GEOMETRIC OPTICS AND INSTABILITY FOR
SEMI-CLASSICAL SCHRÖDINGER EQUATIONS

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Abstract. We prove some instability phenomena for semi-classical (linear or) nonlinear Schrödinger equations. For some perturbations of the data, we show that for very small times, we can neglect the Laplacian, and the mechanism is the same as for the corresponding ordinary differential equation. Our approach allows smaller perturbations of the data, where the instability occurs for times such that the problem cannot be reduced to the study of an o.d.e.

1. Introduction

Consider the semi-classical Schrödinger equation:

\[ i\hbar \partial_t u^h + \frac{\hbar^2}{2} \Delta u^h = |u^h|^2 u^h ; \quad u^h(0, x) = a_0(x), \]

where \( x \in \mathbb{R}^n \), the parameter \( \hbar > 0 \) goes to zero and the initial datum \( a_0 \) is independent of \( \hbar \). We prove that small perturbations of \( a_0 \) cause divergence of the corresponding two solutions on small time intervals. For instance, assume that \( a_0 \) is smooth, \( a_0 \in S(\mathbb{R}^n) \). Consider \( v^h \) solving (1.1) with datum \( a_0 + h^{1-N} a_1 \), with \( a_1 \in S(\mathbb{R}^n), \text{Re}(a_0 a_1) \neq 0 \) and \( N > 0 \). Then there exists \( c > 0 \) independent of \( \hbar \) such that for \( t^h = h^{1/N} \):

\[ \liminf_{\hbar \to 0} \| u^h(t^h) - v^h(t^h) \|_{L^2} \geq c. \]

Such an instability phenomenon goes in the same spirit as the study of G. Lebeau [27] (see also [26], [31]; see [28] for further developments) for the nonlinear wave equation, and followed for instance in [2], [14, 15] (see also the appendix of [3]) and [4] for nonlinear Schrödinger equations. For the above example, in the case \( N < 3 \), our approach relies on the fact that for very small times, the dispersive effects due to the Laplacian are negligible, as in [27] and [15]: a good approximation to the Schrödinger equation is then provided by an ordinary differential equation, which can be solved explicitly. In the case \( N \geq 3 \), the instability mechanism occurs for times such that the action of the Laplacian is no longer negligible, and the equation cannot be reduced to an ordinary differential equation. In that case, our analysis relies on small time properties of solutions of the compressible Euler equation, which describes the semi-classical limit for (1.1).

We also consider weaker nonlinearities in space dimension \( n \geq 2 \), with or without harmonic potential (\( \omega \geq 0 \)):

\[ i\hbar \partial_t u^h + \frac{\hbar^2}{2} \Delta u^h = \omega^2 |x|^2 u^h + h^k |u^h|^2 u^h ; \quad u^h(0, x) = a_0(x). \]

In space dimension three, we can take \( k = 2 \) and \( \omega > 0 \), thus recovering the scaling of [4] corresponding to Bose–Einstein condensation in dimension three with

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repulsive nonlinearity. Unlike in [4] where initial data concentrated at one point
with scale $h$ are considered, we assume that the initial data is independent of $h$,
$v_{1,0} = a_0(x)$. However, the instability mechanism we describe occurs at a
time where the solution is concentrated. The concentration is due to the presence of the
harmonic oscillator, but the rate of concentration when instability occurs is smaller
than in [4] ($h^a$ with $a < 1$, see Section 6 for more details).

So far, we have considered only cubic nonlinearities. As in [20], we extend the
framework to nonlinearities of the form $f(|u|^2)u$ which are smooth, repulsive, and
cubic at the origin:

**Assumptions.** Let $f$ be smooth: $f \in C^\infty(R_+; R)$, with $f(0) = 0$ and $f' > 0$.

**Remark.** The assumption $f(0) = 0$ is neutral, since constant potentials for
Schrödinger equations can be absorbed by an easy change of unknown function.

**Notation.** Let $(\alpha^h)_0<h\leq 1$ and $(\beta^h)_0<h\leq 1$ be two families of positive real numbers.
- We write $\alpha^h \ll \beta^h$ if $\limsup_{h\to0} \alpha^h/\beta^h = 0$.
- We write $\alpha^h \lesssim \beta^h$ if $\limsup_{h\to0} \alpha^h/\beta^h < \infty$.
- We write $\alpha^h \approx \beta^h$ if $\alpha^h \lesssim \beta^h$ and $\beta^h \lesssim \alpha^h$.

**Theorem 1.2.** Let $n \geq 1$, $a_0, \tilde{a}_0 \in S(R^n)$, $\phi_0 \in C^\infty(R^n; R)$, where $a_0$ and $\phi_0$ are
independent of $h$, and $\nabla \phi_0 \in H^s(R^n)$ for every $s \geq 0$. For $\omega \geq 0$, let $u^h$ and $v^h$
solve the initial value problems:

$$ ih\partial_t u^h + \frac{\hbar^2}{2} \Delta u^h = \omega^2 \frac{|x|^2}{2} u^h + f(|u^h|^2) u^h ; \\
\quad u^h(0, x) = a_0(x)e^{i\phi_0(x)/h}.$$  

$$ ih\partial_t v^h + \frac{\hbar^2}{2} \Delta v^h = \omega^2 \frac{|x|^2}{2} v^h + f(|v^h|^2) v^h ; \\
\quad v^h(0, x) = \tilde{a}_0(x) e^{i\phi_0(x)/h}.$$  

Assume that there exists $N \in \mathbb{N}$ and $h^{1-\frac{1}{N}} \ll \delta^h \ll 1$ such that:

$$ \|a_0 - \tilde{a}_0^h\|_{H^s} \approx \delta^h, \quad \forall s \geq 0 ; \quad \limsup_{h\to0} \left\| \frac{\text{Re}(a_0 - \tilde{a}_0^h)a_0}{\delta^h} \right\|_{L^\infty(R^n)} \neq 0. $$

Then we can find $0 < t^h \ll 1$ such that: $\|u^h(t^h) - v^h(t^h)\|_{L^2} \gtrsim t^h \delta^h \gtrsim h$. In particular,

$$ \left\| u^h - v^h \right\|_{L^\infty([0, t^h]; L^2)} \to +\infty \quad \text{as } h \to 0. $$

**Remark.** We state assumptions in $H^s$ for every $s \geq 0$. It will appear in the proof
that choosing $s$ large enough would suffice. Similarly, the assumption $a_0, \tilde{a}_0^h \in S(R^n)$
is not necessary in our proof.

**Remark.** The second part of the assumption [102] can be viewed as a polarization
condition. We could remove it with essentially the same proof as below, up to
demanding $h^{1/2-1/N} \ll \delta^h \ll 1$.

The above result can be applied in the following cases:

**Example 1.** Consider $a_0, b_0 \in S(R^n)$ independent of $h$, such that $\text{Re}(\overline{a_0}b_0) \neq 0$, and
take $\tilde{a}_0^h = a_0 + \delta^h b_0$.

**Example 2.** Consider $a_0 \in S(R^n)$ independent of $h$ and $x^h \in R^n$. We can take
$\tilde{a}_0^h(x) = a_0(x - x^h)$, provided that $|x^h| = \delta^h$ and

$$ \limsup_{h\to0} \left\| \frac{x^h}{|x^h|} \cdot \nabla (|a_0|^2) \right\|_{L^\infty} \neq 0. $$
The above result addresses perturbations which satisfy in particular \( \delta^h \gg h \). This excludes the standard WKB data of the form

\[
u^h |_{t=0} = a_h(\xi)e^{i\phi_h(\xi)/h}, \quad \text{where } a_h \sim a_0 + ha_1 + h^2a_2 + \ldots
\]

In that case, a perturbation of \( a_1 \) is relevant at time \( t^h \approx 1 \), and the previous result is essentially sharp:

**Proposition 1.3.** Let \( n \geq 1, a_0, a_1 \in \mathcal{S}(\mathbb{R}^n), \phi_0 \in C^\infty(\mathbb{R}^n; \mathbb{R}) \) independent of \( h \), with \( \nabla \phi_0 \in H^s(\mathbb{R}^n) \) for every \( s \geq 0 \). Assume that \( \text{Re}(a_0a_1) \neq 0 \). For \( \omega \geq 0 \), let \( u^h \) and \( v^h \) solve the initial value problems:

\[
\begin{align*}
ih\partial_t u^h + \frac{h^2}{2} \Delta u^h &= \omega^2 \frac{|\xi|^2}{2} u^h + f \left( |u^h|^2 \right) u^h; \quad u^h(0, x) = a_0(x)e^{i\phi_0(x)/h}, \\

ih\partial_t v^h + \frac{h^2}{2} \Delta v^h &= \omega^2 \frac{|\xi|^2}{2} v^h + f \left( |v^h|^2 \right) v^h; \quad v^h(0, x) = a_0(x)e^{i\phi_0(x)/h}.
\end{align*}
\]

Then for any \( \tau^h \ll 1 \), \( \|u^h - v^h\|_{L^\infty([0, \tau^h]; L^2)} \ll 1 \), and for \( t > 0 \) independent of \( h \) arbitrarily small: \( \|u^h(t) - v^h(t)\|_{L^2} \gtrsim 1 \).

For weaker nonlinearities, we have the following result. The notation \( \varepsilon \) for the small parameter instead of \( h \) is neither a mistake nor a coincidence (see Section 6).

**Corollary 1.4.** Let \( n \geq 2, 1 < k < n, a_0, a_0 \in \mathcal{S}(\mathbb{R}^n) \) and \( \omega \geq 0 \). Let \( u^\varepsilon \) and \( v^\varepsilon \) solve the initial value problems:

\[
\begin{align*}
i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon &= \omega^2 \frac{|\xi|^2}{2} u^\varepsilon + f \left( \varepsilon^k |u^\varepsilon|^2 \right) u^\varepsilon; \quad u^\varepsilon |_{t=0} = a_0(x)e^{i\phi_0(x)/\varepsilon}, \\
i\varepsilon\partial_t v^\varepsilon + \frac{\varepsilon^2}{2} \Delta v^\varepsilon &= \omega^2 \frac{|\xi|^2}{2} v^\varepsilon + f \left( \varepsilon^k |v^\varepsilon|^2 \right) v^\varepsilon; \quad v^\varepsilon |_{t=0} = a_0(x)e^{i\phi_0(x)/\varepsilon}.
\end{align*}
\]

Assume that:

- Either: there exists \( N \in \mathbb{N} \) and \( \varepsilon^{1-\frac{1}{N}-\frac{1}{k}} \ll \delta^\varepsilon \ll 1 \) such that:

  \[
  \|a_0 - \tilde{a}_0^\varepsilon\|_{H^s} \approx \delta^\varepsilon, \quad \forall s > 0 \quad \limsup_{\varepsilon \to 0} \frac{\text{Re}(a_0 - \tilde{a}_0^\varepsilon)R_{\varepsilon}}{\delta^\varepsilon} \neq 0.
  \]

- Or: \( \tilde{a}_0^\varepsilon = a_0 + \varepsilon^{1-\frac{1}{k}}a_1 \), with \( a_1 \in \mathcal{S}(\mathbb{R}^n), \text{Re}(a_0a_1) \neq 0 \).

