FRANKL-FÜREDI-KALAI INEQUALITIES ON THE $\gamma$-VECTORS OF FLAG NESTOHEDRA

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Abstract. For any flag nestohedron, we define a flag simplicial complex whose $f$-vector is the $\gamma$-vector of the nestohedron. This proves that the $\gamma$-vector of any flag nestohedron satisfies the Frankl-Füredi-Kalai inequalities, partially solving a conjecture by Nevo and Petersen [5]. We also compare these complexes to those defined by Nevo and Petersen in [5] for particular flag nestohedra.

1. Introduction

For any building set $B$ there is an associated simple polytope $P_B$ called the nestohedron (see Section 2 below, [7, Section 7] and [8, Section 6]). When $B = B(G)$ is the building set determined by a graph $G$, $P_{B(G)}$ is the well-known graph-associahedron of $G$ (see [1, Example 2.1], [8, Sections 7 and 12], and [9]). The numbers of faces of $P_B$ of each dimension are conveniently encapsulated in its $\gamma$-polynomial $\gamma(B) = \gamma(P_B)$ defined below.

Recall that for a $d-1$-dimensional simplicial complex $\Delta$, the $f$-polynomial is a polynomial in $\mathbb{Z}[t]$ defined as follows:

$$f(\Delta)(t) := f_0 + f_1 t + \cdots + f_d t^d,$$

where $f_i(\Delta)$ is the number of $(i-1)$-dimensional faces of $\Delta$, and $f_0(\Delta) = 1$. The $h$-polynomial is given by

$$h(\Delta)(t) := (t-1)^d f(\Delta) \left( \frac{1}{t} \right).$$

When $\Delta$ is a homology sphere $h(\Delta)$ is symmetric (this is known as the Dehn-Somerville relations) hence it can be written

$$h(\Delta)(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (1 + t)^{d-2i},$$

for some $\gamma_i \in \mathbb{Z}$. Then the $\gamma$-polynomial is given by

$$\gamma(\Delta)(t) := \gamma_0 + \gamma_1 t + \cdots + \gamma_{\lfloor \frac{d}{2} \rfloor} t^{\lfloor \frac{d}{2} \rfloor}.$$

The vectors of coefficients of the $f$-polynomial, $h$-polynomial and $\gamma$-polynomial are known respectively as the $f$-vector, $h$-vector and $\gamma$-vector. If $P$ is a simple $(d+1)$-dimensional polytope then the dual simplicial complex $\Delta_P$ of $P$ is the boundary complex (of dimension $d$) of the polytope that is polar dual to $P$. The $f$-vector, $h$-vector and $\gamma$-vector of $P$ are defined via $\Delta_P$ as
\[ f(P)(t) := t^d f(\Delta_P)(t^{-1}), \]
so that \( f_i(P) \) is the number of \( i \)-dimensional faces of \( P \), and
\[ h(P)(t) := h(\Delta_P)(t) \]
\[ \gamma(P)(t) := \gamma(\Delta_P)(t). \]

When \( B \) is a building set, we denote the \( \gamma \)-polynomial for \( P_B \) by \( \gamma(B) \).

Recall that a simplicial complex \( \Delta \) is flag if every set of pairwise adjacent vertices is a face. Gal conjectured

**Conjecture 1.1.** [4 Conjecture 2.1.7]. If \( \Delta \) is a flag homology sphere then \( \gamma(\Delta) \) is non negative.

This implies that the \( \gamma \)-vector of any flag polytope has non negative entries. Gal’s conjecture was proven for flag nestohedra by Volodin in [9 Theorem 9]. Frohmader [2 Theorem 1.1] showed that the \( f \)-vector of any flag simplicial complex satisfies the Frankl-Füredi-Kalai inequalities. Nevo and Petersen conjectured the following strengthening of Gal’s conjecture:

**Conjecture 1.2.** [5 Conjecture 6.3]. If \( \Delta \) is a flag homology sphere then \( \gamma(\Delta) \) satisfies the Frankl-Füredi-Kalai inequalities.

They proved this in [5] for the following classes of flag spheres:
- \( \Delta \) is a Coxeter complex (including the simplicial complex dual to \( P_{B(K_n)} \)),
- \( \Delta \) is the simplicial complex dual to an associahedron (= \( P_{B(\text{Path}_n)} \)),
- \( \Delta \) is the simplicial complex dual to a cyclohedron (= \( P_{B(Cyc_n)} \)),
- \( \Delta \) has \( \gamma_1(\Delta) \leq 3 \),

by showing that the \( \gamma \)-vector of such \( \Delta \) is the \( f \)-vector of a flag simplicial complex. In [6] this is proven for the barycentric subdivision of a simplicial sphere.

In this paper we prove Conjecture 1.2 for all flag nestohedra:

**Theorem 1.3.** If \( P_B \) is a flag nestohedron, there is a flag simplicial complex \( \Gamma(B) \) such that \( f(\Gamma(B)) = \gamma(P_B) \). In particular \( \gamma(P_B) \) satisfies the Frankl-Füredi-Kalai inequalities.

Our construction for \( \Gamma(B) \) depends on the choice of a “flag ordering” for \( B \) (see Section 3 below). In the special cases considered by [5] our \( \Gamma(B) \) does not always coincide with the complex they construct.

Here is a summary of the contents of this paper. Section 2 contains preliminary definitions and results relating to building sets and nestohedra. In Section 3 we define the flag simplicial complex \( \Gamma(B) \) for a building set \( B \) and prove Theorem 1.3. In Section 4 we compare the simplicial complexes \( \Gamma(B) \) to the flag simplicial complexes defined in [4], and give combinatorial definitions for \( \Gamma(B) \) when \( B = B(K_n) \) and \( B = B(K_{1,n-1}) \).

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2. Preliminaries

A building set $B$ on a finite set $S$ is a set of non empty subsets of $S$ such that

- For any $I, J \in B$ such that $I \cap J \neq \emptyset$, $I \cup J \in B$.
- $B$ contains the singletons $\{i\}$, for all $i \in S$.

$B$ is connected if it contains $S$. For any building set $B$, $B_{\text{max}}$ denotes the set of maximal elements of $B$ with respect to inclusion. The elements of $B_{\text{max}}$ form a disjoint union of $S$, and if $B$ is connected then $B_{\text{max}} = \{S\}$. Building sets $B_1, B_2$ on $S$ are equivalent, denoted $B_1 \cong B_2$, if there is a permutation $\sigma : S \to S$ that induces a one to one correspondence $B_1 \cong B_2$.

Let $B$ be a building set on $S$ and $I \subseteq S$. The restriction of $B$ to $I$ is the building set

$$B|_I := \{b \mid b \in B, b \subseteq I\}$$

on $I$. The contraction of $B$ by $I$ is the building set

$$B/I := \{b \setminus I \mid b \in B, b \nsubseteq I\}$$

on $S - I$.

