ON SPACELIKE SLANT HELICES IN $H^2_0$ AND $S^2_1$  \\
MESUT ALTINOK, CETIN CAMCI, LEVENT KULA  \\

Abstract. In this paper, we investigate spacelike slant helix in $H^2_0$ and $S^2_1$ and we obtain parametric equation of spherical slant helix in $H^2_0$ and $S^2_1$. Also related examples and their illustrations are given.  

Keywords: Minkowski 3-space, spacelike slant helix, spherical curve.

1. Introduction

Izumiya and Takeuchi, in [5], have introduced the concept of slant helix in Euclidean 3-space. A slant helix in Euclidean space $E^3$ was defined by the property that the principal normal makes a constant angle with a fixed direction. Moreover, Izumiya and Takeuchi showed that $\gamma$ is a slant helix in if and only if the geodesic curvature of the principal normal of a space curve $\gamma$ is a constant function.

In [7], L. Kula and Y. Yayli studied the spherical images under both tangent and binormal indicatrices of slant helices and obtained that the spherical images of a slant helix are spherical helix. In [8], the author characterize slant helices by certain differential equations verified for each one of obtained spherical indicatrix in Euclidean 3-space. Recently, Ali and Lopez, in [1], have studied slant helix in Minkowski 3-space. They showed that the spherical indicatrix of a slant helix in $E^3_1$ are helices. Also in [2], Ali and Turgut, studied the position vector of a timelike slant helix in $E^3_1$.

In [3] we consider the spherical slant helices in $R^3$. We also present the parametric slant helices, their curvatures and torsions. Moreover, we show how could be obtained to a spherical slant helix and we give some slant helix examples in Euclidean 3-space.

In this paper, we investigate spherical spacelike slant helix in Minkowski 3-space $E^3_1$ and we obtain parametric equations of spherical spacelike slant helix.

2. Preliminaries

The Minkowski 3-space $E^3_1$ is the Euclidean 3-space $E^3$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $E^3_1$. Recall that a vector $v \in E^3_1$ is called spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$ and $v \neq 0$. The norm of a vector $v$ is given by
\[ ||v|| = \sqrt{g(v,v)} \] and two vectors \( v \) and \( w \) are said to be orthogonal if \( g(v,w) = 0 \). An arbitrary curve \( \alpha(s) \) in \( \mathbb{E}_1^3 \) can locally be spacelike, timelike or null (lightlike), if all its velocity vectors \( \alpha'(s) \) are spacelike, timelike or null, respectively. Spacelike or a timelike curve \( \alpha \) has unit speed, if \( g(\alpha'(s), \alpha'(s)) = \pm 1 \). A null curve \( \alpha \) is parameterized by pseudo-arc \( s \), if \( g(\alpha''(s), \alpha''(s)) = 1 \) \((\text{[9]})\). For a timelike space curve \( \alpha(s) \) in the space \( \mathbb{E}_1^3 \), the following Frenet formulae are given in \([1]\)

\[
\begin{align*}
T'(s) &= \kappa(s) N(s) \\
N'(s) &= -\varepsilon_0 \varepsilon_1 \kappa(s) T(s) + \tau(s) B(s) \\
B'(s) &= -\varepsilon_1 \varepsilon_2 \tau(s) N(s)
\end{align*}
\]

where \( g(T(s), T(s)) = \varepsilon_0 = \pm 1, g(N(s), N(s)) = \varepsilon_1 = \pm 1 \) and \( g(B(s), B(s)) = \varepsilon_2 = \pm 1 \).

It is well known that, the pseudo-Riemannian sphere with radius \( r = 1 \) and centered at origin is defined by

\[
S^2_1 = \{ p \in \mathbb{E}_1^3 : g(p,p) = 1 \},
\]

the pseudohyperbolic space of radius \( r = 1 \) and centered at origin is defined by

\[
H^2_0 = \{ p \in \mathbb{E}_1^3 : g(p,p) = -1 \}
\]

are the hyperquadrics with dimension 2 and index 1 and with dimension 2 and index 0, respectively,\((\text{[9]}))\).

