Existence of solution for a nonlinear model of thermo-visco-plasticity

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1 | INTRODUCTION AND MAIN RESULT

In this paper, we propose an extension of the nonlinear thermo-visco-elasticity system (the mechanical dissipation is not linearised) to include a plastic effects. Our motivation for current considerations were the result of Blanchard and Guibé. In Blanchard and Guibé, the authors considered the nonlinear thermo-visco-elasticity system and the existence of renormalised solutions for parabolic equations with $L^1$ data was proved (see also Blanchard and Murat). The case of small strains is analysed. Following Chełminski, Chełmiński and Racke, Duvaut and J.L. Lions, Guizada, Klawe and Świerczewska-Gwiazda, Suquet, Temam, and many others, we assume that displacement evolution is so slow that an acceleration term is negligible, so we only look at quasistatic form of the balance of momentum. In the considered model, the inelastic constitutive equation is given by a Lipschitz-continuous function. The mechanical problem is coupled with the heat conduction equation. Changes of the temperature influence the stresses and the domain of plastic behaviour of the considered material. On the other hand, stresses occurring in the body influence the heat production. This type of problems are studied from many points of view (see, eg, previous studies or Roubiček). See also Owczarek and Owczarek, where poroplasticity models are investigated (poroplasticity models have a similar structure to the linear thermo-plasticity). In this paper, we try to tackle a thermomechanically consistent model without the linearisation in neighborhood of the reference temperature (compare previous works).

We consider that the body occupies initially a domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial \Omega$. Let $x \in \Omega$ denote the material point while $t \in \mathbb{R}_+$ the time. Additionally, we will denote by $\mathcal{S}(3)$ the set of $3 \times 3$ symmetric matrices of real entries. The system of equations is written in the following form

\[
\begin{align*}
-\text{div}_x \sigma(t,x) &= b(t,x), \\
\sigma(t,x) &= D(\varepsilon(u(t,x)) - \varepsilon^{pl}(t,x)) + C(\varepsilon(\partial_t u(t,x))) - \phi(\theta(t,x)) II, \\
\partial_t \varepsilon^{pl}(t,x) &= \Lambda(\sigma(t,x), \theta(t,x)), \\
\partial_t \theta(t,x) - \kappa \Delta \theta(t,x) &= -\phi(\theta(t,x)) \text{div}_x \sigma(t,x) + \partial_t \varepsilon^{pl}(t,x) \cdot \sigma(t,x) + C(\varepsilon(\partial_t u(t,x))) \cdot \varepsilon(\partial_x u(t,x)).
\end{align*}
\]

(1)
The first equation describes the balance of momentum in the quasistatic case. The function \( \sigma : \mathbb{R}_+ \times \Omega \to \mathbb{S}(3) \) is the stress tensor. The vector function \( b : \mathbb{R}_+ \times \Omega \to \mathbb{R}^3 \) is a given density of volume forces. The second equation is the generalisation of Hooke's law in Kelvin-Voight constraint. The function \( u : \mathbb{R}_+ \times \Omega \to \mathbb{R}^3 \) describes displacement. We denote by \( \epsilon(u) := \frac{1}{2} (\nabla u + (\nabla u)^T) \) the symmetrical gradient of displacement also called the linear Cauchy strain tensor or simply small-strain tensor. It follows directly from definition that \( \epsilon(u) \in \mathbb{S}(3) \). Additionally, we denote the plastic strain by \( \epsilon^{pl} : \mathbb{R}_+ \times \Omega \to \mathbb{S}(3) \). The function \( \Theta : \mathbb{R}_+ \times \Omega \to \mathbb{R} \) is the temperature. The function \( \phi : \mathbb{R} \to \mathbb{R} \) defines so-called thermal part of stress. We assume that it is a sublinear continuous function, ie, for all \( s \in \mathbb{R} \)

\[
|\phi(s)| \leq |s|^\alpha,
\]

where \( \alpha > 0 \) will be indicated later. By \( \mathbb{I} \) we denote the identity \( 3 \times 3 \) matrix. We assume that operators \( D : \mathbb{S}(3) \to \mathbb{S}(3) \) and \( C : \mathbb{S}(3) \to \mathbb{S}(3) \) are given linear, symmetric and positive definite. In the next formula, the evolution in time of plastic strain is given by an ordinary differential equation. Here, the function \( \Lambda : \mathbb{S}(3) \times \mathbb{R} \to \mathbb{S}(3) \) is Lipschitz continuous with respect to the first argument and Hölder continuous with a respect to the second one, ie, there exist constants \( L_1, L_2 > 0 \) such that for all \( A, B \in \mathbb{S}(3) \) and \( \varphi, \psi \in \mathbb{R} \) it holds that

\[
|\Lambda(A, \varphi) - \Lambda(B, \psi)| \leq L_1|A - B| + L_2|\varphi - \psi|^\beta,
\]

where \( \beta > 0 \) will be indicated later.

The considered model includes nonlinearities that are not globally Lipschitz. This is because on the right-hand side of the heat conduction equation, there are terms \( \phi(\Theta) \text{div}_x \partial_t u \) and \( \partial_t \epsilon^{pl} \cdot \sigma \). They are the worst troublemakers due to their integrability and, in some models, if the acceleration term \( \partial_t^2 u \) is neglected also term \( \partial_t \epsilon^{pl} \cdot \sigma \) is omitted on the right-hand side of the heat conduction equation.

We complete the system above with boundary conditions

\[
\begin{aligned}
\quad & u(t, x) = 0, \quad \text{for } (t, x) \in (0, T) \times \Gamma_0, \\
\quad & \sigma(t, x) \cdot \mathbf{n}(x) = g(t, x), \quad \text{for } (t, x) \in (0, T) \times \Gamma_1, \\
\quad & \frac{\partial u}{\partial n}(t, x) = h(t, x), \quad \text{for } (t, x) \in (0, T) \times \partial \Omega.
\end{aligned}
\]

(BC)

where \( \Gamma_0, \Gamma_1 \subset \partial \Omega \) satysfy \( \Gamma_0 \cap \Gamma_1 = \emptyset \), \( \Gamma_0 \cup \Gamma_1 = \partial \Omega \) and \( H^2(\Gamma_0) > 0 \) (here, \( H^2 \) denotes two-dimensionsal Hausdorff measure). Let \( \Gamma_0 \) is relatively closed and \( \Gamma_1 \) is relatively open in \( \partial \Omega \). Additionally, we will require that the set \( \Omega \cup \Gamma_1 \) is regular in the sense of Gröger, Definition 2. Let \( \omega : U \to \omega(U) \subset \mathbb{R}^3 \) such that \( \omega^{-1} \) is also Lipschitz continuous map and that the set \( \omega(U \cap (\Omega \cup \Gamma_1)) \) takes one of the following forms:

\[
\begin{aligned}
E_1 & := \{ y \in \mathbb{R}^3 : |y| < 1, y_3 < 0 \}, \\
E_2 & := \{ y \in \mathbb{R}^3 : |y| < 1, y_3 \leq 0 \}, \\
E_3 & := \{ y \in E_2 : y_3 < 0 \text{ or } y_1 > 0 \}.
\end{aligned}
\]

Finally, we set initial conditions as follows:

\[
\begin{aligned}
\epsilon^{pl}(0, x) &= \epsilon^{pl}(x), \quad \text{for } x \in \Omega, \\
u(0, x) &= u_0(x), \quad \text{for } x \in \Omega, \\
\theta(0, x) &= \theta_0(x), \quad \text{for } x \in \Omega.
\end{aligned}
\]

(IC)

