THE MODULI SPACE OF 6-DIMENSIONAL 2-STEP NILPOTENT LIE ALGEBRAS

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ABSTRACT. We determine the moduli space of metric 2-step nilpotent Lie algebras of dimension up to 6. This space is homeomorphic to a cone over a 4-dimensional contractible simplicial complex.

Keywords: nilpotent Lie algebra, moduli space

1. Introduction

The geometry of 2-step nilpotent Lie groups with a left-invariant metric is very rich and has been widely studied since the papers of A. Kaplan [9, 10] and P. Eberlein [4] (see, for instance [2, 3, 5, 6, 7, 12, 15] for recent papers on this subject). Important examples of such Lie groups are provided by groups of Heisenberg type [10].

In general, the moduli space of metric Lie algebras of a fixed dimension is a cone with peak the abelian Lie algebra and basis the subset obtained by normalizing the Lie bracket $c$, for instance requiring $\text{Tr}(c^*c) = 2$. In the present paper we determine the moduli space $\mathcal{N}_6$ of 6-dimensional 2-step nilpotent Lie algebras endowed with a metric. We show that $\mathcal{N}_6$ is a cone over an explicitly given contractible 4-dimensional simplicial complex. We also exhibit standard metric representatives of the 7 isomorphism types of 6-dimensional 2-step nilpotent Lie algebras within our picture. This contains all deformations of these Lie algebras, cf. [12].

In [12] J. Lauret identified in a natural way each point of the variety of real Lie algebras with a left-invariant Riemannian metric on a Lie group and studied the interplay between invariant-theoretic and Riemannian aspects of this variety. We show that on a certain subset of $\mathcal{N}_6$ the nullity of the Riemannian curvature tensor singles out products.
The subspace $\mathcal{N}_{n,k} \subset \mathcal{N}_n$ of Lie algebras with $k$-dimensional commutator ideal contains the subspace $\mathcal{D}_{n,k}$ of algebras with isometric $c^*$ as a strong deformation retract. For the algebras $\mathfrak{n}_{(\alpha_+,\alpha_-)} \in \mathcal{D}_{6,2}$ we give the structure equations, write down the curvature tensor and compute their infinitesimal rank, i.e. the minimal nullity of the Jacobi operators. In [15] it was proved that groups of Heisenberg type have infinitesimal rank one. We show that this is also the case for any $\mathfrak{n}_{(\alpha_+,\alpha_-)} \in \mathcal{D}_{6,2}$ with the exception of $\mathfrak{n}_{(1,1)} \cong \mathfrak{h}_3 \oplus \mathfrak{h}_3$ and $\mathfrak{n}_{(1/2,1/2)} \cong \mathfrak{n}_5 \oplus \mathbb{R}$, both endowed with the product metric, whose rank is two.

2. Preliminaries

A Lie algebra $\mathfrak{g}$ is nilpotent if its central series ends, i.e. in the sequence of ideals of $\mathfrak{g}$ recursively defined by $\mathfrak{g}^0 := \mathfrak{g}$, $\mathfrak{g}^{i+1} := [\mathfrak{g}, \mathfrak{g}^i]$ there is an integer $k$ such that $\mathfrak{g}^k = 0$. Then $\mathfrak{g}$ is a $k$-step nilpotent if $\mathfrak{g}^k = 0$ and $\mathfrak{g}^{k-1} \neq 0$. Thus a 2-step nilpotent Lie algebra $\mathfrak{n}$ is a Lie algebra such that its commutator ideal $\mathfrak{n}_1 := [\mathfrak{n}, \mathfrak{n}]$ is contained in its centre.

A left-invariant metric on a (simply connected) 2-step nilpotent Lie group $\mathcal{N}$ is given by a scalar product $\langle \cdot, \cdot \rangle$ on its Lie algebra $\mathfrak{n}$. We will call such a Lie algebra “metric 2-step nilpotent Lie algebra”.

A simply connected 2-step nilpotent Lie group with left-invariant metric is uniquely determined by the triple $(\mathfrak{h}, \mathfrak{z}, j)$ [4, 9], where $\mathfrak{h}$ and $\mathfrak{z}$ are real vector spaces with positive definite scalar product and $j: \mathfrak{z} \to \mathfrak{so}(\mathfrak{h})$ is the homomorphism (of vector spaces, not necessarily of Lie algebras) related to the Lie bracket by

$$\langle [x, y], z \rangle = \langle y, j(z)x \rangle \quad \forall x, y \in \mathfrak{h}, z \in \mathfrak{z}.$$ 

Thus, $j$ is essentially the adjoint of the Lie bracket $c: \Lambda^2\mathfrak{h} \to \mathfrak{z}$. Requiring in addition $j$ to be injective makes this correspondence one to one.

Observe that if one identifies $\mathfrak{z}$ with its dual $\mathfrak{z}^*$ via the metric (and similarly for $\mathfrak{h}$ and $\mathfrak{n}$), then the differential $d: \Lambda^1\mathfrak{z}^* \subset \Lambda^1\mathfrak{n}^* \to \Lambda^2\mathfrak{h}^* \subset \Lambda^2\mathfrak{n}^*$ can be identified with $j$ and $\dim (\text{Im} \ d) = \dim (\mathfrak{n}^3)$.

