Schwartz Function Valued Solutions of the Euler and the Navier–Stokes Equations

Philipp J. di Dio
Department of Mathematics and Statistics, University of Konstanz, Universitätsstraße 10, D-78464 Konstanz, Germany

Abstract
We prove the existence of a solution for the second order system of partial differential equations
\[ \partial_t f = \nu \cdot \Delta f + g \cdot \nabla f + h \cdot f + k \]
by a Montel space version of Arzelà–Ascoli and bound all Schwartz semi-norms. We find that for the Euler and the Navier–Stokes equations the vorticity remains a Schwartz function as long as the classical solution exists. Our approach is not affected by viscosity. It treats the hyperbolic Euler and the parabolic Navier–Stokes equation simultaneously.

Keywords: Euler equation, Navier–Stokes equation, vorticity, Burgers’ equation, Montel space, Schwartz function, breakdown criteria, time-dependent moments
2020 MSC: 35Q30, 76D03, 76D05

1. Introduction
The motion of (incompressible) fluids in \( \mathbb{R}^n \) or \( \mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n \) \( (n = 2, 3) \) are described by the Euler \( (\nu = 0) \) and Navier–Stokes \( (\nu > 0) \) equations
\[
\begin{align*}
\partial_t u(x,t) &= \nu \Delta u(x,t) - u \cdot \nabla u(x,t) - \nabla p(x,t) + F(x,t) \\
\text{div} u(x,t) &= 0
\end{align*}
\]
with initial conditions
\[ u(x,t_0) = u_0(x). \]
Here \( x \in \mathbb{R}^n \) or \( \mathbb{T}^n \) is the position vector and \( t \geq t_0 \) is the time; \( t_0 \in \mathbb{R} \) is the initial starting time and without loss of generality \( t_0 = 0 \). Then \( u(x,t) = (u_1(x,t), \ldots, u_n(x,t))^t \) is the velocity field of the fluid, \( p(x,t) \) is the pressure, and \( F(x,t) = (F_1(x,t), \ldots, F_n(x,t))^t \) are externally applied forces [1]. Reasonable initial conditions [1] are
\[ u_0 = (u_{0,1}, \ldots, u_{0,n})^t \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^n) \quad \text{resp.} \quad C^\infty(\mathbb{T}^n, \mathbb{T}^n), \]
i.e., all $u_{0,i}$ shall be Schwartz functions

$$S(\mathbb{R}^n) := \{ f \in C^\infty(\mathbb{R}^n) \mid \| x^\alpha \cdot \partial^\beta f(x) \|_\infty < \infty \text{ for all } \alpha, \beta \in \mathbb{N}_0^n \},$$

resp. smooth periodic functions. A physically reasonable solution $u$ and $p$ of (1) must fulfill the smoothness condition

$$u_1, \ldots, u_n, p \in C^\infty(\mathbb{R}^n \times [0, \infty)) \quad \text{resp. } C^\infty(\mathbb{T}^n \times [0, \infty))$$

and the bounded energy condition

$$\int u(x, t)^2 \, dx < C \text{ for all } t \geq t_0 \quad [1].$$

With $F = 0$, taking the curl of (1) gives

$$\partial_t \omega(x, t) = \nu \Delta \omega(x, t) - u(x, t) \cdot \nabla \omega(x, t) + \omega(x, t) \cdot \nabla u(x, t)$$

$$\omega(x, t_0) = \omega_0(x) := \text{rot} u_0(x) \quad (2)$$

with the vorticity $\omega(x, t) := \text{rot} u(x, t) \equiv \text{curl} u(x, t) = \nabla \times u(x, t)$, and we have

$$\omega \cdot \nabla u = (\omega_1 \partial_1 + \omega_2 \partial_2 + \omega_3 \partial_3) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \partial_1 u_1 \\ \partial_1 u_2 \\ \partial_1 u_3 \end{pmatrix} + \begin{pmatrix} \partial_2 u_1 \\ \partial_2 u_2 \\ \partial_2 u_3 \end{pmatrix} + \begin{pmatrix} \partial_3 u_1 \\ \partial_3 u_2 \\ \partial_3 u_3 \end{pmatrix} \quad (3)$$

More on the Euler and the Navier–Stokes equations can be found e.g. in [2–34] and references therein.

In this paper we investigate Schwartz function valued vorticity solutions of the Euler and Navier–Stokes equations. Since the initial values fulfill $u \in S(\mathbb{R}^3, \mathbb{R}^3)$ it is interesting if the solution $u$ of (1) resp. the vorticity $\omega$ of (2) stay in $S(\mathbb{R}^3, \mathbb{R}^3)$ or how they leave this space.

The spatial decay (asymptotics) of $u$ and $\omega$ has been investigated before. A classical result is that unless the Dobrokhotov–Shafarevich conditions [21] are fulfilled, the spacial decay in $u$ does not decay faster than $O(|x|^{-4})$. This especially covers the instantaneous breakdown of $u$ being a Schwartz function. For further studies see e.g. [33, 35], for the vorticity $\omega$ such a spreading was never observed, see e.g. [33, Ch. 4.11]. In [29, Prop. 3.1] it was shown that for the Navier–Stokes equation the vorticity remains a Schwartz function for small times. We also want to mention the works [40–46].

In this paper we show that the vorticity for the Euler and the Navier–Stokes equation remains a Schwartz function as long as the smooth solution exists (Theorem 7.1). Our approach is not affected by the Laplace operator and treats the Euler and the Navier–Stokes equations at the same time. It also covers the anisotropic Laplace operator

$$\nu \cdot \Delta := \nu_1 \partial_1^2 + \cdots + \nu_n \partial_n^2$$

with $\nu = (\nu_1, \ldots, \nu_n) \in [0, \infty)^n$.

Let $n, m \in \mathbb{N}$. In what follows $\| f \|_\infty := \sup_{x \in \mathbb{R}^n} |f(x)|$ is the supremum-norm on $\mathbb{R}^n$ and $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, and $|\alpha| := \alpha_1 + \cdots + \alpha_n$ are multi-index notations with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. By $T^* \leq \infty$ we denote the maximal time a classical solution of a PDE exists, i.e., the classical solution exists for all $t \in [0, T^*)$ and $T^*$ is maximal with this property. We denote by

$$C_k^\infty(\mathbb{R}^n, \mathbb{R}^m) := \{ f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m) \mid \| \partial^\alpha f \|_\infty < \infty \text{ for all } \alpha \in \mathbb{N}_0^n \}$$

In this paper we investigate Schwartz function valued vorticity solutions of the Euler and Navier–Stokes equations. Since the initial values fulfill $u \in S(\mathbb{R}^3, \mathbb{R}^3)$ it is interesting if the solution $u$ of (1) resp. the vorticity $\omega$ of (2) stay in $S(\mathbb{R}^3, \mathbb{R}^3)$ or how they leave this space.

The spatial decay (asymptotics) of $u$ and $\omega$ has been investigated before. A classical result is that unless the Dobrokhotov–Shafarevich conditions [21] are fulfilled, the spacial decay in $u$ does not decay faster than $O(|x|^{-4})$. This especially covers the instantaneous breakdown of $u$ being a Schwartz function. For further studies see e.g. [33, 35], for the vorticity $\omega$ such a spreading was never observed, see e.g. [33, Ch. 4.11]. In [29, Prop. 3.1] it was shown that for the Navier–Stokes equation the vorticity remains a Schwartz function for small times. We also want to mention the works [40–46].

In this paper we show that the vorticity for the Euler and the Navier–Stokes equation remains a Schwartz function as long as the smooth solution exists (Theorem 7.1). Our approach is not affected by the Laplace operator and treats the Euler and the Navier–Stokes equations at the same time. It also covers the anisotropic Laplace operator

$$\nu \cdot \Delta := \nu_1 \partial_1^2 + \cdots + \nu_n \partial_n^2$$

with $\nu = (\nu_1, \ldots, \nu_n) \in [0, \infty)^n$.

Let $n, m \in \mathbb{N}$. In what follows $\| f \|_\infty := \sup_{x \in \mathbb{R}^n} |f(x)|$ is the supremum-norm on $\mathbb{R}^n$ and $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, and $|\alpha| := \alpha_1 + \cdots + \alpha_n$ are multi-index notations with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. By $T^* \leq \infty$ we denote the maximal time a classical solution of a PDE exists, i.e., the classical solution exists for all $t \in [0, T^*)$ and $T^*$ is maximal with this property. We denote by

$$C_k^\infty(\mathbb{R}^n, \mathbb{R}^m) := \{ f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m) \mid \| \partial^\alpha f \|_\infty < \infty \text{ for all } \alpha \in \mathbb{N}_0^n \}$$
the set of all smooth bounded functions and by
\[ S(\mathbb{R}^n, \mathbb{R}^m) := \{ f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m) \mid \| x^\alpha \cdot \partial^\beta f(x) \|_\infty < \infty \text{ for all } \alpha, \beta \in \mathbb{N}_0^n \} \]
we denote the set of all Schwartz functions. Here, for functions \( f = (f_1, \ldots, f_n)^t \)
by \( |f| \) we denote \( |f|(x) := \sqrt{\int_1^2(x) + \cdots + f_n(x)^2} \) and \( \| x^\alpha f(x) \|_\infty := \sup_{x \in \mathbb{R}^n} |x^\alpha| \cdot |f|(x) \) for all \( \alpha \in \mathbb{N}_0^n \).

We study (1) and (2) via the initial value problem
\[
\begin{align*}
\partial_t f(x, t) &= \nu \Delta f(x, t) + g(x, t) \cdot \nabla f(x, t) + h(x, t) \cdot f(x, t) + k(x, t) \\
f(x, 0) &= f_0(x),
\end{align*}
\]
and the functions \( g(x, t) = (g_1(x, t), \ldots, g_n(x, t))^t, h(x, t) = (h_{i,j}(x, t))_{i,j=1}^n, \)
and \( k(x, t) = (k_1(x, t), \ldots, k_m(x, t))^t \) with \( n, m \in \mathbb{N} \) are known vector resp.
matrix functions.

