Incomplete neighbourhood multi-granulation decision-theoretic rough set in the hybrid-valued decision system

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Abstract: It is an important subject to mine valuable knowledge from complex and massive data in the era of big data. Rough set theory is a new mathematical tool for dealing with uncertain and inaccurate data, decision-theoretic rough set model (DTRS), as an extension of classical rough set model, is used to analyze decision information systems and multi-granulation decision-theoretic rough set model (MG-DTRS) can analyze and process target concepts from different angles and levels. However, the classical DTRS model exists some limitations in dealing numerical or hybrid-valued data. Considering the different influence of numerical features and symbolic features on decision-making, the paper proposes an incomplete neighborhood multi-granulation decision-theoretic rough set model in hybrid-valued decision system through integrating MG-DTRS with neighbourhood rough sets, and two types of neighborhood multi-granulation decision-theoretic set models are emphatically analysed. Furthermore, taking pessimistic and optimistic neighborhood multi-granulation decision-theoretic rough sets as examples, the implementation algorithms and related properties of the two type of models are studied. Finally, the relationship between the proposed model and other models is analyzed through formula derivation. The model proposed in this paper can effectively solve the decision-making problem of hybrid-valued incomplete information system through multi-angle and multi-level analysis.

1 Introduction

With the rapid development of information technology and data science, data in various fields show the characteristics of enormity, complexity, uncertainty and diversity. How to extract valuable knowledge from these complex data is an important issue in the era of big data. Rough set theory [1, 2] is a new mathematical tool proposed by Pawlak to effectively deal with uncertain and imprecise data, and has been widely used in data mining, knowledge discovery and artificial intelligence.

Decision-theoretic rough set (DTRS) model [3, 4], as a kind of extended probabilistic rough set model, is proposed to analyze decision information systems from the perspective of describing the intersection degree of concepts by conditional probability and risk-cost sensitivity, which is an important part of the rough set theory and has attracted much attention in recent years. DTRS and their extended models have been proposed to meet various requirements [3–9], Yao and Zhao [9] analysed the threshold solution method in detail according to Bayesian minimum risk decision theory, the authors of [5–7] proposed the multi-cost DTRSs from the perspective of cost loss, and attribute reduction methods of DTRS were studied in [9–12]. However, DTRS models mentioned above were only built on the basis of a single granular structure. In the analysis of practical decision-making concepts, it is often necessary to analyse and deal with the concept of objectives from multi-granulation and multi-level. Qian et al. [13] introduced the concept of multiple granulations into rough set model firstly, subsequently, a series of extended models and related algorithms of multi-granulation rough set were presented one after another [14–17], and the research of multi-granulation DTRS (MG-DTRS) models [18–22] has been paid more and more attention since Qian et al. put forward the basic framework of MG-DTRS model in 2014. However, the above research results are all based on symbolic decision information system, and there exist some limitations in dealing decision systems with numerical or hybrid-valued data, which is also another limitation of Pawlak's rough set model. For example, the decision information table is shown in Table 1, which consists of numerical attributes and symbolic attributes, where (a, b, c) are numerical attributes and (d, e) are symbolic attributes, '*'denotes null value, we cannot easily make decisions based on equivalence relations or simple tolerance relations. Some scholars studied the neighbourhood rough set (NRS) model. Lin [23, 24] put forward the NRS model by replacing the equivalence relation with neighbourhood relation. NRS model uses neighbourhood relation to granulate the universe, and defines the upper and lower approximations by neighbourhood particles, it can directly process decision information systems with numerical or hybrid-valued data. Hu et al. [25] studied neighbourhood classifiers from the perspective of metric space, the authors of [26–31] studied the theory of NRS and its extended models, the authors of [32–36] analysed the neighbourhood multi-granulation rough set model from the perspective of granularity computing, Weiwei et al. [37] proposed a DTRS model based on neighbourhood system. The above research results are sufficient to confirm that NRSs have

| Table 1 | Decision table S |
|---------|-----------------|
| U       | a   | b   | c   | d   | e   | D    |
| x1      | 0.1  | 2.1  | 3.0  | 2   | *   | 1    |
| x2      | 0.1  | 2.2  | *   | 2   | 1   | 1    |
| x3      | 0.2  | 2.2  | 3.2  | *   | 2   | 2    |
| x4      | *    | 2.0  | 3.1  | 1   | 2   | 2    |
| x5      | 0.2  | 2.2  | 3.1  | 2   | 1   | 1    |

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better characteristics in dealing with hybrid-valued information systems, but there is a lack of systematic research on the MG-DTRS model based on neighborhood systems. Furthermore, in the research of MG-DTRS, it is usually assumed that the decision cost function is the same under each granularity structure, but the cost of misclassification between numeric and symbolic granularity is often different in practical applications. That is to say, the decision costs made by C1 and C2 in Table 1 are different. In view of the advantages that multi-granularity computing can analyse problems from multiple angles and levels and that NRSs can better process information systems with hybrid-valued data, on the basis of previous studies, considering the different influence of numerical feature granularity and symbolic feature granularity on decision-making, from the perspective of decision cost, an incomplete neighbourhood MG-DTRS (NGM-DTRS) model in hybrid-valued decision system is proposed in this paper.

The paper is organised as follows. First, the conception of rough set theory, incomplete decision system, decision cost, tolerance relation and the upper and lower approximation sets of DTRS model which construct positive regions, boundary regions and negative regions on the basis of two thresholds (α, β) in incomplete decision system are studied. Second, taking the optimistic MG-DTRS and the pessimistic MG-DTRS model as an example, we analyse MG-DTRS model from the perspective of maximum and minimum probabilities for an object x to be partitioned into a target conception X at various granularities, and based on this model, propose an incomplete NGM-DTRS model in hybrid-valued decision system. Especially, considering the difference of misclassification cost between numeric and symbolic granularity, the first type of NGM-DTSSs is proposed, and the second type of NGM-DTSSs is studied by fusing numeric and symbolic granularity. Third, the idea of solving the positive domain of object x under state set P is given, and the related algorithms are designed. Finally, the relationship between the proposed models and other models is analysed through formula derivation.

2 Preliminary knowledge

In this section, we review some existing works on concepts related to Pawlak’s rough set theory, incomplete tolerance relation, DTRS and MG-DTRS.

