Adjoint operators, gauge invariant perturbations, and covariant symplectic structure for black holes in string theory

R. Cartas-Fuentevilla

Instituto de Física, Universidad Autónoma de Puebla, Apartado postal J-48 72570 Puebla Pue., México

Expressions for the general and complete perturbations in terms of Debye potentials of static charged black holes in string theory, valid for curvature below the Planck scale, are derived starting from a decoupled set of equations and using Wald’s method of adjoint operators. Our results cover both extremal and nonextremal black holes and are valid for arbitrary values of the dilaton coupling parameter. The decoupled set is obtained using the Newman-Penrose formulation of the Einstein-Maxwell-dilaton theory and involves naturally field quantities invariant under both ordinary gauge transformations of the electromagnetic potential perturbations and infinitesimal rotations of the perturbed tetrad. Furthermore, using the recent pointed out relationship between adjoint operators and conserved currents, a local continuity law for the field perturbations in terms of the potentials is also obtained. It is shown that such continuity equation implies the existence of conserved quantities and of a covariant symplectic structure on the phase space. Future extensions of the present results are discussed.

PACS numbers: 04.20.Jb, 04.40.Nr

Keywords: Adjoint operators, perturbations, symplectic structure, black holes, string theory, Debye potentials.

Running title: Adjoint operators.....

I. INTRODUCTION

At present, the theories of extended objects such as membranes and strings represent the more viable candidates for the quantum theory of gravity. Particularly, there have been many efforts studying black holes in string theory from different points of view, with the main task of elucidating on the problem of
quantum gravity embedded in them, since such objects appear to play a crucial role in the subject. However, because of the many technical and conceptual difficulties in treating the full theory, the low-energy limit of string theory has been developed as a more pragmatic approach. This low-energy physics emerges as an effective action obtained from the lowest order in the world-sheet and string loop expansion, where the usual Einstein-Hilbert gravity is supplemented by gauge fields, scalar fields such as the axion and the dilaton, which couple in a nontrivial way to the other matter fields. As it is well known, the presence of the dilaton changes drastically the dynamical properties of the systems, and new features arise in this theory due to the nontrivial coupling of this field. In particular, dilaton black holes have shown to have novel thermodynamics properties and to behave like elementary particles in the sense that the excitation spectrum has an energy gap. Besides, it has been explored the viewpoint that quantum black holes are massive excitations of extended objects and also correspond, in this sense, to elementary particles.

On the experimental context, recent investigations attempt to explore a possible experimental evidence of string theory. Since string theory predicts particularly the existence of the dilaton scalar field, the new generation of detectors of gravity waves are sensitive in the presence of a possible scalar component of such waves. Specifically, a scalar component of gravity radiation should excite the monopole mode of new resonant-mass detectors of spherical shape, and should give a specific correlation between an interferometer and the monopole mode of a resonant sphere. Furthermore, the spherical resonant-mass detectors, or an array of interferometers are able, in principle, to determine the spin content of the incoming gravitational waves possibly coupled with their scalar components. In this same context, black holes should be the more typical and possible astrophysical source of gravity waves.

In all issues discussed above, the first-order perturbation analysis plays a fundamental role. Perturbation theory reveals important physical information of the system under study. As we shall see, the adjoint operators approach will cover, in an unified way, various aspects of the same problem (in this case, the perturbation analysis of string black holes), which traditionally have been treated separately. In the remainder of this Introduction, we discuss such aspects, pointing out our aims and successes in the present work, and we make a review of previous works in which the present approach has been employed.

In the scheme of the perturbation theory, the black holes (and other spacetimes) have been studied from different approaches. The traditional approaches consist to try of solving the original set of equations for the field perturbations directly. This approach has several disadvantages and difficulties that can be overcome by means of an alternative and more convenient approach based on the concept of the adjoint of a differential operator (Wald’s method). The reach and differences of this approach with respect to
the usual ones have been already discussed widely in previous works (see for example [12], and references therein). In fact, in the cases where string fields are involved, the approach has been applied successfully in the setting of the Einstein-Maxwell-dilaton-axion (EMDA) theory, which contains the low-energy limit of string theory as a particular case [12, 13]. Additionally, as we shall see, with the connection recently established between adjoint operators and conserved currents, Wald’s method becomes the more convenient and powerful approach for facing the study of perturbations.

At a more general context, the study of conservation laws in field theories involving gravity, becomes particularly interesting because of the lack of conserved currents representing the conservation of energy and momentum. Additionally, in the construction of a covariant symplectic structure on the phase space of classical systems, a bilinear product on first-order deformations of classical solutions on such phase space is required. In both cases, the problem is to find a local expression physically meaningful and coming from some continuity equation. As we shall see, the present adjoint operators scheme allows us to establish a local continuity law with the features described above, from which conserved quantities and a covariant symplectic structure (in terms of Debye potentials) are derived.

It is important to emphasize, at this point, the significance of a covariant symplectic structure in field theory. As well known, Feynman path integral and canonical quantization are the fundamental approaches in quantum field theories. If quantization is carried out by means of path integral, the resultant theory has no necessarily the standard structure in terms of quantum mechanical states and operators. In fact, in string field theory, the existence of such a structure is not obvious [14]. However, Feynman path integral has the great virtue of preserving manifestly the Poincaré invariance. As opposed to path integral, the canonical formalism, with a suitable definition of Poisson brackets, leads to Hamiltonian mechanics of the standard form, which yields a quantum theory of the conventional type (replacing Poisson brackets with commutators). Although this formalism usually is considered that does not preserve the Poincaré invariance, Witten [14], Crncović and Witten [15], and Suckerman [16] have achieved to describe Poisson brackets in terms of a symplectic structure on the classical phase space in a covariant way. In such description, the classical phase space is defined as the space of solutions of the classical equations of motion; such definition is manifestly covariant. The construction of a covariantly conserved two-form $J^\mu$ on such phase space yields a symplectic structure $\omega$ defined as $\omega \equiv \int_\Sigma J^\mu d\Sigma_\mu$, being $\Sigma$ an initial value hypersurface, independent of the choice of $\Sigma$ and, in particular, Poincaré invariant. Additionally, in terms of symplectic structure $\omega$, the fact that Poisson brackets satisfy the Jacoby identity, is equivalent that $\omega$ to be a closed two-form on the phase space, which holds if $J^\mu$ itself is closed. With this properties, $J^\mu$ is known as the symplectic current. One of our goals in the present paper is to establish a local continuity equation that permits to identify, in
a straightforward way, a symplectic current for the solution considered.

In this manner, the purpose of the present work is to perform an analysis of the first-order perturbations of the dilatonic charged black holes employing Wald's method. Previously, it has been demonstrated the self-adjointness of the operator governing the field perturbations in the EMDA theory [12, 13], remaining only the finding of the corresponding decoupled set of equations in the case where the background space-time corresponds to the solution considered, in order to establish our results.

For this purpose, the outline of this paper is as follows. Section II is dedicated to establish the general relationship between adjoint operators and conserved currents, and the extensions of the original Wald's method; some issues on the notation are also discussed in this Section. The relevant information on the background solution is given in Sec. III. In Section IV, a decoupled set of equations for metric, vector potential, and dilaton perturbations is obtained from the original equations for the field perturbations, which are given in Appendix A using the Newman-Penrose formulation. Employing the results of Section IV, the equations for the Debye potentials, and the expressions for the metric, vector potential, and dilaton perturbations in terms of those, are found in Sec. 5.1. In Sec. 5.2, our fundamental continuity equation is established and a symplectic structure is derived in Sec. 5.3. Some additional comments on the role that the Debye potentials play in the present approach, are given in Sec. 5.3. The separation of variables for the equations for the Debye potentials, and for the continuity equation is performed in Sec. VI, such that two conserved quantities are obtained. We conclude this Section with certain differential identities and we comment briefly on their meaning. Appendix B is useful in this section. Finally, we finish with some concluding remarks and future extensions of the present results.

II. ADJOINT OPERATORS

2.1 New branch of adjoint operators: local continuity laws

In Refs. [13] it has been shown that there exists a conserved current associated with any system of homogeneous linear partial differential equations that can be written in terms of a self-adjoint operator. This result is limited for a self-adjoint system, for which the corresponding conserved current depends on a pair of solutions admitted by such a system. However, as we shall see below, there exists a more general possibility that extends for systems of equations that are not self-adjoint necessarily. The demonstration is very easy (see also [18]):
In accordance with Wald’s definition \[19\], if \( E \) corresponds to a linear partial differential operator which maps \( m \)-index tensor fields into \( n \)-index tensor fields, then, the adjoint operator of \( E \), denoted by \( E^\dagger \), is that linear partial differential operator mapping \( n \)-index tensor fields into \( m \)-index tensor fields such that

\[
g^{\rho\sigma...}[E(f_{\mu\nu...})]_{\rho\sigma...} - [E^\dagger(g^{\rho\sigma...})]^{\mu\nu...}f_{\mu\nu...} = \nabla_\mu J^\mu, \tag{1}\]

where \( J^\mu \) is some vector field depending on the fields \( f \) and \( g \). From Eq. (1) we can see that this definition automatically guarantees that, if the field \( f \) is a solution of the linear system \( E(f) = 0 \) and \( g \) a solution of the adjoint system \( E^\dagger(g) = 0 \), then \( J^\mu \) is a covariantly conserved current. This fact means that for any homogeneous equation system, one can always construct a conserved current taking into account the adjoint system. This general result contains the self-adjoint case as a particular one.

In the present work, \( f \) and \( g \) will be associated with the first-order variations of the backgrounds fields. Such field variations will correspond, on the phase space, to one-forms \[15\]. In this manner, the left-hand side of Eq. (1) can be understood as a wedge product on such phase space: \( g \wedge E(f) - E^\dagger(g) \wedge f = \nabla_\mu J^\mu \), and something similar for the bilinear form \( J^\mu \) in its dependence on the fields \( f \) and \( g \) (the operators \( E, E^\dagger, \) and \( \nabla_\mu \) will depend only on the background fields).

It is worth pointing out some issues on the notation. The first-order field variations appearing in Refs. \[12, 13\] are denoted by a superscript \( B \). On the other hand, the field variations coincide, in according to Witten’s interpretation \[15\], with an infinite-dimensional generalization of the usual exterior derivative, which is traditionally represented by the symbol \( \delta \). However, in Refs. \[12, 13\] and present work, the Newman-Penrose formalism is used, in which the symbol \( \delta \) is employed for denoting one of the directional derivatives defined by the null tetrad. In this manner, for avoiding confusion, we will maintain the symbol \( \delta \) as usual in the Newman-Penrose notation, and the superscript \( B \) for the first-order field variations (the exterior derivative of background fields). In the present article, the exterior derivative will not be performed explicitly, and it will be sufficient for our purposes to understand any quantity with the superscript \( B \) as a one-form on the phase space. Quantities without such a superscript will correspond to background fields, which mean zero-forms on the phase space. With these previous considerations, formulae and notation of Refs. \[12, 13\] will be used throughout this paper; the concepts and definitions on differential forms, exterior derivatives, etc, come from Ref. \[15\].

### 2.2 Traditional branch of adjoint operators: decoupled equations and potentials
For completeness, we outline the original idea for introducing the definition (1) in Ref. [19]: reduction of systems of linear partial differential equations to equations for scalar potentials (called Debye potentials), which determine a complete solution of the original system.

If we have the linear system $E(f) = 0$, and there exist linear operators such that $SE = OT$, identically, then the field $S^\dagger(\psi)$ satisfies the equation

$$E^\dagger(S^\dagger(\psi)) = 0,$$

provided that the scalar field $\psi$ satisfies

$$O^\dagger(\psi) = 0.$$

In particular, if $E$ is self-adjoint ($E^\dagger = E$), then $f = S^\dagger(\psi)$ is a solution of $E(f) = 0$. For example, in the case considered in the present work, the (matrix) operator governing the field perturbations in the Einstein-Maxwell-Dilaton theory is, in fact, self-adjoint [12, 13, 17].

Moreover, the existence of operators $S$, $O$, and $T$ satisfying the above identity, is equivalent to the existence of a decoupled system

$$O(\Psi) = 0,$$

obtained from the original system $E(f) = 0$, such that the scalar field $\Psi = T(f)$.

Now, we can mix both branches of the adjoint operators scheme: since the fields $\psi$ and $\Psi$ satisfy equation adjoints to each other, we can establish, in accordance to the first branch, that

$$\psi(O\Psi) - (O^\dagger \psi)\Psi = \nabla_\mu J^\mu(\psi, \Psi),$$

which means that $\nabla_\mu J^\mu(\psi, \Psi) = 0$. Furthermore, since $\Psi$ is finally depending on $\psi \left( \Psi = T(f) = T(S^\dagger(\psi)) \right)$, $J^\mu$ is dependent only on $\psi$ (however, see Section 5.4).

On the other hand, although this result on the existence of conserved currents has been established assuming only tensor fields and the presence of a single equation, such a result can be extended in a direct way to equations involving spinor fields, matrix fields, and the presence of more than one field. Furthermore, this general result can be understood as an important extension of the original Wald’s method: wherever there exists an appropriate decoupled equation, it is not only possible to express the complete solution in terms of scalar potentials, but also to find automatically a corresponding (covariantly) conserved current.
III. BACKGROUND SPACETIME

Static, spherically symmetric solutions of the Einstein-Maxwell-dilaton equations have been found, representing charged black holes for curvature below the Planck scale [2, 3]. The solutions for magnetically charged dilaton black holes have, using the metric convention (+−−−), the line element

\[ ds^2 = \chi^2 dt^2 - \chi^{-2} dr^2 - R^2 d\Omega, \]  

(2)

where \( \chi \) and \( R \) depend only on \( r \):

\[ \chi^2 = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)\left(1-a^2/(1+a^2)\right), \quad R = r\left(1 - \frac{r_+}{r}\right)\left(a^2/(1+a^2)\right), \]  

(3)

where \( r_+ \) and \( r_- \) are the values of the parameter \( r \) at the outer and the inner horizon respectively, and are related to the physical mass (\( M \)) and charge (\( Q \)); \( a \) is the dilaton coupling parameter. The Maxwell and dilaton fields are given by

\[ F = Q \sin \theta d\theta \wedge d\phi, \quad e^{-2\phi} = \left(1 - \frac{r_+}{r}\right)^{2a^2/(a^2+1)}, \quad (\xi = e^{-2\phi}). \]  

(4)

There are also electrically charged solutions which may be obtained by a duality rotation. For more details see [2, 3].

