HAMILTONIAN EVOLUTIONARY GAMES

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Abstract. We introduce a class of o.d.e.'s that generalizes to polymatrix games the replicator equations on symmetric and asymmetric games. We also introduce a new class of Poisson structures on the phase space of these systems, and characterize the corresponding subclass of Hamiltonian polymatrix replicator systems. This extends known results for symmetric and asymmetric replicator systems.

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1. Introduction

State of the art. Evolutionary Game Theory (EGT) originated from the work of John Maynard Smith and George R. Price who applied the theory of strategic games developed by John von Neumann and Oskar Morgenstern to evolution problems in Biology. Unlike Game Theory, EGT investigates the dynamical processes of biological populations.

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Independently A. Lotka and V. Volterra introduced the following class of o.d.e.’s
\[
\frac{dx_i}{dt} = x_i \left( r_i + \sum_{j=1}^{n} a_{ij} x_j \right) \quad (1 \leq i \leq n),
\]
currently known as Lotka-Volterra (LV) systems, and usually taken as models for the time evolution of ecosystems in \( n \) species. Although historically this class of systems preceded EGT they are now considered an integral part of this theory. The entries \( a_{ij} \) represent interactions between different species, while the coefficients \( r_i \) stand for the specie’s natural growth rates. In his studies \cite{19} V. Volterra gave special attention to predator-prey systems and their generalization to food chain systems in \( n \) species, which fall in the category of dissipative and conservative LV systems. Denoting by \( A = [a_{ij}]_{ij} \) its interaction matrix, a LV system is said to be dissipative, resp. conservative, if there exists a positive diagonal matrix \( D \) such that \( AD + DA^t \leq 0 \), resp. \( AD \) is skew symmetric. The matrix \( D \) was interpreted by Volterra as some sort of normalization by the average weights of the different species. If the LV system admits an equilibrium point \( q \in \mathbb{R}^n \) the following function \( H : \text{int}(\mathbb{R}^n_+) \to \mathbb{R} \)
\[
H(x) = \sum_{j=1}^{n} x_j - q_j \log x_j
\]
is either a decreasing Lyapunov function, if the system is dissipative, or else a constant of motion, if the system is conservative. Volterra proved that the dynamics of any \( n \) species conservative LV system can be embedded in a Hamiltonian system of dimension \( 2n \). More recently, in the 1980’s, Redheffer et al. developed further the theory of dissipative LV systems, introducing and studying the class of stably dissipative systems \cite{14-18}. In \cite{2} a re-interpretation was given for the Hamiltonian character of the dynamics of any conservative LV system: there is a Poisson structure on \( \mathbb{R}^n_+ \) which makes the system Hamiltonian. Another interesting fact from \cite{2}, which stresses the importance of studying Hamiltonian LV systems, is that the limit dynamics of any stably dissipative LV system is described by a conservative LV system.

Another class of o.d.e.’s, which plays a central role in EGT, is the replicator equation defined on the simplex \( \Delta^{n-1} = \{ x \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} x_i = 1 \} \) by
\[
\frac{dx_i}{dt} = x_i \left( \sum_{j=1}^{n} a_{ij} x_j - \sum_{k,j=1}^{n} a_{kj} x_k x_j \right) \quad (1 \leq i \leq n).
\]
The coefficients of this o.d.e. are stored in an \( n \times n \) real matrix \( A = [a_{ij}]_{ij} \), that is referred as the pay-off matrix. A game theoretical interpretation for this equation is provided in section \cite{3}. Check \cite{8} on the history of this equation. In \cite{9} J. Hofbauer introduced a change of coordinates, mapping \( \mathbb{R}^n_+ \) to the simplex \( \Delta^n \) minus one face, which conjugates any LV system in \( \mathbb{R}^n_+ \) to a time re-parametrization of a replicator
system in $\Delta^n$, and vice-versa. Thus when a LV system is conservative then the corresponding replicator system is orbit equivalent to a Hamiltonian system. On the other hand, any replicator system on $\Delta^{n-1}$ with skew symmetric pay-off matrix extends to a LV system on $\mathbb{R}^n_+$ with $r_i = 0$, and hence can be viewed as a restriction of a Hamiltonian LV system on $\mathbb{R}^n_+$. Up to our knowledge these are the known subclasses of Hamiltonian replicator systems.

Asymmetric or bimatrix games lead to another fundamental class of models in EGT, the following system of o.d.e.’s whose coefficients are displayed in two pay-off matrices, $A$ of order $n \times m$ and $B$ of order $m \times n$.

$$\begin{align*}
\frac{dx_i}{dt} &= x_i \left( \sum_{j=1}^{m} a_{ij} y_j - \sum_{k=1}^{n} \sum_{j=1}^{m} a_{kj} x_k y_j \right) & i = 1, \ldots, n \\
\frac{dy_j}{dt} &= y_j \left( \sum_{i=1}^{n} b_{ji} x_i - \sum_{k=1}^{m} \sum_{i=1}^{n} b_{ki} y_k x_i \right) & j = 1, \ldots, m
\end{align*}$$

The phase space of this equation is the prism $\Delta^{n-1} \times \Delta^{m-1}$. A game theoretical interpretation is given in section 3. It was remarked by I. Eshel and E. Akin [5] that up to a time re-parametrization these systems always preserve volume. For $\lambda$-zero-sum games ($\lambda < 0$) and $\lambda$-partnership games ($\lambda > 0$), with an interior equilibrium point in the prism $\Delta^{n-1} \times \Delta^{m-1}$, J. Hofbauer proved in [7] that this bimatrix system is orbit equivalent to a Hamiltonian system w.r.t. some Poisson structure in the interior of the prism. Previously, E. Akin and V. Losert [1] had noticed the Hamiltonian character of this model in the zero-sum case.

Polymatrix games, like $n$-player games, generalize the concept of bimatrix games. The main difference between them is that interactions between players are bilateral in the former game but not in the latter. The first reference we could find on the existence of equilibria for these games is the paper of J. Howson [10] who attributes the concept of polymatrix game to E. Yanovskaya (1968). More recently, the structure of Nash equilibria for polymatrix games is studied by L. Quintas in [13].

**Main results.** We introduce a class of o.d.e’s, referred as polymatrix replicator equation, that generalizes to polymatrix games the symmetric and asymmetric replicator equations. We are not aware of any reference on this equation in the literature. The phase space of these systems are finite products of simplexes. We introduce the concept of conservative polymatrix game, which in the case of bimatrix games extends the $\lambda$-zero-sum games ($\lambda < 0$) and the $\lambda$-partnership games ($\lambda > 0$). In Theorem 3.13 we introduce a class of Poisson structures on finite products of simplexes (see (3.4)). We will show that these prisms are stratified Poisson spaces (see section 4). Then in Theorem 3.20 we show that any conservative polymatrix game determines a Hamiltonian polymatrix replicator. This work extends and unifies several known facts on Hamiltonian replicator o.d.e.’s. In the
end of section 3 we compare our results with known facts mentioned in the state of the art subsection.

The paper is organized as follows. In section 2 we introduce the needed concepts from Poisson geometry. In section 3 we state and prove the main results. In section 4 we discuss a method introduced in [6], called singular Poisson reduction, which gives a geometric interpretation of the Poisson structures defined in section 3. In the last section we workout a couple of examples.

2. Generalities on Poisson Structures

In this section we will provide a short introduction to Poisson geometry focused on some dynamical aspects, see any standard textbook on Poisson manifolds and related topics, for example [3,11].

Let $M$ be an $n$-dimensional smooth manifold. We denote by $C^\infty(M)$ the space of smooth functions on $M$. A Poisson structure on $M$ is an $R$-bilinear bracket $\{.,.\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ which satisfies:

i) Anti-symmetry i.e. $\{f,g\} = -\{g,f\}$ for every $f,g \in C^\infty(M)$.

ii) Leibniz’s rule i.e. $\{fg,h\} = f\{g,h\} + g\{f,h\}$ for every $f,g,h \in C^\infty(M)$.

iii) Jacobi identity i.e. $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$

The Leibniz’s rule says that for any smooth function $H : M \to R$ the map $\{.,H\} : f \mapsto \{f,H\}$ is a derivation on $C^\infty(M)$ which in turn yields a vector field $X_H$ on $M$ defined by the equality $\{f,H\} = df(X_H)$. The vector field $X_H$ is called the Hamiltonian vector field associated to $H$ on the Poisson manifold $M$.

The singular distribution $D(x) := \{X_f(x) | f \in C^\infty(M)\}$ is called the characteristic distribution of $M$. As a consequence of the Jacobi identity this distribution integrates to a singular foliation. Denote by $S_x$ the leaf of this foliation through a point $x$. The Poisson structure induces a symplectic form on each leaf $S_x$, passing through arbitrary point $x \in M$, of this foliation defined by $\omega_{S_x}(X_f, X_h) = \{f,h\}$. The foliation $S := \{(S_x,\omega_{S_x}) | x \in M\}$ is called the symplectic foliation of the Poisson manifold $M$.