2.1 Fundamental theory

Pawlak first proposed a rough set theory based on equivalence relation \{1, 2\}. For given information system \(S = (U, A, V, f)\), where \(U = \{x_1, x_2, ..., x_n\}\) is called the universe of objects, \(A = C \cup D (C \cap D = \emptyset)\) denotes set of attributes, where \(C = \{c_1, c_2, ..., c_m\}\) is a set of condition attributes and \(D\) is a set of decision attributes, \(x_i, V_a\) is a non-empty set of values of \(a \in A\), and the information function \(f: U \times A \rightarrow V\) maps an object from \(U\) to exactly one value in \(V_a\), namely \(f(x, a) \in V_a\), \(\forall x \in A, x \in U\).

For any non-empty subset \(B \subseteq A\), the equivalence relation is defined as \(IND(B) = \{(x, y) \in U \times U \mid \forall a \in B, f(x, a) = f(y, a)\}\). The relation \(IND(B)\) partitions \(U\) into a family of pairwise disjoint equivalence classes by \(U/IND(B) = \{[x]_B \mid x \in U\}\), where \([x]_B = \{y \in U \mid (x, y) \in IND(B)\}\). For \(\forall x \in U\), the lower approximations and the upper approximations of \(X\) with respect to \(B\) are defined as \(\text{ap}_B(X) = \{x \in U \mid [x]_B \subseteq X\}\) and \(\text{ap}_B(X) = \{x \in U \mid [x]_B \cap X \neq \emptyset\}\), respectively.

Then, the positive regions, boundary regions, and negative regions of \(X\) are defined, respectively, by

| \(\alpha\) | \(\lambda_{sp}\) | \(\lambda_{ap}\) | \(\lambda_{ps}\) |
|---|---|---|---|
| \(\alpha_p\) | \(\lambda_{sp}(\alpha_p|X)\) | \(\lambda_{ap}(\alpha_p|X)\) | \(\lambda_{ps}(\alpha_p|X)\) |
| \(\alpha_b\) | \(\lambda_{sp}(\alpha_b|X)\) | \(\lambda_{ap}(\alpha_b|X)\) | \(\lambda_{ps}(\alpha_b|X)\) |
| \(\alpha_c\) | \(\lambda_{sp}(\alpha_c|X)\) | \(\lambda_{ap}(\alpha_c|X)\) | \(\lambda_{ps}(\alpha_c|X)\) |

For the information system given above \(S\), suppose \(\exists c \in C, x_i \in U\), satisfying \(f(x, c) = \ast\), where \(\ast\) denotes a null value, i.e. a missing attribute value, then \(S\) is called incomplete decision information system.

According to Kryszkiewicz’s theory \[2, 38\], if

\[S = (U, C \cup D, V, f)\]

is an incomplete decision information system, for any non-empty subset \(P \subseteq C\), the binary tolerance relation \(T_P\) is defined as

\[T_P = \{(x, y) \in U \times U \mid \forall a \in P, f(x, a) = f(y, a)\} \cup \{(x, y) \in U \mid f(x, a) = \ast \land f(y, a) = \ast\}\] \[(1)\]

In this view, for \(\forall x \in U\), the tolerance class \(T_P(x)\) is defined as \(T_P(x) = \{y \in U \mid (x, y) \in T_P\}\). \(T_P(x)\) describes the maximum indistinguishable subset of the object \(x\) under attribute set \(P\). For any subset \(X \subseteq U\), the lower approximations and the upper approximations of \(X\) with respect to tolerance relation \(T_P\) are defined by

\[\text{POS}_P(X) = \text{ap}_{T_P}(X)\]
\[\text{NEG}_P(X) = U - \text{ap}_{T_P}(X)\]
\[\text{BND}_P(X) = \text{ap}_{T_P}(X) - \text{ap}_{T_P}(\text{POS}_P(X))\] \[(1)\]

2.2 Decision-theoretic rough set

For given incomplete information system \(S = (U, C \cup D, V, f)\), suppose \(\Omega = \{\theta_1, \theta_2, ..., \theta_n\}\) denotes \(m\) state sets for an object \(x\), \(B = \{b_1, b_2, ..., b_l\}\) denotes the \(l\) decision sets, for any \(\forall x \in U\), if \(x\) belongs to a certain state, a different action decision will result in different loss. Order \(P[\theta_i]\) represents the conditional probability of object \(x\) under the state \(\theta_i\). \(\lambda_{b_i}(x)\) denotes the risk loss of object \(x\), taking decision \(b_i\) under state \(\theta_i\), according to Bayesian decision procedure, the expected risk loss of object \(x\) taking decision \(b_i\) can be expressed as

\[R_{b_i}(x) = \sum_{j=1}^{m} \lambda(b_i|x) \cdot P(\theta_j|x)\] \[(3)\]

Suppose \(\Omega = \{X, X^\prime\}\) is the state set for an object \(x\), \(B = \{\alpha_p, \alpha_b, \alpha_c\}\) denotes the decision action corresponding to each state, \(\lambda_{sp}, \lambda_{sb}, \lambda_{sr}\) denote the loss function that objects \(x \in \text{POS}(x), x \in \text{NEG}(x, \theta_0)(X)\) and \(x \in \text{BND}(x, \theta_0)(X)\), respectively, when \(x \in X\), \(\lambda_{sp}, \lambda_{sb}, \lambda_{sr}\) denote the loss function that objects \(x \in \text{POS}(x, \theta_0)(X), x \in \text{NEG}(x, \theta_0)(X)\) and \(x \in \text{BND}(x, \theta_0)(X)\), respectively, when \(x \in X\), as shown in Table 2.

For \(x \in U, P \subseteq C\), \(T_P(x)\) denotes the tolerance class of the object \(x\) with respect to \(P\). According to formula (3) and Table 2, the expected loss caused by an object \(x\) adopting three kinds of decisions can be expressed as follows:

| \(\text{POS}_P(X)\) | \(\text{ap}_{T_P}(X)\) | \(\text{ap}_{T_P}(\text{POS}_P(X))\) |
|---|---|---|
| \(\text{NEG}_P(X)\) | \(U - \text{ap}_{T_P}(X)\) | \(\text{ap}_{T_P}(X) - \text{ap}_{T_P}(\text{POS}_P(X))\) |

For the information system given above \(S\), suppose \(\exists c \in C, x_i \in U\), satisfying \(f(x, c) = \ast\), where \(\ast\) denotes a null value, i.e. a missing attribute value, then \(S\) is called incomplete decision information system.

