PARAMETER UNIFORM NUMERICAL METHOD FOR SINGULARLY PERTURBED TWO PARAMETER PARABOLIC PROBLEM WITH DISCONTINUOUS CONVECTION COEFFICIENT AND SOURCE TERM

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Abstract

In this article, we have considered a time-dependent two-parameter singularly perturbed parabolic problem with discontinuous convection coefficient and source term. The problem contains the parameters \(\epsilon\) and \(\mu\) multiplying the diffusion and convection coefficients, respectively. A boundary layer develops on both sides of the boundaries as a result of these parameters. An interior layer forms near the point of discontinuity due to the discontinuity in the convection and source term. The width of the interior and boundary layers depends on the ratio of the perturbation parameters. We discuss the problem for ratio \(\frac{\mu^2}{\epsilon}\). We used an upwind finite difference approach on a Shishkin-Bakhvalov mesh in the space and the Crank-Nicolson method in time on uniform mesh. At the point of discontinuity, a three-point formula was used. This method is uniformly convergent with second order in time and first order in space. Shishkin-Bakhvalov mesh provides first-order convergence; unlike the Shishkin mesh, where a logarithmic factor deteriorates the order of convergence. Some test examples are given to validate the results presented.

Keywords: singularly perturbed, two-parameter, parabolic problem, discontinuous data, boundary and interior layers, Crank-Nicolson method, Shishkin-Bakhvalov mesh

1 Introduction

Here, we have considered the following two-parameter singularly perturbed parabolic boundary value problem (TPSPP-BVP) on the domain \(\Omega = (0, 1) \times (0, T)\):

\[
\mathcal{L}_{\epsilon,\mu} u(x,t) \equiv (\epsilon u_{xx} + \mu au_x - bu - cu_t)(x,t) = f(x,t), \quad (x,t) \in (\Omega^- \cup \Omega^+),
\]

\[u(0,t) = p(t), \quad (0, t) \in \Gamma_l;\]

\[u(1,t) = r(t), \quad (1, t) \in \Gamma_r;\]

\[u(x,0) = q(x), \quad (x, 0) \in \Gamma_b,\]

\[(1.1)\]
where $0 < \epsilon \ll 1$ and $0 < \mu \leq 1$ are two singular perturbation parameters and $\Omega^- = (0, d) \times (0, T) \times (0, T)$, $\Omega^+ = (d, 1) \times (0, T)$. The convection coefficient $a(x, t)$ and source term $f(x, t)$ are discontinuous at $(d, t) \in \Omega \forall t$. Also $a(x, t) \leq -\alpha_1 < 0$, $(x, t) \in \Omega^-$ and $a(x, t) \geq \alpha_2 > 0$, $(x, t) \in \Omega^+$. $\alpha_1, \alpha_2$ are positive constants. The functions $a(x, t)$ and $f(x, t)$ are sufficiently smooth functions on $(\Omega^- \cup \Omega^+)$. In addition, we assume the jumps of $a(x, t)$ and $f(x, t)$ at $(d, t)$ satisfy $|[a](d, t)| < C$, $|[f](d, t)| < C$, where the jump of $\omega$ at $(d, t)$ is defined as $[\omega](d, t) = \omega(d^+, t) - \omega(d^-, t)$. The coefficients $b(x, t)$ and $c(x, t)$ are assumed to be sufficiently smooth functions on $\Omega$ such that $b(x, t) \geq \beta > 0$ and $c(x, t) > \eta > 0$. The boundary data $p(t), r(t)$ and initial data $q(x)$ are sufficiently smooth on the domain and satisfy the compatibility condition. Let $\Gamma^- = (0, d), \Gamma^+ = (d, 1), \Gamma = (0, 1), \Gamma_l = \{(0, t)|0 \leq t \leq T\}, \Gamma_r = \{(1, t)|0 \leq t \leq T\}, \Gamma_b = \{(x, 0)|0 \leq x \leq 1\}$ and $\Gamma_c = \Gamma_l \cup \Gamma_b \cup \Gamma_r$. Under the above assumptions, the problem $[1.1]$ has a unique continuous solution in the domain $\Omega$.

The solution to problem $[1.1]$ contains interior layers at the points of discontinuity $(d, t) , \forall t \in (0, T)$ because of the discontinuity in the convection coefficient and source term. Additionally, the solution displays boundary layers at $\Gamma_l$ and $\Gamma_r$ as a result of the presence of perturbation parameters $\epsilon$ and $\mu$.

Two-parameter singularly perturbed problems was first studied asymptotically by O’Malley in $[20, 21, 22]$. He observed that the solution to these problems depends on the perturbation parameters $\epsilon$ and $\mu$ as well as on their ratio. So, we discuss the solution of Eq. $[1.1]$ under the following cases:

**Case (i):** $\sqrt{\alpha \mu} \leq \sqrt{\rho \epsilon}$, where $\rho = \min_{x \in \Omega} \left\{ \frac{b(x, t)}{a(x, t)} \right\}$, the boundary layers of equal width of $O(\sqrt{\epsilon})$ appear near the boundaries.

**Case (ii):** $\sqrt{\alpha \mu} > \sqrt{\rho \epsilon}$, the solution has boundary layers at both the boundaries of different widths.

In recent years, several authors have developed numerical methods for two-parameter singularly perturbed parabolic problems with smooth data $[3, 11, 13, 18, 19, 24, 27]$. In the case of non-smooth data, the study is very limited. M. Chandru et al. in $[4]$ considered the problem $[1.1]$ and proved that the upwind scheme on space on Shishkin mesh and backward Euler scheme on time is almost first-order accurate. D. Kumar et al. in $[15]$ gave a numerical method with parameter uniform convergence of order two in time and almost order one in space for the problem of the same type. They used the Crank-Nicolson method in time on a uniform mesh and the upwind method in space on a Shishkin mesh. Some work on one parameter parabolic singularly perturbed problem with discontinuous data includes $[2, 23]$.

In this paper, we have used Crank-Nicolson scheme $[7]$ on time on a uniform mesh and upwind scheme on an appropriately defined Shishkin-Bakhvalov mesh $[16, 17]$ in space for case (i). At the point of discontinuity, we used a three-point scheme to resolve it. In the Shishkin-Bakhvalov mesh, we choose transition point as in Shishkin mesh and use graded mesh (as in Bakhvalov $[1]$) in the layer region. In the outer region a uniform mesh is used. We are able to achieve first-order convergence in space due to Shishkin-Bakhvalov mesh and second-order in time due to Crank-Nicolson method.

The article is organised in the order listed below: The *apriori* bounds on the solution and its derivatives are given in section 2. The decomposition of the continuous solution into regular and singular components and bounds on the derivatives of regular and singular components are also discussed here. In section 3, numerical method and mesh construction is discussed. In section 4, parameter uniform error estimates are established for case $\sqrt{\alpha \mu} \leq \sqrt{\rho \epsilon}$. The numerical examples in section 5 support the theoretical results given in previous section. A summary of the work is presented in section 6.

**Notation:** The norm used is the maximum norm given by

$$\|u\|_\Omega = \max_{(x,t) \in \Omega} |u(x, t)|.$$
Throughout the paper, $C, C_1, C_2$ will be denoted as a generic positive constant that is independent of perturbation parameters $\epsilon, \mu$ and mesh size.

2 A Priori Derivatives Bounds of the Solution

**Theorem 2.1.** [15] Suppose that a function $z(x,t) \in C^0(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ satisfies $z(x,t) \geq 0$, $\forall (x,t) \in \Gamma_c$, $\mathcal{L}z(x,t) \leq 0, \forall (x,t) \in (\Omega^- \cup \Omega^+)$ and $|z_x|(d,t) \leq 0, t > 0$ then $z(x,t) \geq 0 \forall (x,t) \in \Omega$.

**Theorem 2.2.** [15] The bounds on $u(x,t)$ is given by

$$\|u\|_\Omega \leq \|u\|_{\Gamma_c} + \frac{\|f\|_{\Omega^+ \cup \Omega^-}}{\beta}.$$

The solution $u(x,t)$ of Eq.(1.1) is divided as in [15] into layers and regular components as $u(x,t) = v(x,t) + w_l(x,t) + w_r(x,t)$. The regular component $v(x,t)$ satisfies the following equation:

$$\mathcal{L}v(x,t) = f(x,t), \ (x,t) \in \Omega^- \cup \Omega^+,$$

$$v(0,t) = u(0,t), v(1,t) = u(1,t), \forall t \in (0,T],$$

$$v(x,0) = u(x,0), \forall x \in \Gamma^- \cup \Gamma^+,$$

$$v(d-,t), v(d+,t) \text{ are chosen suitably } \forall t \in (0,T].$$

$v(x,t)$ can further be decomposed as

$$v(x,t) = \begin{cases} v^-(x,t), & (x,t) \in \Omega^-; \\ v^+(x,t), & (x,t) \in \Omega^+, \end{cases}$$

where $v^-$ and $v^+$ are the left and right regular components respectively.

The singular components $w_l(x,t)$ and $w_r(x,t)$ are the solutions of

$$\mathcal{L}w_l(x,t) = 0, \ (x,t) \in \Omega^- \cup \Omega^+,$$

$$w_l(0,t) = u(0,t) - v(0,t) - w_r(0,t), \forall t \in (0,T],$$

$$w_l(1,t) \text{ are chosen suitably } \forall t \in (0,T],$$

$$w_l(x,0) = u(x,0), \forall x \in \Gamma^- \cup \Gamma^+,$$

and

$$\mathcal{L}w_r(x,t) = 0, \ (x,t) \in \Omega^- \cup \Omega^+,$$

$$w_r(1,t) = u(1,t) - v(1,t) - w_l(1,t), \forall t \in (0,T],$$

$$w_r(0,t) \text{ are chosen suitably } \forall t \in (0,T],$$

$$w_r(x,0) = u(x,0), \forall x \in \Gamma^- \cup \Gamma^+,$$

respectively. Also

$$[w_r](d,t) = -([v] + [w_l])(d,t), \ [(w_r)_x](d,t) = -([v_x] + [(w_l)_x])(d,t), \forall t \in (0,T].$$

Further, the singular components $w_l(x,t)$ and $w_r(x,t)$ are decomposed as

$$w_l(x,t) = \begin{cases} w^-_l(x,t), & (x,t) \in \Omega^-; \\ w^+_l(x,t), & (x,t) \in \Omega^+, \end{cases} \quad w_r(x,t) = \begin{cases} w^-_r(x,t), & (x,t) \in \Omega^-; \\ w^+_r(x,t), & (x,t) \in \Omega^+. \end{cases}$$
where \( w^-_l, w^-_r \) are left components and \( w^+_l, w^+_r \) are right components of the singular component respectively. Hence, the unique solution \( u(x, t) \) to the problem Eq. (1.1) is written as

\[
u(x, t) = \begin{cases} 
(v^- + w^-_l + w^-_r)(x, t), & (x, t) \in \Omega^-, \\
(v^- + w^-_l + w^-_r)(d-, t) = (v^+ + w^+_l + w^+_r)(d+, t), & (x, t) = (d, t), \forall t \in (0, T), \\
(v^+ + w^+_l + w^+_r)(x, t), & (x, t) \in \Omega^+.
\end{cases}
\]

