Experimental Observation of Tensor Monopoles with a Superconducting Qudit

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Monopoles play a center role in gauge theories and topological matter. Examples of monopoles include the Dirac monopole in 3D and Yang monopole in 5D, which have been extensively studied and observed in condensed matter or artificial systems. However, tensor monopoles in 4D are less studied, and their observation has not been reported. Here we experimentally construct a tunable spin-1 Hamiltonian to generate a tensor monopole and then measure its unique features with superconducting quantum circuits. The energy structure of a 4D Weyl-like Hamiltonian with three-fold degenerate points acting as tensor monopoles is imaged. Through quantum-metric measurements, we report the first experiment that measures the Dixmier-Douady invariant, the topological charge of the tensor monopole. Moreover, we observe topological phase transitions characterized by the topological Dixmier-Douady invariant, rather than the Chern numbers as used for conventional monopoles in odd-dimensional spaces.

Introduction.—Monopoles are fundamental topological objects in high-energy physics and condensed matter physics. In 1931, Dirac captured the physical importance of magnetic monopoles (called Dirac monopoles) [1], and proved the quantization of the electric charge. The Dirac monopole was later recognized to be connected to the Berry curvature and Berry phase in quantum mechanics [2]. The topological nature of Dirac monopoles defined in three dimensions (3D) is characterized by the first Chern number as its topological charge. Other monopoles have been identified in gauge theory, such as the ’t Hooft-Polyakov monopole [3, 4] in Yang-Mills theory and the Yang monopole [5]. The Yang monopole is a non-Abelian extension of the Dirac monopole in five dimensions (5D) and is topologically characterized by the second Chern number. Generally, a zoo of monopoles in (2n + 1)-dimensional (n = 1, 2, 3,...) flat spaces can be identified by the n-order Chern numbers, which are given by the integral of the corresponding field strength associated with a monopole’s gauge field [6].

Aside from the Dirac and Yang monopoles that are associated with vector gauge fields, there exists another class of monopoles associated with tensor gauge fields [7–10]. A representative is the so-called “tensor monopole” defined in a four-dimensional (4D) space. The topological charge of a 4D tensor monopole is given by the integral of the tensor gauge field [10–12], known as the Dixmier-Douady (DD) invariant [13, 14]. Tensor monopoles play a key role in string theory, where currents naturally couple to a tensor gauge field [15–17]. Recently, Palumbo and Goldman proposed a realistic three-band model defined over a 4D parameter space to generate tensor monopoles [11, 12], whose topological charges could be extracted from the generalized Berry curvature by measuring the quantum metric [18–22]. The quantum metric in engineered quantum systems can be measured through periodic driving [23, 24], sudden quench [25], and spin-texture [26, 27].

So far, monopoles have not been observed for real particles. However, they can emerge in condensed-matter materials [28, 29] or be engineered in certain artificial systems with effective gauge fields [30–33]. In these systems, monopoles are usually connected to the existence of topological states. For instance, Weyl points in Weyl semimetals can be viewed as fictitious Dirac monopoles in momentum space [29]. The analog Dirac monopoles were created in the synthetic electromagnetic field that arises in the spin texture of atomic spinor condensates [34, 35]. The monopole field and the first Chern number were measured in 3D parameter space of spin-1/2 or spin-1 artificial atoms [36–39]. A quantum-simulated Yang monopole was observed in a 5D parameter space built from an atomic condensate’s internal states, and the second Chern number as its topological charge was measured [40]. However, the 4D tensor monopoles have not been realized or simulated, and the associated topological DD invariant has not been measured. Experimental observation of tensor monopoles can further our understanding of tensor gauge fields [11, 12, 15–17] and advance the search for new exotic topological matter in condensed matter physics and artificial quantum systems [28–33, 41].

In this Letter, we fill this gap by synthesizing tensor monopoles in a 4D parameter space built in superconducting quantum circuits and measuring its topological features. By engineering a tunable 4D Weyl-like spin-1 Hamiltonian, we first image the energy structure with three-fold degenerate points acting as tensor monopoles. By characterizing the associated generalized curvature...
Dirac monopole and tensor monopole. The two are defined as pointlike sources of vector and tensor gauge fields, respectively. The fluxes associated with the field strengths $F_{\mu\nu} \propto r^{-2}$ and $H_{\mu\nu\lambda} \propto r^{-3}$ through the surrounding 2D and 3D spheres ($S^2$ and $S^3$) with radius $r = |q|$ are quantized in terms of two different topological invariants, the first Chern number $C_1 = 1$ and the DD invariant $Q_{DD} = 1$, respectively. The related quantum metric tensors $g_{\mu\nu}$ in $S^2$ and $S^3$ can be measured from the quench scheme.

Tensor monopoles and tensor fields. To establish a basic understanding of the tensor monopole in 4D parameter space, we begin by comparing it with the well-known Dirac monopole in 3D space, both spanned by the parameters $q$, as shown in Fig. 1. For a non-degenerate quantum state $|u_q\rangle$, the geometric property is captured by a quantum geometric tensor $[18, 42, 43]$: $\chi_{\mu\nu} = \langle \partial_{q_\mu} u_q | (1 - |u_q\rangle\langle u_q|) | \partial_{q_\nu} u_q \rangle = g_{\mu\nu} + iF_{\mu\nu}/2$, where the real and imaginary parts define the quantum metric $g_{\mu\nu} = g_{\mu\nu}$ and Berry curvature (gauge field) $F_{\mu\nu} = -F_{\nu\mu}$, respectively. The Berry curvature $F_{\mu\nu} = \partial_{q_\mu} A_{\nu} - \partial_{q_\nu} A_{\mu}$ with the Berry connection $A_{\mu} = i\langle u_q | \partial_{q_\mu} u_q \rangle$ is associated with the Berry phase. The quantum metric $g_{\mu\nu}$ defines the quantum distance between nearby states $|u_q\rangle$ and $|u_{q+\delta q}\rangle$ in the parameter space $[18-22]: ds^2 = 1 - |\langle u_q | u_{q+\delta q} \rangle|^2 = \sum_{\mu\nu} g_{\mu\nu} d\lambda_\mu d\lambda_\nu$, which is related to the wave-function overlap and can thus be directly measured.

