Modified Bargmann-Wigner Formalism (Bosons of Spin 1 and 2)

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Abstract. On the basis of our recent modifications of the Dirac formalism we generalize the Bargmann-Wigner formalism for higher spins to be compatible with other formalisms for bosons. Relations with dual electrodynamics, with the Ogievetskii-Polubarinov notoph and the Weinberg 2(2J+1) theory are found. Next, we introduce the dual analogues of the Riemann tensor and derive corresponding dynamical equations in the Minkowski space. Relations with the Marques-Spehler chiral gravity theory are discussed.

1. Introduction
The equations for higher spins can be derived from the first principles on using modifications of the Bargmann-Wigner formalism. The generalizations of the equations in the \((1/2, 0) \oplus (0, 1/2)\) representation are well known. The Tokuoka-SenGupta-Fushchich formalism is based on the equation \([1, 2, 3, 4]\):

\[
[i\gamma^\mu \partial_\mu + m_1 + m_2\gamma^5]\Psi = 0
\] (1)

If \(m_1^2 \neq m_2^2\) it was claimed \([1]\) that this is simply the change of the representation of \(\gamma^5\)'s. However, the physical consequences are different from those of the Dirac formalism. Fushchich \([5]\) generalized the formalism even further in 1970-72, and, in fact, he connected it with the Gelfand-Tsetlin-Sokolik idea \([6]\) of the 2-dimensional representation of the inversion group. I derived the above parity-violating equation \([4]\) (and its charge-conjugate) by the Sakurai-Gersten method from the first principles. The Barut formalism is based on the equation \([7, 8]\):

\[
[i\gamma^\mu \partial_\mu + \alpha_2 \frac{\partial_\mu \partial_\nu}{m}\gamma^\nu + \kappa]\Psi = 0
\] (2)

It was re-derived from the first principles in \([9, 10]\). Instead of 4 solutions it has 8 solutions with the correct relativistic dispersion \(E = \pm \sqrt{p^2 + m_1^2}\); and, in fact, it describes two mass states \(m_\mu = m_\epsilon(1 + \frac{3}{2\kappa})\), provided that the certain physical condition is imposed on the \(\alpha_2\) parameter \([7]\). One can also generalize the formalism to include the third state, \(\tau\)-lepton, see refs. \([7, 10]\). Barut also indicated at the possibility of including \(\gamma^5\) term. For instance, the equation can look something like this:

\[
[i\gamma^\mu \partial_\mu + a + b\Box + \gamma^5(c + d\Box)]\psi = 0
\] (3)
which cannot be factorized as a product of two Dirac equations with different masses.

The basic principles of the Weinberg-Tucker-Hammer (WTH) formalism in the \((J, 0) \oplus (0, J)\) representation \[11][12] are well-known from my previous works. For spin 1 we have

\[
[\gamma_{\alpha\beta} p_{\alpha} p_{\beta} + A p_{\alpha} p_{\alpha} + B m^2] \Psi = 0, \tag{4}
\]

where \(p_{\mu} = -i \partial_{\mu}\) and \(\gamma_{\alpha\beta}\) are the Barut-Muzinich-Williams covariantly defined \(6 \times 6\) matrices \[20\]. The determinant of \([\gamma_{\alpha\beta} p_{\alpha} p_{\beta} + A p_{\alpha} p_{\alpha} + B m^2]\) is of the 12th order in \(p_{\mu}\). Solutions with \(E^2 - p^2 = m^2\) can be obtained if and only if \(\frac{B}{\Lambda + 1} = 1, \quad \frac{B}{\Lambda - 1} = 1\). The particular cases are:

- \(A = 0, B = 1 \Leftrightarrow\) we have the Weinberg’s equation for \(J = 1\) with 3 solutions \(E = \pm \sqrt{p^2 + m^2}\), 3 solutions \(E = -\sqrt{p^2 + m^2}\), 3 solutions \(E = + \sqrt{p^2 - m^2}\) and 3 solutions \(E = - \sqrt{p^2 - m^2}\).
- \(A = 1, B = 2 \Leftrightarrow\) we have the Tucker-Hammer equation for \(J = 1\). The solutions are only with \(E = \pm \sqrt{p^2 + m^2}\).

Recently we have shown \[13][14\] that one can obtain \textit{four} different equations for antisymmetric tensor fields from the Weinberg \(2(2J + 1)\) component formalism. First of all, we note that \(\Psi\) is, in fact, bivector, \(E_i = -iF_{4i}, B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}\), or \(E_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}, B_i = -iF_{4i}\), or their combination. The four cases are:

- \(\Psi^{(I)} = \begin{pmatrix} E + iB \\ E - iB \end{pmatrix}, P = -1\), where \(E_i\) and \(B_i\) are the components of the tensor.
- \(\Psi^{(II)} = \begin{pmatrix} B - iE \\ B + iE \end{pmatrix}, P = +1\), where \(E_i, B_i\) are the components of the tensor.
- \(\Psi^{(III)} = \Psi^{(I)}, \) but \(\Psi^{(II)} E_i, B_i\) are the corresponding vector and axial-vector components of the dual tensor \(\tilde{F}_{\mu\nu}\).
- \(\Psi^{(IV)} = \Psi^{(II)}, \) where \(E_i\) and \(B_i\) are the components of the \textit{dual} tensor \(\tilde{F}_{\mu\nu}\).