To show the existence of solutions of (4) we split it into the following four
simpler parts and glue them together in a Trotter type \([47]\) fashion.

**Example 1.1** (heat equation). Let \( \nu > 0 \) and for all \( t > 0 \) let \( \Theta_{\nu,t}(x) := \frac{1}{\sqrt{4 \pi t}} \exp \left(-\frac{x^2}{4t}\right) \) be the heat kernel. Then for \( f_0 \in S(\mathbb{R}) \) the convolution
\[
f(x, t) := (\Theta_{\nu,t} * f_0)(x) = \int_{y \in \mathbb{R}} f_0(x-y) \cdot \Theta_{\nu,t}(y) \, dy \quad \in C^1([0, \infty), S(\mathbb{R}, \mathbb{R}))
\]
solves the initial value problem
\[
\begin{align*}
\partial_t f(x, t) &= \nu \cdot \Delta f(x, t) \quad \text{on } \mathbb{R} \times [0, \infty) \\
f(x, 0) &= f_0(x) \quad \text{on } \mathbb{R}.
\end{align*}
\]

In higher dimensions for the anisotropic Laplace operator \( \nu \cdot \Delta = \nu_1 \partial_1^2 + \cdots + \nu_n \partial_n^2 \)
we set \( \Theta_{\nu,t} := \Theta_{\nu_1,t}^{(1)} \cdots \Theta_{\nu_n,t}^{(n)} \), where \( \Theta_{\nu_1,t}^{(i)} \) is the one-dimensional heat kernel
acting resp. depending only on the \( x_i \)-coordinate.

**Example 1.2** (transport equation). Let \( f_0 \in S(\mathbb{R}) \) and \( g \in C([0, \infty), \mathbb{R}) \). Then
\[
f(x, t) := f_0 \left( x + \int_0^t g(s) \, ds \right) \quad \in C^1([0, \infty), S(\mathbb{R}, \mathbb{R}))
\]
is a solution of the initial value problem
\[
\begin{align*}
\partial_t f(x, t) &= g(t) \cdot \nabla f(x, t) \quad \text{on } \mathbb{R} \times [0, \infty) \\
f(x, 0) &= f_0(x) \quad \text{on } \mathbb{R}.
\end{align*}
\]

Note, that if \( g \) also depends on \( x \), then the previous simple formula does not
hold but serves for the ansatz
\[
f(x, t) \approx f_0 \left( x + \int_0^t g(x, s) \, ds \right) \quad \in C^1([0, \infty), S(\mathbb{R}, \mathbb{R}))
\]
for small times \( t \).
Example 1.3 (“stretching equation”). Let \( f_0 \in \mathcal{S}(\mathbb{R}) \) and \( h \in C([0, \infty), \mathcal{C}(\mathbb{R}, \mathbb{R})) \). Then
\[
f(x,t) := \exp \left( \int_0^t h(x,s) \, ds \right) \cdot f_0(x) \in C^1([0, \infty), \mathcal{S}(\mathbb{R}, \mathbb{R}))
\]
is a solution of the initial value problem
\[
\begin{align*}
\partial_t f(x,t) &= h(x,t) \cdot f(x,t) \quad \text{on } \mathbb{R} \times [0, \infty) \\
f(x,0) &= f_0(x) \quad \text{on } \mathbb{R}.
\end{align*}
\]

Example 1.4 (“addition equation”). Let \( k \in C([0, \infty), \mathcal{S}(\mathbb{R})) \) and \( f_0 \in \mathcal{S}(\mathbb{R}) \). Then
\[
f(x,t) = f_0(x) + \int_0^t k(x,s) \, ds \in C^1([0, \infty), \mathcal{S}(\mathbb{R}, \mathbb{R}))
\]
solves the initial value problem
\[
\begin{align*}
\partial_t f(x,t) &= k(x,t) \quad \text{on } \mathbb{R} \times [0, \infty) \\
f(x,0) &= f_0(x) \quad \text{on } \mathbb{R}.
\end{align*}
\]

Each of the initial value problems \( \partial_t f = A_i f \), \( i = 1, \ldots, 4 \), in Examples 1.1 to 1.4 gives a time-evolution or in higher dimensions at least an approximate time evolution \( f(x,t) = E_i(t,t_0)f_0 \). In all four time evolutions, when \( f_0 \in \mathcal{S}(\mathbb{R}) \), then also \( f = E_i(\cdot, t_0)f_0 \in C([0, \infty), \mathcal{S}(\mathbb{R}, \mathbb{R})) \). Our aim is to show that \( (4) \) with \( f_0 \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m) \) also possesses such a Schwartz valued solution and to give explicit bounds on all semi-norms. The four (approximate) solutions \( E_i(t,t_0)f_0 \) will be glued together in the Trotter fashion \( \{4\} \) by using a Schwartz valued version of the Arzelà–Ascoli Theorem (Lemma 2.1), i.e.,
\[
f(\cdot, t) = \lim_{N \to \infty} E_4(t, \frac{N-1}{N}t) \cdots E_1(t, \frac{N-1}{N}t)E_4(\frac{N-1}{N}t, \frac{N-2}{N}t) \cdots E_1(\frac{1}{N}t, 0)f_0
\]
where the convergence is controlled in the Schwartz space \( \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m) \).

The paper is structured as follows. In Section 2 we give for completeness of the paper the Schwartz function valued version of the Arzelà–Ascoli Theorem (Lemma 2.1). In Section 3 the family \( \{f_N\}_{N \in \mathbb{N}} \) of approximate solutions of \( (4) \) is defined. In Section 4 the cover \( \text{cov}_R f \) of a function \( f \) is introduced, i.e., \( |f| \leq \text{cov}_R f \). It is used in Section 5 to prove the main theorem (Theorem 5.1). Theorem 5.1 is applied in Section 6 to Burgers’ equation and in Section 7 to the Euler and the Navier–Stokes equations.

2. The Schwartz Function Valued Arzelà–Ascoli Theorem

A set \( M \subset \mathcal{S}(\mathbb{R}^n) \) is bounded if for all \( \alpha, \beta \in \mathbb{N}_0^n \) there are \( C_{\alpha,\beta} > 0 \) with
\[
\sup_{f \in M} \| x^\alpha \cdot \partial^\beta f(x) \|_\infty \leq C_{\alpha,\beta} < \infty.
\]
\( \mathcal{S}(\mathbb{R}^n) \) is a complete Montel space and every bounded set is relatively compact.
In the proof of the Arzelà–Ascoli Theorem it is crucial that the continuous functions are (real- or) complex-valued to apply the Bolzano–Weierstraß Theorem since \( \mathbb{R} \) and \( \mathbb{C} \) have the Heine–Borel property: Every bounded sequence has a convergent subsequence resp. bounded and closed sets are compact. But every Montel space also has the Heine–Borel property, i.e., the classical proof of the Arzelà–Ascoli Theorem [48–50], see e.g. [51, pp. 85–86], can be literally used for \( \mathcal{S}(\mathbb{R}^n) \). While this was known before, for the sake of completeness of the paper and to make it self-contained we briefly state and prove the result.

Lemma 2.1 (\( \mathcal{S}(\mathbb{R}^n) \)-valued version of Arzelà–Ascoli). Let \( n, m \in \mathbb{N} \), \( T > 0 \), and \( \{ f_n \}_{n \in \mathbb{N}} \subseteq C([0, T], \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m)) \). Assume that

\[
(i) \quad \sup_{n \in \mathbb{N}, t \in [0, T]} \| \partial_x^\alpha \partial^\beta_t f_n(x, t) \|_\infty < \infty \quad \text{for all } \alpha, \beta \in \mathbb{N}_0^n, \text{ and}
\]

\[
(ii) \quad \{ f_n \}_{n \in \mathbb{N}} \text{ is equi-continuous, i.e., for all } \varepsilon > 0 \text{ exists } \delta = \delta(\varepsilon) > 0 \text{ such that for all } N \in \mathbb{N} \text{ we have}
\]

\[
|t - s| < \delta \quad \Rightarrow \quad \| f_N(x, t) - f_N(x, s) \|_\infty \leq \varepsilon.
\]

Then \( \{ f_N \}_{N \in \mathbb{N}} \) is relatively compact in \( C([0, T], \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m)) \).

Proof. It is sufficient to prove the result for \( m = 1 \). Then it holds in one component of \( f_N \) and by choosing subsequences it holds in all components.

Let \( \{ t_k \}_{k \in \mathbb{N}} \subseteq [0, T] \) be a dense countable subset such that for every \( \varepsilon > 0 \) there is a \( k(\varepsilon) \in \mathbb{N} \) with

\[
\sup_{t \in [0, T]} \inf_{1 \leq k \leq k(\varepsilon)} |t - t_k| \leq \varepsilon.
\]

Let \( t \in [0, T] \). Since \( \{ f_N(\cdot, t) \}_{N \in \mathbb{N}} \) is a bounded set in the complete Montel space \( \mathcal{S}(\mathbb{R}^n) \), it has a convergent subsequence. Let \( (N_{i,1})_{i \in \mathbb{N}} \subseteq \mathbb{N} \) be such that \( (f_{N_{i,1}}(\cdot, t_1))_{i \in \mathbb{N}} \) converges in \( \mathcal{S}(\mathbb{R}^n) \). Take a subsequence \( (N_{i,2})_{i \in \mathbb{N}} \) of \( (N_{i,1})_{i \in \mathbb{N}} \) such that \( (f_{N_{i,2}}(\cdot, t_2))_{i \in \mathbb{N}} \) converges in \( \mathcal{S}(\mathbb{R}^n) \). Hence, by the diagonal process of choice we get a subsequence \( (f_{N_i})_{i \in \mathbb{N}} \) with \( N_i := N_{i, i} \) which converges in \( \mathcal{S}(\mathbb{R}^n) \) for all \( t_k \).