Now, let us introduce, following Gröger, Definition 3 (compare also with Herzog et al., Definition 1.8), the function space that is very useful in considering of the mixed boundary problems as follows:

\[
W^{1,q}_r(\Omega) := \{ u |_{\Omega} : u \in C^0(\mathbb{R}^3) \text{ and } \text{supp} u \cap \Gamma_0 = \emptyset \},
\]

where the closure is taken with respect to standard norm in the Sobolev space \( W^{1,q}(\Omega) \), ie, for \( u \in W^{1,q}(\Omega) \), we have

\[
\left\| u \right\|_{W^{1,q}(\Omega)} := \left( \int_{\Omega} |u|^q \, dx + \sum_{i=1}^{3} \int_{\Omega} |\partial_i u|^q \, dx \right)^{\frac{1}{q}}.
\]
The mixed boundary condition for elasticity was studied by Herzog et al. We recall the main fact that we will exploit in the further consideration, i.e., Herzog et al. 24, Theorem 1.1:

**Theorem 1.** Define the following operator \( D_s : W^{1,q}_{\Gamma_0} \to \left( W^{1,q'}_{\Gamma'_0}(\Omega) \right)^* \) as follows:

\[
[D_s(u); v] := \int_{\Omega} D(\mathbf{\varepsilon}(u)) \cdot \mathbf{\varepsilon}(v) \, dx \quad \text{for } u \in W^{1,q}_{\Gamma_0}, v \in W^{1,q'}_{\Gamma'_0}(\Omega).
\]

There exists \( \bar{q} > 2 \) such that for all \( s \in [2, \bar{q}] \) the operator \( D_s \) is continuously invertible. Moreover, the inverse is globally Lipschitz with a Lipschitz constant independent of \( q \in [2, \bar{q}] \).

**Remark 1.** It is worth to notice that it follows from the proof of the theorem mentioned above the constant \( \bar{q} \) depends only on a geometry of the domain \( \Omega \) (also on the partition of the boundary) and entries of the operator \( D \).

In the further investigation, we will treat the constant \( \bar{q} \) as fixed.

Now, we formulate the main framework for further consideration.

Let parameters \( p, r, s, \alpha, \beta \in \mathbb{R} \) satisfy the following relations:

\[
\begin{aligned}
\infty > p, & r > 1, \\
2 \leq q \leq \bar{q}, & \\
\frac{1}{2} > \alpha, & \beta > 0, \\
r \geq \max\{\alpha, \beta\} \cdot q, & \\
p \geq \max\{\alpha, \beta\} \cdot s.
\end{aligned}
\]

For \( p, q, r, s, \alpha, \beta \) indicated above, we introduce assumption on the given data:

**A1** Let \( b \in L^2(0; T; L^3(\Omega; \mathbb{R}^3)) \),

**A2** Let \( g \in L^2(0; T; W^{-1/q}(0, T; \Gamma_1; \mathbb{R}^3)) \),

**A3** Let \( h \in L^p_{p-1/2r}(0, T; L^q(\partial \Omega)) \cap L^q(0, T; W^{1/2-r, 1/2}(\partial \Omega)) \),

**A4** Let \( \mathbf{\varepsilon}_0 \in W^{1,q}(\Omega; \mathcal{S}(3)) \),

**A5** Let \( u_0 \in W^{1,q}(\Omega; \mathbb{R}^3) \),

**A6** Let \( \theta_0 \in B_{r/p}^{2;1-p}(\Omega) \).

Additionally, in the case \( r > 3 \), we have to assume a compatibility condition in the following form:

\[
\frac{\partial \theta_0(x)}{\partial n} = h(0, x), \quad \text{for } x \in \partial \Omega.
\]

The function space \( L^{p-r/2r}(\Omega; \mathcal{S}(3)) \) denotes Lizorkin-Triebel space and \( B_{r/p}^{2;1-p}(\Omega) \) denotes Besov space. For definition and more details, see, e.g., other works. 25,26 In Denk et al. 27 authors proved that such data are enough and sufficient to obtain an existence and optimal regularity of a parabolic equation. We use assumptions \( A3 \) and \( A6 \) to justify lemma 3.

The main theorem of current paper is

**Theorem 2.** Let the assumptions \( A1 \) to \( A6 \) be satisfied. Additionally, if \( r > 3 \), then we assume a compatibility condition (CC). Then for any fixed length of the considered time interval \( T > 0 \), there exists a solution to the system \((TP) + (BC) + (IC)\) \((u, \mathbf{\varepsilon}_p, \theta)\) such that

\[
\begin{align*}
u & \in W^{1,q}(0, T; W^{1,q}(\Omega)), \\
\mathbf{\varepsilon}_p & \in W^{1,q}(0, T; L^q(\Omega)), \\
\theta & \in L^r(0, T; W^{1, r}(\Omega)) \cap W^{1,p}(0, T; L^q(\Omega)).
\end{align*}
\]

Theorem 2 presents one of the first existence result for the nonlinear thermo-visco-plasticity model in which the elastic and plastic constitutive equations depend on the temperature. In the literature, we can find the articles, where the thermo-visco-plastic models are considered, in which the elastic equations do not depend on the temperature effects (see, for example, Bartels and Roubíček 13 and other studies 28,29) or the temperature effects are not included in the plastic flow rules (see, for example, other works 28,30 and Roubíček 37). In the future, we would treat the field \( \Lambda \) in evolution equation...
In this section, we are going to prove that there exists a solution \((u, \varepsilon^p, \theta)\) defined on \((0, T) \times \Omega\). We will handle with the problem of the existence using Schauder fixed point theorem. Before we start with the proof of theorem 2, we prove the following corollary, which can be obtained from theorem 1.

**Corollary 1.** From theorem 1, we can deduce a solvability to a problem of linear elasticity, ie, let \(b \in L^*(0, T; (W_0^{1,q'}(\Omega; \mathbb{R}^3))^*)\) while \(g \in L^*(0, T; W^{-1/q}(0, T; (\Gamma_1; \mathbb{R}^3)))\). Moreover, let \(u_0 \in W_0^{1,q}(\Omega)\). Then there exists a unique solution \(u \in W^{1,q}(0, T; W_0^{1,q}(\Omega))\) to the following problem:

\[
\begin{aligned}
-\text{div}_x D(\varepsilon(u(t,x))) - \text{div}_x C(\varepsilon(\partial_t u(t,x))) &= b(t,x), \text{ for } (t,x) \in (0, T) \times \Omega, \\
\varepsilon(t,x) &= 0, \text{ in } (t,x) \in (0, T) \times \Gamma_0, \\
[D(\varepsilon(u(t,x))) + C(\varepsilon(\partial_t u(t,x)))] \cdot \vec{n}(x) &= g(t,x), \text{ for } (t,x) \in (0, T) \times \Gamma_1, \\
u(0,x) &= u_0(x), \text{ for } x \in \Omega.
\end{aligned}
\]

Further, the solution \(u\) can be estimated as follows:

\[
\|u\|_{W^{1,q}(0, T; W_0^{1,q}(\Omega))} \leq C (T e^{CT} + 1) \left( \|u_0\|_{W_0^{1,q}(\Omega)} + \|b\|_{L^*(0, T; (W_0^{1,q'}(\Omega))^*)} + \|g\|_{L^*(0, T; W^{-1/q}(\Gamma_1))} \right),
\]

where a constant \(C > 0\) depends only on a geometry of the domain \(\Omega\) (also on the partition of the boundary) and entries of operators \(D\) and \(C\).