Remark 2.1. The following are well known properties of Poisson structures:

1) By (i), $dH(X_H) = \{H,H\} = -\{H,H\} = 0$. Thus $H$ is an integral of motion for the vector field $X_H$.

2) The dimension of the linear subspace $D(x)$ is called the rank of the Poisson structure at point $x$, which is equal to the dimension of the leaf $S_x$. Since this leaf is a symplectic manifold on its own it has even dimension.

3) The symplectic foliation $S := \{(S_x,\omega_{S_x}) | x \in M\}$ completely determines the Poisson structure.

4) By definition, it is clear that every symplectic leaf $S_x$ is an invariant submanifold for any Hamiltonian vector field $X_H$. In fact, the restriction of $X_H$ to $S_x$ is Hamiltonian with respect to the symplectic structure $\omega_{S_x}$.
5) Every symplectic manifold \((N, \omega)\) is a Poisson manifold with Poisson bracket defined by \(\{f, g\}_N := \omega(X_f, X_g)\), where \(X_f\) and \(X_g\) are the Hamiltonian vector fields associated to \(f\) and \(g\) by symplectic structure.

6) A function \(f\) is called Casimir if \(\{., f\} = 0\). Note that Casimirs are constants of motion for any Hamiltonian vector field. Furthermore, if \(f_1, f_2\) are two Casimirs then \(\{f_1, f_2\}\) is also a Casimir due to Jacobi identity.

In a local coordinate chart \((U, x_1, ..., x_n)\), or equivalently when \(M = \mathbb{R}^n\), a Poisson bracket takes the form

\[
\{f, g\}(x) = (d_x f)^t \left[ \pi_{ij}(x) \right]_{ij} d_x g = \sum_{i<j} \pi_{ij}(x) \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right),
\]

where \(\pi(x) = [\pi_{ij}(x)]_{ij} = [\{x_i, x_j\}(x)]_{ij}\) is a skew symmetric matrix valued smooth function, and for every function \(f\) we write

\[
d_x f = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix}.
\]

The Jacobi identity translates to:

\[
\sum_{l=1}^n \frac{\partial \pi_{ij}}{\partial x_l} \pi_{lk} + \frac{\partial \pi_{jk}}{\partial x_l} \pi_{li} + \frac{\partial \pi_{ki}}{\partial x_l} \pi_{lj} = 0 \quad \forall i, j, k, \quad (2.1)
\]

or equivalently

\[
\{\{x_i, x_j\}, x_k\} + \{\{x_j, x_k\}, x_i\} + \{\{x_k, x_i\}, x_j\} = 0 \quad \forall i, j, k. \quad (2.2)
\]

Clearly, every skew symmetric matrix valued function \(\pi : \mathbb{R}^n \to \text{Mat}_{n \times n}(\mathbb{R})\) satisfying condition (2.1) defines a Poisson structure on \(\mathbb{R}^n\). In the next section we shall introduce our Poisson structures through their associated skew symmetric matrix valued functions, referred as bivectors \(\pi : \mathbb{R}^n \to \text{Mat}_{n \times n}(\mathbb{R})\). The term bivector means that \(\pi(x)\) is as a linear operator \(\pi(x) : (\mathbb{R}^n)^* \to \mathbb{R}^n\).

**Remark 2.2.** Regarding the function \(\pi\) we have

1) For any function \(H\) the associated Hamiltonian vector field is defined by

\[X_H = \pi dH,\]

2) The characteristic distribution \(D_\pi(x)\) is the one generated by the columns of the matrix \(\pi(x)\).

3) It transforms under a change of variable \(\psi : \mathbb{R}^n \to \mathbb{R}^n\) by

\[
(d_m \psi)(\pi(m))(d_m \psi)^t = \pi(\psi(m)), \quad (2.3)
\]

4) A function \(f \in C^\infty(\mathbb{R}^n)\) is a Casimir if

\[X_f = \pi df = 0.\]

Let \((M, \{., \}_M)\) and \((N, \{., .\}_N)\) be two Poisson manifolds.
Definition 2.3. A smooth map $\psi : M \to N$ will be called a Poisson map if and only if
\[ \{ f \circ \psi, h \circ \psi \}_M = \{ f, h \}_N \circ \psi \quad \forall f, h \in C^\infty(N). \]
In local coordinate, this condition reads as
\[ (d_m\psi)\pi_M(m)(d_m\psi)^t = \pi_N(\psi(m)), \]
where $\pi_M$ and $\pi_N$ are skew symmetric matrix valued functions associated to Poisson structures of $M$ and $N$, respectively, and $d_m\psi$ is the Jacobian matrix of the map $\psi$ at point $m$.

3. Polymatrix games

In this section we introduce the evolutionary polymatrix games to which our main result applies. This class of systems contains both the replicator models and the evolutionary bimatrix games.

Consider a population whose individuals interact with each other using one of $n$ possible pure strategies. The state of the population is described by a probability vector $p = (p_1, \ldots, p_n)$, with the usage frequency of each pure strategy. This vector is a point in the $n-1$-dimensional simplex
\[ \Delta^{n-1} = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \ldots + x_n = 1, x_i \geq 0 \}. \]
A symmetric game is specified by a $n \times n$ pay-off matrix $A = [a_{ij}]_{ij}$, where the entry $a_{ij}$ represents the pay-off of an individual using pure strategy $i$ against another using pure strategy $j$. Given $x \in \Delta^{n-1}$, the value $(Ax)_i = \sum_{j=1}^{n} a_{ij} x_j$ represents the average pay-off of strategy $i$ within a population at state $x$. Similarly, the value $x^t A x = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ stands for the overall average of a population at state $x$, while the difference $(Ax)_i - x^t A x$ measures the relative fitness of strategy $i$ in the population $x$. The replicator model is the following o.d.e. on $\Delta^{n-1}$
\[ \frac{dx_i}{dt} = x_i \left( (Ax)_i - x^t A x \right) \quad 1 \leq i \leq n \]
which says that the logarithmic growth rate of each pure strategy’s frequency equals its relative fitness. The flow of this o.d.e. is complete and leaves the simplex $\Delta^{n-1}$ invariant, as well as every of its faces.

Next we introduce the class of evolutionary asymmetric, or bimatrix games, where two groups of individuals within a population (e.g. males and females), or two different populations, interact using different sets of strategies, say $n$ strategies for the first group and $m$ strategies for the second. The state of this model is a pair of probability vectors in the $(n + m - 2)$-dimensional prism $\Gamma_{n,m} = \Delta^{n-1} \times \Delta^{m-1}$.
There are no interactions within each group. The game is specified by two pay-off matrices: a $n \times m$ matrix $A = [a_{ij}]_{ij}$, where $a_{ij}$ is the pay-off for a member of the first group using strategy $i$ against an individual of the second group using strategy $j$, and a $m \times n$ matrix $B = [b_{ij}]_{ij}$ with the pay-offs for the second group members. Assuming the first and second group states are $x$ and $y$, respectively,
the value \( (Ay)_i \) is the average pay-off for a first group individual using strategy \( i \), the number \( x^t Ay \) is the overall average pay-off for the first group members, and the difference \( (Ay)_i - x^t Ay \) measures the relative fitness of the first group strategy \( i \). Similarly, \( (Bx)_j - y^t Bx \) measures the relative fitness of the second group strategy \( j \) when the group states are \( x \) and \( y \). The bimatrix replicator is the following o.d.e. on the prism \( \Gamma_{n,m} \)

\[
\frac{dx_i}{dt} = x_i \left( (Ay)_i - x^t Ay \right) \quad 1 \leq i \leq n
\]

\[
\frac{dy_j}{dt} = y_j \left( (Bx)_j - y^t Bx \right) \quad 1 \leq j \leq m
\]

which again says that the logarithmic growth rate of each strategy’s frequency equals its relative fitness. The flow of this o.d.e. is complete and leaves the prism \( \Gamma_{n,m} \) invariant, as well as every of its faces.

