IMPROVEMENT ON THE BLOW-UP OF THE WAVE EQUATION WITH THE SCALE-ININVARIANT DAMPING AND COMBINED NONLINEARITIES

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Abstract. We consider in this article the damped wave equation, in the scale-invariant case with combined two nonlinearities, which reads as follows:

\[
(E) \quad u_{tt} - \Delta u + \frac{\mu}{1+t}u_t = |u_t|^p + |u|^q, \quad \text{in } \mathbb{R}^N \times [0, \infty),
\]

with small initial data.

Compared to our previous work [8], we show in this article that the first hypothesis on the damping coefficient \(\mu\), namely \(\mu < \frac{N(q-1)}{2}\), can be removed, and the second one can be extended from \((0, \mu_*/2)\) to \((0, \mu_*)\) where \(\mu_* > 0\) is solution of \((q-1)((N+\mu_*)-1)p-2) = 4\). Indeed, owing to a better understanding of the influence of the damping term in the global dynamics of the solution, we think that this new interval for \(\mu\) characterizes better the threshold between the blow-up and the global existence regions. Moreover, taking advantage of the techniques employed in the problem \((E)\), we also improve the result in [17, 23] in relationship with the Glassey conjecture for the solution of \((E)\) without the nonlinear term \(|u|^q\). More precisely, we extend the blow-up region from \(p \in (1, p_G(N+\sigma)]\), where \(\sigma\) is given by (1.7) below, to \(p \in (1, p_G(N+\mu)]\) giving thus a better estimate of the lifespan in this case.

1. Introduction

We consider the following family of semilinear damped wave equations

\[
(1.1) \quad \begin{cases}
    u_{tt} - \Delta u + \frac{\mu}{1+t}u_t = a|u_t|^p + b|u|^q, & \text{in } \mathbb{R}^N \times [0, \infty), \\
    u(x, 0) = \varepsilon f(x), & u_t(x, 0) = \varepsilon g(x), & x \in \mathbb{R}^N,
\end{cases}
\]

where \(a\) and \(b\) are nonnegative constants and \(\mu \geq 0\). Moreover, the parameter \(\varepsilon\) is a positive number describing the size of the initial data, and \(f\) and \(g\) are positive functions which are compactly supported on \(B_{\mathbb{R}^N}(0, R), R > 0\).

Throughout this article, we suppose that \(p, q > 1\) and \(q \leq \frac{2N}{N-2}\) if \(N \geq 3\).

The corresponding linear equation to (1.1) is given by

\[
(1.2) \quad u_{tt}^L - \Delta u^L + \frac{\mu}{1+t}u_t^L = 0.
\]

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It is well-known that the equation (1.2) is invariant under the following transformation:
\[ \tilde{u}^L(x,t) = u^L(\Omega x, \Omega(1 + t) - 1), \ \Omega > 0. \]
The above scaling justifies the designation of the *scale-invariant* case for (1.1). It is interesting to recall that the scale-invariant case is the interface between the class of parabolic equations (for \( \mu \) large enough) and the one of hyperbolic equations (for small values of \( \mu \)). In fact, in this transition, the parameter \( \mu \) plays a crucial role in determining the behavior of the solution of (1.2), see for example [33].

Let \( \mu = 0 \) and \((a, b) = (0, 1)\) in (1.1), then the equation (1.1) is thus the classical semilinear wave equation for which we have the Strauss conjecture. This case is characterized by a critical power, denoted by \( q_S \), which is solution of the following quadratic equation
\[ (N - 1)q^2 - (N + 1)q - 2 = 0, \]
and is given by
\[ q_S = q_S(N) := \frac{N + 1 + \sqrt{N^2 + 10N - 7}}{2(N - 1)}. \]
More precisely, if \( q \leq q_S \) then there is no global solution for (1.1) under suitable sign assumptions for the initial data, and for \( q > q_S \) a global solution exists for small initial data; see e.g. [15, 27, 34, 35] among many other references.

Now, the case \( \mu = 0 \) and \((a, b) = (1, 0)\) is obeying to the Glassey conjecture which asserts that the critical power \( p_G \) should be given by
\[ p_G = p_G(N) := 1 + \frac{2}{N - 1}. \]
The above critical value, \( p_G \), gives rise to two regions for the power \( p \) ensuring the global existence (for \( p > p_G \)) or the nonexistence (for \( p \leq p_G \)) of a global small data solution; see e.g. [10, 12, 14, 25, 26, 30, 36].

It is worth-mentioning that the presence of two nonlinearities in (1.1) has an interesting effect on the existence or the nonexistence of global in-time solution of (1.1) and its lifespan. In fact, compared to a one single nonlinearity, the presence of mixed nonlinearities produces an additional new blow-up region.

First, we focus on the case \( \mu = 0 \) and \((a, b) \neq 0\), thus, without loss of generality we may assume that \((a, b) = (1, 1)\). It is easy to see that in this case, together with the assumption that the powers \( p \) and \( q \) satisfy \( p \leq p_G \) or \( q \leq q_S \), the blow-up of the solution of (1.1) can be handled in a similar way. Therefore, for \( p > p_G \) and \( q > q_S \), the new
blow-up border is characterized by the following relationship between \( p \) and \( q \):
\[
\lambda(p, q, N) := (q - 1) ((N - 1)p - 2) < 4. \tag{1.6}
\]
We refer the reader to \([5, 9, 11, 32]\) for more details.

Note that it is proven in \([11]\) that, for \( p > p_G \) and \( q > q_S \), the equality in (1.6) yields the global existence of the solution of (1.1) (with \( \mu = 0 \) and \( (a, b) = (1, 1) \)) without going through the intermediate step of “almost global solution”. This is in fact related to the presence of mixed nonlinearities. Naturally, it is interesting to see whether this phenomenon still occurs for the damping case \( \mu > 0 \).

Now, we focus on the case \( \mu > 0 \). In fact, for \( (a, b) = (0, 1) \), it is known in the literature that if the weak damping coefficient \( \mu \) is relatively large, then the equation (1.1) (with \( (a, b) = (0, 1) \)) behaves like the corresponding heat equation; see e.g. \([1, 2, 31]\). Though, if \( \mu \) is small, then the behavior of (1.1) is following the one of the corresponding wave equation. More precisely, for \( \mu \) small, it was proven, in \([19]\) and later on in \([13]\) with a substantial improvement, that the dimension in the critical power \( q_S \) is shifted by \( \mu > 0 \) compared to the one in the case without damping \( (\mu = 0) \), and hence we have for
\[
0 < \mu < \frac{N^2 + N + 2}{N + 2} \quad \text{and} \quad 1 < q \leq q_S(N + \mu),
\]
the blow-up of the solution of (1.1). These blow-up results have been improved in many ways in \([22, 23, 24, 28, 29]\).

In the particular case \( \mu = 2 \) and \( N = 2, 3 \), the same above observation is proven in \([4]\), see also \([3]\). For the global existence in this case \( (\mu = 2) \) we refer the reader to \([3, 4, 20]\).

