SPECTRAL TRIPLES AND FINITE SUMMABILITY ON CUNTZ-KRIEGER ALGEBRAS

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Abstract. We produce a variety of odd bounded Fredholm modules and odd spectral triples on Cuntz-Krieger algebras by means of realizing these algebras as "the algebra of functions on a non-commutative space" coming from a sub shift of finite type. We show that any odd K-homology class can be represented by such an odd bounded Fredholm module or odd spectral triple. The odd bounded Fredholm modules that are constructed are finitely summable. The spectral triples are θ-summable although their bounded transform, when constructed using the sign-function, will already on the level of analytic K-cycles be finitely summable bounded Fredholm modules. Using the unbounded Kasparov product, we exhibit a family of generalized spectral triples, possessing mildly unbounded commutators, whilst still giving well defined K-homology classes.

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Introduction

This paper is a study of how odd $K$-homology classes on Cuntz-Krieger algebras can be realized by explicit cycles; both by means of bounded Fredholm modules (also known as analytic $K$-cycles) and as unbounded Fredholm modules, e.g. spectral triples. We will in this paper use the Poincaré duality constructed for Cuntz-Krieger algebras by Kaminker-Putnam \[36\] to find explicit finitely summable Fredholm modules representing any odd $K$-homology class. This allows us to realize odd $K$-homology class by means of abstract Toeplitz operators and the finite summability of the cycles is proved using the work of the first named author \[31\].

The construction of unbounded Fredholm modules requires a more elaborate construction. We discuss the possibility of using Kasparov products of unbounded Fredholm modules for the fixed point algebra of the gauge action with a well studied unbounded bivariant cycle. In many cases it is difficult to understand cohomological properties of unbounded Fredholm modules on the fixed point algebra, they nevertheless exist in abundance due to \[14\]. Furthermore, in interesting cases such as the Cuntz algebra $O_N$ this will produce $K$-homologically trivial unbounded Fredholm modules.

A more fruitful approach comes from describing the Cuntz-Krieger algebra as the noncommutative quotient of the underlying subshift of finite type, via its groupoid model. This viewpoint is common to noncommutative geometry. The maximal abelian sub algebra corresponding to the unit space in the groupoid plays the rôle of the base space in a fibration. The unbounded Fredholm modules are then obtained by restricting an unbounded bivariant cycle to a "fiber" over the unit space. The bivariant cycle is inspired both by the dynamics of the underlying sub shift of finite type and the structures appearing in Kaminker-Putnam’s Poincaré duality class. This uses the idea of multiplication by real valued functions defined on the groupoid to obtain regular operators as in \[15\]. An explicit construction of the unbounded Kasparov product of this cycle with canonically defined spectral triples on the commutative base from \[6\] yields a generalization of the notion of unbounded $K$-cycle, allowing for unbounded commutators. This generalization is compatible with $K$-homology. These Kasparov products are constructed using the operator space approach to connections initiated by the second named author in \[40\] and developed further in \[9\].

The problem that this work originates from can be formulated as follows. Whenever $B$ is a $C^*$-algebra and $x \in K^*(B)$ is a $K$-homology class, is it possible to find an explicit analytic $K$-cycle or unbounded Fredholm module representing $x$ with "good" analytic properties? By "good", we mainly have various summability properties in mind, e.g. finitely summable or $\theta$-summable. We return to discuss this problem setting more precisely below. The Cuntz-Krieger algebras are interesting in this aspect since results of Connes \[17\], combined with the fact that Cuntz-Krieger algebras admit no traces, imply that it is never possible to have a finitely summable unbounded Fredholm module on a Cuntz-Krieger algebra\[^1\]. In this paper we show that any odd $K$-homology class is represented by a finitely summable $K$-cycle. It should be mentioned that this interesting structure has been shown to appear also on the crossed product of boundary actions of a hyperbolic group \[27\].

As mentioned, obstructions to finite summability of unbounded Fredholm modules are known from \[17\]. There is however (to our knowledge) no example in the literature of a $K$-homology class that can not be represented by an analytic $K$-cycle that is finitely summable on some dense sub-algebra. We provide such an example on a commutative $C^*$-algebra.

**Preliminaries.** Before entering into finite summability issues and the precise formulation of the results in this paper, we recall some concepts of noncommutative geometry. This paper discusses

\[^1\]Not even for the generalized notion of unbounded Fredholm modules alluded to above.
the noncommutative geometry of Cuntz-Krieger algebras from the point of view of Kasparov’s $KK$-theory \cite{37,38}, and the unbounded formulation thereof due to Connes \cite{16} and Baaj-Julg \cite{5}. The central objects in Kasparov’s approach to $KK$-theory are Fredholm modules. Fredholm modules come in two flavors; bounded and unbounded. The bounded Fredholm modules are sometimes referred to as analytic $K$-cycles.

**Definition 1.** Let $A$ be a $C^*$-algebra. A bounded even Fredholm module over $A$ is a triple $(\pi, \mathcal{H}, F)$ consisting of

1. a $\mathbb{Z}/2$ graded Hilbert space $\mathcal{H}$ carrying an even $^*$-representation $\pi : A \rightarrow \mathbb{B}(\mathcal{H})$;
2. an odd operator $F \in \mathbb{B}(\mathcal{H})$ with the property that $\pi(a)(F^2 - 1), \pi(a)(F - F^*)$ and $[F, \pi(a)]$ are all compact operators.

A triple $(\pi, \mathcal{H}, F)$ with the above properties safe for the fact that the Hilbert space $\mathcal{H}$ is graded, defines a bounded odd Fredholm module.

By defining a suitable notion of homotopy, equivalence classes of even Fredholm modules form an abelian group $K^0(A)$, and the odd Fredholm modules are used to build an abelian group $K^1(A)$. The groups $K^0(A)$ and $K^1(A)$ are called the $K$-homology groups of $A$. The $K$-homology groups are homotopy invariants of $A$ and encode index theoretic information. See \cite{34} for an excellent exposition of this theory. Historically, the Fredholm picture of $K$-homology was conceived by Atiyah \cite{3} who introduced it to make the Atiyah-Singer index theorem into a functorial statement. It reached full maturity in the work of Kasparov \cite{37}, culminating in his proof of the Novikov conjecture for a large class of groups \cite{38}. For computational purposes, it is sometimes convenient to work with the unbounded Fredholm modules.

**Definition 2.** An odd (even) unbounded Fredholm module over a $C^*$-algebra $A$ consists of a triple $(\pi, \mathcal{H}, D)$ containing the data:

1. A (graded) Hilbert space $\mathcal{H}$.
2. A (even) $^*$-representation $\pi : A \rightarrow \mathbb{B}(\mathcal{H})$.
3. A selfadjoint (odd) operator $D$ with locally compact resolvents $\pi(a)(D \pm i)^{-1} \in \mathbb{K}(\mathcal{H})$, such that the $^*$-algebra

$$\text{Lip}(\pi, \mathcal{H}, D) := \{a \in A : \pi(a)\text{Dom}(D) \subseteq \text{Dom}(D) \text{ and } [D, \pi(a)] \text{ extends to a bounded operator.}\}$$

is norm dense in $A$. If $\pi$ is faithful and $\mathcal{A} \subseteq \text{Lip}(\pi, \mathcal{H}, D))$ is dense in $\pi(A)$ the triple $(\mathcal{A}, \mathcal{H}, D)$ is called an odd (even) spectral triple.

An unbounded Fredholm module defines a bounded Fredholm module by setting $F := D(1 + D^2)^{-\frac{1}{2}}$, the bounded transform of $D$. It should be noted that for any choice of bounded continuous function $\chi \in C^0(\mathbb{R}, \mathbb{R})$ such that $\chi^2 - 1 \in C_0(\mathbb{R})$ we can associate a bounded Fredholm module by $F_\chi := \chi(D)$ producing an equivalent Fredholm module. Another way of doing this, featured prominently in the present work, is through the phase of $D$, defined as $D|D|^{-1}$. Here $|D|^{-1}$ is defined to be $0$ on the kernel of $D$. Since the spectrum of $D$ is discrete, there is a $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ as above with $\chi' \in C_c(\mathbb{R}, \mathbb{R})$ such that $F_\chi = D|D|^{-1}$. The choice of $\chi$ only affects the bounded Fredholm module by a compact perturbation. Despite this, we will in this paper see several examples of how finer analytic properties do depend on $\chi$.

The foundation of noncommutative geometry is built on the idea that the geometry of a "noncommutative space" is encoded by a spectral triple on the "algebra of functions", i.e. a $C^*$-algebra. Conformal geometry is encoded by a choice of bounded Fredholm module. Homological algebra corresponds to $K$-theory and $K$-homology. Such ideas are in the classical case supported by facts such as
(1) The geodesic distance on a manifold can be reconstructed from any spectral triple defined from a Dirac type operator, see [16], Chapter VI.

(2) The conformal class of a metric is uniquely determined by the bounded transform of a spectral triple defined from a Dirac type operator modulo compact perturbations, see [11].

(3) A Riemannian spin$^c$-manifold can be reconstructed from the spectral triple associated with the spin$^c$-Dirac operator, see [13].

We have made a choice of a distinguishing in terminology between spectral triples and unbounded Fredholm modules as the former corresponds to prescribing a “non-commutative geometry” while the latter is a cycle for a cohomology theory for $C^*$-algebras. Despite this, we abuse the notation by sometimes identifying an unbounded Fredholm module $(\pi, \mathcal{H}, D)$ with the spectral triple $(\pi(\text{Lip}(\pi, \mathcal{H}, D)), \mathcal{H}, D)$ for $\mathcal{A}/\ker \pi$.

Outside the classically geometric setup of manifolds many technical difficulties appear. Under various regularity assumptions there are many computational tools such as cyclic cohomology for computing invariants, e.g., index theory. These regularity assumptions often come in two shapes; externally imposed or canonically constructed from a Fredholm module. The externally imposed conditions appears in an ad hoc manner, e.g., imposing a choice of $\mathcal{M}$ in a spectral triple, and the regularity coming from these conditions are often easier to compute with, for example see [13]. Those canonically constructed from a Fredholm module are often more unwieldy. One of these conditions is that of summability.

**Obstructions to finite summability.** Summability of Fredholm modules is based on the idea of refining the compactness properties in its definition by belonging in a finer symmetrically normed operator ideals. We will mainly use finite summability and $\theta$-summability, defined using Schatten ideals respectively the Li-ideals. We let $\mathcal{H}$ denote a separable Hilbert space throughout the paper. For a compact operator $T$ on $\mathcal{H}$ we let $(\mu_k(T))_{k \in \mathbb{N}} \subseteq \mathbb{R}_+$ denote a decreasing enumeration of the singular values of $T$. Recall that the Schatten ideals are defined as

$$\mathcal{L}^p(\mathcal{H}) := \{ T \in \mathcal{B}(\mathcal{H}) : (\mu_k(T))_{k \in \mathbb{N}} \in \ell^p(\mathbb{N}) \},$$

for $p > 0$. These spaces form ideals of compact operators in $\mathcal{B}(\mathcal{H})$. The homogeneous function

$$\|T\|_{\ell^p} := \sqrt[p]{\text{Tr}((T^*T)^{\frac{p}{2}})} = \|(\mu_k(T))_{k \in \mathbb{N}}\|_{\ell^p(\mathbb{N})},$$

makes $\mathcal{L}^p(\mathcal{H})$ into a Banach $*$-algebra for $p \in [1, \infty)$ and for $p \in (0, 1)$ into a quasi-normed space. For $p \in [1, \infty)$, the space

$$\text{Li}^1(\mathcal{H}) := \{ T \in \mathcal{B}(\mathcal{H}) : \mu_k(T) = O(\log(k)^{-1/p}) \},$$

is an operator ideal as well. We use the notation $\text{Li}(\mathcal{H}) := \text{Li}^1(\mathcal{H})$.

**Definition 3.** Let $(\pi, \mathcal{H}, F)$ be an analytic K-cycle for a $C^*$-algebra $\mathcal{A}$. Then $(\pi, \mathcal{H}, F)$ is said to be $p$-summable if the $*$-algebra

$$\text{Hol}^p(\pi, \mathcal{H}, F) := \{ a \in \mathcal{A} : [F, \pi(a)] \in \mathcal{L}^p(\mathcal{H}), \pi(a)(F^* - F), \pi(a)(F^2 - 1) \in \mathcal{L}^{p/2}(\mathcal{H}) \},$$

is norm dense in $\mathcal{A}$. If $\mathcal{L}^p(\mathcal{H})$ and $\mathcal{L}^{p/2}(\mathcal{H})$ is replaced with $\text{Li}^p(\mathcal{H})$ respectively $\text{Li}^{p/2}(\mathcal{H})$, $(\pi, \mathcal{H}, F)$ is $\theta$-summable. An unbounded Fredholm module $(\pi, \mathcal{H}, D)$ is $p$-summable if $\pi(\alpha)(D \pm i)^{-1} \in \mathcal{L}^p(\mathcal{H})$, for $a$ in a subalgebra of $\text{Lip}(\pi, \mathcal{H}, D)$ dense in $\mathcal{A}$, and $\theta$-summable if $\mathcal{L}^p(\mathcal{H})$ is replaced with $\text{Li}^p(\mathcal{H})$.

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2 Once it is decorated with some minor further structures.
More generally, one can speak of summability relative to any ideal of operators. If \((\pi, \mathcal{H}, F)\) is \(p\)-summable, \(p\) is referred to as the degree of summability. In geometric situations, the degree of summability is often related to the dimension of the underlying space via some type of Weyl law. The notion of \(\theta\)-summability is robust in the sense that \(\theta\)-summable \(K\)-cycles can be lifted to unbounded \(\theta\)-summable Fredholm modules (cf [16, Chapter IV.8.a, Theorem 4]). Particular instances of this phenomenon are known for finite summability as well, notably the paper [58] where a lifting result for the group algebra of a group of polynomial growth was established. The general situation is quite different in the case of finite summability.

The paper [17] shows that the existence of a finitely summable unbounded Fredholm module over a \(C^*\)-algebra \(A\) implies the existence of a tracial state on \(A\). In particular, purely infinite \(C^*\)-algebras do not admit finitely summable unbounded Fredholm modules. Recent results by Emerson-Nica [27] show that certain purely infinite \(C^*\)-algebras arising as boundary crossed product algebras associated to hyperbolic groups are "uniformly summable". I.e. they admit finitely summable bounded Fredholm modules in a strong sense made precise below. Thus, a general lifting construction for Fredholm modules, preserving finite summability, is impossible. In this paper, we show that a result similar to that of [27] holds for Cuntz-Krieger algebras. We furthermore provide a class of examples of \(\theta\)-summable unbounded Fredholm modules on Cuntz-Krieger algebras such that its bounded transform using the sign function in fact is finitely summable. This example indicates not only that lifting is a delicate matter but that the same holds for finer analytic properties of bounded transforms and the choice of \(\chi\).

**Definition 4.** Let \(B\) be a \(C^*\)-algebra. We say that the class \(x \in K^i(B)\) is summable of degree \(p\) if there exists a \(p\)-summable Fredholm module representing \(x\). We define

\[
\deg_{\text{sum}}(x) := \inf \{ p | x \text{ is } p\text{-summable} \} \quad \text{and} \quad \deg_{\text{sum}}^i(A) := \sup \{ \deg_{\text{sum}}(x) | x \in \text{K}^i(A) \}.
\]

Furthermore, we say that the algebra \(A\) has finitely summable even respectively odd \(K\)-homology if \(\deg_{\text{sum}}^0(A) < \infty\) respectively \(\deg_{\text{sum}}^1(A) < \infty\).

Here we apply the convention that the infimum of the empty set if infinite. We say that a \(K\)-homology class is finitely summable if \(\deg_{\text{sum}}(x) < \infty\). The summability degree of a \(C^*\)-algebra is clearly an isomorphism invariant. An interesting question is if it is a homotopy invariant. There is to our knowledge no known obstruction for a \(K\)-homology class to be finitely summable similar to tracial obstructions for finitely summable unbounded Fredholm modules. We also note the terminology uniformly summable from [27]; a \(C^*\)-algebra \(A\) is said to be uniformly summable if there is a \(p > 0\) and a dense \(\mathcal{A} \subseteq A\) such that any \(x \in K^i(A)\) admits a representative that is \(p\)-summable on \(\mathcal{A}\).

**Example.** We have one example of a \(K\)-homology class that is not finitely summable. This result is not to be confused with the interesting results of [53, 54] where a sub algebra of \(A\), on which \(K\)-homology classes are required to be finitely summable, is fixed.

**Lemma 5.** There is a \(K\)-homology class \(x \in K^1\left(\bigoplus_{j=1}^\infty C_0(S^{2j-1})\right)\) such that

\[
\deg_{\text{sum}}(x) = \infty.
\]

**Proof.** Consider the sum of fundamental classes \(x = \sum_{j=1}^\infty [S^{2j-1}]\), that is, \(x\) is represented by \((\pi, \bigoplus_{j=1}^\infty L^2(S^{2j-1}), F)\) where \(\pi\) is action by point wise multiplication and \(F = \bigoplus F_j\) where \(F_j = 2P_j - 1\) and \(P_j\) is the Szegö projection on \(S^{2j-1}\). By the Universal Coefficient Theorem, it holds that \(K^1\left(\bigoplus_{j=1}^\infty C_0(S^{2j-1})\right) \cong \prod_{j=1}^\infty \mathbb{Z}\) and it is a well known fact that \(x|_{\mathbb{C}(S^{2j-1})} \in K^1(C(S^{2j-1})) \cong \mathbb{Z}\) is a generator for any \(j\), see for instance [63].
Moreover, for each class $x$ operator $D$ A Cuntz Krieger algebra Theorem 6. Furthermore, the triple denote the $C$ more in Section 1 below. There is a $C$ summability of a lowers that

Hence we have a contradiction since our assumptions imply that $p \geq 2k - 2$ for all $k$.

Content and organization of the paper. The content of the paper can be summarized in the following Theorem. We use the notation $\Omega_A$ for the space of characters of the standard maximal abelian sub algebra in the Cuntz-Krieger algebra $O_A$ associated with the $N \times N$-matrix $A$, see more in Section $\|$ below. There is a $C(\Omega_A)$-valued conditional expectation on $O_{\omega}$. We let $E^\Omega_A$ denote the $C(\Omega_A)$-Hilbert module closure of $O_{\omega}$ and $\pi^\Omega_A : O_{\omega} \to \text{End}_{C(\Omega_A)}(E^\Omega_A)$ the $O_{\omega}$-action.

**Theorem 6.** A Cuntz Krieger algebra $O_A$ satisfies that

$$\text{deg}^{\text{sum}}_\omega(O_A) = 0.$$ 

Moreover, for each class $x \in K^0(O_A)$ there exists an $\omega_x \in \Omega_A$ and a $\theta$-summable self-adjoint operator $D_x$ on $E^\Omega_A \otimes_{\omega_x} \mathbb{C}$ such that $x$ can be represented by the unbounded Fredholm module

$$(\pi^\Omega_A \otimes_{\omega_x} \text{id}_{\mathbb{C}}, E^\Omega_A \otimes_{\omega_x} \mathbb{C}, D_x)$$

Furthermore, the triple $(\pi^\Omega_A \otimes_{\omega_x} \text{id}_{\mathbb{C}}, E^\Omega_A \otimes_{\omega_x} \mathbb{C}, D_x |D_x^{-1})$ forms a finitely summable analytic $K$-cycle.

Remark 7. The first part of this Theorem should be compared to the results of [27]. The intersection of applications for this paper with [27] lies in the ample supply of examples of discrete hyperbolic groups $\Gamma$ such that $C(\partial \Gamma) \rtimes \Gamma$ is a Cuntz-Krieger algebra. These are always associated with a matrix $A$ such that $A' = A$. E.g. when $\Gamma$ is a free group (see below in Subsubsection 3.3.3). In these cases, the results of [27] are stronger in regards to finite summability of bounded Fredholm modules as they consider also the even $K$-homology.

Remark 8. In the thesis of Whittaker [35], the reader can find constructions similar to those in this paper. The computational approaches differs, but the spirit prevails. It is an interesting question if the results of this paper carries over to general Smale spaces and the situation of [65].

Remark 9. In regards to the discussions above, we can use the same $*$-algebra for all of the Fredholm modules constructed in this paper. Namely that generated by the $C^{*}$-generators of the Cuntz-Krieger algebra.

The paper is organized as follows. In Section $\|$ we recall some well known facts about Cuntz-Krieger algebras, focusing on their origin in the dynamics of sub shifts of finite type. This is encoded by means of a groupoid, first studied by Renault [35] [56]. The possibility to interchange the groupoid picture of Cuntz-Krieger algebras and the standard generator picture, used in the original definition of Cuntz-Krieger [24], is crucial to identifying the $K$-homology classes of the Fredholm modules and spectral triples constructed in this paper.

Section $\|$ contains the proof of the fact that $\text{deg}^{\text{sum}}_\omega(O_A) = 0$, this is stated in Theorem $\|$. To be precise, we recall the construction of Poincaré duality $K^{0}(O_{\omega}) \cong K_{+1}(O_{\omega}^{*})$ from [36]. Using this construction, we can identify exactly which odd $K$-homology cycles we need to prove finite summability for. Any odd $K$-homology class can be represented by a Fredholm module on the GNS-space $L^2(O_{\omega})$. It would be desirable to prove that the duality class $\Delta \in K^{0}(O_{\omega} \otimes \mathbb{C})$ in fact is finitely summable, in which case $\text{deg}^{\text{sum}}_\omega(O_A) = 0$ would follow, but we can in general only prove $\theta$-summability of $\Delta$ (see Theorem $\|$). In certain cases, for instance $SU_q(2)$, we obtain finite summability of a $K$-cycle representing $\Delta$. 
To construct a class of unbounded Fredholm modules on $O_A$ that can represent arbitrary odd $K$-homology classes, we consider two approaches. One using the sub-algebra of fixed points for the gauge action, an AF-algebra, in Section 3 and one using the maximal abelian sub algebra in Section 5. The approach of using the fixed point algebra comes more naturally as there is an unbounded $KK$-cycle naturally associated with the gauge action, called simply the gauge cycle. We prove that the gauge cycle plays the role of a boundary mapping in the Pimsner-Voiculescu six term exact sequence associated with the gauge action. As such, the possible $K$-homology classes of the unbounded Fredholm modules that can be constructed from the fixed point algebra by means of a Kasparov product with the gauge cycle can be computed. In some cases any $K$-homology class is of this form (e.g. the boundary actions of free groups, see Remark 3.3.3), only the odd ones are (e.g. for $SU_q(2)$, see Remark 3.3.5) and in other cases none (e.g. the Cuntz algebra $O_N$, see Remark 3.3.3).

Before considering the approach of constructing unbounded Fredholm modules from the maximal abelian sub algebra in Section 5, we recall some spectral triples in Section 4 considered by Bellissard-Pearson [6]. These spectral triples are interesting since there is an obstruction to extending them to the ambient Cuntz-Krieger algebra coming from a certain groupoid. In particular this shows that the Kasparov product

$$KK_k(O_A, C(\Omega_4)) \times K^0(C(\Omega_4)) \rightarrow K^1(O_A),$$

is surjective. This stands in sharp contrast with the situation where $C(\Omega_4)$ is replaced with the fixed point algebra $F_A$.

In view of this, we end this paper with a construction of an unbounded Kasparov product between the unbounded bivariant $(O_A, C(\Omega_4))$-cycle with the Bellissard-Pearson spectral triples. We compute the class of this Kasparov product in rational $K$-homology. This is achieved in the context of $\varepsilon$-unbounded Fredholm modules, a slight weakening of the notion of unbounded Fredholm modules. This weakening is only hinted at in the literature. We describe the main properties of $\varepsilon$-unbounded Fredholm modules in the Appendix.

1. Groupoids, $C^*$-algebras and dynamics

In this section we will recall some well known facts about the dynamics of sub shifts of finite type, Cuntz-Krieger algebras and the interplay in between them arising from a certain groupoid. The purpose of this section is to set notations and to introduce the underlying classical geometry before describing its noncommutative geometry.

1.1. Subshifts of finite type on the boundary of a tree. In this section we recall basic facts and introduce notation regarding subshifts of finite type. We let $A = (A_{i,j})_{i,j=1}^N$ denote an $N \times N$ matrix with coefficients being 0 or 1. Sometimes we write $A(i,j) = A_{i,j}$. The matrix $A$ can be thought of as defining the admissible paths in a Markov chain, where a jump from $i$ to $j$ is admissible if and only if $A(i,j) = 1$. We always assume that no row nor column of $A$ is zero to guarantee that there is always an allowed jump into as well as out of a letter $j \in \{1, \ldots, N\}$.

There are several well studied geometric objects associated with this Markov chain. The first is the compact space of infinite admissible words:

$$\Omega_A := \{(x_k)_{k \in \mathbb{N}} \in \{1, \ldots, N\}^\mathbb{N} : \forall k : A(x_k, x_{k+1}) = 1\},$$
equipped with the topology induced from the product topology on \(\{1, \ldots, N\}^\mathbb{N}\). The space \(\Theta\) is totally disconnected. There is a natural shift operator
\[
\Theta \to \Theta
\]
\[(x_k)_{k \in \mathbb{N}} \to (x_{k+1})_{k \in \mathbb{N}}.
\]
The pair \((\Theta, \sigma)\) is called a subshift of finite type and is amongst the most well studied systems in dynamics, see for example [39, 47, 52].

We call a sequence of numbers \(\mu = (\mu_j)_{j=1}^M\) with \(\mu_j \in \{1, \ldots, N\}\) a finite word of length \(M\). The length \(M\) of \(\mu\) is denoted by \(|\mu|\). A finite word \(\mu = (\mu_j)_{j=1}^M\) is said to be admissible for \(A\) if \(A(\mu_j, \mu_{j+1}) = 1\) for \(j = 1, \ldots, M - 1\). To simplify notation we often write \(\mu_1 \mu_2 \cdots \mu_M\) for a finite word \(\mu = (\mu_j)_{j=1}^M\). The empty word is defined to be an admissible finite word that we denote by \(\emptyset\). The length of the empty word is defined to be 0. The set of all admissible finite words will be denoted by \(\mathcal{Y}_A\). For \(k \in \mathbb{N}\) we use the notation
\[
\varphi(k) := \#\{\mu \in \mathcal{Y}_A : |\mu| = k\}.
\]
The space \(\Theta\) splits into cylinder sets \(\mathcal{C}_\mu\) associated with finite words \(\mu\):
\[
\mathcal{C}_\mu := \{(x_k)_{k \in \mathbb{N}} : x_1 \cdots x_{|\mu|} = \mu\} \quad \text{and} \quad \Theta = \biguplus_{\mu \in \mathcal{Y}_A} \mathcal{C}_\mu.
\]
A finite word \(\mu\) is admissible if and only if \(\mathcal{C}_\mu \neq \emptyset\). The sets \(\mathcal{C}_\mu\) are clopen subsets of \(\Theta\) and generate the topology. The shift \(\sigma\) is injective on each \(\mathcal{C}_\mu\) if \(|\mu| > 0\). We will abuse the notation by denoting the with \(\sigma\) associated endomorphism of the \(C^*\)-algebra \(C(\Theta)\) also by \(\sigma\).

Whenever \(X \subseteq \Theta\) is a clopen subset, the characteristic function \(\chi_X\) of \(X\) defines a locally constant continuous function. We conclude the following Proposition.

**Proposition 1.1.1.** The algebra of continuous functions \(C(\Theta)\) forms an AF-algebra. The AF-filtration is given by
\[
\mathcal{E}_k := \bigoplus_{|\mu| = k} \mathcal{C}_\mu \cong \mathcal{C}^2(k).
\]
The inclusions \(\mathcal{E}_k \hookrightarrow \mathcal{E}_{k+1}\) are induced from the partition \(\mathcal{C}_\mu = \bigcup_{j=1}^N \mathcal{C}_{\mu^j}\).

The space \(\Theta\) can be viewed as the boundary at infinity of the tree \(\mathcal{Y}_A\) of finite \(A\)-admissible words. The countable set \(\mathcal{Y}_A\) becomes a tree by allowing an edge between \(\mu\) and \(\nu\) whenever \(\nu = \mu \bar{i}\) for some \(i\). By choosing the empty word \(\emptyset\) as the base point, \(\mathcal{Y}_A\) becomes a rooted tree. The space \(\Theta\) is naturally identified with the space of infinite paths starting at \(\emptyset\). As such, \(\Theta\) carries a natural metric,
\[
d_{\Theta}(x, y) := e^{-\min\{|x_i - y_i| : i \in \mathbb{N}\}},
\]
that is, two paths are close when they stay on the same track for a long time. In this metric the cylinder sets satisfy
\[
\text{diam}(\mathcal{C}_\mu) = e^{-|\mu|}.
\]
The tree \(\mathcal{Y}_A\) is in particular a Gromov hyperbolic space, and \(\overline{\mathcal{Y}_A} := \mathcal{Y}_A \cup \Theta\) is a compactification of \(\mathcal{Y}_A\) when given the topology generated by that of \(\mathcal{Y}_A\) and the sets
\[
\mathcal{C}^\epsilon_\mu := \mathcal{C}_\mu \cup \{v \in \mathcal{Y}_A : v \in \mathcal{C}_\mu, d_{\Theta}(\emptyset, v) \geq \epsilon^{-1}\}.
\]
Here \(\mathcal{C}^\epsilon_\mu\) denotes the finite analogue of the cylinder set \(\mathcal{C}_\mu\):
\[
\mathcal{C}^\epsilon_\mu := \{v \in \mathcal{Y}_A : v = \mu \lambda, \text{ for some } \lambda \in \mathcal{Y}_A\} \subset \mathcal{Y}_A.
\]

**Definition 1.1.2.** A function \(t : \mathcal{Y}_A \to \Theta\) is said to satisfy the cylinder condition if
\[
t(\mu) \in \mathcal{C}_\mu \quad \forall \mu \in \mathcal{Y}_A.
\]
Proposition 1.1.3. If \( t : \mathcal{Y}_A \to \Omega_A \) satisfies the cylinder condition, see Definition 1.1.2, then pullback along \( t^*: \mathcal{C}(\Omega_A) \to \mathcal{C}(\mathcal{Y}_A) \) factors over a \( * \)-homomorphism

\[
t^* : \mathcal{C}(\Omega_A) \to \mathcal{C}(\mathcal{Y}_A).
\]

Proof. The Proposition follows once proving that \( t \) factors over a continuous mapping \( \tilde{t} : \mathcal{Y}_A \to \Omega_A \). We define \( \tilde{t} \) by \( \tilde{t}|_{\mathcal{Y}_A} := t \) and \( \tilde{t}|_{\Omega_A} := \text{id}_{\Omega_A} \). It is clear that \( \tilde{t}^{-1}(C_\mu) \subseteq C_\mu \) for some \( \epsilon > 0 \), so \( \tilde{t} \) is continuous.