By [13] there are 34 classes of 6-dimensional nilpotent Lie algebras. Out of these 34 classes, the 2-step nilpotent have the following structure
equations

\[
(0, 0, 0, 12, 13, 23) = h_3^C,
\]
\[
(0, 0, 0, 0, 13 + 42, 14 + 23) = h_3^C,
\]
\[
(0, 0, 0, 0, 12, 14 + 23),
\]
\[
(0, 0, 0, 12, 34) = h_3 \oplus h_3,
\]
\[
(0, 0, 0, 0, 12 + 34) = h_5 \oplus \mathbb{R},
\]
\[
(0, 0, 0, 0, 0, 12) = h_3 \oplus \mathbb{R}^3,
\]

where \( h_3^C \) is the complex 3-dimensional Heisenberg Lie algebra, \( h_3 \) the real 3-dimensional Heisenberg Lie algebra and \( n_5 = (0, 0, 0, 12, 13). \) We use the notation of [14]. For example, \((0, 0, 0, 12, 13, 23)\) denotes the Lie algebra with \( de^i = 0, i = 1, \ldots, 3, \)
\( de^4 = e^1 \wedge e^2, \)
\( de^5 = e^1 \wedge e^3, \)
\( de^6 = e^2 \wedge e^3, \)
where \((e^i)\) is a basis of left-invariant 1-forms.

3. The Moduli space of 2-step nilpotent Lie algebras

We now determine the moduli space of metric 2-step nilpotent Lie algebras. Let \( g \) be a metric Lie algebra of dimension \( n \). We can always choose a linear isometry of \( g \) with Euclidean space \( \mathbb{R}^n \) (endowed with its standard scalar product). The set of Lie brackets on \( \mathbb{R}^n \) is an algebraic subset of \( \text{Hom}(\Lambda^2 \mathbb{R}^n, \mathbb{R}^n) \), whose ideal is given by the Jacobi identity, i.e.,

\[
\hat{L}_n := \{ c \in \text{Hom}(\Lambda^2 \mathbb{R}^n, \mathbb{R}^n) \mid c(c(u, v), w) + c(c(w, u), v) + c(c(v, w), u) = 0 \ \forall u, v, w \in \mathbb{R}^n \}.
\]

The set of 2-step nilpotent Lie brackets on \( \mathbb{R}^n \) is

\[
\hat{N}_n := \{ c \in \text{Hom}(\Lambda^2 \mathbb{R}^n, \mathbb{R}^n) \mid c(c(u, v), w) = 0 \ \forall u, v, w \in \mathbb{R}^n \}.
\]

These sets are invariant under the \( GL(n, \mathbb{R}) \)-action on \( \text{Hom}(\Lambda^2 \mathbb{R}^n, \mathbb{R}^n) \). The moduli space of 2-step nilpotent (metric) \( n \)-dimensional Lie algebras is the space of (isometric) isomorphism classes of such Lie algebras. It inherits its topology as the quotient of \( \hat{N}_n \) by the action of \( GL(n, \mathbb{R}) \) (respectively, \( O(n) \)),

\[
N_n := \hat{N}/O(n) \quad \text{ (resp. } \hat{N}_n = \hat{N}/GL(n, \mathbb{R}) \text{ )}.
\]

For \( k \leq n \) we decompose \( \mathbb{R}^n = \mathbb{R}^{n-k} \oplus \mathbb{R}^k \) orthogonally. A metric \( n \)-dimensional 2-step nilpotent Lie algebra \( n \) with \( \dim n^1 = k \) is isometric to \( (\mathbb{R}^n, c) \) where \( c \) is a 2-step nilpotent Lie bracket of rank \( k = \dim (\text{Im} c) \) such that \( \text{Im} c = \{0\} \oplus \mathbb{R}^k \). We define

\[
\hat{N}_{n,k} := \{ c \in \text{Hom}(\Lambda^2 \mathbb{R}^{n-k}, \mathbb{R}^k) \mid c \text{ surjective} \}.
\]
This space carries an action of $GL(n - k, \mathbb{R}) \times GL(k, \mathbb{R})$. The moduli space of $n$-dimensional (metric) 2-step nilpotent Lie algebras with $k$-dimensional commutator ideal is the quotient

$$\mathcal{N}_{n,k} = \hat{\mathcal{N}}_{n,k}/(O(n - k) \times O(k))$$

(resp. $\hat{\mathcal{N}}_{n,k} = \hat{\mathcal{N}}_{n,k}/(GL(n - k, \mathbb{R}) \times GL(k, \mathbb{R})))$.

Extending $c \in \hat{\mathcal{N}}_{n,k}$ by 0 to all of $\Lambda^2 \mathbb{R}^n$, we may view $\hat{\mathcal{N}}_{n,k} \subset \hat{\mathcal{N}}_n$ and decompose

$$\mathcal{N}_n = \bigcup_{0 \leq k \leq (n - k)} \mathcal{N}_{n,k} \quad \text{(resp. } \hat{\mathcal{N}}_n = \bigcup_{0 \leq k \leq (n - k)} \hat{\mathcal{N}}_{n,k})$$

(3.1)

We denote by $\gamma_{k,V} : \text{Gr}_k(V)$ the tautological vector bundle over the Grassmanian of $k$-planes in a real vector space $V$. We let $S^2_+ \gamma_{k,V} \subset S^2 \gamma_{k,V}$ be the set of positive definite symmetric 2-tensors on $\gamma_{k,V}$.

The adjoint $c^*$ of $c \in \hat{\mathcal{N}}_{n,k}$ is injective on $\mathbb{R}^k$. Pushing forward the standard scalar product $g_{std}$ on $\mathbb{R}^k$, a scalar product on its image is defined. The maps

$$\hat{\mathcal{N}}_{n,k} \xrightarrow{\phi} S^2_+ \gamma_{k, \Lambda^2 \mathbb{R}^{n-k}} \xrightarrow{\pi} \text{Gr}_k(\Lambda^2 \mathbb{R}^{n-k}) \quad \text{(Im}^* c_{std}) \mapsto \text{Im}^*$$

are $O(n - k) \times O(k)$-equivariant. In particular

**Theorem 3.1.** There is a homeomorphism

$$\mathcal{N}_{n,k} \simeq S^2_+ \gamma_{k, \Lambda^2 \mathbb{R}^{n-k}}/O(n - k)$$

and a strong deformation retraction

$$\mathcal{N}_{n,k} \simeq \text{Gr}_k(\Lambda^2 \mathbb{R}^{n-k})/O(n - k) =: \mathcal{D}_{n,k}.$$  

Here $\mathcal{D}_{n,k} \hookrightarrow \mathcal{N}_{n,k}$ is identified with the subset of those 2-step nilpotent Lie algebras with isometric j = $c^*$: $\mathbb{R}^k \rightarrow \mathfrak{so}(n - k)$.

