QUASI-OPTIMAL ADAPTIVE MIXED FINITE ELEMENT METHODS FOR CONTROLLING NATURAL NORM ERRORS

YUWEN LI

ABSTRACT. For a generalized Hodge–Laplace equation, we prove the quasi-optimal convergence rate of an adaptive mixed finite element method controlling the error in the natural mixed variational norm. In particular, we obtain new quasi-optimal adaptive mixed methods for the scalar Poisson, vector Poisson, and Stokes equations. Comparing to existing adaptive mixed methods, the new methods control errors in both two variables.

1. INTRODUCTION

Adaptive finite element method (AFEM) has been an active research area since the pioneering work [5], see, e.g., [45, 7, 37] for a thorough introduction. Comparing to finite element methods using quasi-uniform meshes, AFEMs can achieve quasi-optimal convergence rate by producing a sequence of graded meshes resolving singularity arising from irregular data of differential equations and domains with corners or slits. Typically, AFEM can be described by the feedback loop

\[
\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}.
\]

Given a conforming mesh \( T_0 \), Solve returns the finite element solution \( U_\ell \) of the discrete problem on \( T_\ell \). Estimate returns a collection of error indicators \( \{ E_\ell(U_\ell, T) \}_{T \in T_\ell} \). MARK selects a subset \( M_\ell \) of \( T_\ell \) using the information from \( \{ E_\ell(U_\ell, T) \}_{T \in T_\ell} \). A conforming subtriangulation \( T_{\ell+1} \) is then obtained by applying \text{REFINE} to \( M_\ell \) and \text{SOLVE} is called on \( T_{\ell+1} \). Despite the popularity of AFEMs in practice, [6] for the one-dimensional boundary value problem had been the only convergence result of AFEMs for a long time. Using a bulk chasing marking strategy in \text{MARK}, Dörfler [23] first proved that the Lagrange element solution \( U_\ell \) converges to the exact solution \( U \) in the energy norm for Poisson’s equation in \( \mathbb{R}^2 \) provided the initial mesh is fine enough. Readers are referred to [33, 9, 42, 17] and references therein for further important progress in the analysis of convergence and optimality of AFEMs for symmetric and positive-definite elliptic problems. Of particular relevance in this paper is [25], where the authors used weak convergence technique to prove the quasi-optimal convergence rate of AFEMs for nonsymmetric and nonlinear elliptic problems.

The mixed finite element method (MFEM) is designed to numerically solve systems of partial differential equations arising from elasticity, fluids, electromagnetism, computational geometry etc. In contrast to AFEMs based on positive-definite formulations, the difficulty in convergence and optimality analysis of adaptive mixed finite element methods (AMFEMs) are two-fold. First, the a posteriori
error analysis hinges on delicate decomposition results and possibly bounded commuting quasi-interpolations onto a sequence of finite elements spaces, see, e.g., [1, 39, 22]. In addition, those quasi-interpolations are even required to locally preserve finite element functions when deriving discrete reliability, see, e.g., [21, 46]. Second, the exact solution $U$ of a system of equations is generally only a critical point of some variational principle. Hence $U - U_{t+1}$ is not orthogonal to $U_t - U_{t+1}$ and a technical quasi-orthogonality is indispensable, see, e.g., [18, 8].

Consider the popular model problem for the analysis of MFEMs: Find $(\sigma, u) \in H(\text{div}; \Omega) \times L^2(\Omega)$ such that

\begin{equation}
(\sigma, \tau) - (\text{div} \tau, u) = 0, \quad \tau \in H(\text{div}; \Omega),
\end{equation}

\begin{equation}
(\text{div} \sigma, v) = \langle f, v \rangle, \quad v \in L^2(\Omega).
\end{equation}

In fact (1.1) is the mixed formulation of Poisson’s equation. Let $\{((\sigma_t, u_t, T_t))\}_{t \geq 0}$ be the finite element solutions and meshes produced by some AMFEM for (1.1) using Raviart–Thomas(RT) or Brezzi–Douglas–Marini(BDM) elements, see [38, 11]. Under mild assumptions, it has been shown in e.g., [18, 8, 29, 27] that $\sigma_t$ converges to $\sigma$ in the $L^2$-norm with quasi-optimal convergence rate. As far as we know, existing AMFEMs for Poisson’s equation are not able to control the error $\|\sigma - \sigma_t\|_{H(\text{div})} + \|u - u_t\|$ because most error indicators and quasi-orthogonality in literature are not designed for the natural $H(\text{div}) \times L^2$-norm. This limitation seems not so severe for Poisson’s equation, since $\text{div}(\sigma - \sigma_t)$ is trivially controlled by $f$ and the scalar variable $u$ is practically less important than the flux $\sigma$. However, there are still several works on the a posteriori $H(\text{div}) \times L^2$-error estimates of mixed methods for (1.1), see, e.g., [10, 14].

Poisson’s equation is a special case of the Hodge Laplace equation $(d\delta + \delta d)u = f$, which is the model problem in the theory of finite element exterior calculus (FEEC) developed by Arnold, Falk, and Winther [3, 4]. Here $d$ is the exterior derivative for differential forms and $\delta$ is the adjoint operator of $d$. In general, the Hodge Laplace equation is solved by the mixed method (2.4) in FEEC literature. Adaptivity in FEEC has been an active research area in recent years. Using their regular decomposition and commuting quasi-interpolation, Demlow and Hirani [22] developed the first reliable a posteriori error estimator for controlling the error $\|\sigma - \sigma_t\|_V + \|p - p_t\| + \|u - u_t\|_V$ of the mixed method (2.4). At the same time, Falk and Winther [24] constructed a technical local bounded commuting interpolation connecting the de Rham complex (2.10) and its finite element subcomplex. Using these ingredients, [21, 19, 30, 27] recently developed quasi-optimal AMFEMs for problems posed on the de Rham complex. For the Hodge Laplace equation, we [30] developed an AMFEM for controlling $\|\sigma - \sigma_t\|_V$ with quasi-optimal convergence rate and another AMFEM for controlling $\|\sigma - \sigma_t\|_V + \|p - p_t\| + \|d(u - u_t)\|$ without convergence rate. However, we are not aware of any existing AMFEM for the Hodge Laplace equation for controlling the error $\|\sigma - \sigma_t\|_V + \|u - u_t\|_V$ in the natural $V \times V$ mixed variational norm.

On the other hand, the authors in [12, 13] developed the pseudostress-velocity formulation (5.4) for the Stokes equation, which can be numerically solved by the classical RT and BDM element mixed methods. Let $\sigma$ denote the pseudostress, $u$ the velocity, and $(\sigma_t, u_t)$ the finite element solutions produced by some AMFEM. Following the analysis of AMFEMs for Poisson’s equation, [15, 28] recently developed quasi-optimal AMFEMs for the pseudostress-velocity formulation that
control the error $\|C\tilde{\tau}(\sigma - \sigma_t)\|$, where $C$ is a positive semi-definite operator given in (5.3). Since $\|C\tilde{\tau}\cdot\|$ is only a semi-norm, incorporation of $\|\text{Div}(\sigma - \sigma_t)\|$ is necessary for achieving norm convergence. Unlike Poisson’s equation, the velocity field $u$ in fluids is clearly an important physical quantity. From this perspective, an AMFEM for controlling $\|u - u_t\|$ is favorable.

Motivated by the Hodge Laplace and Stokes equations, this paper is devoted to the quasi-optimal adaptive mixed method for controlling the natural norm error. To this end, we consider the generalized Hodge Laplace equation (2.5), which covers the mixed formulation of Poisson’s equation and the pseudostress-velocity formulation of the Stokes equation. (2.5) also covers the mixed formulation of the Hodge Laplace equation with index $k \leq n - 1$ provided the $k$-th cohomology group $\mathcal{H}^k$ vanishes, e.g., $\Omega$ is simply connected when $k = 1, n = 2$. In Section 5, we will restate our results in the classical context. The contribution of this paper is as follows.

1. Using the Demlow–Hirani regular decomposition and Falk–Winther cochain projection in FEEC, we prove the quasi-optimality of the adaptive algorithm AMFEM for reducing the error in the $V \times V$-norm for the generalized Hodge Laplace equation. In particular, we obtain quasi-optimal AMFEMs for the Hodge Laplace equation. In the special case $k = n, C = \text{id}$, i.e., Poisson’s equation, we obtain an AMFEM that reduces the error in the $H(\text{div}) \times L^2$-norm.

2. By posing the Stokes equation on the de Rham complex of vector-valued differential forms, we modify the aforementioned tools in FEEC to derive a reliable and efficient a posteriori error estimator, and the first quasi-optimal AMFEM for the Stokes equation that reduces the error $\|C\tilde{\tau}(\sigma - \sigma_t)\|^2 + \|\text{Div}(\sigma - \sigma_t)\|^2 + \|u - u_t\|^2$. The authors in [16] used $u_t$ to compute a more accurate postprocessed approximation $u^\ast_t$ and derived an error estimator for $\|u - u^\ast_t\|$. However, the reliability of such estimator depends on the $H^2$-regularity of $\Omega$, e.g., $\Omega$ is convex.

3. Our results for the Poisson and Stokes equations hold on general Lipschitz polyhedral domain $\Omega$. In contrast, existing analysis of AMFEMs in e.g., [18, 8, 15, 28] assumes that the Helmholtz decomposition contains no harmonic vector fields, which hinges on the topology of the domain $\Omega$, e.g., the $(n-1)$-th Betti number of $\Omega$ is 0.

An important ingredient of our convergence analysis is the quasi-orthogonality in Theorem 4.5. We observe that the $H^1$-regular decomposition in [22] yields compact operators $\hat{K}^1_t, \hat{K}^2_t$ in Corollary 3.3 and develop a weak convergence result in Theorem 4.2 for the Petrov-Galerkin method. Note that $\hat{K}^1_t, \hat{K}^2_t$ map weakly convergent sequences to strongly convergent ones. Using this fact and the $L^2$-bounded smoothed projection in [20], we obtain the quasi-orthogonality between $u - u_{t+1}$ and $u_t - u_t$.

Combining it with the quasi-orthogonality between $\sigma - \sigma_{t+1}$ and $\sigma_t - \sigma_{t+1}$ obtained in [30], the quasi-optimal convergence rate follows with a somehow standard procedure using the idea of estimator reduction, see [25]. Feischl et al. first used the weak convergence technique to prove quasi-optimal convergence rate of AFEMs in [25], where they observed that the lower order terms in 2$^{nd}$ order elliptic equation are compact perturbations. As far as we know, there is no convergence analysis of adaptive mixed methods in literature based on the weak convergence technique.
The rest of this paper is organized as follows. In Section 2, we introduce the closed Hilbert complex, de Rham complex, and the generalized Hodge Laplace equation. In Section 3, we derive reliable and efficient a posteriori error estimator for the generalized Hodge Laplace equation on the de Rham complex. Section 4 is devoted to the convergence and optimality analysis of the algorithm AMFEM. In Section 5, we use previous results and correspondence between functions and differential forms to obtain results on scalar Poisson, vector Poisson, and Stokes equations.

2. Hilbert Complex and de Rham Complex

Following the convention of [3, 4], we introduce FEEC in this section.

2.1. Hilbert complex and approximation. Given Hilbert spaces \( X_1, X_2 \), we say \( T : X_1 \rightarrow X_2 \) is a closed, densely-defined operator if the domain \( D(T) = \{ v \in X_1 : Tv \in X_2 \text{ is defined} \} \) is a dense subspace of \( X_1 \), \( T : D(T) \rightarrow X_2 \) is linear, and the graph \( \{(v,Tv) : v \in D(T)\} \) is a closed subset of \( X_1 \times X_2 \). Let \( \langle \cdot, \cdot \rangle_{X_i} \) denote the inner product on \( X_i, i = 1, 2 \). The adjoint operator \( T^* : X_2 \rightarrow X_1 \) is defined to be the operator whose domain is

\[
D(T^*) = \{ v \in X_2 : \exists w \in X_1, \text{ such that } \langle Tu, v \rangle_{X_2} = \langle u, w \rangle_{X_1}, \text{ for all } u \in D(T) \},
\]

in which case \( T^* v := w \). \( T^* \) is also a densely-defined, closed operator. Let \( R(T) \) denote the range of \( T, N(T) \) the kernel of \( T \), the closed range theorem holds:

\[
R(T)^\perp = N(T^*),
\]

where \( \perp \) denotes the orthogonal complement operation.

