WELL-SEPARATING COMMON COMPLEMENTS FOR SEQUENCES OF SUBSPACES OF THE SAME CODIMENSION ARE GENERIC IN HILBERT SPACES

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Abstract. Given a family of subspaces of a Banach or Hilbert space, we investigate existence, quantity and quality of its common complements. In particular, we are interested in common complements for countable families of closed subspaces of finite codimension. For those families, we show that common complements with subexponential decay of quality are generic in Hilbert spaces. Moreover, we prove that the existence of one such complement in a Banach space already implies that they are generic.

1. Introduction

Every subspace $V$ of a vector space $X$ has an algebraic complement, i.e., a subspace $C$ such that $C \cap V = \{0\}$ and $X = C + V$. Given two subspaces $V_1, V_2 \subset X$ the existence of a common complement $C$ for both subspaces simultaneously becomes a more involved question [4,5,11,15]. A necessary requirement for the existence of a common complement is that $V_1$ and $V_2$ have the same codimension. Looking at closed subspaces of finite codimension, in [14] it was proved that this requirement is enough to ensure the existence of common complements in separable Hilbert spaces. More precisely, in [14] it was shown that the set of common complements is dense for any countable family $(V_n)_{n \in \mathbb{N}}$ of closed subspaces of the same, finite codimension.
An equally important and often overlooked aspect besides the existence and the quantity of common complements is the quality of common complements. We measure the quality of a complement in terms of the degree of transversality. It indicates how close a complement is to stop being a complement or to being an optimal complement, which is the orthogonal complement in Hilbert space settings. Looking at common complements \( C \) for families of subspaces \( (V_n)_{n \in \mathbb{N}} \), we want the degree of transversality of each pairing \((C, V_n)\) to be as good as possible. While in general we cannot find a common lower bound on the degree of transversality, as the pairings may become worse with increasing index \( n \), we can describe the rate at which the quality decreases. Common complements with subexponential decay of quality are called \textit{well-separating}.

The following theorem is the first result that combines statements about quantity and quality of common complements for countable families of subspaces in Hilbert spaces. It uses the concept of prevalence [13], which is a measure-theoretic version of genericity that generalizes the notion of “Lebesgue-almost every” to infinite-dimensional vector spaces.

**Theorem.** \textit{Well-separating common complements for sequences of closed subspaces of the same, finite codimension are prevalent in Hilbert spaces.}

Our theorem not only extends existing results about the quantity of common complements, but also provides useful bounds on the quality of common complements. One example in which these bounds have already been implemented pertains the analysis of ergodic quantities in infinite-dimensional dynamical systems. In [12] our theorem helps to find common complements of Oseledets spaces along discrete-time orbits of the dynamical system. Since well-separating common complements have subexponential decay of quality, they do not interfere with the analysis of Oseledets spaces, which describe exponential behavior of linear perturbations. In view of recent advances in the theory of Oseledets spaces for infinite-dimensional settings [6,8,9], especially the identification of coherent structures through transfer operators [6,7], we see even more potential of the above theorem in the analysis of Oseledets spaces and their applications.

While our main result is restricted to Hilbert spaces, many techniques of our proof apply to general Banach spaces. In particular, we show that the existence of well-separating common complements implies their genericity: if there exists at least one well-separating common complement for a countable family of closed subspaces of the same, finite codimension in a Banach space, then well-separating common complements are already generic for this family of subspaces.

The main body of our article consists of two sections. Section 2 briefly discusses common complements for finitely many closed subspaces of codimension 1 and motivates bounds on the quality of common complements.
Section 3 defines well-separating common complements and investigates their existence and genericity for countable families of closed subspaces of finite codimension.

2. Common complements for finitely many hypersubspaces

Using simple geometric tools, we find common complements for finitely many hypersubspaces when \( X = \mathbb{R}^d \) or when \( X \) is an arbitrary Banach space. Those complements have dimension 1 and can be identified with an up to sign unique unit vector. The distances between this vector and the hypersubspaces determine the quality of the complement. Besides the existence of common complements, we investigate their quality and motivate why subexponential decay of the degree of transversality is a natural assumption for common complements for countable families of subspaces.

**Definition 2.1.** Let \((X, \|\cdot\|)\) be a Banach space. The Grassmannian \( \mathcal{G}(X) \) is the set of closed complemented subspaces of \( X \). It contains \( \mathcal{G}_k(X) \), the set of \( k \)-dimensional subspaces, and \( \mathcal{G}^k(X) \), the set of closed subspaces of codimension \( k \). We call elements of \( \mathcal{G}^1(X) \) hypersubspaces.

Moreover, we define

\[
\inf_{x \in C \cap S} d(x, V)
\]

as the *degree of transversality*\(^2\) of \((C, V)\), where \( C \in \mathcal{G}_k(X) \), \( V \in \mathcal{G}^k(X) \) and \( S \) denotes the unit sphere of \( X \).

The degree of transversality takes values between zero and one. It is equal to zero if and only if \( C \) is not a complement of \( V \). If \( X \) is a Hilbert space, then (1) equals one if and only if \( C = V^\perp \).

The next result gives us a geometric tool for finding common complements in \( \mathbb{R}^d \).

**Lemma 2.2.** Let \( Q \subset (\mathbb{R}^d, \|\cdot\|_2) \) be a compact, convex \( d \)-polytope with faces \( (F_i)_{i=1}^m \) and normals \( (f_i)_{i=1}^m \). Moreover, let \( V \subset \mathbb{R}^d \) be a hypersubspace with normal \( v \).
The volume of the orthogonal projection of $Q$ onto $V$ satisfies

$$\text{vol}_{d-1}(P_V Q) = \frac{1}{2} \sum_{i=1}^{m} \text{vol}_{d-1}(F_i) |\langle f_i, v \rangle|.$$ 

**Proof.** This is a known result (see e.g. [2, Theorem 1.1]). The basic ideas are that $P_V Q = P_V \partial Q$ and that the interior of $P_V \partial Q$ is covered twice by the projection of the hull $\partial Q$. Now, one only needs to check that 

$$\text{vol}_{d-1}(P_V F_i) = \text{vol}_{d-1}(F_i) |\langle f_i, v \rangle|$$

for each face. □

**Corollary 2.3.** Let $(V_n)_{n=1}^{N}$ be hypersubspaces of $(\mathbb{R}^d, \| \cdot \|_2)$. There exists a unit vector $x \in \mathbb{R}^d$ with $d(x, V_n) \geq 1/(2Nd)$ for all $n$. 

**Proof.** Let $Q = [-1,1]^d$ and let $v_n$ be the normal of $V_n$. Furthermore, denote by $(e_i)_{i=1}^{d}$ the standard basis of $\mathbb{R}^d$. We have

$$\text{vol}_{d-1}(P_{V_n} Q) = 2^{d-1} \sum_{i=1}^{d} \|\langle e_i, v_n \rangle\| = 2^{d-1} \|v_n\|_1 \leq 2^{d-1} \sqrt{d} \|v_n\|_2 = 2^{d-1} \sqrt{d}.$$ 

Now, let $\delta := 1/(2N \sqrt{d})$. Using the Lebesgue measure $\mu$ on $\mathbb{R}^d$ and the orthogonal decomposition $y = P_{V_n} y + \langle y, v_n \rangle v_n$, we estimate

$$\mu(\{ y \in Q | \exists n : |\langle y, v_n \rangle| \leq \delta \}) \leq \sum_{n=1}^{N} \mu(\{ y \in Q : |\langle y, v_n \rangle| \leq \delta \})$$

$$\leq \sum_{n=1}^{N} \mu(P_{V_n} Q + [-\delta, \delta]v_n) = \sum_{n=1}^{N} 2\delta \text{vol}_{d-1}(P_{V_n} Q) \leq 2\delta N 2^{d-1} \sqrt{d} = 2^{d-1}.$$ 

Since $\text{vol}(Q) = 2^d$, there must be an element $y \in Q$ with $|\langle y, v_n \rangle| \geq \delta$ for all $n$. Writing $x := y/\|y\|_2$ yields $d(x, V_n) = |\langle x, v_n \rangle| \geq \delta/\|y\|_2 \geq 1/(2Nd)$. □

A lower bound better than $1/(2Nd)$ for arbitrary hypersubspaces is possible by looking at intersections of the unit ball and hypersubspaces instead of polytopes and hypersubspaces. In the case $V_n = \{(x_1, \ldots, x_d) | x_n = 0 \}$ with $N = d$ the best possible lower bound is $1/\sqrt{N}$, which corresponds to unit vectors lying in diagonal lines $\{ (\pm t, \ldots, \pm t) | t \in \mathbb{R} \}$.

