Once More on a Colour Ferromagnetic Vacuum State at Finite Temperature

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Abstract

The spontaneous vacuum magnetization at finite temperature is investigated in $SU(2)$ gluodynamics within a consistent effective potential approach including the one-loop and the correlation correction contributions. To evaluate the latter ones the high temperature limits of the polarization operators of charged and neutral gluon fields in a covariantly constant magnetic field and at high temperature are calculated. The radiation mass squared of charged gluons is found to be positive. It is shown that the ferromagnetic vacuum state having a field strength of order $(gH)^{1/2} \sim g^{4/3}T$ is spontaneously generated at high temperature. The vacuum stability and some applications of the results obtained are discussed.

1 Introduction

The problem on generation of magnetic fields in nonabelian gauge theories at finite temperature is of great importance for particle physics and cosmology. Its positive solution, in particular, will give a theoretical basis for investigations of the QCD vacuum at high temperature and the primordial magnetic fields in the early universe [1], [2]. In literature different mechanisms of producing the fields are discussed (see recent papers [3], [4] and references therein). The aim of the present paper is to investigate in more detail one of them - the spontaneous

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magnetization of the vacuum of non-abelian gauge fields at finite temperature. This problem was studied recently in Refs. [1], [5], [6] where the creation of the vacuum magnetic field has been derived. However, a trusty conclusion about the possibility of this phenomenon as well as estimations of the value of the condensed field have not been obtained, yet. In fact, in papers [1], [6] the effective potential (EP) was calculated in one-loop order. But, as it is known, at finite temperature, $T$, the correlation corrections to the EP are of great importance. They may in an essential way influence the properties of a system. In Ref. [5] a number of the corrections has been taken into account. These authors have determined that the possibility of the vacuum magnetization depends on the sign of the next-to-leading term of the polarization operator (PO) of charged gluons, which has to be calculated in the field $H$ and at $T \neq 0$. However, these calculations had not been done there. Besides, in Refs. [1], [5] the temperature mass squared of the unstable mode present in the spectrum of charged gluons was erroneously identified with the Debye one having the order $\sim g^2 T^2$. However, the latter mass is generated for longitudinal modes of gauge fields. The unstable mode $\epsilon^2 (n=0, \sigma=1) = p_3^2 - gH$, where $g$ is the gauge coupling constant, $n$ is the Landau level number, $\sigma$ is the spin projection on the field direction, is the transversal one produced by a spin interaction with the magnetic field. Therefore, the vacuum stabilization condition at $T \neq 0$ has not been studied in detail. Moreover, a part of diagrams describing correlation corrections was missed there. In papers [7], [8] the scale of the magnetic field at $T \neq 0$ was incorrectly estimated and as a consequence the authors came to negative conclusion about the vacuum magnetization.

To summarize the present time situation we recall that in one-loop approximation the Savvidy level of order $(gH)^{1/2} \sim g^2 T$ is generated. But the way of its stabilization and its existence with the correlation corrections been taken into account remain to be investigated.

It will be important for what follows to remind recent results on observation of the gluon magnetic mass in lattice simulations which was found to be of order $m_{mag} \sim g^2 T$ (as it has been expected from nonperturbative calculation in quantum field theory [4], [10]). The mass screens magnetic fields at distances $l > l_m \sim (g^2 T)^{-1}$ but inside the space region $l < l_m$ they may exist. Since the typical values of particle masses at high temperature are $M \sim gT$, the magnetic fields of order $(gH)^{1/2} \sim gT$ or at least $\sim g^2 T$ are to be of interest. They are able to affect all the processes at high temperatures.

In the present paper we investigate the vacuum magnetization at finite temperature within $SU(2)$ gluodynamics. Considering the Abelian covariantly constant chromomagnetic field $H^a = \delta^{a3} H = \text{const}$ and finite temperature as a background we calculate the EP containing the one-loop and the ring diagram contributions of both neutral and charged gluon fields. To find the latter ones the high temperature limits of the PO of gluon fields at the magnetic background are also calculated. It will be shown that in the adopted approximation the Savvidy
level with the field strength \((gH)^{1/2} \sim g^{4/3}T\) is produced. Thus, we come to the conclusion that the correlation corrections make the field strength stronger as compare to the value derived in one-loop approximation \([5],[6]\). This, in particular, means that although the field is screened for \(l > (g^2T)^{-1}\), the spectrum of charged particles, being formed at Larmor’s radius \(r \sim (gH)^{-1/2} \sim (g^{4/3}T)^{-1}\), is located inside this domain \(r << l\) for small \(g\). Moreover, since at high temperatures the particle masses are of order \(\sim gT\), one has to conclude that a constant background magnetic field is a good approximation to investigate processes at high temperature.

The content is as follows. In Sect.2 the one-loop EP (zero and finite temperature parts) is presented in the form convenient for numerical investigations. Here we also present a general expression describing the contribution of the neutral gluon ring diagrams. In Sect.3 we calculate the Debye mass of the neutral gauge field at \(H, T \neq 0\). In Sect.4 the same is done for the charged gluon part. Sect.5 contains calculation of the two-loop diagram taking into account the self-interaction of charged gluon in the vacuum. The investigation of the vacuum magnetization and its stability as well as the discussion of the results obtained are presented in Sect. 6. In the APPENDIX we adduce the necessary high temperature asymptotic formulae.