According to Kryszkiewicz’s theory \[2, 38\], if

\[S = (U, C \cup D, V, f)\]

is an incomplete decision information system, for any non-empty subset \(P \subseteq C\), the binary tolerance relation \(T_P\) is defined as

\[T_P = \{(x, y) \in U \times U \mid \forall a \in P, f(x, a) = f(y, a)\} \cup \{(x, y) \in U \mid f(x, a) = \ast \land f(y, a) = \ast\}\] \[(1)\]

In this view, for \(\forall x \in U\), the tolerance class \(T_P(x)\) is defined as \(T_P(x) = \{y \in U \mid (x, y) \in T_P\}\). \(T_P(x)\) describes the maximum indistinguishable subset of the object \(x\) under attribute set \(P\). For any subset \(X \subseteq U\), the lower approximations and the upper approximations of \(X\) with respect to tolerance relation \(T_P\) are defined by

\[\text{POS}_P(X) = \text{ap}_{T_P}(X)\]
\[\text{NEG}_P(X) = U - \text{ap}_{T_P}(X)\]
\[\text{BND}_P(X) = \text{ap}_{T_P}(X) - \text{ap}_{T_P}(\text{POS}_P(X))\] \[(1)\]
\[
R_d(\alpha P|T_r(x)) = \lambda_{P_P} P(A|T_r(x)) + \lambda_{P_N} P(A^c|T_r(x)) \\
R_d(\alpha N|T_r(x)) = \lambda_{N_P} P(A|T_r(x)) + \lambda_{N_N} P(A^c|T_r(x))
\]

Definition 2: Let \( S = (U, C \cup D, V, f) \) is an incomplete decision information system, \( A_1, A_2, \ldots, A_m \subseteq C \) denotes \( m \) granularity spaces, the tolerance relation cluster \( \{T_{A_1}, T_{A_2}, \ldots, T_{A_m}\} \) induced by the granularity spaces is called \( m \) granularity structure, then for \( \forall X \subseteq U \), the lower and upper approximations based on pessimistic MG-DTRS are defined by \( \sum_{i=1}^{m} A_i^P(X) \) and \( \sum_{i=1}^{m} A_i^N(X) \), respectively,

\[
\sum_{i=1}^{m} A_i^P(X) = \{ x \in U : P(X|T_{A_i}(x)) \}
\]

\[
\sum_{i=1}^{m} A_i^N(X) = \{ x \in U : P(X|T_{A_i}(x)) \}
\]

In the following section, the MG-DTRS model is analysed from the perspective of decision cost. For given incomplete decision information system \( S = (U, C \cup D, V, f) \), \( A_1, A_2, \ldots, A_m \subseteq C \) denote \( m \) granularity spaces, \( \{T_{A_1}, T_{A_2}, \ldots, T_{A_m}\} \) denotes the tolerance relation cluster induced by the granularity spaces. Assuming that the state set is represented by \( \Omega = \{X, X^c\} \), the corresponding decision action set is \( B = \{a_1, a_2, a_3\} \) and the decision loss matrix is the same under each granularity structure, for \( x \in U \), the expected loss of execution decision can be expressed as

\[
R_d(\alpha_P|T_{A_i}(x)) = \lambda_{P_P} P(X|T_{A_i}(x)) + \lambda_{P_N} P(X^c|T_{A_i}(x)) \\
R_d(\alpha_N|T_{A_i}(x)) = \lambda_{N_P} P(X|T_{A_i}(x)) + \lambda_{N_N} P(X^c|T_{A_i}(x))
\]

\[
\sum_{i=1}^{m} A_i^P(X) = \{ x \in U : P(X|T_{A_i}(x)) \}
\]

\[
\sum_{i=1}^{m} A_i^N(X) = \{ x \in U : P(X|T_{A_i}(x)) \}
\]

According to three-way decision semantic rules, \( (P) \) if \( P(x|T_r(x)) \geq \alpha \), then \( x \in \text{POS}(x) \)

\[
(P) \quad \text{if } P(x|T_r(x)) \geq \alpha, \quad \text{then } x \in \text{POS}(x)
\]

\[
(N) \quad \text{if } P(x|T_r(x)) < \beta, \quad \text{then } x \in \text{NEG}(x)
\]

From the above rules, the lower approximation set and the upper approximation set can be expressed as

\[
DT_P(x) = \{ x \in U | P(x|T_r(x)) \geq \alpha \} \\
DT_N(x) = \{ x \in U | P(x|T_r(x)) < \beta \}
\]

\[
\sum_{i=1}^{m} A_i^P(X) = \{ x \in U : P(X|T_{A_i}(x)) \geq \alpha \}
\]

\[
\sum_{i=1}^{m} A_i^N(X) = \{ x \in U : P(X|T_{A_i}(x)) < \beta \}
\]

where \( \sim X \) denotes the complement of \( X \).
According to the Bayesian risk decision rules [8, 9], the following rules can be obtained:

\[(PPI)\text{ if } R_d(a_T|T_A, T_{A'}, \ldots, T_{A_m}) \leq R_d(a_T|T_A, T_{A'}, \ldots, T_{A_m}) \text{ and } R_d(a_T|T_A, T_{A'}, \ldots, T_{A_m}) \leq R_d(a_{\tilde{T}}|T_A, T_{A'}, \ldots, T_{A_m}) \text{ then } x \in POS_{\Sigma_{\tilde{T}}^{\alpha}}(X)\]

\[(PNI)\text{ if } R_d(a_T|T_A, T_{A'}, \ldots, T_{A_m}) \leq R_d(a_T|T_A, T_{A'}, \ldots, T_{A_m}) \text{ and } R_d(a_T|T_A, T_{A'}, \ldots, T_{A_m}) \leq R_d(a_{\tilde{T}}|T_A, T_{A'}, \ldots, T_{A_m}) \text{ then } x \in NEG_{\Sigma_{\tilde{T}}^{\alpha}}(X)\]

\[(PIBI)\text{ if } R_d(a_T|T_A, T_{A'}, \ldots, T_{A_m}) \leq R_d(a_T|T_A, T_{A'}, \ldots, T_{A_m}) \text{ and } R_d(a_{\tilde{T}}|T_A, T_{A'}, \ldots, T_{A_m}) \leq R_d(a_{\tilde{T}}|T_A, T_{A'}, \ldots, T_{A_m}) \text{ then } x \in BND_{\Sigma_{\tilde{T}}^{\alpha}}(X)\]

