. Although higher-order modes may exist, one of which is referred to in Fig. 3, we focus here on the fundamental confined mode with the simplest radial structure.

Since $\omega$ is close to 1 in the 3D theory for reasonable values of $\tilde{Q}$ and with a rigid inner boundary condition, precession rates in the vicinity of 1–3 yr are obtained. Observed cycle times, which can vary significantly even for the same star, are more usually in the range 5–10 yr (Okazaki 1997). This discrepancy might occur because the discs have different temperature or density profiles from those assumed here, either of which would affect the precession period. Non-linearity of the eccentric mode may also increase the precession period by altering the distribution of eccentricity so that it is less peaked in the inner part of the disc. Furthermore, non-isothermal effects or radiative forces might be important. In some cases, including $\gamma$ Cas and 59 Cyg, the presence of a relative close binary companion may also affect the dynamics of the eccentric mode, although it would seem most likely to contribute to the prograde precession and therefore to decrease the precession period.

### 3.7 Physical interpretation of the three-dimensional dynamics

What is the origin of the additional prograde precession that occurs in a three-dimensional disc? As described in Appendix A and Section [4] below, two different types of motion
are involved in an eccentric disc. One (corresponding to the
n = 0 mode in later sections) consists of horizontal veloci-
ties and enthalpy perturbations that are independent of z;
this describes the eccentric orbit motion of the gas. The other
(corresponding to n = 2) involves a vertical velocity propor-
tional to z and an enthalpy perturbation proportional to
z/\frac{\partial^2}{\partial z^2}; this is a vertical 'breathing' mode of the disc.
Coupling of these motions occurs because of the variation
of the vertical gravitational acceleration (or, equivalently,
the vertical frequency \Omega_z, or the scaleheight H) with ra-
dius. Vertical hydrostatic equilibrium cannot be maintained
in an eccentric disc because a fluid element in an elliptical
orbit experiences a vertical gravitational acceleration that
oscillates with the orbital frequency. The breathing mode is
excited and the associated enthalpy perturbation affects the
horizontal dynamics, contributing to the precession of the
eccentric mode. This contribution is found to be always pos-
itive. In the more general situation of a non-isothermal disc
undergoing adiabatic perturbations, the three-dimensional
contribution to \( J \) (in which we compare a 3D-type theory
with a 2DS-type theory) is

\[
\frac{3(\gamma + 1) \frac{P}{\Sigma r^2 \Omega}}{2 \gamma} \tag{31}
\]

where \( \gamma \) is the adiabatic index and \( P \) is the vertically in-
TEGRATED pressure (Ogilvie & Goodchild, in preparation), but
the derivation of this expression is beyond the scope of the
present paper.

4 FULL TREATMENT

4.1 Inviscid dynamics

The dynamical equations governing an isothermal, inviscid
disc are

\[
(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla (\Phi + h), \tag{32}
\]

\[
(\partial_t + \mathbf{u} \cdot \nabla) h = -c_s^2 \mathbf{v} \cdot \mathbf{u}. \tag{33}
\]

When these are linearized about the basic state described
in Section 2, we obtain

\[
-\omega u_r' - 2\Omega u_\phi' = -\partial_z h', \tag{34}
\]

\[
-\omega u_\phi' + \frac{\kappa^2}{2\Omega} u_r' = \frac{im h'}{r}, \tag{35}
\]

\[
-\omega u_z' = -\partial_z h', \tag{36}
\]

\[
-\omega h' + u_r' \partial_r h + u_\phi' \partial_\phi h
= -c_s^2 \left[ \frac{1}{r} \partial_r (ru_r') + \frac{im u_\phi'}{r} + \partial_z u_z' \right], \tag{37}
\]

where \( \omega = \omega - m\Omega \) is the Doppler-shifted wave frequency
and the perturbations, denoted by primed quantities, have the
dependence \exp(-i\omega t + im \phi).

Following Okazaki et al. (1987), we decompose the ver-
tical structure of the mode into the basis of Hermite poly-
nomials defined by

\[
\text{He}_n(\zeta) = e^{c^2/2} \left( -\frac{d}{d\zeta} \right)^n e^{-c^2/2}, \tag{38}
\]

where \( \zeta = \frac{z}{H} \) is a dimensionless vertical coordinate and
\( n = 0, 1, 2, \ldots \). These polynomials satisfy the differential
equation

\[
\text{He}_n''(\zeta) - \zeta \text{He}_n'(\zeta) + n \text{He}_n(\zeta) = 0, \tag{39}
\]

the recurrence relations

\[
\text{He}_n'(\zeta) = n \text{He}_{n-1}(\zeta), \tag{40}
\]

\[
\zeta \text{He}_n(\zeta) = \text{He}_{n+1}(\zeta) + n \text{He}_{n-1}(\zeta), \tag{41}
\]

and the orthogonality relation

\[
\int_{-\infty}^{\infty} e^{-c^2/2} \text{He}_m(\zeta) \text{He}_n(\zeta) d\zeta = (2\pi)^{1/2} n! \delta_{mn}. \tag{42}
\]