### 3. Spherical Spacelike Slant Helix with Spacelike Normal Vector in Minkowski 3-space

In \( \mathbb{E}_1^3 \), the definition of spherical space curve is similar with the Euclidean case but richer than Euclidean case. For example for a timelike curve, if its position vector is a spacelike then the curve lies on the pseudo-Riemannian sphere \( S^2_1 \), if its position vector is a timelike then the curve lies on pseudohyperbolic space \( H^2_0 \). In Minkowski space, for the characterizations of spherical curves, we refer the papers of Petrović-Torgašev and Šućurović, \([10, 11]\) and Inoguchi and Lee \([6]\).

In this section, we give relation between the curvature of spacelike slant helix with spacelike normal vector, the axis of spacelike slant helix with spacelike normal vector and spacelike slant helix with spacelike normal vector in Minkowski 3-space. Moreover, we investigate parametric equation of spacelike slant helix with spacelike normal vector. Here, by \( \varepsilon \) we consider \( \varepsilon = \left\{ \begin{array}{cc} 1 & , \tau^2 - \lambda^2 > 0 \\ -1 & , \tau^2 - \lambda^2 < 0 \end{array} \right\} \).

We will give to "Lemma 3.1" as unproved, since we will use in next sections.

**Lemma 3.1.** Let \( \alpha \) be a spacelike curve with space principal normal vector in \( \mathbb{E}_1^3 \). Geodesic curvature of the spherical image of spacelike principal normal indicatrix \( (N) \) of \( \alpha \) is

\[
\sigma_1(s) = \frac{\kappa^2}{\nu(\kappa^2 - \tau^2)^{3/2}} \left( \frac{\tau}{\kappa} \right)
\]
and geodesic curvature of the spherical image of timelike principal normal indicatrix $(N)$ of $\alpha$ is
\[
\sigma_2(s) = \frac{\kappa^2}{\nu (\tau^2 - \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa}\right),
\]
where $\tau^2 - \kappa^2$ does not vanish.

**Theorem 3.1.** Let $\alpha$ be a spacelike curve with spacelike normal vector in Minkowski space $E^1_3$ with Frenet vectors $T, N, B$ and curvatures $\kappa, \tau$. The following statements are equivalent:

1. $\alpha$ is slant helix
2. $\kappa = \frac{1}{\alpha \nu} \theta' \cosh \theta$
3. $\tau = \frac{1}{\alpha \nu} \theta' \sinh \theta$

\[
\vec{U} = \frac{\tau}{a \sqrt{\varepsilon (\kappa^2 - \tau^2)}} T + N - \frac{\kappa}{a \sqrt{\varepsilon (\kappa^2 - \tau^2)}} B = \text{constant}
\]

**Proof.** $(1 \Rightarrow 3, 4)$: Let $\alpha : I \rightarrow E^1_3$ be a timelike slant helix and the axis of $\alpha$ be $\vec{a}$. Since $\alpha$ is a timelike slant helix, If

\[
\vec{a} = \cos \theta_1 T + \cos \theta_2 N + \cos \theta_3 B,
\]

then

\[
\langle \vec{a}, N \rangle = \cos \theta_2 = \text{constant}.
\]

Differentiating the eq. (3.1), we get

\[
\langle \vec{a}, T \rangle = \frac{\tau}{\kappa} \langle \vec{a}, B \rangle
\]

Differentiating the eq. (3.3), we obtain

\[
\nu \left(\frac{\kappa^2 - \tau^2}{\kappa}\right) \langle \vec{a}, N \rangle = \left(\frac{\tau}{\kappa}\right) \langle \vec{a}, B \rangle.
\]

Since $g(\vec{a}, \vec{a}) = 1$,

\[
\vec{a} = \frac{a}{\sqrt{a^2 - 1}} \left( \frac{\tau}{a \sqrt{\varepsilon (\kappa^2 - \tau^2)}} T + N - \frac{\kappa}{a \sqrt{\varepsilon (\kappa^2 - \tau^2)}} B \right).
\]

