QUASI-PERIODIC AND PERIODIC SOLUTIONS FOR SYSTEMS OF COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider travelling periodic and quasi-periodic wave solutions of a set of coupled nonlinear Schrödinger equations. In fibre optics these equations can be used to model single mode fibers with strong birefringence and two-mode optical fibers. Recently these equations appear as model, which describe pulse-pulse interaction in wavelength-division-multiplexed channels of optical fiber transmission systems. Two phase quasi-periodic solutions for integrable Manakov system are given in terms of two dimensional Kleinian functions. The reduction of quasi-periodic solutions to elliptic functions is discussed. New solutions in terms generalized Hermite polynomials, which are associated with two-gap Treibich-Verder potentials are found.

1. Introduction

We consider the system of two coupled nonlinear Schrödinger equations

\[ iU_t + U_{xx} + (\kappa UU^* + \chi VV^*)U = 0, \]

\[ iV_t + V_{xx} + (\chi UU^* + \rho VV^*)V = 0, \]

(1.1)

where \( \kappa, \chi, \rho \) are some constants. The integrability of this system is proven by Manakov [31]; only for the case \( \kappa = \chi = \rho \), which we shall refer as Manakov system.

The equations (1.1) are important for a number of physical applications when \( \chi \) is positive and all remaining constants equals to 1. For example for two-mode optical fibers \( \chi = 2 \) [13] and for propagation of two modes in fibers with strong birefringence \( \chi = \frac{2}{3} \) [14] and in the general case \( \frac{2}{3} \leq \chi \leq 2 \) for elliptical eigenmodes. The special value

\[ \chi = \frac{2}{3}, \]

The research described in this publication was supported in part by grants from the Civil Research Development Foundation, CRDF grant no. UM1-325, the INTAS grant no. 96-770 (JCE and VZE) and the Engineering and Physical Sciences Research Committee (JCE and VZE and NAK).
$\chi = 1$ (Manakov system) corresponds to at least two possible cases, namely the case of a purely electrostrictive nonlinearity or, in the elliptical birefringence, when angle between the major and minor axes of the birefringence ellipse is approximately $35^\circ$. The experimental observation of Manakov solitons in crystals is reported in [24]. Recently Manakov model appear in the Kerr-type approximation of photorefractive crystals [30]. The pulse-pulse collision between wavelength-division-multiplexed channels of optical fiber transmission systems are described with equations (1.1) $\chi = 2$, [34, 24, 22, 25].

General quasi-periodic solutions in terms of $n$-phase theta functions for integrable Manakov system are derived in [1], while a series of special solutions are given in [3, 38, 36, 37]. The authors of this paper discussed already quasi-periodic and periodic solutions associated to Lamé and Treibich-Verdier potentials for nonintegrable system of coupled nonlinear Schrödinger equations in frames of a special ansatz [12]. We also mention the method of constructing elliptic finite-gap solutions of the stationary KdV and AKNS hierarchy, based on a theorem due to Picard, is proposed in [13, 20, 21] and the method developed by Smirnov in series of publications, the review paper [41] and [43, 42].

In the present paper we investigate integrable Manakov system being restricted to the system integrable in terms of ultraelliptic functions by introducing special ansats, which was recently applied by Porubov and Parker [37] to analyse special classes of elliptic solutions of the Manakov system ($\kappa = \chi = \rho = 1$). More precisely, we seek solution of (1.1) in the form

$$U(x, t) = q_1(x) \exp \left\{ ia_1 t + iC_1 \int_0^x \frac{dx}{q_1^2(x)} \right\},$$

$$V(x, t) = q_2(x) \exp \left\{ ia_2 t + iC_2 \int_0^x \frac{dx}{q_2^2(x)} \right\},$$

where the functions $q_{1,2}(x)$ are supposed to be real and $a_1, a_2, C_1, C_2$ are real constants. Substituting (1.2) into (1.1) we reduce the system to the equations

$$\frac{\partial^2 q_1}{\partial x^2} + \rho q_1^3 + \chi q_1 q_2^2 - a_1 q_1 - \frac{C_1^2}{q_1^3} = 0$$

$$\frac{\partial^2 q_2}{\partial x^2} + \kappa q_2^3 + \chi q_2 q_1^2 - a_2 q_2 - \frac{C_2^2}{q_2^3} = 0.$$
The system (1.3) is the natural hamiltonian two-particle system with the hamiltonian of the form

\[ H = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + \frac{1}{4} (\rho q_1^2 + 2\chi q_1^2 q_2^2 + \kappa q_2^4) - \frac{1}{2} a_1 q_1^2 - \frac{1}{2} a_2 q_2^2 + \frac{1}{2} C_1^2 q_1 + \frac{1}{2} C_2^2 q_2, \]  

(1.4)

where \( p_1(t) = dq_1(t)/dt \).

These equations describe the motion of particles interacting with the quartic potential \( Aq_1^4 + Bq_1^2 q_2^2 + Cq_2^4 \) and perturbed by inverse squared potential. Nowadays four nontrivial cases of complete integrability are known for nonperturbed potential (i) A:B:C = 1:2:1, (ii) A:B:C = 1:12:16, (iii) A:B:C = 1:6:1,(iv) A:B:C = 1:6:8. Cases (i), (ii) and (iii) are separable in respectively ellipsoidal, paraboloidal and Cartesian coordinates, while the case (iv) is separable in general sense [39]. The cases (ii) appears as one of the entries to polynomial hierarchy discussed in [13] the cases (iii) and (iv) are proved to be canonically equivalent under the action of Miura map restricted to the stationary of coupled KdV systems associated with fourth order Lax operator[3]. Moreover all the cases (i)-(iv) permit the deformation of the potential by linear combination of inverse squares and squares with certain limitations on the coefficients [13, 4]. There are also known Lax representations for all these cases which yield hyperelliptic algebraic curves in the cases (i) and (ii) and 4-gonal curve in the cases (iii) and (iv).

