Research Article

New Proofs of Some $q$-Summation and $q$-Transformation Formulas

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Received 13 February 2014; Accepted 11 April 2014; Published 7 May 2014

Academic Editor: Kishin Sadarangani

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We obtain an expectation formula and give the probabilistic proofs of some summation and transformation formulas of $q$-series based on our expectation formula. Although these formulas in themselves are not the probability results, the proofs given are based on probabilistic concepts.

1. Introduction

The probabilistic method is an important tool to derive results in combinatorics, theory of numbers, and other fields (see [1–15]). There have been many applications in the basic hypergeometric series (or $q$-series). For example, Fulman [3] presented a probabilistic proof of Rogers-Ramanujan identity using Markov chain. Chapman [2] proved the Andrews-Gordon identity by using extended Fulman’s methods. Kadell [4] gave a probabilistic proof of Ramanujan’s $\psi_1$ summation based on the order statistics.

Recently, Wang [13, 14] constructed a random variable $X$ and introduced a new probability distribution $W(x; q)$:

$$P\left(X = x^n q^k\right) = p_{n,k} (x; q) = \frac{(-x)^n q^k (x^{n-1} q^{k+1}, x^n q^{k+1}; q)_\infty}{(q, q/x, x; q)_\infty},$$

where

$$p_{n,k} (x; q) > 0, \quad \sum p_{n,k} (x; q) = 1,$$

$$x < 0, \quad 0 < q < 1, \quad n = 0, 1, \quad k = 0, 1, 2, \ldots$$

By applying the above probability distribution, Wang proved the $q$-binomial theorem and $q$-Gauss summation formula and also obtained some new summation formulas and transformation formulas.

One of the most important concepts in probability theory is that of the expectation of a random variable. If $X$ is a discrete random variable having a probability mass function $p(x)$, then the expectation, or the expected value, or the expectation operator of $X$, denoted by $E[X]$, is defined by (e.g., [9, page 125])

$$E[X] = \sum_{p(x) > 0} x p(x).$$

In the following section we introduce some notations, definitions, and formulas of $q$-series. Throughout this paper we suppose $q \in \mathbb{C}, |q| < 1$.

The $q$-shifted factorials are defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - a q^k),$$

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - a q^k), \quad n \geq 1.$$

Clearly,

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty},$$

$$(a^2; q^2)_n = (a; q)_n (-a; q)_n.$$
The following are compact notations for the multiple q-shifted factorials:

\[
(a_1, a_2, \ldots, a_m; q)_n = (a_1; q) (a_2; q) \cdots (a_m; q)_n,
\]

\[
(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.
\]

(6)

The basic hypergeometric series or \(q\)-series are defined by

\[
\phi_s \left( a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} \left(-1\right)^n q^n z^n.
\]

(7)

Heine introduced the \(r+1\) basic hypergeometric series which is defined by

\[
\phi_{r+1} \left( a_1, a_2, \ldots, a_r+1; b_1, b_2, \ldots, b_r; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} z^n.
\]

(8)

Jackson defined the \(q\)-integral (see [17, 18]):

\[
\int_0^d f(t) \, dt = d \left( 1 - q \right) \sum_{n=0}^{\infty} f \left( d q^n \right) q^n,
\]

(9)

\[
\int_c^d f(t) \, dt = \int_0^d f(t) \, dt - \int_0^c f(t) \, dt.
\]

(10)

The following is the Andrews-Askey integral (see [19]) which can be derived from Ramanujan's \(_1\Psi_1\):

\[
\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} \, dt = \frac{\left( 1 - q \right) \left( q, dq/c, c/d, abc/d; q \right)_\infty}{\left( ac, ad, bc, bd; q \right)_\infty} \phi_2 \left( a, b; c, d; q, x \right),
\]

(11)

provided that there are no zero factors in the denominator of the integrals. Recently, Liu and Luo [20] further generalized the above Andrews-Askey integral in the following more general form.

**Lemma 1** (see [21, page 5, (2.5)] [20, Theorem 1]). One has

\[
\int_0^b \frac{(qt/a, qt/b, ct; q)_\infty}{(et, ft, ht; q)_\infty} \, dt = \frac{b \left( 1 - q \right) \left( q, bq/a, a/b, bc, abef; q \right)_\infty}{\left( ae, af, be, bf, bh, q \right)_\infty} \phi_2 \left( a, b; c, d; q, ah \right)
\]

\[
\times \left( be, bf, c, abef, bc \right),
\]

provided \(|ah| < 1\) and \(|q| < 1\), provided that there are no zero factors in the denominator of the integrals.

**Lemma 2** (see [21, page 5, (2.7)]). One has

\[
\int_0^b \frac{(qt/a, qt/b, ct; q)_\infty}{(et, ft, ht; q)_\infty} \, dt = \frac{b \left( 1 - q \right) \left( q, bq/a, a/b, bc; q \right)_\infty}{\left( ae, af, be, bh, q \right)_\infty} \phi_2 \left( a, b; c, d; q, ah \right)
\]

\[
\times \left( be, bf, c, abef, bc \right),
\]

provided \(|ah| < 1\) and \(|q| < 1\), provided that there are no zero factors in the denominator of the integrals.

The aim of the present paper is to give an expectation formula and introduce some probabilistic proofs of the corresponding summation and transformation formulas of \(q\)-series based on an expectation formula. In Section 2 we give an expectation formula of the random variables \((dX; q)_\infty/(aX, bX, cX; q)_\infty\). In Section 3 we show the probabilistic proofs of transformation formulas of \(\phi_2\). In Section 4 we give probabilistic proof of Heine's transformations and Jackson's transformations. In Section 5 we give probabilistic proof of some formulas of \(q\)-series, for example, \(q\)-binomial theorem, \(q\)-Chu-Vandermonde sum formulas, \(q\)-Gauss sum formula, \(q\)-Kummer sum formula, Bailey sum formula, and so forth.

### 2. Main Theorem

In this section we obtain the expectation formulas of some random variables which are very useful to prove the summation and transformation formulas of \(q\)-series.

**Theorem 3.** Let \(X\) denote a random variable with probability distribution \(W(x; q), -1 < x < 0\). Then one has

\[
E \left[ \frac{(dX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] = \frac{(d, abx; q)_\infty}{(a, ax, b, bx, c, q; q)_\infty} \phi_2 \left( a, b; c, d; q, x \right),
\]

(14)

provided that \(\max(|a|, |b|, |c|, |d|) < 1, |c| < 1,\) and \(|q| < 1\).