**Theorem 2.3.** The left layer and right layer components \( w_l(x, t) \) and \( w_r(x, t) \) satisfy the following inequalities for case \( \sqrt{\alpha \mu} \leq \sqrt{\rho \epsilon} \)

\[
\|w_l(x, t)\|_{\Omega^- \cup \Omega^+} \leq \begin{cases} 
C e^{-\theta_1 x}, & x \in \Omega^-, \\
C e^{-\theta_2(x-d)}, & x \in \Omega^+
\end{cases}, \\
\|w_r(x, t)\|_{\Omega^- \cup \Omega^+} \leq \begin{cases} 
C e^{-\theta_2(d-x)}, & x \in \Omega^-, \\
C e^{-\theta_1(1-x)}, & x \in \Omega^+
\end{cases}
\]

where

\[
\theta_1 = \sqrt{\frac{\rho \epsilon}{\epsilon}}, \quad \theta_2 = \sqrt{\frac{\rho \epsilon}{2\epsilon}}.
\] (2.5)

For the case \( \sqrt{\alpha \mu} > \sqrt{\rho \epsilon} \), the bounds of left layer and right layer components satisfy the following inequalities

\[
\|w_l(x, t)\|_{\Omega^- \cup \Omega^+} \leq \begin{cases} 
C e^{-\theta_1 x}, & x \in \Omega^-, \\
C e^{-\theta_2(x-d)}, & x \in \Omega^+
\end{cases}, \\
\|w_r(x, t)\|_{\Omega^- \cup \Omega^+} \leq \begin{cases} 
C e^{-\theta_2(d-x)}, & x \in \Omega^-, \\
C e^{-\theta_1(1-x)}, & x \in \Omega^+
\end{cases}
\]

where

\[
\theta_1 = \frac{\alpha \mu}{\epsilon}, \quad \theta_2 = \frac{\gamma}{2 \mu}.
\] (2.6)

**Proof.** The proof follows from the technique given in [23, 24].

**Theorem 2.4.** Let \( \sqrt{\alpha \mu} \leq \sqrt{\rho \epsilon} \), the singular component \( w_l(x, t) \) and \( w_r(x, t) \) satisfies the following bounds for \( k = 0, 1, 2, 3 \):

\[
\left\| \frac{d^k v^-_l(x, t)}{dx^k} \right\|_{\Omega^-} \leq C \left( 1 + \epsilon \left( \frac{k-3}{2} \right) \right), \quad \left\| \frac{d^k v^+_r(x, t)}{dx^k} \right\|_{\Omega^+} \leq C \left( 1 + \epsilon \left( \frac{k-3}{2} \right) \right),
\]

\[
\left\| \frac{d^k w^-_l(x, t)}{dx^k} \right\|_{\Omega^-} \leq C e^{-\frac{k}{2}}, \quad \left\| \frac{d^k w^+_r(x, t)}{dx^k} \right\|_{\Omega^+} \leq C e^{-\frac{k}{2}},
\]

\[
\left\| \frac{d^k w^-_l(x, t)}{dx^k} \right\|_{\Omega^-} \leq C e^{-\frac{k}{2}}, \quad \left\| \frac{d^k w^+_r(x, t)}{dx^k} \right\|_{\Omega^+} \leq C e^{-\frac{k}{2}}.
\]

For \( \sqrt{\alpha \mu} > \sqrt{\rho \epsilon} \), the singular component \( w_l(x, t) \) and \( w_r(x, t) \) satisfies the following bounds

\[
\left\| \frac{d^k v^-_l(x, t)}{dx^k} \right\|_{\Omega^-} \leq C \left( 1 + \left( \frac{\alpha}{\mu} \right)^{3-k} \right), \quad \left\| \frac{d^k v^+_r(x, t)}{dx^k} \right\|_{\Omega^+} \leq C \left( 1 + \left( \frac{\alpha}{\mu} \right)^{3-k} \right),
\]

\[
\left\| \frac{d^k w^-_l(x, t)}{dx^k} \right\|_{\Omega^-} \leq C \left( \frac{\mu}{\epsilon} \right)^k, \quad \left\| \frac{d^k w^+_r(x, t)}{dx^k} \right\|_{\Omega^+} \leq C \mu^{-k}.
\]

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\[ \left\| \frac{d^k w^-_j(x,t)}{dx^k} \right\|_{\Omega^-} \leq C \mu^{-k}, \quad \left\| \frac{d^k w^+_j(x,t)}{dx^k} \right\|_{\Omega^+} \leq C \left( \frac{\mu}{\epsilon} \right)^k. \]

**Proof.** Using the techniques given in [23, 24], the proof follows. \qed

## 3 Difference scheme

We use the horizontal MOL to discretize the time variable using the Crank-Nicolson method [7], with constant step-size \( \Delta t \), while keeping the variable \( x \) continuous. For a fixed time \( T \), the interval \([0,T]\) is partitioned uniformly as \( \Lambda^M = \{ t_j = j\Delta t : j = 0, 1, \ldots, M, \Delta t = \frac{T}{M} \} \). The semi-discretization yields the following system of linear ordinary differential equations:

\[
\begin{align*}
\epsilon U^{j+\frac{1}{2}}_x(x) + \mu a^{j+\frac{1}{2}}(x) U^{j+\frac{1}{2}}_x(x) - b^{j+\frac{1}{2}}(x) U^{j+\frac{1}{2}}(x) &= f^{j+\frac{1}{2}}(x) + \frac{U^{j+1}(x) - U^j(x)}{\Delta t}, \\
x \in (\Omega^- \cup \Omega^+), &\quad 0 \leq j \leq M - 1, \\
U^{j+1}(0) = u(0, t_{j+1}), &\quad U^{j+1}(1) = u(1, t_{j+1}), \quad 0 \leq j \leq M - 1, \\
U^0(x) = u(x,0), &\quad x \in \Gamma,
\end{align*}
\]

where \( U^{j+1}(x) \) is the approximation of \( u(x,t_{j+1}) \) of Eq. (1.1) at \((j+1)\)-th time level and \( U^{j+\frac{1}{2}} = \frac{U^{j+1}(x) + U^j(x)}{2} \). After simplification, we obtain

\[
\begin{align*}
\hat{\mathcal{L}} U^{j+1}(x) = g(x, t_{j+1}), &\quad x \in (\Gamma^- \cup \Gamma^+), \quad 0 \leq j \leq M - 1, \\
U^{j+1}(0) = u(0, t_{j+1}), U^{j+1}(1) = u(1, t_{j+1}), &\quad 0 \leq j \leq M - 1, \\
U^0(x) = u(x,0), &\quad x \in \Gamma,
\end{align*}
\]

where the operator \( \hat{\mathcal{L}} \) is defined as

\[
\hat{\mathcal{L}} \equiv \epsilon \frac{\delta^2}{\delta x^2} + \mu a^{j+\frac{1}{2}} \frac{\delta}{\delta x} - c^{j+\frac{1}{2}} I,
\]

and

\[
\begin{align*}
g(x,t_{j+1}) &= 2f^{j+\frac{1}{2}}(x) - \epsilon U^{j+\frac{1}{2}}_x(x) - \mu a^{j+\frac{1}{2}}(x) U^{j+\frac{1}{2}}(x) + d^{j+\frac{1}{2}}(x) U^j(x), \\
c^{j+\frac{1}{2}}(x) &= b^{j+\frac{1}{2}}(x) + \frac{2}{\Delta t}, \\
d^{j+\frac{1}{2}}(x) &= b^{j+\frac{1}{2}}(x) - \frac{2}{\Delta t}.
\end{align*}
\]

The error in temporal semi-discretization is defined by \( \epsilon^{j+1} = u(x,t_{j+1}) - \hat{U}^{j+1}(x) \) where \( u(x,t_{j+1}) \) is the solution of Eq. (1.1), \( \hat{U}^{j+1}(x) \) is the solution of semi-discrete equation (3.7), when \( u(x,t) \) is taken instead of \( U^j \) to find solution at \((x,t_{j+1})\).

**Theorem 3.1.** The local truncation error \( T_{j+1} = \hat{\mathcal{L}}(\epsilon^{j+1}) \) satisfies

\[
\|T_{j+1}\| \leq C(\Delta t)^3, \quad 0 \leq j \leq M - 1.
\]

**Proof.** The proof follows from the technique given in [14]. \qed

**Theorem 3.2.** The global error \( E^{j+1} = u(x,t_{j+1}) - U^{j+1}(x) \) is estimated as

\[
\|E^{j+1}\| \leq C(\Delta t)^2, \quad 0 \leq j \leq M - 1.
\]

Here \( U^{j+1}(x) \) is the solution of Eq. (3.7).
Proof. The proof follows from the technique given in \cite{14}.

We will now define the fully discretized scheme. In spatial direction, we use a non-uniform mesh that is graded in the layer region and uniform in the outer region. The semi-discrete problem in (3.7) is discretized using the upwind finite difference method on an appropriately defined Shishkin-Bakhvalov mesh on space. Let the interior points of the spatial mesh are denoted by \( \Gamma^N = \{ x_i : 1 \leq i \leq N \} \cup \{ x_i : \frac{N}{2} + 1 \leq i \leq N - 1 \} \). The \( \Gamma^N = \{ x_i \}_{0}^{N} \cup \{ d \} \) denote the mesh points with \( x_0 = 0 \), \( x_N = 1 \) and the point of discontinuity at point \( x_N = d \). We also introduce the notation \( \Gamma^{N-} = \{ x_i \}_{0}^{N-1} \), \( \Gamma^{N+} = \{ x_i \}_{\frac{N}{2}+1}^{N-1} \), \( \Omega^{N-} = \Gamma^{N-} \times \Lambda^M \), \( \Omega^{N+} = \Gamma^{N+} \times \Lambda^M \) and \( \tilde{\Omega}^{N,M} = \tilde{\Gamma}^{N} \times \Lambda^M \). The domain \([0,1]\) is subdivided into six sub-intervals as

\[
\tilde{\Gamma} = [0, \tau_1] \cup [\tau_1, d - \tau_2] \cup [d - \tau_2, d] \cup [d, d + \tau_3] \cup [d + \tau_3, 1 - \tau_4] \cup [1 - \tau_4, 1].
\]

The transition points are defined as done for Shishkin mesh:

\[
\tau_1 = \frac{4}{\theta_1} \ln N, \quad \tau_2 = \frac{4}{\theta_2} \ln N, \\
\tau_3 = \frac{4}{\theta_2} \ln N, \quad \tau_4 = \frac{4}{\theta_1} \ln N.
\]

where \( \theta_1 \) and \( \theta_2 \) are the same as in (4.16).