The mappings of the WTH equations are:

\[
\partial_\mu \partial_\mu F_{\mu\alpha}^{(I)} - \partial_\beta \partial_\mu F_{\mu\alpha}^{(I)} + \frac{A - 1}{2} \partial_\mu \partial_\mu F_{\alpha\beta}^{(I)} - \frac{B}{2} m^2 F_{\alpha\beta}^{(I)} = 0, \tag{5}
\]

\[
\partial_\mu \partial_\mu F_{\mu\alpha}^{(II)} - \partial_\beta \partial_\mu F_{\mu\alpha}^{(II)} - \frac{A + 1}{2} \partial_\mu \partial_\mu F_{\alpha\beta}^{(II)} + \frac{B}{2} m^2 F_{\alpha\beta}^{(II)} = 0, \tag{6}
\]

\[
\partial_\mu \partial_\mu \tilde{F}_{\mu\alpha}^{(III)} - \partial_\beta \partial_\mu \tilde{F}_{\mu\alpha}^{(III)} - \frac{A + 1}{2} \partial_\mu \partial_\mu \tilde{F}_{\alpha\beta}^{(III)} + \frac{B}{2} m^2 \tilde{F}_{\alpha\beta}^{(III)} = 0, \tag{7}
\]

\[
\partial_\mu \partial_\mu \tilde{F}_{\mu\alpha}^{(IV)} - \partial_\beta \partial_\mu \tilde{F}_{\mu\alpha}^{(IV)} + \frac{A - 1}{2} \partial_\mu \partial_\mu \tilde{F}_{\alpha\beta}^{(IV)} - \frac{B}{2} m^2 \tilde{F}_{\alpha\beta}^{(IV)} = 0. \tag{8}
\]

In the Tucker-Hammer case \((A = 1, B = 2)\) we can recover the Proca theory from \[5\]:

\[
\partial_\alpha \partial_\mu F_{\mu\beta} - \partial_\beta \partial_\mu F_{\mu\alpha} = m^2 F_{\alpha\beta}. \tag{9}
\]

Now we are interested in \textit{parity-violating} equations for antisymmetric tensor fields. We also study the most general mapping of the Weinberg-Tucker-Hammer formulation to the antisymmetric tensor field formulation. Instead of \(\Psi^{(I-IV)}\) we shall try to use now

\[
\Psi^{(A)} = \begin{pmatrix} E + iB \\ B + iE \end{pmatrix} = \frac{1 + \gamma^5}{2} \Psi^{(I)} + \frac{1 - \gamma^5}{2} \Psi^{(II)}. \tag{10}
\]
As a result, the equation for the AST fields is
\[ \partial_\alpha \partial_\mu F_{\mu\beta} - \partial_\beta \partial_\mu F_{\mu\alpha} = \frac{1}{2}(\partial_\mu \partial_\mu)F_{\alpha\beta} + \left[-\frac{A}{2}(\partial_\mu \partial_\mu) + \frac{B}{2} m^2\right]\tilde{F}_{\alpha\beta} \tag{11} \]

The different choice is
\[ \Psi^{(B)} = \left( \begin{array}{c} \mathbf{E} + i \mathbf{B} \\ -\mathbf{B} - i \mathbf{E} \end{array} \right) = \frac{1 + \gamma^5}{2} \Psi^{(I)} - \frac{1 - \gamma^5}{2} \Psi^{(I)} \tag{12} \]

Thus, one has
\[ \partial_\alpha \partial_\mu F_{\mu\beta} - \partial_\beta \partial_\mu F_{\mu\alpha} = \frac{1}{2}(\partial_\mu \partial_\mu)F_{\alpha\beta} + \left[-\frac{A}{2}(\partial_\mu \partial_\mu) + \frac{B}{2} m^2\right]\tilde{F}_{\alpha\beta} \tag{13} \]

Of course, one can also use the dual tensor and obtain analogous equations:
\[ \partial_\alpha \partial_\mu \tilde{F}_{\mu\beta} - \partial_\beta \partial_\mu \tilde{F}_{\mu\alpha} = \frac{1}{2}(\partial_\mu \partial_\mu)\tilde{F}_{\alpha\beta} + \left[-\frac{A}{2}(\partial_\mu \partial_\mu) - \frac{B}{2} m^2\right]F_{\alpha\beta} \tag{14} \]
\[ \partial_\alpha \partial_\mu \tilde{F}_{\mu\beta} - \partial_\beta \partial_\mu \tilde{F}_{\mu\alpha} = \frac{1}{2}(\partial_\mu \partial_\mu)\tilde{F}_{\alpha\beta} + \left[-\frac{A}{2}(\partial_\mu \partial_\mu) - \frac{B}{2} m^2\right]F_{\alpha\beta} \tag{15} \]

They are connected with (11,13) by the dual transformations.

The states corresponding to the new functions \( \Psi^{(A)}, \Psi^{(B)} \) etc are not the parity eigenstates. So, it is not surprising that we have \( F_{\alpha\beta} \) and its dual \( \tilde{F}_{\alpha\beta} \) in the same equations. In total we have already eight equations.