For a Dirac monopole in 3D space $q = (q_x, q_y, q_z)$, in the context of gauge field (electromagnetism), the Berry curvature $F_{\mu\nu}$ can be viewed as the field strength (the Faraday tensor) associated with the flux through the surrounding sphere $S^2$ with radius $r = |q|$. A minimal model realizing a Dirac monopole is the Weyl Hamiltonian $H_{3D} = q \cdot \sigma$, where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. The topological charge of the Dirac monopole at $q = 0$ is then given by the first Chern number $C_1 = \frac{1}{4\pi^2} \int_{S^2} F = 1$, which, as the field is radial, integrates over the sphere’s surface. Notably, the Berry curvature associated with a monopole is related to the determinant of the metric tensor $g_{\mu\nu}$ defined on a sphere with $\mu, \nu = (\theta, \phi)$: $F_{\mu\nu} = 2\epsilon_{\mu\nu} \sqrt{\text{det}(g_{\mu\nu})}$, where $\epsilon_{\mu\nu}$ is the Levi-Civita symbol, $g_{\theta\theta} = 1/4$, $g_{\phi\phi} = \sin^2 \theta/4$, and $g_{\theta\phi} = 0$.

Different from the odd-dimensional monopoles defined with vector fields (Dirac monopoles in 3D and Yang monopoles in 5D), a tensor monopole is defined in even dimensions and associated with tensor fields. A tensor monopole in 4D space $q = (q_x, q_y, q_z)$ takes a (3-form) curvature tensor $F_{\mu\nu\lambda}$, as the generalization of the (2-form) Berry curvature $F_{\mu\nu}$ of the Dirac monopole. A minimal model realizing such a tensor monopole is the three-band Weyl-like Hamiltonian in 4D space $[11]$:

$$H_{4D} = q \cdot \lambda = \begin{bmatrix} 0 & q_x - iq_y & 0 \\ q_x + iq_y & 0 & q_z + iq_w \\ 0 & q_z - iq_w & 0 \end{bmatrix}.$$ (1)

where $\lambda = (\lambda_1, \lambda_2, \lambda_6, \lambda_7)$ are $3 \times 3$ Gell-Mann matrices. The energy spectrum is given by $E_{0,\pm} = 0, \pm |q|$, with a triple-degenerate Weyl-like point at $q = (0, 0, 0, 0)$ in 4D parameter space. Such a Weyl-like node gives a tensor monopole, surrounded by a 3D hypersphere $S^3$. In terms of hyperspherical coordinates $(r, \theta_1, \theta_2, \phi)$ ($\theta_{1,2} \in [0, \pi]$ and $\phi \in [0, 2\pi]$), one has $q_x = r \cos \theta_1$, $q_y = r \sin \theta_1 \cos \theta_2$, $q_z = r \sin \theta_1 \sin \theta_2 \cos \phi$, and $q_w = r \sin \theta_1 \sin \theta_2 \sin \phi$. The generalized curvature tensor as the field strength in $S^3$ is related to the quantum metric $[11]$:

$$H_{\theta_1, \theta_2, \phi} = \epsilon_{\theta_1, \theta_2, \phi} (4 \sqrt{\text{det} g_{\mu\nu}}), \quad \lambda, \nu = (\theta_1, \theta_2, \phi).$$ (2)

Here $H_{4D}$ has $\phi$-rotation symmetry and thus $H_{\theta_1, \theta_2, \phi}$ is independent of $\phi$. For the ground state $|\psi_\nu\rangle$ of the system, all matrix elements of the metric tensor $g$ can be explicitly obtained [see Eqs. (10) in SM $[44]$. The tensor monopole generalizes the Dirac monopole to 4D, and takes a topological charge associated with the generalized curvature tensor $H_{\theta_1, \theta_2, \phi}$:

$$Q_{DD} = \frac{1}{2\pi^2} \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \int_0^{2\pi} d\phi H_{\theta_1, \theta_2, \phi} = 1,$$ (3)

which is the DD invariant $[13, 14]$. Thus, to obtain the topological charge $Q_{DD}$ of a tensor monopole, one can measure $H_{\theta_1, \theta_2, \phi}$ by revealing the quantum metric $g_{\mu\nu}$. 

FIG. 1: (Color online) Pictorial representations of (a) a Dirac monopole in 3D parameter space $q = (q_x, q_y, q_z)$; and (b) a tensor monopole in 4D parameter space $q = (q_x, q_y, q_z, q_w)$. The two are defined as pointlike sources of vector and tensor gauge fields, respectively. The fluxes associated with the field strengths $F_{\mu\nu} \propto r^{-2}$ and $H_{\mu\nu\lambda} \propto r^{-3}$ through the surrounding 2D and 3D spheres ($S^2$ and $S^3$) with radius $r = |q|$ are quantized in terms of two different topological invariants, the first Chern number $C_1 = 1$ and the DD invariant $Q_{DD} = 1$, respectively. The related quantum metric tensors $g_{\mu\nu}$ in $S^2$ and $S^3$ can be measured from the quench scheme.
In parameter space, the quantum distance $ds^2$ is related to the transition probability $P^+$ of the quantum state being excited to other eigenstates after a sudden quench: $P^+ = ds^2$ [18, 19, 25]. One can thus measure the quantum metric via transition probability by the sudden quench method. For a quantum state initially prepared at $\mathbf{q}$, to extract the diagonal components $g_{\mu\nu}$ at this point, one can suddenly quench the system parameter to $\mathbf{q} + \delta \mathbf{e}_\mu$ along the $e_\mu$ direction, and then measure the transition probability $P^+_{\mu\nu} = g_{\mu\nu} \delta q^2 + \mathcal{O}(\delta q^3)$. To extract the off-diagonal components $g_{\mu\vee}$ ($\mu \neq \nu$), we apply a sudden quench to $\mathbf{q} + \delta \mathbf{e}_\mu + \delta \mathbf{e}_\nu$ along the $e_\mu + e_\nu$ direction and then measure the probability $P^+_{\mu\vee}$, which has the relation $P^+_{\mu\nu} - P^+_{\mu\vee} - P^+_{\nu\vee} = 2g_{\mu\nu} \delta \lambda^2 + \mathcal{O}(\delta \lambda^3)$. This sudden quench scheme will be used to measure the quantum metric $g_{\mu\nu}$ in Eq. (2) for quantum-simulated tensor monopoles with a superconducting qudit.