We associate a polytope to a building set as follows. Let $e_1, \ldots, e_n$ denote the standard basis vectors in $\mathbb{R}^n$. Given $I \subseteq [n]$, define the simplex $\Delta_I := \text{ConvexHull}(e_i \mid i \in I)$. Let $B$ be a building set on $[n]$. The nestohedron $P_B$ is a polytope defined in [7] and [8] as the Minkowski sum:

$$P_B := \sum_{I \in B} \Delta_I.$$

A $(d - 1)$-dimensional face of a $d$-dimensional polytope is called a facet. A simple polytope $P$ is flag if any collection of pairwise intersecting facets has non empty intersection, i.e. its dual simplicial complex is flag. We use the abbreviation flag complex in place of flag simplicial complex. A building set $B$ is flag if $P_B$ is flag.

A minimal flag building set $D$ on a set $S$ is a connected building set on $S$ that is flag, such that no proper subset of its elements forms a connected flag building set on $S$. Minimal flag building sets are described in detail in [8, Section 7.2]. They are in bijection with binary trees with leaf set $S$. Given such a tree, the corresponding minimal flag building set consists of the sets of descendants of the vertices of the tree. If $D$ is a minimal flag building set then $\gamma(D) = 1$ (see [8, Section 7.2]).

Let $B$ be a building set. A binary decomposition or decomposition of a non singleton element $b \in B$ is a set $D \subseteq B$ that forms a minimal flag building set on $b$. Suppose that $b \in B$ has a binary decomposition $D$. The two maximal elements $d_1, d_2 \in D - \{b\}$ with respect to inclusion are the maximal components of $b$ in $D$. Propositions [2.1] and [2.2] give alternative characterizations of when a building set is flag.

**Proposition 2.1.** [1, Lemma 7.2]. A building set $B$ is flag if and only if every non singleton $b \in B$ has a binary decomposition.
Proposition 2.2. [1 Corollary 2.6]. A building set $B$ is flag if and only if for every non-singleton $b \in B$, there exist two elements $d_1, d_2 \in B$ such that $d_1 \cap d_2 = \emptyset$ and $d_1 \cup d_2 = b$.

It follows from Proposition 2.2 that a graphical building set is flag.

Lemma 2.3. [1 Lemma 2.7]. Suppose $B$ is a flag building set. If $a, b \in B$ and $a \subsetneq b$, then there is a decomposition of $b$ in $B$ that contains $a$.

Recall the following theorems of Volodin [9]:

Theorem 2.4. [9 Lemma 6]. Let $B$ and $B'$ be connected flag building sets on $S$ such that $B \subseteq B'$. Then $B'$ can be obtained from $B$ by successively adding elements so that at each step the set is a flag building set.

Theorem 2.5. [9 Corollary 1]. See also [1 Lemma 3.3]. If $B'$ is a flag building set on $S$ obtained from a flag building set $B$ on $S$ by adding an element $b$ then

$$
\gamma(B') = \gamma(B) + t\gamma(B'_b)\gamma(B'/b) = \gamma(B) + t\gamma(B^b_b)\gamma(B/b).
$$

3. The Flag Complex $\Gamma(B)$ of a Flag Building Set $B$

For a building set $B$ with maximal components $B_{\text{max}} = \{b_1, \ldots, b_\alpha\}$, let $B_i = B|_{b_i}$ for $i = 1, \ldots, \alpha$. Then we have

$$
P_B = P_{B_1} \times P_{B_2} \times \cdots \times P_{B_\alpha}
$$

which implies that if $\gamma(B_i) = f(\Gamma(B_i))$ for some flag complex $\Gamma(B_i)$, then

$$
\gamma(B) = \gamma(B_1)\gamma(B_2) \cdots \gamma(B_\alpha) = f(\Gamma(B_1) \ast \Gamma(B_2) \ast \cdots \ast \Gamma(B_\alpha)).
$$

Hence to prove Theorem 2.3 we need only consider connected flag building sets.

Suppose that $B$ is a connected flag building set on $[n]$, $D$ is a decomposition of $[n]$ in $B$, and $b_1, b_2, \ldots, b_k$ is an ordering of $B - D$, such that $B_j = D \cup \{b_1, b_2, \ldots, b_j\}$ is a flag building set for all $j$. (Such an ordering exists by Theorem 2.4). We call the pair consisting of such a decomposition $D$ and the ordering on $B - D$ a flag ordering of $B$, denoted $O$, or $(D, b_1, \ldots, b_k)$. For any $b_j \in B - D$, we say an element in $B_{j-1}$ is earlier in the flag ordering than $b_j$, and an element in $B - B_j$ is later in the flag ordering than $b_j$.

For any $j \in [k]$ define

$$
U_j := \{i \mid i < j, b_i \nsubseteq b_j\},
$$

and

$$
V_j := \{i \mid i < j, b_i \subseteq b_j, \exists b \in B_{i-1} \text{ such that } b_i \subseteq b \nsubseteq b_j\}.
$$

If $i \in U_j \cup V_j$ then we say that $b_i$ is non degenerate with respect to $b_j$. If $b_i \in B_{j-1}$ and $i \not\in U_j \cup V_j$ then $b_i$ is degenerate with respect to $b_j$. Degenerate elements with respect to $b_j$ that are not contained in $b_j$ are elements that we need not consider as contributing to the building set $B_j/b_j$. The set of degenerate elements with respect
to $b_j$ that are subsets of $b_j$, together with $b_j$, forms a decomposition of $b_j$ in $B_j/b_j$.

Given a flag building set $B$ with flag ordering $O = (D, b_1, ..., b_k)$ define a graph on the vertex set

$$V_O = \{v(b_1), ..., v(b_k)\},$$

where for any $i < j$, $v(b_i)$ is adjacent to $v(b_j)$ if and only if $i \in U_j \cup V_j$. Then define a flag simplicial complex $\Gamma(O)$ whose faces are the cliques in this graph. If the flag ordering is clear then we denote $\Gamma(O)$ by $\Gamma(B)$. For any $I \subseteq [k]$, we let $\Gamma(O)|_I$ denote the induced subcomplex of $\Gamma(O)$ on the vertices $v(b_i)$ for all $i \in I$.

**Example 3.1.** Consider the flag building set $B(\text{Path}_5)$ on $[5]$. It has a flag ordering $O$ given by

$$D = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{2\}, \{3\}, \{4\}, \{5\}\},$$

and

$$b_1 = \{3, 4\}, \ b_2 = \{2, 3, 4\}, \ b_3 = \{2, 3\}, \ b_4 = \{2, 3, 4, 5\}, \ b_5 = \{3, 4, 5\}, \ b_6 = \{4, 5\}.$$

Then $\Gamma(O)$ has only two edges, namely

$$\{v(b_2), v(b_6)\} \text{ and } \{v(b_3), v(b_4)\}.$$

These are edges because $b_2 = \{2, 3, 4\}$ is the earliest element which has image $\{2, 3\}$ in the contraction by $b_6$, and the element $b_3 = \{2, 3\}$ is a subset of $b_2 = \{2, 3, 4\}$ which is in turn a subset of $b_4$.

