ON SUMMABILITY OF NONLINEAR MAPPINGS: A NEW APPROACH

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Abstract. The main goal of this paper is to characterize arbitrary nonlinear (non-multilinear) mappings $f : X_1 \times \cdots \times X_n \rightarrow Y$ between Banach spaces that satisfy a quite natural Pietsch Domination-type theorem around a given point $(a_1, ..., a_n) \in X_1 \times \cdots \times X_n$. As a consequence of our approach a notion of weighted summability arises naturally, which may be an interesting topic for further investigation.

1. Introduction

The theory of absolutely summing operators was initiated with Grothendieck´s ideas in the 50s but just in the sixties (see [19, 28]) the results were better understood and fully explored (for details we refer to the book [14]). Besides its intrinsic interest, this theory has beautiful applications in Banach space theory and nice connections with the geometry of the Banach spaces involved (see, for example, [8, 19] or [7] for a more recent approach). Due to the success of the linear theory, it is not a surprise that many authors have devoted their interest to the nonlinear setting; the multilinear theory, however, has a longer history, which seems to start with [3, 20]; for recent different nonlinear approaches and applications we mention [10, 11, 12, 13, 18, 22, 23, 24, 26, 27] and references therein.

Pietsch Domination-Factorization Theorems play a central role in the theory of absolutely summing linear operators and provide an unexpected and beautiful measure theoretic taste in the theory (for details we mention the monographs [2, 9, 14, 30]). In the last decade several different nonlinear versions of Pietsch Domination-Factorization Theorem have appeared in the literature (see, for example, [1, 4, 5, 15, 16, 21]); for this reason, in [6], an abstract unified approach to Pietsch-type results was presented as an attempt to show that all the known Pietsch-type theorems were particular cases of a unified general version. The main problem investigated in the present paper is motivated by the Pietsch-Domination Theorem (PDT) for $n$-linear mappings between Banach spaces, which we describe below.

From now on, if $X_1, ..., X_n, Y$ are Banach spaces over a fixed scalar field which can be either $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\text{Map}(X_1, ..., X_n; Y)$ will denote the set of all arbitrary mappings from $X_1 \times \cdots \times X_n$ to $Y$ (no assumption is necessary). The topological dual of a Banach space $X$ will be denoted by $X^*$ and its closed unit ball will be represented by $B_{X^*}$, with the weak-star topology.

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Let $0 < p_1, ..., p_n < \infty$ and $1/p = \frac{1}{n} \sum_{j=1}^{n} 1/p_j$. An $n$-linear mapping $T : X_1 \times \cdots \times X_n \rightarrow Y$ is $(p_1, ..., p_n)$-dominated if there is a constant $C > 0$ so that

$$
(1.1) \quad \left( \sum_{j=1}^{m} \left\| T(x^{(1)}_j, \ldots, x^{(n)}_j) \right\|^p \right)^{1/p} \leq C \prod_{k=1}^{n} \sup_{\varphi \in B_{X_k^*}} \left( \sum_{j=1}^{m} |\varphi(x^{(k)}_j)|^{p_k} \right)^{1/p_k},
$$

regardless of the choice of the positive integer $m$, $x^{(k)}_j \in X_k$, $k = 1, \ldots, n$ and $j = 1, \ldots, m$. The folkloric PDT for $(p_1, ..., p_n)$-dominated multilinear mappings (see [16] or [25] for a detailed proof) asserts that $T$ is $(p_1, ..., p_n)$-dominated if and only if there are Borel probabilities $\mu_k$ on $B_{X_k^*}$, $k = 1, ..., n$, and a constant $C > 0$ such that

$$
(1.2) \quad \left\| T(x^{(1)}, \ldots, x^{(n)}) \right\| \leq C \left( \int_{B_{X_1^*}} \left| \varphi(x^{(1)}) \right|^{p_1} d\mu_1 \right)^{1/p_1} \cdots \left( \int_{B_{X_n^*}} \left| \varphi(x^{(n)}) \right|^{p_n} d\mu_k \right)^{1/p_n},
$$

for all $x^{(j)} \in X_j$, $j = 1, ..., n$.