Consider the closed Hilbert complex \( (W,d) \):

\[
\cdots \rightarrow W^{k-1} \xrightarrow{d^{k-1}} W^k \xrightarrow{d^k} W^{k+1} \xrightarrow{d^{k+1}} \cdots,
\]

i.e., for each index \( k \), \( W^k \) is a Hilbert space equipped with the inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \), \( d^k : W^k \rightarrow W^{k+1} \) is a densely-defined, closed operator, \( R(d^k) \subseteq D(d^{k+1}) \) is closed in \( W^{k+1} \), and \( d^{k+1} \circ d^k = 0 \). Let \( \mathcal{F}^k = N(d^k), \mathcal{B}^k = R(d^{k-1}) \), and \( \mathcal{H}^k = \mathcal{F}^k \cap \mathcal{B}^k \perp \) denote the space of abstract harmonic forms. \( \mathcal{H}^k \) is also called the \( k \)-th cohomology group since \( \mathcal{H}^k \cong \mathcal{F}^k / \mathcal{B}^k \). The cochain complex \( (W,d) \) has the domain complex \( (V,d) \) as a subcomplex:

\[
\cdots \rightarrow V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \xrightarrow{d^{k+1}} \cdots.
\]

Here \( V^k = D(d^k) \) is the domain of \( d^k \), equipped with the \( V \)-inner product

\[
\langle u, v \rangle_V := \langle u, v \rangle + \langle d^k u, d^k v \rangle,
\]

and corresponding \( V \)-norm \( \| \cdot \|_V \). Let \( \mathcal{Z}^{-1, V} = \{ v \in V^k : \langle v, z \rangle = 0 \text{ for all } z \in \mathcal{Z}^k \} \). There exists a constant \( c_p > 0 \), such that

\[
\| v \|_V \leq c_p \| d^k v \| \text{ for all } v \in \mathcal{Z}^{-1, V}.
\]

In FEEC literature, (2.2) is called the Poincaré inequality.

For each index \( k \), choose a finite-dimensional subspace \( V^k_\ell \) of \( V^k \). We assume that \( dV^k_\ell \subseteq V^{k+1}_\ell \) so that \( (V_\ell, d) \) is a subcomplex of \( (V,d) \). Let \( W^k_\ell \) be the same space \( V^k_\ell \) but equipped with the \( W \)-inner product \( \langle \cdot, \cdot \rangle \). Similarly to the continuous case, let \( \mathcal{Z}^{-1, \ell} = N(d^\ell), \mathcal{B}^\ell = R(d^{\ell-1}_\ell), \) and \( \mathcal{H}^\ell = \mathcal{B}^\ell \cap \mathcal{Z}^{-1, \ell} \perp \). Note that in general \( \mathcal{Z}^{-1, \ell} \subseteq \mathcal{Z}^{-1, V} \) and \( \mathcal{H}^\ell \subseteq \mathcal{H}^k \). In order to derive a posteriori error estimate on \( (V_\ell, d) \), we
assume the existence of a bounded cochain projection $\pi_\ell$ from $(V, d)$ to $(V_\ell, d)$. To be precise, for each index $k$, $\pi_\ell^k$ maps $V^k$ onto $V^k_\ell$, $\pi_\ell^k|_{V^k} = \text{id}$, $d^k \pi_\ell^k = \pi_\ell^{k+1} d^k$, and $\|\pi_\ell^k\|_V = \|\pi_\ell^k\|_{V^k \rightarrow V^k} < \infty$ is uniformly bounded with respect to the discretization parameter $\ell$. It has been shown in [4] that the discrete Poincaré inequality holds:

$$\|v\|_V \leq c_\ell \|\pi_\ell^k\|_V \|d^k v\| \text{ for all } v \in V^k_\ell.$$ 

2.2. Generalized Hodge Laplacian and approximation. For each index $k$, let $d_\ell^k$ denote the adjoint operator of $d^{k-1}: W^{k-1} \rightarrow W^k$ and $V^k_\ell = D(d_\ell^k)$. Throughout the rest of this paper, we may drop the superscript or subscript $\ell$ provided no confusion arises. On the closed Hilbert complex $(W, d)$, Arnold, Falk and Winther [4] considered the abstract Hodge Laplace equation

$$(dd^* + d^* d) u = f,$$

where $f \in W^k$ and $u \in V^k \cap V^k_\ell$ satisfy the compatibility condition $f, u \perp \mathcal{H}_k$. Note that $u \in V^k \cap V^k_\ell$ and it is difficult to construct finite element subspaces of $V^k \cap V^k_\ell$. Therefore, the authors in [4] considered the mixed formulation: Find $(\sigma, u, q) \in V^{k-1} \times V^k \times \mathcal{H}_k$, such that

$$\langle \sigma, \tau \rangle - \langle d\tau, u \rangle = 0, \quad \tau \in V^{k-1},$$

$$(2.3)$$

$$\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle v, p \rangle = \langle f, v \rangle, \quad v \in V^k,$$

$$\langle u, q \rangle = 0, \quad q \in \mathcal{H}_k.$$

Using the discrete complex $(V_\ell, d)$, the mixed method for (2.3) seeks $(\sigma_\ell, u_\ell, p_\ell) \in V^{k-1}_\ell \times V^k_\ell \times \mathcal{H}_k^\ell$ such that

$$\langle \sigma_\ell, \tau \rangle - \langle d\tau, u_\ell \rangle = 0, \quad \tau \in V^{k-1}_\ell,$$

$$(2.4)$$

$$\langle d\sigma_\ell, v \rangle + \langle du_\ell, dv \rangle + \langle v, p_\ell \rangle = \langle f, v \rangle, \quad v \in V^k_\ell,$$

$$\langle u_\ell, q \rangle = 0, \quad q \in \mathcal{H}_k^\ell.$$

In general, $\mathcal{H}_k^\ell \not\subseteq \mathcal{H}_k$ and (2.4) is a nonconforming method. For the sake of simplicity, we assume the $k$-th cohomology group $\mathcal{H}_k = \{ 0 \}$ and consider the generalized Hodge Laplacian problem: Find $(\sigma, u) \in V^{k-1} \times V^k$, such that

$$\langle C\sigma, \tau \rangle - \langle d\tau, u \rangle = 0, \quad \tau \in V^{k-1},$$

$$(2.5)$$

$$\langle d\sigma, v \rangle + \langle du, dv \rangle = \langle f, v \rangle, \quad v \in V^k,$$

where $C: W^{k-1} \rightarrow W^{k-1}$ is a self-adjoint, positive semi-definite, continuous linear operator. $C$ is not necessarily positive definite with respect to $\langle \cdot, \cdot \rangle$ and $\|C^{1/2} \cdot \|^2 = \langle C \cdot, \cdot \rangle^{1/2}$ may not define a norm on $W^{k-1}$. We assume that there exists a constant $C_C > 0$ with

$$C_C^{-1} \|\tau\|_V^2 \leq \langle C\tau, \tau \rangle + \|d\tau\|^2 \leq C_C \|\tau\|_V^2$$

for all $\tau \in V^{k-1}$. For $\tau_1, \tau_2 \in V^{k-1}$, let

$$\langle \tau_1, \tau_2 \rangle_{V_C} := \langle C\tau_1, \tau_2 \rangle + \langle d\tau_1, d\tau_2 \rangle.$$

(2.6) shows that $\langle \cdot, \cdot \rangle_{V_C}$ is an inner product on $V^{k-1}$ and the $V_C$-norm $\|\cdot\|_{V_C} = \langle \cdot , \cdot \rangle_{V_C}^{1/2}$ is equivalent to $\|\cdot\|_V$. The mixed method for solving (2.5) is to find $(\sigma_\ell, u_\ell) \in V^{k-1}_\ell \times V^k_\ell$ such that

$$\langle \sigma_\ell, \tau \rangle - \langle d\tau, u_\ell \rangle = 0, \quad \tau \in V^{k-1}_\ell,$$
\( V_{\ell}^{k-1} \times V_{\ell}^{k} \) satisfying

\begin{align}
(2.7a) & \quad \langle C\sigma, \tau \rangle - \langle d\tau, u_{\ell} \rangle = 0, \quad \tau \in V_{\ell}^{k-1}, \\
(2.7b) & \quad \langle d\sigma, v \rangle + \langle du_{\ell}, dv \rangle = \langle f, v \rangle, \quad v \in V_{\ell}^{k}.
\end{align}

Thanks to the cochain projection \( \pi_{\ell} \), we obtain \( \mathcal{S}_{\ell}^{k} = \pi_{\ell}^{k}(\mathcal{S}^{k}) = \{0\} \) and the well-posedness of (2.5) and (2.7), see Theorem 2.1. Assuming \( V_{\ell+1}^{k-1} \subseteq V_{\ell+1}^{k} \) and \( V_{\ell}^{k} \subseteq V_{\ell+1}^{k} \) and using (2.7), we obtain the Galerkin orthogonality

\begin{align}
(2.8a) & \quad \langle C(\sigma_{\ell+1} - \sigma_{\ell}), \tau \rangle - \langle d\tau, u_{\ell+1} - u_{\ell} \rangle = 0, \quad \tau \in V_{\ell}^{k-1}, \\
(2.8b) & \quad \langle d(\sigma_{\ell+1} - \sigma_{\ell}), v \rangle + \langle d(u_{\ell+1} - u_{\ell}), dv \rangle = 0, \quad v \in V_{\ell}^{k}.
\end{align}

Let \( B(\sigma, u; \tau, v) = \langle C\sigma, \tau \rangle - \langle d\tau, u \rangle + \langle d\sigma, v \rangle + \langle du, dv \rangle \). The next theorem shows that \( B \) satisfies the continuous and discrete inf-sup condition, which implies the well-posedness of (2.5) and (2.7). The proof is the same as Theorem 3.2 in [4].

**Theorem 2.1.** Assume \( \mathcal{S}_{\ell}^{k} = \{0\} \). There exists a constant \( \gamma > 0 \) depending only on \( c_{P}, C_{v} \), such that

\[ \gamma(\|\xi\|_{V_{C}} + \|w\|_{V}) \leq \sup_{\tau \in V_{\ell}^{k-1}, v \in V_{\ell}^{k}} \frac{B(\xi, w; \tau, v)}{\|\tau\|_{V_{C}} + \|v\|_{V}} \]

for all \( \xi \in V_{\ell}^{k-1}, w \in V_{\ell}^{k} \). In addition, there exists a constant \( \gamma_{\ell} > 0 \) depending only on \( c_{P}, C_{v}, \|\pi_{\ell}\|_{V}, \) such that

\[ \gamma_{\ell}(\|\xi_{\ell}\|_{V_{C}} + \|w_{\ell}\|_{V}) \leq \sup_{\tau \in V_{\ell}^{k-1}, v \in V_{\ell}^{k}} \frac{B(\xi_{\ell}, w_{\ell}; \tau, v)}{\|\tau\|_{V_{C}} + \|v\|_{V}} \]

for all \( \xi_{\ell} \in V_{\ell}^{k-1}, w_{\ell} \in V_{\ell}^{k} \).

Demlow and Hirani [22] used the continuous inf-sup condition to derive their error estimator for the method (2.4). Using the discrete inf-sup condition, we obtain the discrete upper bound of the abstract natural norm error.