**Proof.** The proof follows the Banach fixed point theorem. We are going to construct a contractive operator \(P : L^*(0, T; W_0^{1,q}(\Omega)) \rightarrow L^*(0, T; W_0^{1,q}(\Omega))\). Let \(v \in L^*(0, T; W_0^{1,q}(\Omega))\) and we look for the solution to the following problem:

\[
\begin{aligned}
-\text{div}_x C(\varepsilon(w)) &= b + \text{div}_x D(\varepsilon(v)), \text{ for } (0, T) \times \Omega, \\
w &= 0, \text{ on } (0, T) \times \Gamma_0, \\
C(\varepsilon(w)) \cdot \vec{n} &= g - D(\varepsilon(v)) \cdot \vec{n}, \text{ on } (0, T) \times \Gamma_1, \\
w|_{t=0} &= u_1, \text{ in } \Omega.
\end{aligned}
\]

That is, for any \(\varphi \in W_0^{1,q}(\Omega)\) and a.e. \(t \in (0, T)\), it holds that

\[
\int_{\Omega} C(\varepsilon(w(t,x))) \varepsilon(\varphi(x)) dx = \langle b(t), \varphi \rangle_{(W_0^{1,q'}(\Omega))^*, W_0^{1,q'}(\Omega)} + \int_{\Gamma_1} g(t,x) \varphi(x) dS(x) + \int_{\Omega} D(\varepsilon(v(t,x))) \varepsilon(\varphi(x)) dx.
\]

Let us observe, since \(v \in L^*(0, T; W_0^{1,q}(\Omega))\), that the right-hand side of (3) defines for a.e. \(t \in (0, T)\) a bounded and linear functional on \(W_0^{1,q}(\Omega)\). By theorem 1, there exists unique solution \(v \in L^*(0, T; W_0^{1,q}(\Omega))\) of the problem (\(\star\)) satisfying

\[
\|w\|_{L^*(0, T; W_0^{1,q}(\Omega))} \leq C \left( \|b\|_{L^*(0, T; (W_0^{1,q'}(\Omega))^*)} + \|g\|_{L^*(0, T; W^{-1/q}(\Gamma_1))} + \|v\|_{L^*(0, T; W_0^{1,q}(\Omega))} \right).
\]
Now, we put
\[ u(t, x) = P(v)(t, x) := \int_0^t w(\tau, x) d\tau + u_0(x). \]

It is easy to observe, that such defined \( u \) belongs to space \( W^{1, s}(0, T; W^{1, q}_r(\Omega)) \). Now we insert \( v_1, v_2 \in L^s(0, T; W^{1, q}_r(\Omega)) \) into system (\( \star \)) and obtain the solutions \( w_1, w_2 \in L^s(0, T; W^{1, q}_r(\Omega)) \). Therefore, the difference \( w_1 - w_2 \) satisfies the following system:
\[
\begin{align*}
-\text{div}_x C(\varepsilon (w_1 - w_2)) &= \text{div}_x D(\varepsilon (v_1 - v_2)), & (0, T) \times \Omega, \\
&w_1 - w_2 = 0, & (0, T) \times \Gamma_0, \\
C(\varepsilon (w_1 - w_2)) \cdot \vec{n} &= -D(\varepsilon (v_1 - v_2)) \cdot \vec{n}, & (0, T) \times \Gamma_1, \\
w_1 - w_2|_{t=0} = 0, & \Omega.
\end{align*}
\]

That is, similarly as previously, for any \( \varphi \in W^{1, q}_r(\Omega) \) and a.e. \( t \in (0, T) \), it holds that
\[
\int_{\Omega} C(\varepsilon (w_1(t, x) - w_2(t, x)) \varepsilon(\varphi(x)) dx = \int_{\Omega} D(\varepsilon (v_1(t, x) - v_2(t, x)) \varepsilon(\varphi(x)) dx. \tag{4}
\]

Again, using theorem 1, since, as previously, the right-hand side of (4) defines a bounded and linear functional on \( W^{1, q}_r(\Omega) \), we conclude that for a.e. \( t \in (0, T) \), it holds that
\[
\|w_1(t) - w_2(t)\|_{W^{1, q}_r(\Omega)} \leq C \|v_1(t) - v_2(t)\|_{W^{1, q}_r(\Omega)}.
\]

And immediately,
\[
\begin{align*}
\|u_1 - u_2\|_{L^s(0, T; W^{1, q}_r(\Omega))} &
\leq \left\| \int_0^t (w_1(\tau, x) - w_2(\tau, x)) d\tau \right\|_{L^s(0, T; W^{1, q}_r(\Omega))} \\
&
\leq C t \|v_1 - v_2\|_{L^s(0, T; W^{1, q}_r(\Omega))}, \tag{5}
\end{align*}
\]

We claim that the operator \( P_1 : L^s((0, T_1); W^{1, q}_r(\Omega)) \to L^s((0, T_1); W^{1, q}_r(\Omega)) \) is a contraction for \( T_1 = \frac{1}{2C} \). Hence, by the Banach fixed point theorem, we obtain that there exists an unique element \( u \in L^s((0, T_1); W^{1, q}_r(\Omega)) \) such that \( P_1(u) = u \). Moreover, from the construction of the operator \( P \), we immediately have that also \( \delta u \in L^s((0, T); W^{1, q}_r(\Omega)) \).

One can see that estimate 5 does not depend on the initial condition; thus, we can repeat the reasoning above to obtain a sequence of contractive operators
\[ P_k : L^s((T_{k-1}, T_k); W^{1, q}_r(\Omega)) \to L^s((T_{k-1}, T_k); W^{1, q}_r(\Omega)), \]

where \( T_k = \frac{k}{2C} \) and corresponding solutions \( u \in W^{1, s}((T_{k-1}, T_k); W^{1, q}_r(\Omega)) \).

It remains to prove the estimate (2). First, we integrate the weak formulation of the system (LE) with respect to time to obtain
\[
\begin{align*}
\int_{\Omega} \int_0^t D(\varepsilon (u(\tau, x))) \varepsilon(\varphi(x)) dx d\tau &+ \int_{\Omega} \int_0^t C(\varepsilon(\partial_t u(\tau, x))) \varepsilon(\varphi(x)) dx d\tau = \\
&\int_{\Omega} \int_0^t \langle b(\tau, \varphi), (w^{1, q}_r(\Omega))^{\gamma; W^{1, q}_r(\Omega)} \rangle d\tau + \int_{\Gamma_1} \int_0^t g(\tau, x) \varphi(x) dS(x) d\tau,
\end{align*}
\]

for any \( \varphi \in W^{1, q}_r(\Omega) \) and \( t \in (0, T) \). Then, using properties of weak derivatives and the initial condition on \( u \), we obtain
\[
\begin{align*}
\int_{\Omega} C(\varepsilon (u(t, x))) \varepsilon(\varphi(x)) dx d\tau &+ \int_{\Omega} \int_0^t C(\varepsilon(u_0(x))) \varepsilon(\varphi(x)) dx d\tau - \int_{\Omega} \int_0^t D(\varepsilon(u(t, x))) \varepsilon(\varphi(x)) dx d\tau \\
&+ \int_{\Gamma_1} \int_0^t \langle b(\tau, \varphi), (w^{1, q}_r(\Omega))^{\gamma; W^{1, q}_r(\Omega)} \rangle d\tau + \int_{\Gamma_1} \int_0^t g(\tau, x) \varphi(x) dS(x) d\tau.
\end{align*}
\]
The right-hand side of the equation above defines a bounded and linear functional on $W^{1,q'}_t(\Omega)$. Therefore, using theorem 1, for a.e. $t \in (0, T)$, we obtain the following estimate:

$$\|u(t)\|_{W^{1,q}_t(\Omega)} \leq C \left( \|u_0\|_{W^{1,q}_t(\Omega)} + \int_0^t \|u(\tau)\|_{W^{1,q}_t(\Omega)} d\tau + \int_0^t \|b(\tau)\|_{L^q(\Omega)} d\tau + \int_0^t \|g(\tau)\|_{W^{-1/q(\Gamma_1)}} d\tau \right).$$

By Gronwall inequality, we obtain that for a.e. $t \in (0, T)$,

$$\|u(t)\|_{W^{1,q}_t(\Omega)} \leq C (e^{CT} + 1) \left( \|u_0\|_{W^{1,q}_t(\Omega)} + \|b(t)\|_{L^q(\Omega)} + \|g(t)\|_{W^{-1/q(\Gamma_1)}} \right).$$

and finally,

$$\|u\|_{L^p(0,T;W^{1,q}_t(\Omega))} \leq CT (e^{CT} + 1) \left( \|u_0\|_{W^{1,q}_t(\Omega)} + \|b\|_{L^p(0,T;L^q(\Omega))} + \|g\|_{L^p(0,T;W^{-1/q(\Gamma_1)})} \right).$$