According to formulae (12)–(14) and the rules (PPIP), (PNI), (PIBI), referring to the deduction method of MG-DTRS model in [22], we can deduce the decision rules based on pessimistic MG-DTRS as follows:

\[(PWM)\text{ if } \sum_{i=1}^{m} \sum_{\alpha=1}^{n} P(X|T_{A_i}(x)) \geq \alpha \text{ then } x \in POS_{\Sigma_{\alpha}}^{\alpha}(X)\]

\[(BWM)\text{ if } \beta \leq \sum_{i=1}^{m} \sum_{\alpha=1}^{n} P(X|T_{A_i}(x)) \text{ then } x \in BND_{\Sigma_{\alpha}}^{\alpha}(X)\]

\[(NWM)\text{ if } \sum_{i=1}^{m} \sum_{\alpha=1}^{n} P(X|T_{A_i}(x)) \leq \beta \text{ then } x \in NEG_{\Sigma_{\alpha}}^{\alpha}(X)\]

From the above rules (PWM), (BWM), (NWM), the lower approximation set and the upper approximation set based on pessimistic MG-DTRS can be expressed as

\[\sum_{i=1}^{m} A^{\alpha}_{\tilde{T}}(X) = \left\{ x \in U \left| \sum_{i=1}^{m} \sum_{\alpha=1}^{n} P(X|T_{A_i}(x)) \geq \alpha \right. \right\}\]

\[\sum_{i=1}^{m} A^{\alpha}_{\tilde{T}}(X) = \left\{ x \in U \left| \sum_{i=1}^{m} \sum_{\alpha=1}^{n} P(X|T_{A_i}(x)) \leq \beta \right. \right\}\]

Binary tuples \(\sum_{i=1}^{m} A^{\alpha}_{\tilde{T}}(X), \sum_{i=1}^{m} A^{\alpha}_{\tilde{T}}(X)\) are called pessimistic MG-DTRSSs of set X.

Using the same analysis method of pessimistic MG-DTRS above, according to the optimistic multi-granulation idea, the lower approximation set and the upper approximation set based on optimistic MG-DTRS can be obtained:

\[\sum_{i=1}^{m} A^{\alpha}_{\tilde{T}}(X) = \left\{ x \in U \left| \sum_{i=1}^{m} \sum_{\alpha=1}^{n} P(X|T_{A_i}(x)) \leq \alpha \right. \right\}\]

\[\sum_{i=1}^{m} A^{\alpha}_{\tilde{T}}(X) = \left\{ x \in U \left| \sum_{i=1}^{m} \sum_{\alpha=1}^{n} P(X|T_{A_i}(x)) \geq \beta \right. \right\}\]

Binary tuples \(\sum_{i=1}^{m} A^{\alpha}_{\tilde{T}}(X), \sum_{i=1}^{m} A^{\alpha}_{\tilde{T}}(X)\) are called optimistic MG-DTRSSs of set X.

### 3 Incomplete NMG-DTRS

In Section 2.3, the MG-DTRS based on tolerance class can make good decision analyses for symbolic information system, but it cannot directly make decision for hybrid-valued or numerical information system. Some studies usually adopt two kinds of methods to overcome this problem in real applications. One is that the numerical data is discretised before applying rough set models and the other one is defining equivalence class based on neighborhood relations [23–25] instead of indiscernibility relation. In this paper, the concept of NRS is introduced to calculate the incomplete neighborhood granularity of symbolic decision systems, the neighborhood granularity is used instead of the conditional equivalence classes of classical MG-DTRS, and the implementation algorithms and related properties of the proposed incomplete NMG-DTRS models are deeply studied.

#### 3.1 NRS and tolerance relation

In this section, we review some basic concepts of Hu's NRS [25], and redefine tolerance relation based on Kryszkiewicz's theory with neighborhood relation from numerical feature granularity and symbolic feature granularity, respectively.

A 4-tuple \(S = ( U, C \cup D, V, f )\) is a decision information system, and \(S\) contains a set of non-empty finite objects, where \(U = \{ x_1, x_2, \ldots, x_n \}\) is the universe of objects, \(B \subseteq C\), for \(\forall x_i \in U\), then the \(\delta\) neighborhood of \(x_i\) in attribute space \(B\) is defined as \(n_{\delta}(x_i) = \{x_j|\Delta_{ \delta}(x_i, x_j) \leq \delta, x_j \in U\}\), where \(\Delta\) is a distance function and \(\delta (\delta \geq 0)\) is the neighborhood radius. At present, the distance functions Minkowski distance, Euclidean distance and Chebyshev distance are commonly used [28, 31]. The Euclidean distance is used in this paper and is defined as

\[\Delta_{\delta}(x_j, x_i) = \left( \sum_{k=1}^{l} (b_i - b_j)^2 \right)^{1/2}\]

when \(x_i\) and \(x_j\) have the same values, then \(\Delta_{\delta}(x_j, x_i) = 0\).

**Definition 3**: Let 5-tuple \(NDIS = ( U, C \cup D, V, f, N )\) is a neighborhood decision system, where \(C\) is a set of condition attributes and \(D\) is a set of decision attributes, \(V = \cup_{i=1}^{m} V_{a}\) denotes a non-empty set of values, \(N\) is a neighborhood relation generated by \(B\). Order the neighborhood radius \(\delta\), for \(\forall B \subseteq C\), \(N_{\delta}\) denotes the neighborhood relation generated by \(B\), then the neighborhood relation \(N_{\delta}\) partitions \(U\) into a group of neighborhood granules by \(U/N_{\delta}\). Namely,

\[U/N_{\delta} = \{ n_{\delta}(x)|x_i \in U, 1 \leq i \leq |U|\}\]

**Definition 4**: Let 5-tuple \(NDIS = ( U, C \cup D, V, f, N )\) is an incomplete neighborhood decision system, \(B \subseteq C\), \(\delta (\delta \geq 0)\) is the neighborhood radius, then the neighborhood tolerance relation induced by \(B\) is defined as

\[N_{T_{\delta}} = \{ (x_j, x_i)|x_i \in U \times U \mid n_{\delta}(x_i, x_j) \leq \delta \}\]

For any subset \(X \subseteq U\), the neighborhood tolerance class induced by \(B\) is defined as

\[N_{T_{\delta}}(x) = \{ x_j \in U \mid (x_j, x_i) \in N_{T_{\delta}}, \forall x_i \in U \}\]
Thus, the lower and upper approximation sets of $X$ are defined as

$$NT_{L}(x) = \{x \mid NT_{B}(x) \subseteq X, x \in U\}$$

$$NT_{U}(x) = \{x \mid NT_{B}(x) \cap X \neq \emptyset, x \in U\}$$

(22)

**Definition 5:** Let 5-tuple $NDIS = (U, C \cup D, V, f, N)$ is an incomplete neighbourhood decision system, $B1(B1 \subseteq C)$ is a numerical attribute set, $B2(B2 \subseteq C)$ is a symbolic attribute set, and $C = B1 \cup B2$ is a mixed attribute set, $\delta > 0$, then the neighbourhood tolerance relation induced by $C$ is defined as (see (23))

The neighbourhood tolerance classes induced by $B1, B2, B1 \cup B2$ can be defined as (see (24)). For any subset $X \subseteq U$, the lower and upper approximation sets of $X$ induced by $B1 \cup B2$ are defined as

$$NT_{B1}(x) = \{x \mid NT_{B1}(x) \subseteq X, x \in U\}$$

$$NT_{B1+B2}(x) = \{x \mid NT_{B1}(x) \cap X \neq \emptyset, x \in U\}$$

(25)

According to formulae (22) and (25), we can easily solve approximation problems based on numerical and hybrid-valued data.