Although each from the system enumerated yield nontrivial classes of solutions of the system (1.1) we shall discuss further only the case (i). The integrability of this case and separability in ellipsoidal coordinates was proved by Wojciechowski [48] (see also [27, 44]). We employ this result to integrate the system in terms of ultraelliptic functions (hyperelliptic functions of the genus two curve) and then execute reduction of hyperelliptic functions to elliptic ones by imposing additional constraints on the parameters of the system.

The paper is organised as follows. In the first section we construct the Lax representation of the system, develop the genus two algebraic curve, which is associated to the system and reduce the problem to solution of the Jacobi inversion problem associated with genus two algebraic curve. In the section two develop the integration of the system in terms of Kleinian hyperelliptic functions which represent a natural generalization of Weierstrass elliptic functions to hyperelliptic curved of higher genera; recently this realization of abelian functions was discussed in [13, 10, 16]. We explain in the section the outline of the Kleinian realization of hyperelliptic functions and give the principle
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formulae for the case of genus two curve. In Section 4 we develop reduction of Kleinian hyperelliptic function to elliptic functions in terms of Darboux coordinates for the curve admitting additional involution. In this way a quasiperiodic solution in terms of elliptic functions is obtained. In the last section we construct a set of elliptic periodic solutions on the basis of application of spectral theory for the Hill equation with elliptic potential.

2. Lax representation

The system $1 : 2 : 1$ ($\kappa = \chi = \rho = 1$) is a completely integrable hamiltonian system

$$\frac{\partial^2 q_1}{\partial x^2} + (q_1^2 + q_2^2)q_1 - a_1q_1 - \frac{C_1^2}{q_1^2} = 0,$$

$$\frac{\partial^2 q_2}{\partial x^2} + (q_1^2 + q_2^2)q_2 - a_2q_2 - \frac{C_2^2}{q_2^2} = 0$$

(2.1)

with the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{2} p_i^2 + \frac{1}{4} (q_1^2 + q_2^2)^2 - \frac{1}{2} a_1q_1^2 - \frac{1}{2} a_2q_2^2 + \frac{1}{2} C_1^2 + \frac{1}{2} C_2^2,$$

(2.2)

where the variables $(q_1, p_1; q_2, p_2)$ are the canonicaly conjugated variables with respect to the standard Poisson bracket, $\{ \cdot ; \cdot \}$.

This system permits the Lax representation (special case of Lax representation given in [23]).

$$\frac{\partial L(\lambda)}{\partial \zeta} = [M(\lambda), L(\lambda)],$$

(2.3)

$$L(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ Q(\lambda) & 0 \end{pmatrix}$$

is equivalent to the (2.1), where $U(\lambda), W(\lambda), Q(\lambda)$ have the form

$$U(\lambda) = -a(\lambda) \left( 1 + \frac{1}{2} \frac{q_1^2}{\lambda - a_1} + \frac{1}{2} \frac{q_2^2}{\lambda - a_2} \right),$$

$$V(\lambda) = -\frac{1}{2} \frac{d}{d\zeta} U(\lambda),$$

$$W(\lambda) = a(\lambda) \left( -\lambda + \frac{q_1^2}{2} + \frac{q_2^2}{2} + \frac{1}{2} \left( p_1^2 + \frac{C_1^2}{q_1^2} \right) \frac{1}{\lambda - a_1} \right).$$
where \( a(\lambda) = (\lambda - a_1)(\lambda - a_2) \).

The Lax representation yields hyperelliptic curve \( V = (\nu, \lambda) \),

\[
\det(L(\lambda) - \frac{1}{2} \nu I_2) = 0,
\]

where \( I_2 \) be \( 2 \times 2 \) unit matrix and is given explicitly as

\[
\nu^2 = 4(\lambda - a_1)(\lambda - a_2)(\lambda^3 - \lambda^2(a_1 + a_2) + \lambda(a_1 a_2 - H) - F)
\]

(2.4) \(- C_i^2(\lambda - a_2)^2 - C_i^2(\lambda - a_1)^2,\)

where \( H \) is the hamiltonian (2.2) and the second independent integral of motion \( F, \{F; H\} = 0 \) is given as

\[
F = \frac{1}{4}(p_1 q_2 - p_2 q_1)^2 + \frac{1}{2}(q_1^2 + q_2^2)(a_1 a_2 - \frac{1}{2} a_2 q_1^2 - \frac{1}{2} a_1 q_2^2)
\]

(2.5) \(- \frac{1}{2} p_1^2 a_2^2 - \frac{1}{2} p_2^2 a_1^2 - \frac{1}{4}(2a_2 - q_2^2) C_i^2 - \frac{1}{4}(2a_1 - q_1^2) C_i^2.\)

We remark, that the parameters \( C_i \) are linked with coordinates of the points \((a_i, \nu(a_i))\) by the formula

(2.6) \( C_i^2 = \frac{\nu(a_i)^2}{(a_i - a_j)^2}, \quad i, j = 1, 2.\)

Let us write the curve (2.4) in the form

(2.7) \( \nu^2 = 4\lambda^5 + \alpha_4 \lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0, \)

where the moduli of the curve \( \alpha_i \) are expressible in terms of physical parameters - level of energy \( H \) and constants \( a_1, a_2, C_1, C_2 \) as follows

\[
\begin{align*}
\alpha_4 &= -8(a_1 + a_2), \\
\alpha_3 &= -4H + 4(a_1 + a_2)^2 + 8a_1 a_2, \\
\alpha_2 &= 4H(a_1 + a_2) - 4F - C_1^2 - C_2^2 - 8a_1 a_2(a_1 + a_2), \\
\alpha_1 &= 4F(a_1 + a_2) - 4a_1 a_2 H + 2C_1^2 a_2 + 2C_2^2 a_1 + 4a_1^2 a_2, \\
\alpha_0 &= -4a_1 a_2 F - C_1^2 a_2^2 - C_2^2 a_1^2.
\end{align*}
\]

