Reading out the final state of qubits is an essential part of any quantum computation scheme. Most commonly, the readout is considered to be projective, meaning that the state of the qubit is projected onto the eigenstates of the measurement operator. This, however, requires strong qubit-detector coupling and fast readout response, which could be practically difficult. Furthermore, strong interaction with the detector may limit the coherence time of the system during quantum operation. For a projective measurement, the dephasing time $\tau_\phi$ due to the readout backaction should be much smaller than the free evolution time of the qubit: $\tau_\phi \ll \Delta^{-1}$, where $\Delta$ is the qubit’s energy splitting. In general, however, it is also possible to measure the qubit’s state with a weaker coupling $(\tau_\phi \gg \Delta^{-1})$ as long as $\tau_\phi \leq \tau_{\text{meas}} \ll \tau_{\text{mix}}$. The measurement time $\tau_{\text{meas}}$ determines when the two states have distinguishable output, and the mixing time $\tau_{\text{mix}}$ shows when the information about the initial state is destroyed due to the interaction with the detector. In this regime (called “Hamiltonian dominated” in Ref. [1]), the qubit is measured in the eigenstates of its own Hamiltonian and the detector’s output is the expectation value of the measurement operator in these eigenstates. The backaction noise causes extra relaxation of the qubit to the ground state, limiting the readout of flux qubits using DC-SQUID in Ref. [3], seems to agree more with this picture than with the projective measurement.

In general, it is desirable to have a long $\tau_{\text{mix}}$ to allow a long readout time with high readout fidelity. In a Quantum Nondemolition (QND) measurement, while the readout process collapses the state of the qubit, it does not demolish it. In other words, the readout measures the eigenstates of the qubit Hamiltonian without changing them (e.g., causing fast relaxation). Mathematically, it means that the measurement operator commutes with the qubit Hamiltonian. Hence, $\tau_{\text{mix}}$ will be limited only by the intrinsic relaxation time of the qubit, which could be long. While increasing the readout fidelity, QND measurement allows reducing the qubit-readout coupling and therefore decreasing unwanted decoherence due to interaction with the detector during the operation time.

Despite the importance of QND measurement, there are only few QND readout schemes for superconducting qubits. In the Saclay’s charge-flux qubit design (quandronium), the measurement operator (flux) commutes with the qubit Hamiltonian only at the flux degeneracy point. At the time of readout, however, the qubit is moved away from this point, hence resulting in a readout that is not a QND type. This may be responsible for the observed small readout fidelity. Another type of charge-flux qubit, which is flux based, was proposed in Ref. [6]. Like in the quandronium case, the qubit can be optimally protected against dephasing due to low frequency noises at the charge-flux double degeneracy (so-called magic) point. In this paper, we suggest a modification to the qubit, to make QND readout possible.

The qubit (Fig. 1) consists of a superconducting loop containing three Josephson junctions threaded by an external flux close to half a flux quantum ($\Phi_x \approx \Phi_0/2 = h/4e$). The Josephson energy and junction capacitance of two of the junctions are the same ($E_{J1} = E_{J2} = E_J$, $C_1 = C_2 = C$) while those of the third junction are slightly smaller ($\alpha E_J$ and $\alpha C$, with $0.5 < \alpha < 1$). In addition, two voltage sources $V_{gA}$ and $V_{gB}$ are capacitively connected to two of the islands, which are also coupled capacitively to an SET or RF-SET as a sensitive charge detector for readout. At an appropriate biasing point, state dependent voltages appear on these islands, which induce a charge on the SET’s island, affecting the current through it.

We define $\phi = (\phi_1 + \phi_2)/2$ and $\theta = (\phi_1 - \phi_2)/2$, where $\phi_{1,2}$ are the phase differences across the two larger junctions. The Hamiltonian of the system is $H = H_Q + H_{\text{SET}} + H_{\text{int}}$, where $H_{\text{SET}}$ is the SET Hamiltonian, and

\[
H_Q = \frac{(p_\phi + n_\phi)^2}{2M_\phi} + \frac{(p_\theta + n_\theta)^2}{2M_\theta} + U(\phi, \theta),
\]

