On the list chromatic index of graphs of
tree-width 3 and maximum degree at least 7

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Abstract

Among other results, it is shown that 3-trees are \( \Delta \)-edge-choosable and
that graphs of tree-width 3 and maximum degree at least 7 are \( \Delta \)-edge-
choosable.

1 Introduction

In this paper we analyse the list chromatic index, \( \chi'(G) \), of simple graphs \( G \) of
tree-width 3 and high maximum degree. Tree-width and path-width are in some
sense measures of how much a graph resembles a tree and a path respectively (see
Section 2 for a proper definition). Our main results are:

**Theorem 1.** Let \( G \) be graph with maximum degree \( \Delta \). It holds that \( \chi'(G) = \Delta \) if \( G \) has

1. tree-width at most 3 and \( \Delta \geq 7 \),
2. path-width at most 3 and \( \Delta \geq 6 \) or
3. path-width at most 4 and \( \Delta \geq 10 \).

A 3-tree is an edge-maximal graph of tree-width 3.

**Theorem 2.** Let \( G \) be a 3-tree with chromatic index \( \chi'(G) \); then \( \chi'(G) = \chi'(G) \).

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1.1 The list colouring conjecture

List colourings are a generalisation of colourings introduced independently by Vizing [14] and Erdős, Rubin and Taylor [5] in the seventies. An instance of list edge-colouring consists of a graph \( G \) and an assignment of lists \( L : E(G) \to \mathcal{P}(\mathbb{N}) \) that maps the edges of \( G \) to lists of colours \( L(e) \). A function \( C : E(G) \to \mathbb{N} \) is called an \( L \)-edge-colouring of \( G \), if \( C(e) \in L(e) \) for each \( e \in E(G) \) and no two adjacent edges receive the same colour. \( G \) is said to be \( k \)-edge-choosable, if for each assignment of lists \( L \), where each list has a size of at least \( k \), there is an \( L \)-edge-colouring of \( G \). The list chromatic index, denoted by \( \chi'(G) \), is the smallest integer \( k \) for which \( G \) is \( k \)-edge-choosable. The following conjecture is one of the central open problems in the field of list colouring.

**Conjecture 1** (List edge-colouring conjecture). For all graphs \( G \) it holds that \( \chi'(G) = \Delta(G) \).

While still open in general, Conjecture 1 has been verified asymptotically by Kahn [9] and also for some particular families of graphs: Galvin proved that \( \chi'(G) = \Delta(G) \) for all bipartite graphs \( G \) [6]. Borodin et al. used this to show that \( \chi'(G) = \Delta(G) \), if the maximum average degree of \( G \) is at most \( \sqrt{2\Delta} \) [1]. Ellingham and Goddyn [4] used a method of Alon and Tarsi to show that that every \( d \)-regular planar graph is \( d \)-edge-choosable. In 1999 Juvan, Mohar and Thomas showed that series-parallel graphs are \( \Delta \)-edge-choosable [8]. This family can also be characterised in terms of tree-width. Series-parallel graphs are exactly the graphs of tree-width at most 2. Bruhn and Meeks suggested that similar ideas could be applied to graphs of tree-width 3 and verified the list colouring conjecture for graphs of path-width at most 3 and maximum degree at least 6 [personal communication in 2012].

The list colouring conjecture has been verified mostly for classes of graphs whose elements have chromatic index \( \Delta(G) \). But this does not hold for graphs of tree-width 3, see Figure 1.1. However, Nakano, Nishizeki and Zhou [15] provided the following result.

**Theorem 3.** Let \( G \) be a graph of tree-width \( k \) and \( \Delta(G) \geq 2k \); then \( \chi'(G) = \Delta(G) \).
Bearing this in mind we focused our research on graphs of tree-width 3 and high maximum degree.

1.2 The approach

Let $G$ be a graph with a subset of edges $F \subseteq E(G)$ and an assignment of lists $L$ to the edges of $G$. For an $L$-edge-colouring $C$ of $G - F$ we call a colour $c$ of the list of an uncoloured edge $e \in F$ available, if no edge adjacent to $e$ has already been coloured with $c$. The set of available colours of the edge $e$ is called list of remaining colours and denoted by $L^c(e)$.

Here is an outline of the proof of Theorem 1. Let a graph $G$ of tree-width 3 and high maximum degree $\Delta$ be given and lists of colours $L$, each of size at least $\Delta$, be assigned to its edges. The tree-width will be used in combination with the maximum degree to locate a suitable substructure in $G$ that consists of edges $F$. We then pursue a Vizing-like approach. More precisely we use an inductive argument to find an $L$-edge-colouring $C$ of the graph $G - F$. Thus in order to extend $C$ to an $L$-edge-colouring of $G$ we have to colour the edges $F$ from the lists of remaining colours $L^c$. We will prove the first two items of Theorem 1 this way. In the proof of the third item we will have to find an $L$-edge-colouring $C$ with certain properties. This is feasible by colouring an auxiliary graph $G^*$ of tree-width 3 and maximum degree $\Delta$. At that point it will important that $G^*$ is $\Delta$-edge-colourable, which is asserted by Theorem 3 if $\Delta$ is at least 6.

The methods presented are used in [10] to prove a list version of Vizing’s theorem for graphs of tree-width 3 and to verify the list colouring conjecture for Halin graphs. The rest of the paper is organised as follows. In Section 2 we will locate certain substructures for which we will solve the related instances of list edge colouring in Section 3. In Section 4 we will combine these efforts to give proofs of the main results.

2 Finding substructures

In this section we will identify some substructures that will arise within the graphs of our interest.

2.1 Bounded tree-width

For a graph $G$ a tree decomposition $(T, \mathcal{V})$ consists of a tree $T$ and a collection $\mathcal{V} = \{V_t : t \in V(T)\}$ of bags $V_t \subseteq V(G)$ such that

- $V(G) = \bigcup_{t \in V(T)} V_t$, 

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A tree decomposition \((T, V)\) of \(G\) has width \(k\), if each bag has a size of at most \(k + 1\). The tree-width of \(G\) is the smallest integer \(k\) for which there exists a width \(k\) tree decomposition of \(G\). As our later proofs are based on minimality it is important to mention that graphs of tree-width at most \(k\) form a minor-closed family. A path decomposition is a tree decomposition \((T, V)\) in which the associated tree is a path, and the path-width of \(G\) is the minimum width over all path decompositions of \(G\).

The next definition presents the general substructure that we are looking for.

**Definition 1.** For a graph \(G\) and integers \(k, l \in \mathbb{N}\) we call a triplet \((V, W, u)\) that consists of two disjoint non-empty subsets \(V, W \subset V(G)\) and a dedicated vertex \(u \in V\) a \((k, l)\)-substructure if

\( a) \) \(W\) is stable and \(N(W) \subset V\),

\( b) \) each vertex of \(W\) is connected to \(u\),

\( c) \) \(\deg(w) \leq k\) for each \(w \in W\),

\( d) \) \(|V| \leq k + 1\),

\( e) \) \(N(u) \subset (V \cup W)\),

\( f) \) \(\deg(u) \geq l + 2 - k\),

\( g) \) \(|W| \geq l + 2 - 2k\) and
Lemma 2.1. For $k$ and for each edge composition $(t \in S)$ call in the next proof. For a tree $T$, the height of any edge $(v, w) \in E(G)$, then $V \leq k$, in the tree decomposition $(T, V)$ associated with $(V, W, u)$ the tree $T$ is a path and the vertex $t \in V(T)$ specified in Definition 4.1 is a leaf.

