New methods for verifying strong periodic detectability and strong periodic D-detectability of discrete-event systems*

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Abstract In this paper, in discrete-event systems modeled by finite-state automata (FSAs), we show new thinking on the tools of detector and concurrent composition and derive two new algorithms for verifying strong periodic detectability (SPD) without any assumption that run in NL; we also reconsider the tool of observer and derive a new algorithm for verifying strong periodic D-detectability (SPDD) without any assumption that runs in PSPACE. These results strengthen the NL upper bound on verifying SPD and the PSPACE upper bound on verifying SPDD for deadlock-free and divergence-free FSAs in the literature.

Keywords discrete-event system, finite-state automaton, strong periodic (D-)detectability, complexity, observer, detector, concurrent composition

1 Introduction

2 Introduction

2.1 Background

Detectability is a basic property of partially-observed dynamical systems: when it holds one can use an observed output/label sequence produced by a system to reconstruct its states [1, 2, 3, 4]. This property plays a fundamental role in many related control problems such as observer design and controller synthesis. Detectability is quite related to another fundamental property diagnosability which implies occurrences of all faulty events could be detected after sufficiently many occurrences of subsequent events [5]. Recently, strong detectability and diagnosability have been unified into one mathematical framework [6] in discrete-event systems (DESs) modeled by finite-state automata (FSAs). On the other hand, detectability is strongly related to many cyber-security properties. For example, the property of opacity, which has been originally proposed to describe information flow security in computer science in the early 2000s [7] can be seen as the absence of detectability.

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For discrete-event systems (DESs) modeled by finite-state automata (FSAs), the verification problems for different definitions of detectability have been widely studied [1, 2, 3, 8, 9, 4, 10, 6], in which several complexity lower bounds and upper bounds for these problems were obtained, but most of the upper bounds depend on two fundamental assumptions that a system is deadlock-free and divergence-free. These requirements are collected in Assumption 1: when it holds, a system will always run and generate an infinitely long label/output sequence. The first verification algorithm for detectability of DESs that does not depend on Assumption 1 was given by us in [3] by developing a technique called concurrent composition, which was used to verify negation of strong detectability. In the current paper, we further develop new methods to obtain verification algorithms for strong periodic detectability and strong periodic D-detectability that do not depend on any assumption, so that complexity upper bounds will be obtained for all FSAs.

We recall basic complexity results used in the paper (see [11, 12]). The symbols NL, P, PSPACE, NPSPACE, and EXPTIME denote the sets of problems solvable in nondeterministic logarithmic space, polynomial time, polynomial space, nondeterministic polynomial space, and exponential time, respectively. coNL and coNPSPACE denote the sets of problems whose complements belong to NL and PSPACE, respectively. It is known that NL ⊂ P ⊂ PSPACE ⊂ EXPTIME, NL = coNL, and PSPACE = NPSPACE = coNPSPACE. It is also known that NL ⊆ PSPACE and P ⊆ EXPTIME, but whether the rest of these containments are strict are long-standing open questions. It is widely conjectured all the other containments are strict. A problem A is NL-hard (resp., PSPACE-hard) if every problem in NL (resp., PSPACE) is log space (resp., polynomial time) reducible to A, for which A has an NL (resp., PSPACE) lower bound. A problem A is NL-complete (resp., PSPACE-complete) if A belongs to NL (resp., PSPACE) and is NL-hard (resp., PSPACE-hard). If a problem A belongs to NL (resp., PSPACE), then A has an NL (resp., PSPACE) upper bound. In this paper, we sometimes say a detectability property has an NL (PSPACE) upper bound for short, which means that the problem of verifying the property in FSAs belongs to NL (PSPACE).

2.2 Literature review on verification of detectability in FSAs

Results based on Assumption 1 In [1], by using an observer\(^1\) method, exponential-time algorithms were given to verify four notions of detectability: strong (periodic) detectability and weak (periodic) detectability. Strong detectability means that there is a time delay \(k\), for each infinite-length event sequence \(s\) generated by an FSA, every prefix of the label/output sequence of \(s\) of length greater than \(k\) allows reconstructing the current state. Weak detectability relaxes strong detectability by replacing each to some. Weak detectability is strictly weaker than strong detectability. Strong periodic detectability implies that at any time, after some observation time delay no greater than a given value, the system states can be determined along each infinite-length transition sequence also by observing the corresponding output sequence. Weak periodic detectability relaxes strong periodic detectability also by changing each to some. Later in [2], by using a detector (obtained from an observer by splitting all its states into subsets of cardinality 2) method, polynomial-time algorithms were designed for verifying strong (periodic) detectability. The problem of verifying weak (periodic) detectability of FSAs was proven to be PSPACE-complete [8] and the problem of verifying strong (periodic)

\(^1\)i.e., the powerset construction used for determinizing nondeterministic finite automata with \(\epsilon\)-transitions [11]
detectability was proven to be NL-complete [9].

In order to make detectability adapt to more scenarios, one can weaken detectability to D-detectability in the sense of not exactly determining the states but making sure that the states cannot contain both states of any pair of states that are previously specified [2]. All above notions of detectability, including strong/weak detectability and strong/weak periodic detectability, can be extended to their D-versions. For example, strong D-detectability can be verified in polynomial time [2], while verifying strong periodic D-detectability is PSPACE-complete [10].

Note that all the above complexity upper bounds were obtained by the verification algorithms designed in [1, 2] based on Assumption 1. For an FSA that does not satisfy Assumption 1, the algorithms may not return a correct answer. In [4, Remark 2], we had given a counterexample to show that neither the observer method [1] nor the detector method [2] correctly verifies its strong detectability. Later in Remark 1 and Remark 2, we will give counterexamples to show that neither of the two methods correctly verifies their strong periodic detectability and strong periodic D-detectability.

Results which do not depend on assumptions The two fundamental assumptions shown in Assumption 1 was for the first time removed by us in [3, 4] by developing a concurrent-composition method and verifying negation of strong detectability. In [4], weak detectability was also verified without any assumption. Later in [6], an NL upper bound was given for the verification problem of strong detectability based on the concurrent-composition method. In addition, decentralized settings of strong detectability, diagnosability, and predictability were unified into one mathematical framework [6]. In [13], strong D-detectability was verified in polynomial time also by the concurrent-composition method.

