Almost–homogeneity of the Universe in Higher–order Gravity

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Abstract

In the \( R + \alpha R^2 \) gravity theory, we show that if freely propagating massless particles have an almost isotropic distribution, then the spacetime is almost Friedmann–Robertson–Walker (FRW). This extends the result proved recently in general relativity (\( \alpha = 0 \)), which is applicable to the microwave background after photon decoupling. The higher–order result is in principle applicable to a massless species that decouples in the early universe, such as a relic graviton background. Any future observations that show small anisotropies in such a background would imply that the geometry of the early universe were almost FRW.

1 Introduction

Recently it has been shown [1] that the observed almost–isotropy of the cosmic microwave background radiation (CMBR) implies that the universe since photon decoupling is almost spatially homogeneous and isotropic. This is proof of the stability under perturbation of the original exact Ehlers–Geren–Sachs (EGS) theorem [2], and shows that almost–isotropy of the CMBR is the foundation of the assertion that the universe is almost a Friedmann–Robertson–Walker (FRW) spacetime. (Indeed the analysis of

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the CMBR via the Sachs–Wolfe effect begins with the assumption that the universe is almost–FRW since decoupling.)

In this paper we extend the result of [1] to a radiation–dominated universe where gravity obeys the higher–order theory with Lagrangian $\mathcal{L} = R + \alpha R^2$. In an earlier paper [3] we showed that the exact EGS theorem holds also in the $\alpha \neq 0$ theory, i.e. if the CMBR is exactly isotropic for a congruence of freely falling observers, then in $R + \alpha R^2$ gravity the spacetime is FRW.

The corrections introduced by the $\alpha R^2$ term in the Lagrangian should be dominant in the very early universe. In fact the higher–order term can generate inflationary expansion without invoking an extraneous inflaton field [4,5]. This paper investigates the effect this term may have on massless particles which decoupled in the early universe. A possible application is the relic graviton radiation, which decoupled around the Planck time and has since not interacted significantly with matter or radiation. In principle the gravitational background radiation provides a link to the spacetime geometry of the very early universe - although in practice any observation of this background radiation seems a very distant possibility.

This radiation would inevitably contain anisotropies that carry an imprint of perturbations at the decoupling time. These anisotropies would be preserved in the subsequent free propagation of the radiation. The result proved here shows that if any gravitational background radiation is observed in the late universe to be almost isotropic, and if gravity is governed by the higher–order theory soon after Planck time, then the spacetime geometry at that time is almost–FRW. If the higher–order corrections are neglected, the result holds for the special case of general relativity ($\alpha = 0$), and represents an extension of [1] from a matter–dominated to a radiation–dominated universe.

A second application, where the observational possibilities are much greater, but the significance of higher–order effects much less, is a (massless) neutrino background after decoupling. Anisotropies in this background would carry information about the spacetime geometry at $t \approx 1$ s. An estimate of the relative size of the higher–order terms at neutrino decoupling depends on the limits placed on the coupling constant $\alpha$. In [5], a cosmological model is constructed with an $\alpha R^2$ inflationary epoch, followed by an oscillatory phase (when re–heating takes place), which leads into the standard Friedmann radiation era. In this model, the inflationary constraints (e.g. on density perturbations) lead to the limits

$$10^{22} \text{GeV}^2 < \alpha^{-1} < 10^{26} \text{GeV}^2.$$
According to these limits, we calculate that the relative correction to the Einstein Lagrangian at the start of the Friedmann era is at most

$$\alpha R \approx 10^{-14}.$$ 

This is effectively negligible, although the value is model-dependent. It is more realistic to take general relativity as holding at neutrino decoupling. Then our result for $\alpha = 0$ shows that an almost isotropic neutrino distribution implies an almost FRW spacetime at neutrino decoupling.

