Let $n$ be a positive integer, and consider the vector space of real-valued continuous functions on the unit sphere in $\mathbb{R}^n$. If $1 \leq p < \infty$, then we can define the $L^p$ norm on this space in the usual manner, by taking the $p$th power of the absolute value of a function on the sphere, integrating it, and taking the $(1/p)$th power of the result. For $p = \infty$ we can use the supremum norm.
norm, which is the maximum of the absolute value of a continuous function on the sphere.

Inside the space of continuous functions on the unit sphere in \( \mathbb{R}^n \) we have the \( n \)-dimensional linear subspace of linear functions, whose value at a point is given by the inner product of that point with some fixed vector in \( \mathbb{R}^n \). For each \( p, 1 \leq p \leq \infty \), the \( p \)-norm mentioned in the previous paragraph applied to a linear function reduces to a constant multiple of the vector used in the inner product defining the linear function.

Thus there is an \( n \)-dimensional inner product space embedded isometrically into the space of real-valued continuous functions on the unit sphere in \( \mathbb{R}^n \) equipped with the \( p \)-norm for any \( p, 1 \leq p \leq \infty \), and similarly one can get \( n \)-dimensional complex inner product spaces in the space of complex-valued continuous functions on the unit sphere in \( \mathbb{C}^n \) equipped with the \( p \)-norm for any \( p, 1 \leq p \leq \infty \). In the first three sections of these notes we shall consider some related constructions in which \( p \)-norms can be compared to norms associated to inner products.

1 Gaussian random variables

Recall that
\[
\int_{-\infty}^{\infty} \exp(-\pi x^2) \, dx = 1.
\]
Indeed, the integral is a positive real number whose square is equal to
\[
\int_{\mathbb{R}^2} \exp(-\pi(x^2 + y^2)) \, dx \, dy,
\]
and this can be rewritten in polar coordinates as
\[
\int_0^{2\pi} \int_0^\infty \exp(-\pi r^2) \, r \, dr \, d\theta
\]
which reduces to
\[
2\pi \int_0^\infty r \exp(-\pi r^2) \, dr.
\]
This last integral can be computed directly, using the fundamental theorem of calculus.

Let \( n \) be a positive integer, and let
\[
\langle v, w \rangle = \sum_{j=1}^{n} v_j w_j
\]
be the standard inner product on $\mathbb{R}^n$, $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n)$.

Let $h(x)$ be a linear function on $\mathbb{R}^n$, so that there is a $v \in \mathbb{R}^n$ such that $h(x) = \langle x, v \rangle$ for all $x \in \mathbb{R}^n$.

It follows from the 1-dimensional computation that

\begin{equation}
\int_{\mathbb{R}^n} \exp(-\pi \langle x, x \rangle) \, dx = 1,
\end{equation}

i.e., the $n$-dimensional integral reduces exactly to a product of 1-dimensional integrals of the same type. If $p$ is a positive real number, consider

\begin{equation}
\left( \int_{\mathbb{R}^n} |h(x)|^p \exp(-\pi \langle x, x \rangle) \, dx \right)^{1/p}.
\end{equation}

This is equal to the product of $|v| = \langle v, v \rangle^{1/2}$ and

\begin{equation}
\left( \int_{\mathbb{R}} |x|^p \exp(-\pi x^2) \, dx \right)^{1/p}.
\end{equation}

Of course this is trivial if $v = 0$, and if $v \neq 0$, then $h(x)$ is constant in the directions orthogonal to $v$. We can integrate out those directions to get a 1-dimensional integral, and after we pull out $|v|$ the result depends only on $p$.

Thus the linear functions on $\mathbb{R}^n$ form an $n$-dimensional linear subspace of real-valued functions more generally, e.g., real-valued continuous functions of polynomial growth. The $p$-norm reduces to a constant multiple of the Euclidean norm for these functions. This does not quite work for $p = \infty$, in the sense that the functions are unbounded. Of course the coordinate functions $x_1, \ldots, x_n$ on $\mathbb{R}^n$, with respect to the probability distribution given by the Gaussian function $\exp(-\pi \langle x, x \rangle)$, are independent random variables with mean 0 and the same Gaussian distribution individually. The linear functions on $\mathbb{R}^n$ are the mean 0 Gaussian random variables which are linear combinations of these $n$ functions.

