Traces of pseudo-differential operators on compact and Hausdorff groups

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Abstract We give a characterization of and a trace formula for trace class pseudo-differential operators on compact Hausdorff groups.

Keywords Pseudo-differential operators · Compact groups

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1 Introduction

Pseudo-differential operators, first developed by Kohn and Nirenberg [8] in 1965 and then is used by Hörmander [7] and others for problems in partial differential equations. Weyl transforms which are a class of pseudo-differential operators have applications in Quantization due to Hermann Weyl [18]. In [21] Weyl transforms on compact Lie groups are introduced and the heat kernels of the Laplacian on the compact Lie group is obtained. In this paper, we look at pseudo-differential operators on compact and Hausdorff groups with $L^2$ symbols. The $L^p$, $1 \leq p \leq \infty$ conditions on the symbols allowing singularities are ideal for a broad spectrum of disciplines ranging from functional analysis to operator algebras to quantization. An analogue of the results in this paper is studied for the compact Lie group $\mathbb{S}^{n-1}$, i.e., the unit sphere.
with center at the origin in [5]. Pseudo-differential operators on the unit sphere are studied extensively in [1–11, 13, 14, 16, 17, 22].

The aim of this paper is to give a characterization of trace class pseudo-differential operators on compact and Hausdorff groups. We give a formula for the trace of pseudo-differential operators in the trace class. The main technique is to obtain a formula for the symbol of the product of two pseudo-differential operators on a compact and Hausdorff group.

In Sect. 2, we give a brief recall of Hilbert-Schmidt and trace class operators. We define pseudo-differential operators on compact and Hausdorff groups by using the space of all unitary and irreducible representations in Sect. 3. A product formula for pseudo-differential operators is given. Then we give a characterization of trace class pseudo-differential operators.

2 Hilbert-Schmidt and trace class operators

Let \( \mathcal{H} \) be a complex and separable Hilbert space in which the inner product and norm are denoted by \((\cdot, \cdot)_{\mathcal{H}}\) and \( \| \cdot \|_{\mathcal{H}} \). Let \( A : \mathcal{H} \rightarrow \mathcal{H} \) be a compact operator. Then the absolute value of \( A \) denoted by \( |A| \) is defined by

\[
|A| = (A^* A)^{1/2}.
\]

The operator \( |A| \) is compact and positive. So, by the spectral theorem, there exists an orthonormal basis \( \{ \varphi_j \}_{j=1}^{\infty} \) of eigenvectors of \( |A| \) with the corresponding eigenvalues \( \{ s_j \}_{j=1}^{\infty} \) of real numbers. The operator \( A \) is said to be in the Hilbert-Schmidt class \( S_2 \) if

\[
\sum_{j=1}^{\infty} s_j^2 < \infty,
\]

and we define its Hilbert-Schmidt norm by

\[
\| A \|_{HS} = \left( \sum_{j=1}^{\infty} s_j^2 \right)^{1/2}.
\]

The operator \( A \) is said to be in the trace class \( S_1 \) if

\[
\sum_{j=1}^{\infty} s_j < \infty,
\]

and we define its trace by

\[
tr(A) = \sum_{j=1}^{\infty} s_j.
\]
We have the following theorem which shows that the Hilbert-Schmidt norm and trace is independent of the choice of orthonormal basis for $\mathcal{H}$, for details see [12] by Reed and Simon.

**Theorem 2.1** Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator. Then

- $A \in S_2$ if and only if there exists an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for $\mathcal{H}$ such that
  \[
  \|A\|_{HS} = \left( \sum_{j=1}^{\infty} \|A\varphi_j\|_{\mathcal{H}}^2 \right)^{1/2} < \infty.
  \]

- $A \in S_1$ if and only if there exists an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for $\mathcal{H}$ such that
  \[
  \text{tr}(A) = \sum_{j=1}^{\infty} (A\varphi_j, \varphi_j)_{\mathcal{H}} < \infty.
  \]

The following theorem will be useful in the next section.