## 2 Basic Formulae and General Considerations

Let us consider the pure Yang-Mills action

\[
S = -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{\mu\nu}_a,
\]

where

\[
F_{\mu\nu}^a = \partial_\mu A_\mu^a - \partial_\nu A_\nu^a + gf^{abc} A_\mu^b A_\nu^c
\]

is the field strength. In the \(SU(2)\) gauge theory the Savvidy vacuum state is characterized by the uniform background classical colour magnetic field

\[
H_i^a = \delta_3^a \delta_i^3 H,
\]

where \(H\) is a constant, \(a = 1, 2, 3\) is the isotopic index.

To introduce this background field one has to decompose the gauge field potential as

\[
A_\mu^a = A_\mu^a \text{ext} + B_\mu^a,
\]

where \(A_\mu^a \text{ext} = \delta_2^a \delta_3^3 H x_1\) and \(B_\mu^a(x)\) is the quantum field. In the EP calculations the background gauge fixing condition

\[
\partial_\mu B_{\mu}^a + g f^{abc} A_\mu^b \text{ext} B_{\mu}^c = 0
\]

will be used.
The thermodynamic potential of the model is
\[ \Omega = -\frac{1}{\beta} \log Z, \] (6)
and
\[ Z = Tr \exp(-\beta \mathcal{H}) \] (7)
is the partition function, \( \mathcal{H} \) is the Hamiltonian of the system, \( \beta = 1/T \) is inverse temperature and the trace is calculated over all physical states.

For what follows it will be convenient to introduce the “charged basis” of fields
\[ W^\pm_\mu = \frac{1}{\sqrt{2}} (A^{1\mu} \pm i A^{2\mu}), \quad A_\mu = A^{3\mu}. \] (8)
In this basis the problem of the vacuum magnetization is reduced to calculation and investigation of the vacuum polarization of fields \( W^+_\mu, W^-_\mu \) in the external field \( A^{ext}_\mu \).

To obtain the EP one has to rewrite eq.(7) as a sum over quantum states calculated near the nontrivial classical solution \( A^{ext}_\mu \). This standard procedure is described in many papers and textbooks (see, for example, Refs. [12],[11],[13],[5]) and the result can be written in the form:
\[ V = -\frac{1}{2} \sum_i Tr \log G_i(0) + V^{(2)}(H,T) + \cdots, \] (9)
where \( i \) marks the type of field (\( W^\pm, A \) and ghosts), \( G_i(0) \) is the corresponding propagator in the external field \( A^{ext}_\mu \). In the present paper to incorporate temperature the imaginary time formalism will be used. In the specified above background field, the trace means summation over the discrete Matsubara frequencies, summation and integration over eigen values of the quantum fields. The first term in Eq.(9) corresponds to the one-loop EP whereas the other ones present the contributions of two-, three-, etc. loop corrections.

Among these terms there are ones responsible for dominant contributions of large distances at high temperature - so-called daisy or ring diagrams (see, for example Ref. [12]). This part of the EP, \( V_{\text{ring}}(H,T) \), is important in the case when massless states appear in a system. The ring diagrams of either neutral and charged gluons have to be calculated when the vacuum magnetization at finite temperature is investigated. Really, one first must assume that the field is nonzero, calculate EP \( V(H,T) \) and after that check whether the minimum of it is located at nonzero \( H \). On the other hand, if one investigates problems in the applied external field, the charged fields become massive with the mass \( \sim (gH)^{1/2} \) and have to be omitted. To find \( V_{\text{ring}}(H,T) \) the one-loop polarization operators of charged, \( \Pi(H,T) \), and neutral, \( \Pi^0(H,T) \), gluons in the external field and at finite temperature have to be calculated in the limit of zero momenta. Then
\( V_{\text{ring}}(H, T) \) is given by the series depicted in figures 1, 2. Here, the dashed lines describe the neutral gluons and the wavy lines represent the charged ones, blobs stand for the one-loop polarization operators. The diagrams with one blob show the two-loop terms of the EP, with two blobs - three-loop ones, etc.

Figure 1: The neutral gluon ring diagrams giving contribution to the effective potential.

Figure 2: The charged gluon ring diagrams giving contribution to the effective potential.