**Example 1:** Given an incomplete neighbourhood decision system $NDIS = (U, C \cup D, V, f, N)$, here, take Table 1 dataset as an example, where $U = \{x_1, x_2, x_3, x_4\}$, $C = \{b, c, d, e\}$ is numerical attributes and $C2 = \{d, e\}$ is symbolic attributes, $D$ is decision attribute. **4** denotes the null value. Suppose neighbourhood radius $\delta = 0.1$. From the data in Table 1, we can get the partition of $U$ on decision attribute $D$, namely, $X = \{x_1, x_2, x_3\}$, $X2 = \{x_1, x_2\}$. According to Definition 5, we can compute the neighbourhood granules of sample. The information granules induced by numerical attributes $C1$ are listed.

$$NT_{C}(x_1) = \{x_1\}$$

$$NT_{C}(x_2) = \{x_2\}$$

$$NT_{C}(x_3) = \{x_3\}$$

(26)

**Example 2:** (Continued from Example 1): As we know, $C1 = \{a, b, c\}$ is the numerical attribute and $C2 = \{d, e\}$ is symbolic attribute. Let $C1, C2 \subseteq C$ are granularity spaces for $C1$, where $C1 = \{a, b\}, C2 = \{c\}, C1' = \{d\}, C2' = \{e\}$, therefore, we have that $NT_{C}(X1) = \{x_1, x_2, x_3\}$ and $NT_{C}(X2) = \{x_3\}$. When consider the comprehensive impact of numerical and symbolic attributes, we can obtain the another and upper approximation sets of the sample induced by $C$ ($C = C1 \cup C2$).

$$NT_{C}(x_1) = \{x_1, x_2, x_3\}$$

$$NT_{C}(x_3) = \{x_3\}$$

$$NT_{C}(X1) = \{x_1, x_2, x_3\}$$

$$U/C2 = \{x_1, x_2, x_3\}, \{x_1, x_2, x_3\}$$ is also a granular structure on $U$.

$$NT_{C}(X1) = \{x_1, x_2, x_3\}$$

$$NT_{C}(X2) = \{x_3\}$$

According to Definitions 1, 2 and formulae (17)-(20) in Section 2.3, by considering the different influence of numerical feature granularity and symbolic feature granularity on decision-making, the first type of NMG-DTRS models is presented below.

**Definition 6:** Let 5-tuple $NDIS = (U, C \cup D, V, f, N)$ is a neighbourhood decision system. $A \subseteq B \subseteq C$ is a symbolic attribute set, order $A_1, A_2, \ldots, A_m \subseteq A$ denotes the $m$ granularity spaces of $A$, the tolerance relation cluster induced by the granularity spaces is $\{T_{A_1}, T_{A_2}, \ldots, T_{A_m}\}$. $B \subseteq C$ is a numerical attribute set, order $B_1, B_2, \ldots, B_m \subseteq B$ denotes the $m$ granularity spaces of $B$, the neighbourhood relation cluster induced by the granularity spaces of $B$ is $\{NT_{B_1}, NT_{B_2}, \ldots, NT_{B_m}\}$. Assuming that the loss function matrix corresponding to each granularity is the same. For $\forall X \subseteq U$, given $a, \beta (1 \geq \alpha \geq \beta \geq 0)$, then the lower and upper approximation sets of $X$ based on the first type of pessimistic NMG-DTRS can be defined as (see (27))

$$BINARY\ TUPLES \ (A+B)_0^0(X), (A+B)_0^0(X)$$

are called the first type of pessimistic NMG-DTRS of set $X$, and abbreviated as $1PNMG-DTRS$.

The lower and upper approximation sets of $X$ based on the first type of optimistic NMG-DTRS can be defined as (see (28)).

$$BINARY\ TUPLES \ (A+B)_0^0(X), (A+B)_0^0(X)$$

are called the first type of optimistic NMG-DTRS of set $X$, and abbreviated as $1ONMG-DTRS$.
\[(A + B)^{\alpha, \beta}_X(X) = \left\{ x \in U \mid \begin{align*}
\vee_{k=1}^m P(x|T_{A_k}(x)) &\geq \alpha \vee \vee_{k=1}^m P(x|N_{T_{B_k}}(x)) - \beta \vee \vee_{k=1}^m P(x|N_{T_{B_k}}(x)) \geq \alpha \end{align*} \right\}
\]

\[
\begin{align*}
(A + B)^{\alpha, \beta}_X(X) = \left\{ x \in U \mid \begin{align*}
\vee_{k=1}^m P(x|T_{A_k}(x)) &\geq \alpha \vee \vee_{k=1}^m P(x|N_{T_{B_k}}(x)) - \beta \vee \vee_{k=1}^m P(x|N_{T_{B_k}}(x)) \geq \alpha \end{align*} \right\}
\end{align*}
\]

\[\tag{28}\]

\[\begin{align*}
&\sum_{i=1}^m P(x|N_{T_{B_i}}(x)) \geq \alpha \vee \sum_{i=1}^m P(x|N_{T_{B_i}}(x)) \geq \alpha
\end{align*}\]

\[\sum_{i=1}^m P(x|N_{T_{B_i}}(x)) \geq \alpha \vee \sum_{i=1}^m P(x|N_{T_{B_i}}(x)) \geq \alpha
\]

\[\tag{29}\]

\[\begin{align*}
&\sum_{i=1}^m P(x|N_{T_{B_i}}(x)) \geq \alpha \vee \sum_{i=1}^m P(x|N_{T_{B_i}}(x)) \geq \alpha
\end{align*}\]

\[\tag{30}\]

\[\begin{align*}
&\sum_{i=1}^m P(x|N_{T_{B_i}}(x)) \geq \alpha \vee \sum_{i=1}^m P(x|N_{T_{B_i}}(x)) \geq \alpha
\end{align*}\]

\[\tag{31}\]

\[\begin{align*}
&\sum_{i=1}^m P(x|N_{T_{B_i}}(x)) \geq \alpha \vee \sum_{i=1}^m P(x|N_{T_{B_i}}(x)) \geq \alpha
\end{align*}\]

\[\tag{32}\]

\[\begin{align*}
&\sum_{i=1}^m P(x|N_{T_{B_i}}(x)) \geq \alpha \vee \sum_{i=1}^m P(x|N_{T_{B_i}}(x)) \geq \alpha
\end{align*}\]
best choice. At this point, we can use the second method described by Definition 7 which combines symbolic and numerical granularity.