2.3 Contribution of the paper

The contributions of the paper are as follows:

1. We use the detector and concurrent composition to derive two new algorithms for verifying strong periodic detectability of FSAs without any assumption, where both algorithms imply an NL upper bound for strong periodic detectability, which strengthens the NL upper bound given in [9] under Assumption 1.

2. We use the observer to derive a new algorithm for verifying strong periodic D-detectability of FSAs without any assumption, where the algorithm implies a PSPACE upper bound for strong periodic D-detectability, which strengthens the PSPACE upper bound given in [10] under Assumption 1. See Tab. 1 for a collection of related results.

Differently from verifying strong periodic detectability itself in [2, 9], we verify its negation. Following such an opposite way, for an FSA, we obtain two conditions on its observer such that at least one of them holds exactly violates its strong periodic detectability. Thus an exponential-time algorithm for verifying strong periodic detectability is obtained without any assumption (Theorem 4.4). Furthermore, by developing a new relationship between the notions of observer and detector (Proposition 4.5), the exponential-time algorithm is reformulated by a detector, resulting in a polynomial-time verification algorithm (Theorem 4.6). Thus, an NL upper bound naturally follows from the
Table 1: Complexity results for verifying different definitions of detectability in FSAs, where * means that the NL and PSPACE upper bounds only apply to FSAs satisfying Assumption 1.

The remainder is structured as follows. In Section 3, basic notation and definitions in FSAs are introduced. In Section 4, the main results are shown. Section 5 ends up this paper with short conclusion.

3 Preliminaries

We introduce necessary notion that will be used throughout this paper. For a finite alphabet $\Sigma$, $\Sigma^*$ and $\Sigma^\omega$ are used to denote the set of finite sequences (called words) of elements of $\Sigma$ including the empty word $\epsilon$ and the set of infinite sequences (called configurations) of elements of $\Sigma$, respectively. $\Sigma^+ := \Sigma^* \setminus \{\epsilon\}$. For a word $s \in \Sigma^*$, $|s|$ stands for its length. For $s \in \Sigma^+$ and natural number $k$, $s^k$ and $s^\omega$ denote the concatenations of $k$-copies and infinitely many copies of $s$, respectively. For a word (configuration) $s \in \Sigma^*(\Sigma^\omega)$, a word $s' \in \Sigma^*$ is called a prefix of $s$, denoted as $s' \sqsubseteq s$, if there exists another word (configuration) $s'' \in \Sigma^*(\Sigma^\omega)$ such that $s = s's''$. For two natural numbers $i \leq j$, $[i, j]$ denotes the set of all integers no less than $i$ and no greater than $j$; and for a set $S$, $|S|$ its cardinality and $2^S$ its power set. As usual, a singleton is defined by a set of cardinality 1. $\subset$ denotes the subset relation.

A DES modeled by an FSA is a sextuple

$$\mathcal{S} = (X, T, X_0, \delta, \Sigma, \ell),$$

where $X$ is a finite set of states, $T$ a finite set of events, $X_0 \subset X$ a set of initial states, $\delta \subset X \times T \times X$ a transition relation, $\Sigma$ a finite set of outputs (labels), and $\ell : T \rightarrow \Sigma \cup \{\epsilon\}$ a labeling function. $\ell$ can be recursively extended to $\ell : T^\ast \cup T^\omega \rightarrow \Sigma^* \cup \Sigma^\omega$ as $\ell(t_1 t_2 \ldots) = \ell(t_1)\ell(t_2)\ldots$ and particularly $\ell(\epsilon) = \epsilon$.

The event set $T$ can be rewritten as disjoint union of observable event set $T_o = \{t \in T|\ell(t) \in \Sigma\}$ and unobservable event set $T_uo = \{t \in T|\ell(t) = \epsilon\}$. Transition relation $\delta$ is recursively extended to $\delta \subset X \times T^\ast \times X$ in the usual way. We call a transition with an observable (unobservable) event an
observable (unobservable) transition. We also denote a transition sequence \((x, s, x') \in \delta\) by \(x \xrightarrow{\delta} x'\), where \(x, x' \in X, s \in T^*\). For \(x \in X\) and \(s \in T^+\), \((x, s, x)\) is called a transition cycle if \((x, s, x) \in \delta\). An observable (resp., unobservable) transition cycle is defined by a transition cycle with at least one (resp., with no) observable transition. Automaton \(S\) is called deterministic if \(|X_0| = 1\) and for all \(x, x', x'' \in X\) and \(t \in T\), \((x, t, x')\), \((x, t, x'')\) \(\in \delta\) imply \(x' = x''\). For deterministic \(S\), for all \(x \in X\) and all \(s \in T^*\), we also denote the unique state \(x' \in X\) (if any) satisfying \(x \xrightarrow{\delta} x'\) by \(\delta(x, s)\). For two states \(x, x' \in X\), we say \(x'\) is reachable from \(x\) if there is \(s \in T^+\) such that \((x, s, x') \in \delta\); we say \(x'\) is reachable if either \(x' \in X_0\) or \(x'\) is reachable from some initial state. Analogously, reachability from a set of states to a state and vice versa could also be defined. Particularly, we call a transition cycle reachable if it is reachable from some initial state.