Let us define new coordinates \( \mu_1, \mu_2 \) as zeros of the entry \( U(\lambda) \) to the Lax operator. Then

(2.8) \( q_1^2 = 2\frac{(a_1 - \mu_1)(a_1 - \mu_2)}{a_1 - a_2}, \quad q_2^2 = 2\frac{(a_2 - \mu_1)(a_2 - \mu_2)}{a_2 - a_1}. \)
The definition of $\mu_1, \mu_2$ in the combination with the Lax representation comes to the equations

$\nu_i = V(\mu_i) = -\frac{1}{2} \frac{\partial}{\partial x} U(\mu_i), \quad i = 1, 2,$

which can be transformed to the equations of the form

$u_1 = \int_{a_1}^{\mu_1} df_1 + \int_{a_2}^{\mu_2} df_1,$

$u_2 = \int_{a_1}^{\mu_1} df_2 + \int_{a_2}^{\mu_2} df_2$

where $du_{1,2}$ denote independent canonical holomorphic differentials

$du_1 = \frac{d\lambda}{\nu}, \quad du_2 = \frac{\lambda d\lambda}{\nu}.$

and $u_1 = a, u_2 = 2x + b$ with the constants $a, b$ defining by the initial conditions. The integration of the problem is then reduces to the solving of the Jacobi inversion problem associated with the curve, which consist in the expression of the symmetric functions of $(\mu_1, \mu_2, \nu_1, \nu_2)$ as function of two complex variables $(u_1, u_2)$.

3. Exact solutions in terms of Kleinian hyperelliptic functions

In this section we give the trajectories of the system under consideration in terms of Kleinian hyperelliptic functions (see, e.g. [4, 10]), being associated with the algebraic curve of genus two (2.7) which can be also written in the form

$\nu^2 = 4 \prod_{i=0}^{4} (\lambda - \lambda_i),$

where $\lambda_i \neq \lambda_i$ are branching points. At all real branching points the closed intervals $[\lambda_{2i-1}, \lambda_{2i}], i = 0, \ldots, 4$ will be referred further as lacunae [49, 32]. Let us equip the curve with a homology basis $(a_1, a_2; b_1, b_2) \in H_1(V, \mathbb{Z})$ and fix the basis in the space of holomorphic differentials as in (2.12). The associated canonical meromorphic differentials of the second kind $dr^T = (dr_1, dr_2)$ have the form

$dr_1 = \frac{\alpha_3 \lambda + 2 \alpha_4 \lambda^2 + 12 \lambda^3}{4 \nu} d\lambda, \quad dr_2 = \frac{\lambda^2}{\nu} d\lambda.$

---

1In what follows we shall denote the integral bounds by the second coordinate of the curve $V = V(\nu, \lambda)$. (2.4)
The $2 \times 2$ matrices of their periods,

\[
2\omega = \left( \int_{a_k}^{b_k} du_l \right)_{k,l=1,2}, \quad 2\omega' = \left( \int_{a_k}^{b_k} du_l \right)_{k,l=1,2},
\]

\[
2\eta = \left( \int_{a_k}^{b_k} dr_l \right)_{k,l=1,2}, \quad 2\eta' = \left( \int_{a_k}^{b_k} dr_l \right)_{k,l=1,2}
\]
satisfy the equations,

\[
\omega'\omega^T - \omega\omega'^T = 0, \quad \eta'\omega^T - \eta\omega'^T = -\frac{i\pi}{2} 1_{2}, \quad \eta'\eta^T - \eta\eta'^T = 0,
\]

which generalizes the Legendre relations between complete elliptic integrals to the case $g = 2$.

The fundamental $\sigma$ function in this case is a natural generalization of the Weierstrass elliptic $\sigma$ function and is defined as follows

\[
\sigma(u) = \frac{\pi}{\sqrt{\det(2\omega)}} \frac{\epsilon}{\prod_{1 \leq i < j \leq 5} (a_i - a_j)} \times \exp \{ u^T \eta (2\omega)^{-1} u \} \theta(\epsilon | (2\omega)^{-1} u| \omega^{-1}),
\]

where $\epsilon^8 = 1$ and $\theta[\epsilon](v|\tau)$ is the $\theta$ function with an odd characteristic $[\epsilon] = \left[ \begin{array}{cc} \epsilon_1 & \epsilon_2 \\ \epsilon'_1 & \epsilon'_2 \end{array} \right]$, which is the characteristic of the vector of Riemann constants,

\[
\theta[\epsilon](v|\tau) = \sum_{m \in \mathbb{Z}^n} \exp \{ i\pi \left( (m + \epsilon)^T \tau (m + \epsilon) + 2(v + \epsilon')^T \tau (m + \epsilon) \right) \}.
\]

Alternatively the $\sigma$ function can be defined by its expansion near $u = 0$

\[
(3.3) \quad \sigma(u) = u_1 + \frac{1}{24} \alpha_2 u_1^3 - \frac{1}{3} \alpha_3 u_2^3 + o(u^5)
\]

and the further terms can be computed with the help of bilinear differential equation [3].

