RELATIVE HOMOLOGICAL ALGEBRA, EQUIVARIANT
DE RHAM COHOMOLOGY AND KOSZUL DUALITY

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Abstract. Let $G$ be a general (not necessarily finite dimensional compact) Lie group, let $\mathfrak{g}$ be its Lie algebra, let $C\mathfrak{g}$ be the cone on $\mathfrak{g}$ in the category of differential graded Lie algebras, and let $\mathcal{G}$ be the functor which assigns to a chain complex $V$ the $V$-valued total de Rham complex of $G$. We describe the $G$-equivariant de Rham cohomology in terms of a suitable relative differential graded Ext, defined on the appropriate category of $(G,C\mathfrak{g})$-modules. The meaning of “relative” is made precise via the dual standard construction associated with the monad involving the functor $\mathcal{G}$ and the associated forgetful functor. The corresponding infinitesimal equivariant cohomology is the relative differential Ext over $C\mathfrak{g}$ relative to $\mathfrak{g}$. The functor $\mathcal{G}$ decomposes into two functors, the functor which determines differentiable cohomology in the sense of Hochschild-Mostow and the functor which determines the infinitesimal equivariant theory, suitably interpreted. This functor decomposition, in turn, entails an extension of a Decomposition Lemma due to Bott. Appropriate models for the differential graded Ext involving a comparison between a suitably defined simplicial Weil coalgebra and the Weil coalgebra dual to the familiar ordinary Weil algebra yield small models for equivariant de Rham cohomology including the standard Weil and Cartan models for the special case where the group $G$ is compact and connected. Koszul duality in de Rham theory results from these considerations in a straightforward manner.

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Introduction

The main result of this paper describes equivariant de Rham theory, in the spirit of Eilenberg and Moore, in terms of a differential graded Ext, defined on an appropriate category: Let $G$ be a Lie group, let $\mathfrak{g}$ be its Lie algebra, let $C\mathfrak{g}$ be the cone on $\mathfrak{g}$ in the category of differential graded Lie algebras, let $\text{Mod}_{(G,C\mathfrak{g})}$ be the category of right $(G,C\mathfrak{g})$-modules where the $G$- and $(C\mathfrak{g})$-actions intertwine in the obvious way, and let $\mathcal{C}$ be the category of chain complexes. Given the $(G,C\mathfrak{g})$-module $W$, we define $\text{Ext}_{((G,C\mathfrak{g});\mathcal{C})}(W, \cdot)$ to be the right derived functor (we view the collection of the various Ext as a single functor) of the functor $\text{Mod}_{(G,C\mathfrak{g})} \to \mathcal{C}$ which assigns
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to the \((G, Cg)\)-module \(V\) the chain complex

\[ \text{Hom}(W, V)^{(G, Cg)} = \text{Hom}_{(G, Cg)}(\mathbb{R}, \text{Hom}(W, V)) \]

of invariants, the real numbers \(\mathbb{R}\) being viewed as a trivial \((G, Cg)\)-module in the obvious way; here the convention is to write \(\text{Ext}^\ast((G, Cg); C)\) rather than \(\text{Ext}^\ast((G, Cg); C)\), the term right derived is interpreted in a suitable relative sense, the term “relative” being made precise by means of the notions of monad and dual standard construction. The requisite categorical language was developed by S. Mac Lane, indeed, the underlying ideas go back to [44] (§3). Thus, let \(\mathcal{G}\) be the functor which assigns to a chain complex \(V\) the familiar \(V\)-valued (totalized) de Rham complex of \(G\). We define the differential graded \(\text{Ext}((G, Cg); C)(\cdot, \cdot)\) via the dual standard construction associated with the monad involving the functor \(\mathcal{G}\) and the corresponding forgetful functor. The de Rham algebra \(\mathcal{A}(X)\) of any smooth \(G\)-manifold \(X\) inherits a \((G, Cg)\)-module structure in an obvious manner via the \(G\)-action and the operations of contraction and Lie derivative. Theorem 2.7.1 below includes the statement that the \(G\)-equivariant de Rham cohomology of a \(G\)-manifold \(X\) is given by the differential graded \(\text{Ext}((G, Cg); C)(\mathbb{R}, \mathcal{A}(X))\). The infinitesimal version of this differential Ext is a suitably defined relative differential \(\text{Ext}((Cg, g); C)(\cdot, \cdot)\) over \(Cg\) relative to \(g\). In (2.3) below, we introduce this infinitesimal theory accordingly via the appropriate monad, and in (2.8) below we spell out the corresponding comonad. Occasionally we refer to \(\text{Ext}((Cg, g); C)(\mathbb{R}, \cdot)\) as the infinitesimal equivariant cohomology (relative to \(G\) or relative to \(g\)).

The exploration of the infinitesimal theory involves notions of Weil coalgebra and simplicial Weil coalgebra associated with a Lie algebra. The Weil coalgebra of the Lie algebra \(g\) is dual to the familiar Weil algebra of \(g\); this coalgebra arises as the ordinary differential graded Cartan-Chevalley-Eilenberg (CCE) coalgebra \(\Lambda'_g[sCg]\) of \(Cg\). It is well known that the functor which assigns to a vector space \(V\) the space of smooth \(V\)-valued maps on \(G\) and the associated forgetful functor combine to a monad which defines, via the corresponding dual standard construction, the differentiable cohomology of \(G\) in the sense of Hochschild-Mostow. For our purposes, a crucial observation is then to the effect that the functor \(\mathcal{G}\) can be written as the composite of the functor which determines differentiable cohomology with the functor which determines the infinitesimal equivariant theory, suitably interpreted. This functor decomposition, in turn, leads to an extension, given as Theorem 2.6.1 below, of Bott’s Decomposition Lemma [4]. Appropriate models for the relative differential graded Ext involving a comparison between the simplicial Weil coalgebra and the Weil coalgebra similar to the classical comparison between the CCE resolution and the bar complex yield small models for equivariant de Rham cohomology including the familiar Weil and Cartan models for the special case where the group is compact and connected. Koszul duality in equivariant de Rham theory then results from these considerations in a straightforward manner. The present paper generalizes in particular a result of Bott’s [4] relating the Chern-Weil construction with differentiable cohomology via a certain spectral sequence. Indeed, our approach recovers equivariant de Rham cohomology in terms of a suitable higher homotopies construction having a spectral sequence of the kind considered by Bott as an invariant and thereby yields in particular, at least in principle, complete information about the higher differentials in Bott’s spectral sequence. See Remark 5.1.17 below for details.

Equivariant cohomology is usually defined by means of the Borel construction. In the de Rham setting, given the smooth \(G\)-manifold \(X\), the appropriate way to
realize this construction is to apply the de Rham functor \( \mathcal{A} \) to the simplicial Borel construction \( N(G, X) \) so that the cosimplicial differential graded algebra \( \mathcal{A}(N(G, X)) \) results; totalization and normalization then yield the chain complex \( |\mathcal{A}(N(G, X))| \) defining the \( G \)-equivariant de Rham theory of \( X \) \([4], [5], [53]\). For the special case where the Lie group \( G \) is compact, older constructions of \( G \)-equivariant cohomology in the literature proceed via the Weil and Cartan models. According to folk lore, the resulting equivariant cohomology is the same as that coming from the Borel construction in that particular case; indeed, in the literature, there are various comparison maps between the Cartan and Weil models and the Borel construction. These comparison maps establish the equivalence between the various theories in the compact case but do not explain why these theories are then equivalent. Our description of equivariant cohomology in terms of the aforementioned differential Ext entails an explanation of the relationship between the Cartan and Weil models and the Borel construction in a conceptual manner: this relationship results as a comparison map for various objects calculating the same derived functor. The ordinary Weil coalgebra then leads to what we refer to as the Weil and Cartan models for the relative differential graded Ext under discussion. When the group \( G \) is compact, any differentiable \( G \)-module is differentiably injective, and the differentiable cohomology is non-zero only in degree zero and boils down to the \( G \)-invariants whence, in view of the aforementioned functor decomposition, the Weyl and Cartan models indeed calculate the \( G \)-equivariant cohomology.

In Section 3 below we shall explore the infinitesimal equivariant cohomology per se. In \([22]\), for a pair \((\mathfrak{a}, \mathfrak{b})\) of ordinary Lie algebras, Hochschild has introduced an acyclic relatively projective CCE complex which yields the relative Lie algebra cohomology of the pair \((\mathfrak{a}, \mathfrak{b})\) in the sense of Chevalley-Eilenberg \([10]\). This CCE complex arises by abstraction from the situation of the invariant de Rham complex of a homogeneous space of compact connected Lie groups. We shall show that the literal translation of that CCE construction, to the pair \((C\mathfrak{g}, \mathfrak{g})\) of differential graded Lie algebras, yields the Weil coalgebra; see Proposition 3.1.7 below. In the situation where the smaller Lie algebra is reductive in the ambient one, Hochschild’s chain complex is actually a relatively projective resolution of the ground ring \([22]\). In our case, a similar result holds. To clarify the situation, extending the idea of a construction which goes back to H. Cartan \([8]\) (exposé 3), in Section 3 below, we introduce the notion of relative construction. Proposition 3.1.7 actually says that the Weil coalgebra \( W'[\mathfrak{g}] \) is a construction for \( U[C\mathfrak{g}] \) relative to \( U[\mathfrak{g}] \) that is \( R \)-acyclic, even \( R \)-contractible. In Theorem 3.4.1 we then show that a result similar to that of Hochschild’s quoted above holds: when \( \mathfrak{g} \) is reductive, the Weil coalgebra \( W'[\mathfrak{g}] \) admits a \( \mathfrak{g} \)-equivariant contracting homotopy.

In the paper \([4]\), Bott communicates a formula which he indicates was inspired by some work of Hochschild, one of the creators of relative homological algebra. Thus our approach explains in particular equivariant cohomology in terms of relative homological algebra and thus closes, perhaps, a circle of ideas. In Section 4 we shall introduce the already mentioned simplicial Weil coalgebra. We will then sometimes refer to the Weil coalgebra as the ordinary Weil coalgebra, in particular when there is a need to distinguish it from the simplicial Weil coalgebra. The ordinary Weil coalgebra leads to a small object calculating the (relative cohomology which yields the) equivariant cohomology. The canonical comparison between the (normalized
chain complex of the) simplicial Weil coalgebra and the ordinary Weil coalgebra, cf. Corollary 4.7 below, then induces a comparison between the object defining equivariant cohomology and a small object calculating this cohomology, interpreted as the relative derived functor in the sense explained before; when a compact connected Lie group is behind, as already hinted at above, this procedure leads eventually to the Weil and Cartan models. Thereby the canonical comparison between the Weil coalgebra and the simplicial Weil coalgebra is formally exactly of the same kind as the classical comparison, spelled out in detail in [9] (chap. XIII), between the CCE complex for an ordinary Lie algebra and the bar complex for its universal enveloping algebra. This relies on the fact, to be established in Theorem 4.5 below, that the (normalized chain complex of the) simplicial Weil coalgebra of the Lie algebra $\mathfrak{g}$ is precisely the (homogeneous form of the) relative bar resolution for the pair $(\mathcal{C}_{\mathfrak{g}} \mathfrak{g}, \mathfrak{g})$.

In Section 5, by means of various HPT techniques which, in [34], we have used to construct small models for ordinary singular equivariant (co)homology, we shall cut to size the defining objects for the various derived functors under discussion. Using the small objects we shall then show in Section 6 that, for a finite dimensional compact group, the ordinary Weil and Cartan models for equivariant cohomology result as special cases. In particular, for a compact group $G$, the $G$-equivariant de Rham cohomology of $X$ is given by the invariants of the relative differential graded Ext$_{(\mathcal{C}_{\mathfrak{g}} \mathfrak{g})(\mathbb{R}, A(X))}$ with respect to the group $\pi_0(G)$ of connected components of $G$, and the standard object calculating this differential graded Ext contracts onto the Cartan model. Pushing the HPT-procedure a bit further, we obtain another (familiar) model which is even smaller than the Cartan model for equivariant cohomology.

In Section 7 we shall exploit the models constructed in the present paper to introduce, via the procedure explained at the end of [34], a certain algebraic duality involving the object which defines the differential graded Ext; what is referred to in the literature as Koszul duality, cf. e. g. [18], is an immediate consequence thereof. This yields a conceptual explanation of Koszul duality for de Rham theory in terms of the extended functoriality of the relevant differential derived functors and places this kind of Koszul duality in the sh-context. The idea behind this extended functoriality goes back to [55] and was pushed further in [21]. For our purposes, the categories of sh-modules and sh-comodules serve as replacements for various derived categories exploited in [18] and elsewhere. In particular, when a Lie group $G$ acts on a smooth manifold $X$, even when the induced action of $H^*_G$ on $H^*_X$ lifts to an action on $A(X)$, in general only an sh-action of $H^*_G$ on $A(X)$ will recover the geometry of the original action.

Given a topological group $G$, in ordinary (singular) (co)homology, the $G$-equivariant (co)homology can be described via suitable differential Tor- and Ext-functors in the sense of Eilenberg and Moore over the chain algebra $C_\ast G$. When $G$ is an algebraic group, from the group multiplication, the algebraic de Rham algebra of $G$ inherits a differential graded coalgebra, in fact Hopf algebra structure, and the $G$-equivariant de Rham theory of a nonsingular algebraic variety is then given by a differential graded Cotor with respect to this differential graded coalgebra structure; a similar observation leads to a description of rational cohomology of algebraic groups. In the smooth setting, such a description is of no avail since the smooth de Rham algebra on a Lie group $G$ does not inherit a diagonal map (in the usual algebraic sense) turning the de Rham algebra into a Hopf algebra. Our approach in terms of the
relative differential Ext explained above entails that a replacement for the non-existent category of comodules over the de Rham complex of $G$ is provided by the category of $(G, C_G)$-modules. Thus our description of equivariant de Rham cohomology in terms of a differential Ext can be seen as a result of the Eilenberg-Moore type. To complete the story we shall show, in (2.10) and (2.11) below, that the rational cohomology of algebraic groups and the algebraic equivariant de Rham theory of nonsingular algebraic varieties relative to an algebraic group can likewise be subsumed under the formalism of monads and dual standard constructions.

We view the present paper as belonging to a certain differential homological algebra tradition which started with Eilenberg-Mac Lane and H. Cartan and was developed further by J. Moore and his school. Within this tradition, the theory takes care of itself and formulas drop out more or less automatically. A typical example is the notion of twisting cochain; once isolated, it explains, in a conceptual way, all sorts of perturbed operators and explicit formulas can then always be derived from structural insight. For example, given an ordinary Lie algebra $\mathfrak{g}$, in terms of (i) the exterior coalgebra $\Lambda'_{\partial}[\mathfrak{s}\mathfrak{g}]$ on the suspension $\mathfrak{s}\mathfrak{g}$ of $\mathfrak{g}$, endowed with the differential determined by the Lie bracket on $\mathfrak{g}$ and of (ii) the universal algebra $U[\mathfrak{g}]$ of $\mathfrak{g}$, the CCE resolution can be written in the form $\Lambda'_{\partial}[\mathfrak{s}\mathfrak{g}] \otimes_{\tau_{\mathfrak{g}}} U[\mathfrak{g}]$ with respect to the corresponding universal twisting cochain $\tau_{\mathfrak{g}}$ from $\Lambda'_{\partial}[\mathfrak{s}\mathfrak{g}]$ to $U[\mathfrak{g}]$; see (1.2.1) below. Another typical example is the idea of a (co)monad. Yet another example is given by the description of the formalism of contraction and Lie derivative in terms of an action of the cone on the corresponding Lie algebra; we shall heavily use this observation in the paper. The familiar Gerstenhaber algebra structure on the CCE complex calculating the homology of a Lie algebra actually amounts to a module structure over the cone on that Lie algebra; see (1.3) below. Despite its flexibility and vast range of possible applications, this differential homological algebra technology has so far hardly been used in differential geometry.

The reader is assumed to be familiar with the notation, terminology, and preliminary material in [34]; this material will not be repeated here. In particular, we will use the HPT-techniques explained in [34] without further explanation. As usual, the group of connected components of a topological group $G$ is written as $\pi_0(G)$. The ground ring is denoted by $R$ and the (real) de Rham functor by $\mathcal{A}$. We treat chain complexes and cochain complexes on equal footing: We consider a cochain complex $(C^*, d)$ as a chain complex $(C_*, d)$ by letting $C_j = C^{-j}$ for $j \in \mathbb{Z}$. An ordinary cochain complex, concentrated in non-negative degrees as a cochain complex, is then a chain complex which is concentrated in non-positive degrees. The identity morphism of an object will occasionally be denoted by the same symbol as that object and the operation of suspension will be written as $s$. For any smooth manifold $N$, we write the tangent bundle as $T N \to N$.

I am much indebted to J. Stasheff for a number of comments on various drafts of the manuscript. I had posted an earlier version to the arxiv under [math.DG/0401161]. Since then, the article [2] has appeared, posted to the arxiv as [math.DG/0406350]. That article contains, for the special case where the group $G$ under discussion is compact, material related to Subsection 3.6, cf. Remark 3.6.10, and to Section 6 below. Publication of our paper has been delayed for personal (non-mathematical) reasons.

The results presented here can be generalized to equivariant Lie-Rinehart coho-
mology arising from a group acting on an arbitrary Lie-Rinehart algebra. This is interesting not only in its own right since this kind of equivariant Lie-Rinehart cohomology arises in arithmetic geometry, equivariant sheaf theory and algebraic K-theory. In a different direction, the theory can, perhaps, be extended to cover equivariant cohomology relative to actions of Lie groupoids rather than just Lie groups. We hope to return to these issues elsewhere.

1. The CCE complex of a Lie algebra

For later reference we describe various pieces of structure on the CCE complex of a Lie algebra which are most easily explained in terms of twisting cochains.

Let \( \mathfrak{g} \) be an \( R \)-Lie algebra which we suppose to be free or at least projective as an \( R \)-module when \( R \) is not a field, so that the CCE complex then has the desired features, cf. [3]. Here is an example of the kind of Lie algebra we have in mind: Let \( G \) be a Lie group, with Lie algebra \( \mathfrak{g} \), let \( \xi: P \to M \) be a principal \( G \)-bundle, and let \( R = C^\infty(M) \), the algebra of smooth functions on \( M \). Then the space of sections \( \mathfrak{g}(\xi) \) of the adjoint bundle \( \mathfrak{g} \times_G P \to M \) acquires in an obvious way an \( R \)-Lie algebra structure. As an \( R \)-module, \( \mathfrak{g}(\xi) \) is projective. This example justifies building the theory over a ground ring more general than a field.

1.1. The cone construction. The cone \( C\mathfrak{h} \) on \( \mathfrak{h} \) in the category of differential graded Lie algebras is the contractible differential graded Lie algebra \( C\mathfrak{h} \) characterized as follows: \( (C\mathfrak{h})_0 = \mathfrak{h}, (C\mathfrak{h})_1 = s\mathfrak{h} \), the differential \( d: (C\mathfrak{h})_1 \to (C\mathfrak{h})_0 \) is determined by the identity \( ds = \mathfrak{h} \) (\( = \text{Id}_\mathfrak{h} \)), the degree 1 constituent \( s\mathfrak{h} \) is abelian, and the action of \( \mathfrak{h} \) on \( s\mathfrak{h} \) is induced from the adjoint action. Thus, as a graded Lie algebra (i.e. when the differential is ignored), \( C\mathfrak{h} \) can be written as the semi-direct product \( C\mathfrak{h} = s\mathfrak{h} \rtimes \mathfrak{h} \). The universal enveloping algebra \( U(C\mathfrak{h}) \) of \( C\mathfrak{h} \) is contractible. As a graded algebra, \( U(C\mathfrak{h}) \) decomposes as a crossed product algebra \( \Lambda[s\mathfrak{h}] \odot U[\mathfrak{h}] \) relative to the obvious action of the Hopf algebra \( U[\mathfrak{h}] \) on \( \Lambda[s\mathfrak{h}] \). In particular, \( \Lambda[s\mathfrak{h}] \) embeds into \( U(C\mathfrak{h}) \) as a graded subalgebra and \( U[\mathfrak{h}] \) embeds into \( U(C\mathfrak{h}) \) as a differential graded subalgebra.

Occasionally we will also use the cone \( \overline{C\mathfrak{h}} \) whose underlying graded \( R \)-module is the same as that of \( C\mathfrak{h} \) but whose differential is the negative of the differential of \( C\mathfrak{h} \). The obvious map which is the identity in degree zero and multiplication by \(-1\) in degree 1 plainly identifies the two cones as differential graded Lie algebras.

Let \( V \) be a projective graded \( R \)-module, concentrated in odd degrees, and consider the graded exterior algebra \( \Lambda[V] \) on \( V \). The diagonal map \( V \to V \oplus V \) is well known to induce a diagonal map for \( \Lambda[V] \) turning the latter into a graded Hopf algebra. We then denote the resulting graded coalgebra by \( \Lambda'[V] \) and, as usual, refer to it as the exterior coalgebra. Whenever a graded exterior coalgebra of the kind \( \Lambda'[V] \) is under discussion, we will suppose throughout that the resulting coalgebra is the graded symmetric coalgebra \( S'[V] \) on \( V \), that is, that the canonical morphism of coalgebras from \( \Lambda'[V] \) to \( S'[V] \) (induced by the canonical projection from \( \Lambda'[V] \) to \( V \)) is an isomorphism of graded coalgebras. This excludes the prime 2 being a zero divisor in the ground ring \( R \). In particular, a field of characteristic 2 is not admitted as ground ring.

1.2. The CCE complex. The algebra \( U[C\mathfrak{h}] \) has the CCE resolution \( K(\mathfrak{h}) \) of \( R \) (in the category of left \( U[\mathfrak{h}] \)-modules and, suitably modified, in that of right \( U[\mathfrak{h}] \)-modules, see (1.3) below) as its underlying differential graded \( U[\mathfrak{h}] \)-module, cf. [9] (Ex. XIII.14 where the ground ring is written as \( K \)), and we will identify \( U[C\mathfrak{h}] \)
and \( K(\mathfrak{h}) \) in notation. In particular, as a graded coalgebra, \( K(\mathfrak{h}) \) amounts to the tensor product \( \Lambda'[\mathfrak{sh}] \otimes S_\Delta[\mathfrak{h}] \) of the graded exterior coalgebra \( \Lambda'[\mathfrak{sh}] \) on \( \mathfrak{sh} \) with the (cocommutative) coalgebra \( S_\Delta[\mathfrak{h}] \) underlying the obvious Hopf algebra structure on the symmetric algebra \( S[\mathfrak{h}] \) on \( \mathfrak{h} \), with the tensor product diagonal. When \( R \) contains the rational numbers as a subring, \( K(\mathfrak{h}) \) is actually a primitively generated differential graded Hopf algebra having \( Ch \) as its space of primitives; furthermore, as a graded coalgebra, the constituent \( S_\Delta[\mathfrak{h}] \) is isomorphic to the symmetric coalgebra \( S'[\mathfrak{h}] \) on \( \mathfrak{h} \), and \( K(\mathfrak{h}) \) is the graded symmetric coalgebra cogenerated by \( Ch \).

The quotient \( K(\mathfrak{h}) \otimes_{U[\mathfrak{h}]} R \) calculates the Lie algebra homology of \( \mathfrak{h} \). This quotient inherits, furthermore, a differential graded coalgebra structure having the ordinary exterior coalgebra \( \Lambda'[\mathfrak{sh}] \) on \( \mathfrak{sh} \) as its underlying graded coalgebra and having as differential the coderivation \( \partial \) corresponding to the Lie bracket of \( \mathfrak{h} \); we denote this differential graded coalgebra by \( \Lambda'_\partial[\mathfrak{sh}] \) and refer to it as the CCE coalgebra of \( \mathfrak{h} \). With respect to the coaugmentation filtration, \( \partial \) is a perturbation of the trivial differential. As a differential graded left \( (\Lambda'_\partial[\mathfrak{sh}]) \)-comodule and right \( (U[\mathfrak{h}]) \)-module, \( K(\mathfrak{h}) \) is isomorphic to the twisted tensor product

\[
\Lambda'_\partial[\mathfrak{sh}] \otimes_{\tau_\mathfrak{h}} U[\mathfrak{h}]
\]

where \( \tau_\mathfrak{h}: \Lambda'[\mathfrak{sh}] \to U[\mathfrak{h}] \) is the twisting cochain induced by the differential in \( Ch \) and in this manner \( K(\mathfrak{h}) \) appears as a free resolution of \( R \) in the category of right \( (U[\mathfrak{h}]) \)-modules. It may, of course, also be rewritten as a free resolution of \( R \) in the category of left \( (U[\mathfrak{h}]) \)-modules.

For later use we recall some of the technical details for the case of a general differential graded Lie algebra where, for simplicity, we suppose that the prime 2 is invertible in the ground ring. This is all we need since later in the paper we shall exclusively work over the reals; see [35] and [36] for the general case: Let \( C \) be a coaugmented differential graded cocommutative coalgebra and \( \mathfrak{g} \) a differential graded Lie algebra which we suppose to be projective as a graded \( R \)-module. We denote the differential of \( C \) and that of \( \mathfrak{g} \) by \( d \). Since \( \mathfrak{g} \) is \( R \)-projective, the symmetric coalgebra \( S'[\mathfrak{s}g] \) on the suspension \( \mathfrak{s}g \) exists; indeed, this is the cofree coaugmented differential graded cocommutative coalgebra on \( \mathfrak{s}g \). Let \( \tau_\mathfrak{g}: S'[\mathfrak{s}g] \to \mathfrak{g} \) be the homogeneous degree \(-1\) morphism (of the underlying graded \( R \)-modules) which is the desuspension \( S'_1[\mathfrak{s}g] = \mathfrak{s}g \to \mathfrak{g} \) from the homogeneous degree 1 constituent of \( S'[\mathfrak{s}g] \) to \( \mathfrak{g} \) and which is zero on the higher degree constituents of \( S'[\mathfrak{s}g] \). Given homogeneous morphisms \( a, b: C \to \mathfrak{g} \), with a slight abuse of the bracket notation \([\cdot, \cdot]\), their cup bracket \([a, b]\) is given by the composite

\[
C \overset{\Lambda}{\to} C \otimes C \overset{a \otimes b}{\longrightarrow} \mathfrak{g} \otimes \mathfrak{g} \overset{[\cdot, \cdot]}{\longrightarrow} \mathfrak{g}.
\]

The cup bracket \([\cdot, \cdot]\) turns \( \text{Hom}(C, \mathfrak{g}) \) into a differential graded Lie algebra. Define the coderivation

\[
\partial: S'[\mathfrak{s}g] \to S'[\mathfrak{s}g]
\]

on \( S'[\mathfrak{s}g] \) by the requirement that the identity

\[
\tau_\mathfrak{g}\partial = \frac{1}{2}[\tau_\mathfrak{g}, \tau_\mathfrak{g}]: S'_2[\mathfrak{s}g] \to \mathfrak{g}
\]
hold in \(\text{Hom}(S'[\mathfrak{s}g], \mathfrak{g})\). Then \(D\partial = (d\partial + \partial d) = 0\) since the Lie algebra structure on \(\mathfrak{g}\) is supposed to be compatible with the differential on \(\mathfrak{g}\). Moreover, the property that the bracket \([\cdot, \cdot]\) on \(\mathfrak{g}\) satisfies the graded Jacobi identity is equivalent to the vanishing of \(\partial\partial\), that is, to \(\partial\) being a coalgebra perturbation of the differential \(d\) on \(S'[\mathfrak{s}g]\), cf. [35], [38]. The resulting differential graded coalgebra \(S'_0[\mathfrak{s}g]\) is the CCE or classifying coalgebra for \(\mathfrak{g}\); cf. e. g. [50] (p. 291) for the case where \(R\) is the field of rational numbers. When the prime 2 is not invertible in the ground ring, by means of suitable squaring operations on \(\mathfrak{g}\) and \(\text{Hom}(S'[\mathfrak{s}g], \mathfrak{g})\), the theory can still be set up but we spare the reader and ourselves these added troubles here; see e. g. [35].

A Lie algebra twisting cochain \(t: C \rightarrow \mathfrak{g}\) is a homogeneous morphism of degree \(-1\) whose composite with the coaugmentation map of \(C\) is zero and which satisfies the equation

\[
Dt = \frac{1}{2}[t, t],
\]

cf. [49], [50], referred to nowadays in the literature as deformation equation or master equation. When the canonical morphism from \(\mathfrak{g}\) to \(U[\mathfrak{g}]\) is injective, the homogeneous degree \(-1\) morphism \(t: C \rightarrow \mathfrak{g}\) is a Lie algebra twisting cochain if and only if the composite of \(t\) with the injection into \(U[\mathfrak{g}]\) is an ordinary twisting cochain. In particular, \(\tau_0: S'_0[\mathfrak{s}g] \rightarrow \mathfrak{g}\) is a Lie algebra twisting cochain. When \(\mathfrak{h}\) is an ordinary Lie algebra (concentrated in degree zero), \(S'_0[\mathfrak{sh}]\) comes down to the ordinary CCE coalgebra of \(\mathfrak{h}\) and, maintaining notation established earlier, we write \(\Lambda'_0[\mathfrak{sh}]\) for the CCE coalgebra. To illustrate our sign conventions we note that, given \(x_1, x_2 \in \mathfrak{h}\),

\[
\frac{1}{2}[\tau_0, \tau_0](sx_1sx_2) = -[\tau_0(sx_1), \tau_0(sx_2)] = [x_2, x_1]
\]

whence \(\partial(sx_1sx_2) = [x_2, x_1]\) etc.