Here are some more definitions and an elemental lemma, that will be used only in the next proof. For a tree $T$, which is rooted in some vertex $r \in V(T)$, we define the height of any $t \in V(T)$ to be the distance from $r$ to $t$. If $(T, V)$ is a tree-decomposition of a graph $G$, then for any $v \in V(G)$, we define $t_v$ to be the unique vertex $t$ of minimum height such that $v \in V_t$. For a connected graph $G$ we call $S \subseteq V(G)$ a cut-set, if the graph $G - S$ is not connected. A proof of the next lemma can be found in [3].

Lemma 2.2. Let $G$ be a connected graph with a tree decomposition $(T, V)$. Then for any edge $t_1 t_2 \in E(T)$ the intersection $V_{t_1} \cap V_{t_2}$ is a cut-set. In other words, if $T_1$ and $T_2$ are the connected components of the forest $T - t_1 t_2$, then the intersection $V_{t_1} \cap V_{t_2}$ separates the vertex sets $\bigcup_{t \in V(T_1)}(V_t)$ and $\bigcup_{t \in V(T_2)}(V_t)$ in $G$.

Proof of Lemma 2.1. By the assumptions we have for all $vw \in E(G)$

\[
deg(v) + \deg(w) \geq \max(l, \deg(v), \deg(w)) + 2 \geq l + 2 \geq 2k + 1. \quad (2.1)
\]

In particular, of any two adjacent vertices, at least one has degree at least $k + 1$ (and $G$ has at least one vertex of degree at least $k + 1$). We define $B \subset V(G)$ to be the (non-empty) set of vertices of degree at least $k + 1$. Then $S := V(G) \setminus B$ is stable.

Fix a width $k$ tree decomposition $(T, V)$ of $G$ and root the associated tree $T$ in an arbitrary vertex $r \in V(T)$. Let $u \in B$ such that $h(t_u) = \max_{v \in B} h(t_v)$. Define $T'$ as the subtree of $T$ rooted at $t_u$, that is, the subgraph of $T$ induced by all vertices $t \in V(T)$ where the path from $t$ to the root $r$ contains $t_u$.

Set $X := \bigcup_{t \in V(T')} V_t$ and $V := N(X) \subseteq V_{t_u}$. Note that $|V| \leq k + 1$. We have $B \cap X \subseteq V$, since any $v \in (B \cap X) \setminus V$ would have $h(t_v) > h(t_u)$, contrary to the choice of $u$. Consequently

\[
X \setminus V \subset S. \quad (2.2)
\]

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By definition of the tree decomposition, no element of \(X \setminus V\) can appear in a bag indexed by a vertex \(t \in V(T - T')\). Since \(S\) is stable this gives

\[
N(X \setminus V) \subset V. \tag{2.3}
\]

By definition of \(t_u\), also \(u\) does not appear in any bag \(V_t\) of a vertex \(t \in T - T'\). So, \(N(u) \subset X\).

Set \(W := N(u) \setminus V\). Then \(W \subset X \setminus U\). We claim that \((V, W, u)\) is the desired substructure. To this end we check if (2.2), (2.3) and as \(N(u) \subset X\), we can guarantee (2).

Using the assumptions of the lemma and (c), we get

\[
\deg(u) \geq l + 2 - \deg(w) \geq l + 2 - k
\]

and thus (f). Since \(N(u) \subset V \cup W\) we obtain

\[
|W| \geq |N(u) \setminus (V \setminus \{u\})| \geq l + 2 - 2k,
\]

which is as desired for (g). Note that the subtree \(T'' = T - (T' - t_u)\) with \(V'' = \bigcup_{t \in T''} V_t\), is a tree-decomposition satisfying (h).

If \(G\) has a path-width of at most \(k\) we can assume that \(T\) is a path and its root \(r\) is a leaf. Let \(t'\) be the neighbour of \(t_u\) in the subpath \(T'\). Without loss we can assume that \(V_{t_u} \neq V_{t'}\). By Lemma 2.2 the vertex set \(V_{t_u} \cap V_{t'}\) separates the vertices of \(X'\) from the remaining vertices of \(G\). Thus \(N(W) \subset V_{t_u} \cap V_{t'}\) and consequently \(|W| = |N(W)| \leq |V_{t_u} \cap V_{t'}| \leq k\).

Finally, we want to show that \((T'', V'')\) is path decomposition of \(G - W\) where \(T''\) starts at \(r\) and ends in \(t_u\). If \(X' \setminus W = \emptyset\), this is true. Otherwise, we need to transfer the vertices of \(X' \setminus W\) into \((T'', V'')\).

To this end write \(X' \setminus W = \{x_1, \ldots, x_m\}\) and set \(P\) to be the path \(r \ldots t_u s_1 \ldots s_{m+1}\) with new vertices \(s_1, \ldots, s_{m+1}\). Define bags \(V''_{s_i} = \{x_i\} \cup V\) for \(1 \leq i \leq m\) and \(V''_{s_{m+1}} = V\). Then the substructure \((V, W, u)\) with path decomposition \((P, V'' \cup \{V''_{s_1}, \ldots, V''_{s_{m+1}}\})\) works as desired.

The next result provides the substructures for the proof of Lemma 4.2 which will be used to prove Theorem 1.

**Lemma 2.3.** Let \(G\) be a graph of tree-width at most 3 and

\[
\deg(v) + \deg(w) \geq \max(7, \deg(v), \deg(w)) + 2
\]

for each edge \(vw \in E(G)\). Then \(G\) has a \((3, 7)\)-substructure \((V, W, u)\) and one of the following holds:

i) \(|W| \geq 4\) or
Figure 2.2: $\deg_G(w_i) = 3$ for $1 \leq i \leq 3$ and $\deg_G(u) = 6$.

Figure 2.3: $\deg_G(w_1) = \deg_G(w_1) = 3$ and $\deg_G(u) = 5$.

ii) $W = \{w_1, w_2, w_3\}$ with $\deg(w_i) = 3$ for each $1 \leq i \leq 3$, $\deg(u) = 6$ and $G$ has one of the subgraphs shown in Figure 2.2.

Proof. By the assumptions $G$ has a $(3, 7)$-substructure $(V, W, u)$ as stated in Lemma 2.1. We will assume that $|W| \leq 3$ and hence $|W| = 3$ by Definition 1(g). We have $\deg(u) \leq |(W \cup V) \setminus \{u\}| = 6$ as $W$ and $V$ are disjoint, which yields $\deg(u) = 6$ by Definition 1(f). So for any $w \in W$ it holds that

$$\deg(w) \geq \max(7, \deg(u), \deg(w)) + 2 − \deg(u) \geq 9 − \deg(u) = 3$$

and thus $\deg(w) = 3$ by Definition 1(c). Either all vertices of $W$ have the same neighbourhood, or exactly two of them have the same neighbourhood, or the neighbourhoods are pairwise distinct. So by symmetry $G$ has one of the subgraphs shown in Figure 2.2. Note that the case $|V| = 2$ can not occur and the case $|V| = 3$ is covered in Figure 2.2a.

Now we will handle the substructures for the Lemma 4.1, which will be used to proof Theorem 1(2).