The first three are \( \text{He}_0(\zeta) = 1, \text{He}_1(\zeta) = \zeta \) and \( \text{He}_2(\zeta) = \zeta^2 - 1. \)

We therefore expand

\[
u_r'(r, z) = \sum_n u_n(r) \text{He}_n(\zeta), \tag{43}
\]

\[
u_\phi'(r, z) = \sum_n v_n(r) \text{He}_n(\zeta), \tag{44}
\]

\[
u_z'(r, z) = \sum_n w_n(r) \text{He}_n(\zeta), \tag{45}
\]

\[h'(r, z) = \sum_n h_n(r) \text{He}_n(\zeta). \tag{46}
\]

with \( u_n = v_n = h_n = 0 \) for \( n < 0 \) and \( w_n = 0 \) for \( n < 1. \)

Bearing in mind that \( H \) depends on \( r \), we have

\[
\partial_r \text{He}_n(\zeta) = -(\partial_r \ln H)[m \text{He}_n(\zeta) + n(n - 1) \text{He}_{n-2}(\zeta)]. \tag{47}
\]
The projected equations are then
\[ -i\omega u_n - 2\Omega v_n = -\partial_r h_n \]
\[ + (\partial_r \ln H)[nh_n + (n + 1)(n + 2)h_{n+2}], \]
\[ -i\omega v_n = \frac{\kappa^2}{2\Omega} u_n = -\frac{inh_n}{r}, \]
\[ -i\omega w_n = \frac{-nh_n}{H}, \]
\[ + (\partial_r \ln H)(u_{n-2} + u_{n-2}) = 0, \]
\[ \text{(51)} \]

where equation (51) is unchanged. These terms act to damp the mode, but have most effect on components of large \( n \). They have no effect on \( u_0 \) and \( v_0 \), which represent horizontal motions independent of \( z \). These viscous terms can also be thought of as providing a coupling between different layers of the disc and thereby encouraging it to adopt a horizontal motion independent of \( z \). We show below that this effect is of considerable importance.

4.3 Numerical solutions

We solve the system of ordinary differential equations in \( r \) for modes with \( m = 1 \) using a Chebyshev collocation (i.e. pseudospectral) method. This approach converts the differential equations and boundary conditions into an algebraic generalized eigenvalue problem for the frequency \( \omega \), which we solve using a standard direct method. Specifically, equations (48)–(50), supplemented by the viscous terms (51), are solved for \( n = 0, 2, 4, \ldots, N \) with \( u_n \), etc., set to zero for \( n > N \). Rigid boundary conditions \( u_n = 0 \) are adopted at both inner and outer boundaries, but it is ensured that the modes obtained are completely insensitive to the value of the outer radius and therefore to the choice of outer boundary condition.

For comparison with the results in Section 3.2, we consider a disc with a midplane density profile \( \rho_0 \propto r^{-\alpha-3/2} \). We also include a shear viscosity corresponding to a constant \( \alpha \) parameter, but no bulk viscosity.

Sample results are shown in Table I. The convergence of the eigenfrequency with increasing truncation order \( N \) of the Hermite polynomial basis is remarkable. The case \( N = 0 \) corresponds exactly to the two-dimensional theory considered by [1993], and therefore agrees well with the 2DO secular approximation. Here \( \tilde{Q} = 20 \) is large enough to support a confined prograde mode. The precession rate is much larger in the case \( N = 2 \) and hardly varies as further Hermite polynomials are included. It agrees reasonably well with the 3D secular theory for an inviscid disc. The slight offset of the precession frequency is attributable partly to errors in the secular approximation, which is valid only to leading order in \( \epsilon \), and partly to the effects of viscosity. As described in Appendix A the viscous damping of vertical motions considered in the full model can be represented within the 3D secular theory by multiplying the coefficient \( 9/4 \) in the three-dimensional expression (8) for \( f \) by \( (1 - i\beta)/(1 + i\beta) \), where \( \beta = \alpha_0 + \frac{i}{2} \). Table I shows that this viscous secular theory gives good agreement with the full model for \( \alpha = 0.1 \).

