Neutrino kinetics in a magnetized dense plasma

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Abstract

The relativistic kinetic equations (RKE) for lepton plasma in the presence of a strong external magnetic field are derived in Vlasov approximation. The new RKE for the electron spin distribution function includes the weak interaction with neutrinos originated by the axial vector current ($\sim c_A$) and provided by the parity nonconservation. In a polarized electron gas Bloch equation describing the evolution of the magnetization density perturbation is derived from the electron spin RKE being modified in the presence of neutrino fluxes. Such modified hydrodynamical equation allows to obtain the new dispersion equation in a magnetized plasma from which the neutrino driven instability of spin waves can be found. It is shown that this instability is more efficient e.g. in a magnetized supernova than the analogous one for Langmuir waves enhanced in an isotropic plasma.

PACS codes: 13.10.+q, 13.15.+g, 97.60.Bw, 52.60.+h
Keywords: Elementary particles (neutrino)– Kinetic equations – Supernova: spin waves, magnetization
1 Introduction

When he conjectured the existence of the neutrino, Pauli imposed very stringent bounds on its electrical neutrality.

Nevertheless, the direct (=weak) interaction of neutrino with electrons and positrons shows a principle possibility of the interaction of neutrino with the electromagnetic field. In vacuum, however, this interaction is negligible. The situation is extremely changed in media with free carriers of the electric charge such as a dense plasma of metals, stars, the lepton plasma of the early Universe, etc.

It was shown in [1, 2, 3] that the electrodynamics of neutrinos is changed in such media comparing with the electrodynamics of neutrinos in vacuum in such a way that the contribution of the neutrino electromagnetic vertex in medium cannot be regarded as a small correction like the case of radiative corrections in vacuum. The main distinction is the appearance of the induced electric charge of neutrino which is proportional to the density of free carriers of electricity \[^1\]. Namely this leads to the new qualitative effect: the appearance of the long-distance forces inevitably leads to the collective interactions of neutrino with an ensemble of charged particles via electromagnetic field.

The approximation of the neutrino propagation in an isotropic plasma considered in [1, 2, 3, 6] and then in [7] has some natural bounds for applications in astrophysics. Therefore one needs to have the self-consistent system of kinetic equations of the most general kind applicable for other astrophysical plasmas including magnetized stars.

Really this turns out to be a new problem unsolved before in the neutrino kinetics while in the one-particle Schrödinger equation approach one can show the importance of the axial vector potential for the neutrino propagation in a magnetized plasma \((V_A \sim <\bar{\psi}e\gamma_i\gamma_5\psi_e> \sim B_i)\) that provides the interaction of neutrino with the magnetic field \(B\) without any neutrino magnetic moment \([8]\) and could explain, e.g. the neutron star kick \([9]\).

In the present work we take into account the spin interactions of electrons that are important for neutrino propagation in a dense degenerate electron gas polarized by the external magnetic field.

In subsection 2.1. we discuss the equilibrium state of an electron gas polarized by the external magnetic field \(B_0 \neq 0\).

Then in subsection 2.2 we present the full set of the coupled Relativistic Kinetic Equations (RKE) for electrons and neutrinos in the lepton plasma with an external magnetic field including the self-consistent electromagnetic field contribution. These RKE’s are derived using the Bogolyubov method analogously to the way described in details for an isotropic plasma in [6].

In subsection 2.3 we complete derivation of RKE’s recasting them for the gauge invariant distribution functions and prove the lepton current conservation, 
\[\partial_j j^{(a)}_\mu(x, t)/\partial x_\mu = 0,\]

in a magnetized plasma both for neutrinos and electrons.

We conclude this section deriving in subsection 2.4 the dispersion equation for the lepton density perturbations in an isotropic plasma that coincides with results \([7, 10]\).

In section 3 we generalize the Bloch equation for the evolution of the magnetization density perturbation in the presence of neutrino fluxes. In section 4, neglecting spatial dispersion we obtain the increment of spin waves excited by the neutrino beam and com-

\[^1\]There appears also induced magnetic moment of neutrino \([4]\), cross-sections of the neutrino scattering off nuclei are modified in plasmas especially in low energy region \([5]\) and so on.
pare it with the case of the neutrino driven instability of plasma waves in an isotropic plasma.

In section 5 we discuss results and give some estimates of relevant parameters in a polarized medium.

2 Relativistic Kinetic Equations for lepton plasma in Vlasov approximation

The kinetic equations in the Standard Model (SM) are derived from the quantum Liouville equation for the non-equilibrium statistical operator $\hat{\rho}(t)$ using the Bogolyubov method with the total interaction Hamiltonian given by the Feynman diagrams for the neutrino scattering off electrons and the usual $ee$-scattering in QED [6]. For a magnetized plasma we do not consider $\nu\nu$-scattering and neglect also weak $ee$-scattering comparing with the electromagnetic interaction of charged particles through the photon exchange.

The one-particle density matrix $\hat{f}_r^{(a)}(p, x, t) = \sum_k e^{ikx} Tr(\hat{\rho}(t)\hat{b}^{+}_r k/2, r, \hat{b}_r k/2, a)$ is determined as the statistical average of the product of creation ($\hat{b}^{+}_r$) and annihilation ($\hat{b}_r$) operators for the lepton $a = e, \nu$. In the case of electrons it takes the form

$$\hat{f}_r^{(e)}(x, t) = f^{(e)}(\mathbf{p}, \mathbf{x}, t) \delta_{\lambda'\lambda} + S^{(e)}(\mathbf{p}, \mathbf{x}, t) \frac{(\sigma_i)_{\lambda'\lambda}}{2},$$

where $\lambda = \pm 1$ is the eigenvalue of the spin projection on an external magnetic field. E.g. for the magnetic field $\mathbf{B} = (0, 0, B_0)$ only the spin component $\Sigma_z$ commutes with the Hamiltonian for electrons, $[\Sigma_z, H^{(e)}] = 0$, hence $(\sigma_z)_{\lambda'\lambda} = \lambda\delta_{\lambda'\lambda}$. In the absence of neutrino magnetic moment the one-particle density matrix for massless Dirac neutrinos corresponds to the active (left-handed, $r = -1$) neutrino distribution function only,

$$f^{(\nu)}(q, x, t) = \frac{(1 - r)\delta_{\nu'}\nu}{2} f^{(\nu)}(q, x, t),$$

where in a uniform medium the helicity $r = \pm 1$ conserves, $(\sigma_i q_i)_{\nu'\nu} = qr \delta_{\nu'\nu}$.

The number density distribution functions are related with the lepton densities $n^e(\mathbf{x}, t) = \int (d^3p/(2\pi)^3) f^{(e)}(\mathbf{p}, \mathbf{x}, t)$ for electrons and $n^\nu(\mathbf{x}, t) = \int (d^3q/(2\pi)^3) f^{(\nu)}(\mathbf{q}, \mathbf{x}, t)$ for neutrinos.

2.1 Equilibrium state in a polarized electron gas

In the linear approximation for a slightly inhomogeneous medium the distribution functions have the form

$$f^{(\nu)}(q, x, t) = f^{(\nu)}_0(q) + \delta f^{(\nu)}(q, x, t),$$
$$f^{(e)}(p, x, t) = f^{(e)}_0(\varepsilon_p) + \delta f^{(e)}(p, x, t),$$
$$S^{(e)}(p, x, t) = S^{(e)}_0(\varepsilon_p) + \delta S^{(e)}(p, x, t),$$

where the neutrino background is given by $f^{(\nu)}_0(q)$ not being in equilibrium with plasma environment (e.g. for the powerful neutrino flux outside of the SN neutrinosphere), and we
consider the uniform polarized equilibrium electron gas for which the one-particle Wigner density matrix takes the form

$$f^{(0e)}_{\nu \lambda}(\varepsilon_p) = \frac{\delta \chi_{\nu \lambda}}{2} f^{(0)}(\varepsilon_p) + \frac{(\sigma_j)_{\nu \lambda}}{2} S_j^{(0)}(\varepsilon_p).$$  (2)

Here $f^{(0)}(\varepsilon_p) = g_e[\exp(\varepsilon_p - \zeta)/T] + 1]^{-1}$ is the equilibrium Fermi function; $g_e = 2$ is the Lande factor;

$$S_j^{(0)}(\varepsilon_p) = -\frac{\mu_B}{\gamma} \frac{B_j^{(0)}}{d\varepsilon_p} f^{(0)}(\varepsilon_p) \varepsilon_p,$$  (3)

is the equilibrium spin distribution; $\gamma = \varepsilon_p/m_e$ is the electron gamma factor, and we use hereafter units $\hbar = c = 1$.

Note that in (3) we assumed the quasi-classical electron spectrum in a realistic external magnetic field $B_0$ obeying the inequality $eB_0 \ll T^2$, when the Landau spectrum

$$\varepsilon(\lambda, p_z, n) = (m_e^2 + p_z^2 + e | B_0(2n + 1 - \lambda))^{1/2}$$  (4)

reduces to

$$\varepsilon_p(\lambda) = \varepsilon_p - \lambda \frac{e | B_0}{2\varepsilon_p},$$  (5)

where $\lambda = \pm 1$ and we changed $| e | B_0(2n + 1) = p_z^2$ for large Landau numbers $n \gg 1$. Thus, the full spectrum (5) contains the small paramagnetic (spin) correction to the continuous spectrum $\varepsilon_p = \sqrt{p^2 + m_e^2}$.