**Proof.** A random variable \(X\) has the distribution \(W(x; q)\). From definitions (9) we have

\[
\int_0^1 \frac{(qt/x, qt/dt; q)_\infty}{(at, bt, ct; q)_\infty} \, dt = \frac{1 - \left( q \left( q^{k+1}/x, q^{k+1}d; q \right)_\infty \right)}{\left( aq^k, bq^k, cq^k; q \right)_\infty},
\]

(12)

\[
\int_0^x \frac{(qt/x, qt/dt; q)_\infty}{(at, bt, ct; q)_\infty} \, dt = \left( 1 - q \right) \sum_{k=0}^{\infty} \frac{(x^{k+1}, xq^{k+1}, dxq^k; q)_\infty}{(axq^k, bxq^k, cxq^k; q)_\infty} q^k,
\]

(15)
Hence, we obtain

\[
\int_0^\infty \frac{(qt/x, qt, dt; q)_\infty}{(at, bt, ct; q)_\infty} d_q t = (1 - q) \sum_{k=0}^\infty \frac{(q^{k+1}/x, q^{k+1}, dq^k; q)_\infty q^k}{(aq^k, bq^k, cq^k; q)_\infty} - x(1-q) \sum_{k=0}^\infty \frac{(q^{k+1}, xq^{k+1}, dxq^k; q)_\infty q^k}{(axq^k, bxq^k, cxq^k; q)_\infty}.
\]

By using the probability distribution \( W(x; q) \) and noting (16) and (12) of Lemma 1, we calculate the expectation of the random variable \( (dX; q)_\infty/(aX, bX, cX; q)_\infty \) as follows:

\[
E \left[ \frac{(dX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] = \frac{1}{(1-q)(q, q/x, x; q)_\infty} \times \left( \frac{dx^n q^n}{(ax^n q^n, bx^n q^n, cx^n q^n; q)_\infty} \right) = \frac{1}{(1-q)(q, q/x, x; q)_\infty} \int_x^\infty \frac{(qt/x, qt, dt; q)_\infty d_q t}{(at, bt, ct; q)_\infty} = \frac{1}{(1-q)(q, q/x, x, d, abx; q)_\infty} \times \left( a, b, \frac{d}{abx}, c ; q, cx \right).
\]

Hence, we obtain

\[
E \left[ \frac{(dX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] = \frac{(d, abx; q)_\infty}{(a, ax, b, bx, c; q)_\infty} \Phi_2 \left( a, b, \frac{d}{abx}, c ; q, cx \right).
\]

Theorem 4. Let \( X \) denote a random variable with probability distribution \( W(x; q), -1 < x < 0 \). Then one has

\[
E \left[ \frac{(dX; q)_\infty}{(aX, bX; q)_\infty} \right] = \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} \Phi_2 \left( a, b, d ; q, dx \right),
\]

provided that \( \max(|a|, |b|, |d|) < 1, |dx| < 1, \) and \( |q| < 1 \).

Proof. By (7) and (8) we have

\[
\Phi_2 \left( a, b, \frac{d}{abx}, c ; q, cx \right) = \sum_{n=0}^\infty \frac{(a; q)_n(b; q)_n(d/c; q)_n(cx)^n}{(q; q)_n(abx; q)_n(d; q)_n}.
\]

By (4) we have

\[
\left( \frac{d}{c} \right)_n c^n = \left( 1 - \frac{d}{c} \right) \left( 1 - \frac{d}{c} q \right) \cdots \left( 1 - \frac{d}{c} q^{n-1} \right) c^n = (c - d) (c - dq) \cdots (c - dq^{n-1}).
\]

Substituting (20) and (21) into the right-hand side of (14), we obtain

\[
E \left[ \frac{(dX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] = \frac{(d, abx; q)_\infty}{(a, ax, b, bx, c; q)_\infty} \sum_{n=0}^\infty \frac{(a; q)_n(b; q)_n x^n}{(q; q)_n(abx; q)_n(d; q)_n} \times (c - d) (c - dq) \cdots (c - dq^{n-1}).
\]

Next, let us replace \( c \) by \( \lambda c \), respectively, and let \( \lambda \to 0 \) in (22); we get

\[
E \left[ \frac{(dX; q)_\infty}{(aX, bX; q)_\infty} \right] = \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} \sum_{n=0}^\infty \frac{(a; q)_n(b; q)_n x^n}{(q; q)_n(abx; q)_n(d; q)_n} \times (-d) (-dq) \cdots (-dq^{n-1}) = \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} \sum_{n=0}^\infty \frac{(a; q)_n(b; q)_n x^n}{(q; q)_n(abx; q)_n(d; q)_n} \times d^n (-1)^n q^{n(n-1)/2} = \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} \sum_{n=0}^\infty \frac{(a; q)_n(b; q)_n dx^n}{(q; q)_n(abx; q)_n(d; q)_n} \times (1-d) (-1)^n q^{n^2/2} = \frac{(d, abx; q)_\infty}{(a, ax, b, bx; q)_\infty} \sum_{n=0}^\infty \frac{(a; q)_n(b; q)_n (dx)^n}{(q; q)_n(abx; q)_n(d; q)_n} \times (1-d) (-1)^n q^{n^2/2}.
\]
**Theorem 5.** Let $X$ denote a random variable with probability distribution $W(x;q)$, $-1 < x < 0$. Then one has

$$
E \left[ \frac{(cx; q)_\infty}{(aX, bX, cX; q)_\infty} \right] = \frac{(cx, abx; q)_\infty}{(ax, bx, a, b, c; q)_\infty} \Phi_2 \left( \begin{array}{c} a, b, x \\ ab, cx, q; cx \end{array} \right),
$$

(24)

provided that $\max(|a|, |b|, |c|) < 1$, $|cx| < 1$, and $|q| < 1$.

**Proof.** Letting $d = cx$ in (14) of Theorem 3, we obtain (24).

**Corollary 6** (see [13, page 463, Theorem 1]). Let $X$ denote a random variable with probability distribution $W(x;q)$, $-1 < x < 0$. Then one has

$$
E \left[ \frac{(cx; q)_\infty}{(aX, bX, cX; q)_\infty} \right] = \frac{(cx, abx; q)_\infty}{(ax, bx, a, b, c; q)_\infty} \Phi_2 \left( \begin{array}{c} a, b, x \\ ab, cx, q; cx \end{array} \right),
$$

(25)

provided that $\max(|a|, |b|, |c|) < 1$.

**Proof.** Using (24) of Theorem 5, we deduce

$$
E \left[ \frac{(cx; q)_\infty}{(aX, bX, cX; q)_\infty} \right] = \frac{(cx, abx; q)_\infty}{(ax, bx, a, b, c; q)_\infty} \Phi_2 \left( \begin{array}{c} a, b, x \\ ab, cx, q; cx \end{array} \right).
$$

Using (31) of Theorem 8, we have

$$
\Phi_2 \left( \begin{array}{c} a, b, x \\ ab, cx, q; cx \end{array} \right) = \Phi_2 \left( \begin{array}{c} a, x, b \\ ab, cx, q; cx \end{array} \right)
$$

(27)

Substituting (27) into the right-hand sides of (26), we have

$$
E \left[ \frac{(cx; q)_\infty}{(aX, bX, cX; q)_\infty} \right] = \frac{(cx, abx; q)_\infty}{(ax, bx, a, b, c; q)_\infty} \Phi_2 \left( \begin{array}{c} a, b, x \\ ab, cx, q; cx \end{array} \right)
$$

(28)

The proof is complete.

**Corollary 7** (see [14, page 245, Lemma 2.4]). Let $X$ denote a random variable with probability distribution $W(x;q)$, $-1 < x < 0$. Then one has

$$
E \left[ \frac{1}{(aX, bX, cX; q)_\infty} \right] = \frac{(abx; q)_\infty}{(ax, bx, a, b, c; q)_\infty} \Phi_2 \left( \begin{array}{c} a, b, 1 \\ ab, d; q, cx \end{array} \right)
$$

(29)

provided that $\max(|a|, |b|) < 1$.