**Experimental system.** We realize a highly tunable spin-1 Hamiltonian with superconducting quantum circuits and observe the energy spectrum and topological charge of the tensor monopole in parameter space. The circuits consist of a superconducting transmon qubit embedded in a 3D aluminum (Al) cavity [38, 39, 45–48]. The resonance frequency of the cavity TE101 mode is 9.0526 GHz. The whole sample package is cooled in a dilution refrigerator to a base temperature of 20 mK. The experimental setup for the qubit control and measurement is well established [38, 39, 45–48]. The coupled transmon qubit and cavity exhibit anharmonic multiple energy levels. In our experiments, the lowest four energy levels labeled $|0\rangle$, $|1\rangle$, $|2\rangle$ and $|3\rangle$ are used, as shown in Fig. 2(a). These levels actually form a qudit system. Microwave fields are applied to couple each energy level. The transition frequencies between them are $\omega_{10}/2\pi = 7.1194$ GHz, $\omega_{12}/2\pi = 6.7747$ GHz and $\omega_{23}/2\pi = 6.3926$ GHz respectively, which are independently determined by saturation spectroscopy [44]. We apply microwave driving along $x$, $y$, and $z$ directions and realize the following effective Hamiltonian in the rotating frame ($\hbar = 1$)

$$H_{\text{exp}} = \frac{1}{2} \begin{pmatrix}
0 & \Omega_x^1 - i\Omega_y^1 & 0 \\
\Omega_x^1 + i\Omega_y^1 & 0 & \Omega_y^2 \\
0 & \Omega_y^2 & 0
\end{pmatrix},$$

(4)

where $\Omega_x^{1(2)}$ ($\Omega_y^{1(2)}$) is the Rabi frequency along the $x$ ($y$) axis of the Bloch sphere spanned by corresponding basis. In our experiments, we simultaneously tune the frequency, amplitude, and phase of the microwaves to create arbitrary three-level Hamiltonians. First, we accurately design microwave fields with calibration of the parameters (using Rabi oscillations and Ramsey fringes). Subsequently, we realize Hamiltonian (1) by mapping it to the parameter space of a superconducting qudit.

**Measuring the energy structure of 4D Weyl-like model.** We obtain the energy structure by measuring the spectrum of the qudit system. By mapping momentum space $\mathbf{k}$ to parameter space $\mathbf{q}$, we can design the Rabi frequencies $\{\Omega_x^1, \Omega_y^1, \Omega_x^2, \Omega_y^2\} = \{\Omega_0(3 + \Lambda - \cos k_x - \cos k_y - \cos k_z - \cos k_w, \Omega_0 \sin k_y, \Omega_0 \sin k_z, \Omega_0 \sin k_w\}$, where the energy unit $\Omega_0$ is set as 5 MHz and the parameter $\Lambda$ can be tuned for topological phase transition. The energy levels used are shown in Fig. 2(a), where $\{1, 2, 3\}$ coupled by applied microwaves are used to construct $H_{\text{exp}}$ and $|0\rangle$ is treated as a reference level for spectrum probing. The dressed states of the qudit under the coupled microwaves are eigenstates of the Hamiltonian (4) labelled $|\psi_0\rangle$ and $|\psi_\pm\rangle$.

