Homology-changing percolation transitions on finite graphs

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We consider homological edge percolation on a sequence \((G_t)\) of finite graphs covered by an infinite (quasi)transitive graph \(H\), and weakly convergent to \(H\). Namely, we use the covering maps to classify 1-cycles on graphs \(G_t\) as homologically trivial or non-trivial, and define several thresholds associated with the rank of thus defined first homology group on the open subgraphs. We identify the growth of the homological distance \(d_t\), the smallest size of a non-trivial cycle on \(G_t\), as the main factor determining the location of homology-changing thresholds. In particular, we show that the giant cycle erasure threshold \(p_0^E\) (related to the conventional erasure threshold for the corresponding sequence of generalized toric codes) coincides with the edge percolation threshold \(p_c(H)\) if the ratio \(d_t/\ln n_t\) diverges, where \(n_t\) is the number of edges of \(G_t\), and we give evidence that \(p_0^E < p_c(H)\) in several cases where this ratio remains bounded, which is necessarily the case if \(H\) is non-amenable.

I. INTRODUCTION

It is the threshold theorem\(^1\)\(^-\)\(^6\) that makes large-scale quantum computation feasible, at least in theory. Related is the notion of quantum channel capacity \(R_Q\), such that for any rational \(R < R_Q\), there exists a quantum error correcting code (QECC) with rate \(R\) which can be used to suppress the logical error probability to any chosen (arbitrarily small) level, but not for \(R > R_Q\). Here the code rate \(R \equiv k/n\) is the ratio of the number \(k\) of the logical (encoded) qubits to the length \(n\) of the code. The precise value of the capacity is not known for most quantum channels of interest, except for the quantum erasure channel with qubit erasure probability \(p\), in which case \(R_Q = \min(0, 1 - 2p)\), see Ref.\(^7\).

In practice, it is often easier to deal with the threshold error probability for a given family (infinite sequence) of QECCs with certain asymptotic code rate \(R\). Depending on the nature of the quantum channel in question, the threshold error probability may be related to the location of a thermodynamical phase transition in certain spin model associated with the codes. In particular, for a family of qubit toric codes on transitive graphs locally isomorphic to a regular euclidean or hyperbolic tiling \(H\) under independent \(\mathbb{Z}\) Pauli errors, the decoding threshold is upper bounded by the position of the multicritical point located at the Nishimori line of the Ising model on \(H\), see Refs.\(^4\)\(^,\)\(^8\) and \(^9\). It is widely believed that the two thresholds coincide, at least for the euclidean tilings like the infinite square lattice and square-lattice toric codes. With a slightly more general model of independent \(X/\mathbb{Z}\) Pauli errors, the threshold is the minimum of the corresponding thresholds for each error type which can be computed independently.

A special case is the relation between quantum erasure errors and percolation\(^10\)\(^-\)\(^12\). An erasure is formed by rendering inoperable all qubits in a known randomly selected set. Information loss happens when erasure covers a logical operator of the code. For certain code families, and for qubit erasure probability \(p\) sufficiently small, \(p < p_E\), the probability to cover a codeword may go to zero as the code length \(n\) is increased to infinity. The corresponding threshold value \(p_E\) is called the erasure threshold associated with the chosen code family or code sequence. With a Calderbank-Shor-Steane (CSS) code\(^13\)\(^,\)\(^14\), one may consider the erasure thresholds for \(X\) and \(Z\) logical operators separately, so that the conventional erasure threshold becomes \(p_E = \min(p_E^X, p_E^Z)\).

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The link between erasure and percolation thresholds is especially simple in the case of toric/surface\cite{14,15,16} and related quantum cycle codes\cite{18} where qubits are labeled by the edges of a graph and, by convention, \( Z \) logical operators are supported on 1-chains in certain equivalence classes, e.g., those connecting two opposite boundaries of a rectangular region, or wrapped around a torus. Then, the erasure threshold \( p_E^Z \) coincides with the discrete version of the homological percolation transition\cite{20,21} for 1-chains. It is also known that for square-lattice toric code the erasure threshold \( p_E^Z \) coincides\cite{22,23} with the edge percolation threshold, \( p_E^Z = p_c(\mathbb{Z}^2) = \frac{1}{2} \). On the other hand, for a family of hyperbolic surface codes based on a given infinite graph \( \mathcal{H} \), a regular tiling on the hyperbolic plane, we only know that the erasure threshold is upper bounded\cite{10,12} by the percolation threshold on \( \mathcal{H} \), \( p_E \leq p_c(\mathcal{H}) \).

Surely, the erasure and the percolation thresholds cannot always coincide. Indeed, percolation threshold is associated with the formation of an infinite cluster; it is defined on an infinite graph, while quantum codes are finite. Further, erasure threshold is not a bulk quantity, as it can be rendered zero by removing a vanishingly small fraction of well-selected qubits. Similarly, many different finite graphs can be associated with a given infinite graph \( \mathcal{H} \), and it is not at all clear that the erasure threshold should remain the same independent of the details.

The goal of this work is to quantify the relation between edge percolation and the stability of quantum cycle codes (QCCs) to erasure errors. Specifically, we consider sequences of finite graphs \( \mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t) \), \( t \in \mathbb{N} \), with a common infinite covering graph \( \mathcal{H} \), and use the covering map \( f_t : \mathcal{H} \to \mathcal{G}_t \) to identify homologically non-trivial cycles on \( \mathcal{G}_t \). The distance \( d_t \) of the corresponding quantum code (the smallest length of a non-trivial cycle) necessarily diverges with \( t \) when the sequence converges weakly to \( \mathcal{H} \). First, we show that it is the scaling of \( d_t \) with the logarithm of the code block length, \( n_t = |\mathcal{E}_t| \), that determines the location of the \( Z \)-erasure threshold, or the 1-chain lower erasure threshold \( p_E^0 \equiv p_E^0 \), the point above which the probability of an open homologically non-trivial 1-cycle remains non-zero in the limit of arbitrarily large graphs \( \mathcal{G}_t \). Roughly, with sublogarithmic distance scaling, \( d_t / \ln n_t \to 0 \) as \( t \to \infty \), \( p_E^0 = 0 \). On the other hand, with superlogarithmic distance scaling, \( d_t / \ln n_t \to \infty \), \( p_E^0 \) coincides with the edge percolation threshold \( p_c(\mathcal{H}) \), so that for \( p < p_c(\mathcal{H}) \), probability to find an open homologically non-trivial 1-cycle be asymptotically zero. We also give an example of a graph family with logarithmic distance scaling, where the inequality in the upper bound is strict, \( p_E^0 < p_c(\mathcal{H}) \), and give numerical evidence that for some regular tilings of the hyperbolic plane, erasure threshold is strictly below the percolation threshold, \( p_E^0 < p_c(\mathcal{H}) \).

Second, the distance \( d_t \) grows at most logarithmically with \( n_t \) when \( \mathcal{H} \) is non-amenable, which is also a necessary requirement to have a finite asymptotic code rate \( k_t/n_t \to R > 0 \), where \( k_t \) is the number of encoded qubits. For such a graph sequence, we define a pair of thermodynamical homological transitions, \( p_H^0 \) and \( p_H^1 \), which characterize singularities in the erasure rate, asymptotic ratio of the expected homology rank of the open subgraph and the number of edges \( n_t \). Namely, erasure rate is zero for \( p < p_H^0 \), it saturates at \( R \) for \( p > p_H^1 \), and it takes intermediate values in the interval \( p_H^0 < p < p_H^1 \) (subsequence construction may be needed in this regime to achieve convergence). We prove that \( p_H^1 - p_H^0 > R \), and, if \( \mathcal{H} \) and its dual, \( \tilde{\mathcal{H}} \), is a pair of transitive planar graphs, we show that \( p_H^0 = p_c(\mathcal{H}) \) and \( p_H^1 = 1 - p_c(\tilde{\mathcal{H}}) \); the latter point coincides with the uniqueness threshold \( p_u(\mathcal{H}) \) on the original graph. We also conjecture that the two homological transitions coincide with the percolation and the uniqueness thresholds, respectively, for any non-amenable (quasi)transitive graph, \( p_H^0 = p_u(\mathcal{H}) \) and \( p_H^1 = p_u(\tilde{\mathcal{H}}) \).

The outline of the paper is as follows. In Sec. \ref{2} we give the necessary notations. We present our analytical results in Sec. \ref{2} and numerical results in Sec. \ref{3} with the proofs collected in the Appendix. In section \ref{4} we give the conclusions and discuss some related open questions.
II. DEFINITIONS

A. Classical binary and quantum CSS codes

A linear binary code with parameters \([n, k, d]\) is a vector space \(C \subseteq \mathbb{F}_2^n\) of length-\(n\) binary strings of dimension \(k\), where the minimum distance \(d\) is the smallest Hamming weight of a non-zero vector in \(C\). Such a code \(C \equiv C_G\) can be specified in terms of a generator matrix \(G\) whose rows are the basis vectors, or in terms of a parity check matrix \(H\), \(C \equiv C_H = \{c \in \mathbb{F}_2^n : Hc^T = 0\}\), where \(C_H^\perp\) denotes the space dual (orthogonal) to \(C_H\). A generator matrix and a parity check matrix of any length-\(n\) code satisfy

\[
GH^T = 0, \quad \text{rank } G + \text{rank } H = n; \tag{1}
\]

such matrices are called mutually dual.

If \(I \subset \{1, \ldots, n\}\) is a set of bit indices, for any vector \(b \in \mathbb{F}_2^n\), we denote \(b[I]\) the corresponding punctured vector with positions outside of \(I\) dropped. Similarly, \(G[I]\) (with columns outside of \(I\) dropped) generates the code \(C_G\) punctured to \(I\), denoted \(C_G[I] = C_G[I']\).

A shortened code is formed similarly, except by puncturing only the vectors supported inside \(I\),

\[
C \text{ shortened to } I = \{c[I] : c \in C \land \text{supp}(c) \subseteq I\}.
\]

We use \(G_I\) to denote a generating matrix of the code \(C_G\) shortened to \(I\). If \(G\) and \(H\) is a pair of mutually dual binary matrices, see Eq. (1), then \(H_I\) is a parity check matrix of the punctured code \(C_G[I]\), and\(^{[24]}\)

\[
\text{rank } G[I] + \text{rank } H_I = |I|; \tag{2}
\]

i.e., matrices \(G[I]\) and \(H_I\) are mutually dual. In addition, if \(\overline{I} = \{1, 2, \ldots, n\} \setminus I\) is the complement of \(I\), then

\[
\text{rank } G[\overline{I}] + \text{rank } G_I = \text{rank } G. \tag{3}
\]

For the present purposes, it is sufficient that an \(n\)-qubit quantum CSS code \(Q = \text{CSS}(G_X, G_Z)\) can be specified in terms of two \(n\)-column binary stabilizer generator matrices with mutually orthogonal rows, \(G_X, G_Z^T = 0\). It is isomorphic to a direct sum of two quotient spaces, \(Q = Q_X \oplus Q_Z\), where \(Q_X = C_{G_Z, X}^\perp / C_{G_X, X}\) and \(Q_Z = C_{G_Z, Z}^\perp / C_{G_X, Z}\). Vectors in \(Q_X\) and \(Q_Z\), respectively, are also called \(X\)- and \(Z\)-logical operators. Explicitly, \(Q_X\) is formed by vectors in \(C_{G_Z, X}^\perp\), with any two vectors that differ by an element of \(C_{G_X, X}\) identified (notice that \(C_{G_X, X} \subset C_{G_Z, Z}^\perp\)). Such a pair of vectors \(c' = c + aG_X\) that differ by a linear combination of the rows of \(G_X\) are called mutually degenerate; we write \(c' \approx c\). The second half of the code, \(Q_Z\), is defined similarly, with the two generator matrices interchanged. For such \(Z\)-like vectors, the degeneracy is defined in terms of the rows of \(G_Z\).

