Lorentz Invariance Breakdown and Constraints from Big-Bang Nucleosynthesis

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The Standard Model Extension formulated by Colladay and Kostelecký is reviewed in the framework of the $^4$He primordial abundance. Upper bounds on coefficients for the Lorentz violation are derived using the present observational data.

Many attempts aimed to construct a quantum theory of gravity have shown that spacetime might have a non trivial topology at the Planck scale. However, the difficulties to build up a complete theory of quantum gravity have motivated the development of semiclassical approaches in which CPT and Lorentz invariance breakdown occur at the level of effective theory. As suggested by Kostelecký and Samuel \cite{1,2}, a departure from Lorentz invariance could manifest itself as an effect of non-locality in string theory: The interactions among tensor fields give rise to non-zero expectation values for the Lorentz tensors that induce spontaneously broken Lorentz symmetry. The general effective field theory describing these Lorentz and CPT violations is the Standard Model Extension (SME) \cite{3}. This model, along with the usual SM and gravitational Lagrangian, includes all possible coordinate-invariant terms constructed with SM, gravitational fields, and violating Lorentz symmetry \cite{4}.

The aim of this paper is to derive, in the framework of Big-Bang Nucleosynthesis (BBN), bounds on parameters of SME. BBN is a cornerstone of standard cosmology: with cosmic background radiation, it provides a strong evidence that during the early phases - i.e. between a fraction of seconds ($\sim 0.01$ s) and few hundred seconds after the BB - the Universe was hot and dense. BBN describes the sequence of nuclear reactions leading to the synthesis of light elements. Our analysis follows the paper by Bernstein, Brown and Feinberg \cite{5}, in which the model of helium synthesis in the early Universe is discussed. During the BBN phase, the geometry of the expanding early Universe is described by the Friedman-Robertson-Walker (FRW) metric $ds^2 = dt^2 - a^2(t)(dX^2 + dY^2 + dZ^2)$. $a(t)$ is the scale factor (the spatial curvature has been taken equal to zero). The dynamical equation for the evolution of the scale factor is $H^2 = \frac{8\pi G}{3} \rho$, where $H = \dot{a}/a$ is the Hubble parameter. The corrections to General Relativity discussed by Kostelecký and Bluhm \cite{4} are not considered here.

As shown by Kostelecký and Lehnert \cite{6}, the dispersion relation of relativistic fermions is given by

$$p\mu p_\mu - m^2 = -2c^{\mu\nu}p_\nu p_\nu + 2a^\mu p_\mu + 2a_\mu c^{\mu\nu}p_\nu p_\nu - a^2 - 2mc^\mu p_\mu + e_\mu e_\nu p^\mu p^\nu + P^2 + A^2$$

(1)

where $\pm$ signs take care of the fermion helicities, and

$$T^\mu = \frac{1}{2}g^{\mu\nu}p_\nu - \frac{H^{\mu\nu}}{2}, \quad \tilde{T}^{\mu\nu} = \frac{1}{2}c^{\nu\alpha\beta}T_{\alpha\beta}.$$ 

All coefficients violate particle Lorentz invariance, and may be flavor depending. Besides, $a_\mu$, $b_\mu$, $e_\mu$, $f_\mu$, $g_{\lambda\mu\nu}$ break CPT. As in \cite{7}, we assume that the only non-vanishing coefficients are $c_{\mu\nu}$ (where $c_{\mu\nu}c^{\mu\nu} = 0$). Due to the symmetries of the FRW metric, we also assume that: a) The spatial coefficients $c_{ij}$ are isotropic, $c_{ij} \simeq c_{ij}$, where $c \simeq c_{XY} \simeq c_{YZ} \simeq c_{ZZ}$ ($c = c_{00}/3$). b) The off-diagonal elements of the coefficients $c_{\mu\nu}$ vanish, i.e. $c_{0i} = 0$, $i = X, Y, Z$. We therefore get $\rho \simeq A\sqrt{E^2 - \tilde{m}^2}$, where $A \equiv 1 + \frac{4\pi c_{00}}{3}$. Moreover, we choose as conventional the photon sector \cite{10}, thus $E_\gamma = k$.

