A VINogradov-TYPE PROBLEM IN ALMOST PRIMES

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Abstract. We generalise the Vinogradov’s three primes theorem to linear equations in almost primes. For \( m \geq 3 \) and fixed positive integers \( c_1, \ldots, c_m, r_1, \ldots, r_m \) we find the asymptotic formula for number of solutions in positive integers of the equation \( c_1n_1 + \cdots + c_mn_m = N \), given that every \( n_i \) has exactly \( r_i \) prime factors. We also assume that at least three of the \( r_i \) are equal to 1. The main novel idea is to introduce a variant of Vinogradov’s theorem with varying coefficients and then, to create almost primes out of them in a combinatorial manner.

1. Introduction

One of the best known problems in additive combinatorics which are already solved is without a doubt so called weak (or ternary) Goldbach conjecture, which can be stated in the following form:

**Theorem 1.1.** Every odd number \( N \) greater than 7 can be represented as a sum of three primes.

The assertion of Theorem 1.1 was proven to be correct for all sufficiently large \( N \) in 1937 by Vinogradov [1]. Later Chen and Wang [2] gave the first effective threshold. Recently, Helfgott [3, 4] gave new bounds, which were strong enough to cover all remaining cases.

The proof of the ineffective version of Theorem 1.1 gives us also the precise asymptotic formula for the number of solutions of the equation \( p_1 + p_2 + p_3 = N \) for \( p_1, p_2, p_3 \) being primes. For technical reasons it is easier to attach a weight \( \log p \) to each power of a prime \( p \) and deal with the sum

\[
\mathcal{R}_3(N) := \sum_{n_1, n_2, n_3} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3),
\]

where \( \Lambda(n) \) denotes the von Mangoldt function.

In this paper we calculate the asymptotic formula for number of solutions in almost primes of the more general equation \( c_1n_1 + \cdots + c_mn_m = N \), where the \( c_i \) are some fixed positive integers (linear equations and systems of linear equations in primes were studied in detail for instance in [5, 6]) and \( m \geq 3 \). Formally, we assume that the number of prime divisors of any \( n_i \) is equal to \( r_i \), where \((r_1, \ldots, r_m)\) is some sequence of fixed positive integers. There are also certain technical obstacles compelling us to assume that at least three of the \( r_i \) are equal to 1. We prove the following result.

**Theorem 1.2.** Fix \( m \geq 3 \). Let \( c_1, \ldots, c_m \) be fixed positive integers satisfying \((c_1, \ldots, c_m) = 1\), and let \( r_1, \ldots, r_m \) be a sequence of fixed positive integers containing at least three elements.
being equal to 1. Then,

\[
\sum_{n_1, \ldots, n_m \geq 1 \atop \forall i \Omega(n_i) = r_i} 1 = \frac{1}{(m-1)!} \frac{1}{(r_1-1)!} \cdots \frac{1}{(r_m-1)!} c_1 \cdots c_m \times \frac{N^{m-1}}{\log^m N} (\log \log N)^{r_1+\cdots+r_m-m} (\mathcal{S}_{c_1, \ldots, c_m}(N) + o(1)),
\]

where

\[
\mathcal{S}_{c_1, \ldots, c_m}(N) := \\
\prod_{p|N} \left( 1 + \frac{\mu(p)}{\varphi(p)} \cdot \cdots \cdot \frac{\mu(p)}{\varphi(p)} \right) \prod_{p|N} \left( 1 - \frac{\mu(p)}{\varphi(p)} \cdot \cdots \cdot \frac{\mu(p)}{\varphi(p)} \right),
\]

and \(\Omega(n)\) denotes the number of prime factors of \(n\) counted with multiplicity.

In Section 2, we present Lemma 2.2 proven via tools acquired from the geometry of numbers. It plays a key role in Sections 3 and 4, where we adapt the circle method to calculate the number of solutions of the equation \(b_1 n_1 + \cdots + b_m n_m = N\) in weighted primes, where we let the \(b_i\) depend on \(N\) to a certain extent (precisely, we assume that \(b_i \ll N^{5/12} (2m - 1)\)). Afterwards, in Section 3 we discard the von Mangoldt weights and replace them with characteristic functions. In the last Section 4 we construct almost primes from primes in a combinatorial manner, which finishes the proof of Theorem 1.2. In Appendix there are four technical lemmas and Lemma A.5 which elaborates a bit on the nature of \(\mathcal{S}_{c_1, \ldots, c_m}\) function.

**Notation.** By log we always denote the natural logarithm. To avoid any disturbances arising from small \(N\) we assume that \(N > 10000\). We use the notation \(N = \{1, 2, 3, \ldots\}\). We also adopt the following notions most of which are common in analytic number theory:

- \(\varphi(n) := |(\mathbb{Z}/n\mathbb{Z})^\times|\) denotes Euler totient function;
- \(\tau(n) := \sum_{d|n} 1\) denotes the divisor function;
- By \((a, b)\) and \([a, b]\) we denote the greatest common divisor and the lowest common multiple, respectively;
- For a logical formula \(\phi\) we define the indicator function \(1_{\phi(x)}\) that equals 1 when \(\phi(x)\) is true, and 0 otherwise;
- By \(\mathbb{R}/\mathbb{Z}\) we denote the appropriate quotient group – in fact, in most cases we simply identify it with the interval \([0, 1]\);
- We make use of the ‘big \(O\)’, the ‘small \(o\)’, and the ‘\(\ll\)’ notation in a standard way. We also consider \(m\) as fixed throughout the paper.

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2. **Geometry of numbers**

Let us define \(J_{b_1, \ldots, b_m}(N)\) to be the number of tuples \((n_1, \ldots, n_m) \in \mathbb{N}^m\) satisfying the equation \(b_1 n_1 + \cdots + b_m n_m = N\) for some \(b_1, \ldots, b_m \in \mathbb{N}\).

We define a lattice as a discrete additive subgroup of \(\mathbb{R}^M\) for some \(M \in \mathbb{N}\). Let also \(v_1, \ldots, v_K \in \mathbb{R}^M\) be linearly independent vectors, and let \(K \in \mathbb{N}\). If \(\Lambda\) is a lattice and

\[
\Lambda = \{a_1 v_1 + \cdots + a_K v_K : a_1, \ldots, a_K \in \mathbb{Z}\},
\]

then \(\Lambda\) is the union of \(K\) translates of \(\mathbb{Z}^M\):

\[
\Lambda = \bigcup_{k=1}^K \left( a_1 v_1 + \cdots + a_K v_K + \mathbb{Z}^M \right).
\]

In this context, a typical example of a lattice \(\Lambda\) is the set of integer \(m\)-vectors \(\left( n_1, \ldots, n_m \right) \in \mathbb{Z}^m\) satisfying the equation \(b_1 n_1 + \cdots + b_m n_m = N\), where \(b_1, \ldots, b_m \in \mathbb{N}\) and \(N \in \mathbb{N}\). Thus, if \(\Lambda\) is a lattice, then for any integer \(n_1, \ldots, n_m \in \mathbb{Z}\):

\[
\sum_{n_1, \ldots, n_m \geq 1 \atop \forall i \Omega(n_i) = r_i} 1 = \frac{1}{(m-1)!} \frac{1}{(r_1-1)!} \cdots \frac{1}{(r_m-1)!} c_1 \cdots c_m \times \frac{N^{m-1}}{\log^m N} (\log \log N)^{r_1+\cdots+r_m-m} (\mathcal{S}_{c_1, \ldots, c_m}(N) + o(1)),
\]

where

\[
\mathcal{S}_{c_1, \ldots, c_m}(N) := \\
\prod_{p|N} \left( 1 + \frac{\mu(p)}{\varphi(p)} \cdot \cdots \cdot \frac{\mu(p)}{\varphi(p)} \right) \prod_{p|N} \left( 1 - \frac{\mu(p)}{\varphi(p)} \cdot \cdots \cdot \frac{\mu(p)}{\varphi(p)} \right),
\]

and \(\Omega(n)\) denotes the number of prime factors of \(n\) counted with multiplicity.
then the collection of points \( \{ v_j \}_{j=1}^K \) is called a basis of the lattice, and the set
\[
\{ t_1 v_1 + \cdots + t_K v_K : t_1, \ldots, t_K \in [0, 1) \}
\]
is called a minimal parallelogram of the lattice. Neither basis nor minimal parallelogram are unique, although the \( K \)-dimensional measure of this parallelogram is, and henceforth it shall be called the determinant of the lattice. We denote it as \( d(\cdot) \).

