RATIONAL MAPS WITHOUT HERMAN RINGS

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ABSTRACT. We prove that a rational map has no Herman ring if it has at most one infinite critical orbit in its Julia set eventually. This criterion derives some known results about the rational maps without Herman rings.

1. Introduction

Let $f$ be a rational map with degree at least two. The Fatou set $F(f)$ of $f$ is defined as the maximal open subset on the Riemann sphere $\mathbb{C}$ in which the sequence of iterations $\{f^n\}_{n \geq 0}$ is normal in the sense of Montel. The Julia set $J(f) := \mathbb{C} \setminus F(f)$ of $f$ is the complement of the Fatou set in $\mathbb{C}$. Each connected component of the Fatou set is called a Fatou component. A Fatou component $U$ is called periodic if there exists an integer $n \geq 1$ such that $f^n(U) = U$. According to Sullivan [Sul], there are five types of periodic Fatou components: attracting basin, superattracting basin, parabolic basin, Siegel disk and Herman ring. Moreover, all Fatou components are iterated to one of these five types periodic Fatou components eventually.

Nowadays, it is relatively easy to construct a family of rational maps such that each one of them has an attracting basin, superattracting basin, parabolic basin or Siegel disk with fixed multiplier since these four types of periodic Fatou components has an associate periodic point (attracting, superattracting or neutral). However, for the case of Herman ring, because there exists no associate periodic point, it is difficult to judge whether a given rational map (or family) has a Herman ring or not.

In [Lyu, §2.4], Lyubich proposed a question: Does there exist a parameter $\omega \in \mathbb{C} \setminus \{0\}$ such that $f_\omega(z) = 1 + \omega/z^2$ has a Herman ring? Later, Shishikura proved that a rational map with degree less than three has no Herman ring [Sh1, Theorem 3] and hence answered Lyubich’s question. The method of Shishikura is analytical and profound. A completely topological method of proving

$$f_\omega(z) = 1 + \omega/z^d \quad (1.1)$$

has no Herman ring was established by Bamón and Bobenrieth, where $d \geq 2$ and $\omega \in \mathbb{C} \setminus \{0\}$ [BB] (See the right picture in Figure 2 for its parameter space). The dynamics of McMullen maps

$$f_\lambda(z) = z^m + \lambda/z^d \quad (1.2)$$

have been studied extensively recently (see [DLU, Ste, QWY] for example), where $m, d \geq 2$. Recently, Xiao and Qiu proved that the McMullen maps have no Herman rings [XQ].

The forward orbit of $z \in \overline{\mathbb{C}}$ under $f$ is defined as $O_f^+(z) := \{f^n(z)\}_{n \geq 0}$. The dynamics of a rational map is dominated by its (forward) critical orbits $\bigcup_{c \in \text{Crit}(f)} O_f^+(c)$, where $\text{Crit}(f)$ is the set of critical points of $f$. The boundary of the Herman ring is accumulated by at least one of the critical orbits in the Julia set [Mi2, Lemma 15.7]. One can believe that
a rational map has no Herman ring provided it has relatively simple critical orbits. The forward orbit $O^+_f(z)$ is called infinite if the cardinal number of $O^+_f(z)$ is equal to $+\infty$. In this paper, we prove the following main theorem.

**Main Theorem.** Let $f$ be a rational map with degree at least two. If there exists at most one infinite critical orbit in the Julia set of $f$ eventually, then $f$ has no Herman ring.

Here ‘eventually’ means that if $c_1, c_2 \in J(f)$ are critical points with infinite forward orbits, then $O^+_f(c_1) \cap O^+_f(c_2) \neq \emptyset$.

There exist some other criterions to judge the non-existence of Herman rings of rational maps. For example, in [Mi1, Appendix A], Milnor proved that a rational map with only two critical points cannot have any Herman rings. Although some similar criterions as stated in the Main Theorem may be known for some experts in this field, we would like to mention that there are no restrictions on the number of critical points in our criterion.

