THE MODULI OF CURVES OF GENUS 6 AND K3 SURFACES

MICHELA ARTEBANI AND SHIGEYUKI KONDO

ABSTRACT. We prove that the coarse moduli space of curves of genus 6 is birational to an arithmetic quotient of a bounded symmetric domain of type IV by giving a period map to the moduli space of some lattice-polarized K3 surfaces.

INTRODUCTION

This paper gives a birational period map between the coarse moduli space of curves of genus six and the moduli space of some lattice-polarized K3 surfaces. This kind of correspondence was given by the second author for curves of genus 3 and genus 4 in [Ko1] and [Ko2]. A part of the results in this paper was announced in [Ko2].

Let \( C \) be a general curve of genus six, then its canonical model is a quadratic section of a unique quintic Del Pezzo surface \( Y \subset \mathbb{P}^5 \) (e.g. [SB]). The double cover of \( Y \) branched along \( C \) is a K3 surface \( X \). By taking the period point of \( X \) we define a period map \( P : \mathcal{M}_6 \rightarrow \mathbb{D}/\Gamma \).

The same construction defines rational period maps

\[
P^* : \mathcal{W}_6^2 \rightarrow \mathbb{D}/\Gamma^*, \quad P^{**} : \tilde{\mathcal{M}}_6 \rightarrow \mathbb{D}/\Gamma^{**}.
\]

The first author was supported by: PRIN 2005: Spazi di moduli e teoria di Lie, Indam (GNSAGA).

The second author is partially supported by Grant-in-Aid for Scientific Research A-18204001 and Houga-20654001, Japan.
the action of subgroups. Afterwards, we prove that $\mathcal{P}, \mathcal{P}^*, \mathcal{P}^{**}$ are birational maps, in particular $\mathcal{P}$ induces an isomorphism

$$M_6 \setminus \{ \text{special curves} \} \cong (D \setminus \mathcal{H})/\Gamma$$

where $\mathcal{H}$ is a divisor defined by hyperplane sections associated to $(-2)$-vectors, called discriminant divisor.

In section 3 we study the discriminant divisor $\mathcal{H}$ and its geometric meaning. We prove that $\mathcal{H}/\Gamma$ has 3 irreducible components which parametrize respectively curves of genus six with a node, pairs $(C, L)$ where $C$ is a plane quintic and $L$ is a line and pairs $(C, D)$ where $C$ is a trigonal curve of genus six and $D \in |K_C - 2g_3^1|$.

In the final section we determine the structure of the boundary of the Satake-Baily-Borel compactification of $D/\Gamma$ and we compare this compactification with the GIT compactification of the space of plane sextics.

*Acknowledgments.* The first author would like to thank R. Laza for several helpful discussions.

*Notation.* A lattice $L$ is a free abelian group of finite rank equipped with a non degenerate bilinear form, which will be denoted by $(\cdot, \cdot)$.
- The discriminant group of $L$ is the finite abelian group $A_L = L^*/L$, where $L^* = \text{Hom}(L, \mathbb{Z})$, equipped with the quadratic form $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$ defined by $q_L(x + L) = (x, x) \mod 2\mathbb{Z}$.
- $O(L)$ and $O(q_L)$ will denote the groups of isometries of $L$ and $A_L$ respectively.
- A lattice is unimodular if $|A_L| = |\det L| = 1$.
- If $M$ is the orthogonal complement of $L$ in a unimodular lattice, then $A_L \cong A_M$ and $q_M = -q_L$.
- We will denote by $U$ the hyperbolic plane and by $A_n^-, D_n, E_n$ the negative definite lattices of rank $n$ associated to the Dynkin’s diagrams of the corresponding types.
- The lattice $L(\alpha)$ is obtained multiplying by $\alpha$ the form on $L$.
- The lattice $L^m$ is the orthogonal direct sum of $m$ copies of the lattice $L$.

We will refer the reader to [N1] for basic facts about lattices.

**1. CURVES OF GENUS SIX AND QUINTIC DEL PEZZO SURFACES**

We start recalling some well-known properties of curves of genus six. By Brill-Noether theory any smooth curve of genus six $C$ has a special divisor $D$ with $\deg(D) = 6$ and $h^0(C, D) = 3$. Let $\varphi_D$ be the morphism assoicated to $D$:

$$\varphi_D : C \longrightarrow \mathbb{P}^2.$$

The curve $C$ will be called *special* if it is either hyperelliptic, trigonal, bi-elliptic or isomorphic to a smooth plane quintic curve. The following is given for example in section A, Ch.V in [ACGH].

**Proposition 1.** Let $C$ be a smooth curve of genus six, then one of the followings holds:

a) $\varphi_D$ is birational and $\varphi_D(C)$ is an irreducible plane sextic having only double points.

b) $C$ is special.

**Case a)** Assume first that $\varphi_D(C)$ is a plane sextic with 4 nodes $p_1, \ldots, p_4$ in general position. The blowing up of $\mathbb{P}^2$ in these points is a quintic del Pezzo surface $Y$ and $C \subset | - 2K_Y |$. In fact, the embedding $C \subset Y \subset \mathbb{P}^5$ is the canonical embedding of $C$ and $Y$ is the unique quintic Del Pezzo
surface containing $C$ (see e.g. [SB]). Let $e_0$ be the class of the pull back of a line and let $e_i$ be the classes of exceptional divisors over the points $p_i$. The surface $Y$ contains 10 lines

$$e_i, e_0 - e_i - e_j, \quad 1 \leq i < j \leq 4.$$  

It is known that the group of automorphisms of the dual graph of the 10 lines is isomorphic to $S_5$. The surface $Y$ admits exactly five birational morphisms to $\mathbb{P}^2$, called blowing down maps, induced by the linear systems:

$$e_0, 2e_0 - \sum_{i=1}^{4} e_i + e_j, \quad j = 1, \ldots, 4.$$  

Note that any such morphism maps $C$ to a plane sextic with 4 nodes. In fact also the converse holds i.e. there is a one-to-one correspondence between the set of blowing down maps for $Y$ and the set $W_6^2(C)$ of $g_6^2$ on $C$. In particular the generic curve of genus 6 has exactly five $g_6^2$. The automorphisms group of $Y$ acts on the blowing down classes, giving a representation $Aut(Y) \to S_5$, which is known to be an isomorphism. The stabilizer of a blowing down model $\phi$ is given by projectivities permuting the 4 points $p_1, \ldots, p_4 \in \mathbb{P}^2$ which are the image of the exceptional divisors of $\phi$, while an element of order five is realized by a quadratic transformation $\alpha$ with fundamental points at $p_1, p_2, p_3$ [Do, Theorem 10.2.2].

If $p_1, \ldots, p_4$ are not in general position then either 3 of them lie on a line or two of them are infinitely near. Note that anything worse is not admitted since $\varphi_D(C)$ is irreducible with at most double points. The blowing up of $\mathbb{P}^2$ in these points is a nodal del Pezzo surface, i.e. $-K_Y$ is nef and big (see [DO]). Equivalently, the anti-canonical model of $Y$ has at most rational double points. In this case the properties of the embedding $C \subset Y$ still hold, in particular $Y$ containing $C$ is unique ([AH, 5.14]). However, the surface $Y$ may have less than five blowing down classes, i.e. $C$ has less than five $g_6^2$.

