$C^1$ actions on manifolds by lattices in Lie groups

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Abstract

In this paper we study Zimmer’s conjecture for $C^1$ actions of lattice subgroup of a higher-rank simple Lie group with finite center on compact manifolds. We show that when the rank of an uniform lattice is larger than the dimension of the manifold, then the action factors through a finite group. For lattices in $\text{SL}(n, \mathbb{R})$, the dimensional bound is sharp.

1. Introduction

Zimmer’s conjecture for actions of higher-rank lattice on compact manifolds says that if the group is large with respect to the dimension of the manifold, then any such action should factor through a finite group. This conjecture is motivated by a long history of research, including the local rigidity results of Selberg [Sel60] and Weil [Wei62] on linear representation theory, the global rigidity results of Mostow [Mos73], the superrigidity theorem of Margulis [Mar91], and the cocycle superrigidity theorem of Zimmer [Zim80]. Since its introduction, Zimmer’s conjecture has attracted considerable interests.

For $C^0$ actions on the circle, the above conjecture is confirmed by Lifschitz, and Witte Morris [LW08, Wit94] for many non-uniform lattices. For $C^1$ actions on the circle, Burger and Monod [BM02] and Ghys [Ghy99] showed similar results for many other cases, including all lattices in higher rank simple Lie groups. For $C^1$ area-preserving actions on closed orientable surface with genus at least 2, Zimmer’s conjecture is proved by Polterovich [Pol02] for non-uniform lattices. His result is then generalized by Franks and Handel in [FH06] to any $C^1$ action which preserves a Borel measure. For analytic actions, Ghys [Ghy99] studied the case where the manifold is a circle; Farb and Shalen [FS99] studied this conjecture under additional assumptions on the group and the manifold. For a very detailed survey on other earlier results on Zimmer’s program, we refer the reader to [Fis11].

In recent breakthroughs [BFH16, BFH20], Brown, Fisher and Hurtado proved the $C^2$ version$^1$ of Zimmer’s conjecture for all co-compact lattices$^2$ in real split simple Lie group and $\text{SL}(n, \mathbb{Z})$ using some previous progress made by Brown, Rodriguez Hertz and Wang in [BRW16b, BRW16a] and Lafforgue, de Laat and de la Salle in [Laf08, dLdlS15, dIS19]. We refer the reader to Fisher’s paper [Fis17] for an excellent survey of the history and recent progress on Zimmer’s conjecture.

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1 Their result can be improved with a bit more work to include $C^{1+\epsilon}$ actions.
2 These results are generalized recently in [BFH21] to all non-uniform lattices.
The purpose of the present paper is to extend the results in [BFH16, BFH20, BFH21] to $C^1$ actions, when the rank of the acting group is sufficiently large. Compared to the previous results, there are two new ideas here. First is that while many results in Non-uniform Hyperbolic Theory fail or remain unknown in the $C^1$ setting, some of them continue to hold under the presence of suitable continuous splitting. In our case, we can apply a variant of Avila and Viana’s invariance principle to an element in the kernel of all Lyapunov functionals to obtain the extra invariance needed to conclude the proof. For $C^2$ action, the idea to use action by an element in the kernel of all fiberwise exponents was originally due to Sebastian Hurtado and appears in the Bourbaki notes of Cantat [Can19]. The second one is that we use the information extracted by using strong property (T) to control the $L^p$ norms of the derivatives for sufficiently large $p$. This allows us to show that $C^1$ action is uniformly bounded under certain Hölder norm. Then we use the resolution of the Hilbert–Smith conjecture for sufficiently Hölder actions to conclude the proof.

2. Statement of the main results

We first recall the statement of Zimmer’s conjecture.

For a real semisimple Lie group $G$ with Lie algebra $g$, let:

- $v(G)$ denote the minimal codimension of proper parabolic subalgebras of $g$;
- $d(G)$ denote the minimal codimension of proper subalgebras of the compact real form of $gc$;
- $n(G)$ denote the minimal dimension of non-trivial real representations of $g$.

It is proved in [Stu91] that $v(G) < n(G)$.

**Conjecture 1.** Let $G$ be a connected real semisimple Lie group with finite center and without almost-simple factors of real rank less than 2. Let $\Gamma < G$ be a lattice, $M$ be a compact manifold, $\alpha : \Gamma \to \text{Diff}(M)$ be an action.

1. If $\dim(M) < v(G)$, then $\alpha$ preserves a Riemannian metric.
2. If $\dim(M) < \min\{v(G), d(G)\}$, then $\alpha(\Gamma)$ is finite.
3. If $\dim(M) < n(G)$ and $\alpha$ preserves a volume density, then $\alpha$ preserves a Riemannian metric.
4. If $\dim(M) < \min\{n(G), d(G)\}$ and $\alpha$ preserves a volume density, then $\alpha(\Gamma)$ is finite.

The main result of this paper is the following generalization of the results in [BFH16, BFH21] to $C^1$ regularity.

**Theorem 1.** Let $M$ be a compact manifold. Let $G$ be an almost simple real Lie group with finite center and with real rank at least 2, and let $\Gamma < G$ be a lattice. Let $\alpha : \Gamma \to \text{Diff}^1(M)$ be a group homomorphism. Assume that $\Gamma$ is an uniform lattice or $\Gamma = \text{SL}(n, \mathbb{Z})$, and assume either that $\dim(M) < \min(\text{rank}_\mathbb{R}(G), d(G))$, or that $\dim(M) \leq \min(\text{rank}_\mathbb{R}(G), d(G) - 1)$ and $\alpha(\Gamma) \subset \text{Diff}^1(M, \text{vol})$. Then $\alpha$ has finite image.

Compared to the main result in [BFH16, BFH21] for almost-simple real Lie groups, in Theorem 1 we have posed a different requirement on the dimension of the manifold. Indeed, we can deduce from [BFH16, Theorem 2.7] that for a group homomorphism $\alpha : \Gamma \to \text{Diff}^2(M)$, the conclusion of Theorem 1 is true if $\text{rank}_\mathbb{R}G$ is replaced by the minimal resonant codimension $r(G)$ (see [BFH16, Definition 2.1]). We remark that under the conditions of Theorem 1, we always have that

$$r(G) \geq \text{rank}_\mathbb{R}G.$$
\textbf{C1 lattice actions}

\textbf{Corollary A.} Let $M$ be a compact manifold. Let $\Gamma < G$ be a lattice. Let $\alpha : \Gamma \rightarrow \text{Diff}^1(M)$ (respectively $\text{Diff}^1(M, \text{vol})$) be a group homomorphism. Assume that $\Gamma$ is an uniform lattice or $\Gamma = \text{SL}(n, \mathbb{Z})$, and assume that one of the following is true:

(1) $G = \text{SL}(n, \mathbb{R})$, dim $M < n - 1$ (respectively $\leq n - 1$) and $n \geq 3$;
(2) $G = \text{Sp}(2n, \mathbb{R})$, dim $M < n$ (respectively $\leq n$) and $n \geq 2$;
(3) $G = \text{SO}(n, n)$, dim $M < n$ (respectively $\leq n$) and $n \geq 4$;
(4) $G = \text{SO}(n, n+1)$, dim $M < n - 1$ (respectively $\leq n - 1$) and $n \geq 3$.

Then $\alpha$ has finite image.

When $\alpha$ is a $C^2$ action, the conclusion of Theorem 1 is already obtained in [BFH16, BFH21]. Moreover, when $G = \text{Sp}(2n, \mathbb{R})$, $\text{SO}(n, n)$ or $\text{SO}(n, n+1)$, the dimension bound in Corollary A is not optimal. However, when $G = \text{SL}(n, \mathbb{R})$, we have

$$r(G) = \text{rank}_\mathbb{R} G = n - 1.$$ 

By considering the actions of $\text{SL}(n, \mathbb{R})$ by projective transformations on $\mathbb{P}(\mathbb{R}^n)$, and by the affine transformations on $\mathbb{T}^n$, we see that Corollary A has optimal bounds for $G = \text{SL}(n, \mathbb{R})$. We note the for $C^0$ action by $\text{SL}(n, \mathbb{Z})$, $(n \geq 3)$ on compact manifold with $\chi(M) \neq 0 \mod 3$, the finite image property of $\alpha$ is proved by Ye in [Ye18].

The proofs of the results in this paper follow closely the strategy in [BFH16]. We recommend the reader to have this paper close at hand as we make many references to these works, although we also repeat some of the main arguments for reader’s convenience. Below we first describe the general strategy of the proofs in [BFH16, BFH20, BFH21], and then we point out the main new ideas and modifications we make here in order to obtain results in $C^1$ regularity.

\textbf{3. Review of the work of Brown, Fisher and Hurtado and outline of the proof}

\textbf{Step 1: Uniform subexponential growth}

We fix a finite set of symmetric generators for $\Gamma$, denoted by $S = \{\gamma_i\}$. For any $\gamma \in \Gamma$, we let $\ell(\gamma)$ denote the word-length distance from $\gamma$ to the identity relative to $S$. In other words, $\ell(\gamma)$ is the smallest integer $k$ such that $\gamma$ may be represented by a product $\zeta_1 \cdots \zeta_k$ where $\zeta_j \in S$ for each $1 \leq j \leq k$.

We first recall the following notion.