For intelligibility we recall the notion of twisted Hom-object, cf. [34] (2.4.1). Let \(A\) be an augmented differential graded algebra, \(C\) a coaugmented differential graded coalgebra, and \(\tau: C \rightarrow A\) a twisting cochain. Given a differential graded right \(A\)-module \(N\) let \(\delta^\tau\) be the operator on \(\text{Hom}(C, N)\) given, for homogeneous \(f\), by \(\delta^\tau(f) = (-1)^{|f|}f \cup \tau\). With reference to the filtration induced by the coaugmentation filtration of \(C\), the operator \(\delta^\tau\) is a perturbation of the differential \(d\) on \(\text{Hom}(C, N)\), and we write the perturbed differential on \(\text{Hom}(C, N)\) as \(d^\tau = d + \delta^\tau\). Likewise, given a differential graded left \(A\)-module \(M\), the operator \(-\tau \cup \cdot\) on \(\text{Hom}(C, M)\) is a perturbation of the differential \(d\) on \(\text{Hom}(C, M)\), and we write the perturbed differential on \(\text{Hom}(C, M)\) as \(d^\tau = d - \tau \cup \cdot\). We refer to \(\text{Hom}^\tau(C, N) = (\text{Hom}(C, N), d^\tau)\) and \(\text{Hom}^\tau(C, M) = (\text{Hom}(C, M), d^\tau)\) as twisted Hom-objects, cf. [34].

With this preparation out of the way, let \(N\) be a right \(\mathfrak{h}\)-module and \(M\) a left \(\mathfrak{h}\)-module. The cohomology of \(\mathfrak{h}\) with values in \(N (M)\) is calculated as the homology of the chain complex \(\text{Hom}_{U[\mathfrak{h}]}(K(\mathfrak{h}), N) \rightarrow \text{Hom}_{U[\mathfrak{h}]}(K(\mathfrak{h}), M)\), the (differential graded) subspace of \(\text{Hom}(K(\mathfrak{h}), N) \rightarrow \text{Hom}(K(\mathfrak{h}), M)\) which consists of \((U[\mathfrak{h}])\)-linear morphisms from \(K(\mathfrak{h})\) to \(N\) (to \(M\)). The assignment to \(\alpha \in \text{Hom}(\Lambda'_0[\mathfrak{sh}], N)\) of

\[
(1.2.2) \quad \Phi_\alpha: \Lambda'_0[\mathfrak{sh}] \otimes_{\tau_0} U[\mathfrak{h}] \rightarrow N, \quad \Phi_\alpha(w \otimes a) = \alpha(w)a, \quad w \in \Lambda'_0[\mathfrak{sh}], \quad a \in U[\mathfrak{h}],
\]

yields an injective chain map

\[
(1.2.3) \quad \text{Hom}^\tau_0(\Lambda'_0[\mathfrak{sh}], N) \rightarrow \text{Hom}(K(\mathfrak{h}), N)
\]
which identifies the twisted Hom-object $\text{Hom}^{\tau_b}(\Lambda'_\partial[s\h], N)$ with $\text{Hom}_{U[\h]}(K(\h), N)$. The same kind of association identifies the twisted Hom-object $\text{Hom}^{\tau_b}(\Lambda'_\partial[s\h], M)$ with $\text{Hom}_{U[\h]}(K(\h), M)$. The chain complex $\text{Alt}(\h, N)$ of $N$-valued alternating forms on $\h$ with the CCE differential is exactly the source $\text{Hom}^{\tau_b}(\Lambda'_\partial[s\h], N)$ of (1.2.3). For $N = R$, we refer to the differential graded algebra $\text{Alt}(\h, R)$ of $R$-valued alternating forms on $\h$ as the CCE algebra of $\h$ or, following [57], as the Maurer-Cartan algebra of $\h$.

1.3. The $(C\h)$-module structures. The Lie algebra $\h$ acts on the CCE coalgebra $\Lambda'_\partial[s\h]$ of $\h$ via the action induced by the adjoint action of $\h$ on itself. This action is well known to be trivial on (co)homology. For later reference, we will now refine this observation.

There is a canonical isomorphism

$$U[C\h] \cong \Lambda'_\partial[s\h] \otimes_{\tau_b} U[\h]$$

of differential graded left $(\Lambda'_\partial[s\h])$-comodules and right $(U[\h])$-modules and, likewise, a canonical isomorphism

$$U[\h] \cong U[\h] \otimes_{\tau_b} \Lambda'_\partial[s\h]$$

of differential graded right $(\Lambda'_\partial[s\h])$-comodules and left $(U[\h])$-modules. The point here is that, for both isomorphisms, $U[\h]$, $\tau_b$, and $\Lambda'_\partial[s\h]$ are the same constituents. The above isomorphisms entail canonical isomorphisms

$$U[C\h] \otimes U[\h], \quad R \otimes_{U[\h]} U[C\h] \cong \Lambda'_\partial[s\h].$$

Consequently the CCE coalgebra $\Lambda'_\partial[s\h]$ of $\h$ acquires a differential graded left $U[C\h]$-module structure and a differential graded right $U[C\h]$-module structure. We write these structures as

$$U[C\h] \times \Lambda'_\partial[s\h] \longrightarrow \Lambda'_\partial[s\h], \quad (a, b) \longmapsto a \cdot b,$$

$$\Lambda'_\partial[s\h] \times U[C\h] \longrightarrow \Lambda'_\partial[s\h], \quad (b, a) \longmapsto b \cdot a.$$ 

It is immediate that, given $Y \in \h$ and $b \in \Lambda[s\h]$,

$$Y \cdot b = \text{ad}_Y(b), \quad b \cdot Y = -\text{ad}_Y(b)$$

$$sY \cdot b = (sY)b, \quad b \cdot sY = b(sY) \quad \text{(exterior multiplication)}.$$ 

As a side remark we note that these $(C\h)$-module and $(\bar{C}\h)$-module structures are actually equivalent to the familiar fact that the Lie algebra homology operator $\partial$ generates the Gerstenhaber bracket $[[\cdot, \cdot]]$ on $\Lambda[s\h]$, that is, for homogeneous $a, b \in \Lambda[s\h]$,

$$\partial(ab) = (\partial a)b + (-1)^{|a|}a\partial b + (-1)^{|a|}[a, b].$$

The ground ring being viewed as a trivial $(C\h)$-module in the obvious way, the induced $(C\h)$-module structure ($(\bar{C}\h)$-module structure) on $\text{Alt}(\h, R)$ is the familiar action via the operations of Lie derivative $\lambda$ and contraction $i$; thus, given $Y \in \h$ and $\alpha \in \text{Alt}(\h, R)$,

$$Y(\alpha) = \lambda_Y(\alpha), \quad (sY)(\alpha) = i_Y(\alpha),$$
and the \((Ch)\)-action on \(\text{Alt}(h, R)\) is well known to be compatible with the multiplicative structure.

1.4. The category of \((G, Cg)\)-modules. Let the ground ring \(R\) be that of the real numbers \(\mathbb{R}\). Let \(G\) be a Lie group and let \(g\) be its Lie algebra. We will use the notion of \textit{differentiable} \(G\)-module in the sense of \([23]\). Henceforth \(\text{"G-module"}\) will mean \(\text{"differentiable G-module"}\). Given a differentiable right \(G\)-module \(V\), we will occasionally write the induced \(g\)-action as

\[
[\cdot, \cdot] : V \times g \to V.
\]

In particular, for \(Y \in g\) and \(b \in V\),

\[
\frac{d}{dt} (b \exp(tY)) \big|_{t=0} = [b, Y],
\]

and the actions intertwine in the sense that, given \(x \in G\),

\[
[bx, Y] = [b, \text{Ad}_x Y]x.
\]

The crucial example of a \((G, Cg)\)-module is the de Rham complex \(\mathcal{A}(X)\) of a smooth \(G\)-manifold \(X\): In this situation, the left \(G\)-action on \(X\) induces an action of the differential graded algebra \(U[Cg]\) on the differential graded de Rham algebra \(\mathcal{A}(X)\) from the right via the operations of contraction and Lie-derivative, evaluated through the infinitesimal anti-action \(g \to \text{Vect}(X)\) of \(g\) on \(X\). The exterior algebra \(\Lambda[s g]\) being canonically a graded subalgebra of \(U[Cg]\) (not a differential graded subalgebra), the \((\Lambda[s g])\)-invariants are then precisely the ordinary \textit{horizontal} elements, that is, the forms \(\alpha\) that are horizontal in the sense that \(\alpha(Y, Y_1, \ldots, Y_m) = 0\) whenever
Y is a fundamental vector field on X, i.e., a smooth vector field on X coming from g via the G-action. For a general \((G, Cg)\)-module, we will therefore refer to a \((\Lambda [sg])\)-invariant element as being horizontal.

1.5. The de Rham complex of a Lie group. Let G be a Lie group, and let g be its Lie algebra, the Lie algebra of left invariant vector fields on G as usual. Let V be a vector space. The V-valued de Rham complex \(\mathcal{A}(G, V)\) of G is well known to amount to the CCE complex calculating the Lie algebra cohomology of g with values in \(\mathcal{A}^0(G, V)\) relative to the g-module structure coming from left translation or, equivalently, relative to that coming from right translation. For later reference, we will now spell out that CCE complex relative to the left translation action of G on itself. The associated fundamental vector field map is the right trivialization of the tangent bundle of G. Beware: This is not the standard identification, which proceeds via the left trivialization of the tangent bundle of G (and will be explored in the next subsection).

Given \(Y \in g\), let \(\nabla Y\) be the associated right invariant vector field, that is, the vector field on G coming from right translation of the associated tangent vector \(Ye\) at the identity element e of G. The fundamental vector field map under discussion is the right trivialization

\[
\mathfrak{g} \times G \longrightarrow TG, \quad (Y, q) \longmapsto \nabla_q Y \quad (Y \in \mathfrak{g}, \ q \in G)
\]

of the tangent bundle of G. Relative to the Lie bracket, the resulting morphism

\[
\mathfrak{g} \longrightarrow \text{Vect}(G), \ Y \longmapsto \nabla Y
\]

is anti-Lie and the induced g-action on \(\mathcal{A}^0(G, V)\) is from the right, that is, \(\mathcal{A}(G, V)\) appears as a right g-module.

To obtain an explicit expression for the identification, in terms of the fundamental vector field isomorphism (1.5.1), of \(\mathcal{A}(G, V)\) with the appropriate CCE complex, given the p-form \(\alpha\) on TG and p vectors \(Y_1, \ldots, Y_p\) in g, let

\[
(\Phi(\alpha))(Y_1, \ldots, Y_p) = \alpha(\nabla_1, \ldots, \nabla_p) \in \mathcal{A}^0(G, V).
\]

**Proposition 1.5.2.** The morphism

\[
\Phi: \mathcal{A}(G, V) \longrightarrow \text{Hom}^{\tau_\mathfrak{g}}(\Lambda^\prime [sg], \mathcal{A}^0(G, V))
\]

is an isomorphism of chain complexes between \(\mathcal{A}(G, V)\) and the CCE complex calculating the Lie algebra cohomology of g with values in the right g-module \(\mathcal{A}^0(G, V)\) (coming from left translation in G). When V is a chain complex, the isomorphism (1.5.3) is compatible with the operators on both sides of (1.5.3) that are induced by the differential of V.

**Proof.** For the left-trivialization of the tangent bundle of G, the corresponding statement is straightforward and classical; see also Proposition 1.6.3 below. The argument translates to the right-trivialization by the standard trick which involves the antipode of the Hopf algebras coming into play, that is, the inversion mapping from G to itself and multiplication by \(-1\) on g. We leave the details to the reader. \(\square\)
1.6. The de Rham complex of a fiber bundle over a homogeneous space.

Let \( H \) be a Lie group, let \( \mathfrak{h} \) be its Lie algebra, let \( G \) be a closed subgroup of \( H \), let \( \mathfrak{g} \) denote the Lie algebra of \( G \), let \( X \) be a left \( G \)-manifold, and consider the de Rham complex \( \mathcal{A}(X) \) of \( X \), with its induced right \((G, C\mathfrak{g})\)-module structure. An obvious adjointness isomorphism
\[
\mathcal{A}(H, \mathcal{A}(X)) \rightarrow \mathcal{A}(H \times X)
\]
identifies the de Rham complex \( \mathcal{A}(H, \mathcal{A}(X)) \) of \( H \) with values in the de Rham complex \( \mathcal{A}(X) \) of \( X \) with the de Rham complex of the product \( H \times X \) in a \( G \)-equivariant manner. Our present aim is to describe the de Rham complex of \( H \times_G X \) in terms of the induced \( G \)- and \((C\mathfrak{g})\)-module structures on a suitable object naturally isomorphic to \( \mathcal{A}(H, \mathcal{A}(X)) \), to be spelled out as the right-hand side of (1.6.3.1) below.

The construction of the quotient \( H \times_G X \) involves the right translation action of \( G \) on \( H \). The fundamental vector field map associated with the \( H \)-action on itself via right translation is the left translation trivialization
\[
H \times \mathfrak{h}_0 \rightarrow TH
\]
of the tangent bundle \( TH \rightarrow H \) of \( H \). Thus, unlike the situation of (1.5) above, given \( Y \in \mathfrak{h} \), the associated fundamental vector field on \( H \) is then simply just \( Y \), viewed as a left invariant vector field, and the resulting injection of \( \mathfrak{h} \) into \( \text{Vect}(H) \) is a morphism of Lie algebras; in fact, this is simply the inclusion of the ordinary Lie algebra \( \mathfrak{h} \) of left-invariant vector fields into the Lie algebra \( \text{Vect}(H) \) of all vector fields on \( H \). Thus, via right translation in \( H \), the chain complex \( \mathcal{A}^0(H, \mathcal{A}(X)) \) of \( \mathcal{A}(X) \)-valued functions on \( H \) acquires a left \( \mathfrak{h} \)-chain complex structure which does not involve \( \mathcal{A}(X) \), and the operator \( \delta^\mathfrak{h} \) determined by the universal Lie algebra twisting cochain \( \tau_\mathfrak{h} : \Lambda^0_\mathfrak{g}[\mathfrak{h}] \rightarrow U[\mathfrak{h}] \), cf. (1.2) above, is defined on \( \text{Hom}(\Lambda^0_\mathfrak{g}[\mathfrak{h}], \mathcal{A}^0(H, \mathcal{A}(X))) \). This operator is a perturbation of the obvious differential on \( \text{Hom}(\Lambda^0_\mathfrak{g}[\mathfrak{h}], \mathcal{A}^0(H, \mathcal{A}(X))) \) coming from \( \partial \) and the differential on \( \mathcal{A}(X) \). For later reference, we spell out the following.

**Proposition 1.6.3.** The fundamental vector field isomorphism (1.6.2) induces an isomorphism
\[
\Phi : \mathcal{A}(H, \mathcal{A}(X)) \rightarrow \text{Hom}^\mathfrak{h}(\Lambda^0_\mathfrak{g}[\mathfrak{h}], \mathcal{A}^0(H, \mathcal{A}(X)))
\]
of chain complexes that is natural in terms of \( H \) and \( X \). This isomorphism admits the following description: Given the \( \mathcal{A}(X) \)-valued \( p \)-form \( \alpha \) on \( TH \) and \( p \) vectors \( Y_1, \ldots, Y_p \) in \( \mathfrak{h} \), viewed as fundamental vector fields on \( H \),
\[
(\Phi(\alpha))(Y_1, \ldots, Y_p) = \alpha(Y_1, \ldots, Y_p) \in \mathcal{A}^0(H, \mathcal{A}(X)).
\]

**Proof.** We leave the details to the reader. We only note that, for the special case where \( X \) is a point, \( \Phi \) amounts to the standard isomorphism of the de Rham complex of \( H \) onto the \( \mathcal{A}^0(H) \)-valued CCE complex of \( \mathfrak{h} \). \( \square \)

To adjust the situation to the standard principal bundle formalism where the action of the structure group is from the right, view the projection from \( H \times X \) to \( H \times_G X \) as a principal right \( G \)-bundle; thus the \( G \)-action on the product \( H \times X \) from the right is given by the association
\[
H \times X \times G \rightarrow H \times X, \quad (q, x, y) \mapsto (qy, y^{-1}x), \quad q \in H, x \in X, y \in G.
\]
In the standard way, this action induces a left \( G \)-action on \( \mathcal{A}(H, \mathcal{A}(X)) \).
Proposition 1.6.5. Rewritten as a right $G$-action

$$\mathcal{A}(H, \mathcal{A}(X)) \times G \longrightarrow \mathcal{A}(H, \mathcal{A}(X)),$$

the action of $G$ on $\mathcal{A}(H, \mathcal{A}(X))$ is given by the assignment to an alternating $(\mathcal{A}(X))$-valued $p$-form $\alpha: (TH)^p \to \mathcal{A}(X)$ on $TH$ ($p \geq 0$) and $y \in G$ of $\alpha \cdot y$, the value $\alpha \cdot y$ on a $p$-tuple $(Z_1, \ldots, Z_p)$ of $p$ vector fields $Z_1, \ldots, Z_p$ on $H$ being given by

$$(\alpha \cdot y)(Z_1, \ldots, Z_p) = (\alpha(Z_1y^{-1}, \ldots, Z_p y^{-1}))y.$$

Proof. Let $q \in H$, let $a$ be a point of $X$, let $Z_q \in T_q H$, and let $U_a \in T_a X$. The association

$$(Z_q, U_a) \mapsto (Z_q \cdot y, y^{-1} \cdot U_a) \ (y \in G)$$

is the canonical extension of the right $G$-action (1.6.4) on $H \times X$ to a right $G$-action on $T(H \times X)$. Now, under the circumstances of Proposition 1.6.5, let $\alpha: (TH)^s \to \mathcal{A}(X)$ be an $\mathcal{A}^s(X)$-valued $p$-form on $H$, let $Z = (Z_1, \ldots, Z_p)$, let $y \in G$, and let $U = (U_1, \ldots, U_n)$ be an $n$-tuple of vector fields on $X$. Then

$$(\alpha \cdot y)_q(Z_q) (U_a) = (\alpha_q y^{-1}(Z_q \cdot y^{-1})) y_a (y \cdot U_a). \quad \square$$

Corollary 1.6.6. On the right-hand side $\text{Hom}^{\tau_h}(\Lambda^s_{\mathfrak{g}}[s\mathfrak{h}], \mathcal{A}^0(H, \mathcal{A}(X)))$ of (1.6.3.1), the right $G$-action is given by the formula

$$(\alpha \cdot y)(Y_1, \ldots, Y_p) = (\alpha(\text{Ad}_y Y_1, \ldots, \text{Ad}_y Y_p)) y, \ y \in G, \ y_j \in \mathfrak{h};$$

here $\alpha$ ranges over $\mathcal{A}^0(H, \mathcal{A}(X))$-valued alternating $p$-forms on $\mathfrak{h}$, $p \geq 0$, and the expression $(\ldots) \cdot y$ refers to the right $G$-action on $\mathcal{A}^0(H, \mathcal{A}(X))$ induced by the right translation action of $G$ on $H$ and by the left $G$-action on $X$.

Proof. In the formula (1.6.5.2), when each vector field $Z_j$ on $H$ is left-invariant, that is, a member of $\mathfrak{h}$, for each $Z_j$, given $y \in G$,

$$(Z_j)_q \cdot y^{-1} = qZ_jy^{-1} = qy^{-1}\text{Ad}_y Z_j = (\text{Ad}_y Z_j)_{qy^{-1}}$$

whence

$$(\alpha \cdot y)_q((Z_1)_q, \ldots, (Z_p)_q)(U_1)_a, \ldots, (U_n)_a) = (\alpha_{qy^{-1}}(\text{Ad}_y Z)_{qy^{-1}}) y_a (y \cdot U_a). \quad \square$$

Thus the $G$-invariant forms relative to the action (1.6.4) are the $\mathcal{A}(X)$-valued $G$-equivariant forms on $H$ and these, in turn, in terms of the right-hand side $\text{Hom}^{\tau_h}(\Lambda^s_{\mathfrak{g}}[s\mathfrak{h}], \mathcal{A}^0(H, \mathcal{A}(X)))$ of (1.6.3.1), amount to the $\mathcal{A}^0(H, \mathcal{A}(X))$-valued $G$-equivariant alternating forms on $\mathfrak{h}$. We will now accordingly characterize the forms that are horizontal relative to the action (1.6.4) in terms of the appropriate equivariance property for $\mathcal{A}^0(H, \mathcal{A}(X))$-valued alternating forms on $\mathfrak{h}$. To this end, we first complete the description of the right $(G, C\mathfrak{g})$-action on the right-hand side of (1.6.3.1), as announced at the beginning of Subsection 1.6. We remind the reader that, as a graded Lie algebra, $C\mathfrak{g} = s\mathfrak{g} \times \mathfrak{g}$, the constituent $s\mathfrak{g}$ being abelian. We will denote the fundamental vector field on $X$ associated with $Y \in \mathfrak{h}$ by $Y_X$ and, accordingly, the fundamental vector field on $H \times X$ associated with $Y \in \mathfrak{h}$ by $Y_H \times X$. The fundamental vector field map associated with (1.6.4) takes the form

$$H \times X \times \mathfrak{g} \longrightarrow TH \times TX, \quad (q, x, Y) \mapsto (Y_q, -(Y_X)_x), \ q \in H, x \in X, Y \in \mathfrak{h}.$$ 

Thus the resulting injection $\mathfrak{g} \rightarrow \text{Vect}(H \times X)$ is given by $Y \mapsto (Y, -Y_X)$ ($Y \in \mathfrak{g}$) and, the $G$-action on $H \times X$ being from the right, the resulting infinitesimal $\mathfrak{g}$-action on $\mathcal{A}(H \times X) \cong \mathcal{A}(H, \mathcal{A}(X))$ via the operation of Lie derivative is from the left, i. e. a morphism of Lie algebras (not anti-Lie).
Corollary 1.6.8. On the right-hand side $\text{Hom}^\tau_\theta(\Lambda'_\theta[sh], \mathcal{A}^0(H, \mathcal{A}(X)))$ of (1.6.3.1), the induced right $(Cg)$-action admits the following description:

(i) The right $g$-action

$$[\cdot, \cdot]: \text{Hom}^\tau_\theta(\Lambda'_\theta[sh], \mathcal{A}^0(H, \mathcal{A}(X))) \times g \rightarrow \text{Hom}^\tau_\theta(\Lambda'_\theta[sh], \mathcal{A}^0(H, \mathcal{A}(X)))$$

is given by the formula

$$(1.6.9) \quad [\alpha, Y](Y_1, \ldots, Y_p) = \sum \alpha(Y_1, \ldots, [Y, Y_j], \ldots, Y_p) + [\alpha(Y_1, \ldots, Y_p), Y],$$

where $Y \in g$ and $Y_j \in h$; here $\alpha$ ranges over $\mathcal{A}^0(H, \mathcal{A}(X))$-valued alternating $p$-forms on $h$, and the right-most expression $[\ldots, Y]$ in (1.6.9) refers to the right $g$-action on $\mathcal{A}^0(H, \mathcal{A}(X))$ induced by the right translation action of $G$ on $H$ and by the left $G$-action on $X$; thus, given $q \in H$,

$$[\alpha(Y_1, \ldots, Y_p), Y](q) = -Y_q(\alpha(Y_1, \ldots, Y_p)) + \lambda_Y((\alpha(Y_1, \ldots, Y_p))(q)).$$

N.B. Given $Y_1, \ldots, Y_p \in h$, the value $\alpha(Y_1, \ldots, Y_p)$ is an $\mathcal{A}(X)$-valued function on $H$.

(ii) Given $z = sY \in sg$ where $Y \in g$ and, furthermore, the $n$-form $\alpha = (\alpha_0, \ldots, \alpha_n)$ in

$$\text{Hom}^\tau_\theta(\Lambda'_\theta[sh], \mathcal{A}^0(H, \mathcal{A}(X))),$$

with components $\alpha_j \in \text{Hom}(\Lambda_j[sh], \mathcal{A}^0(H, \mathcal{A}^{n-j}(X)))$ ($0 \leq j \leq n$), the result $\alpha \cdot z$ relative to the corresponding operation

$$\cdot: \text{Hom}^\tau_\theta(\Lambda'_\theta[sh], \mathcal{A}^0(H, \mathcal{A}(X))) \times sg \rightarrow \text{Hom}^\tau_\theta(\Lambda'_\theta[sh], \mathcal{A}^0(H, \mathcal{A}(X)))$$

is given by

$$(1.6.10) \quad \alpha \cdot z = i_{Y_X} \alpha_0 - i_Y \alpha_1 + i_{Y_X} \alpha_1 - i_Y \alpha_2 + \ldots + i_{Y_X} \alpha_{n-1} - i_Y \alpha_n;$$

here, for $0 \leq j \leq n - 1$, given $q \in H$ and $Y_1, Y_2, \ldots, Y_j \in h$,

$$(i_{Y_X} \alpha_j)(Y_1, Y_2, \ldots, Y_j)(q) = i_{Y_X}((\alpha_j(Y_1, Y_2, \ldots, Y_j))(q)),$$

that is, given the vector fields $U_2, \ldots, U_{n-j}$ on $X$,

$$(i_{Y_X} \alpha_j)(Y_1, Y_2, \ldots, Y_j)(q)(U_2, \ldots, U_{n-j})$$

$$= (-1)^j((\alpha_j(Y_1, Y_2, \ldots, Y_j))(q))(Y_X, U_2, \ldots, U_{n-j}).$$

N.B. Under these circumstances $(\alpha_j(Y_1, Y_2, \ldots, Y_j))(q) \in \mathcal{A}^{n-j}(X)$ ($0 \leq j \leq n$).

Indeed, on the constituent $\mathcal{A}^p(H, \mathcal{A}^{\ell}(X))$ ($p, \ell \geq 0$), the infinitesimal $g$-action

$$\mathcal{A}(H, \mathcal{A}(X)) \times g \rightarrow \mathcal{A}(H, \mathcal{A}(X)), \quad (\alpha, Y) \mapsto [\alpha, Y],$$

on $\mathcal{A}(H, \mathcal{A}(X))$ from the right associated with (1.6.4) is given by the formula

$$[\alpha, Y] = -\lambda_Y \alpha, \quad \alpha: T^pH \rightarrow \mathcal{A}^{\ell}(X),$$
suitably interpreted, in particular, $A^\ell(X)$ is viewed as a right $\mathfrak{g}$-module via the left $G$-action on $X$ and the associated operation of Lie derivative. Explicitly, given the vector fields $Z_1, \ldots, Z_p$ on $H$,

\begin{equation}
(1.6.11) \quad [\alpha, Y](Z_1, \ldots, Z_p) = \sum \alpha(Z_1, \ldots, [Y, Z_j], \ldots, Z_p) + \lambda_{Y\chi}(\alpha(Z_1, \ldots, Z_p))
\end{equation}

(beware the parentheses: $\alpha(Z_1, \ldots, Z_p)$ is an $\ell$-form on $X$), that is, given in addition the vector fields $U_1, \ldots, U_\ell$ on $X$,

\[
[\alpha, Y](Z_1, \ldots, Z_p)(U_1, \ldots, U_\ell) = \sum \alpha(Z_1, \ldots, [Z_j, Y], \ldots, Z_p)(U_1, \ldots, U_\ell)
+ Y_X(\alpha(Z_1, \ldots, Z_p)(U_1, \ldots, U_\ell))
- \sum \alpha(Z_1, \ldots, Z_p)(U_1, \ldots, [Y, U_j], \ldots, U_\ell).
\]

Likewise, on $A(H, A(X))$, the operation of contraction with vectors in $\mathfrak{g}$ induced by (1.6.4) can be described as follows: Given $Y \in \mathfrak{g}$, the operation of contraction

\[
\iota_{Y_{H \times X}}: A(H, A(X)) \cong A(H \times X) \longrightarrow A(H \times X) \cong A(H, A(X))
\]

with the fundamental vector field $\iota_{Y_{H \times X}} = (Y, -Y_X)$ associated with $Y$ is the operation

\[
\iota_{Y_{H \times X}} \alpha = i_Y \alpha - i_{Y_X} \alpha.
\]

An $n$-form $\alpha$ in $A(H, A(X))$ has $n + 1$ components $\alpha_j \in A^j(H, A^{n-j}(X))$ ($0 \leq j \leq n$) and, relative to the right $G$-action (1.6.4) on $H \times X$, given $Y \in \mathfrak{g}$, when $z = sY \in s\mathfrak{g}$,

\begin{equation}
(1.6.12) \quad \alpha \cdot z = -i_{Y_{H \times X}} \alpha = i_{Y_X} \alpha_0 - i_Y \alpha_1 + i_{Y_X} \alpha_1 - i_Y \alpha_2 + \ldots + i_{Y_X} \alpha_{n-1} - i_Y \alpha_n.
\end{equation}

Summing up, we arrive at the following.

**Proposition 1.6.13.** (i) The $n$-form $\alpha = (\alpha_0, \ldots, \alpha_n)$ in $A(H, A(X))$ is horizontal if and only if $\alpha \cdot z$ is zero for every $z \in s\mathfrak{g}$, that is, if and only if

\[
\begin{align*}
\iota_{Y_X} \alpha_0 &= i_Y \alpha_1 \in A^0(H, A^{n-1}(X)) \\
\iota_{Y_X} \alpha_1 &= i_Y \alpha_2 \in A^1(H, A^{n-2}(X)) \\
& \quad \ldots \\
\iota_{Y_X} \alpha_{n-1} &= i_Y \alpha_n \in A^{n-1}(H, A^0(X))
\end{align*}
\]

for every $Y \in \mathfrak{g}$.