If we choose $\vec{a} = \frac{\sigma}{\sqrt{\sigma^2 - 1}} \vec{U}$, then

\[
\vec{U} = \frac{\tau}{a \sqrt{\varepsilon (\kappa^2 - \tau^2)}} T + N - \frac{\kappa}{a \sqrt{\varepsilon (\kappa^2 - \tau^2)}} B.
\]

$\vec{a}$ is constant vector and $\sigma$ is constant, $\vec{U}$ is constant vector, i.e.

\[
\vec{U}' = \left[ \left( \frac{\tau}{a \sqrt{\varepsilon (\kappa^2 - \tau^2)}} \right)' - \nu \kappa \right] T + \left[ - \left( \frac{\kappa}{a \sqrt{\varepsilon (\kappa^2 - \tau^2)}} \right)' + \nu \tau \right] B = 0,
\]
where
\[
\left( \frac{\tau}{\sqrt{\epsilon (\kappa^2 - \tau^2)}} \right)' = a \nu \kappa
\]
(3.7)
\[
\left( \frac{\kappa}{\sqrt{\epsilon (\kappa^2 - \tau^2)}} \right)' = a \nu \tau.
\]

Consequently, by simple calculation,
\[
\kappa = \frac{1}{a \nu} \theta' \cosh \theta
\]
\[
\tau = \frac{1}{a \nu} \theta' \sinh \theta.
\]

(2 \Rightarrow 1): The proof is obvious.

(3 \Rightarrow 1): Let $\vec{U}$ be a constant vector. We can easily show that
\[
\cos \theta_2 = \frac{\sigma}{\sqrt{\sigma^2 - 1}} = \text{constant}
\]
which means that $\alpha$ is a slant helix in $E_3^1$. These complete the proof. \qed

**Theorem 3.2.** Let $\alpha$ be a spacelike curve in $E_3^1$ with equation
\[
x(s) = \frac{1}{\sqrt{-1 + a^2}} (-\sqrt{-1 + a^2} \sinh \theta \sinh \left( \frac{\sqrt{-1 + a^2}}{a} \theta \right) \sinh \left( \frac{\cosh \theta}{a} \right) }
\]
\[
+ \cosh \left( \frac{\sqrt{-1 + a^2}}{a} \theta \right) \left( - \cosh \left( \frac{\cosh \theta}{a} \right) + a \sinh \left( \frac{\cosh \theta}{a} \right) \cosh \theta \right),
\]
\[
y(s) = \frac{1}{\sqrt{-1 + a^2}} \left( \cosh \left( \frac{\cosh \theta}{a} \right) \sinh \left( \frac{\sqrt{-1 + a^2}}{a} \theta \right) \right)
\]
\[
+ \sinh \left( \frac{\cosh \theta}{a} \right) \left( - \sqrt{-1 + a^2} \sinh \theta \cosh \left( \frac{\sqrt{-1 + a^2}}{a} \theta \right) - a \sinh \left( \frac{\sqrt{-1 + a^2}}{a} \theta \right) \cosh \theta \right),
\]
\[
z(s) = \frac{1}{\sqrt{-1 + a^2}} \left( -a \cosh \left( \frac{\cosh \theta}{a} \right) + \sinh \left( \frac{\cosh \theta}{a} \right) \cosh \theta \right),
\]
where $a$ is constant and $\theta = \theta(s)$. $\alpha$ is spacelike slant helix with spacelike normal vector in $S_1^2$.

**Proof.** Let $\alpha$ be a curve in $E_3^1$ with Frenet frame $\{T, N, B\}$, curvature $\kappa$ and torsion $\tau$.

In this case, we will show that $\sigma(s) = \text{constant}$ and $\alpha \in S_1^2$.
By simple calculation, spherical indicatrices $T(s)$, $N(s)$, $B(s)$ of the curve $\alpha$, respectively, are

\[
T(s) = \left( \frac{a \sinh \theta \cosh \left( \sqrt{-1 + a^2} \theta \right)}{\sqrt{-1 + a^2}} - \cosh \theta \sinh \left( \sqrt{-1 + a^2} \theta \right), \right.
\]
\[
\left. - \frac{a \sinh \theta \sinh \left( \sqrt{-1 + a^2} \theta \right)}{\sqrt{-1 + a^2}} + \cosh \theta \cosh \left( \sqrt{-1 + a^2} \theta \right), \right) \quad \frac{\sinh \theta}{\sqrt{-1 + a^2}}.
\]

