Surface expression of a wall fountain: application to subglacial discharge plumes

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(Received ?; revised ?; accepted ?. - To be entered by editorial office)

We investigate the surface expression of a fountain that is released into a homogeneous fluid, adjacent either to a vertical or a sloping wall, that then spreads horizontally at the free surface. We present a model that separates the fountain into two separate regions: a vertical fountain and a horizontal, negatively buoyant jet. The model is compared to laboratory experiments that are conducted over a range of volume fluxes, density differences, and ambient fluid depths. It is shown that the non-dimensionalised size and aspect ratio of the surface expression are dependent on the Froude number, calculated at the start of the negatively buoyant jet. The model is applied to observations of the surface expression from a Greenland subglacial discharge plume. In the case where the discharge plume reaches the surface with negative buoyancy the model can be used to estimate the discharge properties at the base of the glacier.

Key words:

1. Introduction

The mass loss from the Greenland and Antarctic ice sheets is an increasingly significant component of global sea level rise. Interactions between the polar oceans and the ice sheets are an important control on the rate of glacier melting, but are not well understood. An area of recent focus has been the subglacial discharge plumes that are released at the base of Greenland glaciers and that have been linked to elevated melt-rates (e.g. Slater et al. 2016, Carroll et al. 2015, Straneo & Cenedese 2015).

These subglacial discharge plumes originate from surface melting of the ice sheet. The meltwater then flows to the base of the ice shelf where it travels underneath the ice until it is released at the grounding line — the location where the ice shelf becomes afloat. The meltwater forms a turbulent plume that rises up the ice face entraining relatively warm and salty water, transporting heat and salt to the ice and driving rapid melting as demonstrated in laboratory experiments by McConnochie & Kerr (2017). It is typically assumed that the ice face near subglacial plumes is vertical although recent observations have suggested that the ice face can be undercut (Fried et al. 2015).

Greenland fjords typically have an approximately two-layer stratification (Straneo et al. 2011, 2012). Hence the meltwater plume can either intrude at mid-depth, if its density is higher than the upper layer, propagate away from the ice at the free surface, if its...
density is lower than the upper layer, or reach the free surface due to excessive vertical momentum before sinking back down to a level of neutral buoyancy (Beaird et al. 2015; Cenedese & Gatto 2016; Sciascia et al. 2013).

In this study we focus on the third scenario where the meltwater rises through the upper layer as a fountain and then sinks. In contrast to canonical fountains in semi-infinite environments where the flow rises until it has zero vertical momentum and then falls back to a level of neutral buoyancy around the rising fluid (Hunt & Burridge 2015), we focus on flows that reach the free surface with significant vertical momentum. This causes the fluid to spread horizontally at the surface for some distance before sinking down to a level of neutral buoyancy. As a result, the subglacial discharge is often visible as a pool of sediment-laden fluid at the free surface (e.g. How et al. 2017; Mankoff et al. 2016).

Mankoff et al. (2016) photographed a well defined region of turbid fluid in front of Saqqarliup Sermia, Greenland, that was interpreted as a subglacial discharge pool. The pool was triangular in shape, approximately 300 m wide at the glacier face and stretched 300 m away from the glacier. Considering that subglacial discharge plumes are typically assumed to be semi-circular (Slater et al. 2016; Mankoff et al. 2016), the triangular shape is somewhat surprising and as yet, has not been explained.

There are several possible explanations for the triangular shaped surface expression such as a sloping glacier face, a secondary circulation induced by the narrow fjord, and increased melting induced by the plume itself leading to an incised glacier face that redirects the surface outflow. In this paper we use a set of laboratory experiments to investigate several controlling mechanisms on the surface expression of subglacial discharge plumes which we model as a fountain, next to a wall, that reaches the free surface. We consider a fountain rather than a plume as we are interested in a flow that will reach the free surface, spread for some distance and then sink. By considering a fountain we effectively limit the investigation to the region where the flow is above its level of neutral buoyancy.

Turbulent fountains have been extensively studied in the past (see Hunt & Burridge 2015). Many of the previous studies have examined the entrainment of ambient fluid into an axisymmetric fountain (e.g. Burridge & Hunt 2016; Bloomfield & Kerr 1998). Although turbulent fountains and turbulent plumes are dynamically very similar, entrainment into fountains is significantly more complex due to the potential reentrainment of sinking fluid into the rising fountain. The problem of a fountain adjacent to a wall has also been of interest to many authors given its applicability to building fires and enclosed convection (e.g. Goldman & Jaluria 1986; Kapoor & Jaluria 1989).

Despite the previous work on turbulent fountains, there are features of subglacial discharge fountains that have not been fully studied. First, much of the previous work on wall fountains has focused on two-dimensional flows whereas we are interested in wall fountains generated from a point source. In addition, subglacial discharge fountains reach the free surface with a significant vertical momentum that causes them to spread horizontally before sinking. As such the upward and downward flows can be spatially separated, causing the horizontal flow field at the free surface to be important to the overall dynamics. As well as the applicability to subglacial discharge surface expressions, understanding this surface flow could be important in a variety of similar problems as it will control where the source fluid will come to rest and how much entrainment of ambient fluid will occur.

From the definition of a fountain given in Hunt & Burridge (2015), the potential decoupling between the upward and downward flows would suggest that the considered situation is not in fact a fountain but is a vertical, negatively buoyant jet. However, we
retain the term fountain for simplicity, and because there is a continuous transition as the source momentum flux increases (or the source buoyancy decreases) from the vertical flow losing all of its vertical momentum before the free surface to reaching the free surface with excess momentum.

The purpose of this paper is to investigate some of the physical processes that could control the size and shape of the surface expression of a subglacial discharge plume. The hope is that if the processes controlling the surface expression are well understood, then the subglacial discharge properties could be inferred from visual observations of the fjord surface.

In §2 we present a theoretical model of the fountain that results from a subglacial discharge plume and of its surface expression. In §3 we describe the experimental apparatus. §4 and §5 give the experimental results and comparisons with predictions from the theoretical model. Finally in §6 we apply the model to the observations of a surface pool described in Mankoff et al. (2016) and attempt to infer the sub-surface properties of the plume.

2. Theory

We consider a scenario similar to that occurring at Greenland glacier fronts: the steady and vertical release of freshwater, from a single point source located next to a wall, into a relatively deep (many source radii) two-layer stratification. The release of freshwater will produce buoyant fluid that is typically modelled as a semi-circular plume (Slater et al. 2016; Mankoff et al. 2016). Although the initial freshwater discharge is likely to have some horizontal momentum in the geophysical case, it is common to model the discharge as a purely vertical flow as the length scale whereby the discharge attaches to the wall is typically much smaller than the total water depth (e.g. Cowton et al. 2015; Xu et al. 2013).