The standard procedure to incorporate these contribution is to substitute \( G_i^{[0]} \) in Eq. (9) by the inverse full propagator \( G_i^{-1} = [G_i^{[0]}]^{-1} + \Pi(H, T) \) \[12\], \[13\]. In the case under consideration the expression for the EP can be presented as follows \[4\]:

\[
V_{\text{gen}}^{(1)} = \frac{gH}{2\pi \beta} \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp_3}{2\pi} \sum_{n=0,\sigma=\pm 1}^{+\infty} \log[\beta^2(\omega_l^2 + \epsilon_{n,\sigma,p_3}^2 + \Pi(n, T, H, \sigma))] \quad (10)
\]

\[
+ \frac{1}{2\beta} \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d^3p}{(2\pi)^3} \log[\beta^2(\omega_l^2 + \vec{p}^2 + \Pi^0(H, T))].
\]

This generalized EP is written as the sum of energies of the charged gluon field modes in the external magnetic field (\( \omega_l = \frac{2\pi l}{\beta} \) - discrete imaginary energies)

\[
\epsilon_{n,\sigma}^2 = p_3^2 + (2n + 1 - 2\sigma)gH. \quad (11)
\]

It includes the temperature masses \( \Pi(H, n, \sigma, T) \) of the charged modes, which are also dependent on \( H \), the level number \( n = 0, 1, ... \) and the spin projection \( \sigma = \pm 1 \), as well as the one of the neutral gluons, \( \Pi^0(H, T) \). Then, denoting the sum \( \omega_l^2 + \epsilon^2 \) describing the tree level spectrum as \( D_0^{-1}(p_3, H, T) \), let us rewrite the first term of Eq. (10) as follows
\[
V_{\text{gen}}^{(1)} = \frac{gH}{2\pi\beta} \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp_3}{2\pi} \sum_{n,\sigma} \log[\beta^2 D_0^{-1}(p_3, H, T)]
+ \frac{gH}{2\pi\beta} \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp_3}{2\pi} \{\log[1 + (\omega_l^2 + p_3^2 - gH)^{-1} \Pi(H, T)] + \sum_{n \neq 0, \sigma \neq 1} \log[1 + D_0(\epsilon_n^2, H, T)\Pi(H, n, T, \sigma)]\}. \tag{12}
\]

Here, the first term gives the one-loop contribution of the charged gluons, calculated and investigated in detail in Refs.\[1, 5, 6\] the second one is the sum of the ring diagrams of the unstable mode \(\epsilon_n^2 = 0, \sigma = +1 = p_3^2 - gH\) (as it can be easily checked by expanding the logarithm in a series) and the last term describes the sum of ring diagrams of the stable modes.

In general, the incorporation of the polarization operators into EP may lead to incorrect account for the combinatoric factors of the two-loop diagrams. In our case, to be sure in the correctness of the obtaining results the following procedure has been applied. We subtracted the term

\[
V_s = \frac{gH}{2\pi\beta} \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp_3}{2\pi} \sum_{n=0, \sigma = \pm 1} D_0(p_3, H, T)\Pi(H, n, T, \sigma) \tag{13}
\]

from Eq.\(\text{(10)}\) that separates the contributions of the two-loop diagrams of charged gluons from correlation corrections (the first diagram in Fig.2). Then, it is easy to check that the two-loop diagrams containing the one of loops having the neutral gluon lines (the first diagram in Fig.1) can be accounted for within the second term of the EP \(\text{(10)}\). After that the only two-loop diagram describing the self-interaction of charged gluon fields in the vacuum has been straightforwardly computed.

In one-loop order the neutral gluon contribution is a trivial \(H\)-independent constant which can be omitted. However, these fields are long-range states and they do give \(H\)-dependent EP through the correlation corrections depending on the temperature and field. Below, only the longitudinal neutral modes are included because their Debye’s masses \(\Pi_0^0(H, T)\) are nonzero. The corresponding EP is easily calculated and is given by the expression which can be recognized from \(H = 0\) case \[9, 13\]:

\[
V_{\text{ring}} = \frac{1}{24} \Pi_0^0(H, T)T^2 - \frac{T}{12\pi} [\Pi_0^0(H, T)]^{3/2} + \frac{(\Pi_0^0(H, T))^2}{32\pi^2} [\log(\frac{4\pi T}{(\Pi_0^0)^{1/2}}) + \frac{3}{4} - \gamma], \tag{14}
\]
and $\Pi^0(H, T) = \Pi^0_{00}(k = 0, H, T)$ is the zero-zero component of the neutral gluon field polarization operator calculated in the external field at finite temperature and taken at zero momentum, $\gamma$ is Euler's constant. The first term in Eq.(14) has the order $\sim g^2$ in the coupling constant, the second term is of order $\sim g^3$ and the last one $- \sim g^4$. Restricting ourselves by the order $\sim g^3$ it will be omitted in what follows. As usually, for $\Pi(H, T)$ the high temperature limits of the functions have to be substituted into Eqs. (14), (12)[9], [13]. As we noted above and one is able to check, this part of the EP correctly accounts for the contributions of the two-loop diagrams with the neutral gluon lines.