**Lemma 2.2.** For \( 1 \leq k \leq n \), it holds that

\[ \gamma_{\ell+1}(\|\sigma_{\ell+1} - \sigma_{\ell}\|_{V_{C}} + \|u_{\ell+1} - u_{\ell}\|_{V}) \leq \sup_{\tau \in V_{\ell+1}^{k-1}, \|\tau\|_{V_{C}} = 1} \{\langle C\sigma, \tau - \pi_{\ell+1}\tau \rangle - \langle d(\tau - \pi_{\ell+1}\tau), u_{\ell+1} \rangle \} \]

\[ + \sup_{v \in V_{\ell+1}^{k}, \|v\|_{V} = 1} \{\langle f - d\sigma_{\ell}, v - \pi_{\ell}v \rangle - \langle du_{\ell}, dv - \pi_{\ell}dv \rangle \}. \]

**Proof.** Let \( \tau \in V_{\ell+1}^{k-1} \) and \( v \in V_{\ell+1}^{k} \). It follows from (2.8) and (2.7) that

\[ B(\sigma_{\ell+1} - \sigma_{\ell}, u_{\ell+1} - u_{\ell}; \tau, v) = B(\sigma_{\ell+1} - \sigma_{\ell}, u_{\ell+1} - u_{\ell}; \tau - \pi_{\ell}\tau, v - \pi_{\ell}v) \]

\[ = \langle f, v - \pi_{\ell}v \rangle - B(\sigma_{\ell}, u_{\ell}; \tau - \pi_{\ell}\tau, v - \pi_{\ell}v). \]

Combining it with the discrete inf-sup condition in Theorem 2.1 completes the proof. \( \blacksquare \)
2.3. De Rham complex and approximation. The de Rham complex is a canonical example of the closed Hilbert complex. Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded Lipschitz domain. For index \( 0 \leq k \leq n \), let \( \Lambda^k(\Omega) \) denote the space of all smooth \( k \)-forms \( \omega \) which can be uniquely written as
\[
\omega = \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq n} \omega_\alpha dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k},
\]
where each \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k \) is a multi-index, each coefficient \( \omega_\alpha \subseteq C^\infty(\Omega) \) and \( \wedge \) is the wedge product. For \( \eta \in \Lambda^k(\Omega) \) with \( \eta = \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq n} \eta_\alpha dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k} \), the inner product of \( \omega \) and \( \eta \) is
\[
\langle \omega, \eta \rangle := \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq n} \int_\Omega \omega_\alpha \eta_\alpha dx.
\]
The exterior derivative \( d = d^k : \Lambda^k(\Omega) \rightarrow \Lambda^{k+1}(\Omega) \) is given by
\[
d\omega = \sum_{1 \leq \alpha_1 < \cdots < \alpha_k \leq n} \sum_{j=1}^n \frac{\partial \omega_\alpha}{\partial x^j} dx^j \wedge dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}.
\]
Let \( \| \cdot \| \) denote the \( L^2 \)-norm given by \( \langle \cdot, \cdot \rangle \) and \( L^2 \Lambda^k(\Omega) \) the space of \( k \)-forms with \( L^2 \)-coefficients. Then \( d \) can be understood in the distributional sense. Let \( D(d^k) := H \Lambda^k(\Omega) = \{ \omega \in L^2 \Lambda^k(\Omega) : d\omega \in L^2 \Lambda^{k+1}(\Omega) \} \). The following cochain complex
\[
L^2 \Lambda^0(\Omega) \xrightarrow{d} L^2 \Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} L^2 \Lambda^{n-1}(\Omega) \xrightarrow{d} L^2 \Lambda^n(\Omega)
\]
is an example of the closed Hilbert complex \((W, d)\). The \( L^2 \)-de Rham complex [corresponds to \((V, d)\)] is
\[
H \Lambda^0(\Omega) \xrightarrow{d} H \Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H \Lambda^{n-1}(\Omega) \xrightarrow{d} H \Lambda^n(\Omega).
\]
In order to characterize the adjoint of \( d \), we need the Hodge star operator \( * : L^2 \Lambda^k(\Omega) \rightarrow L^2 \Lambda^{n-k}(\Omega) \) determined by \( \int_\Omega \omega \wedge \mu = \langle *, \omega, \mu \rangle \) for all \( \mu \in L^2 \Lambda^{n-k}(\Omega) \). The coderivative \( \delta : \Lambda^k(\Omega) \rightarrow \Lambda^{k-1}(\Omega) \) is then determined by \( *\delta \omega = (-1)^k d* \omega \). \( d \) and \( \delta \) are related by the integrating by parts formula
\[
\langle d\omega, \mu \rangle = \langle \omega, \delta \mu \rangle + \int_{\partial \Omega} \text{tr} \omega \wedge \text{tr} * \mu, \quad \omega \in \Lambda^k(\Omega), \quad \mu \in \Lambda^{k+1}(\Omega),
\]
where the trace operator \( \text{tr} \) on \( \partial \Omega \) is the pullback for differential forms induced by the inclusion \( \partial \Omega \rightarrow \overline{\Omega} \). If \( \Omega \) is replaced by \( T \) in (2.11), \( \text{tr} \) denotes the trace on \( \partial T \) by abuse of notation. We make use of the spaces \( H^* \Lambda^k(\Omega) = \{ \omega \in L^2 \Lambda^k(\Omega) : \text{tr} \omega = 0 \} \) and \( \tilde{H} \Lambda^k(\Omega) = \{ \omega \in H \Lambda^k(\Omega) : \text{tr} \omega = 0 \} \). The next lemma characterizes the adjoint operator of \( d \).

**Theorem 2.3 (Theorem 4.1 in [4])**. Let \( d \) be the exterior derivative viewed as a densely-defined, closed operator \( d : L^2 \Lambda^{k-1}(\Omega) \rightarrow L^2 \Lambda^k(\Omega) \) with domain \( H \Lambda^{k-1}(\Omega) \). Then the adjoint \( d^* \), as a densely-defined, closed operator \( L^2 \Lambda^k(\Omega) \rightarrow L^2 \Lambda^{k-1}(\Omega) \), has domain \( H^* \Lambda^k(\Omega) := \{ \omega \in H \Lambda^k(\Omega) : \text{tr} \omega = 0 \} \), and coincides with \( \delta \).

The generalized Hodge Laplacian problem (2.5) on the de Rham complex uses \( V^{k-1} = H \Lambda^{k-1}(\Omega), V^k = H \Lambda^k(\Omega) \). A more compact form of (2.5) is
\[
C \sigma = \delta u, \quad d \sigma + \delta du = f.
\]
\[ (d\delta + \delta d) u = f \text{ in } \Omega, \]

\[ \text{tr} \ast u = 0, \quad \text{tr} \ast du = 0 \text{ on } \partial \Omega. \]

Let \( \mathcal{T}_0 \leq \mathcal{T}_1 \leq \cdots \leq \mathcal{T}_\ell \leq \cdots \) be a sequence of nested conforming simplicial triangulations of \( \Omega \), where \( \mathcal{T}_\ell \leq \mathcal{T}_{\ell+1} \) means \( \mathcal{T}_{\ell+1} \) is a refinement of \( \mathcal{T}_\ell \). For \( T \in \mathcal{T}_\ell \), let \( |T| \) denote the volume of \( T \) and \( h_T := |T|^{\frac{1}{N}} \). We assume that \( \{ \mathcal{T}_\ell \}_{\ell \geq 0} \) is shape regular, namely,

\[ \sup_{\ell \geq 0} \max_{\substack{T \in \mathcal{T}_\ell \\rho_T \leq C \tau_0 < \infty,}} \]

where \( r_T \) and \( \rho_T \) are radii of circumscribed and inscribed spheres of the simplex \( T \), respectively. Let \( \mathcal{P}_r \Lambda^k(T) \) denote the space of \( k \)-forms on \( T \) with polynomial coefficients of degree \( \leq r \). Let

\[ \mathcal{P}_r \Lambda^k(\mathcal{T}_\ell) = \{ v \in H\Lambda^k(\Omega) : v|_T \in \mathcal{P}_r \Lambda^k(T) \text{ for all } T \in \mathcal{T}_\ell \}, \]

\[ \mathcal{P}_{r+1} \Lambda^k(\mathcal{T}_\ell) = \mathcal{P}_r \Lambda^k(\mathcal{T}_\ell) + \kappa \mathcal{P}_r \Lambda^{k+1}(\mathcal{T}_\ell), \]

where \( \kappa : L^2 \Lambda^k(\Omega) \to L^2 \Lambda^{k-1}(\Omega) \) is the interior product given by

\[ (\kappa \omega)_x(v_1, \ldots, v_{k-1}) = \omega_x(X(x), v_1, \ldots, v_{k-1}), \quad v_1, \ldots, v_{k-1} \in \mathbb{R}^n, \]

with \( X(x) = (x_1, \ldots, x_n)^T \) being the position vector field. For \( r \geq 0 \), let

\[ \mathcal{V}_\ell^{k-1} = \mathcal{P}_{r+1} \Lambda^{k-1}(\mathcal{T}_\ell) \text{ or } \mathcal{P}_{r+1}^{-1} \Lambda^{k-1}(\mathcal{T}_\ell), \]

\[ \mathcal{V}_\ell^k = \mathcal{P}_{r+1} \Lambda^k(\mathcal{T}_\ell) \text{ or } \mathcal{P}_r \Lambda^k(\mathcal{T}_\ell), \]

Other spaces \( \mathcal{V}_\ell^j \) with \( j \neq k, k-1 \) are chosen in the same way. In \( \mathbb{R}^n \), there are \( 2^{n-1} \) different discrete subcomplexes \( \mathcal{V}_\ell \) on a simplicial triangulation \( \mathcal{T}_\ell \).

3. A Posteriori Error Estimate

In the rest of this paper, \( A \leq B \) provided \( A \leq C \cdot B \) and \( C \) is a generic constant depending only on \( C_\ell, \tau_0 \) and \( \Omega \). Let \( H^r \Lambda^k(\Omega) \) be the space of \( k \)-forms whose coefficients are in \( H^r(\Omega) \). Formula (2.11) still holds for \( \omega \in H^1 \Lambda^k(\Omega) \) and \( \mu \in H^1 \Lambda^{k+1}(\Omega) \). Let \( \| \cdot \|_{H^r(\Omega)} \) denote the \( H^r(\Omega) \) norm with some \( l \) depending on the context. Let \( \langle \cdot, \cdot \rangle_T \) denote the \( L^2 \)-inner product restricted to \( T \), \( \| \cdot \|_T \) and \( \| \cdot \|_{\partial T} \) the \( L^2 \)-norms restricted to \( T \) and \( \partial T \), respectively.

To derive discrete upper bounds on the de Rham complex, we need the local \( V \)-bounded cochain projection \( \pi_\ell \) developed by Falk and Winther [24]. The existence of \( \pi_\ell \) implies the discrete inf-sup condition by Theorem 2.1.

**Theorem 3.1** (local \( V \)-bounded cochain projection). For each index \( 0 \leq k \leq n \), there exists a projection \( \pi_\ell^k : H\Lambda^k(\Omega) \to \mathcal{V}_\ell^k \) commuting with the exterior derivative, i.e., \( \pi_\ell^k|_{\mathcal{V}_\ell^k} = \text{id} \) and \( d^k \pi_\ell^k = \pi_\ell^{k+1} d^k \). For \( T \in \mathcal{T}_\ell \), let \( D_T = \{ T' \in \mathcal{T}_\ell : T' \cap T \neq \emptyset \} \) and \( \mathcal{V}_\ell^k|_{D_T} = \{ v|_{D_T} : v \in \mathcal{V}_\ell^k \} \). Then

\[ v - \pi_\ell^k v = 0 \text{ on } T \text{ for } v \in \mathcal{V}_\ell^k|_{D_T}, \]

\[ \| \pi_\ell^k v \|_{H\Lambda^k(T)} \lesssim \| v \|_{H\Lambda^k(D_T)} \text{ for } v \in H\Lambda^k(D_T). \]
In addition, for \( v \in H^1\Lambda^k(D_T) \),
\[
|\pi_T^k v|_{H^1(T)} + h_T^{-1} \| v - \pi_T^k v \|_{T} + h_T^{-\frac{3}{2}} \| \text{tr}(v - \pi_T^k v) \|_{\partial T} \lesssim |v|_{H^1(D_T)}.
\]

**Proof.** Properties (3.1a) and (3.1b) are given by Falk and Winther in [24]. Let \( \mathcal{I}_\ell : H\Lambda^k(\Omega) \to V^k_T \) be the interpolation given by applying the Scott–Zhang interpolation (cf. [41]) on \( T_\ell \) to each coefficient of \( \omega \in H\Lambda^k(\Omega) \). It follows from the property (3.1a) and the same property of \( \mathcal{I}_\ell \) that
\[
\pi_T^k v - \mathcal{I}_\ell v = 0 \text{ on } T \text{ for } v \in V^k_T|_{D_T},
\]
and thus
\[
|\pi_T^k v - \mathcal{I}_\ell v|_{T} \lesssim h_T |v|_{H^2(D_T)}
\]
by the Bramble–Hilbert lemma. In addition, it is well-known that
\[
|\mathcal{I}_\ell v|_{H^1(T)} \lesssim |v|_{H^1(D_T)}, \quad |v - \mathcal{I}_\ell v|_{T} \lesssim h_T |v|_{H^1(D_T)}.
\]
Then using the triangle inequality, (3.3) and (3.4), we obtain
\[
|\pi_T^k v|_{H^1(T)} \lesssim |\pi_T^k v - \mathcal{I}_\ell v|_{H^1(T)} + |\mathcal{I}_\ell v|_{H^1(T)} \lesssim |v|_{H^1(D_T)},
\]
\[
|v - \pi_T^k v|_{T} \lesssim |v - \mathcal{I}_\ell v|_{T} + |\mathcal{I}_\ell v - \pi_T^k v|_{T} \lesssim h_T |v|_{H^1(D_T)}.
\]
Combining it with the trace inequality verifies
\[
h_T^{-\frac{3}{2}} \| \text{tr}(v - \pi_T^k v) \|_{\partial T} \lesssim |v|_{H^1(D_T)}. \quad \square
\]
Using (3.2) and the bounded overlapping property of \( D_T \), we obtain
\[
|\pi_T v|_{H^1(\Omega)}^2 + \sum_{T \in T_\ell} h_T^{-2} \|v - \pi_T v\|_T^2 + h_T^{-1} \| \text{tr}(v - \pi_T v) \|_{\partial T}^2 \lesssim |v|_{H^1(\Omega)}^2.
\]

In addition to \( \pi_\ell \), we need an \( H^1 \)-regular decomposition result, see Lemma 5 in [22]. The proof therein hinges on the technical \( H^1 \)-solution regularity of the equation \( d\varphi = g \) under the Dirichlet boundary condition, see e.g., [31, 40]. In our convergence analysis, the linearity of such regular decomposition is also required. Since only the natural boundary condition is considered, we give a simple proof of the regular decomposition below. For convenience, let \( H\Lambda^{-1}(\Omega) = \{0\} \).