One can also consider the most general case
\[ \Psi^{(W)} = \left( \begin{array}{c} aF_{4\mu} + b\tilde{F}_{4\mu} + c\epsilon_{ijk}F_{jk} + d\epsilon_{ijk}\tilde{F}_{jk} \\ eF_{4\mu} + f\tilde{F}_{4\mu} + g\epsilon_{ijk}F_{jk} + h\epsilon_{ijk}\tilde{F}_{jk} \end{array} \right) \tag{16} \]

So, we have dynamical equations for \( F_{\alpha\beta} \) and \( \tilde{F}_{\alpha\beta} \) with additional parameters \( a, b, c, d, \ldots \in \mathbb{C} \).

We have a lot of antisymmetric tensor fields here.

The Bargmann-Wigner formalism for constructing of high-spin particles has been given in [15, 16]. However, they claimed explicitly that they constructed \((2J+1)\) states (the Weinberg-Tucker-Hammer theory has essentially \(2(2J+1)\) components). The standard Bargmann-Wigner formalism for \( J = 1 \) is based on the following set
\[ [i\gamma_\mu \partial_\mu + m]_{\alpha\beta} \Psi_{\beta\gamma} = 0, \tag{17} \]
\[ [i\gamma_\mu \partial_\mu + m]_{\gamma\beta} \Psi_{\alpha\beta} = 0 \tag{18} \]

If one has
\[ \Psi_{\{\alpha\beta\}} = (\gamma_\mu R)_{\alpha\beta}A_\mu + (\sigma_{\mu\nu} R)_{\alpha\beta}F_{\mu\nu} \tag{19} \]
with
\[ R = e^{i\varphi} \begin{pmatrix} \Theta & 0 \\ 0 & -\Theta \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{20} \]

in the spinorial representation of \( \gamma \)-matrices we obtain the Duffin-Proca-Kemmer equations:
\[ \partial_\alpha F_{\alpha\mu} = \frac{m}{2} A_\mu, \tag{21} \]
\[ 2mF_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{22} \]
After the corresponding re-normalization $A_\mu \to 2mA_\mu$, we obtain the standard textbook set:

$$\partial_\alpha F_{\alpha \mu} = m^2 A_\mu, \quad (23)$$
$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (24)$$

It gives the equation (9) for the antisymmetric tensor field. How can one obtain other equations following the Weinberg-Tucker-Hammer approach? The third equation can be obtained in a simple way: use, instead of $(\sigma_{\mu \nu} R) F_{\mu \nu}$, another symmetric matrix $(\gamma^5 \sigma_{\mu \nu} R) F_{\mu \nu}$, see [17]. And what about the second and the fourth equations? I suggest:

- to use, see above and [1]:
  $$[i\gamma_\mu \partial_\mu + m] \Psi = 0 \Rightarrow [i\gamma_\mu \partial_\mu + m_1 + m_2 \gamma_5] \Psi = 0; \quad (25)$$

- to use the Barut extension:
  $$[i\gamma_\mu \partial_\mu + a - b \Box - \gamma_5 (c - d \Box)] \alpha \beta \Psi_{\beta \gamma} = 0 \quad \text{and} \quad \gamma_5 (c - d \Box) \phi = 0. \quad (32)$$

In such a way we can enlarge the set of possible states.

2. Modified Bargmann-Wigner Formalism

We begin with

$$[i\gamma_\mu \partial_\mu + a - b \Box + \gamma_5 (c - d \Box)] \alpha \beta \Psi_{\beta \gamma} = 0, \quad (27)$$
$$[i\gamma_\mu \partial_\mu + a - b \Box - \gamma_5 (c - d \Box)] \alpha \beta \Psi_{\beta \gamma} = 0. \quad (28)$$

$\Box$ is the d’Alembertian. Thus, we obtain the Proca-like equations:

$$\partial_\nu A_\lambda - \partial_\lambda A_\nu - 2(a + b \partial_\mu \partial_\mu) F_{\nu \lambda} = 0, \quad (29)$$
$$\partial_\mu F_{\mu \lambda} = \frac{1}{2} (a + b \partial_\mu \partial_\mu) A_\lambda + \frac{1}{2} (c + d \partial_\mu \partial_\mu) \tilde{A}_\lambda. \quad (30)$$

$\tilde{A}_\lambda$ is the axial-vector potential (analogous to that used in the Duffin-Kemmer set for $J = 0$). Additional constraints are:

$$i\partial_\lambda A_\lambda + (c + d \partial_\mu \partial_\mu) \tilde{\phi} = 0, \quad (31)$$
$$\epsilon_{\lambda \mu \nu \rho} \partial_\mu F_{\nu \rho} = 0, (c + d \partial_\mu \partial_\mu) \phi = 0. \quad (32)$$

The spin-0 Duffin-Kemmer equations are:

$$(a + b \partial_\mu \partial_\mu) \phi = 0, i \partial_\mu \tilde{A}_\mu - (a + b \partial_\mu \partial_\mu) \tilde{\phi} = 0, \quad (33)$$
$$(a + b \partial_\mu \partial_\mu) \tilde{A}_\nu + (c + d \partial_\mu \partial_\mu) A_\nu + i(\partial_\nu \tilde{\phi}) = 0. \quad (34)$$

The additional constraints are:

$$\partial_\mu \phi = 0, \partial_\nu \tilde{A}_\lambda - \partial_\lambda \tilde{A}_\nu + 2(c + d \partial_\mu \partial_\mu) F_{\nu \lambda} = 0 \quad (35)$$

