GRAPH C*-ALGEBRAS WITH A T₁ PRIMITIVE IDEAL SPACE

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Abstract. We give necessary and sufficient conditions which a graph should satisfy in order for its associated C*-algebra to have a T₁ primitive ideal space. We give a description of which one-point sets in such a primitive ideal space are open, and use this to prove that any purely infinite graph C*-algebra with a T₁ (in particular Hausdorff) primitive ideal space is a c₀-direct sum of Kirchberg algebras. Moreover, we show that graph C*-algebras with a T₁ primitive ideal space canonically may be given the structure of a C(𝕋)-algebra, and that isomorphisms of their 𝕋-filtered K-theory (without coefficients) lift to E(𝕋)-equivalences, as defined by Dadarlat and Meyer.

1. Introduction

When classifying non-simple C*-algebras a lot of focus has been on C*-algebras with finitely many ideals. However, Dadarlat and Meyer recently proved in [2] a Universal Multicoefficient Theorem in equivariant E-theory for separable C*-algebras over second countable, zero-dimensional, compact Hausdorff spaces. In particular, together with the strong classification result of Kirchberg [7], this shows that any separable, nuclear, Cₘ-infinite C*-algebra with a zero-dimensional, compact Hausdorff primitive ideal space, for which all simple subquotients are in the classical bootstrap class, is strongly classified by its filtered total K-theory. This suggests and motivates the study of C*-algebras with infinitely many ideals, in the eyes of classification.

In this paper we consider graph C*-algebras with a T₁ primitive ideal space, i.e. a primitive ideal space in which every one-point set is closed. Clearly our main interest are such graph C*-algebras with infinitely many ideals, since any finite T₁ space is discrete. In Section 2 we recall the definition of graph C*-algebras and many of the related basic concepts. In particular, we give a complete description of the primitive ideal space of a graph C*-algebra. In Section 3 we find necessary and sufficient condition which a graph should satisfy in order for the induced C*-algebra to have a T₁ primitive ideal space. In Section 4 we prove that a lot of subsets of such primitive ideal spaces are both closed and open. In particular, we give a complete description of when one-point sets are open. We use this to show that any purely infinite graph C*-algebra with a T₁ primitive ideal space is a c₀-direct sum of Kirchberg algebras. Moreover, we show that any graph C*-algebra with a T₁ primitive ideal space may be given a canonical structure of a (not necessarily continuous) C(𝕋)-algebra, where 𝕋 is the one-point compactification of 𝕋. As an

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ending remark, we prove that $\tilde{n}$-filtered $K$-theory classifies these $C(\tilde{n})$-algebras up to $E(\tilde{n})$-equivalence, as defined by Dadarlat and Meyer in [2].

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2. Preliminaries

We recall the definition of a graph $C^*$-algebra and many related definitions and properties. Let $E = (E^0, E^1, r, s)$ be a countable directed graph, i.e. a graph with countably many vertices $E^0$, countably many edges $E^1$ and a range and source map $r, s: E^1 \to E^0$ respectively. A vertex $v \in E^0$ is called a sink if $s^{-1}(v) = \emptyset$ and an infinite emitter if $|s^{-1}(v)| = \infty$. A graph with no infinite emitters is called row-finite.

We define the graph $C^*$-algebra of $E$, $C^*(E)$, to be the universal $C^*$-algebra generated by a family of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries with mutually orthogonal ranges $\{s_e : e \in E^1\}$, subject to the following Cuntz-Krieger relations

1. $s^*_es_e = p_{r(e)}$ for $e \in E^1$,
2. $s_es^*_e \leq p_{s(e)}$ for $e \in E^1$,
3. $p_v = \sum_{e \in s^{-1}(v)} s_es^*_e$ for $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$.

By universality there is a gauge action $\gamma : \mathbb{T} \to \text{Aut}(C^*(E))$ such that $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = zs_e$ for $v \in E^0, e \in E^1$ and $z \in \mathbb{T}$. An ideal in $C^*(E)$ is said to be gauge-invariant if it is invariant under $\gamma$. All ideals are assumed to be two-sided and closed.

If $\alpha_1, \ldots, \alpha_n$ are edges such that $r(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \ldots, n-1$, then we say that $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a path, with source $s(\alpha) = s(\alpha_1)$ and range $r(\alpha) = r(\alpha_n)$. A loop is a path of positive length such that the source and range coincide, and this vertex is called the base of the loop. A loop $\alpha$ is said to have an exit, if there exist $e \in E^1$ and $i = 1, \ldots, n$ such that $s(e) = s(\alpha_i)$ but $e \neq \alpha_i$. A loop $\alpha$ is called simple if $s(\alpha_i) \neq s(\alpha_j)$ for $i \neq j$. A graph $E$ is said to have condition (K) if each vertex $v \in E^0$ is the base of no (simple) loop or is the base of at least two simple loops. It turns out that a graph $E$ has condition (K) if and only if every ideal in $C^*(E)$ is gauge-invariant if and only if $C^*(E)$ has real rank zero.

For $v, w \in E^0$ we write $v \geq w$ if there is a path $\alpha$ with $s(\alpha) = v$ and $r(\alpha) = w$. A subset $H$ of $E^0$ is called hereditary if $v \geq w$ and $v \in H$ implies that $w \in H$. A subset $H$ of $E^0$ is called saturated if whenever $v \in E^0$ satisfies $0 < |s^{-1}(v)| < \infty$ and $r(s^{-1}(v)) \subseteq H$ then $v \in H$. If $X$ is a subset of $E^0$ then we let $\Sigma H(X)$ denote the smallest hereditary and saturated set containing $X$. If $H$ is hereditary and saturated we define

$$
H^\infty_{\text{fin}} = \{ v \in E^0 \setminus H : |s^{-1}(v)| = \infty \text{ and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty \},
$$
$$
H^0_{\infty} = \{ v \in E^0 \setminus H : |s^{-1}(v)| = \infty \text{ and } s^{-1}(v) \cap r^{-1}(E^0 \setminus H) = \emptyset \}.
$$

By [1] Theorem 3.6 there is a one-to-one correspondence between pairs $(H, B)$, where $H \subseteq E^0$ is hereditary and saturated and $B \subseteq H^\infty_{\text{fin}}$, and the gauge-invariant ideals of $C^*(E)$. In fact, this is a lattice isomorphism when the different sets are given certain lattice structures. The ideal corresponding to $(H, B)$ is denoted $J_{H,B}$ and if $B = \emptyset$ we denote it by $J_H$. 
A non-empty subset $M \subseteq E^0$ is called a maximal tail if the following three conditions are satisfied.