1. \( \omega = 0 \): let \( \phi_0(x) = -|\xi|^2/2 \). There exist \( T^\varepsilon \to 1^- \) and \( 0 < \tau^\varepsilon \ll 1 \) such that:

  \[
  \|u^\varepsilon - v^\varepsilon\|_{L^\infty([0,T^\varepsilon];L^2)} \ll 1 \quad \|u^\varepsilon - v^\varepsilon\|_{L^\infty([0,T^\varepsilon+\tau^\varepsilon];L^2)} \gtrsim 1.
  \]

2. \( \omega > 0 \): let \( \phi_0 \equiv 0 \). There exist \( T^\varepsilon \to \frac{\pi}{2\omega^-} \) and \( 0 < \tau^\varepsilon \ll 1 \) such that \( \text{[13]} \) holds.

**Example 3.** If \( n = 3, k = 2 \) and the nonlinearity is cubic, we consider:

\[
i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = \omega^2 \frac{|\xi|^2}{2} u^\varepsilon + \varepsilon^2 |u^\varepsilon|^2 u^\varepsilon.
\]

Then perturbations of order \( \delta^\varepsilon \) with \( 1 \gg \delta^\varepsilon \gg \varepsilon^{1/3-1/N} \) cause instability. On the other hand, this phenomenon does not occur for the same equation in space dimension two, and there is stability for a large class of initial data (see \([5, 9]\)).

**Remark 1.5.** We have \( T^\varepsilon \to 1^- \) in the first case because of the initial quadratic oscillations. In the linear case, such oscillations cause focusing at the origin at time \( t = 1 \) (see e.g. \([3]\)). We will see that the instability mechanism occurs when the solution is no longer of order \( O(1) \) and is already concentrated at scale \( 1 - T^\varepsilon \). In the case \( \omega > 0 \), a similar phenomenon occurs without initial phase because the action of the harmonic oscillator is similar. In both cases, taking \( \phi_0(x) = -b|x|^2 \) and modulating \( b \), we could have the instability mechanism occur near any time \( T > 0 \), and not only near \( \frac{\pi}{2\omega^-} \).
Remark 1.6. In Corollary 1.4, the assumption \( k < n \) is crucial. When \( k = n \), the above result is no longer true (see [23, 15, 8, 12] for an homogeneous nonlinearity, with or without harmonic potential). From the point of view of geometrical optics, assuming \( k < n \) amounts to considering a super-critical régime if a caustic reduced to a point appears. This goes in the spirit of the formal computations of [21], and of the papers [22, 23, 24], [27, 31], [5, 7, 8].

The above results show in particular that computing directly semi-classical limits of nonlinear Schrödinger equations by numerical methods is highly challenging.

Compare with other results on instability. In [31, 15, 13, 4], the perturbation are of order \( |\ln h|^{-\theta} \) for some \( \theta > 0 \). Then instability occurs at time of order \( h |\ln h|^{\theta/2} \). Our analysis allows smaller perturbations, and the instability occurs a little later.

The paper is organized as follows. Section 2 is devoted to a general discussion on WKB methods for nonlinear Schrödinger equations. In Section 3, we give heuristic arguments to prepare the proof of Theorem 1.2 and Corollary 1.4. The proof of Theorem 1.2 is completed in Section 5. Corollary 1.4 is shown in Section 6. In a first appendix, we exhibit some results of [14, 15] can be recovered from semi-classical analysis. Finally in Appendix 3, we establish the following result:

**Corollary 1.7.** Let \( n \geq 3 \). Consider the cubic, defocusing Schrödinger equation:

\[
(1.4) \quad i\partial_t u + \frac{1}{2}\Delta u = |u|^2 u ; \quad u|_{t=0} = u_0 .
\]

Denote \( s_c = \frac{n}{2} - 1 \). Let \( 0 < s < s_c \). We can find a family \( (u_0^\varepsilon)_{0<\varepsilon \leq 1} \) in \( S(\mathbb{R}^n) \) with

\[
\|u_0^\varepsilon\|_{H^s(\mathbb{R}^n)} \to 0 \quad \text{as} \quad \varepsilon \to 0 ,
\]

and \( 0 < t^\varepsilon \ll 1 \) such that the solution \( u^\varepsilon \) to (1.4) associated to \( u_0^\varepsilon \) satisfies:

\[
\|u^\varepsilon(t^\varepsilon)\|_{H^k(\mathbb{R}^n)} \to +\infty \quad \text{as} \quad \varepsilon \to 0 , \quad \forall k \in \left[ \frac{s}{2} - s , s \right] .
\]

In particular, for any \( t > 0 \), the map \( u_0 \mapsto u(t) \) given by (1.4) fails to be continuous at \( 0 \) from \( H^s \) to \( H^k \).

Remark 1.8. This result can be viewed as a weak version of the analog result to [28] for the cubic, defocusing Schrödinger equation. An important difference though is that we consider a sequence of initial data, while in [28], G. Lebeau considers the weak solution associated to a fixed initial data. What prevents us from filling this gap is the finite speed of propagation which is used in [28] for the wave equation, and is not available for Schrödinger equations.

2. WKB METHODS FOR NONLINEAR SCHRODINGER EQUATIONS

Consider the initial value problem, for \( x \in \mathbb{R}^n \):

\[
(2.1) \quad i\hbar \partial_t u^\hbar + \frac{\hbar^2}{2}\Delta u^\hbar = h^\kappa |u^\hbar|^2 u^\hbar ; \quad u^\hbar|_{t=0} = a_0^\hbar(x) e^{i\phi_0(x)/\hbar} .
\]

The aim of WKB methods is to describe \( u^\hbar \) in the limit \( \hbar \to 0 \), when \( \phi_0 \) does not depend on \( \hbar \), and \( a_0^\hbar \) has an asymptotic expansion of the form:

\[
a_0^\hbar(x) \sim a_0(x) + h a_1(x) + h^2 a_2(x) + \ldots
\]

The parameter \( \kappa \geq 0 \) describes the strength of a coupling constant, which makes nonlinear effects more or less important in the limit \( \hbar \to 0 \); the larger \( \kappa \), the weaker the nonlinear interactions. Note that since we consider an homogeneous
nonlinearity, this amounts to considering the case where the coupling constant is 1, with initial data of order $\hbar^{\kappa/2}$.

An interesting feature of (2.1) is that one does not expect the creation of harmonics. The WKB methods consist in seeking an approximate solution to (2.1) of the form:

$$u^\hbar(t, x) \sim \left(a^{(0)}(t, x) + h a^{(1)}(t, x) + h^2 a^{(2)}(t, x) + \ldots \right) e^{i\phi(t, x)/\hbar}.$$  

For such an expansion to be available with profiles $a^{(j)}$ independent of $\hbar$, it is reasonable to assume that $\kappa$ is an integer, $\kappa \in \mathbb{N}$. One must not expect this approach to be valid when caustics are formed: roughly speaking, when a caustic appears, all the terms $\phi$, $a^{(0)}$, $a^{(1)}$, ... become singular. In this paper, we always consider times preceding this break-up.

2.1. Notion of criticality. If $\kappa \geq 2$, then nonlinear effects are negligible at leading order in WKB methods. On the other hand, if $\kappa = 1$ (weakly nonlinear geometric optics), then nonlinear effect are relevant at leading order. The present discussion is formal, its aim being to prepare the study of the case $\kappa = 0$.

When $\kappa \geq 2$, plugging the asymptotic expansion (2.2) into (2.1) yields formally:

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0 ; \quad \phi|_{t=0} = \phi_0.$$  

$$\partial_t a^{(0)} + \nabla \phi \cdot \nabla a^{(0)} + \frac{1}{2} a^{(0)} \Delta \phi = 0 ; \quad a^{(0)}|_{t=0} = a_0.$$  

The first equation is the well known eikonal equation, which describes the geometry of the propagation. If $\phi_0$ is smooth, it has a smooth solution, locally in time. This solution may become singular in finite time, this phenomenon being the formation of a caustic.

The second equation is a transport equation, which is simply an ordinary differential equation for the leading order amplitude along the rays of geometrical optics. To see this, introduce a parametrization of these rays:

$$\frac{d}{dt} X_t(x) = \nabla \phi(t, X_t(x)) ; \quad X_0(x) = x,$$

and the Jacobi determinant: $J_t(x) = \det \nabla X_t(x)$. It is well defined and smooth so long as no caustic appears. The break-up time $t_c > 0$, if any, is such that there exists $x_c$ such that $J_t(x_c) = 0$. The transport equation for $a^{(0)}$ is the trivial ordinary differential equation:

$$\frac{d}{dt} \left(a^{(0)}(t, X_t(x)) \sqrt{J_t(x)} \right) = 0.$$  

From this, we easily see that when a caustic appears, not only $\phi$ becomes singular, but also $a^{(0)}$, since $J_t(x)$ goes to zero at the caustic.

The value $\kappa = 1$ is critical as far as leading order phenomena are concerned: the transport equation for $a^{(0)}$ is then nonlinear,

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0 ; \quad \phi|_{t=0} = \phi_0.$$  

$$\partial_t a^{(0)} + \nabla \phi \cdot \nabla a^{(0)} + \frac{1}{2} a^{(0)} \Delta \phi = -i \left| a^{(0)} \right|^2 a^{(0)} ; \quad a^{(0)}|_{t=0} = a_0.$$  

On the other hand, the eikonal equation is still the same as in the linear case, hence the term "weakly nonlinear" (see also [33] and references therein). The correctors $(a^{(j)})_{j \geq 2}$ solve linear transport equations. With the above notations, the nonlinear transport equation is again an ordinary differential equation along rays:

$$\frac{d}{dt} \left(a^{(0)}(t, X_t(x)) \sqrt{J_t(x)} \right) = -i \left| a^{(0)}(t, X_t(x)) \right|^2 a^{(0)}(t, X_t(x)) \sqrt{J_t(x)}.$$  

This ordinary differential equation is of the form $\dot{y} = iVy$, where the nonlinear potential $V$ is real-valued. In particular, the modulus of $y$ is constant, and we just have to solve a linear differential equation. Thus, leading order nonlinear effects are measured by a (nonlinear) phase shift, which may be compared to the phenomenon of phase self-modulation in laser physics (see e.g. [35, 116]).

