Incidences with curves in three dimensions

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July 9, 2020

Abstract

We study incidence problems involving points and curves in $\mathbb{R}^3$. The current (and in fact only viable) approach to such problems, pioneered by Guth and Katz [23, 24], requires a variety of tools from algebraic geometry, most notably (i) the polynomial partitioning technique, and (ii) the study of algebraic surfaces that are ruled by lines or, in more recent studies [25], by algebraic curves of some constant degree. By exploiting and refining these tools, we obtain new and improved bounds for point-curve incidence problems in $\mathbb{R}^3$.

Incidences of this kind have been considered in several previous studies, starting with Guth and Katz’s work on points and lines [24]. Our results, which are based on the work of Guth and Zahl [25] concerning surfaces that are doubly ruled by curves, provide a grand generalization of most of the previous results. We reconstruct the bound for points and lines, and improve, in certain significant ways, recent bounds involving points and circles (in [36]), and points and arbitrary constant-degree algebraic curves (in [35]). While in these latter instances the bounds are not known (and are strongly suspected not) to be tight, our bounds are, in a certain sense, the best that can be obtained with this approach, given the current state of knowledge.

As an application of our point-curve incidence bound, we show that the number of triangles spanned by a set of $n$ points in $\mathbb{R}^3$ and similar to a given triangle is $O(n^{15/7})$, which improves the bound of Agarwal et al. [1]. Our results are also related to a study by Guth et al. (work in progress), and have been recently applied in Sharir et al. [42] to related incidence problems in three dimensions.

Keywords. Combinatorial geometry, incidences, the polynomial method, infinitely ruled surfaces, algebraic geometry.

1 Introduction

1.1 The main results

The paper studies incidences between points and constant-degree algebraic curves in $\mathbb{R}^3$. It derives several results that yield improved bounds, and have several significant additional advantages over previous work.

The introduction states the various new results, discusses the relevant background, and introduces and discusses various parameters and constructs that control the sharpness of the derived bounds. To help the

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*Work on this paper by Noam Solomon and Micha Sharir was supported by Grants 892/13 and 260/18 from the Israel Science Foundation. Work by Micha Sharir was also supported by Grant 2012/229 from the U.S.–Israel Binational Science Foundation, by Grant G-1367-407.6/2016 from the German-Israeli Foundation for Scientific Research and Development, by the Israeli Centers of Research Excellence (I-CORE) program (Center No. 4/11), by the Blavatnik Research Fund in Computer Science at Tel Aviv University and by the Hermann Minkowski-MINERVA Center for Geometry at Tel Aviv University. A preliminary version of an expanded version of the paper has appeared in Proc. 28th ACM-SIAM Symposium on Discrete Algorithms (2017), 2456–2475.

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reader, we list some of the main results right upfront, somewhat prematurely, and supplement them with informal and brief explanation of the setups that we consider. Full and rigorous details are given later in the introduction. (The numbering of the theorems below is their numbering in the main body of the introduction.)

**Incidences with algebraic curves: An informal preview.** We consider infinite families \( C_0 \) of algebraic curves in \( \mathbb{R}^3 \) of constant degree \( E \), and assume that the curves in \( C_0 \) have \( k \) degrees of freedom—any \( k \) points determine only \( O(1) \) curves from \( C_0 \) incident to all of them, and any pair of curves intersect at only \( O(1) \) points.

An algebraic surface \( V \) is infinitely ruled by curves from \( C_0 \) if each generic point \( p \in V \) is incident to infinitely many curves from \( C_0 \) that are fully contained in \( V \). We then have:

**Theorem 1.4.** Let \( P \) be a set of \( m \) points and \( C \) a set of \( n \) irreducible algebraic curves of constant degree \( E \) in \( \mathbb{R}^3 \), taken from a family \( C_0 \) with \( k \) degrees of freedom, such that no surface that is infinitely ruled by curves of \( C_0 \) contains more than \( q \) curves of \( C \), for a parameter \( q < n \). Then

\[
I(P, C) = O \left( m^{k/E} n^{3k-3} + m n^{2k-1} q^{k-1} + m + n \right),
\]

where the constant of proportionality depends on \( k \), \( E \), and the complexity of the family \( C_0 \).

See later for a precise definition of the complexity of a family \( C_0 \).

We say that \( C_0 \) is a family of reduced dimension \( s \) if, for each surface \( V \) that is infinitely ruled by curves of \( C_0 \), the subfamily of the curves of \( C_0 \) that are fully contained in \( V \) is \( s \)-dimensional; loosely, and not as general as the precise definition given later, this means that each curve in this subfamily can be specified algebraically in terms of \( s \) real parameters. For example, the ‘full’ dimension of the family of circles in \( \mathbb{R}^3 \) is 6, but the reduced dimension of the subfamily of circles that lie on some fixed plane or sphere (which are the only surfaces that are infinitely ruled by circles \([28]\)) is only 3. In this case we obtain the following variant of Theorem 1.4.

**Theorem 1.5.** Let \( P \) be a set of \( m \) points and \( C \) a set of \( n \) irreducible algebraic curves of constant degree \( E \) in \( \mathbb{R}^3 \), taken from a family \( C_0 \) that has \( k \) degrees of freedom and reduced dimension \( s \), such that no surface that is infinitely ruled by curves of \( C_0 \) contains more than \( q \) of the curves of \( C \). Then

\[
I(P, C) = O \left( m^{k/E} n^{3k-3} + O_{\varepsilon} \left( m^{2/3} n^{1/3} q^{1/3} + m^{2s/3} n^{s-4} q^{s-4} \varepsilon + m + n \right) \right),
\]

for any \( \varepsilon > 0 \), where the first constant of proportionality depends on \( k \), \( s \), \( E \), and the maximum complexity of any subfamily of \( C_0 \) consisting of curves that are fully contained in some surface that is infinitely ruled by curves of \( C_0 \), and the second constant also depends on \( \varepsilon \).

**Incidences with circles: An informal preview.** Theorem 1.4 yields the fundamental incidence bound of Guth and Katz \([24]\), reviewed later in the introduction, for points and lines in \( \mathbb{R}^3 \), because lines have \( k = 2 \) degrees of freedom, and the only surfaces that are infinitely ruled by lines are planes; see below for more details. Theorem 1.5 yields an improved and refined bound for incidences between points and circles in \( \mathbb{R}^3 \), as compared with the previous result of \([36]\) (see also another bound in \([40]\), which will be discussed later in the introduction):

**Theorem 1.6.** Let \( P \) be a set of \( m \) points and \( C \) a set of \( n \) circles in \( \mathbb{R}^3 \), so that no plane or sphere contains more than \( q \) circles of \( C \). Then

\[
I(P, C) = O \left( m^{3/7} n^{6/7} + m^{2/3} n^{1/3} q^{1/3} + m^{6/11} n^{5/11} q^{4/11} \log^{2/11} (m^3/q) + m + n \right).
\]

\(^1\)Genericity is a standard notion in algebraic geometry; see, e.g., Cox et al. \([13\) Definition 3.6] and also \([37\) Section 2.1].

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We now proceed to the full introduction, in which these results, and several others, will be given, along with the background and detailed discussion of the various parameters and notions mentioned above.

A preliminary version of the paper has appeared as part of the conference paper [39].

The setup: Incidences between points and curves in three dimensions. Let \( P \) be a set of \( m \) points and \( C \) a set of \( n \) irreducible algebraic curves of constant degree in \( \mathbb{R}^3 \). We consider the problem of obtaining sharp incidence bounds between the points of \( P \) and the curves of \( C \). This is a major topic in incidence geometry since the groundbreaking work of Guth and Katz [24] on point-line incidences in \( \mathbb{R}^3 \), with many follow-up studies, some of which are reviewed below. Building on the recent work of Guth and Zahl [25], which bounds the number of 2-rich points determined by a set of bounded-degree algebraic curves in \( \mathbb{R}^3 \) (i.e., points incident to at least two of the given curves), we are able to generalize Guth and Katz’s point-line incidence bound to a general bound on the number of incidences between points and bounded-degree irreducible algebraic curves that satisfy certain natural assumptions, discussed in detail below.

1.2 Background

Points and curves, the planar case. The case of incidences between points and curves has a rich history, starting with the case of points and lines in the plane, studied in the seminal paper of Szemerédi and Trotter [45], and later also in [12, 44], where the worst-case tight bound on the number of incidences is \( \Theta(m^{2/3}n^{2/3} + m + n) \), where \( m \) is the number of points and \( n \) is the number of lines. Still in the plane, Pach and Sharir [32] extended this bound to incidence bounds between points and curves with \( k \) degrees of freedom. These are curves with the property that, for each set of \( k \) points, there are only \( \mu = O(1) \) curves that pass through all of them, and each pair of curves intersect in at most \( \mu \) points; \( \mu \) is called the multiplicity (of the degrees of freedom).

**Theorem 1.1** (Pach and Sharir [32]). Let \( P \) be a set of \( m \) points in \( \mathbb{R}^2 \) and let \( C \) be a set of \( n \) bounded-degree algebraic curves in \( \mathbb{R}^2 \) with \( k \) degrees of freedom and with multiplicity \( \mu \). Then

\[
I(P, C) = O\left(m^{\frac{1}{k-1}}n^{\frac{2k-3}{k-1}} + m + n\right),
\]

where the constant of proportionality depends on \( k \) and \( \mu \).

**Remark.** The result of Pach and Sharir holds for more general families of curves, not necessarily algebraic, but, since algebraicity will be assumed in three dimensions, we assume it also in the plane.

Note that the Szemerédi–Trotter bound is a special case of Theorem 1.1 since lines have two degrees of freedom. However, except for the case of lines (and \( k = 2 \)), the bound is not known, and is strongly suspected not, to be tight in the worst case. Indeed, in a series of papers during the 2000’s [2, 5, 29], an improved bound has been obtained for incidences with circles, parabolas, or other families of curves with certain properties (see [2] for the precise formulation). Specifically, for a set \( P \) of \( m \) points and a set \( C \) of \( n \) circles, or parabolas, or similar curves, we have

\[
I(P, C) = O\left(m^{2/3}n^{2/3} + m^{6/11}n^{9/11} \log^{2/11}(m^3/n) + m + n\right).
\]

Some further (slightly) improved bounds, over the bound in Theorem 1.1 for more general families of curves in the plane have been obtained by Chan [10, 11] and by Bien [7]. They are, however, considerably weaker than the bound in (1).

