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Second-Order Non-Canonical Neutral Differential Equations with Mixed Type: Oscillatory Behavior

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Abstract: In this paper, we establish new sufficient conditions for the oscillation of solutions of a class of second-order delay differential equations with a mixed neutral term, which are under the non-canonical condition. The results obtained complement and simplify some known results in the relevant literature. Example illustrating the results is included.

Keywords: non-canonical differential equations; second-order; neutral delay; mixed type; oscillation criteria

1. Introduction

This paper discusses the oscillatory behavior of solutions of second-order functional differential equation with a mixed neutral term of the form

\[
(r(l) \left( (y(l) + p_1(l)y(l)) + p_2(l)y(l) \right))' + q(l)y(\sigma(l)) = 0, \tag{1}
\]

where \( l \geq l_0 \). Throughout this paper, we assume the following:

1. \( \gamma \in Q_{odd} := \{a/b : a, b \in \mathbb{Z}^+ \text{ are odd} \} \) and \( r \in C([l_0, \infty), (0, \infty)) \);
2. \( p_1, p_2, \sigma \in C([l_0, \infty), \mathbb{R}) \), \( p_1(l) \leq l \leq p_2(l), \sigma(l) \leq l \) and \( p_1, p_2, \sigma \to \infty \) as \( l \to \infty \);
3. \( p_1, p_2, q \in C([l_0, \infty), [0, \infty)) \) and \( q(l) \) is not identically zero for large \( l \).

Let \( y \) be a real-valued function defined for all \( l \) in a real interval \([l_y, \infty)\), \( l_y \geq l_0 \), and having a second derivative for all \( l \in [l_y, \infty) \). The function \( y \) is called a solution of the differential Equation (1) on \([l_y, \infty)\) if \( y \) satisfies (1) on \([l_y, \infty)\). A nontrivial solution \( y \) of any differential equation is said to be oscillatory if it has arbitrary large zeros; otherwise, it is said to be nonoscillatory. We will consider only those solutions of (1) which exist on some half-line \([l_b, \infty)\) for \( l_b \geq l_0 \) and satisfy the condition \( \sup \{|y(l)| : l_c \leq l < \infty\} > 0 \) for any \( l_c \geq l_b \).

A delay differential equation of neutral type is an equation in which the highest order derivative of the unknown function appears both with and without delay. During the last decades, there is a great interest in studying the oscillation of solutions of neutral differential equations. This is due to the fact that such equations arise from a variety of applications including population dynamics, automatic control, mixing liquids, and vibrating masses attached to an elastic bar, biology in explaining self-balancing of the human body, and in robotics in constructing biped robots, it is easy to notice the emergence of models of the neutral delay differential equations, see [1,2].

In the following, we review some of the related works that dealt with the oscillation of the neutral differential equations of mixed-type.
Grammatikopoulos et al. [3] established oscillation criteria for the equation

\[ (r(l)\psi'(l))' + q(l)y(\sigma(l)) = 0, \tag{2} \]

where

\[ z(l) = y(l) + p_1(l)y(l - \sigma_1) + p_2(l)y(l + \sigma_2), \]

with constant mixed arguments:

\[ r(l) = 1, p_2(l) = 0, 0 \leq p_1 \leq 1, \text{ and } q(l) \geq 0. \]

Ruan [4] obtained some oscillation criteria for the Equation (2) by employing Riccati technique and averaging function method, when \( p_2(l) = 0 \) and \( \sigma(l) = \hat{l} - \sigma \). Arul and Shobha [5] studied the oscillatory behavior of solution of (2), when \( 0 \leq p_1(l) \leq p_1 < \infty \) and \( 0 \leq p_2(l) \leq p_2 < \infty \).

Dzurina et al. [6] presented some sufficient conditions for the oscillation of the second-order equation

\[ \left( \frac{1}{r(l)}y'(l) \right)' + p(l)y(\tau(l)) + q(l)y(\sigma(l)) = 0. \]

Li [7] and Li et al. [8] studied the oscillation of solutions of the second-order equation with constant mixed arguments:

\[ (r(l)z'(l))' + q_1(l)y(l - \sigma_3) + q_2(l)y(l + \sigma_4) = 0. \tag{3} \]

Arun and Shobha [5] established some sufficient conditions for the oscillation of all solutions of Equation (3) in the canonical case, that is,

\[ \int_{l_0}^{\infty} r^{-1}(\theta)d\theta = \infty, \]

Thandapani et al. [9] studied the oscillation criteria for the differential equation of the form

\[ (z^a(l))'' + q(l)y^b(l - \tau_1) + p(l)y^\gamma(l + \tau_1) = 0. \]