Let \( \varepsilon > 0 \). By the equi-continuity of \( \{ f_N \}_{N \in \mathbb{N}} \) there is a \( \delta = \delta(\varepsilon) > 0 \) such that \( |t - s| < \delta \) implies \( \| f_N(x, t) - f_N(x, s) \|_\infty \leq \varepsilon \). Hence, for every \( t \in [0, T] \) there exists a \( k \) with \( k \leq k(\varepsilon) \) such that

\[
\| f_{N_i}(x, t) - f_{N_j}(x, t) \|_\infty \leq \| f_{N_i}(x, t) - f_{N_i}(x, t_k) \|_\infty + \| f_{N_i}(x, t_k) - f_{N_j}(x, t_k) \|_\infty
\]

\[
+ \| f_{N_j}(x, t_k) - f_{N_j}(x, t) \|_\infty
\]

\[
\leq 2\varepsilon + \| f_{N_j}(x, t_k) - f_{N_j}(x, t_k) \|_\infty.
\]

Thus \( \lim_{i, j \to \infty, t \in [0, T]} \sup_{i \leq k \leq k(\varepsilon)} \| f_{N_i}(x, t) - f_{N_j}(x, t) \|_\infty \leq 2\varepsilon \) and since \( \varepsilon > 0 \) was arbitrary we have

\[
\lim_{i, j \to \infty, t \in [0, T]} \sup_{i \leq k \leq k(\varepsilon)} \| f_{N_i}(x, t) - f_{N_j}(x, t) \|_\infty = 0.
\]

So for every \( x \in \mathbb{R}^n \) the sequence
for all $\alpha, \beta$. Hence by construction $f(\cdot, t_k) \in \mathcal{S}(\mathbb{R}^n)$ for all $t_k$ dense in $[0, T]$. But
\[
\|x^\alpha \cdot \partial^\beta f(x, t)\|_{L^\infty} \leq \sup_{s \in [0, T], N \in \mathbb{N}} \|x^\alpha \cdot \partial^\beta f_N(x, s)\|_{L^\infty} < \infty
\]
for all $\alpha, \beta \in \mathbb{N}_0^n$ implies $f(\cdot, t) \in \mathcal{S}(\mathbb{R}^n)$ for all $t \in [0, T]$. \hfill \Box

### 3. The Approximate Solutions $f_N$

**Definition 3.1.** Let $N \in \mathbb{N}$ and $T > 0$. A decomposition $Z_N$ of $[0, T]$ is a set $Z_N = \{t_0, t_1, \ldots, t_N\}$ with $t_0 = 0 < t_1 < \cdots < t_N = T$ and we set $\Delta Z_N := \max_{i=1, \ldots, N} |t_i - t_{i-1}|$.

We use the following type of functions.

**Definition 3.2.** Let $d \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $T > 0$. We denote by
\[
C^d([0, T], \mathcal{S}(\mathbb{R}^n))
\]
all functions $f : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ such that
\begin{enumerate}[(i)]  
  
  \item for every $x \in \mathbb{R}^n$ we have $f(x, \cdot) \in C^d([0, T], \mathbb{R})$,  
  
  \item $f(\cdot, t), \partial_t f(\cdot, t), \ldots, \partial_t^d f(\cdot, t) \in \mathcal{S}(\mathbb{R}^n)$ for all $t \in [0, T]$, and  
  
  \item $\partial_i^\alpha f \in C(\mathbb{R}^n \times [0, T], \mathbb{R})$ for all $i = 0, 1, \ldots, d$ and $\alpha \in \mathbb{N}_0^n$.  
\end{enumerate}

We will always denote by $\partial_t$ the time derivative and we will therefore abbreviate the spatial derivatives $\partial^\alpha = \partial^\alpha_x$. (iii) implies by the Theorem of Schwarz that the order of the derivatives $\partial_t, \partial_1, \ldots, \partial_n$ can be arbitrary.

**Definition 3.3.** Let $n, m \in \mathbb{N}$, $f_0 \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m)$, and $t_0 \in \mathbb{R}$. We define the following time evolutions $E_1, \ldots, E_4$ for all $t \in [t_0, \infty)$:
\begin{enumerate}[(i)]  
  
  \item Let $\nu = (\nu_1, \ldots, \nu_n) \in [0, \infty)^n$. Then we define
    \[
    E_1(t, t_0) f_0 := \Theta_{\nu, t - t_0} \ast f_0
    \]
    with $\Theta_{\nu, t} := \Theta_{\nu, t}^{(1)} \cdots \Theta_{\nu, t}^{(n)}$ where $\Theta_{\nu, t}^{(i)}$ is the one-dimensional heat kernel acting resp. depending only on the $x_i$-coordinate.  
  
  \item For $g = (g_1, \ldots, g_n) \in C([t_0, \infty), C^\infty_0(\mathbb{R}^n, \mathbb{R}^n))$ we define
    \[
    E_2(t, t_0) f_0 := f_0 \left( x + \int_{t_0}^t g(\cdot, s) \, ds \right).
    \]
  
  \item For $h = (h_{i,j})_{i,j=1}^{m} \in C([t_0, \infty), C^\infty_0(\mathbb{R}^n, \mathbb{R}^{m \times m}))$ we define
    \[
    E_3(t, t_0) f_0 := \left( 1 + \int_{t_0}^t h(\cdot, s) \, ds \right) \cdot f_0.
    \]
\end{enumerate}
(iv) For $k = (k_1, \ldots, k_m) \in \mathbb{C}([t_0, \infty), \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m))$ we define

$$E_4(t, t_0)f_0 := f_0 + \int_{t_0}^t k(\cdot, s) \, ds.$$

We see that the $E_4$’s are (approximate) time evolutions $\partial_t f = A_i f$ with respect to the operators $A_1 = \nu \cdot \Delta$, $A_2 = g \cdot \nabla$, the multiplication operator $A_3 = h$, and the addition operator $A_4 = + k$. By the Trotter approach [47] as an approximate solution of $\partial_t f = (A_1 + \cdots + A_4) f$ we can therefore take

$$E_{4}(t_4, t_{N-1})E_{3}(t_3, t_{N-1})E_{2}(t_2, t_{N-1})E_{1}(t_1, t_{N-1})E_{4}(t_{N-1}, t_{N-2}) \cdots E_{1}(t_1, t_0)f_0.$$  

**Definition 3.4.** Let $n, m \in \mathbb{N}$, $T > 0$, $f_0 \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m)$, and let $\nu, g, h$, and $k$ be as in Definition [47]. For each $N \in \mathbb{N}$ let $Z_N$ be a decomposition of $[0, T]$. We define the functions $f_N : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^m$ piece-wise on each interval $[t_i, t_{i+1}]$ by the following:

(i) For $t \in [t_0, t_1]$ we set

$$f_N(\cdot, t) := E_{4}(t, t_0)E_{3}(t, t_0)E_{2}(t, t_0)E_{1}(t, t_0)f_0.$$  

(ii) For $t \in [t_i, t_{i+1}]$ with $i = 1, \ldots, N-1$ we set

$$f_N(\cdot, t) := E_{4}(t, t_i)E_{3}(t, t_i)E_{2}(t, t_i)E_{1}(t, t_i)f_N(\cdot, t_i).$$  

The following ensures that we can apply the Arzelà–Ascoli Theorem to the family $\{f_N\}_{N \in \mathbb{N}}$.

**Lemma 3.5.** $f_N \in C([0, T], \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m))$ and $C^1$ in $t \in (t_i, t_{i+1})$.

**Proof.** For all $f \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m)$ and $t \geq t' \geq 0$ we have $E_j(t, t')f \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m)$, $j = 1, \ldots, 4$, and from Definition [47] we also have $C^1$ in $t \in (t_i, t_{i+1})$. 

4. The Cover of a Function

For the approximate solutions $f_N$’s we need to bound all semi-norms

$$\|x^\alpha \cdot \partial^\beta f_N(x, t)\|_{\infty}$$

to apply the Arzelà–Ascoli Theorem. To handle these extensive calculations, we introduce the cover cov$_R f$ of a function $f$, i.e., $|f_N(\cdot, t)| \leq \text{cov}_{R} f_N(\cdot, t)$.

**Definition 4.1.** Let $n, m \in \mathbb{N}$ and $R \geq 0$. We introduce the cover $\text{cov}_{R}$ of a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m)$ as the map

$$\text{cov}_{R} : \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m) \rightarrow C(\mathbb{R}^n, \mathbb{R})$$

defined by

$$ (\text{cov}_{R} f)(x) := \max_{y \in \mathbb{R}^n : ||y||_2 \geq ||x||_2 - R} \|f(y)\|_2.$$
Lemma 4.2. Let \( n, m \in \mathbb{N} \), \( R, R' \geq 0 \), \( a, b \in \mathbb{R} \), and \( f, f' \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m) \). The cover \( \text{cov} \) has the following general properties:

(i) \( \text{cov}_R f \) is non-negative, radial symmetric, decreases with increasing \( \|x\|_2 \), and fulfills \( |f| \leq \text{cov}_0 f \).

(ii) If \( |f| \leq |f'| \), then \( \text{cov}_R f \leq \text{cov}_R f' \).

(iii) \( \text{cov}_R f \leq \text{cov}_{R+R'} f \).

(iv) \( \text{cov}_R (af + bf') \leq |a| \cdot \text{cov}_R f + |b| \cdot \text{cov}_R f' \).

(v) \( \text{cov}_R (\text{cov}_R f) = \text{cov}_{R+R'} f \).

(vi) For \( d \in \mathbb{N} \) let \( C_d > 0 \) be such that \( |f|(x) \leq \frac{C_d}{1 + \|x\|_{2}^{d}} \) for all \( x \in \mathbb{R}^n \). Then

\[
(\text{cov}_0 f)(x) \leq \frac{C_d}{1 + \|x\|_{2}^{d}}
\]

for all \( x \in \mathbb{R}^n \).