**Case b)** The following characterization holds:

**Proposition 2.** A curve of genus six $C$ is special if and only if $\dim W_6^2(C) > 0$.

**Proof.** We have seen that if $C$ is not special, then $\dim W_6^2(C) = 0$ and contains at most five points. We now see what happens for special curves ([ACGH]).

- If $C$ is trigonal then it has two types of $g_6^2$:
  $$D = 2g_3^1 \quad \text{and} \quad D(p) = K_C - g_3^1 - p, \quad p \in C.$$  

Hence $W_6^2(C)$ is one dimensional and has two irreducible components. The plane model $\varphi_D(C)$ is a triple conic and $\varphi_{D(p)}(C)$ is a plane sextic with a triple point and a node.

- If $C$ is isomorphic to a plane quintic then any $g_6^2$ on $C$ is of type: $D(p) = g_5^2 + p, \quad p \in C$. Hence $W_6^2(C) \cong C$. The plane model $\varphi_{D(p)}(C)$ is a plane quintic.

- If $C$ is bi-elliptic i.e. there exists $\pi : C \to E$, where $E$ is an elliptic curve, then any $g_6^2$ corresponds to $\phi \circ \pi$ where $\phi$ is a $g_3^1$ on $E$. The plane model of $C$ is a double cubic.

- If $C$ is hyperelliptic then any $g_6^2$ is of type:
  $$D(p, q) = K_C - g_3^1 - p - q, \quad p, q \in C.$$  

Hence $W_6^2(C) \cong Sym^2(C)$. In fact $D = K_C - 2g_3^1$ is a singular point of $W_6^2(C)$. The plane model $\varphi_D(C)$ is a double rational cubic and $\varphi_{D(p, q)}(C)$ is a double conic.
Lemma 1. \( H^2(X, \mathbb{Z})^\pm = \{ x \in H^2(X, \mathbb{Z}) : \sigma^*(x) = \pm x \} \).

**Lemma 2.** \( H^2(X, \mathbb{Z})^+ \cong A_1(-1) \oplus A_1^1, \) \( H^2(X, \mathbb{Z})^- \cong U \oplus U \oplus E_8 \oplus A_1^5. \)

**Proof.** By definition, the lattices \( H^\pm = H^2(X, \mathbb{Z})^\pm \) are 2-elementary, i.e. their discriminant groups are 2-elementary abelian groups. By [N1] Theorem 3.6.2 the isomorphism class of a 2-elementary even indefinite lattice \( L \) is determined uniquely by the triple \((s, \ell, \delta)\), where \( s \) is the signature, \( \ell \) is the minimal number of generators of \( A_L \) and \( \delta = 0 \) (resp. 1) if the quadratic form on \( A_L \) always assumes integer values (resp. otherwise). On the other hand [N2] Theorem 4.2.2 shows that \( H^+ \) has \( s = (1, 4), \ell = 5, \delta = 1 \). Since \( H^- \) is the orthogonal complement of \( H^+ \) in the unimodular lattice \( H^2(X, \mathbb{Z}), \) \( s = (2, 15), \ell = 5, \delta = 1. \) Hence it is enough to check that the lattices in the right hand sides have the same triple of invariants.

2.1. **The geometric construction.** Let \( C \subset \mathbb{P}^5 \) be the canonical model of a non-special smooth curve of genus six. By the remarks in the previous section, there is a unique nodal Del Pezzo surface \( Y \) such that \( C \) lies in the anti-canonical model of \( Y \) in \( \mathbb{P}^5. \) Let \( Y' \to Y \) be the canonical resolution of rational double points of \( Y. \) Since \( C \in |-2K_Y|, \) there exists a double cover
\[
\pi : X \to Y'
\]
branched along \( C \) and \( X \) is a \( K3 \) surface. It is well known that \( H^2(X, \mathbb{Z}), \) together with the cup product, is an even unimodular lattice of signature \((3, 19)\). The covering involution \( \sigma \) of \( \pi \) acts on this lattice with eigenspaces
\[
H^2(X, \mathbb{Z})^\pm = \{ x \in H^2(X, \mathbb{Z}) : \sigma^*(x) = \pm x \}.
\]

Let \( S_X \) be the Picard lattice of \( X \) and let \( T_X \) be its transcendental lattice:
\[
S_X = H^2(X, \mathbb{Z}) \cap \omega^\perp_X, \quad T_X = S_X^\perp.
\]

Note that the invariant lattice \( H^2(X, \mathbb{Z})^+ \) coincides with the pull-back of the Picard lattice of \( Y, \) hence
\[
H^2(X, \mathbb{Z})^+ \subset S_X, \quad T_X \subset H^2(X, \mathbb{Z})^-.
\]

If \( \omega_X \) is a nowhere vanishing holomorphic 2-form on \( X, \) then \( \omega_X \in T_X \otimes \mathbb{C}, \) hence \( \sigma^*(\omega_X) = -\omega_X. \)

**Lemma 2.** There are no \((-2)\)-vectors in \( S_X \cap H^2(X, \mathbb{Z})^-. \)

**Proof.** Assume that \( r \) is such a vector. By Riemann-Roch theorem we may assume that \( r \) is effective. Then \( \sigma^*(r) = -r \) is also effective. This is a contradiction.

2.2. **Lattices.** We will denote by \( L_K3 \) an even unimodular lattice of signature \((3, 19)\). This is known to be unique up to isomorphisms (see e.g. [N1] Theorem 1.1.1)), hence the lattice \( H^2(X, \mathbb{Z}) \) is isomorphic to \( L_K3. \) Let \( e_0, e_1, \ldots, e_4 \) be the pull-backs of the classes \( e_0, e_1, \ldots, e_4 \) under \( \pi^*. \) These generate a sublattice of \( S_X \) isometric to \( A_1(-1) \oplus A_1^4. \) Let
\[
S = A_1(-1) \oplus A_1^4, \quad T = U \oplus U \oplus E_8 \oplus A_1^5.
\]
Denote by \( s_0, s_1, \ldots, s_4 \) an orthogonal basis for \( S \) with \( s_0^2 = 2, s_i^2 = -2, \ i = 1, \ldots, 4 \) and denote by \( r_1, \ldots, r_5 \) an orthogonal basis for the \( A_1^5 \) component of \( T \).