Now $D/b_k$ is a decomposition of $[n] - b_k$, and we have an induced ordering of $(B/b_k) - (D/b_k)$, where the $i$th element is $b'_u := b_u \setminus b_k$ if $u_i$ is the $i$th element of $U_k$ (listed in increasing order). Then for all $i$, $D/b_k \cup \{b'_u\} \subseteq \{b'_u\}$ is a flag building set. Hence we can also define a flag complex $\Gamma(B/b_k)$. We label the vertices of $\Gamma(B/b_k)$ by $v(b_{u_{i_1}}), v(b_{u_{i_2}}), ..., v(b_{u_{i_k}})$.

**Claim 3.2.** Let $B$ be a connected flag building set with flag ordering $(D, b_1, ..., b_k)$. For all $b \in B$ let $b' = b \setminus b_k$. If $b' \neq \emptyset$, $j \in U_k$, and $b \in B_{j-1}$ then $b \subseteq b_j$ if and only if $b' \subseteq b'_j$.

**Proof.** $\Rightarrow$: It is clear that $b \subseteq b_j$ implies $b' \subseteq b'_j$.

$\Leftarrow$: Suppose for a contradiction that $b' \subseteq b'_j$ and $b \nsubseteq b_j$. Then $b \cap b_j \neq \emptyset$ and $b \cup b_j \neq b_j$, which implies that (since $B_j$ is a building set) $b \cup b_j \in B_{j-1}$. We also have that $(b \cup b_j)' = b'_j$, which implies that $b_j$ is degenerate with respect to $b_k$, a contradiction.

**Proposition 3.3.** Let $B$ be a connected flag building set with flag ordering given by $(D, b_1, ..., b_k)$. Then $\Gamma(B/b_k) \cong \Gamma(B)|_{U_k}$. The map on the vertices is given by $v(b'_i) \mapsto v(b_i)$.\[\square\]
Proof: $\Gamma(B)|_{U_k}$ is a flag complex with vertex set $v(b_{u_1}), v(b_{u_2}), ..., v(b_{u_{|U_k|}})$ and $\Gamma(B/b_k)$ is a flag complex with vertex set $v(b'_{u_1}), v(b'_{u_2}), ..., v(b'_{u_{|U_k|}})$. Suppose that $i < j$ where $i, j \in U_k$. We need to show that $\{v(b'_i), v(b'_j)\} \in \Gamma(B/b_k)$ if and only if $\{v(b_j), v(b_i)\} \in \Gamma(B)|_{U_k}$. We will show the following:

(1) If $b_i \subseteq b_j$ (by Claim 3.2 equivalently $b'_i \subseteq b'_j$) then $\{v(b'_i), v(b'_j)\} \in \Gamma(B/b_k)$ if and only if $\{v(b_j), v(b_i)\} \in \Gamma(B)|_{U_k}$.

(2) If $b_i \not\subseteq b_j$ (by Claim 3.2 equivalently $b'_i \not\subseteq b'_j$) then $\{v(b'_i), v(b'_j)\} \in \Gamma(B/b_k)$ if and only if $\{v(b_j), v(b_i)\} \in \Gamma(B)|_{U_k}$.

(1) $\Rightarrow$: Suppose that $\{v(b'_i), v(b'_j)\} \in \Gamma(B/b_k)$, so that there exists $b \in B_{i-1}$ such that $b'_i \subseteq b' \subseteq b'_j$. By Claim 3.2 $b \subseteq b_j$ and since $b_i \subseteq b_j$ this implies $b \cup b_i \subseteq b_j$. Since $b \cap b_i \neq \emptyset$ we have $b \cup b_i \in B_{i-1}$. Hence $b_i \subseteq b \cup b_i \subseteq b_j$ which implies $\{v(b_i), v(b_j)\} \in \Gamma(B)|_{U_k}$.

$\Leftarrow$: Suppose $\{v(b_i), v(b_j)\} \in \Gamma(B)|_{U_k}$, so that there exists $b \in B_{i-1}$ such that $b \subseteq b_j$. Then $b'_i \subseteq b' \subseteq b'_j$, and $b' \neq b'_i$ or $b'_j$ since $i, j \in U_k$, so that $b'_i \not\subseteq b' \not\subseteq b'_j$. Hence $\{v(b'_i), v(b'_j)\} \in \Gamma(B/b_k)$.

(2) $\Rightarrow$: Suppose that $\{v(b'_i), v(b'_j)\} \in \Gamma(B/b_k)$, and suppose for a contradiction that $\{v(b_i), v(b_j)\} \not\in \Gamma(B)|_{U_k}$. Then there exists $b \in B_{i-1}$ such that $bd' = b'_i \cup b'_j$. Then $bd' = b'_i \cup b'_j$ which implies the contradiction that $\{v(b'_i), v(b'_j)\} \not\in \Gamma(B/b_k)$.

$\Leftarrow$: We will prove the contrapositive that $\{v(b'_i), v(b'_j)\} \not\in \Gamma(B/b_k)$ implies that $\{v(b_i), v(b_j)\} \not\in \Gamma(B)|_{U_k}$. $\{v(b'_i), v(b'_j)\} \not\in \Gamma(B/b_k)$ implies there exists $m \in B_{i-1}$ such that $m \cap b'_j = b'_i \cap b'_j$.

- Assume that $m \subseteq b_i$, and for this case refer to Figure 5.1. Let $R := b_k \cap (b_i \setminus (m \cup b_j))$, and let $J := b_i \setminus (m \cup b_k)$. Since $m \subseteq b_i$, by Lemma 2.3 there exists a decomposition of $b_i$ in $B_j$ that contains $m$. Hence $m$ is contained in a maximal component $d'$ of this decomposition. Let $d'$ be the other maximal component. If $d' \cap R = \emptyset$ then $\{v(b_i), v(b_j)\} \not\in \Gamma(B)|_{U_k}$ since $d' \cap b_j = b'_i \cup b'_j$, hence the desired condition holds. If $d' \cap J = \emptyset$ then $b_i \setminus b_k = d \setminus b_k$ which contradicts $i \in V_k$. If $d' \cap J \neq \emptyset$ and $d' \cap R \neq \emptyset$ then $(d' \cap b_j \setminus b_k = b'_i \setminus b_k$ which contradicts $j \in V_k$. 


Figure 3.1. A picture of the sets in case (2), assuming $m \subseteq b_i$.
Note that $b_i \setminus (m \cup b_j \cup b_k) = \emptyset$ by the definition of $m$.

