An isospin lattice-gas model, which is a spin-1 Ising model, is employed to investigate the liquid-gas phase transition in asymmetric nuclear matter. We consider nuclear matter as a lattice where each lattice site can be either empty or occupied by a proton or a neutron, with a nearest-neighbor interaction among the nucleons. With the Bragg-Williams mean field approximation, we calculate various thermodynamic properties of nuclear matter for different densities and different proton-neutron asymmetry parameter $s$. Our model exhibits liquid-gas phase transition below a critical temperature $T_c$, and predicts a monotonic decreasing of $T_c$ as the magnitude of $s$ is increased. The dependence of the nuclear matter isotherms on the asymmetry parameter $s$ is discussed.
1. Introduction

The study of the equation of state (EOS) of asymmetric nuclear matter has been, in the last few years, a subject of renewed interest particularly in connection with astrophysical problems\cite{1,2}, such as supernovae explosions and the evolution of neutron stars. The EOS for the latter samples a range of densities and isospin asymmetry which are different from those for supernovae. Thus these two physical systems provide a unique laboratory where the EOS of nuclear matter can be critically investigated and give the possibility of obtaining related but not identical information about the EOS.

Several laboratories have studied the nuclear matter EOS by way of heavy-ion collisions, such as the recent $^{197}$Au+$^{197}$Au reaction\cite{3} investigated at GSI-Darmstadt. The proton-neutron ratio in $^{197}$Au is approximately 2/3. Clearly, the nuclear matter formed via such reactions should be significantly asymmetric, lending further support to the need for carrying out theoretical studies of the EOS for asymmetric nuclear matter.

In the past, asymmetric nuclear matter calculations have been done using standard many-body methods, such as the HF calculations using the Skyrme interactions\cite{4,5} and the Gogny interaction\cite{6}. Brueckner-Hartree-Fock calculations for asymmetric nuclear matter, using realistic nucleon-nucleon (NN) interactions have also been performed\cite{7,8,9}.

We would like to consider a different approach for the EOS of asymmetric nuclear matter. In statistical physics, phase transitions in extended systems are generally studied using lattice (Ising) models. For instance, the lattice gas model of Lee and Yang\cite{10}, where a gas with one type of atoms is mapped into an Ising model for spin-$\frac{1}{2}$ particles, has successfully described the liquid-gas phase transitions for atomic systems. It should be of interest to explore the feasibility of describing the nuclear-matter EOS using lattice-gas model. Indeed, several authors\cite{11,12,13,14,15} have employed lattice-gas models to study the liquid-gas phase transition in symmetric nuclear matter. In this paper we present a lattice-gas model for investigating the liquid-gas phase transitions in asymmetric nuclear matter.
2. Formulation

We revisit the lattice Hamiltonian for nuclear matter proposed by Kuo et al. [11, 15],

\[ H_{int} = - \sum_{<ij>} J_{ij} \tau_i \tau_j - h \sum_i \tau_i \]  

(1)

The summation index \(<ij>\) implies nearest neighbor interaction and \(h\) is some applied external field. \(J_{ij}\) are interaction strength parameters: \(J_s\) for symmetric pairs (pp and nn) and \(J_d\) for asymmetric type pairs (pn pairs). The \(J_{ij}\) have the property

\[ J_{ij} = \begin{cases} J_{ij} & \text{for nearest neighbor distance} = a \\ \infty & \text{for neighbor distance} = 0 \\ 0 & \text{otherwise} \end{cases} \]  

(2)

where \(a\) is the lattice spacing. The isospin \(\tau_z\) is

\[ \tau_z = \begin{cases} -1 & \text{for neutron} \\ 0 & \text{for vacancy} \\ 1 & \text{for proton} \end{cases} \]  

(3)

If \(N_{++}, N_{--}\) and \(N_{+-}\) represent respectively the nearest neighbor pairs of proton-proton, neutron-neutron and proton-neutron, the interaction Hamiltonian may be written as,

\[ H_{int} = -J_s(N_{++} + N_{--}) + J_dN_{+-} - h(N_+ - N_-) \]  

(4)

There is no interaction between vacancy and nucleon.