\[
N(s) = \left( \frac{\cosh \left( \sqrt{-1 + a^2} \theta \right)}{\sqrt{-1 + a^2}}, - \frac{\sinh \left( \sqrt{-1 + a^2} \theta \right)}{\sqrt{-1 + a^2}}, \frac{a}{\sqrt{-1 + a^2}} \right),
\]

\[
B(s) = \left( - \frac{a \cosh \theta \cosh \left( \sqrt{-1 + a^2} \theta \right)}{\sqrt{-1 + a^2}} + \sinh \theta \sinh \left( \sqrt{-1 + a^2} \theta \right), \right.
\]
\[
\left. \frac{a \cosh \theta \sinh \left( \sqrt{-1 + a^2} \theta \right)}{\sqrt{-1 + a^2}} - \sinh \theta \cosh \left( \sqrt{-1 + a^2} \theta \right), - \frac{\cosh \theta}{\sqrt{-1 + a^2}} \right).
\]

We can easily see that $g(T, T) = 1$, $g(N, N) = 1$ and $g(B, B) = -1$, that is, $T$ is a spacelike vector, $N$ is a spacelike vector and $B$ is a timelike vector. Moreover, by the formulae of the curvature and the torsion for a general parameter, we can calculate that

\[
\kappa(s) = \sec h \left( \frac{\cosh \theta}{a} \right),
\]

\[
\tau(s) = \tanh \theta \sec h \left( \frac{\cosh \theta}{a} \right).
\]

Therefore,

\[
\sigma(s) = a = \text{constant}
\]

which means that $\alpha$ is a slant helix with spacelike normal vector in $S^2_1$.

Finally,

\[
g(\alpha(s), \alpha(s)) = 1.
\]

Then, $\alpha$ is slant helix in $S^2_1$. Thus the proof of theorem is completed.
Theorem 3.3. Let $\alpha$ be a spacelike curve in $E^3_1$ with equation

$$
\begin{align*}
x(s) &= \frac{1}{\sqrt{-1 + a^2}}(-\sqrt{-1 + a^2} \sinh \theta \sinh \left[\frac{\sqrt{-1 + a^2}}{a} \theta\right] \sinh \left[\frac{\cosh \theta}{a}\right] \\
&\quad + \cosh \left[\frac{\sqrt{-1 + a^2}}{a} \theta\right] \left(-\cosh \left[\frac{\cosh \theta}{a}\right] + a \sinh \left[\frac{\cosh \theta}{a}\right] \cosh \theta\right)), \\
y(s) &= \frac{1}{\sqrt{-1 + a^2}}\left(-a \cosh \left[\frac{\cosh \theta}{a}\right] + \sinh \left[\frac{\cosh \theta}{a}\right] \cosh \theta\right), \\
z(s) &= \frac{1}{\sqrt{-1 + a^2}}\left(\cosh \left[\frac{\cosh \theta}{a}\right] \sinh \left[\frac{\sqrt{-1 + a^2}}{a} \theta\right] \\
&\quad + \sinh \left[\frac{\cosh \theta}{a}\right] \left(-\sqrt{-1 + a^2} \sin \theta \cosh \left[\frac{\sqrt{-1 + a^2}}{a} \theta\right] - a \sinh \left[\frac{\sqrt{-1 + a^2}}{a} \theta\right] \cosh \theta\right)\right),
\end{align*}
$$

where $a$ is constant and $\theta = \theta(s)$. $\alpha$ is spacelike slant helix with spacelike normal vector in $S^2_1$.