There may be other massless species that decoupled before neutrinos, interact very weakly, and have yet to be detected. Finally, there is also a more mathematical motivation for proving an almost–EGS theorem when $\alpha \neq 0$. We can consider the higher–order Lagrangian as a ‘perturbation’ of the general relativity Lagrangian, and we show that the almost–EGS result is ‘stable’ under this kind of perturbation, having previously shown [3] that the exact EGS result is.

For convenience, we summarise the theorems derived in [1–3]. Consider a congruence of freely falling observers in an expanding universe, measuring freely propagating radiation (massless particles).

(i) EGS Theorem: If the radiation is exactly isotropic, then in general relativity the spacetime is exactly FRW [2].

(ii) Almost–EGS Theorem: If the radiation is almost isotropic in a matter–dominated universe, then in general relativity the spacetime is almost FRW [1].

(iii) Higher–order EGS Theorem: If the radiation is exactly isotropic, then in $R + \alpha R^2$ gravity the spacetime is exactly FRW [3].

In section 4 we prove:

(iv) Higher–order Almost–EGS Theorem: If the radiation is almost isotropic in a radiation–dominated universe, then in $R + \alpha R^2$ gravity the spacetime is almost FRW.

The proof of (iii) above rests on showing that the scalar curvature is spatially homogeneous. This follows in [3] from the field equations only after detailed calculation. Consequently, the proof of (iv) rests on the requirement that the Ricci scalar curvature is almost spatially homogeneous. This is the main part of the generalisation of the general relativity theorem and is shown in sections 4.1 and 4.2. Consequently we can invoke theorem (iii) in our proof that the metric is almost isotropic and spatially homogeneous in section 4.3. In order to make the discussion as self–contained as possi-
ble, we briefly review in section 2 the necessary facts from the covariant formalism for analysing small anisotropies in radiation [1], and in section 3 we summarise the main points from [3] about the higher–order equations.

**Notation:** We follow [1]. The metric $g_{ab}$ has signature $(-,+,+,+)$. Einstein’s gravitational constant and the speed of light in vacuum are 1. Round brackets on indices denote symmetrisation, square brackets anti–symmetrisation. $\nabla_a$ is the covariant derivative defined by $g_{ab}$. Given a four–velocity $u^a$, the associated projection tensor is $h_{ab} = g_{ab} + u_a u_b$, and the comoving time derivative and spatial gradient are

$$\dot{Q}_{a...b} \equiv u^c \nabla_c Q_{a...b},$$

$$^3 \nabla_c Q_{a...b} \equiv h_c^d h_a^e \ldots h_b^f \nabla_d Q_{e...f}$$

for any tensor $Q_{a...b}$. Given a smallness parameter $\epsilon$, $O[N]$ denotes $O(\epsilon^N)$ and $A \simeq B$ means $A - B = O[2]$.

## 2 Covariant analysis of radiation anisotropy

Following [1], we do not assume a background metric, but start from the real spacetime with almost isotropic radiation, and proceed to show that the real metric is close to an FRW metric. Of course, it follows from the higher–order EGS theorem (iii) that if the anisotropy vanishes, then the metric is FRW.

We assume that the universe is radiation–dominated at the time of de-coupling of the massless particles whose distribution is almost isotropic. The process of decoupling leads to anisotropies in the decoupled species, and probably also in the other massless or ultra–relativistic species. After decoupling, the freely propagating species preserves its anisotropy, while any anisotropy in the other species is rapidly removed by collisions.

A unique physical four–velocity $\tilde{u}^a$ is defined as the unit normal field to the surfaces $\{\bar{\mu} = \text{constant}\}$, where $\bar{\mu}$ is the energy density of the thermalised massless (or ultra–relativistic) species. The distribution of the decoupled species will appear as almost isotropic to ‘observers’ comoving with $\tilde{u}^a$. This field is irrotational, but in general not geodesic. Inspection of the proof in [1] and of the higher–order equations in [3] shows that it is easier to work with a geodesic but rotating four–velocity $u^a$, which may be chosen close to $\tilde{u}^a$ on an initial surface. Then the difference between the two vectors will remain $O[1]$ provided that the spatial gradient of the energy density $\bar{\mu}$ relative to
$u^a$ is $O[1]$, which is proved in section 4 (see (24)). So the anisotropy measured by freely falling ‘observers’ comoving with $u^a$ is also small. Thus, although $u^a$ is not uniquely and physically defined, the $O[1]$ nature of anisotropies relative to $u^a$, and conclusions based on that, are covariant.