3. Thermodynamics

Let \(N\) denote the total number of lattice sites, and \(N_+, N_-, N_0\) the number of proton, neutron and vacancy sites respectively. The relative emptiness \((r)\) and proton-neutron asymmetry parameter \((s)\) are defined as,

\[ r = \frac{N_0}{N}, \quad s = \frac{N_+ - N_-}{N}, \quad N = N_+ + N_- + N_0 \]  

(5)
The nuclear-matter density $\rho$ is proportional to $(1 - r)$ in our model.

For atomic systems, where the spin-$1/2$ lattice gas models have been used with remarkable success in describing phase transitions, Huang [16] suggests adding the ideal gas pressure to the lattice gas grand potential in order to describe the isotherms. The ideal gas pressure comes from the kinetic energy. This motivated us to include a kinetic-energy term in our hamiltonian itself, namely

$$H_{\text{gas}} = -J_\text{s}(N_{++} + N_{--}) + J_\text{d}N_{+-} + N\kappa(1 - r)^{5/3} - Nh s$$  \hspace{1cm} (6)

where we have assumed the kinetic energy per particle to be proportional to $\rho^{2/3}$ as per the Fermi gas model. $\kappa$ is a constant which will be discussed later. From the above hamiltonian we can write down the partition function. For the evaluation of the partition function, we have employed the Bragg-Williams mean field approximation [16], namely

$$\frac{N^2}{N^2} \approx \frac{N_{++}}{N\gamma/2}, \quad \frac{N^2}{N^2} \approx \frac{N_{+-}}{N\gamma/2}, \quad \frac{N^2}{N^2} \approx \frac{N_{00}}{N\gamma/2}$$ \hspace{1cm} (7)

where $\gamma$ denotes the number of nearest neighbors to any given site, and $N\gamma/2$ is the total number of pairs. For three dimensional simple cubic lattice, $\gamma = 6$. If one counts the number of bonds between a site and all its nearest neighbors, then for (say) all the proton sites one obtains

$$\gamma N_+ = 2N_{++} + N_{+-} + N_{-0}$$ \hspace{1cm} (8)

and similarly for the neutron and vacant sites.

Using the above we can eliminate $N_{+-}$ as,

$$N_{+-} = \gamma(N_+ + N_-) - (N_{++} + N_{--}) + N_{00} - \gamma N/2$$ \hspace{1cm} (9)

The hamiltonian of eq. (6) can then be rewritten as

$$H_{\text{gas}}(r, s, N) = -C_1Ns^2 - C_2N(1 - r)^2 + N\kappa(1 - r)^{5/3} - Nh s$$ \hspace{1cm} (10)

where

$$C_1 = \frac{(J_\text{s} + J_\text{d})\gamma}{4}, \quad C_2 = \frac{(J_\text{s} - J_\text{d})\gamma}{4}$$ \hspace{1cm} (11)
Unlike the symmetric case, $s$ is now a parameter and not a variable. The canonical partition function is therefore given by

$$Q_{gas} = \sum_{r=0}^{1} g(r, s, N) e^{-\beta H_{gas}}$$  \hspace{1cm} (12)

where the multiplicity factor \cite{17} $g(r, s, N)$ is given by

$$g(r, s, N) = \frac{N!}{N_0!N_+!N_-!} = \frac{N!}{(N r)![\frac{N}{2}(1 + r + s)]![\frac{N}{2}(1 - r - s)]!}$$  \hspace{1cm} (13)

To study the system at varying densities, we need to work with the grand partition function with a chemical potential $\lambda$

$$Q_G = \sum_r g(r, s, N) z^{(N_+ + N_-)} e^{-\beta H_{gas}}$$

$$= \sum_r g(r, s, N) z^{(N_+ + N_-)} e^{-\beta H_{gas}}$$  \hspace{1cm} (14)

where $z = e^{\beta \lambda}$ is the fugacity.