The next theorem is a well-known result in the context of the Banach–Mazur compactum. As a consequence of John’s theorem [10] about ellipsoids, the (multiplicative) distance of any Banach space of dimension $d$ to the standard euclidean space $(\mathbb{R}^d, \| \cdot \|_2)$ is at most $\sqrt{d}$:

**Theorem 2.4.** Let $X$ be a Banach space of dimension $d$. There exists an isomorphism $T: (X, \| \cdot \|) \to (\mathbb{R}^d, \| \cdot \|_2)$ such that $\|T\|\|T^{-1}\| \leq \sqrt{d}$. 

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By scaling such an isomorphism with a positive constant, we can assure that either \( \|T\| = 1 \) or \( \|T^{-1}\| = 1 \) holds additionally.

**Corollary 2.5.** Let \((V_n)_{n=1}^N\) be hypersubspaces of a Banach space \(X\). There exists a unit vector \(x \in X\) such that \(d(x, V_n) \geq 1/(4N^{2/5})\) for all \(n\).

**Proof.** Set \(V := V_1 \cap \cdots \cap V_N\) and \(Y := X/V\). As a quotient space, \(Y\) is a Banach space of dimension \(d \leq N\). The quotient map \(\pi : X \to Y\) sends \((V_n)_{n=1}^N\) to hypersubspaces of \(Y\). By Corollary 2.3 we find \(z \in \mathbb{R}^d\) with \(\|z\|_2 = 1\) and \(d(z, T\pi V_n) \geq 1/(2Nd)\) for all \(n\). Let \(y := T^{-1}z\). It holds \(\|y\|_Y \leq 1\) and

\[
\frac{1}{2Nd} \leq \inf_{v_n \in V_n} \|z - T\pi v_n\|_2 \leq \inf_{v_n \in V_n} \|T\|\|y - \pi v_n\|_Y \leq \sqrt{d} d(y, \pi V_n).
\]

Take \(x' \in X\) with \(\pi x' = y\). Since \(\inf_{v \in V} \|x' - v\| = \|y\|_Y \leq 1\), we find \(v' \in V\) with \(\|x' - v'\| \leq 2\). Set \(x := (x' - v')/\|x' - v'\|\). One readily checks that \(d(y, \pi V_n) = d(x', V_n) = \|x' - v'\| d(x, V_n)\). The claim follows. \(\square\)

Given \(N\) hypersubspaces of a Banach space, Corollary 2.5 implies that there exists a common complement such that the degree of transversality between each pair is bounded from below by \(1/(4N^{2/5})\). On the other hand, there are cases where \(1/\sqrt{N}\) is the best that can be achieved. Hence, as the number of hypersubspaces is increased to infinity, we cannot hope for a common complement with a degree of transversality bounded away from zero in general. Instead, we ask for complements such that the degree of transversality decays at most subexponentially.

### 3. Well-separating common complements

**Definition 3.1.** Let \(X\) be a Banach space and let \((V_n)_{n \in \mathbb{N}} \subset \mathcal{G}^k(X)\) be given. A common complement \(C \in \mathcal{G}^k(X)\) for \((V_n)_{n \in \mathbb{N}}\) is called well-separating with respect to \((V_n)_{n \in \mathbb{N}}\) if

\[
\lim_{n \to \infty} \frac{1}{n} \log \inf_{c \in C \cap S} d(c, V_n) = 0,
\]

where \(S\) denotes the unit-sphere of \(X\).

More specifically, given a sequence \(\delta = (\delta_n)_{n \in \mathbb{N}}\) with

\[
\delta_n > 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \log \delta_n = 0,
\]

\(C\) is called \(\delta\)-well-separating if

\[
\inf_{c \in C \cap S} d(c, V_n) \geq \delta_n.
\]
3.1. Existence. In this subsection we prove the existence of well-separating common complements in Hilbert spaces. So far, an existence result for Banach spaces has not been achieved. The only remaining hurdle for a similar result as Theorem 3.2 would be to generalize Lemma 3.4 to Banach spaces.\footnote{If $X$ is separable, then the problem reduces to solving Lemma 3.4 for $X = l^1$. Indeed, every separable Banach space is isomorphic to a quotient of $l^1$. Now, let $\pi : l^1 \to l^1 / A$ be a quotient map. Then, $\pi$ induces a map $\mathcal{G}^1(l^1 / A) \to \mathcal{G}^1(l^1)$ by $V \mapsto \pi^{-1} V$. It holds $d(x, \pi^{-1} V) = d(\pi x, V)$ for $x \in l^1$. Hence, well-separating common complements in $l^1$ project onto well-separating common complements in $l^1 / A$.}

**Theorem 3.2.** Let $H$ be a Hilbert space and let $(V_n)_{n \in \mathbb{N}} \subset \mathcal{G}^k(H)$. There exists a well-separating common complement $C \in \mathcal{G}^k(H)$ for $(V_n)_{n \in \mathbb{N}}$.

If $\dim H < \infty$, the claim of Theorem 3.2 for $k = 1$ follows from Proposition 3.8. For the case $\dim H = \infty$, we need the following two lemmata:

**Lemma 3.3.** Let $(v_n)_{n=1}^d \subset (\mathbb{R}^d, \|\cdot\|_2)$ be unit vectors such that $v_n \in \mathbb{R}^n \times \{0\}$. There are an absolute constant $c > 0$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with $|x_i| \leq 1/i^2$ and $|\langle x, v_n \rangle| \geq c/n^5$.

**Proof.** Let $Q = \prod_{i=1}^d [-i^{-2}, i^{-2}]$ and let $V_n$ be the hypersubspace orthogonal to $v_n$. By Lemma 2.2 we have

$$\text{vol}_d(P_{V_n} Q) = \sum_{i=1}^d \left( \prod_{j=1, j \neq i}^d 2j^{-2} \right) |\langle e_i, v_n \rangle|$$

$$= \frac{1}{2} \text{vol}_d(Q) \sum_{i=1}^d i^2 |\langle e_i, v_n \rangle| = \frac{1}{2} \text{vol}_d(Q) \sum_{i=1}^d i^2 |\langle e_i, v_n \rangle| \leq \frac{1}{2} \text{vol}_d(Q) n^3.$$ 

Now, let $\delta_n := 3/(\pi^2 n^5)$. We estimate

$$\mu(\{ y \in Q \mid \exists n : |\langle y, v_n \rangle| \leq \delta_n \}) \leq \sum_{n=1}^d \mu(\{ y \in Q : |\langle y, v_n \rangle| \leq \delta_n \})$$

$$\leq \sum_{n=1}^d \mu(P_{V_n} Q + [-\delta_n, \delta_n] v_n) = \sum_{n=1}^d 2\delta_n \text{vol}_{d-1}(P_{V_n} Q)$$

$$\leq \text{vol}_d(Q) \sum_{n=1}^d \delta_n n^3 \leq \text{vol}_d(Q) \frac{3}{\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} = \frac{1}{2} \text{vol}_d(Q).$$

Thus, there must be an element $y \in Q$ with $|\langle y, v_n \rangle| \geq \delta_n$ for all $n$. Since $\|y\|_2^2 \leq \sum_{n=1}^\infty 1/n^4 = \pi^4/90$, writing $x := y/\|y\|_2$ yields $|\langle x, v_n \rangle| \geq \delta_n/\|y\|_2 \geq c/n^5$ with $c := 3\sqrt{90}/\pi^4$. \hfill \Box
Lemma 3.4. Let $H$ be a Hilbert space of infinite dimension and let $(\varphi_n)_{n \in \mathbb{N}} \subset H'$ be a sequence of bounded linear functionals of norm 1. There exist a sequence $(\delta_n)_{n \in \mathbb{N}} \subset \mathbb{R}_{>0}$ and a unit vector $x \in H$ such that
\[
\lim_{n \to \infty} \frac{1}{n} \log \delta_n = 0
\]
and $|\varphi_n(x)| \geq \delta_n$ for all $n$.