Now we can give several applications of the Main Theorem. Note that the set of critical points of $f_\omega(z) = 1 + \omega/z^d$ is $\{0, \infty\}$, where $d \geq 2$. Then $f_\omega$ has only one critical orbit $0 \mapsto \infty \mapsto 1 \mapsto 1 + \omega \mapsto \cdots$ eventually. As an immediate corollary of the Main Theorem, we have following result.

**Corollary 1.1** (Bamón and Bobenrieth). The rational maps $f_\omega(z) = 1 + \omega/z^d$ have no Herman rings, where $d \geq 2$ and $\omega \in \mathbb{C} \setminus \{0\}$.

Let $P$ be a non-constant cubic complex polynomial. The rational map $N_P : \mathbb{C} \to \mathbb{C}$ defined as

$$N_P : z \mapsto z - P(z)/P'(z)$$

is called *Newton’s method* for $P$. The iteration of $N_P$ is used to find the roots of $P$. Except $\infty$, all fixed points (three in all) of $N_P$ are superattracting and they are exactly the roots of $P$. Since each attracting periodic orbit attracts at least one critical orbit [Mi2, Theorem 8.6], there exists at most one infinite critical orbit in the Julia set of $P$. Therefore, we have following corollary.

**Corollary 1.2** (Shishikura-Tan). The Newton’s method for a cubic polynomial has no Herman ring.

Actually, Shishikura proved that the Julia set of the Newton’s method for every polynomial is connected and hence they have no Herman rings [Sh3, Corollary II] and Tan proved the case of degree three in [Tan, Proposition 2.6].

This paper is organized as follows: In §2, we first show that a rational map with exactly one infinite critical orbit eventually cannot have fixed Herman ring. Then we show this rational map cannot have Herman ring with period greater than one and complete the proof of the Main Theorem. In §3, we show that there are lots of other rational maps satisfying the condition in the Main theorem, including the holomorphic family of rational maps with exactly one free critical point. In §4, we give another proof that the McMullen maps have no Herman rings by using the ideas in the proof of the Main Theorem.

## 2. Proof of the Main Theorem

The proof of the Main Theorem is based on quasiconformal surgery. One can refer [Sh1, Sh2] for the basic ideas.

Let $f$ be a rational map with degree at least two. It is known that the boundary of the Herman ring has two connected components and each of them is an infinite set. If all the critical orbit in $J(f)$ is finite, then $J(f)$ has no Herman ring since the boundary of Herman
ring is contained in the closure of the critical orbit in \( J(f) \). Therefore, in the following, we always assume that \( f \) has exactly one infinite critical orbit in its Julia set eventually. Recall that ‘eventually’ introduced in the introduction means that if \( c_1, c_2 \in J(f) \) are critical points with infinite orbits, then \( O_j^+(c_1) \cap O_j^+(c_2) \neq \emptyset \). Or in other words, \( c_1 \) and \( c_2 \) have the same grand orbits.

For a Jordan curve \( \gamma \subset \overline{\mathbb{C}} \setminus \{ \infty \} \), we use \( \gamma_{\text{ext}} \) to denote the component of \( \overline{\mathbb{C}} \setminus \gamma \) containing \( \infty \) and \( \gamma_{\text{int}} \) the other. A continuous map \( f : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \) is called quasiregular if it can be written as \( f = \varphi \circ g \circ \psi \), where \( g \) is rational and \( \varphi, \psi \) are both quasiconformal. Let \( T_a := \{ z : |z| = s \} \) be the circle centered at the origin with radius \( s > 0 \) and \( A_r := \{ z : 0 < r < |z| < 1 \} \) the annulus with inner radius \( r \).

**Theorem 2.1.** Let \( f \) be a rational map with exactly one infinite critical orbit in \( J(f) \) eventually, then \( f \) has no Herman ring with period one.

**Proof.** We give the proof by contradiction. Let \( U \) be a fixed Herman ring of \( f \). After making a rotation on the Riemann sphere, we assume that \( \infty \not\in \overline{U} \). By the definition of Herman ring, there exists a conformal map \( \varphi : U \to \mathbb{A}_r \) and an irrational number \( \theta \in (0, 1) \) such that \( \varphi \circ f \circ \varphi^{-1}(z) = e^{2\pi i \theta} z \). Let \( \eta = \varphi^{-1}(T_a) \) and \( \gamma = \varphi^{-1}(T_b) \), where \( r < a < b < 1 \). It is easy to see \( \gamma, \eta \subset \overline{\mathbb{C}} \) are both analytic curves.