Grace et al. [10] studied the oscillatory behavior of solutions of the equation

\[ (r(l) \left( y(l) + p_1(l)y^{\delta_1}(\sigma_1(l)) + p_2(l)y^{\delta_2}(\sigma_2(l)) \right)')' + q(l)y^\gamma(\tau(l)) = 0, \]

and considered the two cases

\[ \int_{l_0}^{\infty} r^{-1/\gamma}(\theta)d\theta = \infty, \tag{4} \]

and

\[ \int_{l_0}^{\infty} r^{-1/\gamma}(\theta)d\theta < \infty. \tag{5} \]

In [11], Tunc et al. studied the oscillatory behavior of the differential Equation (1) under the condition (4). Moreover, they considered the two following cases: \( p_1(l) \geq 0, p_2(l) \geq 1, \text{ and } p_2(l) \neq 1 \) eventually; \( p_2(l) \geq 0, p_1(l) \geq 1, \text{ and } p_2(l) \neq 1 \) eventually.

For the third-order equations, Han et al. [12] studied the oscillation and asymptotic properties of the third-order equation

\[ (a(l)z''(l))' + q_1(l)y(l - \tau_3) + q_2(l)y(l + \tau_4) = 0, \]

and established two theorems which guarantee that the above equation oscillates or tends to zero. Moaaz et al. [13] discussed the oscillation and asymptotic behavior of solutions of the third-order equation

\[ \left( r(l)(x''(l))^a \right)' + q_1(l)f_1(y(\sigma_1(l))) + q_2(l)f_2(y(\sigma_2(l))) = 0. \]
where $x(l) = y(l) + p_1(l)y(\tau_1(l)) + p_2(l)y(\tau_2(l))$. For further results, techniques, and approaches in studying oscillation of the delay differential equations, see in [14–24].

In this paper, we study the oscillatory behavior of solutions of the second-order differential equation with a mixed neutral term (1) under condition (5). We follow a new approach based on deducing a new relationship between the solution and the corresponding function. Assume that $y$ is a positive solution of (1) on $[l_0, \infty)$. Therefore, there exists a $l_0 \geq 0$ such that, for all $l \geq l_0$, $\psi(l) > 0$ and $(r(l)(\psi'(l))^\gamma) \leq 0$. From (1), we see that

$$
\left( r(l)(\psi'(l))^\gamma \right)' = -q(l)y^\gamma(\sigma(l)) \leq 0.
$$

Obviously, $\psi$ is either eventually decreasing or eventually increasing. Let $\psi$ be a decreasing function on $[l_1, \infty)$. Then, $\lim_{l \to \infty} \psi(l) < \infty$, and so

$$
\psi(l) \geq -\int_l^\infty r^{-1/\gamma}(\theta)r^{1/\gamma}(\theta)\psi'(\theta)d\theta \geq -\kappa(l,\infty)r^{1/\gamma}(l)\psi'(l).
$$

2. Main Results

We adopt the following notation for a compact presentation of our results:

$$
\psi(l) := y(l) + p_1(l)y(\rho_1(l)) + p_2(l)y(\rho_2(l)),
$$

$$
\kappa(u,v) := \int_u^v r^{-1/\gamma}(\delta)d\delta,
$$

$$
B_1(l) := 1 - p_1(l)\frac{\kappa(\rho_1(l),\infty)}{\kappa(l,\infty)} - p_2(l)
$$

and

$$
B_2(l) := 1 - p_1(l) - p_2(l)\frac{\kappa(l_1,\rho_2(l))}{\kappa(l_1,l)}.
$$

Lemma 1. Assume that $\Theta(\theta) := A\theta - B(\theta - C)^{(\gamma+1)/\gamma}$, where $A$, $B$, and $C$ are real constants; $B > 0$; and $\gamma \in \mathbb{Q}_{odd}$. Then,

$$
\Theta(\theta^*) \leq \max_{\theta \in \mathbb{R}} \Theta(\theta) = AC + \frac{\gamma}{(\gamma + 1)^{\gamma+1}} A^{\gamma+1} B^{-\gamma}.
$$

Lemma 2. Assume that $y$ is a positive solution of (1) on $[l_0, \infty)$. If $\psi$ is a decreasing positive function for $l \geq l_1$ large enough, then

$$
\left( \frac{\psi(l)}{\kappa(l,\infty)} \right)' \geq 0, \text{ for } l \geq l_1.
$$

(6)