The finite words come with a shift defined as a map

\[
\sigma_v : \mathcal{Y}_A \to \mathcal{Y}_A, \quad \mu = \mu_1\mu_2 \cdots \mu_N \mapsto \mu_2 \cdots \mu_N.
\]

The endomorphism \( \sigma : \mathcal{C}(\Omega_A) \to \mathcal{C}(\Omega_A) \) has an associated transfer operator

\[
(1.3) \quad L_{\sigma}(f)(x) := \sum_{y \in \sigma^{-1}(x)} f(y).
\]

This operator extends to an operator \( \tilde{L}_{\sigma} : \mathcal{C}(\mathcal{Y}_A) \to \mathcal{C}(\mathcal{Y}_A) \) by setting \( \tilde{L}_{\sigma}(f)(\mu) := \sum_{v \in \sigma^{-1}(\mu)} f(v) \) for \( \mu \in \mathcal{Y}_A \). Via the Riesz Representation Theorem, the induced operator \( \tilde{L}_{\sigma}^* : \mathcal{C}(\mathcal{Y}_A)^* \to \mathcal{C}(\mathcal{Y}_A)^* \) can be viewed as an operator on the Borel measures on \( \mathcal{Y}_A \).

Definition 1.1.4. A Borel measure \( \mu \) on \( \Omega_A \) is called \textit{conformal of dimension} \( \delta_A \) if \( \tilde{L}_{\sigma}^*(\mu) = e^{\delta_A} \mu \).

There is a canonical \( \sigma \)-conformal measure \( \mu \) on \( \Omega_A \), which can be constructed explicitly. Denote by \( \delta_A \) the \textit{upper Minkowski dimension} (sometimes called the \textit{upper box dimension}, see e.g. [30]) of \( \Omega_A \).

Theorem 1.1.5 (cf. [6], Theorem 2). Let \( s > 0 \). The series \( \sum_{v \in \mathcal{Y}_A} e^{-|v|} \) is convergent for all \( s > \delta_A \) and divergent for \( 0 < s \leq \delta_A \). Consequently \( \delta_A := \inf\{s : \sum_{v \in \mathcal{Y}_A} e^{-|v|} < \infty\} \).

A direct corollary of this Theorem goes as follows. Recall the definition of \( \varphi \) from (1.1).

Corollary 1.1.6. There is a positive sequence \( C_s \in \ell^1(\mathbb{N}) \) such that \( \varphi(k) \leq C_s(k)e^{sk} \) whenever \( s > \delta_A \).

Consider the measures

\[
\mu_s := \sum_{v \in \mathcal{Y}_A} e^{-|v|} \delta_v, \quad \sum_{v \in \mathcal{Y}_A} e^{-|v|} \delta_v,
\]

viewed as an element of \( \mathcal{C}(\mathcal{Y}_A)^* \). Subsequently define \( \mu_A = \text{w}^*\lim_{s \to \delta_A} \mu_s \), which is to be interpreted as a weak* limit in \( \mathcal{C}(\mathcal{Y}_A)^* \). Since the series of Theorem 1.1.5 diverges at \( \delta_A \), the measure \( \mu_A \) is supported only on the boundary \( \Omega_A \). For \( f \in \mathcal{C}(\Omega_A) \), \( \int_{\Omega_A} f d\mu_A \) can be computed by choosing an extension \( f \rightarrow \mathcal{Y}_A \), since any two such extensions differ by a function supported in \( \mathcal{Y}_A \).

Theorem 1.1.7 (cf. [19], Theorem 8.3). The measure \( \mu_A \) is \( \sigma \)-conformal of dimension \( \delta_A \).

Proof. First we compute

\[
\int_{\Omega_A} f dL_{\sigma}^* \mu_s = \frac{\sum_{v \in \mathcal{Y}_A} e^{-|v|} L_{\sigma} f(v)}{\sum_{v \in \mathcal{Y}_A} e^{-|v|}} = \frac{\sum_{v \in \mathcal{Y}_A} e^{-|v|} \sum_{\lambda \in \sigma^{-1}(v)} f(\lambda)}{\sum_{v \in \mathcal{Y}_A} e^{-|v|}} = \frac{\sum_{v \in \mathcal{Y}_A} e^{-|v| - 1} f(v)}{\sum_{v \in \mathcal{Y}_A} e^{-|v|}} = e^s \frac{\sum_{v \in \mathcal{Y}_A} e^{-|v|} f(v)}{\sum_{v \in \mathcal{Y}_A} e^{-|v|}}.
\]
and then, using that \( \sum_{v \in \gamma_i} e^{-|v|} \) diverges at \( s = \delta_A \), we take the limit

\[
\lim_{t \to 0} \int_{\Omega_k} f \ dL^*_c \mu_t = \lim_{t \to 0} e^{t} \frac{\sum_{v \in \gamma_i} e^{-|v|} f(v)}{\sum_{v \in \gamma_i} e^{-|v|}} = \lim_{t \to 0} e^{t} \frac{\sum_{v \in \gamma_i} e^{-|v|} f(v)}{\sum_{v \in \gamma_i} e^{-|v|}} = e^{\delta_A} \int_{\Omega_k} f \ d\mu.
\]

\[\Box\]

1.2. Groupoids, \( C^* \)-algebras and modules. Groupoids are an intermediate structure between spaces and groups. The \( C^* \)-algebras constructed from groupoids form a rich source of noncommutative \( C^* \)-algebras, and the groupoid origin provides a geometric description of those.

**Definition 1.2.1.** A groupoid is a small category \( \mathcal{G} \) in which all morphisms are invertible.

The requirement of being small is of a set-theoretical nature, and entails that morphisms in \( \mathcal{G} \) form a set. We denote the set of objects by \( \mathcal{G}^{(0)} \) and the set of morphisms by \( \mathcal{G}^{(1)} \). There is an inclusion \( \mathcal{G}^{(0)} \to \mathcal{G}^{(1)} \) as identity morphisms. Consequently we often write \( \mathcal{G} \) for \( \mathcal{G}^{(1)} \). The domain and range maps are denoted \( d, r: \mathcal{G}^{(1)} \to \mathcal{G}^{(0)} \) and the set of composable pairs is

\[
\mathcal{G}^{(2)} := \{(\xi, \eta) \in \mathcal{G} \times \mathcal{G} : d(\xi) = r(\eta)\}.
\]

This is itself a groupoid with domain and range maps the coordinate projections, and composition

\[
(\xi_1, \eta_1) \circ (\eta_1, \xi_2) := (\xi_1, \xi_2).
\]

If \( \mathcal{G} \) carries a locally compact Hausdorff topology for which the maps \( r, d \) and composition \( \mathcal{G}^{(2)} \to \mathcal{G} \) are continuous, then \( \mathcal{G} \) is said to be a locally compact Hausdorff groupoid.

**Definition 1.2.2.** A locally compact Hausdorff groupoid \( \mathcal{G} \) is étale if the fibers of the range map \( r : \mathcal{G} \to \mathcal{G}^{(0)} \) are discrete.

An étale groupoid \( \mathcal{G} \) carries a canonical *Haar system* (cf. [55]), consisting of counting measure in each fibre of \( r \). This allows for the definition of the *convolution product* on \( C_c(\mathcal{G}) \), defined by

\[
f * g(\eta) = \sum_{\xi \in r^{-1}(\eta)} f(\xi)g(\xi^{-1}\eta),
\]

which is a finite sum because \( f \) is compactly supported and \( r^{-1}(\eta) \) is discrete.

There is a locally compact Hausdorff étale groupoid \( \mathcal{G}_A \) encoding the dynamics of the totally disconnected compact space \( \Omega_A \) and the self mapping \( \sigma \). The unit space of \( \mathcal{G}_A \) is defined as \( \mathcal{G}_A^{(0)} := \Omega_A \) and the morphism space by

\[
\mathcal{G}_A^{(1)} := \{(x, n, y) \in \Omega_A \times \mathbb{Z} \times \Omega_A : \exists k \in \mathbb{N} \text{ s.t. } \sigma^{n+k}(x) = \sigma^n(y)\}.
\]

The range and source mappings are defined by

\[
r(x, n, y) = x \quad \text{respectively} \quad d(x, n, y) = y.
\]

The composition is given by

\[
(x, n, y)(y, m, z) = (x, m + n, z).
\]

It can be given a locally compact étale topology in the following way (cf. [55] [56]). Let \( U \) be an open set on which \( \sigma^m \) is injective, and \( V \) an open set on which \( \sigma^n \) is injective. The basic open sets for the topology are then given by

\[
(U, m, n, V) := \{(x, m - n, y) : \sigma^m(x) = \sigma^n(y)\}.
\]

Since this groupoid is étale it admits a natural Haar system \( \nu^+ \) given by counting measure in the fibers.
Recall that a measure \( \mu \) on \( \mathcal{G}^{(0)} \) is called quasi-invariant if the induced measure \( d\mu(\xi) = dy^*(\xi)d\mu(x) \) is equivalent to its inverse \( d\mu(\xi^{-1}) \). The Radon-Nikodym derivative \( \Delta := \frac{d\mu^{-1}}{d\mu} \) is a measurable 1-cocycle on \( \mathcal{G} \) called the modular function. If \( \mathcal{G} \) is an étale groupoid and \( U \subset \mathcal{G} \) an open set on which both \( r \) and \( d \) are injective, define \( T : r(U) \to d(U) \) by \( x \mapsto d(r^{-1}(x) \cap U) \). A measure \( \mu \) on \( \mathcal{G}^{(0)} \) is quasi-invariant with modular function \( \Delta \) if for every such \( U \) we have
\[
\frac{dT^*\mu}{d\mu}(x) = \Delta(r^{-1}(x) \cap U),
\]

\( \text{cf.} \ [55] \) Remark 3.22.

**Proposition 1.2.3.** The measure \( \mu_\Lambda \) is a quasi-invariant measure on \( \Omega_\Lambda \) with modular function \( \Delta(x,n,y) = e^{-\delta_{x,n}} \).

**Proof.** The maps \( r \) and \( d \) are injective on the basic open sets \( (U,m,n,V) \). For \( n > m \geq 0 \) we have
\[
T^*f(x) = \sum_{y \in r^{-1}(x)} f(y) = L_n^*f(x).
\]
We conclude that
\[
\int_U f dT^*\mu_\Lambda = \int_U f dL_n^*\mu_\Lambda = e^{(n-m)\delta} \int_U f d\mu_\Lambda.
\]
For \( n-m < 0 \)
\[
\int_U f d\mu_\Lambda = \int_U f dT^{-1}\mu_\Lambda = \int_U f dL_n^*\mu_\Lambda = e^{(m-n)\delta} \int_U f d\mu_\Lambda,
\]
so in this case \( \int_U f dT^*\mu_\Lambda = e^{(n-m)\delta} \int_U f d\mu_\Lambda \) as well. \( \square \)

The reduced \( C^* \)-algebra of an étale groupoid \( \mathcal{G} \) is a certain \( C^* \)-algebra completion of the algebra that \( C_c(\mathcal{G}) \) forms under the convolution product \( \langle \cdot, \cdot \rangle \). There is a conditional expectation \( \rho : C_c(\mathcal{G}) \to C_0(\mathcal{G}^{(0)}) \) given by restriction of functions to \( \mathcal{G}^{(0)} \). To construct \( C^*_r(\mathcal{G}) \), define the \( C_0(\mathcal{G}^{(0)}) \)-valued inner product
\[
\langle f, g \rangle(x) := \sum_{\xi \in r^{-1}(x)} \overline{f(\xi^{-1})g(\xi^{-1})} = \rho(f^* \cdot g),
\]
which is \( C_0(\mathcal{G}^{(0)}) \)-linear for multiplication from the right. The completion of \( C_c(\mathcal{G}) \) in the norm induced from \( \langle \cdot, \cdot \rangle \) is a \( C^* \)-module \( \mathcal{E}^\mathcal{G} \), the Haar module, on which \( C_c(\mathcal{G}) \) acts, via convolution, by adjointable operators. Its completion in the operator norm is \( C_r^*(\mathcal{G}) \). The map \( \rho \) above extends to a conditional expectation
\[
\rho : C_r^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)}).
\]
This intrinsic construction of \( C_r^*(\mathcal{G}) \) was first considered in \([11]\).

For a closed subgroupoid \( \mathcal{H} \subset \mathcal{G} \) we can do a similar construction. Denote by \( \rho_{\mathcal{H}} : C_c(\mathcal{G}) \to C_c(\mathcal{H}) \) the restriction map. This extends to a conditional expectation \( \rho_{\mathcal{H}} : C_r^*(\mathcal{G}) \to C_r^*(\mathcal{H}) \), \( \text{cf.} \ [55] \). Relative to the closed subgroupoid \( \mathcal{G}^{(0)} \subset \mathcal{G} \), the inner product \( \langle f, g \rangle \) can be expressed as \( \langle f, g \rangle = \rho(f^* \cdot g) \). For a closed subgroupoid \( \mathcal{H} \subset \mathcal{G} \) there is a right \( C_c(\mathcal{H}) \)-module structure on \( C_c(\mathcal{G}) \) given by
\[
g \cdot h(\eta) := \sum_{\xi \in r^{-1}(d(\eta)) \cap \mathcal{H}} g(\eta \xi) h(\xi^{-1}) = \rho_{\mathcal{H}}(f^* \cdot g),
\]
and the formula for the inner product is similar to \( \langle \cdot, \cdot \rangle \):
\[
\langle f, g \rangle(\eta) := \sum_{\xi \in r^{-1}(r(\eta))} \overline{f(\xi^{-1} \eta)} g(\xi^{-1}) \quad \text{for} \quad \eta \in \mathcal{H}.
\]
The completion of $C_c(\mathcal{G})$ with respect to this inner product is a $C^*$-module $E^g_{\mathcal{H}}$ over $C^*_r(\mathcal{H})$, carrying a left action of $C^*_r(\mathcal{G})$, defined by convolution. The algebra of compact operators on such modules can be easily described.

Consider the right action of $\mathcal{H}$ on $\mathcal{G}$ and its associated quotient space

$$\mathcal{G}/\mathcal{H} = \{[\xi] : \xi \in \mathcal{G}, [\xi_1] = [\xi_2] \iff \exists \eta \in \mathcal{H} \xi_1 \eta = \xi_2\}.$$  

The space $\mathcal{G} \ltimes \mathcal{G}/\mathcal{H} := \{(\xi, [\eta]) : d(\xi) = r(\eta)\}$, comes with a map

$$\mathcal{G} \ltimes \mathcal{G}/\mathcal{H} \rightarrow \mathcal{G}/\mathcal{H} \quad (\xi, [\eta]) \mapsto [\xi \eta].$$

It can be made into a groupoid by defining domain and range maps

$$r(\xi, [\eta]) := [\xi \eta], \quad d(\xi, [\eta]) = [\eta],$$

composition

$$(\xi_1, [\eta]) \circ (\xi_2, [\xi_2^{-1} \eta]) := (\xi_1 \xi_2, [\xi_2^{-1} \eta]),$$

and inversion

$$(\xi, [\eta])^{-1} := ([\eta]^{-1}, [\xi \eta]).$$

This groupoid is étale because both $\mathcal{G}$ and $\mathcal{H}$ are. The above construction is a special case of an action of the groupoid $\mathcal{G}$ on a space, which in this case is $\mathcal{G}/\mathcal{H}$. In that context, the map $[\eta] \mapsto r(\eta)$, viewed as a map $\mathcal{G}/\mathcal{H} \rightarrow \mathcal{G}^{(0)}$ is called the moment map of the action. For the general theory of groupoid actions, its relation to $C^*$-algebras and modules, and further references see [45, 48, 62].

**Theorem 1.2.4** (cf. [51, 60]). Let $\mathcal{G}$ be an étale groupoid and $\mathcal{H} \subset \mathcal{G}$ a closed subgroupoid. The mapping

$$\pi^g_{\mathcal{H}} : C^*_r(\mathcal{G} \ltimes \mathcal{G}/\mathcal{H}) \rightarrow \mathcal{K}_{C^*_r(\mathcal{H})}(E^g_{\mathcal{H}})$$

defined on $a \in C_c(\mathcal{G} \ltimes \mathcal{G}/\mathcal{H}) \subseteq C^*_r(\mathcal{G} \ltimes \mathcal{G}/\mathcal{H})$ and $f \in C_c(\mathcal{G}) \subseteq E^g_{\mathcal{H}}$ by

$$\pi^g_{\mathcal{H}}(a)f(\eta) := \sum_{\xi \in r^{-1}(r(\eta))} a(\xi, [\xi^{-1} \eta])f(\xi^{-1} \eta) \quad \text{for} \quad \eta \in \mathcal{G},$$

is an isomorphism.

The fact that $C^*_r(\mathcal{G} \ltimes \mathcal{G}/\mathcal{H}) \cong \mathcal{K}_{C^*_r(\mathcal{H})}(E^g_{\mathcal{H}})$ follows from the Morita equivalence $\mathcal{H} \sim \mathcal{G} \ltimes \mathcal{G}/\mathcal{H}$ of groupoids and the results in [51, 60]. The explicit formula for the isomorphism can be found in [28, Equation (11)].

1.3. **Cuntz-Krieger algebras.** Let $O_A$ be the Cuntz-Krieger algebra associated with the $N \times N$ matrix $A = (A_{ij})$. Recall our assumption on $A$; no row nor column in $A$ is 0. The $C^*$-algebra $O_A$ was defined in [24] as the universal $C^*$-algebra generated by elements $S_i$ satisfying the relations

$$S_i^* S_i = \sum_{j=1}^N A_{ij} S_j S_i^*, \quad i = 1, \ldots, N$$

$$\sum_{i=1}^N S_i S_i^* = 1,$$

$$S_i S_j^* S_k S_i^* = S_i S_j^* S_k S_i^* = \delta_{ij} S_k S_i^*.$$

This completes the description of the Cuntz-Krieger algebra $O_A$. The relations (1.9) and (1.10) imply

$$S_i^* S_i = \sum_{j=1}^N A_{ij} S_j S_i^*.$$
Following the notation \[24\], for the source projections we write \(Q_i := S_i^*S_i\) and for the range projections \(P_i := S_iS_i^*\). The relations \([1.9]-[1.11]\) become
\[
P_iP_j = \delta_{ij}P_i \quad \text{and} \quad Q_i = \sum_{j=1}^{N} A_{ij}P_j.
\]

For any finite word \(\mu = \mu_1\mu_2 \cdots \mu_k\) we let \(S_\mu \in O_A\) denote the elements \(S_{\mu_1}S_{\mu_2} \cdots S_{\mu_k}\). The relation \([1.12]\) guarantees that the element \(S_\mu\) is non-zero if and only if \(\mu\) is an admissible word.

**Proposition 1.3.1** (Lemma 1.1 of [13]). The following computation holds:
\[
S_i^*S_j = \begin{cases}
S_{\beta}, & \text{if } \gamma = \nu\beta \\
Q_{\alpha}, & \text{if } \nu = \gamma = (i_1, \ldots, i_k) \\
S_{\gamma}^*, & \text{if } \nu = \beta\gamma \\
0, & \text{otherwise}
\end{cases}
\]

Every non-zero word in \(S_i\) and \(S_j^*\) can be written as a finite sum of terms of the form \(S_\mu S_i^*\) where the admissible \(\mu = (\mu_1, \ldots, \mu_k)\) and \(\nu = (\nu_1, \ldots, \nu_l)\) satisfy that \(\mu_k = \nu_l\).

The following fundamental result is due to Renault.

**Theorem 1.3.2** ([23] [56]). There is a canonical isomorphism between the groupoid \(C^*-\)algebra \(C_r^*(\mathcal{G}_\alpha)\) and the universal \(C^*-\)algebra \(O_A\).

The isomorphism is implemented by mapping \(S_i\) to the characteristic function of the set
\[
X_i := \{(x, 1, \sigma(x)) : x \in \Omega_A\}.
\]

Then the \(S_i\) satisfy the Cuntz-Krieger relations, defining the homomorphism \(O_A \to C_r^*(\mathcal{G}_\alpha)\). For more details on the proof see [56].

Recall the following condition, usually referred to as condition \((I)\), on the \(N \times N\)-matrix \(A\). A finite admissible word \(\nu = \nu_1 \cdots \nu_R\) is a loop based in \(j \in \{1, \ldots, N\}\) if \(\nu_1 = \nu_R = j\) and \(\nu_k \neq j\) for \(k = 2, \ldots, R-1\). If any \(j = 1, \ldots, N\) satisfies that there is an admissible finite word \(\mu = \mu_1 \cdots \mu_M\) with \(\mu_1 = j\) and there are two different loops based in \(\mu_M\), we say that \(A\) satisfies condition \((I)\). The matrix \(A\) satisfies condition \((I)\) if and only if \(\Omega_A\) has no isolated points. Examples when condition \((I)\) is satisfied is if \(A\) is irreducible but not a permutation matrix.

**Theorem 1.3.3** (Theorem 2.13–2.14 of [24], Proposition 4.3 of [2]). The Cuntz-Krieger algebra \(O_A\) satisfies the following:

1. If \(A\) is irreducible, \(O_A\) is simple.
2. If \(A\) satisfies \((I)\), then \(O_A\) is purely infinite.
3. If \(A\) satisfies \((I)\), then \(O_A\) is unique up to isomorphism.\(^3\)

The quasi-invariant measure \(\mu_A\) induces a functional
\[
\varphi_A : C_r^*(\mathcal{G}_\alpha) \to \mathbb{C}, \quad f \mapsto \int_{\Omega_A} f \, d\mu_A,
\]
which extends to a state on \(C_r^*(\mathcal{G})\). The GNS-representation of \(O_A\) on \(L^2(O_A, \varphi_A)\) is canonically isomorphic to the convolution representation of \(C_r^*(\mathcal{G}_\alpha)\) on \(L^2(\mathcal{G}_\alpha, \mu_A)\). We will refer to this as the fundamental representation.

\(^3\)In the sense that whenever \(\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_K\) are operators satisfying the Cuntz-Krieger relations \([19]-[21]\), the mapping \(S_i \mapsto \hat{S}_i\) extends to an isomorphism of \(O_A\) onto the \(C^*-\)algebra generated by \(\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_K\).
1.3.1. The algebra $O_N$. Also known as the Cuntz algebra, was first introduced in [20]. The algebra $O_N$ is the universal C*-algebra generated by $N$ orthogonal isometries. The algebra $O_N$ is the Cuntz-Krieger algebra associated with the symmetric $N \times N$-matrix giving by $A_{ij} = 1$ for all $i, j$. The geometry of $O_N$ takes a very simple form, since any word is admissible it holds that $\nu_{O_N} = \cup_{k \in \mathbb{N}}\{1, \ldots, N\}^k$ and $\nu(k) = N^k$. In this special case, the KMS-state $\varphi_{O_N}$ can be computed as

$$\varphi_{O_N}(S^*_{\nu}S^*_\mu) = \delta_{\nu,\mu}N^{-|\nu|}.$$  

1.4. The fixed point algebra of the circle action. The Cuntz-Krieger groupoid comes with a natural circle action. We describe the action in both pictures of $O_A$. First of all the map

$$c_A : \mathcal{G}_A \to \mathbb{Z}$$

$$(x, n, y) \mapsto n,$$

is a continuous homomorphism, or 1-cocycle. Note that $\ln \Delta = -\delta_{A|c_A}$, with $\Delta$ as in Proposition [1.2.3]. This induces a disjoint union decomposition

$$\mathcal{G} = \bigcup_{n \in \mathbb{Z}} \mathcal{G}_n,$$

where $\mathcal{G}_n = c_A^{-1}(n)$. Its kernel

$$\mathcal{H}_A := \ker c_A = c_A^{-1}(0) = \{(x, 0, y) : \exists k, \sigma^k(x) = \sigma^k(y)\},$$

is a closed subgroupoid. We denote $F_A := C^*_c(\mathcal{H}_A) \subset C^*_c(\mathcal{G}_n)$. We remark that by [24, Remark on the end of page 3], the algebra $F_A$ is simple if $A$ is aperiodic. There is a $U(1)$-action on $C^*_c(\mathcal{G}_n)$ (cf. Definition 3.15 and Proposition 5.4 of [55]) constructed from the cocycle $c$ via

$$\alpha_t(f)(\xi) := e^{itc(\xi)}f(\xi).$$

We refer to this action as the gauge action. The fixed point algebra for this action is exactly $F_A$. The following result is well known.

**Proposition 1.4.1** (cf. Definition 3.15 and Proposition 5.4 of [55]). The state $\varphi_A$ (1.14) satisfies the KMS-condition at inverse temperature $\beta_A$ with respect to the gauge action.

A third way of describing $F_A$ comes from the generators $S_\alpha$. Observe that in terms of the linearly spanning elements $S_\mu S^*_\nu$ coming from Proposition 1.3.1

$$\alpha_t(S_\mu S^*_\nu) = e^{(\mu | -\nu)t}S_\mu S^*_\nu.$$  

Hence $F_A$ is the C*-algebra generated by $S_\mu S^*_\nu$ for $|\mu| = |\nu|$. We define $F^l_A$ to be the span of all non-zero $S_\mu S^*_\nu$ where $|\mu| = |\nu| = l + 1$. As was computed in the proof of [24, Proposition 2.3]; for a fixed $j$, the elements $S_\mu S^*_\nu$ where $|\mu| = |\nu| = l + 1$ and $\mu_{i+1} = \nu_{i+1} = j$ form a set of matrix units; whenever $l + 1 = |\mu| = |\nu| = |\mu'| = |\nu'|$,

$$S_\mu S^*_\nu S_\mu S^*_\nu = \delta_{\nu,\mu}S_\mu S^*_\nu S_{\nu} S^*_\nu = \delta_{\nu,\mu}S_\mu S^*_\nu.$$  

These identities follows from Proposition 1.3.1. We can conclude the following Proposition.

**Proposition 1.4.2** (Proposition 2.3 of [24]). The space $F^l_A$ is closed under multiplication and adjoint. In particular,

$$F_A = \cup_{l \in \mathbb{N}} F^l_A$$

is an AF-algebra.
The stabilization $F_A \otimes K$ admits yet another description in terms of groupoids. It follows from [18 Lemma 3.4] that $\mathcal{G}_A/\mathcal{H}_A \cong \Omega_1 \times \mathbb{Z}$. The moment mapping $\mathcal{G}_A/\mathcal{H}_A \to \Omega_1$ is projection onto the first coordinate under the above homeomorphism. Hence we can identify $\mathcal{G}_A \ltimes \mathcal{G}_A/\mathcal{H}_A = \mathcal{G}_A \times \mathbb{Z}$. We will denote elements of $\mathcal{G}_A \ltimes \mathcal{G}_A/\mathcal{H}_A$ by $(x, k, y, l)$. The range and domain mappings are given by 

$$r, d : \mathcal{G}_A \ltimes \mathcal{G}_A/\mathcal{H}_A = \mathcal{G}_A \times \mathbb{Z} \to \mathcal{G}_A/\mathcal{H}_A = \Omega_1 \times \mathbb{Z}$$

are given by 

$$r(x, k, y, l) = (x, l) \quad \text{and} \quad d(x, k, y, l) = (x, k + l)$$

The groupoid multiplication in $\mathcal{G}_A \ltimes \mathcal{G}_A/\mathcal{H}_A$ is given by 

$$(x, k, y, l)(y, m, z, k + l) = (x, k + m, z, l)$$

The next Proposition follows by a standard argument, left to the reader.

**Proposition 1.4.3.** The mapping 

$$\beta : \mathcal{G}_A \ltimes \mathcal{G}_A/\mathcal{H}_A \to \mathcal{G}_A \ltimes \mathcal{G}_A/\mathcal{H}_A, \quad (x, k, y, l) \mapsto (x, k, y, l - 1),$$

is a groupoid automorphism. There is an isomorphism $C^*(\mathcal{G}_A \ltimes \mathcal{G}_A/\mathcal{H}_A) \cong O_1 \times U(1)$ under which $\beta$ corresponds to the dual $\mathbb{Z}$-action. In particular, 

$$C^*(\mathcal{G}_A \ltimes \mathcal{G}_A/\mathcal{H}_A) \times_\beta \mathbb{Z} \cong O_1 \times U(1).$$

**Remark 1.4.4.** A consequence of Theorem [12, 34] and Proposition [1.4.3] is that there is a $\mathbb{Z}$-action on $F_A \otimes K$ such that $O_1 \otimes K \cong (F_A \otimes K) \ltimes \mathbb{Z}$.

The restriction map $\rho : C_1(\mathcal{G}_A) \to C_1(\mathcal{H}_A)$ is a conditional expectation. The associated $C^*$-module is denoted $E^\tau$. Under the isomorphism $O_1 \cong C_1(\mathcal{G}_A)$, $\rho$ corresponds to the map $\tau : O_1 \to F_A$ defined by 

$$\tau(a) := \frac{1}{2\pi} \int_{U(1)} a_t(\omega) dt.$$ 

The mapping $\tau$ defines an $F_A$ valued inner product on $O_1$. The completion $E^\tau$ of $O_1$ in the norm associated to this inner product is a $\mathbb{Z}$-graded $C^*$-module over $F_A$.

**Proposition 1.4.5.** The isomorphism $C_1(\mathcal{G}_A) \cong O_1$ is $U(1)$-equivariant and induces a $\mathbb{Z}$-graded isomorphism $E^\tau \cong E^\omega$.

1.4.1. **The quantum group $SU_q(2)$**. Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The partial isometries $S_1$ and $S_2$ generating $O_q$ satisfy the relations 

$$S_1^*S_2 = 0, \quad S_2S_1^* = S_1^*S_2, \quad S_1S_1^* + S_2S_2^* = 1 \quad \text{and} \quad S_2S_1 = 1.$$ 

This condition guarantees that $O_q \cong C(SU_q(2))$ for any $q \in [0, 1)$, see more in [35]. The compact quantum group $SU_q(2)$ is well studied and we merely describe it here as an interesting example, instead of deriving anything new. Any admissible sequence $\mu \in \mathcal{G}_A$ has the form 

$$\mu = \{1, 11, 12, \ldots, 122, \ldots, 2\},$$

that is, if the letter 2 starts appearing in a word, it will continue with 2:s. It does in particular follow that 

$$|l| = \#\{\mu \in \mathcal{G}_A| |\mu| = l\} = l + 1.$$ 

**Proposition 1.4.6.** There is an isomorphism $C(SU_q(2))^{U(1)} \cong \mathbb{K}$ – the unitalization of the compact operators on a separable infinite dimensional Hilbert space.
Proof. We use the notation \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( F_A \) for the fixed point algebra in \( C(SU_q(2)) \). An admissible sequences of length \( l \) is determined by how many 1's appear in the beginning. In particular, it holds that

\[
F_A^l \cong \mathbb{C} \oplus M_l(\mathbb{C}).
\]

The first summand is spanned by \( S_{11-1} S_{11-1}^* \) and the second summand spanned by \( S_\mu S_\nu^* \) where \( \mu \) and \( \nu \) are of length \( l \) and not ending in 1. Since

\[
S_{11-1} S_{11-1}^* = S_{11-1} + S_{11-1}^* + S_{11-1} S_{11-1}^*.
\]

It also holds that

\[
S_{11-22} S_{11-22}^* = S_{11-22} + S_{11-22}^* + S_{11-22} S_{11-22}^*.
\]

so the embedding of the second factors \( M_l(\mathbb{C}) \to M_{l+1}(\mathbb{C}) \) is a corner embedding. Hence the mappings \( \mathbb{C} \oplus M_l(\mathbb{C}) \to \mathbb{C} \oplus M_{l+1}(\mathbb{C}) \) are unital. It follows that \( \lim_l (\mathbb{C} \oplus M_l(\mathbb{C})) \cong \mathbb{K}. \)

2. Finite summability of Fredholm modules

We will in this section investigate the finite summability of odd \( K \)-homology classes on Cuntz-Krieger algebras. The central idea when treating the \( K \)-homology of Cuntz-Krieger algebras is the usage of Kaminker-Putnam's Poincaré duality class for Cuntz-Krieger algebras. After recalling its construction we will prove the following Theorem:

**Theorem 2.0.7.** Any class in \( K^1(\mathcal{O}_A) \) admits a \( p \)-summable representative for any \( p > 0 \).

To be precise, we prove that any class in \( K^1(\mathcal{O}_A) \) can be represented by a \( K \)-cycle that is finite rank summable on the \( * \)-algebra generated by the generators of \( \mathcal{O}_A \). We return to the proof of this Theorem in the end of Subsection 2.2.