2.2. Super-critical case. In the super-critical case $\kappa = 0$, the nonlinearity is present in the eikonal equation: the hierarchy of the case $\kappa = 1$ is shifted, so that the corrector $a^{(1)}$ is present in the transport equation for $a^{(0)}$. As noted in [17], the system for the phase $\phi$ and the amplitudes $a^{(0)}, a^{(1)}, \ldots$ is not closed (see also [11, 13, 12]). For instance, we find:

\[
\begin{align*}
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + |a^{(0)}|^2 &= 0, \\
\partial_t a^{(0)} + \nabla \phi \cdot \nabla a^{(0)} + \frac{1}{2} a^{(0)} \Delta \phi &= -2i a^{(0)} \text{Re} \left( a^{(0)} \overline{a^{(1)}} \right), \\
\partial_t a^{(1)} + \nabla \phi \cdot \nabla a^{(1)} + \frac{1}{2} a^{(1)} \Delta \phi &= \frac{i}{2} \Delta a^{(0)} - i \nabla |a^{(1)}|^2 a^{(0)} \\
&\quad - 2i a^{(1)} \text{Re} \left( a^{(0)} \overline{a^{(1)}} \right) - 2i a^{(0)} \text{Re} \left( a^{(0)} \overline{a^{(0)}} \right). 
\end{align*}
\]

(2.4)

However, as pointed out in [17], the phase $\phi$ can be found when considering:

$$\begin{pmatrix} \rho, v \end{pmatrix} = \left( |a^{(0)}|^2, \nabla \phi \right).$$

Indeed, it solves the compressible, isentropic Euler equation:

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho v) &= 0 ; \\
\partial_t v + v \cdot \nabla v + \nabla \rho &= 0 ; \\
\rho_{t=0} &= |a_0|^2. 
\end{align*}
\]

(2.5)

For smooth initial data decaying to zero at infinity, this system as a smooth solution locally in time [25, 30, 10]. In general, finite time blowup occurs [30, 10, 34], but not always [13]: the known results depend on the propagation of the initial velocity by the (multi-dimensional) Burgers’ equation.

Once $(\rho, v)$ is determined, $\partial_t \phi$ is given by the eikonal equation; this yields $\phi$. Note that knowing $(\rho, v)$ suffices to compute important quadratic quantities such as Wigner measures. To complete the closure of the system, and provided that the leading order amplitude $a_0$ is nowhere zero, one may consider a generalized Madelung transform (see [17]), which we do not describe here.

2.3. Justification on small time intervals. Justifying geometric optics in the super-critical case is, in general, an open problem. However, as noticed in [17] and exploited in many other works (see e.g. [25, 31, 13, 8, 4]), if one studies this limit on time intervals of the form $[0,c_0|\ln h|]$ for some $c_0 > 0$, then the problem is simpler. Consider the more general nonlinear Schrödinger equation in $\mathbb{R}^n$:

\[
ih \partial_t u^h + \frac{h^2}{2} \Delta u^h = \omega |u^h|^{2\sigma} u^h ; \quad u^h(0,x) = a_0(x),
\]

(2.6)

where $\omega \in \mathbb{R} \setminus \{0\}$ and $\sigma \in \mathbb{N} \setminus \{0\}$. We consider the case $\phi_0 \equiv 0$ to prove that in this case, one can choose an approximate which is even simpler than the one given by (2.6). Formally, $u^h$ is formally approximated by $ae^{i\phi/h}$ where:

\[
\begin{align*}
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + \omega |a|^{2\sigma} &= 0 ; \\
\partial_t a + \nabla \phi \cdot \nabla a + \frac{1}{2} a \Delta \phi &= 0 ; \\
\phi_{t=0} &= 0, \\
\partial_t a_{t=0} &= a_0.
\end{align*}
\]

Looking at Taylor expansions for $\phi$ and $a$ as $t \to 0$, we see that

\[
\begin{align*}
\phi(t,x) &= a_0(x) + O(t^2) ; \\
a(t,x) &= -t \omega |a_0(x)|^{2\sigma} + O(t^3).
\end{align*}
\]
We prove that \( a(t,x) e^{i\phi(t,x)/h} \) can approximated by \( a_0(x) e^{-it\omega |a_0(x)|^2\sigma/h} \) on some time interval of the form \([0,c_0 h |\ln h|]\). Call \( \psi^h \) the latter function. It solves the ordinary differential equation:

(2.7) \[ ih\partial_t \psi^h = \omega |\psi^h|^{2\sigma} \psi^h; \quad \psi^h|_{t=0} = a_0. \]

We prove that if \( c_0 \) is sufficiently small, then \( \psi^h \) is a good approximation of \( u^h \) on \([0,c_0 h |\ln h|]\). Let \( u^h = u - \psi^h \). It solves:

(2.8) \[ ih\partial_t u^h + \frac{h^2}{2} \Delta u^h = \omega \left( F(w^h + \psi^h) - F(\psi^h) \right) + O(h^2) + O(t^2) + O(ht), \]

with \( u^h|_{t=0} = 0 \), where we have set \( F(z) = |z|^{2\sigma} z \). The \( O(h^2) \) term corresponds to the fact that we consider only the first two terms of a WKB analysis, the term \( O(t^2) \) stems from the approximation of the phase for small times, and \( O(ht) \) from the approximation of the amplitude for small times. To be more precise, we must say that these source terms are measured in \( L^2 \cap L^\infty(\mathbb{R}^n) \). When measured in \( H^k \), they must be multiplied by a factor of order \( 1 + (t/h)^k \), due to the differentiation of the phase. For \( k \geq 0 \), we have:

\[
\|w^h\|_{L^\infty([0,t];H^k)} \leq \frac{1}{h^k} \left\| F(w^h + \psi^h) - F(\psi^h) \right\|_{L^1([0,t];H^k)} + \frac{1}{h} \int_0^t \left( h^2 + s^2 + hs \right) \left\langle \frac{s}{h} \right\rangle^k ds.
\]

At least for \( \sigma \) integer, we have, when \( k > n/2 \):

\[
\|F(w^h(t) + \psi^h(t)) - F(\psi^h(t))\|_{H^k} \lesssim \left( \|w^h(t)\|_{H^k}^{2\sigma} + \|\psi^h(t)\|_{H^k}^{2\sigma} \right) \|w^h(t)\|_{H^k} \lesssim \left( \|w^h(t)\|_{H^k}^{2\sigma} + \left\langle \frac{t}{h} \right\rangle^{2\sigma k} \right) \|w^h(t)\|_{H^k}.
\]

On any time interval where we have, say, \( \|w^h\|_{H^k} \leq 1 \), we infer:

\[
\|w^h\|_{L^\infty([0,t];H^k)} \leq C \int_0^t \left\langle \frac{s}{h} \right\rangle^{2\sigma k} \|w^h(s)\|_{H^k} ds + C_1 \int_0^t \left( h + \frac{s^2}{h} + s \right) \left\langle \frac{s}{h} \right\rangle^k ds.
\]

Gronwall lemma yields:

\[
\|w^h\|_{L^\infty([0,t];H^k)} \lesssim \int_0^t \left( h + \frac{s^2}{h} + s \right) \left\langle \frac{s}{h} \right\rangle^k \exp \left( \frac{C}{h} \int_s^t \left\langle \frac{\tau}{h} \right\rangle^{2\sigma k} d\tau \right) ds.
\]

Let \( t^h = c_0 h |\ln h|^\theta \):

\[
\|w^h\|_{L^\infty([0,t^h];H^k)} \lesssim \exp \left( Cc_0 |\ln h|^\theta (|\ln h|)^{2\sigma k\theta} \right) \int_0^{t^h} \left( hs + \frac{s^3}{h} + s^2 \right) \left\langle \frac{s}{h} \right\rangle^k ds \lesssim \exp \left( Cc_0 |\ln h|^\theta (c_0 |\ln h|)^{2\sigma k\theta} \right) h^2 |\ln h|^{4\theta}.
\]

For \( \theta = (1 + 2\sigma k)^{-1} \) and \( c_0 \) sufficiently small, this yields:

\[
\|w^h\|_{L^\infty([0,c_0 h|\ln h|^\vartheta];H^k)} \lesssim |\ln h|^{-1 - \frac{1}{\vartheta}}.
\]

We can then conclude with a continuity argument, for \( h \) sufficiently small:

**Proposition 2.1.** Let \( n \geq 1 \), \( \omega \in \mathbb{R} \setminus \{0\} \), \( \sigma > 0 \) an integer, and \( a_0 \in \mathcal{S}(\mathbb{R}^n) \). Fix \( k > n/2 \). Then we can find \( c_0, c_1, \theta > 0 \) independent of \( h \in [0,1] \) such that \( u^h \) and \( \psi^h \), solutions to (2.7) and (2.8) respectively, satisfy:

\[
\|u^h - \psi^h\|_{L^\infty([0,c_0 h|\ln h|^\vartheta];H^k)} \lesssim h |\ln h|^{c_1}.
\]
3. Instability: formal computations

In this section, we show how to reduce the proof of Theorem 1.2 to the justification of super-critical nonlinear geometric optics on a time interval which is independent of $h$. This formal approach would remain valid for a larger class of nonlinearities, not necessarily defocusing and cubic at the origin.

3.1. The o.d.e. mechanism. Consider the general Schrödinger equation with data independent of $h$:

\[ i\hbar \partial_t u^h + \frac{\hbar^2}{2} \Delta u^h = f \left( |u^h|^2 \right) u^h ; \quad u^h(0, x) = a_0(x). \]

The instability mechanism we sketch in this section is valid for initial data which are not highly oscillatory: $\phi_0 \equiv 0$. We study a more general framework, corresponding to Theorem 1.2 in Section 3.2 below.

Following WKB methods, seek $u^h$ such that $u^h \sim ae^{i\phi/h}$ as $h \to 0$. Plugging this ansatz into (3.1) and canceling $O(1)$ and $O(h)$ terms yields:

\[ \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + f(|a|^2) = 0 \quad; \quad \phi(0, x) = 0. \]

\[ \partial_t a + \nabla \phi \cdot \nabla a + \frac{1}{2} a \Delta \phi = 0 \quad; \quad a(0, x) = a_0(x). \]

As $t \to 0$, approximate $\phi$ and $a$ by their Taylor expansion:

\[ \phi(t, x) \sim \sum_{j \geq 1} t^{2j-1} \phi_j(x) \quad; \quad a(t, x) \sim \sum_{j \geq 0} t^{2j} a_j(x). \]

Note that for $j = 0$, the notations are consistent. The fact that only odd (resp. even) powers of $t$ appear in the expansion for $\phi$ (resp. $a$) is due to the assumption $\phi_{|t=0} \equiv 0$. Plugging these formal series into (3.2), we find:

\[ \phi_1 = -f(|a_0|^2) \quad; \quad 2a_1 = -\nabla \phi_1 \cdot \nabla a_0 - \frac{1}{2} a_0 \Delta \phi_1. \]

Thus, $a_1$ is the first term where the presence of the Laplacian becomes relevant: let $u_1^h(t, x) = a_0(x) \exp(it\phi_1(x)/h)$. It solves the ordinary differential equation:

\[ i\hbar \partial_t u_1^h = f \left( |u_1^h|^2 \right) u_1^h \quad; \quad u_1^h(0, x) = a_0(x), \]

where $x$ is now just a parameter. Assume that for some time interval $[0, T^h]$, WKB method provides a good approximation for $u^h$ in $L^2(\mathbb{R}^n)$:

\[ \left\| u^h - ae^{i\phi/h} \right\|_{L^\infty([0, T^h]; L^2)} \to 0 \text{ as } h \to 0. \]