While if $\psi$ is a increasing positive function for $l \geq l_1$, then

$$
\left( \frac{\psi(l)}{\kappa(l_1,l)} \right)' \leq 0, \text{ for } l \geq l_1.
$$

(7)

Proof. Assume that (1) has a positive solution $y$ on $[l_0, \infty)$. Therefore, there exists a $l_1 \geq l_0$ such that, for all $l \geq l_1$, $\psi(l) \geq y(l) > 0$ and $(r(l)(\psi'(l))^\gamma) \leq 0$. From (1), we see that

$$
\left( r(l)(\psi'(l))^\gamma \right)' = -q(l)y^\gamma(\sigma(l)) \leq 0.
$$

Obviously, $\psi$ is either eventually decreasing or eventually increasing. Let $\psi$ be a decreasing function on $[l_1, \infty)$. Then, $\lim_{l \to \infty} \psi(l) < \infty$, and so

$$
\psi(l) \geq -\int_l^\infty r^{-1/\gamma}(\theta)r^{1/\gamma}(\theta)\psi'(\theta)d\theta \geq -\kappa(l,\infty)r^{1/\gamma}(l)\psi'(l).
$$

(8)
Thus,
\[
\left( \psi(l) \over \kappa(l, \infty) \right)' = \frac{\kappa(l, \infty) \psi'(l) + r^{-1/\gamma}(l) \psi(l)}{(\kappa(l, \infty))^2} \geq 0.
\]

Let \( \psi \) be an increasing function on \([l_1, \infty)\). Then, we obtain
\[
\psi(l) \geq \int_{l_1}^{l} r^{-1/\gamma}(\theta)^{1/\gamma}(\theta) \psi'(\theta) d\theta \geq \kappa(l_1, l) r^{1/\gamma}(l) \psi(l),
\]
and so
\[
\left( \frac{\psi(l)}{\kappa(l_1, l)} \right)' = \frac{\kappa(l_1, l) \psi'(l) - r^{-1/\gamma}(l) \psi(l)}{\kappa^2(l_1, l)} \leq 0.
\]
Thus, the proof is complete. \( \Box \)

**Theorem 1.** Assume that \( B_2(l) \geq B_1(l) > 0 \). If
\[
\limsup_{l \to \infty} \int_{l_1}^{l} \frac{1}{r^{1/\gamma}(\beta)} \left( \int_{l_1}^{\beta} q(\delta) B_1^\gamma(\sigma(\delta)) \kappa^\gamma(\sigma(\delta), \infty) d\delta \right)^{1/\gamma} d\beta = \infty,
\]
then, all solutions of (1) are oscillatory.

**Proof.** Assume the contrary that Equation (1) has a positive solution \( y \) on \([l_0, \infty)\). Then, \( y(\rho_1(l)), y(\rho_2(l)) \) and \( y(\sigma(l)) \) are positive for all \( l \geq l_1 \), where \( l_1 \) is large enough. Thus, from (1) and the definition of \( \psi \), we note that \( \psi(l) \geq y(l) > 0 \) and \( r(l)(\psi'(l))^\gamma \) is nonincreasing. Therefore, \( \psi' \) is either eventually negative or eventually positive. Let \( \psi'(l) < 0 \) on \([l_1, \infty)\). By using Lemma 2, we have
\[
\psi(\rho_1(l)) \leq \frac{\kappa(\rho_1(l), \infty)}{\kappa(l, \infty)} \psi(l),
\]
based on the fact that \( \rho_1(l) \leq l \). Therefore,
\[
y(l) = \psi(l) - p_1(l) y(\rho_1(l)) - p_2(l) y(\rho_2(l)) \geq \psi(l) - p_1(l) \psi(\rho_1(l)) - p_2(l) \psi(\rho_2(l)) \geq \left( 1 - p_1(l) \frac{\kappa(\rho_1(l), \infty)}{\kappa(l, \infty)} - p_2(l) \right) \psi(l) = B_1(l) \psi(l).
\]
Therefore, (1) becomes
\[
\left( r(l)(\psi'(l))^\gamma \right)' \leq -q(l) B_1^\gamma(\sigma(l)) \psi^\gamma(\sigma(l)). \tag{10}
\]
As \( (r(l)(\psi'(l))^\gamma)' \leq 0 \), we have
\[
r(l)(\psi'(l))^\gamma \leq r(l_1)(\psi'(l_1))^\gamma := -L < 0, \tag{11}
\]
for all \( l \geq l_1 \), from (8) and (11), we have
\[
\psi^\gamma(l) \geq L \kappa^\gamma(l, \infty) \text{ for all } l \geq l_1. \tag{12}
\]
Combining (10) with (12) yields
\[
\left( r(l)(\psi'(l))^\gamma \right)' \leq -L q(l) B_1^\gamma(\sigma(l)) \kappa^\gamma(\sigma(l), \infty), \tag{13}
\]