Finally we introduce the class of polymatrix replicators. Consider \( p \) different populations, or else a single population stratified in \( p \) groups. We shall use greek letters like \( \alpha \) and \( \beta \) to denote these groups. Assume that for each group \( \alpha \in \{1, \ldots, p\} \), there are \( n_\alpha \) pure strategies for interacting with members of another group, including its own. Let us call signature of the game to the vector \( n = (n_1, \ldots, n_p) \). The total number of strategies is therefore \( n = n_1 + \ldots + n_p \). The polymatrix game is specified by a single \( n \times n \) matrix \( A = [a_{ij}]_{ij} \) with the pay-off \( a_{ij} \) for a user of strategy \( i \), member of one group, against a user of strategy \( j \), member of another group, possibly the same. The main difference between polymatrix games and the symmetric game, also specified by a single matrix \( A \), is that in the polymatrix game competition is restricted to members of the same group. This means that the relative fitness of each strategy refers to the overall average pay-off of strategies within the same group. To be more precise we need to introduce some notation. We decompose \( A \) in blocks, \( A = [A^{\alpha,\beta}]_{\alpha,\beta} \), where each block \( A^{\alpha,\beta} = [a_{ij}^{\alpha,\beta}]_{ij} \) is a \( n_\alpha \times n_\beta \) matrix. Similarly we decompose each vector \( x \in \mathbb{R}^n \) as \( x = (x^\alpha)_\alpha \), where \( x^\alpha \in \mathbb{R}^{n_\alpha} \). We say that a strategy \( i \) belongs to a group \( \alpha \), and write \( i \in \alpha \), if and only if \( n_1 + \ldots + n_{\alpha-1} < i \leq n_1 + \ldots + n_\alpha \). Similarly we write \( (i, j) \in \alpha \times \beta \) when \( i \in \alpha \) and \( j \in \beta \). With this notation we have

\[\begin{align*}
(a) & \quad x^\alpha_i = x_i \quad \text{if } i \in \alpha, \\
(b) & \quad a^{\alpha,\beta}_{ij} = a_{ij} \quad \text{if } (i, j) \in \alpha \times \beta.
\end{align*}\]

Hence the difference \( (Ax)_i - \sum_{\beta=1}^p (x^\alpha)^t A^{\alpha,\beta} x^\beta \) represents the relative fitness of a strategy \( i \in \alpha \) within the group \( \alpha \). The polymatrix replicator is the o.d.e.

\[
\frac{dx^\alpha_i}{dt} = x^\alpha_i \left( (Ax)_i - \sum_{\beta=1}^p (x^\alpha)^t A^{\alpha,\beta} x^\beta \right) \quad \forall \ i \in \alpha, \ \alpha \in \{1, \ldots, p\} ,
\]

\[\text{(3.3)}\]
which once more says that the logarithmic growth rate of each pure strategy’s frequency equals its relative fitness. The flow of this o.d.e. is complete and leaves the prism \( \Gamma_n = \Delta^{n_1-1} \times \ldots \times \Delta^{n_p-1} \) invariant. The underlying vector field on \( \Gamma_n \) will be denoted by \( X_A \). The pair \( G = (n, A) \) will be referred as a polymatrix game, and the dynamical system determined by \( X_A = X_{(n,A)} \) as the associated polymatrix replicator on \( \Gamma_n \).

**Remark 3.1.** When \( p = 1 \), \( \Gamma_n = \Delta^{n-1} \) and the evolutionary polymatrix game (3.3) coincides with the replicator o.d.e. (3.1).

**Remark 3.2.** When \( p = 2 \) and \( A^{1,1} = 0 \), \( A^{2,2} = 0 \) system (3.3) coincides with the bimatrix replicator (3.2) on \( \Gamma_n = \Delta^{n_1-1} \times \Delta^{n_2-1} \).

The proofs of the following three propositions are easy exercises.

**Proposition 3.3** (Identity). The correspondence \( A \mapsto X_{(n,A)} \) is linear and its kernel is formed by matrices \( A \in \text{Mat}_{n \times n}(\mathbb{R}) \) such that the block matrix \( A^{\alpha,\beta} \) has equal rows for all \( \alpha, \beta = 1, \ldots, p \). Thus, two matrices \( A, B \in \text{Mat}_{n \times n}(\mathbb{R}) \) determine the same vector field \( X_{(n,A)} = X_{(n,B)} \) on \( \Gamma_n \) iff the block matrix \( A^{\alpha,\beta} - B^{\alpha,\beta} \) has equal rows for all \( \alpha, \beta = 1, \ldots, p \).

**Definition 3.4.** Given a signature \( n = (n_1, \ldots, n_p) \) and matrices \( A, B \in \text{Mat}_{n \times n}(\mathbb{R}) \), we say that the polymatrix games \((n, A)\) and \((n, B)\) are equivalent, and write \((n, A) \sim (n, B)\), if and only if \( A^{\alpha,\beta} - B^{\alpha,\beta} \) has equal rows for all \( \alpha, \beta = 1, \ldots, p \).

Equivalent matrices determine the same evolutionary polymatrix game on \( \Gamma_n \). In other words \((n, A) \sim (n, B)\) iff \( X_{(n,A)} = X_{(n,B)} \).

**Proposition 3.5** (Equilibria). A point \( q \in \Gamma_n \) is an equilibrium of \( X_{(n,A)} \) if and only if \( (AQ_j) = (AQ)_j \) for all \( \alpha = 1, \ldots, p \) and every \( i, j \in \alpha \).

**Definition 3.6.** Given a signature \( n = (n_1, \ldots, n_p) \), we define the set

\[
\mathcal{J}_n := \{ I \in \{1, \ldots, n\} : \#(I \cap \alpha) \geq 1, \ \forall \alpha = 1, \ldots, p \},
\]

where \( I \cap \alpha := I \cap [n_1 + \ldots + n_{\alpha-1} + 1, n_1 + \ldots + n_\alpha] \). A set \( I \in \mathcal{J}_n \) determines the face \( \sigma_I := \{ x \in \Gamma_n : x_j = 0, \ \forall j \notin I \} \) of \( \Gamma_n \).

The correspondence between sets in \( \mathcal{J}_n \) and faces of \( \Gamma_n \) is bijective.

**Definition 3.7.** Consider a polymatrix game \( G = (n, A) \). Given a set \( I \in \mathcal{J}_n \) the pair \( G|_I = (n^I, A_I) \), where \( n^I = (n^I_1, \ldots, n^I_p) \) with \( n^I_\alpha = \#(I \cap \alpha) \), and \( A_I = [a_{ij}]_{i,j \in I} \) is called the restriction of the polymatrix game \( G \) to the face \( I \).

The following proposition says that the restriction of a polymatrix replicator to a face is another polymatrix replicator.
Proposition 3.8 (Inheritance). Consider the system (3.3) associated to the polymatrix game \( G = (n, A) \). Given \( I \in \mathbb{I}_n \), the face \( \sigma_I \) of \( \Gamma_n \) is invariant under the flow of \( X_{(n, A)} \) and the restriction of (3.3) to \( \sigma_I \) is the polymatrix replicator associated to the restricted game \( G|_I \).

We set some notation in order to produce neater formulas. In any matrix equality the vectors in \( \mathbb{R}^n \), or \( \mathbb{R}^{n_\alpha} \), should be identified with column matrices. We set \( \mathbf{1} = \mathbf{1}_n = (1, 1, \ldots, 1) \in \mathbb{R}^n \) and will omit the subscript \( n \) whenever the dimension of this vector is clear from the context. Similarly, we write \( \mathbf{1} = \mathbf{1}_n \) for the \( n \times n \) identity matrix, and we omit the subscript \( n \) whenever its value is clear. Given \( x \in \mathbb{R}^n \), we denote by \( D_x \) the \( n \times n \) diagonal matrix
\[
D_x = \text{diag}(x)\rlap{,}
\]
and set \( T_x \) to be the \( n \times n \) block diagonal matrix
\[
T_x = \text{diag}(T_{x, i, j})_{i, j}
\]
for each \( (i, j) \in \alpha \times \beta \).

Given a polymatrix game \( G = (n, A) \), we define the matrix valued mapping \( \pi_A : \mathbb{R}^n \to \text{Mat}_{n \times n}(\mathbb{R}) \)
\[
\pi_A(x) = (-1) T_x D_x A D_x T_x^t.
\]
We have \( D_x A D_x = [D_{x, i, j}^A A_{i, j}^\alpha A_{i, j}^\beta]_{\alpha, \beta} \) where \( D_{x, i, j}^A A_{i, j}^\alpha A_{i, j}^\beta = \left[ a_{i, j}^\alpha^\beta x_i^\alpha x_j^\beta \right]_{i, j} \). Simple calculations show that \( \pi_A(x) = \sum_{i, j} \pi_{A, i, j}(x) \) where for all \( (i, j) \in \alpha \times \beta \)
\[
\pi_{A, i, j}(x) = x_i^\alpha x_j^\beta \left( -a_{i, j}^\alpha^\beta + (A_{i, j}^\alpha A_{i, j}^\beta) + ((A_{i, j}^\alpha A_{i, j}^\beta)^t x_{i, j}^\alpha - (x_{i, j}^\alpha)^t A_{i, j}^\alpha A_{i, j}^\beta) \right).
\]
These computations reduce to the simple case \( p = 1, n_1 = n \) where
\[
\pi_A(x) = (-1) (x 1^t - I) D_x A D_x (1 x^t - I),
\]
and
\[
\pi_{A, i, j}(x) = x_i x_j (-a_{i, j} + (Ax)_i + (A^t x)_j - x^t A x).
\]

Remark 3.9. Notice that \( \pi_A(x) \) is a skew symmetric matrix valued map whenever \( A \) is a skew symmetric matrix.

Definition 3.10. A formal equilibrium of a polymatrix game \( G = (n, A) \) is any vector \( q \in \mathbb{R}^n \) such that
(a) \( (A q)_i = (A q)_j \) for all \( i, j \in \alpha \), and all \( \alpha = 1, \ldots, p \),
(b) \( \sum_{j \in \alpha} q_j = 1 \) for all \( \alpha = 1, \ldots, p \).

