MODULUS OF CONTINUITY OF SOLUTIONS TO COMPLEX HESSIAN EQUATIONS

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ABSTRACT. We give a sharp estimate of the modulus of continuity of the solution to the Dirichlet problem for the complex Hessian equation of order $m$ ($1 \leq m \leq n$) with a continuous right hand side and a continuous boundary data in a bounded strongly $m$-pseudoconvex domain $\Omega \subset \mathbb{C}^n$. Moreover when the right hand side is in $L^p(\Omega)$, for some $p > n/m$ and the boundary value function is $C^{1,1}$ we prove that the solution is Hölder continuous.

1. INTRODUCTION

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with smooth boundary and $m$ be an integer such that $1 \leq m \leq n$. Given $\varphi \in C(\partial \Omega)$ and $0 \leq f \in C(\overline{\Omega})$. We consider the Dirichlet problem for complex Hessian equation:

\begin{equation}
\begin{cases}
  u \in SH_m(\Omega) \cap C(\overline{\Omega}) \\
  (dd^c u)^m \wedge \beta^{n-m} = f \beta^n \quad \text{in } \Omega \\
  u = \varphi \quad \text{on } \partial \Omega
\end{cases}
\end{equation}

where $SH_m(\Omega)$ denote the set of all $m$-subharmonic functions in $\Omega$ and $\beta := dd^c|z|^2$ is the standard Kähler form in $\mathbb{C}^n$.

In the case $m = 1$, this equation corresponds to the Poisson equation which is classical. The case $m = n$ corresponds to the complex Monge-Ampère equation which was intensively studied these last decades by several authors (see [BT76], [CP92], [CK94], [Bl96], [Ko98], [GKZ08]).

The complex Hessian equation is a new subject and is much more difficult to handle than the complex Monge-Ampère equation (e.g. the $m$-subharmonic functions are not invariant under holomorphic change of variables, for $m < n$). Despite that, the pluripotential theory which was developed in ([BT82], [D89], [Ko98]) for the complex Monge-Ampère equation can be adapted to the complex Hessian equation.

The Dirichlet problem (1.1) was considered by Li in [Li04]. He proved that if $\Omega$ is a bounded strongly $m$-pseudoconvex domain with smooth boundary

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(see the definition below), $\varphi \in C^\infty(\partial \Omega)$ and $0 < f \in C^\infty(\overline{\Omega})$ then there exists a unique smooth solution to the Dirichlet problem (1.1).

The existence of continuous solution for the homogenous Dirichlet problem in the unit ball was proved by Z. Blocki [Bl05]. The Hölder continuity of the solution when the right hand side and the boundary data are Hölder continuous was proved by H.C. Lu [Lu12]. Recently, S. Dinew and S. Kolodziej proved in [DK11] that there exists a unique continuous solution to (1.1) when $f \in L^p(\Omega)$, $p > n/m$. The Hölder continuity of the solution in this case has been studied independently by H.C. Lu [Lu12] and Nguyen [N13].

A viscosity approach to the complex Hessian equation has been developed by H.C. Lu in [Lu13b]. A potential theory for the complex Hessian equation was developed by Sadullaev and Abdullaev in [SA12] and H.C. Lu [Lu12] at the same time.

Our first main result in this paper gives a sharp estimate for the modulus of continuity of the solution to the Dirichlet problem for the complex Hessian equation (1.1).

More precisely, we will prove the following result.

**Theorem 1.1.** Let $\Omega$ be a smoothly bounded strongly $m$-pseudoconvex domain in $\mathbb{C}^n$, assume that $0 \leq f \in C(\Omega)$ and $\varphi \in C(\partial \Omega)$. Then the modulus of continuity $\omega_U$ of the solution $U$ satisfies the following estimate

$$\omega_U(t) \leq \tau (1 + \|f\|^{1/m}_{L^\infty(\overline{\Omega})}) \max\{\omega_\varphi(t^{1/2}), \omega_{f^{1/m}}(t), t^{1/2}\}$$

where $\tau \geq 1$ is a constant depending only on $\Omega$.

In the case of the complex Monge-Ampère equation, Y. Wang gave a control on the modulus of continuity of the solution assuming the existence of a subsolution and a supersolution with the given boundary data ([W12]).

Here we do not assume the existence of a subsolution and a supersolution. Actually the main argument in our proof consists in constructing adequate barriers for the Dirichlet problem for the complex Hessian equation (1.1).

For the case when the density $f \in L^p(\Omega)$ with $p > n/m$, C.H. Nguyen [N13] proved the Hölder continuity of the solution when the boundary data is in $C^{1,1}(\partial \Omega)$ and the density $f$ satisfies a growth condition near the boundary of $\Omega$.

For the case $m = n$, we proved recently ([Ch14]) that the solution to the Dirichlet problem (1.1) is Hölder continuous on $\Omega$ without any condition near the boundary. Using the same idea we can prove a similar result for the Hessian equation. More precisely, we have the following theorem.

**Theorem 1.2.** Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly $m$-pseudoconvex domain with smooth boundary, $\varphi \in C^{1,1}(\partial \Omega)$ and $0 \leq f \in L^p(\Omega)$, for some $p > n/m$. Then the solution $U$ to (1.1) belongs to $C^{0,\alpha}(\overline{\Omega})$ for any $0 < \alpha < \gamma_1$. 

Moreover, if \( p \geq 2n/m \) then the solution to the Dirichlet problem \( U \in C^{0,\alpha}(\Omega) \) for any \( 0 < \alpha < \min\{\frac{1}{2}, 2\gamma_1\} \), where \( \gamma_r = \frac{r}{r+m+\frac{m(q-p)}{p-m}} \) and \( r \geq 1 \).

## 2. Preliminaries

In this section, we briefly recall some facts from linear algebra and basic results from potential theory for \( m \)-subharmonic functions. We refer the reader to [Bl05, DK11, Lu13a, Lu13c, SA12] for more details.

Let us set

\[
H_m(\lambda) = \sum_{1 \leq j_1 < \cdots < j_m \leq n} \lambda_{j_1} \cdots \lambda_{j_m},
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \).

Thus \( (t + \lambda_1) \cdots (t + \lambda_n) = \sum_{m=0}^{n} H_m(\lambda)t^{n-m} \) for \( t \in \mathbb{R} \), where \( H_0(\lambda) = 1 \).

We denote \( \Gamma_m \) the closure of the connected component of \( \{H_m > 0\} \) containing \( (1, 1, \ldots, 1) \). One can show that

\[
\Gamma_m = \{\lambda \in \mathbb{R}^n : H_m(\lambda_1 + t, \ldots, \lambda_n + t) \geq 0, \forall t \geq 0\}.
\]

It follows from the identity

\[
H_m(\lambda_1 + t, \ldots, \lambda_n + t) = \sum_{p=0}^{m} \binom{n-p}{m-p} H_p(\lambda)t^{m-p},
\]

that

\[
\Gamma_m = \{\lambda \in \mathbb{R}^n : H_j(\lambda) \geq 0, \forall 1 \leq j \leq m\}.
\]

It is clear that \( \Gamma_n \subset \Gamma_{n-1} \subset \ldots \subset \Gamma_1 \), where \( \Gamma_n = \{\lambda \in \mathbb{R}^n : \lambda_i \geq 0 \forall i\} \).

By the paper of Gårding [G59], the set \( \Gamma_m \) is a convex cone in \( \mathbb{R}^n \) and \( H_0^{1/m} \) concave on \( \Gamma_m \). By Maclaurin inequality, we get

\[
\binom{n}{m}^{-1/m} H_m^{1/m} \leq \binom{n}{p}^{-1/p} H_p^{1/p}, \quad 1 \leq p \leq m \leq n.
\]

Let \( \mathcal{H} \) denote the vector space over \( \mathbb{R} \) of complex hermitian \( n \times n \) matrices. For any \( A \in \mathcal{H} \), let \( \lambda(A) = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) be the eigenvalues of \( A \). We set

\[
\tilde{H}_m(A) = H_m(\lambda(A)).
\]

Now, we define the cone

\[
\tilde{\Gamma}_m := \{A \in \mathcal{H} : \lambda(A) \in \Gamma_m \} = \{A \in \mathcal{H} : \tilde{H}_j(A) \geq 0, \forall 1 \leq j \leq m\}.
\]

Let \( \alpha \) be a real \((1,1)\)-form determined by

\[
\alpha = \frac{i}{2} \sum_{i,j} a_{ij} dz_i \wedge d\bar{z}_j
\]

where \((a_{ij})\) is a hermitian matrix. After diagonalizing the matrix \( A = (a_{ij}) \), we see that

\[
\alpha^m \wedge \beta^{n-m} = \tilde{S}_m(\alpha)\beta^n,
\]
where $\beta$ is the standard Kähler form in $\mathbb{C}^n$ and $\tilde{S}_m(\alpha) = m!(n-m)!\tilde{H}_m(A)$. The last equality allows us to define

$$\hat{\Gamma}_m := \{ \alpha \in \mathbb{C}_{(1,1)} : \alpha \wedge \beta^{n-1} \geq 0, \alpha^2 \wedge \beta^{n-2} \geq 0, \ldots, \alpha^m \wedge \beta^{n-m} \geq 0 \}.$$ 

where $\mathbb{C}_{(1,1)}$ is the space of real $(1,1)$-forms with constant coefficients.

Let $M : \mathbb{C}^{m}_{(1,1)} \to \mathbb{R}$ be the polarized form of $\tilde{S}_m$, i.e. $M$ is linear in every variable, symmetric and $M(\alpha, \ldots, \alpha) = \tilde{S}_m(\alpha)$, for any $\alpha \in \mathbb{C}_{(1,1)}$.