for all $l \geq l_1$. Integrating (13) from $l_1$ to $l$, we obtain
\[
\begin{align*}
  r(l)(\psi'(l))^\gamma &\leq r(l_1)(\psi'(l_1))^\gamma - L \int_{l_1}^{l} q(\delta)B_1^\gamma(\sigma(\delta))\kappa'(\sigma(\delta),\infty) \, d\delta \\
  &\leq -L \int_{l_1}^{l} q(\delta)B_1^\gamma(\sigma(\delta))\kappa'(\sigma(\delta),\infty) \, d\delta.
\end{align*}
\]
Integrating the last inequality from $l_1$ to $l$, we get
\[
\psi(l) \leq \psi(l_1) - L^{1/\gamma} \int_{l_1}^{l} \frac{1}{r^{1/\gamma}(\beta)} \left( \int_{l_1}^{l} q(\delta)B_1^\gamma(\sigma(\delta))\kappa'(\sigma(\delta),\infty) \, d\delta \right)^{1/\gamma} \, d\beta.
\]
At $l \to \infty$, we get a contradiction with (9).

Let $\psi'(l) > 0$ on $[l_1, \infty)$. From Lemma 2, we arrive at
\[
\psi(\rho_2(l)) \leq \frac{\kappa(l_1,\rho_2(l))}{\kappa(l_1,1)} \psi(l). \tag{14}
\]
From the definition of $\psi$, we obtain
\[
y(l) = \psi(l) - p_1(l)y(\rho_1(l)) - p_2(l)y(\rho_2(l)) \\
\geq \psi(l) - p_1(l)y(\rho_1(l)) - p_2(l)y(\rho_2(l)). \tag{15}
\]
Using that (14) and $\psi(\rho_1(l)) \leq \psi(l)$ where $\rho_1(l) < l$ in (15), we obtain
\[
y(l) \geq \psi(l) \left( 1 - p_1(l) - p_2(l) \frac{\kappa(l_1,\rho_2(l))}{\kappa(l_1,1)} \right) \\
\geq B_2(l)\psi(l). \tag{16}
\]
Thus, (1) becomes
\[
\left( r(l)(\psi'(l))^\gamma \right)' \leq -q(l)B_2^\gamma(\sigma(l))\psi'(\sigma(l)). \tag{17}
\]
Now, from (9) and (C2), we have that $\int_{l_1}^{l} q(\theta)B_1^\gamma(\sigma(\theta))\kappa'(\sigma(\theta),\infty) \, d\theta$ is unbounded. Therefore, as $\kappa'(l,\infty) < 0$, we obtain that
\[
\int_{l_1}^{l} q(\theta)B_1^\gamma(\sigma(\theta)) \, d\theta \to \infty \text{ as } l \to \infty. \tag{18}
\]
Integrating (17) from $l_2$ to $l$, we get
\[
\begin{align*}
  r(l)(\psi'(l))^\gamma &\leq r(l_2)(\psi'(l_2))^\gamma - \int_{l_2}^{l} q(\theta)B_2^\gamma(\sigma(\theta))\psi'(\sigma(\theta)) \, d\theta \\
  &\leq r(l_2)(\psi'(l_2))^\gamma - \psi'(\sigma(l_2)) \int_{l_2}^{l} q(\theta)B_2^\gamma(\sigma(\theta)) \, d\theta.
\end{align*}
\]
As $B_2(l) > B_1(l)$, we get
\[
\begin{align*}
  r(l)(\psi'(l))^\gamma &\leq r(l_2)(\psi'(l_2))^\gamma - \psi'(\sigma(l_2)) \int_{l_2}^{l} q(\theta)B_1^\gamma(\sigma(\theta)) \, d\theta \tag{19}
\end{align*}
\]
From (18) and (19), we get a contradiction with the positivity of $\psi'(l)$. Therefore, the proof is complete. \qed
Theorem 2. Assume that $B_2(l) \geq B_1(l) > 0$. If
\[
\limsup_{l \to \infty} \int_{l_1}^{l} q(\theta)B_{1}^{\gamma}(\theta)\,d\theta > 1,
\]
then, all solutions of (1) are oscillatory.