In our routine, we execute the spectrum-like measurement and the resonant peaks of microwave absorption are detected [44]. The frequency of the resonant peak is a function of $k_{x,y,z,w}$, and we are able to extract the energy structure of the 4D Weyl-like cone, as illustrated in the right panel of Fig. 2(a). To demonstrate the topological properties, we set $k_y = k_w = k_z = 0$ to emphasize the $E$-$k_y$ plane, where the phase transition can be clearly observed. The system has two different phases determined by the parameter $\Lambda$, as shown in Fig. 2(b): the 4D Weyl-like semimetal with a pair of 4D Weyl points in the bands when $|\Lambda| < 1$ and the trivial gapped insulator when $|\Lambda| > 1$ [11, 12]. At the critical point $|\Lambda| = 1$, the two gapless points merge and then disappear at the band center, indicating a phase transition. The extracted energy structures for $\Lambda = 0, 1, 2$ are illustrated in Fig. 2(b), which capture the features of the theoretical prediction with two gapless points located at $k_x = \pm \pi/2$ ($k_{y,z,w} = 0$) when $\Lambda = 0$, giving the tensor monopoles.
Measuring quantum metric by sudden quench.—We now measure the quantum metric \( g_{\mu \nu} \) (\( \mu, \nu = (\theta_1, \theta_2, \phi) \)) of the simulated tensor monopole using the sudden quench scheme. To do this, we reconstruct the Hamiltonian in hyper-sphere coordinates, and the parameters in Eq. (4) become \( \Omega_{1} = \Omega_{0} \cos \theta_{1}, \Omega_{y} = \Omega_{0} \sin \theta_{1} \cos \theta_{2}, \Omega_{z} = \Omega_{0} \sin \theta_{1} \sin \theta_{2} \cos \phi, \Omega_{x} = \Omega_{0} \sin \theta_{1} \sin \theta_{2} \sin \phi \). The system is initially prepared in the ground state \( |\psi_{-}\rangle \) in the parameter space \( q = \{\theta_{1}, \theta_{2}, \phi\} \) with \( \phi = 0 \). The Hamiltonian is then rapidly swept to \( H(q + \delta q) \), followed by state tomography to obtain the transition probability. We set the quench parameter to \( q(t) = q + t/T \delta q \) along the \( e \) direction, where quench time \( T = 9 \) ns and \( \delta q = \pi/8 \) or \( \pi/16 \) [44]. For the diagonal term \( g_{\mu \mu} \), only one parameter ramps linearly in each quench with \( e = \{e_{\theta_1}, e_{\theta_2}, e_{\phi}\} \), respectively. For the off-diagonal term \( g_{\mu \nu} (\mu \neq \nu) \), the parameters \( \mu \) and \( \nu \) ramp simultaneously, with \( e = \{e_{\theta_1} + e_{\theta_2}, e_{\theta_1} + e_{\phi}, e_{\theta_2} + e_{\phi}\} \). These ramp procedures are illustrated in Fig. 1(b). From the final state’s tomography, we extract the quantum metric at \( q \) from the measured transition probability: \( g_{\mu \nu} \approx P_{\mu \nu}/\delta q^{2} \) and \( g_{\mu \nu} \approx (P_{\mu \nu} - P_{\mu \mu} - P_{\nu \nu})/2\delta q^{2} \). The measured quantum metric \( g_{\mu \nu} \) as a function of \( \theta_{1} \) and \( \theta_{2} \) for the ground state \( |\psi_{-}\rangle \) are shown in Figs. 3(a) (diagonal) and 3(b) (off-diagonal), which agree well with theoretical results [see Eqs. (10) in SM [44]].

\(|\psi_{-}\rangle \) is usually a superposition state of \( |0\rangle, |1\rangle \) and \( |2\rangle \), which is the function of \( \theta_{1} \) and \( \theta_{2} \). State preparation and tomography bring extra errors to practical 3-level experiments. To simplify state initialization and increase measurement fidelity, we rotate the frame axis with \( U_{R} \) in the experiment to maintain the eigenstate of the initial Hamiltonian at the energy level \( |0\rangle \) [44]. The extra benefit of this process is dramatic reduction of effect from system decoherence. Consequently, the ramping Hamiltonian transforms to \( H(q) = U_{R} H(q) U_{R}^{\dagger} \). To increase the accuracy of our experimental data, we repeat the measurements 16000 times and obtain the density matrix of the qudit using the least square method.

**Observing topological phase transitions.**—To further study the tensor monopole, we observe topological phase transition characterized by the tensor monopole charge in our superconducting circuits. By designing microwave fields on the qudit, we modify Eq. (4) by adding a tunable offset \( \Lambda \) into the \( \Omega_{1} \) term, such that \( \Omega_{1} = \Omega_{0} (\cos \theta_{1} + \Lambda), \) while other terms remain unchanged (without breaking the \( \phi \)-rotation symmetry). By measuring the metric tensor with the sudden-quench approach, we can obtain the generalized curvature \( H_{\theta_{1}, \theta_{2}, \phi} \) and then integrate it to derive the topological charge \( Q_{DD} \). For offset \( \Lambda = 0 \), the extracted \( H_{\theta_{1}, \theta_{2}, \phi} \) as a function of parameters \( \theta_{1} \) and \( \theta_{2} \) is shown in Fig. 4(a). Experimental data (left) agree with theoretical results (right). Consequently, we can calculate the \( Q_{DD} \) using Eq. (3) and obtain \( Q_{DD} = 0.92 \pm 0.15 \) for \( \Lambda = 0 \).

To study the topological phase transition, we execute the protocol with varying \( \Lambda \). The extracted DD invariant as a function of \( \Lambda \) is shown in Fig. 4(b). When \( |\Lambda| = 0 \), the manifold of the parameter space \( S^{3} \) surrounds the tensor monopole in the center. With the in-
crease of $|\Lambda|$, the tensor monopole moves along the $q_x$ axis. $Q_{DD} \approx 1$ when $|\Lambda| < 1$ for the $S^3$ sphere surrounding the tensor monopole. $Q_{DD} \approx 0$ when $|\Lambda| > 1$ since the monopole moves outside the hyper-sphere manifold, indicating that the system has transformed to the trivial insulator phase. $Q_{DD}$ declines rapidly to around 0 in the vicinity of $\Lambda = \pm 1$, which indicates a topological phase transition. The accuracy of the topological charge of tensor monopoles extracting from the sudden quench routine depends on the ramp step. In Fig. 4(b), the numerical simulation results with $\delta q = \pi/1024$ are plotted, which are very close to the expected integer values. However, such a small step is not feasible to implement in practice due to limitation of readout fidelity. With a larger $\delta q$, measurement obtained from the sudden quench routine will deviate from ideal values. For comparison, we perform the routine with $\delta q = \pi/8$ and $\pi/16$, as demonstrated in Fig. 4(b). When $\delta q$ decreases, the deviation from the ideal quantized values becomes smaller.

**Conclusion.** In summary, we have created tensor monopoles in 4D parameter space and explored their unique properties using superconducting circuits. Our experimental observation contributes to exploring tensor gauge fields in quantum mechanics and creates a unique approach in the search for exotic topological matter in condensed matter physics and artificial systems.