Let \( C_1 = A \cup B = \{a_1, a_3\} \) and \( C_2 = A \cup B = \{a_2, a_3\} \). Suppose neighbourhood radius \( \delta = 0.2 \) and \( \alpha = 0.6 \). According to Table 4, we can calculate the partition of \( U \) on decision attribute \( d \) as \( X_1 = \{x_1, x_2, x_3, x_4, x_5\} \) and \( X_2 = \{x_1, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}\} \).

We can obtain granularity structures of \( U \) by \( U/C_1 \) and \( U/C_2 \) (see (33)) . By Definition 7, the lower and upper approximation sets of \( X_1 \) and \( X_2 \) based on 2PNMG-DTRS can be obtained, respectively, as

\[
\begin{align*}
\sum_{i=1}^{m} NT_{C}^{P, \alpha}(X_1) &= \{x_1, x_2, x_3, x_4, x_5\} \\
\sum_{i=1}^{m} NT_{C}^{P, \beta}(X_1) &= \{x_1, x_2, x_3, x_4, x_5\} \\
\sum_{i=1}^{m} NT_{C}^{Q, \alpha}(X_2) &= \{x_1, x_4, x_5, x_6, x_8, x_9, x_{10}, x_{11}\} \\
\sum_{i=1}^{m} NT_{C}^{Q, \beta}(X_2) &= \{x_2, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}
\end{align*}
\]

(34)

### 3.3 Related properties of NMG-DTRS

According to the definition of the lower and upper approximation sets of the first type of pessimistic and optimistic NMG-DTRS models, the following properties can be obtained.

**Theorem 1:** Let \( \text{NDIS} = (U, C \cup D, V, f, N) \) is an incomplete neighbourhood decision system, \( A \cup B \subseteq C, A \subseteq C \) is a symbolic attribute set and granular structure \( A, A_1, \ldots, A_m \subseteq A, B \subseteq C \) is a numerical attribute set and granular structure \( B, B_1, \ldots, B_m \subseteq B \), assume \( 1 \geq \alpha > \beta \geq 0 \), for any \( \forall X, Y \subseteq U \), the following properties can be obtained:

(i) \( \sum_{i=1}^{m} NT_{C}^{P, \alpha}(\emptyset) = \sum_{i=1}^{m} NT_{C}^{P, \beta}(\emptyset) = \emptyset (A + B)_0 (\emptyset) = \emptyset \)

(ii) \( (A + B)_0 (U) = (A + B)_0 (U) = (A + B)_0 (U) = U \)

(iii) \( (A + B)_0 (X) \subseteq X ; (A + B)_0 (X) \subseteq X \)

(iv) Assume \( X \subseteq Y \), the following formulas are established:

\[
\begin{align*}
(A + B)_0 (X) &\subseteq (A + B)_0 (Y) \\
(A + B)_0 (X) &\subseteq (A + B)_0 (Y) \\
(A + B)_0 (X) &\subseteq (A + B)_0 (Y)
\end{align*}
\]

(35)

\[
\begin{align*}
(A + B)_0 (X) &\subseteq (A + B)_0 (Y) \\
(A + B)_0 (X) &\subseteq (A + B)_0 (Y) \\
(A + B)_0 (X) &\subseteq (A + B)_0 (Y)
\end{align*}
\]

(36)

For \( 1 \geq \alpha > \beta \geq 0.5 \), then

\[
\begin{align*}
(A + B)_0 (X) &\subseteq (A + B)_0 (Y) \\
(A + B)_0 (X) &\subseteq (A + B)_0 (Y) \\
(A + B)_0 (X) &\subseteq (A + B)_0 (Y)
\end{align*}
\]

\[
\begin{align*}
(A + B)_0 (X) &\subseteq (A + B)_0 (Y) \\
(A + B)_0 (X) &\subseteq (A + B)_0 (Y) \\
(A + B)_0 (X) &\subseteq (A + B)_0 (Y)
\end{align*}
\]

(37)

\[
\begin{align*}
(A + B)_0 (X) &\subseteq (A + B)_0 (Y) \\
(A + B)_0 (X) &\subseteq (A + B)_0 (Y) \\
(A + B)_0 (X) &\subseteq (A + B)_0 (Y)
\end{align*}
\]

(38)

**Table 4** Discretised neighbourhood decision table

| \( U \) | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( d \) |
|---|---|---|---|---|
| \( x_1 \) | 0 | 0.528 | 0 | 1 |
| \( x_2 \) | 0.121 | 0.472 | * | 2 |
| \( x_3 \) | 0.89 | 0 | 1 | 1 |
| \( x_4 \) | * | 0.283 | 1 | 2 |
| \( x_5 \) | 0.6 | 0.566 | 2 | 2 |
| \( x_6 \) | 0.495 | * | 2 | 2 |
| \( x_7 \) | 0.956 | 0.509 | 0 | 1 |
| \( x_8 \) | 0.934 | * | 1 | 2 |
| \( x_9 \) | * | 0.547 | 1 | 1 |
| \( x_{10} \) | 1 | 1 | 2 | 2 |
| \( x_{11} \) | 0.165 | * | 0 | 1 |
| \( x_{12} \) | 0.912 | 0.396 | 1 | 2 |

According to the definition of the lower and upper approximation sets of the second type of pessimistic and optimistic neighbourhood MG-DTRS models, the following properties can be obtained.