Proof. By using the method in the above theorem the proof of the theorem is obvious. \qed

4. SPHERICAL SPACELIKE SLANT HELIX WITH TIMELIKE NORMAL VECTOR IN MINKOWSKI 3-SPACE

In this section, we give relation between the curvature of spacelike slant helix with timelike normal vector, the axis of spacelike slant helix with timelike normal vector and spacelike slant helix with timelike normal vector in Minkowski 3-space. Moreover, we investigate parametric equation of spacelike slant helix with timelike normal vector in $H^3_0$.

We will give to "Lemma 4.1" as unproved, since we will use in next sections.

Lemma 4.1. Let $\alpha$ be a spacelike curve with space principal normal vector in $E^3_1$. Geodesic curvature of the spherical image of timelike principal normal indicatrix $(N)$ of $\alpha$ is

$$
\sigma(s) = \frac{\kappa^2}{(\tau^2 + \kappa^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)'
$$

where $\tau^2 + \kappa^2$ does not vanish.

Theorem 4.1. Let $\alpha$ be a spacelike curve with spacelike normal vector in Minkowski space $E^3_1$ with Frenet vectors $T, N, B$ and curvatures $\kappa, \tau$. The following statements are equivalent:

1. $\alpha$ is slant helix
2. $\kappa = \frac{1}{a} \nu \theta' \sin \theta$
3. $\tau = \frac{1}{a} \nu \theta' \cos \theta$
By simple calculation, spherical indicatrices

\[ g = \frac{\kappa}{a \sqrt{\varepsilon (\kappa^2 + \tau^2)}} B = \text{constant} \]

**Proof.** By using the method in the Theorem 3.2 the proof of the theorem is obvious.

**Theorem 4.2.** Let \( \alpha \) be a spacelike curve in \( E^3_a \) with equation

\[
\begin{align*}
    x(s) &= \frac{1}{\sqrt{1 + a^2}} \left( \alpha \cos \left[ \frac{\sin \theta}{a} \right] - \sinh \left[ \frac{\sin \theta}{a} \right] \sin \theta \right), \\
    y(s) &= \frac{1}{\sqrt{1 + a^2}} \left( - \cosh \left[ \frac{\sin \theta}{a} \right] \sin \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right] \\
    &\quad + \sin \left[ \frac{\sin \theta}{a} \right] \left( \sqrt{1 + a^2} \cos \theta \cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right] + a \sin \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right] \sin \theta \right) \right), \\
    z(s) &= \frac{1}{\sqrt{1 + a^2}} \left( - \cosh \left[ \frac{\sin \theta}{a} \right] \cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right] \\
    &\quad + \sin \left[ \frac{\sin \theta}{a} \right] \left( - \sqrt{1 + a^2} \cos \theta \sin \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right] + a \cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right] \cos \theta \right) \right),
\end{align*}
\]

where \( a \) is constant and \( \theta = \theta(s) \). \( \alpha \) is spacelike slant helix with timelike normal vector \( H^2_0 \).

**Proof.** Let \( \alpha \) be a curve in \( E^3_a \) with Frenet frame \( \{ T, N, B \} \), curvature \( \kappa \) and torsion \( \tau \).

In this case, we will show that \( \sigma(s) = \text{constant} \) and \( \alpha \in H^2_0 \)

By simple calculation, spherical indicatrices \( T(s), N(s), B(s) \) of the curve \( \alpha \), respectively, are

\[
\begin{align*}
    T(s) &= \left( - \frac{\cos \theta}{\sqrt{1 + a^2}} - \frac{a \cos \theta \sin \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right]}{\sqrt{1 + a^2}} + \sin \theta \cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right], \\
    &\quad \frac{a \cos \theta \cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right]}{\sqrt{1 + a^2}} + \sin \theta \sin \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right] \right), \\
    N(s) &= \left( \frac{a}{\sqrt{1 + a^2}} \cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right], \quad - \frac{\sin \theta}{\sqrt{1 + a^2}} \right), \\
    B(s) &= \left( \frac{\sin \theta}{\sqrt{1 + a^2}} \sin \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right] + \cos \theta \cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right], \\
    &\quad \frac{a \sin \theta \cos \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right]}{\sqrt{1 + a^2}} + \cos \theta \sin \left[ \frac{\sqrt{1 + a^2}}{a} \theta \right] \right).
\end{align*}
\]