Proof. By Riesz’s representation theorem, we can write $\varphi_n = \langle v_n, \cdot \rangle$ for unit vectors $v_n \in H$. Now, take an orthonormal set $(c_n)_{n \in \mathbb{N}} \subset H$ with $v_n \in \text{span}(c_1, \ldots, c_n)$. We get maps $\pi_n: H \to \mathbb{R}^n$ defined through
\[
\pi_n(x) := \begin{pmatrix}
\langle x, c_1 \rangle \\
\vdots \\
\langle x, c_n \rangle
\end{pmatrix}.
\]
By construction $(\pi_n(v_i))_{i=1}^n \subset \mathbb{R}^n$ are unit vectors such that $\pi_n(v_i) \in \mathbb{R}^i \times \{0\}$. In particular, by Lemma 3.3, there exists an element $\alpha \in \prod_{k=1}^n [-k^{-2}, k^{-2}]$ with $|\langle \alpha, \pi_n(v_i) \rangle| \geq c/i^5 =: \tilde{\delta}_i$ for all $i = 1, \ldots, n$. Let $A_n$ be the set of all such $\alpha$:
\[
A_n := \left\{ \alpha \in \prod_{k=1}^n [-k^{-2}, k^{-2}] \mid \forall i \leq n : \left| \sum_{k=1}^n \alpha_k \langle v_i, c_k \rangle \right| \geq \tilde{\delta}_i \right\}.
\]
We know that $A_n$ is a nonempty, closed subset of $\mathbb{R}^n$. For $\alpha \in A_n$, we can define $y := \sum_{k=1}^n \alpha_k c_k$. Since $\|y\|^2 \leq \sum_{k=1}^\infty 1/k^4 = \pi^4/90$, it holds
\[
|\varphi_i(x)| = \|y\|^{-1} |\langle v_i, y \rangle| = \|y\|^{-1} \left| \sum_{k=1}^n \alpha_k \langle v_i, c_k \rangle \right| \geq \sqrt{90} \pi^{-2} \tilde{\delta}_i =: \delta_i
\]
for $i \leq n$, where $x := y/\|y\|$. Thus, every $\alpha \in A_n$ induces an element $x \in H$ fulfilling the claim for $\varphi_1, \ldots, \varphi_n$. The rest of this proof treats the transition $n \to \infty$.

By Tychonoff’s theorem the space $B := \prod_{k=1}^\infty [-k^{-2}, k^{-2}]$ equipped with the product topology is compact. Since the product topology is the coarsest topology such that the canonical projections $\text{pr}_k: B \to [-k^{-2}, k^{-2}]$ are continuous, we find that $B_n := (\text{pr}_1 \times \cdots \times \text{pr}_n)^{-1} A_n$ are nonempty, closed subsets of $B$. The sets $B_n$ can be written as
\[
B_n = \left\{ \alpha \in B \mid \forall i \leq n : \left| \sum_{k=1}^\infty \alpha_k \langle v_i, c_k \rangle \right| \geq \tilde{\delta}_i \right\}.
\]
From this form is becomes obvious that $B_1 \supset B_2 \supset \cdots$ is a decreasing sequence of nonempty, closed subsets of $\mathcal{B}$. As $\mathcal{B}$ is compact, the intersection of all $B_n$ must be nonempty. Thus, we find some $\alpha$ in
\[
\bigcap_{n=1}^{\infty} B_n = \left\{ \alpha \in \mathcal{B} \mid \forall i \in \mathbb{N} : \left| \sum_{k=1}^{\infty} \alpha_k \langle v_i, c_k \rangle \right| \geq \tilde{\delta}_i \right\}.
\]
Similar to above, we set $y := \sum_{k=1}^{\infty} \alpha_k c_k$. Again, it holds $\|y\|^2 \leq \pi^4 / 90$. Defining $x := y / \|y\|$, we get
\[
|\varphi_n(x)| = \|y\|^{-1} |\langle v_n, y \rangle| = \|y\|^{-1} \left| \sum_{k=1}^{\infty} \alpha_k \langle v_n, c_k \rangle \right| \geq \sqrt{90} \pi^{-2} \tilde{\delta}_n = \delta_n.
\]
for $n \in \mathbb{N}$. □

The proof shows that $\delta_n$ can be chosen as $c/n^5$ for some constant $c > 0$. Improvements of the exponent of $n$ are possible. For instance, one may use $1/n^{1+\varepsilon}$ instead of $1/n^2$ to define the polytope in Lemma 3.3. However, since our goal is only to find an at most polynomially decaying lower bound, we aimed for better readability at the cost of a worse estimate.

So far it is not known to the author if Lemma 3.4 is true for Banach spaces instead of Hilbert spaces. However, the remainder of this section holds for arbitrary Banach spaces $(X, \|\|)$.

**Lemma 3.5.** For $x \in X$ and $\varphi \in X'$, we have
\[
|\varphi(x)| = \|\varphi\| d(x, \ker \varphi).
\]

**Proof.** Assume $\varphi \neq 0$. The first inequality follows immediately since
\[
|\varphi(x)| = \inf_{v \in \ker \varphi} |\varphi(x - v)| \leq \|\varphi\| \inf_{v \in \ker \varphi} \|x - v\| = \|\varphi\| d(x, \ker \varphi).
\]
For the converse inequality, let $\varepsilon \in (0, 1)$ and let $y \in X$ with $\|y\| = 1$ such that $|\varphi(y)| \geq (1 - \varepsilon)\|\varphi\|$. It holds
\[
x - \frac{\varphi(x)}{\varphi(y)} y \in \ker \varphi
\]
and
\[
d(x, \ker \varphi) \leq \left\| x - \left( x - \frac{\varphi(x)}{\varphi(y)} y \right) \right\| = \frac{|\varphi(x)|}{|\varphi(y)|} \leq \frac{\|\varphi\|}{1 - \varepsilon}.
\]
Since $(1 - \varepsilon)\|\varphi\| d(x, \ker \varphi) \leq |\varphi(x)|$ holds for all $\varepsilon \in (0, 1)$, we get the second inequality needed to prove the lemma. □

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Proof of Theorem 3.2 for $k = 1$. By the Hahn–Banach theorem there are bounded linear functionals $(\varphi_n)_{n \in \mathbb{N}} \subset X'$ of norm 1 such that \( \ker \varphi_n = V_n \). Assume that we find $(\delta_n)_{n \in \mathbb{N}}$ and $x \in X$ as described by Lemma 3.4. Since $|\varphi_n(x)| = d(x, V_n)$ by Lemma 3.5, the subspace spanned by $x$ is a $\delta$-well-separating common complement for $(V_n)_{n \in \mathbb{N}}$. □

To prove Theorem 3.2 for arbitrary $k$ we need the following lemma:

Lemma 3.6. Let $(X, \|\|)$ be a Banach space. Furthermore, assume $x_1, x_2 \in B_X(0, 1)$ are vectors with $\|x_1\| \geq \mu_1$ and $d(x_2, \text{span}(x_1)) \geq \mu_2$ for some numbers $0 < \mu_1, \mu_2 \leq 1$. Then

$$\inf_{t \in \mathbb{R}} \|tx_1 + (1 - t)x_2\| \geq \frac{1}{2\sqrt{5}} \mu_1 \mu_2.$$ 

Proof. The argument can be restricted to $\text{span}(x_1, x_2)$. Thus, assume that $\dim X = 2$. First, we look at $X = \mathbb{R}^2$ equipped with $\|\|_2$. After a rotation we may assume $x_1 = (\alpha_1, 0)$ with $\alpha_1 \geq \mu_1$. Now, the assumption on $x_2$ implies that its second coordinate has at least size $\mu_2$. Let $L$ be the line passing through $x_1$ and $x_2$ (see Fig. 1). We want to estimate the distance between $L$ and the origin. Clearly, the distance becomes smallest if $L$ intersects the unit circle at $(-\sqrt{1 - \mu_2^2}, \pm \mu_2)$. Hence, the task reduces to finding $\delta$ in Fig. 2. After applying Pythagoras’ theorem to find the diagonal $d$ of the big triangle and comparing ratios between both pairs of legs opposite to $\alpha$ and the hypotenuses, we get

$$\delta = \frac{\mu_1 \mu_2}{d} = \frac{\mu_1 \mu_2}{\sqrt{\mu_2^2 + (\sqrt{1 - \mu_2^2} + \mu_1)^2}} \geq \frac{1}{\sqrt{5}} \mu_1 \mu_2.$$
Fig. 2: The triangle reduction from the proof of Lemma 3.6

Thus, the claim holds for the euclidean case.

Now, let \( X \) be any 2-dimensional Banach space. By Theorem 2.4 there exists an isomorphism \( T \) from \((X, \| \cdot \|)\) to \((\mathbb{R}^2, \| \cdot \|_2)\) with \( \| T \| \leq 1 \) and \( \| T^{-1} \| \leq \sqrt{2} \). Let \( x_1, x_2 \in X \) be as in the claim. It holds \( Tx_1, Tx_2 \in B_2(0, 1) \), \( \| Tx_1 \|_2 \geq \mu_1/\sqrt{2} \) and \( d(Tx_2, \text{span}(Tx_1)) \geq \mu_2/\sqrt{2} \). From the euclidean case we get

\[
\inf_{t \in \mathbb{R}} \| tx_1 + (1-t)x_2 \| \geq \inf_{t \in \mathbb{R}} \| tTx_1 + (1-t)Tx_2 \|_2 \geq \frac{1}{2\sqrt{5}} \mu_1 \mu_2. \quad \square
\]

**Proof of Theorem 3.2 for arbitrary \( k \).** The proof is done by induction over \( k \). Assume that the claim holds for \( k \geq 1 \). Let \((V_n)_{n \in \mathbb{N}} \subset G^{k+1}(X)\) be as in the claim and define \( \pi_n : X \to X/V_n \) to be the associated quotient maps. We embed \((V_n)_{n \in \mathbb{N}}\) into two different sequences of closed complemented subspaces of \( X \), one having codimension \( k \) and the other having codimension 1. Summing their well-separating common complements will yield a well-separating common complement for our initial sequence.

First, take any \((V_1^n)_{n \in \mathbb{N}} \subset G^k(X)\) with \( V_1^n \supset V_n \). According to the codimension \( k \) case we find a \( \delta^1 \)-well-separating common complement \( C_1 \in G_k(X) \) for \((V_1^n)_{n \in \mathbb{N}}\). It holds \( \| \pi_n x_1 \| \geq d(x_1, V_1^n) \geq \delta^1_n \) for all \( x_1 \in C_1 \) of norm 1.

Next, let \( V_2^n := V_n \oplus C_1 \). Then, \((V_2^n)_{n \in \mathbb{N}} \subset G^1(X)\) is a sequence of closed, complemented subspaces of codimension 1. Hence, we find a \( \delta^2 \)-well-separating common complement \( C_2 \in G_1(X) \). Let \( x_2 \) be one of the two unit vectors of \( C_2 \). We have \( d(\pi_n x_2, \pi_n C_1) \geq \delta^2_n \).

Let \( C := C_1 \oplus C_2 \). To check if \( C \) is well-separating, we need to find a lower bound of \( \| \pi_n x \| \) with \( x \in C \) of norm 1. We scale \( x \) so that it intersects with an element of the boundary of the double cone

\[
\Delta := \{ c \in C \mid c = tx_1 + (1-t)(\pm x_2), \ t \in [0,1], \ x_1 \in B_{C_1}(0,1) \},
\]

which is contained in \( B_C(0,1) \) (see Fig. 3). The boundary \( \partial \Delta \) is made up of line segments connecting unit vectors \( x_1 \in C_1 \) with one of the two apexes
$\pm x_2 \in C_2$. By Lemma 3.6 the image of each line segment under $\pi_n$ is far
enough from the origin, i.e., we have

$$\inf_{t \in [0,1]} \left\| t\pi_n x_1 + (1 - t)(\pm \pi_n x_2) \right\| \geq \frac{1}{2\sqrt{5}} \delta_n^1 \delta_n^2 =: \delta_n.$$ 

Since every $x \in C$ of norm 1 can be written as $x = \lambda c$ for some $\lambda \geq 1$ and
$c \in \partial \Delta$, it holds $d(x, V_n) = \| \pi_n x \| = \lambda \| \pi_n c \| \geq \lambda \delta_n \geq \delta_n$. Thus, $C$ is a $\delta$-well-
separating common complement for $(V_n)_{n \in \mathbb{N}}$. □

**Remark 3.7.** In view of Corollary 2.5, the above proof guarantees the
existence of a common complement $C \in \mathcal{G}_k(X)$ for a finite family of subspaces
$(V_n)_{n=1}^N \subset \mathcal{G}_1(X)$ with degree of transversality at least $1/((2\sqrt{5})^{k-1} 4^k N^{\frac{2k}{2}})$.

### 3.2. Prevalence

In finite dimensions it is a simple task to show that
almost every vector spans a well-separating common complement for a given
family of countably many hypersubspaces:

**Proposition 3.8.** Let $X$ be a Banach space of finite dimension and
let $(V_n)_{n \in \mathbb{N}} \subset \mathcal{G}_1(X)$ be hypersubspaces. Almost every $x \in X$ spans a well-
separating common complement for $(V_n)_{n \in \mathbb{N}}$.

**Proof.** Since well-separating common complements are retained when
changing to an equivalent norm, we may assume $(X, \| \cdot \|) = (\mathbb{R}^d, \| \cdot \|_2)$. Furthermore, we can restrict ourselves to $x \in B_d(0,1)$. Define $\delta_n^\varepsilon := \varepsilon/n^2$ for
$\varepsilon > 0$. We estimate

$$\mu \{ x \in B_d(0,1) \mid \exists n : d(x, V_n) \leq \delta_n^\varepsilon \}$$

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Fig. 4: A common complement for ten randomly chosen hypersubspaces of $\mathbb{R}^3$. The hypersubspaces are represented by unit circles resulting from intersecting the hypersubspaces with the unit sphere. Each circle on the unit sphere is equipped with a neighborhoods of size $\varepsilon/n^2$, where $n = 1, \ldots, 10$ is the number of the corresponding hypersubspace. In the proof of Proposition 3.8 we seek common complements, here plotted as an orange line, which do not intersect the blue set for some $\varepsilon > 0$. Since the blue set becomes arbitrary small for $\varepsilon \to 0$, almost every 1-dimensional subspace fulfills this assumption.