- Assume that $m \not\subseteq b_i$. For this case refer to Figure 3.2. Let $H := b_i \setminus (b_j \cup b_k)$. In $(b_j/b_k)/b_i'$ both $b_i'$ and $m'$ have the same image that is given by $H$, and $H \neq \emptyset$ since $H = \emptyset$ implies $b_i' \subseteq b_j'$. Let $K := m \setminus (b_k \cup b_i)$. Then $K \neq \emptyset$ since $K = \emptyset$ implies $b_i \setminus b_k = m \setminus b_k$, which contradicts $i \in V_k$. Let $L := m \setminus (b_i \cup b_j)$. $L = \emptyset$ implies $\{v(b_i), v(b_j)\} \notin \Gamma(B)_{\bar{U}_k}$ since $m \setminus b_j = b_i \setminus b_j$, so the desired condition holds. Suppose now $L \neq \emptyset$. Then $m$ intersects each of $H, K$ and $L$. Let $b$ be a minimal (for inclusion) element in in $B_{i-1}$ that intersects $H, K$ and $L$. Then $|b| \geq 3$ and at least one of the elements in the decomposition of $b$ (in $B_{i-1}$) must intersect exactly two of $K, H$ and $L$. Denote such an element by $\hat{d}$. If $\hat{d}$ intersects $K$ and $L$ then $(b_j \cup \hat{d}) \setminus b_k = b_j \setminus b_k$ which contradicts $j \in V_k$. If $\hat{d}$ intersects both $K$ and $H$ then $\{v(b_i), v(b_j)\} \notin \Gamma(B)_{\bar{U}_k}$ since $(b_i \cup \hat{d}) \setminus b_j = b_i \setminus b_j$, so the desired condition holds. If $\hat{d}$ intersects $L$ and $H$ then $(b_i \cup \hat{d}) \setminus b_k = b_i \setminus b_k$, which contradicts $i \in V_k$.

Figure 3.2. A picture of the sets in case (2), assuming $m \not\subseteq b_i$.
Note that $b_i \setminus (m \cup b_j \cup b_k) = \emptyset$ by the definition of $m$.
We now consider the flag building set $B|b_k$. It is not necessarily true that $D|b_k$ is a decomposition of $b_k$. Let

$$D_k := D|b_k \cup \{b_j \mid b_j \subseteq b_k, j \not\in V_k\}.$$ 

Then $D_k$ is a decomposition of $b_k$ in $B$, and for any $j$ we have that $D_k \cup \{b_i \mid i \leq j$ and $i \in V_k\}$ is a connected flag building set on $b_k$. We define $\Gamma(B|b_k)$ to be the flag complex $\Gamma(O)$ with respect to the flag ordering $O$ of $B|b_k$ with decomposition $D_k$ and ordering of $B|b_k - D_k$ given by $b_{v_1}, b_{v_2}, ..., b_{v_{i|V_k}}$ where $v_j$ is the $j$th element of $V_k$ listed in increasing order. We label the vertices of $\Gamma(B|b_k)$ by $v(b_{v_1}), ..., v(b_{v_{i|V_k}})$ rather than by their index in $V_k$. In keeping with the notation that $B_j$ is the flag building set obtained after adding elements indexed up to $j$, we let $(B|b_k)_j$ denote the flag building set $D_k \cup \{b_i \mid i \leq j$ and $i \in V_k\}$, so that $\Gamma((B|b_k)_j)$ is defined. Note then that for any $j$, $B_j|b_k \subseteq (B|b_k)_j$.

**Proposition 3.4.** Let $B$ be a connected flag building set with flag ordering given by $(D,b_1,\ldots,b_k)$. Then $\Gamma(B|b_k) = \Gamma(B)|V_k$.

**Proof:** Both $\Gamma(B|b_k)$ and $\Gamma(B)|V_k$ are both flag complexes with the vertex set $v(b_{v_1}), v(b_{v_2}), ..., v(b_{v_{i|V_k}})$. We need to show that for any $i,j \in V_k$ where $i < j$, $\{v(b_i), v(b_j)\} \in \Gamma((B)|V_k)$ if and only if $\{v(b_i), v(b_j)\} \in \Gamma(B|b_k)$.

$\Rightarrow$: Suppose that $\{v(b_i), v(b_j)\} \in \Gamma((B)|V_k)$. First assume that $b_i \subseteq b_j$. Then there is some $b \in B_{i-1}$ such that $b_i \subseteq b \subseteq b_j$. Since $b \in B_{i-1}|b_k$ and $B_{i-1}|b_k \subseteq (B|b_k)_{i-1}$ this implies that $\{v(b_i), v(b_j)\} \in \Gamma(B|b_k)$.

Now suppose that $b_i \not\subseteq b_j$. Suppose for a contradiction that $\{v(b_i), v(b_j)\} \not\in \Gamma(B|b_k)$. Then there exists some $d \in D_k - D|b_k$, $d \not\in B_{i-1}$, such that $d \cup b_j = b_i \cup b_j$. Since $i \in V_k$ there exists some $b \in B_{i-1}$ such that $b_i \subseteq b \subseteq b_j$. Since $\{v(b_i), v(b_j)\} \in \Gamma((B)|V_k)$ we have that $b \backslash (b_i \cup b_j) \not= \emptyset$. Since the index of $d$ is not in $V_k$, every element in the restriction to $b_k$ that is earlier than $d$ in the flag ordering is a subset of it or does not intersect it. This implies $d \subseteq b$, so $d \backslash (b_i \cup b_j) \not= \emptyset$, which contradicts $d \cup b_j \not= b_i \cup b_j$.

$\Leftarrow$: Suppose that $\{v(b_i), v(b_j)\} \not\in \Gamma(B|b_k)$. First assume that $b_i \subseteq b_j$, so that there is some $d \in (B|b_k)_{i-1}$ such that $b_i \subseteq d \subseteq b_j$. If $d \in B_{i-1}|b_k$ then clearly $\{v(b_i), v(b_j)\} \not\in \Gamma((B)|V_k)$ as desired. If $d \not\in B_{i-1}|b_k$ then $d \in D_k - D|b_k$. Since $i \in V_k$ there exists some $b \in B_{i-1}$ such that $b_i \subseteq b \subseteq b_k$. Since the index of $d$ is not in $V_k$ we have that $b_i \not\subseteq b \subseteq d$. This is because $d$ either contains or does not intersect elements that are earlier in the flag ordering and contained in $b_k$. Then since $d \subseteq b_j$ this implies $b_i \subseteq b_j$ and since $b \in B_{i-1}$ and $b_i \not\subseteq b \subseteq b_j$ this implies $\{v(b_i), v(b_j)\} \not\in \Gamma((B)|V_k)$.