The congruence is necessarily expanding since it must ‘track’ the expanding $\hat{u}^a$. Thus $u^a$ satisfies

$$u^a \equiv \dot{u}^a = 0, \quad \Theta \equiv u^a_{\cdot a} > 0.$$  (1)

The rate of expansion defines a Hubble rate $H$ and average scale factor $S$ by \(\Theta = 3H = 3\dot{S}/S\). The kinematics of the congruence is determined by $\Theta$ and the shear $\sigma_{ab}$ and vorticity $\omega_{ab}$.

Now $u^a$ defines an invariant 3+1 splitting of tensors [1,6]. In particular, for a massless particle four-momentum

$$p^a = E(u^a + e^a), \quad e_a u^a = 0, \quad e_a e^a = 1,$$

where $E$ is the energy and $e_a$ the direction of momentum, relative to the ‘observers’ above. After decoupling, the total radiation distribution function is

$$f_{\text{tot}}(x^c, E, e^d) = \bar{f}(x^c, E) + f(x^c, E, e^d)$$

where $\bar{f}$ is a collision–dominated Planckian distribution describing the thermalised species, while $f$ is the distribution function of the decoupled species. This distribution function may be expanded as [1,7]

$$f(x^c, E, e^d) = F(x^c, E) + F_a(x^c, E)e^a + F_{ab}(x^c, E)e^ae^b + \ldots$$  (2)

where the covariant multipole moments $F_{a_1\ldots a_L}(x^c, E)$ for $L \geq 1$ are symmetric trace–free tensors orthogonal to $u^a$, that provide a measure of the deviation of $f$ from exact isotropy (as measured by $u^a$). If the decoupled radiation is almost isotropic, then [1]:

$$F, \quad \dot{F} = O[0], \quad F_{a_1\ldots a_L}, \quad \nabla_b F_{a_1\ldots a_L} = O[1] \quad (L \geq 1).$$  (3)

Energy integrals of the first three moments define the decoupled radiation energy density, energy flux and anisotropic stress:

$$\mu = 4\pi \int_0^\infty E^3 F dE = 3p = O[0],$$  (4)

$$q_a = \frac{4\pi}{3} \int_0^\infty E^3 F_a dE = O[1],$$  (5)

$$\pi_{ab} = \frac{8\pi}{15} \int_0^\infty E^3 F_{ab} dE = O[1].$$  (6)
(In [1] $\mu_R$ is used for $\mu$.) We will also need the integral of the octopole moment:

$$\xi_{abc} = \frac{8\pi}{35} \int_0^\infty E^3 F_{abc} dE = O[1],$$

(7)

The total radiation energy–momentum tensor is

$$T_{ab} = (\mu + \mathbf{u}_a \mathbf{u}_b + \frac{1}{3}(\mu + \mathbf{h}_{ab}) + \pi_{ab} + 2\mathbf{u}_{(a}q_{b)}.$$

(8)

Since the decoupled and thermalised species are non–interacting, they separately obey the conservation equations:

$$\dot{\mu} + \frac{4}{3} \mu \Theta = 0,$$

(9)

$$\dot{\mu} + \frac{1}{3} \mu \Theta + \pi_{ab} \omega^{ab} + 3 \nabla_a q^a = 0,$$

(10)

$$\dot{q}_a + (\omega_{ab} + \sigma_{ab} + \frac{1}{3} \Theta h_{ab}) q^b + \frac{1}{3} (3 \nabla_a \mu) + 3 \nabla_b \pi^{ab} = 0,$$

(11)

where we have used (1).