The distances \(d_X\) and \(d_Z\) of a CSS code are the minimum weights of non-trivial vectors in \(Q_X\) and \(Q_Z\), respectively, e.g., \(d_X = \text{min}\{\text{wgt } c : c \in C_{G_Z, X}^\perp \setminus C_{G_X, X}\}\). Any minimum-weight codeword is always irreducible, that is, it cannot be written as a sum of two vectors with disjoint supports, one of them being a codeword\(^{[29]}\). The conventional distance, the minimum weight of a logical operator in \(Q\), is \(d = \text{min}(d_X, d_Z)\). The dimension \(k\) of a CSS code is the dimension of the vector space \(Q_X\) (it is the same as the dimension of \(Q_Z\)), the number of linearly independent and mutually non-degenerate vectors that can be used to form a basis of \(Q_X\). For a length-\(n\) code with stabilizer generator matrices \(G_X\) and \(G_Z\),

\[
k = n - \text{rank } G_X - \text{rank } G_Z. \tag{4}
\]

The parameters of a quantum CSS code are commonly written as \([[n, k, (d_X, d_Z)]\]) or just \([[n, k, d]]\).
Any CSS code formed by matrices $G_X$ and $G_Z$ of respective dimensions $r_X \times n$ and $r_Z \times n$ also defines a binary chain complex with three non-trivial vector spaces,

$$A : \ldots \leftarrow \{0\} \xrightarrow{\partial_0} A_0 \xrightarrow{\partial_1} A_1 \xrightarrow{\partial_2} A_2 \xrightarrow{\partial_3} \{0\} \leftarrow \ldots,$$

(5)

where the spaces $A_i = \mathbb{F}_2^{a_i}$ have dimensions $a_0 = r_X$, $a_1 = n$, and $a_2 = r_Z$, and the non-trivial boundary operators are expressed in terms of the generator matrices $\partial_1 = G_X$, $\partial_2 = G_Z^T$. This guarantees the defining property of a chain complex, $\partial \partial = 0$. Then, the code $Q_Z$ is defined identically to the first homology group $H_1(A) = \ker(\partial_1)/\text{im}(\partial_2)$, where elements of $\text{im}(\partial_2)$ called cycles are linear combinations of the columns of $\partial_2 = G_Z^T$, while elements of $\ker(\partial_1)$ called boundaries are vectors orthogonal to the rows of $\partial_1 = G_X$. The other definitions also match. In particular, the dimension $k$ of the quantum code is the rank of the first homology group, $k = \text{rank} H_1(A)$, while the definition of the homological distance $d_1(A)$ matches that of $d_Z$. The other code, $Q_X$, corresponds to the co-homology group defined in the co-chain complex $\check{A}$ formed similarly but with the two matrices interchanged.

Let us now consider the structure of the homology group where the space $A_i$ is restricted so that only components with indices in the index set $I \subset \{1, 2, \ldots, n\}$ be non-zero. Respectively, the spaces $\ker \partial_1 = \mathcal{C}_{G_X}$ and $\text{im} \partial_2 = \mathcal{C}_{G_Z}$ should be replaced with the corresponding reduced spaces. The result is isomorphic to a chain complex $A'_I$ where the two boundary operators are obtained by puncturing and shortening, respectively: $\partial_1' = G_X[I]$ and $\partial_2' = (G_Z)^T_I$. The dimension of thus defined restricted homology group is given by

$$k'_I \equiv \text{rank} H_1(A'_I) = |I| - \text{rank} G_X[I] - \text{rank}(G_Z)_I.$$

(6)

Using Eq. (5), we also get

$$k'_I = |I| - \text{rank} G_Z - \text{rank} G_X[I] + \text{rank} G_Z[I].$$

(7)

The corresponding result for the rank $\check{k}'_I$ of the restricted co-homology group can be found by exchanging the matrices $G_X$ and $G_Z$; this gives the duality relation

$$k'_I + \check{k}'_I = k.$$

(8)

B. Graphs, cycles, and cycle codes

We consider only simple graphs with no loops or multiple edges. A graph $G = (\mathcal{V}, \mathcal{E})$ is specified by its sets of vertices $\mathcal{V} \equiv \mathcal{V}_G$, also called sites, and edges $\mathcal{E} \equiv \mathcal{E}_G$. Each edge $e \in \mathcal{E}$ is a set of two vertices, $e = \{u, v\}$; it can also be denoted with a wave, $u \sim v$. For every vertex $v \in \mathcal{V}$, its degree $\text{deg}(v)$ is the number of edges that include $v$. An infinite graph $G$ is called quasi-transitive if there is a finite subset $\mathcal{V}_0 \subset \mathcal{V}_G$ of its vertices, such that for every vertex $v \in \mathcal{V}$ there is an automorphism (symmetry) of $G$ mapping $v$ to an element of $\mathcal{V}_0$. A transitive graph is a quasi-transitive graph where the subset $\mathcal{V}_0$ of vertex classes contains only one element. All vertices in a transitive graph have the same degree.

We say that vertices $u$ and $v$ are connected on $G$ if there is a path $P \equiv P(u_0, u_\ell)$ between $u \equiv u_0$ and $v \equiv u_\ell$, a set of edges which can be ordered and oriented to form a walk, a sequence of vertices starting with $u$ and ending with $v$, with each directed edge in $P$ matching the corresponding pair of neighboring vertices in the sequence,

$$P(u_0, u_\ell) = \{u_0 \sim u_1, u_1 \sim u_2, \ldots, u_{\ell-1} \sim u_\ell\} \subseteq \mathcal{E}.$$

(9)

We call such a path open if $u_0 \neq u_\ell$, and closed otherwise. The path is called self-avoiding (simple) if $u_i \neq u_j$ for any $0 \leq i < j \leq \ell$, except that $u_0$ and $u_\ell$ coincide if the path is closed. The length of the path is the number of edges in the set, $\ell = |P|$. The distance $d(u, v)$ between vertices $u$ and $v$ is the smallest length of a path between them. Given a vertex $v \in \mathcal{V}$ and a natural $r \in \mathbb{N}$, a ball $B(v, r; G)$ is the subgraph of $G$ induced by the vertices $u \in \mathcal{V}$ such that $d(v, u) \leq r$. 


The edge boundary $\partial U$ of a set of vertices $U \subseteq V$ is the set of edges connecting $U$ and its complement $\overline{U} \equiv V \setminus U$. Given an exponent $\alpha \leq 1$, we define the isoperimetric constant of a graph,

$$b_\alpha = \inf_{\emptyset \neq U \subseteq V, |U| \neq \infty} \frac{|\partial U|}{\min \{|U|, |\overline{U}|\}^{\alpha}}.$$  \hfill (10)

For an infinite graph, or a set of finite graphs that includes graphs of arbitrarily large size, particularly important is the largest $\alpha$ such that the corresponding $b_\alpha > 0$. Such a graph (or graph family) is called an $\alpha$-expander; when $\alpha < 1$, the related parameter $\delta = (1 - \alpha)^{-1}$ is called the isoperimetric dimension. Isoperimetric dimension of any regular $D$-dimensional lattice is $\delta = D$. When $\alpha = 1$, the isoperimetric constant $b_1$ of a graph $G$ is called its Cheeger constant, $h(G) = b_1$. An infinite graph with a non-zero Cheeger constant is called non-amenable.

A set of edges $C \subseteq E$ is called a cycle if the degree of each vertex in the subgraph induced by $C$, $G' = (V, C)$, is even. The set of all cycles on a graph $G$, with the symmetric difference defined as $A \oplus B \equiv (A \setminus B) \cup (B \setminus A)$ used as the group operation, forms an abelian group, the cycle group of $G$, denoted $\mathcal{C}(G)$. Clearly, a closed path is a cycle. A simple cycle is a self-avoiding closed path.

A graph $\mathcal{H}$ is called a covering graph of $G$ if there is a function $f$ mapping $\mathcal{V}_\mathcal{H}$ onto $\mathcal{V}_G$, such that an edge $(u, v) \in \mathcal{E}_\mathcal{H}$ is mapped to the edge $(f(u), f(v)) \in \mathcal{E}_G$, with an additional property that $f$ be invertible in the vicinity of each vertex, i.e., for a given vertex $u' \in \mathcal{V}_\mathcal{H}$ and an edge $(f(u'), v) \in \mathcal{E}_G$, there must be a unique edge $(u', v') \in \mathcal{E}_\mathcal{H}$ such that $f(v') = v$. As a result, given a path $P$ connecting vertices $u$ and $v$ on $G$ and a vertex $u' \in \mathcal{V}_\mathcal{H}$ such that $f(u') = u$, there is a unique path $P'$ on $\mathcal{H}$, the lift of $P$, such that $f$ maps the sequence of vertices $u_1 = u, u', \ldots$ in $P'$ to that in $P$. To simplify the notations, we will in some cases write a covering map as a map between the graphs, $f : \mathcal{H} \to G$.

A set of vertices $u'$ with the same covering map image $u, f(u') = u$, is called the fiber of $u$. A lift of a closed path starting and ending with $u$ is either a closed path, or an open path connecting two different vertices in the fiber of $u$. We call a simple cycle on $G$ homologically trivial if all its lifts are simple cycles (of the same length). A cycle on $G$ is trivial if it is a union of edge-disjoint homologically trivial simple cycles. The set of trivial cycles on $G$, with “$\oplus$” used for group operation, is a subgroup of the cycle group on $G$. We denote such a group $C_0(H; f)$. The corresponding group quotient, $H_1(f) \equiv \mathcal{C}(G)/C_0(H; f)$, is the (first) homology group associated with the map $f$; its elements are equivalence classes formed by sets of cycles whose elements differ by an addition of a trivial cycle. Namely, cycles $C$ and $C'$ are equivalent, $C' \sim C$, if $C' = C \oplus C_0$, with $C_0 \in C_0(H; f)$.

The cycle space of a graph $G = (V, E)$ with $n = |E|$ edges can be defined algebraically in terms of the vertex-edge incidence matrix $J \equiv J_G$. Namely, it is isomorphic to the binary code $C_+^1 \subset \mathbb{F}_2^n$ whose parity check matrix is the incidence matrix $J$, $\mathcal{C}(G) \cong C_+^1$. On the other hand, the code $C_f$ generated by the incidence matrix is isomorphic to the cut space of the graph. Elements of the cut space are edge boundaries $\partial U$ of different partitions defined by sets of vertices $U \subset V$.

In principle, any set $C' \subset \mathcal{C}(G)$ of cycles on $G$ can be used to construct a binary matrix $K$ with the rows orthogonal to $J$, $JK^T = 0$; the code $C_{K} \subset \mathbb{F}_2^n$ is isomorphic to the subspace of the cycle space generated by elements of $C'$. In particular, given the covering map $f : \mathcal{H} \to G$, such a matrix $K$ can be constructed using a basis set of homologically trivial cycles $C_0(H; f)$. Thus, such a covering map has a chain complex \( [5] \) associated with it, where $\mathcal{A}_0, \mathcal{A}_1,$ and $\mathcal{A}_2$ are spaces generated by sets of vertices, edges, and homologically trivial cycles, respectively. In particular, the support $\text{supp}(a)$ of any vector $a \in A_1$ corresponds to a set of edges. The boundary operators are given by the constructed matrices $\partial_1 = J$, $\partial_2 = K^T$. Equivalently, the same matrices can be used to define a stabilizer code $\text{CSS}(J, K)$ with generators $G_X = J$ and $G_Z = K$. We will denote such a quantum cycle code associated with the covering map $f : \mathcal{H} \to G$ as $\mathcal{Q}(H; f)$. The length of the code is $n = |E|$, the number of encoded qubits $k = \text{rank} H_1(A)$ is the rank of the first homology group associated with covering map, and the distances $d_Z, d_X$, respectively, are the homological distances $a_1(A), a_1(A)$.
clusters; and (c) associated with the number of infinite clusters. Most generally, we expect

For a quasi-transitive graph, one has \( p_{T} \leq p_{c} \leq p_{u} \). For a quasi-transitive graph, one has

### C. Percolation transitions

We only consider Bernoulli edge percolation, where each edge \( e \in \mathcal{E} \) of a graph \( \mathcal{H} = (\mathcal{V}, \mathcal{E}) \) is independently labeled as open or closed, with probabilities \( p \) and \( 1 - p \), respectively. We are focusing on the subgraph \( [\mathcal{H}]_{p} \) remaining after removal of all closed edges; connected components of \( [\mathcal{H}]_{p} \) are called clusters. For a given \( v \in \mathcal{V} \), the cluster which contains \( v \) is denoted \( \mathcal{K}_{v} \subseteq [\mathcal{H}]_{p} \). If \( \mathcal{K}_{v} \) is infinite, for some \( v \), we say that percolation occurs.

