Localization of the states of a $PT$-symmetric double well

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October 31, 2014

Abstract
We make a nodal analysis of the processes of level crossings in a model of quantum mechanics with a $PT$-symmetric double well. We prove the existence of infinite crossings with their selection rules. At the crossing, before the $PT$-symmetry breaking and the localization, we have a total $P$-symmetry breaking of the states.

1 Introduction
The interest on simple quantum mechanical models is also given by certain similarities with quantum field theory. In particular, is of great interest the possibility of summing divergent perturbation series. This problem is related to the existence of singularities of the levels as functions of the perturbation parameter. Such singularities, due to the level crossings, are not so easy to study in a rigorous way. The semiclassical theory provided good qualitative and quantitative results for lower semiclassical parameter up to the crossing
The exact semiclassical method [13] has given good qualitative and quantitative results for larger values of the parameter [11], [12]. We believe that only the nodal analysis, begun in the papers [7], [8], [9], [2], can give a clear and exhaustive analysis of the level crossings.

In recent times the main interest focused on the $PT$-symmetric Hamiltonians [6], [16]. In particular it was of some interest to prove the reality and the analyticity of the spectrum of certain oscillators [1], [18]. André Martínez and one of us (V. G.) in the paper [2] have proved the Padé summability of the perturbation series to the perturbative levels $\tilde{E}_n(\beta)$ of the imaginary cubic oscillator,

$$H(\beta) = p^2 + x^2 + i\sqrt{\beta}x^3, \quad p^2 := -\frac{d^2}{dx^2}, \quad \beta \neq 0, \quad |\arg(\beta)| < \pi. \quad (1)$$

In this paper [2], was also used the semiclassical method, but the exact results was mostly given by the control of the nodes of the states. Our program is to extend the analysis of the perturbative levels to the other regions of $\beta$ where the level crossings are expected.

By changing representation, we study the spectrum of a semiclassical Hamiltonian, the closed, $PT$-symmetric operator,

$$H_\hbar := \hbar^2p^2 + V(x), \quad V(x) := i(x^3 - x), \quad \hbar > 0, \quad (2)$$

where the derivative of the potential $V'(x)$ has two real zeros: $x_{\pm} = \pm 1/\sqrt{3}$. In some sense, (2) is a $PT$-symmetric double well Hamiltonian. The large $\hbar$ behavior of the levels $E_n(\hbar)$, corresponding to the perturbative levels $\tilde{E}_n(\beta)$, is studied by the other Hamiltonian,

$$K(\alpha) = p^2 + W(\alpha, x), \quad W(\alpha, x) = i(x^3 + \alpha x), \quad \alpha \geq 0. \quad (3)$$

The level $\tilde{E}_m(\alpha)$, real analytic for $\alpha > 0$ [11], [2], is real analytic also for $-\alpha > 0$ small. Since we know the absence of singularities of $E_n(\hbar)$ for small
in certain sectors, we define two other types of levels for small \( \bar{\hbar} > 0 \), by the analytic continuation of \( \hat{E}_n(\alpha) \) on the complex plane, along paths starting from \( \alpha = \bar{\hbar}^{-4/5} \), continuing with \( \langle \alpha \rangle = \bar{\hbar}^{-4/5} \), and arriving to \( \alpha^\pm := \exp(\pm i\pi)\bar{\hbar}^{-4/5} \), respectively. Thus, we define the levels,

\[
E_n^\pm(\bar{\hbar}) := \bar{\hbar}^{4/5} \hat{E}_n(\alpha^\mp), \quad n \in \mathbb{N}, \quad \alpha^\pm = \exp(\pm i\pi)\bar{\hbar}^{-4/5}, \quad \bar{\hbar} > 0.
\]

All such levels are extendible as multi-valued functions, to the sector on the \( \bar{\hbar} \) complex plane:

\[
C^0 = \{ \bar{\hbar} \in \mathbb{C}; \bar{\hbar} \neq 0, |\arg(\bar{\hbar})| < \pi/4 \}.
\]