\textbf{Definition 1.} Let $\alpha : \Gamma \rightarrow \text{Diff}^1(M)$ be an action of $\Gamma$ on a compact manifold $M$ by $C^1$ diffeomorphisms. We fix a background $C^\infty$ Riemannian metric on $M$. We say that $\alpha$ has \textit{uniform subexponential growth of derivatives} if for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that for all $\gamma \in \Gamma$ we have

$$\|D\alpha(\gamma)\| \leq C_\varepsilon e^{\varepsilon \ell(\gamma)}.$$ 

It is clear that the above definition is independent of the choice of the metric on $M$ or the generating set $S$.

The main result of Step 1 is the following.

\textbf{Proposition 1.} Let $M$ be a compact manifold, and let $G$ be a connected, almost-simple real Lie group with finite center and whose real rank is at least 2. Let $\Gamma < G$ be a lattice. Let $\alpha : \Gamma \rightarrow \text{Diff}^1(M)$ be a group homomorphism. Assume $\Gamma$ is an uniform lattice or $\Gamma = \text{SL}(n, \mathbb{Z})$, and assume either that $\dim M < \text{rank}_\mathbb{R}(G)$, or that $\dim M \leq \text{rank}_\mathbb{R}(G)$ and $\alpha(\Gamma) \subset \text{Diff}^1(M, \text{vol})$. Then $\alpha$ has uniform subexponential growth of derivatives.
We prove Proposition 1 following the same scheme in [BFH16]. As in [BFH16], we define the suspension space $M^\alpha$ as the quotient of $G \times M$ by $\Gamma$-action $(g, x) \mapsto (g\gamma, \alpha(\gamma^{-1})x)$. We recall that $M^\alpha$ is a fiber bundle over $G/\Gamma$ with fibers modeled on $M$. Moreover $M^\alpha$ is equipped with a left $G$-action, denoted by $\tilde{\alpha}$, by diffeomorphisms which preserves the foliation into fibers. We present the construction of $M^\alpha$ and its further properties in § 4.1.

As the $G$-action preserves the foliation into fibers of $M^\alpha$, we may consider the restriction of $D\tilde{\alpha}$ to the subbundle $E^F := \text{Ker}(D\pi)$ tangent to the fibers of $M^\alpha$. Let $A$ be the maximal split torus of $G$, and let $\mu$ be an $A$-ergodic $A$-invariant measure on $M^\alpha$. We can associate to $\mu$ and the derivative $A$-cocycle $D\tilde{\alpha}|_{E^F}$ a set of fiberwise Lyapunov functionals $\lambda_i^F : \text{Lie}(A) \to \mathbb{R}$, $1 \leq i \leq k$ by the higher-rank Oseledec’s theorem (see, e.g. [BRW16b, Part I, Theorem 2.4]). We refer the reader to [BFH16, Proposition 3.3] for the definition and properties of Lyapunov functionals. The maximal fiberwise Lyapunov exponent for $a \in A$ with respect to an $a$-invariant probability measure $\mu$ is defined as

$$\lambda_i^F(a, \mu) = \inf_{n \to \infty} \frac{1}{n} \int \log \| D\tilde{\alpha}(a^n) \|_{E^F(x)} \, d\mu(x).$$

We have the following.

**Proposition 2.** Suppose that $\Gamma$ is an uniform lattice or $\Gamma = \text{SL}(n, \mathbb{Z})$, and $\alpha$ fails to have uniform subexponential growth of derivatives. There exists an $s \in A$ and an $A$-invariant Borel probability measure $\mu$ on $M^\alpha$ with $\lambda_i^F(s, \mu) > 0$ such that $\pi_*\mu$ is the Haar measure on $G/\Gamma$.

When $\Gamma$ is an uniform lattice, the above proposition is just [BFH16, Proposition 3.7]. When $\Gamma = \text{SL}(n, \mathbb{Z})$, the above proposition is proved in [BFH20] even though it is not explicitly stated as a single proposition. Indeed, we can define the measure $\mu$ in Proposition 2 by [BFH20, Proposition 5.10] as a limit of a sequence $\mu_n$; and by [BFH20, Proposition 5.6] and the two paragraphs below it, we see that $\pi_*\mu$ is the Haar measure on $G/\Gamma$.

This is the only place where we have used the hypothesis that $\Gamma$ is an uniform lattice or $\Gamma = \text{SL}(n, \mathbb{Z})$. In a recent paper of Brown, Fisher and Hurtado [BFH21, Proposition 8.1], they have generalized Proposition 2 to any lattice in $G$. Admitting their results, all of the results in the present paper hold for arbitrary lattices.

To complete the proof of Proposition 1, it remains to show the following.

**Proposition 3.** Let $\mu$ be an $A$-invariant Borel probability measure on $M^\alpha$ such that $\pi_*\mu$ is the Haar measure on $G/\Gamma$. If either we have that $\text{rank}_\mathbb{R} G > \dim M$, or we have that $\text{rank}_\mathbb{R} G \geq \dim M$ and $\alpha(\Gamma) \subset \text{Diff}^1(M, \text{vol})$, then $\mu$ is $G$-invariant.

Let $a \in G$ be a $\mathbb{R}$-semisimple element. The unstable, respectively stable, subgroup for $a$ are respectively,

$$H^u := \left\{ g \mid \lim_{n \to +\infty} a^n ga^{-n} = e \right\},$$

$$H^s := \left\{ g \mid \lim_{n \to -\infty} a^n ga^{-n} = e \right\}.$$

**Proposition 4.** Let $a \in A$ be an $\mathbb{R}$-semisimple element. Suppose $\mu$ is an $a$-invariant $a$-ergodic probability measure on $M^\alpha$ such that:

1. $\pi_*\mu$ is the Haar measure on $G/\Gamma$; and
2. all fiberwise Lyapunov exponents of $D\tilde{\alpha}(a)$ are non-positive.

Then $\mu$ is $H^u$-invariant.
\[ \text{C}^1 \text{ lattice actions} \]

The proof of Proposition 4 will be given in § 4. We are ready to deduce Proposition 3 from Proposition 4.

**Proof of Proposition 3.** We can assume without loss of generality that \( \mu \) is \( A \)-ergodic, otherwise we may replace \( \mu \) by any one of its \( A \)-ergodic components. This is because any \( A \)-ergodic component of \( \mu \) projects to some \( A \)-ergodic component of \( \pi_* \mu \); while by hypothesis \( \pi_* \mu \) is the Haar measure on \( G/\Gamma \) which is itself \( A \)-ergodic by Moore’s ergodicity theorem (see for instance \cite{Moo66} or \cite[Theorem 2.2.6]{Zim84}). This allows us to define fiberwise Lyapunov functionals. We denote by \( \lambda^F_1, \ldots, \lambda^F_k \) the total collection of different fiberwise Lyapunov functionals. Then under the condition of the proposition, we can pick an arbitrary element \( a \in (\bigcap_{i=1}^k \exp(\ker(\lambda^F_i))) \setminus \{e\} \) such that

\[ \lambda^F_{+}(a, \mu) = \lambda^F_{+}(a^{-1}, \mu) = 0. \]

Then all \( a \)-ergodic components of \( \mu \) have vanishing fiberwise Lyapunov exponents. By Proposition 4, we deduce that \( \mu \) is \( H^\alpha \)-invariant. By symmetry, we also have that \( \mu \) is \( H^\alpha \)-invariant. As \( G \) is almost-simple, \( G \) is generated by \( H^u \) and \( H^s \). Consequently, \( \mu \) is \( G \)-invariant. \( \square \)

**Proof of Proposition 1.** Assume that \( \alpha \) fails to have uniform subexponential growth of derivatives. Then by Proposition 2, there is a \( s \in A \) and an \( A \)-invariant measure \( \mu \) such that \( \lambda^F_+(s, \mu) > 0 \) and \( \pi_* \mu \) is the Haar measure on \( G/\Gamma \). By Proposition 3, we deduce that \( \mu \) is \( G \)-invariant. Recall that \( n(G) > \text{rank}_G G \) where \( n(G) \) denotes the minimal dimension of a non-trivial real representation of the Lie algebra of \( G \). By Zimmer’s cocycle superrigidity theorem (we use the version by Fisher and Margulis in \cite[Theorem 1.4]{FM03}; we refer the reader to \cite{Zim80, Zim84, ZW08} for some earlier results), the \( G \)-action preserves a measurable metric on \( E^F \). This contradicts that \( \lambda^F_{+}(s, \mu) > 0 \). Thus \( \alpha \) must have uniform subexponential growth of derivatives. \( \square \)

**Step 2: Strong property (T) and averaging**

In this step, we follow \cite{BFH16} to construct a \( \Gamma \)-invariant continuous distance by using the strong property (T) of \( \Gamma \) proved by Lafforgue, de Laat and de la Salle in \cite{La08, dLdlS15, dlS19}. The main result of this step is the following proposition whose proof will be given in § 5.

**Proposition 5.** If \( \alpha \) has uniform subexponential growth of derivatives, then there exists a distance \( \tilde{d} : M \times M \to [0, \infty) \) that is invariant by the \( \Gamma \)-action \( \alpha \). Moreover, for any \( \beta \in (0, 1) \), the set \( \alpha(\Gamma) \) is precompact in \( \text{Hol-Homeo}^\beta(M) \), the space of \( \beta \)-bi-Hölder homeomorphisms of \( M \) with respect to the background Riemannian distance.