**Theorem 3.2** (regular decomposition). Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \). For \( 0 \leq k \leq n \), there exist bounded linear operators \( K^k_1 : H\Lambda^k(\Omega) \to H^1\Lambda^{k-1}(\Omega) \) and \( K^k_2 : H\Lambda^k(\Omega) \to H^{1-k}(\Omega) \), such that for \( v \in H\Lambda^k(\Omega) \),
\[
v = dK^k_1 v + K^k_2 v.
\]

**Proof.** When \( k = 0 \), \( H\Lambda^0(\Omega) = H^1(\Omega) \) and \( K^0_1 v = 0 \), \( K^0_2 v = v \). When \( k = n \), \( d^{n-1} \) is identified with the divergence operator \( \text{div} \), see Section 5. In this case, let \( K^k_2 v = 0 \) and \( K^k_1 \) be the \( H^1 \)-regular right inverse of \( \text{div} \), i.e., \( dK^k_1 v = v \), cf. Theorem 2.4 in [3].

Assume \( 1 \leq k \leq n - 1 \). Let \( \overline{\Omega} \) be a compact subset of a ball \( B \subset \mathbb{R}^n \). There exists a linear bounded extension operator \( E : H\Lambda^k(\Omega) \to H\Lambda(B) \), see e.g., [32] and Lemma 5 in [22]. Consider the exterior derivative on \( B \):
\[
d_B = d : L^2\Lambda^k(\Omega) \to L^2\Lambda^{k-1}(\Omega), \quad D(d_B) = H\Lambda^{k-1}(B).
\]
For \( v \in H\Lambda^k(\Omega) \), we can take \( z \in N(d_B)^\perp \cap H\Lambda^{k-1}(B) \) such that \( dz = dE v \). Due to (2.1), Theorem 2.3, and \( \delta \circ \delta = 0 \), we have
\[
N(d_B)^\perp = R(d_B^* ) = \delta(\hat{H}^* \Lambda^k(B)) \subset \hat{H}^* \Lambda^{k-1}(B).
\]
Therefore \( z \in \tilde{H}^1 \Lambda^{k-1}(B) \cap H^1 \Lambda^{k-1}(B) \subset H^1 \Lambda^k(B) \), where the following Sobolev embedding (cf. \cite{26}) is used:

\[
\|z\|_{\tilde{H}^1(B)} \lesssim \|z\|_{L^2(B)} + \|dz\|_{L^2(B)} + \|\delta z\|_{L^2(B)}.
\]

Using the Poincaré inequality (2.2) and \( \delta z = 0 \), the above estimate reduces to

\[
(3.6) \quad \|z\|_{\tilde{H}^1(B)} \lesssim \|dz\|_{L^2(B)} = \|dEv\|_{L^2(B)} \lesssim \|v\|_{H^\omega(\Omega)}.
\]

Since \( B \) is contractible and \( d(Ev - z) = 0 \), there exists \( \varphi \in H^\omega(\Omega) \), such that \( d\varphi = Ev - z \). The \((k - 1)\)-form \( \varphi \) can be chosen in \( H^1 \Lambda^{k-1}(B) \) as in the proof of \( H^1 \)-regularity of \( z \). In addition, the \( H^1 \)-norm of \( \varphi \) can be controlled by

\[
(3.7) \quad \|\varphi\|_{H^1(B)} \lesssim \|d\varphi\|_{L^2(B)} = \|Ev - z\|_{L^2(B)} \lesssim \|v\|_{H^\omega(\Omega)}.
\]

Taking \( K^k_1 v = \varphi|_\Omega, K^k_2 v = z|_\Omega \) and using (3.6) and (3.7) completes the proof. \( \square \)

Although \( \pi_\ell \) is \( V \)-bounded, \( \pi_\ell v \) is not defined for all \( v \in L^2 \Lambda^k(\Omega) \). Instead, Christiansen and Wintner \cite{20} constructed an \( L^2 \)-bounded smooth projection \( \bar{\pi}_\ell \), i.e., \( K^k_\ell : L^2 \Lambda^k(\Omega) \to V^k_\ell \) is a linear operator for each \( k \), \( \bar{\pi}_\ell \) commutes with \( d \), \( \bar{\pi}_\ell |_{V^k_\ell} = id \), and

\[
\|\bar{\pi}_\ell^k\| := \sup_{v \in L^2 \Lambda^k(\Omega), \|v\| = 1} \|\bar{\pi}_\ell^k v\| \lesssim 1.
\]

Built upon \( \bar{\pi}_\ell \) and \( K_1, K_2 \), we immediately obtain a discrete version of Theorem 3.2 and compact operators which are crucial for proving quasi-orthogonality.

**Corollary 3.3.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \). For each index \( k \), let

\[
i_1 : H^1 \Lambda^{k-1}(\Omega) \rightarrow L^2 \Lambda^{k-1}(\Omega), \quad i_2 : H^1 \Lambda^k(\Omega) \rightarrow L^2 \Lambda^k(\Omega)
\]

denote the natural inclusions. Then \( K^k_1 = i_1 \circ K^k_1 : H^\omega(\Omega) \rightarrow L^2 \Lambda^{k-1}(\Omega) \) and \( K^k_2 = i_2 \circ K^k_2 : H^\omega(\Omega) \rightarrow L^2 \Lambda^k(\Omega) \) are compact operators. In addition, for \( v \in V^k_\ell \),

\[
v = d\bar{\pi}_\ell K^k_1 v + \bar{\pi}_\ell K^k_2 v.
\]

**Proof.** Using the Rellich–Kondrachov lemma, the inclusions \( i_1 \) and \( i_2 \) are compact operators. Since \( K^k_1 \) and \( K^k_2 \) are bounded, the compositions

\[
K^k_1 \circ i_1 = i_1 \circ K^k_1, \quad K^k_2 \circ i_2 = i_2 \circ K^k_2
\]

must be compact. \( v = d\bar{\pi}_\ell K^k_1 v + \bar{\pi}_\ell K^k_2 v \) directly follows from Theorem 3.2, \( v = \bar{\pi}_\ell v \) and that \( \bar{\pi}_\ell \) commutes with the exterior derivative \( d \). \( \square \)

### 3.1. Error indicator.

On the de Rham complex, we still use \( \| \cdot \|_V \) to denote the \( H^\omega(\Omega) \) norm for some \( l \). Let

\[
H^1 \Lambda^l(\mathcal{T}_\ell) := \{ \omega \in L^2 \Lambda^l(\Omega) : \omega|_T \in H^1 \Lambda^l(T) \text{ for all } T \in \mathcal{T}_\ell \}
\]

and \( \mathcal{S}_\ell \) be the set of \((n - 1)\)-faces in \( \mathcal{T}_\ell \). For each interior face \( S \in \mathcal{S}_\ell \) and \( \omega \in H^1 \Lambda^l(\mathcal{T}_\ell) \), let \( \|\text{tr} \omega\|_S := \text{tr}_{\mathcal{S}_\ell} \omega|_{T_1} - \text{tr}_{\mathcal{S}_\ell} \omega|_{T_2} \) denote the jump of \( \text{tr} \omega \) on \( S \), where \( T_1 \) and \( T_2 \) are the two simplexes sharing \( S \) as an \((n - 1)\)-face. On each boundary face \( S \), \( \|\text{tr} \omega\| := \text{tr}_{\mathcal{S}_\ell} \omega \).

Let \( \delta_{ij} = 0 \) if \( i = j \), otherwise \( \delta_{ij} = 1 \). For \( 1 \leq k \leq n - 1 \), let \( f \in H^1 \Lambda^k(\mathcal{T}_\ell) \) and

\[
E_{\ell}(T) = \delta_{k1} h^2 \|\delta \omega|_T\|_T^2 + h_T \|\text{tr} \omega|_T\|_{\partial T}^2 + h_T \|\text{tr} \omega|_{\partial T}\|_{\partial T}^2 + h_T \|\text{tr} \omega|_{\partial T}\|_{\partial T}^2 + h_T \|\text{tr} \omega|_{\partial T}\|_{\partial T}^2 + h_T \|\text{tr} \omega|_{\partial T}\|_{\partial T}^2.
\]
When $k = n$, let
\[
\mathcal{E}_\ell(T) = h^2 T \| \delta \sigma \|_T^2 + h T \| \text{tr} \star \sigma \|_T^2 + \sum_{T' \in \mathcal{T}_\ell \cap T} \| f - d \sigma \|_{\partial T}^2
\] + h^2 T \| \nabla^2 \sigma \|_T^2 + h T \| \text{tr} \star u \|_T^2.
\]
The estimator $\mathcal{E}_\ell = \sum_{T \in \mathcal{T}_\ell} \mathcal{E}_\ell(T)$ controls the error $\| \sigma - \sigma \|_V^2 + \| u - u \|_V^2$. $\mathcal{E}_\ell$ for the standard Hodge Laplace equation was first introduced in [22]. Let
\[
\mathcal{E}_\ell(\mathcal{M}) = \sum_{T \in \mathcal{M}} \mathcal{E}_\ell(T), \quad \mathcal{M} \subseteq \mathcal{T}_\ell.
\]

Define the enriched collection of refinement elements:
\[
\mathcal{R}_\ell = \{ T \in \mathcal{T}_\ell : T \cap T' \neq \emptyset \text{ for some } T' \in \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1} \}.
\]
The next theorem confirms the discrete and continuous reliability of $\mathcal{E}_\ell$. It is noted that Demlow [21] used similar technique when deriving the discrete reliability of his AFEM for computing harmonic forms.

**Theorem 3.4 (discrete and continuous upper bounds).** Assume $f \in H^1 \Lambda^k(\mathcal{T}_\ell)$ with $1 \leq k \leq n - 1$ or $f \in L^2 \Lambda^n(\Omega)$ when $k = n$. There exist a constant $C_{up}$ depending solely on $C_\ell$, $\mathcal{T}_0$ and $\Omega$, such that
\[
\| \sigma_{\ell+1} - \sigma \|_V^2 + \| u_{\ell+1} - u \|_V^2 \leq C_{up} \mathcal{E}_\ell(\mathcal{R}_\ell), \quad (3.8)
\]
\[
\| \sigma - \sigma \|_V^2 + \| u - u \|_V^2 \leq C_{up} \mathcal{E}_\ell, \quad (3.9)
\]

**Proof.** For $T \in V_{\ell+1}^{k-1}$ with $\| \tau \|_V = 1$, let $\tau = d\varphi_1 + z_1$, where $\varphi_1 = K_1^{k-1} \tau$ and $z_1 = K_2^{k-1} \tau$ in Theorem 3.2 satisfying the following bounds
\[
\| \varphi_1 \|_{H^1(\Omega)} + \| z_1 \|_{H^1(\Omega)} \lesssim 1.
\]
Similarly, for $v \in V_{\ell+1}^k$ with $\| v \|_V = 1$, we have $v = d\varphi_2 + z_2$ with
\[
\| \varphi_2 \|_{H^1(\Omega)} + \| z_2 \|_{H^1(\Omega)} \lesssim 1.
\]
Thanks to the local property (3.1a),
\[
\text{supp}(\tau - \pi_{\ell} \tau) \subseteq \bigcup_{T \in \mathcal{R}_\ell} T, \quad \text{supp}(v - \pi_{\ell} v) \subseteq \bigcup_{T \in \mathcal{R}_\ell} T.
\]
Since $\pi_{\ell+1}$ is a cochain projection, we have $\tau = \pi_{\ell+1} \tau = d\pi_{\ell+1} \varphi_1 + \pi_{\ell+1} z_1$ and $v = \pi_{\ell+1} v = d\pi_{\ell+1} \varphi_2 + \pi_{\ell+1} z_2$. Hence
\[
\tau - \pi_{\ell} \tau = dE_{\varphi_1} + E_{z_1}, \quad v - \pi_{\ell} v = dE_{\varphi_2} + E_{z_2}, \quad (3.11)
\]
where
\[
E_{\varphi_1} = \pi_{\ell+1} \varphi_1 - \pi_{\ell} \pi_{\ell+1} \varphi_1, \quad E_{z_1} = \pi_{\ell+1} z_1 - \pi_{\ell} \pi_{\ell+1} z_1, \quad i = 1, 2.
\]
Here $E_{\varphi_1} = 0$ when $k = 1$. Using (3.10), (3.11), and the integration by parts formula (2.11), we have
\[
\langle \mathcal{C} \sigma_\ell, \tau - \pi_{\ell} \tau \rangle - \langle d(\tau - \pi_{\ell} \tau), u_\ell \rangle
\]
\[
= \sum_{T \in \mathcal{R}_\ell} \langle \mathcal{C} \sigma_\ell, dE_{\varphi_1} \rangle_T + \langle \mathcal{C} \sigma_\ell, E_{z_1} \rangle_T - \langle dE_{z_1}, u_\ell \rangle_T
\]
\[
+ \sum_{T \in \mathcal{R}_\ell} \langle \delta \mathcal{C} \sigma_\ell, E_{\varphi_1} \rangle_T + \langle \mathcal{C} \sigma_\ell - \delta u_\ell, E_{z_1} \rangle_T
\]
\[
+ \int_{\partial T} \text{tr} E_{\varphi_1} \wedge \mathcal{C} \sigma_\ell - \int_{\partial T} \text{tr} E_{z_1} \wedge \text{tr} u_\ell.
\]
Let \( S(\mathcal{R}_k) = \{ S \in \mathcal{S}_k : S \subset \partial T \text{ for some } T \in \mathcal{R}_k \} \). Since \( E_{\varphi_i} \in V_{k+1}^{k-2} \) and \( E_{z_i} \in V_{k+1}^{k-1} \) are finite element differential forms, their traces \( \text{tr} E_{\varphi_i} \) and \( \text{tr} E_{z_i} \) are well-defined polynomials on each side \( S \in \mathcal{S}_k \). Therefore using the previous equation, we obtain