The interval \([0, \tau_1]\) is subdivided into \( \frac{N}{8} \) sub-intervals by inverting the function \( e^{-\theta_1 x} \) linearly in it, so for \( i = 0, \ldots, \frac{N}{8} \),

\[
e^{-\theta_1 x_i/8} = Ai + B
\]

such that \( x_0 = 0 \), \( x_{\frac{N}{8}} = \tau_1 \). Thus, we obtain

\[
x_i = -\frac{8}{\theta_1} \log \left( 1 + \frac{8i}{N} \left( \frac{1}{\sqrt{N}} - 1 \right) \right), \quad 0 \leq i \leq \frac{N}{8}.
\]

In interval \([d - \tau_2, d]\), we invert the function \( e^{-\theta_2 (x-d)} \) linearly to obtain \( \frac{N}{8} + 1 \) mesh points. Thus, we obtain

\[
x_i = d + \frac{8}{\theta_2} \log \left( \frac{8i}{N} \left( \frac{1}{\sqrt{N}} - 1 \right) + \frac{4}{\sqrt{N}} - 3 \right), \quad \frac{3N}{8} \leq i \leq \frac{N}{2}.
\]

where \( x_{\frac{3N}{8}} = d - \tau_2 \) and \( x_{\frac{N}{2}} = d \). Similarly by inverting the function \( e^{-\theta_2 (d-x)} \) linearly in the interval \([d, d + \tau_3]\), we obtain the following \( \frac{N}{8} + 1 \) mesh points for \( \frac{N}{2} \leq i \leq \frac{5N}{8} \):

\[
x_i = d - \frac{8}{\theta_2} \log \left( \frac{8i}{N} \left( \frac{1}{\sqrt{N}} - 1 \right) + 5 - \frac{4}{\sqrt{N}} \right), \quad \frac{N}{2} \leq i \leq \frac{5N}{8}
\]

where \( x_{\frac{N}{2}} = d, x_{\frac{5N}{8}} = d + \tau_3 \). Also, the interval \([1 - \tau_4, 1]\) can be subdivided into \( \frac{N}{8} \) sub-intervals by inverting the function \( e^{-\theta_1 (1-x)} \) linearly in it, so, we have

\[
x_i = 1 + \frac{8}{\theta_1} \log \left( \frac{8i}{N} \left( \frac{1}{\sqrt{N}} - 1 \right) + \frac{8}{\sqrt{N}} - 7 \right), \quad \frac{7N}{8} \leq i \leq N.
\]

A uniform mesh consists of \( \left( \frac{N}{2} + 1 \right) \) mesh points is employed between intervals \([\tau_1, d - \tau_2]\) and \([d + \tau_3, 1-\tau_4]\).
The mesh points in space are given by

\[
x_i = \begin{cases} 
- \frac{8}{\theta_1} \log \left( 1 + \frac{8i}{N} \left( \frac{1}{\sqrt{N}} - 1 \right) \right), & 0 \leq i \leq \frac{N}{8}, \\
\tau_1 + \frac{(d - \tau_1 - \tau_2)(\xi_i - \frac{1}{8})}{4}, & \frac{N}{8} \leq i \leq \frac{3N}{8}, \\
d - \frac{8}{\theta_2} \phi_2(\xi_i), & \frac{3N}{8} \leq i \leq \frac{N}{2}, \\
d + \frac{8}{\theta_2} \phi_3(\xi_i), & \frac{N}{2} \leq i \leq \frac{5N}{8}, \\
d + \tau_3 + \frac{(1 - d - \tau_3 - \tau_4)(\xi_i - \frac{5}{8})}{4}, & \frac{5N}{8} \leq i \leq \frac{7N}{8}, \\
1 - \frac{8}{\theta_1} \phi_4(\xi_i), & \frac{7N}{8} \leq i \leq N,
\end{cases}
\]

with \( \xi_i = \frac{i}{N} \). The functions \( \phi_1 \) and \( \phi_3 \) are monotonically increasing on \([0, \frac{1}{8}]\) and \([\frac{1}{8}, \frac{5}{8}]\) respectively. The functions \( \phi_2 \) and \( \phi_4 \) are monotonically decreasing on \([\frac{3}{8}, \frac{1}{2}]\) and \([\frac{7}{8}, 1]\) respectively. The assumptions and lemma for resolving layers properly are as follows:

**Assumption 3.1.** Here we assume for \( \sqrt{\alpha \mu} \leq \sqrt{\rho \epsilon} \), \( \sqrt{\epsilon} < N^{-1} \) to resolve the layers properly.

**Lemma 3.1.** Here, the mesh-generating functions \( \phi_1, \phi_2, \phi_3 \) and \( \phi_4 \) are piecewise differentiable and satisfy the following conditions:

\[
\max_{\xi \in [0, \frac{1}{8}]} |\phi_1'(\xi)| \leq CN, \quad \max_{\xi \in [\frac{3}{8}, \frac{1}{2}]} |\phi_2'(\xi)| \leq CN,
\]

\[
\max_{\xi \in [\frac{1}{8}, \frac{5}{8}]} |\phi_3'(\xi)| \leq CN, \quad \max_{\xi \in [\frac{7}{8}, 1]} |\phi_4'(\xi)| \leq CN,
\]

and

\[
\int_0^{\frac{1}{8}} \{\phi_1'(\xi)\}^2 d\xi \leq CN, \quad \int_\frac{3}{8}^{\frac{1}{2}} \{\phi_2'(\xi)\}^2 d\xi \leq CN,
\]

\[
\int_\frac{5}{8}^1 \{\phi_3'(\xi)\}^2 d\xi \leq CN, \quad \int_{\frac{7}{8}}^1 \{\phi_4'(\xi)\}^2 d\xi \leq CN.
\]
\[ \int_{\frac{1}{2}}^{\frac{3}{2}} \{\phi_3'(\xi)\}^2 d\xi \leq CN, \quad \int_{\frac{1}{2}}^{1} \{\phi_4'(\xi)\}^2 d\xi \leq CN. \]

**Proof.** The mesh-generating functions \( \phi_1(\xi) = -\log \left[ 1 - 8\xi \left( \frac{1}{\sqrt{N}} - 1 \right) \right], \quad \xi \in [0, \frac{2}{8}]. \)

Therefore,
\[ |\phi_1'(\xi)| \leq \frac{8\sqrt{N}}{\sqrt{N} + (1 - \sqrt{N})} \leq 8\sqrt{N} \leq CN. \]

Similarly, we can prove the bounds for \( \phi_2', \phi_3' \) and \( \phi_4' \) in the intervals \([\frac{3}{8}, \frac{1}{2}], [\frac{1}{2}, \frac{7}{8}] \) and \([\frac{7}{8}, 1] \) respectively.

Also,
\[ \int_{0}^{\frac{3}{8}} \{\phi_1'(\xi)\}^2 d\xi \leq \int_{0}^{\frac{3}{8}} (8\sqrt{N})^2 d\xi \leq CN. \]

For other bounds, we can follow a similar procedure.

Using the Lemma (3.1) and Assumption (3.1) we see that for \( 1 \leq i \leq \frac{N}{8} \),
\[ h_i = x_i - x_{i-1} = \frac{8}{\theta_1} \{\phi_1(\xi_i) - \phi_1(\xi_{i-1})\} \leq \frac{8}{\theta_1} (\xi_i - \xi_{i-1}) \max_{\xi \in [0, \frac{1}{8}]} |\phi_1'(\xi)| \leq \frac{C}{\theta_1} \leq CN^{-1}. \]

Similarly, in the interval \( \frac{3N}{2} \leq i \leq \frac{N}{2}, \frac{N}{2} \leq i \leq \frac{5N}{8} \) and \( \frac{7N}{8} \leq i \leq N \) we can bound \( h_i \) by using different \( \phi_i' s, i = 2, 3, 4 \) to obtain that
\[ h_i \leq CN^{-1}, \quad \forall \ x_i \in [d - \tau_2, d] \cup [d, d + \tau_3] \cup [1 - \tau_4, 1]. \]

The fully discretized scheme is given by: Find \( U^{j+1}(x_i) = U(x_i, t_{j+1}) \) such that
\[
\begin{cases}
    L^N U^{j+1}(x_i) = \bar{g}(x_i, t_{j+1}), & x_i \in \Gamma^N_\text{in} \cup \Gamma^N_\text{out}, 0 \leq j \leq M - 1, \\
    U^{j+1}(0) = u(0, t_{j+1}), U^{j+1}(1) = u(1, t_{j+1}), & 0 \leq j \leq M - 1, \\
    U^0(x_i) = u(x_i, 0), & i = 0, \ldots, N,
\end{cases}
\]

(3.8)

where the operator \( L^N \) is defined as
\[ L^N \equiv (\epsilon \delta_x^2 + \mu a^{j+\frac{1}{2}} D_x^s - \epsilon^{j+\frac{1}{2}} I) \]

and \( \bar{g}(x_i, t_{j+1}) = 2 f^{j+\frac{1}{2}}(x_i) - \epsilon \delta_x^2 U^j(x_i) - \mu a^{j+\frac{1}{2}} D_x^s U^j(x_i) + d^{j+\frac{1}{2}}(x) U^j(x_i). \)

At the point of discontinuity, we have used a three-point formula
\[ D_x^+ U^{j+1}(x_{\frac{N}{2}}) = D_x^- U^{j+1}(x_{\frac{N}{2}}), \quad \forall \ 0 \leq j \leq M - 1, \]

where
\[ D_x^+ U^{j+1}(x_i) = \frac{U^{j+1}(x_{i+1}) - U^{j+1}(x_i)}{x_{i+1} - x_i}, \quad D_x^- U^{j+1}(x_i) = \frac{U^{j+1}(x_i) - U^{j+1}(x_{i-1})}{x_i - x_{i-1}}. \]

The matrix associated with the above discrete scheme (3.8) is monotone and irreducibly diagonally dominant. It is an M-matrix and hence invertible.
4 Convergence and Stability Analysis

**Theorem 4.1.** Suppose that a mesh function $Y(x_i,t_j)$ satisfies $Y(x_0,t_j) \geq 0, Y(x_N,t_j) \geq 0$ and $D_x^+ Y(x_{\frac{N}{2}},t_j) - D_x^- Y(x_{\frac{N}{2}},t_j) \leq 0$ for all $j = 0, \ldots, M$. If $L^N Y(x_i,t_j) \leq 0$ for all $(x_i,t_j) \in \Omega^N$ then $Y(x_i,t_j) \geq 0, \forall (x_i,t_j) \in \Omega^{N,M}$.