The conformal map \( \varphi : \gamma_{\text{int}} \cap \eta_{\text{ext}} \to \{ z : a < |z| < b \} \) between these two annuli can be extended to a quasiconformal mapping defined from \( \gamma_{\text{int}} \) to \( \{ z : |z| < b \} \), which we denote also by \( \varphi \). Define

\[
g(z) = \begin{cases} f(z) & \text{if } z \in \overline{\mathbb{C}} \setminus \gamma_{\text{int}}, \\ \varphi^{-1}(e^{2\pi i \theta} \varphi(z)) & \text{if } z \in \gamma_{\text{int}}.
\end{cases}
\]

It is easy to see \( g \) is continuous on \( \gamma \) and hence quasiregular. Let \( \sigma_0 \) be the standard complex structure on \( \overline{\mathbb{C}} \). We use \( \sigma \) to denote the \( g \)-invariant complex structure such that

\[
\sigma = \begin{cases} \varphi^* \sigma_0 & \text{in } \gamma_{\text{int}}, \\ \sigma_0 & \text{outside of } \bigcup_{n \geq 0} g^{-n}(\gamma_{\text{int}}).
\end{cases}
\]

By the Measurable Riemann Mapping Theorem, there exists a map \( h \) integrating the almost complex structure \( \sigma \) such that \( h^* \sigma_0 = \sigma \) and \( h : (\overline{\mathbb{C}}, \sigma) \to (\overline{\mathbb{C}}, \sigma_0) \) is an analytic isomorphism. This means that \( F := h \circ g \circ h^{-1} \) is a rational map.

Note that the conformal map \( \varphi \circ h^{-1} : h(\overline{U \setminus \gamma_{\text{int}}}) \to \{ z : b < |z| < 1 \} \) conjugates \( F \) to the rigid rotation \( z \mapsto e^{2\pi i \theta} z \). It follows that \( h(U \setminus \gamma_{\text{int}}) \) is contained in a Siegel disk or Herman ring of \( F \). Since \( F(h(\overline{\gamma_{\text{int}}})) = h(\overline{\gamma_{\text{int}}}) \), it means that \( \{ F^n \}_{n \geq 0} \) is normal on \( h(\overline{\gamma_{\text{int}}}) \). Therefore, the Fatou component containing \( h(U \cup \gamma_{\text{int}}) \) is contained in a fixed Siegel disk of \( F \). In fact, by the construction of \( g \), it is easy to see \( h(U \cup \gamma_{\text{int}}) \) is just the Siegel disk of \( F \).

The images of the critical points in the Fatou set of \( f \) under \( h \) are still in the Fatou set of \( F \) (Note that the images of some critical points of \( f \) under \( h \) are not critical points of \( F \) any more because of the surgery). This means that there exists at most one infinite critical orbit in \( J(F) \) eventually. Since \( F \) has a Siegel disk, it follows that \( F \) has exactly one infinite critical orbit in \( J(F) \) eventually. Moreover, there exists a critical point \( c' \) of \( F \) with infinite orbit in \( J(F) \) such that \( c' = h(c) \), where \( c \) is a critical point of \( f \) with infinite orbit in \( J(f) \).

Note that \( \partial U \subset \overline{O_j^+(c)} \). There exists a smallest \( k > 0 \) such that \( f^{ok}(c) \in \gamma_{\text{int}} \). Then we have

\[
F^{ok}(c') = F^{ok} \circ h(c) = h \circ g^{ok}(c) = h \circ f^{ok}(c).
\]
However, $F^{ok}(c')$ lies in the Julia set of $F$ while $h \circ F^{ok}(c) \in h(\gamma_{int})$ lies in a Siegel disk of $F$. This is a contradiction. The proof is complete. \hfill \Box

Let $A \subset \overline{\mathbb{C}}$ be an annulus such that $\infty \notin \overline{A}$. We use $\partial_+ A$ and $\partial_- A$ to denote its outer and inner boundary components, respectively. Define the topological disks $A^{\text{ext}} := \gamma_{\text{ext}}$ and $A^{\text{int}} := \gamma_{\text{int}}$, where $\gamma$ is the core curve of $A$. Recall that the core curve of an annulus $A$ is defined as $\varphi^{-1}(\sqrt{r})$, where $\varphi : A \to \mathbb{A}_r$ is the conformal isomorphism. From the proof of Theorem 2.1, we have following two corollaries.