2.1. Kaminker-Putnam’s Poincaré duality class. Whenever \( \mu \in \bigcup_{k \in \mathbb{N}} \{ 1, \ldots, N \}^k \), we let \( \delta_\mu \in \ell^2(\mathcal{F}_A) \) denote the delta function in \( \mu \) if \( \mu = \gamma_A \) and \( \delta_\mu = 0 \) if \( \mu \notin \gamma_A \). We obtain an ON-basis \( \{ \delta_\mu | \mu \in \gamma_A \} \) for \( \ell^2(\mathcal{F}_A) \). We use the notation \( e_1, \ldots, e_N \) for the standard ON-basis of \( \mathbb{C}^N \). If \( \mu = (\mu_1, \ldots, \mu_k) \in \{ 1, \ldots, N \}^k \) we use the notation \( e_\mu := e_{\mu_1} \otimes \cdots \otimes e_{\mu_k} \in (\mathbb{C}^N)^{\otimes k} \). Let \( \mathcal{F} \) denote the Hilbert space completion of \( \bigoplus_{k=0}^{\infty} (\mathbb{C}^N)^{\otimes k} \), with \( (\mathbb{C}^N)^{\otimes 0} = \mathbb{C} \), in the scalar product

\[
\langle e_\mu, e_\nu \rangle_\mathcal{F} = \delta_{\mu, \nu}.
\]

There is a natural isometric embedding \( \ell^2(\mathcal{F}_A) \to \mathcal{F} \) whose range is the closed linear span of the set \( \{ e_\mu | \mu \in \gamma_A \} \), and we often identify \( \ell^2(\mathcal{F}_A) \) with its image under this embedding, that is, we identify \( e_\mu \) with \( \delta_\mu \) if \( \mu \in \gamma_A \). We also let \( P_\delta : \mathcal{F} \to \ell^2(\mathcal{F}_A) \) denote the orthogonal projection, in particular \( P_\delta e_\mu = \delta_\mu \) for any finite word \( \mu \). Define the bounded operators

\[
L_\delta^A : \ell^2(\mathcal{F}_A) \to \ell^2(\mathcal{F}_A), \quad \delta_\mu \mapsto e_\mu \mapsto P_\delta(e_\mu) = \delta_\mu.
\]

There is a bijection of sets \( \gamma_A \to \gamma_{A'} \) given by

\[
\mu = \mu_1 \mu_2 \cdots \mu_{k-1} \mu_k \mapsto \bar{\mu} := \mu_k \mu_{k-1} \cdots \mu_2 \mu_1,
\]

i.e. the word \( \mu \) ordered in the opposite way. We define the unitary isomorphism

\[
J_\gamma : \ell^2(\mathcal{F}_A) \to \ell^2(\mathcal{F}_A'), \quad \delta_\mu \mapsto \delta_{\bar{\mu}}.
\]

Consider the operators \( R_\delta := J_\gamma L_\delta^A J_\gamma \), which act as \( R_\delta \delta_\mu = \delta_{\bar{\mu}} \).

We let \( \{ S_i | i = 1, \ldots, N \} \) and \( \{ T_i | i = 1, \ldots, N \} \) denote the generators of \( \mathcal{O}_A \) respectively \( \mathcal{O}_{A'} \). We define the \( * \)-homomorphisms

\[
\beta_A := \mathcal{O}_A \to \mathcal{K}(\ell^2(\mathcal{F}_A)), \quad S_i \mapsto L_\delta^A \mod \mathbb{K}(\ell^2(\mathcal{F}_A))
\]

and
Proposition 2.1.2. (Consequence of [36])

The data

\[ \mathcal{E}(\ell^2(\gamma_A)) \to \mathcal{E}(\ell^2(\gamma_{Ab})), \quad \mathcal{H}_i \to R^i \mod \mathbb{K}(\ell^2(\gamma_A)). \]

Here \( q : \mathbb{B}(\ell^2(\gamma_A), \ell^2(\gamma_{Ab})) \to \mathbb{B}(\ell^2(\gamma_A), \ell^2(\gamma_{Ab})) / \mathbb{K}(\ell^2(\gamma_A), \ell^2(\gamma_{Ab})) \) denotes the quotient mapping. The fact that \( \beta_A^T \) is a \( \ast \)-homomorphism for any \( A \) is shown in [36, Proposition 2.1.2]. It also follows from Lemma 4.2.1 and Proposition 4.2.2 below. A short computation shows that

\[ [L^A_i, R^A_i] = 0 \quad \text{and} \quad [(L^A_i)^*, R^A_i] = \delta_{i,j} P_j, \]

where \( P_j \) denotes the orthogonal projection onto \( C\delta_{i,j} \). See more in [36, Proposition 4.2]. It follows that the algebra \( \beta_A(O_{AB}) \) commutes with \( \beta_A^T(O_{AB}) \) in \( \mathcal{E}(\ell^2(\gamma_A)) \). Since \( O_A \) and \( O_A^\prime \) are nuclear we obtain a \( \ast \)-homomorphism

\[ \beta_{K^P} := \beta_A \otimes \beta_A^T : O_A \otimes O_{A^\prime} \to \mathcal{E}(\ell^2(\gamma_A)). \]

By standard constructions, see [40, Chapter 3.3] and its proof relies on the Stinespring Theorem which guarantees a completely positive splitting of the short exact sequence (2.19) that has the following form. There is a Hilbert space \( \mathcal{H} \), a representation \( \pi : O_A \otimes O_{A^\prime} \to \mathbb{B}(\mathcal{H}) \) and an isometry \( W : \ell^2(\gamma_A) \to \mathcal{H} \) such that

\[ \beta_{K^P}(a) = q(W^\ast \pi(a)W) \quad \text{for any} \quad a \in O_A \otimes O_{A^\prime}. \]

(2) The image of \( [\beta_{K^P}] \) under \( \text{Ext}(O_A \otimes O_{A^\prime}, \mathbb{K}(\ell^2(\gamma_A))) \to K^1(O_A \otimes O_{A^\prime}) \) is represented by the odd analytic \( K \)-cycle \((\pi, \mathcal{H}, 2W W^\ast - 1)\).

The data \( \pi, \mathcal{H} \) and \( W \) is difficult to construct in general. Further, the problem of finite summability on a dense sub algebra is not made easier by the abstract construction from the Stinespring Theorem. We will return to this problem in the next section. First we recall the construction of Poincaré duality from the image \( \Delta \in K^1(O_A \otimes O_{A^\prime}) \) of the extension class \( [\beta_{K^P}] \).

Theorem 2.1.1 (Consequence of [36]). The mapping

\[ K_\ast(O_{A^\prime}) \to K^{*+1}(O_A), \quad [e] \mapsto (1_{O_A} \otimes [e]) \otimes_{O_A \otimes O_{A^\prime}} \Delta \]

is an isomorphism.

In order to use Theorem 2.1.1 we will need to compute Kasparov products in this special case. Computation of this type are well known to experts in the field, we include them for the sake of completeness. Throughout this subsection, we let \( A \) and \( B \) denote unital \( \ast \)-algebras and \((\pi, \mathcal{H}, F)\) an odd analytic \( K \)-cycle for \( A \otimes B \).

Proposition 2.1.2. Let \( e \in B \otimes M_N(\mathbb{C}) \) be a projection and set

\[ \mathcal{H}_e := \pi(1 \otimes e)(\mathcal{H} \otimes \mathbb{C}^N). \]

There is an odd analytic \( K \)-cycle \((\pi_e, \mathcal{H}_e, F_e)\) on \( A \) defined by

\[ \pi_e : A \to \mathbb{B}(\mathcal{H}_e), \quad a \mapsto \pi(a \otimes e), \quad \text{and} \quad F_e := \pi(1 \otimes e)F\pi(1 \otimes e). \]
Proof. Since $F$ commutes with $\pi(1 \otimes e)$ up to compacts,
\[ F^2 - \pi(1 \otimes e)F^2\pi(1 \otimes e) \in \mathbb{K}(\mathcal{H}_e). \]
And since $F^2 - 1$ is compact, so is $F^2 - 1$. Furthermore $F^* = \pi(1 \otimes e)F^*\pi(1 \otimes e)$ so $F^* - F \in \mathbb{K}(\mathcal{H}_e)$. Finally we have for any $a \in A$ that
\[ [F_e, \pi_e(a)] = [(1 \otimes e)F(1 \otimes e), \pi_e(a \otimes e)] = \pi(1 \otimes e)[F, \pi_e(a \otimes 1)](1 \otimes e) \in \mathbb{K}(\mathcal{H}_e). \]

\[ \square \]

Lemma 2.1.3. If $e \in \mathcal{B} \otimes M_n(\mathbb{C})$ is a projection, the Kasparov product $(1_A \otimes [e]) \otimes_{\mathcal{A} \mathcal{B} \mathcal{B}} [\pi, \mathcal{H}, F]$ can be represented by the Fredholm module $(\pi_e, \mathcal{H}_e, F_e)$.

Proof. The $K$-theory class $1_A \otimes [e]$ can be represented by the $A-\mathcal{A} \mathcal{B} \mathcal{B}$ Kasparov module $(\mathcal{A} \otimes e \mathcal{B}^N, 0)$ with its obvious $A$-action on the left and the structure of an $\mathcal{A} \otimes \mathcal{B}$-Hilbert module comes from the inclusion $A \otimes e \mathcal{B}^N \subseteq A \otimes \mathcal{B}^N$. It is clear that as $A-\mathcal{C}$-Hilbert bi-modules
\[ \mathcal{H}_e = (A \otimes e \mathcal{B}^N) \otimes_{\mathcal{A} \mathcal{B} \mathcal{B}} \mathcal{H}. \]

Since $(\pi_e, \mathcal{H}_e, F_e)$ is a Fredholm module on the right Hilbert space, to verify that it is a Kasparov product between $[\pi, \mathcal{H}, F]$ and $(\mathcal{A} \otimes e \mathcal{B}^N, 0)$ it suffices to prove that $F_e$ is an $F$-connection, see [40, Definition 2.2.4]. The other conditions on a Kasparov product is automatic as the Kasparov operator in $(\mathcal{A} \otimes e \mathcal{B}^N, 0)$ is 0, see [40, Definition 2.2.7]. Recall that $F_e$ is an $F$-connection if for $x \in A \otimes e \mathcal{B}^N$, the linear mapping
\[ \xi \mapsto x \otimes_{\mathcal{A} \mathcal{B} \mathcal{B}} (F\xi) - F_e(x \otimes_{\mathcal{A} \mathcal{B} \mathcal{B}} \xi) \]
is compact. However, since $(1 \otimes e)x = x$ this fact follows from the identity
\[ x \otimes_{\mathcal{A} \mathcal{B} \mathcal{B}} (F\xi) - F_e(x \otimes_{\mathcal{A} \mathcal{B} \mathcal{B}} \xi) = \pi(x)F\xi - \pi(1 \otimes e)F\pi(1 \otimes e)\pi(x)\xi = \pi(1 \otimes e)[\pi(x), F]\xi. \]

\[ \square \]

Remark 2.1.4. The mapping $K^1(A) \to \text{Ext}(A, \mathbb{K})$ is defined by mapping a cycle $x := (\pi, \mathcal{H}, F)$ to the extension associated with the Busby invariant
\[ \beta_F : A \to \mathcal{C}(\mathcal{H}), \quad \beta_F(a) := q(P_F\pi(x)P_F) \quad \text{where} \quad P_F := (F + 1)/2 \]
and $q : \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ denotes the quotient mapping. If $F^2 = 1$, the Hilbert space in fact be reduced to $P_F\mathcal{H} = \ker(F - 1)$. The Busby invariant is degenerately equivalent to $\beta_F : A \to \mathcal{C}(P_F\mathcal{H})$, $\beta_F(a) := q(P_F\pi(x)P_F)$. We especially note that the Busby invariant of the $K$-cycle $(\pi_e, \mathcal{H}_e, F_e)$ constructed in Lemma 2.1.3 is
\[ \beta_e(a) := \beta_F(a \otimes e). \]

We end this subsection with a Proposition on finite summability concerning Poincaré dualities whose proof is carried out mutatis mutandis to that of Proposition 2.1.2. We let $\mathcal{S}$ denote a symmetrically normed operator ideal, see [59, Chapter 1.7]. Assume that $\mathcal{S} \subseteq A$ and $\mathcal{B} \subseteq B$ are unital dense $\ast$-sub algebras.

Proposition 2.1.5. Let $e \in \mathcal{B} \otimes M_n(\mathbb{C})$ is a projection and assume that $(\pi, \mathcal{H}, F)$ is $\mathcal{S}$-summable on the $\ast$-subalgebra $\mathcal{S} \otimes \mathbb{C} \subseteq A \otimes \mathcal{B}$, then $(\pi_e, \mathcal{H}_e, F_e)$ is $\mathcal{S}$-summable on $\mathcal{S}$. 

Remark 2.1.6. We note the following important consequence of Proposition 2.1.4. Assume that $(\pi, \mathcal{H}, F)$ is $\mathcal{J}$-summable on $\mathcal{A} \otimes_{\text{alg}} \mathcal{B}$. Then any element in the image of the mapping

$$K_0(B) \rightarrow K^1(A), \quad x \mapsto (1_A \otimes x) \otimes_{\text{alg}} [\pi, \mathcal{H}, F],$$

is $\mathcal{J}$-summable on $\mathcal{A}$. A slight modification of the argument above implies that the same holds true for elements in the image of the analogously defined mapping $K_1(B) \rightarrow K^0(A)$. This fact follows from [10 Proposition 3.12] which allows us to assume $\mathcal{B} \subseteq B$ to be holomorphically closed, and the mapping $K_1(\mathcal{B}) \rightarrow K_1(B)$ induced from the inclusion $\mathcal{B} \hookrightarrow B$ to be an isomorphism.

2.2. Finite summability in $K^1(O_A)$. To deal with the finite summability problem for $O_A$ we note an important relation between linear splittings and finite summability based on [31]. The observation will reduce the problem of finite summability for odd $K$-homology class to finding such $\pi$, $\mathcal{H}$ and $W$ described above, in the paragraph preceding Theorem 2.1.1, that behaves well only on generators. Whenever $\{x_i\}_{i \in I}$ is a set of elements in a $*$-algebra, we let $\mathbb{C}^+[x_i| i \in I]$ denote the $*$-algebra generated by $\{x_i\}_{i \in I}$.

Lemma 2.2.1. Let $\mathcal{J}, \mathcal{J} \subseteq \mathcal{B}$ be symmetric operator ideals such that $a^*a \in \mathcal{J}$ implies $a \in \mathcal{J}$. Suppose that

$$0 \rightarrow \mathbb{K}(\mathcal{H}_0) \rightarrow E \rightarrow A \rightarrow 0$$

is a short exact sequence of $C^*$-algebras with Bushy invariant $\beta_E$. Assume the following:

1. The $C^*$-algebra $A$ contains a dense $*$-sub algebra generated by a set $\{x_i\}_{i \in I} \subseteq A$, where $I$ is an index set, and that there is a set $\{x_i\}_{i \in I} \subseteq \mathbb{B}(\mathcal{H}_0)$ of pre images of $\{\beta_E(x_i)\}_{i \in I}$ under the quotient mapping $q : \mathbb{B}(\mathcal{H}_0) \rightarrow \mathcal{C}(\mathcal{H}_0)$ such that the mapping

$$\mathbb{C}^+[x_i| i \in I] \rightarrow \mathbb{B}(\mathcal{H}_0)/\mathcal{J}(\mathcal{H}_0), \quad x_i \mapsto x_i \mod \mathcal{J}(\mathcal{H}_0),$$

is a well defined $*$-homomorphism.

2. There is Hilbert space $\mathcal{H}$, a $*$-representation $\pi : A \rightarrow \mathbb{B}(\mathcal{H})$ and an isometry $W : \mathcal{H}_0 \rightarrow \mathcal{H}$ such that

$$X_i - W^*\pi(x_i)W \in \mathcal{J}(\mathcal{H}_0).$$

Then $[\beta_E]$ defines an invertible class in $\text{Ext}(A, \mathbb{K}(\mathcal{H}_0))$ whose image in $K^1(A)$ is represented by the $K$-cycle $(\pi, \mathcal{H}, 2WW^* - 1)$ which is $\mathcal{J}$-summable on the dense $*$-sub algebra $\mathbb{C}^+[x_i | i \in I] \subseteq A$.

The proof is closely modeled on the structure in the refined extension invariant of [31] that is adapted for extensions of Schatten class ideals. Compare to for instance [31, Theorem 3.2]. The examples of ideals to keep in mind is the finitely summable case $\mathcal{J} = \mathcal{L}^p$ and $\mathcal{J} = \mathcal{L}^q$ or the $\theta$-summable case $\mathcal{J} = L_0$ and $\mathcal{J} = L_{1/2}$.

Proof. It follows by the construction of the isomorphism $\text{Ext}(A, \mathbb{K})^{-1} \cong K^1(A)$ that $[\beta_E]$ is represented by the $K$-cycle $(\pi, \mathcal{H}, 2WW^* - 1)$. The $\mathcal{J}$-summability statement requires a more algebraic analysis.

To simplify notation we set $\mathcal{A} := \mathbb{C}^+[x_i | i \in I]$. We can define a linear mapping $\tau : \mathcal{A} \rightarrow \mathbb{B}(\mathcal{H}_0), \quad a \mapsto W^*\pi(a)W$. The assumptions of the Lemma guarantees that we can define the $*$-algebra

$$\mathcal{E} := \{(a, T) \in \mathcal{A} \oplus \mathbb{B}(\mathcal{H}_0) : \tau(a) - T \in \mathcal{J}(\mathcal{H}_0)\}.$$ 

There is a natural mapping $\sigma_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{A}$ given by $(a, T) \mapsto a$ which admits a linear splitting $\tilde{\tau}(a) := (a, \tau(a))$. 


The mapping $\tau$ induces a $*$-homomorphism $\beta_\varepsilon : \mathcal{A} \to \mathbb{B}(\mathcal{H}_0)/\mathcal{I}(\mathcal{H}_0)$. The pullback of the universal $\mathcal{J}$-summable extension along $\beta_\varepsilon$ places $\varepsilon$ in a diagram of $*$-algebras with exact rows:

$$
0 \longrightarrow \mathcal{J}(\mathcal{H}_0) \longrightarrow \varepsilon \longrightarrow \mathcal{A} \longrightarrow 0
$$

where $\varepsilon \to \mathbb{B}(\mathcal{H}_0)$ is defined by $(a, T) \mapsto T$.

We set $P := WW^*$. The operator $U := W^*|_{p}\mathcal{H} : p\mathcal{H} \to \mathcal{H}_0$ is a unitary isomorphism. We now turn to the $*$-algebra $\mathcal{E}$ defined from the diagram

$$
0 \longrightarrow \mathcal{J}(\mathcal{H}_0) \longrightarrow \mathbb{B}(\mathcal{H}_0) \longrightarrow \mathbb{B}(\mathcal{H}_0)/\mathcal{J}(\mathcal{H}_0) \longrightarrow 0
$$

By construction, the linear mapping $\tilde{\varepsilon}(a) := \pi(a)P = U^* \tau(a)U \in \mathcal{E}$ defines a splitting of the lower row. In particular, for any $a, b \in \mathcal{A}$ it holds that

$$
\tilde{\varepsilon}(ab) - \tilde{\varepsilon}(a)\tilde{\varepsilon}(b) \in \mathbb{B}(\mathcal{H}).
$$

It follows that $[R, \pi(a)] \in \mathcal{J}(\mathcal{H})$ for all $a \in \mathcal{A}$ by an algebraic manipulation, see [31, Lemma 3.7].

**Remark 2.2.2.** If the mapping $\beta_\varepsilon : \mathcal{A} \to \mathbb{B}(\mathcal{H}_0)/\mathcal{I}(\mathcal{H}_0)$ in the proof of Lemma 2.2.1 is injective, the mapping $\mathcal{E} \to \mathbb{B}(\mathcal{H}_0)$ is injective. Hence there is an isomorphism of $*$-algebras

$$\mathcal{E} \cong \{T \in \mathbb{B}(\mathcal{H}) : T \mod \mathcal{J}(\mathcal{H}_0) \in \text{im } \beta_\varepsilon \}.$$

Let us return to the algebra $O_A$. Recall the definition of the KMS-state $\phi_A$ on $O_A$ from [1.13], the associated GNS-space $L^2(O_A, \phi_A)$ and the fundamental representation $\pi_A$. By the results of Subsection 1.3 there is an isomorphism $L^2(O_A, \phi_A) \cong L^2(\mathcal{Y}_0)$ intertwining the $O_A$-action with the $C^*(\mathcal{Y}_0)$-action under the isomorphism $O_A \cong C^*(\mathcal{Y}_0)$ from Theorem 1.3.2.

Fix a finite admissible word $w \in \mathcal{Y}_A$. Define $\mathcal{H}_0$ as the closed linear span of all elements $S_{\mu} \in L^2(O_A, \phi_A)$. For any two finite words $\mu, \nu \in \mathcal{Y}_A$, Proposition 1.3.1 implies that

$$
(S_{\mu}, S_{\nu})_{L^2(O_A, \phi_A)} = \delta_{\mu, \nu} \phi_A(S_{\mu}^* S_{\mu}) = \delta_{\mu, \nu} \sum_{j=1}^N A_{\mu, j} \text{vol}(C_j),
$$

where $k := |\mu|$. For any finite word $\mu$, admissible or not, we set

$$
c_{\mu} := \left(\sum_{j=1}^N A_{\mu, j} \text{vol}(C_j)\right)^{-1/2}.
$$

In particular, it holds that $c_{\mu}$ only depends on the last letter of $\mu$. It follows from the computation above that the non-zero elements of $\{c_{\mu} S_{\mu} | \mu \in \mathcal{Y}_A\}$ forms an ON-basis for $\mathcal{H}_0$. Define $\mathcal{P}_\lambda : L^2(O_A, \phi_A) \to \mathcal{H}_0$ to be the orthogonal projection. We also define the unitary isomorphism $U : \mathcal{H}_0 \to L^2(\mathcal{Y}_A)$ by $c_{\mu} S_{\mu} \mapsto \delta_{\mu}$. Finally, we can define the partial isometry $W_\lambda : L^2(\mathcal{Y}_A) \to L^2(O_A, \phi_A)$ from

$$
W_\lambda^* := U \mathcal{P}_\lambda : L^2(O_A, \phi_A) \to L^2(\mathcal{Y}_A).
$$

The coimage of $W_\lambda$ will consist of the closed subspace $L^2(\mathcal{Y}_A)$ spanned by the basis vectors associated with the words

$$
\mathcal{Y}_A := \{\mu \lambda \in \mathcal{Y}_A | \mu \in \mathcal{Y}_A\}. 
$$
Proposition 2.2.3. The partial isometry $W_\lambda$ satisfies that

$$W_\lambda^* \pi_{\Delta}(S)W_\lambda = L_i^\Delta W_\lambda^* W_\lambda, \quad i = 1, \ldots, N.$$ 

Proof. It suffices to prove that $W_\lambda^* \pi_{\Delta}(S)W_\lambda^* \delta_{\mu\lambda} = L_i^\Delta \delta_{\mu\lambda}$ since the vectors $\delta_{\mu\lambda}$ span the range of $W_\lambda$. A direct computation goes as follows:

$$W_\lambda^* \pi_{\Delta}(S)W_\lambda^* \delta_{\mu\lambda} = c_{\mu\lambda} W_\lambda^* \pi_{\Delta}(S_i)S_{\mu\lambda} = c_{\mu\lambda} W_\lambda^* S_{\mu\lambda} = \delta_{\mu\lambda} = L_i^\Delta \delta_{\mu\lambda},$$

since $c_{\mu\lambda} = c_{\mu\lambda}$. □

Remark 2.2.4. The projection $\Psi_\lambda W_\lambda^* W_\lambda^*$ can be constructed in the groupoid picture as follows. We note that for any finite word $\mu$, $S_\mu$ corresponds to the characteristic function of the set $\{(x, |\mu|, \sigma^{[\mu]}(x)) \in \mathcal{G}_\mu | x \in C_\mu\}$. It holds that

$$P_\lambda f := \sum_{\mu \in \mathcal{G}_\lambda} c_{\mu\lambda} S_{\mu\lambda} \int_{\Omega_\lambda} \nu(S_{\mu\lambda} f) d\mu_\lambda,$$

with $\nu$ as in (1.7), defines a projection in $L^2(\mathcal{G}_\mu, \mu_\lambda)$ corresponding to $\Psi_\lambda$ under the isomorphism $L^2(\mathcal{G}_\mu, \mu_\lambda) \cong L^2(\mathcal{G}_\mu, \mu_\lambda)$.

Proposition 2.2.5. The extension defined from the Busby invariant

$$\beta_i : O_X \to \mathcal{C}(\ell^2(\mathcal{G}_\lambda)), \quad a \mapsto \beta_i^a\,(a \otimes T_i^*)$$

can be represented by the odd analytic K-cycle $(\pi_{\Delta}, L^2(O_X, \varphi), 2W_i W_i^* - 1)$ which is p-summable for any $p > 0$ on the dense sub algebra $\mathcal{R}$ generated by $\{S_\mu S_\nu^* | \mu, \nu \in \mathcal{G}_\lambda\}$.

Proof. For $j = 1, \ldots, n$, the operators $X_j := L^\Delta R_i^0(R_i^0)^* \rho_i(S_{j})$ lifts $\beta_i(S_j)$. By Lemma 1.2.1 and Equation (2.13) the operators $X_j$ satisfy the Cuntz-Krieger relations modulo finite rank operators, so $S_j \mapsto X_j$ mod $L^p(\ell^2(\mathcal{G}_\lambda))$ defines a $*$-homomorphism from $\mathcal{R}$ for any $p > 0$, even modulo finite rank operators. It also holds that $W_i^* W_i = R_i^0(R_i^0)^*$. In particular, $X_j = W_i^* \pi_{\Delta}(S_i)W_i$, Hence, the Proposition follows from Lemma 2.2.1. □

We recall the following description of $K_0(O_{\hat{\mathcal{R}}})$ from [21] Proposition 3.1.

Proposition 2.2.6 (Proposition 3.1 of [21]). The mapping

$$\mathbb{Z}^N \to K_0(O_{\hat{\mathcal{R}}}), \quad (k_j)_{j=1}^N \mapsto \sum_{j=1}^N k_j [T_j T_j^*]$$

is surjective with kernel being $(1 - A)\mathbb{Z}^N$.

Proof of Theorem 2.0.7. By Theorem 2.1.1 Remark 2.1.4 and Proposition 2.2.6 any K-homology class on $O_X$ can be represented by an extension class of the form $\sum_{j=1}^N k_j [\beta_j]$. The Theorem follows from Proposition 2.2.6. □

2.3. A representative for $\Delta$. As previously indicated (see Proposition 2.1.3), any summability property of a K-cycle representative for $\Delta$ would carry over to any K-homology class for $O_X$. The problem is to represent $\Delta$ in a reasonable way. We will in this subsection construct a $\theta$-summable representative for $\Delta$.

We will use the notation $\mathcal{H}_{\Lambda}^\Delta$ for the closed linear span of $\{T_\mu | \mu \in \mathcal{G}_\lambda\}$ in $L^2(\mathcal{R}_\lambda, \varphi_{\Lambda})$. We also let $P_0^\mu : L^2(\mathcal{R}_\lambda, \varphi_{\Lambda}) \to \mathcal{H}_{\Lambda}^\Delta$ denote the orthogonal projection. Just as for $O_X$, there are constants...
c^T_\mu > 0 only depending on the first word of \mu such that \{c^T_\mu T_\mu | \mu \in \mathcal{V}_\lambda\} forms an ON-basis for \mathcal{H}_0^T.

Define the linear mapping \mathcal{W}_0 by

\mathcal{W}^* : \mathcal{H}_0 \otimes \mathcal{H}_0^T \rightarrow L^2(\mathcal{V}_\lambda), \quad c_\mu c^T_\nu S_\mu \otimes T_\nu \mapsto (|\mu\nu| + 1)^{-1/2} \delta_{\mu\nu}

The operator \mathcal{W}_0 is an isometry since

\mathcal{W}^* \mathcal{W}_0 \delta_\lambda = \sum_{\mu\nu = \lambda} (|\mu\nu| + 1)^{-1/2} \delta_{\mu\nu} = \delta_\lambda.

We will use the isometry

\mathcal{W} := (P_0 \otimes P_0^T) \mathcal{W}_0 : L^2(\mathcal{V}_\lambda) \rightarrow L^2(O_0, \mathcal{V}_\lambda) \otimes L^2(O_0, \mathcal{V}_\lambda^T).

Recall the definition of \varphi from \([11]\).

**Lemma 2.3.1.** For any \lambda it holds that

\[ L^4 - \mathcal{W}^*[\pi_\lambda(S_\lambda) \otimes 1_{O_0^T}] \mathcal{W}^* \in \mathcal{L}(L^2(\mathcal{V}_\lambda)). \]

If there is a \lambda > 0 such that \varphi(l) \lesssim l^p, it holds that

\[ L^4 - \mathcal{W}^*[\pi_\lambda(S_\lambda) \otimes 1_{O_0^T}] \mathcal{W}^* \in \mathcal{L}^{p,\infty}(L^2(\mathcal{V}_\lambda)). \]

Here we use the standard notation \mathcal{L}^{p,\infty} for the symmetrically normed ideal defined for \lambda > 1 as

\[ \mathcal{L}^{p,\infty}(\mathcal{H}) := \{ T \in \mathcal{H}(\mathcal{H}) | \mu_k(T) = O(k^{-1/p}) \} \subseteq \cap_{q>p} \mathcal{L}^q(\mathcal{H}). \]

The definition for \lambda = p differs slightly.