We extend all the states,

\[
\psi_n^\pm(\bar{\hbar}, z), \quad n \in \mathbb{N}, \quad \bar{\hbar} > 0 \text{ small}, \quad \psi_m(\bar{\hbar}, z), \quad m \in \mathbb{N}, \quad \bar{\hbar} > 0 \text{ large}, \quad z \in \mathbb{C},
\]

for fixed \( \bar{\hbar} \), as entire functions on the complex \( z \) plane. For large fixed \( \bar{\hbar} > 0 \), the representation of the state \( \psi_m(z) \) is taken \( P_x T \)-symmetric, where \( P_x \psi_m(x + iy) = \psi_m(-x + iy) \). In this case, the set of the nodes, as the set of the other zeros, of \( \psi_m(z) \) is \( P_x \) symmetric. For small fixed \( \bar{\hbar} > 0 \), we have \( \psi_n^\pm(z) = P_x T \psi_n^\pm(z) \), so that the set of nodes \( S_n^\pm \) of the state \( \psi_n^\pm \) is the \( P_x \) transform of \( S_n^\pm \). For small \( \bar{\hbar} > 0 \), both the levels \( E_n^\pm(h) \), \( n \in \mathbb{N} \), \( E_n^\pm(h) = \hat{E}_n^\pm(\bar{\hbar}) \), are non-real up to the first crossing. In particular, the perturbative levels have the following semiclassical behavior,

\[
E_n^\pm(h) = \pm i \frac{2}{3\sqrt{3}} + \sqrt{\pm i \sqrt{3}(2n + 1) \bar{\hbar} + O(\bar{\hbar}^2)}, \quad \bar{\hbar} > 0,
\]

in the limit \( \bar{\hbar} \to 0^+ \). These behaviors correspond to the behaviors of the perturbative levels of the Hamiltonian \( \Pi \). In the same limit, all the nodes of the states \( \psi_n^\pm(h) \) shrink to the centers of the wells,

\[
x_{\pm} \in \mathbb{C}^\pm = \{ z \in \mathbb{C}; \pm \Re z > 0 \},
\]

respectively. Now, we prove that the zeros, and in particular the nodes, of a state \( \psi_n^\pm(h) \) cannot reach and cross the imaginary axis (Lemma 5). Moreover, the nodes on \( \mathbb{C}^\pm \) respectively, cannot disappear going to infinite (Lemma 4). This means that the number \( n \) of the nodes of a state \( \psi_n^\pm(h) \) in \( \mathbb{C}^\pm \) are stable for small \( \bar{\hbar} \) up to the first crossing at \( h_n = (-\alpha_n)^{-5/4} > 0 \). At the limit of \( \bar{\hbar} \to \bar{\hbar}_n^- \), the energy levels \( E_n^\pm(h) \) have the limit \( E_n^\pm > 0 \) and the set of...
the zeros of \( \psi_n^\pm(h) \) becomes \( P_x \)-symmetric. The critical state \( \psi_n^c \), has a \( P_x \)-symmetric set of \( 2n \) nodes in \( C^+ \cup C^- \) (Lemma 7). The discontinuity of the number of nodes at \( h_n, n > 0 \), is due to the \( PT \)-symmetry breaking of the states.

Now we look for a pair of positive analytic levels \( E_j(h), E_k(h), j, k \in \mathbb{N}, j \neq k \), with limits \( E_j(h_n^+) = E_k(h_n^+) = E_n^c \) and corresponding states with the limit value \( \psi_j(h_n^+) = \psi_k(h_n^+) = \psi_n^c \). This is possible if both the corresponding \( P_x \)-symmetric sets of nodes \( S_j(h), S_k(h) \) contain \( 2n \) nodes stable in \( C^+ \). But what can be said about the imaginary nodes? In order to distinguish a possible node in the imaginary axis from the other zeros, we consider the limit \( h \to +\infty \) corresponding to the limit \( \alpha \to 0^- \). We prove (Lemma 6) that the nodes in this limit are confined in the lower half plane. Since the imaginary turning point, for a level \( E > 0 \), is \( I_0 = i\tilde{y}, \tilde{y} > 0 \), we define as an imaginary node a zero in \( \Sigma = i(-\infty, \tilde{y}) \). We also prove that the number of imaginary nodes can be zero or one. Since there are in any case \( 2n \) non imaginary nodes, the two independent states are necessarily the states \( (\psi_{2n}(h), \psi_{2n+1}(h)) \), with levels \( (E_{2n}(h) < E_{2n+1}(h)) \) for \( h > h_n \). Actually, exists \( h_n^p > h_n \) such that the state \( \psi_{2n+1}(h) \) has an imaginary node for \( h > h_n^p \). In Lemma 9 we give the crossing selection rules (24):

\[
E_n^\pm(h_n^-) = E_m(h_n^+) := E_n^c > 0,
\]

\[
\psi_n^\pm(h_n^-) = \psi_m(h_n^+) := \psi_n^c, \quad \forall m \in \mathbb{N}, \ [m/2] = n.
\]  

(9)