**Theorem 2:** Let \( \text{NDIS} = (U, C \cup D, V, f, N) \) is an incomplete neighbourhood decision system, \( A \cup B \subseteq C, A \subseteq C \) is a symbolic attribute set and granular structure \( A, A_1, \ldots, A_m \subseteq A, B \subseteq C \) is a numerical attribute set and granular structure \( B, B_1, \ldots, B_m \subseteq B \), assume \( 1 \geq \alpha > \beta \geq 0 \), for any \( \forall X, Y \subseteq U \), the following properties can be obtained:

(i) \( \sum_{i=1}^{m} NT_{C}^{P, \alpha}(\emptyset) = \sum_{i=1}^{m} NT_{C}^{P, \beta}(\emptyset) = \emptyset \)

(ii) \( \sum_{i=1}^{m} NT_{C}^{P, \alpha}(U) = \sum_{i=1}^{m} NT_{C}^{P, \beta}(U) = \emptyset \)

(iii) \( \sum_{i=1}^{m} NT_{C}^{P, \alpha}(X) \subseteq X ; \sum_{i=1}^{m} NT_{C}^{P, \beta}(X) \subseteq X \).

(iv) Assume \( X \subseteq Y \), the following formulas are established:

\[
\begin{align*}
\sum_{i=1}^{m} NT_{C}^{P, \alpha}(X) &\subseteq \sum_{i=1}^{m} NT_{C}^{P, \beta}(Y) \\
\sum_{i=1}^{m} NT_{C}^{P, \alpha}(X) &\subseteq \sum_{i=1}^{m} NT_{C}^{P, \beta}(Y) \\
\sum_{i=1}^{m} NT_{C}^{P, \alpha}(X) &\subseteq \sum_{i=1}^{m} NT_{C}^{P, \beta}(Y)
\end{align*}
\]

(39)

\[
\begin{align*}
\sum_{i=1}^{m} NT_{C}^{P, \alpha}(X) &\subseteq \sum_{i=1}^{m} NT_{C}^{P, \beta}(Y) \\
\sum_{i=1}^{m} NT_{C}^{P, \alpha}(X) &\subseteq \sum_{i=1}^{m} NT_{C}^{P, \beta}(Y) \\
\sum_{i=1}^{m} NT_{C}^{P, \alpha}(X) &\subseteq \sum_{i=1}^{m} NT_{C}^{P, \beta}(Y)
\end{align*}
\]

(40)

(v) For \( 1 \geq \alpha > \beta \geq 0.5 \), then \( \sum_{i=1}^{m} NT_{C}^{P, \alpha}(X) \subseteq \sum_{i=1}^{m} NT_{C}^{P, \beta}(X) \).

(vi) For \( 0.5 > \beta \geq 0 \), then \( X \subseteq U \)

\[
\begin{align*}
\sum_{i=1}^{m} NT_{C}^{P, \alpha}(X) &\subseteq \sum_{i=1}^{m} NT_{C}^{P, \beta}(X) \\
\sum_{i=1}^{m} NT_{C}^{P, \alpha}(X) &\subseteq \sum_{i=1}^{m} NT_{C}^{P, \beta}(X) \\
\sum_{i=1}^{m} NT_{C}^{P, \alpha}(X) &\subseteq \sum_{i=1}^{m} NT_{C}^{P, \beta}(X)
\end{align*}
\]

(41)

(42)

3.4 Algorithm implementation

According to the deduction of the MG-DTRS model in Section 2.3, namely formulae (17)-(20), it can be seen that the model describes
the upper and lower approximation sets mainly by the maximum probability and minimum probability that object \( x \) belongs to a certain objective concept \( X \) under each granularity. When dealing with symbolic information systems, the model has the same decision cost matrix under each granularity structure and plays the same role in solving decision problems. However, when dealing with hybrid-valued information systems, symbolic and numerical granularities often play different roles.

The first type of NMG-DTRS (1NMG-DTRS) model proposed in this paper fully considers the difference between symbolic granularity and numerical granularity in decision-making. When both numerical granularity and symbolic granularity are required to satisfy the conditions, it is called a pessimistic neighbourhood MG-DTRS model; when one of the numerical granularity or symbolic granularity meets the conditions, it is called optimistic neighbourhood MG-DTRS, as described in Definition 6. Fig. 1 depicts the idea of a model, the specific implementation algorithm is shown in Algorithm 1 (see Fig. 2).

The second type of NMG-DTRS (2NMG-DTRS) model proposed in this paper fully considers the interaction and fusion of symbolic granularity and numerical granularity. By dividing the granularity in light of the neighbourhood tolerance relation, the model takes into account the role of the probability that object \( x \) belongs to a target concept \( X \) in decision-making under various mixed-value granularities. When the probability of object \( x \) belongs to target concept \( X \) under each mixed granularity satisfies the condition at the same time, the model is called the second type of pessimistic NMG-DTRS model. When the probability of object \( x \) belongs to target concept \( X \) under any mixed granularity satisfies the condition, it is called the second type of optimistic NMG-DTRS model, as described in Definition 7. Fig. 3 gives the positive domain solution of the 2NMG-DTRS model. The specific implementation algorithm is shown in Algorithm 2 (see Fig. 4).

4 Relationships between NMG-DTRS and other models

In this section, in this paper, the authors mainly the first type of NMG-DTRS model proposed in Section 3, focusing on the relationships between the first type of model and other models.

In the NMG-DTRS model, the probability value, the thresholds \( \alpha \) and \( \beta \), and the size of granularity decide its detailed form of rough sets. Based on the definition of the lower and upper approximation sets mainly by the maximum probability and minimum probability that object \( x \) belongs to a certain objective concept \( X \) under each granularity. When dealing with symbolic information systems, the model has the same decision cost matrix under each granularity structure and plays the same role in solving decision problems. However, when dealing with hybrid-valued information systems, symbolic and numerical granularities often play different roles.

The first type of NMG-DTRS (1NMG-DTRS) model proposed in this paper fully considers the difference between symbolic granularity and numerical granularity in decision-making. When both numerical granularity and symbolic granularity are required to satisfy the conditions, it is called a pessimistic neighbourhood MG-DTRS model; when one of the numerical granularity or symbolic granularity meets the conditions, it is called optimistic neighbourhood MG-DTRS, as described in Definition 6. Fig. 1 depicts the idea of a model, the specific implementation algorithm is shown in Algorithm 1 (see Fig. 2).