Recently, Sharir and Zahl [41] have considered general families of constant-degree algebraic curves in the plane that belong to an \( s \)-dimensional family of curves. Extending the definition given above, this
means that each curve in that family can be represented by a constant number of real parameters, so that, in this parametric space, the points representing the curves lie in an s-dimensional algebraic variety $F$ of some constant degree (to which we refer as the “complexity” of $F$). See [11] for more details.

**Theorem 1.2** (Sharir and Zahl [11]). Let $C$ be a set of $n$ algebraic plane curves that belong to an s-dimensional family $F$ of curves of maximum constant degree $E$, no two of which share a common irreducible component, and let $P$ be a set of $m$ points in the plane. Then, for any $\varepsilon > 0$, the number $I(P, C)$ of incidences between the points of $P$ and the curves of $C$ satisfies

$$I(P, C) = O\left( m^{\frac{2s-4}{5s-4} + \frac{5s-6}{5s-4} + \varepsilon} + m^{2/3}n^{2/3} + m + n \right),$$

where the constant of proportionality depends on $\varepsilon$, $s$, $E$, and the complexity of the family $F$.

Except for the factor $O(n^\varepsilon)$, this is a significant improvement over the bound in Theorem 1.1 (for $s \geq 3$), in cases where the assumptions in Theorem 1.2 imply (as they often do) that $C$ has $k = s$ degrees of freedom. Concretely, when $k = s$, we obtain an improvement, except for the factor $n^\varepsilon$, which is an artifact of the proof of Theorem 1.2 for the entire “meaningful” range $n^{1/s} \leq m \leq n^2$, in which the bound is superlinear. The factor $n^\varepsilon$ makes the bound in [11] slightly weaker only when $m$ is close to the lower end $n^{1/s}$ of that range. Note also that for circles (where $s = 3$), the bound in Theorem 1.2 nearly coincides with the slightly more refined bound [11].

**Incidences with curves in three dimensions.** The seminal work of Guth and Katz [24] establishes the sharper bound $O(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}q^{1/3} + m + n)$ on the number of incidences between $m$ points and $n$ lines in $\mathbb{R}^3$, provided that no plane contains more than $q$ of the given lines. This has lead to many recent works on incidences between points and lines or other curves in three and higher dimensions; see [9, 25, 35, 36, 37, 40, 46] for a sample of these results. Most relevant to our present study are the works of Sharir, Sheffer, and Solomon [35] on incidences between points and curves in any dimension, the work of Sharir, Sheffer, and Zahl [36] on incidences between points and circles in three dimensions, the work of Sharir and Solomon [37] on incidences between points and lines in four dimensions, and the recent work of Do and Sheffer [14] that presents more general bounds for incidences with curves and surfaces in any dimension, as well as several other studies of point-line incidences by the authors [38, 40].

Of particular significance is the recent work of Guth and Zahl [25] on the number of 2-rich points in a collection of curves, namely, points incident to at least two of the given curves. For the case of lines, Guth and Katz [24] have shown that the number of such points is $O(n^{3/2})$, when no plane or regulus contains more than $O(n^{1/2})$ lines. Guth and Zahl obtain the same asymptotic bound for general algebraic curves, under analogous (but stricter) restrictive assumptions.

The new bounds that we will derive require the extension to three dimensions of the notions of having $k$ degrees of freedom and of being an $s$-dimensional family of curves. The definitions of these concepts, as given above for the planar case, extend, more or less verbatim, to three (or higher) dimensions, but, even in typical situations, these two concepts do not coincide anymore. For example, lines in three dimensions have two degrees of freedom, but they form a 4-dimensional family of curves (this is the number of parameters needed to specify a line in $\mathbb{R}^3$).

1.3 Our results

Before we state our results, we discuss three notions that are used in these statements. These are the notions of $k$ degrees of freedom (already mentioned above), of constructibility, and of surfaces infinitely ruled by curves.

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2This bound is not explicitly mentioned in [24], but it immediately follows from the analysis given there.
**k degrees of freedom.** Let $C_0$ be an infinite family of irreducible algebraic curves of constant degree $E$ in $\mathbb{R}^3$. Formally, in complete analogy with the planar case, we say that $C_0$ has $k$ degrees of freedom with multiplicity $\mu$, where $k$ and $\mu$ are constants, if (i) for every tuple of $k$ points in $\mathbb{R}^3$ there are at most $\mu$ curves of $C_0$ that are incident to all $k$ points, and (ii) every pair of curves of $C_0$ intersect in at most $\mu$ points. As in [32], the bounds that we derive depend more significantly on $k$ than on $\mu$—see below.

**Constructibility.** In the statements of the following theorems, we also assume that $C_0$ is a constructible family of curves. This notion generalizes the notion of being algebraic, and is discussed in detail in Guth and Zahl [25]. Informally, a set $Y \subseteq \mathbb{C}^d$ is constructible if it is a Boolean combination of algebraic sets. The formal definition goes as follows (see, e.g., Harris [26, Lecture 3]). For $z \in \mathbb{C}$, define $v(0) = 0$ and $v(z) = 1$ for $z \neq 0$. Then $Y \subseteq \mathbb{C}^d$, for some fixed $d$, is a constructible set if there exist an integer $J_Y$, a finite set of polynomials $f_j : \mathbb{C}^d \rightarrow \mathbb{C}$, for $j = 1,\ldots,J_Y$, and a subset $B_Y \subset \{0,1\}^{J_Y}$, so that $x \in Y$ if and only if $(v(f_1(x)), \ldots, v(f_{J_Y}(x))) \in B_Y$.

When we apply this definition to a set of curves, we think of them as points in some parametric (complex) $d$-space, where $d$ is the number of parameters needed to specify a curve. When $J_Y = 1$ we get all the algebraic hypersurfaces (that admit the implied $d$-dimensional representation) and their complements. An $s$-dimensional family of curves, for $s < d$, is obtained by taking $J_Y = d - s$ and $B_Y = \{0\}^{J_Y}$. In doing so, we assume that the curves that we obtain are complete intersections. Following Guth and Zahl (see also a comment to that effect in the appendix), this involves no loss of generality, because every curve is contained in a curve that is a complete intersection. In what follows, when we talk about constructible sets, we implicitly assume that the ambient dimension $d$ is constant.

The constructible sets form a Boolean algebra. This means that finite unions and intersections of constructible sets are constructible, and the complement of a constructible set is constructible. Another fundamental property of constructible sets is that, over $\mathbb{C}$, the projection of a constructible set is constructible; this is known as Chevalley’s theorem (see Harris [26, Theorem 3.16] and Guth and Zahl [25, Theorem 2.3]). If $Y$ is a constructible set, we define the complexity of $Y$ to be $\min(\deg f_1 + \cdots + \deg f_{J_Y})$, where the minimum is taken over all representations of $Y$, as described above. As just observed, constructibility of a family $C_0$ of curves extends the notion of $C_0$ being $s$-dimensional. One of the main motivations for using the notion of constructible sets (rather than just $s$-dimensionality) is the fact, established by Guth and Zahl [25, Proposition 3.3], that the set $C_{3,E}$ of irreducible curves of degree at most $E$ in complex 3-dimensional space (either affine or projective) is a constructible set of constant complexity that depends only on $E$. Moreover, Theorem [1.7] one of the central technical tools that we use in our analysis (see below for its statement and the appendix for its proof), holds for constructible families of curves.

**The connection between degrees of freedom and constructibility/dimensionality.** Loosely speaking, in the plane the number of degrees of freedom and the dimensionality of a family of curves tend to be equal. In three dimensions the situation is different. This is because the constraint that a curve $\gamma$ passes through a point $p$ imposes two equations on the parameters defining $\gamma$. We therefore expect the number of degrees of freedom to be half the dimensionality. A few instances that illustrate this connection are:

(i) Lines in three dimensions have two degrees of freedom, and they form a 4-dimensional family of curves (this is the number of parameters needed to specify a line in $\mathbb{R}^3$).

(ii) Circles in three dimensions have three degrees of freedom, and they form a 6-dimensional family of curves (e.g., one needs three parameters to specify the plane containing the circle, two additional parameters to specify its center, and a sixth parameter for its radius).

(iii) Conic sections have five degrees of freedom, but they form an 8-dimensional family of curves, as is
easily checked (we need three parameters for the plane containing the curve, and five parameters to define a conic section in that plane). This discrepancy (for conic sections) is explained by noting that four points are not sufficient to define the curve, because the first three determine the plane containing it, so the fourth point, if at all coplanar with the first three, only imposes one constraint on the parameters of the curve.

**Remark.** The definition of constructibility is given over the complex field $\mathbb{C}$. This is in accordance with most of the basic algebraic geometry tools, which have been developed over the complex field. Some care has to be exercised when applying them over the reals. For example, Theorem 1.7 one of the central technical tools that we use in our analysis, as well as the related results of Guth and Zahl [25], apply over the complex field, but not over the reals. On the other hand, when we apply the partitioning method of [24] (as in the proof of Theorems 1.4) or when we use Theorem 1.2 we (have to) work over the reals.

It is a fairly standard practice in algebraic geometry that handles a real algebraic variety $V$, defined by real polynomials, by considering its complex counterpart $V_{\mathbb{C}}$, namely the set of complex points at which the polynomials defining $V$ vanish. The rich toolbox that complex algebraic geometry has developed allows one to derive various properties of $V_{\mathbb{C}}$, which, with some care, can usually be transported back to the real variety $V$.