The $\sigma$-function possesses the following periodicity property: put

\[
E(m, m') = \eta m + \eta' m', \quad \Omega(m, m') = \omega m + \omega' m',
\]

where $m, m' \in \mathbb{Z}^n$, then

\[
\sigma[\epsilon](z + 2\Omega(m, m'), \omega, \omega') = \exp \{ 2E^T(m, m')(z + \Omega(m, m')) \}
\]

\[
\times \exp \{ -\pi i m^T m' - 2\pi i \epsilon^T m' \} \sigma[\epsilon](z, \omega, \omega')
\]

As modular function the Kleinian $\sigma$-function is invariant under the transformation of the symplectic group, what represents the important characteristic feature.
We introduce the Kleinian hyperelliptic functions as second logarithmic derivatives
\[ \wp_{11}(u) = -\partial_{u_1}^2 \ln \sigma(u), \quad \wp_{12}(u) = -\partial_{u_1} \partial_{u_2} \ln \sigma(u), \]
\[ \wp_{22}(u) = -\partial_{u_2}^2 \ln \sigma(u). \]

The multi-index symbols \( \wp_{i,j,k} \) etc. are defined as logarithmic derivatives by the variable \( u_i, u_j, u_k \) on the corresponding indices \( i, j, k \) etc.

The principal result of the theory is the formula of Klein, which reads in the case of genus two as follows
\[
2 \sum_{k,l=1}^{2} \wp_{kl} \left( \int_{\infty}^{\mu} du - \int_{\infty}^{\mu_1} du - \int_{\infty}^{\mu_2} du \right) \mu^{k-1} \mu_i^{l-1} = \frac{F(\mu, \mu_i) + 2\nu \nu_i}{4(\mu - \mu_i)^2}, \quad i = 1, 2,
\]
(3.4)

where
\[
F(\mu_1, \mu_2) = \sum_{r=0}^{2} \mu_1^r \mu_2^r [2\alpha_{2r} + \alpha_{2r+1}(\mu_1 + \mu_2)].
\]
(3.5)

By expanding these equalities in the vicinity of the infinity we obtain the complete set of the relations for hyperelliptic functions.

The first group of the relations represents the solution of the Jacobi inversion problem in the form
\[
\lambda^2 - \wp_{22}(u) \lambda - \wp_{12}(u) = 0,
\]
(3.6)

that is, the pair \((\mu_1, \mu_2)\) is the pair of roots of (3.6). So we have
\[
\wp_{22}(u) = \mu_1 + \mu_2, \quad \wp_{12}(u) = -\mu_1 \mu_2.
\]
(3.7)

The corresponding \( \nu_i \) is expressed as
\[
\nu_i = \wp_{22}(u) \mu_i + \wp_{12}(u), \quad i = 1, 2.
\]
(3.8)

The functions \( \wp_{22}, \wp_{12} \) are called basis functions. The function \( \wp_{11}(u) \) can be also expressed as symmetric function of \( \mu_1, \mu_2 \) and \( \nu_1, \nu_2 \):
\[
\wp_{11}(u) = \frac{F(\mu_1, \mu_2) - 2\nu \nu_2}{4(\mu_1 - \mu_2)^2},
\]
(3.9)

where \( F(\mu_1, \mu_2) \) is given in (3.5).
Further from (3.8) we have

\[ \begin{align*}
\varphi_{222}(u) &= \frac{\nu_1 - \nu_2}{\mu_1 - \mu_2}, \\
\varphi_{221}(u) &= \frac{\mu_1 \nu_2 - \mu_2 \nu_1}{\mu_1 - \mu_2}, \\
\varphi_{211}(u) &= -\frac{\mu_2 \nu_2 - \mu_1 \nu_1}{\mu_1 - \mu_2}, \\
\varphi_{111}(u) &= \frac{\nu_2 \psi(\mu_1, \mu_2) - \nu_1 \psi(\mu_2, \mu_1)}{4(\mu_1 - \mu_2)^3},
\end{align*} \]

(3.10)

where

\[ \psi(\mu_1, \mu_2) = 4\alpha_0 + \alpha_1(3\mu_1 + \mu_2) + 2\alpha_2\mu_1(\mu_1 + \mu_2) + \alpha_3\mu_1^2(\mu_1 + 3\mu_2) + 4\alpha_4\mu_1^2\mu_2 + 4\mu_1^2\mu_2(3\mu_1 + \mu_2). \]

The next group of the relations, which can be derived by the expansion of the equations (3.4) are the pairwise products of the \( \varphi_{ijk} \) functions being expressed in terms of \( \varphi_{22}, \varphi_{12}, \varphi_{11} \) and constants \( \alpha_s \) of the defining equation (3.1). We give here only basis equations

\[ \begin{align*}
\varphi_{222}^{(2)} &= 4\varphi_{22}^2 + 4\varphi_{12}\varphi_{22} + \alpha_4\varphi_{22}^2 + 4\varphi_{11} + \alpha_3\varphi_{22} + \alpha_2, \\
\varphi_{222}\varphi_{122} &= 4\varphi_{12}\varphi_{22}^2 + \varphi_{12}^2 - 2\varphi_{11}\varphi_{22} + \alpha_4\varphi_{12}\varphi_{22} + \varphi_{12} + \varphi_{12}^2 + \alpha_2, \\
\varphi_{122}^2 &= 4\varphi_{22}\varphi_{12}^2 - 4\varphi_{11}\varphi_{12} + \alpha_4\varphi_{12}^2 - \alpha_0.
\end{align*} \]

All such expressions may be rewritten in the form of an extended cubic relation as follows. For arbitrary \( l, k \in \mathbb{C}^4 \) the following formula is valid [3]

\[ (3.11) \quad l^T \pi \pi^T k = -\frac{1}{4} \det \begin{pmatrix} H & l \\ k^T & 0 \end{pmatrix}, \]

where \( \pi^T = (\varphi_{22}, -\varphi_{22}, \varphi_{211}, -\varphi_{111}) \) and \( H \) is the 4 × 4 matrix:

\[ (3.12) \quad H = \begin{pmatrix}
\alpha_0 & \frac{1}{2}\alpha_1 & -2\varphi_{11} & -2\varphi_{12} \\
\frac{1}{2}\alpha_1 & \alpha_2 + 4\varphi_{11} & \frac{1}{2}\alpha_3 + 2\varphi_{12} & -2\varphi_{22} \\
-2\varphi_{11} & \frac{1}{2}\alpha_3 + 2\varphi_{12} & \alpha_4 + 4\varphi_{22} & 2 \\
-2\varphi_{12} & -2\varphi_{22} & 2 & 0
\end{pmatrix}. \]

The vector \( \pi \) satisfies the equation \( H\pi = 0 \), and so the functions \( \varphi_{22}, \varphi_{12} \) and \( \varphi_{11} \) are related by the equation

\[ (3.13) \quad \det H = 0. \]

The equation (3.13) defines the quartic Kummer surface \( K \) in \( \mathbb{C}^3 \) [23].