\[ \leq \sum_{n=1}^{\infty} \mu(\{ x \in B_d(0,1) \mid d(x, V_n) \leq \delta^\varepsilon_n \}) \]

\[ \leq \sum_{n=1}^{\infty} 2\delta^\varepsilon_n \, \text{vol}_{d-1}(B_{d-1}(0,1)) = \varepsilon \frac{\pi^2}{3} \, \text{vol}_{d-1}(B_{d-1}(0,1)) \xrightarrow{\varepsilon \to 0} 0. \]

Hence, for almost every $x \in B_d(0,1)$, there is an $\varepsilon > 0$ such that $\text{span}(x)$ is a $\delta^\varepsilon$-well-separating common complement for $(V_n)_{n \in \mathbb{N}}$ (see Fig. 4 for a conceptual representation of the proof). \qed

Since there is no equivalent of the Lebesgue measure for arbitrary Banach or Hilbert spaces, the proof of Proposition 3.8 does not generalize to infinite dimensions. Even the notion of “almost every” in the claim is not clear a priori. Instead of “Lebesgue almost every” we will use the concept of prevalence:

**Definition 3.9** [13]. A Borel subset $E \subset X$ of a Banach space is called **prevalent** if there exists a Borel measure $\mu$ on $X$ such that

1. $0 < \mu(C) < \infty$ for some compact set $C \subset X$, and
2. $E + x$ has full $\mu$-measure for all $x \in X$. 

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A general subset $F \subset X$ is called prevalent if it contains a prevalent Borel set. We say that almost every element $x \in X$ lies in $F$.

**Proposition 3.10 [13].** The following are true:
1. $F$ prevalent $\implies$ $F$ dense in $X$,
2. $L \supset G$, $G$ prevalent $\implies$ $L$ prevalent,
3. countable intersections of prevalent sets are prevalent,
4. translations of prevalent sets are prevalent, and
5. $G \subset \mathbb{R}^d$ is prevalent $\iff$ $G$ has full Lebesgue measure, i.e., its complement has Lebesgue measure zero.

The last point implies that the notions of “almost every” in the sense of Lebesgue and in the sense of prevalence coincide in finite-dimensional Banach spaces.

To identify prevalent sets in infinite-dimensional spaces it is convenient to use probe spaces. A probe is a finite-dimensional subspace $P \subset X$ of a Banach space with which we can test for prevalence of a subset $F \subset X$. By identification with the standard euclidean space we can equip $P$ with a Borel measure $\lambda_P$. This measure induces a Borel measure $\mu_P$ on $X$ by $\mu_P(A) := \lambda_P(A \cap P)$ for Borel sets $A \subset X$. With $\mu_P$ in Definition 3.9 we get the following:

**Definition 3.11 [13].** A finite-dimensional subspace $P \subset X$ is called a probe for $F \subset X$ if there exists a Borel set $E \subset F$ such that $E + x$ has full $\mu_P$-measure for every $x \in X$.

**Proposition 3.12 [13].** The existence of a probe for $F \subset X$ implies that $F$ is prevalent.

With the additional terminology we are ready to state our result about the genericity of well-separating common complements in Hilbert spaces:

**Theorem 3.13.** Let $H$ be a Hilbert space and let $(V_n)_{n \in \mathbb{N}} \subset \mathcal{G}^k(H)$. The set of all $(x_1, \ldots, x_k) \in H^k$, such that span$(x_1, \ldots, x_k)$ is a well-separating common complement for $(V_n)_{n \in \mathbb{N}}$, is prevalent.

In Proposition 3.18 we show that the existence of one well-separating common complement already implies that they are prevalent for the given family of subspaces. In particular, this proves Theorem 3.13. However, before beginning with the proof we need a few elementary and technical lemmata.

**Lemma 3.14.** Let $X$ be a Banach space and $U \subset X$ an open subset. If $f : U \times (\mathbb{R}^k \setminus \{0\}) \to \mathbb{R}$ is continuous, then the mapping $g : U \to \mathbb{R}$ defined by $g(x) := \min_{\|\alpha\|_2 = 1} f(x, \alpha)$ is continuous as well.
Proof. Let \( \varepsilon > 0 \) be given. For each \( (x, \alpha) \in U \times (\mathbb{R}^k \setminus \{0\}) \), we find \( \delta_{(x, \alpha)} > 0 \) such that
\[
\| (x, \alpha) - (y, \beta) \| < \delta_{(x, \alpha)} \quad \implies \quad |f(x, \alpha) - f(y, \beta)| < \varepsilon
\]
for \( (y, \beta) \in U \times (\mathbb{R}^k \setminus \{0\}) \). Fix \( x \in U \). Since the set \( \{x\} \times \{\|\alpha\|_2 = 1\} \) is compact, it is covered by finitely many balls of radius \( \delta_{(x, \alpha)} \) with \( \alpha \) from \( \{\|\alpha\|_2 = 1\} \). Thus, we find \( \delta_x > 0 \) such that
\[
\| (x, \alpha) - (y, \beta) \| < \delta_x \quad \implies \quad |f(x, \alpha) - f(y, \beta)| < \varepsilon
\]
for \( (y, \beta) \in U \times (\mathbb{R}^k \setminus \{0\}) \) with \( \|\alpha\|_2 = 1 \). Now, if \( \|x - y\| < \delta_x \), then
\[
g(x) \leq \min_{\|\alpha\|_2 = 1} (f(y, \alpha) + |f(x, \alpha) - f(y, \alpha)|) \leq g(y) + \varepsilon. \quad \square
\]

Lemma 3.15. The set of all tuples spanning well-separating common complements for \( (V_n)_{n \in \mathbb{N}} \subset G^k(X) \) is a Borel subset of \( X^k \).

Proof. First, define the map \( s: X^k \to \mathbb{R} \) by
\[
s(c) := \min_{\|\alpha\|_2 = 1} \left\| \sum_{i=1}^k \alpha_i c_i \right\|.
\]
With the help of Lemma 3.14 it is easily seen that \( s \) is continuous. In particular, the set \( U := s^{-1}(0, \infty) \) of all linearly independent tuples is open in \( X^k \). Next, let \( \pi_n: X \to X/V_n \) be the quotient map associated to \( V_n \). We apply Lemma 3.14 again to see that the maps \( g_n: U \to \mathbb{R} \) given by
\[
g_n(c) := \min_{\|\alpha\|_2 = 1} \left\| \sum_{i=1}^k \alpha_i \pi_n c_i \right\| \left\| \sum_{i=1}^k \alpha_i c_i \right\|
\]
are continuous. Slightly rewriting \( g_n \) reveals that
\[
g_n(c) = \inf_{x \in \text{span}(c_1, \ldots, c_k) \cap S} d(x, V_n)
\]
has the form as in Definition 3.1. In particular, \( \text{span}(c_1, \ldots, c_k) \) is a well-separating common complement if, and only if, \( c \in U \), \( g_n(c) > 0 \), and
\[
\lim_{n \to \infty} \frac{1}{n} \log g_n(c) = 0.
\]
Let \( f_n: U_n \to \mathbb{R} \) be given by \( f_n(c) := (1/n) \log g_n(c) \), where \( U_n := g_n^{-1}(0, \infty) \subset X^k \) is open. Then, \( f_n \) is continuous and bounded from above by zero.
(since \( g_n(c) \) is bounded from above by 1). Finally, the set of tuples spanning well-separating common complements can be expressed as

\[
\bigcap_{l>0} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \left\{ c \in U_n \ \big| \ f_n(c) > -\frac{1}{l} \right\},
\]

which is a Borel set. \( \square \)

**Lemma 3.16.** Let \((A_n)_{n \in \mathbb{N}} \subset \mathbb{R}^{k \times k}\) be a sequence of matrices. For almost every \(A \in \mathbb{R}^{k \times k}\), there exists \(\varepsilon > 0\) such that

\[
|\det(A + A_n)| \geq \frac{\varepsilon}{n^2} \quad \text{for all } n \in \mathbb{N}.
\]

