Anomalous energy losses in fractal medium

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Abstract

We derive equation describing distribution of energy losses of the particle propagating in fractal medium with quenched and dynamic heterogeneities. We show that in the case of the medium with fractal dimension $2 < D < 3$ the losses of energy are described by the Mittag-Leffler renewal process. The average energy loss of the particle experiences anomalous drift $\Delta \sim x^\alpha$ with power-law dependence on the distance $x$ from the surface and exponent $\alpha = D - 2$. 
1 Introduction

In this paper we study the problem of energy losses of the particle propagating in disordered medium. We consider both a model of discrete energy loss, when the particle collides with randomly placed scattering centers in the medium, so that during intervals between such scattering events the particle has ballistic trajectory and the energy of the particle does not change, and its continuum limit, corresponding to the description of the energy loss process in terms of kinetic equation.

There exists a close analogy between this problem and the diffusion of a particle through the medium: the energy loss is proportional to the number of scattering events \( n(x) \) at the distance \( x \) from the surface of the medium, and \( x \) plays the role of the time \( t \) in the diffusion problem. The propagation of a particle in the regular homogeneous medium, with the number of scattering events \( \langle n \rangle \sim x \), is analogous to the standard Brownian motion with the average number of events \( \langle n \rangle \sim t \). Anomalous power-law coordinate dependence \( \langle n \rangle \sim x^\alpha \) is related to the presence of power-law spatial correlations in the density distribution of scattering centers in the random medium. Such correlations are typical both for fractal systems with quenched disorder and for systems with dynamic heterogeneities at the critical point of a phase transition.

In diffusion problems anomalous time dependence of the number of events is often observed near glass transition\[1,2\], for sub-diffusion-limited reactions\[3\], in turbulent regime\[4\] and for some transport problems in disordered systems.\[5,6\]

Our consideration of the discrete energy loss model with power law correlations is done at the level analogous to the Continuous Random Walk (CTRW) \[7\]. The main distinction of the stochastic process describing the energy loss and the CTRW one is that the energy loss problem corresponds to a sum of random positive quantities – energy losses at scattering events. Of substantial interest is a continuous limit of the discrete energy loss model corresponding to some effective kinetic description. The highly nontrivial and interesting transition from the discrete to kinetic description in the usual CTRW case was discussed in \[8\].

In section 2 we study the energy losses in weakly heterogeneous medium. We demonstrate that average energy losses in such medium can anomalously grow with the distance \( x \) from the surface, \( \langle \Delta(x) \rangle \sim x^{1/\beta} \), with super-diffusional exponent \( \beta^{-1} > 1 \). In section 3 we study energy losses in strongly
heterogeneous medium. We first formulate simple model of anomalous scattering, describing the loss of energy by the particle during its propagation in random medium (section 3.1). In section 3.2 this process is generalized to the case of general distribution of energy losses during individual collisions using formalism of CTRW. We derive equation for the distribution function of energy losses and show that the energy of the particle experiences anomalous drift with average energy losses growing with the distance $x$ as $\langle \Delta(x) \rangle \sim x^\alpha$, with sub-diffusional exponent $\alpha < 1$. In section 3.3 we demonstrate, that the loss of the energy because of scattering on fractal structure is described by the Mittag-Leffler renewal process\cite{9}, which may be considered as fractional generalization of the well known Poisson renewal process. Main results of this work are summarized in section 4.

2 Weakly heterogeneous medium

The loss of the energy by the particle propagating in homogeneous medium is described, in the continuous limit, by the Landau kinetic equation for the distribution function of energy losses at point $x$\cite{10}

$$\frac{\partial f(\Delta,x)}{\partial x} = \frac{1}{a} \int_0^\infty d\varepsilon w(\varepsilon) [f(\Delta - \varepsilon, x) - f(\Delta, x)]$$

(1)

where $w(\varepsilon)$ is the probability distribution of energy loss at scattering event and $a^{-1}$ is linear density of scattering centers separated by a distance $a$.

In the random medium the probability distribution of energy loss becomes a random function of the coordinate $x$, $w(\varepsilon) \rightarrow w(\varepsilon|x)$. In principle one could imagine arbitrary variations of the form of $w(\varepsilon|x)$ from one point to another that could be correlated over spatial domains with the size controlled by the corresponding correlation length $\xi$.