**Proof.** This is yet another proof by computation. Choose a finite word \lambda \in \mathcal{V}_\lambda. It holds that

\[
\mathcal{W}^*[\pi_\lambda(S_\lambda) \otimes 1_{O_0^T}] \mathcal{W}^\delta_\lambda = \mathcal{W}^\delta_\lambda \left( \sum_{\mu\nu = \lambda} c_\mu c^T_\nu (|\mu\nu| + 1)^{-1/2} S_\mu \otimes T_\nu \right) = \frac{(|\lambda| + 1)^{-1/2}}{(i|\lambda| + 1)^{-1/2}} \delta_{\lambda\lambda} = L^4 \delta_\lambda + \left( \frac{|\lambda| + 2}{|\lambda| + 1} - 1 \right) \delta_{\lambda\lambda}
\]

and

\[
\mathcal{W}^*[1_{O_0} \otimes \pi_\lambda(T_\lambda)] \mathcal{W}^\delta_\lambda = \mathcal{W}^\delta_\lambda \left( \sum_{\mu\nu = \lambda} c_\mu c^T_\nu (|\mu\nu| + 1)^{-1/2} S_\mu \otimes T_\nu \right) = \frac{(|\lambda| + 1)^{-1/2}}{(i|\lambda| + 1)^{-1/2}} \delta_{\lambda\lambda} = R^4 \delta_\lambda + \left( \frac{|\lambda| + 2}{|\lambda| + 1} - 1 \right) R^4 \delta_{\lambda\lambda},
\]

since \(c_\mu\) only depend on the last letter of \mu and \(c^T_\nu\) only depend on the first letter of \nu. We define \(\Gamma \in \mathcal{B}(L^2(\mathcal{V}_\lambda))\) by

\[ \Gamma \delta_\lambda := \left( \frac{|\lambda| + 2}{|\lambda| + 1} - 1 \right) \delta_\lambda \]

and reformulate the above identities as

\[ \mathcal{W}^*[\pi_\lambda(S_\lambda) \otimes 1_{O_0^T}] \mathcal{W} - L^4 = L^4 \Gamma \]

and \(\mathcal{W}^*[1_{O_0} \otimes \pi_\lambda(T_\lambda)] \mathcal{W} - R^4 = R^4 \Gamma.\)

We recall the elementary asymptotics

\[ \sqrt{\frac{|\lambda| + 2}{|\lambda| + 1} - 1} = \frac{1}{2|\lambda|} + O \left( \frac{1}{|\lambda|^2} \right), \text{ as } |\lambda| \to \infty. \]

It holds in general that \(\varphi(l) \lesssim e^s \delta_\lambda\) for \(s > \delta_\lambda\) by Corollary \([11,6]\) so \(\Gamma \in \mathcal{L}(\mathcal{H}_0).\) On the other hand \(\varphi(l) \lesssim l^p\) implies \(\Gamma \in \mathcal{L}^{p,\infty}(L^2(\mathcal{V}_\lambda)).\) \(\square\)
From Lemma 2.2.1 and Lemma 2.3.1 we may conclude a summability result for the duality class $\Delta$. This result is by no means a surprise. There is to the authors’ knowledge no known counter examples to the $\theta$-summability problem for unbounded Fredholm modules, so in effect there are no counter examples to representing $K$-homology classes by $\theta$-summable Fredholm modules on $O$. The gauge cycle.

$\rho_a$ a finitely summable representative. Products of bivariant cycles with unbounded Fredholm modules on one of the subalgebras $O$ modules on $(\pi_\lambda \otimes \pi_\lambda, \mathcal{L}(\mathcal{T}_1), \mathcal{L}(\mathcal{T}_1), 2n\mathcal{F}^\mathcal{F} - 1)$, which is $\theta$-summable on the dense $*$-sub algebra of $O \otimes O_L$ generated by $S_i \otimes 1$ and $1 \otimes T_i$ for $i = 1, \ldots, n$. If there is a $\theta > 0$ such that $\varphi(1) \leq \theta^2$, this is a $\mathcal{L}^{\infty, \infty}$-summable $K$-cycle.

We can from this Theorem and (1.14) conclude that any $K$-homology class on $SU_q(2)$ admits a finitely summable representative.

3. Unbounded $(O_A, F_A)$-cycles

We will in this section start approaching the problem of constructing unbounded Fredholm modules on $O_A$. It is natural to try and construct unbounded Fredholm modules as Kasparov products of bivariant cycles with unbounded Fredholm modules on one of the subalgebras $C(\Omega_A) \subseteq F_A \subseteq O_A$. We will construct classes in $KK_1(O_A, F_A)$ and $KK_1(O_A, C(\Omega_A))$, in the current section respectively Section 5 that behave analogously to those studied in Section 2 apart from the difficulties of being bivariant. A problem with using the fixed point algebra is that, despite there being a well studied bivariant $(O_A, F_A)$-cycle that is naturally constructed, the unbounded Fredholm modules on $F_A$ are difficult to construct and understand topologically. E.g. for the Cuntz algebra $O_N$ it holds that the even $K$-homology of the fixed point algebra vanishes but its odd $K$-homology is an uncountable group (cf. Proposition 3.3.2).

3.1. The gauge cycle. To this end we first describe the structure of the module $E_\xi$ associated with the conditional expectation $\rho_\xi : O_A \rightarrow F_A^\mathcal{F}$ coming from the restriction mapping $C_c(\mathcal{A}_\xi) \rightarrow C_c(\mathcal{H}_\xi)$. By [13], pointwise multiplication by the cocycle $\mathcal{C}_\xi$ induces a selfadjoint regular operator $D_\xi$ on the $C^*$-module $E_\xi$, giving $(E_\xi, D_\xi)$ the structure of an odd unbounded $KK$-cycle for $KK_1(O_A, F_A)$. We will refer to this cycle as the gauge cycle. The construction of the gauge cycle was considered in a more general setup in [13] that we make use of in the next subsection.

We assume that $B$ is a $C^*$-algebra with a strongly continuous $U(1)$-action satisfying the spectral subspace assumption [12] Definition 2.2]. We let $F \subseteq B$ denote the fixed point algebra for the $U(1)$-action.

There is a positive expectation value $E : B \rightarrow F$ given by $a \mapsto \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta}(a) d\theta$, this expectation coincides with $\rho_\xi$ for $O_A$. After completion of $B$ with respect to the associated $F$-valued scalar product we obtain a $B - F$-Hilbert bi module that we will denote $\hat{B} E_F$. We can also define the operator

$$D_\xi y = i \frac{d}{d\theta} \left(e^{-i\theta}, y\right) \big|_{\theta=0},$$

which is a densely defined $F$-linear operator on $\hat{B} E_F$. Since $U(1)$ is abelian, $D_\xi$ commutes with the circle action on $\hat{B} E_F$ giving a $U(1)$-equivariant operator.

**Proposition 3.1.1.** Whenever the $U(1)$-action on $B$ satisfies the spectral subspace assumption, the pair $(D_\xi, \hat{B} E_F)$ forms a $U(1)$-equivariant unbounded $B - F$-Kasparov module.

For a proof, see [12] Proposition 2.9]. We now describe the structure of the module $E_\xi$. It is clear that $E_\xi$ decomposes as a direct sum of $F$-modules:

$$E_\xi = \bigoplus_{n \in \mathbb{Z}} E_\xi_n,$$
corresponding to the disjoint union decomposition $\mathcal{G} = \bigcup_{n \in \mathbb{Z}} \mathcal{G}_n$, where $\mathcal{G}_n = c_A^{-1}(n)$. We show below that each $\mathcal{E}_n^c$ is a finite projective $F$-module, and consequently $\mathcal{E}_n^c$ is isomorphic to a direct sum of finite projective $F_A$-modules.

**Lemma 3.1.2.** Let $n > 0$. The column vectors $v_n := (S^*_\mu)_{\mu \in \mathcal{G}_n} \in \text{Hom}^*_F(\mathcal{E}_n^c, F^2(n))$ have the property that $v_n^* v_n = 1$. In particular, $\mathcal{E}_n^c$ is a finitely generated projective $F_A$-module for $n > 0$.

**Proof.** We have

$$v_n^* v_n = \sum_{|\mu|=n} S^*_\mu S^*_\mu = 1,$$

which follows from successively applying the relation (1.10).

For an element $a \in F_A$ of degree $n$, the vector $v_n^* a$, constructed by coordinatewise multiplication by $a$, is an element of $F^2(n)$. Therefore, for positive $n$, the map

$$\mathcal{E}_n^c \rightarrow F^2(n)$$

$$a \mapsto v_n^* a,$$

is an isometry onto its image. This image is equal to $p_n F^2_A$, with $p_n := v_n^* v_n$, and hence $\mathcal{E}_n^c$ is a finite projective module. $\square$

We denote the projections $S_j S^*_j$ by $P_j$. The projections $P_j$ are of degree 0, and $S_j P_j$ is of degree 1 for any $i, j$. Recall that we assume that neither row nor column of $A$ is composed of only zeroes. Hence the numbers

$$N_j := \sum_{i=1}^N A_{ij},$$

satisfy $0 < N_j \leq N$. For two finite words $\mu$ and $\nu$, not necessarily admissible, of the same length $n > 0$ we set

$$R_{\mu, \nu} := \frac{1}{\sqrt{N_{\mu_1} \cdots N_{\mu_n} N_{\nu_1} \cdots N_{\nu_n}}} S_{\mu_1} P_{\nu_1} \cdots S_{\mu_n} P_{\nu_n}$$

We use the notation $\tilde{\varphi}(n) := \#\{(\mu, \nu) : |\mu| = |\nu| = n, R_{\mu, \nu} \neq 0\}$. It is clear that $\tilde{\varphi}(n) \leq \varphi(n)^2$.

**Lemma 3.1.3.** Let $n > 0$. The column vectors $w_n := (R_{\mu, \nu})_{\mu, \nu \in \mathcal{G}_n} \in \text{Hom}^*_F(\mathcal{E}_n^c, F^2(n))$ have the property that $w_n^* w_n = 1$. In particular, $\mathcal{E}_n^c$ is a finitely generated projective $F_A$-module for $n > 0$.

**Proof.** We have

$$w_n^* w_n = \sum_{|\mu|=|\nu|=n} \prod_{\nu \in \mathcal{G}_n} P_{\nu} S^*_\mu \cdots P_{\nu} S^*_\mu S^*_\mu P_{\nu} \cdots S^*_\mu P_{\nu}$$

$$= \sum_{\mu, \nu \in \mathcal{G}_n} \prod_{\nu \in \mathcal{G}_n} P_{\nu} S^*_\mu \left( \sum_{|\mu|=|\nu|} \prod_{\nu \in \mathcal{G}_n} P_{\nu} S^*_\mu S^*_\mu \cdots \right) S_{\mu} P_{\nu}$$

$$= \sum_{\mu, \nu \in \mathcal{G}_n} \prod_{\nu \in \mathcal{G}_n} P_{\nu} S^*_\mu w_{n-\nu}^* \nu S_{\mu} P_{\nu}.$$
Hence, the result follows by induction once proven for \( n = 1 \). In that case, the equation becomes
\[
w_1^*w_1 = \sum_{i,j=1}^k \frac{1}{k_j} p_i^* s_i^* p_j = \sum_{i,j=1}^k \left( \frac{1}{k_j} A_{ij} p_i^* s_i^* p_j \right) \quad \text{by (1.9)}
\]
\[
= \sum_{j=1}^k \frac{1}{k_j} A_{ij} s_j^* \quad \text{by (1.11)}
\]
\[
= \sum_j s_j s_j^* = 1 \quad \text{by (1.10)}.
\]

\[ \Box \]

As in Section 2 we can construct \( KK \)-cycles using the finite admissible words. A difference here is that one has to work with partial isometries in \( C^* \)-modules.

**Theorem 3.1.4.** The \( F_A \)-linear adjointable mapping
\[
v : \ell^2(\gamma_A) \otimes F_A \to \mathcal{E}^c, \quad \text{defined by } v : \delta_\mu \otimes a \mapsto S_\mu a,
\]
is a partial isometry and the projection \( vv^* \in \text{End}_{\ell^2(\mathcal{E})}^1(\mathcal{E}^c) \) has compact commutators with \( O_A \). It consequently defines a \( U(1) \)-equivariant \( (O_A, F_A) \)-Kasparov module \( (\mathcal{E}^c, 2vv^* - 1) \) whose class in \( KK^U(1)(O_A, F_A) \) coincides with the class \([\mathcal{E}^c, D_c]\) of the gauge cycle \( (\mathcal{E}^c, D_c) \).

**Proof.** Observe that \( v \) is adjointable by lemma 3.1.2 and 3.1.3. It is clear that \( v \) is a partial isometry because the elements \( S_\mu \) are mutually orthogonal in the module \( \mathcal{E}^c \) and \( v^*(S_\mu a) = \delta_\mu \otimes S_\mu^* S_\mu a \), so both \( vv^* \) and \( vv^* \) are projections. The statement that the isometries define Kasparov modules is proved as in the previous section. To see that \( vv^* \) defines the gauge cycle, one only needs to observe that it is exactly the projection onto the positively graded part of the module \( \mathcal{E}^c \).

\[ \Box \]

**Remark 3.1.5.** In a similar way as in Theorem 3.1.4 we can define a partial isometry
\[
w : \ell^2(\gamma_A) \otimes F_A \otimes F_{A'} \to \mathcal{E}^c \otimes F_{A'}^c,
\]
\[
\delta_\mu \otimes a \otimes b \mapsto \sum_{\lambda = \mu} \frac{1}{\sqrt{\lambda| + 1}} S_\mu a \otimes T_\mu b.
\]

It can be proven in the same way as in Theorem 3.1.4 that the projection \( ww^* \) has compact commutators with \( O_A \otimes O_{A'} \). Consequently we obtain an odd \( U(1) \)-equivariant \( (O_A \otimes O_{A'}, F_A \otimes F_{A'}) \)-Kasparov module. Compare to the construction of Subsection 2.3.

3.2. **The Pimsner-Voiculescu sequence for the Cuntz-Krieger algebra.** What we wish to do in this section is to relate the cohomological properties of the Cuntz-Krieger algebra with the fixed point algebra. The standard procedure, found in [21] for instance, is to apply the Pimsner-Voiculescu sequence. In this section we briefly recall the proof of the Pimsner-Voiculescu in \( KK \) following [25] and prove that the gauge cycle appears as the boundary mapping. We summarize the ambition of this subsection in the following Theorem:

**Theorem 3.2.1.** The gauge element \([\mathcal{E}^c, D_c] \in KK(\mathcal{O}_A, F_A)\), the \( \mathbb{Z} \)-action \( \beta \) on \( F_A \otimes \mathbb{K} \) of Proposition 1.4.3 and the inclusion \( \iota : F_A \to \mathcal{O}_A \) fits into a distinguished triangle in \( KK \):

\[
\begin{array}{ccc}
F_A & \xrightarrow{1-\beta} & F_A \\
\downarrow \iota & & \downarrow \\
[\mathcal{E}^c, D_c] & \xrightarrow{\delta} & \mathcal{O}_A.
\end{array}
\]
The triangulated structure of $KK$ is explained in [50]. In practice, it ensures that for any separable $C^*$-algebra $B$ there are the following six term exact sequences:

\[
\begin{array}{cccc}
KK_0(B, F_A) & \xrightarrow{1-\beta} & KK_0(B, F_A) & \xrightarrow{1} & KK_0(B, O_A) \\
\otimes [\varepsilon', D_\cdot] & & \otimes [\varepsilon', D_\cdot] & & \otimes [\varepsilon', D_\cdot] \\
KK_1(B, O_A) & \xrightarrow{1-\beta} & KK_1(B, F_A) & \xrightarrow{1-\beta} & KK_1(B, F_A) \\
KK_0(O_A, B) & \xrightarrow{1-\beta} & KK_0(F_A, B) & \xrightarrow{1-\beta} & KK_0(F_A, B) \\
[K, D_\cdot] & & \otimes [\varepsilon', D_\cdot] & & \otimes [\varepsilon', D_\cdot] \\
KK_0(F_A, B) & \xrightarrow{1-\beta} & KK_0(F_A, B) & \xrightarrow{1-\beta} & KK_0(O_A, B) \\
\end{array}
\]

The Pimsner-Voiculescu sequence can be derived in many ways. We will here consider the Toeplitz extension approach due to Cuntz. Assume that $A$ is a unital $C^*$-algebra and that $\beta$ is an automorphism of $A$. The restriction that $A$ is unital makes the semantics easier, but can be lifted. Let $\mathcal{T}(A)$ denote the $C^*$-algebra generated by $A$ and an isometry $v_A$ satisfying the relation $v_A^*v_A = \beta(a)$.

One can represent $\mathcal{T}(A)$ in $\text{End}_A^\ast(\oplus_{k \in \mathbb{Z}} A)$ by extending the mappings

$$A \ni a \mapsto \oplus_k \beta^k(a) \in \text{End}_A^\ast(\oplus_{k \in \mathbb{Z}} A) \quad \text{and} \quad v_A(x_k)_{k \in \mathbb{Z}} := (x_{k-1})_{k \in \mathbb{Z}}.$$

There is a $U(1)$-action on $\mathcal{T}(A)$ induced from the grading on $\oplus_{k \in \mathbb{Z}} A$, i.e. the $U(1)$-action is defined by $z(v_A) := zv_A$.

Let us realize $A \rtimes \mathbb{Z}$ as the universal algebra generated by $A$ and a unitary $u_A$ satisfying

$$u_Au_A^* = \beta(a).$$

There is a $*$-homomorphism $\sigma_A \colon \mathcal{T}(A) \to A \rtimes \mathbb{Z}$ given by extending $v_A \mapsto u_A$. Since $\sigma_A$ respects the grading it is clear that $\sigma_A$ is $U(1)$-equivariant with respect to the dual $U(1)$-action on $A \rtimes \mathbb{Z}$. There is an isomorphism of $A$-Hilbert modules $\oplus_{k \in \mathbb{Z}} A \cong A \otimes \ell^2(\mathbb{N})$.

**Lemma 3.2.2.** The morphism $\sigma_A$ is well defined and fits into a $U(1)$-equivariant semi split short exact sequence

\[
0 \to \mathbb{K}_A(A \otimes \ell^2(\mathbb{N})) \to \mathcal{T}(A) \xrightarrow{\sigma_A} A \rtimes \mathbb{Z} \to 0.
\]

**Proof.** We can identify $A \rtimes \mathbb{Z}$ with the $C^*$-subalgebra of $\text{End}_A^\ast(A \otimes \ell^2(\mathbb{Z}))$ generated by the image of

$$A \ni a \mapsto \oplus_k \beta^k(a) \in \text{End}_A^\ast(\oplus_{k \in \mathbb{Z}} A) \quad \text{and the unitary} \quad u_A(x_k)_{k \in \mathbb{Z}} := (x_{k-1})_{k \in \mathbb{Z}}.$$

Let $P : A \otimes \ell^2(\mathbb{Z}) \to A \otimes \ell^2(\mathbb{N})$ denote the orthogonal projection. The adjointable operator $P$ is $U(1)$-equivariant. It is clear that the $A$-linear mapping

$$\mathcal{T} : A \rtimes \mathbb{Z} \to \mathcal{T}(A), \quad b \mapsto PbP$$

is a $U(1)$-equivariant completely positive splitting of $\sigma_A$. Let $q : \text{End}_A^\ast(A \otimes \ell^2(\mathbb{N})) \to \mathcal{L}_A(A \otimes \ell^2(\mathbb{N})) := \text{End}_A^\ast(A \otimes \ell^2(\mathbb{N}))/\mathbb{K}_A(A \otimes \ell^2(\mathbb{N}))$ denote the quotient mapping. Once we prove that $q \circ \mathcal{T}$ is a $*$-homomorphism, the Lemma follows. The operator $P$ commutes with the $A$-action on $A \otimes \ell^2(\mathbb{Z})$. Furthermore, if we let $e_k$ denote the standard basis for $\ell^2(\mathbb{Z})$

$$[P_i u_A] x \otimes e_k = \begin{cases} 0, & k \neq -1 \\ x \otimes e_0, & k = -1. \end{cases}$$
In particular, \([P, u_A] \in 1_A \otimes K(\ell^2(\mathbb{Z})) \subseteq K(A \otimes \ell^2(\mathbb{Z}))\). It follows that \([P, b] \in K(A \otimes \ell^2(\mathbb{Z}))\) for any \(b \in A \times \mathbb{Z}\). Hence \(\varphi \circ \mathcal{T}\) is a \(\ast\)-homomorphism. \(\square\)

**Lemma 3.2.3.** There is a \(U(1)\)-equivariant homotopy \(\mathcal{T}(A) \sim_A A\) with trivial \(U(1)\)-action on \(A\).

For a proof, see [22]. Let \(\iota_A : A \to A \times \mathbb{Z}\) denote the embedding.

**Corollary 3.2.4.** The morphism \([\mathcal{T}] \in KK_1^{U(1)}(A \times \mathbb{Z}, A)\) defined from the invertible extension class \((3.23)\) fits into a distinguished triangle in \(KK^{U(1)}\):

\[
\begin{array}{ccc}
A & \xrightarrow{1-\beta} & A \\
\downarrow{[\mathcal{T}]} & & \downarrow{\iota_A} \\
A \times \mathbb{Z}, & & \\
\end{array}
\]

using the homotopy of Proposition 3.2.5 and the Morita equivalence \(K(A \otimes \ell^2(\mathbb{N})) \sim_M A\).

As a consequence of the Corollary, what remains to prove Theorem 3.2.1 is to show that \([\mathcal{T}]\) coincides with the gauge element in \(KK_1(O_A, F_A)\) after Takesaki-Takai duality \(O_A \otimes \mathbb{K} \cong (F_A \otimes \mathbb{K}) \times \mathbb{Z}\). We first construct an unbounded representative for the Pimsner-Voiculescu element \([\mathcal{T}] \in KK_1^{U(1)}(A \times \mathbb{Z}, A)\).

Before constructing this, let us make a series of minor remarks placing the algebra above in a more analytic framework. The Fourier transform induces an isomorphism \(L^2(S^1) \cong \ell^2(\mathbb{Z})\) which in turn produces an isomorphism \(C(S^1) \cong C^*(\mathbb{Z})\) intertwining the point wise action of the former with the left regular representation of the latter. The image of \(\ell^2(\mathbb{N})\) under the Fourier transform is \(H^2(S^1)\)-the Hardy space consisting of functions in \(L^2(S^1)\) with a holomorphic extension to the interior of \(S^1 \subseteq \mathbb{C}\). The analogy of the projection \(P\) in this picture is the projection of \(A \otimes L^2(S^1)\) onto those \(A\)-valued functions on \(S^1\) with a holomorphic extension to the interior. We note that

\[
A \otimes L^2(S^1) = \left\{ (a_k)_{k \in \mathbb{Z}} \in \prod_{k \in \mathbb{Z}} A : \sum_{k \in \mathbb{Z}} a_k^* a_k < \infty \right\}.
\]

**Proposition 3.2.5.** There is a natural unitary isomorphism of \(A \times \mathbb{Z} - A\)-Hilbert bimodules

\[
A \otimes L^2(S^1) \cong \overline{A \times \mathbb{Z}},
\]

where the closure is taken in \(A\)-valued scalar product \(\langle a, b \rangle := \int_{U(1)} z \cdot (a^* b)\).

We define

\[
W^{1,2}(S^1, A) := \left\{ (a_k)_{k \in \mathbb{Z}} \in \prod_{k \in \mathbb{Z}} A : \sum_{k} k^2 a_k^* a_k < \infty \right\},
\]

and the \(U(1)\)-equivariant \(A\)-linear unbounded operator \(D_{A \otimes L^2(S^1)}\) on \(A \otimes L^2(S^1)\) on an elementary tensor by

\[
D_{A \otimes L^2} (x \otimes e_k) := k x \otimes e_k
\]

and extending it to the domain \(W^{1,2}(S^1, A)\) by continuity. If \(y \in W^{1,2}(S^1, A)\) then

\[
D_{A \otimes L^2} y = i \frac{d}{d\theta} \left( e^{-i\theta} \cdot y \right) \bigg|_{\theta = 0}.
\]

**Proposition 3.2.6.** The operator \(D_{A \otimes \mathbb{Z}}\) with domain \(W^{1,2}(S^1, A)\) gives a \(U(1)\)-equivariant unbounded \(KK\)-cycle for \((A \times \mathbb{Z}, A)\).

For a proof, see [12, Section 2].
Lemma 3.2.7. The bounded transform of $D_{A_{\mathbb{Z}^2}}$ is a compact perturbation of the $(A \otimes \mathbb{Z}, A)$-Kasparov module $\langle 2P - 1, A \otimes L^2(S^1) \rangle$. Especially: $[D_{A_{\mathbb{Z}^2}}] = [\mathcal{F}] \in KK_1^{(1)}(A \otimes \mathbb{Z}, A)$.

Proof. We have that

$$D_{A_{\mathbb{Z}^2}} (1 + D_{A_{\mathbb{Z}^2}})^{-1/2} (x \otimes e_k) := k(1 + k^2)^{-1/2} x \otimes e_k.$$ 

Since $k(1 + k^2)^{-1/2} - \text{sign}(k) \sim -(2k)^{-1}$ as $|k| \to \infty$ and $A \otimes \mathbb{Z}$ satisfies the spectral subspace condition (see [12, Definition 2.2]) the Proposition follows from [12, Lemma 2.4]. □

It is possible to, in the Pimsner-Voiculescu sequence of the Cuntz-Krieger algebra, replace the Toeplitz element $[\mathcal{F}]$ by the gauge cycle. Recall its definition from above. We denote the class associated with the gauge cycle $(D_{g}, g \mathcal{E}_g)$ by $[D_g] \in KK_1^{(1)}(B, F)$. We note the following Proposition whose proof is left to the reader.

Proposition 3.2.8. The image of $[D_g]$ under the isomorphism

$$j_{\mathbb{K}(L^2(S^1))} : KK_1^{(1)}(B, F) \to KK_1^{(1)}(B \otimes \mathbb{K}(L^2(S^1)), F)$$

associated with the $U(1)$-equivariant Morita equivalence $B \cong M B \otimes \mathbb{K}(L^2(S^1))$, where the right hand side is equipped with the diagonal $U(1)$-action, coincides with the class of

$$(D_g \otimes 1)_{L^2(S^1)} = \mathcal{E}_g \otimes L^2(S^1))$$

where the $B \otimes \mathbb{K}(L^2(S^1)) - F$-Hilbert bi module $\mathcal{E}_g \otimes L^2(S^1)$ is equipped with the diagonal $U(1)$-action.

The $B$-action on $\mathcal{E}_g$ is by construction equivariant, hence there is an action of $B \times U(1)$ on $\mathcal{E}_g$. By [12, Lemma 2.4.ii], the spectral subspace assumption guarantees that this action induces a $*$-homomorphism $B \times U(1) \to \mathcal{E}_g$ via $g \cdot x \mapsto g x$. We let $\mathcal{G}_{\mathcal{E}_g}$ denote the associated $B \times U(1) - F$-Hilbert bimodule. We denote the associated class in $KK_1^{(1)}$ by $[\mathcal{G}_{\mathcal{E}_g}] \in KK_1^{(1)}(B \times U(1), F)$. Let us remark that by $[\mathcal{G}_{\mathcal{E}_g}]$, as Banach spaces and as $B - F$-bimodules we have the equality

$$B \otimes \mathcal{G}_{\mathcal{E}_g} = \mathcal{E}_g \otimes L^2(S^1).$$

Lemma 3.2.9. If $B$ satisfies the spectral subspace assumption, then

$$j_{\mathbb{K}(L^2(S^1))} [D_g] = [\mathcal{G}_{\mathcal{E}_g}] \in KK_1^{(1)}(B \otimes \mathbb{K}(L^2(S^1)), F),$$

where $[\mathcal{G}_{\mathcal{E}_g}] \in KK_1^{(1)}(B \otimes \mathbb{K}(L^2(S^1)), B \times U(1))$ is the element constructed in Proposition 3.2.8.

Proof. To simplify notation, we set $A := B \times U(1)$ which is a $\mathbb{Z} - C^*$-algebra under the dual action. The class $[D_{A_{\mathbb{Z}^2}}] \otimes [\mathcal{G}_{\mathcal{E}_g}]$ can be represented by the $U(1)$-equivariant $A \times \mathbb{Z} - F$-Kasparov cycle

$$\langle D_{A_{\mathbb{Z}^2}} \otimes _A 1, (A \otimes L^2(S^1)) \otimes A \mathcal{G}_g \rangle.
$$

Using Proposition 3.2.8 we have an isomorphism of $A \times \mathbb{Z} - F$-Hilbert bimodules

$$\langle A \otimes L^2(S^1) \rangle \otimes _A \mathcal{G}_g \cong A \mathcal{G}_g \otimes L^2(S^1).$$

Define the unitary $U \in \text{End}_{_A \mathcal{G}_g} (A \mathcal{G}_g \otimes L^2(S^1))$ by representing the unitary $U_0 \in \mathcal{M}(C^*(U(1)) \otimes C(U(1)))$ which is defined as an operator on $L^2(U(1)) \otimes L^2(U(1))$ via

$$U_0 f(g, h) = f(gh, h).$$

The unitary $U$ implements Takesaki-Takai duality giving an isomorphism of $A \times \mathbb{Z} - F$-Hilbert bimodules

$$A \mathcal{G}_g \otimes L^2(S^1) \cong \mathcal{E}_g \otimes L^2(S^1).$$
Corollary 3.2.10. Under the mapping $KK_1(O_A, F_A) \to KK_1((F_A \otimes \mathbb{K}) \times \mathbb{Z}, F_A \otimes \mathbb{K})$ induced from the Morita equivalence $O_A \sim_M (F_A \otimes \mathbb{K}) \times \mathbb{Z}$, the gauge element $[D_B]$ is mapped to the Toeplitz element $[D_{\beta}]$. This Corollary follows directly from that $O_{SU(1)} \mathcal{G}_F$ is an imprimitivity bimodule implementing the Morita equivalence $O_A \times U(1) \sim_M F_A$. In general, we can conclude the following Corollary which implies Theorem 3.2.1.

Corollary 3.2.11. If $B$ has a circle action satisfying the spectral subspace assumption such that $B_{SU(1)} \mathcal{G}_F$ is a Morita equivalence, the following triangle is distinguished in $KK^{U(1)}$

$$
\begin{array}{ccc}
F & \xrightarrow{1-\beta} & F \\
\downarrow{\iota_\beta} & & \downarrow{\iota_\beta} \\
B_1 & & B_1
\end{array}
$$

where $\iota_\beta : F \to B$ denotes the inclusion and $\beta \in KK_0(F, F)$ is Morita equivalent to the $\mathbb{Z}$-action dual to the $U(1)$-action on $B$.

3.3. Computations and problems with the approach using the fixed point algebra. In this subsection we will compute $K$-groups in some examples of Cuntz-Krieger algebras and their fixed point algebras. These computations are well known, and are provided only as a basis for discussion regarding possibilities of constructing unbounded Fredholm modules with prescribed $K$-homology classes. In order to do so, we will require a Proposition giving a general formula for the $K$-theory and $K$-homology of the fixed point algebra.