The critical state \( \psi_n^c \) is orthogonal to its \( P \)-transform, or, in other words, is totally \( P \)-asymmetric (Lemma 10). The crossing corresponds to a square root singularity of this pair of analytic functions, positive analytic for \( h > h_n \). For the value of \( h_n^p \) and \( E_{2n+1}(h_n^p) := E_n^p \), we have only numerical results. Our numerical computations (Table 1) show that \( h_n^p \to 0 \), and \( E_{2n+1}(h_n^p) := E_n^p \to E^p > 0 \) as \( n \to \infty \) with \( E_n^p - E^p = O((h_n^p)^2) \), where \( E^p \sim 0, 352268.. \) is the unique energy such that the imaginary turning point [12] is on the short Stokes line [3, 9]. The semiclassical state corresponding to the energy \( E^p \) is considered a bilocalized state, the transition state between the delocalized and the localized state.

The unicity of the crossing for each pair \( E_n^\pm \) is taken as a conjecture (Conjecture 1) assumed in order to simplify the notations and the discussion. We give the structure of the Riemann sheet of the levels \( E_m(h) \) from large \( h > 0 \) to all the real axis, with the values at the borders of the cut \((0, h_n] \)
(Theorem 1).
In Sec. 2 we prove the positivity of the spectrum for large $\hbar$ and the reality of the states on the imaginary axis; in Sec. 3 we consider the appearance of an imaginary node for the odd states; In Sec. 4, we follow the process of crossing; in Sec. 5 we prove that for small $\hbar > 0$ the imaginary axis is free of zeros and the nodes are bounded; in Sec. 6 we prove a confinement of the nodes for large $\hbar > 0$; in Sec. 7 we prove the quantization rules, the continuity and the boundedness of the levels; in Sec. 8 we prove the total P-symmetry breaking at the crossing; in Sec. 9 we give the local structure of the Riemann sheets of the positive levels with the cuts directed toward 0.

| $n$ | $h^p_n$ | $E^p_n$ |
|-----|---------|---------|
| 8   | 0.043835| 0.3519±0.0010 |
| 9   | 0.030683| 0.3514±0.0011 |
| 10  | 0.023605| 0.3518±0.0013 |
| 11  | 0.013060| 0.3522±0.0002 |

Table 1: The values of $h^p_n$, and $E^p_n$ with the errors, for different values of $n = 8 - 11$.

2 Positivity of the levels and reality of the states on the imaginary axis for large $\hbar > 0$

The level $\hat{E}_m(\alpha)$, $m \in \mathbb{N}$ of $K(\alpha)$ is analytic in a neighborhood of the origin $U \subset \mathbb{C}$ \[2, 20\]. Since it is real analytic for $\alpha < 0$ it is real analytic also in $U \cup \mathbb{R}$ \[11\]. The positivity comes from the positivity of the kinetic energy,

$$\Re \hat{E}_m(\alpha) = \Re < \hat{\psi}_m(\alpha), K(\alpha) \hat{\psi}_m(\alpha)> = < \hat{\psi}_m(\alpha), p^2 \hat{\psi}_m(\alpha)> > 0,$$

where $\psi_m(\alpha)$ is the corresponding normalized state. Also the level $E_m(\hbar)$ is real analytic and positive for $\hbar > 0$ large enough. Thus, we have proved:

**Lemma 1**
The level $E_m(\hbar)$, $m \in \mathbb{N}$, is real analytic and positive for $\hbar > 0$ large enough.

We now extend the analysis of the analytic states on the complex plane.
Let us consider \( y \in \mathbb{R} \) and the translation \( f(x) \rightarrow f(x + iy) \), so that the \( PT \)-symmetric Hamiltonian becomes the other isospectral \( PT \)-symmetric Hamiltonian on the \( \mathcal{H}_y \) representation:

\[
H_h(y) := \hbar^2 p^2 + i(x^3 - (3y^2 + 1)x) - (3yx^2 - y^3 - y) \sim H_h. \tag{10}
\]

The eigenfunction \( \psi_{n,y}(x) := \psi_n(x + iy) \) on the \( \mathcal{H}_y \) representation, with real eigenvalue \( E_n \), can be taken \( PT \)-symmetric:

\[
PT \psi_{n,y}(x) := \overline{\psi_{n,y}(-x)}, \tag{11}
\]

and in particular

\[
\psi_{n,y}(0) = \overline{\psi_{n,y}(0)} = \psi(iy). \tag{12}
\]

Thus, we have proved the following,

**Lemma 2**

For large \( \hbar > 0 \), the level \( E_m(\hbar) \), \( m \in \mathbb{N} \), is positive and, for a choice of the gauge, the state \( \psi_m(\hbar) \), extended to the complex plane as an entire function, is \( P_x T \)-symmetric:

\[
(P_x T \psi_m)(x + iy) := \overline{\psi_m(-x + iy)} = \psi_m(x + iy), \quad \forall x, y \in \mathbb{R}, \tag{12}
\]

and, in particular, the state is real on the imaginary axis,

\[
\exists \psi_m(iy) = 0, \quad \forall y \in \mathbb{R}. \tag{13}
\]

The set of all the zeros of the state is \( P_x \)-symmetric.