**Lemma 2.4.** Let $G$ be a graph of path-width at most 3 and

$$\deg(v) + \deg(w) \geq \max(6, \deg(v), \deg(w)) + 2$$

for each edge $vw \in E(G)$. Then $G$ has a $(3, 6)$-substructure $(V, W, u)$ and one of the following holds:

i) $|W| \geq 3$ or
Figure 2.4: $\deg_G(w_1) = 4$ and $\deg_G(w_i) = 2$ for $2 \leq i \leq 4$.

ii) $|W| = 2$, $\deg(w) = 3$ for each $w \in W$, $\deg(u) = 5$ and $G$ has a subgraph as shown in Figure 2.3.

Proof. By the assumptions $G$ has a $(3, 6)$-substructure $(V, W, u)$ as stated in Lemma 2.1. There is a vertex in $V$ to which no element of $W$ is connected and therefore $|N(W)| \leq 3$. We will assume that $|W| \leq 2$ thus $|W| = 2$ by Definition 1(g). As $W$ and $V$ are disjoint, $\deg(u) \leq |(W \cup V) \setminus \{u\}| = 5$ which yields $\deg(u) = 5$ by Definition 1(f). So for any $w \in W$ it holds that

$$\deg(w) \geq \max(6, \deg(u), \deg(w)) + 2 - \deg(u) \geq 8 - \deg(u) = 3.$$  

As $|N(W)| \leq 3$, the elements of $W$ share the same neighbourhood which results in the subgraph shown in Figure 2.3.

Finally, here are the substructures for the Lemma 4.3 which implies Theorem 1(3).

Lemma 2.5. Let $G$ be a graph of path-width at most 4 and

$$\deg(v) + \deg(w) \geq \max(10, \deg(v), \deg(w)) + 2$$

for each edge $vw \in E(G)$. Then $G$ has a $(4, 10)$-substructure $(V, W, u)$ and there is a subset $W' \subset W$ of size 4 for which $|N(W')| \leq 4$ and one of the following holds:

i) Each vertex of $W'$ has a degree of at least 3,

ii) each vertex of $W'$ has a degree of at most 3,

iii) there are two vertices of degree 2 in $W'$ with the same neighbourhood or

iv) $G$ has a subgraph as shown in Figure 2.4.

Proof. By the assumptions $G$ has a $(4, 10)$-substructure $(V, W, u)$ as stated in Lemma 2.1. There is one vertex $v' \in V$ to which none of the vertices of $W$ are connected. So we have $|N(W)| \leq 4$ and $|W| \geq 4$ by Definition 1(g). We will assume that (i)–(iii) do not hold. Because (i) does not hold and $|N(W)| \leq 4$, there
is at least one vertex $w_1 \in W$ of degree 2. The hypothesis of the lemma implies $\deg(w) \geq 2$ for every vertex $w$. This yields that

$$\deg(u) \geq \max(10, \deg(u), \deg(w_4)) + 2 - \deg(w_4) \geq 12 - \deg(w_4) = 10.$$ 

As $V$ and $W$ are disjoint, $|V| = 5$ and by Definition 1(e), we have $|W| \geq |N(u) \setminus (V \setminus \{u\})| \geq \deg(u) - 4 \geq 6$. On the one hand, as [ii] does not hold there is at least one vertex $w_1 \in W$ of degree 4. On the other hand, as [i] does not hold and $|W| \geq 6$, there are two more vertices $w_2$ and $w_3 \in W \setminus \{w_1, w_4\}$ of degree 2. As $w_2$, $w_3$ and $w_4 \in N(u)$ and [iii] does not hold, we get the subgraph shown in Figure 2.4.

2.2 $K$-trees

A $k$-tree is a graph that can be constructed from a complete graph on $k+1$ vertices by iteratively adding a new vertex and connecting it to all vertices of a complete subgraph of order $k$. It is easy to see that a $k$-tree has tree-width $k$. In fact each graph of tree-width $k$ is a subgraph of some $k$-tree. So as said in the introduction, $k$-trees are edge-maximal graphs of tree-width $k$ (see [7] for details). The following lemma characterises 3-trees of maximum degree at most 6. We will colour these in the next section and use this with Theorem 1 to prove Theorem 2 in Section 4.

Lemma 2.6. Let $G$ be a 3-tree of maximum degree at most 6. Then $G$ has path-width 3 or is isomorphic to one of the graphs shown in Figure 2.5.

Proof. We will construct 3-trees and see that the maximum degree grows quickly larger than 6. We start with the $K_4$ in Figure 2.5a. If we add a vertex, we get the graph shown in Figure 2.5b. Adding another vertex yields the graph in Figure 2.5c or 2.5d by symmetry. As any triangle in the last two graphs contains a vertex of degree 5, adding any other vertex will raise the maximum degree to 6. Therefore the graphs shown in the Figures 2.5a, 2.5c, and 2.5d are exactly the 3-trees of maximum degree at most 6. Now let $G$ be 3-tree of maximum degree 6 with a width 3 tree decomposition $(T, V)$, such that each bag has exactly four vertices and for all $tt' \in E(T)$ we have $|V_t \cap V_{t'}| = 3$. If $T$ is a path, we are done. Consequently let $t \in V(T)$ be a vertex with neighbours $t_1$, $t_2$ and $t_3 \in V(T)$. Set $X = V_t \cup V_{t_1} \cup V_{t_2} \cup V_{t_3}$. Then $|V_t \cap V_{t_1} \cap V_{t_2} \cap V_{t_3}| = 1, 2$ or 3. In these cases the graph induced by $X$, $G[X]$, is isomorphic to the graph shown in Figure 2.5e, 2.5f, or 2.5g. Observe that any triangle in the graphs of Figure 2.5e and 2.5f has at least one vertex of degree 6, which yields $G[X] = G$ as $\Delta(G) = 6$. So let $G[X]$ be the graph shown in Figure 2.5g and assume that there is at least one more vertex $v_8 \in V(G) \setminus X$. Up to symmetry there are only two triangles in $G[X]$, which do not already contain a vertex of degree 6 and to whose vertices another vertex $v_8$
Figure 2.5: Some 3-trees
can be connected without raising the maximum degree. This results in one of the graphs shown in Figures 2.5h and 2.5i. As all triangles in these graphs contain at least one vertex of degree 6, the graphs of Figure 2.5e–2.5i cover all 3-trees of maximum degree at most 6 that may not have a width 3 path decomposition.

3 Colouring substructures

In this section we will solve the instances of list edge-colouring related to the substructures we have just found. We will generally assume that the size of a list is exactly the size of its respective lower bound. We can always try to colour $G$ semi-greedyly, by iteratively colouring an edge with a smallest list of remaining colours with an arbitrary available colour. The following result has already been mentioned. We will apply it several times.

**Theorem 4** (Galvin, 1994). Let $G$ be a bipartite graph; then $\chi'(G) = ch'(G)$.

For a graph $G$ with an assignment of lists $L$ to the edges of $G$ and $e, f \in E(G)$ we call two colours $c_1 \in L(e)$ and $c_2 \in L(f)$ compatible if $c_1 = c_2$ or for each edge $g$ that is adjacent to both $e$ and $f$ the list $L(g)$ contains at most one of the two colours $c_1$ and $c_2$. The following lemma turns out to be quite useful in order to solve instances of list edge-colouring with small graphs. The idea of the proof can be extracted from [2].

**Lemma 3.1.** Let $G$ be a graph with an assignment of lists $L$ to the edges of $G$ and let $v_1v_2, w_1w_2 \in E(G)$ be two non-adjacent edges. If

$$|L(v_1v_2)||L(w_1w_2)| > \sum_{v_iw_j \in E(G)} \left\lfloor \frac{|L(v_iw_j)|}{2} \right\rfloor \left\lceil \frac{|L(v_iw_j)|}{2} \right\rceil,$$

then there are compatible colours $c_1 \in L(v_1v_2)$ and $c_2 \in L(w_1w_2)$.