FIG. 1: Three-Josephson-junction charge-flux qubit coupled to an SET for QND readout.
is the qubit Hamiltonian. Here
\[ U(\phi, \theta) = E_J[-2 \cos \phi \cos \theta + \alpha \cos(2\pi f + 2\theta)] \]
is the potential energy, \( P_\phi = -i \partial/\partial \phi \) and \( P_\theta = -i \partial/\partial \theta \) are the momenta conjugate to \( \phi \) and \( \theta \), \( M_\phi = (C(1 + \gamma)/2e^2, M_\theta = C(1 + \gamma + 2\alpha)/2e^2, \gamma = C_\Sigma/C \), and \( n = n_{gA} \pm n_{gB} \), with \( n_{gA,B} = V_{gA,B}C_\Sigma/2e \) being the normalized gate charges. Throughout this paper we take \( f = \Phi_0/\Phi_0 - 1/2 = 0 \), leading to a \( U(\phi, \theta) \) with degenerate minima at \( \phi = 0, \theta = \pm \arccos(1/2\alpha) \).

The interaction Hamiltonian is written as
\[ H_{\text{int}} = \left( \frac{\kappa_+}{M_\phi} P_\phi + \frac{\kappa_-}{M_\theta} P_\theta \right) \hat{n}. \tag{1} \]
where \( \kappa_\pm = (C_{cA} \pm C_{cB})/C_\Sigma \), \( \hat{n} \) represents the normalized charge (operator) of the SET island, and \( C_\Sigma \) is its effective capacitance.

The commutation relation
\[ [H_{\text{int}}, H_Q] = -i(\sin \phi \cos \theta) \frac{\kappa_+}{M_\phi} \hat{n} - i(\cos \phi \sin \theta - 2\alpha \sin 2\theta) \frac{\kappa_-}{M_\theta} \hat{n} \]
is clearly nonzero, meaning that in general the readout can change the eigenstates of the qubit Hamiltonian. For a qubit well in the flux regime, however, the qubit wave function is confined within a unit cell in the phase space. The fluctuations of \( \phi \) is therefore small and, for the lowest energy eigenstates, \( \phi \approx 0 \). Thus, \( [H_{\text{int}}, H_Q] \propto \kappa_- \) vanishes if \( C_{gA} = C_{gB} \), suggesting a scheme for QND measurement. However, inter-unit-cell tunneling, which is important for operation in the charge-flux regime, will delocalize \( \phi \), conflicting with the above picture. Therefore, it is not possible to have \( H_{\text{int}} \) commute with \( H_Q \). As we shall see however, it is possible to make them commute in the Hilbert space reduced to the first two energy levels, hence satisfying a much weaker requirement: (1) \( H_{\text{int}} | 0 \rangle = 0 \), yet sufficient for QND measurement. The backaction noise in that case will not affect the rate of relaxation between the two lowest energy states (\( | 0 \rangle \) and \( | 1 \rangle \)) of the qubit during readout, i.e. will not reduce \( \tau_{\text{mix}} \).

To study in more detail, we calculate the relaxation rate due to the SET’s backaction on the qubit:
\[ \tau_{\text{mix}}^{-1} = \left| \frac{\kappa_+}{M_\phi} P_{\phi,10} + \frac{\kappa_-}{M_\theta} P_{\theta}^{10} \right|^2 |S(\Delta) + S(-\Delta)| \tag{2} \]
where \( P_{\phi,\theta}^{10} \equiv \langle i|P_{\phi,\theta}|1 \rangle \), and
\[ S(\omega) = \int e^{i\omega \tau} \langle \hat{n}(\tau)|\hat{n}(0)\rangle d\tau \tag{3} \]
is the noise spectral density. Using the orthodox theory for a normal SET\[11\] the spectral density \( S(\omega) \) at small bias current is given by\[12\]
\[ S(\omega) = \frac{2I/e}{\omega^2 + (I_1 + I_2)^2}, \tag{4} \]
where \( I = e\Gamma_1 \Gamma_2/(\Gamma_1 + \Gamma_2) \) is the average current through the SET and \( \Gamma_1 (\Gamma_2) \) is the rate of tunnelling of an electron into (out of) the island via the first (second) SET junction:
\[ \Gamma_{1,2} = \frac{1}{e^2 R} \frac{\Delta E_{1,2}/h}{1 - e^{-\Delta E_{1,2}/k_B T}}, \tag{5} \]
\[ \Delta E_{1,2} = \pm 2E_{C_\Sigma}(n_{gs} - 1/2) + (1/2)eV_{DS}. \tag{6} \]
Here \( R \) is the normal resistance of the junctions, \( E_{C_\Sigma} \) is the charging energy, \( n_{gs} = C_{gA}V_{gs}/e \) is the normalized gate charge, and \( V_{DS} \) is the voltage between drain and source of the SET.