For each \(\sigma \in \Sigma^*\), we denote by \(M(S, \sigma)\) the current-state estimate, i.e., the set of states that the system can be in after \(\sigma\) has been observed, i.e., \(M(S, \sigma) := \{x \in X | (\exists x_0 \in X_0)(\exists s \in T^*)(|s| = \sigma)\land (x_0 \xrightarrow{\delta} x)\}\). We use \(L(S) = \{s \in T^* | (\exists x_0 \in X_0)(\exists x \in X)[x_0 \xrightarrow{\delta} x]\}\) to denote the set of finite-length event sequences generated by \(S\), we also use \(L^*(S) = \{t_1t_2\ldots \in T^* | (\exists x_0 \in X_0)(\exists x_1, x_2, \ldots \in X)[x_0 \xrightarrow{\delta} x_1 \xrightarrow{\delta} x_2 \ldots]\}\) to denote the set of infinite-length event sequences generated by \(S\). Analogously, we use \(L(S)\) denotes the language generated by \(S\), i.e., \(L(S) := \{\sigma \in \Sigma^* | M(S, \sigma) \neq \emptyset\}\), we also use \(L^\omega(S)\) to denote the \(\omega\)-language generated by \(S\), i.e., \(L^\omega(S) := \{\sigma \in \Sigma^* | (\exists s \in L^\omega(S))(|s| = \sigma)\}\).

For a state \(x \in X\), its unobservable reach is defined by \(UR(x) := \{x' \in X | (\exists s \in (T_{uo})^*)(|x, s, x'\rangle \in \delta)\}\). For a subset \(X' \subset X\), \(UR(X') = \bigcup_{x \in X'} UR(x)\). Hence \(UR(X_0) = M(S, \epsilon)\). For a state \(x \in X\), its observable reach under \(\sigma \in \Sigma\) is defined by \(Reach_{\sigma}(x) := \{x' \in X | (\exists t \in T)[((x, t, x') \in \delta) \land (\sigma = |t|)]\}\). Analogously, for a subset \(X' \subset X\), \(Reach_{\sigma}(X') = \bigcup_{x \in X'} Reach_{\sigma}(x)\).

The following two assumptions are commonly used in detectability studies (cf. [1, 2, 9, 10]), but are not needed in the current paper based on our new thinking of the tools of observer, detector, and concurrent composition.

**Assumption 1** An FSA \(S\) as in (1) satisfies

(A) \(S\) is deadlock-free, i.e., for each reachable state \(x \in X\), there exist \(t \in T\) and \(x' \in X\) such that \((x, t, x') \in \delta\);

(B) \(S\) is prompt or divergence-free, i.e., for every reachable state \(x \in X\) and every nonempty unobservable event sequence \(s \in (T_{uo})^+\), there exists no transition sequence \(x \xrightarrow{\delta} x\) in \(S\).

One sees (A) implies \(L^\omega(S) \neq \emptyset\) if \(X_0 \neq \emptyset\); while (B) implies for all \(s \in L^\omega(S)\), \(|s| \in \Sigma^\omega\); hence (A) and (B) together imply \(L^\omega(S) = \emptyset\) if \(X_0 \neq \emptyset\), but not vice versa.

### 4 Main results

#### 4.1 Preliminary results

The definitions of strong detectability, strong periodic detectability, and strong periodic D-detectability for FSAs are as follows [2].
Definition 1 (SD) An FSA $S$ as in (1) is called strongly detectable if there exists a positive integer $k$ such that for each infinite-length event sequence $s \in L^\omega(S)$ generated by $S$, for each prefix $s' \sqsupseteq s$, if $|\ell(s')| > k$ then $|\mathcal{M}(S, \ell(s'))| = 1$.

Definition 2 (SPD) An FSA $S$ is called strongly periodically detectable if there exists a positive integer $k$ such that for each $s \in L^\omega(S)$ and each $s' \sqsupseteq s$, there is $s'' \in T^*$ such that $|\ell(s'')| < k$, $s's'' \sqsupseteq s$, and $|\mathcal{M}(S, \ell(s's''))| = 1$.

In order to formulate strong periodic D-detectability, we specify a set

$$T_{\text{spec}} \subset X \times X$$

of crucial state pairs that should be separated.

Definition 3 ($T_{\text{spec}}$-SPDD) An FSA $S$ is called strongly periodically D-detectable with respect to $T_{\text{spec}}$ if there exists a positive integer $k$ such that for each $s \in L^\omega(S)$ and each $s' \sqsupseteq s$, there is $s'' \in T^*$ such that $|\ell(s'')| < k$, $s's'' \sqsupseteq s$, and $(\mathcal{M}(S, \ell(s's'')) \times \mathcal{M}(S, \ell(s's''))) \cap T_{\text{spec}} = \emptyset$.

In order to verify detectability of an FSA $S$, an observer

$$S_{\text{obs}} := (2^X \setminus \{\emptyset\}, \Sigma, \mathcal{M}(S, \epsilon), \delta_{\text{obs}})$$

as a deterministic FSA was constructed in [1], where $\mathcal{M}(S, \epsilon)$ is the unique initial state; for all $X' \in 2^X$ and $\sigma \in \Sigma^*$, $\delta_{\text{obs}}(\mathcal{M}(S, \epsilon), \sigma) = X'$ if and only if $X' = \mathcal{M}(S, \sigma)$. The size of $S_{\text{obs}}$ is exponential of that of $S$.

Later in [2], a detector

$$S_{\text{det}} := (Q, \Sigma, \mathcal{M}(S, \epsilon), \delta_{\text{det}})$$

that is a nondeterministic FSA was used to provide polynomial-time algorithms for verifying strong detectability and strong periodic detectability under Assumption 1, where $Q \subset 2^X \setminus \{\emptyset\}$ consists of $\mathcal{M}(S, \epsilon)$ and subsets of $X$ with cardinality $\leq 2$; for all $q, q' \in Q$, and $\sigma \in \Sigma$, $(q, \sigma, q') \in \delta_{\text{det}}$ if and only if either (1) $|(\text{UR} \circ \text{Reach}_\sigma)(q)| > 1$, $q' \in (\text{UR} \circ \text{Reach}_\sigma)(q)$, and $|q'| = 2$, or (2) $|(\text{UR} \circ \text{Reach}_\sigma)(q)| = 1$ and $q' = (\text{UR} \circ \text{Reach}_\sigma)(q)$. The size of $S_{\text{det}}$ is polynomial of that of $S$. The results obtained in [2] are as follows.

