Initial-boundary value problems for a fourth-order dispersive nonlinear Schrödinger equation in condensed-matter physics and biophysics

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Abstract

In this paper, we start with the spectral analysis of the Lax pair, and use the Fokas method to construct the Riemann-Hilbert (RH) problem of the fourth-order dispersive nonlinear Schrödinger (FODNLS) equation on the half-line. By solving this matrix RH problem, we obtain the potential function solution $u(x,t)$ of the FODNLS equation on the half-line that satisfies the initial value and boundary values. When the parameter $\gamma = 0$, which can be reduced to the RH problem of the classical NLS equation on the half-line. Unlike the classical NLS equation, the jump curve has changed in the complex $\zeta$-plane, which leads to the difference of the bounded analytical region of function $D(x,t,\zeta)$.

Keywords: Riemann-Hilbert problem; Fokas method; fourth-order dispersive nonlinear Schrödinger equation; initial-boundary value problem.

PACS numbers: 02.30.Ik, 02.30.Jr, 03.65.Nk

AMS Subject Classification: 35G31, 35Q15, 35Q51, 37K10, 37K15

1. Introduction

For a long time, finding a method to solve integrable equations has been a very important research topic in theory and application. The display of integrable equations with exact solutions and some special solutions can provide important guarantees for the analysis of its various properties. However, there is no unified method to solve all integrable equations. With the in-depth study of integrable systems by scholars, a series of methods to solve the classic integrable development equation have emerged. For example, inverse scattering transform (IST) method\textsuperscript{[1]}, Hirota bilinear method\textsuperscript{[2]}, Bäcklund transform method\textsuperscript{[3]}, Darboux transform (DT) method\textsuperscript{[4]} and so on. Among them, the IST method is the main analytical method for the exact solution of nonlinear integrable systems. However, due to the IST method is suitable for the limitations of the initial value conditions at infinity, and it is almost only used to study the pure initial value problem of integrable equations, many real-world phenomena and some studies in the fluctuation process not only need to consider the initial value conditions, but the boundary value conditions also need to be considered. Naturally, people need to replace the initial value problems with the initial-boundary value (IBV) problems in the research process.
In 1997, Fokas proposed a unified transformation method from the initial value problem to the IBV problems based on the IST method idea, which is called the Fokas method. This method can be used to study the IBV problems of partial differential equation [5], and in the past 22 years, the Fokas method has been used to study the IBV problems of some classical integrable equations, such as the modified Korteweg-de Vries (MKdV) equation [6], the nonlinear Schrödinger (NLS) equation [7], the Kaup-Newell equation [8], the stationary axisymmetric Einstein equations [9], the Ablowitz-Ladik system [10], the Kundu-Eckhaus equation [11], the Hirota equation [12, 13]. In 2012, Lenells extended the Fokas method to the integrable equation with higher-order matrix Lax pairs [16, 17, 18, 19, 20, 21, 22, 23, 24]. The authors have also done a slice of works on the application of the Fokas method to an integrable equation [25, 26, 27].

In this paper, our work is related to the fourth-order dispersive nonlinear Schrödinger (FODNLS) equation [28, 29] expressed as:

\[ \begin{align*}
    &i\partial_u + \alpha_1 u_{xx} + \alpha_2 u|u|^2 + \frac{\epsilon^2}{12} (\alpha_3 u_{xxxx} + \alpha_4 u^2 u_{xx} + \alpha_5 u^2 \overline{u}_{xx} + \alpha_6 u^2 \overline{u} + \alpha_7 u|u|^2 + \alpha_8 |u|^4 u) = 0, \\
\end{align*} \]

(1.1)

where \( u \) represents the amplitude of the slowly varying envelope of the wave, \( x \) and \( t \) are the normalized space and time variables, \( \epsilon^2 \) is a dimensionless small parameter representing the high-order linear and nonlinear strength, and \( \alpha_j (j = 1, 2, \ldots, 8) \) is the real parameter. The Eq. (1.1) is mainly derived from fiber optics and magnetism. On the one hand, in optics, Eq. (1.1) can simulate the nonlinear propagation and interaction of ultrashort pulses in high-speed fiber-optic transmission systems [30]. On the other hand, in magnetic mechanics, Eq. (1.1) can be used to describe the nonlinear spin excitation of a one-dimensional Heisenberg ferromagnetic chain with octupole and dipole interactions [31]. In particular, when the parameter value is \( \alpha_1 = \alpha_3 = 1, \alpha_2 = \alpha_5 = 2, \alpha_4 = 8, \alpha_6 = \alpha_8 = 6, \alpha_7 = 4, \) and let \( \gamma = \frac{\epsilon^2}{12} \) the Eq. (1.1) can be written as

\[ \begin{align*}
    i\partial_u + u_{xx} + 2u|u|^2 + \gamma (u_{xxxx} + 8u^2 u_{xx} + 2u^2 \overline{u}_{xx} + 6u^2 \overline{u} + 4u|u|^2 + 6|u|^4 u) = 0, \\
\end{align*} \]

(1.2)

which is an integrable model, and many properties have been widely studied, such as, the Lax pair, the infinite conservation laws [32], the breather solution, and the higher-order rogue wave solution based on the DT method [33, 34, 35], the bilinear form and the N-soliton solution with help of the Hirota bilinear method [36, 37], the multi-soliton solutions by using RH approach [38], the bright and dark solitary waves, and rogue wave solution by using phase plane analysis method [39]. However, as far as we know, the FODNLS (1.2) on the half-line has not been studied, and in the following work, we utilize Fokas method to discuss the IBV problems of the FODNLS equation (1.2) on the half-line domain \( \Omega = \{ (x, t) : 0 < z < \infty, 0 < t < T \} \).

The paper is organized as follows. In section 2, we will introducing eigenfunction to spectral analysis of the Lax pair. In section 3, a slice of spectral functions \( y(\zeta), z(\zeta), Y(\zeta), Z(\zeta) \) are further discussed. In section 4, an important theorem is proposed. And the last section is devoted to conclusions.

2. The spectral analysis

Base on Ablowitz-Kaup-Newell-Segur scheme, the Lax pair of Eq. (1.2) is expressed as

\[ \Psi_t = (-i\zeta \Lambda + P)\Psi, \]

(2.1a)
$$
\Psi_t = [(8i\gamma \zeta^4 - 2i\zeta^2)\Lambda - 8i\gamma \zeta^3 P - 4i\gamma \zeta^2 A_1 - 2\zeta A_2 + iA_3] \Psi, 
$$

(2.1b)

where $\Psi = (\Psi_1, \Psi_2)^T$ is the vector eigenfunction, $\zeta$ is a complex spectral parameter, the $2\times 2$ matrices $\Lambda = \text{diag}[1, -1]$, and $P, A, B$ and $C$ are defined by

$$
P = \begin{pmatrix}
0 & u \\
-\bar{u} & 0
\end{pmatrix}, A_1 = \begin{pmatrix}
|u|^2 & u_x \\
\bar{u}_x & -|u|^2
\end{pmatrix}, A_2 = \begin{pmatrix}
\gamma(\bar{u}_x - \bar{u} u_x) & -\gamma u_{xx} - (2\gamma|u|^2 + 1)u \\
-\gamma u_{xx} - (2\gamma|u|^2 + 1)\bar{u} & -\gamma(\bar{u}_x - \bar{u} u_x)
\end{pmatrix},
$$

$$
A_3 = \begin{pmatrix}
\gamma(3|u|^4 - |u_x|^2 + \bar{u} u_{xx} + \bar{u}u_{xx}) + |u|^2 & \gamma u_{xx} + (6\gamma|u|^2 + 1)u_x \\
\bar{u}_{xx} + (6\gamma|u|^2 + 1)\bar{u}_x & -\gamma(3|u|^4 - |u_x|^2 + \bar{u} u_{xx} + \bar{u}u_{xx}) + |u|^2
\end{pmatrix}.
$$