**Proof.** Letting $d = c$ or $d = a = c$ in (14) of Theorem 3, then we have

$$
E \left[ \frac{1}{(aX, bX, cX; q)_\infty} \right] = \frac{(abx; q)_\infty}{(ax, bx, a, b, c; q)_\infty} \Phi_2 \left( \begin{array}{c} a, b, 1 \\ ab, d; q, cx \end{array} \right)
$$

(30)

The proof is complete.

### 3. Probabilistic Proofs of Transformation Formulas of $\Phi_2$

Sears’ $\Phi_2$ transformation formula is widely applied to the special functions. In this section we will introduce probabilistic proofs of transformation of $\Phi_2$.

**Theorem 8** (see [17, page 359. III. 9, III. 10]). One has

$$
\Phi_2 \left( \begin{array}{c} a, b, c \\ d, e ; q, \frac{de}{abc} \end{array} \right) = \Phi_2 \left( \begin{array}{c} a, b, 1 \\ d, e; q, \frac{e'}{a} \end{array} \right)
$$

(31)

$$
= \Phi_2 \left( \begin{array}{c} a, b, c \\ d, e; q, \frac{de}{ab} \end{array} \right)
$$

(32)

**Proof.** Interchanging $b$ and $c$ in (14), then we have

$$
E \left[ \frac{(dX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] = \frac{(d, acx; q)_\infty}{(ax, a, c, x; q)_\infty} \Phi_2 \left( \begin{array}{c} a, c, 1 \\ d, e; q, bx \end{array} \right)
$$

(33)

Interchanging $a$ and $c$ in (14), then we have

$$
E \left[ \frac{(dX; q)_\infty}{(aX, bX, cX; q)_\infty} \right] = \frac{(d, acx; q)_\infty}{(ax, a, c, x; q)_\infty} \Phi_2 \left( \begin{array}{c} a, c, 1 \\ d, e; q, bx \end{array} \right)
$$

(34)
By (14) and (33), we obtain
\[ 3\phi_2 \left( a, b, \frac{d}{c} q, c, x \right)_{\infty} = \frac{(bx, acx; q)_{\infty}}{(ax, bx; q)_{\infty}} 3\phi_2 \left( a, c, \frac{d}{bc} ; q, bx \right)_{\infty}, \] (35)
and, replacing \((a, b, d/c, d, abx)\) by \((a, b, c, d, e)\) in (35), we obtain a 3\(\phi_2\) transformation formula
\[ 3\phi_2 \left( a, b, c, \frac{de}{abc} q, de/abc \right)_{\infty} = \frac{e/abc}{e, abc, z; q, ab} \frac{(ax, bxc; q)_{\infty}}{(ax, bx, ab; q)_{\infty}} 3\phi_2 \left( a, b, c, \frac{de}{abc} q, bx \right)_{\infty}, \] (36)
By (14) and (34) and then replacing \((a, b, d/c, d, abx)\) by \((a, b, c, d, e)\), we obtain (31).

### 4. Probabilistic Proof of Heine and Jackson’s Transformations

Heine [22] derived transformation formulas for \(2\phi_1\) and also proved Euler's transformation formula. A basic hypergeometric representation for a given function is by no means unique. There are groups of transformation between various hypergeometric representations of the same function. We will first prove the classical Heine's transformation formula which will be useful in proving many other formulas. In this section we give the probabilistic proofs of Heine and Jackson's transformations.

**Theorem 9** (see [17, page 359, III. 1, III. 2, III. 3]). **Heine's transformation formulas for \(2\phi_1\) are**
\[ 2\phi_1 \left( a, x, \frac{b}{c} ; q, ax \right) = \frac{(b, az; q)_{\infty}}{(ax, bx; q)_{\infty}} 2\phi_1 \left( a, az, x; q, b \right). \] (37)
and, replacing \((c, b, d/a, d, bxc)\) by \((a, b, c, d, e)\) in (37), we obtain (32). The proof is complete.

### Proof.
Comparing (24) of Theorem 5 and (25) of Corollary 6, we obtain
\[ \frac{(acx, bxc, x; q)_{\infty}}{(ax, bx, c, cx; q)_{\infty}} 3\phi_2 \left( a, b, \frac{c}{ax} q, c, x \right)_{\infty} = \frac{(c, bx; q)_{\infty}}{(ax, bx, ab; q)_{\infty}} 3\phi_2 \left( a, b, x; q, cx \right)_{\infty}, \] (41)
or, equivalently, that
\[ 3\phi_2 \left( \frac{ab, c, cx}{acx, bxc}; q, x \right)_{\infty} \] (42)
Setting \(b = 0\) in (42), we have
\[ 2\phi_1 \left( c, cx; q, x \right) = \frac{(cx, cx; q)_{\infty}}{(acx, cx; q)_{\infty}} 2\phi_1 \left( a, x; q, cx \right) \] (43)
Replacing \((c, cx, x)\) by \((a, b, c, z)\) in (43), we get
\[ 2\phi_1 \left( a, b; q, \frac{c}{az} \right) = \frac{(az, bx, ax; q)_{\infty}}{(ax, bx, ab; q)_{\infty}} 2\phi_1 \left( a, b; q, az \right) \] (44)
for \(|b| < 1, |z| < 1\).

which is just (38).

Setting \(d = 0\) and \(a = b\) and replacing \(c\) by \(b\) in (14), we have
\[ \mathbb{E} \left[ \frac{1}{(aX, aX, bX; q)_{\infty}} \right] = \frac{(a^2 x; q)_{\infty}}{(ax, ax, a, b; q)_{\infty}} 2\phi_1 \left( a, a^2 x; q, bx \right) \] (45)
Setting \(d = 0\) and \(a = c\) in (14) of Theorem 3, we have
\[ \mathbb{E} \left[ \frac{1}{(aX, aX, bX; q)_{\infty}} \right] = \frac{(ax, bx, ab; q)_{\infty}}{(ax, bx, ab; q)_{\infty}} 2\phi_1 \left( a, ab; q, ax \right) \] (46)
Comparing (45) and (46), we obtain
\[ 2\phi_1 \left( b, a; ax, \frac{xb}{ax} q \right) = \frac{(bx, a^2 x; q)_{\infty}}{(abx, ax, cx; q)_{\infty}} 2\phi_1 \left( a, a; q, bx \right) \] (47)
Replacing \((a, b, x)\) by \((b, a, z/b)\), we get
\[ 2\phi_1 \left( a, b; az, q, z \right) = \frac{(az, bz; q)_{\infty}}{(az, z; q)_{\infty}} 2\phi_1 \left( b, b; q, ab \right) \] (48)
Letting \( c = az \) in (48) gives
\[
\phi_1\left(a, b, c \mid q, z\right) = \frac{(c/b, bz; q)^a}{(c, z; q)^a} \phi_1\left(\frac{a}{b} \frac{c}{z} \mid q, \frac{z}{b}\right).
\]

(49)

We get (39). From (39) we can deduce (40).