We can easily see that \( g(T, T) = 1, g(N, N) = -1 \) and \( g(B, B) = 1 \), that is, \( T \) is a spacelike vector, \( N \) is a timelike vector and \( B \) is a timelike vector. Moreover,
by the formulae of the curvature and the torsion for a general parameter, we can calculate that

\[
\kappa(s) = \sec h \left[ \frac{\sin \theta}{a} \right],
\]
\[
\tau(s) = \cot \theta \sec h \left[ \frac{\sin \theta}{a} \right].
\]

Therefore,

\[
\sigma(s) = a = \text{constant}
\]

which means that \( \alpha \) is a spacelike slant helix \( \alpha \) with timelike normal vector in \( H^2_0 \).

Finally,

\[
g(\alpha(s), \alpha(s)) = -1.
\]

Then, \( \alpha \) is slant helix in \( H^2_0 \). Thus the proof of theorem is completed. \( \Box \)

5. Example

In this section we give an example of spacelike slant helix in \( S^2_1 \), spacelike slant helix in \( H^2_0 \) and draw its pictures and its tangent indicatrix, normal indicatrix, and binormal indicatrix by using Mathematica.

Example 5.1. We consider a spacelike slant helix \( \alpha \) with spacelike normal vector in \( S^2_1 \) is defined by

\[
x(s) = -\sinh \theta \sinh \left[ \frac{\cosh \theta}{3} \right] \sinh \left[ \frac{2\sqrt{2}}{3} \theta \right] \\
+ \frac{1}{2\sqrt{2}} \cosh \left[ \frac{2\sqrt{2}}{3} \theta \right] \left( -\cosh \left[ \frac{\cosh \theta}{3} \right] + 3 \sinh \left[ \frac{\cosh \theta}{3} \right] \cosh \theta \right),
\]
\[
y(s) = \frac{1}{2\sqrt{2}} \left( \cosh \left[ \frac{\cosh \theta}{3} \right] \sinh \left[ \frac{2\sqrt{2}}{3} \theta \right] \\
+ \sinh \left[ \frac{\cosh \theta}{3} \right] \left( 2\sqrt{2} \sinh \theta \cosh \left[ \frac{2\sqrt{2}}{3} \theta \right] - 3 \sinh \left[ \frac{2\sqrt{2}}{3} \theta \right] \cosh \theta \right) \right),
\]
\[
z(s) = \frac{1}{2\sqrt{2}} \left( -3 \cosh \left[ \frac{\cosh \theta}{3} \right] + \sinh \left[ \frac{\cosh \theta}{3} \right] \cosh \theta \right).
\]

The picture of the curve \( \alpha \) is rendered in Figure 1.

The parametrization of the tangent indicatrix \( T = (T_1, T_2, T_3) \) of the spacelike slant helix \( \alpha \) with spacelike normal vector is
For \( a = 3 \), spacelike slant helix \( \alpha \) with spacelike normal vector in \( S_1^2 \).

\[
T_1(s) = \frac{3}{2\sqrt{2}} \cosh \left[ \frac{2\sqrt{2}}{3} \theta \right] \sinh \theta - \sinh \left[ \frac{2\sqrt{2}}{3} \theta \right] \cosh \theta,
\]

\[
T_2(s) = -\frac{3}{2\sqrt{2}} \sinh \left[ \frac{2\sqrt{2}}{3} \theta \right] \sinh \theta + \cosh \left[ \frac{2\sqrt{2}}{3} \theta \right] \cosh \theta,
\]

\[
T_3(s) = \frac{1}{2\sqrt{2}} \sinh \theta.
\]

The picture of the tangent indicatrix is rendered in Figure 2 (a).

The parametrization of the normal indicatrix \( N = (N_1, N_2, N_3) \) of the spacelike slant helix \( \alpha \) with spacelike normal vector is

\[
N_1(s) = \frac{1}{2\sqrt{2}} \cosh \left[ \frac{2\sqrt{2}}{3} \theta \right],
\]

\[
N_2(s) = -\frac{1}{2\sqrt{2}} \sinh \left[ \frac{2\sqrt{2}}{3} \theta \right],
\]

\[
N_3(s) = \frac{3}{2\sqrt{2}}.
\]

The picture of the normal indicatrix is rendered in Figure 2 (b).