The detailed calculations of the one-loop EP $V^{(1)}(H, T)$ have been carried out in Refs. [5], [6]. The EP can be written in the form:

$$V^{(1)} = \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^2} e^{-i\mu^2 s} \sum_{l=-\infty}^{+\infty} \exp(il^2\beta^2/4s) \left[ \frac{gH \cos(2gH s)}{\sin(gH s)} - \frac{1}{s} \right],$$

(15)

where an auxiliary mass parameter $\mu^2 - i\epsilon, \epsilon \to 0$, is introduced which regulates an infrared region and plays the role of the normalization point in the field. It is useful for analytic continuations from weak fields $gH \leq \mu^2$ to the field strengths $gH > \mu^2$ when an imaginary part of the EP is calculated. Then it can be set to zero. The term with $l = 0$ gives the well studied zero temperature part, while other terms describe the statistical part. The proper time ($s$-representation) is used. By integrating over $s$ we present Eq.(15) in the form convenient for numeric investigations [6]:

$$V^{(1)}(H, T) = V^{(1)}(H) + V^{(1)}_\tau(H, T),$$

$$V^{(1)}(H) = \frac{1}{2} H^2 + \frac{(gH)^2}{4\pi^2} \left[ \frac{11}{12} \log\left(\frac{gH}{\mu^2}\right) - i \frac{1}{8\pi} \right],$$

$$V^{(1)}_\tau(H, T) = -\frac{g^2}{\pi^2} \frac{(gH)^{3/2}}{\beta} \sum_{l=1}^{\infty} \frac{2K_1(l\beta(gH)^{1/2}(1 + 2p)^{1/2})}{l} - \frac{\pi}{2} Y_1(l\beta(gH)^{1/2})$$

$$- K_1(l\beta(gH)^{1/2}) - i \frac{1}{2\pi} \frac{(gH)^{3/2}}{\beta} \sum_{l=1}^{\infty} \frac{1}{l} J_1(l\beta(gH)^{1/2}),$$

(16)

where $K_1(x), Y_1(x)$ and $J_1(x)$ are Bessel's functions. Remind that $\mu$ is a subtraction point in the field $H$. The imaginary part of the EP is signaling the vacuum instability. To have a more transparent expression let us calculate the high temperature limit $T >> (gH)^{1/2} >> \mu$ of the $V^{(1)}(H, T)$. By means of the Mellin transformation technique we obtain [5], [6]:

$$V^{(1)}(H, T) = \frac{H^2}{2} + \frac{11 g^2}{48 \pi^2} H^2 \log \frac{T^2}{\mu^2} - \frac{1}{3} \left( \frac{(gH)^{3/2}}{\pi} - \frac{2}{\pi} \right) + O(g^2 H^2).$$

(17)

1This type of the EP was used in Ref.[3] to account for the correlation corrections of charged gluons. However, the corresponding correct expression is quite different (see Eq.(20)).
Note the important cancellation of the $H$-dependent logarithms entering the vacuum and the statistical parts.

Now, let us consider the second term in Eq.(12). An elementary integration gives

$$V_{\text{unstable}} = \frac{gHT}{2\pi} [\Pi(H, T, n = 0, \sigma = +1) - gH]^{1/2} + \frac{i(gH)^{3/2}T}{2\pi}.$$  \hspace{1cm} (18)

From Eqs.(17) and (18) it is seen that imaginary terms are cancelled out in the total. The final EP is real if the relation $\Pi_{\text{unstable}}(H, T) > gH$ holds. To check it the case or not one has to calculate both the field spontaneously generated in the vacuum and the radiation masses of charged gluons. The former value can be immediately calculated by differentiating Eq.(17) with respect to $H$ and setting the result to zero [5],[6]:

$$\left(gH\right)^{1/2} = \frac{g^2}{2\pi} T.$$ \hspace{1cm} (19)

To answer the question whether this vacuum is stable or not one has to compute $\Pi(H, T, n, \sigma)$, which is the average value of the gluon PO taken in the tree level state of the charged gluon, and consider the effective mass squared $M^2(H_c, T) = \Pi(H_c, T, n = 0, \sigma = +1) - gH_c$. If it is positive, one has to conclude that radiation corrections stabilize the vacuum (in one-loop order), otherwise it remains unstable.

## 3 Debye Mass of Neutral Gluons

First, let us calculate the $H$-dependent Debye mass of the neutral gluons. The following procedure will be applied. We calculate the one-loop EP due to vacuum polarization of charged gluons in the external field $H$ and some chemical potential, $\rho$, which plays the role of an auxiliary parameter. Since the EP is the generating functional of the one-particle irreducible Green’s functions of the field $\rho$, one has by differentiating it twice with respect to $\rho$ and setting $\rho = 0$ to obtain the mass squared $m_D^{\text{neutral}} = \Pi_{00}(H, T, \rho_0 = 0)$.