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Supplemental Materials

Quantum metric and topological charge of a tensor monopole

For a generic Hamiltonian \( H(q) \) in parameter space \( q = (q_1, q_2, \cdots, q_N) \in \mathcal{M} \), one has the eigen-energies \( E_n(q) \) and eigen-states \( |u_n(q)\rangle \) at each point of the manifold of the quantum state \( \mathcal{M} \). In the absence of energy degeneracies, the quantum geometric tensor associated with \( |u_\lambda\rangle \) is defined as \[18, 20\]

\[
\chi_{\mu \nu} = \langle \partial_{q_\mu} u_q | \partial_{q_\nu} u_q \rangle - \langle \partial_{q_\mu} u_q | u_q \rangle \langle u_q | \partial_{q_\nu} u_q \rangle = \langle \partial_{\lambda_\mu} u_q (1 - |u_q\rangle \langle u_q|) \partial_{\lambda_\nu} u_q \rangle. \tag{5}
\]

A generalized quantum geometric tensor can be defined for degenerate cases \[21, 22\]. The real part of this geometric tensor is symmetry and defines the quantum metric \( g_{\mu \nu} = \Re[\chi_{\mu \nu}] = g_{\mu \nu} \), which is the so-called Fubini-Study metric on the projective Hilbert space \( \mathcal{P}H(\lambda) = H(\lambda)/U(1) \), required by the principle of gauge invariance \[? \]. The imaginary part is related to the well-known anti-symmetry Berry curvature \( F_{\mu \nu} = -2\Im[\chi_{\mu \nu}] = -F_{\nu \mu} \). The quantum metric \( g_{\mu \nu} \) measures the quantum distance between nearby states \( |u_q\rangle \) and \( |u_q+\delta q\rangle \) as

\[
ds^2 = P^+ = 1 - |\langle u_q|u_{q+\delta q}\rangle|^2 = \sum_{\mu \nu} \chi_{\mu \nu} d\lambda_\mu d\lambda_\nu + \mathcal{O}(|\delta q|^3) = \sum_{\mu \nu} g_{\mu \nu} d\lambda_\mu d\lambda_\nu + \mathcal{O}(|\delta q|^3), \tag{6}
\]

where \( ds^2 \) is determined by the wave-function overlap (a maximal overlap of 1 corresponds to the zero distance \( ds^2 = 0 \), while the orthogonal states correspond to the maximal distance \( ds^2 = 1 \), and \( P^+ \) is the probability to excite the system to other eigenstates after a quantum quench where the parameter suddenly changes from \( q \) to \( q + \delta q \) \[25\].

For the three-band Weyl-like Hamiltonian in the 4D parameter space \[11\]:

\[
H_{4D} = \begin{bmatrix}
0 & q_x - i q_y & 0 \\
q_x + i q_y & 0 & q_z + i q_w \\
0 & q_z - i q_w & 0
\end{bmatrix}, \tag{7}
\]

the energy spectrum is given by \( E_{0,\pm} = 0, \pm |q| \) for three eigenstates \( |\psi_0\rangle \) and \( |\psi_\pm\rangle \), respectively. At \( q = (0, 0, 0, 0) \) in 4D parameter space, a triple-degenerate Weyl-like point acts as a tensor monopole, which is surrounded by a 3D hypersphere \( S^3 \). In terms of hyperspherical coordinates \( \{r, \theta_1, \theta_2, \phi\} \) \( (\theta_{1,2} \in [0, \pi] \text{ and } \phi \in [0, 2\pi]) \), one has

\[
q_x = r \cos \theta_1, \\
q_y = r \sin \theta_1 \cos \theta_2, \\
q_z = r \sin \theta_1 \sin \theta_2, \\
q_w = r \sin \phi, \tag{8}
\]

in Eq. (1). We consider the ground state \( |\psi_-\rangle = (\cos \theta_1 - i \cos \theta_2 \sin \theta_1, -1, \sin \theta_1 \sin \theta_2 e^{-i \phi})^T \) with \( T \) denoting the transposition of matrix, the \( 3 \times 3 \) quantum metric tensor \( g \) in the \( S^3 \) is given by

\[
g = \begin{bmatrix}
g_{\theta_1, \theta_1} & g_{\theta_1, \theta_2} & g_{\theta_1, \phi} \\
g_{\theta_2, \theta_1} & g_{\theta_2, \theta_2} & g_{\theta_2, \phi} \\
g_{\phi, \theta_1} & g_{\phi, \theta_2} & g_{\phi, \phi}
\end{bmatrix}, \tag{9}
\]

where the three diagonal components and six off-diagonal components are derived as

\[
g_{\theta_1, \theta_1} = \frac{1}{8}(3 - \cos 2 \theta_1), \\
g_{\theta_2, \theta_2} = \frac{1}{4} \sin^2 \theta_1 [2 \cos^2 \theta_2 - (\cos^2 \theta_1 - 2) \sin^2 \theta_2], \\
g_{\phi, \phi} = -\frac{1}{4} \sin^2 \theta_1 \sin^2 \theta_2 (\sin^2 \theta_1 \sin^2 \theta_2 - 2), \\
g_{\theta_1, \theta_2} = g_{\theta_2, \theta_1} = \frac{1}{4} \cos \theta_1 \cos \theta_2 \sin \theta_1 \sin \theta_2, \\
g_{\theta_1, \phi} = g_{\phi, \theta_1} = -\frac{1}{4} (\cos \theta_2 \sin^2 \theta_1 \sin^2 \theta_2), \\
g_{\theta_2, \phi} = g_{\phi, \theta_2} = \frac{1}{4} \cos \theta_1 \sin^2 \theta_1 \sin^3 \theta_2. \tag{10}
\]
It has been shown that the quantum metric is related to the generalized curvature tensor $H_{\theta_1 \theta_2 \phi}$ as the field strength of the tensor monopole in $S^3$ [11, 12]:

$$H_{\theta_1 \theta_2 \phi} = e_{\theta_1 \theta_2 \phi} (4 \sqrt{\det g_{\mu \nu}}), \quad \mu, \nu = \{\theta_1, \theta_2, \phi\}. \tag{11}$$