**Theorem 2.2** Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded operator. Then $A \in S_1$ if and only if $A = BC$, where $B$ and $C$ are in $S_2$.

### 3 Pseudo-differential operators on compact and Hausdorff groups

Let $G$ be a locally compact and Hausdorff group. Let $\mathcal{H}$ be a separable and complex Hilbert space. We denote the group of all unitary operators by $U(\mathcal{H})$. A group homomorphism $\pi : G \rightarrow U(\mathcal{H})$ is said to be a unitary representation of $G$ on $\mathcal{H}$ if it is strongly continuous, i.e., the mapping $g \mapsto \pi(g)h$ is a continuous mapping for all $h \in \mathcal{H}$. The dimension of $\mathcal{H}$ is known as the degree or the dimension of the representation $\pi$.

A closed subspace $M$ of $\mathcal{H}$ is said to be invariant with respect to the unitary representation $\pi : G \rightarrow U(\mathcal{H})$ if

\[
\pi(g)M \subseteq M, \quad \forall g \in G.
\]

A unitary representation $\pi : G \rightarrow U(\mathcal{H})$ is called irreducible if it has only the trivial invariant subspaces, i.e., $\{0\}$ and $\mathcal{H}$. The following theorem is crucial in defining pseudo-differential operators on compact Hausdorff groups. For its proof see [6,20].

**Theorem 3.1** Let $G$ be a compact and Hausdorff group. Then any irreducible and unitary representation of $G$ on a complex and separable Hilbert space is finite dimensional.

Let $\mathbb{C}$ be a compact and Hausdorff group with the left (and right) Haar measure is denoted by $\mu$. Let $\hat{G}$ be the set of all irreducible and unitary representations of $\mathbb{C}$. 
Let \( \pi \in \hat{G} \). Then \( \pi \) is finite dimensional. Let \( a_\pi \) be the degree of \( \pi \) and let \( \mathcal{H}_\pi \) be its representation space. We denote the inner product and the norm in \( \mathcal{H}_\pi \) by \( (\cdot, \cdot)_{\mathcal{H}_\pi} \) and \( \| \cdot \|_{\mathcal{H}_\pi} \) respectively. Let \( \{ \psi_1, \psi_2, \ldots, \psi_{a_\pi} \} \) be an orthonormal basis for \( \mathcal{H}_\pi \). Then for all \( j, k = 1, 2, \ldots \), we define \( \pi_{jk} \) by
\[
\pi_{jk}(g) = \sqrt{a_\pi (\pi(g) \psi_k, \psi_j)}_{\mathcal{H}_\pi}
\]

**Theorem 3.2** (Peter-Weyl Theorem) \( \{ \pi_{jk} : \pi \in \hat{G}, j, k = 1, 2, \ldots, a_\pi \} \) forms an orthonormal basis for \( L^2(\mathbb{G}) \).

By Peter-Weyl theorem, every \( f \in L^2(\mathbb{G}) \) can be expressed as
\[
f = \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} (f, \pi_{jk})_{L^2(\mathbb{G})} \pi_{jk}.
\]

have the following Plancherel theorem.

**Theorem 3.3** For all \( f \in L^2(\mathbb{G}) \),
\[
\|f\|_{L^2(\mathbb{G})} = \left\{ \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} |(f, \pi_{jk})_{L^2(\mathbb{G})}|^2 \right\}^{1/2}.
\]

Let \( \sigma \) be a measurable function on \( \mathbb{G} \times \hat{G} \times \mathbb{N} \times \mathbb{N} \). For all measurable functions \( f \) on \( \mathbb{G} \), we define \( T_\sigma f \) on \( \mathbb{G} \) formally by
\[
(T_\sigma f)(g) = \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} \sigma(g, \pi, j, k) (f, \pi_{jk})_{L^2(\mathbb{G})} \pi_{jk}(g), \quad g \in \mathbb{G}.
\]

\( T_\sigma \) is called the pseudo-differential operator on \( \mathbb{G} \) corresponding to the symbol \( \sigma \). For pseudo-differential operators on Euclidean spaces see for example [15, 19].