**Proof.** If the lists of the edges $v_1v_2$ and $w_1w_2$ share a colour $c$ we are done. Therefore assume that $L(v_1v_2) \cap L(w_1w_2) = \emptyset$. This yields that there are $|L(v_1v_2)||L(w_1w_2)|$ pairs of distinct colours $(c, c)$ with $c \in L(v_1v_2)$ and $c' \in L(w_1w_2)$. But an edge $v_iw_j \in E(G)$ for $1 \leq i, j \leq 2$ can contain both colours of at most $\left\lfloor \frac{|L(v_iw_j)|}{2} \right\rfloor \left\lceil \frac{|L(v_iw_j)|}{2} \right\rceil$ of those pairs. So if (3.1) holds the desired compatible colours $c_1 \in L(v_1v_2)$ and $c_2 \in L(w_1w_2)$ exist.

Remark that (3.1) holds, if all involved lists have a size of exactly $k$, where $k$ is an odd number. It also holds if all involved lists have the same size and at least one of the four edges $v_iw_j$ is missing from $G$. 

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3.1 Small 3-trees

We now analyse the list chromatic index of the 3-trees, which we have found in Lemma 2.6.

**Lemma 3.2.** Let $G$ be one of the graphs shown in Figure 2.5. Then $\chi'(G) = \chi'(G)$.

**Proof.** Let $G_a$ be the graph shown in Figure 2.5a with an assignment of lists $L_a$ to the edges of $G_a$, where each edge has list of size at least $\Delta(G_a) = 3$. For two non-adjacent edges $e, f \in G_a$, use Lemma 3.1 to pick two compatible colours $c_1 \in L_1(e)$ and $c_2 \in L_1(f)$ and colour $e$ and $f$ with them. The rest of the graph forms a $K_{2,2}$ whose edges retain enough available colours to apply Theorem 4.

Let $G_b$ be the graph shown in Figure 2.5b. As it has 5 vertices, out of 3 edges at least 2 are adjacent. Since the total number of edges is 9, we have $\chi'(G_b) = \Delta + 1 = 5$. So for a given assignment of lists $L_b$ to the edges of $G_b$, where each list has a size of at least 5, pick two compatible colours $c_3 \in L_b(v_1v_3)$ and $c_4 \in L_b(v_2v_4)$ and colour these edges with them. We apply Lemma 3.1 another time to pick two compatible and available colours $c_5 \in L_b(v_1v_2)$ and $c_6 \in L_b(v_3v_5)$, colour the respective edges with these colours and finish semi-greedily.

Let $G_c$ be the graph shown in Figure 2.5c with lists of colours $L_c$, each of size at least $\Delta(G_c) = 5$, assigned to its edges. Colour the triangle $v_1, v_2, v_3$ semi-greedily and apply Theorem 4 to the rest.

Let $G_d$ be the graph shown in Figure 2.5d with lists $L_d$, each of size at least $\Delta(G_d) = 5$, assigned to its edges. Use Lemma 3.1 to pick two compatible colours $c_1 \in L_d(v_1v_3)$ and $c_2 \in L_d(v_2v_4)$ and colour the respective edges with them. Apply Lemma 3.1 again to pick two compatible and available colours $c_3 \in L_d(v_1v_2)$ and $c_4 \in L_d(v_3v_5)$. Use Lemma 3.1 a last time to pick two compatible and available colours $c_5 \in L_d(v_1v_2)$ and $c_6 \in L_d(v_3v_5)$, colour the respective edges with these colours and finish semi-greedily.

Let $G$ be one of the graphs shown in Figure 2.5e, 2.5f, 2.5g or 2.5i with lists $L$, each of size at least $\Delta(G) = 6$, assigned to its edges. The vertices $v_1, v_2, v_3$ and $v_4$ induce a $K_4$ in $G$. Colour its edges as shown in the case of Figure 2.5a and observe that the rest of the graph forms a bipartite graph whose edges are adjacent to already coloured edges on at most one end. So the lists of remaining colours retain sizes big enough to apply Theorem 4. For the graph $G_h$ shown in Figure 2.5h, we can apply the same argument to colour $G_h - v_6v_8$ and then finish semi-greedily afterwards.

3.2 Bipartite graphs

In the following we will discuss some instances of list edge-colouring with bipartite graphs. Throughout this subsection $G$ will be a bipartite graph with biparts
(a) The integers indicate the minimum sizes of the respective lists. 
(b) The integers indicate the colours given to the edges.

Figure 3.1

V, W of the same size. Vertices denoted as v, v' etc. will be assumed to lie in V, vertices denoted as w, w' will lie in W. We will apply the following refined version of Theorem [1]. It can be found in [13] and [I].

**Theorem 5.** Let $G$ be a bipartite graph with an edge-colouring $C$ and an assignment of lists $L$ to the edges of $G$ such that

$$|L(w'v')| \geq |\{(wv') \in E(G) ; C(wv') > C(w'v')\}| + |\{(w'v) \in E(G) ; C(w'v) < C(w'v')\}| + 1$$

for each edge $w'v' \in E(G)$. Then there is an $L$-edge-colouring of $G$.

As a first application we get the following lemma. We will use it later in the proof of Lemma [3.17].

**Lemma 3.3.** Let $G$ be the bipartite graph shown in Figure 3.1a with lists of colours assigned to the edges, where the minimum sizes of the lists are indicated by the integers on the edges. Then there is an $L$-edge-colouring of $G$.

**Proof.** Use Theorem [5] with the edge-colouring in Figure 3.1b.

We call an assignment of lists $L$ to the edges of a bipartite graph $G V$-dominated if $|L(wv)| \geq \deg(v)$ for each $v \in V$. We say that $G$ is $V$-choosable, if every $V$-dominated assignment of lists $L$ permits an $L$-edge-colouring. The next lemma can also be found in [I].

**Lemma 3.4.** Let $G$ be a bipartite graph with an assignment of lists $L$ to the edges of $G$ such that $L(wv) = \{1, \ldots, \deg(v)\}$ for $w \in W$ and $v \in V$. Then $G$ is $V$-choosable if and only if it has an $L$-edge-colouring.
Lemma 3.5. Let $G$ be a bipartite graph in which $\deg(w) \leq 3$ for each $w \in W$. Then $G$ is $V$-choosable if it has a 2-regular spanning subgraph $H$.

Proof. By Lemma 3.4 it suffices to show that $G$ has an $L$-edge-colouring from the lists $L(wv) = \{1, \ldots, \deg(v)\}$ for $w \in W$ and $v \in V$. By Theorem 4 we can colour the edges of the subgraph $H$ with colours 1 and 2. As the remaining edges form stars with their centres in $V$, we can finish semi-greedily.

Lemma 3.6. Let $G$ be a bipartite graph with biparts $V$ and $W$, of size $|V| \leq |W| = 3$. Then we can find a subset $W' \subset W$ such that the graph $G[W' \cup N(W')]$ is $N(W')$-choosable.