**Proof.** See [15] for proof. $\square$

To find the error estimates for the scheme (3.8) defined above, we first decompose the discrete solution $U^{j+1}(x_i)$ into the regular and singular components. Let

$$U^{j+1}(x_i) = V^{j+1}(x_i) + W^{j+1}(x_i)$$

The regular components is

$$V^{j+1}(x_i) = \begin{cases} V^{-(j+1)}(x_i), & (x_i,t_j) \in \Omega^N, \\ V^{+(j+1)}(x_i), & (x_i,t_j) \in \Omega^N \end{cases}$$

(4.9)

where $V^{-(j+1)}(x_i)$ and $V^{+(j+1)}(x_i)$ approximate $v^-(x_i,t_{j+1})$ and $v^+(x_i,t_{j+1})$ respectively. They satisfy the following equations:

$$L^N V^{-(j+1)}(x_i) = g(x_i,t_{j+1}), \forall (x_i,t_{j+1}) \in \Omega^N, \quad V^{-(j+1)}(x_0) = v^-(0,t_{j+1}), \quad V^{-(j+1)}(x_N) = v^-(d-,t_{j+1}), \quad \forall j = 0,1,\ldots,M-1$$

(4.10)

$$L^N V^{+(j+1)}(x_i) = g(x_i,t_{j+1}), \forall (x_i,t_{j+1}) \in \Omega^N, \quad V^{+(j+1)}(x_0) = v^+(d+,t_{j+1}), \quad V^{+(j+1)}(x_N) = v^+(1,t_{j+1}), \quad \forall j = 0,1,\ldots,M-1$$

(4.11)

The singular components $W^{j+1}_l(x_i)$ and $W^{j+1}_r(x_i)$ are also decomposed as:

$$W^{j+1}_l(x_i) = \begin{cases} W^{-(j+1)}_l(x_i), & (x_i,t_j) \in \Omega^N, \\ W^{+(j+1)}_l(x_i), & (x_i,t_j) \in \Omega^N \end{cases}$$

(4.12)

where $W^{-(j+1)}_l(x_i)$, $W^{+(j+1)}_l(x_i)$ approximates the left layer components $w^-(x_i,t_{j+1})$ and $w^+(x_i,t_{j+1})$ respectively. These components satisfies the following equations

$$L^N W^{-(j+1)}_l(x_i) = 0, \forall (x_i,t_{j+1}) \in \Omega^N, \quad W^{-(j+1)}_l(x_0) = w^-(0,t_{j+1}), W^{-(j+1)}_l(x_N) = w^-(d,t_{j+1}), \quad \forall j = 0,1,\ldots,M-1$$

(4.13)

$$L^N W^{+(j+1)}_l(x_i) = 0, \forall (x_i,t_{j+1}) \in \Omega^N, \quad W^{+(j+1)}_l(x_N) = w^+(d,t_{j+1}), \quad \forall j = 0,1,\ldots,M-1$$

(4.14)
\[ L^N W_r^{-(j+1)}(x_i) = 0, \quad \forall (x_i, t_j+1) \in \Omega^N, \]
\[ W_r^{+(j+1)}(x_{\frac{N}{2}}) = 0, \quad W_r^{+(j+1)}(x_N) = w_r(1, t_j+1), \quad \forall j = 0, 1, \ldots, M - 1. \]  

Hence, the discrete solution \( U^{j+1}(x_i) \) is defined as
\[
U^{j+1}(x_i) = \begin{cases} 
(V^{-j+1} + W_l^{-j+1} + W_r^{-j+1})(x_i), & (x_i, t_j+1) \in \Omega^{N-}, \\
(V^{-j+1} + W_l^{-j+1} + W_r^{-j+1})(x_{\frac{N}{2}}), & (x_i, t_j+1) = (d, t_j+1), \\
(V^{-j+1} + W_l^{+(j+1)} + W_r^{+(j+1)})(x_i), & (x_i, t_j+1) \in \Omega^{N+}. 
\end{cases}
\]

**Theorem 4.2.** Let \( \sqrt{\alpha \mu} \leq \sqrt{7} \epsilon \), the singular component \( W_l^{-(j+1)}(x_i), W_l^{+(j+1)}(x_i), W_r^{-(j+1)}(x_i) \) and \( W_r^{+(j+1)}(x_i) \) satisfy the following bounds
\[
|W_l^{-(j+1)}(x_i)| \leq C \gamma_l^{-(j+1)}, \quad \gamma_l^{-(j+1)} = \prod_{n=1}^{i} (1 + \theta_1 h_n)^{-1}, \quad \gamma_l^{-(j+1)} = C_1, \quad i = 0, 1, \ldots, \frac{N}{2},
\]
\[
|W_l^{+(j+1)}(x_i)| \leq C \gamma_l^{+(j+1)}, \quad \gamma_l^{+(j+1)} = \prod_{n=\frac{N}{2}+1}^{i} (1 + \theta_2 h_n)^{-1}, \quad \gamma_l^{+(j+1)} = C_1, \quad i = \frac{N}{2} + 1, \ldots, N,
\]
\[
|W_r^{-(j+1)}(x_i)| \leq C \gamma_r^{-(j+1)}, \quad \gamma_r^{-(j+1)} = \prod_{n=i+1}^{\frac{N}{2}} (1 + \theta_2 h_n)^{-1}, \quad \gamma_r^{-(j+1)} = C_1, \quad i = 0, 1, \ldots, \frac{N}{2},
\]
\[
|W_r^{+(j+1)}(x_i)| \leq C \gamma_r^{+(j+1)}, \quad \gamma_r^{+(j+1)} = \prod_{n=i+1}^{\frac{N}{2}} (1 + \theta_1 h_n)^{-1}, \quad \gamma_r^{+(j+1)} = C_1, \quad i = \frac{N}{2} + 1, \ldots, N,
\]
\[
\theta_1 = \frac{\sqrt{\rho \alpha}}{2\sqrt{\epsilon}}, \quad \theta_2 = \frac{\sqrt{\rho \alpha}}{2\sqrt{\epsilon}}.
\]  

**Proof.** Let us define the barrier function for the left layer term \( W_l^{-(j+1)}(x_i) \) in \( \Omega^{N-} \) as
\[
\psi_l^{-(j+1)}(x_i) = \gamma_l^{-(j+1)} \pm W_l^{-(j+1)}(x_i)
\]
where
\[
\gamma_l^{-(j+1)} = \begin{cases} 
\prod_{n=1}^{i} (1 + \theta_1 h_n)^{-1}, & 1 \leq i \leq \frac{N}{2}, \\
C_1, & i = 0,
\end{cases}
\]
and \( \theta_1 \) is defined in (4.16) of Section 4. For large \( C \), \( \psi_l^{-(j+1)}(x_0) \geq 0 \) and \( \psi_l^{-(j+1)}(x_{\frac{N}{2}}) \geq 0 \), \( \forall j = 0, \ldots, M - 1. \)
Consider,
\[
\mathcal{L}^N_l \psi^{-\gamma_{(j+1)}}(x_i) = \left\{ e^{\theta^2 \gamma_{(j+1)} h_{i+1}} + \mu a^{j+\frac{1}{2}}(x_i) D_x \gamma_{(j+1)} - c^{j+\frac{1}{2}}(x_i) \gamma_{(j+1)} \right\} \\
= e^{\theta^2 \gamma_{(j+1)} h_{i+1}} + \mu a^{j+\frac{1}{2}}(x_i) (-\theta_1 \gamma_{(j+1)}(-1) - c^{j+\frac{1}{2}}(x_i) \gamma_{(j+1)}) \\
= \gamma_{(j+1)} \left[ e^{\theta^2 \gamma_{(j+1)} h_{i+1}} - 2 \theta_1 \gamma_{(j+1)} h_{i+1} - c^{j+\frac{1}{2}}(x_i) \theta_1 h_{i+1} \right] \\
\leq \gamma_{(j+1)} \left( 2e \theta_1 - \mu a^{j+\frac{1}{2}}(x_i) \theta_1 - c^{j+\frac{1}{2}}(x_i) \right) \\
\leq \gamma_{(j+1)} \left( 2e \rho \alpha 4 \epsilon - \mu a^{j+\frac{1}{2}}(x_i) \sqrt{\rho \alpha} 2 \epsilon - c^{j+\frac{1}{2}}(x_i) \right) \\
\leq \gamma_{(j+1)} \left( \rho a^{j+\frac{1}{2}}(x_i) - c^{j+\frac{1}{2}}(x_i) \right) \leq 0
\]

Therefore, by the discrete minimum principle defined in [24], for the continuous case, we prove that
\[
\psi^{-\gamma_{(j+1)}}(x_i) \geq 0, \forall \, x \in \Omega^N.
\]

Similarly, we define the barrier function for right layer term \( W^{-\gamma_{(j+1)}}(x_i) \) in \( \Omega^N \) as
\[
\psi^{-\gamma_{(j+1)}}(x_i) = C \gamma^{-\gamma_{(j+1)}}(x_i) \pm W^{-\gamma_{(j+1)}}(x_i)
\]

where
\[
\gamma^{-\gamma_{(j+1)}} = \left\{ \begin{array}{ll}
\prod_{n=1+1}^{N} (1 + \theta_2 h_n)^{-1}, & 0 \leq i \leq N - \frac{1}{2} - 1, \\
C_1, & i = N - \frac{1}{2},
\end{array} \right.
\]

where \( \theta_2 \) is defined in (4.16) of Section 4. For large \( C \), \( \psi^{-\gamma_{(j+1)}}(x_0) \geq 0 \) and \( \psi^{-\gamma_{(j+1)}}(x_{\frac{N}{2}}) \geq 0 \), \( \forall \, j = 0, \ldots, M - 1 \).

Consider
\[
\mathcal{L}^N_l \psi^{-\gamma_{(j+1)}}(x_i) = C \left\{ e^{\theta^2 \gamma_{(j+1)} h_{i+1}} + \mu a^{j+\frac{1}{2}}(x_i) D_x \gamma_{(j+1)} - c^{j+\frac{1}{2}}(x_i) \gamma_{(j+1)} \right\} \\
= C \left\{ e^{\theta^2 \gamma_{(j+1)} h_{i+1}} + \mu a^{j+\frac{1}{2}}(x_i) \theta_2 \gamma_{(j+1)} - c^{j+\frac{1}{2}}(x_i) \gamma_{(j+1)} \right\} \\
= \gamma_{(j+1)} \left[ e^{\theta^2 \gamma_{(j+1)} h_{i+1}} - 2 \theta_1 \gamma_{(j+1)} h_{i+1} - c^{j+\frac{1}{2}}(x_i) \theta_2 h_{i+1} \right] \\
\leq \gamma_{(j+1)} \left( 2e \theta_1 + \mu a^{j+\frac{1}{2}}(x_i) \theta_2 - c^{j+\frac{1}{2}}(x_i) \theta_2 h_{i+1} \right) \\
\leq \gamma_{(j+1)} \left( 2e \rho \alpha 4 \epsilon + \mu a^{j+\frac{1}{2}}(x_i) \frac{\sqrt{\rho \alpha}}{2 \epsilon} - c^{j+\frac{1}{2}}(x_i) \theta_2 h_{i+1} \right) \\
\leq \gamma_{(j+1)} \left( \rho a^{j+\frac{1}{2}}(x_i) - c^{j+\frac{1}{2}}(x_i) \right) \leq 0
\]

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Therefore, by the discrete minimum principle defined in [24] for the continuous case, we prove that \( \psi_i^{-(j+1)}(x_i) \geq 0, \forall x \in \Omega^{N-} \). Similarly, by defining the corresponding barrier functions \( \psi_i^{+(j+1)}(x_i) \) and \( \psi_i^{+(j+1)}(x_i) \) for \( W_i^{+(j+1)}(x_i) \) and \( W_i^{+(j+1)}(x_i) \) we get the remaining two inequalities for the left and right layer term in \( \Omega^{N+} \).