Recall the following theorem from [9, Chapter 2.10.4, Theorem 4].

**Theorem 2.1.** In every lattice \( L \) there exists a basis \( \{ v_j \}_{j=1}^K \) which satisfies
\[
\prod_{j=1}^K \| v_j \| \ll_K d(L).
\]

Our goal in this chapter is to prove the following result.

**Lemma 2.2.** Let \( \delta \in \mathbb{R}_+ \). Consider integers \( 1 \leq b_1, \ldots, b_m \leq N^\delta \) such that \( (b_1, \ldots, b_m) = 1 \). Then,
\[
J_{b_1, \ldots, b_m}(N) = \frac{N^{m-1}}{(m-1)! b_1 \cdots b_m} + O \left( N^{2\delta(m-1)+m-2} \right).
\]

**Proof.** Define two more lattices:
\[
\Lambda := \mathbb{Z}^m,
\]
\[
\tilde{\Lambda} := \{(n_1, \ldots, n_m) \in \mathbb{Z}^m : n_1 + \cdots + n_m = 0\}.
\]

We can transform the condition \( b_1 n_1 + \cdots + b_m n_m = 0 \) into the conjunction of \( n_1 + \cdots + n_m = 0 \) and \( b_1 | n_1, \ldots, b_m | n_m \). Define yet another lattices:
\[
\Lambda^* := \{(n_1, \ldots, n_m) \in \mathbb{Z}^m : b_1 | n_1, \ldots, b_m | n_m \},
\]
\[
\tilde{\Lambda}^* := \{(n_1, \ldots, n_m) \in \mathbb{Z}^m : b_1 | n_1, \ldots, b_m | n_m, n_1 + \cdots + n_m = 0 \}.
\]

Obviously, \( \Lambda^* \subset \Lambda \) and \( \tilde{\Lambda}^* \subset \tilde{\Lambda} \). We have
\[
\begin{bmatrix}
 b_1 b_m & 0 & \cdots & 0 & -b_1 b_m \\
 0 & b_2 b_m & \cdots & 0 & -b_2 b_m \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & b_{m-1} b_m & -b_{m-1} b_m
\end{bmatrix}
\]

Hence, the following set of linearly independent vectors (not necessarily a basis of \( \tilde{\Lambda}^* \)):
\[
(0, \ldots, b_i b_m, \ldots, 0, -b_i b_m) : i = 1, \ldots, m - 1 \subset \tilde{\Lambda}^*,
\]
generates a non-degenerated parallelogram of dimension \( m - 1 \). Its volume expressed as a square root of the modulus of the Gram’s matrix can be estimated from above by the Hadamard’s inequality:
\[
\sqrt{\det [b_i b_j b_m^2 (1 + 1_{i=j})]_{i,j=1}^{m-1}} \leq 2^{m-1} (m-1)^{\frac{m-1}{2}} N^{2\delta(m-1)}.
\]

Let us denote the \( \mathbb{Z} \)-module generated by the set of vectors from (2.1) by \( V \).

A submodule of a finitely generated free module of rank \( n \) over a principal ideal domain is also free. Moreover, the rank of this submodule cannot exceed \( n \). Therefore, the lattices \( V \), \( \Lambda^* \), and \( \tilde{\Lambda} \), all treated as \( \mathbb{Z} \)-modules, have to be free. Consequently, the chain of inclusions
\[
\mathbb{Z}^{m-1} \cong V \subset \tilde{\Lambda}^* \subset \tilde{\Lambda} \cong \mathbb{Z}^{m-1}
\]
implies \( \tilde{\Lambda}^* \cong \mathbb{Z}^{m-1} \).
By (2.2) and Theorem 2.1 we can produce a basis \( \{v_1, \ldots, v_{m-1}\} \subset \Lambda^* \) satisfying
\[
\prod_{j=1}^{m-1} \|v_j\| \ll d(\Lambda^*) \ll N^{2\delta(m-1)}.
\]

On the other hand, the distance between each two points of \( \Lambda^* \) is not smaller than \( \sqrt{2} \), which implies \( \|v_i\| \gg 1 \) for all \( 1 \leq i \leq m - 1 \). Hence,
\[
\|v_j\| \ll N^{2\delta(m-1)}
\]
for \( j = 1, \ldots, m - 1 \).

Consider the minimal parallelogram
\[
P := \{ t_1v_1 + \cdots + t_{m-1}v_{m-1} : t_1, \ldots, t_{m-1} \in [0,1) \}.
\]

We would like to know, how many points from the lattice \( \tilde{\Lambda} \) are contained in every parallelogram of the form \( x + P \) with \( x \in \Lambda^* \). To achieve this, let us consider the sequence of maps
\[
\Lambda \xrightarrow{\pi} \tilde{\Lambda} \xrightarrow{\rho} \Lambda^*/\Lambda^*,
\]
where \( \pi \) is defined as
\[
\pi: (n_1, \ldots, n_{m-1}, n_m) \mapsto (n_1, \ldots, n_{m-1}, -(n_1 + \cdots + n_{m-1}))
\]
and \( \rho \) is the module division. The kernel of \( (\rho \circ \pi) \) consists of those \( (n_1, \ldots, n_m) \) satisfying
\[
n_1 \equiv 0 \pmod{b_1},
\]
\[
\vdots
\]
\[
n_{m-1} \equiv 0 \pmod{b_{m-1}},
\]
\[
n_1 + \cdots + n_{m-1} \equiv 0 \pmod{b_m}.
\]
Let us count the number of solutions of (2.4) meeting conditions 1 \( \leq n_i \leq b_i b_m \) for all \( 1 \leq i \leq m - 1 \), and also \( n_m = 0 \). This problem can be rephrased a bit. We may equivalently ask about the number of solutions \( (g_1, \ldots, g_{m-1}) \in \mathbb{Z}_{b_m}^{m-1} \) of
\[
g_1 b_1 + \cdots + g_{m-1} b_{m-1} \equiv 0 \pmod{b_m}.
\]
All these solutions form a submodule of \( \mathbb{Z}_{b_m}^{m-1} \). According to Bézout’s lemma, our assumption \( (b_1, \ldots, b_m) = 1 \) implies that there exist such \( g_1', \ldots, g_{m-1}' \in \mathbb{Z}_{b_m} \) that
\[
g_1' b_1 + \cdots + g_{m-1}' b_{m-1} \equiv 1 \pmod{b_m}.
\]
Therefore, the whole module \( \mathbb{Z}_{b_m}^{m-1} \) can be decomposed into union of \( b_m \) equinumeroius cosets, each defined by equation
\[
g_1' b_1 + \cdots + g_{m-1}' b_{m-1} \equiv g \pmod{b_m},
\]
with an appropriate choice of \( 0 \leq g \leq b_m - 1 \). It implies that equation (2.5) has exactly \( b_m^{-2} \) solutions. Therefore, (2.4) also has \( b_m^{-2} \) solutions satisfying 1 \( \leq n_i \leq b_i b_m \) for all \( 1 \leq i \leq m - 1 \), and \( n_m = 0 \). On the other hand, we know that if we replaced (2.4) by the empty set of conditions and left only the extra assumption that \( n_m = 0 \), we would have \( b_1 \cdots b_{m-1} b_m^{-1} \) solutions. Thus,
\[
\#(\Lambda/\ker(\rho \circ \pi)) = b_1 \cdots b_m.
\]
We have the isomorphism
\[
\tilde{\Lambda}/\Lambda^* \cong \Lambda/\ker(\rho \circ \pi),
\]
so \( \tilde{\Lambda}/\Lambda^* \) has exactly \( b_1 \cdots b_m \) elements. Consequently, every parallelogram of the form \( x + P \) with \( x \in \Lambda^* \) also contains exactly \( b_1 \cdots b_m \) elements.
Define the following subsets
\[ \tilde{\Lambda}_N := \{(n_1, \ldots, n_m) \in \mathbb{Z}^m : n_1 + \cdots + n_m = N\}, \]
\[ \tilde{\Lambda}_N^* := \{(n_1, \ldots, n_m) \in \mathbb{Z}^m : n_1 + \cdots + n_m = N, \ b_1 | n_1, \ldots, b_m | n_m\}. \]
If \( n = (n_1, \ldots, n_m) \in \mathbb{Z}^m \) is any solution of the equation \( b_1 n_1 + \cdots + b_m n_m = N \) (which certainly exists, because \((b_1, \ldots, b_m) = 1\)), then we can write \( \tilde{\Lambda}_N = n + \Lambda \) and \( \tilde{\Lambda}_N^* = n + \tilde{\Lambda}^*. \) According to this, for each \( x \in \tilde{\Lambda}_N^* \) the parallelogram of the form \( x + P \) contains exactly \( b_1 \cdots b_m \) points from \( \tilde{\Lambda}_N \).