Proof. Assume the contrary that Equation (1) has a positive solution $y$ on $[l_0, \infty)$. Then, $y(\rho_1(l)), y(\rho_2(l))$ and $y(\sigma(l))$ are positive for all $l \geq l_1$, where $l_1$ is large enough. Thus, from (1) and the definition of $\psi$, we note that $\psi(l) \geq y(l) > 0$ and $r(l)(\psi'(l))^\gamma$ is nonincreasing. Therefore, $\psi'$ is either eventually negative or eventually positive.

Let $\psi'(l) < 0$ on $[l_1, \infty)$. Integrating (10) from $l_1$ to $l$, we get
\[
\begin{align*}
& r(l)(\psi'(l))^\gamma \leq r(l_1)(\psi'(l_1))^\gamma - \int_{l_1}^{l} q(\theta)B_{1}^{\gamma}(\sigma(\theta))\psi(\sigma(\theta))\,d\theta \\
& \leq -\psi^\gamma(\sigma(l)) \int_{l_1}^{l} q(\theta)B_{1}^{\gamma}(\theta)\,d\theta.
\end{align*}
\]
Using $\psi(\sigma(l)) \geq \psi(l)$ and (8) in (21), we obtain
\[
- r(l)(\psi'(l))^\gamma \geq -r(l)(\psi'(l))^\gamma \kappa^\gamma(l, \infty) \int_{l_1}^{l} q(\theta)B_{1}^{\gamma}(\theta)\,d\theta.
\]
Divide both sides of inequality (22) by $-r(l)(\psi'(l))^\gamma$ and taking the limsup, we get
\[
\limsup_{l \to \infty} \int_{l_1}^{l} q(\theta)B_{1}^{\gamma}(\theta)\,d\theta \leq 1.
\]
Thus, we get a contradiction with (20).

Let $\psi' > 0$ on $[l_1, \infty)$. From (20) and the fact that $\kappa(l, \infty) < \infty$, we have that (18) holds. Then, this part of proof is similar to that of Theorem 1. Therefore, the proof is complete. \(\square\)

Theorem 3. Assume that $B_2(l) > 0$, $B_1(l) > 0$ and $r' > 0$. If there exist positive functions $\mu, \delta \in C^1([l_0, \infty))$ and $l_1 \in [l_0, \infty)$ such that
\[
\limsup_{l \to \infty} \left\{ \frac{\kappa^\gamma(l, \infty)}{\delta(l)} \int_{l_1}^{l} \left( \delta(\theta)q(\theta)B_{1}^{\gamma}(\sigma(\theta)) - \frac{r(\theta)}{(\gamma + 1)^{\gamma+1}} (\delta(\theta))^{\gamma+1} \right) \,d\theta \right\} > 1
\]
and
\[
\limsup_{l \to \infty} \int_{l_1}^{l} \left( \mu(\theta)q(\theta)B_{1}^{\gamma}(\sigma(\theta)) - \frac{1}{(\gamma + 1)^{\gamma+1}} \frac{r(\theta)(\mu(\theta))^{\gamma+1}}{(\mu(\theta)(\sigma(\theta)))^{\gamma+1}} \right) \,d\theta = \infty,
\]
then, all solutions of (1) are oscillatory.

Proof. Assume the contrary that Equation (1) has a positive solution $y$ on $[l_0, \infty)$. Then, $y(\rho_1(l)), y(\rho_2(l))$ and $y(\sigma(l))$ are positive for all $l \geq l_1$, where $l_1$ is large enough. Thus, from (1) and the definition of $\psi$, we note that $\psi(l) \geq y(l) > 0$ and $r(l)(\psi'(l))^\gamma$ is nonincreasing. Therefore, $\psi'$ is either eventually negative or eventually positive.

Let $\psi' < 0$ on $[l_1, \infty)$. As in proof of Theorem 1, we arrive at (10). Now, we define the function
\[
\omega(l) = \delta(l) \left( \frac{r(l)(\psi'(l))^\gamma}{\psi^\gamma(l)} + \frac{1}{\kappa^\gamma(l, \infty)} \right) \text{ on } [l_1, \infty).
\]
From (8), we have that $\omega \geq 0$ on $[l_1, \infty)$. Differentiating (25), we get

$$\omega'(l) = \frac{\delta'(l)}{\delta(l)} \omega(l) + \delta(l) \left( \frac{r(l)(\varphi'(l))^{\gamma'}}{\varphi'(l)} \right)' - \gamma \delta(l) r(l) \left( \frac{\varphi'(l)}{\varphi(l)} \right)^{\gamma+1}$$

$$+ \frac{\gamma \delta(l)}{r^{\gamma/\gamma}(l)} \kappa^{\gamma+1}(l, \infty).$$