\[
\langle C\sigma, \tau - \pi_\ell \tau \rangle - \langle d(\tau - \pi_\ell \tau), u_\ell \rangle = \sum_{T \in \mathcal{R}_k} \langle \partial C\sigma_i, E_{\varphi_i} \rangle_T + \langle C\sigma_i - \delta u_\ell, E_{z_i} \rangle_T
\]

\[
+ \sum_{S \in S(\mathcal{R}_k)} \int_S \text{tr} E_{\varphi_i} \wedge \| \text{tr} \star C\sigma \| - \int_S \text{tr} E_{z_i} \wedge \| \text{tr} \star u_\ell \|.
\]

By the same argument, it holds that for \( 1 \leq k \leq n-1 \),

\[
\langle f - d\sigma_\ell, v - \pi_\ell v \rangle = \sum_{T \in \mathcal{R}_k} \langle f - d\sigma_\ell, dE_{\varphi_2} + E_{z_2} \rangle_T - \langle du_\ell, dE_{z_2} \rangle_T
\]

\[
= \sum_{T \in \mathcal{R}_k} \langle d(f - d\sigma_\ell), E_{\varphi_2} \rangle_T + \langle f - d\sigma_\ell - \delta du_\ell, E_{z_2} \rangle_T
\]

\[
+ \sum_{S \in S(\mathcal{R}_k)} \int_S \text{tr} E_{\varphi_2} \wedge \| \text{tr} \star (f - d\sigma_\ell) \| - \int_S \text{tr} E_{z_2} \wedge \| \text{tr} \star du_\ell \|.
\]

Combining (3.12), (3.13) with Lemma 2.2 and using the Cauchy–Schwarz inequality, we obtain for \( 1 \leq k \leq n-1 \),

\[
\| \sigma_{k+1} - \sigma_i \|_V + \| u_{k+1} - u_\ell \|_V
\]

\[
\lesssim \mathcal{E}(\mathcal{R}_k)^{\frac{1}{2}} \left( \sum_{i=1}^{2} h_T^{-2} \| E_{\varphi_i} \|_{T}^{2} + h_T^{-1} \| \text{tr} E_{\varphi_i} \|_{\mathcal{R}_T}^{2} + h_T^{-2} \| E_{z_i} \|_{T}^{2} + h_T^{-1} \| \text{tr} E_{z_i} \|_{\partial T}^{2} \right)^{\frac{1}{2}}.
\]

The discrete upper bound (3.8) then follows together with the approximation property (3.5) and the bounds \( \sum_{i=1}^{2} \| \varphi_i \|_{H^1(\Omega)} + \| z_i \|_{H^1(\Omega)} \lesssim 1 \).

When \( k = n \), we simply have \( du_\ell = 0 \), \( \langle f - d\sigma_\ell, \pi_\ell v \rangle = 0 \), and thus

\[
\langle f - d\sigma_\ell, v - \pi_\ell v \rangle = \langle du_\ell, d(v - \pi_\ell v) \rangle = \langle f - d\sigma_\ell, v \rangle.
\]

Combining it with Lemma 2.2, (3.12), (3.5) still yields (3.8).

Let \( \mathcal{T}_{k+1} \) be a uniform refinement of \( \mathcal{T}_k \). In this case \( \mathcal{E}(\mathcal{R}_k) = \mathcal{E}_k \). In addition, \( (\sigma_{k+1}, u_{k+1}) \) converges to \( (\sigma, u) \) in \( V^{k-1} \times V^k \) as \( \max_{T \in \mathcal{T}_{k+1}} h_T \to 0 \). Therefore passing to the limit in (3.8) yields the continuous upper bound (3.9). \( \square \)

For an integer \( p \geq 0 \), let \( Q^p_T \) denote the \( L^2 \)-projection onto \( \mathcal{P}_p \Lambda^l(T) \) and \( Q^p_{\partial T} \) the \( L^2 \)-projection onto \( \mathcal{P}_p \Lambda^l(\partial T) \) with appropriate \( l \). Here \( \mathcal{P}_p \Lambda^l(\partial T) \) is the space of \( l \)-forms on \( \partial T \) whose restriction to each face of \( T \) are polynomial \( l \)-forms of degree \( \leq p \). Let \( \Pi^p_T = \text{id} - Q^p_T \) and \( \Pi^p_{\partial T} = \text{id} - Q^p_{\partial T} \). The efficiency of \( \mathcal{E}_k \) directly follows from the Verfürth bubble function technique used in [22] and the proof is skipped.

**Theorem 3.5 (efficiency).** Assume \( C(V^{k-1}_k) \subseteq \mathcal{P}_q \Lambda^{k-1}(\mathcal{T}_k) \) for some \( q \geq 0 \). There exists a constant \( C_{\text{low}} > 0 \) depending solely on \( p, q, C, \mathcal{T}_0, \Omega \), such that

\[
C_{\text{low}} \mathcal{E}_k \leq \| \sigma - \sigma_k \|_{V^k}^2 + \| u - u_\ell \|_V^2 + \text{osc}^2_{\ell}(f),
\]
where \( \text{osc}_\ell^2(f) = \sum_{T \in \mathcal{T}_\ell} \text{osc}_\ell^2(f, T) \) with
\[
\text{osc}_\ell^2(f, T) = \delta_{kn} \{ h_{\ell}^2 \| \Pi_T f - d\sigma - \delta d\mu \|_T^2 \\
+ h_{\ell}^2 \| \Pi_T^2 \delta(f - d\sigma) \|_T^2 + h_T \| \Pi_{GT} \| \text{tr} * (f - d\sigma) \|_{\delta T}^2 \}.
\]

From the definitions of \( \mathcal{E}_\ell \) and \( \text{osc}_\ell \), it can be observed that
\[
(3.15) \quad \text{osc}_\ell(f) \leq \mathcal{E}_\ell.
\]

4. QUASI-OPTIMALITY

The adaptive algorithm AMFEM is based on the standard “SOLVE \rightarrow ESTIMATE \rightarrow MARK \rightarrow REFINE” feedback loop. In the procedure REFINE, we use the newest vertex bisection in \( \mathbb{R}^2 \) and its generalization in \( \mathbb{R}^n \) (denoted by NVB), see, e.g., [44, 43]. For simplicity of presentation, we set the error tolerance to be 0 so that AMFEM produces an infinite sequence \( \{\sigma_\ell, u_\ell, \mathcal{T}_\ell\}_{\ell \geq 0} \). To compute the estimator \( \mathcal{E}_\ell \), we assume that \( f \in H^1 \Lambda^k(\Omega_0) \) when \( 1 \leq k \leq n - 1 \), that is, the discontinuity of \( f \) is aligned with the initial mesh \( \Omega_0 \).

Algorithm 4.1. AMFEM. Input an initial mesh \( \Omega_0 \) and a marking parameter \( \theta \in (0, 1) \). Set \( \ell = 0 \).

SOLVE: Solve (2.7) on \( \mathcal{T}_\ell \) to obtain the finite element solution \( (\sigma_\ell, u_\ell) \).

ESTIMATE: Compute error indicators \( (\mathcal{E}_\ell(T))_{T \in \mathcal{T}_\ell} \) and \( \mathcal{E}_\ell = \sum_{T \in \mathcal{T}_\ell} \mathcal{E}_\ell(T) \).

If \( \mathcal{E}_\ell = 0 \), let \( (\sigma_j, u_j) = (\sigma_\ell, u_\ell) \) and \( \mathcal{T}_j = \mathcal{T}_\ell \) for all \( j \geq \ell \); return.

MARK: Select a subset \( \mathcal{M}_\ell \) of \( \mathcal{T}_\ell \) with \( \mathcal{E}_\ell(\mathcal{M}_\ell) \geq \theta \mathcal{E}_\ell \).

REFINE: Refine all elements in \( \mathcal{M}_\ell \) and necessary neighboring elements by NVB to get a conforming mesh \( \mathcal{T}_{\ell + 1} \). Set \( \ell = \ell + 1 \). Go to SOLVE.

4.1. Contraction. In this subsection, we prove the contraction of AMFEM. To this end, we first prove a weak convergence result for Petrov–Galerkin methods. Given Hilbert spaces \( U, V \) equipped with norm \( \| \cdot \| \), a continuous bilinear form \( B : U \times V \rightarrow \mathbb{R} \) and a continuous linear functional \( F : V \rightarrow \mathbb{R} \), consider the variational formulation: Find \( U \in U \) such that
\[
B(U, V) = F(V), \quad V \in V.
\]

Given subspaces \( U_\ell \subseteq U \) and \( V_\ell \subseteq V \), the abstract Petrov–Galerkin method seeks \( U_\ell \in U_\ell \) such that
\[
(4.1) \quad B(U_\ell, V) = F(V), \quad V \in V_\ell.
\]

The method (4.1) is well-posed provided
\[
\inf_{W \in U, \|W\| = 1} \sup_{V \in V, \|V\| = 1} B(W, V) = \inf_{V \in V, \|V\| = 1} \sup_{W \in U, \|W\| = 1} B(W, V) = \beta > 0,
\]
where \( \beta \) is a constant independent of \( \ell \). We say \( x_\ell \rightharpoonup x \) in \( U \) if \( \{x_\ell\}_{\ell \geq 0} \) weakly converges to \( x \) in \( U \) as \( \ell \rightarrow \infty \). The next theorem shows that the normalized error of (4.1) always weakly converges to 0.

Theorem 4.2 (weak convergence). Let \( \{U_\ell\}_{\ell \geq 0} \) and \( \{V_\ell\}_{\ell \geq 0} \) be sequences of subspaces of \( U \) and \( V \), respectively. Assume \( U_\ell \subseteq U_{\ell + 1} \) and \( V_\ell \subseteq V_{\ell + 1} \) for all \( \ell \geq 0 \).
Let

\[ \xi_\ell = \begin{cases} \frac{U_{\ell+1} - U_\ell}{\|U_{\ell+1} - U_\ell\|} & \text{if } \|U_{\ell+1} - U_\ell\| \neq 0, \\ 0 & \text{if } \|U_{\ell+1} - U_\ell\| = 0. \end{cases} \]

We have

\[ \xi_\ell \rightarrow 0 \text{ in } U. \]

**Proof.** Since \( \|\xi_\ell\| \leq 1 \), there exists a weakly convergent subsequence \( \{\xi_{\ell_j}\}_{j \geq 0} \):

\[ \xi_{\ell_j} \rightharpoonup \xi \text{ in } U. \]

For any \( \ell \geq 0 \) and \( V_\ell \in V_\ell \), it follows from the continuity of \( B \) that \( B(\cdot, V_\ell) \) is a continuous linear functional on \( U \). Hence using the weak convergence and the Galerkin orthogonality

\[ B(U_{\ell_{j+1}} - U_{\ell_j}, V_\ell) = 0 \text{ when } \ell_{j+1} \geq \ell, \]

we have

(4.2) \[ B(\xi_{\ell_j}, V_\ell) = \lim_{j \to \infty} B(\xi_{\ell_j}, V_\ell) = 0 \text{ for all } V_\ell \in V_\ell \text{ and } \ell \geq 0. \]

Given \( W \subseteq U \) (resp. \( V \)), let \( W \subseteq U \) (resp. \( V \)) denote the closed subspace spanned by \( W \).

\[ U_\infty = \bigcup_{\ell \geq 0} U_\ell, \quad V_\infty = \bigcup_{\ell \geq 0} V_\ell. \]

Since \( U_\infty \) is closed, convex and \( \{\xi_{\ell_j}\}_{j \geq 0} \subseteq U_\infty \), the weak limit \( \xi \) is contained in \( U_\infty \). It has been shown in (4.5), [34] that the discrete inf-sup condition holds:

(4.3) \[ \inf_{W \in U_\infty, \|W\|=1} \sup_{V \in V_\infty, \|V\|=1} B(W, V) \geq \beta > 0. \]

On the other hand, it follows from (4.2) and the continuity of \( B \) that

(4.4) \[ B(\xi, V) = 0 \text{ for all } V \in V_\infty. \]

Therefore using (4.3) and (4.4), we obtain

\[ \|\xi\| \leq \beta^{-1} \sup_{V \in V_\infty, \|V\|=1} B(\xi, V) = 0. \]

We have indeed shown that each subsequence of \( \{\xi_{\ell_j}\} \) has a further subsequence that weakly converges to 0, which apparently implies \( \xi \rightharpoonup 0 \text{ in } U. \)

**Remark 4.3.** Let the sequence \( \{U_\ell\}_{\ell \geq 0} \) be produced by some AFEM and

\[ \zeta_\ell = \begin{cases} \frac{U - U_\ell}{\|U - U_\ell\|} & \text{if } \|U - U_\ell\| \neq 0, \\ 0 & \text{if } \|U - U_\ell\| = 0. \end{cases} \]

Under very mild assumptions, it has been shown in [34] that \( U_\infty = U \), see also [25] for AFEMs using Dörfler marking. In this case, \( \zeta \in U_\infty \), and the weak convergence \( \zeta \rightharpoonup 0 \) in \( U \) follows from the same proof of Theorem 4.2.