**Theorem 4.3.** The discrete regular component \( V_j^{j+1}(x_i) \) defined in (4.9) and \( v(x,t) \) is solution of the problem (2.2). So, the error in the regular component satisfies the following estimate for \( \sqrt{\alpha \mu} \leq \sqrt{p} \):

\[
\| V - v \|_{\Omega^{N-} \cup \Omega^{N+}} \leq C(N^{-1} + (\Delta t)^2).
\]

**Proof.** Using the truncation error in domain \( \Omega^{N-} \), we get

\[
|\mathcal{L}^N(V^{-(j+1)} - v^{-(j+1)})(x_i)| = |\mathcal{L}^N(V^{-(j+1)}(x_i) - \mathcal{L}^N v^{-(j+1)}(x_i)|, (x_i, t_{j+1}) \in \Omega^{N-}
\]

\[
\leq \epsilon \left( \delta^2 - \frac{d^2}{dx^2} \right) v^{-(j+1)}(x_i) + \mu |a^i_{(j+\frac{1}{2})} (x_i)\left( D_x^0 \frac{d}{dx} v^{-(j+1)}(x_i) \right)
\]

\[
+ r_{(j+1)}(x_i) - \frac{d}{dt} u^{-(j+\frac{1}{2}) (x_i)}
\]

\[
\leq C_1 \max_{0 \leq t \leq \frac{T}{N}} h_i (\epsilon ||v_{x}||_{\Omega^+} + \mu ||v_{xx}||_{\Omega^+}) + C_2 (\Delta t)^2
\]

\[
\leq C(N^{-1} + (\Delta t)^2).
\]

Define the barrier function

\[
\psi^{j+1}(x_i) = C(N^{-1} + (\Delta t)^2) \pm (V^{-(j+1)} - v^{-(j+1)})(x_i), (x_i, t_{j+1}) \in \Omega^{N-}.
\]

It is clear that \( \psi^{j+1}(x_0) \geq 0 \) and \( \psi^{j+1}(x_N) \geq 0 \). For large \( C \), we obtain \( \mathcal{L}^N \psi^{j+1}(x_i) \leq 0 \). Applying discrete minimum principle [24], we get \( \psi^{j+1}(x_i) \geq 0 \).

\[
\| V^+ - v^+ \|_{\Omega^{N+}} \leq C(N^{-1} + (\Delta t)^2), (x_i, t_{j+1}) \in \Omega^{N+}.
\]

The error estimate in the domain \( \Omega^{N+} \) is derived similarly.

Combining the above two equations [4.17] and [4.18], we obtain the desired result. \( \square \)

**Lemma 4.1.** Let \( \sqrt{\alpha \mu} \leq \sqrt{p} \) and \( W_l^{-(j+1)}(x_i), w_i^{-(j+1)}(x,t) \) are solution of the problem (4.12) and (2.3) respectively. The left singular component of the truncation error satisfies the following estimate in \( \Omega^{N-} \):

\[
\| W_l^+ - w_l^+ \|_{\Omega^{N+}} \leq C(N^{-1} + (\Delta t)^2).
\]

**Proof.** We first calculate the truncation error in the outer region \( \Omega_l \): In \( \Omega_l \), the left layer component satisfies the following bound given in theorem (2.3):

\[
|w_l^{-(j+1)}(x_i)| \leq C \exp^{-\theta_1 x_i} \leq C \exp^{-\theta_1 r_1} \leq C \alpha^{-1}, 0 \leq j \leq M - 1.
\]

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Also $W_t^{-(j+1)}(x_i)$ is decreasing function in $[\tau_1, d) \times (0, T]$, so

$$|W_t^{-(j+1)}(x_i)| \leq |W_t^{-(j+1)}(x_{\frac{N}{8}})|$$

$$= \prod_{n=1}^{\frac{N}{8}} (1 + \theta_1 h_n)^{-1}$$

$$\log |W_t^{-(j+1)}(x_i)| \leq \log \left( \prod_{n=1}^{\frac{N}{8}} (1 + \theta_1 h_n)^{-1} \right)$$

$$= - \sum_{n=1}^{\frac{N}{8}} \log(1 + \theta_1 h_n)$$

As $\log(1 + t^2) \geq t - \frac{t^2}{2}$ for $t \geq 0$.

So, $\sum_{n=1}^{\frac{N}{8}} \log(1 + \theta_1 h_n) \geq \sum_{n=1}^{\frac{N}{8}} \theta_1 h_n - \sum_{n=1}^{\frac{N}{8}} \left( \frac{\theta_1 h_n}{2} \right)^2$

$$\geq 4 \ln N - \sum_{n=1}^{\frac{N}{8}} \left( \frac{\theta_1 h_n}{2} \right)^2.$$

To calculate $\sum_{n=1}^{\frac{N}{8}} \left( \frac{\theta_1 h_n}{2} \right)^2$, let $1 \leq n \leq \frac{N}{8}$,

$$h_n = x_n - x_{n-1} = \frac{8}{\theta_1} (\phi_1(\xi_n) - \phi_1(\xi_{n-1}))$$

$$= \frac{8}{\theta_1} \int_{\xi_{n-1}}^{\xi_n} \phi_1'(\xi) d\xi$$

$$\frac{\theta_1 h_n}{8} = \int_{\xi_{n-1}}^{\xi_n} \phi_1'(\xi) d\xi$$

$$\Rightarrow \left( \frac{\theta_1 h_n}{8} \right)^2 \leq (\xi_n - \xi_{n-1}) \int_{\xi_{n-1}}^{\xi_n} \phi_1'(\xi)^2 d\xi \quad \text{(Using Holder’s inequality)}$$

$$\sum_{n=1}^{\frac{N}{8}} \left( \frac{\theta_1 h_n}{8} \right)^2 \leq N^{-1} \int_0^{\frac{1}{8}} \phi_1'(\xi)^2 d\xi$$

$$\leq C \quad \text{(from Lemma 3.1)}.$$
We use truncation error analysis to find error in the inner region $(0, \tau_1) \times (0, T]$. Using the derivative bounds for the left layer component $w_-$ from the theorem (2.4), we obtain

\[
|\mathcal{L}^N(W_i^{-(j+1)} - w_i^{-(j+1)})(x_i)| \leq e\left|\left(\delta_x^2 - \frac{d}{dx^2}\right)w_i^{-(j+1)}(x_i)\right| + \mu|a^{j+\frac{1}{2}}(x_i)|\left(D_x - \frac{d}{dx}\right)w_i^{-(j+1)}(x_i)
\]

\[
+ \left|\frac{D_x w_i^{-(j+1)}(x_i)}{\delta_t} - \frac{\delta_t w_i^{-(j+1)}(x_i)}{\delta_t}\right|
\]

\[
\leq C_1 \max_{0 \leq i \leq N} h_i(\epsilon\|w_{i{xx}}\|_{\Omega^-} + \mu\|w_{i{x}}\|_{\Omega^-}) + C_2(\Delta t)^2
\]

\[
\leq C_1 \left(\frac{1}{\sqrt{\epsilon}}\right) + C_2(\Delta t)^2 \quad \text{(from theorem (2.4))}
\]

\[
\leq C\left(\frac{N-1}{\sqrt{\epsilon}} + (\Delta t)^2\right).
\]

Choosing the barrier function for the layer component as

\[
\psi^{j+1}(x_i) = C_1\left(\frac{x_i N^{-1}}{\sqrt{\epsilon \ln N}} + (\Delta t)^2\right) \pm (W_i^{-(j+1)} - w_i^{-(j+1)})(x_i), \quad (x_i, t_{j+1}) \in (0, \tau_1) \times (0, T].
\]

For sufficiently large $C_1$, we have $\mathcal{L}^N\psi^{j+1}(x_i) \leq 0$, $(x_i, t_{j+1}) \in (0, \tau_1) \times (0, T]$. Also $\psi^{j+1}(x_0) \geq 0$ and $\psi^{j+1}(x, T) \geq 0$. Hence by discrete minimum principle [24], we can obtain the following bounds:

\[
|(W_i^{-(j+1)} - w_i^{-(j+1)})(x_i)| \leq C_1\left(\frac{x_i N^{-1}}{\sqrt{\epsilon \ln N}} + (\Delta t)^2\right)
\]

\[
= \left(\frac{\tau_1 N^{-1}}{\sqrt{\epsilon \ln N}} + (\Delta t)^2\right)
\]

\[
|(W_i^{-(j+1)} - w_i^{-(j+1)})(x_i)| \leq C(N^{-1} + (\Delta t)^2).
\]

Hence, by combining equation (4.21) and (4.22), we have bounds for the left layer component

\[
||W_i^- - w_i^-||_{\Omega_{N-}} \leq C(N^{-1} + (\Delta t)^2).
\]

(4.23)

**Lemma 4.2.** Let $\sqrt{\alpha \mu} \leq \sqrt{\rho \epsilon}$ and $W_i^{+(j+1)}(x_i), w_i^{+}(x, t)$ be the solution of the problem (4.13) and (2.3) respectively. The left singular component of the truncation error satisfies the following estimate in $\Omega_{N+}^-$

\[
||W_i^+ - w_i^+||_{\Omega_{N+}} \leq C N^{-1} + (\Delta t)^2
\]

**Proof.** We first calculate the truncation error in outer region $[d + \tau_3, 1) \times (0, T]$

In $[d + \tau_3, 1) \times (0, T]$ i.e. $\frac{5N}{8} \leq i < N$, the left layer component satisfies the following bound given in theorem (2.3):

\[
|w_i^{+(j+1)}(x_i)| \leq C e^{-\theta_2(x_i-d)} \leq C e^{-\theta_2 \tau_3} \leq C N^{-4}, \quad (x_i, t_{j+1}) \in [d + \tau_3, 1) \times (0, T].
\]

(4.24)

Also $W_i^{+(j+1)}(x_i)$ is a decreasing function in $[d + \tau_3, 1) \times (0, T]$, so

\[
|W_i^{+(j+1)}(x_i)| \leq |W_i^{+(j+1)}(x_{\frac{5N}{8}})| = \prod_{n=\frac{5N}{8}}^{\frac{3N}{8}} (1 + \theta_2 h_n)^{-1}
\]