Let \( L^2(\mathbb{G} \times \hat{G} \times \mathbb{N} \times \mathbb{N}) \) be the space all measurable functions \( \sigma \) on \( \mathbb{G} \times \hat{G} \times \mathbb{N} \times \mathbb{N} \) for which
\[
\|\sigma\|_{L^2(\mathbb{G} \times \hat{G} \times \mathbb{N} \times \mathbb{N})} = \left\{ \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} \int_{\mathbb{G}} |\sigma(g, \pi, j, k) \pi_{jk}(g)|^2 \ d\mu(g) \right\}^{1/2} < \infty.
\]

We have the following result on the \( L^2 \)-boundedness of pseudo-differential operators on \( \mathbb{G} \).

**Theorem 3.4** Let \( \sigma \in L^2(\mathbb{G} \times \hat{G} \times \mathbb{N} \times \mathbb{N}) \). Then \( T_\sigma : L^2(\mathbb{G}) \to L^2(\mathbb{G}) \) is a bounded operator and
\[
\|T_\sigma\|_* \leq \|\sigma\|_{L^2(\mathbb{G} \times \hat{G} \times \mathbb{N} \times \mathbb{N})},
\]
where \( \| \cdot \|_* \) is the norm in the \( C^* \)-algebra of all bounded linear operators from \( L^2(\mathbb{G}) \) into \( L^2(\mathbb{G}) \).
Proof Let \( f \in L^2(\mathbb{G}) \). By Minkowski’s inequality and the Schwarz inequality, we have

\[
\|T_\sigma f\|_{L^2(\mathbb{G})} = \left( \int_{\mathbb{G}} |(T_\sigma f)(g)|^2 \, d\mu(g) \right)^{1/2} \\
= \left( \int_{\mathbb{G}} \left| \sum_{\pi \in \hat{\mathbb{G}}} \sum_{j,k=1}^{a_\pi} \sigma(g, \pi, j, k) (f, \pi, j, k)_{L^2(\mathbb{G})} \pi_j g \right|^2 \, d\mu(g) \right)^{1/2} \\
\leq \sum_{\pi \in \hat{\mathbb{G}}} \left( \int_{\mathbb{G}} \left( \sum_{j,k=1}^{a_\pi} |(f, \pi, j, k)_{L^2(\mathbb{G})}|^2 \right) \left( \sum_{j,k=1}^{a_\pi} |\sigma(g, \pi, j, k)|^2 \right) \, d\mu(g) \right)^{1/2} \\
= \sum_{\pi \in \hat{\mathbb{G}}} \left( \int_{\mathbb{G}} \left( \sum_{j,k=1}^{a_\pi} |(f, \pi, j, k)_{L^2(\mathbb{G})}|^2 \right)^{1/2} \left( \sum_{j,k=1}^{a_\pi} |\sigma(g, \pi, j, k)|^2 \right)^{1/2} \, d\mu(g) \right)^{1/2} \\
\leq \left( \sum_{\pi \in \hat{\mathbb{G}}} \left( \sum_{j,k=1}^{a_\pi} |(f, \pi, j, k)_{L^2(\mathbb{G})}|^2 \right)^{1/2} \left( \sum_{\pi \in \hat{\mathbb{G}}} \int_{\mathbb{G}} \left( \sum_{j,k=1}^{a_\pi} |\sigma(g, \pi, j, k)|^2 \right) \, d\mu(g) \right)^{1/2} \\
= \|f\|_{L^2(\mathbb{G})} \|\sigma\|_{L^2(\hat{\mathbb{G}} \times \mathbb{N} \times \mathbb{N})}. \\
\]

Now we are ready to give a necessary and sufficient condition on \( \sigma \) to guarantee Hilbert-Schmidt properties of \( T_\sigma \).