On the other hand for \( \mu > 0 \) and \( (a, b) = (1, 0) \), the authors prove in \([17]\) a blow-up result for the solution of (1.1) (with \( (a, b) = (1, 0) \)) and they give an upper bound of the lifespan. We stress the fact that in this case there is no restriction for \( \mu \) in the blow-up region for \( p \), namely \( p \in (1, p_G(N + 2\mu)] \). Recently, Palmieri and Tu proved in \([23]\), among many other interesting results, a more accurate blow-up interval for \( p \) in relationship with the solution of (1.1) with \( (a, b) = (1, 0) \), one time-derivative nonlinearity (that is (1.9) below) and a mass term. More precisely, it is proven that the solution of this problem blows up in finite time for \( p \in (1, p_G(N + \sigma)] \) where
\[
\sigma = \begin{cases} 
2\mu & \text{if } \mu \in [0, 1), \\
2 & \text{if } \mu \in [1, 2), \\
\mu & \text{if } \mu \geq 2.
\end{cases} \tag{1.7}
\]
Of course the problems studied in \([23]\) are more general, but, we want to point out here the improvement obtained for (1.9) below.
In this article, we are interested in the study of the following Cauchy problem which is related to the scale-invariant wave equation with combined nonlinearities. More precisely, we consider

\begin{equation}
\begin{aligned}
&u_{tt} - \Delta u + \frac{\mu}{1 + t} u_t = |u_t|^p + |u|^q, \quad \text{in } \mathbb{R}^N \times [0, \infty), \\
&u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), \quad x \in \mathbb{R}^N,
\end{aligned}
\end{equation}

where \( \mu > 0, \ N \geq 1, \ \varepsilon > 0 \) is a sufficiently small parameter and \( f, g \) are positive functions chosen in the energy space with compact support.

The emphasis in our work is the study of the Cauchy problem (1.8) for \( \mu > 0 \) and the influence of the parameter \( \mu \) on the blow-up result and the lifespan estimate. For the analogous system of (1.8) with \( (\mu/(1 + t))u_t \) being replaced by \( (\mu/(1 + t)^\beta)u_t \) and \( \beta > 1 \), which corresponds to the scattering case, Lai and Takamura proved in [18] that, comparing to the wave equation without damping, the scattering damping term has no influence in the dynamics. However, in the scale-invariant case \( (\beta = 1) \) the situation is different. Indeed, as expected, the combination of a weak damping term and the two mixed nonlinearities is playing here a crucial role. This fact has been shown in our previous work [8] where the focus were on the obtaining of the equation of the hyperbola part for the blow-up region. In the present work, owing to a better comprehension of the role of the weak damping term in (1.8) in the dynamics and following the technique used in [29] (the method is based on the use of some test functions which closely describe the solution of the linear part of (1.8)), we will here improve the bound of the blow-up region in the hyperbola part. We precisely show in this article that (1.6) holds for \( \lambda(p, q, N + \mu) \) instead of \( \lambda(p, q, N + 2\mu) \) which was obtained in [8]. Hence, the new hypothesis on \( p \) and \( q \) \( (\lambda(p, q, N + \mu) < 4) \) constitutes a shift by \( \mu \) of the dimension \( N \). Naturally, we obtain a better bound for the lifespan. We face here a situation similar to the Strauss exponent case, \( q_S \), as explained above. However, thanks to a good choice of the functional as in (4.2) below, we succeed in this article to remove one of the smallness hypotheses on \( \mu \), namely \( \mu < \frac{N(q-1)}{2} \), which was assumed in [8, Theorem 2.3].

Furthermore, keeping in mind all the hypotheses as for (1.8), we consider here the following equation with only one time-derivative nonlinearity, namely

\begin{equation}
\begin{aligned}
&u_{tt} - \Delta u + \frac{\mu}{1 + t} u_t = |u_t|^p, \quad \text{in } \mathbb{R}^N \times [0, \infty), \\
&u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), \quad x \in \mathbb{R}^N.
\end{aligned}
\end{equation}

Taking advantage of the techniques used for (1.8), we will improve the blow-up interval,\( p \in (1, p_G(N + \sigma)) \) \( (\sigma \text{ is given by (1.7))}, \) obtained in [23], which is itself an improvement of [17], to reach the interval \( p \in (1, p_G(N + \mu)) \), for \( \mu \in (0, 2) \). However, for \( \mu \geq 2 \), our result for (1.9) coincides with the one in [23]. We may conjecture that the obtained upper bound exponent is the critical one in the sense that it gives the threshold between
the blow-up and the global existence regions.

The article is organized as follows. We start in Section 2 by introducing the weak formulation of (1.8) in the energy space. Then, we state the main theorems of our work. Some technical lemmas are thus proven in Section 3. These auxiliary results, among other tools, are used to conclude the proof of the main results in Sections 4 and 5. More precisely, in Section 4 (resp. Sec. 5), we prove that the solution of (1.8) (resp. (1.9)) for \( p \) and \( q \) satisfying \( \lambda(p, q, N + \mu) < 4 \) (resp. for \( p \) verifying \( p \in (1, p_G(N + \mu)) \)) blows up in finite time.

2. Main Results

This section is aimed to state our main results. For that purpose, we first start by giving the definition of the solution of (1.8) in the corresponding energy space. More precisely, the weak formulation associated with (1.8) reads as follows:

**Definition 2.1.** We say that \( u \) is a weak solution of (1.8) on \([0, T]\) if

\[
\begin{aligned}
\left\{ \begin{array}{ll}
u \in C([0, T), H^1(\mathbb{R}^N)) \cap C^1([0, T), L^2(\mathbb{R}^N)), \\
u 
\end{array} \right.
\end{aligned}
\]

satisfies, for all \( \Phi \in C^\infty_0(\mathbb{R}^N \times [0, T]) \) and all \( t \in [0, T] \), the following equation:

\[
\begin{aligned}
&\int_{\mathbb{R}^N} u_t(x,t)\Phi(x,t)dx - \int_{\mathbb{R}^N} u_t(x,0)\Phi(x,0)dx \\
&- \int_0^t \int_{\mathbb{R}^N} u_t(x,s)\Phi_t(x,s)dx
ds + \int_0^t \int_{\mathbb{R}^N} \nabla u(x,s) \cdot \nabla \Phi(x,s)dx
ds \\
&+ \int_0^t \int_{\mathbb{R}^N} \frac{\mu}{1+s} u_t(x,s)\Phi(x,s)dx
ds = \int_0^t \int_{\mathbb{R}^N} \{ |u_t(x,s)|^p + |u(x,s)|^q \} \Phi(x,s)dx
ds.
\end{aligned}
\]

Obviously, the weak formulation corresponding to (1.9) can be also given by (2.1) with simply ignoring the nonlinear term \( |u|^q \) with the necessary modifications accordingly.

The following theorems state the main results of this article.

**Theorem 2.2.** Let \( p, q \) and \( \mu > 0 \) such that

\[
\lambda(p, q, N + \mu) < 4,
\]

where the expression of \( \lambda \) is given by (1.6), and \( p > p_G(N + \mu) \) and \( q > q_S(N + \mu) \).