Jackson's transformations formula is an important formula in basic hypergeometric series, and now we give a probabilistic proof of Jackson's transformation formulas for \( \phi_1 \) and \( \phi_2 \).

**Theorem 10** (see [17, page 359, III.4]). Jackson's transformation of \( \phi_1, \phi_2 \) series are
\[
\phi_1\left(a, b, c \mid q, z\right) = \frac{(az; q)^a}{(z; q)^a} \phi_2\left(\frac{a}{z}, \frac{c}{az} \mid q, \frac{z}{c}\right).
\]

(50)

**Proof.** This includes employing two different forms of
\[
E[(dX; q)\_\infty]/(aX, bx; q)\_\infty].
\]

Letting \( b = 0 \) in (14) of Theorem 3 and then replacing \( c \) by \( b \), we get
\[
E \left[ \frac{(dX; q)\_\infty}{(a, ax, b, bx, q)\_\infty} \phi_1\left(\frac{a}{b}, d \mid q, bx\right) \right] = \frac{(d, q)\_\infty}{(a, ax, bx, q)\_\infty} \phi_2\left(\frac{a}{b}, \frac{d}{bx} \mid q, bx\right).
\]

(51)

Comparing (51) and (19) of Theorem 4 gives
\[
\frac{(d, q)\_\infty}{(a, ax, bx, q)\_\infty} \phi_1\left(\frac{a}{b}, d \mid q, bx\right) = \frac{(d, abx, q)\_\infty}{(a, ax, bx, q)\_\infty} \phi_2\left(\frac{a}{b}, \frac{abx}{bx} \mid q, dx\right).
\]

Then we obtain
\[
\phi_1\left(\frac{a}{b}, d \mid q, bx\right) = \frac{(abx, q)\_\infty}{(bx, q)\_\infty} \phi_2\left(\frac{a}{b}, \frac{abx}{bx} \mid q, dx\right).
\]

(53)

Replacing \( (a, d/b, c, ax, bx) \) by \( (a, b, c, z) \) gives
\[
\phi_1\left(\frac{a}{b}, c \mid q, z\right) = \frac{(az, q)\_\infty}{(z, q)\_\infty} \phi_2\left(\frac{a}{b}, \frac{c}{az} \mid q, bz\right).
\]

(54)

This completes the proof.

\[
5. \textbf{Probabilistic Proofs of Some Formulas of } q\text{-Series}
\]

The \( q \)-binomial theorem is an important mathematical result which has been widely applied in the special functions, physics, quantum algebra, and quantum statistics. The \( q \)-binomial theorem was derived by Cauchy [23], Heine [22], and Jacobi [24] concerning the nonterminating form. There are many proofs of the \( q \)-binomial theorem to show the corresponding references; for example, a better and simpler proof, by using the method of the finite difference, was obtained by Askey (see [25]); a nice proof of the \( q \)-binomial theorem based on combinatorial considerations was given by Joichi and Stanton (see [26]). In 1847, Heine [22] derived a \( q \)-analog of Gauss's summation formula which is important in \( q \)-series. Joichi and Stanton [26] gave a bijective proof of the \( q \)-Gauss summation formula based on combinatorial considerations. Rahman and Suslov [27] used the method of the first order linear difference equations to prove the \( q \)-Gauss summation formula. By analytic continuation, the terminating case, when \( a = q^{-n} \), reduces to \( q \)-analogs of Vandermonde's formula. Bailey and Daum independently discovered the \( q \)-Kummer summation formula.

In this section we will introduce probabilistic proof of some formulas of \( q \)-series, for example, \( q \)-binomial theorem, \( q \)-Chu-Vandermonde, \( q \)-Gauss summation formula and \( q \)-Kummer summation formula, and so forth.

**Theorem 11** (see [16, page 488, Theorem 10.2.1] [17, page 354, II. 3]). The \( q \)-binomial theorem is
\[
_1\phi_0\left(a \mid q, z\right) = \sum_{n=0}^{\infty} \frac{(a; q)\_n}{(q; q)\_n} z^n = \frac{(az, q)\_\infty}{(z, q)\_\infty}
\]

for \(|z| < 1\), \(|q| < 1\).

**Proof.** Below we give two proofs of (55).

Setting \( d = a \) and replacing \( b \) and \( c \) by \( a \) and \( b \) in (14), we obtain
\[
E \left[ \frac{1}{(aX, bx, q)\_\infty} \phi_0\left(\frac{a}{b} \mid q, bx\right) \right] = \frac{1}{(ax, a, b, q)\_\infty} \phi_0\left(\frac{a}{b} \mid q, bx\right).
\]

(56)

Comparing (56) and (29) of Corollary 7, we have
\[
\frac{1}{(ax, a, b, q)\_\infty} \phi_0\left(\frac{a}{b} \mid q, bx\right) = \frac{(abx; q)\_\infty}{(ax, bx, q)\_\infty}.
\]

(57)

Then we obtain
\[
\phi_0\left(\frac{a}{b} \mid q, bx\right) = \frac{(abx; q)\_\infty}{(bx, q)\_\infty}.
\]

(58)

Replacing \( bx \) by \( z \), we can get
\[
\phi_0\left(\frac{a}{b} \mid q, z\right) = \frac{(az, q)\_\infty}{(z, q)\_\infty};
\]

(59)

that is,
\[
_1\phi_0\left(a \mid q, z\right) = \sum_{n=0}^{\infty} \frac{(a; q)\_n}{(q; q)\_n} z^n = \frac{(az; q)\_\infty}{(z; q)\_\infty}
\]

for \(|z| < 1\).

(60)
Setting \( d = a = 0 \) and \( b = c \) and replacing \( b \) by \( a \) in (14), we obtain
\[
\mathbb{E} \left[ \frac{1}{(ax; q)_{\infty}^2} \right] = \frac{1}{(ax, a; q)_{\infty}} \phi_0 \left( \frac{a}{ax; q} \right).
\] 
(61)

Letting \( a = b = d \) or \( a = b \) and \( d = c \) in (14) of Theorem 3, we obtain
\[
\mathbb{E} \left[ \frac{1}{(ax, ax; q)_{\infty}^2} \right] = \frac{1}{(ax, ax, a, a; q)_{\infty}} \phi_0 \left( \frac{a^2x}{ax, ax; q} \right) = \frac{a^2x; q}{(ax, ax, a, a; q)_{\infty}}
\] 
(62)

Comparing (61) and (62) gives
\[
\frac{1}{(ax, a, a; q)_{\infty}} \phi_0 \left( \frac{a}{ax; q} \right) = \frac{a^2x; q}{(ax, a; q)_{\infty}}.
\] 
(63)

Then we obtain
\[
\phi_0 \left( \frac{a}{ax; q} \right) = \frac{(ax, a; q)_{\infty}}{(a; q)_{\infty}}.
\] 
(64)

Replacing \( ax \) by \( z \) gives
\[
\phi_0 \left( \frac{a}{z; q} \right) = \frac{(az, q)_{\infty}}{(z; q)_{\infty}},
\] 
(65)

that is,
\[
\phi_0 \left( \frac{a}{z; q} \right) = \sum_{n=0}^{\infty} \frac{(az; q)_{n} q^{n}}{(z; q)_{n}} = \frac{(az; q)_{\infty}}{(z; q)_{\infty}} \quad \text{for} \ |z| < 1.
\] 
(66)

