Certain Class of Analytic Functions Based on \( q \)-difference operator

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Abstract

In this paper, we considered a generalized class of star-like functions defined by Kanas and Răducanu\cite{4} to obtain integral means inequalities and subordination results. Further, we obtain the for various subclasses of starlike functions.

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1 Introduction

Let \( \mathcal{A} \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open disc \( \mathbb{U} = \{z : |z| < 1\} \). Also denote by \( \mathcal{T} \) a subclass of \( \mathcal{A} \) consisting functions of the form

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad z \in \mathbb{U}
\]
introduced and studied new subclasses namely, the class of starlike functions of order $\alpha$ denoted by $T^*(\alpha)$ and convex functions of order $\alpha$ denoted by $K(\alpha)$ of $\mathcal{T}$ in [7].

Now, we refer to a notion of $q$-operators i.e. $q$-difference operator that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of $q$-calculus was initiated by Jackson [3] (also see [1, 6, 4]). Kanas and Răducanu [4] have used the fractional $q$-calculus operators in investigations of certain classes of functions which are analytic in $\mathbb{U}$.

For $0 < q < 1$ the Jackson’s $q$-derivative of a function $f \in \mathcal{A}$ is, by definition, given as follows [3]

$$D_q f(z) = \begin{cases} 
\frac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0, \\
\frac{f'(0)}{1-q} & \text{for } z = 0,
\end{cases}$$

(1.3)

and $D_q^2 f(z) = D_q(D_q f(z))$. From (1.3), we have $D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]a_n z^{n-1}$, where

$$[n] = \frac{1 - q^n}{1 - q},$$

(1.4)

is sometimes called the basic number $n$. If $q \to 1^-$, $[n] \to n$. For a function $h(z) = z^m$, we obtain $D_q h(z) = D_q z^m = \frac{1 - q^m}{1 - q} z^{m-1} = [m] z^{m-1}$, and $\lim_{q \to 1^-} D_q h(z) = \lim_{q \to 1^-} ([m] z^{m-1}) = m z^{m-1} = h'(z)$, where $h'$ is the ordinary derivative. Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Let the $q$-generalized Pochhammer symbol be defined as $[t]_n = [t][t+1][t+2]...[t+n-1]$ and for $t > 0$ let the $q$-gamma function be defined by

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t), \quad \Gamma_q(1) = 1.$$ 

(1.5)

Now we recall the definition of Ruscheweyh $q$-differential operator defined and discussed by Kanna and Răducanu[4].

**Definition 1.1.** [4] For $f \in \mathcal{A}$ let the Ruscheweyh $q$-differential operator be

$$R^\lambda_q f(z) = f(z) * F_{q,\lambda+1}(z) \quad (z \in \mathbb{U}, \lambda > 1) \quad (1.6)$$
where
\[ F_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)} z^n = z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{n-1}}{[n-1]!} z^n \]  
and * stands for the Hadamard product (or convolution).

From (1.6) we note that
\[ R_{q}^{0} f(z) = f(z); \quad R_{q}^{1} f(z) = z D_{q} f(z) \]
and
\[ R_{q}^{1} f(z) = z \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)} a_n z^{n-1} \]
(1.7)

Making use of (1.6) and (1.7), we have
\[ R_{q}^{\lambda} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)} a_n z^n \quad (z \in \mathbb{U}) \]  
(1.8)

Note that as \( \lim_{q \to 1} \) we get
\[ F_{q,\lambda+1}(z) = z (1 - z)^{\lambda+1} \quad \text{and} \quad R_{q}^{\lambda} f(z) = f(z) * \frac{z}{(1 - z)^{\lambda+1}}. \]

As a consequence of (1.3), for \( f \in \mathcal{A} \) we obtain
\[ D_{q}(R_{q}^{\lambda} f(z)) = 1 + \sum_{n=2}^{\infty} \frac{[n][\lambda]}{[n-1]!\Gamma_q(1+\lambda)} a_n z^{n-1}. \]  
(1.9)

where
\[ \Psi_q(n, \lambda) = \frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)} \]  
(1.10)

Now we recall the following definition of the function class \( ST_{q}^{\lambda}(k, \alpha) \) and its characterization property due to Kanas and Raducanu [4].

For \( 0 \leq \alpha < 1, \ k \geq 0 \) and \( \lambda \geq -1 \), let \( ST_{q}^{\lambda}(k, \alpha) \) be the subclass of \( \mathcal{A} \) consisting of functions of the form (1.1) and satisfying the analytic criterion
\[ \Re \left( \frac{z D_{q}(R_{q}^{\lambda} f(z))}{R_{q}^{\lambda} f(z)} - \alpha \right) > k \left| \frac{z D_{q}(R_{q}^{\lambda} f(z))}{R_{q}^{\lambda} f(z)} - 1 \right|, \quad z \in \mathbb{U}, \]  
(1.11)

where \( D_{q}(R_{q}^{\lambda} f(z)) \) is given by (1.9).