For each \( \ell \geq 0 \), let

\[ e_{d,\sigma,\ell} = \|d(\sigma - \sigma_\ell)\|^2, \quad e_{\sigma,\ell} = \|\sigma - \sigma_\ell\|^2_{V_\ell}, \quad e_{u,\ell} = \|u - u_\ell\|^2_{V_\ell}, \]

\[ E_{d,\sigma,\ell} = \|d(\sigma_{\ell+1} - \sigma_\ell)\|^2, \quad E_{\sigma,\ell} = \|\sigma_{\ell+1} - \sigma_\ell\|^2_{V_\ell}, \quad E_{u,\ell} = \|u_{\ell+1} - u_\ell\|^2_{V_\ell}, \]
Theorem 4.5. For $e = e_{\sigma, \ell} + e_{u, \ell}$ and $E_\ell = E_{\sigma, \ell} + E_{u, \ell}$. Let
\[ E_\ell^{-\frac{1}{2}}(\sigma_{\ell+1} - \sigma_\ell) := 0 \text{ and } E_\ell^{-\frac{1}{2}}(u_{\ell+1} - u_\ell) := 0 \text{ if } E_\ell = 0. \]
Theorem 4.2 immediately implies the following weak convergence result of (2.7).

Corollary 4.4. For $1 \leq k \leq n$, it holds that
\[ E_\ell^{-\frac{1}{2}}(\sigma_{\ell+1} - \sigma_\ell) \to 0 \text{ in } H\Lambda^{k-1}(\Omega), \]
\[ E_\ell^{-\frac{1}{2}}(u_{\ell+1} - u_\ell) \to 0 \text{ in } H\Lambda^k(\Omega). \]

The next lemma deals with error reduction on two nested meshes $T_\ell \leq T_{\ell+1}$.

Theorem 4.5. For $\ell \geq 0$ and $\varepsilon \in (0, 1)$, it holds that
\[ e_{d\sigma, \ell+1} = e_{d\sigma, \ell} - E_{d\sigma, \ell}, \]
\[ (1 - \varepsilon)e_{\sigma, \ell+1} \leq e_{\sigma, \ell} - (1 - \varepsilon)E_{\sigma, \ell} + C_\varepsilon E_{d\sigma, \ell}, \]
where $C_\varepsilon = \varepsilon^{-1}C_\sigma$, and $C_\sigma$ depends only on $C_C, T_0$ and $\Omega$. For any $\varepsilon > 0$, there exists $\ell_0 = \ell_0(\varepsilon) \in \mathbb{N}$, such that whenever $\ell \geq \ell_0$,
\[ e_{u, \ell+1} \leq e_{u, \ell} - \frac{1}{2}E_{u, \ell} + \frac{1}{2}E_{\sigma, \ell} + \varepsilon e_{\ell+1}. \]

Proof. (4.5) follows from $(d(\sigma - \sigma_{\ell+1}), d(\sigma_\ell - \sigma_{\ell+1})) = 0$. (4.6) has been proved in Lemma 4.1, [30]. Using Corollary 3.3, we have
\[ v_\ell := E_\ell^{-\frac{1}{2}}(u_\ell - u_{\ell+1}) = d\varphi_\ell + z_\ell, \]
where
\[ \varphi_\ell = \pi_{\ell+1}\bar{K}_1^k v_\ell \in V_{\ell+1}^{k-1}, \quad z_\ell = \pi_{\ell+1}\bar{K}_2^k v_\ell \in V_{\ell+1}^k. \]

On the other hand, it follows from Corollary 4.4 that
\[ v_\ell \to 0 \text{ in } H\Lambda^k(\Omega). \]

Therefore the compact operators
\[ \bar{K}_1^k : H\Lambda^k(\Omega) \to L^2\Lambda^{k-1}(\Omega), \quad \bar{K}_2^k : H\Lambda^k(\Omega) \to L^2\Lambda^k(\Omega) \]
produce strongly convergent sequences in the $L^2$-norm:
\[ \|\bar{K}_1^k v_\ell\| \to 0 \text{ and } \|\bar{K}_2^k v_\ell\| \to 0 \text{ as } \ell \to \infty. \]

Due to $\|\pi_{\ell+1}\| \lesssim 1$, we obtain
\[ \|\bar{\varphi}_\ell\| \to 0 \text{ and } \|z_\ell\| \to 0 \text{ as } \ell \to \infty. \]

In particular, for any $\varepsilon > 0$, there exists $\ell_0 = \ell_0(\varepsilon)$, such that
\[ \|C_1^k \bar{\varphi}_\ell\| \leq \frac{\varepsilon}{4} \text{ and } \|z_\ell\| \leq \frac{\varepsilon}{4} \text{ whenever } \ell \geq \ell_0. \]

Using (4.8), (4.9), and
\[ \langle C(\sigma - \sigma_{\ell+1}), \varphi_\ell \rangle = \langle d\varphi_\ell, u_\ell - u_{\ell+1} \rangle, \]
we have
\[ \langle u_\ell - u_{\ell+1}, u_\ell - u_{\ell+1} \rangle = \langle C(\sigma - \sigma_{\ell+1}), \varphi_\ell \rangle E_\ell^{\frac{1}{2}} + \langle u_\ell - u_{\ell+1}, z_\ell \rangle E_\ell^{\frac{1}{2}} \]
\[ \leq \frac{\varepsilon}{4} \left( \|C_1^k (\sigma - \sigma_{\ell+1})\| + \|u_\ell - u_{\ell+1}\| \right) E_\ell^{\frac{1}{2}} \leq \frac{1}{8}E_\ell + \frac{1}{4}e_{\ell+1}, \quad \ell \geq \ell_0. \]
Similarly, for $\ell \geq \ell_0$,

$$
\langle d(u - u_{\ell+1}), d(u_\ell - u_{\ell+1}) \rangle = -\langle d(\sigma - \sigma_{\ell+1}), z_\ell \rangle E_\ell^2
\leq \frac{\varepsilon}{4} \|d(\sigma - \sigma_{\ell+1})\| E_\ell^2 \leq \frac{1}{8} E_\ell + \frac{\varepsilon}{8} e_{d\sigma,\ell+1}.
$$

(4.11)

Combining (4.10), (4.11) with $e_{u,\ell+1} = e_{u,\ell} - E_{u,\ell} + 2\langle u - u_{\ell+1}, u_\ell - u_{\ell+1} \rangle_V$, we obtain (4.7). 

The following estimator reduction result is standard, see, e.g., Corollary 3.4, [17].

**Lemma 4.6 (estimator reduction).** There exist constants $0 < \zeta < 1$ and $C_{\text{re}} > 0$ depending only on $\theta, n, T_0, C, \Omega$, such that

$$
E_{\ell+1} \leq \zeta E_\ell + C_{\text{re}} E_\ell.
$$

With the above preparations, we are able to prove the contraction of the estimator $E_\ell$ in AMFEM using the quasi-orthogonality results in Theorem 4.5, the continuous upper bound in Theorem 3.4, and Lemma 4.6. The authors in [25] first developed the estimator contraction technique based on the following-type quasi-orthogonality

$$
\|U - U_{\ell+1}\|^2 \leq \frac{1}{1 - \varepsilon} \|U - U_\ell\|^2 - \|U_{\ell+1} - U_\ell\|^2, \quad \ell \geq \ell_0(\varepsilon)
$$

for nonsymmetric 2nd order elliptic problems.

**Theorem 4.7 (estimator contraction).** There exist constants $0 < \gamma < 1$ and $C_{\text{conv}} > 0$ depending solely on $\zeta, C_{\text{up}}, C_{\text{re}}, C_{\varepsilon}$, and $\ell_0 = \ell_0(\varepsilon)$ with

$$
\varepsilon = \min\left(\frac{1 - \zeta}{16 C_{\text{up}} C_{\text{re}}}, \frac{1}{4}\right),
$$

such that for all $\ell \geq 0$ and $m \geq 1$, it holds that

$$
E_{\ell+m} \leq C_{\text{conv}} \gamma^m E_\ell.
$$

**Proof.** For any $J \geq \ell + 1$ and $\alpha = \frac{1 - \zeta}{2}$, it follows from Lemma 4.6 and (3.9) that

$$
\sum_{j=\ell+1}^{J} E_j \leq \sum_{j=\ell+1}^{J} \zeta E_{j-1} + C_{\text{re}} E_{j-1}
\leq \sum_{j=\ell+1}^{J} (\zeta + \alpha) E_{j-1} + C_{\text{re}} (E_{j-1} - \alpha C_{\text{re}}^{-1} C_{\text{up}}^{-1} e_{j-1}),
$$

and thus

$$
(1 - \zeta - \alpha) \sum_{j=\ell+1}^{J} E_j \leq (\zeta + \alpha) E_\ell + C_{\text{re}} \sum_{j=\ell}^{J-1} (E_j - \beta e_j),
$$

where $\beta = \alpha C_{\text{up}}^{-1} C_{\text{re}}^{-1}$. Setting $\varepsilon = \min(\frac{\beta}{2}, \frac{1}{4})$ in Theorem 4.5 and using (4.6)+(4.7), we obtain

$$
(1 - 2\varepsilon) e_{j+1} \leq e_j - \frac{1}{4} E_j + C_{\varepsilon} E_{d\sigma,j} \text{ for } j \geq \ell_0 = \ell_0(\varepsilon).
$$
Hence using the above quasi-orthogonality and (4.5), we have for $\ell \geq \ell_0$,

$$\sum_{j=\ell}^{J-1} (E_j - \beta e_j) \leq \sum_{j=\ell}^{J-1} (4e_j - 4(1-2\varepsilon)e_{j+1} + 4C_e e_{\delta,\sigma,j} - \beta e_j)$$

$$\leq (4 - \beta) \sum_{j=\ell}^{J-1} (e_j - e_{j+1}) + 4C_e \sum_{j=\ell}^{J-1} (e_{\delta,\sigma,j} - e_{\delta,\sigma,j+1})$$

$$\leq (4 - \beta)e_\ell + 4C_e e_{\delta,\sigma,\ell} \leq (4 - \beta + 4C_e)e_\ell.$$  

A combination of (4.12), (4.13), and $e_\ell \leq C_{up}E_\ell$ shows that for $\ell \geq \ell_0$,

$$\sum_{j=\ell+1}^\ell \mathcal{E}_j \leq C_1 \mathcal{E}_\ell,$$

where $C_1 = \zeta + \alpha + (4 - \beta + 4C_e)C_{up}C_{re}$. For $\ell < \ell_0$, $\mathcal{E}_\ell = 0$ implies that the algorithm AMFEM terminates at step $\ell$ and $\mathcal{E}_{\ell+1} = \mathcal{E}_{\ell+2} = \cdots = 0$. Hence we can simply take

$$C_{sup} := \max_{0 \leq \ell < \ell_0, \mathcal{E}_\ell \neq 0} \mathcal{E}_\ell^{-1} \sum_{j=\ell+1}^{\ell_0} \mathcal{E}_j < \infty$$

and obtain

$$\sum_{j=\ell+1}^\ell \mathcal{E}_j \leq C_{sup}\mathcal{E}_\ell.$$  

Let $C_2 := C_{sup} + (1 - \zeta - \alpha)^{-1}C_1$. Using (4.14) and (4.15), we have

$$\sum_{j=\ell+1}^\infty \mathcal{E}_j \leq C_2 \mathcal{E}_\ell$$

for all $\ell \geq 0$.