**Corollary 2.2.** Let $U$ be a fixed Herman ring of $f$ such that $\infty \notin \overline{U}$. Then $\partial U$ is contained in the closure of at least two disjoint infinite critical orbits. In particular, there exist two critical points $c_1 \in U^{\text{ext}} \cap J(f)$ and $c_2 \in U^{\text{int}} \cap J(f)$ with different grand orbits, such that $\partial_+ U \subset \overline{O_f^+(c_1)}$ and $\overline{O_f^+(c_1)} \cap U^{\text{int}} = \emptyset$, $\partial_- U \subset \overline{O_f^+(c_2)}$ and $\overline{O_f^+(c_2)} \cap U^{\text{ext}} = \emptyset$.

**Corollary 2.3.** Let $U_1$ and $U_2$ be two fixed Herman ring of $f$ such that $\infty \notin U_1^{\text{int}} \cup U_2^{\text{int}}$. Then there exists a critical point $c \in U_1^{\text{ext}} \cap U_2^{\text{int}} \cap J(f)$, such that $\partial_- U_2 \subset \overline{O_f^+(c)}$ (or $\partial_+ U_1 \subset \overline{O_f^-(c)}$) and $\overline{O_f^+(c)} \cap (U_1^{\text{int}} \cup U_2^{\text{ext}}) = \emptyset$.

If $f$ has a Herman ring with period $p > 1$, we cannot use the surgery showed in Theorem 2.1 directly. However, we can use Corollaries 2.2 and 2.3 after considering the iteration $f^{p}$. Note that the number of disjoint infinite critical orbits of $f^{p}$ in $J(f^{p}) = J(f)$ is greater than one.

**Theorem 2.4.** Let $f$ be a rational map with exactly one infinite critical orbit in $J(f)$ eventually, then $f$ has no Herman ring with period $p > 1$.

**Proof.** Suppose that $f$ has a periodic orbit of Herman rings $U_0 \mapsto U_1 \mapsto \cdots \mapsto U_{p-1} \mapsto U_0$ with period $p > 1$. After making a rotation on $\mathbb{C}$, we assume that $\infty \notin \bigcup_{i=0}^{p-1} U_i$. Consider the iteration of $f$, then each $U_i$ is fixed by $f^{p}$. By the definition, there exists a conformal map $\varphi_i : U_i \to \mathbb{A}_r$ and an irrational number $\theta \in (0, 1)$ such that $\varphi_i \circ f^{p} \circ \varphi_i^{-1}(z) = e^{2\pi i \theta} z$, where $0 \leq i < p$.

Let $E_0, E_1, \ldots, E_q$ be the sequence of disjoint nonempty open sets, such that

1. $E_0 \cup E_1 \cup \cdots \cup E_q = \bigcup_{i=0}^{p-1} U_i$;
2. $E_0 = U_0$, $E_j = \bigcup_{1 \leq k \leq j} U_{j_k}$, where $0 < j_k < p$ and $j_k' \neq j_k''$ if $k' \neq k''$; and
3. any component $U_{j_{k+1}}$ of $E_{j+1}$ is nested inside some component $U_{j_k'}$ of $E_j$, where $0 \leq j < q$.

This means that the union of periodic components of Herman rings $\bigcup_{i=0}^{p-1} U_i$ has been divided into $q + 1$ levels and $1 + \sum_{j=1}^{q} l_j = p$ (see Figure 1).