Proposition 4.1 ([2]) Consider an FSA $S$. Under Assumption 1, $S$ is strongly detectable if and only if in $S_{\text{det}}$, any state reachable from any reachable transition cycle is a singleton; $S$ is strongly periodically detectable if and only if in $S_{\text{det}}$, every reachable transition cycle contains at least one singleton; $S$ is strongly periodically D-detectable if and only if in $S_{\text{obs}}$, every reachable transition cycle contains at least one state $q$ such that $(q \times q) \cap T_{\text{spec}} = \emptyset$.

In [3], in order to verify (delayed) strong detectability of $S$, the self-composition

$$CC_A(S) = (X', T', X'_0, \delta')$$

of $S$ (i.e., the concurrent composition of $S$ and itself) was constructed as follows:
• $X' = X \times X$;
• $T' = T'_o \cup T'_{uo}$, where $T'_o = \{(i, \bar{y})|i, \bar{y} \in T, \ell(i) = \ell(\bar{y}) \in \Sigma\}$, $T'_{uo} = \{(i, \epsilon)|i \in T, \ell(i) = \epsilon\} \cup \{(\epsilon, \bar{i})|\bar{i} \in T, \ell(\bar{i}) = \epsilon\};$
• $X'_0 = X_0 \times X_0$;
• for all $(\bar{x}_1, \bar{x}'_1), (\bar{x}_2, \bar{x}'_2) \in X'$, $(\bar{i}, \bar{y}) \in T'_o$, $(\bar{i}', \epsilon) \in T'_{uo}$, and $(\epsilon, \bar{m}) \in T'_{uo},$
  
  $- ((\bar{x}_1, \bar{x}'_1), (\bar{i}, \bar{y}), (\bar{x}_2, \bar{x}'_2)) \in \delta' \text{ if and only if } (\bar{x}_1, \bar{i}, \bar{x}_2),$
  
  $(\bar{x}'_1, \bar{i}', \bar{x}'_2) \in \delta,$
  
  $- ((\bar{x}_1, \bar{x}'_1), (\bar{i}', \epsilon), (\bar{x}_2, \bar{x}'_2)) \in \delta' \text{ if and only if } (\bar{x}_1, \bar{i}', \bar{x}_2) \in \delta, \bar{x}'_1 = \bar{x}'_2,$
  
  $- ((\bar{x}_1, \bar{x}'_1), (\epsilon, \bar{m}), (\bar{x}_2, \bar{x}'_2)) \in \delta' \text{ if and only if } \bar{x}_1 = \bar{x}_2, (\bar{x}'_1, \bar{m}, \bar{x}'_2) \in \delta.$

For an event sequence $s' \in (T')^*$, $s'(L)$ and $s'(R)$ denote its left and right components, respectively. Similarly for $x' \in X'$, denote $x' = (x'(L), x'(R))$. In addition, for every $s' \in (T')^*$, $\ell(s')$ denotes $\ell(s'(L))$ or $\ell(s'(R))$, since $\ell(s'(L)) = \ell(s'(R))$. In the above construction, $CC_\lambda(S)$ aggregates every pair of transition sequences of $S$ producing the same label sequence. The size of $CC_\lambda(S)$ is polynomial of that of $S$.

### 4.2 Verifying strong periodic detectability

In order to verify strong periodic detectability without any assumption, we first characterize its negation. By directly observing Definition 2, the following result follows.

**Proposition 4.2** An FSA $S$ is not strongly periodically detectable if and only if for every positive integer $k$, there exists $s_k \in L^\omega(S)$ and prefix $s' \subset s_k$ such that for all $s'' \in T^*$, $s's'' \subset s_k$ and $|\ell(s'')| < k$ imply $|M(S, \ell(s')| > 1$.

By Proposition 4.2, the following proposition holds.

**Proposition 4.3** An FSA $S$ is not strongly periodically detectable if and only if at least one of the following two conditions holds.

(i) There exists $\gamma \in L(S)$ and $x \in M(S, \gamma)$ such that $|M(S, \gamma)| > 1$ and there is a transition sequence $x \xrightarrow{s_1} x' \xrightarrow{s_2} x''$ for some $s_1 \in (T_{uo})^*$, $s_2 \in (T_{uo})^+$, $x' \in X$.

(ii) There exists $\alpha \beta \in L(S)$ such that $|\beta| > 0$, $M(S, \alpha) = M(S, \alpha \beta)$, and $|M(S, \alpha \beta)| > 1$ for all $\beta \subset \bar{\beta}$.

**Proof** “if”: Assume (i) holds. Then there exists a transition sequence $x_0 \xrightarrow{s_1} x_1 \xrightarrow{s_2} x'$ such that $x_0 \in X_0$ and $\ell(s_2) = \gamma$. For every positive integer $k$, choose $s_k = s_\gamma s_1(s_2)^\omega \in L^\omega(S)$, then for every $s'' \subset s_1(s_2)^{\omega}$, one has $\ell(s'') = \epsilon$ and $|M(S, \ell(s_\gamma s''))| = |M(S, \gamma)| > 1$, which violates strong periodic detectability by Proposition 4.2.

Assume (ii) holds. Then $\alpha \beta \omega \in L^\omega(S)$. For every positive integer $k$, choose $s_k = s_\alpha s_\beta \in L^\omega(S)$ such that $\ell(s_\alpha) = \alpha$ and $\ell(s_\beta) = \beta \omega$. Then for every $s'_\beta \subset s_\beta$, one has $|M(S, \ell(s_\alpha s'_\beta))| > 1$, which also violates strong periodic detectability by Proposition 4.2.
“only if”: Assume \( S \) is not strongly periodically detectable and (ii) does not hold, next we prove (i) holds.