Therefore,

$$(1 + C_2^{-1}) \sum_{j=\ell+1}^\infty \mathcal{E}_j \leq \sum_{j=\ell+1}^\infty \mathcal{E}_j + \mathcal{E}_\ell = \sum_{j=\ell}^\infty \mathcal{E}_j$$

for all $\ell \geq 0$.

Using (4.17) and (4.16), it holds for $m \geq 1$ and $\ell \geq 0$ that

$$\mathcal{E}_{\ell+m} \leq \sum_{j=\ell+1}^\infty \mathcal{E}_j \leq (1 + C_2^{-1})^{-m+1} \sum_{j=\ell+1}^\infty \mathcal{E}_j \leq (1 + C_2^{-1})^{-m+1} C_2 \mathcal{E}_\ell.$$

Taking $\gamma = (1 + C_2^{-1})^{-1}$ and $C_{conv} = (1 + C_2^{-1})C_2$ completes the proof. \hfill \Box

It follows from (3.15), (3.9), and Theorem 3.5 that

$$\mathcal{E}_\ell \approx \|\sigma - \sigma_\ell\|_{V_\ell}^2 + \|u - u_\ell\|_{V_\ell}^2 + \text{osc}_{\ell+1}(f).$$

Therefore due to Theorem 4.7, there exists a constant $\tilde{C}_{conv}$ depending only on $C_C, T_0, \Omega$, and $C_{conv}$, such that for $\ell \geq 0, m \geq 1$,

$$\|\sigma - \sigma_{\ell+m}\|_{V_\ell}^2 + \|u - u_{\ell+m}\|_{V_\ell}^2 + \text{osc}_{\ell+m}(f)$$

$$\leq \tilde{C}_{conv}\gamma^m (\|\sigma - \sigma_\ell\|_{V_\ell}^2 + \|u - u_\ell\|_{V_\ell}^2 + \text{osc}_{\ell}(f)).$$
4.2. **quasi-optimality.** Let \( T \) denote the collection of all subtriangulations of \( T_0 \) produced by NVB. For \( T_\ell \in T \), and \( \tau \in V^{k-1}_\ell, v \in V^k_\ell \), define the total error
\[
E_\ell(\tau, v) = ||\sigma - \tau||_V^2 + ||u - v||_V^2 + \text{osc}_F^2(f),
\]
For \( s > 0 \), define the semi-norm
\[
||\sigma, u, f||_s = \sup_{N > 0} \left\{ \min_{T_\ell \in T} \min_{\tau, v} \min_{\ell \in V_\ell^{k-1}, v \in V_\ell^k} E_\ell(\tau, v)^{1 \over 2} \right\},
\]
and the approximation class
\[
A_s := \{(\tau, v, g) \in V^{k-1} \times V^k \times W^k : |(\tau, v, g)|_s < \infty\}.
\]
To specify the dependence of \( E_\ell \) on the \( \sigma_\ell, u_\ell \), let \( E_\ell(T) = E_\ell(\sigma_\ell, u_\ell, T) \) and \( E_\ell(\tau, v, T) \) be given by replacing \((\sigma_\ell, u_\ell)\) with \((\tau, v)\) in \( E_\ell(T) \). The following estimator perturbation result is standard, see, e.g., Proposition 3.3 in [17]. There exists a constant \( C_{\text{stab}} \) depending only on \( T_0 \), such that for \( \tau \in V^{k-1}_{\ell+1}, v \in V^{k}_{\ell+1} \) and \( \varepsilon > 0 \),
\[
\sum_{T \in T_\ell \cap T_{\ell+1}} E_\ell(\sigma_\ell, u_\ell, T) \leq (1 + \varepsilon) \sum_{T \in T_\ell \cap T_{\ell+1}} E_\ell(\tau, v, T) + (1 + \varepsilon^{-1})C_{\text{stab}}(||\sigma_\ell - \tau||_V^2 + ||u_\ell - v||_V^2).
\]
The marking parameter \( \theta \) is required to be below the threshold
\[
\theta_* = \frac{1}{1 + C_{\text{up}}C_{\text{stab}}},
\]
see, e.g., Lemma 5.5 in [25]. To prove optimality of AFEMs, Stevenson [42] assumed the collection of marked elements has minimal cardinality:
\[(4.19) \quad \text{Procedure MARK selects a subset } M_\ell \text{ with minimal cardinality.}\]
Recall that \( \{T_\ell\}_{\ell \geq 0} \) produced by AMFEM is a sequence of meshes generated by NVB. Assuming a matching condition on the initial mesh \( T_0 \), it has been shown in [9, 43] that the accumulated cardinality of marked elements can be controlled by
\[(4.20) \quad \#T_\ell - \#T_0 \leq \sum_{j=0}^{\ell-1} M_j.\]
Built upon the assumptions on MARK, marking parameter and using the contraction result, we obtain the quasi-optimal convergence rate of \( \{E_\ell\}_{\ell \geq 0} \), see, e.g., Theorem 5.3 in [25].

**Theorem 4.8** (quasi-optimality). Let \( \{(\sigma_\ell, u_\ell, T_\ell)\}_{\ell \geq 0} \) be a sequence of finite element solutions and meshes generated by Algorithm AMFEM. Assume \((\sigma, u, f) \in A_s \), \( 0 < \theta < \theta_* \), (4.19), (4.20), and the condition in Theorem 3.5 hold. There exists a constant \( C_{\text{opt}} \) depending only on \( \theta, \theta_*, T_0, C_C, \Omega \), and \( \gamma, C_{\text{conv}} \) such that
\[
E_\ell^{1 \over 2} \leq C_{\text{opt}}||((\sigma, u, f)||_s (\#T_\ell - \#T_0)^{-s}.
\]
Using the equivalence (4.18) and Theorem 4.8, we obtain the quasi-optimal convergence rate of AMFEM, that is, there exists \( \tilde{C}_{\text{opt}} \) depending only on \( C_C, T_0, \Omega \) and \( C_{\text{opt}} \), such that
\[
(||\sigma - \sigma_\ell||_V^2 + ||u - u_\ell||_V^2 + \text{osc}_F^2(f))^{1 \over 2} \leq \tilde{C}_{\text{opt}}||((\sigma, u, f)||_s (\#T_\ell - \#T_0)^{-s}.
\]
5. Applications

In this section, we present several important applications of the results obtained in Sections 3 and 4.

5.1. Hodge Laplace equation. For $1 \leq i \leq n$, let

$$ e_i = (-1)^{i+1} dx_1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n, $$

where $\widehat{dx^i}$ means that $dx^i$ is suppressed. There is a correspondence $j$ between $(n-1)$-forms and $\mathbb{R}^n$-valued functions:

$$ \sum_{i=1}^n \tau_i e_i \mapsto (\tau_1, \tau_2, \ldots, \tau_n). $$

On the other hand, an $n$-form can be identified with a scalar-valued function by

$$ v dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \mapsto h \mapsto v. $$

Using the bijections $j$, $h$ and the definition (2.9), $d^{n-1}$ is identified with the divergence operator $\text{div}$, the adjoint $\delta^n$ becomes the negative gradient $-\nabla$.

Given a vector space $V$, let $L^2(\Omega; V)$ denote the space of $V$-valued $L^2$-functions on $\Omega$. $V^{n-1}$ is isometric to $H(\text{div}; \Omega) = \{ \tau \in L^2(\Omega; \mathbb{R}^n) : \text{div} \tau \in L^2(\Omega) \}$ via $j$ and $V^n$ is isometric to $L^2(\Omega)$ via $h$. The Hodge Laplace equation (2.12) with index $k = n$ reads

$$ \begin{align*}
\sigma &= -\nabla u & \text{in } \Omega, \\
\text{div } \sigma &= f & \text{in } \Omega, \\
\sigma &= 0 & \text{on } \partial\Omega.
\end{align*} $$

Since $\text{div} : H(\text{div}; \Omega) \to L^2(\Omega)$ is surjective, $\mathcal{S}^n = \{0\}$ always vanishes on bounded Lipschitz domain $\Omega$.

Again using $j$ and $h$, $\mathcal{P}_r \Lambda^n(T_0) = \mathcal{P}_r^{-1} \Lambda^n(T_0)$ is identified with the space of piecewise polynomials of degree $\leq r$ without any continuity constraint,

$$ \mathcal{P}_r \Lambda^{n-1}(T_0) = \{ \tau \in H(\text{div}; \Omega) : \tau|_T \in \mathcal{P}_r(T; \mathbb{R}^n) \text{ for all } T \in T_0 \}, $$

where $\mathcal{P}_p(T; \mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued polynomials of degree $\leq p$. The mixed method (2.7) with index $k = n$ and $V^{n-1}_T \times V^n_T = \mathcal{P}_r \Lambda^{n-1}(T_0) \times \mathcal{P}_r \Lambda^n(T_0)$ or $V^{n-1}_T \times V^n_T = \mathcal{P}_r \Lambda^{n-1}(T_0) \times \mathcal{P}_r \Lambda^n(T_0)$ is indeed the RT or BDM element method, respectively. The $V^{n-1}$-norm is the $H(\text{div})$-norm and $V^n$-norm is simply the $L^2$-norm. Therefore Theorem 4.8 shows that $(\sigma, u)$ converges to $(-\nabla u, u)$ with quasi-optimal convergence rate in the $H(\text{div}) \times L^2$-norm:

$$ (\|\sigma - \sigma_e\|_{H(\text{div})}^2 + \|u - u_e\|^2)^{1/2} \lesssim \mathcal{E}^2, \quad \mathcal{E} \lesssim C_{\text{opt}}(\|\sigma, u, f\|_{s}(|\#T_{0} - \#T_{0})^{-s}). $$

The identification of $V^k$ and $d^k, \delta^{k+1}$ with $k \leq n-2$ depends on the dimension of $\mathbb{R}^n$. For example, the $(n-2)$-forms in $\mathbb{R}^2$ are automatically scalar-valued functions; the $(n-2)$-forms in $\mathbb{R}^3$ are realized by

$$ v_1 dx^1 + v_2 dx^2 + v_3 dx^3 \mapsto (v_1, v_2, v_3). $$

Using $s$ and $j$, we have $d^{n-2} = \text{curl}, \delta^{n-1} = \text{rot}$, where

$$ \text{curl } v = \begin{cases} \\
\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1} \text{ (if } n = 2), \\
\nabla \times v \text{ (if } n = 3) \end{cases}, \quad \text{rot } \tau = \begin{cases} \\
\frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2} \text{ (if } n = 2), \\
\nabla \times \tau \text{ (if } n = 3) \end{cases}. $$
$V^{n-2}$ is identified with

$$H^{1}(\Omega) = \begin{cases} H^{1}(\Omega), & n = 2, \\ \{ \phi \in L^{2}(\Omega; \mathbb{R}^{3}) : \text{curl} \phi \in L^{2}(\Omega; \mathbb{R}^{3}) \}, & n = 3. \end{cases}$$

When $n = 2$ or 3, the $L^{2}$-de Rham complex (2.10) reduces to the well-known complexes

$$H^{1}(\Omega) \xrightarrow{\text{curl}} H^{1}(\text{div}; \Omega) \xrightarrow{\text{div}} L^{2}(\Omega) \text{ in } \mathbb{R}^{2},$$

$$H^{1}(\Omega) \xrightarrow{\nabla} H^{1}(\text{curl}; \Omega) \xrightarrow{\text{curl}} H^{1}(\text{div}; \Omega) \xrightarrow{\text{div}} L^{2}(\Omega) \text{ in } \mathbb{R}^{3}.$$ 

Given a face $S$ in $T_{t}$, $\text{tr}_{|S \times T}^{n} = T_{t}$ is the tangential trace of $T$, where

$$T_{t} = T \times n \text{ when } n = 2, \quad T_{t} = T \times n \text{ when } n = 3.$$ 

Here $t$ and $n$ are unit tangent and normal to the face $S$, respectively. The error indicator $\mathcal{E}_{T}$ for Poisson's equation reads

$$\mathcal{E}_{T}(T) = h^{2}_{T} \| \sigma_{T}^{e} + \nabla u_{T} \|_{T}^{2} + h_{T} \| u_{T} \|_{T}^{2} + h_{T} \| \sigma_{T}^{e} \|_{T}^{2} + \| f - \text{div} u_{T} \|_{T}^{2},$$

which controls the error $\| \sigma - \sigma_{T}^{e} \|_{H(\text{div})}^{2} + \| u - u_{T} \|^{2}$. Readers are referred to [10, 14] for other error estimators controlling the $H(\text{div}) \times L^{2}$-error.