### 3 The nodal analysis of the process of crossing

Let \( E_m(\hbar) \), for \( \hbar > 0 \) large enough, be a positive level of the Hamiltonian \( H_h \) with a corresponding state \( \psi_m(\hbar) \). Now, by the complex dilation \( z \rightarrow iz \), we consider the Hamiltonian on the imaginary axis:

\[
H_h^r = -\hbar^2 \frac{d^2}{dy^2} + \tilde{V}(y) \sim -H_h, \quad \tilde{V}(y) := -y^3 - y, \tag{14}
\]

well defined by the \( L^2 \) condition on the \( x \)-axis, here playing the role of the imaginary axis. The Hamiltonian \( H_h^r \) has the same spectrum as \( -H_h \), so
that $-E := -E_m(\hbar) < 0$ is one of its eigenvalues. The corresponding state $\phi_m(y) := \psi_m(iy)$ can be taken real. Actually, since we can have the $P_xT$-symmetry of the entire state $\psi_m(z)$, we can have the reality of $\psi(iy)$ for real $y$: $\psi_m(x+iy) = \overline{\psi}_m(-x+iy)$, $\psi_m(iy) = \overline{\psi}_m(iy)$. We consider together the two states $\psi_m(z)$, $[m/2] = n \in \mathbb{N}$, for a fixed $\hbar \geq h_n$ (24). Both the states have $n$ nodes on both the half-planes $\mathbb{C}^\pm$ and are distinguished by the number of imaginary nodes for $\hbar > 0$ large. All the process of crossing for $\hbar \geq h_n$ can be studied by the behaviors of the states $\psi_m(z)$, with energy $E = E_m$, $[m/2] = n$ on the imaginary semi-axis,

$$\Sigma(E) = \{z = iy; -\infty < y < \tilde{y}(E)\}, \quad (15)$$

where the imaginary turning point is $I_0 = i\tilde{y}(E)$. This means that we consider the two states $\phi_m(y)$, of (14), with energies $-E = -E_m$, $[m/2] = n$ for $y \leq \tilde{y}(E)$.

For $\hbar > 0$ large, we have two possible behaviors of the state $\phi(y)$ of (14) with level $-E$. Let us recall that if, for $y$ in a bounded interval of the semi-axis $-\infty < y < \tilde{y}(E)$, a state $\phi(y)$ is positive, it is convex; if it is negative, it is concave. On the other side, for $y > \tilde{y}(E)$, where an eigenfunction $\phi(y)$ is positive it is also concave, and where it is negative it is also convex.

Since we can consider $\phi(y)$ positive decreasing for $y << \tilde{y}(E)$, there are only two cases:

a) the existence of one node on $\Sigma(E)$,

b) the absence of nodes on $\Sigma(E)$.

Let us remark that $\tilde{y}(E) > 0$ so that a possible node on the imaginary axis should be in $\Sigma(E)$ for large $\hbar$. Thus, we have the result:

**Lemma 3**

The state $\psi_m(\hbar)$, $m \in \mathbb{N}$, with corresponding positive level $E = E_m(\hbar)$, have at most one zero in $\Sigma(E)$. This zero, considered a node, exists for $\hbar > h_n$ large enough if $m = 2n + 1$, but don’t exists if $m = 2n$.

4 **Non real levels: imaginary axis free of zeros and bounded nodes**

Let us consider the general case with a level $E \in \mathbb{C}$, and the corresponding state $\psi(z)$, with $\psi(iy) = \phi(y)$, $z, y \in \mathbb{C}$. We transform the Hamiltonian by
imaginary translations:

\[ H'_h(x) = -\hbar^2 \frac{d^2}{dy^2} + \tilde{V}(y - ix), \]

\[ \tilde{V}(y - ix) = -(y - ix)^3 - (y - ix) = -y^3 + 3x^2y - y + i(x(3y^2 + 1) - x^3) = \]

\[ = \Re \tilde{V}(y - ix) + i\Im \tilde{V}(y - ix), \]

where \( \Im \tilde{V}(y - ix) = (x(3y^2 + 1) - x^3) \) with level \(-E\), for a fixed \( x \neq 0 \), and we consider a state,

\[ \phi_x(y) := \phi(y - ix), \quad n \in \mathbb{N}, \]

with the well known asymptotic behaviors,

\[ |\phi_x(y)|^2 \sim C|y|^{-3/2} \text{ for } y \to +\infty, \]

\[ |\phi_x(y)|^2 \sim C|y|^{-3/2} \exp(-2y^{5/2}/\hbar) \text{ for } y \to -\infty. \quad (16) \]