The next group of the equations, which is derived as the result of expansion of the equalities (3.4) are the expressions of four index symbols
These equations can be identified with completely integrable partial differential equations and dynamical systems, which are solved in terms of Abelian functions of hyperelliptic curve of genus two. In particular, the first two equations represent the KdV hierarchy with “times” $(t_1, t_2) = (u_2, u_1) = (x, t)$,

\begin{equation}
X_{k+1}[U] = RX_k[U] \tag{3.19}
\end{equation}

where $R = \partial_x^2 - U + c - \frac{1}{2}U_x\partial^{-1}$, $c = \alpha_4/12$ is the Lenard recursion operator. The first two equations from the hierarchy are

\begin{align}
U_{t_1} &= U_x, & U_{t_2} &= \frac{1}{2}(U_{xxx} - 6U_x U) \tag{3.20}
\end{align}

the second equation is the KdV equation, which is obtained from (3.14) as the result of differentiation by $x = u_2$ and setting $U = 2\varphi_{22} + \alpha_4/6$. The equation (3.14) plays role of the stationary equation in the hierarchy and is obtained as the result of action of the recursion operator.

Let us introduce finally the Baker-Akhiezer function, which in the frames of the formalizm developed is expressible in terms of the Kleinian $\sigma$-function as follows

\begin{equation}
\Psi(\lambda, u) = \frac{\sigma \left( \int_{\infty}^{\lambda} d\nu \sim u - u \right)}{\sigma(u)} \exp \left\{ \int_{\infty}^{\lambda} d\nu^T u \right\}, \tag{3.21}
\end{equation}

where $\lambda$ is arbitrary and $u$ is the Abel image of arbitrary point $(\nu_1, \mu_1) \times (\nu_2, \mu_2) \in V \times V$. It is straighforward to show by the direct calculation, being bases on the usage of the relations for three and four-index Kleinian $\varphi$-functions that $\Psi(\lambda, u)$ satisfy to the Schrödinger equation

\begin{equation}
\left( \frac{\partial^2}{\partial u_2^2} - 2\varphi_{22}(u) \right) \Psi(\lambda, u) = \left( \lambda + \frac{1}{4}\alpha_4 \right) \Psi(\lambda, u) \tag{3.22}
\end{equation}

for all $(\nu, \mu)$.

Now we are in position to write the solution of the of the system in terms of Kleinian $\sigma$-functions and identify the constants in terms of
the moduli of the curve. Using (3.7),(2.8) the solutions of (2.1) have the following form in terms of Kleinian functions 
\[ q_1^2 = 2 \frac{a_1^2 - \wp_{22}(u)}{a_1 - a_2}, \]
\[ q_2^2 = 2 \frac{a_2^2 - \wp_{22}(u)}{a_2 - a_1}, \]
where the vector \( u^T = (a, 2x + b) \).

4. Periodic solutions expressed in terms of elliptic functions of different moduli

We consider in this section the reduction Jacobi (see e.g. [29]) of hyperelliptic integrals to elliptic ones, when the hyperelliptic curve \( V \) has the form
\[ w^2 = z(z - 1)(z - \alpha)(z - \beta)(z - \alpha \beta) \]
(4.1)

The curve (4.1) covers two-sheetedly two tori
\[ \pi^\pm : V = (w, z) \rightarrow E^\pm = (\eta^\pm, \xi^\pm), \]
(4.2)
with Jacobi moduli
\[ k^2_\pm = \frac{(\sqrt{\alpha} \mp \sqrt{\beta})^2}{(1 - \alpha)(1 - \beta)}, \]
(4.3)
The covers \( \pi^\pm \) are described by the formulae
\[ \eta^\pm = -\sqrt{(1 - \alpha)(1 - \beta)} \frac{z \mp \sqrt{\alpha \beta}}{(z - \alpha)^2(z - \beta)^2} w, \]
(4.4)
\[ \xi = \xi^\pm = \frac{(1 - \alpha)(1 - \beta)}{(z - \alpha)(z - \beta)}. \]
(4.5)
The following formula is valid for the reduction of holomorphic hyperelliptic differential to the elliptic ones:
\[ \frac{d\xi^\pm}{\eta^\pm} = -\sqrt{(1 - \alpha)(1 - \beta)}(z \mp \sqrt{\alpha \beta}) \frac{dz}{w}. \]
(4.6)
Suppose that the spectral curve (2.7) admits the symmetry of the (4.1) and apply the discussed reduction case to the problem. Then the equations of the Jacobi inversion problem (2.11) can be rewritten in the form
\[ \sum_{i=1}^{2} \int_{x_0}^{z_i} (z - \sqrt{\alpha \beta}) \frac{dz}{w} = 2u_+ , \]

\[ \sum_{i=1}^{2} \int_{x_0}^{z_i} (z + \sqrt{\alpha \beta}) \frac{dz}{w} = 2u_- . \]

with \((\nu_i, \mu_i) = (2w_i, z_i)\) and

\[ u_\pm = -\sqrt{(1 - \alpha)(1 - \beta)}(u_2 \pm \sqrt{\alpha \beta}u_1) \]

Reduce in (4.7,4.8) hyperelliptic integrals to elliptic ones according to (4.4,4.5).