Now, once again, using the weak formulation of (LE), for any $A_{\eta} \in W^{1,q}_t(\Omega)$ and a.e. $t \in (0, T)$, we get

$$\int_{\Omega} C(\varepsilon(\partial_t u(t,x))) \varepsilon(\varphi(x)) dx = -\int_{\Omega} D(\varepsilon(u(t,x))) \varepsilon(\varphi(x)) dx + \langle b(t), \varphi \rangle_{(W^{1,q}_t(\Omega)),W^{1,q}_t(\Omega)} + \int_{\Gamma_1} g(t,x) \varphi(x) dS(x).$$

We observe, as previously, that the right-hand side of the equation above defines a bounded and linear functional on $W^{1,q}_t(\Omega)$ and using theorem 1, for a.e. $t \in (0, T)$, we obtain the following estimate on the time derivative of $u$:

$$\|\partial_t u(t)\|_{W^{1,q}_t(\Omega)} \leq C \left( \|u(t)\|_{W^{1,q}_t(\Omega)} + \|b(t)\|_{L^q(\Omega)} + \|g(t)\|_{W^{-1/q(\Gamma_1)}} \right);$$

therefore,

$$\|\partial_t u\|_{L^q(0,T;W^{1,q}_t(\Omega))} \leq CT \left( e^{CT} + 1 \right) \left( \|u_0\|_{W^{1,q}_t(\Omega)} + \|b\|_{L^q(0,T;L^q(\Omega))} + \|g\|_{L^q(0,T;W^{-1/q(\Gamma_1)})} \right),$$

and the proof is completed. \(\square\)

### 2.1 Fixed point

We shall construct the compact operator

$$T : L^p(0,T;L^q(\Omega)) \rightarrow L^p(0,T;L^q(\Omega)).$$

Let us fix a function $\theta^* \in L^p(0,T;L^q(\Omega))$ and consider the first auxiliary problem:

\[
\begin{align*}
- \text{div}_x \sigma(t,x) &= b(t,x), \\
\sigma(t,x) &= D(\varepsilon(u(t,x))) - \varepsilon_0(t,x) + C(\varepsilon(\partial_t u(t,x))) - \phi(\theta^*(t,x))I, \\
\partial_t \varepsilon_0(t,x) &= \Lambda(\sigma(t,x), \theta^*(t,x)), \\
\partial_t \varepsilon_0(t,x)|_{\Gamma_{\infty}(0,T)} &= 0, \\
\sigma(t,x) \cdot \vec{n}(x)|_{\Gamma_{\infty}(0,T)} &= g(t,x), \\
\varepsilon_0(t,0,x) &= \varepsilon_0^0(x), \\
u(0,x) &= u_0(x).
\end{align*}
\]

(\text{AP1})

**Lemma 1.** Assume that $A_1$ to $A_2$ and $A_5$ to $A_6$ are satisfied and moreover $\theta^* \in L^p(0,T;L^q(\Omega))$. Then there exists the unique solution $(\sigma, \varepsilon_0, u)$ to (AP1) satisfying $\sigma \in L^p(0,T;L^q(\Omega))$ and $\varepsilon_0 \in W^{1,2}(0,T;L^q(\Omega))$ while $u \in W^{1,2}(0,T;W^{1,q}(\Omega;\mathbb{R}^3))$. Additionally, the following estimate holds

\[
\begin{align*}
\|\sigma\|_{L^p(0,T;L^q(\Omega))} + \|\varepsilon_0\|_{W^{1,2}(0,T;L^q(\Omega))} + \|u\|_{W^{1,2}(0,T;W^{1,q}(\Omega))} \\
\leq E(T) \left( \|\theta^*\|_{L^p(0,T;L^q(\Omega))} + \|\theta^*\|_{L^p(0,T;L^q(\Omega))} + |\Omega| \|\Lambda(0,0)\| + \|\varepsilon_0\|_{L^q(\Omega)} \right),
\end{align*}
\]

\[(6)\]
Proof. We will again use the Banach fixed point theorem. Let \( \epsilon^{pl,*} \in L^1(0, T; L^q(\Omega; S(3))) \), and we solve the linear elasticity problem in the form

\[
\begin{cases}
- \text{div}_x D(\epsilon(w)) - \text{div}_x C(\epsilon(\partial_t w)) = - \text{div}_x D(\epsilon^{pl,*}) - \nabla_x \phi(\theta^*) + b, & \text{on } (0, T) \times \Omega, \\
\epsilon = 0, & \text{on } (0, T) \times \Gamma_0, \\
[D(\epsilon(w)) + C(\epsilon(\partial_t w))] \cdot n = [D(\epsilon^{pl,*}) + ||\phi(\theta^*)||] \cdot n + g, & \text{on } (0, T) \times \Gamma_1.
\end{cases}
\]

(7)

More precisely, for any function \( \phi \in W^{1,q}_1(\Omega) \) and a.e. \( t \in (0, T) \), it holds:

\[
\int_\Omega D(\epsilon(w(t,x)))\epsilon(\phi(x))dx + \int_\Omega C(\epsilon(\partial_t w(t,x)))\epsilon(\phi(x))dx = \\
\int_\Omega D(\epsilon^{pl,*}(t,x))\epsilon(\phi(x))dx + \int_\Omega \epsilon(\theta^*(t,x))\nabla_x \phi(x)dx + \langle b(t), \phi \rangle_{(W^{1,q}_1(\Omega))^\ast; W^{1,q}_1(\Omega)} + \int g(t,x)\phi(x)dS(x).
\]

The right-hand side of the expression above defines a bounded and linear functional on \( W^{1,q}_1(\Omega) \), and by corollary 1, we obtain that there exists a unique solution \( w \in W^{1,0}(0, T; W^{1,q}_1(\Omega; \mathbb{R}^3)) \) satisfying the following estimate:

\[
\|w\|_{W^{1,0}(0, T; W^{1,q}_1(\Omega))} \leq CT \left( e^{CT} + 1 \right) \left( \left\| \epsilon^{pl,*} \right\|_{L^1(0, T; L^q(\Omega))} + \left\| \phi(\theta^*) \right\|_{L^1(0, T; L^q(\Omega))} + \left\| \theta^* \right\|_{L^1(0, T; L^q(\Omega))} + \left\| b \right\|_{L^1(0, T; L^q(\Omega))} + \left\| g \right\|_{L^1(0, T; L^q(\Omega))} \right),
\]

where the constant \( C > 0 \) depends on entries of operators \( D, C \) and the geometry of the set \( \Omega \). Let us denote \( D(T) := CT \left( e^{CT} + 1 \right) \) and let us put

\[
\sigma = D(\epsilon(w) - \epsilon^{pl,*}) + C(\epsilon(\partial_t w)) - \phi(\theta^*);
\]

thus, obviously,

\[
\|\sigma\|_{L^1(0, T; L^q(\Omega))} \leq C \left( \|w\|_{W^{1,0}(0, T; W^{1,q}_1(\Omega))} + \left\| \epsilon^{pl,*} \right\|_{L^1(0, T; L^q(\Omega))} \right) + D(T) \left( \left\| \epsilon^{pl,*} \right\|_{L^1(0, T; L^q(\Omega))} + \left\| \phi(\theta^*) \right\|_{L^1(0, T; L^q(\Omega))} + \left\| \theta^* \right\|_{L^1(0, T; L^q(\Omega))} + \left\| b \right\|_{L^1(0, T; L^q(\Omega))} + \left\| g \right\|_{L^1(0, T; L^q(\Omega))} \right).
\]

We define \( \epsilon^{pl}(t,x) := \int_0^t \Lambda(\sigma(x, \tau), \theta^*(x, \tau))d\tau + \epsilon^0(x) \), so we obtain the following estimate

\[
|\epsilon^{pl}(t,x)| \leq \int_0^t |\Lambda(\sigma(x, \tau), \theta^*(x, \tau))| d\tau + |\epsilon^0(x)| \\
\leq \int_0^t [\Lambda(\sigma(x, \tau), \theta^*(x, \tau)) - \Lambda(0, 0)] d\tau + t|\Lambda(0, 0)| + |\epsilon^0(x)| \\
\leq \int_0^t L_1 |\sigma(x, \tau)| + L_2 |\theta^*(x, \tau)| |d\tau + t|\Lambda(0, 0)| + |\epsilon^0(x)|.
\]