In such a way the spin states are mixed through the 4-vector potentials. After elimination of the 4-vector potentials we obtain the equation for the AST field of the second rank:

$$\partial_\mu \partial_\nu \alpha \beta \Psi_{\alpha \beta} + \left[ (c^2 - a^2) - 2(ab - cd) \partial_\mu \partial_\mu + (d^2 - b^2)(\partial_\mu \partial_\mu)^2 \right] F_{\mu \lambda} = 0, \quad (36)$$
which should be compared with our previous equations which follow from the Weinberg-like formulation. Just put:

\[ c^2 - a^2 \Rightarrow \frac{-Bm^2}{2}, \quad c^2 - a^2 \Rightarrow \frac{+Bm^2}{2}, \]  
\[ -2(ab - cd) \Rightarrow \frac{A - 1}{2}, \quad +2(ab - cd) \Rightarrow \frac{A + 1}{2}, \]  
\[ b = \pm d. \]  

(37)  

(38)  

(39)  

Of course, these sets of algebraic equations have solutions in terms \( A \) and \( B \). We found them and restored the equations, see above.

The parity violation and the spin mixing are \textit{intrinsic} possibilities of the Proca-like theories. One can go in a different way: instead of modifying the equations, consider the spin basis rotations. In the helicity basis we have (see also [18, 19], where it was claimed explicitly that helicity states cannot be parity eigenstates):

\[ \epsilon_\mu(p, \lambda = +1) = \frac{1}{\sqrt{2}} \frac{e^{i\phi}}{p} \left( 0, \frac{p_x p_z - ip_y p}{\sqrt{p_x^2 + p_y^2}}, \frac{p_y p + ip_x p}{\sqrt{p_x^2 + p_y^2}} - \sqrt{p_x^2 + p_y^2} \right), \]  
\[ \epsilon_\mu(p, \lambda = -1) = \frac{1}{\sqrt{2}} \frac{e^{-i\phi}}{p} \left( 0, -\frac{p_x p_z - ip_y p}{\sqrt{p_x^2 + p_y^2}}, \frac{p_y p + ip_x p}{\sqrt{p_x^2 + p_y^2}} + \sqrt{p_x^2 + p_y^2} \right), \]  
\[ \epsilon_\mu(p, \lambda = 0) = \frac{1}{m} \left( p_x - \frac{\phi}{p} p_x - \frac{\gamma}{p} p_y, -\frac{\gamma}{p} p_x \right), \epsilon_\mu(p, \lambda = 0) = \frac{1}{m} \left( E, -p_x, -p_y, -p_z \right) \]  

(40)  

(41)  

(42)  

and

\[ \mathbf{E}(p, \lambda = +1) = -\frac{iEp_x}{\sqrt{2}p_y} \mathbf{p} - \frac{E}{\sqrt{2}p_y} \mathbf{p}, \quad \mathbf{B}(p, \lambda = +1) = -\frac{p_x}{\sqrt{2}p_y} \mathbf{p} + \frac{ip_y}{\sqrt{2}p_y} \mathbf{p}, \]  
\[ \mathbf{E}(p, \lambda = -1) = \frac{iEp_x}{\sqrt{2}p_y} \mathbf{p} - \frac{E}{\sqrt{2}p_y} \mathbf{p}^*, \quad \mathbf{B}(p, \lambda = -1) = -\frac{p_x}{\sqrt{2}p_y} \mathbf{p} - \frac{ip_y}{\sqrt{2}p_y} \mathbf{p}^*, \]  
\[ \mathbf{E}(p, \lambda = 0) = \frac{im}{p} \mathbf{p}, \quad \mathbf{B}(p, \lambda = 0) = 0, \]  

(43)  

(44)  

(45)  

with \( \mathbf{p} = column(p_y, -p_x, -ip) \).

In fact, there are several modifications of the BW formalism. One can came to the following set:

\[ [i\gamma_\mu \partial_\mu + \epsilon_1 m_1 + \epsilon_2 m_2 \gamma_5]_{\alpha\beta} \Psi_{\beta\gamma} = 0, \]  
\[ [i\gamma_\mu \partial_\mu + \epsilon_3 m_1 + \epsilon_4 m_2 \gamma_5]_{\alpha\beta} \Psi_{\gamma\beta} = 0, \]  

(46)  

(47)  

where \( \epsilon_i \) are the sign operators. So, at first sight, we have 16 possible combinations for the AST fields. We first come to

\[ [i\gamma_\mu \partial_\mu + m_1 A_1 + m_2 A_2 \gamma_5]_{\alpha\beta} \{ (\gamma_\lambda R)_{\beta\gamma} A_\lambda + (\sigma_{\lambda\nu} R)_{\beta\gamma} F_{\lambda\nu} \} + \]  
\[ + [m_1 B_1 + m_2 B_2 \gamma_5] \left\{ R_{\beta\gamma} \varphi + (\gamma_5 R)_{\beta\gamma} \tilde{\phi} + (\gamma_5 \gamma_5 R)_{\beta\gamma} \tilde{A}_\lambda \right\} = 0, \]  
\[ [i\gamma_\mu \partial_\mu + m_1 A_1 + m_2 A_2 \gamma_5]_{\alpha\beta} \{ (\gamma_\lambda R)_{\alpha\beta} A_\lambda + (\sigma_{\lambda\nu} R)_{\alpha\beta} F_{\lambda\nu} \} - \]  
\[ - [m_1 B_1 + m_2 B_2 \gamma_5] \left\{ R_{\alpha\beta} \varphi + (\gamma_5 R)_{\alpha\beta} \tilde{\phi} + (\gamma_5 \gamma_5 R)_{\alpha\beta} \tilde{A}_\lambda \right\} = 0, \]  