(vii) For \( d \in \mathbb{N} \) let \( C_d > 0 \) be such that \( |f(x)| \leq \frac{C_d}{1 + \|x\|_{2}^{d}} \) for all \( x \in \mathbb{R}^n \). Then

\[
(\text{cov}_R f)(x) \leq \frac{C_d}{1 + \|x\|_{2}^{d}} \quad \text{with} \quad r := \max\{0, \|x\|_{2} - R\}
\]

for all \( x \in \mathbb{R}^n \).

Proof. (i)-(iv): Follows immediately from the Definition 4.1.

(v): By (i) we have that \( \text{cov}_R f \) is radial symmetric and decreases with increasing \( \|x\|_2 \). Hence, for fixed \( x \in \mathbb{R}^n \) there is a \( y \in \mathbb{R}^n \) with \( \|y\|_2 \geq \|x\|_2 - R' \) such that \( |f(y')| = (\text{cov}_R f)(x) \). But in \( \text{cov}_{R+R'} f \) we have the restriction \( \|y\|_2 \geq \|x\|_2 - R'' \), i.e., a larger range for \( y \) and hence the inequality holds.

(vi): Since \( a_d(x) := \frac{C_d}{1 + \|x\|_{2}^{d}} \) is non-negative, radial symmetric, and decreases with increasing \( \|x\|_2 \) we have \( \text{cov}_0 a_d = a_d \) and since \( |f| \leq a_d \) we have that (ii) implies \( \text{cov}_0 f \leq \text{cov}_0 a_d = a_d \).

(vii): Set \( a_d(x) := \frac{C_d}{1 + \|x\|_{2}^{d}} \). Then

\[
(\text{cov}_R f)(x) \leq \text{cov}_R (\text{cov}_0 f) \quad \text{(v)}
\]

and since \( a_d \) is non-negative, radial symmetric, decreases with increasing \( \|x\|_2 \) we have \( \text{cov}_R a_d = \frac{C_d}{1 + r'} \) with \( r := \max\{0, \|x\|_{2} - R\} \).

We use the cover \( \text{cov} \) to bound the Schwartz semi-norms for the approximate time evolution from \( E_1, \ldots, E_4 \). Hence, we collect in the following the special properties connected to the \( E_i \)'s.
Lemma 4.3. Let $n, m \in \mathbb{N}$, $R, R' \geq 0$, $t_0, t \in \mathbb{R}$ with $t \geq t_0$, and $f \in S(\mathbb{R}^n, \mathbb{R}^m)$. Let $\nu, g, h, \text{ and } k$ be as in Definition 3.3. The cover $\text{cov}$ has the following special properties:

(i) $\text{cov}_{R}(E_1(t, t_0)f) \leq E_1(t, t_0)\text{cov}_{R}f$.

(ii) With $G(t, t_0) := \int_{t_0}^{t} \|g(\cdot, s)\|_{\infty} \, ds$ we get

$$|E_2(t, t_0)f| \leq \text{cov}_{G(t, t_0)}f.$$ 

(iii) With $H(t, t_0) := \int_{t_0}^{t} \|h(\cdot, s)\|_{\infty} \, ds$ we have

$$\text{cov}_{R}(E_3(t, t_0)f) \leq (1 + H(t, t_0)) \cdot \text{cov}_{R}f.$$ 

(iv) $\text{cov}_{R}(E_4(t, t_0)f) \leq \text{cov}_{R}f + \int_{t_0}^{t} \text{cov}_{R}k(\cdot, s) \, ds$.

Proof. (i): Follow immediately from the fact that $E_1(t, t_0)$ is the convolution with the non-negative heat kernel $\Theta_{\nu, t-t_0}$.

(ii): Since $E_2(t, t_0)$ is a translation, each point $x \in \mathbb{R}^n$ with $\|x\|_2 = r$ is moved to some $x' \in \mathbb{R}^n$ with $\|x'\|_2 \leq r + G(t, t_0)$ which proves the inequality.

(iii): Follows immediately from taking the supremum of $1 + \int_{t_0}^{t} h(\cdot, s) \, ds$.

(iv): We have

$$\text{cov}_{R}(E_4(t, t_0)f) = \text{cov}_{R} \left( f + \int_{t_0}^{t} k(\cdot, s) \, ds \right)$$

$$\leq \text{cov}_{R}f + \int_{t_0}^{t} \text{cov}_{R}k(\cdot, s) \, ds.$$ 

$\square$

We want to bound all semi-norms in the Schwartz space. Hence, we also have to look at derivatives of the approximate solutions $f_N$.

Lemma 4.4. Let $n, m \in \mathbb{N}$, $d \in \mathbb{N}_0$, and $f_0 \in S(\mathbb{R}^n, \mathbb{R}^m)$. Let $\nu, g, h, \text{ and } k$ be as in Definition 3.3. Set

$$f(x, t) := E_4(t, 0)E_3(t, 0)E_2(t, 0)E_1(t, 0)f_0$$

$$= \left( 1 + \int_{0}^{t} h(x, s) \, ds \right) (\Theta_{\nu, t} * f_0) \left( x + \int_{0}^{t} g(x, s) \, ds \right) + \int_{0}^{t} k(x, s) \, ds \quad (10)$$

with $F := \Theta_{\nu, t} * f_0$, $H := \int_{0}^{t} h(\cdot, s) \, ds$, $G := \int_{0}^{t} g(\cdot, s) \, ds$, $G_j := \int_{0}^{t} g_j(\cdot, s) \, ds$, and $K := \int_{0}^{t} k(\cdot, s) \, ds$. Then for $i_1, \ldots, i_d \in \{1, \ldots, n\}$ we have

$$\partial_{i_1} \ldots \partial_{i_d} f(x, t) = \left( 1 + H + \sum_{r=1}^{d} \partial_{i_r} G_i \right) \cdot (\partial_{i_1} \ldots \partial_{i_d} F)(x + G) + \partial_{i_1} \ldots \partial_{i_d} K$$

$\text{9}$
Assume (11) holds for some \( \gamma \) with \( o(1) \).

We prove (11) by induction over \( d \).

Proof. We prove (11) by induction over \( d \in \mathbb{N}_0 \).

- \( d = 0 \): Clear, since (10) = (11).
- \( d \to d + 1 \): Assume (11) holds for some \( d \in \mathbb{N}_0 \). Let \( i_1, \ldots, i_{d+1} \in \{1, \ldots, n\} \). Then we have

\[
\begin{align*}
\partial_{i_1} \cdots \partial_{i_d} F &= \partial_{i_{d+1}} [\partial_{i_1} \cdots \partial_{i_d} F] \\
&= \partial_{i_{d+1}} \left[ \left( 1 + H + \sum_{r=1}^{d} \partial_{i_r} G_{i_r} \right) \cdot (\partial_{i_1} \cdots \partial_{i_d} F)(x + G) \\
&\quad + \sum_{r=1, j_r \neq i_r}^{d} \partial_{i_r} G_{j_r} \cdot (\partial_{i_1} \cdots \partial_{i_{r-1}} \partial_{i_r} \partial_{i_{r+1}} \cdots \partial_{i_1} F)(x + G) \\
&\quad + \partial_{i_d} \cdots \partial_{i_1} K + O(t^2) + o(d - 1, t) \right] \\
&= \left( \partial_{i_{d+1}} H + \sum_{r=1}^{d} \partial_{i_{d+1}} \partial_{i_r} G_{i_r} \right) \cdot (\partial_{i_1} \cdots \partial_{i_d} F)(x + G) \\
&\quad + \left( 1 + H + \sum_{r=1}^{d} \partial_{i_r} G_{i_r} \right) \cdot \left( \sum_{j_{d+1}=1}^{n} (\partial_{j_{d+1}} + \partial_{i_{d+1}} G_{j_{d+1}}) \right) \\
&\quad \times (\partial_{i_d} \cdots \partial_{i_1} F)(x + G) \\
&\quad + \sum_{r=1, j_r \neq i_r}^{d} \partial_{i_r} G_{j_r} \cdot \left( \partial_{i_1} \cdots \partial_{i_{r-1}} \partial_{i_r} \partial_{i_{r+1}} \cdots \partial_{i_1} F \right)(x + G) \\
&\quad + \partial_{i_{d+1}} \cdots \partial_{i_1} K + O(t^2) + o(d - 1, t)
\end{align*}
\]

where \( o(d - 1, t) \) is the set of functions \( \{ \text{derivatives of } h \text{ or } g \} \cdot O_{v,t} \cdot (\partial^\gamma f_0) \) with \( \gamma \in \mathbb{N}^n_0 \) and \( |\gamma| \leq d - 1 \), i.e., the growth in \( t \) is at most linear.

Therefore, \( (\partial_{i_1} \cdots \partial_{i_d} F \in O(t^2) + o(d - 1, t)) \) or \( (\partial_{i_d} \cdots \partial_{i_1} K \in O(t^2) + o(d - 1, t)) \) for \( j_{d+1} = i_{d+1} \); for \( j_{d+1} \neq i_{d+1} \).
\[
\begin{align*}
= & \left(1 + H + \sum_{r=1}^{d} \frac{\partial_r G_{tr}}{t} \right) \cdot \left(1 + \partial_{\lambda+1} G_{\lambda+1} \right) \cdot (\partial_{\lambda+1} \cdot \partial_1 F)(x + G) \\
+ & \sum_{j=1}^{n} \frac{\partial_0 G_{jr}}{t} \cdot \partial_{\lambda+1} G_{\lambda+1} \cdot (\partial_{\lambda+1} \cdot \partial_1 F)(x + G) \\
+ & \partial_{\lambda+1} \cdot \partial_1 F + O(t^2) + o(d, t)
\end{align*}
\]

which proves (11) for \(d + 1\) and hence by induction (11) for all \(d \in \mathbb{N}_0\).