A related question, not covered by the abstract approach presented in [6], arises:

**Problem 1.1.** If $(a_1, ..., a_n) \in X_1 \times \cdots \times X_n$, what kind of mappings $f \in \text{Map}(X_1, ..., X_n; Y)$ satisfy, for some $C > 0$ and Borel probabilities $\mu_k$ on $B_{X_k^*}$, $k = 1, ..., n$, the inequality

$$
(1.3) \quad \left\| f(a_1 + x^{(1)}, \ldots, a_n + x^{(n)}) - f(a_1, \ldots, a_n) \right\| \leq C \prod_{k=1}^{n} \left( \int_{B_{X_k^*}} \left| \varphi(x^{(k)}) \right|^{p_k} d\mu_k \right)^{1/p_k},
$$

for all $x^{(j)} \in X_j$, $j = 1, ..., n$?

In the next section we solve Problem 1.1.

2. Main Result

Let $0 < p_1, ..., p_n < \infty$ and $1/p = \frac{1}{n} \sum_{j=1}^{n} 1/p_j$. We will say that $f \in \text{Map}(X_1, ..., X_n; Y)$ is $(p_1, ..., p_n)$-dominated at $(a_1, ..., a_n) \in X_1 \times \cdots \times X_n$ if there is a $C > 0$ and there are Borel probabilities $\mu_k$ on $B_{X_k^*}$, $k = 1, ..., n$, such that (1.3) is valid for all $x^{(j)} \in X_j$, $j = 1, ..., n$.

It is worth mentioning that Pietsch’s original proof of his domination theorem uses Ky Fan Lemma instead of the usual Hahn-Banach separation theorem (see [29]). The use of Hahn-Banach theorem seems to be not adequate for proving our main result; for this task Pietsch’s original idea of using Ky Fan Lemma will be very useful. It is in some sense a nice surprise that Pietsch’s first argument conceived for linear maps has shown to be the more adequate when dealing with a very general and fully nonlinear context.

**Lemma 2.1 (Ky Fan).** Let $K$ be a compact Hausdorff topological space and $F$ be a concave family of functions $f : K \rightarrow \mathbb{R}$ which are convex and lower semicontinuous. If for each $f \in F$ there is a $x_f \in K$ so that $f(x_f) \leq 0$, then there is a $x_0 \in K$ such that $f(x_0) \leq 0$ for every $f \in F$.
For the proof of our main theorem we will need the following lemma (see [17, Page 17]):

**Lemma 2.2.** Let $0 < p_1, \ldots, p_n, p < \infty$ be so that $1/p = \sum_{j=1}^{n} 1/p_j$. Then

$$\frac{1}{p} \prod_{j=1}^{n} q_j^p \leq \sum_{j=1}^{n} \frac{1}{p_j} q_j^p$$

regardless of the choices of $q_1, \ldots, q_n \geq 0$.

**Theorem 2.3.** A map $f \in Map(X_1, \ldots, X_n; Y)$ is $(p_1, \ldots, p_n)$-dominated at $(a_1, \ldots, a_n) \in X_1 \times \cdots \times X_n$ if and only if there is a $C > 0$ such that

$$(2.1) \quad \left( \sum_{j=1}^{m} \left( \left| b_{j}^{(1)} \ldots b_{j}^{(n)} \right| \left\| f(a_1 + x_{j}^{(1)}, \ldots, a_n + x_{j}^{(n)}) - f(a_1, \ldots, a_n) \right\| \right)^{p} \right)^{1/p} \leq C \prod_{k=1}^{n} \sup_{\varphi \in B_{X_k^{*}}} \left( \sum_{j=1}^{m} \left( \left| b_{j}^{(k)} \right| \left\| \varphi(x_{j}^{(k)}) \right\| \right)^{p_k} \right)^{1/p_k}$$

for every positive integer $m$, $(x_{j}^{(k)}, b_{j}^{(k)}) \in X_k \times \mathbb{K}$, with $(j, k) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$.