This proof is complete. \( \square \)

**Theorem 12** (see [17, page 354, II. 7]). The \( q \)-Chu-Vandermonde sums are
\[
\binom{a, q^{-n}}{c, q^{-n}} = \frac{(c/a; q)_{n}}{(c; q)_{n}}.
\] 
(67)

**Proof.** The below are two proofs of the \( q \)-Chu-Vandermonde.

(i) First proof: setting \( d = a = b \) and replacing \( c \) by \( b \) in (14), we have
\[
\mathbb{E} \left[ \frac{1}{(ax, bx; q)_{\infty}} \right] = \frac{1}{(ax, ax, a, b; q)_{\infty}} \phi_1 \left( \frac{a, b, q, bx}{a^2x} \right).
\] 
(68)

Replacing \( (a, b, x) \) by \( (a, aq^n, c/a^2) \) in (68), then we have
\[
\mathbb{E} \left[ \frac{1}{(ax, bx; q)_{\infty}} \right] = \frac{1}{(ax, ax, a, b; q)_{\infty}} \phi_1 \left( \frac{a, b, q, bx}{a^2x} \right).
\]
(69)

\[\begin{align*}
\phi_1 \left( \frac{a, b, q, bx}{a^2x} \right) &= \frac{(c/a^2; q)_{\infty}^{n} q^{k} (c/a^2)^{n-1} q^{k+1}, (c/a^2)^{n} q^{k+1}; q)_{\infty}^{k}}{(q, a^{2}q/c, c/a^2; q)_{\infty}^{k}} \\
&= \frac{(-c/a^2)^{n} q^{k} (c/a^2)^{n-1} q^{k+1}, (c/a^2)^{n} q^{k+1}; q)_{\infty}^{k}}{(q, a^{2}q/c, c/a^2; q)_{\infty}^{k}}.
\end{align*}\]
(70)

Hence,
\[
\mathbb{E} \left[ \frac{1}{(ax, bx; q)_{\infty}} \right] = \frac{1}{(ax, ax, a, b; q)_{\infty}} \phi_1 \left( \frac{a, b, q, bx}{a^2x} \right).
\]
(71)

By using the probability distribution \( W(c/a^2; q) \) and employing Andrews-Askey \( q \)-integral (11), now we calculate the expectation of the random variables \( 1/(aY, aq^nY; q)_{\infty} \) as follows:
\[
\mathbb{E} \left[ \frac{1}{(ax, bx; q)_{\infty}} \right] = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-c/a^2)^{n} q^{k} (c/a^2)^{n-1} q^{k+1}, (c/a^2)^{n} q^{k+1}; q)_{\infty}^{k}}{(q, a^{2}q/c, c/a^2; q)_{\infty}^{k}} \frac{1}{(1-q) (q, a^{2}q/c, c/a^2; q)_{\infty}}
\]
(72)
Comparing (71) and (72) gives
\[
\frac{(c;q)\infty}{(c/a,c/a,a,aq^n;q)\infty} 2\phi_1 \left( \frac{a,q^{-n}}{c}, \frac{cq^n}{a}; q\right) = \frac{(cq^n;q)\infty}{(c/a,cq^n/a,a,aq^n;q)\infty}
\]
(73)

Then we obtain
\[
2\phi_1 \left( \frac{a,q^{-n}}{c}, \frac{cq^n}{a}; q\right) = \frac{(cq^n;q)\infty}{(c/a,cq^n/a,a,aq^n;q)\infty} \frac{(c/a,c/a,a,aq^n;q)\infty}{(c;q)\infty} \frac{(c;aq^n/a;q)\infty}{(c;q)\infty}
\]
(74)
\[
= \frac{(c/a;q)\infty}{(c;q)\infty} \frac{(c/a,c/a,a,aq^n;q)\infty}{(c;q)\infty} \frac{(c;aq^n/a;q)\infty}{(c;q)\infty}
\]
which is just \(q\)-Vandermonde sums (67).

(ii) Second proof: replacing \((a, b, x)\) by \((a, aq^n, c/a^2)\) in (29), we have
\[
E \left[ \frac{1}{(aY, aq^nY;q)\infty} \right] = \frac{(cq^n;q)\infty}{(c/a,cq^n/a,a,aq^n;q)\infty} \frac{(c/a,c/a,a,aq^n;q)\infty}{(c;q)\infty} \frac{(c;aq^n/a;q)\infty}{(c;q)\infty}
\]
(75)

Comparing (71) and (75), we obtain
\[
\frac{(c;aq^n/a;q)\infty}{(c/a,cq^n/a,a,aq^n;q)\infty} \frac{(c/a,c/a,a,aq^n;q)\infty}{(c;q)\infty} \frac{(c;aq^n/a;q)\infty}{(c;q)\infty}
\]
(76)
\[
= \frac{(c/a;q)\infty}{(c;q)\infty} \frac{(c/a,c/a,a,aq^n;q)\infty}{(c;q)\infty} \frac{(c;aq^n/a;q)\infty}{(c;q)\infty}
\]
Hence,
\[
2\phi_1 \left( \frac{a,q^{-n}}{c}, \frac{cq^n}{a}; q\right) = \frac{(cq^n;q)\infty}{(c/a,cq^n/a,a,aq^n;q)\infty} \frac{(c/a,c/a,a,aq^n;q)\infty}{(c;q)\infty} \frac{(c;aq^n/a;q)\infty}{(c;q)\infty}
\]
(77)
\[
= \frac{(c/a,q)\infty}{(c;q)\infty}
\]
which is just \(q\)-Vandermonde sums (67).

\[\square\]

**Theorem 13** (see [16, page 522, Corollary 10.9.2] or [17, page 354, II. 8]). The \(q\)-Gauss sum is
\[
2\phi_1 \left( \frac{a,b}{c}, \frac{c}{ab}; q, \frac{b}{a}; q\right) = (c/a,c/b;q)\infty \frac{(c,a,b;q)\infty}{(c,c/ab;q)\infty}
\]
(78)

**Proof.** Letting \(d = a = b\) and replacing \(c\) by \(b\) in (14), we obtain
\[
E \left[ \frac{1}{(aX,bX;q)\infty} \right] = \frac{(aq/b)^n q^k}{(ax,ax,a,b;q)\infty} 2\phi_1 \left( \frac{a,b}{a^2x}, \frac{c}{ab}; q, bx\right).
\]
(79)

Comparing (29) and (79) gives
\[
\frac{(a^2x;q)\infty}{(ax,ax,a,b;q)\infty} 2\phi_1 \left( \frac{a,b}{a^2x}; q, bx\right) = \frac{(abx;q)\infty}{(ax,bx,a,b;q)\infty}
\]
(80)

hence we get
\[
2\phi_1 \left( \frac{a,b}{a^2x}; q, bx\right) = \frac{(abx,ax;q)\infty}{(a^2x,bx;q)\infty}
\]
(81)

Replacing \((a, a/b, a^2x)\) by \((a, b, c)\) in the above formula, we obtain
\[
2\phi_1 \left( \frac{a,b,c}{c}; q, \frac{c}{ab}; q\right) = \frac{(c,a,c/b;q)\infty}{(c,c/ab;q)\infty}
\]
(82)
which is just the \(q\)-Gauss sum (78).