On the other hand, we can approximate $ae^{i\phi/h}$ by $u_1^h$ if

\[ \left\| ae^{i\phi/h} - a_0 e^{it\phi_1/h} \right\|_{L^\infty([0, T^h]; L^2)} \to 0 \text{ as } h \to 0. \]

If $T^h \to 0$, which we may assume in view of Th. 1.2, then approximating $a$ by $a_0$ is not a problem. We have to be more careful with the phase, because of the division by $h$. Formally, the above limit holds if

\[ \exp \left( it^3/h \right) \to 1 \text{ as } h \to 0. \]

If $t^h \leq T^h$ is such that $(t^h)^3 \ll h$, then we expect:

\[ \left\| u^h - u_1^h \right\|_{L^\infty([0, t^h]; L^2)} \to 0 \text{ as } h \to 0. \]
Now let \( v^h \) solve (3.1) with initial data \( v^h(0, x) = \tilde{a}_0^h(x) \), where \( \tilde{a}_0^h \) satisfies (1.2); the assumption on \( \delta^h \) will appear later. Let \( v_1^h \) be the solution of the corresponding ordinary differential equation. Similarly, we expect:

\[
\left|\|v^h - v_1^h\|_{L^\infty([0,t^h];L^2)}\right| \to 0 \text{ as } h \to 0.
\]

An instability like in Th. 1.2 then stems from an instability at the o.d.e. level:

\[
v_1^h(t, x) = \tilde{a}_0^h(x) \exp \left(i t \tilde{\phi}_1^h(x)/h\right),
\]

where \( \tilde{\phi}_1^h = -f \left( |\tilde{a}_0^h|^2 \right) \). We have obviously

\[
\left|v_1^h - a_0 \exp \left(i \tilde{\phi}_1^h/h\right)\right|_{L^\infty([0,t^h];L^2)} \to 0 \text{ as } h \to 0,
\]

as soon as \( \delta^h \ll 1 \). Instability comes from the phase:

\[
u_1^h(t, x) - v_1^h(t, x) \sim a_0(x) \left(e^{i t \tilde{\phi}(x)/h} - e^{i \tilde{\phi}_1^h(x)/h}\right).
\]

Using Taylor formula for \( f \), we have:

\[
\tilde{\phi}_1^h = -f \left( |a_0|^2 \right) - 2 \delta^h \Re \left((a_0 - \tilde{a}_0^h)|\tilde{a}_0^h| f'(|a_0|^2) + O(\delta^h)^2\right).
\]

Since \( f' > 0 \), we infer from (1.2):

\[
\tilde{\phi}_1^h(x) = \phi_1(x) + \delta^h c(x) + o(\delta^h),
\]

where the function \( c \) does not depend on \( h \) and is not identically zero on the support of \( a_0 \). For \( t \delta^h \approx h \), we infer:

\[
|u_1^h(t, x) - v_1^h(t, x)| \sim |a_0(x)| \left|e^{i t \delta^h c(x)/h} - 1\right|.
\]

This has a nonzero limit as \( h \to 0 \) since \( t \delta^h \approx h \). The only constraint we imposed so far was \( t^h \ll h^{1/3} \), so taking \( h^{2/3} \ll \delta^h \ll 1 \) predicts an instability as stated in Th. 1.2. To prove Th. 1.2 we must establish (3.4) and (3.5) for suitable \( t^h \).

Remark 3.1. In view of [4], introduce the complex projective distance:

\[
\rho(u_1, u_2) := \arccos \left(\frac{|\langle u_1, u_2 \rangle|}{\|u_1\|_{L^2}\|u_2\|_{L^2}}\right).
\]

Then we can check that up to demanding \( t^h \ll h^{1/3} \) and \( h^{2/3} \delta^h \gg h \) (these conditions can be satisfied for \( h^{2/3} \ll \delta^h \ll 1 \)), and provided that (3.4) and (3.5) hold:

\[
\frac{d_{\rho}(u^h(t^h, v^h(t^h)))}{d_{\rho}(u^h(0), v^h(0))} \to +\infty \quad \text{as } h \to 0.
\]

Remark 3.2. We prove in Appendix A a result in a similar spirit for linear equations.

Remark 3.3. We show in Appendix B how this analysis and Proposition 2.1 yield ill-posedness properties for the nonlinear Schrödinger equation, established in [14, 15].

3.2. Another instability mechanism. We now consider the general assumptions Theorem 1.2 (in particular, we no longer assume \( \phi_0 \equiv 0 \)). Seeking \( u^h \sim a e^{i \phi/h} \) as \( h \to 0 \), we now find:

\[
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + f \left(|a|^2\right) = 0 \quad ; \quad \phi(0, x) = \phi_0(x).
\]

\[
\partial_t a + \nabla \phi \cdot \nabla a + \frac{1}{2} a \Delta \phi = 0 \quad ; \quad a(0, x) = a_0(x).
\]

As \( t \to 0 \), approximate \( \phi \) and \( a \) by their Taylor expansion:

\[
\phi(t, x) \sim \sum_{j \geq 0} \hat{t}^j \phi_j(x) \quad ; \quad a(t, x) \sim \sum_{j \geq 0} \hat{t}^j a_j(x).
\]
Now all the powers of \( t \) must be taken into account. Using (3.4), we see that \((\phi_{j+1}, a_{j+1})\) is given recursively by \((\phi_1, a_1)_{0 \leq i \leq j}\). Define

\[
\begin{align*}
\tilde{u}_k^h(t, x) & = a_0(x) \exp \left( \frac{i}{h} \sum_{j=0}^{k} t^j \phi_j(x) \right), \\
\tilde{v}_k^h(t, x) & = a_0(x) \exp \left( \frac{i}{h} \sum_{j=0}^{k} t^j \phi_j^h(x) \right),
\end{align*}
\]

where \((\phi_j^h, a_j^h)_{j \geq 1}\) is constructed like \((\phi_j, a_j)_{j \geq 1}\) with \(a_0\) replaced by \(a_0^h\). By induction, we have:

**Lemma 3.4.** Under the assumption (1.2),

\[
\|\phi_1 - \phi_1^h\|_{H^s} \approx \delta^h, \quad \forall s \geq 0,
\]

**Lemma 3.4.** Under the assumption (1.2),

and for any \( j \geq 2 \) and every \( s \geq 0 \),

\[
\|\phi_j - \phi_j^h\|_{H^s} + \|a_{j-1} - a_{j-1}^h\|_{H^s} \lesssim \delta^h.
\]

For any \( t^h \ll 1 \), we have

\[
|u_k^h(t^h, x) - u_k^h(t^h, x)| \sim |a_0(x)| \left| \exp \left( \frac{i}{h} \phi(t^h, x) - \frac{i}{h} \sum_{j=1}^{k} (t^h)^j \phi_j(x) \right) - \right|.
\]

This goes to zero provided that \((t^h)^{k+1} \ll h\). On the other hand,

\[
|u_k^h(t^h, x) - v_k^h(t^h, x)| \sim |a_0(x)| \left| \exp \left( \frac{i}{h} \sum_{j=1}^{k} (t^h)^j (\phi_j(x) - \phi_j^h(x)) \right) - \right|.
\]

Since \(\phi_0^h = \phi_0\), and from Lemma 3.2 the main term in the exponential is

\[
\frac{t^h}{h} (\phi_1(x) - \phi_1^h(x)) \approx \frac{t^h \delta^h}{h}.
\]

All the other terms are negligible from Lemma 3.2 since \( t^h \ll 1 \). We then have an instability if:

\[
(t^h)^{k+1} \ll h ; \quad t^h \delta^h \gtrsim h ; \quad (t^h)^2 \delta^h \ll h.
\]

All these conditions can be satisfied if we take \( k + 1 \geq N \). We conclude this paragraph by showing that in general, the mechanism is not the same as in the previous section.

First, if \( \phi_0 \neq 0 \), then trivially \( \phi_1 \) depends on \( \nabla \phi_0 \). If \( \phi_0 \equiv 0 \), we have, for \( k \geq 2 \),

\[
|u_k^h - u_k^h| = |a_0| \left| \exp \left( \frac{i}{h} \sum_{j=2}^{k} t^j \phi_j(x) \right) - \right|.
\]

If \( \phi_0 \equiv 0 \), then \( \phi_2 \equiv 0 \), but in general, \( \phi_3 \neq 0 \). So if the above instability occurs for \( t^h \gtrsim h^{1/3} \), then it is not an o.d.e. mechanism. We check that if \( \delta^h = h^{2/3} \), then \( u_2^h \) and \( u_3^h \) diverge before the instability; therefore, so do \( u^h \) and \( u_1^h \).
3.3. Strong nonlinearities with harmonic potential. Introduce an isotropic harmonic potential:
\[ i\hbar \partial_t u^h + \frac{\hbar^2}{2} \Delta u^h = \frac{|x|^2}{2} u^h + f \left( (|u^h|^2) u^h \right) ; \quad u^h(0, x) = a_0(x)e^{i\phi_0(x)/\hbar}. \]

Following ideas used in the linear case \[32\], we remove the potential by posing:
\[ U^h(t, x) = \frac{1}{(1 + t^2)^{n/4}} e^{\frac{|x|^2}{2(1 + t^2)}} u^h \left( \arctan t, \frac{x}{\sqrt{1 + t^2}} \right). \]

Then \( U^h \) solves:
\[ \begin{aligned}
  &i\hbar \partial_t U^h + \frac{\hbar^2}{2} \Delta U^h = \frac{1}{1 + t^2} f \left( (1 + t^2)^{n/2} |U^h|^2 \right) U^h, \\
  &U^h(0, x) = a_0(x)e^{i\phi_0(x)/\hbar}.
\end{aligned} \]

We can then proceed as above. The only difference is the presence of time in the nonlinearity, which changes very little at the formal level.

3.4. Weaker nonlinearities. We come to the framework of Corollary \[13\]
\[ i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = f \left( (|u^\varepsilon|^2) u^\varepsilon \right) ; \quad u^\varepsilon|_{t=0} = a_0(x)e^{-i\varepsilon^2}, \]
where \( n \geq 2, 1 < k < n \). Following \[2\], denote \( \gamma = k/n \) and introduce
\[ u^\varepsilon(t, x) = \frac{1}{(1 - t)^{n/2}} \psi^\varepsilon \left( \frac{\varepsilon^\gamma}{1 - t} \frac{x}{1 - t} \right) e^{\frac{i\varepsilon^2}{2(1 - t)}}. \]

This can be viewed as a “semi-classical” conformal transform, as compared to the “usual” case introduced in \[18\]. Then with \( \hbar = \varepsilon^{1 - \gamma} \), which goes to zero by assumption, and denoting \( t_0^h = \frac{\varepsilon^\gamma}{1 - \varepsilon} \), \( \psi(t, x) \) solves:
\[ \begin{aligned}
  &i\hbar \partial_t \psi^h + \frac{\hbar^2}{2} \Delta \psi^h = t^{-2} f \left( (t^n)|\psi^h|^2 \right) \psi^h, \\
  &\psi^h|_{t=t_0^h} = a_0(x).
\end{aligned} \]

We can then adapt the preceding approach. This explains the different notation \( \varepsilon \) for the semi-classical parameter. Note that the apparently singular factor \( t^{-2} \) is harmless as \( t \to 0 \), since we assumed \( n \geq 2 \) and \( f(0) = 0 \) (this is where this assumption comes into play).