Now assume that $b_i \not\subseteq b_j$. Suppose for a contradiction that $\{v(b_i), v(b_j)\} \not\in \Gamma((B)|V_k)$. Then there exists $b \in B_{i-1}|b_k$ such that $b \cup b_j = b_i \cup b_j$. Since $B_{i-1}|b_k \subseteq (B|b_k)_{i-1}$ this contradicts $\{v(b_i), v(b_j)\} \in \Gamma((B)|V_k)$.

\]

**Theorem 3.5.** Let $B$ be a connected flag building set with flag ordering $O$. Then $\gamma(B) = f(\Gamma(O))$. 


Proof. This is a proof by induction on the number of elements of $B - D$. The result holds for $k = 0$ since $f(\Gamma(D)) = 1 = \gamma(D)$. So we assume $k \geq 1$ and that the result holds for all connected flag building sets with a smaller value of $k$.

By Propositions 3.3 and 3.4 and the inductive hypothesis we have $f(\Gamma(B)|_{U_k}) = f(\Gamma(B)/b_k) = \gamma(B/b_k)$, and $f(\Gamma(B)|_{V_k}) = f(\Gamma(B)|_{b_k}) = \gamma(B|_{b_k})$.

Suppose that $u \in U_k$ and $w \in V_k$. Then $\{v(b_u), v(b_w)\} \in \Gamma(B)$, for suppose for a contradiction that $\{v(b_u), v(b_w)\} \not\in \Gamma(B)$. Suppose that $u < w$. Then there is some element $b \in B_{u-1}$ such that $b \cup b_w = b_u \cup b_w$. This implies that $b \cup b_k = b_u \cup b_k$ which contradicts $u \in U_k$. Suppose that $w < u$. Then either $b_u \cap b_w = \emptyset$ or $b_w \subseteq b_u$ (otherwise $b_u \cup b_w$ makes $b_u$ degenerate with respect to $b_k$). Suppose that $b_w \cap b_u = \emptyset$. Then since $\{v(b_u), v(b_w)\} \not\in \Gamma(B)$, there exists $b \in B_{w-1}$ such that $b \cup b_u = b_w \cup b_u$, and $b \cap b_k \neq \emptyset$. Then $b \cup b_u$ makes $b_u$ degenerate with respect to $b_k$, a contradiction. Suppose that $b_w \subseteq b_u$. Now $w \in V_k$ implies there is some $b \in B_{w-1}$ such that $b_w \subseteq b \subseteq b_k$. Also, $b \subseteq b_u$ else $b \cup b_u$ makes $b_u$ degenerate with respect to $b_k$. However, this implies the contradiction that $\{v(b_u), v(b_w)\} \in \Gamma(B)$ since $b_w \subseteq b \subseteq b_u$.

Hence

$$\Gamma(B)|_{U_k \cup V_k} = \Gamma(B)|_{U_k} \ast \Gamma(B)|_{V_k},$$

and therefore

$$f(\Gamma(B)|_{U_k \cup V_k}) = f(\Gamma(B)|_{U_k})f(\Gamma(B)|_{V_k}) = \gamma(B/b_k)\gamma(B|_{b_k}).$$

Since the vertex $v(b_k)$ is adjacent to the vertices indexed by elements in $U_k \cup V_k$ we have

$$f(\Gamma(B)) = f(\Gamma(B_{k-1})) + t\gamma(B/b_k)\gamma(B|_{b_k}).$$

By the induction hypothesis this implies that

$$f(\Gamma(B)) = \gamma(B_{k-1}) + t\gamma(B|_{b_k})\gamma(B/b_k),$$

which implies that $f(\Gamma(B)) = \gamma(B)$ by Theorem 2.5.

For two flag orderings $O_1$, $O_2$ of a connected flag building set $B$, it is not necessarily true that the flag complexes $\Gamma(O_1)$, $\Gamma(O_2)$ are equivalent (up to change of labels on the vertices) even if they have the same decomposition. The following example provides a counterexample.

Example 3.6. Let $B = B(Cyc_3)$, and let

$$D = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{2, 3, 4, 5\}, \{3, 4, 5\}, \{3, 4\}, \{3, 4, 5, 1\}, \{4, 5, 1, 2\}, \{5, 1, 2, 3\}, \{4, 5, 1\}, \{5, 1, 2\}, \{1, 5\}.\$$

Let $O_1$ be the flag ordering with decomposition $D$ and the following ordering of $B - D$:

$$\{2, 3\}, \{2, 3, 4\}, \{2, 3, 4, 5\}, \{4, 5\}, \{3, 4, 5\}, \{3, 4\}.$$

Let $O_2$ be the flag ordering with decomposition $D$ and the following ordering of $B - D$:

$$\{2, 3\}, \{2, 3, 4\}, \{2, 3, 4, 5\}, \{3, 4\}, \{3, 4, 5\}, \{4, 5\}, \{3, 4, 5\}, \{3, 4\}.$$
\{3, 4, 5, 1\}, \{4, 5, 1, 2\}, \{5, 1, 2, 3\}, \{4, 5, 1\}, \{5, 1, 2\}, \{1, 5\}.

Then \(\Gamma(O_1)\) and \(\Gamma(O_2)\) are depicted in Figure 3.3.

**Figure 3.3.** \(\Gamma(O_1)\) is on the left, and \(\Gamma(O_2)\) is on the right.

4. The flag complexes of Nevo and Petersen

In this section we compare the flag complexes that we have defined to those defined for certain graph-associahedra by Nevo and Petersen [5]. They define flag complexes \(\Gamma(\hat{S}_n)\), \(\Gamma(\hat{S}_n(312))\) and \(\Gamma(P_n)\) such that

- \(\gamma(B(K_n)) = f(\Gamma(\hat{S}_n))\),
- \(\gamma(B(\text{Path}_n)) = f(\Gamma(\hat{S}_n(312)))\),
- \(\gamma(B(\text{Cyc}_n)) = f(\Gamma(P_n))\).

We show that there is a flag ordering for \(B(\text{Path}_n)\) so that

\[\Gamma(B(\text{Path}_n)) \cong \Gamma(\hat{S}_n(312)),\]

and that the analogous statement is not true for \(B(K_n)\) and \(B(\text{Cyc}_n)\).

4.1. The flag complexes \(\Gamma(B(K_n))\) and \(\Gamma(\hat{S}_n)\). The permutohedron is the nestohedron \(P_{B(K_n)}\). Note that \(B(K_n)\) consists of all nonempty subsets of \([n]\). The \(\gamma\)-polynomial of \(P_{B(K_n)}\) is the descent generating function of \(\hat{S}_n\), which denotes the set of permutations with no double descents or final descent (see [3] Theorem 11.1). First we recall the definition of \(\Gamma(\hat{S}_n)\) given by Nevo and Petersen [5] Section 4.1.

