A Control-Model-Based Approach for Reinforcement Learning

Yingdong Lu, Mark S. Squillante, Chai W. Wu
Mathematical Sciences
IBM Research
Yorktown Heights, NY 20198, USA
{yingdong, mss, cwwu}@us.ibm.com

Abstract

We consider a new form of model-based reinforcement learning methods that directly learns the optimal control parameters, instead of learning the underlying dynamical system. This includes a form of exploration and exploitation in learning and applying the optimal control parameters over time. This also includes a general framework that manages a collection of such control-model-based reinforcement learning methods running in parallel and that selects the best decision from among these parallel methods with the different methods interactively learning together. We derive theoretical results for the optimal control of linear and nonlinear instances of the new control-model-based reinforcement learning methods. Our empirical results demonstrate and quantify the significant benefits of our approach.

1 Introduction

Over the past many years, reinforcement learning (RL) has proven to be very successful in solving a wide variety of learning and decision making under uncertainty problems. This includes problems such as those related to game playing (e.g., Tesauro [20], Togelius et al. [21]), bicycle riding (e.g., Randlov and Alstrom [13]), and robotic control (e.g., Riedmiller et al. [15]). Many different RL approaches, with varying levels of success, have been developed to address these problems [8, 19, 18]. Among these different approaches, model-free RL has been demonstrated to learn and solve various problems without any prior knowledge (e.g., Randlov and Alstrom [13] and Mnih et al. [11]). Such model-free approaches, however, often suffer from high sample complexity that can require an inordinate amount of samples for some problems which can be prohibitive in practice, especially for those problems limited by time or other constraints. Model-based RL has been demonstrated to significantly reduce sample complexity and has been shown to outperform model-free approaches for various problems (e.g., Deisenroth and Rasmussen [6] and Meger et al. [10]). Such model-based approaches, however, can often suffer from the difficulty of learning an appropriate model and from worse asymptotic performance than model-free approaches due to model bias from inherently assuming the learned system dynamics model accurately represents the true system environment (e.g., Atkeson and Santamara [3], Schneider [17], and Schaal [16]). While model-free approaches can asymptotically achieve better performance because they are not limited by the accuracy of the system model, this again comes at the potential expense of significantly higher sample complexity. Some recent work has sought to address these issues by initializing a model-free approach with a proposed model-based approach (e.g., Nagabandi et al. [12]).

In this paper we consider a novel general approach for RL comprising two key aspects. The first aspect is our proposal of alternative model-based RL methods that, instead of learning a system dynamics model, learn an optimal control model for a general underlying (unknown) dynamical system and directly apply the corresponding optimal control from the model. Many traditional
model-based RL methods, after learning the system dynamics model which is often of high complexity and dimensionality, then use this system dynamics model to compute an optimal solution of a corresponding dynamic programming problem, often applying model predictive control (e.g., Nagabandi et al. [12]). In contrast, our new alternative model-based RL methods learn the parameters of an optimal control model, often of lower complexity and dimensionality, from which the optimal solution is directly obtained. Furthermore, we establish that our control-model-based (CMB) RL approach converges to the optimal solution analogous to model-free RL approaches while eliminating the problems of model bias in traditional model-based RL approaches.

The second aspect of our model-based RL approach is a general framework that supports multiple CMB RL methods running in parallel. This framework includes a metacontroller that continually selects the best decision from among the different CMB RL methods over time and that manages the interactions among the different CMB RL methods through the novel concept of a control-table which further allows the methods to interactively learn from each other. In doing so, our framework aims to reap the advantages of each of the different CMB RL methods running in parallel while minimizing any disadvantages of these different CMB RL methods. The resulting benefits include closer and more general interactions among the different RL methods than those based on initializing a model-free approach with a model-based approach as in [12].

To the best of our knowledge, this paper presents the first proposal and derivation of such general CMB RL methods and general frameworks for parallel CMB RL methods, an approach that should be exploited to a much greater extent in the RL literature. In the remainder of this paper, we first devise our new CMB RL methods in which the parameters of both linear and nonlinear optimal control models are learned, including the convergence of our CMB methods to the optimal solution. We then present our general framework that manages multiple CMB RL methods running in parallel. Lastly, we present empirical results for a couple of classical problems in the OpenAI Gym framework [5] that demonstrate and quantify the significant benefits of our general approach over existing algorithms. We will release on GitHub the corresponding python code and make it publically available.