Hence, we can estimate as follows:

\[
\left\| \epsilon^{pl} \right\|_{L^1(0, T; L^q(\Omega))} \leq CT \left( L_1 \|\sigma\|_{L^1(0, T; L^q(\Omega))} + |\Omega| |\Lambda(0, 0)| \right) + L_2 \|\theta^*\|_{L^1(0, T; L^q(\Omega))} + |\epsilon^0(x)|_{L^1(\Omega)}.
\]

Now, we define the operator \( R : L^1(0, T; L^q(\Omega; S(3))) \to L^1(0, T; L^q(\Omega; S(3))) \) as \( R(\epsilon^{pl,*}) := \epsilon^{pl} \). We claim that we can choose such a short time interval that the operator \( R \) is a contraction. Indeed, suppose that \( \epsilon_1^{pl,*}, \epsilon_2^{pl,*} \in L^1(0, T; L^q(\Omega; S(3))) \), while \( w_1, w_2 \) respectively are solutions to 7, then we can obtain similarly as previously
where \( \sigma_i = D(\epsilon(w_i) - \epsilon^i_{pl,*}) + C(\epsilon(\partial_i w)) - \phi(\theta^*) \) for \( i = 1, 2 \). We subtract \( \epsilon^i_{pl} - \epsilon^i_{pl} \) to obtain

\[
\| \epsilon^i_{1}(t) - \epsilon^i_{2}(t) \|_{L^2(\Omega)} \leq \| \Lambda(\sigma_1(\tau), \theta^*(\tau)) - \Lambda(\sigma_2(\tau), \theta^*(\tau)) \|_{L^2(\Omega)} \int_0^t \| \sigma_1(\tau) - \sigma_2(\tau) \|_{L^2(\Omega)} d\tau;
\]

hence,

\[
\| \epsilon^i_{1} - \epsilon^i_{2} \|_{L^2(0,T;L^2(\Omega))} \leq L_1 \| \sigma_1 - \sigma_2 \|_{L^2(0,T;L^2(\Omega))}.
\]

Since \( w_1 \) and \( w_2 \) are solutions to (7) with terms \( \epsilon^i_{pl,*} \) and \( \epsilon^i_{pl,*} \) on the right-hand side, respectively, it follows from the linearity of the problem (7) that the difference \( w_1 - w_2 \) satisfies

\[
\begin{cases}
-\text{div}_x D(\epsilon(w_1 - w_2)) - \text{div}_x C(\epsilon(\partial_i w_1 - w_2)) = -\text{div}_x D(\epsilon^{pl,*}_{1} - \epsilon^{pl,*}_{2}), & \text{in } (0,T) \times \Omega, \\
\epsilon(w_1 - w_2) = 0, & \text{on } (0,T) \times \Gamma_0, \\
(D(\epsilon(w_1 - w_2)) + C(\epsilon(\partial_i w_1 - w_2))) \cdot \bar{n} = \epsilon^{pl,*}_{1} - \epsilon^{pl,*}_{2} \cdot \bar{n}, & \text{on } (0,T) \times \Gamma_1, \\
\epsilon(w_1 - w_2) = 0, & \text{on } \Omega \times \{t=0\}
\end{cases}
\]

and the following inequality holds

\[
\| w_1 - w_2 \|_{W^{1,2}(0,T;W^{1,2}(\Omega))} \leq D(T) \| \epsilon^{pl,*}_{1} - \epsilon^{pl,*}_{2} \|_{L^2(0,T;L^2(\Omega))},
\]

where the constant \( D(T) \) is the same as previously. Hence,

\[
\| \sigma_1 - \sigma_2 \|_{L^2(0,T;L^2(\Omega))} \leq C \left( \| w_1 - w_2 \|_{W^{1,2}(0,T;W^{1,2}(\Omega))} + \| \epsilon^{pl,*}_{1} - \epsilon^{pl,*}_{2} \|_{L^2(0,T;L^2(\Omega))} \right)
\]

\[
\leq CD(T) \| \epsilon^{pl,*}_{1} - \epsilon^{pl,*}_{2} \|_{L^2(0,T;L^2(\Omega))}.
\]

Considering the difference \( R(\epsilon^{pl,*}_{1}) - R(\epsilon^{pl,*}_{2}) \), we can obtain

\[
\| R(\epsilon^{pl,*}_{1}) - R(\epsilon^{pl,*}_{2}) \|_{L^2(0,T;L^2(\Omega))} = \| \epsilon^{pl}_{1} - \epsilon^{pl}_{2} \|_{L^2(0,T;L^2(\Omega))} \leq CTD(T) \| \epsilon^{pl,*}_{1} - \epsilon^{pl,*}_{2} \|_{L^2(0,T;L^2(\Omega))},
\]

where the constant \( C > 0 \) does not depend on time \( T \) and the initial data while \( D(T) \) is the same as previously and also does not depend on initial data. Choosing properly small \( T_1 > 0 \), we get that \( R(\epsilon) \) is a contraction on \( L^2(0,T_1;L^4(\Omega;S(3))) \). Therefore, by the Banach fixed point theorem, we obtain that there exists \( \epsilon^{pl} \in L^2(0,T_1;L^4(\Omega;S(3))) \) such that \( R(\epsilon^{pl}) = \epsilon^{pl} \) and the problem (AP1) possesses the unique solution \( (\sigma,\epsilon^{pl},u) \in L^2(0,T_1;L^4(\Omega;S(3))) \times L^2(0,T_1;L^4(\Omega)) \times W^{1,2}(0,T;L^4(\Omega)) \) on the time interval \( (0,T_1) \). Using the reasoning analogous to the proof of corollary 1, we can extend our solution to the whole interval \( (0,T) \) since the estimate (8) is independent of the initial data.

It remains to prove the estimate (6). Thus, using the second equation in (AP1), we obtain the following inequalities

\[
\| \sigma(t) \|_{L^2(\Omega)} \leq C \left( \| u(t) \|_{W^{1,2}(\Omega)} + \| \partial_i u(t) \|_{W^{1,2}(\Omega)} + \| \epsilon^{pl}(t) \|_{L^2(\Omega)} + \| \phi(\theta^*(t)) \|_{L^2(\Omega)} \right)
\]

\[
\leq D(T) \left( \| \epsilon^{pl}(t) \|_{L^2(\Omega)} + \| \phi(\theta^*(t)) \|_{L^2(\Omega)} + \| u_0 \|_{W^{1,2}(\Omega)} + \| b(t) \|_{L^2(\Omega)} + \| g(t) \|_{W^{1/2,4}_f(\Gamma_1)} \right).
\]

\[
\leq D(T) \left( L_1 \int_0^t \| \sigma(\tau) \|_{L^2(\Omega)} + L_2 \| \theta^*(\tau) \|_{L^2(\Omega)} + t|\Omega| \Lambda(0,0) \Bigg) \right.
\]