The Gårding inequality (see [G59]) asserts that

$$M(\alpha_1, \alpha_2, \ldots, \alpha_m) \geq \tilde{S}_m(\alpha_1)^{1/m} \ldots \tilde{S}_m(\alpha_m)^{1/m}, \quad \alpha_1, \alpha_2, \ldots, \alpha_m \in \hat{\Gamma}_m.$$ 

**Proposition 2.1.** ([Bl05]). If $\alpha_1, \ldots, \alpha_p \in \hat{\Gamma}_m$, $1 \leq p \leq m$, then we have

$$\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_p \wedge \beta^{n-m} \geq 0.$$ 

Let us set

$$\Sigma_m := \{ \alpha \in \hat{\Gamma}_m \text{ of constant coefficients such that } \tilde{S}_m(\alpha) = 1 \}.$$ 

Recall the following elementary lemma and we include its proof for the convenience of the reader.

**Lemma 2.2.** Let $\alpha \in \hat{\Gamma}_m$. Then the following identity holds

$$\tilde{S}_m(\alpha)^{1/m} = \inf \left\{ \frac{\alpha \wedge \alpha_1 \wedge \ldots \wedge \alpha_{m-1} \wedge \beta^{n-m}}{\beta^n} ; \alpha_i \in \Sigma_m, \forall i \right\}.$$ 

**Proof.** Let $M$ be a polarized form of $\tilde{S}_m$ define by

$$M(\alpha, \alpha_1, \ldots, \alpha_{m-1}) = \frac{\alpha \wedge \alpha_1 \wedge \ldots \wedge \alpha_{m-1} \wedge \beta^{n-m}}{\beta^n},$$

for $\alpha_1, \ldots, \alpha_{m-1} \in \Sigma_m, \alpha \in \hat{\Gamma}$. By Garding inequality (2.1) we have

$$M(\alpha, \alpha_1, \ldots, \alpha_{m-1}) \geq \tilde{S}_m(\alpha)^{1/m}.$$ 

Then we obtain that

$$\tilde{S}_m(\alpha)^{1/m} = \inf \left\{ \frac{\alpha \wedge \alpha_1 \wedge \ldots \wedge \alpha_{m-1} \wedge \beta^{n-m}}{\beta^n} ; \alpha_i \in \Sigma_m, \forall i \right\}.$$ 

Now, setting $\alpha_1 = \ldots = \alpha_{m-1} = \frac{\alpha}{S_m(\alpha)^{1/m}}$, we can ensure that

$$M(\alpha, \alpha_1, \ldots, \alpha_{m-1}) = \tilde{S}_m(\alpha)^{1/m}.$$ 

This complete the proof of lemma. \(\square\)

**Aspects about $m$-subharmonic functions.** Let $\Omega \subset \mathbb{C}^n$ is a bounded domain. Let also $\beta := dd^c|z|^2$ is the standard Kähler form in $\mathbb{C}^n$. 

Definition 2.3. ([Bl05]). Let $u$ be a subharmonic function in $\Omega$.

1) For smooth case, $u \in C^2(\Omega)$ is said to be $m$-subharmonic (briefly $m$-sh) if the form $dd^c u$ belongs pointwise to $\tilde{\Gamma}_m$.

2) For non-smooth case, $u$ is called $m$-sh if for any collection $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \tilde{\Gamma}_m$, the inequality

$$dd^c u \wedge \alpha_1 \wedge \ldots \wedge \alpha_{m-1} \wedge \beta^{n-m} \geq 0$$

holds in the weak sense of currents in $\Omega$.

We denote $SH_m(\Omega)$ the set of all $m$-sh functions. We will recall some properties of $m$-sh functions.

Proposition 2.4. ([Bl05]).

1) $PSH = SH_n \subset SH_{n-1} \subset \ldots \subset SH_1 = SH$.

2) If $u, v \in SH_m(\Omega)$ then $\lambda u + \eta v \in SH_m(\Omega)$, $\forall \lambda, \eta \geq 0$.

3) If $u \in SH_m(\Omega)$ and $\gamma : \mathbb{R} \to \mathbb{R}$ is convex increasing function then $\gamma \circ u \in SH_m(\Omega)$.

4) If $u \in SH_m(\Omega)$ then the standard regularizations $u_\epsilon = u * \rho_\epsilon$ are also $m$-sharmonic in $\Omega_\epsilon := \{ z \in \Omega | dist(z, \partial \Omega) > \epsilon \}$.

5) Let $U$ be a non-empty proper open subset of $\Omega$, if $u \in SH_m(\Omega)$, $v \in SH_m(U)$ and $\lim_{z \to y} v(z) \leq u(y)$ for every $y \in \partial U \cap \Omega$, then the function

$$w = \begin{cases} 
\max(u, v) & \text{in } U \\
u & \text{in } \Omega \setminus U 
\end{cases}$$

is $m$-sh in $\Omega$.

6) Let $\{u_\alpha\} \subset SH_m(\Omega)$ be locally uniformly bounded from above and $u = \sup u_\alpha$. Then the upper semi-continuous regularization $u^*$ is $m$-sh and equal to $u$ almost everywhere.

For locally bounded $m$-subharmonic functions, we can inductively define a closed nonnegative current (following Bedford and Taylor for plurisubharmonic functions).

$$dd^c u_1 \wedge \ldots \wedge dd^c u_p \wedge \beta^{n-m} := dd^c(u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_p \wedge \beta^{n-m}),$$

where $u_1, u_2, \ldots, u_p \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$, $p \leq m$.

In particular, we define the nonnegative Hessian measure of $u \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$ to be

$$H_m(u) := (dd^c u)^m \wedge \beta^{n-m}.$$ 

Let us define the differential operator $L_\alpha : SH_m(\Omega) \cap L^\infty_{loc}(\Omega) \to \mathcal{D}'(\Omega)$ such that

$$dd^c u \wedge \alpha_1 \wedge \ldots \wedge \alpha_{m-1} \wedge \beta^{n-m} = L_\alpha u \beta^m,$$

where $\alpha_1, \ldots, \alpha_{m-1} \in \Sigma_m$. In appropriate complex coordinates this operator is the Laplace operator.

Example 2.5. Using Garding inequality (2.1), one can note that $L_\alpha(\langle z \rangle^2) \geq 1$ for any $\alpha_i \in \Sigma_m, 1 \leq i \leq m - 1$. 


We will prove the following essential proposition by applying ideas from the viscosity theory developed in \[\text{EGZ11}\] for the complex Monge-Ampère equation and extended to the complex Hessian equation by H.C.Lu (\[\text{Lu12}, \text{Lu13b}\]).

**Proposition 2.6.** Let \(u \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)\) and \(0 \leq f \in C(\Omega)\). The following conditions are equivalent:

1) \(L_\alpha u \geq f^{1/m}, \forall \alpha_1, \ldots, \alpha_{m-1} \in \Sigma_m\).

2) \((dd^c u)^m \wedge \beta^{n-m} \geq f\beta^n\) in \(\Omega\).

**Proof.** First observe that if \(u \in C^2(\Omega)\), then by Lemma 2.2 we can see that (1) is equivalent to

\[
\tilde{S}_m(\alpha)^{1/m} \geq f^{1/m},
\]

where \(\alpha = dd^c u\) is a real \((1,1)\)-form belongs to \(\hat{\Gamma}_m\).

The last inequality corresponds to

\[
(dd^c u)^m \wedge \beta^{n-m} \geq f\beta^n\ 	ext{in} \ \Omega.
\]

(1 \(\Rightarrow\) 2) We consider the standard regularization \(u_\epsilon\) of \(u\) by convolution with a smoothing kernel. We then get

\[
L_\alpha u_\epsilon \geq (f^{1/m})_\epsilon.
\]

Since \(u_\epsilon\) is smooth, by the observation above, we have

\[
(dd^c u_\epsilon)^m \wedge \beta^{n-m} \geq ((f^{1/m})_\epsilon)^m \beta^n.
\]

Letting \(\epsilon \to 0\), by the convergence theorem for the Hessian operator under decreasing sequence, we get

\[
(dd^c u)^m \wedge \beta^{n-m} \geq f\beta^n\ 	ext{in} \ \Omega.
\]

(2 \(\Rightarrow\) 1) Fix \(x_0 \in \Omega\) and \(q\) is \(C^2\)-function in a neighborhood \(V \subset \Omega\) of \(x_0\) such that \(u \leq q\) in this neighborhood and \(u(x_0) = q(x_0)\). We will prove that

\[
(dd^c q)^m_{x_0} \wedge \beta^{n-m} \geq f(x_0)\beta^n.
\]

First step: we claim that \(dd^c q_{x_0} \in \hat{\Gamma}_m\).

If \(u\) is smooth, we note that \(x_0\) is a local minimum point of \(q - u\), then \(dd^c (q - u)_{x_0} \geq 0\). Hence we see that \((dd^c q)^k \wedge \beta^{n-k} \geq 0\) in \(x_0\), for \(1 \leq k \leq m\). This gives that \(dd^c q_{x_0} \in \hat{\Gamma}_m\).

If \(u\) non smooth, let \(u_\epsilon\) is the standard smooth regularization of \(u\), then \(u_\epsilon\) is \(m\)-sh, smooth and \(u_\epsilon \searrow u\). Now let us fix \(\delta > 0\) and \(\epsilon_0 > 0\) such that the neighborhood of \(x_0\), \(V \subset \Omega_{\epsilon_0}\). For each \(\epsilon < \epsilon_0\), let \(y_\epsilon\) be the maximum point of \(u_\epsilon - q - \delta|x - x_0|^2\) on \(B \subset V\) (where \(B\) is a small ball centered at \(x_0\)). Then we have

\[
u_\epsilon(x) - q(x) - \delta|x - x_0|^2 \leq u_\epsilon(y_\epsilon) - q(y_\epsilon) - \delta|y_\epsilon - x_0|^2.
\]

Assume that \(y_\epsilon \to y \in \bar{B}\). Let us put \(x = x_0\) an passing to the limit in the last inequality, we obtain

\[
0 \leq u(y) - q(y) - \delta|y - x_0|^2,
\]
but \( q \geq u \) in \( V \), then we can conclude that \( y = x_0 \).