Instability occurs for \( t_0^h \approx t^h \) where \( t^h \) and \( \delta^h \) satisfy conditions in the same vein as above. When an isotropic potential is incorporated, we can essentially superimpose the above two changes of unknown functions.

4. Proof of Theorem \[12\]

4.1. Case with no potential. In this section, we complete the proof of Theorem \[12\] in the case \( \omega = 0 \). Let \( n \geq 1, a_0^h \in S(\mathbb{R}^n) \) bounded in \( H^s \) uniformly in \( h \in [0, 1] \) for every \( s > 0 \), and \( \phi_0 \) as in Theorem \[12\]. Consider the initial value problem:
\[ i\hbar \partial_t w^h + \frac{\hbar^2}{2} \Delta w^h = f \left( (|w^h|^2) w^h \right) ; \quad w^h(0, x) = a_0^h(x)e^{i\phi_0(x)/\hbar}. \]

We recall the method of \[20\]. It somehow boils down to seeking WKB approximation “the other way round”: first write the solution as \( w^h = \alpha^h e^{i\varphi^h/\hbar} \) (no approximation at this stage), and then study the behavior of \( (\alpha^h, \varphi^h) \) as \( h \to 0 \), to recover what the usual WKB methods yield formally. Seek \( w^h = \alpha^h e^{i\varphi^h/\hbar} \), with:
\[ \begin{aligned}
  &\partial_t \varphi^h + \frac{1}{2} |\nabla \varphi^h|^2 + f \left( |\alpha^h|^2 \right) = 0, \\
  &\varphi^h|_{t=0} = \phi_0, \\
  &\partial_t \alpha^h + \nabla \varphi^h \cdot \nabla \alpha^h + \frac{1}{2} \alpha^h \Delta \varphi^h = \frac{i\hbar}{2} \Delta \alpha^h, \\
  &\alpha^h|_{t=0} = a_0^h.
\end{aligned} \]
Introducing the "velocity" \( v^h = \nabla \varphi^h \), \( \ref{07} \) yields
\[
\partial_t v^h + v^h \cdot \nabla v^h + 2f' (|\alpha^h|^2) \text{Re} \left( \overline{\alpha^h} \nabla \alpha^h \right) = 0 \quad ; \quad v^h \big|_{t=0} = \nabla \phi_0 ,
\]
(4.3)
\[
\partial_t \alpha^h + v^h \cdot \nabla \alpha^h + \frac{1}{2} \alpha^h \text{div} v^h = i \frac{\hbar}{2} \Delta \alpha^h \quad ; \quad \alpha^h \big|_{t=0} = \alpha_0^h .
\]
Separate real and imaginary parts of \( \alpha^h \), \( \alpha^h = \alpha_1^h + i \alpha_2^h \). Then we have
\[
\partial_t u^h + \sum_{j=1}^n A_j(u^h) \partial_j u^h = \frac{\hbar}{2} L u^h ,
\]
with \( u^h = \begin{pmatrix} \alpha_1^h \\ \alpha_2^h \\ v_1^h \\ \vdots \\ v_n^h \end{pmatrix} \), \( L = \begin{pmatrix} 0 & -\Delta & 0 & \cdots & 0 \\ \Delta & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \),
and \( A(u, \xi) = \sum_{j=1}^n A_j(u) \xi_j = \begin{pmatrix} v \cdot \xi & 0 & \frac{\alpha_1^h \xi}{2} \\ 0 & v \cdot \xi & \frac{\alpha_2^h \xi}{2} \\ \frac{2f' \alpha_1^h \xi}{2} & \frac{2f' \alpha_2^h \xi}{2} & v \cdot \xi I_n \end{pmatrix} \),
where \( f' \) stands for \( f'(|\alpha_1^h|^2 + |\alpha_2^h|^2) \). The matrix \( A(u, \xi) \) can be symmetrized by
\[
S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{1}{2} I_n \end{pmatrix} ,
\]
which is symmetric and positive since \( f' > 0 \). For an integer \( s > 2 + n/2 \), we bound \( \langle S \partial_x^a u^h, \partial_x^a u^h \rangle \) where \( \alpha \) is a multi index of length \( \leq s \), and \( \langle \cdot, \cdot \rangle \) is the usual \( L^2 \) scalar product. We have
\[
\frac{d}{dt} \langle S \partial_x^a u^h, \partial_x^a u^h \rangle = \langle \partial_t S \partial_x^a u^h, \partial_x^a u^h \rangle + 2 \langle S \partial_t \partial_x^a u^h, \partial_x^a u^h \rangle ,
\]
since \( S \) is symmetric. For the first term, we must consider the lower \( n \times n \) block:
\[
\langle \partial_t S \partial_x^a u^h, \partial_x^a u^h \rangle \leq \frac{1}{f'} \left\| \partial_t \left( f' (|\alpha_1^h|^2 + |\alpha_2^h|^2) \right) \right\|_{L^\infty} \left\langle S \partial_x^a u^h, \partial_x^a u^h \right\rangle .
\]
So long as \( \|u^h\|_{L^\infty} \leq 2 a_0^h \), we have:
\[
f' (|\alpha_1^h|^2 + |\alpha_2^h|^2) \geq \inf \left\{ f'(y) ; \ 0 \leq y \leq 4 \limsup \|a_0^h\|_{L^\infty} \right\} = \delta_n > 0 ,
\]
where \( \delta_n \) is now fixed, since \( f' \) is continuous with \( f' > 0 \). We infer,
\[
\frac{1}{f'} \left\| \partial_t \left( f' (|\alpha_1^h|^2 + |\alpha_2^h|^2) \right) \right\|_{L^\infty} \lesssim \|u^h\|_{H^s} ,
\]
where we used Sobolev embeddings and \( \ref{04} \). For the second term we use
\[
\langle S \partial_t \partial_x^a u^h, \partial_x^a u^h \rangle = \frac{\hbar}{2} \langle SL(\partial_x^a u^h), \partial_x^a u^h \rangle - \left( S \partial_x^a \left( \sum_{j=1}^n A_j(u^h) \partial_j u^h \right), \partial_x^a u^h \right) .
\]
We notice that \( SL \) is a skew-symmetric second order operator, so the first term is zero. For the second term, use the symmetry of \( SA_j(u^h) \) and usual estimates on commutators to get finally:
\[
\frac{d}{dt} \sum_{|\alpha| \leq s} \langle S \partial_x^a u^h, \partial_x^a u^h \rangle \leq C \left( \|u^h\|_{H^s} \right) \sum_{|\alpha| \leq s} \langle S \partial_x^a u^h, \partial_x^a u^h \rangle ,
\]
for \( s > 2 + d/2 \). Gronwall lemma along with a continuity argument yield the counterpart of \( \cite{20} \) Theorem 1.1:
**Proposition 4.1.** Let $a_0^h \in S(\mathbb{R}^n)$ bounded in $H^m$ uniformly in $h \in [0,1]$ for every $m > 0$, and let $s > 2 + n/2$. Then there exist $T > 0$ independent of $h \in [0,1]$ and $w^h(t,x) = \alpha^h(t,x)e^{i\varphi^h(t,x)/h}$ solution to (4.1) on $[0,T]$. Moreover, $\alpha^h$ and $\varphi^h$ are bounded in $L^\infty([0,T];H^s)$, uniformly in $h \in [0,1]$.

The solution to (4.2) formally “converges” to the solution of:

$$
\begin{align*}
\partial_t \varphi^h + \frac{1}{2} |\nabla \varphi^h|^2 + f(|\alpha^h|^2) &= 0 \quad ; \quad \varphi^h|_{t=0} = \varphi_0, \\
\partial_t \alpha^h + \nabla \varphi^h \cdot \nabla \alpha^h + \frac{1}{2} h^2 \Delta \varphi^h &= 0 \quad ; \quad \alpha^h|_{t=0} = a_0^h.
\end{align*}
$$

(4.5)

The term “converges” may not seem appropriate, since the initial data keeps depending on $h$. Yet, under our assumptions on $a_0^h$, (4.3) has a unique solution $(\alpha^h, \varphi^h)$, uniformly bounded in $L^\infty([0,\tau];H^s)$ for any $m > 0$ for some $\tau > 0$ independent of $h \in [0,1]$ (see e.g. [20]). We infer:

**Proposition 4.2.** Let $s \in \mathbb{N}$. There exists $C_s$ independent of $h$ such that for every $0 \leq t \leq \min(T,\tau)$,

$$
\|\alpha^h(t) - a^h(t)\|_{H^s} + \|\varphi^h(t) - \varphi^h(t)\|_{H^s} \leq C_s h t.
$$

**Proof.** We keep the same notations as above, (4.3). Denote by $v^h$ the analog of $u^h$ corresponding to $(\alpha^h, \varphi^h)$. We have

$$
\partial_t (u^h - v^h) + \sum_{j=1}^n A_j(u^h)\partial_j (u^h - v^h) + \sum_{j=1}^n (A_j(u^h) - A_j(v^h)) \partial_j v^h = \frac{h}{2} Lu^h.
$$

Keeping the symmetrizer $S$ corresponding to $u^h$, we can do similar computations to the previous ones. Note that we know that $u^h$ and $v^h$ are bounded in $L^\infty([0,\min(T,\tau)];H^s)$. Denoting $w^h = u^h - v^h$, we get, for $s > 2 + n/2$:

$$
\frac{d}{dt} \sum_{|\alpha| \leq s} (S\partial_\alpha^h w^h, \partial_\alpha^h w^h) \leq \sum_{|\alpha| \leq s} (S\partial_\alpha^h w^h, \partial_\alpha^h w^h) + h\|w^h(t)\|_{H^s}.
$$

We conclude with Gronwall lemma.

This result shows that for small times, WKB solution in the sense of (4.5) provides a good approximation for the exact solution. Note that since we have to divide phases by $h$, we can deduce such a result only for times $\ll 1$. The following corollary is a straightforward consequence of Proposition 4.1:

**Corollary 4.3.** Under the assumptions of Proposition 4.1 denote $w_{\text{app}}^h = a^h e^{i\varphi^h/h}$ where $(a^h, \varphi^h)$ solves (4.5). Then for any $0 < t^h \ll 1$,

$$
\|w^h - w_{\text{app}}^h\|_{L^\infty([0,t^h];L^2)} \ll 1.
$$

We now study small time properties of $(a^h, \varphi^h)$.