In the thermodynamic limit ($N \to \infty$), the sum in eq. (14) can be replaced by its most dominant term \cite{17} (assuming the dominant term to be non-degenerate). We have verified that this approximation is acceptable by numerically checking the relative size of the different leading order terms. Using Stirling’s formula one obtains

$$\ln Q_G = \beta C_1 N s^2 + \beta C_2 N(1 - r)^2 + \beta h N s + \beta \lambda N (1 - r) - \beta \kappa N (1 - r)^{5/3}$$

$$-N[\frac{(1 - r + s)}{2}]\ln(1 - r + s) + \frac{(1 - r - s)}{2}\ln(1 - r - s)$$

$$+r\ln r - (1 - r)\ln 2]$$  \hspace{1cm} (15)

with the extremum condition

$$\frac{\partial (\ln Q_G)}{\partial r} \equiv -\frac{N}{2}\ln[\frac{4r^2}{(1 - r)^2 - s^2}] - 2\beta NC_2(1 - r) + \frac{5}{3}\beta \kappa N (1 - r)^{2/3}$$

$$+\beta N \lambda = 0$$  \hspace{1cm} (16)

In a previous work \cite{15} it was shown that $h = 0$ and $s = 0$ results in a global maximum for this term. In the present case however we allow $s$ to vary parametrically. Factoring out the common terms eq. (16) is written as

$$k_B T \ln[\frac{4r^2}{1 - r^2 - s^2}] + 2C_2(1 - r) - \frac{5}{3}\kappa (1 - r)^{2/3} + \lambda = 0.$$  \hspace{1cm} (17)

A graphical analysis of the above equation reveals the existence of a condensed phase.
4. Phase transition and critical temperature

Let us consider the $\lambda=0$ case first. In this case we rewrite eq. (17) as

$$\chi(r, T) = k_B T \cdot f(r, T) - g(r) = 0$$

with

$$f(r) \equiv \frac{1}{2} \ln\left[\frac{4r^2}{(1-r)^2 - s^2}\right]$$

$$g(r) \equiv \frac{5\kappa}{3}(1-r)^{2/3} - 2C_2(1-r)$$

In the region $r \in [0, 1]$, $k_B T \cdot f(r)$ is a monotonically increasing, unbounded function of $r$. It has one point of inflection where it intersects the $r$ axis and this is at $r = \frac{1}{3}$ when $s = 0$. In the same domain $g(r)$ however, is a bounded function with a negative curvature and one maximum in the region $r \in [0, 1]$. Thus there is a possibility of $g(r)$ intersecting with $k_B T \cdot f(r)$ more than once below a suitable temperature $T_c$. $k_B T \cdot f(r)$ intersects the $r$ axis at $r = \frac{-1+\sqrt{1+3(1-s^2)}}{3}$ which depends on $s$ but is independent of the value of $T$. On the other hand $g(r)$ intersects the $r$ axis at two points: One is at $r = 1$ which is independent of the values of the parameters $C_2$, $\kappa$, and $s$, and the second point of intersection depends on $s$ and the ratio $\alpha \equiv 5\kappa/6C_2$. For a suitable choice of the parameter $\alpha$, the functions $k_B T \cdot f(r)$ and $g(r)$ intersect the $r$ axis at a common point given by $r = \frac{-1+\sqrt{1+3(1-s^2)}}{3}$. The $T_c$ is then determined by that value of temperature for a given $s$ at which the curves $k_B T \cdot f(r)$ and $g(r)$ are tangential at their common zero point (point of intersection with the $r$ axis).

In Fig. 1 we display some typical behaviours of $k_B T \cdot f(r)$ and $g(r)$. The points of intersection of the two curves are shown for a family of $k_B T \cdot f(r)$ and $g(r)$ with $k_B T = 5, 10, 15$ MeV. This is done for several values of the asymmetry parameter.