To produce a surface expression of the plume that is denser than the upper layer, we set the ambient density profile and subglacial discharge characteristics such that the plume is initially positively buoyant but, due to entrainment of the lower layer fluid, it becomes more dense than the upper layer. However, the vertical momentum at the interface between the lower and upper layers is assumed to be sufficient that the plume will rise to the free surface and spread horizontally for some distance before sinking back to the interface depth. This is shown schematically in figure 1.

To simplify the experiments (§3), we will ignore the positively buoyant plume in the lower layer and consider a dynamically equivalent system only comprising the lighter upper layer. The initial discharge in the simplified system is equivalent to the plume at the density interface in the full geophysical system. The experimental system is shown as the unhatched region in figure 1. In the following section we consider only the simplified system but note that the same equations that are used for the fountain in the upper layer can be applied to the buoyant plume that forms in the lower layer.

The flow can be considered in two separate regions: first, a fountain that rises up the wall and second, a horizontal negatively buoyant jet that is generated at the free surface. As the flow transitions from the first to the second region we assume that all vertical momentum is converted to horizontal momentum. In the appendix we show how the vertical momentum is converted into horizontal momentum in the transition region. Details of the two separate regions and the transition are described below.
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Figure 1. Schematic showing the flow being considered. Relatively fresh fluid is released into a two-layer stratification with upper layer density $\rho_u$ and lower layer density $\rho_l$. Labels 1, 2, and T show the two regions of the flow and the transition from vertical to horizontal flow. The experiments only consider the upper (unhatched) layer. Therefore the interface between the two density layers on this figure is the base of the experimental tank. $s_y$ shows the size of the surface expression in the $y$ (wall-perpendicular) direction as measured in our experiments and $\delta$ is the thickness of the horizontal negatively buoyant jet.

2.1. Wall fountain

The canonical equations for a buoyant plume from Morton et al. (1956) are used in region 1. Following Slater et al. (2016) and Ezhova et al. (2018) we have adjusted the plume model to account for a semi-circular geometry caused by the wall. The fountain volume flux $Q$, momentum flux $M$, and buoyancy flux $B$ are calculated from

$$\frac{dQ}{dz} = \pi \alpha bw,$$

$$\frac{dM}{dz} = \pi b^2 g' \frac{g'}{2} - 2C_d bw^2,$$

and

$$\frac{dB}{dz} = \frac{d}{dz} \left( \frac{\pi b^2 w g'}{2} \right) = 0$$

where $z$ is the height above the source, $b$ is the top-hat fountain radius, $w$ is the top-hat fountain velocity, $g'$ is the top-hat reduced gravity of the fountain, $C_d$ is the drag coefficient assumed to be 0.0025 (Cowton et al. 2015) and $\alpha = w_e/w$ is the entrainment coefficient with $w_e$ being the velocity with which ambient fluid is entrained into the fountain. The value for $\alpha$ is determined by the experiments described in §4. It is assumed that the drag against the wall is negligible compared to the buoyancy forces. Using a drag coefficient of 0.0025, the drag force is estimated to be approximately 5% of the buoyancy force in the laboratory experiments and typically < 3% of the buoyancy force for a geophysical scenario. Equations (2.1)-(2.3) are initialised at $z = 0$ using the values of the volume, momentum and buoyancy fluxes and the discharge area at the source. These equations are used to determine the vertical fluxes at the free surface.

2.2. Transition from vertical to horizontal fluxes

Following Ezhova et al. (2018), we expect the fountain to have roughly Gaussian time-averaged density and velocity profiles with a greater width in the wall-parallel direction than in the wall-perpendicular direction. The more rapid spreading in the wall-parallel direction than in the wall-perpendicular direction is explained by Launder & Rodi (1983) in the context of wall jets by the interaction with the wall leading to non-isotropic
turbulent fluctuations — eddies normal to the wall cannot be as large as eddies parallel to the wall.

Although the density maximum will be located at the wall, the velocity maximum will be offset some distance due to the no-slip boundary condition imposed by the wall. The distance of this offset, \( y_0 \), is taken from direct numerical simulations of a wall plume (Ezhova et al. 2018) and we approximate the velocity profile between the wall and the maximum velocity location as linear in the \( y \) direction and Gaussian in the \( x \) direction.

The velocity and density profiles at the free surface can then be described as

\[
w(x, y) = \begin{cases} 
  w_0 \exp \left[ -\frac{1}{2} \left( \frac{x^2}{m^2} + \frac{(y - y_0)^2}{n^2} \right) \right], & y > y_0 \\
  \left( \frac{y w_0}{y_0} \right) \exp \left[ -\frac{x^2}{2m^2} \right], & y < y_0 
\end{cases}
\]

and

\[
g' = g'_0 \exp \left[ -\frac{1}{2} \left( \frac{x^2}{m^2} + \frac{y^2}{n^2} \right) \right],
\]

where \( w_0 \) and \( g_0 \) are the maximum values of the fountain velocity and reduced gravity, \( x \) and \( y \) are the distances in the wall-parallel and wall-perpendicular direction, and \( m \) and \( n \) define the size of the fountain in the \( x \) and \( y \) directions, respectively. \( w_0 \) and \( g_0 \) are taken from the free surface values of §2.1 and depend on the source conditions and ambient fluid depth. \( m \) and \( n \) are taken from experiments that measured the width of the fountain before it reached the surface (§4). \( x \) and \( y \) are defined such that \((x, y) = (0, 0)\) gives the centre of the fountain in the \( x \) direction and the position of the wall in the \( y \) direction.

The exact form of the velocity field for the region \( y < y_0 \) is relatively unimportant to the spreading of the surface expression. The fluid between \( y_0 \) and the wall spreads parallel to the wall rather than away from the wall, so it has almost no effect on the length of the surface expression in the wall-perpendicular direction. The velocity profile only has a small effect on the width of the surface expression in the wall-parallel direction due to the small fluxes in this region. As the Reynolds number increases this region will contain a smaller proportion of the total fluxes so in a geophysical setting, with a much higher Reynolds number, it is most likely completely insignificant.

We assume that the fountain’s momentum causes the free surface to rise a small amount so that the flow can be treated as equivalent to the solution for the flow around a 90° corner. This assumption is justified in Appendix A. The pressure at the free surface \( p_s \) then leads to a height change according to

\[
Z(x, y) = \frac{p_s(x)}{g} = \frac{w^2 - u_s(x)^2/2}{g},
\]

where \( g \) corresponds to the acceleration due to gravity and not the reduced gravity. \( u_s(x) \) is the horizontal velocity at the surface and the second part of this equation follows from Bernoulli’s principle.