2 Control-Model-Based RL

The dynamic programming formulation for the RL problems of interest can be expressed as follows

$$\min_{u_1, \ldots, u_T} \sum_{t=1}^{T} \mathbb{E}[c(x_t, u_t)], \quad \text{s.t.} \quad x_t = f(x_{t-1}, u_{t-1}), \quad (1)$$

where \(x_t\) represents the state of the system, \(u_t\) represents the control variables, \(f(\cdot, \cdot)\) represents the evolution function of the dynamical system characterizing the system state given the previous state and the taken action together with unknown uncertainty, and \(c(\cdot, \cdot)\) represents a cost-based objective function of both the system state and control action. Alternatively, the dynamic programming formulation associated with the RL problem of interest can be expressed as

$$\max_{u_1, \ldots, u_T} \sum_{t=1}^{T} \mathbb{E}[r(x_t, u_t)], \quad \text{s.t.} \quad x_t = f(x_{t-1}, u_{t-1}), \quad (2)$$

where \(r(\cdot, \cdot)\) represents a reward-based objective function of both the system state and control action, with all other variables and functions as given above.

We note that (1) and (2) can represent a wide variety of RL problems and corresponding dynamic programming problems based on the different forms taken by the evolution function \(f(\cdot, \cdot)\). For example, a linear system dynamics model results when \(f(\cdot, \cdot)\) takes the form of linear transformations. The function \(f(\cdot, \cdot)\) can also characterize the discretized evolutionary system dynamics governed by (partial) differential equations. In addition, the cost function \(c(\cdot, \cdot)\) and reward function \(r(\cdot, \cdot)\) are also allowed to take on various general forms, and thus can represent any combination of cumulative and terminal costs or rewards, respectively. Both formulations (1) and (2) can also be analogously defined in continuous time. On the other hand, most classical RL formulations assume the dynamics evolve in discrete time. When the underlying system is based on a continuous-time model, then a discretization operator such as forward Euler discretization is used to generate the discrete time samples.

Traditional model-based RL methods seek to first learn the system dynamics model of high complexity and dimensionality, and then incur the additional overhead of computing the optimal solution to (1)
We then need to learn the matrix \( B \). We begin with a derivation of a particular general class of our CMB RL methods. By taking a (w.r.t. a given state, an action taken, and a corresponding outcome) as part of the learning process, thus the search with random restarts and hierarchical regression-based fitting of the control matrices against the learning process, we focus in this paper on a combination of local and collaborative learning. While our methods can be identified in several ways based on various forms of exploration and exploitation, we introduce the novel concept of a control table, or \( c\)-table, that is somewhat analogous to the \( Q\)-table but different in several important ways including its construction and usage. Alternative forms of RL can be applied to determine the best parameters for the matrix \( B \) or the matrices \( B, C, D \). In particular, we can exploit the low complexity and dimensionality of learning the parameters of the optimal control model, especially relative to the high complexity and dimensionality of learning the system dynamics model, to solve the corresponding optimization problem after a relatively small number of sample measurements. Hence, the learning problem is reduced to solving a small stochastic optimization problem in which uncertainty has to be sampled. Many different algorithms, including a combination of local search with random restarts and regression-based fitting against the \( c\)-table, can be deployed to solve these optimization problems and learn the desired control parameters based on forms of exploration and exploitation.

We note that higher-order Taylor series approximations can be considered in an analogous manner. Furthermore, although the above derivation focuses on a deterministic optimal control method, a corresponding set of stochastic optimal control methods are the subject of ongoing research.

Our general CMB RL approach can be summarized as follows (see also Algorithm 1 of Appendix A).

1. Identify the reduced complexity and dimensionality of the control model based on the dimensions of the state and action vectors, possibly exploiting additional prior knowledge.
2. Initialize the matrices \( B^* = B_0 \) (linear) or \( B^* = B_0 \), \( C^* = C_0 \), \( D^* = D_0 \) (nonlinear).
3. Execute the system for the \( e \)-th episode of the task using the current control parameters \( B_e \) (linear) or \( B_e, C_e, D_e \) (nonlinear) generating a sequence of state, action, reward tuples.
4. Exploit the \( c\)-table to record the best possible reward for each state-action pair.
5. When the episode yields an improvement in the total reward, update the control-model parameters \( B^* = B_e \) (linear) or \( B^* = B_e, C^* = C_e, D^* = D_e \) (nonlinear).
6. Identify an alternative candidate for the control model \( B_{e+1} \) (linear) or \( B_{e+1}, C_{e+1}, D_{e+1} \) (nonlinear), based on one of several available options.
7. Increment \( e \) and repeat from Step 3 until \( B^* \) or \( B^*, C^*, D^* \) satisfy tolerance.