\[
+ \| u_0 \|_{W^{1,2}(\Omega)} + \| \epsilon^{pl}(0) \|_{L^2(\Omega)} + \| b(t) \|_{L^2(\Omega)} + \| g(t) \|_{W^{1/2,4}_f(\Gamma_1)} \Bigg).
\]
The Gronwall inequality applied to the expression above implies
\[
\|\sigma(t)\|_{L^p(\Omega)} \leq E_1(T) \left( \|\theta^*\|_{L^p(0,T;L^q(\Omega))}^{\beta} + \|\theta^*\|_{L^\infty(0,T;L^q(\Omega))}^{\alpha} \right) + \|\Omega\|_p |\Lambda(0,0)| + \|\varepsilon_0^p\|_{L^p(\Omega)}
\]
\[
+ \|u_0\|_{W^{1,q}_0(\Omega)} + \|b\|_{L^p(0,T;L^q(\Omega))} + \|g\|_{L^p(0,T;W^{-\frac{1}{2},q}(\Gamma_1))}
\]
and immediately, we obtain the required estimate for \(\|\sigma\|_{L^p(0,T;L^q(\Omega))}\). To estimate term \(\|\partial_t \varepsilon^p\|_{L^p(0,T;L^q(\Omega))}\), we treat as follows:
\[
\|\partial_t \varepsilon^p\|_{L^p(0,T;L^q(\Omega))} \leq E_2(T) \left( L_1 \|\sigma\|_{L^p(0,T;L^q(\Omega))} + \|\Omega\|_p |\Lambda(0,0)| + L_2 \|\theta^*\|_{L^p(0,T;L^q(\Omega))}^{\beta} + \|\varepsilon_0^p\|_{L^p(\Omega)}
\]
\[
+ \|u_0\|_{W^{1,q}_0(\Omega)} + \|b\|_{L^p(0,T;L^q(\Omega))} + \|g\|_{L^p(0,T;W^{-\frac{1}{2},q}(\Gamma_1))} \right),
\]
To obtain the estimate for \(\|u\|_{W^{1,q}_0(\Omega)}\), we once again use corollary 1 as in problem 7. As \(u\) satisfies (AP1), we have for any function \(\varphi \in W^{1,q}_0(\Omega)\) and a.e. \(t \in (0,T)\) that
\[
\int_{\Omega} D(\varepsilon(u(t,x)) \varepsilon(\varphi(x))) dx + \int_{\Omega} C(\varepsilon(\partial_t u(t,x))) \varepsilon(\varphi(x)) dx =
\]
\[
\int_{\Omega} D(\varepsilon^p(t,x)) \varepsilon(\varphi(x)) dx + \int_{\Omega} \int_{\Gamma_1} \partial_t \varphi^* (t,x) \varepsilon^p(t,x) dx + \int_{\Omega} \int_{\Gamma_1} \varepsilon^p(t,x) \varepsilon(\varphi(x)) dx + \int_{\Gamma_1} g(t,x) \varphi(x) dS(x).
\]
As previously, the right-hand side of the expression above defines a bounded and linear functional on \(W^{1,q}_0(\Omega)\) and by corollary 1, we obtain that the following estimate holds true:
\[
\|u\|_{W^{1,q}_0(\Omega)} \leq D(T) \left( \|\varepsilon^p\|_{L^p(0,T;L^q(\Omega))} + \|\varphi^*\|_{L^\infty(0,T;L^q(\Omega))}^{\alpha} + \|u_0\|_{W^{1,q}_0(\Omega)}
\]
\[
+ \|b\|_{L^p(0,T;L^q(\Omega))} + \|g\|_{W^{1,q}_0(\Omega)} \right) + \|\Omega\|_p |\Lambda(0,0)| + \|\varepsilon_0^p\|_{L^p(\Omega)}
\]
\[
+ \|u_0\|_{W^{1,q}_0(\Omega)} + \|b\|_{L^p(0,T;L^q(\Omega))} + \|g\|_{L^p(0,T;W^{-\frac{1}{2},q}(\Gamma_1))} \right). \tag{12}
\]

The inequalities (10) together with (11) and (12) give us (6). \(\square\)

Next for \(\theta^* \in L^p(0,T;L'(\Omega))\) and \((\sigma, \varepsilon^p, u)\) the corresponding solution to (AP1) we solve the second auxiliary problem formulated as follows:
\[
\left\{
\begin{align*}
\partial_t \theta(t, x) - \kappa \Delta \theta(t,x) &= -\varphi(\theta^*(t,x)) \text{div}_x \partial_t u(t,x) + \partial_t \varepsilon^p(t,x) \cdot \sigma(t,x), \\
&+ C(\varepsilon(\partial_t u(t,x))) \cdot \varepsilon(\partial_t u(t,x)), \\
\frac{\partial}{\partial n}(t, x)|_{\partial \Omega(0,T)} &= h(t,x), \\
\theta(x,0) &= \theta_0(x).
\end{align*}
\right.
\tag{AP2}
\]

The problem above is the linear heat conduction equation and its solvability is proved for example in Denk et al.\(^{27}\) We only need to estimate a norm of the following term:
\[
RHS := -\varphi(\theta^*) \text{div}_x \partial_t u + \partial_t \varepsilon^p \cdot \sigma + C(\varepsilon(\partial_t u)) \cdot \varepsilon(\partial_t u)
\]
in the space \(L^p(0,T;L'(\Omega))\).
Lemma 2. For RHS defined as above, it holds that
\[ \| \text{RHS} \|_{L^p(0,T;L^q(\Omega))} \leq D(T) \left( \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{2\beta} + \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{2\alpha} + \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\beta + \delta} \right), \]
where constant \( D(T) > 0 \) depends only on a length of a time interval \([0,T]\) and given data.

Proof. Using estimates that were obtained before, we can calculate as follows:
\[
\| \text{RHS} \|_{L^p(0,T;L^q(\Omega))} \leq \| \phi(\theta^*) \|_{L^p(0,T;L^r(\Omega))} + \| \partial_t \mathbf{E} \cdot \sigma \|_{L^p(0,T;L^r(\Omega))} + \| C(\varepsilon(e_i u)) \cdot \varepsilon(\partial_t u) \|_{L^p(0,T;L^r(\Omega))} \\
\leq \| \phi(\theta^*) \|_{L^p(0,T;L^r(\Omega))} + \| \partial_t \mathbf{E} \cdot \sigma \|_{L^p(0,T;L^r(\Omega))} + C \| \mathbf{\nabla} \partial_t u \|_{L^p(0,T;L^r(\Omega))}^2 \\
\leq \left( \| \theta^* \|_{L^p(0,T;L^r(\Omega))} + \| \partial_t \mathbf{E} \cdot \sigma \|_{L^p(0,T;L^r(\Omega))} \right) + C(T) \| \theta^* \|_{L^p(0,T;L^r(\Omega))} \bigg( \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\beta} + \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\alpha} + |\Omega| |\Lambda(0,0)| \bigg) + \| \epsilon_0 \|_{L^p(\Omega)}^2 \\
+ \| u_0 \|_{H^1(\Omega)}^2 + \| b \|_{W^{1,p}(0,T;L^q(\Omega))} + \| g \|_{W^{1,p}(0,T;L^q(\Omega))} \bigg)^2 \left( \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\beta} + \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\alpha} + \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\beta + \delta} \right) + \| \epsilon_0 \|_{L^p(\Omega)}^2 \\
+ \| u_0 \|_{H^1(\Omega)}^2 + \| b \|_{W^{1,p}(0,T;L^q(\Omega))} + \| g \|_{W^{1,p}(0,T;L^q(\Omega))} \bigg)^2 \left( \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\beta} + \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\alpha} + \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\beta + \delta} \right) \\
\leq \left( \| \theta^* \|_{L^p(0,T;L^r(\Omega))} + \| \theta^* \|_{L^p(0,T;L^r(\Omega))} \bigg( \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\beta} + \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\alpha} + \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\beta + \delta} \right) + \| \epsilon_0 \|_{L^p(\Omega)}^2 \\
+ \| u_0 \|_{H^1(\Omega)}^2 + \| b \|_{W^{1,p}(0,T;L^q(\Omega))} + \| g \|_{W^{1,p}(0,T;L^q(\Omega))} \bigg)^2 \left( \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\beta} + \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\alpha} + \| \theta^* \|_{L^p(0,T;L^r(\Omega))}^{\beta + \delta} \right) \right). \]

Lemma 3. Assume that \( A3 \) and \( A6 \) are satisfied and \( \theta^* \in L^p(0,T;L'(\Omega)) \). Additionally, if \( r > 3 \) we assume the compatibility condition CC. Moreover, let \( \sigma \in L^1(0,T;L^q(\Omega;\mathbb{S}(3))) \), \( \epsilon^p \in W^{1,q}(0,T;L^q(\Omega;\mathbb{S}(3))) \), and \( u \in W^{1,1}(0,T;W^{1,q}(\Omega;\mathbb{R}^n)) \) be the solution to (AP1). Then the problem (AP2) possesses a unique solution \( \theta \in L^p(0,T;W^{2,r}(\Omega)) \cup W^{1,p}(0,T;L'(\Omega)). \)

Furthermore, the following estimate holds:
\[ \| \theta \|_{L^p(0,T;W^{2,r}(\Omega))} + \| \theta \|_{W^{1,p}(0,T;L'(\Omega))} \leq C(T) \left( \| \text{RHS} \|_{L^p(0,T;L^r(\Omega))} + \| \theta_0 \|_{H^{2\beta-1/\alpha}(\Omega)} + \| h \|_{L^p(0,T;L^r(\Omega))} \right) \]
\[ + \| h \|_{L^p(0,T;W^{1-1/r,p}(\Omega))}. \]  

Remark 3. Lemma 3 follows from Herzog et al.24, Theorem 2.3 and Lemma 2.