**Proof.** In order to simplify notation, from now on we will write

$$f(b_{j}^{(k)}, x_{j}^{(k)})_{k=1}^{n} := \left( \left( b_{j}^{(1)} \ldots b_{j}^{(n)} \right| \left\| f(a_1 + x_{j}^{(1)}, \ldots, a_n + x_{j}^{(n)}) - f(a_1, \ldots, a_n) \right\| \right)^{p}.$$ 

Assume the existence of such measures $\mu_1, \ldots, \mu_n$ satisfying [1,3]. Then, given $m \in \mathbb{N}$, $x_{j}^{(l)} \in E_l$ and $b_{j}^{(l)} \in \mathbb{K}$, with $(j, l) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, we have, using Hölder Inequality,

$$\sum_{j=1}^{m} f(b_{j}^{(k)}, x_{j}^{(k)})_{k=1}^{n} \leq C_{p} \prod_{k=1}^{n} \left( \int_{B_{X_k^{*}}} \left( \left| b_{j}^{(k)} \right| \left\| \varphi(x_{j}^{(k)}) \right\| \right)^{p_k} d\mu_k \right)^{\frac{p}{p_k}} \leq C_{p} \prod_{k=1}^{n} \left( \sup_{\varphi \in B_{X_k^{*}}} \left( \sum_{j=1}^{m} \left( \left| b_{j}^{(k)} \right| \left\| \varphi(x_{j}^{(k)}) \right\| \right)^{p_k} \right)^{\frac{p}{p_k}} \right).$$