One can easily check that in this quasi-classical limit the exact expression for magnetization of the electron gas,

$$M_j^{(0)} = | \mu_B | < \psi e^{i \gamma_j \gamma_5} \psi | = | \mu_B | \sum_{n=0}^{\infty} \frac{| e | B_0}{(2\pi)^2} \int_{-\infty}^{+\infty} dp_z \mathrm{Tr}[\sigma_j f^{(0e)}(\varepsilon(\lambda, p_z, n))]| =$$

$$= | \mu_B | \sum_{n=0}^{\infty} \frac{| e | B_0 \delta_{jz}}{(2\pi)^2} \int_{-\infty}^{+\infty} dp_z \sum_{\lambda} \lambda f_{\lambda \lambda}^{(0e)}(\varepsilon(\lambda, p_z, n)),$$  (6)

takes the quasi-classical form

$$M_j^{(0)} = | \mu_B | \int \frac{d^3p}{(2\pi)^3} S_j^{(0)}(\varepsilon_p) = -2\mu_B B_0 j \int D(\varepsilon_p) \frac{df^{(0)}(\varepsilon_p)}{d\varepsilon_p} d\varepsilon_p,$$  (7)

where the spin distribution $S_j^{(0)}(\varepsilon_p)$ is given by eq. (3) and $D(\varepsilon_p) = pm_e/(2\pi)^2$. In particular, in the non-relativistic (NR) limit this background magnetization corresponds to the spin paramagnetism of the free electron gas in metal [1].

Note that for the degenerate electron gas from the quasi-classical Eq. (7) one obtains the same value $M_z^{(0)} = | \mu_B | | e | B_0 p_{Fz}/2\pi^2$ as it follows from the exact quantum eq. (3) for an arbitrary strong magnetic field for which electrons populating the main Landau level (n=0) contribute only.

Note also that this magnetization determines the axial vector potential of a probing neutrino in a magnetized plasma too [3],

$$V_A = -G_F \sqrt{2} e_A \frac{q_j}{q} \frac{M_j^{(0)}}{\mu_B},$$

4
that changes the spectrum of the ultrarelativistic neutrino, \( \varepsilon_q = q + V + V_A \), comparing with the standard one in an isotropic medium (e.g. in the Sun), \( \varepsilon_q = q + V \), and modifies the neutrino oscillations in a magnetized SN described by the one-particle Schrödinger equation for two neutrino species [8, 9],

\[
i \left( \begin{array}{c} \dot{\nu}_a \\ \dot{\nu}_b \end{array} \right) = \left( \begin{array}{cc} V + V_A - c_2 \delta & s_2 \delta \\ s_2 \delta & 0 \end{array} \right) \left( \begin{array}{c} \nu_a \\ \nu_b \end{array} \right),
\]

where \( c_2 = \cos 2\theta \), \( s_2 = \sin 2\theta \), \( \delta = \Delta m^2/4E \) are the neutrino mixing parameters; \( \nu_b \) is the sterile neutrino wave function.

In our kinetic approach for the SM lepton plasma we do not consider oscillations \( (s_2 = 0) \) assuming the massless spectrum \( \varepsilon_q = q \) for active neutrino of the given kind \( \nu_a \), \( a = e, \mu, \tau \).

2.2 The master kinetic equations for perturbations in plasma

Within the linear approximation [1] the RKE for the Lorentz-invariant Wigner number density distribution functions \( \delta f^{(a)}(p, x, t) = Tr[\delta f^{(a)}(p, x, t)] \) take the covariant forms

\[
q_\mu \frac{\partial \delta f^{(e)}(q, x, t)}{\partial x_\mu} + G_F \sqrt{2} e \int \frac{d^3 p_e (p^\mu_e q_\mu) \partial \delta f^{(e)}(p_e, x, t) \partial f^{(e)}(q)}{p_e} = 0,
\]

\[
\frac{G_F C_A q^\mu}{\sqrt{2}} \int \frac{d^3 e}{(2\pi)^3} \frac{\partial e \delta f^{(e)}(p_e, x, t) \partial f^{(e)}(q)}{p_e} = 0,
\]

for neutrinos and

\[
p_\mu \frac{\partial \delta f^{(e)}(p, x, t)}{\partial x_\mu} + e \delta F^{(e)}(p, x, t) p^\mu \frac{\partial f^{(e)}(E_p)}{\partial p_j} + e F^{(0)}(p, x, t) p^\mu \frac{\partial f^{(e)}(E_p)}{\partial p_j} +
\]

\[
+ G_F C_V \sqrt{2} \int \frac{d^3 q}{(2\pi)^3} \frac{q_{\mu} q_{\mu}}{\varepsilon} \frac{\partial \delta f^{(e)}(p, x, t) \partial f^{(e)}(E_p)}{\partial p_j} +
\]

\[
+ G_F C_A \sqrt{2} e \int \frac{d^3 q}{(2\pi)^3} \frac{q_{\mu} q_{\mu}}{\varepsilon} \frac{\partial \delta f^{(e)}(p, x, t)}{\partial p_j} = 0,
\]

for electrons where \( C_V = 2\xi \pm 0.5 \), \( C_A = \mp 0.5 \) are the vector and axial weak couplings with upper (lower) sign for electron (muon or tau) neutrinos correspondingly, \( \xi = \sin^2 \theta_W \) is the Weinberg parameter; the tensor \( F^{(0)} = e_{jkl} B_{0l} \) corresponds to the external magnetic field term.

Here we have introduced in eq. (11) the 4-vector \( a^{(e)}_\mu(p, x, t) = a^{(c)}_\mu(p) S^{(e)}(p, x, t) \) that is the statistical generalization [1] of the Pauli-Lubański 4-vector [14]

\[
a^{(e)}_\mu(p) = (a^{(c)}_\mu(p), \zeta) = \left[ \frac{p c}{m_e}; \zeta + \frac{p(p c)}{m_e(\varepsilon_p + m_e)} \right]
\]

and has the components

\[
a^{(e)}_\mu(p, x, t) = \left[ \frac{p S^{(e)}(p, x, t)}{m_e}; S^{(e)}(p, x, t) + \frac{p(p S^{(e)}(p, x, t))}{m_e(\varepsilon_p + m_e)} \right].
\]
Note that in the presence of the external magnetic field $B_0$ the neutrino parts of these Vlasov equations differ from the analogous ones used in [7] due to the inclusion of axial vector interactions ($\sim c_A$) in the total re-scattering amplitude that allows us to account for collective interactions of $\nu_\mu$ and $\nu_\tau$-neutrinos too for which the vector coupling $c_V$ is small, $c_V = 2\xi - 0.5 \approx 0$.

To check general symmetries we temporarily refuse the linearization Eq. (4) and write down the RKE for the total electron spin distribution function $S^{(e)}(p, x, t)$ accounting for the weak interaction of electrons with neutrinos:

$$
\frac{\partial S^{(e)}(p, x, t)}{\partial t} + v \frac{\partial S^{(e)}(p, x, t)}{\partial x} + eF_{\mu}(x, t) \frac{\partial S^{(e)}(p, x, t)}{\partial p_\mu} + 2\mu_B \left[ E_i(x, t)(vS^{(e)}(p, x, t)) - \nu_i(E(x, t)S^{(e)}(p, x, t)) + \frac{[S^{(e)}(p, x, t) \times B(x, t)]}{\gamma} \right] +
$$

$$
+ 2G_F \sqrt{2} A \mu e \varepsilon_{ikl} \int \frac{d^3 q}{(2\pi)^3} \left\langle q_\mu a_\mu^k(p) \right\rangle f^{(\nu)}(q, x, t) S_i^{(e)}(p, x, t) +
$$

$$
+ G_F \sqrt{2} A \mu e \varepsilon_{ijkl} \int \frac{d^3 q}{(2\pi)^3} \left\langle q_\mu q_\nu \right\rangle \frac{\partial S^{(e)}(p, x, t)}{\partial p_\mu} \frac{\partial f^{(\nu)}(q, x, t)}{\partial x_\nu} +
$$

$$
+ G_F \sqrt{2} A \mu e \varepsilon_{ijkl} \int \frac{d^3 q}{(2\pi)^3} \left\langle q_\mu q_\nu \right\rangle \frac{\partial S^{(e)}(p, x, t)}{\partial p_\mu} \frac{\partial f^{(\nu)}(q, x, t)}{\partial x_\nu} +
$$

$$
+ G_F \sqrt{2} A \mu e \varepsilon_{ijkl} \int \frac{d^3 q}{(2\pi)^3} \left\langle q_\mu q_\nu \right\rangle \frac{\partial S^{(e)}(p, x, t)}{\partial p_\mu} \frac{\partial f^{(\nu)}(q, x, t)}{\partial x_\nu} +
$$

$$
- \epsilon_{\mu \nu \rho \sigma} \int \frac{d^3 q}{(2\pi)^3} \left\langle q_\rho q_\sigma \right\rangle \frac{\partial f^{(\nu)}(q, x, t)}{\partial x_\rho} \frac{\partial f^{(\nu)}(q, x, t)}{\partial x_\sigma} \right\rangle S^{(e)}(p, x, t) = 0 .
$$

This is the relativistic generalization in SM of the kinetic equation for spin waves in nonferromagnetic metals [13]. But it should be kept in mind that we consider here the Fermi gas of free electrons in contrast with the quasi-particle approach for metals and also neglected exchange interactions (neglecting exchange Feynman diagrams for the e-scattering means here that the long-range forces are dominant).