(2.2)

### 2.1. The exact one-form

The Lax pair equations (2.1a)-(2.1b) are rewritten as follows

$$
\Psi_x + i\zeta \Lambda \Psi = P(x,t,\zeta)\Psi, 
$$

(2.3a)

$$
\Psi_t - (8i\gamma \zeta^4 - 2i\zeta^2)\Lambda \Psi = R(x,t,\zeta)\Psi, 
$$

(2.3b)

where

$$
R(x,t,\zeta) = -8\gamma \zeta^3 P - 4i\gamma \zeta^2 A_1 - 2\zeta A_2 + iA_3
$$

$$
= -8\gamma \zeta^3 P + 4i\gamma \zeta^2(P^2 + P)\Lambda + 2\gamma(PP_x - P_x P - P_{xx} + 2P^3)\Lambda - 2\zeta P\Lambda \\
+ i\gamma(3P^4 + P_x^2 - P_{xx} P - PP_{xx} - 6P^2 P_x - P_3)\Lambda - iP^2.
$$

For the convenience of later calculation, we record $\theta = 8\gamma \zeta^4 - 2\zeta^2$ and introduce the following function transformation

$$
\Psi(x,t,\zeta) = G(x,t,\zeta)e^{i\theta t - \zeta^3 x}A, 0 < x < \infty, 0 < t < T.
$$

(2.4)

Then, we get

$$
G_x + i\zeta[\Lambda, G] = PG, 
$$

(2.5a)

$$
G_t - i\theta[\Lambda, G] = RG, 
$$

(2.5b)

which can be expressed as the following full differential

$$
d(e^{i(\theta t - \zeta^3 x)\Lambda} G(x,t,\zeta)) = F(x,t,\zeta),
$$

(2.6)

where exact one-form $F(x,t,\zeta)$ is given by

$$
F(x,t,\zeta) = e^{i(\theta t - \zeta^3 x)\Lambda}(P(x,t,\zeta)dx + R(x,t,\zeta)dt)G(x,t,\zeta),
$$

(2.7)

and $\Lambda$ represents a matrix operator acting on a second order matrix $\Lambda$, i.e. $\Lambda P = [\Lambda, P]$ and $e^{\Lambda P} = e^{\Lambda}Pe^{-\Lambda}$.
On the other hand, it follows from the Eq.(2.8) that the first column of $G_\zeta \in C$, we can calculate the bounded analytic region of the eigenfunctions $[G_j(x, t, \zeta)]_1$ that is $\zeta$ must satisfies

\[
\begin{align*}
[G_1]_1(x, t, \zeta) & : \{\text{Im}\zeta \geq 0 \} \cap \{\text{Im}(4\gamma\zeta^4 - \zeta^2) \geq 0\}, \\
[G_2]_1(x, t, \zeta) & : \{\text{Im}\zeta \geq 0 \} \cap \{\text{Im}(4\gamma\zeta^4 - \zeta^2) \leq 0\}, \\
[G_3]_1(x, t, \zeta) & : \{\text{Im}\zeta \leq 0\}.
\end{align*}
\]
Similarly, it follows from Eq. (2.8) that the second column of $G_j(x, t, \zeta)$ contains $e^{-2i(1-\zeta) + 2i(\theta_t - \tau)}$. Then, for $\zeta \in \mathbb{C}$, we can also calculate the bounded analytic region of the eigenfunctions $[G_j(x, t, \zeta)]_2$, that means $\zeta$ must satisfies

$$[G_1]_2(x, t, \zeta) : [\text{Im}\zeta \leq 0] \cap [\text{Im}(4\gamma\zeta^2 - \zeta^3) \leq 0],$$  

$$[G_2]_2(x, t, \zeta) : [\text{Im}\zeta \leq 0] \cap [\text{Im}(4\gamma\zeta^2 - \zeta^3) \geq 0],$$  

$$[G_3]_2(x, t, \zeta) : [\text{Im}\zeta \geq 0].$$  

where the $[G_j]_k(x, t, \zeta)$ denotes the $k$-columns of $G_j(x, t, \zeta)$. After calculation, we get the bounded analytic region of $G_j(x, t, \zeta)$ as follows

$$G_1(x, t, \zeta) = ([G_1]_1^{\leq \zeta} (x, t, \zeta), [G_1]_2^{\leq \zeta} (x, t, \zeta)),$$

$$G_2(x, t, \zeta) = ([G_2]_1^{\leq \zeta} (x, t, \zeta), [G_2]_2^{\leq \zeta} (x, t, \zeta)),$$

$$G_3(x, t, \zeta) = ([G_3]_1^{\leq \zeta} (x, t, \zeta), [G_3]_2^{\leq \zeta} (x, t, \zeta)).$$

where $G_j^k(x, t, \zeta)$ represents the bounded analytic region of $[G_j(x, t, \zeta)]^k$ is $\zeta \in L_i, i = 1, 2, \ldots, 8$, and $L_i, i = 1, 2, \ldots, 8$ are shown in Figure 2.

To establish the RH problem of the FODNLS equation (1.2), we must also define two special functions $\psi(\zeta)$ and $\phi(\zeta)$ with the eigenfunction $[G_j(x, t, \zeta)]^1$ as follows

$$G_3(x, t, \zeta) = G_1(x, t, \zeta)e^{i(\theta_t - \lambda)}\psi(\zeta),$$  

$$G_2(x, t, \zeta) = G_1(x, t, \zeta)e^{i(\theta_t - \lambda)}\phi(\zeta).$$  

Set $(x, t) = (0, 0)$ in Eq. (2.14a), and let $(x, t) = (0, T)$ in Eq. (2.14b), we obtain the following relationship

$$\psi(\zeta) = G_3(0, 0, \zeta), \quad \phi(\zeta) = G_2(0, 0, \zeta) = [e^{i\tau \Lambda} G_1(0, T, \zeta)]^{-1},$$  

then, we get

$$G_3(x, t, \zeta) = G_1(x, t, \zeta)e^{i(\theta_t - \lambda)}G_3(0, 0, \zeta),$$  

and

$$G_2(x, t, \zeta) = G_1(x, t, \zeta)e^{i(\theta_t - \lambda)}[e^{-i\tau \Lambda} G_1(0, T, \zeta)]^{-1},$$
it follows from the Eqs. (2.16), (2.17) that

$$G_2(x, t, \zeta) = G_3(x, t, \zeta)e^{\theta(x-\zeta)x\lambda}(\phi(\Lambda))^{-1}\phi(\Lambda).$$  \hspace{1cm} (2.18)