Hence we have (2.1). Conversely, suppose (2.1) and consider the sets $P(B_{X_k^{*}})$ of the probability measures in $C(B_{X_k^{*}})^{*}$, for all $k = 1, \ldots, n$. It is well-known that each $P(B_{X_k^{*}})$ is compact when each $C(B_{X_k^{*}})^{*}$ is endowed with the weak-star topology. For each $(x_{j}^{(l)})_{j=1}^{m} \in E_l$ and $(b_{j}^{(l)})_{j=1}^{m} \in \mathbb{K}$, with
In fact, since each \( B \) is compact, \( g \in \mathcal{F} \) is concave. In fact, let \( N \) be a positive integer, \( g_k \in \mathcal{F} \) and \( \alpha_k \geq 0 \), \( k = 1, \ldots, N \), so that \( \alpha_1 + \cdots + \alpha_N = 1 \). We have
\[
\sum_{k=1}^{N} \alpha_k g_k ((\mu_i)_{i=1}^{n}) \leq g \left( (x_{i,j}^{(j)})_{j,k=1}^{m}, (b_{j}^{(j)})_{j,k=1}^{m}, (\alpha_k b_{j}^{(j)})_{j,k=1}^{m}, (s,l) \in \{1, \ldots, n \} \times \{1, \ldots, m \} \right).
\]
One can also easily prove that each \( g \in \mathcal{F} \) is convex and continuous. Besides, for each \( g \in \mathcal{F} \) there are measures \( \mu_k^g \in P(B_{X_k}^*) \), \( k = 1, \ldots, n \), so that
\[
g(\mu_1^g, \ldots, \mu_n^g) \leq 0.
\]
In fact, since each \( B_{X_k}^* \) is compact \( (k = 1, \ldots, n) \) there are \( \varphi_k \in B_{X_k}^* \) so that
\[
\sum_{j=1}^{m} \left( b_{j}^{(k)} \left| \varphi_k(x_{j}^{(k)}) \right| \right)^{p_k} = \sup_{\varphi \in B_{X_k}^*} \sum_{j=1}^{m} \left( b_{j}^{(k)} \left| \varphi(x_{j}^{(k)}) \right| \right)^{p_k}.
\]
Now, consider the Dirac measures \( \mu_k^g = \delta_{\varphi_k} \), \( k = 1, \ldots, n \), and hence
\[
g(\mu_1^g, \ldots, \mu_n^g) = \sum_{j=1}^{m} \left[ \frac{1}{p} f(b_{j}, x_{j}) \left| x_{j}^{(k)} \right| \right]_{k=1}^{n} - C_p \sum_{k=1}^{n} \frac{1}{p_k} \int_{B_{X_k}^*} \sum_{j=1}^{m} \left( b_{j}^{(k)} \left| \varphi(x_{j}^{(k)}) \right| \right)^{p_k} d\mu_k^g
\]
\[
= \sum_{j=1}^{m} \left[ \frac{1}{p} f(b_{j}, x_{j}) \left| x_{j}^{(k)} \right| \right]_{k=1}^{n} - C_p \sum_{k=1}^{n} \frac{1}{p_k} \left[ \left( \sup_{\varphi \in B_{X_k}^*} \sum_{j=1}^{m} \left( b_{j}^{(k)} \left| \varphi(x_{j}^{(k)}) \right| \right)^{p_k} \right)^{\frac{1}{p_k}} \right]
\]
\[
\leq \sum_{j=1}^{m} \left[ \frac{1}{p} f(b_{j}, x_{j}) \left| x_{j}^{(k)} \right| \right]_{k=1}^{n} - C_p \frac{1}{p_k} \prod_{k=1}^{n} \left[ \left( \sup_{\varphi \in B_{X_k}^*} \sum_{j=1}^{m} \left( b_{j}^{(k)} \left| \varphi(x_{j}^{(k)}) \right| \right)^{p_k} \right)^{\frac{1}{p_k}} \right]
\]
\[
\leq 0,
\]
where in (*) we have used Lemma 2.2 and in (**) we invoked (2.1). So Ky Fan Lemma applies and we obtain \( \overline{\mu_k} \in P(B_{X_k}^*) \), \( k = 1, \ldots, n \), so that
\[
g(\overline{\mu_1}, \ldots, \overline{\mu_n}) \leq 0
\]
for all \( g \in \mathcal{F} \). Hence
\[
\sum_{j=1}^{m} \left[ \frac{1}{p} f(b_{j}, x_{j}) \left| x_{j}^{(k)} \right| \right]_{k=1}^{n} - C_p \sum_{k=1}^{n} \frac{1}{p_k} \int_{B_{X_k}^*} \sum_{j=1}^{m} \left( b_{j}^{(k)} \left| \varphi(x_{j}^{(k)}) \right| \right)^{p_k} d\overline{\mu_k} \leq 0
\]
and making \( m = 1 \) we get (for every \( b^{(k)} \in \mathbb{K} \) and \( x^{(k)} \in X_k, \ k = 1, \ldots, n \))

\[
\frac{1}{p} \left( \left| b^{(1)} \ldots b^{(n)} \right| \left\| f(a_1 + x^{(1)}, \ldots, a_n + x^{(n)}) - f(a_1, \ldots, a_n) \right\| \right)^p \leq C^p \sum_{k=1}^{n} \frac{1}{p_k} \int_{B_{X_k}^*} \left( \left| b^{(k)} \right| \left| \varphi(x^{(k)}) \right| \right)^{p_k} d\mu_k.
\]

Let \( x^{(1)}, \ldots, x^{(n)} \) and \( b^{(1)}, \ldots, b^{(n)} \neq 0 \) be given and, for \( k = 1, \ldots, n \), define

\[
\tau_k := \left( \int_{B_{X_k}^*} \left( \left| b^{(k)} \right| \left| \varphi(x^{(k)}) \right| \right)^{p_k} d\mu_k \right)^{1/p_k}.
\]