By Corollary 2.2, there exists a critical point of $f^{p}$ in $J(f)$, say $c_{-1} \in U_0^{\text{ext}}$, such that $\partial_+ U_0 \subset \overline{O_{f^p}(c_{-1})}$ and $\overline{O_{f^p}(c_{-1})} \cap U_0^{\text{int}} = \emptyset$. In particular, $\overline{O_{f^p}(c_{-1})} \cap \bigcup_{0 \leq i < p} U_i = \emptyset$. By Corollary 2.3, there exists a critical point of $f^{p}$ in $J(f)$, say $c_0 \in U_0^{\text{int}}$, such that $\partial_- U_0 \subset \overline{O_{f^p}(c_0)}$ and $\overline{O_{f^p}(c_0)} \cap \bigcup_{1 \leq k \leq l_1} U_k^{\text{int}} = \emptyset$. In particular, $\overline{O_{f^p}(c_0)} \cap \bigcup_{0 \leq i < p} \partial_- U_i = \emptyset$.

Similarly, there exists a critical point of $f^{p}$ in $J(f)$, say $c_1 \in U_{1_k}^{\text{int}}$, where $1 \leq k \leq l_1$, such that $\partial_- U_{1_k} \subset \overline{O_{f^p}(c_{1_k})}$ and $\overline{O_{f^p}(c_{1_k})} \cap \bigcup_{1 \leq k \leq l_1} U_k^{\text{int}} = \emptyset$ and $\overline{O_{f^p}(c_{1_k})} \cap \bigcup_{1 \leq k \leq l_2} U_k^{\text{int}} = \emptyset$ if $q \geq 2$. In particular, $\overline{O_{f^p}(c_{1_k})} \cap \bigcup_{2 \leq j \leq q} \bigcup_{1 \leq k' \leq l_j} \partial_- U_{j_k'} = \emptyset$ if $q \geq 2$.

Induce on the level of $E_j$, it follows that there needs at least $2 + \sum_{j=1}^{q} l_j = p + 1$ critical points $c_{-1}, c_0, c_1, \ldots, c_{p-1}$ of $f^{p}$ with disjoint infinite critical orbits in $J(f)$ such that $\bigcup_{i=0}^{p-1} \partial U_i \subset \bigcup_{j=1}^{p-1} \overline{O_{f^p}(c_j)}$. 
However, \( f^{op} \) has exactly \( p \) infinite critical orbits in \( J(f) \) eventually since \( f \) has exactly one. This is a contradiction. The proof is complete. \( \square \)

**Remark 2.5.** As far as we know, the integer \( q \) appeared in the proof of 2.4 is always equal to 1. We don’t know the examples of periodic Herman rings such \( q > 1 \).

**Proof of the Main Theorem.** Combine Theorems 2.1 and 2.4. \( \square \)

### 3. Other examples having no Herman rings

In this section, we give some examples such that one can apply the Main Theorem. In complex dynamics, one may want to study the rational maps in family. Most of the time, the family has only one parameter such this family has only one free critical point (or essentially has one, such as McMullen maps) since it is convenient to define the corresponding mandelbrot set or connected locus (see Figure 2). More precisely, let \( f_\lambda : \Lambda \times \mathbb{C} \rightarrow \mathbb{C} \) be a holomorphic family of rational maps parameterized by \( \Lambda \). Assume further that each critical point of \( f_\lambda \) can be parametered by \( \lambda \). This family is called that it *has only one free critical point* if all but one critical points are constants and the non-constant critical point is called the *free critical point*.

**Theorem 3.1.** Suppose that a holomorphic family of rational maps \( f_\lambda : \Lambda \times \mathbb{C} \rightarrow \mathbb{C} \) has only one free critical point and all the non-free critical points have finite forward orbits. Then \( f_\lambda \) has no Herman ring for all \( \lambda \in \Lambda \).