Prior knowledge (e.g., expert opinion, mathematical models), when available, can also be exploited in Step 1 to bound the degree of freedom of the problem; in such cases, we simply take advantage of the additional prior knowledge and boundary conditions available in many real-world problems. Initial conditions can be determined from prior solutions of similar problems, determined mathematically from a simple model, or chosen randomly. Step 3 evaluates the total reward of episode \( e \) and updates the \( c\)-table based on the path of the episode. An alternative candidate for the control model can be identified in several ways based on various forms of exploration and exploitation. While our algorithm can exploit many approaches of interest, we focus in this paper on a combination of local search with random restarts and hierarchical regression-based fitting of the control matrices against the \( c\)-table. To start, we use the local search with random restart to find new candidates and fill-in the \( c\)-table. Then, as the score (\( R \)-value) of the hierarchical regression reaches a level of sufficient
accuracy, we switch to using a combination of local search with random restarts and hierarchical regression-based fitting, where the regression score is used to determine the candidate control model.

Local search has been previously considered in linear policy optimization [9]. Our approach, however, combines local search together with hierarchical regression fitting against the control table to efficiently and effectively learn the control model parameters. The choice between the local-search and hierarchical-regression candidates can be formulated as a multi-bandit problem with a balance between exploration (local search) and exploitation (hierarchical regression). Although our algorithm can substitute linear regression for the hierarchical regression, we note that the latter has the advantage of partitioning the control matrices into submatrices with different parameters for the control submatrices that best fit the corresponding region of the c-table; then the control model parameters are based on the regions of the control submatrices traversed by the path in the current episode. In particular, when the c-table is adequately populated, it can be beneficial to identify control models by conducting linear regressions that treat the state and optimal-action pair as observations of independent and dependent variables. To allow different linear controls for different regions of states (e.g., these regions of states could depend on the geometric or algebraic properties of the states such as the distance to the landing zone or permutations of the directions in the lunar lander problem), hierarchical regression can be run with the introduction of new variables. Specifically, a two-level hierarchical linear regression typically takes the form \( Y_{ij} = \beta_{0,j} + \beta_{1,j}X_{ij} + r_{ij} \), where \( i \) indexes the individual state and \( j \) indexes the different domains of the states; \( X \) and \( Y \) represent the independent and dependent variables, \( \beta_{0,j} \) and \( \beta_{1,j} \) are the slopes and intercepts that can be different for different domains, and \( r_{ij} \) represents the errors. For additional details on hierarchical regression, refer to [14]. Note that software implementations of hierarchical regression are readily available in many statistical packages (e.g., R, SAS, and SPSS), as well as machine learning packages such as Stan.

2.1 Convergence

As noted in the introduction and discussed in the literature (e.g., [12]), one advantage of model-free RL over traditional model-based RL is that the Bellman equation in the former guarantees convergence of \( Q \)-learning to the optimal policy, whereas traditional model-based RL that learns the system dynamics model and then uses this model to compute an optimal solution of the corresponding dynamic programming problem (often applying model predictive control) may not guarantee convergence, especially if the learned system dynamics model deviates from the true dynamics of the system. On the other hand, there is no clear bound on the number of iterations that model-free RL may require to reach the optimal policy and these approaches can often exhibit very slow convergence behavior in practice for some RL problems. This degree of high sample complexity requiring an inordinate amount of samples can be prohibitive in practice for such problems when each trial is costly or time-consuming or there are simply only a limited number of trials available to compute the action policy. W.r.t our CMB RL method, we establish the following theoretical result on guaranteed convergence to an optimal stabilizing feedback control.

**Theorem 2.1.** Under the assumption that optimal control matrices exist, the CMB RL approach will asymptotically converge to a set of optimal control matrices.

**Proof.** Consider the dynamics of a system given by

\[ \dot{x} = f(x, t) \]

for the continuous-time case or by

\[ x_{n+1} = f(x_n, n) \]

for the discrete-time case. Suppose \( f \) is Lipschitz continuous with Lipschitz constant \( L \) and the goal of the task at hand is to ensure that the trajectory of the system converges toward an equilibrium point of the system \( x_0 \). Lyapunov stability analysis and known results from time-varying linear systems [7] show that

\[ u = Bx \]

is a linear stabilizing feedback provided that

\[ Df(x_0, t) + B \]

has eigenvalues with negative real parts bounded away from the origin and slowly varying, where \( Df(\cdot, \cdot) \) is the Jacobian matrix of \( f(\cdot, \cdot) \). In particular, \(-\alpha I\) is a stabilizing feedback matrix where \( \alpha > L \).

\[ \square \]
The proof implies that a binary search on $\alpha$ will result in a stabilizing feedback. A more optimal search can be performed in practice to find feedback matrices that satisfy bounds, such as spectral bounds, using both gradient-based and gradient-free methods, such as genetic algorithms, differential evolution, and local-search methods. The eigenvalue conditions show that a feedback matrix within a small ball around $B$ will still stabilize the system, and therefore such instances of our CMB optimization method will guarantee to find a solution given sufficient time to search the space, analogous to the guaranteed convergence under model-free RL via the Bellman equation. We next illustrate these approaches using two well known examples of RL control.