(26)

Combining (10) and (26), we have

$$\omega'(l) \leq - \frac{\gamma}{(\delta(l) r(l))^1/\gamma} \left( \omega(l) - \frac{\delta(l)}{\kappa^\gamma(l, \infty)} \right)^{(\gamma+1)/\gamma} - \delta(l) q(l) B^\gamma_1(\varphi(l)) \frac{\varphi'(l)}{\varphi(l)}$$

$$+ \frac{\gamma \delta(l)}{r^{\gamma/\gamma}(l) \kappa^{\gamma+1}(l, \infty)} + \frac{\delta'(l)}{\delta(l)} \omega(l).$$

(27)

Using Lemma 1 with $A := \delta'(l)/\delta(l)$, $B := \gamma(\delta(l) r(l))^{-1/\gamma}$, $C := \delta(l)/\kappa^\gamma(l, \infty)$ and $\theta := \omega$, we get

$$\frac{\delta'(l)}{\delta(l)} \omega(l) - \frac{\gamma}{(\delta(l) r(l))^1/\gamma} \left( \omega(l) - \frac{\delta(l)}{\kappa^\gamma(l, \infty)} \right)^{(\gamma+1)/\gamma} \leq \frac{1}{(\gamma+1)^{\gamma+1}} r(l) \frac{(\delta'(l))^{\gamma+1}}{(\delta(l))^\gamma}$$

As $l \geq \sigma(l)$, we arrive at

$$\psi(\sigma(l)) \geq \psi(l),$$

(28)

which, in view of (27) and (28), gives

$$\omega'(l) \leq \frac{\delta'(l)}{\kappa^\gamma(l, \infty)} + \frac{1}{(\gamma+1)^{\gamma+1}} r(l) \frac{(\delta'(l))^{\gamma+1}}{(\delta(l))^\gamma} - \delta(l) q(l) B^\gamma_1(\varphi(l)) \frac{\varphi'(l)}{\varphi(l)}$$

$$+ \frac{\gamma \delta(l)}{r^{\gamma/\gamma}(l) \kappa^{\gamma+1}(l, \infty)} + \frac{\delta'(l)}{\delta(l)} \omega(l).$$

(29)

Integrating (29) from $l_2$ to $l$, we arrive at

$$\int_{l_2}^{l} \left( \frac{\delta'(l)}{\kappa^\gamma(l, \infty)} - \frac{r(l)(\varphi'(l))^{\gamma'}}{(\gamma+1)^{\gamma+1}} \right) d\theta \leq \left( \frac{\delta(l)}{\kappa^\gamma(l, \infty)} - \omega(l) \right)_{l_2}$$

$$\leq - \left( \frac{\delta(l) r(l)(\varphi'(l))^{\gamma'}}{(\gamma+1)^{\gamma+1}} \varphi(l) \right)_{l_2}. (30)$$

From (8), we have

$$- \frac{r^{1/\gamma}(l) \varphi'(l)}{\psi(l)} \leq \frac{1}{\kappa(l, \infty)},$$

which, in view of (30), implies

$$\frac{\kappa^\gamma(l, \infty)}{\delta(l)} \int_{l_2}^{l} \left( \frac{\delta'(l) q(l) \varphi(l)}{\psi(l)} - \frac{r(l)(\varphi'(l))^{\gamma'}}{(\gamma+1)^{\gamma+1}} \right) d\theta \leq 1.$$
Thus, we get a contradiction with (23).

Let \( \psi'(l) > 0 \) on \([l_1, \infty)\). As in proof of Theorem 1, we arrive at (17). Now, we define the function

\[
\varphi(l) = \mu(l) \frac{r(l)(\psi'(l))^{\gamma}}{\psi'(\sigma(l))}.
\]

(31)

Therefore, we have that \( \omega \geq 0 \) on \([l_1, \infty)\). Differentiating (31), we find

\[
\varphi'(l) = \frac{\mu'(l)}{\mu(l)} \varphi(l) + \mu(l) \frac{(r(l)(\psi'(l))^{\gamma})'}{\psi'(\sigma(l))} - \gamma \mu(l) r(l) \frac{(\psi'(l))^{\gamma}(\psi'(\sigma(l)))^{\gamma+1}}{\psi'(\sigma(l))^{\gamma+1}}.
\]

(32)

Combining (17) and (32), we have

\[
\varphi'(l) \leq \frac{\mu'(l)}{\mu(l)} \varphi(l) - \mu(l) q(l) B_2^2(\sigma(l)) - \gamma \mu(l) r(l) \frac{(\psi'(l))^{\gamma+1}}{\psi'(\sigma(l))^{\gamma+1}}.
\]