\[ \int_0^{\sqrt{\xi(\mu_1)}} \frac{dx}{\sqrt{(1 - x^2)(1 - k_+^2 x^2)}} + \int_0^{\sqrt{\xi(\mu_2)}} \frac{dx}{\sqrt{(1 - x^2)(1 - k_-^2 x^2)}} = u_\pm , \]

One can further express the symmetric functions of \(\mu_1, \mu_2, \nu_1, \nu_2\) on \(V \times V\) in term of elliptic functions of tori \(E_\pm\). To this end we introduce the Darboux coordinates (see [23], p.105)

\[ X_1 = \text{sn}(u_+, k_+)\text{sn}(u_-, k_-) , \]
\[ X_2 = \text{cn}(u_+, k_+)\text{cn}(u_-, k_-) , \]
\[ X_3 = \text{dn}(u_+, k_+)\text{dn}(u_-, k_-) , \]

where \(\text{sn}(u_\pm, k_\pm), \text{cn}(u_\pm, k_\pm), \text{dn}(u_\pm, k_\pm)\) are standard Jacobi elliptic functions.

We apply further the addition theorem for Jacobi elliptic functions,

\[ \text{sn}(u_1 + u_2, k) = \frac{s_1^2 - s_2^2}{s_1 c_2 d_2 - s_2 c_1 d_1} , \]
\[ \text{cn}(u_1 + u_2, k) = \frac{s_1 c_2 d_2 - s_2 c_1 d_1}{s_1 c_2 d_2 - s_2 c_1 d_1} , \]
\[ \text{dn}(u_1 + u_2, k) = \frac{s_1 c_2 d_2 - s_2 c_1 d_1}{s_1 c_2 d_2 - s_2 c_1 d_1} , \]

where we denoted \(s_i = \text{sn}(u_i, k), c_i = \text{cn}(u_i, k), d_i = \text{dn}(u_i, k), i = 1, 2\) and formulae (3.7,3.8) for the Kleinian hyperelliptic functions.

The straightforward calculations lead to the formulae

\[ X_1 = -\frac{(1 - \alpha)(1 - \beta)(\alpha \beta + \varphi_{12})}{(\alpha + \beta) \varphi_{12} - \alpha \beta \varphi_{22} + \varphi_{11}} , \]
\[ X_2 = -\frac{(1 + \alpha \beta)(\alpha \beta - \varphi_{12}) - \alpha \beta \varphi_{22} - \varphi_{11}}{(\alpha + \beta)(\varphi_{12} - \alpha \beta) + \alpha \beta \varphi_{22} + \varphi_{11}} , \]
\[ X_3 = \frac{\alpha \beta \varphi_{22} - \varphi_{11}}{(\alpha + \beta)(\varphi_{12} - \alpha \beta) + \alpha \beta \varphi_{22} + \varphi_{11}} . \]
The formulae (4.11) can be inverted as follows

\[ \wp_{11} = (B - 1) \frac{A(X_2 + X_3) - B(X_3 + 1)}{X_1 + X_2 - 1}, \]  
(4.12) \[ \wp_{12} = (B - 1) \frac{1 + X_1 - X_2}{X_1 + X_2 - 1}, \]  
(4.13) \[ \wp_{22} = \frac{A(X_2 - X_3) + B(X_3 - 1)}{X_1 + X_2 - 1}, \]  
(4.14)

where \( A = \alpha + \beta, \) \( B = 1 + \alpha \beta. \)

The obtained results permit to present few solutions in elliptic functions of the initial problem, which are quasi-periodic in \( \zeta. \) Using (4.13) and (4.14) for solutions of the (2.1) in the form (3.23) we have

\[ q_1^2 = 2 \frac{1}{a_1 - a_2} \left( a_1^2 - \frac{A(X_2 - X_3) + B(X_3 - 1)}{X_1 + X_2 - 1} a_1 \right), \]
\[ q_2^2 = 2 \frac{1}{a_2 - a_1} \left( a_2^2 - \frac{A(X_2 - X_3) + B(X_3 - 1)}{X_1 + X_2 - 1} a_2 \right), \]

where

\[ u_{\pm} = -2 \sqrt{(1 - \alpha)(1 - \beta)}(x \mp c) \]  
(4.15)

and \( c \) is the constant depending on initial conditions.

We also remark, that the derived quasi periodic solution was associated with the Jacobi reduction case in which the ultraelliptic integrals were reduced to elliptic ones by the aid of second order substitution. This means on the language of two-dimensional \( \theta \)-functions, that the associated period matrix is equivalent to the matrix with the off-diagonal element \( \tau_{12} = \frac{1}{2}. \) Such the reduction case was considered in various places (see e.g. [7]). Solutions of this type for nonlinear Schrödinger equation \((\sigma = 0)\) are recently obtained in [11].

The analogous technique can be carried out for other well documented case of reduction , when \( \tau_{12} = 1/N \) and the \( N = 3, 4, \ldots. \) In general such the reduction can be carried out for covers of arbitrary degree within the Weierstrass-Poincaré reduction theory (see e.g. [29, 7]).

5. Elliptic periodic solutions

In this section we develop a method (see also [27, 18, 14]) which allows us to construct periodic solutions of (2.1) in a straightforward
way based on the application of spectral theory for the Schrödinger equation with elliptic potentials \[2, 32\]. We start with the formula (3.14) and with equation for Baker function $\Psi(\lambda; u)$.

\[
\frac{d^2}{dx^2} \Psi(\lambda, u) - U \Psi(x, u) = (\lambda + \frac{\alpha_4}{4}) \Psi(\lambda, u),
\]

where we identify the potential $U = 2\wp_2 + \frac{1}{6}\alpha_4$.