**Lemma 3.** Let \( \xi_i = r_i/2 \), then the discriminant group \( A_T \) consists of the following vectors:

\[
\begin{align*}
q(x) = 0 : & \quad 0, \sum_{i\neq j} \xi_i, \quad 1 \leq j \leq 5 \\
q(x) = 1 : & \quad \xi_i + \xi_j, \quad 1 \leq i < j \leq 5 \\
q(x) = -1/2 : & \quad \xi_i, \quad 1 \leq i \leq 5, \sum_{i=1}^{5} \xi_i \\
q(x) = -3/2 : & \quad \sum_{i\neq j,k} \xi_i, \quad 1 \leq j < k \leq 5.
\end{align*}
\]

It follows from [NT Theorem 1.14.4] that \( S \) can be embedded uniquely in \( L_{K_3} \) and \( T \) is isomorphic to its orthogonal complement. Since \( L_{K_3} \) is unimodular,

\[ A_S \cong A_T \cong \mathbb{F}_2^5, \quad q_S \cong -q_T \]

and an isomorphism from \( A_S \) to \( A_T \) is given by

\[ s_0/2 \mapsto \xi_1, \quad (2s_0 - \sum_{i=1}^{4} s_i + s_j)/2 \mapsto \xi_{j+1}, \quad j = 1, \ldots, 4. \]

**Lemma 4.** There are isomorphisms \( O(q_S) \cong O(q_T) \cong S_5 \) and the natural maps

\[ O(T) \to O(q_T), \quad O(S) \to O(q_S) \]

are surjective.

**Proof.** The first statement follows from [MS]. Note that \( O(q_T) \) acts on \( A_T \) by permuting the \( \xi_i \)'s. The surjectivity statement for \( T \) is obvious, since clearly exist isometries of \( T \) permuting the \( r_i \)'s. On the other hand, the automorphism group \( S_5 \) of \( Y \) acts on \( S \) as isometries. These isometries act on \( A_S \) as \( S_5 \). More concretely, the isometries of \( S \) permuting the \( s_i \)'s \( (1 \leq i \leq 4) \) and the isometry

\[ s_0 \mapsto 2s_0 - s_1 - s_2 - s_3, \quad s_1 \mapsto s_0 - s_1 - s_3, \quad s_2 \mapsto s_4, \quad s_3 \mapsto s_0 - s_2 - s_3, \quad s_4 \mapsto s_0 - s_1 - s_2 \]

generate \( O(q_S) \).

In the following we will consider three arithmetic groups acting on \( T \):

\[ \Gamma = O(T), \quad \Gamma^* = \{ \gamma \in O(T) : \gamma(\xi_1) = \xi_1 \}, \quad \Gamma^{**} = \{ \gamma \in O(T) : \gamma|A_T = 1 \}. \]

Note that \( \Gamma/\Gamma^{**} \cong O(q_T) \cong S_5 \).

**Lemma 5.** Let \( O_T = \{ \gamma \in O(L_{K_3}) : \gamma(T) = T \} \). Then the restriction homomorphisms

\[ O_T \to \Gamma, \quad \{ \gamma \in O_T : \gamma(s_0) = s_0 \} \to \Gamma^* \text{ and } \quad \{ \gamma \in O_T : \gamma|S = 1_S \} \to \Gamma^{**} \]

are surjective.

**Proof.** Let \( \gamma \in \Gamma \). By Lemma [4] there exists \( \beta \in O(S) \) such that \( \beta = \gamma \) on \( A_S \cong A_T \). Then the isometry \( \beta \oplus \gamma \) on \( S \oplus T \) lifts to an isometry in \( O_T \). If \( \gamma \in \Gamma^* \) or \( \Gamma^{**} \) then \( \beta \) can be chosen such that \( \beta(s_0) = s_0 \) or \( \beta = 1_S \), respectively (see the proof of Lemma [4]).

| Remark 2. | There are two orbits of vectors with \( q(x) = -1/2 \) under the action of \( O(q_T) \):

\[ O_1 = \{ \sum_{i=1}^{5} \xi_i \}, \quad O_2 = \{ \xi_1, \ldots, \xi_5 \}. \]
2.3. Moduli spaces. Since both \( S \) and \( T \) are 2-elementary lattices, the isometry \((1_S, -1_T)\) on \( S \oplus T \) can be extended to an isometry \( \iota \) of \( L_{K3} \). Let \( \alpha : H^2(X, \mathbb{Z}) \to L_{K3} \) be an isometry satisfying \( \alpha(H^2(X, \mathbb{Z})^+) = S \). Then \( \iota \circ \alpha = \alpha \circ \sigma^* \). Since \( \sigma^*(\omega_X) = -\omega_X \) then the period
\[
p_X(\alpha) = \alpha_C(\omega_X)
\]
belongs to the set
\[
\mathcal{D} = \{ \omega \in \mathbb{P}(T \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \},
\]
called the period domain of \( S \)-polarized K3 surfaces. By Lemma 2, there are no \((-2)\)-vectors orthogonal to the period, hence \( p_X(\alpha) \) belongs to the complement of the divisor
\[
\mathcal{H} = \bigcup_{r \in T, r^2 = -2} \mathcal{H}_r \quad \text{where} \quad \mathcal{H}_r = \{ \omega \in \mathcal{D} : (r, \omega) = 0 \}.
\]
Consider the orbit spaces
\[
\mathcal{M} = \mathcal{D}/\Gamma, \quad \mathcal{M}^* = \mathcal{D}/\Gamma^*, \quad \mathcal{M}^{**} = \mathcal{D}/\Gamma^{**}.
\]
Let \( \mathcal{W}_6^2 \) be the moduli space of pairs \((C, D)\) where \( C \) is a smooth curve of genus 6 and \( D \in W_6^2(C) \) (see [ACGH]) and let \( \mathcal{M}_6 \) be the moduli space of plane sextics with four ordered nodes.

**Theorem 1.** The geometric construction in [2.1] defines a birational map
\[
\mathcal{P}^{**} : \mathcal{M}_6 \dashrightarrow \mathcal{M}^{**}.
\]
The map \( \mathcal{P}^{**} \) is equivariant for the natural action of \( S_5 \), taking quotients for this action and for the action of a subgroup isomorphic to \( S_4 \) gives birational maps
\[
\mathcal{P} : \mathcal{M}_6 \dashrightarrow \mathcal{M}, \quad \mathcal{P}^* : \mathcal{W}_6^2 \dashrightarrow \mathcal{M}^*.
\]
In fact it induces an isomorphism
\[
\mathcal{M}_6 \setminus \{ \text{special curves} \} \cong \mathcal{M} \setminus (\mathcal{H}/\Gamma).
\]

**Proof.** Let \( C \) be a plane sextic with 4 ordered nodes. The construction in [2.1] associates to \( C \) a K3 surface \( X \) which is birational to the the double cover of \( \mathbb{P}^2 \) branched along the plane sextic. If \( C \) is general, then \( S_X = H^2(X, \mathbb{Z})^+ \) is the pull-back of the Picard lattice of \( Y \) and \( \{ \bar{e}_0, \bar{e}_1, \ldots, \bar{e}_4 \} \) gives an ordered basis of \( S_X \).

In general, by using Lemma [5] choose a marking \( \alpha : H^2(X, \mathbb{Z}) \to L_{K3} \) such that \( \alpha(H^2(X, \mathbb{Z})^+) \subset S \) and \( \alpha(\bar{e}_i) = s_i \), \( 0 \leq i \leq 4 \). By Lemma [2] \( \alpha_C(\omega_X) \in \mathcal{D} \setminus \mathcal{H} \). Moreover, if \( \alpha_1, \alpha_2 \) are two markings of this type, then \( \alpha_2 \alpha_1^{-1} \) preserves the ordered basis \( \{ s_i \} \), hence its restriction to \( T \) belongs to \( \Gamma^{**} \). Thus we can associate to \( C \) a point in \( \mathcal{D}/\Gamma^{**} \), i.e. we defined a rational map \( \mathcal{P}^{**} : \mathcal{M}_6 \dashrightarrow \mathcal{M}^{**} \).