For our present purpose, it is more convenient to specify the line element (2) by the null tetrad

\[ D \equiv l^\mu \partial_\mu = \frac{1}{\chi^2} \partial_t + \partial_r, \quad \Delta \equiv n^\mu \partial_\mu = \frac{1}{2}(\partial_t - \chi^2 \partial_r), \]  

\[ \delta \equiv m^\mu \partial_\mu = \frac{1}{\sqrt{2R}}(\partial_\theta + \text{icsc} \theta \partial_\phi), \quad \overline{\delta} \equiv \overline{m}^\mu \partial_\mu = \frac{1}{\sqrt{2R}}(\partial_\theta - \text{icsc} \theta \partial_\phi). \]  

(5)

Using the commutation relations of the tetrad (5), the nonvanishing spin coefficients can be conveniently expressed as

\[ \rho = D \ln R^{-1}, \quad \mu = \Delta \ln R, \quad \gamma = \Delta \ln \chi^{-1}, \]  

\[ \beta = \delta \ln \sin^{1/2} \theta, \quad \alpha = -\overline{\delta} \ln \sin^{1/2} \theta, \]  

(6)

where \( \rho \), \( \mu \), and \( \gamma \) depend only on \( r \), and \( \beta \) and \( \alpha \) on both \( r \) and \( \theta \).

On the other hand, considering the first of Eqs. (4) and the definitions \( \varphi_0 \equiv l^\mu m^\nu F_{\mu\nu} \), \( \varphi_1 \equiv \frac{1}{2}(l^\nu n^\nu + m^\mu m^\nu)F_{\mu\nu} \), and \( \varphi_2 \equiv m^\mu n^\nu F_{\mu\nu} \), the Newman-Penrose components of the electromagnetic field are given by

\[ \varphi_0 = 0 = \varphi_2, \quad \varphi_1(r) = \frac{iQ}{2R^2}. \]  

(7)
Note that $\varphi_1 + \varphi_1 = 0 = \delta \phi_1$, which will be used implicitly below. On the other hand, from Eqs. (4) and (5), the only nonvanishing derivatives of the dilaton field are $D\phi$ and $\Delta \phi$, which depend only on $r$, and

$$\delta \phi = 0 = \overline{\delta \phi}.$$  \hfill (8)

Thus, the only nonvanishing Ricci scalars are (see Appendix of Ref. [12])

$$\Phi_{00} = -(D\phi)^2, \quad \Phi_{22} = -(\Delta \phi)^2,$$

$$\Phi_{11} = -\frac{1}{2}(D\phi)(\Delta \phi) - 2\xi \varphi_1^2, \quad \Lambda = -\frac{1}{6}(D\phi)(\Delta \phi),$$  \hfill (9)

and the only nonvanishing component of the Weyl spinor can be expressed as

$$\Psi_2(r) = 2\gamma \rho - \frac{2}{3}D\phi \Delta \phi.$$  \hfill (10)

Furthermore, the background Maxwell’s equations take the form [12]

$$(D - 2\rho)\varphi_1 = 0, \quad (\Delta + 2\mu)\varphi_1 = 0,$$  \hfill (11)

and similarly, the background dilaton equation is

$$D\Delta \phi + 2\mu D\phi - 2a\xi \varphi_1^2 = 0.$$  \hfill (12)

Additionally, using Eqs. (4)-(9) and the commutation relations, we can find the following relations:

$$(D + pp)(\delta + q\beta) = (\delta + q\beta)[D + (p + 1)\rho],$$

$$(\Delta + p' \gamma + p' \mu)(\delta + q\beta) = (\delta + q\beta)[\Delta + p' \gamma + (p' - 1)\mu],$$  \hfill (13)

where $p$, $q$, and $p'$ are three arbitrary constants.

In the Newman-Penrose formalism, the adjoints of the tetrad components (5) are given, in general, by Eqs. (16) of Ref. [12], which reduce to

$$D^\dagger = -(D - 2\rho), \quad \Delta^\dagger = -(\Delta - 2\gamma + 2\mu), \quad \delta^\dagger = -(\delta + 2\beta), \quad \overline{\delta}^\dagger = -(\overline{\delta} + 2\overline{\beta}),$$  \hfill (14)

for this background solution. These equations will be used below.

**IV. DECOUPLED SET OF EQUATIONS FOR GAUGE INVARIANT PERTURBATIONS**
The notation, conventions, and Appendix of Ref. [12] will be used extensively throughout this paper. In particular, the metric, vector potential, and dilaton variations are represented by $h_{\mu\nu}$, $b_\mu$, and $\phi^B$, respectively. The metric and vector potential perturbations are defined modulo gauge transformations. Since, the dilaton is a fundamental physical field, there no exists gauge invariance associated with this field.

On the other hand, it is well known that when the perturbation analysis is performed using the Newman-Penrose formalism, one is faced with the perturbed tetrad gauge freedom. The traditional approaches make use of this gauge freedom in order to simplify the equations for the perturbations ([12] and references therein). However, we shall see that in the present case, although including string fields, there is no need to invoke perturbed tetrad rotations, but that appropriate combinations of the perturbed quantities, which are independent on the perturbed tetrad gauge freedom, lead in a natural way, to a decoupled set of equations from the original set. Such combinations prove to be also independent on the ordinary gauge transformations of the electromagnetic potential perturbations.

For example, let us consider the first-order perturbations of the spin coefficient $\sigma^B$:

$$\sigma^B \equiv -(l^\mu m^\nu \nabla_\nu m_\mu)^B = l^\mu m^\nu m_\gamma (\Gamma^\gamma_{\mu\nu})^B - l^\mu m^\nu \nabla_\nu m_\mu^B - (l^\mu m^\nu)^B \nabla_\nu m_\mu$$

(15)

where it has been considered that the only nonvanishing spin coefficients in the background are given in Eq. (6); $(\Gamma^\gamma_{\mu\nu})^B = \frac{1}{2} g^{\gamma\rho} (\nabla_\mu h_{\nu\rho} + \nabla_\nu h_{\mu\rho} - \nabla_\rho h_{\mu\nu})$, corresponds to the variations of the connection, and in this manner, the first term in the above equation is defined completely in terms of $h_{\mu\nu}$. On the order hand, $l^\mu m_\mu^B$ is dependent on the perturbed tetrad gauge freedom. Furthermore, from the definition $\phi_0 \equiv l^\mu m^\nu F_{\mu\nu}$, we have that

$$\phi_0^B = l^\mu m^\nu F_{\mu\nu}^B + 2\phi_1(l^\mu m_\mu^B),$$

(16)

where Eq. (7) have been considered; $F_{\mu\nu}^B = \partial_\mu b_\nu - \partial_\nu b_\mu$, and thus the first term of Eq. (16) is defined completely in terms of $b_\mu$. Therefore, from Eqs. (15), and (16) we can see easily that the perturbed quantity $\tilde{\sigma}^B \equiv \sigma^B + (\delta - 2\beta)(\phi_0^B/2\phi_1)$, is independent on the perturbed tetrad gauge freedom and defined completely in terms of $h_{\mu\nu}$ and $b_\mu$. Furthermore, since the field perturbation $F_{\mu\nu}^B$ is invariant under the ordinary gauge transformation $b_\mu \rightarrow b_\mu + \nabla_\mu \varepsilon$, where $\varepsilon$ is an arbitrary scalar field, $\phi_0^B$ in Eq. (16) is also invariant under the transformation and, in this manner $\tilde{\sigma}^B(h_{\mu\nu}, b_\mu) = \tilde{\sigma}^B(h_{\mu\nu}, b_\mu + \nabla_\mu \varepsilon)$. The remaining quantities with similar invariance properties involved in our present analysis, are given in Appendix A.

For obtaining our first perturbation equation, we apply $(\delta - 2\beta)$ to the first of Eqs. (A22), and using the commutation relations (13), we can use the first and second of Eqs. (A21), and first of Eq. (A23), for eliminating the resultant terms $(\delta - 2\beta)\kappa^B$, $(\delta - 2\beta)\pi^B$, and $(\delta - 2\beta)\bar{\Psi}^B$ respectively, in favor of terms
including $\Psi_0^B$, $\sigma^B$, $\lambda^B$, and $\phi^B$, and to obtain, after grouping suitably, the second-order differential equation:  

$$
O_{11}^B \Psi_0^B + O_{13}^B \sigma^B - (D\phi) F_1^\lambda^B + F_1(\delta - 2\beta)\phi^B = S_{11} T_{\mu\nu}^B,
$$  

(17)

where  

$$
O_{11} = (D - 5\rho)\bar{\Phi} + (\delta - 2\beta)(\bar{\Phi} + 4\beta) - (3\Psi_2 + 2\Phi_{11} + 2D\phi \Delta\phi),
$$  

$$
O_{13} = -8\xi\bar{\phi}_1 \bar{\sigma} - 4D\phi(\gamma D\phi - 3a\xi\bar{\phi}_1),
$$  

$$
F_1 = 8\chi^{-2} D\phi \left( \gamma + a\xi\bar{\phi}_1^2 \right),
$$  

(18)

and  

$$
S_{11} = 2(\delta - 2\beta)(D - 3\rho)\bar{m}_\mu \bar{m}_\nu - [(D - 5\rho)(D - \rho) + \Phi_{00}] \bar{m}_\mu \bar{m}_\nu - (\delta - 2\beta)\delta \bar{m}_\mu \bar{m}_\nu.
$$  

(19)

Similarly, applying $(\delta - 2\beta)$ to the second of Eqs. (A22), using the commutation relations (13), the fourth, fifth of Eqs. (A21), and second of Eqs. (A23) for eliminating the resultant terms $(\delta - 2\beta)\bar{\tau}^B$, $(\delta - 2\beta)\bar{\nu}^B$, and $(\delta - 2\beta)\bar{\Psi}_3^B$, respectively, in favor of terms involving $\Psi_0^B$, $\sigma^B$, $\lambda^B$, and $\phi^B$, one obtains another second-order differential equation:  

$$
O_{22}^B \Psi_4^B + \frac{\lambda^4}{4} \Delta \phi \bar{\sigma}^B + O_{24}^B \lambda^B + \frac{\lambda^4}{4} F_1(\delta - 2\beta)\phi^B = S_{21} T_{\mu\nu}^B,
$$  

(20)

where  

$$
O_{22} = (\Delta + 2\gamma + 3\mu)(D - \rho) - (\delta - 2\beta)(\bar{\Phi} + 4\beta) - (3\Psi_2 + 2D\phi \Delta\phi - 2\Phi_{11}),
$$  

$$
O_{24} = 8\xi\bar{\phi}_1(\Delta + 2\gamma) + 4\Delta\phi(\gamma D\phi - 3a\xi\bar{\phi}_1),
$$  

(21)

and  

$$
S_{21} = 2(\delta - 2\beta)(\Delta + 2\gamma + 3\mu)\bar{n}^\mu \bar{n}_\nu - [(\Delta + 2\gamma + 3\mu)(\Delta + \mu) + \Phi_{22}] \bar{m}_\mu \bar{m}_\nu - (\delta - 2\beta)\delta \bar{n}_\mu \bar{n}_\nu.
$$  

(22)

With the purpose of obtaining perturbation equations which involve only the perturbation quantities appearing in Eqs. (17) and (20), we substitute directly $\bar{\phi}^B$ and $\bar{\phi}^B_1$ from Eqs. (A6) and (A7) respectively into Eq. (A12), and then substituting the resultant term $D\bar{r}^B$ from the third of Eqs. (A21), we obtain:  

$$
-2\bar{\Psi}_1^B - (\Delta - 4\gamma - \mu)\bar{\chi}^B + (\bar{\delta} + 4\bar{\beta})\bar{r}^B + aD\phi(\bar{\xi}^B + \bar{z}^B) - a(D - \rho)\bar{\phi}^B = \frac{1}{2\bar{\phi}_1} [\delta(\xi^{-1} \bar{m}_\mu \bar{m}_\nu) - (D - 3\rho)(\xi^{-1} m_\mu \bar{m}_\nu)],
$$  

(23)
further, applying \((\delta - 2\beta)\) to the above equation, using the commutation relations (13), and substituting the resultant terms \((\delta - 2\beta)\tilde{\psi}^B_1\), \((\delta - 2\beta)\tilde{\pi}^B\), \((\delta - 2\beta)\tilde{\tau}^B\), and \((\delta - 2\beta)\tilde{\phi}^B\) from first of Eqs. (A23), first, second, and fourth of Eqs. (A21) respectively, we obtain

\[
O_{31}\tilde{\psi}^B_0 + O_{33}\tilde{\pi}^B + O_{34}(\delta - 2\beta)\tilde{\phi}^B = S_{31}(T_{\mu\nu}) + S_{32}(j_\mu),
\]  

where

\[
O_{31} = \Delta - 4\gamma + 2\mu,
\]

\[
O_{33} = (\Delta - 4\gamma)(D - 2\rho) - aD\phi(\Delta - 2\gamma) - (\delta - 2\beta)(\overline{\phi} + 4\overline{\pi}) - 2(3\Phi_2 + 2\Phi_{11}),
\]

\[
O_{34} = -aD\phi D + 2\Phi_{00},
\]

\[
O_{35} = a(D - 2\rho) - 2D\phi,
\]

and

\[
S_{31} = 2(\delta - 2\beta)l^\mu m^\nu - 2(D - \rho)m^\mu m^\nu,
\]

\[
S_{32} = \frac{1}{2\phi_1}(\delta - 2\beta)[(D - 3\rho)\xi^{-1}m^\mu - \delta(\xi^{-1}n^\mu j_\mu)].
\]

Similarly, following the above procedure for obtaining the equation (24), we substitute \(\hat{\phi}^B_1\) and \(\hat{\phi}^B_2\) from Eqs. (A6) and (A7) into Eq. (A14), and then substituting the resultant term \(\Delta\tilde{\pi}^B\) from the sixth of Eqs. (A21), we obtain:

\[
2\tilde{\psi}^B_3 - (D + \rho)\tilde{\nu}^B + (\overline{\phi} + 4\overline{\pi})\tilde{\lambda}^B + a\Delta\phi(\tilde{\pi}^B + \tilde{\tau}^B) + a(\Delta + \mu)\tilde{\phi}^B = \frac{1}{2\phi_1}[(\Delta + 3\mu)(\xi^{-1}m^\mu j_\mu) - \delta(\xi^{-1}n^\mu j_\mu)],
\]

now, applying \((\delta - 2\beta)\) to Eq. (27), using the commutation relations (13), and substituting the resultant terms \((\delta - 2\beta)\tilde{\psi}^B_1\), \((\delta - 2\beta)\tilde{\pi}^B\), \((\delta - 2\beta)\tilde{\tau}^B\), and \((\delta - 2\beta)\tilde{\phi}^B\) from second of Eqs. (A23), fifth, second, and fourth of Eqs. (A21), respectively, we obtain:

\[
O_{42}\tilde{\psi}^B_4 + O_{43}\tilde{\pi}^B + O_{44}(\delta - 2\beta)\tilde{\phi}^B = S_{41}(T_{\mu\nu}) + S_{42}(j_\mu),
\]

where

\[
O_{42} = D - 2\rho,
\]

\[
O_{43} = a\Delta\phi(\Delta - 2\gamma) - 2\Phi_{22},
\]

\[
O_{44} = -D(\Delta + 2\gamma + 2\mu) + a\Delta\phi D + (\delta - 2\beta)(\overline{\phi} + 4\overline{\pi}) + 2(3\Phi_2 + 2\Phi_{11}),
\]

\[
O_{45} = a(\Delta + 2\mu) - 2\Delta\phi,
\]

and
\[ S_{41} = 2(\delta - 2\beta)\mu m^\nu - 2(\Delta + \mu)m^\mu m^\nu, \]
\[ S_{42} = \frac{1}{2\varphi_1}(\delta - 2\beta)[(\Delta + 3\mu)\xi^{-1}m^\mu - \delta\xi^{-1}n^\mu]. \]  