Figure 2 shows the modelled free surface deviation \( Z \), normalised by the maximum value. A threshold free surface deviation of \( Z = 0.01Z_{\text{max}} \) is used to define the outside edge of the fountain (blue line on figure 2). Following Zghieb et al. (2015), we separate the fountain into independent sectors. The sectors are defined such that the arc angle that the sector boundaries make with the centre of the fountain is constant and that the sector edges follow the maximum gradient in the free surface (black dashed lines on figure 2). As such, all sectors start from the centre of the fountain, follow the steepest gradient of the free surface and finish at the fountain boundary at uniformly spaced angles. Due
Figure 2. The modelled free surface deviation caused by the fountain impacting the free surface. The blue line shows the defined edge of the fountain and the black dashed lines show the sector boundaries. $\beta$ is the angle normal to the fountain edge that the jet will propagate in. $W$ is the width of the sector. In this case, the value of $y_0$ is 0.71 cm.

to the asymmetric Gaussian velocity profiles, this results in sector boundaries that are slightly curved rather than being straight lines as in Zgheib et al. (2015). All of the fluid between $y_0$ and the wall will travel in the wall-parallel direction so this region is treated as two sectors: one in the positive $x$ direction and the other in the negative $x$ direction.

The vertical volume, momentum and buoyancy fluxes entering each sector are calculated from the velocity and reduced gravity profiles given in (2.4) and (2.5). The fluxes are combined with the sector width $W$ (see figure 2) to calculate the top-hat velocity $u$, reduced gravity $g'$, and thickness $\delta$ of the negatively buoyant jets that leave the sector horizontally:

$$u = \frac{\dot{M}}{\dot{Q}},$$  
$$g' = \frac{\dot{B}}{\dot{Q}},$$  
$$\delta = \frac{\dot{Q}^2}{MW},$$

where $\dot{\cdot}$ refers to vertical fluxes integrated over an individual sector. The horizontal velocity leaving a given sector $u$ is assumed to be in the direction of maximum free-surface gradient at the centre of the sector as shown by the arrow on figure 2 and consistent with the definition of the sector boundaries.

2.3. Horizontal negatively buoyant jet

The second region is composed of a series of negatively buoyant jets, directed horizontally, and emanating from the outside boundary of the sectors shown in figure 2. The velocity $u$, reduced gravity $g'$, thickness $\delta$, width $W$, and direction $\beta$ of the jet are all obtained from \[2.2\] It is envisaged that a separate jet is leaving from each sector. The jets have a cross-sectional area given by the thickness $\delta$ and the sector width $W$ and are bounded on the top by the free surface, on either side by the neighbouring jets (or the wall), and ambient fluid on the base. As such they only entrain ambient fluid through the base. The
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Equations that govern the propagation of each jet are similar to those given in (2.1)–(2.3) but are adapted for the different geometry and orientation:

\[
\frac{dQ}{ds} = \alpha u W, \tag{2.10}
\]

\[
\frac{dM_{x,y}}{ds} = 0, \tag{2.11}
\]

\[
\frac{dM_z}{ds} = W \delta g', \tag{2.12}
\]

and

\[
\frac{dB}{ds} = 0, \tag{2.13}
\]

where \( s \) is the distance along the path length of each jet, and \( M_{x,y} \) and \( M_z \) are the momentum fluxes in the horizontal and vertical directions, respectively. An entrainment coefficient of \( \alpha = 0.1 \) is used in this region.

The jet centreline position, \( s = (s_x, s_y, s_z) \), is also tracked over time as:

\[
\frac{ds_x}{ds} = \cos \beta \cos \left[ \tan^{-1} \left( \frac{M_z}{M_{x,y}} \right) \right], \tag{2.14}
\]

\[
\frac{ds_y}{ds} = \sin \beta \cos \left[ \tan^{-1} \left( \frac{M_z}{M_{x,y}} \right) \right], \tag{2.15}
\]

\[
\frac{ds_z}{ds} = -\sin \left[ \tan^{-1} \left( \frac{M_z}{M_{x,y}} \right) \right], \tag{2.16}
\]

where \( \beta \) is the horizontal angle of jet propagation taken from §2.2 and measured from a plane that is parallel to the wall, as shown on figure 2. (\( \tan^{-1} \left( \frac{M_z}{M_{x,y}} \right) \)) gives the angle of jet propagation in the vertical plane, measured from the horizontal and increasing downwards (i.e. \( \gamma \) is initially zero and increases as the horizontal jet becomes a vertical plume). \( s_z \) is constrained such that the distance between the jet centreline and the free surface is at least half the jet thickness. Equations (2.10)–(2.16) are evolved in \( s \) from the edge of the fountain (blue line on figure 2) until the upper surface of the jet is deeper than a predetermined threshold based on the experimental setup, at which point \( s_x \) and \( s_y \) define the outside edge of the fountain surface expression for each sector.

Different sectors have different momentum and buoyancy fluxes due to both the fountain asymmetry and the offset between the Gaussian velocity and density profiles. The fountain asymmetry results in the edge of the fountain (blue line on figure 2) having a larger radius of curvature in the centre \( (x \approx 0) \) than near the wall \( (y \rightarrow y_0) \). This causes the jet to decelerate more rapidly in the wall-parallel direction than in the wall-perpendicular direction. The offset in the velocity and density profiles results in the sectors in the centre of the fountain having a lower reduced gravity and higher velocity than the sectors near the wall. Both of these factors will cause the surface expression to spread further in the wall-perpendicular direction than in the wall-parallel direction.

3. Experiments

Experiments were conducted in a glass tank that was 61.5 cm wide in the horizontal directions and 40 cm high. A section of perspex, almost as wide as the tank, was attached to the base of the tank, approximately 1 cm from one wall, via a hinge. The perspex could be rotated to represent a vertical or sloping ice face.
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A point source was installed at the centre of the hinged wall and 10 cm above the base of the tank. The source had a radius of 0.27 cm and was designed such that the discharge was turbulent from the point of release, as described in Kaye & Linden (2004). The source rotated with the hinged wall such that the discharge was always parallel to the wall. The fountain typically became attached to the wall after a few centimetres.

The tank was initially filled with a mixture of oceanic salt water and fresh water to provide a predetermined density. The density was measured using an Anton Parr densimeter to an accuracy of \(10^{-6} \text{ g cm}^{-3}\). The temperature of the ambient fluid was thermally equilibrated at room temperature by resting in a storage drum for at least 12 hours prior to filling the tank. Residual motions caused by filling the experimental tank were left to decay for at least 30 minutes before the experiment was started.

The flow rate discharging from the source was controlled with a pump. The pump could provide flow rates from \(2.5–7.5 \text{ cm}^3 \text{s}^{-1}\). Lower flow rates would have been possible but were avoided to ensure that the discharge was turbulent. The source fluid was oceanic salt water with a small amount of rhodamine dye added for visualisation. Similarly to the ambient fluid, the density was measured prior to an experiment with an Anton Parr densimeter and the fluid was allowed at least 12 hours to thermally equilibrate.

