The Polychromatic Number of Small Subsets of the Integers Modulo \( n \)

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Abstract
If \( S \) is a subset of an abelian group \( G \), the \textit{polychromatic number} of \( S \) in \( G \) is the largest integer \( k \) so that there is a \( k \)-coloring of the elements of \( G \) such that every translate of \( S \) in \( G \) gets all \( k \) colors. We determine the polychromatic number of all sets of size 2 or 3 in the group of integers mod \( n \).

Keywords Polychromatic coloring • Abelian group • Group tiling • Complement set

1 Introduction
Throughout this paper \( G \) will denote an arbitrary abelian group. Given \( S \subseteq G \), \( a \in G \), \( a + S = \{a + s | s \in S\} \). Any set of the form \( a + S \) is called a \textit{translate} of \( S \). A \( k \)-coloring of the elements of \( G \) is \( S \)-polychromatic if every translate of \( S \) contains an element of each of the \( k \) colors. The \textit{polychromatic number} of \( S \) in \( G \), denoted \( p_G(S) \), is the largest number of colors such that there exists an \( S \)-polychromatic coloring of \( G \). The notation \( p(S) \) is used when \( G \) is the set of integers, \( \mathbb{Z} \), and \( p_n(S) \) is used when \( G = \mathbb{Z}_n \), the group of integers mod \( n \). In this paper, \( p_n(S) \) is determined

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for all \( n \geq 3 \) and \(|S| = 2 \) or 3. The techniques used may be useful in determining \( p_n(S) \) for larger sets \( S \) and for other coloring problems.

The notions of polychromatic colorings and polychromatic number for sets in abelian groups can be extended. If \( G \) is any structure and \( H \) is a family of substructures then a \( k \)-coloring of \( G \) is \( H \)-polychromatic if every member of \( H \) gets all \( k \) colors, and the polychromatic number \( p_G(H) \) of \( H \) in \( G \) is the largest \( k \) such that there is an \( H \)-polychromatic coloring with \( k \) colors. In this paper, \( G \) is \( \mathbb{Z}_n \) and \( H \) is the family of all translates of a subset \( S \). Alon et al. [1], Bialostocki [4], Offner [9], and Goldwasser et al. [6] considered the case when \( G \) is an \( n \)-cube and \( H \) is the family of all sub-\( d \)-cubes for some fixed \( d \leq n \). Axenovich et al. [2] considered the case where \( G \) is the complete graph on \( n \) vertices and \( H \) is the family of all perfect matchings or Hamiltonian cycles or 2-factors.

If \( S \) and \( T \) are subsets of an abelian group \( G \), we say \( T \) is a blocking set for \( S \) if \( G \setminus T \) contains no translate of \( S \). Blocking sets are of interest in extremal combinatorics, because if \( T \) is a minimum size blocking set for \( S \) then \( G \setminus T \) is a maximum size subset of \( G \) with no translate of \( S \), so is the solution to a Turán-type problem. It is well known [3, 7, 10] that \( T \) is a complement set for \( S \) if and only if \(-T\) is a blocking set for \( S \). Clearly each color class in an \( S \)-polychromatic coloring is a blocking set for \( S \).

In [3], Axenovich et al. considered the situation when \( G \) is the group of integers and \( H \) is the family of all translations of a set \( S \) of 4 integers. They showed that the polychromatic number of any set \( S \) of 4 integers in \( \mathbb{Z} \) is at least 3, by finding a particular value of \( n \) such that \( 3 \leq p_n(S) \). That implies that any set \( S \) of size 4 has a blocking set in \( \mathbb{Z} \) of density at most 1/3, proving a conjecture of Newman about densities of complement sets.

Whereas in [3] it was shown that for each set \( S \) of integers of size 4, there exists an integer \( n \) such that \( 3 \leq p_n(S) \), such an inequality does not hold for all \( S \) and \( n \). For example, if \( S = \{0, 1, 3, 6\} \) and \( n = 11 \), then \( p_n(S) = 2 \). It would be difficult to determine \( p_n(S) \) for all values of \( n \) and all sets \( S \) of size 4, but in this paper these values are determined for all sets \( S \) of size 3.

**Example 1.1** Let \( S = \{0, a, b\} \) be a subset of \( \mathbb{Z}_n \) where \( n \) is divisible by 3, \( a \equiv 1 \pmod{3} \), and \( b \equiv 2 \pmod{3} \). Then \( p_n(S) = 3 \) as the coloring RGYRGY... is obviously \( S \)-polychromatic.

![Fig. 1 Fano plane and an incidence matrix](image)
Example 1.2 If $S = \{0, 1, 3\}$ and $n = 7$ then $p_n(S) = 1$. Consider Fig. 1 and note that the $7 \times 7$ circulant matrix is an incidence matrix for the Fano plane. It is well known (and it is easy to check) that in any 2-coloring of the vertices of the Fano plane there is a monochromatic edge, which implies there is no $S$-polychromatic 2-coloring, so $p_7(S) = 1$.

The main result of this paper is that Examples 1.1 and 1.2 are essentially the only examples of sets $S$ of size three such that $p_n(S)$ is not equal to 2.