Proposition 3.3.1. The $K$-theory of $F_A$ is given by

$$K_0(F_A) \cong \varprojlim(\mathbb{Z}^N, A^1) \quad \text{and} \quad K_1(F_A) \cong 0.$$ 

The $K$-homology of $F_A$ is given by

$$K^0(F_A) \cong \varprojlim A^1\mathbb{Z}^N \quad \text{and} \quad K^1(F_A) \cong \hat{\mathbb{Z}}_A^N/\varprojlim A^1\mathbb{Z}^N.$$ 

Here $\hat{\mathbb{Z}}_A^N := \varprojlim A^1\mathbb{Z}^N$ is the $A$-adic completion of $\mathbb{Z}^N$.

The computation of the $K$-homology of the fixed point algebra might not be as well known as the corresponding result in $K$-theory so we will sketch the proof. For a detailed proof in a special case, we refer to the note [33]. The proof relies on a result of Schochet-Rosenberg (see [57, Theorem 1.14]) stating that if $B = \varprojlim B_i$ there is a graded short exact sequence

$$0 \to \varprojlim K^{++1}(B_i) \to K^*(B) \to \varprojlim K^*(B_i) \to 0.$$ 

We can directly conclude from (3.25) and the AF-structure of $F_A$ that

$$K^0(F_A) \cong \varprojlim(\mathbb{Z}^N, A) \quad \text{and} \quad K^1(F_A) \cong \varprojlim(\mathbb{Z}^N, A).$$

These isomorphisms are simplified further using the explicit construction of derived projective limits in the category of abelian groups, see for instance [61 Chapter 3.5].
3.3.1. The algebra $O_N$. The algebra $O_N$, also known as the Cuntz algebra, was recalled above in Subsubsection 1.3.1. We let $F_N$ denote the fixed point algebra in $O_N$.

Proposition 3.3.2. It holds that

$$K_0(F_N) \cong \mathbb{Z} \left[ \frac{1}{N} \right], \quad K^1(F_N) \cong \mathbb{Z}/N\mathbb{Z} \quad \text{and} \quad K_1(F_N) \cong K^0(F_N) \cong 0.$$ 

We use the notation $\mathbb{Z} \left[ \frac{1}{N} \right]$ for the ring generated by $\frac{1}{N}$ and $\mathbb{Z}/N\mathbb{Z}$ for the $N$-adic completion of $\mathbb{Z}$.

Proof. We let $w := (1, 1, \ldots, 1)^T \in \mathbb{Z}^N$ and $\ell := (1, 1, \ldots, 1) \in \text{Hom}(\mathbb{Z}^N, \mathbb{Z})$. It holds that $A = w \otimes \ell$. For any $k \in \mathbb{N}_+$, $A^k x = N^{k-1} \ell(x)w$ for any $x \in \mathbb{Z}^N$. Hence $K_0(F_N) \cong \lim_K(\mathbb{Z}/N) = \mathbb{Z}[N^{-1}]$. Similarly, $K^0(F_N) \cong \lim_K(\mathbb{Z}/N) = 0$. It also follows that $\mathbb{Z}^N/A^k\mathbb{Z}^N = \mathbb{Z}^{N-1} \oplus \mathbb{Z}/N^{k-1}\mathbb{Z}$. Hence

$$\mathbb{Z}^N = \lim_K(\mathbb{Z}^N/A^k\mathbb{Z}^N) = \mathbb{Z}^{N-1} \oplus \mathbb{Z}/N = K^1(F_N) \cong (\mathbb{Z}^{N-1} \oplus \mathbb{Z}/N)/\mathbb{Z}^N = \mathbb{Z}/N.$$ 

The isomorphism $K_0(F_N) \cong \mathbb{Z} \left[ \frac{1}{N} \right]$ is implemented by the tracial state $\varphi_N : F_N \to \mathbb{C}$ given by restricting the KMS-state on $O_N$ to $F_N$.

Remark 3.3.3. The well known computations

$$K_0(O_N) \cong K^1(O_N) \cong \mathbb{Z}/(N-1)\mathbb{Z} \quad \text{and} \quad K_1(O_N) \cong K^0(O_N) \cong 0,$$

follow from Proposition 3.3.2 and the Pimsner-Voiculescu sequence (Theorem 3.2.1). In particular, we arrive at the short exact sequence for the only non-vanishing $K$-homology group $K^1(O_N)$:

$$0 \to K^1(O_N) \to \mathbb{Z}/N \to \mathbb{Z}/N \to 0.$$ 

It follows that the Kasparov product with the gauge class of $O_N$ on $K$-homology vanishes.

3.3.2. The quantum group $SU_q(2)$. Recall that $C(SU_q(2))$ is isomorphic to the Cuntz-Krieger algebra constructed from the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, as in Subsubsection 1.4.1.

Proposition 3.3.4. When $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, it holds that

$$K_0(F_A) \cong K^0(F_A) \cong \mathbb{Z}^2 \quad \text{and} \quad K_1(F_A) \cong K^1(F_A) \cong 0.$$ 

This Proposition follows directly from Proposition 1.4.6 or the computation of the $K$-groups, in Proposition 3.3.1 since $A$ is invertible. The $K$-theory and $K$-homology for $O_A$ is in this case given by

$$K_0(O_A) \cong K^1(O_A) \cong K^0(O_A) \cong K^1(O_A) \cong \mathbb{Z},$$

as can be seen from the Pimsner-Voiculescu sequences

$$0 \to K_1(O_A) \xrightarrow{\otimes[D]} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \mathbb{Z}^2 \to K_0(O_A) \to 0,$$

$$0 \to K^0(O_A) \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{[D]} \mathbb{Z}^2 \to K^1(O_A) \to 0.$$
Remark 3.3.5. We conclude that the Kasparov product with the gauge class surjects onto the odd K-homology group of $SU_r(2)$. As the fixed point algebra is the unitalization of the algebra of compacts, see Proposition 3.4.3, the problems we consider in this paper verge on trivial for this case; it is not difficult to construct unbounded Fredholm modules on the unitalization of the algebra of compacts with both good analytic and topological properties.

3.3.3. The crossed product $C(\partial F_d) \rtimes F_d$. Let $F_d$ denote the free group on $d$ generators that we denote by $\{\gamma_1, \ldots, \gamma_d\}$. The boundary of $F_d$ consists of infinite words in the alphabet given by the generators $\{\gamma_1, \ldots, \gamma_d, \gamma_1^{-1}, \ldots, \gamma_d^{-1}\}$ subject to the condition that for any $i$, the letters $\gamma_i$ and $\gamma_i^{-1}$ cannot succeed each other. The group $F_d$ is a hyperbolic group thus acting amenably on its boundary $\partial F_d$, see for instance [11]. Hence $C(\partial F_d) \rtimes F_d \cong C(\partial F_d) \rtimes_r F_d$.

Proposition 3.3.6. The crossed product $C(\partial F_d) \rtimes F_d$ is the Cuntz-Krieger algebra constructed from the symmetric $2d \times 2d$-matrix

\[
A := \begin{pmatrix}
1 & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 1 & 1 & 1 & \cdots & \vdots & \\
1 & 1 & 0 & 1 & \cdots & \vdots & \\
1 & 1 & 0 & 1 & \cdots & \vdots & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \\
1 & 1 & \cdots & 1 & 0 \\
1 & 1 & 1 & \cdots & 0 & 1
\end{pmatrix}
\]

In particular, $\mathcal{V}_A$ is the Cayley graph of $F_d$.

It suffices to provide an isomorphism of groupoids $\varphi : \partial F_d \rtimes F_d \xrightarrow{\sim} \mathcal{G}_A$ for this specific choice of matrix $A$. Such an isomorphism is given by $\varphi(x, \gamma) := (x, n(x, \gamma), x\gamma)$ where $n(x, \gamma)$ is the difference of the word length of $\gamma$ and the number of reductions necessary in $x\gamma$ to put it into reduced form.

Proposition 3.3.7. It holds that

\[
K^*((C(\partial F_d) \rtimes F_d)^{(2d)(1)}) \cong \begin{cases}
\mathbb{Z}^{d-1}, & *=0, \\
\mathbb{Z}_{2d-1}, & *=1
\end{cases}
\quad \text{and}
\]

\[
K^*(C(\partial F_d) \rtimes F_d) \cong \begin{cases}
0, & *=0, \\
(\mathbb{Z}/2\mathbb{Z})^{2(d-1)} \oplus \mathbb{Z}/4(d-1)\mathbb{Z}, & *=1.
\end{cases}
\]

The proof of Proposition 3.3.7 follows from Proposition 3.3.1 and Theorem 3.2.1 after a lengthier exercise in linear algebra.

Remark 3.3.8. Writing out the Pimsner-Voiculescu sequence of Theorem 3.2.1 using the computations of Proposition 3.3.7, we arrive at a commuting diagram whose rows are exact:

\[
\begin{array}{c}
0 \to K^0(\mathcal{O}_A) \to K^0(\gamma A) \to K^0(\mathcal{O}_A) \to 0 \\
\| \| \| \| \\
0 \to \mathbb{Z}^{d-1} \to \mathbb{Z}^{d-1} \to (\mathbb{Z}/2\mathbb{Z})^{2(d-1)} \oplus \mathbb{Z}/4(d-1)\mathbb{Z} \to \mathbb{Z}_{2d-1} \to \mathbb{Z}_{2d-1} \to 0
\end{array}
\]

Since the $2d-1$-adic numbers $\mathbb{Z}_{2d-1}$ is a torsion-free group, it follows that the mapping $K^1(\mathcal{O}_A) \to K^1(\gamma A)$ vanishes. We conclude that the Kasparov product with the gauge class in fact surjects onto the K-homology of the Cuntz-Krieger algebra $C(\partial F_d) \rtimes F_d$. 
4. An even spectral triple on the algebra $C(\Omega_A)$

In [6] a family of even spectral triple was defined for boundaries of trees. While the space of finite words $\mathcal{Y}_A$ is a tree and $\Omega_A$ is its boundary, even spectral triples for $C(\Omega_A)$ can be obtained in this way. We will in this section recall the construction of [6] and prove that the spectral triples obtained in this way pair non-degenerately with many elements in $K_0(C(\Omega_A))$. They encode geometric, measure-theoretic and dynamical data (see [6, 61]), and have the interesting property that the class $2[1_{0,r}] \in K_0(O_{A'})$ obstructs the extension of these spectral triples to $O_A$ (see Proposition 4.2.4). We also interpret these spectral triples as secondary invariants for the triples obtained in this way pair non-degenerately with many elements in $K_0(C(\Omega_A))$ (see Remark 4.2.7). In the next section we will consider generalized unbounded Fredholm modules for $O_A$ constructed through the unbounded Kasparov product, using the present spectral triples as the base.

4.1. The Bellissard-Pearson spectral triples. The geometric setup of [6] was to define a spectral triple on the boundary of a tree by means of interior properties in the tree. In our situation the tree is $\mathcal{Y}_A$ and its boundary is $\Omega_A$. The key geometric idea, that transfers geometry on the interior to that on the boundary, is that of a choice function.

**Definition 4.1.1** (Choice functions on finite words). Let $t, t' : \mathcal{Y}_A \to \Omega_A$ denote functions. We say that the pair $\tau = (t, t')$ is comparable if there is a constant $C > 0$ such that $\tau$ satisfies that

$$d_{\Omega_A}(t(\mu), t'(\mu)) \leq C \text{diam}(\mu).$$

If the inequality is an equality with $C = 1$ for all $\mu$, we say that $t$ and $t'$ are strictly comparable. A comparable pair of functions satisfying the cylinder condition, see Definition 1.1.2, is called a weak choice function. If the comparison is strict we say $\tau$ is a choice function.

For a function $t : \mathcal{Y}_A \to \Omega_A$, we let $\pi_t : C(\Omega_A) \to \mathcal{B}(\ell^2(\mathcal{Y}_A))$ denote the composition of the pullback homomorphism $t^* : C(\Omega_A) \to C_b(\mathcal{Y}_A)$ with the representation given by point wise multiplication $C_b(\mathcal{Y}_A) \to \mathcal{B}(\ell^2(\mathcal{Y}_A))$. Cf. Proposition 1.1.3 if $t$ satisfies the cylinder condition.

**Definition 4.1.2** (The Bellissard-Pearson spectral triple [6]). Let $\tau = (\tau_+, \tau_-) : \mathcal{Y}_A \to \Omega_A \times \Omega_A$ be a comparable pair. The associated even Bellissard-Pearson spectral triple $BP^{\tau_+}(\tau) := (\pi_{\tau}, \ell^2(\mathcal{Y}_A, \mathbb{C}^2), D_{BP}^{\tau_+})$ consists of

1. The Hilbert space $\ell^2(\mathcal{Y}_A, \mathbb{C}^2)$ is graded by the decomposition

$$\ell^2(\mathcal{Y}_A, \mathbb{C}^2) = \ell^2(\mathcal{Y}_A) \oplus \ell^2(\mathcal{Y}_A).$$

2. The even representation $\pi_{\tau} : C(\Omega_A) \to \mathcal{B}(\ell^2(\mathcal{Y}_A, \mathbb{C}^2))$ is given by

$$\pi_{\tau} := \pi_{\tau_+} \oplus \pi_{\tau_-}.$$

3. The self-adjoint operator $D^{BP}_{\tau_+}$ is defined on its core $C_c(\mathcal{Y}_A, \mathbb{C}^2)$ by

$$D^{BP}_{\tau_+} \left( \begin{array}{c} \varphi_+ \\ \varphi_- \end{array} \right)(\mu) := \text{diam}(\mu)^{-1} \cdot \left( \begin{array}{c} \varphi_-(\mu) \\ \varphi_+(\mu) \end{array} \right) = e^{i|\mu|} \left( \begin{array}{c} \varphi_-(\mu) \\ \varphi_+(\mu) \end{array} \right).$$

We also define the logarithmic family of Bellissard-Pearson spectral triples

$$BP_s(\tau) := (\pi_{\tau}, \ell^2(\mathcal{Y}, \mathbb{C}^2), D_{BP_s})$$

for $s \in (0, 1]$, where the operator $D_{BP_s}$ is defined on its core $C_c(\mathcal{Y}_A, \mathbb{C}^2)$ by the expression

$$D_{BP_s} \left( \begin{array}{c} \varphi_+ \\ \varphi_- \end{array} \right)(\mu) := (-\log \text{diam}(\mu))^{s} \cdot \left( \begin{array}{c} \varphi_-(\mu) \\ \varphi_+(\mu) \end{array} \right) = |\mu|^s \left( \begin{array}{c} \varphi_-(\mu) \\ \varphi_+(\mu) \end{array} \right).$$
Remark 4.1.3. The construction of spectral triples in [6] was only carried out for choice functions and the logarithmic version was not considered. The results in [6] regarding these spectral triples were more concerned with metric and measure-theoretic properties. The motivation to lax the conditions on \( \tau \) stems from the wish to obtain a larger variety of K-homology classes that pairs in the index pairing non-degenerately with “many” K-theory elements. The introduction of the logarithmic version of the spectral triple is a matter we return to in the next subsection and Subsection 5.4.

Proposition 4.1.4. The logarithmic and the ordinary even Bellissard-Pearson spectral triples of a comparable pair \( \tau = (\tau_+, \tau_-) \) form even unbounded Fredholm modules. For \( s \geq \frac{1}{2} \), \( BP_{\tau}(\tau) \) is \( \theta \)-summable, whereas \( BP_{\exp}(\tau) \) is finitely summable. If the image of \( \tau \) is dense\(^4\) the Bellissard-Pearson spectral triples indeed form spectral triples.

Proof. It was proven in [6] Proposition 8 that \((\tau_+, \ell^2(\mathcal{Y}_\Omega), \mathcal{D}_{BP}^+)\) is a well defined spectral triple on the algebra \( \text{Lip}(\Omega_\Lambda, \mathcal{D}_{BP}) \) consisting of functions \( f : \Omega_\Lambda \to \mathbb{C} \) that are Lipschitz in the metric on \( \Omega_\Lambda \) defined in (1.2). From the estimate

\[
\| [D_{\mathcal{Y}_\Omega}, \pi_\tau(f)] \|_{\mathcal{B}(\ell^2(\mathcal{Y}_\Omega, \mathbb{C}^2))} = \sup_{\mu \in \mathcal{Y}_\Omega} |\mu|^p \left\| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} f(\tau_+(\mu)) & 0 \\ 0 & f(\tau_-(\mu)) \end{bmatrix} \right\|_{M_2(\mathbb{C})} \leq \sup_{\mu \in \mathcal{Y}_\Omega} e^{|\mu|^p} \left\| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} f(\tau_+(\mu)) & 0 \\ 0 & f(\tau_-(\mu)) \end{bmatrix} \right\|_{M_2(\mathbb{C})},
\]

it follows that the same holds for \( BP_\tau(\tau) \). Since \( \text{diam}(C_\mu) = e^{\frac{|\mu|}{\delta}} \), it follows that

\[
\text{Tr}(e^{-tD_{\mathcal{Y}_\Omega}^+}) = \sum_{\mu \in \mathcal{Y}_\Omega} e^{-t|\mu|^p} = \sum_{k=0}^{\infty} \sum_{|\mu| = k} e^{-tk^2}.
\]

By Corollary 4.1.6 the operator \( e^{-tD_{\mathcal{Y}_\Omega}^+} \) is trace class if \( s \geq \frac{1}{2} \) and \( t > \delta_{\mathcal{Y}_\Omega} \).

Remark 4.1.5. For \( s = \frac{1}{2} \) the trace (4.27) equals the Poincaré series from theorem 1.6.5. After introducing a power in the metric defined in Equation (1.2) and in the expression defining \( \mathcal{D}_{BP}^+ \), one can obtain arbitrarily low degree of finite summability. Further, if there are constants \( C, p > 0 \) such that \( \varphi(k) \leq Ck^p \) for all \( k \), also \( BP_\tau(\tau) \) is finitely summable. This holds for instance for \( SU_q(2) \) by (1.17). This is not possible if \( \Omega_\Lambda \) contains no isolated points as the following Proposition shows.

Proposition 4.1.6. If the matrix \( A \) satisfies condition (1), there are \( C, \varepsilon > 0 \) such that

\[
\varphi(k) \geq Ce^{\varepsilon k}.
\]

Remark 4.1.7. For any point \( x \in \Omega_\Lambda \), let \( \varepsilon_x : \text{C}(\Omega_\Lambda) \to \mathbb{C} \) denote point evaluation in \( x \). Let \([x] \in K^0(\text{C}(\Omega_\Lambda))\) denote the K-homology class associated with \( \varepsilon_x \). Formally, we may realize the K-homology class that the Bellissard-Pearson spectral triple defines as the formal difference of the sum of all \([x]\), where \( x \) ranges over \( \tau_+(\mathcal{Y}_\Lambda) \), and the sum of all \([x]\), where \( x \) ranges over \( \tau_-(\mathcal{Y}_\Lambda) \).

This can be made sense of in a more rigorous way. For \( \mu \in \mathcal{Y}_\Lambda \), the difference \([\tau_+(\mu)] - [\tau_-(\mu)] \in K^0(\text{C}(\Omega_\Lambda))\) can be represented by the unbounded Fredholm module

\[
\mathcal{E}_{\mu, \tau} = \left( \varepsilon_{\tau_+(\mu)} \oplus \varepsilon_{\tau_-(\mu)}, \mathbb{C}^2, \begin{bmatrix} 0 & |\mu|^p \\ |\mu|^p & 0 \end{bmatrix} \right).
\]

\(^4\)E.g., when \( \tau \) satisfies the cylinder condition.
The direct sum
\[
\bigoplus_{\mu \in \mathcal{Y}_s} \mathcal{G}_{\mu,s} = \left( \bigoplus_{\mu \in \mathcal{Y}_s} \mathbb{C}^1, \bigoplus_{\mu \in \mathcal{Y}_s} \mathbb{C}^2, \bigoplus_{\mu \in \mathcal{Y}_s} \begin{pmatrix} 0 & |\mu| \nu \\ 0 & 0 \end{pmatrix} \right) = BP_s(\tau)
\]

is well defined once making suitable closures and choices of domains.

4.2. **Obstructions to extending to Cuntz-Krieger algebras.** Our motivation for introducing the logarithmic version of the Bellissard-Pearson spectral triple is that it extends to a slightly larger algebra related to the Cuntz-Krieger algebra, but not equal to it. The deficiency between that algebra and the Cuntz-Krieger algebra comes from an obstruction in $K_0(\mathcal{O}_\Lambda)$.

We let $V_\sigma \in \mathcal{B}(\ell^2(\mathcal{Y}_A))$ be defined by
\[
V_\sigma f(\nu) = \begin{cases} f(\sigma_\nu(v)) & \text{if } \nu \neq \sigma_0 \\ 0 & \text{if } \nu = \sigma_0. \end{cases}
\]

A direct computation gives the identity
\[
V_\sigma^* V_\sigma = S,
\]
where $S f(x) = |\sigma_\nu^{-1}(x)| f(x)$. We will henceforth apply the convention that
\[
\sigma_\nu^{-1}(\sigma_\nu(\nu)) = 0.
\]

Assume that $t : \mathcal{Y}_A \to \Omega_A$ is function satisfying the cylinder condition (see Definition 1.1.2). We define the operators $s_{i,t} \in \mathcal{B}(\ell^2(\mathcal{Y}_A))$ for $i = 1, \ldots, n$ by
\[
s_{i,t} := \pi_t(\chi_C) V_\sigma.
\]

We also let $P_\sigma : \ell^2(\mathcal{Y}_A) \to \ell^2(\mathcal{Y}_A)$ denote the orthogonal projection onto the space spanned by $\delta_{\sigma_0}$.

**Lemma 4.2.1.** Let $t : \mathcal{Y}_A \to \Omega_A$ be a function satisfying the cylinder condition. The operators $s_{i,t}$ are partial isometries satisfying the relations
\[
(4.28) \quad s_{i,t}^* s_{k,t} = \delta_{i,k} \sum_{j=1}^N A_{i,j} s_{j,t} s_{j,t}^* + P_\sigma,
\]
for any $i$ and $k$.

**Proof.** To prove that $s_{i,t}$ are partial isometries it suffices to prove that $s_{i,t} s_{i,t}^*$ are projections. For $f \in \ell^2(\mathcal{Y}_A)$ it holds that
\[
\pi_t(\chi_C) V_\sigma^* \pi_t(\chi_C) f(\mu) = \chi_C(t(\mu)) \sum_{v \in \sigma_\nu^{-1}(\sigma_\nu(\mu))} \chi_C(t(v)) f(v) = \chi_C(t(\mu)) f(\mu),
\]
since the cylinder condition guarantees that if $(\mu) \in \mathcal{C}_i$, $\mu$ is the unique pre image of $\sigma_\nu(\mu)$ under $\sigma_\nu$ with first letter $i$. As a consequence
\[
s_{i,t}^* s_{i,t} = \pi_t(\chi_C) V_\sigma^* \pi_t(\chi_C) V_\sigma \pi_t(\chi_C) = \pi_t(\chi_C) V_\sigma V_\sigma^* \pi_t(\chi_C) = s_{i,t} s_{i,t}^*.
\]
Since $\chi_C \chi_C = 0$ if $i \neq j$, it follows that
\[
s_{i,t}^* s_{j,t} = V_\sigma^* \pi_t(\chi_C) V_\sigma = 0
\]
i \neq j$. Given $f \in \ell^2(\mathcal{Y}_\alpha)$ we have that
\[
s_{t, t}^* s_{t, t} f(\mu) = V_{C}^* \sigma_{t} (\chi_{C}) f(\sigma_{t} (\mu)) = \sum_{v \in \sigma_{t}^{-1}(\mu)} \chi_{C}(t(v)) f(\sigma_{t}(v)) = \sum_{v \in \sigma_{t}^{-1}(\mu), t(v) \in C} f(\mu) = \sum_{j=1}^{N} A_{ij} \chi_{C}(t(\mu)) f(\mu),
\]

since the word $v = i \mu \in \sigma^{-1}(\mu)$ is admissible only when $A_{i \mu, i} \neq 0$. On the other hand,
\[
s_{t} s_{t}^* f(\mu) = \pi_{t} (\chi_{C}) V_{\nu} V_{C}^* \sigma_{t} (\chi_{C}) f(\mu) = \sum_{v \in \sigma_{t}^{-1}(\nu)} \chi_{C}(t(\mu)) \chi_{C}(t(v)) f(v) = \begin{cases} \chi_{C}(t(\mu)) f(\mu), & \text{if } \mu \neq o_{A}, \\ 0 & \text{if } \mu = o_{A}. \end{cases}
\]

Rewriting this, we obtain the identity
\[
s_{t, t}^* s_{t, t} = \begin{cases} \pi_{t} (\chi_{C}) - P_{A}, & \text{if } t(\nu) \in C, \\ \pi_{t} (\chi_{C}), & \text{if } t(\nu) \notin C. \end{cases}
\]
Since there is only one $i$ for which $t(\nu) \in C$, Equation (4.28) holds true. \qed

Remark 4.2.2. A geometric consequence of Lemma 4.2.1 for $\mathcal{B}_A$. Later we will prove that for any function $t$ with the cylinder condition, the linear mapping $S_{t} \to s_{t, t}$ cannot be compactly perturbed to a $*$-homomorphism $O_{A} \to \mathbb{B}(\ell^2(\mathcal{Y}_\alpha))$ if $[1] \neq 0$ in the $K$-theory group $K_0(O_{A'})$. This is related to Kaminker-Portman’s Poincaré duality $K^*(O_{A}) \cong K_{*+1}(O_{A'})$. See more in Remark 2.1.3 Proposition 4.2.9 and Proposition 4.2.4. In particular, it proves impossible for a function $t : \mathcal{Y}_\alpha \to O_{A}$ satisfying the cylinder condition to be viewed as the moment map of a $\mathcal{B}_A$-action on the finite words $\mathcal{Y}_\alpha$ since if that was the case, it would extend to a $*$-homomorphism $O_{A} \cong C^*(\mathcal{B}_A) \to \mathbb{B}(\ell^2(\mathcal{Y}_\alpha))$.

In order to understand the role of the operators $\{s_{i, t} : t \in \mathcal{T}_{i, t}\}$ we need to relate them to a similar set of operators appearing above in Subsection 2.1, cf. [47].

Proposition 4.2.3. If $t$ satisfies the cylinder condition, it holds that $L_{t}^4 = s_{t, t}$.

Proof. For any finite word $\mu \in \mathcal{Y}_\alpha$,
\[
s_{i, t} \delta_{i \mu} = \sum_{v \in \sigma_{t}^{-1}(\mu)} \chi_{C}(t(v)) \delta_{i \nu} = \delta_{i \mu}.
\]
since the cylinder condition (see Definition 1.1.2) guarantees that $v \neq i \mu$ is the unique word in $
\sigma^{-1}_{v}(\mu)$ such that $\chi_{C}(t(v)) \neq 0$. \qed

A consequence of Proposition 4.2.3 is that the operators $s_{i, t}$ do not depend on the choice of $t$. The computations of Lemma 4.2.1 follow from Proposition 4.2.3 and [47] Proposition 4.2. By the element $[1_{O_{A'}}] \in K_{0}(O_{A'})$ is 2-torsion, the $K$-homological obstruction to lifting the mapping $S_{i} \to s_{i} \mod \mathbb{K} \in \mathcal{E}(\ell^2(\mathcal{Y}_\alpha, \mathbb{C}^2))$ to a $*$-homomorphism $O_{A} \to \mathbb{B}(\ell^2(\mathcal{Y}_\alpha, \mathbb{C}^2))$, vanishes. Recall that $\beta_{A} : O_{A} \to \mathcal{E}(\ell^2(\mathcal{Y}_\alpha))$ was defined by $S_{i} \to L_{i}^{A} \mod \mathbb{K}(\ell^2(\mathcal{Y}_\alpha))$ using Proposition 1.2.2. In light of Proposition 4.2.3 $[\beta_{A}] \in \text{Ext}(O_{A}, \mathbb{K})$ is Poincaré dual to the element $[1_{O_{A'}}] \in K_{0}(O_{A'})$. Hence we can conclude the following.

Proposition 4.2.4. Assume that $k$ is such that $k[1_{O_{A'}}] \neq 0$, then for functions $t_{1}, \ldots, t_{k} : \mathcal{Y}_\alpha \to O_{A}$ satisfying the cylinder condition the representation
\[
\Phi_{j=1}^{k} \pi_{i_{j}} : C(\mathcal{Y}_\alpha) \to \mathbb{B}(\ell^2(\mathcal{Y}_\alpha, \mathbb{C}^{k})),
\]
or any compact perturbation of it, does not extend to a representation of $O_{A}$. 

Remark 4.2.5. If $K^0(F_A) = 0$, it follows from Theorem 3.2.4 that $K^1(O_A) \to K^1(F_A)$ is injective. This happens for instance for the algebra $O_A$ as we saw above in Remark 3.2.8. In this particular case, the obstruction mentioned in Proposition 1.2.4 to lifting the representation of $C(O_A)$ in the Bellissard-Pearson spectral triples remains for $F_A$.

**Proposition 4.2.6.** If $t : \mathcal{Y}_A \to \Omega_A$ is a function satisfying the cylinder condition, the representation $\pi_t$ of $C(\Omega_A)$ satisfies that

$$q \circ \pi_t = \beta_A|_{C(\Omega_A)},$$

and hence $[\beta_A]|_{C(\Omega_A)} = 0$ in $K^1(C(\Omega_A))$.

**Remark 4.2.7.** An interesting interpretation of this Proposition is that the Bellissard-Pearson spectral triples should be thought of as an invariant of the homological triviality of the Toeplitz extension associated with the restriction of $\beta_A$ to $C(\Omega_A)$, i.e. a secondary invariant.

**Proposition 4.2.8.** For $i = 1, \ldots, n$, the operator $s_i := s_{i, \tau_s} \oplus s_{i, \tau_s}$ and its adjoint

1. preserve $C_0(\mathcal{Y}_A, \mathbb{C}^2)$;
2. admit bounded commutators with $D_{\mathcal{Y}_A}$;
3. there is a sequence $(f_k) \subseteq C_c(\mathcal{Y}_A, \mathbb{C}^2)$ such that $\|f_k\| = 1$ but $\|[D_{\mathcal{Y}_A}, s_i]f_k\| \to \infty$.

**Proof.** Property (1) is clear from the definition of $s_i$. To prove (2), we note that Proposition 1.2.2 implies that

$$[D_{\mathcal{Y}_A}, s_i] \begin{pmatrix} \delta_u \\ \delta_v \end{pmatrix} = \begin{pmatrix} (|u|^2 - |v|^2) \delta_{iv} \\ (|\mu|^2 - |\xi|^2) \delta_{i\mu} \end{pmatrix}.$$

Since $\mu \mapsto -\log \text{diam}(C_\mu) = |\mu|$ grows linearly in $|\mu|$, $\mu \mapsto |i\mu|^2 - |\mu|^2$ is a bounded function for $0 < s \leq 1$. Hence $[D_{\mathcal{Y}_A}, s_i]$ is bounded.