For most of the experiments (§5), a green LED light sheet was placed horizontally near the free surface of the tank with a thickness of approximately 0.5–1 cm in the region of the source fluid. We expect the negatively buoyant jet to be visible near the free surface until its upper surface falls below the base of the light sheet (i.e. 0.5–1 cm below the free surface). For experiments that were designed to measure the spreading rate of the fountain (§4) the light sheet was placed horizontally at varying depths in the water column. For these experiments the light sheet had a thickness of approximately 0.5 cm in the region of the source fluid.

The fountain surface expression was recorded with a Nikon camera placed directly above the tank. The camera recorded a video of the entire experiment that was later processed using ‘Streams’ (Nokes 2014). Approximately 30 s of video was time averaged to remove the turbulent fluctuations in the surface expression. Letting the experiment continue for longer times resulted in sinking dense fluid entraining back into the fountain. A reference image from before the fountain was started was subtracted from the averaged experimental image to remove any effects from inconsistent lighting. Finally the intensity of red light was calculated for each pixel of the averaged and subtracted image and a threshold was applied to the resulting intensity field to determine the edge of the dyed fluid (e.g. figure 3).

4. Fountain spreading rate

A small set of experiments was conducted to measure the rate at which the fountain spreads horizontally due to entrainment of ambient fluid. From Ezhova et al. (2018) we expect the flow to spread more rapidly in the wall-parallel direction than the wall-perpendicular direction. As such, these experiments had two purposes. First, to measure the relative size of the fountain in the wall-parallel and the wall-perpendicular directions and second, to measure a bulk entrainment coefficient for use in the fountain model described in §2.1.

We assume that the wall fountain is a semi-ellipse and calculate the top-hat radius of an equivalent semi-circular fountain with the same cross-sectional area as

\[ b = \sqrt{r_1 r_2}, \]  \hspace{1cm} (4.1)
where \( r_1 \) and \( r_2 \) are the measured half-widths of the rising fountain in the wall-parallel and wall-perpendicular directions.

Figure 3 shows two images from the spreading rate experiments. The images have been processed as described in §3 to show the normalised light intensity. The top image shows the fountain shape at a height of 4.8 cm above the source while the second image shows the fountain 10.5 cm above the source. It is clear that the fluid has spread much more rapidly in the wall-parallel direction than in the wall-perpendicular direction leading to an increasing asymmetry with height. The black lines on figure 3 show the fountain edge based on a threshold intensity of 52% of the maximum value. 52% is used as representative of a top-hat profile where the concentration field spreads slightly more rapidly than the velocity field (Turner 1973).

Figure 4 shows all measurements of the top-hat fountain half-widths in the wall-parallel and wall-perpendicular directions as a function of height above the source. Also shown is the equivalent radius of a semi-circular fountain. A linear regression has been used to calculate the spreading rate of the fountain which is used to calculate an entrainment coefficient:

\[
\alpha = \frac{5}{6} \frac{db}{dz}.
\] (4.2)

For the range of heights that were tested, the semi-circular fountain radius is given by

\[
b = 0.12z + 0.57
\] (4.3)

where \( b \) and \( z \) are measured in cm and \( z \) is measured from the source. Thus, the bulk entrainment coefficient that we used in (2.1) is \( \alpha = 0.10 \). The ratio of half-widths in the wall-parallel and wall-perpendicular directions is given by

\[
\frac{r_1}{r_2} = 0.11z + 0.71.
\] (4.4)
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Figure 4. Experimental measurements of fountain spreading as a function of height above the source. The top panel shows the measured half widths in the wall-parallel and wall-perpendicular directions, the middle panel shows the top-hat radius of an equivalent semi-circular fountain and the bottom panel shows the ratio of half widths in the wall-parallel ($r_1$) to the wall-perpendicular ($r_2$) direction.

This ratio, as well as the calculated fountain radius from (2.1)–(2.3), is used to calculate $m$ and $n$ in (2.4) and (2.5). Since our source is circular we would expect the initial ratio of half widths to be 1. The lower value of 0.71 is likely due to the fountain being drawn towards the wall by the Coandă effect (Wille & Fernholz 1965) whereby entraining flows create a low pressure region near boundaries and hence are attracted to the boundary.

We note that far away from the source it is expected that the aspect ratio would reach a constant value given by the ratio of the spreading rate in the wall-parallel direction to that in the wall-perpendicular direction. Thus, although the aspect ratio is seen to increase with height for the experimental fluid depths used in the present study, we would...
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Table 1. Experimental volume flux $Q$, reduced gravity $g'$, ambient fluid depth $z$, and calculated values of $\delta$ and $\text{Fr}$ for all of the surface expression experiments with a vertical wall. $z$ is measured from the fountain source rather than the base of the tank.

| $Q$ (mL s$^{-1}$) | $g'$ (cm s$^{-2}$) | $z$ (cm) | $\delta_x$ (cm) | $\delta_y$ (cm) | $\text{Fr}_x$ (-) | $\text{Fr}_y$ (-) |
|------------------|-------------------|--------|----------------|----------------|-----------------|----------------|
| 2.71             | 3.54              | 10.0   | 1.23           | 1.52           | 1.17            | 1.44            |
| 5.89             | 12.10             | 11.2   | 1.24           | 1.50           | 1.43            | 1.81            |
| 4.30             | 7.00              | 10.0   | 1.08           | 1.33           | 1.69            | 2.08            |
| 5.89             | 10.61             | 11.0   | 1.10           | 1.38           | 1.82            | 2.25            |
| 3.42             | 3.32              | 10.0   | 0.98           | 1.22           | 2.11            | 2.86            |
| 5.89             | 9.18              | 11.0   | 1.05           | 1.31           | 2.13            | 2.66            |
| 4.30             | 3.73              | 10.0   | 0.96           | 1.19           | 2.59            | 3.09            |
| 4.71             | 2.80              | 13.1   | 1.14           | 1.32           | 3.05            | 4.08            |
| 5.09             | 4.06              | 10.0   | 0.89           | 1.10           | 3.83            | 4.72            |
| 5.89             | 4.57              | 11.0   | 0.93           | 1.16           | 3.87            | 4.82            |
| 5.89             | 4.44              | 10.0   | 0.88           | 1.08           | 4.36            | 5.42            |
| 5.89             | 3.32              | 10.0   | 0.86           | 1.06           | 5.28            | 6.52            |
| 7.48             | 4.56              | 10.0   | 0.85           | 1.06           | 5.85            | 7.13            |
| 6.39             | 3.61              | 9.0    | 0.78           | 0.99           | 6.23            | 7.44            |
| 6.68             | 2.92              | 10.0   | 0.85           | 1.05           | 6.60            | 8.13            |
| 6.72             | 3.68              | 8.5    | 0.67           | 0.95           | 7.28            | 8.18            |

We expect it to be constant at the greater fluid depths relevant to geophysical situations. We estimate this constant aspect ratio from the upper panel of figure 4 as

\[
\frac{\left(\frac{dr_1}{dz}\right)}{\left(\frac{dr_2}{dz}\right)} \approx 4. \tag{4.5}
\]

We note that the ratio of spreading rates is very similar to that from numerical simulations of a buoyant plume next to a vertical wall (Ezhova et al. 2018).