Proposition 5 replaces \cite[Theorem 2.9]{BFH16}. In \cite{BFH16}, the authors study a \( C^2 \) action of \( \Gamma \), and the induced \( \Gamma \)-action on \( W^{1,p}(S^2(T^*M)) \), the Sobolev space of all the sections \( \varphi \) of the bundle of symmetric two forms \( S^2(T^*M) \) such that both \( \varphi \) and \( D\varphi \) are \( L^p \) with respect to the Lebesgue measure. Then the strong property (T) and the uniform subexponential growth of derivatives give us the \( \Gamma \)-invariant section in \( W^{1,p}(S^2(T^*M)) \) which is continuous if \( p \) is sufficiently large. The above method can be adapted to the case where the action is \( C^{1+\epsilon} \) for any \( \epsilon > 0 \).

In our case, \( \alpha \) is only \( C^1 \), and consequently \( \alpha \) does not induce a \( \Gamma \)-action on \( W^{1,p}(S^2(T^*M)) \). We consider instead the induced \( \Gamma \)-action on \( L^p(S^2(T^*M)) \), and obtain a \( L^p(\alpha \)-invariant section of \( S^2(T^*M) \). We use the exponential convergence inherited from the strong property (T) and Cauchy inequality to bound the Sobolev norms of the \( \Gamma \)-action.
To make use of Proposition 5, we also need the solution of the Hilbert–Smith conjecture for sufficiently Hölder actions proved in [RS97, Mal97]. We recall the statement here.

**Lemma 1.** For any $\beta \in (\dim M/(\dim M + 1), 1)$ the following is true. Let $H$ be a compact topological group which admits a faithful action on $M$ by $\beta$-Hölder homeomorphisms; then $H$ is a Lie group.

**Corollary B.** Let $G, \Gamma, \mu, \alpha$ be as in Theorem 1. Assume either that $\dim M < \text{rank}_R(G)$, or that $\dim M \leq \text{rank}_R(G)$ and $\alpha(\Gamma) \subset \text{Diff}^1(M, \text{vol})$. Then $\alpha$ factors through a compact Lie group. That is, there exist: a compact Lie group $H$; an injective group homomorphism $\iota : H \to \text{Homeo}(M)$; and a group homomorphism $\phi : \Gamma \to H$ such that $\alpha = \iota \circ \phi$.

**Proof.** By Proposition 1, the action $\alpha$ has uniform subexponential growth of derivatives. We fix any $\beta \in (\dim M/(\dim M + 1), 1)$. By Proposition 5, the closure of $\alpha(\Gamma)$ in $\text{Hol-Homeo}^\beta(M)$, denoted by $K_0$, is a compact topological subgroup of $\text{Homeo}(M)$. By Lemma 1, we see that $K_0$ is a compact Lie group. □

**Step 3: Margulis superrigidity with compact codomain**

After Steps 1 and 2, we can apply precisely the same method as in [BFH16] to show the finite image property. We refer the reader to [BFH16, §7] for details.

**Proof of Theorem 1.** The proof is essentially contained in [BFH16, §7]. We reproduce it below for the convenience of the reader.

Let $H$ be the compact Lie group given by Corollary B, and let $\iota : H \to \text{Homeo}(M)$ and $\phi : \Gamma \to H$ be the associated group homomorphisms. Assume that $\alpha = \iota \circ \phi$ has infinite image. Then by Margulis’ arithmeticity theorem and superrigidity theorem, each almost simple factor of $H$ is a compact form of $G$. Since $\iota : H \to \text{Homeo}(M)$ is injective, there is some $x \in M$ such that $\iota(H)x$ contains a compacta homeomorphic to $K/C$ where $K$ is a compact form of $G$ and $C$ is a closed proper subgroup of $K$. This is impossible since by hypothesis $\dim(K/C) \geq d(G) > \dim M$. □

### 4. Proof of Proposition 4

#### 4.1 Suspension space

In this subsection, we recall the suspension construction and the induced $G$-action in [BRW16a, S2].

Let $\alpha$ be a $\Gamma$-action on $M$ by $C^1$ diffeomorphisms, i.e. $\alpha(gh) = \alpha(g)\alpha(h)$. We consider the right action by $\Gamma$ on $G \times M$ defined as

$$(g, x) \cdot \gamma = (g\gamma, \alpha(\gamma^{-1})(x)), \quad \forall \gamma \in \Gamma$$

and the left $G$-action

$$a \cdot (g, x) = (ag, x), \quad \forall a \in G.$$ 

Define the quotient manifold $M^\alpha := (G \times M)/\Gamma$. Since the left $G$-action commutes with the right $\Gamma$-action, the left $G$-action descends to a left $G$-action on $M^\alpha$, denoted by $\tilde{\alpha}$. Since $\alpha$ is a $C^1$ action, $M^\alpha$ is naturally equipped with a $C^1$ manifold structure. The action $\tilde{\alpha}$ is given by $C^1$ diffeomorphisms of $M^\alpha$. Moreover, denote by $\pi : M^\alpha \to G/\Gamma$ the projection induced by $G \times M \to G$, then $M^\alpha$ is a $C^1$ fiber bundle over $G/\Gamma$ induced by $\pi$ with fibers diffeomorphic to $M$.

With a slight abuse of notation, we use $d(\cdot, \cdot)$ to denote both the right-invariant metric on $G$, and the quotient metric on $G/\Gamma$. We denote by $\nu$ the normalized left Haar measure on $G/\Gamma$.  

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By the construction in [BFH20, §2.2] (see also [BRW16a, §2.1] for the details), there exists a $C^1$ Riemannian metric $\langle \cdot, \cdot \rangle$ on $G \times M$ with the following properties:

1. $\langle \cdot, \cdot \rangle$ is invariant under the right $\Gamma$-action;
2. for each $(g, x) \in G \times M$, under the canonical identification of the $G$-orbit of $(g, x)$ with $G$, the restriction of $\langle \cdot, \cdot \rangle$ to the $G$-orbit of $(g, x)$ coincides with $d_G$;
3. there exist a Siegel fundamental set $D \subset G$ for the right $\Gamma$-action (see [Mar91, VIII.1] for the definition) containing the identity $e \in G$, and a constant $C_1 > 1$ such that for any $g_1, g_2 \in D$, the map $(g_1, x) \mapsto (g_2, x)$ distorts the restrictions of $\langle \cdot, \cdot \rangle$ to $\{g_1\} \times M$ and $\{g_2\} \times M$ by at most $C_1$.

We use $\langle \cdot, \cdot \rangle_g$ to denote the restriction of $\langle \cdot, \cdot \rangle$ to $\{g\} \times M$, and view it as a metric on $M$. By item (1) above, we can equip $M^\alpha$ with the quotient metric of $\langle \cdot, \cdot \rangle$.

We fix $\{\gamma_i\}$, a finite symmetric generating set for $\Gamma$. Let $\ell$ denote the word-length distance on $\Gamma$ relative to $\{\gamma_i\}$. Given a fundamental domain $F_D \subset D$ for the right $\Gamma$-action on $G$, i.e. $G = F_D \Gamma$ and $F_D \gamma \cap F_D = \emptyset$ for all $\gamma \in \Gamma \setminus \{e\}$, the return cocycle $\beta : G \times G/\Gamma \to \Gamma$ associated to $F_D$ is defined as follows. For any $g \in G$, $x \in G/\Gamma$, we set $\beta(g, x)$ to be the unique element $\gamma \in \Gamma$ such that $g\bar{x} \in F_D \gamma$, where $\bar{x}$ is the lift of $x$ in $F_D$. The following are from [BFH20] whose proofs rely on [LMR00].

**Lemma 2.** If $F_D \subset D$ is a fundamental domain for the right $\Gamma$-action on $G$ such that $e \in F_D$, then there is a constant $C > 0$ such that for any $g \in G$, any $x \in G/\Gamma$,

$$\ell(\beta(g, x)) < C d(g, e) + C d(x, \Gamma) + C.$$ 

**Lemma 3.** There is a constant $C > 0$ such that the following is true. For any $g \in G$, any $x \in G/\Gamma$, any $p \in \pi^{-1}(x)$ we have

$$\log \|D_p \alpha(g)\| < C d(g, e) + C d(x, \Gamma) + C.$$ 

### 4.2 Smooth cocycle

Let $a$ be as in Proposition 4. In various statements about typical points in $G/\Gamma$ in this rest of this section, we will always refer to the Haar measure $\nu$.

We first recall several basic definitions from measure theory following [CFS92, Appendix 1]. A partition of a Lebesgue space $(Y, \mathcal{Y}, \mu_Y)$ (for the definition, see [CFS92, Appendix 1, Definition 4]) is a family $\xi = \{C\}$ of non-empty disjoint measurable subsets $C$ such that $\bigcup_{C \in \xi} C = M$. A subset $A \in \mathcal{Y}$ is said measurable with respect to $\xi$ if $A$ is a union of elements of $\xi$. The partition $\xi$ is said to be measurable if there exists a countable collection of sets $\{B_i \mid i \in I\}$ which are measurable with respect to $\xi$ such that for any $C_1, C_2 \in \xi$ we can find an $i \in I$ such that either $C_1 \cap B_i = C_2 \cap B_i$ or $C_2 \cap B_i = C_1 \cap B_i$. To any measurable partition $\xi$ we can assign a complete $\sigma$-algebra $\mathcal{B}_\xi \subset \mathcal{Y}$ consisting of the sets $A \in \mathcal{Y}$ which coincide modulo $\mu_Y$-null sets with one of the sets which is measurable with respect to $\xi$. In fact such correspondence is bijective (see [CFS92, Appendix 1, §3]).

