Gap nodes induced by coexistence with antiferromagnetism in iron-based superconductors

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We investigate the pairing in iron pnictides in the coexistence phase, which displays both superconducting and antiferromagnetic orders. By solving the pairing problem on the Fermi surface reconstructed by long-range magnetic order, we find that the pairing interaction necessarily becomes angle-dependent, even if it was isotropic in the paramagnetic phase, which results in an angular variation of the superconducting gap along the Fermi surfaces. We find that the gap has no nodes for a small antiferromagnetic order parameter $M$, but may develop accidental nodes for intermediate values of $M$, when one pair of the reconstructed Fermi surface pockets disappear. For even larger $M$, when the other pair of reconstructed Fermi pockets is gapped by long-range magnetic order, superconductivity still exists, but the quasiparticle spectrum becomes nodeless again. We also show that the application of an external magnetic field facilitates the formation of nodes. We argue that this mechanism for a nodeless-nodal-nodeless transition explains recent thermal conductivity measurements of hole-doped (Ba$_{1-x}$K$_x$)Fe$_2$As$_2$ [J-Ph. Read et al arXiv:1105.2232].

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I. INTRODUCTION

Exploring the different regions in the phase diagram of iron based superconductors (FeSCs) is an important step towards developing a unified understanding of the physics of superconductivity in these systems. A typical phase diagram of a FeSC in the $(T, x)$ plane, where $x$ is doping, shows a metallic antiferromagnetic order, also called spin density wave (SDW), below $T_N$ at $x = 0$. Upon doping, the SDW order parameter is suppressed and superconductivity (SC) emerges with maximum $T_c(x)$ near the point where $T_c(x)$ exceeds $T_N(x)$ (an "optimal doping"). The gap symmetry at optimal doping is most likely $s^+$, but the structure of the $s$-wave gap varies from one material to another: the gap is nodeless in Ba(Fe$_{1-x}$Co$_x$)$_2$As$_2$ and (Ba$_{1-x}$K$_x$)Fe$_2$As$_2$ systems, but has nodes in BaFe$_2$(As$_{1-x}$P$_x$)$_2$. In this article we investigate the behavior of the $s^+$ SC gap in the underdoped regime of the Ba(Fe$_{1-x}$Co$_x$)$_2$As$_2$ and (Ba$_{1-x}$K$_x$)Fe$_2$As$_2$ systems, where $T_c(x) < T_N(x)$, and superconductivity emerges in a continuous fashion from a pre-existing SDW order. The microscopic coexistence of SC and SDW below $T_c(x)$ has been reported by magnetization and NMR experiments.

The electronic structure of FeSCs in the paramagnetic phase consists of near-circular hole pockets and elliptical electron pockets. We assume, following earlier works, that the interaction leading to SDW order is independent on the angle along the FS. In this situation, the Brillouin Zone (BZ) is reduced once the SDW order sets in, and the Fermi surface (FS) is reconstructed into four banana-like pockets (right panel in Fig. 1). As the SDW order parameter $M$ increases, the reconstructed pockets shrink and eventually disappear. This happens in two stages: first, at $M = M_1$, the number of pockets shrinks from four to two, (middle panel in Fig. 1), and then the remaining two pockets vanish at a larger value $M = M_2$ (left panel in Fig. 1). Because these reconstructed FSs are formed by mixing electron and hole bands, on which the $s^+$ SC order parameter has different signs in the absence of SDW order, one could...
naively expect that the SC gap on the reconstructed FSs alternates between plus and minus signs, implying that it must have nodes. However, calculations show\textsuperscript{13} that this is not the case because SDW order, with e.g. $\langle S_z \rangle \neq 0$, mixes hole and electron FSs with opposite $\sigma_z$ components of electron spins. Because a spin-singlet s-wave gap changes sign under $\sigma_z \rightarrow -\sigma_z$, the sign change imposed by the $s^+-$ gap structure is compensated by the sign change due to the flip of $\sigma_z$, and, as a result, the gaps on the reconstructed FSs retain the same nodeless structure as in the absence of magnetic order.

Experiments near the onset of the co-existence phase, where $M$ is small, are in agreement with this reasoning. For instance, the ratio $\kappa/T$, where $\kappa$ is the in-plane thermal conductivity, tends to zero in the $T \rightarrow 0$ limit, as it is expected for a superconductor with a nodeless gap. However, recent thermal conductivity measurements on ($\text{Ba}_1_{-x}\text{K}_x$)Fe$_2$As$_2$ deep in the coexistence phase\textsuperscript{22} found that there is a range of dopings where $\kappa/T$ remains finite at $T = 0$, as it happens in a superconductor with line nodes. At even smaller dopings, $\kappa/T$ again vanishes at $T = 0$. Application of a magnetic field makes the doping dependence more smooth and extends the range of nodal behavior.

In this article we explain the sequence of nodeless-nodal-nodeless behavior as a transition from a nodeless $s^+-$ gap in the region where all four reconstructed FS pockets are present, to an $s^+-$ gap with accidental nodes in the region where two FS pockets are present, to the fully gapped quasiparticle spectrum in the region where the remaining two reconstructed FS pockets are also gapped (see Fig. I). We show that the pairing interaction in the coexistence phase acquires an additional angular dependence as it gets dressed by angle-dependent coherence factors associated with the SDW order. The angular dependence of the interactions in turn gives rise to an angular dependence of the superconducting gap. This effect is weak in the limit of small $M$, considered in Ref. \textsuperscript{13} when all four reconstructed Fermi pockets are present, but is strong and gives rise to gap nodes when $M$ becomes large enough to gap out one pair of Fermi pockets. At even larger $M$, deeper in the coexistence region, the remaining pair of Fermi pockets is gapped out. In this situation, superconductivity still develops over some range of dopings\textsuperscript{11}, but the quasiparticle excitation spectrum remains gapped irrespective of the structure of the SC gap, and $\kappa/T$ again vanishes at $T = 0$. Concurrently, the superconducting transition temperature $T_c$ shows a minimum when the remaining pockets vanish. The theoretical behavior across the SDW-SC coexistence region is shown schematically in Fig. I. It agrees well with the experimental results of Refs. 22 and 23.

The paper is organized as follows. In Sec. II we describe the formalism listing out the details of the FS geometries, the nature of the reconstructed FSs, and the generic form of the SC gap in the coexistence region. In Sec. III we discuss the solutions for the gap obtained within this formalism, show when and how nodes arise in the coexistence region, and provide analytical explanation of the results. We also briefly discuss the role played by bands that do not participate in the formation of SDW. In Sec. IV we show that the doping range where the SC gap has nodes is enhanced by applying a magnetic field. We state our final conclusions in Sec. V.

II. THE PAIRING PROBLEM ON THE RECONSTRUCTED FERMI SURFACE

Our point of departure is a microscopic band model of interacting fermions located near hole or electron pockets. The electronic structure of FeSCs consists of two or three hole pockets, centered at $(0,0)$ in the folded zone (two Fe atoms per unit cell), and two hybridized electron pockets centered at $(\pi, \pi)$. The interactions between low-energy fermions are generally angle-dependent already in the normal state, before either SDW or SC order sets in, which gives rise to angle-dependent SDW and SC gaps even outside the coexistence region.