We assume, without losing generality, that the associated curve has the property $\alpha_4 = 0$. To make this assumption applicable to the initial curve of the system (2.1) being derived from the Lax representation, we undertake the shift of the spectral parameter,

\[
\lambda \rightarrow \lambda + \Delta, \quad \Delta = \frac{2}{5}a_1 + \frac{2}{5}a_2.
\]

Suppose, that $U$ be two gap Lamé or two gap Treibich-Verdier potential, what means, that

\[
U(x) = 2 \sum_{i=1}^{N} \wp(x - x_i),
\]

where $\wp(x)$ is standard Weierstrass elliptic functions with periods $2\omega, 2\omega'$ and numbers $x_i$ takes the values from the set \{0, $\omega_1 = \omega, \omega_2 = \omega + \omega', \omega_3 = \omega'$\}. It is known, that the set of such the potentials is exhausted by six potentials \[15, 17\]

\[
\begin{align*}
U_3(x) &= 6\wp(x), \\
U_4(x) &= 6\wp(x) + 2\wp(x + \omega_i), \quad i = 1, 2, 3, \\
U_5(x) &= 6\wp(x) + 2\wp(x + \omega_i) + 2\wp(x + \omega_j), \quad i \neq j = 1, 2, 3, \\
U_6(x) &= 6\wp(x) + 6\wp(x + \omega_i), \quad i = 1, 2, 3, \\
U_8(x) &= 6\wp(x) + 2 \sum_{i=1}^{3} \wp(x + \omega_i), \\
U_{12}(x) &= 6\wp(x) + 6 \sum_{i=1}^{3} \wp(x + \omega_i),
\end{align*}
\]

where the subscript shows the number of $2\wp$ functions involved and display the degree of the cover of the associated genus two curve over elliptic curve. Because the last three potentials can be obtained from the first three by Gauss transform we shall call the first three as basis potentials. The potential (5.4) is two gap Lamé potential, which is
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associated with three sheeted cover of elliptic curve; the potentials (5.19) are Treibich-Verdier potentials [46, 45] associated with four and five sheeted cover correspondingly.

To display the class of periodic solutions of system (2.1) we introduce the generalized Hermite polynomial $F(x, \lambda)$ by the formula

$$F(x, \lambda) = \lambda^2 - \pi_{22}(x)\lambda - \pi_{12}(x)$$

with $\pi_{22}(x)$ and $\pi_{12}(x)$ given as follows

$$\pi_{22}(x) = \sum_{j=1}^{N} \wp(x - x_j) + \frac{1}{3} \sum_{j=1}^{5} \lambda_j,$$

$$\pi_{12}(x) = -3 \sum_{i<j} \wp(x - x_i)\wp(x - x_j) - \frac{Ng_2}{8}$$

$$- \frac{1}{6} \sum_{i<j} \lambda_i\lambda_j + \frac{1}{6} \left( \sum_{j=1}^{5} \lambda_j^2 \right)$$

where $x_i$ are half-periods and $N$ is the degree of the cover (see for example [18]). The introduction of this formula is based on the possibility to compute the symmetric function $\mu_{1,2}$ in terms of differential polynomial of the first one with the help of the equation (3.14), which serves in this context as the “trace formula” [49].

The solutions of the system (2.1) are then given as

$$q_i^2(x) = \frac{2F(x, a_i - \Delta)}{a_i - a_2}, \quad q_2^2(x) = \frac{2F(x, a_2 - \Delta)}{a_2 - a_1}.$$  

The final formula for the solutions of the system (1.1) then reads

$$\mathcal{U}(x, t) = \sqrt{2\frac{F(x, a_1 - \Delta)}{a_1 - a_2}} \exp \left\{ ia_1 t - \frac{1}{2} \nu(a_1 - \Delta) \int_{x}^{x} \frac{dx}{F(x, a_1 - \Delta)} \right\},$$

$$\mathcal{V}(x, t) = \sqrt{2\frac{F(x, a_2 - \Delta)}{a_2 - a_1}} \exp \left\{ ia_2 t - \frac{1}{2} \nu(a_2 - \Delta) \int_{x}^{x} \frac{dx}{F(x, a_2 - \Delta)} \right\},$$

where we used (5.3) and (2.6).

It is important to remark for our consideration, that if the potential is known, then the associated algebraic curve of genus two can be described with the help of the Novikov equation [35]. Let us consider the two-gap potential for normalized by its expansion near the singular
Then the algebraic curve associated with this potential has the form
\[ \nu^2 = \lambda^5 - \frac{5 \cdot 7}{2} a \lambda^3 + \frac{3^2 \cdot 7}{2} b \lambda^2 \]
\[ + \left( \frac{3^4 \cdot 7}{8} a^2 + \frac{3^3 \cdot 11}{4} c \right) \lambda - \frac{3^4 \cdot 17}{4} ab + \frac{3^2 \cdot 11 \cdot 13}{2} d. \]

(5.12)

We shall consider below examples of genus two curves, which are associated with the two gap elliptic potentials (5.4), (5.19) and (5.6).

Consider the potential \( U_3 \) and construct the associated curve (5.12)
\[ L^2 = (\lambda^2 - 3g_2)(\lambda + 3e_1)(\lambda + 3e_2)(\lambda + 3e_3), \]

(5.13)

The Hermite polynomial \( F_3(\wp(x), \lambda) \) associated to the Lame potential (5.4), which is already normalized as in (5.11) has the form
\[ F_3(\wp(x), \lambda) = \lambda^2 - 3\wp(x)\lambda + 9\wp^2(x) - \frac{9}{4} g_2. \]

Then the finite and real solution to the system (2.1) is given by the formula (5.9) with the Hermite polynomial depending in the argument \( x + \omega' \) (the shift in \( \omega' \) provides the holomorphy of the solution). The solution is real under the choice of the arbitrary constants \( a_1, a_2 \) in such way, that the constants \( a_{1,2} - \Delta \) lie in different lacunae. According to (2.6) the constants \( C_i \) are then given as
\[ C_i^2 = \frac{4\nu^2(a_i - \Delta)}{(a_i - a_j)^2}, \]

where \( \Delta \) is the shift (5.2) and \( \nu \) is the coordinate of the curve (5.13) and the integrals \( H \) and \( F \) have the following form
\[ H = \frac{1}{25} \, (a_1 + a_2)^3 + \frac{21}{4} g_2, \]
\[ F = \frac{1}{25} \, (a_1 + a_2)^3 - \frac{1}{4} C_1^2 - \frac{1}{4} C_2^2 - \frac{27}{4} g_3 - \frac{21}{20} g_2 (a_1 + a_2). \]