(48)  

(49)
where \( A_1 = \frac{\epsilon_1 + \epsilon_3}{2} \), \( A_2 = \frac{\epsilon_1 - \epsilon_3}{2} \), \( B_1 = \frac{\epsilon_2 + \epsilon_4}{2} \), and \( B_2 = \frac{\epsilon_2 - \epsilon_4}{2} \). Thus for spin 1 we have
\[
\begin{align*}
\partial_\mu A_\lambda - \partial_\lambda A_\mu + 2m_1 A_1 F_{\mu\lambda} + i m_2 A_2 \epsilon_{\alpha\beta\mu\lambda} F_{\alpha\beta} &= 0, \\
\partial_\lambda F_{\kappa\lambda} - \frac{m_1}{2} A_1 A_\kappa - \frac{m_2}{2} B_2 \tilde{A}_\kappa &= 0,
\end{align*}
\]
with constraints
\[
\begin{align*}
-i \partial_\mu A_\mu + 2m_1 B_1 \phi + 2m_2 B_2 \tilde{\phi} &= 0, \\
i \epsilon_{\mu\nu\kappa\lambda} \partial_\mu F_{\nu\kappa} - m_2 A_2 A_\lambda - m_1 B_1 \tilde{A}_\lambda &= 0, \\
m_1 B_1 \phi + m_2 B_2 \tilde{\phi} &= 0.
\end{align*}
\]
If we remove \( A_\lambda \) and \( \tilde{A}_\lambda \) from this set, we come to the final results for the AST field. Actually, we have twelve equations, see [14]. One can go even further. One can use the Barut equations for the BW input. So, we can get \( 16 \times 16 \) combinations (depending on the eigenvalues of the corresponding sign operators), and we have different eigenvalues of masses due to \( \partial_\mu^2 = km^2 \).

Why do I think that the shown arbitrariness of equations for the AST fields is related to 1) spin basis rotations; 2) the choice of normalization? In the common-used basis the three 4-potentials have parity eigenvalues \(-1\) and one time-like (or spin-0 state), \(+1\); the fields \( E \) and \( B \) have also definite parity properties in this basis. If we transfer to other basis, e.g., to the helicity basis we can see that the 4-vector potentials and the corresponding fields are superpositions of the vector and the axial-vector. Of course, they can be expanded in the fields in the “old” basis.

So, we conclude: the addition of the Klein-Gordon equation to the \((J,0) \oplus (0,J)\) equations may change physical content even on the free level. In the \((1/2,0) \oplus (0,1/2)\) representation it is possible to introduce the parity-violating frameworks. We found the mappings between the Weinberg-Tucker-Hammer formalism for \( J = 1 \) and the AST fields of the 2nd rank of at least eight types. Four of them include both \( F_{\mu\nu} \) and \( \tilde{F}_{\mu\nu} \), which tells us that the parity violation may occur during the study of the corresponding dynamics. If we want to take into account the \( J = 1 \) solutions with different parity properties, the Bargmann-Wigner (BW), the Proca and the Duffin-Kemmer-Petiau (DKP) formalisms are to be generalized. We considered the most general case, introducing eight scalar parameters. In order to have covariant equations for the AST fields, one should impose constraints on the corresponding parameters. It is possible to get solutions with mass splitting. We found the 4-potentials and fields in the helicity basis. They have different parity properties comparing with the standard (“parity”) basis (cf. [18] [19]). The discussion induced us to generalize the BW, the Proca and the Duffin-Kemmer-Petiau formalisms. Higher-spin equations may actually describe various spin, mass, helicity and parity states. The states of different parity, helicity, and mass may be present in the same equation. On the basis of generalizations of the BW formalism, finally, we obtained twelve equations for the AST fields. A hypothesis was presented that the obtained results are related to the spin basis rotations and to the choice of normalization.

3. Standard Formalism (Spin 2)
The general scheme for derivation of higher-spin equations was given in [15]. A field of rest mass \( m \) and spin \( j \geq 1/2 \) is represented by a completely symmetric multispinor of rank \( 2j \). The particular cases \( j = 1 \) and \( j = 3/2 \) were given in the textbooks, e. g., ref. [16]. The spin-2 case can also be of some interest because it is generally believed that the essential features of the gravitational field are obtained from transverse components of the \((2,0) \oplus (0,2)\) representation.
of the Lorentz group. Nevertheless, questions of the redundant components of the higher-spin relativistic equations are not yet understood in detail [21].

In this section we use the commonly-accepted procedure for the derivation of higher-spin equations. We begin with the equations for the 4-rank symmetric spinor:

\[ i\gamma^\mu \partial_\mu - m \alpha \alpha' \Psi_{\alpha' \gamma} = 0, \]
\[ i\gamma^\mu \partial_\mu - m \gamma_\gamma \Psi_{\alpha \beta \gamma} = 0, \]
\[ i\gamma^\mu \partial_\mu - m \delta_\delta \Psi_{\alpha \beta \delta} = 0. \]

(55)

(56)

The massless limit (if one needs) should be taken in the end of all calculations.