**Proof.** This is the immediate conclusion of the Main Theorem. \( \square \)

We can apply Theorem 3.1 to prove that some holomorphic families of rational maps have no Herman rings. Let

\[
S_\lambda(z) = \left( \frac{(z + \lambda - 1)^d + (\lambda - 1)(z - 1)^d}{(z + \lambda - 1)^d - (z - 1)^d} \right)^d,
\]

where \( d \geq 2 \) is an integer and \( \lambda \neq 0 \) is a complex parameter. This family of rational maps are actually the renormalization transformation of the generalized diamond hierarchical Potts model [Qia]. The special cases named *standard diamond lattice* \( (d = 2) \) was first studied in [DSI].
Corollary 3.2. The renormalization transformation $S_\lambda$ has no Herman ring.

Proof. For every $\lambda \in \mathbb{C} \setminus \{0\}$, we have $S_\lambda = T_\lambda \circ T_\lambda$, where

$$T_\lambda(z) = \left(\frac{z + \lambda - 1}{z - 1}\right)^d.$$ 

A directly calculation shows that the set of all critical points of $T_\lambda$ is $\text{Crit}(T_\lambda) = \{1, 1 - \lambda\}$ and both with multiplicity $d - 1$. Under the iteration of $T_\lambda$, we have following critical orbits:

$$1 \mapsto \infty \mapsto 1 \mapsto \infty \mapsto \cdots \text{ and } 1 - \lambda \mapsto 0 \mapsto (1 - \lambda)^d \mapsto \cdots.$$ 

From the first orbit, we know that 1 lies in the Fatou set of $T_\lambda$. This means that there exists at most one infinite critical orbit in $J(T_\lambda) = J(S_\lambda)$. By the Main Theorem or Theorem 3.1, it follows that $T_\lambda$ and hence $S_\lambda$ has no Herman ring. \qed

4. The McMullen maps have no Herman rings

In this section, we give another proof of the nonexistence of Herman rings of McMullen maps by using the idea in the proof of Theorem 2.1. For each $m, d \geq 2$, let

$$f_\lambda(z) = z^m + \lambda/z^d$$

be the McMullen map with parameter $\lambda \in \mathbb{C} \setminus \{0\}$. The map $f_\lambda$ has a critical point at the origin with multiplicity $d - 1$, a critical point at $\infty$ with multiplicity $m - 1$, and a simple critical points at $\omega_j = \delta^j(\lambda d/m)^{1/(d+m)}$, where $\delta = e^{2\pi i/(d+m)}$ and $0 \leq j \leq d + m - 1$.

By a straightforward calculation, we have $f_\lambda(\delta z) = \delta^m f_\lambda(z)$. Therefore, $\{f_\lambda^n\}_{n \geq 0}$ is normal at $z$ if and only if it is normal at $\delta z$. This means that the Fatou set and hence Julia set of $f_\lambda$ are invariant under the rotation $z \mapsto \delta z$.

Since $f_\lambda$ has a critical orbit $0 \mapsto \infty \mapsto \infty \mapsto \cdots$ in the Fatou set of $f_\lambda$. In the following, we always suppose the orbit of $\omega_j$ is bounded for every $0 \leq j \leq d + m - 1$ since these $d + m$ critical points tend to infinity or have bounded orbits simultaneously.
Let $X$ be a subset of $\mathbb{C}$, we use $aX$ to denote the set $\{ax : x \in X\}$, where $a \in \mathbb{C} \setminus \{0\}$. Compare the following two orbits:

$$z \mapsto f_\lambda(z) \mapsto f_\lambda^2(z) \mapsto f_\lambda^3(z) \mapsto \cdots$$

and

$$\delta z \mapsto \delta^m f_\lambda(z) \mapsto \delta^m f_\lambda^2(z) \mapsto \delta^m f_\lambda^3(z) \mapsto \cdots.$$

It follows that they satisfy $O_f^+(\delta z) \subset \bigcup_{j=0}^{d+m-1} \delta^j O_f^+(z)$. Therefore, for every $0 \leq j \leq d + m - 1$, the closure $\overline{O_f^+(\delta z)}$ is contained in $\bigcup_{j=0}^{d+m-1} \delta^j O_f^+(z)$.

**Theorem 4.1** (Xiao and Qiu, [XQ]). The McMullen maps have no Herman rings.

**Proof.** Suppose that there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $f_\lambda$ has a periodic orbit of Herman rings $U_0 \mapsto U_1 \mapsto \cdots \mapsto U_{p-1} \mapsto U_p = U_0$ with period $p \geq 1$. They are contained in $\mathbb{C}$ since $\infty$ is superattracting.