1. If $v \in E^0$, $w \in M$ and $v \geq w$ then $v \in M$.
2. If $v \in M$ and $0 < |s^{-1}(v)| < \infty$ then there exists $e \in E^1$ such that $s(e) = v$ and $r(e) \in M$.
3. For every $v, w \in M$ there exists $y \in M$ such that $v \geq y$ and $w \geq y$.

Note that $E^0 \setminus M$ is hereditary by (1) and saturated by (2). Moreover, by (3) it follows that $(E^0 \setminus M)^\infty_\infty$ is either empty or consists of exactly one vertex. We let $\mathcal{M}(E)$ denote the set of all maximal tails in $E$, and let $\mathcal{M}_\gamma(E)$ denote the set of all maximal tails $M$ in $E$ such that each loop in $M$ has an exit in $M$. We let $\mathcal{M}_\tau(E) = \mathcal{M}(E) \setminus \mathcal{M}_\gamma(E)$.

If $X \subseteq E^0$ then define

$$\Omega(X) = \{w \in E^0 \setminus X : w \not\geq v \text{ for all } v \in X\}.$$ 

Note that if $M$ is a maximal tail, then $\Omega(M) = E^0 \setminus M$. For a vertex $v \in E^0$, $E \setminus \Omega(v)$ is a maximal tail if and only if $v$ is a sink, an infinite emitter or if $v$ is the base of a loop.

We define the set of breaking vertices to be

$$BV(E) = \{v \in E^0 : |s^{-1}(v)| = \infty \text{ and } 0 < |s^{-1}(v)\setminus r^{-1}(\Omega(v))| < \infty\}.$$ 

Hence an infinite emitter $v$ is a breaking vertex if and only if $v \in \Omega(v)^\infty_\infty$.

In [6] they define for each $N \in \mathcal{M}_\tau(E)$ and $t \in \mathbb{T}$ a (primitive) ideal $R_{N,t}$, and prove that there is a bijection

$$\mathcal{M}_\gamma(E) \cup BV(E) \cup (\mathcal{M}_\tau(E) \times \mathbb{T}) \to \text{PrimC}^*(E)$$

given by

$$\mathcal{M}_\gamma(E) \ni M \mapsto J_{\Omega(M), \Omega(M)^\infty_\infty} \quad BV(E) \ni v \mapsto J_{\Omega(v), \Omega(v)^\infty_\infty \setminus \{v\}}$$

and

$$\mathcal{M}_\tau(E) \times \mathbb{T} \ni (N, t) \mapsto R_{N,t}.$$ 

In [6], Hong and Szymański give a complete description of the hull-kernel topology on $\text{PrimC}^*(E)$ in terms of the maximal tails and breaking vertices. In order to describe this we use the following notation. Whenever $M \in \mathcal{M}_\tau(E)$ there is a unique (up to cyclic permutation) simple loop in $M$ which generates $M$, and we denote by $L^0_M$ the set of all vertices in this. If $Y \subseteq \mathcal{M}_\tau(E)$ we let

$$Y_{\text{min}} := \{U \in Y : \text{for all } U' \in Y, U' \neq U \text{ there is no path from } L^0_U \text{ to } L^0_{U'}\},$$
$$Y_{\infty} := \{U \in Y : \text{for all } V \in Y_{\text{min}} \text{ there is no path from } L^0_V \text{ to } L^0_U\}.$$ 

Due to a minor mistake in [6] the description of the topology is however not entirely correct. We will give a correct description below and explain what goes wrong in the original proof in Remark [11].

**Theorem 1** (Hong-Szymański). Let $E$ be a countable directed graph. Let $X \subseteq \mathcal{M}_\gamma(E), W \subseteq BV(E), Y \subseteq \mathcal{M}_\tau(E)$, and let $D(U) \subseteq \mathbb{T}$ for each $U \in Y$. If $M \in \mathcal{M}_\gamma(E), v \in BV(E), N \in \mathcal{M}_\tau(E)$, and $z \in \mathbb{T}$, then the following hold.

1. $M \subseteq X$ if and only if one of the following three conditions holds.
   
   (i) $M \subseteq X$,
   (ii) $M \subseteq \bigcup X$ and $\Omega(M)^\infty_\infty = \emptyset$,
   (iii) $M \subseteq \bigcup X$ and $|s^{-1}(\Omega(M)^\infty_\infty) \cap r^{-1}(\bigcup X)| = \infty$.  

However, this is not the case. If both \( w / \) are infinite then \( J_v / \) \( \Omega( ) = 1 \) \( \end{array} \), and every Hausdorff space is a \( T_1 \) space. For a \( C^* \)-algebra \( A \) the primitive ideal space \( \text{Prim}A \) is \( T_1 \) exactly if every primitive ideal is a maximal ideal. All of our ideals are assumed to be two-sided and closed.

As shown in [1], every gauge-invariant primitive ideal of a graph \( C^* \)-algebra may be represented by a maximal tail or by a breaking vertex. The following lemma shows that we only need to consider maximal tails.

Recall that a topological space is said to satisfy the separation axiom \( T_1 \) if every one-point set is closed. In particular, every Hausdorff space is a \( T_1 \) space. For a \( C^* \)-algebra \( A \) the primitive ideal space \( \text{Prim}A \) is \( T_1 \) exactly if every primitive ideal is a maximal ideal. All of our ideals are assumed to be two-sided and closed.