We apply the Loeffel-Martin method \cite{19} generalized to the case of diverging integrals:

\[ \hbar^2 \Im [\overline{\phi_x(y)} \frac{\partial}{\partial y} \phi_x(y)] = \]

\[ = \hbar^2 \Im [\overline{\phi_x(y)} \frac{\partial}{\partial y} \phi_x(y)] + \int_y^\infty (x(3s^2 + 1) - x^3 + \Im E)|\phi_x(s)|^2 ds \to +\infty, \quad (17) \]

where \( x(3s^2 + 1) - x^3 = \Im \tilde{V}(s - ix) \), as \( y \to +\infty \) for fixed \( \tilde{y}, x \in \mathbb{R}, x \neq 0 \). We know that the zeros, for \( |z| \) large, have the asymptotic direction \( \arg z \to \pi/2 \) \cite{2}. By (17) we prove a stronger condition on the asymptotics of the zeros:

**Lemma 4**

Let \( E \) be a level with state \( \psi(z) \) of the Hamiltonian \( H_h \) for a fixed \( h > 0 \). Consider a generic zero \( Z_j = X_j + iY_j \) of \( \psi(z) \). Exists \( M > 0 \), such that \( \pm X_j > 0 \) if \( \mp \Im E > 0 \), \( Y_j > M \). In case of real level, \( \Im E = 0 \), the large zeros are purely imaginary.

**Proof**

The integral in (17) don’t diverge for \( y \to +\infty \) only if \( x \) depends on \( y \) such that,

\[ x(y) \to 0, \quad \pm x(y) \geq \frac{|\Im E|}{3y^2 + 1}, \quad \mp \Im E > 0, \text{ as } y \to +\infty. \quad (18) \]
Actually, condition \((18)\) is necessary for having a change of sign on the integrand in \((17)\). Otherwise, the integral in \((17)\) diverge. We have the state \(\phi_E(y) := \psi_E(iy)\) with corresponding level \(E\) of the Hamiltonian \(H_h\). We consider the Loeffel-Martin formula in order to generalize to our problem the expression of the imaginary part of a shape resonance:

\[
h^2 \Im (\phi(y)\partial_y \phi(y)) = \Im E \int_{-\infty}^{y} |\phi(s)|^2 ds \neq 0, \quad \forall y \in \mathbb{R}, \tag{19}\]

where the integral in \((19)\) exists bounded for the semiclassical behavior. Thus, we state the result:

**Lemma 5** Let us consider the level \(E = \hat{E}_m(0), m \in \mathbb{N}\), of \(K(\alpha)\) at \(\alpha = 0\), corresponding to the level \(E_m(h)\) of \(H_h\) at the limit of \(h = +\infty\), because of the relation \((6)\), \(E_n(h) = h^{6/5} \hat{E}_n(-h^{-4/5})\). It is relevant that the scaling used for this relation \((6)\) is a regular one with a positive scale \(\lambda = h^{2/5}\) (even if infinite) respecting the angles on the complex plane. It is known that the level \(\hat{E}_m(\alpha)\) is positive for \(\alpha \geq 0\) \([1]\). We prove now a confinement of the zeros which allows us to distinguish the nodes from the other zeros.

We consider the operator \(K(0)\) \((3)\) translated by \(x \rightarrow x + iy\),

\[
K_y(0) = p^2 + i(x + iy)^3 = p^2 + i(x^2 - 3y^2)x + y^3 - 3yx^2 := p^2 + V_y(x) .
\]

We apply the Loeffel-Martin method \([19]\) to a level \(E = \hat{E}_m(0)\), with \(E > 0\):

\[
-\Im [\bar{\psi}(x+iy)\partial_x \psi(x+iy)] = \int_{x}^{\infty} \Im V_y(s) |\psi(s+iy)|^2 ds = \int_{x}^{\infty} (s^2 - 3y^2)s|\psi(s+iy)|^2 ds =
\]

\[
= -\int_{-\infty}^{x} (s^2 - 3y^2)s|\psi(s + iy)|^2 ds \neq 0,
\]

for \(\pm x \geq \sqrt{3}|y|, \quad y \in \mathbb{R}\). In this case we have a rigorous confinement, extendible to all \(\alpha > 0\), of the region of the nodes,

\[
C_\sigma = \{z = x + iy; y < 0, |x| < -\sqrt{3}y\} \subset C_- = \{z = x + iy; y < 0\}.
\]
Since the same confinement extends to all $\alpha > 0$, we have that the $m$ zeros of the state $\tilde{\psi}_m(\alpha)$ on $C_-$ are stable in the limit $\alpha \to +\infty$, i. e. are nodes by definition. Previous computations of the nodes [17] suggest that the present confinement may be sharp. Thus, we state a result:

**Lemma 6**

*All the nodes of the state $\psi_m(h, z)$, $m \in \mathbb{N}$, for $h > 0$ large enough, are in $C_-$.***

**6 The quantization rules**

Suppose the existence of a continuation of each level $E^\pm := E^\pm_n(h)$ from $\bar{h} < h_n$ to $\bar{h} \geq h_n$. For the moment, we keep the same names $E^\pm_n(h)$ for the continuations of the levels even if such names are no more specific. We have two kinds of quantization rules for a fixed $\bar{h} > 0$ small, giving the eigenvalues $E^\pm_n(h)$, respectively. Exist two regular circuits

$$\gamma^\pm,$$

such that, $P_\gamma \gamma^+ = \gamma^-$,

$$\gamma^\pm = \partial D^\pm,$$

where $D^\pm$ is a regular region large enough, with

$$D^\pm \subset C^\pm := \{x + iy, \pm x > 0, y \in \mathbb{R}\};$$

and,

$$\frac{1}{2i\pi} \oint_{\gamma^\pm} \frac{\psi'(\bar{h}, E^\pm, z)}{\psi(\bar{h}, E^\pm, z)} dz = n, \quad (20)$$

where $\pm \Im E^\pm \geq 0$. In particular, for small $h > 0$, we have the semiclassical quantization condition,

$$\frac{1}{2i\pi} \oint_{\gamma^\pm} p_0(E^\pm, z) dz = h(n + \frac{1}{2}) + O(h^2). \quad (21)$$

This quantization conditions are still valid for all $h > 0$, but, for large $h \geq h_n$, are both satisfied by both the new states $\psi_m(h)$, $m \in \mathbb{N}$, $[m/2] = n$ and the critical state $\psi^c_n$. We now prove that the state $\psi^c_n$ has the set of $2n$ nodes
in $C^+ \cup C^-$. At the limit of $\bar{h} \to h_0^-$, the energy levels $E_n^c(\bar{h})$ have the limit $E_n^c > 0$ and the set of the zeros of $\psi_n^\pm(\bar{h})$ becomes $P_x$-symmetric. Let a node $N_j \in C^+$ of $\psi_n^+(\bar{h})$ to have a limit $N_j^c$ as $\bar{h} \to h_0^-$. Because of the $P_x$-symmetry of the set of all the zeros at the limit $\bar{h} \to h_0^-$, does exist a zero $Z_k$ such that $Z_k \to P_xN_j^c$ as $\bar{h} \to h_0^-$. It is possible to disprove the possibility that $\Re N_j^c = 0$. Actually, in this case $N_j^c$ would be a double zero of the state $\psi_n^c := \psi_{M}^\pm(\bar{h})$, but we know that the zeros are simple. Thus, the sets $S_n^\pm(\bar{h})$ of the $n$ nodes of the states $\psi_n^\pm(\bar{h})$ have limits $S_n^\pm(h_0^-) \subset C^\pm$, respectively, as $\bar{h} \to h_0^-$. Both the states $\psi_m(h), m \in \mathbb{N}, \lfloor m/2 \rfloor = n$ for $\bar{h} > h_0$, have $2n$ nodes in $C^{+\pm}$. Actually, these noses are stable: cannot become imaginary for the symmetry and the simplicity of the spectrum and cannot diverge along the imaginary axis. We have proved the:

**Lemma 7**
The critical state $\psi_n^c$, has a $P_x$-symmetric set of $2n$ nodes in $C^{+\pm} := C^+ \cup C^-$,

$$S_n^c = S_n^+(h_0^-) \cup P_xS_n^-(h_0^-) = S_n^+(h_0^-) \cup S_n^-(h_0^-).$$  \hspace{1cm}(22)

Both the states $\psi_m(h), m \in \mathbb{N}, \lfloor m/2 \rfloor = n$ for $\bar{h} > h_0$, have $2n$ nodes in $C^{+\pm}$.