As we said, we follow earlier works and consider a simplified model in which we neglect the angular dependencies of the interactions in the normal state, and approximate the normal-state interactions as constants.\textsuperscript{13,12} We further neglect the hybridization and perform calculations in the unfolded zone (one Fe atom per unit cell), in which electron pockets are ellipses centered at $(0,0)$ and $(\pi,0$) and hole pockets are at $(0,0)$ and $(\pi, \pi)$. Finally, we neglect the third hole pocket and only consider two circular hole pockets centered at $(0,0)$ in the unfolded zone (in orbital notations these are $d_{x z}/d_{y z}$ pockets). Earlier works have found\textsuperscript{24} that SDW magnetic order emerges in this model with ordering vector $Q = (0, \pi)$ or $Q = (\pi, 0)$, mixing one hole and one electron pocket. For definiteness we set $Q = (0, \pi)$ in which case one hole pocket at $(0,0)$ and one electron pocket at $(0, \pi)$ are mixed up. The other two pockets are not participating in SDW and remain intact once the SDW order sets in. In line with our goal, we first consider only the 2-pocket model with one circular hole and one elliptical electron pocket, which get reconstructed in the SDW phase. We discuss the role of the other pockets later in the next section.

To quadratic order in the fermions, the Hamiltonian of the 2-pocket model in the SDW phase is $H_0 + H_{SDW}$, where

$$H_0 = \sum_{k,\sigma} \left( \varepsilon_{k}^{c} \sigma_{k}^{c} c_{k\sigma}^{\dagger} + \varepsilon_{k}^{f} f_{k+Q}\sigma^{\dagger} f_{k+Q}\sigma \right)$$

$$H_{SDW} = \sum_{k\sigma} M \left( \sigma_{k}\sigma^{\dagger} f_{k+Q}\sigma + \text{h.c.} \right)$$

Here $c$ and $f$ are fermionic operators for hole and electron states, respectively, $\sigma$ is $\pm 1$, and $M$ is the SDW order parameter, which we treat below as a variable. In practice, larger $M$ correspond to smaller dopings, whereas smaller $M$ correspond to doping close to the optimal one. The fermionic dispersions are
the eigenstates of (1) are coherent superpositions of electrons and holes. For definiteness, we set the interatomic spacing to one and use \( \mu_c = \mu_f = \mu, m = 1/(2\mu), m_x = 0.5/(2\mu), \) and \( m_y = 1.5/(2\mu) \). These parameters are chosen to give the FS geometry as in Fig. 2 We will also measure \( M \) in units of \( 2\mu \).

The \( M \) term in (1) couples \( c \) and \( f \) operators such that the eigenstates of (1) are coherent superpositions of electrons and holes. These are described by new fermionic operators \( a \) and \( b \), with dispersion \( E_{k}^{ab} \) that vanishes at the reconstructed FSs. The transformation to the new operators is

\[
\begin{pmatrix}
\sigma a_{k\sigma} \\
b_{k\sigma}
\end{pmatrix}
= \begin{pmatrix}
-\sigma \cos \theta_k & \sin \theta_k \\
-\sin \theta_k & -\sigma \cos \theta_k
\end{pmatrix}
\begin{pmatrix}
c_{k\sigma} \\
f_{k+Q\sigma}
\end{pmatrix}
\tag{3}
\]

where

\[
\cos \theta_k = \frac{M}{\sqrt{M^2 + (E^-)^2}}
\]

\[
\sin \theta_k = \frac{E^-}{\sqrt{M^2 + (E^-)^2}}
\tag{4}
\]

and

\[
E^- = \frac{\varepsilon^c - \varepsilon^f}{2} - \sqrt{\left(\frac{\varepsilon^c - \varepsilon^f}{2}\right)^2 + M^2}
\tag{5}
\]

such that

\[
\cos 2\theta_k = \frac{\varepsilon^c - \varepsilon^f}{\sqrt{M^2 + (\varepsilon^c - \varepsilon^f)^2}}
\]

\[
\sin 2\theta_k = -\frac{M}{\sqrt{M^2 + (\varepsilon^c - \varepsilon^f)^2}}
\tag{6}
\]

In terms of the new operators,

\[
H_0 + H_{SDW} = \sum_{k,\sigma} (E_k^{ab} a_{k\sigma} a_{k\sigma} + E_k^{ab} b_{k\sigma} b_{k\sigma}) \tag{7}
\]

where

\[
E_{k}^{ab} = \frac{\varepsilon^c + \varepsilon^f}{2} \pm \sqrt{\left(\frac{\varepsilon^c - \varepsilon^f}{2}\right)^2 + M^2}
\tag{8}
\]

are the quasi-particle excitation energies of the SDW state (the plus sign corresponds to \( a \) fermions, the minus sign corresponds to \( b \) fermions).

The condition \( E_{k}^{ab} = 0 \) defines the new reconstructed FSs. For small enough \( M \), there are two pairs of banana-shaped reconstructed pockets, the \( a \)-pockets and the \( b \)-pockets (see Fig. 2), with the latter larger than the former. At the critical \( M = M_1 \) given by

\[
M_1 = \frac{|m - m_y|}{4 \sqrt{mm_y}} \tag{9}
\]

the \( a \)-pockets disappear while the \( b \)-pockets remain (Fig. 2 right). At even larger \( M = M_2 \) given by

\[
M_2 = \frac{|m - m_x|}{4 \sqrt{mm_x}} \tag{10}
\]

the \( b \)-pockets also disappear, i.e. all electronic states become gapped by SDW. For our set of parameters \( M_1 \approx 0.102 \) and \( M_2 \approx 0.177 \).

We now turn to the issue of the pairing on the reconstructed FS. There are four residual interactions between the original fermions located near the hole and electron FSs \( \Omega \): the interaction between fermionic densities near hole and electron pockets \( \left[ U_1 c_{\sigma} c_{\sigma}^\dagger f_{\sigma}^\dagger f_{\sigma} \right] \), the exchange interaction between hole and electron pockets \( \left[ U_2 c_{\sigma}^\dagger f_{\sigma} f_{\sigma}^\dagger c_{\sigma} \right] \), the umklapp pair-hopping interaction in which two fermions near a hole pocket are converted into two fermions near an electron pocket and vice-versa \( \left[ U_3/2(c_{\sigma}^\dagger f_{\sigma} c_{\sigma} f_{\sigma}^\dagger + f_{\sigma}^\dagger c_{\sigma} f_{\sigma} c_{\sigma}^\dagger) \right] \), and the density-density interaction within each pocket \( \left[ U_4/2(c_{\sigma}^\dagger c_{\sigma}^\dagger c_{\sigma} c_{\sigma} + f_{\sigma}^\dagger f_{\sigma} f_{\sigma}^\dagger f_{\sigma}) \right] \). In the absence of SDW order, only \( U_3 \) and \( U_4 \) contribute to the pairing channel (\( U_3 \) must be larger than \( U_4 \) for \( s^+ \) pairing). In the SDW phase, \( c \) and \( f \) operators are mixed up, and all four interactions contribute to pairing vertices for \( a \) and \( b \) fermions. The interactions \( U_3 \) and \( U_4 \) do not give rise to new physics and only renormalize the values of the pairing vertices obtained from the \( U_1 \) and \( U_3 \) terms. To shorten the formulas, we neglect the \( U_2 \) and \( U_4 \) terms and approximate \( H_{int} \) by