Particularly, in the eigenfunction $G_j(x, t, \zeta)$, $j = 1, 2$, when $x = 0$, we have

$$G_1(0, t, \zeta) = ([G_1]_{1}^{L(0,t);L(0,\zeta)}(0, 0, \zeta), [G_1]_{2}^{L(0,t);L(0,\zeta)}(0, 0, \zeta))$$

$$= I + \int_{0}^{t} e^{\theta(x-\tau)x\lambda}(RG_1)(0, \tau, \Lambda)d\tau,$$  \hspace{1cm} (2.19a)

$$G_2(0, t, \zeta) = ([G_2]_{1}^{L(0,t);L(0,\zeta)}(0, 0, \zeta), [G_2]_{2}^{L(0,t);L(0,\zeta)}(0, 0, \zeta))$$

$$= I - \int_{0}^{t} e^{\theta(x-\tau)x\lambda}(RG_2)(0, \tau, \Lambda)d\tau,$$  \hspace{1cm} (2.19b)

and in the eigenfunction $G_1(x, t, \zeta), G_3(x, t, \zeta)$, when $t = 0$, we have

$$G_1(x, 0, \zeta) = ([G_1]_{1}^{L(x,0);L(x,\zeta)}(x, 0, \zeta), [G_1]_{2}^{L(x,0);L(x,\zeta)}(x, 0, \zeta))$$

$$= I + \int_{0}^{x} e^{-\xi(x-\xi)x\lambda}(PG_1)(\xi, 0, \zeta)d\xi,$$  \hspace{1cm} (2.20a)

$$G_3(x, 0, \zeta) = ([G_3]_{1}^{L(x,0);L(x,\zeta)}(x, 0, \zeta), [G_3]_{2}^{L(x,0);L(x,\zeta)}(x, 0, \zeta))$$

$$= I - \int_{0}^{x} e^{-\xi(x-\xi)x\lambda}(PG_3)(\xi, 0, \zeta)d\xi,$$  \hspace{1cm} (2.20b)

Assuming that $u_0(x) = u(x, t = 0)$ is an initial data of the functions $u(x, t)$, and $v_0(t) = u(x = 0, t)$, $v_1(t) = u_x(x = 0, t)$, $v_2(t) = u_{xx}(x = 0, t), v_3(t) = u_{xxx}(x = 0, t)$ are boundary datas of the functions $u_x(x, t), u_{xx}(x, t), u_{xxx}(x, t)$, at this time, the matrix $P(x, 0, \zeta)$ and $R(0, t, \zeta)$ have the following matrix forms, respectively.

$$P(x, 0, \zeta) = \left(\begin{array}{c} 0 \\ -\bar{\Pi}_0 \end{array}\right), \hspace{1cm} R(0, t, \zeta) = \left(\begin{array}{cc} R_{11}(0, t, \zeta) & R_{12}(0, t, \zeta) \\ R_{21}(0, t, \zeta) & -R_{11}(0, t, \zeta) \end{array}\right),$$

with

$$R_{11}(0, t, \zeta) = -4\gamma^2\bar{\zeta}v_0^2 + 2\gamma\zeta v_0 - v_1, R_{12}(0, t, \zeta) = -4\gamma^2\bar{\zeta}v_0 + 2\gamma\zeta v_1 + 2\gamma v_2 + \frac{i\gamma v_3}{2} + i\gamma v_4 + (6\gamma |v_0|^2 + 1)v_1$$

$$R_{21}(0, t, \zeta) = -8\gamma^3\bar{\zeta}v_0^2 - 4\gamma^2 v_1^2 + 2\gamma^2 v_2 + \frac{2\gamma v_3}{2} + 1\gamma v_0 + i\gamma v_4 + (6\gamma |v_0|^2 + 1)v_1$$

$$R_{22}(0, t, \zeta) = -8\gamma^3\bar{\zeta}v_0 + 2\gamma^2 v_1 + 2\gamma^2 v_2 + \frac{2\gamma v_3}{2} + 1\gamma v_0 + i\gamma v_4 + (6\gamma |v_0|^2 + 1)v_1$$

2.3. The other properties of the eigenfunctions

**Proposition 2.1.** The matrix-value functions $G_j(x, t, \zeta) = ([G_j]_{1}(x, t, \zeta), [G_j]_{2}(x, t, \zeta))(j = 1, 2, 3)$ are given in Eq. (2.8) enjoy the following properties:

- $\det G_j(x, t, \zeta) = 1$;
- The $[G_1]_{1}(x, t, \zeta)$ is an analytic function for $\zeta \in L_1 \cup L_3$, and the $[G_1]_{2}(x, t, \zeta)$ is also an analytic function for $\zeta \in L_6 \cup L_8$;
- The $[G_2]_{1}(x, t, \zeta)$ is an analytic function for $\zeta \in L_2 \cup L_4$, and the $[G_2]_{2}(x, t, \zeta)$ is also an analytic function for $\zeta \in L_5 \cup L_7$;
- The $[G_3]_{1}(x, t, \zeta)$ is an analytic function for $\zeta \in L_5 \cup L_6 \cup L_7 \cup L_8$, and the $[G_3]_{2}(x, t, \zeta)$ is also an analytic function for $\zeta \in L_3 \cup L_2 \cup L_5 \cup L_4$.  

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In terms of the Eq. (2.15) and Eqs. (2.22a)-(2.22b), we known that the following properties are true,

Proposition 2.2. The spectral functions $\psi(\zeta)$, and $\phi(\zeta)$ are defined in Eqs. (2.14a) or Eq. (2.15) is expressed as

\[
\psi(\zeta) = 1 - \int_0^\infty e^{i\xi\zeta} (PG\zeta)(\xi, 0; \zeta) d\xi, \quad (2.22a)
\]

\[
\phi^{-1}(\zeta) = 1 + \int_0^T e^{i\eta\zeta} (RG\zeta)(0, \tau; \zeta) d\tau. \quad (2.22b)
\]

It follows from the symmetry properties of $P(x, t, \zeta)$ and $R(x, t, \zeta)$ that

\[
(G_j(x, t, \zeta))_{11} = (G_j(x, t, \zeta))_{11}, \quad (G_j(x, t, \zeta))_{21} = -(G_j(x, t, \zeta))_{12},
\]

then

\[
\psi_{11}(\zeta) = \overline{\psi_{11}(\zeta)}, \quad \psi_{21}(\zeta) = -\overline{\psi_{12}(\zeta)},
\]

\[
\phi_{11}(\zeta) = \overline{\phi_{11}(\zeta)}, \quad \phi_{21}(\zeta) = -\overline{\phi_{12}(\zeta)},
\]

and assume that the $\psi(\zeta)$ and $\phi(\zeta)$ admits the matrix form as follows

\[
\psi(\zeta) = \begin{bmatrix} y(\zeta) \\ z(\zeta) \end{bmatrix}, \quad \phi(\zeta) = \begin{bmatrix} Y(\zeta) \\ Z(\zeta) \end{bmatrix}. \quad (2.23)
\]

In terms of the Eq. (2.15) and Eqs. (2.22a)-(2.22b), we known that the following properties are true,