The number of points of the form \((n_1, \ldots, n_m) \in \mathbb{N}^m\), satisfying \( n_1 + \cdots + n_m = N \), equals

\[
\binom{N-1}{m-1} = \frac{N^{m-1}}{(m-1)!} + O(N^{m-2}).
\]

For every \( x \in \tilde{\Lambda}_N^* \) the parallelogram \( x + P \) contains exactly one point from \( \tilde{\Lambda}_N^* \). Therefore, \( J_{b_1, \ldots, b_m}(N) \) is equal to the number of parallelograms of this form contained in the simplex

\[
T := \text{conv}\{(N,0,\ldots,0), \ldots, (0,\ldots,0,N,0,\ldots,0), \ldots, (0,\ldots,0,N)\}
\]
up to the number of those having non-empty intersections with \( \partial T \). Let us define \( R := \text{diam} \; P \). Observe that (2.3) implies

\[
R \leq \sum_{j=1}^{m-1} \|v_j\| \ll N^{2(m-1)}.
\]

From \( x + P \subset B(x, R) \) and (2.8) we obtain

\[
|\{x \in \tilde{\Lambda}_N^* : \text{dist}(x, \partial T) \leq R\}| \ll |\{x \in \Lambda : \text{dist}(x, \partial T) \leq R\}| \ll R^m |\partial T| \ll N^{2\delta m(m-1)+m-2}.
\]

We conclude that there are at most \( O(N^{2\delta m(m-1)+m-2}) \) parallelograms of the form \( x + P \) for \( x \in \tilde{\Lambda}_N^* \) having a non-empty intersection with \( \partial T \). Hence, the claim follows. Also note that the main term dominates the error term for each \( \delta < \frac{1}{2m(m-1)} \) which is good enough for our purposes.  

3. Major arcs

Put \( Q := \log B \; N \) for any real \( B > 0 \). Then, for integers \( q \leq Q \) and \( a \), such that \((a, q) = 1\), we set

\[
\mathcal{M}_{a,q} = \left\{ \alpha \in \mathbb{R}/\mathbb{Z} : \|\alpha - a/q\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{Q}{N} \right\},
\]
where \( \| \cdot \|_{\mathbb{R}/\mathbb{Z}} \) denotes the distance to the nearest integer. We define the union of all major arcs as

\[
\mathcal{M} := \bigcup_{q \leq Q} \bigcup_{(a,q)=1}^{q} \mathcal{M}_{a,q},
\]
and the union of all minor arcs as \( m := R/Z \setminus M \). The name ‘minor arc’ may be a bit misleading since the measure of \( m \) actually approaches 1 as \( N \to \infty \). We put
\[
S_i(N, \alpha) := \sum_{n \leq N/b_i} \Lambda(n) e(nb_i\alpha),
\]
\[
S(N, \alpha) := \sum_{n \leq N} \Lambda(n) e(n\alpha),
\]
\[
u_i(y) := \sum_{n \leq N/b_i} e(nb_iy),
\]
\[

u(y) := \sum_{n \leq N} e(ny).
\]

Recall a useful lemma from [7, Lemma 3.1].

**Lemma 3.1.** There is a positive constant \( C \) such that whenever \( 1 \leq a \leq q \leq Q \), \( (a, q) = 1 \), \( \alpha \in M_{a,q} \) one has
\[
S(N, \alpha) = \frac{\mu(q)}{\varphi(q)} u(y) + O \left( N \exp \left( -c \sqrt{\log N} \right) \right),
\]
where \( \alpha := \frac{a}{q} + y \).

It is worth mentioning that this lemma is ineffective due to its reliance on the Siegel–Walfisz theorem. It also makes the whole proof of Theorem 1.2 ineffective.

Now, we prove the following result.

**Theorem 3.2.** Fix \( c_1, \ldots, c_m \in N \). Let \( b_i := c_i \eta_i \) for \( i = 1, \ldots, m \), where each \( \eta_i \) is some positive integer with prime divisors greater than \( Q \). Let us further assume that \( (b_1, \ldots, b_m) = 1 \), and that \( b_1, \ldots, b_m \leq N^\delta \) for some \( \delta \in (0, \frac{5}{12m(2m-1)}) \). Then, for every \( \varepsilon > 0 \) we have
\[
\sum_{b_1 n_1 + \cdots + b_m n_m = N} \Lambda(n_1) \cdots \Lambda(n_m) = \frac{1}{(m-1)!} b_1 \cdots b_m \Theta_{c_1, \ldots, c_m}(N)
\]
\[
+ O \left( \frac{N^{m-1}}{b_1 \cdots b_m Q^{m-2-\varepsilon}} \right) + \int_m \prod_{i=1}^m S_i(N, \alpha) e(-N\alpha) \, d\alpha.
\]

**Proof.** Apply Lemma 3.1 and replace \( \alpha \mapsto b_i \alpha \), \( y \mapsto b_i y \), and \( a \) by an integer \( \tilde{a} \) satisfying \( 1 \leq \tilde{a} \leq q \) and \( \tilde{a} \equiv b_i \alpha \mod q \). Thus, we obtain
\[
S_i(N, \alpha) = \frac{\mu(q)}{\varphi(q)} \left( \frac{a}{(b_i,q)} \right) u_i(y) + O \left( \frac{N}{b_i \log N} \right)
\]
for some positive constant \( C \). We are ready to estimate the contribution of a single major arc to the integral:
\[
\int_{M_{a,q}} \prod_{i=1}^m S_i(N, \alpha) e(-N\alpha) \, d\alpha = \prod_{i=1}^m \frac{q}{\varphi(q)} \left( \frac{a}{(b_i,q)} \right) e \left( -\frac{aN}{q} \right) \times
\]
\[
\int_{-N}^{N} \prod_{i=1}^m u_i(y) e(-Ny) \, dy + O \left( \sum_{\omega \in \{0,1\}^m} \int_{-N}^{N} f_{\omega_1}(y) \cdots f_{\omega_m}(y) e(-Ny) \, dy \right),
\]
where
\[ f_{\omega_j}(y) := \begin{cases} u_i(y) \times \mu \left( \frac{a_{(b_i,q)}}{q} \right) / \varphi \left( \frac{a_{(b_i,q)}}{q} \right) & \text{if } \omega_j = 1 \\ \frac{N}{\varepsilon} \exp \left( -C \sqrt{\log N} \right) & \text{if } \omega_j = 0 \end{cases} \]

We have the obvious inequality \( |u_i(y)| \leq \frac{N}{\varepsilon} \). Thus, if at least one coordinate of \( \omega \) is equal to 0, then for such \( \omega \) we get
\[
\int f_{\omega_1}(y) \cdots f_{\omega_m}(y) e(-Ny) dy \ll \frac{1}{b_1 \cdots b_m} \exp((C - \varepsilon) \sqrt{\log N})
\]

Summing over every possible \( a, q \) one gets
\[
\int \prod_{i=1}^{m} S_i(N,a) e(-Na) da = \sum_{q \leq Q} \prod_{i=1}^{m} \mu \left( \frac{a_{(b_i,q)}}{q} \right) \sum_{a \leq q} e \left( -\frac{aN}{q} \right) \int f_{\omega_i}(y) e(-Ny) dy
\]
\[
+ O \left( \frac{N^{m-1}}{b_1 \cdots b_m} \exp((C - 3\varepsilon) \sqrt{\log N}) \right).
\]

Put \( c_q(N) := \sum_{a \leq q} e \left( -\frac{aN}{q} \right) \). Due to multiplicativity of Ramanujan’s sums we conclude that
\[
\sum_{q \leq Q} \prod_{i=1}^{m} \mu \left( \frac{a_{(b_i,q)}}{q} \right) c_q(N) = \prod_{p \mid N} \left( 1 + \prod_{i=1}^{m} \mu \left( \frac{p}{(c_i,p)} \right) (p - 1) \right) \prod_{p \mid N} \left( 1 - \prod_{i=1}^{m} \mu \left( \frac{p}{(c_i,p)} \right) \right)
\]
\[
+ O \left( \sum_{q > Q} \prod_{i=1}^{m} \frac{\varphi(a_{(c_i,p)})}{\varphi(a_{(b_i,q)})} c_q(N) \right) = \mathcal{S}_{c_1,\ldots,c_m}(N) + O \left( \frac{1}{Q^{m-1}} \right).
\]