Assume that \( f \in H^1(\mathbb{R}^N) \) and \( g \in L^2(\mathbb{R}^N) \) are non-negative functions which are compactly supported on \( B_{R_N}(0, R) \), and do not vanish everywhere. Let \( u \) be an energy solution of (1.8) on \([0, T]\) such that \( \text{supp}(u) \subset \{(x, t) \in \mathbb{R}^N \times [0, \infty) : |x| \leq t + R\} \). Then, there exists a constant \( \varepsilon_0 = \varepsilon_0(f, g, N, R, p, q, \mu) > 0 \) such that \( T_\varepsilon \) verifies

\[
T_\varepsilon \leq C \varepsilon^{-\frac{2(p-1)}{2\lambda(p, q, N + \mu)}},
\]
where $C$ is a positive constant independent of $\varepsilon$ and $0 < \varepsilon \leq \varepsilon_0$.

**Theorem 2.3.** Let $\mu > 0$. Assume that $f \in H^1(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N)$ are non-negative functions which are compactly supported on $B_{\mathbb{R}^N}(0, R)$, and do not vanish everywhere. Let $u$ be an energy solution of (1.9) on $[0, T_\varepsilon)$ such that $\text{supp}(u) \subset \{(x, t) \in \mathbb{R}^N \times [0, \infty) : |x| \leq t + R\}$. Then, there exists a constant $\varepsilon_0 = \varepsilon_0(f, g, N, R, p, \mu) > 0$ such that $T_\varepsilon$ verifies

$$T_\varepsilon \leq \begin{cases} \frac{C}{\varepsilon^{\frac{2(p-1)}{2(N+4)-(p-1)}}} & \text{for } 1 < p < p_G(N+\mu), \\ \exp(C\varepsilon^{-(p-1)}) & \text{for } p = p_G(N+\mu), \end{cases}$$

where $C$ is a positive constant independent of $\varepsilon$ and $0 < \varepsilon \leq \varepsilon_0$.

**Remark 2.1.** It is well-known that the assumption (1.6) with $p > p_G(N)$ and $q > q_S(N)$ still yields the blow-up of the corresponding undamped equation to (1.8) (with $\mu = 0$), see [9, 11, 18]. More precisely, it is proven in [18, Remark 2.3] the existence of a pair $(p_0(N), q_0(N))$ verifying (1.6), $p_0(N) > p_G(N)$ and $q_0(N) > q_S(N)$. Consequently, for $\mu > 0$, we have $(p_0(N+\mu), q_0(N+\mu))$ which satisfy (2.2), $p_0(N+\mu) > p_G(N+\mu)$ and $q_0(N+\mu) > q_S(N+\mu)$. Hence, the hypothesis on $p$ and $q$ in Theorem 2.2 makes sense.

**Remark 2.2.** Theorem 2.3 asserts that the critical exponent for $p$ is greater than $p_G(N+\mu)$. We believe that the limiting value $p_G(N+\mu)$ coincides with the critical one. Of course one has to rigorously confirm this assertion. This will be the subject of a forthcoming work.

**Remark 2.3.** Unlike the case with only one nonlinearity ($|u_t(x,s)|^p$ or $|u(x,s)|^q$), one can note, in addition to the two blow-up regions $p \leq p_G(N)$ and $q \leq q_S(N)$, the obtaining of another blow-up region, characterized by (1.6) together with $p > p_G(N)$ and $q > q_S(N)$. The new region is in fact due to the presence of two mixed nonlinearities in (1.8), see [11] for the problem (1.8) with $\mu = 0$. This observation still holds in our case but with (1.6) being replaced by (2.2), otherwise $p_G(N)$ being replaced by $p_G(N+\mu)$ and $q_S(N)$ by $q_S(N+\mu)$, for $\mu$ small. It was previously conjectured that $q_S(N+\mu)$ constitutes the critical value for $q$ between the blow-up and the global existence zones, and Theorem 2.2 gives a first assertion for this conjecture. To complete the whole picture, the global existence in-time of the solution of (1.8) will be studied in a subsequent work.

**Remark 2.4.** The assumption (2.2) can be seen as a smallness condition for $\mu$, namely $\mu \in [0, \mu_*]$ where $\mu_* \mu_s$ satisfies the equality in (2.2) (otherwise $\mu_* := \frac{2(q+1)}{p(q-1)} - N + 1$).

**Remark 2.5.** Note that the results in Theorems 2.2 and 2.3 hold true after replacing the linear damping term in (1.8) $\mu \frac{u_t}{1+t}u_t$ by $b(t)u_t$ with $[b(t) - \mu(1+t)^{-1}]$ belongs to $L^1(0, \infty)$. The proof of this generalized damping case can be obtained by following the same steps as in the proofs of Theorems 2.2 and 2.3 with the necessary modifications.
3. SOME AUXILIARY RESULTS

We define the following positive test function

\[
\psi(x,t) := \rho(t)\phi(x); \quad \phi(x) := \begin{cases} \int_{S^{N-1}} e^{x \cdot \omega} d\omega & \text{for } N \geq 2, \\ e^x + e^{-x} & \text{for } N = 1, \end{cases}
\]

where \(\phi(x)\) is introduced in [34] and \(\rho(t), [21, 24, 28, 29]\), is solution of

\[
\frac{d^2 \rho(t)}{dt^2} - \rho(t) - \frac{d}{dt} \left( \frac{\mu}{1 + t} \rho(t) \right) = 0.
\]

The expression of \(\rho(t)\) reads as follows (see the Appendix for more details):

\[
\rho(t) = (t + 1)^{\frac{\mu + 1}{2}} K_{\frac{\mu - 1}{2}}(t + 1),
\]

where

\[
K_\nu(t) = \int_0^\infty \exp(-t \cosh \zeta) \cosh(\nu \zeta) d\zeta, \quad \nu \in \mathbb{R}.
\]

Moreover, the function \(\phi(x)\) verifies

\[
\Delta \phi = \phi.
\]

Note that the function \(\psi(x,t)\) satisfies the conjugate equation corresponding to (1.2), namely we have

\[
\partial_t^2 \psi(x,t) - \Delta \psi(x,t) - \frac{\partial}{\partial t} \left( \frac{\mu}{1 + t} \psi(x,t) \right) = 0.
\]

Throughout this article, we will denote by \(C\) a generic positive constant which may depend on the data \((p, q, \mu, N, R, f, g)\) but not on \(\varepsilon\) and whose the value may change from line to line. Nevertheless, we will precise the dependence of the constant \(C\) on the parameters of the problem when it is necessary.

The following lemma holds true for the function \(\psi(x,t)\).

**Lemma 3.1** ([34]). Let \(r > 1\). There exists a constant \(C = C(N, R, p, r) > 0\) such that

\[
\int_{|x| \leq t + R} (\psi(x,t))^r \leq C \rho^*(t) e^{rt}(1 + t)^{\frac{(2-r)(N-1)}{2}}, \quad \forall \ t \geq 0.
\]

As in the non-perturbed case, we define here the functionals that we will use to prove the blow-up criteria later on:

\[
G_1(t) := \int_{\mathbb{R}^N} u(x,t)\psi(x,t)dx,
\]

and

\[
G_2(t) := \int_{\mathbb{R}^N} \partial_t u(x,t)\psi(x,t)dx.
\]
The next two lemmas give the first lower bounds for $G_1(t)$ and $G_2(t)$, respectively. More precisely, we will prove that $G_1(t)$ and $G_2(t)$ are two coercive functions. This is the first observation which will be used to improve the main results of this article. Indeed, in our previous work [8], and compared to the results in (3.8) and (3.20) below, we obtained weaker lower bounds for the functionals, $G_1(t)$ and $G_2(t)$, of size $\varepsilon/(1 + t)^{\mu/2}$ (instead of $\varepsilon$ here); see Lemmas 3.2 and 3.3 in [8].