Since \( S \) is not strongly periodically detectable, by Proposition 4.2, choose integer \( k > |2^X|, \)
\( s_k \in L^\omega(S) \), and prefix \( s' \supseteq s_k \) such that for all \( s'' \in T^* \), \( s's'' \supseteq s_k \) and \( |\ell(s'')| < k \) imply \( |M(S, \ell(s''))| > 1 \). Since (ii) does not hold, one has \( \ell(s_k) \in \Sigma^* \) and \( |\ell(s_k)| < k + |\ell(s')| \). Otherwise if \( |\ell(s_k)| \geq k + |\ell(s')| \) or \( \ell(s_k) \in \Sigma^* \), we can choose \( s'' \) such that \( s's'' \supseteq s_k \) and \( |\ell(s'')| = k \), then by the Pigeonhole Principle and \( k > |2^X| \), there exist \( s''_1, s''_2 \supseteq s'' \) such that \( |\ell(s''_1)| < |\ell(s''_2)| \) and \( M(S, \ell(s''_1)) = M(S, \ell(s''_2)) \), that is, (ii) holds. Then \( s_k = s's''_1s''_2 \), where \( s''_1 \in T^* \), \( s''_2 \in (T_{uo})^\omega \). Moreover, one has \( |M(S, \ell(s''_1))| > 1 \), and also by the Pigeonhole Principle there exists a transition sequence \( x_0 \xrightarrow{s''_1} x \xrightarrow{s''_2} x' \) for some \( x_0 \in X_0 \), \( x, x' \in X \), \( s''_1 \in (T_{uo})^* \), and \( s''_2 \in (T_{uo})^+ \), i.e., (i) holds.

**Theorem 4.4** An FSA \( S \) is not strongly periodically detectable if and only if in its observer \( S_{obs} \) as in (2), at least one of the two following conditions holds.

(iii) There is a reachable state \( q \in 2^X \) in \( S_{obs} \) and \( x \in q \) such that \( |q| > 1 \) and there is a transition sequence \( x \xrightarrow{s''_1} x' \xrightarrow{s''_2} x' \) in \( S \) for some \( s_1 \in (T_{uo})^* \), \( s_2 \in (T_{uo})^+ \), \( x' \in X \).

(iv) There is a reachable transition cycle such that no state in the cycle is a singleton.

**Proof** By definition, one sees that for all \( \sigma \in L(S) \), \( M(S, \sigma) = \delta_{obs}(M(S, \epsilon), \sigma) \). Then (iii) (resp., (iv)) of this theorem is equivalent to (i) (resp., (ii)) of Proposition 4.3.

Theorem 4.4 provides an exponential-time algorithm for verifying strong periodic detectability of \( S \). Next we obtain a polynomial-time verification algorithm by simplifying Theorem 4.4. To this end, we need to prove a relationship between \( S_{det} \) and \( S_{obs} \).

**Proposition 4.5** Consider an FSA \( S \). For every transition \( (q, \sigma, q') \in \delta_{obs} \), for every \( q \neq q' \) satisfying \( |\bar{q}| = 2 \) if \( |q'| \geq 2 \), there is \( \bar{q} \subset q \) such that \( (\bar{q}, \sigma, \bar{q}') \in \delta_{det} \), where \( |\bar{q}| = 2 \) if \( |q| \geq 2 \).

**Proof** We only need to prove the case \( |q| \geq 2 \) and \( |q'| \geq 2 \), the other cases hold similarly. Arbitrarily choose \( \{x_1, x_2\} = q' \subset q' \) such that \( x_1 \neq x_2 \). By definition, either (1) there exists \( x_3 \in X, t_1, t_2 \in T_o, s_1, s_2 \in (T_{uo})^* \) such that \( (x_3, t_1s_1, x_1), (x_3, t_2s_2, x_2) \in \delta \) and \( \ell(t_1) = \ell(t_2) = \sigma \), or (2) there exist \( x_4, x_5 \in X, t_1, t_2 \in T_o, s_1, s_2 \in (T_{uo})^* \) such that \( x_4 \neq x_5, (x_4, t_1s_1, x_1), (x_5, t_2s_2, x_2) \in \delta \) and \( \ell(t_1) = \ell(t_2) = \sigma \). If (1) holds, we choose \( \bar{q} = \{x_3, x_6\} \), where \( x_6 \in q \setminus \{x_3\} \); if (2) holds, we choose \( \bar{q} = \{x_4, x_5\} \). By definition, no matter (1) or (2) holds, one has \( (\bar{q}, \sigma, \bar{q}') \in \delta_{det} \).

**Theorem 4.6** An FSA \( S \) is not strongly periodically detectable if and only if in its detector \( S_{det} \) as in (3), at least one of the two following conditions holds.

(v) There is a reachable state \( q' \in Q \) and \( x \in q' \) such that \( |q'| > 1 \) and there is a transition sequence \( x \xrightarrow{s_1} x' \xrightarrow{s_2} x' \) in \( S \) for some \( s_1 \in (T_{uo})^* \), \( s_2 \in (T_{uo})^+ \), \( x' \in X \).

(vi) There is a reachable transition cycle such that all states in the cycle have cardinality 2.

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Proof We use Theorem 4.4 and Proposition 4.5 to prove this result.

First, by Definition 4.3, we consider two FSAs $S_1$ and $S_2$ shown in Fig. 2. One sees that $S_1$ satisfies Assumption 1. However, $S_2$ does not satisfy Assumption 1, as $x_1$ is a deadlock (violating (A) of Assumption 1), and there is a reachable unobservable transition cycle $x_2 \xrightarrow{t} x_2$ (violating (B) of Assumption 1).

Their detectors $S_{1det}$ and $S_{2det}$ are shown in Fig. 2. One sees $S_{1det}$ satisfies (vi) of Theorem 4.6 because there is a self-loop on reachable state $\{x_1, x_2\}$, but does not satisfy (v) because $\{x_1, x_2\}$ is
the unique reachable state of cardinality 2 and in $S_1$ there is no infinitely long unobservable transition sequence starting at $x_1$, the same for $x_2$. $S_{2\text{det}}$ satisfies (v) because $\{x_1, x_2\}$ is reachable in $S_{2\text{det}}$ and in $S_2$, starting at $x_2$ there is an infinite-length unobservable transition sequence, but does not satisfy (vi) because there is no cycle all of whose states are of cardinality 2. Hence by Theorem 4.6, neither $S_1$ nor $S_2$ is strongly periodically detectable.

