Analytical model for solving ship maneuvering response under continuous linear steering

Weiling Zhang\textsuperscript{a}, Feng Cai\textsuperscript{b} and Bo Yang\textsuperscript{*}
Dalian Naval Academy, Dalian, China

*Corresponding author e-mail: 877146322@qq.com, \textsuperscript{a}18868112697@163.com, \textsuperscript{b}820437895@qq.com

Abstract. In this paper, the analytical model of the nonlinear differential equations is solved by combining the characteristics of the first-order and second-order KT steering response equations. In order to make the model as close as possible to the actual situation, the steering time is taken into account in the modelling, and the problem of sudden change of steering speed in linear steering is solved, and the analytical model expression of the KT response equation under continuous linear steering is obtained. The analytical model simulation results, Runge-Kutta numerical simulation results and real ship Z-shaped test data are compared to verify the effectiveness of the analytical model.

1. Introduction
Ships sailing in vast waters are most concerned with heading course \cite{1}. For the forecasting and simulation of the heading angle, there are three mainstream maneuvering mathematical models-Abkowitz model, MMG model and KT maneuvering response model. Compared with the previous two models, the KT maneuvering response model responds to the steering angle at the heading angle. There are great advantages in terms. In recent years, with the continuous optimization of parameter identification methods, the identification accuracy has been continuously improved \cite{2-6}, and the KT maneuvering response model has been widely used in ship motion simulation and heading prediction. However, most scholars have applied the equation to the form of their differential equations, using the Runge-Kutta method for numerical solution \cite{2, 3}, and rarely explored and applied them in depth.

Fan Shangyu \cite{1} has solved the analytical model of steering to stable rudder angle by superimposing two linear steerings; Gan Xionglang \cite{7} simplified the steering time and discussed the analytical model of single-step steering. In order to meet the actual ship condition as much as possible, this paper takes the steering time into account and solves the KT steering response analytical model under continuous linear steering. The steering response model is returned to the ships’ operator’s most concerned heading angle and rudder angle. Relationship between. Finally, the analytical model simulation results, Runge-Kutta numerical simulation results and real ship Z-shaped test data are compared to verify the effectiveness of the analytical model.
2. The Model Statement

2.1. Analytical model of first-order KT response equation

The first-order KT maneuver response equation is expressed as follows [8]:

\[ T\ddot{r} + r = K\delta \]  \hspace{1cm} (1)

Where: \( r, \dot{r}, \delta \): angular velocity, angular acceleration, rudder angle value; 
\( T, K \): equation parameters.

Change it appropriately into the following form:

\[ \ddot{r} + \frac{1}{T} r = \frac{K\delta}{T} \]  \hspace{1cm} (2)

The equation is a first-order non-homogeneous equation, where the non-homogeneous term is \( \frac{K\delta}{T} \).

Order \( q(t) = \frac{K\delta}{T} \), \( p(t) = \frac{1}{T} \). Then the general solution of the equation can be expressed as follows:

\[ r(t) = e^{-\int p(t)dt} \left( C + \int q(t)e^{\int p(t)dt} \right) \]  \hspace{1cm} (3)

\[ r(t) = e^{-\int \frac{1}{T} dt} \left( C + \int \frac{K}{T} \delta(t)e^{\int \frac{1}{T} dt} \right) \]  \hspace{1cm} (4)

\[ \int \frac{K}{T} \delta(t)e^{\int \frac{1}{T} dt} = \frac{K}{T} (T\delta(t)e^{\frac{1}{T}} - T^2 \dot{\delta}(t)e^{\frac{1}{T}}) + C \]  \hspace{1cm} (5)

In the actual handling process, the rudder angle is constantly changing. Assume that the rudder angle changes linearly, that is, the rate of change of the rudder angle is constant during a certain maneuver.