\[
H_{int} = \sum_{[1234]} U_1 c_{\sigma}^\dagger f_{2\sigma}^\dagger f_{3\sigma} c_{4\sigma} + \frac{U_3}{2} \left( c_{\sigma}^\dagger c_{\sigma} f_{3\sigma} f_{4\sigma} + h.c. \right) \tag{11}
\]

where \( \sum_{[1234]} \) denotes the sum over all momenta subject to \( k_1 + k_2 = k_3 + k_4 \), and the summation over repeated spin indices is implied. Converting to new operators via Eq. (9) and keeping only the interactions which contribute to the pairing, we obtain from Eq. (11)
Note, however, that the three angular components of \( \Delta_{ab} \) are not orthogonal, since e.g. \( \int_a dq_0 \cos 2\theta_q \) does not vanish. Substituting these expressions into the gap equation, we re-express it as a matrix equation for \( T_c \) and \( g_i \):

\[
\begin{pmatrix}
g_1 \\
g_2 \\
g_3
\end{pmatrix} = \frac{U_3 L}{2} \begin{pmatrix} N & N_c & N_s \\ -N_c & -N_{cc} & -N_{cs} \\ -sN_s & -sN_{cs} & -sN_{ss} \end{pmatrix} \begin{pmatrix} g_1 \\
g_2 \\
g_3
\end{pmatrix}
\]

where

\[
\begin{align*}
N &= 2 \int_a \frac{dk}{4\pi^2 v_F} + 2 \int_b \frac{dk}{4\pi^2 v_F} \\
N_c &= 2 \int_a \frac{dk}{4\pi^2 v_F} \cos 2\theta_k - 2 \int_b \frac{dk}{4\pi^2 v_F} \cos 2\theta_k \\
N_s &= 2 \int_a \frac{dk}{4\pi^2 v_F} \sin 2\theta_k - 2 \int_b \frac{dk}{4\pi^2 v_F} \sin 2\theta_k \\
N_{cc} &= 2 \int_a \frac{dk}{4\pi^2 v_F} \cos^2 2\theta_k + 2 \int_b \frac{dk}{4\pi^2 v_F} \cos^2 2\theta_k \\
N_{ss} &= 2 \int_a \frac{dk}{4\pi^2 v_F} \sin^2 2\theta_k + 2 \int_b \frac{dk}{4\pi^2 v_F} \sin^2 2\theta_k \\
N_{cs} &= 2 \int_a \frac{dk}{4\pi^2 v_F} \cos 2\theta_k \sin 2\theta_k \\
&\quad + 2 \int_b \frac{dk}{4\pi^2 v_F} \cos 2\theta_k \sin 2\theta_k
\end{align*}
\]

Without SDW order, \( \sin 2\theta_k = 0, \cos 2\theta_k = \text{sign}(\epsilon - \epsilon_f) \), and \( N_s, N_{sc}, \text{and } N_{ss} \) all vanish. Then \( g_3 = 0 \), while \( g_2/g_1 = -N_c/(N + \sqrt{N^2 - N^2_s}) \). As a result, the gaps on the inner and outer sides of the \( a \) and \( b \) pockets reduce to \( g_1(1 + N_c/(N + \sqrt{N^2 - N^2_s})) \) and \( g_1(1 - N_c/(N + \sqrt{N^2 - N^2_s})) \). These two coincide with the gaps on the \( c \) and \( f \)-Fermi surfaces at \( T_c \) in the paramagnetic phase, which we set in our model to be angle-independent (we recall that the sign of the two gaps is the same because we attributed an extra minus sign to the gap on the \( f \)-Fermi surface by flipping the spins of \( f \)-fermions). The magnitudes of the gaps on \( c \) and
The quantities $N_j/N$, where $N_j = N_c, N_s, N_{cs}$, $N_{ss}$, plotted as functions of $M$. The dashed line indicates the value $M = M_1$ at which the $a$-pockets disappear. The ellipsis indicate the area around $M_1$ where the ratios rapidly evolve. Note that $N_c/N$ remains small for all $M$, while $N_s/N$ and $N_{ss}$ increase with $M$ and become positive and of order one at $M \geq M_1$. We plot the same ratios for $M = 0.09$ and $M = 0.12$. The key result here is that for $s \geq 1$, the magnitude of $g_3/g_1$ becomes greater than 1 for $M = 0.12$. This gives rise to accidental nodes of the gap on the remaining FS pocket (see text).

For arbitrary $M$, $g_2/g_1$ and $g_3/g_1$ are rather complex functions of the ratios $N_j/N$ and also $s$. We plot $N_j/N$ in Fig.\ 8. We see that $N_{cs}/N$ and $N_{ss}/N$, are small because the integrands for $N_c$ and $N_{cs}$ contain $\cos 2\theta_k$, which changes sign between the inner and outer sides of the $a$ and $b$-pockets (Fig.\ 9). At the same time, the magnitudes of $N_s/N$ and $N_{ss}/N$ increase with increasing $M$ and become of order one.

In Fig.\ 10 we plot $\cos 2\theta_k$ and $\sin 2\theta_k$ along the $a$ and

![Figure 3](image1.png)

Figure 3: The quantities $N_j/N$, where $N_j = N_c, N_s, N_{cs}$, $N_{ss}$, plotted as functions of $M$. The dashed line indicates the value $M = M_1$ at which the $a$-pockets disappear. The ellipsis indicate the area around $M_1$ where the ratios rapidly evolve. Note that $N_c/N$ remains small for all $M$, while $N_s/N$ and $N_{ss}$ increase with $M$ and become positive and of order one at $M \geq M_1$. We plot the same ratios for $M = 0.09$ and $M = 0.12$. The key result here is that for $s \geq 1$, the magnitude of $g_3/g_1$ becomes greater than 1 for $M = 0.12$. This gives rise to accidental nodes of the gap on the remaining FS pocket (see text).

![Figure 4](image2.png)

Figure 4: The behavior of $\cos 2\theta_k$ (left column) and $\sin 2\theta_k$ (right column) along the $a$ and $b$ pockets for $M = 0.04$ and $M = 0.09$. The gap is given by Eq. \[13\]. The "length along the pocket" refers to the numbers along each pocket specified in Fig.\ 2.

$f-$ FSs are generally unequal as $N_c \neq 0$ because of the different geometry of hole and electron pockets. Numerically, however, for our set of parameters $N_c/N \ll 1$, i.e., at $M = 0$ the gaps on $a$ and $b$ FSs are essentially isotropic. Once SDW order sets in, all $N_1$ and $N_{j1}$ become non-zero, $g_3$ becomes finite, $\cos 2\theta_k$ and $\sin 2\theta_k$ become smooth functions along the pockets, and the gaps $\Delta_a(q)$ and $\Delta_b(q)$ acquire smooth angular variations. The magnitudes of the variations depend on the two parameters: the ratios $g_2/g_1$ and $g_3/g_1$ and the actual variation of $\cos 2\theta_k$ and $\sin 2\theta_k$ along $a-$ and $b-$ Fermi surfaces.