(30)

Similarly, substituting \( \hat{\varphi}_1^B \), and \( \check{\varphi}_1^B \) from Eqs. (A6) and (A7) into Eq. (A20), then applying \((\delta - 2\beta)\) to the resultant equation (and performing substitutions such as in the above equations for \((\delta - 2\beta)\tilde{\Psi}_3^B\), \((\delta - 2\beta)\tilde{T}_B\), \((\delta - 2\beta)\tilde{\tau}_B\), and \((\delta - 2\beta)\tilde{\kappa}_B\)), we obtain:

\[
\frac{\chi^4}{8}F_1\Psi_0^B + \frac{1}{2}F_1\Psi_3^B + C_{53}^B + C_{54}^B + C_{55}(\delta - 2\beta)\tilde{\phi}^B = S_{51}(T_{\mu\nu}) + S_{52}(j_\mu) + S_{53}(\phi_s),
\]

(31)

where

\[
O_{53} = \frac{-\chi^4}{8}F_1(D - 2\rho) + [\Delta\phi(D - \rho) - 2a\xi\varphi_1^2][\Delta - 2\gamma + \mu] - \Delta\phi(\delta - 2\beta)(\delta + 4\beta)
\]
\[- \mu F_2 - \Phi_{22}D\phi,
\]

\[
O_{54} = (F_2 - \mu D\phi)(D - \rho) - (D - 3\rho)D\phi(\Delta + 2\gamma + 2\mu) + D\phi(\delta - 2\beta)(\delta + 4\beta)
\]
\+[\Delta\phi(D - 2a\xi\varphi_1^2)]\rho + D\phi(3\Psi_2 + 2\Phi_{11}),

\[
O_{55} = (D - 3\rho)(\Delta + 3\mu) - (\delta - 2\beta)(\delta + 4\beta) - 3\Psi_2 + 2\mu\rho - 3D\phi\Delta\phi - 4(a^2 - 1)\xi\varphi_1^2,
\]

\[
F_2 \equiv 2D\phi \left( \mu + \frac{2\xi\varphi_1^2}{D\phi} \right),
\]

(32)

and

\[
S_{51} = 2D\phi(\delta - 2\beta)\mu m^\nu + 2\Delta\phi(\delta - 2\beta)\xi m^\nu + (4a\xi\varphi_1^2 - \Delta\phi D - D\phi\Delta)m^\mu m^\nu,
\]
\[
S_{52} = -4a\varphi_1(\delta - 2\beta)m^\mu,
\]
\[
S_{53} = \frac{1}{2}(\delta - 2\beta)\delta.
\]

(33)

Hence, we have finally a system of five second-order linear partial differential equations (17), (20), (24), (28), and (31), for five unknowns: \( \Psi_0^B, \Psi_3^B, \tilde{\sigma}, \tilde{\lambda} \), and \((\delta - 2\beta)\tilde{\phi}^B\) (in Ref. [12], a similar system was obtained for the equations governing the perturbations of the solution that represents waves bound to collisions in the same scheme of the Einstein-Maxwell-dilaton theory). This system of equations can be expressed in the following matrix form:

\[
\mathcal{O}(\Psi^B) = \mathcal{S}\begin{pmatrix} T_{\mu\nu} \\ j_\mu \\ \phi_s \end{pmatrix},
\]

(34)
where $\mathcal{O}$ is the $5 \times 5$ matrix

\[
\mathcal{O} \equiv \begin{pmatrix}
\mathcal{O}_{11} & 0 & \mathcal{O}_{13} & -F_1 D\phi & F_1 \\
0 & \mathcal{O}_{22} & \hat{\Delta} \phi F_1 & \mathcal{O}_{24} & \hat{\Delta} \phi F_1 \\
\mathcal{O}_{31} & 0 & \mathcal{O}_{33} & \mathcal{O}_{34} & \mathcal{O}_{35} \\
0 & \mathcal{O}_{42} & \mathcal{O}_{43} & \mathcal{O}_{44} & \mathcal{O}_{45} \\
\frac{\hat{\alpha}^4}{2} F_1 & \frac{1}{2} F_1 & \mathcal{O}_{53} & \mathcal{O}_{54} & \mathcal{O}_{55}
\end{pmatrix}, \quad (35)
\]

and $\mathcal{S}$ the $5 \times 3$ matrix:

\[
\mathcal{S} \equiv \begin{pmatrix}
\psi_B^1 \\
\psi_B^2 \\
\sigma^B \\
\tilde{\lambda}^B \\
(\delta - 2\alpha)\tilde{\phi}_B^B
\end{pmatrix}, \quad (36)
\]

Note that both $\mathcal{O}$ and $\mathcal{S}$ depend only on the background fields. As mentioned previously, a gauge-fixing condition on the perturbed tetrad is unnecessary for obtaining the complete system (34). Furthermore, the entries of the matrix $(\Psi^B)$ are automatically independent on the gauge transformations of the vector potential variations $b_\mu$ (see paragraph after Eq. (16)): $(\Psi^B)(h_\mu^\nu, b_\mu) = (\Psi^B)(h_\mu^\nu, b_\mu + \nabla_\mu \epsilon)$. In this manner, the invariance under the gauge freedoms of the matter fields and the perturbed tetrad is guaranteed. This issue will be particularly important below, when we discuss the bilinear forms on the reduced phase space.

In the traditional approach, the field perturbations are separated in polar and axial perturbations (and some gauge-fixing conditions are imposed) with the purpose of reducing the equations governing the perturbations to Schrödinger-type equations, and then to apply semiclassical methods based on the Hermiticity of such system of equations. However, as shown in Ref. [18], such treatment is unnecessary, and for many aims one can obtain essentially the same physical results working directly with the original non-Hermitian system of equations. In fact, when string fields are involved, such as the present case, those reductions seem to be very difficult to carry out, or when possible, the interaction matrix is too complex to be displayed in explicit form $\mathcal{B}$. Therefore, Eqs. (34) in its original form, without separations nor reductions, are sufficient for our present purposes.
V. LOCAL CONTINUITY LAWS ON THE PHASE SPACE AND DEBYE POTENTIALS

5.1 Equations for the Debye potentials

Following the ideas of Section II (see for example that made in Ref. [12]), if the matrix potential \((\psi)\) satisfies \(\mathcal{O}^\dagger(\psi) = 0\), with

\[
\begin{pmatrix}
\psi_G \\ \psi_H \\ \psi_E \\ \psi_F \\ \psi_D
\end{pmatrix} = (\psi)
\]

(38)

then the metric, vector potential, and dilaton real variations are given by

\[
\begin{pmatrix}
-\frac{1}{2}h_{\mu\nu} \\ 2b_{\mu} \\ \phi_{\nu}^{*}
\end{pmatrix} = \mathcal{S}_{\mu}^\dagger(\psi) = \begin{pmatrix}
S_{11}^\dagger & S_{21}^\dagger & S_{31}^\dagger & S_{41}^\dagger & S_{51}^\dagger \\
0 & 0 & S_{32}^\dagger & S_{42}^\dagger & S_{52}^\dagger \\
0 & 0 & 0 & 0 & S_{53}^\dagger
\end{pmatrix} \begin{pmatrix}
\psi_G \\ \psi_H \\ \psi_E \\ \psi_F \\ \psi_D
\end{pmatrix}
\]

\begin{align}
S_{11}^\dagger &= 2l_{(\mu}m_{\nu)}(D + \rho)(\delta + 4\beta) - \mu_{\nu}m_{\nu}[(D - \rho)(D + 3\rho) + \Phi_{00}] - \mu_{\nu}m_{\nu}(\delta + 2\beta)(\delta + 4\beta), \\
S_{21}^\dagger &= 2n_{(\mu}m_{\nu)}(\Delta - 4\gamma - \mu)(\delta + 4\beta) - \mu_{\nu}m_{\nu}[(\Delta - 2\gamma + \mu)(\Delta - 4\gamma - 3\mu) + \Phi_{22}] - \mu_{\nu}m_{\nu}(\delta + 2\beta)(\delta + 4\beta), \\
S_{31}^\dagger &= 2m_{\mu}m_{\nu}(D - \rho) - 2l_{(\mu}m_{\nu)}(\delta + 4\beta), \\
S_{41}^\dagger &= 2m_{\mu}m_{\nu}(\Delta - 2\gamma + \mu) - 2n_{(\mu}m_{\nu)}(\delta + 4\beta), \\
S_{51}^\dagger &= -2D\phi_{\mu}m_{\nu})(\delta + 4\beta) - 2\Delta\phi_{(\mu}m_{\nu)}(\delta + 4\beta) + m_{\nu}m_{\nu}(8\alpha\varphi_{1}^{2} + \Delta\phi D + D\phi\Delta), \\
S_{12}^\dagger &= \frac{1}{2\xi}[m_{\mu}(D + \rho) - l_{(\mu}(2\beta)])(\delta + 4\beta)\frac{1}{\varphi_{1}}, \\
S_{22}^\dagger &= \frac{1}{2\xi}[m_{\mu}(\Delta - 2\gamma - \mu) - n_{\mu}(\delta + 2\beta)](\delta + 4\beta)\frac{1}{\varphi_{1}}.
\end{align}

from Eqs. (14), (19), (22), (26), (30), and (33) we have explicitly that,
\[ S_{52}^{\dagger} = 4a_{1} m_{\mu}(\delta + 4\beta), \]
\[ S_{53}^{\dagger} = \frac{1}{2}(\delta + 2\beta)(\delta + 4\beta). \]

In this manner, the complete field variations are given by Eqs. (39) in terms of the Debye potentials, which satisfy a system of five second-order linear partial differential equations:

\[
O^{1}(\psi) = \begin{pmatrix}
O_{11}^{1} & 0 & 0 & \frac{1}{2} F_{1} \\
0 & O_{22}^{1} & 0 & \frac{1}{2} F_{1} \\
O_{13}^{1} & \frac{\lambda}{2} \Delta \phi F_{1} & O_{33}^{1} & O_{43}^{1} & O_{53}^{1} \\
-F_{1} D\phi & O_{24}^{1} & O_{34}^{1} & O_{44}^{1} & O_{54}^{1} \\
F_{1} & \frac{\lambda}{2} F_{1} & O_{15}^{1} & O_{45}^{1} & O_{55}^{1}
\end{pmatrix}
\begin{pmatrix}
\psi_{C} \\
\psi_{H} \\
\psi_{E} \\
\psi_{F} \\
\psi_{D}
\end{pmatrix} = 0, \tag{41}
\]

where

\[
O_{11}^{1} = (\Delta + 2\gamma + \mu)(D + 3\rho) - (\delta - \Delta\beta)(\delta + 4\beta) - (3\Psi_{2} - 2\Phi_{11} + 2D\phi\Delta\phi),
\]
\[
O_{13}^{1} = 8\xi\phi_{1}^{2}(D + 2\rho) + F_{1}\Delta\phi,
\]
\[
O_{22}^{1} = (D - \rho)(\Delta - 4\gamma - 3\mu) - (\delta - \Delta\beta)(\delta + 4\beta) - (3\Psi_{2} - 2\Phi_{11} + 2D\phi\Delta\phi),
\]
\[
O_{24}^{1} = -8\xi\phi_{1}^{2}(\Delta - 4\gamma - 2\mu) + \frac{\lambda^{2}}{2} F_{1}\Delta\phi,
\]
\[
O_{31}^{1} = -(\Delta + 2\gamma),
\]
\[
O_{33}^{1} = D(\Delta + 2\gamma + 2\mu) + aD\phi(\Delta + 2\mu) - (\delta - \Delta\beta)(\delta + 4\beta) - 2(3\Psi_{2} + 2\Phi_{11}) + a\Delta D\phi,
\]
\[
O_{34}^{1} = aD\phi(D - 2\rho) + aD^{2}\phi + 2\Phi_{90},
\]
\[
O_{35}^{1} = -aD - 2D\phi, \quad O_{42}^{1} = -D, \quad O_{43}^{1} = -a\Delta\phi(\Delta + 2\mu) - a\Delta^{2}\phi - 2\Phi_{22},
\]
\[
O_{44}^{1} = -(\Delta - 4\gamma + a\Delta\phi)(D - 2\rho) + (\delta - \Delta\beta)(\delta + 4\beta) + 2(3\Psi_{2} + 2\Phi_{11}) - aD\Delta\phi,
\]
\[
O_{45}^{1} = -(a(\Delta - 2\gamma) - 2\Delta\phi,
\]
\[
O_{53}^{1} = \frac{1}{8} D\chi \xi F_{1} + (\Delta + \mu)(4a\xi\phi_{1}^{2} - \mu D\phi + \Delta\phi D) - \Delta\phi(\delta - \Delta\beta)(\delta + 4\beta) - \Phi_{22} D\phi - \mu F_{2},
\]
\[
O_{54}^{1} = -(D - \rho)(F_{2} - \mu D\phi) - (\Delta - 4\gamma) D\phi(D + \rho) + D\phi(\delta - \Delta\beta)(\delta + 4\beta) + [(\Delta\phi D - 2a\xi\phi_{1}^{2})\rho] + D\phi(3\Psi_{2} + 2\Phi_{11}),
\]
\[
O_{55}^{1} = (\Delta - 2\gamma - \mu)(D + \rho) - (\delta - \Delta\beta)(\delta + 4\beta) - 3\Psi_{2} + 2\mu\rho - 3D\phi\Delta\phi + 4(a^{2} - 1)\xi\phi_{1}^{2},
\]

and Eqs. (14), (18), (21), (25), (29), and (32) have been used. Eqs. (41) are our fundamental equations since, as we shall see, all conserved quantities and bilinear forms on the phase space are defined in terms of the Debye potentials. Although these equations admit separable solutions in a simple way, we will use them first in the form (41) in order to establish a covariant conservation law, and subsequently to carry out such
5.2 Covariant continuity equation and bilinear forms on the phase space