**Theorem 1.1.** [4] Let \( f(z) \in \mathcal{A} \) of the form (1.1). If the equality
\[ \sum_{n=2}^{\infty} ([n](1+k) - k - \alpha) \Psi_q(n, \lambda) |a_n| \leq 1 - \alpha, \]  
(1.12)
holds true for some $0 \leq k < \infty$, $\lambda \geq -1$, $0 \leq \alpha < 1$, and $\Psi_q(n, \lambda)$ is given by (1.10), then $f \in \mathcal{ST}_q^\lambda(k, \alpha)$. The result is sharp for the function

$$f_n(z) = z - \frac{1 - \alpha}{\Phi_n(\lambda, \alpha, k)} z^n,$$

(1.13)

where

$$\Phi_n(\lambda, \alpha, k) = ([n](1 + k) - k - \alpha) \Psi_q(n, \lambda)$$

(1.14)

and $\Psi_q(n, \lambda)$ given by (1.10).

It is interest to note that in [4], Kannas and Raducanu, state that the condition (1.12) is also necessary for functions $f_n$ given by (1.13) is in $\mathcal{ST}_q^\lambda(k, \alpha)$.

Making use of the above necessary and sufficient conditions for $f \in \mathcal{ST}_q^\lambda(k, \alpha)$, in this paper, we obtain integral means inequalities and subordination results.

## 2 Integral Means Inequalities

**Definition 2.1.** (Subordination Principle) For analytic functions $g$ and $h$ with $g(0) = h(0)$, $g$ is said to be subordinate to $h$, denoted by $g \prec h$, if there exists an analytic function $w$ such that $w(0) = 0$, $|w(z)| < 1$ and $g(z) = h(w(z))$, for all $z \in \mathbb{U}$.

**Lemma 2.1.** [5] If the functions $f$ and $g$ are analytic in $\mathbb{U}$ with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,

$$\int_0^{2\pi} |g(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta.$$  

(2.1)

In [7], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$ and applied this function to resolve his integral means inequality, conjectured in [8] and settled in [9], that

$$\int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\eta d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$. In [9], Silverman also proved his conjecture for the subclasses $\mathcal{T}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of $\mathcal{T}$. 

4
Applying Lemma 2.1 and Theorem 1.1, we prove the following result.

**Theorem 2.1.** Suppose \( f \in ST_\eta^{\lambda}(k, \alpha) \), \( \eta > 0 \), \( 0 \leq k < \infty \), \( \lambda \geq -1 \), \( 0 \leq \alpha < 1 \), and \( f_2(z) \) is defined by
\[
f_2(z) = z - \frac{1 - \alpha}{\Phi_2(\lambda, \alpha, k)} z^2,
\]
where
\[
\Phi_2(\lambda, \alpha, k) = \left( [2](1 + k) - k - \alpha \right) \Psi_q(2, \lambda) \tag{2.2}
\]
and from (1.10), \( \Psi_q(2, \lambda) \) is given by
\[
\Psi_q(2, \lambda) = \frac{\Gamma_q(2 + \lambda)}{[2 - 1]! \Gamma_q(1 + \lambda)} \tag{2.3}
\].

Then for \( z = re^{i\theta}, 0 < r < 1 \), we have
\[
\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta. \tag{2.4}
\]

**Proof.** For \( f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \), (2.4) is equivalent to proving that
\[
\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \right|^{\eta} d\theta \leq \int_0^{2\pi} \left| 1 - \frac{1 - \alpha}{\Phi_2(\lambda, \alpha, k)} z \right|^{\eta} d\theta.
\]

By Lemma 2.1, it suffices to show that
\[
1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1 - \alpha}{\Phi_2(\lambda, \alpha, k)} z.
\]

Setting
\[
1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1 - \alpha}{\Phi_n(\lambda, \alpha, k)} w(z), \tag{2.5}
\]
and using (1.12), we obtain \( w(z) \) is analytic in \( U \), \( w(0) = 0 \), and
\[
|w(z)| \leq \sum_{n=2}^{\infty} \frac{\Phi_2(\lambda, \alpha, k)}{1 - \alpha} |a_n| |z|^{n-1} \leq |z| \sum_{n=2}^{\infty} \frac{\Phi_n(\lambda, \alpha, k)}{1 - \alpha} |a_n| \leq |z|,
\]
where \( \Phi_n(\lambda, \alpha, k) \) is as given in (1.14). This completes the proof by Theorem 2.1.
3 Subordination Results

Following Frasin [2] and Singh [10], we obtain subordination results for the new class \( \mathcal{ST}^\lambda_q(k, \alpha) \).