(ii) The $n$-form $\alpha$ in $A(H, A(X))$ is basic in the sense that it descends to an $n$-form on $H \times_G X$, i.e. is horizontal and $G$-invariant, if and only if it is $(G, C\mathfrak{g})$-invariant. \(\square\)

1.7. The diagonal structure on the de Rham complex with values in a $(G, C\mathfrak{g})$-module. Abstracting from the material in Subsection 1.6, we now replace the de Rham complex $A(X)$ with a general $(G, C\mathfrak{g})$-module $V$. Right translation in $H$ and the $G$-action on $V$ induce a right $(G, C\mathfrak{g})$-module structure

\begin{equation}
(1.7.1) \quad A(H, V) \times (G, C\mathfrak{g}) \longrightarrow A(H, V)
\end{equation}
on \( \mathcal{A}(H, V) \). An explicit description thereof is given by the formulas (1.6.5.1), (1.6.11) and (1.6.12), with \( V \) substituted for \( \mathcal{A}(X) \).

Similarly as before, via right translation in \( H \), the chain complex \( \mathcal{A}^0(H, V) \) of \( V \)-valued functions on \( H \) acquires a left \( \mathfrak{h} \)-chain complex structure that does not involve \( V \), and the operator \( \delta' \) determined by the universal Lie algebra twisting cochain \( \tau_\mathfrak{h} : \Lambda'_\mathfrak{h}[\mathfrak{h}] \to \mathbb{U}[\mathfrak{h}] \), cf. \( (1.2) \) above, is defined on \( \text{Hom}(\Lambda'_\mathfrak{h}[\mathfrak{h}], \mathcal{A}^0(H, V)) \). This operator is a perturbation of the obvious differential on \( \text{Hom}(\Lambda'_\mathfrak{h}[\mathfrak{h}], \mathcal{A}^0(H, V)) \) coming from \( \partial \) and the differential on \( V \). Now, the formulas (1.6.7), (1.6.9) and (1.6.10), with \( V \) instead of \( \mathcal{A}(X) \), yield a right \((G, C\mathfrak{g})\)-module structure

\[
\text{Hom}^{\tau_\mathfrak{h}}(\Lambda'_\mathfrak{h}[\mathfrak{h}], \mathcal{A}^0(H, V)) \times (G, C\mathfrak{g}) \to \text{Hom}^{\tau_\mathfrak{h}}(\Lambda'_\mathfrak{h}[\mathfrak{h}], \mathcal{A}^0(H, V))
\]
on \( \text{Hom}^{\tau_\mathfrak{h}}(\Lambda'_\mathfrak{h}[\mathfrak{h}], \mathcal{A}^0(H, V)) \).

For completeness, we spell out the result of the action with an element \( z = sY \) of the constituent \( s \mathfrak{g} \) of \( C\mathfrak{g} = s \mathfrak{g} \times \mathfrak{g} \) where \( Y \in \mathfrak{g} \). To this end, for \( 0 \leq j \leq n-1 \), given \( q \in H \), vector fields \( Y_1, Y_2, \ldots, Y_j \) on \( H \), and the \( V^{n-j} \)-valued \( j \)-form \( \alpha_j \in \mathcal{A}^j(H, V^{n-j}) \), the value \((i_{Y_1} \alpha_j)(Y_1, Y_2, \ldots, Y_j)(q)\) is given by

\[
(i_{Y_1} \alpha_j)(Y_1, Y_2, \ldots, Y_j)(q) = i_{Y_1}((\alpha_j(Y_1, Y_2, \ldots, Y_j))(q)) = ((\alpha_j(Y_1, Y_2, \ldots, Y_j))(q)) \cdot z.
\]

N.B. The value \((\alpha_j(Y_1, Y_2, \ldots, Y_j))(q)\) lies in \( V^{n-j} \), and \(((\alpha_j(Y_1, Y_2, \ldots, Y_j))(q)) \cdot z \) lies in \( V^{n-j-1} \). With this preparation out of the way, given the \( n \)-form \( \alpha = (\alpha_0, \ldots, \alpha_n) \) \( \mathcal{A}(H, V) \), with components \( \alpha_j \in \mathcal{A}^j(H, V^{n-j}) \) \( (0 \leq j \leq n) \), the value \( \alpha \cdot z \) relative to the corresponding operation

\[
\cdot : \text{Hom}^{\tau_\mathfrak{h}}(\Lambda'_\mathfrak{h}[\mathfrak{h}], \mathcal{A}^0(H, V)) \times s \mathfrak{g} \to \text{Hom}^{\tau_\mathfrak{h}}(\Lambda'_\mathfrak{h}[\mathfrak{h}], \mathcal{A}^0(H, V))
\]
is given by

\[
\alpha \cdot z = i_{Y_1} \alpha_0 - i_Y \alpha_1 + i_{Y_2} \alpha_1 - i_{Y_2} \alpha_2 + \ldots + i_{Y_n} \alpha_{n-1} - i_{Y_n} \alpha_n.
\]

**Proposition 1.7.3.** The fundamental vector field isomorphism (1.6.2) induces an isomorphism

\[
(1.7.3.1) \quad \Phi : \mathcal{A}(H, V) \to \text{Hom}^{\tau_\mathfrak{h}}(\Lambda'_\mathfrak{h}[\mathfrak{h}], \mathcal{A}^0(H, V))
\]
of right \((G, C\mathfrak{g})\)-modules that is natural in terms of \( H \) and \( V \). This isomorphism admits the following explicit description: Given the \( V \)-valued \( p \)-form \( \alpha \) on \( TH \) and \( p \) vectors \( Y_1, \ldots, Y_p \) in \( \mathfrak{h} \), viewed as fundamental vector fields on \( H \),

\[
(\Phi(\alpha))(Y_1, \ldots, Y_p) = \alpha(Y_1, \ldots, Y_p) \in \mathcal{A}^0(H, V).
\]

**Proof.** The reasoning is exactly the same as that for Corollary 1.6.6 and Corollary 1.6.8. \( \square \)

2. Relative differential homological algebra

2.1. Adjunctions and (co)monads. Before going into details we note that we avoid the terminology “triple” etc. and exclusively use the monad-comonad terminology.
An adjunction is well known to determine a monad and a comonad [46]: Let \( \mathcal{D} \) and \( \mathcal{M} \) be categories, let \( \mathcal{G}: \mathcal{D} \to \mathcal{M} \) be a functor, suppose that the functor \( \Box: \mathcal{M} \to \mathcal{D} \) is left-adjoint to \( \mathcal{G} \), and let

\[
\mathcal{T} = \mathcal{G} \Box: \mathcal{M} \to \mathcal{M}.
\]

Let \( \mathcal{I} \) denote the identity functor, let \( \eta: \mathcal{I} \to \mathcal{T} \) be the unit, \( \varepsilon: \Box \mathcal{G} \to \mathcal{I} \) the counit of the adjunction, and let \( \mu \) be the natural transformation

\[
\mu = \mathcal{G} \varepsilon \Box: \mathcal{G} \Box \Box = \mathcal{T}^2 \to \mathcal{T} = \mathcal{G} \Box.
\]

The data \((\mathcal{T}, \eta, \mu)\) constitute a monad over the category \( \mathcal{M} \). The dual standard construction, cf. [13], [17] ("construction fondamentale" on p. 271), [46], then yields the cosimplicial object

\[
(\mathcal{T}^{n+1}, \varepsilon^j: \mathcal{T}^{n+1} \to \mathcal{T}^{n+2}, \eta^j: \mathcal{T}^{n+2} \to \mathcal{T}^{n+1})_{n \in \mathbb{N}};
\]

here, for \( n \geq 0 \),

\[
\varepsilon^j = \mathcal{T}^j \eta^j \mathcal{T}^{n-j+1}: \mathcal{T}^{n+1} \to \mathcal{T}^{n+2}, \quad j = 0, \ldots, n + 1,
\]

\[
\eta^j = \mathcal{T}^j \mu^j \mathcal{T}^{n-j}: \mathcal{T}^{n+2} \to \mathcal{T}^{n+1}, \quad j = 0, \ldots, n.
\]

Thus, given an object \( V \) of \( \mathcal{M} \),

\[
\mathcal{T}(V) = (\mathcal{T}^{n+1}(V), \varepsilon^j, \eta^j)
\]

is a cosimplicial object in \( \mathcal{M} \); here we do not distinguish in notation between the natural transformations \( \eta^j \) and \( \varepsilon^j \) and the morphisms they induce after evaluation of the corresponding functors in an object.

Under suitable circumstances, e. g. when \( \mathcal{M} \) is a category of modules, the associated chain complex \(|\mathcal{T}(V)|\) is a relatively injective resolution of \( V \), more precisely, a resolution of \( V \) in the category \( \mathcal{M} \) that is injective relative to the category \( \mathcal{D} \). We will use this construction to introduce and exploit various relative differential \( \text{Ext}_{(\mathcal{M}, \mathcal{D})} \)-functors over suitable categories \( \mathcal{M} \) and \( \mathcal{D} \). Examples will be given shortly. For sheaves, this kind of construction goes back to [17] (pp. 270–279).

Likewise, let \( \mathcal{F}: \mathcal{D} \to \mathcal{M} \) be a functor, suppose that the functor \( \Box: \mathcal{M} \to \mathcal{D} \) is right-adjoint to \( \mathcal{F} \), and let

\[
\mathcal{L} = \mathcal{F} \Box: \mathcal{M} \to \mathcal{M}.
\]

Let \( \eta: \mathcal{I} \to \Box \mathcal{F} \) be the unit, \( \varepsilon: \mathcal{L} \to \mathcal{I} \) the counit of the adjunction, and let \( \delta \) be the natural transformation

\[
\delta = \mathcal{F} \eta \Box: \mathcal{L} = \mathcal{F} \Box \to \mathcal{F} \Box \mathcal{F} = \mathcal{L}^2.
\]

The data \((\mathcal{L}, \varepsilon, \delta)\) constitute a comonad over the category \( \mathcal{M} \). The standard construction then yields the simplicial object

\[
(\mathcal{L}^{n+1}, d_j: \mathcal{L}^{n+2} \to \mathcal{L}^{n+1}, s_j: \mathcal{L}^{n+1} \to \mathcal{L}^{n+2})_{n \in \mathbb{N}};
\]
here, for $n \geq 0$,

$$
\begin{align*}
    d_j &= L^n \varepsilon L^{n-j+1} : L^{n+2} \to L^{n+1}, \quad j = 0, \ldots, n + 1, \\
    s_j &= L^n \delta L^{n-j} : L^{n+1} \to L^{n+2}, \quad j = 0, \ldots, n.
\end{align*}
$$

Thus, given an object $W$ of $\mathcal{M}$,

$$
\mathbf{L}(W) = (L^{n+1}(W), d_j, s_j)_{n \in \mathbb{N}}
$$

is a simplicial object in $\mathcal{M}$, the \textit{standard object associated with $W$ and the comonad};

again, we do not distinguish in notation between the natural transformations $d_j$ and

$s_j$ and the morphisms they induce after evaluation of the corresponding functors in

an object. Under suitable circumstances, the associated chain complex $|\mathbf{L}(W)|$ is a

relatively projective resolution of $W$, more precisely, a resolution of $W$ in the category

$\mathcal{M}$ that is projective relative to the category $\mathcal{D}$. We will use this construction to

introduce and exploit certain relative differential $\text{Tor}^{(\mathcal{M}, \mathcal{D})}$- and $\text{Ext}^{(\mathcal{M}, \mathcal{D})}$-functors

over certain categories $\mathcal{M}$ and $\mathcal{D}$.

\section{Differentiable cohomology.}

Let the ground ring be that of the reals, $\mathbb{R}$. Recall that $\mathcal{A}$ refers to the de Rham functor on smooth manifolds. Thus $\mathcal{A}^0$ refers to ordinary smooth functions. Let $G$ be a Lie group. It is well known that, contrary to what is the case for projective resolutions, the mechanism of \textit{injective} resolutions can be adapted to take account of additional structure, here that of \textit{differentiability} of a $G$-module. Indeed, the appropriate way to resolve an object of the category of differentiable $G$-modules is by means of a \textit{differentiably injective resolution} [23]:

Let $\mathcal{D} = \text{Vect}$, the category of real vector spaces, $\mathcal{M} = \text{Mod}_G$ that of (differentiable) right $G$-modules, and let $\mathcal{G}_G : \text{Vect} \to \text{Mod}_G$ be the functor which assigns to the real vector space $V$ the $G$-representation

$$
\mathcal{G}_G V = \mathcal{A}^0(G, V),
$$

endowed with the right $G$-module structure coming from left translation on $G$. For our purposes it would be more appropriate to endow $\mathcal{A}^0(G, V)$ with the right module structure coming from right translation in $G$ combined with the inversion mapping of $G$, but to arrive at formulas consistent with what is in the literature we proceed with the left translation action of $G$ on itself. We use the font $\mathcal{G}$ merely for convenience since this is reminiscent of the notation $G$ in [46] for this kind of functor; this usage of the font $\mathcal{G}$ has nothing to do with our usage of the notation $G$ for the group variable. The functor $\mathcal{G}_G$ is right adjoint to the forgetful functor $\Box : \text{Mod}_G \to \text{Vect}$ and hence defines a monad $(\mathcal{T}, \eta, \mu)$ over the category $\text{Mod}_G$. Given a $G$-module $V$, the chain complex arising from the \textit{dual standard construction} $\mathcal{T}(V)$ associated with $V$ is the standard differentiably injective resolution of $V$ in the category of $G$-modules which defines the differentiable cohomology $H^*_{\text{cont}}(G, V)$. Here we write $H^*_{\text{cont}}$ since the differentiable cohomology with coefficients in a differentiable module coincides with the continuous cohomology with coefficients in that module. See [23] for details. An explicit description of this resolution will be given in (2.5) below. The same kind of construction works for continuous cohomology but, in this paper, we shall exclusively exploit the differentiable version.
2.3. **The relative differential Ext**$_{(C_\mathfrak{g}, \mathfrak{g})}$. Let $R$ be an arbitrary commutative ring with 1 and $\mathfrak{g}$ an $R$-Lie algebra, projective as an $R$-module as throughout the paper. Let $\mathcal{D} = C_\mathfrak{g}$, the category of right $\mathfrak{g}$-chain complexes, let $\mathcal{M} = \text{Mod}_{C_\mathfrak{g}}$, and let $\mathcal{G}^\mathfrak{g}_{C_\mathfrak{g}}: C_\mathfrak{g} \to \text{Mod}_{C_\mathfrak{g}}$ be the functor given by

$$\mathcal{G}^\mathfrak{g}_{C_\mathfrak{g}}(V) = \text{Hom}_\mathfrak{g}(U[C_\mathfrak{g}], V) \cong \text{Hom}^r(\Lambda^*_\mathfrak{g}[\mathfrak{g}], V) \cong (\text{Alt}(\mathfrak{g}, V), d),$$

the total object arising from the bicomplex having $\text{Alt}^*(\mathfrak{g}, V_*)$ as underlying bigraded $R$-module; here $V$ ranges over right $\mathfrak{g}$-chain complexes, $(\text{Alt}(\mathfrak{g}, V), d)$ is endowed with the obvious right $(C_\mathfrak{g})$-module structure coming from the obvious left $(U[C_\mathfrak{g}])$-module structure on itself or, equivalently, that given by the operations of contraction and Lie derivative on the CCE complex $(\text{Alt}(\mathfrak{g}, V), d)$, cf. (1.3) above. The functor $\mathcal{G}^\mathfrak{g}_{C_\mathfrak{g}}$ is right adjoint to the forgetful functor $\square: \text{Mod}_{C_\mathfrak{g}} \to C_\mathfrak{g}$ and hence defines a monad $(\mathcal{T}, \eta, \mu)$ over the category $\text{Mod}_{C_\mathfrak{g}}$. Given a right $(C_\mathfrak{g})$-module $V$, the chain complex $|\mathcal{T}(V)|$ arising from the dual standard construction $\mathcal{T}(V)$ associated with $V$ is a resolution of $V$ in the category of $(C_\mathfrak{g})$-modules that is injective relative to the category $C_\mathfrak{g}$ of right $\mathfrak{g}$-chain complexes. Given a right $(C_\mathfrak{g})$-module $W$, the relative differential $\text{Ext}_{(C_\mathfrak{g}, \mathfrak{g})}(W, V)$ is the homology of the chain complex

$$\text{Hom}_{C_\mathfrak{g}}(W, |\mathcal{T}(V)|).$$

In particular, for $W = R$, the relative differential graded $\text{Ext}_{(C_\mathfrak{g}, \mathfrak{g})}(R, V)$ is the homology of the chain complex $|\mathcal{T}(V)|^C_\mathfrak{g}$.

2.4. **The relative differential Ext**$_{(G, C_\mathfrak{g}), C}$. As before, view the group $G$ as a left $G$-manifold via left translation. Given the chain complex $V$, let $\mathcal{A}(G, V)$, the $V$-valued (totalized) de Rham complex $\mathcal{A}(G, V)$ of $G$ be the chain complex arising from the operation of totalization applied to the bicomplex $(\mathcal{A}^*(G, V_1), \delta, d)$, where $\delta$ refers to the de Rham complex operator and $d$ to the differential induced by the differential of $V$, and endow $\mathcal{A}(G, V)$ with the right $(G, C_\mathfrak{g})$-module structure explained above. Let $\mathcal{C}$ be the category of real chain complexes. Consider the pair of categories $(\mathcal{M}, \mathcal{D}) = (\text{Mod}_{(G, C_\mathfrak{g})}, \mathcal{C})$ and let

$$\mathcal{G}_{(G, C_\mathfrak{g})}: \mathcal{C} \to \text{Mod}_{(G, C_\mathfrak{g})}$$

be the functor which assigns to the chain complex $V$ the right $(G, C_\mathfrak{g})$-module

$$\mathcal{G}_{(G, C_\mathfrak{g})}V = \mathcal{A}(G, V).$$

**Proposition 2.4.3.** The functor $\mathcal{G}_{(G, C_\mathfrak{g})}$ is right adjoint to the forgetful functor $\square: \text{Mod}_{(G, C_\mathfrak{g})} \to \mathcal{C}$ and hence defines a monad $(\mathcal{T}, \eta, \mu)$ over the category $\text{Mod}_{(G, C_\mathfrak{g})}$.

**Proof.** Let $W$ be a vector space, $\mathbf{V}$ a right $(G, C_\mathfrak{g})$-module, and denote the graded vector space which underlies $\mathbf{V}$ by $\mathbf{V}^\mathfrak{g}$ The obvious linear map

$$\text{Hom}(\mathbf{V}^\mathfrak{g}, W) \to \text{Hom}_{(G, C_\mathfrak{g})}(\mathbf{V}, \mathcal{A}(G, W))$$

sends the homogeneous linear map $\varphi: \mathbf{V}^\mathfrak{g} \to W$ to the $(G, C_\mathfrak{g})$-linear morphism

$$\Phi: \mathbf{V} \to \mathcal{A}(G, W) \cong \text{Hom}_\mathfrak{g}(U[C_\mathfrak{g}], \mathcal{A}^0(G, W))$$
determined by the requirement that, for a homogeneous member \( v \) of \( V \) of degree \(-k \leq 0\), the value \( \Phi(v) \) be the \( W \)-valued \( k \)-form on \( G \) such that, given \( Y_1, \ldots, Y_k \in \mathfrak{g} \),

\[
\Phi(v)(sY_1 \ldots sY_k) = \Phi(vsY_1 \ldots sY_k)
\]

is the smooth \( W \)-valued function on \( G \) given by

\[
\Phi(vsY_1 \ldots sY_k)(x) = \varphi(vsY_1 \ldots sY_kx), \quad x \in G.
\]

Here the juxtaposition \((vsY_1 \ldots sY_k, x) \mapsto vsY_1 \ldots sY_kx\) refers to the \( G \)-action on \( V \). The linear map (2.4.4) is an isomorphism of vector spaces. Replacing \( W \) with a chain complex and taking the totalized object \( A(G,W) \), we arrive at the desired adjunction

\[
\text{Hom}_C(\square V, W) \rightarrow \text{Hom}_{(G,\mathfrak{C}_g)}(V, A(G,W)).
\]

In view of the general theory reproduced in (2.1) above, the adjunction spelled out in Proposition 2.4.3 yields a monad \((T, \eta, \mu)\) over the category \( \text{Mod}_{(G,\mathfrak{C}_g)} \).

Let \( \mathcal{C}_G \) be the category of right \( G \)-chain complexes. On the category \( \mathcal{C}_G \), the obvious variant of the functor \( \mathcal{G}_G^\mathfrak{g} \), cf. (2.3.1), takes the form of the functor

\[
(2.4.5) \quad \mathcal{G}_G^\mathfrak{g} : \mathcal{C}_G \rightarrow \text{Mod}_{(G,\mathfrak{C}_g)} ; \quad \mathcal{G}_G^\mathfrak{g}(V) = \text{Hom}^\tau_\mathfrak{g}(\Lambda_0'[\mathfrak{s}g], V) \cong (\text{Alt}(\mathfrak{g}, V), d)
\]

where \( V \) ranges over right \( G \)-chain complexes.

**Proposition 2.4.6.** The functor \( \mathcal{G}_{(G,\mathfrak{C}_g)} \) admits the decomposition

\[
\mathcal{G}_{(G,\mathfrak{C}_g)} = \mathcal{G}_G^\mathfrak{g} \circ \mathcal{G}_G : \mathcal{C}_G \rightarrow \text{Mod}_{(G,\mathfrak{C}_g)}.
\]

**Proof.** This is an immediate consequence of Proposition 1.5.2. \( \square \)

Let \( V \) be a right \((G,\mathfrak{C}_g)\)-module. The chain complex \( |T(V)| \) arising from the dual standard construction \( T(V) \) associated with the monad \((T, \eta, \mu)\) and the right \((G,\mathfrak{C}_g)\)-module \( V \) is a resolution of \( V \) in the category of right \((G,\mathfrak{C}_g)\)-modules that is injective relative to the category of chain complexes. Given a right \((G,\mathfrak{C}_g)\)-module \( W \), the differential graded \( \text{Ext}_{((G,\mathfrak{C}_g);\mathcal{C})}(W, V) \) is the homology of the chain complex

\[
\text{Hom}_{(G,\mathfrak{C}_g)}(W, |T(V)|).
\]

In particular, relative to the obvious trivial \((G,\mathfrak{C}_g)\)-module structure on \( \mathbb{R} \), the differential graded \( \text{Ext}_{((G,\mathfrak{C}_g);\mathcal{C})}(\mathbb{R}, V) \) is the homology of the chain complex \( |T(V)|^{(G,\mathfrak{C}_g)} \) of \((G,\mathfrak{C}_g)\)-invariants in \( |T(V)| \).

2.5. **The Borel construction.** As before, \( G \) denotes a Lie group, neither necessarily compact nor necessarily finite dimensional. Let \( X \) be a left \( G \)-manifold. The
simplicial Borel construction takes the form of either a nonhomogeneous construction or of a homogeneous construction.

Recall that any object $Y$ of a symmetric monoidal category endowed with a cocommutative diagonal—we will take the categories of spaces, of smooth manifolds, of groups, of vector spaces, of Lie algebras, etc., defines two simplicial objects in the category, the trivial object which, with an abuse of notation, we still write as $Y$, and the total object $EY$ (“total object” not being standard terminology in this generality); the trivial object $Y$ has a copy of $Y$ in each degree and all simplicial operations are the identity while, for $p \geq 0$, the degree $p$ constituent $EY_p$ of the total object $EY$ is a product of $p+1$ copies of $Y$ with the familiar face operations given by omission and degeneracy operations given by insertion. See e. g. [4] and [34] (1.1). When $Y$ is an ordinary $R$-module, the simplicial $R$-module associated with $Y$ is in fact the result of application of the Dold-Kan functor $DK$ from chain complexes to simplicial $R$-modules, cf. e. g. [11] (3.2 on p. 219).

When $G$ is substituted for $Y$, the resulting simplicial object is a simplicial group $EG$, and the diagonal injection $G \to EG$ turns $EG$ into a simplicial principal right (or left) $G$-space. The simplicial manifold $N(G,X) = EG \times_G X$ is what we refer to as the homogeneous Borel-construction. The term “homogeneous” is intended to hint at the fact that the formulation uses the group structure only for the $G$-action on $EG$ and not for the simplicial structure on $EG$; cf. [45] (IV.5 p. 119) where this distinction is discussed relative to the bar resolution.

Let $V$ be a right $(G,Cg)$-module. The simplicial structure of $EG$ induces a cosimplicial structure on the $V$-valued de Rham complex

$$\mathcal{A}(EG,V),$$

and the degreewise right diagonal $(G,Cg)$-module structures (1.7.1), relative to the right translation $G$-action on $EG$ where $G$ is viewed as a subgroup of $EG$ and relative to the $(G,Cg)$-module structure on $V$, turn $\mathcal{A}(EG,V)$ into a cosimplicial object in the category of right $(G,Cg)$-modules. Our aim is to prove the following.

**Theorem 2.5.2.** Given the $(G,Cg)$-module $V$, the dual standard construction $T(V)$ associated, over the category $\text{Mod}_{G,Cg}$, with the monad $(T,\eta,\mu)$ spelled out in (2.4) above and the $(G,Cg)$-module $V$, is naturally isomorphic, as a cosimplicial object in the category of right $(G,Cg)$-modules, to $\mathcal{A}(EG,V)$.

We begin with the preparations for the proof. At the risk of making a mountain out of a molehill we first explain briefly the right and left nonhomogeneous versions $(EG)^{\text{right}}$ and $(EG)^{\text{left}}$ of the Borel construction; we need them both to arrive at consistent formulas at a later stage. Indeed, the dual standard construction $T(V)$ associated, over the category $\text{Mod}_{G,Cg}$, with the monad $(T,\eta,\mu)$ spelled out in (2.4) above and the $(G,Cg)$-module $V$, is more naturally identified with $\mathcal{A}((EG)^{\text{left}},V)$, cf. Proposition 2.5.7 below, whereas, given the left $G$-manifold $X$, within the framework of comonads and standard construction, the ordinary Borel construction leads to the simplicial manifold $(EG)^{\text{right}} \times_G X$.

The nonhomogeneous right (left) Borel-construction arises as follows: Let Smooth be the category of smooth manifolds, Smooth$_G$ (Smooth$_G$) that of smooth right (left) $G$-manifolds, and let $F: \text{Smooth} \to \text{Smooth}_G$ ($F: \text{Smooth} \to F(\text{Smooth})$) be the functor which assigns to the smooth manifold $Z$ the smooth right $G$-manifold $Z \times G$...
HOMOLOGICAL ALGEBRA AND EQUIVARIANT DE RHAM COHOMOLOGY

(Left $G$-manifold $G \times Z$), endowed with the obvious right (left) $G$-action induced by right (left) translation in $G$. This functor is left adjoint to the forgetful functor $\Box: \text{Smooth}_G \to \text{Smooth}$ ($\Box: \text{GSmooth} \to \text{Smooth}$), and the standard construction applied to the resulting comonad and the right (left) $G$-manifold $Z$ yields a simplicial manifold $\mathcal{E}(Z, G)$ ($\mathcal{E}(G, Z)$) endowed with a free right (left) $G$-action. For $Z$ a point $o$, we will write $(EG)^\text{right} = \mathcal{E}(o, G)$ ($(EG)^\text{left} = \mathcal{E}(G, o)$). This is the nonhomogeneous version of the total simplicial $G$-object for $G$ in the category of right (left) $G$-manifolds.