This issue arises time and again in this paper. Roughly speaking, we approach it as follows. We apply the polynomial partitioning technique to the given sets of points and of curves or surfaces, in the original real (affine) space, as we should. Within the cells of the partitioning we then apply some field-independent argument, based either on induction or on some ad-hoc combinatorial argument. Then we need to treat points that lie on the zero set of the partitioning polynomial. We can then switch to the complex field, when it suits our purpose, noting that this step preserves all the real incidences; at worst, it might add additional incidences involving the non-real portions of the variety and of the curves. Hence, the bounds that we obtain for this restrictive setup transport, more or less verbatim, to the real case too.

**Surfaces infinitely ruled by curves.** Back in three dimensions, a surface $V$ is (singly, doubly, or infinitely) ruled by some family $\Gamma$ of curves of degree at most $E$, if every generic point $p \in V$ is incident to (at least one, at least two, or infinitely many) curves of $\Gamma$ that are fully contained in $V$. The connection between ruled surface theory and incidence geometry goes back to the pioneering work of Guth and Katz [24] and shows up in many subsequent works. See Guth’s survey [21] and book [22], and Kollár [27] for details.

In most of the previous works, only singly-ruled and doubly-ruled surfaces have been considered. Looking at infinitely-ruled surfaces adds a powerful ingredient to the toolbox, as will be demonstrated in this paper.

We recall that the only surfaces that are infinitely ruled by lines are planes (see, e.g., Fuchs and Tabachnikov [16 Corollary 16.2]), and that the only surfaces that are infinitely ruled by circles are spheres and planes (see, e.g., Lubbes [28] Theorem 3) and Schicho [34]. see also Skopenkov and Krasauskas [33] for recent work on celestials, namely surfaces doubly ruled by circles, and Nilov and Skopenkov [31], proving that a surface that is ruled by a line and a circle through each (generic) point is a quadric). It should be noted that, in general, for this definition to make sense, it is important to require that the degree $E$ of the ruling curves be smaller than deg($V$). Otherwise, every variety $V$ is infinitely ruled by, say, the curves $V \cap h$, for hyperplanes $h$, having the same degree as $V$. A challenging open problem is to characterize all the surfaces that are infinitely ruled by algebraic curves of degree at most $E$ (or by certain special classes thereof). However, the following result of Guth and Zahl provides a useful necessary condition for this property to hold.

**Theorem 1.3** (Guth and Zahl [25]). Let $V$ be an irreducible surface, and suppose that it is doubly ruled by curves of degree at most $E$. Then $\text{deg}(V) \leq 100E^2$.

In particular, an irreducible surface that is infinitely ruled by curves of degree at most $E$ is doubly ruled.
ruled by these curves, so its degree is at most \(100E^2\). Therefore, if \(V\) is irreducible of degree \(D\) larger than this bound, \(V\) cannot be infinitely ruled by curves of degree at most \(E\). This leaves a gray zone, in which the degree of \(V\) is between \(E\) and \(100E^2\). We would like to conjecture that if \(V\) is an irreducible variety that is infinitely ruled by degree-\(E\) curves then its degree is \(O(E)\). Being unable to establish this conjecture, we leave it as a challenging open problem for further research.

**Our results.** We can now state our main results on point-curve incidences.

**Theorem 1.4.** Let \(P\) be a set of \(m\) points and \(C\) a set of \(n\) irreducible algebraic curves of constant degree \(E\), taken from a constructible family \(C_0\), of constant complexity, with \(k\) degrees of freedom (and some multiplicity \(\mu\)) in \(\mathbb{R}^3\), such that no surface that is infinitely ruled by curves of \(C_0\) contains more than \(q\) curves of \(C\), for a parameter \(q < n\). Then

\[
I(P,C) = O\left(m^{\frac{k}{3k-2}}n^{\frac{3k-3}{3k-2}} + m^{\frac{k}{2k-1}}n^{\frac{k-1}{2k-1}}q^{\frac{k-1}{q^{2k-1}}} + m + n\right),
\]

where the constant of proportionality depends on \(k, \mu, E\), and the complexity of the family \(C_0\).

**Remarks.** (1) In certain favorable situations, such as in the cases of lines or circles, discussed above, the surfaces that are infinitely ruled by curves of \(C_0\) have a simple characterization. In such cases the theorem has a stronger flavor, as its assumption on the maximum number of curves on a surface has to be made only for this concrete kind of surfaces. For example, as already noted, for lines (resp., circles) only need to require that no plane (resp., no plane or sphere) contains more than \(q\) of the curves. In general, as mentioned, characterizing infinitely-ruled surfaces by a specific family of curves is a difficult task. Nevertheless, we can overcome this issue by replacing the assumption in the theorem by a more restrictive one, requiring that no surface that is infinitely ruled by curves of degree at most \(E\) contain more than \(q\) curves of \(C\). By Theorem 1.3 any infinitely ruled surface of this kind must be of degree at most \(100E^2\). Hence, an even simpler (albeit weaker) formulation of the theorem is to require that no surface of degree at most \(100E^2\) contains more than \(q\) curves of \(C\). This can indeed be much weaker: In the case of circles, say, instead of making this requirement only for planes and spheres, we now have to make it for every surface of degree at most 400. See also [24] for a similar approach.

(2) In several recent works (see [19, 35, 36]), the assumption in the theorem is replaced by a much more restrictive assumption, that no surface of degree at most \(c_\varepsilon\) contains more than \(q\) given curves, where \(c_\varepsilon\) is a constant that depends on another prespecified parameter \(\varepsilon > 0\) (where \(\varepsilon\) appears in the exponents in the resulting incidence bound), and is typically very large (and increases as \(\varepsilon\) becomes smaller). Getting rid of such an \(\varepsilon\)-dependent constant (and of the \(\varepsilon\) in the exponent) is a significant feature of Theorem 1.4.

(3) As already mentioned, Theorem 1.4 generalizes the incidence bound of Guth and Katz [24], obtained for the case of lines. In this case, lines have \(k = 2\) degrees of freedom, they certainly form a constructible (in fact, a 4-dimensional) family of curves, and, as just noted, planes are the only surfaces in \(\mathbb{R}^3\) that are infinitely ruled by lines. Thus, in this special case, both the assumptions and the bound in Theorem 1.4 are identical to those in Guth and Katz [24]. That is, if no plane contains more than \(q\) input lines, the number of incidences is \(O(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}q^{1/3} + m + n)\).

**Improving the bound.** The bound in Theorem 1.4 can be further improved, if we also throw into the analysis the dimensionality \(s\) of the family \(C_0\). Actually, as will follow from the proof, the dimensionality that will be used is only that of any subset of \(C_0\) whose members are fully contained in some variety that is infinitely ruled by curves of \(C_0\). As just noted, such a variety must be of constant degree (at most \(100E^2\), or smaller as in the cases of lines and circles), and the additional constraint that the curves be contained in the variety can typically be expected to reduce the dimensionality of the family.
For example, if $C_0$ is the collection of all circles in $\mathbb{R}^3$, then, since the only surfaces that are infinitely ruled by circles are spheres and planes, the subfamily of all circles that are contained in some sphere or plane is only 3-dimensional (as opposed to the entire $C_0$, which is 6-dimensional).

We capture this setup by saying that $C_0$ is a family of reduced dimension $s$ if, for each surface $V$ that is infinitely ruled by curves of $C_0$, the subfamily of the curves of $C_0$ that are fully contained in $V$ is $s$-dimensional. We then obtain the following variant of Theorem 1.4.

**Theorem 1.5.** Let $P$ be a set of $m$ points and $C$ a set of $n$ irreducible algebraic curves of constant degree $E$, taken from a constructible family $C_0$ with $k$ degrees of freedom (and some multiplicity $\mu$) in $\mathbb{R}^3$, such that no surface that is infinitely ruled by curves of $C_0$ contains more than $q$ of the curves of $C$, and assume further that $C_0$ is of reduced dimension $s$. Then

$$I(P, C) = O\left( m^{\frac{k}{3k-2}n^{\frac{3k-3}{3k-2}}} + O_\varepsilon \left( n^{\frac{2s-2}{3s-4}}q^{\frac{2s-2}{3s-4}+\varepsilon} + m + n \right) \right),$$

for any $\varepsilon > 0$, where the first constant of proportionality depends on $k$, $\mu$, $s$, $E$, and the maximum complexity of any subfamily of $C_0$ consisting of curves that are fully contained in some surface that is infinitely ruled by curves of $C_0$, and the second constant also depends on $\varepsilon$.

**Remarks.** (1) Theorem 1.5 is an improvement of Theorem 1.4 when $s \leq k$ and $m > n^{1/k}$, in cases where $q$ is sufficiently large so as to make the second term in (2) dominate the first term; for smaller values of $m$ the bound is always linear. This is true except for the term $q^\varepsilon$, which is an artifact of the proof of the theorem, and which affects the bound only when $m$ is very close to $n^{1/k}$ (when $s = k$). When $s > k$ we get a threshold exponent $\beta = \frac{5s-4k-2}{ks-4k+2}$ (which becomes $1/k$ when $s = k$), so that the bound in Theorem 1.5 is stronger (resp., weaker) than the bound in Theorem 1.4 when $m > n^\beta$ (resp., $m < n^\beta$), again, up to the extra factor $q^\varepsilon$.

(2) The bounds in Theorems 1.4 and 1.5 improve, in three dimensions, the recent result of Sharir, Sheffer, and Solomon [35], in three significant ways: (i) The leading terms in both bounds are essentially the same, but our bound is sharper, in that it does not include the factor $O(n^\varepsilon)$ appearing in [35]. (ii) The assumption here, concerning the number of curves on a low-degree surface, is much weaker than the one made in [35], where it was required that no surface of some (constant but potentially very large) degree $c_\varepsilon$, that depends on $\varepsilon$, contains more than $q$ curves of $C$. (See also Remark (2) following Theorem 1.4) (iii) The two variants of the non-leading terms here are significantly smaller than those in [35], and, in a certain sense (that will be elaborated following the proof of Theorem 1.5) are best possible.

(3) The results and techniques in this paper have recently been applied in Sharir et al. [42] to incidence problems in three dimensions with curves that have ‘almost two degrees of freedom’ (a notion defined in that paper).