2 Simplifying Assumptions and the Main Theorem

The polychromatic number of a set $S$ in $\mathbb{Z}_n$ is unchanged under certain operations involving translation, multiplication, and scaling. If $|S| = 3$ we can use those operations to convert a set $S$ to a set $S'$ which has the same polychromatic number, and has one of two specific forms.

Lemma 2.1 If $1 \leq d, t, n \in \mathbb{Z}$, $S = \{a_1, a_2, \ldots, a_t\} \subseteq \mathbb{Z}_n$, and $S' = \{da_1, da_2, \ldots, da_t\}$, then $p_d(S') = p_n(S)$.

Proof Any $S$-polychromatic coloring of $\mathbb{Z}_n$ can clearly be copied on the subgroup $\langle d \rangle$ of $\mathbb{Z}_d$, and then duplicated on all the cosets of $\langle d \rangle$, to get an $S'$-polychromatic coloring of $\mathbb{Z}_d$. Going the other way, in any $S'$-polychromatic coloring of $\mathbb{Z}_d$, the restricted coloring on $\langle d \rangle$ can be copied on $\mathbb{Z}_n$ to get an $S$-polychromatic coloring. Hence we can simply divide out a common factor of $n$ and the elements of $S$ without changing the polychromatic number. Since we can also take any translation of $S$ without changing the polychromatic number, from now on we will assume that every set $S$ of size 3 in $\mathbb{Z}_n$ has the form $S = \{0, a, b\}$ where $\gcd(a, b, n) = 1$.

Lemma 2.2 Let $1 \leq d, t, n \in \mathbb{Z}$ such that $d < n$ and $\gcd(d, n) = 1$. If $S' = \{da_1, da_2, \ldots, da_t\}$ and $S = \{a_1, a_2, \ldots, a_t\}$, then $p_n(S) = p_n(S')$.

Proof If $\chi'$ is $S'$-polychromatic, the coloring $\chi$ defined by $\chi(y) = \chi'(dy)$ is clearly $S$-polychromatic. This argument can be reversed since $d$ is invertible in $\mathbb{Z}_n$.

Definition 2.1 If $S = \{a_1, a_2, \ldots, a_t\} \subseteq \mathbb{Z}_n$ and $S' = \{da_1 + c, da_2 + c, \ldots, da_t + c\}$, where $c, d \in \mathbb{Z}_n$ and $\gcd(d, n) = 1$, then we say that $S$ and $S'$ are equivalent sets in $\mathbb{Z}_n$.

Thus, Lemma 2.2 says that equivalent sets in $\mathbb{Z}_n$ have the same polychromatic number.

Lemma 2.3 For all $b \in \mathbb{Z}_n$ with $3 \leq n$ there exists $b' \in \mathbb{Z}_n$ so that $b' \leq \left[\frac{n}{2}\right]$ and $p(\{0, 1, b\}) = p(\{0, 1, b'\})$. 

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Proof Since, $n - 1$ is always relatively prime to $n$ for $3 \leq n$, $p_n(S) = p_n(-S)$ for all $S \subseteq \mathbb{Z}_n$ by Lemma 2.2. If $\left\lceil \frac{n}{2} \right\rceil < b$, then let $b' = n - b + 1 \leq \frac{n}{2}$. Therefore, $p(\{0, 1, b\}) = p(\{-1, 0, -b\}) = p(\{0, 1, b + 1\}) = p(\{0, 1, n - b + 1\})$. 

Proposition 2.1 Let $S = \{0, a, b\} \subseteq \mathbb{Z}_n$ where $\gcd(a, b, n) = 1$. Then at least one of the following occurs.

(i) $S$ is equivalent to a set $S' = \{0, 1, b'\}$ where $b' \leq \frac{n}{2}$.

(ii) $\gcd(a, n) \neq 1$, $\gcd(b, n) \neq 1$, $a \notin \langle b \rangle$ and $b \notin \langle a \rangle$.

Proof If $\gcd(a, n) = 1$ then $a$ is invertible in $\mathbb{Z}_n$, so $S$ is equivalent to a set $\{0, 1, c\}$, for some $c$ ($d = a^{-1}$ in Definition 2.1), and then to $S'$ by Lemma 2.3. Similarly if $\gcd(b, n) = 1$. Now suppose neither $\gcd(a, n)$ nor $\gcd(b, n)$ is equal to 1. If $b$ is a multiple of $a$ then, since $\gcd(a, b, n) = 1$, $\gcd(a, n)$ must equal 1, a contradiction, so $b$ is not a multiple of $a$. Similarly, $a$ is not a multiple of $b$. 

We remark that if Case ii occurs and $\gcd(b - a, n) = 1$, then Case i also occurs. However, in our proof we just need that at least one of them occurs. We will treat Case i in Sect. 5 and Case ii in Sect. 6. The following theorem is the main result of this paper.