**Proof.** Let \(M > 0\) and \(\tilde{A}_n := (1/M)A_n\). Assume the claim holds for almost every \(\tilde{A} \in B(0, 1)^k\) with respect to the sequence \((\tilde{A}_n)_{n \in \mathbb{N}}\), where \(B(0, 1)^k \subset \mathbb{R}^{k \times k}\) is a product of unit balls in \((\mathbb{R}^k, \|\cdot\|_2)\) each representing a column. Setting \(A := MA\) for any such \(\tilde{A}\) yields

\[
|\det(A + A_n)| = M^k|\det(\tilde{A} + \tilde{A}_n)| \geq M^k \frac{\varepsilon}{n^2}
\]

for some \(\tilde{\varepsilon} > 0\). In particular, almost every \(A \in B(0, M)^k\) fulfills the required estimate with respect to \((A_n)_{n \in \mathbb{N}}\). Exhausting \(\mathbb{R}^{k \times k}\) with \(B(0, M)^k\) for \(M \to \infty\) implies that the claim holds for almost every \(A \in \mathbb{R}^{k \times k}\). Thus, it remains to prove that the claim holds for almost every \(A \in B(0, 1)^k\).

For \(A = (a_1, \ldots, a_k)\), it holds

\[
|\det A| = |\det(a_1, P_{\text{span}(a_1)} a_2, \ldots, P_{\text{span}(a_{i-1})} a_i)|
\]

\[
= \text{vol}_k(Q(a_1, P_{\text{span}(a_1)} a_2, \ldots, P_{\text{span}(a_{i-1})} a_i))
\]

\[
= ||a_1||_2 \left| P_{\text{span}(a_1)} a_2 \right|_2 \cdots \left| P_{\text{span}(a_{i-1})} a_i \right|_2,
\]

where \(Q(v_1, \ldots, v_k) \subset \mathbb{R}^k\) denotes the parallelepiped spanned by vectors \(v_1, \ldots, v_k \in \mathbb{R}^k\). Using this representation, we will derive an estimate of the form

\[
(2) \quad \mu\left( \left\{ A \in B(0, 1)^k : |\det(A + \tilde{A})| \leq \eta \right\} \right) \leq c \eta
\]

for all \(\eta > 0\) independent of \(\tilde{A}\), where \(c > 0\) is a constant only depending on \(k\).

To this end fix \(\tilde{A} = (\tilde{a}_1, \ldots, \tilde{a}_k)\) and define

\[
t_i(a_1, \ldots, a_i) := ||a_1 + \tilde{a}_1||_2 \left| P_{\text{span}(a_1 + \tilde{a}_1)} (a_2 + \tilde{a}_2) \right|_2
\]

\[
\cdots \left| P_{\text{span}(a_1 + \tilde{a}_1, \ldots, a_{i-1} + \tilde{a}_{i-1})} (a_i + \tilde{a}_i) \right|_2
\]

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for \( i = 1, \ldots, k \). Set \( t_0 := 1 \). To arrive at an estimate as in (2) we split the integral

\[
\int_{B(0,1)^k} \chi_{\{A: |\det(A+A)| \leq \eta\}}(A) \, dA
\]

using Fubini’s theorem column by column. The inner integral becomes

\[
I := \int_{B(0,1)} \chi\{a_k: \|P_{\text{span}(a_1+\tilde{a}_1, a_{k-1}+\tilde{a}_{k-1})} (a_k+\tilde{a}_k)\|_2 \leq \eta t_{k-1}^{-1}\}(a_k) \, da_k,
\]

where \( t_{k-1}^{-1} \) depends on \( a_1, \ldots, a_{k-1} \) and might be \( \infty \). If it is \( \infty \), then the inner integral is \( \text{vol}_k(B(0,1)) \). In the other case \( a_1 + \tilde{a}_1, \ldots, a_{k-1} + \tilde{a}_{k-1} \) must be linearly independent. Hence, their linear span is of dimension \( k - 1 \) and we find an orthogonal transformation \( T \) that maps \( e_1, \ldots, e_{k-1} \) into their span and maps \( e_k \) into the orthogonal complement. After applying the transformation to \( I \), we have

\[
I = \int_{B(0,1)} \chi\{b_k: \|P_{\text{span}(a_1+\tilde{a}_1, a_{k-1}+\tilde{a}_{k-1})} (Tb_k+\tilde{a}_k)\|_2 \leq \eta t_{k-1}^{-1}\}(b_k) \, db_k.
\]

Writing \( b_k = (\beta_{1k}, \ldots, \beta_{kk})^T \) and \( \tilde{b}_k = (\tilde{\beta}_{1k}, \ldots, \tilde{\beta}_{kk})^T \) for \( \tilde{b}_k := T^{-1}\tilde{a}_k \), we get

\[
I = \int_{B(0,1)} \chi\{b_k: |\beta_{kk}+\tilde{\beta}_{kk}| \leq \eta t_{k-1}^{-1}\}(b_k) \, db_k
\]

\[
\leq 2^{k-1} \int_{-1}^1 \chi\{\beta_{kk}: |\beta_{kk}+\tilde{\beta}_{kk}| \leq \eta t_{k-1}^{-1}\}(\beta_{kk}) \, d\beta_{kk}
\]

\[
\leq 2^{k-1} \int_{-1}^1 \chi\{\beta_{kk}: |\beta_{kk}| \leq \eta t_{k-1}^{-1}\}(\beta_{kk}) \, d\beta_{kk} = 2^k \min(1, \eta t_{k-1}^{-1}) \leq 2^k \eta t_{k-1}^{-1}.
\]

For the first inequality, we embedded \( B(0,1) \) into \([-1,1]^k\). Now, we have an estimate of \( I \) depending on \( a_1, \ldots, a_{k-1} \) that also holds when \( t_{k-1}^{-1} = \infty \). In the following we show that

\[
\int_{B(0,1)^{k-1}} t_{k-1}^{-1} \, d(a_1, \ldots, a_{k-1}) \leq c'
\]

for some constant \( c' \) by proving that

\[
\int_{B(0,1)} t_{k-i}^{-1} \, da_{k-i} \leq c'_i t_{k-(i+1)}^{-1}
\]

for some constants \( c'_i \) for \( i = 1, \ldots, k-1 \). Ultimately, it follows that we can set \( c' := c'_1 \cdots c'_{k-1} \) and \( c := 2^k c' \) to reach the desired estimate in (2).