**Proof.** Two Lie brackets $c, c' \in \hat{\mathcal{N}}_{n,k} \subset \text{Hom}(\Lambda^2 \mathbb{R}^{n-k}, \mathbb{R}^k)$ are equivalent in $\mathcal{N}_{n,k}$ if there are $A \in O(k)$ and $T \in O(n - k)$ such that $A c T^{-1} = c'$. Equivalent formulations are

$$(T^{-1})^* c^* A^* = c'^*,$$

$$c^* A^* (c'^*|_{\text{Im}^*})^{-1} = T^*,$$

that is to say, $T^*: \text{Im} c'^* \rightarrow \text{Im} c^*$ is isometric with respect to the metrics pushed forward by $c, c'$. Thus, the map $\hat{\mathcal{N}}_{n,k} \xrightarrow{\phi} S^2_+ \gamma_{k, \Lambda^2 \mathbb{R}^{n-k}}$ in (3.2) induces an homeomorphism on both quotients by $O(n - k) \times O(k)$.

Let $g_0$ be an $O(n - k)$-invariant scalar product on $\Lambda^2 \mathbb{R}^{n-k}$ (for instance the opposite of the Cartan-Killing form on $\Lambda^2 \mathbb{R}^{n-k} = \mathfrak{so}(n - k)$).
Thus all the spaces of Lie algebras above appear in \( N_{s} \) (3.1) becomes 4-dimensional simplicial complex pictured in Figure 1. The decomposition \( \cdot \) of \( \varnothing \) that 0 \( \leq \lambda_{1} \leq \ldots \leq \lambda_{\lfloor \frac{n-1}{2} \rfloor} \). Hence \( \text{Gr}_{1}(\Lambda^{2}\mathbb{R}^{n-1})/O(n-1) \approx \Delta^{\lfloor \frac{n-1}{2} \rfloor-1} \) is homeomorphic to a \((\lfloor \frac{n-1}{2} \rfloor - 1)\)-simplex and 
\[
\mathcal{N}_{n,1} \approx \Delta^{\lfloor \frac{n-1}{2} \rfloor-1} \times \mathbb{R}^{+}.
\]
For odd \( n \) and \( \lambda_{1} = \ldots = \lambda_{\frac{n-1}{2}} = 1 \) we recover the \( n \)-dimensional Heisenberg algebras \( \mathfrak{h}_{n} \in \mathcal{N}_{n,1} \).

- If \( k = \binom{n-k}{2} \), then \( \text{Gr}_{k}(\Lambda^{2}\mathbb{R}^{n-k}) \) is homeomorphic to a point and 
\( \mathcal{N}_{n,k} \approx S_{+}^{2}(\Lambda^{2}\mathbb{R}^{n-k})^{*}/O(n-k) \) is a quotient of the cone \( S_{+}^{2}(\Lambda^{2}\mathbb{R}^{n-k})^{*} \).

4. **Metric 2-step nilpotent Lie algebras of dimension \( \leq 6 \)**

In this section we study in detail the case of Lie algebras of dimension up to 6. We denote by \( \mathcal{N}_{s,*}^{0,*} \) the subspace of Lie algebras with \( \text{Tr}(j^{*}j) = 2 \). The whole space \( \mathcal{N}_{s} \) is a cone over \( \mathcal{N}_{s}^{*} \) whose peak is the abelian Lie algebra. Clearly, for \( m \leq 2 \), \( \mathcal{N}_{m} \) is a point, the abelian Lie algebra. For \( m = 3, 4 \) we get \( \mathcal{N}_{3}^{0} = \{ \mathfrak{h}_{3} \} \) and \( \mathcal{N}_{4}^{0} = \{ \mathfrak{h}_{3} \oplus \mathbb{R} \} \). For \( m = 5 \) we have \( \mathcal{N}_{5} = \mathcal{N}_{5,0} \cup \mathcal{N}_{5,1} \cup \mathcal{N}_{5,2} \). By remark \( \Pi \) \( \mathcal{N}_{5,1}^{0} \) is homeomorphic to an interval with endpoints the Lie algebras \( \mathfrak{h}_{3} \oplus \mathbb{R}^{2} \) and \( \mathfrak{n}_{5} \). Let now \( \mathfrak{n}_{5} \in \mathcal{N}_{5,2} \) denote a Lie algebra with isometric \( j: \mathbb{R}^{2} \rightarrow \mathfrak{so}(3) \); all such Lie algebras are isometrically isomorphic. We will see later that the closure \( \overline{\mathcal{N}_{5,2}^{0}} \) is homeomorphic to an interval with endpoints \( \mathfrak{h}_{3} \oplus \mathbb{R}^{2} \) and \( \mathfrak{n}_{5} \). For any \( m \leq n \) there are embeddings \( \mathcal{N}_{m,k} \hookrightarrow \mathcal{N}_{n,k} \), \( \mathfrak{n} \mapsto \mathfrak{n} \oplus \mathbb{R}^{n-m} \). Thus all the spaces of Lie algebras above appear in \( \mathcal{N}_{6} \).