Conversely, let \( \omega \in \mathcal{D} \setminus \mathcal{H} \). By the surjectivity theorem of the period map ([Ku, PP]) there exists a marked K3 surface \((X, \alpha)\) such that \( \alpha_C(\omega_X) = \omega \). Then \( \iota(\omega) = -\omega \) and there exist no \((-2)\)-vectors in \( T \cap \omega^\perp \) since \( \omega \notin \mathcal{H} \), hence \( \iota \) preserves an ample class. It now follows from the Torelli theorem [Na, Theorem 3.10] that \( \iota \) is induced by an automorphism \( \sigma \) on \( X \).

By [N2, Theorem 4.2.2] the fixed locus of \( \sigma \) is a smooth curve \( C \) of genus six. The quotient surface \( Y = X/(\sigma) \) is smooth and the image of \( C \) belongs to \( | -2K_Y | \). Hence \(-K_Y \) is nef and big with \( K_Y^2 = 5 \), i.e. \( Y \) is a nodal quintic del Pezzo surface. In fact, the pull back of \( \text{Pic}(Y) \) is exactly \( \alpha^{-1}(S) \subset S_X \).

If we choose \( \omega \in \mathcal{D} \setminus \mathcal{H} \) up to the action of \( \Gamma^{**} \) then, by Lemma [5] we get \( \alpha \) up to an isometry in \( \text{O}_T \) which preserves an ordered basis \( \{ s_i \} \). Hence this gives a K3 surface \( X \) with a class \( \alpha^{-1}(s_i) \in \text{O}_T \).
Let \( \Delta \) be the set of vectors \( r \in T \) with \( r^2 = -2 \), then

- the group \( \Gamma \) has three orbits in \( \Delta \):
  \[ \Delta_1 = \{ r \in \Delta : r/2 \not\in T^* \}, \quad \Delta_2 = \{ r \in \Delta : r/2 \in O_1 \}, \quad \Delta_3 = \{ r \in \Delta : r/2 \in O_2 \}; \]
- the group \( \Gamma^* \) has 4 orbits in \( \Delta \): \( \Delta_1, \Delta_2 \) and two orbits decomposing \( \Delta_3 \)
  \[ \Delta_{3a} = \{ r \in \Delta : r/2 = \xi_2 \}, \quad \Delta_{3b} = \{ r \in \Delta : r/2 = \xi_1 \}. \]

**Proof.** Given a vector \( r \in \Delta \) we will classify the embeddings of \( \Lambda = \langle r \rangle \) in \( T \) up to the action of \( \Gamma \) by applying [NI] Proposition 1.15.1. We first need to give an isometry \( \alpha \) between a subgroup of \( A_\Lambda \) and a subgroup of \( A_T \cong \mathbb{F}_2^5 \). If \( H \) is such a subgroup, then either \( H = 0 \) or \( H = \mathbb{F}_2 \). Note that \( H = \mathbb{F}_2 \) if and only if \( r/2 \in T^* \).

In case \( H = 0 \), since there is a unique a lattice \( K \) with \( q_K = q_\Lambda \oplus (-q_T) \) and \( \text{O}(K) \to \text{O}(q_K) \) is surjective by [NI] Theorem 1.14.2, then by [NI] Proposition 1.15.1 there is a unique embedding of \( \Lambda \) in \( T \) such that \( \Lambda \oplus A^\perp = T \).

In case \( H = \mathbb{F}_2 \) there are two different embeddings of \( \Lambda \), according to the choice of \( \alpha(r/2) \) in \( O_1 \) or \( O_2 \). This gives the first assertion.

The second assertion can be proved in a similar way, by observing that \( \Gamma^* \) has three orbits on the set of vectors \( x \in A_T \) with \( q(x) = -1/2 \).

For \( r \in \Delta_i \), let \( T_i = \{ x \in T : (x, r) = 0 \} \) and denote by \( S_i \) the orthogonal complement of \( T_i \) in \( L_{K3} \). Then we have:

**Lemma 7.**

\[
\begin{align*}
S_1 &\cong A_1(-1) \oplus A_1^5; \quad T_1 \cong U \oplus U \oplus E_7 \oplus A_1^5, \\
S_2 &\cong U(2) \oplus D_4; \quad T_2 \cong U \oplus U(2) \oplus E_8 \oplus D_4, \\
S_3 &\cong U \oplus A_1^4; \quad T_3 \cong U \oplus U \oplus E_8 \oplus A_1^4.
\end{align*}
\]

**Proof.** Because of Lemma 6 the isomorphism class of \( T_i \) does not depend on the choice of \( r \in \Delta_i \). If \( r \in \Delta_1 \) or \( \Delta_3 \) then we can assume \( r \) to be one generator of \( E_8 \) or respectively one generator of \( A_1 \) in a decomposition \( T = U \oplus U \oplus E_8 \oplus A_1^5 \). If \( r \in \Delta_2 \) we can assume \( r \) to be a generator of \( A_1 \) in a decomposition \( T = U \oplus U(2) \oplus E_8 \oplus D_4 \oplus A_1 \). In all these cases the orthogonal complement of \( r \) in \( T \) can be easily computed. The lattices \( S_i \) can be computed by applying [NI] Theorem 3.6.2.

**Corollary 1.** The divisor \( \mathcal{H}/\Gamma \) has 3 irreducible components \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \) and \( \mathcal{H}/\Gamma^* \) has 4 irreducible components \( \mathcal{H}_1^*, \mathcal{H}_2^*, \mathcal{H}_{3a}, \mathcal{H}_{3b} \) such that
Let $\sigma$ be the isometry of $L_{K3}$ defined by $\iota_i S_i = 1_{S_i}$ and $\iota_i T_i = -1_{T_i}$. The following can be proved by means of Torelli theorem, as in the proof of Theorem 1.

**Lemma 8.** There exists a K3 surface $X_i$ such that $S_{X_i} \cong S_i$ and carrying an involution $\sigma_i$ of $X_i$ with $\sigma_i^* = \iota_i$.

### 3.2. Curves of genus six with a node

Let $C_1$ be a generic plane sextic with five nodes. The blowing up of the projective plane at the nodes is a quartic del Pezzo surface $Y_1$ and its double cover branched along the strict transform of $C_1$ is a K3 surface $X$. Alternatively, if we blow up the plane at four nodes, we get a quintic del Pezzo surface on which the strict transform of $C_1$ is a curve of genus six with a node. The pull-back of Pic($Y_1$) is a sublattice of the Picard lattice of $X$ isomorphic to $S_1$. We now show that also the converse is true.

**Proposition 3.** The K3 surface $X_1$ is birational to the double cover of a quintic del Pezzo surface branched along a generic curve of genus six with a node or, equivalently, to a double plane branched along a generic sextic with 5 nodes.