\[\square\]

**Theorem 14** (see [17, page 354, II. 9]). The \(q\)-Kummer sum formula is
\[
2\phi_1 \left( \frac{a,b}{c}, \frac{c}{ab}; q, -\frac{q}{b}\right) = \frac{(-q;q)\infty}{(a^2x,bx;q)\infty} \frac{aq^2/b^2;q^2)\infty}{(-q/b,aq/b;q)\infty}
\]
(83)

**Proof.** Letting \(b = 0\) in (14) and then replacing \(c\) by \(b\), we have
\[
E \left[ \frac{(dX;q)\infty}{(aX,bX;q)\infty} \right] = \frac{(d;q)\infty}{(a,ax,b;q)\infty} 2\phi_1 \left( \frac{a,d}{d}; q, bx\right).
\]
(84)

Replacing \((a, b, d, x)\) by \((a, aq/b^2, aq/b, -b/a)\) in (84), we write
\[
P \left( Z = \left( -\frac{b}{a} \right)^n q^k \right) = p_{nk} \left( -\frac{b}{a}; q\right)
\]
(85)
\[
= p_{nk} \left( -\frac{b}{a}; q\right) = (b/a)^n q^k \frac{(a/b)^{n-1} q^k, (-b/a)^n q^k; q)\infty}{(q,-aq/b,-b/a;q)\infty}
\]
\[
= p_{nk} \left( -\frac{b}{a}; q\right) > 0, \quad \sum p_{nk} \left( -\frac{b}{a}; q\right) = 1,
\]
(86)

where
\[
-\frac{b}{a} < 0, \ 0 < q < 1, \ n = 0, 1, k = 0, 1, 2, \ldots.
\]

Hence, we obtain
\[
E \left[ \frac{(aq/b)^n Z;q)\infty}{(aZ,(aq/b^2)^n Z;q)\infty} \right]
\]
(87)
By using the probability distribution \( W(-b/a; q) \) and Lemma 2, we calculate the expectation of the random variables \( ((a/b)Z; q)_{\infty}/(aZ, (a/b)Z; q)_{\infty} \) as follows:

\[
E \left[ \frac{((a/b)Z; q)_{\infty}}{(aZ, (a/b)Z; q)_{\infty}} \right] = E \left[ \frac{((a/q)Z; q)_{\infty}}{(aZ, (a/q)Z; q)_{\infty}} \right] = \frac{1}{(1-q)(q, -aq/b, -b/a; q)_{\infty}}
\]

\[
\times \left\{ \begin{array}{l}
(1-q) \sum_{k=0}^{\infty} \left[ (aq/b) \frac{q^k}{q^{k+1}} \right] (aq/bq; q)_{\infty} q_k \\
- \left( \frac{b}{a} \right) (1-q) \sum_{k=0}^{\infty} \left[ (aq/bq)^{k+1} \right] (aq/bq; q)_{\infty} q_k \\
\end{array} \right\}
\]

\[
= \frac{1}{(1-q) (q, -aq/b, -b/a; q)_{\infty}} \times \left\{ \begin{array}{l}
(1-q) \sum_{k=0}^{\infty} \left[ (aq/b) \frac{q^k}{q^{k+1}} \right] (aq/bq; q)_{\infty} q_k \\
- \left( \frac{b}{a} \right) (1-q) \sum_{k=0}^{\infty} \left[ (aq/bq)^{k+1} \right] (aq/bq; q)_{\infty} q_k \\
\end{array} \right\}
\]

\[
= \frac{1}{(1-q) (q, -aq/b, -b/a; q)_{\infty}} \times \frac{1}{(1-q) \sum_{k=0}^{\infty} \left[ (aq/bq) \frac{q^k}{q^{k+1}} \right] (aq/bq; q)_{\infty} q_k} \times \frac{1}{(1-q) \sum_{k=0}^{\infty} \left[ (aq/bq)^{k+1} \right] (aq/bq; q)_{\infty} q_k}
\]

\[
= \frac{1}{(1-q) (q, -aq/b, -b/a; q)_{\infty}} \times \frac{1}{(1-q) \sum_{k=0}^{\infty} \left[ (aq/bq) \frac{q^k}{q^{k+1}} \right] (aq/bq; q)_{\infty} q_k} \times \frac{1}{(1-q) \sum_{k=0}^{\infty} \left[ (aq/bq)^{k+1} \right] (aq/bq; q)_{\infty} q_k}
\]

Comparing (87) and (88), we have

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{b}{a} \right)^n q^k \left( \frac{-b}{a} \right)^{n-1} q^{k+1} (a/bq; q)_{\infty} (aq/bq; q)_{\infty} \times \left( \frac{aq/bq}{b} ; q, -b \right)
\]

\[
= \frac{(-b/q)_{\infty}}{(-q/b; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{b}{a} \right)^n q^k \left( \frac{-b}{a} \right)^{n-1} q^{k+1} (a/bq; q)_{\infty} (aq/bq; q)_{\infty} \times \left( \frac{aq/bq}{b} ; q, -b \right).
\]

Using Heine's transformation and q-binomial theorem, we have

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{b}{a} \right)^n q^k \left( \frac{-b}{a} \right)^{n-1} q^{k+1} (a/bq; q)_{\infty} (aq/bq; q)_{\infty} \times \left( \frac{aq/bq}{b} ; q, -b \right)
\]

\[
= \frac{(-b/q)_{\infty}}{(-q/b; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{b}{a} \right)^n q^k \left( \frac{-b}{a} \right)^{n-1} q^{k+1} (a/bq; q)_{\infty} (aq/bq; q)_{\infty} \times \left( \frac{aq/bq}{b} ; q, -b \right).
\]

Hence, we obtain (83).
Hence, we have

\[
\mathbb{E} \left[ \frac{(-qR; q)_{\infty}}{(aR, (q/a)R; q)_{\infty}} \right] = \frac{(-q, b; q)_{\infty}}{(a, ab/q, q/a, b/a; q)_{\infty}} 2\phi_2 \left( \frac{a, q}{b, -q}; q, -b \right). \tag{95}
\]

By using the probability distribution \( W(b/q; q) \) and Lemma 2, we calculate the expectation of the random variables \((-qR; q)_{\infty}/(aR, (q/a)R; q)_{\infty}\) as follows:

\[
\begin{aligned}
&\mathbb{E} \left[ \frac{(-qR; q)_{\infty}}{(aR, (q/a)R; q)_{\infty}} \right] \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left( \frac{-b}{q} \right)^n q^k \left( \frac{b}{q} \right)^{n-1} q^{k+1}, \left( \frac{b}{q} \right)^n q^k, \left( \frac{q}{a} \right)^n q^k ; q \right)_{\infty} \\
&\times \left( \left( \frac{q, q^2}{b, -q}, a \left( \frac{b}{q} \right)^n q^k, \left( \frac{q}{a} \right)^n q^k ; q \right)_{\infty} \right)^{-1} \\
&= \frac{1}{(1-q) (q, q^2 / b, b/q; q)_{\infty}} \times \left( \frac{1-q}{a q^k, (q/a) q^k ; q} \right)_{\infty} \\
&\times \left( \frac{b - q}{q} \right) (1-q) \\
&\times \sum_{k=0}^{\infty} \left( \frac{a(b/q) q^k, (q/a)(b/q) q^k ; q} \right)_{\infty} \\
&= \frac{1}{(1-q) (q, q^2 / b, b/q; q)_{\infty}} \times \left( \frac{1-q}{a q^k, (q/a) q^k ; q} \right)_{\infty} \\
&\times \left( \frac{a - q}{q} ; q, b/v; q \right)_{\infty} 2\phi_1 \left( \frac{a - q}{b/v} ; q, b/v ; q \right). \tag{96}
\end{aligned}
\]

Comparing (95) and (96), we have

\[
2\phi_2 \left( \frac{a, q}{a, -q, b} ; q, -b \right) = \left( \frac{b/a; q}{(b, q)} \right)_{\infty} 2\phi_1 \left( \frac{a - q}{b} ; q, -b \right) \tag{97}
\]

Hence, we get (91).