In the sequel we will show that \( \mathcal{N}_{6} \) is a cone over a contractible 4-dimensional simplicial complex pictured in Figure 1. The decomposition \( (5.4) \) becomes 
\[
\mathcal{N}_{6} = \mathcal{N}_{6,0} \cup \mathcal{N}_{6,1} \cup \mathcal{N}_{6,2} \cup \mathcal{N}_{6,3}.
\]
From remark 1 we have
\[
\mathcal{N}_{6,0} \approx \ast, \\
\mathcal{N}_{6,1} \approx [0, 1] \times \mathbb{R}^+, \\
\mathcal{N}_{6,3} \approx S^2_1(\Lambda^2\mathbb{R}^3)^*/O(3).
\]

4.1. Invariants for \( \mathcal{N}_6 \). The subsequent simultaneous description of the pieces of \( \mathcal{N}_6 \) and their glueing relies on the isomorphism of Lie algebras
\[
(\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{su}_+(2) \oplus \mathfrak{su}_-(2) = \mathbb{R}^3_+ \oplus \mathbb{R}^3_-).
\]
Under the identification \( (4.3) \), the action of \( \text{SO}(4) \) on \( \mathfrak{so}(4) \) translates to the (dual of the) usual action of \( \text{SO}(3) \times \text{SO}(3) \) on \( \mathbb{R}^3_+ \oplus \mathbb{R}^3_- \). The whole orthogonal group in addition contains an element \( \tau \in O(4) \) of determinant \(-1\) which interchanges the factors. Explicitly, the isomorphism \( (4.3) \) is given by mapping
\[
\xi e_1^\pm + \psi e_2^\pm + \chi e_3^\pm = \begin{pmatrix}
i&i&i\
-\psi+i\chi&\psi+i\chi-i\xi
\end{pmatrix} \in \mathfrak{su}_\pm(2)
\]
to
\[
\begin{pmatrix}0&\xi&\psi&\chi
-\xi&0&-\chi&\psi
-\psi&\chi&0&-\xi
-\chi&-\psi&\xi&0
\end{pmatrix}
\]
and
\[
\begin{pmatrix}0&\xi&\psi&\chi
-\xi&0&\chi&-\psi
-\psi&-\chi&0&\xi
-\chi&\psi&-\xi&0
\end{pmatrix}
\]
for "-" and "+" respectively. The diagonal matrix \( \text{diag}(-1, 1, 1, 1) \) acts as the involution \( \tau \). We denote the two components of \( j \) by \( j_\pm : \mathbb{R}^2 \to \mathbb{R}^3_\pm \). The spectra of \( j_\pm^* j_\pm \) and the trace of \( j_\pm^* j_- j_\pm^* j_+ \) are invariant under the \( O(2) \times O(4) \)-action, up to interchanging \( \pm \).

We claim that these data suffice to determine the equivalence class of \( j \) under the \( O(2) \times O(4) \)-action:

Clearly, the entire matrices \( j_+ j_- \) and \( j_+^* j_- \) determine \( j \) up to the action of \( O(4) \). If both \( j_+^* j_- \) and \( j_+^* j_+ \) have two identical eigenvalues, then both matrices are diagonal for any orthonormal basis of \( \mathbb{R}^2 \). Otherwise, after possibly using \( \tau \) to permute \( \pm \), we may assume that \( j_+^* j_- \) has two different eigenvalues \( \alpha_- \), \( \beta_- \) and that \( e_1, e_2 \) are the respective eigenvectors. If \( j_+^* j_+ = \begin{pmatrix}x & z \\
z & y\end{pmatrix} \), then \( \text{Tr}(j_+^* j_+) = x + y \), \( \det(j_+^* j_+) = xy - z^2 \) and \( \text{Tr}(j_+^* j_- j_+^* j_-) = \alpha_- x + \beta_- y \) determine \( x, y \geq 0 \) and \( z \) up to sign. Since the sign of \( z \) can be changed by conjugation with \( \begin{pmatrix}-1 & 0 \ \ 0 & 1\end{pmatrix} \), we may assume \( z \geq 0 \). Thus, all of \( j_+^* j_+ \) is determined by the above invariants.
Let \( \text{Spec}(j^*_\pm j_\pm) = \{ \alpha_\pm, \beta_\pm \} \), with \( 0 \leq \alpha_\pm \leq \beta_\pm \) and \( \text{Tr}(j^*_\pm j_\pm j^*_\pm j^-) = t \). The possible range for \( t \) in dependence of \( \alpha_\pm, \beta_\pm \) is obtained by solving

\[
t = \alpha_- x + \beta_- y, \quad x + y = \alpha_+ + \beta_+, \quad xy - z^2 = \alpha_+ \beta_+, \quad z \geq 0.
\]

We get

\[
t \in I_{\alpha_\pm, \beta_\pm} = [\alpha_- \beta_+ + \alpha_+ \beta_-, \alpha_- \alpha_+ + \beta_- \beta_+].
\]

Let \( S \) denote the set of all 5-tuples \((\alpha_\pm, \beta_\pm, t)\) satisfying the above conditions and subject to the relation induced from \( \tau \), i.e.

\[
S := \left\{ (\alpha_-, \beta_-, \alpha_+, \beta_+, t) : (0 \leq \alpha_- \leq \beta_-, \quad 0 \leq \alpha_+ \leq \beta_+) \quad \text{and} \quad t \in I_{\alpha_\pm, \beta_\pm} \right\} / \sim
\]

with the identification

\[
(\alpha_-, \beta_-, \alpha_+, \beta_+, t) \sim (\alpha_+, \beta_+, \alpha_-, \beta_-, t).
\]