The system of RKE’s is completed by the Maxwell equations for the electromagnetic field $F_{\mu \nu}(x, t) = \delta F_{\mu \nu}(x, t) + F_{\mu \nu}^{(0)}$ accounting for the spin wave contribution,

$$
\frac{1}{c} \frac{\partial B(x, t)}{\partial t} = [\nabla \times E(x, t)] ,
$$

and

$$
[\nabla \times [\nabla \times E(x, t)]] + \frac{1}{c^2} \frac{\partial^2 E(x, t)}{\partial t^2} = - \frac{4\pi}{c^2} \frac{\partial}{\partial t} \left( e \int \frac{d^3 p}{(2\pi)^3} \sqrt{\gamma} f^{(e)}(p, x, t) +
$$

$$
+ \mu_B \left[ \int \frac{d^3 p}{(2\pi)^3} \frac{\nabla \times S^{(e)}(p, x, t)}{\gamma} - \int \frac{d^3 p}{(2\pi)^3} \left( \nabla \times S^{(e)}(p, x, t) \right) \right] \right) .
$$

In eqs. (5)–(13) $e = - |e|$ is the electric charge of the electron; $G_F$ is the Fermi constant; $\mu_B = \hbar/2m_e c$ is the Bohr magneton; $v = p/\varepsilon_p$ is the electron velocity; the latin indices run $i, j = 1, 2, 3$ and the greek ones are $\mu = 0, 1, 2, 3$ for scalars written in the Feynman metrics, $A_\mu B^\mu = A_0 B_0 - A_i B_i$. 

6
In the non-relativistic (NR) limit $|\mathbf{v}| \ll 1$ ($\gamma \rightarrow 1$) the last term drops out and eq. (13) coincides with the Maxwell equation in a magnetized medium:

$$[\nabla \times (\mathbf{B} - 4\pi \mathbf{M})] = 4\pi \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t},$$

where $\mathbf{M}(\mathbf{x}, t) = |\mu_B| \int d^3p \mathbf{S}(\mathbf{p}, \mathbf{x}, t)/(2\pi)^3$ is the magnetization density of NR plasma.

Multiplying eq. (12) by the energy $\varepsilon_p$ and the spin distribution function $S_i(\mathbf{p}, \mathbf{x}, t)$ we obtain the covariant RKE for the Lorentz-invariant product $a_\mu(\mathbf{p}, \mathbf{x}, t) a^\mu(\mathbf{p}, \mathbf{x}, t) = -(\mathbf{S}(\mathbf{p}, \mathbf{x}, t))^2$ where the 4-vector $a_\mu^{(e)}(\mathbf{p}, \mathbf{x})$ is given by (11).

Note that for the uniform electron beam $(S_i^{(e)}(\mathbf{p}, t) = S_i(t)(2\pi)^3 n_\gamma \delta_3(\mathbf{p} - \mathbf{p}_0))$ omitting the neutrino term and integrating the spin RKE (12) over $d^3p$ we obtain the one-particle spin evolution equation

$$\frac{d\mathbf{S}(t)}{dt} = \frac{2\mu_B}{\gamma_0} [\mathbf{S} \times \mathbf{B}] + \frac{2\mu_B}{\gamma_0 + 1} [\mathbf{S} \times (\mathbf{E} \times \mathbf{v}_0)],$$

that turns out to be exactly the Bargman-Mishel-Telegdi equation for the electron spin motion with the normal magnetic moment $\mu_B$ [14]. Here $\mathbf{v}_0 = \mathbf{p}/\varepsilon_p$ is the velocity of the electron beam, $\gamma_0 = \varepsilon_p/m_e$ is its $\gamma$-factor.

In general, it is possible to generalize eq. (12) adding the electromagnetic scattering of electrons through anomalous magnetic moment (Schwinger correction $\mu' = (\alpha/2\pi)\mu_B$) that could lead to additional terms similar to electromagnetic terms in the neutrino spin RKE [13].

Below we consider some important particular cases of the master equations checking their consistency with the known results [7, 10, 12].

### 2.3 Lepton current conservation \( \partial j^{(a)}_\mu(\mathbf{x}, t)/\partial x_\mu = 0 \)

Let us rewrite eq. (9) as the classical equation

$$\frac{\partial \delta f^{(\nu)}(\mathbf{q}, \mathbf{x}, t)}{\partial t} + \mathbf{\dot{x}} \frac{\partial \delta f^{(\nu)}(\mathbf{q}, \mathbf{x}, t)}{\partial \mathbf{x}} + \mathbf{\dot{q}} \frac{\partial f^{(\nu)}(\mathbf{q})}{\partial \mathbf{q}} = 0,$$

where $\mathbf{\dot{x}} = \mathbf{n} = \mathbf{q}/q$ is the velocity of massless neutrino, the derivative $\mathbf{\dot{q}}$ is given by

$$\mathbf{\dot{q}} = + \nabla \left[ G_F \sqrt{2} c_V \delta n^{(e)}(\mathbf{x}, t) + \frac{G_F}{\sqrt{2}} c_A \delta A_0(\mathbf{x}, t) \right] - G_F \left[ \sqrt{2} c_V \nabla [\mathbf{n} \delta j^{(e)}(\mathbf{x}, t)] + \frac{c_A}{\sqrt{2}} \nabla [\mathbf{n} \delta A^{(e)}(\mathbf{x}, t)] \right].$$

Here $\delta j^{(e)}(\mathbf{x}, t) = (\delta n^{(e)}(\mathbf{x}, t); \delta j^{(e)}(\mathbf{x}, t)) = \int (d^3p/(2\pi)^3) (p_\mu/\varepsilon_p) \delta f^{(e)}(\mathbf{p}, \mathbf{x}, t)$ is the four-vector of the electron current density perturbation;

$\delta A^{(e)}_\mu(\mathbf{x}, t) = m_e \int (d^3p/(2\pi)^3) \delta a^{(e)}_\mu(\mathbf{p}, \mathbf{x}, t)/\varepsilon_p$ is the axial four-vector of the spin density perturbation; the four-vector spin distribution $\delta a^{(e)}_\mu(\mathbf{p}, \mathbf{x}, t)$ is given by (11); $\nabla \equiv \partial_i = \partial/\partial x^i = -\partial^\gamma$.

\footnote{In NR plasma the spin density is given by the 3-vector component only, $\delta a^{(e)}_\mu(\mathbf{x}, t) = (0, \delta \mathbf{S}(\mathbf{x}, t))$.}
Analogously the electron RKE (10) can be rewritten as the quasi-classical one,

\[
\frac{\partial \delta f^{(e)}(p, x, t)}{\partial t} + \mathbf{x} \frac{\partial \delta f^{(e)}(p, x, t)}{\partial \mathbf{x}} + \text{Tr} \left( \hat{p} \frac{\partial f^{(0e)}(\varepsilon_p)}{\partial \hat{p}} \right) = 0 ,
\]

(17)

where the particle number distribution function is obtained via the summing over spin quantum numbers, \( \delta f^{(e)}(p, x, t) = \text{Tr}(\delta f^{(e)}(p, x, t)) \); \( \mathbf{x} = v = p/\varepsilon_p \) is the electron velocity;

\[
\dot{\hat{p}} = \left( -e (\mathbf{E}(x, t) + [\mathbf{v} \times \mathbf{B}(x, t)]) + G_F \sqrt{2} c_V \left[ \frac{p^\mu}{\varepsilon_p} \nabla \delta j_\mu^{(e)}(x, t) \right] \right) \delta_{\lambda\lambda'} + G_F \sqrt{2} c_A \left[ \frac{m_e}{\varepsilon_p} \nabla \delta j_\mu^{(e)}(x, t) a_k^{(e)}(p) \sigma_k \right] \delta_{\lambda\lambda'}
\]

(18)

is the force matrix which accounts for the Lorentz force with electromagnetic fields \( \mathbf{E} \), \( \mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B} \) as well as the weak interaction terms \( \sim G_F \).

\[
\dot{f}^{(0e)}(\varepsilon_p) = \frac{\delta_{\lambda\lambda'} f^{(0e)}(\varepsilon_p)}{2} + \frac{\left( \frac{\delta b^{(0)}}{\lambda \lambda'} \right)}{2} \delta^{(0e)}(\varepsilon_p)
\]

(19)

is the equilibrium density matrix as given in [2] with the ort \( \hat{b}^{(0)} = \mathbf{B}_0/B_0 \) separated from the isotropic distribution function \( \delta^{(0e)}(\varepsilon_p) = -\left( | \mu_B | B_0/\gamma \right) df^{(0e)}/d\varepsilon_p \) given by [2].