**Point-circle incidences in $\mathbb{R}^3$.** Theorem 1.5 yields a new bound for the case of incidences between points and circles in $\mathbb{R}^3$, which improves over the previous bound of Sharir, Sheffer, and Zahl [36]. Specifically, we have:

**Theorem 1.6.** Let $P$ be a set of $m$ points and $C$ a set of $n$ circles in $\mathbb{R}^3$, so that no plane or sphere contains more than $q$ circles of $C$. Then

$$I(P, C) = O\left( m^\frac{3}{7}n^\frac{6}{7} + m^{2/3}n^{1/3}q^{1/3} + m^{6/11}n^{5/11}q^{4/11} \log^{2/11}(m^3/q) + m + n \right).$$

Note that the bound in Theorem 1.6 is slightly better than the bound obtained from Theorem 1.5 for $s = 3$, in that it replaces the factor $q^\varepsilon$ by the factor $\log^{2/11}(m^3/q)$. This is a consequence of the proof technique and of the improved point-circle incidence bound in the plane [1]. See Section 3 for details.
Here too we have the three improvements noted in Remark (2) above. In particular, in the sense of part (iii) of that remark, the new bound is, in a rather vague sense, “best possible” with respect to the best known bound\(^1\) for the planar or spherical cases. See Section 3 for details. We remark, though, that a recent work of Zahl\(^4\) gives a different point-circle incidence bound, namely,

\[ O^* \left( m^{1/2} n^{3/4} + m^{2/3} n^{13/15} + m^{1/3} n^{8/9} + nq^{2/3} + m \right) \]

where the notation \(O^*\) hides subpolynomial factors, which is better for a certain range of \(m, n, \) and \(q.\)

Theorem 1.6 has an interesting application to the problem of bounding the number of similar triangles spanned by a set of \(n\) points in \(\mathbb{R}^3.\) It yields the bound \(O(n^{15/7})\), which improves the bound of Agarwal et al.\(^1\). Again, see Section 3 for details.

### 1.4 The main techniques

There are three main ingredients used in our approach. The first ingredient, already mentioned in the context of planar point-curve incidences, is the techniques of Pach and Sharir\(^32\) (given in Theorem 1.1), and of Sharir and Zahl\(^41\) (Theorem 1.2) concerning incidences between points and algebraic curves in the plane.

The second ingredient, relevant to the proof of Theorems 1.4 and 1.5, is the polynomial partitioning technique of Guth and Katz\(^24\), and its more recent extension by Guth\(^20\), which yields a divide-and-conquer mechanism via space decomposition by the zero set of a suitable polynomial. This will produce subproblems that will be handled recursively, and will leave us with the overhead of analyzing the incidence pattern involving the points that lie on the zero set itself, which in turn can be handled using Theorem 1.7 stated below. We assume familiarity of the reader with these results; more details will be given in the applications of this technique in the proofs of the aforementioned theorems.

The third ingredient arises in the proof of Theorems 1.4 and 1.5, where we argue that a “generic” point on a variety \(V\), that is not infinitely ruled by constant-degree curves of some given family, as in the statement of the theorems, is incident to at most a constant number of the given curves that are fully contained in \(V.\) Moreover, we can also control the number and structural properties of “non-generic” points.

Before formally stating, in detail, the technical properties that we need, we review a few notations.

Fix a constructible subfamily \(C_0\) of the family \(C_{3,E}\) of the irreducible curves of degree at most \(E\) in 3-dimensional space, and a trivariate polynomial \(f.\) Following Guth and Zahl\(^25\) Section 9], we call a point \(p \in Z(f)\) a \((t, C_0, r-)\)-flecnodes, if there are at least \(t\) curves \(\gamma_1, \ldots, \gamma_t \in C_0,\) such that, for each \(i = 1, \ldots, t,\) (i) \(\gamma_i\) is incident to \(p,\) (ii) \(p\) is a non-singular point of \(\gamma_i,\) and (iii) \(\gamma_i\) osculates to \(Z(f)\) to order \(r\) at \(p.\) This is a generalization of the notion of a flecnodal point, due to Salmon\(^33\) Chapter XVII, Section III] (see also\(^24\) \(37\) for more details). Our analysis requires the following theorem. It is a consequence of the analysis of Guth and Zahl\(^25\) Corollary 10.2,\) which itself is a generalization of the Cayley–Salmon theorem on surfaces ruled by lines (see, e.g., Guth and Katz\(^24\)), and is closely related to Theorem 1.3 (also due to Guth and Zahl\(^25\)). The novelty in this theorem is that it addresses surfaces that are infinitely ruled by certain families of curves, whereas the analysis in\(^25\) only handles surfaces that are doubly ruled by such curves.

**Theorem 1.7.** (a) For given integer parameters \(c\) and \(E,\) there are constants \(c_1 = c_1(c, E),\) \(r = r(c, E),\) and \(t = t(c, E),\) such that the following holds. Let \(f\) be a complex irreducible polynomial of degree\(^3\) \(D \gg E,\) and let \(C_0 \subset C_{3,E}\) be a constructible set of complexity at most \(c.\) If there exist at least \(c_1 D^2\) curves of \(C_0,\) such

\(^3\)See the appendix for the precise relationship between \(D\) and \(E.\)
that each of them is contained in \(Z(f)\) and contains at least \(c_1 D\) points on \(Z(f)\) that are \((t, C_0, r)\)-flecnodes, then \(Z(f)\) is infinitely ruled by curves from \(C_0\).

(b) In particular, if \(Z(f)\) is not infinitely ruled by curves from \(C_0\) then, except for at most \(c_1 D^2\) exceptional curves, every curve in \(C_0\) that is fully contained in \(Z(f)\) is incident to at most \(c_1 D^2\) \(t\)-rich points, namely points that are incident to at least \(t\) curves in \(C_0\) that are also fully contained in \(Z(f)\).

Note that (b) follows from (a) because, by definition, a \(t\)-rich point is a \((t, C_0, r)\)-flecnode, for any \(r \geq 1\) (a curve fully contained in the variety osculates to it to any order).

Note that, by making \(c_1\) sufficiently large (specifically, choosing \(c_1 > E\)), the assumption that each of the \(c_1 D^2\) curves in the premises of the theorem is fully contained in \(Z(f)\) follows (by Bézout’s theorem) from the fact that each of them contains at least \(c_1 D\) points on \(Z(f)\). Although the theorem is a (heretofore unstated) corollary of the work of Guth and Zahl in [25], we review (in the appendix) the machinery needed for its proof, and sketch a brief version of the proof itself, for the convenience of the reader and in the interest of completeness.

2 Proofs of Theorems 1.4 and 1.5

The proofs of both theorems are almost identical, and they differ in only one step in the analysis. We will give a full proof of Theorem 1.4 and then comment on the few modifications that are needed to establish Theorem 1.5.

**Proof of Theorem 1.4.** Since the family \(C\) has \(k\) degrees of freedom with multiplicity \(\mu\), the incidence graph \(G(P, C)\), as a subgraph of \(P \times C\), does not contain \(K_{k, \mu + 1}\) as a subgraph. The Kővári-Sós-Turán theorem (e.g., see [30, Section 4.5]) then implies that \(I(P, C) = O(mn^{1 - 1/k} + n)\), where the constant of proportionality depends on \(k\) (and \(\mu\)). We refer to this as the naive bound on \(I(P, C)\). In particular, when \(m = O(n^{1/k})\), we get \(I(P, C) = O(n)\). We may thus assume that \(m \geq a'n^{1/k}\), for some absolute constant \(a'\).

The proof proceeds by double induction on \(n\) and \(m\), and establishes the bound

\[
I(P, C) \leq A \left( m^{k - 2} n^{3k - 3} + m^{k - 1} n^{2k - 1} q^{k - 1} + m + n \right),
\]

for a suitable constant \(A\) that depends on \(k, \mu, E\), and the complexity of \(C_0\).

The base case for the outer induction on \(n\) is \(n \leq n_0\), for a suitable sufficiently large constant threshold \(n_0\) that will be set later. The bound (4) clearly holds in this case if we choose \(A \geq n_0\).

The base case for the inner induction on \(m\) is \(m \leq a'n^{1/k}\), in which case the naive bound implies that \(I(P, C) = O(n)\), so (4) holds with a sufficiently large choice of \(A\). Assume then that the bound (4) holds for all sets \(P', C'\) with \(|C'| < n\) or with \(|C'| = n\) and \(|P'| < m\), and let \(P\) and \(C\) be sets of sizes \(|P| = m, |C| = n|, such that \(n > n_0\), and \(m \geq a'n^{1/k}\).

It is instructive to notice that the two terms \(m^{k - 1} n^{3k - 3}\) and \(m\) in (4) compete for dominance; the former (resp., latter) dominates when \(m \leq n^{3/2}\) (resp., \(m \geq n^{3/2}\)). One therefore has to treat these two cases somewhat differently; see below and also in earlier works [24, 38].

**Applying the polynomial partitioning technique.** We construct a partitioning polynomial \(f\) for the set \(C\) of curves, as in the recent variant of the polynomial partitioning technique, due to Guth [20].
Specifically, we choose a degree

\[
D = \begin{cases} 
    cm^{\frac{k}{3k-2}}/n^{\frac{1}{3k-2}}, & \text{for } a'n^{1/k} \leq m \leq an^{3/2}, \\
    cn^{1/2}, & \text{for } m > an^{3/2}, 
\end{cases}
\]  

(5)

for suitable constants \( c, a \), and the previously introduced constant \( a' \) (all of whose values will be set later), and obtain a polynomial \( f \) of degree at most \( D \), such that each of the \( O(D^3) \) (open) connected components of \( \mathbb{R}^3 \setminus Z(f) \) is crossed by at most \( O(n/D^2) \) curves of \( C \), where the former constant of proportionality is absolute, and the latter one depends on \( E \). Note that in both cases \( 1 \leq D \ll n^{1/2} \), if \( a, a' \), and \( c \) are chosen appropriately. Denote the cells of the partition as \( \tau_1, \ldots, \tau_u \), for \( u = O(D^3) \). For each \( i = 1, \ldots, u \), let \( C_i \) denote the set of curves of \( C \) that intersect \( \tau_i \), and let \( P_i \) denote the set of points that are contained in \( \tau_i \). We set \( m_i = |P_i| \) and \( n_i = |C_i| \), for \( i = 1, \ldots, u \), put \( m' = \sum_i m_i \leq m \), and notice that, by construction, \( n_i = O(n/D^2) \), for each \( i \). An obvious property (which is a consequence of the generalized version of Bézout’s theorem [17]) is that each curve of \( C \) intersects at most \( ED + 1 = O(D) \) cells of \( \mathbb{R}^3 \setminus Z(f) \).