5. Surface expression

The majority of the experiments were designed to examine the surface expression of the fountain. The initial volume flux and reduced gravity of the fountain, as well as the ambient fluid depth, were all varied over the experiments. Burridge & Hunt (2017) considered a two-dimensional dense jet released horizontally at the free surface and found that the non-dimensionalised distance that the jet stays at the surface increases linearly with the source Froude number defined as $\text{Fr} = \frac{u(g'\delta)^{-1/2}}{2}$. Although the flow that we are considering is significantly different from a two-dimensional surface jet, we expect that the size of the surface expression in our experiments will have a similar dependence on $\text{Fr}$.

Values of the experimental parameters and calculated values of $\delta$ and $\text{Fr}$ are provided for each of the experiments in table 1. $\delta$ and $\text{Fr}$ are calculated at the transition to the horizontal negatively buoyant jet region. Since the value of $\delta$ and $\text{Fr}$ are different for each sector in the model we have shown the two extreme values: the wall-parallel direction $x$, and the wall-perpendicular direction $y$.

5.1. Size of the surface expression

Figure 5 shows processed images of the surface expression for three experiments. The edge of the surface expression, defined by the normalised 10% light intensity contour, is
Figure 5. Normalised light intensity as measured from typical experiments focused on the surface expression. The particular experiments are those with Fr = 1.44 (top), 4.82 (middle), and 8.18 (bottom) as given in table 1. The black line shows the normalised 10% light intensity threshold which was used to determine the edge of the surface expression and the blue line shows the predicted shape of the surface expression based on the model presented in §2. The reduction in observed light intensity is caused by a variety of processes and is not correlated directly with the concentration of dye within the surface expression. The relatively low sensitivity of the observed surface expression size to changes in the intensity threshold defining the edge of the surface expression (described later) suggests that the primary process reducing the observed light intensity is the advective sinking of dyed fluid.
below the light sheet. However, there are a number of secondary processes that could also cause the observed reduction of the light intensity. The most important of these are the attenuation of the light sheet as it passes through the dyed fluid, averaging the temporal variability when producing the light intensity figures, and dilution of the surface expression dyed fluid due to mixing. Dilution of the surface expression is increasingly important for experiments with larger Froude numbers. Since the experiments were not designed to measure dye concentration (which would have required a camera with a larger dynamic range and careful calibration) we are unable to quantify the effects of dilution in the experiments. However, based on our model of the surface expression flow, we expect the dye to dilute to 85% and 43% of its maximum value across the surface expression for the top and bottom panel of figure 5, respectively. At high Fr numbers both the dilution and the attenuation of light are significant which could help to explain why the observed surface expression is smaller than the model predictions for large Fr.

Figure 6 shows the experimental and predicted size of the surface expression, non-dimensionnalised by \( \delta \), for each experiment given in table 1. The error bars for the model predictions are calculated by using a depth threshold for the negatively buoyant jet of 0.5 cm and 1 cm to reflect the uncertain position of the bottom of the light sheet during the experiments. The error bars for the experimental results are estimated to be 1 cm for all experiments. This is based on processing the experimental data with intensity thresholds of 5% and 20% when defining the edge of the surface expression for a selection of experiments. The relatively low sensitivity of the measured surface expression size to the chosen intensity threshold suggests that the light intensity is decreasing due to the surface expression dyed fluid sinking below the light sheet rather than due to mixing or attenuation of the light sheet. Once the surface expression fluid starts sinking, the light intensity will quickly decrease as the dyed fluid sinks below the thin light sheet. In contrast, both mixing and light attenuation will reduce the observed light intensity continually with a initially rapid decrease followed by a slower decay.

A linear fit is plotted through the model predictions shown in figure 6. The linear fit is applied to the model predictions rather than the experimental values to test the similarity of our negatively buoyant jet model to that of Burridge & Hunt (2017). The applicability of the model presented by Burridge & Hunt (2017) is not obvious a priori. Burridge & Hunt (2017) give the applicability of their model to flows where the Froude number is greater than 12 whereas the Froude numbers in the present study vary between 1 and 9. Additionally, the flows being considered are significantly different. Burridge & Hunt (2017) considered a two-dimensional jet that only spreads vertically due to entrainment from the base, while we consider a jet that also spreads radially as it propagates. However, figure 6 shows that the model presented in 2 also results in the non-dimensionnalised surface expression being linearly dependent on Fr (with a different constant of proportionality compared to the results of Burridge & Hunt (2017)). The size of the surface expression is accurately predicted by the model presented in 2 across most of the parameter space with a root mean square error of 2.95 and 1.30 for \( s_x/\delta_x \) and \( s_y/\delta_y \), respectively.

5.2. Aspect ratio of the surface expression

We define an aspect ratio of the surface expression as the width parallel to the wall (\( s_x \)) divided by the length perpendicular to the wall (\( s_y \)). Therefore, if the surface expression was semi-circular it would have an aspect ratio of 2. Figure 7 shows the experimental and predicted aspect ratio of the surface expression for the experiments given in table 1. The different symbols indicate different fluid depths. The predicted values are shown in
Figure 6. Experimental measurements of the fountain surface expression in the wall-parallel ($s_x$, top panel) and the wall-perpendicular ($s_y$, lower panel) directions plotted as a function of $Fr_x$ and $Fr_y$, respectively. The size of the surface expression has been non-dimensionalised by the jet thickness $\delta$ in the corresponding direction. Also shown are the model predictions for each experiment with a linear fit.

Figure 7 shows that the aspect ratio decreases as $Fr$ increases. For low values of $Fr$ the surface expression is largely defined by the shape of the fountain below the surface (figure 3, bottom panel) and the aspect ratio is larger than 2. As $Fr$ increases, the negatively buoyant jet travels further away from the wall before sinking below the light sheet and the initial asymmetry in fountain shape becomes less important. Instead, the lower radius of curvature at the middle of the fountain and the offset velocity and density profiles lead to the jet travelling further away from the wall than along the wall ($\S 2.3$) and the aspect ratio decreases.
5.3. The effect of a sloping wall

A supplementary set of experiments was undertaken to investigate the effect of a sloping wall on the surface expression. Experiments similar to those described in §4 showed that the entrainment coefficient and fountain asymmetry were not significantly affected by a sloping wall. Due to refraction of light from the perspex wall, visualising the fountain beneath the surface was much more challenging for a sloping wall than a vertical wall. As such, the results were not used to determine the entrainment coefficient, only to confirm that the value is not significantly different from the vertical case. Furthermore, we assume that the distance that the maximum velocity is offset from the wall is unaffected by the slope.