![Figure 1. Schematic diagram of steering.](image-url)
Assuming that the time vector of the steering is \( t_i (i = 1, 2, \ldots), \) the corresponding rudder angle value is \( \delta_i (i = 1, 2, \ldots), \) for \( t \in \left[ t_i, t_{i+1} \right), \) there is
\[
\delta(t) = \frac{\delta_{i+1} - \delta_i}{t_{i+1} - t_i} (t - t_i) + \delta_i.
\]

For the entire maneuvering process, because \( \delta(t) \) is discontinuous, piecewise integration is required when solving the integral of (5). First, make the following definition:

\[
F_1(t) \triangleq T \delta(t) e^\frac{t}{T} - T^2 \dot{\delta}(t) e^\frac{t}{T}
\]

\[
M_{i-1} \triangleq \sum_{j=1}^{i-1} (F_1(t_{j+1}) - F_1(t_j))
\]

Where \( M_0 = 0. \)

Then (4) can be expressed as follows:

\[
r(t) = \frac{K}{T} e^{\frac{-t}{T}} (C + F_1(t) - F_1(t_i) + M_{i-1})
\]

Assume that the ship initially sails in a stable heading, when \( t=0, \ r(0) = C = 0. \) Therefore, the angular velocity expression for continuous steering is:

\[
r(t) = \frac{K}{T} e^{\frac{-t}{T}} (F_1(t) - F_1(t_i) + M_{i-1})
\]

In the process of solving the heading angle, the integral of the angular velocity also requires piecewise integration.

\[
\varphi(t) = K \delta t + \dot{\delta} K (\frac{1}{2} t^2 - t_i t) - TK \dot{\delta} t + K (F_1(t_i) - M_{i-1}) e^{\frac{t}{T}} + C
\]

\[
F_2(t) \triangleq K \delta t + \dot{\delta} K (\frac{1}{2} t^2 - t_i t) - TK \dot{\delta} t + K (F_1(t_i) - M_{i-1}) e^{\frac{t}{T}}
\]

\[
N_{i-1} \triangleq \sum_{j=1}^{i-1} (F_2(t_{j+1}) - F_2(t_j)), \ t \in \left[ t_i, t_{i+1} \right)
\]

Where \( N_0 = 0. \)

The heading angle is expressed as follows:

\[
\varphi(t) = F_2(t) - F_2(t_i) + N_{i-1} + \varphi_0, \ t \in \left[ t_i, t_{i+1} \right)
\]
Where $\phi_0$ indicates the initial heading of the maneuvering process.

2.2. Analytical model of second-order KT response equation

The second-order KT maneuvering response equation is expressed as follows [9]:

$$T_1T_2\ddot{r} + (T_1 + T_2)\dot{r} + r = K\delta + KT_3\dot{\delta}$$

(14)

Where: $r$, $\dot{r}$, $\ddot{r}$: angular velocity, angular acceleration, angular acceleration
$
\delta$, $\dot{\delta}$: rudder angle value, rudder angle change rate;

$T_1$, $T_2$, $T_3$, $K$: equation parameter.

Change it appropriately into the following form:

$$\dot{r} + \frac{(T_1 + T_2)}{T_1T_2} \ddot{r} + \frac{1}{T_1T_2} r = \frac{K\delta}{T_1T_2} + \frac{KT_3\dot{\delta}}{T_1T_2}$$

(15)

And $-\frac{1}{T_1}$ and $-\frac{1}{T_2}$ are the solutions of the characteristic equation below corresponding to the above formula.

$$\lambda^2 + \frac{(T_1 + T_2)}{T_1T_2} \lambda + \frac{1}{T_1T_2} = 0$$

(16)

So, the homogeneous differential equation corresponding to (15) has a general solution [10]:

$$r = C_1e^{-\frac{1}{T_1}t} + C_2e^{-\frac{1}{T_2}t}$$

(17)

Due to linear steering, for $\forall t \in [t_i, t_{i+1})$, the non-homogeneous term of (15) is a first-order polynomial. Suppose its special solution is $r = at + b$, substituting into (15) which can be solved:

$$\begin{cases} a = K\delta \\ b = K\delta_0 - K\dot{\delta}(T_1 + T_2 - T_3) \end{cases}$$