**Theorem 16** (see [17, page 354, II. 11]). The Gauss sum formula is

\[
2\phi_2 \left( \frac{a^2, b^2}{abq^{1/2}, -abq^{1/2}} ; q, -q \right) = \left( \frac{a^2 q, b^2 q}{q^2} \right)_{\infty} \left( \frac{q, a^2 q}{q^2} \right)_{\infty}. \tag{98}
\]

**Proof.** By (14), we have

\[
\mathbb{E} \left[ \frac{(dX; q)_{\infty}}{(aX, bX; q)_{\infty}} \right] = \frac{(d, abx; q)_{\infty}}{(a, abx, d; q)_{\infty}} 2\phi_2 \left( \frac{a, b}{d, abx, d ; q, dx} \right). \tag{99}
\]

Replacing \((a, b, d, x)\) by \((a^2, b^2, -abq^{1/2}, q^{1/2}/ab)\) in (99) gives

\[
P \left( S = \left( \frac{q^{1/2}}{ab} \right)^n \right) = p_{n,k} \left( \frac{q^{1/2}}{ab} ; q \right) \tag{100}
\]

where

\[
\frac{1}{(a, abq^{1/2}, q^{1/2}/ab ; q)_{\infty}} \quad \frac{(-q^{1/2}/ab)^n q^k \left( \left( q^{1/2}/ab \right)^{n-1} q^{k+1}, \left( q^{1/2}/ab \right)^n q^{k+1} ; q \right)_{\infty}}{(q, abq^{1/2}, q^{1/2}/ab ; q)_{\infty}}, \tag{101}
\]

\[
p_{n,k} \left( \frac{q^{1/2}}{ab} ; q \right) > 0, \quad \sum p_{n,k} \left( \frac{q^{1/2}}{ab} ; q \right) = 1, \tag{101}
\]

\[
\frac{q^{1/2}}{ab} < 0, \quad 0 < q < 1, \quad n = 0, 1, k = 0, 1, 2, \ldots
\]
Hence,

\[
E \left[ (-aq^{1/2}; q)_\infty \right] \frac{(-abq^{1/2}, abq^{1/2}; q)_\infty}{(a^2, aq^{1/2}; b, b^2, bq^{1/2}; q)_\infty} z_\Phi_2
\times \left( \frac{a^2, b^2}{abq^{1/2}, -abq^{1/2}; q, -q} \right).
\]

By using the probability distribution \(W(q^{1/2} / ab; q)\) and employing Andrews-Askey \(q\)-integral (13) of Lemma 2, we calculate the expectation of the random variables \((-aq^{1/2}; q)_\infty / (a^2, b^2; q)_\infty\) as follows:

\[
E \left[ (-aq^{1/2}; q)_\infty \right] \frac{(-abq^{1/2}, abq^{1/2}; q)_\infty}{(a^2, aq^{1/2}; b, b^2, bq^{1/2}; q)_\infty} \times \left( (-aq^{1/2}, abq^{1/2}; q)_\infty \right) \times \left( \frac{a^2, b^2}{abq^{1/2}, -abq^{1/2}; q, -q} \right)
\]

\[
= (1 - q) (q, abq^{1/2}; q^{1/2} / ab; q)_\infty \times \left( \frac{a^2, b^2}{abq^{1/2}, -abq^{1/2}; q, -q} \right)
\]

Comparing (102) and (103), we have

\[
\phi_2 \left( \frac{a^2, b^2}{abq^{1/2}, -abq^{1/2}; q, -q} \right)
\]

Using Heine's transformation formula, we have

\[
\phi_1 \left( \frac{a^2, -aq^{1/2}}{b, -bq^{1/2}; q, -q} \right)
\]

Substituting (105) into (104) yields

\[
\phi_2 \left( \frac{bq^{1/2}; b, bq^{1/2}; q, -q} {abq^{1/2}; q, -abq^{1/2}; q, -q} \right)
\]

\[
\times \left( \frac{aq^{1/2}; -aq^{1/2}, b}{bq^{1/2}; -bq^{1/2}, q, -q} \right)
\]

\[
= (b^2, -q; q)_\infty \times \phi_1 \left( \frac{aq^{1/2}; -aq^{1/2}, b}{bq^{1/2}; -bq^{1/2}, q, -q} \right)
\]

Substituting (105) into (104) yields

\[
\phi_2 \left( \frac{a^2, b^2}{abq^{1/2}, -abq^{1/2}; q, -q} \right)
\]

\[
= \phi_1 \left( \frac{aq^{1/2}; -aq^{1/2}, b}{bq^{1/2}; -bq^{1/2}, q, -q} \right)
\]

\[
\times \left( \frac{aq^{1/2}; -aq^{1/2}, b}{bq^{1/2}; -bq^{1/2}, q, -q} \right)
\]

\[
= \phi_1 \left( \frac{aq^{1/2}; -aq^{1/2}, b}{bq^{1/2}; -bq^{1/2}, q, -q} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{aq^{1/2}; -aq^{1/2}, b}{bq^{1/2}; -bq^{1/2}, q, -q} \right) (b^2)^n
\]
\begin{align*}
&= \frac{\left(b^2, -q; q\right)_{\infty}}{(abq^{1/2}, -abq^{1/2}; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a^2q/b^2; q^2)_{n}}{(q^2; q^2)_{n}} (b^2)^n \\
&= \frac{(b^2, -q; q)_{\infty}}{(abq^{1/2}, -abq^{1/2}; q)_{\infty}} \frac{(a^2q; q^2)_{\infty}}{(b^2; q^2)_{\infty}} \\
&= (a; q)_{\infty} \frac{(a; q)_{\infty}}{(a, a^2; q)_{\infty}} \phi_2 \left( \frac{a^2q, b^2q^2; q^2}{a^2b^2q^2; q^2}; \frac{b^2; q^2}{(a, a^2; q)_{\infty}} \right). \\
&\text{Replacing } (a, a^2x) \text{ by } (a, c), \text{ we get} \\
&\phi_1 \left( \frac{a, c}{c, a} ; q, a \right) = \frac{(c/a; q)_{\infty}}{(c; q)_{\infty}}. \\
&\text{This proof is complete.} \qedhere
\end{align*}