Following [LS82] and [MT94, §9.3], we may find a measurable partition $\xi$ of $G/\Gamma$ with the following properties.

1. $\xi$ is subordinate to the partition of $G/\Gamma$ into $H^u$ orbits: for a.e. $x \in G/\Gamma$:
   a. the atom $\xi(x)$ is contained in the orbit $H^u \cdot x$;
   b. the atom $\xi(x)$ is precompact in the orbit $H^u \cdot x$;
   c. the atom $\xi(x)$ contains a neighborhood of $x$ in the orbit $H^u \cdot x$.
2. $\xi$ is $a$-decreasing, i.e. $a(\xi) \leq \xi$. 

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We also require that \( \xi \) satisfies the following additional property.

(3) There is a compact set \( W \subset H^u \) such that for a.e. \( x \),

\[
\xi(x) \subset W \cdot x.
\]

To build a partition \( \xi \) satisfying (1)–(3), we first let \( \xi_0 \) be a partition satisfying (1) and (2). Select a \( \xi_0 \)-measurable subset \( S \subset G/\Gamma \) with positive \( \nu \)-measure such that the diameter of \( \xi_0(x) \) is uniformly bounded in the \( H^u \cdot x \)-orbit for all \( x \in S \). It is well known that \( a \) is ergodic with respect to the Haar measure \( \nu \). Thus for a.e. \( x \in G/\Gamma \), the following number is well defined:

\[
n_x = \inf \{ n \in \mathbb{N} \mid a^n \cdot x \in S \}.
\]

We set

\[
\xi(x) = a^{-n_x} \xi_0(a^{n_x} \cdot x).
\]

Then \( \xi \) still satisfies (1) and (2). Since \( \text{Ad}(a^{-1}) \) is a contraction restricted to the Lie algebra of \( H^u \), \( \xi \) also satisfies (3).

Since \( \xi \) is measurable, we may apply [AV10, Lemma 4.6] to find a measurable selection: there is a measurable map \( \psi : G/\Gamma \to G/\Gamma \) such that \( \psi \) is constant on every atom of \( \xi \), and \( \psi(x) \in \xi(x) \) for \( \nu \)-a.e. \( x \). Recall our choice of a Siegel fundamental set \( D \subset G \) and fix a fundamental domain \( F_D \subset D \) such that \( e \in F_D \). Let \( \tilde{\psi} : G/\Gamma \to G \) be the map that assigns \( x \in G/\Gamma \) the unique \( g \in F_D \) with \( \psi(x) = g \Gamma \). Note that \( \tilde{\psi} \) is \( \xi \)-measurable.

Since \( H^u \) is horospherical for \( a \), for a.e. \( x \in G/\Gamma \) the map \( H^u \to G/\Gamma, h \mapsto h \cdot x \) is injective. Indeed, for a \( \mu \)-typical \( x \in G/\Gamma \), there is a sequence \( \{ t_m \}_{m \geq 0} \) of positive numbers that tends to infinity such that \( \{ a^{-t_m} \cdot x \}_{m \geq 0} \) is precompact. Then \( h \mapsto h \cdot x \) must be injective on \( H^u \) since each \( H^u \)-orbit is contracted by the backward iterates of \( a \), and \( G \to G/\Gamma \) is a local homeomorphism. For any such \( x \), we let \( W_x \) be the inverse image of \( \xi(x) \) under the map \( H^u \to G/\Gamma, h \mapsto h \cdot \psi(x) \); and let \( \xi_1(x) = W_x \tilde{\psi}(x) \). Notice that by definition \( \pi(\xi_1(x)) = \xi(x) \), and \( \xi_1(x) \cap F_D \neq \emptyset \).

As \( F_D \) is a fundamental domain contained in \( D \), we can choose a Borel trivialization associated to \( F_D \), denoted by

\[
\iota : M^\alpha \to F_D \times M,
\]

where for each \( x \in G/\Gamma \), we identify \( \iota|_{\pi^{-1}(x)} \) with a diffeomorphism \( \iota_x : \pi^{-1}(x) \to M \). Moreover, by the construction of the metric \( \langle \cdot, \cdot \rangle \) on \( D \times M \), we may assume that \( \| \iota_x \|_{C^1} \) is uniformly bounded over all \( x \in G/\Gamma \).

Given a typical \( x \in G/\Gamma \), let \( u_x \in H^u \) be such that \( x = u_x \cdot \psi(x) \). Set \( g_x : \pi^{-1}(x) \to \pi^{-1}(\psi(x)) \) to be

\[
g_x(y) = \tilde{\alpha}(u_x^{-1})(y).
\]

Given \( x \in G/\Gamma \), set \( F_x : M \to M \) to be

\[
F_x(y) = \iota_{\psi(a^{-1} \cdot x)}(g_{a^{-1} \cdot \psi(x)}(\tilde{\alpha}(a^{-1})(\iota_{\psi(x)}^{-1}(y)))).
\]  \hfill (4.1)

Let \( F : G/\Gamma \times M \to G/\Gamma \times M \) be the measurable map

\[
F(x, y) = (a^{-1} \cdot x, F_x(y)).
\]  \hfill (4.2)

Using \( \{ g_x \} \), we define a measurable map \( \Phi : M^\alpha \to G/\Gamma \times M \) as follows:

\[
\Phi(y) = (\pi(y), \iota_{\psi(\pi(y))}g_{\pi(y)}(y)).
\]  \hfill (4.3)

Let \( \mu \) be the a-ergodic a-invariant measure in Proposition 4, let \( \mu^* = \Phi_* \mu \).
**C^1 lattice actions**

**Claim 1.** \(Φ\) is a Borel isomorphism. Moreover, for \(μ\)-a.e. \(x \in G/Γ\), \(Φ\) is a \(C^1\) diffeomorphism from \(π^{-1}(x)\) to \(M\), and we have

\[F \cdot Φ = Φ \cdot ˜α(a^{-1}).\]

**Proof.** We set \(x = π(z)\). Then we have

\[π(a^{-1} \cdot z) = a^{-1} \cdot π(z) = a^{-1} \cdot x.\]

Then

\[FΦ(z) = (a^{-1} \cdot x, t_{ψ(a^{-1},x)}g_{a^{-1},ψ(x)}(α(a^{-1})(g_x(z))))\]

and

\[Φ(α(a^{-1}))(z) = (a^{-1} \cdot x, t_{ψ(a^{-1},x)}g_{a^{-1},x}(α(a^{-1})(z))).\]

Then by definition, it suffices to show that

\[a_π^{-1} \cdot x = u_π aπ^{-1} \cdot π(x).\]

By definition,

\[a_π^{-1} \cdot π(x) = a \cdot a^{-1} \cdot x = x.\]

We also notice that \(a^{-1} \cdot ψ(x) \in a^{-1} \cdot ξ(x) \subset ξ(a^{-1} \cdot x)\). Thus

\[ψ(a^{-1} \cdot ψ(x)) = ψ(a^{-1} \cdot x).\]

Then

\[u_π aπ^{-1} \cdot ψ(x) \cdot ψ(a^{-1} \cdot x) = u_π a \cdot a^{-1} \cdot ψ(x) = x.\]

This completes the proof. \(□\)

Let \(\{μ_x^\ast\}\) be the disintegration of \(μ^\ast\) with respect to the partition of \(G/Γ \times M\) into fibers. The following properties follow immediately from the above constructions and observations.

**Proposition 6.** We have the following.

1. For a.e. \(x \in G/Γ\) and every \(x' \in ξ(x)\), \(F_x = F_{x'}\); in particular, \(x \mapsto F_x\) is \(ξ\)-measurable.
2. The function \(x \mapsto log\|F_x^{-1}\|_{C^1}\) is in \(L^1(G/Γ, ν)\).
3. \(Φ\) is a measurable conjugacy between the dynamics of \(a^{-1}\) on \(M^α\) and of \(F\) on \(G/Γ \times M\).
4. The fiberwise Lyapunov exponents for \(Dα\) with respect to \(μ\) are all non-positive if, and only if, the fiberwise Lyapunov exponents of \(F\) with respect to \(μ^\ast\) are all non-negative.
5. \(μ\) is \(H^α\)-invariant if and only if the map \(x \mapsto μ_x^\ast\) is \(ξ\)-measurable.

**Proof.** Part (1) follows immediately from the construction. Part (3) is given by Claim 1. Part (4) follows from part (3) and our hypothesis on \(a\) in Proposition 4: all fiberwise Lyapunov exponents of \(Dα\) are non-positive.