In the small $M$ limit, considered previously in Ref. \[13\], $\sin 2\theta_k$ is small except for narrow ranges near the tips of the bananas. Then $N_s, N_{cs},$ and $N_{ss}$ are small, and $N_{cs}/N \approx 1$. Solving Eq. \[13\] in this limit, we find that $g_3 \ll g_2 \ll g_1$, hence $\Delta_a$ and $\Delta_b$ remain almost constant along the $a-$ and $b-$pockets, including the tips of the bananas.

![Figure 5](image3.png)

Figure 5: Top panel: the ratios $g_2/g_1$ (blue line) and $g_3/g_1$ (red line) as function of $s$ for $M = 0.09$ (solid lines) and $M = 0.12$ (dashed lines). These values are slightly below and slightly above the critical $M_1 = 0.102$ at which two out of four pockets disappear. Both ratios flip sign between $M = 0.09$ and $M = 0.12$. The key result here is that for $s \geq 1$, the magnitude of $g_3/g_1$ becomes larger than 1 for $M = 0.12$. This gives rise to accidental nodes of the gap on the remaining FS pocket (see text). Bottom panel: the plot of the same ratios as functions of $M$ for $s = 5$. The magnitude of $g_2/g_1$ remains relatively small, while the magnitude of $g_3/g_1$ becomes greater than 1 for $M \geq M_1$. For arbitrary $M$, $g_2/g_1$ and $g_3/g_1$ are rather complex functions of the ratios $N_j/N$ and also $s$. We plot $N_j/N$ in Fig.\ 8. We see that $N_{cs}/N$ and $N_{ss}/N$, are small because the integrands for $N_c$ and $N_{cs}$ contain $\cos 2\theta_k$, which changes sign between the inner and outer sides of the $a$ and $b$-pockets (Fig.\ 9). At the same time, the magnitudes of $N_s/N$ and $N_{ss}/N$ increase with increasing $M$ and become of order one.
b-pockets. We recall that the prefactors of the \( \cos 2\theta_k \) terms in \( \Delta_\alpha \) and \( \Delta_\delta \) are already small because \( g_2/g_1 \) is small. Fig. 4 shows that the variation of \( \cos 2\theta_k \) along each of the pockets decreases with increasing \( M \), making the \( g_3 \cos 2\theta_k \) term even less relevant. On the other hand, the range where \( |\sin 2\theta_k| \) is not small widens up with increasing \( M \). Over some range of \( M \), up to \( M_{11} = 0.5(m_y - m)/(m_y + m) < M_1 \) on the a-pocket and up to \( M_{22} = 0.5(m - m_x)/(m + m_x) < M_2 \) on the b-pocket, \( \sin 2\theta_k \) varies between \( -1 \) and some negative value. At \( M_{11} < M < M_1 \), \( \sin 2\theta_k \) on the a-pocket varies in a more narrow interval, does not reach \( -1 \) and eventually becomes

\[
\sin 2\theta_{k_a,M\rightarrow M_1} = -2\sqrt{m m_y/m + m_y} = -0.98. \tag{16}
\]

The same behavior holds for \( \sin 2\theta_k \) on the \( b \)-pocket for \( M_{22} < M < M_2 \), but with

\[
\sin 2\theta_{k_b,M\rightarrow M_2} = -2\sqrt{m m_x/m + m_x} = -0.94. \tag{17}
\]

### III. GAP STRUCTURE IN THE COEXISTENCE STATE AS FUNCTION OF \( M \)

#### A. From small to intermediate \( M \): nodeless-nodal transition

We now discuss in more detail the solution of the matrix equation \( \langle \rangle \) and the structure of the SC gap. We found in the previous section that the angular dependence of \( \cos 2\theta_k \) and \( \sin 2\theta_k \) along the \( a \) and \( b \)-pockets depends on \( M \), while the solution of the matrix equation for \( g_\ell \) depends on \( M \) and also on \( s \) - which, we remind, is the ratio of the interactions \( s = U_1/U_3 \).

The solution of the matrix equation can be easily analyzed in the limits of small and large \( s \). At small \( s \), \( g_3 \) is small compared to \( g_2 \) (\( g_3/g_2 = O(s) \)), while \( g_2/g_1 \) is given by

\[
\frac{g_2}{g_1} \approx -\frac{N_{\alpha}}{N + N_{\alpha}} \tag{18}
\]

This ratio is always small because \( N_{\alpha}/N \) is small for all \( M \) (see Fig. 3). As a result, the gaps \( \Delta_\alpha(q) \) and \( \Delta_\delta(q) \) remain essentially constants regardless of how strong the SDW order is.

In the opposite limit \( s \gg 1 \) the behavior is different. Only in a narrow range of the smallest \( M \ll 1/s \) the gaps remain almost angle-independent. Outside this range \( g_2 \) is small compared to \( g_3 \) (\( g_2/g_3 = O(1/s) \)), while \( g_3/g_1 \) is given by

\[
\frac{g_3}{g_1} \approx -\frac{N_s}{N_{ss}} \tag{19}
\]

Both \( N_s \) and \( N_{ss} \) become non-zero at a finite \( M \) and their ratio is of order one, i.e., \( g_3 \sim g_1 \). This leads to sizable angular dependencies of \( \Delta_\alpha(q) \) and \( \Delta_\delta(q) \). For small \( M < M_1 \), \( N_\alpha \) is smaller in magnitude than \( N_{ss} \) (see Fig. 3), hence \( |\Delta_\alpha| < 1 \) and the angular variations of the gaps do not give rise to nodes. However, as \( M \) increases and approaches \( M_1 \), the ratio \( N_\alpha/N_{ss} \) changes sign, becomes negative and its magnitude exceeds one (see Fig. 3). The combination of this behavior and the fact that for \( M \sim M_1 \) \( \cos 2\theta_k \) reaches \(-1\) at the tip of the bananas implies that the gap along the \( b \)-pocket develops accidental nodes. This happens even before \( M \) reaches \( M_1 \) because at \( M = M_1 \) (when the \( a \)-pocket disappears), \( |N_\alpha/N_{ss}| \) is already larger than one, as one can immediately see from Eq. 15 if one neglects in this equation the contributions from the \( a \)-pocket. The nodes in \( \Delta_\alpha(q) \) obviously survive up to \( M_{22} \) because at \( M < M_{22} \) the minimal value of \( \sin 2\theta_k \) remains \(-1\). At larger \( M_{22} < M < M_2 \) this minimal value becomes smaller than one, but we checked numerically that the nodes still survive and remain present up to \( M = M_2 \).

To understand the gap structure in between the limits \( s \ll 1 \) and \( s \gg 1 \), we solve the \( 3 \times 3 \) gap equation \( \langle \rangle \) numerically for several \( s \) with \( M \) as a running parameter, and for several \( M \) with \( s \) as a running parameter. In the two panels in Fig. 4 we plot the ratios \( g_2/g_1 \) and \( g_3/g_1 \) as function of \( s \) for two representative values of \( M \), \( M = 0.09 < M_1 \) and \( M_1 < M = 0.12 < M_2 \), and as function of \( M \) for \( s = 5 \). We see that \( |g_2/g_1| \) is always small, while \( |g_3/g_1| \) increases with increasing \( s \) and for \( M = 0.12 \) becomes larger than \( 1 \) above a certain \( s \). Once this happens, the gap on the \( b \)-pocket develops accidental nodes. Overall, these results show that the behavior that we obtained analytically at large \( s \) extends to all \( s \geq 1 \). For these values of \( s \), the gap along the \( b \)-pocket, which survives up to larger \( M \), necessarily develops accidental nodes around \( M = M_1 \), where the \( a \)-pocket disappears.