Since the decoupled system and the system of equations for the Debye potentials are adjoints to each other, in accordance with the results of Section II we have that

\[ (\psi) \wedge \mathcal{O}(\Psi^B) - \mathcal{O}^\dagger(\psi) \wedge (\Psi^B) = \nabla_\mu J^\mu(\psi, \Psi^B). \]  

(43)

The left-hand side contains terms of the form \( \psi_G \wedge \mathcal{O}_{11} \Psi^B_0 - \mathcal{O}_{11}^\dagger \psi_G \wedge \Psi^B_0 \) (see Eqs. (35) and (41)), which can be expressed in the following form, considering the explicit forms of the operators \( \mathcal{O}_{11} \), and \( \mathcal{O}_{11}^\dagger \) given in Eqs. (18), and (42) respectively, that \( D \equiv l^\nu \partial_\mu \), \( \Delta \equiv n^\nu \partial_\mu \), \( \delta \equiv m^\nu \partial_\mu \), and that they are acting on scalar fields:

\[
\begin{align*}
\psi_G \wedge \mathcal{O}_{11} \Psi^B_0 - \mathcal{O}_{11}^\dagger \psi_G \wedge \Psi^B_0 &= \nabla_\mu [l^\nu \psi_G \wedge (\Delta - 4\gamma + \mu) \Psi^B_0 - n^\nu (D + 3\rho) \psi_G \wedge \Psi^B_0] \\
-m^\nu \psi_G \wedge (\delta + 4\beta) \Psi^B_0 - m^\nu (\delta + 4\beta) \psi_G \wedge \Psi^B_0,
\end{align*}
\]

(44)

and similarly for the remaining terms:
are also dependent only on the background fields. In next section, we will demonstrate that
5.3 Covariant symplectic structure on the phase space

\[ + m^\mu \psi_F \wedge (\delta + 4 \beta) \lambda^B - m^\mu (\delta + 4 \beta) \psi_F \wedge \lambda^B, \]
\[ \psi_F \wedge O_{45} (\delta - 2 \beta) \delta^B - O_{45}^1 \psi_F \wedge (\delta - 2 \beta)^2 B = \nabla_\mu [m^\mu \psi_F \wedge (\delta - 2 \beta) \delta^B], \]
\[ \psi_D \wedge O_{53} \sigma^B - O_{53}^1 \psi_D \wedge \sigma^B = \nabla_\mu [-n^\mu (\Delta n D - \mu D \phi + 4a \xi \phi^2) \psi_D \wedge \sigma^B \]
\[ + l^\mu \Delta \psi_D \wedge (\Delta + 2 \gamma + \frac{2a \xi \phi^2}{D\phi}) \sigma^B - \Delta m^\mu \psi_D \wedge (\delta + 3 \beta) \sigma^B + \Delta m^\mu \sigma^D \wedge (\delta + 2 \beta) \psi_D \wedge \sigma^B], \]
\[ \psi_D \wedge O_{54} \lambda^B - O_{54}^1 \psi_D \wedge \lambda^B = \nabla_\mu [D m^\mu (D + \rho) \psi_D \wedge \lambda^B - D m^\mu \psi_D \wedge (\Delta + 2 \gamma + \frac{2a \xi \phi^2}{D\phi}) \lambda^B \]
\[ + D m^\mu \psi_D \wedge (\delta + \beta) \lambda^B - D m^\mu \sigma^D \wedge (\delta + 4 \beta) \psi_D \wedge \lambda^B], \]
\[ \psi_D \wedge O_{55} (\delta - 2 \beta) \delta^B - O_{55}^1 \psi_D \wedge (\delta - 2 \beta)^2 B = \nabla_\mu [-n^\mu (D + \rho) \psi_D \wedge (\delta - 2 \beta) \delta^B \]
\[ + l^\mu \psi_D \wedge (\Delta + 3 \mu) (\delta - 2 \beta) \delta^B - m^\mu \psi_D \wedge (\delta + 3 \beta) (\delta - 2 \beta) \delta^B \]
\[ + m^\mu (\delta + 4 \beta) \psi_D \wedge (\delta - 2 \beta) \delta^B]. \]

(45)

Moreover, from Eqs. (34), and (41) \(O(\Psi^B) = 0\), and \(O^I(\psi) = 0\); hence, from Eq. (43) we have the local continuity law:

\[ \nabla_\mu J^\mu (\Psi^B, \psi) = 0, \]
\[ J^\mu = J_{11}^\mu + J_{13}^\mu + J_{24}^\mu + J_{31}^\mu + J_{33}^\mu + J_{34}^\mu + J_{43}^\mu + J_{44}^\mu + J_{42}^\mu + J_{43}^\mu + J_{44}^\mu + J_{53}^\mu + J_{54}^\mu + J_{55}^\mu, \]

and, of course, the \(J_{ij}^\mu\)’s \((i, j = 1, 2, 3, 4, 5)\) are the components coming from Eqs. (44), and (45); for example, \(J_{54}^\mu = -a D \phi l^\mu \psi_E \wedge \lambda^B\). Thus, \(J^\mu\) is a covariantly conserved current. We will discuss now the properties and physical meaning of \(J^\mu\).

It is easy to verify that, such as \(\Psi^B\) in Eq. (36), the matrix potential \(\psi\) in Eq. (38) is made out of one-forms. Eqs. (39) give the field variations \(b_\mu, \psi_\mu, \) and \(\phi^B\) (one-forms), in terms of \(\psi\). Since the operator \(S^I\) is dependent only on background fields (zero-forms), thus \(\psi\) corresponds to one-forms. This implies automatically that \(J^\mu = J^\mu (\Psi^B, \psi)\) in Eq. (46) is a (non-degenerate) two-form on the corresponding phase space of the solution considered (the matrix operators \(O\) and \(O^I\) involved in the construction of \(J^\mu\) are also dependent only on the background fields). In next section, we will demonstrate that \(J^\mu\) is a closed two-form on the phase space, from which a symplectic structure will be constructed.

5.3 Covariant symplectic structure on the phase space

The presence of an inhomogeneous term corresponding to the additional sources of the field variations in Eqs. (34), is only a knack for finding the operator \(S\). Finally we set \(T^\mu_\nu = 0, j_\mu = 0, \phi_5 = 0\).
For demonstrating that $J^\mu$ is a closed two-form, we need rewrite the $J^\mu_{ij}$'s in Eq. (46). For example, $J^\mu_{11}$ (see Eq. (44)) can be rewritten as:

$$l^\mu \psi_G \wedge (\Delta - 4\gamma + \mu) \Psi_0^B - n^\mu (D + 3\rho) \psi_G \wedge \Psi_0^B - m^\mu \psi_G \wedge (\delta + 4\beta) \Psi_0^B + \overline{m}^\mu (\delta + 4\beta) \psi_G \wedge \Psi_0^B$$

$$= -[l^\mu \psi_G (\Delta - 4\gamma + \mu) \Psi_0^B] + [n^\mu (D + 3\rho) \psi_G \Psi_0^B] + [m^\mu \psi_G (\delta + 4\beta) \Psi_0^B] - \overline{m}^\mu (\delta + 4\beta) \psi_G \Psi_0^B,$$

(47)

where we have considered that $\Psi_0$ vanishes at the background, and the Leibniz rule for the exterior derivative.

Eq. (47) implies that $J^\mu_{11}$ is an exact two-form, and automatically a closed two-form. Similarly, using the fact that $\Psi_B^1, \tilde{\sigma}_B, \tilde{\lambda}_B$, and $(\delta - 2\beta) \tilde{\phi}_B$ can be expressed as variations of vanishing background fields, and the property of exterior derivative used above, we can find that:

$$(J^\mu_{ij})^B = 0,$$

(48)

which makes that $J^\mu$ itself to be closed. In this manner, the geometrical structure defined as $\omega \equiv \int_{\Sigma} J^\mu d\Sigma^\mu$, where $\Sigma$ is an initial value hypersurface, corresponds to a symplectic structure on the phase space. As $J^\mu$ is conserved, $\omega$ is independent of the choice of $\Sigma$ and, in particular, is Poincaré invariant. Since $(\Psi_B^B)$ is invariant under gauge transformation of $b_\mu$ (see paragraph after Eq. (16)), $J^\mu$ and $\omega$ have the same invariance properties. Hence, we have constructed a gauge-invariant closed two-form $\omega$ on the reduced phase space, which means the phase space modulo gauge transformations. Similarly, $J^\mu$ and $\omega$ are independent of the perturbed tetrad gauge freedom.

### 5.4 Debye potentials as fundamental geometrical structures

As we have seen, the bilinear forms $J^\mu$ and $\omega$ depend on the background fields and the solutions admitted by the decoupled system for $(\Psi_B^B)$ and its adjoint system for the Debye potentials. However, the components of $(\Psi_B^B)$, as described in the Appendix A, are defined completely in terms of the field variations $h_{\mu\nu}, b_\mu,$ and $\phi^B$, which in turn, are defined in terms of the Debye potentials (see Eqs. (39)). Therefore, $J^\mu$ and $\omega$ can be expressed finally in terms of a single solution of the equations for Debye potentials. However, in the more general case, if $(\psi)_1$ is a solution admitted by the equations for the potentials, the matrix $(\Psi_B^B)$ can be expressed in terms of a second solution $(\psi)_2$, in general different of $(\psi)_1$, and thus, $J^\mu$ and $\omega$ are defined in terms of a pair of solutions for those equations. Therefore, the Debye potentials, which correspond to one-forms on the phase space, become the fundamental geometrical objects. The analysis of the structure
of the phase space (and the perturbation analysis) has been reduced to the study of scalar equations for the potentials, which is a relatively simple issue. As we will see below, conserved quantities will be also expressed completely in terms of the same potentials.

VI. SEPARATION OF VARIABLES AND CONSERVED QUANTITIES

Our fundamental equations for the Debye potentials (41) and the continuity equation (46), admit separation of variables in terms of harmonic time and the spin-weighted spherical harmonics. The first ones are reduced to a system of ordinary differential equations for the radial parts of the potentials, the second one yields two conserved quantities expressed in terms of such radial parts.

6.1 Separable solutions for the potentials

An advantage of using the Newman-Penrose formalism is that each quantity has a type, and its corresponding boost weight and spin weight. This property suggests the separable solutions more convenient for the equations under study.

More specifically, if \( \eta \) is a quantity of type \( \{p, q\} \), the effect of the (relevant) Geroch-Held-Penrose operators on \( \eta \) is given by \( \partial \eta \equiv (\delta - p\beta - q\alpha)\eta \), and \( \partial \eta \equiv (\delta - p\alpha - q\beta)\eta \), which, using Eqs. (5) and (6), reduce to

\[
\partial \eta = \frac{\sin s}{\sqrt{2R}} \left( \partial_y + i \csc \theta \partial_x \right) \sin^{-s} \theta \eta,
\]

\[
\partial \eta = \frac{\sin^{-s} \theta}{\sqrt{2R}} \left( \partial_y - i \csc \theta \partial_x \right) \sin^s \theta \eta,
\]

where \( s \equiv (p - q)/2 \) is the spin weight of \( \eta \). In the particular case that \( \eta = sY_{lm} \), which means the spin-weighted spherical harmonics:

\[
\partial \eta Y_{lm} = (\delta - 2s\beta) Y_{lm} = \frac{1}{\sqrt{2R}} \left( 1 - s \right) Y_{l+1,m},
\]

\[
\partial \eta Y_{lm} = (\delta + 2s\beta) Y_{lm} = \frac{1}{\sqrt{2R}} \left( 1 + s \right) Y_{l-1,m}.
\]

On the other hand, from Eqs. (41), it is easy to determine that the potentials \( \psi_G, \psi_H, \psi_E, \psi_F, \) and \( \psi_D \) have types \( \{-4, 0\}, \{0, 4\}, \{-3, 1\}, \{-1, 3\}, \) and \( \{-2, 2\} \) respectively. Therefore, all potentials have spin weight \(-2\).
Making use of the fact that the background solution is static and spherically symmetric, we seek for solutions for the potentials of the form:

$$\psi_I = \psi_i(r) - 2Y_{lm}(\theta, \varphi)e^{-i\omega t},$$  \hspace{1cm} (51)

where the subscript $I = G, H, E, F, D$, and $i = g, h, e, f, d$ respectively. Since $(\delta - 2\beta)(\delta + 4\beta)$ is the only operator appearing in Eqs. (41), and (42) that involves angular variables, we only need to know that:

$$(\delta - 2\beta)(\delta + 4\beta)\psi_I = -\frac{L^2}{2R^2}\psi_I, \hspace{1cm} L = [(l - 1)(l + 2)]^{1/2},$$  \hspace{1cm} (52)

where Eqs. (50), and (51) have been employed. The remaining terms correspond to functions and differential operators involving only radial and time variables. In fact, from Eqs. (5), and (51) we have that:

$$D\psi_I = D\psi_i, \hspace{1cm} \Delta\psi_I = -\frac{\chi^2}{2}D\psi_i, \hspace{1cm} (D^2\psi_I = D\psi_i, \hspace{1cm} \Delta\psi_I = -\frac{\chi^2}{2}D\psi_i),$$  \hspace{1cm} (53)

where

$$D = \partial_r - \frac{i\omega}{\chi^2}, \hspace{1cm} \overline{D} = \partial_r + \frac{i\omega}{\chi^2}. \hspace{1cm} (54)$$

In this manner, it suffices to substitute the operators $D$ and $\Delta$, in according to Eqs. (53), by $D$ and $-\frac{\chi^2}{2}\overline{D}$ respectively, $(\delta - 2\beta)(\delta + 4\beta)$ by $-\frac{L^2}{2R^2}$ (in according to Eq. (52)), and $\psi_I$ by $\psi_i$ (the corresponding radial part) into Eqs. (41), for reducing them to an system of ordinary equations for the radial parts $\psi_i$’s of the potentials. Hence, the separation of variables proposed in Eq. (51) applies in a natural and straightforward way.

### 6.2 Separation of variables for the continuity equation

In this section we will see that the covariant continuity equation (46), together with the separable solutions admitted for the potentials (Eq. (51)), and the corresponding separation of variables for the field variations (Appendix B), lead to the existence of two conserved quantities.