Remark 1 By Example 4.8, one sees that (v) and (vi) do not imply each other. So they cannot take the place of each other when verifying strong periodic detectability. Let us compare Theorem 4.6 with Proposition 4.1. One directly sees that the equivalent condition for strong periodic detectability under Assumption 1 shown in Proposition 4.1 is exactly negation of (vi). By Proposition 4.1, $S_2$ is strongly periodically detectable vacuously. Then Proposition 4.1 does not always work correctly if Assumption 1 is not satisfied.

Next we show that a slight variant of the concurrent-composition structure can also provide an NL upper bound for strong periodic detectability. The concurrent-composition structure has essentially different features compared with the detector structure. On the one hand, a detector tracks output sequences and collects all states between only unobservable transitions and divides them into subsets of cardinality 2. So a detector does not reflect information in unobservable transitions. However, the concurrent-composition structure can do that. On the other hand, a concurrent composition collects all pairs of transition sequences generating the same output sequence, but sometimes does not collect different transitions starting at the same state. However, a detector can do that. For example, consider states $x_1, x_2, x_3, x_4$ such that $x_1 \neq x_2$ and $x_3 \neq x_4$, there exist transitions $x_1 \xrightarrow{t_1} x_3$, $x_1 \xrightarrow{t_2} x_4$ satisfying $\ell(t_1) = \ell(t_2) \neq \epsilon$, but there is no transition $x_2 \xrightarrow{t_3} x_4$ satisfying $\ell(t_3) = \ell(t_1)$. Then in $S_{\text{det}}$ there is a transition $\langle x_1, x_2 \rangle \xrightarrow{\ell(t_1)} \{x_3, x_4\}$, but in $\text{CC}_A(S)$ there is no transition $(x_1, x_2) \xrightarrow{t'} (x_3, x_4)$ for any $t' \in T'$ satisfying $\ell(t') = \ell(t_1)$. Next we add additional transitions into $\text{CC}_A(S)$ to remove this drawback of $\text{CC}_A(S)$ so that a verification algorithm for strong periodic detectability could be derived.
Consider an FSA $S$ as in (1) and its self-composition $CC_A(S)$ as in (4). We construct a variant

$$CC_A^<\epsilon(S) = (X', T' \cup \{\epsilon\}, X'_0, \delta'_\epsilon)$$

(5)

from $CC_A(S)$ as follows: For all $x_1, x_2, x_3, x_4 \in X$, and $t' \in T'$ such that $x_1 \neq x_2$, $((x_1, x_1), t', (x_3, x_4)) \in \delta'$ (resp., $((x_2, x_2), t', (x_3, x_4)) \in \delta'$), but $((x_1, x_2), \overline{t}, (x_3, x_4)) \notin \delta'$ for any $\overline{t} \in T'$, add transition $((x_1, x_2), \epsilon, (x_1, x_1))$ (resp., $((x_1, x_2), \epsilon, (x_2, x_2))$), where we let $\ell(\epsilon) = \epsilon$. We call $CC_A^<\epsilon(S)$ $\epsilon$-extended self-composition of $S$.

One can see the following proposition.

**Proposition 4.9** Consider an FSA $S$ as in (1), its observer $S_{\text{obs}}$ as in (2), its detector $S_{\text{det}}$ as in (3), and its $\epsilon$-extended self-composition $CC_A^<\epsilon(S)$ as in (5). Assume states $x_1, x_2, x_3, x_4 \in X$ such that $x_1 \neq x_2$ and $x_3 \neq x_4$. The following hold.

(vii) For every transition $\{x_1, x_2\} \xrightarrow{\sigma} \{x_3, x_4\}$ in $S_{\text{det}}$, there is an observable transition sequence $(x_1, x_2) \xrightarrow{\sigma'} (x_3, x_4)$ or $(x_1, x_2) \xrightarrow{\sigma'} (x_4, x_3)$ in $CC_A^<\epsilon(S)$ such that $\ell(s') = \sigma$.

(viii) For every transition $\{x_1, x_2\} \xrightarrow{\sigma} \{x_3\}$ in $S_{\text{det}}$, there is an observable transition sequence $(x_1, x_2) \xrightarrow{\sigma'} (x_3, x_3)$ in $CC_A^<\epsilon(S)$ such that $\ell(s') = \sigma$.

(ix) For every transition $\{x_1\} \xrightarrow{\sigma} \{x_3, x_4\}$ in $S_{\text{det}}$, there is an observable transition sequence $(x_1, x_1) \xrightarrow{\sigma'} (x_3, x_3)$ in $CC_A^<\epsilon(S)$ such that $\ell(s') = \sigma$.

(x) For every transition $\{x_1\} \xrightarrow{\sigma} \{x_3\}$ in $S_{\text{det}}$, there is an observable transition sequence $(x_1, x_1) \xrightarrow{\sigma'} (x_3, x_3)$ in $CC_A^<\epsilon(S)$ such that $\ell(s') = \sigma$.

(xi) In $CC_A^<\epsilon(S)$, consider an arbitrary transition sequence $x'_0 \xrightarrow{\sigma'_0} x'_1 \xrightarrow{t'_1} x'_2 \xrightarrow{\sigma'_2} \ldots \xrightarrow{t'_n} x'_n \xrightarrow{\sigma'_n} x'_{2n+1}$, where $x'_0 \in X'_0$, $x'_1, \ldots, x'_{2n+1} \in X'$, $s'_0, \ldots, s'_n \in (T'_{uo} \cup \{\epsilon\})^*$, $t'_1, \ldots, t'_n \in T'_{\circ}$. For every $i \in [0, n]$, denote the union of all states of unobservable transition sequence $x_{2i} \xrightarrow{\sigma'_i} x_{2i+1}$ by $q_i$, then we obtain a sequence $q_0 \xrightarrow{t'_0} \ldots \xrightarrow{t'_i} q_i$ for every $i \in [1, n]$, there exists $q_0 \supset q_i$ such that $M(S, \epsilon) \xrightarrow{t'_0} q_0 \xrightarrow{t'_1} \ldots \xrightarrow{t'_i} q_i$ is a transition sequence of $S_{\text{obs}}$.