**Proof.** Consider the involution $\sigma_1$ on $X_1$ as in Lemma 8. By [N2] Theorem 4.2.2] the fixed locus of $\sigma_1$ is a smooth curve $C_1$ of genus 5. The quotient of $X_1$ by $\sigma_1$ is a smooth rational surface $Y_1$ and the image of $C_1$ belongs to $|−2K_{Y_1}|$, hence $Y_1$ is a del Pezzo surface of degree 4.

Any $−1$-curve $e$ on $Y_1$ intersects the image of $C_1$ at two points since $|−K_{Y_1}, e| = 1$. Hence, contracting one $−1$-curve of $Y_1$ we get a quintic del Pezzo surface where the image of $C_1$ is a curve of genus six with a node, and contracting five disjoint $−1$-curves $C_1$ is mapped to a plane sextic with five nodes.

**Corollary 2.** The divisor $H_1$ is birational to the moduli space of curves of genus six with one node and $H_1^*$ to the moduli space of plane sextics with 5 nodes, with one marked.

**Proof.** Taking the quotient of $H_1$ for the action of $\Gamma$, we identify two markings on $X_1$ which give the same embedding of $\alpha^{-1}(S)$ in Pic($X_1$). This data identifies a $−1$-curve on $Y_1$, whose contraction gives a quintic Del Pezzo surface and a curve of genus 6 with a node. The group $\Gamma^*$, instead, identifies two markings on $X_1$ if they also give the same embedding of $\alpha^{-1}(h)$ in the Picard lattice. This class gives a blowing down map on $Y_1$ with a distinguished exceptional divisor.

Using these remarks and Proposition 3, the result follows as in the proof of Theorem 1.

### 3.3. Plane quintics

Let $C_2$ be a smooth plane quintic and let $L$ be a line transversal to $C_2$. The minimal resolution of the double plane branched along $C_2 \cup L$ is a K3 surface $X$. The Picard lattice of $X$ contains five disjoint $−2$-curves, coming from the resolution of singularities, and a $−2$-curve which is the proper transform of $L$. These rational curves generate a lattice which is isomorphic to $S_2$.

**Proposition 4.** The surface $X_2$ is birational to a double plane branched along the union of a plane quintic and a line.

**Proof.** This was proved in [L] Ch.6.

**Corollary 3.** The divisor $H_2$ is birational to the moduli space of pairs $(C, L)$ where $C$ is a plane quintic and $L$ is a line, while $H_2^*$ parametrizes triples $(C, L, p)$ where $p \in C \cap L$.


Proof: The first statement is [[19]|Corollary 6.21]. The second statement can be proved similarly to Corollary 2. □

3.4. Trigonal curves of genus six. Let \( C \subset \mathbb{P}^5 \) be the canonical model of a trigonal curve of genus 6. Any 3 points in the \( g^3_1 \) lie on a line by Riemann-Roch theorem and the closure of the union of all these lines is a quadric \( Q \) such that the curve \( C \) belongs to \(|4f + 3e|\), where \( e, f \) are the rulings of \( Q \). The minimal resolution of the double cover of \( Q \) branched along the union of \( C \) with a line \( L \in |e| \) is a K3 surface \( X \). The ruling \( f \), the proper transform of \( L \) and the exceptional divisors over the four points in \( C \cap L \) generate a sublattice of \( S_X \) isomorphic to \( S_3 \).

As before, we now prove a converse statement.

Proposition 5. The surface \( X_3 \) is birational to:

- the double cover of a quadric \( Q \) branched along a line and a trigonal curve of genus six.
- the double cover of a Hirzebruch surface \( \mathbb{F}_4 \) branched along a curve with 4 nodes in \(|3h|\) and the rational curve in \(|s|\), where \( h^2 = 4, \ s^2 = -4, \ (h, s) = 0 \).

Proof. By [[2]|Theorem 4.2.2] the set of fixed points of \( \sigma_3 \) on \( X_3 \) is the disjoint union of a smooth curve \( C \) of genus 6 and a smooth rational curve \( L \). Since \( S_3 \subset \text{Pic}(X_3) \), \( X_3 \) admits an elliptic fibration \( \pi \) with a section and four singular fibers of Kodaira type \( I_2 \) or type \( III \). Since any fiber of \( \pi \) is preserved by \( \sigma_3 \), then \( L \) is a section of \( \pi \) and \( C \) intersects each fiber in 3 points. Hence \( C \) has a triple cover to \( \mathbb{P}^1 \) and its ramification points are the singular points of irreducible fibers of \( \pi \).

We will denote by \( F_1, \ldots, F_4 \) the singular fibers of \( \pi \) of type \( I_2 \) or \( III \), by \( E_i \) the component of \( F_i \) meeting \( L \) and by \( E_i' \) the other component. Let \( p: X_3 \to Y_3 \) be the quotient by the involution \( \sigma_3 \). Note that \( p(E_i) \) and \( p(E_i') \) are \((-1)-\)curves.

By contracting the curves \( p(E_i) \), we get a smooth quadric surface. This gives the first assertion.

On the other hand, contracting the curves \( p(E_i') \), we get a Hirzebruch surface \( \mathbb{F}_4 \) (note that the image of \( L \) has self-intersection \(-4\)). Since \( C \) intersects the ruling in 3 points, each \( E_i' \) at two points and it does not intersect \( L \), then its image in \( \mathbb{F}_4 \) has 4 nodes and belongs to the class \( 3h \). This gives the second assertion. □

Corollary 4.

- The divisor \( H_3 \) is birational to the moduli space of pairs \((C, L)\) where \( C \) is a trigonal curve of genus 6 and \( L \in |K_C - 2g^1_3| \).
- The divisor \( H_{3a} \) parametrizes pairs \((C, p)\) where \( C \) is trigonal and \( p \in C \) or, equivalently, plane sextics with a node and a triple point.
- The divisor \( H_{3b} \) is birational to the moduli space of curves in \(|3h|\) of \( \mathbb{F}_4 \) with 4 nodes.

Proof. The first statement follows from Proposition 5 and the remarks at the beginning of this subsection since, by adjunction formula, the restriction of \( e \) to \( C \) coincides with \( K_C - 2g^1_3 \).

Given a trigonal curve \( C \subset Q \) of genus six and \( p \in C \), there exists a unique line \( L \in |e| \) through \( p \). This determines a K3 surface \( X \) with \( S_3 \subset S_X \) as before. Moreover, the projection of \( C \) from \( p \) is a plane sextic with a triple point and a double point. The hyperplane class of \( \mathbb{P}^2 \) induces the linear system \( K_C - g^1_3 - p \) on \( C \) and its pull-back to \( X \) is a nef class \( h \) with \( h^2 = 2 \).

Conversely, a generic point in \( H_{3a} \cup H_{3b} \) gives a K3 surface \( X \) with \( S_X \cong S_3 = U \oplus A_1^4 \) and a degree two polarization \( h \). Let \( e, f \) be a basis of \( U \) and \( e_1, \ldots, e_4 \) an orthogonal basis of \( A_1^4 \). Up to an isometry of \( S_3 \) we can assume that \( r = e - f \) and that \( f \) gives an elliptic fibration on \( X \). The orthogonal
complement $S_3 \cap r^\perp \cong S$ has two types of degree two polarizations: $h_j = 2(e + f) - \sum_{i=1}^4 e_i + e_j$ or $h = e + f$.