As we have seen, at each spacetime point, $J^\mu$ in Eq. (46) is a two-form on the phase space. Regardless of the last interpretation, we can maintain $J^\mu$ as a bilinear product on field perturbations on the spacetime manifold. In this manner, the covariantly conserved current (46) can be rewritten, grouping conveniently its components on the null tetrad, in the form:

$$J^\mu = Vl^\mu + Vn^\mu + Vm^\mu + \overline{Vm}^\mu, \hspace{1cm} (55)$$
where

\[ V_i \equiv \psi_C(\Delta - 4\gamma + \mu)\Psi_B^0 - 8\xi\phi_1^2\psi_C\delta^B - \Psi_1^0(\Delta - 4\gamma - 3\mu)\psi_H - \delta^B(\Delta + 2\gamma + 2\mu)\psi_E \]

\[ -aD\phi\psi_E\lambda^B + a\psi_E(\delta - 2\beta)\phi^B + \psi_F\Psi_1^0 - \psi_F(\Delta + 2\gamma + 2\mu - a\Delta\phi)\lambda^B \]

\[ +\Delta\phi\psi_d\left(\Delta + \mu + \frac{2a\xi\phi_1^2}{D\phi}\right)\sigma^B - D\phi\psi_d\left(\Delta + 2\gamma + \mu - \frac{2a\xi\phi_1^2}{D\phi}\right)\lambda^B \]

\[ +\psi_d(\Delta + 3\mu)[\delta - 2\beta]\hat{\phi}^B, \]

\[ V_n \equiv -\Psi_0^B(D + \rho)\psi_C + \psi_H(D - \rho)\Psi_1^1 + 8\xi\phi_1^2\psi_H\lambda^B + \psi_E\Psi_0^B + \psi_E(D - 2\rho - aD\phi)\sigma^B \]

\[ +a\Delta\phi\psi_E\delta^B + \lambda^B(D - 2\rho)\psi_E + a\psi_E(\delta - 2\beta)\phi^B - \delta^B[4a\xi\phi_1^2 - \mu D\phi + \Delta\phi]D\psi_D \]

\[ +D\phi\lambda^B(D + \rho)\psi_D - [(D + \rho)\psi_D]D(\delta - 2\beta)\phi^B, \]

\[ V_m \equiv -\psi_C(\delta + 4\beta)\Psi_0^B - \psi_H(\delta + 4\beta)\Psi_1^1 - \psi_E(\delta + 4\beta)\lambda^B + \psi_F(\delta + 4\beta)\lambda^B - \Delta\phi\psi_D(\delta + 4\beta)\sigma^B \]

\[ +D\phi\lambda^B(\delta + 4\beta)\lambda^B - \psi_D(\delta + 4\beta)D(\delta - 2\beta)\phi^B, \]

\[ V_{\text{nr}} \equiv \Psi_0^B(\delta + 4\beta)\psi_C + \Psi_1^1(\delta + 4\beta)\psi_H + \sigma(\delta + 4\beta)\psi_E - \lambda^B(\delta + 4\beta)\psi_E + \Delta\phi\sigma^B(\delta + 4\beta)\psi_D \]

\[ -D\phi\lambda^B(\delta + 4\beta)\psi_D + (\delta + 4\beta)\psi_D(\delta - 2\beta)\phi^B. \]

(56)

Therefore, considering that in the Newman-Penrose formalism \(\partial_\mu l^\mu = -2\rho, \partial_\mu n^\mu = 2\mu - 2\gamma, \partial_\mu m^\mu = 2\beta\), the continuity equation (46) can be rewritten in the following form:

\[ \partial_\mu(V_i l^\mu + V_n n^\mu + V_m m^\mu + V_{\text{nr}} n^\mu) = (D - 2\rho)V_i + (\Delta + 2\mu - 2\gamma)V_n + (\delta + 2\beta)V_m + (\delta + 2\beta)V_{\text{nr}} = 0. \]

(57)

However, there is an immediate reduction in the terms involving \(V_m\) and \(V_{\text{nr}}\) in Eq. (57). Considering that all components of \((\Psi^B)\) have spin weight 2 (see Eqs. (36), (B8), and (B9)), we can obtain an equation analogous to Eq. (52):

\[ (\delta - 2\beta)(\delta + 4\beta)(\Psi^B) = -\frac{L^2}{2R^2}(\Psi^B); \]

(58)

Furthermore, from the explicit forms of \(V_m\) and \(V_{\text{nr}}\) in Eqs. (56), \((\delta + 2\beta)V_m + (\delta + 2\beta)V_{\text{nr}}\) in Eq. (57) contains terms of the form \(- (\delta + 2\beta)[\psi_1(\delta + 4\beta)\Psi^B] + (\delta + 4\beta)[\Psi^B(\delta + 4\beta)\psi_1]\), which, using Eqs. (52) and (58), vanish:

\[ -(\delta + 2\beta)[\psi_1(\delta + 4\beta)\Psi^B] + (\delta + 4\beta)[\Psi^B(\delta + 4\beta)\psi_1] = -\psi_1(\delta - 2\beta)(\delta + 4\beta)\Psi^B \]

\[ +\Psi^B(\delta - 2\beta)(\delta + 4\beta)\psi_1 = -\psi_1(\delta - 2\beta)(\delta + 4\beta)\Psi^B \]

\[ = -\psi_1\left[-\frac{L^2}{2R^2}\Psi^B\right] + \Psi^B\left[-\frac{L^2}{2R^2}\psi_1\right] = 0. \]

In this manner \((\delta + 2\beta)V_m + (\delta + 2\beta)V_{\text{nr}} = 0\), is satisfied identically, and Eq. (57) reduces to:

\[ (D - 2\rho)V_i + (\Delta + 2\mu - 2\rho)V_n = 0. \]

(59)
Thus, the whole physical information about our conserved quantities is contained in \( V_i \) and \( V_n \). Furthermore, direct substitutions of the separable solutions for the potentials (Eq. (51)), and field variation (Eqs. (B8) and (B9)) into the expressions for the bilinear products \( V_i \) and \( V_n \) given in Eqs. (56), lead to a splitting of such products in terms of the form \( e^0 \) and \( e^{-2i\omega t} \):

\[
V_n = [V_n^+ + \frac{i\omega}{\chi} G^+] \cdot 2Y_{lm} -2Y_{lm} + e^{-2i\omega t}[V_n^- + \frac{i\omega}{\chi} G^-] \cdot 2Y_{lm} 2Y_{lm},
\]

\[
V_i = [V_i^+ + \frac{i\omega}{2} G^+] \cdot 2Y_{lm} -2Y_{lm} + e^{-2i\omega t}[V_i^- - \frac{i\omega}{2} G^-] \cdot 2Y_{lm} 2Y_{lm},
\]

(60)

where

\[
V_n^\pm \equiv \Psi_0^\pm [\psi_e - R^3 \partial_r (R^{-3} \psi_e)] + \psi_h [R^{-1} \partial_r (R \Psi_4^\pm) + 8\xi \varphi^2 \lambda^\pm]
\]

\[
+ \psi_e R^{-2} \xi^{-1/2} \partial_r (R^2 \xi^{1/2} \sigma^\pm) + a\psi[r \Delta \phi \lambda^\pm + \tilde{\phi}^\pm] + \lambda^\pm R^2 \partial_r \partial_r (R^2 \psi_e)
\]

\[
- \sigma^\pm [\Delta \phi R \partial_r (R^{-1} \psi_d) + 4a\xi \varphi^2 \psi_d] + R [d \phi \lambda^\pm - \tilde{\sigma}^\pm \partial_r (R^{-1} \psi_d)],
\]

\[
V_i^\pm = \Psi_4^\pm \left[ \psi_f + \frac{1}{2} R^{-1} \partial_r (R \chi^2 \psi_e) \right] - \psi_g \left[ \frac{1}{2} R^{-1} \partial_r (R \chi^2 \Psi_0^\pm) \right] + \frac{2a\xi \varphi^2}{D\phi} \lambda^\pm
\]

\[
+ \Delta \phi \psi_d \left[ \frac{1}{2} R^{-1} \psi_d (R \sigma^\pm) + \frac{2a\xi \varphi^2}{D\phi} \lambda^\pm \right] - \frac{1}{2} R^3 \partial_r \partial_r (R \lambda^\pm),
\]

(61)

and

\[
G^+ \equiv \psi_g \Psi_0^B + \psi_h \Psi_4^B + \psi_e \sigma^B - \psi_f \lambda^B + \psi_d [\Delta \phi \lambda^B + \tilde{\phi}^B],
\]

\[
G^- \equiv \psi_g \Psi_0^B - \psi_h \Psi_4^B - \psi_e \sigma^B - \psi_f \lambda^B + \psi_d [\Delta \phi \lambda^B - \tilde{\phi}^B] = \frac{L^2 l (l+1)}{8R^3} \psi^2_d,
\]

(62)

are only functions of \( r \), and the relations (B10) have been used for reducing \( G^- \). Since the components \((\Psi^B)^-\) (see Eqs. (B8) and (B9)) are directly proportional to the potentials, \( V_n^- \) and \( V_i^- \) in Eqs. (61) have remarkable reductions (unlike \( V_n^+ \) and \( V_i^+ \)):

\[
V_n^- = - \frac{L^2 l (l+1)}{8R^4} \left[ -2\psi_g \psi_h \partial_r \ln R^3 + R \psi_d \psi_e \left( \frac{\psi_d}{R} \right) \right],
\]

\[
V_i^- = - \frac{L^2 l (l+1)}{16R^4} \left[ 2\psi_g \psi_h \partial_r \ln R^3 + R \psi_d \psi_e \left( \frac{\psi_d}{R} \right) \right],
\]

(63)

therefore, from Eqs. (62) and (63) is very easy to show that:

\[
V_n^+ + 2\chi^{-2} V_i^- + R^{-2} \partial_r (R^2 G^-) = 0,
\]

\[
V_i^- - \frac{\chi^2}{2} V_n^- = - \frac{L^2 l (l+1) \chi^2}{4R^4} (\partial_r \ln R^3) \psi_g \psi_e,
\]

(64)
which will be useful below.

6.3 Conserved quantities

Substituting expressions (60) into Eq. (59), using the explicit form for $D$, $\Delta$, $\rho$, $\mu$, and $\gamma$ we obtain, after some simplification and suitably grouping, that:

$$
\frac{1}{R^2} \partial_t R^2 \left[ V_t^+ - \frac{\chi^2}{2} V_n^+ \right] - 2Y_{lm} \bar{V}_{lm} + \frac{e^{-2i\omega t}}{R^2} \partial_t R^2 \left[ V_t^- - \frac{\chi^2}{2} V_n^- \right] - 2Y_{lm} \bar{Y}_{lm} = 0,
$$

(65)

the last term vanishes in accordance to the first of Eqs. (64), thus Eq. (65) reduces to:

$$
\partial_t R^2 \left[ V_t^+ - \frac{\chi^2}{2} V_n^+ \right] - 2Y_{lm} \bar{V}_{lm} + e^{-2i\omega t} \partial_t R^2 \left[ V_t^- - \frac{\chi^2}{2} V_n^- \right] - 2Y_{lm} \bar{Y}_{lm} = 0,
$$

(66)

which implies (using the linear independence of terms of the form $e^{i\omega t}$ and $e^{-i\omega t}$) that there exist two conserved quantities, which we denote by $K^{(\pm)}$:

$$
R^2 \left[ V_t^{(\pm)} - \frac{\chi^2}{2} V_n^{(\pm)} \right] \equiv K^{(\pm)}.
$$

(67)

Although $K^+$ has a complicated form in terms of the potentials, $K^-$ has a remarkably simple form, in accordance with the last expression in Eq. (64):

$$
K^- \equiv R^2 \left[ V_t^- - \frac{\chi^2}{2} V_n^- \right] = -\frac{L^2(l+1) \chi^2 (\partial_t \ln R^2)}{4 R^2} \psi_b \bar{\psi}_h.
$$

(68)

Note that, since $(\Psi_B)^+$ depends on $(\psi_i)$, $K^+$ depends on $(\psi_i)$ and $(\overline{\psi_i})$, whereas $K^-$ directly on the potentials without involving its complex conjugates.

The existence of these two conserved quantities deserves some important comments. First: although the equations used for obtaining such quantities are not Hermitian ones (for which the constancy of the Wronskian yields traditionally conserved quantities), one can obtain, without any restrictions and full generality, conserved quantities, provided that the original system of equations and its adjoint system to be used. Second: as we have seen, if the potentials have a time dependence of the form $e^{-i\omega t}$, the field perturbations appearing in the decoupled system contain terms proportional to $e^{-i\omega t}$ and $e^{i\omega t}$ (in the classical cases, unlike the present case involving string fields, only terms proportional to $e^{i\omega t}$ are present \cite{18}), which lead finally to two conserved quantities. In the classical cases, only a conserved quantity analogous to the
present $K^+$ is obtained. In fact, the bilinear terms depending on $\Psi_{B+}^0$ and $\psi_g$ in the expression for $K^+$ (see the explicit forms for $V_{\alpha}^+$ and $V_\ell^+$ in Eqs. (61)), yield a conservation relation for the energy of gravitational perturbations in the classical Schwarzschild black hole (and something similar for electromagnetic perturbations) [18]. In this manner, it is possible that $K^+$ has the same physical meaning for the present string black hole: the conservation of the energy for the coupled field perturbations. However, this question will require a long asymptotic analysis and, will be studied in a subsequent work. On the other hand, $K^-$ is a novel conserved quantity apparently without classical analogous; it is also an open question to investigate its physical meaning.

6.4 Differential identities

As mentioned, $(\Psi^B)$ in the decoupled system can be expressed essentially in the form $(\Psi^B) = (\Psi^B)^+ \frac{1}{-2Y_{lm}} e^{i\omega t} + (\Psi^B)^- 2Y_{lm} e^{-i\omega t}$. Thus, the decoupled system $\mathcal{O}((\Psi^B)^+) = 0$, can be reduced (again, using the linear independence of the terms of the form $e^{i\omega t}$ and $e^{-i\omega t}$) to $\mathcal{O}((\Psi^B)^+) = 0$, and $\mathcal{O}((\Psi^B)^-) = 0$. The adjoint system for the potentials is the same, coming from both above equations: $\mathcal{O}^\dagger(\psi) = 0$. In this manner, the two conserved quantities constructed in Section 6.3, can be obtained separately: $K^+$ will become from the equation $(\psi)\mathcal{O}((\Psi^B)^+) - \mathcal{O}^\dagger(\psi)((\Psi^B)^+) = \nabla_\mu J^\mu$, and $K^-$ will from the equation $(\psi)\mathcal{O}((\Psi^B)^-) - \mathcal{O}^\dagger(\psi)((\Psi^B)^-) = \nabla_\mu J^\mu$.