Let us first consider the case that the readout is coupled only to one of the islands (as in Ref.\[7\]: \( C_{cB} = 0 \), therefore \( \kappa_+ = \kappa_- = C_{cA}/C_\Sigma \). An SET, operating in the quasiparticle branch, typically has \( I \approx 1 \) nA and \( V_{DS} \approx 400 \) μV, which lead to \( \Gamma_{1,2} \approx 10^{-10} \) s\(^{-1} \). For a typical qubit with \( E_C/E_J = 0.1 \) and \( \alpha = 0.75 \), as suggested in Ref.\[7\] we find: \( E_C \approx 5 \) GHz, \( \Delta \approx 7 \) GHz, \( M_{\phi}^{-1} \approx 2.5M_{\theta}^{-1} \approx 4E_C \approx 20 \) GHz, \( P_{\phi}^{10} \approx 0.4 \), and \( P_{\theta}^{10} \approx 0.5 \). Substituting into \( \Theta \) and \( \Phi \) and using \( C_{cA}/C_\Sigma \approx 0.2 \), we find a mixing time of \( \tau_{\text{mix}} \approx 20 \) ns, which is probably too short to read out the qubit. It is therefore important to use symmetric coupling (see below) or operate the SET at the double Josephson quasiparticle point\[11\] where the SET’s backaction noise (but also its sensitivity) is significantly smaller. Notice that \( P_{\phi}^{10} \approx P_{\theta}^{10} \) (see also Fig.\[4\]), hence the second term in \( \Theta \) has much larger contribution to the backaction noise than the first one.

In the symmetric coupling case \( (C_{cA} = C_{cB}) \), \( \kappa_- \) vanishes and therefore the second term in \( \Theta \) is cancelled. The relaxation rate will then only depend on \( P_{\phi}^{10} \), which is typically much smaller than \( P_{\theta}^{10} \). As we shall see, there even exist special points at which \( P_{\phi}^{10} = 0 \), hence \( \Phi \) vanishes, making QND readout possible. Let us write
\[ P_{\phi}^{10} = \frac{1}{2} \int d\phi \int d\theta \Psi_1(\phi, \theta)(-i\partial_\theta)\Psi_0(\phi, \theta), \tag{7} \]
where \( \Psi_{0,1} \) are the first two energy eigenfunctions. When \( n_{gA} = n_{gB} = 0 \), the wave functions have the following symmetries
\[ \Psi_{0,1}(\phi, -\theta) = \pm \Psi_{0,1}(\phi, \theta) \tag{8} \]
Either of these is enough to make \( \Theta \) vanish. The output however will be zero at this “magic” point. It is therefore necessary to move the qubit away from this point at the time of readout. As we shall see, unlike in the quaternion case, it is possible to find points at which \( \Theta \) vanishes while the measurement output is finite.