The various constructions are isomorphic as simplicial free right (left) $G$-manifolds. Indeed, $(EG)^\text{right}$ is the simplicial group having the iterated semi-direct product

$$(EG)^\text{right}_n = G \rtimes G \rtimes \ldots \rtimes G \ (n + 1 \text{ copies of } G)$$

as degree $n$ constituent, with $G$-action from the right via right translation on the rightmost copy of $G$. The nonhomogeneous face operators $\partial_j$ are given by the expressions

$$(2.5.3.r) \quad \partial_0(x_0, x_1, \ldots, x_n) = (x_1, \ldots, x_n),$$
$$\partial_j(x_0, x_1, \ldots, x_n) = (x_0, \ldots, x_{j-2}, x_{j-1}x_j, x_{j+1}, \ldots, x_n) \ (1 \leq j \leq n)$$

and, likewise, the nonhomogeneous degeneracy operators $s_j$ are given by

$$(2.5.4) \quad s_j(x_0, x_1, \ldots, x_n) = (x_0, \ldots, x_{j-1}, e, x_j, \ldots, x_n) \ (0 \leq j \leq n).$$

The notation $X$ being maintained for the left $G$-manifold at the beginning of the present subsection, let

$$\mathcal{N}(G, X) = (EG)^\text{right} \times_G X.$$ 

The associations

$$(x_0, x_1, \ldots, x_n) \mapsto (x_0x_1x_2 \ldots x_n, \ldots, x_{n-1}x_n, x_n), \ x_j \in G,$$

as $n$ ranges over the natural numbers, induce an isomorphism of simplicial principal right $G$-manifolds

$$(2.5.5.r) \quad (EG)^\text{right} \to EG,$$

in fact, an isomorphism of simplicial groups, and thence an isomorphism of simplicial manifolds from $\mathcal{N}(G, X)$ onto $N(G, X) = EG \times_G X$. In degree $n$, the inverse of (2.5.5.r) is plainly given by the association

$$(y_1, \ldots, y_n, x) \mapsto (x_0, x_1, \ldots, x_n) = (y_1y_2^{-1}, y_2y_3^{-1}, \ldots, y_{n-1}y_n^{-1}, y_nx^{-1}, x).$$

This association explains, in a somewhat more down to earth manner than the elegant categorical explanation on p. 107 of [52], the formulas in [12] and [52], cf. also p. 573 of [29], for the projection from the homogeneous total simplicial object $EG$, written there as $\mathcal{N}G$ (the nerve of a suitably defined category $\mathcal{G}$ associated with $G$) to the base $(EG)^\text{right}/G$ of the universal simplicial $G$-bundle, written out in nonhomogeneous form so that the simplicial structure on the base becomes more perspicuous.
Likewise, \((EG)^\text{left}\) is the simplicial group having the iterated semi-direct product
\[
(EG)^\text{left}_n = G \times G \times \ldots \times G \quad (n + 1 \text{ copies of } G)
\]
as degree \(n\) constituent, with \(G\)-action from the left via left translation on the leftmost copy of \(G\). The \textit{nonhomogeneous} face operators \(\partial_j\) are given by the familiar expressions
\[
\begin{align*}
\partial_j(x_0, x_1, \ldots, x_n) &= (x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \quad (0 \leq j < n) \\
\partial_n(x_0, x_1, \ldots, x_n) &= (x_0, \ldots, x_{n-1})
\end{align*}
\]
and the \textit{nonhomogeneous} degeneracy operators \(s_j\) are still given by (2.5.4). The associations
\[
(x_0, x_1, \ldots, x_n) \mapsto (x_0, x_0x_1, \ldots, x_0x_1x_2 \ldots x_{n-1}, x_0x_1x_2 \ldots x_n), \quad x_j \in G,
\]
as \(n\) ranges over the natural numbers, induce an isomorphism of simplicial principal left \(G\)-manifolds
\[
(EG)^\text{left} \rightarrow EG,
\]
in fact, an isomorphism of simplicial groups. In degree \(n\), the inverse of (2.5.5.1) is plainly given by the association
\[
(x, y_1, \ldots, y_n) \mapsto (x_0, x_1, \ldots, x_n) = (x, x^{-1}y_1, y_1^{-1}y_2, \ldots, y_{n-1}^{-1}y_n).
\]

Let \(V\) be a right \(G\)-module. We will now consider \(A^0((EG)^\text{left}, V)\) as a cosimplicial right \(G\)-module, the right \(G\)-module structure being the diagonal structure relative to left \(G\)-translation on \((EG)^\text{left}\) and the cosimplicial structure being induced from the simplicial structure on \((EG)^\text{left}\); we recall that the diagonal structure is given by the association
\[
A^0((EG)^\text{left}, V) \times G \rightarrow A^0((EG)^\text{left}, V), \quad (\alpha, x) \mapsto \alpha \cdot x,
\]
where \((\alpha \cdot x)y = (\alpha(xy))x, \quad x \in G, \quad y \in (EG)^\text{left}\); here \(\alpha\) ranges over smooth functions from \((EG)^\text{left}\) to \(V\). We remind the reader that we write various forgetful functors as \(\Box\). Thus, given the \(G\)-representation \(V\), the notation \(\Box V\) refers to the vector space which underlies \(V\), possibly endowed with trivial \(G\)-action.

**Proposition 2.5.7.** Let \(T^0(V)\) be the dual standard construction associated, over the category \(\text{Mod}_G\), with the monad \((\mathcal{T}, \eta, \mu)\) spelled out in (2.2) above and the \(G\)-module \(V\). Relative to the diagonal \(G\)-action on \(A^0((EG)^\text{left}, V)\), the morphism
\[
\Phi^0 = (\varphi_0, \ldots): A^0((EG)^\text{left}, V) \rightarrow T^0(V)
\]
of graded \(R\)-modules which, in degree \(n\), is given by the association
\[
\varphi_n: A^0(G^{\times(n+1)}, V) \rightarrow A^0(G^{\times(n+1)}, \Box V),
\]
\[
\varphi_n(\alpha)(x_0, \ldots, x_n) = (\alpha(x_0, \ldots, x_n)) \cdot x_0 \cdot \ldots \cdot x_n, \quad x_0, \ldots, x_n \in G,
\]
where \( \alpha \) ranges over smooth maps from \( G^{\times (n+1)} \) to \( V \), is an isomorphism of cosimplicial right \( G \)-modules.

**Proof.** This is certainly folk lore; a direct argument comes down to a tedious but straightforward verification. We leave the details to the reader. \( \square \)

N.B. In view of \( G \)-equivariance, the isomorphism \( \varphi_0 \) is determined by the requirement

\[
(\varphi_0(\alpha))(e) = \alpha(e), \quad \alpha : G \to V.
\]

Let \( V \) be a right \((G, Cg)\)-module. We will now consider \( \mathcal{A}((EG)^{\text{left}}, V) \) as a cosimplicial right \((G, Cg)\)-module, the cosimplicial structure being induced from the simplicial structure on \((EG)^{\text{left}}\) and the right \((G, Cg)\)-module structure being the diagonal structure relative to left \( G \)-translation on \((EG)^{\text{left}}\) and the right \((G, Cg)\)-module structure on \( V \) (beware: in (2.5.1) above that kind of structure was considered relative to right translation on \( EG \) via the explicit description (1.7.1)).

For intelligibility, we spell out this diagonal structure explicitly: The right \( G \)-module structure on \( \mathcal{A}((EG)^{\text{left}}, V) \) is given by the extension of the association (2.5.6) above to the present situation. That is, in a cosimplicial degree \( n \), given the \( V \)-valued \( p \)-form \( \alpha \) on \((EG)^{\text{left}}_n\), the \( p \)-tuple \( Z_1, \ldots, Z_p \) of vector fields on \((EG)^{\text{left}}_n\), and \( x \in G \), let

\[
(\alpha \cdot x)(Z_1, \ldots, Z_p) = (\alpha(xZ_1, \ldots, xZ_p))x;
\]

here the notation \( xZ_j \) \((1 \leq j \leq p)\) refers to the induced left \( G \)-action on the vector space of smooth vector fields \( \text{Vect}((EG)^{\text{left}}_n) \) on \((EG)^{\text{left}}_n\). The right \( G \)-module structure on \( \mathcal{A}((EG)^{\text{left}}_n, V) \) is given by the pairing

\[
\mathcal{A}((EG)^{\text{left}}_n, V) \times G \to \mathcal{A}((EG)^{\text{left}}_n, V), \quad (\alpha, x) \mapsto \alpha \cdot x, \quad x \in G,
\]

where \( \alpha \) ranges over \( V \)-valued \( p \)-forms on \((EG)^{\text{left}}_n\), for \( p \geq 0 \). Furthermore, let \( Y \in g \); given the \( V \)-valued \( p \)-form \( \alpha \), the \( p \)-form \([\alpha, Y]\), evaluated at the \( p \)-tuple \( Z_1, \ldots, Z_p \) of vector fields on \((EG)^{\text{left}}_n\), is given by

\[
[\alpha, Y](Z_1, \ldots, Z_p) = \sum \alpha(Z_1, \ldots, [Y, Z_j], \ldots, Z_p) + [\alpha(Z_1, \ldots, Z_p), Y].
\]

Likewise, let \( z = sY \) where \( Y \in g \). Let \( \alpha = (\alpha_0, \ldots, \alpha_m) \) be an \( m \)-form in \( \mathcal{A}((EG)^{\text{left}}_n, V) \), with components \( \alpha_j \in \mathcal{A}^j((EG)^{\text{left}}_n, V^{m-j}) \) \((0 \leq j \leq m)\). Now, for \( 0 \leq j \leq m-1 \), given \( q \in H \) and the vector fields \( Y_1, Y_2, \ldots, Y_j \) on \( H \), the value \((i_Y \alpha_j)(Y_1, Y_2, \ldots, Y_j)(q)\) is given by

\[
(i_Y \alpha_j)(Y_1, Y_2, \ldots, Y_j)(q) = i_Y((\alpha_j(Y_1, Y_2, \ldots, Y_j))(q)) = ((\alpha_j(Y_1, Y_2, \ldots, Y_j))(q)) \cdot z.
\]

N.B. The value \((\alpha_j(Y_1, Y_2, \ldots, Y_j))(q)\) lies in \( V^{m-j} \), and \(((\alpha_j(Y_1, Y_2, \ldots, Y_j))(q)) \cdot z \) lies in \( V^{m-j-1} \). With this preparation out of the way, \( \alpha \cdot z \) is given by

\[
\alpha \cdot z = i_Y \alpha_0 - i_Y \alpha_1 + i_Y \alpha_1 - i_Y \alpha_2 + \ldots + i_Y \alpha_{m-1} - i_Y \alpha_m.
\]
Proposition 2.5.10. Let $T(V)$ be the dual standard construction associated, over the category $\text{Mod}_{(G,Cg)}$, with the monad $(T,\eta,\mu)$ spelled out in (2.4) above and the $(G,Cg)$-module $V$. Relative to the diagonal $(G,Cg)$-action on $\mathcal{A}((EG)^{\text{left}}, V)$, the unique extension

$$\Phi^{\text{left}}: \mathcal{A}((EG)^{\text{left}}, V) \longrightarrow T(V)$$

of the morphism $\Phi^0$ given as (2.5.8) above is an isomorphism of cosimplicial right $(G,Cg)$-modules. This extension is characterized as follows: In a simplicial degree $n$, given the $V$-valued $j$-form on $(EG)^{\text{left}}$, $Y_1,\ldots,Y_j \in g$, and $z_j = sY_j \in sg$,

$$(\Phi\alpha)(Y_1,\ldots,Y_j)(e) = (\alpha \cdot z_1 \cdot \ldots \cdot z_j)(e) \in V.$$

We will now consider the simplicial left $G$-manifold $(EG)^{\text{left}}$ as a simplicial right $G$-manifold via the pairing

$$\left( (EG)^{\text{left}} \times G \rightarrow (EG)^{\text{left}}, (y,x) \mapsto x^{-1}y, \ y \in (EG)^{\text{left}}, x \in G. \right)$$

Proposition 2.5.13. The diffeomorphisms

$$G^{\times (n+1)} \rightarrow G^{\times (n+1)}, \ (x_0,\ldots,x_n) \mapsto (x_1,\ldots,x_n,x_n^{-1}x_{n-1}^{-1}\ldots x_0^{-1}) \quad (n \geq 0)$$

induce an isomorphism

$$\left( (EG)^{\text{left}} \rightarrow (EG)^{\text{right}} \right)$$

of simplicial right $G$-manifolds.

We can now complete the proof of Theorem 2.5.2: The isomorphism (2.5.14), together with (2.5.11), induces an isomorphism

$$\Phi^{\text{right}}: \mathcal{A}((EG)^{\text{right}}, V) \longrightarrow T(V)$$

of cosimplicial right $(G,Cg)$-modules. This isomorphism, in turn, combined with the isomorphism induced by the isomorphism (2.5.5.r), yields the desired isomorphism between (2.5.1) and the dual standard construction $T(V)$ associated with the monad $(T,\eta,\mu)$ under discussion over the category $\text{Mod}_{(G,Cg)}$ and the $(G,Cg)$-module $V$. This proves Theorem 2.5.2.

Henceforth we shall no longer distinguish in notation between the homogeneous version $EG$ and the nonhomogeneous versions $(EG)^{\text{left}}$ and $(EG)^{\text{right}}$.

2.6. Extension of Bott’s Decomposition Lemma. Let $V$ be a right $(G,Cg)$-module. Application of the functor $\mathcal{A}^0$ to $EG$ yields the cosimplicial algebra $\mathcal{A}^0(EG)$; likewise application of that functor to $EG$ and $V$ yields the cosimplicial chain complex $\mathcal{A}^0(EG,V)$, and the latter inherits a cosimplicial differential graded $(\mathcal{A}^0(EG))$-module structure. Just as for (2.5.1) above, the degreewise right diagonal $(G,Cg)$-module structures (1.7.2), relative to the right translation $G$-action on $EG$ where $G$ is viewed as a subgroup of $EG$ and relative to the $(G,Cg)$-module structure on $V$, turn $\text{Hom}^{\text{left}}(\mathcal{A}_{sEg}^0,\mathcal{A}^0(EG,V))$ into a cosimplicial object in the category of right $(G,Cg)$-modules.

The decomposition of the functor $\mathcal{G}_{(G,Cg)}$ spelled out in Proposition 2.4.6 translates to a decomposition for the corresponding standard constructions. The explicit description thereof leads to the following:
Theorem 2.6.1. (Extended decomposition lemma) The degreewise left trivialization of the tangent bundle of the simplicial group \( \text{EG} \), that is, the morphism (1.6.3.1), evaluated degreewise, yields an isomorphism

\[
\Phi : \mathcal{A}(\text{EG}, \mathbf{V}) \to \text{Hom}^{\tau E_g}(\Lambda'_{\partial}[sE_g], \mathcal{A}^0(\text{EG}, \mathbf{V}))
\]

of cosimplicial right \((G, Cg)\)-modules from the cosimplicial chain complex \( \mathcal{A}(\text{EG}, \mathbf{V}) \) onto the differential graded cosimplicial diagonal object on the right-hand side of (2.6.2).

**Proof.** In a degree \( p \geq 0 \), the cosimplicial diagonal object on the right-hand side of (2.6.2) comes down to

\[
\text{Hom}^{\tau E_g}(\Lambda'_{\partial}[sE_g], \mathcal{A}^0(\text{EG}, \mathbf{V}))_p = \text{Hom}^{\tau(E_g)_p}(\Lambda'_{\partial}[s(E_g)_p], \mathcal{A}^0(\text{EG}, \mathbf{V})_p);
\]

since \((E_g)_p = g^{p+1}\) and \((EG)_p = G^{p+1}\), in view of Proposition 1.7.3, the isomorphism (1.6.3.1), with \( H = G^{p+1}, h = g^{p+1}\), and \( V = V\), identifies this cosimplicial diagonal object with the \( V \)-valued de Rham complex of \( G^{p+1} \). \( \square \)

**Remark 2.6.3.** For the special case where \( V \) is the ground field \( \mathbb{R} \) and \( G \) finite dimensional, a version of the isomorphism (2.6.2) (in a language different from ours) is given in the Decomposition Lemma in [4].

2.7. **EQUIVARIANT DE RHAM THEORY AS A DIFFERENTIAL EXT.** As before, \( G \) denotes a Lie group (neither necessarily compact nor necessarily finite dimensional) and \( X \) a left \( G \)-manifold. Application of the de Rham functor to the simplicial Borel construction relative to \( G \) and \( X \) yields a cosimplicial differential graded algebra whose total object defines the \( G \)-equivariant de Rham cohomology of \( X \).

Exploiting the monad \((T, \eta, \mu)\) over the category \( \text{Mod}_{(G, CG)} \) introduced in Subsection 2.4 above we can now spell out the de Rham theory replacement for Theorem 3.1 in [34] (which, in turn, refers to ordinary cohomology). For our purposes, this de Rham theory replacement reduces equivariant de Rham theory to ordinary homological algebra and thereby provides a high amount of flexibility. In a sense we will explore this flexibility in the rest of the paper.

**Theorem 2.7.1.** The cosimplicial chain complex \( \mathcal{A}(N(G, X)) \) associated with the nonhomogeneous simplicial Borel construction \( N(G, X) \) is canonically isomorphic to the cosimplicial chain complex \( \text{Hom}(\mathcal{A}(X))\text{Hom}(G, CG) \), the \((G, CG)\)-invariants of the chain complex associated with the dual standard construction relative to the monad \((T, \eta, \mu)\) over the category \( \text{Mod}_{(G, CG)} \) and the \((G, CG)\)-module \( \mathcal{A}(X) \). Consequently the \( G \)-equivariant de Rham cohomology of \( X \) is canonically isomorphic to the differential Ext(\( (G, CG); \mathcal{C})\text{Hom}(\mathbb{R}, \mathcal{A}(X)) \).

**Proof.** This is a consequence of Theorem 2.5.2, combined with the observation, cf. Proposition 1.6.13, that the projection \( \text{EG} \times X \to N(G, X) \) identifies the cosimplicial differential graded algebra \( \mathcal{A}(N(G, X)) \cong \mathcal{A}(NG, \mathcal{A}(X)) \) with the cosimplicial differential graded subalgebra \( \mathcal{A}(\text{EG}, \mathcal{A}(X))\text{Hom}(G, CG) \) of \( G \)-invariant \( \mathcal{A}(X) \)-valued horizontal forms on \( \text{EG} \). \( \square \)

Thus the category of \((G, CG)\)-modules serves, in the smooth category, as a replacement for the non-existent category of comodules relative to the de Rham complex.
of \( G \). Since the differential \( \text{Ext}_{((G,C\mathfrak{g});C)}(\mathbb{R}, \cdot) \) yields \( G \)-equivariant de Rham cohomology, we occasionally refer to the associated infinitesimal object, that is, to the relative differential \( \text{Ext}_{(C\mathfrak{g},\mathfrak{g})}(\mathbb{R}, \cdot) \) isolated in (2.3) above, as \textit{infinitesimal }\mathfrak{g}\text{-equivariant cohomology}.

The present approach to equivariant de Rham theory is completely formal; suitably rephrased it works perfectly well in similar situations and yields e. g. the \textit{equivariant sheaf cohomology} of a complex manifold relative to a holomorphic action of a complex Lie group.

2.8. The relative differential \( \text{Tor}^{(C\mathfrak{g},\mathfrak{g})} \). Here again the ground ring \( R \) is a general commutative ring with 1. Any pair of rings \((\mathcal{R},\mathcal{S})\) with \( \mathcal{R} \supset \mathcal{S} \) gives rise to a resolvent pair of categories, cf. [45] (IX.6), that is to say, the functor \( \mathcal{F}:\text{Mod}_{\mathcal{S}} \to \text{Mod}_{\mathcal{R}} \) which assigns to the right \( \mathcal{S} \)-module \( N \) the induced \( \mathcal{R} \)-module \( \mathcal{F}(N) = N \otimes_{\mathcal{R}} \mathcal{S} \) is left adjoint to the forgetful functor \( \square:\text{Mod}_{\mathcal{R}} \to \text{Mod}_{\mathcal{S}} \). Relative (co)homology is then defined and calculated in terms of a relatively projective resolution in the sense of [22]. Given a right \( \mathcal{R} \)-module \( N \), the \textit{standard construction} \( L(N) \) arising from \( N \) and the \textit{comonad} \((\mathcal{L},\varepsilon,\delta)\) associated with the adjunction is a simplicial object whose associated chain complex \( |L(N)| \) coincides with the standard relatively projective resolution of \( N \) in the sense of [22].

In our case where \( (\mathcal{R},\mathcal{S}) = (U[C\mathfrak{g}],U[\mathfrak{g}]) \), the functor

\[
\mathcal{F}:C_{\mathfrak{g}} \to \text{Mod}_{C_{\mathfrak{g}}}
\]

calculates the Lie algebra homology of \( \mathfrak{g} \) with coefficients in \( N \) (suitably interpreted relative to the chain complex structure on \( N \)), and the differential \( \text{Tor}^{(C\mathfrak{g},\mathfrak{g})} \) and \( \text{Ext}^{(C\mathfrak{g},\mathfrak{g})} \)-functors are defined on the category of \((C\mathfrak{g})\)-modules. The functor \( \text{Ext}^{(C\mathfrak{g},\mathfrak{g})} \) is the same as that introduced in Subsection 2.3 above.

2.9. Lie-Rinehart algebras and Lie algebroids. Even though this is not relevant later in the paper, we spell out briefly a generalization of the situation of (2.3) to illustrate the flexibility of the present formal approach.

Let \((A,L)\) be a Lie-Rinehart algebra and let \( U(A,L) \) be the universal algebra associated with \( A \) and \( L \); see e. g. [31] or [32] for details. Let \( \mathcal{F} : A\text{Mod} \to U(A,L)\text{Mod} \) be the functor which assigns to the \( A \)-module \( M \) the induced \( U(A,L) \)-module

\[
\mathcal{F}(M) = U(A,L) \otimes_A M
\]

and, as before, denote the forgetful functor by \( \square : U(A,L)\text{Mod} \to A\text{Mod} \). The resulting relative \( \text{Ext}_{(U(A,L),A)} \) is precisely the cohomology theory introduced in [51] by means of a generalized CCE complex adapted to Lie-Rinehart algebras. In particular, when \( L \) is projective as an \( A \)-module, the relative \( \text{Ext}_{(U(A,L),A)} \) is an absolute \( \text{Ext}_{U(A,L)} \). In particular, when \( L \) is the \((R,A)\)-Lie algebra \( D_{\{ \cdot,\cdot \}} \) associated with a Poisson structure \( \{ \cdot,\cdot \} \) on \( A \) [30], the relative \( \text{Ext}_{(U(A,L),A)}(A,A) \) coincides with the \textit{Poisson cohomology} of \( A \), cf. [30].

The situation in (2.3) does not extend directly to Lie-Rinehart algebras since the \textit{cone on a Lie-Rinehart algebra is ill-defined}. Indeed, given the \((A,L)\)-module \( M \), for
a \in A and \alpha \in L, the operations \iota of contraction and \lambda of Lie derivative satisfy the familiar identity

\begin{equation}
\lambda_{\alpha \iota}(\omega) = a\lambda_{\iota}(\omega) + da \cup i_{\alpha}(\omega) \quad (a \in A, \alpha \in L, \omega \in \text{Alt}_A(L,M))
\end{equation}

involving the term \( da \cup i_{\alpha}(\omega) \) which does not arise for an ordinary Lie algebra. Thus the values of the functor \( G \) on the category \( (A,L)\text{-Mod} \) of \( (A,L) \)-modules which assigns the Rinehart complex (generalized de Rham complex)

\[ G(V) = (\text{Alt}_A(L,V), d) \]

to the \( (A,L) \)-module \( V \) lie in a certain category \( \mathcal{M} \) of \( (A,L) \)-modules which are also endowed with an action of the ordinary cone \( C_L \) on \( L \) in the category of Lie algebras (beware: not Lie-Rinehart algebras), subject to certain identities including (2.9.1). The resulting adjunction defines a monad and the corresponding dual standard construction yields a relative differential graded Ext. We believe that this is a formally correct approach to phrase developments like the BRST-complex, the variational bicomplex, and the Noether identities. The corresponding constructions for Lie algebroids can then presumably be globalized via Lie groupoids by a suitable comonadic construction. We hope to come back to these issues at another occasion.

We conclude our discussion with two more examples which illustrate the universality of the present relative homological algebra approach to equivariant cohomology. While these examples are not strictly needed for the rest of the paper, they will make it clear that the dual standard construction in (2.2) above which defines differentiable cohomology as well as that which defines the differential Ext\(_{(G,C_G);C} \) in (2.4) above are both versions of completed cobar constructions.

2.10. Rational cohomology of algebraic groups. Let \( k \) be a field, let \( k\text{-Vect} \) be the category of \( k \)-vector spaces, let \( G \) be an algebraic group defined over \( k \), and let \( k[G] \) be the coordinate ring of \( G \). The group structure turns \( k[G] \) into a Hopf algebra. A rational \( G \)-representation is, by definition, a \( k[G] \)-comodule. Let \( k[G] \text{-Comod} \) be the category of \( k[G] \)-comodules or, equivalently, rational \( G \)-representations, and let \( G: k\text{-Vect} \to k[G] \text{-Comod} \) be the functor which assigns to the \( k \)-vector space \( V \) the induced comodule \( k[G] \otimes V \). This functor is right adjoint to the forgetful functor \( \Box: k[G] \text{-Comod} \to k\text{-Vect} \) whence the two functors define a monad, and the resulting dual standard construction yields the appropriate cobar construction which defines the Cotor\(_{k[G]};\cdot,\cdot \) and in particular the rational cohomology of \( G \) with coefficients in a rational \( G \)-module.

2.11. Equivariant de Rham cohomology for algebraic varieties. Let \( k \) be a field and let \( G \) be an algebraic group defined over \( k \). The algebraic de Rham algebra \( \mathcal{A}[G] \) of \( G \) acquires a differential graded Hopf algebra structure. Let \( \mathcal{A}[G] \text{-Comod} \) be the category of \( \mathcal{A}[G] \)-comodules, \( k\mathcal{C} \) that of \( k \)-chain complexes, and let \( G: k\mathcal{C} \to \mathcal{A}[G] \text{-Comod} \) be the functor which assigns to the \( k \)-chain complex \( V \) the induced comodule \( \mathcal{A}[G] \otimes V \), appropriately totalized. This functor is right adjoint to the forgetful functor \( \Box: \mathcal{A}[G] \text{-Comod} \to k\mathcal{C} \) whence the two functors define a monad, and the resulting dual standard construction yields the appropriate cobar construction which defines the functor Cotor\(_{\mathcal{A}[G]};\cdot,\cdot \). Given a non-singular \( G \)-variety \( X \), the algebraic de Rham algebra \( \mathcal{A}[X] \) acquires an \( \mathcal{A}[G] \)-comodule structure, and the algebraic \( G \)-equivariant de Rham cohomology of \( X \) is given by Cotor\(_{\mathcal{A}[G]}(k,\mathcal{A}[X]) \).
3. Infinitesimal equivariant (co)homology

In this section we will explore the *infinitesimal equivariant cohomology funnel* \( \text{Ext}_{(C\mathfrak{g},\mathfrak{g})}(\mathcal{R}, \cdot) \) by means of homological algebra techniques.

The ordinary Weil algebra of a Lie algebra was introduced as an object which arises from abstraction of the operations of contraction and Lie derivative and serves as the principal tool for the description of the Chern-Weil map and of the Weil and Cartan models for equivariant cohomology relative to a compact Lie group.

In \([9]\) (Ex. XIII.14), the CCE resolution is denoted by \( V(\mathfrak{h}) \) and the universal differential graded algebra \( U[\mathcal{C}\mathfrak{h}] \) which, as recalled above, reproduces the CCE resolution, is written as \( W(\mathfrak{h}) \). Is the usage of the letter \( W \) just a notational incidence or, at the time, was the notation \( W \) intended to hint at the relationship with the Weil algebra we are about to explain?

3.1. The Weil coalgebra. We will show that a classical construction in \([22]\), adapted to our situation, leads to what we refer to as the *ordinary Weil coalgebra*. We have introduced that Weil coalgebra in in \([31]\).

As before, let \( \mathcal{R} \) be a commutative ring and \( \mathfrak{g} \) an \( \mathcal{R} \)-Lie algebra, which we suppose throughout to be projective as an \( \mathcal{R} \)-module. As noted earlier, the CCE coalgebra \( S'_\mathfrak{g}[sC\mathfrak{g}] \) is defined for the differential graded Lie algebra \( C\mathfrak{g} \) (the cone on \( \mathfrak{g} \)). We will write this differential graded coalgebra as \( W'[\mathfrak{g}] = S'_\mathfrak{g}[sC\mathfrak{g}] \). Thus \( d + \partial \) turns \( W'[\mathfrak{g}] \) into a differential graded coalgebra which we refer to as the *Weil coalgebra* of \( \mathfrak{g} \). The dual \( \text{Hom}(W'[\mathfrak{g}], \mathcal{R}) \) is the ordinary *Weil algebra* of \( \mathfrak{g} \). For later reference, we will now spell out some additional structure on the Weil coalgebra.

(3.1.1) Since, as a graded \( \mathcal{R} \)-module, \( sC\mathfrak{g} = s^2\mathfrak{g} \oplus \mathfrak{s}\mathfrak{g} \), as a graded coalgebra, the Weil coalgebra decomposes canonically as

\[
W'[\mathfrak{g}] \cong S'[s^2\mathfrak{g}] \otimes \Lambda'[\mathfrak{s}\mathfrak{g}].
\]

(3.1.2) The underlying graded Lie algebra of \( C\mathfrak{g} \) acts on \( S'[s^2\mathfrak{g}] \) in a canonical way through the projection from \( C\mathfrak{g} \) to \( \mathfrak{g} \) (not a differential graded projection). The graded \( (C\mathfrak{g}) \)-module structures on \( \Lambda'[\mathfrak{s}\mathfrak{g}] \) and \( S'[s^2\mathfrak{g}] \) combine to a graded right \((C\mathfrak{g})\)-module structure

\[
W'[\mathfrak{g}] \otimes C\mathfrak{g} \rightarrow W'[\mathfrak{g}]
\]

which is, in fact, a differential graded right \((C\mathfrak{g})\)-module structure since the construction of the differential graded CCE coalgebra is functorial in the differential graded Lie algebra variable.

(3.1.3) Relative to the \( \mathfrak{g} \)-action on the right of \( S'[s^2\mathfrak{g}] \) coming from the adjoint action of \( \mathfrak{g} \) on itself,

\[
(W'[\mathfrak{g}], \partial) = S'[s^2\mathfrak{g}] \otimes_{\mathfrak{g}} \Lambda'_\mathfrak{g}[\mathfrak{s}\mathfrak{g}] \cong S'[s^2\mathfrak{g}] \otimes_{\mathfrak{g}} U[C\mathfrak{g}],
\]

that is, \((W'[\mathfrak{g}], \partial)\) is precisely the standard complex computing the Lie algebra homology of \( \mathfrak{g} \) with values in \( S'[s^2\mathfrak{g}] \), viewed as a right \( \mathfrak{g} \)-module.