In the simplest case of finite correlation length $\xi$ the basic kinetic equation for the averaged distribution function $f(\Delta,x)$ on distances $x \gg \xi$ takes the form

$$\frac{\partial f(\Delta,x)}{\partial x} = \frac{1}{a} \int_0^\infty d\varepsilon \bar{w}(\varepsilon) [f(\Delta - \varepsilon, x) - f(\Delta, x)]$$

(2)

with

$$\bar{w}(\varepsilon) = \langle w(\varepsilon|x) \rangle.$$

(3)

The averaging in Eq. (3) is over the randomness of the distribution function $w(\varepsilon|x)$. 

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As an illustration, let us consider the distribution $w(\varepsilon|\sigma)$ used in Eq. (1), where $\sigma$ denotes the set of parameters characterizing this distribution. The simplest way of introducing spatial randomness of the distribution $w(\varepsilon|\sigma)$ is to consider the parameters $\sigma$ as random fields depending on spatial coordinates, $w(\varepsilon|\sigma) \to w(\varepsilon|\sigma(x))$. At distances larger than the correlation length $\xi$ the fields $w(\varepsilon|\sigma(x))$ are effectively uncorrelated and the distribution $\bar{w}(\varepsilon)$ is simply given by

$$\bar{w}(\varepsilon) = \int d\sigma p(\sigma)w(\varepsilon|\sigma)$$

(4)

where $p(\sigma)$ is the probability distribution describing the local fluctuations of $\sigma(x)$. The properties of the resulting distribution $\bar{w}(\varepsilon)$ can substantially differ from those of the initial distribution $w(\varepsilon|\sigma)$. Of special importance is the influence of randomness in $\sigma$ on the asymptotic behavior of $\bar{w}(\varepsilon)$ at large $\varepsilon$. In particular, the power-like decay of $p(\sigma)$ at large $\sigma$ induces the power-like decay of $\bar{w}(\varepsilon)$ even for well-localized exponentially decaying $w(\varepsilon|\sigma)$.

The solution of Eq. (2) is most conveniently obtained by solving for the Laplace-transformed distribution function

$$\tilde{f}(p,x) \equiv \int_0^\infty d\Delta e^{-p\Delta} f(\Delta,x) = \exp[-\omega(p)x]$$

(5)

where

$$\omega(p) = \frac{1}{a} \int_0^\infty d\varepsilon \bar{w}(\varepsilon) \left(1 - e^{-p\varepsilon}\right)$$

(6)

Calculating the inverse Laplace transform of Eq. (5) we find that the distribution function $f(\Delta,x)$ can have different form depending on the large energy asymptotes of the distribution $\bar{w}(\varepsilon)$:

a) If this function $\bar{w}(\varepsilon)$ decays at large $\varepsilon$ faster than $1/\varepsilon^3$, the distribution function $f(\Delta,x)$ has Gaussian form with the center at $\Delta = \Delta_1 x$ and the width $\sqrt{\Delta_2 x}$, where $\Delta_k$ are corresponding moments of the distribution $\bar{w}(\varepsilon)$:

$$\Delta_k = \int_0^\infty d\varepsilon \varepsilon^k \bar{w}(\varepsilon)$$

(7)

b) If the function $\bar{w}(\varepsilon)$ decays as $1/\varepsilon^{1+\beta}$ with $1 < \beta < 2$ the distribution function $f(\Delta,x)$ is still centered at $\Delta = \Delta_1 x$ but is not Gaussian, and its width grows as power law $x^{1/\beta}$ of $x$. In both cases a) and b) the distribution $f(\Delta,x)$ becomes sharper with the rise of $x$. 

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c) If the function $\bar{w}(\varepsilon)$ decays slower than $1/\varepsilon^2$ then the first moment $\Delta_1$ diverges. In the case of power distribution $\bar{w}(\varepsilon) \sim \varepsilon^{-1-\beta}$ with exponent $0 < \beta < 1$ we find from Eq. (6) $\omega(p) = Cp^\beta$ with certain constant $C \sim \Delta_1^\beta/\alpha$. Calculating the inverse Laplace transformation of Eq. (5) we get

$$f(\Delta,x) = \beta \frac{Cx}{\Delta^{1+\beta}} W_{\beta} \left( \frac{Cx}{\Delta^{\beta}} \right),$$

(8)

where $W_{\beta}(z)$ is the Wright type function (see Appendix B). The center of this distribution and its width are on the same order of value and grow as the power law $\langle \Delta(x) \rangle \sim x^{1/\beta}$ of the distance $x$ from the surface. Such dependence is familiar for Lévy flight processes characterized by the exponent $1/\beta > 1$.[11]

3 Strongly heterogeneous medium

In this section we consider energy losses in strongly heterogeneous medium with infinite correlation radius $\xi$ (or on distances from the surface small with respect to $\xi$). Such situation takes place in the case of fractal medium or for the system at critical point of phase transition.