**Theorem 3.5** Let \( \sigma \) be a measurable function on \( \mathbb{G} \times \hat{\mathbb{G}} \times \mathbb{N} \times \mathbb{N} \). Then \( T_\sigma \) is a Hilbert-Schmidt operator if and only if \( \sigma \in L^2(\mathbb{G} \times \hat{\mathbb{G}} \times \mathbb{N} \times \mathbb{N}) \) and

\[
\|T_\sigma\|_{HS} = \|\sigma\|_{L^2(\mathbb{G} \times \hat{\mathbb{G}} \times \mathbb{N} \times \mathbb{N})}. \\
\]

Proof For \( \tilde{\pi} \in \hat{\mathbb{G}} \) and \( j_0, k_0 = 1, 2, \ldots, a_{\tilde{\pi}} \), we have

\[
(T_\sigma \tilde{\pi}_{j_0 k_0})(g) = \sum_{\pi \in \hat{\mathbb{G}}} \sum_{j,k=0}^{a_\pi} \sigma(g, \pi, j, k) (\tilde{\pi}_{j_0 k_0}, \pi_j)_{L^2(\mathbb{G})} \pi_j(g) \\
= \sum_{\pi \in \hat{\mathbb{G}}} \sum_{j,k=0}^{a_\pi} \sigma(g, \pi, j, k) (\tilde{\pi}_{j_0 k_0}, \pi_j)_{L^2(\mathbb{G})} \pi_j(g) \\
= \sigma(g, \tilde{\pi}, k_0, j_0) \tilde{\pi}_{j_0 k_0}(g), \quad g \in \mathbb{G}. \\
\]

Hence
\[
\sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_{\pi}} \left\| T_{\sigma} \pi_{jk} \right\|_{L^2(G)}^2 = \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_{\pi}} \left( \int_{G} \left| T_{\sigma} \pi_{jk}(g) \right|^2 d\mu(g) \right) = \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_{\pi}} \int_{G} \left| \sigma(g, \pi, j, k) \pi_{jk}(g) \right|^2 d\mu(g) = \|\sigma\|_{L^2(G \times \hat{G} \times \mathbb{N} \times \mathbb{N})}^2.
\]
and the proof is complete. \(\square\)

Let \(\sigma\) and \(\tau\) measurable functions on \(G \times \hat{G} \times \mathbb{N} \times \mathbb{N}\). Then we define \(\sigma \otimes \tau\) by

\[
\sigma \otimes \tau(g, \xi, l, m) = \left( \int_{\hat{G}} \tau(w, \xi, l, m) \xi_{lm}(w) \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_{\pi}} \sigma(g, \pi, j, k) \pi_{jk}(w) \pi_{jk}(g) \ d\mu(w) \right)^{-1}
\]

for all \(g \in G, \xi \in \hat{G}\) and \(l, m = 1, \ldots, a_{\xi}\). The following theorem shows that the composition of two pseudo-differential operators on \(G\) is again a pseudo-differential operator.

**Theorem 3.6** Let \(\sigma\) and \(\tau\) be measurable functions on \(G \times \hat{G} \times \mathbb{N} \times \mathbb{N}\). Then

\[
T_{\sigma} T_{\tau} = T_{\lambda}
\]

where \(\lambda = \sigma \otimes \tau\).

**Proof** Let \(f\) be in \(L^2(G)\). Then for all \(g \in G\)

\[
(T_{\sigma} T_{\tau} f)(g) = \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_{\pi}} \sigma(g, \pi, j, k) \left( T_{\tau} f, \pi_{jk} \right)_{L^2(G)} \pi_{jk}(g)
\]

\[
= \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_{\pi}} \sigma(g, \pi, j, k) \left( \int_{\hat{G}} (T_{\tau} f)(w) \overline{\pi_{jk}(w)} \ d\mu(w) \right) \pi_{jk}(g)
\]