In order to select a single state it is sufficient to use the inequality $E_{2n+1}(\bar{h}) > E_{2n+1}(\bar{h})$. One more node of $\psi_{2n+1}(\bar{h})$ lies on the imaginary axis. This is clear for large $\bar{h} > 0$, where all the nodes have a negative imaginary part. In this case it is possible to give a unique quantization rule,

$$\frac{1}{2i\pi} \oint_\Gamma \frac{\psi'(h,E,z)}{\psi(h,E,z)} dz = m,$$

where $m = 2n$ or $2n + 1$, and $\Gamma = \partial\Omega$ for $\Omega \subset C_-, C_-= \{ z \in C; \Im z < 0 \}$.

For $\bar{h} > 0$ large, it is convenient to use the scaling of the operator $K(\alpha)$ in order to have energy and nodes uniformly bounded.

**Lemma 8**
For each $n \in \mathbb{N}$, does exist $h_n > 0$ and a crossing:

$$E_n^\pm(h_n) = E_m(h_0^+) := E_n^c > 0,$$

$$\psi_n^\pm(h_n^-) = \psi_m(h_0^+) := \psi_n^c, \forall m \in \mathbb{N}, \lfloor m/2 \rfloor = n.$$  \hspace{1cm}(24)

**Proof**
The existence of this crossing is necessary because of the positivity of the
analytic functions $E_m(h)$ for large $h > 0$, and the non reality of the analytic functions $E^\pm_n(h)$ for small $h > 0$. The relation between the integer $n$ and the integers $m$ is due to the doubling of the nodes at $h_n^-$ (Lemma 4) because of the $P_x$ symmetry of the set of nodes. This nodes are in $C^+$ and are stable for $h \geq h_n$. The differentiation of the number of nodes of the two states in necessary for large $h > 0$. Actually, for $h > h_o > h_n$ $\psi_{2n+1}(h)$ has imaginary node $N_0 = iy$ with $y < \tilde{y}$, where the imaginary turning point is $I_0 = i\tilde{y}$.

**Conjecture 1**

We assume that the crossing between the pair $E^\pm_n$, $\forall n \in \mathbb{N}$, is unique.

This conjecture is justified by numerical, semiclassical and exact semiclassical results [10] [14], [12]. This conjecture allow us to simplify the notations. Now we prove that each level is bounded for bounded parameter $h > 0$.

**Lemma 9**

Let $E(h)$, be any level of the pair $E^\pm_n(h)$, for $h < h_n$, or any level of the pair $E_m(h)$, $m \in \mathbb{N}$, $[m/2] = n$ for $h > h_n$. Does not exists a $h^c \geq 0$, such that $E(h)$ diverge as $h \to h^c$.

**Proof**

We prove by absurd, and we consider the case $E = E_m(h)$ for a fixed $m \in \mathbb{N}$, and $h$ near $h^c > 0$. The extension to the general case is simple. We consider the operator

$$\frac{H_h - E(h)}{|E|(h)} \sim \hat{h}^2 p^2 + ix^3 - i\alpha x - \eta,$$

by a scaling $x \to \lambda x$, $\lambda = |E|^{1/3}$, where $\hat{h} = h|E^{-5/3}|$, $\alpha = |E|^{-2/3}$, $\eta = E/|E|$, $|\eta| = 1$. For small $\hat{h} > 0$, by simply putting $\alpha = 0$, we have the semiclassical quantization condition,

$$\frac{1}{2i\pi} \oint C \sqrt{|\eta - iz^3|}dz = \hat{h}(m + 1/2) + O((\hat{h})^2).$$

which can be valid only if $\eta \to 0$ as $\hat{h} \to 0$.

7 At the crossing the state is orthogonal to its P-transform

We have a crossing of $E^\pm_n(h)$ at $h = h_n$, when $\exists E^\pm_n(h) = 0$. For $0 < h < h_n$, the two clamped points of $\psi^\pm_n$ are $(I_+, I_0)$ respectively. At the crossing, we
have $P_x$ symmetry of the turning points, so that $I_- = I_+^\ast$, $I_0 = -I_0$.