A point in $\mathcal{H}_{3b}^*$ gives a polarization $h_b$ such that $h_b/2 = r/2$ in $A_T \cong A_S$, hence $h_b = h$. The class $h_b$ contains $r$ in the base locus and $2h_b$ maps $X$ onto a cone over a rational normal quartic. In fact, the morphism associated to $2h_b$ is exactly the contraction of the curves $p(E_i')$ and the image of the curve $L$ described in the proof of Proposition [5].

A point in $\mathcal{H}_{3a}^*$ gives a polarization $h_a = h_j$ for some $j = 1, \ldots, 4$. In this case $h_a$ has no base locus and gives a generically 2:1 map $X \to \mathbb{P}^2$. The branch locus of this map is a plane sextic with a triple point (in the image of $r$) and a node (in the image of $e_j$). The line through the two singular points intersects the sextic in one more point $p$. Hence this gives a pair $(C, p)$, where $C$ is trisecant and $p \in C$.

**Remark 3.** The two irreducible components in $\mathcal{M}^*$ over $\mathcal{H}_3$ correspond to the components in $\mathcal{W}_6^2$ over the trigonal divisor in $\mathcal{M}_6$. With the notation in the proof of Proposition [2], the divisor $\mathcal{H}_{3a}^*$ corresponds to pairs $(C, D(p))$ and $\mathcal{H}_{3b}^*$ to $(C, 2g_3^1)$. This agrees with [Sh], where it is proved that the triple conic, which is the plane model of $C$ associated to $2g_3^1$ (Proposition 2), “represents” K3 surfaces with a degree two polarization with a fixed component.

**Remark 4.** Let $C$ be a plane sextic with four nodes $p_1, \ldots, p_4$ such that $p_1, p_2, p_3$ lie on a line $L$. The blowing up of the plane in these points is a nodal del Pezzo surface $Y$ (see section 1) and the double cover of $Y$ branched along the proper transform of $C$ is a K3 surface $X$. The pencil of lines through $p_4$ induces an elliptic fibration on $X$ with general fiber $f$, 3 fibers of type $I_2$ and two sections $s_1, s_2$, given by the two (disjoint) inverse images of the line $L$. In particular, the Picard lattice of $X$ contains the sublattice $S' = U + A_3^1 < -4 >$, where $U$ is generated by the fiber $f$ and $s_1, A_3^1$ by the reducible components in each fiber and $< -4 >$ by $2f + s_1 - s_2$.

Conversely, let $r \in T$ be a primitive vector with $r^2 = -4$ such that $r/2 \in A_T$, then its orthogonal complement in $T$ is isomorphic to $T' = U \oplus U \oplus E_8 \oplus A_3^1 < -4 >$ and $T'^\perp \cong S'$.

By choosing a different blow-down map for $Y$ we get a plane sextic with a tacnode and two nodes. In fact, the elliptic fibration described above is induced by the pencil of lines through the tacnode.

### 4. Compactifications

#### 4.1. Satake-Baily-Borel compactification

The moduli spaces $\mathcal{M}, \mathcal{M}^*$ are quasi-projective algebraic varieties. Since they are arithmetic quotients of a symmetric bounded domain, we can consider their Satake-Baily-Borel (SBB) compactifications $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}^*$ (see [BB] and [Sc], § 2).

It is known that boundary components of the SBB compactification are in bijection with primitive isotropic sublattices of $T$ up to $\Gamma$ and $\Gamma^*$ respectively, such that $k$-dimensional boundary components correspond to rank $k + 1$ isotropic sublattices. Since $T$ has signature $(2, 15)$, the boundary components will be either 0 or 1 dimensional.

**Lemma 9.** Let $\mathcal{I}$ be the set of primitive isotropic vectors in $T$. There are two orbits in $\mathcal{I}$ with respect to the action of $\Gamma$:

$$\mathcal{I}_1 = \{v \in \mathcal{I} : (v, T) = \mathbb{Z}\} \quad \mathcal{I}_2 = \{v \in \mathcal{I} : (v, T) = 2\mathbb{Z}\}.$$

There are three orbits with respect to $\Gamma^* : \mathcal{I}_1$ and two orbits decomposing $\mathcal{I}_2$.

**Proof.** By Proposition 4.1.3 in [Sc] there is a bijection between orbits of isotropic vectors in $T$ modulo $\Gamma$ ($\Gamma^*$) and isotropic vectors in $A_T$ modulo the induced action of $\Gamma$ ($\Gamma^*$). By Lemma 4 the map...
\[ \Gamma \to O(q_T) \text{ is surjective and clearly the image of } \Gamma^* \text{ is given by elements of } O(q_T) \text{ fixing } \xi_1. \text{ Then it follows from Lemma 3 that there are exactly two orbits of isotropic vectors in } A_T \text{ for the action of } \Gamma \text{ and three for the action induced by } \Gamma^*. \square \]

**Corollary 5.** The boundaries of \( \mathcal{M} \) and \( \mathcal{M}^* \) contain two and three zero-dimensional components respectively.

We will denote by \( p, q \) the zero-dimensional boundary components of \( \mathcal{M} \) corresponding to the orbits \( I_1, I_2 \) in Lemma 9 respectively and with \( q_1, q_2 \) the zero-dimensional boundary components of \( \mathcal{M}^* \) corresponding to the orbits of \( \Gamma^* \) decomposing \( I_2 \).

**Remark 5.** By [N1] Theorem 3.6.2 there is also an isomorphism
\[ T \cong U \oplus U(2) \oplus A_1 \oplus D_4 \oplus E_8. \]

In the following we will denote by \( e, f \) and \( e', f' \) the standard bases of \( U \) and \( U(2) \), by \( \beta \) a generator of \( A_1 \), by \( \gamma_1, \ldots, \gamma_4 \) and \( \alpha_1, \ldots, \alpha_8 \) the standard root bases of \( D_4 \) and \( E_8 \). Note that \( e, f \in I_1 \) and \( e', f' \in I_2 \).

We now classify one dimensional boundary components in \( \mathcal{M} \) by studying \( \Gamma \)-orbits of primitive isotropic planes in \( T \). We will say that such a plane is of type \( (i, j) \), \( i, j = 1, 2 \) if it is generated by a vector in \( I_i \) and one in \( I_j \).

Let \( \mathcal{G}_1 \) be the genus of \( E_8 \oplus A_1^5 \) and let \( \mathcal{G}_2 \) be the genus of \( E_8 \oplus A_1 \oplus D_4 \). If \( N \) is a lattice in \( \mathcal{G}_1 \), then \( T \cong U \oplus U \oplus N \) by [N1] Theorem 3.6.2. By taking two isotropic vectors, each in one copy of \( U \), we get an isotropic plane in \( T \) of type \( (1, 1) \). Similarly, if \( N_2 \in \mathcal{G}_2 \) then \( T \cong U \oplus U(2) \oplus N_2 \) and the plane generated by a generator of \( U \) and one of \( U(2) \) is isotropic of type \( (1, 2) \).