Concerning (3), it follows from Proposition 1.2.3 that

$$[D_{\mathcal{Y}_A}^p, s_i] \begin{pmatrix} \delta_u \\ \delta_v \end{pmatrix} = \begin{pmatrix} (\text{diam}(C_u) - \text{diam}(C_v)) \delta_{iv} \\ (\text{diam}(C_{i\mu}) - \text{diam}(C_{i\xi})) \delta_{i\mu} \end{pmatrix}.$$

Take a sequence $(\mu_k)_{k=1}^{\infty} \subseteq \mathcal{Y}_A$ such that $|\mu_k| = k$ and $i\mu_k$ is admissible for all $k$. Set $f_k := (\delta_{\mu_k}, 0)^T$. It trivially holds that $f_k \in C_c(\mathcal{Y}_A, \mathbb{C}^2)$ and that $\|f_k\| = 1$. Since $\text{diam}(C_\mu) = e^{-|\mu|}$, there is an $\varepsilon > 0$ for which

$$\frac{\text{diam}(C_\mu)}{\text{diam}(C_{i\mu})} > 1 + \varepsilon.$$

We conclude that

$$\|[D_{\mathcal{Y}_A}^p, s_i]f_k\| \geq \varepsilon \text{diam}(C_\mu)^{-1} \to \infty, \quad \text{as} \quad |\mu| \to \infty.$$

Consider the $C^*$-algebra

$$E_{BP} := C^*(s_i | i = 1, \ldots, n) + K(\ell^2(\mathcal{Y}_A)) \subseteq \mathbb{K}(\ell^2(\mathcal{Y}_A)).$$

As a consequence of Proposition 1.2.8, the operator $D_{\mathcal{Y}_A}$ defines a spectral triple on $E_{BP}$ which is $\theta$-summable for $s \geq 1/2$. Yet another consequence is that $D_{\mathcal{Y}_A}^p$ does not define a spectral triple on $E_{BP}$ such that $s_{i,t}$ is in the Lipschitz algebra. In the light of Theorem 1.3.3 and Proposition 1.1.3 this result does not come as a surprise as in that case we would obtain a finitely summable spectral triple on $E_{BP}$. We do however note that there is no obvious obstruction to finitely summable spectral triples on $E_{BP}$ since it is never purely infinite by [14, Proposition V.2.2.23] and the fact that $\mathbb{K}(\ell^2(\mathcal{Y}_A)) \subseteq E_{BP}$. By arguments similar to those in Subsection 2.1, $E_{BP}/\mathbb{K}(\ell^2(\mathcal{Y}_A)) \cong O_A$. We can conclude the following Proposition from Remark 2.1.4.
**Proposition 4.2.9.** The extension $E_{bp}$ represent the image of $[1_{O_A}] \in K_0(O_A)$ under the isomorphism $K_0(O_A) \to K^1(O_A)$ of Theorem 4.2.11. We end this subsection by a comparison of various description of the extension dual to $[1_{O_A}] \in K_0(O_A)$ in the special case of a free group. This example, described above in Subsection 3.3.3, falls into the category of extensions studied by Emerson-Nica [24]. The extension constructed in [24] is defined from the short exact sequence

$$0 \to C_0(F_d) \times F_d \to C(F_d) \times F_d \to C(\partial F_d) \times F_d \to 0,$$

which is exact since the $F_d$-action on its boundary is amenable. Using the isomorphism $C_0(F_d) \times F_d \approx \mathbb{K}(t^2(F_d))$ we obtain an extension $E_{en}$ whose class was proven in [27] to be dual to $[1_{O_A}] \in K_0(O_A)$. In [27] an explicit finitely summable analytic $K$-cycle representing this extension class was prescribed. Recall the measure $\mu_A$ on $\partial F_d$ constructed as in Subsection 1.11. Let $P_{en}$ be the orthogonal projection onto the image of the isometric embedding $t^2(F_d) \to t^2(F_d, L^2(\partial F_d, \mu_A))$ as constant functions on $\partial F_d$. By [27] Theorem 1.1 the class $[E_{en}]$ is represented by the finitely summable analytic cycle $(\pi_{F_d}, t^2(\partial F_d, L^2(\partial F_d, \mu_A)), 2P_{en} - 1)$, where $\pi_{F_d}$ is the crossed product representation associated with the covariant $C(\partial F_d)$-representation on $t^2(F_d, L^2(\partial F_d, \mu_A))$. One can check that this construction of $P_{en}$ corresponds to the construction of $\psi_\alpha$ in Remark 2.2.4. By our considerations above we have the following Proposition.

**Proposition 4.2.10.** If $A$ is the $2d \times 2d$-matrix from Subsection 3.3.3, the following diagram with exact rows commute:

$$
\begin{array}{cccc}
0 & \to & \mathbb{K}(t^2(F_d)) & \to & E_{bp} & \to & O_A & \to & 0 \\
0 & \to & \mathbb{K}(t^2(F_d)) & \to & E_{en} & \to & O_A & \to & 0
\end{array}
$$

Furthermore, under the unitary equivalence $L^2(O_A, \varphi_\alpha) \cong t^2(F_d, L^2(\partial F_d, \mu_A))$ induced by the isomorphism of groupoids $G_A \cong \partial F_d \rtimes F_d$ it holds that

$$(\pi_{A}, L^2(O_A, \varphi_\alpha), 2W_0W_0^* - 1) = (\pi_{F_d}, t^2(F_d, L^2(\partial F_d, \mu_A)), 2P_{en} - 1)$$

**Remark 4.2.11.** For a general $N \times N$-matrix $A$, there are several other equivalent ways of constructing extensions equivalent to $E_{bp}$ in a geometric way from the short exact sequence

$$0 \to C_0(Y_\alpha) \to C(\overline{Y}_\alpha) \to C(\Omega_\alpha) \to 0.$$  

For instance, using crossed products by partial actions of the free group $F_N$ on $\Omega_\alpha$ (see [25]) or a crossed product by the shift endomorphism (see [29]).

### 4.3. K-homology classes

We now turn to the study of the index theory of the Bellissard-Pearson spectral triples. Whenever $(\mathcal{A}, \mathcal{H}, D)$ is an unbounded Fredholm module on a C$^*$-algebra $A$, we let $[\mathcal{A}, \mathcal{H}, D] \in K^0(A)$ denote the K-homology class associated with its bounded transform. We will throughout this subsection let $\tau = (\tau^+, \tau^-)$ denote a comparable pair of functions $\varphi_\alpha \in \Omega_\alpha$ and for the most of the section, $\tau$ will be a weak choice function.

**Lemma 4.3.1.** For $0 < s \leq 1$, the bounded transforms of the logarithmic and the ordinary even Bellissard-Pearson spectral triples coincide in K-homology:

$$[BP^{exp}_\tau(\tau)] = [BP_\tau(\tau)] \in K^0(C(\Omega_\alpha)).$$

Further, the class $[BP_\tau(\tau)] \in K^0(C(\Omega_\alpha))$ of a comparable pair $\tau$ can be represented by the analytic $K$-cycle

$$\left(\pi_{\tau}, t^2(\mathcal{V}_\alpha, \mathbb{C}^2), F\right), \text{ where } F := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
If \( \tau \) is a weak choice function this \( K \)-cycle is \( p \)-summable, for any \( p > 0 \), in the dense \(*\)-sub algebra generated by functions inside \( C(\Omega_\lambda) \).

**Proof.** It is clear that \([BP^{sp}(\tau)] = [BP_\pi(\tau)]\). That \([BP_\pi(\tau)] \in K^0(C(\Omega_\lambda))\) is represented by the \( K \)-cycle [3.30] is clear from that \( F = D_{BP}^{sp}|D_{BP}^{sp}|^{-1} \). To verify the \( p \)-summability claim, take a finite word \( \mu \in \mathcal{V}_\Lambda \) and consider the locally constant function \( \chi_c_\mu \in C(\Omega_\lambda) \). For any \( \nu, \nu' \in \mathcal{V}_\Lambda \) it holds that

\[
[F, \pi_-(\chi_c_\mu)] = \left( \begin{array}{c} \chi_c_\nu(\tau_-(\nu')) - \chi_c_\nu(\tau_+(\nu')) \\ \chi_c_\nu(\tau_+(\nu)) - \chi_c_\nu(\tau_-(\nu)) \end{array} \right).
\]

If both \( \tau_+ \) and \( \tau_- \) satisfies the cylinder condition,

\[
\chi_c_\nu(\tau_+(\nu)) - \chi_c_\nu(\tau_-(\nu)) = 0 \quad \text{if} \quad |\nu| \geq |\mu|.
\]

The latter statement holds true since \( \chi_c_\nu(\tau_+(\nu)) \) is non-zero if and only if \( \tau_\mu(\nu) \in C_\mu \) and whenever \(|\nu| \geq |\mu|\) the cylinder condition and \( \tau_\mu(\nu) \in C_\mu \) implies that there is a finite word \( \lambda \) with \( \nu = \mu \lambda \), hence \( \tau_\mu(\nu) \in C_\mu \) if and only if \( \tau_\mu(\nu) \in C_\mu \). It follows that \([F, \pi_-(\chi_c_\nu)]\) is an operator of rank at most \( \sum_{k|\nu|} \varphi(k) \) and hence \( p \)-summable for any \( p > 0 \). The linear span of the cylinder functions \( \{\chi_c_\nu | \mu \in \mathcal{V}_\Lambda \} \) forms a dense sub algebra of \( C(\Omega_\lambda) \) and the Lemma follows. \( \square \)

Lemma 4.3.1 gives us a description of the class \([BP(\tau)]\) by means of the quasi-homomorphism \( (\pi_+, \pi_-) \), cf. [23]. To understand the index pairing of the Bellissard-Pearson spectral triples with \( K \)-theory, we first need to divulse ourselves into a computation of the \( K \)-theory of \( C(\Omega_\lambda) \).

**Lemma 4.3.2.** The \( K \)-theory group \( K_*(C(\Omega_\lambda)) \) is given by

\[
K_*(C(\Omega_\lambda)) = \begin{cases} \mathbb{C}(\Omega_\lambda, \mathbb{Z}), & * = 0, \\ \mathbb{C}(\Omega_\lambda, \mathbb{C}), & * = 1. \end{cases}
\]

**Proof.** We write \( C(\Omega_\lambda) = \lim_{\to k} \mathcal{S}_k \) as in Proposition 1.1. Continuity of \( K \)-theory under direct limits implies that

\[
K_*(C(\Omega_\lambda)) = \lim_{\to k} K_*(\mathcal{S}_k) = \begin{cases} \lim_{\to k} K_0(\mathcal{S}_k), & * = 0, \\ \lim_{\to k} K_1(\mathcal{S}_k), & * = 1. \end{cases}
\]

\( \square \)

**Definition 4.3.3.** We say that a word \( \mu \in \mathcal{V}_\Lambda \) is minimal if

\[
(4.31) \quad \text{If } \nu_0, \lambda_0 \in \mathcal{V}_\Lambda \text{ satisfies } \mu = \nu_0 \lambda_0 \Rightarrow \chi_c_\mu \neq \chi_c_\nu_0.
\]

**Lemma 4.3.4.** Let \( \mu \in \mathcal{V}_\Lambda \) and let \( \nu_0 \) be the longest minimal word such that \( \mu = \nu_0 \lambda_0 \) for some \( \lambda_0 \). Then there is a weak choice function \( \tau = (\tau_+, \tau_-) \) such that whenever \( \nu, \lambda \in \mathcal{V}_\Lambda \setminus \{\mathcal{O}_\Lambda\} \) are such that \( \mu = \nu \lambda \), then

1. It holds that \( \tau_+(\nu) \in C_\mu \) if and only if \( \tau_-(\nu) \in C_\mu \) for \( |\lambda| \neq |\lambda_0| + 1 \)
2. It holds that \( \tau_-(\nu) \in C_\mu \) and \( \tau_+(\nu) \in C_\mu \) if \( |\lambda| = |\lambda_0| + 1 \).

**Proof.** Let \( \tau^0 \) be any weak choice function. We will redefine \( \tau^0 \) on the set of \( \nu \) such that there exists a \( \lambda \in \mathcal{V}_\Lambda \setminus \{\mathcal{O}_\Lambda\} \) with \( \nu = \nu \lambda \). Since \( \chi_c_\mu = \chi_c_\nu \) we can equally well assume \( \mu = \nu_0 \) and \( |\lambda_0| = 0 \).

Whenever \( \mu = \nu \lambda \), we divide into the four cases

A) \( |\lambda| > 1 \) and \( \tau^0_\nu(\nu) = \tau^0_\nu(\nu). \)
B) \( |\lambda| = 1 \) and \( \tau^0_\nu(\nu) \neq \tau^0_\nu(\nu). \)
C) \( |\lambda| > 1 \) and \( \tau^0_\nu(\nu) \neq \tau^0_\nu(\nu). \)
D) \( |\lambda| = 1 \) and \( \tau^0_\nu(\nu) = \tau^0_\nu(\nu). \)
If $\nu$ satisfies $A)$, we do not alter $\tau^0(\nu)$. If $\nu$ satisfies $B)$, and $\tau_+(\nu) \in C_\mu$ we do not alter $\tau^0(\nu)$. If $\nu$ satisfies $B)$, and $\tau_-(\nu) \in C_\mu$ we redefine $\tau_+(\nu) := \tau^0_+(\nu)$. If $\nu$ satisfies $B)$ and $\tau^0_+(\nu), \tau^0_-(\nu) \not\in C_\mu$, we do not alter $\tau^0_+(\nu)$ but define $\tau_+(\nu) := \tau^0_+(\nu)$. If $D)$ holds true, the minimality assumption $(4.31)$ guarantees that there is a finite word $\lambda'$ such that $|\lambda'| = |\lambda|$, $\lambda' \neq \lambda$ and $v \lambda'$ is admissible. Define $\tau_+(\nu) := \tau^0_+(\mu) \in C_\mu$ and $\tau_-(\nu) := \tau^0_-(\nu)$. It remains to be proven that the new function $\tau$ is a weak choice function. The functions $\tau_\pm$ satisfy the cylinder condition by construction. Since we are altering a comparable pair in finitely many points, the new pair is again comparable. $\square$

Our main result of this subsection indicates the topological importance of the Bellissard-Pearson spectral triples.

**Lemma 4.3.5.** For any non-empty word $\mu \in \gamma_\Lambda \setminus \{\alpha_\Lambda\}$ there is a weak choice function $\tau_\mu$ such that

$$\langle [\chi_{C_\mu}], [BP(\tau_\mu)] \rangle = 1.$$  

**Proof.** It suffices to prove the Lemma for finite words $\mu$ satisfying the minimality assumption $(4.31)$. A straightforward index manipulation gives the identities

$$\langle [\chi_{C_\mu}], [BP(\tau_\mu)] \rangle = \text{ind}(\tau^*_+(\chi_{C_\mu}) \chi_{C_\mu}) \chi_{C_\mu}) \rightarrow \text{ind}(\tau^*_+(\chi_{C_\mu}) \chi_{C_\mu}) \chi_{C_\mu}) =$$

$$= \text{ind}(\tau^*_+(\chi_{C_\mu}) \chi_{C_\mu}) \chi_{C_\mu}),$$

where the last index denotes the relative index of the pair of projections $(\tau^*_+(\chi_{C_\mu}), \tau^*_-(\chi_{C_\mu}))$. Using $(4.32)$ Proposition 2.2, it follows that

$$\langle [\chi_{C_\mu}], [BP(\tau_\mu)] \rangle = \text{Tr}_{\rho_\mu(g)}(\tau^*_+(\chi_{C_\mu}) - \tau^*_-(\chi_{C_\mu})) = \sum_{|v|<|\mu|} [\chi_{C_\mu}(\tau_+(v)) - \chi_{C_\mu}(\tau_-(v))]$$

$$(4.32) = \# \left\{ v \mid \tau_+(v) \in C_\mu, \tau_-(v) \not\in C_\mu \right\} - \# \left\{ v \mid \tau_-(v) \in C_\mu, \tau_+(v) \not\in C_\mu \right\}.$$  

The Lemma follows from $(4.32)$ and Lemma 4.3.4. $\square$

5. Unbounded $(O_\Lambda, C(\Omega_\Lambda))$-cycles and the associated spectral triples

In this section we will construct classes over the commutative base by combining the philosophies of Section 2 and Section 3. The advantage of using $C(\Omega_\Lambda)$ is that there are several well behaved $K$-homology classes, e.g. point evaluations and Bellissard-Pearson spectral triples. We will use these to construct unbounded Fredholm modules on Cuntz-Krieger algebras $O_\Lambda$ and prove that such unbounded Fredholm modules exhaust $K^1(O_\Lambda)$. After that we will describe the Kasparov products of the $(O_\Lambda, C(\Omega_\Lambda))$-cycles with the Bellissard-Pearson spectral triples.

5.1. An unbounded $(O_\Lambda, C(\Omega_\Lambda))$-cycle. We start this subsection with a structure analysis for the Haar module $\mathcal{U}_\Lambda^0$ over the commutative algebra $C(\Omega_\Lambda)$. Consider the filtration of $\mathcal{F}_\Lambda$ given by

$$(5.33) \quad \mathcal{F}_\Lambda^k := \{(x, n, y) \in \mathcal{F}_\Lambda : \sigma^{k+n}(x) = \sigma^k(y)\},$$

which forms a filtration by subsets such that:

1. Each set $\mathcal{F}_\Lambda^k$ is closed under under composition.
2. Inversion is a filtered operation in the sense that if $\xi = (x, n, y) \in \mathcal{F}_\Lambda^k$, then $\xi^{-1} \in \mathcal{F}_\Lambda^{n+k}$.
3. The filtering respects the cocycle grading: $\mathcal{F}_\Lambda^k = \bigcup_{n \in \mathbb{Z}} \mathcal{F}_n^k$ where $\mathcal{F}_n^k := \mathcal{F}_\Lambda^k \cap \mathcal{C}_\Lambda^{-1}(n)$.

In fact we can further decompose this filtration into a grading.
Lemma 5.1.1. The function

\[ \kappa : \mathcal{A} \to \mathbb{N} \]

\[ (x, n, y) \mapsto \min\{k : \sigma^{k+n}(x) = \sigma^k(y)\}, \]

is locally constant and hence continuous.

Proof. Recall the definition of the basic open sets from (1.5). Let \( \kappa(x, n, y) = k \) and take \( (U, n + k, k, V) \) with \( U := C_{x_{n+k}} \) and \( V = C_{y_{-n+k}} \). Since \( k \) is minimal, it follows that \( x_{n+k} \neq y_k \). Therefore it is clear that for any \( (x', n, y') \in (U, n + k, k, V) \), \( \kappa(x', n, y') = k \). So \( \kappa \) is locally constant. \( \square \)

Because \( \kappa \) is continuous, the sets \( \kappa^{-1}(k) \) are clopen in \( \mathcal{A} \). Therefore each \( \mathcal{A}_k \) decomposes as a disjoint union

\[ \mathcal{A}_k = \bigcup_{i=0}^{k} \kappa^{-1}(i), \]

compatible with the cocycle grading. Writing \( \mathcal{E} \)

\[ \mathcal{E} = \bigcup_{n \in \mathbb{Z}} \bigcup_{k \in \mathbb{N}} \mathcal{A}_k \]

\[ \text{and } C_n(\mathcal{A}) = \bigoplus_{n \in \mathbb{Z}} C_n(\mathcal{A}_n), \]

where the former is a disjoint union and the latter is a decomposition into \( C(\Omega) \)-submodules. For if \( f \in C_n(\mathcal{A}) \) and \( g \in C(\Omega) \), then

\[ f \ast g(x, n, y) = f(x, n, y)g(y), \]

so \( \text{supp}(f \ast g) \subseteq \text{supp} f \). If \( n + k < 0 \), \( \mathcal{A}_n^k = \emptyset \) and we use the convention \( C_n(\mathcal{A}_n^k) = 0 \) if \( n + k < 0 \).

After completion this gives a decomposition of the \( C^* \)-module \( \mathcal{E}_n^\Omega \) as

\[ \mathcal{E}_n^\Omega = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{k \in \mathbb{N}} \mathcal{E}_n^k. \]

We will now proceed to show that each \( \mathcal{E}_n^k \) is a finitely generated projective \( C(\Omega) \) module. Define the sets

\[ \mathcal{X}_{n, \mu} = \{ (x, n, y) : \kappa(x, n, y) = k, x \in \mathcal{C}_\mu, |\mu| = k + n \}, \]

whose characteristic function we denote by \( \chi_{n, \mu} \). We set

\[ \varphi_\lambda(l) = \#\{ \mu \in \gamma_\lambda : \mu \lambda \in \mathcal{A}, |\mu| = l - |\lambda| \}. \]

Recall the conditional expectation \( \rho : O_\lambda \to C(\Omega) \) defined in (1.7).

Lemma 5.1.2. For any finite word \( \lambda \in \gamma_\lambda \), and \( f \in C_n(\mathcal{X}_n^k) \) the column vectors

\[ v_{n,k,\lambda} := \left( \left( \chi_{n,\mu}^{k \lambda} \right)^* \right)_{|\mu|=n+k-|\lambda|} \in \left( C_n(\mathcal{X}_n^k)^{\varphi_{k+n}(\sigma^k)} \right)^* \subseteq C_n(\mathcal{X}_{n-k}^{|\mu|} \varphi_{k+n}), \]

satisfy

\[ v_{n,k,\lambda} \rho(v_{n,k,\lambda} \ast f) = \chi_{(\sigma^{-k} \lambda)^{-1} \varphi_{k+n}} \ast f. \]

In particular, under the inclusion \( \left( C_n(\mathcal{X}_n^k)^{\varphi_{k+n}} \right)^* \subseteq \text{Hom}_{C(\Omega)}(\mathcal{E}_n^k, C(\Omega)^{\varphi_{k+n}}) \), the following \( C(\Omega) \)-linear operators define isometries

\[ v_{n,k} := v_{n,k,\lambda} \in \text{Hom}_{C(\Omega)}(\mathcal{E}_n^k, C(\Omega)^{\varphi_{k+n}}). \]
First we compute, for arbitrary \( \mu \),

\[
\rho \left( \left( \chi^k_{\mu, \nu} \right)^\ast f \right)(x,0,x) = \sum \left( \chi^k_{\mu, \nu} \right)^\ast(x, \ell, z)f(z, -\ell, x)
\]

and subsequently

\[
\chi^k_{\mu, \nu} \rho \left( \left( \chi^k_{\mu, \nu} \right)^\ast f \right)(x, m, y) = \sum \chi^k_{\mu, \nu}(x, \ell, z) \rho \left( \left( \chi^k_{\mu, \nu} \right)^\ast f \right)(z, m - \ell, y)
\]

Therefore we have

\[
\sum_{|\mu| = n+k-|\lambda|} \chi^k_{\mu, \nu} \rho \left( \left( \chi^k_{\mu, \nu} \right)^\ast f \right)(x, m, y) = \sum_{|\mu| = n+k-|\lambda|} \chi^k_{\mu, \nu}(x, m, y) f(x, m, y)
\]

\[
= \chi^k_{\mu, \nu}(x, m, y) f(x, m, y) = (\chi^k_{\mu, \nu} \ast f)(x, m, y).
\]

\[\square\]

Proposition 5.1.3. The Haar module \( \mathcal{E}_A^\Omega \) is the direct sum of the finitely generated projective \( C(\Omega) \)-modules \( \mathcal{E}_n^k \).

Proof. It is clear from Lemma 5.1.2 that the image of the isometries \( \nu_{n,k} \) equals the range of the projections \( p_{n,k} = \nu_{n,k}^\ast \nu_{n,k} \). Hence \( \mathcal{E}_n^k = p_{n,k} C(\Omega, \mathcal{G}^k) \) are finitely generated projective \( C(\Omega) \)-modules. The Proposition follows from the fact that \( \mathcal{E}_A^\Omega = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{k \in \mathbb{N}} \mathcal{E}_n^k \).

Define an operator \( D_k : C_c(\mathcal{G}) \to C_c(\mathcal{G}) \) via pointwise multiplication \( D_k f(\xi) = \kappa(\xi) f(\xi) \).

Proposition 5.1.4. The operator \( D_k \) is essentially selfadjoint and regular in \( \mathcal{E}_A^\Omega \). Moreover, it commutes up to bounded operators with the generators \( S_i \).

Proof. The operator \( D_k \) is obviously symmetric. Moreover \( D_k \pm i \) maps the submodule \( C_c(\mathcal{G}^k) \) surjectively onto itself, and the union \( \bigcup_k C_c(\mathcal{G}^k) \) is dense in \( \mathcal{E}_A^\Omega \). Therefore \( D_k \pm i \) have dense range, and the closure of \( D_k \) is selfadjoint and regular in \( \mathcal{E}_A^\Omega \). That \( D_k \) commutes with the \( S_i \) follows by direct computation:

\[
[D_k, S_i] g(x, n, y) = \sum \kappa(x, n, y) S_i(x, \ell, z) g(z, n - \ell, y) - S_i(x, \ell, y) (\kappa g)(z, n - \ell, y)
\]

\[
= \sum \kappa(x, n, y) - \kappa(z, n - \ell, y) S_i(x, \ell, z) g(z, n - \ell, y)
\]

\[
= (\kappa(x, n, y) - \kappa(\sigma(x), n-1, y)) g(\sigma(x), n-1, y)
\]

\[
= S_i p_- g(x, n, y),
\]

where \( p_- \) denotes the projection onto negative part of the grading, since \( \kappa(x, n, y) = \kappa(\sigma(x), n-1, y) \) when \( n > 0 \) and \( \kappa(x, n, y) = \kappa(\sigma(x), n-1, y) + 1 \) when \( n \leq 0 \). \[\square\]
The factorization and decomposition statements now follow directly from Lemma 5.1.2. We note that \( Y_\lambda := \{ (x, n, y) \in \mathcal{G}_A^0 : |\lambda| \leq n \text{ and } \sigma^{n-|\lambda|}(x) = \lambda y \} \).

We let \( p_1 \in \text{End}^*_{C(\Omega_A)}(\mathcal{E}_A^n) \) denote the projection given by point wise multiplication by the characteristic function of \( Y_\lambda \). We write \( \mathcal{E}_{n,\lambda}^0 \) for the the completion of the submodule \( C_\lambda(Y_\lambda \cap \mathcal{C}^{-1}(n)) \), and \( \mathcal{E}_{n,\lambda}^1 \) for the completion of \( C_\lambda(\mathcal{G}_A^0 \setminus Y_\lambda \cap \mathcal{C}^{-1}(n)) \). Recall the notation \( \mathcal{G}_A^\lambda = \{ \mu \lambda \in \mathcal{G}_A \} \).

**Proposition 5.1.5.** The projection \( p_\lambda \) projects onto the closed \( C(\Omega_A) \)-submodule of \( \mathcal{E}_A^0 \) generated by \( \{ S_\mu \mu \lambda \} \), and factors as \( p_\lambda f = \sum_{n=0}^\infty \nu_{n,\lambda} f(\nu_{n,\lambda} * f) \). In particular, for any finite word \( \lambda \), the Haar module \( \mathcal{E}_{n,\lambda}^0 \) decomposes as a direct sum of finitely generated projective \( C(\Omega_A) \)-modules

\[
\mathcal{E}_A^n = \bigoplus_{n=0}^\infty \mathcal{E}_{n,\lambda}^0 \oplus \bigoplus_{n=0}^\infty \mathcal{E}_{n,\lambda}^1 \oplus \bigoplus_{k=1}^\infty \mathcal{E}_n^k
\]

**Proof.** It suffices to prove that \( p_\lambda S_\mu \mu \lambda = S_\mu \mu \lambda \) and that \( p_\lambda S_\mu S_\nu \nu = 0 \) if and only if \( \mu \neq \mu_0 \nu \) for all \( \mu_0 \in \mathcal{G}_A^\lambda \). Since \( S_\mu \mu \lambda \) is defined from the characteristic function of the set \( \{ (x, |\mu| + |\lambda|, y) \in \mathcal{G}_A^0 | x \in C_{\mu \lambda}, \sigma^{|\mu|+|\lambda|}(x) = y \} \), it follows that \( p_\lambda S_\mu \mu \lambda = S_\mu \mu \lambda \). The element \( S_\mu S_\nu \nu \) is defined from the characteristic function of the set \( \{ (x, |\mu| - |\nu|, y) \in \mathcal{G}_A^0 | x \in C_{\mu \nu}, y \in C_{\nu}, \sigma^{|\mu| - |\nu| - |\lambda|}(x) = \lambda y \} \).

The Proposition follows from the fact that \( p_\lambda S_\mu S_\nu \nu \) is the characteristic function of the set \( \{ (x, |\mu| - |\nu|, y) \in \mathcal{G}_A^0 | x \in C_{\mu \nu}, y \in C_{\nu}, |\lambda| + |\nu| \geq |\mu|, \sigma^{|\mu|}(x) = \sigma^{|\nu|}(y), \sigma^{|\mu| - |\nu| - |\lambda|}(x) = \lambda y \} = \begin{cases} \emptyset, & \text{ if } \mu \neq \mu_0 \nu \forall \mu_0 \in \mathcal{G}_A^\lambda \\ \{(x, |\mu| - |\nu|, y) \in \mathcal{G}_A^0 | x \in C_{\mu \nu}, y \in C_{\nu}, \sigma^{|\mu|}(x) = \sigma^{|\nu|}(y)\}, & \text{ if some } \mu_0 \in \mathcal{G}_A^\lambda, \mu = \mu_0 \nu. \end{cases} \)

The factorization and decomposition statements now follow directly from Lemma 5.1.2.

Recall that \( \kappa_{|\eta|} = k \in \{0, 1, 2, \ldots\} \) and that \( \epsilon_{|\eta|} + k \geq 0 \). Now consider the function \( \psi_\lambda : \mathcal{G}_A \to \mathbb{Z} \) given by

\[
\psi_\lambda(x, n, y) = \begin{cases} n & \text{ when } (x, n, y) \in Y_\lambda \\ -n & \text{ when } (x, n, y) \in \mathcal{G}_A^0 \setminus Y_\lambda, \\ -|n| - \kappa(x, n, y) & \text{ when } (x, n, y) \in \mathcal{G}_A \setminus \mathcal{G}_A^0, \end{cases}
\]

The function \( \psi_\lambda \) is clearly continuous. Define an operator \( D_\lambda : C(\mathcal{G}_A) \to C(\mathcal{G}_A) \) by pointwise multiplication by \( \psi_\lambda \), i.e. \( D_\lambda f(x, n, y) = \psi_\lambda(x, n, y) f(x, n, y) \).

**Theorem 5.1.6.** The operator \( (\mathcal{E}_A^n, D_\lambda) \) is an odd unbounded KK-cycle for \( (O_A, C(\Omega_A)) \), which is in the same class as the \( (O_A, C(\Omega_A)) \)-Kasparov module \( (\mathcal{E}_A^n, 2p_\lambda - 1) \).