The viscous damping rate of the modes is considerable. Although the dominant motion is horizontal and independent of \( z \), and therefore does not incur any viscous forces in our approximation, the accompanying vertical motion is damped. To excite eccentric modes in a three-dimensional disc, this damping must be overcome. Viscous overstability may be able to do this, but detailed calculations are required and there is uncertainty in the applicability of a Navier–Stokes viscosity to turbulent stresses in the disc. A simple estimate can be made as follows. In a two-dimensional, isothermal, Keplerian shearing sheet with constant kinematic shear viscosity \( \nu \) and no bulk viscosity, the maximum local growth rate of the overstability is \( \approx 0.034 \alpha \Omega \) and occurs for a radial wavelength \( \approx 13 H \) [Latter & Ogilvie 2006]. Although
our eigenfunctions do not have an obviously wavelike form, this suggests that overstability may be able to compensate for the damping rate found in Table 1 which corresponds to only \( \approx 0.0018 \alpha f \) for the parameters adopted there.

A sufficiently large viscosity is required to couple different layers in the disc effectively. If the viscosity is reduced to \( \alpha = 0.01 \), with other parameters as in Table 1, a frequency eigenvalue of \( \tilde{\omega} = 2.7831 - 0.1793 i \) is obtained, and a slightly larger value of \( N \) is required to obtain the same convergence. While the precession rate agrees well with the case of \( \alpha = 0.1 \), the damping rate is now larger than predicted by the viscous secular 3D theory. This happens because of the increasing \( z \)-dependence of the horizontal motion in the absence of a strong coupling between layers: although \( u_2 \) is still much smaller than \( u_0 \) for \( \alpha = 0.01 \), the ratio \( u_2/u_0 \) is several times larger than in the case \( \alpha = 0.1 \). The viscous damping of this shearing motion is more important, for \( \alpha = 0.01 \), than that of the vertical motion considered in the viscous secular theory.

If \( \alpha \) is reduced further, the precession rate starts to deviate from the 3D secular theory and more vertical structure develops in the eigenfunction, requiring a larger value of \( N \) for convergence. Similar behaviour was found by Latter & Ogilvie (2004). The behaviour of a three-dimensional inviscid eccentric disc could be very difficult to describe.

We also note that the stresses associated with tangled magnetic fields in a disc in which the magnetorotational instability occurs may provide an elastic or viscoelastic coupling between different layers (Ogilvie 2001). The associated damping rate may be smaller than that estimated on the basis of a Navier–Stokes viscosity.

## 5 CONCLUSIONS

In this paper we have examined the linear dynamics of one-armed oscillation modes in the circumstellar discs of Be stars. A three-dimensional effect, first identified by Ogilvie (2001) but neglected in previous treatments of Be stars, makes a crucial positive contribution to the precession rates of such modes. It allows confined prograde modes to be obtained for all reasonable disc temperatures and stellar rotation rates. This property allows us to give a unified description of eccentric discs of Be stars of all stellar types without introducing uncertain radiative forces or modifying the inner boundary condition. While we do not deny that in some cases radiative forces might be important, or that the inner boundary condition might differ from a rigid one, we have shown that these innovations are unnecessary. We obtained these results using a secular theory of eccentric discs and confirmed them using a spectral treatment of the full linearized equations for three-dimensional isothermal discs including viscous terms that couple the horizontal motions at different altitudes. In order to make these modes grow, viscous damping must be overcome by an excitation mechanism such as viscous overstability, which will be investigated in a subsequent paper.

The three-dimensional dynamics that we have described may also have important consequences for the behaviour of eccentric discs in other circumstances. For example, in cataclysmic variable stars exhibiting superhumps, the relation between the precession rate of the disc and the binary mass ratio is an important observational property that is not fully explained by current theoretical models (Goodchild & Ogilvie 2006; Smith et al. 2007).

The physical model adopted in this paper is idealized. To explain in detail the observed cyclical behaviour of the emission lines of Be stars within this theoretical framework is likely to require a treatment of non-isothermal and nonlinear effects as well as a better understanding of the time-dependent behaviour of the density distribution of the circumstellar disc. In addition, future work should attempt to identify and assess mechanisms, such as viscous overstability, by which one-armed oscillation modes may be excited in circumstellar discs. Nevertheless, the effect investigated in this paper enhances the credibility of the one-armed oscillation model by providing a natural explanation of confined prograde modes.