Remark 3.11. A formal equilibrium of \( G = (n, A) \) is an equilibrium of the natural extension of \( X_{(n, A)} \) to the affine subspace spanned by \( \Gamma_n \).
Next proposition says that the existence of a formal equilibrium is a sufficient condition for the vector field \( X_{[n,A]} \) of system \([3.3]\) to be a gradient of a simple function \( H \) with respect to \( \pi_A \). We denote by \( \Gamma^n_n \) the topological interior of \( \Gamma^n_n \) in the affine subspace of \( \mathbb{R}^n \) spanned by \( \Gamma^n_n \).

**Proposition 3.12.** Given \( A \in \text{Mat}_{n \times n}(\mathbb{R}) \), assume there exists a formal equilibrium \( q \in \mathbb{R}^n \) of \( G = (n, A) \). Then, setting \( H(x) = \sum_{i=1}^n q_i \log x_i \),

\[
X_{[n,A]}(x) = \pi_A(x) d_2 H \quad \text{for every} \quad x \in \Gamma^n_n .
\]

**Proof.** Consider the vector field \( Z = \pi_A dH \). For any \( \alpha \) and \( i \in \alpha \), denote by \( Z^\alpha_i (x) \) the \( i \)-th component of \( Z(x) \). Using that \( \sum_{j \in \beta} q_{ij}^\beta = 1 \) we have

\[
Z^\alpha_i (x) = \left( \sum_{\beta=1}^k \pi_A^{\alpha,\beta}(x) \frac{q_{ij}^\beta}{x_j^\beta} \right) = \sum_{\beta=1}^k \left( \sum_{j \in \beta} \pi_A^{\alpha,ij}(x) \frac{q_{ij}^\beta}{x_j^\beta} \right)
\]

\[
= x^\alpha_i \sum_{\beta=1}^k \left[ (A^\alpha,\beta x^\beta)_i - (x^\alpha)^t A^\alpha,\beta x^\beta \left( \sum_{j \in \beta} q_{ij}^\beta \right) + (-A^\alpha,\beta q^\beta)_i + (x^\alpha)^t A^\alpha,\beta q^\beta \right]
\]

\[
= x^\alpha_i \left[ (Ax)_i - \sum_{\beta=1}^k (x^\alpha)^t A^\alpha,\beta x^\beta + \left( -Aq \right)_i + \sum_{i \in \alpha} x^\alpha_i (Aq)_i \right]
\]

\[
= x^\alpha_i \left[ (Ax)_i - \sum_{\beta=1}^k (x^\alpha)^t A^\alpha,\beta x^\beta \right] = X^\alpha_{A,ij}(x),
\]

where the vanishing term follows from \( q \) being an equilibrium point and \( x^\alpha \in \Delta_n^{n-1} \). This completes the proof. \( \Box \)

For every \( \alpha = 1, \ldots, p \) consider the \((n_\alpha - 1) \times n_\alpha \) matrix

\[
E_\alpha := \begin{bmatrix}
-1 & 0 & \cdots & 0 & 1 \\
0 & -1 & \cdots & 0 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -1 & 1 
\end{bmatrix}
\]

and set

\[
E := \text{diag}(E_1, \ldots, E_p),
\]

\[
B := (-1)EA^t E^t .
\]

Note that \( E \in \text{Mat}_{(n-p) \times n}(\mathbb{R}) \) and \( B \in \text{Mat}_{(n-p) \times (n-p)}(\mathbb{R}) \). Next we introduce a mapping \( \phi : \mathbb{R}^{n-p} \to \Gamma^n_n \). We write a vector \( u \in \mathbb{R}^{n-p} = \mathbb{R}^{n_1-1} \times \ldots \times \mathbb{R}^{n_p-1} \) as \( u = (u^\alpha)_\alpha \), where \( u^\alpha := (u^\alpha_1, \ldots, u^\alpha_{n^\alpha-1}) \), and the components of \( \phi \) as \( \phi(u^\alpha)_\alpha := \)
(φ^α(u^α))_α, where each φ^α : ℝ^{n_α} → (Δ^{n_α-1}) is the map defined by

φ^α(u^α) := \left( \begin{array}{c}
e u^α_i \\
1 + \sum_{i=1}^{n_α-1} e u^α_{i} \\
\end{array} \right) , \ldots , \left( \begin{array}{c}
e u^α_i \\
1 + \sum_{i=1}^{n_α-1} e u^α_{i} \\
\end{array} \right) .

The following is our main result. Consider a polymatrix game G = (n, A).

**Theorem 3.13.** If A is skew symmetric then the mapping π_A in [3.4] defines a stratified Poisson structure on Γ^n. Moreover the mapping φ : ℝ^n → Γ^n is a Poisson diffeomorphism if we endow ℝ^n with the constant Poisson structure associated to the skew symmetric matrix B defined in (3.6).

**Proof.** The map φ : ℝ^n → Γ^n is a diffeomorphism whose inverse is easily computed. If A is skew symmetric then so is B. Hence this matrix induces a constant Poisson structure on ℝ^n. We want to prove that π_A determines a Poisson structure on Γ^n which makes φ a Poisson map. By (2.3) we just need to show that for every u ∈ ℝ^n and x = φ(u),

(duφ)B(duφ)^T = (−1)T_x D_x A D_x T_x = π_A(x) .

(3.7)

The fact that π_A also determines a stratified Poisson structure on Γ^n, and on ℝ^n, will be proved later. See Remark 3.15. In order to prove (3.7), it is enough to see that for every x = φ(u)

(duφ)E = T_x D_x .

Writting the components of φ^α as φ^α(u^α) = (φ^α_1(u^α), \ldots, φ^α_n(u^α)) we compute for every i = 1, \ldots, n_α and j = 1, \ldots, n_α - 1,

∂φ^α_i ∂u^α_j = δ_ij φ^α_i - φ^α_i φ^α_j .

Hence if x = φ(u), the Jacobian of φ at the point u is

duφ = diag(J_α(x^α))_α,

where for every α = 1, \ldots, p,

J_α(x_1, \ldots, x_n_α) := \left[ \begin{array}{cccc}
x_1 - x_1^2 & -x_1 x_2 & -x_1 x_3 & \ldots & -x_1 x_n_α-1 \\
-x_1 x_1 & x_2 - x_2^2 & -x_2 x_3 & \ldots & -x_2 x_n_α-1 \\
-x_3 x_1 & -x_3 x_2 & x_3 - x_3^2 & \ldots & -x_3 x_n_α-1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-x_n_α-1 x_1 & -x_n_α-1 x_2 & -x_n_α-1 x_3 & \ldots & x_n_α-1 - x_n_α-1^2 \\
-x_n_α x_1 & -x_n_α x_2 & -x_n_α x_3 & \ldots & -x_n_α x_n_α-1 \\
\end{array} \right].

A simple multiplication of matrices, using the relation x_1 + \ldots + x_n_α = 1, shows that J_α(x^α)E_α = T_x D_x for every α = 1, \ldots, p. Therefore

(duφ)E = diag(J_α(x^α))E_α = diag(T_x D_x)E_α = T_x D_x ,
which completes the proof. □

The next corollary gives a complete description of the symplectic foliation of \((Γ^o_n, π_A)\). Two examples will be given in section 5.

**Corollary 3.14 (Symplectic Foliation).** The symplectic leaves of \((Γ^o_n, π_A)\) are the images of the symplectic leaves of \((\mathbb{R}^{n-p}, B)\) under the diffeomorphism \(φ\). The symplectic leaf \(S_u\) of \((\mathbb{R}^{n-p}, B)\) is the (even dimensional) affine subspace through \(u\) parallel to the subspace generated by the columns of \(B\).

**Remark 3.15.** Given a face \(I ∈ J_n\) consider the payoff matrix \(A_I\), see Definition 3.7. Applying Theorem 3.13 to any face \(σ_I\) of \(Γ_n\) we see that \(σ_I\) is a Poisson manifold on its own with the Poisson structure \(π_{A_I}\). Moreover \((σ_I, π_{A_I})\) is the restriction of \((Γ_n, π_A)\) in the sense that the inclusion map \(i: σ_I → Γ_n\) is a Poisson map. Hence the interiors of the faces of \(Γ_n\), regarded as Poisson manifolds, give \((Γ_n, π_A)\) the structure of a Poisson stratified space. In addition it will be shown that \(π_A\) defines a Poisson structure on \(\mathbb{R}^n\). On section 4 we provide a geometric explanation for these facts.

**Proposition (3.12) together with Theorem (3.13) yields the following corollary.**

**Corollary 3.16.** If \(A\) is skew symmetric and \(q ∈ \mathbb{R}^n\) is a formal equilibrium of \(G = (n, A)\) then \(X_{(n, A)}\) is a Hamiltonian vector field, with Hamiltonian \(H(x) = \sum_{i=1}^n q_i \log x_i\), w.r.t. the Poisson structure \(π_A\) in \(Γ^o_n\).