- \[
\begin{bmatrix} x(\zeta) \\ y(\zeta) \end{bmatrix} = [G_3]^{L_1L_2L_3L_4}(0, 0, \zeta) = \begin{bmatrix} (G_3)_{12}^{L_1L_2L_3L_4}(0, 0, \zeta) \\ (G_3)_{22}^{L_1L_2L_3L_4}(0, 0, \zeta) \end{bmatrix},
\]

where the vector function $[G_3]^{L_1L_2L_3L_4}(x, 0, \zeta)$ satisfy the ordinary differential equation as follows

\[
\frac{\partial}{\partial x} [G_3]^{L_1L_2L_3L_4}(x, 0, \zeta) + 2i\xi \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [G_3]^{L_1L_2L_3L_4}(x, 0, \zeta) = P(x, 0, \zeta)[G_3]^{L_1L_2L_3L_4}(x, 0, \zeta), \quad 0 < x < \infty,
\quad (2.24)
\]

where $P(x, 0, \zeta)$ is given in Eq. (2.21) and

\[
\lim_{x \to \infty} [G_3]^{L_1L_2L_3L_4}(x, 0, \zeta) = (0, 1)^T.
\]

- \[
\begin{bmatrix} -e^{-2i\theta} Z(\zeta) \\ Y(\zeta) \end{bmatrix} = [G_1]^{L_1L_2L_3L_4}(0, T, \zeta) = \begin{bmatrix} (G_1)_{12}^{L_1L_2L_3L_4}(0, t, \zeta) \\ (G_1)_{22}^{L_1L_2L_3L_4}(0, t, \zeta) \end{bmatrix},
\]

where the vector function $[G_1]^{L_1L_2L_3L_4}(0, t, \zeta)$ satisfy the ordinary differential equation as follows

\[
\frac{\partial}{\partial t} [G_1]^{L_1L_2L_3L_4}(0, t, \zeta) - 2i\theta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [G_1]^{L_1L_2L_3L_4}(0, t, \zeta) = R(0, t, \zeta)[G_1]^{L_1L_2L_3L_4}(0, t, \zeta), \quad 0 < t < T,
\quad (2.25)
\]

where $R(0, t, \zeta)$ is given in Eq. (2.21) and

\[
[G_1]^{L_1L_2L_3L_4}(0, 0, \zeta) = (0, 1)^T.
\]
\[ y(-\zeta) = y(\zeta), \quad z(-\zeta) = -z(\zeta), \]
\[ Y(-\zeta) = Y(\zeta), \quad Z(-\zeta) = -Z(\zeta). \]

For \( \zeta \in \mathbb{R} \), \( \det \psi(\zeta) = |y(\zeta)|^2 + |z(\zeta)|^2 = 1. \)

For \( \zeta \in \mathbb{C} \), \( \det \psi(\zeta) = Y(\zeta)Y(\zeta) + Z(\zeta)Z(\zeta) = 1, \quad (\zeta \in \Gamma_m, \text{ if } T = \infty). \)

where curve \( \Gamma_m, m = 1, 2, 3, 4 \) are given in Eq. (2.30).

\[ y(\zeta) = 1 + O(\frac{1}{\zeta}), \quad z(\zeta) = O(\frac{1}{\zeta}), \text{ as } \zeta \to \infty, \]
\[ Y(\zeta) = 1 + O(\frac{1}{\zeta}) + O\left(\frac{e^{-2\theta T}}{\zeta}\right), \quad Z(\zeta) = O(\frac{1}{\zeta}) + O\left(\frac{e^{-2\theta T}}{\zeta}\right), \text{ as } \zeta \to \infty. \]

2.4. The basic Riemann-Hilbert problem

In order to facilitate calculation and formula representation, we introduce the symbolic assumptions as follows

\[ \omega(x, t, \zeta) = \xi x - \theta t, \quad \theta = 8y_3^2 - 2\zeta^2, \quad (2.26a) \]
\[ \rho(\zeta) = y(\zeta)Y(\zeta) + z(\zeta)Z(\zeta), \quad (2.26b) \]
\[ \kappa(\zeta) = y(\zeta)Z(\zeta) - z(\zeta)Y(\zeta), \quad (2.26c) \]
\[ \delta(\zeta) = \frac{z(\zeta)}{y(\zeta)}, \quad \Delta(\zeta) = -\frac{Z(\zeta)}{y(\zeta)\rho(\zeta)}, \quad (2.26d) \]

then, we have

\[ \overline{Z(\zeta)} = y(\zeta)\kappa(\zeta) + z(\zeta)\rho(\zeta), \]
\[ \rho(\zeta)\rho(\zeta) - \kappa(\zeta)\kappa(\zeta) = 1, \]
\[ \rho(\zeta) = 1 + O(\frac{1}{\zeta}), \quad \kappa(\zeta) = O(\frac{1}{\zeta}) \text{ as } \zeta \to \infty, \]
\[ \rho(-\zeta) = \rho(\zeta), \quad \kappa(-\zeta) = -\kappa(\zeta). \]

and the matrix function \( D(x, t, \zeta) \) is defined by

\[ D_+(x, t, \zeta) = \left( \frac{[G_{1}]_1^{L_1\cup L_3}(x, t, \zeta), [G_{3}]_3^{L_1\cup L_3}(x, t, \zeta)}, y(\zeta) \right), \zeta \in L_1 \cup L_3, \quad (2.27a) \]
\[ D_-(x, t, \zeta) = \left( \frac{[G_{1}]_1^{L_2\cup L_4}(x, t, \zeta), [G_{3}]_3^{L_2\cup L_4}(x, t, \zeta)}, \rho(\zeta) \right), \zeta \in L_2 \cup L_4, \quad (2.27b) \]
\[ D_+(x, t, \zeta) = \left( \frac{[G_{1}]_1^{L_5\cup L_7}(x, t, \zeta), [G_{3}]_3^{L_5\cup L_7}(x, t, \zeta)}, \rho(\zeta) \right), \zeta \in L_5 \cup L_7, \quad (2.27c) \]
\[ D_-(x, t, \zeta) = \left( \frac{[G_{1}]_1^{L_6\cup L_8}(x, t, \zeta), [G_{3}]_3^{L_6\cup L_8}(x, t, \zeta)}, y(\zeta) \right), \zeta \in L_6 \cup L_8. \quad (2.27d) \]

Obviously, the above definitions indicates that

\[ \det D(x, t, \zeta) = 1, \quad D(x, t, \zeta) \to I, \text{ as } \zeta \to \infty. \quad (2.28) \]
Theorem 2.3. Suppose that the matrix function \(D(x,t,\zeta)\) is defined by Eqs. (2.27a)-(2.27d) and the potential function \(u(z,t)\in S\), then, the matrix function \(D(x,t,\zeta)\) admits the jump relation on the curve \(\Gamma_m, m=1,\ldots,4\) as follows

\[D_+(x,t,\zeta) = D_-(x,t,\zeta)Q(x,t,\zeta), \zeta \in \Gamma_m, m = 1, \ldots, 4, \quad (2.29)\]