5. The Existence of a Schwartz Function Valued Solution

The next result is the main theorem of this article. It completely solves the Schwartz function regularity problem of (11) with explicit bounds for the Schwartz function semi-norms \(\|x^n \cdot \partial^\beta f(x, t)\|_\infty\)

**Theorem 5.1.** Let \(n, m \in \mathbb{N}, \ d \in \mathbb{N}_0\), and \(\nu = (\nu_1, \ldots, \nu_n) \in [0, \infty)^n\). Furthermore, let \(g \in C^d([0, \infty), C^\infty_b(\mathbb{R}^n, \mathbb{R}^n))\), \(h \in C^d([0, \infty), C^\infty(\mathbb{R}^n, \mathbb{R}^{m \times m}))\), and \(k \in C^d([0, \infty), \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m))\). Set \(H(t_1, t_0) := \int_{t_0}^{t_1} \|h(\cdot, s)\|_\infty \, ds\), \(G(t_1, t_0) := \int_{t_0}^{t_1} \|g(\cdot, s)\|_\infty \, ds\), and \(G'(t_1, t_0) := \int_{t_0}^{t_1} \|\nabla g(\cdot, s)\|_\infty \, ds\) for all \(t_1 \geq t_0 \geq 0\). For \(f_0 \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m)\) the initial value problem

\[
\begin{align*}
\partial_t f &= \nu \cdot \Delta f + g \cdot \nabla f + h \cdot f + k \\
f(\cdot, 0) &= f_0
\end{align*}
\]
has a solution \( f \in C^{d+1}([0, \infty), \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m)) \) with the covers
\[
|\partial^\beta f(\cdot, t)| \leq B_{|\beta|}(\cdot, t) := \exp \left( H(t, 0) + b^2 \cdot G'(t, 0) \right)
\times E_1(t, 0) \sup_{\beta} \left( \max \partial^\beta f_0 \right)
+ \int_0^t \exp \left( H(t, s) + b^2 \cdot G'(t, s) \right)
\times E_1(t, s) \sup_{\beta} \left( \max \partial^\beta k(\cdot, s) \right) ds
+ B_{d-1}(\cdot, t)
\]
for all \( \beta \in \mathbb{N}_0 \) such that
\[
\|x^\alpha \cdot \partial^\beta f(x, t)\|_\infty \leq \max_{x \in \mathbb{R}^n} \|x\|_2^{|\alpha|} \cdot B_{|\beta|}(x, t) < \infty
\]
for all \( \alpha, \beta \in \mathbb{N}_0^d \) and \( t \geq 0 \). Therein, \( B_d(\cdot, t) \) is a linear combination of bounds \( B_j \) for \( j \leq d - 1 \) with coefficients as integrals over \( \partial^\gamma g \) and \( \partial^\gamma h \) with \( |\gamma| \leq d \).

**Proof.** Let \( N \in \mathbb{N}, T > 0 \), and \( \Delta Z_N \) be a decomposition of \([0, T]\). Take the \( f_N \)'s from Definition 3.4, i.e., \( f_N \in C([0, T], \mathcal{S}(\mathbb{R}^n, \mathbb{R}^m)) \) by Lemma 3.3.

We look at the family \( \{f_N\}_{N \in \mathbb{N}} \) and want to use Lemma 2.1 to find an accumulation point \( f \) for \( N \to \infty \) with \( \Delta Z_N \to 0 \). Since \( T > 0 \) is arbitrary, it is sufficient that we bound the semi-norms at \( t = T \). Additionally, since
\[
|\partial^\beta f_N(\cdot, T)| \leq \sup_{\beta} \left( \max |\partial^\beta f_0(\cdot, T)\right)
\]
we have
\[
\|x^\alpha \cdot \partial^\beta f_N(x, T)\|_\infty \leq \|x^\alpha \cdot \partial^\beta f_0(x, T)\|_\infty.
\]
We will proceed via induction over \( b = |\beta| \in \mathbb{N}_0 \).

**b = 0:** We have
\[
|f_N(\cdot, t_N)|
\]
\[
\text{Lem. (2.2)}
\]
\[
\leq \text{Dfn. (3.4)}
\]
\[
\leq \exp(H(t, t_{N-1})) \cdot \exp(E_2(t, t_{N-1}) E_1(t, t_{N-1}) f_N(\cdot, t_{N-1})
+ \int_{t_{N-1}}^{t_N} \sup_{k(\cdot, s)} ds
\]
\[
\text{Lem. (2.3)}
\]
\[
\leq \exp(H(t, t_{N-1})) \cdot \exp(E_2(t, t_{N-1}) E_1(t, t_{N-1}) f_N(\cdot, t_{N-1})
+ \int_{t_{N-1}}^{t_N} \sup_{k(\cdot, s)} ds
\]
\[
\text{Lem. (2.3)}
\]
\[
\leq \exp(H(t, t_{N-1})) \cdot \exp(E_2(t, t_{N-1}) E_1(t, t_{N-1}) f_N(\cdot, t_{N-1})
+ \int_{t_{N-1}}^{t_N} \sup_{k(\cdot, s)} ds
\]
\[
\text{Lem. (2.3)}
\]
\[
\leq \exp(H(t, t_{N-1})) \cdot \exp(E_2(t, t_{N-1}) E_1(t, t_{N-1}) f_N(\cdot, t_{N-1})
+ \int_{t_{N-1}}^{t_N} \sup_{k(\cdot, s)} ds
\]

12
\[ + \int_{t_{N-1}}^{t_N} \text{cov} k(\cdot, s) \, ds \]  

Lemma 4.3(i) \leq \exp(H(t_N, t_{N-1})) \cdot E_1(t_N, t_{N-1}) \text{cov} G(t_N, t_{N-1}) f_N(\cdot, t_N) 

and applying Definition 3.4 and Lemma 4.3(i-iv) on the time interval \([t_{N-1}, t_N] \) in the same way gives

\[ \leq \exp(H(t_N, t_{N-2})) \cdot E_1(t_N, t_{N-2}) \text{cov} G(t_N, t_{N-2}) f_N(\cdot, t_{N-2}) \]

\[ + \int_{t_{N-2}}^{t_N} E_1(t_N, t_{N-2}) \text{cov} k(\cdot, s) \, ds \]

and proceeding gives finally

\[ \leq \exp(H(t_N, t_0)) \cdot E_1(t_N, t_0) \text{cov} G(t_N, t_0) f_0 \]

\[ + \sum_{i=0}^{N} \exp(H(t_N, t_{N-i})) \cdot \int_{t_{N-i-1}}^{t_{N-1}} E_1(t_{N-i}, t_{N-i}) \text{cov} G(t_{N-i}, t_{N-i}) k(\cdot, s) \, ds \]

which converges by Riemann integration for \( N \to \infty \) with \( \Delta Z_N \to 0 \) to

\[ \to B_0(x, T) = \exp(H(T, 0)) \cdot E_1(T, 0) \text{cov} G(T, 0) f_0 \]

\[ + \int_0^T \exp(H(T, s)) \cdot E(T, s) \text{cov} G(T, s) k(\cdot, s) \, ds. \]

By Lemma 4.2(vi) and (vii) we have that

\[ \| x^\alpha \cdot f_N(x, T) \|_\infty \leq \sup_{x \in \mathbb{R}^n} \| x \|_2^{|\alpha|} \cdot B_0(x, T) < \infty \]

for all \( \alpha \in \mathbb{N}_0^n \).

\[ d \to d + 1: \] Assume for all \( i = 0, \ldots, d \) we have bounds \( B_i \) with

\[ | \partial^\beta f_N(x, T) | \leq B_{|\beta|}(x, T) \]

such that

\[ \| x^\alpha \cdot \partial^\beta f_N(x, T) \|_\infty \leq \sup_{x \in \mathbb{R}^n} \| x \|_2^{|\alpha|} \cdot B_{|\beta|}(x, T) < \infty \]

for all \( N \in \mathbb{N} \) and all \( \alpha, \beta \in \mathbb{N}_0^n \) with \( |\beta| \leq d \). We show that such a bound \( B_{d+1} \) also exists.

From Lemma 4.3 we have that \( \partial^\beta f(\cdot, t_N) \) gives an induction from \( t_i \) to \( t_{i-1} \) where the derivative \( \partial^\beta \) applies to \( f_N \) again, gives contributions of order \( \Delta Z_N^2 \),

13
contributions from lower derivatives linear in \( \Delta Z_N \). In the limit \( \Delta Z_N \to 0 \) by Riemann integration (14) the \( \Delta Z_N^2 \) contributions vanish and the \( o(d, \Delta Z_N) \) contributions become a sum over the covers \( B_0, \ldots, B_d \). Hence, to shorten the calculations, we only drag \( \mathcal{O}(\Delta Z_N^2) \) and \( o(d, \Delta Z_N) \) through the calculations:

\[
\max_{\beta \in N^d} |\partial_\beta f_N(\cdot, t_N)| \\
\text{Lem. 5.2(i)} \\
\leq \text{coo}(\max_\beta \partial_\beta f_N(\cdot, t_N)) \\
\text{Def. 5.2(iii)} \\
\text{coo}(\max_\beta \partial_\beta E_4(t_N, t_{N-1}) \ldots E_1(t_N, t_{N-1}) f_N(\cdot, t_{N-1})) \\
\text{Lem. 1.3(iv)} \\
\text{coo}(\max_\beta \partial_\beta E_3(t_N, t_{N-1}) \ldots E_1(t_N, t_{N-1}) f_N(\cdot, t_{N-1})) \\
+ \int_{t_{N-1}}^{t_N} \text{coo}(\max_\beta \partial_\beta k(\cdot, s)) \, ds \\
\text{Lem. 1.3} \\
\leq (1 + H(t_N, t_{N-1}) + b \cdot G'(t_N, t_{N-1})) \\
\times \text{coo}(\max_\beta \partial_\beta E_2(t_N, t_{N-1}) E_1(t_N, t_{N-1}) f_N(\cdot, t_{N-1})) \\
+ b(b-1) \cdot G'(t_N, t_{N-1}) \\
\times \text{coo}(\max_\beta \partial_\beta E_2(t_N, t_{N-1}) E_1(t_N, t_{N-1}) f_N(\cdot, t_{N-1})) \\
+ \int_{t_{N-1}}^{t_N} \text{coo}(\max_\beta \partial_\beta k(\cdot, s)) \, ds + \mathcal{O}(\Delta Z_N^2) + o(d, \Delta Z_N) \\
b + b(b-1) = b^2 \\
\leq (1 + H(t_N, t_{N-1}) + b^2 \cdot G'(t_N, t_{N-1})) \\
\times \text{coo}(\max_\beta \partial_\beta E_2(t_N, t_{N-1}) E_1(t_N, t_{N-1}) f_N(\cdot, t_{N-1})) \\
+ \int_{t_{N-1}}^{t_N} \text{coo}(\max_\beta \partial_\beta k(\cdot, s)) \, ds + \mathcal{O}(\Delta Z_N^2) + o(d, \Delta Z_N) \\
\text{Lem. 1.3} \\
\leq (1 + H(t_N, t_{N-1}) + b^2 \cdot G'(t_N, t_{N-1})) \\
\times E_1(t_N, t_{N-1}) \text{coo}(\max_\beta \partial_\beta f_N(\cdot, t_{N-1})) \\
+ \int_{t_{N-1}}^{t_N} \text{coo}(\max_\beta \partial_\beta k(\cdot, s)) \, ds + \mathcal{O}(\Delta Z_N^2) + o(d, \Delta Z_N) \\
1 + y \leq \exp(H(t_N, t_{N-1}) + b^2 \cdot G'(t_N, t_{N-1})) \\
\times E_1(t_N, t_{N-1}) \text{coo}(\max_\beta \partial_\beta f_N(\cdot, t_{N-1})) \\
+ \int_{t_{N-1}}^{t_N} \text{coo}(\max_\beta \partial_\beta k(\cdot, s)) \, ds + \mathcal{O}(\Delta Z_N^2) + o(d, \Delta Z_N) \\
\text{and proceeding with this on each interval } [t_i, t_{i-1}] \text{ we finally get} \\
\leq \exp(H(t_N, t_0) + b^2 \cdot G'(t_N, t_0)) \\
14
\[ 
\times E_1(t_N, t_0) \text{cov}_{G(t_N, t_0)} \left( \max_{\beta} \partial^\beta f_N(\cdot, t_0) \right) 
\]

\[ 
+ \sum_{i=1}^{N} \exp \left( H(t_N, t_{N+1-i}) + b^2 \cdot G'(t_N, t_{N+1-i}) \right) 
\times \int_{t_{N-i}}^{t_{N+1-i}} E_1(t_N, t_{N+1-i}) \text{cov}_{G(t_N, t_{N+1-i})} \left( \max_{\beta} \partial^\beta k(\cdot, s) \right) ds 
\]

\[ 
+ O(\Delta Z^2_N) + o(\Delta Z_N) 
\]

which converges for \( N \to \infty \) with \( \Delta Z_N \to 0 \) by Riemann integration to

\[ 
\to \exp \left( H(T, 0) + b^2 \cdot G'(T, 0) \right) \cdot E_1(T, 0) \text{cov}_{G(T, 0)} \left( \max_{\beta} \partial^\beta f_0 \right) 
\]

\[ 
+ \int_0^T \exp \left( H(T, s) + b^2 \cdot G'(T, s) \right) 
\times E_1(T, s) \text{cov}_{G(T, s)} \left( \max_{\beta} \partial^\beta k(\cdot, s) \right) ds 
\]

\[ 
+ B_d(\cdot, T) 
\]

which is the bound \( B_{d+1} \) with

\[ 
\| x^\alpha \cdot \partial^\beta f_N(\cdot, T) \|_\infty \leq \| x^\alpha \cdot B_{d+1}(x) \|_\infty < \infty 
\]

for all \( \alpha, \beta \in \mathbb{N}_0^n \) with \( |\beta| = d + 1 \). In summary, we have shown that for the family \( \{f_N\}_{N \in \mathbb{N}} \) on \([0, T]\) Lemma 2.1(i) is fulfilled.

It remains to show that condition (ii) of Lemma 2.1 is fulfilled. Since all \( f_N \) are piece-wise differentiable, it is sufficient to show that \( \partial_t f_N \) is bounded. But this follows immediately from the bounds \( B_0, B_1, \ldots \) and hence there exists a constant \( L > 0 \) such that

\[ 
\sup_{t \in [0, T]} \| \partial_t f_N(\cdot, t) \|_\infty \leq L < \infty 
\]

for all \( N \in \mathbb{N} \), i.e., \( f_N \) are all Lipschitz in \( t \in [0, T] \) with a Lipschitz constant independent on \( N, x, \) and \( t \). Condition (ii) in Lemma 2.1 is then fulfilled since Lipschitz continuity implies equi-continuity.

Since conditions (i) and (ii) of Lemma 2.1 are fulfilled \( \{f_N\}_{N \in \mathbb{N}} \) is relatively compact. Hence, there exists a subsequence \( (N_i)_{i \in \mathbb{N}} \subseteq \mathbb{N} \) such that \( f_N \), converges on \( \mathbb{R}^n \times [0, T] \) to a function \( f \in C([0, T], S(\mathbb{R}^n, \mathbb{R}^m)) \), i.e.,

\[ 
\sup_{t \in [0, T]} \| x^\alpha \cdot \partial^\beta f_N(x, t) - x^\alpha \cdot \partial^\beta f(x, t) \|_\infty \xrightarrow{i \to \infty} 0 \quad (14) 
\]

for all \( \alpha, \beta \in \mathbb{N}_0^n \).

We now show that the accumulation point \( f \) solves (4) and is in \( C^{d+1} \) in \( t \), i.e., \( f \in C^{d+1}([0, T], S(\mathbb{R}^n, \mathbb{R}^m)) \). By Definition 4.4 of the \( f_N \) we have that each \( f_N \) is piece-wise differentiable in \( t \) and taking the derivative \( \partial_t f_N \) we find
\[ \left\| x^\alpha \cdot \partial_t \partial^\beta f_{N_i}(x,t) - x^\alpha \cdot \partial^\beta \left[ \nu \Delta f(x,t) + (g(x,t) \cdot \nabla) f(x,t) \\
+ h(x,t) \cdot f(x,t) + k(x,t) \right] \right\|_\infty \xrightarrow{i \to \infty} 0 \quad (15) \]

for all \( \alpha, \beta \in \mathbb{N}_0^n \) uniformly in \( t \in [0,T] \). Hence, with \( \alpha = \beta = 0 \) we have that \( f \) solves \((13)\) and for every fixed \( x \in \mathbb{R}^n \) the function \( G(t) = f(x,t) \) is continuous in \( t \). Now let \( x \in \mathbb{R}^n \) and \( \alpha = \beta = 0 \), then \((14)\) implies
\[
f(x,t) = \lim_{i \to \infty} f_{N_i}(x,t) \quad (16a)
\]
since the \( f_{N_i} \) are piece-wise differentiable \( \partial_t f_{N_i}(x,t) \) is Riemann integrable in \( t \)
\[
= \lim_{i \to \infty} \int_0^t \partial_t f_{N_i}(x,s) \, ds \quad (16b)
\]
where \((14)\) implies we can interchange integration and the limit \( i \to \infty \)
\[
= \int_0^t \lim_{i \to \infty} \partial_t f_{N_i}(x,s) \, ds \quad (16c)
\]
and then \((15)\) implies
\[
= \int_0^t \nu \Delta f(x,s) + (g(x,s) \cdot \nabla) f(x,s) + h(x,s) \cdot f(x,s) + k(x,s) \, ds \quad (16d)
\]
for all \( t \in [0,T] \). Since \( g, h, \) and \( k \) are \( C^d \) in \( t \) and \( f \) is \( C^0 \), we have that the integrand (I) is \( C^0 \) in \( t \). Hence, \( f \) is an integral over a \( C^0 \) function and therefore \( C^1 \) in \( t \). Continuing this argument we see that \( f \) is \( C^{d+1} \) in \( t \) since the integrand (I) is at least \( C^d \) in \( t \).

Since \( T > 0 \) was arbitrary, \( \{f_{N_i}\}_{N \in \mathbb{N}} \) has an accumulation point \( f \) for all \( T > 0 \) and all accumulation points fulfill \((13)\) with \( f \) is \( C^{d+1} \) in \( t \). \( \square \)

**Remark 5.2.** In the standard approach via weak solutions and Sobolev theory one usually works with spaces \( L^p([0,T], L^q(\mathbb{R}^n)) \), \( 1 \leq p, q \leq \infty \), or more generally \( L^p([0,T], X) \) \( X \) a Banach space with norm \( \| \cdot \|_X \). Then \( L^p([0,T], X) \) is equipped with the \( L^p \)-norm \( \| F \|_{L^p} \) of \( F(t) := \| f(\cdot, t) \|_X \) and therefore it is also a Banach space. For regularity in \( t \) one then has additional work to do.