**Definition 3.17.** A polymatrix game \(G = (n, A)\) is said to be conservative iff

(a) \(G\) has a formal equilibrium,

(b) there are matrices \(A_0, D ∈ \text{Mat}_{n×n}(\mathbb{R})\) such that

(i) \(A ∼ A_0D\),

(ii) \(A_0\) is a skew symmetric,

(iii) \(D = \text{diag}(λ_1 I_{n_1}, ..., λ_p I_{n_p})\) with \(λ_β \neq 0\) for every \(β = 1, ..., p\).

The matrix \(A_0\) will be referred as a skew symmetric model for \(G\), and \((λ_1, ..., λ_p) ∈ (\mathbb{R}\{0\})^p\) as a scaling vector.

**Remark 3.18.** Given a skew symmetric matrix \(A_0 ∈ \text{Mat}_{n×n}(\mathbb{R})\), a signature \(n\) and a point \(q ∈ \mathbb{R}^n\) such that

(a) \((A_0 q)_i = (A_0 q)_j\) for all \(i, j ∈ \alpha\), and all \(α = 1, ..., p\),

(b) \(∑_{j ∈ α} q_j ≠ 0\) for all \(α = 1, ..., p\),

then \(G = (n, A_0D)\) is a conservative polymatrix game, where \(D = \text{diag}(λ_α I_{n_α})\) with \(λ_α := ∑_{j ∈ α} q_j\), and \(q = D^{-1}q\) is a formal equilibrium of \(G\).

It follows from the previous remark that any generic skew symmetric matrix can be taken as a model for a conservative polymatrix game. More precisely,

**Proposition 3.19.** Given a signature \(n = (n_1, ..., n_p)\) with \(∑_{α=1}^p n_α = n\), the set of skew symmetric matrices \(A_0 ∈ \text{Mat}_{n×n}(\mathbb{R})\) such that \(G = (n, A_0D)\) is a
conservative polymatrix game for some diagonal matrix $D$ is an open and dense subset of the space of skew symmetric matrices.

Next theorem basically says that the replicator system (3.3) is Hamiltonian for every conservative polymatrix game.

**Theorem 3.20.** Consider a conservative polymatrix game $G = (n, A)$ with formal equilibrium $q$, skew symmetric model $A_0$ and scaling co-vector $(\lambda_1, \ldots, \lambda_p)$. Then $X_G$ is Hamiltonian in the interior of the Poisson stratified space $(\Gamma, \pi_A)$, with Hamiltonian function

$$H(x) = \sum_{\beta=1}^{p} \lambda_{\beta} \sum_{j \in \beta} q_{j}^{\beta} \log x_{j}^{\beta}. \quad (3.8)$$

**Proof.** In view of definition 3.4 we can assume that $A = A_0 D$. For every $\alpha, \beta$,

$$T^{\alpha}_{x} D^{\alpha}_{x} A_{0}^{\alpha, \beta} D_{x}^{\beta} (T^{\beta}_{x} x^{\beta}) \lambda_{\beta} q_{x}^{\beta} = T^{\alpha}_{x} D^{\alpha}_{x} A^{\alpha, \beta} D_{x}^{\beta} (T^{\beta}_{x} x^{\beta}) \frac{q^{\beta}}{x^{\beta}},$$

where $q^{\beta}/x^{\beta}$ stands for the componentwise division of the vectors. Adding up in $\beta$, and using Proposition 3.12, we get

$$\pi_{A_0}(x) d_x H = \pi_A(x) d_x \left( \sum_{j=1}^{n} q_j \log x_j \right) = X_G(x).$$

\[\square\]

In the next paragraphs we compare our results with previously known facts. Given a skew symmetric matrix $A \in \text{Mat}_{n \times n}(\mathbb{R})$, since $x^t A x = 0$ for all $x \in \mathbb{R}^n$, the replicator equation (3.1) reduces to a Lotka-Volterra equation with growth rates $r_i = 0$

$$\frac{dx_i}{dt} = x_i (A x)_i \quad 1 \leq i \leq n. \quad (3.9)$$

For any $q \in \mathbb{R}^n$ such that $A q = 0$ the function $H(x) = \sum_{j=1}^{n} x_j - q_j \log x_j$ is a constant of motion for (3.9). A Poisson structure on $\mathbb{R}^n$ defined by the bivector $\hat{\pi}_A(x) = D_x A D_x$ was introduced in [2]. System (3.9) is Hamiltonian in the interior of $\mathbb{R}^n_+$ w.r.t. $\hat{\pi}_A$ having $H$ as Hamiltonian function. Like $\hat{\pi}_A$ the Poisson structure $\pi_A$ introduced here can be extended to $\mathbb{R}^n$, but unlike $\pi_A$ the structure $\hat{\pi}_A$ does not restrict to a Poisson structure on the simplex $\Delta^{n-1}$. Using the Poisson structure $\pi_A$ we can now say, if there exists $q \in \mathbb{R}^n$ such that $A q = 0$ and $\sum_{j=1}^{n} q_j \neq 0$, that the system (3.9) is Hamiltonian in the interior of the simplex $\Delta^{n-1}$. Furthermore, here we study the replicator equation itself and not a topologically equivalent LV system.

Consider now a bimatrix game with signature $(n_1, n_2)$ and matrix

$$A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}.$$
If $\lambda > 0$, resp. $\lambda < 0$, the polymatrix game $((n_1, n_2), A)$ is conservative with scaling vector $(1, \lambda)$ if and only if it has a formal equilibrium and the bimatrix game $(A_{12}, A_{21})$ is $\lambda$-zero-sum game, resp. $\lambda$-partnership game, (see definitions in section 11.2 of [8]). Theorem 3.20 generalizes the main result (section 5) in [7], which says that the evolutionary system (3.2) associated to a $\lambda$-zero-sum or $\lambda$-partnership game is orbit equivalent to a bipartite Lotka-Volterra system that is Hamiltonian w.r.t. some Poisson structure. This leads to the same constant of motion (3.8), but from the work [7] we only derive the existence of a Poisson structure in the interior of the prism $\Delta^{n_1-1} \times \Delta^{n_2-1}$ for which some time re-parametrization of system (3.2) is Hamiltonian w.r.t. that Poisson structure. On the other hand here we provide a Poisson structure on the full prism that makes the original system Hamiltonian in the interior of the prism.

We finish this section with an extension of the class of Hamiltonian polymatrix replicators. Given $p$ smooth functions $\lambda_\alpha : \Gamma_n \rightarrow \mathbb{R}\{0\}$, $\alpha = 1, \ldots, p$, consider the matrix valued smooth function $D : \Gamma_n \rightarrow \text{Mat}_{n \times n}(\mathbb{R})$, $D(x) = \text{diag}(\lambda_\alpha(x)I_n)$, and the system of o.d.e.’s

$$
\frac{dx_\alpha^i}{dt} = x_\alpha^i \left( (AD(x))_i - \sum_{\beta=1}^p \lambda_\beta(x) (x^\alpha)^t A^{\alpha,\beta} x^\beta \right) \quad \forall \ i \in \alpha, \ 1 \leq \alpha \leq p \quad (3.10)
$$

associated with the vector field $Y(x) = X_{(\Gamma_n, AD(x))}(x)$ on $\Gamma_n$.

**Proposition 3.21.** Let $A \in \text{Mat}_{n \times n}(\mathbb{R})$ be a skew symmetric matrix, $q \in \mathbb{R}^n$ a formal equilibrium of $G = (\Gamma_n, A)$, and consider the 1-form

$$\xi(x) = \sum_{\alpha=1}^p \sum_{j \in \alpha} \lambda_\alpha(x) q^j \frac{dx^\alpha}{x^j}.$$ 

Then system (3.10) is the gradient of the 1-form $\xi$ w.r.t. the Poisson structure $\pi_A$ in the interior of $\Gamma_n$, i.e.,

$$Y(x) = \pi_A(x) \xi(x).$$

System (3.10) is Hamiltonian if the form $\xi$ is exact, i.e., there exists a smooth function $H$ such that $\xi = dH$. But even if $\xi$ is not exact, the dynamics of $Y$ leaves invariant the symplectic foliation of $(\Gamma_n^\circ, \pi_A)$.

**Proof.** The proof is similar to that of Theorem 3.20. $\square$

The previous model (3.10) contains the following class of o.d.e.’s introduced by J. Maynard Smith as an extension of the asymmetric replicator equation (3.2).

$$
\frac{dx_i}{dt} = x_i \left( (A_{12} y_i) - x^t A_{12} y \right) m_1(x, y) \quad 1 \leq i \leq n \quad (3.11)
$$

$$
\frac{dy_j}{dt} = y_j \left( (A_{21} x_j) - y^t A_{21} x \right) m_2(x, y) \quad 1 \leq j \leq m
$$
See appendix J of [20], and system (9.1) in [7]. Taking

\[
A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} \quad \text{and} \quad D(x) = \begin{bmatrix} m_2(x,y) I_m \\ 0 \\ 0 \\ m_1(x,y) I_n \end{bmatrix}
\]

system (3.11) reduces to (3.10). Since system (3.11) has a dissipative character for certain choices of the functions \(m_1(x,y)\) and \(m_2(x,y)\) it would be interesting to investigate analogous properties of system (3.10).