Let us then define
\[
\tilde{q} := q + \delta \|x - x_0\|^2 + u_\epsilon(y_\epsilon) - q(y_\epsilon) - \delta \|y_\epsilon - x_0\|^2.
\]

which is \( C^2 \)-function in \( B \) and satisfies \( u_\epsilon(y_\epsilon) = \tilde{q}(y_\epsilon) \) and \( \tilde{q} \geq u_\epsilon \) in \( B \), then the following inequality holds in \( y_\epsilon \)
\[
(dd^c \tilde{q})^k \wedge \beta^{n-k} \geq 0 \text{ for } 1 \leq k \leq m,
\]

that is
\[
(dd^c q + \delta \beta)^k \wedge \beta^{n-k} \geq 0 \text{ for } 1 \leq k \leq m.
\]

Letting \( \epsilon \) tend to 0, we get
\[
(dd^c q + \delta \beta)^k \wedge \beta^{n-k} \geq 0 \text{ for } 1 \leq k \leq m.
\]

Since the last inequality holds for any \( \delta > 0 \), we can get that \( dd^c q_{x_0} \in \tilde{\Gamma}_m \).

Second step: assume that there exist a point \( x_0 \in \Omega \) and a \( C^2 \)-function \( q \)
satisfy \( u \leq q \) in a neighborhood of \( x_0 \) and \( u(x_0) = q(x_0) \) such that
\[
(dd^c q)^m_{x_0} \wedge \beta^{n-m} < f(x_0)\beta^n.
\]

Let us put
\[
q^\epsilon(x) = q(x) + \epsilon(\|x - x_0\|^2 - \frac{r^2}{2})
\]

which is \( C^2 \)-function and for \( 0 < \epsilon \ll 1 \) small enough we have
\[
0 < (dd^c q^\epsilon)^m_{x_0} \wedge \beta^{n-m} < f(x_0)\beta^n.
\]

Since \( f \) is continuous on \( \Omega \), there exists \( r > 0 \) such that
\[
(dd^c q^\epsilon)^m \wedge \beta^{n-m} \leq f\beta^n \text{ in } \mathbb{B}(x_0, r).
\]

Hence, we get
\[
(dd^c q^\epsilon)^m \wedge \beta^{n-m} \leq (dd^c u)^m \wedge \beta^{n-m} \text{ in } \mathbb{B}(x_0, r)
\]
and \( q^\epsilon = q + \epsilon r^2/2 > q \geq u \) on \( \partial \mathbb{B}(x_0, r) \). From the comparison principle ( see \([BL05, Lu12]\)), it follows that \( q^\epsilon \geq u \) in \( \mathbb{B}(x_0, r) \), but this contradicts that \( q^\epsilon(x_0) = u(x_0) - \epsilon r^2/2 < u(x_0) \).

We have shown that for every point \( x_0 \in \Omega \), and every \( C^2 \)-function \( q \) in a neighborhood of \( x_0 \) such that \( u \leq q \) in this neighborhood and \( u(x_0) = q(x_0) \),
we have \((dd^c q)^m_{x_0} \wedge \beta^{n-m} \geq f(x_0)\beta^n\), hence we have \( L_0 q_{x_0} \geq f^{1/m}(x_0) \).

Final step to prove 1). Assume that \( f > 0 \) is smooth function. Then there exists a \( C^\infty \)-function, say \( g \) such that \( L_0 g = f^{1/m} \). Hence Theorem 3.2.10' in \([H94]\) implies that \( \varphi = u - g \) is \( L_0 \)-subharmonic, consequently \( L_0 u \geq f^{1/m} \).

In the case \( f > 0 \) is only continuous. Note that
\[
f = \sup\{w \in C^\infty(\bar{\Omega}), 0 < w \leq f\}.
\]
Since \((dd^c u)^m \wedge \beta^{n-m} \geq f \beta^n\), we get \((dd^c u)^m \wedge \beta^{n-m} \geq w \beta^n\). As \(w > 0\) smooth, we can see that \(L_\alpha u \geq w^{1/m}\), therefore \(L_\alpha u \geq f^{1/m}\).

In general case, \(0 \leq f \in C(\bar{\Omega})\). We observe that \(u_\varepsilon(z) = u(z) + \varepsilon |z|^2\) satisfies
\[
(dd^c u_\varepsilon)^m \wedge \beta^{n-m} \geq (f + \varepsilon |z|^2) \beta^n.
\]
By the last step, we get \(L_\alpha u_\varepsilon \geq (f + \varepsilon |z|^2)^{1/m}\), therefore the wanted result follows by letting \(\varepsilon\) tend to 0.

**Definition 2.7.** Let \(\Omega \subset \mathbb{C}^n\) be a smoothly bounded domain, we say that \(\Omega\) is strongly \(m\)-pseudoconvex if there exist a defining function \(\rho\) of \(\Omega\) (i.e., a smooth function in a neighborhood \(U\) of \(\bar{\Omega}\) such that \(\rho < 0\) on \(\Omega\), \(\rho = 0\) and \(d\rho \neq 0\) on \(\partial \Omega\)) and \(A > 0\) such that
\[
(dd^c \rho)^k \wedge \beta^{n-k} \geq A \beta^n \text{ in } U, \text{ for } 1 \leq k \leq m.
\]

The existence of a solution \(\mathbb{U}\) to Dirichlet problem \([11]\) was proved in [DK11]. This solution can be given by the upper envelope of subsolutions to the Dirichlet problem as in [BT76] for the complex Monge-Ampère equation.

\[
(2.2) \quad \mathbb{U} = \sup\{v \in SH_m(\Omega) \cap C(\bar{\Omega}); d|_{\partial \Omega} \leq \varphi \text{ and } (dd^c v)^m \wedge \beta^{n-m} \geq f \beta^n\}.
\]
However, thanks to Lemma 2.6, we can describe the solution as the following
\[
(2.3) \quad \mathbb{U} = \sup\{v \in \mathcal{V}_m(\Omega, \varphi, f)\},
\]
where the nonempty family \(\mathcal{V}_m(\Omega, \varphi, f)\) is defined as
\[
\mathcal{V}_m = \{v \in SH_m(\Omega) \cap C(\bar{\Omega}); v|_{\partial \Omega} \leq \varphi \text{ and } L_\alpha v \geq f^{1/m}, \forall \alpha_i \in \Sigma_m, 1 \leq i \leq m-1\}.
\]
Observe that the description of the solution in formula \((2.3)\) is more convenient, since it deals with subsolutions with respect to a family of linear elliptic operators.

### 3. The Modulus of Continuity of the Solution

Recall that a real function \(\omega\) on \([0, l], 0 < l < \infty\), is called a modulus of continuity if \(\omega\) is continuous, subadditive, nondecreasing and \(\omega(0) = 0\).

In general, \(\omega\) fails to be concave, we denote \(\tilde{\omega}\) to be the minimal concave majorant of \(\omega\). For more details, we refer the reader to [T63, Kor82]. We denote \(\omega_\psi\) the optimal modulus of continuity of the continuous function \(\psi\) which is defined by
\[
\omega_\psi(t) = \sup_{|x-y| \leq t} |\psi(x) - \psi(y)|.
\]
The following lemma is one of the useful properties of \(\tilde{\omega}\).

**Lemma 3.1.** Let \(\omega\) be a modulus of continuity on \([0, l]\) and \(\tilde{\omega}\) be the minimal concave majorant of \(\omega\). Then \(\omega(\eta t) \leq \tilde{\omega}(\eta t) < (1 + \eta)\omega(t)\) for any \(t > 0\) and \(\eta > 0\).
Proof. Fix $t_0 > 0$ such that $\omega(t_0) > 0$. We claim that
\[
\frac{\omega(t)}{\omega(t_0)} \leq 1 + \frac{t}{t_0}, \quad \forall t \geq 0.
\]
For $0 < t \leq t_0$, since $\omega$ is nondecreasing, we have
\[
\frac{\omega(t)}{\omega(t_0)} \leq 1 + \frac{t}{t_0}.
\]
Otherwise, if $t_0 \leq t \leq l$, by the Euclid’s Algorithm, we write $t = kt_0 + \alpha$, $0 \leq \alpha < t_0$ and $k$ is natural number with $1 \leq k \leq t/t_0$. Using the subadditivity of $\omega$, we observe that
\[
\omega(t) \omega(t_0) \leq k\omega(t_0) + \omega(\alpha) \omega(t_0) \leq k + 1 \leq 1 + \frac{t}{t_0}.
\]
Let $l(t) := \omega(t_0) + \frac{t}{t_0} \omega(t_0)$ is a straight line, then $\omega(t) \leq l(t)$ for all $0 < t \leq l$. Therefore
\[
\bar{\omega}(t) \leq l(t) = \omega(t_0) + \frac{t}{t_0} \omega(t_0)
\]
for all $0 < t \leq l$. Hence, for any $\eta > 0$ we have
\[
\omega(\eta t) < \bar{\omega}(\eta t) < (1 + \eta)\omega(t).
\]

□

In the following proposition, we establish a barrier to the problem and estimate its modulus of continuity.