\section{Rademacher functions}

For each positive integer $\ell$ let $\mathcal{B}_\ell$ denote the set of sequences $(x_1, \ldots, x_\ell)$ such that each $x_j$ is either $-1$ or $1$. Thus $\mathcal{B}_\ell$ has $2^\ell$ elements.

We can think of $\mathcal{B}_\ell$ as a commutative group with respect to coordinatewise multiplication. In other words, $\mathcal{B}_\ell$ is a product of $\ell$ copies of the group with 2 elements.
For each \( j = 1, \ldots, \ell \) define \( r_j \) to be the function on \( B_\ell \) given by \( r_j(x) = x_j \). One can think of \( r_j \) as a homomorphism from \( B_\ell \) into the group \( \{\pm 1\} \). One can also think of the \( r_j \)'s as random variables, with respect to the uniform probability distribution on \( B_\ell \), so that the \( r_j \)'s are independent random variables which have mean 0 and are identically distributed as fair coin tosses.

If \( I \) is a subset of \( \{1, \ldots, \ell\} \), let us write \( w_I \) for the function on \( B_\ell \) which is the product of the \( r_j \)'s with \( j \in I \), where this is interpreted as being the function which is equal to 1 everywhere on \( B_\ell \) when \( I = \emptyset \). The \( r_j \)'s are called Rademacher functions, and the \( w_I \)'s are called Walsh functions.

For a pair of real-valued functions \( f_1(x), f_2(x) \) on \( B_\ell \), one can define their inner product to be

\[
2^{-\ell} \sum_{x \in B_\ell} f_1(x) f_2(x).
\]

The inner product of a Walsh function with itself is equal to 1, since \( w_I(x)^2 = 1 \) for all subsets \( I \) of \( \{1, \ldots, \ell\} \) and all \( x \in B_\ell \). If \( I_1, I_2 \) are distinct subsets of \( \{1, \ldots, \ell\} \), then one can show that the inner product of the corresponding Walsh functions on \( B_\ell \) is equal to 0.

Thus the Walsh functions form an orthonormal basis for the vector space of real-valued functions on \( B_\ell \) with respect to the inner product that we have defined. That is, they are orthonormal by the remarks in the previous paragraph, and they form a basis because there are \( 2^\ell \) Walsh functions and the vector space of functions on \( B_\ell \) has dimension equal to \( 2^\ell \).

The Rademacher functions \( r_1, \ldots, r_\ell \) are special cases of Walsh functions, corresponding to subsets \( I \) of \( \{1, \ldots, \ell\} \) with exactly one element. The linear combinations of the Rademacher functions form a very interesting \( \ell \)-dimensional subspace of the vector space of real-valued functions on \( B_\ell \).

For \( 0 < p < \infty \), consider the quantity

\[
\left( 2^{-\ell} \sum_{x \in B_\ell} |f(x)|^p \right)^{1/p},
\]

where \( f(x) \) is a real-valued function on \( B_\ell \). When \( p = 2 \) this is the norm associated to the inner product on functions on \( B_\ell \). This quantity is monotone increasing in \( p \), as a result of the fact that \( t^r \) defines a convex function on the nonnegative real numbers when \( r \geq 1 \).

For each \( p \geq 2 \) there is a positive real number \( C(p) \) such that

\[
\left( 2^{-\ell} \sum_{x \in B_\ell} |f(x)|^p \right)^{1/p} \leq C(p) \left( 2^{-\ell} \sum_{x \in B_\ell} |f(x)|^2 \right)^{1/2}
\]

(2.3)
when \( f \) is a linear combination of Rademacher functions. To see this, it suffices to consider the case where \( p \) is an even integer, by monotonicity in \( p \).