4. Singular Poisson Reduction

This section is devoted to elaborate Remark (3.15). We will review the singular Poisson reduction introduced in [6] and use it to show that the phase space of an evolutionary game with a skew symmetric payoff matrix is a Poisson stratified space.

A smooth action of a Lie group \(G\) on the manifold \(M\) is a smooth map

\[\mathcal{A}: G \times M \to M\]

such that for every \(g, h \in G\) and \(m \in M\) one has \(\mathcal{A}(gh, m) = \mathcal{A}(g, \mathcal{A}(h, m))\) and \(\mathcal{A}(e, m) = m\), where \(e\) is the identity element of \(G\). For every \(g \in G\), \(\mathcal{A}_g\) denotes the diffeomorphism defined by \(m \mapsto \mathcal{A}(g, m)\).

The action is said to be proper if the map

\[G \times M \to M \times M \quad (g, m) \mapsto (m, \mathcal{A}(g, m)),\]

is proper. Recall that a map is called proper if the preimage of any compact subset is compact. The stabilizer of a point \(m \in M\) is

\[G_m := \{g \in G | \mathcal{A}(g, m) = m\}\]

The action is called free if \(G_m = \{e\}\) for every \(m \in M\). The set

\[\mathcal{O}_m := \{\mathcal{A}(g, m) | g \in G\}\]

is called the orbit passing through the point \(m \in M\) and the set

\[M/G := \{\mathcal{O}_m | m \in M\}\]

is called the orbit space of the action. The map \(\pi_G : M \to M/G\) sending every point \(m\) to its orbit \(\mathcal{O}_m\) is called the projection map of the action. The orbit space \(M/G\) can be given a topology by \(U \subset M/G\) being open if and only if \(\pi_G^{-1}(U)\) is open in \(M\). With this topology, it is a Hausdorff topological space if the action is proper. We will only consider proper actions so the orbit space shall always be a Hausdorff topological space all over this section.

For any subgroup \(H\) of \(G\) the \(H\)-isotropy type submanifold of \(M\) is

\[M_H := \{m \in M | G_m = H\},\]

and the \((H)\)-orbit type submanifold is

\[M_{(H)} := \{m \in M | G_m \in (H)\},\]
where \((H)\) denotes the conjugacy class of \(H\) in \(G\). Notice that the action of \(G\) restricts to \(M_{(H)}\) and \(M_{(H)}/G\) can be defined and is called the \((H)\)-orbit type reduced space.

We recall the definition of a smooth stratified space, see [4,12].

**Definition 4.1.** Let \(X\) be a paracompact Hausdorff topological space. A *smooth stratification* of \(X\) is a locally finite partition of \(X\) into locally closed connected smooth submanifolds \(S_i\) \((i \in I)\), called the strata of the stratification, such that for a pair of submanifolds \(S_i, S_j\) if \(S_i \cap \overline{S_j} \neq \emptyset\) then \(S_i \subset \overline{S_j}\). When this happens \(S_i\) is called incident to \(S_j\) or a boundary piece of \(S_j\).

The following proposition is a well-known result in the theory of Lie group actions, see e.g. [4,12] for the proof.

**Proposition 4.2.** If the action of the Lie group \(G\) on \(M\) is proper then the orbit space \(M/G\) is a smooth stratified space. Furthermore, if the action is free then \(M/G\) can be equipped with a smooth manifold structure such that the projection map \(\pi_G\) becomes a submersion.

A function \(f \in C^\infty(M)\) is called \(G\)-invariant if and only if
\[
f \circ A_g(m) = f(m) \quad \forall g \in G, m \in M.
\]

Any \(G\)-invariant function reduces to a function on \(M/G\) so we define:

**Definition 4.3.** The algebra of smooth functions on the orbit space \(M/G\) is
\[
C^\infty(M/G) := \{f \in C^0(M/G) \mid f \circ \pi_G \in C^\infty(M)\},
\]
where \(C^0(M/G)\) denotes the algebra of continuous functions on the topological space \(M/G\).

**Definition 4.4.** An action of \(G\) on the Poisson manifold \((M, \{., .\})\) is called Poisson if \(A_g\) is a Poisson diffeomorphism for every \(g \in G\).

Notice that if the action is Poisson i.e.
\[
\{f \circ A_g, h \circ A_g\} = \{f, g\} \circ A_g \quad \forall f, h, g
\]
then Poisson bracket of any two \(G\)-invariant function is again \(G\)-invariant. Using this fact, a bracket can be defined on the algebra of smooth functions on \(M/G\) by
\[
\{f, h\}_{M/G}(O_m) = \{f \circ \pi_G, h \circ \pi_G\}(m'), \quad (4.1)
\]
where \(m'\) is an arbitrary element of \(O_m\). In the case of free proper Poisson action this bracket is a Poisson bracket on the manifold \(M/G\). Clearly \(\pi_G\) is a Poisson map between \((M, \{., .\})\) and \((M/G, \{., .\})_{M/G}\).

It is clear that \((C^\infty(M/G), \{., .\})_{M/G}\) is a Poisson algebra. Recall that a Poisson algebra is an algebra equipped with a skew symmetric bracket satisfying Leibniz’s rule and Jacobi identity. We state Theorem 2.12 of [6] which will be used to show that the phase space of an evolutionary polymatrix game with a skew symmetric payoff matrix is a Poisson stratified space.
Theorem 4.5 (Singular Poisson Reduction). Let \( A : G \times M \to M \) be a proper Poisson action. Then the connected components of the orbit type reduced space \( M_{(H)/G} \) form a Poisson stratification \( \{ S_i \}_{i \in I} \) of \( (M/G, \{ . \}, M/G) \) i.e. a smooth stratification such that

- Each Strata \( S_i, i \in I, \) is a Poisson manifold.
- The inclusions \( i : S_i \to M/G \) are Poisson maps.

The following is, basically, the example which is presented in [6, Section 2.5]. Let

\[
M = \mathbb{C}^{n_1} \setminus \{0\} \times \ldots \times \mathbb{C}^{n_p} \setminus \{0\},
\]

where \( n_1, \ldots, n_p \) are integers such that \( n = n_1 + \ldots + n_p \). We will consider \( M \) as a real \( 2n \) dimensional manifold with coordinates \((\xi, \eta) \in \mathbb{R}^{2n}\) where \( z_i = \xi_i + i\eta_i \) for \( i = 1, \ldots, n \). Equip \( M \) with the quadratic Poisson structure defined by:

\[
\pi_M(dw_i, dw_j) = \{w_i, w_j\}_M = \frac{1}{4} a_{ij} w_i w_j,
\]

where \( i, j = 1, \ldots, n \) for \( w = \xi, \eta \) and \( A \) is a skew symmetric matrix. In the language of bivectors:

\[
\pi_M(\xi, \eta) = \frac{1}{4} \left( D_\xi AD_\xi D_\eta AD_\eta \right).
\]

We shall denote by \( \mathbb{C}^* \) the group of non zero complex numbers. The group \( (\mathbb{C}^*)^n \) acts on \( M \) by component-wise multiplication. Denote this action by

\[
A(\lambda, z) = (\lambda_1 z_1, \ldots, \lambda_n z_n) \quad (4.2)
\]

Lemma 4.6. The action of \( (\mathbb{C}^*)^n \) on \( M \) is Poisson i.e. for any \( \lambda \in (\mathbb{C}^*)^n \) the linear map \( A_\lambda : M \to M \) defined by

\[ z \mapsto (\lambda_1 z_1, \ldots, \lambda_n z_n) \]

is a Poisson map.

Proof. In real coordinates, we denote \( \lambda = (\xi_0, \eta_0) \) and \( z = (\xi, \eta) \). By this notation

\[ L_\lambda(z) = (\xi_0 \xi - \eta_0 \eta, \eta_0 \xi + \xi_0 \eta), \]

where \( \xi_0 \xi \) stands for component-wise multiplication of these vectors. Similarly for \( \eta_0 \eta, \eta_0 \xi \) and \( \xi_0 \xi \). We need to check condition \( (2.3) \). Clearly,

\[ d_{(\xi, \eta)} A_\lambda = \begin{pmatrix} D_{\xi_0} & -D_{\eta_0} \\ D_{\eta_0} & D_{\xi_0} \end{pmatrix}. \]

Simple calculation shows

\[
\frac{1}{4} (d_{(\xi, \eta)} A_\lambda) \begin{pmatrix} D_{\xi} AD_\xi & D_{\xi} AD_\eta \\ D_{\eta} AD_\xi & D_{\eta} AD_\eta \end{pmatrix} (d_{(\xi, \eta)} A_\lambda)^t = \frac{1}{4} \begin{pmatrix} D_{(\xi_0 \xi - \eta_0 \eta)} AD_{(\xi_0 \xi - \eta_0 \eta)} & D_{(\xi_0 \xi - \eta_0 \eta)} AD_{(\eta_0 \xi + \xi_0 \eta)} \\ D_{(\eta_0 \xi + \xi_0 \eta)} AD_{(\xi_0 \xi - \eta_0 \eta)} & D_{(\eta_0 \xi + \xi_0 \eta)} AD_{(\eta_0 \xi + \xi_0 \eta)} \end{pmatrix} = \pi_M(A_\lambda(\xi, \eta))
\]
Consider the subgroup $G$ of $(\mathbb{C}^*)^n$ defined by,

$$G := \{(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n : |\lambda_i| = |\lambda_j|, \forall i, j \in \alpha, \forall \alpha = 1, \ldots, p\}$$

The Poisson action of $(\mathbb{C}^*)^n$ on $M$ restricts to a Poisson action of $G$ on $M$. Clearly, this action is proper.