For $T < T_c$, the curves $k_B T \cdot f(r)$ and $g(r)$ have three intersection points, as denoted by $A$, $B$, and $C$ for the $k_B T = 5$ MeV case. It is readily checked that the middle intersection point, i.e. $B$, corresponds to a minimum of $\ln Q_G$ and hence it is
not the physically interesting solution that we are looking for. The intersection points A and C are the physical solutions.

At some critical temperature \( T_c \) the curves \( k_B T_c \cdot f(r) \) and \( g(r) \) are tangent to each other at their common zero point. Above this critical temperature \( T_c \), the two curves intersect only once at their common point of intersection. Below \( T_c \) however, the slope of \( k_B T \cdot f(r) \) at its zero point is less than that of \( g(r) \) and two more points of intersection appear which represent the densities of the two different phases. Thus the critical vacancy \( r_0 \) and the critical temperature \( T_c \) are determined by

\[
k_B T_c \cdot f(r_0) = g(r_0)
\]
\[
k_B T_c \cdot f'(r_0) = g'(r_0)
\] (19)

As the magnitude of the asymmetry parameter \( s \) increases, the curve \( T \cdot f(r) \) becomes steeper and multiple intersection with \( g(r) \) becomes possible only at lower temperatures. For pure neutron matter \( (s = -1.0) \) the critical temperature from our model is equal to 0 and there is only one phase as is expected. (Fig. 3)

5. Equation of state and isotherms

The equation of state is obtained as

\[
P(\bar{r}, T) \equiv \frac{k_B T}{N} \ln Q_G
\]
\[
= C_1 s^2 + h s + C_2 (1 - \bar{r})^2 + \lambda(1 - \bar{r}) - \kappa(1 - \bar{r})^{5/3} - k_B T \left[ \frac{(1 - \bar{r} + s)}{2} \ln(1 - \bar{r} + s) + \frac{(1 - \bar{r} - s)}{2} \ln(1 - \bar{r} - s) \right] + \bar{r} \ln \bar{r} - (1 - \bar{r}) \ln 2 \] (20)

\[
\frac{1}{v} \equiv \frac{z}{N} \frac{\partial}{\partial z} \ln Q_G = (1 - \bar{r})
\] (21)

Note that the first two terms contribute just an additive constant to the pressure for any given \( s \). We calculate the p-v isotherms using eqs.(18), (20) and (21). Our results are shown in Figs. 4 to 6.

For a given temperature \( T \) the isotherm is obtained as a parametric plot of specific volume versus pressure with \( \lambda \) its generating parameter. For \( T \geq T_c \) one gets a single
smooth curve by varying $\lambda$. The phase boundary corresponds to $\lambda = 0$. The isotherms to the left of this boundary are obtained with positive $\lambda$, those to the right with negative $\lambda$. The figure suggests the existence of three regions below $T_c$, the dense (small $r$, large $\rho$) liquid-like phase, the rare (large $r$, small $\rho$) gas-like phase and the coexistence phase in between.

For $T$ below $T_c$, it is of interest to note that no isotherms are obtained in the intermediate region, i.e. the coexistence region. The above is because when $T < T_c$, eq. (18) has no solutions in the coexistence region. With the introduction of an infinitesimal $\lambda$ the system chooses one of the two values of $\bar{r}$ (or $v = \frac{1}{1-r}$) admissible for the given $T < T_c$, depending on the sign of $\lambda$. This phenomena is reminiscent of the spontaneous symmetry breaking in ferromagnetism.

The isotherms for different asymmetry parametered systems are shown in Fig. 4, 5 and 6. These graphs are obtained by a very densely chosen set of points where eq. (18) is solved and the corresponding pressure and volume determined. To avoid falling into the trap of seeing a smooth isotherm when it is not, we intentionally chose not to use spline, or any higher polynomial fit. A very densely packed set of points with short straight line connects generated the isotherms. The absence of jaggedness that can be seen is coming from the physics and calculation itself, and not due to an artifact of any particular curve fitting technique.