### 2.2 Examples

In this subsection we apply our CMB RL methods to two different problems from the OpenAI Gym \[5\], namely Lunar Lander and Cart Pole, which also comprise the basis for our experimental results in Section \[4\]. While exact solution can be obtained for both problems with complete information using optimal control techniques (e.g., Anderson & Moore \[2\]), we seek to obtain the optimal control models based solely on the known dimensions of the state and action vectors, with all other aspects of the problem unknown. In the case of cart pole, as an illustrative example, we further assume that a small amount of partial, domain-based, information is additionally known about the problem at hand, which we exploit within the context of a hybrid version of our CMB RL method.

**Lunar Lander.** This problem is discussed in Brockman et al. \[5\]. In each state, characterized by an 8-dimensional state vector, there are four possible discrete actions (left, right, vertical or no thrusters). The goal is to maximize the cumulative reward comprising positive points for successful degrees of landing and negative points for fuel usage and crashing.

Under the assumption that the system dynamics are unknown and that a simple linear or nonlinear state feedback control is sufficient to solve the Lunar Lander problem, the goal then becomes learning the parameters of the corresponding optimal control model. Here our rationale is that the control matrix is of a simpler complexity and dimensionality than the system dynamics as the number of controls inputs is smaller. In particular, we assume a (unknown) dynamical system model $\dot{x} = f(x)$, and further assume that a linear control model \((B, b)\) exists such that $\dot{x} = f(x) + Bx + b$ will solve the problem of landing the spacecraft; we also analogously consider a simple nonlinear (quadratic) control model of the form $\dot{x} = f(x) + Bx + b + Cx'Dx$. W.r.t. complexity and dimensionality, in the Lunar Lander problem, the system is nonlinear and the state space is of dimension 6, implying that each linearized vector field is of size $6 \times 6$, has 36 elements, and depends on the current state. This is in comparison with there being only two control dimensions (left/right and vertical), and thus the linear control matrix $B$ and vector $b$ (which do not depend on the state) are of size $6 \times 2$ and $1 \times 2$, respectively, having a total of only 14 elements. Similarly, the simple nonlinear control uses the scalar $x^2_7$ in addition to the 6-dimensional state, which results in a $7 \times 2$ matrix and a $1 \times 2$ vector, totaling 16 elements. This representative example illustrates how the complexity and dimensionality of the system dynamics model tends to be higher than that for the optimal control model, given that the physics of the problem is well known to be complex. Any additional knowledge, as we exploit next, can further restrict the degrees of freedom of the optimal control model.

**Cart Pole.** This problem is discussed in Barto et al. \[4\]. In each state, characterized by a 4-dimensional state vector, there are two possible discrete actions (push left, push right). The goal is to maximize the score representing the number of steps that the cart pole stays upright before either falling over or going out of bounds. The task is solved when the number of steps reaches 200.

We consider two approaches for this problem. The first is analogous to the approach used for Lunar Lander, where we assume a linear control model of the form $u = Bx + b$. Since the actions are 1-dimensional (left, right), the matrix $B$ and vector $b$ are of size $1 \times 4$ and $1 \times 1$, respectively, totaling 5 elements. The solution proceeds along the lines of the above Lunar Lander description.

For the second approach, we assume that the form of the system dynamics equations are known, but not the system parameters and physical constants such as the gravitational acceleration $g$, the mass $m$ and length $l$ of the pole, the mass $M$ of the cart, and the force $F$ that is applied. The state vector is 4-dimensional, consisting of $x, \dot{x}, \theta$ and $\dot{\theta}$, and satisfies the equations \[3\] and \[4\] in Appendix \[B\]. When linearized around the origin, which is the goal position, the function $f(x)$ is reduced to a linearized form $Ax + FW$, with $A$ and $W$ respectively given in \[5\] and \[6\] of Appendix \[B\].
We obtain estimates of \( A \) via the forward Euler approximation of the state equation that is returned by the state transition function \( x_{t+1} = x_t + h(Ax_t + Fw) \), where we further assume the discretization constant \( h \) to be unknown. Hence, with \( p \) observations of the next state given the current state, we have \( p \) nonlinear equations with unknown parameters \( g, m, l, F \) and \( h \). On the other hand, in order to find the optimal control, we do not need to know the values of these parameters, rather we solely need to know the value of the matrices \( A \) and \( W \). We therefore can use a least squares approach to solve for the elements of \( A \) and \( W \) and then use these learned parameters to construct an optimal control near the origin with a Linear Quadratic Regulator (LQR) [2]. Since a discrete action is assumed, the vector is quantized to a discrete action.