As \( (r(l)(\psi'(l))^{\gamma})' < 0 \) and \( \sigma(l) \leq l \), we arrive at

\[
\varphi'(l) \leq \frac{\mu'(l)}{\mu(l)} \varphi(l) - \mu(l) q(l) B_2^2(\sigma(l)) - \gamma \mu(l) r(l) \frac{(\psi'(l))^{\gamma+1}}{\psi'(\sigma(l))^{\gamma+1}}.
\]

From (31), we have

\[
\varphi'(l) \leq \frac{\mu'(l)}{\mu(l)} \varphi(l) - \mu(l) q(l) B_2^2(\sigma(l)) - \gamma \frac{\varphi'(l)}{\mu^{\gamma}(r^{\gamma})(l)} \frac{r(l)(\mu'(l))^{\gamma+1}}{\mu^{\gamma}(l)(\sigma'(l))^{\gamma}}.
\]

(34)

Using the inequality

\[
K \varphi - L \varphi^{(\gamma+1)/\gamma} \leq \frac{\gamma \gamma}{(\gamma + 1)\gamma+1} \frac{K^{\gamma+1}}{L^{\gamma}}, \quad L > 0,
\]

(33)

with \( K = \mu'(l)/\mu(l), L = \gamma \sigma'(l)/\mu^{1/\gamma}(l) r^{1/\gamma}(l) \) and \( \nu = \varphi \), we have

\[
\varphi'(l) \leq -\mu(l) q(l) B_2^2(\sigma(l)) + \frac{1}{(\gamma + 1)^{\gamma+1}} \frac{r(l)(\mu'(l))^{\gamma+1}}{\mu^{\gamma}(l)(\sigma'(l))^{\gamma}}.
\]

Integrating (34) from \( l_2 \) to \( l \), we arrive at

\[
\int_{l_2}^{l} \left( \frac{\mu(\theta)q(\theta)B_2^2(\sigma(\theta))}{\sigma'(\theta)^{\gamma}} - \frac{1}{(\gamma + 1)^{\gamma+1}} \frac{r(\theta)(\mu'(\theta))^{\gamma+1}}{\mu^{\gamma}(\theta)(\sigma'(\theta))^{\gamma}} \right) d\theta \leq \varphi(l_2).
\]

Taking the \( \text{lim sup} \) on both sides of this inequality, we have a contradiction with (24).

The proof of the theorem is complete. \( \square \)

**Example 1.** Consider the second-order neutral differential equation

\[
\left( l^2 \left( y(l) + p_0 y \left( \frac{1}{A} \right) + p_\ast y(l) \lambda l \right) \right)' + q_0 y(\sigma_0 l) = 0,
\]

(35)

where \( \lambda > 1, \sigma_0 \in (0, 1) \) and \( (\lambda p_0 + p_\ast) \in (0, 1) \). We note that \( r(l) = l^2, p_1(l) = p_0 \), \( p_2(l) = p_\ast \), \( p_1(l) = 1/\lambda, p_2(l) = \lambda l, q(l) = q_0 l \) and \( \sigma(l) = \sigma_0 l \). It is easy to verify that

\[
B_3(l) = 1 - \lambda p_0 - p_\ast,
\]
and
\[ B_2(l) = 1 - p_0 - p_*(l - T), \]
and so \( B_2 > B_1 > 0 \). Now, we see that
\[
\limsup_{l \to \infty} \int_1^l \frac{1}{r^{1/\gamma}(\beta)} \left( \int_1^\beta q(\delta) B_1^{\gamma}(\sigma(\delta), \infty) d\delta \right)^{1/\gamma} d\beta
\]
\[
= \limsup_{l \to \infty} \int_1^l \frac{1}{\beta^2} \left( \int_1^\beta q_0(1 - \lambda p_0 - p_*) \frac{1}{c_0}\right) d\beta = \infty.
\]

Then, by Theorem 1, we have that (35) is oscillatory.

3. Conclusions

In this work, new criteria to test the oscillation of the solutions of second-order non-canonical neutral differential equations with mixed type were presented. These criteria are to further complement and simplify relevant results in the literature.