The corresponding temperature dependent part of EP is

$$V^{(1)} = -\frac{gH}{4\pi^2} \sum_{l=1}^{\infty} \int_0^\infty ds \frac{1}{s^2} \exp\left(-l^2\beta^2/4s\right)\left[\frac{1}{\sinh(gHs)} + 2\sinh(gHs)\right] \cosh(\beta l \rho).$$ \hspace{1cm} (20)

All the notations are obvious. Then, after the differentiations we get

$$-m_D^2 = \frac{\partial^2 V^{(1)}}{\partial \rho^2} \bigg|_{\rho=0} = \frac{gH}{\pi^2} \beta^2 \frac{\partial}{\partial \beta^2} \sum_{l=1}^{\infty} \int_0^\infty ds \frac{1}{s} \exp\left(-l^2\beta^2eH/4s\right)\left[\frac{1}{\sinh(s)} + 2\sinh(s)\right].$$ \hspace{1cm} (21)
Expanding $\sinh^{-1}s$ in series over Bernoulli’s polynomials,

$$\frac{1}{\sinh s} = \frac{e^{-s}}{s} \sum_{k=0}^{\infty} \frac{B_k}{k!} (-2s)^k,$$

(22)

and carrying out integration over $s$ in accordance with the standard formula

$$\int_0^\infty ds s^{n-1} \exp(-as - \frac{b}{s}) = 2(a)^{n/2}K_n(2\sqrt{ab}),$$

(23)

$a, b > 0$, we obtain in the high temperature limit $(gH)^{1/2}/T << 1$,

$$-m_D^2 = \frac{gH}{\pi^2} \beta^2 \frac{\partial}{\beta^2} \sum_{l=1}^{\infty} \left[ \frac{8K_1((gH)^{1/2}\beta l)}{l(gH)^{1/2}\beta} + 4K_0((gH)^{1/2}\beta l) + O(\beta) \right].$$

(24)

Hence, summing up series by means of Mellin’s transformation (see the Appendix) and differentiating with respect to $\beta^2$, one finds the Debye mass of neutral gluons,

$$m_D^2 = \frac{2}{3} g^2 T^2 - \frac{gH}{\pi} T - \frac{1}{4\pi^2} (gH)^{1/2} + O((gH)^2/T^2).$$

(25)

Here, the first term is the well known temperature mass squared of gluon and other ones give field-dependent contributions. As it is seen, they are negative. This is important for what follows. It is also interesting that spin does not contribute to the Debye mass in the leading order. Substituting expression (24) into equation (14), we obtain the correlation corrections due to neutral gluons.

### 4 Ring Diagrams for Charged Gluons

Now, we are going to calculate $\Pi_{unstable}(H, T)$ and $\Pi(H, T, n, \sigma)$ which cannot be found from any effective potential and require an explicit calculation of the mass operator of the charged gluon field. This is a separate and sufficiently complicate problem which is discussed in detail in the other publication [15]. Here, we only adduce the necessary for our present purpose results - the high temperature limits of $\Pi_{unstable}(H, T)$ :

$$\Pi_{unstable}(H, T) = \Pi(n = 0, \sigma = 1 | Re\Pi_{\mu\nu}^{charged} | n = 0, \sigma = 1) = 15.62 \frac{g^2}{4\pi} (gH)^{1/2} T,$$

(26)

and of excited states $\Pi(n \neq 0, \sigma \neq +1)$,

$$Re\Pi(p_4 = 0, n, p_3 = 0, H, T, \sigma = +1) = \frac{g^2}{4\pi} (gH)^{1/2} T (15.62 + 4n),$$

(27)

$$Re\Pi(p_4 = 0, n, p_3 = 0, H, T, \sigma = -1) = \frac{g^2}{4\pi} (gH)^{1/2} T (11.44 + 4n).$$
where the average values of the PO in the states of the spectrum \((\prod)\) are calculated. These formulae correspond to the limit \(gH/T^2 << 1\). The operator contains also an imaginary part which describes the decay of the states due to the transitions to lower energy levels. But for the problem under consideration only the real part is needed, since it describes the radiation mass.

Let us note the most important features of the expressions \((26),(27)\). It is seen, at \(H = 0\) no screening magnetic mass is produced in one-loop order, as it should \([9]\). Second, the masses squared of the modes are positive and act to stabilize the spectrum of charged gluons at high temperatures. Thus, one has to conclude that in a nonzero chromomagnetic field the charged transversal gluons become massive at finite temperature.\(^2\)

The Debye mass of charged gluons is found to be \([15]\)

\[
\Pi_{00}(k_4 = 0, k_3 = 0, H, T) = \frac{2}{3}g^2T^2 + \frac{g^2}{4\pi}(gH)^{1/2}T(6 + 4n),
\]  

(28) where again only the real part is presented. As it is seen, the next-to-leading term is the positive growing function of \(n\). It is interesting to note that the \(\Pi(p_4 = 0, p_3 = 0, n, \sigma, H, T)\) is more informative as compared the function \(\Pi(p_4 = 0, \vec{p} = 0, \sigma, T)\) used at zero field. This is because it exactly accounts for the transversal momenta described by \(n\) variable:

\[
p_2^2 + p_1^2 = (2n + 1)gH.
\]

Now, substituting the the expressions \((26),(27), (28)\) into Eq.(12) and integrating over momentum and calculating the sums in \(n\), we obtain the ring diagram contribution of charged gluon fields. The result can be expressed in terms of the generalized \(\xi\)-function and looks as follows,

\[
V_{ch}^{\text{ring}} = \frac{gHT}{2\pi}\{\sqrt{2gHD}[\zeta(-\frac{1}{2}, a_+)+\zeta(-\frac{1}{2}, a_-)+2\zeta(-\frac{1}{2}, a_D)]
\]