The energy density of relativistic particles ($T \gg m$, $\mu$, where $\mu$ is the chemical potential) filling up the early Universe is given by $\rho = \frac{\pi^2}{2} \int E n_E E^2 d^3p$ \cite{8}, where $g_\mu$ represent the degeneracy factors for particle species involved ($g_e = 2$, $g_\nu = 4$, $g_\nu = 2$), and $n_E$ is the number density of particles. Since $c_{\mu\nu}^2 \ll 1$, the total energy density can be written as

$$\rho = \rho_b + \rho_f \simeq \rho^{(0)} + \rho^{(LIV)}, \quad \rho^{(0)} = \frac{\pi^2}{30} g T^4$$

is the standard energy density, and

$$\rho^{(LIV)} = \frac{\pi^2}{30} \left[ \frac{7}{2} c_{00} g_e + \frac{21}{2} c_{00} g_\nu \right] T^4.$$ 

(2)
\[
g \equiv g_s + \frac{1}{2} g_F = \frac{43}{4} \quad (g_F = g_e + 3g_\mu = 10)
\]
is the effective number of degrees of freedom (it is implicitly assumed that muon and tau neutrinos have a small mass compared to the effective temperature, and that no other massless species are present).

The formation of the primordial \(^4\)He occurs at the early time when the temperature of the Universe was \(T \sim 100\) MeV and the energy and number density were dominated by relativistic particles: leptons (electrons, positrons, neutrinos) and photons. At this stage of the Universe evolution, the smattering of neutrons and protons does not contribute in a relevant way to the total energy density. All these particles are in thermal equilibrium owing to their rapid collisions. Besides, protons and neutrons are kept in thermal equilibrium by their interactions with leptons
\[
\begin{align*}
\nu_e + n &\leftrightarrow p + e^- \quad (3) \\
e^+ + n &\leftrightarrow p + \bar{\nu}_e \quad (4) \\
n &\leftrightarrow p + e^- + \bar{\nu}_e \quad (5)
\end{align*}
\]

To estimate the neutron abundance in the expanding Universe, one has to compute the conversion rate of protons into neutrons, \(\lambda_{pn}(T)\), and its inverse \(\lambda_{np}(T)\). At enough high temperature, the weak interaction rate is given by
\[
\Lambda(T) = \lambda_{np}(T) + \lambda_{pn}(T).
\]

\(\lambda_{np}\) and \(\lambda_{pn}\) are related by \(\lambda_{np}(T) = e^{-Q/T}\lambda_{pn}(T)\), with \(Q = \tilde{m}_n - \tilde{m}_p\). \(\lambda_{np}(T)\) is expressed as the sum of the rates associated to the individual processes (3)-(5)
\[
\lambda_{np} = \lambda_{n+\nu_e \rightarrow p+e^-} + \lambda_{n+e^+ \rightarrow p+\bar{\nu}_e} + \lambda_{n \rightarrow p+e^- + \bar{\nu}_e}.
\]

During the freeze-out period, the following approximations can be done [5]: 1. The temperatures involved are the same, i.e. \(T_\nu = T_e = T_\gamma = T\). 2. The temperature \(T\) is low with respect to the typical energies \(E\) that contribute to the integrals entering in the definition of the rates, \(\text{one then replaces the Fermi-Dirac distribution with the Boltzmann distribution, } n_E \simeq e^{-E/T}\). 3. The electron mass \(m_e\) can be neglected as compared with the electron and neutrino energies, i.e. \(m_e \ll E_{\nu}, E_e\).

The processes (3)-(5) are described by weak interactions which involve the gauge boson \(W\) as mediator. According to the SME’s prescription [3], the interaction \(npW\) may be described in terms of an effective weak gauge theory in which \(p\) and \(n\) are arranged in the doublet \((n \ p)\), and one coefficient corresponding to each Right-component, \(n_R\) and \(p_R\).

The combination of these coefficients leads to the coefficients \(e_\mu\), for \(n\), \(c_\mu\), for \(p\), and \(e_{\mu\nu}\), for the vertex \(npW\) [11]. \(e_\mu\) and \(c_\mu\) enter also in the propagators of \(n\) and \(p\), respectively. For the lepton sector \(e-\nu_e\), \(\bar{e}_\nu\), and \(\bar{e}_{\mu\nu}\) enter in the propagators of \(e\) and \(\nu_e\), respectively, whereas \(\bar{e}_{\mu\nu}\) enter in the vertex \(e\nu_eW\).