Three observables are usually associated with percolation: the probability that vertex \( v \) is in an infinite cluster,

\[
\theta_{v} \equiv \theta_{v}(\mathcal{H}, p) = \mathbb{P}_{p}(|\mathcal{K}_{v}| = \infty),
\]

the connectivity function,

\[
\tau_{u,v} \equiv \tau_{u,v}(\mathcal{H}, p) = \mathbb{P}_{p}(u \in \mathcal{K}_{v}),
\]

the probability that vertices \( u \) and \( v \) are in the same cluster, and the local cluster susceptibility,

\[
\chi_{v} \equiv \chi_{v}(\mathcal{H}, p) = \mathbb{E}_{p}(|\mathcal{K}_{v}|),
\]

the expected size of the cluster connected to \( v \). Equivalently, cluster susceptibility can be defined as the sum of probabilities for individual vertices to be in the same cluster as \( v \), i.e., as a sum of connectivities,

\[
\chi_{v} = \sum_{u \in \mathcal{V}} \tau_{v,u}.
\]

The critical probability \( p_{c} \), the percolation threshold, is associated with the formation of an infinite cluster. There is no percolation, \( \theta_{v} = 0 \), for \( p < p_{c} \), but \( \theta_{v} > 0 \) for \( p > p_{c} \). An equivalent definition is based on the existence of an infinite cluster anywhere on \( [\mathcal{H}]_{p} \); the probability of finding such a cluster is zero at \( p < p_{c} \), and one at \( p > p_{c} \), see, e.g., Theorem (1.11) in Ref. [26] (the same proof works for any infinite connected graph).

Similarly, the critical probability \( p_{T} \) is associated with divergence of site susceptibilities: \( \chi_{v} \) is finite for \( p < p_{T} \) but not for \( p > p_{T} \). Again, in a connected graph, this definition does not depend on the choice of \( v \in \mathcal{V} \). If percolation occurs (i.e., with probability \( \theta_{v} > 0 \), \( |\mathcal{K}_{v}| = \infty \)), then clearly \( \chi_{v} = \infty \). This implies \( p_{c} \geq p_{T} \). The reverse is known to be true for percolation on quasi-transitive graphs [27,28]. \( \chi_{v} = \infty \) can only happen inside or on the boundary of the percolation phase. Thus, for a quasi-transitive graph, \( p_{c} = p_{T} \).

An important question is the number of infinite clusters on \( [\mathcal{H}]_{p} \), in particular, whether an infinite cluster is unique. For infinite quasi-transitive graphs, there are only three possibilities: (a) almost surely there are no infinite clusters; (b) there are infinitely many infinite clusters; and (c) there is only one infinite cluster [29,31]. A third critical probability, \( p_{u} \), is associated with the number of infinite clusters. Most generally, we expect \( p_{T} \leq p_{c} \leq p_{u} \). For a quasi-transitive graph, one has

\[
0 < p_{T} = p_{c} \leq p_{u}.
\]
Here, $p_u$ is the uniqueness threshold, such that there can be only one infinite cluster for $p > p_u$, whereas for $p < p_u$, the number of infinite clusters may be zero, or infinite. For an amenable quasi-transitive graph, $p_c = p_u$\cite{12,13} it was conjectured by Benjamini and Schramm\cite{20} that $p_c < p_u$ for non-amenable quasi-transitive graphs. Among other examples, the conjecture has been recently verified for a large class of Gromov-hyperbolic graphs\cite{35}\cite{36}.

In order for the uniqueness threshold to be non-trivial, $p_u < 1$, the graph $H$ has to have only one end. That is, it can not be separated into two or more infinite components by removing a finite number of edges.

In addition to uniqueness of the infinite cluster, the same threshold $p_u$ can be characterized in terms of the connectivity function\cite{35}: Namely, $\inf_{u,v \in V} \tau_{u,v}(p) > 0$ for $p > p_u$ and it is zero for $p < p_u$. Further, for planar transitive graphs, the uniqueness threshold is related to the percolation threshold on the dual graph,

$$p_u(H) = 1 - p_c(\tilde{H}),$$

see the proof of Theorem 7.1 in Ref. 30. In the case of planar amenable graphs where $p_c(H) = p_u(H)$, the duality [16] is between the two percolation transitions\cite{30}.

III. HOMOLOGY-CHANGING TRANSITIONS

A. Weakly converging sequences of graphs with a common cover

Consider a finite graph $G = (V_G, E_G)$ covered by an infinite graph $H = (V, E)$. While the graph $H$ needs not be quasi-transitive, the set of vertex degrees of $H$ is finite and matches that of $G$; in particular, the two graphs have the same maximal degree $\Delta_{\text{max}}$. The covering map $f : V \rightarrow V_G$ also defines a quantum cycle code $Q(H; f)$ with parameters $[(n, k, (d_X, d_Z))]$, where $n = |E_G|$ the number of edges in $G$, and $k = \text{rank } H_1(f)$ the dimension of the first homology group associated with the map $f$. We are particularly interested in the case where the graphs $G$ and $H$ look identically on some scale. Formally, this is formulated in terms of the injectivity radius, defined as the largest integer $r_f$ such that the map $f$ is one-to-one in any ball $B(v, r_f; H)$. Necessarily, for any covering map $f$, the injectivity radius $r_f \geq 1$. We start by giving lower bounds for the distances $d_X$, $d_Z$ in terms of the injectivity radius.

First, an injectivity radius $r_f$ implies that no two vertices located at distance $r_f$ or smaller from any vertex on $H$ map to the same vertex on $G$. On the other hand, any simple cycle $C \subset G$ of length $\ell$ is for sure covered by a ball of radius $r = \lceil \ell/2 \rceil$ centered on a vertex in $C$. This gives (formal proofs are given in the Appendix):

**Lemma 1.** Consider a finite graph $G$ covered by an infinite graph $H$, with the injectivity radius $r_f$. Then the minimum weight $d_Z$ of a non-trivial cycle on $G$ satisfies the inequality $2r_f + 1 \leq d_Z \leq 2r_f + 3$.

Second, the minimum distance $d_X$ is the minimum size of a homologically non-trivial co-cycle, a set of edges on $G$ which has even overlap with any homologically trivial cycle, but is not a cut of $G$. A lower bound for $d_X$ requires some additional assumptions:

**Lemma 2.** Consider a finite graph $G$ covered by an infinite one-ended graph $H$, with the injectivity radius $r_f$. Assume that the cycle group of $H$ can be generated by cycles of weight not exceeding $\omega \geq 3$. Then, the minimum weight of a non-trivial co-cycle on $G$ satisfies the inequality $d_X > r_f/\omega$.

In addition, it will be important that for any covering map $f : H \rightarrow G$, the vertices of $G$ can be lifted in such a way that they induce a connected subgraph of $H$, just as a square-lattice torus with periodic boundary conditions becomes a rectangular piece of the square lattice after cutting two rows of edges.
Lemma 3. Let $\mathcal{G}$ be a finite connected graph, $\mathcal{H}$ its cover with the covering map $f : \mathcal{V} \to \mathcal{V}_f$ and the injectivity radius $r_f$. For any $v' \in \mathcal{V}$ let $v = f(v') \in \mathcal{V}_f$ be its image. Then there exists a set of vertices $\mathcal{V}_f \subset \mathcal{V}$ which contains a unique representative from the fiber of every vertex of $\mathcal{V}_f$, such that the subgraph $\mathcal{H}_f = \mathcal{H}$ induced by $\mathcal{V}_f$ be connected and contains the ball $B(v', r_f; \mathcal{H})$.

In the following, we consider not a single graph $\mathcal{G}$, but a sequence $(\mathcal{G}_t)_{t \in \mathbb{N}}$ of finite graphs $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$ sharing an infinite connected covering graph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, with the covering maps $f_t : \mathcal{V} \to \mathcal{V}_t$. If the corresponding sequence of injectivity radii $r_t \equiv r_f$ diverges, we say that the sequence $(\mathcal{G}_t)_{t \in \mathbb{N}}$ weakly converges to $\mathcal{H}$. Such a convergent sequence can be constructed, e.g., as a sequence of finite quotients of the graph $\mathcal{H}$ with respect to a sequence of subgroups of its symmetry group, which requires $\mathcal{H}$ to be quasitransitive. We do not know whether quasitransitivity of $\mathcal{H}$ is necessary to have a sequence of finite graphs covered by $\mathcal{H}$ and weakly convergent to $\mathcal{H}$. By this reason, in the following, we specify (quasi)transitivity only when necessary for the corresponding proof.

Given a graph sequence with a common covering graph $\mathcal{H}$, we use $\mathcal{Q}_t$ to denote the CSS code with parameters $[[n_t, k_t, (dX_t, dZ_t)]]$ associated with the covering map $f_t$. We also denote the “flattened” subgraphs from Lemma 3 as $\mathcal{H}_t \equiv \mathcal{H}_{f_t} \subset \mathcal{H}$. When the sequence $(r_t)_{t \in \mathbb{N}}$ diverges, we can always construct a subsequence $(t_s)_{s \in \mathbb{N}}$, $t_{s+1} > t_s$, such that the corresponding sequence of graphs $(\mathcal{H}_{t_s})_{s}$ be increasing, $\mathcal{H}_{t_{s+1}} \subset \mathcal{H}_{t_s}$. To this end, it is sufficient to take $r_{t_{s+1}} > n_{t_s}$, regardless of the particular spanning trees used in the construction of the graphs $\mathcal{H}_t$.

### B. Homology erasure thresholds

Coming back to percolation, let $H_1(f_t, p)$ denote the first homology group formed by classes of homologically non-trivial cycles on the open subgraph $[\mathcal{G}_t]_{p}$. We will consider several observables that quantify the changes in homology in the open subgraphs at large $t$ as the probability $p$ is increased. The first two, defined by analogy with corresponding quantities for 1-cycle proliferation in continuum percolation\cite{21}, are designed to detect any changes in homology compared to the empty graphs at $p = 0$, and the graphs with all edges present at $p = 1$. Respectively, we define the probability that a homologically non-trivial cycle exists in the open subgraph,

$$P_E(t, p) \equiv \mathbb{P}_p(\text{rank } H_1(f_t, p) \neq 0),$$

and the probability that not all homologically non-trivial cycles are covered in the open subgraph,

$$P_A(t, p) \equiv \mathbb{P}_p(\text{rank } H_1(f_t, p) \neq k_t).$$

Equivalently, $P_A(t, p)$ is the probability that the open subgraph at $\bar{p} = 1 - p$ covers a homologically non-trivial co-cycle. In terms of the associated CSS code $\mathcal{Q}_t$, $P_E(t, p)$ and $P_A(t, 1 - p)$ are the erasure probabilities for a $Z$- and an $X$-type codeword, respectively. These quantities do not necessarily characterize bulk phase(s), as they may be sensitive to the state of a sublinear number of edges.

As $p$ is increasing from 0 to 1, $P_E(t, p)$ is monotonously increasing from 0 to 1 while $P_A(t, p)$ is monotonously decreasing from 1 to 0. Thus, a version of the subsequence construction can be used to ensure the existence of their $t \to \infty$ limits almost everywhere on the interval $p \in [0, 1]$. Instead, we define the (lower) cycle erasure threshold for any given graph sequence,

$$p^0_E = \sup \left\{ p \in [0, 1] : \lim_{t \to \infty} P_E(t, p) = 0 \right\}.$$  

Because of monotonicity of $P_E(t, p)$ as a function of $p$, a zero limit at some $p = p_0 > 0$ ensures the limit exists and remains the same everywhere on the interval $p \in [0, p_0]$. Further,
the absence of convergence of the sequence $P_E(t,p)$ at some $p = p_1$ implies that the superior and the inferior limits at $t \to \infty$ must be different, which, in turn, implies the existence of a subsequence convergent to the non-zero limit given by $\lim \sup_{t \to \infty} P_E(t,p_1) > 0$.