where

\[Q(x,t,\zeta) = \begin{cases} Q_1(x,t,\zeta), & \zeta \in \Gamma_1, \\bar{\zeta} \in (L_1 \cup L_3) \cap (L_2 \cup L_4), \\ Q_2(x,t,\zeta) = Q_3^{-1}Q_4, & \zeta \in \Gamma_2, \\bar{\zeta} \in (L_2 \cup L_4) \cap (L_1 \cup L_3), \\ Q_3(x,t,\zeta), & \zeta \in \Gamma_3, \\bar{\zeta} \in (L_1 \cup L_2) \cap (L_3 \cup L_4), \\ Q_4(x,t,\zeta), & \zeta \in \Gamma_4, \\bar{\zeta} \in (L_3 \cup L_4) \cap (L_1 \cup L_2), \end{cases} \quad (2.30)\]

and

\[Q_1(x,t,\zeta) = \begin{pmatrix} 1 & 0 \\ \Delta(\zeta)e^{2i\omega(\zeta)} & 1 \end{pmatrix}, \quad Q_2(x,t,\zeta) = \begin{pmatrix} 1 - (\delta(\zeta) + \Delta(\zeta))(\Delta(\zeta) + \delta(\zeta)) & (\delta(\zeta) + \Delta(\zeta))e^{-2i\omega(\zeta)} \\ 0 & 1 \end{pmatrix}, \quad (2.31a)\]

\[Q_3(x,t,\zeta) = \begin{pmatrix} 1 & 0 \\ -\Delta(\zeta)e^{2i\omega(\zeta)} & 1 + i\delta(\zeta)^2 \end{pmatrix}, \quad Q_4(x,t,\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.31b)\]

Proof In terms of the Eqs. (2.14a)-(2.14b) and Eq. (2.23), we have

\[\gamma(\zeta)[G_1]_{1 \rightarrow 1}(x,t,\zeta) - \gamma(\bar{\zeta})e^{2i\omega(\zeta)}[G_1]_{1 \rightarrow 1}(x,t,\zeta) = [G_3]_{1 \rightarrow 1}(x,t,\zeta), \quad (2.32a)\]

\[\gamma(\zeta)[G_1]_{2 \rightarrow 1}(x,t,\zeta) + \gamma(\bar{\zeta})[G_1]_{2 \rightarrow 1}(x,t,\zeta) = [G_3]_{2 \rightarrow 1}(x,t,\zeta), \quad (2.32b)\]

and

\[\overline{\gamma(\zeta)}[G_1]_{1 \rightarrow 1}(x,t,\zeta) - \overline{\gamma(\bar{\zeta})e^{2i\omega(\zeta)}}[G_1]_{1 \rightarrow 1}(x,t,\zeta) = [G_2]_{1 \rightarrow 1}(x,t,\zeta), \quad (2.33a)\]

\[\overline{\gamma(\zeta)}[G_1]_{2 \rightarrow 1}(x,t,\zeta) + \overline{\gamma(\bar{\zeta})}[G_1]_{2 \rightarrow 1}(x,t,\zeta) = [G_2]_{2 \rightarrow 1}(x,t,\zeta), \quad (2.33b)\]

according to the Eqs. (2.31a)-(2.32b) and Eqs. (2.26a)-(2.26d) yields

\[\rho(\zeta)[G_3]_{1 \rightarrow 1}(x,t,\zeta) - \eta(\bar{\zeta})e^{2i\omega(\zeta)}[G_3]_{1 \rightarrow 1}(x,t,\zeta) = [G_2]_{1 \rightarrow 1}(x,t,\zeta), \quad (2.34a)\]

\[\rho(\zeta)[G_3]_{2 \rightarrow 1}(x,t,\zeta) + \eta(\bar{\zeta})[G_3]_{2 \rightarrow 1}(x,t,\zeta) = [G_2]_{2 \rightarrow 1}(x,t,\zeta). \quad (2.34b)\]

By the Eqs. (2.27a)-(2.27d) and Eq. (2.29), one have

\[
\begin{align*}
\frac{[G_1]_{1 \rightarrow 1}(x,t,\zeta)}{\gamma(\zeta)} & = \left(\frac{[G_2]_{1 \rightarrow 1}(x,t,\zeta)}{\rho(\zeta)}\right)Q_1(x,t,\zeta), \\
\frac{[G_3]_{1 \rightarrow 1}(x,t,\zeta)}{\gamma(\zeta)} & = \left(\frac{[G_2]_{1 \rightarrow 1}(x,t,\zeta)}{\rho(\zeta)}\right)Q_2(x,t,\zeta), \\
\frac{[G_1]_{2 \rightarrow 1}(x,t,\zeta)}{\eta(\zeta)} & = \left(\frac{[G_2]_{2 \rightarrow 1}(x,t,\zeta)}{\rho(\zeta)}\right)Q_3(x,t,\zeta), \\
\frac{[G_3]_{2 \rightarrow 1}(x,t,\zeta)}{\eta(\zeta)} & = \left(\frac{[G_2]_{2 \rightarrow 1}(x,t,\zeta)}{\rho(\zeta)}\right)Q_4(x,t,\zeta).
\end{align*}
\]

Therefore, we can derive from the Eqs. (2.34a)-(2.34d) that the jump matrices \([Q_i(x,t,\zeta)]^\dagger\) meets the Eq. (2.30).
Proposition 2.5. The matrix function $D(x, t, \zeta)$ is defined by Eqs. (2.27a)-(2.27d) meets the following residue conditions:

\[
\text{Res}[[D(x, t, \zeta)]_1, \zeta_j] = \frac{1}{z(\zeta_j)\dot{y}(\zeta_j)} e^{2i\nu(\zeta_j)} [D(z, t, \zeta_j)]_2, j = 1, \cdots, 2n_1. \tag{2.35a}
\]

\[
\text{Res}[[D(x, t, \zeta)]_2, \zeta_j] = -\frac{1}{z(\zeta_j)\dot{y}(\zeta_j)} e^{-2i\nu(\zeta_j)} [D(z, t, \zeta_j)]_1, j = 1, \cdots, 2n_2. \tag{2.35b}
\]

\[
\text{Res}[[D(x, t, \zeta)]_1, \eta_j] = -\frac{Z(\eta_j)}{y(\eta_j)\dot{y}(\eta_j)} e^{2i\nu(\eta_j)} [D(z, t, \eta_j)]_1, j = 1, \cdots, 2N_1. \tag{2.35c}
\]

\[
\text{Res}[[D(x, t, \zeta)]_2, \eta_j] = \frac{Z(\eta_j)}{y(\eta_j)\dot{y}(\eta_j)} e^{-2i\nu(\eta_j)} [D(z, t, \eta_j)]_2, j = 1, \cdots, 2N_2. \tag{2.35d}
\]

where $\rho(A) = \frac{\delta \rho}{\delta x}$.

**Proof** We only manifest that the residue relationship Eq. (2.35a) as follows:

Due to $D(x, t, \zeta) = \left( [G]_1 \right)_{1,1}^{1,1}(x, t, \zeta) \left( [G]_3 \right)_{2,2}^{1,1}(x, t, \zeta)$, which means that the zeros $\{\zeta_j\}_1^{2n}$ of $y(\zeta)$ are the poles of $\frac{[G]_1}{y(\zeta)}$.