We could replace \( b_i \) by \( c_i \) because \( (b_i,q) = (c_i,q) \) for each \( q \leq Q \) and each \( 1 \leq i \leq m \). Moreover,
\[
|\mathcal{S}_{c_1,\ldots,c_m}(N)| < \prod_p \left( 1 + \frac{1}{\varphi(p)^{m-1}} \right) \ll 1.
\]

We are left with the integral on the right hand side of (3.5) to estimate. Let us consider the following subsets of \( \mathbb{R}/\mathbb{Z} \):
\[
J_k^{(j)} = \left[ \frac{k}{b_j} - \frac{1}{b_jN^{1/3}}, \frac{k}{b_j} + \frac{1}{b_jN^{1/3}} \right],
\]
\[
l_k^{(j)} = \left[ \frac{k}{b_j} - \frac{1}{b_jN^{1/2}}, \frac{k}{b_j} + \frac{1}{b_jN^{1/2}} \right],
\]
for \( j = 1,\ldots,m \) and \( k = 0,\ldots,b_j - 1 \). The distance between two distinct fractions of the form \( k/b_j \) satisfies
\[
|k_1/b_j - k_2/b_j| \geq \frac{1}{b_j} \quad \text{for } N > 2^{3m/2}, \text{ and arbitrary } k_1, k_2 \in \mathbb{Z}.
\]
Consequently, we can assume that \( N \) is so large that each pair of intervals of the form \( J_k^{(j)} \) centered in different points \( k/b_j \) have empty intersection.
Applying it, we obtain

\[ \exists \text{ such an index } j \]

(3.12)

where \( b := \min\{b_1, \ldots, b_m\} \) and \( S \) denotes the set \( \mathbb{R}/\mathbb{Z} \setminus \bigcup_{j=1}^{m} \bigcup_{k=0}^{b_j-1} J_k^{(j)} \). Now, we recall the basic Dirichlet kernel estimation:

\[ u_i(y) \leq \frac{1}{1 - e(b_i y)} \leq \frac{1}{\|b_i y\|_{\mathbb{R}/\mathbb{Z}}}. \]

Applying it, we obtain

(3.9)

The assumption \( (b_1, \ldots, b_m) = 1 \) implies that for each \( y_0 \in (0, 1) \) at most \( m - 1 \) elements from the set \( \{b_1 y_0, \ldots, b_m y_0\} \) can be integers. Hence, for every \( y \in \mathbb{R}/\mathbb{Z} \setminus [-\frac{1}{bN^{1/3}} + \frac{1}{bN^{1/3}}, \frac{1}{bN^{1/3}}] \) there exists such an index \( j \in \{1, \ldots, m\} \) that none of the \( J_k^{(j)} \) contains \( y \). Thus,

(3.10)

From (3.10) we calculate

(3.11)

Combining (3.8)–(3.11), and recalling that \( \delta < \frac{5}{12m(2m-1)} \leq \frac{1}{12m} \), we conclude that

(3.12)

The idea regarding the \( J_k^i \) and the \( I_k^{(j)} \) intervals is heavily inspired by the comment of user Tal H from mathoverflow.net.
From (3.5), (3.6), (3.12), and Lemma 2.2 we get

(3.13) \[
\int \prod_{i=1}^{m} S_i(N, \alpha) e(-N\alpha) d\alpha = \frac{1}{(m-1)!b_1 \cdots b_m} \Phi_{c_1, \ldots, c_m}(N) + O \left( \frac{1}{b_1 \cdots b_m N^{m-2-\varepsilon}} \right).
\]

\[\square\]

4. Minor arcs

In this section we estimate the contribution of the integral

(4.1) \[
\int \prod_{i=1}^{m} S_i(N, \alpha) e(-N\alpha) d\alpha.
\]

Usually, some variations of the Vinogradov’s lemma (Lemma 4.4 in our case) are exploited to establish results of this kind. Firstly, let us recall yet another lemma.

**Lemma 4.1** (R. C. Vaughan). Let \(X, Y, \alpha \in \mathbb{R}\) where \(X, Y \geq 1\). Assume that \(|\alpha - \frac{a}{q}| \leq \frac{1}{Xq^2}\) for some \(a, q \in \mathbb{N}\) such that \((a, q) = 1\). Thus,

\[
\sum_{n \leq X} \min \left\{ \frac{XY}{n}, \frac{1}{n|\alpha|} \right\} \ll \left( \frac{XY}{q} + X + q \right) \log(2Xq).
\]

**Proof.** See [7, Lemma 2.2]. \[\square\]

We need the following variation of the Vaughan’s identity [8].

**Lemma 4.2.** For any real \(x, \alpha\), and for \(2 \leq U, V \leq x\) we have the identity

\[
S(x, \alpha) = S_{I,1} - S_{I,2} - S_{II} + S_0,
\]

where

\[
S_{I,1} = \sum_{d \leq U} \mu(d) \sum_{n \leq \frac{x}{d}} (\log n) e(n\alpha),
\]

\[
S_{I,2} = \sum_{d \leq V} \Lambda(d) \sum_{\delta \leq U} \mu(\delta) \sum_{n \leq \frac{x}{\delta d}} e(n\alpha),
\]

\[
S_{II} = \sum_{d > U} \left( \sum_{\delta \leq U} \mu(\delta) \right) \sum_{n > V} \Lambda(n) e(n\alpha),
\]

\[
S_0 = \sum_{n \leq V} \Lambda(n) e(n\alpha).
\]

We aim to perform our minor arc estimation by applying the following key lemma.

**Lemma 4.3.** Assume that \(1 \leq b_i \leq N\) and \(b_i \in \mathbb{N}\) for each \(i = 1, \ldots, m\). Let \(|\alpha - \frac{a}{q}| \leq \frac{1}{Xq^2}\) for some \(a, q \in \mathbb{N}\) such that \((a, q) = 1\) and \(q \leq N\). Hence,

\[
S_i(N, \alpha) \ll \log^4 N \left( \frac{N}{\sqrt{q}} + N^{\frac{1}{2}} b_i^2 + \sqrt{Nq} \right).
\]
Proof. We follow the reasoning described in \textsuperscript{[7], Chapter 3}. Let us apply Lemma \textsuperscript{4.2} with $x := \frac{N}{b_i}$ and $b_i \alpha$ instead of $\alpha$.

The inner sum in $S_{I,1}$ equals

\[
\sum_{n \leq \frac{N}{b_i}} e(n dB_i \alpha) \int_1^n \frac{dy}{y} = \int_1^{\frac{N}{b_i}} \sum_{n \leq \frac{N}{b_i}} e(n dB_i \alpha) 1_{y < n} \frac{dy}{y} = \int_1^{\frac{N}{b_i}} \sum_{y < n \leq \frac{N}{b_i}} e(n dB_i \alpha) \frac{dy}{y},
\]

which gives

\begin{equation}
S_{I,1} \ll \log N \min_{d \leq U} \left\{ \frac{N}{d b_i}, \frac{1}{\|d b_i \alpha\|_{\mathbb{R}/\mathbb{Z}}} \right\} = \log N \min_{d \leq U b_i} \left\{ \frac{N}{d}, \frac{1}{\|d \alpha\|_{\mathbb{R}/\mathbb{Z}}} \right\}.
\end{equation}

Furthermore, one can see that

\begin{equation}
S_{I,2} = \sum_{d \leq U} \sum_{\frac{N}{d} \leq n \leq \frac{N}{b_i}} \sum_{\delta_1 \leq U, \delta_2 \leq V} \mu(\delta) \Lambda(\delta) e(\delta dB_i \alpha)
\end{equation}

\[
= \sum_{d \leq U} \sum_{\delta_1 \leq U, \delta_2 \leq V} \mu(\delta_1) \Lambda(\delta_2) \ll (U V) \sum_{d \leq U} \min_{\frac{N}{d} \leq n \leq \frac{N}{b_i}} \left\{ \frac{N}{d b_i}, \frac{1}{\|d b_i \alpha\|_{\mathbb{R}/\mathbb{Z}}} \right\} = \log(U V) \sum_{d \leq U} \min_{\frac{N}{d} \leq n \leq \frac{N}{b_i}} \left\{ \frac{N}{d}, \frac{1}{\|d \alpha\|_{\mathbb{R}/\mathbb{Z}}} \right\}.
\]