After decoupling, the radiation obeys the equilibrium Boltzmann equation

$$L(f_{tot}) = 0,$$

where $L(\bar{f})$ vanishes because of detailed balancing of the collisions in the thermalised component, and $L(f)$ vanishes because the decoupled species is collision–free:

$$L(f) \equiv p^a \frac{\partial f}{\partial x^a} - \Gamma_{bc}^a p^b p^c \frac{\partial f}{\partial p^a} = 0.$$

(12)

Consequently $q_a$ and $\pi_{ab}$ in (8) are not dissipative quantities, but measure the deviation of $f$ from isotropy. From (3), (5) and (6), it follows that

$$\dot{q}_a \ , \ 3 \nabla_b q_a = O[1] \ , \ \dot{\pi}_{ab} \ , \ 3 \nabla_c \pi_{ab} = O[1].$$

(13)

The Liouville equation (12) may also be covariantly decomposed into multipole moments. The monopole moment is (10), the dipole moment is (11), and the quadropole moment involves $\xi_{abc}$ and is given in section 4 in linearised form (see (19)).

### 3 Higher–order equations

The field equations and Ricci and Bianchi identities in higher–order gravity are given in general 3+1 form in [3]. Quantum corrections to the gravitational Lagrangian which
yield the simplest higher–order field equations are of the form

\[ \mathcal{L} = R + \alpha R^2, \]

where \( \alpha \) is a coupling constant. The field equations derived from this are

\[ R_{ab} - \frac{1}{2} R g_{ab} + 2\alpha \left[ R(R_{ab} - \frac{1}{2} R g_{ab}) + g_{ab} \Box R - R_{ab} \right] = T_{ab}, \]

where \( \Box = g^{ab} \nabla_a \nabla_b \) and \( T_{ab} \) is the radiation energy–momentum tensor (8). Their 3+1 splitting is

\[ R_{ab} u^a u^b = \left( 1 + 2\alpha R \right)^{-1} \left[ (\bar{\mu} + \mu) - \frac{1}{2} R(1 + \alpha R) + 2\alpha \Box R + 2\alpha R_{ab} u^a u^b \right], \]

\[ R_{ab} u^a h^b_c = \left( 1 + 2\alpha R \right)^{-1} \left[ -q_c + 2\alpha R_{ab} u^a h^b_c \right], \]

\[ R_{ab} h^a c h^b d = \left( 1 + 2\alpha R \right)^{-1} \left[ \left\{ \frac{1}{3}(\bar{\mu} + \mu) + \frac{1}{2} R(1 + \alpha R) - 2\alpha \Box R \right\} h_{cd} + \right. \]

\[ \left. + \pi_{cd} + 2\alpha R_{ab} h^a c h^b_d \right], \]

while their trace is (using (8))

\[ R = 6\alpha \Box R. \]

Together with the Ricci and Bianchi identities, these give the constraint and evolution equations governing a radiation spacetime in higher–order gravity. These equations are given in full in [3], and will be quoted when needed.

The FRW metric is

\[ ds^2 = -dt^2 + S^2(t) \left( \frac{dr^2}{(1 - kr^2)} + r^2 d\Omega^2 \right), \]

where \( k = 0, \pm 1 \). The field equations for this metric are [5]

\[ \frac{2\dot{S}}{S} + \frac{(\dot{S}^2 + k)}{S^2} = -\frac{1}{3}(\bar{\mu} + \mu) - 2\alpha \left[ 9 \left\{ \frac{\dot{S}}{S} + \frac{(\dot{S}^2 + k)}{S^2} \right\}^2 + \ddot{R} + \frac{2\dot{S}\dot{R}}{S} \right], \]

\[ \frac{3(\dot{S}^2 + k)}{S^2} = (\bar{\mu} + \mu) - 2\alpha \left[ 9 \left\{ \frac{(\dot{S}^2 + k)^2}{S^4} - \frac{\dot{S}^2}{S^2} \right\} + \frac{3\dot{S}\dot{R}}{S} \right]. \]

Equation (16) is the higher–order Friedmann equation for radiation and is compatible with (15) through the conservation equations (9), (10). In general relativity, the FRW
spacetime is uniquely characterised by the existence of a geodesic, expanding four-velocity with zero vorticity and shear. In higher-order gravity, this is also true for radiation, but not in general (i.e. not without further conditions) [3]. The four-velocity is then the unique four-velocity normal to the spatial hypersurfaces and with respect to which the radiation is exactly isotropic.