Substituting (18), (19) into (17) one can easily check that the electron RKE (10) takes the form

\[
\frac{\partial \delta f^{(e)}(p, x, t)}{\partial t} + \mathbf{v} \frac{\partial \delta f^{(e)}(p, x, t)}{\partial \mathbf{x}} + e \delta F_{\mu}(x, t) \left( \frac{p^\mu}{\varepsilon_p} \frac{\partial f^{(e)}(\varepsilon_p)}{\partial p_j} \right) + e F_{\mu}^{(0)} \frac{p^\mu}{\varepsilon_p} \frac{\partial f^{(e)}(p, x, t)}{\partial p_j} + G_F \sqrt{2} c_V \left( \frac{\partial \delta n^{(e)}(x, t)}{\partial x_j} - v_k \left( \frac{\partial \delta j^{(e)}_k(x, t)}{\partial x_j} \right) \left( \frac{\partial f^{(e)}(\varepsilon_p)}{\partial p_j} \right) \right)
\]

\[
\left. + G_F \sqrt{2} c_A m_e \left( \frac{\partial \delta n^{(e)}(x, t)}{\partial x_j} a_0(p) \right) \frac{1}{\varepsilon_p} \left( \frac{\partial \delta j^{(e)}_k(x, t)}{\partial x_j} \right) \right) \frac{\partial S^{(0e)}(\varepsilon_p)}{\partial p_j} = 0 ,
\]

(20)

where we introduced the unit polarization four-vector \( a^\mu(p) = a^\mu_i(p) \hat{b}^{(0)}_i \), \( a_{\mu} a^\mu = -1 \).

On first glance both the neutrino (15) and electron (17) RKE’s do not obey the lepton number conservation law, \( \partial j_\mu^{(a)}(x, t)/\partial x_\mu \neq 0 \), (due to the last terms in (15) and the last three terms in (20) correspondingly).

This non-conservation is true for the gauge non-invariant Wigner distributions \( f^{(a)}(p, x, t) \equiv \tilde{f}^{(a)}(p, x, t) = \int e^{i\mathbf{p} \cdot \mathbf{y}} \tilde{f}^{(a)}(x - y/2, x + y/2, t) \) we used above, for which the distribution function in the coordinate representation (within integrand),

\[
\tilde{f}^{(a)}(x_1, x_2, t) = \text{Tr} \left( \hat{\rho}(t) \hat{\psi}^{(a)\dagger}(x_2) \hat{\psi}^{(a)}(x_1) \right) ,
\]

\( \tilde{f}^{(a)}(x_1, x_2, t) = \text{Tr} \left( \hat{\rho}(t) \hat{\psi}^{(a)\dagger}(x_2) \hat{\psi}^{(a)}(x_1) \right) \),

3 Such terms can be also obtained from the weak interaction Hamiltonian for a probe electron moving in a neutrino medium

\[
\hat{H}_{\text{weak}} = -\hat{L}_{\text{weak}} = G_F \sqrt{2} \bar{u}_\lambda(p) \gamma^\mu u_\lambda(p) \delta j_\mu^{(e)}(x, t) + G_F \sqrt{2} c_A \bar{u}_\lambda(p) \gamma^\mu \gamma^5 u_\lambda(p) \delta j_\mu^{(e)}(x, t) .
\]

\[\hat{H}_{\text{weak}} = -\hat{L}_{\text{weak}} = G_F \sqrt{2} \bar{u}_\lambda(p) \gamma^\mu u_\lambda(p) \delta j_\mu^{(e)}(x, t) + G_F \sqrt{2} c_A \bar{u}_\lambda(p) \gamma^\mu \gamma^5 u_\lambda(p) \delta j_\mu^{(e)}(x, t) .\]
is not invariant with respect to gauge transformation of the corresponding interaction fields.

For example, in QED plasma the kinetic equation for the gauge non-invariant distribution of charged particles \( \tilde{f}^{(e)}(x_1, x_2, t) \) derived from the quantum Liouville equation by the same Bogolyubov method does not obey the electric current conservation law since the force term depends on the electromagnetic potentials \( A_\mu(x, t) \) which do not enter as combinations expressed via field strengths, \( E, B \).

The recasting of such RKE for the gauge-invariant Wigner distribution

\[
f^{(e)}(p, x, t) = \tilde{f}^{(e)}(p + eA(x, t), x, t) = \int d^3y e^{ipy} f^{(e)}(x - y/2, x + y/2, t) ,
\]

where the gauge invariant distribution in the coordinate representation \( f^{(e)}(x_1, x_2, t) \) is connected with the gauge non-invariant one,

\[
\tilde{f}^{(e)}(x_1, x_2, t) = Tr \left( \hat{\rho}(t) \hat{\Psi}^{(e)+}(x_2) \hat{\Psi}^{(e)}(x_1) \right),
\]

by the important phase factor \( \hat{\chi} \):

\[
f^{(e)}(x_1, x_2, t) = \exp \left[ ie(x_2 - x_1) \right] \int_0^1 d\xi A(x_2 + \xi(x_1 - x_2), t) \tilde{f}^{(e)}(x_1, x_2, t) ,
\]

allows to obtain the usual Lorentz force term in the standard Boltzmann equation for charged particles \( \hat{\chi} \):

\[
\frac{\partial f^{(e)}(p, x, t)}{\partial t} + v \frac{\partial f^{(e)}(p, x, t)}{\partial x} + e(E + [v \times B]) \frac{\partial f^{(e)}(p, x, t)}{\partial p} = 0 ,
\]

for which, of course, the electric current \( j^{(e)}_\mu(x, t) = \int d^3p (p_\mu/\varepsilon_p) f^{(e)}(p, x, t)/(2\pi)^3 \) is conserved, \( \partial j^{(e)}_\mu / \partial x_\mu = 0 \).

Such conservation is stipulated by the presence of the phase factor in the distribution \( \hat{\Psi}^{(e)}(x_1) \rightarrow e^{-ie\chi(x_1, t)} \hat{\Psi}^{(e)}(x_1) , \)

\[
\hat{\Psi}^{(e)+}(x_2) \rightarrow e^{ie\chi(x_2, t)} \hat{\Psi}^{(e)}(x_2) ,
\]

\[
A(x_2 + \xi(x_1 - x_2), t) \rightarrow A(x_2 + \xi(x_1 - x_2), t) - \frac{\partial \chi}{\partial x_2}(x_2 + \xi(x_1 - x_2), t) ,
\]

or, equivalently, this arbitrary phase \( \chi(x, t) \) cancels in \( \tilde{f}^{(e)} \). Such invariance is crucial for macroscopic physics since it provides the physical sense of the Wigner function \( f^{(e)} \) and the conservation of the macroscopic electric current.

Deriving electron RKE \( \hat{\Psi}^{(e)}(x_1) \rightarrow e^{-ie\chi(x_1, t)} \hat{\Psi}^{(e)}(x_1) \), we used such gauge transformation while for the weak interaction terms we did not.

Thus, a recipe of the gauge invariance restoration in RKE’s \( \hat{\Psi}^{(e)}(x_1) \rightarrow e^{-ie\chi(x_1, t)} \hat{\Psi}^{(e)}(x_1) \), \( \Delta \) should be similar to the change of arguments in \( f^{(e)} \): we should substitute the generalized (conjugate) neutrino momentum,

\[
Q = q + G_F \sqrt{2} e \nu \hat{\nu}_\nu(x, t) + \frac{G_F}{\sqrt{2}} e \nu A^{(e)}(x, t) ,
\]

for the neutrino gauge invariant Wigner distribution

\[
f^{(\nu)}(q, x, t) = \tilde{f}^{(\nu)}(Q, x, t) ,
\]
and the matrix of the generalized momentum
\[ \hat{P} = [p + G_F \sqrt{2} c_V \delta j^{(\nu)}(x, t)] \delta_{\lambda \lambda'} - G_F \sqrt{2} c_A \delta n^{(\nu)}(x, t) \delta_{\lambda \lambda'} \],
for the electron gauge invariant Wigner distribution
\[ f^{(e)}(p, x, t) = \tilde{f}^{(e)}(p, x, t) \],
where \( P = 2^{-1} \text{Tr} \hat{P} \).