6. Analysis and results

To have an attractive nearest-neighbor interaction, $J_s > 0$ and $J_d < 0$. In this case $C_2$ is positive. The magnitude of $J_s$ and $J_d$ is comparable to the average potential energy in nuclear matter, about $-40$ MeV. Hence $C_2 \simeq 125$ MeV, recalling that $C_2$ has been defined in eq. (11). The parameter $\kappa$ may be estimated from the average kinetic energy given by the Fermi gas model.

Near the critical point, there is a subtle dependence of the solutions of eq. (18) on the ratio $\alpha \equiv \frac{5\kappa}{6C_2}$. For a choice of $\alpha = (2/3)^{1/3}$, the three intersection points below $T_c$ ($A$, $B$ and $C$ of Fig. 1), all merge together at the critical point as in the $s = 0$ case[13].
Then the phase diagram near $T_c$ has the familiar smooth shape. For a different choice of $\alpha$, the merging would generally take place in two steps, first involving two intersection points and then the third. This will lead to a phase diagram with a *cusp* shape near the critical point, which seems rather unconventional. It would be interesting to investigate these unconventional phase boundaries further for different values of $\alpha$. For the present study however, we have chosen $\alpha$ so as to get a smooth phase boundary. This choice of $\alpha$ is one for which the curves $k_B T \cdot f(r)$ and $g(r)$ intersect the $r$ axis at a common point. With this $\alpha$ and $C_2=125$, we have $\kappa=131.037$, which has been used in obtaining the results presented in our Figs. 1 to 6.

For fixed $T$, the solutions of eq. (17) determine the $r$ values where the grand partition function is maximum. Very similar to what was observed in the $s = 0$ case, this maximum term is an overwhelmingly dominant one. (Numerical simulation for finite lattices has shown that the magnitude of the maximum term is typically a factor of $\sim 10^{80}$ greater than the other terms.) Hence the $r$ values given by eq. (18) are just the average values (strictly speaking the modal value) of $r$, denoted by $\bar{r}$, for the system at temperature $T$, asymmetry $s$ and $\lambda=0$ for which $\ln Q_G$ is maximum. The $\bar{r}$ values for $\lambda \neq 0$ are given by the solution of eq. (17).

We find support of our earlier observation [15] in our present calculation as well, that a term of Fermi gas type kinetic energy is needed for appropriate phase structure to appear. If there is no kinetic energy term, $\kappa = 0$ and hence $\alpha = 0$. Then the requirement that at $T < T_c$, $g(r)$ intersect $T \cdot f(r)$ more than once cannot be met, and our model would have no phase transitions.

**Conclusion**

Our simple model exhibits the existence of a liquid gas phase transition in asymmetric nuclear matter. The $p$-$v$ isotherms obtained by our model look surprisingly similar to those given by the van der Waals theory, except for one crucial difference that for $T < T_c$ our isotherms do not have the metastable states in the coexistence region. Hence with our model one does not need to determine the phase boundary by way of
a Maxwell construction.

The liquid-gas critical temperature obtained is dependent on the asymmetry parameter $s$, $T_c$ decreases with increasing neutron density as shown in Fig. 3. The value is around $T_c = 17 - 20$ for typical values of $s$ which is fairly close to the results given by earlier calculations[4, 18, 19, 15]. The existence of a liquid-gas phase transition together with the determination of its critical temperature on the basis of a simple model that assumes only a phenomenological two-body, nearest-neighbor interaction is quite remarkable.
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FIGURE CAPTIONS

**Fig. 1** Graphical solution of eq.(18) for $s=-0.2$.

**Fig. 2** Graphical solution of eq.(18) for $s=-0.4$.

**Fig. 3** Critical Temperature versus asymmetry parameter.

**Fig. 4** Nuclear matter $p-v$ isotherms for $s=-0.2$.

**Fig. 5** Nuclear matter $p-v$ isotherms for $s=-0.4$.

**Fig. 6** Nuclear matter $p-v$ isotherms for $s=-0.6$.
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