By Theorem 4.5, the quotient space $M/G$ is a Poisson stratified space. The quotient space $\pi M/G$ can be identified by $\Gamma_{\pi}$. The identification is obtained via the map

$$\pi^{-1}(x^1, \ldots, x^n) : \mathbb{C}^{n_1} \setminus \{0\} \times \ldots \times \mathbb{C}^{n_p} \setminus \{0\} \rightarrow \Delta^{n_1-1} \times \ldots \times \Delta^{n_p-1},$$

where $x^\alpha : \mathbb{C}^{n_\alpha} \setminus \{0\} \rightarrow \Delta^{n_\alpha-1}$ defined by

$$x^\alpha(z_1^\alpha, \ldots, z_{n_\alpha}^\alpha) = \left(\frac{|z_1^\alpha|^2}{|z_1^\alpha|^2 + \ldots + |z_{n_\alpha}^\alpha|^2}, \ldots, \frac{|z_{n_\alpha}^\alpha|^2}{|z_1^\alpha|^2 + \ldots + |z_{n_\alpha}^\alpha|^2}\right)$$

Above and in the sequel $0 = (0, \ldots, 0)$ denotes a zero vector, while $0$ stands for a zero scalar. The strata of $\Gamma_{\pi}$ are identified with the faces of $\Gamma_{\pi}$, see Theorem 4.5, are identified with the faces of $\Gamma_{\pi}$. Let $e$ be the identity element of $G$, then the $(\{e\})$-orbit type reduced space is $\Gamma_{\pi}^e$ the interior of $\Gamma_{\pi}$.

For any $\alpha = 1, \ldots, p, i \in \alpha$

$$dx_i^\alpha = -2x_i^\alpha \frac{1}{r^\alpha}(0, \ldots, \xi_i^\alpha, \ldots, 0, 0, \ldots, \eta_i^\alpha, \ldots, 0)^t + \frac{2}{r^\alpha}(0, \ldots, (0, \ldots, \xi_i^\alpha, \ldots, 0), \ldots, 0, \ldots, (0, \ldots, \eta_i^\alpha, \ldots, 0), \ldots, 0)^t,$$

where $r^\alpha = |z_1^\alpha|^2 + \ldots + |z_{n_\alpha}^\alpha|^2$. Let $\beta = 1, \ldots, p$ and $j \in \beta$ also, by definition, see (4.1), we have

$$\{x_i^\alpha, x_j^\beta\}_{\Gamma_{\pi}^e} = \frac{1}{4}(dx_i^\alpha)^t \begin{pmatrix} D_{\xi A} \theta D_{\xi} & D_{\xi A} \theta D_{\eta} \\ D_{\eta A} \theta D_{\xi} & D_{\eta A} \theta D_{\eta} \end{pmatrix} (dx_j^\beta) = (W^\alpha)^t \begin{pmatrix} D_{\xi A} \alpha \beta D_{\xi} & D_{\xi A} \alpha \beta D_{\eta} \\ D_{\eta A} \alpha \beta D_{\xi} & D_{\eta A} \alpha \beta D_{\eta} \end{pmatrix} W^\beta,$$

where

$$W^\alpha := \frac{1}{r^\alpha} \left[-x_i^\alpha \xi_i^\alpha + (0, \ldots, \xi_i^\alpha, \ldots, 0), -x_i^\alpha \eta_i^\alpha + (0, \ldots, \eta_i^\alpha, \ldots, 0)\right]$$

A straightforward calculations show:

$$\{x_i^\alpha, x_j^\beta\}_{\Gamma_{\pi}^e} = (\pi_A)^{\alpha, \beta}_{i, j},$$

where $\pi_A$ is defined at (3.4). This shows that $\{x_i^\alpha, x_j^\beta\}_{\Gamma_{\pi}^e}$ is the same Poisson structure on $\Gamma_{\pi}^e$ that was considered in Section 3. The same holds for all the faces of $\Gamma_{\pi}$, which justifies Remark 3.15.
Notice that in our case the condition (2.2) is an algebraic equality which holds on \( \Gamma^2 \). The equality (2.2) is invariant w.r.t. multiplication of \( x_i, x_j, x_k \) with constant numbers. Hence our algebraic equality must hold on the open subset \( \mathbb{R}^n \), which in turn yields that it is satisfied all over \( \mathbb{R}^n \), i.e., \( \pi_A \) is actually a Poisson structure on \( \mathbb{R}^n \).

5. Examples

It is possible to fully classify the dynamics of 2D and 3D conservative polymatrix replicator systems, but in this section we just briefly describe two examples of 3D polymatrix replicators.

**First Example.** Consider the signature \( n = (2, 2, 2) \), take the skew symmetric matrix

\[
A_0 = \begin{bmatrix}
0 & -1 & 0 & \frac{1}{2} & 0 & 1 \\
1 & 0 & 0 & -\frac{1}{2} & -1 & \frac{1}{2} \\
0 & 0 & 0 & 0 & \frac{1}{2} & -1 \\
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 1 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\
-1 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 \\
\end{bmatrix},
\]

and the point \( p = (\frac{7}{4}, \frac{3}{4}, \frac{5}{4}, 1, 1, 1) \) such that \( A_0 p = (\frac{3}{4}, \frac{3}{4}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{8}, -\frac{3}{8}) \). Consider the matrix \( A = A_0 D \), where \( D = \text{diag}(\frac{5}{2}, \frac{5}{2}, \frac{9}{4}, \frac{9}{4}, 2, 2) \). This matrix is

\[
A = \begin{bmatrix}
0 & -\frac{5}{2} & 0 & 9/8 & 0 & 2 \\
\frac{5}{2} & 0 & 0 & -9/8 & -2 & 1 \\
-\frac{5}{4} & \frac{5}{4} & 0 & 0 & 0 & 0 \\
0 & 5/2 & -9/8 & 0 & 0 & -1 \\
-\frac{5}{2} & -\frac{5}{4} & 9/4 & 0 & 1 & 0 \\
\end{bmatrix}.
\]

By remark 3.18, \((2, 2, 2), A\) is a conservative polymatrix game. The phase space of the associated replicator system is the cube

\[ \Gamma_{(2,2,2)} = \Delta^1 \times \Delta^1 \times \Delta^1 \equiv [0,1]^3. \]

In the model \([0,1]^3\), the equilibrium point \( q = D^{-1} p \) has coordinates \( q = (\frac{7}{10}, \frac{5}{10}, \frac{1}{10}) \), and hence is an interior point. The line through \( q \) with direction \( v = (\frac{6}{5}, -\frac{5}{3}, -1) \) intersects the cube \([0,1]^3\) along the set \( \Sigma \) of equilibria of this replicator system. This set \( \Sigma \) is a line segment joining two points in the faces \( \{x = 1\} \) and \( \{z = 1\} \).

To compute the symplectic foliation of \([0,1]^3\) consider the matrix

\[
E = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
\end{bmatrix}
\]
and define $B = -EA_0E'$. A simple calculation gives

$$B = \begin{bmatrix}
0 & 1 & -\frac{1}{2} \\
-1 & 0 & -\frac{3}{2} \\
\frac{1}{2} & \frac{3}{2} & 0
\end{bmatrix}.$$ 

**Figure 1.** Phase portraits of 3D polymatrix replicators

The vector $w = \left(-\frac{3}{2}, \frac{1}{2}, 1\right)$ is orthogonal to the space spanned by the columns of $B$. The symplectic leaves of the constant Poisson structure on $\mathbb{R}^3$ defined by the skew symmetric matrix $B$ are the planes orthogonal to $w$. Thus, if we consider the Poisson diffeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\phi(u_1, u_2, u_3) = \left(\frac{e^{u_1}}{1 + e^{u_1}}, \frac{e^{u_2}}{1 + e^{u_2}}, \frac{e^{u_3}}{1 + e^{u_3}}\right),$$

the symplectic leaves on $\mathbb{R}^3$ are the $\phi$ images of these planes. Inverting the map $\phi$, the symplectic leaves are given by the equations

$$\left(\frac{x}{1-x}\right)^{-3/2} \left(\frac{y}{1-y}\right)^{1/2} \left(\frac{z}{1-z}\right) = e^c \Leftrightarrow (1 - x)^{3/2}y^{1/2}z = e^c x^{3/2}(1 - y)^{1/2}(1 - z),$$

with $c \in \mathbb{R}$. Let $U_+$, resp. $U_-$, be the union of the faces $\{x = 1\}$, $\{y = 0\}$, $\{z = 0\}$, resp. $\{x = 0\}$, $\{y = 1\}$, $\{z = 1\}$. On the interiors of these two open subsets of the cube’s boundary the equation above is never satisfied. Therefore the closure of every symplectic leaf intersects the cube’s boundary along the closed curve $C = \partial U_+ = \partial U_- \subset \partial [0, 1]^3$. Because $\Sigma$ intersects both $U_-$ and $U_+$, it follows that every symplectic leaf must intersect $\Sigma$, hence having a unique equilibrium.
The orbits of our polymatrix replicator foliate each symplectic leaf into closed curves around that equilibrium point. We can also check that $C$ is a heteroclinic cycle of the vector field $X_{(2,2,2),A}$. See Figure 1(a).