To show part (2), we first notice that by (4.1) for \(ν\)-a.e. \(x \in G/Γ\), we have

\[\|F_x^{-1}\|_{C^1} \leq ∥\tilde{α}(u_{α^{-1},ψ(x)})∥ \|π^{-1}(ψ(ab^{-1},x))\|_{C^1} ∥\tilde{α}(a)∥ \|π^{-1}(α^{-1},ψ(x))\|_{C^1}.\]

By Lemma 3, we have

\[log ∥\tilde{α}(a)∥ \|π^{-1}(α^{-1},ψ(x))\|_{C^1} \leq Cd(a, e) + Cd(a^{-1} \cdot ψ(x), x_0) + C,\]

\[log ∥\tilde{α}(u_{α^{-1},ψ(x)})∥ \|π^{-1}(ψ(ab^{-1},x))\|_{C^1} \leq C sup \{d(b, e) + Cd(ψ(a^{-1}, x), x_0) + C.\}

Note that there are \(u_1, u_2 \in W\) such that

\[a^{-1} \cdot ψ(x) = a^{-1}u_1^{-1} x, \quad ψ(a^{-1} \cdot x) = u_2^{-1}a^{-1} x.\]
Then we have
\[ d(a^{-1} \cdot \psi(x), x_0), d(\psi(a^{-1} \cdot x), x_0) \leq Cd(x, x_0) + C' \]
for some \( C \) depending only on \( G, \Gamma \), and some \( C' \) depending only on \( W \) and \( a \). We now briefly explain why we have that
\[ (x \mapsto d(x, x_0)) \in L^1(G/\Gamma, \nu). \] (4.4)
Let \( \rho : G \to \text{SL}(N, \mathbb{R}) \) be an embedding given in [Mar91, Chap. VIII, §1]. By [LMR00, 3.5 (*)], we have
\[ d(g\Gamma, ag\Gamma) \leq d(g, ag) \leq C(1 + \log \|\rho(a)\|). \]
We remind the reader that the first \( d \) denotes the quotient metric on \( X \), and the second \( d \) denotes the right-invariant metric on \( G \). We deduce (4.4) by [Mar91, Chap. VIII, §1, Proposition 1.2]. Then part (2) follows suit.

The ‘only if’ part of part (5) follows by definition. We assume that \( x \mapsto \mu^x_\ast \) is \( \xi \)-measurable. Then for \( \mu \)-a.e. \( x \), for any \( h \in H^u \) such that \( h(\pi(x)) \in \xi(\pi(x)) \), we have \( \tilde{\alpha}(h)_\ast \mu_{\pi(x)} = \mu_{h(\pi(x))} \) where \( \{\mu_z\}_{z \in G/\Gamma} \) is the disintegration of \( \mu \) along the fibers. Moreover by Claim 1, we see that \( x \mapsto \mu^x_\ast \) is \( a^n(\xi) \)-measurable for any \( n \geq 1 \). We can use the above argument for \( a^n(\xi) \) instead of \( \xi \) (for all \( n \geq 1 \)) to show that \( \mu \) is \( H^u \)-invariant.

\[ \square \]

4.3 Avila and Viana’s invariance principle
We will use a variant of [AV10, Theorem B] to conclude the proof of Proposition 4. Let us first briefly recall the setting in [AV10].

Let \((X, \mathcal{B}, \mu)\) be a probability space, and let \( \hat{f} : \hat{X} \to \hat{X} \) be an invertible \( \hat{\mu} \)-preserving measurable transformation. Let \( N \) be a compact Riemannian manifold. We set \( \hat{\mathcal{E}} = \hat{X} \times N \), and denote by \( \hat{P} : \hat{\mathcal{E}} \to \hat{X} \) the projection to the first coordinate. We say that a \( \hat{B} \otimes \mathcal{B}_N \)-measurable transformation \( \hat{F} : \hat{\mathcal{E}} \to \hat{\mathcal{E}} \) is a smooth cocycle over \( \hat{f} \) if \( \hat{F} \) is of form \( \hat{F}(\hat{x}, \hat{y}) = (\hat{f}(\hat{x}), \hat{F}_\hat{x}(\hat{y})) \), where \( \hat{F}_\hat{x} \) is a diffeomorphism of \( N \) for each \( \hat{x} \). We also assume the following:
\[ \int |\log(\sup_{\hat{y}} \|D\hat{F}_\hat{x}(\hat{y})^{-1}\|)| d\hat{\mu}(\hat{x}) < \infty. \] (4.5)
In the following, for any integer \( k \), for any \( \hat{x} \in \hat{X} \) we define
\[ \hat{F}_\hat{x}^k = \begin{cases} \hat{F}_\hat{x}^{k-1}(\hat{x}) \cdots \hat{F}_\hat{x} & k \geq 0, \\ (\hat{F}_\hat{f}^{-k}(\hat{x}))^{-1} \cdots (\hat{F}_\hat{f}^{-1}(\hat{x}))^{-1} & k < 0. \end{cases} \]
We warn the reader not to confuse the above notation with \( (\hat{F}_\hat{x})^k \).

In this case, for any \( \hat{F} \)-invariant probability measure \( \hat{m} \) on \( \hat{\mathcal{E}} \) that projects to \( \hat{\mu} \) under \( \hat{P} \), the minimal Lyapunov exponent is a well-defined quantity at \( \hat{m} \)-a.e. \( (\hat{x}, \hat{y}) \) by the following formula:
\[ \lambda_-(\hat{F}, \hat{x}, \hat{y}) = \lim_{n \to \infty} \frac{1}{n} \log \|D\hat{F}_\hat{x}^{\hat{f}}(\hat{y})^{-1}\|^{-1}. \]

The following theorem, whose proof is deferred to Appendix A, is a variant of [AV10, Theorem B].

**Theorem 2.** Let \( \hat{m} \) be an \( \hat{F} \)-invariant measure on \( \hat{\mathcal{E}} \) which projects down to \( \hat{\mu} \). Let \( \mathcal{B}_0 \subset \hat{\mathcal{B}} \) be a \( \sigma \)-algebra which generates \( \hat{\mathcal{B}} \mod 0 \) under \( \hat{f} \). Assume that both \( \hat{f} \) and \( \hat{x} \mapsto \hat{F}_\hat{x} \) are \( \mathcal{B}_0 \)-measurable mod 0, and \( \lambda_-(\hat{F}, \hat{x}, \hat{y}) \geq 0 \) for \( \hat{m} \)-a.e. \( (\hat{x}, \hat{y}) \), then the disintegration \( \hat{x} \mapsto m_{\hat{x}} \) of the measure \( \hat{m} \) is \( \mathcal{B}_0 \)-measurable modulo 0.
4.4 Completing the proof

We can now finish the proof of Proposition 4.

Proof of Proposition 4. By Proposition 6, the hypothesis of Theorem 2 is satisfied with \((X, N, B_0, \bar{\mu}, \bar{m}, \bar{f}, \bar{F})\) being \((G/\Gamma, M, B_\xi, \mu, \nu, a^{-1}, F)\). Here \(B_\xi\) denotes the complete \(\sigma\)-algebra generated by the partition \(\xi\). Then by Theorem 2, the map \(x \mapsto \mu_x^*\) is \(\xi\)-measurable. Proposition 4 then follows from Proposition 6(5).

5. Proof of Proposition 5

Recall that we fixed a finite set of symmetric generators \(\{\gamma_i\}\) for \(\Gamma\). The word distance \(\ell\) on \(\Gamma\) is defined in §4.1.

Proof of Proposition 5. We let \(\|\cdot\|_g\) denote the background Riemannian metric \(g\) on \(TM\), and let \(\Vol_g\) denote the volume form induced by \(\|\cdot\|_g\). There is a \(C^\infty\) Riemannian metric on \(S^2(T^*M)\) associated to \(\|\cdot\|_g\). We denote by \(L^p(M, \Vol_g, S^2(T^*M))\) the space of \(L^p\) sections of the tensor bundle \(S^2(T^*M)\) with respect to \(\Vol_g\).

Since \(\alpha\) has uniform subexponential growth of derivatives, by the strong property (T) of the lattice \(\Gamma\) (proved in [Laf08, dLdIS15, dIS19]), we can adapt the argument in [BFH16] to show that there exist:

1. constants \(C_p^{\mu\sigma}, \sigma_p > 0\) for every \(1 \leq p < \infty\);
2. \(\bar{g} \in L^p(M, \Vol_g, S^2(T^*M))\) for all \(1 \leq p < \infty\), which is non-degenerate, i.e. \(\|v\|_{\bar{g}} > 0\) for \(\Vol_g\)-a.e. \(x \in M\), and every non-zero \(v \in T_x M\);
3. a sequence of probability measures on \(\Gamma\), denoted by \(\{\omega_n\}\), satisfying \(\supp(\omega_n) \subset B_{\text{word}}(e, n) \subset \Gamma\) for every \(n\), where \(B_{\text{word}}(e, n)\) denotes the radius \(n\) open ball in \(\Gamma\) centered at \(e\) with respect to the word distance,

such that, setting \(g_n = \int \alpha(\gamma)^* g \, d\omega_n(\gamma)\), then we have

\[
\|g_n - \bar{g}\|_{L^p} < C_p^{\mu\sigma} e^{-n\sigma_p}, \quad \forall 1 \leq p < \infty.
\]  

As a consequence, if we denote by \(\Vol_{\bar{g}}\) the measurable volume form induced by \(\|\cdot\|_{\bar{g}}\), then the measure \(d\Vol_{\bar{g}}\) is absolutely continuous with respect to \(d\Vol_g\), and the density function \(d\Vol_{\bar{g}}/d\Vol_g\) has full support.

