On the rms-radius of the proton

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Abstract

We study the world data on elastic electron-proton scattering in order to determine the proton charge \( \text{rms} \)-radius. After accounting for the Coulomb distortion and using a parameterization that allows to deal properly with the higher moments we find a radius of \( 0.895 \pm 0.018 \text{ fm} \), which is significantly larger than the radii used in the past.
**Introduction.** The root-mean-square (rms) radius of the proton is a quantity of great interest for an understanding of the proton; it describes the most important integral property concerning its size. Accurate knowledge of the rms-radius of the charge distribution is needed for the interpretation of high-precision measurements of transitions in hydrogen atoms, studied in connection with measurements of fundamental constants [1]; these measurements recently have made great progress, and are now limited by the accuracy with which the proton radius is known [2]. The radius is also needed for the planned measurements of muonic X-ray transitions [3]; these experiments can only scan a narrow frequency range, which must be chosen according to the best value of the rms-radius presently known.

The proton rms-radius in the past in general has been determined from elastic electron-proton scattering. The usual approach has been to employ the most accurate cross sections at low momentum transfer $q$, perform an experimental separation of longitudinal (L, charge) and transverse (T, magnetic) contributions. The resulting charge data as a function of $q^2$ are then fit with an appropriate function to get the rms-radius, i.e. the $q^2 = 0$ slope of the form factor $^1$. Alternative approaches have included theory-motivated fits such as given by the Vector Dominance Model (VDM) in combination with dispersion relations.

**Past results.** The initial electron scattering experiments on the proton were performed some 40 years ago by the Hofstadter group at Stanford [4, 5]. This data, mainly at medium $q$ and not low $q$, was fitted using multi-pole form factors. From the parameters of the fit an rms-radius could be calculated. The resulting value of $0.81 \text{ fm}$, which is still quoted in the literature, should have long been superseded by values coming from more precise data at lower $q$ which are indeed sensitive to the rms-radius.

In the seventies, accurate low-$q$ data, mainly measured at the Mainz electron accelerator, became available [6]-[9]. After an L/T-separation, the data were usually fitted with a polynomial expansion of the form factor

$$G_e(q) = 1 - q^2 \langle r^2 \rangle / 6 + q^4 \langle r^4 \rangle / 120 - \ldots$$

(1)

and, in general, a floating normalization of the individual data sets in order to produce the lowest $\chi^2$. The most prominent result was probably the one obtained by Simon et al. [8], $r_{\text{rms}} = 0.862 \pm 0.012 \text{ fm}$.

$^1$This quantity can be determined without making use of the nonrelativistic notion of the charge density as a Fourier transform of the form factor.
Occasionally, fits with 2- or 4-pole expressions \cite{21} were performed, and significantly bigger values, \textit{i.e.} $0.88 \pm 0.02 \text{fm}$ and $0.92 \pm 0.02 \text{fm}$ were found as compared to values determined at very low $q$ \cite{18}. The difference was partly understood \cite{21} as a consequence of different treatments of the $\langle r^4 \rangle$ term.

In parallel, fits based on dispersion relations and the VDM \cite{22, 23} were performed by several groups. These fits included much more theory input, and were constrained by the need to fit all four nucleon form factors. The most recent value resulting from such fits is the one of Mergell \textit{et al.}, $0.847 \pm 0.009 \text{fm}$. The average, $0.854 \pm 0.012 \text{fm}$, of this radius and the one of Simon \textit{et al} is quoted as the ”best” value in the compilation of Mohr and Taylor \cite{24}.

Recent studies have provided additional insight: even for a system as light as the proton, Coulomb distortion of the electron waves needs to be accounted for \cite{25, 26}. This Coulomb distortion was shown to solve a long standing puzzle with the deuteron \textit{rms}-radius, and Rosenfelder demonstrated \cite{27} that it also increases the proton \textit{rms}-radius. Using a restricted set of data and the above mentioned polynomial expansion he showed that the radius increases by about $0.01 \text{fm}$ when accounting for Coulomb distortion.

\textbf{Model-independent radii?}  In general, the groups studying the proton data have tried to extract a \textit{rms}-radius that is model-independent. This is possible when using as in eq.\(1\) the expansion of $G_e(q)$ in terms of the moments $\langle r^2 \rangle$, $\langle r^4 \rangle$, ... . At very low $q$, one could hope that the $q^4\langle r^4 \rangle$-term is small, such that the $\langle r^2 \rangle$- term can be determined without using a specific model for $G_e(q)$.