When \( a'n^{1/k} \leq m \leq an^{3/2} \) (where the left inequality holds by the induction assumption), we use, within each cell \( \tau_i \) of the partition, for \( i = 1, \ldots, u \), the naive bound

\[
I(P_i, C_i) = O(m_i n_i^{1-1/k} + n_i) = O\left(m_i (n/D^2)^{1-1/k} + n/D^2\right),
\]

and, summing over the \( O(D^3) \) cells, we get a total of

\[
O\left(\frac{mn^{1-1/k}}{D^{2(1-1/k)}} + nD\right).
\]

With the above choice of \( D \), we deduce that the total number of incidences within the cells is

\[
O\left(m^{\frac{k}{3k-2}} n^{\frac{3k-3}{3k-2}}\right).
\]

When \( m > an^{3/2} \), within each cell \( \tau_i \) of the partition we have \( n_i = O(n/D^2) = O(1) \), so the number of incidences within \( \tau_i \) is at most \( O(m_i n_i) = O(m_i) \), for a total of \( O(m) \) incidences. Putting these two alternative bounds together, we get a total of

\[
O\left(m^{\frac{k}{3k-2}} n^{\frac{3k-3}{3k-2}} + m\right) \tag{6}
\]

incidences within the cells.

**Incidences within the zero set** \( Z(f) \). It remains to bound incidences with points that lie on \( Z(f) \). Set \( P^* := P \cap Z(f) \) and \( m^* := |P^*| = m - m' \). Let \( C^* \) denote the set of curves that are fully contained in \( Z(f) \), and set \( C' := C \setminus C^* \), \( n^* := |C^*| \), and \( n' := |C'| = n - n^* \). Since every curve of \( C' \) intersects \( Z(f) \) in at most \( ED = O(D) \) points, we have (for either choice of \( D \))

\[
I(P^*, C') = O(nD) = O\left(m^{\frac{k}{3k-2}} n^{\frac{3k-3}{3k-2}} + m\right). \tag{7}
\]

Finally, we consider the number of incidences between points of \( P^* \) and curves of \( C^* \). Decompose \( f \) into (complex) irreducible factors \( f_1, \ldots, f_t \), for \( t \leq D \) (this order of the factors is arbitrary but fixed), and assign each point \( p \in P^* \) (resp., curve \( \gamma \in C^* \)) to the first irreducible factor \( f_i \), such that \( Z(f_i) \) contains \( p \) (resp., fully contains \( \gamma \); such a component always exists). The number of “cross-incidences”, between points and curves assigned to different factors, is easily seen, arguing as above, to be \( O(nD) \), which satisfies our bound. In what follows, we recycle the symbols \( m_i \) (resp., \( n_i \)), to denote the number of points (resp., curves) assigned to \( f_i \), and put \( D_i = \deg(f_i) \), for \( i = 1, \ldots, t \). We clearly have \( \sum_i m_i = |P^*| = m^* \), \( \sum_i n_i = |C^*| = n^* \), and \( \sum_i D_i \leq \deg(f) = D \).
For each \( i = 1, \ldots, t \), there are two cases to consider.

**Case 1:** \( Z(f_i) \) is infinitely ruled by curves of \( \mathcal{C}_0 \). By assumption, there are at most \( q \) curves of \( C \) on \( Z(f_i) \), implying that \( n_i \leq q \). Put \( m_{\text{inf}} := \sum_i m_i \), summed over the infinitely ruled components \( Z(f_i) \). We project the points of \( P \) and the curves of \( C_i \) onto some generic plane \( \pi_0 \). A suitable choice of \( \pi_0 \) guarantees that (i) no pair of intersection points or points of \( P \) project to the same point, (ii) if \( p \) is not incident to a curve \( \gamma \in C_i \) then the projections of \( p \) and of \( \gamma \) remain non-incident, (iii) no pair of curves in \( C_i \) have overlapping projections, and (iv) no curve of \( C_i \) contains any segment orthogonal to \( \pi_0 \).

Moreover, the number of degrees of freedom does not change in the projection, if one uses a restricted version of this notion, defined shortly (see Sharir et al. [35]). Before discussing this, we note that this statement is false for the standard definition of degrees of freedom. Consider for example the family \( \mathcal{C} \) of circles in \( \mathbb{R}^3 \), whose projections onto some plane forms the family \( \mathcal{C}^* \) of all ellipses in the plane. In this case \( \mathcal{C} \) has three degrees of freedom, but \( \mathcal{C}^* \) has five degrees of freedom. However, the property that the number of degrees of freedom does not increase in the projection holds if we define it only with respect to the projection \( P^* \) of the given point set \( P \) and the projection \( C^* \) of the given set \( C \) of curves, and assume that the projection plane is sufficiently generic. Specifically, we say that the projected set \( C^* \) of \( C \) has \( k \) degrees of freedom with respect to the given point set \( P^* \) if for any \( k \)-tuple of distinct points of \( P^* \) there are at most \( k \) curves of \( C^* \) that pass through all of them.

We claim that if the original family \( \mathcal{C}_0 \) in \( \mathbb{R}^3 \) has \( k \) degrees of freedom then the projection \( C^* \) of any finite set \( C \subset \mathcal{C}_0 \) does indeed have \( k \) degrees of freedom with respect to the projection \( P^* \) of any finite point set \( P \subset \mathbb{R}^3 \), assuming that the projection plane \( \pi_0 \) is sufficiently generic, as above. Indeed, let \( p_1^*, \ldots, p_k^* \) be a \( k \)-tuple of distinct points of \( P^* \), and let \( \gamma^* \in C^* \) be the projection of some curve \( \gamma \) of \( C \) that passes through all these points (if there is no such curve, there is nothing to argue about). For each \( j = 1, \ldots, k \), let \( p_j \) be the unique point of \( P \) that projects to \( p_j^* \). By property (ii) of the genericity of \( \pi_0 \), \( p_j \in \gamma \). Since there are at most \( k \) curves of \( C \) that pass through \( p_1, \ldots, p_k \), the claim follows. See also [35] for a related argument.

We remark that, as is easily checked, the above property also holds for the infinite family of all the curves of \( \mathcal{C}_0 \) that are contained in some fixed algebraic surface, which is the only context in which we use this property (where the surface is an infinitely ruled component of \( Z(f) \)).

The number of incidences for the points and curves assigned to the same \( Z(f_i) \) is therefore equal to the number of incidences between the projected points and curves, which, by Theorem [11] is \(^4\)

\[
O \left( m_i^{\frac{k}{2k-1}} n_i^{\frac{k-1}{2k-1}} + m_i + n_i \right) = O \left( m_i^{\frac{k}{2k-1}} n_i^{\frac{k-1}{2k-1}} q^{\frac{k-1}{2k-1}} + m_i + n_i \right).
\]

Summing over \( i = 1, \ldots, t \), and using Hölder’s inequality, we get the bound

\[
O \left( m^{\frac{k}{2k-1}} n^{\frac{k-1}{2k-1}} q^{\frac{k-1}{2k-1}} + m_{\text{inf}} + n \right),
\]

which, by making \( A \) sufficiently large, is at most

\[
A \left( m^{\frac{k}{2k-1}} n^{\frac{k-1}{2k-1}} q^{\frac{k-1}{2k-1}} + m_{\text{inf}} + n \right).
\]

**Remark.** This is the only step in the proof where being of reduced dimension \( s \), for \( s \) sufficiently small, might yield an improved bound (over the one in (8)); see below, in the follow-up proof of Theorem [15] where this observation is exploited, for details.

**Case 2:** \( Z(f_i) \) is not infinitely ruled by curves of \( \mathcal{C}_0 \). In this case, Theorem [17](b) implies that there exist suitable constants \( c_1, t \) that depend on \( E \) and on the complexity of \( \mathcal{C}_0 \), such that there are at most

\(^4\)An inspection of Pach and Sharir [32] shows that the analysis there too only requires that the given family of curves have \( k \) degrees of freedom with respect to the given point set.
$c_1D_i^2$ exceptional curves, namely, curves that contain at least $c_1D_i$ $t$-rich points. Therefore, by choosing $c$ (in the definition of $D$) sufficiently small, we can ensure that, in both cases (of medium-range $m$ and large $m$), $\sum_i D_i^2 \leq (\sum_i D_i)^2 = D^2 \leq n/(64c_1)$, say. This allows us to apply induction on the number of curves, to handle the exceptional curves.