We note here that the proof of Lemma 3.2 is known in the literature; see e.g. [21, 28, 29]. However, we choose to include all the details about the proof of this lemma, on the one hand, to make the article self-contained and, on the other hand, to make later use of some computations therein. Nevertheless, Lemma 3.3 constitutes a novelty in this work and its utilization in the proofs of Theorems 2.2 and 2.3 is fundamental.

**Lemma 3.2.** Assume the existence of an energy solution $u$ of the system (1.8) with initial data satisfying the assumptions in Theorem 2.2. Then, there exists $T_0 = T_0(\mu) > 1$ such that

$$(3.8) \quad G_1(t) \geq C_{G_1} \varepsilon, \quad \text{for all } t \geq T_0,$$

where $C_{G_1}$ is a positive constant which depends on $f$, $g$, $N$, $R$ and $\mu$.

**Proof.** Let $t \in [0, T)$. Using Definition 2.1 and performing an integration by parts in space in the fourth term in the left-hand side of (2.1), we obtain

$$(3.9) \quad \int_{\mathbb{R}^N} u_t(x, t) \Phi(x, t) dx - \varepsilon \int_{\mathbb{R}^N} g(x) \Phi(x, 0) dx$$

$$- \int_0^t \int_{\mathbb{R}^N} \{u_t(x, s) \Phi_t(x, s) + u(x, s) \Delta \Phi(x, s)\} dx ds + \int_0^t \int_{\mathbb{R}^N} \frac{\mu}{1 + s} u_t(x, s) \Phi(x, s) dx ds$$

$$= \int_0^t \int_{\mathbb{R}^N} \{|u_t(x, s)|^p + |u(x, s)|^q\} \Phi(x, s) dx ds, \quad \forall \Phi \in C_0^\infty(\mathbb{R}^N \times [0, T)).$$

Now, substituting in (3.9) $\Phi(x, t)$ by $\psi(x, t)$, we infer that

$$(3.10) \quad \int_{\mathbb{R}^N} u_t(x, t) \psi(x, t) dx - \varepsilon \int_{\mathbb{R}^N} g(x) \psi(x, 0) dx$$

$$- \int_0^t \int_{\mathbb{R}^N} \{u_t(x, s) \psi_t(x, s) + u(x, s) \Delta \psi(x, s)\} dx ds + \int_0^t \int_{\mathbb{R}^N} \frac{\mu}{1 + s} u_t(x, s) \psi(x, s) dx ds$$

$$= \int_0^t \int_{\mathbb{R}^N} \{|u_t(x, s)|^p + |u(x, s)|^q\} \psi(x, s) dx ds.$$
Performing an integration by parts for the first and third terms in the second line of (3.10) and utilizing (3.1) and (3.4), we obtain
\[
\int_{\mathbb{R}^N} \left[ u_t(x, t)\psi(x, t) - u(x, t)\psi_t(x, t) + \frac{\mu}{1 + t}u(x, t)\psi(x, t) \right] dx
\]
\[
= \int_0^t \int_{\mathbb{R}^N} \{ |u_t(x, s)|^p + |u(x, s)|^q \} \psi(x, s) dx \, ds + \varepsilon C(f, g),
\]
where
\[
C(f, g) := \rho(0) \int_{\mathbb{R}^N} \left[ (\mu - \frac{\rho'(0)}{\rho(0)}) f(x)\phi(x) + g(x)\phi(x) \right] dx.
\]
We notice that the constant \( C(f, g) \) is positive thanks to (6.2) and the fact that the function \( K_\nu(t) \) is positive (see (6.3) in the Appendix).
Hence, using the definition of \( G_1 \), as in (3.6), and (3.1), the equation (3.11) yields
\[
G_1'(t) + \Gamma(t)G_1(t) = \int_0^t \int_{\mathbb{R}^N} \{ |u_t(x, s)|^p + |u(x, s)|^q \} \psi(x, s) dx \, ds + \varepsilon C(f, g),
\]
where
\[
\Gamma(t) := \frac{\mu}{1 + t} - 2\frac{\rho'(t)}{\rho(t)}.
\]
Multiplying (3.13) by \( \frac{(1 + t)^\mu}{\rho'(t)} \) and integrating over \((0, t)\), we deduce that
\[
G_1(t) \geq G_1(0) \frac{\rho^2(t)}{(1 + t)^\mu} + \varepsilon C(f, g) \frac{\rho^2(t)}{(1 + t)^\mu} \int_0^t (1 + s)^\mu \, ds.
\]
Using (3.3) and the fact that \( G_1(0) > 0 \), the estimate (3.15) yields
\[
G_1(t) \geq \varepsilon C(f, g)(1 + t)K_{\nu + 1}^2(t)K_{\nu + 1}^2(t + 1) \int_{t/2}^t \frac{1}{(1 + s)K_{\nu + 1}^2(s + 1)} \, ds.
\]
From (6.3), we have the existence of \( T_0 = T_0(\mu) > 1 \) such that
\[
(1 + t)K_{\nu + 1}^2(t + 1) > \frac{\pi}{4}e^{-2(t + 1)} \quad \text{and} \quad (1 + t)^{-1}K_{\nu - 1}^{-2}(t + 1) > \frac{1}{\pi}e^{2(t + 1)}, \quad \forall \, t \geq T_0/2.
\]
Hence, we have
\[
G_1(t) \geq \varepsilon C(f, g)e^{-2t} \int_{t/2}^t e^{2s} ds \geq \frac{\varepsilon}{8}C(f, g)e^{-2t}(e^{2t} - e^t), \quad \forall \, t \geq T_0.
\]
Finally, using \( e^{2t} > 2e^t, \forall \, t \geq 1 \), we deduce that
\[
G_1(t) \geq \frac{\varepsilon}{16}C(f, g), \quad \forall \, t \geq T_0.
\]
This ends the proof of Lemma 3.2.
\]
Now we are in a position to prove the following lemma.
Lemma 3.3. For any energy solution $u$ of the system (1.8) with initial data satisfying the assumptions in Theorem 2.2, there exists $T_1 = T_1(\mu) > 0$ such that

$$G_2(t) \geq C_{G_2} \varepsilon, \quad \text{for all } t \geq T_1,$$

where $C_{G_2}$ is a positive constant which depends on $f$, $g$, $N$ and $\mu$.