(3.1.4) Let \( \tau' : S'[s^2\mathfrak{g}] \rightarrow \Lambda[\mathfrak{s}\mathfrak{g}] \) be the standard universal twisting cochain. Relative to the decomposition \( S'[s^2\mathfrak{g}] \otimes \Lambda[\mathfrak{s}\mathfrak{g}] \) of \( W'[\mathfrak{g}] \), the differential \( d \) is the operator
\( \partial^{r'g} = -(r^{g'} \cap \cdot): S'[s_2^g] \otimes \Lambda[s\mathfrak{g}] \to S'[s^g_2] \otimes \Lambda[s\mathfrak{g}], \) so \( \Lambda[s\mathfrak{g}] \) is viewed as fiber and \( S'[s_2^g] \) as base of the corresponding twisted tensor product whence, as chain complexes,

\[
(W'[\mathfrak{g}], d) = (S'[s_2^g] \otimes \Lambda[s\mathfrak{g}], d) = S'[s_2^g] \otimes_{r^{g'}} \Lambda[s\mathfrak{g}].
\]

(3.1.5) The graded right \((C\mathfrak{g})\)-module structures on \( \Lambda'[s\mathfrak{g}] \) and \( S'[s_2^g] \) combine to a differential graded right \((C\mathfrak{g})\)-module structure on \((W'[\mathfrak{g}], \partial)\) as well (on \( W'[\mathfrak{g}] \) endowed merely with the operator \( \partial \)), that is, the pairing (3.1.2.1), written out in the form

\[
(S'[s_2^g] \otimes_{r_\partial} \Lambda'_{\partial}[s\mathfrak{g}]) \otimes C\mathfrak{g} \to S'[s_2^g] \otimes_{r_\partial} \Lambda'_{\partial}[s\mathfrak{g}]
\]

is compatible with the differentials. This reflects the familiar fact that the effect of the adjoint action on Lie algebra homology is trivial, cf. (1.3) above.

Henceforth we will denote by \( S'_{2p}[s_2^g] \) the \( k \)-th homogeneous constituent of the graded symmetric coalgebra \( S'[s_2^g] \). In [22], for a pair \((a, b)\) of ordinary Lie algebras, Hochschild has introduced an acyclic relatively projective CCE complex which yields the relative Lie algebra cohomology of the pair \((a, b)\) in the sense of Chevalley-Eilenberg [10]. This CCE complex arises by abstraction from the situation of the invariant de Rham complex of a homogeneous space of compact connected Lie groups. Hochschild has furthermore shown that, when \( b \) is reductive in \( a \), that CCE complex is a relatively projective resolution of the ground ring, that is, that CCE complex admits a \( b \)-equivariant contracting homotopy. The literal translation of that CCE construction, to the pair \((C\mathfrak{g}, \mathfrak{g})\) of differential graded Lie algebras, yields the following:

**Proposition 3.1.7.** The Weil coalgebra \( W'[\mathfrak{g}] \), written out in the form

\[
\ldots \xrightarrow{d} S'_{2p}[s_2^g] \otimes_{r_\partial} \Lambda'_{\partial}[s\mathfrak{g}] \xrightarrow{d} \ldots \xrightarrow{d} S'_{2}[s_2^g] \otimes_{r_\partial} \Lambda'_{\partial}[s\mathfrak{g}] \xrightarrow{d} \Lambda'_{\partial}[s\mathfrak{g}],
\]

is a relatively projective complex in the category of right \((U[C\mathfrak{g}])\)-modules which, augmented by the obvious augmentation map \( \varepsilon: \Lambda'_{\partial}[s\mathfrak{g}] \to R \) (the counit of \( \Lambda'_{\partial}[s\mathfrak{g}] \)), is an acyclic complex. Here, for each \( p \geq 1 \), the right \((U[C\mathfrak{g}])\)-module structure is the obvious one on

\[
S'_{2p}[s_2^g] \otimes_{r_\partial} \Lambda'[s\mathfrak{g}] = S'_{2p}[s_2^g] \otimes_{U[\mathfrak{g}]} U[C\mathfrak{g}].
\]

**Proof.** In view of the preparatory steps (3.1.1)–(3.1.5) this is straightforward and left to the reader. \( \square \)

We do not claim that (3.1.8) has a \( \mathfrak{g} \)-linear contracting homotopy. Thus we do not assert that (3.1.8) is a relatively projective resolution of \( R \) in the category of right \((U[C\mathfrak{g}])\)-modules.

To spell out the appropriate structure that the Weil coalgebra \( W'[\mathfrak{g}] \) acquires, we return to the situation of (2.8) above: Thus, consider a pair of algebras \((\mathcal{R}, \mathcal{S})\) with \( \mathcal{R} \supset \mathcal{S} \) and suppose that \( \mathcal{R} \) is actually an augmented differential graded algebra and that \( \mathcal{S} \) is an augmented differential graded subalgebra with zero differential. Given a chain complex \( V \) we denote by \( V^\sharp \) the graded \( \mathcal{R} \)-module underlying \( V \). Let \( M^\sharp \) be a differential graded right \( \mathcal{S} \)-module. We will refer to an augmented differential graded right \( \mathcal{R} \)-module \( M^\sharp \) is an induced
graded module of the kind $M^\sharp = M^\sharp \otimes S R^\sharp$ as a construction for $\mathcal{R}$ relative to $S$ provided the induced isomorphism

$$\overline{M} \rightarrow M \otimes_\mathcal{R} R$$

of graded $R$-modules is an isomorphism of chain complexes. When $S$ is the ground ring this notion of construction comes down to the usual notion of construction in the sense of H. Cartan, cf. [48].

Proposition 3.1.7 plainly says that the Weil coalgebra $W'[\mathfrak{g}]$ is an $R$-acyclic, even $R$-contractible construction for $U[g]$ relative to $U[\mathfrak{g}]$.

**Theorem 3.1.9.** Suppose that the ground ring $R$ is a field of characteristic zero and that $\mathfrak{g}$ is reductive. For any right $(C\mathfrak{g})$-module $N$, the relative differential graded $\text{Ext}_{(C\mathfrak{g},\mathfrak{g})}(R,N)$ is the homology of the chain complex

$$\text{Hom}(W'[\mathfrak{g}], N)^{C\mathfrak{g}}.$$ (3.1.10)

Likewise for any left $(C\mathfrak{g})$-module $M$, the relative differential graded $\text{Tor}_{(C\mathfrak{g},\mathfrak{g})}(R,M)$ is the homology of the chain complex

$$W'[\mathfrak{g}] \otimes_{C\mathfrak{g}} M.$$ (3.1.11)

We will prove Theorem 3.1.9 in (3.3.6) and in (3.4) below. Under the circumstances of Theorem 3.1.9, we will refer to (3.1.10) as the Weil model for the differential graded $\text{Ext}_{(C\mathfrak{g},\mathfrak{g})}(R,N)$. In the language of ordinary differential geometry, (3.1.10) consists of the basic elements of $\text{Hom}(W'[\mathfrak{g}], N)$: indeed, $C\mathfrak{g} = (s\mathfrak{g}) \times \mathfrak{g}$, the invariants relative to the constituent $\mathfrak{g}$ are the invariant elements (in the usual sense) and the invariants relative to the constituent $s\mathfrak{g}$ are the horizontal elements whence the $(C\mathfrak{g})$-invariants are the elements which are horizontal and invariant. In Section 6 below we shall show that a variant of (3.1.10) yields the familiar Weil model for the equivariant cohomology relative to a finite-dimensional compact connected Lie group.

Recall that, for purely formal reasons, $\text{Ext}_{(C\mathfrak{g},\mathfrak{g})}(R,R)$ acquires a graded commutative algebra structure.

**Corollary 3.1.12.** When the ground ring $R$ is a field of characteristic zero and when $\mathfrak{g}$ is reductive, as a graded commutative algebra, $\text{Ext}_{(C\mathfrak{g},\mathfrak{g})}(R,R)$ is canonically isomorphic to the algebra $\text{Hom}(S'[s^2\mathfrak{g}], R)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants of the algebra $\text{Hom}(S'[s^2\mathfrak{g}], R)$, that is, to the algebra of $\mathfrak{g}$-invariants of the symmetric algebra $S[s^{-2}\mathfrak{g}^*]$ on the double desuspension $s^{-2}\mathfrak{g}^*$ of the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. Likewise, $\text{Tor}_{(C\mathfrak{g},\mathfrak{g})}(R,R)$ then acquires a graded coalgebra structure and, as a graded coalgebra, is canonically isomorphic to the coalgebra $S'[s^2\mathfrak{g}] \otimes_\mathfrak{g} R$ of $\mathfrak{g}$-coinvariants of the graded coalgebra $S'[s^2\mathfrak{g}]$. □

**Remark 3.1.13.** When $\mathfrak{g}$ is abelian, the Weil coalgebra $W'[\mathfrak{g}]$, written out as a chain complex as in (3.1.8), plainly comes down to the ordinary Koszul resolution of the ground ring in the category of $(\Lambda[s\mathfrak{g}])$-modules.

### 3.2. The relative bar resolution
Given a right $(C\mathfrak{g})$-module, this resolution is the standard resolution of that module associated with the comonad $(\mathcal{L}, \varepsilon, \delta)$ mentioned in Subsection 2.8 above; cf. [13] and [46] for this notion of standard resolution.
Let $\beta_{U[g]}(U[Cg])$ denote the two-sided simplicial bar resolution of $U[Cg]$ relative to the category of $(U[g])$-modules; this is a simplicial $(U[g])$-bimodule. Normalization yields the associated two-sided normalized relative bar resolution $B_{U[g]}(U[Cg])$ of $U[Cg]$ in the category of $(U[Cg])$-bimodules.

Let $M$ be a right $(U[g])$-module. The simplicial right $(U[Cg])$-module

\[(3.2.1) \quad \beta_{U[g]}(M, U[Cg], U[Cg]) = M \otimes_{U[Cg]} \beta_{U[g]}(U[Cg])\]

is the simplicial bar resolution of $M$ in the category of right $(U[Cg])$-modules relative to the category of $(U[g])$-modules. This is precisely the standard resolution of $M$ relative to the comonad $(\varepsilon, \delta)$. The normalized chain complex of the simplicial object $M \otimes_{U[Cg]} \beta_{U[g]}(U[Cg])$ yields the normalized relative bar resolution $B_{U[g]}(M, U[Cg], U[Cg])$ of $M$ in the category of right $(U[Cg])$-modules, plainly a resolution of $M$ in the category of right $(U[Cg])$-modules that is projective relative to the category of $(U[g])$-modules. In particular, when $M = R$, viewed as a trivial right $(U[Cg])$-module, $R \otimes_{U[Cg]} B_{U[g]}(U[Cg])$ leads to the normalized relative bar resolution $B_{U[g]}(R, U[Cg], U[Cg])$ of $R$ in the category of right $(U[Cg])$-modules. Our notation for the functor $B_{U[g]}$ etc. is that in [20], with $U[g]$ substituted for the ground ring. With the notation $IA$ for the augmentation ideal of the exterior algebra $\Lambda$, the resolution $B_{U[g]}(R, U[Cg], U[Cg])$ has the form

\[(3.2.2) \quad \ldots \xrightarrow{d} (\Lambda[s][g])^n \otimes \Lambda[s][g], \partial \xrightarrow{d} \ldots \xrightarrow{d} (\Lambda[s][g] \otimes \Lambda[s][g], \partial) \xrightarrow{d} (\Lambda[s][g], \partial)\]

where $d$ is the differential arising from the operation of normalization. Here the (differential graded) right $(U[Cg])$-module structure is induced by the obvious pairing map

\[(\Lambda[s][g])^n \otimes \Lambda[s][g] \otimes (U[g] \otimes \Lambda[s][g]) \rightarrow (\Lambda[s][g])^n \otimes \Lambda[s][g]\]

coming from the obvious diagonal action of $U[g]$ on the right of $(\Lambda[s][g])^n \otimes \Lambda[s][g]$ and from the right multiplication action of $\Lambda[s][g]$ on itself.

To spell out the operator $\partial$ explicitly, we note that the augmentation map $\varepsilon$ of $\Lambda[s][g]$ yields the counit

\[\varepsilon: \Lambda'[s][g] \rightarrow R\]

of the CCE coalgebra $\Lambda'[s][g]$ of $g$ as well; this kernel is plainly a differential graded cocommutative coalgebra without counit and without coaugmentation and, whenever this coalgebra structure is under discussion, we use the notation $IA'[s][g]$ for the kernel of $\varepsilon$ and refer to it as the augmentation coalgebra of $\Lambda'[s][g]$. For $n \geq 1$, the iterated twisted tensor product

\[(3.2.3) \quad (IA'[s][g]) \otimes_{\tau_0} (IA'[s][g]) \otimes_{\tau_0} \ldots \otimes_{\tau_0} (IA'[s][g])\]

of $n$ copies of $IA'[s][g]$ is still defined, e. g. as a subcomplex of the twisted tensor product

\[\Lambda'[s][g] \otimes_{\tau_0} \Lambda'[s][g] \otimes_{\tau_0} \ldots \otimes_{\tau_0} \Lambda'[s][g]\]

of $n$ copies of $\Lambda'[s][g]$, and the chain complex $((IA'[s][g])^n \otimes \Lambda[s][g], \partial)$ in the resolution (3.2.2) amounts to the iterated twisted tensor product

\[(3.2.4) \quad ((IA'[s][g]) \otimes_{\tau_0} (IA'[s][g]) \otimes_{\tau_0} \ldots \otimes_{\tau_0} (IA'[s][g])) \otimes_{\tau_0} \Lambda'[s][g]\]
of $n$ copies of $I\Lambda'[\mathfrak{g}]$ with a single copy of $\Lambda'[\mathfrak{g}]$.

Following Mac Lane [45], we will use the notation $|\cdot|^*$ for the condensation functor. Thus condensation transforms the resolution (3.2.2) into the corresponding construction

$$|R \otimes_{U[C\mathfrak{g}]} \beta_{U[\mathfrak{g}]}(U[C\mathfrak{g}])|^* = \left(T'[sI\Lambda[\mathfrak{g}]] \otimes \Lambda[\mathfrak{g}], d \otimes \text{Id}_{\Lambda[\mathfrak{g}]} - \tau^B \cap \cdot + \partial \right)$$

in the relative sense explained above, and we will use the notation

$$\text{(3.2.5)} \quad BA_\theta[\mathfrak{g}] = |R \otimes_{U[C\mathfrak{g}]} \beta_{U[\mathfrak{g}]}(U[C\mathfrak{g}])|^*$$

Thus $BA_\theta[\mathfrak{g}]$ is a construction for $U[C\mathfrak{g}]$ relative to $U[\mathfrak{g}]$. The ordinary bar construction contracting homotopy is a $\mathfrak{g}$-linear contracting homotopy for this construction.

Pushing a bit further, we observe that, for each $n \geq 1$, via the obvious morphism $s^\otimes n : (I\Lambda[\mathfrak{g}])^\otimes n \to (sI\Lambda[\mathfrak{g}])^\otimes n$, the operator $\partial$ given by the chain complex (3.2.3) induces an operator

$$\partial : (sI\Lambda[\mathfrak{g}])^\otimes n \to (sI\Lambda[\mathfrak{g}])^\otimes n$$

(where the notation $\partial$ is abused again). These operators assemble to an operator $\partial$ on the graded tensor coalgebra $T'[sI\Lambda[\mathfrak{g}]]$, and we will write

$$\overline{B}A_\theta[\mathfrak{g}] = (T'[sI\Lambda[\mathfrak{g}]], \partial).$$

The object $\overline{B}A_\theta[\mathfrak{g}]$ carries an obvious right $\mathfrak{g}$-module structure, and

$$(T'[sI\Lambda[\mathfrak{g}]] \otimes \Lambda[\mathfrak{g}], \partial) = \overline{B}A_\theta[\mathfrak{g}] \otimes_{\tau_\theta} \Lambda_\theta'[\mathfrak{g}].$$

Thus, relative to the operator $\partial$, the object under discussion appears as a twisted tensor product relative to $\Lambda_\theta'[\mathfrak{g}]$ as base and $\overline{B}A_\theta[\mathfrak{g}]$ as fiber; cf. (3.14) above where the corresponding twisted tensor product decomposition is spelled out relative to the differential $d$.

It is obvious that, when $\mathfrak{g}$ is abelian, the normalized relative resolution comes down to the ordinary bar resolution of $R$ in the category of right $(\Lambda[\mathfrak{g}])$-modules.

**Theorem 3.2.6.** **Relative to the tensor product coalgebra structure on $T'[sI\Lambda[\mathfrak{g}]] \otimes \Lambda'[\mathfrak{g}]$, the condensed object $BA_\theta[\mathfrak{g}]$ (cf. (3.2.5)) is a differential graded coalgebra.**

**Proof.** This comes down to a tedious verification. The statement is also a consequence of Theorem 4.5 below, see Remark 4.6.

### 3.3. Comparison between the Weil coalgebra and the relative bar construction.

The bracket on $\mathfrak{g}$ being momentarily ignored, let $\tau^{S'} : S'[s^2\mathfrak{g}] \to \Lambda[\mathfrak{g}]$ be the obvious acyclic twisting cochain; its adjoint is the canonical injection $\overline{\tau^{S'}} : S'[s^2\mathfrak{g}] \to \overline{BA}[\mathfrak{g}]$ of differential graded coalgebras. Let

$$\iota = \overline{\tau^{S'}} \otimes \text{Id} : S'[s^2\mathfrak{g}] \otimes_{\tau^{S'}} \Lambda[\mathfrak{g}] \to \overline{BA}[\mathfrak{g}] \otimes_{\tau^{\Pi}} \Lambda[\mathfrak{g}].$$

This is a morphism

$$\text{(3.3.1)} \quad \iota : S'[s^2\mathfrak{g}] \otimes_{\tau^{S'}} \Lambda[\mathfrak{g}] \to BA[\mathfrak{g}]$$

of differential graded coalgebras. The canonical comparison between $W'[\mathfrak{g}]$ and $BA_\theta[\mathfrak{g}]$ is achieved by the following.
Theorem 3.3.2. The perturbation \( \partial \) determined by the Lie bracket on \( \mathfrak{g} \) being taken into account, \( \iota \) is a morphism

\[
\iota: W'[\mathfrak{g}] \rightarrow B\Lambda_\partial[\mathfrak{g}]
\]

of differential graded coalgebras which is, furthermore, compatible with the right \((Cg)\)-module structures.

This comparison is formally exactly of the same kind as the classical comparison, spelled out in detail in [9] (chap. XIII), between the CCE complex for an ordinary Lie algebra \( \mathfrak{g} \) and the bar complex for \( U[\mathfrak{g}] \).

The proof will rely on Lemma 3.3.4 below. To prepare for it, we note that, as a graded coalgebra,

\[
B\Lambda_\partial[\mathfrak{g}] = B\Lambda[\mathfrak{g}] = \mathcal{B}\Lambda[\mathfrak{g}] \otimes \Lambda'[\mathfrak{g}] = T'[s\Lambda[\mathfrak{g}]] \otimes \Lambda'[\mathfrak{g}]
\]

and that, as a graded algebra, \( U[C\mathfrak{g}] = \Lambda[\mathfrak{g}] \otimes U[\mathfrak{g}] \), the crossed product algebra. Abusing the notation \( \tau^\mathcal{B} \) and \( \tau^\mathfrak{g} \), slightly, write

\[
\tau^\mathcal{B} = \tau^\mathcal{B} \otimes \eta \varepsilon: B\Lambda[\mathfrak{g}] = T'[s\Lambda[\mathfrak{g}]] \otimes \Lambda'[\mathfrak{g}] \rightarrow \Lambda[\mathfrak{g}] \otimes U[\mathfrak{g}] = U[C\mathfrak{g}]
\]

and, likewise, write

\[
\tau^\mathfrak{g} = \eta \varepsilon \otimes \tau^\mathfrak{g}: B\Lambda[\mathfrak{g}] = T'[s\Lambda[\mathfrak{g}]] \otimes \Lambda'[\mathfrak{g}] \rightarrow \Lambda[\mathfrak{g}] \otimes U[\mathfrak{g}] = U[C\mathfrak{g}].
\]

Lemma 3.3.4. The sum \( \tau^\mathcal{B} + \tau^\mathfrak{g} \) is a twisting cochain

\[
\tau^\mathcal{B} + \tau^\mathfrak{g}: B\Lambda_\partial[\mathfrak{g}] \rightarrow U[C\mathfrak{g}].
\]

Proof. We must prove that

\[
D(\tau^\mathcal{B} + \tau^\mathfrak{g}) = (\tau^\mathcal{B} + \tau^\mathfrak{g}) \cup (\tau^\mathcal{B} + \tau^\mathfrak{g})
\]

that is,

\[
D\tau^\mathcal{B} + D\tau^\mathfrak{g} = \tau^\mathcal{B} \cup \tau^\mathcal{B} + \tau^\mathfrak{g} \cup \tau^\mathfrak{g} + \tau^\mathfrak{g} \cup \tau^\mathcal{B} + \tau^\mathfrak{g} \cup \tau^\mathfrak{g}.
\]

We note first that

\[
D\tau^\mathfrak{g} = \tau^\mathfrak{g}(\partial - \tau^\mathfrak{g})
\]

\[
D\tau^\mathcal{B} = d\tau^\mathcal{B} + \tau^\mathcal{B}(d + \partial - \tau^\mathfrak{g}).
\]

By construction, \( \Lambda'_\partial[\mathfrak{g}] \) is a differential graded subcoalgebra of \( B\Lambda_\partial[\mathfrak{g}] \), the algebra \( U[\mathfrak{g}] \) is a differential graded subalgebra of \( U[C\mathfrak{g}] \), and the restriction of \( \tau^\mathcal{B} + \tau^\mathfrak{g} \) to \( \Lambda'_\partial[\mathfrak{g}] \) amounts to the composite of \( \tau^\mathfrak{g}: \Lambda'_\partial[\mathfrak{g}] \rightarrow U[\mathfrak{g}] \) with the injection of \( U[\mathfrak{g}] \) into \( U[C\mathfrak{g}] \). Consequently

\[
\tau^\mathfrak{g}\partial = \tau^\mathfrak{g} \cup \tau^\mathfrak{g}.
\]
Furthermore, $\tau^B$ is the bar construction twisting cochain for $\Lambda[s\mathfrak{g}]$ whence

$$\tau^B d = \tau^B \cup \tau^B$$

and, likewise,

$$d\tau^B - \tau_g(\tau^B \cap) = 0.$$ 

The identities established so far in particular show that $\tau$ is a twisting cochain in the special case where the bracket is zero.

It remains to show that

$$\tau^B \cup \tau_g + \tau_g \cup \tau^B = \tau^B \partial.$$ 

Given the elements $x$ and $y$ of $\mathfrak{g}$,

$$\partial(s^2 y \otimes sx) = -s\partial(sy \otimes sx) = s(s[y,x]) = -s^2[y,x]$$

whereas, since $\tau_g(sx) = x$,

$$(\tau^B \cup \tau_g + \tau_g \cup \tau^B)(s^2 y \otimes sx) = -s[x,y]$$

whence, indeed,

$$\left(\tau^B \cup \tau_g + \tau_g \cup \tau^B\right)(s^2 y \otimes sx) = -s[x,y] = \tau^B \partial(s^2 y \otimes sx). \quad \Box$$

**Proof of Theorem 3.3.2.** The adjoint

$$\overline{\tau^B + \tau_g} : B\Lambda[s\mathfrak{g}] \longrightarrow \overline{\mathbb{B}[C\mathfrak{g}]}$$

is a morphism of differential graded coalgebras, necessarily injective. The ordinary Weil coalgebra $W'[\mathfrak{g}]$ has been defined as the CCE coalgebra $S'[sC\mathfrak{g}]$ for the differential graded Lie algebra $C\mathfrak{g}$. The composite

$$(\tau^B + \tau_g) \circ \iota : W'[\mathfrak{g}] \longrightarrow \overline{\mathbb{B}[C\mathfrak{g}]}$$

plainly coincides with the adjoint

$$\overline{\tau_C} : W'[\mathfrak{g}] \longrightarrow \overline{\mathbb{B}[C\mathfrak{g}]}$$

of the universal twisting cochain $\tau_C : W'[\mathfrak{g}] \longrightarrow \mathbb{B}[C\mathfrak{g}]$. Consequently $\iota$ is compatible with the structure as asserted. \quad \Box

**Remark 3.3.5.** The classical fact that, relative to the zero bracket, the canonical injection $S'[s^2\mathfrak{g}] \rightarrow B\Lambda[s\mathfrak{g}]$ is an isomorphism on homology corresponds to, indeed, is equivalent to the statement “$H(\Lambda^q\Sigma s\mathfrak{g}^*) = S^q\mathfrak{g}^*$ (in dim $q$)” in Lemma 3.1 of [4]. This statement, in turn, is established there by means of the observation that the appropriate Dold-Puppe derived functor [11] of the $q$-th exterior power functor $\Lambda^q$ is the $q$-th symmetric power functor $S^q$, suitably shifted.
3.3.6. A SPECTRAL SEQUENCE PROOF OF THEOREM 3.1.9. Let $N$ be a right $(Cg)$-module. The comparison $\iota$ plainly induces a morphism

$$\iota^*: \text{Hom}(W'[g], N)^{Cg} \to \text{Hom}(\text{BA}_0[sg], N)^{Cg}$$

of chain complexes. As a morphism of the underlying graded objects, $\iota^*$ can be written as

$$\text{Hom}(S'[s^2g], N)^g \to \text{Hom}(\text{BA}_0[sg], N)^g.$$ 

The coaugmentation filtrations of $S'[s^2g]$ and $\text{BA}_0[sg]$ induce Serre filtrations on both sides of $\iota^*$ and $\iota^*$ is compatible with the filtrations whence it induces a morphism between the associated spectral sequences. At the $E_0$-level, the comparison comes down to the standard comparison, between the complexes induces by the bar and Koszul resolutions for the exterior algebra $\Lambda[sg]$, but restricted to the $g$-invariants, and thence this comparison has the form

$$\text{Hom}(S'[s^2g], N)^g \to \text{Hom}(\text{BA}_0[sg], N)^g.$$ 

Since $g$ is reductive, from the $E_1$-level on, $\iota^*$ induces an isomorphism of spectral sequences. Consequently $\iota^*$ is an isomorphism on homology whence the relative differential graded Ext$_{Cg}(R, N)$ is the homology of the chain complex (3.1.10) as asserted. The same kind of reasoning shows that, for any left $(Cg)$-module $M$, the relative differential graded Tor$_{Cg}(R, M)$ is the homology of the chain complex (3.1.11). This proves Theorem 3.1.9.

3.4. THE WEIL COALGEBRA $W'[g]$ AS A RELATIVE $U[g]$-CONTRACTIBLE CONSTRUCTION IN THE REDUCTIVE CASE. Suppose that the ground ring $R$ is a field of characteristic zero and let $g$ be a reductive Lie algebra. The diagonal map of $g$ induces a graded commutative algebra structure on $H^*(g)$ and, furthermore, a graded cocommutative coalgebra structure on $H_*(g)$. The projection $\pi: \Lambda'_0[sg] \to \Lambda'_0[sg] \otimes g R$, restricted to the invariants $\Lambda'[sg]^g$, is an isomorphism whence the differential on $\Lambda'_0[sg] \otimes g R$ is zero, and we will write $\Lambda'[sg] \otimes g R$ rather than $\Lambda'_0[sg] \otimes g R$. This quotient is naturally isomorphic to the homology $H_+(g)$; further, as a chain complex, $\Lambda'_0[sg]$ decomposes as

$$\Lambda'_0[sg] = \ker(\pi) \oplus \Lambda'_0[sg]^g \cong \ker(\pi) \oplus \Lambda'[sg] \otimes g R,$$

and $\ker(\pi)$ is a contractible chain complex. The quotient $\Lambda'[sg] \otimes g R \cong H_+(g)$ acquires a graded cocommutative coalgebra structure for purely formal reasons; the resulting coalgebra structure on $H_+(g)$ is the one induced by the diagonal of $g$.

The kernel of the composite of the projection $U[g] \to \Lambda'_0[sg]$ with $\pi$ is the (differential graded) two-sided ideal $\langle g \rangle$ in $U[g]$ generated by $g$ whence $H_+(g)$ acquires a graded commutative algebra structure which combines with the coalgebra structure to a Hopf algebra structure. Dually the cohomology $H^+(g)$ acquires a coalgebra structure which combines with its algebra structure to a Hopf algebra structure. As an algebra, $H^+(g)$ is the exterior algebra $\Lambda[\text{Prim}(g)]$ generated by the primitives $\text{Prim}(g) \subseteq H^+(g)$ relative to the coalgebra structure. The dual $I(g)$ of $\text{Prim}(g)$ is the module of indecomposables relative to the algebra structure on $H_+(g)$, the injection of $\text{Prim}(g)$ into $H^+(g)$ dualizes to the canonical projection $H_+(g) \to I(g)$.
(defining the indecomposables) and, as a graded coalgebra, $H_\ast(g)$ is the exterior coalgebra $\Lambda'[I(g)]$ cogenerated by $I(g)$.

As a graded coalgebra, $\text{Tor}^{C\otimes g}(R,R)$ is the cofree graded cocommutative coalgebra $S'[sI(g)]$ cogenerated by the suspension $sI(g)$ of $I(g)$. Pick a section $j:I(g) \rightarrow \Lambda[I(g)]$ for the projection $\Lambda[I(g)] \rightarrow I(g)$. Then the composite

$$\tau:S'[sI(g)] \xrightarrow{pr} sI(g) \xrightarrow{s^{-1}} I(g) \xrightarrow{j} \Lambda[I(g)]$$

is a transgression twisting cochain. By *transgression twisting cochain* we mean a twisting cochain which induces the transgression in the corresponding spectral sequence; cf. [45] for the notion of transgression in a spectral sequence. We will write $\Lambda = \Lambda[I(g)] (= H_\ast(g))$ and $S' = S'[sI(g)]$.

**Theorem 3.4.1.** The Lie algebra $g$ being reductive, the Weil coalgebra $W'[g]$ admits a $g$-equivariant contracting homotopy.