As shown in [1], every gauge-invariant primitive ideal of a graph \( C^* \)-algebra may be represented by a maximal tail or by a breaking vertex. The following lemma shows that we only need to consider maximal tails.

\[
(2) \quad v \in X \text{ if and only if } v \in \bigcup X \text{ and } |s^{-1}(v) \cap r^{-1}(\bigcup X)| = \infty.
\]

\[
(3) \quad (N, z) \in X \text{ if and only if } N \subseteq \bigcup X.
\]

\[
(4) \quad M \in \mathbb{W} \text{ if and only if either}
\]

\[
(i) \quad M \subseteq \mathcal{E}_0 \backslash \bigcap_{w \in W} \Omega(0) \text{ and } \Omega(M)_{0} = \emptyset, \text{ or}
\]

\[
(ii) \quad M \subseteq \mathcal{E}_0 \backslash \bigcap_{w \in W} \Omega(0) \text{ and } |s^{-1}(\Omega(M)_{0}) \cap r^{-1}(\mathcal{E}_0 \backslash \bigcap_{w \in W} \Omega(0))| = \infty,
\]

\[
(5) \quad v \in \mathbb{W} \text{ if and only if either}
\]

\[
(i) \quad v \in \mathcal{W}, \text{ or}
\]

\[
(ii) \quad v \in \mathcal{E}_0 \backslash \bigcap_{w \in W} \Omega(w) \text{ and } |s^{-1}(v) \cap r^{-1}(\mathcal{E}_0 \backslash \bigcap_{w \in W} \Omega(w))| = \infty.
\]

\[
(6) \quad (N, z) \in \mathbb{W} \text{ if and only if } N \subseteq \mathcal{E}_0 \backslash \bigcap_{w \in W} \Omega(w).
\]

\[
(7) \quad M \text{ is in the closure of } \{(U, t) : U \in Y, t \in D(U)\} \text{ if and only if one of the following four conditions holds.}
\]

\[
(i) \quad M \subseteq \bigcup Y_{\infty} \text{ and } \Omega(M)_{\infty} = \emptyset,
\]

\[
(ii) \quad M \subseteq \bigcup Y_{\infty} \text{ and } |s^{-1}(\Omega(M))_{\infty} \cap r^{-1}(\bigcup Y_{\infty})| = \infty,
\]

\[
(iii) \quad M \subseteq \bigcup Y_{\min} \text{ and } \Omega(M)_{\infty} = \emptyset,
\]

\[
(iv) \quad M \subseteq \bigcup Y_{\min} \text{ and } |s^{-1}(\Omega(M))_{\infty} \cap r^{-1}(\bigcup Y_{\min})| = \infty.
\]

\[
(8) \quad v \text{ is in the closure of } \{(U, t) : U \in Y, t \in D(U)\} \text{ if and only if either}
\]

\[
(i) \quad v \in \bigcup Y_{\infty} \text{ and } |s^{-1}(v) \cap r^{-1}(\bigcup Y_{\infty})| = \infty, \text{ or}
\]

\[
(ii) \quad v \in \bigcup Y_{\min} \text{ and } |s^{-1}(v) \cap r^{-1}(\bigcup Y_{\min})| = \infty.
\]

\[
(9) \quad (N, z) \text{ is in the closure of } \{(U, t) : U \in Y, t \in D(U)\} \text{ if and only one of the following three conditions holds.}
\]

\[
(i) \quad N \subseteq \bigcup Y_{\infty},
\]

\[
(ii) \quad N \notin Y_{\min} \text{ and } N \subseteq \bigcup Y_{\min},
\]

\[
(iii) \quad N \in Y_{\min} \text{ and } z \in D(N).
\]

**Remark 1.** The minor mistake in the original proof of Theorem 1 is an error which occurs in the proofs of Lemma 3.3 and Theorem 3.4 of [1]. We will explain what goes wrong. Suppose that \( M \) is a maximal tail, \( K \) is a hereditary and saturated set such that \( K \subseteq \Omega(M) \), and that \( B \subseteq K_{\infty} \). Note that \( B' \Omega(M)_{\infty} \subseteq \Omega(M) \cup \Omega(M)_{\infty} \). Hence if \( w \in \Omega(M)_{\infty} \) then \( J_{K,B} \subseteq J_{\Omega(M),\Omega(M)_{\infty}} \) if and only \( w \notin B \), since \( w \notin \Omega(M) \cup \Omega(M)_{\infty} \). In the cases we consider we have that \( w \in B \) if and only if \( w \in K_{\infty} \). Now it is claimed that \( w \notin K_{\infty} \) if and only \( \bigcap s^{-1}(w) \cap r^{-1}(K) \) is finite. However, this is not the case. If both \( s^{-1}(w) \cap r^{-1}(K) \) and \( s^{-1}(w) \cap r^{-1}(E_0 \backslash K) \) are infinite then \( w \notin K_{\infty} \). The correct statement would be that \( w \notin K_{\infty} \) if and

\[
(3) \quad T_1 \text{ primitive ideal space}
\]

Recall that a topological space is said to satisfy the separation axiom \( T_1 \) if every one-point set is closed. In particular, every Hausdorff space is a \( T_1 \) space. For a \( C^* \)-algebra \( A \) the primitive ideal space \( \text{Prim}A \) is \( T_1 \) exactly if every primitive ideal is a maximal ideal. All of our ideals are assumed to be two-sided and closed.

As shown in [1], every gauge-invariant primitive ideal of a graph \( C^* \)-algebra may be represented by a maximal tail or by a breaking vertex. The following lemma shows that we only need to consider maximal tails.
**Lemma 1.** Let $E$ be a graph such that $\text{Prim}(C^*(E))$ is $T_1$. Then $E$ has no breaking vertices.

*Proof.* Suppose $E$ has a breaking vertex $v$. Then

$$J_{\Omega(v),\Omega(v)^{fin}_\infty \setminus \{v\}} \text{ and } J_{\Omega(v),\Omega(v)^{fin}_\infty}$$

are primitive ideals of $C^*(E)$, the former being a proper ideal of the latter by [1, Corollary 3.10]. Hence

$$J_{\Omega(v),\Omega(v)^{fin}_\infty} \in \{J_{\Omega(v),\Omega(v)^{fin}_\infty \setminus \{v\}}\}$$

and thus $C^*(E)$ cannot have a $T_1$ primitive ideal space. $\square$

It turns out that it might be helpful to consider gauge-invariant ideals which are maximal in the following sense.

**Definition 1.** Let $E$ be a countable directed graph and let $J$ be a proper ideal of $C^*(E)$. We say that $J$ is a maximal gauge-invariant ideal if $J$ is gauge-invariant and if $J$ and $C^*(E)$ are the only gauge-invariant ideals containing $J$.

The following theorem gives a complete description of the graphs whose induced $C^*$-algebras have a $T_1$ primitive ideal space.