The interaction rate for the process (3) [12]
\[
d\lambda_{n+\nu_e \rightarrow p+e^-} = \frac{|M|^2}{2\tilde{m}_n} \frac{d^3p_e}{(2\pi)^3E_e} \frac{d^3p_{\nu_e}}{(2\pi)^3E_{\nu_e}} \frac{d^3p_p}{(2\pi)^3E_p} (2\pi)^4 \delta(4) (p_n + p_{\nu_e} - p_p - p_e) n_{E_e}(1 - n_{E_e}),
\]
where
\[
M = \left(\frac{g_w}{8M_W}\right)^2 \left[\bar{\nu}_e \Sigma_{\mu \nu} \nu_e\right],
\]
\(\Sigma^\mu = (\gamma^\mu + c_{\mu\nu} \gamma^\nu)(c_V - c_A \gamma^5), \quad \Sigma^\mu = (\gamma^\mu + c_{\mu\nu} \gamma^\nu)(1 - \gamma^5)\). In (9), we used the fact that the transferred momentum \(q^\mu = p_n^\mu - p_p^\mu\) satisfies the condition \(q^2 \ll M_W^2\), thus the boson propagator is \(i\eta_{\mu}\Sigma_{\mu\nu}\) (corrections induced by the Lorentz violating (anti-symmetric) coefficients \((k_{\mu\nu})_{\mu\nu}\) [13] are discarded).

In the nucleosynthesis phase, the energy recoil of nucleons can be neglected [5], i.e. \(p_n^\mu = (m_n, 0), p_p^\mu = (m_p, 0)\). After lengthy computations, Eq. (8) becomes
\[
\lambda_{n+\nu_e \rightarrow p+e^-} = \tilde{A} A_{\nu_e}^3 I_y, \quad I_y = \int_y^\infty \frac{1}{\epsilon - Q'} \sqrt{\epsilon^2 - y^2} n_{e-Q'} (1 - n_e) \, d\epsilon,
\]
\(y = \frac{\tilde{m}_e}{T}, \quad Q' = \frac{\tilde{m}_n - \tilde{m}_p}{T}, \quad \tilde{A} = A(1 + B), \quad A = \frac{g^2 + 3g^2}{2\pi^3} = 1.02 \times 10^{-47} eV^{-4}\).
κ_{n+e^{-} \rightarrow p+\bar{\nu}_{e}} = \frac{3}{2} \frac{\lambda_{n+e^{-} \rightarrow p+\bar{\nu}_{e}}}{\lambda_{n+e^{-} \rightarrow p} + \lambda_{n}^{(LIV)}},

(11)

where \lambda_{n+e^{-} \rightarrow p+\bar{\nu}_{e}} = AT^{3}Q^{2} \left[ 4! \left( \xi_{00}^{0} + \xi_{00}^{0} + \frac{B}{4} \right) \frac{T^{2}}{Q^{2}} + 2 \times 3! \left( \xi_{00}^{0} + \xi_{00}^{0} + \frac{B}{4} \right) \frac{Q_{m}}{Q} + 2! \left( \xi_{00}^{0} + \xi_{00}^{0} + \frac{B}{4} + \frac{Q_{m}}{2Q} \right) \right]

Here \(Q_{m} = m_{p}^{n} - m_{n}^{n} \). From (6) and (7), it follows \(\Lambda(T) \approx 2\lambda_{np} = \Lambda_{LIV} \) (\(\Lambda_{LIV} = 4 \lambda_{n+e^{-} \rightarrow p+\bar{\nu}_{e}}\)).

To estimate the primordial mass fraction of \(^4\)He we employ the expression [14]

\[ Y_{p} = \Lambda \left( \frac{2x(t_{f})}{1+x(t_{f})} \right), \]

(12)

where \(\Lambda = e^{-(t_{n}-t_{f})/\tau} \), \(t_{f}\) and \(t_{n}\) are the time of freeze-out of the weak interactions and of the nucleosynthesis, respectively, and \(x(t_{f}) = \exp[-Q/T(t_{f})]\) is the neutron to proton equilibrium ratio. \(\lambda(t_{f})\) is the fraction of neutrons that in the time \(t \in [t_{f}, t_{n}]\) decays into protons. At the radiation era [14,5], \(T(t) \sim (t/sec)^{-1/2}\) MeV, one obtains the deviation from the fractional mass \(Y_{p}\) due to the variation of the freezing temperature \(T_{f}\)

\[ \delta Y_{p} = Y_{p} \left( 1 - \frac{Y_{p}}{2\Lambda} \right) \ln \left( \frac{2\Lambda}{Y_{p}} - 1 \right) - \frac{2t_{f}}{\tau} \frac{\delta T_{f}}{T_{f}}. \]