Similarly, we define the upper cycle erasure threshold,

$$p_E^* = \inf \left\{ p \in [0,1] : \lim_{t \to \infty} P_A(t,p) = 0 \right\},$$

as the smallest $p$ such that open subgraphs preserve the full-rank homology group with probability approaching one in the limit of the sequence.

Existence of a homologically non-trivial cycle not covered by open edges implies that closed edges must cover a conjugate codeword, a non-trivial co-cycle. The related threshold on an infinite graph can be interpreted in terms of a transition dual to percolation, proliferation of the boundaries at the complementary edge configuration, with all closed edges replaced by open edges, and v.v., so that the open edge probability becomes $\bar{p} = 1 - p$. On a locally planar graph, like a tiling of a two-dimensional manifold, the dual transition maps to the usual percolation on the dual graph.

We also notice that the usual erasure threshold $p_E$ for a family (or a sequence) of quantum codes corresponds to a non-zero probability of an erasure, a configuration where a codeword is covered by erased qubits. For a CSS code, this implies a non-zero probability that either an $X$- or a $Z$-type codeword be covered. For codes $Q_t$ associated with covering maps $f_t : \mathcal{H} \to \mathcal{G}_t$ in the sequence $(\mathcal{G}_t)_{t \in \mathbb{N}}$, the conventional erasure threshold can be found in terms of the thresholds for cycles and co-cycles,

$$p_E = \min(p_E^0, 1 - p_E^*).$$

The following lower bound constructed using a Peierls-style counting argument is adapted from Ref. [23].

**Statement 4.** Consider a sequence of finite graphs $(\mathcal{G}_t)_{t \in \mathbb{N}}$ with a common covering graph $\mathcal{H}$. Let $\Delta_{\text{max}}$ be the maximum degree of $\mathcal{H}$, and assume that for some $t_0 > 0$, the injectivity radius $r_t$ associated with the maps $f_t : \mathcal{H} \to \mathcal{G}_t$ at $t \geq t_0$ scales at least logarithmically with the number of edges $n_t$, $r_t \geq A \ln n_t$, with some $A > 0$. The cycle erasure threshold for the corresponding sequence of CSS codes $(Q_t)_{t \in \mathbb{N}}$ satisfies the lower bound $p_E^0 \geq e^{-1/(2A)/(\Delta_{\text{max}} - 1)}$.

It follows from the fact that $Q_t = \text{CSS}(J_t, K_t)$, where $J_t$ is the vertex-edge incidence matrix of $\mathcal{G}_t$, with row weights given by the vertex degrees, and Lemma [1].

We would like to ensure that the conventional erasure threshold [21] also be non-trivial, which requires that $p_E^* < 1$. To construct such an upper bound, which becomes a lower bound in terms of $\bar{p} = 1 - p$ in the dual representation, it is sufficient [23] that rows of the trivial-cycle-edge adjacency matrix $K_t$ have bounded weights, and that the distance $d_{X_t}$ diverges logarithmically or faster with $n_t$. Notice that here we do not rely on Lemma [2] which gives a rather weak lower bound for the distance but, instead, directly assume desired scaling of the minimum weight $d_{X_t}$ of a non-trivial co-cycle with $n_t$. We have

**Statement 5.** Consider a sequence of finite graphs $(\mathcal{G}_t)_{t \in \mathbb{N}}$ with a common covering graph $\mathcal{H}$, with the cycle group $C(\mathcal{H})$ generated by cycles of weight not exceeding $\omega > 1$. Further, assume that the minimum weight $d_{X_t} = d_X(\mathcal{H}; f_t)$ of a non-trivial co-cycle associated with the map $f_t : \mathcal{H} \to \mathcal{G}_t$ grows at least logarithmically with the number of edges $n_t$, $d_{X_t} \geq A' \ln n_t$, for sufficiently large $t \geq t_0$ and some $A' > 0$. The upper erasure threshold for the corresponding sequence of CSS codes $(Q_t)_{t \in \mathbb{N}}$ satisfies the bound $1 - p_E^1 \geq e^{-1/A'}/(\omega - 1)$.

Let us now relate the cycle erasure threshold $p_E^0$ with the bulk percolation threshold. Most generally, it serves as an upper bound:

**Theorem 6.** Consider a sequence of finite graphs $(\mathcal{G}_t)_{t \in \mathbb{N}}$ covered by an infinite graph $\mathcal{H}$. Then, $p_E^0 \leq p_c(\mathcal{H})$. 

This includes the case where the sequence of the injectivity radii remains bounded (no weak convergence to $\mathcal{H}$), in which case, obviously, $p_E^0 = 0$. More precise results for $p_E^0$ are available with additional assumptions, including the scaling of the injectivity radius with the logarithm of the graph size.

**Theorem 7.** Consider a sequence of finite transitive graphs $(\mathcal{G}_t)_{t \in \mathbb{N}}$ covered by an infinite graph $\mathcal{H}$. If the homological distance $d_{Zt}$ scales sublogarithmically with graph size, 
\[
\lim_{t \to \infty} \frac{d_{Zt}}{\ln n_t} = 0, \text{ then } p_E^0 = 0.
\]

**Theorem 8.** Consider a sequence of finite graphs $(\mathcal{G}_t)_{t \in \mathbb{N}}$ covered by an infinite quasitransitive graph $\mathcal{H}$. If the injectivity radius scales superlogarithmically with the graph size, 
\[
\lim_{t \to \infty} \frac{r_t}{\ln n_t} = \infty, \text{ then } p_E^0 = p_c.
\]

Information about the other threshold, $p_E^1$, can be obtained in the planar case with the help of duality:

**Corollary 9.** Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be a pair of mutually dual infinite quasitransitive planar graphs. Consider a sequence of finite graphs $(\mathcal{G}_t)_{t \in \mathbb{N}}$ weakly convergent to $\mathcal{H}$, a cover of the graphs in the sequence. Then,
1. $p_E^1 \geq 1 - p_c(\tilde{\mathcal{H}})$. In addition,
2. if the graphs $\mathcal{G}_t$ in the sequence are transitive, $t \in \mathbb{N}$, and the injectivity radius grows sublogarithmically with the graph size, then $p_E^1 = 1$;
3. if the injectivity radius grows superlogarithmically, then $p_E^1 = 1 - p_c(\tilde{\mathcal{H}})$.

Notice that for a superlogarithmic scaling of the injectivity radius, the graph must be amenable, in which case $p_a(\mathcal{H}) = p_c(\mathcal{H})$. We also believe that under conditions of the Corollary, the duality gives $p_a(\mathcal{H}) = 1 - p_c(\tilde{\mathcal{H}})$, see Eq. (16), although we only found the proof for the case where the graph $\mathcal{H}$ is transitive. Whenever such a duality relation holds, the upper cycle erasure threshold is bounded below by the uniqueness threshold, $p_E^1 \geq p_a(\mathcal{H})$; with superlogarithmic scaling of the injectivity radius, the sequence of thresholds collapses to a single point, $p_E^0 = p_E^1 = p_c(\mathcal{H}) = p_a(\mathcal{H})$.

These results leave out an important case of percolation with logarithmic distance scaling. It is easy to see that logarithmic distance scaling does not necessarily imply that $p_E^0$ and $p_c(\mathcal{H})$ be equal:

**Example 10** (Anisotropic square-lattice toric codes). Consider a sequence of tori $\mathcal{G}_t = T_{L_x(t), L_y(t)}$ obtained from the infinite square lattice $\mathcal{H}$ by identifying the vertices at distances $L_x(t)$ and $L_y(t)$ along the edges in $x$ and $y$ directions, respectively. For some $A > 0$, consider the scaling $L_x(t) = t$, $L_y(t) = e^{t/A}$. This gives $d_{Zt} = t$ and $n_t = e^{t/A}$, so that $d_{Zt} = A \ln n_t$. The cycle erasure threshold $p_E^0$ for this graph sequence satisfies $e^{-1/A}/3 < p_E^0 \leq e^{-1/A}$.

The upper bound follows from considering $L_y(t)$ independent non-trivial cycles of length $t$, while the lower bound is given by Statement 4. In comparison, for edge percolation on infinite square lattice, $p_c = 1/2$.

In addition to Example 10 in Sec. IV we give numerical evidence that $p_E^0 < p_c(\mathcal{H})$ for several families of hyperbolic codes based on regular $\{f, d\}$ tilings of the hyperbolic plane (here $2df > d + f$; these are known to have a finite asymptotic rate $R = 1 - 2/d - 2/f$).

**C. Erasure rate thresholds**

Logarithmic scaling of the minimum distance $d_{Zt}$ associated with the first homology group is the largest one may hope for in the important case when the covering graph $\mathcal{H}$
is non-amenable. We specifically focus on the case of a graph sequence with extensive homology rank scaling, i.e., where the associated codes have an asymptotically finite rate, $R \equiv \lim_{t \to \infty} k_t/n_t > 0$. For such graph sequences, we also consider the expected dimension of the erased subspace per edge, or the erasure rate,

$$R_E(t, p) \equiv n_t^{-1} \mathbb{E}_p(\text{rank} \ H_1(f_t, p)).$$

(22)

Analogous quantity was analyzed in detail by Delfosse and Zémor. Unlike the probabilities $P_E$ and $P_A$, the erasure rate $R_E$ is a bulk quantity which can be used to define a thermodynamical transition in the usual sense. For any $t \in \mathbb{N}$, the erasure rate $R_E(t, p)$ is a monotonously increasing function of $p \in [0, 1]$, bounded by the values at the ends of the interval,

$$0 \leq R_E(t, p) \leq R_t \equiv k_t/n_t \leq 1.$$

(23)

Let us now consider the thresholds associated with the erasure rate (22). We define the lower $p^0_H$ and the upper $p^1_H$ critical points as the values of $p$ where $R_E(t, p)$ in the limit of large $t$ starts to deviate from 0 and from $R$, respectively:

$$p^0_H = \sup\{p \in [0, 1] : \lim_{t \to \infty} R_E(t, p) = 0\},$$

(24)

$$p^1_H = \inf\{p \in [0, 1] : \lim_{t \to \infty} R_E(t, p) = R\}.$$

(25)

We call these, respectively, the lower and the upper homological thresholds. Evidently, $p^0_E \leq p^0_H \leq p^1_H \leq p^1_E$. The critical point $p^0_H$ was discussed in Refs. 10 and 11. Our first result, an analogue of the corresponding inequality for the Ising model, Eq. (34) in Ref. 9, gives a lower bound on the gap between the two homological thresholds:

**Theorem 11.** Consider a sequence of finite graphs $(\mathcal{G}_i)_{i \in \mathbb{N}}$ weakly convergent to an infinite graph $\mathcal{H}$, a cover of the graphs in the sequence, with rate-$R$ extensive homology rank. Then there is a finite gap between the two homological thresholds,

$$p^1_H - p^0_H \geq R.$$

(26)

Second, we prove an “easy” inequality relating the lower homological threshold with the percolation threshold on the covering graph:

**Theorem 12.** For a sequence of finite graphs $(\mathcal{G}_i)_{i \in \mathbb{N}}$ weakly convergent to an infinite graph $\mathcal{H}$, a cover of the graphs in the sequence with extensive homology rank, $p_c(\mathcal{H}) \leq p^0_H$.

The remaining analytical result is obtained with the help of the usual duality between locally planar graphs, and is therefore limited to planar graphs $\mathcal{H}$:

**Theorem 13.** Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be a pair of infinite mutually dual transitive planar graphs. Consider a sequence of finite graphs $(\mathcal{G}_i)_{i \in \mathbb{N}}$ weakly convergent to $\mathcal{H}$, a cover of the graphs in the sequence with extensive homology rank. Then,

(i) $p^0_H = p_c(\mathcal{H})$, (ii) $p^1_H = 1 - p_c(\tilde{\mathcal{H}}) = p_u(\mathcal{H})$.