By solving problems (AP1) and (AP2), we can define the operator
\[
\mathcal{T} : L^p(0,T;L'(\Omega)) \to L^p(0,T;W^{2,r}(\Omega)) \cap W^{1,p}(0,T;L'(\Omega)); \\
\mathcal{T} : \theta^* \mapsto \theta.
\]

Proposition 1. The operator \( \mathcal{T} : L^p(\mathbb{Q}_T) \to L^p(0,T;W^{2,r}(\Omega)) \cap W^{1,p}(0,T;L'(\Omega)) \) defined above is continuous.

Proof. To prove continuity of the operator \( \mathcal{T} \), we choose \( \theta^*_1, \theta^*_2 \in L^p(\Omega) \) and we study the difference \( \mathcal{T}(\theta^*_1) - \mathcal{T}(\theta^*_2) \).

First, we consider \((\sigma_1, \epsilon^p_1, u_1)\) and \((\sigma_2, \epsilon^p_2, u_2)\) solutions to (AP1) corresponding to \( \theta^*_1, \theta^*_2 \) with the same given data.
b.f.g. Let denote $\theta^*_\Delta := \theta_1^* - \theta_2^*$ and, respectively, $\sigma_\Delta := \sigma_1 - \sigma_2, \varepsilon_{\Delta}^p := \varepsilon_1^p - \varepsilon_2^p$ and $u_\Delta := u_1 - u_2$. Notice that the functions $\sigma_\Delta, \varepsilon_{\Delta}^p, u_\Delta$ satisfy the following system of the equations:

$$\begin{align*}
-\text{div}_x \sigma_\Delta(t, x) & = 0, \\
\sigma_\Delta(t, x) & = D(\varepsilon(u_\Delta(t, x)) - \varepsilon_{\Delta}^p(t, x)) + C(\varepsilon(\partial_t u_\Delta(t, x))) \\
& - (\phi(\theta_1^*(t, x)) - \phi(\theta_2^*(t, x))), \\
\partial_t \varepsilon_{\Delta}^p(t, x) & = \Lambda(\sigma_1, \theta_1^*) - \Lambda(\sigma_2, \theta_2^*), \\
u_\Delta(t, x)|_{\Gamma_{0,T}} & = 0, \\
\sigma_\Delta(t, x) \cdot \vec{n}(x)|_{\Gamma_{0,T}} & = 0, \\
u_\Delta(0, x) & = 0, \\
\varepsilon_{\Delta}^p(x, 0) & = 0.
\end{align*}$$

Hence, we can estimate as follows:

$$|\varepsilon_{\Delta}^p(t, x)| \leq L_1 \int_0^t |\sigma_\Delta(x, \tau)| d\tau + L_2 \int_0^t |\theta^*_\Delta(t, \tau)|^p d\tau.$$  

Additionally, $u_\Delta$ satisfies the following:

$$\begin{align*}
\text{div}_x D(\varepsilon(u_\Delta)) + \text{div}_x C(\varepsilon(\partial_t u_\Delta)) & = \text{div}_x D(\varepsilon_{\Delta}^p) + \nabla \cdot (\phi(\theta_1^*) - \phi(\theta_2^*)), \text{ on } \Omega \times (0, T) \\
u_\Delta & = 0, \text{ on } \Gamma_{0,T} \\
(D(\varepsilon(u_\Delta)) + C(\varepsilon(\partial_t u_\Delta))) \cdot \vec{n} & = \left( D(\varepsilon_{\Delta}^p) + (\phi(\theta_1^*) - \phi(\theta_2^*)) \right) \cdot \vec{n}, \text{ on } \Gamma_1 \times (0, T)
\end{align*}$$

and similarly as in the proof of lemma 1, we can estimate

$$\begin{align*}
\|u_\Delta\|_{W^{1,1}(0, T; W^{1,1}_0(\Omega))} & \leq C(T) \left( \|\varepsilon_{\Delta}^p\|_{W^{1,1}(0, T; L^1(\Omega))} + \|\phi(\theta_1^*) - \phi(\theta_2^*)\|_{L^1(0, T; L^1(\Omega))} \right) \\
& \leq C(T) \left( \|\sigma_\Delta\|_{L^1(0, T; L^1(\Omega))} + \|\theta^*_\Delta\|_{L^{1,\infty}(0, T; L^{1,\infty}(\Omega))} \right)
\end{align*}$$

Next, similar calculations as in (10) and (11) lead us to the following:

$$\begin{align*}
\|\sigma_\Delta\|_{L^1(0, T; L^1(\Omega))} + \|\varepsilon_{\Delta}^p\|_{L^1(0, T; L^1(\Omega))} & \leq C(T) \left( \|\theta^*_\Delta\|_{L^{1,\infty}(0, T; L^{1,\infty}(\Omega))} + \|\phi(\theta_1^*) - \phi(\theta_2^*)\|_{L^1(0, T; L^1(\Omega))} \right).
\end{align*}$$

The estimate (14) together with (15) gives us

$$\begin{align*}
\|u_\Delta\|_{W^{1,1}(0, T; W^{1,1}_0(\Omega))} + \|\sigma_\Delta\|_{L^1(0, T; L^1(\Omega))} + \|\varepsilon_{\Delta}^p\|_{L^1(0, T; L^1(\Omega))} & \leq C(T) \left( \|\theta^*_\Delta\|_{L^{1,\infty}(0, T; L^{1,\infty}(\Omega))} + \|\phi(\theta_1^*) - \phi(\theta_2^*)\|_{L^1(0, T; L^1(\Omega))} \right).
\end{align*}$$

Moreover, one can observe that operator defined by function $\phi$ as $L^p(0, T; L^q(\Omega)) \ni \theta \mapsto \phi(\theta) \in L^p(0, T; L^q(\Omega))$ is bounded and continuous. Therefore, for any $\theta_1 \in L^p(0, T; L^q(\Omega))$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\theta^*_\Delta\|_{L^p(0, T; L^q(\Omega))} < \delta$ then it holds that

$$\begin{align*}
\|u_\Delta\|_{W^{1,1}(0, T; W^{1,1}_0(\Omega))} + \|\sigma_\Delta\|_{L^1(0, T; L^1(\Omega))} + \|\varepsilon_{\Delta}^p\|_{L^1(0, T; L^1(\Omega))} & \leq \varepsilon.
\end{align*}$$