Here $H_{\theta_1 \theta_2 \phi}$ is independent on $\phi$ as $H_{4D}$ has $\phi$-rotation symmetry. The tensor monopole has a topological charge (the DD invariant) associated with the curvature $H_{\theta_1 \theta_2 \phi}$:

$$Q_{DD} = \frac{1}{2\pi^2} \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \int_0^{2\pi} d\phi H_{\theta_1 \theta_2 \phi} = \frac{1}{2\pi^2} \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \int_0^{2\pi} d\phi \sin^2 \theta_1 \sin \theta_2 = 1. \tag{12}$$

Thus, by revealing the quantum metric $g_{\mu \nu}$ through sudden quench scheme, we can measure $H_{\theta_1 \theta_2 \phi}$ and then obtain the topological charge $Q_{DD}$ of the tensor monopole. See the main text for the detailed procedures of ramping parameters. The obtained numerical results of $Q_{DD}$ as a function of an additional offset $\Lambda$ for varying ramp step $\delta q$ are shown in Fig. 5. When $\delta q$ decreases, the deviation from the ideal quantized values ($Q_{DD} = 0$ when $|\Lambda| > 1$ and $Q_{DD} = 1$ when $|\Lambda| < 1$) becomes smaller. For $\delta q = \pi/256$, the obtained $Q_{DD}$ is very close to the integer values.

![Topological Charge vs Lambda](image)

**FIG. 5:** Simulation results for the topological charge $Q_{DD}$ as a function of offset $\Lambda$ for different $\delta q$.

**Experimental setup and qubit calibration**

The sample used in our experiments is a 3D transmon, which consists of a superconducting qubit embedded in a 3D aluminium cavity [45]. The contribution of the cavity to our experiments is providing a convenient method to manipulate and measure qubit. We employed an experimental setup for manipulating and measuring of the 3D transmon. Basically, there are two SMA connectors on the 3D cavity. One is for microwave input and one for output. The input (output) quality factor is adjusted to be about $5 \times 10^5$ ($2 \times 10^8$). Microwave pulses for manipulating and reading out qubit are sent in through input connector after appropriate attenuation and isolation. A microwave generator combined with an inphase and quadrature (IQ) mixer can produce microwave pulses for qubit manipulating. By adjusting the voltage of the IQ mixer we can control the phase (i.e. X and Y components) of microwave. To read out qubit states, we use ordinary microwave heterodyne setup. The output microwave is pre-amplified by HEMT at 4 K stage in the dilution refrigerator and further amplified by two low noise amplifiers at room temperature. The microwave is then tuned into 50 MHz and collected by ADCs. In order to simplify our experiments procedures and data analysis while maintaining sufficient signal-to-noise ratio, we choose “high power readout” scheme [49]. Simply speaking, we send in a strong microwave on-resonance with the cavity, the transmitted amplitude of the microwave will reflect the state of qubit due to the non-linearity of the cavity QED system.

We first use saturation spectroscopies to determine the transmon parameters. The resonant peaks indicate that the transition frequencies between $|i\rangle$ to $|j\rangle$ is $\omega_{01}/2\pi = 7.1194$ GHz, $\omega_{12}/2\pi = 6.7747$ GHz, and $\omega_{23}/2\pi = 6.3926$ GHz. From these we obtain the Josephson energy $E_J/h \sim 2\pi \times 20.71$ GHz and the charge energy $E_C/h \sim 2\pi \times 0.341$ GHz. The bare resonant frequency of the cavity is 9.0526 GHz. We measure the energy relaxation times of energy level $|1\rangle$, $|2\rangle$, and $|3\rangle$ using pump and decay method. It is found that $T_{11}^{01} \sim 15 \mu s$, $T_{12}^{12} \sim 12 \mu s$, and $T_{23}^{23} \sim 10 \mu s$, respectively. The dephasing times are obtained from Ramsey measurement, which are $T_{21}^{02} \sim 6.0 \mu s$, $T_{21}^{01} \sim 4.5 \mu s$ and $T_{21}^{01} \sim 3.1 \mu s$, respectively.

In the experiments, we have to accurately design the magnitude, frequency and phase of the microwave, which can be controlled by the waveform pulses applied to IQ mixer. We also calibrate the amplitude, phase and offset of the
shown in Fig. 2(a) of the main text and construct the Hamiltonian  

\[ H \]

which is widely used in qubit experiments [38]. First of all, the whole system is initialized in the ground state, which

\[ |\psi_0\rangle \]

and sideband mirror. Tomography results indicate that the performance of our IQ mixer is very good.

Then we turn on the probe microwave. The widths of the construct and probe microwaves are 200 µs. By adjusting these parameters carefully, we can significantly suppress the leakage of the LO signal and sideband mirror. As described in the text, we only measure populations of the dressed states \(|\psi_0\rangle\) and \(|\psi_\pm\rangle\) distributed at the bare state \(|0\rangle\).

In our experiments, \(|1\rangle\), \(|2\rangle\) and \(|3\rangle\) of transmon form an artificial spin-1 particle. The corresponding energy levels of the Hamiltonian are obtained by measuring the eigenenergies of the Hamiltonian. Measuring eigenenergies of the microwave driven three-level system is similar to that of the spectroscopy measurement with saturation microwave, which is widely used in qubit experiments [38]. First of all, the whole system is initialized in the ground state, which is \(|0\rangle\). The microwaves with frequencies \(\omega_{12}\) and \(\omega_{23}\) are applied to generate the transitions \(\Omega_1\), \(\Omega_2\), \(\Omega_3\) and \(\Omega_4\) as shown in Fig. 2(a) of the main text and construct the Hamiltonian \(H_{exp}\). Thus, the driven qutrit forms dressed states \(|\psi_0\rangle\) and \(|\psi_\pm\rangle\), which can be written as

\[
|\psi_+\rangle = \frac{1}{\sqrt{2}} \left( (\Omega_x^2 + i\Omega_y^2)/\Omega \right) |\psi_0\rangle, \quad |\psi_-\rangle = \frac{1}{\sqrt{2}} \left( (\Omega_x^2 + i\Omega_y^2)/\Omega \right) |\psi_0\rangle
\]