Proof. Let $t \in [0, T)$. Then, using the definition of $G_1$ and $G_2$, given respectively by (3.6) and (3.7), (3.1) and the fact that

$$G_2(t) \geq C_{G_2} \varepsilon,$$

the equation (3.13) yields

$$G_2' + \left( \frac{\mu}{1 + t} - \frac{\rho'(t)}{\rho(t)} \right) G_1(t)
= \int_0^t \int_{\mathbb{R}^n} \{|u_t(x, s)|^p + |u(x, s)|^q\} \psi(x, s)dx ds + \varepsilon C(f, g).$$

Differentiating the equation (3.22) in time and ignoring the nonnegative term in the right-hand side of the obtained equation, we get

$$G_2' = \left( \frac{\mu}{1 + t} - \frac{\rho'(t)}{\rho(t)} \right) G_1' - \left( \frac{\mu''(t)}{(1 + t)^2} + \frac{\rho''(t)\rho(t) - (\rho'(t))^2}{\rho^2(t)} \right) G_1(t) \geq 0.
(3.23)$$

Using (3.2) and (3.21), the identity (3.23) becomes

$$G_2' + \left( \frac{\mu}{1 + t} - \frac{\rho'(t)}{\rho(t)} \right) G_2(t) - G_1(t) \geq 0.
(3.24)$$

Remember the definition of $\Gamma(t)$, given by (3.14), we obtain

$$G_2' + \frac{3\Gamma(t)}{4} G_2(t) \geq \Sigma_1(t) + \Sigma_2(t),
(3.25)$$

where

$$\Sigma_1(t) = \left( -\frac{\rho'(t)}{2\rho(t)} - \frac{\mu}{4(1 + t)} \right) \left( G_2(t) + \left( \frac{\mu}{1 + t} - \frac{\rho'(t)}{\rho(t)} \right) G_1(t) \right),
(3.26)$$

and

$$\Sigma_2(t) = \left( 1 + \left( \frac{\rho'(t)}{2\rho(t)} + \frac{\mu}{4(1 + t)} \right) \left( \frac{\mu}{1 + t} - \frac{\rho'(t)}{\rho(t)} \right) \right) G_1(t).
(3.27)$$

Now, using (3.22) and (6.4), we deduce that there exists $\bar{T}_1 = \bar{T}_1(\mu) \geq T_0$ such that

$$\Sigma_1(t) \geq C \varepsilon, \quad \forall \ t \geq \bar{T}_1.
(3.28)$$

Moreover, form Lemma 3.2 and (6.4), we conclude the existence of $\bar{T}_2 = \bar{T}_2(\mu) \geq \bar{T}_1(\mu)$ such that

$$\Sigma_2(t) \geq C \varepsilon, \quad \forall \ t \geq \bar{T}_2.
(3.29)$$
Combining (3.25), (5.3) and (3.29), we obtain

\[(3.30) \quad G_2'(t) + \frac{3\Gamma(t)}{4} G_2(t) \geq C \varepsilon, \quad \forall \ t \geq \tilde{T}.
\]

Multiplying (3.30) by \(\frac{(1+t)^{3\mu/4}}{\rho^{3/2}(t)}\) and integrating over \((\tilde{T}, t)\), we deduce that

\[(3.31) \quad G_2(t) \geq G_2(\tilde{T}) \frac{\rho^{3/2}(t)(1 + \tilde{T})^{3\mu/4}}{\rho^{3/2}(\tilde{T})(1 + t)^{3\mu/4}} + C \varepsilon \frac{\rho^{3/2}(t)}{(1 + s)^{3\mu/4}} \int_{\tilde{T}}^{t} \frac{(1 + s)^{3\mu/4}}{\rho^{3/2}(s)} ds, \quad \forall \ t \geq \tilde{T}.
\]

Now, observe that \(G_2(t) = \rho(t)e^{t}F_2(t)\) where \(F_2(t)\) is given by (3.4) in [8]. Hence, using [8, Lemma 3.3] we infer that \(G_2(t) \geq 0\) for all \(t \geq 0\).

Therefore, using the above observation and (3.3), we deduce that

\[(3.32) \quad G_2(\tilde{T}) \frac{\rho^{3/2}(t)(1 + \tilde{T})^{3\mu/4}}{\rho^{3/2}(\tilde{T})(1 + t)^{3\mu/4}} \geq 0, \quad \forall \ t \geq 0.
\]

Employing (3.17) and (3.32), the estimate (5.7) yields, for all \(t \geq 2\tilde{T}\),

\[(3.33) \quad G_2(t) \geq C \varepsilon e^{-3t/2} \int_{t/2}^{t} e^{3s/2} ds.
\]

Hence, we have

\[(3.34) \quad G_2(t) \geq C \varepsilon, \quad \forall \ t \geq T_1 := 2\tilde{T}.
\]

This concludes the proof of Lemma 3.3. \(\Box\)

**Remark 3.1.** Notice that in the proofs of Lemmas 3.2 and 3.3 we only used the positivity of each one of the two nonlinearities \(|u_t|^p\) and \(|u|^q\). Indeed, the results in these lemmas are based on the comprehension of the dynamics in the linear part and, thus, the same conclusions can be handled similarly for any positive nonlinearity of the form \(F(u, u_t)\) instead of \(|u_t|^p + |u|^q\).

**Remark 3.2.** Obviously, the results of Lemmas 3.2 and 3.3 naturally hold true when we consider one nonlinearity \(|u_t|^p\) or \(|u|^q\) as it is the case for (1.9). However, to prove Theorem 2.3, and due to the nature of the equation, we make use of the linear part of (1.9) together with the nonlinear term while estimating \(G_2\). The approach used to estimate \(G_2\) in the proof of Theorem 2.3 is inspired from the computations carried out in [17] for \(F_2(t) = G_2(t)/(e^t \rho(t))\).

### 4. Proof of Theorem 2.2

In this section, we will give the proof of the first (main) theorem in this article which states the blow-up result and the lifespan estimate of the solution of (1.8). For
that purpose, we will make use of the lemmas proven in Section 3 and a Kato’s lemma type.

First, using the hypotheses in Theorem 2.2, we recall that supp(u) ⊂ \{(x, t) ∈ \mathbb{R}^N × [0, \infty) : |x| ≤ t + R\}.

Let t ∈ [0, T). Then, we set

\begin{equation}
F(t) := \int_{\mathbb{R}^N} u(x, t) dx,
\end{equation}

and

\begin{equation}
G(t) := \zeta(t) F(t) \text{ with } \zeta(t) = (1 + t)^{\frac{n}{2}}.
\end{equation}

Now, by choosing the test function Φ in (2.1) such that Φ ≡ 1 in \{(x, s) ∈ \mathbb{R}^N × [0, t] : |x| ≤ s + R\}\(^2\), we get

\begin{equation}
\int_{\mathbb{R}^N} u_t(x, t) dx - \int_{\mathbb{R}^N} u_t(x, 0) dx + \int_0^t \mu \int_{\mathbb{R}^N} u_t(x, s) dx ds = \int_0^t \int_{\mathbb{R}^N} \{|u_t(x, s)|^p + |u(x, s)|^q\} dx ds.
\end{equation}

Using the definition of F(t), the equation (4.3) can be written as

\begin{equation}
F'(t) + \int_0^t \frac{\mu}{1 + s} F'(s) ds = F'(0) + \int_0^t \int_{\mathbb{R}^N} \{|u_t(x, s)|^p + |u(x, s)|^q\} dx ds.
\end{equation}

Differentiating (4.4) in time, we obtain

\begin{equation}
F''(t) + \frac{\mu}{1 + t} F'(t) = \int_{\mathbb{R}^N} \{|u_t(x, t)|^p + |u(x, t)|^q\} dx.
\end{equation}

Now, we introduce the following multiplier

\begin{equation}
m(t) := (1 + t)^{\mu}.
\end{equation}

Multiplying (4.5) by m(t) and integrating over (0, t), we infer that

\begin{equation}
m(t) F'(t) = F'(0) + \int_0^t m(s) \int_{\mathbb{R}^N} \{|u_t(x, s)|^p + |u(x, s)|^q\} dx ds.
\end{equation}