A peak of a permutation \(w = w_1 \ldots w_n\) in \(\hat{S}_n\) is a position \(i \in [1, n-1]\) such that \(w_{i-1} < w_i > w_{i+1}\), (where \(w_0 := 0\)). We denote a peak at position \(i\) with a bar \(w_1 \ldots w_i | w_{i+1} \ldots w_n\). A descent of a permutation \(w = w_1 \ldots w_n\) is a position \(i \in [n-1]\) such that \(w_{i+1} < w_i\). Let \(\hat{S}_n\) denote the set of permutations in \(\hat{S}_n\) with no double (i.e. consecutive) descents or final descent, and let \(\tilde{S}_n\) denote the set of permutations in \(\hat{S}_n\) with one peak. Then \(\hat{S}_n \cap \tilde{S}_n\) consists of all permutations of the form

\[w_1 \ldots w_i | w_{i+1} \ldots w_n\]
where \( 1 \leq i \leq n - 2, \ w_1 < \cdots < w_i, \ w_i > w_{i+1}, \ w_{i+1} < \cdots < w_n. \)

Define the flag complex \( \Gamma(\mathcal{E}_n) \) on the vertex set \( \mathcal{E}_n \cap \mathcal{E}_n \) where two vertices

\[
u = u_1 | u_2
\]

and

\[
u = v_1 | v_2
\]

with \( |u_1| < |v_1| \) are adjacent if there is a permutation \( w \in \mathcal{S}_n \) of the form

\[
u = u_1 | a | v_2.
\]

Equivalently, if \( v_2 \subseteq u_2, |v_2 - v_2| \geq 2, \ min(u_2 - v_2) < \max(v_1) \) and \( \max(u_2 - v_2) > \min(v_2) \). (Since there must be two peaks in \( w \) this implies \( |a| \geq 2 \).) The faces of \( \Gamma(\mathcal{E}_n) \) are the cliques in this graph.

**Example 4.1.** Taking only the part after the peak, \( \mathcal{E}_5 \cap \mathcal{E}_5 \) can be identified with the set of subsets of \( [5] \) of sizes 2, 3 and 4 which are not \( \{4,5\}, \{3,4,5\}, \) or \( \{2,3,4,5\} \). Then the edges of \( \Gamma(\mathcal{E}_5) \) are given by:

\[
\{1,2,3,4\} \text{ is adjacent to each of } \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}.
\]

\[
\{1,2,3,5\} \text{ is adjacent to each of } \{1,2\}, \{1,3\}, \{1,5\}, \{2,3\}, \{2,5\}.
\]

\[
\{1,2,4,5\} \text{ is adjacent to each of } \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \text{ and}
\]

\[
\{1,3,4,5\} \text{ is adjacent to each of } \{3,4\}, \{3,5\}.
\]

**Proposition 4.2.** There is no flag ordering of \( B(K_5) \) so that

\[
\Gamma(B(K_5)) \cong \Gamma(\mathcal{E}_5).
\]

**Proof.** Suppose for a contradiction that there is some flag ordering of \( B(K_5) \) with decomposition \( D \) such that \( \Gamma(B(K_5)) \cong \Gamma(\mathcal{E}_5) \). Then there is some vertex \( v(b_j) \in \Gamma(B(K_5)) \) of degree 5. We consider the following three cases:

1. \( |b_j| = 2 \).
2. \( |b_j| = 3 \).
3. \( |b_j| = 4 \).

Note that \( D \) can only be one of the following three building sets (up to the order reversing permutation of \( B \)):

\[
\begin{align*}
\{\{1,\ldots,5\}, \{2\}, \{3\}, \{4\}, \{5\}\}, \\
\{\{1,\ldots,5\}, \{1,2\}, \{3,4\}, \{4,5\}\}, \\
\{\{1,\ldots,5\}, \{2,3\}, \{4,5\}, \{5\}\}.
\end{align*}
\]

(1) Suppose that \( |b_j| = 2 \). Then \( V_j = \emptyset \) and \( |U_j| \leq 2 \) (using the fact that \( D/b_j \) includes at least one 2-element subset). So there are \( \geq 3 b_k \)'s with \( k > j \) and \( j \in U_k \cup V_k \) (i.e. \( v(b_j) \) is adjacent to \( v(b_k) \)). Such \( b_k \)'s must be two element sets not intersecting \( b_j \) or four element sets that contain \( b_j \). Without loss of generality (WLOG for short), let \( b_j = \{4,5\} \).

(1a) Suppose that no three element set containing \( b_j \) occurs earlier than \( b_j \).

Then the case of 4-element \( b_k \)'s cannot occur, so \( \{1,2\}, \{1,3\}, \{2,3\} \) are the \( b_k \)'s. Since there is a 2-element set in \( D \), we have (WLOG) \( \{3,5\} \in D \), implying that \( \{3,4,5\} \) is earlier than \( \{4,5\} \), a contradiction.
(1b) Suppose that exactly one 3-element set containing \( b_j \), WLOG \( \{3, 4, 5\} \), occurs earlier than \( b_j \). Then \( \{1, 3\}, \{2, 3\}, \{1, 2, 4, 5\} \) can’t occur among the \( b_k \)’s, so \( \{1, 2\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\} \) are the \( b_k \)’s. Hence \( |U_j|=2 \), so \( B_{j-1}/b_j \) consists of all non-empty subsets of \( \{1, 2, 3\} \). So \( \exists b \in B_{j-1} \) such that \( b/b_j = \{2, 3\} \). But then \( b \in B_{j-1}, \{3, 4, 5\} \in B_{j-1} \) implies \( b \cup \{2, 3, 4\} = \{2, 3, 4, 5\} \in B_{j-1} \), a contradiction.

(1c) Suppose that there are at least two 3-element sets containing \( b_j \), WLOG \( \{2, 4, 5\} \) and \( \{3, 4, 5\} \), that occur earlier than \( b_j \). Then \( \{2, 3, 4, 5\} \) occurs earlier than \( b_j \) and \( \{1, 2\}, \{1, 3\}, \{2, 3\} \) can’t occur among the \( b_k \)’s, so we have a contradiction.

(2) Suppose that \( |b_j|=3 \). It is easy to see that \( v(b_j) \) is not adjacent to any vertices \( v(b_i) \) where \( i < j \), i.e. \( U_j = V_j = \emptyset \). Hence there must be 5 elements \( b_k, k > j \), such that \( j \in V_k \cup U_k \), and these elements must be of size 2. Suppose WLOG, that \( b_j = \{1, 2, 3\} \), and that the five elements \( b_k \) are
\[
\{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 5\}, \{2, 5\}.
\]
There is one two element subset of \( b_j \) that is earlier than \( b_j \) in the flag ordering since \( b_j \) requires a decomposition, and this element must have the same image in the contraction by one of \( \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 5\}, \{2, 5\} \) as \( \{1, 2, 3\} \), hence this case cannot occur.