- \(\sqrt{2gH}[3\zeta(-\frac{1}{2}, \frac{1}{2})+\zeta(-\frac{1}{2}, \frac{3}{2})]\)

+ (\(\Pi(H, T, n = 0, \sigma = +1)^{1/2}\),

(29)

where the first term in the first squared brackets corresponds to the spin projection \(\sigma = +1\), the second term - \(\sigma = -1\) and the last one describes the part due to longitudinal charged gluons. The terms in the second square brackets give the independent of \(\Pi(H, T)\) part of eq.(12). The last term in the curly brackets is due to the radiation mass of the unstable mode. This expression is real for sufficiently high temperatures. The notations are introduced:

\[
g_{H_D} = gH + \frac{g^2}{2\pi}(gH)^{1/2}T, \, a_- = \frac{1}{2} + \frac{g^2}{4\pi}11.44(gH)^{1/2}T, \, a_+ = \frac{1}{2} + \frac{g^2}{4\pi}19.62(gH)^{1/2}T
\]

and

\[
a_D = \left(\frac{3}{2}g^2T^2 + \frac{3g^2}{2\pi}(gH)^{1/2}T + gH\right)/2gH_D.
\]

\(^2\)This conclusion is in obvious contradiction with the result of Refs. \([7], [8]\) where no radiation mass for the state with \(n = 0, \sigma = +1\) has been derived. These authors have included the dependence of \(\Pi(n = 0, \sigma = +1, H, T)\) on \(H\) through the eigen states of the tree-level spectrum, only. As the operator they used the zero field expression in the high temperature limit. But for transversal modes at \(p_3 = 0\) this is zero \([9]\).
Substituting the expression (25) into eq.(14) and gathering all other contributions (16), (29), we obtain the consistent expression for the EP. To correctly account for the two-loop diagram contributions we have to subtract from it the terms (13), which will be calculated below.

5 Two-Loop Contribution of Charged Gluons

In this section, to complete the calculation of the EP, the contribution of the two-loop vacuum diagram taking into account the self-interaction of charged gluons is computed. As before, we use the Feynmann gauge and the Furry picture for Green’s functions in the coordinate representation. If one denotes the propagator of the charged gluons in the external field \( H \) as \( iG_{\mu\nu}(x, y) \), the contribution of the diagram to the EP can be written as

\[
V_{ch}^{(2)} = -\frac{g^2}{2} \left[ G_{\mu\mu}(x, x)G_{\nu\nu}(x, x) + G_{\mu\nu}(x, x)G_{\nu\mu}(x, x) - 2G_{\mu\nu}(x, x)G_{\mu\nu}(x, x) \right],
\]  

(30)

where the Green function is \([14] \)

\[
G_{\mu\nu}(x, y) = -\frac{1}{16\pi^2} \exp \left\{ -ig \int \frac{y}{x} \left[ A(z) + \frac{1}{2} F \cdot (z - y) \right] \right\}
\]

\[
\int_0^{\infty} \frac{ds}{s^2} \exp \left\{ -\frac{1}{2} \log \frac{\sinh gFs}{gFs} - \frac{i}{4} (x - y) \cdot gF \coth gFs \cdot (x - y) \right\} e^{-2gFs}.
\]

(31)

Here, \( A_\mu(x) \) and \( F_{\mu\nu} \) are the potential and the field strength tensor of the external field. The proper time (s-representation) is introduced. To incorporate temperature in Eq. (31) the method of Ref. \([16] \) will be used. After rotation to the Euclidean space the Matsubara-Green function is expressed in terms of the Green function at zero temperature by the formula

\[
G(x, y; T) = \sum_{n=-\infty}^{+\infty} (-1)^{n+|x|}\lambda G(x - [x]\beta u, y - n\beta u),
\]

(32)

where the corresponding Green function at \( T = 0 \) enters the right-hand side, \( \beta = \frac{1}{T}, u = (0, 0, 0, 1), [x] \) denotes the integer part of \( x_4/\beta \) and we use the parameter \( \lambda = 1 \) for fermions, \( \lambda = 0 \) for boson and ghost fields.

Substituting Eq.(32), where \( G \) in the right-hand side is given by Eq.(31), into expression (30), we obtain the contribution \( V_{ch}^{(2)}(H, T) \). By calculating the trace and setting \( y = x \) in \( G_{\mu\nu}(x, y, T) \) the temperature dependent part of it can be written as follows,

\[
V_{ch}^{(2)}(H, T) = \frac{g^2}{(4\pi)^4} \sum_{n=1}^{\infty} \left[ I_1^2(n, H, \beta) + I_2(n, H, \beta) - 2I_3(n, H, \beta) \right],
\]

(33)
where the notation are introduced:

\[
I_1(n, H, \beta) = 2i \int_0^\infty \frac{ds}{s} \frac{gH}{\sinh gHs} e^{-\frac{s^2 \beta^2}{4}} (1 + \cosh 2gHs)
\]

\[
I_2 - 2I_3 = 2 \left[ \int_0^\infty ds \frac{e^{-\frac{s^2 \beta^2}{4}} gH}{\sinh gHs} \right]^2 + 2 \left[ \int_0^\infty ds \frac{e^{-\frac{s^2 \beta^2}{4}} gH \coth gHs}{\sinh gHs} \right] - 6 \left[ \int_0^\infty ds \frac{e^{-\frac{s^2 \beta^2}{4}} gH}{\sinh gHs} \right]^2.
\]

The zero temperature part is given by the term with \(n = 0\) in the sum over the discrete frequencies (see Ref.\[16\]). It contains divergences and require a renormalization. The statistical part is finite.