In Fig. 5 we show the variations of \( \Delta_\alpha(q) \) and \( \Delta_\beta(q) \) for two different values of \( s \), \( s = 1 \) and \( s = 5 \), and three different values of \( M \): \( M \ll M_1 \), \( M \ll M_1 \), \( M_1 < M < M_2 \). For \( s = 1 \), the gaps \( \Delta_\alpha \) and \( \Delta_\beta \) have no nodes for all \( M \), consistent with \( |g_3/g_1| < 1 \) in Fig. 3. The only effect of the disappearance of the \( a \)-pockets is the switch between the maxima and the minima of the gap function along the remaining \( b \)-pockets. This switch is a consequence of the sign change of both \( g_2/g_1 \) and \( g_3/g_1 \) (see Fig. 3).

The situation is different for larger \( s = 5 \). Even for small \( M \ll M_1 \), the gap variation along both \( a \)- and \( b \)-pockets become substantial. Once \( M \) becomes large enough to (almost) gap out the \( a \)-pockets, the nodes appear near the tips of the \( b \)-pocket bananas. This behavior is in full agreement with the analysis of \( g_3/g_1 \) earlier in this section.

We therefore conclude that the two conditions to obtain nodes in the superconducting gap in the coexistence phase with stripe antiferromagnetism are: (i) relatively large density-density interaction \( U_1 \) (leading to
magnetism) compared to the pair-hopping interaction $U_3$, and (ii) the disappearance of one of the two pairs of reconstructed Fermi pockets. Our analysis agrees with previous works that found fully gapped quasi-particle excitations in the coexistence state in the cases of small $M$ (Ref. 13) and $U$ (Ref. 13).

The other interactions, which we listed in Sec. II but did not include into $H_{\text{int}}$ in Eq. (11), modify the value of $s$ and therefore affect whether or not the nodes appear upon increasing $M$. In particular, the exchange term $\left[U_2 c_{\sigma}^+ f_{\sigma} f_{\sigma'} c_{\sigma'}\right]$ changes $s$ to $s = \frac{U_1 - U_2}{U_2}$. If $U_2$ is negative, it makes the development of nodes more likely, while if it is positive and smaller than $U_1$, it decreases $s$ and may eliminate the nodes. The inclusion of this term also opens up the somewhat exotic possibility of a negative $s$. Although negative $s$ is unlikely for FeSCs, we analyzed the $s < 0$ case for completeness and show the results in Fig. 6. We see that now the behavior is non-monotonic with $s$: the nodes appear at some intermediate $|s|$ and disappear at larger $|s|$, already at relatively small $M$, when all four pockets are present.

**Figure 6:** The gap structure for two values of $s = U_1/U_3$ and three different $M$ corresponding to the cases when the $a$ and $b$-pockets are of comparable size (left column), when the $a$ pockets are about to disappear (middle column), and when only the $b$-pockets are left (right column). Note the appearance of gap nodes for $s = 5$ and $M = 0.12$, when one pair of reconstructed pockets disappear.

**B. From intermediate to large $M$: nodal-nodeless transition**

In the previous section we showed that nodes appear at large enough $s$ at intermediate values of $M \sim M_1$, where one of the two pairs of reconstructed pockets (the $a$-pocket) disappears. We now analyze what happens when $M$ becomes larger than $M_2$ and $b$-pockets also disappear.

We find that the results depend on whether we restrict the pairing to the FS or include into the pairing problem also the states which are already gapped out by SDW. If we restrict the pairing to the FS, as we did before, we find that the nodes in $\Delta_0$ are present for all $M < M_2$. At $M = M_2$ the remaining FS disappears and $T_c$ vanishes. If we do not restrict the pairing to the FS and take into consideration the states already gapped by SDW, SC persists into the region $M > M_2$ (see Refs. 13, 16, 24). In this last region, all states are gapped already above $T_c$, and the opening of an additional SC gap only moves states further away from zero energy.
The thermal conductivity and the penetration depth in this last regime show exponential behavior, typical for a superconductor with a full gap. From this perspective, the evolution of the system response around \( M = M_2 \) mimics the transformation from a nodal to a nodeless gap, irrespective of whether the actual SC gap by itself evolves around \( M_2 \).

On a more careful look, we find that the gap \( \Delta_b \) does change its structure near \( M = M_2 \) and becomes nodeless. At around the same \( M \), \( T_c \) is likely non-monotonic and passes through a minimum.

The argument goes as follows. Consider first the pairing confined to the FS and solve for \( T_c \). Neglecting \( N_c \) and \( N_{cs} \) in Eq. (14), which are small for all \( M \) and \( s \), and solving for the linearized gap equation for \( L = \log \left( \frac{\Lambda}{T_c} \right) \), we obtain

\[
1 + \frac{U_3}{2} (sN_{ss} - N) L + \left( \frac{U_3}{2} \right)^2 s (N_s^2 - NN_{ss}) L^2 = 0
\]

(20)

One can verify that \( NN_{ss} - N_s^2 \) is positive for all \( M > M_2 \). For \( s \geq 1 \), when \( \Delta_b \) has nodes at \( M \sim M_1 \), we also have \( sN_{ss} > N \). In this situation, there is a single solution of (20) with \( L > 0 \):

\[
L = \frac{1}{sU_3(NN_{ss} - N_s^2)} \times \left( \sqrt{(sN_{ss} + N)^2 - 4sN_s^2} + (sN_{ss} - N) \right)
\]

(21)

As long as \( N_{s}^2 \ll NN_{ss}, L \approx 2/U_3N \), i.e. \( T_c \) does not depend strongly on \( M \). However, near \( M = M_2 \), \( \sin \theta \) on the \( b \)-pockets tends to a constant value \( c \), see Eq. (17), \( (c = -0.94 \text{ for our parameters}) \) and \( N_{s}^2 \) and \( NN_{ss} \) both tend to the same value \( c^2N^2 \). As a result, \( L \) diverges as \( (2/U_3N)(sN_{ss} - N)/(NN_{ss} - N_s^2) \), i.e. \( T_c \) vanishes (see Fig 8). One can straightforwardly show that \( L \) diverges at \( M = M_2 \) even if we solve the full \( 3 \times 3 \) linearized matrix gap equation, without neglecting the terms \( N_c \) and \( N_{cs} \).

In the same limit \( NN_{ss} \approx N_{s}^2 \), we also have

\[
\frac{g_1}{g_3} \sin 2\theta_q = -\frac{N_s}{NN_{ss}} \sin 2\theta_q \rightarrow -\frac{c}{c^2} = 1.
\]

(22)

We see that the angle-independent and the \( \sin 2\theta \) terms in \( \Delta_b \) become identical and cancel each other, i.e. \( \Delta_b \) vanishes (see Eq. (13)). This is consistent with the vanishing of \( T_c \).