We would like to point out the important difference here between this hybrid CMB approach and traditional model-based RL. Even though we use a form of the system dynamics equations, this model form is exploited solely to derive the structural properties of the matrices \( A \) and \( W \) in order to reduce the degrees of freedom of the optimal control model. Unlike traditional model-based RL, our hybrid CMB approach directly learns the system matrices \( A \) and \( W \), together with the optimal control model, and does not learn the system dynamics parameters such as \( M \) and \( l \); whereas the traditional approach learns these individual parameters and then computes a control policy.

**Summary.** A main thesis of our general CMB RL approach is that there exists a continuum of knowledge about the system that we could exploit in order to reduce the complexity of the optimal control model and speed up the learning of the optimal solution where these different modeling and learning techniques can benefit from each other (as discussed in the next section). With no knowledge of the system dynamics, we can directly learn the control model parameters such as \( (B, b) \) for linear control and \( (B, C, D, b) \) for simple nonlinear control. Then, with different degrees of knowledge of the form of the system dynamics equations such as knowing the state equations are due to Newton’s laws, we can additionally infer the quasi-companion form of the relatively sparse matrix \( A \) and the structural form of \( W \) to reduce the degrees of freedom of the optimal control model.

### 3 General Framework

Our general framework supports the running of multiple CMB RL methods simultaneously. This framework seeks to gain the advantages of different CMB RL methods and to minimize any disadvantages of these different RL methods by allowing the multiple RL methods to run in parallel and learn from each other primarily through the \( c \)-table. Each CMB RL method operates exactly as in its standard RL environment, providing the next action that should be taken given the current state of the system. A metacontroller then determines which action to actually take given the set of next action recommendations provided by each of the RL methods. The \( c \)-table and each CMB RL method are then updated in an appropriate manner with the information they maintain based on the action actually taken and the resulting outcome. Figure 3 illustrates our general framework.

The design of the metacontroller for our general framework includes a higher level RL problem in which the metacontroller must determine which next action to take given the set of next-action recommendations from the collection of CMB RL methods running in parallel. This can include factors such as changes in the operating environment. We focus here on having the metacontroller determine the next best action from a linear optimal control model and a simple nonlinear optimal control model, both for a general (unknown) dynamical system underlying the problem of interest. The metacontroller initially starts by using the linear-CMB RL method described in the previous section and applies its next-action recommendation for an episode. The sequence of actions taken and resulting outcomes are then recorded in the \( c \)-table and made available to both the linear-CMB RL method and the nonlinear-CMB RL method described in the previous section, each of which can update its internal information which is also visible to the metacontroller. Iterations of the general RL framework continue in this manner, during which time the metacontroller monitors both the hierarchical regression score and the current best total reward for each CMB RL method. The metacontroller then selects the CMB RL method that currently is performing best and applies its control model parameters for the next episode. To provide an additional level of exploration and exploitation, the metacontroller will switch to one of the alternative CMB RL methods with some probability. While our focus here is on the design of a metacontroller over the combination of these two linear-control and nonlinear-control models — for which our experimental results in Section 6 demonstrate significant performance improvements — we note that this can be extended in a similar manner to the design of a metacontroller across a broader set of CMB RL methods.
We note that the design of the metacontroller also includes support for managing the interactions among the different RL methods that allows the different methods to closely interact and learn from each other. This is achieved through the updating of the $c$-table based on each episode and the use of hierarchical regression to efficiently and effectively fit the corresponding control models against the $c$-table, as described above. More specifically, once the metacontroller selects one of the CMB RL methods and applies its recommended next-actions for the current episode, the $c$-table is updated accordingly and each CMB RL method updates its control model parameters based on the revised $c$-table. The metacontroller therefore provides support that allows the linear-CMB RL method and the simple nonlinear-CMB RL method to operate in parallel in a cooperative manner which enables interactions and learning from each other.

![Figure 1: Comparison of CMB RL with Classical Q-Learning for the Lunar Lander problem.](image)

Our general metacontroller approach for managing the interactions among the different CMB RL methods can be summarized as follows. First, we initialize the $c$-table comprising the four dimensions of state, next state, action, and reward. Second, at the start of each episode, the metacontroller selects a control from among the set of control models. Third, at the end of each episode, the $c$-table is updated according to the sequence of observation of the system under the actions of the current control model; and the control model parameters are updated based on their best fit against the latest version of the $c$-table that maximizes the total reward.

As noted above, hybrid approaches are possible in which our new CMB RL methods can be exploited to learn both the optimal control model parameters and structural properties of the system dynamics equations. Specifically, in Sections 2 and 4, we consider one instance of such an approach that consists of simultaneously learning a simplified lower-dimensional version of the system dynamics model and learning the control parameters of linear-control and nonlinear-control models.