**Proposition 3.2.** Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly $m$-pseudoconvex domain with smooth boundary, assume that $\omega_\varphi$ is the modulus of continuity of $\varphi \in C(\partial \Omega)$ and $0 \leq f \in C(\overline{\Omega})$. Then there exists a subsolution $v \in V_m(\Omega, \varphi, f)$ such that $v = \varphi$ on $\partial \Omega$ and the modulus of continuity of $v$ satisfies the following inequality
\[
\omega_v(t) \leq \lambda \max\{\omega_\varphi(t^{1/2}), t^{1/2}\},
\]
where $\lambda = \eta(1 + \|f\|_{L^\infty(\overline{\Omega})}^{1/m})$ and $\eta \geq 1$ is a constant depending on $\Omega$.

**Proof.** Fix $\xi \in \partial \Omega$, we will prove that there exists $v_\xi \in V_m(\Omega, \varphi, f)$ such that $v_\xi(\xi) = \varphi(\xi)$.

We claim that there exists a constant $C > 0$, depending only on $\Omega$, such that for every point $\xi \in \partial \Omega$ and $\varphi \in C(\partial \Omega)$, there is a function $h_\xi \in SH_m(\Omega) \cap C(\Omega)$ such that
\begin{enumerate}
  \item $h_\xi(z) \leq \varphi(z), \forall z \in \partial \Omega$
  \item $h_\xi(\xi) = \varphi(\xi)$
  \item $\omega_{h_\xi}(t) \leq C\omega_\varphi(t^{1/2})$
\end{enumerate}

Assume that the claim is proved. Fix a point $z_0 \in \Omega$ and choose $K_1 \geq 0$ such that $K_1 = \sup_{\Omega} f^{1/m}$, hence $L_\alpha(K_1|z - z_0|^2) = K_1 L_\alpha|z - z_0|^2 \geq f^{1/m}(z)$ for
all \( \alpha_i \in \Sigma_m, 1 \leq i \leq m-1 \) and let us set \( K_2 = K_1|\xi - z_0|^2 \). Then for the continuous function
\[
\tilde{\varphi}(z) := \varphi(z) - K_1|z - z_0|^2 + K_2.
\]
we have \( h = h_\xi \) such that 1),2)and 3) hold.

Then the desired function \( v_\xi \in V_m(\Omega, \varphi, f) \) is given by
\[
v_\xi(z) := h(z) + K_1|z - z_0|^2 - K_2.
\]
Indeed, \( v_\xi \in SH_m(\Omega) \cap C(\bar{\Omega}) \) and we have
\[
h(z) \leq \tilde{\varphi}(z) = \varphi(z) - K_1|z - z_0|^2 + K_2 \text{ on } \partial\Omega.
\]
So that \( v_\xi(z) \leq \varphi(z) \) on \( \partial\Omega \) and \( v_\xi(\xi) = \varphi(\xi) \). It is clear that
\[
L_\alpha v_\xi = L_\alpha h + K_1L_\alpha|z - z_0|^2 \geq f^{1/m} \text{ in } \Omega.
\]
Moreover, by the hypothesis , we can get an estimate for the modulus of continuity of \( v_\xi \)
\[
\omega_v(t) = \sup_{|z-y| \leq t} |v(z) - v(y)| \leq \omega_h(t) + K_1\omega_{|z-z_0|^2}(t)
\]
\[
\leq C\omega_{\tilde{\varphi}}(t^{1/2}) + 4d^{3/2}K_1t^{1/2}
\]
\[
\leq C\omega_{\varphi}(t^{1/2}) + 2dK_1(C + 2d^{1/2})t^{1/2}
\]
\[
\leq (C + 2d^{1/2})(1 + 2dK_1)\max\{\omega_{\varphi}(t^{1/2}), t^{1/2}\}.
\]

Then we have
\[
\omega_{v_\xi}(t) \leq \eta(1 + K_1)\max\{\omega_{\varphi}(t^{1/2}), t^{1/2}\},
\]
where \( \eta := (C + 2d^{1/2})(1 + 2d) \) is a constant depending on \( \Omega \) and \( d := \text{diam}(\Omega) \).

Now, we prove the claim. Since \( \rho \) is smooth, we can choose \( B > 0 \) large enough such that the function
\[
g(z) = B\rho(z) - |z - \xi|^2
\]
is \( m \)-subharmonic on \( \Omega \). Let us set
\[
\chi(t) = -\tilde{\omega}_{\varphi}((-t)^{1/2}),
\]
for \( t \leq 0 \) which is convex nondecreasing function on \([-d^2, 0]\). Now, fix \( r > 0 \) so small that \( |g(z)| \leq d^2 \) in \( B(\xi, r) \cap \Omega \) and define for \( z \in B(\xi, r) \cap \bar{\Omega} \) the function
\[
h(z) = \chi \circ g(z) + \varphi(\xi).
\]
It is clear that \( h \) is continuous \( m \)-subharmonic function on \( B(\xi, r) \cap \Omega \) and one can observe that \( h(z) \leq \varphi(z) \) if \( z \in B(\xi, r) \cap \partial\Omega \) and \( h(\xi) = \varphi(\xi) \). Moreover, by the subadditivity of \( \tilde{\omega}_{\varphi} \) and Lemma 3.4 we have
\[ \omega_h(t) = \sup_{|z-y| \leq t} |h(z) - h(y)| \]
\[ \leq \sup_{|z-y| \leq t} \tilde{\omega}_\varphi \left[ \left| |z| - \xi \right|^2 - |y - \xi|^2 - B(\rho(z) - \rho(y)) \right]^{1/2} \]
\[ \leq \sup_{|z-y| \leq t} \tilde{\omega}_\varphi \left[ ((2d + B_1)|z - y|^2 \right]^{1/2} \]
\[ \leq C \omega_\varphi(t^{1/2}), \]

where \( \tilde{C} := 1 + (2d + B_1)^{1/2} \) is a constant depending on \( \Omega \).

Recall that \( \xi \in \partial \Omega \) and fix \( 0 < r_1 < r \) and \( \gamma_1 \geq d/r_1 \) such that

\[ -\gamma_1 \tilde{\omega}_\varphi \left[ \left| |z| - \xi \right|^2 - B(\rho(z)) \right]^{1/2} \leq \inf_{\partial \Omega} \varphi - \sup_{\partial \Omega} \varphi, \]

for \( z \in \partial \Omega \cap \partial B(\xi, r_1) \). Let us set \( \gamma_2 = \inf_{\partial \Omega} \varphi \), then it follows that

\[ \gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi) \leq \gamma_2 \text{ for } z \in \partial B(\xi, r_1) \cap \tilde{\Omega}. \]

Let us put

\[ h_\xi(z) = \begin{cases} \max[\gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi), \gamma_2] & ; z \in \tilde{\Omega} \cap (B(\xi, r_1) \\ \gamma_2 & ; z \in \tilde{\Omega} \setminus B(\xi, r_1). \end{cases} \]

This is a well defined \( m \)-subharmonic function on \( \Omega \) and continuous on \( \tilde{\Omega} \). Moreover, it satisfies \( h_\xi(z) \leq \varphi(z) \) for all \( z \in \partial \Omega \). Since on \( \partial \Omega \cap B(\xi, r_1) \), we have

\[ \gamma_1(h(z) - \varphi(\xi)) + \varphi(\xi) = -\gamma_1 \tilde{\omega}_\varphi(|z - \xi|) + \varphi(\xi) \leq -\tilde{\omega}_\varphi(|z - \xi|) + \varphi(\xi) \leq \varphi(z). \]

Furthermore, the modulus of continuity of \( h_\xi \) satisfies

\[ \omega_{h_\xi}(t) \leq C \omega_\varphi(t^{1/2}), \]

where \( C := \gamma_1 \tilde{C} \) depends on \( \Omega \). Hence it is clear that \( h_\xi \) satisfies the three conditions above.

We have just proved that for each \( \xi \in \partial \Omega \), there is a function

\[ v_\xi \in \mathcal{V}_m(\Omega, \varphi, f), \ v_\xi(\xi) = \varphi(\xi), \text{ and } \omega_{v_\xi}(t) \leq \lambda \omega_\varphi(t^{1/2}), \]

where \( \lambda := \eta(1 + K_1) \). Let us set

\[ v(z) = \sup \{ v_\xi(z); \xi \in \partial \Omega \}. \]

We have \( 0 \leq \omega_v(t) \leq \lambda \omega_\varphi(t^{1/2}) \), then \( \omega_v(t) \) converges to zero when \( t \) converges to zero. Consequently, we get \( v \in C(\tilde{\Omega}) \) and \( v = v^* \in \text{SH}_m(\Omega) \).

Thanks to Choquet lemma, we can choose a nondecreasing sequence \( (v_j) \), where \( v_j \in \mathcal{V}_m(\Omega, \varphi, f) \), converging to \( v \) almost everywhere, so that

\[ L_\alpha v = \lim_{j \to \infty} L_\alpha v_j \geq f^{1/m}, \forall \alpha_i \in \Sigma_m. \]

It is clear that \( v(\xi) = \varphi(\xi), \forall \xi \in \partial \Omega \). Finally, we get \( v \in \mathcal{V}_m(\Omega, \varphi, f), v = \varphi \) on \( \partial \Omega \) and \( \omega_v(t) \leq \lambda \omega_\varphi(t^{1/2}) \). 

\[ \square \]
Corollary 3.3. Under the same assumption of Proposition 3.2, There exists a $m$-superharmonic function $\tilde{v} \in C(\Omega)$ such that $\tilde{v} = \varphi$ on $\partial \Omega$ and
\[
\omega_{\tilde{v}}(t) \leq \lambda \max\{\omega_{\varphi}(t^{1/2}), t^{1/2}\},
\]
where $\lambda > 0$ as in Proposition 3.2.