**Definition 4.4.** If $T > 0$, $(\phi^h_j)_{j \geq 0}$ is a sequence in $H^\infty(\mathbb{R}^n) := \cap_{s \geq 0} H^s(\mathbb{R}^n)$, and $\phi^h \in C([0,T];H^s(\mathbb{R}^n))$ for every $s > 0$, the asymptotic relation

$$
\phi^h(t,x) \sim \sum_{j \geq 0} t^j \phi^h_j(x) \quad \text{as } t \to 0
$$

means that for every integer $J \geq 0$ and every $s > 0$,

$$
\left\| \phi^h(t,\cdot) - \sum_{j=0}^J t^j \phi^h_j \right\|_{H^s(\mathbb{R}^n)} = o(t^J) \quad \text{as } t \to 0.
$$
Proposition 4.5. Under the assumptions of Proposition 4.1, there exist sequences $(\phi_j^h)_{j \geq 0}$ and $(a_j^h)_{j \geq 1}$ in $H^\infty(\mathbb{R}^n)$ (uniformly in $h \in [0,1]$), such that the solution of (4.5) satisfies
\[ \phi^h(t,x) \sim \sum_{j \geq 0} t^j \phi_j^h(x) , \quad \text{and} \quad a^h(t,x) \sim \sum_{j \geq 0} t^j a_j^h(x) \quad \text{as} \quad t \to 0. \]

Moreover, $\phi_0^h = \phi_0$, and $\phi_1^h$ is given by $\phi_1^h = -f(|a_0^h|^2)$.

Plugging such asymptotic series into (4.5), a formal computation yields a source term which is $O(t^\infty)$ as $t \to 0$. The result then follows with the same approach as in the proof of Proposition 4.1 and Borel lemma (see e.g. [33]). Taking Corollary 4.3 into account, we find:

Corollary 4.6. Let $N \in \mathbb{N} \setminus \{0\}$. Under the assumptions of Proposition 4.1, for any $0 < t^h \ll h^{1/(N+1)}$, we have
\[ \| w^h - w^h_N \|_{L^\infty([0,t^h];L^2)} \ll 1, \]
where $w^h_N$ is given by
\[ w^h_N(t,x) = a_0^h(x) \exp \left( \frac{i}{h} \sum_{j=0}^N j^j \phi_j^h(x) \right). \]

Applying Corollary 4.6 to $u^h$ and $v^h$ respectively yields Theorem 1.2 when $\omega = 0$. 4.2. With an harmonic potential. Now suppose $\omega > 0$. Up to a dilation of the coordinates, we can assume that $\omega = 1$. Let $a_0^h \in \mathcal{S}(\mathbb{R}^n)$ bounded in $H^s$ uniformly in $h \in [0,1]$ for every $s > 0$. Consider the initial value problem:
\[ \text{i}h \partial_t u^h + \frac{h^2}{2} \Delta u^h = \frac{1}{2} |x|^2 |w^h|^2 + f(|w^h|^2) w^h ; \quad w^h \big|_{t=0} = a_0^h(x). \]

The change of unknown functions (3.7) leads to Equation (3.8) with initial data $a_0^h$. We can then follow every line of Section 4.1. The presence of time in the nonlinearity does not need special care: for the symmetrizer $S$, we can take
\[ S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{(1+t^2)^{1-n/2}}{4f'} I_n \end{pmatrix}, \quad \text{where} \quad f' \quad \text{stands for} \quad f' \left( (1+t^2)^{n/2} (|a_1^h|^2 + |a_2^h|^2) \right). \]

The presence of time does not perturb the analysis (we always consider bounded times). We obtain the analogue of (4.5):
\[ \partial_t \phi^h + \frac{1}{2} |\nabla \phi^h|^2 + \frac{1}{1+t^2} f \left( (1+t^2)^{n/2} |\phi^h|^2 \right) = 0 ; \quad \phi^h \big|_{t=0} = 0, \]
\[ \partial_t a^h + \nabla \phi^h \cdot \nabla a^h + \frac{1}{2} a^h \Delta \phi^h = 0 ; \quad a^h \big|_{t=0} = a_0^h. \]

The conclusions of Proposition 4.5 remain: $\phi_1^h$ is given by the same formula, but the formulae giving $(\phi_j^h, a_j^h)_{j \geq 2}$ are different because of time in the nonlinearity. Since the change of unknown functions (3.7) is unitary on $L^2(\mathbb{R}^n)$, the end of the proof of Theorem 1.2 follows.

5. Proof of Proposition 4.3

We study the case with no harmonic potential, $\omega = 0$, the case $\omega > 0$ is a straightforward consequence as explained in Section 4.2.

As we noted in Section 4.1, the solution to (4.5) yields a good approximation of the solution to (4.4) only for small times (see Corollary 4.3). The reason is the same as that mentioned in Section 2.2, the shift in the cascade of equations in
WKB methods is such that initial corrections of order \( h \) become relevant for times of order 1.

For \( a_0, a_1 \in S(\mathbb{R}^n) \) independent of \( h \), consider the initial value problem:

\[
(5.1) \quad \frac{i}{\hbar}\partial_tw^h + \frac{\hbar^2}{2}\Delta w^h = f \left( |w^h|^2 \right) w^h ; \quad w^h(0,x) = (a_0(x) + ha_1(x)) e^{i\phi_0(x)/\hbar}.
\]

We proved in Section 4.1 that there exists \( T_0 \) independent of \( h \in [0,1] \) such that

\[
\text{w}^h = \alpha^h e^{i\phi^h}/\hbar, \text{with } \alpha^h, \phi^h \in L^\infty(0, T_0; H^s) \text{ for every } s \geq 0, \text{ uniformly for } h \in [0,1].
\]

Moreover, Proposition 1.3 yields \((\alpha^h, \phi^h) = (a, \phi) + \mathcal{O}(ht)\), where \((a, \phi)\) solves (3.6).

Pursuing the analysis of [20], we have:

**Proposition 5.1.** Let \( n \geq 1, a_0, a_1 \in S(\mathbb{R}^n) \), \( \phi_0 \in C^\infty(\mathbb{R}^n; \mathbb{R}) \) independent of \( h \), with \( \nabla \phi_0 \in H^s(\mathbb{R}^n) \) for every \( s \geq 0 \). Let \( w^h \) solve (5.1). Define \((a^{(1)}, \phi^{(1)})\) by

\[
\begin{align*}
\partial_t a^{(1)} + \nabla \phi \cdot \nabla a^{(1)} + 2\text{ Re} \left( \mu a^{(1)} \right) f'(|a|^2) & = 0 ; \quad \phi^{(1)}(0, x) = 0, \\
\partial_t \phi^{(1)} + \nabla a^{(1)} \cdot \nabla \phi^{(1)} + \nabla a^{(1)} \cdot \nabla \phi^{(1)} + \frac{1}{2} a^{(1)} \Delta \phi + \frac{1}{2} a \Delta \phi^{(1)} & = \frac{i}{2} \Delta a ; \quad a^{(1)}(0, x) = a_1.
\end{align*}
\]

Then \((a^{(1)}, \phi^{(1)}) \in L^\infty(0, T_0; H^s) \) for every \( s \geq 0 \), and

\[
\|a^h - a_0\|_{L^\infty(0, T_0; H^s)} + \|\phi^h - \phi - \phi_0\|_{L^\infty(0, T_0; H^s)} \lesssim h^2, \quad \forall s \geq 0.
\]

The pair \((a, \phi)\) is given by (3.6), and does not depend on \( a_1 \).

The proof is a straightforward consequence of the analysis of Section 4 and is given in [20]. Despite the notations, it seems unadapted to consider \( \phi^{(1)} \) as being part of the phase. Indeed, we infer from Proposition 5.1 that

\[
\|w^h - ae^{i\phi^{(1)}} e^{i\phi_0/h}\|_{L^\infty(0, T_0; L^2)} = \mathcal{O}(h).
\]

Relating this information to the WKB methods presented in Section 2, we have:

\[
a^{(0)} = ae^{i\phi^{(1)}}.
\]

Since \( \phi^{(1)} \) depends on \( a_1 \) while \( a \) does not, we retrieve the fact that in super-critical régimes, the leading order amplitude in WKB methods depends on the initial first corrector \( a_1 \). Now Proposition 5.1 is straightforward, since \((a^{(1)}, \phi^{(1)})\) solves a linear system, and

\[
\begin{align*}
\partial_t \phi^{(1)} \big|_{t=0} & = -2 \text{ Re} (\pi_0 a_1) f'(|a_0|^2) .
\end{align*}
\]

6. **Proof of Corollary 1.3**

We indicate how to adapt the analysis of Section 4 when the nonlinearity is attenuated by a power of the small parameter. By an obvious change of unknown functions, this is equivalent to considering solutions of (5.1) with data of order \( h^{k/2} \).

6.1. **Case with no potential.** Assume \( \omega = 0 \). For \( n \geq 2, 1 < k < n, a_0^h \in S(\mathbb{R}^n) \) bounded in \( H^s \) uniformly in \( \varepsilon \in [0,1] \) for every \( s > 0 \), consider:

\[
(6.1) \quad \frac{i}{\hbar} \partial_t w^\varepsilon + \frac{\varepsilon^2}{2} \Delta w^\varepsilon = f \left( \varepsilon^k |w^\varepsilon|^2 \right) w^\varepsilon ; \quad w^\varepsilon \big|_{t=0} = a_0^h(x) e^{-i|x|^2/2\varepsilon}.
\]

Introduce \( \psi \) given by

\[
w^\varepsilon(t, x) = \frac{1}{(1 - t)^{n/2}} \psi^\varepsilon \left( \frac{\varepsilon \gamma}{1 - t} \frac{x}{1 - t} \right) e^{i\frac{|x|^2}{2(1 - t)}}.
\]

Denoting \( \gamma = k/n, h = \varepsilon^{1-\gamma} \) and \( t_0^h = h^{\gamma/(1-\gamma)} \), \( \psi(t, x) \) solves:

\[
(6.1) \quad \frac{i}{\hbar} \partial_t \psi^h + \frac{\hbar^2}{2} \Delta \psi^h = t^{-2} f \left( t^n |\psi^h|^2 \right) \psi^h ; \quad \psi^h \big|_{t=t_0^h} = a_0^h(x),
\]
Proposition 6.1. Let $n \geq 2$, $a_0^b \in \mathcal{S}(\mathbb{R}^n)$ bounded in $H^m$ uniformly in $h \in [0,1]$ for every $m > 0$, and let $s > 2 + n/2$. Then there exist $T > 0$ independent of $h \in [0,1]$ and $\psi^b(t,x) = \alpha^b(t,x)e^{i\varphi^b(t,x)/h}$ solution to (6.1) on $[t_0^b, t_0^b + T]$. Moreover, $\alpha^b$ and $\varphi^b$ are bounded in $L^\infty([t_0^b, t_0^b + T]; H^s)$, uniformly in $h \in [0,1]$.