It is instructive to start with the case where \( p = 4 \). If \( f(x) \) is a linear combination of Rademacher functions, then one can expand \( f(x)^4 \) as a linear combination of certain Walsh functions, many of which have sum equal to 0, and the nonzero sums can be estimated in terms of the \( p = 2 \) case. Similar computations apply for larger \( p \)’s.

One can also show that for each positive real number \( q \leq 2 \) there is a positive real number \( C(p) \) such that

\[
(2.4) \quad \left( 2^{-\ell} \sum_{x \in B_\ell} |f(x)|^2 \right)^{1/2} \leq C(p) \left( 2^{-\ell} \sum_{x \in B_\ell} |f(x)|^q \right)^{1/q}.
\]

One can derive this from the earlier result using Hölder’s inequality, i.e., the quantity for \( p = 2 \) can be estimated in terms of the product of fractional powers of the quantity for \( p = 4 \) and the quantity for \( q < 2 \) by Hölder’s inequality, and therefore the quantity for \( p = 2 \) can be estimated in terms of the quantity for \( q < 2 \) since the quantity for \( p = 4 \) can be estimated in terms of the quantity for \( p = 2 \).

## 3 Lacunary series

Suppose that \( a_0, \ldots, a_n \) are complex numbers. Because

\[
(3.1) \quad \frac{1}{2\pi} \int_T z^j \bar{z}^l |dz|
\]

is equal to 0 when \( j \neq l \) and equal to 1 when \( j = l \), where \( T \) denotes the unit circle in the complex plane, we have that

\[
(3.2) \quad \frac{1}{2\pi} \int_T \left| \sum_{j=0}^n a_j z^j \right|^2 |dz| = \sum_{j=0}^n |a_j|^2.
\]

Now suppose that \( c_0, \ldots, c_m \) are complex numbers, and consider the function \( f(z) = \sum_{j=0}^m c_j z^{2^j} \). To estimate

\[
(3.3) \quad \frac{1}{2\pi} \int_T |f(z)|^4 |dz|,
\]

one can write \( |f(z)|^4 \) as \( f(z)^2 \bar{f}(z)^2 \) and expand the sums. Many of the terms integrate to 0.
Indeed, the integral of
\[ z^{2j_1} z^{2j_2} \overline{z}^{2l_1} \overline{z}^{2l_2} \]
over the unit circle is equal to 0 unless
\[ 2^{j_1} + 2^{j_2} = 2^{l_1} + 2^{l_2}. \]

One can check that this happens for nonnegative integers \( j_1, j_2, l_1, l_2 \) if and only if \( j_1 = l_1 \) and \( j_2 = l_2 \) or \( j_1 = l_2 \) and \( j_2 = l_1 \).

As a result one can show that the integral of \( |f(z)|^4 \) over the unit circle is bounded by a constant times the square of the integral of \( |f(z)|^2 \). There are many variants and extensions of this observation.

### 4 Some matrix norms

Fix positive integers \( m, n \). Suppose that \( A \) is a linear mapping from \( \mathbb{R}^m \) to \( \mathbb{R}^n \) associated to the \( m \times n \) matrix \((a_{j,l})\) of real numbers. Thus for \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m \), the \( l \)th component of \( A(x) \) is equal to \( \sum_{j=1}^m a_{j,l} x_j \).

If \( y = A(x) \) for some \( x \in \mathbb{R}^m \), then
\[ |y_1| + \cdots + |y_n| \leq \left( \sum_{j=1}^m \sum_{l=1}^n |a_{j,l}| \right) \max(|x_1|, \ldots, |x_m|). \]

This inequality is optimal if the \( a_{j,l} \)'s are all nonnegative real numbers, as one can see by taking \( x_j = 1 \) for all \( j \).

Now suppose that \( T \) is a linear mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) associated to the \( n \times m \) matrix \((t_{p,q})\) of real numbers in the same way as before. If \( x = T(y) \), then
\[ \max(|x_1|, \ldots, |x_m|) \leq \max \{|t_{p,q}| : 1 \leq p \leq n, 1 \leq q \leq m \} \left( |y_1| + \cdots + |y_n| \right), \]
and it is easy to see that this inequality is sharp.