\[
= \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_{\pi}} \sigma(g, \pi, j, k) \left( \sum_{\xi \in \hat{G}} \sum_{l,m=1}^{a_{\xi}} \tau(w, \xi, l, m) (f, \xi_{lm})_{L^2(G)} \xi_{lm}(w) \overline{\pi_{jk}(w)} \ d\mu(w) \right) \pi_{jk}(g)
\]
By the Schwarz inequality and Plancheral’s theorem, \( \text{Traces of pseudo-differential operators on compact and Hausdorff groups} \)

\[
\sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} \sigma(g, \pi, j, k) \left( \sum_{\xi \in \hat{G}} \sum_{l,m=1}^{a_\xi} (f, \xi_{lm})_{L^2(G)} \int_G \tau(w, \xi, l, m)\xi_{lm}(w)\pi_{jk}(w) d\mu(w) \right) \pi_{jk}(g)
\]

\[
= \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} \left( \int_G \tau(w, \xi, l, m)\xi_{lm}(w) \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} \sigma(g, \pi, j, k)\pi_{jk}(w) d\mu(w) \right) \times (\xi_{lm}(g))^{-1} (f, \xi_{lm})_{L^2(G)} \xi_{lm}(g)
\]

\[
= \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} \lambda(g, \pi, j, k) (f, \xi_{lm})_{L^2(G)} \xi_{lm}(g), \quad (3.2)
\]

where \( \lambda = \sigma \otimes \tau \). Thus,

\[
T_\lambda = T_\sigma T_\tau
\]

\[\Box\]

Now we are ready to give a characterization of trace class pseudo-differential operators on \( \hat{G} \).

**Theorem 3.7** Let \( \lambda \) be a measurable function in \( \hat{G} \times \hat{G} \times \mathbb{N} \times \mathbb{N} \). Then \( T_\lambda \in S_1 \) if and only if there exist symbols \( \sigma \) and \( \tau \) in \( L^2(\hat{G} \times \hat{G} \times \mathbb{N} \times \mathbb{N}) \) such that

\[
\lambda = \sigma \otimes \tau,
\]

and

\[
\text{tr}(T_\lambda) = \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} \int_G \lambda(g, \pi, j, k)|\pi_{jk}(g)|^2 d\mu(g)
\]

\[
= \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} \sum_{\xi \in \hat{G}} \sum_{l,m=1}^{a_\xi} \left( \tau(\cdot, \xi, l, m)\xi_{lm} \pi_{jk} \right)_{L^2(G)} \left( \sigma(\cdot, \pi, j, k)\pi_{jk}, \xi_{lm} \right)_{L^2(G)}.
\]

**Proof** The first part of the theorem follows from Theorem 2.2, Theorem 3.5 and Theorem 3.6. Now we check the absolute convergence of the series

\[
\sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} \sum_{\xi \in \hat{G}} \sum_{l,m=1}^{a_\xi} \left( \tau(\cdot, \xi, l, m)\xi_{lm} \pi_{jk} \right)_{L^2(G)} \left( \sigma(\cdot, \pi, j, k)\pi_{jk}, \xi_{lm} \right)_{L^2(G)}
\]

By the Schwarz inequality and Plancheral’s theorem,

\[
\sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} \sum_{\xi \in \hat{G}} \sum_{l,m=1}^{a_\xi} \left| \left( \tau(\cdot, \xi, l, m)\xi_{lm} \pi_{jk} \right)_{L^2(G)} \right| \left| \left( \sigma(\cdot, \pi, j, k)\pi_{jk}, \xi_{lm} \right)_{L^2(G)} \right|
\]