Therefore, by dividing (4.7) by m(t), integrating over (0, t) and using the positivity of F(0) and F'(0), we deduce that

\begin{equation}
F(t) ≥ \int_0^t \frac{1}{m(s)} \int_0^s m(\tau) \int_{\mathbb{R}^N} \{|u_t(x, \tau)|^p + |u(x, \tau)|^q\} dx d\tau ds.
\end{equation}

\(^2\) The choice of a test function Φ which is identically equal to 1 is possible thanks to the fact that the initial data f and g are supported on \(B_{\mathbb{R}^N}(0, R)\).
By Hölder’s inequality and the estimates (3.5) and (3.20), we can bound the nonlinear term as follows:

\[
\left(\int_{\mathbb{R}^N} |u_t(x,t)|^p dx \right)^{1/p} \geq C_{2}^{p}(t) \left( \int_{|x| \leq t+R} \left( \psi(x,t) \right)^{\frac{p}{p-1}} dx \right)^{1/(p-1)} \geq C \rho^{-p}(t) e^{-pt} \psi^{1/(p-2)}, \quad \forall \ t \geq T_1.
\]

Using (3.3) and (3.17), we get

\[
\rho(t)e^t \leq C(t+1)^{\frac{N}{2}}, \quad \forall \ t \geq T_0/2.
\]

Consequently, we have

\[
\int_{\mathbb{R}^N} |u_t(x,t)|^p dx \geq C \rho^{-p}(t) e^{-pt} \psi^{1/(p-2)}, \quad \forall \ t \geq T_1.
\]

Plugging the above inequality into (4.8), we obtain

\[
F(t) \geq C \rho^{-p}(t) e^{-pt} \psi^{1/(p-2)}, \quad \forall \ t \geq T_1.
\]

Hence, by (4.2), the estimate (4.12) implies that

\[
G(t) \geq C \rho^{-p}(t) e^{-pt} \psi^{1/(p-2)}, \quad \forall \ t \geq T_1.
\]

On the other hand, we have

\[
\left( \int_{\mathbb{R}^N} u(x,t) dx \right)^q \leq \int_{|x| \leq t+R} |u(x,t)|^q dx \left( \int_{|x| \leq t+R} dx \right)^{q-1},
\]

and consequently we deduce that

\[
G^q(t) \leq C(t+1)^{N(q-1)+\frac{N}{2}} \int_{|x| \leq t+R} |u(x,t)|^q dx.
\]

Now, by differentiating (4.7) with respect to time, we obtain

\[
(m(t)F(t))' = m(t) \int_{\mathbb{R}^N} \{ |u_t(x,t)|^p + |u(x,t)|^q \} dx \geq m(t) \int_{\mathbb{R}^N} |u(x,t)|^q dx.
\]

Combining (4.15) in (4.16) and dividing the obtained equation (from (4.16)) by \( \zeta(t) = \sqrt{m(t)} \), we infer that

\[
G''(t) + \frac{\mu(2-\mu)}{4(1+t)^2} G(t) \geq C \frac{G^q(t)}{(1+t)^{N+\frac{N}{2}}(q-1)}, \quad \forall \ t > 0.
\]

At this level, we distinguish two cases depending on the value of the parameter \( \mu \).

**First case** \( (\mu \geq 2) \).

For this value of \( \mu \), the estimate (4.17) yields

\[
G''(t) \geq C \frac{G^q(t)}{(1+t)^{(N+\frac{N}{2})(q-1)}}, \quad \forall \ t > 0.
\]
Thanks to the fact that $G(t) = \zeta(t) F(t)$, (4.7) and (4.8) we have $G'(t) > 0$. Then, multiplying (4.18) by $G'(t)$ implies that

$$
(4.19) \quad \left\{ \left( G'(t) \right)^2 \right\}' \geq C \frac{(G^{q+1}(t))'}{(1 + t)^{(N + \frac{p}{2})(q-1)}}, \quad \forall \ t > 0.
$$

Integrating the above inequality, we obtain

$$
(4.20) \quad \left( G''(t) \right)^2 \geq C \frac{G^{q+1}(t)}{(1 + t)^{(N + \frac{p}{2})(q-1)}}, \quad \forall \ t > 0.
$$

Observe that the last term in the right-hand side of (4.20) is positive since we consider here small initial data, and more precisely this holds for $\varepsilon$ small enough.

Hence, (4.20) yields

$$
(4.21) \quad \frac{G''(t)}{G^{1+\delta}(t)} \geq C \frac{G^{q+1}(t)}{(1 + t)^{(N + \frac{p}{2})(q-1)-2}}, \quad \forall \ t > 0,
$$

for $\delta > 0$ small enough.

**Second case ($\mu < 2$).**

First, we recall that, as observed above, we have $G'(t) > 0$. Then, multiplying (4.17) by $(1 + t)^2 G'(t)$ gives

$$
(4.22) \quad \frac{(1 + t)^2}{2} \left( (G'(t))^2 \right)' + \frac{\mu(2 - \mu)}{8} (G^2(t))' \geq C \frac{(G^{q+1}(t))'}{(1 + t)^{(N + \frac{p}{2})(q-1)-2}}, \quad \forall \ t > 0.
$$

Integrating the above inequality and using the fact that $t \mapsto 1/(1 + t)^{(N + \frac{p}{2})(q-1)-2}$ is a decreasing function (since $N(q - 1) - 2 > 0$ because $q > 1 + \frac{2}{N}$ since we consider here $q > q_S(N + \mu)^3$), we have

$$
(4.23) \quad \frac{(1 + t)^2}{2} (G'(t))^2 + \frac{\mu(2 - \mu)}{8} G^2(t) \geq C_1 \frac{G^{q+1}(t)}{(1 + t)^{(N + \frac{p}{2})(q-1)-2}} + G^2(0) \left( \frac{\mu(2 - \mu)}{8} - C G^{q-1}(0) \right), \quad \forall \ t > 0.
$$

Observe that the last term in the right-hand side of (4.23) is positive since we consider here small initial data, and more precisely this holds for $\varepsilon$ small enough. Hence, we have

$$
(4.24) \quad \frac{(1 + t)^2}{2} (G'(t))^2 + \frac{\mu(2 - \mu)}{8} G^2(t) \geq C_1 \frac{G^{q+1}(t)}{(1 + t)^{(N + \frac{p}{2})(q-1)-2}}.
$$

We now aim to get rid of the second term in the left-hand side of (4.24). Indeed, in the present case $\mu < 2$, we will show that $\frac{\mu(2 - \mu)}{8} G^2(t)$ can be absorbed by the term in

\footnote{It is clear that if $q \leq q_S(N + \mu)$ the blow-up result can be proven by only considering the nonlinearity $|u(x, s)|^q$.}
the right-hand side of (4.24), namely $C_1 G^{q+1}(t)/(1 + t)^{(N+\frac{2}{q})(q-1)-2}$. In other words, our target consists in obtaining the following estimate:

$$
(4.25) \quad \frac{G^{q-1}(t)}{(1 + t)^{(N+\frac{2}{q})(q-1)-2}} > \frac{\mu(2 - \mu)}{4C_1},
$$

for all $t \geq T_2$ where $T_2 = T_2(\varepsilon, \mu) > 0$ is given by (4.27) below.