3.1 Simple model with constant losses

Let us first consider the simple one-dimensional microscopic model for energy loss in heterogeneous medium in which the loss takes place at through a sequence of discrete events in which the incident particle looses an equal amount of energy $\Delta_1$. This model describes the eikonal energy loss of a high energy particle loosing energy in small portions so that one can, in the first approximation, consider its trajectory to be a straight line. The model generalizes the simplest model of discrete random walk with the fixed elementary step to the case of the positive variable. The assumption of constant energy loss will be relaxed in the next paragraph in which we will consider a more general model. More precisely, in the simple model we consider a particle entering the medium at the point $x = 0$ and loosing energy at random points $x_1, \ldots, x_n$ lying on the trajectory composed by the intervals of length $l_1, \ldots, l_n$, see Fig. 1.

The cumulative energy loss at some point $x$ for the given event is, evidently, equal $\Delta(x) = \Delta_1 n(x)$, where $n(x)$ is the number of scattering events
that the particle experienced along its trajectory from the initial point to the point under consideration. The probabilistic description of the process of random energy loss is thus fully specified by the probabilistic properties of \( n(x) \). Let consider the case in which these properties are fully described by the probability density \( \psi(l) \) of the spatial distance \( l \equiv x - x' \) between the points \( x \) and \( x' < x \) at which two subsequent scattering events took place and the distribution \( \varphi(l_1) \) characterizes the probability density of the position \( l_1 \) of the first scattering event, see Fig. 1. In what follows we will simply assume that \( \varphi(l_1) = \psi(l_1) \). The function \( \psi(l) \) determines the survival probability \( \Psi(r) \) that the particle experiences no collisions at the distance \( r = x - x' \) after the collision at point \( x' \):

\[
\Psi(r) = \int_{r}^{\infty} \psi(l) dl
\]

To develop a quantitative description of the energy loss process it is convenient to introduce the probability density \( \psi_n(x - x') \) for the \( n \)-th collision to take place at the distance \( x - x' \) from the collision at point \( x' \). Then the probability density \( f_n(x) \) for the number \( n \) of scattering events along the trajectory leading to the point \( x \) reads

\[
f_n(x) = \int_{0}^{x} dx' \Psi(x - x') \psi_n(x')
\]

In turn, the probability density \( \psi_n(x) \) is determined by recurrence relation

\[
\psi_n(x) = \int_{0}^{x} dx' \psi(x - x') \psi_{n-1}(x')
\]

with \( \psi_1(x) = \psi(x) \).

The equations (10,11) are easily solved by using the Laplace transform

\[
\tilde{f}_n(q) = \int_{0}^{\infty} f_n(x) e^{-qx} dx
\]
We have
\[ \tilde{f}_n (q) = \tilde{\psi}^n (q) \frac{1 - \tilde{\psi} (q)}{q} \]  \hspace{1cm} (12)
where \( \tilde{\psi} (q) \) is the Laplace transform of the probability density \( \psi (l) \).

Of special interest is the case of the medium characterized by power-law scale-invariant fluctuations of the positions of the scattering centers. An example of this situation is provided by the fractal medium. In the considered one-dimensional case the relevant correlations are determined by projecting the full three-dimensional correlation pattern on the axis of particle propagation. In the scale-invariant case the function \( \psi (l) \) should read
\[
\psi (l) = \begin{cases} 
0 & \text{at } l < a \\
\alpha a^\alpha / l^{1+\alpha} & \text{at } l > a
\end{cases}
\]  \hspace{1cm} (13)
where, as shown in Appendix A, exponent \( \alpha \) is related to the fractal dimension \( D \) of the scattering medium as:
\[
\alpha = D - 2.
\]  \hspace{1cm} (14)

Notice, that there are both upper and low boundaries of the fractal dimension \( D \) for the problem of anomalous scattering. Upper fractal dimension \( D = 3 \) describes homogeneous medium and at lower critical fractional dimension \( D = 2 \) the scattering cluster is too sparse to effectively scatter the propagating particle.