**Proof.** Since $g$ is reductive, Hodge theory yields a contraction

$$(3.4.1.1) \quad \left( S' \otimes \Lambda' \xleftarrow{\pi_1} S'[s^2 g] \otimes_{\tau_0} \Lambda_0'[s g], h_1 \right)$$

in the category of $g$-modules, the $g$-actions on $S'$ and $\Lambda'$ being trivial. See e.g. [43] for details. Recall that $W'[g] = (S'[s^2 g] \otimes_{\tau_0} \Lambda_0'[s g], d)$. Relative to the Serre filtrations, the contraction (3.4.1.1) is a filtered contraction, and an application of the perturbation lemma transforms the contraction (3.4.1.1) into the (filtered) contraction

$$(3.4.1.2) \quad \left( S' \otimes_{\tau} \Lambda' \xleftarrow{\pi_2} W'[g], h_2 \right)$$

in the category of $g$-modules. The twisted tensor product $S' \otimes_{\tau} \Lambda'$ is contractible; thus let

$$(3.4.1.3) \quad \left( R \xleftarrow{\eta} S' \otimes_{\tau} \Lambda', h_3 \right)$$

be a contraction of $S' \otimes_{\tau} \Lambda'$ onto the ground ring $R$, necessarily $g$-equivariant, the $g$-actions being trivial; beware: the contracting homotopy $h_3$ is not unique. Further, this kind of contraction is not a filtered one relative to the Serre filtration of $S' \otimes_{\tau} \Lambda'$, though. Combining the two contractions, we obtain the contraction

$$(3.4.1.4) \quad \left( R \xleftarrow{\eta} W'[g], h \right)$$

in the category of $g$-modules. □

Theorem 3.1.9 is a consequence of Theorem 3.4.1. Indeed, the Lie algebra being reductive, pick a $g$-equivariant contracting homotopy of $W'[g]$ of the kind constructed in Theorem 3.4.1. Then the canonical comparison, cf. [45] (Theorem IX.6.2 on p. 267), [48], yields a $(U[Cg])$-linear morphism $\alpha_\partial: \text{BA}_0'[sg] \rightarrow W'[g]$ and homogeneous $(U[Cg])$-linear operators $h_\partial$ on $\text{BA}_0'[sg]$ and $h_W$ on $W'[g]$ of degree 1 such that

$$(3.4.2) \quad Dh_\partial = \text{Id} - \omega_\partial, \quad Dh_W = \text{Id} - \alpha_\partial.$$
Thus the data

\[(3.4.3) \quad \left( h_W, W'[\mathfrak{g}] \xrightarrow{\alpha} B\Lambda[\mathfrak{g}], h_\partial \right) \]

continue a filtered chain equivalence which is, furthermore, \((U[C\mathfrak{g}])\)-linear. The notion of filtered chain equivalence was introduced in [37]; in the present paper we shall exclusively use the defining property (3.4.2), though, and no reference to [37] will be made. The standard reasoning then immediately establishes Theorem 3.1.9.

3.5. THE CARTAN MODEL. Return momentarily to a general ground ring \(R\). Let \(\mathfrak{g}\) be an \(R\)-Lie algebra which, as an \(R\)-module, is projective. Given the right \((C\mathfrak{g})\)-module \(N\), the chain complex \(\text{Hom}(W'[\mathfrak{g}], N)\) \(C\mathfrak{g}\), cf. (3.1.10) above, is still defined. We will now rewrite this chain complex as a twisted object.

When the Lie bracket on \(\mathfrak{g}\) is ignored, as a differential graded right \((\Lambda[\mathfrak{g}])\)-module, \(W'[\mathfrak{g}]\) has the form \(S'[s^2\mathfrak{g}] \otimes \Lambda[\mathfrak{g}]\), the differential being the operator

\[
\partial_{\tau^{S'}} = - (\tau^{S'} \cap \cdot): S'[s^2\mathfrak{g}] \otimes \Lambda[\mathfrak{g}] \rightarrow S'[s^2\mathfrak{g}] \otimes \Lambda[\mathfrak{g}]
\]

relative to the universal twisting cochain \(\tau^{S'}: S'[s^2\mathfrak{g}] \rightarrow \Lambda[\mathfrak{g}]\). With reference to the graded right \((\Lambda[\mathfrak{g}])\)-module structure on \(N\), when the differential on \(N\) is ignored, the twisted Hom-object \(\text{Hom}(S'[s^2\mathfrak{g}], N)\) is defined; we remind the reader that the operator \(\delta\) on \(\text{Hom}(S'[s^2\mathfrak{g}], N)\), cf. (1.2) above and [34] (2.4.1), is defined by \(\delta(f) = (-1)^{|f|} f \cup \tau^{S'}\), the argument \(f\) being a homogeneous morphism. With the differential on \(N\) and the Lie bracket on \(\mathfrak{g}\) incorporated, on the \(\mathfrak{g}\)-invariants, the operator \(\delta\) on \(\text{Hom}(S'[s^2\mathfrak{g}], N)\) is still a perturbation of the differential on \(\text{Hom}(S'[s^2\mathfrak{g}], N)\) (coming from that on \(N\)), and we write the resulting twisted object as

\[(3.5.1) \quad \text{Hom}(S'[s^2\mathfrak{g}], N)^{\mathfrak{g}}.\]

The exterior algebra \(\Lambda = \Lambda[\mathfrak{g}]\) is a Hopf algebra. Recall that, in terms of the notation \(\Lambda = \eta \varepsilon + \iota\), the antipode \(S\) of \(\Lambda\) can be written as

\[S = \eta \varepsilon - \iota + \iota \cup \iota - \iota \cup \iota + \ldots = \sum (-1)^j \iota \cup \iota^j: \Lambda \rightarrow \Lambda.\]

At the risk of notational confusion with our notation for a symmetric algebra, we use here the standard notation \(S\) for the antipode. This notation for the antipode is not used elsewhere in the paper.

Let \(\mathcal{A}\) be a general graded Hopf algebra and let \(M\) and \(N\) be ordinary right \(\mathcal{A}\)-modules. Then \(\mathcal{A}\) acts on the right of \(\text{Hom}(M, N)\) in various ways:

The \(\mathcal{A}\)-actions on \(M\) and \(N\) induce the \(\mathcal{A}\)-actions

\[
\mu_M: \text{Hom}(M, N) \otimes \mathcal{A} \rightarrow \text{Hom}(M, N) \\
\mu_N: \text{Hom}(M, N) \otimes \mathcal{A} \rightarrow \text{Hom}(M, N)
\]

on the right of \(\text{Hom}(M, N)\), and the two pairings \(\mu_M\) and \(\mu_N\) combine to an \(\mathcal{A}\)-action

\[
\mu_{M, N}: \text{Hom}(M, N) \otimes \mathcal{A} \rightarrow \text{Hom}(M, N)
\]
on \( \text{Hom}(M, N) \) given as the composite of the following two morphisms:

\[
\begin{align*}
\text{Hom}(M, N) \otimes A \xrightarrow{\text{Hom}(M, N) \otimes \Delta} & \text{Hom}(M, N) \otimes A \otimes A \\
\text{Hom}(M, N) \otimes A \otimes A \xrightarrow{\mu_M \otimes A} & \text{Hom}(M, N) \otimes A \xrightarrow{\mu_N} \text{Hom}(M, N).
\end{align*}
\]

We will now take \( M \) to be \( A \) itself, viewed as a right \( A \)-module via right multiplication. The following is well known and classical.

**Lemma 3.5.2.** The association \( \alpha \mapsto \alpha \cup A \) induces an isomorphism

\( \psi: (\text{Hom}(A, N), \mu_A) \rightarrow (\text{Hom}(A, N), \mu_A, N) \)

of right \( A \)-modules. The inverse isomorphism

\( \phi: (\text{Hom}(A, N), \mu_A, N) \rightarrow (\text{Hom}(A, N), \mu_A) \)

is given by the association \( \beta \mapsto \beta \cup S \). □

**Theorem 3.5.3.** [Cartan] The assignment to a homogeneous \( \alpha \in \text{Hom}(S'[s^2 g], N)^g \) of

\[
(3.5.4) \quad \Phi_\alpha: S'[s^2 g] \otimes \Lambda[s g] \rightarrow N, \quad \Phi_\alpha(w \otimes a) = \alpha(w)a, \ w \in S'[s^2 g], \ a \in \Lambda[s g],
\]

yields an injective morphism

\[
(3.5.5) \quad \text{Hom}(S'[s^2 g], N)^g \rightarrow \text{Hom}(S'[s^2 g], \text{Hom}(\Lambda'[sg], N))
\]

of bigraded \( R \)-modules, and the composite of (3.5.5) with the morphism

\[
(3.5.6) \quad \phi_*: \text{Hom}(S'[s^2 g], \text{Hom}(\Lambda'[sg], N)) \rightarrow (S'[s^2 g], \text{Hom}(\Lambda'[sg], N))
\]

of bigraded \( R \)-modules induced by \( \phi: \text{Hom}(\Lambda'[sg], N) \rightarrow \text{Hom}(\Lambda'[sg], N) \), combined with the adjointness isomorphism

\[
(S'[s^2 g], \text{Hom}(\Lambda'[sg], N)) \cong \text{Hom}(W'[g], N),
\]

yields an injective chain map

\[
(3.5.7) \quad \text{Hom}^- (S'[s^2 g], N)^g \rightarrow \text{Hom}(W'[g], N)
\]

which identifies the source (3.5.1) of (3.5.7) with \( \text{Hom}(W'[g], N)^{Cg} \)

**Proof.** In the special case where the ground ring is that of the reals and where \( g \) is compact, this goes back to Cartan [6], and the reasoning in the general case is formally the same. □

We will now suppose that \( R \) is a field of characteristic zero and that \( g \) is reductive. We then refer to the twisted object (3.5.1) as the Cartan model for the differential graded \( \text{Ext}_{(Cg, g)}(R, N) \) and to a morphism of the kind (3.5.6) as a Cartan twist. The following is immediate.
Corollary 3.5.8. The chain map (3.5.7) identifies the source (3.5.1) of (3.5.7) with (3.1.10). Thus the Cartan model indeed calculates the differential graded \( \text{Ext}(C_g, g)(R, N) \). Consequently \( \text{Ext}(C_g, g)(R, R) \) is canonically isomorphic to the algebra of \( g \)-invariants of the algebra \( \text{Hom}(S'[s^2g], R) \), that is, to the algebra of \( g \)-invariants of the symmetric algebra \( S[s^{-2}g^*] \). \( \square \)

Corollary 3.5.9. The differential graded \( \text{Ext}(C_g, g)(R, N) \) acquires the structure

\[
(3.5.10) \quad \text{Hom}(S'[s^2g], R)^g \otimes \text{Ext}(C_g, g)(R, N) \to \text{Ext}(C_g, g)(R, N)
\]

of a \( \text{Hom}(S'[s^2g], R)^g \)-module via the induced \( (S'[s^2g]) \)-comodule structure on \( W'[g] \). \( \square \)

3.6. Cutting the Cartan model to size in the reductive case. Suppose that \( R \) is a field of characteristic zero and let \( g \) be a reductive Lie algebra. Let \( V \) be a differential graded right \( (Cg) \)-module. The space \( V^g \) of invariants is manifestly a \( (Cg) \)-submodule and the induced \( (Cg) \)-action on \( V^g \), restricted to \( g \), is plainly trivial. Consequently the \( (U[Cg]) \)-action on \( V^g \), restricted to the two-sided differential graded ideal \( (g) \) in \( U[Cg] \) generated by \( g \), is trivial whence the action passes through an action of the quotient algebra \( U[Cg]/\langle g \rangle \) on \( V^g \), and this action is compatible with the differentials. Hence the action of \( U[Cg] \) on \( V \) then passes to an action

\[
(3.6.1) \quad H_*(g) \otimes V^g \to V^g
\]

of \( H_*(g) \) on \( V^g \) which is compatible with the differential on \( V^g \).

Through the projection \( U[Cg] \to H_*(g) = \Lambda \) of differential graded algebras, the twisted object \( S' \otimes_{\tau} \Lambda \) acquires a canonical right \( (Cg) \)-module structure. Exploiting the Hopf algebra structure of \( U[Cg] \), we endow \( S' \otimes_{\tau} \Lambda \otimes_{\tau} W'[g] \) with the diagonal right \( (Cg) \)-module structure and, likewise exploiting the Hopf algebra structure of \( S' \), we endow \( S' \otimes_{\tau} \Lambda \otimes_{\tau} W'[g] \) with the diagonal left \( S' \)-comodule structure. To spell out the latter, we write the multiplication map of \( S' \) as \( \mu: S' \otimes S' \to S' \), the comodule structure maps as \( \Delta: S' \otimes \Lambda \to S' \otimes S' \otimes \Lambda \) and \( \Delta: W'[g] \to S' \otimes W'[g] \), and the twist map as \( T: \Lambda \otimes S' \to S' \otimes \Lambda \). With these preparations out of the way, the structure map of the diagonal left \( S' \)-comodule structure is the composite of the following three morphisms

\[
S' \otimes \Lambda \otimes W'[g] \xrightarrow{\Delta \otimes \Delta} S' \otimes S' \otimes \Lambda \otimes S' \otimes W'[g] \\
S' \otimes S' \otimes \Lambda \otimes S' \otimes W'[g] \xrightarrow{S' \otimes S' \otimes T \otimes W'[g]} S' \otimes S' \otimes S' \otimes \Lambda \otimes W'[g] \\
S' \otimes S' \otimes S' \otimes \Lambda \otimes W'[g] \xrightarrow{\mu \otimes S' \otimes \Lambda \otimes W'[g]} S' \otimes S' \otimes \Lambda \otimes W'[g].
\]

This left \( S' \)-comodule structure is compatible with the differentials.

The \( (Cg) \)-linear projections

\[
(3.6.2) \quad \varepsilon \otimes \varepsilon \otimes W'[g]: S' \otimes_{\tau} \Lambda \otimes_{\tau} W'[g] \to W'[g] \\
(3.6.3) \quad S' \otimes \Lambda \otimes \varepsilon: S' \otimes_{\tau} \Lambda \otimes_{\tau} W'[g] \to S' \otimes_{\tau} \Lambda
\]

are morphisms of right \( (Cg) \)-modules and left \( S' \)-comodules; furthermore, (3.6.2) and (3.6.3) are chain equivalences since the objects involved are contractible.
Proposition 3.6.4. The projections (3.6.2) and (3.6.3) induce chain equivalences

\[(3.6.5) \quad \text{Hom}(W'[\mathfrak{g}], V)^{Cg} \to \text{Hom}(S' \otimes \Lambda \otimes \tau W'[\mathfrak{g}], V)^{Cg}\]

\[(3.6.6) \quad \text{Hom}(S' \otimes \Lambda, V^0)^{Cg} \to \text{Hom}(S' \otimes \Lambda \otimes \tau W'[\mathfrak{g}], V)^{Cg}\]

that are compatible with the induced differential graded \(\text{Hom}(S', R)\)-module structures. As a graded algebra, \(\text{Hom}(S', R)\) is therefore canonically isomorphic to \(\text{Ext}_{(Cg, g)}(R, R)\) and, for general \(V\), the twisted \(\text{Hom}\)-object \(\text{Hom}^{\tau}(S', V^0)\) is a small model for the differential graded \(\text{Ext}_{(Cg, g)}(R, V)\) that is compatible with the bundle structures in the sense that the resulting pairing

\[(3.6.7) \quad \text{Hom}(S', R) \otimes \text{Hom}^{\tau}(S', V^0) \to \text{Hom}^{\tau}(S', V^0)\]

induces the \(\text{Hom}(S', R)\)-module structure (3.5.10) on \(\text{Ext}_{(Cg, g)}(R, V)\).

Proof. A spectral sequence comparison argument shows that (3.6.5) and (3.6.6) are isomorphisms on homology and hence chain equivalences.

Adjointness yields the isomorphism

\[\text{Hom}^{\tau}(S', V^0) \to \text{Hom}(S' \otimes \Lambda, V^0)^{\Lambda} = \text{Hom}(S' \otimes \Lambda, V)^{Cg}\]

which combines with (3.6.6) to the chain equivalence

\[(3.6.8) \quad \text{Hom}^{\tau}(S', V^0) \to \text{Hom}(S' \otimes \Lambda \otimes \tau W'[\mathfrak{g}], V)^{Cg}\]

Since (3.6.5) is a chain equivalence as well, the left-hand side of (3.6.8) is a small model for the differential graded \(\text{Ext}_{(Cg, g)}(R, V)\) as asserted. The verification of the compatibility with the bundle structures is left to the reader. □

We will refer to \(\text{Hom}^{\tau}(S', V^0)\) as the small Cartan model for the differential graded \(\text{Ext}_{(Cg, g)}(R, V)\). The following is an immediate consequence of Proposition 3.6.4.

Theorem 3.6.9. The canonical map

\[\text{Ext}_{\Lambda}(R, V^0) \to \text{Ext}_{(Cg, g)}(R, V)\]

induced by the projection \(U[Cg] \to \Lambda = H_*(\mathfrak{g})\) is an isomorphism. □

Remark 3.6.10. The construction of an explicit section for (3.6.2) is not entirely obvious whence the construction of an explicit map between \(\text{Hom}^{\tau}(S', V^0)\) and the Weil model requires some care. Incomplete reasoning and faulty constructions aiming at comparing the Weil model (or Cartan model) with the small Cartan model led to a certain activity in the literature [1], [2], [18], [47], see in particular the introduction of [2].

4. The simplicial Weil coalgebra

As before, let \(R\) be a commutative ring and \(\mathfrak{g}\) an ordinary \(R\)-Lie algebra which is projective as an \(R\)-module. In the previous section, we explored the nonhomogeneous form of the relative bar resolution for the pair \((U[Cg], U[\mathfrak{g}])\). The present aim is to show that totalization carries a suitably defined simplicial Weil coalgebra associated
with \( g \) to the *homogeneous* form of the relative bar resolution under discussion. This homogeneous bar resolution will enable us to introduce small models for the corresponding relative differential Ext-functors.

A cosimplicial version of the Weil algebra associated with a Lie algebra has been introduced in [40] and [41] (p. 59), see also [42]. The dual of our simplicial Weil coalgebra does not coincide with the cosimplicial version of the Weil algebra explored in [40]–[42], though. Yet we prefer to stick to our terminology since Corollary 4.7 below will establish a canonical comparison between the ordinary Weil coalgebra and the total object associated with the simplicial Weil coalgebra in our sense.

As before, let \( g \) be an \( R \)-Lie algebra which is projective as an \( R \)-module. We remind the reader that the notation \( \text{DK} \) refers to the Dold-Kan functor, cf. (2.5) above. The simplicial \( R \)-module \( \text{DK} g \) acquires a simplicial Lie algebra structure. We will often discard the symbol \( \text{DK} \) in notation and thus view \( g \) as a simplicial Lie algebra whenever necessary, each structure map being the identity. The total object \( E g \) associated with \( g \) in the category of Lie algebras relative to the obvious monoidal structure is a simplicial Lie algebra as well. We refer to the resulting simplicial differential graded CCE coalgebra \( \Lambda'_\partial[sEg] \) as the *simplicial Weil coalgebra* associated with \( g \).

In the same vein, consider the symmetric monoidal category of coaugmented differential graded cocommutative coalgebras, with the differential graded coalgebra tensor product \( \otimes \) as monoidal structure, and with the coalgebra diagonal as diagonal structure for the category—it is here where the requirement that the coalgebras be graded cocommutative is needed. In this category, the total object \( E^\otimes \Lambda'_\partial[sg] \) is a simplicial differential graded coalgebra. By functoriality, the differential graded \((Cg)\)-action on \( \Lambda'_\partial[sg] \) given in (1.3) above induces a differential graded \((Cg)\)-action on \( E^\otimes \Lambda'_\partial[sg] \) that is compatible with the coalgebra structure.

**Lemma 4.1.** As a simplicial differential graded coalgebra, the CCE coalgebra \( \Lambda'_\partial[sEg] \), carried out for the simplicial Lie algebra \( E g \), that is, for the total object associated with \( g \), is canonically isomorphic to the total object \( E^\otimes \Lambda'_\partial[sg] \) associated with the CCE coalgebra \( \Lambda'_\partial[sg] \) for \( g \) in the category of differential graded coalgebras.

**Proof.** In a simplicial degree \( p \geq 0 \),

\[
(Eg)_p = g \times \cdots \times g \quad (p + 1 \text{ factors})
\]

whence, as \( p \) ranges over the natural numbers,

\[
(\Lambda'_\partial[sEg])_p = (\Lambda'_\partial[(sg)^{(p+1)}]) \cong (\Lambda'_\partial[sg])^{(p+1)} \cong (E^\otimes \Lambda'_\partial[sg])_p.
\]

These isomorphisms are compatible with the simplicial operations. \( \square \)

We will denote by \( \|\Lambda'_\partial[sEg]\| \) the complex

\[
(4.2) \quad d_1 \to \|\Lambda'_\partial[sEg]\|_n \to \cdots \to \|\Lambda'_\partial[sEg]\|_1 \to \|\Lambda'_\partial[sg]\|
\]

of right \((Cg)\)-modules, necessarily contractible, arising from the simplicial differential graded coalgebra \( \Lambda'_\partial[sEg] \) by normalization. Our next aim is to identify the complex (4.2) with the relative bar resolution (3.2.2). To this end, we recall the (right)
nonhomogeneous version of $E_g$. The situation is formally the same as that in (2.5) above: View momentarily $E_g$ merely as a graded $R$-module and consider the familiar automorphism

$$\Phi: E_g \longrightarrow E_g$$

of graded $R$-modules which, in degree $n$, that is, on $(E_g)_n = g^{\times (n+1)}$, is given by the formula

$$\Phi(x_0, x_1, \ldots, x_n) = (x_0 + x_1 + x_2 + \ldots + x_n, \ldots, x_{n-1} + x_n, x_n).$$

In degree $n$, the inverse of $\Phi$ is plainly given by the association

$$(y_1, \ldots, y_n, x) \mapsto (x_0, x_1, \ldots, x_n) = (y_1 - y_2, y_2 - y_3, \ldots, y_{n-1} - y_n, y_n - x, x).$$

The nonhomogeneous face operators $\partial_j$ are given by the familiar expressions

$$\partial_0(x_0, x_1, \ldots, x_n) = (x_1, \ldots, x_n),$$

$$\partial_j(x_0, x_1, \ldots, x_n) = (x_0, \ldots, x_{j-2}, x_{j-1} + x_j, x_{j+1}, \ldots, x_n) \quad (1 \leq j \leq n)$$

and, likewise, the nonhomogeneous degeneracy operators $s_j$ are given by

$$s_j(x_0, x_1, \ldots, x_n) = (x_0, \ldots, x_{j-1}, 0, x_j, \ldots, x_n) \quad (0 \leq j \leq n);$$

we will denote the resulting simplicial $R$-module by $(E_g)^{\text{right}}$. The automorphism $\Phi$ of graded $R$-modules is well known to be an isomorphism

$$\Phi: (E_g)^{\text{right}} \longrightarrow E_g$$

of simplicial $R$-modules. Here the chosen nonhomogeneous formulas reflect the $(C_g)$-operations being from the right, and our constructions are written as tensor product of “base” times “fiber”. The formulas in the literature for objects with operators from the left differ from the above ones. See e. g. p. 75 of [20].

**Theorem 4.5.** The isomorphism $\Phi$ of simplicial $R$-modules induces an isomorphism

(4.5.1)  $$\beta_{U[g]}(R, U[C_g], U[C_g]) \longrightarrow \Lambda_g'[sE_g]$$

of simplicial right $(U[C_g])$-complexes. Consequently the complex $|\Lambda_g'[sE_g]|$ of right $(U[C_g])$-modules comes down to the homogeneous form of the normalized relative bar resolution of $R$ in the category of right $(U[C_g])$-modules.

**Proof.** In terms of the nonhomogeneous description, in a fixed degree $n \geq 0$, the Lie algebra $(E_g)^{\text{right}}_n$ simply comes down to the iterated semi-direct product

$$g \rtimes g \rtimes \ldots \rtimes g \quad (n + 1 \text{ copies of } g),$$

the formulas (4.3) and (4.4) for the simplicial structure still being valid. A special case thereof is the observation that the linear map

$$g \rtimes g \longrightarrow g \times g, \quad (x_0, x_1) \mapsto (x_0 + x_1, x_1)$$
is an isomorphism of Lie algebras with inverse mapping given by the assignment to \((y, x)\) of \((y - x, x)\). Consequently, for a fixed \(n \geq 0\), the degree \(n\) differential graded coalgebra \(\Lambda'_g[s(Eg)^{\text{right}}]_n\) of the simplicial differential graded coalgebra \(\Lambda'_g[s(Eg)^{\text{right}}]\) takes the form of an iterated twisted tensor product

\[
\Lambda'_g[sg] \otimes_{\tau_0} \Lambda'_g[sg] \otimes_{\tau_0} \ldots \otimes_{\tau_0} \Lambda'_g[sg]
\]
of \(n + 1\) copies of the CCE coalgebra \(\Lambda'_g[sg]\) of \(g\) relative to the appropriate canonical actions of the corresponding copy of \(g\) on the right of that part of \(\Lambda'_g[s(Eg)^{\text{right}}]_n\) left to \(g\) in the tensor product decomposition. For fixed \(n\), the differential graded coalgebra \(\Lambda'_g[s(Eg)^{\text{right}}]_n\) is precisely the corresponding constituent (3.2.2) of the relative simplicial bar construction. In particular, each face and degeneracy operator is manifestly a morphism of differential graded coalgebras, and these operators are exactly the same as those in the simplicial bar construction. Hence \(\Phi\) induces an isomorphism from the complex (3.2.2) onto (4.2). This observation establishes Theorem 3.2.6

**Corollary 4.7.** The comparison (3.3.3), combined with the induced morphism \(|4.5.1|\) between the condensed objects, yields a morphism

\[
W'[g] \longrightarrow |\Lambda'_g[s(Eg)]|
\]
of differential graded coalgebras between the ordinary Weil coalgebra and the total object associated with the simplicial Weil coalgebra for \(g\). □

5. **Cutting the defining object for \(\text{Ext}_{((G, Cg): C)}\) to size**

As before, let \(R\) be a commutative ring and \(g\) an \(R\)-Lie algebra which is projective as an \(R\)-module.

5.1. **The model for \(\text{Ext}_{((G, Cg): C)}\) arising from the Weil coalgebra.** The realization \(|Eg|\) of the total simplicial Lie algebra \(Eg\) is a differential graded Lie algebra, and the simplicial twisting cochain

\[
\tau_{Eg}: \Lambda'_g[SEg] \to U[Eg]
\]

the constituents of which in each simplicial degree are given in (1.2) above induces, via the twisted Eilenberg-Zilber theorem [19], an acyclic twisting cochain

\[
\tau_{|Eg|}: \text{BA}_g[sg] = |\Lambda'_g[SEg]|^* \to U[|Eg|].
\]

More precisely, as chain complexes, \(|sEg| \cong s|Eg|\), and the canonical projection from the differential graded coalgebra \(|\Lambda'_g[SEg]|^*\) to \(|sEg|\) determines, via the universal property of the differential graded CCE coalgebra \(S'_g[s|Eg|]\) for the differential graded Lie algebra \(|Eg|\), a unique morphism

\[
\text{BA}_g[sg] = |\Lambda'_g[SEg]|^* \to S'_g[s|Eg|]
\]
of differential graded coalgebras which, combined with the universal twisting cochain for the CCE construction of \(|Eg|\), yields the asserted twisting cochain \(\tau_{|Eg|}\).

We now take the ground ring to be that of the reals, \(\mathbb{R}\). Until the end of the present section, we take \(G\) to be a Lie group, \(g\) its Lie algebra, an \(V\) a right \((G, Cg)\)-module. In view of Proposition 2.5.7, the chain complex \(|\mathcal{A}^0(EG, V)|\) is the standard injective resolution of \(V\) in the category of (differentiable) right \(G\)-modules and, in particular, carries a canonical right \(G\)-module structure. Furthermore, the obvious componentwise actions of the constituents of the simplicial Lie algebra \(Eg\) on the constituents of \(\mathcal{A}^0(EG, V)\) induce an action of the differential graded Lie algebra \(|Eg|\) on \(|\mathcal{A}^0(EG, V)|\); this action does not involve \(V\).

\[
(5.1.2) \quad \mathcal{B}^*_*(\mathbb{R}, G, V) = \text{Hom}^{\tau_{|Eg|}}(\mathcal{B}A\partial[sg], |\mathcal{A}^0(EG, V)|),
\]

the resulting twisted Hom-object. Since the \(|Eg|\)-action does not involve \(V\), the twisting cochain \(\tau_{|Eg|}\) does not involve \(V\). The twisted Hom-object (5.1.2) inherits a canonical \(G\)-action. Furthermore, the assignment to \(G\) and \(V\) of \(\mathcal{B}^*_*(\mathbb{R}, G, V)\) is plainly a functor in the group variable and, given the group \(G\), in the \((G, Cg)\)-module variable as well. The functor \(\mathcal{B}^*_*(\mathbb{R}, G, V)\) is, in a somewhat generalized sense, a \textit{dualized unreduced bar construction} for the category of \((G, Cg)\)-modules. Indeed, the object (5.1.2) acquires a natural \((G, Cg)\)-module structure. For \(\mathcal{B}A\partial[sg]\) and \(V\) inherit graded (not differential graded) \((\Lambda[sg])\)-module structures from their differential graded \((Cg)\)-module structures, and these \((\Lambda[sg])\)-module structures induce a graded (not differential graded) \((\Lambda[sg])\)-module structure on the corresponding untwisted object

\[
(5.1.3) \quad \text{Hom}(\mathcal{B}A\partial[sg], |\mathcal{A}^0(EG, V)|).
\]

On the twisted object (5.1.2), the induced differential graded \(g\)-module structure (the diagonal structure coming from that on \(\mathcal{B}A\partial[sg]\) and the diagonal structure on \(|\mathcal{A}^0(EG, V)|\)) and the graded \((\Lambda[sg])\)-module structure combine to a differential graded right \((Cg)\)-module structure. Thus \(\mathcal{B}^*_*(\mathbb{R}, G, V)\) is a \((G, Cg)\)-module functor, that is, a functor having as range the category of \((G, Cg)\)-modules.