We have

**Theorem 4.1.** The closure of \( \mathcal{N}_{6,2} \) is homeomorphic to \( S \) under the map

\[
\Psi: \overline{\mathcal{N}_{6,2}} \to S
\]

\[
j = (j_-, j_+) \mapsto (\text{Spec}(j^-_\pm j_-), \text{Spec}(j^+_\pm j_+), \text{Tr}(j^+_\pm j_+ j^-_\pm j_-))
\]

**Proof.** We have already shown that the above map is bijective. It is continuous since the spectrum of a matrix depends continuously on its entries. Since

\[
\text{Tr}(j^*_j) = \text{Tr}(j^*_j^- j_-) + \text{Tr}(j^*_j^+ j_+) = \alpha_-(j) + \beta_-(j) + \alpha_+(j) + \beta_+(j),
\]

we get that, for all \( r > 0 \), \( \Psi \) defines a continuous bijection

\[
\{ j : \text{Tr}(j^*_j) \leq r \}/_{O(2) \times O(4)} \leftrightarrow S \cap \{ (\alpha_-, \beta_-, \alpha_+, \beta_+, t) : (0 \leq \alpha_- \leq \beta_- + \alpha_+ + \beta_+ \leq r) \}/_{\sim}
\]

which is a homeomorphism since these sets are compact. It follows that \( \Psi \) is a homeomorphism. \( \square \)

The spaces \( \mathcal{N}_{6,1} \) and \( \mathcal{N}_{6,3} \) are treated similarly. For \( \mathcal{N}_{6,1} \) we have to deal with maps \( j: \mathbb{R} \to \mathfrak{so}(5) \). Any such map is conjugate to some \( j: \mathbb{R} \to \mathfrak{so}(4) \subset \mathfrak{so}(5) \). Extending \( j \) by \( 0 \) to a map \( \mathbb{R}^2 \to \mathfrak{so}(4) \), we can identify \( \mathcal{N}_{6,1} \) with a subset of \( \partial \overline{\mathcal{N}_{6,2}} \). In the terminology above, both components \( j_+ \) and \( j_- \) have only one nonvanishing eigenvalue, \( 0 \leq \beta_+ \leq \beta_- \) respectively. Moreover \( \text{Tr}(j^*_+ j_+ j^-_+ j_-) = \beta_- \beta_+ \) gives no new invariant on \( \mathcal{N}_{6,1} \).

For \( \mathcal{N}_{6,3} \), we observe that the imbedding \( \mathfrak{so}(3) \hookrightarrow \mathfrak{so}(4) \) induced from \( \mathbb{R}^3 \hookrightarrow \mathbb{R}^4 \) translates under \([13]\) to the skew-diagonal map \( \mathfrak{so}(3) \ni X \mapsto \)
\( \frac{1}{2} (X, -X) \in \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{so}(4) \). Thus, for \( j = (j_-, j_+) \in \mathcal{N}_{6,3} \) we have \( j_+ = -j_- \). Hence, \( \text{Spec}(j_-^* j_+) = \text{Spec}(j_-^* j_-) = \{ \omega, \alpha, \beta \} \) with \( 0 \leq \omega \leq \alpha = \alpha_- = \alpha_+ \leq \beta = \beta_- = \beta_+ \) and \( t = \omega^2 + \alpha^2 + \beta^2 \).

As a whole, \( \mathcal{N}_6 \) is a cone over the set \( \mathcal{N}_6^0 \) of those isometric isomorphism classes with \( \text{Tr}(j^* j) = \text{Tr}(j_-^* j_-) + \text{Tr}(j_+^* j_+) = \alpha_- + \beta_- + \alpha_+ + \beta_+ = 2 \). Its peak is the abelian Lie algebra \( \mathbb{R}^6 \). The following picture illustrates the set \( \mathcal{N}_6^0 \), where the invariant \( t \) is omitted over the interior of \( \mathcal{N}_{6,2} \). We have chosen a fundamental domain for the \( \tau \)-action such that the parameters \( \alpha_\pm, \beta_\pm, t \) always satisfy \( \beta_- - \alpha_- \geq \beta_+ - \alpha_+ \). We then only need to identify \( (\alpha_\pm, \beta_\pm, t) \sim (\alpha_\mp, \beta_\mp, t) \) if \( \beta_- - \alpha_- = \beta_+ - \alpha_+ \). On the right hand face of \( \mathcal{N}_{6,2} \) this requires to identify the two triangles by the reflection indicated by the two arrows ↔. The dots • mark standard representatives for the seven different isomorphism classes of Lie algebras and are identified in the next section.

**Figure 1.** \( \mathcal{N}_6^0 \)

### 4.2. Classification of 6-dimensional 2-step nilpotent Lie algebras

Next, we determine the isomorphism classes (disregarding the metric) of 2-step nilpotent 6-dimensional Lie algebras and exhibit canonical representatives. To this end, we compute the action of \( \text{GL}(2, \mathbb{R}) \times \text{GL}(4, \mathbb{R}) \) on our invariants.
**Remark 2.** Using the \( \text{GL}(2, \mathbb{R}) \)-action for any \( j \) we find an equivalent one which is an isometric monomorphism.

**Remark 3.** For isometric \( j \), we can simultaneously diagonalize \( j^*_+j_+ \) and \( j^*_-/j_- \). For \( \mathcal{N}_{6,2} \), this yields the relations \( \alpha_- + \beta_+ = 1 = \alpha_+ + \beta_- \) and \( t = \alpha_- \beta_+ + \alpha_+ \beta_- \). For \( \mathcal{N}_{6,3} \), we have \( \omega = \alpha = \beta = \frac{1}{2} \) and for \( \mathcal{N}_{6,1} \) we get \( \alpha_+ = \beta_+ = 0, \beta_+ + \beta_- = 1 \) and \( t = \beta_+ \beta_- \).