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## APPENDIX A: DERIVATION OF THE SECULAR THEORY

We consider equations (45)–(51) together with the viscous effects described in Section 4.2. When \( v_n \) and \( w_n \) are eliminated, we have

\[
\frac{(\tilde{\omega}_n^2 - c_s^2)}{\tilde{\omega}_n^2} u_n = -\partial_r h_n + \frac{2m\Omega}{r\tilde{\omega}_n} h_n + (\partial_r \ln H)[nh_n + (n + 1)(n + 2)h_{n+2}],
\]

\[
\left(-i\tilde{\omega} - \frac{n\Omega^2}{\tilde{\omega}_n} - \frac{m^2c_s^2}{r} \right) \frac{1}{\tilde{\omega}_n} \frac{h_n}{c_s^2} + \frac{mn^2}{2r\tilde{\omega}_n} u_n + \frac{1}{r\Sigma} \partial_r (r\Sigma u_n) + (\partial_r \ln H)du_n = 0,
\]

where \( \tilde{\omega}_n = \tilde{\omega} + in\Omega z \) and \( \tilde{\omega}_n = \tilde{\omega} + i(n-1)\Omega z \) are the Doppler-shifted wave frequency modified to allow for the viscous damping of horizontal and vertical motions, respectively. We are interested in the case \( m = 1 \).
This system of equations is not closed because the \( u_0 \) equation refers to \( h_2 \), which depends on \( u_2 \). In the turn, the \( u_2 \) equation refers to \( h_4 \), which depends on \( u_4 \), and so on. However, suppose for the time being that \( u_2 \) can be neglected in the \( h_2 \) equation (i.e. equation (A3) with \( n = 2 \)). Then we have a closed system

\[
\left( \frac{\dot{\omega}^2 - \kappa^2}{\kappa^2} \right) u_0 = -\partial_r h_0 + \frac{2\Omega}{r^2} h_0 + 2(\partial_r \ln H) h_2, \quad (A3)
\]

\[
\left( \frac{i\dot{\omega} - c_s^2 \frac{1}{r^2} \frac{1}{i\kappa^2}}{c_s^2} \right) h_0 + \frac{\kappa^2}{2 r^2 \omega} u_0 + \frac{1}{r \Sigma} \partial_r (r \Sigma u_0) = 0, \quad (A4)
\]

\[
\left( \frac{i\dot{\omega} - 2 \Omega^2}{i \omega \Sigma^2} - c_s^2 \frac{1}{r^2} \frac{i}{i \kappa^2} \right) h_2 + \partial_r (\ln H) u_0 \approx 0. \quad (A5)
\]

These equations agree with a two-dimensional theory except for the additional \( h_2 \) term in equation (A3), which can be related to \( u_0 \) through the approximate equation (A2).

In the secular approximation we neglect the differences between \( \Omega, \kappa, \Omega_s \) and \( \Omega_K \) except where essential, so that \( \Omega \propto r^{-3/2} \) and \( H \propto r^{3/2} \), and we assume a low frequency of \( |\omega| \ll \Omega \). The leading approximations to the above equations in the inviscid case are then

\[
2i \left( \frac{\Omega^2 - \kappa^2}{2\Omega} - \omega \right) u_0 = -\partial_r h_0 - \frac{2h_0}{r} + \frac{3h_2}{f}, \quad (A6)
\]

\[
i\Omega \frac{\partial h_0}{c_s^2} - \frac{u_0}{2r} + \frac{1}{r \Sigma} \partial_r (r \Sigma u_0) = 0, \quad (A7)
\]

\[
-i \Omega \frac{h_2}{c_s^2} + \frac{3U_0}{2r} = 0. \quad (A8)
\]

These combine to give equation (8) for \( u = u_0 \), with the three-dimensional expression (6) for \( f \). If \( h_2 \) is neglected altogether, equation (8) is obtained, but with the two-dimensional expression (7) for \( f \). It can be seen from the above equations that \( h_2 \) always makes a positive contribution to the precession rate \( \omega \).

Neglecting \( u_2 \) compared to \( u_0 \) is equivalent to assuming that the eccentricity is independent of \( z \), since \( H u_0 (\xi) = 1 \) and \( H u_2 (\xi) = \xi^2 - 1 \). Under what conditions is this assumption reasonable? Rough estimates based on equations (A3) and (A2) show that \( u_2 \) can be neglected in the \( h_2 \) equation if \( \alpha \Omega \) is much larger than the precession frequencies \( \omega \) or \( f \), meaning that the shear viscosity prevents the significant development of a \( z \)-dependent horizontal motion. This condition is readily satisfied in the circumstellar discs of Be stars for reasonable values of \( \alpha \). Ultimately, however, the validation of this approximation comes from the numerical solution of the full system of linearized equations.

If viscosity is retained in this analysis, within the secular approximation, then the coefficient of \( h_2 \) in equation (A3) is multiplied by \( (1 + i\beta)/(1 - i\beta) \), where \( \beta = \alpha \Omega + \frac{1}{2} \omega \). In this case, equation (8) is again obtained, but in expression (6) for \( f \) the coefficient \( 9/4 \) of the three-dimensional term is multiplied by \( (1 - i\beta)/(1 + i\beta) \). In this case the mode decays because of the viscous damping of the vertical motion. For sufficiently small \( \alpha \), however, the decay rate is enhanced by the viscous damping of the \( z \)-dependent horizontal motion.

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