Indeed, employing the estimate (4.13), the expression of $G(t)$, the definition of $\lambda(p, q, N)$ (given by (1.6)) and the expression of $\zeta(t)$ (given by (4.2)), we deduce that

$$
(4.26) \quad \frac{G^{q-1}(t)}{(1 + t)^{(N+\frac{2}{q})(q-1)-2}} > C_2 \varepsilon^{p(q-1)}(1 + t)^{2-\frac{\lambda(p,q,N+\mu)}{2}}, \forall t \geq T_1.
$$

Now, we choose $T_2$ such that

$$
(4.27) \quad T_2 = \max \left( C_3^{-\frac{2}{4-\lambda(p,q,N+\mu)}}, \varepsilon^{-\frac{2p}{4-\lambda(p,q,N+\mu)}}, T_1(\mu) \right),
$$

where $C_3 = 4C_1 C_2/(\mu(2 - \mu))$. Note that for $\varepsilon$ small enough

$$
T_2 = T_2(\varepsilon) := C_3^{-\frac{2}{4-\lambda(p,q,N+\mu)}} \varepsilon^{-\frac{2p}{4-\lambda(p,q,N+\mu)}}.
$$

Therefore (4.25) is now proven, and by combining (4.25) in (4.24), we get

$$
(4.28) \quad (1 + t)^2 (G'(t))^2 \geq C_1 \frac{G^{q+1}(t)}{(1 + t)^{(N+\frac{2}{q})(q-1)-2}}, \forall t \geq T_2,
$$

that we rewrite as follows:

$$
(4.29) \quad \frac{G'(t)}{G^{1+\delta}(t)} \geq C \frac{G^{q-1-\delta}(t)}{(1 + t)^{(2N+\mu)(q-1)}} - \delta, \forall t \geq T_2,
$$

for $\delta > 0$ small enough.

Therefore, for the two cases $\mu \geq 2$ and $\mu < 2$ we end up with almost the same estimates (4.25 and 4.29, respectively); they only differ by the starting times 0 and $T_2$, respectively. Hence, the estimate (4.29) holds true for both cases for all $t \geq T_2$ where $T_2$ is given by (4.27).

Now integrating the inequality (4.29) on $[t_1, t_2]$, for all $t_2 > t_1 \geq T_2$, and using (4.13), we obtain

$$
(4.30) \quad \frac{1}{\delta} \left( \frac{1}{G^\delta(t_1)} - \frac{1}{G^\delta(t_2)} \right) \geq C(\varepsilon^p)^{\frac{1}{2}-\frac{\delta}{2}} \int_{t_1}^{t_2} \frac{1}{(1 + s)^{(2N+\mu)(q-1)}} ds, \forall t_2 \geq t_1 \geq T_2.
$$
Neglecting the second term in the left-hand side of (4.30) and using the definition of $\lambda(p,q,N)$ (given by (1.6)) yield

\begin{equation}
\frac{1}{G^\delta(t_1)} \geq C\delta \varepsilon \frac{\theta(p,q-1)}{2} - p\delta \int_{t_1}^{t_2} (1 + s) \frac{\lambda(p,q,N+\mu)}{4} - \delta \left(2 - \frac{\mu(p-1)+(N-1)(p-2)}{2}\right) ds.
\end{equation}

Employing the hypothesis (2.2), we have $-\frac{\lambda(p,q,N+\mu)}{4} + 1 > 0$. Hence, we choose $\delta = \delta_0$ small enough such that $\gamma := -\frac{\lambda(p,q,N+\mu)}{4} - \delta_0 \left(2 - \frac{\mu(p-1)+(N-1)(p-2)}{2}\right) > -1$. Then, the estimate (4.31) implies that

\begin{equation}
\frac{1}{G^{\delta_0}(t_1)} \geq C\varepsilon \frac{\theta(p,q-1)}{2} - p\delta_0 \left((1 + t_2)^{\gamma+1} - (1 + t_1)^{\gamma+1}\right), \quad \forall \ t_2 > t_1 \geq T_2.
\end{equation}

Now, using (4.13), we infer that

\begin{equation}
\varepsilon \frac{\theta(p,q-1)}{2} \left((1 + t_2)^{\gamma+1} - (1 + t_1)^{\gamma+1}\right) \leq C_4 \left(1 + t_1\right)^{-\delta_0 \left(2 - \frac{\mu(p-1)+(N-1)(p-2)}{2}\right)}, \quad \forall \ t_2 > t_1 \geq T_2.
\end{equation}

Consequently, we have

\begin{equation}
\varepsilon \frac{\theta(p,q-1)}{2} (1 + t_2)^{\gamma+1} \leq C_4 (1 + t_1)^{-\delta_0 \left(2 - \frac{\mu(p-1)+(N-1)(p-2)}{2}\right)} + \varepsilon \frac{\theta(p,q-1)}{2} (1 + t_1)^{\gamma+1}, \quad \forall \ t_2 > t_1 \geq T_2.
\end{equation}

At this level, since $-\frac{\lambda(p,q,N+\mu)}{4} + 1 > 0$, then for all $\varepsilon > 0$, we choose $\tilde{T}_3$ such that

\begin{equation}
\tilde{T}_3 = C_4^{-\frac{\theta(p,q-1)}{2}} \varepsilon^{-\frac{\theta(p,q-1)}{4}}.
\end{equation}

Finally, we set $t_1 = \max(T_2, \tilde{T}_3)$ Hence, using (4.35), we deduce from (4.34) that

\begin{equation}
t_2 \leq 2 \frac{1}{\gamma+1} (1 + t_1) \leq C\varepsilon^{-\frac{\theta(p,q-1)}{4}}.
\end{equation}

This achieves the proof of Theorem 2.2. \hfill \Box

5. PROOF OF THEOREM 2.3.

This section is devoted to the proof of Theorem 2.3 which is somehow related to the determining of the critical exponent associated with the nonlinear term in the problem (1.9). First, we follow the computations already done in Section 3 where we gain a better understanding of the linear problem associated with (1.9) which is the same as the linear problem associated with (1.8). More precisely, we aim here to take advantage of the techniques used in Section 3. We first note that Lemmas 3.2 and 3.3 remain true for the solution of (1.9) instead of (1.8) (see Remark 3.1 and the beginning of Remark 3.2) since we only used the positivity of the nonlinear terms and not their types. In fact, we proved in Lemma 3.3 that $G_2(t)$ is a coercive function. This is a crucial observation that we will use to improve the blow-up result of (1.9). To this end, we use similar computations, as in [17], and the new lower bound for the functional $G_2(t)$ obtained in
between the blow-up and the global existence regions.

Thanks to the observation described above, we obtain an improvement of the blow-up result in [23] (respec. [17]) passing from \( p \in (1, p_G(N + \sigma)] \) where \( p_G(N) \) is the Glassey exponent given by (1.5) and \( \sigma \) is given by (1.7) (respec. \( p_G(N + 2\mu) \)) to \( p_G(N + \mu) \).

More precisely, our result for (1.9) enhances the corresponding one in [23], for \( \mu \in (0, 2) \), and coincides with it for \( \mu \geq 2 \).

We believe that the obtained critical exponent \( p_G(N + \mu) \) may reach the threshold between the blow-up and the global existence regions.