Concretely, denote by $m_{\text{rich}}$ (resp., $m_{\text{poor}}$) the number of $t$-rich (resp., $t$-poor) points assigned to components of $Z(f)$ that are not ruled by curves of $C_0$. We thus have $m_{\text{rich}} + m_{\text{poor}} + m_{\text{inf}} = m^*$. Incidences with the $t$-poor points are easy to bound, because each such point is incident to at most $t$ curves of $C^*$ (that are assigned to the same component as the point), for a total of $tm_{\text{poor}} = O(m_{\text{poor}})$ incidences. We thus continue the analysis with the $t$-rich points only. For each $i$ (for which $Z(f_i)$ is not infinitely ruled by curves of $C_0$), we have an inductive instance of the problem involving at most $m_i t$-rich points and at most $c_1D_i^2 \leq n/64$ curves of $C$. By the induction hypothesis, the corresponding incidence bound is at most

$$A \left( m_i^{3k-\alpha_1} (c_1D_i^2)^{\frac{3k-3}{3k-2}} + m_i^{\frac{k}{2k-1}} (c_1D_i^2)^{\frac{k-1}{2k-1}} q^{\frac{k-1}{2k-1}} + m_i + c_1D_i^2 \right).$$

We now sum over $i$. For the first terms, we bound each $m_i$ by $m$, and, for the fourth terms too, use the fact that $\sum_i D_i^2 \leq D^\alpha$ for any $\alpha \geq 1$. For the third terms, we bound $\sum_i m_i$ by $m_{\text{rich}}$. For the second terms, we use Hölder’s inequality. Overall, we get the incidence bound

$$A \left( m^{3k-\alpha_1} (c_1D^{2\alpha_1})^{\frac{3k-3}{3k-2}} + m^{\frac{k}{2k-1}} (c_1D^2)^{\frac{k-1}{2k-1}} q^{\frac{k-1}{2k-1}} + m_{\text{rich}} + c_1D^2 \right) \leq A \left( m^{3k-\alpha_1} (n/64)^{\frac{3k-3}{3k-2}} + m^{\frac{k}{2k-1}} (n/64)^{\frac{k-1}{2k-1}} q^{\frac{k-1}{2k-1}} + m_{\text{rich}} + n/64 \right),$$

which can be upper bounded (since $k \geq 2$) by

$$A \frac{4}{3} \left( m^{3k-\alpha_1} n^{\frac{3k-3}{3k-2}} + m^{\frac{k}{2k-1}} n^{\frac{k-1}{2k-1}} q^{\frac{k-1}{2k-1}} + n \right) + Am_{\text{rich}}. \tag{9}$$

Except for these incidences, for each $f_i$, each non-exceptional curve in $C^*$ that is assigned to $Z(f_i)$ is incident to at most $c_1D_i$ $t$-rich points; the total number of incidences of this kind involving the $n_i$ curves assigned to $Z(f_i)$ and their incident $t$-rich points is at most $n_i c_1D_i = O(n_i D_i)$. Other incidences involving the non-exceptional curves in $C$ that are assigned to $Z(f_i)$ only involve $t$-poor points that are assigned to $Z(f_i)$; as argued, the overall number of such point-curve incidences is $O(m_{\text{poor}})$. Therefore, when $Z(f_i)$ is not infinitely ruled by curves of $C^*$, the number of incidences with the $t$-rich points assigned to $Z(f_i)$ is $O(n_i D_i)$, plus terms that are accounted for by the induction. Summing over these components $Z(f_i)$, we get the bound $O(nD)$ plus the inductive bounds in (9), and, choosing $A$ to be sufficiently large, these bounds will collectively be at most

$$A \frac{4}{3} \left( m^{3k-\alpha_1} n^{\frac{3k-3}{3k-2}} + m^{\frac{k}{2k-1}} n^{\frac{k-1}{2k-1}} q^{\frac{k-1}{2k-1}} + n \right) + Am_{\text{rich}} + m_{\text{poor}}. \tag{10}$$

Adding up all the bounds obtained so far, and choosing $A$ to be a sufficiently large constant, the number of incidences satisfies the inequality in (11), thus establishing the induction step, and thereby completing the proof.

**Proof of Theorem 1.5** The proof proceeds by the same double induction on $n$ and $m$, and establishes the bound, for any prespecified $\varepsilon > 0$,

$$I(P, C) \leq Am^{3k-\alpha_1} n^{\frac{3k-3}{3k-2}} + A_\varepsilon \left( m^{\frac{2k}{5k-4}} n^{\frac{3k-4}{5k-4}} q^{\frac{2k-2}{5k-4}} + m^{2/3} n^{1/3} q^{1/3} + m + n \right), \tag{11}$$

for a suitable constant $A$ that depends on $k$, $\mu$, $s$, $E$, and the complexity of $C_0$, and another constant $A_\varepsilon$ that also depends on $\varepsilon$. The flow of the proof is very similar to that of the preceding proof. The main
difference is in the case where some component $Z(f_i)$ of $Z(f)$ is infinitely ruled by curves from $C_0$. Again, in this case it contains at most $q$ curves of $C^*$.

We take the points of $P^*$ and the curves of $C^*$ that are assigned to $Z(f_i)$, and project them onto some generic plane $\pi_0$ (the same, sufficiently generic plane can be used for all such components), as in the proof of Theorem 1.14 and get the same properties (i)–(iv) of the projected points and curves. Let $P_i$ and $C_i$ denote, respectively, the set of projected points and the set of projected curves; the latter is a set of $n_i$ plane irreducible algebraic curves of constant maximum degree $DE$. Moreover, as in the preceding proof, the contribution of $Z(f_i)$ to $I(P^*, C^*)$ is equal to the number $I(P_i, C_i)$ of incidences between $P_i$ and $C_i$.

We can now apply Theorem 1.2 to the rest of the proof proceeds as the previous proof, more or less verbatim, except that we need a more careful (albeit straightforward) separate handling of the leading term, multiplied by $\epsilon$. The induction step then establishes the bound in (11) in much the same way as above.

Remark. (1) As already mentioned in the introduction, the “lower-order” terms

$$
O \left( m^{2s} n^{3s-4} q^{2s-4} + m^{2/3} n^{1/3} + m + n \right)
$$

in the bound are (almost) “best possible” in the following sense. If the bound in Theorem 1.2 were optimal, or nearly optimal, in the worst case, for points and curves of $C_0$ that lie in a constant-degree surface $V$ that is infinitely ruled by such curves, the same would also hold for the lower-order terms in the bound in

---

5A projection preserves irreducibility and does not increase the degree; see, e.g., Harris [26] for a reference to these facts.
Theorem 1.5. This is shown by a simple packing argument, in which we take \( n/q \) generic copies of \( V \), and place on each of them \( mq/n \) points and \( q \) curves, so as to obtain

\[
\Omega \left( (mq/n)^{2s/3} q^{5s/4} + (mq/n)^{2/3} q^{2/3} + mq/n + q \right)
\]

incidences on each copy, for a total of

\[
(n/q) \cdot \Omega \left( (mq/n)^{2s/3} q^{5s/4} + (mq/n)^{2/3} q^{2/3} + mq/n + q \right) = \Omega \left( \frac{2s}{s-k} n^{2s/3} q^{5s/4} + \frac{2s-2}{s-k} n^{1/3} q^{1/3} + m + n \right)
\]

incidences. (This construction works when \( m > n/q \). Otherwise, the bound is linear, and clearly best possible. Also, we assume that the lower bound does not involve the factor \( q^2 \), to simplify the reasoning.) In particular, this remark applies to the case of points and circles, as discussed in Theorem 1.6.

(2) There is an additional step in the proof in which the fact that \( C_0 \) is of some constant (not necessarily reduced) dimension \( s' \) could lead to an improved bound. This is the base case \( m = O(n^{1/k}) \), where we use the Kővári-Sós-Turán theorem to obtain a linear bound on \( I(P,C) \). Instead, we can use the result of Fox et al. [17 Corollary 2.3], and the fact that the incidence graph does not contain \( K_{k,\mu+1} \) as a subgraph, to show that, when \( m = O(n^{1/s'}) \), the number of incidences is linear. The problem is that here we need to use the dimension \( s' \) of the entire \( C_0 \), rather than the reduced dimension \( s \) (which, as we recall, applies only to subsets of curves of \( C_0 \) that lie on a variety that is infinitely ruled by curves of \( C_0 \)). Typically, as already noted, \( s \) is larger than \( k \) (generally twice as large as \( k \)), making this bootstrapping bound inferior to what we have. Still, in cases where \( s' \) happens to be smaller than \( k \), this would lead to a further improved incidence bounds, in which the leading term is also smaller.

**Rich points.** Theorems 1.4 and 1.5 can easily be restated as bounding the number of \( t \)-rich points for a set \( C \) of curves with \( k \) degrees of freedom (and or reduced dimension \( s \)) in \( \mathbb{R}^3 \), when \( t \) is at least some sufficiently large constant. (Here richness is defined with respect to the entire set \( C \).) The case \( t = 2 \) is treated in Guth and Zahl [25], and the same bound that they obtain holds for larger values of \( t \) (albeit without an explicit dependence on \( t \)), smaller than the threshold in the following corollary.

Corollary 2.2. (a) Let \( C \) be a set of \( n \) irreducible algebraic curves, taken from some constructible family \( C_0 \), of constant complexity, of irreducible curves of degree at most \( E \) and with \( k \) degrees of freedom (with some multiplicity \( \mu \)) in \( \mathbb{R}^3 \), and assume that no surface that is infinitely ruled by curves of \( C_0 \), or, alternatively, by curves of degree at most \( E \), contains more than \( q \) curves of \( C \) (e.g., make this assumption for all surfaces of degree at most \( 100E^2 \)). Then there exists some constant \( t_0 \), depending on \( k \) (and \( \mu \)) and on \( C_0 \), or, more generally, on \( E \), such that, for any \( t \geq t_0 \), the number of \( t \)-rich points, namely points that are incident to at least \( t \) curves of \( C \), is

\[
O \left( \frac{n^{3/2}}{t^{2\kappa-2}} + \frac{nq}{t^{2\kappa-1}} + \frac{n}{t} \right),
\]

where the constant of proportionality depends on \( k \), \( E \), \( \mu \), and the complexity of \( C_0 \).

(b) If \( C_0 \) is also of reduced dimension \( s \), the bound on the number of \( t \)-rich points becomes

\[
O \left( \frac{n^{3/2}}{t^{2\kappa-2}} + \frac{nq^{2s-2}}{t^{2s-1}} + \frac{n}{t} \right),
\]

where the constant of proportionality now also depends on \( s \) and \( \varepsilon \). (Actually, the first and third terms come with a constant that is independent of \( \varepsilon \).)
Proof. Denoting by \( m_t \) the number of \( t \)-rich points, the corollary is obtained by combining the upper bound in Theorem 1.4 or Theorem 1.5 with the lower bound \( tm_t \).