At distances much larger than the ultraviolet cutoff \( a \) the Laplace transform of the function in Eq. (13) reads
\[
\tilde{\psi} (q) = \int_0^\infty \psi (x) e^{-qx} dx \simeq 1 - (aq)^\alpha, \quad aq \ll 1 \]  \hspace{1cm} (15)
In the case \( \alpha < 1 \) the average scattering length \( \bar{l} \) diverges leading to anomalous losses of the particle energy. Calculating the inverse Laplace transform of Eq. (12), we get at large \( x \gg a \) the distribution function of energy losses:
\[
f_n (x) = (a/x)^\alpha W_\alpha [(a/x)^\alpha n], \]  \hspace{1cm} (16)
where \( W_\alpha (z) \) is the Wright type function, defined in Eq. (40) of Appendix B.
Using this distribution, we can calculate the average energy loss at the distance \( x \) from the surface, which is monotonically increasing function of \( x \):
\[
\langle \Delta (x) \rangle = \Delta_1 \bar{n} (x) = \frac{\Delta_1}{\Gamma (1 + \alpha)} \left( \frac{x}{a} \right)^\alpha, \quad 0 < \alpha < 1
\]  \hspace{1cm} (17)
At $\alpha = 1$ the average scattering length $\bar{l}$ is finite and in average the losses of particle energy $\langle \Delta(x) \rangle = \Delta_1 x / \bar{l}$ grow linearly with the distance $x$, similar to the case of homogeneous system \[10\].

### 3.2 Continuous Time Random Walk (CTRW)

In this section we consider the case, when energy loss $\varepsilon$ at each collision with scattering center is random and is described by the distribution function $w(\varepsilon)$. By analogy with CTRW \[7\] the process of energy losses can be considered as subordinated to the anomalous scattering process shown in Fig. 1.

Let us stress once again that the model under consideration differs from the usual CTRW in considering the random walk of the positive variable $\varepsilon$. The distribution function of energy losses can be presented as the convolution of the distribution function of the number of collisions $f_n(x)$ studied in previous section and the distribution function of energy loss $w_n(\Delta)$ for the given number of collisions $n$:

$$ f(\Delta, x) = \sum_{n} w_n(\Delta) f_n(x). \quad (18) $$

Due to the additive nature of the energy loss the distribution function $w_n(\Delta)$ is determined by the inverse Laplace transform of the power $\tilde{w}_n(p)$ of the Laplace transform $\tilde{w}(p)$ of the distribution $w(\varepsilon)$. Explicit expression \[18\] for the distribution function $f(\Delta, x)$ is not very convenient to use, so below we derive the kinetic equation for it.

Simple probabilistic consideration of the losses of the energy by the particle propagating in random medium characterized by the probability distribution $w(\varepsilon)$ shows that this random process can be described by equation

$$ f(\Delta, x) = \delta(\Delta) \Psi(x) + \int_{0}^{x} dx' \psi(x-x') \int_{0}^{\infty} d\varepsilon w(\varepsilon) f(\Delta-\varepsilon, x') \quad (19) $$

where $\Psi(x)$ is the survival probability at the distance $x$ from the surface, Eq. \[9\]. To solve Eq. \[19\] it is convenient to introduce the double Laplace transform of $f(\Delta, x)$ over both variables $\Delta$ and $x$

$$ \tilde{f}(p, q) \equiv \int_{0}^{\infty} d\Delta e^{-p\Delta} \int_{0}^{\infty} dx e^{-qx} f(\Delta, x) $$