We define Lebesgue measurable functions \(\bar{R}, R : M \to \mathbb{R}_+\) as follows. Set

\[
\bar{R}(x) = \sup_{v \in T_x M, \|v\|_{\bar{g}}=1} \|v\|_{\bar{g}}, \quad R(x) = \inf_{v \in T_x M, \|v\|_{\bar{g}}=1} \|v\|_{\bar{g}}.
\]

We can see directly that for \(\Vol_g\)-a.e. \(x \in M\),

\[
\frac{d\Vol_{\bar{g}}}{d\Vol_g}(x) < \bar{R}^{-\dim M}(x), \quad \frac{d\Vol_{\bar{g}}}{d\Vol_g}(x) < R^{-\dim M}(x).
\]

We have the following lemma.

Lemma 4. For every \(1 \leq p < \infty\), there is \(C_p > 0\) such that

\[
\int \bar{R}^{-p} \, d\Vol_g < C_p, \quad \int R^{-p} \, d\Vol_g < C_p.
\]

Proof. The second inequality follows immediately from the fact that \(\bar{g} \in L^p(M, \Vol_g, S^2(T^*M))\). It remains to prove the first inequality.

4 We obtain (1)–(3) for \(p \in (1, \infty)\) by strong property (T), and then the case for \(p = 1\) follows from Cauchy’s inequality.
We define for every \( n \geq 1 \),
\[
R_n(x) = \inf_{v \in T_x M, \|v\|_g = 1} \|v\|_{g_n}, \quad \forall x \in M,
\]
and
\[
\Omega_n = \{ x \mid R(x) \geq \frac{1}{2} R_n(x) \}.
\]
For the convenience of the notation, we set \( \Omega_0 = \emptyset \). It is clear that \( \bigcup_n \Omega_n \) is a \( d\text{Vol}_g \)-conull subset of \( M \).

By the uniform subexponential growth of derivatives, for every \( \varepsilon > 0 \) there is \( C_\varepsilon'' > 0 \) such that
\[
\sup_{x \in M} (R_n(x)^{-1}) < C_\varepsilon'' e^{n \varepsilon}, \quad \forall n \geq 1. \tag{5.2}
\]

By (5.1) and (5.2), for every \( \varepsilon > 0 \) we have
\[
\text{Vol}_g(\Omega^c_n) \leq \text{Vol}_g(\{ x \mid |R(x) - R_n(x)| > \frac{1}{2} R_n(x) \})
\leq 2 \sup_{x \in M} (R_n(x)^{-1}) \int |R(x) - R_n(x)| \text{dVol}_g(x)
\leq 2C_\varepsilon'' C_1 e^{n \varepsilon - n \sigma_1}.
\]

Then for each \( 1 \leq p < \infty \), we take \( \varepsilon = \sigma_1/(10p) \), and we obtain
\[
\int M (\bar{R}(x))^{-p} \text{dVol}_g(x) \leq 2^p \sum_{n=0}^{\infty} \int_{\Omega_{n+1} \setminus \Omega_n} R_{n+1}(x)^{-p} \text{dVol}_g(x)
\leq 2^p \sum_{n=0}^{\infty} \sup_x (R_{n+1}(x)^{-p}) \text{Vol}_g(\Omega^c_n)
\leq 2^{p+1} (C_\varepsilon'')^{p+1} C_1 \sum_{n=0}^{\infty} e^{(n+1)p \varepsilon - n (\sigma_1 - \varepsilon)} := C_p < \infty.
\]

**Lemma 5.** For every \( 1 \leq p < \infty \), there exists \( D_p > 0 \) such that for every \( \gamma \in \Gamma \),
\[
\int M \|D_x \alpha(\gamma)\|_g^p \text{dVol}_g(x) \leq D_p.
\]

**Proof.** Take an arbitrary \( \gamma \in \Gamma \), and set \( F = \alpha(\gamma) \). We recall that \( F \) preserves \( \bar{g} \). That is, for \( d\text{Vol}_{\bar{g}} \)-a.e. \( x \), for every \( v \in T_x M \), we have \( \|v\|_{\bar{g}} = \|D_x F(v)\|_{\bar{g}} \). Hence the measure \( d\text{Vol}_{\bar{g}} \) is \( F \)-invariant.

Notice that for \( d\text{Vol}_{\bar{g}} \)-a.e. \( x \in M \),
\[
\|D_x F\|_{\bar{g}} = \sup_{v \in T_x M, \|v\|_{\bar{g}} = 1} \|D_x F(v)\|_{\bar{g}}
= \sup_{v \in T_x M, \|v\|_{\bar{g}} = 1} \|D_x F(v)\|_{\bar{g}} \frac{\|D_x F(v)\|_{\bar{g}}}{\|D_x F(v)\|_{\bar{g}}}
= \sup_{v \in T_x M, \|v\|_{\bar{g}} = 1} \|v\|_{\bar{g}} \frac{\|D_x F(v)\|_{\bar{g}}}{\|D_x F(v)\|_{\bar{g}}}
\leq \bar{R}(x) \bar{R}(F(x))^{-1}.
\]

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Then by Cauchy’s inequality,

$$\int \|D_x F\|_g^p \, d\text{Vol}_g(x) \leq \left( \int \tilde{R}(x)^{2p} \, d\text{Vol}_g(x) \right)^{1/2} \left( \int \tilde{R}(F(x))^{-2p} \, d\text{Vol}_g(x) \right)^{1/2}. $$

Also

$$\int \tilde{R}(F(x))^{-2p} \, d\text{Vol}_g(x) = \int \tilde{R}(F(x))^{-2p} \frac{d\text{Vol}_g}{d\text{Vol}_g}(x) \, d\text{Vol}_g(x)$$

$$\leq \left( \int \tilde{R}(F(x))^{-4p} \, d\text{Vol}_g(x) \right)^{1/2} \left( \int \left( \frac{d\text{Vol}_g}{d\text{Vol}_g}(x) \right)^2 \, d\text{Vol}_g(x) \right)^{1/2}$$

$$\leq \left( \int \tilde{R}(x)^{-4p} \, d\text{Vol}_g(x) \right)^{1/2} \left( \int \tilde{R}^{-\dim M}(x) \, d\text{Vol}_g(x) \right)^{1/2}$$

and

$$\int \tilde{R}(x)^{-4p} \, d\text{Vol}_g(x) = \int \tilde{R}(x)^{-4p} \frac{d\text{Vol}_g}{d\text{Vol}_g}(x) \, d\text{Vol}_g(x)$$

$$\leq \left( \int \tilde{R}(x)^{-8p} \, d\text{Vol}_g(x) \right)^{1/2} \left( \int \left( \frac{d\text{Vol}_g}{d\text{Vol}_g}(x) \right)^2 \, d\text{Vol}_g(x) \right)^{1/2}$$

$$\leq \left( \int \tilde{R}(x)^{-8p} \, d\text{Vol}_g(x) \right)^{1/2} \left( \int \tilde{R}^{2\dim M}(x) \, d\text{Vol}_g(x) \right)^{1/2}.$$

By Lemma 4,

$$\int \|D_x F\|_g^p \, d\text{Vol}_g(x) \leq C_{2p}^{1/2} C_{8p}^{1/8} C_{2\dim M}^{1/4} C_{\dim M}^{1/4}.$$

Since $\gamma$ is chosen arbitrarily, we can conclude the proof by taking $D_p$ to be the right hand side of the last inequality.

We fix an embedding $\iota : M \to \mathbb{R}^N$ for some integer $N$. Let $\pi_i : \mathbb{R}^N \to \mathbb{R}$ be the projection to the $i$th coordinate. We have seen that for every $1 \leq p < \infty$, there exists a constant $C'_p > 0$ such that for every $1 \leq i \leq N$, for every $\gamma \in \Gamma$,

$$\int \|D_x (\pi_i \iota \alpha(\gamma))\|_g^p \, d\text{Vol}_g(x) < C'_p.$$ 

Take $p > \dim M/(1 - \beta)$. Then by Sobolev’s embedding theorem, we can see that the set $\{ \alpha(\gamma) \mid \gamma \in \Gamma \}$ is pre-compact in Hol-Homeo$^\beta(M)$. We know that any pre-compact subset of Hol-Homeo$^\beta(M)$ is equicontinuous in Homeo($M$). Thus the closure of $\alpha(\Gamma)$ in Homeo($M$) is a compact topological group $K_0$, and we can directly verify by definition that $K_0 \subset$ Hol-Homeo$^\beta(M)$. It is then direct to construct a $\Gamma$-invariant continuous distance on $M$ by averaging.