Theorem 2.1 Let $S = \{0, a, b\} \subseteq \mathbb{Z}_n$ and $\gcd(a, b, n) = 1$, then

$$p_n(S) = \begin{cases} 3 & \text{if } 3 \mid n \text{ and } a \text{ and } b \text{ are in different nonzero mod } 3 \text{ congruence classes} \\ 1 & \text{if } n = 7 \text{ and } \{0, a, b\} \text{ is equivalent to } \{0, 1, 3\} \\ 2 & \text{otherwise.} \end{cases}$$

If we do not make the assumption that $\gcd(a, b, n) = 1$, then we get the following theorem, which is clearly equivalent to Theorem 2.1:

Theorem 2.2 If $3 \leq n$, $a, b \in \mathbb{Z}_n$, and $a \neq b$, then

$$p_n(\{0, a, b\}) = \begin{cases} 3 & \text{if } n \equiv 0 \mod 3^{i+1}, a = 3^i m_a, b = 3^i m_b, \\ & m_a, m_b \neq 0 \mod 3, \text{and } m_a + m_b \equiv 0 \mod 3 \\ 1 & \text{if } n \equiv 0 \mod 7, |\langle a \rangle| = 7, \text{and } b = 3a \text{ or } 5a \\ 2 & \text{otherwise.} \end{cases}$$

3 Sets of Size 2

For the following proposition we assume without loss of generality that 0 is in the chosen subset of $\mathbb{Z}_n$.

Proposition 3.1 If $S = \{0, b\} \subseteq \mathbb{Z}_n$ where $\gcd(b, n) = 1$ then
\[ p_n(S) = \begin{cases} 1 & \text{if } |\langle b \rangle| \text{ is odd} \\ 2 & \text{if } |\langle b \rangle| \text{ is even.} \end{cases} \]

**Proof** Clearly there will be an \( S \)-polychromatic 2-coloring of the multiples of \( b \) if and only if \(|\langle b \rangle|\) is even. \( \square \)

## 4 Sets That Tile

Given a set \( S \subseteq G \) where \( G \) is an abelian group, a set \( T \subseteq G \) is a **complement set** for \( S \) if \( S + T = G \). \( S \) tiles \( G \) by translation if \( T \) is a complement set for \( S \) and if \( s_1, s_2 \in S, t_1, t_2 \in T \), and \( s_1 + t_1 = s_2 + t_2 \) implies \( s_1 = s_2 \) and \( t_1 = t_2 \). The notation \( S \oplus T \) is used when \( S \) tiles \( G \) by translation. Without loss of generality, \( 0 \in S, T \) for all of the following arguments.

Newman [8] proved necessary and sufficient conditions for a finite set \( S \) to tile \( \mathbb{Z} \) if \( |S| \) is a power of a prime.

**Theorem 4.1** [8] Let \( S = \{s_1, \ldots, s_k\} \) be distinct integers with \( |S| = p^a \) where \( p \) is prime and \( a \) is a positive integer. For \( 1 \leq i < j \leq k \) let \( p^{e_{ij}} \) be the highest power of \( p \) that divides \( s_i - s_j \). Then \( S \) tiles \( \mathbb{Z} \) if and only if \( |\{e_{ij} : 1 \leq i < j \leq k\}| \leq a \).

The characterization of sets \( S \) of size 3 such that \( p_n(S) = 3 \) (Theorem 2.1 and Proposition 4.1) follows immediately from Newman’s theorem (Theorem 4.1). When commenting on this theorem in [8] Newman says: “Surely the special case [when \( |S| = 3 \)] deserves to have a completely trivial proof - but we have not been able to find one.”

If there is an \( S \)-polychromatic \( k \)-coloring of \( \mathbb{Z}_n \), then clearly there is an \( S \)-polychromatic \( k \)-coloring of \( \mathbb{Z} \) with period \( n \). If there is an \( S \)-polychromatic \( k \)-coloring of \( \mathbb{Z} \) for a finite set \( S \), then there is an \( S \)-polychromatic \( k \)-coloring of \( \mathbb{Z}_n \) for some \( n \). To see this, let \( d \) equal the largest difference between two elements in \( S \). If \( \chi \) is an \( S \)-polychromatic \( k \)-coloring of \( \mathbb{Z} \), there are only \( k^{(d+1)} \) possibilities for the coloring on \( d + 1 \) consecutive integers, so two such strings must be identical. If \( n \) is the difference between the first integers in these two strings, then we can “wrap around” the coloring \( \chi \) to get an \( S \)-polychromatic \( k \)-coloring of \( \mathbb{Z}_n \).

Suppose \( S = \{0, a, b\} \) and \( \chi \) is an \( S \)-polychromatic \( 3 \)-coloring of \( \mathbb{Z} \). By the above remark there exists an \( S \)-polychromatic \( 3 \)-coloring of \( \mathbb{Z}_n \) for some \( n \). By Proposition 4.1, \( a \) and \( b \) are in different nonzero mod 3 congruence classes, which fulfills Newman’s wish to have a simple proof of his theorem for the special case when \( |S| = 3 \).

Later Coven and Meyerowitz [5] gave necessary and sufficient conditions for \( S \) to tile \( \mathbb{Z} \) when \( |S| = p_1^{a_1} p_2^{a_2} \), where \( p_1 \) and \( p_2 \) are primes. The following characterization of tiling by translation in an abelian group was obtained in [3].