Proof. We can do the same construction as in the proof of Proposition 3.2 for the function $\varphi_1 = -\varphi \in C(\partial \Omega)$, then we get $v_1 \in V_m(\Omega, \varphi_1, f)$ such that $v_1 = \varphi_1$ on $\partial \Omega$ and $\omega_{v_1}(t) \leq \lambda \max\{\omega_{\varphi}(t^{1/2}), t^{1/2}\}$. Hence, we set $\bar{v} = -v_1$ which is a $m$-superharmonic function on $\Omega$, continuous on $\bar{\Omega}$ and satisfies $\bar{v} = \varphi$ on $\partial \Omega$ and $\omega_{\bar{v}}(t) \leq \lambda \max\{\omega_{\varphi}(t^{1/2}), t^{1/2}\}$. □

Proof of Theorem 3.1 Thanks to Proposition 3.2 we obtain a subsolution $v \in V_m(\Omega, \varphi, f)$, $v = \varphi$ on $\partial \Omega$ and $\omega_v(t) \leq \lambda \max\{\omega_{\varphi}(t^{1/2}), t^{1/2}\}$. Observing Corollary 3.3 we construct a $m$-superharmonic function $\tilde{v} \in C(\Omega)$ such that $\tilde{v} = \varphi$ on $\partial \Omega$ and $\omega_{\tilde{v}}(t) \leq \lambda \max\{\omega_{\varphi}(t^{1/2}), t^{1/2}\}$, where $\lambda = \eta(1 + \|f\|^1_{L_\infty(\partial \Omega)})$ and $\eta > 0$ depends on $\Omega$.

Applying the comparison principle (see [Bl05, Lu12]), we get that
\[
v(z) \leq U(z) \leq \tilde{v}(z) \text{ for all } z \in \bar{\Omega}.
\]
Let us set $g(t) = \max(\lambda \max\{\omega_{\varphi}(t^{1/2}), t^{1/2}\}, \omega_{f_1/m}(t))$, then
\[
|U(z) - U(\xi)| \leq g(|z - \xi|); \forall z \in \Omega, \forall \xi \in \partial \Omega
\]
because,
\[
-g(|z - \xi|) \leq v(z) - \varphi(\xi) \leq U(z) - \varphi(\xi) \leq \bar{v}(z) - \varphi(\xi) \leq g(|z - \xi|).
\]
Let us fix a point $z_0 \in \Omega$, For any small vector $\tau \in C^n$, we define
\[
V(z, \tau) = \begin{cases} U(z) & ; z + \tau \notin \Omega, z \in \bar{\Omega} \\ \max(U(z), v_1(z)) & ; z, z + \tau \in \bar{\Omega} \end{cases}
\]
where $v_1(z) = U(z + \tau) + g(||\tau||)|z - z_0|^2 - d^2 g(||\tau||) - g(||\tau||) .

Observe that if $z \in \Omega, z + \tau \in \partial \Omega$, we have
\[
v_1(z) - U(z) \leq g(||\tau||) + g(||\tau||)|z - z_0|^2 - Ag(||\tau||) - g(||\tau||) \leq 0 \quad (*)
\]
hence $v_1(z) \leq U(z)$ for $z \in \Omega, z + \tau \in \partial \Omega$. In particular, $V(z, \tau)$ is well defined and belongs to $SH_m(\Omega) \cap C(\Omega)$.

We assert that $L_0 V \geq f^{1/m}$ for all $\alpha \in m^*$, indeed
\[
L_0 v_1(z) \geq f^{1/m}(z + \tau) + g(||\tau||) L_0 (|z - z_0|^2) \\
\geq f^{1/m}(z + \tau) + g(||\tau||) \\
\geq f^{1/m}(z + \tau) + |f^{1/m}(z + \tau)| - f^{1/m}(z) \\
\geq f^{1/m}(z),
\]
for all $\alpha \in \Sigma_m, 1 \leq i \leq m - 1$.

If $z \in \partial \Omega, z + \tau \notin \Omega$, then $V(z, \tau) = U(z) = \varphi(z)$; On the other hand, $z \in \partial \Omega, z + \tau \in \Omega$, by (*) we get $V(z, \tau) = \max(U(z), v_1(z)) = U(z) = \varphi(z)$.

Then $V(z, \tau) = \varphi(z)$ on $\partial \Omega$, hence $V \in V_m(\Omega, \varphi, f)$. 

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Consequently, \( V(z, \tau) \leq U(z); \forall z \in \bar{\Omega} \). This implies that if \( z \in \Omega \), \( z + \tau \in \bar{\Omega} \), we have
\[
U(z + \tau) + g(\|\tau\|)|z - z_0|^2 - d^2g(\|\tau\|) - g(\|\tau\|) \leq U(z)
\]
Hence,
\[
U(z + \tau) - U(z) \leq (d^2 + 1)g(\|\tau\|) - g(\|\tau\|) \cdot |z - z_0|^2 \leq (d^2 + 1)g(\|\tau\|).
\]
Reversing the roles of \( z + \tau \) and \( z \), we get
\[
|U(z + \tau) - U(z)| \leq (d^2 + 1)g(\|\tau\|); \forall z \in \bar{\Omega}.
\]
Thus,
\[
\omega_U(t) \leq (d^2 + 1)g(t).
\]
Finally
\[
\omega_U(t) \leq \gamma \max\{\omega_\varphi(t^{1/2}), \omega_{f^{1/m}}(t), t^{1/2}\},
\]
where \( \gamma := \tau(1 + \|f\|_{L^\infty(\Omega)}) \) and \( \tau \geq 1 \) is a constant depending on \( \Omega \). \( \square \)

Theorem 1.1 has the following consequence.

**Corollary 3.4.** Let \( \Omega \) be a smoothly bounded strongly \( m \)-pseudoconvex domain in \( \mathbb{C}^n \). Let \( \varphi \in \mathcal{C}^{2\alpha}(\partial \Omega) \) and \( 0 \leq f^{1/m} \in \mathcal{C}^\alpha(\bar{\Omega}), \ 0 < \alpha \leq 1/2 \). Then the solution of Dirichlet problem \( U \) belongs to \( \mathcal{C}^\alpha(\bar{\Omega}) \).

This result was proved by Nguyen in [N13] for the homogeneous case \((f \equiv 0)\) (see also [Lu12, Lu13b]).

We now give examples to point out that there is a loss in the regularity up to the boundary and show that our result is optimal.

**Example 3.5.** Let \( \psi \) be a concave modulus of continuity on \([0, 1]\) and
\[
\varphi(z) = -\psi[\sqrt{(1 + \Re z_1)/2}], \ \text{for} \ z = (z_1, z_2, ..., z_n) \in \partial \mathbb{B} \subset \mathbb{C}^n.
\]
We can show that \( \varphi \in \mathcal{C}(\partial \mathbb{B}) \) with modulus of continuity \( \omega_\varphi(t) \leq C\psi(t) \) for some \( C > 0 \).

We consider the following Dirichlet problem:
\[
\begin{align*}
\left\{ u \in \mathcal{S}\mathcal{H}_m(\Omega) \cap \mathcal{C}(\bar{\Omega}) \\
(\ddc u)^m \wedge \beta^{n-m} = 0 \quad \text{in} \ \mathbb{B} \\
u = \varphi \quad \text{on} \ \partial \mathbb{B},
\end{align*}
\]
where \( 2 \leq m \leq n \) be an integer. Then by the comparison principle, \( U(z) := -\psi[\sqrt{(1 + \Re z_1)/2}] \) is the unique solution to this problem. One can observe by a radial approach to the boundary point \((-1, 0, ..., 0)\) that
\[
C_1\psi(t^{1/2}) \leq \omega_U(t) \leq C_2\psi(t^{1/2}),
\]
for some \( C_1, C_2 > 0 \).
4. Hölder continuity of the solution when $f \in L^p(\Omega)$.

4.1. Preliminaries and known results. The existence of a weak solution to complex Hessian equation in domains of $\mathbb{C}^n$ was established in the work of Dinew and Kolodziej [DK11]. More precisely, let $\Omega \subset \mathbb{C}^n$ be a smoothly $(m-1)$-pseudoconvex domain, $\varphi \in C(\partial \Omega)$ and $0 \leq f \in L^p(\Omega)$ for some $p > n/m$, then there exist $U \in SH_m(\Omega) \cap C(\bar{\Omega})$ such that $(dd^c U)^m \wedge \beta^{n-m} = f \beta^n$ in $\Omega$ and $U = \varphi$ on $\partial \Omega$.

Recently, N.C. Nguyen in [N13] proved that the Hölder continuity of this solution under some additional conditions on the density near the boundary and on the boundary data, that is for $f \in L^p(\Omega)$, $p > n/m$ bounded near the boundary $\partial \Omega$ or $f \leq C|\rho|^{-m\nu}$ there and $\varphi \in C^{1,1}(\partial \Omega)$.

Here we follow the approach proposed in [GKZ08] for the complex Monge-Ampère equation. A crucial role in this approach is played by an a priori weak stability estimate of the solution. This approach has been adapted to the complex Hessian equation in [N13] and [Lu12]. Here is the precise statement.

**Theorem 4.1.** Fix $0 \leq f \in L^p(\Omega)$, $p > n/m$. Let $\varphi, \psi \in SH_m(\Omega) \cap L^\infty(\bar{\Omega})$ be such that $(dd^c \varphi)^m \wedge \beta^{n-m} = f \beta^n$ in $\Omega$, and let $\varphi \geq \psi$ on $\partial \Omega$. Fix $r \geq 1$ and $0 \leq \gamma < \gamma_r$, where $\gamma_r = \frac{r}{r+mq+\frac{p(1-m)}{m}}$ and $1/p + 1/q = 1$. Then there exists a uniform constant $C = C(\gamma, \|f\|_{L^p(\Omega)}) > 0$ such that

$$
\sup_{\Omega} (\psi - \varphi) \leq C(\|\psi - \varphi\|_{L^r(\Omega)})^{\gamma}
$$

where $(\psi - \varphi)_+ := \max(\psi - \varphi, 0)$.