(18)

Then the general solution of (15) is:
By integrating the angular velocity, the heading angle expression can be got:

$$\phi(t) = -T_1C_1e^{-\frac{1}{T_1}(t-t_0)} - T_2C_2e^{-\frac{1}{T_2}(t-t_0)} + \frac{1}{2}K\dot{\delta}(t-t_0)^2 + \left[K\delta_0 - K\dot{\delta}(T_1 + T_2 - T_3)\right]t + C$$

(20)

Because $\dot{\delta}(t)$ is discontinuous in the process of continuous steering, diagonal speed requires segmentation integration.

$$F_3(t) \triangleq -T_1C_1e^{-\frac{1}{T_1}(t-t_0)} - T_2C_2e^{-\frac{1}{T_2}(t-t_0)} + \frac{1}{2}K\dot{\delta}(t-t_0)^2 + \left[K\delta_0 - K\dot{\delta}(T_1 + T_2 - T_3)\right]t$$

(21)

$$L_{i-1} \triangleq \sum_{j=1}^{i-1} \left(F_3(t_{j+1}) - F_3(t_j)\right), \ t \in [t_i, t_{i+1})$$

(22)

Where $L_0 = 0$.

Then the heading angle expression is as follows:

$$\phi(t) = F_3(t) - F_3(t_i) + L_{i-1} + \phi_0, \ t \in [t_i, t_{i+1})$$

(23)

3. The Model Verification

Based on the data of one of the 10°/10° Z-shaped tests, the analytical model was verified by simulation. During the test, the steering time vector and the corresponding rudder angle value are shown in equation (24), and the test data is shown in Figure 2.

$$\begin{align*}
    t &= (0 \ 5.51 \ 31.46 \ 39.84 \ 92.08 \ 101.07 \ 164.59 \ 173.29 \ 226.54 \ 234.63 \ 300.04) \\
    \delta &= (0 10 -10 -10 10 10 -10 -10 10 10) 
\end{align*}$$

(24)

Substitute the data into the first-order analytical models (9) and (13), where the equation parameters, and the simulation step size is 0.1 s. At the same time, the equation is numerically solved by the fourth-
order classical Runge-Kutta method, and the analytical model results, numerical simulation results and experimental data are compared, as shown in Figure 3 and Figure 4.

Substitute the data of (24) into the second-order analytical models (19) and (23), in which the parameters of the equation are $K = 6.12 \times 10^{-2} , T_1 = 7.95 , T_2 = 5.21 , T_3 = 5.91 \times 10^{-3}$, and the simulation step size is 0.1s. At the same time, the equation is numerically solved by the fourth-order classic Runge-Kutta method. The analytical model results, numerical simulation results and experimental data are compared, and the results are shown in Figure 5-7.

Figure 2. Test Data.  
Figure 3. First-order model heading angle verification.  
Figure 4. First-order model angular velocity verification.  
Figure 5. Second-order model heading angle verification.
Figure 6. Second-order model angular velocity verification.

Figure 7. Second-order model angular acceleration verification.

Table 1. Model error analysis table.

|                      | Heading angle error | Angular velocity error | Angular acceleration error |
|----------------------|---------------------|------------------------|----------------------------|
|                      | Maximum error (°)   | Average error (°)      | Maximum error (°)          | Average error (°) | Maximum error (°) | Average error (°) |
| First-order model    | 1.3994              | 0.7824                 | 0.1052                     | 0.0287            |
| analytical simulation|                     |                        |                            |                   |
| Second-order model   | 1.8773              | 0.7836                 | 0.0882                     | 0.0304            | 0.0184            | 0.0049            |
| analytical simulation|                     |                        |                            |                   |

Table 2. Analytical model and numerical model deviation analysis table.