To find the high temperature limit of these expressions we first carry out the integration over the parameter \(s\), as in Sect.3. That can easily be done by expanding \(\sinh^{-1}(gHs)\) and \(\coth(gHs)\) in series over the Bernoulli polynomials and integrating over \(s\) in accordance with formula (23). The calculation of sums in \(n\) can be carried out by means of the Mellin transformation technique (see, for example, Ref.\[6\]). The necessary formulae are adduced in the Appendix. In this way we obtain for the leading terms of the limit of interest:

\[
V_{ch}^{(2)}(H, T)|_{T \to \infty} = -\frac{g^2}{4\pi} (gH)^{1/2} T^3 + O(gHT^2).
\]

The negative sign is important for what follows.

The high temperature limit of the term (35) is

\[
V_s(H, T)|_{T \to \infty} = \frac{g^2}{6\sqrt{2\pi}} \zeta\left(\frac{1}{2}, -\frac{1}{2}\right) (gH)^{1/2} T^3 + O(gHT^2).
\]

It was also calculated by the Mellin transformation method. This term must be subtracted from the total EP and therefore gives the negative contribution to the leading terms of the asymptotic expansion because \(\zeta\left(\frac{1}{2}, -\frac{1}{2}\right) = 0.8093\). It is worth to mention that these are the longitudinal modes that determine the high temperature behaviour of \(V^{(2)}(H, T)\). Having obtained the two-loop corrections to the EP, one is able to investigate the spontaneous magnetization of the vacuum at high temperature.

6 Discussion

The derived EP is expressed in terms of the well known special functions. Therefore, it can easily be investigated numerically for any range of parameters entering. As usually, it is convenient to introduce the dimensionless variables: the
field $\phi = (gH)^{1/2}/T$ and the EP $v(\phi, g) = V(H, T)/T^4$. The vacuum magnetization at high temperatures, $T >> (gH)^{1/2}$, $\phi \to 0$, will be investigated within the following limiting form of the EP:

$$v_{\text{total}}(\phi, g)|_{\phi \to 0} = \frac{\phi^4}{2g^2} + \frac{11 \phi^4}{48 \pi^2} \log\left(\frac{T^2}{\mu^2}\right) - \frac{1}{3 \pi} \frac{\phi^3}{\pi} - \frac{g^2}{24\pi} \phi$$

$$- \frac{g^3}{3\sqrt{3} \phi} \frac{1}{2\pi} - \frac{g^2}{4\pi} \phi - \frac{g^2}{6\sqrt{2}\pi} \cdot 0, 8093 \phi + O(g^3).$$

(37)

The logarithmic term is signalling the asymptotic freedom of $g^2(T)$ at high temperatures [19], [6]. It includes explicitly the dependence on the scale parameter $\mu$. Other terms present, respectively, the high temperature asymptotics of the one-loop EP, the neutral gluon and the charged gluon rings and the two-loop diagram contributions. To obtain the term due to $V_{\text{ring}}^{ch}$ the asymptotic expansion of the Zeta-function [20]

$$\zeta\left(-\frac{1}{2}, a_D\right)|_{a_D \to \infty} = -\frac{2}{3} a_D^{3/2} + a_D^{1/2} - \frac{1}{48} a_D^{-1/2} + O(a_D^{-3/2})$$

(38)

has been used. Zeta-functions with $a_+, a_-$ do not contribute in leading order. Since we are searching for the fields $\phi$ of order greater then $g^2$, we can omit the term $g^3$ and obtain for the condensed field

$$(gH)^{1/2}c = \frac{0.6}{\pi^{1/3}} g^{4/3} T.$$ 

(39)

Thus, we come to the conclusion that the ferromagnetic vacuum state does exist at high temperatures. The correlation corrections increase the field strength as compared to the value $(gH)^{1/2}_c \sim g^2 T$ generated in one-loop order.