(3) Suppose that \( |b_j|=4 \). Note that \( U_j = \emptyset \).

(3a) Suppose that no three element subset of \( b_j \) occurs earlier than \( b_j \).

Then \( V_j = \emptyset \), so there are at least five \( b_k, k > j \) such that \( j \in U_k \cup V_k \). These \( b_k \)’s are clearly 2-element subsets of \( b_j \), but for \( b_j \) to have a decomposition in \( B_j \), two of the 2-element subsets of \( b_j \) must occur earlier than \( b_j \), a contradiction.

(3b) Suppose WLOG that \( b_j = \{1, 2, 3, 4\} \) and that \( \{1, 2, 3\} \) occurs earlier than \( b_j \). Since \( B_{j-1} \) is a building set no other 3-element subset of \( b_j \) occurs before \( b_j \). If \( v(b_j) \) is adjacent to \( v(b_i) \) then either \( k < j \) which forces \( b_k \) to be a 2-element subset of \( \{1, 2, 3\} \), or \( k > j \) which also forces \( b_k \) to be a two element subset of \( \{1, 2, 3\} \) (so that \( \{1, 2, 3\} \backslash b_k \neq \{1, 2, 3, 4\} \backslash b_k \) and \( \{1, 2, 3, 4, 5\} \backslash b_k \neq \{1, 2, 3, 4\} \backslash b_k \)). So \( v(b_j) \) is adjacent to at most three vertices, a contradiction.

Since we have shown that none of the cases (1), (2) or (3) can occur we have a contradiction, as desired.

We will now give a combinatorial description of \( \Gamma(B(K_n)) \) for a particular flag ordering. Let \( O \) be the flag ordering of \( B = B(K_n) \) with decomposition
\[
D = \{\{1\}, \{2\}, \ldots, \{n\}, \{2\}, \{3\}, \ldots, \{n\}\}
\]
where elements \( a, b \in B - D \) are ordered so that \( a \) is earlier than \( b \) if:
- \( \max(a) < \max(b) \), or
- \( \max(a) = \max(b) \) and \( |a| > |b| \), or
- \( \max(a) = \max(b) \), \( |a| = |b| \) and \( \min(a \vartriangle b) \in a \)
where $\nabla$ denotes the symmetric difference between two sets.

Then in $\Gamma(O)$, vertices corresponding to elements $a, b \in B - D$ are adjacent if either:

- $a \subseteq b$ and $\min(b - a) < \max(a)$,
- $\max(a) \notin b$ and $|a| \geq 2$ and $\min(b \setminus a) > \max(a)$.

**Example 4.3.** The edges of $\Gamma(B(K_3))$ are between the consecutive vertices in the following three sequences, which form cycles:

$v(\{1, 4\}), v(\{1, 2, 4, 5\}), v(\{2, 4\}), v(\{2, 3, 4, 5\}), v(\{3, 4\}), v(\{1, 3, 4, 5\}), v(\{1, 4\})$

and

$v(\{1, 3\}), v(\{1, 2, 3, 5\}), v(\{2, 3\}), v(\{4, 5\}), v(\{1, 3\})$

and

$v(\{1, 2, 4\}), v(\{1, 5\}), v(\{1, 3, 4\}), v(\{3, 5\}), v(\{2, 3, 4\}), v(\{2, 5\}), v(\{1, 2, 4\})$.

**4.2. The flag complexes** $\Gamma(B(\text{Path}_n))$ and $\Gamma(\mathcal{E}_n(312))$. The associahedron is the nestohedron $P_{B(\text{Path}_n)}$. Note that $B(\text{Path}_n)$ consists of all intervals $[j, k]$ with $1 \leq j \leq k \leq n$. The $\gamma$-polynomial of the associahedron is the descent generating function of $\mathcal{E}_n(312)$, which denotes the set of 312-avoiding permutations with no double or final descents (see [5, Section 10.2]). We now describe the flag complex $\Gamma(\mathcal{E}_n(312))$ defined by Nevo and Petersen [5, Section 4.2].

Given distinct integers $a, b, c, d$ such that $a < b$ and $c < d$, the pairs $(a, b), (c, d)$ are non-crossing if either

- $a < c < d < b$ (or $c < a < b < d$), or
- $a < b < c < d$ (or $c < d < a < b$).

Define $\Gamma(\mathcal{E}_n(312))$ to be the flag complex on the vertex set

$$V_n := \{(a, b) \mid 1 \leq a < b \leq n - 1\},$$

with faces the sets $S$ of $V_n$ such that if $(a, b) \in S$ and $(c, d) \in S$ then $(a, b)$ and $(c, d)$ are non-crossing.

Let $O$ denote the flag ordering of $B = B(\text{Path}_n)$ with decomposition $D = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{5\}\}$, where elements $a, b \in B - D$ are ordered so that $a$ is earlier than $b$ if:

- $\max(a) < \max(b)$, or
- $\max(a) = \max(b)$ and $|a| > |b|$.

**Proposition 4.4.** For the flag ordering $O$ of $B = B(\text{Path}_n)$ described above, $\Gamma(O) \cong \Gamma(\mathcal{E}_n(312))$ where the bijection on the vertices is given by $v([a+1, b+1]) \mapsto (a, b)$.

**Proof.** Since $B - D = \{[j, k] \mid 2 \leq j < k \leq n\}$, it is clear that the stated map on vertices is a bijection. Let $[l, m], [j, k]$ be distinct elements of $B - D$ with $[l, m]$ occurring before $[j, k]$. Then $m \leq k$, and if $m = k$ we have $l < j$. If $[l, m] \not\subseteq [j, k]$ then $v([l, m])$ is adjacent to $v([j, k])$ if and only if $m < j$. If $[l, m] \subseteq [j, k]$ (which entails $m < k$), then $v([l, m])$ is adjacent to $v([j, k])$ if and only if $j < l$. So in either
case \(v([l, m])\) is adjacent to \(v([j, k])\) if and only if \((l - 1, m - 1)\) and \((j - 1, k - 1)\) are non-crossing.

\[\square\]

4.3. The flag complexes \(\Gamma(B(Cyc_n))\) and \(\Gamma(P_n)\). The cyclohedron is the nestohedron \(B(Cyc_n)\). Note that \(B(Cyc_n)\) consists of all sets \(\{i, i+1, i+2, ..., i+s\}\) where \(i \in [n], s \in \{0, 1, ..., n-1\}\), and the elements are taken mod \(n\). By [8, Proposition 11.15] \(\gamma_r(B(Cyc_n)) = \binom{n}{r, r, n-2r}\). We now describe the flag complex \(\Gamma(P_n)\) defined by Nevo and Petersen [5, Section 4.3].