|                      | Heading angle error | Angular velocity error | Angular acceleration error |
|----------------------|---------------------|------------------------|----------------------------|
|                      | Maximum error (°)   | Average error (°)      | Maximum error (°)          | Average error (°) | Maximum error (°) |
| First-order model    | 0.1273              | 0.0583                 | 0.0110                     | 0.0023            |
| analytical simulation|                     |                        |                            |                   |
| Second-order model   | 0.4211              | 0.1486                 | 0.0104                     | 0.0023            | 0.0014           | 1.9918e-04        |
| analytical simulation|                     |                        |                            |                   |

4. Conclusion

It can be seen from the error analysis table that the errors of the first-order and second-order analytical models are very low. The maximum heading angle error is within 2°, the average absolute error is within 1°, and the simulation effect is good. It can be seen from the graph that the curve of the analytical model is almost completely coincident with the fourth-order classical Runge-Kutta numerical simulation curve. The deviation between the two is very small, which further proves the validity of the analytical model constructed in this paper.

For the ship KT maneuvering response model, it is no longer necessary to use the Runge-Kutta method for numerical solution. After the analytical model is established, the amount of calculation is greatly reduced, and there is no need to consider the steps of the numerical solution, error and convergence. Using the analytical model to directly solve the ship KT maneuvering response model can
solve and predict the heading angle, angular velocity and angular acceleration information to the greatest extent accurately and quickly.

Acknowledgments
This work was financially supported by the fund: Thirteenth Five-Year Equipment Pre-research Project (41407010301).

References
[1] Liu Yang, Liu Lin, Guo Dabin, Hu Yan. How to navigate safely under the conditions of heavy wind and waves [J]. China Water Transport (2nd Half), 2017, 17 (08): 35 - 36.
[2] Zhang Heng, Zhu Jun. Evaluation Method of Surface Ship Combat Capability Considering the Influence of Wind and Wave Environment [J]. Chinese Ship Research, 2017, 12 (3): 36 - 42.
[3] Ying Rongrong, Shi Aiguo, Cai Feng, Wang Xiao, Yang Baozhang. Assessment of Navigation Risk Level of Ships with Large Wind Waves [J]. China Shipping, 2009, 32 (04): 49 - 52+ 76.
[4] Jiao Jialong, Sun Shuzheng, Ren Hui. Fuzzy Comprehensive Evaluation Method for Environmental Adaptability of Surface Ships [J]. Journal of Harbin Engineering University, 2014, 35 (06): 667 - 673.
[5] Yang Hu. Research on environmental adaptability assessment of surface ship based on multiobjective comprehensive evaluation [D]. Harbin Engineering University, 2016.
[6] Liu Meng, Zhang Xinyu, Wang Xiao, Wang Zuichao, Shi Aiguo. Risk Assessment of Ship's Large Wind Wave Navigation [J]. Ship Science and Technology, 2014, 36 (10): 9 - 12.
[7] Wu Yueqin, Fu Yun, Ao Liang. Evaluation Method of Equipment Environmental Adaptability under Typical Environmental Conditions [J]. Environmental Engineering, 2010, 7(06): 109-112.
[8] Han Zhiguo. Research on ship evaluation index system and evaluation method [D]. Dalian University of Technology, 2008.
[9] Liu Yuanfeng, Tang Xingli. Ship Navigation Safety Evaluation Based on Fuzzy Comprehensive Evaluation Method [J]. Journal of Chongqing Jiaotong University, 2004 (03): 123 - 126.
[10] Yan Ruixia, Liu Jinliang, Yao Bingxue. A Method and Application of Expert Weight Determination in Group Decision Making [J]. Statistics and Decision, 2007 (23): 84 - 86.
[11] Ding Qiong, Sun Meilin. Improvement and Empirical Study on Evaluation Model of Probability Weight Efficacy Coefficient [J]. Statistics and Decision, 2015 (04): 93 - 95.
[12] Jiao Jialong. Experimental study on large-scale model motion and load response of ships in actual wave environment [D]. Harbin Engineering University, 2016.