The second result gives the Hölder continuity under some additional hypothesis.

**Theorem 4.2.** ([N13]). Let $0 \leq f \in L^p, p > n/m$ and $\varphi \in C(\partial \Omega)$. Let $U$ be the continuous solution to (1.1). Suppose that there exists $v \in SH_m(\Omega) \cap C^{0,\nu}(\bar{\Omega})$, for $0 < \nu < 1$ such that $v \leq U$ in $\Omega$ and $v = U$ on $\partial \Omega$.

1) If $\nabla U \in L^2(\Omega)$ then $U \in C^{0,\alpha}(\Omega)$ for any $\alpha < \min\{\nu, \gamma_2\}$. 
2) If the total mass of $\Delta U$ is finite in $\Omega$ then $U \in C^{0,\alpha}(\Omega)$ for any $\alpha < \min\{\nu, 2\gamma_1\}$, where $\gamma_r = \frac{r}{r+mq+\frac{p(1-m)}{m}}$ for $r \geq 1$.

4.2. Construction of Hölder barriers. The remaining problem is to construct a Hölder continuous barrier with the right exponent which guarantees one of the conditions in Theorem 4.2.

Using the interplay between the real and the complex Monge-Ampère measures as suggested by Cegrell and Persson in [CP92], we will construct Hölder continuous $m$-subharmonic barriers for the problem (1.1) when $f \in L^p(\Omega)$, $p \geq 2n/m$. 
We recall that if $u$ is a locally convex smooth function in $\Omega$, its real Monge-Ampere measure is defined by

$$Mu := \det \left( \frac{\partial^2 u}{\partial x_j \partial x_k} \right) dV_{2n}.$$ 

When $u$ is only convex then $Mu$ can be defined following Alexandrov [A55] by means of the gradient image as a nonnegative Borel measure on $\Omega$ (see [Gut01], [RT77], [Gav77]).

**Proposition 4.3.** Let $0 \leq f \in L^p(\Omega)$, $p \geq 2n/m$ and $u$ be a locally convex function in $\Omega$ and continuous in $\bar{\Omega}$. If the real Monge-Amp`ere measure $Mu \geq f^{2n/m}dV_{2n}$ then the complex Hessian measure satisfies the inequality $(dd^c u)^n \wedge \beta^{n-m} \geq f \beta^n$ in the weak sense of measures on $\Omega$.

**Proof.** First step, we claim that $(dd^c u)^n \geq f^{n/m} \beta^n$ in the sense of measures. Indeed, for a smooth function $u$, we have

$$|\det(\partial^2 u/\partial z_j \partial \bar{z}_k)|^2 \geq \det(\partial^2 u/\partial x_j \partial x_k),$$

hence we get immediately that $(dd^c u)^n \geq f^{n/m} \beta^n$ (see [CP92]).

Moreover, it is well known for smooth convex function that

$$(Mu)^{1/n} = \inf \Delta_H u,$$

where $\Delta_H u := \sum_{j,k} h_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k}.$

for any symmetric positive definite matrix $H = (h_{jk})$ with $\det H = n^{-n}$ (see [Gav77], [Bl97]). In general case, we will prove that $(dd^c u)^n \geq f^{n/m} \beta^n$ weakly in $\Omega$. Indeed, the problem being local, we can assume that $u$ is defined and convex in a neighborhood of a ball $\bar{B} \subset \Omega$. For $\delta > 0$ we put $\mu_\delta := Mu * \rho_\delta$ then $\mu_\delta \geq g_\delta$ where $g_\delta := f^{2n/m} * \rho_\delta$ (without loss of generality we assume $g_\delta > 0$), we may assume that $u$ and $\mu_\delta$ are defined in this neighborhood of $\bar{B}$. Let $\varphi_\delta$ be a sequence of smooth function on $\partial B$ converging uniformly to $u$ there. Let $u^\delta$ is a smooth convex function such that $Mu^\delta = \mu_\delta$ in $B$ and $u^\delta = \varphi_\delta$ on $\partial B$. Let $\tilde{u}$ convex continuous on $\bar{B}$ such that $Mu = 0$ and $\tilde{u} = \varphi_\delta$ on $\partial B$. Moreover, let $v^\delta$ convex continuous on $\bar{B}$ such that $Mv^\delta = \mu_\delta$ and $v^\delta = 0$ on $\partial B$.

From the comparison principle for the real Monge-Ampère operator (see [RT77]), we can see that

$$(u^\delta)^x \leq \tilde{u} \leq u^\delta.$$ 

It follows from Lemma 3.5 in [RT77] that

$$(v^\delta(x))^x \leq c_n(diam B)^{-n-1} \text{dist}(x, \partial B)Mv^\delta(B), \quad x \in B.$$ 

Then we conclude that $\{u^\delta\}$ is uniformly bounded on $B$, hence there exists a subsequence $\{u^{\delta_j}\}$ converging locally uniformly on $B$.

Moreover, (4.3) and (4.4) imply that $\{u^{\delta_j}\}$ is uniformly convergent on $\bar{B}$. 


From the comparison principle it follows that $u^{δ_j}$ converges uniformly to $u$. Since $u^{δ_j} ∈ C^∞(B)$ and $Mu^{δ_j} ≥ f^{2n/m} * ρ_{δ_j} dV_{2n}$, we get that

$$(dd^c u^{δ_j})^n ≥ (f^{2n/m} * ρ_{δ_j})^{1/2} β^n$$

Finally, as $u^{δ_j}$ converges uniformly to $u$ and by the convergence theorem of Bedford and Taylor, we get that

$$(dd^c u)^n ≥ f^{n/m} β^n.$$  

Second step, let $v = |z|^2 ∈ PSH(Ω)$. Since $(dd^c u)^n ≥ f^{n/m} β^n$ and $(dd^c v)^n = β^n$, by Theorem 1.1 in [Di09] we get that

$$(dd^c u)^m ∧ β^{n−m} ≥ f β^n.$$  

We recall the theorem of existence of convex solution to the Dirichlet problem for the real Monge-Ampère equation, this theorem is due to Rauch and Taylor.

**Theorem 4.4. ([RT77]).** Let $Ω$ is strictly convex domain. Let $φ ∈ C(∂Ω)$ and $µ$ is a non negative Borel measure on $Ω$ with $µ(Ω) < ∞$. Then there is a unique convex $u ∈ C(Ω)$ such that $Mu = µ$ in $Ω$ and $u = φ$ on $∂Ω$.

The following result gives the existence of a $1/2$-Hölder continuous $m$-subharmonic barrier for the problem (1.1) when $f ∈ L^p(Ω)$, $p ≥ 2n/m$.

**Theorem 4.5.** Let $φ ∈ C^{0,1}(∂Ω)$ and $f ∈ L^p(Ω)$, $p ≥ 2n/m$. Then there exists $v ∈ SH_m(Ω) ∩ C^{0,1/2}(Ω)$ such that $v = φ$ on $∂Ω$ and $(dd^c v)^m ∧ β^{n−m} ≥ f β^n$ in the weak sense of currents, hence $v ≤ u$ in $Ω$.

**Proof.** Let $B$ a big ball contains $Ω$ and let $f$ be the function defined by $f = f$ on $Ω$ and $f = 0$ on $B \setminus Ω$. Then $f ∈ L^p(B)$, $p ≥ 2n/m$. Let $µ := f^{2n/m} β^n$. This is a nonnegative Borel measure on $B$ with $µ(B) < ∞$. Thanks to Theorem 1.4 there exists a unique convex function $u ∈ C(Ω)$ such that $Mu = µ$ in $B$ and $u = 0$ on $∂B$. Hence $u$ is lipschitz continuous on $Ω$. By Proposition 4.3 we have $(dd^c u)^m ∧ β^{n−m} ≥ f β^n$ in $Ω$.

We will construct the required barrier as follows. Let $h_{φ−u}$ be the upper envelope of $V_m(Ω, φ − u, 0)$. Then, thanks to Theorem 1.1, $h_{φ−u}$ is Hölder continuous of exponent $1/2$ in $Ω$. Now it is easy to check that $v := u + h_{φ−u}$ is $m$-sh in $Ω$ and satisfies $v = φ$ in $∂Ω$ and $(dd^c v)^m ∧ β^{n−m} ≥ f β^n$ on $Ω$, hence $v ≤ u$ in $Ω$.

The last theorem provides us with a Hölder continuous barrier for the Dirichlet problem (1.1) with better exponent.

However, when $f ∈ L^p(Ω)$ for $p > n/m$, we can find a Hölder continuous barrier with small exponent less than $γ_1$.