\[
\leq \left\{ \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{a_\pi} \sum_{\xi \in \hat{G}} \sum_{l,m=1}^{a_\xi} \left| \left( \tau(\cdot, \xi, l, m)\xi_{lm} \pi_{jk} \right)_{L^2(G)} \right|^2 \right\}^{1/2}
\]
\[
\left\{ \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{\infty} \left( \sigma(\cdot, \pi, j, k) \pi_{jk} \xi_{lm} \right)_{L^2(G)} \right\}^{1/2}
= \left\{ \sum_{\xi \in \hat{G}} \sum_{l,m=1}^{\infty} \int_{G} |\tau(g, \xi, l, m)\xi_{lm}(g)|^2 \, d\mu(g) \right\}^{1/2}
= \left\{ \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{\infty} \int_{G} |\sigma(g, \pi, j, k)\pi_{jk}(g)|^2 \, d\mu(g) \right\}^{1/2}
\]
\[
= \| \sigma \|_{L^2(G \times \hat{G} \times \mathbb{N} \times \mathbb{N})} \| \tau \|_{L^2(G \times \hat{G} \times \mathbb{N} \times \mathbb{N})} < \infty.
\]

Since \( \bigcup_{\pi \in \hat{G}} \{ \pi_{jk} : j, k = 1, 2, \ldots, a_\pi \} \) forms an orthonormal basis for \( L^2(G) \), it follows that
\[
\text{tr}(T_\xi) = \sum_{\xi \in \hat{G}} \sum_{l,m=1}^{\infty} (T_\lambda \xi_{lm}, \xi_{lm})_{L^2(G)}
= \sum_{\xi \in \hat{G}} \sum_{l,m=1}^{\infty} \int_{G} \lambda(g, \xi, l, m)\xi_{lm}(g)\xi_{lm}(g) \, d\mu(g)
= \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{\infty} \left( \int_{G} \tau(w, \xi, l, m)\xi_{lm}(w) \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{\infty} \sigma(g, \pi, j, k)\pi_{jk}(w)\pi_{jk}(g) \, d\mu(w) \right) \xi_{lm}(g) \, d\mu(g)
\]
\[
= \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{\infty} \sum_{\xi \in \hat{G}} \sum_{l,m=1}^{\infty} \int_{G} \tau(w, \xi, l, m)\xi_{lm}(w)\pi_{jk}(w)\pi_{jk}(g) \, d\mu(w) \int_{G} \sigma(g, \pi, j, k)\pi_{jk}(g) \, d\mu(g) \xi_{lm}(g) \, d\mu(g)
= \sum_{\pi \in \hat{G}} \sum_{j,k=1}^{\infty} \sum_{\xi \in \hat{G}} \sum_{l,m=1}^{\infty} \left( \tau(\cdot, \xi, l, m)\xi_{lm}, \pi_{jk} \right)_{L^2(G)} \left( \sigma(\cdot, \pi, j, k)\pi_{jk}, \xi_{lm} \right)_{L^2(G)}.
\]

\( \square \)

We can now look at the special case when \( G = S^1 \).

**Example 3.8** Let \( S^1 \) be the unit circle centered at the origin. For all \( n \in \mathbb{Z} \), we define \( \pi_n : S^1 \to U(1) \) by
\[
\pi_n(\theta) = (2\pi)^{-1/2} e^{in\theta}, \quad \theta \in [-\pi, \pi].
\]

Then
\[
\hat{S}^1 = \{ \pi_n : n \in \mathbb{Z} \}.
\]
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and $\hat{\mathbb{S}} \simeq \mathbb{Z}$. Hence for any symbol $\sigma$ on $S^1 \times \mathbb{Z}$ we can define pseudo-differential operator $T_{\sigma}$ on $L^2(S^1)$ by

$$ (T_{\sigma} f)(\theta) = \sum_{n \in \mathbb{Z}} \sigma(\theta, n)( f, \pi_n)_{L^2(S^1)} \pi_n(\theta), \quad f \in L^2(S^1), \quad \theta \in [-\pi, \pi]. $$

Hence,

$$ (T_{\sigma} f)(\theta) = (2\pi)^{-1} \sum_{n \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{in(\theta - \varphi)} \sigma(n, \theta) f(\varphi) d\varphi. $$

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