Then, we have

\[
\text{Res} \left[ \frac{[G]_1}{y(\zeta)} \right]_{1,1}(x, t, \zeta), \zeta_j \right] = \lim_{\zeta \rightarrow \zeta_j} \left[ \frac{[G]_1}{y(\zeta)} \right]_{1,1}(x, t, \zeta) = \frac{[G]_1}{y(\zeta_j)} \left[ [G]_2 \right]_{1,2}^{1,1}(x, t, \zeta). \tag{2.36}
\]

Taking $\zeta = \zeta_j$ into the second equation of Eqs. (2.33a)–(2.33b) yields

\[
\left[ [G]_1 \right]_{1,1}^{1,1}(x, t, \zeta_j) = \frac{1}{y(\zeta_j)} e^{2i\nu(\zeta_j)} \left[ [G]_3 \right]_{2,2}^{1,1}(x, t, \zeta_j). \tag{2.37}
\]

According to the Eq. (2.36) and Eq. (2.37), we get

\[
\text{Res} \left[ \frac{[G]_1}{y(\zeta)} \right]_{1,1}(x, t, \zeta), \zeta_j \right] = \frac{1}{z(\zeta_j)\dot{y}(\zeta_j)} e^{2i\nu(\zeta_j)} \left[ [G]_3 \right]_{2,2}^{1,1}(x, t, \zeta_j). \tag{2.38}
\]

Therefore, the Eq. (2.38) can lead to the Eq. (2.35a), and the remaining three residue relationships Eqs. (2.35b)–(2.35d) can be similarly proved.

2.5. The global relation

In this subsection, we give the spectral functions are not independent but meet a nice global relation. In fact, at the boundary of the region $(\xi, \tau) : 0 < \xi < \infty, 0 < \tau < t$, the integral of the one-form $F(x, t, \zeta)$ defined by the Eq. (2.27) is vanished. If we assume $G(x, t, \zeta) = G_3(x, t, \zeta)$ in the one-form $F(x, t, \zeta)$ defined by the Eq. (2.27), one can get

\[
\int_0^\infty e^{i\theta^1}(PG_3)(\xi, 0, \zeta) d\xi + \int_0^0 e^{-i\theta^1}(RG_3)(0, \tau, \zeta) d\tau + e^{-i\theta^1} \times \int_0^\infty e^{i\theta^1}(PG_3)(\xi, t, \zeta) d\xi = \lim_{\zeta \rightarrow \infty} e^{i\theta^1} \int_0^\infty e^{-i\theta^1}(RG_3)(x, \tau, \zeta) d\tau. \tag{2.39}
\]
On the one hand, according to the definition of $\psi(\zeta)$ in Eq. (2.15) and together with the Eq. (2.20b), we known that the first term of the Eq. (2.39) is

$$\phi(\zeta) - I.$$  

Let $x = 0$ in the Eq. (2.16) to get

$$G_3(0, \tau, \zeta) = G_1(0, \tau, \zeta)e^{i\theta \hat{\Lambda}}\psi(\Lambda),$$  

therefore

$$e^{-i\theta \hat{\Lambda}}(RG_3)(0, \tau, \zeta) = [e^{-i\theta \hat{\Lambda}}(RG_1)(0, \tau, \zeta)]\psi(\Lambda).$$  

On the other hand, the Eq. (2.41) and Eq. (2.19a) means that the second term of the Eq. (2.39) is

$$[e^{-i\theta \hat{\Lambda}}RG_1(0, t, \zeta) - I]\psi(\Lambda).$$  

For $x \to \infty$, setting $u(x, t) \in S$, then, the Eq. (2.39) is equivalent to

$$\phi^{-1}(t, \Lambda)\psi(\Lambda) + e^{-i\theta \hat{\Lambda}} \times \int_0^\infty e^{i\theta \hat{\Lambda}}(PG_3)(\xi, t, \zeta)d\xi = I,$$  

where the first column of the Eq. (2.42) is valid for $\zeta \in L_5 \cup L_6 \cup L_7 \cup L_8$ and the second column of the Eq. (2.42) is valid for $\zeta \in L_1 \cup L_2 \cup L_3 \cup L_4$, and $\phi(t, \Lambda)$ is given by

$$\phi^{-1}(t, \Lambda) = e^{-i\theta \hat{\Lambda}}G_1(0, t, \zeta).$$  

Owing to $\phi(\Lambda) = \phi(T, \Lambda)$ and letting $t = T$, then, the Eq. (2.42) is equivalent to

$$\phi^{-1}(\Lambda)\psi(\Lambda) + e^{-i\theta \hat{\Lambda}} \times \int_0^\infty e^{i\theta \hat{\Lambda}}(PG_3)(\xi, T, \zeta)d\xi = I.$$  

Hence, the (12)th-component of the Eq. (2.43) equals

$$\gamma(\Lambda)Z(\Lambda) - Y(\Lambda)\gamma(\Lambda) = e^{-2i\theta \hat{\Lambda}}E(\zeta),$$  

where $E(\zeta)$ expressed as

$$E(\zeta) = \int_0^\infty e^{i\theta \hat{\Lambda}}(PG_3)_{12}(\xi, T, \zeta)d\xi,$$  

which is the so-called global relation.

### 3. The spectral functions

**Definition 3.1. (Related to $\gamma(\zeta), \gamma(\zeta)$)** Let $u_0(x) = u(x, 0) \in S$, the map

$$H_1 : \{u_0(x)\} \to \{y(\zeta), z(\zeta)\},$$

is defined by

$$\begin{pmatrix} z(\zeta) \\ y(\zeta) \end{pmatrix} = \begin{pmatrix} G_3^{L_5 \cup L_6 \cup L_7 \cup L_8}(0; \zeta) \\ (G_3^{L_5 \cup L_6 \cup L_7 \cup L_8})_2(0; \zeta) \end{pmatrix} \left( \begin{pmatrix} G_3^{L_5 \cup L_6 \cup L_7 \cup L_8}(0; \zeta) \\ (G_3^{L_5 \cup L_6 \cup L_7 \cup L_8})_2(0; \zeta) \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} \gamma(\Lambda) \\ \gamma(\Lambda) \end{pmatrix} \right),$$

where $G_3(x, \zeta)$ with $P(x, \zeta)$ are given by Eq. (2.20b) and Eq. (2.21), respectively.
**Proposition 3.2.** The $y(\zeta)$ and $z(\zeta)$ satisfies the properties as follows
(i) For $\Im \zeta < 0$, $y(\zeta)$ and $z(\zeta)$ are analytic functions,
(ii) $y(\zeta) = 1 + O(\frac{1}{\zeta})$, $z(\zeta) = O(\frac{1}{\zeta})$, as $\zeta \to \infty$,
(iii) For $\zeta \in \mathbb{R}$, $\det \psi(\zeta) = |y(\zeta)|^2 + |z(\zeta)|^2 = 1$.
(iv) $S_1 = H_1^{-1} : \{y(\zeta), z(\zeta)\} \to \{u_0(x)\}$, it's defined as follows
\[ u_0(x) = 2i \lim_{\zeta \to \infty} (\zeta D^{(i)}(x, \zeta))_{12}, \]
where $D^{(i)}(x, \zeta)$ meet the following RH problem.