**Proof.** The operator \( D_\lambda \) is \( C(\Omega_A) \)-linear. The operators \( D_\lambda \pm i : C(\mathcal{G}_A) \to C(\mathcal{G}_A) \) are bijective since \( D_\lambda \) is defined via multiplication by a real valued function. Thus \( D_\lambda \) extends to a selfadjoint regular operator in the module \( \mathcal{E}_A^n \). To prove that \( D_\lambda \) has compact resolvent, we observe that the restriction of \( D_\lambda^2 \) to \( \mathcal{E}_A^n \) acts as multiplication by \((|n| + k)^2\), so by finite projectivity of the \( \mathcal{E}_n^k \), the resolvent \((1 + D_\lambda^2)^{-1}\) is compact.
It remains to show that \( D_\lambda \) has bounded commutators with the generators \( S_i \), which we do by first computing
\[
[D_\lambda, S_i] f(x, n, y) = (\psi_\lambda(x, n, y) - \psi_\lambda(\sigma(x), n - 1, y)) g(\chi_\lambda(x)f(\sigma(x), n, y))
\]
and then analyzing the expression \( \psi_\lambda(x, n, y) - \psi_\lambda(\sigma(x), n - 1, y) \) case by case:
\[
\psi_\lambda(x, n, y) - \psi_\lambda(\sigma(x), n - 1, y) = \begin{cases} 
1 & n > 0, (x, n, y) \notin \mathcal{Y}_\lambda \\
-1 & n > 0, (x, n, y) \in \mathcal{Y}_\lambda \setminus \mathcal{Y}_\lambda \\
2 & n \leq 0.
\end{cases}
\]

Since for \( n > 0 \) it holds that \( \kappa(x, n, y) = \kappa(\sigma(x), n - 1, y) \) whereas for \( n \leq 0 \) we have \( \kappa(x, n, y) = \kappa(\sigma(x), n - 1, y) - 1 \). Combining this with \(|\kappa| - |\kappa - 1| = -1\) for \( n \leq 0 \) yields the statement above. Therefore
\[
[D_\lambda, S_i] = p_{>0} \left( 1 - 2 \chi_\mathcal{Y}_\lambda \right) S_i + 2p_{\leq 0} S_i,
\]
where \( p_{>0} \) is the projection onto the strictly positive part of the cocycle grading, \( p_{\leq 0} = 1 - p_{>0} \) and we identify the function \( \chi_\mathcal{Y}_\lambda \) with the operator it induces via point wise multiplication. Since \( \psi_\lambda \) is positive exactly on \( \mathcal{Y}_\lambda \), the class of this unbounded cycle coincides with that of \( p_\lambda \) using Proposition 5.1.6.

Remark 5.1.7. It is also possible to construct even classes over \( C(\Omega_\lambda) \) in this way. On the direct sum \( E^n_\lambda \oplus E^n_\mu \), consider the \( O_\lambda \) representation determined by \( S_i \to S_i \oplus S_i \) and the unbounded symmetric operator
\[
D^\nu := \begin{pmatrix}
0 & D_\nu + iD_\nu \\
D_\nu - iD_\nu & 0
\end{pmatrix}.
\]

The pair \( (E^n_\lambda \oplus E^n_\mu, D^\nu) \) defines a cycle for \( KK_0(O_\lambda, C(\Omega_\lambda)) \).

We end this subsection with a computational result for the special case of \( SU_q(2) \). Recall the construction from Subsubsection 5.1.1.

Proposition 5.1.8. If \( \tau = (\tau_+, \tau_-) : \mathcal{Y}_\Omega(2) \to \Omega_{SU_q(2)} \times \Omega_{SU_q(2)} \) is a weak choice function such that \( \tau_+ \in \mathcal{C}_2 \) and \( \tau_- \in \mathcal{C}_1 \), then the class \([E^n_{SU_q(2)}; D_\nu] \otimes_{C(\Omega_{SU_q(2)})} \mathcal{K}^1(C(SU_q(2)))\) generates \( \mathcal{K}_1(C(SU_q(2))) \).

We will use the identification \( \mathcal{Y}_\Omega(2) \cong \mathbb{N} \times \mathbb{N} \) given by the mapping that maps \((k, l)\) to the word \( 1 \cdots 12 \cdots 2 \) of \( k \) 1’s and \( l \) 2’s.

Proof. It is well-known (see more in Equation 8.326), that \( K_1(C(SU_q(2))) \cong \mathbb{Z} \cong K^1(C(SU_q(2))) \). Hence the Universal Coefficient Theorem implies that the index pairing
\[
K_1(C(SU_q(2))) \otimes K^1(C(SU_q(2))) \to \mathbb{Z}
\]
is non-degenerate and in fact an isomorphism. Thus it suffices to construct is a unitary \( u \in C(SU_q(2)) \) such that the class \( x := [u] \otimes_{C(SU_q(2))} [E^n_{SU_q(2)}; D_\nu] \) satisfies that \( x \otimes_{C(\Omega_{SU_q(2)})} [\text{BP}(\tau)] = 1 \).

Consider the unitary \( u := S_2 + 1 - S_2 S_2^* S_2 + S_1 S_1^* \). We set \( T := p_2 u p_2 \in \text{End}_{C(\Omega_{SU_q(2)})}^* (p_2 E^n_{SU_q(2)}) \), so \( x = \text{ind}_{C(\Omega_{SU_q(2)})}(T) \). It holds that \( p_2 E^n_{SU_q(2)} \) is generated over \( C(\Omega_{SU_q(2)}) \) by the elements \( \{S(k, l) : l > 0\} \). A direct computation gives that
\[
TS(k, l) = \begin{cases} 
S(k, l), & k > 0, \\
S_{0, l+1}, & k = 0,
\end{cases}
\]
and
\[
T^* S(k, l) = \begin{cases} 
S(k, l), & k > 0, \\
S_{0, l-1}, & k = 0, l > 1, \\
0, & k = l - 1 = 0.
\end{cases}
\]
Lemma 5.2.1. Let \( \Lambda \) be the unbounded Fredholm modules, we need a lemma whose notation will come in handy. Recall the notation \( \jmath \) starts in \((5.38)\). The identity of Theorem 5.1.6 to "fibers" over points in \( K \) sign-function coincides up to unitary equivalence and a finite rank perturbation with the finitely restricted to a fiber. 5.2. Theorem 5.2.3. Let \( \theta \) be a character on \( \mathcal{C}(\Omega_{\Lambda}) \) such that the source projection is the orthogonal projection onto \( \bigoplus_{\ell(j,k)\neq 0} \ell^2(\gamma_k) \) and \[
abla_{\mu\nu} = S_{\mu\nu} \otimes \mathbb{C}.
\]

\[
\langle \iota_{\nu} \delta_{\mu}, \iota_{\nu} \delta_{\nu} \rangle_{\mathcal{E}_A, \mathbb{C}} = \sum_{\ell(j,k)\neq 0} p_k \delta_{\mu}, \delta_{\nu} = \begin{cases} \delta_{\mu,\nu}, & \text{if } A(j,k) \neq 0, \\ 0, & \text{otherwise}. \end{cases}
\]

Let \( \mu = \mu_0 \). A direct computation shows that \[
\langle \iota_{\nu} \delta_{\mu}, \iota_{\nu} \delta_{\nu} \rangle_{\mathcal{E}_A, \mathbb{C}} = e(S_{\mu}^* S_{\nu}) = \delta_{\mu,\nu} e(S_{\mu}^* S_{\nu}) = \delta_{\mu,\nu} \sum_{i=1}^N A_{ik} e(S_i S_i^*) = \begin{cases} \delta_{\mu,\nu}, & \text{if } A(j,k) \neq 0, \\ 0, & \text{otherwise}. \end{cases}
\]

Remark 5.2.2. We note that already on an algebraic level it holds that \[
\iota_{\nu}(\delta_{\mu}) = S_{\mu} \otimes \mathbb{C}.
\]

Let \( \lambda \in \mathcal{Y}_{\Lambda} \) be a finite word starting in \( \lambda_1 \). We define the partial isometry \( W_{\lambda,\epsilon} : \ell^2(\mathcal{Y}_{\Lambda}) \rightarrow \mathcal{E}_A \otimes \mathbb{C} \) by \[
W_{\lambda,\epsilon} := \iota_{\epsilon}(\iota_{\gamma_{\lambda}}).
\]

By Lemma 5.2.1 it holds that \( W_{\lambda,\epsilon} \) is an isometry if \( A(j,\lambda_1) = 1 \) and \( W_{\lambda,\epsilon} = 0 \) if \( A(j,\lambda_1) = 0 \). We let \( \pi_{\Lambda} : \mathcal{O}_A \rightarrow \text{End}(\mathcal{C}(\Omega_{\Lambda}) \mathcal{E}_A) \) denote the left \( \mathcal{O}_A \)-action.

Theorem 5.2.3. Let \( \epsilon : \mathcal{C}(\Omega_{\Lambda}) \rightarrow \mathbb{C} \) be a character starting in \( j \). For a finite word \( \lambda \in \mathcal{Y}_{\Lambda} \) the unbounded Fredholm module
\[(5.38) \quad \mathcal{E}_\lambda(D_\Lambda) = (\pi_{\Lambda}\otimes\epsilon, \mathcal{E}_\lambda \otimes \mathbb{C}, D_\Lambda \otimes \epsilon, 1),
\]
is \( \theta \)-summable and if \( \varphi(1) \leq C k^p \) for some \( C, p > 0 \) then it is \( \mathcal{L}^{p+1,\infty} \)-summable. Furthermore, it holds that the bounded transform of the unbounded Fredholm module \( (5.38) \) defined using the sign-function coincides up to unitary equivalence and a finite rank perturbation with the finitely summable analytic \( K \)-cycle:
\[
(\pi_{\Lambda} \otimes \epsilon, \mathcal{E}_\lambda \otimes \mathbb{C}, 2W_{\lambda,\epsilon} W_{\lambda,\epsilon}^* - 1).
\]
In particular, if \( A(j, \lambda_1) = 0 \), where \( \lambda_1 \) is the last letter of \( \lambda \), the bounded transform of the unbounded Fredholm module \( \mathcal{E}_\lambda \) defined using the sign-function is a degenerate cycle and \( K \)-homologically trivial. On the level of \( K \)-homology, it holds that
\[
\varepsilon_e[\mathcal{E}_\lambda^0 \otimes D_\lambda] = A(j, \lambda_1)[\beta_{\lambda_1}] \quad \text{in} \quad K^1(\mathcal{O}_\lambda),
\]
where \( \lambda_1 \) denotes the first letter of \( \lambda \).

Recall the notation \( \beta_k \) from Proposition 5.1.3. We wish to remark that since \( (\mathcal{E}_\lambda^0, D_\lambda) \) is an unbounded \( KK \)-cycle, functoriality of unbounded \( KK \)-cycles guarantees that \( \varepsilon_e(\mathcal{E}_\lambda^0, D_\lambda) \) is an unbounded Fredholm module. As such, the proof consists of proving \( \theta \)-summability and identifying its bounded transform. We structure the proof of the later in a Proposition.

**Proposition 5.2.4.** Let \( \varepsilon \) be a character on \( C(\Omega_\lambda) \), \( \lambda \in \mathcal{C}_\lambda \) and define \( K_{\lambda}^{\lambda} \) as the closed linear span of \( \{S_\mu \otimes \varepsilon_1 | \mu = \mu_0 \lambda \} \subseteq \mathcal{E}_\lambda^0 \otimes \varepsilon \mathbb{C} \). It holds that the non-negative spectral projection of \( D_\lambda \otimes \varepsilon \) is the orthogonal projection onto \( K_{\lambda}^{\lambda} \subseteq \mathcal{E}_\lambda^0 \otimes \varepsilon \mathbb{C} \). In particular, if \( \varepsilon \) starts in \( \delta \) and \( A(j, \lambda_1) = 0 \), where \( \lambda_1 \) is the last letter of \( \lambda \), then \( K_{\lambda}^{\lambda} = 0 \).

The proof of the first part of Proposition 5.2.4 is clear from Proposition 5.1.6 and the proof of Theorem 5.2.1. The second part follows from the first part and Lemma 5.2.1 (cf. Remark 5.2.2).

**Proof of Theorem 5.2.3.** It follows from Proposition 5.2.4 and Lemma 5.2.1 that the projection onto the non-negative spectrum of \( D_\lambda \otimes \varepsilon \) coincides with \( W_{\lambda, \varepsilon}^* W_{\lambda, \varepsilon}^* \) outside a finite-dimensional space, and modulo a finite rank operator
\[
\frac{D_\lambda \otimes \varepsilon 1}{|D_\lambda \otimes \varepsilon 1|} = 2W_{\lambda, \varepsilon}^* W_{\lambda, \varepsilon}^* - 1.
\]

Hence, if \( A(j, \lambda_1) = 0 \) Equation (5.39) follows. To prove Equation (5.39) if \( A(j, \lambda_1) = 1 \), we apply the ideas of Subsection 2.2 after computing
\[
W_{\lambda, \varepsilon}^* \left[ (\pi \otimes \varepsilon_0)(S_\mu) \right] W_{\lambda, \varepsilon} = L_1^i(\varepsilon_1), \quad i = 1, \ldots, N.
\]

The identity (5.39) and finite summability follows mutatis mutandis to the proof of Proposition 5.2.4 using the fact that \( \ell^2(\varepsilon \mathcal{C}_\lambda) = R_1^0(R_1^0)^* \ell^2(\varepsilon \mathcal{C}_\lambda) \) and in the \( K \)-theory of \( O_\varepsilon \) it holds that
\[
T_\lambda T_\lambda^* \sim T_\lambda^* T_\lambda = T_\lambda^* T_\lambda \sim T_\lambda^* T_\lambda.
\]

It remains to prove \( \theta \)-summability, i.e. that \( e^{-(D_\lambda \otimes \varepsilon)^2} \) is trace class. Applying the computations of Proposition 5.1.3 and the definition of \( D_\lambda \), we have that
\[
\mathcal{E}_\lambda^0 \otimes \varepsilon \mathbb{C} = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{l \in \mathbb{N}} e(p_{n,l}) \mathcal{C}^{(l+n)} \quad \text{and in this decomposition} \quad (D_\lambda \otimes \varepsilon)^2 = \bigoplus_{n \in \mathbb{N}} \bigoplus_{l \in \mathbb{N}} e(p_{n,l}).
\]

It follows from Corollary 5.1.3 that \( e^{-(D_\lambda \otimes \varepsilon)^2} \) is trace class. Assuming that \( \varphi(l) \leq C l_p \) for some \( p \) implies that \( |D_\lambda \otimes \varepsilon 1|^{-1} \in \mathcal{L}^{p+1, \infty}(\mathcal{E}_\lambda^0 \otimes \varepsilon \mathbb{C}) \) and in this case it is a \( \mathcal{L}^{p+1, \infty} \)-summable unbounded Fredholm module.

**Remark 5.2.5.** In particular, Theorem 5.2.3 implies that for a choice of characters \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N \) such that each \( \varepsilon_k \) starts in \( \delta_k \) with \( A(j_k, k) = 1 \), the mapping
\[
\mathbb{Z}^N \to K^1(O_\lambda), \quad (l_1, l_2, \ldots, l_N) \to \sum_{k=1}^N l_k \left[ (\varepsilon_k)_e(\mathcal{E}_\lambda^0, D_\lambda) \right]
\]
is surjective.

This gives an explicit proof of the fact that in odd degree, the Kasparov product
\[
KK_*(O_\lambda, C(\Omega_\lambda)) \times KK_*(C(\Omega_\lambda)) \to K^*(O_\lambda),
\]

\footnote{For the sake of mental peace of the reader.}
is surjective.

Remark 5.2.6. If the matrix $A$ has property (I), Theorem [15,3] implies that the unbounded Fredholm modules $e_i(D, \pi_\phi(D))$ in fact are spectral triples.

5.3. A connection on the Haar module. The point localizations of the previous section are a simple case of the Kasparov product in $KK$-theory. We will now employ the general form of this construction in order to construct an $\epsilon$-unbounded Fredholm module (see appendix A) from any cycle $(E, D, A)$, with $\lambda$ a finite word, and any Belissard-Pearson spectral triple $(\pi, \ell^2(\mathcal{V}_A, \mathbb{C}^2), D)_{\phi, s}$ for $s \in (0, 1)$. We briefly recall the techniques developed in [9].

Definition 5.3.1. Let $(\pi, \mathcal{H}, D)$ be an unbounded Fredholm module. Its Lipschitz algebra is the $*$-algebra

\[ \mathcal{A}_D = \text{Lip}(\pi, \mathcal{H}, D) := \{ a \in A : [D, a] \in \mathbb{B}(\mathcal{H}) \}. \]

Note that we take the maximal subalgebra with the property that $[D, a]$ is bounded. The algebra $\mathcal{A}$ can be topologised by the representation

\[ \tilde{\pi}_D := \text{id} \oplus \pi_D : \mathcal{A} \to A \oplus \mathbb{B}(\mathcal{H} \oplus \mathcal{H}) \]

where $\pi_D : a \mapsto \begin{pmatrix} \pi(a) & 0 \\ [D, \pi(a)] & \pi(a) \end{pmatrix}$, realising $\mathcal{A}$ as a closed subalgebra of $A \oplus \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$. As such it is an operator algebra. The reader can consult [9] for an exposition of the general theory of nonselfadjoint operator algebras. The involution in $A$ induces an involution in $\mathcal{A}$, which is well behaved with respect to the representation $\tilde{\pi}_D$. Indeed,

\[ \pi_D(a^*) = v^* \pi_D(a)^* v, \quad v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \]

which implies that the involution is completely isometric for the norm induced by $\tilde{\pi}_D$. Operator algebras with completely bounded involution are called involutive operator algebras [9]. The main feature of involutive operator algebras is that there is a class of modules over them, which in many ways behave like $C^*$-modules. We recall the theory for Lipschitz algebras.

Definition 5.3.2 (9). Let $\mathcal{B}_D$ be a Lipschitz algebra. The standard free module over $\mathcal{B}$ is the module

\[ \mathcal{H}_{\mathcal{B}} := \{ (b_i)_{i \in \mathbb{N}} : \sum_{i \in \mathbb{N}} \tilde{\pi}_D(b_i)^* \tilde{\pi}_D(b_i) < \infty \}. \]

A Lipschitz module over $\mathcal{B}$ is a closed submodule $\mathcal{E} \subset \mathcal{H}_{\mathcal{B}}$ which is the range of a densely defined (possibly unbounded) projection $p : \text{Dom} p \to \mathcal{H}_{\mathcal{B}}$.

The module $\mathcal{H}_{\mathcal{B}}$ carries an inner product, but this inner product does not define the norm. A projection is an operator satisfying $p^2 = p^* = p$. The existence of unbounded projections in $\mathcal{H}_{\mathcal{B}}$ is due to the fact that norm and inner product are not related in the same way as they are in $C^*$-modules. The algebra $\mathcal{E}(\mathcal{E})$ is defined to be the cb-norm closure of the finite rank operators.

Proposition 5.3.3 (9). For each $i \in \mathbb{Z}$, let $p_i \in M_n(\mathcal{B})$ be a projection and $\mathcal{E}_i := p_i \mathcal{B}^0 \subset \mathcal{B}^0$. Then the direct sum $\bigoplus_{i \in \mathbb{Z}} \mathcal{E}_i$ is a Lipschitz module.

The main feature of Lipschitz modules is the existence of connections on them. Recall that the space of 1-forms associated to $(\pi, \mathcal{H}, D)$ is

\[ \Omega_1^D := \{ \sum_i \pi(a_i)[D, \pi(b_i)] : a_i \in B, b_i \in \mathcal{B}_D \} \subset \mathbb{B}(\mathcal{H}), \]
where the sums converges in operator norm. The operator space $\Omega^1_B$ is a left $B$-module and a right $\mathcal{B}$-module. The map $b \mapsto [D, b]$ is a completely bounded derivation of $\mathcal{B}$ into this $(B, \mathcal{B})$-bimodule. A $D$-connection on a Lipschitz module $\mathcal{E}$ is a completely bounded map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_B$, where $\mathcal{E}$ denotes the Haagerup tensor product, satisfying the Leibniz rule $\nabla(eb) = \nabla(e)b + e \otimes [D, b]$. By [9], connections on Lipschitz modules always exist, since the Grassmann connection $p[D, p]$ is completely bounded.

**Lemma 5.3.4.** Let $(\tau, \ell^2(\mathcal{Y}_0, \mathbb{C}^2), D_{\mathcal{Y}_0})$ be a logarithmic Bellissard-Pearson spectral triple. The projections $p_{n,k,\lambda} : = v_{n,k,\lambda} v_{n,k,\lambda}^* \in M_\varphi(\mathcal{C}(\Omega))$ are in fact elements of $M_\varphi(\mathcal{C}(\Omega), d_{\Omega^n})$, and therefore $[D_{\mathcal{Y}_0}, p_{n,k,\lambda}] \in \ell^2(\mathcal{Y}_0, \mathbb{C}^2)$.

**Proof.** The projection $v_{n,k,\lambda} v_{n,k,\lambda}^*$ has entries $\rho(\chi^k_{n,\mu}^* \chi^k_{n,\mu}(x))$ which equal $0$ if $\mu \neq \nu$ and for $\mu = \nu$ the convolution product gives

$$\left(\chi^k_{n,\mu}\right)^* \chi^k_{n,\mu}(x) = \sum \chi^k_{n,\mu}(\varepsilon, n, x) = \begin{cases} 1 & \text{if } A_{n,+,x_{n+1}} = 1 \text{ and } \mu_{n+1} \neq x_k \\ 0 & \text{otherwise} \end{cases}$$

Thus, for $k = 0$ this function equals the projection

$$\sum_{i=1}^N A_{n,+,x_i} \chi^i_C,$$

whereas for $k > 0$ we get

$$\sum_{j=1}^N \sum_{i \neq j} A_{n,+,x_i} \chi^i_C (e^{k-1}).$$

Since these are sums shifted cylinder functions, they are Lipschitz in the metric $d_{\Omega^n}$.

In view of this fact, Proposition 5.1.3 and Lemma 5.3.4 implies that the module $\mathcal{L}_k^\tau$ admits a submodule $\mathcal{E}_k^n$ with the structure of a projective operator module over the involutive operator algebra

$$\operatorname{Lip}^\tau_\tau(\Omega) := \operatorname{Lip}^\tau(\pi, \ell^2(\mathcal{Y}_0, \mathbb{C}^2), D_{\mathcal{Y}_0})$$

$$= \left\{ f \in C(\Omega) \left| \begin{array}{cc} \pi_{\tau}(f) & 0 \\ \left[D_{\mathcal{Y}_0}, \pi_{\tau}(f)\right] & \pi_{\tau}(f) \end{array} \right. \right\} \in \ell^2(\mathcal{Y}_0, \mathbb{C}^2) \otimes \ell^2(\mathcal{Y}_0, \mathbb{C}^2).$$

This uses the fact that Proposition 5.1.3 implies that there is a continuous inclusion $\operatorname{Lip}(\Omega, d_{\Omega^n}) \hookrightarrow \operatorname{Lip}^\tau(\Omega)$ for any $s \in (0, 1]$. Denote by $\mathcal{E} \subset \mathcal{E}$ the submodule

$$\mathcal{E} := \{ f \in \mathcal{E} : \sum \tilde{\pi}_\tau(\rho(\chi^k_{n,\mu} f)) \tilde{\pi}_\tau(\rho(\chi^k_{n,\mu} f)) \less \infty \},$$

which is complete in the norm

$$(5.41) \quad \|f\|_{\mathcal{E}}^2 := \left\| \sum \tilde{\pi}_\tau(\rho(\chi^k_{n,\mu} f)) \tilde{\pi}_\tau(\rho(\chi^k_{n,\mu} f)) \right\|.$$
whose direct sum extends to a connection

$$\nabla : \mathcal{E} \to E \hat{\otimes}_{C(\Omega)} \Omega^1_z.$$  

**Proposition 5.3.5.** The module $\mathcal{E}$ is dense $\mathcal{E}$ and $\mathcal{E}$ is a Lipschitz module in the norm $\| \cdot \|_{\mathcal{E}}$. The operator $D_\lambda$ restricts to a selfadjoint regular operator in $\mathcal{E}$, and $(D_\lambda \pm i)^{-1} \in \mathbb{K}(\mathcal{E})$. Moreover $[D_\lambda, \nabla] = 0$.

**Proof.** To see that $\mathcal{E}$ is dense in $\mathcal{E}$, observe that

$$\mathcal{E}^k_n := \{ f \in \mathcal{E}^k_n : \left( \chi^k_n \right)^*(f) \in \text{Lip}_1(\Omega_n)^{(n+k)} \} \subset \mathcal{E},$$

each $\mathcal{E}^k_n$ is dense in $\mathcal{E}^k_n$ and $\mathcal{E}$ contains the algebraic direct sum of the $\mathcal{E}^k_n$. Since the norm $\| \cdot \|_{\mathcal{E}}$ comes from the embedding

$$\nu : \mathcal{E} \to \bigoplus_{n,k,\mu} \text{Lip}_1^i(\Omega_n)^{(n+k)} \cong \mathcal{H}_{\text{Lip}}(\Omega_n)$$

$$f \to (\rho(\chi^k_n f))_{n,k,\mu},$$

$\mathcal{E}$ is a Lipschitz module. We now prove that the resolvents $(D_\lambda \pm i)^{-1}$ are completely bounded for the Lipschitz norm. The Lipschitz norm is given by $\| \cdot \|_{\mathcal{E}}$,

$$\| (D_\lambda \pm i)^{-1} f \|_{\mathcal{E}}^2 = \sum_{n,k,\mu} \pi_D(\rho(\chi^k_n (D_\lambda \pm i)^{-1} f))^* \pi_D(\rho(\chi^k_n (D_\lambda \pm i)^{-1} f)),$$

and this is compatible with the projective module decomposition $\| \cdot \|_{\mathcal{E}}$. Thus, although $D_\lambda$ depends on whether $k = 0$ or $k > 0$ and $\mu \in \mathcal{T}_\lambda$ or not, for fixed $n, k, \mu$ we have

$$\pi_D(\rho(\chi^k_n (D_\lambda \pm i)^{-1} f))^* \pi_D(\rho(\chi^k_n (D_\lambda \pm i)^{-1} f)) \leq (1 + n^2 + k^2)^{-1} \pi_D(\rho(\chi^k_n f))^* \pi_D(\rho(\chi^k_n f)),$$

by definition of $\psi_\lambda$, see Equation (5.35). This shows that $\| (D_\lambda \pm i)^{-1} f \|_{\mathcal{E}}^2 \leq \| f \|_{\mathcal{E}}^2$. The same computation shows that the resolvent $(D_\lambda \pm i)^{-1}$ is completely contractive. Moreover, they also show that the resolvents are cb-norm limits of finite rank operators, and hence $(D_\lambda \pm i)^{-1} \in \mathbb{K}(\mathcal{E})$.

By construction the connection satisfies $[\nabla, D_\lambda] = 0$.  

For an element $e \in \mathcal{E}$, the operator $1 \otimes_D D_{\lambda,\gamma}$ acts as

$$(1 \otimes_D D_{\lambda,\gamma})(e \otimes \left( \varphi^+ - \varphi^- \right))(v) = \sum_{k=0}^{\infty} \sum_{n=-k}^{\infty} \sum_{\mu=0}^{\infty} \chi^k_n \otimes \left( |v|^0 \pi_-(\rho(\chi^k_n e)) \varphi^- |v|^0 \pi_+ (\rho(\chi^k_n e)) \varphi^+ \right)(v).$$

**Theorem 5.3.6.** For any logarithmic Bellissard-Pearson spectral triple with grading operator $\gamma$ and any finite word $\lambda$, the operator

$$D_{\lambda,\gamma} := D_\lambda \otimes \gamma + 1 \otimes_D D_{\lambda,\gamma},$$

is selfadjoint and has compact resolvent in $H(\tau) := C^2(\mathcal{T}, \mathbb{C}^2)$.  

**Proof.** The unbounded KK-cycle $(\mathcal{E}^0_\lambda, D_\lambda)$ admits a compatible Lipschitz structure $(\mathcal{E}, \nabla, D_\lambda)$ for any Bellissard-Pearson spectral triple. Therefore the operator $1 \otimes_D D_{\lambda,\gamma}$ is selfadjoint by [9] Theorem 2.25]. Because $(D_\lambda \pm i)^{-1} \in \mathbb{K}(\mathcal{E})$, it follows that

$$\text{im}(D_\lambda \otimes \gamma \pm i)^{-1} (1 \otimes_D D_{\lambda,\gamma} \pm i)^{-1} = \text{im}(1 \otimes_D D_{\lambda,\gamma} \pm i)^{-1} (D_\lambda \otimes \gamma \pm i)^{-1},$$

and $D_\lambda \otimes \gamma$ and $1 \otimes_D D_{\lambda,\gamma}$ anticommute on this subspace. Cf. [19] Theorem 6.1.8], the sum $D_\lambda \otimes \gamma + 1 \otimes_D D_{\lambda,\gamma}$ is selfadjoint on the intersection of the domains. The resolvent products are compact by construction, and hence by [19] Lemma 6.3.2, the resolvent of the sum is compact as well.  

□
5.4. Kasparov products with the Bellissard-Pearson spectral triples. We now proceed to show that \((\mathcal{H}(\tau), D_{\lambda,\tau})\) constitutes an \(\varepsilon\)-bounded Fredholm module representing the Kasparov product

\[
KK_1(O_\mathcal{A}, C(\Omega_\mathcal{A})) \times K^0(C(\Omega_\mathcal{A})) \to K^1(\Omega_\mathcal{A})
\]

\[
[D_{\lambda}] \times [BP_\tau] \to [D_{\lambda}] \otimes_{C(\Omega_\mathcal{A})} [BP_\tau].
\]

Here the classes \([D_{\lambda}]\) are described in the previous subsection, and \([BP_\tau]\) are the classes associated with the logarithmic Bellissard-Pearson spectral triples, with \(s < 1\), from Section 4. The reader is referred to the appendix for the notion of \(\varepsilon\)-bounded unbounded Fredholm module.

**Lemma 5.4.1.** Let \(k + n > 0\) and \(\mu\) be a nonempty word. Then

1. \( S_i x_{n,\mu}^k = A_{i,\mu_1} x_{n+1,\mu}^k \);
2. \( (x_{n,\mu}^k)^* S_i = \delta_{i,\mu} \left( x_{n-1,\sigma_i(\mu)}^k \right)^* \).

**Proof.** We compute

\[
S_i x_{n,\mu}^k (x, m, y) = \sum S_i (x, \ell, z) x_{n,\mu}^k (z, m - \ell, y) = \chi_C (x) x_{n,\mu}^k (\sigma(x), m - 1, y),
\]

which is nonzero only if \(m = n + 1\), \(x_1 = i\), \(\sigma(x) \in C_\mu\) and \(\kappa(\sigma(x), n, y) = k\). This holds if and only if \(x \in C_\mu\) and \(\kappa(x, n + 1, y) = k\), proving 1.) For 2.) we compute again

\[
(\chi_{n,\mu}^k)^* S_i (x, m, y) = \sum (\chi_{n,\mu}^k)^* (x, \ell, z) S_i (z, m - \ell, y) = \sum \chi_{n,\mu}^k (z, -\ell, x) S_i (z, m - \ell, y) = A_{i,\mu_1} x_{n,\mu}^k (iy, 1 - m, x),
\]

and this is nonzero only if \(m = -(n - 1)\), \(\mu_1 = i\), \(y \in C_{\sigma_i(\mu)}\) and \(\kappa(iy, n, x) = k\). This holds only if \(\kappa(y, n - 1, x) = k\), proving 2.)