The corresponding equation for $\tilde{f}(p, q)$ reads:

$$ \tilde{f}(p, q) = \frac{1 - \tilde{\psi}(q)}{q} + \tilde{\psi}(q) \tilde{w}(p) \tilde{f}(p, q) \quad (20) $$
Introducing the function
\[ \tilde{g}(q) = \frac{\tilde{\psi}(q)}{1 - \psi(q)} \] (21)
we can rewrite the above equation (21) in the form
\[ \tilde{f}(p, q) = \frac{1}{q} + \tilde{g}(q) [\tilde{w}(p) - 1] \tilde{f}(p, q) \] (22)
which is equivalent to
\[ f(\Delta, x) = \delta(\Delta) + \int_0^x dx' g (x - x') \times \int_0^\infty d\varepsilon w(\varepsilon) [f(\Delta - \varepsilon, x') - f(\Delta, x')] \] (23)
Eq. (23) can be considered as generalization of Eq. (1) for homogeneous medium corresponding to constant density of scattering centers, \( g(r) = 1/a \).
The function \( g(r) \) is found by inverse Laplace transform of the function \( \tilde{g}(q) \), Eq. (21). Using Eq. (12) \( \tilde{g}(q) \) can be related to the Laplace transform of the average number of scattering events:
\[ \tilde{n}(q) = \sum_n n \tilde{f}_n(q) = \frac{\tilde{g}(q)}{q}, \] (24)
We conclude, that \( g(r) \) has the meaning of average density of scattering events along the direction \( \mathbf{e}_x \) of particle propagation at the distance \( r \) from the scattering event:
\[ g(r) = d\bar{n}(r) / dr \] (25)
In general, the function \( g(r) \) depends on characteristics of the medium, and it can be related
\[ g(r) = a^2 G(\mathbf{e}_x r), \] (26)
to the so-called structure function of the medium
\[ G(r) = \left\langle \sum_{n \neq 0} \delta(x_i - x_{i+n} - r) \right\rangle, \] (27)
\( x_i \) are coordinates of the \( i \)-th scattering center. In Eq. (26) \( a^2 \) is the scattering area of the particle and \( \mathbf{e}_x \) is unit vector along the trajectory of the particle.
This relation can be established rewriting Eq. (27) in the form

\[ G(r) = \sum_{n \neq 0} \psi_n'(r), \quad \int \psi_n'(r) dr = 1 \]  \hspace{1cm} (28)

\[ \psi_n'(r) = \langle \delta(x_i - x_{i+n} - r) \rangle \]

Although the sum in Eq. (28) is going over all scattering centers in the medium, only centers along the trajectory of the particle enter into Eq. (26). Expanding Eq. (21) in powers of \( \tilde{\psi}(q) \) and taking the inverse Laplace transform of each term of the obtained series, we get

\[ g(r) = \sum_{n \neq 0} \psi_n(x), \quad \int \psi_n(r) dr = 1 \]  \hspace{1cm} (29)

where \( \psi_n(r) = \langle \delta(x_i - x_{i+n} - r) \rangle \) is the probability distribution of the distance \( r = x_{i+n} - x_i \) between \( n \) consequent collisions along the trajectory of the propagating particle (see Eq. (11)). Comparing Eq. (28) and (29) term by term, we reproduce relation (26) between functions \( g(r) \) and \( G(r) \).

### 3.3 Continuous limit

In the case of scattering medium with fractal dimension \( D \) the Laplace transform of the function \( g(r) \) in the long wavelength limit \( aq \ll 1 \) has the form:

\[ \tilde{g}(q) = (aq)^{-\alpha}, \quad \alpha = D - 2 \]  \hspace{1cm} (30)

In Appendix C we derive asymptotic solutions of Eq. (23) close to the surface and far from it. Close enough to the surface the distribution of energies will have large peak at \( \Delta = 0 \), describing non-scattered particle

\[ f(\Delta, x) \approx \delta(\Delta) \Psi(x), \]  \hspace{1cm} (31)

where the function \( \Psi(x) \) is the survival probability that the particle does not scatter at the depths smaller than \( x \). Using large \( p \) asymptotes of the function \( \omega(p) \approx 1/a \) defined in Eq. (6), we find from Eq. (51) of Appendix C

\[ \Psi(x) = E_\alpha [(x/a)^\alpha] \]  \hspace{1cm} (32)

where the function \( E_\alpha \) is defined in Eq. (46) of Appendix B.
The process with survival probability \( \Psi (r) = E_\alpha (r) \) is known as the Mittag-Leffler renewal process, that is described by fractional differential equation

\[
\frac{\partial^\alpha \Psi (r)}{\partial r^\alpha} = -\Psi (r)
\]  

(33)

where \( \partial^\alpha /\partial x^\alpha \) is the Caputo fractional derivative

\[
\frac{\partial^\alpha \Psi (r)}{\partial r^\alpha} = \frac{1}{\Gamma (1 - \alpha)} \int_0^r \frac{\Psi' (x)}{(r - x)^\alpha} dx, \quad 0 < \alpha < 1.
\]

(34)

It was introduced by Caputo in the later 1960s for modeling the energy dissipation in the rheology of the Earth\[12\].