We proceed expanding the field function in the set of symmetric matrices (as in the spin-1 case, cf. ref. [4a]). In the beginning let us use the first two indices:

\[ \Psi_{\{\alpha \beta\}}^{\gamma \delta} = (\gamma_\mu R)_{\alpha \beta} \Psi_{\mu}^{\gamma \delta} + (\sigma_{\mu \nu} R)_{\alpha \beta} \Psi_{\mu \nu}^{\gamma \delta}. \]

(57)

We would like to write the corresponding equations for functions \( \Psi_{\gamma \delta}^{\mu} \) and \( \Psi_{\gamma \delta}^{\mu \nu} \) in the form:

\[ \frac{2}{m} \partial_\mu \Psi_{\gamma \delta}^{\mu \nu} = -\Psi_{\gamma \delta}^{\nu}, \]
\[ \Psi_{\gamma \delta}^{\mu} = \frac{1}{2m} \left[ \partial^\mu \Psi_{\gamma \delta}^{\mu \nu} - \partial^\nu \Psi_{\gamma \delta}^{\mu \mu} \right]. \]

(58)

Constraints \( (1/m)\partial_\mu \Psi_{\gamma \delta}^{\mu \nu} = 0 \) and \( (1/m)\epsilon^{\mu \nu \alpha \beta} \partial_\nu \Psi_{\gamma \delta}^{\alpha \beta} = 0 \) can be regarded as a consequence of Eqs. (58). Next, we present the vector-spinor and tensor-spinor functions as

\[ \Psi_{\{\alpha \beta\}}^{\gamma \delta} = (\gamma^\mu R)_{\alpha \beta} T_{\gamma \delta}^{\mu \nu} + (\sigma^\mu R)_{\alpha \beta} R_{\gamma \delta}^{\mu \nu}. \]

(59)

\[ \Psi_{\{\gamma \delta\}}^{\mu \nu} = (\gamma^\mu R)_{\gamma \delta} T_{\mu \nu}^{\gamma \delta} + (\sigma^\mu R)_{\gamma \delta} R_{\mu \nu}^{\gamma \delta}. \]

(60)

i. e., using the symmetric matrix coefficients in indices \( \gamma \) and \( \delta \). Hence, the total function is

\[ \Psi_{\{\alpha \beta\}}^{\{\gamma \delta\}} = (\gamma_\mu R)_{\alpha \beta} (\gamma^\nu R)_{\gamma \delta} G_{\mu \nu} + (\gamma_\mu R)_{\alpha \beta} (\sigma_\mu R)_{\gamma \delta} F^{\mu \nu} + (\sigma_\mu R)_{\alpha \beta} (\sigma_{\mu \nu} R)_{\gamma \delta} R_{\mu \nu}^{\gamma \delta} ; \]

(61)

and the resulting tensor equations are:

\[ \frac{2}{m} \partial_\mu T_{\mu \nu}^{\gamma \delta} = -G_{\gamma \delta}^{\nu}, \]
\[ \frac{2}{m} \partial_\mu R_{\mu \nu}^{\gamma \delta} = -F^{\gamma \delta \nu}, \]
\[ T_{\mu \nu}^{\gamma \delta} = \frac{1}{2m} \left[ \partial^\mu G_{\gamma \delta}^{\nu} - \partial^\nu G_{\gamma \delta}^{\mu} \right], \]
\[ R_{\mu \nu}^{\gamma \delta} = \frac{1}{2m} \left[ \partial^\mu F_{\gamma \delta}^{\nu} - \partial^\nu F_{\gamma \delta}^{\mu} \right]. \]

(62)

(63)

(64)

The constraints are re-written to

\[ \frac{1}{m} \partial_\mu G_{\mu} = 0, \]
\[ \frac{1}{m} \partial_\mu F_{\mu} = 0, \]
\[ \frac{1}{m} \epsilon_{\alpha \beta \mu \nu} \partial^\alpha T_{\beta \nu} = 0, \]
\[ \frac{1}{m} \epsilon_{\alpha \beta \mu \nu} \partial^\alpha R_{\beta \nu} = 0. \]

(65)

(66)

However, we need to make symmetrization over these two sets of indices \( \{\alpha \beta\} \) and \( \{\gamma \delta\} \). The total symmetry can be ensured if one contracts the function \( \Psi_{\{\alpha \beta\}}^{\{\gamma \delta\}} \) with antisymmetric matrices \( R^{-1}_{\beta \gamma}, (R^{-1}_{\gamma} R^{5})_{\beta \gamma} \) and \((R^{-1}_{\gamma} R^{5} R^{\gamma})_{\beta} \gamma\) and equate all these contractions to zero (similar
to the \( j = 3/2 \) case considered in ref. [16] p. 44. We obtain additional constraints on the tensor field functions:

\[
\begin{align*}
G_\mu^\mu & = 0, \\
G_{[\nu,\mu]} & = 0, \\
G^{\mu\nu} & = \frac{1}{2}g^{\mu\nu}G_\nu^\nu, \\
F_{\kappa\mu} & = F_{\mu\kappa}^\mu = 0, \\
\epsilon^{\kappa\tau\mu\nu}F_{\kappa\tau,\mu\nu} & = 0, \\
T_\mu^\mu & = T_{\mu,\kappa}^\mu = 0, \\
\epsilon^{\kappa\tau\mu\nu}T_{\kappa,\tau\mu\nu} & = 0, \\
F^{\kappa\tau,\mu} & = T_{\mu,\kappa}^{\kappa\tau}, \\
\epsilon^{\kappa\tau\mu\lambda}(F_{\kappa\tau,\mu\lambda} + T_{\kappa,\tau\mu}) & = 0, \\
R_{\kappa\mu}^{\mu\nu} & = R_{\nu\kappa}^{\nu\mu} = R_{\kappa\mu}^{\nu\nu} = R_{\kappa\mu}^{\nu\nu} = R_{\kappa\mu}^{\mu\mu} = 0, \\
\epsilon^{\mu\nu\alpha\beta}(g_{\beta\kappa}R_{\mu\kappa,\nu\alpha} - g_{\beta\kappa}R_{\nu\alpha,\mu\kappa}) & = 0 \\
\epsilon^{\kappa\tau\mu\nu}R_{\kappa,\tau\mu\nu} & = 0.
\end{align*}
\]

Thus, we encountered with the known difficulty of the theory for spin-2 particles in the Minkowski space. We explicitly showed that all field functions become to be equal to zero. Such a situation cannot be considered as a satisfactory one (because it does not give us any physical information) and can be corrected in several ways.

[1]

4. Generalized Formalism (Spin 2)

We shall modify the formalism [17]. The field function is now presented as

\[
\Psi_{\{\alpha\beta\}\gamma\delta} = \alpha_1(\gamma\mu R)_{\alpha\beta}\Psi_{\gamma\delta}^\mu + \alpha_2(\sigma_{\mu\nu} R)_{\alpha\beta}\Psi_{\gamma\delta}^{\mu\nu} + \alpha_3(\gamma_5\sigma_{\mu\nu} R)_{\alpha\beta}\tilde{\Psi}_{\gamma\delta}^{\mu\nu},
\]

with

\[
\begin{align*}
\Psi_{\{\gamma\delta\}\gamma\delta} & = \beta_1(\gamma_{\tau} R)_{\gamma\delta}G_\kappa^\kappa + \beta_2(\sigma_{\kappa\tau} R)_{\gamma\delta}F^\kappa_{\mu\nu} + \beta_3(\gamma_{5\tau} \sigma_{\kappa\tau} R)_{\gamma\delta}\tilde{F}_{\mu\nu}, \\
\Psi_{\{\gamma\delta\}\gamma\delta} & = \beta_4(\gamma_{\tau} R)_{\gamma\delta}T^\kappa_{\mu\nu}, \\
\tilde{\Psi}_{\{\gamma\delta\}\gamma\delta} & = \beta_5(\gamma_{5\tau} R)_{\gamma\delta}\tilde{T}^\kappa_{\mu\nu} + \beta_6(\sigma_{\kappa\tau} R)_{\gamma\delta}\tilde{D}^\kappa_{\mu\nu} + \beta_7(\gamma_{5\tau} \sigma_{\kappa\tau} R)_{\gamma\delta}\tilde{D}^\kappa_{\mu\nu}.
\end{align*}
\]

Hence, the function \( \Psi_{\{\alpha\beta\}\gamma\delta} \) can be expressed as a sum of nine terms:

\[
\begin{align*}
\Psi_{\{\alpha\beta\}\gamma\delta} & = \alpha_1\beta_1(\gamma_{\mu} R)_{\alpha\beta}(\gamma^5 R)_{\gamma\delta}G_\kappa^\kappa + \alpha_1\beta_2(\gamma_{\mu} R)_{\alpha\beta}(\sigma^\kappa R)_{\gamma\delta}F^\kappa_{\mu\nu} + \\
 & + \alpha_1\beta_3(\gamma_{\mu} R)_{\alpha\beta}(\gamma^5 \sigma^\kappa R)_{\gamma\delta}\tilde{F}^\kappa_{\mu\nu} + \alpha_2\beta_1(\sigma_{\mu\nu} R)_{\alpha\beta}(\gamma^\kappa R)_{\gamma\delta}T^\kappa_{\mu\nu} + \\
 & + \alpha_2\beta_2(\sigma_{\mu\nu} R)_{\alpha\beta}(\sigma^\kappa R)_{\gamma\delta}R^\kappa_{\mu\nu} + \alpha_2\beta_3(\sigma_{\mu\nu} R)_{\alpha\beta}(\gamma^5 \sigma^\kappa R)_{\gamma\delta}\tilde{R}^\kappa_{\mu\nu} + \\
 & + \alpha_3\beta_1(\gamma^5 \sigma_{\mu\nu} R)_{\alpha\beta}(\gamma^\kappa R)_{\gamma\delta}\tilde{T}^\kappa_{\mu\nu} + \alpha_3\beta_2(\gamma^5 \sigma_{\mu\nu} R)_{\alpha\beta}(\sigma^\kappa R)_{\gamma\delta}\tilde{D}^\kappa_{\mu\nu} + \\
 & + \alpha_3\beta_3(\gamma^5 \sigma_{\mu\nu} R)_{\alpha\beta}(\gamma^5 \sigma^\kappa R)_{\gamma\delta}\tilde{D}^\kappa_{\mu\nu}.
\end{align*}
\]