4 Almost–EGS theorem in higher–order gravity

Following the approach of [1], and given theorem (iii), we must show that if the deviations from isotropy in $f$ are small, i.e. if (3) holds, then: (a) all covariant quantities that vanish in FRW spacetime are small; (b) the metric can be put into a perturbed FRW form.

It is clear that any covariant quantity which is non-zero in the exactly isotropic (and therefore FRW) case is an $O[0]$ quantity. However, if a covariant quantity vanishes in the exactly isotropic (FRW) case, it is not necessarily $O[1]$. That is precisely what has to be proven.

We follow the arguments of [1], explaining the deviations in the proof of the theorem caused by the higher–order equations. We will only summarise the proof where the equations are independent of the field equations used, and include details where they are not.

4.1 Almost–FRW kinematics

The nature of the radiation kinematic quantities is independent of the field equations and follows from the covariant multipole decomposition of the Liouville equation (12). The zero and first moments are the energy and momentum conservation equations (10) and (11) respectively. The momentum conservation equation (11), together with (13), implies that

$$3\nabla_a \mu = O[1],$$

and any of its derivatives are also at least $O[1]$. The definition of $3\nabla_a$ leads to the following identity [1]:

$$\left( 3\nabla_a 3\nabla_b - 3\nabla_b 3\nabla_a \right) \mu = -2\omega_{ab} \dot{\mu}.$$
By (3) and (17) this implies that
\[ \omega_{ab} = O[1] . \]  
(18)
The quadrupole moment of (12) is the evolution equation for the anisotropic stress tensor \( \pi_{ab} \), which has linearised form [1]
\[ \dot{\pi}_{ab} + \frac{4}{3} \Theta \pi_{ab} + \frac{8}{15} \mu \sigma_{ab} + 2 \left\{ 3 \nabla_{<a} q_{b>} \right\} + 3 \nabla_{c} \xi_{ab} \approx 0 , \]  
(19)
where the angled brackets denote the spatially projected, trace–free and symmetrised part of the enclosed indices, i.e.
\[ S_{<ab>} \equiv \left[ h_{(a} h_{b)}^d - \frac{1}{3} h_{a b} h^{c d} \right] S_{c d} , \]
for any \( S_{ab} \). Using (3), (13) and (18), (19) implies
\[ \sigma_{ab} = O[1] , \]  
(20)
and similarly for all of its derivatives. The linearised conservation equations are thus
\[ \dot{\mu} + \frac{4}{3} \mu \Theta + 3 \nabla_{a} q^{a} \approx 0 , \]  
(21)
\[ \dot{q}_{a} + \frac{4}{3} \Theta q_{a} + \frac{1}{3} \left( 3 \nabla_{a} \mu \right) + 3 \nabla_{b} \pi^{b}_{a} \approx 0 . \]  
(22)
Taking spatial gradients of (21) and using (4), (5) and (17), we obtain
\[ 3 \nabla_{a} \Theta = O[1] . \]  
(23)
Equation (23) and the spatial gradient of (9) then lead to
\[ 3 \nabla_{a} \dot{\mu} = O[1] . \]  
(24)
Equation (23) is crucial to proving almost–homogeneity of the scalar curvature. The higher–order \( R_{b c} u^{b} h^{c a} \) field equation is the constraint equation [3]
\[ h^{ab} \left( \frac{2}{3} \Theta_{,b} - \sigma_{b,c d} h^{c d} \right) - \eta^{abcd} u_{b} \omega_{c d} = (1 + 2 \alpha R)^{-1} \left[ q^{a} - 2 \alpha h^{a b} R_{,b c} u^{c} \right] , \]
where \( \omega^{a} = \frac{1}{2} \eta^{abcd} u_{b} \omega_{c d} \) and we have used (1). This equation implies, by (5), (18), (20) and (23), that
\[ h_{a}^{b} R_{;b c} u^{c} = O[1] . \]  
(25)
Equation (25) is insufficient to show that the spatial gradient of the scalar curvature is $O[1]$. This requires the higher–order Raychaudhuri equation [3], whose linearised form is