**Lemma 5.4.2.** Let \(n = -k\). Then

1. \( (x_{-k,\beta}^k)^* S_i = \sum_{j=1}^N A_{ij} \left( \chi_{C_j} * x_{-(k+1),\beta}^{k+1} \right)^* \);
2. \( S_i x_{-k,s}^k = x_{-k+1,s}^k + \chi_{C_j} * x_{-k+1,s}^k \).

**Proof.** For 1.) compute

\[
(\chi_{-k,s}^k)^* S_i (x, m, y) = \sum (\chi_{-k,s}^k)^* (x, \ell, z) S_i (z, m - \ell, y) = \sum \chi_{-k,s}^k (z, -\ell, x) S_i (z, m - \ell, y) = A_{i,\mu_1} x_{-k,s}^k (iy, 1 - m, x) = \begin{cases} 1 & \text{when } m = k + 1, A_{i,\mu_1} = 1, \kappa(iy, -k, x) = k \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{when } m = k + 1, A_{i,\mu_1} = 1, \kappa(y, -(k + 1), x) = k + 1 \\ 0 & \text{otherwise} \end{cases} = \sum_{j=1}^N A_{ij} \left( \chi_{C_j} * x_{-(k+1),\beta}^{k+1} \right)^* (x, m, y).
\]
Lemma 5.4.3. We have the identities

1) \( \chi_{\mathcal{C}_i} \ast \chi_{-k,\sigma}^k = \chi_{-k,\sigma}^k \ast (\chi_{\mathcal{C}_i} \circ \sigma^k) \);

2) \( (\chi_{\mathcal{C}_i} \circ \sigma^k) (\chi_{-k-1,\sigma}^{k+1}) = \sum_{j=1}^{N} A_{ij} (\chi_{-k-1,\sigma}^{k+1}) \ast \chi_{\mathcal{C}_i} \).

Proof. This is again verified by direct computation.

\[
\begin{align*}
\chi_{\mathcal{C}_i} \ast \chi_{-k,\sigma}^k (x, m, y) &= \chi_{\mathcal{C}_i}(x) \chi_{-k,\sigma}^k(x, m, y) \\
&= \begin{cases} 
1 & \text{when } x \in \mathcal{C}_i, m = -k, \kappa(x, -k, y) = k \\
0 & \text{otherwise}
\end{cases} \\
&= \begin{cases} 
1 & \text{when } x \in \mathcal{C}_i, m = -k, x = \sigma^k(y) \\
0 & \text{otherwise}
\end{cases} \\
&= \begin{cases} 
1 & \text{when } \sigma^k(y) \in \mathcal{C}_i, m = -k, \kappa(x, -k, y) = k \\
0 & \text{otherwise}
\end{cases} \\
&= \chi_{-k,\sigma}^k \ast (\chi_{\mathcal{C}_i} \circ \sigma^k)(x, m, y).
\end{align*}
\]

\[
\begin{align*}
(\chi_{\mathcal{C}_i} \circ \sigma^k) (\chi_{-k-1,\sigma}^{k+1})^* (x, m, y) &= \chi_{\mathcal{C}_i}(\sigma^k(x)) (\chi_{-k-1,\sigma}^{k+1})^* (x, m, y) \\
&= \chi_{\mathcal{C}_i}(\sigma^k(x)) \chi_{-k-1,\sigma}^{k+1}(y, -m, x) \\
&= \begin{cases} 
1 & \text{when } \sigma^k(x) \in \mathcal{C}_i, m = k + 1, \kappa(y, -k - 1, x) = k + 1 \\
0 & \text{otherwise}
\end{cases} \\
&= \begin{cases} 
1 & \text{when } \sigma^k(x) \in \mathcal{C}_i, m = k + 1, y = \sigma^{k+1}(x) \\
0 & \text{otherwise}
\end{cases} \\
&= \begin{cases} 
1 & \text{when } y \in \sigma(\mathcal{C}_i), m = k + 1, \kappa(y, -k - 1, x) = k + 1 \\
0 & \text{otherwise}
\end{cases} \\
&= \sum_{j=1}^{N} A_{ij} (\chi_{-k-1,\sigma}^{k+1})^* \chi_{\mathcal{C}_i}(x, m, y).
\end{align*}
\]

Proposition 5.4.4. Let \((\pi_{\mathcal{A}}, \ell^2(\mathcal{A}, \mathbb{C}^2), D_{\mathcal{A}_+})\) be a logarithmic Bellissard-Pearson spectral triple. The operators \(S_i\) preserve the algebraic tensor product

\[
\bigoplus_{n,k} \phi_n^k \otimes_{\text{alg}} \mathcal{C}_i(\mathcal{A}, \mathbb{C}^2).
\]
which is a core for $D_s \otimes \gamma + 1 \otimes \varphi$ and $[1 \otimes \varphi D_{\gamma,s}, S_1]$ is given by the sum

$$\sum_{k=0}^{\infty} \chi_{k,0} \otimes [D_{\gamma,s}, \pi_{s}(\chi_{1,0} \circ \sigma^k)] \pi_{s}(\rho(\chi_{k-1,0}^s e)) \left( \varphi_+ \right) \left( \varphi_- \right)$$

with only finitely many non-zero terms, on this space.

Proof. Since $S_1 : \mathfrak{g}^k \rightarrow \mathfrak{g}^k$, the operator $S_1$ preserves the algebraic direct sum of the $\mathfrak{g}_n$ and hence a common core for $D_s \otimes 1$ and $1 \otimes \varphi D_{\gamma,s}$. The commutator $[S_1, 1 \otimes \varphi D_{\gamma,s}]$ is computed as

$$[S_1, 1 \otimes \varphi D_{\gamma,s}](e \otimes \left( \varphi_+ \right) \left( \varphi_- \right))$$

$$= \sum_{n,k,\mu} S_1 \chi_{n,0}^k \otimes \left( \left| v \right| \pi_{-}(\rho(\chi_{n,0}^k e)) \varphi_- \right) \left( \left| v \right| \pi_{+}(\rho(\chi_{n,0}^k e)) \varphi_+ \right)$$

$$- \sum_{n,k,\mu} \chi_{n,0}^k \otimes \left( \left| v \right| \pi_{-}(\rho(\chi_{n,0}^k e)) \varphi_- \right) \left( \left| v \right| \pi_{+}(\rho(\chi_{n,0}^k e)) \varphi_+ \right)$$

(by lemmas 5.3.1 and 5.4.2)

$$= \sum_{n,k,\mu} S_1 \chi_{n,0}^k \otimes \left( \left| v \right| \pi_{-}(\rho(\chi_{n,0}^k e)) \varphi_- \right) \left( \left| v \right| \pi_{+}(\rho(\chi_{n,0}^k e)) \varphi_+ \right)$$

Subsequently we exchange $\pi_+$ and $\pi_-$ at the expense of a commutator to obtain

$$\sum_{n,k,\mu} \chi_{n,0}^k \otimes \left( \left| v \right| \pi_{+}(\rho(\chi_{n,0}^k e)) \varphi_+ \right) \left( \left| v \right| \pi_{-}(\rho(\chi_{n,0}^k e)) \varphi_- \right)$$

(5.44)

$$+ \sum_{n,k,\mu} S_1 \chi_{n,0}^k \otimes [D_{\gamma,s}, \pi_{s}(\rho(\chi_{n,0}^k e))] \left( \varphi_+ \right) \left( \varphi_- \right)\left( \varphi_- \right)$$

(5.43)

and to the term (5.43) we apply lemma 5.4.1 to obtain

$$\sum_{n,k,\mu} \chi_{n,0}^k \otimes \left( \left| v \right| \pi_{+}(\rho(\chi_{n,0}^k e)) \varphi_+ \right) \left( \left| v \right| \pi_{-}(\rho(\chi_{n,0}^k e)) \varphi_- \right)$$

(5.44)

$$= \sum_{n,k,\mu} \chi_{n,0}^k \otimes \left( \left| v \right| \pi_{+}(\rho(\chi_{n,0}^k e)) \varphi_+ \right) \left( \left| v \right| \pi_{-}(\rho(\chi_{n,0}^k e)) \varphi_- \right)$$

which vanishes by Lemma 5.4.3. The remaining term (5.44) further simplifies to (5.42). □

Recall the notation $\mathcal{H}(\tau)$ and $D_{\lambda, \tau,s}$ from Theorem 5.3.6

Theorem 5.4.5. For $s \in (0, 1)$ the pair $(\mathcal{H}(\tau), D_{\lambda, \tau,s})$ is a well defined $(1-s)$-unbounded Fredholm module on $O_{\lambda}$ that represents the Kasparov product $[E, D_s] \otimes_{C(O_{\lambda})} [BP(\tau)]$. 


Proof. The operator $D_{\lambda,\tau,s} = D_\lambda \otimes \gamma + 1 \otimes \delta D_{\gamma,s}$ is selfadjoint with compact resolvent by Theorem 5.3.6. We will show that the operators

$$[1 \otimes \delta D_{\gamma,s}, S_j](1 + D^2_{\lambda,\tau,s})^{-\frac{1}{2}}, \quad (1 + D^2_{\lambda,\tau,s})^{-\frac{1}{2}}[1 \otimes \delta D_{\gamma,s}, S_j]$$

are bounded. By Proposition 5.4.3, $S_j$ preserves a core for $D_\lambda \otimes \gamma + 1 \otimes \delta D_{\gamma,s}$ and the commutator $[D_{\gamma,s}, \pi_\tau(\chi_1 \circ \sigma^k)]$ vanishes whenever $|v| > k$ since $\tau_+(v)k_+ = \tau_-(v)k_+$ so $\chi_1(\tau_+(v)) = \chi_1(\tau_-(v))$, here the subscripts indicates the $(k+1)$st letter. Therefore

$$\|[D_{\gamma,s}, \pi_\tau(\chi_1 \circ \sigma^k)]\| \leq k^4.$$ 

The operator $1 + D^2_{\lambda,\tau,s} = 1 + (D_\lambda \otimes \gamma)^2 + (1 \otimes \delta D_{\gamma,s})^2$ preserves the subspace $\mathcal{E}_k \otimes \text{Lip}(\Omega) C_c(\mathcal{Y}_A, \mathbb{C}^2)$, and from the form of 5.12, it follows that

$$(1 + D^2_{\lambda,\tau,s})^{-\frac{1}{2}}[1 \otimes \delta D_{\gamma,s}, S_j] : \mathcal{E}_k \otimes \text{Lip}(\Omega) C_c(\mathcal{Y}_A, \mathbb{C}^2) \to \mathcal{E}_k \otimes \text{Lip}(\Omega) C_c(\mathcal{Y}_A, \mathbb{C}^2).$$

By Equation 5.12 it suffices to show that

$$\|\sum_j \frac{1}{\sqrt{2}} (1 + D^2_{\lambda,\tau,s})^{-\frac{1}{2}} x_{-1,0} \otimes [D_{\gamma,s}, \pi_\tau(\chi_1 \circ \sigma^k)] \pi_\tau(\rho(\chi^k_{-1,0} e_j)) (\varphi^0_j, \varphi^0_j) \|_{\mathcal{H}(\tau)}$$

$$\leq \|\sum_j \frac{1}{\sqrt{2}} x_{-1,0} \otimes \pi_\tau(\rho(\chi^k_{-1,0} e_j)) (\varphi^0_j, \varphi^0_j) \|_{\mathcal{H}(\tau)},$$

for every $k \geq 0$. This is a straightforward computation using the fact that $D_\lambda \otimes 1$ acts as multiplication by $-2k$ on $\mathcal{E}_k \otimes \text{Lip}(\Omega) C_c(\mathcal{Y}_A, \mathbb{C}^2)$ and the inequality (5.45):

$$\|\sum_j \frac{1}{\sqrt{2}} (1 + 4k^2 + (1 \otimes \delta D_{\gamma,s})^2)^{-\frac{1}{2}} x_{-1,0} \otimes [D_{\gamma,s}, \pi_\tau(\chi_1 \circ \sigma^k)] \pi_\tau(\rho(\chi^k_{-1,0} e_j)) (\varphi^0_j, \varphi^0_j) \|_{\mathcal{H}(\tau)}$$

$$\leq (1 + k^2)^{-\frac{1}{2}} k^4 \|\sum_j \frac{1}{\sqrt{2}} \pi_\tau(\rho(\chi^k_{-1,0} e_j)) (\varphi^0_j, \varphi^0_j) \|_{\mathcal{H}(\tau)}$$

$$= \|\sum_j \frac{1}{\sqrt{2}} x_{-1,0} \otimes \pi_\tau(\rho(\chi^k_{-1,0} e_j)) (\varphi^0_j, \varphi^0_j) \|_{\mathcal{H}(\tau)},$$

which proves that $(1 + D^2_{\lambda,\tau,s})^{-\frac{1}{2}}[1 \otimes \delta D_{\gamma,s}, S_j]$ is bounded. Boundedness of the reverse product follows from a similar computation. Now Lemma 5.8 implies that the commutators $[D_\lambda, S_j]$ are $\varepsilon$-bounded. Thus, by Proposition 5.5 D has $\varepsilon$-bounded commutators with the $*$-subalgebra of $O_k$ generated by the operators $S_j$, which is dense in $O_k$. Thus we have an $\varepsilon$-unbounded Fredholm module with $\varepsilon = 1 - s$. To see that this $\varepsilon$-unbounded Fredholm module represents the Kasparov product one uses Theorem 5.7 to check that the connection condition 1.), the domain condition 2.) and the semiboundedness condition 3.) are satisfied by construction.

Lastly, we identify the classes of these Kasparov products in rational $K$-homology.

Theorem 5.4.6. In $K^1(O_k) \otimes Q$ we have

$$[\varepsilon, D_\lambda] \otimes \text{Lip}(\Omega) [BP(\tau)] \otimes Q = (A(j_+, \lambda_1) - A(j_-, \lambda_1)) \beta_{j_k} \otimes Q,$$

where $j_k$ is the first letter of $\tau_+(e_k)$ and $\lambda_1$ and $\lambda_1$ are the first respectively last letter of $\lambda$.

Proof. The computation of the class $[\varepsilon, D_\lambda] \otimes \text{Lip}(\Omega) [BP(\tau)]$ in $K^1(O_k) \otimes Q$ relies on the fact that $O_k$ is in the bootstrap class with finitely generated $K$-theory and $K$-homology, so $K^1(O_k) \otimes Q \cong$
Hom\(_{\text{gr}}(K_*(O_\lambda), Q)\) and rational classes are determined by their index pairing. Furthermore, using Remark 5.4.7 we can write

\[ \mathcal{H}(\tau) = \bigoplus_{\mu \in \mathcal{Y}_\lambda} \mathcal{E}_\lambda^0 \otimes \varepsilon_{\tau_{-(\mu)}} \otimes \varepsilon_{\tau_{-(\mu)}} \mathbb{C}^2 \quad \text{and} \quad D_{\lambda, \tau, s}[D_{\lambda, \tau, s}]^{-1} = \bigoplus_{\mu \in \mathcal{Y}_\lambda} F_\mu. \]

Since in each fixed summand \(\mathcal{E}_\lambda^0 \otimes \varepsilon_{\tau_{-(\mu)}} \otimes \varepsilon_{\tau_{-(\mu)}} \mathbb{C}^2 \subseteq \mathcal{H}(\tau)\), \(D_{\lambda, \tau, s}\) is a bounded perturbation of the operator \(D_{\lambda, \tau_{-(\mu)}} \otimes \varepsilon_{\tau_{-(\mu)}}(1 \oplus -1)\), it follows from [13] Appendix A, Theorem 8, and an argument similar to the proof of Proposition 5.2.3 that for any \(\mu\)

\( F_\mu - (2p_\lambda - 1) \otimes \varepsilon_{\tau_{-(\mu)}}(1 \oplus -1) \in \mathcal{K}(\mathcal{E}_\lambda^0 \otimes \varepsilon_{\tau_{-(\mu)}} \mathbb{C}^2) \)

and \( \|F_\mu - (2p_\lambda - 1) \otimes \varepsilon_{\tau_{-(\mu)}}(1 \oplus -1)\|_{\mathcal{K}(\mathcal{E}_\lambda^0 \otimes \varepsilon_{\tau_{-(\mu)}} \mathbb{C}^2)} \leq 2 + |\mu|^p. \)

For any \(x \in K_1(O_\lambda)\), represented by a unitary \(u\), there is a finite set \(F_u \subseteq \mathcal{Y}_\lambda\) such that

\[ x \otimes_{\mathcal{O}_\lambda} [\mathcal{E}_\lambda^0, D_{\lambda}] \otimes_{C(O_\lambda)} [BP_\tau(\tau)] = \sum_{\mu \in F_u} x \otimes_{\mathcal{O}_\lambda} [\mathcal{E}_\lambda^0, D_{\lambda}] \otimes_{C(O_\lambda)} [\mathcal{E}_\lambda^0, D_{\lambda}] \]

\[ = \sum_{\mu \in F_u} x \otimes (\varepsilon_{\tau_{-(\mu)}} - \varepsilon_{\tau_{-(\mu)}})[\mathcal{E}_\lambda^0, D_{\lambda}]. \]

If such a finite set \(F_u\) does not exist, the index pairing \(x \otimes_{\mathcal{O}_\lambda} [\mathcal{E}_\lambda^0, D_{\lambda}] \otimes_{C(O_\lambda)} [BP_\tau(\tau)]\) cannot be well defined. The cylinder condition implies that for any nonempty word \(\mu\) it holds that the first letter of \(\tau_{+(\mu)}\) is the same as that of \(\tau_{-(\mu)}\). Hence

\[ x \otimes_{\mathcal{O}_\lambda} [\mathcal{E}_\lambda^0, D_{\lambda}] \otimes_{C(O_\lambda)} [BP_\tau(\tau)] = x \otimes (\varepsilon_{\tau_{-(\lambda)}} - \varepsilon_{\tau_{-(\lambda)}})[\mathcal{E}_\lambda^0, D_{\lambda}]. \]

The computation \([\mathcal{E}, D_{\lambda}] \otimes_{C(O_\lambda)} [BP_\tau(\tau)] \otimes Q = (A_{j, \lambda_i} - A_{j, \lambda_i})[\beta_{\lambda_i}] \otimes Q\) follows from Theorem 5.2.3.

**Remark 5.4.7.** It would be interesting to compute the integral class \([\mathcal{E}_\lambda^0, D_{\lambda}] \otimes_{C(O_\lambda)} [BP_\tau(\tau)] \in K^1(O_\lambda)\) explicitly. It is to the authors unclear if the rational computations of Theorem 5.4.5 resists a direct K-homological proof (that would hold in the integral case) due to analytic difficulties or a deeper homological obstruction.

**Appendix A. \(\epsilon\)-unbounded KK-cycles and the Kasparov product**

We describe a weakening of the definition of unbounded KK-cycle [5]. This notion, and in particular Theorem A.6 below, originated from discussions of the second author with A. Rennie. One of the key observations in the proof of this theorem appears in [32] Lemma 51. Related notions are anticipated in the literature, (*eg. [13] [43]*) but to the authors’ knowledge, a concise exposition as in this appendix has not appeared before. The main idea here is to relax the requirement on the commutators \([D, a]\) to be bounded by only asking for \(\epsilon\)-boundedness of these operators.

**Definition A.1.** An operator \(a \in \text{End}_b^*(\mathcal{E})\) has \(\epsilon\)-bounded commutators with the selfadjoint regular operator \(D\) if

1. \(a \text{Dom} D \subset \text{Dom} D;\)
2. \([D, a](1 + D^2)^{\frac{1-\epsilon}{\epsilon}}\) and \((1 + D^2)^{\frac{1-\epsilon}{\epsilon}} [D, a]\) extend to operators in \(\text{End}_b^*(\mathcal{E})\).

In short we say that \([D, a]\) is \(\epsilon\)-bounded. We write \(\delta := \frac{\epsilon}{2}\) throughout this section.

Let us give a geometric example of \(\epsilon\)-bounded commutators. Let \(D\) be a self-adjoint elliptic pseudodifferential operator of order \(m \in (0, 2)\) acting on a vector bundle \(E \to M\) on a closed manifold \(M\). The Hilbert space is \(\mathcal{H} = L^2(M, E)\). The domain of \(D\) is the Sobolev space \(W^{2,m}(M, E)\). If \(a \in C^\infty(M)\), then \([D, a]\) is a pseudodifferential operator of order \(m - 1\). Hence \((1 + D^2)^{\frac{1-\epsilon}{\epsilon}} [D, a]\)
Definition A.2. Let $A$ and $B$ be $C^*$-algebras and $\varepsilon > 0$. An $\varepsilon$-KK-cycle is a pair $(\mathcal{E}, D)$ where $\mathcal{E}$ is a $C^* - (A,B)$-bimodule and $D$ a selfadjoint regular operator such that

1. $a(1 + D^2)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{E});$
2. the space $\operatorname{Lip}^r(\mathcal{E}, D) := \{ a \in A : [D,a] \text{ is } \varepsilon\text{-bounded} \}$ is dense in $A$.

If $B = C^*$ we call an $\varepsilon$-KK-cycle an $\varepsilon$-unbounded Fredholm module.

Remark A.3. Although the definition of $\varepsilon$-boundedness allows for larger classes of unbounded Fredholm modules, obstructions to finite summability remains. The reader can check that the proof of [17, Theorem 8] implies the following statement: if $A$ is a $C^*$-algebra and $(\pi, \mathcal{H}, D)$ is an $\varepsilon$-unbounded Fredholm module with $(1 + D^2)^{-p} \in \mathcal{L}^p(\mathcal{H})$, for some $p \in [1, \infty)$, then there is a tracial state on $A$.

An $\varepsilon$-cycle is an $\varepsilon'$-cycle for any $\varepsilon' \leq \varepsilon$. All of the proofs below rely on the integral representation formula and the estimates in the following lemma.

Lemma A.4. For any $0 < r < 1$

\[(A.47) \quad (1 + D^2)^{-r} = \frac{\sin(r \pi)}{\pi} \int_0^\infty \lambda^{-r}(1 + D^2 + \lambda)^{-1} d\lambda,\]

is a norm convergent integral. Moreover we have the estimates

1. $\| (1 + D^2 + \lambda)^{-r} \| \leq \frac{1}{(1 + \lambda)^r}$;
2. $\| D(1 + D^2 + \lambda)^{-\frac{1}{2}} \| \leq 1$;
3. $\| D^2(1 + D^2 + \lambda)^{-1} \| \leq 1$

The integral formula has been known since the work of Baaj-Julg [5]. A detailed treatment can be found in [13, Appendix A, Remark 3]. The estimates can be found in [13, Appendix A, Remark 5].

Proposition A.5. If $a, b \in \operatorname{Lip}^r(\mathcal{E}, D)$ then $a^*, b^*, ab \in \operatorname{Lip}^r(\mathcal{E}, D)$. In particular $\operatorname{Lip}^r(\mathcal{E}, D)$ is a $\ast$-algebra.

Proof. The statement about the adjoint follows directly from the definition. For the product, we write

$$[D,ab](1 + D^2)^{-\frac{1}{2} + \delta} = a[D,b](1 + D^2)^{-\frac{1}{2} + \delta} + [D,a]b(1 + D^2)^{-\frac{1}{2} + \delta},$$

and observe that the first summand is bounded. For the second one we use the integral expression

$$[b,(1 + D^2)^{-\frac{1}{2} + \delta}] = \frac{\sin(\frac{1}{2} - \delta)\pi}{\pi} \int_0^\infty \lambda^{-\frac{1}{2} + \delta}(1 + D^2 + \lambda)^{-1} ([b,D]D + D[D,b])(1 + D^2 + \lambda)^{-1} d\lambda.$$

Multiplying with $[D,a]$ and estimating the relevant parts of the integral gives

$$\|[D,a]\lambda^{-\frac{1}{2} + \delta}(1 + D^2 + \lambda)^{-1} [b,D]D(1 + D^2 + \lambda)^{-1}\| \leq \frac{C_{a,\delta}C_{b,\delta}}{\lambda^{\frac{1}{2} - \delta}} \| (1 + D^2 + \lambda)^{-r} \| \| (1 + D^2 + \lambda)^{-\frac{1}{2}} \| \leq \frac{C_{a,\delta}C_{b,\delta}}{\lambda^{1 + \delta}},$$

and similarly

$$\|[D,a]\lambda^{-\frac{1}{2} + \delta}(1 + D^2 + \lambda)^{-1} D[D,b](1 + D^2 + \lambda)^{-1}\| \leq \frac{C_{a,\delta}C_{b,\delta}}{\lambda^{1 + \delta}}.$$
We now come to the main result about \(\varepsilon\)-KK-cycles, concerning the bounded transform and the relation to KK-theory.

**Theorem A.6** (cf. [32]). *The bounded transform \((E, D(1 + D^2)^{-\frac{1}{2}})\) of an \(\varepsilon\)-KK-cycle is an \((A, B)\) Kasparov module and hence defines a class in \(KK_0(A, B)\).*

We note that the proof of this Theorem is carried out analogously to the proof of [92] Lemma 51.

**Proof.** The proof of this relies on the integral formula (A.47) to show that the commutators are compact. Using the estimates from Lemma A.4 we find that

\[
[D(1 + D^2)^{-\frac{1}{2}}, a] = [D, a](1 + D^2)^{-\frac{1}{2}} + D[(1 + D^2)^{-\frac{1}{2}}, a],
\]

the first term being compact since \((1 + D^2)^{-\frac{1}{2}}\) is compact and \([D, a](1 + D^2)^{-\frac{1}{2}}\) is bounded. For the second term we expand

\[
\begin{align*}
D[(1 + D^2)^{-\frac{1}{2}}, a] &= \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} (1 + D^2 + \lambda)^{-1} [a, D] D(1 + D^2 + \lambda)^{-1} \\
&\quad + D(1 + D^2 + \lambda)^{-1} [a, D]^2 (1 + D^2 + \lambda)^{-1} d\lambda.
\end{align*}
\]

(A.48)

Using the estimates from Lemma A.4 we find that

\[
\|\lambda^{-\frac{1}{2}} (1 + D^2 + \lambda)^{-1} [a, D] D(1 + D^2 + \lambda)^{-1}\| \leq \frac{C_{a, \varepsilon}}{2\lambda^{1+\varepsilon}},
\]

and

\[
\|\lambda^{-\frac{1}{2}} D(1 + D^2 + \lambda)^{-1} [a, D]^2 (1 + D^2 + \lambda)^{-1}\| \leq \frac{C_{a, \varepsilon}}{\lambda^{1+\varepsilon}},
\]

where \(C_{a, \varepsilon} := \|[D, a](1 + D^2)^{-\frac{1}{2}}\|\). We conclude that the integral formula converges in norm and the commutators are compact. \(\square\)

Kucerovský [32] gives sufficient conditions for a triple of cycles to represent a Kasparov product.

**Theorem A.7** (cf. [32]). *Let \((E, S)\), \((\mathcal{F}, T)\) and \((E \otimes_B \mathcal{F}, D)\) be a triple of \(\varepsilon\)-unbounded KK-cycles such that

1. for all \(x\) in a dense subspace of \(AE\) the operator

\[
\begin{pmatrix}
D & 0 \\
0 & T
\end{pmatrix} \begin{pmatrix}
0 & T_x \\
T_x^* & 0
\end{pmatrix}
\]

defined on \(\text{Dom} D \oplus \text{Dom} T\), extends to an operator in \(\text{End}_\mathbb{B}(E \otimes_B \mathcal{F})\);

2. \(\text{Dom} D \subset \text{Dom} S \otimes 1\);

3. there is \(\lambda \in \mathbb{R}\) such that \((Dx, S \otimes 1x) + (S \otimes 1x, Dx) \geq -\lambda(x, x)\).

Then \((E \otimes_B \mathcal{F}, D)\) represents the Kasparov product of \((E, S)\) and \((\mathcal{F}, T)\).*

**Proof.** As in [32], conditions 2.) and 3.) imply the positivity condition for the bounded transforms. The proof that condition 1.) implies the bounded connection condition is the same as the proof that an \(\varepsilon\)-unbounded KK-cycle gives a Fredholm module. \(\square\)

The following lemma describes a weakening of the domain preservation condition, and is useful in practice for proving \(\varepsilon\)-boundedness.

\[
(2) \quad \text{Dom} D \subset \text{Dom} S \otimes 1.
\]

\[
(3) \quad \text{there is } \lambda \in \mathbb{R} \text{ such that } (Dx, S \otimes 1x) + (S \otimes 1x, Dx) \geq -\lambda(x, x).
\]

Then \((E \otimes_B \mathcal{F}, D)\) represents the Kasparov product of \((E, S)\) and \((\mathcal{F}, T)\). \(\square\)
Lemma A.8. Suppose a maps a core for $D$ into $\text{Dom} \, D$ and $[D,a](1 + D^2)^{-\frac{1}{2} + \delta}$ and $(1 + D^2)^{-\frac{1}{2} + \delta}[D,a]$ extend to operators in $\text{End} \, _{\pi}(\mathcal{E})$. Then the commutator $[(1 + D^2)^{-\frac{1}{2}}, a]$ maps $\mathcal{E}$ into $\text{Dom} \, D$. Consequently a preserves $\text{Dom} \, D$ and $[D,a](1 + D^2)^{-\frac{1}{2} + \delta}$ and $(1 + D^2)^{-\frac{1}{2} + \delta}[D,a]$ are bounded on $\text{Dom} \, D$.

Proof. Denote the core respected by $a$ by $X$. By \cite{13} Lemma 2.3 and the discussion succeeding it, we can write

\begin{equation}
[a,(1 + D^2)^{-\frac{1}{2}}] = \frac{1}{\pi} \int_0^\infty (1 + D^2 + \lambda)^{-1} D[a]((1 + D^2 + \lambda)^{-1}) d\lambda
\end{equation}

as a norm convergent integral on $X$. The integral expression \cite{13} for $D[(1 + D^2)^{-\frac{1}{2}}, a]$ converges in norm on $X$. Since $X$ is a core, it is of the form $(1 + D^2)^{-\frac{1}{2}} Y$ for some dense $Y \subset \mathcal{E}$. For a Cauchy sequence $y_n \in Y$, with limit $e \in \mathcal{E}$, the integrals \cite{13} and \cite{13} converge in norm at $y_n - y_m$. Thus $[(1 + D^2)^{-\frac{1}{2}}, a]y_n \in \text{Dom} \, D$ is Cauchy for the graph norm and therefore $[(1 + D^2)^{-\frac{1}{2}}, a]e \in \text{Dom} \, D$. From this it follows that for a sequence $y_n \rightarrow e$ we have

\[ a(1 + D^2)^{-\frac{1}{2}} \ y_n = [a,(1 + D^2)^{-\frac{1}{2}}] y_n + (1 + D^2)^{-\frac{1}{2}} a y_n, \]

and thus $a(1 + D^2)^{-\frac{1}{2}} e \in \text{Dom} \, D$. \hfill \qed

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