Now, let us compare the results of investigations in Refs. [1], [5], [6] with that of the present paper. In Ref. [6] the one-loop EP was calculated. In equation (37) this corresponds to the first and third terms which determine the vacuum magnetization of order $(gH)^{1/2} \sim g^2 T$. The authors of Ref. [1] missed the important cancellation of the logarithmically dependent on $H$ terms presenting in both the vacuum and the statistical parts of the EP. Hence, their estimate of the magnetic field strength generated at the GUT scale is actually based on the zero temperature EP calculated by Savvidy [17]. As one can see, there are no dependent on $\log(gH)$ terms in Eq. (37). Further, in Ref. [1] the Debye mass $\sim g^2 T^2$ as the temperature dependent radiation one of the transversal charged gluons has been substituted, that is an incorrect assumption (although, in principle, it is not important from the point of view of the result of our investigation because the radiation mass of charged transversal modes has been found to stabilize the spectrum at high temperature). The Debye mass for the transversal modes was also used in Ref. [6]. Therein the mentioned cancellation of the logarithms was established and in one-loop approximation the condensed field of order $(gH)^{1/2}c \sim g^2 T$
has been obtained. However, the temperature mass \( m^2_D \) was used twice. First, as a heuristic factor providing the vacuum stabilization at high temperature. Second, in the structure of the EP responsible for the ring diagram contribution of the unstable mode. The latter is the main point for the final result. Besides, the contribution of the neutral gluons was missed and the dependence on the Landau level number, \( n \), has not been assumed at all. Therefore, the EP used in Ref.\[5\] gave no possibility to calculate the correct value of the condensed field.

The speculations on impossibility of the spontaneous vacuum magnetization in Refs. \([7],[8]\) are based on the results of Ref.\[1\] and the observations of the gluon magnetic mass at finite temperature of order \( \sim g^2 T \). Since the field at finite temperature was identified with the zero temperature value \((gH)^{1/2}(T = 0) \ll g^2 T\) whereas the spectrum of the charged particles is formed at distances \( l \sim (gH)^{-1/2} \) much longer then the screening magnetic length the absence of the magnetization has been claimed.\(^3\) Our EP is free from these shortcomings. It includes all the relevant terms in leading order and the temperature masses with the correct dependences on \( H \) and \( T \) for all modes. With these improvements made we come to the conclusion that the ferromagnetic vacuum state is spontaneously generated at high temperature and the field strength is found to be of order \((gH)^{1/2}_c \sim g^{4/3} T\).

For small \( g(T) \) this is stronger then the field determined in one-loop order.

To better understand the situation with the vacuum stability, let us first consider the one-loop case. As the effective mass squared we find for the condensed field \((gH)^{1/2}_c = (g^2/2\pi)T\): \[ M^2(H, T) = \Pi(H_c, T, n = 0, \sigma = +1) - gH_c > 0. \]

Thus, the vacuum stabilization is observed. But if one substitutes instead this value the field strengths of order \((gH)^{1/2}_c \sim g^{4/3} T\) described by equation \([21]\) the effective mass squared becomes negative. The gluon one-loop radiation mass does not stabilize the true vacuum magnetic field.

Nevertheless, the one-loop result makes hopeful the idea to have the stable vacuum due to radiation corrections to the charged gluon spectrum. Naturally, to investigate this possibility the gluon polarization operator with the correlation correction included should be calculated. This problem requires an additional investigation. Other interesting possibility is the formation at high temperatures of the gluon electrostatic potential, so-called \( A_0 \) condensate (see survey \([21]\)), which also acts as a stabilizing factor \([3]\). To realize the latter scenario consistently the simultaneous spontaneous generation of both the \( A_0 \) condensate and the magnetic field should be investigated. If again the homogeneous vacuum field will be found to be unstable with these improvements made, the inhomogeneous fields of the lattice type discussed in papers \([3],[22],[23]\) may be created. Since the condensed magnetic field is strong at high temperatures, the lattice structures having the cells of order \( \sim 1/(g^{4/3} T) \ll 1/(g^2 T) \) are located inside the region

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\(^3\)This estimate is used in Ref.\[18\] devoted to investigation of cosmic magnetic fields in inflationary universe. However, in connection with the results presented above some of numeric estimations has to be corrected.
where the fields are not screened by the gluon magnetic mass.

These results may found applications in problems of cosmology, in particular, in studying the primordial magnetic fields in the early universe. In a few recent years the scenario of the evolution of the universe in external hypermagnetic field \[4\], \[24\] became popular. Our calculation of the vacuum magnetization unambiguously determines the possibility of the presence of strong magnetic fields in the hot universe. Other application is the high temperature QCD. Here, one also should take into account the generation of the chromomagnetic field in the deconfining phase. As a conclusion we stress again that Savvidy’s mechanism of the vacuum magnetization does work at high temperature, although a number of questions concerning the vacuum stability has to be investigated in order to derive a final picture.

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Appendix

To calculate the the sums in \( n \) in Sects. 3, 5 the Mellin transformation was used. It is described in detail in Refs. \[25\], \[1\]. Here, we adduce the results for summations of series appeared in the considered cases. In high temperature limit \( gH/T^2 \ll 1 (\omega \to 0) \) one obtains:

\[ \sum_{n=1}^{\infty} \frac{1}{n} K_1(n\omega) = \frac{\zeta(2)}{\omega} - \frac{\pi}{2} + \frac{\omega}{4} (\log \frac{4\pi}{\omega} - \gamma + 1/2) + O(\omega^2), \]

\[ \sum_{n=1}^{\infty} K_0(n\omega) = \frac{1}{2} (\gamma - \log \frac{4\pi}{\omega}) + O(\omega), \]

where \( \gamma \) is Euler’s constant.

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