(27)

This is an easy consequence of two previous results: the expression for the expected homology rate in terms of the average inverse cluster sizes on the graph and its dual, and the exponential decay of the size of finite clusters away from the percolation point on transitive graphs.

Notice that in Theorem 13, the lower and the higher homological thresholds, respectively, are actually associated with the percolation and the uniqueness thresholds on the infinite graph $\mathcal{H}$. We believe this is not a coincidence, and put forward

**Conjecture 14.** Consider a sequence of finite graphs $(\mathcal{G}_i)_{i \in \mathbb{N}}$ weakly convergent to a quasitransitive infinite graph $\mathcal{H}$, a cover of the graphs in the sequence with extensive homology rank. Then,

(i) $p^0_H = p_c(\mathcal{H})$, (ii) $p^1_H = p_u(\mathcal{H})$.

(28)
Such a result makes sense, since neither the percolation nor the uniqueness thresholds can be seen locally, by examining a finite subgraph of $\mathcal{H}$. Similarly, the homological transitions require changes in cycles of length exceeding the injectivity radius, which diverges without a bound.

IV. NUMERICAL RESULTS FOR LOCALLY PLANAR HYPERBOLIC CODES

In addition to analytical results, we also evaluated the erasure and the percolation thresholds numerically for several families of planar hyperbolic codes, as well as for a planar euclidean family of square lattice toric codes. Each family corresponds to a particular infinite graph $\mathcal{H}_{f,d}$, regular tiling of the hyperbolic or euclidean plane, parameterized by the Schläfli symbol $\{f,d\}$, with $2/d + 2/f \leq 1$. In such a graph, $d$-gons meet in each vertex. The finite graphs are constructed as finite quotients of the corresponding graph $\mathcal{H}_{f,d}$ with respect to subgroups of the symmetry group.

The parameters of the graphs used in the calculations are listed in Tab. I where $\{f,d\}$ is the Schläfli symbol of the corresponding tiling, $n$ is the number of edges, and $d_Z$ and $d_X$, respectively, are the distances of the corresponding CSS codes. The smaller graphs with $n < 10^3$ edges are from N. P. Breuckmann We generated the remaining graphs with a custom GAP program, which constructs coset tables of freely presented groups obtained from the infinite van Dyck group $D(d, f, 2) = \langle a, b | a^d, b^f, (ab)^2 \rangle$ [here $a$ and $b$ are group generators, while the remaining arguments are relators which correspond to imposed conditions, $a^d = b^f = (ab)^2 = 1$] by adding one more relator obtained as a pseudo random string of generators to obtain a suitable finite group $D$, a quotient of the original infinite group $D(d, f, 2)$. Then, the vertices, edges, and faces are enumerated by the right cosets with respect to the subgroups $\langle a \rangle$, $\langle ab \rangle$, and $\langle b \rangle$, respectively. The vertex-edge and face-edge incidence matrices $J$ and $K$ are obtained from the coset tables. Namely, non-zero matrix elements are in the positions where the corresponding pair of cosets share an element. Finally, the distance $d_Z$ of the CSS code CSS($J, K$) was computed using the covering set algorithm, which has the advantage of being extremely fast when distance is small and additionally verified by comparing the number of cycles through a given vertex on the finite graph $\mathcal{G}$ and on a sufficiently large subgraph of the infinite covering graph $\mathcal{H}_{f,d}$ (or the corresponding dual graphs in the case of $d_X$).

To analyze percolation, we used a version of the Newman–Ziff (NZ) Monte Carlo algorithm. The original version of the algorithm simultaneously draws from a sequence of canonical ensembles with $x = 1, 2, \ldots$ open edges, by starting with all closed edges and randomly adding one open edge at a time, with the acceleration due to a lower cost of statistics update. To find the rank $k'$ of the first homology group associated with the open subgraph, we used the formula

$$k' = x - |V| + |K'| - |\overline{K}'| + 1, \quad (29)$$

where $x \equiv |E'|$ is the number of open edges, and $|K'|$ and $|\overline{K}'|$, respectively, are the numbers of connected components in the open subgraph of $\mathcal{G}$ and in the closed subgraph of the corresponding dual graph $\mathcal{G}'$. Eq. (29) is a consequence of Eq. (7). It can also be derived with the help of the cycle rank Euler formula and the fact that any trivial open cycle is a cut for the corresponding dual graph. Respectively, in our version of the NZ algorithm, we simultaneously evolve a pair of dual subgraphs, starting with all closed edges on $\mathcal{G}$ and all open edges on $\mathcal{G}'$, and adding an open edge to $\mathcal{G}'$ and removing the corresponding open edge from $\mathcal{G}'$ at each step. In addition, for each set of average quantities $A_x$ computed in the canonical ensemble with $x \in \{0, \ldots, n\}$ edges open, we calculated the corresponding grand-canonical quantity

$$A_p = \sum_{x=0}^n \binom{n}{x} p^x (1-p)^{n-x} A_x. \quad (30)$$
TABLE I. Parameters of the hyperbolic graphs used to calculate critical $p$ values. Given a Schl"afli symbol $\{f,d\}$, finite graphs $G$ and their dual graphs $\tilde{G}$ are parameterized by the number of edges $n$; they have $2n/d$ and $2n/f$ vertices, respectively, and the first homology groups of rank $k = 2 + (1 - 2/d - 2/f)n$. Distances $d_Z$ and $d_X$ are the lengths of the shortest homologically non-trivial cycles on $G$ and $\tilde{G}$, respectively. Numbers in bold indicate the smallest graph found with such a distance; only such graphs were used for calculating the cycle erasure threshold $p_E^0$, see Figs. 2 and 4. Percolation transition critical point $p_c$ on the infinite hyperbolic graph $H_{f,d}$ (same as the giant cluster transition) was calculated using all graphs in the corresponding family, with the exception of graphs whose distances are shown in gray.
For the sake of numerical efficiency, we restricted the summation to terms with $|x - pn| < M\sqrt{np(1-p)}$, with $M = 10^2$. We verified that the results do not change when $M$ is increased by a factor of two. For each graph, we run $10^6$ Newman–Ziff sweeps and saved the grand-canonical averages of observables for $10^3$ values of $p$ with intervals of $\Delta p = 10^{-3}$. In addition, to get an independent estimate of the errors, each threshold calculation was repeated three times.

A de-facto standard way for estimating the erasure threshold $p_E^0$ is the crossing point method. The method is based on the expectation that the block error probability is asymptotically zero for any $p < p_E^0$ and is equal to one for $p > p_E^0$, with the crossover region small for large codes. Respectively, when the erasure probability found numerically for several graphs is plotted as a function of $p$, the corresponding lines are expected to cross in a single point, which is identified as the pseudothreshold.

This works well for codes with power-law distance scaling. An example is shown in Fig. 1 where the homological error probability (17) for several square lattice toric codes with parameters $[2d^2, 2, d]$ and $d$ ranging from 60 to 220 is plotted as a function of the open edge probability $p$. Visually, a beautiful crossing point close to $p_E^0 = 1/2$ is observed. To find the corresponding erasure pseudothreshold, the data was fitted collectively with polynomials of $\xi = p - p^0$. The polynomials had different coefficients for different graphs, except the zeroth order coefficient used to find the ordinate of the crossing point. With the fit range $0.49 < p < 0.51$, the degree of the polynomials was adjusted by hand to minimize the standard deviation of $p^0$, the abscissa of the crossing point extracted from the data. For the square-lattice graphs, using 6th degree polynomials, we obtained $p^0(\{4, 4\}) = 0.500004 \pm 0.000002$, very close to the square lattice percolation threshold $p_c(\{4, 4\}) = 1/2$, as expected from Refs. 22 and 23 and Theorem 8. The corresponding linear terms $A_{1n}$ (the derivative at the crossing point) have a power law $A_{1n} = bn^\alpha$ scaling (not shown), with the exponent $\alpha = 0.375 \pm 0.003$, consistent with the expectation of a sharp threshold in the large-$n$ limit.

![Fig. 1](image-url)

FIG. 1. (Color online) Finding the erasure pseudothreshold for square lattice toric codes. Symbols show the homological error probability (17) evaluated numerically for different graphs labeled by the number of edges $n = 2d^2$, with distances $d$ ranging from 60 to 220, plotted as a function of open edge probability $p$. As expected, beautiful crossing point close to $p_E^0 = 1/2$ is observed. Lines are the polynomials $f_n(\xi) = A_0 + A_{1n}\xi + A_{2n}\xi^2 + \ldots$ of $\xi = p - p^0$ of degree 6 obtained by fitting the data collectively in the range $0.49 < p < 0.51$. The vertical dashed line indicates the square lattice percolation threshold $p_c = 1/2$.

We used a similar technique to process the homological error probability data for hyperbolic graphs. A sample of the corresponding plots is shown in the top portions of Figs. 2 and 4. These plots have two significant differences with that in Fig. 1. First, the crossing points are significantly below the percolation transitions indicated by the vertical dashed lines. Second, despite smaller scales, the convergence near the crossing crossing points does not look as nice. Empirically, deviations in the position of the curves are associated with
the differences in the ratio $\ln n/d$, cf. the bounds in Statements 4, 5 and Example 10. To reduce the corresponding errors, in the calculation of the erasure thresholds we only used the “optimal” graphs, the smallest graphs with the corresponding distances; such graphs are indicated in Tab. I with the distance shown in bold.

Yet, using only the optimal graphs was not sufficient to completely eliminate the finite-size variation. Much better crossing points are obtained by introducing a vertical shift $B \ln n/d$, where $B$ is an additional global fit parameter (see bottom plots in Figs. 2 and 4).

![Graph](image)

**FIG. 2.** (Color online) Top: as in Fig. 1 but for the hyperbolic code family $\{5, 5\}$. The green arrow indicates the position of the crossing point found by the fit; it is significantly below the percolation threshold for the corresponding infinite lattice (vertical red dashed line). In addition, the data for the graph with $n = 15 350$ is shifted upward, which we associate with a slightly smaller ratio $d/\ln n$, see Fig. 3. This is verified in the bottom plot, where an additional vertical shift proportional to $\ln n/d$ is added, which substantially improves the convergence at the crossing point.

In comparison, the crossing point method does not work for measuring the location of the homological transition $p_0^H$, even though the variation between the graphs is not expected to matter that much here. Main reason for the difference is that the erasure rate (22) retains a finite slope in the infinite graph limit, which makes the crossing point analysis unreliable.

To check for spurious errors, in our simulations we have also measured the conventional percolation characteristics, in particular, the average sizes $S_j = \langle K_j \rangle$, $j = 1, 2, 3$ of the three largest clusters. We have used several finite-size scaling techniques to extract the location of the percolation transition which coincides with the giant-cluster transition, see Theorem 1.3 in Ref. 46. All techniques, including the cluster-size ratio technique $^{47, 48}$, give transition points in a reasonable agreement with the values expected from invasion percolation simulations in Ref. 49. For hyperbolic graphs $\mathcal{H}_{f,d}$ with $(f + d)/fd < 2$, we found that the most accurate values of $p_c$ are found using the technique based on the expectation of cluster size scaling similar to that for random graphs $^{50, 51}$, $S_j \propto n^{2/3}$ near
FIG. 3. Homological distance $d \equiv d_Z$ associated with non-trivial cycles for optimal graphs in \{7, 3\} and \{5, 5\} families vs. the graph size $n$ (number of edges) with the logarithmic scale. Numbers also indicate the graph sizes. Smaller relative distances $d/\ln n$ result in larger erasure probabilities in Figs. 2 and 4 (top); this can be compensated to some extent by using the correction term as in bottom plots in Figs. 2 and 4.