The corresponding dynamical equations are given by the set

\[
\begin{align*}
\frac{2\alpha_2\beta_1}{m}\partial_{\mu\nu}T^\kappa_{\kappa\tau\mu\nu} + \frac{i\alpha_3\beta_1}{m}\epsilon^{\mu\nu\alpha\beta}\partial_{\mu\nu}\tilde{T}^\kappa_{\kappa\alpha\beta} & = \alpha_1\beta_1G_\kappa^\kappa; \\
\frac{2\alpha_2\beta_2}{m}\partial_{\mu\nu}R^\kappa_{\mu\nu} + \frac{i\alpha_3\beta_2}{m}\epsilon^{\mu\nu\alpha\beta}\partial_{\mu\nu}\tilde{R}^\kappa_{\kappa\alpha\beta} + \frac{i\alpha_3\beta_3}{m}\epsilon^{\mu\nu\alpha\beta}\partial_{\mu\nu}\tilde{D}^\kappa_{\kappa\alpha\beta} & = -\frac{\alpha_3\beta_1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_{\mu\nu}D^\kappa_{\kappa\alpha\beta},
\end{align*}
\]

1 The reader can compare our results of this Section with those of G. Marques and D. Spehler, Mod. Phys. Lett. A13 (1998) 553-569. I consider their discussion of the standard formalism in the Sections I and II, as insufficient.
The essential constraints are:

\[
\begin{align*}
&\alpha_1\beta_1 G_{\mu\nu} = 0, \quad \alpha_1\beta_1 G_{(\varepsilon\mu\nu)} = 0; 2\alpha_1\beta_2 F_{\alpha\mu} + \alpha_1\beta_3 \varepsilon_{\varepsilon\mu\nu} F_{\varepsilon\kappa\mu} = 0; \\
&2\alpha_1\beta_3 \tilde{T}_{\mu} - \alpha_2\beta_4 \varepsilon_{\varepsilon\mu\nu} T_{\varepsilon\kappa\mu} = 0; \\
&\varepsilon_{\varepsilon\mu\nu} \alpha_3\beta_5 R_{\mu\nu} + 2\alpha_3\beta_6 D_{\mu\nu} + 2\alpha_3\beta_7 D_{\mu\nu} = 0; \\
&\varepsilon_{\varepsilon\mu\nu} \alpha_3\beta_8 R_{\mu\nu} + 2\alpha_3\beta_9 D_{\mu\nu} + 2\alpha_3\beta_4 D_{\mu\nu} = 0; \\
&\varepsilon_{\varepsilon\mu\nu} \alpha_3\beta_6 D_{\mu\nu} + 2\alpha_3\beta_7 D_{\mu\nu} + 2\alpha_3\beta_8 D_{\mu\nu} + 2\alpha_3\beta_9 D_{\mu\nu} = 0; \\
&\varepsilon_{\varepsilon\mu\nu} \alpha_3\beta_4 D_{\mu\nu} + 2\alpha_3\beta_5 D_{\mu\nu} + 2\alpha_3\beta_6 D_{\mu\nu} + 2\alpha_3\beta_7 D_{\mu\nu} + 2\alpha_3\beta_8 D_{\mu\nu} + 2\alpha_3\beta_9 D_{\mu\nu} = 0; \\
&\varepsilon_{\varepsilon\mu\nu} \alpha_3\beta_5 D_{\mu\nu} + 2\alpha_3\beta_6 D_{\mu\nu} + 2\alpha_3\beta_7 D_{\mu\nu} + 2\alpha_3\beta_8 D_{\mu\nu} + 2\alpha_3\beta_9 D_{\mu\nu} = 0.
\end{align*}
\]

They are the results of contractions of the field function \(\{77\}\) with three antisymmetric matrices, as above. Furthermore, one should recover the relations \(\{67\}\) in the particular case when \(\alpha_3 = \beta_3 = \beta_6 = \beta_9 = 0\) and \(\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \beta_4 = \beta_7 = \beta_8 = 1\).

As a discussion we note that in such a framework we already have physical content because only certain combinations of field functions would be equal to zero. In general, the fields \(F_{\kappa\mu}, R_{\kappa\mu}, T_{\kappa\mu}, D_{\kappa\mu}, D_{\kappa\mu}\) can correspond to different physical states and the equations above describe oscillations one state to another. Furthermore, from the set of equations \(\{78\}\) one obtains the second-order equation for symmetric traceless tensor of the second rank (\(\alpha_1 \neq 0, \beta_1 \neq 0\)):

\[
\frac{1}{m^2} \left[ \partial_\nu \partial^\mu G_\kappa - \partial_\nu \partial^\mu G_\kappa \right] = G_\kappa^\mu.
\]
After the contraction in indices $\kappa$ and $\mu$ this equation is reduced to the set
\[\partial_{\mu}G^{\mu\kappa} = F_{\kappa}\]
\[\frac{1}{m^2}\partial_{\mu}F^{\kappa} = 0,\]

i.e., to the equations connecting the analogue of the energy-momentum tensor and the analogue of the 4-vector potential. Further investigations may provide additional foundations to “surprising” similarities of gravitational and electromagnetic equations in the low-velocity limit, refs. \[22, 23.\]

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