Proof. We can assume that each $w \in W$ has $\deg(w) \geq 2$. Otherwise $\{w\}$ would be $N(\{w\})$-choosable. If there is a vertex $v \in V$ of degree 1, say $vw \in W$, we set $W' = W \setminus \{w\}$ and apply Theorem 4. So assume that $G$ has minimum degree 2. If $V$ has less than 3 vertices any two vertices of $W$ will work by Theorem 4. If $G$ has 6 edges it is a cycle and we can apply Theorem 4 again. If $G$ has 7 edges, there are exactly two vertices $w \in W$ and $v \in V$ of degree 3 and $wv \in E(G)$. As $G - wv$ is 2-regular, we are done by Lemma 3.5. If $G$ has 8 edges, there are four vertices $w_1, w_2 \in W$ and $v_1, v_2 \in V$ of degree 3 and $w_1v_1, w_2v_2 \in E(G)$. As $G - w_1v_1 - w_2v_2$ is 2-regular, we are done by Lemma 3.5. If $G$ has 9 edges it is isomorphic to $K_{3,3}$ and we can apply Theorem 4.

For the proof of the next lemma we define a $k$-vertex to be a vertex of degree $k$ and a matching to be a set of pairwise non-adjacent edges.

Lemma 3.7. Let $G$ be a bipartite graph of minimum degree 2 with biparts $V$ and $W$ such that $|V| = |W| = 4$ and $\deg(w) \leq 3$ for each $w \in W$. Then $G$ is $V$-choosable, except for the case where $G$ is isomorphic to the graph in Figure 3.2.

Proof. We will show that $G$ has a 2-regular spanning subgraph, from which the result follows immediately by Lemma 3.5. Write $W = \{w_1, w_2, w_3, w_4\}$ and $V = \{v_1, v_2, v_3, v_4\}$. If there are no 3-vertices in $W$, we are done. Say the 3-vertices of $W$ are $w_1, \ldots, w_k$. We first assume that $k = 1$ and the other 3-vertex is $v_4$. If $w_1$ is adjacent to $v_4$ then $G - w_1v_4$ is a 2-regular spanning subgraph; otherwise,
that the graph $G$ of degree 1. If there is a vertex $v \in V$ such that $\deg(v) = 1$, then $G - v$ is a 2-regular spanning subgraph; otherwise, we may assume that $v_1$ and $v_2$ are 3-vertices, and there is a matching $M$ of two edges between $\{v_1, v_2\}$ and $\{w_1, w_2\}$, since each of these vertices is adjacent to at least one vertex in the other set; then $G - M$ is a 2-regular spanning subgraph.

If $k = 3$, let $v_i$ be a vertex of degree at least 3 that is adjacent to $w_3$, and note that $G - w_3v_i$ satisfies the hypothesis of the lemma. As in $G - w_3v_i$, $W$ has only two 3-vertices, we can find a 2-regular spanning subgraph as in the above lines. The case of $k = 4$ follows the same way.

Lemma 3.8. Let $G$ be a bipartite graph with biparts $V$ and $W$ such that $|V| \leq |W| = 4$ and $\deg(w) \leq 3$ for each $w \in W$. Then we can find a subset $W' \subset W$ such that the graph $G[W' \cup N(W')]$ is $N(W')$-choosable, except for the case where $G$ is isomorphic to the graph in Figure 3.2.

Proof. If $V$ has a size of less than 4, Lemma 3.6 applies and we are done. So $|V| = 4$ and we may assume that for each proper subset $W' \subset W$ we have $|W'| < |N(W')|$ as we could apply Lemma 3.6 otherwise. As before there is no vertex $w \in W$ of degree 1. If there is a vertex $v \in V$ of degree 1, say $vw \in E(G)$, the result holds with $W' = W \setminus \{w\}$. This yields that $G$ has minimum degree at least 2. By Lemma 3.7, $G$ is $V$-choosable or isomorphic to the graph in Figure 3.2.

Lemma 3.9. Let $G$ be a bipartite graph with biparts $V$ and $W$ such that $|V| \leq |W| = 4$ and $\deg(w) \geq 3$ for each $w \in W$. Further let there be one vertex $u \in V$ to which each vertex of $W$ is connected. Then we can find a subset $W' \subset W$ such that the graph $G[W' \cup N(W')]$ is $N(W')$-choosable.

Proof. If $V$ has a size of less than 4, Lemma 3.6 applies and we are done. So $|V| = 4$ and as before we may assume that for each proper subset $W' \subset W$ we have $|W'| < |N(W')|$ as we could apply Lemma 3.6 otherwise. This implies that $G$ has minimum degree 2 as seen in the proof of Lemma 3.6. Denote by $k$ the number 4-vertices in $W$. We have $|E(G)| = 4k + 3(4 - k) = 12 + k$ and therefore there are $8 + k$ edges incident to the vertices of $V \setminus \{u\}$. Thus there are at least as many 4-vertices in $V$ as there are in $W$. Let $L$ be an assignment of lists to the edges $vw \in E(G)$ such that $L(vw) = \{1, \ldots, \deg(v)\}$. We will show that $G$ has an $L$-edge-colouring and is hence $V$-choosable by Lemma 3.4. Denote by $X$ the 4-vertices of $V$. Let $Y$ be the set of $|X|$ vertices with of largest degree in $W$. Note that each 4-vertex of $W$ is included in $Y$. As every vertex of $X$ is connected to every vertex of $Y$, there is a matching $M$ of size $|X|$ between $X$ and $Y$. Colour
the edges of $M$ with colour 4. The graph $G - M$ retains a minimum degree of at least 2 and has a maximum degree of at most 3. Thus we can use Lemma 3.7 to colour the edges of $G - M$ from the lists of remaining colours.

**Lemma 3.10.** Let $G$ be the graph shown in Figure 3.3 with an assignment of lists $L$ to the edges of $G$ such that the size of each list $L(w_i, v_j)$ is at least $\text{deg}(v_i)$ for $1 \leq i, j \leq 4$. There is an $L$-edge-colouring of $G$, if there is an $L^1$-colouring $C$ of the complete bipartite graph $K_{2,4}$ with bipartition classes $\{p_1, p_2\}$ and $\{q_1, q_2, q_3, q_4\}$, from the assignment of lists $L^1$, where $L^1(p_i,q_j) = L(v_jw_i)$ for $1 \leq i \leq 2$ and $1 \leq j \leq 4$ (see Figure 3.3b).

**Proof.** If $C(p_1q_1) \notin L(v_1w_2)$ colour the edges $v_iw_1$ with $C(p_1q_i)$ for $1 \leq i \leq 4$ and finish semi-greedily. The case $C(p_2q_1) \notin L(v_1w_2)$ can be handled the same way. So we can assume that $L(v_1w_1) = L(v_1w_2)$ and by symmetry the same for $v_2$ and $v_3$, hence:

$$L(v_iw_1) = L(v_iw_1+1)$$  \hspace{1cm} (3.2)

for $1 \leq i \leq 3$. Now we colour the edges

- $v_iw_1$ with $C(p_1q_i)$,
- $v_4w_1$ with $C(p_1q_4)$ and
- $v_iw_i+1$ with $C(p_2q_i)$

for $1 \leq i \leq 3$. If we can not finish this edge-colouring, then by Lemma 3.12 there are colours $c_1$ and $c_2$ such that

$$L(v_4w_i+1) = \{c_1, c_2, C(p_1q_4), C(p_2q_i)\}$$  \hspace{1cm} (3.3)

for $1 \leq i \leq 3$. By (3.2) and (3.3), we can colour the edges as follows

- $v_iw_1$ with $C(p_2q_i)$,
• $v_4w_{i+1}$ with $C(p_2q_i)$,
• $v_iw_{i+1}$ with $C(p_1q_i)$ and
• $v_4w_1$ with $C(p_2q_4)$

for $1 \leq i \leq 3$ to obtain an $L$-edge-colouring.