**Theorem 4.2** [3] Let \( G \) be an abelian group and \( S \) a finite subset of \( G \). \( S \) tiles \( G \) by translation if and only if \( p(S) = |S| \). Moreover, if \( \chi \) is an \( S \)-polychromatic coloring of \( G \) with \( |S| \) colors and \( T \) is a color class of \( \chi \), then \( S \oplus T = G \).
Lemma 4.1 Suppose \( S = \{0, a, b\} \) where \( \gcd(a, b, n) = 1 \), \( S \oplus T = \mathbb{Z}_n \) and \( 0 \in T \).
If \( x \in T \), then \( x + \langle a + b \rangle \subseteq T \).

Proof Note that because \( S \oplus T = \mathbb{Z}_n \), every element of \( \mathbb{Z}_n \) belongs to exactly one of the sets \( T, a + T, b + T \).

Suppose \( x \in T \). If \( x + a + b \in b + T \), then \( x + a \in T \). However, \( x + a \in a + T \).
If \( x + a + b \in a + T \), then \( x + b \in T \). Hence \( x + a + b \in T \) and, repeating the argument, \( x + \langle a + b \rangle \subseteq T \).

Proposition 4.1 Let \( S = \{0, a, b\} \subseteq \mathbb{Z}_n \) where \( \gcd(a, b, n) = 1 \). Then \( p_n(S) = 3 \) if and only if \( 3|n \) and \( a \) and \( b \) are in different nonzero mod \( 3 \) congruence classes.

Proof If \( 3|n \) and \( a \) and \( b \) are in different nonzero mod \( 3 \) congruence classes then clearly the alternating coloring \( RGYRGY\ldots \) is polychromatic, so \( p_n(S) = 3 \).
Conversely, suppose \( p_n(S) = 3 \). Hence, by Theorem 4.2, \( S \) tiles \( \mathbb{Z}_n \).
Let \( T \subseteq \mathbb{Z}_n \) such that \( \mathbb{Z}_n = \{0, a, b\} \oplus T \) and \( 0 \in T \subseteq \mathbb{Z}_n \). Therefore, \( n = 3|T| \) which implies \( n \equiv 0 \mod 3 \). By Lemma 4.1, for any \( x \in T \), the coset \( x + \langle a + b \rangle \) is a subset of \( T \), so \( T \) is the disjoint union of cosets of \( \langle a + b \rangle \). Therefore, there is some integer \( q \) such that \( q|\langle a + b \rangle| = |T| = n \frac{2}{3} \). Also, \( |\langle a + b \rangle| = \frac{n}{\gcd(a+b,n)} \). Thus, \( q \frac{n}{\gcd(a+b,n)} = n \frac{2}{3} \), which implies \( 3q = \gcd(a+b,n) \). Hence, \( 3|\langle a + b \rangle \). Since 3 cannot divide both \( a \) and \( b \), it follows that \( a \) and \( b \) are in different nonzero mod \( 3 \) congruence classes.

5 Subsets of the Form \( \{0,1,b\} \)

As shown in Proposition 2.1, every set \( S \) of size 3 is equivalent to a set \( S' \) with two possible forms. In this section we will consider case i of Proposition 2.1, that \( S' \) contains 0 and 1.

Lemma 5.1 If \( n \) is odd, \( 5 \leq n \), and \( n \neq 7 \), then there exists a \( \{0,1,3\} \)-polychromatic coloring of \( \mathbb{Z}_n \) with two colors.

Proof It is easy to check that each integer greater than 3, except 7, is the sum of an even number of 2s and 3s. We color \( \mathbb{Z}_n \) by alternating colors of strings of 2 or 3 consecutive elements with the same color. Of course there must be an even number of strings. For example, 9=2+2+2+3, so the coloring would be \( RRRYRRYY \); 11=2+3+3+3+3, so the coloring would be \( RRYYRRRRYY \). Clearly any translate of \( S \) hits two consecutive strings, so gets both colors.

As will be seen in the proof of Theorem 5.1, it is easy to show that \( p_n(\{0,1,b\}) \geq 2 \) if \( b \) or \( n \) is even. Lemma 5.3 will take care of the more difficult case. Our proof of Lemma 5.3 depends on the following elementary result about sums of integers.

Lemma 5.2 Let \( b \) be an odd integer greater than or equal to 5. Then any odd integer \( n \) greater than or equal to \( 2b - 1 \) can be expressed as the sum of an odd
number of summands, each of which is between \( \frac{(b-1)}{2} \) and \( b - 1 \) inclusive, with the exception that no more than half of them can be equal to \( \frac{(b-1)}{2} \).