Let \( \lambda_1, \ldots, \lambda_m \) denote the linear mappings from \( \mathbb{R}^m \) into \( \mathbb{R} \) which take a given vector in \( \mathbb{R}^m \) to its \( m \) coordinates. Also let \( e_1, \ldots, e_n \) denote the standard basis vectors in \( \mathbb{R}^n \), so that \( e_j \) has \( j \)th coordinate equal to 1 and all other coordinates equal to 0. We can express \( A \) as a sum of rank-1 operators in the standard way,
\[ A(x) = \sum_{j=1}^m \sum_{l=1}^n a_{j,l} \lambda_l(x) e_j. \]
The composition of $T \circ A$ defines a linear mapping from $\mathbb{R}^m$ to itself, and the trace of the linear mapping is equal to

\begin{equation}
\sum_{j=1}^{m} \sum_{l=1}^{n} a_{j,l} t_{l,j}.
\end{equation}

(4.4)

The absolute value of the trace of $T \circ A$ is less than or equal to the product of $\sum_{j=1}^{m} \sum_{l=1}^{n} |a_{j,l}|$ and $\max\{|t_{p,q}| : 1 \leq p \leq n, 1 \leq q \leq m\}$, and this inequality is sharp.

## 5 Grothendieck’s inequality

Let $m, n$ be positive integers, and let $(a_{j,l})$ be an $m \times n$ matrix of real numbers. Let us assume that this matrix is restricted in the sense that

\begin{equation}
|\sum_{j=1}^{m} \sum_{l=1}^{n} a_{j,l} v_l w_l| \leq 1
\end{equation}

(5.1)

whenever $v = (v_1, \ldots, v_m), w = (w_1, \ldots, w_n)$ satisfy

\begin{equation}
|v_1|, \ldots, |v_m|, |w_1|, \ldots, |w_n| \leq 1.
\end{equation}

(5.2)

Equivalently, $(a_{j,l})$ is restricted if

\begin{equation}
|y_1| + \cdots + |y_n| \leq \max(|x_1|, \ldots, |x_m|)
\end{equation}

(5.3)

for all $x \in \mathbb{R}^m$, where $y \in \mathbb{R}^n$ is given by $y_l = \sum_{j=1}^{m} a_{j,l} x_j$.

Let $V$ be a finite-dimensional real vector space equipped with an inner product $\langle v, w \rangle$, which one can simply take to be a Euclidean space with the standard inner product. Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be vectors in $V$. Grothendieck’s inequality states that there is a universal constant $k > 0$ so that if

\begin{equation}
\|v_1\|, \ldots, \|v_m\|, \|w_1\|, \ldots, \|w_n\| \leq 1,
\end{equation}

(5.4)

where $\|u\| = \langle u, u \rangle^{1/2}$ is the norm of $u \in V$ associated to the inner product, then

\begin{equation}
|\sum_{j=1}^{m} \sum_{l=1}^{n} a_{j,l} \langle v_l, w_l \rangle| \leq k.
\end{equation}

(5.5)

See volume 1 of [LinT2] for a proof, with $k = (\exp(\pi/2) - \exp(-\pi/2))/2$. 

7
6 Maximal functions

As in Section 2, for each positive integer \( \ell \) let \( B_\ell \) denote the set of sequences \( x = (x_1, \ldots, x_\ell) \) of length \( \ell \) such that \( x_j \in \{ \pm 1 \} \) for all \( j \). In this section and the next one, for each real-valued function \( f \) on \( B_\ell \), we put

\[
\| f \|_p = \left( 2^{-\ell} \sum_{x \in B_\ell} |f(x)|^p \right)^{1/p}
\]

when \( 1 \leq p < \infty \) and

\[
\| f \|_\infty = \max \{|f(x)| : x \in B_\ell \}.
\]

If \( x \in B_\ell \) and \( k \) is an integer with \( 0 \leq k \leq \ell \), let \( N_k(x) \) denote the set of \( y \in B_\ell \) such that \( y_j = x_j \) when \( j \leq k \). Thus \( N_0(x) = B_\ell \), \( N_\ell(x) = \{ x \} \), and \( N_k(x) \) has \( 2^{\ell-k} \) elements for each \( k \).