To see explicitly how the symmetries \( \Theta \) and \( \Phi \) change at finite \( n_{gA} \) and \( n_{gB} \), we first consider the simpler problem of a one dimensional particle in a symmetric...
periodic potential $V(x)$
\[
H = \frac{1}{2m}(P + eA/c)^2 + V(x),
\]
\[
V(x + d) = V(x) = V(-x),
\]
where $A$ is a uniform vector potential. The eigenfunctions of this system are described by the Bloch wave functions. The ground state of the system has zero crystal momentum and is therefore the solution of (10) with periodic boundary condition $\Psi_0(x + d) = \Psi_0(x)$. When $\alpha = 0$, the ground state is symmetric: $\Psi_0(-x) = \Psi_0(x)$. At finite $A$, one can perform a gauge transformation to remove $A$ from (10). The boundary condition then becomes $\Psi_0(x + d) = e^{i\chi}\Psi_0(x)$, where $\chi = \int (e/\hbar c)A dx = eAd/\hbar c$. When the wave function is confined within the wells, a standard tight binding type of calculation can be used to show that a small antisymmetric term, proportional to $\sin \chi$, will be added to the otherwise symmetric ground state. In general, the wave function of the system in the $i$-th state, has the form
\[
\Psi_i(x) = \Psi_0^i(x) + \Psi_1^i(x) \sin \chi,
\]
where $\Psi_1^i(x)$ is a small term with opposite symmetry as that of the main term $\Psi_0^i(x)$. $\Psi_1^i(x)$ is dominantly determined by the geometry of the well and therefore is independent of $\chi$. For example, when the two lowest levels of the system are well separated from the rest, $\Psi_0^i(x) \propto \Psi_1^0(x)$. Let us now return to our qubit system. The two dimensional potential energy of the system is $2\pi$ periodic in both $\phi_1$ and $\phi_2$ directions. In terms of $\phi$ and $\theta$, the boundary condition reads
\[
\Psi(\phi + \pi, \theta \pm \pi) = e^{i\pi(n_+ \pm n_-)}\Psi(\phi, \theta).
\]
A straightforward generalization of the above argument yields
\[
\Psi_i(\phi, \theta) = \Psi_0^i(\phi, \theta) + \Psi_1^i(\phi, \theta) \sin n_+ + \Psi_1^i(\phi, \theta) \sin n_- + \Psi_1^i(\phi, \theta) \sin n_+ \sin n_-.
\]
For the ground (first excited) state $|i = 0(1), \Psi_0^i$ and $\Psi_1^i$ are symmetric (antisymmetric) functions of $\phi$, while $\Psi_0^i$ and $\Psi_1^i$ are antisymmetric (symmetric). Likewise, $\Psi_0^i$ and $\Psi_1^i$ are symmetric (antisymmetric) in $\theta$, while $\Psi_0^i$ and $\Psi_1^i$ are antisymmetric (symmetric). The difference between $\phi$ and $\theta$ stems from the fact that the potential has double-well structure only in $\theta$ direction. Neglecting the small contribution from the decay region, equation (11) gives
\[
P_{0}^{10}=Z\sin \pi n_+ \sin \pi n_- = \frac{1}{2}Z(\cos 2\pi n_B - \cos 2\pi n_B),
\]
where $Z = A_{00}^{10} + A_{11}^{10} + A_{22}^{10} + A_{33}^{10}$, and
\[
A_{n\beta}^{ij} = \frac{1}{2} \int d\phi \int d\theta \Psi_i^\phi(\phi, \theta)(-i\partial_\phi)\Psi_j^\phi(\phi, \theta).
\]
Similarly, for $P_{\theta}$ we find
\[
P_{\theta}^{10} = B_{00}^{10} + B_{11}^{10} + B_{22}^{10} + B_{33}^{10} \sin^2 \pi n_+ + B_{22}^{10} \sin^2 \pi n_- + B_{33}^{10} \sin^2 \pi n_+ \sin^2 \pi n_-,
\]
where
\[
B_{ij}^{n\beta} = \frac{1}{2} \int d\phi \int d\theta \Psi_i^\phi(\phi, \theta)(-i\partial_\phi)\Psi_j^\phi(\phi, \theta).
\]
While $P_{0 \theta}^{10}$ is always finite, $P_{\theta \theta}^{10}$ vanishes when either $n_+$ or $n_-$ is an integer number. At these special points, $\langle 1|H_{\text{int}}|0 \rangle = 0$ if $\kappa = 0$ in (1). The latter can be easily achieved by symmetrically coupling the two islands of the qubit to the readout SET ($C_{\text{qA}} = C_{\text{qB}}$).

The above argument is valid beyond the approximations used to derive (11) and (12) (see Fig. 2). Indeed, symmetries (8) and (9) always hold when respectively $n_+$ and $n_-$ are integer numbers, causing (13) to vanish. A simple way to access these symmetry points is to adjust the gate voltages so that $n_B = \pm n_B$, which correspond to $n_2 = 0$, respectively.

Figure 2 shows the result of numerical calculation of $|P_{0 \theta}^{10}|$ for different values of $\alpha$. When $\alpha$ is small, the wave function of the qubit is localized within a unit cell. As a result, the $|P_{0 \theta}^{10}|$ curve agrees very well with (14). A larger $\alpha$ delocalizes the states, causing deviation from a simple sinusoidal behavior. When $n_B = 0.25$ (Fig. 2a), $|P_{0 \theta}^{10}|$ vanishes at $n_B = 0.25 (n_- = 0)$ and $n_B = 0.75 (n_+ = 1)$, in agreement with the above argument. For $n_B = 0.5$ (Fig. 2b) both points coincide at $n_B = 0.5 (n_- = 0$ and $n_+ = 1$).