The parametrization of the binormal indicatrix \( B = (B_1, B_2, B_3) \) of the spacelike slant helix \( \alpha \) with spacelike normal vector is
\( B_1(s) = -\frac{3}{2\sqrt{2}} \cosh\left[\frac{2\sqrt{2}}{3}\theta\right] \cosh\theta + \sinh\left[\frac{2\sqrt{2}}{3}\theta\right] \sinh\theta, \)

\( B_2(s) = \frac{3}{2\sqrt{2}} \sinh\left[\frac{2\sqrt{2}}{3}\theta\right] \cosh\theta - \cosh\left[\frac{2\sqrt{2}}{3}\theta\right] \sinh\theta, \)

\( B_3(s) = -\frac{1}{2\sqrt{2}} \cosh\theta. \)

The picture of the binormal indicatrix is rendered in Figure 2 (c).

![Figure 2](image)

**Figure 2.** For \( a = 1 \), tangent indicatrix of the slant helix \( \alpha \) in \( S_1^2 \) (a), normal indicatrix of the slant helix \( \alpha \) in \( S_1^2 \) (b) and binormal indicatrix of the slant helix \( \alpha \) \( H_2^2 \) (c).

**Example 5.2.** We consider a spacelike slant helix \( \alpha \) with timelike normal vector in \( H_2^2 \) is defined by

\[
\begin{align*}
x(s) &= \frac{1}{2\sqrt{2}} \left(3 \cosh\left[\frac{\sin\theta}{3}\right] - \sinh\left[\frac{\sin\theta}{3}\right] \sin\theta\right), \\
y(s) &= \frac{1}{2\sqrt{2}} \left(-\cosh\left[\frac{\sin\theta}{3}\right] \sin\left[\frac{2\sqrt{2}}{3}\theta\right]\right)
+ \sinh\left[\frac{\sin\theta}{3}\right] \left(2\sqrt{2} \cos\theta \cos\left[\frac{2\sqrt{2}}{3}\theta\right] + 3 \sin\left[\frac{2\sqrt{2}}{3}\theta\right] \sin\theta\right), \\
z(s) &= \frac{1}{2\sqrt{2}} \left(-\cosh\left[\frac{\sin\theta}{3}\right] \cos\left[\frac{2\sqrt{2}}{3}\theta\right]\right)
+ \sinh\left[\frac{\sin\theta}{3}\right] \left(-2\sqrt{2} \cos\theta \sin\left[\frac{2\sqrt{2}}{3}\theta\right] + 3 \sin\left[\frac{2\sqrt{2}}{3}\theta\right] \cos\theta\right).
\end{align*}
\]

The picture of the curve \( \alpha \) is rendered in Figure 3.

The parametrization of the tangent indicatrix \( T = (T_1, T_2, T_3) \) of spacelike slant helix \( \alpha \) with timelike normal vector is
Figure 3. For $a = 3$, spacelike slant helix $\alpha$ with timelike normal vector in $H^2_0$.

\[
T_1(s) = -\frac{1}{2\sqrt{2}} \cos \theta,
\]
\[
T_2(s) = -\frac{3}{2\sqrt{2}} \sin \left[\frac{2\sqrt{2}}{3} \theta\right] \cos \theta + \cos \left[\frac{2\sqrt{2}}{3} \theta\right] \sin \theta,
\]
\[
T_3(s) = \frac{3}{2\sqrt{2}} \cos \left[\frac{2\sqrt{2}}{3} \theta\right] \cos \theta + \sin \left[\frac{2\sqrt{2}}{3} \theta\right] \sin \theta.
\]

The picture of the tangent indicatrix is rendered in Figure 4 (a).