Let $H := H_h$, $H_h^\ast = \bar{H} := H_{\bar{h}}$, with two eigenvalues $E_j = \bar{E}_j$, and eigenvectors $\psi_j$, $j = 1, 2$. We have

$$H \psi_1 = E_1 \psi_1, \quad \bar{H} \bar{\psi}_2 = E_2 \bar{\psi}_2,$$

so that

$$< \bar{\psi}_2, H \psi_1 > = E_1 < \bar{\psi}_2, \psi_1 >, \quad (26)$$

$$< H \psi_1, \bar{\psi}_2 > = < \psi_1, \bar{H} \bar{\psi}_2 > = E_2 < \psi_1, \bar{\psi}_2 >,$$

and, by complex conjugation,

$$< \bar{\psi}_2, H \psi_1 > = E_2 < \bar{\psi}_2, \psi_1 >. \quad (27)$$

By subtraction of the two equations (26) (27), we get,

$$0 = (E_2 - E_1) < \psi_1, \bar{\psi}_2 >.$$ 

Let now to vary the semiclassical parameter $\hbar$, so that:

$$0 = (\bar{E}_2(\bar{\hbar}) - E_{\hbar}) < \psi_{\hbar}, \bar{\psi}(\bar{\hbar}) >,$$

for $\hbar > 0$. If $E_1(\hbar) \neq E_2(\hbar)$ for $\hbar > h_n$, and $E_1(h_n^+) = E_2(h_n^+) = E$, $\psi_1(h_n^+) = \psi_2(h_n^+) = \psi$, we have

$$0 = < \psi, \bar{\psi} >= < \psi, P \psi > = \int_{\mathbb{R}} \psi^2(x) dx. \quad (28)$$

Thus, we have proved the following:

**Lemma 10**

*The PT-symmetric state at the crossing point,*

$$\psi_n^c = \psi_{2n+1}(h_n^+) = \psi_{2n}(h_n^+) = PT \psi_n^c,$$

*is completely P-asymmetric, i.e. is orthogonal to its P-transform:*

$$\int_{\mathbb{R}} \psi_n^c(x)^2 dx = < \psi_n^c, P \psi_n^c > = 0.$$
8 The Riemann surface near the real axis

Let us consider the sector on the $\bar{h}$ complex plane (7):

$$C^0 = \{h \in \mathbb{C}; h \neq 0, \text{arg}(h) < \pi/4\},$$

and the Riemann sheet $C^0_m$ of the level $E_m(h)$, $n = [m/2]$, defined in $C^0$, with a square root singularity at $h_n$ and a cut, $\gamma_{n,n} = (0, h_n]$. Thus, we prove the following:

Theorem 1
The levels $(E_{2n}(h), E_{2n}(h))$, $n \in \mathbb{N}$, are analytic functions defined on the Riemann sheets $(C^0_{2n}, C^0_{2n+1})$, respectively, both with only the cut $\gamma_{n,n} = (0, h_n]$ on the real axis. The positive analytic functions $(E_{2n+1}(h), E_{2n}(h))$, with $E_{2n+1}(h) > E_{2n}(h)$ on $(h_n, +\infty)$ have the following values at the borders of the cut:

$$E_{2n}(h \pm i0^+) = E_n^\pm(h), \quad E_{2n+1}(h \pm i0^+) = E_{n}^{\mp}(h), \quad \forall \ 0 < h < h_n. \quad (29)$$

Proof
Since both the functions $(E_{2n}(h), E_{2n}(h))$ have a square root singularity at $h_n$, and

$$E_{2n+1}(h_n + \epsilon) - E_{2n}(h_n + \epsilon) = O(\sqrt{\epsilon}) > 0,$$

for $\epsilon > 0$ small,

$$\pm \Im(E_{2n+1}(h_n + \exp(\pm i\pi)\epsilon) - E_{2n}(h_n + \exp(\pm i\pi)\epsilon)) < 0,$$

and $\mp \Im E_n^\pm(h) > 0$, for $h < h_n$, necessarily we have,

$$E_{2n+1}(h_n + \exp(\pm i\pi)\epsilon) := E_n^\mp(h_n - \epsilon),$$

$$E_{2n}(h_n + \exp(\pm i\pi)\epsilon) := E_n^\pm(h_n - \epsilon).$$

Remark
The Riemann sheet $C^0_0$ of the fundamental level has only one cut $\gamma_{0,0} = [0, h_0]$ on $R$ [12], and the discontinuity on the cut is defined by the rule,

$$E_0(h \pm i0^+) = E_0^\pm(h), \quad \forall \ h, \ 0 < h < h_0. \quad (30)$$
We recall, for instance, that $E_0^\pm (\hbar) := E_0(h + i0)$, is defined as the limit from above for small $\hbar > 0$. This definition extends directly to all $\hbar > 0$ in absence of complex singularities. Formula (30) means that it is possible the absence of other singularities involving the function $E_0(h)$. Thus, using the principle of maximal analyticity, we assume that in $C_0^0$ there is only the cut $\gamma_{0,0}$.

**Acknowledgments**

It is a pleasure to thanks Professor André Martinez for long and useful discussions at the beginning of this research.

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