For intelligibility we recall that, given the group \(H\) (we will then substitute \(EG\) for \(H\)) and the subgroup \(G\), the twisted object

\[
\text{Hom}^{\tau_{h}}(\Lambda'\partial[sh], |\mathcal{A}^0(H, V)|)
\]

is defined relative to the universal Lie algebra twisting cochain \(\tau_{h}: \Lambda'\partial[sh] \to U[h]\). Here \(|\mathcal{A}^0(H, V)|\) is a left \(h\)-module via right translation in \(H\); this structure does not involve \(V\), and the operator

\[
\delta^{\tau_{h}}: \text{Hom}(\Lambda'[sh], |\mathcal{A}^0(H, V)|) \to \text{Hom}(\Lambda'[sh], |\mathcal{A}^0(H, V)|)
\]

determined by the universal Lie algebra twisting cochain \(\tau_{h}: \Lambda'\partial[sh] \to U[sh]\), cf. (1.2) above, is defined; cf. (1.6.3.1) above where this is explained for the special case where \(V\) is the de Rham complex \(\mathcal{A}(X)\) of a \(G\)-manifold \(X\). Thus the operator

\[
\delta^{\tau_{|Eg|}}: \text{Hom}(\mathcal{B}A\partial[sg], |\mathcal{A}^0(EG, V)|) \to \text{Hom}(\mathcal{B}A\partial[sg], |\mathcal{A}^0(EG, V)|)
\]
determined by the universal Lie algebra twisting cochain $\tau_{E\theta}: B\Lambda_\theta[s\mathfrak{g}] \to U[|E\mathfrak{g}|]$ does not involve $V$.

Define the functors $\overline{B}^*_\cdot(\mathbb{R}, \cdot, \cdot)$, $\overline{B}^*_\cdot(\cdot)$ and $\overline{B}^*_\cdot(\cdot)$ by

$$\overline{B}^*_\cdot(\mathbb{R}, G, V) = \overline{B}^*_\cdot(\mathbb{R}, G, V)^{(G, C\mathfrak{g})}$$

(5.1.4)

$$\overline{B}^*_\cdot(G) = \overline{B}^*_\cdot(G, G)$$

$$\overline{B}^*_\cdot(G) = \overline{B}^*_\cdot(G, G, \mathbb{R}).$$

We have chosen the notation $\mathcal{B}^*$ and $\overline{B}^*$ since $\mathcal{B}^*$ and $\overline{B}^*$ are, in a somewhat generalized sense, dualized respective unreduced and reduced bar constructions, with reference to the category $\text{Mod}_{(G, C\mathfrak{g})}$ and, accordingly, we refer to $\mathcal{B}^*$ and $\overline{B}^*$ as unreduced and reduced constructions, respectively. In particular, $\mathcal{B}^*_\cdot(G)$ and $\overline{B}^*_\cdot(G)$ are differential graded algebras; further, $\overline{B}^*_\cdot(G, G, V)$ is a $\overline{B}^*_\cdot(G, G)$-module and $\overline{B}^*_\cdot(G, G, V)$ is a $\overline{B}^*_\cdot(G, G)$-module in an obvious manner.

**Theorem 5.1.5.** The differential graded $\text{Ext}_{(G, C\mathfrak{g})}(\mathbb{R}, V)$ is canonically isomorphic to the homology of the twisted object $\overline{B}^*_\cdot(\mathbb{R}, G, V)$.

Theorem 5.1.5 is an immediate consequence of the following lemma.

**Lemma 5.1.6.** The left trivialization of the tangent bundle of $G$ induces a contraction of the totalized complex $|A(E\mathfrak{g}, V)^{(G, C\mathfrak{g})}|$ onto the twisted object $\overline{B}^*_\cdot(\mathbb{R}, G, V)$.

**Proof.** The de Rham theory Eilenberg-Zilber theorem yields the contraction

$$(\mathcal{B}^*_\cdot(\mathbb{R}, G, V) \overset{\nabla^*}{\leftrightarrow} |\text{Hom}^{\tau_{E\theta}}(\Lambda_\theta[sE\mathfrak{g}], A^0(E\mathfrak{g}, V)|, h^2),$$

necessarily $G$- and $(C\mathfrak{g})$-equivariant. Exploiting Theorem 2.6.1 (the extended decomposition lemma), we replace $|\text{Hom}^{\tau_{E\theta}}(\Lambda_\theta[sE\mathfrak{g}], A^0(E\mathfrak{g}, V))|$ with $|A(E\mathfrak{g}, V)|$ and, thereafter, we take $(G, C\mathfrak{g})$-invariants. This yields the contraction

(5.1.7) $$(\overline{B}^*_\cdot(\mathbb{R}, G, V) \overset{\nabla^*}{\leftrightarrow} |A(E\mathfrak{g}, V)^{(G, C\mathfrak{g})}|, h^*).$$

Thus the homology of the twisted object $\overline{B}^*_\cdot(\mathbb{R}, G, V)$ coincides with the differential graded $\text{Ext}_{(\mathfrak{g}, C\mathfrak{g})}(\mathbb{R}, V)$ whence Theorem 5.1.5.

Recall that $\iota: W'/[\mathfrak{g}] \to B\Lambda_\theta[s\mathfrak{g}]$ refers to the canonical comparison (3.3.3), cf. also Corollary 4.7. The composite of $\iota$ with the acyclic twisting cochain (5.1.1) is plainly an acyclic twisting cochain

$$\tau_{E\theta} \circ \iota: W'/[\mathfrak{g}] \to U[|E\mathfrak{g}|].$$

**Proposition 5.1.8.** (i) The comparison $\iota$ induces a homology isomorphism

(5.1.9) $$\overline{B}^*_\cdot(\mathbb{R}, G, V) \longrightarrow \text{Hom}^{\tau_{E\theta}\circ\iota}(W'/[\mathfrak{g}], |A^0(E\mathfrak{g}, V)|)^{(G, C\mathfrak{g})}$$
between (5.1.2) and

\[
(5.1.10) \quad \text{Hom}^{\tau|E_0|\omega}(W'[g], |A^0(EG, V)|)^{(G,Cg)}.
\]

(ii) When \( G \) is reductive, the chain equivalence (3.4.3) induces a \((G,Cg)\)-equivariant chain equivalence of the kind

\[
(5.1.11) \quad \left( h^2_W, \text{Hom}^{\tau|E_0|\omega}(W'[g], |A^0(EG, V)|) \right) \leftarrow \leftarrow \left( B_{(G,Cg)}^{*}(\mathbb{R}, G, V), h^2 \right)
\]

and, taking \((G,Cg)\)-invariants, we obtain the chain equivalence

\[
(5.1.12) \quad \left( h^2_W, \text{Hom}^{\tau|E_0|\omega}(W'[g], |A^0(EG, V)|)^{(G,Cg)} \right) \leftarrow \leftarrow \left( B_{(G,Cg)}^{*}(\mathbb{R}, G, V), h^2 \right).
\]

Taking \((G,Cg)\)-invariants means taking \((\Lambda[sg])\)- and \(G\)-invariants; the \((\Lambda[sg])\)-invariants are the horizontal elements in a sense explained earlier.

**Proof.** Filtering the twisted objects (5.1.2) and (5.1.10) by the degree complementary to the \(G\)-resolution degree we obtain spectral sequences \((E_r, d_r)\) and \((E_r, d_r)\) converging to \(\text{Ext}_{((G,Cg):C)}(\mathbb{R}, V)\) and the total object of (5.1.10) respectively, and the comparison \(\iota\) induces a morphism

\[
(E_r(5.1.2), d_r) \rightarrow (E_r(5.1.10), d_r)
\]

of spectral sequences. The bigraded \(R\)-module

\[
\text{Hom}(W'[g], |A^0(EG, V)|)^{(G,Cg)},
\]

endowed with the bar complex operator alone, amounts to the chain complex

\[
|A^0(EG, \text{Hom}(W'[g], V)^{\Lambda[sg]})|^G \cong |A^0(EG, \text{Hom}(S'[s^2g], V))|^G
\]

whence

\[
E_1(5.1.10) \cong H^*_{\text{cont}} \left( G, \text{Hom}(W'[g], V)^{\Lambda[sg]} \right) = H^*_{\text{cont}} \left( G, \text{Hom}(S'[s^2g], V) \right),
\]

the \((\Lambda[sg])\)-action being given by contraction, and the operator \(d_1\) is the Koszul resolution operator, induced by the operator \(\partial^{\tau|E_0|\omega}\) on \(\text{Hom}(S'[s^2g], V)\). Consequently

\[
E_2(5.1.10) \cong H^*_{\text{cont}} \left( G, \text{Ext}_{\Lambda[sg]}(\mathbb{R}, V) \right).
\]

Likewise, the bigraded \(R\)-module

\[
\text{Hom}(\Lambda[B\Lambda][sg], |A^0(EG, V)|)^{(G,Cg)},
\]

endowed with the bar complex operator alone, amounts to the chain complex

\[
|A^0(EG, \text{Hom}(\Lambda[B\Lambda][sg], V)^{\Lambda[sg]})|^G \cong |A^0(EG, \text{Hom}(\overline{B}[\Lambda[sg], V])|^G
\]

whence

\[
E_1(5.1.2) \cong H^*_{\text{cont}} \left( G, \text{Hom}(\Lambda[B\Lambda][sg], V)^{\Lambda[sg]} \right) = H^*_{\text{cont}} \left( G, \text{Hom}(\overline{B}[\Lambda[sg], V]) \right),
\]

the \((\Lambda[sg])\)-action being given by contraction, and the operator \(d_1\) is the bar resolution operator. Consequently

\[
E_2(5.1.2) \cong H^*_{\text{cont}} \left( G, \text{Ext}_{\Lambda[sg]}(\mathbb{R}, V) \right).
\]

Hence the ordinary spectral sequence comparison establishes the assertion (i). Assertion (ii) is immediate. \(\square\)
Theorem 5.1.13. Via the comparison \( \iota \), the differential graded \( \text{Ext}_{(G,C\mathbb{G})}^{(\mathbb{R},V)} \) is canonically isomorphic to the homology of the twisted object \( (5.1.10) \), viz. of
\[
\text{Hom}^{\tau|_{E\theta|\mathcal{O}^t}}(W'[g], |\mathcal{A}^0(EG, V)|)^{(G,C\mathbb{G})}.
\]

Proof. This is an immediate consequence of Proposition 5.1.8. \( \Box \)

The twisted object \( (5.1.10) \) has somewhat the form of a Weil model, with \( |\mathcal{A}^0(EG, V)| \) instead of the module \( V \) itself in the ordinary Weil model. There is also a corresponding object which takes the form of a Cartan model: The composite of the injective morphism
\[
\text{Hom}(S'[s^2g], |\mathcal{A}^0(EG, V)|)^G \rightarrow \text{Hom}(S'[s^2g], \text{Hom}(\Lambda'[s\mathfrak{g}], |\mathcal{A}^0(EG, V)|))
\]
of bigraded \( R \)-modules with the Cartan twist
\[
\phi_*: \text{Hom}(S'[s^2g], \text{Hom}(\Lambda'[s\mathfrak{g}], |\mathcal{A}^0(EG, V)|)) \rightarrow (S'[s^2g], \text{Hom}(\Lambda'[s\mathfrak{g}], |\mathcal{A}^0(EG, V)|)),
\]

cf. (3.5.6) above, combined with the adjointness isomorphism
\[
(S'[s^2g], \text{Hom}(\Lambda'[s\mathfrak{g}], |\mathcal{A}^0(EG, V)|)) \cong \text{Hom}(W'[g], |\mathcal{A}^0(EG, V)|),
\]
yields an injective morphism of graded vector spaces
\[
\text{Hom}(S'[s^2g], |\mathcal{A}^0(EG, V)|)^G \rightarrow \text{Hom}(W'[g], |\mathcal{A}^0(EG, V)|)
\]
which induces an isomorphism from
\[
(5.1.14) \quad \text{Hom}^{\tau_{S'}, \tau|_{E\theta|\mathcal{O}^t}}(S'[s^2g], |\mathcal{A}^0(EG, V)|)^G
\]
on to the twisted object \( (5.1.10) \). The reasoning is essentially the same as that which establishes Theorem 3.5.3. The total differential of (5.1.14) has the form \( d + \partial \) of a perturbed differential: The operator \( d \) is the naive differential on \( \text{Hom}(S'[s^2g], |\mathcal{A}^0(EG, V)|)^G \) coming from the bar complex operator \( \delta \) and the differential on \( V \). Furthermore,
\[
(5.1.15) \quad \partial = \delta^{\tau_{S'}} + \delta|_{E\theta|\mathcal{O}^t},
\]
where \( \delta^{\tau_{S'}} \) is the operator defined with reference to the action of \( \Lambda[s\mathfrak{g}] \) on \( |\mathcal{A}^0(EG, V)| \) coming from the action of \( \Lambda[s\mathfrak{g}] \) on \( V \), and where \( \delta|_{E\theta|\mathcal{O}^t} \) is the operator induced from the twisting cochain \( \tau_{E\theta}| \circ \iota \); see (1.2) above for details. In view of Theorem 5.1.5, the twisted object \( (5.1.14) \) calculates the differential graded \( \text{Ext}_{((G,C\mathbb{G});C)}^{(\mathbb{R},V)} \); the twisted object \( (5.1.14) \) is somewhat smaller than the original object defining \( \text{Ext}_{((G,C\mathbb{G});C)}^{(\mathbb{R},V)} \). By adjointness, we may rewrite the graded object which underlies \( (5.1.14) \) as
\[
(5.1.16) \quad |\mathcal{A}^0(EG, \text{Hom}(S'[s^2g], V))|^G.
\]
Since $EG$ is contractible, the chain complex $|\mathcal{A}^0(EG, \text{Hom}(S'[s^2g], V))|$, endowed with the operator $\delta$, is a differentiably injective resolution of $\text{Hom}(S'[s^2g], V)$, and taking $G$-invariants we obtain precisely the object $(5.1.16)$ endowed merely with the bar complex operator $\delta$, which therefore calculates the differentiable cohomology of $G$ with coefficients in $\text{Hom}(S'[s^2g], V)$. For the special case where $g$ is finite dimensional and $V$ the real numbers with trivial action, this is exactly Theorem 1 in [4].

Remark 5.1.17. (Relationship with the Bott spectral sequence) Suppose that $G$ is a finite-dimensional Lie group and let $X$ be a left $G$-manifold. Substituting $\mathcal{A}(X)$ for $V$ in the spectral sequence $(E_r(5.1.2), d_r)$, we obtain a spectral sequence $(E_r(G, X), d_r)$ having

$$E_2 = H^*_{\text{cont}}(G, \text{Ext}_{\mathcal{A}[\mathfrak{g}]}(\mathbb{R}, \mathcal{A}(X))).$$

For $X$ a point, this is the spectral sequence explored by Bott in [4]. In particular, $\text{Ext}_{\mathcal{A}[\mathfrak{g}]}(\mathbb{R}, \mathbb{R}) = \text{Hom}(S'[s^2g], \mathbb{R})$ and, in Theorem 1 in [4], the object which corresponds to our $\text{Hom}(S'[s^2g], \mathbb{R})$ is written as $Sg^*$. Likewise, the spectral sequence $(E_r(5.1.10), d_r)$ has the form of a van Est spectral sequence. Indeed, for a finite-dimensional connected Lie group $G$ and a $G$-representation $V$, the complex $\mathcal{A}(G, V)$ of $V$-valued forms on $G$ can be written as $\text{Alt}(g, \mathcal{A}^0(G, V))$ and the $G$-invariant sub-complex $\mathcal{A}(G, V)^G$ amounts to the CCE complex $\text{Alt}(g, V)$ calculating $H^*(g, V)$ where $V$ is viewed as a $g$-module. However, since $G/K$ is contractible, the cohomology of $\mathcal{A}(G/K, \mathcal{A}(G, V))^G$ amounts to the cohomology of $\mathcal{A}(G, V)^G$. The spectral sequence of the form degree filtration of $\mathcal{A}(G/K, \mathcal{A}(G, V))^G$ relative to $G/K$ is the van Est spectral sequence [56]. This spectral sequence has

$$E_2 = H^*_{\text{cont}}(G, H^\text{top}_r(G, V))$$

and converges to $H^*(g, V)$. See also Theorem 2.10 in [41].

5.2. A small object for $\text{Ext}_{((G, Cg); c)}$ in the strictly exterior case. At the present stage, the Lie group $G$ is not supposed to be reductive. To simplify the exposition somewhat, define the functor

$$t^*: \text{Mod}_{(G, Cg)} \to \mathcal{B}_{(G, Cg)}(G)\text{Mod}$$

by the assignment to a (right) $(G, Cg)$-module $N$ of the twisted object

$$t^*(N) = \mathcal{B}_{(G, Cg)}(\mathbb{R}, G, N)$$

which, in turn, calculates the differential graded $\text{Ext}_{((G, Cg); c)}(\mathbb{R}, N)$. This functor is one of two (Koszul) duality functors; we shall come back to this duality in Section 7 below.

Theorem 5.2.2. Suppose that $G$ is of strictly exterior type in such a way that $H^*(G)$ is the exterior Hopf algebra $\Lambda[y_1, \ldots]$ in suitable universally transgressive generators $y_1, y_2, \ldots$ and, for each $y_j$ in $H^*G$, choose a cycle in $\mathcal{B}^*$ such that $y_j$ transgresses to the class of this cycle. This choice determines a differential $D$ such that

$$\text{(5.2.3)} \quad (\text{Hom}(H_*(BG), V), D)$$

is a small model calculating $\text{Ext}_{((G, Cg); c)}(\mathbb{R}, V)$.
At this stage, the group $G$ is a general Lie group of strictly exterior type, possibly infinite dimensional. The hypothesis of Theorem 5.2.2 is of course automatically satisfied when $G$ is finite dimensional and connected. In this particular case, an explicit description of the differential $D$ will be given later.

We now begin with the preparations for the proof of Theorem 5.2.2. For simplicity, we will momentarily write $B^\ast(G,C_g)(G)$ as $B^\ast$ and $\overline{B}((G,C_g),g)(G)$ as $\overline{B}$. We define the completed tensor product $B^\ast\hat{\otimes}B^\ast$ by

$$B^\ast\hat{\otimes}B^\ast = \text{Hom}^\tau(B\Lambda_\partial[s\mathfrak{g}] \otimes B\Lambda_\partial[s\mathfrak{g}], A^0(E(G \times G)))$$

where

$$\tau^\otimes = \tau|_{E\mathfrak{g}} \otimes \eta\varepsilon + \eta\varepsilon \otimes \tau|_{E\mathfrak{g}}: B\Lambda_\partial \otimes B\Lambda_\partial \to U[[E\mathfrak{g}]] \otimes U[[E\mathfrak{g}]] \cong U[[E(\mathfrak{g} \times \mathfrak{g})]].$$

Likewise, we define the reduced completed tensor product $\overline{B}^\ast\hat{\otimes}\overline{B}^\ast$ by

$$\overline{B}^\ast\hat{\otimes}\overline{B}^\ast = (B^\ast\hat{\otimes}B^\ast)((G,C_g) \times (G,C_g)).$$

We will now exploit the graded coalgebra structure which underlies the graded Hopf algebra $H^\ast(G)$.

**Lemma 5.2.4.** For each $y_j$ in $H^\ast G$, choose a cycle in $\overline{B}^\ast$ such that $y_j$ transgresses to the class of this cycle. This choice determines an acyclic twisting cochain

$$\zeta_{\overline{B}^\ast}: H^\ast G \to \overline{B}^\ast.$$  

**Proof.** Given two twisting cochains $\sigma:C \to \overline{B}^\ast$ and $\sigma':C' \to \overline{B}^\ast$ defined on graded cocommutative coalgebras $C$ and $C'$, the values of the twisting cochain $\sigma \otimes \eta\varepsilon + \eta\varepsilon \otimes \sigma'$ lie in $\overline{B}^\ast\hat{\otimes}\overline{B}^\ast$. Moreover, the Eilenberg-Zilber theorem furnishes a contraction of the kind

$$\left(\overline{B}((G,C_g))(G)\otimes\overline{B}((G,C_g))(G) \xrightarrow{\alpha^\ast} \overline{B}((G \times G,C(\mathfrak{g} \times \mathfrak{g}))(G \times G),h^\ast)\right).$$

The inductive construction of the twisting cochain (3.2.1*) in the proof of Lemma 3.2* in [34] dualizes, with [34] (2.2.1*) instead of [34] (2.2.1*). We leave the details to the reader. □

**Proof of Theorem 5.2.2.** Suppose that a choice of cycle in $\overline{B}^\ast$ for each $y_j$ has been made so that the resulting twisting cochain $\zeta_{\overline{B}^\ast}$ is available. We first construct a contraction from $(H^\ast G)\otimes\zeta_{\overline{B}^\ast}$ onto $V$. To this end we note first that, in view of the naturality of the functor $\overline{B}$ in the group variable, restriction of the construction to the trivial group induces a canonical projection $t^\ast(V) \to V$ which thus forgets the $(G,C_\mathfrak{g})$-structure. This projection extends to a projection $\alpha$ from $(H^\ast G)\otimes t^\ast(V)$ to $V$, necessarily a chain equivalence. A section for this projection which is compatible with the differentials includes an sh-comodule structure over $H^\ast G$ on $V$. To construct such a section, we consider $V$ momentarily endowed with the trivial $(G,C_\mathfrak{g})$-structure
which we refer to by the notation $V(0)$. Then the obvious injection of $V$ into $t^*(V)$ which, for $v \in V$, assigns $\psi_v : \overline{\Lambda}_\partial \to A^0(EG, V)$ to $v \in V$ given by

$$
\psi_v(w)(x) = \varepsilon(w)xv, \quad w \in \overline{\Lambda}_\partial, \ x \in EG,
$$

induces an injective chain map $j(0) : V(0) \to (H^*G) \otimes_{\zeta t^*} t^*(V(0))$, and there is an obvious extension

$$(5.2.7) \quad \left( V(0) \xrightarrow{\alpha} (H^*G) \otimes_{\zeta t^*} t^*(V(0)), h(0) \right)$$

of the data to a contraction. Incorporating the non-trivial $(G,Cg)$-structure on $V$ amounts to perturbing the differential on the right-hand side via an operator $\partial$ which lowers the filtration coming from the coaugmentation filtration of $\overline{\Lambda}_\partial[sg]$ and the simplicial degree filtration with reference to $EG$. Application of the perturbation lemma, cf. [34] (2.4), yields the contraction

$$(5.2.8) \quad \left( \operatorname{Hom}(H_*(BG), V) \xrightarrow{\alpha} \operatorname{Hom}(H_*(BG), (H^*G) \otimes_{\zeta t^*} t^*(V)), h \right)$$

in the obvious manner where the notation $\dot{j}$, $\alpha$, $h$ is abused somewhat. Let $\tau : H_*(BG) \to H_*G$ be the transgression twisting cochain. Application of the perturbation lemma, cf. [34] (2.3), yields the contraction

$$(5.2.9) \quad \left( (\operatorname{Hom}(H_*(BG), V), D) \xrightarrow{\alpha} \operatorname{Hom}(H_*(BG), (H^*G) \otimes_{\zeta t^*} t^*(V)), h_\tau \right)$$

The left-hand side of (5.2.9) yields the desired small model calculating $\operatorname{Ext}_{((G,Cg); C)}(\mathbb{R}, V)$. □

5.3. THE WEIL AND CARTAN MODELS FOR COMPACT $G$. Recall the obvious acyclic twisting cochain $\tau^{S'} : S'[s^2g] \to \Lambda[sg]$. With reference to the graded right $(\Lambda[sg])$-module structure on $V$, when the differential on $V$ is ignored, the twisted Hom-object $\operatorname{Hom}^{\tau^{S'}}(S'[s^2g], V)$ is defined; we remind the reader that the operator $\delta^{\tau^{S'}}$ on $\operatorname{Hom}(S'[s^2g], V)$, cf. (1.2) above and [34] (2.4.1), is defined by $\delta^{\tau^{S'}}(f) = (-1)^{|f|}f \cup \tau^{S'}$, the argument $f$ being a homogeneous morphism. Further we will write the differential on $V$ and the differential it induces on $\operatorname{Hom}^{\tau^{S'}}(S'[s^2g], V)$ as $d$, with an abuse of notation. A slight extension of the reasoning for Theorem 3.5.3 establishes the following.
Proposition 5.3.1. For a general Lie group $G$, the canonical morphism of graded objects from $\text{Hom}(S'[s^2g], V)$ to $\text{Hom}(W'[g], V)$, followed by the Cartan twist (3.5.6) and the appropriate adjointness isomorphism, induces an isomorphism

$$\text{Hom}^{\tau^S'}(S'[s^2g], V)^G \to \text{Hom}(W'[g], V)^{(G,Cg)}$$

of graded $R$-modules such that the differential on the right-hand side passes to the sum $\delta^S + d$, restricted to the $G$-invariants. \(\square\)

Until the end of the present subsection we suppose $G$ to be finite-dimensional and compact.

Theorem 5.3.2. The chain complex $\text{Hom}(W'[g], V)^{(G,Cg)}$ and the twisted object $\text{Hom}^{\tau^S'}(S'[s^2g], V)^G$ are small models for $\text{Ext}_{((G,Cg);C)}(\mathbb{R}, V)$, and these models are compatible with the bundle structures in the sense that the obvious pairings

$$\text{Hom}(W'[g], \mathbb{R})^{(G,Cg)} \otimes \text{Hom}(W'[g], V)^{(G,Cg)} \to \text{Hom}(W'[g], V)^{(G,Cg)}$$

and

$$\text{Hom}(S'[s^2g], \mathbb{R})^G \otimes \text{Hom}^{\tau^S'}(S'[s^2g], V)^G \to \text{Hom}^{\tau^S'}(S'[s^2g], V)^G$$

induce the $(H^*(BG))$-module structure on $\text{Ext}_{((G,Cg);C)}(\mathbb{R}, V)$.

Proof. For a general Lie group $G$, the canonical injection $\overline{j}$ of right $G$-modules extends to a contraction

$$(5.3.3) \quad \left( V \xleftarrow{j} \left| A^0(EG, V) \right|, \overline{h} \right)$$

of chain complexes; this contraction encapsulates the fact that $|A^0(EG, V)|$ is a differentiably injective $G$-resolution of $V$. The contraction (5.3.3), in turn, induces a contraction

$$(5.3.4) \quad \left( \text{Hom}(W'[g], V) \xleftarrow{\alpha} \text{Hom}^{\tau_{|Eg|}^0}(W'[g], |A^0(EG, V)|), \overline{h} \right)$$

of chain complexes where $\overline{j}$ is a morphism of $(G,Cg)$-modules.

The group $G$ being compact, integration over $G$ transforms the contraction (5.3.4) into the $(G,Cg)$-equivariant contraction

$$(5.3.5) \quad \left( \text{Hom}(W'[g], V) \xleftarrow{\alpha} \text{Hom}^{\tau_{|Eg|}^0}(W'[g], |A^0(EG, V)|), h \right).$$

Taking $(G,Cg)$-invariants on both sides, we obtain the contraction

$$(5.3.6) \quad \left( \text{Hom}(W'[g], V)^{(G,Cg)} \xleftarrow{\alpha} \text{Hom}^{\tau_{|Eg|}^0}(W'[g], |A^0(EG, V)|)^{(G,Cg)}, h \right)$$

where the notation $\alpha$ and $h$ is abused somewhat; notice that $\overline{j}$ remains unchanged under integration, though, since it was already $G$-equivariant. Proposition 5.3.1 implies that $\text{Hom}^{\tau^S'}(S'[s^2g], V)^G$ is a small model for $\text{Ext}_{((G,Cg);C)}(\mathbb{R}, V)$ as well. \(\square\)

We will refer to $\text{Hom}(W'[g], V)^{(G,Cg)}$ as the Weil model for $\text{Ext}_{((G,Cg);C)}(\mathbb{R}, V)$ and to the twisted object $\text{Hom}^{\tau^S'}(S'[s^2g], V)^G$ as the Cartan model for $\text{Ext}_{((G,Cg);C)}(\mathbb{R}, V)$ associated with $G$ and $V$. 
Corollary 5.3.7. Passing to \( G \)-invariants in the chain complex \( \text{Hom}^\tau(S'[s^2g], V)^G \) calculating \( \text{Ext}_{(G,\mathcal{G})}(\mathbb{R}, V) \) induces a canonical isomorphism

\[
\text{Ext}_{((G,\mathcal{G});\mathcal{C})}(\mathbb{R}, V) \cong \text{Ext}_{(G,\mathcal{G})}(\mathbb{R}, V)^{\pi_0(G)}. \quad \square
\]

Combining this corollary with Theorem 3.6.4 we arrive at the following.