\begin{theorem}[see [17, page 354, II. 1, II. 2]] The two \( q \)-exponential functions are
\begin{align*}
e_q(z) &= \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_{\infty}} \text{ for } |z| < 1, \\
E_q(z) &= \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} = (-z; q)_{\infty}. \\
\end{align*}
\end{theorem}

\begin{proof}
Setting \( a = b = d = 0 \) and then replacing \( c \) by \( a \) in (14), we obtain
\begin{align*}
E \left[ \frac{1}{(aX; q)_{\infty}} \right] &= \frac{1}{(ax, a; q)_{\infty}} \\
\phi_0 \left( 0 ; q, ax \right) &= \frac{1}{(ax, a; q)_{\infty}}.
\end{align*}
Comparing (117) and (118) gives
\begin{equation*}
\frac{1}{(a; q)_{\infty}} \phi_0 \left( 0 ; q, ax \right) = \frac{1}{(ax, a; q)_{\infty}}.
\end{equation*}
From the above formula, we obtain
\begin{equation*}
\phi_0 \left( 0 ; q, ax \right) = \frac{1}{(ax; q)_{\infty}}.
\end{equation*}
Replacing \( ax \) by \( z \) gives
\begin{equation*}
\phi_0 \left( 0 ; q, z \right) = \frac{1}{(z; q)_{\infty}};
\end{equation*}
that is,
\begin{equation*}
e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_{\infty}} \text{ for } |z| < 1.
\end{equation*}
Similarly, setting \( d = a \) and \( b = 0 \) in (19) of Theorem 4, we obtain
\begin{equation*}
E[0] = \frac{1}{(ax; q)_{\infty}} \phi_0 \left( 0 ; q, ax \right).
\end{equation*}
And obviously letting \( d = c \) and \( a = b = 0 \) in (14) of Theorem 3, we obtain
\begin{equation*}
E[1] = 1.
\end{equation*}
Comparing (123) and (124), we obtain
\[ \frac{1}{(ax;q)_\infty} \phi_0 \left( \frac{-}{-}; q, ax \right) = 1. \] (125)
Replacing $ax$ by $-z$, we get
\[ \phi_0 \left( \frac{-}{-}; q, -z \right) = (-z;q)_\infty, \] (126)
that is,
\[ E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} z^n}{(q;q)_n} = (-z;q)_\infty. \] (127)

Remark 19. In the present paper we obtain the part transformation and sum formulas of the $q$-series by applying the probabilistic method. We hope to find and construct another probability distribution in order to prove the transformation and sum formulas of the bilateral basic hypergeometric series, for example, Ramanujan and Bailey sum formulas and so forth.

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments
The authors express their sincere gratitude to the referee for many valuable comments and suggestions. The present investigation was supported by the Natural Science Foundation Project of Chongqing, China, under Grant CSTC2011JJA00024, Research Project of Science and Technology of Chongqing Education Commission, China, under Grant KJ120625, and Fund of Chongqing Normal University, China, under Grants 10XLR017 and 2011XZ07.

References
[1] G. Chang and C. Xu, “Generalization and probabilistic proof of a combinatorial identity,” The American Mathematical Monthly, vol. 118, no. 2, pp. 175–177, 2011.
[2] R. Chapman, “A probabilistic proof of the Andrews-Gordon identities,” Discrete Mathematics, vol. 290, no. 1, pp. 79–84, 2005.
[3] J. Fulman, “A probabilistic proof of the Rogers-Ramanujan identities,” Bulletin of the London Mathematical Society, vol. 33, no. 4, pp. 397–407, 2001.
[4] K. W. J. Kadell, “A probabilistic proof of Ramanujan’s $q$-series sum,” SIAM Journal on Mathematical Analysis, vol. 18, no. 6, pp. 1539–1548, 1987.
[5] P. A. Lee, “Probability distribution and a self-reciprocal relation involving confluent hypergeometric functions,” Journal of the Franklin Institute, vol. 334, no. 1, pp. 19–22, 1997.
[6] S. H. Ong, “Probabilistic proof of a Hankel transform of Laguerre polynomials,” Journal of the Franklin Institute, vol. 336, no. 6, pp. 1007–1011, 1999.
[7] M. N. Pascu, “A probabilistic proof of the fundamental theorem of algebra,” Proceedings of the American Mathematical Society, vol. 133, no. 6, pp. 1707–1711, 2005.
[8] A. Rosalsky, “A simple and probabilistic proof of the binomial theorem,” The American Statistician, vol. 61, no. 2, pp. 161–162, 2007.
[9] S. M. Ross, A First Course in Probability, Prentice Hall, Upper Saddle River, NJ, USA, 8th edition, 2010.
[10] H. M. Srivastava and C. Vignat, “Probabilistic proofs of some relationships between the Bernoulli and Euler polynomials,” European Journal of Pure and Applied Mathematics, vol. 5, no. 2, pp. 97–107, 2012.
[11] S. J. Miller, “A probabilistic proof of Wallis’ formula for $\pi$,” The American Mathematical Monthly, vol. 115, no. 8, pp. 740–745, 2008.
[12] P. Sun, “Moment representation of Bernoulli polynomial, Euler polynomial and Gegenbauer polynomials,” Statistics & Probability Letters, vol. 77, no. 7, pp. 748–751, 2007.
[13] M. Wang, “An expectation formula with applications,” Journal of Mathematical Analysis and Applications, vol. 379, no. 1, pp. 461–468, 2011.
[14] M. Wang, “A new probability distribution with applications,” Pacific Journal of Mathematics, vol. 247, no. 1, pp. 241–255, 2010.
[15] A. Xia, “A probabilistic proof of Stein’s factors,” Journal of Applied Probability, vol. 36, no. 1, pp. 287–290, 1999.
[16] G. E. Andrews, R. Askey, and R. Roy, Special Functions, Cambridge University Press, Cambridge, UK, 1999.
[17] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, UK, 2004.
[18] F. H. Jackson, “On $q$-definite integrals,” The Quarterly Journal of Pure and Applied Mathematics, vol. 50, pp. 101–112, 1901.
[19] G. E. Andrews and R. Askey, “Another $q$-extension of the beta function,” Proceedings of the American Mathematical Society, vol. 81, no. 1, pp. 97–100, 1981.
[20] D. F. Liu and Q. M. Luo, “An extension for Andrews-Askey integral formula,” In press.
[21] M. Rahman and A. Verma, “A $q$-integral representation of Rogers’ $q$-ultraspherical polynomials and some applications,” Constructive Approximation, vol. 2, no. 1, pp. 1–10, 1986.
[22] E. Heine, “Untersuchungen über die Reihe,” Journal für die reine und angewandte Mathematik, vol. 1847, no. 34, pp. 285–328, 1847.
[23] A.-L. Cauchy, “Mémoire sur les fonctions dont plusieurs valeurs sont liées entre elles par une équation linéaire, et sur diverses transformations de produits composés d’un nombre indéfini de facteurs,” Comptes Rendus de l’Académie des Sciences, vol. 17, p. 523, 1843, in Oeuvres de Cauchy, vol. 8, 1st série, pp. 42–50, Gauthier-Villars, Paris, France, 1893.