The second type of NMG-DTRS (2NMG-DTRS) model proposed in this paper fully considers the interaction and fusion of symbolic granularity and numerical granularity. By dividing the granularity in light of the neighbourhood tolerance relation, the model takes into account the role of the probability that object \( x \) belongs to a target concept \( X \) in decision-making under various mixed-value granularities. When the probability of object \( x \) belongs to target concept \( X \) under each mixed granularity satisfies the condition at the same time, the model is called the second type of pessimistic NMG-DTRS model. When the probability of object \( x \) belongs to target concept \( X \) under any mixed granularity satisfies the condition, it is called the second type of optimistic NMG-DTRS model, as described in Definition 7. Fig. 3 gives the positive domain solution of the 2NMG-DTRS model. The specific implementation algorithm is shown in Algorithm 2 (see Fig. 4).
approximation sets of the 1NMG-DTRS model, when $\alpha = 1$, $\beta = 0$, we have that (see (39))
(see (40))
(see (41))
(see (42))
(see (43))
From formulas (21) and (42) deduced above, we can see that the lower and upper approximation sets of the 1PNMG-DTRS model are consistent with those in the classical pessimistic multi-granulation rough set for hybrid-valued decision systems. Hence, when $\alpha = 1$, $\beta = 0$, the first type of pessimistic NMG-DTRS (1PNMG-DTRS) will degenerate into the first type of hybrid-valued pessimistic multi-granulation rough set (1HP-MGRS).

\[ (A + B)^\alpha_{\beta} = \begin{cases} 
\{ x \in U \mid \lambda^\alpha_{\beta} \langle x \rangle \subseteq X \text{ and } T_{A_{\alpha}}(x) \subseteq X \text{ and } \lambda^\alpha_{\beta} \langle x \rangle \subseteq X \} \quad |A| = 0 \\
\{ x \in U \mid \lambda^\alpha_{\beta} \langle x \rangle \subseteq X \text{ and } T_{A_{\alpha}}(x) \subseteq X \text{ and } \lambda^\alpha_{\beta} \langle x \rangle \subseteq X \} \quad |B| = 0 
\end{cases} \]  
(40)

\[ (B + A)^\beta_{\alpha} = \begin{cases} 
\{ x \in U \mid \lambda^\alpha_{\beta} \langle x \rangle > 0 \} \\
\{ x \in U \mid \lambda^\alpha_{\beta} \langle x \rangle > 0 \} 
\end{cases} \]  
(41)

\[ \begin{cases} 
\{ x \in U \mid \lambda^\alpha_{\beta} \langle x \rangle \subseteq X \} \\
\{ x \in U \mid \lambda^\alpha_{\beta} \langle x \rangle \subseteq X \} 
\end{cases} \]  
(42)

\[ \begin{cases} 
\{ x \in U \mid \lambda^\alpha_{\beta} \langle x \rangle \subseteq X \} \\
\{ x \in U \mid \lambda^\alpha_{\beta} \langle x \rangle \subseteq X \} 
\end{cases} \]  
(43)
Formula (40) shows that when $|B| = 0$, all attribute values in decision information system are symbolic data, and then the model will degenerate into the pessimistic multi-granulation rough set (1P-MGRS). When $|A| = 0$, all attribute values in decision information system are numerical data. In this case, the model will degenerate into the pessimistic multi-granulation set (1PN-MGRS). Hence (see (45))

Formulas (45) and (46) show that when $m = 1$, $\left( A + B \right)^m(X) = \left( A + B \right)^m_0(X)$ will degenerate into hybrid-valued pessimistic multi-granulation DTRS (1HPMG-DTRS). When $|B| = 0$, all attribute values in decision information system are symbolic data, then the model will become DTRSs. When $|A| = 0$, all attribute values in decision information system are numerical data, and the model will degenerate into neighbourhood DTRSs. When $|A| = 0$, all attribute values in decision information system are numerical data, and the model will degenerate into numerical DTRSs, as shown in Fig. 5.

Similarly, based on the definition of upper and lower approximation sets of the 1ONMG-DTRS model, when $\alpha = 1$, $\beta = 0$, we have that

$$\left( A + B \right)^m_0(X) = \left\{ x \in U \mid \frac{\vee_{k=1}^{\alpha} P_r(X|T_{Ak}(x)) + \vee_{k=1}^{\beta} P_r(X|T_{Ak}(x)) - \vee_{k=1}^{\alpha} P_r(X|NT_{Bk}(x))}{1 + \vee_{k=1}^{\alpha} P_r(X|T_{Ak}(x)) - \vee_{k=1}^{\alpha} P_r(X|NT_{Bk}(x))} \geq \alpha \right\}$$

(45)

According to the reasoning method of formula (42), we can get that (see (48)). Formula (47) shows that $\left( A + B \right)^m_0(X) = \left( A + B \right)^m_0(X)$ will degenerate into the hybrid-valued optimistic multi-granulation rough set (1HOMG-DTRS). When $m = 1$, based on the same reasoning method of formulae (40) and (43). When $|B| = 0$, all attribute values in decision information system are symbolic data, then the model will become DTRSs. When $|A| = 0$, all attribute values in decision information system are numerical data, and the model will degenerate into N-DTRS, as shown in Fig. 6.

5 Conclusion

Multiple granularity goal decision problems are an important problem of knowledge mining under a large data environment. Decision analysis of hybrid-valued information system is an important content of MG-DTRS model. In this paper, based on the research of hybrid-valued incomplete decision information systems and MG-DTRSs, we proposed an incomplete NMG-DTRS model, two types of neighbourhood multi-granulation decision-theoretic set models and related properties are emphatically analysed. Meanwhile, taking pessimistic and optimistic NMG-DTRSs as examples, the implementation algorithms and the relationship between the proposed model and other models are studied. The use of NMG-DTRS model to solve the decision problem of multi-label data and dynamic data will be the goal of the next research.
\[
(A + B)_m^{10}(X) = \left\{ x \in U \left| \frac{\bigwedge_{k=1}^m P_i(X | T_{A_k}(x))}{1 + \bigwedge_{k=1}^m P_i(X | T_{A_k}(x))} \geq 1 \right. \right\} = \left\{ x \in U \left| (1 - \bigvee_{k=1}^m P(x | T_{A_k}(x)) \leq 0 \right. \right\}
\]

(47)

\[
(A + B)_m^{10}(X) = \left\{ x \in U \left| \frac{\bigwedge_{k=1}^m P_i(X | T_{A_k}(x))}{1 + \bigwedge_{k=1}^m P_i(X | T_{A_k}(x))} \geq 1 \right. \right\} = \left\{ x \in U \left| (1 - \bigvee_{k=1}^m P(x | T_{A_k}(x)) \leq 0 \right. \right\}
\]

(48)

Fig. 6 Relationships between 1ONMG-DTRS and other models

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