If \( f(x) \) is a real-valued function on \( B_\ell \) and \( 0 \leq k \leq \ell \), define \( E_k(f) \) to be the function on \( B_\ell \) whose value at a given point \( x \) is the average of \( f \) on \( N_k(x) \), i.e.,

\[
E_k(f)(x) = 2^{-\ell+k} \sum_{y \in N_k(x)} f(y).
\]

Thus \( E_k(f)(x) \) is a function on \( B_\ell \) which really only depends on the first \( k \) coordinates of \( x \). If \( h(x) \) is a function on \( B_\ell \) which depends on only the first \( k \) coordinates of \( x \), then

\[
E_k(h) = h,
\]

and in fact

\[
E_k(f h) = h E_k(f)
\]

for all functions \( f \) on \( B_\ell \).

Notice that if \( 0 \leq k < \ell \) and \( f \) is a function on \( B_\ell \) such that \( f(x) \) depends only on the first \( k + 1 \) coordinates of \( x \), then \( f \) can be expressed uniquely as \( f_1 + r_{k+1} f_2 \), where \( r_{k+1}(x) = x_{k+1} \) as in Section 2 and \( f_1(x), f_2(x) \) depend only on the first \( k \) coordinates of \( x \). Moreover, \( E_k(f) = f_1 \) in this situation.

For each real-valued function \( f \) on \( B_\ell \) the associated maximal function is given by

\[
M(f)(x) = \max \{|E_k(x)| : 0 \leq k \leq \ell \},
\]

\( x \in B_\ell \). Observe that

\[
\| M(f) \|_\infty \leq \| f \|_\infty
\]
for all functions $f$ on $\mathcal{B}_\ell$, and in fact we have equality, since $E_\ell(f) = f$.

Fix a function $f$ on $\mathcal{B}_\ell$. Let $\lambda$ be a positive real number, and consider the set

$$A_\lambda = \{x \in \mathcal{B}_\ell : M(f) > \lambda\}. \tag{6.8}$$

Thus $A_\lambda$ is the set of $x \in \mathcal{B}_\ell$ such that $|E_k(f)(x)| > \lambda$ for some $k$, $0 \leq k \leq \ell$.

If $x \in A_\lambda$, so that $|E_k(f)(x)| > \lambda$ for some $k$, $0 \leq k \leq \ell$, then we automatically have that $|E_k(f)(y)| > \lambda$ for all $y \in N_k(x)$, since $E_k(f)$ is constant on $N_k(x)$. We can describe $A_\lambda$ as the union of the subsets of $\mathcal{B}_\ell$ of the form $N_k(x)$ on which the absolute value of the average of $f$ is larger than $\lambda$.

If $x, x'$ are elements of $\mathcal{B}_\ell$ and $k, k'$ are integers such that $0 \leq k, k' \leq \ell$, then

$$N_k(x) \subseteq N_{k'}(x'), \text{ or } N_{k'}(x') \subseteq N_k(x), \text{ or } N_k(x) \cap N_{k'}(x') = \emptyset. \tag{6.9}$$

This is easy to check from the definitions. We can think of $A_\lambda$ as the union of the maximal subsets of $\mathcal{B}_\ell$ of the form $N_k(x)$ on which the absolute value of the average of $f$ is larger than $\lambda$, and the maximality of these subsets implies that they are pairwise disjoint.

Let us write $|A_\lambda|$ for $2^{-\ell}$ times the number of elements of $A_\lambda$. In other words, if $a_\lambda(x)$ is the function on $\mathcal{B}_\ell$ such that $a_\lambda(x) = \lambda$ when $x \in A_\lambda$ and $a_\lambda(x) = 0$ otherwise, then

$$\|a_\lambda\|_1 = \lambda |A_\lambda|. \tag{6.10}$$

A key point now is that

$$\lambda |A_\lambda| \leq \|f\|_1. \tag{6.11}$$

This follows from the fact that $A_\lambda$ is a disjoint union of sets of the form $N_k(x)$ on which the absolute value of the average of $f$ is larger than $\lambda$.