$$\dot{\Theta} + \frac{1}{3} \Theta^2 + (1 + 2\alpha R)^{-1}(\bar{\mu} + \mu) - \alpha(1 + 2\alpha R)^{-1} \left[ \frac{1}{2} R^2 + \Box R - 2R_{;ab}u^a u^b \right] \simeq 0 \ , \quad (26)$$

where we have used (18) and (20). We take the spatial gradient of (26), and use (14) together with (17), (18), (20), (23) and (25), to obtain

$$\left[ \alpha \left( \frac{8}{3} \dot{\Theta} + \frac{2}{3} \Theta^2 - R - \frac{1}{6} \right) \right] \nabla_c R + (1 + 2\alpha R)^3 \nabla_c (\dot{\Theta} + \frac{1}{3} \Theta^2) + 2\alpha \left[ h^{d}_{c} R_{ab} u^{a} u^{b} + R_{;ab} (h^{d}_{c} u^{a} u^{b} + u^{a} h^{d}_{c} u^{b} ; d) \right] + 3 \nabla_c (\bar{\mu} + \mu) \simeq 0 .$$

This implies, using (17), (18), (20), (23), (24) and (25), that

$$\left[ \alpha \left( \frac{8}{3} \dot{\Theta} + \frac{2}{3} \Theta^2 - R - \frac{1}{6} \right) \right] \nabla_c R = O[1].$$

Since they are non–zero in the exactly isotropic (and thus FRW) case, $R$, $\Theta$ and $\dot{\Theta}$ are $O[0]$. Hence we deduce that

$$3 \nabla_a R = O[1] \text{ and thus } R_{;<cd>} = O[1] . \quad (27)$$

The definition of the magnetic part of the Weyl tensor $H_{ab}$ in terms of the kinematic quantities (18) and (20) is independent of the field equations so that, as in [1]

$$H_{ab} = O[1] . \quad (28)$$

However, to prove that the electric part $E_{ab}$ of the Weyl tensor is $O[1]$ requires the higher–order shear propagation equation [3]:

$$h^{c}_{a} h^{d}_{b} \dot{\sigma}_{cd} + \omega_{a} \omega_{b} + \sigma_{ac} \sigma^{c}_{b} + \frac{2}{3} \Theta \sigma_{ab} - \frac{1}{3} h_{;ab} [\omega^2 + 2\sigma^2] + E_{ab} - \frac{1}{2} (1 + 2\alpha R)^{-1} \pi_{ab} - \alpha (1 + 2\alpha R)^{-1} R_{;<cd>} = 0 , \quad (29)$$

on using (1). Then, by (18), (20), (27) and (29)

$$E_{ab} = O[1] . \quad (30)$$

This establishes that the kinematic quantities for the radiation and geometry are almost–FRW, i.e. all the covariant quantities that vanish in the exactly isotropic (FRW) case are at least $O[1]$. The higher–order terms in the field equations thus have no $O[0]$ effect on any of the kinematic quantities.
4.2 Almost–FRW dynamics