**Proof** Let \( b \geq 5 \) and \( k \geq 3 \) be odd integers. Let \( m(b, k) \) denote the smallest odd integer \( n \) which is the sum of \( k \) summands between \( \frac{(b-1)}{2} \) and \( b - 1 \) inclusive with less than half of them equal to \( \frac{(b-1)}{2} \) and let \( M(b, k) \) be the largest such odd integer \( n \). Then it is not hard to check that

\[
m(b, k) = \begin{cases} 
\left( \frac{(b-1)}{2} \right) \left( \frac{(k-1)}{2} \right) + \left( \frac{(b+1)}{2} \right) \left( \frac{(k+1)}{2} \right) = \frac{(bk+1)}{2} & \text{if } b \equiv k \pmod{4} \\
\left( \frac{(b-1)}{2} \right) \left( \frac{(k-3)}{2} \right) + \left( \frac{(b+1)}{2} \right) \left( \frac{(k+3)}{2} \right) = \frac{(bk+3)}{2} & \text{if } b \not\equiv k \pmod{4}
\end{cases}
\]

and

\[
M(b, k) = (b-1)(k-1) + (b-2) = (b-1)k - 1.
\]

It suffices to show that for fixed odd \( b \geq 5 \), the union of the intervals \([m(b, k), M(b, k)]\) over all odd \( k \geq 3 \) contains all odd integers \( n \geq 2b - 1 \). This will follow for each odd \( b \geq 5 \) if we show that \( 2b - 1 \in [m(b, 3), M(b, 3)] \) and that \( M(b, k - 2) + 2 \geq m(b, k) \) for all odd \( k \geq 5 \).

The first of these holds because \( \frac{(3b+3)}{2} \leq 2b - 1 \leq 3(b-1) - 1 \) for all \( b \geq 5 \). By simple algebra the second will hold provided

\[
b \geq \begin{cases} 
2 + \frac{3}{k-4} & \text{if } b \equiv k \pmod{4} \\
2 + \frac{5}{k-4} & \text{if } b \not\equiv k \pmod{4}.
\end{cases}
\]

These inequalities clearly hold if \( b \) or \( k \) is at least 7, and also if \( b = k = 5 \). \( \square \)

As in the last proof, for example, suppose \( b = 5 \). With \( k = 3 \) the sum can be \( 3 + 3 + 3 = 9 \) or \( 4 + 4 + 3 = 11 \). With \( k = 5 \) the sum can be any odd integer between \( 3 + 3 + 3 + 2 + 2 = 13 \) and \( 4 + 4 + 4 + 4 + 3 = 19 \). With \( k = 7 \) the sum can be any odd integer between \( 3 + 3 + 3 + 3 + 2 + 2 = 19 \) and \( 4 + 4 + 4 + 4 + 4 + 4 + 3 = 27 \). With \( k = 9 \) the sum can be any odd integer between \( 3 + 3 + 3 + 3 + 2 + 2 + 2 = 23 \) and \( 4 + 4 + 4 + 4 + 4 + 4 + 4 + 3 = 35 \), and so on. Clearly it is possible to get any odd sum greater than or equal to \( 2b - 1 = 9 \).

**Lemma 5.3** Let \( 9 \leq n, b \) and \( n \) both be odd, and \( S = \{0, 1, b\} \subset \mathbb{Z}_n \). There exists an \( S \)-polychromatic coloring of \( \mathbb{Z}_n \) with two colors.

**Proof** By Lemmas 2.3 and 5.1 we can assume \( 5 \leq b \leq \frac{(n+1)}{2} \). We will use Lemma 5.2 to construct an \( S \)-polychromatic 2-coloring using mostly alternating \( R \)
and Y, but some “double letters.” The summands from Lemma 5.2 tell you where to put the double letters. For example, if \( b = 9 \) and \( n = 27 \) then \( 27 = 4 + 5 + 5 + 6 + 7 \) (there are many other possibilities), so to get a polychromatic coloring we put the double letters in positions \((0, 1), (4, 5), (9, 10), (14, 15), (20, 21)\), the summands telling you the differences between the positions of the first letter of the double letters.

The point is that if \((x, x+1)\) is a double letter, then the interval \([x, x+b]\) will contain precisely one other double letter because of the size of the summands, as long as you do not put two summands of size \((b-1)/2\) next to each other. Since \( b \) is odd, \( x \) and \( x+b \) will have different colors. The reason there has to be an odd number of summands is that then there must be an even number of even summands. Note what happens when you have an even summand, say 6. Then you would have double letters at, say, \((0, 1)\) and \((6, 7)\) That gives you

\[
|RRYRYR|YY\ldots
\]

so the first segment has \( RR \) and the second segment has \( YY \), that is the double letter changes. In a cycle there must be an even number of changes of the double letter, hence an even number of even summands and an odd number of odd summands.

**Theorem 5.1** Let \( 3 \leq n \) and \( S = \{0, 1, b\} \subseteq \mathbb{Z}_n \). If \( n \neq 7 \) or \( b \neq 3 \) or 5, there is an \( S \)-polychromatic coloring of \( \mathbb{Z}_n \) with two colors.

**Proof** If \( n \) is even then alternating colors \( RYRY\ldots \) is clearly an \( S \)-polychromatic coloring. If \( n \) is odd and \( b \) is even then the coloring \( RRYRYRYR\ldots \) which has one repeated color, and otherwise alternates colors, is \( S \)-polychromatic. If \( n \) and \( b \) are both odd then an \( S \)-polychromatic 2-coloring exists by Lemmas 5.1 and 5.3, except in the exceptional case when \( n = 7 \).