Next, we consider the heat conduction equation (in fact the problem similar to (AP2)) for RHS defined by $u_i, \sigma_i, \varepsilon_{\Delta}^p, \theta_i^*$ where $i = 1, 2$

$$\begin{align*}
\partial_t \theta_i(t, x) - \kappa \Delta \theta_i(t, x) & = -\phi(\theta_1^*(t, x)) \text{div}_x \sigma_i(t, x) + \phi(\theta_2^*(t, x)) \cdot \sigma_i(t, x) \\
& + C(\varepsilon(\partial_t u_i(t, x))) \cdot \varepsilon(\partial_t u_i(t, x)), \\
\frac{\partial \theta_i}{\partial n}(t, x)|_{\partial \Omega_{\Delta}(0, T)} & = h(t, x), \\
\theta_i(x, 0) & = \theta_0(x), \quad i = 1, 2.
\end{align*}$$
Obviously, the difference $\theta_\Delta$ solves the following problem:

\[
\begin{aligned}
\begin{cases}
\partial_t \theta_\Delta(t, x) - \kappa \Delta \theta_\Delta(t, x) &= -\phi(\theta_1^*(t, x)) \nabla_x \theta_1 u_1(t, x) + \phi(\theta_2^*(t, x)) \nabla_x \theta_2 u_2(t, x) \\
&+ \partial_\ell \psi_1(t, x) \cdot \sigma_1(t, x) - \sigma_2(t, x) + C(\epsilon(\partial_t u_1(t, x))) \cdot \epsilon(\partial_t u_1(t, x)) - C(\epsilon(\partial_t u_2(t, x))) \cdot \epsilon(\partial_t u_2(t, x)), \\
\frac{\partial \theta_\Delta}{\partial t}(t, x)|_{\partial \Omega(x, T)} &= 0, \\
\theta_\Delta(x, 0) &= 0.
\end{cases}
\end{aligned}
\]

Hence, we can estimate

\[
\|\theta_\Delta\|_{L^p(0, T; W^{1, p}(\Omega))} + \|\theta_\Delta\|_{W^{1, p}(0, T; L^p(\Omega))} 
\leq C(T) \left( \|\phi(\theta_1^*) \nabla_x \theta_1 u_1 - \phi(\theta_2^*) \nabla_x \theta_2 u_2\|_{L^p(0, T; L^1(\Omega))} + \left\| \partial_\ell \psi_1 \cdot \sigma_1 - \partial_\ell \psi_2 \cdot \sigma_2 \right\|_{L^p(0, T; L^1(\Omega))} \\
+ \|\epsilon(\partial_t u_1)\cdot \epsilon(\partial_t u_1) - C(\epsilon(\partial_t u_2))\cdot \epsilon(\partial_t u_2)\|_{L^p(0, T; L^1(\Omega))} \right)
\]

\[
\leq C(T) \left( \|\phi(\theta_1^*) \nabla_x \theta_1 u_1\|_{L^2(0, T; L^2(\Omega))} + \left\| \partial_\ell \psi_1 \cdot \sigma_1 \right\|_{L^2(0, T; L^2(\Omega))} + \left\| \partial_\ell \psi_2 \cdot \sigma_2 \right\|_{L^2(0, T; L^2(\Omega))} \\
+ \|\epsilon(\partial_t u_1)\|_{L^2(0, T; L^2(\Omega))} + \|\epsilon(\partial_t u_2)\|_{L^2(0, T; L^2(\Omega))} \right).
\]

Now, we can see that for $\delta > 0$ small enough, it holds that $\|\theta_\Delta\|_{L^p(0, T; L^p(\Omega))} \leq 2\|\theta_1^*\|_{L^p(0, T; L^p(\Omega))}$ if only $\|\theta_\Delta\|_{L^p(0, T; L^p(\Omega))} \leq \delta$. Hence, using estimates 6 and 16, we can write the following:

\[
\|\theta_\Delta\|_{L^p(0, T; W^{1, p}(\Omega))} + \|\theta_\Delta\|_{W^{1, p}(0, T; L^p(\Omega))} 
\leq C(T) \left( \|\nabla_x \partial_t u_1\|_{L^2(0, T; L^2(\Omega))} + \|\phi(\theta_1^*) - \phi(\theta_2^*)\|_{L^2(0, T; L^2(\Omega))} \\
+ \|\sigma_1\|_{L^2(0, T; L^2(\Omega))} + \left\| \partial_\ell \psi_1 \right\|_{L^2(0, T; L^2(\Omega))} \right)
\]

\[
\leq C(T) \left( \|\nabla_x \partial_t u_1\|_{L^2(0, T; L^2(\Omega))} + \|\phi(\theta_1^*) - \phi(\theta_2^*)\|_{L^2(0, T; L^2(\Omega))} \\
+ \|\sigma_1\|_{L^2(0, T; L^2(\Omega))} + \left\| \partial_\ell \psi_1 \right\|_{L^2(0, T; L^2(\Omega))} \right)
\]

\[
\leq C(T) \epsilon \leq \epsilon.'
\]

where constant $C(T) > 0$ depends on a length of the time interval $(0, T)$, norms of given data, and the norm $\|\theta_1^*\|_{L^p(0, T; L^p(\Omega))}$. For any given $\epsilon > 0$, we can choose such $\delta > 0$ that the estimate above holds true. Therefore, operator $T$ is continuous.

\[\Box\]

**Proposition 2.** The operator $T : L^p(0, T; L'(\Omega)) \to L^p(0, T; L'(\Omega))$ is compact.

The reason of the fact above is the Aubin-Lions lemma, i.e.,

$L^p(0, T; W^{2, p}(\Omega)) \cap W^{1, p}(0, T; L'(\Omega)) \hookrightarrow\hookrightarrow L^p(0, T; L'(\Omega))$.
The embedding above together with proposition 1 give us a compactness for the operator $T$ in required spaces.

**Lemma 4.** There exists $\theta \in L^p(0, T; L'(\Omega))$ such that $T(\theta) = \theta$.

**Proof.** We use the Schauder fixed point theorem. Let us fix $\theta^* \in L^p(0, T; L'(\Omega))$ such that $\|\theta^*\|_{L^p(0, T; L'(\Omega))} \leq M$ for some $M > 0$, which will be indicated later. We are going to prove that $\|T(\theta^*)\|_{L^p(0, T; L'(\Omega))} \leq M$. From estimates, lemma 2, and lemma 3, we immediately obtain that

$$
\|\theta\|_{L^p(0, T; L'(\Omega))} \\
\leq \|\theta\|_{L^p(0, T; W^{2,p}\Omega)} + \|\theta\|_{W^{1,p}\Omega}
$$

$$
\leq C(T) \left( \|\theta^*\|_{L^p(0, T; L'(\Omega))}^{2\beta} + \|\theta^*\|_{L^p(0, T; L'(\Omega))}^{2\alpha} + \|\theta^*\|_{L^p(0, T; L'(\Omega))}^{\beta+a} \\
+ |\Omega|^2 |\Lambda(0, 0)|^2 + \|\epsilon_0\|_{L^p(\Omega)}^2 + \|\beta\|_{L^p(\Omega)}^2 + \|g\|_{L^p(\Omega)}^2 + \|\theta_0\|_{L^p(\Omega)} + \|\theta^*\|_{L^p(0, T; W^{1,p}\Omega)} \\
+ \|\theta^*\|_{W^{1,p}(\Omega)} + \|\theta^*\|_{W^{1,p}(\Omega)} \right) \leq C(T) \left( M^{2\beta} + M^{2\alpha} + M^{\beta+a} + 1 \right)
$$

$$
\leq E(T) + \frac{1}{2} M
$$

$$
\leq M
$$

if only

$$
M \geq 2E(T).
$$

Here, the constant $E(T) > 0$ does depend only on given boundary and initial data, entries of the operators $D$ and $C$, geometry of the domain $\Omega$ and the length of the time interval $(0, T)$.

Hence, for such a constant $M$, the compact operator $T (B(0, M)) \subset B(0, M)$, where

$$
B(0, M) := \{ \xi \in L^p(0, T; L'(\Omega)) | \|\xi\|_{L^p(0, T; L'(\Omega))} \leq M \}.
$$

Finally, we use the Schauder fixed point theorem to end the proof. \hfill \Box

**Corollary 2.** The consideration above proves the main result of this paper, ie, theorem 2.

**Remark 4.** It is possible to consider the model with nonhomogeneous boundary condition in Dirichlet part, ie,

$$
u(t, x) = f(t, x) \quad \text{for} \ (t, x) \in (0, T) \times \Gamma_0.
$$

It is enough to assume that the function $f$ can be expanded to a function

$$
F \in W^{1,\beta}(0, T; W^{1,\beta}(\Omega; \mathbb{R}^3)).
$$

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