Let $f'(x)$ be the function on $\mathcal{B}_\ell$ which is equal to $f(x)$ when $|f(x)| > \lambda$ and which is equal to 0 otherwise. Of course

$$|f(x) - f'(x)| \leq \lambda \tag{6.12}$$

for all $x \in \mathcal{B}_\ell$. Hence

$$M(f)(x) \leq M(f')(x) + \lambda \tag{6.13}$$

for all $x \in \mathcal{B}_\ell$. 

9
Therefore \( A_{2\lambda} \) is contained in the set of \( x \in B_\ell \) such that \( M(f') > \lambda \), and the previous estimate implies that
\[
\lambda |A_{2\lambda}| \leq \|f'\|_1.
\]
Using this one can show that for each \( p > 1 \), \( \|M(f)\|_p \) is bounded by a constant depending only on \( p \) times \( \|f\|_p \).

7 Square functions

If \( f_1, f_2 \) are real-valued functions on \( B_\ell \), let us write \( (f_1, f_2) \) for their inner product,
\[
(f_1, f_2) = 2^{-\ell} \sum_{x \in B_\ell} f_1(x)f_2(x).
\]
Thus the norm \( \|f\|_2 \) is the same as the norm associated to the inner product, which is to say that \( \|f\|^2 = (f, f) \) for all real-valued functions \( f \) on \( B_\ell \).

For any function \( f \) on \( B_\ell \), we can decompose \( f \) into the sum
\[
f = E_0(f) + \sum_{k=1}^{\ell} (E_k(f) - E_{k-1}(f)).
\]
The functions \( E_0(f), E_k(f) - E_{k-1}(f), 1 \leq k \leq \ell \), are pairwise orthogonal with respect to the inner product just defined.

In particular,
\[
\|f\|_2^2 = \|E_0(f)\|^2 + \sum_{k=1}^{\ell} \|E_k(f) - E_{k-1}(f)\|^2,
\]
where actually \( E_0(f) \) is a constant and its norm reduces to its absolute value. Define the square function associated to \( f \) by
\[
S(f)(x) = \left(|E_0(f)(x)|^2 + \sum_{k=1}^{\ell} |E_k(f)(x) - E_{k-1}(f)(x)|^2\right)^{1/2}.
\]
The previous formula for \( \|f\|_2 \) can be rewritten as
\[
\|S(f)\|_2 = \|f\|_2.
\]
As a special case, suppose that \( f = \sum_{j=1}^{\ell} a_j r_j \), where \( a_1, \ldots, a_\ell \) are real numbers and \( r_j(x) = x_j \). Then \( E_0(f) = 0 \), \( E_k(f) - E_{k-1}(f) = a_k r_k \), and \( S(f) \) is the constant \((\sum_{j=1}^{\ell} a_j^2)^{1/2}\).

Now let \( f \) be any real-valued function on \( B_\ell \). A very cool fact is that \( \|S(f)\|_4 \) is bounded by a constant times \( \|f\|_4 \).

It is easy enough to write \( S(f)^4 \) explicitly, by multiplying out the sums. One can rewrite this as a sum over \( k \), where each part in the sum is a product of something around level \( k \) times a sum involving the next levels. When one sums over \( x \in B_\ell \), these sums over levels \( \geq k \) can be analyzed using orthogonality.

The basic conclusion is that \( \|S(f)\|_4 \) can be estimated, using the Cauchy–Schwarz inequality, in terms of the product of \( \|S(f)\|_4^{1/2} \) and \( \|M(f^2)\|_2 \). Because we can estimate \( \|M(f^2)\|_2 \) in terms of \( \|f^2\|_2 \), which is the same as \( \|f\|_4^{1/2} \), we can estimate \( \|S(f)\|_4 \) in terms of \( \|f\|_4 \), as desired.

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