These results are in complete agreement with solutions obtained in [37] by introducing the ansatz of the form
\[ q_i(x) = \sqrt{A_i \wp(x)^2 + B_i \wp(x)} + C_i, \quad i = 1, 2 \]

with the constants \( A_i, B_i, C_i \) which are defined from the compatibility condition of the ansatz with the equations of motion. In the foregoing
examples we are considering the solutions of the form
\[ q_i(x) = \sqrt{Q_i(\wp(x))}, \]
where \( Q_i \) are rational functions of \( \wp(x) \).

Consider with this purpose the Treibich Verdier potential
\[ U_4(x) = 6\wp(x) + 2\wp(x + \omega_1) - 2e_1, \]
associated with four sheeted cover. The potential is normalized according to \( (5.11) \). The associated spectral curve is of the form
\[ \nu^2 = 4(\lambda + 6e_1) \prod_{k=1}^{4}(\lambda - \lambda_k) \]
(5.16)
\[ \lambda_{1,2} = e_3 + 2e_2 \pm 2\sqrt{(5e_3 + 7e_2)(2e_3 + e_2)} \]
\[ \lambda_{3,4} = e_2 + 2e_3 \pm 2\sqrt{(5e_2 + 7e_3)(2e_2 + e_3)}, \]

The Hermite polynomials are given by the formula
\[ F(x, \lambda) = \lambda^2 - (3\wp(x) + \wp(x + \omega_1) + e_1)\lambda \]
\[ + 9\wp(x)(\wp(x) + \wp(x + \omega) - e_1) - 3e_1\wp(x + \omega_1) \]
\[ + \frac{9}{4}g_2 - 51e_1^2; \]

The finite real solution of \( (2.1) \) results the substitution this Hermite polynomial \( F(x + \omega', \lambda) \) into \( (5.9) \) depending in shifted by imaginary half period argument into the formula (answer). To provide the reality of the solution we shall fix the parameters \( a_i - \Delta \) in the permitted zones. The constants \( C_i \) are computed by the formula \( (5) \) at which \( \nu \) means the coordinate of the curve \( (5.16) \).

Consider further the Treibich Verdier potential
\[ U_5(x) = 6\wp(x) + 2\wp(x + \omega_2) + 2\wp(x + \omega_3) + 2e_1, \]
associated with four sheeted cover. The potential is normalized according to \( (5.11) \). The associated spectral curve is of the form
\[ \nu^2 = (\lambda + 6e_2 - 3e_3)(\lambda + 6e_3 - 3e_2) \]
\[ \times \left[ \lambda^3 + 3e_1\lambda^2 - (29e_2^2 - 22e_2e_3 + 29e_3^2)\lambda \right. \]
\[ + 159(e_2^3 + e_3^3) - 51e_2e_3(e_2 + e_3) \right] \]
(5.20)
The associated Hermite polynomials are given by the formula
\[ F(x, \lambda) = \lambda^2 - (3\wp(x) + \wp(x + \omega_2)) + \wp(x + \omega_3) + e_1)\lambda \]
\[ + 9\wp(x)(\wp(x) + \wp(x + \omega_2) + \wp(x + \omega_3)) + 3\wp(x + \omega_2)\wp(x + \omega_3) \]
\[ + 3e_1(3\wp(x) + \wp(x + \omega_2)) + \wp(x + \omega_3)) - \frac{39}{2}g_2 + 54e_1^2. \]
The solution of the system results the substitution of these expressions to (5.9) as before, but this solution is blowing up.

We remark, that since McKean, Moser and Airlault paper [2] is well known, that all elliptic potentials of the Schrödinger equations and their isospectral transformation under the action of the KdV flow has the form,

$$U(x) = 2 \sum_{i=1}^{N} \wp(x - x_i(t)),$$

(5.21)

The number $N$ is a positive integer $N > 2$ (the number of “particles”) and the numbers $x = (x_1(t), \ldots, x_N(t))$ belongs to the locus $L_N$, i.e., the geometrical position of the points given by the equations

$$L_N = \left\{ (x); \sum_{i \neq j} \wp'(x_i(t) - x_j(t)) = 0, \ j = 1, \ldots N \right\}.$$

(5.22)

If the evolution of the particles $x_i$ over the locus is given by the equations,

$$\frac{dx_i}{dt} = 6 \sum_{j \neq i} \wp(x_i(t) - x_j(t))$$

then the potential (5.21) is the elliptic solution to the KdV equation. Henceforth the elliptic potentials discuss can serve as input for the isospectral deformation along the locus. Moreover these elliptic potential do not exhausted all the variety of elliptic potential; we can mention here the elliptic potentials of Smirnov ([40, 41]) for which the shifts $x_i$ are not half periods. The involving of these objects to the subject can enlarge the classes of elliptic solutions to the system (1.1).

6. Conclusions

In this paper we have described a family of elliptic solutions for the coupled nonlinear Schrödinger equations using Lax pair method and the general method of reduction of Abelian functions to elliptic functions. Our approach is systematic in the sense that special solutions (periodic, soliton etc.) are obtained in a unified way. We considered only the family of elliptic solutions associated with the integrable case $1 : 2 : 1$ of quartic potential, the approach developed can be applied to other integrable cases being enumerated in the introduction.

In fiber optics applications of the periodic and quasi-periodic waves are of interest in optical transmission systems.
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