This is true in principle, but very hard in practice. At small $q$ also the $q^2\langle r^2 \rangle/6$-term is small, and it is difficult to determine it accurately from the experimental form factors which are proportional to $1-q^2 \langle r^2 \rangle/6+...$. Small systematic errors in the normalization of the cross sections have a strong influence on the small $q^2\langle r^2 \rangle/6$-term. When ”eliminating” problems with the normalization of the data by floating them much of the sensitivity to the \textit{rms}-radius gets lost and the norm-determining (implicit) extrapolation to $q = 0$ becomes very sensitive to small $q$-dependent systematic errors in the data (which are always ignored).

In practice, one therefore has to include data at not-so-low $q$ which are also sensitive to the higher moments. The problem with these moments is particularly detrimental for the proton. The proton has approximately an exponential charge density (or, more accurately speaking, a form factor of
the dipole shape, \( G_e(q) = (1 + q^2 0.055 fm^2)^{-2} \), the Fourier transform of which gives an exponential). For such a density (form factor) the higher moments are increasing with order, \( \langle r^4 \rangle = 2.5 \langle r^2 \rangle^2, \langle r^6 \rangle = 11.6 \langle r^2 \rangle^3 \) etc, hence giving a large contribution to \( G(q) \).

The consequence: there is no \( q \)-region where the \( \langle r^2 \rangle \) term dominates the finite size effect to \( >98\% \) and the finite size effect is sufficiently big compared to experimental errors to allow a, say, 2\% determination of the \( rms \)-radius. There is also no region of \( q \) where the \( \langle r^4 \rangle \) moment can be determined accurately without getting into difficulty with the \( \langle r^6 \rangle \) term. Towards higher \( q \), the polynomial expansion is seriously restricted by the convergence radius of \( \sim 1.4 fm^{-1} \).

Figure 1: The figure shows the contribution of the \( q^n \) terms to the finite size effect, calculated using the moments from the CF parameterization. The black curve gives the total finite-size effect.

This situation is illustrated in fig 1 which shows the contribution of the various \( q^n \) terms to the finite size effect. This problematic situation with the higher moments is at the origin of the difficulties of determining a model-independent proton \( rms \)-radius.
Continued-fraction expansion. Continued Fraction (CF) expansions

\[ G_\epsilon(q) = \frac{1}{1 + \frac{q^2 b_1}{1 + \frac{q^2 b_2}{1 + \cdots}}} \]  

are a subclass of Padé approximants which have initially been introduced to solve the "problem of moments", i.e. to find a function \( f(z) \) specified by its moments \( \langle z^n \rangle \) \cite{28} and to accelerate the convergence of poorly converging series \cite{29}. The radius of convergence of the CF expansion is much larger than the one of the polynomial expansion, although within the convergence radius of the latter it agrees exactly with it.

The moments of interest are directly linked to the coefficients \( b_1, b_2, \ldots b_N \), i.e. the coefficients of \( q^2, q^4, \ldots \) are given by \( b_1^2, b_1^2 + b_1 b_2, \ldots \). An important advantage, already exploited in fits of the deuteron form factor \cite{30}, is the fact that the parameters \( b_1, b_2 \) for exponential-type densities are well decoupled. This is a consequence of the fact that the CF is the natural parameterization for form factors resulting from exchange-poles at \( q^2 < 0 \), the physical mechanism exploited in the VDM.

Tests of CF-expansion. In order to study the dependence introduced by the usage of the CF expansion with given number \( N \) of terms and given \( q_{\text{max}} \), we have used pseudo-data. These cross sections were generated using parameterized expressions for the form factors (dipole form, or the dispersion relation parameterization of Hoehler et al. \cite{22}). The pseudo data were generated at the energies and angles of the experimental data, with the error bars of the experimental data. In the fits, the pseudo data were used as calculated from the parameterization, or with random fluctuations calculated from the experimental error bars superimposed.