**Theorem 2.** Let $E$ be a countable directed graph. The following are equivalent.

1. $C^*(E)$ has a $T_1$ primitive ideal space,
2. $E$ has no breaking vertices, and whenever $M$ and $N$ are maximal tails such that $M$ is a proper subset of $N$, then $\Omega(M)_{\infty}^0$ is non-empty, and

$$|s^{-1}(\Omega(M)_{\infty}^0) \cap r^{-1}(N)| < \infty,$$

3. $E$ has no breaking vertices, and $J_{\Omega(M),\Omega(M)_{\infty}^{fin}}$ is a maximal gauge-invariant ideal in $C^*(E)$ for any maximal tail $M$,
4. $E$ has no breaking vertices, and the map $M \mapsto J_{\Omega(M),\Omega(M)_{\infty}^{fin}}$ is a bijective map from the set of maximal tails of $E$ onto the set of all maximal gauge-invariant ideals of $C^*(E)$.

The last condition in (2) of the theorem may look complicated but it is easy to describe. It says, that if $M \subset N$ are maximal tails then $M$ must contain an infinite emitter $v$ which only emits edges out of $M$, and only emits finitely many edges to $N$. Note that this is equivalent to $v \in \Omega(N)_{\infty}^{fin}$.

*Proof.* We start by proving (1) $\iff$ (2). By Lemma 1 we may restrict to the case where $E$ has no breaking vertices. The proof is just a translation of Theorem 1 into our setting. We have four cases.

Case 1: Let $M, N \in \mathcal{M}_r(E)$. By Theorem 1 we have $M \in \{N\}$ if and only if one of the following three holds: (i) $M = N$, (ii) $M \subset N$ and $\Omega(M)_{\infty}^0 = \emptyset$, (iii) $M \subset N$, $\Omega(M)_{\infty}^0 \neq \emptyset$ and

$$|s^{-1}(\Omega(M)_{\infty}^0) \cap r^{-1}(N)| = \infty.$$

We eliminate the possibilities (ii) and (iii) exactly by imposing the conditions in (2).

Case 2: Let $(M, z) \in \mathcal{M}_r(E) \times \mathbb{T}$ and $N \in \mathcal{M}_r(E)$. By Theorem 1 (4) $(M, z) \in \{N\}$ if and only if $M \subset N$. Since $M \in \mathcal{M}_r(E)$ it follows that $\Omega(M)_{\infty}^0 = \emptyset$ and thus the conditions in (2) says $M \nsubseteq N$. 

Case 3: Let \((N, t) \in \mathcal{M}(E) \times T\) and \(M \in \mathcal{M}(E)\). Note that \(\{N\}_{\min} = \{N\}\) and \(\{N\}_{\infty} = \emptyset\). By Theorem 17 we have \(M \in \{(N, t)\}\) if and only if one of the following two holds: (i) \(M \subseteq N\) and \(\Omega(M)_{\infty}^0 = \emptyset\), (ii) \(M \subseteq N, \Omega(M)_{\infty}^0 \neq \emptyset\) and 

\[\left| s^{-1}(\Omega(M)_{\infty}^0) \cap \tau^{-1}(N) \right| = \infty.\]

Conditions (i) and (ii) do not hold exactly when assuming the conditions of (2).

Case 4: Let \((M, z) \in \mathcal{M}(E) \times T\). By Theorem 19 we have \((M, z) \in \{(N, t)\}\) if and only if either \(M \subseteq N\) or \(M = N\) and \(z = t\). Note that condition 9(i) of the theorem can never be satisfied. Since the maximal tail \(M\) satisfies \(\Omega(M)_{\infty}^0 = \emptyset\) the conditions of (2) say \(M \subseteq N\) if and only if \(M = N\) thus finishing (1) \(\iff\) (2).

We will prove (1) \(\Rightarrow\) (3). In order to simplify matters, we replace \(E\) with its desingularisation \(F\) (see [4]) thus obtaining a row-finite graph without sinks. Since \(E\) has no breaking vertices by Lemma 1 there is a canonical one-to-one correspondence between \(\mathcal{M}(E)\) and \(\mathcal{M}(F)\) and a lattice isomorphism between the ideal lattices of \(C^*(E)\) and \(C^*(F)\) such that \(M' \mapsto M\) implies \(J_{\Omega(M') \Omega(M')_{\infty}^0} \mapsto J_{\Omega(M)}\). In this case \(J_{\Omega(M') \Omega(M')_{\infty}^0}\) is a maximal gauge-invariant ideal if and only if \(J_{\Omega(M)}\) is a maximal gauge-invariant ideal and thus it suffices to prove that \(J_{\Omega(M)}\) is a maximal gauge-invariant ideal in \(C^*(F)\) for \(M \in \mathcal{M}(F)\).

Suppose \(J_{\Omega(M)} \subseteq J_H\) for some hereditary and saturated set \(H \neq F^0\). Since \(F\) is row-finite without sinks we may find an infinite path \(\alpha\) in \(F\setminus H\). Let 

\[N = \{v \in F: v \geq s(\alpha_j)\text{ for some } j\}\]

which is a maximal tail such that \(N \subseteq F^0\setminus H\). Hence \(\Omega(M) \subseteq H \subseteq \Omega(N)\) which implies \(N \subseteq M\). Since \(F\) is row-finite, \(\Omega(N)_{\infty}^0\) is empty, and thus since (1) \(\iff\) (2), \(M = N\). Hence \(H = \Omega(M)\) and thus (1) \(\Rightarrow\) (3).

We will prove (3) \(\Rightarrow\) (4). Again, we let \(F\) be the desingularisation of \(E\) and note that (4) holds for \(F\) if and only if it holds for \(E\). Note that (3) implies that the map in (4) is well-defined, and this is clearly injective. Let \(H\) be a hereditary and saturated set in \(F\) such that \(J_H\) is a maximal gauge-invariant ideal in \(C^*(F)\). As above, we may find a maximal tail \(M\) such that \(H \subseteq \Omega(M)\) which implies \(J_H \subseteq J_{\Omega(M)}\). Since \(J_H\) is a maximal gauge-invariant ideal, \(H = \Omega(M)\) which proves surjectivity of the map and finishes (3) \(\Rightarrow\) (4).