To estimate the value of $S_{II}$ let us define the set

\[ \mathcal{Y} := \{ U, 2U, 4U, \ldots, 2^k U : 2^k U < \frac{N}{b_i} \leq 2^{k+1} U \} . \]

Thus,

\[ S_{II} = \sum_{Z \in \mathcal{Y}} S(Z), \]

where

\[ S(Z) := \sum_{Z \leq d \leq 2Z} \left( \sum_{\delta \leq U} \mu(\delta) \right) \sum_{V < n \leq \frac{N}{b_i}} \Lambda(n) e(n dB_i \alpha). \]

By the Cauchy–Schwarz inequality

\[ |S(Z)|^2 \leq \left( \sum_{Z \leq d \leq 2Z} \tau(e)^2 \right) \sum_{Z \leq d \leq 2Z} \sum_{V < n \leq \frac{N}{b_i}} \Lambda(n) e(n dB_i \alpha), \]
From Lemma A.2 we get
\[
|S(Z)|^2 \ll Z \log^3 N \sum_{V < n_1, n_2 \leq \frac{N}{Z}} \Lambda(n_1) \Lambda(n_2) \sum_{Z < d \leq 2Z} e((n_1 - n_2)db_1\alpha)
\]
\[
\ll Z \log^5 N \sum_{n_1, n_2 \leq \frac{N}{Z}} \min \left\{ Z, \frac{1}{\|(n_1 - n_2)b_1\alpha\|_{R/Z}} \right\}
\]
(4.4)
\[
= Z \log^5 N \sum_{n \leq \frac{N}{Z}} \sum_{d \leq \frac{N}{Z}} \min \left\{ Z, \frac{1}{\|d\alpha\|_{R/Z}} \right\}
\]
\[
\ll \frac{N}{b_i} \log^5 N \left( Z + \sum_{d \leq \frac{N}{Z}} \min \left\{ \frac{N}{d}, \frac{1}{\|d\alpha\|_{R/Z}} \right\} \right),
\]

Let us combine (4.2)–(4.4) and put \( U, V := \left( \frac{N}{Z} \right)^\frac{b_i}{d} \). Assume that \( |\alpha - \frac{a}{q}| \leq \frac{1}{q^2} \) for some \( a, q \in \mathbb{N} \) such that \( (a, q) = 1, q \leq N \). Applying Lemma 4.1 one gets

\[
(4.5)
\sum_{n \leq \frac{N}{Z}} \min \left\{ \frac{N}{n}, \frac{1}{\|n\alpha\|_{R/Z}} \right\} \ll \left( \frac{N}{q} + \frac{N \sqrt{b_i}}{q} + q \right) \log N,
\]

\[
Z + \sum_{d \leq \frac{N}{Z}} \min \left\{ \frac{N}{d}, \frac{1}{\|d\alpha\|_{R/Z}} \right\} \ll \left( \frac{N}{q} + \frac{N}{Z} + q \right) \log N.
\]

Note that the restrictions \( b_i|d \) under summands from (4.2), (4.3), and (4.4) were simply discarded, so there might be some room for improvement here. Now, (4.2)–(4.5) imply

\[
S_{I,1} \ll \left( \frac{N}{q} + \frac{N \sqrt{b_i}}{q} + q \right) \log^2 N,
\]

\[
S_{I,2} \ll \left( \frac{N}{q} + \frac{N \sqrt{b_i}}{q} + q \right) \log^2 N.
\]

Since \( |Y| \ll \log N \) and
\[
|S(Z)|^2 \ll \frac{N}{b_i} \log^6 N \left( \frac{N}{q} + \frac{N}{Z} + q \right),
\]
on one also gets

\[
(4.7) \quad S_{II} \ll \frac{\log^3 N}{\sqrt{b_i}} \sum_{Z \in Y} \left( \frac{N}{\sqrt{q}} + \frac{N}{\sqrt{Z}} + \sqrt{Nq} + \sqrt{NZ} \right) \ll \frac{\log^4 N}{\sqrt{b_i}} \left( \frac{N}{\sqrt{q}} + \frac{N \sqrt{b_i}}{q} + \sqrt{Nq} \right).
\]

The claim follows from (4.6)–(4.7) merged with the inequality \( S_0 \ll N^{\frac{d}{2}} \).

Recall that \( Q := \log^B N \), and that \( \alpha \in \mathfrak{m} \) means that \( \alpha \) does not satisfy the approximation
\[
|\alpha - \frac{a}{q}| \leq \frac{Q}{N}
\]
for any \( 1 \leq q \leq Q \) and any integer \( a \) coprime to \( q \). We present the following lemma.
Lemma 4.4. Fix real $B, \varepsilon > 0$. Let $b_i \leq N^{1-\varepsilon}$ and $b_i \in \mathbb{N}$ for each $i = 1, \ldots, m$. Hence, for any $\alpha \in \mathbb{m}$ we have
\[ S_i(N, \alpha) \ll \frac{N}{\log \frac{2}{3} - 4} N. \]

Proof. By the Dirichlet’s approximation theorem, for any $\alpha \in \mathbb{R}$ there exists a positive integer $q \leq \frac{N}{\varepsilon}$ such that $|\alpha - \frac{a}{q}| \leq \frac{1}{qN}$ for some $a \in \mathbb{N}$ satisfying the condition $(a, q) = 1$. On the other hand, it has to be $q > Q$, because otherwise $\alpha \in \mathcal{M}$ would follow. Hence, by Lemma 4.3 the claim is true. \hfill \Box

Now, we summarize the section by proving the following result.

Lemma 4.5. Fix a real $B > 0$. Under the assumptions of Theorem 3.2 and the extra assumption $\eta_1, \eta_2, \eta_3 = 1$ we have
\[ \int \prod_{i=1}^{m} S_i(N, \alpha)e(-N\alpha) \, d\alpha \ll \frac{1}{b_1 \cdots b_m} \frac{N^{m-1}}{\log \frac{2}{3} - 3} N. \]

Proof. We have
\begin{align*}
\int \prod_{i=1}^{m} S_i(N, \alpha)e(-N\alpha) \, d\alpha & \leq \prod_{i=1}^{m} \max_{\alpha \in \mathbb{m}} |S_i(N, \alpha)| \int_{\mathbb{R}/\mathbb{Z}} |S_1(N, \alpha)S_2(N, \alpha)| \, d\alpha \\
& \leq \frac{N^{m-3}}{b_1 \cdots b_m} \left( \frac{\max_{\alpha \in \mathbb{m}} |S_1(N, \alpha)|}{N} \right)^{1/2} \left( \int_{\mathbb{R}/\mathbb{Z}} |S_1(N, \alpha)|^2 \, d\alpha \right)^{1/2} \\
& \ll \frac{N^{m-3}}{b_1 \cdots b_m \log \frac{2}{3} - 4} \frac{N}{b_1 b_2} \frac{N}{\log(N/b_1) \log(N/b_2)} \ll \frac{N^{m-3}}{b_1 \cdots b_m \log \frac{2}{3} - 3} N,
\end{align*}
which follows from the Cauchy–Schwarz inequality, the Plancherel identity, the basic estimation $N \ll x \log x$ valid for $x \geq 1$, Lemma 4.3 and the fact that $b_1, b_2, b_3$ are considered fixed. \hfill \Box

We combine the results from the last two sections and establish the following result.

Theorem 4.6. Fix a real $B > 6$ and $c_1, \ldots, c_m \in \mathbb{N}$. Let $b_i := c_i \eta_i$ for $i = 1, \ldots, m$, where each $\eta_i$ is some positive integer with prime divisors greater than $Q$. Let us further assume that $(b_1, \ldots, b_m) = 1$, that $b_1, \ldots, b_m \leq N^\delta$ for some $\delta \in (0, \frac{5}{12m(2m-1)})$, and that $\eta_1, \eta_2, \eta_3 = 1$. Hence,
\[ \sum_{n_1, \ldots, n_m \leq N, b_1 n_1 + \cdots + b_m n_m = N} \Lambda(n_1) \cdots \Lambda(n_m) = \frac{1}{(m-1)! b_1 \cdots b_m} \Theta_{c_1, \ldots, c_m}(N) + O \left( \frac{N^{m-1}}{b_1 \cdots b_m \log^4 N} \right), \]
where $A = (B - 6)/2$.