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Applying the similar arguments as in Lemma (4.1), we get

\[
|W_l^{+(j+1)}(x_i)| \leq CN^{-4}, \quad (x_i, t_{j+1}) \in [d + \tau_3, 1) \times (0, T].
\]  

(4.25)

Hence, by combining equation (4.24) and (4.25) we get

\[
|(W_l^{+(j+1)} - w_l^{+(j+1)})(x_i)| \leq |W_l^{+(j+1)}(x_i)| + |w_l^{+(j+1)}(x_i)| \leq CN^{-4}, \quad (x_i, t_{j+1}) \in [d + \tau_3, 1) \times (0, T].
\]  

(4.26)

We use truncation error analysis to find errors in the inner region \((d, d + \tau_3) \times (0, T)\). Using the derivative bounds for the left layer component \(w_l^+\) from the theorem (2.4), we obtain

\[
|\mathcal{L}^N(W_l^{+(j+1)} - w_l^{+(j+1)})(x_i)| \leq \epsilon \left| \left( \frac{\delta^2 - \frac{d^2}{dx^2}}{\delta^2} \right) w_l^{+(j+1)}(x_i) \right| + \mu|\delta^{j+\frac{1}{2}}(x_i)| \left( \frac{D_x^- - \frac{d}{dx}}{\delta^2} \right) w_l^{+(j+1)}(x_i) \\
+ \left| D_t^- w_l^{+(j+1)}(x_i) - \frac{\delta}{\delta t} w_l^{+(j+\frac{1}{2})}(x_i) \right|
\]

\[
\leq C_1 \max_{\frac{1}{2} \leq i \leq N} h_i(\epsilon\|w_{xx}^+\|\Omega^+ + \mu\|w_{xx}^+\|\Omega^+) + C_2(\Delta t)^2
\]

\[
\leq C_1 \left( \frac{1}{\sqrt{\epsilon}} \right) + C_2(\Delta t)^2 \leq C \left( \frac{N^{-1}}{\sqrt{\epsilon}} + (\Delta t)^2 \right).
\]

Choosing the barrier function for the layer component as

\[
\psi^{j+1}(x_i) = C_1 \left( \frac{(d + \tau_3 - x_i)N^{-1}}{\sqrt{\epsilon} \ln N} \right) + (\Delta t)^2 \pm (W_l^{+(j+1)} - w_l^{+(j+1)})(x_i), \quad (x_i, t_{j+1}) \in (d, d + \tau_3) \times (0, T]
\]

For sufficiently large \(C_1\), we have \(\mathcal{L}^N\psi^{j+1}(x_i) \leq 0\), \((x_i, t_{j+1}) \in (d, d + \tau_3) \times (0, T]\). Also \(\psi^{j+1}(x_{\frac{3N}{8}}) \geq 0\) and \(\psi^{j+1}(x_{\frac{5N}{8}}) \geq 0\). Hence, by the discrete minimum principle [24], we can obtain the following bounds:

\[
|(W_l^{+(j+1)} - w_l^{+(j+1)})(x_i)| \leq C_1 \left( \frac{(d + \tau_3 - x_i)N^{-1}}{\sqrt{\epsilon} \ln N} \right) + (\Delta t)^2
\]

\[
\leq C_1 \left( \frac{\tau_3N^{-1}}{\sqrt{\epsilon} \ln N} + (\Delta t)^2 \right)
\]

\[
|(W_l^{+(j+1)} - w_l^{+(j+1)})(x_i)| \leq C(N^{-1} + (\Delta t)^2), \quad (x_i, t_{j+1}) \in (d, d + \tau_3) \times (0, T].
\]  

(4.27)

Hence, by combining the equation (4.26) and (4.27), we have bounds for the left layer component

\[
\|W_l^+ - w_l^+\|_{\Omega^+} \leq C(N^{-1} + (\Delta t)^2).
\]  

(4.28)

\[
\square
\]

**Lemma 4.3.** Let \(\sqrt{\alpha\mu} \leq \sqrt{\epsilon}\) and \(W_r^{-(j+1)}(x_i), w_r^-(x, t)\) are solution of the problem (4.14) and (2.4) respectively. The right singular component of the truncation error satisfies the following estimate in \(\Omega^-\)

\[
\|W_r^- - w_r^-\|_{\Omega^-} \leq C(N^{-1} + (\Delta t)^2)
\]

**Proof.** We first calculate the truncation error in the outer region \((0, d - \tau_2) \times (0, T]\): For \(1 \leq i \leq \frac{3N}{8}\), the left layer component satisfies the following bound given in theorem (2.3):

\[
|w_r^{-(j+1)}(x_i)| \leq Ce^{-\theta_2(d-x_i)} \leq Ce^{-\theta_2\tau_2} \leq CN^{-4}, \quad (x_i, t_{j+1}) \in (0, d - \tau_2) \times (0, T].
\]  

(4.29)
Also $W_r^{-j+1}(x_i)$ is a increasing function in $(0, d - \tau_2] \times (0, T]$,

\[
|W_r^{-j+1}(x_i)| \leq |W_r^{-j+1}(x_{3N})| = \prod_{n=3N+1}^{N} (1 + \theta_2 h_n)^{-1}.
\]

Applying the similar arguments as in Lemma (4.1), we get

\[
|W_r^{-j+1}(x_i)| \leq CN^{-4}, (x_i, t_{j+1}) \in (0, d - \tau_2] \times (0, T].
\]

Hence, by combining the equation (4.29) and (4.30), we get

\[
|(W_r^{-j+1} - w_r^{-j+1})(x_i)| \leq |W_r^{-j+1}(x_i)| + |w_r^{-j+1}(x_i)| \leq CN^{-4}, (x_i, t_{j+1}) \in (0, d - \tau_2] \times (0, T].
\]

We use truncation error analysis to find an error in the inner region $(d - \tau_2, d) \times (0, T]$. Using the derivative bounds for the left layer component $w_r^{-}$ from theorem (2.4), we obtain

\[
|\mathcal{L}^N(W_r^{-j+1} - w_r^{-j+1})(x_i)| \leq \epsilon \left(\frac{\delta^2_2 - \frac{\partial^2}{\partial x^2}}{\partial x^2} w_r^{-j+1}(x_i)\right) + \mu|a^{j+\frac{1}{2}}(x_i)| \left(\frac{D_x - \frac{d}{dx}}{D_x} w_r^{-j+1}(x_i)\right)
\]
\[
+ \left|D_t w_r^{-j+1}(x_i) - \frac{\delta t}{\delta t} w_r^{-j+1}(x_i)\right|
\]
\[
\leq C_1 \max_{0 \leq i \leq N} h_i(\epsilon) \|w_{xx}^{-}\|_{\Omega} + \mu \|w_{x}^{-}\|_{\Omega} + C_2(\Delta t)^2
\]
\[
\leq C_1 \frac{1}{\theta_2} + C_2(\Delta t)^2 \leq C\left(\frac{N^{-1}}{\sqrt{\epsilon}} + (\Delta t)^2\right).
\]

Choosing the barrier function for the layer component in $(x_i, t_{j+1}) \in (d - \tau_2, d) \times (0, T]$ as

\[
\psi^{j+1}(x_i) = C_1 \left(\frac{(x_i - (d - \tau_2))N^{-1}}{\sqrt{\epsilon} \ln N} + (\Delta t)^2\right) \pm (W_r^{-j+1} - w_r^{-j+1})(x_i)
\]

We can choose sufficiently large $C_1$ so that $\mathcal{L}^N \psi^{j+1}(x_i) \leq 0, \forall (x_i, t_{j+1}) \in (d - \tau_2, d) \times (0, T]$. Also \(\psi^{j+1}(x_{3N}) \geq 0\) and \(\psi^{j+1}(x_{N}) \geq 0\). Hence, by the discrete minimum principle [24], we can obtain the following bounds:

\[
|(W_r^{-j+1} - w_r^{-j+1})(x_i)| \leq C_1 \left(\frac{(x_i - (d - \tau_2))N^{-1}}{\sqrt{\epsilon} \ln N} + (\Delta t)^2\right)
\]
\[
\leq C_1 \left(\frac{\tau_2 N^{-1}}{\sqrt{\epsilon} \ln N} + (\Delta t)^2\right)
\]

\[
|(W_r^{-j+1} - w_r^{-j+1})(x_i)| \leq C(N^{-1} + (\Delta t)^2), (x_i, t_{j+1}) \in (d - \tau_2, d) \times (0, T].
\]

Hence, by combining the equation (4.31) and (4.32), we have bounds for the right layer component

\[
\|W_r^+ - w_r^+\|_{\Omega^+} \leq C(N^{-1} + (\Delta t)^2).
\]

\[
\Box
\]

**Lemma 4.4.** Let $\sqrt{\alpha \mu} \leq \sqrt{\rho \epsilon}$ and $W_r^{+(j+1)}(x_i), w_r^+(x, t)$ are solution of the problem (4.15) and (2.4) respectively. The right singular component of the truncation error satisfies the following estimate in $\Omega^N$:

\[
\|W_r^+ - w_r^+\|_{\Omega^N} \leq C(N^{-1} + (\Delta t)^2)
\]
\textbf{Proof.} We first calculate the truncation error in the outer region \((d, 1 - \tau_1] \times (0, T] \): For \(\frac{N}{2} < i \leq \frac{7N}{8}\), the left layer component satisfies the following bound given in the theorem \([2, 3]\):

\[|w_r^{(j+1)}(x_i)| \leq Ce^{-\theta_1(1-x_i)} \leq Ce^{-\theta_1 \tau_1} \leq CN^{-4}, \quad (x_i, t_{j+1}) \in (d, 1 - \tau_1] \times (0, T].\quad (4.34)\]

Also \(W_r^{(j+1)}(x_i)\) is a increasing function in \((d, 1 - \tau_2] \times (0, T]\), so

\[|W_r^{(j+1)}(x_i)| \leq |W_r^{(j+1)}(x_{\frac{7N}{8}})| = \prod_{n = \frac{7N}{8} + 1}^{N} (1 + \theta_1 h_n)^{-1}. \]

Applying the similar arguments as in Lemma \([4, 1]\), we get

\[|W_r^{(j+1)}(x_i)| \leq CN^{-4}, \quad \forall (x_i, t_{j+1}) \in (d, 1 - \tau_1] \times (0, T].\quad (4.35)\]

Hence for all \((x_i, t_{j+1}) \in (1, d - \tau_2] \times (0, T]\) from equation \((4.34)\) and \((4.35)\), we get

\[|\psi_{i}^{(j+1)}(x_i)| \leq |W_r^{(j+1)}(x_i)| + |w_r^{(j+1)}(x_i)| \leq CN^{-4}. \quad (4.36)\]