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Appendix A

We now give the proof of Theorem 2 in this appendix. We recall the construction in [AV10, §3]. There is a Lebesgue space \((X, \mathcal{B}, \mu)\) obtained by identifying any two points of \(\hat{X}\) which are not distinguished by any element of \(\mathcal{B}\); and a projection \(\pi : \hat{X} \to X\) such that \(\mathcal{B} = \pi_* \mathcal{B}_0\) and \(\mu = \pi_* \hat{\mu}\). Since \(\hat{f}\) is \(\mathcal{B}_0\)-measurable modulo 0, there exists a \(\mathcal{B}\)-measurable modulo 0 transformation \(f : X \to X\) such that \(\pi \circ \hat{f} = f \circ \pi\). Let \(\mathcal{E} = X \times \mathbb{N}\) and \(P : \mathcal{E} \to X\) the canonical projection. Since \(\hat{F}\) is \(\mathcal{B}_0\)-measurable modulo 0, we may write \(F_{\pi(\hat{x})} = \hat{F}_{\hat{x}}\) for some \(\mathcal{B}\)-measurable modulo 0 fiber bundle morphism \(F : \mathcal{E} \to \mathcal{E}\) over \(f\). The measure \(m = (\pi \times id)_* \hat{m}\) is \(F\)-invariant and projects down to \(\mu\). Denote by \(\{m_x\}_{x \in X}\) the measure disintegration of \(m\) corresponding to the partition of \(\mathcal{E}\) into the fibers. By the \(F\)-invariance of \(m\) we deduce that for \(\mu\text{-a.e. } x \in X\),

\[
 m_{f(x)} = \int (F_{x'})_* m_{x'} \, d\mu^{-1}_x(B_0)(x') . \tag{A.1}
\]

For any integer \(l \geq 0\), we define \(J_l : \mathcal{E} \to [0, \infty)\) by considering the Lebesgue decomposition of \((F^{-l}_x)_* m_{f(x)}\) relative to \(m_x\):

\[
(F^{-l}_x)_* m_{f(x)} = J_l(x, \cdot) m_x + \eta^{(l)}_x .
\]

We abbreviate \(J_1\) as \(J\).

Define

\[
h(F, m) = \int - \log J \, dm .
\]

Following the proof of [AV10, Theorem B], we will show that \(m(\{J = 0\}) = 0\) and in addition the following is true.

**Proposition A.1.** We have

\[
0 \leq h(F, m) \leq - \dim N \int \min\{0, \lambda_-(\hat{F})\} \, d\hat{m} .
\]

The statement of Proposition A.1 is the same as [AV10, Proposition 3.1], except that we are now assuming (4.5), while in [AV10] the authors assume that \(\log \| D\hat{F}_\hat{x}(\hat{y})^{-1} \|, \log \| DHx(\hat{y}) \|\) and \(\log \| D\hat{H}_\hat{x}(\hat{y})^{-1} \|\) are all uniformly bounded, and the dependences of \(D\hat{F}_\hat{x}(\hat{y})\), \(D\hat{H}_\hat{x}(\hat{y})\) on \(\hat{x}\), \(\hat{y}\) are uniformly continuous. Thus we will need to make some adjustments to the proof in [AV10] (see also [Led86]).

Under the hypothesis of Theorem 2, we can conclude by Proposition A.1 that \(h(F, m)\) vanishes. Once we know that \(h(F, m)\) vanishes, we can apply [AV10, Proposition 3.2] to conclude the proof of Theorem 2. We recall the statement below.

**Proposition A.2.** If \(h(F, m) = 0\) then \(\hat{x} \mapsto \hat{m}_\hat{x}\) is \(\mathcal{B}_0\)-measurable modulo 0.

The proof of Proposition A.2 is rather general and the condition (4.5) suffices. Now it suffices to give the proof of Proposition A.1. The proof here follows essentially the scheme in [Led86].

**Proof of Proposition A.1.** By the same argument in [AV10, §3.2], we may assume without loss of generality that \(\hat{m}\) is ergodic for \(\hat{F}\). In this case, \(\min(0, \lambda_-(\hat{F}))\) is a constant \(\hat{m}\)-almost everywhere, and is denoted by \(-\lambda \leq 0\).

For any integer \(k\), for any \((x, \xi) \in X \times N\) we define

\[
F^k_x = \begin{cases} F_{f^{-k}(x)} \cdots F_{x} & k \geq 0, \\ F_{f^{-k}(x)}^{-1} \cdots F_{f^{-1}(x)}^{-1} & k < 0, \end{cases}
\]

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\[ L_k(x, \xi) = \| D_\xi F_{f_k(x)}^{-k} \|, \quad C_k(x) = \sup_{\xi \in \mathcal{N}} L_k(x, \xi), \quad \tilde{C}_k(x, \xi) = C_k(x). \]

Notice that we have
\[ 0 \leq \log \tilde{C}_k(x, \xi) \leq \sum_{i=0}^{k-1} \log \tilde{C}_i(F^i(x, \xi)). \quad (A.2) \]

Given \((x, \xi) \in X \times \mathcal{N}\), we denote by \(B(\xi, \delta)\) the ball in \(\mathcal{N}\) centered at \(\xi\) of radius \(\delta > 0\) and write \(B((x, \xi), \delta) = \{x\} \times B(\xi, \delta)\).

For each integer \(l \geq 0\), we write
\[ J_l(x, \xi; \delta) = \frac{(F_{f_l(x)}^{-l})_* m_{f_l(x)}(B(\xi, \delta))}{m_x(B(\xi, \delta))} \]
and
\[ J_l^*(x, \xi) = \max_{\delta > 0} J_l(x, \xi; \delta). \]

It is clear that \(J_l \geq 0\) and \(J_l^* \geq 1\).

We fix some \(\epsilon > 0\). Then there is \(\beta_1 = \beta_1(\epsilon) > 0\) so that for any set \(A \subset X \times \mathcal{N}\) with \(m(A) < \beta_1\), we have
\[ \int_A \log \tilde{C}_1 d\mu < \epsilon. \quad (A.3) \]

Fix some integer \(l > 0\) such that the measurable set \(\Lambda_1 \subset X \times \mathcal{N}\) defined by
\[ \Lambda_1 = \{(x, \xi) \mid L_l(x, \xi) \leq e^{(\lambda + \epsilon)l}\} \]
satisfies \(m(\Lambda_1) > 1 - \beta_1/2\). Then there is a subset \(\Lambda \subset \Lambda_1\) with
\[ m(\Lambda) > 1 - \beta_1 \quad (A.4) \]
such that the derivatives \(D_\xi F_x\) are uniformly continuous in \(\xi\) over all \(x \in \Lambda\), and for some \(\delta_1 = \delta_1(\epsilon, l, \Lambda) > 0\), for any \(x \in \Lambda\), for any \(\delta \in (0, \delta_1(\epsilon))\) we have
\[ F_{f_l(x)}^{-l}(B(\xi, \delta)) \subset B(F_{f_l(x)}^{-l}(\xi), e^{(\lambda + 2\epsilon)\delta}). \quad (A.5) \]

We denote by \(E_l\) the collection of ergodic component of \(m\) for \(F^l\). Since \(\mu\) is \(F\)-ergodic, we deduce that \(E_l\) is finite and \(F\) induces a cyclic permutation of \(E_l\). Moreover, for \(m\)-almost every \((x, \xi)\) we denote by \(m_{(x, \xi)}\) the ergodic component at \((x, \xi)\).

By [Led86, Proposition 5], we know that
\[ \log J_l^* \in L^1(\mathcal{E}, m). \]

More precisely, we have the following. \(\square\)

**Lemma A.1.** For \(m\)-almost every \((x, \xi)\), we have
\[ J_l(x, \xi) = \prod_{i=0}^{l-1} J(F_i(x, \xi)). \]

Consequently, for any \(m' \in E_l\), we have
\[ lh(F, m) = -\int \log J_l(x, \xi) dm'(x, \xi). \quad (A.6) \]
Proof. It is clear that the first equality holds when \( l = 1 \). For any \( l > 1 \), we have

\[
(F_{f^{(l)}}^{-1})_\ast m_{f(x)} = (F_{x}^{-1})_\ast (F_{f^{(l-1)}}^{-1})_\ast m_{f(x)}
\]

\[
= (F_{x}^{-1})_\ast (J_{l-1}(F(x, \xi))m_{f(x)} + \eta_{f(x)}^{(l-1)})
\]

\[
= J_{l-1}(F(x, \xi)) \cdot J(x, \xi)m_{x} + J_{l-1}(F(x, \xi))\eta_{x} + (F_{x}^{-1})_\ast \eta_{f(x)}^{(l-1)}.
\]

Here \( \eta_{f(x)}^{(l-1)} \) is the singular component of \( (F_{f^{(l)}}^{-1})_\ast m_{f(x)} \) with respect to \( m_{f(x)} \). By definition, \( \eta_{x} \) is singular with respect to \( m_{x} \). Moreover, \( (F_{x}^{-1})_\ast \eta_{f(x)}^{(l-1)} \) is also singular with respect to \( m_{x} \) for \( m \)-a.e. \( x \). Otherwise, we would know that \( \eta_{f(x)}^{(l-1)} \) is not singular with respect to \( (F_{x})_\ast m_{x} \); then by (A.1) we would know that \( \eta_{f(x)}^{(l-1)} \) is not singular with respect to \( m_{f(x)} \), which is a contradiction. Consequently, we see that

\[
J_{l} = J_{l-1} \circ F \cdot J.
\]

We then conclude the proof of the first equality by induction. The equality (A.6) in the lemma is an immediate consequence of the first equality, and the fact that \( m = (1/l) \sum_{i=0}^{l-1}(F^{i})_\ast m' \) for any \( m' \in E_{l} \).