Proof. Follows from Theorem 3.2 and Lemma 4.5 upon taking $\varepsilon = \frac{1}{7}$. \hfill \Box
5. Reducing the logarithmic weights

Firstly, for each $k > 1$ we would like to discard the contribution of the numbers of the form $p^k$ from the expression appearing in the left-hand side of the equation from the statement of Theorem 4.6 For the sake of convenience, let us introduce the following notation:

\[ \sum_{n_1, \ldots, n_m} b := \sum_{\substack{n_1, \ldots, n_m \text{ being prime} \ \ b_1 n_1 + \cdots + b_m n_m = N}} 1, \]

so for instance

\[ r_m(N; b_1, \ldots, b_m) := \sum_{\substack{n_1 \leq \frac{N}{b_1}, \ldots, n_m \leq \frac{N}{b_m} \text{ are prime}}} 1. \]

Define

\[ \theta(n) = \begin{cases} \log n, & \text{if } n \text{ is prime} \\ 0, & \text{otherwise} \end{cases} \]

Hence, for some $A > 0$ one has

\[ \sum_{\substack{n_1, \ldots, n_m \leq N \ \ b_1 n_1 + \cdots + b_m n_m = N}} 1 \leq \log N \sum_{i=1}^{m} \prod_{k=1}^{m} \log \frac{N}{b_k} \cdot \sum_{k_1 \leq \sqrt{N}, k_2, \ldots, k_{m-1} \leq N} 1 \]

\[ \leq N^{m - \frac{3}{2}} \log^m N \leq \frac{N^{m - \frac{3}{2} + \delta_m} \log^m N}{b_1 \cdots b_m}. \]

We are going to study the asymptotic behaviour of the function

\[ r_m(N; b_1, \ldots, b_m) := \sum_{\substack{n_1 \leq \frac{N}{b_1}, \ldots, n_m \leq \frac{N}{b_m}}} 1. \]

Define also

\[ \tilde{R}_m(N; b_1, \ldots, b_m) := \sum_{n_1 \leq \frac{N}{b_1}, \ldots, n_m \leq \frac{N}{b_m}} \theta(n_1) \cdots \theta(n_m). \]

Let us prove the following result.

**Lemma 5.1.** Under the assumptions of Theorem 4.7 we have

\[ r_m(N; b_1, \ldots, b_m) = \frac{1}{(m-1)! b_1 \cdots b_m} \frac{N^{m-1}}{\prod_{i=1}^{m} \log \frac{N}{b_i}} (G_{c_1, \ldots, c_m}(N) + o(1)). \]

**Proof.** Obviously $\tilde{R}_m = 0$ if and only if $r_m = 0$ and in such a case theorem follows trivially. Note that

\[ \tilde{R}_m(N; b_1, \ldots, b_m) \leq r_m(N; b_1, \ldots, b_m) \prod_{i=1}^{m} \log \frac{N}{b_i}. \]

On the other hand, for every $\epsilon > 0$ one has
\( \widetilde{R}_m(N; b_1, \ldots, b_m) \geq \sum_{b_1}^b \theta(n_1) \cdots \theta(n_m) \geq (1 - \varepsilon)^m \left( \prod_{i=1}^m \log \frac{N}{b_i} \right) \sum_{b_1}^b 1. \)

From Lemma 2.2, the last sum in (5.8) differs from \( r_m \) by at most

\[
(5.9) \quad \ll \sum_{j=1}^m \sum_{n_i \leq \frac{N}{b_j} \atop 1 \leq i \leq m} 1 = \sum_{j=1}^m \sum_{n_j \leq \left( \frac{N}{b_j} \right)^{1-\varepsilon}} J_{b_1, \ldots, b_j-1, b_{j+1}, \ldots, b_m} (N - b_j n_j) \ll N^{m-1} \frac{(1-\delta)\varepsilon}{b_1 \cdots b_m}.
\]

Now, we combine (5.7)–(5.9) and we let \( \varepsilon \) very slowly approach 0. In consequence

\[
(5.10) \quad \widetilde{R}_m(N; b_1, \ldots, b_m) = (1 + o(1)) \left( r_m(N; b_1, \ldots, b_m) \prod_{i=1}^m \log \frac{N}{b_i} \right) + O \left( \frac{N^{m-1-1-(1-\delta)\varepsilon}}{b_1 \cdots b_m} \right).
\]

Combining this with Theorem 4.6 and (5.4), we complete the proof. \( \square \)

6. Cutting off

Recall that \((c_1, \ldots, c_m) = 1\). Let \( N \) be large enough so that \( Q := \log^B N > \max \{c_1, \ldots, c_m\} \).

In this section we finish the proof of Theorem 1.2 by making the key transition from primes to almost primes. Firstly, let us introduce the following simplifications. From this point forward we put

\[
n_j := n_j^{(1)} \cdots n_j^{(r_j)} \quad \text{and} \quad n_j := n_j^{(2)} \cdots n_j^{(r_j)}
\]

for all \( j = 1, \ldots, m \). We have to calculate the following expression:

\[
\sum_{n_j^{(i)} \leq \frac{N}{c_j} \atop 1 \leq j \leq m, 1 \leq i \leq r_j} 1.
\]

Notice that for each choice of \( n_j \) only one of the \( n_j^{(i)} \) can be larger than \( \sqrt{N} \). Therefore, our problem is equivalent (up to some overlapping cases when every of the \( n_j^{(i)} \) is smaller than \( \sqrt{N} \); the appropriate error term is calculated in (6.10)) to calculating yet another expression:

\[
(6.1) \quad \sum_{n_1^{(1)} \leq \frac{N}{c_1}, \ldots, n_m^{(1)} \leq \frac{N}{c_m} \atop n_j^{(i)} \leq \sqrt{N} ; 1 \leq j \leq m, 1 \leq i \leq r_j \atop \text{all of the } n_j^{(i)} \text{ being prime}} 1_{c_1 n_1 + \cdots + c_m n_m = N}.
\]
We also adopt two new notions:

\[
\sum^\sharp := \sum \sum_{\substack{1 \leq j \leq m, 1 \leq i \leq r_j \leq m \text{ all of the } n_j^{(i)} \text{ being prime} \\ c_1 n_1 + \cdots + c_m n_m = N}} n_j^{(i)},
\]

(6.2)

\[
\sum':= \sum \text{some conditions} \text{ only prime indices}
\]

The general strategy is to show that even after attaching some stronger conditions to the sum (6.1), its asymptotic behaviour remains unchanged. Let

\[ r := \max\{r_1, \ldots, r_m\}. \]

For \(1 \leq i \leq m \) and \(2 \leq j \leq r_i \) the range of indices \( n_j^{(i)} \) from (6.1) shall be cut from \([1, \sqrt{N}] \cap \mathbb{N}\) to \([Q, N^\gamma] \cap \mathbb{N}\) for some real \(0 < \gamma < 1/2\). These further restrictions do not hurt the asymptotic behaviour of expression (6.1), and simultaneously make it calculable via Lemma 5.1.

Let us focus on the details. Put \( \gamma := \frac{1}{12m(2m-1)} \). The expression (6.1) can be decomposed as

\[
\sum^\dagger + O\left(\sum_{\ell=1}^{m} \sum_{k=2}^{r_{\ell}} \left( \sum_{1}^{(\ell,k)} + \sum_{2}^{(\ell,k)} + \sum_{3} \right) \right),
\]

where

\[
\sum^\dagger := \sum \sum_{\substack{1 \leq j \leq m \leq \frac{N}{\eta_j} \text{ \ Q< } n_j^{(i)} \text{ \ N: } 1 \leq j \leq m, 2 \leq i \leq r_j \\ (\eta_1, \ldots, \eta_m)=1}} n_j^{(i)}.
\]

(6.3)

\[
\sum_{1}^{(\ell,k)} := \sum \sum_{\substack{1 \leq j \leq m \leq \frac{N}{\eta_j} \\ n_j^{(i)} \leq \sqrt{N} \text{ \ N: } 1 \leq j \leq m, 2 \leq i \leq r_j \\ n_j^{(i)} > N^\gamma \text{ \ n_j^{(i)} \ N}}} n_j^{(i)}.
\]

(6.4)

\[
\sum_{2}^{(\ell,k)} := \sum \sum_{\substack{1 \leq j \leq m \leq \frac{N}{\eta_j} \\ n_j^{(i)} \leq \frac{N}{\eta_j} \text{ \ N: } 1 \leq j \leq m, 2 \leq i \leq r_j \\ n_j^{(i)} \leq Q \text{ \ n_j^{(i)} \ Q}}} n_j^{(i)}.
\]

\[
\sum_{3} := \sum \sum_{\substack{1 \leq j \leq m \leq \frac{N}{\eta_j} \text{ \ Q< } n_j^{(i)} \text{ \ N: } 1 \leq j \leq m, 2 \leq i \leq r_j \\ (\eta_1, \ldots, \eta_m)>1}} 1.
\]