We briefly describe other Hodge Laplace equations in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$. The Hodge Laplacian problem (2.13) with index $k \leq n - 1$ and $n = 2, 3$ reduces to the vector Laplacian problem:

$$\Delta u = f \text{ in } \Omega,$$

$$u \cdot t = 0, \text{div } u = 0 \text{ on } \partial \Omega \text{ when } k = 2, n = 3 \text{ or } k = 1, n = 2,$$

$$u \cdot n = 0, \text{curl } u \cdot t = 0 \text{ on } \partial \Omega \text{ when } k = 1, n = 3.$$ 

In the case that $\mathcal{S}^{k}_{T} = \{ 0 \}$, Theorem 4.8 confirms the quasi-optimal convergence rate of AMFEM for solving (5.2) using pairs (2.15). $P_{r}^{-} \Lambda^{0}(T_{t}) = P_{r} \Lambda^{0}(T_{t})$ is the nodal finite element space of degree $r$. In $\mathbb{R}^{3}$, $P_{r+1} \Lambda^{1}(T_{t})$ and $P_{r+1} \Lambda^{1}(T_{t})$ are called Nédélec edge finite element spaces [35, 36] in the classical context.

5.2. Pseudostress-velocity formulation of the Stokes equation. Given $f \in L^{2}(\Omega; \mathbb{R}^{n})$, the Stokes problem is to find $u$ and $p$ with $u|_{\partial \Omega} = 0$, $\int_{\Omega} p \, dx = 0$ satisfying

$$-\Delta u + \nabla p = f \text{ in } \Omega,$$

$$\text{div } u = 0 \text{ in } \Omega.$$ 

The pseudostress is $\sigma := -\nabla u + p I_{n}$, where $\nabla$ denotes the row-wise gradient and $I_{n}$ is the $n \times n$ identity matrix. The operator $\mathcal{C}$ is given by

$$\mathcal{C} \sigma := \sigma - \frac{1}{n} \text{Tr}(\sigma) I_{n},$$

where $\text{Tr}$ is the trace operator for square matrices. It is readily checked that $\mathcal{C}$ is continuous and self-adjoint. Let $\text{Div}$ be the row-wise divergence for matrix-valued functions. The Stokes problem is equivalent to the pseudostress-velocity formulation (see, e.g., [13])

$$\mathcal{C} \sigma = -\nabla u \text{ in } \Omega,$$

$$\text{Div } \sigma = f \text{ in } \Omega,$$
where \( u|_{\partial \Omega} = 0 \) and \( \sigma \) satisfies the compatibility condition \( \int_{\Omega} \tau \cdot \nabla \sigma \, dx = 0 \). Let

\[
V^{n-1} = \{ \tau \in L^2(\Omega; \mathbb{R}^{n \times n}) : \text{Div} \tau \in L^2(\Omega; \mathbb{R}^n), \int_{\Omega} \text{Tr}(\tau) \, dx = 0 \}
\]

and \( V^n = L^2(\Omega; \mathbb{R}^n) \). The mixed variational formulation seeks find \( \sigma \in V^{n-1} \) and \( u \in V^n \) satisfying

\[
\begin{align*}
\langle C \sigma, \tau \rangle - \langle \text{Div} \tau, u \rangle &= 0, \quad \tau \in V^{n-1}, \\
\langle \text{Div} \sigma, v \rangle &= \langle f, v \rangle, \quad v \in V^n.
\end{align*}
\] (5.4)

It has been shown in [2] that

\[
\| \sigma \| \lesssim \| C^2 \sigma \| + \| \text{Div} \sigma \|_{H^{-1}(\Omega)}.
\]

Combining it with \( \| \text{Div} \sigma \|_{H^{-1}(\Omega)} \leq \| \text{Div} \sigma \| \) and the continuity of \( C \), the assumption (2.6) is verified.

For a vector- or matrix-valued function \( v \), let \( v_i \) denote the \( i \)-th entry or \( i \)-th row of \( v \), respectively. Let \( V^n = \{ v \in L^2(\Omega; \mathbb{R}^n) : v_i \in V^n, i = 1, \ldots, n \} \) and \( V^{n-1} = \{ \tau \in L^2(\Omega; \mathbb{R}^{n \times n}) : \tau_i \in V^{n-1}, i = 1, \ldots, n, \int_{\Omega} \tau \cdot dx = 0 \} \), where \( V^n \) and \( V^{n-1} \) are given in Subsection 5.1. The mixed method for (5.4) seeks \( \sigma_i \in V^{n-1}_i \) and \( u_i \in V^n \) satisfying

\[
\begin{align*}
\langle C \sigma_i, \tau_i \rangle - \langle \text{Div} \tau_i, u_i \rangle &= 0, \quad \tau_i \in V^{n-1}_i, \\
\langle \text{Div} \sigma_i, v_i \rangle &= \langle f, v_i \rangle, \quad v_i \in V^n.
\end{align*}
\] (5.5)

Let \( L^2 \Lambda^k(\Omega; \mathbb{R}^n) \) denote the space of all \( \mathbb{R}^n \)-valued \( k \)-forms \( \omega \), namely,

\[
\omega = \sum_{1 \leq \alpha_1 < \ldots < \alpha_k \leq n} \omega_{\alpha} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k},
\]

where each \( \omega_{\alpha} \in L^2(\Omega; \mathbb{R}^n) \). The theory of de Rham complex in subsection 2.3 can be directly extended to the vector-valued case. There is a natural correspondence \( j \) between \( \mathbb{R}^n \)-valued \((n-1)\)-forms and \( n \times n \) matrix-valued functions:

\[
\sum_{i=1}^n \tau_i e_i \begin{pmatrix} \cdots \end{pmatrix} \rightarrow (\tau_1, \tau_2, \ldots, \tau_n)^T.
\]

Let \( d^{n-2} : L^2 \Lambda^{n-2}(\Omega; \mathbb{R}^n) \rightarrow L^2 \Lambda^{n-1}(\Omega; \mathbb{R}^n) \) denote the exterior derivative for vector-valued forms, \( D = j \circ d^{n-2} \) and

\[
V^{n-2} = \{ v \in L^2 \Lambda^{n-2}(\Omega; \mathbb{R}^n) : \text{Div} v \in V^{n-1} \}.
\]

As in the scalar case, it is readily checked that the following is a closed Hilbert complex:

\[
V^{n-2} \xrightarrow{D} V^{n-1} \xrightarrow{\text{Div}} V^n.
\] (5.6)

Let \( V^{n-2}_i = \{ v \in V^{n-2} : v_i \in V^{n-2}_i, i = 1, \ldots, n \} \). The discrete subcomplex reads

\[
V^{n-2}_i \xrightarrow{D} V^{n-1}_i \xrightarrow{\text{Div}} V^n_i.
\] (5.7)

The surjectivity of \( \text{Div} : V^{n-1} \rightarrow V^n \) implies the \( n \)-th cohomology group \( H^n \) vanishes. The regular decomposition in Theorem 3.2 can be applied to each row of test functions in \( V^n \) and \( V^{n-1} \).

To apply the theory in Sections 3 and 4, it suffices to construct a local \( V \)-bounded cochain projection \( \Pi_\ell \) from (5.6) to (5.7) as well as an \( L^2 \)-bounded cochain projection \( \Pi_\ell \) from

\[
L^2 \Lambda^{n-2}(\Omega; \mathbb{R}^n) \xrightarrow{D} L^2(\Omega; \mathbb{R}^{n \times n}) \xrightarrow{\text{Div}} L^2(\Omega; \mathbb{R}^n)
\]
to (5.7); compare with Theorem 3.1 and Corollary 3.3. In fact, \(\Pi_\ell\) and \(\bar{\Pi}_\ell\) can be contracted from \(\pi_\ell\) and \(\bar{\pi}_\ell\), respectively. As described in subsection 5.1, \(\pi_\ell^{n-1}\) and \(\pi_\ell^n\) can be applied to functions in \(H(\text{div}; \Omega)\) and \(L^2(\Omega)\), respectively. Let \(\pi_\ell^{n-1}\) and \(\pi_\ell^n\) denote the row-wise version of \(\pi_\ell^{n-1}\) and \(\pi_\ell^n\), respectively. Let \(\Pi_\ell^n = \pi_\ell^n : V^n \rightarrow V^n_\ell\) and \(\Pi_\ell^{n-1} : V^{n-1} \rightarrow V^{n-1}_\ell\) be defined by

\[
\Pi_\ell^{n-1} \tau = \pi_\ell^{n-1} \tau - \frac{I_n}{n|\Omega|} \int_\Omega \text{Tr}(\pi_\ell^{n-1} \tau) dx
\]

so that \(\Pi_\ell^{n-1}(V^{n-1}) = V^{n-1}_\ell\). Note that \(L^2\Lambda^{n-2}(\Omega; \mathbb{R}^n) = [L^2\Lambda^{n-2}(\Omega)]^n\), the Cartesian product of \(L^2\Lambda^{n-2}(\Omega)\) with \(n\) copies. We can take \(\pi_\ell^{n-2} : L^2\Lambda^{n-2}(\Omega; \mathbb{R}^n) \rightarrow [V^{n-2}_\ell]^n\) to be the component-wise version of \(\pi_\ell^{n-2}\). For \(w \in V^{n-2}\), let

\[
\Pi_\ell^{n-2} w := \pi_\ell^{n-2} w - \frac{\mu}{n(n-1)|\Omega|} \int_\Omega \text{Tr}(D\pi_\ell^{n-2} w) dx,
\]

where \(\mu = (\kappa e_1, \kappa e_2, \ldots, \kappa e_n)^T\) with \(\kappa\) given in (2.14). For example,

\[
\mu = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \quad \text{when } n = 2, \quad \mu = \frac{1}{2} \begin{pmatrix} x_2 dx^3 - x_3 dx^2 \\ x_3 dx^1 - x_1 dx^3 \\ x_1 dx^2 - x_2 dx^1 \end{pmatrix} \quad \text{when } n = 3.
\]

Using the formula \((dx + \kappa d)e_i = (n-1)e_i\) (cf. Theorem 3.1 in [3]) and \(de_i = 0\) with \(1 \leq i \leq n\), we have \(d^{n-2} \mu = (n-1)(e_1, e_2, \ldots, e_n)^T\) and thus

\[
D\mu = (n-1)I_n, \quad \int_\Omega \text{Tr}(D\Pi_\ell^{n-2} w) dx = 0.
\]

Combining it with \(\kappa e_i \in P_1^- \Lambda^{n-2}(T_i) \subseteq V^{n-2}_\ell\), we have \(\Pi_\ell^{n-2} w \in V^{n-2}_\ell\). Note that \(\Pi_\ell\) is simply obtained by subtracting a global constant or differential form from \(\pi_\ell\). Therefore \(\Pi_\ell\) is a local \(V\)-bounded cochain projection satisfying the properties in Theorem 3.1. The \(L^2\)-bounded projection \(\Pi_\ell\) can be constructed in the same way using \(\bar{\pi}_\ell\).

Let

\[
E_\ell(T) = h_T^2 \left| C\sigma_\ell + \nabla u_\ell \right|_{L^2}^2 + h_T \left| \left\| u_\ell \right\|_{L^2} \right|_{L^2}^2 + h_T^2 \left| D^* C\sigma_\ell \right|_{L^2}^2 + h_T \left| \text{tr} x^{-1}(\sigma_\ell) \right|_{L^2}^2 + \left| \text{Div} \sigma_\ell \right|_{L^2}^2,
\]

where \(D^*\) is the adjoint of \(D\). In \(\mathbb{R}^2\) or \(\mathbb{R}^3\), \(D^* = \text{Rot}\), where Rot is the row-wise version of rot; \(\text{tr} x^{-1}(\sigma_\ell)\) is the row-wise tangential trace of \(\sigma_\ell\). It follows from Theorems 3.4 and 3.5 that \(E_\ell\) is reliable and efficient for controlling \(\|C^{1/2}(\sigma - \sigma_\ell)\|^2 + \|\text{Div}(\sigma - \sigma_\ell)\| + \|u - u_\ell\|^2\). Theorem 4.8 shows that AMFEM for (5.4) has quasi-optimal convergence rate:

\[
\left( \|C^{1/2}(\sigma - \sigma_\ell)\|^2 + \|\text{Div}(\sigma - \sigma_\ell)\| + \|u - u_\ell\|^2 \right)^{1/2} \leq E_\ell^{1/2} \leq C_{\text{opt}} \left( \|\sigma, u, f\|_{s} (\# T_\ell - \# T_0)^{-s} \right).
\]

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