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So, let us prove the above inductive formula (4). We write
\[ \int_{B(0,1)} t_{k-1}^{-1} da_{k-i} \]
\[ = t_{k-(i+1)}^{-1} \int_{B(0,1)} \left\| P_{\text{span}(a_1+\tilde{a}_1, \ldots, a_{k-(i+1)}+\tilde{a}_{k-(i+1)})} (a_{k-i} + \tilde{a}_{k-i}) \right\|_2^{-1} da_{k-i}. \]
As before, we distinguish between the cases \( t_{k-(i+1)}^{-1} = \infty \) and \( t_{k-(i+1)}^{-1} < \infty \).
In the first case, the inductive formula (4) is obviously satisfied. In the second case, we again apply a transformation \( T \) which rotates the first \( k-(i+1) \) vectors of the standard basis to \( \text{span}(a_1 + \tilde{a}_1, \ldots, a_{k-(i+1)} + \tilde{a}_{k-(i+1)}) \) and the remaining basis vectors to its orthogonal complement. Similar to before, writing \( b_{k-i} = (\beta_1(k-i), \ldots, \beta_{k-k-i})^T \) and \( \tilde{b}_{k-i} = (\tilde{\beta}_1(k-i), \ldots, \tilde{\beta}_{k-k-i})^T \) for \( \tilde{b}_{k-i} := T^{-1} \tilde{a}_{k-i} \), we get
\[ \int_{B(0,1)} \left\| P_{\text{span}(a_1+\tilde{a}_1, \ldots, a_{k-(i+1)}+\tilde{a}_{k-(i+1)})} (a_{k-i} + \tilde{a}_{k-i}) \right\|_2^{-1} da_{k-i} \]
\[ = \int_{B(0,1)} \left\| P_{\text{span}(a_1+\tilde{a}_1, \ldots, a_{k-(i+1)}+\tilde{a}_{k-(i+1)})} (Tb_{k-i} + \tilde{a}_{k-i}) \right\|_2^{-1} db_{k-i} \]
\[ = \int_{B(0,1)} \left\| (\beta_{k-i}(k-i)) + (\tilde{\beta}_{k-i}(k-i)), \ldots, \beta_{k-k-i} + (\tilde{\beta}_{k-k-i})^T \right\|_2^{-1} db_{k-i}. \]
Let \( \beta_{k-i} := (\beta_{k-i}(k-i), \ldots, \beta_{k-k-i})^T \) and \( \tilde{\beta}_{k-i} := (\tilde{\beta}_{k-i}(k-i), \ldots, \tilde{\beta}_{k-k-i})^T \).
Embedding \( B(0,1) \subset \mathbb{R}^k \) into \([-1,1]^{k-(i+1)} \times B(0,1) \subset \mathbb{R}^{k-(i+1)} \times \mathbb{R}^{i+1} \) shows that the above integral can be estimated by
\[ 2^{k-(i+1)} \int_{B(0,1)} \left\| \beta_{k-i} + \tilde{\beta}_{k-i} \right\|_2^{-1} d\beta_{k-i} \]
\[ \leq 2^{k-(i+1)} \int_{B(0,1)} \left\| \beta_{k-i} \right\|_2^{-1} d\beta_{k-i} : = c'_i < \infty. \]
Tracing back the steps, this concludes the proof of (4), which in turn gives us (3) and (2). Having (2), we set \( \eta := \varepsilon/n^2 \) and \( \tilde{A} := A_n \). It holds
\[ \mu\left( \{ A \in B(0,1)^k \mid \exists n : |\det(A + A_n)| \leq \varepsilon n^{-2} \} \right) \]
\[ \leq \sum_{n=1}^\infty \mu\left( \{ A \in B(0,1)^k : |\det(A + A_n)| \leq \varepsilon n^{-2} \} \right) \leq \sum_{n=1}^\infty c\varepsilon n^{-2} = \frac{c\pi^2}{6} \varepsilon \rightarrow 0. \]
Hence, for almost every \( A \in B(0,1)^k \), there is \( \varepsilon > 0 \) such that we have \( |\det(A + A_n)| \geq \varepsilon/n^2 \) for all \( n \in \mathbb{N} \). □
LEMMA 3.17. Let \((A_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^{k \times k}\) be a sequence of matrices such that \(\|A_n\|_2 \leq 1/\delta_n\) with \(0 < \delta_n \leq 1\). For almost every \(A \in \mathbb{R}^{k \times k}\), there is \(\varepsilon > 0\) with
\[
\|(A + A_n)^{-1}\|_2^{-1} \geq \varepsilon n^{-2}\delta_n^{k-1} \quad \text{for all } n \in \mathbb{N}.
\]

PROOF. Let \(A\) be as in Lemma 3.16. Using the adjugate, we write
\[
(A + A_n)^{-1} = \det(A + A_n)^{-1}(A + A_n)^{\text{ad}}.
\]
Hence, we have
\[
\|(A + A_n)^{-1}\|_2^{-1} = |\det(A + A_n)| \|(A + A_n)^{\text{ad}}\|_2^{-1}.
\]
According to Lemma 3.16 the determinant part can be estimated from below by \(\tilde{\varepsilon}/n^2\). For the adjugate part, we remark that the spectral norm and the max norm on \(\mathbb{R}^{k \times k}\) are equivalent. Thus, there are constants \(c_1, c_2 > 0\) with \(c_1\|\cdot\|_{\text{max}} \leq \|\cdot\|_2 \leq c_2\|\cdot\|_{\text{max}}\). Moreover, the entries of the adjugate consist of determinants of \((k - 1) \times (k - 1)\)-matrices with entries from \(A + A_n\). As a simple corollary of Hadamard’s inequality, we can estimate these determinants using the max norm to obtain
\[
\|(A + A_n)^{\text{ad}}\|_2 \leq c_2\|(A + A_n)^{\text{ad}}\|_{\text{max}}
\]
\[
\leq c_2\|A + A_n\|_{\text{max}}^{k-1}(k-1)^{-\frac{k-1}{2}} \leq c_2(k-1)^{-\frac{k-1}{2}}c_1^{-(k-1)}\|A + A_n\|_2^{k-1}
\]
\[
\leq c_2(k-1)^{-\frac{k-1}{2}}c_1^{-(k-1)}(\|A\|_2 + \|A_n\|_2)^{k-1}
\]
\[
\leq c_2(k-1)^{-\frac{k-1}{2}}c_1^{-(k-1)}(\|A\|_2 + 1)^{k-1}\delta_n^{-(k-1)} =: c\delta_n^{-(k-1)}.
\]
Now, we set \(\varepsilon := \tilde{\varepsilon}/c\) to obtain the result. \(\square\)

PROPOSITION 3.18. Let \(X\) be a Banach space. Assume there exists a well-separating common complement for \((V_n)_{n\in\mathbb{N}} \subseteq \mathcal{G}^k(X)\). Then, the set of all \((x_1, \ldots, x_k) \in X^k\), such that \(\text{span}(x_1, \ldots, x_k)\) is a well-separating common complement for \((V_n)_{n\in\mathbb{N}}\), is prevalent.

PROOF. According to Proposition 3.12 the set
\[
\Lambda := \{(x_1, \ldots, x_k) \in X^k \mid \text{span}(x_1, \ldots, x_k)\}
\]
is prevalent if there exists a probe for \(\Lambda\). Given any \(\delta\)-well-separating common complement \(C\) for \((V_n)_{n\in\mathbb{N}}\), we will show that \(C^k\) is a probe for \(\Lambda\). Since \(\Lambda\) is a Borel set (Lemma 3.15) and by Definition 3.11, \(C^k\) is a probe for \(\Lambda\) if,
and only if, \( \Lambda + (x_1, \ldots, x_k) \) has full \( \mu_{C_k} \)-measure for every \( (x_1, \ldots, x_k) \in X^k \), or equivalently, if the set

\[
\{ (c_1, \ldots, c_k) \in C^k \mid \text{span}(c_1 + x_1, \ldots, c_k + x_k) \}
\]

is a well-separating common complement for \( (V_n)_{n \in \mathbb{N}} \).

has full Lebesgue measure in \( C^k \) for every \( (x_1, \ldots, x_k) \in X^k \). To get a notion of Lebesgue measure on \( C^k \) we identify a basis \( (b_1, \ldots, b_k) \) of \( C \) with the standard basis \( (e_1, \ldots, e_k) \) of \( \mathbb{R}^k \). Let us denote this identification by \( I: C \to \mathbb{R}^k \). We naturally get an isomorphism \( I^k: C^k \to \mathbb{R}^{k \times k} \) mapping elements of \( C^k \) to matrices column by column. Thus, we need to check for the measure of all coefficient matrices yielding well-separating common complements. At this point, let us note that the norm on \( X^k \) is given by

\[
\| (x_1, \ldots, x_k) \|_{X^k} := \| x_1 \| + \cdots + \| x_k \|.
\]

Fix a translation \( (x_1, \ldots, x_k) \). For each \( n \in \mathbb{N} \), we can write \( x_i = c'_i,n + v'_i,n \) according to the splitting \( X = C \oplus V_n \). The translation contributed by \( (c'_1,n, \ldots, c'_k,n) \) boils down to a translation on \( \mathbb{R}^{k \times k} \) by \( A_n := I^k(c'_1,n, \ldots, c'_k,n) \). We are interested in the extend of this translation with increasing \( n \). Assuming \( \|c'_i,n\| > 0 \), it holds

\[
\frac{\| x_i \|}{\| c'_i,n \|} = \frac{\| c'_i,n \|}{\| c'_i,n \| + \| v'_i,n \|} \geq d\left( \frac{c'_i,n}{\| c'_i,n \|}, V_n \right) \geq \delta_n.
\]

Thus, we have \( \| c'_i,n \| \leq (1/\delta_n) \max_i \| x_i \| \). Switching to the coefficient space, we get \( \| A_n \|_2 \leq \| I^k \| (1/\delta_n) \max_i \| x_i \| \), which can be estimated further by \( 1/\tilde{\delta}_n := \max (1, \| I^k \| (1/\delta_n) \max_i \| x_i \|) \).