For (4) \(\Rightarrow\) (1) we may again replace \(E\) by its desingularisation \(F\). Since (1) \(\iff\) (2) and \(F\) is row-finite, (1) is equivalent to the following: if \(M \subseteq N\) are maximal tails then \(M = N\), since \(\Omega(M)_{\infty}^0 = \emptyset\) for every maximal tail \(M\). Let \(M \subseteq N\) be maximal tails in \(F\). Then \(J_{\Omega(N)} \subseteq J_{\Omega(M)}\) are maximal gauge-invariant ideals and thus \(N = M\), which finishes the proof.

\[\square\]

**Definition 2.** Let \(E\) be a countable directed graph. If \(E\) satisfies one (and hence all) of the conditions in Theorem 2 then we say that \(E\) is a \(T_1\) graph.

For row-finite graphs the above theorem simplifies significantly.

**Corollary 1.** Let \(E\) be a row-finite graph. The following are equivalent.

1. \(E\) is a \(T_1\) graph,
2. if \(M \subseteq N\) are maximal tails, then \(M = N\),
3. \(J_{\Omega(M)}\) is a maximal gauge-invariant ideal in \(C^*(E)\) for any maximal tail \(M\),
(4) the map $M \mapsto J_{\Omega(M)}$ is a bijective map from the set of maximal tails of $E$ onto the set of all maximal gauge-invariant ideals of $C^*(E)$.

Proof. Since $E$ is row-finite it has no breaking vertices and $\Omega(M)^0_\infty$ is empty for any maximal tail $M$. Hence it follows from Theorem 2. \hfill \Box

We will end this section by constructing a class of graph $C^*$-algebras, all of which have a non-discrete $T_1$ primitive ideal space.

**Example 1.** Let $B$ be a simple AF-algebra and let $F$ be a Bratteli diagram of $B$ as in [3], such that the vertex set $F^0$ is partitioned into vertex sets $F^0_n = \{w^1_n, \ldots, w^k_n\}$ and every edge with a source in $F^0_n$ has range in $F^0_{n+1}$. Let $G_1, G_2, \ldots$ be a sequence of graphs all of which have no non-trivial hereditary and saturated sets. Construct a graph $E$ as follows:

$$
E^0 = F^0 \cup \bigcup_{n=1}^{\infty} G^0_n,
$$

$$
E^1 = F^1 \cup \bigcup_{n=1}^{\infty} G^1_n \cup \bigcup_{n=1}^{\infty} \{e^1_n, \ldots, e^k_n\}
$$

where the range and source maps do not change on $F^1 \cup \bigcup_{n=1}^{\infty} G^1_n$ and where $s(e^i_n) = w^i_n$ and $r(e^i_n) \in G^0_n$.

Using that $F$ and each $G_n$ have no non-trivial hereditary and saturated sets we get that the maximal tails of $E$ are

$$
M_n = \bigcup_{k=1}^{n} F^0_k \cup G^0_n,
$$

$$
M_\infty = \bigcup_{k=1}^{\infty} F^0_k = F^0.
$$

Hence no maximal tail is contained in another and thus the primitive ideal space of $C^*(E)$ is $T_1$. For any of these maximal tails $M$ each vertex in $M$ emits only finitely many edges to $\Omega(M)$ and thus $\Omega(M^\text{fin}_\infty)$ is empty. The quotients $C^*(E)/J_{\Omega(M_n)}$ are Morita equivalent $C^*(G_n)$ and $C^*(E)/J_{\Omega(M_\infty)} = C^*(F)$ which is Morita equivalent to $B$.

If, in addition, each $G_n$ has condition (K) then one can verify that $\text{Prim} C^*(E)$ is homeomorphic to $\tilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, the one-point compactification of $\mathbb{N}$. Such a homeomorphism may be given by

$$
\tilde{\mathbb{N}} \ni n \mapsto J_{\Omega(M_n)} \in \text{Prim} C^*(E).
$$

4. Clopen maximal gauge-invariant ideals

Whenever a subset of a topological space is both closed and open, then we say that the set is clopen. In this section we give a description of which one-point sets in the primitive ideal space of a $T_1$ graph are clopen. In fact, we describe which maximal gauge-invariant ideals in the primitive ideal space correspond to clopen sets. We use this description to show that every purely infinite graph $C^*$-algebra with a $T_1$ primitive ideal space is a $c_0$-direct sum of Kirchberg algebras. Moreover, we prove that graph $C^*$-algebras with a $T_1$ primitive ideal space are canonically
C(\(\hat{\mathbb{N}}\))-algebras, which are classified up to \(E(\hat{\mathbb{N}})\)-equivalence by their \(\hat{\mathbb{N}}\)-filtered \(K\)-theory.

In order to describe the clopen maximal gauge-invariant ideals, we need a notion of when a maximal tail distinguishes itself from all other maximal ideals in a certain way.

**Definition 3.** Let \(E\) be a \(T_1\) graph and let \(M\) be a maximal tail in \(E\). We say that \(M\) is isolated if either

1. \(M\) contains a vertex which is not contained in any other maximal tail, or
2. \(\Omega(M)^\emptyset\) is non-empty and

\[
|s^{-1}(\Omega(M)^\emptyset) \cap r^{-1}\left(\bigcup_{N \in \mathcal{M}(E)} N\right)| < \infty.
\]

where \(\mathcal{M}(E)\) denotes the set of all maximal tails \(N\) such that \(M \subseteq N\).

This definition may look strange but it turns out that a maximal tail corresponds to a clopen maximal gauge-invariant ideal if and only if it is isolated, see Theorem 3.

**Remark 2.** For a row-finite \(T_1\) graph \(E\) the above definition simplifies, since \(\Omega(M)^\emptyset\) is empty for any maximal tail \(M\). Hence, in this case, a maximal tail is isolated if and only if it contains a vertex which is not contained in any other maximal tail.