While the points $n_B = n_B$ and $n_B = -n_B$ both result in $P_{0 \theta}^{10} = 0$, only the former is suitable for readout. To see this, notice that the detector measures the expectation value of the sum of the island voltages in the $|0, 1 \rangle$ states: $\langle \partial H/\partial n_+ \rangle = P_{\phi \phi}^{10}/M_{\phi \phi}$, $i = 0, 1$. However
\[
P_{0 \phi}^{00} = \sin \pi n_+ [A_{00}^{00} + A_{11}^{00} + (A_{22}^{00} + A_{33}^{00}) \sin^2 \pi n_-],
\]
\[
P_{0 \phi}^{11} = \sin \pi n_+ [A_{01}^{11} + A_{10}^{11} + (A_{21}^{11} + A_{32}^{11}) \sin^2 \pi n_-],
\]
both vanish when \( n_+ = 0 \) (\( n_{gA} = -n_{gB} \)), resulting in indistinguishable (zero) outputs. Physically, the voltages of the two islands will be exactly equal but with opposite signs, cancelling each other’s effects on the output. Thus, the qubit should be read out at \( n_+ = 0 \) (or any integer number). The difference in the output between the two states is then proportional to

\[
P^\alpha_\phi - P_\phi^{00} = \sin \pi n_+ [A_{11}^{11} + A_{11}^{00} - A_{10}^{00}] = \frac{\pi n_+}{\sqrt{2}} \sin \theta,
\]

which vanishes at integer \( n_+ \), as expected.

Figure 3 displays the result of numerical calculation of \( P^\alpha_\phi - P_\phi^{00} \) for different values of \( \alpha \), for the same set of parameters as in Fig. 2. The plot is vs \( n_+ \) while \( n_- = 0 \) (\( n_{gA} = n_{gB} \)). The sinusoidal-type behavior, is in agreement with (16). Maximum output is achieved when \( n_+ \approx 0.5 \) (\( n_{gA} = n_{gB} \approx 0.25 \)). This is indeed the optimal point for reading out the qubit.

None of the above arguments depended on the qubit’s regime of operation, although we mainly focussed on the flux regime. The only restriction is that the qubit island voltages should not be too small to be detectable by the SET; the qubit should not be too far in the flux regime. It is indeed interesting to see how the scheme works when the qubit is in charge regime \((E_C \gg E_J)\). Let us write the qubit wave function in charge basis \(|n_A, n_B\rangle\), where \( n_{A,B} \) is the number of Cooper pairs on the two islands. When \( n_{gA} = n_{gB} (n_- = 0) \), the states \(|n, n+1\rangle\) and \(|n+1, n\rangle\) will be degenerate. The presence of the Josephson term of the third junction will remove this degeneracy, making the rotated states \(|n, \pm\rangle = (|n, n+1\rangle \pm |n+1, n\rangle) / \sqrt{2} \) preferred for the qubit. In the Hamiltonian matrix, \(|n, +\rangle\) has nonzero off-diagonal matrix elements with other charge states, hence will mix after complete diagonalization. The matrix elements of \(|n, -\rangle\) with other states, however, are all zero as long as \( E_{J1} = E_{J2} \). It fact, \(|n, -\rangle\) is an eigenstate of the qubit Hamiltonian, as well as an eigenvector of \( H_{\text{int}} \) when \( k = 0 \). Thus, \( H_{\text{int}} \) will not mix \(|n, -\rangle\) with any other states. It is easy to show that for all values of \( n_+ \), one of the two lowest energy levels of the qubit will be a \(|n, -\rangle\) state (with an appropriate value of \( n \) depending on \( n_+ \)), hence \( \langle 1|H_{\text{int}}|0\rangle = 0 \).

To summarize, we have shown that QND measurement of a three Josephson junction qubit is possible if both island charges are measured symmetrically at a symmetric biasing point. The scheme is applicable for a wide range of parameters from charge to flux regime as long as charge measurement is possible. QND readout allows long readout time with high fidelity and small qubit-detector coupling to prevent decoherence due to the interaction with the measurement device during quantum operation. It also makes it possible to operate the SET in the quasiparticle branch benefitting from larger charge sensitivity. Experimental study of such a system can shine light on the intrinsic mechanisms of relaxation and decoherence sources in charge or charge-flux qubits.

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**FIG. 3:** \( P^{11}_\phi - P^{00}_\phi \) vs \( n_+ \) for the qubit system of Fig. 2 when \( n_- = 0 \). Different lines correspond to different \( \alpha \) the way described in the caption of Fig. 2.