The RKE's for gauge invariant \([23], [24]\) are obtained from \([13], [17]\) as
\[ \frac{\partial \delta \tilde{f}^{(\nu)}(Q, x, t)}{\partial t} + \frac{\partial \tilde{f}^{(\nu)}(Q, x, t)}{\partial x} + Q \frac{\partial \tilde{f}^{(\nu)}(Q)}{\partial Q} = 0 \],
and
\[ \frac{\partial \delta \tilde{f}^{(e)}(P, x, t)}{\partial t} + \frac{\partial \tilde{f}^{(e)}(P, x, t)}{\partial x} + Tr \left[ \hat{P} \frac{\partial \tilde{f}^{(e)}(\varepsilon_P)}{\partial P} \right] = 0 \],
where generalized momenta \( Q, \hat{P} \) are given by Eqs. \([24]\) and \([26]\) correspondingly; the background distributions \( \tilde{f}_0^{(\nu)} \) can be changed to \( f_0^{(\nu)}(q) \) for neutrinos and \( \tilde{f}_0^{(e)}(\varepsilon_p) \) \([13]\) for electrons since we retain the lowest order \( \sim G_F \) in the corresponding weak interaction terms.

Substituting in last RKE's the momenta \([24]\) and \([28]\), for which we take into account both the derivatives \([13], [18]\) and the total time derivatives \( \delta \dot{j}^{(e)}(x, t)/\partial t = \delta \dot{j}^{(e)}(x, t)/\partial t + (n \nabla) \delta j^{(e)}(x, t), \delta \dot{A}^{(e)}(x, t)/\partial t = \delta \dot{A}^{(e)}(x, t)/\partial t + (n \nabla) \delta A^{(e)}(x, t) \) in the neutrino RKE, \( \delta j^{(e)}(x, t)/\partial t = \delta \dot{j}^{(e)}(x, t)/\partial t + (v \nabla) \delta j^{(e)}(x, t) \) in the electron one, and then using the identities
\[ (n \nabla) \delta j^{(e)}(x, t) = \nabla(n \delta j^{(e)}(x, t)), \]
\[ (n \nabla) \delta A^{(e)}(x, t) = \nabla(n \delta A^{(e)}(x, t)), \]
\[ (v \nabla) \delta j^{(e)}(x, t) = \nabla(v \delta j^{(e)}(x, t)), \]
\[ (v \nabla) \delta n^{(e)}(x, t) = \nabla(v \delta n^{(e)}(x, t)), \]
we arrive to the final forms of the lepton RKE's.

Namely, accounting for the definition of the neutrino gauge invariant distribution \([23]\), the neutrino RKE \([13]\) which is equivalent to the master \([9]\) takes finally in the Vlasov approximation the form
\[ \frac{\partial \delta f^{(\nu)}(q, x, t)}{\partial t} + n \frac{\partial \delta f^{(\nu)}(q, x, t)}{\partial x} + \delta F^{(V)}_{j \mu}(x, t) q^\mu \frac{\partial f_0^{(\nu)}(q)}{\partial q_j} + \delta F^{(A)}_{j \mu}(x, t) q^\mu \frac{\partial f_0^{(\nu)}(q)}{\partial q_j} = 0 \],
where the antisymmetric tensors \( \delta F^{(V,A)}_{j k}(x, t) \) entering effective Lorentz force terms are given by the vector and axial vector currents correspondingly,
\[ \delta F^{(V)}_{j \mu}(x, t)/G_F \sqrt{2} c_V = -\nabla j^{(e)}(x, t) - \frac{\partial \delta j^{(e)}(x, t)}{\partial t}, \]
\[ \delta F^{(V)}_{j k}(x, t)/G_F \sqrt{2} c_V = e_{j kl}(\nabla \times \delta j^{(e)}(x, t))_l, \]
\[ \sqrt{2} \delta F^{(A)}_{j \mu}(x, t)/G_F c_A = -\nabla j^{(e)}(x, t) - \frac{\partial \delta A^{(e)}(x, t)}{\partial t}, \]
\[ \sqrt{2} \delta F^{(A)}_{j k}(x, t)/G_F c_A = e_{j kl}(\nabla \times \delta A^{(e)}(x, t))_l \],
\[ \text{for} \quad j = 0, 1, 2, \ldots, k = 0, 1, 2, \ldots \]
and $\partial/\partial Q = \partial/\partial q$ from the relation (24).

First three terms in RKE (28) were derived in [7, 10]. In the paper [12] analogous vector coupling terms ($\sim c_V$ for electron neutrinos) were obtained in cold hydrodynamics.

Analogously for the electron gauge invariant distribution (27) we obtain finally from (20) (equivalent to the master equation (10)) the electron RKE

$$\frac{\partial \delta f^e(p, x, t)}{\partial t} + v \frac{\partial \delta f^e(p, x, t)}{\partial x} + e\delta F_{\mu j}(x, t) \frac{p^\mu}{\varepsilon_p} \frac{\partial \delta f^e(p, x, t)}{\partial p_j} + eF^{(0)}_{\mu j} \frac{\partial \delta f^e(p, x, t)}{\partial p_j} +$$

$$+ F^{(weak)}_{j\mu}(x, t) \frac{p^\mu}{\varepsilon_p} \frac{\partial \delta f^e(p, x, t)}{\partial p_j} - G_F \sqrt{2}\varepsilon_p \left( \frac{\partial \delta n^{(e)}(x, t)}{\partial t} - (v \times \nabla \times \delta n^{(e)})(x, t) \hat{b}^{(0)} + \right)$$

$$+ \frac{m_e}{\varepsilon_p} \nabla (a(p) \delta j^{(e)}(x, t)) \right) \frac{\partial S^{(0e)}(\varepsilon_p)}{\partial p_j} = 0,$$

(30)

where the weak vector term ($\sim c_V$) has the Lorentz structure with the tensor components

$$F^{(weak)}_{j0}(x, t) / G_F \sqrt{2}c_V = -\nabla_j \delta n^{(e)}(x, t) - \frac{\partial \delta j^{(e)}(x, t)}{\partial t},$$

$$F^{(weak)}_{jk}(x, t) / G_F \sqrt{2}c_V = \varepsilon_{jkl} (\nabla \times \delta j^{(e)}(x, t))_l;$$

(31)

$\partial/\partial P = \partial/\partial p$ from the relation (26); and the axial vector term does not contribute to the continuity equation since the last term with the 3-vector component of the four-vector $a_\mu(p)$ introduced after (20),

$$a(p) = \hat{b}^{(0)} + \frac{p(p \cdot \hat{b}^{(0)})}{m_e(\varepsilon_p + m_e)},$$

vanishes under integration $\int d^3p (a_k(p)/\varepsilon_p) \partial S^{(0e)}(\varepsilon_p)/\partial p_j$ due to the isotropy of the background and the odd power of momenta $p$ under the integral.

Obviously, RKE’s (28), (30) obey the lepton current conservation law

$$\frac{\partial j^{(a)}_\mu(x, t)}{\partial x^\mu} = 0,$$

(32)

or the lepton currents are conserved in a magnetized plasma.

### 2.4 Dispersion equation for lepton density perturbations in isotropic plasma

Here checking master equations (3), (10) against known results for an isotropic plasma in the absence of an external magnetic field, $B_0 = 0$, hence neglecting spin waves, we show that accounting for the lepton current conservation (12) such observables like the spectra of lepton density perturbations do not depend whether we apply initial RKE’s for gauge non-invariant distribution functions (3), (4), or same equations written in the completed forms, (28), (30).

The situation is similar to the case of standard plasma where the initial RKE for the gauge non-invariant Wigner distribution $\tilde{f}^e(p, x, t)$ is often more suitable for the obtaining of concrete results than Boltzmann equation with the Lorentz force (14). Nevertheless, the electric current should be written in the gauge invariant form obeying (12).
As well as authors [7, 10] we consider the electron plasma waves (EPW) driven by the intense neutrino flux, neglecting also the transverse wave contribution, i.e. retaining only the electric field in the Lorentz force for electrons.

Using the Fourier representation in the RKE’s (9), (10) one can easily obtain the algebraic system for the current perturbations

\[
\delta j_{\nu}^{(\nu)}(\omega, k) = \int d^3q (q_\mu/q) \delta f^{(\nu)}(q, k, \omega)/(2\pi)^3, \\
\delta j_{\mu}^{(e)}(\omega, k) = \int d^3p (p_\mu/\varepsilon_p) \delta f^{(e)}(p, k, \omega)/(2\pi)^3,
\]

\[
\delta n^{(\nu)}(\omega, k) + G_F \sqrt{2} c \nu \left[ \delta n^{(e)}(\omega, k) \right] \int \frac{d^3q}{(2\pi)^3} \frac{k_j \partial f_0^{(\nu)}(q)/\partial q_j}{\omega - kn} -
\]

\[-\delta f_k^{(e)}(\omega, k) k_j \int \frac{d^3q}{(2\pi)^3} \frac{n_k \partial f_0^{(\nu)}(q)/\partial q_j}{\omega - kn} = 0,
\]

\[
\delta n^{(e)}(\omega, k) \left[ 1 + \chi_e(\omega, k) \right] + G_F \sqrt{2} c \nu \left[ \delta n^{(\nu)}(\omega, k) \right] \int \frac{d^3p}{(2\pi)^3} \frac{k_j \partial f_0^{(e)}(\varepsilon_p)/\partial p_j}{\omega - kv} -
\]

\[-\delta f_k^{(\nu)}(\omega, k) k_j \int \frac{d^3p}{(2\pi)^3} \frac{v_k \partial f_0^{(e)}(\varepsilon_p)/\partial p_j}{\omega - kv} = 0,
\]

(33)

where \( n = q/q \) is the velocity of the massless neutrino and the term \( \chi_e(\omega, k) \) is connected with the longitudinal permittivity,

\[
\chi_e(\omega, k) = \varepsilon_l(\omega, k) - 1 = \frac{4\pi e^2}{k^2} \int \frac{d^3p}{(2\pi)^3} \frac{k_j \partial f_0^{(e)}(\varepsilon_p)/\partial p_j}{\omega - kv}.
\]

(34)

Note that starting instead of (9), (10) from the corresponding RKE’s (28), (30) derived in the previous subsection for the gauge invariant Wigner distribution functions \( f^{(\nu)}(Q, x, t), f^{(e)}(P, x, t) \) we obtain the same algebraic system (33) omitting axial terms \( \sim c_A \) for \( B_0 = 0 \).