### 4 Experimental Results

In this section we present experiments for the two problems from OpenAI Gym [5] discussed above, i.e., Lunar Lander and Cart Pole. For each case the state space is continuous and in order to use $Q$-learning, it is discretized to a finite set of states, where each dimension is partitioned into equally spaced bins and the number of bins depends on both the problem to be solved and the reference codebase that is used. For our CMB methods the continuous state is used. The experiments for each problem from OpenAI Gym with model-free RL methods were executed using the existing code found at [22] exactly as is; our CMB RL methods described in Section 2 were implemented on top of this same existing code base, namely our algorithm starts with a combination of local search and random restarts, building up the $c$-table, and then alternates between local search with random restarts and hierarchical regression-based fitting of the control matrices against the contents of the $c$-table.

#### 4.1 Lunar Lander

Recall that the state vector is 8-dimensional with a total of four possible actions, and the score represents the cumulative reward comprising positive points for successful degrees of landing and negative points for fuel usage and crashing. In our $Q$-learning experiments, the 6 continuous state variables are each discretized into 4 bins. Our algorithm is applied to the space of all control matrices of size 14 (linear control) and 16 (simple nonlinear control) to find the optimal control directly derived from the control matrices except for a special condition when one of the legs makes contact. The code change to the existing $Q$-learning codebase is straightforward and consists of replacing the function that uses a quantizer and the $Q$-table to determine the next action with a linear or simple nonlinear function that maps the (unquantized) state to a control vector that is quantized to an action. The total reward of an episode is used as the objective function to be maximized by our algorithm.
Figure 1 plots the corresponding score results, averaged over 20 trials, from our CMB RL methods in comparison with the corresponding results from the classical Q-learning approach using the Bellman operator. The results plotted for our CMB RL methods represent those obtained with local search and random restarts; further improvements are obtained, consisting of some additional increases in average scores and significant reductions in the variance of scores, when regression-based fitting is included after a few thousand or so episodes. Observe from these results that, while Q-learning continues to realize a negative score on average after 12,000 episodes, our CMB algorithm finds a control matrix for the simple nonlinear control model that achieves a mean score above 200 after a few hundred episodes. Note that the simple nonlinear control method achieves a mean score above 200 sooner than the linear control method, though both CMB RL methods perform very well. Further note that constructing the table for Q-learning over a high dimensional space can be prohibitive because of the large number of grid points and the way the table is updated (in contrast to the c-table).

Figure 2: Comparison of CMB RL methods with Classical Q-Learning for the Cart Pole problem.

### 4.2 Cart Pole

Recall that the state vector is 4-dimensional with two actions possible in each state, and the score represents the number of steps where the cart pole stays upright before either falling over or going out of bounds. With a score of 200, the problem is considered solved and the simulation ends, i.e., no score is above 200. In our Q-learning experiments, the position and velocity are discretized into 8 bins whereas the angle and angular velocity are discretized into 10 bins. We will consider two versions of our CMB RL methods to find the feedback control policy. The first method is analogous to the method for Lunar Lander, which we use to directly find the linear control model matrices B and b. In Figure 2a, we plot the corresponding score results, averaged over 20 trials. Observe that our CMB RL approach is able to find the optimal control that solves the problem within a few hundred episodes, whereas Q-learning still oscillates well below the maximal score of 200.

The second version of our methods consists of expanding the base CMB RL approach to also obtain an estimate of $A$ via the forward Euler approximation returned by the state transition function given in the cart pole example of Section 2.2. We therefore have $p$ nonlinear equations w.r.t. any $p$ observations of the next state given the current state. The unknown parameters are $g$, $m$, $M$, $l$, $F$, and $h$, i.e., the time discretization constant $h$ is also assumed to be unknown. Once again, our only assumption here is that we know the structural equations of the systems, but not the exact details of the physics since the timescale $h$ and the gravitational acceleration on the Earth’s surface $g$ are considered unknown. As described above, rather than estimating these parameters, we estimate the nonzero entries of the matrices $A$ and $W$ directly. Applying a least squares approach on a sliding window of the past observations near the origin, we can solve for the nonzero and nonconstant parameters of $A$ and $W$ and update the current estimate of $A$ and $W$ as $A_{new} = \alpha A_{old} + (1 - \alpha) A_{est}$ and $W_{new} = \alpha W_{old} + (1 - \alpha) W_{est}$. These estimates are then used to construct an optimal control near the origin based on an LQR. Since a discrete control is assumed, the control vector is quantized. In Figure 2b, we plot the score of this hybrid instance of our base CMB RL method over 200 episodes in comparison with the classical Q-learning approach using the Bellman operator. We observe that the CMB RL method quickly solves the problem in contrast with Q-learning which continues to take many episodes to converge to the optimal solution. For both CMB methods, the code changes are analogous to the Lunar Lander case described above, with the exception that in the second method the local search is replaced with the LQR derived from past observations.