The model that was presented in §2 is slightly adapted to account for a sloping wall. In the vertical case, all of the fountain momentum that entered into a sector was converted to horizontal momentum with a direction normal to the sector boundary. For a sloping wall, the same process is done for the vertical component of the fountain momentum, but the horizontal momentum is retained in the wall-perpendicular direction. This adjustment results in the size of the surface expression being increased in the wall-perpendicular direction and decreased in the wall-parallel direction. The effect in the wall-perpendicular direction is small as the sector boundary in the centre of the surface expression is approximately parallel to the wall. As such, both the vertical and horizontal components of the fountain momentum leave the sector in the wall-perpendicular direction ($\beta = 90^\circ$ on figure 2). In contrast, for the sector next to the wall, the vertical component of the fountain momentum will be directed along the wall while the horizontal component will be directed in the wall-perpendicular direction. More significant changes to the wall-perpendicular direction would require transfer of momentum between sectors and it is not clear how this should be done.

The experimental parameters for the sloping experiments are given in table 2. Figure 8.
shows the light intensity field for two experiments with similar discharge characteristics but a vertical and a sloping wall. It can be seen that in the case of a sloping wall the fountain fluid travels further in the wall-perpendicular direction than in the case of a vertical wall.

Figure 9 gives the experimental and model results for experiments conducted with a sloping wall and shows that the model predictions do not deviate significantly from the linear dependence on Fr that was determined for a vertical wall. We note that although the model results shown on figure 9 suggest that the dependence of the surface expression size on Fr is not significantly affected by the presence of a sloping wall, the modifications described above can have a large impact on the value of Fr. As an example, the experiment shown in table 1 for a vertical wall with $Q = 5.89 \text{ ml s}^{-1}$ and $g' = 4.57 \text{ cm s}^{-2}$ has Froude number values of $Fr_x = 3.87$ and $Fr_y = 5.65$. In contrast, a similar experiment with a $55^\circ$ slope angle ($Q = 5.89 \text{ ml s}^{-1}$ and $g' = 4.98 \text{ cm s}^{-2}$ on table 2) has Froude number values of $Fr_x = 3.26$ and $Fr_y = 6.64$. However, even with the modified treatment of the fountain momentum through the transition region the model under predicts the size of the surface expression in the wall-perpendicular direction for both $55^\circ$ and $70^\circ$ angles.

Figure 10 shows the measured and predicted values of the surface expression aspect ratio for the sloping wall experiments. Similarly to figure 7, the value of Fr that is shown is the mean value of $Fr_x$ and $Fr_y$. Since the model systematically under predicts the size of the surface expression in the wall-perpendicular direction, the predicted aspect ratio is always too large. The discrepancy between the predicted and measured aspect ratio is larger for the $55^\circ$ experiments than for the $70^\circ$ experiments as the underestimation of the wall-perpendicular surface expression size is larger. The discrepancy is reduced for larger values of Fr as the surface expression becomes larger and the effect of an approximately constant error in the wall-perpendicular direction is reduced.

### 6. Application to observations

In this section we apply the model presented in §2 to observations of a subglacial discharge plume from Saqqarliup Fjord, Greenland (Mankoff et al. 2016). Photographs of the fjord surface show a triangular surface expression that extends approximately

| $\theta$ ($^\circ$) | $Q$ (ml s$^{-1}$) | $g'$ (cm s$^{-2}$) | $z$ (cm) | $\delta_x$ (cm) | $\delta_y$ (cm) | $Fr_x$ | $Fr_y$ |
|-------------------|-----------------|-----------------|-------|-----------------|-----------------|--------|--------|
| 55                | 3.63            | 4.25            | 10.0  | 1.52            | 1.34            | 1.14   | 2.03   |
| 55                | 4.30            | 4.32            | 10.0  | 1.38            | 1.22            | 1.62   | 3.26   |
| 55                | 5.13            | 3.52            | 10.0  | 1.27            | 1.11            | 2.55   | 5.20   |
| 55                | 5.89            | 4.25            | 10.0  | 1.26            | 1.11            | 2.71   | 5.47   |
| 55                | 5.89            | 3.52            | 10.0  | 1.24            | 1.09            | 3.07   | 6.23   |
| 70                | 5.89            | 4.98            | 10.0  | 1.23            | 1.08            | 3.26   | 6.64   |
| 55                | 7.48            | 4.32            | 10.0  | 1.22            | 1.07            | 3.65   | 7.41   |
| 70                | 3.54            | 4.25            | 10.0  | 1.27            | 1.24            | 1.43   | 2.50   |
| 70                | 4.30            | 3.52            | 10.0  | 1.12            | 1.09            | 2.46   | 4.32   |
| 70                | 5.13            | 3.52            | 10.0  | 1.07            | 1.05            | 3.21   | 5.60   |
| 70                | 5.89            | 4.57            | 10.0  | 1.07            | 1.04            | 3.24   | 5.69   |
| 70                | 7.48            | 4.25            | 10.0  | 1.03            | 1.01            | 4.61   | 8.03   |

Table 2. Experimental slope angle $\theta$, volume flux $Q$, reduced gravity $g'$, ambient fluid depth $z$, and calculated values of $\delta$ and Fr for all of the surface expression experiments with a sloping wall. $z$ is measured from the fountain source rather than the base of the tank.
Figure 8. Normalised light intensity as measured from two typical experiments. The upper panel shows an experiment with a vertical wall and $Q = 5.89 \, \text{ml s}^{-1}$, $g' = 3.32 \, \text{cm s}^{-2}$ while the lower panel shows an experiment with a 55° sloping wall and $Q = 5.89 \, \text{ml s}^{-1}$, $g' = 3.52 \, \text{cm s}^{-2}$. The low light intensity around $y = 5\,\text{cm}$ on the lower panel is an artefact of the top of the perspex wall and is not physically meaningful.

300 m along the front of the glacier and 300 m into the fjord. Accompanying these aerial photographs of the surface expression are observations of the water properties within the fjord. The oceanographic observations show an approximately two layer density profile with warmer and fresher water overlying cooler and saltier water. However, within 300 m of the ice face the water column has a more uniform density due to the subglacial discharge plume bringing dense water towards the surface and displacing the lighter layer. For the remainder of this section, the warmer and fresher water that is displaced down the fjord will be referred to as the upper layer and the rest of the ambient fjord water will be referred to as the lower layer despite the lower layer being carried up to the surface near the ice face.