Really additional terms coming from the total time derivatives,

\[
\frac{\partial \delta j^{(e)}(x, t)}{\partial t} + (n\nabla) \delta j^{(e)}(x, t)
\]

for \( \dot{Q} \) via the second term in (24), and

\[
\frac{\partial \delta j^{(\nu)}(x, t)}{\partial t} + (v\nabla) \delta j^{(\nu)}(x, t)
\]

for \( \dot{P} \) via the second term in (24), are proportional in the Fourier representation to the Čerenkov factors \( \omega - kn, \omega - kv \) correspondingly that exactly cancels with the poles within integrands in (33) leading to zero contributions in brackets for both equations (33): \( \sim \delta j^{(e)}(\omega, k) \int (d^3q/(2\pi)^3) \partial f^{(\nu)}(q)/\partial q_j = 0 \), and \( \sim \delta j^{(\nu)}(\omega, k) \int (d^3p/(2\pi)^3) \partial f^{(e)}(\varepsilon_p)/\partial p_j = 0 \).

Now using the symmetrical tensor entering the second line in (33) (and the analogous one in the fourth line)

\[
\int \frac{d^3q}{(2\pi)^3} \frac{n_k \partial f_0^{(\nu)}(q)/\partial q_j}{\omega - kn} = A\delta_{kj} + B\frac{k_k k_j}{k^2},
\]
where the scalars $A$, $B$ are given by

\begin{align*}
A &= \frac{1}{2} \int \frac{d^3q}{(2\pi)^3(\omega - kn)} \left( n_i \frac{\partial f_0^{(\nu)}(q)}{\partial q_i} - \frac{\omega}{k^2} k_i \frac{\partial f_0^{(\nu)}(q)}{\partial q_i} \right) \\
B &= \frac{1}{2} \int \frac{d^3q}{(2\pi)^3(\omega - kn)} \left( 3 \omega k_i \frac{\partial f_0^{(\nu)}(q)}{\partial q_i} - n_i \frac{\partial f_0^{(\nu)}(q)}{\partial q_i} \right)
\end{align*}

we obtain the factor multiplying the electron current $\delta j_k^{(e)}$ in the second line (and the analogous one in the fourth line in (33))

\[
k_j \int \frac{d^3q}{(2\pi)^3} \frac{n_k \delta f_0^{(\nu)}(q)/\partial q_j}{\omega - kn} = k_k(A + B) = \frac{k_k \omega}{k^2} \int \frac{d^3q}{(2\pi)^3(\omega - kn)} k_i \frac{\partial f_0^{(\nu)}(q)}{\partial q_i}.
\]

Substituting eqs. (34), (35) we rewrite the system (33) as

\[
\delta n^{(\nu)}(\omega, k) + G_F \sqrt{2} c_V \left[ \delta n^{(e)}(\omega, k) - \frac{\omega}{k^2} k_k \delta j_k^{(e)}(\omega, k) \right] \times
\]

\[
\times \int \frac{d^3q}{(2\pi)^3(\omega - kn)} k_i \frac{\partial f_0^{(\nu)}(q)}{\partial q_i} = 0,
\]

\[
G_F \sqrt{2} c_V \left[ \delta n^{(\nu)}(\omega, k) - \frac{\omega}{k^2} k_k \delta j_k^{(\nu)}(\omega, k) \right] k^2 \chi_e(\omega, k) \frac{k^2 \chi_e(\omega, k)}{4\pi e^2} +
\]

\[
+ \delta n^{(e)}(\omega, k) \left( 1 + \chi_e(\omega, k) \right) = 0.
\]

Using the lepton current conservation (32), $k_k \delta j_k^{(e)}(\omega, k) = \omega \delta n^{(e)}(\omega, k)$, we obtain from (36) the dispersion equation in the lowest order over the Fermi constant ($\sim G_F^2$) (7, 10):

\[
1 + \chi_e(\omega, k) - \Delta_0 \frac{k^2 c_T^2}{\omega^2} \left( 1 - \frac{\omega^2}{k^2} \right)^2 \chi_e(\omega, k) \int \frac{d^3q}{(2\pi)^3(\omega - kn)} k_i \frac{\partial f_0^{(\nu)}(q)}{\partial q_i} = 0,
\]

where $\Delta_0 = 2 G_F^2 n_{e0} m_{e0}/m_e$ is the weak parameter introduced in [7]; $f_0^{(\nu)} = f_0^{(\nu)}(q)/n_{\nu e}$ is the normalized neutrino background distribution function; $\omega_{pe} = \sqrt{4\pi n_{e0}/m_e}$ is the non-relativistic expression for the plasma frequency.

Let us stress the importance of additional terms in RKE’s (28), (30) discussed after eq. (34) which did not contribute to (33) while they become important when we claim the lepton current conservation deriving dispersion equation (37). We can conclude that the master RKE’s (7), (10) plus the lepton current conservation laws provided by corresponding eqs. (28), (30) lead to the correct issues known in literature, in particular, to the dispersion equation (37).

The prediction of the streaming instability driven by the neutrino beam in a dense plasma (i.e., outside the neutrinosphere of a supernova) (7) was criticized in (14), (18), (19). In this section, however, we do not touch these issues following from eq. (37).

Let us turn to the case of magnetized plasma.

## 3 Bloch equation in the presence of neutrino beam

Substituting the linear decomposition (1) into eq. (12), then integrating the latter over $d^3p$ and multiplying by $\mu_B$ we have generalized Bloch equation for the magnetization.
density perturbations \( \mathbf{m}(x, t) = |\mu_B| \int d^3p \delta \mathbf{S}(p, x, t)/(2\pi)^3 \) in a polarized NR electron gas that is the base of the theory of the electron paramagnetic resonance in the absence of weak interactions \(^\text{[20]}\) while in SM with neutrinos such equation takes the form

\[
\frac{\partial m_j(x, t)}{\partial t} + 2\mu_B \left[ \left( \mathbf{m}(x, t) - \chi_0 b(x, t) \right) \times \mathbf{B}_0 \right]_j + \frac{\partial \Sigma_{kj}(x, t)}{\partial x_k} + \frac{G_F \sqrt{2} c_A \mu_B n_{0e}}{m_e} \int \frac{d^3q}{(2\pi)^3} \frac{\partial \delta f^{(\nu)}(q, x, t)}{\partial x_j} = 0 . \tag{38}
\]

Here \( \chi_0 = -2\mu_B^2 \int d^3p (df^{(0)}/d\varepsilon_p)/(2\pi)^3 \) is the static susceptibility of the polarized electron gas \(^\text{[1]}\) \( b(x, t) \) is the magnetic field perturbation in the total field \( \mathbf{B}(x, t) = \mathbf{B}_0 + b(x, t) \); \( n_{0e} = \int d^3p f^{(0)}(\varepsilon_p)/(2\pi)^3 \) is the density of the electron gas. The pseudotensor \( \Sigma_{kj}(x, t) = \mu_B \int d^3p v_k \delta S_j(p, x, t)/(2\pi)^3 \) describes spatial inhomogeneity of the magnetization and the last new vector term \( \sim c_A \) corresponds to the parity violation for the evolution of the macroscopic axial vector \( m_j \) in SM.

Note that last two lines in the complicated spin RKE \((12)\) are the relativistic corrections and do not contribute to \((38)\) in NR plasma. Moreover, we omitted the small term \(-G_F \sqrt{2} c_A \chi_0 \int [v^j \delta S_j(p, x, t)]/\varepsilon_p \) originated from the third line (without derivatives) in \((12)\) when comparing it with the third term in the Bloch equation \((38)\) stipulated by the standard spin precession in QED plasma (arising due to the last term in the second line in \((12)\)).

We neglected also the fifth term in \((12)\) that is proportional to the vector coupling \( \sim c_V \) since the background polarization is small in our WKB approximation, \( f^{(0)}/\varepsilon_p \ll f^{(0)}(\varepsilon_p) \), and this term is much less than the previous one in \((12)\) which is proportional to the axial coupling \( \sim c_A \) retained in \((38)\). This is in agreement with estimates of the small polarization \( <\lambda> \approx n_{0e}/n_e \approx 0.01 \) made in \([8]\) (Nunokawa et al) even for strong magnetic field in SN where \( n_{0e} \) is the electron density on the main Landau level, \( n_e \) is the total electron density.