FIG. 4. (Color online) As in Fig. 2 but for the hyperbolic code family \{7, 3\}. The convergence at the crossing point is much better than that in Fig. 2 (top), which we associate with substantially higher ratios $d/\ln n$ for the graphs in this family. Bottom plot: addition of the additional vertical shift $B \ln n/\ln d$ causes a substantial shift of the crossing point position without visibly improving the convergence.
$p_c$, with the critical region of width $\Delta p \sim n^{-1/3}$. Respectively, when interpolated values of $p$ such that the expected size of the largest cluster satisfies $S_1(p) = \omega n^{2/3}$ are plotted for $\omega \in \{1/4,1/2,1\}$ as a function of $x \equiv n^{-1/3}$, the data for graphs with different $n$ fit nicely, and can be extrapolated to $x = 0$ (infinite graph size) using polynomial fits, see Fig. 5. Notice that while this technique works well for hyperbolic graphs and for random graphs, in our simulations it failed dramatically for the planar $\{4,4\}$ graph family, as can be seen from the corresponding value of $p_c(2/3)$ in Tab. II.

| $\{f, d\}$ | $p_c(2/3)$ | $n_c(2/3)$ | $deg_p^{(C)}$ | $n_c^{(C)}$ | $deg_p^{(shift)}$ | $n_c^{(shift)}$ | $B$ |
|------------|-------------|-------------|---------------|-------------|---------------|-------------|-----|
| $\{3, 7\}$ | 0.1993565(5) | 0.1999(8) | 0.7 | 2 | 0.1941(2) | -26 | 6 | 0.19318(9) | -69 | 0.081(5) |
| $\{7, 3\}$ | 0.5305264(8) | 0.5320(5) | 3.0 | 2 | 0.52109(8) | -120 | 4 | 0.52042(5) | -200 | 0.087(2) |
| $\{3, 8\}$ | 0.1601555(2) | 0.160(2) | -0.08 | 3 | 0.1519(4) | -21 | 7 | 0.1524(1) | -78 | 0.26(1) |
| $\{8, 3\}$ | 0.5136441(4) | 0.513(2) | -0.3 | 3 | 0.5032(2) | -52 | 6 | 0.5026(1) | -110 | 0.32(3) |
| $\{4, 5\}$ | 0.2689195(3) | 0.2695(6) | 1.0 | 2 | 0.2581(2) | -54 | 5 | 0.2547(2) | -71 | 0.306(8) |
| $\{5, 4\}$ | 0.3512228(3) | 0.3519(7) | 1.0 | 2 | 0.3415(4) | -24 | 2 | 0.3412(4) | -25 | 0.18(9) |
| $\{4, 6\}$ | 0.20714787(9) | 0.2076(4) | 2.3 | 1 | 0.1964(4) | -290 | 5 | 0.1949(3) | -41 | -0.08(3) |
| $\{6, 4\}$ | 0.3389049(2) | 0.3395(1) | 6.0 | 1 | 0.3271(4) | -29 | 2 | 0.3275(3) | -38 | 0.14(4) |
| $\{5, 5\}$ | 0.25416087(31) | 0.2545(7) | 0.5 | 2 | 0.2437(4) | -26 | 5 | 0.2453(2) | -44 | 0.88(6) |

TABLE II. Critical $p$ values found for graph families characterized by Schlafli symbols $\{f, d\}$, where $f = \infty$ stands for random graphs of degree $d$. Here $p_c$ is the percolation threshold (using invasion percolation data from Ref. 49 or exact values where known), $p_c(2/3)$ is the percolation threshold using random-graph-like cluster size scaling (see Fig. 5), $p_c^{(C)}$ is the cycle erasure pseudothreshold obtained using the crossing point method (see Fig. 1 and top plots in Figs. 2 and 4), and $p_c^{(shift)}$ is the same from the crossing point with an additional shift as in the bottom plots in Figs. 2 and 4 with $B$ the shift coefficient. Numbers in the parenthesis indicate the standard deviation $\sigma$ in the units of the last significant digit, so that, e.g., $0.14(4) = 0.14 \pm 0.04$. Values of $n_c \equiv (p - p_c)/\sigma$ give the “number of sigmas” for the deviation of the corresponding critical value found from the invasion percolation or exact threshold value if available. Numbers in columns labeled “deg” give the degrees of the polynomials used to interpolate the data; polynomials of the same degrees were used to obtain $p_c^{(C)}$ and $p_c^{(shift)}$.

The obtained critical values $p_c(2/3)$, $p_c^{(C)}$, and $p_c^{(shift)}$ for different graph families are summarized in Tab. II where they are compared with the corresponding percolation thresholds from Ref. 49 obtained from invasion percolation simulations, or exact values where available. Numerical data indicates that the erasure (pseudo)threshold is substantially below $p_c$ for hyperbolic graphs with logarithmic distance scaling, with the variation of the ratio $\ln n/d$ having a significant effect on the quality of the crossing point. In contrast, for graphs from the $\{4, 4\}$ family where $d \propto n^{1/2}$, the cycle erasure (pseudo)threshold is very close to the bulk percolation threshold, as generally expected from Refs. 22 and 23 and Theorem 8.

Our results also indicate that for expander graphs, most accurate results for percolation transition critical point are obtained using the random-graph-like scaling, see Fig. 5 although this technique is not at all applicable when the limiting graph is a tiling of the euclidean plane. Detailed comparison of the performance of different extrapolation methods for percolation transition critical point for various amenable and non-amenable graph families will be published elsewhere.

V. CONCLUSIONS

In this work we focused on critical points associated with homology-changing percolation transitions in a sequence of finite graphs weakly convergent to an infinite graph $\mathcal{H}$, a covering graph of the graphs in the sequence. We also quantified the relation between these critical points and edge percolation threshold on $\mathcal{H}$.

The position of the homological 1-cycle erasure threshold $p_c^{(E)}$ is governed by the scaling of the homological distance $d$ with $\log n$, where $d$ is the size of a smallest non-trivial cycle
FIG. 5. (Color online) Using the random-graph-like scaling for locating the percolation transition for hyperbolic graphs in the \{(5, 5)\} (top) and \{(7, 3)\} (middle) families, and for degree-5 random graphs (bottom). Values of the open bond probability \(p\) where the expected size of the largest cluster equals \(\omega n^{2/3}\) are plotted as a function of \(n^{-1/3}\), for values of \(\omega\) as indicated. Here \(n\) is the number of edges in the graph. The lines intersect close to the percolation transition point, as indicated by horizontal dashed lines.

and \(n\) is the graph size (number of edges). Generally, \(p^0_E \leq p_c(\mathcal{H})\), where the equality is reached for superlogarithmic distance scaling, while \(p^0_E = 0\) is expected for sublogarithmic distance scaling. In the case of logarithmic distance scaling where the quantity \(d/\ln n\) remains bounded away from 0 and from infinity, the cycle erasure threshold \(p^0_E\) remains strictly positive as long as \(\mathcal{H}\) is a bounded-degree graph, and we expect \(p^0_E\) to be strictly below \(p_c\).

For an amenable graph \(\mathcal{H}\) with a finite isoperimetric dimension, an easy upper bound
on the distance can be constructed by considering a ball with the radius equal to the injectivity radius, giving a power-law scaling of the distance with \( n \). Generically, we expect that a sequence of covering maps with superlogarithmic distance scaling can be constructed when such a graph is quasitransitive, resulting in \( p_E^0 = p_c(\mathcal{H}) \). In particular, this is the case for any periodic lattice in dimension \( D > 1 \), since covering maps can be constructed by using periodic boundary conditions along each axis.

On the other hand, logarithmic scaling of the distance is the most one can expect when \( \mathcal{H} \) is non-amenable. For such a graph the uniqueness threshold is expected to be strictly higher than the percolation threshold, \( \Delta p \equiv p_u(\mathcal{H}) - p_c(\mathcal{H}) > 0 \), which gives a non-trivial upper bound for the asymptotic rate, \( R \leq \Delta p \), where \( R > 0 \) corresponds to an extensive scaling of the homology rank associated with non-trivial 1-cycles. For any graph sequence with \( R > 0 \), we also introduced a pair of homological thresholds \( p_H^0 \) and \( p_H^1 \), associated with the points where asymptotic erasure rate \( (22) \) deviates from the values at \( p = 0 \) and \( p = 1 \), respectively. Generally, \( p_H^0 \geq p_c \); for planar transitive graphs we proved \( p_H^0 = p_c(\mathcal{H}) \) and \( p_H^1 = p_u(\mathcal{H}) \). We conjecture this to be the case more generally.

A number of open questions remain. First, related to the sequences of finite graphs both weakly convergent to an infinite graph \( \mathcal{H} \), and covered by \( \mathcal{H} \). What are the properties of \( \mathcal{H} \) necessary for such a sequence to exist, in particular, is it necessary that \( \mathcal{H} \) be quasitransitive? Second, is it true that with a logarithmic distance scaling, the strict inequality holds \( p_E^0 < p_c(\mathcal{H}) \)?

Finally, an important open question is to what extent present results can be extended to other models, in particular, Ising and, more generally, \( q \)-state Potts model on various graphs. Indeed, successful decoding probability in qubit quantum LDPC codes can be mapped to ratios of partition functions of associated random-bond Ising models\(^{(4,9,52)}\). In the clean (no-disorder) limit, these can be rewritten in terms of Fortuin-Kasteleyn (FK) random-cluster models. For such a model with \( q \geq 1 \), Hutchcroft\(^{(53)}\) has recently proved the exponential decay of cluster size distribution in the subcritical regime. In particular, this could help fixing the location of the boundary of the decodable region for certain families of graph-based quantum CSS codes in the weak-noise limit.

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**APPENDIX: THE PROOFS**

1. **Proof of Lemma 1**

**Lemma 1.** Consider a finite graph \( \mathcal{G} \) covered by an infinite graph \( \mathcal{H} \), with the injectivity radius \( r_f \). Then the minimum weight \( d_Z \) of a non-trivial cycle on \( \mathcal{G} \) satisfies the inequality \( 2r_f + 1 \leq d_Z \leq 2r_f + 3 \).

**Proof.** Let \( C \subset \mathcal{E}_G \) be a non-trivial cycle of weight \( d_Z \), and \( v \in \mathcal{V}_G \) a vertex on \( C \). Let \( v' \in \mathcal{V} \) be a vertex from the fiber of \( v \), then the ball \( B \equiv B(v', r_f; \mathcal{H}) \) is mapped one-to-one by \( f \).

Since \( C \) is non-trivial, it must contain at least one edge outside of the image of \( B \). Since \( C \) is also a minimum-weight non-trivial cycle, it must be self-avoiding, i.e., it should contain two edge-disjoint paths connecting \( v \) to the boundary of the image of \( B \). Necessarily, \( d_Z > 2r_f \).

Conversely, consider a ball \( B_1 \) of radius \( r_f + 1 \) which covers a non-trivial cycle \( C_1 \) on \( \mathcal{G} \) of weight \( w \), the shortest cycle among those covered by \( B_1 \). At most two vertices of \( C_1 \) are at the distance \( r_f + 1 \) from \( u \) (otherwise a shorter cycle could be constructed), which gives \( d_Z \leq w \leq 2r_f + 3 \). \( \square \)
2. Proof of Lemma 2

Lemma 2. Consider a finite graph \( \mathcal{G} \) covered by an infinite one-ended graph \( \mathcal{H} \), with the injectivity radius \( r_f \). Assume that the cycle group of \( \mathcal{H} \) can be generated by cycles of weight not exceeding \( \omega \geq 3 \). Then, the minimum weight of a non-trivial co-cycle on \( \mathcal{G} \) satisfies the inequality \( d_X > r_f/\omega \).

Proof. The statement is trivial if \( r_f < \omega \), since \( d_X \geq 1 \) by definition. Assume \( r_f \geq \omega \), so that any generator of the cycle group on \( \mathcal{H} \) is mapped one-to-one. Thus, any (finite) cycle on \( \mathcal{H} \) is mapped to a homologically trivial cycle, where we assume that the symmetric set difference "@" is used when an edge is encountered in the image more than once. Consequently, a lift of a walk cycling around a simple non-trivial cycle \( C \) on \( \mathcal{G} \) cannot be closed; instead, it must be a portion of a semi-infinite self-avoiding path on \( \mathcal{H} \). Respectively, for any edge \( e_0 \in C \) and its lift \( e'_0 \in \mathcal{E} \) such that \( f(e'_0) = e_0 \), we denote \( C' \equiv C'(C, e'_0) \equiv e'_0 \), the extended lift of \( C \), the union of lifts of all walks on \( C \) starting with \( e'_0 \) and \( e_0 \), respectively; \( C' \) is an infinite self-avoiding path.