Since the lower layer is being brought to the surface by the plume, the situation in Saqqarliup Fjord, when the observations were made, is not exactly comparable to our experiments. In the experiments the fountain entrained fluid from the lighter layer whereas in Saqqarliup Fjord the subglacial discharge plume will only entrain denser fluid corre-
sponding to the lower layer (hatched region in figure [1]). In addition, observations from Saqarliup Fjord suggest that the subglacial discharge plume was positively buoyant when it reached the surface whereas in our experiments the fountain was always negatively buoyant. However, the observations presented in Mankoff et al. (2016) are the only observations of a subglacial discharge surface expression that we are aware of.

Observations slightly down-fjord of a subglacial surface expression have recently been
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reported in Jackson et al. (2017). Observations were made 500 m and 2000 m away from a subglacial discharge plume and in both transects a clear subsurface intrusion was observed. The velocity of this intrusion was considerably lower than those in the surface flow reported in Mankoff et al. (2016) and the volume fluxes in the intrusion suggest that there is little mixing with the ambient fluid. The observations presented in Jackson et al. (2017) highlight that the dynamics governing the subsurface intrusion are significantly different from those governing the surface expression observed in Mankoff et al. (2016) and are not included in our model or experiments.

Hence, we will compare our model only to the observations in Mankoff et al. (2016) and will do so in two separate ways. First, we will apply our model to the two layer stratification that is observed away from the ice face. This stratification is expected to occur at the beginning of summer when the subglacial discharge has been absent or very weak for some time. Second, we will apply our model to the approximately linear stratification observed within 300 m of the ice face and then assume that the extent of the surface expression is given by a balance between the velocity of the horizontal jet away from the ice and a theoretical velocity of the upper layer towards the ice face.

For both of these stratifications we will attempt to infer subglacial discharge properties at the ice face. We are not aware of any direct measurements of subglacial discharge fluxes or channel size. To obtain such measurements would be challenging and would likely require an autonomous underwater vehicle close to the active discharge channel. As such, the ability to infer discharge properties based only on the size of the surface expression would enable an increased understanding of subglacial discharges.

Based on water mass properties, Mankoff et al. (2016) estimated that the subglacial discharge at the time of their observations was $105–140 \text{ m}^3 \text{s}^{-1}$ which compares favourably with the estimated runoff for the 5 days prior to observation from the RACMO model of $101 \text{ m}^3 \text{s}^{-1}$. Mankoff et al. (2016) assumed the radius of the source to be $5–15 \text{ m}$.

6.1. A two-layer stratification

We assume that the subglacial discharge is purely fresh water and use three values of the source radius: 5, 10, and 15 m. The radii provide the initial condition for $b$ in (2.1)–(2.3) and, with the prescribed volume flux, define the initial vertical velocity $w$. We note that although we have previously only considered the fountain region (the unhatched region in figure 1), here we consider the entire fjord depth. This manifests in (2.1)–(2.3) as a discontinuity in $g'$ and $B$ at the interface between the upper and lower layers. Following previous studies (Cowton et al. 2015; Xu et al. 2013), and as mentioned in §2, we ignore the initial transient where the subglacial discharge flux is released horizontally and treat the discharge as vertical at the source. Based on (4.5) we assume that the fountain has an aspect ratio of 4 before reaching the surface.

The fjord is assumed to be 150 m deep and to contain two uniform layers. The lower layer is 130 m thick and has a density of $1026 \text{ kg m}^{-3}$ while the upper layer is 20 m thick and has a density of $1024 \text{ kg m}^{-3}$ (Mankoff et al. 2016). We vary the source volume flux of the subglacial discharge across a range of possible values for each radius to find an initial volume flux that gives the observed surface expression length of approximately 300 m.

Figure 11 shows the model results plotted for the three different discharge radii as a function of the source volume flux. The solid line shows the observed size of the surface expression. Figure 11 shows that regardless of the source radius, the plume needs to be very close to its neutral density when it reaches the surface for the surface expression to be 300 m long ($s_p \rightarrow \infty$ when the plume reaches a neutral density at the surface).

Although figure 11 gives a discharge radius ($r=10 \text{ m}$) and volume flux ($Q \approx 140 \text{ m}^3 \text{s}^{-1}$)
that are consistent with that derived from oceanographic measurements in [Mankoff et al. (2016)], it seems unlikely that the size of the surface expression would be as sensitive to small changes in volume flux as predicted by the model. For example, for a radius of 10 m, if the volume flux decreased by 0.2% from 141.8 m$^3$s$^{-1}$, the size of the surface expression would be expected to reduce by over 20%. This sensitivity is due to the reduced gravity of the fountain being very close to zero at the surface and, as a result, the Froude number being highly sensitive to small changes in \(g'\).

6.2. A linear stratification

In an attempt to reduce the sensitivity of the size of the surface expression on \(Q\) we assume a different density structure in the fjord. Next to the ice face we consider a linearly stratified water column with a bottom density of 1027 kg m$^{-3}$ and a surface density of 1025 kg m$^{-3}$. Further downfjord we once again consider a two-layer stratification such as was described in §6.1. This density structure is consistent with the observations within 300 m of the ice face presented in Mankoff et al. (2016) (their figure 7).

In this new arrangement we expect that a portion of the upper layer, with similar depth to the surface expression, will be displaced downfjord by the rising plume. The displaced portion of the upper layer will behave like a gravity current and be forced towards the ice by its own buoyancy. We calculate a theoretical velocity of the displaced fluid as \(u_f \sim \sqrt{g_f' \delta_f} = 0.54 \text{ m s}^{-1}\), where \(g_f' = 1.9 \times 10^{-2} \text{ m s}^{-2}\) is based on the density difference between the low density water in the upper layer and the underlying ambient water and \(\delta_f = 15 \text{ m}\) is the thickness of the surface expression.

The introduction of a flow towards the ice in the upper layer provides an additional limit on the surface expression size. Instead of the surface expression sinking below the surface some distance from the ice face, it will become arrested when its velocity is less than the upper layer velocity \(u_f\). As such, we consider the jet becoming arrested at the location where its velocity decays to \(u_f\) and for it to then be subducted between the upper and lower layers. Unlike the laboratory experiments and the case described in §6.1, the density of the surface expression now has almost no effect on its size. The size of the surface expression is controlled purely by its velocity and in particular, when the velocity is equal to \(u_f\).

Figure 12 shows the distance from the ice where the jet velocity and \(u_f\) are equal for...
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Figure 12. Calculation of the distance away from the ice front where the horizontal jet velocity is equal to the upper layer velocity $u_f$. The solid line shows the observed size of the surface expression and dashed lines show the estimated subglacial discharge flux from Mankoff et al. (2016).

three source radii and a range of discharge flow rates. We note that for many of these discharge configurations, the flow is positively buoyant at the surface relative to the ambient fluid density so it would not sink if it weren’t for the upper layer of low density water downstream in the fjord.

The predicted size of the surface expression shown on figure 12 is consistent with the observations described in Mankoff et al. (2016). In addition, the reduced gravity of the plume no longer needs to be very close to zero at the free surface which reduces the sensitivity of the surface expression size to changes in the subglacial discharge volume flux compared to the case of a two-layer stratification described in §6.1.