But in our approach we do not work in a Banach space \( X \) but in a Montel space, i.e., we do not have a single norm \( \| \cdot \|_X \) but a family of semi-norms, here \( \| x^\alpha \cdot \partial^\beta f(x,t) \|_\infty \). The convergence of our approximation is \((14)\), i.e., uniform on \( \mathbb{R}^n \times [0,T] \) for all derivatives. \((14)\) takes care of all spatial derivatives and therefore by \((13)\) also of the time derivatives. We demonstrated this explicitly in \((16)\) for clarity but this argument also follows from \([52\text{, Thm. 7.17}]. \)

**Remark 5.3.** From \((12)\) we see that \( \| x^\alpha \cdot \partial^\beta f(\cdot, t) \|_\infty \) depend only on \( \| x^\gamma \cdot \partial^\delta f_0 \|_\infty \) for all \( \gamma, \delta \in \mathbb{N}_0^n \) with \( |\gamma| \leq |\alpha| \) and \( |\delta| \leq |\beta| \). Hence, weaker conditions on the initial value \( f_0 \) is possible since \( S(\mathbb{R}^n) \) is dense in any \( W^{p,k}(\mathbb{R}^n) \) with \( k \in \mathbb{N}_0 \) and \( p \in [1, \infty) \). \( \diamond \)
6. Burgers’ Equation

For Burgers’ equation we have from Theorem 5.1 the following.

**Theorem 6.1.** Let $u_0 \in S(\mathbb{R}, \mathbb{R})$. Then there exist maximal $T_1, T_2 > 0$ such that Burgers’ equation

$$\partial_t u = -u \cdot \partial_x u$$

$$u(\cdot, 0) = u_0$$

(17)

has a unique classical solution $u \in C^\infty((-T_1, T_2), \mathcal{S}(\mathbb{R}, \mathbb{R}))$. $(-T_1, T_2)$ is the maximal interval such that $u \in C((-T_1, T_2), C^\infty_b(\mathbb{R}, \mathbb{R}))$.

**Proof.** The $C^\infty$ solution of Burgers’ equation is unique, i.e., there exists a maximal time interval $(-T_1, T_2)$ such that $u \in C((-T_1, T_2), C^\infty_b(\mathbb{R}, \mathbb{R}))$. Set $n = m = 1, \nu = 0, g = u, h = 0,$ and $k = 0$ in (4). Then Theorem 5.1 shows that there exists a $f \in C^\infty((-T_1, T_2), \mathcal{S}(\mathbb{R}, \mathbb{R}))$ that solves (4). By uniqueness of $u$ from Burgers’ equation we have $f = u$.

Since for Burgers’ equation we have $u(\cdot, t) \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ we can in theory calculate all moments of $u$ for all times $t \in (-T_1, T_2)$. The simplicity of (17) allows us to calculate the time-dependent moments explicitly.

**Theorem 6.2.** Let $u_0 \in \mathcal{S}(\mathbb{R}, \mathbb{R})$. Then for all $p \in \mathbb{N}$ and $k \in \mathbb{N}_0$ the time-dependent moments

$$s_{k,p}(t) := \int_\mathbb{R} x^k u(x, t)^p \, dx$$

of the solution $u$ of Burgers’ equation (17) are

$$s_{k,p}(t) = \sum_{i=0}^{k} \frac{s_{k-i,p+i}(0)}{i!} \cdot t^i \cdot \prod_{j=0}^{i-1} \frac{(p+j)(k-j)}{1+(p+j)^2} \in \mathbb{R}[t].$$

**Proof.** We proceed via induction over $k \in \mathbb{N}_0$.

$k = 0$: We have

$$\partial_t s_{0,p}(t) = \partial_t \int_\mathbb{R} u(x, t)^p \, dx = -p \int_\mathbb{R} u(x, t)^p \cdot \partial_x u(x, t) \, dx$$

with partial integration since $u(\cdot, t)$ is a Schwartz function

$$= p \int_\mathbb{R} \partial_x [u(x, t)^p] \cdot u(x, t) \, dx = p^2 \int_\mathbb{R} u(x, t)^p \cdot \partial_x u(x, t) \, dx$$

$$= -p \cdot \partial_t s_{0,p}(t)$$

which gives $\partial_t s_{0,p}(t) = 0$ and therefore $s_{0,p}(t) = s_{0,p}(0)$.

$k - 1 \rightarrow k$: We have

$$\partial_t s_{k,p}(t) = \partial_t \int_\mathbb{R} x^k \cdot u(x, t)^p \, dx$$

$$= -p \int_\mathbb{R} x^k \cdot u(x, t)^p \cdot \partial_x u(x, t) \, dx$$

$$= -p \cdot \partial_t s_{k,p}(t)$$
\[ p \int_{\mathbb{R}} \partial_x (x^k \cdot u(x,t)^p) \cdot u(x,t) \, dx \]
\[ = p \cdot k \int_{\mathbb{R}} x^{k-1} \cdot u(x,t)^{p+1} \, dx + p^2 \int_{\mathbb{R}} x^k \cdot u(x,t)^p \cdot \partial_x u(x,t) \, dx \]
\[ = p \cdot k \cdot s_{k-1,p+1}(t) - p^2 \cdot \partial_t s_{k,p}(t) \]
\[ = \frac{p \cdot k}{1 + p^2} \cdot s_{k-1,p+1}(t) \]

and solving this induction gives

\[ s_{k,p}(t) = s_{k,p}(0) + \frac{p \cdot k}{1 + p^2} \int_0^t s_{k-1,p+1}(\tau_1) \, d\tau_1 \]
\[ = s_{k,p}(0) + \frac{p \cdot k}{1 + p^2} \int_0^t \left[ s_{k-1,1}(0) + \frac{(p+1)(k-1)}{1 + (p+1)^2} \int_0^{\tau_1} s_{k-2,p+2}(\tau_2) \, d\tau_2 \right] \, d\tau_1 \]
\[ \vdots \]
\[ = \sum_{i=0}^{k} \frac{s_{k-i,i+p+1}(0)}{i!} \cdot t^i \cdot \prod_{j=0}^{i-1} \frac{(p+j) \cdot (k-j)}{1 + (p+j)^2} \]

which proves the statement. \qed

In Burgers’ equation as a transport equation when \( u_0 \geq 0 \) then the classical solution remains non-negative. But from the moments in Theorem 6.2 we observe the following.

**Example 6.3.** For \( p = 1 \) we have the following three explicit time-dependent moments from Theorem 6.2

\[ \int_{\mathbb{R}} u(x,t) \, dx = s_{0,1}(t) = s_{0,1}(0), \]
\[ \int_{\mathbb{R}} x \cdot u(x,t) \, dx = s_{1,1}(t) = s_{1,1}(0) + s_{0,2}(0) \cdot t, \]
\[ \int_{\mathbb{R}} x^2 \cdot u(x,t) \, dx = s_{2,1}(t) = s_{2,1}(0) + s_{1,2}(0) \cdot t + \frac{2s_{0,3}(0)}{5} \cdot t^2. \]

For the function

\[ u_0(x) := \begin{cases} 1 + x & \text{for } x \in [-1,0], \\ 1 - x & \text{for } x \in [0,1], \\ 0 & \text{else} \end{cases} \]

we have \( s_{0,1}(0) = 1, s_{1,1}(0) = 0, s_{0,2}(0) = \frac{2}{3}, s_{2,1}(0) = \frac{1}{6}, s_{1,2}(0) = 0, s_{0,3} = \frac{1}{2} \)

and therefore

\[ \int_{\mathbb{R}} (x - t)^2 \cdot u(x,t) \, dx = L_s(t)((x - t)^2) = \frac{1}{6} - \frac{2}{15} t^2 \quad t \to \pm \infty, \quad -\infty. \quad (18) \]
Since \( u_0 \notin \mathcal{S}(\mathbb{R}) \) using a mollifier we get \( u^\varepsilon_0 := S_\varepsilon * u_0 \in C^\infty_0(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}) \) for any \( \varepsilon > 0 \). We can chose by continuity of the \( s_{p,k}(0) \) on \( \varepsilon \) an \( \varepsilon > 0 \) small such that the coefficient of \( t^2 \) in (18) remains negative. Hence, non-negativity in the assumed classical solution is not preserved, i.e., we have a finite breakdown. \( \diamond \)

Let \( k \in \mathbb{N} \) and \( k \geq 2 \). For

\[
\partial_t u = u^k \cdot \partial_x u
\]  

(19)

multiply (19) with \( k \cdot u^{k-1} \) to get \( \partial_t (u^k) = u^k \cdot \partial_x (u^k) \). This is Burgers’ equation with \( v = u^k \). If \( u_0 \geq 0 \) we can allow \( k \in [1, \infty) \) in (19).

7. Euler and Navier–Stokes Equations

By Beale–Kato–Majda [16] the classical solutions of the Euler and the Navier–Stokes equations \( u \) and \( \omega \) exist as long as \( ||\omega(\cdot,t)||_\infty < \infty \). A finite breakdown in time can therefore be observed through a breakdown of \( ||\omega||_\infty \). For the Euler and the Navier–Stokes equations we have from Theorem 5.1 the following.

**Theorem 7.1.** Let \( \nu \in [0, \infty) \), \( u_0 \in C^\infty_b(\mathbb{R}^3, \mathbb{R}^3) \) with \( \text{div} \ u_0 = 0 \) and \( \omega_0 := \text{rot} \ u_0 \in \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3) \), and \( T^* > 0 \) be maximal such that \( u \) is the solution of the Euler (\( \nu = 0 \)) resp. Navier–Stokes (\( \nu > 0 \)) equation (3) with \( ||\omega(\cdot,t)||_\infty < \infty \) for all \( t \in [0, T^*) \), i.e., the unique smooth solution \( u \) exists for all \( t \in [0, T^*) \). Then \( \omega \in C^\infty([0, T^*), \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)) \).