Similarly, we have the analogue of Corollary 4.3 with:

$$
\partial_t \phi^b + \frac{1}{2} |\nabla \phi^b|^2 + t^{-2} f'(t^n|\alpha^h|^2) = 0 \quad ; \quad \phi^b|_{t=0} = 0 ,
$$

(6.2)

$$
\partial_t \alpha^h + \nabla \phi^b \cdot \nabla \alpha^h + \frac{1}{2} \alpha^h \Delta \phi^b = 0 \quad ; \quad \alpha^h|_{t=0} = a_0^h .
$$

Like before, this system has a smooth solution on $[0, \tau]$ for some $\tau > 0$ independent of $h \in [0,1]$. Something must be explained about this approximate system: the time where data are prescribed is now $t = 0$. This seems reasonable since $t_0^b \to 0$ as $h \to 0$, but there is a price to pay. First, we have the analogue of Proposition 4.5 with different powers of $t$ due to the presence of time in the nonlinearity, and our assumption $\phi_{j=0}^h = 0$:

(6.3) \[ \phi^b(t,x) \sim \sum_{j \geq 1} t^{j-1} \phi_{j}^h(x) , \quad \text{and} \quad a^h(t,x) \sim \sum_{j \geq 0} t^n a_{j}^h(x) \quad \text{as} \quad t \to 0 . \]

To prove the analogue of Corollary 4.3, we compare $(\alpha^h, \phi^h)|_{t=t_0^b}$ with $(\alpha^h, \phi^h)|_{t=0}$ thanks to the above relations. Roughly speaking, the error is of order $(t_0^b)^{s-1}$. This yields the following result, whose proof can be found in [7]:

Proposition 6.2. Let $n \geq 2$, $a_0^b \in \mathcal{S}(\mathbb{R}^n)$ bounded in $H^m$ uniformly in $h \in [0,1]$ for every $m > 0$. Let $s \in \mathbb{N}$. There exists $C$ independent of $h$ such that for every $0 \leq t \leq \min(T, \tau)$,

$$
\|a^h(t) - \alpha^h(t)\|_{H^s} + \|\phi^h(t) - \varphi^h(t)\|_{H^s} \leq C \left( h t + h^{\frac{s-1}{s-1}} \right) .
$$

where we changed the notations $\psi^c$ and $a_0^h$ to $\psi^h$ and $a_0^h$ to keep in mind that these functions depend on the small parameter. Equation (6.1) differs from (4.1) by two aspects: the presence of time in the nonlinearity, and the data are prescribed at time $t = t_0^h$ instead of $t = 0$. We explain how the computations of Section 4 can be adapted to this case. Seeking $\psi^h = \alpha^h e^{i\varphi^h/h}$ (6.3) becomes:

$$
\partial_t \psi^h + \psi^h \cdot \nabla \psi^h + 2t^{n-2} f'(t^n|\alpha^h|^2) \Re(\overline{\nabla \alpha^h}) = 0 \quad ; \quad \psi^h|_{t=t_0^h} = 0 ,
$$

(6.2)

$$
\partial_t \alpha^h + \psi^h \cdot \nabla \alpha^h + \frac{1}{2} \alpha^h \div \psi^h = \frac{h}{2} \Delta \alpha^h \quad ; \quad \alpha^h|_{t=t_0^h} = a_0^h .
$$

As a symmetrizer, we take:

$$
S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{t^2-n}{t^2} I_n \end{pmatrix} , \quad \text{where} \quad f' \text{ stands for } f'(t^n(\alpha_1^h|^2 + \alpha_2^h|^2)) .
$$

Unlike in Section 4.2 we must be careful with the powers of $t$: the term $t^{2-n}$ in the lower block is singular. When computing $\partial_t S$ in the energy estimate, differentiating $t^{2-n}$ on the numerator of the lower block yields a non-positive term: once again, the assumption $n \geq 2$ is necessary for our proof to work. When differentiating the denominator, we can factor out $(S\partial_a^u \alpha^h, \partial_a^u \psi^h)$, times

$$
\left\| \frac{1}{t^n} \partial_t \left( f'(t^n(\alpha_1^h|^2 + \alpha_2^h|^2)) \right) \right\|_{L^\infty(\mathbb{R}^n)} .
$$

Thus the singular term $t^{2-n}$ is finally harmless. Apart from that remark, the computations are similar, and we refer to [7] for more details. We infer:
The last term is o(h) as soon as k > 1, hence this assumption. Then we have the analogue of Corollary 4.3. Using (6.3), we infer the analogue of Corollary 4.6:

**Corollary 6.3.** Let \( N \in \mathbb{N} \setminus \{0\} \). Under the assumptions of Proposition 6.2, \( \| \psi^h - \Psi^h_N \|_{L^\infty([t_0^h,t^h];L^2)} \ll 1 \) for any \( t^h_0 \leq t^h \ll h^{(N+1)/m-1} \), where \( \Psi^h_N \) is given by:

\[
\Psi^h_N(t,x) = a^h_0(x) \exp \left( \frac{i}{h} \sum_{j=1}^{N} t^{n-1} \phi^h_j(x) \right).
\]

We infer Corollary 4.4 in the case \( \omega = 0 \), in the first case concerning \( \delta^\varepsilon \). Back to the initial variables, the instability occurs for

\[
\frac{\varepsilon_\gamma}{1-t^\varepsilon} \approx \frac{\varepsilon^{1-\gamma}}{\delta^\varepsilon},
\]

and the solution at that time is concentrated at scale \( 1-t^\varepsilon \); \( u^\varepsilon \) is of order \( (1-t^\varepsilon)^{-n/2} \). If \( \delta^\varepsilon \approx \varepsilon^{1-\gamma-\frac{1}{25}} \), the rate of concentration is then

\[
1-t^\varepsilon \approx \varepsilon^{\gamma-\frac{1}{25}}.
\]

**6.2. With an harmonic potential.** With \( a^0_0 \) as above, consider now:

\[
i\varepsilon \partial_t w^\varepsilon + \frac{\varepsilon^2}{2} \Delta w^\varepsilon = \frac{|x|^2}{2} w^\varepsilon + f(e^k|w^\varepsilon|^2) w^\varepsilon; \quad w^\varepsilon \big|_{t=0} = a^\varepsilon_0(x).
\]

The harmonic potential causes focusing of the linear solution \( f \equiv 0 \) in the limit \( \varepsilon \to 0 \) at time \( t = \pi/2 \). Use the transform (3.7) to remove the harmonic potential:

\[
W^\varepsilon(t,x) = \frac{1}{(1+t^2)^{n/4}} e^{\frac{|x|^2}{1+t^2}} w^\varepsilon \left( \arctan t, \frac{x}{\sqrt{1+t^2}} \right).
\]

Now the focusing phenomenon occurs for \( W^\varepsilon \) when time goes to infinity. To “compactify” time, we use another semi-classical conformal transform:

\[
w^\varepsilon(t,x) = \frac{1}{(1-t)^{n/2}} W^\varepsilon \left( \frac{t}{1-t}, \frac{x}{1-t} \right) e^{i \frac{|x|^2}{2(1-t)^n}},
\]

The focusing for \( w^\varepsilon \) occurs for times close to 1. It is natural to use (3.9):

\[
w^\varepsilon(t,x) = \frac{1}{(1-t)^{n/2}} \psi^\varepsilon \left( \frac{x}{1-t}, \frac{x}{1-t} \right) e^{i \frac{|x|^2}{2(1-t)^n}}, \quad \text{where } \gamma = \frac{k}{n} < 1.
\]

We thus have

\[
\psi^\varepsilon(t,x) = W^\varepsilon \left( \frac{t}{\varepsilon^\gamma} - 1, x \right)
\]

Keep the notations \( h = \varepsilon^{1-\gamma} \) and \( t^h_0 = h^{\gamma/(1-\gamma)} \). The function \( \psi \) solves:

\[
\begin{cases}
\frac{i h}{\partial_t} \psi^h + \frac{h^2}{2} \Delta \psi^h = \frac{1}{(t^h_0)^2 + (t-t^h_0)^2} f \left( \left((t^h_0)^2 + (t-t^h_0)^2 \right)^{n/2} |\psi^h|^2 \right) \psi^h, \\
\psi^h \big|_{t=t^h_0} = a^h_0.
\end{cases}
\]

We have the same equation as (6.1), with \( t \) in the nonlinearity replaced by

\[
\left((t^h_0)^2 + (t-t^h_0)^2 \right)^{1/2}.
\]

We can reproduce the analysis of Section 6.1 with again (6.2) as a limiting system, since \( t^h_0 \to 0 \) as \( h \to 0 \). The price to pay is the same: we have an error estimate like in Proposition 6.2, so we must assume \( k > 1 \) to approximate the phases.
Back to the initial variables, the instability occurs for
\[ t^\varepsilon \approx \frac{\pi}{2} - \arctan \left( \frac{\delta^2}{\varepsilon^{1-2\gamma}} \right), \]
and the solution at that time is concentrated at scale \( \cos t^\varepsilon \); \( u^\varepsilon \) is of order \((\cos t^\varepsilon)^{-n/2}\).
If \( \delta^2 \approx \varepsilon^{1-\gamma} \frac{1}{2} \), the rate of concentration is then
\[ \cos t^\varepsilon \approx \sin \arctan \left( \frac{\delta^2}{\varepsilon^{1-2\gamma}} \right) \approx \varepsilon^{1-\gamma} \frac{1}{2}. \]
In particular, this rate of concentration is large compared to the one studied in [3], which is \( \varepsilon \), while the authors consider the case \( n = 3 \) and \( k = 2 \) (see also Remark 3.1).
Finally, in the case \( a_0^2 = a_0 + \varepsilon^{1-\gamma} a_1 \), Corollary 3.3 stems from Proposition 1.2 in the same fashion as above.

**APPENDIX A. LINEAR EQUATION**

In the linear case, justifying WKB methods is rather easy, and we prove:

**Proposition A.1.** Let \( n \geq 1 \), \( a_0 \in \mathcal{S}(\mathbb{R}^n) \), and \( V, V_1 \in C^\infty(\mathbb{R}^n; \mathbb{R}) \) be smooth sub-quadratic potentials:
\[ \partial^\alpha V, \partial^\alpha V_1 \in L^\infty(\mathbb{R}^n), \quad \forall \alpha \in \mathbb{N}^n \text{ such that } |\alpha| \geq 2. \]
Assume also that \( V_1 \neq 0 \) on \( \text{supp} \ a_0 \). Let \( u^h \) and \( v^h \) solve the initial value problems:
\[
\begin{align*}
    i \hbar \partial_t u^h + \frac{\hbar^2}{2} \Delta u^h &= V(x) u^h \quad ; \quad u^h|_{t=0} = a_0(x), \\
    i \hbar \partial_t v^h + \frac{\hbar^2}{2} \Delta v^h &= (V(x) + \delta V_1(x)) v^h \quad ; \quad v^h|_{t=0} = a_0(x).
\end{align*}
\]
Assume that \( h^{2/3} \ll \hbar \ll 1 \). Then we can find \( 0 < \nu^h < h^{1/3} \) such that:
\[ \|u^h - v^h\|_{L^\infty([0, \nu^h]; L^2)} \gtrsim 1. \]