**Proof** (vii) We need to consider four different cases of transition sequences in S (shown in Figs. 3, 4) that form the transition $\{x_1, x_2\} \xrightarrow{\sigma} \{x_3, x_4\}$ in $S_{\text{det}}$, where in these figures, $t_1, t_2 \in T_{\circ}$, $\ell(t_1) = \ell(t_2) = \sigma$, $s_1, s_2, s_3, s_4 \in (T_{uo})^*$, $x_5, x_6, x_7, x_8 \in X$.

![Figure 3: Case 1 (left). Case 2 (right).](image)

We need to prove for each case, there is an observable transition sequence $(x_1, x_2) \xrightarrow{\sigma'} (x_3, x_4)$ in $CC_A^<\epsilon(S)$ such that $\ell(s') = \sigma$. We only need to consider the most complex Case 4, all the other cases can be dealt with similarly. For Case 4, by definition, the corresponding observable transition
There is a reachable transition cycle sequence is \((s_1, s_2) \xrightarrow{s} (s_1, s_2) \xrightarrow{(t_1, t_2)} (s_5, s_6) \xrightarrow{s'_1} (s_7, s_8) \xrightarrow{s'_2} (s_3, s_4)\), where \(s'_1(L) = s_1, s'_1(R) = s_2, s'_2(L) = s_3, s'_2(R) = s_4\).

(viii), (ix), and (x) can be proved similarly.

(xi) directly follows from definition.

\[\square\]

**Example 4.10** Consider FSA \(S_3\), its detector \(S_{\text{det}}\), and its \((\varepsilon\text{-extended})\) self-composition \(CC_A(S_3)\) (\(CC_A^\varepsilon((S_3))\)) shown in Fig. 5. There is a transition \(\{x_1, x_2\} \xrightarrow{b} \{x_1, x_2\}\) in \(S_{\text{det}}\), but there is neither transition sequence \((x_1, x_2) \xrightarrow{s'} (x_1, x_2)\) nor \((x_1, x_2) \xrightarrow{s''} (x_2, x_1)\) such that \(\ell(s') = b\) in \(CC_A(S_3)\). However, in \(CC_A^\varepsilon(S_3)\), there is a transition sequence \((x_1, x_2) \xrightarrow{1} (x_1, x_1) \xrightarrow{(t_3, t_4)} (x_1, x_2)\) such that \(\ell(\varepsilon(t_3, t_4)) = b\).

![Figure 5: FSA \(S_3\) (upper left), its detector \(S_{\text{det}}\), its \((\varepsilon\text{-extended})\) self-composition \(CC_A(S_3)\) (right, dotted transitions excluded), and its \(\varepsilon\text{-extended}\) self-composition \(CC_A^\varepsilon(S_3)\) (right).](image)

With these properties, we are ready to give a new polynomial-time algorithm for verifying strong periodic detectability by using \(CC_A^\varepsilon(S)\).

**Theorem 4.11** An FSA \(S\) is not strongly periodically detectable if and only if in its \(\varepsilon\text{-extended self-composition}\) \(CC_A^\varepsilon(S)\) as in (5), at least one of the two following conditions holds.

(xii) There is a reachable state \((x, \bar{x})\) such that \(x \neq \bar{x}\) and there is a transition sequence \(x \xrightarrow{s_1} x' \xrightarrow{s_2} x'\) in \(S\) for some \(s_1 \in (T_{\text{obs}})^*, s_2 \in (T_{\text{obs}})^+, x' \in X\).

(xiii) There is a reachable transition cycle \((x_1, \bar{x}_1) \xrightarrow{s'_1} \cdots \xrightarrow{s'_n} (x_{n+1}, \bar{x}_{n+1})\) for some positive integer \(n\) such that \((x_1, \bar{x}_1) = (x_{n+1}, \bar{x}_{n+1}), x_i \neq \bar{x}_i, \text{ and } \ell(s'_i) \in \Sigma \text{ for all } i \in [1, n]\).
Proof We use Theorem 4.6 and Propositions 4.5 and 4.9 to prove this result.
We first check (xii) is equivalent to (v).

“⇒”: Assume (xii) holds. By (xi) of Proposition 4.9 and Proposition 4.5, for every reachable state
\( (x, x') \) of \( \text{CC}_A^\text{es} (S) \) such that \( x \neq x' \), either \( \{x, x'\} \subseteq \mathcal{M}(S, \epsilon) \) or \( \{x, x'\} \) is reachable in \( S_{\text{det}} \). Hence (v) holds.

“⇐”: Assume (v) holds. If \( q' = \mathcal{M}(S, \epsilon) \), then (xii) holds. Otherwise (i.e., in case \( |q'| = 2 \) and \( q' \neq \mathcal{M}(S, \epsilon) \)), by (vii), (viii), (ix), (x) of Proposition 4.9, one has (xii) holds.

We second check (xiii) is equivalent to (vi).

“⇒”: Assume (xiii) holds. By (xi) of Proposition 4.9 and the Pigeonhole Principle, there is a reacha-
ble transition cycle in \( S_{\text{obs}} \) none of whose states is a singleton. The by Proposition 4.5, (vi)
holds.

“⇐”: Assume (vi) holds. By (vii) of Proposition 4.9, (xiii) holds.

Similarly to the case that Theorem 4.6 implies Theorem 4.7, Theorem 4.11 also implies an NL
upper bound for strong periodic detectability of FSAs without any assumption.

Example 4.12 We next use one example to compare Theorem 4.6 with Theorem 4.11. Reconsider \( S_3 \)
in Example 4.10 (shown in Fig. 5, upper left). The existence of reachable transition cycle \( \{x_1, x_2\} \overset{b}{\to} \{x_1, x_2\} \) in \( S_{\text{det}} \) implies that \( S_3 \) is not strongly periodically detectable by Theorem 4.6 (satisfying (vi)). The existence of reachable transition cycle \( (x_1, x_2) \overset{\epsilon}{\to} (x_1, x_1) \overset{(t_3, t_1)}{\to} (x_1, x_2) \) such that \( \ell(\epsilon(t_3, t_1)) = b \in \Sigma \) in \( \text{CC}_A^\text{es} (S) \) also implies that \( S_3 \) is not strongly periodically detectable, by Theorem 4.11 (satisfying (xiii)).