In order to break the chain of hydrodynamical equations for magnetization moments, \( m_j \), \( \Sigma_{ij} \), etc, one can consider long-wave spin waves with the wave lengths \( \lambda \) that are much bigger than the electron Larmour radius, \( \lambda_\alpha \gg \lambda_H \). In such case the spin perturbation function is given by the first two moments:

\[
\mu_B \delta S_j(p, x, t) = \frac{df^{(0)}}{d\varepsilon_p} \int \frac{d^3p}{(2\pi)^3} \frac{df^{(0)}}{d\varepsilon_p}^{-1} \times \left( m_j(x, t) + \frac{3v_j}{\nu^2} \Sigma_{ij}(x, t) \right),
\]

that allows us to complete the hydrodynamical system by the equation for the pseudotensor \( \Sigma_{ij} \),

\[
\frac{\partial \Sigma_{ij}(x, t)}{\partial t} + \Omega_v \left( e_{ij} \dot{\mathbf{n}}^p B \Sigma_{il}(x, t) - e_i 0 \frac{\partial}{\partial x_l} \left( m_j(x, t) - \chi_0 b_j(x, t) \right) \right) + \frac{\partial}{\partial x_i} \left( m_j(x, t) - \chi_0 b_j(x, t) \right) - \frac{G_F c_A \mu_B n_{0e}}{\sqrt{2} m_e} \int \frac{d^3q}{(2\pi)^3} n_j(q) \frac{\partial \delta f^{(\nu)}(q, x, t)}{\partial x_j} = 0 . \tag{39}
\]

\(^4\) The static susceptibility is small in a degenerate electron gas, \( \chi_0 = \alpha v_F / 4\pi^2 \ll 1 \), in contrast with the varying one \( \chi(t) \) (see below eq \((41)\)). This is the reason why the static magnetic induction \( B_0 = (1 + 4\pi \chi_0) H_0 \) and the magnetic field strength \( H_0 \) practically coincide there.
Here $\Omega_e = eB_0/m_e$ is the electron cyclotron frequency;
$<v^2> = \left[ \int d^3p d\varepsilon(0)/d\varepsilon \right]^{-1} \int d^3p d\varepsilon(0)/d\varepsilon$ is the average of the velocity squared; $\hat{n}^B = B_0/B_0$ is the unit pseudovector and last true tensor term describes the parity violation through weak interactions. Note that eq. (39) would be important accounting for the large spatial dispersion $k < v > \sim \omega$ outside the region (40) and claims an inclusion of exchange interactions (from the ee-scattering exchange Feynman diagram) we omitted here.

4  Neutrino driven streaming instability of spin waves

In this section we derive from generalized Bloch equation (38) the dispersion equation in magnetized plasma in the presence of neutrino beam.

Apparently master RKE’s for number density distribution functions (9), (10) should be consistent with the spin RKE (12) and its consequence (38). Therefore we are checking below whether more general (28), (30) are necessary. As we find below there is no difference between the use of these equations for the spin wave propagation in plasma.

Let us consider for simplicity the case of long-wave perturbations with the spectrum $\omega(k)$ obeying the inequalities
$$\frac{k^2}{\omega} \geq \omega \gg k < v > .$$

As the mean electron velocity $<v>$ is small in NR plasma, $<v> \ll 1$, and keeping in mind the Maxwell equation written in the Fourier representation $b = [k \times \delta E]/\omega$ one finds from the condition $k/\omega \gg k < v >$ that the electric field contribution in the spin RKE (12) ($\sim \delta \mathbf{E}$) is negligible comparing with the magnetic one even for the maximum frequency in whole space-like region $\omega \leq k$ relevant to the Čerenkov resonance with neutrinos, $\omega = k\mathbf{n}$. Without electric fields the transversal components of the permeability and the susceptibility tensors $(i,j = x,y)$ appearing in the second term of (38) are diagonal in plasma, $\mu_{ij}(\omega,k) = \mu(\omega,k)\delta_{ij}$, $\chi_{ij}(\omega,k) = \chi(\omega,k)\delta_{ij}$, that differs this medium from ferromagnets.

Moreover, the latter inequality in (40), $\omega \gg k < v >$, means the high-frequency approximation, for which the spatial uniform susceptibility $\chi(t)$ and the permeability $\mu(t) = 1 + 4\pi\chi(t)$ can be considered instead of more general ones, $\chi(x,t)$ and $\mu(x,t)$, that allows us to neglect the complicated pseudotensor term $\Sigma_{ij}$ in (38) as well as the spatial dispersion in the perturbations $m_\pm(x,t) = \int dt' \chi(t-t')h_\pm(x,t')$ and $b_\pm(x,t) = \int dt' \mu(t-t')h_\pm(x,t')$ correspondingly where $h_\pm(x,t) = h_\mp(x,t) = ih_y(x,t)\pm h_x(x,t)$ is the magnetic field strength perturbation.

Under such conditions we find from the linearized spin RKE (12) the susceptibility in QED plasma neglecting neutrinos,
$$\chi(\omega) = \frac{\pm \Omega_e \chi_0}{\omega \pm \Omega_e(1 - 4\pi \chi_0)} ,$$
that may be finite at the paramagnetic resonance $\omega = \mp \Omega_e$ given by the equation $1 + 4\pi \chi(\omega) = 0$. Latter follows from the shortened Maxwell equation $\mathbf{k} \times (\mathbf{b} - 4\pi \mathbf{m}) = 0$ when one neglects electric field terms.
Then substituting into (38) the solution of the neutrino RKE (3) obtained in the same linear approximation (1),

\[
\delta f^{(\nu)}(q, k, \omega) = G_F \sqrt{2c_V} \frac{k_k}{(\omega - kn)} \frac{\partial f_0^{(\nu)}(q)}{\partial q_k} \int \frac{d^3p_e}{(2\pi)^3} (1 - v_e n) \delta f^{(e)}(p_e, k, \omega) +
\]

\[
+ \frac{G_F c_A}{\sqrt{2}} \frac{k_k n_i(q)}{(\omega - kn)} \frac{\partial f_0^{(\nu)}(q)}{\partial q_k} \int \frac{d^3p_e}{(2\pi)^3} \delta S_{ij}(p_e, k, \omega),
\]

we can rewrite the generalized Bloch equation (38) in the Fourier representation as

\[
- \omega m_j(k, \omega) + 2\mu_B [(m(k, \omega) - \chi_0 b(k, \omega)) \times B_0]_j + \frac{e_A^2 \Delta \nu}{2} A_i^{(\nu)}(\omega, k) m_i(k, \omega) ik_j +
\]

\[
+ \frac{e_A c_V \Delta \nu k_i}{8\pi m_e} \left( A_i^{(\nu)}(\omega, k) - \frac{B^{(\nu)}(\omega, k)}{\omega} k_i \right) \times (e_{ik} k_k [b_i(k, \omega) - 4\pi m_i(k, \omega)] + \omega \delta E_i) = 0.
\]

Here the factor \(\Delta \nu\) is given after eq. (37); the vector \(A_i^{(\nu)} \equiv A_i^{(\nu)}(\omega, k)\) and the scalar \(B^{(\nu)} \equiv B^{(\nu)}(\omega, k)\) depend on the neutrino background distribution \(f_0^{(\nu)}(q)\),

\[
A_i^{(\nu)} = \int \frac{d^3q}{(2\pi)^3} f_0^{(\nu)}(q) \left( \frac{(kn^2 - k^2) n_i(q)}{(\omega - kn)^2} + \frac{(kn)n_i - k_i}{\omega - kn} \right),
\]

\[
B^{(\nu)} = \int \frac{d^3q}{(2\pi)^3} f_0^{(\nu)}(q) \left( \frac{(kn^2 - k^2)}{(\omega - kn)^2} \right).
\]

Obtaining last term in eq. (43) \((\sim c_A c_V)\) we used the exact Maxwell equation (14) when we substituted the convection current \(\delta j_i(k, \omega) = \int d^3p \nu \delta f^{(e)}(p, k, \omega)\) in the first line of eq. (43).

Note that the electromagnetic current \(j^{(e)}(x, t)\) conserves automatically since we can neglect weak interactions in the electron RKE (10) while retaining them in the neutrino RKE (1). In turn the neutrino influence the electron spin evolution coming from the solution (12) is not changed if we would substitute the solution of the complete equation (28) instead of the initial one (3).

Really in the last case there appear the two additional terms in (12),

\[
- G_F \sqrt{2c_V} \frac{\partial f_0^{(\nu)}(q)}{\partial q_k} \delta f_k^{(e)}(\omega, k) - \frac{G_F}{\sqrt{2} c_A} \frac{\partial f_0^{(\nu)}(q)}{\partial q_k} \delta m_k(\omega, k) \]

which do not contribute to the generalized Bloch equation (28).