Corollary 5.3.8. The twisted \( \text{Hom} \)-object \( \text{Hom}^\tau(S', V^G) \) is a small model for \( \text{Ext}_{((G,\mathcal{G});\mathcal{C})}(\mathbb{R}, V) \) that is compatible with the bundle structures in the sense that the obvious pairing

\[
\text{Hom}(S', \mathbb{R}) \otimes \text{Hom}^\tau(S', V^G) \rightarrow \text{Hom}^\tau(S', V^G)
\]

induces the \( H^*(BG) \)-module structure on \( \text{Ext}_{((G,\mathcal{G});\mathcal{C})}(\mathbb{R}, V) \). \quad \square

Let \( G_0 \) be the connected component of the identity. It is worthwhile noting that, under the circumstances of Corollary 5.3.8, \( \Lambda = H_*(G_0) \) and \( S' = H_*(BG_0) \).

We will refer to the twisted object \( \text{Hom}^\tau(S', V^G) \) as the \textit{small Cartan model} for \( \text{Ext}_{((G,\mathcal{G});\mathcal{C})}(\mathbb{R}, V) \) associated with \( G \) and \( V \).

6. Small models in equivariant de Rham theory

Let \( G \) be a reductive Lie group and \( X \) a left \( G \)-manifold.

Substitution of \( \mathcal{A}(X) \) for \( V \) in (5.1.14) yields the model

\[
(6.1) \quad \text{Hom}^\tau(S'[s^2g], |\mathcal{A}(E_G, \mathcal{A}(X))|)^G.
\]

for the \( G \)-equivariant de Rham cohomology of \( X \). An explicit chain equivalence between \( |\mathcal{A}(N(G, X))| \) and (6.1) arises from combination of the contraction (5.1.7) and the chain equivalence (5.1.12) together with the Cartan twist, with \( \mathcal{A}(X) \) being substituted for \( V \).

Suppose that \( G \) is compact. Substitution of \( \mathcal{A}(X) \) for \( V \) in the Weil model \( \text{Hom}(W'[g], V)^{(G,\mathcal{G})} \) for \( \text{Ext}_{((G,\mathcal{G});\mathcal{C})}(\mathbb{R}, V) \) associated with \( G \) and \( V \), cf. (5.3) above, then yields the \textit{Weil model} for the \( G \)-equivariant de Rham cohomology of \( X \). Likewise, substitution of \( \mathcal{A}(X) \) for \( V \) in the Cartan model \( \text{Hom}^\tau(S'[s^2g], V^G) \) for \( \text{Ext}_{((G,\mathcal{G});\mathcal{C})}(\mathbb{R}, V) \) associated with \( G \) and \( V \), cf. (5.3) above, then yields the \textit{Cartan model}

\[
(6.2) \quad \text{Hom}^\tau(S'[s^2g], \mathcal{A}(X))^G
\]

for the \( G \)-equivariant de Rham cohomology of \( X \). An explicit chain equivalence between \( |\mathcal{A}(N(G, X))| \) and the Cartan model arises from combination of the above chain equivalence between \( |\mathcal{A}(N(G, X))| \) and (6.1) with (5.3.6) together with the Cartan twist, with \( \mathcal{A}(X) \) being substituted for \( V \).

In the same vein, substitution of \( \mathcal{A}(X) \) for \( V \) in the small Cartan model \( \text{Hom}^\tau(S', V^G) \) for \( \text{Ext}_{((G,\mathcal{G});\mathcal{C})}(\mathbb{R}, V) \) associated with \( G \) and \( V \), cf. (5.3) above, yields the \textit{small model}

\[
(6.3) \quad \text{Hom}^\tau(H_*(BG), \mathcal{A}(X))^G
\]

for the \( G \)-equivariant de Rham cohomology of \( X \).
Remark 6.4. In [18] (Section 8 and thereafter) a small model for equivariant de Rham theory of the kind (6.3) is explored. Incomplete reasoning and faulty usage of this model led to a certain activity in the literature [1], [2], [18], [47], see in particular the introduction of [2]; cf. Remark 3.6.10 above.

6.5. Homogeneous spaces. For illustration, suppose that $G$ is a closed subgroup of a compact connected Lie group $K$. The $G$-equivariant cohomology of $K$ equals the cohomology of the homogeneous space $K/G$. In the small model (6.3), with $X = K$, we may replace the de $G$-invariant de Rham algebra $A(K)^G$ with the cohomology $H^*(K)$ which, in fact, sits inside $A(K)$ as the graded subalgebra of biinvariant forms. The resulting model for the cohomology of $G/K$ has the form $\text{Hom}_\vartheta(H^*(BG), H^*(K))$, the twisting cochain $\vartheta$ being given as the composite of the induced morphism from $H^*(BG)$ to $H^*(BK)$ with the transgression twisting cochain from $H^*(BK)$ to $H^*(K)$. This is the Cartan model for the de Rham cohomology of the homogeneous space $K/G$ [7].

6.6. Multiplicative cohomology generators. The group $G$ being supposed compact and connected, for the special case where $V = \mathbb{R}$, consider the canonical injection

\[(6.6.1) \quad H^*(BG) \to \overline{B}^{\vartheta}(G, C_g)(G)\]

which is the composite of the injections in (5.1.12) and (5.3.6) for $V = \mathbb{R}$; the injection (6.6.1) is plainly an isomorphism on cohomology. The composite of (6.6.1) with the injection $\alpha^*$ in (5.1.7) for the special case where $V = \mathbb{R}$ yields an injection

\[(6.6.2) \quad H^*(BG) \to |A(NG)|,\]

manifestly a cohomology isomorphism. This injection involves no choices at all and is, in particular, natural in $G$; it is certainly not multiplicative unless $H^*(BG)$ is trivial or a polynomial algebra in a single generator. This injection includes the construction of representatives in $|A(NG)|$ of the multiplicative cohomology generators of $H^*(BG)$, similar to that given in [12] and [53] via the simplicial Chern-Weil construction. It is interesting to note that the present construction of the injection (6.6.2) does not involve curvature arguments.

7. Duality
Suppose that the group $G$ is of finite homological type (that is, its homology is finite in each degree). Recall that the functor

\[t^*: \text{Mod}_{(G, C_g)} \to \overline{B}^{\vartheta}(G, C_g)(G)\text{ Mod}\]

has been defined above, cf. (5.2.1). Define the functor

\[(7.1) \quad h^*: \overline{B}^{\vartheta}(G, C_g)(G)\text{ Mod} \to \text{Mod}_{(G, C_g)}\]

by the assignment to a $(\overline{B}^{\vartheta}(G, C_g)(G))$-module $M$ of the twisted object

\[(7.2) \quad h^*(M) = B^*(G, C_g)(\mathbb{R}, G, M)\overline{B}^{\vartheta}(G, C_g)(G).\]
The latter inherits a canonical $(G,C\mathfrak{g})$-module structure: Indeed,

$$\mathcal{B}^r_{(G,C\mathfrak{g})}(\mathbb{R}, G, M) = \text{Hom}^{T_{\mathfrak{g}}\{\mathfrak{g}\}}(\text{BA}_{\mathfrak{g}}[\mathfrak{g}], A^0(EG, M))$$

inherits a $(G,C\mathfrak{g})$-module structure from the $G$-actions on $\text{BA}_{\mathfrak{g}}[\mathfrak{g}]$ and on $EG$; further, given the $(\mathcal{B}^r_{(G,C\mathfrak{g})}(G))$-module $M$, the action of

$$\mathcal{B}^r_{(G,C\mathfrak{g})}(G) = \mathcal{B}^r_{(G,C\mathfrak{g})}(\mathbb{R},G,M)^{(G,C\mathfrak{g})}$$

on $\mathcal{B}^r_{(G,C\mathfrak{g})}(\mathbb{R}, G, M)$ preserves the $(G,C\mathfrak{g})$-module structure whence the chain complex of $(\mathcal{B}^r_{(G,C\mathfrak{g})}(G))$-invariants inherits a $(G,C\mathfrak{g})$-module structure. Since $G$ is of finite homological type, the twisted object $h^*(M)$ calculates the differential graded $\text{Tor}^{\mathcal{B}^r_{(G,C\mathfrak{g})}(G)}(\mathbb{R},M)$. The functors $t^*$ and $h^*$ are formally different from those denoted by $t^*$ and $h^*$ in [34] (4.1); indeed, since the category of $(G,C\mathfrak{g})$-modules is not one of modules over a chain algebra the framework of [34] (4.1) is not directly applicable.

**Proposition 7.3.** The functors $t^*$ and $h^*$ are homotopy inverse to each other.

**Proof.** Given a $(G,C\mathfrak{g})$-module $N$, the canonical injection $N \rightarrow h^*(t^*(N))$ is a morphism of $(G,C\mathfrak{g})$-modules; given a $\mathcal{B}^r_{(G,C\mathfrak{g})}(G)$-module $M$, the canonical injection $M \rightarrow t^*(h^*(M))$ is a morphism of $\mathcal{B}^r_{(G,C\mathfrak{g})}(G)$-modules. As a $(G,C\mathfrak{g})$-module, $h^*(t^*(N))$ amounts to

$$\mathcal{B}^r_{(G,C\mathfrak{g})}(\mathbb{R}, G, N) = \text{Hom}^{T_{\mathfrak{g}}\{\mathfrak{g}\}}(\text{BA}_{\mathfrak{g}}[\mathfrak{g}], A^0(EG, N)),$$

with the diagonal $(G,C\mathfrak{g})$-module structure. This corresponds to the fact that, for a space $Y$ over $BG$, the Borel construction, applied to the total space $P_Y$ of the induced $G$-bundle, yields the space $EG \times Y$. Likewise, as a $\mathcal{B}^r_{(G,C\mathfrak{g})}(G)$-module, $t^*(h^*(M))$ amounts to

$$\mathcal{B}^r_{(G,C\mathfrak{g})}(\mathbb{R}, G, M) = \text{Hom}^{T_{\mathfrak{g}}\{\mathfrak{g}\}}(\text{BA}_{\mathfrak{g}}[\mathfrak{g}], A^0(EG, M)),$$

with the diagonal $\mathcal{B}^r_{(G,C\mathfrak{g})}(G)$-module structure. This corresponds to the fact that, for a $G$-space $X$, the total space $P_Y$ of the induced $G$-bundle over the Borel construction $Y = EG \times_G X$ amounts to $EG \times X$. Furthermore, since $\text{BA}_{\mathfrak{g}}[\mathfrak{g}]$ is contractible (cf. Section 3 above) and since, for any chain complex $V$, $A^0(EG,V)$ contracts onto $V$, cf. (5.3.3), for any chain complex $V$, the injection of $V$ into $\mathcal{B}^r_{(G,C\mathfrak{g})}(\mathbb{R}, G, V)$ is a chain equivalence. Hence the injections $M \rightarrow t^*(h^*(M))$ and $N \rightarrow h^*(t^*(N))$ are chain equivalences, in fact, may be extended to contractions in a canonical way. □

We will now apply the duality spelled out in Proposition 7.3 to spaces. Let $Y$ be a simplicial space $Y$ over the simplicial space $NG$ and consider the fiber square

$$\begin{array}{ccc}
P_Y & \longrightarrow & EG \\
\downarrow & & \downarrow \\
Y & \longrightarrow & NG
\end{array}$$

(7.4)
of simplicial spaces, the left-hand arrow being the induced simplicial principal $G$-bundle over $Y$. Via the induced morphism of differential graded algebras from $|A(NG)|$ to $|A(Y)|$, the chain complex $|A(Y)|$ inherits a differential graded $|A(NG)|$-module structure and the de Rham cohomology of the fiber $P_Y$ is canonically isomorphic to the differential graded

$$(7.5) \quad \text{Tor}^{|A(NG)|}(R, |A(Y)|)$$

which, by definition, is the homology of the bar construction

$$(7.6) \quad B(R, |A(NG)|, |A(Y)|) = B|A(NG)| \otimes_{\tau B} |A(Y)|.$$ 

The duality spelled out in Proposition 7.3 does not apply directly, since the $|A(NG)|$-module structure on $|A(Y)|$ does not factor through a $(B^*(G,Cg)(G))$-module structure. Now the contraction (5.1.7), with $V = R$, takes the form

$$(7.7) \quad \left( B^*(G,Cg)(G) \overset{\Sigma^*}{\leftarrow} |A(NG)|, h^* \right)$$

and, in the category of sh-algebras, $|A(NG)|$ and $B^*(G,Cg)(G)$ are isomorphic via the contraction (7.7); this notion of isomorphism is explained in Section 6 of [34]. Hence $|A(Y)|$ inherits an sh-module structure over $B^*(G,Cg)(G)$, unique up to homotopy. We now make this explicit.

To this end, we apply the construction [34] (2.2.1*) to the contraction (7.7), the bar construction twisting cochain from $B^*(G,Cg)(G)$ to $B^*(G,Cg)(G)$ being substituted for the twisting $\sigma$ in [34] (2.2.1*). This yields the acyclic twisting cochain

$$\xi: B^*(G,Cg)(G) \to |A(NG)|.$$ 

Via the adjoint

$$\xi: \Omega B^*(G,Cg)(G) \to |A(NG)|,$$

we view henceforth any $|A(NG)|$-module as an $\Omega B^*(G,Cg)(G)$-module, that is, as an sh-module over $B^*(G,Cg)(G)$.

On the category $B^*(G,Cg)(G)\text{Mod}^\infty$ of sh-modules over $B^*(G,Cg)(G)$, consider the functor

$$H^*_\infty: B^*(G,Cg)(G)\text{Mod}^\infty \to B^*(G,Cg)(G)\text{Comod}$$

which assigns to an arbitrary sh-module $(M, \tau_{\Omega B})$ over $B^*(G,Cg)(G)$ the twisted object

$$(7.8) \quad H^*_\infty(M, \tau_{\Omega B}) = B^*B^*(G,Cg)(G) \otimes_{\tau_{\Omega B}} M;$$

here $\tau_{\Omega B}$ refers to the universal twisting cochain from $B$ to $\Omega B$. In particular, the twisting cochain $\xi$ induces an sh-structure on $|A(Y)|$, and

$$H^*_\infty(|A(Y)|, \tau_{\Omega B}) = B^*B^*(G,Cg)(G) \otimes \xi |A(Y)|.$$
we will simplify the notation and write

\[ H^*_\infty(\lvert \mathcal{A}(Y) \rvert, \xi) = H^*_\infty(\lvert \mathcal{A}(Y) \rvert, \tau_{\mathbb{P}^1}). \]

Since the induced bundle morphism

\[ (7.9) \quad \xi \otimes \text{Id}: \mathbb{B}^*_{\mathbb{C}}(G) \otimes \xi \lvert \mathcal{A}(Y) \rvert \to \mathbb{B}^* \lvert \mathcal{A}(NG) \rvert \otimes \tau_{\mathbb{P}^1(NG)} \lvert \mathcal{A}(Y) \rvert \]

is a chain equivalence, in fact, can be extended to a contraction via the perturbation lemma, cf. [34] (2.3), the twisted object \( H^*_\infty(\lvert \mathcal{A}(Y) \rvert, \xi) \) calculates the differential graded Ext

\[ \Lambda \otimes \mathbb{C} \to \mathcal{A}(Y) \]

Lemma, cf. [34] (2.3), the twisted object

\[ \mathcal{A}(G) \text{ and the missing coalgebra structure thereupon. By functoriality, for an arbitrary sh-module } (M, \tau_{\mathbb{P}^1}) \text{ over } \mathbb{B}^*_{\mathbb{C}}(G), \text{ the obvious right } (G, C_B) \text{-module structure on } \mathbb{B}^*_{\mathbb{C}}(G) \text{ induces a } (G, C_B) \text{-module structure on the twisted object } \mathbb{B}^*_{\mathbb{C}}(G) \otimes \tau_{\mathbb{P}^1} M. \]

In this fashion, we view \( H^*_\infty \) as a functor of the kind

\[ (7.10) \quad H^*_\infty: \mathbb{B}^*_{\mathbb{C}}(G) \text{Mod}^\infty \to \text{Mod}(G, C_B). \]

The duality between the two functors \( t^* \) and \( h^* \) spelled out in Proposition 7.3 above entails the following:

**Theorem 7.11.** On the category of left \( G \)-manifolds, the functor \( h^* \circ t^* \circ \mathcal{A} \) is chain-equivalent to the functor \( \mathcal{A} \) as \( (G, C_B) \)-module functors; and on the category of simplicial manifolds over \( NG \), the functor \( t^* \circ H^*_\infty \circ \lvert \mathcal{A} \rvert \) is chain-equivalent to the functor \( \lvert \mathcal{A} \rvert \) as sh-module functors over \( \mathbb{B}^*_{\mathbb{C}}(G) \). In particular, application of the functor \( h^* \) to the twisted object \( t^*(\mathcal{A}(X)) = \mathbb{B}^*_{\mathbb{C}}(\mathbb{R}, G, \mathcal{A}(X)) \) calculating the \( G \)-equivariant de Rham cohomology of \( X \) (in view of Theorem 2.7.1) reproduces an object calculating the ordinary de Rham cohomology of \( X \); and application of the functor \( t^* \) to the twisted object \( H^*_\infty(\lvert \mathcal{A}(Y) \rvert, \xi) \) reproduces an object calculating the de Rham cohomology of the simplicial space \( Y \). □

7.12. Koszul duality. Let \( \Lambda = H_s G, S' = H_s (BG), \Lambda' = H^* G, S = H^* (BG), \) and let \( \tau: S' \to \Lambda \) be the transgression twisting cochain. Ordinary Koszul duality involves the two functors

\[ t^*: \text{Mod}_\Lambda \to \text{sMod}, \quad t^*(N) = \text{Hom}^\tau(S', N) \]

\[ h^*: \text{sMod} \to \text{Mod}_\Lambda, \quad h^*(M) = \text{Hom}^\tau(\Lambda, M). \]

The former assigns to a (right) \( \Lambda \)-module \( N \) the twisted Hom-object \( t^*(N) \) which calculates the differential graded \( \text{Ext}_{\Lambda}(\mathbb{R}, N) \), and the latter assigns to a (left) \( S \)-module \( M \) the twisted Hom-object \( h^*(M) \) which, since \( S \) is of finite type, calculates the differential graded \( \text{Tor}^S(\mathbb{R}, M) \). These functors are chain homotopy inverse to each other in an obvious manner.

Replace \( \lvert \mathcal{A}(X) \rvert \) with the ordinary \( \Lambda \)-module

\[ (7.12.1) \quad \Lambda' \otimes_{\zeta_B} t^*(\lvert \mathcal{A}(X) \rvert) = \Lambda' \otimes_{\zeta_B} \mathbb{B}^*_{\mathbb{C}}(\mathbb{R}, G, \lvert \mathcal{A}(X) \rvert), \]
cf. (5.2) above. In the same vein, given a simplicial space $Y$ over $NG$, we replace $|A(Y)|$ with an ordinary $S$-module as follows where we write

$$\mathcal{B}^* = \mathcal{B}^*_{(G,C^g)}(\mathbb{R},G,\mathbb{R})$$

for simplicity: Extend the adjoint $\overline{\zeta}_S$ of the twisting cochain (5.2.5) to a contraction

$$\overset{(7.12.2)}{\Lambda' \xrightarrow{\overline{\zeta}_S} \mathcal{B}^*, h}.$$ 

The construction [34] (2.2.1*), applied to (7.12.2) and the (acyclic) transgression twisting cochain $\tau^*: \Lambda' \to S$ (the dual of the transgression twisting cochain $\tau: S' \to \Lambda$) yields the acyclic twisting cochain

$$\overset{(7.12.3)}{\zeta^\mathcal{B}^*: \mathcal{B}^* \to S}.$$ 

This twisting cochain determines the twisted object

$$\overset{(7.12.4)}{S \otimes \zeta^\mathcal{B}^* H^\infty_*(|A(Y)|, \xi)}$$

which, for our purposes, is the appropriate replacement for $|A(Y)|$. This twisted object is, in particular, an ordinary $S$-module. With these twisted objects, a version of Koszul duality is given by the functors $t^*$ and $h^*$ between the categories $\text{sMod}$ and $\text{Mod}_\Lambda$: The functor $h^*$ reconstructs the ordinary cohomology of a $G$-manifold $X$ from a model of the kind (7.12.4) for the $G$-equivariant cohomology (where the construction (7.12.4) is carried out for $Y = N(G,X)$); and the functor $t^*$ reconstructs the equivariant cohomology of a $G$-manifold from a model of the kind (7.12.1) for the ordinary cohomology. This corresponds to the procedure employed in [18] (cf. e. g. p. 29) which consists in replacing the naive cochain complexes, where the $\Lambda$- and $S$-actions are not defined, by equivalent cochain complexes where the actions are defined.

7.13. Koszul duality when $G$ is finite dimensional, compact and connected, cf. e. g. [18]. We recall it, to establish the link with the theory built up above. As noted in (5.3) above, cf. Corollary 5.3.8, given $X$, the algebra $A(X)^G$ of invariants inherits now a $(H_*(G))$-module structure, and the model (6.3) for the $G$-equivariant de Rham cohomology is exactly $t^*(A(X)^G)$. The functor $h^*$ reconstructs the ordinary cohomology of $X$ from $t^*(A(X)^G)$. Under our general circumstances (where $G$ is a general, possibly infinite dimensional Lie group), (7.12.1) is a replacement for (6.3) and the functor $t^*$ applies, for a general simplicial space over $NG$, to the model (7.12.4).

References

1. C. Allday and V. Puppe, On a conjecture of Goresky, Kottwitz and MacPherson, Canad. J. of Mathematics 51 (1999), 3-9.
2. A. Alexeiev and E. Meinrenken, Equivariant cohomology and the Maurer-Cartan equation, Duke Math. J. 130 (2005), 479–521, math.DG/0406350.
3. M. Barr, Cartan-Eilenberg cohomology and triples, Journal of Pure and Applied Algebra 112 (1996), 219–238.
4. R. Bott, *On the Chern-Weil homomorphism and the continuous cohomology of Lie groups*, Advances 11 (1973), 289–303.

5. R. Bott, H. Shulman, and J. Stasheff, *On the de Rham theory of certain classifying spaces*, Advances 20 (1976), 43–56.

6. H. Cartan, *Notions d'algèbre différentielle; applications aux groupes de Lie et aux variétés où opère un groupe de Lie*, Bruxelles, Coll. Topologie Algébrique (1950), 15–28.

7. ______, *La transgression dans un groupe de Lie et dans un espace fibré principal*, Bruxelles, Coll. Topologie Algébrique (1950), 57–72.

8. ______, *Algèbres d'Eilenberg–Mac Lane et homotopie*, exposés 2–11, Séminaire H. Cartan 1954/55, Ecole Normale Superieure, Paris, 1956.

9. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956.

10. C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. 63 (1948), 85–124.

11. A. Dold und D. Puppe, *Homologie nicht-additiver Funktoren. Anwendungen*, Annales de l’Institut Fourier 11 (1961), 201–313.

12. J. L. Dupont, *Simplicial de Rham cohomology and characteristic classes of flat bundles*, Topology 15 (1976), 233–245.

13. J. Duskin, *Simplicial methods and the interpretation of “triple” cohomology*, Memoirs Amer. Math. Soc. 163 (1975).

14. S. Eilenberg and J. C. Moore, *Foundations of relative homological algebra*, Memoirs AMS 55 (1965), Amer. Math. Soc., Providence, Rhode Island.

15. M. Franz, *Koszul duality and equivariant cohomology for tori*, Int. Math. Res. Not. 42 (2003), 2255–2303, [math.AT/0301083](http://arxiv.org/abs/math.AT/0301083).

16. ______, *Koszul duality and equivariant cohomology*, [math.AT/0307115](http://arxiv.org/abs/math.AT/0307115).

17. R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1958.

18. M. Goersky, R. Kottwitz, and R. Mac Pherson, *Equivariant cohomology, Koszul duality and the localization theorem*, Invent. Math. 131 (1998), 25–83.

19. V.K.A.M. Gugenheim, *On the chain complex of a fibration*, Illinois J. of Mathematics 16 (1972), 398–414.

20. V.K.A.M. Gugenheim and J.P. May, *On the theory and applications of differential torsion products*, Memoirs of the Amer. Math. Soc. 142 (1974).

21. V.K.A.M. Gugenheim and H. J. Munkholm, *On the extended functoriality of Tor and Cotor*, J. of Pure and Applied Algebra 4 (1974), 9–29.

22. G. Hochschild, *Relative homological algebra*, Trans. Amer. Math. Soc. 82 (1956), 246–269.

23. G. Hochschild and G. D. Mostow, *Cohomology of Lie groups*, Illinois J. of Math. 6 (1962), 367–401.

24. J. Huebschmann, *Perturbation theory and small models for the chains of certain induced fibre spaces*, Habilitationsschrift, Universität Heidelberg, 1984, Zbl 576.55012.

25. ______, *Perturbation theory and free resolutions for nilpotent groups of class 2*, J. of Algebra 126 (1989), 348–399.

26. ______, *Cohomology of nilpotent groups of class 2*, J. of Algebra 126 (1989), 400–450.
27. Loday, The mod p cohomology rings of metacyclic groups, J. of Pure and Applied Algebra 60 (1989), 53-105.
28. Loday, Cohomology of metacyclic groups, Trans. Amer. Math. Soc. 328 (1991), 1-72.
29. Loday, Extended moduli spaces, the Kan construction, and lattice gauge theory, Topology 38 (1999), 555-596, dg-ga/9505005, dg-ga/9506006.
30. Loday, Poisson cohomology and quantization, J. reine angew. Math. 408 (1990), 57–113.
31. Loday, Extensions of Lie-Rinehart algebras and the Chern-Weil construction, in: Festschrift in honor of J. Stasheff's 60-th birthday, Cont. Math. 227 (1999), Amer. Math. Soc., Providence R. I., 145–176, math.DG/9706002.
32. Loday, Lie-Rinehart algebras, descent, and quantization, in: Galois theory, Hopf algebras, and semiabelian categories, Fields Institute Communications 43 (2004), Amer. Math. Soc., Providence R. I., 295–316, math.SG/0303016.
33. Loday, Minimal free multi models for chain algebras, in: Chogoshvili Memorial, Georgian Math. J. 11 (2004), 733–752, math.AT/0405172.
34. Loday, Homological perturbations, equivariant cohomology, and Koszul duality, math.AT/0401160.
35. Loday, The Lie algebra perturbation lemma, in: Festschrift in honor of M. Gerstenhaber's 80-th and Jim Stasheff's 70-th birthday, Progress in Math., Birkhäuser-Verlag (to appear), arXiv:0708.3977.
36. Loday, The sh-Lie algebra perturbation lemma, arXiv:0710.2070.
37. J. Huebschmann and T. Kadeishvili, Small models for chain algebras, Math. Z. 207 (1991), 245–280.
38. J. Huebschmann and J. D. Stasheff, Formal solution of the master equation via HPT and deformation theory, math.AG/9906036. Forum mathematicum 14 (2002), 847–868.
39. D. Husemoller, J. C. Moore, and J. D. Stasheff, Differential homological algebra and homogeneous spaces, J. of Pure and Applied Algebra 5 (1974), 113–185.
40. F. W. Kamber and Ph. Tondeur, Algèbres de Weil semi simpliciales, C. R. Acad. Sci. Paris Sér. A-B 276 (1973), A1407–1410.
41. Loday, Characteristic invariants of foliated bundles, Manuscripta Math. 11 (1974), 51–89.
42. Loday, Semi-simplicial Weil algebras and characteristic classes, Tôhoku Math. J. (2) 30 (1978), 373-422.
43. B. Kostant, Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the $\rho$-decomposition $C(\mathfrak{g}) = \text{End}_V \otimes C(P)$, and the $\mathfrak{g}$-module structure of $\wedge \mathfrak{g}$, Adv. in Math. 125 no. 2 (1997), 275–350.
44. S. Mac Lane, Homologie des anneaux et des modules, in: Colloque de topologie algébrique, Louvain (1956), 55–80.
45. S. Mac Lane, Homology, Die Grundlehren der mathematischen Wissenschaften No. 114, Springer, Berlin · Göttingen · Heidelberg, 1963.
46. Loday, Categories for the Working Mathematician, Graduate Texts in Mathematics, vol. 5, Springer, Berlin · Göttingen · Heidelberg, 1971.
47. T. Maszczyk and A. Weber, Koszul duality for modules over Lie algebras, Duke Math. J. 112 (2002), 111–120, math.AG/0101180.
48. J. C. Moore, *Cartan’s constructions*, Colloque analyse et topologie, en l’honneur de Henri Cartan, Astérisque 32–33 (1976), 173–221.

49. , *Differential homological algebra*, Actes, Congres intern. math. Nice, 1970 (1971), Gauthiers-Villars, Paris, 335–339.

50. D. Quillen, *Rational homotopy theory*, Ann. of Math. 90 (1969), 205–295.

51. G. Rinehart, *Differential forms for general commutative algebras*, Trans. Amer. Math. Soc. 108 (1963), 195–222.

52. G. B. Segal, *Classifying spaces and spectral sequences*, Publ. Math. I. H. E. S. 34 (1968), 105–112.

53. H. B. Shulman, *Characteristic classes and foliations*, Ph. D. Thesis, University of California, 1972.

54. J. D. Stasheff, *Continuous cohomology of groups and classifying spaces*, Bull. Amer. Math. Soc. 84 (1978), 513–530.

55. J. D. Stasheff and S. Halperin, *Differential algebra in its own rite*, Proc. Adv. Study Alg. Top. August 10–23, 1970, Aarhus, Denmark, 567–577.

56. W. T. Van Est, *Une application d’une méthode de Cartan-Leray*, Nederl. Akad. Wetensch. Proc. Ser. A 58 = Indag. Math. 17 (1955), 542–544.

57. , *Algèbres de Maurer-Cartan et holonomie*, Ann. Fac. Sci. Toulouse Math. 5 (suppl.), 93–134.