If \( \tau_k = 0 \) for every \( k \) then, from (2.2) we conclude that

\[
\left( \left| b^{(1)} \ldots b^{(n)} \right| \left\| f(a_1 + x^{(1)}, \ldots, a_n + x^{(n)}) - f(a_1, \ldots, a_n) \right\| \right)^p = 0
\]

and we obtain (1.3), as planned. Let us now suppose that \( \tau_j \) is not zero for some \( j \in \{1, \ldots, n\} \). Consider

\[
V = \{ j \in \{1, \ldots, n\}; \tau_j \neq 0 \}
\]

and for each \( \beta > 0 \) define

\[
\vartheta_{\beta,j} = \begin{cases} \left( \tau_j \beta^p \right)^{-1} & \text{if } j \in V \\ 1 & \text{if } j \notin V. \end{cases}
\]

So, from (2.2), we have

\[
\frac{1}{p} f(\vartheta_{\beta,k} b^{(k)}, x^{(k)})_{k=1}^n \leq C^p \sum_{k=1}^{n} \frac{1}{p_k} \int_{B_{X_k}^*} \left( \left| \vartheta_{\beta,k} b^{(k)} \right| \left| \varphi(x^{(k)}) \right| \right)^{p_k} d\mu_k
\]

\[
\leq C^p \sum_{k \in V} \frac{1}{p_k} \vartheta_{\beta,k}^{p_k} \int_{B_{X_k}^*} \left( \left| b^{(k)} \right| \left| \varphi(x^{(k)}) \right| \right)^{p_k} d\mu_k
\]

\[
\leq C^p \sum_{k \in V} \frac{1}{p_k} \left( \tau_k \beta^p \right)^{-p_k} \tau_k^{p_k}
\]

\[
= C^p \sum_{k \in V} \frac{1}{p_k} \frac{1}{\beta^p}
\]

\[
\leq C^p \frac{1}{\beta^p}.
\]

Hence

\[
\vartheta_{\beta,1} \ldots \vartheta_{\beta,n} \frac{1}{p} f(b^{(k)}, x^{(k)})_{k=1}^n \leq C^p \frac{1}{\beta^{1/p}}
\]
and we have
\[
(2.3) \quad f(b^{(k)}_1, x^{(k)}_1) \leq C_p \beta^{-1/p} \left( \varphi_{\beta,1}^{p} \ldots \varphi_{\beta,n}^{p} \right)^{-1} n_k = 1 \leq C_p \beta^{-1/p} \prod_{j \in V} \left( \tau_j \beta^{p_j} \right)^{p} = C_p \beta^{-1/p} \prod_{j \in V} \tau_j^{p}. \]

If \( V \neq \{1, \ldots, n\} \), then
\[
\frac{1}{p} - \sum_{j \in V} \frac{1}{p_j} > 0.
\]
Letting \( \beta \to \infty \) in (2.3) we get
\[
f(b^{(k)}_1, x^{(k)}_1) n_k = 1 = 0
\]
and we again reach (1.3). If \( V = \{1, \ldots, n\} \), from (2.3) we conclude the proof, since
\[
\left( \left| b^{(1)}_j \ldots b^{(n)}_j \right| \left| f(a_1 + x^{(1)}_j, \ldots, a_n + x^{(n)}_j) - f(a_1, \ldots, a_n) \right| \right)^{p} = f(b^{(k)}_1, x^{(k)}_1) n_k = 1 \leq C_p \prod_{j=1}^{n} \tau_j^{p}.
\]

Note that inequality (2.1) seems to arise an idea of weighted summability. We interpret as each \( x^{(k)}_j \) has a “weight” \( b^{(k)}_j \) and in this context the respective sum
\[
\left\| f(a_1 + x^{(1)}_j, \ldots, a_n + x^{(n)}_j) - f(a_1, \ldots, a_n) \right\|
\]
inherits a weight \( \left| b^{(1)}_j \ldots b^{(n)}_j \right| \). It is easy to note that if \( f \) is \( n \)-linear and \( a_1 = \ldots = a_n = 0 \), then inequality (2.1) coincides with the usual non-weighted inequality. So, the concept of weighted summability can be viewed as a natural extension of the multilinear concept to nonlinear (non-multilinear) maps.

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