However, $\mathcal{O}((\Psi^B)^-) = 0$ is essentially the same equations for the potentials $\mathcal{O}^\dagger(\psi) = 0$ (remembering that the components of $(\Psi^B)^-$ are directly proportional to $(\psi)$). In fact, after separation of variables, the first row of equations $\mathcal{O}((\Psi^B)^-) = 0$ corresponds to the second equation for the potentials (which means, the second row of $\mathcal{O}^\dagger(\psi) = 0$), satisfying the following differential identities between components of the operators $\mathcal{O}$ and $\mathcal{O}^\dagger$: $R^4\mathcal{O}_{11} = \mathcal{O}_{12}^\dagger$, and $\frac{1}{8\pi^2}(\mathcal{O}_{13} - F_1\Delta\phi)\xi^{-1} = \mathcal{O}_{12}^\dagger$. Similarly, the second of those equations, corresponds to the first equation for the potentials satisfying the relations $R^4\mathcal{O}_{12} = \mathcal{O}_{11}^\dagger$, and $-\frac{1}{8\pi^2}(\mathcal{O}_{24} + \frac{1}{4}F_1 D\phi)\xi^{-1} = \mathcal{O}_{13}^\dagger$. The third and fourth of the decoupled equations correspond to the following combinations of the equations for the potentials: (fourth one) + $D\phi$ (fifth one) and, (third one) – $\Delta\phi$ (fourth one) respectively. In these cases, the following differential identities are satisfied:

\begin{equation}
\begin{align*}
2Q^2\xi\mathcal{O}_{31} = & \mathcal{O}_{34}^\dagger + \frac{1}{4}F_1 D\phi, \\
-\frac{a}{2}F_1 = & \mathcal{O}_{34}^\dagger + D\phi\mathcal{O}_{35}^\dagger, \\
\xi(\mathcal{O}_{43} - \mathcal{O}_{35}\Delta\phi)\xi^{-1} = & -\mathcal{O}_{44}^\dagger + D\phi\mathcal{O}_{15}^\dagger, \\
Q^2\xi\mathcal{O}_{55} = & \mathcal{O}_{54}^\dagger + D\phi\mathcal{O}_{55}^\dagger.
\end{align*}
\end{equation}

(69)
and

\[-2Q^2 \xi O_{42} \frac{1}{R^4} = O_{13}^\dagger - F_1 \Delta \phi,\]
\[\frac{a \chi^4}{8} F_1 = O_{43}^\dagger - \Delta \phi O_{45}^\dagger,\]
\[\xi (O_{44} + O_{45} D \phi) \xi^{-1} = -(O_{33}^\dagger - \Delta \phi O_{35}^\dagger),\]
\[-Q^2 \xi O_{45} \frac{1}{R^4} = O_{53}^\dagger - \Delta \phi O_{55}^\dagger,\]  \hspace{1cm} (70)

respectively. Finally, the fifth of the decoupled equations corresponds to the fifth of the equations for the potentials, and the corresponding differential identities are:

\[R^4 O_{55} \frac{1}{R^4} = O_{55}^\dagger,\]
\[\frac{1}{4 \varphi^4} (O_{54} + O_{55} D \phi) \xi^{-1} = -O_{35}^\dagger,\]
\[\frac{1}{4 \varphi^4} (O_{53} - O_{55} \Delta \phi) \xi^{-1} = O_{45}^\dagger.\]  \hspace{1cm} (71)

What do such differential identities mean? The answer is that they map solutions of the equations for the (radial parts) of the potentials into solutions for the (radial parts) of the field variations appearing in the decoupled set of equations, and conversely.

As we have demonstrated, if

\[(\psi)(r) = \begin{pmatrix} \psi_g \\ \psi_h \\ \psi_e \\ \psi_f \\ \psi_d \end{pmatrix},\]

is the radial part of a solution of the form \((\psi) = (\psi)(r) - 2 Y_{lm} e^{-i \omega t}\) admitted by \(O\dagger(\psi) = 0\), then

\[(\Psi^B)^-(r) = \begin{pmatrix} \frac{1}{R^4} \psi_h \\ \frac{1}{R^4} \psi_g \\ -\frac{1}{2Q \xi} \psi_f \\ \frac{1}{2Q \xi} \psi_e \\ \frac{1}{2Q \xi} (D \phi \psi_e + \Delta \phi \psi_f) \end{pmatrix},\]

is the radial part of a solution of the form \((\Psi^B) = (\Psi^B)^- (r) - 2 Y_{lm} e^{-i \omega t}\) for the decoupled system \(O(\Psi^B) = 0\).

If in the preceding expression for \((\Psi^B)\), \(\omega\) is replaced by \(-\omega\), then \((\Psi^B) = (\Psi^B)^- (r) - 2 Y_{lm} e^{i \omega t}\) satisfies \(O(\Psi^B) = 0\)
with

\[
(\Psi_B)^-(r) = \left( \begin{array}{c}
\frac{i}{\sqrt{4\pi}} \overline{\psi}_h \\
\frac{i}{\sqrt{4\pi}} \overline{\psi}_g \\
-\frac{1}{2Q_2} \xi \overline{\psi}_f \\
\frac{1}{2} \left( \overline{\psi}_d \right) + \frac{1}{2Q_2} (D\phi \overline{\psi}_e + \Delta \phi \overline{\psi}_f) 
\end{array} \right).
\]

(74)

On the other hand, \((\Psi_B)^+\) in Eq. (B8) and (B9) is also the radial part of a solution of the form \(e^{i\omega t}\) for the decoupled system. Thus, \((\Psi_B)^+ = C(\Psi_B)^-\), being \(C\) a constant. This relation of proportionality would lead to differential identities analogous to the Teukolsky-Starobinsky identities found in the study of classical black holes [21]. However, this subject will be extended in a subsequent work.

### VII. CONCLUDING REMARKS

We summarize some questions that remain open and will be the subject of forthcoming works.

First: although string black holes are considered as classical black holes plus Planck-scale corrections, they are not actually authentic quantum black holes. Hence, for example, the thermodynamics properties argued in Refs. [2, 3] are limited in this sense; a proper quantization will give a more complete and satisfactory description of such objects (see the paragraph before final comments of Ref. [5]). The idea is, of course, that the symplectic structure constructed in the present work, to be the starting point for such a proper (canonical) quantization, which will give us a consistent quantum extension of string black holes.

Second: as mentioned, the physical meaning of the conserved quantities obtained in the present work, remains to be worked out. This subject will include the calculation of physical quantities such as scattering amplitudes, reflection and transmission coefficients, etc. The differential identities established here, will be useful in this task; they will permit to relate the outcoming flux of energy to the incoming flux of energy for the coupled field perturbations [21].

Third: the results established in Sec. II can be considered in the formal context of differential equations. The possible applications of these very general results in other cases (and other areas of physics) are open questions.

Finally, beyond the specific application presented in this work, adjoint operators scheme gives a new approach for covariant canonical quantization [22], which represents a subject of permanent and wide interest in physics. The possible implications by using this approach in this matter is also a problem for the future.
ACKNOWLEDGMENTS

This work was supported by CONACYT and the Sistema Nacional de Investigadores (México).

Appendix A: Gauge invariant perturbations

In order to construct quantities with invariance properties similar those of $\sigma^B$, which are useful in our approach, we follow Eqs. (15) and (16), and we find the following expression for the variations of the vanishing background Newman-Penrose quantities:

$$\kappa^B \equiv -(l^\nu l^\mu \nabla_\nu m_\mu)^B = l^\nu l^\mu m_\mu (\Gamma^\nu_{\mu\gamma})^B - (D - \rho)(l^\mu m_\mu)^B,$$

$$\pi^B \equiv -(m^\alpha l^\nu \nabla_\alpha n_\mu)^B = m^\alpha l^\nu n_\gamma (\Gamma^\alpha_\nu_{\mu\gamma})^B - D(m^\alpha n_\mu^B) + \mu(l^\mu m_\mu)^B,$$

$$\lambda^B \equiv -(m^\mu m^\nu \nabla_\gamma n_\mu)^B = m^\mu m^\nu n_\gamma (\Gamma^\gamma_{\mu\nu})^B + \mu m^\mu m^\nu h_{\mu\nu} - (\delta - 2\beta)(m^\mu n_\mu^B),$$

$$\pi^B \equiv -(m^\nu n^\mu \nabla_\nu m_\mu)^B = m^\nu m^\mu n_\alpha (\Gamma^\gamma_{\mu\alpha})^B + \mu m^\nu n^\mu h_{\mu\nu} - (\Delta + 2\gamma + \mu)(m^\nu n_\mu^B),$$

$$\varphi_1^B \equiv -(l^\mu n^\nu \nabla_\nu m_\mu)^B = (\Delta - 2\gamma)(l^\mu m_\mu)^B + \rho(n^\nu m_\mu^B),$$

$$\varphi_2^B \equiv (m^\mu n^\nu F_{\mu\nu}^B)^B = m^\mu n^\nu F_{\mu\nu}^B - 2\varphi_1(n^\nu m_\mu^B).$$

where $m^\mu n_\mu^B$, $n^\mu m_\mu^B$, $m^\nu l_\mu^B$, and $l^\mu m_\mu^B$ are dependent on the perturbed tetrad gauge freedom and Eqs. (6)-(8) for the background quantities have been considered. Note that

$$2\varphi_1^B = (l^\mu n^\nu + m^\mu m^\nu) F_{\mu\nu}^B - 2\varphi_1[m_\mu(m^\mu)^B + \overline{m}_\mu(m^\mu)^B] = (l^\mu n^\nu + m^\mu m^\nu) F_{\mu\nu}^B + 2\varphi_1 m^\mu m^\nu h_{\mu\nu},$$

which means that $\varphi_1^B = \varphi_1^B(h_{\mu\nu}, b_p)$, is defined completely in terms of $h_{\mu\nu}$ and $b_p$, and independent on the perturbed tetrad gauge freedom. Thus, from Eqs. (A1) and (A2) we can find easily the following quantities, independent on both, perturbed tetrad gauge freedom and gauge transformations of the vector potential.
variations:

\[ \dot{\sigma}^B \equiv \sigma^B + (\delta - 2\beta) \frac{\dot{\varphi}_0^B}{2\varphi_1}, \]
\[ \kappa^B \equiv \kappa^B + (D - \rho) \frac{\dot{\varphi}_0^B}{2\varphi_1}, \]
\[ \hat{\pi}^B \equiv \hat{\pi}^B + D \left( \frac{\varphi_2^B}{2\varphi_1} \right) - \mu \frac{\varphi_0^B}{2\varphi_1}, \]
\[ \hat{\lambda}^B \equiv \hat{\lambda}^B + (\delta - 2\beta) \frac{\varphi_2^B}{2\varphi_1}, \]
\[ \dot{\nu}^B \equiv \dot{\nu}^B + (\Delta + 2\gamma + \mu) \frac{\varphi_2^B}{2\varphi_1}, \]
\[ \hat{\tau}^B \equiv \hat{\tau}^B + (\Delta - 2\gamma) \frac{\varphi_0^B}{2\varphi_1} + \mu \frac{\varphi_2^B}{2\varphi_1}, \]
\[ \dot{\varphi}_1^B \equiv (\delta \varphi_1)^B + \mu \varphi_0^B + \rho \varphi_2^B, \quad \dot{\varphi}_1^B \equiv (\delta \varphi_1)^B - \mu \varphi_0^B - \rho \varphi_2^B, \]
\[ \phi^B \equiv (\delta \phi)^B - \Delta \phi \frac{\varphi_0^B}{2\varphi_1} + D \phi \frac{\varphi_2^B}{2\varphi_1}. \tag{A3} \]

The variations of the Weyl scalars \( \Psi_{B0}^B \) and \( \Psi_{B4}^B \) turn out to be directly, independent on the perturbed tetrad gauge freedom, similar to the perturbed quantity in Eq. (A2). Finally, we can find the following gauge invariant quantities, related to the Weyl scalar variations and electromagnetic field variations:

\[ \dot{\Psi}_{B3}^B \equiv \Psi_{B3}^B + 3 \Psi_2 \left( \frac{\varphi_2^B}{2\varphi_1} \right), \]
\[ \dot{\Psi}_1^B \equiv \Psi_1^B - 3 \Psi_2 \left( \frac{\varphi_0^B}{2\varphi_1} \right). \tag{A4} \]

In this manner, the field quantities in Eqs. (A3), and (A4) (and \( \Psi_{B0}^B \), and \( \Psi_{B4}^B \) in according to first and fifth of Eqs. (A21)) are determined completely in terms of \( h_{\mu\nu} \), \( b_\mu \), and \( \phi^B \).

With the purpose of finding the equations governing the gauge invariant variations, let us take first-order variations of Eq. (A3) of Ref. [12], and we obtain the following equation involving no gauge invariance quantities:

\[ (\Delta - 2\gamma + \mu - \Delta \phi) \varphi_0^B - (\delta \varphi_1)^B + 2 \varphi_1 \tau^B + a D \phi \varphi_2^B + 2 a \varphi_1 (\delta \phi)^B = \xi^{-1} m^\mu j_\mu, \tag{A5} \]

where the background solution for the static charged black holes of Sec. II has been considered and a source \( j_\mu \) for the electromagnetic perturbations has been included [12]. However, using the expressions (A3) we can substitute \( (\delta \varphi_1)^B \), \( \tau^B \), and \( (\delta \phi)^B \) in favor of \( \varphi_1^B \), \( \tau^B \), and \( \phi^B \), into Eq. (A5), and to obtain easily the equation

\[ 2 \varphi_1 \tau^B + 2 a \varphi_1 \phi^B - \varphi_1^B = \xi^{-1} m^\mu j_\mu, \tag{A6} \]
involving only gauge invariant quantities. Similarly, from the complex conjugate of Eq. (A4) of Ref. [12] we obtain

\[ -2\phi_1 \bar{\pi}^B + 2a\phi_1 \bar{\sigma}^B + \bar{\phi}_1^B = \xi^{-1} m^\nu j_\mu. \]  

(A7)

The remaining two Maxwell equations (A1) and (A2) of Ref. [12], require a more elaborate procedure in order to avoid the appearance of undesirable perturbed quantities. Before considering the variations, we apply \( \delta \) to Eq. (A1) of Ref. [12] and we obtain

\[ \delta(\delta + 2\bar{\alpha})\phi_0 - \delta D \phi_1 + 2\bar{\delta}(\rho \phi_1) - \delta(\kappa \varphi_2) - a\delta[\bar{\varphi}_0 \bar{\pi} + \bar{\varphi}_0 \delta - (\varphi_1 + \varphi_1) D] \phi = 0, \]  

(A8)

using the commutation relations, the second term can be expressed as

\[ \delta D \phi_1 = (D - \bar{\sigma} - \epsilon + \bar{\tau})\delta \phi_1 + (\bar{\alpha} + \beta - \bar{\pi})D \phi_1 + \kappa \Delta \phi_1 - \sigma \bar{\delta} \phi_1, \]  

and considering the background solution, we have from the above equation that

\[ (\delta D \phi_1)^B = (D - \rho)(\delta \phi_1)^B + D \phi_1(\bar{\alpha} + \beta - \bar{\pi})^B + \Delta \phi_1 K^B, \]  