**Proposition 4.6.** Let $φ ∈ C^{0,1}(∂Ω)$ and $f ∈ L^p(Ω)$, $p > n/m$. Then there exists $v ∈ SH_m(Ω) ∩ C^{α,1}(Ω)$ for $α < γ_1 = \frac{1}{1+mq+\frac{pq(n−m)}{p−m}}$ such that $v = φ$ on $∂Ω$ and $v ≤ u$ in $Ω$. 
Proof. Let us fix a large ball $B \subset \mathbb{C}^n$ such that $\Omega \Subset B \subset \mathbb{C}^n$. We define $\tilde{f} = f$ in $\Omega$ and $\tilde{f} = 0$ in $B \setminus \Omega$. Let $h_1$ to the Dirichlet problem in $B$ with density $\tilde{f}$ and zero boundary values. Since $\tilde{f} \in L^p(\Omega)$ is bounded near $\partial B$, $h_1$ is Hölder continuous on $B$ with exponent $\alpha < 2\gamma_1$ (see [N13]). Now let $h_2$ denote the solution to the Dirichlet problem in $\Omega$ with boundary values $\varphi - h_1$ and the zero density. Thanks to Theorem [13] we see that $h_2 \in C^{0,\alpha_2}(\Omega)$ where $\alpha_2 = \alpha_1/2$. Therefore, the required barrier will be $v = h_1 + h_2$. It is clear that $v \in \text{SH}_m(\Omega) \cap C(\bar{\Omega})$, $v|_{\partial \Omega} = \varphi$ and $(dd^c v)^m \wedge \beta^n - m \geq f \beta^n$ in the weak sense in $\Omega$. Hence, by the comparison principle we get that $v \leq u$ in $\Omega$ and $v = u = \varphi$ on $\partial \Omega$. Moreover we have $v \in C^{0,\alpha}(\Omega)$ for any $\alpha < \gamma_1$. \hfill $\square$

We will need in the sequel the following elementary lemma.

**Lemma 4.7.** Let $u, v \in \text{SH}_m(\Omega) \cap C(\bar{\Omega})$ such that $v \leq u$ on $\Omega$ and $u = v$ on $\partial \Omega$. Then

$$\int_\Omega dd^c u \wedge \beta^{n-1} \leq \int_\Omega dd^c v \wedge \beta^{n-1}.$$ 

We recall the definition of the class $E^0_m(\Omega)$ (see [Lu13c]).

**Definition 4.8.** We denote $E^0_m(\Omega)$ the class of bounded functions $v$ in $\text{SH}_m(\Omega)$ such that $\lim_{z \rightarrow \partial \Omega} v(z) = 0$ and $\int_\Omega (dd^c v)^m \wedge \beta^{n-m} < +\infty$.

The following proposition was proved by induction in [Ce04] for plurisubharmonic functions and we can do the same argument for $m$-sh functions.

**Proposition 4.9.** Suppose that $h, u_1, u_2 \in E^0_m(\Omega)$, $p, q \geq 1$ such that $p+q \leq m$ and $T = dd^c g_1 \wedge ... \wedge dd^c g_{m-p-q} \wedge \beta^{n-m}$ where $g_1, ..., g_{m-p-q} \in E^0_m(\Omega)$. Then we get

$$\int_\Omega -h(dd^c u_1)^p \wedge (dd^c u_2)^q \wedge T \leq \left[ \int_\Omega -h(dd^c u_1)^{p+q} \wedge T \right]^{\frac{p}{p+q}} \left[ \int_\Omega -h(dd^c u_2)^{p+q} \wedge T \right]^{\frac{q}{p+q}}.$$ 

**Proof.** We first prove the statement for $p = q = 1$. Thanks to the Cauchy-Schwarz inequality (see [Lm13c]), we have

$$\int_\Omega -hdd^c u_1 \wedge dd^c u_2 \wedge T = \int_\Omega -u_1 dd^c u_2 \wedge dd^c h \wedge T \leq \left[ \int_\Omega -u_1 dd^c u_1 \wedge dd^c h \wedge T \right]^{1/2} \left[ \int_\Omega -u_2 dd^c u_2 \wedge dd^c h \wedge T \right]^{1/2} = \left[ \int_\Omega -h(dd^c u_1)^2 \wedge T \right]^{1/2} \left[ \int_\Omega -h(dd^c u_2)^2 \wedge T \right]^{1/2}.$$ 

The general case follows by induction in the same way as in [Ce04]. \hfill $\square$

We will only need the following particular case.

**Corollary 4.10.** Let $u_1, u_2 \in E^0_m(\Omega)$. Then we have

$$\int_\Omega dd^c u_1 \wedge (dd^c u_2)^{m-1} \wedge \beta^{n-m} \leq \left[ \int_\Omega (dd^c u_1)^m \wedge \beta^{n-m} \right]^{\frac{1}{m}} \left[ \int_\Omega (dd^c u_2)^{m} \wedge \beta^{n-m} \right]^{-\frac{1}{m}}.$$
4.3. Proof of Theorem 4.2. Let $u_0$ the solution to the Dirichlet problem (1.1) with zero boundary values and the density $f$. We first claim that the total mass of $\Delta u_0$ is finite in $\Omega$. Indeed, let $\rho$ be the defining function of $\Omega$, then by Corollary 4.10 we have that (4.5)

$$
\int_{\Omega} dd^c u_0 \wedge (dd^c \rho)^{m-1} \wedge \beta^{n-m} \leq \left( \int_{\Omega} (dd^c u_0)^m \wedge \beta^{n-m} \right)^{\frac{1}{m}} \left( \int_{\Omega} (dd^c \rho)^m \wedge \beta^{n-m} \right)^{\frac{m-1}{m}}.
$$

Since $\Omega$ is a bounded strongly $m$-pseudoconvex domain, there exists a constant $c > 0$ such that $(dd^c \rho)^2 \wedge \beta^{n-j} \geq c \beta^n$ in $\Omega$ and we can find $A \gg 1$ such that $A \rho - |z|^2$ is $m$-sh function. Now it is easy to see that

$$
\int_{\Omega} dd^c u_0 \wedge \beta^{n-1} \leq \int_{\Omega} dd^c u_0 \wedge (Add^c \rho)^m-1 \wedge \beta^{n-m}.
$$

Hence the inequality (4.5) yields

$$
\int_{\Omega} dd^c u_0 \wedge \beta^{n-1} \leq A^{m-1} \left[ \int_{\Omega} (dd^c u_0)^m \wedge \beta^{n-m} \right]^\frac{1}{m} \left[ \int_{\Omega} (dd^c \rho)^m \wedge \beta^{n-m} \right]^{\frac{m-1}{m}}.
$$

Now we note that the total mass of complex Hessian measures of $\rho$ and $u_0$ are finite in $\Omega$. Therefore, the total mass of $\Delta u_0$ is finite in $\Omega$.

Let $\tilde{\varphi}$ be a $C^{1,1}$-extension of $\varphi$ to $\Omega$ such that $\| \varphi \|_{C^{1,1}(\Omega)} \leq C \| \varphi \|_{C^{1,1}(\partial \Omega)}$ for some $C > 0$. Now, let $v = A \rho + \tilde{\varphi} + u_0$ where $A \gg 1$ such that $A \rho + \tilde{\varphi} \in SH_m(\Omega) \cap C(\bar{\Omega})$. By the comparison principle we see that $v \leq u$ in $\Omega$ and $v = u = \varphi$ on $\partial \Omega$. Since $\rho$ is smooth in a neighborhood of $\Omega$ and $\| \Delta u_0 \|_{\Omega} < +\infty$, we get that $\| \Delta v \|_{\Omega} < +\infty$. Then by Lemma 4.7 we have $\| \Delta u \|_{\Omega} < +\infty$.

When $p > n/m$, we can get by Theorem 4.6 and Theorem 4.2 that $u \in C^{0,\alpha}(\Omega)$ where $\alpha < \gamma_1$.

Moreover, if $p \geq 2n/m$, the Proposition 4.11 gives the existence of a $1/2$-H"older continuous barrier to the Dirichlet problem. Then using Theorem 4.2 we obtain that $u \in C^{0,\alpha}(\Omega)$ where $\alpha < \min\{1/2, 2\gamma_1\}$.

Finally, in the particular case when $f \in L^p(\Omega)$, for $p > n/m$ and satisfies some condition near the boundary $\partial \Omega$, we obtain a better exponent.

**Proposition 4.11.** Let $\Omega \subset \mathbb{C}^n$ be a strongly $m$-pseudoconvex bounded domain with smooth boundary, suppose that $\varphi \in C^{1,1}(\partial \Omega)$ and $0 \leq f \in L^p(\Omega)$, for some $p > n/m$, and

$$
f(z) \leq (h \circ \rho(z))^m \text{ near } \partial \Omega,
$$

where $\rho$ is the defining function on $\Omega$ and $0 \leq h \in L^2([-A,0])$, with $A \geq \sup_{\Omega} |\rho|$, be an increasing function. Then the solution $u$ to (1.1) is H"older continuous with exponent $\alpha < \gamma_2$.

**Proof.** Let $\chi : [-A,0] \to \mathbb{R}^-$ be the primitive of $h$ such that $\chi(0) = 0$. It is clear that $\chi$ is a convex increasing function. By the H"older inequality, we
see that
\[ |\chi(t_1) - \chi(t_2)| \leq \|h\|_{L^2} |t_1 - t_2|^{1/2}, \]
for all \( t_1, t_2 \in [-A, 0] \). From the hypothesis, there exists a compact \( K \subseteq \Omega \) such that
\[ f(z) \leq (h \circ \rho(z))^m \text{ for } z \in \Omega \setminus K. \]

Then the function \( v = \chi \circ \rho \) is \( m \)-subharmonic in \( \Omega \), continuous in \( \bar{\Omega} \) and satisfies
\[ dd^c \chi \circ \rho = \chi''(\rho) d\rho \wedge dd^c \rho + \chi'(\rho) dd^c \rho \geq \chi'(\rho) dd^c \rho, \]
in the sense of currents on \( \Omega \).