The assumption \( \eta_1, \eta_2, \eta_3 = 1 \) instantly gives

\[ \sum_{3} = 0. \]

6.1. Estimating \( \sum^\dagger \). We can rewrite the studied sum in the following form:

\[
\sum^\dagger = \sum_{Q< n_j^{(i)} \leq N^\gamma; \quad 1 \leq j \leq m, 2 \leq i \leq r_j} \left( \sum_{1 \leq j \leq m \leq \frac{N}{\eta_j} \text{ \ N: } 1 \leq j \leq m, 2 \leq i \leq r_j} \frac{1}{c_2 \eta_1 n_1^{(i)} + \cdots + c_m \eta_m n_m^{(i)} = N} \right).
\]
We have $\eta_j \leq N^{(r-1)c}$ for all $j = 1, \ldots, m$. For $N$ sufficiently large this gives $c_j \eta_j \leq N^{rc}$. The sum in the parenthesis equals $\tau_m(N; c_1 \eta_1, \ldots, c_m \eta_m)$. Hence, the assumption $\eta_1, \eta_2, \eta_3 = 1$ and Lemma 3.1 imply that we can transform the expression above into:

\[
\sum'_{Q < n_j^{(1)} \leq N^{\gamma}} \frac{1}{\eta_1 \ldots \eta_m} \frac{1}{\log(c_1 \eta_1) \ldots \log(c_m \eta_m)}^1_{(\eta_1, \ldots, \eta_m) = 1}.
\]

Given that $1_{(\eta_1, \ldots, \eta_m) = 1}$, we calculate

\[
\prod_{j=1}^{m} \sum'_{Q < n_j^{(1)} \leq N^{\gamma}} \frac{1}{\eta_j \log(N/c_j)} =
\]

\[
\prod_{j=1}^{m} \sum'_{Q < n_j^{(1)} \leq N^{\gamma}} \left( \sum'_{Q < n_j^{(2)} \leq N^{\gamma}} \left( \sum'_{Q < n_j^{(3)} \leq N^{\gamma}} \cdots \right) \right) \frac{1}{n_j^{(r_j)}}.
\]

From $\eta_j^{(r_j)} < N^{1-c}$ for sufficiently large $N$ and from Lemma A.3 applied $r_j - 1$ times we conclude that the expression (6.6) equals

\[
(1 + o(1)) \prod_{j=1}^{m} \frac{(\log N)^{r_j-1}}{\log(N/c_j)} = (1 + o(1)) \frac{(\log N)^{r_1 + \cdots + r_m - m}}{\log^m N}.
\]

Combining (6.5)–(6.7) we arrive at

\[
\sum' = \frac{1}{(m-1)! c_1 \ldots c_m} \frac{N^{m-1}}{\log^m N} (\tau_m(N; c_1 \eta_1, \ldots, c_m \eta_m) + o(1)).
\]

6.2. Upper bounds on $\sum_1^{(\ell, k)}$ and $\sum_2^{(\ell, k)}$. We deal with these two sums in exactly the same way, so only the calculations for the first one are presented in detail. We can assume that $\ell \geq 4$ without loss of generality.

The first sum can be rewritten in the following manner

\[
\sum' = \frac{1}{(m-1)! c_1 \ldots c_m} \frac{N^{m-1}}{\log^m N} (\tau_m(N; c_1 \eta_1, \ldots, c_m \eta_m) + o(1)) \times
\]

\[
\sum'_{n_1^{(1)} \leq N^{\gamma}, \ldots, n_m^{(1)} \leq N^{\gamma}} \sum'_{n_1^{(1)} \leq N^{\gamma}, \ldots, n_m^{(1)} \leq N^{\gamma}} \sum'_{n_1^{(1)} \leq N^{\gamma}, \ldots, n_m^{(1)} \leq N^{\gamma}} 1_{\eta_1 \eta_2 \eta_3 \eta_4 \cdots \eta_m \eta_4 = N - c_4 \eta_4 n_4^{(1)} - \cdots - c_m \eta_m n_m^{(1)}},
\]

From $\eta_1 = \eta_2 = \eta_3 = 1$ and Lemma A.1 the inner sum from (6.9) equals

\[
\tau_m(N - c_4 \eta_4 n_4^{(1)} - \cdots - c_m \eta_m n_m^{(1)}; c_1, c_2, c_3) \ll \frac{N^2}{\log^2 N}.
\]
Thus, we obtain

\[
\sum_1^{(\ell,k)} 1 \ll \frac{N^2}{\log^2 N} \sum_{\substack{n_j^{(1)}, \ldots, n_m^{(1)} \leq N \\
_j^{(1)} \leq \sqrt{N} : 4 \leq j \leq m, 2 \leq i \leq r_j \\
_j^{(1)} \leq \sqrt{N} : 4 \leq j \leq m, 2 \leq i \leq r_j}} 1
\]

\[
\sum_{n_j \leq N : 4 \leq j \leq m, j \neq \ell} \left( \prod_{\ell \neq n_j} \mathbf{1}_{\Omega(n_j) = r_j} \right) \sum_{n_j \leq N : 4 \leq j \leq m, j \neq \ell} 1.
\]

From Theorem A.4 we have

\[
\sum_{n_j \leq N : 4 \leq j \leq m, j \neq \ell} \mathbf{1}_{\Omega(n_j) = r_j} \ll \frac{N (\log \log N)^{r_{\ell} - 2}}{\log N} \sum_{n_j \leq \sqrt{N}} 1 \ll \frac{N (\log \log N)^{r_{\ell} - 2}}{\log N},
\]

and for \(j \neq \ell\) we have

\[
\sum_{n_j \leq N} \mathbf{1}_{\Omega(n_j) = r_j} \ll \frac{N (\log \log N)^{r_j - 1}}{\log N},
\]

which gives

\[
\sum_1^{(\ell,k)} 1 \ll \frac{N^{m-1} (\log \log N)^{r_1 + \cdots + r_m - m - 1}}{\log^m N}.
\]

Analogously, we can obtain the following upper bound for the second sum

\[
\sum_2^{(\ell,k)} \ll \frac{N^{m-1} (\log \log N)^{r_1 + \cdots + r_m - m - 1} (\log \log \log N)}{\log^m N}.
\]

The \(\log \log \log N\) term appears because this time \(n_j^{(k)}\) is an index supported on \([1, Q] \cap \mathbb{Z}\) instead of \((N^{\gamma}, N^{1/2}] \cap \mathbb{Z}\) like in the first case; we easily get

\[
\sum_{n_j^{(k)} \leq Q} \frac{1}{n_j^{(k)}} \ll \log Q \ll \log \log \log N.
\]

6.3. Proof of Theorem 1.2  Now, we can summarize the discussion from this section. Under the assumptions of Theorem 1.2, we have

\[
\sum_{\substack{n_j^{(1)} \leq N \\
_j^{(1)} \leq \sqrt{N} : 1 \leq j \leq m, 2 \leq i \leq r_j}} 1 = \frac{1}{(m - 1)!} \frac{N^{m-1} (\log \log N)^{r_1 + \cdots + r_m - m}}{\log^m N} (\mathcal{E}_{c_1, \ldots, c_m}(N) + o(1)).
\]
We wish to discard the $n_j^{(i)} \leq \sqrt{N}$ restriction from the sum above. Notice that every $\eta_j$ has at most one term greater than $\sqrt{N}$. Hence,

\begin{equation}
\sum_{n_j^{(i)} \leq \sqrt{N} : 1 \leq j \leq m, 1 \leq i \leq r_j} 1 = r_1 \cdots r_m \sum_{n_j^{(i)} \leq \sqrt{N}, 1 \leq j \leq m, 2 \leq i \leq r_j} 1 + O \left( \sum_{K=1}^{m} \sum_{j=1}^{r_\ell} \left( \sum_{1}^{(\ell,k)} + \sum_{2}^{(\ell,k)} \right) \right).
\end{equation}

We assumed that $\eta_1 = \eta_2 = \eta_3 = 1$. The main term in the expression above equals

\begin{equation}
r_1 \cdots r_m \frac{N^{m-1} (\log \log N)^{r_1 \cdots r_m - m}}{(m-1)! c_1 \cdots c_m} \left( \mathcal{E}_{c_1, \ldots, c_m}(N) + o(1) \right).
\end{equation}

Let us study the error term from (6.14). We have

\begin{equation}
\sum_{n_1^{(i)} \leq \sqrt{N}, \ldots, n_K^{(i)} \leq \sqrt{N}, n_m^{(i)} \leq \sqrt{N}} 1
= \sum_{Q \leq n_j^{(i)} \leq N : 1 \leq j \leq m, 2 \leq i \leq r_j} 1 + O \left( \sum_{m} \sum_{\ell=1}^{r_\ell} \left( \sum_{1}^{(\ell,k)} + \sum_{2}^{(\ell,k)} \right) \right).
\end{equation}

The 'big O' term has order at most as large as the error term from (6.15). The new main term (which is essentially only the error term in (6.14)) can be bounded as $\leq N^{m-2 + \gamma r}$.