**Definition 3.1.** (Subordinating Factor Sequence) A sequence \( \{b_n\}_{n=1}^\infty \) of complex numbers is said to be a subordinating sequence if, whenever \( f(z) = \sum_{n=1}^\infty a_n z^n, \ a_1 = 1 \) is regular, univalent and convex in \( U \), we have
\[
\sum_{n=1}^\infty b_n a_n z^n \prec f(z), \ z \in U. \tag{3.1}
\]

**Lemma 3.1.** [11] The sequence \( \{b_n\}_{n=1}^\infty \) is a subordinating factor sequence if and only if
\[
\Re \left( 1 + 2 \sum_{n=1}^\infty b_n z^n \right) > 0, \ z \in U. \tag{3.2}
\]

**Theorem 3.1.** Let \( f \in \mathcal{ST}^\lambda_q(k, \alpha) \) and \( g(z) \) be any function in the usual class of convex functions \( CV \), then
\[
\Phi_2(\lambda, \alpha, k) \frac{2}{2[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} (f * g)(z) \prec g(z) \tag{3.3}
\]
where \( 0 \leq k < \infty, \lambda \geq -1, \ 0 \leq \alpha < 1 \), with \( \Psi_q(2, \lambda) \) given in (2.3)
\[
\Re (f(z)) > -\frac{1 - \alpha + \Phi_2(\lambda, \alpha, k)}{\Phi_2(\lambda, \alpha, k)}, \ z \in U. \tag{3.4}
\]
The constant \( \frac{\Phi_2(\lambda, \alpha, k)}{2[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} \) is the best estimate.

**Proof.** Let \( f \in \mathcal{ST}^\lambda_q(k, \alpha) \) and suppose that \( g(z) = z + \sum_{n=2}^\infty c_n z^n \in CV \). Then
\[
\Phi_2(\lambda, \alpha, k) \frac{2}{2[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} (f * g)(z)
\]
\[
= \Phi_2(\lambda, \alpha, k) \frac{2}{2[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} \left( z + \sum_{n=2}^\infty c_n a_n z^n \right). \tag{3.5}
\]
Thus, by Definition 3.1, the subordination result holds true if
\[
\left( \frac{\Phi_2(\lambda, \alpha, k)}{2[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} \right)_{n=1}^\infty
\]
is a subordinating factor sequence, with \( a_1 = 1 \). In view of Lemma 3.1, this is equivalent to the following inequality
\[
\Re \left( 1 + \sum_{n=1}^\infty \frac{\Phi_2(\lambda, \alpha, k)}{[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} a_n z^n \right) > 0, \quad z \in \mathbb{U}. \tag{3.6}
\]
Now, for \(|z| = r < 1\), we have
\[
\Re \left( 1 + \frac{\Phi_2(\lambda, \alpha, k)}{[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} \sum_{n=1}^\infty a_n z^n \right)
= \Re \left( 1 + \frac{\Phi_2(\lambda, \alpha, k)}{[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} z + \frac{\sum_{n=2}^\infty \Phi_2(\lambda, \alpha, k)a_n z^n}{[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} \right)
\geq 1 - \frac{\Phi_2(\lambda, \alpha, k)}{[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} r - \frac{\sum_{n=2}^\infty \Phi_n(\lambda, \alpha, k)a_n r^n}{[1 - \alpha + \Phi_2(\lambda, \alpha, k)]}
= 1 - \frac{\Phi_2(\lambda, \alpha, k)}{[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} r - \frac{\sum_{n=2}^\infty \Phi_n(\lambda, \alpha, k)a_n r^n}{[1 - \alpha + \Phi_2(\lambda, \alpha, k)]}
> 0,
\]
and by noting the fact that \( \Phi_n(\lambda, \alpha, k) \) is increasing function for \( n \geq 2 \). Thus we have also made use of the assertion (1.12) of Theorem 1.1. This evidently proves the inequality (3.6) and hence also the subordination result (3.3) asserted by Theorem 3.1. The inequality (3.4) follows from (3.3) by taking \( g(z) = \frac{z}{1 - z} = z + \sum_{n=2}^\infty z^n \in CV \). Next we consider the function \( F(z) := z - \frac{1 - \alpha}{\Phi_2(\lambda, \alpha, k)} z^2 \) where \( 0 \leq k < \infty, \lambda \geq -1, 0 \leq \alpha < 1 \), and \( \Psi_q(2, \lambda) \) is given by (2.3). Clearly \( F \in ST^\lambda_q(k, \alpha) \). For this function (3.3) becomes
\[
\frac{\Phi_2(\lambda, \alpha, k)}{2[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} F(z) \prec \frac{z}{1 - z}.
\]
It is easily verified that
\[
\min \left\{ \Re \left( \frac{\Phi_2(\lambda, \alpha, k)}{2[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} F(z) \right) \right\} = -\frac{1}{2}, \quad z \in \mathbb{U}.
\]
This shows that the constant \( \frac{\Phi_2(\lambda, \alpha, k)}{2[1 - \alpha + \Phi_2(\lambda, \alpha, k)]} \) is best possible. \(\square\)
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