In what follows we will use the equations and the estimates from (3.22) to (3.25) with omitting the nonlinear term \( |u(x,t)|^q \) and keeping the other one related to the nonlinearity \( |u_t(x,t)|^p \). Hence, the analogous of the estimate (3.25) reads

\[
G'_2(t) + \frac{3\Gamma(t)}{4}G_2(t) \geq \Sigma_1(t) + \Sigma_2(t) + \Sigma_3(t), \quad \forall \, t > 0,
\]

where \( \Sigma_1(t) \) and \( \Sigma_2(t) \) are given, respectively, by (3.26) and (3.27), and

\[
\Sigma_3(t) := \int_{\mathbb{R}^N} |u_t(x,t)|^p \psi(x,t) dx.
\]

From (3.22) and (6.4), we have

\[
\Sigma_1(t) \geq C \varepsilon + \left( -\frac{\rho'(t)}{2\rho(t)} - \frac{\mu}{4(1+t)} \right) \int_0^t \int_{\mathbb{R}^N} |u_t(x,s)|^p \psi(x,s) dxds, \quad \forall \, t \geq \tilde{T}_1.
\]

Now, using (6.4), we deduce that there exists \( \tilde{T}_3 = \tilde{T}_3(\mu) \geq \tilde{T}_2 \) such that

\[
\Sigma_1(t) \geq C \varepsilon + \frac{1}{4} \int_0^t \int_{\mathbb{R}^N} |u_t(x,s)|^p \psi(x,s) dxds, \quad \forall \, t \geq \tilde{T}_3.
\]

Plugging (5.4) together with (5.2) and (3.29) in (5.1), we deduce that

\[
G'_2(t) + \frac{3\Gamma(t)}{4}G_2(t) \geq \frac{1}{4} \int_0^t \int_{\mathbb{R}^N} |u_t(x,s)|^p \psi(x,s) dxds
\]

\[
+ \int_{\mathbb{R}^N} |u_t(x,t)|^p \psi(x,t) dx + C_5 \varepsilon, \quad \forall \, t \geq \tilde{T}_3.
\]

Setting

\[
H(t) := \frac{1}{8} \int_{\tilde{T}_4}^t \int_{\mathbb{R}^N} |u_t(x,s)|^p \psi(x,s) dxds + \frac{C_6 \varepsilon}{8},
\]

where \( C_6 = \min(C_5, 8C_{G_2}) \) (\( C_{G_2} \) is defined in Lemma 3.3) and \( \tilde{T}_4 > \tilde{T}_3 \) is chosen such that \( \frac{1}{4} - \frac{3\Gamma(t)}{32} > 0 \) and \( \Gamma(t) > 0 \) for all \( t \geq \tilde{T}_4 \) (this is possible thanks to (3.14) and (6.4)), and let

\[
\mathcal{F}(t) := G_2(t) - H(t).
\]
Hence, we have
\[
\mathcal{F}'(t) + \frac{3\Gamma(t)}{4} \mathcal{F}(t) \geq \left( \frac{1}{4} - \frac{3\Gamma(t)}{32} \right) \int_{\tilde{T}_4}^{t} \int_{\mathbb{R}^N} \left| u_t(x,s) \right|^p \psi(x,s) \, dx \, ds \\
+ \frac{7}{8} \int_{\mathbb{R}^N} \left| u_t(x,t) \right|^p \psi(x,t) \, dx + C_6 \left( 1 - \frac{3\Gamma(t)}{32} \right) \varepsilon \\
\geq 0, \quad \forall \, t \geq \tilde{T}_4.
\]

Multiplying (5.6) by \( \frac{(1+t)^{3\mu/4}}{\rho^{3/2}(t)} \) and integrating over \((\tilde{T}_4, t)\), we deduce that
\[
\mathcal{F}(t) \geq \mathcal{F}(\tilde{T}_4) \left( 1 + \frac{\tilde{T}_4}{4} \right)^{3\mu/4} \frac{\rho^{3/2}(t)}{(1+t)^{3\mu/4}}, \quad \forall \, t \geq \tilde{T}_4,
\]
where \( \rho(t) \) is defined by (3.3).

Therefore we have \( \mathcal{F}(\tilde{T}_4) = G_2(\tilde{T}_4) - \frac{C_6 \varepsilon}{8} \geq G_2(\tilde{T}_4) - C_{G_2} \varepsilon \geq 0 \) thanks to Lemma 3.3 and the fact that \( C_6 = \min(C_5, 8C_{G_2}) \leq 8C_{G_2} \).

Then, we have
\[
G_2(t) \geq H(t), \quad \forall \, t \geq \tilde{T}_4.
\]

By Hölder’s inequality and the estimates (3.5) and (3.20), we can bound the nonlinear term as follows:
\[
\int_{\mathbb{R}^N} \left| u_t(x,t) \right|^p \psi(x,t) \, dx \geq G_2^p(t) \left( \int_{|x| \leq t+R} \psi(x,t) \, dx \right)^{-(p-1)} \\
\geq C G_2^p(t) \rho^{-(p-1)(t)}(1+t)^{-\frac{(N-1)(p-1)}{2}}, \quad \forall \, t \geq \tilde{T}_4.
\]

Using (4.10), we obtain
\[
\int_{\mathbb{R}^N} \left| u_t(x,t) \right|^p \psi(x,t) \, dx \geq C G_2^p(t)(1+t)^{-\frac{(N+\mu-1)(p-1)}{2}}, \quad \forall \, t \geq \tilde{T}_4.
\]

From the above estimate and (5.8), we infer that
\[
H'(t) \geq C H^p(t)(1+t)^{-\frac{(N+\mu-1)(p-1)}{2}}, \quad \forall \, t \geq \tilde{T}_4.
\]

Since \( H(\tilde{T}_4) = C_0 \varepsilon /8 > 0 \), we can easily get the upper bound of the lifespan estimate in Theorem 2.3.

6. Appendix

In this appendix, we will recall some properties of the function \( \rho(t) \), the solution of (3.2). Hence, following the computations in [29] (with \( \eta = 1 \)), we can write the expression of \( \rho(t) \) as follows:
\[
\rho(t) = (t+1)^{\frac{\mu-1}{2}} K_{\mu-1}(t+1),
\]
where
where
\[ K_{\nu}(t) = \int_0^{\infty} \exp(-t \cosh \zeta) \cosh(\nu \zeta) d\zeta, \ \nu \in \mathbb{R}. \]

Using the property of \( \rho(t) \) in the proof of Lemma 2.1 in [29] (with \( \eta = 1 \)), we infer that
\[ \frac{\rho'(t)}{\rho(t)} = \frac{\mu}{1 + t} - \frac{K_{\mu+1}(t+1)}{K_{\nu+1}(t+1)}. \]  
(6.2)

From [7], we have the following property for the function \( K_{\mu}(t) \):
\[ K_{\mu}(t) = \sqrt{\frac{\pi}{2t}} e^{-t} (1 + O(t^{-1}), \ \text{as } t \to \infty). \]  
(6.3)

Combining (6.2) and (6.3), we infer that
\[ \frac{\rho'(t)}{\rho(t)} = -1 + O(t^{-1}), \ \text{as } t \to \infty. \]  
(6.4)

Finally, we refer the reader to [6] for more details about the properties of the function \( K_{\mu}(t) \).

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