A pair \( (T, S) \in \text{GL}(2, \mathbb{R}) \times \text{GL}(4, \mathbb{R}) \) acts on \( j \) by replacing \( j(z) \) with \( S^*j(Tz)S \). In the bases \( e_i^\pm \) for \( \mathbb{R}_\pm^3 = \mathfrak{so}_\pm(3) \subset \mathfrak{so}(4) \) and \( (e_1, e_2) \) for \( \mathbb{R}_2^2 \), we can put \( j \) in the form

\[
(4.4) \quad j = \begin{pmatrix}
  a_- & 0 & 0 & 0 \\
  0 & b_- & 0 & 0 \\
  p & r & 0 & 0 \\
  0 & 0 & q & 0 
\end{pmatrix}
\]

with \( 0 \leq a_- \leq b_-, 0 \leq p, q, r \), using only the \( \text{O}(2) \times \text{O}(4) \subset \text{GL}(2, \mathbb{R}) \times \text{GL}(4, \mathbb{R}) \) action. The coefficients \( a_-, b_-, p, q, r \in \mathbb{R}_0^+ \) are determined from the invariants by solving the equations

\[
a_-^2 = \alpha_-, \quad b_-^2 = \beta_-, \quad p^2 + q^2 + r^2 = \alpha_+ + \beta_+, \\
p^2 q^2 = \alpha_+ \beta_+, \quad \alpha_- p^2 + \beta_- (q^2 + r^2) = t.
\]

We first assume that \( j \) is isometric, i.e. \( r = 0, a_-^2 + p^2 = 1 = b_-^2 + q^2 \) and \( p^2 = \beta_+, q^2 = \alpha_+ \). Possibly interchanging \( \pm \) we may also suppose \( \beta_- \geq \beta_+ \). Then, the only free invariants for this case are \( \alpha_\pm \) and satisfy the conditions \( \alpha_- \geq \alpha_+ \) and \( \alpha_+ \leq 1 - \alpha_- \).

By means of the isomorphism \( [13] \) \( j \) defines the homomorphism \( j: \mathbb{R}^2 \to \mathfrak{so}(4) \) given by

\[
j(u, v) = \begin{pmatrix}
  0 & (a_- + p)u & (b_- + q)v & 0 \\
  -(a_- + p)u & 0 & (b_- - q)v & 0 \\
  -(b_- - q)v & 0 & 0 & (a_- + p)u \\
  0 & (q - b_-)v & (a_- - p)u & 0 
\end{pmatrix}
\]

In case \( b_- - q \neq 0 \), this is equivalent to the matrix with coefficients \( (a_-, b_-, p, 0, 0), b_- = \sqrt{(b_- + q)(b_- - q)} \), via the matrix

\[
S := \begin{pmatrix}
  \lambda^{-1} & 0 & 0 & 0 \\
  0 & \lambda & 0 & 0 \\
  0 & 0 & \lambda^{-1} & 0 \\
  0 & 0 & 0 & \lambda 
\end{pmatrix}
\]
with \( \lambda = \left( \frac{b_- + q}{b_- - q} \right)^{1/4} \). By rescaling \( v \) we can keep \( j \) isometric. Similarly, in case \( a_- - p > 0 \), we use a matrix

\[
T := \begin{pmatrix}
\lambda^{-1} & 0 & 0 & 0 \\
0 & \lambda^{-1} & 0 & 0 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & \lambda
\end{pmatrix}
\]

with \( \lambda = \left( \frac{a_- + p}{a_- - p} \right)^{1/4} \) to see that any \( j \) with coefficients \( (a_-, b_-, p, q, 0) \) is equivalent to one with coefficients \( (a'_-, b_-, 0, q, 0) \) where \( a'_- = \sqrt{(a_- + p)(a_- - p)} \).

In case \( a_- - p < 0 \) we can replace \( (a_-, b_-, p, q, 0) \) by \( (0, b_-, p, q, 0) \) by means of the above matrix \( T \) with \( \lambda = \left( \frac{a_- + p}{p-a_-} \right)^{1/4} \). In order to keep \( j \) isometric, we rescale \( u \).

The diagram below visualizes the subset of \( \mathcal{N}_{6,2} \) represented by isometric \( j \). With respect to the action of \( \text{GL}(2, \mathbb{R}) \times \text{GL}(4, \mathbb{R}) \) it decomposes into four isomorphism classes of Lie algebras indicated by the components in the picture.

\[
\alpha_- = \alpha_+ = 1/2 \\
\alpha_- = \alpha_+ = 0 \\
\alpha_- = 1/2, \; \alpha_+ = 0 \\
\alpha_- = 1, \; \alpha_+ = 0
\]

Figure 2. \( \mathcal{D}_{6,2} \)

Remark 4. If \( j \) is not isometric and has invariants \( (\alpha_- \mp, \beta_\mp, t) \) we first compute the coefficients \( (a_-, b_-, p, q, r) \) to write \( j \) in the shape \( (4.4) \). With

\[
A := \begin{pmatrix}
1 & \frac{-pr}{(a_-^2 + p^2)} \\
0 & 1
\end{pmatrix}
\]


and $B = \text{diag}(1/\|jAe_1\|, 1/\|jAe_2\|)$ we get that $jAB$ is isometric. Computing $\alpha_\pm(jAB)$, the isomorphism type of $j$ can be determined.

In $\mathcal{N}_6$ we get the following isomorphism types, where the parameters are given for isometric $j$.