Far enough from the surface the loss of the energy is determined by multiple scattering processes, when we can expand

\[
\omega (p) \simeq p \frac{\Delta_1}{a}, \quad \Delta_1 = \int_0^\infty d\varepsilon \varepsilon w (\varepsilon)
\]

Calculating the inverse Laplace transform of Eq. (52) in Appendix C, we get

\[
f (\Delta, x) \simeq \frac{1}{\Delta_1} \left( \frac{a}{x} \right)^\alpha W_\alpha \left[ \frac{\Delta}{\Delta_1} \left( \frac{a}{x} \right)^\alpha \right]
\]

(35)

where the Wrize function \( W_\alpha \) is defined in Eq. (40) of Appendix B.

The function \( f (\Delta, x) \) (35) is the solution of the space-fractional drift equation of the order \( \alpha \):

\[
\frac{\partial f (\Delta, x)}{\partial \Delta} = -a^\alpha \frac{\partial^\alpha f (\Delta, x)}{\Delta_1 \partial x^\alpha}
\]

(36)

This distribution function (35) describes pure renewal process with anomalous exponent \( \alpha \).\[9\] This process can be modelled as the series of jumps of the energy \( \Delta \) with the amplitude \( \Delta_1 \) each happened at renewal points \( x \) separated by random discrete intervals \( l \) distributed according to the power law: \( \psi (l) \sim l^{1-\alpha} \), see Fig. I.

In the continuous limit we can generalize the simple model of section 3.1 using the thinning procedure\[13\]. In this procedure for each positive \( n \) a decision is made: the scattering event is maintained with probability \( p \) or it is deleted with probability \( 1 - p \). In the limit \( p \to 0 \) the amplitude of scattering become smaller and smaller, their number in a given span of space
larger and larger, and the ballistic trajectories between scattering events smaller and smaller. In this limit there are no ballistic trajectories anymore and we come to continuous medium.

When the energy distribution \( w(\varepsilon) \) decays quicker than \( \varepsilon^{-2} \) energy losses for the thinning procedure model are described by asymptotically universal Mittag–Leffler distribution. This distribution is characterized by the spatial scale \( b \gg a \) at which the particle is scattering with the probability about 1/2. In the random medium with power-low correlations the parameter \( b \) determines the amplitude of the Laplace transform of the function \( g(r) \):

\[
\tilde{g}(q) = (bq)^{-\alpha} \quad aq \ll 1
\]

(37)

The Mittag–Leffler asymptotic distribution corresponds to the model of constant energy losses \( \Delta = \Delta_1 n \) (section 3.1) with asymptotic function \( \tilde{g}(q) \) (37). Calculating the inverse Laplace transform of expression (33) in Appendix C and renormalizing the spatial scale \( a \to b \) we get:

\[
f_n(x) = \frac{(x/b)^{\alpha n}}{n!} E^{(n)}_{\alpha} \left( \left( \frac{x}{b} \right)^{\alpha} \right).
\]

(38)

where \( E^{(n)}_{\alpha} \) is \( n \)-th derivative of the function \( E_{\alpha} \), Eq. (46) of Appendix B and the average over this distribution \( \tilde{n}(x) = (x/b)^{\alpha}/\Gamma(1+\alpha) \) (17) determines average losses at distance \( x \) from the surface, \( \langle \Delta(x) \rangle = \Delta_1 \tilde{n}(x) \). At \( n = 0 \) the Mittag–Leffler distribution turns to the survival probability, Eq. (31), while in the limit \( x \gg b \) it turns to the renormalized function (35). In the case \( \alpha = 1 \) the distribution (38) takes the well known Poisson form

\[
f_n(x) = \frac{\tilde{n}^n}{n!} e^{-\tilde{n}}, \quad \tilde{n} = \frac{x}{b}
\]

and may be considered as generalization of Poisson distribution of scattering events for the case of fractal medium.