One can easily see that in NR plasma with a low electron density \(n_0e \ll m_e^3 \approx 2 \times 10^{31} \text{ cm}^{-3}\) the general condition of the macroscopic description \(2\pi/k \gg (n_0)^{-1/3}\) means that long wave lengths exceeding the Compton one, \(2\pi/k \gg m_e^{-1}\) are possible only, and due to (12) frequencies obey the inequality \(\omega \leq k \ll m_e\).

Accounting for low \(\omega\) and \(k\) and comparing the term \(\sim c_V c_A\) with the previous one in eq. (13) \((\sim c_A^2)\) one finds that the convection current gives a negligible contribution. Note that for \(\nu_{\mu, \tau}\)-neutrinos with \(|c_V| \ll 1\) such correction becomes even less than for \(\nu_e\).

Omitting also the small term \(\sim \chi_0\) we arrive to the shortened form of (13),

\[
- \omega m_j(k, \omega) + 2\mu_B [(m(k, \omega) \times B_0]_j + \frac{e_A^2 \Delta \nu}{2} A_i^{(\nu)}(\omega, k) m_i(k, \omega) ik_j = 0.
\]
Thus, from the generalized Bloch equation (47) we have derived finally the dispersion equation in SM,

\[ (\omega^2 - \Omega_e^2) \left( \omega - \frac{c^2 \Delta (\nu)}{2} A_{z}^{(\nu)} k_z \right) - \frac{\omega^2 c^2 \Delta (\nu)}{4} (A_{z}^{(\nu)} k_+ + A_{z}^{(\nu)} k_-) = 0, \]  

(48)

with the vector \( A_{z}^{(\nu)} \) as given in eq. (44).

In the particular case of the neutrino beam, \( \dot{v}_0^{(\nu)} (q) = (2\pi)^3 \delta^{(3)} (q - q_0) \), \( n_0 = q_0 / q_0 \), for the transversal neutrino propagation \( n_0 \perp B_0 \) with the beam direction along x-axis, \( n_0 = (1, 0, 0) \), and the magnetic field \( B_0 = (0, 0, B_0) \) we obtain from eq. (48) the resonant excitation of spin waves \( \omega = kn_0 + i\delta = \Omega_e + i\delta \) with the increment \( \delta \),

\[ \delta \simeq \Omega_e^{1/3} \sqrt{3} \left( \frac{\Delta (\nu)}{q_0} \right)^{1/3} (\sqrt{2} | c_A | k \sin \theta_{q_0})^{2/3} \geq \Omega_e \sqrt{3} \left( \frac{\Delta (\nu)}{q_0} \right)^{1/3} (\sqrt{2} | c_A | \sin \theta_{q_0})^{2/3}, \]

(49)

where we substituted the scalar product \( kn_0 = k \cos \theta_{q_0} = k \sin \theta \cos \phi \) denoting \( \theta_{q_0} \) as the angle between the neutrino beam direction and the wave vector \( k \), \( \theta \) is the angle of \( k \) with respect to the magnetic field \( B_0 \). Note that we relied in the last inequality on the frequency approximation \( k \geq \omega \simeq \Omega_e \) assumed above.

Let us compare this increment with the fastest one in the case of isotropic plasma following from the dispersion equation (77) (7),

\[ \delta_{\text{weak}} = \frac{\sqrt{3}}{2} \omega_{pe} \left( \frac{\Delta \nu \sin^6 \theta_{q_0}}{q_0 \cos^4 \theta_{q_0}} \right)^{1/3}. \]

(50)

One can easily see the advantage of the excitation of collective modes by the intense neutrino flux in a polarized electron gas. For a strong magnetic field in a dense plasma obeying \( \Omega_e \gtrsim \omega_{pe} \), the angular dependence in eq. (49) \( \sim (\sin \theta_{q_0})^{2/3} \) gives a less suppression of the increment for the small angles \( \theta_{q_0} \leq \arccos(<v>/<c>) \) for which Landau damping for growing modes is absent. This is due to the absence of the factor \((1 - \omega^2/k^2)^2\) in the new dispersion equation (48). It is obvious that such angular dependence would be especially dangerous and important for the relativistic plasma case.

There is the second advantage of spin waves enhanced via the weak axial vector currents (\( \sim c_A = \mp 0.5 \)) instead of the case of plasma waves excited via the weak vector currents with the small vector coupling in the case of muon and tau neutrinos (choosing the lower sign in \( c_V = 2\xi \pm 0.5 \) where \( \xi \simeq 0.23 \) is the Weinberg parameter ). This is the reason why authors (7) considered the case of electron neutrinos only and put for them \( c_V \rightarrow 1 \). Note that during the main neutrino burst in SN all neutrino species are produced in the hot SN core via the pair annihilation \( e^+ e^- \rightarrow \nu_\alpha \bar{\nu}_\alpha, \alpha = e, \mu, \tau \).

On the other hand, there are other arguments against streaming instability driven by neutrino beams in SN (e.g. not collimated beam) (10) to be relevant for neutrino propagation in a magnetized medium. Nevertheless, we have just showed that in a polarized electron gas the dispersion equations are quite different from the case of the isotropic plasma (6, 7, 14) that stimulates a future exploration of collective plasma phenomena in the presence of intense neutrino fluxes.


5 Discussion and conclusions

The above derivation shows that the Bogolyubov method starting from QFT Feynman diagrams remains a power and straightforward tool to obtain RKE's. Being supported with an additional gauge restoring transformation similar to [17] it leads to the same results obtained within Hamiltonian approach of [7] for the case of neutrinos propagating in an isotropic plasma.

In the case of a polarized electron gas the master RKE’s derived above by the Bogolyubov method allow to analyse a new phenomenon- spin wave propagation in NR plasma enhanced by the neutrino beam. Note that complete RKE’s (28), (30) are not necessary to derive the dispersion equation (48) for spin waves as we showed in (46).

The violation of parity in the SM lepton plasma given by axial vector currents ($c_A$) leads to the growth of spin eigen modes through their excitation by the intense neutrino flux. These spin waves are generally coupled to the magnetosonic ones analogously to the case of spin waves in ferromagnets [21] or can transfer their energy to electromagnetic and plasma waves at the cross of spectra that finally could lead to the heating of ions and the background plasma.

The possible explanation of the shock revival in SN by different collective mechanisms including the neutrino driven streaming instability seems to be very perspective goal for future studies in the case of polarized electron gas.

The application of these mechanisms in the magnetized plasma behind the shock and outside the SN neutrinosphere is self-consistent with plasma parameters expected there. Really, we do not consider neutrino collisions with matter within this region using the collisionless neutrino RKE [3] and may also neglect the electron-ion collision frequency $\nu_{ei}$ comparing with the cyclotron frequency at the paramagnetic resonance $\omega = \Omega_e$. Hence collisionless Vlasov approximation should be valid for the spin equation (12) as well as for the electron RKE (10) since the Debye number is large, $N_D = n_0e r_D^3 \gg 1$. Indeed, in the field $B_0 = 10^{12}$ Gauss the cyclotron frequency reaches $\Omega_e = 1.7 \times 10^7 B_0 \sim 1.7 \times 10^{19}$ sec$^{-1}$. This frequency is comparable with the plasma one at the density $n_{e0} \sim 10^{29}$ cm$^{-3}$, $\omega_{pe} = 1.8 \times 10^{19}$ sec$^{-1}$, and turns out to be larger than e.g. the electron-proton collision frequency $\nu_{ep} = 50 n_{e0} (cm^{-3})/(T_e(K))^{3/2}$ sec$^{-1} \sim 1.6 \times 10^{17}$ sec$^{-1}$ in the surrounding NR plasma with the temperature $T_e \sim 10^9$ K. For these parameters the Debye radius $r_D \sim 10^{-9}$ cm corresponds to the large plasma parameter $N_D \sim 100 \gg 1$.

Under such conditions for the mean neutrino energy $q_0 \sim 10$ MeV the increment (49) reaches the maximum value $\delta \sim 10^{10}$ sec$^{-1}$ as in previous optimistic estimates [7, 10]. Note that this means too sharp collimated neutrino beam with the spread of directions $\Delta k n_0$ for the fixed wave number $k$ not exceeding very small value $\Delta k n_0/\Omega_e \lesssim \delta/\Omega_e \sim 10^{-9}$. If neutrinos move along radii beyond the neutrinosphere $r > R_\nu$ this estimate is too optimistic since there is a spread of the angular distribution of neutrino trajectories that damages simple model with parallel rays assumed here [10].

Nevertheless, we think the simple dispersion equation (48) based on the enhancement of pure magnetic field perturbations is the only particular case of the general kinetic equations derived here.

Note that even in this approximation there are some advantages of our model comparing with the isotropic plasma case discussed in [3] [4, 18, 19] as we have just shown in previous section (after eq. (50)).

A more general case with the overlap of electromagnetic eigen modes in a polarized medium and accounting for the spatial dispersion in such plasma seems to be a more
realistic model for a magnetized SN while it is beyond of the scope of the present work.

We would like to acknowledge discussions with A.S. Volokitin and A.I. Rez. This work was supported for V.S. by the RFBR grant 00-02-16271.

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