1. Any Lie algebra in $\mathcal{N}_{6,1}$ is isomorphic to a Lie algebra with parameters $\alpha_\pm = 0 = \beta_+, \beta_- = 1, t = 0$ or $\alpha_\pm = 0, \beta_+ = \beta_- = 1/2, t = 1/4$. The first type is the product $(0, 0, 0, 0, 12 + 34) = \mathbb{R}^3 \oplus h_3$. The second type is a product $(0, 0, 0, 0, 12) = h_5 \oplus \mathbb{R}$.

2. In $\mathcal{N}_{6,2}$ we have four isomorphism types corresponding to
   (a) $\alpha_- < 1/2$, which gives $(0, 0, 0, 0, 12, 34) = h_3 \oplus h_3$
   (b) $\alpha_+ < 1/2 = \alpha_-, (0, 0, 0, 12, 14 + 23)$
   (c) $\alpha_+ = 1/2 = \alpha_-, (0, 0, 0, 12, 13) = n_5 \oplus \mathbb{R}$ with $n_5 \in \mathcal{N}_{5,2}$
   (d) $\alpha_+ < 1/2 < \alpha_-, (0, 0, 0, 0, 13 + 42, 14 + 23) = h_3^C$

3. Any Lie algebra in $\mathcal{N}_{6,3}$ is isomorphic to $(0, 0, 0, 12, 13, 23)$ with $\omega = \alpha = \beta = 1/2$.

**Figure 3.** Isomorphism classes in $\mathcal{D}_{6,2}$

**Remark 5.** A Lie algebra $c \in \tilde{\mathcal{L}}_n \subset \text{Hom}(\Lambda^2 \mathbb{R}^n, \mathbb{R}^n)$ is said to degenerate to another Lie algebra $\tilde{c}$, if $\tilde{c}$ is represented by a structure which lies in the Zariski closure of the $\text{GL}(n, \mathbb{R})$-orbit of a structure which represents $c$. In this case the entire $\text{GL}(n, \mathbb{R})$-orbit of $\tilde{c}$ in $\tilde{\mathcal{L}}_n$ lies in the closure of the orbit of $c$. Recall that $c$ degenerates to $\tilde{c}$ if there exist $g_s \in \text{GL}(n, \mathbb{R})$ such that $\lim_{s \to 0} g_s \cdot c = \tilde{c}$. Using this, it is easy to see that the Lie algebras $h_3^C, h_3 \oplus h_3, (0, 0, 0, 0, 12, 14 + 23)$ all degenerate to $n_5 \oplus \mathbb{R}$ (the top point in Figure 3).
Remark 6. Using Remark \[1\] one can determine the structure equations for any 6-dimensional 2-step nilpotent Lie algebra. As an example, from the isomorphism
\[
\mathfrak{so}(4) \cong \Lambda^2 \mathbb{R}^4
\]
we will give the structure equations for the Lie algebras in \( \mathcal{D}_{6,2} \).

Indeed, if one fixes a non-zero element \( w \in \Lambda^4 \mathbb{R}^4 \), one can consider the bilinear form \( \phi \) of signature (3,3) on \( \Lambda^2 \mathbb{R}^4 \) defined by \( \sigma \wedge \tau = \phi(\sigma, \tau)w \).

Given an orientation and a metric \( g \) on \( \mathbb{R}^4 \) (and so on \( \Lambda^2 \mathbb{R}^4 \)), there is an \( SO(4) \)-decomposition
\[
\Lambda^2 \mathbb{R}^4 = \Lambda^2_+ \oplus \Lambda^2_-,
\]
where \( \Lambda^2_\pm \) are the eigenspaces of the conformally invariant involution \(*\) of \( \Lambda^2 \mathbb{R}^4 \) for which \( \phi(\ast \sigma, \tau) = g(\sigma, \tau) \). From a representation-theoretic point of view, \( (4.6) \) is equivalent to the Lie algebra splitting \( (4.3) \).

If one chooses a basis \( \{e^1, e^2, e^3, e^4\} \) of \( \mathbb{R}^4 \) such that \( w = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \), then
\[
\Lambda^2_+ = \text{span} \{e^1 \wedge e^2 + e^3 \wedge e^4, e^1 \wedge e^3 + e^2 \wedge e^4, e^1 \wedge e^4 + e^2 \wedge e^3\},
\]
\[
\Lambda^2_- = \text{span} \{e^1 \wedge e^2 - e^3 \wedge e^4, e^1 \wedge e^3 - e^2 \wedge e^4, e^1 \wedge e^4 - e^2 \wedge e^3\}.
\]

Using \( (4.5) \) and the embeddings \( (4.3) \) one has the following identifications
\[
e^+_1 \sim e^1 \wedge e^2 + e^3 \wedge e^4,
\]
\[
e^+_2 \sim e^1 \wedge e^3 - e^2 \wedge e^4,
\]
\[
e^-_1 \sim e^1 \wedge e^2 - e^3 \wedge e^4,
\]
\[
e^-_2 \sim e^1 \wedge e^3 + e^2 \wedge e^4
\]
and thus \( n_{(\alpha_+,\alpha_-)} \) has structure equations
\[
de^i = 0, \quad i = 1, \ldots, 4,
\]
\[
de^5 = (a_- + p) e^1 \wedge e^2 + (a_- - p) e^3 \wedge e^4,
\]
\[
de^6 = (b_- + q) e^1 \wedge e^3 - (b_- - q) e^2 \wedge e^4,
\]
where
\[
a_- = \sqrt{\alpha_-}, \quad b_- = \sqrt{\beta_-}, \quad p = \sqrt{\beta_+}, \quad q = \sqrt{\alpha_+}.
\]

Remark 7. Next, we compute the infinitesimal rank of a Lie algebra in \( \mathcal{D}_{6,2} \).