**Example 2.** Consider the two graphs

\[
\begin{array}{ccc}
 & w & \\
v_1 & \downarrow & \downarrow \\
v_2 & \rightarrow & \rightarrow \\
v_3 & \rightarrow & \rightarrow \\
& \ldots \\
\end{array}
\]

The latter graph is the desingulisation of the former but without changing sinks to tails. The maximal tails of the former graph are given by \(N_n = \{w, v_n\}\) and \(N_\infty = \{w\}\). The maximal tails of the latter graph are

\[
M_n = \{w_1, \ldots, w_n, v_n\},
\]

\[
M_\infty = \{w_1, w_2, \ldots\}.
\]

Hence both graphs are easily seen to be \(T_1\) graphs. All the maximal tails \(N_n\) and \(M_n\) for \(n \in \mathbb{N}\) are easily seen to be isolated, and by Remark 2, \(M_\infty\) is not isolated. Since \(\Omega(N_\infty) = \{w\}\) and

\[
|s^{-1}(\Omega(N_\infty)^\emptyset) \cap r^{-1}\left(\bigcup_{N \in \mathcal{M}(E)_{\infty}} N\right)| = \infty
\]

we note that \(N_\infty\) is not isolated. In fact, by Corollary 4 below, \(N_\infty\) would be isolated if and only if \(M_\infty\) was isolated.

The latter graph is an example of a graph in Example 1, with \(B = \mathbb{C}\) and each \(G_n\) consisting of one vertex and no edges. Since the graph has condition (K), the primitive ideal space is homeomorphic to \(\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}\), the one-point compactification of \(\mathbb{N}\), by the map

\[
\hat{\mathbb{N}} \ni n \mapsto J_{\Omega(M_n)}.
\]
It turns out that many maximal tails are isolated, as can be seen in the following lemma.

**Lemma 2.** Let $E$ be a $T_1$ graph and let $M$ be a maximal tail which contains a sink or a loop. Then $M$ is isolated.

**Proof.** Let $v \in M$ be the sink or the base of a loop in $M$, and note that $\Omega(v)_0 = \emptyset$. If $N$ is a maximal tail such that $v \in N$ then $E^0 \setminus \Omega(v) \subseteq N$ and since $\Omega(v)_0$ is empty, $N = E^0 \setminus \Omega(v)$ by Theorem 2. Hence $v$ is not contained in any other maximal tail than $M$ and thus $M$ is isolated. □

The following is the main theorem of this section, mainly due to all the corollaries following it.

**Theorem 3.** Let $E$ be a countable directed graph for which the primitive ideal space of $C^*(E)$ is $T_1$, and let $M$ be a maximal tail in $E$. Then

$$\{ p \in \text{Prim}C^*(E) : J_{\Omega(M),\Omega(M)_0^I} \subseteq p \} \subseteq \text{Prim}C^*(E)$$

is a clopen set if and only if $M$ is isolated.

In particular, if $M \in \mathcal{M}_r(E)$, then the one-point set

$$\{ J_{\Omega(M),\Omega(M)_0^I} \} \subseteq \text{Prim}C^*(E)$$

is clopen if and only if $M$ is isolated, and if $M \in \mathcal{M}_r(E)$ then

$$\{ R_{M,t} : t \in \mathbb{T} \} \subseteq \text{Prim}C^*(E)$$

is a clopen set homeomorphic to the circle $S^1$.

**Proof.** To ease notation define

$$U_M := \{ p \in \text{Prim}C^*(E) : J_{\Omega(M)} \subseteq p \}.$$

By definition $U_M$ is closed. By [6, Lemma 2.6] it follows that if $J$ is a gauge-invariant ideal, $M \in \mathcal{M}_r(E)$ and $t \in \mathbb{T}$, then $J \subseteq R_{M,t}$ if and only if $J \subseteq J_{\Omega(M),\Omega(M)_0^I}$. We will use this fact several times throughout the proof, without mentioning it.

Suppose $U_M$ is clopen. If $M \in \mathcal{M}_r(E)$ then $M$ contains a loop and is thus isolated by Lemma 2. Hence we may suppose $M \in \mathcal{M}_r(E)$ for which $U_M = \{ J_{\Omega(M),\Omega(M)_0^I} \}$. Since $U_M$ is open there is a unique ideal $J$ such that

$$\{ J_{\Omega(M),\Omega(M)_0^I} \} = \{ p \in \text{Prim}C^*(E) : J \not\subseteq p \}.$$

Suppose $J$ is not gauge-invariant. Then we can find a $z \in \mathbb{T}$ such that $\gamma_z(J) \neq J$. Note that $\gamma_z(J) \not\subseteq \gamma_z(J_{\Omega(M),\Omega(M)_0^I}) = J_{\Omega(M),\Omega(M)_0^I}$. For any primitive ideal $p \in \text{Prim}C^*(E) \setminus \{ J_{\Omega(M),\Omega(M)_0^I} \}$, we have $\gamma_z(J) \subseteq \gamma_z(p)$, since $J \subseteq p$. Since $\gamma_z$ fixes $J_{\Omega(M),\Omega(M)_0^I}$ it induces a bijection from $\text{Prim}C^*(E) \setminus \{ J_{\Omega(M),\Omega(M)_0^I} \}$ to itself and thus $\gamma_z(J) \not\subseteq p$ for any primitive ideal $p \neq J_{\Omega(M),\Omega(M)_0^I}$. However, this contradicts the uniqueness of $J$, and thus $J$ must be gauge-invariant.

Since $J$ is gauge-invariant, $J = J_{H,B}$ for a hereditary and saturated set $H$ and $B \subseteq \Omega_0^I$. If $H \not\subseteq \Omega(M)$ then any vertex $v \in H$ such that $v \in M$ is not contained in any other maximal tail, since $J_{H,B} \subseteq J_{\Omega(N),\Omega(N)_0^I}$ for any maximal tail $N \neq M$. Hence we may restrict to the case where $H \subseteq \Omega(M)$. Since $J_{H,B} \not\subseteq J_{\Omega(M),\Omega(M)_0^I}$, $B \not\subseteq \Omega(M) \cup \Omega(M)_0^I$. It is easily observed that $B \setminus \Omega(M)_0^I \subseteq \Omega(M) \cup \Omega(M)_0^I$. Thus $B \setminus \Omega(M)_0^I = \Omega(M)$ and hence follows that $\Omega(M)_0^I = \{ w \}$ for some vertex $w$ and that $w \in B$. Recall that $\mathcal{M}(E) = \{ N \in \mathcal{M}(E) : M \subseteq N \}$. Since $w \in \Omega_0^I$, we have

$$|s^{-1}(w) \cap r^{-1}(E^0 \setminus H)| < \infty.$$
and since $\bigcup_{N \in \mathcal{M}(E)} N \subseteq E^0 \setminus H$ it follows that

$$|s^{-1}(w) \cap r^{-1}(\bigcup_{N \in \mathcal{M}(E)} N)| < \infty.$$  

Thus $M$ is isolated.