Define the vertex set

\[V_{P_n} := \{(l, r) \in [n-1] \times [n-1] \mid l \neq r\}.\]

\(\Gamma(P_n)\) is the flag complex on the vertex set \(V_{P_n}\) where vertices \((l_1, r_1), (l_2, r_2)\) are adjacent in \(\Gamma(P_n)\) if and only if \(l_1, l_2, r_1, r_2\) are all distinct and either \(l_1 < l_2\) and \(r_1 < r_2\), or \(l_2 < l_1\) and \(r_2 < r_1\).

**Example 4.5.** \(\Gamma(P_5)\) is the flag complex on vertices

\(V_{P_5} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}\)

with edges

\[
\{(1, 3), (2, 4)\}, \{(3, 1), (4, 2)\}, \{(1, 2), (3, 4)\}, \\
\{(1, 2), (4, 3)\}, \{(2, 1), (4, 3)\}, \{(2, 1), (3, 4)\}.
\]

Note that \(\Gamma(P_5)\) has exactly two vertices of degree two, and has six connected components, four of which contain more than one vertex.

**Proposition 4.6.** There is no flag ordering of \(B(Cyc_5)\) so that \(\Gamma(B(Cyc_5)) \cong \Gamma(P_5)\).

**Proof.** Suppose for a contradiction that there is some flag ordering of \(B = B(Cyc_5)\) with decomposition \(D\) such that \(\Gamma(B(Cyc_5)) \cong \Gamma(P_5)\). It is not too hard to show that if vertices \(v(a)\) and \(v(b)\) are adjacent then at least one of \(a\) or \(b\) is a 2-element set. Therefore there must be at least one vertex that corresponds to a building set element of size two in each of the four non-singleton connected components of \(\Gamma(B(Cyc_5))\). Since there must be one two element subset in \(D\) this implies that there is exactly one vertex corresponding to a two element set in each non-singleton connected component, and these include the vertices of degree two.

The possibilities for \(D\) (up to a cyclic permutation of \(B\)) are

\[
D_1 = \{[5], [4], [3], [2], \{1\}, ..., \{5\}\}, \\
D_2 = \{[5], [2], \{5, 1, 2\}, \{5, 1, 2, 3\}, \{1\}, ..., \{5\}\}, \\
D_3 = \{[5], [4], \{1, 2\}, \{3, 4\}, \{1\}, ..., \{5\}\}, \\
D_4 = \{[5], [3], \{4, 5\}, \{1, 2\}, \{1\}, ..., \{5\}\}.
\]

The flag ordering must have decomposition \(D_1\) or \(D_2\) since there are four elements of size two in \(B(Cyc_5) - D\). We will show that if \(D\) is \(D_1\) or \(D_2\) then there must be two vertices in \(\Gamma(B(Cyc_5))\) that are adjacent that correspond to building
set elements of size two, a contradiction.

The size two elements in $B - D_1$ and $B - D_2$ are $\{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}$. If $b_j \in B - D_j$ is earliest in the flag ordering than every one of these size two elements, then $b_j$ must contain $\{1, 2\}$ since otherwise it would not have a decomposition in $B_j$. So the only elements of $B - D_1$ that can be earlier in the flag ordering than every element of size two are

$S_1 = \{\{5, 1, 2, 3\}, \{5, 1, 2\}, \{4, 5, 1, 2\}\}$.

Similarly, the elements of $B - D_2$ that can be earlier in the flag ordering than every element of size two are

$S_2 = \{\{1, 2, 3\}, \{4, 5, 1, 2\}, \{1, 2, 3, 4\}\}$.

Consider which of the size two elements in $B - D_1$ is earliest. Suppose that $\{1, 5\}$ is earliest in the flag order. Then $v(\{3, 4\})$ is adjacent to $v(\{1, 5\})$ since $\{1, 5\} \not\subseteq (D_1 \cup S_1)/\{3, 4\} = (D_2 \cup S_2)/\{3, 4\}$.

Suppose that $\{2, 3\}$ is earliest in the flag order. Then $v(\{4, 5\})$ is adjacent to $v(\{2, 3\})$ since $\{2, 3\} \not\subseteq (D_1 \cup S_1)/\{4, 5\} = (D_2 \cup S_2)/\{4, 5\}$.

Suppose that $\{3, 4\}$ is earliest in the flag order. Then $v(\{1, 5\})$ is adjacent to $v(\{3, 4\})$ since $\{3, 4\} \not\subseteq (D_1 \cup S_1)/\{1, 5\} = (D_2 \cup S_2)/\{1, 5\}$.

Suppose that $\{4, 5\}$ is earliest in the flag order. Then $v(\{2, 3\})$ is adjacent to $v(\{4, 5\})$ since $\{4, 5\} \not\subseteq (D_1 \cup S_1)/\{2, 3\} = (D_2 \cup S_2)/\{2, 3\}$.

4.4. The flag complex $\Gamma(B(K_{1,n}^-))$. Here we give a combinatorial description of $\Gamma(B(K_{1,n}^-))$ for a particular flag ordering. $B = B(K_{1,n}^-)$ is the graphical building set for the graph $K_{1,n}^-\{1\}$ where we assume the vertex of degree $n - 1$ is labelled 1. So $B(K_{1,n}^-)$ consists of all subsets of $[n]$ containing 1, together with $\{2\}, \{3\}, ..., \{n\}$. Let $O$ be the flag ordering with decomposition

$D = \{[n], [n - 1], ..., [2], \{1\}, \{2\}, ..., \{n\}\}$,

where $a, b \in B - D$ are ordered so that $a$ is earlier than $b$ if:

- $\max(a) < \max(b)$, or
- $\max(a) = \max(b)$ and $|a| > |b|$, or
- $\max(a) = \max(b)$ and $|a| = |b|$ and $\min(a \backslash b) \in a$.

Then in $\Gamma(O)$, vertices corresponding to elements $a, b \in B - D$ are adjacent if either:

- $a \subseteq b$ and $\min(b - a) < \max(a)$,
- $\max(a) \not\subseteq b$ and $|a \setminus b| \geq 2$ and $\min(b \setminus a) > \max(a)$.

Example 4.7. The edges of $\Gamma(B(K_{1,4}))$ are:

$\{v(\{1, 5\}), v(\{1, 2, 4\}), v(\{1, 5\}), v(\{1, 3, 4\})\}$
$\{v(\{1, 3, 4, 5\}), v(\{1, 4\})\}$,
$\{v(\{1, 2, 4, 5\}), v(\{1, 4\})\}$.
In fact, for this flag ordering, the restriction of $\Gamma(K_{1,n-1})$ to the vertices corresponding to sets of size $\geq 3$ is isomorphic to $\Gamma(K_{n-1})$ for the flag ordering defined in Section 4.1.

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