**Remark 3.3.** Assume that
\[
D_+(x, \zeta) = \begin{pmatrix} [G_{11}]_{L_1 \cup L_2 \cup L_4}(x, \zeta) & [G_{12}]_{L_1 \cup L_2 \cup L_4}(x, \zeta) \end{pmatrix}, \quad \Im \zeta \geq 0,
D_-(x, \zeta) = \begin{pmatrix} [G_{11}]_{L_4 \cup L_6 \cup L_8}(x, \zeta) & [G_{12}]_{L_4 \cup L_6 \cup L_8}(x, \zeta) \end{pmatrix}, \quad \Im \zeta \leq 0,
\]

hence, $D^{(i)}(x, \zeta)$ admits the RH problem as:
- $D^{(i)}(x, \zeta) \to I$, $\zeta \to \infty$.
- $y(\zeta)$ possess $2n$ simple zeros $\{\zeta_j\}_{j=1}^{2n}$, $2n = 2n_1 + 2n_2$, let us pretend that $\{\zeta_j\}_{j=1}^{2n_1}$ be part of $L_1 \cup L_2 \cup L_3 \cup L_4$, then, $\{\bar{\zeta}_j\}_{j=1}^{2n_2}$ be part of $L_5 \cup L_6 \cup L_7 \cup L_8$.
- $[D_+^{(i)}](x, \zeta)$ enjoy simple poles for $\zeta = \{\zeta_j\}_{j=1}^{2n_1}$ and the $[D_-^{(i)}](x, \zeta)$ enjoy simple poles for $\zeta = \{\bar{\zeta}_j\}_{j=1}^{2n_2}$. In this case, the residue relations define by
\[
\text{Res}([D_+^{(i)}](x; \zeta)), \zeta_j = \frac{e^{2i\zeta_j x}}{z(\zeta) y(\zeta)} [D_+^{(i)}(x; \zeta_j)]_2, \]
\[
\text{Res}([D_-^{(i)}](x; \bar{\zeta}), \bar{\zeta}_j = \frac{e^{-2i\zeta_j x}}{z(\bar{\zeta}) y(\bar{\zeta})} [D_-^{(i)}(x; \bar{\zeta}_j)]_1.
\]

**Definition 3.4.** (Related to $Y(\zeta), Z(\zeta)$). Set $v_0(t), v_1(t), v_2(t), v_3(t) \in \mathcal{S}$, the map
\[
H_2 : \{v_0(t), v_1(t), v_2(t), v_3(t)\} \to \{Y(\zeta), Z(\zeta)\},
\]
is defined by
\[
\begin{pmatrix} Y(\zeta) \\ Z(\zeta) \end{pmatrix} = [G_{21}]_{L_1 \cup L_2 \cup L_4 \cup L_8}(t, \zeta) = \begin{pmatrix} (G_{21})_{L_1 \cup L_2 \cup L_4 \cup L_8}(t, \zeta) \\ (G_{22})_{L_1 \cup L_2 \cup L_4 \cup L_8}(t, \zeta) \end{pmatrix},
\]
where $G_{2}(t, \zeta)$ with $R(t, \zeta)$ are given by Eq. (2.18) and Eq. (2.19), respectively.
Proposition 3.5. The $Y(\zeta)$ and $Z(\zeta)$ satisfies the properties as follows

(i) For $\text{Im} \theta \leq 0$, $Y(\zeta), Z(\zeta)$ are analytic functions,

(ii) $Y(\zeta) = 1 + O(e^{\frac{2\pi i}{\zeta}}), Z(\zeta) = O(e^{\frac{2\pi i}{\zeta}})$, as $\zeta \to \infty$,

(iii) For $\zeta \in C$, $\det \Phi(\zeta) = Y(\zeta)Y(\zeta) + Z(\zeta)Z(\zeta) = 1$, ($\theta \in R$, if $T = \infty$),

(iv) $S_2 = H_2^{-1} : [Y(\zeta), Z(\zeta)] \to \{v_0(t), v_1(t), v_2(t), v_3(t)\}$, is defined by

\[ v_0(t) = 2i(D^{(1)}(t))_{12} = 2i \lim_{\zeta \to \infty} (\zeta D^{(i)}(t, \zeta))_{12}, \]

\[ v_1(t) = 4(D^{(2)}(t))_{12} + 2iv_0(t)(D^{(1)}(t))_{12} = \lim_{\zeta \to \infty} [4(\zeta^2 D^{(i)}(t, \zeta))_{12} + 2iv_0(t)(\zeta D^{(i)}(t, \zeta))_{22}], \]

\[ \gamma v_2(t) = -8i\gamma(D^{(3)}(t))_{12} + 4\gamma v_0(t)(D^{(2)}(t))_{22} + 2i\gamma v_0(t)(D^{(1)}(t))_{12} + v_1(t)(D^{(1)}(t))_{22} \]

\[ + 2i(D^{(1)}(t))_{12} - (2\gamma v_0^2(t) + 1)v_0(t) \]

\[ = \lim_{\zeta \to \infty} [-8i\gamma(\zeta^2 D^{(i)}(t, \zeta))_{12} + 4\gamma v_0(t)(\zeta^2 D^{(i)}(t, \zeta))_{22} + 2i\gamma v_0(t)(\zeta D^{(i)}(t, \zeta))_{12} + v_1(t)(\zeta D^{(i)}(t, \zeta))_{22}] \]

\[ + 2i(D^{(1)}(t, \zeta))_{12} - (2\gamma v_0^2(t) + 1)v_0(t), \]

\[ \gamma v_3(t) = -16i\gamma(D^{(4)}(t))_{12} + 4(D^{(2)}(t))_{22} - 8iv_0(t)(D^{(3)}(t))_{22} + 4\gamma v_0(t)(D^{(2)}(t))_{12} + v_1(t)(D^{(2)}(t))_{22} \]

\[ + i(66v_0^2(t) + 1)v_1(t) - 2i[\gamma v_0(t)v_1(t) - v_0(t)v_1(t)(D^{(1)}(t))_{12} - (\gamma v_2(t) + 2\gamma v_0^2(t) + 1)v_0(t)(D^{(1)}(t))_{22}] \]

\[ = \lim_{\zeta \to \infty} [-16i\gamma(\zeta^4 D^{(i)}(t, \zeta))_{12} + 4(\zeta^2 D^{(i)}(t, \zeta))_{12} - 8iv_0(t)(\zeta^2 D^{(i)}(t, \zeta))_{22} \]

\[ + 4\gamma v_0(t)(\zeta^2 D^{(i)}(t, \zeta))_{12} + v_1(t)(\zeta^2 D^{(i)}(t, \zeta))_{22} + i(66v_0^2(t) + 1)v_1(t) \]

\[ - 2i[\gamma v_0(t)v_1(t) - v_0(t)v_1(t)](\zeta D^{(i)}(t, \zeta))_{12} - (\gamma v_2(t) + 2\gamma v_0^2(t) + 1)v_0(t)(\zeta D^{(i)}(t, \zeta))_{22}]], \]

where $D^{(1)}(t, \zeta), D^{(2)}(t, \zeta), D^{(3)}(t, \zeta), D^{(4)}(t, \zeta)$ meets the following asymptotic expansion

\[ D^{(i)}(t, \zeta) = 1 + \frac{D^{(i)}(t, \zeta)}{\zeta} + \frac{D^{(2)}(t, \zeta)}{\zeta^2} + \frac{D^{(3)}(t, \zeta)}{\zeta^3} + \frac{D^{(4)}(t, \zeta)}{\zeta^4} + O(\frac{1}{\zeta^5}), \text{ as } \zeta \to \infty, \]

where $D^{(i)}(t, \zeta)$ meet the following RH problem:

Remark 3.6. Assume that

\[ D^{(i)}(t, \zeta) = \left[ \frac{[G_1]_{1}^{L_1 \cup L_4 \cup L_5 \cup L_8}(t, \zeta)}{Y(\Lambda)}, \frac{[G_3]_{1}^{L_1 \cup L_3 \cup L_6 \cup L_7}(t, \zeta)}{Y(\Lambda)} \right], \text{ Im} \theta \geq 0, \]

\[ D^{(i)}(t, \zeta) = \left[ \frac{[G_1]_{1}^{L_1 \cup L_4 \cup L_5 \cup L_8}(t, \zeta)}{Y(\Lambda)}, \frac{[G_3]_{1}^{L_1 \cup L_3 \cup L_6 \cup L_7}(t, \zeta)}{Y(\Lambda)} \right], \text{ Im} \theta \leq 0, \]

hence, $D^{(i)}(t, \zeta)$ admits the RH problem as follows.