It follows from section 4.1 that the $O[0]$ equations governing the dynamics are just those of an FRW universe. Thus by (26) and the conservation equation (21), the evolution of the scale factor $S$ is to $O[0]$ that of a higher–order radiation FRW universe. The first integral of (26) is the almost–homogeneous analogue of the higher–order Friedmann equation (16)

$$\frac{3k}{S^2} \simeq -3\frac{\dot{S}^2}{S^2} + (\bar{\mu} + \mu) - 2\alpha \left[ 9 \left\{ \left( \frac{\dot{S}^2 + k}{S^2} \right)^2 - \left( \frac{\ddot{S}}{S} \right)^2 \right\} + \frac{3\dot{S}\dot{R}}{S} \right], \quad \dot{k} = O[1].$$ (31)

Then the dynamical equations differ from those of the exactly isotropic (FRW) metric at $O[1]$ only. Because the kinematic quantities are just those of a perturbed FRW universe, it is now possible to linearise the covariant equations about the FRW values in the usual manner [8], and to consider the evolution of density inhomogeneities. The background metric will obey the FRW equations, and consequently we can use the usual covariant FRW perturbation analysis.

4.3 Almost–FRW metric

By returning to the frame $\mathring{u}^a$ described in section 2, we now show that the metric may be given in an almost–FRW form. In [1], the choice of $\mathring{u}^a$ is based on surfaces of constant matter density, since for applications to the CMBR, the universe is matter–dominated. In our case, $\mathring{u}^a$ is based on the surfaces of constant thermalised radiation density ($\bar{\mu}$), and

$$\tilde{\omega}_{ab} = 0, \quad \mathring{u}^a \neq 0.$$

(All kinematic quantities of the $\mathring{u}^a$ congruence are denoted by a hat.)

We are able to choose the geodesic $u^a$ to be close to $\mathring{u}^a$ because of (24), which ensures that the angle between $\mathring{u}^a$ and $u^a$ will be $O[1]$. Clearly the energy flux and anisotropic stress relative to $\mathring{u}^a$ remain $O[1]$. Changes to all $O[1]$ quantities will be $O[2]$, but changes to $O[0]$ quantities will be $O[1]$, and will introduce effective energy fluxes with respect to $\mathring{u}^a$.

The congruence $\mathring{u}^a$ will be normal to cosmic time surfaces (coinciding with surfaces of constant $\bar{\mu}$) in the perturbed model, and the spacetime metric will be [1]:

$$ds^2 = -A^2(x^a)dt^2 + \mathring{h}_{\alpha\beta}dx^\alpha dx^\beta,$$ (32)
where $\hat{u}^a = A^{-1} \delta^a_0$, $\hat{h}_{ab} = g_{ab} + \hat{u}_a \hat{u}_a$, and Greek indices run from 1 to 3. The kinematic quantities for $\hat{u}^a$ are [1]

$$\hat{\alpha}^a \equiv \hat{u}^a_{;b} \hat{u}^b = -\hat{h}^{ab} (\log A)_{,b}, \quad \hat{\omega}_{ab} = 0, \quad \hat{\theta}_{ab} = \frac{1}{2} A \hat{h}_{ab,0}.$$ (33)

If we express the non–coincidence of the flows of $\hat{u}^a$ and $u^a$ by [1]

$$\hat{u}^a = u^a - V_a + O[1], \quad u_a V^a = 0, \quad V_a V^a = O[1],$$ (34)

then the kinematic and dynamic quantities relative to $\hat{u}^a$ are given in terms of those relative to $u^a$ by

$$\hat{\theta} = \Theta + O[1], \quad \hat{\sigma}_{ab} = \sigma_{ab} + O[2], \quad \hat{\alpha}^a = O[1]$$

$$\hat{\mu} = \mu + O[1], \quad \hat{\pi}^a = q^a + \frac{4}{3} \mu V^a + O[2], \quad \hat{\pi}_{ab} = \pi_{ab} + O[2],$$ (35)

where $\hat{\mu}$ is the energy density of the decoupled species as measured by ‘observers’ comoving with the thermalised species. Thus the kinematic results (17–30) hold in this frame, and in particular

$$\hat{\nabla}_a \hat{\mu} = O[1], \quad \hat{\nabla}_a \hat{\theta} = O[1].$$ (36)