The parametrization of the normal indicatrix $N = (N_1, N_2, N_3)$ of the spacelike slant helix $\alpha$ with timelike normal vector is

\[
N_1(s) = \frac{3}{2\sqrt{2}},
\]
\[
N_2(s) = \frac{1}{2\sqrt{2}} \sin \left[\frac{2\sqrt{2}}{3} \theta\right],
\]
\[
N_3(s) = \frac{1}{2\sqrt{2}} \cos \left[\frac{2\sqrt{2}}{3} \theta\right].
\]

The picture of the normal indicatrix is rendered in Figure 4 (b).

The parametrization of the binormal indicatrix $B = (B_1, B_2, B_3)$ of the spacelike slant helix $\alpha$ with timelike normal vector is

\[
B_1(s) = \frac{1}{2\sqrt{2}} \sin \theta,
\]
\[
B_2(s) = \frac{3}{2\sqrt{2}} \sin \left[\frac{2\sqrt{2}}{3} \theta\right] \sin \theta + \cos \left[\frac{2\sqrt{2}}{3} \theta\right] \cos \theta,
\]
\[
B_3(s) = -\frac{3}{2\sqrt{2}} \cos \left[\frac{2\sqrt{2}}{3} \theta\right] \sin \theta + \sin \left[\frac{2\sqrt{2}}{3} \theta\right] \cos \theta.
\]

The picture of the binormal indicatrix is rendered in Figure 4 (c).
Figure 4. For $a = 1$, tangent indicatrix of the slant helix $\alpha$ in $S^2_{11}$ (a), normal indicatrix of the slant helix $\alpha$ in $H^2_0$ (b) and binormal indicatrix of the slant helix $\alpha$ in $S^2_{11}$ (c).

References

[1] Ali, Ahmad T. and López, R., Slant helices in Minkowski space $\mathbb{E}^3_{11}$, J. Korean Math. Soc. 48 (2011), no. 1, 159–167.
[2] Ali, Ahmad T. and Turgut, M., Position vector of a time-like slant helix in Minkowski 3-space, Journal of Mathematical Analysis and Applications, Volume 365, Issue 2 (2010), Pages 559-569.
[3] Camcı, C., Kula, L., and Altınok, M., On Spherical Slant Helices in Euclidean 3-space, http://arxiv.org/abs/1308.5532 (2013).
[4] İlarslan, K., Some special curves on non-Euclidean manifolds, Doctoral thesis, Ankara University, Graduate School of Natural and Applied Sciences, 2002
[5] Izumiya, S. and Tkuchi, N., New special curves and developable surfaces, Turk J. Math 28 (2004), 153-163.
[6] Inoguchi, J. and Lee, S., Null curves in Minkowski 3-space, Int. Electron. J. Geom. 1 (2008), no. 2, 40–83.
[7] Kula, L. and Yayli, Y., On slant helix and its spherical indicatrix, Applied Mathematics and Computation, 169 (2005), 600-607.
[8] Kula, L.; Ekmekeci, N. Yaylı, Y. and İlarslan, K., Characterizations of slant helices in Euclidean 3-space, Turkish J. Math. 34 (2010), no. 2, 261–273.
[9] O’Neill, B. Semi-Riemannian geometry with applications to relativity, Academic Press, New York, 1983.
[10] Petrović-Torgašev, M. and Šućurović, E., Some characterizations of the Lorentzian spherical timelike and null curves, Mat. Vesnik 53 (2001), no 1-2, 21–27.
[11] Petrović-Torgašev, M. and Šućurović, E., Some characterizations of the spacelike, the timelike and the null curves on the pseudohyperbolic space $H^2_0$ in $\mathbb{E}^3_{11}$, Kragujevac J. Math. 22 (2000), 71–82.

(Mesut Altınok) Ahi Evran University, Faculty of Sciences and Arts, Department of Mathematics, Kirsehir, Turkey

E-mail address: altnokmesut@gmail.com

(Cetin Camcı) Onsekiz Mart University, Faculty of Sciences and Arts, Department of Mathematics, Canakkale, Turkey

E-mail address: ccamci@comu.edu.tr

(Levent Kula) Ahi Evran University, Faculty of Sciences and Arts, Department of Mathematics, Kirsehir, Turkey

E-mail address: lkula@ahievran.edu.tr