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### 6 Subsets Not Equivalent to \( \{0, 1, b\} \)

For certain values of \( a \) and \( b \) in \( S = \{0, a, b\} \), we will construct a matrix which we will use to produce \( S \)-polychromatic colorings with two colors. Consider the \( s \times t \) matrix

\[
M = \begin{bmatrix}
x_{00} & x_{01} & \cdots & x_{0(t-1)} \\
x_{10} & x_{11} & \cdots & x_{1(t-1)} \\
\vdots & \vdots & \ddots & \vdots \\
x_{(s-1)0} & x_{(s-1)1} & \cdots & x_{(s-1)(t-1)}
\end{bmatrix}
\]

An \( L \)-tile of \( M \) is a subset of entries of \( M \) consisting of entries of a \( 2 \times 2 \) submatrix without the lower right entry:

\[
\begin{array}{c|c}
x_{ij} & x_{i(j+1)} \\
\hline
x_{(i+1)j} & \end{array}
\]
The indices are read mod $s$ and mod $t$, so L-tiles are allowed to ’wrap around’ ($i = s - 1$ or $j = t - 1$). An L-tile 2-coloring of $M$ is a coloring of the entries of $M$ with two colors such that both colors appear in every L-tile of $M$.

**Lemma 6.1** If $2 \leq s, t$, then every $s \times t$ matrix has an L-tile 2-coloring.

**Proof** If $s$ is even, then define $\chi$ such that

$$\chi(x_{ij}) = \begin{cases} R & \text{if } i \equiv 0 \text{ mod } 2 \\ Y & \text{if } i \equiv 1 \text{ mod } 2. \end{cases}$$

Also, a similar coloring that alternates the colors of the columns works when $t$ is even.

If $s$ and $t$ are both odd, then define $\chi$ such that

$$\chi(x_{ij}) = \begin{cases} R & \text{if } i \equiv j \text{ mod } 2 \text{ and } (i, j) \neq (0, t - 1), (s - 1, 0) \\ Y & \text{otherwise.} \end{cases}$$

If $s$ and $t$ are both odd, then a “checker-board” coloring would assign the same color, say $R$, to all four corner entries, and the L-tile with entries $x_{s-1,t-1}, x_{s,t-1}$, and $x_{s-1,0}$ would be monochromatic. The coloring $\chi$ avoids this problem by changing the color of entries $x_{0,t-1}$ and $x_{s-1,0}$ from $R$ to $Y$, without creating any other monochromatic L-tiles (just changing the color of one of them would suffice as well).

The goal now is to create matrices with elements from $\mathbb{Z}_n$ such that all of the translates of $S$ correspond to L-tiles. The matrices can then be colored by using Lemma 6.1, which will create $S$-polychromatic colorings.

**Lemma 6.2** Let $S = \{0, a, b\} \subseteq \mathbb{Z}_n$, where $\gcd(a, b, n) = 1$ but $\gcd(a, n)$ and $\gcd(b, n)$ are both greater than 1. Then $p_n(S) \geq 2$.

**Proof** If $n$ is even then either $a$ or $b$ is odd, so the alternating coloring $RYRYRY\ldots$ is polychromatic, so we can assume $n$ is odd. Let $s = \gcd(a, n), t = \gcd(b, n)$, and $M = [m_{ij}]$ be the $\frac{n}{s} \times \frac{n}{t}$ matrix with entries in $\mathbb{Z}_n$ where $m_{ij} = ai + bj$, $0 \leq i \leq \frac{n}{s} - 1$, $0 \leq j \leq \frac{n}{t} - 1$. Note that $\langle a \rangle = \frac{n}{s}$, $\langle b \rangle = \frac{n}{t}$, and $\gcd(s, t) = 1$.

If $m_{ij} = m_{i'j'}$ with $0 \leq i' \leq i \leq t - 1$ and $0 \leq j, j' \leq \frac{n}{t} - 1$, then $a(i - i') = b(j' - j)$. Therefore, $\langle a \rangle = \langle i - i' \rangle$ since $\gcd(a, b) = 1$, which implies $i = i'$ because $0 \leq i - i' < t$. Since $a(i - i') = b(j' - j)$ it is also the case that $j = j'$. Therefore, each element of $\mathbb{Z}_n$ will be an entry somewhere in the first $t$ rows of $M$. In fact, the first $t$ rows of $M$ are just the $t$ cosets of $\langle b \rangle$ in $\mathbb{Z}_n$.

Now let $M'$ be the $\frac{n}{s} \times \frac{n}{t}$ block matrix created by partitioning $M$ into $s \times t$ blocks. Let $A_{ij}$ be the $i$th block of $M'$. Note that $A_{i+1,j} = A_{ij} + at$ and $A_{i,j+1} = A_{ij} + bs$ and $\langle at \rangle = \langle bs \rangle = \frac{n}{at}$, so the matrix $A_{ij} + k(bs)$ appears as a block in the $i$th row of $M'$ for each integer $k$. Furthermore, $a = ps$ for some $p \in \mathbb{Z}$ and $bq \equiv t \mod n$ for some $q \in \mathbb{Z}$ since $t = \gcd(b, n)$. Therefore, $A_{i+1,j} = A_{ij} + (pq)bs$, so $A_{i+1,j}$ is equal to some block in the $i$th row of $M'$.