in the basis \(|1\rangle, |2\rangle, |3\rangle\), where \(\Omega = \sqrt{(\Omega_x^2 + (\Omega_y)^2)^2 + (\Omega_3^2 + \Omega_4^2)^2 + (\Omega_2^2 + \Omega_5^2)^2}\). The corresponding eigenenergies are \(E_0 = \omega_{01}, E_+ = \omega_{01} + \Omega/2\), and \(E_- = \omega_{01} - \Omega/2\), respectively. Fig. 6(a) is the example of selected amplitude of microwave. Then we turn on the probe microwave. The widths of the construct and probe microwaves are 200 µs and 100 µs, which are much longer than the decoherence time of our transmon. We sweep the frequency of probe microwave. When the probe frequency matches the energy difference between an eigenstate and \(|0\rangle\), the system will be excited to the corresponding eigenstate. After turn off the construct and probe microwaves, a readout microwave pulse is sent to the cavity to measure the states of the system. Resonant peaks with frequencies representing the eigenenergies have been observed as shown in Fig. 6(b). Positions of resonant peaks indicate values of eigenenergies, while heights of resonant peaks reflect each components of eigenstates at \(|1\rangle\). Then we change the parameters \(k_x\) to collect the spectrum with different resonant peaks. In the Fig. 2 of the main text, we shifted the energy zero point to \(\omega_{01}\).

We here discuss the spectral brightness distribution. In our experiments, the spectra we have measured actually reflect the populations of the dressed states \(|\psi_0\rangle, |\psi_\pm\rangle\) at the bare state \(|0\rangle\). The distributions of the dressed states at the bare states \(|2\rangle\) and \(|3\rangle\) are not necessary to measure, and furthermore they are more complicated to measure since the two/three photon procedure is included. From Eq. (??) we know that the brightness should be proportional to

\[
P_\pm = |\langle 1 |\psi_\pm\rangle|^2 = ((\Omega_x^2)^2 + (\Omega_y^2)^2)/2\Omega^2, \quad P_0 = |\langle 1 |\psi_0\rangle|^2 = ((\Omega_x^2)^2 + (\Omega_y^2)^2)/\Omega^2.
\]

Therefore, the ratio of the brightness are given by

\[
P_+/P_0 = P_-/P_0 = ((\Omega_x^2)^2 + (\Omega_y^2)^2)/2((\Omega_x^2)^2 + (\Omega_y^2)^2).
\]

Energy structure measurement in a multi-level system

![Fig. 6: (a) The waveform of construct microwave sending to the transmon for the typical parameters \(k_x = \pi/3\) and \(k_z \approx 0\). The blue (black) pulse corresponds the \(\Omega_1\) (\(\Omega_2\)) of driving microwave. (b) An example of the spectrum of a fixed \(k_x\) and \(k_z\). The driving of construct microwave transforms the bare states (|1\rangle, |2\rangle, and |3\rangle) to the eigen-states of driven system (i.e., dressed states) (|\psi_0\rangle, |\psi_+\rangle, and |\psi_-\rangle). By sweeping the frequency of the probe microwave (horizontal axis), we can observe resonant peaks at frequencies corresponding eigenenergies of dressed states, from which the energy structure can be extracted. As described in the text, we only measure populations of the dressed states |\psi_0\rangle and |\psi_\pm\rangle distributed at the bare state |0\rangle.]

In our experiments, \(|1\rangle\), \(|2\rangle\) and \(|3\rangle\) of transmon form an artificial spin-1 particle. The corresponding energy levels of the Hamiltonian are obtained by measuring the eigenenergies of the Hamiltonian. Measuring eigenenergies of the microwave driven three-level system is similar to that of the spectroscopy measurement with saturation microwave, which is widely used in qubit experiments [38]. First of all, the whole system is initialized in the ground state, which is \(|0\rangle\). The microwaves with frequencies \(\omega_{12}\) and \(\omega_{23}\) are applied to generate the transitions \(\Omega_1\), \(\Omega_2\), \(\Omega_3\) and \(\Omega_4\) as shown in Fig. 2(a) of the main text and construct the Hamiltonian \(H_{exp}\). Thus, the driven qutrit forms dressed states \(|\psi_0\rangle\) and \(|\psi_\pm\rangle\), which can be written as

\[
|\psi_+\rangle = \frac{1}{\sqrt{2}} \left( (\Omega_x^2 + i\Omega_y^2)/\Omega \right) |\psi_0\rangle, \quad |\psi_-\rangle = \frac{1}{\sqrt{2}} \left( (\Omega_x^2 + i\Omega_y^2)/\Omega \right) |\psi_0\rangle
\]

in the basis \(|1\rangle, |2\rangle, |3\rangle\), where \(\Omega = \sqrt{(\Omega_x^2 + (\Omega_y)^2)^2 + (\Omega_3^2 + \Omega_4^2)^2 + (\Omega_2^2 + \Omega_5^2)^2}\). The corresponding eigenenergies are \(E_0 = \omega_{01}, E_+ = \omega_{01} + \Omega/2\), and \(E_- = \omega_{01} - \Omega/2\), respectively. Fig. 6(a) is the example of selected amplitude of microwave. Then we turn on the probe microwave. The widths of the construct and probe microwaves are 200 µs and 100 µs, which are much longer than the decoherence time of our transmon. We sweep the frequency of probe microwave. When the probe frequency matches the energy difference between an eigenstate and \(|0\rangle\), the system will be excited to the corresponding eigenstate. After turn off the construct and probe microwaves, a readout microwave pulse is sent to the cavity to measure the states of the system. Resonant peaks with frequencies representing the eigenenergies have been observed as shown in Fig. 6(b). Positions of resonant peaks indicate values of eigenenergies, while heights of resonant peaks reflect each components of eigenstates at \(|1\rangle\). Then we change the parameters \(k_x\) to collect the spectrum with different resonant peaks. In the Fig. 2 of the main text, we shifted the energy zero point to \(\omega_{01}\).