**Proof.** The (vorticity formulation of the) Euler and the Navier–Stokes equations have a unique smooth solution \( u, \omega \in C^\infty([0, T^*), C^\infty_b(\mathbb{R}^3, \mathbb{R}^3)) \). Set \( n = m = 3 \), \( g = u, \ h = \nabla u \), and \( k = 0 \) in (12). Then by Theorem 5.1 we have a solution \( f \in C^\infty([0, T^*), \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)) \). But by the uniqueness of \( \omega \) we have \( f = \omega \). \( \square \)

A breakdown in \( ||\omega||_\infty \) provides \( T^* < \infty \) [16]. Now by Theorem 7.1 a breakdown in any \( ||x^\alpha \partial^\beta \omega \gamma||_\infty \) or \( ||x^\alpha \partial^\beta \omega \gamma||_{L^p(\mathbb{R}^3)} \) with \( \alpha, \beta, \gamma \in \mathbb{N}_0^3 \), and \( \omega^\gamma := \omega_1 \gamma_1 \cdot \omega_2 \gamma_2 \cdot \omega_3 \gamma_3 \), \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \neq 0 \), also provides \( T^* < \infty \). By Remark 5.3 weaker conditions on \( \omega_0 \) are possible. Unfortunately, similar calculations as in Theorem 6.2 or Example 6.3 for Burgers’ equation are not yet accessible to us for the Euler or Navier–Stokes equations. With \( k = \text{rot} \ F \in C^\infty([0, \infty), \mathcal{S}(\mathbb{R}^3, \mathbb{R}^3)) \) in Theorem 5.1 we have that Theorem 7.1 also holds with external forces.

**Acknowledgments**

We thank Tarek Elgindi for valuable remarks and fruitful discussions on the paper. We thank Lorenzo Brandolese for valuable remarks, fruitful discussions, and for providing additional references.

The author and this project are financed by the Deutsche Forschungsgemeinschaft DFG with the grant DI-2780/2-1 and his research fellowship at the Zukunftskolleg of the University of Konstanz, funded as part of the Excellence Strategy of the German Federal and State Government.
References

[1] C. L. Fefferman, The millennium prize problems, Clay Math. Inst., Cambridge, MA, 2006, Ch. Existence and smoothness of the Navier–Stokes equation, pp. 57–67.

[2] L. Euler, Principes généraux du mouvement des fluides, Mémoires de l’académie des sciences de Berlin 11 (1757) 274–315.

[3] C.-L. Navier, Mémoire sur les lois du mouvement des fluides, Mémoires de l’Acad. des Sciences de l’Institut de France 6 (1827) 389–440.

[4] G. G. Stokes, On the theory of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids, Trans. Cambridge Phil. Soc. 8 (1849) 287–319.

[5] C. W. Oseen, Sur les formules de Green généralisées qui se présentent dans l’hydrodynamique et sur quelques unes de leurs applications, Acta Math. 34 (1911) 205–284.

[6] C. W. Oseen, Methoden und Ergebnisse in der Hydrodynamik, Akademische Verlagsgesellschaft, Leipzig, 1927.

[7] J. Leray, Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l’hydrodynamique, J. Math. Pures Appl. 12 (1933) 1–82.

[8] J. Leray, Essai sur le mouvement d’un liquide visqueux emplissant l’espace, Acta Math. 63 (1934) 193–248.

[9] J. Leray, Essai sur le mouvements plans d’un liquide visqueux que limitent des parois, J. Math. Pures Appl. 13 (1934) 331–418.

[10] E. Hopf, Über die Anfangswertaufgabe für die hydrodynamische Grundgleichung, Math. Nachr. 4 (1951) 213–231.

[11] O. A. Ladyzhenskaya, The mixed problem for a hyperbolic equation, Gosudarstv. Izdat., Moskow (in Russian), 1953.

[12] A. A. Kiselev, O. A. Ladyzhenskaya, On the existence and uniqueness of solutions of the non-stationary problem for a viscous incompressible fluid, Izv. Akad. Nauk SSR Ser. Mat. 21 (1957) 655–680, English transl., Amer. Math. Soc. Transl. 24 (1963), 79–106.

[13] O. A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, New York, 1963.

[14] O. A. Ladyzhenskaya, Example of nonuniqueness in the Hopf class of weak solutions for the Navier–Stokes equation, Izv. Akad. Nauk SSR Ser. Mat. 33 (1969) 240–247, English transl., Math. USSR-Izv. 3 (1969), 229–236.
[15] L. Caffarelli, R. Kohn, L. Nirenberg, Partial Regularity of Suitable Weak Solutions of the Navier–Stokes Equations, Comm. Pure and Appl. Math. 35 (1982) 771–831.

[16] J. T. Beale, T. Kato, A. Majda, Remarks on the Breakdown of Smooth Solutions for the 3D Euler Equations, Commun. Math. Phys. 94 (1984) 61–66.

[17] R. Temam, Navier–Stokes Equations, American Mathematical Society, Providence, Rhode Island, 1984.

[18] W. von Wahl, The Equations of Navier–Stokes and Abstract Parabolic Equations, Friedr. Vieweg & Sohn, Braunschweig, Wiesbaden, 1985.

[19] P. Constantin, C. Foias, Navier–Stokes Equations, The University of Chicago Press, Chicago, US, 1988.

[20] H.-O. Kreiss, J. Lorenz, Initial-Boundary Value Problems and the Navier–Stokes Equations, Academic Press, Inc., Boston, 1989.

[21] S. Dobrokhotov, A. Shafarevich, Some integral identities and remarks on the decay at infinity of the solutions to the Navier–Stokes equations in the entire space, Russ. J. Math. Phys. 2 (1994) 133–135.

[22] R. Temam, Navier–Stokes Equations and Nonlinear Functional Analysis, 2nd Edition, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1995.

[23] R. Temam, Navier–Stokes equations: Theory and numerical analysis, American Mathematical Society, Providence, Rhode Island, 2001.

[24] C. Foias, O. Manley, R. Rosa, R. Temam, Navier–Stokes Equations and Turbulence, Cambridge University Press, Cambridge, UK, 2001.

[25] A. J. Majda, A. L. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, UK, 2002.

[26] P. G. Lemarié-Rieusset, Recent developments in the Navier–Stokes problem, Chapman & Hall/CRC, Boca Raton, Florida, 2002.

[27] O. Darrigol, Between hydrodynamics and elasticity theory: the five first births of the Navier–Stokes equation, Arch. Hist. Exact. Sci. 56 (2002) 95–150.

[28] O. A. Ladyzhenskaya, Sixth problem of the millennium: Navier–Stokes equations, existence and smoothness, Russian Math. Surveys 58 (2003) 251–286.

[29] L. Brandolese, Atomic decomposition for the vorticity of a viscous flow in the whole space, Math. Nachr. 273 (2004) 28–42.
[30] T. Tao, Nonlinear Dispersive Equations - Local and Global Analysis, American Mathematical Society, Providence, Rhode Island, 2006.

[31] H. Behouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Springer-Verlag, Berlin, Heidelberg, 2011.

[32] F. Boyer, P. Fabrie, Mathematical Tools for the Study of the Incompressible Navier–Stokes Equation and Related Models, Springer, New York, 2013.

[33] P. G. Lemarié-Rieusset, The Navier–Stokes Problem in the 21st Century, CRC Press, Boca Raton, FL, 2016.

[34] J. C. Robinson, J. L. Rodrigo, W. Sadowski, The Three-Dimensional Navier–Stokes Equations: Classical Theory, Cambridge University Press, Cambridge, UK, 2016.

[35] L. Brandolese, Localisations, oscillations et comportement asymptotique pour les équations de Navier–Stokes, Ph.D. thesis, ENS Chachan (2001).

[36] L. Brandolese, Y. Meyer, On the instantaneous spreading for the Navier–Stokes system in the whole space, ESIAM Contr. Optim. Calc. 8 (2002) 273–285.

[37] L. Brandolese, F. Vigneron, New asymptotic profiles of nonstationary solutions of the Navier–Stokes system, J. Math. Pures Appl. (2004).

[38] H.-O. Bae, L. Brandolese, On the effect of external forces on the motion of incompressible flows at large distances, Ann. Univ. Ferrara 55 (2009) 225–238.

[39] L. Brandolese, Fine properties of self-similar solutions of the Navier–Stokes equations, Arch. Ration. Mech. Anal. 192 (2009) 375–401.

[40] L. Ting, On the application of the integral invariants and decay laws of vorticity distributions, J. Fluid Mech. 127 (1983) 497–506.

[41] R. Danchin, Analyse numérique et harmonique d’un problème de mécanique des fluides, Ph.D. thesis, École Polytechnique, Palaiseau, France (1996).

[42] T. Miyakawa, On space-time decay properties of nonstationary incompressible Navier-Stokes flows in $\mathbb{R}^n$, Funkcial. Ekvac. 43 (2000) 541–557.

[43] Y. Meyer, Oscillating Patterns in Image Processing and Nonlinear Evolution Equations: The Fifteenth Dean Jacqueline B. Lewis Memorial Lectures, Vol. 22 of University Lecture Series, American Mathematical Society, 2001.

[44] T. Gallay, C. E. Wayne, Long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbb{R}^3$, Philos. Trans. Roy. Soc. A 360 (2002) 2155–2188.
[45] R. McOwen, P. Topalov, Spatial asymptotic expansions in the incompressible Euler equation, Geom. Func. Anal. 27 (2017) 637–675.

[46] S. Sultan, P. Topalov, On the asymptotic behavior of solutions of the 2d Euler equation, J. Diff. Eq. 269 (2020) 5280–5337.

[47] H. F. Trotter, On the product of semi-groups of operators, Proc. Amer. Math. Soc. 10 (1959) 545–551.

[48] C. Arzelà, Un’osservazione intorno alle serie di funzioni, Rend. Dell’Accad. R. Delle Sci. Dell’Istituto di Bologna (1882/83) 142–159.

[49] G. Ascoli, Le curve limite di una varietà data di curve, Atti della R. Accad. Dei Lincei Memorie della Cl. Sci. Fis. Mat. Nat. 18 (1883/84) 521–586.

[50] C. Arzelà, Sulle funzioni di linee, Mem. Accad. Sci. Ist. Bologna Cl. Sci. Fis. Mat. 5 (1895) 55–74.

[51] K. Yosida, Functional Analysis, 2nd Edition, Springer-Verlag, Berlin, 1968.

[52] W. Rudin, Principles of Mathematical Analysis, Mc-Graw-Hill, New York, 1976.