The bound in (b) is an improvement, for \( s = k \), when \( q > t^{k+\varepsilon'} \), for another arbitrarily small parameter \( \varepsilon' > 0 \), which is linear in the prespecified \( \varepsilon \). (To be more precise, this is an improvement at all only when the second term dominates the bound.)

It would be interesting to close the gap, by obtaining a \( t \)-dependent bound also for values of \( t \) between 3 and \( t_0 \). It does not seem that the technique in Guth and Zahl [25] extends to this setup.

3 Incidences between points and circles and similar triangles in \( \mathbb{R}^3 \)

We first briefly discuss the fairly straightforward proof of Theorem 1.6. As already mentioned in the introduction, we have \( k = s = 3 \), for the case of circles, so we can apply Theorem 1.5 in the context of circles, and obtain the bound

\[
I(P, C) = O \left( m^{3/7} n^{6/7} + m^{2/3} n^{1/3} q^{1/3} + m^{6/11} n^{5/11} q^{4/11 + \varepsilon} + m + n \right),
\]

for any \( \varepsilon > 0 \), where \( q \) is the maximum number of the given circles that are coplanar or cospherical. In fact, the extension of the planar bound [1] to higher dimensions, due to Aronov et al. [4], asserts that, for any set \( C \) of circles in any dimension, we have

\[
I(P, C) = O \left( m^{2/3} n^{2/3} + m^{6/11} n^{9/11} \log^{2/11} (m^3 / n) + m + n \right),
\]

which is slightly better than the general bound of Sharir and Zahl [11] (given in Theorem 1.2). If we use this bound, instead of that in Theorem 1.5 in the proof of Theorem 1.6 (specialized for the case of circles), we get the slight improvement (in which the two constants of proportionality are now absolute)

\[
I(P, C) = O \left( m^{3/7} n^{6/7} + m^{2/3} n^{1/3} q^{1/3} + m^{6/11} n^{5/11} q^{4/11} \log^{2/11} (m^3 / q) + m + n \right),
\]

which establishes Theorem 1.6.

The number of similar triangles. Theorem 1.6 has the following interesting application. Let \( P \) be a set of \( n \) points in \( \mathbb{R}^3 \), and let \( \Delta = abc \) be a fixed given triangle. The goal is to bound the number, denoted as \( S_{\Delta}(P) \), of triangles spanned by \( P \) and similar to \( \Delta \). The best known upper bound for \( S_{\Delta}(P) \), obtained by Agarwal et al. [1], is \( O(n^{13/6}) \), and the proof that establishes this bound in [1] is fairly involved. Using Theorem 1.6, we obtain the following simple and fairly straightforward improvement.

Theorem 3.1. \( S_{\Delta}(P) = O(n^{15/7}) \).

Proof. Following a standard strategy, fix a pair \( p, q \) of points in \( P \), and consider the locus \( \gamma_{pq} \) of all points \( r \) such that the triangle \( pqr \) is similar to \( \Delta \) (when \( p, q, r \) correspond to \( a, b, c \), respectively). Clearly, \( \gamma_{pq} \) is a circle whose axis (line passing through the center of \( \gamma_{pq} \) and perpendicular to its supporting plane) passes through \( p \) and \( q \). Moreover, there exist at most two (ordered) pairs \( p, q \) and \( p', q' \) for which \( \gamma_{pq} = \gamma_{p'q'} \). Let \( C \) denote the set of all these circles (counted without multiplicity). Then \( S_{\Delta}(P) \) is at most two thirds of the number \( I(P, C) \) of incidences between the \( n \) points of \( P \) and the \( N = O(n^2) \) circles of \( C \).

By Theorem 1.6 we thus have

\[
S_{\Delta}(P) = O \left( n^{3/7} (n^2)^{6/7} + n^{2/3} (n^2)^{1/3} q^{1/3} + n^{6/11} (n^2)^{5/11} q^{4/11} \log^{2/11} n + n^2 \right),
\]
where \( q \) is the maximum number of circles in \( C \) that are either coplanar or cospherical. That is, we have

\[
S_\Delta(P) = O \left( n^{15/7} + n^{4/3} q^{1/3} + n^{16/11} q^{4/11} \log^{2/11} n + n^2 \right). \tag{13}
\]

We claim that \( q = O(n) \). This is easy for coplanarity, because, for any fixed plane \( \pi \), each point \( p \in P \) can generate at most one circle \( \gamma_{pq} \) in \( C \) that is contained in \( \pi \). Indeed, the axis of such a circle is perpendicular to \( \pi \) and passes through \( p \). This fixes the center of \( \gamma_{pq} \) (on \( \pi \)), and it is easily checked that the radius is also fixed. A similar argument holds for cospherical circles. Here too, for a fixed sphere \( \sigma \), each point \( p \in P \) that is not the center \( o \) of \( \sigma \) can generate at most two circles \( \gamma_{pq} \) in \( C \) that are contained in \( \sigma \). This is because the axis of such a circle must pass through \( o \), which fixes the axis \( \lambda \). The circle is then the intersection of \( \sigma \) with the cone with apex \( p \) and axis \( \lambda \) whose generators form angle \( \angle bac \) with \( \lambda \). Since this cone intersects \( \sigma \) in at most two circles, the claim follows. For \( p = o \) there are at most \( n - 1 \) additional such circles.

Hence, plugging \( q = O(n) \) into (13), we get \( S_\Delta(P) = O(n^{15/7}) \), as asserted. \( \square \)

We remark that Zahl’s recent bound [46] on point-circle incidences in \( \mathbb{R}^3 \), as reviewed in the introduction, yields the weaker bound \( O(n^{12/5}) \).

4 Discussion

In this paper we have made significant progress on incidence problems involving points and fairly general families of algebraic curves in three dimensions. The study in this paper raises several interesting open problems.

(i) As remarked above, a challenging open problem is to characterize all the surfaces that are infinitely ruled by algebraic curves of degree at most \( E \) (or by certain classes thereof), extending the known characterizations for lines and circles. A weaker, albeit still hard problem is to reduce the upper bound \( 100E^2 \) on the degree of such a surface, perhaps all the way down to \( E + 1 \), or at least to \( O(E) \).

(ii) It would also be interesting to find additional applications of the results of this paper, like the one with an improved bound on the number of similar triangles in \( \mathbb{R}^3 \), given in Section 3. One direction to look at is the analysis of other repeated patterns in a point set, such as higher-dimensional congruent or similar simplices, which can sometimes be reduced to point-sphere incidence problems; see [1, 3].

As already mentioned, the results of this paper have recently been used in Sharir et al. [42] for bounding incidences between points and curves with almost two degrees of freedom.

(iii) A potentially weak issue in our analysis, manifested in the proof of our main theorems, is that in order to bound the number of incidences between points and curves on some variety \( V \) of constant degree, we project the points and curves on some generic plane and use a suitable planar bound, from Theorem 1.1 or Theorem 1.2 to bound the number of incidences between the projected points and curves. It would be very interesting if one could obtain an improved bound, exploiting the fact that the points and curves lie on a variety \( V \) in \( \mathbb{R}^3 \), under suitable (natural) assumptions on \( V \). Note that we only need to apply this argument for surfaces \( V \) that are infinitely ruled by the family \( C_0 \) of the given curves. Perhaps this restricted setup could aid in improving the analysis.

(iv) Finally, it would be challenging to extend the results of this paper to higher dimensions, extending the result by the authors in [37], obtained for the case of lines, to more general families of curves.
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**A On surfaces ruled by curves**

In this appendix we review, and sketch the proofs, of several tools from algebraic geometry that are required in our analysis, the main one of which is Theorem 1.7. These tools are presented in Guth and Zahl [25, Section 6], but we reproduce them here, in a somewhat sketchy form, for the convenience of the reader and in the interest of completeness. We work over the complex field $\mathbb{C}$, but the results here also apply to our setting over the real numbers (see [25, 37] and a preceding remark for discussions of this issue).

A subset of $\mathbb{C}^N$ described by some polynomial equalities and one non-equality, of the form

$$\{p \in \mathbb{C}^N \mid f_1(p) = 0, \ldots, f_r(p) = 0, g(p) \neq 0\}, \quad \text{for } f_1, \ldots, f_r, g \in \mathbb{C}[x_1, \ldots, x_N],$$

is called locally closed. We recall that the (geometric) degree of an algebraic variety $V \subset \mathbb{C}^N$ is defined as the number of intersection points of $V$ with the intersection of $N - \dim(V)$ hyperplanes in general position (see, e.g., Harris [26, Definition 18.1]). Locally closed sets have the following property.

**Theorem A.1** (Bézout’s inequality; Bürgisser et al. [8, Theorem 8.28]). Let $V$ be a nonempty locally closed set in $\mathbb{C}^N$, and let $H_1, \ldots, H_r$ be algebraic hypersurfaces in $\mathbb{C}^N$. Then

$$\deg(V \cap H_1 \cap \cdots \cap H_r) \leq \deg(V) \cdot \deg(H_1) \cdots \deg(H_r).$$
A constructible set $C$ is easily seen to be a union of locally closed sets. Moreover, one can decompose $C$ uniquely as the union of irreducible locally closed sets (namely, sets that cannot be written as the union of two nonempty and distinct locally closed sets). By Bürgisser et al. \[8, \text{Definition 8.23}\], the degree of $C$ is the sum of the degrees of its irreducible locally closed components. Theorem \[A.1\] implies that when a constructible set $C$ has complexity $O(1)$, its degree is also $O(1)$. We also have the following corollaries.

**Corollary A.2.** Let

$$X = \{ p \in \mathbb{C}^N \mid f_1(p) = 0, \ldots, f_r(p) = 0, g(p) \neq 0 \}, \quad \text{for } f_1, \ldots, f_r, g \in \mathbb{C}[x_1, \ldots, x_N].$$

If $X$ contains more than $\deg(f_1) \cdots \deg(f_r)$ points, then $X$ is infinite.