In the case of slowly decaying energy distribution at scattering events \( w(\varepsilon) \sim \varepsilon^{-1-\beta} \) \( (0 < \beta < 1) \) the scattering on fractal structures has the form of convolution of Lévy flight processes characterized by the exponent \( \beta \) with anomalous scaling processes characterized by the exponent \( \alpha \). Such combined process is described by fractional space-energy differential equation, and average energy losses grow with the distance \( x \) from the surface as \( \langle \Delta(x) \rangle \sim x^{\alpha/\beta} \). The effective exponent \( \alpha/\beta \) of this process can be smaller or larger than 1, depending on relation between exponents \( \alpha \) and \( \beta \).
4 Conclusion

We study the loss of energy of the particle moving in fractal media and in the system with dynamic heterogeneities formed at the critical point of phase transition. We show that when the distribution of energy loss during collisions $w(\varepsilon)$ quickly decays with the energy $\varepsilon$ the distribution function of particle energies is universal and depends only on fractal dimension $D$ of the medium. In the case $D = 3$ spacial heterogeneities only weakly affect the scattering process, which can be described by the classical theory.\cite{10} In the case $2 < D < 3$ spacial heterogeneities change the character of the scattering, which can be described by fractional differential equations of order $\alpha = D - 2$. Nonlocal character of fractional derivatives (see Eq. (34)) reflects power-law correlations existing in the fractal system. We show that the loss of the energy in fractal medium can be described by the Mittag-Leffler renewal process of order $\alpha$, which is fractional generalization of the Poisson process corresponding to the case $\alpha = 1$ of absence of such correlations.

One of the most important applications of this theory is propagation of particle through the percolation scattering structure with fractal dimension $D \simeq 2.49$, when the exponent $\alpha \simeq 0.49$. Similar exponent $\alpha = 1/2$ is obtained for fractals with lattice animals structure – a set of randomly connected sites on a lattice.\cite{14} The lower boundary $D = 2$ (corresponding to random walk structures) of applicability of our consideration equal to the dimension $d = 2$ of the interface. Therefore in the case of fractal dimension $D < 2$ of scattering clusters the losses of the particle energy per unit area decrease with the rise of the interface area.

We also derived general kinetic equation (23) for the average distribution function of energy losses in random fractal medium, that can be considered as generalization of the Landau equation (11) for homogeneous medium with constant density of scattering centers, $g(r) = 1/a$. In heterogeneous medium the integral kernel $g(r)$ of this equation is proportional to the structure function $G(r)$ of the medium. The non-local character of the kinetic equation (23) is related to the presence of strong non-local correlations in a fractal medium. The consideration of scattering of paricles in turbulent medium needs additional study because of multifractal structure of turbulent flows.
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A  Estimation of exponent $\alpha$ for fractal medium

Here we present simple scaling estimation of the survival probability $\Psi(r)$, Eq. (9), in the case of scattering of particle in random medium with fractal dimension $D$. We consider all particle trajectories colliding with the same scattering center as identical. Average number of such trajectories starting from one scattering center can be found by draw the sphere of radius $r$ around this center, see Fig. 2. The total number of scattering centers inside this sphere is $N(r) \sim (r/a)^D$. Note, that scaling consideration can be applied to fractal objects with some care: to get the right scaling on the scale $r$ we have to place all these scattering centers randomly at a distance $\sim r$ from the center of the sphere. The propagating particle collides (first time, as in the case of the first passage problem, see Fig. 2) only with small survival part $\Psi(r)$ of these centers. Therefore, the total number of different ballistic trajectories is $N(r)\Psi(r)$ and their total area on the sphere is $a^2 N(r)\Psi(r)$. On the other hand, projections of these trajectories on the sphere cover the whole area $\sim r^2$ of this sphere:

$$a^2 N(r) \Psi(r) \simeq r^2$$

Solving this equation with respect to the survival probability we find

$$\Psi(r) \simeq \frac{r^2}{a^2 N(r)} \simeq \left(\frac{a}{r}\right)^{D-2}, \quad r > a. \quad (39)$$

B  Wrize type functions

The Wrize function is defined as

$$W_{\alpha}(z) = \sum_{l=0}^{\infty} \frac{(-z)^l}{l! \Gamma(1-\alpha-\alpha l)} \quad (40)$$

We can present this function in integral form using corresponding presentation of Gamma function for non-integer $-z$

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_C (-t)^{-z} e^{-t} dt \quad (41)$$
Figure 2: Scaling estimation of survival probability $\Psi(r)$. The number of scattering centers inside the sphere of radius $r$ is $N(r) \simeq (r/a)^D$. The particle collides first time with only $N(r)\Psi(r)$ centers. The total surface area of these centers $a^2N(r)\Psi(r) \simeq r^2$ equal to the area of the sphere.