(A9)

thus, from Eqs. (A8) and (A9) and considering again the background solution, one obtains the linearized equation

\[ \delta(\delta + 2\bar{\sigma})\varphi_0^B - (D - 3\rho - aD\phi)(\delta \varphi_1)^B + 2\mu \varphi_1 K^B + 2\rho \varphi_1 \bar{\pi}^B + aD\phi(\delta \varphi_1)^B \]

\[ + 2\varphi_1[(\delta \rho) - \rho(\bar{\alpha} + \beta)] = \delta(\xi^{-1} l^\nu j_\mu), \]  

(A10)

however, from the Ricci identities we can find additionally the linearized equation

\[ (\delta \rho)^B - \rho(\bar{\alpha} + \beta)^B - (\bar{\sigma} + 4\bar{\beta})\sigma^B + \Psi_1^B - D\phi(\delta \phi)^B - 2\xi \varphi_1 \varphi_0^B = l^\mu m^\nu T_{\mu\nu}, \]  

(A11)

where we have included an additional source for the gravitational perturbations, \( T_{\mu\nu} \) [12], and \( \Phi_{01}^B = D\phi(\delta \phi)^B + 2\xi \varphi_1 \varphi_0^B \) (see Eqs. (A8) of Ref. [12]). Therefore, we have finally, from Eqs. (A10), (A11) and from direct substitutions of the relations (A3) and (A4), that

\[ -(D - 3\rho - aD\phi) \varphi_1^B + aD\phi \varphi_1^B + 2\mu \varphi_1 K^B + 2\rho \varphi_1 \bar{\pi}^B + 2\bar{\varphi}_1 D \phi \bar{\sigma}^B + 2\varphi_1 \bar{\Psi}_1^B \]

\[ + 2\varphi_1 D \bar{\phi} \bar{\phi}^B = \delta(\xi^{-1} l^\nu j_\mu) - 2\bar{\varphi}_1 l^\mu m^\nu T_{\mu\nu}, \]  

(A12)

which involves only gauge invariant quantities. Similarly, from Eq. (A2) of Ref. [12] and using the linearized equation

\[ -(\delta \bar{\alpha})^B - \mu(\bar{\alpha} + \beta)^B + (\bar{\sigma} + 4\bar{\beta})\bar{\alpha}^B + \bar{\Psi}_0^B - \Delta \phi(\delta \phi)^B + 2\xi \varphi_1 \varphi_2^B = m^\mu m^\nu T_{\mu\nu}, \]  

(A13)
coming from the Ricci identities, we can obtain the equation

\[(\Delta + 3\mu - a\Delta\phi)\varphi_1^B - a\Delta\phi\varphi_1^B + 2\mu\varphi_1 \tilde{\tau}^B + 2\rho\varphi_1\tilde{\nu}^B - 2\varphi_1(\tilde{\tau} + 4\tilde{\eta})\tilde{\lambda}^B - 2\varphi_1\tilde{\psi}_1^B + 2\varphi_1\Delta\phi\tilde{B} = \delta(\xi^{-1}n^\mu j_\mu) - 2\varphi_1n^\nu m^\nu T_{\mu\nu}. \] (A14)

In the case of the dilaton equation, we apply again \(\delta\) to Eq. (A5) of Ref. [12], before considering the variations:

\[D\phi(\delta\tilde{\eta}) + \tilde{n}\delta D\phi + \delta[(D + \epsilon + \tilde{\tau} - \rho)\Delta\phi] - (\delta\tilde{\eta})\tilde{\phi} - \pi\delta\tilde{\phi} + \delta[(-\tilde{\eta} + \alpha - \tilde{\tau} - \pi)\delta\phi] + \frac{a}{4}\delta(\xi F^2) = 0. \] (A15)

Moreover, using the commutation relations (see Eq. (A9)) one finds that

\[\delta D\phi = (D - \rho)(\delta\phi)^B + \Delta\phi\kappa^B + D\phi(\tilde{\tau} + \beta - \tilde{\eta})^B, \]

\[\delta\Delta\phi = (\Delta + \mu)(\delta\phi)^B - D\phi(\tilde{\tau} + \beta - \til{\eta})^B, \] (A16)

where the background solution has been considered. Furthermore,

\[(\delta F^2)^B = -8\xi\varphi_1[(\delta\varphi_1)^B - (\delta\til{\varphi}_1)^B - 2\alpha\varphi_1(\delta\phi)^B], \] (A17)

where Eq. (A7) of Ref. [12] has been used. Similarly,

\[\delta(D + \epsilon + \til{\tau} - \rho)\Delta\phi = (D - 2\rho)(\delta\Delta\phi)^B + D\Delta\phi(\til{\tau} + \beta - \til{\eta})^B + \Delta^2\phi\kappa^B + \Delta\phi[\delta(\epsilon + \til{\tau})^B - (\delta\rho)^B]. \] (A18)

On the other hand, from the Ricci identities

\[(D - \rho)(\til{\tau} + \beta)^B - \delta(\epsilon + \til{\tau})^B + (\mu + 2\gamma)\kappa^B - \rho\kappa^B - \til{\psi}_1^B - D\phi(\delta\phi)^B - 2\xi\varphi_1^B = \mu m^\nu T_{\mu\nu}. \] (A19)

Thus, by linearizing Eq. (A15), considering Eqs. (A16)–(A19) and direct substitutions of \((\delta\rho)^B, (\delta\til{\eta})^B,\)
\(\delta(\epsilon + \til{\tau})^B\) from Eqs. (A11), (A13), and (A19), we have, after some simplification and grouping suitably, that

\[\mu(D - \rho) + (D - 2\rho)(\Delta + \mu) - \delta(\til{\tau} + 2\til{\eta}) - 3D\phi\Delta\phi - 4\alpha^2\xi\varphi_1^B + [\Delta^2\phi + 2(\mu + \gamma)\Delta\phi]\til{\kappa}^B \]

\[-(D\phi)\til{\til{\tau}}^B + D\phi(\til{\tau} + 4\til{\eta})\til{\lambda}^B - \Delta\phi(\til{\tau} + 4\til{\eta})\til{\sigma}^B - (D - 2\rho)D\phi\til{\nu}^B + (D - 2\rho)\Delta\phi\til{\til{\nu}}^B \]

\[+(D\phi)\til{\til{\psi}}_1^B + 2\alpha\varphi_1(\varphi_1^B - \varphi_1^B) = \frac{1}{2}\delta\phi_\alpha + D\phi m^\nu T_{\mu\nu} + 2\Delta\phi m^\nu T_{\mu\nu}, \] (A20)

where \(\phi_\alpha\) represents a source for the dilaton field perturbations, and the relations (A3) and (A4) have been considered. The above equation involves, as wanted, only gauge invariant quantities.

The system of equations (A6), (A7), (A12), (A14), and (A20) comes from the linearization of the matter field equations (A1)-(A5) of Ref. [12], considering that the background solution corresponds to dilatonic
charged black holes. This system is completed by linearizing Ricci identities:

\[
\bar{\Psi}^B_0 + (\delta - 2\beta)\bar{\kappa}^B - (D - 2\rho)\bar{\sigma}^B = 0,
\]
\[
(D - \rho)\bar{\lambda}^B - (\delta - 2\beta)\bar{\pi}^B - \mu\bar{\sigma}^B = m^\mu m^\nu T_{\mu\nu},
\]
\[
-\bar{\Psi}^B_1 - (\Delta - 4\gamma)\bar{\kappa}^B + (D - \rho)\bar{\tau}^B - \rho\bar{\sigma}^B - D\phi\bar{\tau}^B = \nu^\mu m^\nu T_{\mu\nu},
\]
\[
(\delta - 2\beta)\bar{\tau}^B - (\Delta - 2\gamma + \mu)\bar{\sigma}^B - \rho\bar{\lambda}^B = m^\mu m^\nu T_{\mu\nu},
\]
\[
\bar{\Psi}^B_4 - (\delta - 2\beta)\nu^B + (\Delta + 2\gamma + 2\mu)\bar{\lambda}^B = 0,
\]
\[
-\bar{\Psi}^B_3 + D\bar{\nu}^B - (\Delta + \mu)\bar{\pi}^B - \Delta\phi\bar{\nu}^B = n^\mu m^\nu T_{\mu\nu},
\]
\[
(A21)
\]

and linearizing Bianchi identities:

\[
(\bar{\sigma} + 4\bar{\beta})\bar{\Psi}^B_0 - (D - 4\rho)\bar{\Psi}^B_1 - (3\Psi_2 + 2D\phi\Delta\phi - 2\Phi_{11})\bar{\kappa}^B
\]
\[
+ (D\phi)^2\bar{\pi}^B - (D\phi)^2D(\bar{\phi}^B/D\phi) = -(D - 2\rho)\nu^\mu m^\nu T_{\mu\nu} + \delta l^\mu n^\nu T_{\mu\nu},
\]
\[
(\bar{\sigma} + 4\bar{\beta})\bar{\Psi}^B_4 - (\Delta + 2\gamma + 4\mu)\bar{\Psi}^B_3 + (3\Psi_2 + 2\Phi_{11} + 2D\phi\Delta\phi)\bar{\tau}^B - (\Delta\phi)^2\bar{\tau}^B
\]
\[
- (D\phi\Delta\phi)\Delta(\bar{\phi}^B/D\phi) = -(\Delta + 2\gamma + 2\mu)m^\mu m^\nu T_{\mu\nu} + \delta (n^\mu m^\nu T_{\mu\nu}).
\]
\[
(A22)
\]

Thus, we have finally a complete system of thirteen equations (A6), (A7), (A12), (A14), (A20)-(A22) for thirteen unknowns, the nine ones given in (A3), plus \( \bar{\Psi}^B_0, \bar{\Psi}^B_1, \bar{\Psi}^B_4, \) and \( \bar{\Psi}^B_3. \) All the other equations appear to be a consequence of them. It is worth to point out that if one considers directly perturbation equations such as (A5), (A10), (A11), and (A19), without involving gauge invariance quantities, then, one obtains a system of equations in which the number of unknowns exceed highly the number of possible equations. Therefore, apparently there is a direct physical meaning behind the existence of the complete system obtained here; it is what may be obtained in a form that involves only certain natural gauge invariant perturbed field quantities.

However, the system for thirteen unknowns, will be no used as obtained, but a more manageable system is obtained from it in Sec. III. For this purpose, the two following equations are useful, which come from the combinations of Eqs. (A21), or directly from linearizing Ricci identities:

\[
(\Delta - 4\gamma + \mu)\bar{\Psi}^B_0 - (\delta - 2\beta)\bar{\Psi}^B_1 - (3\Psi_2 + 2\Phi_{11})\bar{\sigma}^B - D\phi(\delta - 2\beta)\bar{\tau}^B + (D\phi)^2\bar{\lambda}^B
\]
\[
= (\delta - 2\beta)\nu^\mu m^\nu T_{\mu\nu} - (D - \rho)m^\mu m^\nu T_{\mu\nu},
\]
\[
(D - \rho)\bar{\Psi}^B_4 - (\delta - 2\beta)\bar{\Psi}^B_3 + (3\Psi_2 + 2\Phi_{11})\bar{\lambda}^B - \Delta\phi(\delta - 2\beta)\bar{\nu}^B - (\Delta\phi)^2\bar{\nu}^B
\]
\[
= (\delta - 2\beta)n^\mu m^\nu T_{\mu\nu} - (\Delta + \mu)m^\mu m^\nu T_{\mu\nu}.
\]
\[
(A23)
\]
Appendix B: Separation of variables for the field variations

The separation of variables for the potentials in Eq. (51) implies a separation for the components of the field variations. For example, from Eqs. (39) and (40) (considering that the only nonvanishing contractions of the tetrad (l_μ, n_μ, m_μ, n_μ) are \( l^\mu n_\mu = 1 = -m^\mu m_\mu \), \( l^\mu b_\mu = -\frac{1}{4\xi}[(\delta + 2\beta)(\delta + 4\beta)\frac{\psi_E}{\varphi_1} + \text{c.c.}] \), which reduces to

\[
l^\mu b_\mu = \frac{iL[l(l + 1)]^{1/2}}{4\xi} \left[ \psi_f Y_{lm} e^{-i\omega t} - \text{c.c.} \right],
\]

where we have employed the second of Eqs. (7), Eq. (51), and repeatedly the first of Eqs. (50). From Eq. (B1), and using again the first of Eqs. (50) we obtain the following useful expression

\[
(\delta - 2\beta)\delta(l^\mu b_\mu) = \frac{iL^2[l(l + 1)]}{8\xi R^2} \left[ \psi_f 2Y_{lm} e^{-i\omega t} - \overline{\psi_f} e^{-i\omega t} \right],
\]

and similarly for the other components of the electromagnetic field variations:

\[
\begin{align*}
r^\mu b_\mu &= -\frac{1}{4\xi} \left[ (\delta + 2\beta)(\delta + 4\beta)\frac{\psi_E}{\varphi_1} + \text{c.c.} \right] = \frac{iL[l(l + 1)]^{1/2}}{4\xi} \left[ \psi_e Y_{lm} e^{-i\omega t} - \text{c.c.} \right], \\
(\delta - 2\beta)\delta(n^\mu b_\mu) &= \frac{iL^2[l(l + 1)]}{8\xi R^2} \left[ \psi_e 2Y_{lm} e^{-i\omega t} - \overline{\psi_e} 2Y_{lm} e^{i\omega t} \right], \\
m^\mu b_\mu &= -\frac{1}{4\xi} \left[ (D + \rho)(\delta + 4\beta)\frac{\psi_E}{\varphi_1} + (\Delta - 2\gamma - \mu)(\delta + 4\beta)\overline{\psi_E} - 8a\varphi_1(\delta + 4\beta)\overline{\psi_p} \right], \\
(\delta - 2\beta)(m^\mu b_\mu) &= \frac{iL^2}{4\xi R^2} \left[ \overline{\psi_e} - \left( \frac{\chi^2}{2} D + 2\gamma \right)\overline{\psi_f} + 8a\varphi_1\overline{\psi_d} \right] - \overline{Y_{lm}} e^{i\omega t}.
\end{align*}
\]