Note that the instance shown in Figure 3.2 can be solved in a very similar way.

3.3 Other instances

Now we will deal with some general instances that are related to the substructures that may appear within graphs of bounded tree-width, as it was shown in Section 2.

We start with a classic result that can be found in [5].

Lemma 3.11 (Erdős, Rubin and Taylor, 1979). Let $G$ be a cycle with an assignment of lists $L$ to the edges of $G$, where the size of each list is at least 2. There is an $L$-edge-colouring of $G$, unless it is an odd cycle, all lists have size two and are identical.

Lemma 3.12. Let $G$ be the graph with vertices $V(G) = \{w_1, w_2, w_3, v\}$ and edges $E(G) = \{vw_1, vw_2, vw_3\}$. Let $L$ be an assignment of lists, where each list has a size of at least 2. Then there is an $L$-edge-colouring of $G$ if either

• one of the lists has size at least 3 or
• the lists are not pairwise identical.

Proof. In both cases we can assume that by symmetry there is a $c \in L(vw_1) \setminus L(vw_2)$. Colour $vw_1$ with $c$ and finish semi-greedily.

Lemma 3.13. Let $G$ be a cycle with edges $e_1, \ldots, e_n$ and an additional edge $f$ that is adjacent exactly to the vertex of $C$ that $e_1$ and $e_n$ share. Then there is an $L$-edge-colouring of $G$ for each assignment of lists $L$, where the lists have a size of at least 2 and $|L_{e_1}| \geq 3$.

Proof. If there is a colour $c \in L(e_n) \setminus L(f)$, colour $e_n$ with $c$ and finish semi-greedily. This yields $L(f) = L(e_n)$ and hence there is a colour $c \in L(e_1) \setminus (L(f) \cup L(e_n))$. Colour $e_1$ with $c$ and finish semi-greedily.

Lemma 3.14. Let $G$ be a graph consisting of two cycles $V$ and $W$ with edges $E(V) = \{g_1, \ldots, g_n\}$ and $E(W) = \{f_1, \ldots, f_m\}$ respectively, that share exactly one vertex $v \in g_1 \cap g_n \cap f_1 \cap f_m$. Then for any assignment of lists $L$, where each list has a size of at least 2 and and the size of $L(g_n)$ and $L(f_m)$ is at least 4, there is an $L$-edge-colouring of $G$.
Figure 3.4: The integers indicate the minimum sizes of the respective lists.

Proof. If there is a colour $c \in L(g_1) \setminus L(f_1)$, colour $g_1$ with $c$, colour the edges $g_2, \ldots, g_{m-1}$ semi-greedily and finish as shown in Lemma 3.13. So we can assume that $L(g_i) = L(f_i)$ for some $1 \leq i \leq n-1$. In that case we can colour $g_i$ with $c$ and finish semi-greedily. Therefore let $L(g_i) = L(g_{i+1})$ for $1 \leq i \leq n-1$ and by symmetry $L(f_i) = L(f_{i+1})$ for $1 \leq i \leq m-1$. Bearing this in mind we can simply colour the graph semi-greedily.

The next lemma will cover the instance related to the graph shown in Figure 2.3.

**Lemma 3.15.** Let $G$ be the graph shown in Figure 3.4 with lists of colours $L$ assigned to the edges, where the minimal sizes of the lists are indicated by the integers on the edges. Then there is an $L$-edge-colouring of $G$.

Proof. If there is a colour $c \in L(v_1u) \cap L(v_2w_1)$, colour both edges with $c$ and finish semi-greedily. So we may assume

$$L(v_1u) \cap L(v_2w_1) = \emptyset. \quad (3.4)$$

If there is a colour $c \in L(v_2w_1) \cap L(v_1w_2)$, colour both edges with $c$, note that $c \notin L(v_1u) \cup L(v_2u)$ by (3.4) (symmetry) and finish as in Lemma 3.13. Therefore we have

$$L(v_2w_1) \cap L(v_1w_2) = \emptyset. \quad (3.5)$$

If there is a colour $c \in L(uw_1) \cap L(v_1w_2)$, colour both edges with $c$, note that $c \notin L(v_1u) \cup L(v_2w_1)$ by (3.4) and (3.5) (symmetry) and $c$ is on at most one of the lists of $L(v_1w_1)$ and $L(v_2w_2)$ by (3.5) (symmetry). In both cases we can we can finish semi-greedily. So we have

$$L(uw_1) \cap L(v_1w_2) = \emptyset. \quad (3.6)$$

If there is a colour $c \in L(v_1w_2) \cap L(v_1w_1)$, colour $L(v_1w_2)$ with $c$, note that by (3.4), (3.5) and (3.6) (symmetry) the only other edge that can have $c$ on its list is $v_1u$ and finish semi-greedily. Hence

$$L(v_1w_2) \cap L(v_1w_1) = \emptyset. \quad (3.7)$$

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Figure 3.5: The integers indicate the minimum sizes of the respective lists.

If there is a colour \( c \in L(v_1u) \cap L(v_1w_2) \), colour \( L(v_1u) \) with \( c \), observe that by (3.4), (3.5), (3.6) and (3.7) (symmetry) the only edge that can have \( c \) on its list is \( uw_2 \) and finish semi-greedily. Consequently we have

\[
L(v_1u) \cap L(v_1w_2) = \emptyset. \tag{3.8}
\]

Colour the edges \( v_1u, v_2u, uw_1 \) and \( uw_2 \) semi-greedily. The remaining edges form a 4-cycle that retain at least two available colours by (3.8). We finish by applying Theorem 4.

In the next two lemmas we will colour the instances related to the substructures shown in Figure 2.2b and 2.2c.

**Lemma 3.16.** Let \( G \) be the graph shown in Figure 3.5a with lists of colours assigned to the edges, where the minimum sizes of the lists are indicated by the integers on the edges. Then there is an \( L \)-edge-colouring of \( G \).

**Proof.** The same argument as in the proof of Lemma 3.1 allows us to pick colours \( c_1 \in L(v_3u) \) and \( c_2 \in L(v_2w_2) \) such that either \( c_1 = c_2 \) or the lists of \( v_2u \) and \( v_3w_2 \) contain each at most one of the colours \( c_1 \) and \( c_2 \). Colour \( v_3u \) with \( c_1 \), \( v_2w_2 \) with \( c_2 \) and \( v_3w_2 \) semi-greedily. Colour the edge \( uw_2 \) with a colour \( c \in L(uw_2) \setminus L(v_1u) \) and apply Lemma 3.15 to the rest.

**Lemma 3.17.** Let \( G \) be the graph shown in Figure 3.5b with lists of colours \( L \) assigned to the edges, where the minimum sizes of the lists are indicated by the integers on the edges. Then there is an \( L \)-edge-colouring of \( G \).