These results, however, hold only as long as we restrict the integrals for \( N \), \( N_s \), and \( N_{ss} \) to the FS. Once we include the contributions from the gapped states, \( N_{s}^2 \) and \( NN_{ss} \) no longer tend to the same value. The contributions from the gapped states contain, instead of \( L = \log \Lambda/T_c \), the factor \( L_D = \log \Lambda/\sqrt{D^2 + T_c^2} \), where \( D \) is the gap in the absence of SC. In a generic weak coupling case, these contributions would be small in \( L_D/L \) compared to the contributions from the FS, but in our case the FS contribution to \( NN_{ss} - N_{s}^2 \) vanishes at \( M_2 \), and the contribution from the gapped states become the dominant ones. Similarly, contributions from the gapped state break the cancellation between the angle-independent and the \( \sin 2\theta \) terms in \( \Delta_b \).

Figure 7: Gap structure for negative \( s \) and \( M = 0.09 \). As \( |s| \) increases, the gap on the \( b \) pocket is pushed down, acquires the nodes, and eventually reverses sign and again becomes nodeless. For all three values of \( s \) the FS contains both \( a \) and \( b \) pockets.

Figure 8: a) Schematic behavior of \( T_c \) as a function of \( M \), if \( T_c \) is obtained by restricting to contributions only from the FSs. This \( T_c \) drops to zero as \( M \rightarrow M_2 \). The arrow indicates the direction along which \( M \) increases. b) The actual \( T_c \) (solid red line) is a superposition of contributions from the FS and from the gapped states (dashed lines). This \( T_c \) initially decreases when \( M \) approaches \( M_2 \) but reverses the trend and remains finite at \( M_2 \) and even in some range of \( M > M_2 \). The dip in \( T_c \) is in the region where the contributions from the FS and from the gapped states become comparable to each other. This non-monotonic behavior of \( T_c \) around \( M_2 \) is captured by Eq. (21) (see text).
gapped states to $N_i$, we have near $M = M_2$,
\[
\begin{align*}
N &= N_0 \left( 1 + \gamma \frac{L_D}{L} \right) \\
N_s &= N_0 \left( c_s - \gamma_s \frac{L_D}{L} \right) \\
N_{ss} &= N_0 \left( c_{ss} + \gamma_{ss} \frac{L_D}{L} \right) 
\end{align*}
\]
(23)
where $c_s \approx -c > 0$ and $c_{ss} \approx c^2$. One can introduce $\epsilon \equiv c_{ss} - c_s^2$ as a measure of deviation from $M_2$, with $\epsilon = 0$ at $M_2$. The constants $\gamma_s$ are all positive, at least if they predominantly come from gapped $a$-pockets. Since $\sin 2q$ along the $a$-pockets is quite close to $-1$ (see Fig. 3), we have $\gamma_s^2 = \gamma_{ss}$ to a good accuracy. Substituting these forms into the result for $L$, Eq. (21), and assuming for simplicity that $s \gg 1$, we obtain for small $\epsilon$
\[
\begin{align*}
\epsilon \tilde{L}^2 + K_1 \tilde{L} - K_2 &= 0 \\
K_1 &= (\gamma c^2 + \gamma_{ss} + 2\gamma_s |c|) \tilde{L}_D - c^2 \\
K_2 &= \gamma_{ss} \tilde{L}_D > 0
\end{align*}
\]
(24)
where $\tilde{L} = \tilde{L}(N_0U_3/2)$, $\tilde{L}_D = \tilde{L}(N_0U_3/2)$. If we set $\tilde{L}_D = 0$, i.e., neglect the contributions from the gapped states, $K_2 = 0$, $K_1 = -c^2$, and we obtain $\tilde{L} = c^2/\epsilon$, as before. If instead we set $\epsilon = 0$, we find that $\tilde{L}$ remains finite and positive, and to leading order in $\tilde{L}_D^2 < 1$ is given by
\[
\tilde{L}_{\epsilon=0} \approx \frac{\gamma_{ss}}{\gamma_{ss} + \gamma c^2 + 2\gamma_s |c|}
\]
(25)
The behavior of $\tilde{L}$ at small but finite $\epsilon$ is rather involved, but its main features can be understood directly from Eq. (24). Far from $M_2$, $T_c$ is not small, implying that $\tilde{L}_D$ is small, i.e. and $K_1 < 0$ and $K_2$ can be neglected. As a result, $\tilde{L}$ increases as $\tilde{L} \sim 1/\epsilon$, leading to a decrease in $T_c$ and, consequently, to an increase in $L_D$. With increasing $\tilde{L}_D$, $K_1$ crosses zero and changes sign. At this point, $K_2 = O(1)$, and $\tilde{L} \sim K_2/\sqrt{\epsilon} \sim 1/\sqrt{\epsilon}$, i.e., it still increases with decreasing $\epsilon$, but more slowly then before. Because $\tilde{L}_{\epsilon=0} \sim O(1)$ given by (25) is certainly smaller than $1/\sqrt{\epsilon}$, $\tilde{L}$ necessary passes through a maximum near the point where $K_1$ changes sign, and then decreases towards $\tilde{L}_{\epsilon=0}$. Accordingly, $T_c = \tilde{L}_{\epsilon=0} \sim \epsilon^{-2} U_3 N_0$ passes through a minimum at some $M \leq M_2$, as shown schematically in Fig. 8.

Contributions from the gapped states also affect the interplay between the constant and the $\sin 2\theta_q$ terms in $\Delta_b$. If we include these terms, we obtain, near $M = M_2$, instead of Eq. (22),
\[
\frac{g_3}{g_1} \sin 2\theta_q = \frac{1 + \gamma_{ss} \frac{L_D}{L}}{1 + \gamma_{ss} \frac{L_D}{L}}
\]
(26)
where, we remind, in our case $\gamma_{ss} > 0$ and $c = -0.94$. We see that the numerator gets smaller than one and the denominator gets larger than one. As a result, $\sin 2\theta_q$ becomes smaller than one, and the gap $\Delta_b$ recovers its nodeless form in the vicinity of $M_2$. Very likely $\Delta_b$ remains gapless also at larger $M$, before $T_c$ finally vanishes.

We also note that our consideration is self-consistent in the sense that the transformation from nodal to nodeless gap and non-vanishing of $T_c$ at $M = M_2$, where $\epsilon = 0$, are consistent with each other. Indeed, once the gap $\Delta_b$ becomes nodeless, the pairing problem at $M \geq M_2$ is qualitatively similar to the one for fermions with the dispersion
\[
\epsilon_{\text{gapped}}(k) = \sqrt{D^2 + [\mathbf{v}_F \cdot (k - k_F)]^2},
\]
(27)
where $D$ is the pre-existing gap in the excitation spectrum. For such model, $T_c$ is non-zero at $D = 0$ (equivalent of $M = M_2$) and extends into the region where $D \neq 0$:
\[
T_c(D) = T_c(0) F \left[ \frac{D}{T_c(0)} \right]
\]
(28)
where $F(x)$ is a decreasing function of $x$ subject to $F(x \to 0) = 1 - x^2$ and $F(x) = -1/\ln(1.76 - x)$ near the critical $x = 1.76$.

The outcome of this analysis is that nodal SC gap exists only in relatively narrow range of $M$, from $M \leq M_1$ to $M \leq M_2$. At larger $M$ the gap is again nodeless, and, furthermore, for $M > M_2$ all low-energy states are gapped already above $T_c$. The SC transition temperature $T_c$ is non-zero at $M_2$ and decreases into $M > M_2$ region. Before that, $T_c$ has a minimum roughly where the gap changes from nodal to nodeless.