We here discuss the spectral brightness distribution. In our experiments, the spectra we have measured actually reflect the populations of the dressed states \(|\psi_0\rangle, |\psi_\pm\rangle\) at the bare state \(|0\rangle\). The distributions of the dressed states at the bare states \(|2\rangle\) and \(|3\rangle\) are not necessary to measure, and furthermore they are more complicated to measure since the two/three photon procedure is included. From Eq. (??) we know that the brightness should be proportional to

\[
P_\pm = |\langle 1 |\psi_\pm\rangle|^2 = ((\Omega_x^2)^2 + (\Omega_y^2)^2)/2\Omega^2, \quad P_0 = |\langle 1 |\psi_0\rangle|^2 = ((\Omega_x^2)^2 + (\Omega_y^2)^2)/\Omega^2.
\]

Therefore, the ratio of the brightness are given by

\[
P_+/P_0 = P_-/P_0 = ((\Omega_x^2)^2 + (\Omega_y^2)^2)/2((\Omega_x^2)^2 + (\Omega_y^2)^2).
\]
To ensure initial Hamiltonian system, which is composed by an octant of a unit sphere and a torus. Rotation in Hilbert space of qutrit system, which is composed by an octant of a unit sphere and a torus\cite{50, 51}. Rotation $U_R$ of frame axis is applied in experiment to ensure initial Hamiltonian $H_0'$ lying along $\lambda_3$ axis.

Measurement of quantum metric using sudden quench

Without loss of generality, we implement a rotation $U_R$ on frame axis to ensure the initial state in every quench remains at $|0\rangle$ \cite{25}, as shown in Fig. 7. The Hamiltonian in each quench is modified to $H_0 = U_R H_0 U_R^\dagger$. Procedure of state initialization can be skipped in experiment, which results in an increase of measurement fidelity in practice. In practice, the $U_R$ can be decomposed as $U_R = \tilde{R}(\beta_1)_{02} \tilde{R}(\alpha_1)_{03} \tilde{R}(\alpha_2)_{01}$, where operator $\tilde{R}$ denotes rotation along axis $\hat{n}$ in the Bloch sphere spanned by the basis $\{|i\rangle, |j\rangle\}$. For an initial state $\frac{1}{\sqrt{2}}[\cos \theta_1 - i \sin \theta_1 \cos \theta_2, -1, \sin \theta_1 \sin \theta_2]^T$, tan $\alpha_1 = -\sin \theta_1 \sin \theta_2/|\cos \theta_1 - i \sin \theta_1 \cos \theta_2|$, tan $\alpha_2 = -(\sin \theta_1 \sin \theta_2)^2 - |\cos \theta_1 - i \sin \theta_1 \cos \theta_2|^2$, tan $\beta_1 = \sin \theta_1 \cos \theta_2/\cos \theta_1$ and $\beta_2 = -\beta_1/2$. If we write $H_0$ in the form as

$$H_0 = \frac{1}{2} \begin{pmatrix} 0 & \Omega^1(t)e^{i\phi(t)} & 0 \\ \Omega^1(t)e^{-i\phi(t)} & 0 & \Omega^2(t) \\ 0 & \Omega^2(t) & 0 \end{pmatrix},$$

then we can obtain the modified Hamiltonian

$$H_0' = \frac{1}{2} \begin{pmatrix} (A e^{2i\beta_2} + A e^{-2i\beta_2}) \sin \alpha_2 \cos \alpha_2 & -A e^{2i\beta_2} \sin^2 \alpha_2 + A e^{-2i\beta_2} \cos^2 \alpha_2 & -B e^{i\beta_2} \sin \alpha_2 \\ -A e^{2i\beta_2} \cos^2 \alpha_2 - A e^{-2i\beta_2} \sin^2 \alpha_2 & -(A e^{2i\beta_2} + A e^{-2i\beta_2}) \sin \alpha_2 \cos \alpha_2 & B e^{i\beta_2} \cos \alpha_2 \\ -B e^{-i\beta_2} \sin \alpha_2 & B e^{-i\beta_2} \cos \alpha_2 & 0 \end{pmatrix},$$

where $A = \Omega^1(t) \cos \alpha_1 e^{-i(\phi(t) - \beta_1)} - \Omega^2(t) \sin \alpha_1 e^{-i\beta_1}$ and $B = \Omega^1(t) \sin \alpha_1 e^{i(\phi(t) - \beta_1)} + \Omega^2(t) \cos \alpha_1 e^{i\beta_1}$. For the Hamiltonian $H_{exp}$ with offset term $\Lambda$, we execute the same rotate procedure with modified parameters $\alpha_1(\Lambda), \alpha_2(\Lambda), \beta_1(\Lambda)$ and $\beta_2(\Lambda)$.

To measure the probability of excited states, we have to perform quantum tomography of qutrit, which is realized by measuring the density matrix $\rho$. Reconstruction of the full density matrix needs to do a set of rotations, $I$, $(\pm \frac{\pi}{2})_x \otimes I$, $(\pm \frac{\pi}{2})_z \otimes I$, $(\pm \pi)_x \otimes I$, $(\pm \pi)_z \otimes I$, $(\pm \pi)_y \otimes I$, $(\pi)_x \otimes I$, $(\pi)_y \otimes I$, and $(\pm \pi)_{x,y,z}$, where $I$ denotes identical operation and $\theta^\phi_{ij}$ denotes rotation along axis $a$ with angle $\theta$ on $ij$ transition, which contains more procedure than qubit tomography. After measuring the density matrix $\rho$, we calculated the probability of eigen-states using

$$P = \langle \psi_- | \rho | \psi_- \rangle,$$

where $| \psi_- \rangle$ is the ground state of the driving Hamiltonian.