The contour $C$ of integration encircles positive axis of complex variable $t$. Substituting this expression into Eq. (40) and calculating the sum over $l$ we find

$$W_\alpha(z) = \frac{i}{2\pi} \int_C (-t)^{\alpha-1} e^{-t-z(-t)\alpha} dt \quad (42)$$

The function $W_\alpha(z)$ is normalized by the condition

$$\int_0^\infty W_\alpha(z) \, dz = \frac{i}{2\pi} \int_C (-t)^{-1} e^{-t} \, dt = 1 \quad (43)$$

Calculating the integral (42) by the steepest descent method, we find its asymptotic behavior at large $z \gg 1$:

$$W_\alpha(z) \simeq \frac{1}{\sqrt{\pi} (1-\alpha)} (z\alpha)^{-\frac{1-2\alpha}{2(1-\alpha)}} e^{-\frac{1-\alpha}{\alpha}(z\alpha)^{1-\alpha}} \quad (44)$$

Explicit expressions can be obtained for some particular cases:

$$W_\alpha(z) = \begin{cases} e^{-z} & \text{for } \alpha = 0 \\ \frac{1}{\sqrt{\pi}} e^{-z^2/4} & \text{for } \alpha = 1/2 \\ \delta(z - 1) & \text{for } \alpha = 1 \end{cases} \quad (45)$$

The one-parameter Mittag-Leffler function is defined by the series

$$E_\alpha(z) = \sum_{l=0}^{\infty} \frac{(-z)^l}{\Gamma(1+l\alpha)}. \quad (46)$$
Its integral presentation can be found similar to Eq. (42):

\[
E_\alpha (z) = \frac{i}{2\pi} \int_C \frac{(-t)^{\alpha-1}}{z + (-t)^\alpha} e^{-t} dt
\]

and at large \( z \gg 1 \) and \( \alpha < 1 \) it decays as:

\[
E_\alpha (z) \simeq \frac{1}{\Gamma (1 - \alpha) z}
\]

We show also known explicit expressions for this function:

\[
E_\alpha (z) = \begin{cases} 
(1 + z)^{-1} & \text{for } \alpha = 0 \\
\exp (z^2 \text{erfc} (z)) & \text{for } \alpha = 1/2 \\
e^{-z} & \text{for } \alpha = 1
\end{cases}
\]

\[(49)\]

C Solution of CTRW

The solution of Eq. (22) with function \( \tilde{g} (q) \) (30) has the form:

\[
\tilde{f} (p, q) = \frac{1}{q + \omega (p) (aq)^{1-\alpha}}
\]

where \( \omega (p) = [1 - \tilde{w} (p)] / a \). Calculating the inverse Laplace transform over \( q \)

\[
\tilde{f} (p, x) \equiv \int_0^{\infty} d\Delta e^{-p\Delta} f (\Delta, x)
\]

we find in the limits of small and large \( x \):

a) At small \( x \) one can expand Eq. (50) in powers of small \( q^{-\alpha} \), and we get simple analytical form of the Laplace transform of the distribution function

\[
\tilde{f} (p, x) = E_\alpha \left[ \omega (p) a^{1-\alpha} x^\alpha \right]
\]

where \( E_\alpha \) is the one-parameter Mittag-Leffler function, defined in Eq. (47) of Appendix B.

b) At large \( x \) one can expand Eq. (50) in powers of small \( q^\alpha \), and we find

\[
\tilde{f} (p, x) = \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{\Gamma (1 - \alpha l)} \frac{1}{[a \omega (p) (x/a)^\alpha]^l}
\]

\[(52)\]
In the case of constant losses of particle energy $\Delta = \Delta_{1}n$ the function
\[ \omega(p) = \frac{1 - e^{-p\Delta_{1}}}{a}, \]
and we find from Eq. (50)
\[ \tilde{f}_{n}(q) = \frac{(aq)^{(1-\alpha)n}}{[q + (aq)^{1-\alpha}]^{n+1}} \tag{53} \]