C. Role of the non-reconstructed pockets

So far, we have restricted our consideration of the pairing problem to one hole and one electron pocket reconstructed by SDW. There are additional pockets which do not participate in the SDW state. In this subsection we analyze to what extent these additional pockets affect our results. Specifically, we add another elliptical electron pocket centered at $(\pi, 0)$, denote fermions near this pocket by $d_k$, and solve the set of coupled linearized gap equations for $\Delta_d(q)$, $\Delta_b(q)$, and $\Delta_a(q)$. The interactions involving fermions near $(0, 0)$ and near the electron pocket at $(\pi, 0)$ are the same as the interactions between $(0, 0)$ and $(0, \pi)$, i.e., the couplings are the same $U_1$, $U_2$, $U_3$, and $U_4$. As before, we consider the model with only $U_3$ and $U_1 = sU_3$. We also neglect direct interaction between electron pockets.

The effective pairing interactions in the $a - b - d$ space are obtained by dressing the interactions by coherence factors associated with the transformation from $c$ and $f$ to $a$ and $b$ operators for the pockets at $(0, 0)$ and at $(0, \pi)$. Since the expressions for the vertices are long, we
refrain from presenting them. Yet, it is straightforward to obtain and solve the set of linearized gap equations for \( \Delta_a(q) \), \( \Delta_b(q) \), and \( \Delta_b(q) \). It is essential that the "dressing" does not involve \( d^- \) -fermions, hence the interaction has no dependence on the angle along the \( d^- \) -FS. As a consequence, \( \Delta_a(q) \) remains angle-independent. This constant \( \Delta_a(q) \), however, affects the interplay between the angle-independent and the \( 2\theta_k^b \) and \( \sin 2\theta_k^b \) terms for \( \Delta_a(q) \) and \( \Delta_b(q) \).

In Fig. 7 we show the gap structure obtained from the solution of the pairing problem for \( s = 5 \) and the same three values of \( M \) as in Fig. 6. Comparing Figs. 6 and Fig. 7 we see that the non-reconstructed \( d^- \) -pocket has only a minor effect on the \( a - b \) gap structure – the gap on the \( b \) pocket still becomes more angle-dependent with increasing \( M \) and develops accidental nodes at \( M \geq M_1 \). In view of this result, it is very likely that the physics of gap variation with \( M \) is fully captured already within the two-band model of one hole and one electron pocket.

**IV. EFFECT OF AN EXTERNAL MAGNETIC FIELD**

We now consider how the SC gap structure is affected by an external magnetic field \( \mathbf{H} \). The specific goal is to verify whether the field increases or reduces the tendency towards the development of nodes in \( \Delta_b \). This issue is related to experiments on \((\text{Ba}_{1-x} \text{K}_x)\text{Fe}_2\text{As}_2\), particularly to recent measurements of thermal conductivity in a field\(^{22}\).

We direct the field along \( z \) and include into the Hamiltonian the Zeeman coupling between the field and the \( z \) -component of the spin of a fermion \( S_z = (1/2)\sigma_z^a \sigma_z^b \). For an isotropic system, the SDW vector \( \mathbf{M} \) is oriented transverse to \( \mathbf{H} \) and for definiteness we direct it along \( x \). At a finite \( H \), the system also develops a non-zero uniform magnetization, which rotates a spin at a given site towards the field and creates a canting two-sublattice structure, with orthogonal antiferromagnetic SDW component \( \mathbf{M} = M_x \) and ferromagnetic component along \( z \).

The quadratic part of the Hamiltonian in the presence of the field is

\[
H = \sum_{k, \sigma} \left[ \left( \epsilon^c_k + \sigma \hbar \right) c_{k \sigma}^\dagger c_{k \sigma} + \left( \epsilon^f_k + \sigma \hbar \right) f_{k+Q, \sigma}^\dagger f_{k+Q, \sigma} \right] + \sum_k M \left( c_{k \uparrow}^\dagger f_{k+Q, \downarrow} + c_{k \downarrow}^\dagger f_{k+Q, \uparrow} + \text{h.c.} \right)
\]

where \( \hbar = \mu_B H \). We will measure \( h \) in units of \( 2\mu \), as with other observables.

The transformation to the new operators is now given by

\[
\begin{pmatrix}
\sigma a_{k \sigma} \\
\delta a_{k \sigma}
\end{pmatrix} = \begin{pmatrix}
\pm \sigma \cos \theta_k & \pm \sin \theta_k \\
\pm \sin \theta_k & -\sigma \cos \theta_k
\end{pmatrix} \begin{pmatrix}
c_{k \sigma} \\
f_{k+Q, \sigma}
\end{pmatrix}
\]

where

\[
\cos \theta_k = \frac{M}{\sqrt{M^2 + (E_\sigma^-)^2}}
\]

\[
\sin \theta_k = \frac{E_\sigma^-}{\sqrt{M^2 + (E_\sigma^-)^2}}
\]

and

\[
E_\sigma^- = \sqrt{\left( \frac{\epsilon^c - \epsilon^f}{2} + \sigma \hbar \right)^2 - \left( \frac{\epsilon^c - \epsilon^f}{2} + \sigma \hbar \right)^2} + M^2
\]

Substituting the transformation into \( H_{\text{int}} \) in Eq. (11) and taking care of the fact that \( \cos \theta_{k, \uparrow} \) and \( \cos \theta_{k, \downarrow} \) are now different, we obtain Eq. (12) but with new \( U_{aa}^{ab} \), \( U_{ab}^{bb} \) and \( U_{ab}^{ab} \) given by

\[
U_{aa}^{ab} = U_{bb}^{ab} = -\frac{U_3}{2} \left( C_{k \uparrow} C_{-q \downarrow} S_{k \uparrow} S_{-q \downarrow} + S_{k \uparrow} S_{-q \downarrow} C_{k \uparrow} C_{-q \downarrow} \right) + U_1 C_{k \uparrow} C_{-q \downarrow} C_{k \uparrow} C_{-q \downarrow}
\]

\[
U_{ab}^{ab} = \frac{U_3}{2} \left( C_{k \uparrow} C_{-q \downarrow} S_{k \uparrow} S_{-q \downarrow} + S_{k \uparrow} S_{-q \downarrow} C_{k \uparrow} C_{-q \downarrow} \right) - U_1 C_{k \uparrow} C_{-q \downarrow} C_{k \uparrow} S_{q \downarrow}
\]

where \( C_{k, \sigma} \equiv \cos \theta_k \sigma \) and \( S_{k, \sigma} \equiv \sin \theta_k \sigma \). The gap structure consistent with these \( U_{ij}^{ab} \) is of the form

\[
\Delta_a(q) = g_1 C_{k \uparrow} C_{-q \downarrow} + g_2 S_{k \uparrow} S_{-q \downarrow} + g_3 C_{k \uparrow} S_{-q \downarrow} + g_4 S_{k \uparrow} C_{-q \downarrow}
\]

\[
\Delta_b(q) = g_1 C_{k \uparrow} C_{-q \downarrow} + g_2 S_{k \uparrow} S_{-q \downarrow} - g_3 C_{k \uparrow} S_{-q \downarrow} - g_4 S_{k \uparrow} C_{-q \downarrow}
\]

We constructed the set of coupled linearized gap equations for \( g_i \) by standard means and solved them for various \( M, s = U_1/U_3 \) and \( h \). We found that the field enhances the angle variation of the gaps \( \Delta_a \) and \( \Delta_b \), such that nodes appear at smaller \( M \) and smaller \( s \). To illustrate this, in Fig. 10 we compare the gap structure for a given \( M = 0.04 \) and \( s = 1 \) (same as in the upper left panel in Fig. 8) without a field and with a field. We clearly see that the gap variations along the reconstructed FSs grow with the field, and for large enough field nodes appear
well before $M$ reaches the value at which the $a-$pockets disappear. The implication is that the range where the gap has nodes widens up with the application of a field and, in particular, extends to smaller $M$ (larger dopings), where without a field the system was a superconductor with a nodeless gap.