**Proof.** Assume that $X$ is finite. Put $V = \{ p \in \mathbb{C}^N \mid g(p) \neq 0 \}$. By Bézout’s inequality (Theorem \[A.1\]), we have

$$\deg(X) \leq \deg(V) \cdot \deg(f_1) \cdots \deg(f_r) = \deg(f_1) \cdots \deg(f_r),$$

where the equality $\deg(V) = 1$ follows by the definition of the degree of locally closed sets (see, e.g., Bürgisser et al. \[8, \text{Definition 8.23}\]). When $X$ is finite, i.e., zero-dimensional, its degree is equal to the number of points in it, counted with multiplicities. This implies that the number of points in $X$ is at most $\deg(f_1) \cdots \deg(f_r)$, contradicting the assumption of the theorem. Therefore, $X$ is infinite. \[\square\]

As an immediate consequence, we also have:

**Corollary A.3.** Let $C \subset \mathbb{C}^N$ be a constructible set and write it as the union of locally closed sets $\bigcup_{i=1}^t X_i$, where

$$X_i = \{ p \in \mathbb{C}^N \mid f_1^i(p) = 0, \ldots, f_{r_i}^i(p) = 0, g^i(p) \neq 0 \}, \quad \text{for } f_1^i, \ldots, f_{r_i}^i, g^i \in \mathbb{C}[x_1, \ldots, x_N].$$

If $C$ contains more than $\sum_{i=1}^t \deg(f_1^i) \cdots \deg(f_{r_i}^i)$ points, then $C$ is infinite.

For a constructible set $C$, let $d(C)$ denote the minimum of $\sum_{i=1}^t \deg(f_1^i) \cdots \deg(f_{r_i}^i)$, as in Corollary \[A.3\] over all possible decompositions of $C$ as the union of locally closed sets. By Bézout’s inequality (Theorem \[A.1\]), it follows that $\deg(C) \leq d(C)$. Corollary \[A.3\] implies that if $C$ contains more than $d(C)$ points, then it is infinite.

Following Guth and Zahl \[25, \text{Section 4}\], we call an algebraic curve $\gamma \subset \mathbb{C}^3$ a **complete intersection** if $\gamma = Z(P, Q)$ for some pair of polynomials $P, Q$. We let $\mathbb{C}[x, y, z]_{\leq E}$ denote the space of complex trivariate polynomials of degree at most $E$, and choose an identification of $\mathbb{C}[x, y, z]_{\leq E}$ with $\mathbb{C}(E^3_3)$. We use the variable $\alpha$ to denote an element of $(\mathbb{C}[x, y, z]_{\leq E})^2$, and write

$$\alpha = (P_\alpha, Q_\alpha) \in (\mathbb{C}[x, y, z]_{\leq E})^2 = (\mathbb{C}(E^3_3))^2.$$

Given an irreducible curve $\gamma$, we associate with it a choice of $\alpha \in (\mathbb{C}(E^3_3))^2$ such that $\gamma$ is contained in $Z(P_\alpha, Q_\alpha)$, and the latter is a curve (one can show that such an $\alpha$ always exists; see Guth and Zahl \[25, \text{Lemma 4.2}\] and also Basu and Sombra \[6\]). Let $x \in \gamma$ be a non-singular point of $\gamma$; we say that $\alpha$ is associated to $\gamma$ at $x$, if $\alpha$ is associated to $\gamma$, and $\nabla P_\alpha(x)$ and $\nabla Q_\alpha(x)$ are linearly independent. We refer the reader to Guth and Zahl \[25, \text{Definition 4.1 and Lemma 4.2}\] for details. This is analogous to the works of Guth and Zahl.\footnote{Here $(E^3_3)$ is the maximum number of monomials of the polynomials that we consider. For obvious reasons, the actual representation should be in the complex projective space $\mathbb{C}P(E^3_3)$, but we use the many-to-one representation in $\mathbb{C}(E^3_3)$ for convenience.} In particular, we get a one-to-one correspondence between the set of points in $\gamma$ and the set of polynomials $f_1, f_2$ that vanish on $\gamma$ such that $\nabla f_1(x)$ and $\nabla f_2(x)$ are linearly independent.\footnote{Given an irreducible curve in $\mathbb{R}^3$, a point $x \in \gamma$ is non-singular if there are polynomials $f_1, f_2$ that vanish on $\gamma$ such that $\nabla f_1(x)$ and $\nabla f_2(x)$ are linearly independent.}
Katz [24] and of Sharir and Solomon [37] for the special cases of parameterizing lines in three and four dimensions, respectively.

In what follows, we fix a constructible set $C_0 \subset C_{3,E}$ of irreducible curves of degree at most $E$ in 3-dimensional space (recall that the entire family $C_{3,E}$ is constructible). Following [25] Section 9, we call a point $p \in Z(f)$, for a given polynomial $f \in \mathbb{C}[x,y,z]$, a $(t,C_0,r)$-flecnode, if there are at least $t$ curves $\gamma_1, \ldots, \gamma_t \in C_0$, such that, for each $i = 1, \ldots, t$, (i) $\gamma_i$ is incident to $p$, (ii) $p$ is a non-singular point of $\gamma_i$, and (iii) $\gamma_i$ osculates to $Z(f)$ to order $r$ at $p$. This is a generalization of the notion of a flecnode point, due to Salmon [33, Chapter XVII, Section III] (see also [24, 37] for details).

With all this machinery, we can now present a (sketchy) proof of Theorem 1.7. The theorem is stated in Section 11, and we recall it here. It is adapted from Guth and Zahl [25, Corollary 10.2], serves as a generalization of the Cayley–Salmon theorem on surfaces ruled by lines (see, e.g., Guth and Katz [24]), and is closely related to Theorem 1.3 (also due to Guth and Zahl [25]). We only consider here the first part of the theorem, as the second part is an easy consequence.

**Theorem 1.7.** For given integer parameters $c,E$, there are constants $c_1 = c_1(c,E)$, $r = r(c,E)$, and $t = t(c,E)$, such that the following holds. Let $f$ be a complex irreducible polynomial of degree $D \gg E$, and let $C_0 \subset C_{3,E}$ be a constructible set of complexity at most $c$. If there exist at least $c_1 D^2$ curves of $C_0$, such that each of them is contained in $Z(f)$ and contains at least $c_1 D$ points on $Z(f)$ that are $(t,C_0,r)$-flecnodes, then $Z(f)$ is infinitely ruled by curves from $C_0$. In particular, if $Z(f)$ is not infinitely ruled by curves from $C_0$ then, except for at most $c_1 D^2$ exceptional curves, every curve in $C_0$ that is fully contained in $Z(f)$ is incident to at most $c_1 D$ $t$-rich points (points that are incident to at least $t$ curves in $C_0$ that are also fully contained in $Z(f)$).

**Proof.** For the time being, let $r$ be arbitrary. By Guth and Zahl [25, Lemma 8.3 and Equation (8.1)], since $f$ is irreducible, there exist $r$ polynomials $h_j(\alpha,p) \in \mathbb{C}[\alpha,x,y,z]$, for $j = 1, \ldots, r$, of degree at most $b_j$ in $\alpha$ (where $b_j$ is a constant depending on $j$ and on $E$), and of degree $O(D)$ in $p = (x,y,z)$, with the following property: let $\gamma$ be an irreducible curve, let $p$ be a non-singular point of $\gamma$, and let $\alpha$ be associated to $\gamma$ at $p$, then $\gamma$ osculates to $Z(f)$ to order $r$ at $p$ if and only if $h_j(\alpha,p) = 0$ for $j = 1, \ldots, r$. (These polynomials are suitable representations of the first $r$ terms of the Taylor expansion of $f$ at $p$ along $\gamma$; see [25, Section 6.2] for this definition, and also [24, 37] for the special cases of lines in $\mathbb{R}^2$ and $\mathbb{R}^4$, respectively.)

Regarding $p$ as fixed, the system $h_j(\alpha,p) = 0$, for $j = 1, \ldots, r$, in conjunction with the constructible condition that $\alpha \in C_0$, defines a constructible set $C_\alpha$. By definition, we have $d(C_\alpha) \leq \left( \prod_{j=1}^{r} b_j \right) \cdot d(C_0)$, which is a constant that depends only on $r$ and $E$. By Corollary A.3, $C_\alpha$ is either infinite or contains at most $d(C_\alpha) = O(1)$ points. By Guth and Zahl [25, Corollary 12.1], there exist a Zariski open set $\mathcal{O}$, and a sufficiently large constant $r_0$, that depend on $C_\alpha$ and $E$ (recall that $E$ is also assumed to be a constant, and see [25, Theorem 8.1] for the way $r_0$ is obtained), such that if $p \in \mathcal{O}$ is a $(t,C_0,r)$-flecnode, with $r \geq r_0$, there are at least $t$ curves that are incident to $p$ and are fully contained in $Z(f)$. Since, by assumption, there are at least $c_1 D^2$ curves, each containing at least $c_1 D$ $(t,C_0,r)$-flecnodes, it follows from [25, Proposition 10.2] that there exists a Zariski open subset $\mathcal{O}$ of $Z(f)$, all of whose points are $(t,C_0,r)$-flecnodes. As noted above, [25, Corollary 12.1] then implies that every point of $\mathcal{O}$ is incident to at least $t$ curves of degree at most $E$ that are fully contained in $Z(f)$. As observed above, when $t \geq \left( \prod_{j=1}^{r} b_j \right) \cdot d(C_0)$, a constant depending only on $C_0$ and $E$, $Z(f)$ is infinitely ruled on this Zariski open set. By a simple argument (a variant of which is given in [40, Lemma 6.1]), we can conclude that $Z(f)$ is infinitely ruled by curves from $C_0$, thus completing the proof. \(\Box\)

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9To say that $h_j$ is a polynomial in $\alpha$ (and $p$) means that it is a polynomial in the $2^{(E+3)}$ coefficients of the monomials of the two polynomials in the pair $\alpha$ (and in the coordinates $(x,y,z)$ of $p$).

10For this proposition we require that $r$, and therefore $E$ too, be considered as constants compared with $D$. 

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