Now suppose that $M$ is an isolated maximal tail. If $M$ contains a vertex $v$ which is not contained in any other maximal tail, then $J_{\Sigma H(v)} \not\subseteq J_{\Omega(M),\Omega(M)_{\infty}^0}$ and $J_{\Sigma H(v)} \subseteq J_{\Omega(N),\Omega(N)_{\infty}^0}$ for any maximal tail $N \neq M$. Hence

$$U_M = \{ p \in \text{Prim}^*(E) : J_{\Sigma H(v)} \not\subseteq p \}$$

and thus $U_M$ is clopen. Now suppose that every vertex of $M$ is contained in some other maximal tail. Let $H = \bigcap_{N \in \mathcal{M}(E)} \Omega(N)$ which is hereditary and saturated. Since $M$ is isolated, $\Omega(M)_{\infty}^0 = \{ w \}$ for some vertex $w$ and moreover $w \in H_{\infty}^\infty$. Hence $J_{H,\{w\}} \not\subseteq J_{\Omega(M),\Omega(M)_{\infty}^0}$ and $J_{H,\{w\}} \subseteq J_{\Omega(N),\Omega(N)_{\infty}^0}$ for any $N \in \mathcal{M}(E)$ by Theorem 2. Now, as above, $J_{\Sigma H(w)} \not\subseteq J_{\Omega(N),\Omega(N)_{\infty}^0}$ for any $N \notin \mathcal{M}(E)$ and $J_{\Sigma H(w)} \subseteq J_{\Omega(N),\Omega(N)_{\infty}^0}$ for $N \in \mathcal{M}(E)$. Hence

$$U_M = \{ p : J_{H,\{w\}} \not\subseteq p \} \cap \{ p : J_{\Sigma H(w)} \not\subseteq p \}$$

is the intersection of two open sets, and is thus clopen.

For the ‘in particular’ part note that if $M \in \mathcal{M}_r(E)$ then $U_M = \{ J_{\Omega(M),\Omega(M)_{\infty}^0} \}$. If $M \in \mathcal{M}_r(E)$ then $M$ contains a loop and is thus isolated by Lemma 2. Hence

$$U_M = \{ R_{M,t} : t \in \mathbb{T} \}$$

is clopen. By Theorem 1 it follows that this set is homeomorphic to the circle $S^1$. \hfill \Box

**Corollary 2.** Let $E$ be a $T_1$ graph and $p \in \text{Prim}^*(E)$ be a primitive ideal. Then $\{ p \}$ is clopen if and only if $p = J_{\Omega(M),\Omega(M)_{\infty}^0}$ for an isolated maximal tail $M \in \mathcal{M}_r(E)$.

**Corollary 3.** Let $E$ be a $T_1$ graph and suppose that every maximal tail in $E$ is isolated. Then

$$\text{Prim}^*(E) \cong \bigcup_{M \in \mathcal{M}_r} \bigstar \bigcup_{M \in \mathcal{M}_r} S^1$$

is a disjoint union, where $\bigstar$ is a one-point topological space and $S^1$ is the circle.

In particular, if $E$ in addition has condition (K) then $\text{Prim}^*(E)$ is discrete.

If two graphs $E$ and $F$ have Morita equivalent $C^*$-algebras, then the corresponding ideal lattices are canonically isomorphic. Hence, if $E$ and $F$ have no breaking vertices, there is an induced one-to-one correspondence between the maximal tails in $E$ and $F$. The following corollary is immediate from Theorem 3.

**Corollary 4.** Let $E$ and $F$ be $T_1$ graphs such that $C^*(E)$ and $C^*(F)$ are Morita equivalent. Then a maximal tail in $E$ is isolated if and only if the corresponding maximal tail in $F$ is isolated.

Our main application of the above theorem is the following corollary.

**Corollary 5.** Any purely infinite graph $C^*$-algebra with a $T_1$ (in particular Hausdorff) primitive ideal space is isomorphic to a $c_0$-direct sum of Kirchberg algebras.
Thus the smallest ideal containing all $J$ gets the following result. Let $\text{Remark 3.}$ By an analogous argument as given in the proof of Corollary 6, we which is not contained in any other maximal tail. Hence $J$ is the base of a loop, is in $V$ contained in exactly one maximal tail. For any isolated maximal tail $A$ summand in $T$ primitive ideal space is $A/J$ contained in a loop. Then $\text{Remark 4.}$ Let $\text{Corollary 6.}$ Let $\text{Proof.}$ Note that the ideal is well-defined by Theorem 3, since $A/J$ is an AF-algebra. Hence we may assume that there is a row-finite graph $E$ such that $C^*(E) = A$. Let $V$ denote the set of all vertices which are contained in exactly one maximal tail. For any isolated maximal tail $M$, the direct summand in $A$ which corresponds to $A/J_{\Omega(M)}$ is $J_{\Sigma H(v)}$ where $v$ is any vertex in $M$ which is not contained in any other maximal tail. Hence $J = J_{\Sigma H(V)}$ since this is the smallest ideal containing all $J_{\Sigma H(v)}$ for $v \in V$. By Lemma 2 any vertex which is the base of a loop, is in $V$. Hence the graph $E \setminus \Sigma H(V)$ contains no loops and thus $A/J = C^*(E \setminus \Sigma H(V))$ is an AF-algebra. 

\text{Remark 3.} By an analogous argument as given in the proof of Corollary 6 we get the following result. Let $A$ be a real rank zero graph $C^*$-algebra for which the primitive ideal space is $T_1$. Then $A$ contains a (unique) purely infinite ideal $J$ such that $A/J$ is an AF-algebra.

In fact, we could define $V$ in the proof of Corollary 6 to be the set of all vertices which are the base of some loop. Then $J = J_{\Sigma H(V)}$ would be the direct sum of all simple purely infinite ideals in $A$, and $A/J$ would again be an AF-algebra.