We emphasize that the extension of the nodal region to smaller $M$ is a different effect than the field-induced changes in $M_1$ and $M_2$. These last changes are due to the modification of the quasiparticle dispersion, which in the presence of the field becomes 4-band dispersions and takes the form

$$E_{k}^{a,b} = \frac{\varepsilon_{c} + \varepsilon_{f}}{2} \pm \sqrt{\left(\frac{\varepsilon_{c} - \varepsilon_{f}}{2} \pm h\right)^2 + M^2}$$  \hspace{1cm} (33)

The two $a$-pockets are split by $\pm h$ into $a^+$ and $a^-$ pockets, and the two $b$-pockets are split by $\pm h$ into $b^+$ and $b^-$ pockets.

The presence of $\pm h$ under the square root splits $M_1$ and $M_2$ into $M_1^+, M_1^-$ and $M_2^+, M_2^-$, where $M_{1,2}^+ = M_{1,2} (1 + 2h)$ and $M_{1,2}^- = M_{1,2} (1 - 2h)$, with $M_1$ and $M_2$ given by Eqs (9) and (11) (we recall that $h$ is measured in units of $2\mu$). For $h = 0.08$ used in Fig. 11 we get $M_{1}^+ = 0.118, M_{1}^- = 0.086$ instead of $M_1 = 0.102$, and $M_{2}^+ = 0.205, M_{2}^- = 0.149$ instead of $M_2 = 0.177$. Clearly, the field-induced changes in the form of reconstructed FSs are small compared to the changes of the SC gap structure associated with the field-induced change of the pairing interaction. The small effect of the field on the reconstructed FS is in agreement with previous works on the cuprates.

V. DISCUSSION AND CONCLUDING REMARKS

We showed that the superconducting gap of an $s^+$ SC in the co-existence phase with an SDW order acquires additional angular dependence compared to the case when SDW order is not present. In particular, an isotropic pairing interaction becomes angle-dependent inside the SDW state. At small SDW order parameter $M$, or when the density-density interaction between the hole and electron pockets is small compared to pair hopping interaction term, this extra dependence is weak, in agreement with previous studies. However, when density-density and pair hopping interactions are comparable and $M$ is large enough to gap out one out of two pairs of reconstructed FS pockets, the angular dependence gets quite strong and the SC gap on the remaining FS banana-like pocket develops accidental nodes near the tips of the bananas. At even larger $M$, the system bounces back into the fully gapped SC state which extends into the regime where large enough $M$ gaps out the remaining pair of FS pockets. The SC transition temperature $T_c$ has a dip near the point where nodal SC state becomes nodeless. In the presence of a magnetic field, the width of the nodal region expands, and, in particular, the nodes appear at smaller $M$ (larger doping).

Our results offer a consistent explanation for the recent experimental observations in underdoped (Ba$_{1-x}$K$_x$)Fe$_2$As$_2$. Performing in-plane thermal conductivity measurements, Reid et al. found a small range of doping in the SC-SDW coexistence region where the ratio $\kappa/T$ is finite at $T = 0$, i.e., $\kappa$ is linear in $T$ at small $T$. This linearity is generally viewed as a strong indication for the presence of the nodes in the SC gap. At smaller and larger dopings Reid et al. found that $\kappa/T$ vanishes at $T = 0$, as it is expected for a fully gapped supercon-
ductors. They also observed that the doping range where $\kappa/|T|$ is finite at $T = 0$ expands upon the application of an external magnetic field.

In terms of our model, the nodeless-nodal-nodeless transition observed by Reid et al. can be attributed to the following sequence of events: at the optimal doping, which roughly coincides with the onset of the coexistence range, the SC gaps in (Ba$_{1-x}$K$_x$)$_2$Fe$_2$As$_2$ are almost isotropic, and the excitation spectrum is fully gapped. Decreasing $x$ and moving to the underdoped region in which SC and SDW start to coexist is equivalent to making $M$ nonzero in our model. In this situation, the gaps become anisotropic, but remain nodeless over some range of $M$. Moving deeper inside the coexistence region is equivalent to making $M$ larger. Eventually, $M$ becomes large enough such that one of the reconstructed pairs of pockets vanishes. In this situation, we found (for $s \geq 1$) that the gap on the remaining FS pockets develops nodes, what leads to a finite $\kappa/|T|$ at $T = 0$. As the doping level decreases even further, $M$ continues to increase and the gap structure bounces back to nodeless, since for such pairing state $T_c$ remains non-zero even when the remaining pair of reconstructed pockets disappears. The quasiparticle spectrum becomes fully gapped again, and $\kappa/|T|$ vanishes at $T = 0$.

In this explanation the nodeless-nodal transition is roughly associated with the vanishing of one pair of reconstructed FS pockets. An SDW-driven electronic transition has been observed by ARPES as well as by transport measurements in electron-doped Ba(Fe$_{1-x}$Co$_x$)$_2$As$_2$. In that case, however, the reconstructed pocket that disappears is a much smaller hole-like pocket associated with additional details of the band dispersion not captured by our simplified two-band model. It would be interesting to verify experimentally whether the transition from four to two pockets takes place in the hole-doped (Ba$_{1-x}$K$_x$)$_2$Fe$_2$As$_2$ materials, and what is their relationship to the SC gap structure. Such transitions should have distinct signatures not only in the band dispersions, but also in transport coefficients. Furthermore, as discussed in Refs. 27,28 the onset of accidental nodes should also affect the low-temperature behavior of several thermodynamic quantities, giving rise to peculiar scaling relations. Another result of our analysis is the observation of a dip in the doping dependence of $T_c$ near the doping where the system undergoes the nodal-nodeless transition.

Our model indeed does not include all aspects of the physics of the coexistence state. In particular, doping with holes or electrons changes the chemical potential, which was kept constant in our calculation. The feedback from this change affects the values $M_1$ and $M_2$ at which reconstructed FSs can be present even for zero doping. The angular dependence of the magnetic interaction may also preserve FS pockets even for large $M$. Still, we believe that the physics described by our model is quite generic and should hold for more realistic models.

Finally, we point out that the disappearance of two out of four pockets is not the only mechanism that can give rise to nodes in the coexistence state. When the exchange interaction between unreconstructed hole and electron pockets is strong enough, it can induce nodes even in the absence of any dramatic change of the reconstructed FSs. Conversely, there is also a region in parameter space (small $s$) in which the coexistence with SDW does not generate nodes in the SC gap.

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The actual situation may be more complex as at least in some orbital models SDW gap turns out to be angular dependent and vanishes along particular directions. Near these directions FS survives even when SDW order parameter is large.