This means that the $(i + 1)$st row of $M'$ is the $i$th row of $M'$ shifted by $pq$, for all $i$. So coloring each matrix in $M'$ with the same L-tile 2-coloring from Lemma 6.1 will...
ensure that $M$ is a well-defined L-tile 2-coloring. It is well-defined since every element is colored and each time an element appears it receives the same color. It is an L-tile 2-coloring because it is periodic using an L-tile 2-coloring that ‘wraps around’. This yields an S-polychromatic coloring of $\mathbb{Z}_n$ with two colors. □

Here is an example of how to get the coloring of $\mathbb{Z}_{105}$ when $a = 18$ and $b = 25$. The matrix $M$ is

$$
\begin{bmatrix}
0 & 25 & 50 & 75 & 100 & 20 & 45 & 70 & 95 & 15 & 40 & 65 & 90 & 10 & 35 & 60 & 85 & 5 & 30 & 55 & 80 \\
18 & 43 & 68 & 93 & 13 & 38 & 63 & 88 & 8 & 33 & 58 & 83 & 3 & 28 & 53 & 78 & 103 & 23 & 48 & 73 & 98 \\
36 & 61 & 86 & 6 & 31 & 56 & 81 & 1 & 26 & 51 & 76 & 101 & 21 & 46 & 71 & 96 & 16 & 41 & 66 & 91 & 11 \\
54 & 79 & 104 & 24 & 49 & 74 & 99 & 19 & 44 & 69 & 94 & 14 & 39 & 64 & 89 & 9 & 34 & 59 & 84 & 4 & 29 \\
72 & 97 & 17 & 42 & 67 & 92 & 12 & 37 & 62 & 87 & 7 & 32 & 57 & 82 & 2 & 27 & 52 & 77 & 102 & 22 & 47 \\
90 & 10 & 35 & 60 & 85 & 5 & 30 & 55 & 80 & 0 & 25 & 50 & 75 & 100 & 20 & 45 & 70 & 95 & 15 & 40 & 65 \\
9 & 3 & 84 & 11 & 11 & 46 & 71 & 96 & 16 & 41 & 66 & 91 & 11 & 36 & 61 & 86 & 21 & 46 & 71 & 96 & 16 \\
6 & 31 & 56 & 81 & 1 & 26 & 51 & 76 & 101 & 21 & 46 & 71 & 96 & 16 & 41 & 66 & 91 & 11 & 36 & 61 & 86 \\
2 & 27 & 52 & 77 & 102 & 22 & 47 & 72 & 97 & 17 & 42 & 67 & 92 & 12 & 37 & 62 & 87 & 7 & 32 & 57 & 82 \\
57 & 82 & 2 & 27 & 52 & 77 & 102 & 22 & 47 & 72 & 97 & 17 & 42 & 67 & 92 & 12 & 37 & 62 & 87 & 7 & 32 \\
75 & 100 & 20 & 45 & 70 & 95 & 15 & 40 & 65 & 60 & 85 & 5 & 30 & 55 & 80 & 0 & 25 & 50 & 75 & 100 & 20 \\
78 & 103 & 23 & 48 & 73 & 98 & 18 & 43 & 68 & 93 & 13 & 38 & 63 & 88 & 8 & 33 & 58 & 83 & 3 & 28 & 53 \\
96 & 16 & 41 & 66 & 91 & 11 & 36 & 61 & 86 & 6 & 31 & 56 & 81 & 1 & 26 & 51 & 76 & 101 & 21 & 46 & 71 \\
9 & 34 & 59 & 84 & 4 & 29 & 54 & 79 & 104 & 24 & 49 & 74 & 99 & 19 & 44 & 69 & 94 & 14 & 39 & 64 & 89 \\
27 & 52 & 77 & 102 & 22 & 47 & 72 & 97 & 17 & 42 & 67 & 92 & 12 & 37 & 62 & 87 & 7 & 32 & 57 & 82 & 2 \\
45 & 70 & 95 & 15 & 40 & 65 & 90 & 10 & 35 & 60 & 85 & 5 & 30 & 55 & 80 & 0 & 25 & 50 & 75 & 100 & 20 \\
63 & 88 & 8 & 33 & 58 & 83 & 3 & 28 & 53 & 78 & 103 & 23 & 48 & 73 & 98 & 18 & 43 & 68 & 93 & 13 & 38 \\
81 & 1 & 26 & 51 & 76 & 101 & 21 & 46 & 71 & 96 & 16 & 41 & 66 & 91 & 11 & 36 & 61 & 86 & 6 & 31 & 56 \\
9 & 99 & 19 & 44 & 69 & 94 & 14 & 39 & 64 & 89 & 9 & 34 & 59 & 84 & 4 & 29 & 54 & 79 & 104 & 24 & 49 \\
12 & 37 & 62 & 87 & 7 & 32 & 57 & 82 & 2 & 27 & 52 & 77 & 102 & 22 & 47 & 72 & 97 & 17 & 42 & 67 & 92 \\
30 & 55 & 80 & 0 & 25 & 50 & 75 & 100 & 20 & 45 & 70 & 95 & 15 & 40 & 65 & 90 & 10 & 35 & 60 & 85 & 5 \\
48 & 73 & 98 & 18 & 43 & 68 & 93 & 13 & 38 & 63 & 88 & 8 & 33 & 58 & 83 & 3 & 28 & 53 & 78 & 103 & 23 \\
66 & 91 & 11 & 36 & 61 & 86 & 6 & 31 & 56 & 81 & 1 & 26 & 51 & 76 & 101 & 21 & 46 & 71 & 96 & 16 & 41 \\
84 & 4 & 29 & 54 & 79 & 104 & 24 & 49 & 74 & 99 & 19 & 44 & 69 & 94 & 14 & 39 & 64 & 89 & 9 & 34 & 59 \\
102 & 22 & 47 & 72 & 97 & 17 & 42 & 67 & 92 & 12 & 37 & 62 & 87 & 7 & 32 & 57 & 82 & 2 & 27 & 52 & 77 \\
15 & 40 & 65 & 90 & 10 & 35 & 60 & 85 & 5 & 30 & 55 & 80 & 0 & 25 & 50 & 75 & 100 & 20 & 45 & 70 & 95 \\
33 & 58 & 83 & 3 & 28 & 53 & 78 & 103 & 23 & 48 & 73 & 98 & 18 & 43 & 68 & 93 & 13 & 38 & 63 & 88 & 8 \\
51 & 76 & 101 & 21 & 46 & 71 & 96 & 16 & 41 & 66 & 91 & 11 & 36 & 61 & 86 & 6 & 31 & 56 & 81 & 1 & 26 \\
69 & 94 & 14 & 39 & 64 & 89 & 9 & 34 & 59 & 84 & 4 & 29 & 54 & 79 & 104 & 24 & 49 & 74 & 99 & 14 & 44 \\
87 & 7 & 32 & 57 & 82 & 2 & 27 & 52 & 77 & 102 & 22 & 47 & 72 & 97 & 17 & 42 & 67 & 92 & 12 & 37 & 62 \\
\end{bmatrix}
$$