Let $\gamma$ be a core curve of an annulus $A \subset \mathbb{C}$. Recall that the topological disk $A^\text{ext} := \gamma^\text{ext}$ is the component of $\mathbb{C} \setminus \gamma$ containing $\infty$ and $A^\text{int} := \gamma^\text{int}$ is another. Suppose that there exists $0 \leq i \leq p - 1$ such that $U_i$ surrounds the origin. Let $\gamma_i$ be its core curve. Then $\gamma_{i+1} = f_\lambda(\gamma_i)$ is the core curve of $U_{i+1}$. The image of $U_i^\text{int}$ under $f_\lambda$ covers $\mathbb{C}$ in $d - 1$ times and $U_{i+1}^\text{ext}$ in $d$ times since $f_\lambda$ maps $0$ to $\infty$ with degree $d$. Moreover, the image of $U_i^\text{ext}$ under $f_\lambda$ covers $\mathbb{C}$ in $m - 1$ times but covers $U_1^\text{ext}$ in $m$ times. This means that $U_i^\text{int}$ is covered only $m + d - 2$ times by $f_\lambda$, which is a contradiction. Therefore, 0 cannot be surrounded by any $U_i$.

Since $0 \in U_i^\text{ext}$ for every $0 \leq i \leq q - 1$, we have orbit $\partial_{\pm} U_0 \mapsto \partial_{\pm} U_1 \mapsto \cdots \mapsto \partial_{\pm} U_{p-1} \mapsto \partial_{\pm} U_0$. Note that this is not always correct for other rational maps (see [Sh1, Theorem 5(B)]). Suppose that $\bigcup_{i=0}^{p-1} \partial_i U_i \subset \overline{O_f^+(\omega_k)}$ for some critical point $\omega_k$. We claim that $\bigcup_{i=0}^{p-1} \partial_i U_i \cap \overline{O_f^+(\omega_k)} = \emptyset$. In fact, after replacing the dynamics in $\bigcup_{i=0}^{p-1} U_i^\text{int}$ by a periodic orbit of rotation domains and straightening the corresponding complex structure, we can obtain a rational map with a periodic cycle of Siegel disks with period $p$. If $\bigcup_{i=0}^{p-1} \partial_i U_i \cap \overline{O_f^+(\omega_k)} \neq \emptyset$, this will derive that $\omega_k$ is not a critical point of $f_\lambda$ by the same argument in the last part of Theorem 2.1. Similarly, we have $\overline{O_f^+(\omega_k)} \cap \partial U' = \emptyset$, where $U'$ is any other periodic Herman ring not in the cycle $U_0 \mapsto U_1 \mapsto \cdots \mapsto U_{p-1} \mapsto U_0$.

Note that each component of $\bigcup_{j=0}^{d+m-1} \bigcup_{i=0}^{p-1} \delta^i U_i$ is a periodic Herman ring since the Fatou set of $f_\lambda$ is invariant under the rotation $z \mapsto \delta z$. We have $\overline{O_f^+(\omega_k)} \cap \bigcup_{j=0}^{d+m-1} \bigcup_{i=0}^{p-1} \delta^i \partial U_i = \emptyset$. It follows that $\bigcup_{j=0}^{d+m-1} \delta^j O_f^+(\omega_k) \cap \bigcup_{j=0}^{d+m-1} \bigcup_{i=0}^{p-1} \delta^i \partial U_i = \emptyset$. On the other hand, $\bigcup_{j=0}^{d+m-1} \delta^j O_f^+(\omega_k) \subset \bigcup_{j=0}^{d+m-1} \delta^j \overline{O_f^+(\omega_k)}$. This means that the orbit $\partial U_0 \mapsto \partial U_1 \mapsto \cdots \mapsto \partial U_{p-1} \mapsto \partial U_0$ is disjoint with the closure of the critical orbits of $f_\lambda$, which is a contradiction. The proof is complete. □

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