Now, take a binary vector \( b \) with \( \text{wgt}(b) = d_X \) such that \( B \equiv \text{supp}(b) \subset \mathcal{E}_\mathcal{G} \) be a minimum-weight non-trivial co-cycle on \( \mathcal{G} \). Then, it must be irreducible, which implies that \( B \) must be cycle-connected, i.e., for any pair of edges \( e_i \neq e_f \) in \( B \), it should also contain a connecting edge sequence \( S = (e_1 = e_i, e_2, \ldots, e_{m-1}, e_m = e_f) \subseteq B \), with any pair of neighboring edges sharing an image of a basis cycle on \( \mathcal{H} \). Given such a sequence of length \( m \), the conventional graph distance between any pair of vertices from the union \( e_i \cup e_f \) must be strictly smaller than \( \omega m \).

To prove the contrary, let us assume that \( d_X \leq r_f/\omega \). Then, a minimum-weight co-cycle \( B \subset \mathcal{E}_\mathcal{G} \) must have a diameter strictly smaller than \( r_f \), i.e., there be a ball \( B_r \subset \mathcal{G} \) of radius \( r \leq r_f \) such that \( B \subset B_r \). Indeed, with \( \text{wgt}(B) = d_X \), any connecting sequence contains at most \( m = d_X \) edges, which implies the conventional distance between any pair of vertices on \( B \) smaller than \( d_X \omega \leq r_f \). This implies that any lift \( B' \) of \( B \) should be mapped one-to-one by \( f \).

To finish the proof, let \( C \subset \mathcal{E}_\mathcal{G} \) be an irreducible cycle conjugate to \( B \), i.e., the corresponding binary vectors satisfy \( bC = 1 \), which implies the existence of an edge \( e_0 \subset B \cap C \). Irreducibility of \( C \) implies that it must be a simple cycle on \( \mathcal{G} \). Given \( e'_0 \) such that \( f(e'_0) = e_0 \), let \( B' \equiv e'_0 \) be a lift of \( B \) and \( C' \equiv C'(C, e'_0) \) an extended lift of \( C \), an infinite self-avoiding path on \( \mathcal{H} \). Since \( B' \) is mapped one-to-one by \( f \), it has odd-weight intersection with \( C' \) and even-weight intersection with any basis cycle on \( \mathcal{H} \). Respectively, \( B' \) must have an odd-weight intersection with any deformation \( C'' \equiv C' \oplus M \) of \( C' \), where \( M \subset \mathcal{C}_f \) is a finite cycle on \( \mathcal{H} \). Thus, \( B' \) is a finite-size cut splitting \( \mathcal{H} \) into infinite portions, which cannot be the case since \( \mathcal{H} \) is assumed one-ended.

3. Proof of Lemma 3

Lemma 3. Let \( \mathcal{G} \) be a finite connected graph, \( \mathcal{H} \) its cover with the covering map \( f : \mathcal{V} \to \mathcal{V}_\mathcal{G} \) and the injectivity radius \( r_f \). For any \( v' \in \mathcal{V} \) let \( v \equiv f(v') \in \mathcal{V}_\mathcal{G} \) be its image. Then there exists a set of vertices \( \mathcal{V}_f \subset \mathcal{V} \) which contains a unique representative from the fiber of every vertex of \( \mathcal{V}_\mathcal{G} \), such that the subgraph \( \mathcal{H}_f \subset \mathcal{H} \) induced by \( \mathcal{V}_f \) be connected and contains the ball \( B(v', r_f; \mathcal{H}) \).

Proof. Consider a graph \( \mathcal{G}' \) obtained from \( \mathcal{G} \) by removing the ball \( B \equiv B(v, r_f; \mathcal{G}) \). Construct a connected graph \( \mathcal{G}'' \) from a union of \( B \) and spanning trees of every connected component of \( \mathcal{G}' \), by sequentially adding bridge bonds connecting individual components so that no new cycles are introduced. Such a subgraph contains all vertices of \( \mathcal{G} \) and can be lifted to \( \mathcal{H} \) starting with \( v' \); let \( \mathcal{V}_f \subset \mathcal{V}_\mathcal{H} \) be the corresponding vertex set. By construction, \( f \) acts one-to-one on \( \mathcal{V}_f \). It is also easy to check that \( \mathcal{H}_f \), the subgraph of \( \mathcal{H} \) induced by \( \mathcal{V}_f \), is connected.
4. Proof of Theorem 6

**Theorem 6.** Consider a sequence of finite graphs \((G_t)_{t \in \mathbb{N}}\) covered by an infinite graph \(\mathcal{H}\). Then, \(p^0_E \leq p_c(\mathcal{H})\).

**Proof.** If \(p_c(\mathcal{H}) = 1\), the statement of the theorem is trivial. In the following, assume \(p_c(\mathcal{H}) < 1\) and take \(p\) such that \(p_c(\mathcal{H}) < p < 1\). For some \(t \in \mathbb{N}\), a chosen \(v' \in \mathcal{V}_t\) and \(v \equiv f_t(v')\), we connect percolation on \(\mathcal{H}\) and on \(G_t\) using a set-up similar to invasion percolation[22].

Namely, we start with single-site zeroth generation clusters \(K^{(0)}_\nu = \{v\} \subset \mathcal{V}_t\) and \(K^{(0)}_\nu = \{v'\} \subset \mathcal{V}_t\), with no edges labeled open or closed. Given a generation-\(j\) cluster \(K^{(j)}_\nu \subset \mathcal{V}_t\), every previously unlabeled edge adjacent to a vertex in \(K^{(j)}_\nu\) is labeled open with independent probability \(p\) and otherwise closed. The next generation cluster \(K^{(j+1)}_\nu\) is formed by adding any vertices connected to those in \(K^{(j)}_\nu\) by newly open edges. Let us denote by \(P_j(p; \mathcal{H})\) the probability that the process can be continued after step \(j\), i.e., there be one or more unlabeled edges incident on the \(j\)-th generation cluster. Clearly, \(P_0(p; \mathcal{H}) = 1\) and \(P_j(p; \mathcal{H})\) is strictly decreasing as a function of \(j\), with \(\lim_{j \to \infty} P_j(p; \mathcal{H}) = \theta(p; \mathcal{H})\).

Let us now look at thus constructed percolation process on \(\mathcal{G}_t\). As long as the image of no vertex connected to \(K^{(j)}_\nu\) by a so far unlabeled edge coincides with the image of a vertex in \(K^{(j)}_\nu\) (we call such a cluster “flat”), we can use the map \(f_t\) to make the labels on \(\mathcal{G}_t\) match those on \(\mathcal{H}\). Clearly, all clusters are flat for \(j < r_t\), the injectivity radius; for such \(j\) the probabilities that the percolation process may be continued match exactly on the two graphs, \(P_j(p; \mathcal{G}_t) = P_j(p; \mathcal{H})\). On the other hand, \(P_j(p; \mathcal{G}_t) = 0\) for \(j \geq |\mathcal{V}_t|\). The percolation processes necessarily decouple whenever a cluster \(K^{(j)}_\nu\) ceases to be flat, i.e., there be an unlabeled edge on \(\mathcal{G}_t\) connecting a pair of vertices in \(K^{(j)}_\nu \subset \mathcal{V}_t\). Given such a cluster, we can assign the remaining unlabeled edges on \(\mathcal{G}_t\) all at once; the resulting open subgraph of \(\mathcal{G}_t\) contains a homologically non-trivial cycle with probability greater than or equal to \(p\). At the same time, the cluster \(K^{(j)}_\nu \subset \mathcal{V}_t\) is removed from the percolation process on \(\mathcal{H}\). Since it is not certain that a descendant of a given cluster be infinite, we get the lower bound

\[
P(K_{\nu} \text{ contains a non-trivial cycle}) \geq p \theta(p; \mathcal{H}),
\]

which is positive for any \(p > p_c\), thus \(p^0_E \leq p_c\). \(\square\)

5. Proof of Theorem 7

**Theorem 7.** Consider a sequence of finite transitive graphs \((G_t)_{t \in \mathbb{N}}\) covered by an infinite graph \(\mathcal{H}\). If the homological distance \(d_{2\ell}\) scales sublogarithmically with graph size, \(\lim_{t \to \infty} \frac{d_{2\ell}}{\ln n_t} = 0\), then \(p^0_E = 0\).

**Proof.** To set up independent erasure events, cut \(\mathcal{G}_t\) into non-overlapping regions, images of non-overlapping balls on \(\mathcal{H}\) of radius \(\rho_t = 1 + \lfloor d_{2\ell}/2\rfloor\). Given the maximum graph degree \(\Delta_{\max}\), we can cut out at least

\[
N_t \geq |\mathcal{V}_t|/|\mathcal{B}_0(2\rho_t, \mathcal{G}_t)| > |\mathcal{V}_t|/\Delta_{\max}^{2+d_{2\ell}}
\]

such balls. By transitivity of \(\mathcal{G}_t\) and Lemma 2 each ball contains a homologically non-trivial cycle of length \(d_{2\ell}\), which is open with probability \(P_1 = p^{d_{2\ell}}\). Now, probability that a homology is covered in none of the \(N_t\) balls can be upper bounded as

\[
P_{\text{none}} = |1 - P_1|^{N_t} \leq |1 - p^{d_{2\ell}}|^{N_t} \leq \exp(-N_t p^{d_{2\ell}}) \leq \exp\left(-|\mathcal{V}_t|/\Delta_{\max}^{2+d_{2\ell}}\right),
\]

which is guaranteed to converge to zero for any \(p > p_c\) since \(d_{2\ell}\) scales sublogarithmically with \(|\mathcal{V}_t|\). (Notice that \(|\mathcal{V}_t| \geq 2n_t/\Delta_{\max}\) by a version of the hand-shaking lemma.) \(\square\)
Notice that the requirement of transitivity for the graphs $G_t$ can be relaxed a bit, namely, by assuming that the number of vertex classes [defined by distinct vertex orbits connected by elements of $\text{Aut}(G_t)$] remains uniformly bounded for the graphs $G_t$. In that case, the balls need to be taken of radius $\rho_t = \lfloor d_{Z_t}/2 \rfloor + m$, where $m$ is the maximum number of vertex classes. The proof is completed with the following lemma:

**Lemma 15.** Consider a connected graph $\mathcal{H}$, with $m \geq 1$ vertex classes. Any ball of radius $m$ contains representative(s) of all classes.

*Proof.* Consider a class connectivity graph $\mathcal{G}$ corresponding to $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, with $m$ vertices (one per class) and an edge between two vertices if $\mathcal{H}$ contains an edge between a pair of vertices in these classes. Necessarily, $\mathcal{G}$ is connected. Further, given a vertex $v \in \mathcal{V}$, any spanning tree on $\mathcal{G}$ can be lifted to a tree on $\mathcal{H}$ that contains $v$; such a tree contains a representative from every vertex class. Further, the diameter of the tree cannot exceed $m$; such a tree is contained in a ball $B(v, m; \mathcal{H})$. The proof is complete since the choice of $v$ is arbitrary. \qed

6. Proof of Theorem 8

**Theorem 8.** Consider a sequence of finite graphs $(G_t)_{t \in \mathbb{N}}$ covered by an infinite quasi-transitive graph $\mathcal{H}$. If the injectivity radius scales superlogarithmically with the graph size, 

$$\lim_{t \to \infty} \frac{r_t}{\ln n_t} = \infty,$$

then $p^0_E = p_c$.