Therefore,
Now Lemma 6.1 can be used to color each $A_{i,j}$ in the following way:

\[ \chi(A_{i,j}) = \begin{bmatrix} R & Y & Y \\ Y & R & Y \\ Y & Y & R \end{bmatrix}. \]

Of course only the first row of $M'$ is needed to get the coloring of $\mathbb{Z}_n$.

Finally, we give a proof of Theorem 2.1.

**Proof** By Proposition 2.1 we have either Case i, where $S$ is equivalent to a set of the form \{0, 1, b\}, or Case ii, where $\gcd(a,n)$ and $\gcd(b,n)$ are both greater than 1. Proposition 4.1 characterizes the sets $S$ for which $p_n(S) = 3$, and Theorem 5.1 shows show that $p_n(S) = 2$ for all other sets $S$ in Case i, except when $n = 7$ and $b = 3$. Finally Lemma 6.2 takes care of Case ii. \(\square\)

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**Declarations**

**Conflict of Interest** The authors declare that they have no conflict of interest.

**References**

1. Alon, N., Krech, A., Szabó, T.: Turán’s theorem in the hypercube. SIAM J. Discret. Math. 21, 66–72 (2007)
2. Axenovich, M., Goldwasser, J., Hansen, R., Lidický, B., Martin, R.R., Offner, D., Talbot, J., Young, M.: Polychromatic colorings of complete graphs with respect to 1-, 2-factors. J. Graph Theory 87(4), 660–671 (2018)
3. Axenovich, M., Goldwasser, J., Lidický, B., Martin, R.R., Offner, D., Talbot, J., Young, M.: Polychromatic colorings on the integers. Integers 19, A18 (2019)
4. Bialostocki, A.: Some Ramsey type results regarding the graph of the $n$-cube. Ars Comb. 16(A), 39–48 (1983)
5. Coven, E.M., Meyerowitz, A.: Tiling the integers with translates of one finite set. J. Algebra 212, 161–174 (1999)
6. Goldwasser, J., Lidický, B., Martin, R., Offner, D., Talbot, J., Young, M.: Polychromatic colorings on the hypercube. J. Comb. 9(4), 631–657 (2018)
7. Newman, D.J.: Complements of finite sets of integers. Mich. Math. J. 14, 481–486 (1967)
8. Newman, D.J.: Tesselation of integers. J. Number Theory 9(1), 107–111 (1977)
9. Offner, D.: Polychromatic colorings of subcubes of the hypercube. SIAM J. Discret. Math. 22(2), 450–454 (2008)
10. Stein, S.: Tiling, packing, and covering by clusters. Rocky Mt. J. Math. 16(2), 277–321 (1986)

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