**Proof.** If there is a colour \( c \in L(v_2u) \setminus L(v_2w_1) \), colour \( v_2u \) with \( c \) and colour the edges \( v_1u \) and \( v_3u \) semi-greedily. We can apply Lemma 3.3 to colour the rest and
get an $L$-edge-colouring. This and symmetry yield
\[ L(v_2u) = L(v_2w_1) = L(v_2w_2). \] (3.9)

If there is a colour $c \in L(v_2u)$ that is not on $L(uw_1)$ use it to colour $v_2u$. If $c \in L(v_3u)$, then we have $c \in L(v_3w_2)$ by (3.9) and symmetry. Colour $v_3w_2$ with that colour. If $c \notin L(v_3u)$ colour $v_3w_2$ semi-greedily. In both cases the list of $v_3u$ retains at least two available colours and we continue by colouring the edges $v_2w_2$ and $v_2w_1$ semi-greedily. Apply Lemma \[3.13\] to colour the edges $v_1w_1$, $v_1u$, $v_1w_3$, $v_3u$ and $v_3w_3$ and finish semi-greedily to get an $L$-edge-colouring. By this and symmetry we have
\[ L(v_2u) \subset (L(w_1u) \cap L(uw_2) \cap L(uw_3)). \] (3.10)

By the size of the lists, (3.10) and symmetry, there is a colour $c \in L(v_2u)$ that is also in $L(v_3u)$ or $L(v_1u)$. By symmetry we can assume the latter. By (3.9) we have also $c \in L(v_2w_1)$. Colour $v_1u$ and $v_2w_1$ with $c$ and colour the edges $v_1w_1$, $v_1w_3$ semi-greedily. Apply Lemma \[3.13\] to colour the edges $v_3w_3$, $v_3u$, $v_2u$, $v_2w_2$ and $v_3w_2$ and finish semi-greedily to get an $L$-edge-colouring of $G$.

\[ \square \]

4 Proofs of the main results

In this section we will combine the results of the two previous sections in order to give proofs of Theorem \[1\] and \[2\]. The size of a graph $G$ is $|V(G)| + |E(G)|$. $H$ is smaller than $G$ if its size is less than the size of $G$. For a subset of vertices $W \subset V(G)$, we denote by $G(W)$ the graph with vertex set $W \cup N(W)$ and edge set $E(G) \setminus E(G - W)$.

Let $G$ be a graph with an assignment of lists $L$ such that for a fixed $l \in \mathbb{N}$ each list $L(vw)$ has a size of at most $\max(l, \deg_G(v), \deg_G(w))$. Suppose that for some proper subset of vertices $W \subset V(G)$, we can find an $L$-edge-colouring $C$ of the graph $G - W$. In order to extend $C$ to an $L$-edge-colouring of $G$ we need find an $L^C$-colouring of $G(W)$. For an edge $w_1w_2 \in E(G)$ with $w_1, w_2 \in W$ we have
\[ |L^C(w_1w_2)| = |L(w_1w_2)|. \] (4.1)

Since $\max(\deg_G(v), \deg_G(w), l) \geq \deg_G(v) = \deg_{G-W}(v) + \deg_{G(W)}(v)$, we have for an edge $vw \in E(G)$ with $w \in W$ and $v \in V(G) \setminus W$
\[ |L^C(vw)| \geq |L(vw)| - \deg_{G-W}(v) \geq \deg_{G(W)}(v) \] (4.2)

Our proofs will be based on minimality. In most cases we want to prove for some subgraph-closed family of graphs and a fixed $l \in \mathbb{N}$ that for each member
$G$ with lists of colours $L(vw)$ assigned to the edges of size at least $|L(vw)| \geq \max(l, \deg_G(v), \deg_G(w))$ there is always an $L$-edge-colouring of $G$. Suppose this does not hold for a graph $G$, but for all graphs that are smaller than $G$ and let $L$ be an assignment of lists of above described sizes for which there is no $L$-edge-colouring of $G$. Let $W \subseteq V(G)$ be some non-empty subset of vertices. As 

\[ \text{ch}'(H) \leq \text{ch}'(G) \text{ for every subgraph } H \subseteq G, \text{ G - W is smaller than G and for each } v \in V(G - W) \text{ it holds that } \deg_{G - W}(v) \leq \deg_G(W), \text{ we can find an } L\text{-edge-colouring } C \text{ of } G - W. \]

This yields immediately that $G$ has no isolated vertices and more interestingly for every edge $vw$ we have

\[ \deg(v) + \deg(w) \geq \max(l, \deg_G(v), \deg_G(w)) + 2. \] (4.3)

Otherwise colour $G - vw$ by minimality and observe that $L(vw)$ retains at least one available colour by (4.2), which contradicts the assumptions on $G$.

### 4.1 Proofs

The next three lemmas imply Theorem 1

**Lemma 4.1.** Let $G$ be a graph of path-width at most 3 with an assignment of lists $L$ to the edges of $G$, such that each list $L(vw)$ has a size of at least $\max(6, \deg(v), \deg(w))$. Then $G$ has an $L$-edge-colouring.

**Proof.** We will assume that the lemma is wrong and obtain a contradiction. So let $G$ be a smallest counterexample to the lemma with lists $L(vw)$ of size at least max($6, \deg(v), \deg(w)$) assigned to the edges, such that there is no $L$-edge-colouring of $G$. By (4.3) $G$ has minimum degree 2 and a (3,6)-substructure $(V,W,u)$ as stated in Lemma 2.4. If $G$ has the substructure of (1) of Lemma 2.4 and hence $|W| \geq 3$, choose a subset $W_1 \subseteq W$ of size 3 and an $L$-edge-colouring $C_1$ of $G - W_1$ by minimality. Inequality (4.2) asserts that the lists of remaining colours of the graph $G\langle W_1 \rangle$ retain sizes big enough to apply Lemma 3.6, to extend $C_1$ to an $L$-edge-colouring of $G$. Thus we can assume that $G$ has the substructure of Lemma 2.4 (1) (see Figure 2.3). Set $W_2 = W \cup \{u\}$ and use as before the minimality of $G$ to find an $L$-edge-colouring $C_2$ of the graph $G - W_2$. By (4.1) and (4.2), the lists of remaining colours of the graph $G\langle W_2 \rangle$ retain sizes big enough to colour $uv_3$ semi-greedily and then apply Lemma 3.15 to extend $C_2$ to an $L$-edge-colouring of $G$. A contradiction.

**Lemma 4.2.** Let $G$ be a graph of tree-width at most 3 with an assignment of lists $L$ to the edges of $G$, such that each list $L(vw)$ has a size of at least $\max(7, \deg(v), \deg(w))$. Then $G$ has an $L$-edge-colouring.
Proof. We will assume that the lemma is wrong and obtain a contradiction. So let $G$ be a smallest counterexample to the lemma with lists $L(vw)$ of size at least $\max(7, \deg(v), \deg(w))$ assigned to the edges, such that there is no $L$-edge-colouring of $G$. We can assume that $G$ is connected and non-empty. By (4.3), $G$ has minimum degree 2 and a $(3, 7)$-substructure $(V, W, u)$ as stated in Lemma 2.3. If $G$ has the substructure of Lemma 2.3 and hence $|W| \geq 4$, choose a subset $W_1 \subset W$ of size 4. By minimality we can find an $L$-edge-colouring $C_1$ of the graph $G - W_1$. Inequality (4.2) asserts that the size of the lists of remaining colours for the graphs $G(W_1)$ are big enough to apply Lemma 3.8 to extend $C_1$ to an $L$-edge-colouring of $G$. Thus we can assume that $G$ has one of the substructures of Lemma 2.3. Set $W_2 = W \cup \{u\}$ and use as before the minimality of $G$ to find an $L$-edge-colouring $C_2$ of the graph $G - W_2$. By (4.1) and (4.2), the lists of remaining colours of the graph $G(W_2)$ retain sizes big enough to apply Lemma 3.6, 3.16 or 3.17 respectively to extend $C_2$ to an $L$-edge-colouring of $G$. A contradiction. □

We get Theorem 2 as a corollary.