**Second Example.** Consider the signature $n = (3, 2)$, take the skew symmetric matrix

$$A_0 = \begin{pmatrix}
0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 \\
0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0
\end{pmatrix},$$

and the point $p = \left(\frac{9}{10}, -\frac{8}{5}, \frac{1}{2}, 0, 1\right)$ such that $A_0 p = \left(-\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}, -\frac{3}{20}, -\frac{3}{20}\right)$. Consider the matrix $A = A_0 D$, where $D = \text{diag} \left(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, 1, 1\right)$. This matrix is

$$A = \begin{pmatrix}
0 & 0 & -\frac{1}{10} & \frac{1}{2} & -1 \\
0 & 0 & \frac{1}{10} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{10} & -\frac{1}{10} & 0 & 1 & \frac{1}{2} \\
\frac{1}{10} & \frac{1}{10} & \frac{1}{2} & 0 & 0 \\
-\frac{1}{5} & -\frac{1}{5} & 0 & 0 & 0
\end{pmatrix}.$$

By remark 3.18 $((3, 2), A)$ is a conservative polymatrix game. The phase space of the associated replicator system is the prism $\Gamma_{(3,2)} = \Delta^2 \times \Delta^1 \equiv \{(x, y, z) : 0 \leq x, y, z \leq 1, x + y \leq 1\} =: P$.

In the model $P \subset \mathbb{R}^3$ the equilibrium point $q = D^{-1} p$ has coordinates $q = (-\frac{9}{2}, 8, 0)$, and hence is not interior to $P$. The line of equilibria goes through $q$ with direction $v = (-\frac{5}{2}, 5, -1)$ and does not intersect the prism $P$. To compute the symplectic foliation of $P^o$ consider the matrix

$$E = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

and define $B = -E A_0 E^t$. A simple calculation gives

$$B = \begin{pmatrix}
0 & 1 & -\frac{1}{2} \\
-1 & 0 & -1 \\
\frac{1}{2} & 1 & 0
\end{pmatrix}.$$
the symplectic leaves on $P^o$ are the $\phi$ images of these planes. Inverting the map $\phi$, the symplectic leaves are given by the equations

\[
\left(\frac{x}{1-x-y}\right)^{-1} \left(\frac{y}{1-x-y}\right)^{1/2} \left(\frac{z}{1-z}\right) = e^c
\]

\[
\iff (1 - x - y)^{1/2} y^{1/2} z = e^c x (1 - z) ,
\]

with $c \in \mathbb{R}$. Let $U_+$, resp. $U_-$, be the union of the faces $\{x + y = 1\}$, $\{y = 0\}$, $\{z = 0\}$, resp. $\{x = 0\}$, $\{z = 1\}$. On the interiors of these two open subsets of the prism’s boundary the equation above is never satisfied. Therefore the closure of every symplectic leaf intersects the prism’s boundary along the closed curve $C = \partial U_+ = \partial U_- \subset \partial P$. The points $r = (1, 0, 0)$ and $s = (0, 0, 1)$ on $C$ are respectively a global repeller and a global sink of the polymatrix replicator, and every symplectic leaf is foliated into orbits flowing from the repeller $r$ to the sink $s$. The closed curve $C$ is also the union of two heteroclinic chains from $r$ to $s$. See Figure 1(b). Note that this dynamical behaviour does not contradict the Hamiltonian character of the system because the area of each symplectic leaf is infinite.

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References

[1] Ethan Akin and Viktor Losert, Evolutionary dynamics of zero-sum games, J. Math. Biol. 20 (1984), no. 3, 231–258, DOI 10.1007/BF00275987. MR765812 (86g:92024a)

[2] Pedro Duarte, Rui L. Fernandes, and Waldyr M. Oliva, Dynamics of the attractor in the Lotka-Volterra equations, J. Differential Equations 149 (1998), no. 1, 143–189, DOI 10.1006/jdeq.1998.3443. MR1643678 (99h:34075)

[3] Jean-Paul Dufour and Nguyen Tien Zung, Poisson structures and their normal forms, Progress in Mathematics, vol. 242, Birkhäuser Verlag, Basel, 2005.

[4] J. J. Duistermaat and J. A. C. Kolk, Lie groups, Universitext, Springer-Verlag, Berlin, 2000. MR1738431 (2001j:22008)

[5] I. Eshel and E. Akin, Coevolutionary instability and mixed Nash solutions, J. Math. Biol. 18 (1983), no. 2, 123–133, DOI 10.1007/BF00280661. MR723584 (85d:92023)

[6] Rui Loja Fernandes, Juan-Pablo Ortega, and Tudor S. Ratiu, The momentum map in Poisson geometry, Amer. J. Math. 131 (2009), no. 5, 1261–1310, DOI 10.1353/ajm.0.0068. MR2555841 (2011f:53199)

[7] Josef Hofbauer, Evolutionary dynamics for bimatrix games: a Hamiltonian system?, J. Math. Biol. 34 (1996), no. 5-6, 675–688, DOI 10.1007/s002850050025. MR1393843 (97h:92011)
[8] Josef Hofbauer and Karl Sigmund, *Evolutionary games and population dynamics*, Cambridge University Press, Cambridge, 1998. MR1635735 (99h:92027)

[9] Josef Hofbauer, *On the occurrence of limit cycles in the Volterra-Lotka equation*, Nonlinear Anal. 5 (1981), no. 9, 1003–1007, DOI 10.1016/0362-546X(81)90059-6. MR633014 (83c:92063)

[10] Joseph T. Howson Jr., *Equilibria of polymatrix games*, Management Sci. 18 (1971/72), 312–318. MR0392000 (52 #12818)

[11] A. M. Perelomov, *Integrable systems of classical mechanics and Lie algebras. Vol. I*, Birkhäuser Verlag, Basel, 1990. Translated from the Russian by A. G. Reyman [A. G. Reîman], MR1048350 (91g:58127)

[12] Markus J. Pflaum, *Analytic and geometric study of stratified spaces*, Lecture Notes in Mathematics, vol. 1768, Springer-Verlag, Berlin, 2001. MR1869601 (2002m:58007)

[13] L. G. Quintas, *A note on polymatrix games*, Internat. J. Game Theory 18 (1989), no. 3, 261–272, DOI 10.1007/BF01254291. MR1024957 (91a:90188)

[14] Ray Redheffer, *Volterra multipliers. I, II*, SIAM J. Algebraic Discrete Methods 6 (1985), no. 4, 592–611, 612–623, DOI 10.1137/0606059. MR800991 (87j:15037a)

[15] __________, *A new class of Volterra differential equations for which the solutions are globally asymptotically stable*, J. Differential Equations 82 (1989), no. 2, 251–268, DOI 10.1016/0022-0396(89)90133-2. MR1027969 (91f:34058)

[16] Ray Redheffer and Wolfgang Walter, *Solution of the stability problem for a class of general Volterra prey-predator systems*, J. Differential Equations 52 (1984), no. 2, 245–263, DOI 10.1016/0022-0396(84)90179-7. MR741270 (85k:92068)

[17] Ray Redheffer and Zhi Ming Zhou, *Global asymptotic stability for a class of many-variable Volterra prey-predator systems*, Nonlinear Anal. 5 (1981), no. 12, 1309–1329, DOI 10.1016/0362-546X(81)90108-5. MR646217 (83h:92074)

[18] __________, *A class of matrices connected with Volterra prey-predator equations*, SIAM J. Algebraic Discrete Methods 3 (1982), no. 1, 122–134, DOI 10.1137/0603012. MR644963 (83m:15020)

[19] Vito Volterra, *Leçons sur la théorie mathématique de la lutte pour la vie*, Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics], Éditions Jacques Gabay, Sceaux, 1990 (French). Reprint of the 1931 original. MR1189803 (93k:92011)

[20] John Maynard Smith, *Evolution and the Theory of Games*, Cambridge University Press, 1982.