*Proof.* Only a cluster with $s \geq d_{Z_t} > r_t$ vertices can cover a homology. For a graph $\mathcal{G}$, let $M_v(s; \mathcal{G})$ denote the probability that vertex $v$ is in an open cluster with exactly $s$ vertices on $[\mathcal{G}]_p$. On the quasi-transitive graph $\mathcal{H}$, this probability has an exponential bound, $M_v(s; \mathcal{H}) < M(s) = e^{-\gamma(p)s}$, for some $\gamma(p)$ non-zero in the subcritical region, $\gamma(p) > 0$ for $p < p^c$. Note also $\sum_{s \geq 1} M_v(s; \mathcal{G}) = 1$ on any finite graph; below percolation threshold this is also true for infinite graphs. Also, for any $v \in \mathcal{V}_t$, finding a cluster of size $s \leq r_t$ attached to $v$ on $\mathcal{G}_t$ has the same probability as that attached to a vertex $v'(v)$ from the fiber of $v$ on $\mathcal{H}$. Use the union bound for the probability of finding a cluster of size $r_t + 1$ or larger on $\mathcal{G}_t$,

$$P_{\text{one}} \leq \sum_{v \in \mathcal{V}_t} \sum_{s > r_t} s^{-1} M_v(s; \mathcal{G}_t)$$

$$< \sum_{v \in \mathcal{V}_t} \sum_{s > r_t} M_v(s, \mathcal{G}_t)$$

$$= \sum_{v \in \mathcal{V}_t} \left(1 - \sum_{1 \leq s \leq r_t} M_v(s; \mathcal{G}_t)\right)$$

$$= \sum_{v \in \mathcal{V}_t} \left(1 - \sum_{1 \leq s \leq r_t} M_{v'(v)}(s; \mathcal{H})\right)$$

$$= \sum_{v \in \mathcal{V}_t} \sum_{s > r_t} M_{v'(v)}(s; \mathcal{H})$$

$$< |\mathcal{V}_t| \sum_{s > r_t} e^{-\gamma(p)s} = \frac{|\mathcal{V}_t| e^{-\gamma(p)r_t}}{e^{\gamma(p)} - 1}, \quad (33)$$

which goes to zero with $t \to \infty$ whenever $\gamma(p) > 0$ since $r_t$ is assumed to be superlogarithmic in $n_t \geq |\mathcal{V}_t| - 1$. This proves $p^0_E \geq p_c$; the statement of the Theorem is obtained with the help of Theorem 6. \qed
7. Proof or Corollary 9

Corollary 9. Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be a pair of mutually dual infinite quasitransitive planar graphs. Consider a sequence of finite graphs $(\mathcal{G}_t)_{t \in \mathbb{N}}$ weakly convergent to $\mathcal{H}$, a cover of the graphs in the sequence. Then,

(i) $p_E^1 \geq 1 - p_c(\tilde{\mathcal{H}})$. In addition,

(ii) if the graphs $\mathcal{G}_t$ in the sequence are transitive, $t \in \mathbb{N}$, and the injectivity radius grows sublogarithmically with the graph size, then $p_E^1 = 1$;

(iii) if the injectivity radius grows superlogarithmically, then $p_E^1 = 1 - p_c(\tilde{\mathcal{H}})$.

Proof. Since $\tilde{\mathcal{H}}$ is quasitransitive, it has a finite maximum degree, which is the maximum size of a face of $\mathcal{H}$. Thus, with injectivity radius large enough, $f_t$ must be invertible on the union of any face and its adjacent faces on $\mathcal{H}$. This guarantees that (with $t$ sufficiently large, $t > t_0$), $\mathcal{G}_t$ be locally planar, so that we can construct the locally planar dual graph $\tilde{\mathcal{G}}_t$ whose cover is $\tilde{\mathcal{H}}$. Further, for any open edge configuration, the ranks of the homology groups on the open subgraph of $\mathcal{G}_t$ and on the closed subgraph of $\tilde{\mathcal{G}}_t$ add to $k_t$, the number of inequivalent homologically non-trivial cycles on $\mathcal{G}_t$ [Eq. (8)]. Thus, the two erasure thresholds are simply interchanged by duality, $p_E^1 = 1 - p_E^0$ and $p_E^0 = 1 - p_E^1$, so that the inequality $p_E^1 \geq 1 - p_c(\tilde{\mathcal{H}})$ follows immediately from Theorem 6.

The identities in (ii) and (iii) similarly follow from Theorems 7 and 8 with the help of Lemmas 4 and 2, which guarantee that the injectivity radii on the sequence of mutually dual graphs scale simultaneously in a sub-logarithmic, logarithmic, or superlogarithmic fashion. \hfill \Box

8. Proof of Theorem 11

Theorem 11. Consider a sequence of finite graphs $(\mathcal{G}_t)_{t \in \mathbb{N}}$ weakly convergent to an infinite graph $\mathcal{H}$, a cover of the graphs in the sequence, with rate-$R$ extensive homology rank. Then there is a finite gap between the two homological thresholds,

$$p_H^1 - p_H^0 \geq R. \tag{26}$$

Proof. Let $k_t = \text{rank } H_1(f_t)$ be the number of non-trivial independent cycles on $\mathcal{G}_t$. Consider any open edge configuration on $\mathcal{G}_t$, with homology rank $k_t' \leq k_t$, and another edge configuration obtained by removing some open edges, with homology rank $k''_t \leq k'_t$. Such a change in homology requires removing at least $\Delta k_t = k'_t - k''_t$ open edges. Considering these as random edge configurations at $p' > p_H^1$ and $p'' < p_H^0$, averaging, and dividing by the total number of edges $n_t$, we obtain

$$p' - p'' \geq R_E(t, p') - R_E(t, p'');$$

in the limit $t \to \infty$ this becomes $p' - p'' \geq R$. Taking infimum over $p' > p_H^1$ and supremum over $p'' < p_H^0$, we obtain the claimed inequality. \hfill \Box

9. Proof of Theorem 12

Theorem 12. For a sequence of finite graphs $(\mathcal{G}_t)_{t \in \mathbb{N}}$ weakly convergent to an infinite graph $\mathcal{H}$, a cover of the graphs in the sequence with extensive homology rank, $p_c(\mathcal{H}) \leq p_H^0$.

Proof. Take $p > p_H^0$, then the limit in Eq. (24) is either strictly positive or does not exist. In either case, since terms in the sequence are bounded, $R_E(t, p) < 1$, the superior limit $f_p = \lim \sup_{t \to \infty} R_E(t, p)$ exists and is strictly positive, $f_p > 0$ at $p > p_H^0$. This implies the existence of a convergent subsequence, e.g., specified by an increasing sequence of indices $(t_j)_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} R_E(t_j, p) = f_p$. \hfill \Box
Because of the existence of the limit, whenever \( f_p > 0 \), for any \( \epsilon > 0 \) and a sufficiently large \( j \), clusters covering homologically non-trivial cycles are expected to occupy at least \((f_p - \epsilon)n_t\) edges, where \( t \equiv t_j \). Thus, if we choose \( \epsilon = f_p/2 \), a cluster \( K_v \subset [G_t]_p \) connected to a randomly chosen vertex \( v \in V_t \) covers a homologically non-trivial cycle with probability \( P_{non-triv} \geq f_p n_t/(2|V_t|) \). Using a map like that in the proof of Theorem 13 at sufficiently large \( t \), a cluster covering a non-trivial cycle on \([G_t]_p\) corresponds to an infinite cluster on \([\mathcal{H}]_p\), which gives

\[
\theta_v(p) \geq \lim_{j \to \infty} f_p n_t_j / (2|V_t_j|) \geq f_p/2 > 0,
\]

thus \( p > p_c(\mathcal{H}) \).

\[\square\]

10. Proof of Theorem 13

**Theorem 13.** Let \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) be a pair of infinite mutually dual transitive planar graphs. Consider a sequence of finite graphs \( (G_t)_{t \in \mathbb{N}} \) weakly convergent to \( \mathcal{H} \), a cover of the graphs in the sequence with extensive homology rank. Then,

\[
\begin{align*}
(I) \quad p^0_{\mathcal{H}} &= p_c(\mathcal{H}), \\
(II) \quad p^1_{\mathcal{H}} &= 1 - p_c(\tilde{\mathcal{H}}) = p_u(\mathcal{H}).
\end{align*}
\]

**Proof.** As in the proof of Corollary 9 at sufficiently large \( t \) the graph \( G_t = (V_t, E_t) \) is necessarily locally planar, which implies the existence of the corresponding dual graph \( \tilde{G}_t = (\tilde{V}_t, \tilde{E}_t) \), with the dual-graph sequence weakly convergent to the dual infinite graph \( \tilde{\mathcal{H}} \).

The proof relies on the relation between the expected homology rank of the open subgraph and the expected inverse cluster sizes on an open subgraph of \( \mathcal{H} \) and a closed subgraph of \( \tilde{\mathcal{H}} \). While the argument goes back to the work of Sykes and Essam, we give a complete derivation here. Consider a configuration of open/closed edges on \([G_t]_p\) with \( E' \leq n_t = |E_t| \) open edges, \( K' \) clusters, and the cycle group of rank \( C' = C_{triv} + K' \), where \( K' \) is the number of non-trivial basis cycles. According to Euler’s theorem, \( K' = |V_t| - E' - C' \). The other hand, duality matches any simple trivial cycle on \( G_t \) to a cut on the dual graph \( \tilde{G}_t \), which gives \( C_{triv} = K' - 1 \), with \( K' \) being the number of clusters on the dual graph in the dual edge configuration, with open and closed edges interchanged. This gives

\[
k' = K' - K' + E' - |V_t| + 1.
\]

Taking the average over the edge configurations on \([G_t]_p\) we obtain for \( k_p^{(t)} \equiv \mathbb{E}_p(\text{rank } H_{1}(f_t, p)) \)

\[
k_p^{(t)} = \sum_{v \in V_t} \kappa_v(p; G_t) - \sum_{v \in V_t} \kappa_v(\bar{p}; \bar{G}_t) + pn_t - |V_t| + 1.
\]

Here \( \kappa_v(p; G_t) \equiv \mathbb{E}_p(|K_v|^{-1}) \) is the expected inverse size of a cluster containing vertex \( v \) on \([G_t]_p\), and \( \kappa_v(\bar{p}; \bar{G}_t) \) is the corresponding quantity on the dual graph, averaged over the dual edge configurations, which is equivalent to \( \bar{p} = 1 - p \). Introducing the corresponding vertex-average quantities, e.g., \( \kappa(p; G_t) \equiv |V_t|^{-1} \sum_{v \in V_t} \kappa_v(p; G_t) \), we get

\[
k_p^{(t)} = |V_t| \kappa(p; G_t) - |V_t| \kappa(1 - p; \bar{G}_t) + pn_t - |V_t| + 1.
\]

To obtain the asymptotic erasure rate (22) we divide the obtained result by \( n_t \) and notice that very large clusters give no contribution to the total while (at sufficiently large \( t \)) any finite cluster on \([G_t]_p\) has the same probability as an equivalent cluster on \([\mathcal{H}]_p\). Further, assuming the transitive graphs \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) of degrees \( d \) and \( f \), respectively, the graphs \( G_t \) and \( \tilde{G}_t \) respectively have the same degrees, and the hand-shaking lemma gives \( |V_t| = 2n_t/d \), \( |V_t| = 2n_t/f \). This proves both the existence and the value of the following limit at any \( p \),

\[
R_E(p) \equiv \lim_{t \to \infty} R_E(t, p) = \frac{2}{d} \kappa(p; \mathcal{H}) - \frac{2}{f} \kappa(\bar{p}; \tilde{\mathcal{H}}) + p - \frac{2}{d},
\]

(34)
Finally, we notice that for non-amenable transitive graphs $\mathcal{H}$ and $\tilde{\mathcal{H}}$, the quantities $\kappa(p; \mathcal{H})$ and $\kappa(p; \tilde{\mathcal{H}})$ are analytic functions of $p$ in the vicinity of any $p \in (0, 1)$ such that $p \neq p_c(\mathcal{H})$ and $\tilde{p} \neq p_c(\tilde{\mathcal{H}})$, respectively. Thus, the r.h.s. of Eq. (34) is an analytic function of $p$ for

$$p \in (0, 1) \setminus \{p_c(\mathcal{H}), 1 - p_c(\tilde{\mathcal{H}})\},$$

where $1 - p_c(\tilde{\mathcal{H}}) = p_a(\mathcal{H}) > p_c(\mathcal{H})$. On the other hand, $R_p(c)$ cannot be analytic in the lower and upper homological thresholds $0 < p_H^l < p_H^u < 1$, which gives the two equalities.

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