Proof of Theorem 3. Let $G$ be a 3-tree. If $\Delta(G) \geq 7$ we have $\chi'(G) = \chi(G)$ by Lemma 4.2. If $\Delta(G) \leq 6$, then by Lemma 2.6 $G$ has either path-width 3 or is isomorphic to one of the graphs shown in Figure 2.5. In the first case we can apply Lemma 4.1 and otherwise we are done by Lemma 3.2. □

Lemma 4.3. Let $G$ be a graph of tree-width at most 4 with an assignment of lists $L$ to the edges of $G$, such that each list $L(vw)$ has a size of at least $\max(10, \deg(v), \deg(w))$. Then $G$ has an $L$-edge-colouring.

Proof. We will assume that the lemma is wrong and obtain a contradiction. So let $G$ be a smallest counterexample to the lemma with lists $L(vw)$ of size at least $\max(10, \deg(v), \deg(w))$ assigned to the edges, such that there is no $L$-edge-colouring of $G$. We can assume that $G$ is connected and non-empty. By (4.3) $G$ has minimum degree 2 and a $(4, 10)$-substructure $(V, W, u)$ with a dedicated subset $W_1 \subset W$ of size 4 as stated in Lemma 2.3. If $G$ has the substructure of Lemma 2.3 and thus each element of $W_1$ has a degree of at least 3, pick an $L$-edge-colouring of the graph $G - W_1$ by minimality of $G$. By (4.1) the size of the lists of remaining colours retain sizes big enough to find an $L$-edge-colouring of the graph $G(W_1)$ by applying Lemma 3.9. If $G$ has the substructure of Lemma 2.3 and hence each vertex of $W_1$ has a degree of at most 3, we can find an $L$-edge-colouring of $G$ as seen in the proof of Lemma 4.2 with Lemma 3.8. If $G$ has the substructure of Lemma 2.3 and therefore two vertices $w_1, w_2 \in W_1$ of degree 2 have the same neighbourhood, we can find an $L$-edge-colouring of $G - w_1 - w_2$ by minimality and extend this to an $L$-edge-colouring of $G$ by applying Theorem 4 to the graph $G\{w_1, w_2\}$ with lists of remaining colours. So we can assume that $G$
Figure 4.1: Two bipartite graph that are not \( \{v_1, v_2, v_3, v_4\}\)-choosable.

has the substructure of Lemma 2.5 and by consequence \( W_1 \) contains exactly one vertex \( w_1 \) of degree 4 and three vertices \( w_2, w_3, w_4 \) of degree 2 with pairwise distinct neighbourhoods. Let \( V = \{v_1, v_2, v_3, u\} \) as shown in Figure 2.4. Denote by \( G_1 \) the graph \( G - W_1 \) and let \( G^* \) be the graph obtained from \( G_1 \) by adding two new vertices \( p_1, p_2 \) and connecting both to each vertex of \( V \). Observe that \( \Delta(G^*) \leq \Delta(G) \) and \( |V(G^*)| + |E(G^*)| < |V(G)| + |E(G)| \). Further \( G^* \) has path-width 4. Lemma 2.5 provides a width 4 path decomposition of \( G_1 \) where \( V = V_t \) for some vertex \( t \) of the associated tree. We can extend this to a width 4 path decomposition of \( G^* \) as in the proof of Lemma 2.1. Let \( L^* \) be an assignment of lists to the edges of \( G^* \) with

- \( L^*(e) = L(e) \) if \( e \in E(G_1) \),
- \( L^*(up_j) = L(uw_1) \) and
- \( L^*(v_ip_j) = L(v_iw_1) \)

for \( 1 \leq j \leq 2 \) and \( 1 \leq i \leq 3 \). By minimality there is an \( L^*\)-colouring \( C^* \) of \( G^* \)

We can extract an \( L\)-edge-colouring \( C \) of the graph \( G_1 \) by setting \( C(e) = C^*(e) \) for each edge \( e \in E(G_1) \). As the graph \( G(W_1) \) with the lists of remaining colours \( L^C \) fulfils the conditions of Lemma 3.10, we can extend \( C \) to an \( L\)-edge-colouring of \( G \). A contradiction.

4.2 Remarks

The case of tree-width 3 and maximum degree 6 has been studied, but not resolved. Partial results can be found in [11]. It would be nice to have more general versions of the lemmas concerning bipartite substructures in section 3.2. These could be used in combination with Lemma 2.1 to colour graphs where the maximum degree is some linear function of the tree-width. Of course there may occur substructures that do not permit edge-colouring from the lists of remaining colours. For example

\[ \text{At this point it is important that } \text{ch}'(G^*) \leq \text{ch}'(G) \text{ as it was discussed briefly in Section 4.} \]
the graphs shown in Figure 4.1a and 4.1b are not $V$-choosable, while they do appear as substructures of graphs of path-width 4 and maximum degree 10. But we can overcome these obstacles, by further analysis of the graph structure as seen in Lemma 2.5 and using refined methods to colour the substructures as explained in Lemma 3.10.

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References

[1] O. V. Borodin, A. V. Kostochka, and D. R. Woodall. List Edge and List Total Colourings of Multigraphs. *Journal of Combinatorial Theory, Series B*, 71(2):184–204, 1997.

[2] D. Cariolaro and K-W. Lih. The Edge-Choosability of the Tetrahedron.

[3] R. Diestel. *Graph theory*. Springer-Verlag, New York, 4 edition, 2010.

[4] M. N. Ellingham and L. A. Goddyn. List Edge Colourings of Some 1-Factorable Multigraphs. *Combinatorica*, 16(3):343–352, 1996.

[5] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in Graphs. *Congressus Numerantium*, 26:125–157, 1979.

[6] F. Galvin. The List Chromatic Index of a Bipartite Multigraph. *Journal of Combinatorial Theory, Series B*, 63:153–158, 1995.

[7] M. Grötschel and G. O. H. Katona. *Building bridges: between mathematics and computer science*. Springer Verlag, Berlin, Heidelberg, New York, 2008.

[8] M. Juvan, B. Mohar, and R. Thomas. List Edge-Colorings Of Series-Parallel Graphs. *Electronic Journal of Combinatorics*, 6, 1999.

[9] J. Kahn. Asymptotically Good List-Colorings. *Journal of Combinatorial Theory, Series A*, 73(1):1–59, 1996.

[10] R. Lang. A note on list-edge colourings of halin graphs. *Submitted*.

[11] R. Lang. On the list chromatic index of graphs of tree-width 3 and maximum degree 7. Master’s thesis, Freie Universität Berlin, 2012.
[12] K. Meeks and A. Scott. The Parameterised Complexity of List Problems on Graphs of Bounded Treewidth. *Computing Research Repository*, abs/1110.4077, 2011.

[13] T. Slivnik. Short Proof of Galvin’s Theorem on the List-chromatic Index of a Bipartite Multigraph. *Combinatorics, Probability & Computing*, 5:91–94, 1996.

[14] V. G. Vizing. Vertex colorings with given colors. *Diskret. Analiz.* 29: 3–10, 1976.

[15] X. Zhou, S. Nakano, and T. Nishizeki. Edge-Coloring Partial k-Trees. *Journal of Algorithms*, 21(3):598–617, 1996.