We use truncation error analysis to find error in the inner region \((1 - \tau_4, 1) \times (0, T]\). Using the derivative bounds for the left layer component \(w_r^+\) from theorem \([2, 4]\), we obtain \[|\mathcal{L}^N(W_r^{(j+1)} - w_r^{(j+1)})(x_i)| \leq \epsilon \left|d_x \frac{d^2}{dx^2} w_r^{(j+1)}(x_i)\right| + \mu |a^{j+\frac{1}{2}}(x_i)| \left(D_x^+ - \frac{d}{dx} w_r^{(j+1)}(x_i)\right)
\]

\[+ \left|D_i^- w_r^{(j+1)}(x_i) - \frac{\delta}{\delta t} w_r^{(j+\frac{1}{2})}(x_i)\right| \leq C_1 \max_{\frac{N}{2} \leq i \leq N} h_i(\epsilon \|w_{x_{(}}\|_{\Omega^+} + \mu \|w_{x_{(}}\|_{\Omega^+}) + C_2(\Delta t)^2) \leq C_1 \left(\frac{1}{\sqrt{\epsilon}}\right) + C_2(\Delta t)^2 \leq C\left(\frac{N-1}{\sqrt{\epsilon}} + (\Delta t)^2\right).\]

Choosing the barrier function in the domain \((1 - \tau_4, 1) \times (0, T]\) for the layer component as \[\psi_{i}^{(j+1)}(x_i) = C_1 \left(\frac{1 - x_i}{\sqrt{\epsilon} \ln N} + (\Delta t)^2\right) \pm (W_r^{(j+1)} - w_r^{(j+1)})(x_i),\]

For sufficiently large \(C_1\) \(\mathcal{L}^N \psi_{i}^{(j+1)}(x_i) \leq 0, \forall (x_{i,j}^+), \in (1 - \tau_4, 1) \times (0, T]\). Also \(\psi_{i}^{(j+1)}(x_{\frac{7N}{8}}) \geq 0\) and \(\psi_{i}^{(j+1)}(x_N) \geq 0\). Hence, by the discrete minimum principle \([24]\), we can obtain the following bounds:

\[|\psi_{i}^{(j+1)}| \leq C_1 \left(\frac{1 - x_i}{\sqrt{\epsilon} \ln N} + (\Delta t)^2\right) \leq C_1 \left(\frac{\tau_4 N^{-1}}{\sqrt{\epsilon} \ln N} + (\Delta t)^2\right) \leq C\left(\frac{N^{-1}}{\sqrt{\epsilon}} + (\Delta t)^2\right). \quad (4.37)\]

Hence, by combining the equation \((4.36)\) and \((4.37)\), we have bounds for the left layer component

\[\|W_r^+ - w_r^+\|_{\Omega^+} \leq C(\Delta t)^2. \quad (4.38)\]
Theorem 4.4. The error \( e^{j+1}(x_{\frac{N}{2}}) \) estimated at the point of discontinuity \( (x_{\frac{N}{2}}, t_{j+1}) = (d, t_{j+1}), 0 \leq j \leq M - 1 \) satisfies the following estimate for \( \sqrt{\alpha\mu} \leq \sqrt{\rho\epsilon} \):

\[
(D^+ - D^-)(U^{j+1}(x_{\frac{N}{2}})) - u^{j+1}(x_{\frac{N}{2}})) \leq C \frac{\bar{h}}{\epsilon}
\]

where \( \bar{h} = \max\{h_N, h_{N+1}\} \).

Proof. Consider

\[
|\frac{d}{dx} - D^+|u^{j+1}(d)| + |\frac{d}{dx} - D^-|u^{j+1}(d)| \leq C_1 \frac{\hat{h}}{\epsilon}
\]

Now

\[
|\frac{d}{dx} - D^+|u^{j+1}(d)| + |\frac{d}{dx} - D^-|u^{j+1}(d)| \leq C \frac{\hat{h}}{\epsilon}
\]

(3.8)

respectively, then,

\[
\|U - u\|_{\Omega} \leq C(N^{-1} + \Delta t^2)
\]

where \( C \) is a constant independent of \( \epsilon, \mu \) and discretization parameter \( N, M \).

Proof. Combining the lemmas (4.1), (4.2), (4.3) and (4.4), we obtain the following bound for \((x_i, t_{j+1}) \neq (d, t_{j+1})\)

\[
|(U - u)(x_i, t_{j+1})| \leq C(N^{-1} + \Delta t^2), \quad \forall (x_i, t_{j+1}) \in \Omega^{N-} \cup \Omega^{N+}.
\]

(4.39)

To obtain error at the point of discontinuity \( (x_i, t_{j+1}) = (x_{\frac{N}{2}}, t_{j+1}) \) for the first case \( \sqrt{\alpha\mu} \leq \sqrt{\rho\epsilon} \), we consider the discrete barrier function \( \phi^{j+1}_1(x_i) = \psi^{j+1}_1(x_i) \pm e^{j+1}(x_i) \) in the interval \((d - \tau_2, d + \tau_3)\) where

\[
\psi^{j+1}_1(x_i) = C_1(\Delta t)^2 + \frac{C_2\tau}{\epsilon N(\log N)^2}
\]

\[
\left\{ \begin{array}{ll}
\left| x_i - (d - \tau_2) \right|, & (x_i, t_{j+1}) \in \Omega^{N,M} \cap (d - \tau_2, d), \\
|d + \tau_3 - x_i|, & (x_i, t_{j+1}) \in \Omega^{N,M} \cap [d, d + \tau_3].
\end{array} \right.
\]

where \( \tau = \max\{\tau_2, \tau_3\} \). It could be seen that \( \phi^{j+1}_1(d - \tau_2) \) and \( \phi^{j+1}_1(d + \tau_3) \) are non negative and \( \mathcal{L}_N \phi^{j+1}_1(x_i) \leq 0 \), \( (x_i, t_{j+1}) \in \Omega^{N,M} \), \( |(D^+ - D^-)\phi^{j+1}_1(x_{\frac{N}{2}})| \leq 0 \).

Hence, by applying the discrete minimum principle, we get \( \phi^{j+1}_1(x_i) \geq 0 \).

Therefore, for \((x_i, t_{j+1}) \in (d - \tau_2, d + \tau_3)\)

\[
|e^{j+1}(x_i)| = |(U - u)(x_i, t_{j+1})| \leq C_1(\Delta t)^2 + \frac{C_2\tau^2}{\epsilon N(\log N)^2} \leq C(N^{-1} + (\Delta t)^2).
\]

Combining the equation (4.39) and (4.40), we get the required result. \( \square \)
5 Numerical Examples

In this section, we examine two test problems with discontinuous convection coefficients and discontinuous source terms to validate the proposed method.

Example 5.1.

\[ \epsilon u_{xx} + \mu a(x,t)u_x - b(x,t)u - u_t = f(x,t) \quad (x,t) \in (0,.5) \cup (0.5,1) \times (0,1], \]

\[ u(0,t) = u(1,t) = u(x,0) = 0, \]

with

\[ a(x,t) = \begin{cases} 
-(1 + x(1 - x)), & 0 \leq x \leq 0.5, t \in (0,1], \\
1 + x(1 - x), & 0.5 < x \leq 1, t \in (0,1], 
\end{cases} \]

and

\[ f(x,t) = \begin{cases} 
-2(1 + x^2)t, & 0 \leq x \leq 0.5, t \in (0,1], \\
2(1 + x^2)t, & 0.5 < x \leq 1, t \in (0,1]. 
\end{cases} \]

\[ b(x,t) = 1 + \exp(x), \quad \text{and} \quad c(x,t) = 1. \]

Example 5.2.

\[ \epsilon u_{xx} + \mu a(x,t)u_x - b(x,t)u - u_t = f(x,t) \quad (x,t) \in (0,.5) \cup (0.5,1) \times (0,1], \]

\[ u(0,t) = u(1,t) = u(x,0) = 0, \]

with

\[ a(x,t) = \begin{cases} 
-(1 + x(1 - x)), & 0 \leq x \leq 0.5, t \in (0,1], \\
1 + x(1 - x), & 0.5 < x \leq 1, t \in (0,1], 
\end{cases} \]

and

\[ f(x,t) = \begin{cases} 
-2(1 + x^2)t, & 0 \leq x \leq 0.5, t \in (0,1], \\
3(1 + x^2)t, & 0.5 < x \leq 1, t \in (0,1]. 
\end{cases} \]

\[ b(x,t) = 1 + \exp(x), \quad \text{and} \quad c(x,t) = 1. \]

As the exact solutions of eg. 5.1 and eg. 5.2 is unknown, to calculate the maximum point-wise error and the rate of convergence, we use the double mesh principle \[8,9\]. The double mesh difference is defined by

\[ E_{N,M}^{N,M} = \max_j \left( \max_i |U_{2i,2j}^{2N,2M} - U_{i,j}^{N,M}| \right) \]

where \( U_{i}^{N,M} \) and \( U_{2i}^{2N,2M} \) are the solutions on the mesh \( \Omega_{N,M} \) and \( \Omega_{2N,2M} \) respectively. The order of convergence is given by

\[ R_{N,M}^{N,M} = \log_2 \left( \frac{E_{N,M}^{N,M}}{E_{2N,2M}^{2N,2M}} \right). \]

In the numerical examples, we have taken \( N = M \).

The Figure represents the surface plot of numerical solution and error graph for \( \epsilon = 10^{-12}, \mu = 10^{-8} \) and \( N = 64 \). We note that the maximum point-wise error is at the point of discontinuity. In Table we
Table 1: Maximum point-wise error $E_{\epsilon,\mu}^{N,M}$ and approximate orders of convergence $R_{\epsilon,\mu}^{N,M}$ for Example 5.1 when $\epsilon = 10^{-8}$

| $\mu$  | Number of mesh points N |   |   |   |
|--------|-------------------------|---|---|---|
|        | 64                      | 128 | 256 | 512 |
| $10^{-6}$ | 0.039036               | 0.019471 | 0.009595 | 0.004722 |
| Order  | 1.0035                  | 1.0210 | 1.0229 |
| $10^{-7}$ | 0.039041               | 0.019476 | 0.009597 | 0.004723 |
| Order  | 1.0033                  | 1.0211 | 1.0230 |
| $10^{-8}$ | 0.039042               | 0.019477 | 0.009597 | 0.004722 |
| Order  | 0.7480                  | 0.8690 | 0.9254 |
| $10^{-9}$ | 0.039042               | 0.019477 | 0.009597 | 0.004722 |
| Order  | 1.0032                  | 1.0211 | 1.0230 |
| $10^{-10}$ | 0.039042               | 0.019477 | 0.009591 | 0.004722 |
| Order  | 1.0032                  | 1.0211 | 1.0230 |
| $10^{-11}$ | 0.039042               | 0.019477 | 0.009597 | 0.004722 |
| Order  | 1.0032                  | 1.0211 | 1.0230 |
| $10^{-12}$ | 0.039043               | 0.019487 | 0.009599 | 0.004732 |
| Order  | 1.0025                  | 1.0215 | 1.0203 |
| $10^{-13}$ | 0.039041               | 0.019492 | 0.009560 | 0.004751 |
| Order  | 1.0024                  | 1.0274 | 1.0083 |
| $10^{-14}$ | 0.039042               | 0.019477 | 0.009597 | 0.004722 |
| Order  | 1.0024                  | 1.0274 | 1.0083 |
| $10^{-15}$ | 0.039042               | 0.019477 | 0.009597 | 0.004722 |
| Order  | 1.0024                  | 1.0274 | 1.0083 |
| $10^{-16}$ | 0.039042               | 0.019477 | 0.009597 | 0.004722 |
| Order  | 1.0024                  | 1.0274 | 1.0083 |

Figure 1: (a) and (b): Plots of numerical solution and errors for $\epsilon = 10^{-12}, \mu = 10^{-8}$ when $N = 64$ for Example 5.1

present the maximum point-wise error $E_{\epsilon,\mu}^{N,M}$ and approximate orders of convergence $R_{\epsilon,\mu}^{N,M}$ for Example 5.1 for various values of $\mu$ when $\epsilon = 10^{-8}$.  