From the definition of \( \rho \) (see the Definition 2.7), there is a constant \( A > 0 \) such that
\[ (dd^c \rho)^m \wedge \beta^{n-m} \geq A \beta^n, \]
thus the inequality (4.6) yields
\[ (dd^c v)^m \wedge \beta^{n-m} \geq A (h \circ \rho)^m \beta^n \geq A f \beta^n \text{ in } \Omega \setminus K. \]

Now consider a \( C^{1,1} \) extension \( \tilde{\varphi} \) of \( \varphi \) to \( \bar{\Omega} \) and choose \( B \gg 1 \) large enough so that \( \tilde{\varphi} + B \rho \) is \( m \)-subharmonic in \( \Omega \) and
\[ \tilde{v} := B v + \tilde{\varphi} + B \rho \leq U \text{ in a neighborhood of } K. \]

Then \( \tilde{v} \) is \( m \)-subharmonic in \( \Omega \) and if \( B \geq 1/A \), it follows from (4.7) that
\[ (dd^c \tilde{v})^m \wedge \beta^{n-m} \geq f \beta^n \text{ in } \Omega \setminus K. \]

By the comparison principle (see [Bl05, Lu12]), we have \( \tilde{v} \leq U \) on \( \Omega \setminus K \), hence we get \( \tilde{v} \leq U \) on \( \Omega \), \( \tilde{v} = \varphi \) on \( \partial \Omega \) and \( \tilde{v} \in C^{0,1/2}(\bar{\Omega}) \).

We claim that \( \nabla \tilde{v} \in L^2(\Omega) \). Indeed, it is enough to observe that \( \tilde{\varphi} + B \rho \) is Lipschitz in \( \bar{\Omega} \) and
\[ \int_{\Omega} dv \wedge d\tilde{v} \wedge \beta^{n-1} = \int_{\Omega} (h \circ \rho)^2 d\rho \wedge d\rho \wedge \beta^{n-1} < +\infty, \]
since \( h \circ \rho \in L^2(\Omega) \).

Therefore \( v \leq U \) and \( \tilde{v} = \varphi \) on \( \partial \Omega \). Then an easy integration by parts shows that
\[ \int_{\Omega} dU \wedge d\tilde{v} \wedge \beta^{n-1} \leq \int_{\Omega} d\tilde{v} \wedge d\tilde{v} \wedge \beta^{n-1} < +\infty, \]
hence \( \nabla U \in L^2(\Omega) \) (see [GKZ08, N13]).

By Theorem 4.2 we get that \( U \in C^{0,\alpha}(\bar{\Omega}) \) for any \( \alpha < \min\{1/2, \gamma_2\} = \gamma_2 \).

As an example of application of the last result, fix \( p > n/m \), take \( h(t) := t^{-\alpha} \) with \( 0 < \alpha < 1/(pm), t < 0 \) and define \( f := (h \circ \rho)^m \).
4.4. Hölder continuity for radially symmetric solution. Here we consider the case when the right hand side and the boundary data are radial. In this case, Yong and Lu [YL10] gave an explicit formula for the radial solution of the Dirichlet problem \( (1.1) \) with \( f \in C(\mathbb{B}) \). Moreover, they studied higher regularity for radial solutions (see also [DD12]).

Here, we will extend this explicit formula to the case when \( f \in L^p(\mathbb{B}) \), for \( p > n/m \), is a radial non-negative function and \( \varphi \equiv 0 \) on \( \partial \mathbb{B} \). Then we prove Hölder continuity of the radially symmetric solution with a better exponent which turns out to be optimal.

**Theorem 4.12.** Let \( f \in L^p(\mathbb{B}) \) be a radial function, where \( p > n/m \). Then the unique solution \( U \) for \( (1.1) \) with zero boundary value is given by the explicit formula

\[
U(r) = -B \int_r^1 \frac{1}{t^{2n/m-1}} \left( \int_0^t \rho^{2n-1}f(\rho) d\rho \right)^{1/m} dt,
\]

where \( B = \left( \frac{c_n}{2n/m+1} \right)^{-1/m} \). Moreover, \( U \in C^{0,2-\frac{2n}{mp}}(\mathbb{B}) \) for \( n/m < p < 2n/m \) and \( U \in Lip(\mathbb{B}) \) for \( p \geq 2n/m \).

**Proof.** Let \( f_k \in C(\mathbb{B}) \) positive radial symmetric function such that \( \{f_k\} \) converges to \( f \) in \( L^p(\mathbb{B}) \). Then there exists a unique solution \( U_k \in C(\mathbb{B}) \) for \( \text{Dir}(\mathbb{B},0,f_k) \) (see [YL10]) given by the following formula:

\[
U_k(r) = -B \int_r^1 \frac{1}{t^{2n/m-1}} \left( \int_0^t \rho^{2n-1}f_k(\rho) d\rho \right)^{1/m} dt.
\]

It is clear that \( U_k \) converges in \( L^1(\mathbb{B}) \) to the function \( \tilde{u} \) given by the same formula i.e.

\[
\tilde{u}(r) = -B \int_r^1 \frac{1}{t^{2n/m-1}} \left( \int_0^t \rho^{2n-1}f(\rho) d\rho \right)^{1/m} dt.
\]

We claim that the sequence \( \{U_k\} \) is uniformly bounded and equicontinuous in \( \mathbb{B} \). Indeed, let \( 0 < r < r_1 \leq 1 \), we have

\[
|U_k(r_1) - U_k(r)| = B \int_r^{r_1} \frac{1}{t^{2n/m-1}} \left( \int_0^t \rho^{2n-1}f_k(\rho) d\rho \right)^{1/m} dt \\
\leq B \int_r^{r_1} \frac{1}{t^{2n/m-1}} \left( \int_0^t \rho^{(2n-1)/q} \rho^{(2n-1)/p} f_k(\rho) d\rho \right)^{1/m} dt \\
\leq C \|f_k\|_{L^p(\mathbb{B})} \int_r^{r_1} \frac{1}{t^{2n/m-1}} \left( \int_0^t \rho^{2n-1} d\rho \right)^{1/mq} dt \\
\leq C \|f_k\|_{L^p(\mathbb{B})} (r_1^{2-\frac{2n}{mp}} - r^{2-\frac{2n}{mp}}).
\]

Since \( f_k \) converges to \( f \) in \( L^p(\mathbb{B}) \), we get \( \|f_k\|_{L^p(\mathbb{B})} \leq C \) where \( C > 0 \) does not depend on \( k \), hence \( U_k \) is equicontinuous on \( \mathbb{B} \). By Arzelà-Ascoli theorem, there exists a subsequence \( U_{k_j} \) converges uniformly to \( \tilde{u} \).

Consequently, \( \tilde{u} \in SH_m(\mathbb{B}) \cap C(\mathbb{B}) \) and thanks to the convergence theorem...
for the Hessian operator (see [Lu12]) we can see that \((dd^c \tilde{u})^n = f \beta^n\) in \(\mathbb{B}\).

Passing to the limit in the inequality

\[ |U_k(r_1) - U_k(r)| \leq C\|f_k\|_{L^p(\mathbb{B})}^{1/m} (r_1^{2 - \frac{2n}{mp}} - r^{2 - \frac{2n}{mp}}), \]

we get that

\[ |\tilde{u}(r_1) - \tilde{u}(r)| \leq C\|f\|_{L^p(\mathbb{B})}^{1/m} (r_1^{2 - \frac{2n}{mp}} - r^{2 - \frac{2n}{mp}}). \]

Hence, for \(p \geq 2n/m\) we get \(\tilde{u} \in \text{Lip}(\overline{\mathbb{B}})\) and for \(n/m < p < 2n/m\), we have \(\tilde{u} \in C^{0,2 - \frac{2n}{mp}}(\overline{\mathbb{B}})\).

□

We give an example which illustrates that the Hölder exponent \(2 - \frac{2n}{mp}\) given by the Theorem 4.12 is optimal.

Example 4.13. Let \(p \geq 1\) a fixed exponent. Take \(f_\alpha(z) = \frac{1}{|z|^\alpha}\), with \(0 < \alpha < 2n/p\). Then it is clear that \(f_\alpha \in L^p(\mathbb{B})\). The unique radial solution to the Dirichlet problem (1.1) with right hand side \(f_\alpha\) and zero boundary value is given by

\[ U_\alpha(z) = c(r^{2-\alpha/m} - 1), r := |z| \leq 1, \]

where \(c = \left(\frac{C^{2m/m}}{2m-m}\right)^{-1/m} \left(\frac{1}{2n-\alpha}\right)^{1/m} \frac{m}{2m-\alpha} \). Then we have

1. If \(p > n/m\) then \(0 < \alpha < 2m\) and the solution \(U_\alpha\) is \((2 - \frac{2n}{mp} + \delta)\) Hölder with \(\delta = (2n/p - \alpha)/m\). Since \(\alpha\) can be chosen arbitrarily close to \(2n/p\), this implies that the optimal Hölder exponent is \(2 - \frac{2n}{mp}\).

2. Observe that when \(1 \leq p < n/m\) and \(2m < \alpha < 2n\), then the solution \(U_\alpha\) is unbounded.

The next example shows that in Theorem 4.12 \(n/m\) is the critical exponent in order to have a continuous solution.

Example 4.14. Consider the density \(f\) given by the formula

\[ f(z) := \frac{1}{|z|^{2m(1 - \log|z|)\gamma}}, \]

where \(\gamma > m/n\) is fixed.

It is clear that \(f \in L^{n/m}(\mathbb{B}) \setminus L^{n/m+\delta}(\mathbb{B})\) for any \(\delta > 0\). An elementary computation shows that the corresponding solution \(U\) given by the explicit formula (4.8) can be estimated by

\[ U(z) \leq C(1 - (1 - \log|z|)^{1-\gamma/m}), \]

where \(C > 0\) depends only on \(n, m\) and \(\gamma\). Hence we see that if \(m/n < \gamma < m\) then \(U\) goes to \(-\infty\) when \(z\) goes to \(0\). In this case the solution \(U\) is unbounded.

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