We choose some \( \beta_{2} = \beta_{2}(\epsilon, l) > 0 \) such that for \( m' \in E_{l} \), and for every \( A \subset X \times N \) with \( m'(A) < \beta_{2} \), we have

\[
\int_{A} (\log J_{l}^* + \log J_{l}) dm' < \epsilon. \tag{A.7}
\]

We define

\[
Z = \{(x, \xi) \mid J_{l}(x, \xi) = 0\}, \quad G = Z^c = \{(x, \xi) \mid J_{l}(x, \xi) > 0\}.
\]

We fix a large constant \( D > 0 \). Given a constant \( \delta > 0 \), we define

\[
G(\delta) = \{(x, \xi) \in G \mid \log J_{l}(x, \xi; \delta') \leq \log J_{l}(x, \xi) + \epsilon, \forall \delta' \in (0, \delta)\},
\]

\[
Z(\delta) = \{(x, \xi) \mid \log J_{l}(x, \xi; \delta') \leq -D, \forall \delta' \in (0, \delta)\}.
\]

We fix some \( \delta_{2} = \delta_{2}(\epsilon, l) > 0 \) sufficiently small so that for every \( m' \in E_{l} \),

\[
m'(G \setminus G(\delta_{2})) < \beta_{2}. \tag{A.8}
\]

We take an arbitrary \( \delta_{0} \in (0, \min(\delta_{1}, \delta_{2})) \). Given \( (x, \xi) \in X \times N \), define \( \delta_{l}(x, \xi; 0) = \delta_{0} \) and for \( k \geq 1 \) recursively define

\[
\delta_{l}(x, \xi; k) = \begin{cases} 
 e^{(-\lambda-2\epsilon)k}\delta_{l}(x, \xi; k-1) & F^{kl}(x, \xi) \in \Lambda, \\
 [\tilde{C}l(F^{kl}(x, \xi))]^{-1}\delta_{l}(x, \xi; k-1) & F^{kl}(x, \xi) \notin \Lambda.
\end{cases}
\]

Observe that we have

\[
\delta_{l}(x, \xi; k+1) \leq \delta_{l}(x, \xi; k) \leq \delta_{0}, \quad \forall k \geq 0
\]

and by (A.5) and the definition of \( \tilde{C} \) we deduce that

\[
F^{-l}(B(F^{(k+l)})(x, \xi), \delta_{l}(x, \xi; k+1))) \subset B(F^{kl}(x, \xi), \delta_{l}(x, \xi; k)), \quad k \geq 0.
\]

**Lemma A.2.** For \( m \)-almost every \( (x, \xi) \), we have

\[
\liminf_{n \to \infty} \frac{1}{n} \log \delta_{l}(x, \xi; n) \geq (-\lambda - 3\epsilon)l.
\]
Proof. By definition, we have

\[
\log \delta_l(x, \xi; n) = \log \delta_0 + \sum_{k=0}^{n-1} (l(-\lambda - 2\epsilon)1_{F^{kl}(x, \xi) \in A} - \log \tilde{C}_l(F^{kl}(x, \xi))1_{F^{kl}(x, \xi) \notin A}).
\]

Then by pointwise ergodic theorem, we have

\[
\lim inf_{n \to \infty} \frac{1}{n} \log \delta_l(x, \xi; n) \geq - (\lambda + 2\epsilon)m(x, \xi)(A) - \int_{\Lambda^c} \log \tilde{C}_l \, dm(x, \xi).
\]

By (A.2), (A.3) and (A.4), we obtain

\[
\int_{\Lambda^c} \log \tilde{C}_l \, dm(x, \xi) \leq \sum_{i=0}^{l-1} \int_{\Lambda^c} \log \tilde{C}_1 \circ F^i \, dm(x, \xi) \leq \sum_{i=0}^{l-1} \int_{\Lambda^c} \log \tilde{C}_1 d(F^i)_* m(x, \xi) = l \int_{\Lambda^c} \log \tilde{C}_1 \, dm \leq l\epsilon.
\]

The equality follows from \( m = \frac{1}{l} \sum_{i=0}^{l-1} (F^i)_* m(x, \xi) \) for \( m \)-almost every \((x, \xi)\). Then

\[
\lim inf_{n \to \infty} \frac{1}{n} \log \delta_l(x, \xi; n) \geq -(\lambda + 3\epsilon)l. \quad \square
\]

Recall that, by [Led86, Proposition 5], for any Borel probability measure \( \nu \) on \( N \), we have

\[
\lim sup_{r \to 0} \frac{\log \nu(B(\xi, r))}{\log r} \leq \dim N, \quad \nu \text{-a.e. } \xi.
\]

Thus we may pick a subset \( \Omega_1 \subset X \times N \) such that

\[
\lim sup_{r \to 0} \frac{\log \inf_{(x, \xi) \in \Omega_1} m_x (B(\xi, r))}{\log r} \leq \dim M
\]

and \( m'(\Omega_1) > 0 \) for every \( m' \in E_l \). Then for \( m \)-almost every \((x, \xi)\), there is an infinite sequence of \( n \) such that \( F^{nl}(x, \xi) \in \Omega_1 \). Then for all sufficiently large \( n \) in such sequence we have

\[
\frac{1}{n} \log m_{F^{nl}(x)}(B(F^{nl}(x, \xi), \delta_l(x, \xi; n))) \geq \frac{1}{n} \log \delta_l(x, \xi; n)(\dim N + \epsilon) \geq (-\lambda - 4\epsilon)(\dim N + \epsilon)l. \quad (A.9)
\]
On the other hand, we have
\[ m_f^{nl}(x)(B(F^{nl}(x, \xi), \delta_l(x, \xi; n))) \]
\[ = m_x(B(x, \delta_l(x, \xi; 0))) \prod_{j=0}^{n-1} \frac{m_{f_{(j+1)l}}(B(F^{(j+1)l}(x, \xi), \delta_l(x, \xi; j + 1)))}{m_{f_{jl}}(B(F^{jl}(x, \xi), \delta_l(x, \xi; j)))} \]
\[ \leq \prod_{j=0}^{n-1} \frac{m_{f_{(j+1)l}}(B(F^{(j+1)l}(x, \xi), \delta_l(x, \xi; j + 1)))}{m_{f_{jl}}(B(F^{jl}(x, \xi), \delta_l(x, \xi; j)))} \]
\[ \leq \prod_{j=0}^{n-1} \frac{m_{f_{(j+1)l}}(B(F^{(j+1)l}(x, \xi), \delta_l(x, \xi; j)))}{m_{f_{jl}}(B(F^{jl}(x, \xi), \delta_l(x, \xi; j)))} \]
\[ = \prod_{j=0}^{n-1} J_l(F^{jl}(x, \xi), \delta_l(x, \xi; j)). \]

Take an arbitrary \( m' \in E_l \). Then for \( m' \)-almost every \( (x, \xi) \) we have
\[ \limsup \frac{1}{n} \log m_{f^{nl}(x)}(B(F^{nl}(x, \xi), \delta_l(x, \xi; n))) \]
\[ \leq \limsup \frac{1}{n} \sum_{j=0}^{n-1} \log J_l(F^{jl}(x, \xi), \delta_l(x, \xi; j)) \]
\[ \leq \int_{Z(\delta_0)} \log J_l dm' + \int \log J_l^* dm' \]
\[ \leq -Dm'(Z(\delta_0)) + l \int \log J_l^* dm. \]

The last inequality follows from Lemma A.1 and the definition of \( Z(\delta_0) \). Combining the above inequality with (A.9), we conclude that
\[ m'(Z(\delta_0)) \leq \frac{l(\int \log J_l^* dm + (\lambda + 4\epsilon)(\dim N + \epsilon))}{D}. \]

Since the above holds for any \( \delta_0 \) sufficiently small and for any \( m' \in E_l \), we deduce that
\[ m(Z) \leq \limsup_{\delta_0 \to 0} m(Z(\delta_0)) \leq \frac{l(\int \log J_l^* dm + (\lambda + 4\epsilon)(\dim N + \epsilon))}{D}. \]

By letting \( D \) tend to infinity, we conclude that \( m(Z) = 0 \), and consequently \( m(G) = 1 \).

Now notice that for \( m' \)-almost every \( (x, \xi) \) we have
\[ \limsup \frac{1}{n} \log m_{f^{nl}(x)}(B(F^{nl}(x, \xi), \delta_l(x, \xi; n))) \]
\[ \leq \limsup \frac{1}{n} \sum_{j=0}^{n-1} \log J_l(F^{jl}(x, \xi), \delta_l(x, \xi; j)) \]
\[ \leq \int_{G(\delta_0)} (\log J_l + 4\epsilon) dm' + \int_{G(\delta_0)^c} \log J_l^* dm' \]
\[ \leq \int \log J_l dm' + 5\epsilon. \]
The last inequality follows from (A.8), \( G(\delta_2) \subset G(\delta_0) \) and (A.7). Combining the above inequality with (A.9) and (A.6) in Lemma A.1, we obtain

\[
h(F, m) - 5\epsilon \leq (\lambda + 4\epsilon)(\dim N + \epsilon).
\]

Since \( \epsilon \) is arbitrary, we conclude the proof of Proposition A.1. \( \square \)

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