Since the expansions are equal (up to $O[1]$), the scale factor $\hat{S}(t, x^\alpha)$, defined by

$$A^{-1} (\log \hat{S})_{,0} = \frac{1}{3} \hat{\theta},$$

will satisfy the higher–order Friedmann equation (31). From (36) and this definition of $\hat{S}$ we can write the spatial metric in (32) as

$$\hat{h}_{ab}(t, x^\gamma) dx^a dx^b = \hat{S}^2(t, x^\gamma) f_{ab}(t, x^\gamma) dx^a dx^b,$$ (37)

where most of the time variation is in $\hat{S}$, and from the result [1]

$$\hat{\sigma}_{ab} = \frac{1}{2} A \hat{S}^2 f_{ab,0} = O[1],$$

most of the spatial coordinate dependence is in $f_{ab}$. Equations (33) and (35) imply that the function $A(t, x^\alpha)$ is only weakly dependent on the spatial coordinates, and we can consequently set $A(t, x^\alpha) = 1 + O[1]$ by rescaling the time coordinate.
It remains only to show that the embedding of the 3–surfaces is almost–isotropic. The higher–order Gauss–Codazzi equations [3] for the surfaces \{\tilde{\mu} = \text{const}\} give the trace–free part of the Ricci tensor of these surfaces as

$$3\hat{R}_{ab} - \frac{1}{3}(3\hat{R})\hat{h}_{ab} = h_a^\epsilon h_b^d \left[\hat{\alpha}_{(c;d)} - \hat{\sigma}_{cd;e} \tilde{\alpha}^e\right] - \hat{\Theta}\hat{\sigma}_{ab} + \hat{a}_a\hat{a}_b - \frac{1}{3}\hat{h}_{ab}\hat{\alpha}^{c;e} +$$

$$\left(1 + 2\alpha R\right)^{-1} \left[\hat{\tau}_{ab} + 2\alpha(h_a^e h_b^d - \frac{1}{3}\hat{h}_{ab}\hat{h}^{cd})R_{cd}\right]. \quad (38)$$

The scalar curvature is independent of the four–velocity, and consequently (27) gives

$$\left(h_a^e h_b^d - \frac{1}{3}\hat{h}_{ab}\hat{h}^{cd}\right)R_{cd} = O[1]. \quad (39)$$

Given (39) and the almost–FRW conditions on the kinematic quantities (35), equation (38) implies

$$3\hat{R}_{ab} - \frac{1}{3}(3\hat{R})\hat{h}_{ab} = O[1], \quad (40)$$

where, from [3]

$$3\hat{R} = -\frac{2}{3}\hat{\Theta}^2 + \left(1 + 2\alpha R\right)^{-1} \left[2\hat{\mu} + \alpha R^2 + 4\alpha\hat{h}^{ab}R_{ab}\right] + O[1]. \quad (41)$$

Because \(R, \hat{\Theta}\) and \(\hat{\mu}\) are almost spatially homogeneous, by (36) and (41), so is \(3\hat{R}\). Then the 3–spaces are spaces of almost–constant curvature:

$$3\hat{R}_{abcd} = \frac{1}{6}(3\hat{R}) \left[h_{ac}\hat{h}_{bd} - \hat{h}_{ad}\hat{h}_{bc}\right] + O[1].$$

Thus, as in [1], we have recovered (to \(O[0]\)) all of the standard relations governing an FRW universe, and in particular, we have shown that there exists an almost–FRW metric in higher–order gravity, given in almost–comoving coordinates.

We have shown that in a radiation–dominated universe, an almost–isotropic background radiation implies an almost–FRW spacetime geometry in the higher–order theory of gravity (including general relativity as a special case). In effect, together with the higher–order exact EGS theorem of [3], we have now shown that the EGS theorem is stable under perturbations of both the Lagrangian and the background metric.
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