Combining the results from this subsection we can state that

\begin{equation}
\sum_{n_j^{(i)} \leq \sqrt{N} : 1 \leq j \leq m, 1 \leq i \leq r_j} 1
= \frac{r_1 \cdots r_m}{(m-1)! c_1 \cdots c_m} \frac{N^{m-1} (\log \log N)^{r_1 \cdots r_m - m}}{\log^m N} \left( \mathcal{E}_{c_1, \ldots, c_m}(N) + o(1) \right).
\end{equation}

By Lemma A.1 and Theorem A.4 we can repeat the trick from the previous subsection to obtain the following upper bound for some $1 \leq \ell \leq m$ and $1 \leq k_1, k_2 \leq r_\ell$ for which $k_1 \neq k_2$:

\begin{equation}
\mathcal{E}_{\ell,k_1,k_2}(N) \leq \frac{N^{m-1} (\log \log N)^{r_1 \cdots r_m - m}}{\log^m N}.
\end{equation}
Therefore, we have the following identity

\[
(6.19) \sum_{1 \leq j \leq m, 1 \leq i \leq r_j} \delta_{n_j(\cdot)}^{(i)} \sum_{c_1 n_1 + \cdots + c_m n_m = N} \prod_{i=1}^{m} \delta_{\Omega(n_i)} = r_i + O\left(\left(\frac{N^{m-1}(\log \log N)^{r_1 + \cdots + r_m - 1}}{\log^m N}\right)\right).
\]

This finally closes the proof of Theorem 1.2.

**Appendix A.**

**Lemma A.1.** Let \(b_1, \ldots, b_m \leq N\) be positive integers such that \(b_1, b_2, b_3\) are fixed. Then,

\[
t_m(N; b_1, \ldots, b_m) \ll \frac{N^{m-1}}{b_1 \cdots b_m \prod_{j=1}^m \log(N/b_j)}.
\]

**Proof.** From Theorem 4.6 we have

\[
t_3(N; b_1, b_2, b_3) \ll \frac{N^2}{\log^3 N}
\]

for sufficiently large integer \(N\).

We can present \(t_m(N; b_1, \ldots, b_m)\) as

\[
\sum_{n_4 \leq \frac{N}{b_1}, \ldots, n_m \leq \frac{N}{b_m}} \left( \sum_{n_1 \leq \frac{N}{b_1}, \ldots, n_3 \leq \frac{N}{b_3}} 1_{b_1 n_1 + b_2 n_2 + b_3 n_3 = N - b_4 n_4 - \cdots - b_m n_m} \right).
\]

The term inside the bracket equals \(t_3(N - b_4 n_4 - \cdots - b_m n_m; b_1, b_2, b_3)\), so one gets

\[
t_m(N; b_1, \ldots, b_m) \ll \frac{N^2}{b_1 b_2 b_3 \log^3 N} \sum_{n_4 \leq \frac{N}{b_1 b_2 b_3}, \ldots, n_m \leq \frac{N}{b_1 b_2 \cdots b_m}} 1
\]

\[
\ll \frac{N^2}{b_1 b_2 b_3 \log^3 N} \prod_{j=4}^{m} \frac{N}{b_j \log(N/b_j)} \ll \frac{N^{m-1}}{b_1 \cdots b_m \prod_{j=1}^m \log(N/b_j)}.
\]

\[\square\]

**Lemma A.2.** For \(x \geq 2\) we have

\[
\sum_{n \leq x} \tau(n)^2 \ll x \log^3 x.
\]

**Proof.** See \([10]\). \[\square\]

**Lemma A.3.** Let \(x, y \in \mathbb{R}\). Then, for any \(\delta \in (0, 1)\) we have

\[
\sum_{p \leq x} \frac{1}{p \log \frac{x}{p}} = (1 + o_\delta(1)) \frac{\log \log x}{\log x}.
\]

**Proof.** Apply summation by parts and the prime number theorem. \[\square\]

**Theorem A.4** (Landau). For \(k \geq 1\) we have

\[
\sum_{n \leq x} 1_{\Omega(n) = k} = (1 + o(1)) \frac{x (\log \log x)^{k-1}}{(k-1)! \log x}.
\]

**Proof.** See \([11]\). \[\square\]
Lemma A.5. Fix positive integers $c_1, \ldots, c_m$ satisfying $(c_1, \ldots, c_m) = 1$. It is true that $\mathcal{S}_{c_1, \ldots, c_m}(N) \neq 0$ if and only if $c_1 + \cdots + c_m + N \equiv 0 \mod 2$ and

$$(N, c_2, \ldots, c_m) = N = (c_1, \ldots, c_{i-1}, N, c_{i+1}, \ldots, c_m) = N = (c_1, \ldots, c_{m-1}, N) = 1.$$  

Moreover, $\mathcal{S}_{c_1, \ldots, c_m}(N) \gg 1$ for every $N$ such that $\mathcal{S}_{c_1, \ldots, c_m}(N) \neq 0$.

Proof. Let us prove the 'moreover' part first. We may assume that no factor of $\mathcal{S}_{c_1, \ldots, c_m}(N)$ equals 0. Hence, if $p | N$, then at least 2 of the $c_i$ are not divisible by $p$, because otherwise either the term corresponding to $p$ would vanish or $p | (c_1, \ldots, c_m)$. Therefore,

$$1 + (p-1) \frac{\mu(p)}{\varphi(c_1p)} \cdots \frac{\mu(p)}{\varphi(c_mp)} \geq 1 - \frac{1}{p-1},$$

$$1 - \frac{\mu(p)}{\varphi(c_1p)} \cdots \frac{\mu(p)}{\varphi(c_mp)} \geq 1 - \frac{1}{p-1}.$$  

The second inequality works because we are always guaranteed to have at least one of the $c_i$ that is not divisible by $p$. It is also true that there exist at most finitely many primes dividing $\prod_{i=1}^m c_i$. Let $z$ be any fixed real number greater than all of them. Hence,

(A.1)  

$$\mathcal{S}_{c_1, \ldots, c_m}(N) \geq 2 \prod_{2 < p \leq z} \left(1 - \frac{1}{p-1}\right) \prod_{p > z} \left(1 - \frac{1}{(p-1)m-1}\right) \gg 1.$$  

Let us consider the ' implies' case. From the previous part we already know that $\mathcal{S}_{c_1, \ldots, c_m}(N)$ equals 0 if and only if one of its factors vanishes. For any prime $p \geq 3$ the factor

$$1 + (p-1) \frac{\mu(p)}{\varphi(c_1p)} \cdots \frac{\mu(p)}{\varphi(c_mp)} \geq 1 - (-1)^{\# \{1 \leq i \leq m : 2 | c_i\}}$$

related to the $p | N$ case, vanishes if and only if every of the $c_i$ is divisible by $p$ except exactly one of them. The second factor actually never vanishes except possibly when $p = 2$. In the $p = 2$ case we note that factors related to $N$ being even and $N$ being odd simplify to

$$1 + (-1)^{\# \{1 \leq i \leq m : 2 | c_i\}} \quad \text{and} \quad 1 - (-1)^{\# \{1 \leq i \leq m : 2 | c_i\}},$$

respectively. It means that whenever $N$ has different parity than the sum $c_1 + \cdots + c_m$, one of these has to vanish.

The ' implies' case is now straightforward, because if $\mathcal{S}_{c_1, \ldots, c_m}(N) = 0$, then one of its factors vanishes. Consequently, either parities of $c_1 + \cdots + c_m$ and $N$ do not match or there exists some prime $p \geq 3$ dividing $N$ and each of the $c_i$ except exactly one.  

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