**Lemma 10.** The isomorphism classes of lattices in \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are given in the following table.

| \( R \)       | \( \mathcal{G}_1 \)                                              | \( \mathcal{G}_2 \)                        |
|--------------|---------------------------------------------------------------|------------------------------------------|
| a \( E_8^3 \) | \( E_7 \oplus D_4 \oplus A_1^2, D_6^2 \oplus A_1, E_8 \oplus A_1^5 \) | \( A_1 \oplus E_8 \oplus D_4 \) |
| b \( E_7 \oplus D_{10} \) | \( E_7 \oplus A_1^6, D_6 \oplus D_4 \oplus A_1^5, D_8 \oplus D_4 \oplus A_1, D_8 \oplus A_1^5, D_{10} \oplus A_1^3 \) | \( E_7 \oplus D_6, D_{10} \oplus A_1^3 \) |
| c \( D_{16} \oplus E_8 \) | \( D_8 \oplus A_1^5 \)                                        | \( D_{12} \oplus A_1 \)                  |
| d \( A_{17} \oplus E_7 \) | \( (A_1^4)\perp \) in \( A_{17} \)                           |                                          |

**TABLE 1.** One dimensional boundary components

**Proof.** The orthogonal complements of \( E_8 \oplus A_1^5 \) and \( E_8 \oplus A_1 \oplus D_4 \) in \( E_8^3 \) are isomorphic to \( R_1 = E_7 \oplus A_1^4 \) and \( R_2 = E_7 \oplus D_4 \) respectively. By Proposition 6.1.1, [SC] the isomorphism classes in \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) can be obtained by taking the orthogonal complements of primitive embeddings of \( R_1 \) and respectively \( R_2 \) into even negative definite unimodular lattices of rank 24, i.e. Niemeier lattices. These lattices are uniquely determined by their root sublattice \( R \), hence they are denoted by \( N(R) \) (see [CS], Chap. 18). In order to determine all lattices in the \( \mathcal{G}_i \) we first classify all primitive embeddings of \( R_1, R_2 \) into \( R \) and take their orthogonal complements \( R_i^\perp \) in \( R \). Then we take the primitive overlattice.
$R_i^\perp$ of $R_i^\perp$ in $N(R)$ which contains $R_i^\perp$ as a subgroup of index at most 2. Here we have used the classification of embeddings between root lattices due to Nishiyama [Ni]. This gives isomorphism classes $R_i^\perp$ in $\mathcal{G}_i$. In Table I all root lattices $R_i$ appear such that $R_i$ can be embedded in $N(R)$ and the corresponding lattices in $\mathcal{G}_1$ and $\mathcal{G}_2$. If $R_i^\perp$ is primitive in $N(R)$, then we omit the overline.

□

**Theorem 2.** The boundary of $\overline{\mathcal{M}}$ contains 14 one dimensional components $B_1, \ldots, B_{14}$ where the closure of $B_i$, $i = 1, \ldots, 10$ contains only $p$ and the closure of $B_j$, $j = 11, \ldots, 14$ contains both $p$ and $q$.

*Proof.* As remarked before, to the lattices in $\mathcal{G}_1$ we can associate isotropic planes of type $(1, 1)$ in $T$ which are not $\Gamma$-equivalent. Conversely, by Lemma 5.2 in [Sc], any isotropic plane $E$ of type $(1, 1)$ can be embedded in $U \oplus U$ and $T \cong U \oplus U \oplus E^\perp/E$ where $E^\perp/E \in \mathcal{G}_1$. Hence, boundary components containing only $p$ are in one-to-one correspondence with lattices in $\mathcal{G}_1$.

The proof is more subtle for isotropic planes of type $(1, 2)$. Note that if $v \in T$ is a primitive isotropic vector of type 2 and $E$ is an isotropic plane containing $v$, then $E$ determines a primitive vector in $M_v = v^\perp/\mathbb{Z}v$. Hence, isotropic planes of type $(1, 2)$ correspond to orbits of isotropic vectors in $M_v$. In this case $M_v \cong U \oplus E_8 \oplus D_4 \oplus A_1$ and orbits of isotropic vectors can be determined by Vinberg’s algorithm (see § 1.4 [V] or § 4.3 [St]).

By [Ni] Theorem 0.2.3, the Weyl group $W(M_v)$ has finite index in $O(M_v)$. This implies that the algorithm will finish in a finite number of steps. To start the algorithm we fix the vector $\bar{x} = e + f$. Then at each step we have to choose roots $x \in M_v$ such that the height

$$h = (x, \bar{x}) \sqrt{x^2}$$

is minimal and $(x_i, x_j) \geq 0$ for $j = 1, \ldots, i - 1$. In our case we get:

1) $(x, \bar{x}) = 0$: $u := e - f$, $\alpha_1, \ldots, \alpha_8, \gamma_1, \ldots, \gamma_4, \beta$.

2) $(x, \bar{x}) = 1$: $\alpha := f + \bar{\alpha}_8$, $\gamma := f + \bar{\gamma}_1$, $\beta' := f - \beta$.

3) $(x, \bar{x}) = 4$: $\delta_i := 2(e + f) - \beta + \bar{\alpha}_1 + \bar{\gamma}_i$, $j = 2, 3, 4$.

4) $(x, \bar{x}) = 12$: $\alpha' := 6(e + f) - 3\beta + 2\bar{\alpha}_4 + \bar{\gamma}_2 + \bar{\gamma}_3 + \bar{\gamma}_4$

where $\bar{\alpha}_1, \ldots, \bar{\alpha}_8$ and $\bar{\gamma}_1, \ldots, \bar{\gamma}_4$ are the dual bases of $E_8$ and $D_4$. We now draw the Dynkin diagram associated to these roots. Let $g_{ij} = (e_i, e_j)/\sqrt{e_i^2 e_j^2}$. Then two vertices $i, j$ corresponding to vectors $e_i, e_j$ are connected by

- • • if $g_{ij} = 0$,
- •—• if $g_{ij} = 1/2$,
- ——• if $g_{ij} = 1$,
- ———if $g_{ij} > 1$.

The diagram in our case is given in Figure 1 (see also Figure 5, [Ko]). Note that the symmetry group of the diagram is $\mathbb{Z}_2 \times S_3$ and it can be easily seen that all symmetries can be realized by isometries in $\Gamma$. The maximal parabolic subdiagrams of rank 13 are of four types:
Note that each type is an orbit for the action of $\Gamma$. These subdiagrams correspond to non-equivalent isotropic vectors in $M_v$. Hence, we get $4$ isotropic planes in $T$ containing a vector in $I_2$ and a direct analysis shows that all of them are of type $(1, 2)$. □

It follows from the proof of Theorem 2 that the boundary components of $\overline{M}$ are in one-to-one correspondence with the lattices in $G_1$ and $G_2$. These lattices appear in connection to degenerations of K3 surfaces as explained for example in [Sc]. This allows to compare the SBB compactification with more geometrically meaningful compactifications, as the ones obtained by means of geometric invariant theory.