Now, let \( A \) be as in Lemma 3.17 with respect to the sequence \( (A_n)_{n \in \mathbb{N}} \). We will show that \( A \) induces a well-separating common complement. Let \( (c_1, \ldots, c_k) := (I^k)^{-1}A \) and let \( c \in \text{span}(c_1 + x_1, \ldots, c_k + x_k) \) with \( \| c \| = 1 \). We express \( c \) in terms of coefficients

\[
c = \sum_{i=1}^{k} \gamma_i(c_i + x_i) = \sum_{i=1}^{k} \gamma_i \sum_{j=1}^{k} (\alpha_{ij} + \alpha_{ji,n}) b_j + \sum_{i=1}^{k} \gamma_i v'_{i,n},
\]

where \( A = (\alpha_{ij})_{ij} \) and \( A_n = (\alpha_{ij,n})_{ij} \). Since \( A + A_n \) is invertible by Lemma 3.17 and \( (b_1, \ldots, b_k) \) is a basis, the vectors \( \sum_{j=1}^{k} (\alpha_{ij} + \alpha_{ji,n}) b_j \) for \( i = 1, \ldots, k \) form a basis of \( C \). In particular, the double sum in (5) does not vanish. Using the fact that \( C \) is \( \delta \)-well-separating, we compute

\[
d(c, V_n) = d\left( \sum_{i,j} \gamma_i (\alpha_{ij} + \alpha_{ji,n}) b_j, V_n \right)
\]
\[
\sum_{i,j} \gamma_i (\alpha_{ji} + \alpha_{ji,n}) b_j \geq \sum_{i,j} \gamma_i (\alpha_{ji} + \alpha_{ji,n}) b_j \delta_n.
\]

We transfer further norm estimates onto the coefficient space. It holds
\[
\left\| \sum_{i,j} \gamma_i (\alpha_{ji} + \alpha_{ji,n}) e_j \right\|_2 \leq \|I\| \left\| \sum_{i,j} \gamma_i (\alpha_{ji} + \alpha_{ji,n}) b_j \right\|_2.
\]

Let \( \gamma := (\gamma_1, \ldots, \gamma_k)^T \). Using the identity \( \gamma = (A + A_n)^{-1}(A + A_n)\gamma \), we get
\[
\left\| \sum_{i,j} \gamma_i (\alpha_{ji} + \alpha_{ji,n}) e_j \right\|_2 = \|(A + A_n)\gamma\|_2
\]
\[
\geq \| (A + A_n)^{-1} \|_2^{-1} \| \gamma \|_2 \geq \varepsilon n^{-2} \delta_n^{-1} \| \gamma \|_2
\]
with \( \varepsilon > 0 \) from Lemma 3.17. As \((c_i + x_i)_{i=1}^k\) are linearly independent, the norm of \( \gamma \) such that \( \| \sum_{i=1}^k \gamma_i (c_i + x_i) \| = 1 \) for fixed \( c_i \) and \( x_i \) is bounded from below by a positive constant \( \eta > 0 \).

Let \( \delta_n' := \varepsilon n^{-2} \delta_n^{-1} \| \gamma \|_2 \). Putting everything together, we have shown that
\[
\inf_{c \in \text{span}(c_1 + x_1, \ldots, c_k + x_k) \cap S} d(c, V_n) \geq \delta_n'
\]
for all \( n \in \mathbb{N} \), which tells us that \( \text{span}(c_1 + x_1, \ldots, c_k + x_k) \) is a \( \delta_n' \)-well-separating common complement for \((V_n)_{n \in \mathbb{N}}\). Hence, given an arbitrary translation by \((x_1, \ldots, x_k) \in X^k\), almost every \( A \in \mathbb{R}^{k \times k} \) induces a well-separating common complement. \( \square \)

Tracking \( \delta_n' \) in the Hilbert space setting reveals that almost every tuple yields a common complement such that the degree of transversality decays at most polynomially with \( c/n^{5k^2+2} \) for some \( c > 0 \) depending on the tuple. A better general rate of decay can be obtained by carefully refining the proofs.

References

[1] A. Blumenthal, A volume-based approach to the multiplicative ergodic theorem on Banach spaces, *Discrete Contin. Dyn. Syst.*, 36 (2016), 2377–2403.

[2] T. Burger, P. Gritzmann, and V. Klee, Polytope projection and projection polytopes, *Amer. Math. Monthly*, 103 (1996), 742–755.

*Analysis Mathematica* 48, 2022
[3] F. Deutsch, The angle between subspaces of a Hilbert space, in: Approximation Theory, Wavelets and Applications, S. P. Singh, ed., Springer Netherlands (Dordrecht, 1995), pp. 107–130.

[4] D. Drivaliaris and N. Yannakakis, Subspaces with a common complement in a Banach space, *Studia Math.*, 182 (2007), 141–164.

[5] D. Drivaliaris and N. Yannakakis, Subspaces with a common complement in a separable Hilbert space, *Integral Equations Operator Theory*, 62 (2008), 159–167.

[6] G. Froyland, S. Lloyd, and A. Quas, A semi-invertible oseledets theorem with applications to transfer operator cocycles, *Discrete Contin. Dyn. Syst.*, 33 (2013), 3835–3860.

[7] C. González-Tokman, Multiplicative ergodic theorems for transfer operators: Towards the identification and analysis of coherent structures in non-autonomous dynamical systems, in: Contributions of Mexican Mathematicians Abroad in Pure and Applied Mathematics, Contemporary Mathematics, vol. 709, Aportaciones Mat., American Mathematical Society (Providence, RI, 2018), pp. 31–52.

[8] C. González-Tokman and A. Quas, A semi-invertible operator Oseledets theorem, *Ergodic Theory Dynam. Systems*, 34 (2014), 1230–1272.

[9] C. González-Tokman and A. Quas, A concise proof of the multiplicative ergodic theorem on Banach spaces, *J. Modern Dynamics*, 9 (2015), 237–255.

[10] F. John, Extremum problems with inequalities as subsidiary conditions, in: Traces and Emergence of Nonlinear Programming, G. Giorgi and T. H. Kjeldsen, eds., Birkhäuser (Basel, 2014), pp. 197–215.

[11] M. Lauzon and S. Treil, Common complements of two subspaces of a Hilbert space, *J. Funct. Anal.*, 212 (2004), 500–512.

[12] F. Noethen, Computing covariant Lyapunov vectors in Hilbert spaces, *J. Comput. Dyn.*, 8 (2021), 325–352.

[13] W. Ott and J. A. Yorke, Prevalence, *Bull. Amer. Math. Soc.*, 42 (2005), 263–290.

[14] L. Rodman, On global geometric properties of subspaces in Hilbert space, *J. Funct. Anal.*, 45 (1982), 226–235.

[15] A. R. Todd, Covers by linear subspaces, *Math. Mag.*, 63 (1990), 339–342.