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References

1. Hale, J.K. Partial neutral functional differential equations. Rev. Roum. Math. Pures Appl. 1994, 39, 339–344.
2. MacDonald, N. Biological Delay Systems: Linear Stability Theory; Cambridge University Press: Cambridge, UK, 1989.
3. Grammatikopoulos, M.K.; Ladas, G.; Meimaridou, A. Oscillation of second order neutral delay differential equations. Rad. Math. 1985, 1, 267–274.
4. Ruan, S.G. Oscillations of second order neutral differential equations. Can. Math. Bull. 1993, 36, 485–496. [CrossRef]
5. Arul, R.; Shobha, V.S. Oscillation of second order neutral differential equations with mixed neutral term. Int. J. Pure Appl. Math. 2015, 104, 181–191. [CrossRef]
6. Dzurina, J.; Busha, J.; Airyan, E.A. Oscillation criteria for second-order differential equations of neutral type with mixed arguments. Differ. Equ. 2002, 38, 137–140. [CrossRef]
7. Li, T. Comparison theorems for second-order neutral differential equations of mixed type. Electron. J. Differ. Equ. 2020, 1–7.
8. Li, T.; Baculíková, B.; Džurina, J. Oscillation results for second-order neutral differential equations of mixed type. Tatra Mt. Math. Publ. 2011, 48, 101–116. [CrossRef]
9. Thandapani, E.; Selvarangam, S.; Vijaya, M.; Rama, R. Oscillation results for second order nonlinear differential equation with delay and advanced arguments. Kyungpook Math. J. 2016, 56, 137–146. [CrossRef]
10. Grace, S.R.; Graef, J.R.; Jadlovská, I. Oscillation criteria for second-order half-linear delay differential equations with mixed neutral terms. Math. Slovaca 2019, 69, 1117–1126. [CrossRef]
11. Tunc, E.; Ozdemir, O. On the oscillation of second-order half-linear functional differential equations with mixed neutral term. J. Taibah Univ. Sci. 2019, 13, 481–489. [CrossRef]
12. Han, Z.; Li, T.; Zhang, C.; Sun, S. Oscillatory behavior of solutions of certain third-order mixed neutral functional differential equations. Bull. Malays. Math. Sci. Soc. 2012, 35, 611–620.
13. Moaaz, O.; Chalishajar, D.; Bazighifan, O. Asymptotic behavior of solutions of the third order nonlinear mixed type neutral differential equations. Mathematics 2020, 8, 485. [CrossRef]
14. Agarwal, R.; Shieh, S.L.; Yeh, C.C. Oscillation criteria for second order retard differential equations. Math. Comput. Model. 1997, 26, 1–11. [CrossRef]
15. Agarwal, R.P.; Zhang, C.; Li, T. Some remarks on oscillation of second order neutral differential equations. *Appl. Math. Compt.* 2016, 274, 178–181. [CrossRef]

16. Baculikova, B.; Dzurina, J. Oscillation theorems for second-order nonlinear neutral differential equations. *Comput. Math. Appl.* 2011, 62, 4472–4478.

17. Bohner, M.; Grace, S.R.; Jadlovska, I. Oscillation criteria for second-order neutral delay differential equations. *Electron. J. Qual. Theory Differ. Equ.* 2017, 2017, 1–12. [CrossRef]

18. Chatzarakis, G.E.; Dzurina, J.; Jadlovska, I. New oscillation criteria for second-order half-linear advanced differential equations. *Appl. Math. Comput.* 2019, 347, 404–416. [CrossRef]

19. Moaaz, O.; Elabbasy, E.M.; Qaraad, B. An improved approach for studying oscillation of generalized Emden–Fowler neutral differential equation. *J. Inequal. Appl.* 2020, 2020, 69. [CrossRef]

20. Moaaz, O.; Muhib, A. New oscillation criteria for nonlinear delay differential equations of fourth-order. *Appl. Math. Comput.* 2020, 377, 125192. [CrossRef]

21. Sun, Y.G.; Meng, F.W. Note on the paper of Dzurina and Stavroulakis: “Oscillation criteria for second-order delay differential equations” [Appl. Math. Comput. 2003, 140, 445–453]. *Appl. Math. Comput.* 2006, 174, 1634–1641.

22. Xu, R.; Meng, F. Some new oscillation criteria for second order quasi-linear neutral delay differential equations. *Appl. Math. Comput.* 2006, 182, 797–803. [CrossRef]

23. Zhang, C.; Agarwal, R.P.; Bohner, M.; Li, T. New results for oscillatory behavior of even-order half-linear delay differential equations. *Appl. Math. Lett.* 2013, 26, 179–183. [CrossRef]

24. Zhang, C.; Li, T.; Suna, B.; Thandapani, E. On the oscillation of higher-order half-linear delay differential equations. *Appl. Math. Lett.* 2011, 24, 1618–1621. [CrossRef]