For the components of the metric variations, using Eqs. (39) and (40), we have the expressions:

\[
\begin{align*}
\frac{1}{2}l^\mu l^\nu h_{\mu\nu} &= (\delta + 2\beta)(\delta + 4\beta)\psi_H + \text{c.c.} = \frac{L[l(l + 1)]^{1/2}}{2R^2} \left[ \psi_h Y_{lm} e^{-i\omega t} + \text{c.c.} \right], \\
(\delta - 2\beta)\delta(l^\mu l^\nu h_{\mu\nu}) &= \frac{L^2[l(l + 1)]}{2R^4} \left[ \psi_h 2Y_{lm} e^{-i\omega t} + \overline{\psi_h} 2Y_{lm} e^{i\omega t} \right], \\
\frac{1}{2}n^\mu n^\nu h_{\mu\nu} &= (\delta + 2\beta)(\delta + 4\beta)\psi_G + \text{c.c.} = \frac{L[l(l + 1)]^{1/2}}{2R^2} \left[ \psi_g Y_{lm} e^{-i\omega t} + \text{c.c.} \right], \\
(\delta - 2\beta)\delta(n^\mu n^\nu h_{\mu\nu}) &= \frac{L^2[l(l + 1)]}{2R^4} \left[ \psi_g 2Y_{lm} e^{-i\omega t} + \overline{\psi_g} 2Y_{lm} e^{i\omega t} \right], \\
\frac{1}{2}l^\mu m^\nu h_{\mu\nu} &= (\Delta - 4\gamma - \mu)(\delta + 4\beta)\overline{\psi_H} - (\delta + 4\beta)^2\overline{\psi_p} - D\varphi(\delta + 4\beta)\overline{\psi_D} \\
&= -\frac{L}{\sqrt{2}} \left[ \left( \frac{\chi^2}{2} D + 4\gamma + \mu \right)\overline{\psi_h} + \overline{\psi_f} + D\varphi_{\text{c.c.}} \right] - Y_{lm} e^{i\omega t},
\end{align*}
\]
\[(\delta - 2\beta)(l^m m^\nu h_{\mu\nu}) = \frac{L^2}{R^2} \left[ \left( \frac{\chi}{2} \mathcal{D} + 4\gamma + 2\mu \right) \overline{\psi}_h + \overline{\psi}_f + D\phi \overline{\psi}_d \right] - 2Y_{\text{Im}} e^{i\omega t}, \]
\[
\frac{1}{2} m^m m^\nu h_{\mu\nu} = (D + \rho)(\delta + 4\beta) \overline{\psi}_G - (\delta + 4\beta) \overline{\psi}_E - \Delta \phi (\delta + 4\beta) \overline{\psi}_D \\
= \frac{L}{\sqrt{2}} \left[ (\mathcal{D} + \rho) \frac{\psi_G}{R} - \frac{\psi_E}{R} - \Delta \phi \frac{\psi_D}{R} \right] - 1Y_{\text{Im}} e^{i\omega t}, \]
\[
(\delta - 2\beta)(n^m m^\nu h_{\mu\nu}) = -\frac{L^2}{R^2} [(\mathcal{D} + 2\rho) \overline{\psi}_G - \overline{\psi}_E - \Delta \phi \overline{\psi}_D] - 2Y_{\text{Im}} e^{i\omega t}, \]
\[
\frac{1}{2} m^m m^\nu h_{\mu\nu} = [(D - \rho)(D + 3\rho) + \Phi_{00}] \overline{\psi}_G + [(\Delta - 2\gamma + \mu)(\Delta - 4\gamma - 3\mu) + \Phi_{22}] \overline{\psi}_H \\
-2(D - \rho) \overline{\psi}_E - 2(\Delta - 2\gamma + \mu) \overline{\psi}_G - [8a \xi \rho_1^2 + \Delta \phi D + D\phi \Delta] \overline{\psi}_D \\
= \left\{ [(\mathcal{D} - \rho)(\mathcal{D} + 3\rho) + \Phi_{00}] \overline{\psi}_G + [\frac{\chi^2 - 2\Delta}{2} D - 2\gamma + \mu] \overline{\psi}_f \\
+ \Phi_{22} \overline{\psi}_h - 2(\mathcal{D} - \rho) \overline{\psi}_d - 2(\frac{\chi^2}{2} D - 2\gamma + \mu) \overline{\psi}_f \\
- [8a \xi \rho_1^2 + \Delta \phi D + \frac{\chi^2}{2} D\phi \Delta] \overline{\psi}_d \right\} - 1Y_{\text{Im}} e^{i\omega t} , \quad (B4) \]

and
\[
\phi^B = \frac{1}{2} (\delta + 2\beta)(\delta + 4\beta) \overline{\psi}_D + c.c. = \frac{L[l(l + 1)]^{1/2}}{4R^2} \left[ \psi_d Y_{\text{Im}} e^{-i\omega t} + \overline{\psi}_d - 2Y_{\text{Im}} e^{i\omega t} \right], \quad (B5) \]

for dilaton field variations.

As we have seen, all gauge invariant variations of the Newman-Penrose quantities are defined in terms of the components of the field variations given in Eqs. (B1)–(B5). Particularly, from Eqs. (A1)–(A3) we have that
\[
\nu^B = -(\Delta + 2\gamma + \mu)(m^m m^\nu h_{\mu\nu}) + \frac{1}{2} \delta(n^m n^\nu h_{\mu\nu}) \right. + \frac{1}{2}(\Delta + 2\gamma + \mu) \frac{1}{\varphi_1} [\delta(n^m b_\nu) - (\Delta + \mu)(m^m b_\nu)], \]
\[
\kappa^B = (D - \rho)(l^m m^\nu h_{\mu\nu}) - \frac{1}{2} \delta(l^m n^\nu h_{\mu\nu}) + \frac{1}{2}(D - \rho) \frac{1}{\varphi_1} [(D - \rho)(m^m b_\nu) - \delta(l^m b_\nu)], \]
\[
\hat{\sigma}^B = D \left( \frac{1}{2} m^m m^\nu h_{\mu\nu} \right) + \frac{1}{2\varphi_1} (\delta - 2\beta) [(D - \rho)(m^m b_\nu) - \delta(l^m b_\nu)], \]
\[
\tilde{\lambda}^B = -(\Delta + 2\gamma + \mu)(m^m m^\nu h_{\mu\nu}) + \frac{1}{2\varphi_1} (\delta - 2\beta) [\delta(n^m b_\nu) - (\Delta + \mu)(m^m b_\nu)], \quad (B6) \]

and from Eqs. (A21)
\[
\Psi_0^B = -(\Delta - 2\beta) \tilde{\lambda}^B + (D - 2\rho) \delta^B = (D - 2\rho) D \left( \frac{1}{2} m^m m^\nu h_{\mu\nu} \right) - (D - 2\rho) [(\delta - 2\beta)(l^m m^\nu h_{\mu\nu})] \\
+ \frac{1}{2} (\delta - 2\beta) \delta(l^m l^\nu h_{\mu\nu}), \]
\[
\Psi_4^B = (\delta - 2\beta) \nu^B - (\Delta + 2\gamma + 2\mu) \tilde{\lambda}^B = (\Delta + 2\gamma + 2\mu) \Delta \left( \frac{1}{2} m^m m^\nu h_{\mu\nu} \right) 
\]

33
\[-(\Delta + 2\gamma + 2\mu)[(\delta - 2\beta)(n^\mu n^\nu h_{\mu\nu})] + \frac{1}{2}(\delta - 2\beta)\delta(n^\mu n^\nu h_{\mu\nu}). \quad (B7)\]

Hence, substituting directly Eqs. (B1)–(B4) into Eqs. (B6) and (B7), we have the following expressions for the quantities appearing in the decoupled system:

\[
\Psi_0^B = \left[\overline{T_0} - 2\rho\right]\overline{T_0} \left(\frac{1}{2}m^\mu m^\nu h_{\mu\nu}\right)(r) - \left(\overline{T_0} - 2\rho\right)[(\delta - 2\beta)(l^\mu m^\nu h_{\mu\nu})](r) + \frac{L^2l(l+1)}{4R^4} \tilde{\psi}_h \right]_{-2Y_{lm}} e^{-i\omega t}
+ \frac{L^2l(l+1)}{4R^4} \tilde{\psi}_h 2Y_{lm} e^{-i\omega t} \equiv \Psi_0^{B\text{\dagger}} Y_{lm} e^{-i\omega t} + \Psi_0^{B-} Y_{lm} e^{-i\omega t},
\]

\[
\Psi_4^B = \left\{ -\frac{1}{2}\left(\frac{\chi}{2}D + 2\gamma + 2\mu\right)\chi D \left(l^\mu m^\nu h_{\mu\nu}\right)(r) - \left(\frac{\chi}{2}D + 2\gamma + 2\mu\right)[(\delta - 2\beta)(n^\mu m^\nu h_{\mu\nu})](r) + \frac{L^2l(l+1)}{4R^4} \tilde{\psi}_{\gamma} \right\}_{-2Y_{lm}} e^{-i\omega t} + \frac{L^2l(l+1)}{4R^4} \tilde{\psi}_{\gamma} 2Y_{lm} e^{-i\omega t} \equiv \Psi_4^{B\text{\dagger}} Y_{lm} e^{-i\omega t} + \Psi_4^{B-} Y_{lm} e^{-i\omega t},
\]

\[
\sigma^B = \left\{ \frac{1}{2}\left(\frac{\chi}{2}m^\mu m^\nu h_{\mu\nu}\right)(r) + \frac{1}{2\chi} \left(\overline{T_0} - 2\rho\right)[(\delta - 2\beta)m^\mu b_{\mu}](r) + \frac{L^2l(l+1)}{8Q^2\xi} \tilde{\psi}_{\delta} \right\}_{-2Y_{lm}} e^{-i\omega t} + \frac{L^2l(l+1)}{8Q^2\xi} \tilde{\psi}_{\delta} 2Y_{lm} e^{-i\omega t} \equiv \sigma^{B\text{\dagger}} Y_{lm} e^{-i\omega t} + \sigma^{B-} Y_{lm} e^{-i\omega t},
\]

\[
\tilde{\lambda}^B = \left\{ \frac{1}{2}\left(\frac{\chi}{2}m^\mu m^\nu h_{\mu\nu}\right)(r) - \frac{1}{2\chi} \left(\overline{T_0} - 2\rho\right)[(\delta - 2\beta)m^\mu b_{\mu}](r) - \frac{L^2l(l+1)}{8Q^2\xi} \tilde{\psi}_{\lambda} \right\}_{-2Y_{lm}} e^{-i\omega t} + \frac{L^2l(l+1)}{8Q^2\xi} \tilde{\psi}_{\lambda} 2Y_{lm} e^{-i\omega t} \equiv \tilde{\lambda}^{B\text{\dagger}} Y_{lm} e^{-i\omega t} + \tilde{\lambda}^{B-} Y_{lm} e^{-i\omega t}, \quad (B8)
\]

where \((r)\) denotes the radial part of the corresponding quantity. For example, from Eqs. (B4), \([(\delta - 2\beta)(n^\mu m^\nu h_{\mu\nu})](r) = \frac{L^2}{2R^2}[(\overline{T_0} - 2\rho)\tilde{\psi}_{\gamma} - \overline{\psi}_{\gamma} - \Delta \phi \overline{\psi}_{\gamma}], and similarly for \([(\delta - 2\beta)(l^\mu m^\nu h_{\mu\nu})](r), (\frac{1}{2}m^\mu m^\nu h_{\mu\nu})(r), and \([(\delta - 2\beta)m^\mu b_{\mu}](r)\) from Eqs. (B4) and (B3). Moreover, the second equalities are only for defining in a compact way the radial parts of the form \(e^{i\omega t}\) and \(e^{-i\omega t}\) of the corresponding quantity. Finally, from the last of Eqs. (A3) and (B5) we have that

\[
(\delta - 2\beta)\phi^B = (\delta - 2\beta)\phi^B - \frac{1}{2\chi} \left[\Delta \phi(D - 2\rho) + D\phi(\Delta + 2\mu)\right](\delta - 2\beta)(m^\mu b_{\mu})
+ \frac{1}{2\chi} \left[\Delta \phi(D - 2\rho) + D\phi(\Delta + 2\mu)\right](\delta - 2\beta)(m^\mu b_{\mu})
\]

\[
\equiv \phi^{B\text{\dagger}} Y_{lm} e^{i\omega t} + \phi^{B-} Y_{lm} e^{-i\omega t}. \quad (B9)
\]
From Eqs. (B8) and (B9) we can obtain the following useful relations:

\[
\psi_\gamma \Psi^B_0 - \psi_h \Psi^B_\delta = 0,
\]
\[
\psi_\varepsilon \Psi^B_\delta + \psi_\gamma \Psi^B_\delta = 0,
\]
\[
\Delta \phi \psi_\delta \sigma^B - D\phi \psi_\delta \lambda^B + \psi_\delta \phi^B = \frac{L^2 l (l + 1)}{8 R^4} \psi_\delta^2.
\]
(B10)

Note that \((\Psi^B)^+\) depends on \((\psi_i)\), whereas \((\Psi^B)^-\) on \((\psi_i)\).

References

[1] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, 1987).
[2] G. W. Gibbons and K. Maeda, Nucl. Phys. B298, 741 (1988).
[3] D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D 43, 3140 (1991); 45, 3888(E) (1992).
[4] J. Presskill, P. Schwarz, A. Shapere, S. Trivedi and F. Wilczek, Mod. Phys. Lett. A 6, 2353 (1991).
[5] C. F. E. Holzhey and F. Wilczek, Nucl. Phys. B380, 447 (1992).
[6] R. Kallosh, A. Linde, T. Ortín, A. Peet and A. Van Proeyen, Phys. Rev. D 46, 5278 (1992).
[7] B. Harms and Y. Leblanc, Ann. Phys. (N.Y.) 244, 262 (1995); 244, 272 (1995); 242, 265 (1995).
[8] M. Brunetti, E. Coccia, V. Fafone, and F. Fucito, Phys. Rev. D 59, 044027 (1999).
[9] M. Maggiore and A. Nicolis, Phys. Rev. D 62, 024004 (2000).
[10] M. Bianchi, E. Coccia, C. N. Colacino, V. Fafone, and F. Fucito, Class. Quantum Grav. 13, 2865 (1996).
[11] M. Shibata, K. Nakao, and T. Nakamura, Phys. Rev. D 50, 7304 (1994).
[12] R. Cartas-Fuentevilla, Phys. Rev. D, 56, 7700 (1997).
[13] R. Cartas-Fuentevilla, Phys. Rev. D, 57, 3433 (1998).
[14] E. Witten, N. Phys., B276, 291 (1986).
[15] C. Crnčović and E. Witten, in Three Hundred Years of Gravitation, edited by W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987).
[16] E. Zuckerman, in Mathematical Aspects of String Theory, edited S. T. Yau (World Scientific, Singapore, 1986), p. 259.
[17] R. Cartas-Fuentevilla, Phys. Rev. D 57, 3443 (1998); J. Math. Phys. 40, 4622 (1999).
[18] R. Cartas-Fuentevilla, J. Math. Phys. 41, 7521 (2000).
[19] R. M. Wald, Phys. Rev. Lett. 41, 203 (1978).
[20] E. T. Newman and R. Penrose, J. Math. Phys. 7, 863 (1966).
[21] G. T. Torres del Castillo, in Field Theory, Integrable Systems and Symmetries, edited by F. Khanna and L. Vinet (CRM, Montreal, 1997), p. 203–213.

[22] R. Cartas-Fuentevilla, J. Math. Phys. 43, 644 (2002).