- $D^{(i)}(t, \zeta) \in L_1 \cup L_4 \cup L_5 \cup L_8$ is a sectionaly analytic function.

- $D^{(i)}(t, \zeta) = D^{(i)}(t, \zeta)Q^{(i)}(t, \zeta), \zeta \in \Gamma_n$, $n = 1, 2, 3, 4$, and

\[ Q^{(i)}(t, \zeta) = \left\{ \begin{array}{ll} \frac{1}{Y(\Lambda)}e^{-2\pi it}, & \text{if } \text{Im} \theta \geq 0, \\ \frac{1}{Y(\Lambda)}e^{2\pi it}, & \text{if } \text{Im} \theta \leq 0. \end{array} \right. \]

- $D^{(i)}(t, \zeta) \to 1, \zeta \to \infty$.  


• \(Y(\zeta)\) possesses \(2k\) simple zeros \(\{\nu_j\}_{1}^{2k}, 2k = 2k_1 + 2k_2\), let us pretend that \(\{\nu_j\}_{1}^{2k_1}\) be part of \(L_1 \cup L_3\), then, \(\{\nu_j\}_{1}^{2k_2}\) be part of \(L_2 \cup L_4\).

• \([D_{ij}^{(0)}](t, \zeta)\) enjoy simple poles for \(\zeta = \{\nu_j\}_{1}^{2k_1}\) and the \([D_{ij}^{(0)}]\_2(t, \Lambda)\) enjoy simple poles for \(\zeta = \{\nu_j\}_{1}^{2k_2}\). In this case, the residue relations define by

\[
\text{Res}([D_{ij}^{(0)}(t, \zeta)]_1, \nu_j) = \frac{e^{-4i(4j+1)^2\nu\gamma}}{Z(\nu_j)Y(\nu_j)}[D_{ij}^{(0)}(t, \nu_j)]_2, \quad (3.7a)
\]

\[
\text{Res}([D_{ij}^{(0)}(t, \zeta)]_2, \nu_j) = \frac{e^{4i(4j+1)^2\nu\gamma}}{Z(\nu_j)Y(\nu_j)}[D_{ij}^{(0)}(t, \nu_j)]_1, \quad (3.7b)
\]

4. The Riemann-Hilbert problem

In this part, we give two important results in theorem form.

**Theorem 4.1.** Set \(u_0(x) \in S(R^+), \) the spectral functions \(\phi(\zeta)\) and \(\phi(\zeta)\) are defined in terms of \(y(\zeta), z(\zeta), Y(\zeta), Z(\zeta)\) are showed in Eq. (2.22), respectively. And the \(y(\zeta), z(\zeta), Y(\zeta), Z(\zeta)\) denotes by functions \(u_0(\zeta), v_j(t), j = 0, 1, 2, 3\) are showed in Definition 3.1 and Definition 3.4. Assume that the function \(y(\zeta)\) possess the possible simple zeros are \(\{\nu_j\}_{1}^{2m}\), and the function \(\nu(\zeta)\) possess the possible simple zeros are \(\{\eta_j\}_{1}^{2k}\). Then, the matrix function \(D(x, t, \zeta)\) is the solution of the Riemann-Hilbert problem as follows:

• \(D(x, t, \zeta)\) is a piecewise analytic function for \(\zeta \in C \setminus \Gamma_m (m = 1, \ldots, 4)\).

• \(D(x, t, \zeta)\) jump appears on the curves \(\Gamma_m\), which meets the jump conditions as

\[
D_+(x, t, \zeta) = D_-(x, t, \zeta)Q(x, t, \zeta), \quad \zeta \in \Gamma_m, m = 1, \ldots, 4 \quad (4.1)
\]

• \(D(x, t, \zeta) = 1 + O(\frac{1}{\zeta}), \quad \zeta \to \infty\).

• \(D(x, t, \zeta)\) possesses residue relationship are showed in Proposition 2.5.

Thus, the matrix function \(D(x, t, \zeta)\) exists and is unique. Furthermore, the potential function

\[
u(x, t) = 2i \lim_{\zeta \to \infty} (\zeta D(x, t, \zeta))_{12}, \quad (4.2)
\]

Then \(u(x, t)\) solves the FODNLS equation (1.2). Furthermore,

\[
u(x, 0) = u_0, \quad u(0, t) = v_0, \quad u_{x}(0, t) = v_1, \quad u_{xx}(0, t) = v_2, \quad u_{xxx}(0, t) = v_3.
\]

**Proof.** In fact, the manifest of this RH problem by following [7].

**Theorem 4.2.** (The vanishing theorem) If the matrix function \(D(x, t, \zeta) \to 0 (\zeta \to \infty)\), then, the RH problem in Theorem 4.1 possess only the zero solution.

**Proof.** Indeed, the derivation of this vanishing theorem is given in [7].

**Remark 4.3.** So far, we have obtained the RH problem of the FODNLS equation (1.2) on the half-line, when \(\gamma = 0\), that is the IBV problems of the classical NLS equation on the half-line [7]. Different from the classical NLS equation, where the bounded analytical region and the jump curve of the FODNLS equation (1.2) are different, which the jump curve contains not only the coordinate axis, but also the hyperbola on the \(x\)-axis, and the analytical region is not symmetrical.
5. Conclusions

In this paper, we utilize Fokas method to solve the IBV problems of the FODNLS equation (1.2), which can simulate the nonlinear transmission and interaction of ultrashort pulses in the high-speed optical fiber transmission system, and describe the nonlinear spin excitation phenomenon of one-dimensional Heisenberg ferromagnetic chain with eight poles and dipole interaction. Introduce a slice of important functions to spectral analysis of the Lax pair, established the basic RH problem, and the global relationship between spectral functions is also given. Furthermore, we can analyze the integrable FODNLS equation (1.2) on a finite interval, and also discuss the asymptotic behavior for the solution of the integrable FODNLS equation (1.2). These two questions will be studied in our future research work.

Acknowledgements

This work has been partially supported by the NSFC (Nos. 11601055, 11805114 and 11975145), the NSF of Anhui Province (No.1408085QA06), the University Excellent Talent Fund of Anhui Province (No. gxyq2019096), the Natural Science Research Projects in Colleges and Universities of Anhui Province (No. KJ2019A0637).

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