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The Figure 2 represents the surface plots of numerical solution and error graph for \( \epsilon = 10^{-8}, \mu = 10^{-8} \) and \( N = 128 \). In Table 2, we present the maximum point-wise error and \( E_{\epsilon,\mu}^{N,M} \) and approximate orders of convergence \( R_{\epsilon,\mu}^{N,M} \) for Example 5.2 for various values of \( \mu \) when \( \epsilon = 10^{-12} \). In both examples, we note that the overall order of convergence is one as \( N = M \).

Table 2: Maximum point-wise error \( E_{\epsilon,\mu}^{N,M} \) and approximate orders of convergence \( R_{\epsilon,\mu}^{N,M} \) for Example 5.2 when \( \epsilon = 10^{-12} \)

| \( \mu \) | \( 64 \) | \( 128 \) | \( 256 \) | \( 512 \) |
| --- | --- | --- | --- | --- |
| \( 10^{-7} \) | 0.048709 | 0.024266 | 0.011958 | 0.005588 |
| Order | 1.00522 | 1.02091 | 1.02206 | |
| \( 10^{-8} \) | 0.048775 | 0.024330 | 0.011989 | 0.005900 |
| Order | 1.00337 | 1.02100 | 1.02291 | |
| \( 10^{-9} \) | 0.048782 | 0.024337 | 0.011992 | 0.005901 |
| Order | 1.00318 | 1.02101 | 1.02301 | |
| \( 10^{-10} \) | 0.048782 | 0.024337 | 0.011992 | 0.005901 |
| Order | 1.00316 | 1.02101 | 1.02301 | |
| \( 10^{-11} \) | 0.048782 | 0.024337 | 0.011993 | 0.005901 |
| Order | 1.00316 | 1.02101 | 1.02302 | |
| \( 10^{-12} \) | 0.048782 | 0.024338 | 0.011993 | 0.005901 |
| Order | 1.00316 | 1.02101 | 1.02302 | |
| \( 10^{-13} \) | 0.048782 | 0.024338 | 0.011993 | 0.005901 |
| Order | 1.00316 | 1.02101 | 1.02302 | |
| \( 10^{-14} \) | 0.048783 | 0.024338 | 0.01199301 | 0.0059016 |
| Order | 1.00316 | 1.02101 | 1.02302 | |
| \( 10^{-15} \) | 0.048783 | 0.024338 | 0.01199301 | 0.0059016 |
| Order | 1.00316 | 1.02101 | 1.02302 | |
| \( 10^{-16} \) | 0.048783 | 0.024338 | 0.01199301 | 0.0059016 |
| Order | 1.00316 | 1.02101 | 1.02302 | |
| \( 10^{-17} \) | 0.048783 | 0.024338 | 0.01199301 | 0.0059016 |
| Order | 1.00316 | 1.02101 | 1.02302 | |

6 Conclusions

Here, we have considered a two-parameter singularly perturbed parabolic problem with a discontinuous convection co-efficient and source term. The solution to this problem exhibits boundary layers near the boundaries and internal layers near the point of discontinuity due to the discontinuity of the convection coefficient and the source term. We have proposed a parameter uniform numerical scheme for the case \( \sqrt{\alpha \mu} \leq \sqrt{\rho \epsilon} \). In the temporal direction, we employed the Crank-Nicolson method on a uniform mesh, and in the spatial direction, we used an upwind finite difference scheme on a Shishkin-Bakhvalov mesh. At the point of discontinuity, we used a three-point formula. The use of Crank-Nicolson method and Shishkin-Bakhvalov mesh helped to obtain second-order accuracy in time and first-order accuracy in space; unlike the Shishkin mesh, where a logarithmic factor deteriorates the order of convergence. The theoretical analysis was supported by the numerical results.
Figure 2: (a) and (b): Plots of numerical solution and errors for $\epsilon = 10^{-8}, \mu = 10^{-8}$ when $N = 128$ for Example 5.2.

Declaration of interests

\[ \square \] The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] N. S. Bakhvalov, Towards optimization of methods for solving boundary value problems in the presence of a boundary layer, (in Russian) Zh. Vychisl. Mat. Mat. Fiz. 9 (1969) 841-859.

[2] T. A. Bullo, G. A. Degla, G. F. Duressa, Uniformly convergent higher-order finite difference scheme for singularly perturbed parabolic problems with non-smooth data, J. Appl. Math. Comput. Mech. 20(1) (2021) 5-16.

[3] T. A. Bullo, G. Duressa, G. DEGLA, Higher order fitted operator finite difference method for two-parameter parabolic convection-diffusion problems, Int. J. Eng. Technol. Manag. Appl. Sci. 11(4) (2019) 455-467. https://doi.org/10.24107/ijeas.644160.

[4] M. Chandru, P. Das, H. Ramos, Numerical treatment of two-parameter singularly perturbed parabolic convection diffusion problems with non-smooth data, Math. Methods Appl. Sci. 41(14) (2018) 5359-5387.

[5] M. Chandru, T. Prabha, P. Das, V. Shanthi, A numerical method for solving boundary and interior layers dominated parabolic problems with discontinuous convection coefficient and source terms, Differ. Equ. Dyn. Syst. 27 (1-3) (2019) 91-112.

[6] C. Clavero, J. L. Gracia, G. I. Shishkin, L. P. Shishkina, An efficient numerical scheme for 1d parabolic singularly perturbed problems with an interior and boundary layers, J. Comput. Appl. Math. 318 (2017) 634-645.

[7] J. Crank, P. Nicolson, A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type, Math. Proc. Camb. Philos. Oc. 43 (1947) 50-67. https://doi.org/10.1017/S0305004100023197.
[8] P. Das, V. Mehrmann, Numerical solution of singularly perturbed convection-diffusion-reaction problems with two small parameters, BIT Numer. Math. 56(1) (2016) 51-76.

[9] P. Das, A higher order difference method for singularly perturbed parabolic partial differential equations, J. Differ. Equ. Appl. 24(3) (2018) 452-477.

[10] J. L. Gracia, E. O’Riordan, M. L. Pickett, A parameter robust second order numerical method for a singularly perturbed two-parameter problem, Appl. Num. Math. 56(7) (2006) 962-980.

[11] V. Gupta, M. K. Kadalbajoo, R. K. Dubey, A parameter-uniform higher order finite difference scheme for singularly perturbed time-dependent parabolic problem with two small parameters, Int. J. Comput. Math. 96(3) (2018) 1-29.

[12] A. Jha, M. K. Kadalbajoo, A robust layer adapted difference method for singularly perturbed two parameter parabolic problems, Int. J. Comput. Math. 92(6) (2015) 1204-1221.

[13] M. K. Kadalbajoo, A. S. Yadaw, Parameter-uniform finite element method for two-parameter singularly perturbed parabolic reaction-diffusion problems, Int. J. Comput. Methods 9(4) (2012). doi:10.1142/S0219876212500478.

[14] M. K. Kadalbajoo, L. P. Tripathi, A. Kumar, A cubic B-spline collocation method for a numerical solution of the generalized Black-Scholes equation, Math. Comput. Model. 55 (2012) 1483-1505.

[15] D. Kumar, P. Kumari, Uniformly convergent scheme for two-parameter singularly perturbed problems with non-smooth data, Numer. Methods Partial Differ. Equ. 37(1) (2021) 796-817.

[16] T. Linß, An upwind difference scheme on a novel Shishkin-type mesh for a linear convection diffusion problem, J. Comput. Appl. Math. 110(1) (1999) 93-104.

[17] T. Linß, Analysis of a Galerkin finite element method on a Bakhvalov-Shishkin mesh for a linear convection-diffusion problem, IMA J. Numer. Anal. 20(4) (2000) 621-632.

[18] T. B. Mekonnen, G. Duressa, Computational method for singularly perturbed two-parameter parabolic convection-diffusion problems, Cogent Math. Stat. 7(1) (2020) 1829277.

[19] J. B. Munyakazi, A robust finite difference method for two-parameter parabolic convection-diffusion problems, Appl. Math. Inf. Sci. 9 (2015) 2877-2883.

[20] R. E. O’Malley Jr, Two-parameter singular perturbation problems for second order equations, J. Math. Mech. 16 (1967) 1143-1164.

[21] R. E. O’Malley Jr, Introduction to Singular Perturbations, Academic Press, New York, (1974).

[22] R. E. O’Malley Jr, Singular Perturbation Methods for Ordinary Differential Equations, Springer New York (1990).

[23] E. O’Riordan, G. I. Shishkin, Singularly perturbed parabolic problems with non-smooth data, J. Comput. Appl. Math. 166 (2004) 233-245.

[24] E. O’Riordan, G. I. Shishkin, M. L. Picket, Parameter-uniform finite difference schemes for singularly perturbed parabolic diffusion-convection-reaction problems, Math. Comput. 75 (2006) 1135-1154.

[25] G. I. Shishkin, Discrete Approximation of Singularly Perturbed Elliptic and Parabolic Equations, Russian Academy of Sciences, Russian Acad. Sci., Ural Branch, Ekaterinburg (1992) (in Russian).
[26] N. S. Yadav, K. Mukherjee, Uniformly convergent new hybrid numerical method for singularly perturbed parabolic problems with interior layers, Int. J. Appl. Comput. Math. 6 (2020) (53). https://doi.org/10.1007/s40819-020-00804-7.

[27] W. K. Zahra, M. S. El-Azab, A. M. El Mhlawy, Spline difference scheme for two-parameter singularly perturbed partial differential equations, Int. J. Appl. Math. 32(1-2) (2014) 185-201.