The rank of a geodesic in a Riemannian manifold \( M \) is the dimension of the real vector space of parallel Jacobi fields along it. The rank \( \text{rk}(M) \) of \( M \) is the minimum of the ranks of all its geodesics. Recall that the Jacobi-operator \( R_v \) is the endomorphism of \( T_p M \) given by \( w \mapsto R_{v,w}v \). The infinitesimal rank
infrk(M) of M is the minimal dimension of the kernels of its Jacobi-operators \[15\]. A Riemannian manifold M has higher (infinitesimal) rank if \(\text{infrk}(M) \geq 2\).

First, we use the structure equations \[14\] to compute the curvature tensor with respect to the metric \(g\) for which the forms \((e^i)\) are dual to an orthonormal basis \((e_i)\).

The non vanishing components \(R_{ijkl} = g(R_{e_i,e_j,e_k,e_l})\) of the Riemannian curvature tensor are:

\[
\begin{align*}
R_{1212} &= -\frac{3}{4}(a_- + p)^2, \\
R_{1234} &= -\frac{1}{4}a_-^2 + \frac{1}{2}p^2 + \frac{1}{2}b_+^2 - \frac{1}{4}q^2 = R_{3412}, \\
R_{1313} &= -\frac{3}{4}(b_- - q)^2, \\
R_{1324} &= -\frac{1}{4}a_-^2 + \frac{1}{2}p^2 + \frac{1}{2}b_-^2 - \frac{1}{4}q^2 = R_{2413}, \\
R_{1423} &= \frac{1}{4}a_-^2 - \frac{1}{2}p^2 + \frac{1}{2}b_-^2 - \frac{1}{4}q^2 = R_{2314}, \\
R_{1456} &= \frac{1}{2}pq - \frac{1}{2}a_- b_- = R_{5614}, \\
R_{1515} &= \frac{1}{4}(a_- + p)^2 = R_{2525}, \\
R_{1546} &= -\frac{1}{4}(b_- + q)(a_- - p) = R_{4615}, \\
R_{1616} &= \frac{1}{4}(b_- + q)^2 = R_{3636}, \\
R_{1645} &= \frac{1}{4}(b_- - q)(a_- + p) = R_{4516}, \\
R_{2356} &= -\frac{1}{4}a_- b_- - \frac{1}{2}pq = R_{5623}, \\
R_{2424} &= -\frac{1}{4}(b_- - q)^2, \\
R_{2536} &= \frac{1}{4}(a_- - p)(b_- - q) = R_{3625}, \\
R_{2626} &= \frac{1}{4}(b_- - q)^2 = R_{4646}, \\
R_{2635} &= \frac{1}{4}(b_- + q)(a_- + p) = R_{3526}, \\
R_{3434} &= -\frac{1}{4}(a_- - p)^2, \\
R_{3535} &= \frac{1}{4}(a_- - p)^2 = R_{4545}.
\end{align*}
\]

The infinitesimal rank is 1 for any \(a_-, p, b_-, q\) (with respect to the above metric), except for \((a_-, b_-) = (1, 0)\) and \((a_-, b_-) = (\sqrt{2}/2, \sqrt{2}/2)\). Indeed the Jacobi operator:

\[R_{e_1+e_6} : X \mapsto R_{e_1+e_6,x}e_1 + e_6\]

whose associated matrix is

\[
\begin{pmatrix}
\frac{1}{4}\eta^2 & 0 & 0 & 0 & 0 & -\frac{1}{4}\eta^2 \\
0 & -\frac{1}{4}[3\nu^2 - (b_- - q)^2] & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{4}\eta^2 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4}(b_- - q)^2 & -\rho & 0 \\
0 & 0 & 0 & 0 & \rho & \frac{1}{4}\nu^2 \\
-\frac{1}{4}\eta^2 & 0 & 0 & 0 & 0 & \frac{1}{4}\eta^2
\end{pmatrix}
\]

(with \(\nu = a_- + p, \eta = b_- + q, \rho = \frac{1}{4}[3pq - 3a_- b_- + pb_- - a_- q]\)) has one dimensional kernel, except for the following cases:

1. \(a_- = q, b_- = p\);
2. \(a_- = p = \sqrt{2}/2\).
If one considers, in addition, the Jacobi operator:

\[ R_{e_2+e_5} : X \mapsto R_{e_2+e_5} \cdot X e_2 + e_5 \]

its associated matrix is

\[
\begin{pmatrix}
-\frac{1}{2} \nu^2 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} \nu^2 & 0 & 0 & 0 & -\frac{1}{4} \nu^2 \\
0 & 0 & \frac{1}{4} (a_- - p)^2 & 0 & -\zeta & 0 \\
0 & -\frac{1}{4} \nu^2 & 0 & \frac{1}{4} [3(b_- - q)^2 - (a_- - p)] & 0 & 0 \\
0 & 0 & -\zeta & 0 & 0 & \frac{1}{4} \nu^2 \\
0 & 0 & \frac{1}{4} (b_- - q) & 0 & 0 & 0
\end{pmatrix},
\]

(with \( \zeta = \frac{1}{4} [3 a_- b_- + 3pq + a_-q + pb_-] \)). Again, the dimension of the kernel of \( R_{e_2+e_5} \) is generically one. Both \( R_{e_1+e_5} \) and \( R_{e_2+e_5} \) have kernel of dimension bigger than one if \( b_- = 0 = p \) and \( a_- = b_- = \sqrt{2}/2 \). These two cases correspond to the Lie algebras \( \mathfrak{n}_{(1,1)} \cong \mathfrak{h}_3 \oplus \mathfrak{h}_3 \) and \( \mathfrak{n}_{1/2,1/2} \cong \mathfrak{n}_5 \oplus \mathbb{R} \), which are both Riemannian products of (infinitesimal) rank one Lie algebras. Thus, their (infinitesimal) rank is two.

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