Observe from Figure 2b that, in comparison with Figure 2a, the hybrid version of our CMB RL method solves the cart pole problem faster than the base linear CMB RL method to directly obtain
the optimal feedback control. This is due to the fact that the hybrid linear CMB RL method takes advantage of partial knowledge of the physical equations of the system in consideration, which therefore results in a lower complexity (degrees of freedom) model (i.e., the matrices $A$ and $W$ are not dense) and further supports our thesis that we should exploit as much accurate prior knowledge as possible in building our optimal control model. Lastly, we note that additional improvements are obtained with both versions of our CMB RL methods in Figure 2, consisting of some additional increases in average scores and significant reductions in the variance of scores, when regression-based fitting is included together with the LQR after a few thousand or so episodes.

5 Conclusions

In this paper we considered a novel general approach for RL comprising: (1) a new form of model-based RL methods that, instead of learning a system dynamics model, directly learns optimal control model parameters; (2) a general framework that supports multiple CMB RL methods running in parallel and interactively learning together. We presented derivations of instances of our new CMB RL methods and results on the convergence of our CMB RL approach to the optimal solution. Empirical results also demonstrated and quantified the significant benefits of our general approach.

References

[1] M. Alzantot. Solution of mountaincar OpenAI Gym problem using Q-learning. https://gist.github.com/malzantot/9d1d3fa4fde4a101be48a135d8f9a289, 2017.
[2] B. D. O. Anderson and J. B. Moore. Optimal Control: Linear Quadratic Methods. Prentice-Hall, Inc., Upper Saddle River, NJ, USA, 1990.
[3] C. G. Atkeson and J. C. Santamaria. A comparison of direct and model-based reinforcement learning. In Proc. International Conference on Robotics and Automation, 1997.
[4] A. G. Barto, R. S. Sutton, and C. W. Anderson. Neuronlike adaptive elements that can solve difficult learning control problems. IEEE Transactions on Systems, Man, and Cybernetics, SMC-13(5):834–846, Sept. 1983.
[5] G. Brockman, V. Cheung, L. Pettersson, J. Schneider, J. Schulman, J. Tang, and W. Zaremba. OpenAI Gym. CoRR, abs/1606.01540, 2016.
[6] M. Deisenroth and C. Rasmussen. PILCO: a model-based and data-efficient approach to policy search. In Proc. International Conference on Machine Learning, 2011.
[7] A. Ichmann, D. Owens, and D. Pratzel-Wolters. Sufficient conditions for stability of linear time-varying systems. Systems & Control Letters, 9(2):157–163, 1987.
[8] L. Kaelbling, M. Littman, and A. Moore. Reinforcement learning: A survey. Journal of Artificial Intelligence Research, 4:237–285, 1996.
[9] H. Mania, A. Guy, and B. Recht. Simple random search provides a competitive approach to reinforcement learning. In NIPS, 2018.
[10] D. Meger, J. Higuera, A. Xu, P. Giguere, and G. Dudek. Learning legged swimming gaits from experience. In Proc. International Conference on Robotics and Automation, 2015.
[11] V. Mnih, K. Kavukcuoglu, D. Silver, A. Graves, I. Antonoglou, D. Wierstra, and R. Riedmiller. Playing Atari with deep reinforcement learning. In NIPS Workshop on Deep Learning, 2013.
[12] A. Nagabandi, G. Kahn, R. Fearing, and S. Levine. Neural network dynamics for model-based deep reinforcement learning with model-free fine-tuning. ArXiv e-prints, December 2017.
[13] J. Randlov and P. Alstrom. Learning to drive a bicycle using reinforcement learning and shaping. In Proc. International Conference on Machine Learning, 1998.
[14] S. Raudenbush, S. Bryk, A. Bryk, and B. coaut. Hierarchical Linear Models: Applications and Data Analysis Methods. Advanced Quantitative Techniques in the Social Sciences. SAGE Publications, 2002.
[15] M. Riedmiller, T. Gabel, R. Hafner, and S. Lange. Reinforcement learning for robot soccer. Autonomous Robots, 27, 2009.
[16] S. Schaal. Learning from demonstration. *Advances in Neural Information Processing Systems*, 1997.

[17] J. Schneider. Exploiting model uncertainty estimates for safe dynamic control learning. *Advances in Neural Information Processing Systems*, 1997.

[18] R. S. Sutton and A. G. Barto. *Reinforcement Learning: An Introduction*. MIT Press, 2011.

[19] C. Szepesvari. Algorithms for reinforcement learning. In *Synthesis Lectures on Artificial Intelligence and Machine Learning*, volume 4.1, pages 1–103. Morgan & Claypool, 2010.