(13)

In (13) we set \(\delta T_{n} = \delta T(t_{n}) = 0\) being \(T_{n}\) fixed by the deuterium binding energy [15]. From \(\Lambda \approx H = \sqrt{8\pi G / 3} \rho\), one derives the freeze-out temperature \(T = T_{f} \left( 1 + \delta T_{f} / T_{f} \right)\), where \(T_{f} \approx 0.6\) MeV, and

\[ \frac{\delta T_{f}}{T_{f}} \approx -1.83 \xi_{00}^{0} - 1.7 \xi_{00}^{0} + 6.5 \times 10^{-3} \xi_{00}^{0} + 236.16 \xi_{00}^{0} - 236.61 \xi_{00}^{0}. \]

(14)

Estimations on mass fraction \(Y_{p}\) of baryons converted to \(^4\)He during BBN are reported in Table I. By using \(Y_{p} = 0.2476\) and \(\delta Y_{p} < 10^{-4}\), Eq. (13) gives

\[ \frac{\delta T_{f}}{T_{f}} < 4.7 \times 10^{-4}. \]

(15)

Eqs. (14) and (15) lead to upper bounds on coefficients for the Lorentz violation

\[ |\xi_{00}^{0} + 0.93 \xi_{00}^{0} - 3.5 \times 10^{-3} \xi_{00}^{0} + 129.3 \xi_{00}^{0} - 129.05 \xi_{00}^{0} | < 2.56 \times 10^{-4}. \]

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[1] V.A. Kostelecký, S. Samuel, Phys. Rev. D 39, 683 (1989); Phys. Rev. Lett. 63, 224 (1989).
In the SME, the Lagrangian density for quarks is constructed by adding to the usual SM the most general CPT-violating terms [3], which encode the coefficients \((c_Q)_{\mu\nu AB}, (c_U)_{\mu\nu AB}\) and \((c_D)_{\mu\nu AB}\) \((A, B = 1, 2, 3)\). The coefficients \((c_Q)_{\mu\nu AB}\) fix the propagator and the interactions with gauge fields of left-handed quarks, while \((c_U)_{\mu\nu AB}\) and \((c_D)_{\mu\nu AB}\) fix the propagator and the interactions of right-handed quarks. The combination of the coefficients \((c_Q)_{\mu\nu 11}\) and \((c_U)_{\mu\nu 11}\) enters in the propagator of the quark \(u\), while \((c_Q)_{\mu\nu 11}\) and \((c_D)_{\mu\nu 11}\) enter in the propagator of the quark \(d\). The propagators of bounded states of quarks (constituting neutrons and protons) contain combinations of these coefficients. In the effective weak gauge theory, we denote them with \(c_{\mu\nu}^d\) and \(c_{\mu\nu}^u\). Moreover, the vertex of the interaction \(dW\) (hence \(npW\)) is controlled by the coefficients \((c_Q)_{\mu\nu 11}\). We set \((c_Q)_{\mu\nu 11} \equiv c_{\mu\nu}^Q\).

A comment is in order. The available phase space for the final fermion state is [7] \(d\Pi = d^3p/(2\pi)^3 2EN(p)\). In computing the cross sections or decay rates, the normalization factors \(N(p)\) cancels out with ones coming from the transition probability \(|\langle M|^2\rangle\), once the completeness relations of spinors are used. A boost of \(d\Pi\) to another inertial frame is typically non trivial since one has to consider the redefinition of the spinor \(\psi \rightarrow (1 - c_{\mu\nu}^Q \gamma^\mu \gamma^\nu/2)\psi\). Nevertheless, in our context, we are setting \(c_{\mu\nu}^Q = 0\), so that the spinor \(\psi\) is redefined up to a constant factor, which does not alter the final results.

Table I. Values of \(Y_\rho\) and \(\delta Y_\rho\).

| \(Y_\rho\) | \(\delta Y_\rho\) | Ref. |
|-----------|-----------------|-----|
| 0.2476    | \(\pm 0.0010\)  | [17]|
| 0.2452    | \(\pm 0.0015\)  | [18]|
| 0.2429    | \(\pm 0.0009\)  | [19]|
| 0.2479    | \(\pm 0.0004\)  | [20]|
| 0.238     | \(\pm 0.002\)   | [21]|
| 0.244     | \(\pm 0.002\)   | [22]|
| 0.234     | \(\pm 0.003\)   | [23]|