In case of K3 surfaces with a degree two polarization this is well-understood ([Sh], [F], [Lo2]). Table 2 describes the correspondence between type II boundary components of the GIT compactification of plane sextics and one dimensional boundary components of the Baily-Borel compactification for degree two K3 surfaces. The lattice appearing in the SBB column is $E_\perp/E$, where $E$ is the isotropic lattice associated to the boundary component.

**Remark 6.** In the proof of Theorem 2 we showed that boundary components of $\overline{M}$ containing only $p$ in their closure correspond to primitive embeddings of the lattice $E_7 \oplus A_{1}^4$ into Neimeier lattices. Equivalently, they correspond to primitive embeddings of the lattice $A_{1}^4$ in the root lattices $E_8 \oplus E_8$, $E_7 \oplus D_{10}$, $D_{16}$, $A_{17}$. Note that a double cover branched over a node has an $A_1$ singularity hence, embedding $A_{1}^4$ in the root lattices is equivalent to choose a distribution of the 4 nodes on the corresponding configurations in Table 2 (where more than one node can “collapse” to the same singular point of the configuration).
For example, let $q_1, q_2$ be the two singular points in the IIa configuration. We can either embed one node in $q_1$ and 3 nodes in $q_2$ (this gives the root lattice $E_7 \oplus D_1 \oplus A_1$), two nodes in $q_1$ and two in $q_2$ (this gives the root lattice $D_2^6 \oplus A_1$) or 4 nodes in $q_1$ (this gives the root lattice $E_8 \oplus A_5^5$).

Similarly, boundary components containing both $p$ and $q$ in their closure correspond to embeddings of the lattice $D_4$ into the previous root lattices. Note that a double cover branched over a triple point has a $D_4$ singularity.

In fact we conjecture that a one dimensional boundary component $B$ of $\overline{M}$ of type a, b, c or d (see Table 1) corresponds to a boundary component of type IIa, IIb, IIc or IId respectively with

- 4 marked nodes (eventually collapsing) if $q \not\in B$
- a marked triple point if $q \in B$.

Note that the configuration IId has no triple points, in fact there is no one-dimensional boundary component of type d containing $q$ in its closure.

**Remark 7.** By corollaries 3 and 4 the moduli space $M$ contains two divisors which are birational to $\mathbb{P}^2$ and $\mathbb{P}^1$ fibrations over the locus of plane quintics and trigonal curves respectively. This suggests that we need to blow-up the moduli space of curves of genus six in order to extend the period map to these loci.

Bi-elliptic and hyperelliptic curves of genus six are mapped to one dimensional boundary components of $\overline{M}$. In fact, the configuration IIc is a plane model for hyperelliptic curves and case IId is the plane model of a bi-elliptic curve of genus six (see §1).

### References

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris. *Geometry of Algebraic Curves*, I, Springer.

[AH] E. Arbarello, J. Harris. Canonical curves and quadrics of rank 4, Compositio Mathematica, 43, n.2, (1981) 145–179.

[BB] W. L. Baily and A. Borel. Compactifications of arithmetic quotients of bounded symmetric domains. *Ann. Math.*, 2 (84) (1966), 442–528.

[CS] J. H. Conway, N. J. A. Sloane. *Sphere Packings, Lattices and Groups*, Springer.

[DO] I. Dolgachev, D. Ortland. Point sets in projective spaces and theta functions, Asterisque 165 (1988).

[Do] I. Dolgachev. *Topics in classical algebraic geometry*, I. www.math.lsa.umich.edu/~idolga/lecturenotes.html.

[F] R. Friedman. A new proof of the global Torelli theorem for K3 surfaces. *Ann. Math.*, 120, n.2 (1984), 237–269.

[H] E. Horikawa. Surjectivity of the period map of K3 surfaces of degree 2. *Math. Ann.*, 228 (2) (1977), 113–146.

[Ko] S. Kondō. Algebraic surfaces with finite automorphism groups, *Nagoya Math. J.*, 116 (1989), 1–15.

[Ko1] S. Kondō. A complex hyperbolic structure for the moduli space of curves of genus three, *J. Reine Angew. Math.*, 525 (2000), 219–232.
S. Kondō. The moduli space of curves of genus 4 and Deligne-Mostow’s complex reflection groups, *Adv. Studies Pure Math.*, **36** (2002), Algebraic Geometry 2000, Azumino, 383–400.

V. S. Kulikov. Degenerations of K3 surfaces and Enriques surfaces, *Izv. Akad. Nauk SSSR Ser. Mat.*, **41** (1977), 1008–1042.

R. Laza. *Deformations of singularities and variations of GIT quotients*, Ph.D thesis, Columbia University, [math.AG/0607003](http://arxiv.org/abs/math.AG/0607003).

E. Looijenga. Compactifications defined by arrangements. II. Locally symmetric varieties of type IV, *Duke Math. J.*, **119**, n.3 (2003), 527–588.

E. Looijenga. New compactifications of locally symmetric varieties, in Proceedings of the 1984 Vancouver Conference in Algebraic Geometry, ed. J. Carrell, A. V. Geramita, and P. Russell, CMS Conf. Proc. 6, Amer. Math. Soc., Providence (1986), 341–364.

D. Morrison, M-H. Saito. *Cremona transformations and degree of period maps for K3 surfaces with ordinary double points*, Algebraic Geometry, Sendai 1985, Adv. Studies Pure Math., **10** (1987), 477–513.

Y. Namikawa. Periods of Enriques surfaces, *Math. Ann.*, **270** (1985), 201–222.

V. V. Nikulin. Integral symmetric bilinear forms and its applications, *Math. USSR Izv.*, **14** (1980), 103–167.

V. V. Nikulin. Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections, *J. Soviet Math.*, **22** (1983), 1401–1475.

K. Nishiyama. The Jacobian fibrations on some K3 surfaces and their Mordell-Weil groups, *Japanese J. Math.*, **22** (1996), 293–347.

U. Persson and H. Pinkham. Degenerations of surfaces with trivial canonical bundle, *Annals of Math.*, **113** (1981), 45–66.

I. Piatetski-Shapiro, I. R. Shafarevich. A Torelli theorem for algebraic surfaces of type K3, *Math. USSR Izv.*, **5** (1971), 547–587.

N. I. Shepherd-Barron. Invariant theory for S5 and the rationality of M6, *Compositio Math.*, **70** (1989), 13–25.

F. Scattone. On the compactification of moduli spaces for algebraic K3 surfaces, *Memoirs of A. M. S.*, **70** (1987), No. 374.

J. Shah. A complete moduli space for K3 surfaces of degree 2, *Ann. of Math.*, **112** (1980), 485–510.

H. Sterk. Compactifications of the period space of Enriques surfaces, I, II, *Math. Z.*, **207** (1991), 1–36, ibid **220** (1995), 427–444.

E. B. Vinberg. Some arithmetic discrete groups in Lobachevskii spaces, in “Discrete subgroups of Lie groups and applications to moduli”, Tata-Oxford (1975), 323–348.

**DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDAD DE CONCEPCIÓN, CASILLA 160-C, CONCEPCIÓN, CHILE**

E-mail address: martebani@udec.cl

**GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, NAGOYA, 464-8602, JAPAN**

E-mail address: kondo@math.nagoya-u.ac.jp