7. Conclusion

We have presented a model and experiments that examine the surface expression of a fountain released adjacent to a wall that is either vertical or sloping. The model separates the flow into vertical and horizontal regions. A transition where all of the vertical momentum is converted to horizontal momentum through a free-surface pressure gradient connects the two regions. The model predicts that the size of the free-surface expression of the fountain, non-dimensionalised by the thickness of the horizontal jet, will be linearly dependent on Fr, which is consistent with previous studies of two-dimensional negatively buoyant surface jets (Burridge & Hunt 2017).

Experiments are first used to examine the shape and spreading rate of the subsurface fountain. Similar to Ezhova et al. (2018), we find that the fountain spreads more rapidly in the wall-parallel direction than in the wall-perpendicular direction. Neither the rate of spreading nor the asymmetry are significantly affected by a sloping wall.

Experimental measurements of the size of the surface expression generally agree with the model for a vertical wall. For a sloping wall, the model slightly under predicts the size of the surface expression in the wall-perpendicular direction.

Finally, the model is applied to observations of a subglacial discharge plume in front of Saqqaqriup Sermia (Mankoff et al. 2016). The model was used to estimate properties
at the discharge location which are difficult to measure. We apply the model to the fjord density profile close to and away from the ice face. When using the two-layer density profile away from the ice, the model predicts that the reduced gravity of the fountain is close to zero at the free surface resulting in the size of the surface expression being extremely sensitive to small changes in the volume flux. When the linear density profile close to the ice is used, the size of the surface expression is determined based on equating the opposing velocities of the surface expression and the upper layer that has been displaced down the fjord. In this case, the predicted size of the surface expression agrees with observations and is relatively insensitive to small changes in the discharge properties.

This approach provides a method of using the surface expression of subglacial discharges to infer properties at the base of a glacier. The estimates of subglacial discharge contain significant uncertainty based on the channel radius and the size of the surface expression. However, the uncertainty in the discharge flux is similar to the observational estimates presented in Mankoff et al. (2016). As such, challenging measurements near active discharge channels can be replaced with either oceanographic measurements from the surface water in front of the glacier or aerial photographs.

We gratefully acknowledge technical assistance from Anders Jensen. CM thanks the Weston Howard Jr. Scholarship for funding. Support to CC was given by NSF project 1434041 and OCE-1658079. We thank the anonymous reviewers for improving the manuscript throughout the review process.

Appendix A.

This appendix justifies the assumption made in §2.2 that the transition from vertical to horizontal flow can be treated as the flow around a 90° corner with no loss of momentum or mass fluxes. The following calculation assumes incompressible and two-dimensional flow. Within the vertical plane through the centreline, and in the vicinity of the corner, these assumptions are expected to be valid.

We note that within this appendix $x, y, z,$ and $w$ are used differently to the rest of the paper. Instead we use the standard nomenclature of complex variables and Lamb (1916). The nomenclature within the appendix should be considered to be self-contained and independent from the remainder of the manuscript.

The inner boundary of the fountain is in contact with the stationary ambient fluid and, after subtracting the hydrostatic pressure, at high Reynolds number it can be regarded as a free surface (i.e. the pressure is zero). There is a general method using complex variables due to Lamb (1916) §73 which can be used for solving these problems in the inviscid, irrotational case. That is it conserves momentum and mass which should be a good approximation near the surface.

In our geometry there is a single right angle bend corresponding to a power of $1/2$, therefore following Lamb (1916) we must have

$$\frac{dz}{dw} = \sqrt{\coth w}, \quad (A1)$$

where $z = x + iy$ is the complex coordinate, $w = \phi + i\psi$, $\phi$ is the velocity potential and $\psi$ is the stream function. $\psi = 0$ is the streamline along the wall and the horizontal free surface and $\psi = \pi/4$ is the streamline along the inner boundary of the fountain.
Integrating (A 1) gives

\[ z = \cot^{-1} \sqrt{\coth w} + \coth^{-1} \sqrt{\coth w}, \]  

an implicit equation for the stream function \( \psi \). Applying Bernoulli’s theorem the pressure is given by

\[ p = C - \frac{u^2 + v^2}{2} = C - \frac{1}{2} \left| \frac{dw}{dz} \right|^2 = \frac{1}{2} - \frac{1}{2} |\tanh w|, \]  

where we have chosen the pressure to be zero at the inner boundary of the fountain. The pressure along the vertical wall and the horizontal free surface is given by taking \( w = s \), where \( s \) is real and \( w = 0 \) is the corner:

\[ p_w = \frac{1}{1 + e^{2s}}. \]  

The connection between \( s \) and world coordinates comes from (A 2).

The equation for the inner boundary of the fountain is given implicitly by substituting \( w = \pi/4 + s \) into (A 2). In the vicinity of the bend we can write a parametric power series solution

\[ x = \frac{\pi + \log(3 + 2\sqrt{2})}{4} + \frac{1}{\sqrt{2}} \left( +s + \frac{1}{2} s^2 - \frac{1}{6} s^3 - \frac{5}{24} s^4 + \frac{17}{120} s^5 \right), \]  

\[ y = \frac{\pi + \log(3 + 2\sqrt{2})}{4} + \frac{1}{\sqrt{2}} \left( -s + \frac{1}{2} s^2 + \frac{1}{6} s^3 - \frac{5}{24} s^4 - \frac{17}{120} s^5 \right). \]

If we define \( R = \left[ \pi + \log(3 + 2\sqrt{2}) \right] / 2 \) we can write this solution to the same order fully
implicitly as

\[ R^2 = (x + y)^2 + (x - y)^2 / \sqrt{2} + \frac{1}{8} \left( \frac{R}{\sqrt{18}} - 1 \right) (x - y)^4 + \cdots. \]  

(A 7)

We can also look far from the corner where we have

\[ y = \frac{\pi}{4} + \exp(\pi/2 - 2x), \quad \text{or} \quad x = \frac{\pi}{4} + \exp(\pi/2 - 2y). \]  

(A 8)

Thus the solution corresponds to a jet of thickness \( \pi/4 \) and uniform incoming and outgoing speed 1. The solution can trivially be scaled to any velocity and size.

Figure [13] shows a time-averaged normalised light intensity image of the fountain viewed from the side of the tank with the solution for flow around a corner superimposed. The edge of the fountain is identified as the location where the normalised light intensity falls to less than 20% of the maximum value at each height. The edge of the fountain is only used for comparison with the solution for flow around a corner and as such the exact threshold is not significant.

It can be seen that the solution given by (A 7) agrees well with the contour of light intensity near the free surface where the solution is expected to be valid, justifying the assumption of inviscid, irrotational flow in vicinity of the corner.

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