**Proof.** For \( \delta \in [0, 1] \), let \( w^h_\delta \) solve
\[
    i \hbar \partial_t w^h_\delta + \frac{\hbar^2}{2} \Delta w^h_\delta = (V(x) + \delta V_1(x)) w^h_\delta \quad ; \quad w^h_\delta|_{t=0} = a_0(x).
\]
WKB method yields \( w^h_\delta \sim w^h_0 = A_\delta e^{i\Phi_\delta/h} \), where:
\[ (A.1) \quad \partial_t \Phi_\delta + \frac{1}{2} \left( |\nabla_x \Phi_\delta|^2 \right) + V(x) + \delta V_1(x) = 0 \quad ; \quad \Phi_\delta(0, x) = 0. \]
\[ (A.2) \quad \partial_t A_\delta + \nabla_x \Phi_\delta \cdot \nabla_x A_\delta + \frac{\hbar^2}{2} A_\delta \Delta \Phi_\delta = 0 \quad ; \quad A_\delta(0, x) = a_0(x). \]
Since the difference \( r^h_\delta := w^h_0 - w^h_\delta \) solves:
\[
    i \hbar \partial_t r^h_\delta + \frac{\hbar^2}{2} \Delta r^h_\delta = (V(x) + \delta V_1(x)) r^h_\delta + e^{i\Phi_\delta/h} \frac{\hbar^2}{2} \Delta A_\delta \quad ; \quad r^h_\delta|_{t=0} = 0,
\]
standard energy estimates yield:
\[ \|r^h_\delta\|_{L^\infty([0, T]; L^2)} \lesssim h \|\Delta A_\delta\|_{L^1([0, T]; L^2)}. \]
Since \( V \) and \( V_1 \) are smooth and sub-quadratic, there exists \( T > 0 \) such that for every \( \delta \in [0, 1] \), \( A.1 \) has a smooth solution \( \Phi_\delta \in C^\infty([0, T] \times \mathbb{R}^n) \), and \( A.2 \) has a smooth solution such that \( A_\delta \in L^\infty([0, T]; H^2) \), and:
\[ \|A_\delta\|_{L^\infty([0, T]; H^2)} \leq C, \text{ where } C \text{ is independent of } \delta \in [0, 1]. \]
Moreover, plugging Taylor expansion in time for \( \Phi_\delta \) and \( A_\delta \), we find:
\[ \Phi_\delta(t, x) = -t(\delta V_1(x)) + \mathcal{O}(t^3) \quad ; \quad A_\delta(t, x) = a_0(x) + \mathcal{O}(t) \text{ as } t \to 0. \]
This implies that for $0 < t^h \ll h^{1/3}$,
\[
\|u^h - v^h\|_{L^\infty([0,t^h];L^2)} = \|u^h - v^h\|_{L^\infty([0,t^h];L^2)} + o(1) \text{ as } h \to 0,
\]
where $u^h$ and $v^h$ solve the ordinary differential equations:
\[
ih \partial_t u^h = V(x) u^h; \quad \ih \partial_t v^h = (V(x) + \delta^h V_1(x)) v^h; \quad u^h|_{t=0} = v^h|_{t=0} = a_0(x).
\]
We infer:
\[
\|u^h - v^h\|_{L^\infty([0,t^h];L^2)} = \|a_0(x) \left( e^{it^h V_1(x)/h} - 1 \right) \|_{L^\infty([0,t^h];L^2)} + o(1).
\]
By assumption, we can make the right hand side $\gtrsim 1$ for times $0 < t^h \ll h^{1/3}$ such that $t^h \delta^h \gtrsim h$, and the proposition follows.

\section*{Appendix B. Application: ill-posedness results}

As a consequence of the analysis of Section 3.1, we retrieve some results established in [14, 15] concerning ill-posedness issues for the nonlinear Schrödinger equation without a small parameter.

\begin{proposition}[14, 15] Let $n \geq 1$, $\omega \in \mathbb{R} \setminus \{0\}$ and $\sigma > 0$ an integer. Consider the nonlinear Schrödinger equation in $\mathbb{R}^n$:
\begin{equation}
B.1 \quad i \partial_t u + \frac{1}{2} \Delta u = \omega |u|^{2\sigma} u; \quad u|_{t=0} = u_0.
\end{equation}
\begin{itemize}
\item Ill-posedness. Let $s < \frac{n}{2} - \frac{1}{\sigma}$. Then \((B.1)\) is not locally well-posed in $H^s(\mathbb{R}^n)$: for any $\delta > 0$, we can find families $(u^0_0)_0 \ll 1$ and $(u^0_2)_0 \ll 1$ with $u^0_0, u^0_2 \in \mathcal{S}(\mathbb{R}^n)$ such that $u^0_0 \parallel H_\sigma, \parallel u^0_2 \parallel H_\sigma \leq \delta$, $\parallel u^0_1 - u^0_2 \parallel H_\sigma \ll 1$, such that if $u^1_1$ and $u^1_2$ denote the solutions to \((B.1)\) with these initial data, there exists $0 < t^0 \ll 1$ such that $\parallel u^1_1(t^0) - u^1_2(t^0) \parallel H_\sigma \gtrsim 1$.
\item Norm inflation. Assume $0 < s < \frac{n}{2} - \frac{1}{\sigma}$. We can find $(u^0)_0 \ll 1$ solving \((B.1)\), such that $u^0 \in \mathcal{S}(\mathbb{R}^n)$ and:
\[
\|u^0_0\|_{H^s} \ll 1 \quad \text{ and } \exists \varepsilon \ll 1, \quad \|u^0(t^0)\|_{H^s} \gg 1.
\]
\end{itemize}
\end{proposition}

\begin{proof}
This result is a straightforward consequence of WKB analysis for small time, as in Proposition 2.1. For $a_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\|a_0\|_{H^s} \leq \delta/2$, and $\lambda > 0$, consider $u$ solving \((B.1)\) with:
\[
u_0(x) = \lambda^{-\frac{n}{2}+\sigma} a_0 \left( \frac{a}{\lambda} \right).
\]
Using the parabolic scaling and the scaling of $H^s$, define $u_\lambda$ by:
\[
u_\lambda(t, x) = \lambda^{-\frac{n}{2}-s} u \left( \lambda^2 t, \lambda x \right).
\]
It solves:
\[
i \partial_t u_\lambda + \frac{1}{2} \Delta u_\lambda = \omega \lambda^{2-n+2s} |u_\lambda|^{2\sigma} u_\lambda; \quad u_\lambda|_{t=0} = a_0.
\]
Let $h = \lambda^{\frac{n}{2}-1-\sigma}$: $\lambda$ and $h$ go to zero simultaneously since $s < \frac{n}{2} - \frac{1}{\sigma}$. Define
\[
\nu^h(t, x) = u_\lambda(\lambda t, x) = \lambda^{-\frac{n}{2}-s} u \left( \lambda^{\frac{n}{2}+1-\sigma} t, \lambda x \right).
\]
It solves:
\begin{equation}
B.2 \quad i\h \partial_t \nu^h + h^2 \Delta \nu^h = \omega |\nu^h|^{2\sigma} \nu^h; \quad \nu^h|_{t=0} = a_0.
\end{equation}
We go back to $u$ via the formula:
\[
u(t, x) = \lambda^{-\frac{n}{2}+\sigma} \nu^h \left( \frac{t}{\lambda^{\frac{n}{2}+1-\sigma}}, \frac{x}{\lambda} \right).
\]
\end{proof}
Ill-posedness. Let $\tilde{\psi}^h$ solve (B.2) with a slightly different initial data:
$$\tilde{\psi}^h_{t=0} = (1 + \delta^h) a_0,$$
with $\delta^h = |\ln h|^{-\theta} \ll 1$, where $\theta > 0$ stems from Proposition 2.1. We infer from Proposition 2.1 and the discussion of Section 3.1 that for $t^h = c_0 h |\ln h|^\theta \ll 1$, we have:
$$\|\psi^h(t^h) - \tilde{\psi}^h(t^h)\|_{H^{s'}} \gtrsim 1.$$
Back to the function $u$, this yields the first part of Proposition B.1.

Norm inflation. In [15], this phenomenon appears as a transfer of energy from low to high Fourier modes. It corresponds to the apparition of rapid oscillations in a super-critical WKB régime, which can be viewed as a particular case of the above statement: even though $\psi^h$ is not $h$-oscillatory initially, rapid oscillations appear instantly. Note that a similar phenomenon was shown recently in the context of Euler equations by C. Cheverry and O. Guès [12].

Still from Proposition 2.1 and the discussion of Section 3.1 with $t^h = c_0 h |\ln h|^\theta$, we have
$$\psi^h(t^h) \sim a_0(x) e^{i t^h \phi_1(x)} = a_0(x) e^{-i \omega |a_0(x)|^2 (\ln \frac{1}{h})^\theta}.$$  
Even though $\psi^h$ is not yet $h$-oscillatory, “rapid” oscillations have appeared already. Now as in [3], we may replace $a_0$ with $|\ln \lambda|^{-\theta} a_0$ to complete the proof of Proposition B.1. □

APPENDIX C. ON THE FLOW MAP FOR THE CUBIC, DEFocusing NLS

In the previous section, ill-posedness results were established thanks to a justification of WKB analysis for very small times, of order $h |\ln h|^\theta$. For the cubic, defocusing Schrödinger equation, we saw that a rigorous WKB analysis was available for times of order $O(1)$.

Proof of Corollary [14]. Mimicking the previous section, for $a_0 \in S(\mathbb{R}^n)$, let
$$u_0(x) = \lambda^{-\frac{2}{s} - s} a_0 \left( \frac{x}{\lambda} \right).$$
Let $h = \lambda^{\frac{2}{s} - 1 - s}$: $h$ and $\lambda$ go simultaneously to zero, since $s < s_c$. Define
$$\psi^h(t, x) = u^\lambda (ht, x) = \lambda^{\frac{2}{s} - s} u \left( \lambda^{\frac{2}{s} + 1 - s} t, \lambda x \right).$$
It solves:
$$i h \partial_t \psi^h + \frac{h^2}{2} \Delta \psi^h = |\psi^h|^2 \psi^h; \quad \psi^h_{t=0} = a_0(x).$$
(C.1)

The idea of the proof is that for times of order $O(1)$, $\psi^h$ is $h$-oscillatory. This result is expected to be true not only for cubic defocusing nonlinearities, but it seems this is the only framework where it has been proved [20].

We infer from Proposition 6.1 that there exist $T > 0$ independent of $h \in [0, 1]$, and $a, \phi, \phi_1 \in C([0, T]; H^m)$ for any $m \geq 0$, such that:
$$\|\psi^h - a e^{i \phi_1 e^{i \phi/h}}\|_{L^\infty([0, T]; H^m)} \leq C_m h^{1-m}.$$  
Since the $H^m$-norm of $a e^{i \phi_1 e^{i \phi/h}}$ is of order $h^{-m}$ (when $\phi$ is not stationary), we deduce that there exists $t \in [0, T]$ such that for any $m \geq 0$:
$$\|\psi^h(t)\|_{H^m} \approx h^{-m}.$$  
This implies:
$$\|u \left( \lambda^{\frac{2}{s} + 1 - s} t \right)\|_{H^k} \approx \lambda^{s-k} \|\psi^h(t)\|_{H^k} \approx \lambda^{s-k} h^{-k} = \lambda^{s-k} (\frac{\lambda}{h})^{1-s}.$$
The result then follows when considering the limit $\lambda \to 0$. As in the previous section, we get exactly the statement of the corollary by replacing $a_0$ by $|\ln \lambda|^{-1} a_0$ for instance.

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