\text{Remark 4.} Let $\tilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$. We may give any graph $C^*$-algebra $A$ with a $T_1$ primitive ideal space a canonical structure of a $C(\tilde{\mathbb{N}})$-algebra. In fact, list all of the direct summands in $A$ corresponding to $A/J_{\Omega(M),\Omega(M)}^{\infty}$ for $M$ an isolated maximal tail, as $J_1, J_2, \ldots$. By letting

$$A(\{n\}) = J_n, \text{ and } A(\{n, n+1, \ldots, \infty\}) = A/\bigoplus_{k=1}^{n-1} J_k,$$

then $A$ gets the structure of a $C^*$-algebra over $\tilde{\mathbb{N}}$ which is the same as a (not necessarily continuous) $C(\tilde{\mathbb{N}})$-algebra (see e.g. [5]). This structure is unique up to an automorphism functor $\sigma_\ast$ on $\mathcal{C}^\ast\text{-alg}(\tilde{\mathbb{N}})$, the category of $C(\tilde{\mathbb{N}})$-algebras, where $\sigma : \tilde{\mathbb{N}} \to \tilde{\mathbb{N}}$ is a homeomorphism. Moreover, by Corollary 6 the fibre $A_\infty$ is an AF-algebra.

Using the structure of a $C(\tilde{\mathbb{N}})$-algebra we may construct an $\tilde{\mathbb{N}}$-filtered $K$-theory functor as in [2]. In fact, let $C(\mathbb{N}, \mathbb{Z})$ be the ring of locally constant maps $\mathbb{N} \to \mathbb{Z}$. If $A$ is a $C(\tilde{\mathbb{N}})$-algebra then the $K$-theory $K_n(A)$ has the natural structure as a $\mathbb{Z}/2$-graded $C(\mathbb{N}, \mathbb{Z})$-module. Similarly, let $\Lambda$ be the ring of B"ockstein operation, and let $C(\tilde{\mathbb{N}}, \Lambda)$ be the ring of locally constant maps $\tilde{\mathbb{N}} \to \Lambda$. If $A$ is a $C(\tilde{\mathbb{N}})$-algebra
then the total $K$-theory $K(A)$ has the natural structure as a $C(\mathbb{N}, \Lambda)$-module. It is this latter invariant, that Dadarlat and Meyer proved a UMCT for.

We end this paper by showing that for $T_1$ graph $C^*$-algebras given $C(\mathbb{N})$-algebra structures as in Remark 4 an isomorphism of $\mathbb{N}$-filtered $K$-theory (without coefficients) lifts to an $E(\mathbb{N})$-equivalence. Note that this is not true in general by [2, Example 6.14].

**Proposition 1.** Let $A$ and $B$ be graph $C^*$-algebras with $T_1$ primitive ideal spaces, and suppose that these have the structure of $C(\mathbb{N})$-algebras as in Remark 4 Then $K_*(A) \cong K_*(B)$ as $\mathbb{Z}/2$-graded $C(\mathbb{N}, \mathbb{Z})$-modules if and only if $A$ and $B$ are $KK(\mathbb{N})$-equivalent.

In addition, if $A$ and $B$ are continuous $C(\mathbb{N})$-algebras, then $K_*(A) \cong K_*(B)$ as $\mathbb{Z}/2$-graded $C(\mathbb{N}, \mathbb{Z})$-modules if and only if $A$ and $B$ are $KK(\mathbb{N})$-equivalent.

**Proof.** Clearly an $E(\mathbb{N})$-equivalence induces an isomorphism of $\mathbb{N}$-filtered $K$-theory. Suppose that $\phi = (\phi_0, \phi_1): K_*(A) \to K_*(B)$ is an isomorphism of $\mathbb{Z}/2$-graded $C(\mathbb{N}, \mathbb{Z})$-modules. By the UMCT of Dadarlat and Meyer, [2, Theorem 6.11], it suffices to lift $\phi$ to an isomorphism of $\mathbb{N}$-filtered total $K$-theory. Since the $K_1$-groups are free, $K_0(D; \mathbb{Z}/n) = K_0(D) \otimes \mathbb{Z}/n$ for $D \in \{A, B\}$. Hence define

$$\phi^n_0 = \phi_0 \otimes \text{id } /n: K_0(A; \mathbb{Z}/n) \to K_0(B; \mathbb{Z}/n)$$

which are isomorphisms for each $n \in \mathbb{N}$. Since the fibres $A_\infty$ and $B_\infty$ are AF-algebras by Corollary [3] $K_1(A_\infty; \mathbb{Z}/n) = K_1(B_\infty; \mathbb{Z}/n) = 0$ for each $n \in \mathbb{N}$. Since the map $K_0(D; \mathbb{Z}/n) \to K_0(D_\infty; \mathbb{Z}/n)$ is clearly surjective, and $K_1(D_\infty; \mathbb{Z}/n) = 0$, it follows by six-term exactness that

$$K_1(D; \mathbb{Z}/n) \cong K_1(D(\mathbb{N}); \mathbb{Z}/n) \cong \bigoplus_{k \in \mathbb{N}} K_1(D_k; \mathbb{Z}/n)$$

for $D \in \{A, B\}$ and $n \in \mathbb{N}$. Since $\phi_*: K_*(A) \to K_*(B)$ is an isomorphism of $\mathbb{Z}/2$-graded $C(\mathbb{N}, \mathbb{Z})$-modules, $\phi_*$ restricts to an isomorphism $\phi_{*,k}: K_*(A_k) \to K_*(B_k)$ for each $k \in \mathbb{N}$. Lift these to isomorphisms of the total $K$-theory $\phi_{*,k}: K(A) \to K(B_k)$. Now define the group isomorphisms $\phi_0: K_0(A) \to K_0(B)$ to be the isomorphism induced by $\phi_0$ and each $\phi^n_0$, and $\phi_1: K_1(A) \to K_1(B)$ to be the composition

$$K_1(A) \cong \bigoplus_{k \in \mathbb{N}} K_1(A_k) \xrightarrow{\bigoplus \phi^n_1} \bigoplus_{k \in \mathbb{N}} K_1(B_k) \cong K_1(B),$$

where $K_1(A) = K_1(D) \oplus \bigoplus_{n \in \mathbb{N}} K_1(D; \mathbb{Z}/n)$. It is straight forward to check that $\phi = (\phi_0, \phi_1): K(A) \to K(B)$ is an isomorphism of $C(\mathbb{N}, \Lambda)$-modules.

If $A$ and $B$ are continuous $C(\mathbb{N})$-algebras then $E(\mathbb{N})$- and $KK(\mathbb{N})$-theory agree by [2, Theorem 5.4].

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