[20] G. Tesauro. Temporal difference learning and td-gammon. *Communications of the ACM*, 38, 1995.

[21] J. Togelius, S. Karakovskiy, J. Koutnik, and J. Schmidhuber. Super Mario evolution. In *Proc. Symposium on Computational Intelligence and Games*, 2009.

[22] V. M. Vilches. Basic reinforcement learning tutorial 4: Q-learning in OpenAI Gym. https://github.com/vmayoral/basic_reinforcement_learning/blob/master/tutorial4/README.md, May 2016.
A Algorithm for General Control-Model-Based Reinforcement Learning

Algorithm 1: General CMB RL Method

**Input**: Initial control matrices \((B_0, C_0, D_0)\), Initial c-table \(c(\cdot, \cdot)\);  
**Output**: Set of best control matrices \(C_p^* = (B^*, C^*, D^*)\), corresponding c-table \(c^*(\cdot, \cdot)\);

Initialize \(C_p = newC_p = (B_0, C_0, D_0)\)

for episode \(e \in \{1, \ldots\}\) do
  \((x_i, a_i, r_i)_{i \in [T]} = Run_Episode\( (newC_p)\)
  \(c(\cdot, \cdot), C_p = Update_Table_and_Model\((x_i, a_i, r_i)_{i \in [T]}, c(\cdot, \cdot), newC_p, C_p)\)
  \(newC_p = Find_Next_Control_Model\((C_p, c(\cdot, \cdot))\)
  if tolerance satisfied then
    \(\text{return } C_p, c(\cdot, \cdot)\)
  end
end

**Function Run_Episode**

**Data**: Current control matrices \(newC_p\) for episode  
**Result**: Sequence of \((x_i, a_i, r_i)_{i \in [T]}\) comprising current episode under \(newC_p\)

Execute the real system (or simulation thereof) using \(newC_p\) to obtain the corresponding sample measurements for the episode.

**Function Update_Table_and_Model**

**Data**: Sequence \((x_i, a_i, r_i)_{i \in [T]}\), Current c-table \(c(\cdot, \cdot)\), Current control matrices \(newC_p\) and \(C_p\)  
**Result**: Updated c-table \(c(\cdot, \cdot)\), Updated \(C_p\)

for \(t\) ranging from \(T\) to 1 do
  \(r_t(t) = r(t) + \gamma r_t(t+1)\)
  if \(r_t(t) > c(x(t), a(t))\) then
    \(c(x(t), a(t)) = r_t(t)\)
  else
    \(r_t(t) = c(x(t), a(t))\)
end

Update \(C_p\) with \(newC_p\) if \(newC_p\) performs better.

**Function Find_Next_Control_Model**

**Data**: Current best control matrices \(C_p\), Current c-table \(c(\cdot, \cdot)\)  
**Result**: Next control matrix \(newC_p\)

Apply hierarchical regression to obtain best fit for \((B, C, D)\) based on \(c(\cdot, \cdot)\)

Compute overall score \(\delta\) based on hierarchical regression

if \(\delta < \mu\) then
  Set \(newC_p\) based on local search with random restart;
else
  Set \(newC_p\) based on \((B, C, D)\);
end

**return** \(newC_p\)

B Form of System Dynamics Equations: Cart Pole

For the second approach considered in the cart pole example of Section 2.2, we assume that the form of the system dynamics equations are known, but not the system parameters and physical constants such as the gravitational acceleration \(g\), the mass \(m\) and length \(l\) of the pole, the mass \(M\) of the cart, and the force \(F\) that is applied. The state vector is 4-dimensional, consisting of \(x, \dot{x}, \theta\) and \(\dot{\theta}\), and
satisfies the equations
\[ \ddot{\theta} = \frac{g \sin(\theta) - \cos(\theta) \left( F + m l \dot{\theta}^2 \sin(\theta) \right)}{l(4/3 - (m \cos^2(\theta))/(m + M))}, \]
(3)
\[ \dot{x} = \frac{(F + m l \dot{\theta}^2 \sin(\theta)) - m l \ddot{\theta} \cos \theta}{m + M}. \]
(4)

When linearized around the origin, which is the goal position, the function \( f(x) \) is reduced to a linearized form \( A x + F W \), where
\[ A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{g m}{m - 4/3(M + m)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g(M + m)}{l(m - 4/3(M + m))} & 0 \end{pmatrix}, \]
(5)
\[ W = \begin{pmatrix} 0 \\ -\frac{3m/4 + (M + m)}{l(M - 4/3(M + m))} \\ 0 \\ 1 \end{pmatrix}. \]
(6)

C General Framework for Control-Model-Based Reinforcement Learning

![Figure 3: General RL Framework](image)