4.3 Verifying strong periodic D-detectability

We also first characterize negation of \( T_{\text{spec}} \)-strong periodic D-detectability. The following result di-
rectly follows from Definition 3.

Proposition 4.13 An FSA \( S \) is not strongly periodically D-detectable with respect to \( T_{\text{spec}} \) if and only if for each positive integer \( k \), there exist \( s_k \in L^\omega (S) \) and \( s' \sqsubset s_k \) such that for every \( s'' \in T^* \), \( |\ell(s'')| < k \) and \( s's'' \sqsubset s \) imply \( (\mathcal{M}(S, \ell(s'')) \times \mathcal{M}(S, \ell(s's''))) \cap T_{\text{spec}} \neq \emptyset \).

Similarly to Theorem 4.4, we can prove the following result. We omit the similar proof.

Theorem 4.14 An FSA \( S \) is not strongly periodically D-detectable with respect to \( T_{\text{spec}} \) if and only if in its observer \( S_{\text{obs}} \) as in (2), at least one of the two following conditions holds.

(xiv) There is a reachable state \( q \in 2^X \) in \( S_{\text{obs}} \) and \( x \in q \) such that \( (q \times \emptyset) \cap T_{\text{spec}} \neq \emptyset \) and there is a transition sequence \( x \xrightarrow{a} x' \xrightarrow{b} x'' \) in \( S \) for some \( s_1 \in (T_{\text{uo}})^* \), \( s_2 \in (T_{\text{uo}})^+ \), \( x' \in X \).

(xv) There is a reachable transition cycle such that each state \( q \) of the cycle satisfies \( (q \times \emptyset) \cap T_{\text{spec}} \neq \emptyset \).

Theorem 4.15 The problem of verifying strong periodic D-detectability with respect to \( T_{\text{spec}} \) belongs to PSPACE.
Proof Condition (xiv) can be checked by guessing \( q \in 2^X \), \( x \in q \), and \( x' \in X \) and doing the corresponding checks by nondeterministic research. Since each state \( q \) of \( S_{obs} \) is bounded by the number of states of \( S \), and \((q \times q) \cap T_{spec} \neq \emptyset \) can be checked in time quadratic in the number of states of \( S \), (xiv) can be checked in \( \text{NPSPACE} \).

Condition (xv) can be checked by nondeterministically guessing a sequence of label sequence and checking whether the sequence leads \( S_{obs} \) to such a transition cycle. Hence, (xiv) can also be checked in \( \text{NPSPACE} \).

Hence by Theorem 4.14, the problem of verifying strong periodic D-detectability with respect to \( T_{spec} \) belongs to \( \text{coNPSPACE} \), i.e., \( \text{PSPACE} \).

Remark 2 One directly sees that the equivalent condition for strong periodic D-detectability of FSAs under Assumption 1 given in [2, Theorem 9] (collected in Proposition 4.1) is exactly negation of (xv) in Theorem 4.14. So the algorithm induced from [2, Theorem 9] usually does not work correctly without Assumption 1. See the following example.

Reconsider \( S_2 \) (shown in Fig. 1, right) and its observer \( S_{2obs} \) (shown in Fig. 2, right). As shown in Example 4.8, \( S_2 \) violates Assumption 1. Now choose \( T_{spec} = \{(x_1, x_2)\} \). For every positive integer \( k \), choose \( s_k = t_2(t_4)^w \in L^w(S_2) \), then for all \((t_4)^n\), where \( n \geq 0 \), one has \( t_2(t_4)^n \sqsubseteq s_k \), \( \ell((t_4)^n) = 0 < k \), \( \ell(t_2(t_4)^n) = a \), and \((M(S_2, a) \times M(S_2, a)) \cap T_{spec} = T_{spec} \neq \emptyset \). That is, \( S_2 \) is not strongly periodically D-detectable with respect to \( T_{spec} \) by definition. However, since there is no cycle in \( S_{2obs} \), the condition “every reachable transition cycle contains at least one state \( q \) such that \((q \times q) \cap T_{spec} = \emptyset \)” in Proposition 4.1 is satisfied vacuously. Thus, \( S_{2obs} \) is strongly periodically D-detectable with respect to \( T_{spec} \) by Proposition 4.1, which is incorrect.

Example 4.16 We next illustrate Theorem 4.14. Reconsider FSA \( S_3 \) in Fig. 5 (upper left) and its observer \( S_{3obs} \) in Fig. 5 (lower left). If we choose \( T_{spec}^1 = \{(x_1, x_2)\} \), the existence of self-loop \( \{(x_1, x_2) \mapsto (x_1, x_2)\} \) in \( S_{3obs} \) satisfies (xv) of Theorem 4.14 (i.e., \( \{(x_1, x_2) \times \{x_1, x_2\}\} \cap T_{spec}^1 = T_{spec}^1 \neq \emptyset \)), hence \( S_3 \) is not strongly periodically D-detectable with respect to \( T_{spec}^1 \). If we choose \( T_{spec}^2 = \{(x_0, x_2)\} \), then by \( S_{3obs} \), one sees neither (xiv) nor (xv) is satisfied, hence \( S_3 \) is strongly periodically D-detectable with respect to \( T_{spec}^2 \).

5 Conclusion

In this paper, we obtained an NL upper bound for verifying strong periodic detectability of FSAs without any assumption, strengthening the related results given in [2, 9] under two assumptions of deadlock-freeness and divergence-freeness. We also obtained a PSPACE upper bound for verifying strong periodic D-detectability of FSAs without any assumption, strengthening the related result given in [2, 10] also under the two assumptions.

As shown in our previous paper [3], the self-composition method can be used to verify (delayed) strong detectability of FSAs without any assumption, but the detector method cannot. In this paper, we showed that both the detector method and a variant of the self-composition method can be used to verify strong periodic detectability of FSAs without any assumption. It is an interesting future topic to study the intrinsic relationships between the two methods.
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