Fits of these pseudo-data were performed with the CF expansion with a variable number \( N \) of terms, and with variable \( q_{\text{max}} \) of the points fitted. We have studied the scatter of the resulting fitted \( \langle r^2 \rangle \) values, and their deviation from the known radius used in the generation of the pseudo-data. In these tests, we have been rather generous in accepting fits, i.e. by including fits with \( \chi^2 \leq 1.2 \chi^2_{\text{min}} \).

When using the region \( 1 \text{fm}^{-1} < q_{\text{max}} < 5 \text{fm}^{-1} \) and 2 to 5 terms in the CF-expansion, we find a scatter of the fitted \( \text{rms-radius} \) of \( \pm 0.010 \text{fm} \) around the true (input) values. This scatter we take as representative of the
uncertainty due to the choice of $N$ and $q_{\text{max}}$; it covers the statistical error (which for pseudo- and real data is the same by construction) as well.

**Analysis of world data.** In order to determine the proton $\text{rms}$-radius we use the world cross sections [4]-[20] for $q < 4 fm^{-1}$. The most precise data relevant for the radius determination have been measured at Mainz [6]-[9]. These data are *absolute*, that is they have small systematic uncertainties in the absolute normalization. This type of data is the most useful one for a determination of the $\text{rms}$-radius.

We use for our fits the primary cross sections. When parameterizing both $G_e(q)$ and $G_m(q)$ with the CF expansion and fitting $G_e$ and $G_m$ simultaneously to the cross sections, the L/T-separation is automatically performed, with superior quality as compared to the standard approach of separating L and T for each individual experiment.

The Coulomb corrections are calculated in second-order Born approximation according to [26] using an exponential charge density. These corrections are applied to the cross section data, such that the subsequent fit can be performed in PWIA as has been done in the past.

In the fits we use all data with their standard random uncertainties. The error matrix is used to compute the random uncertainty of derived quantities. In order to evaluate the effect of the systematic uncertainties (normalization uncertainties) the individual data sets are changed by their quoted uncertainties, refitted and the resulting changes quadratically added.

In the fits one finds experimental data sets (for instance the 40 years old Stanford data) that have much too large a $\chi^2$; these points, however, do not inappropriately influence the final result, so we have not increased their error bars just to get a good-looking $\chi^2$. We also find small discrepancies in the overall normalization of some data sets (e.g. the data set of ref. [9] seems $\sim 1\%$ high). We have chosen to keep the norm at the experimental value, and not float the data. For such precision experiments more than half the effort has gone into the determination of the overall normalization; ignoring this effort by floating the norm (or greatly mitigating its influence by treating the normalization as just one further data point) does not do justice to the experiments and leads to loss of much information. Again, the effects upon the $\text{rms}$-radius of the observed ”discrepancies” have been found to be small and are covered by the quoted uncertainty.

As a check we have also used the polynomial expansion, with $q_{\text{max}} = 1.2 fm^{-1}$ and the $q^4$ coefficient taken from a fit that explains the higher-$q$ data. We find the same $\text{rms}$-radius as with the CF fit, but a larger
uncertainty and a higher sensitivity to the \( q_{\text{max}} \) employed.

The quality of the fits is quite good. We show in fig.2 the ratio of experimental cross sections and fit for the CF parameterization and 5 CF coefficients. The \( \chi^2 \) is 512 for 310 data points \(^2\). The resulting \( r_{\text{rms}} \)-radius is 0.895 fm. The uncertainty due to N, \( q_{\text{max}} \) and statistics is \( \pm 0.010 \) fm, the systematic uncertainty 0.013 fm. This yields as the final result for the charge radius of the proton \( r_{\text{rms}}^e = 0.895 \pm 0.018 \) fm. This radius is significantly larger than the values generally cited in the literature. It agrees with the most accurate value derived from atomic transitions \(^2\) 0.890 \( \pm 0.014 \) fm.

Figure 2: The figure shows the ratio of experimental and fit cross sections for the CF parameterization.

**Differences to previous determinations.** It may be interesting to understand why previous analyses gave smaller radii. Simon *et al.* \(^8\) \((r_{\text{rms}} = 0.862 \text{ fm})\) used the polynomial expansion up to \( q^4 \) and \( q_{\text{max}} = 1.2 \text{ fm}^{-1} \), but found a \( \langle r^4 \rangle \)-moment that was a factor of ten smaller than given by fits that explain the proton data to higher \( q \); this difference comes from very small systematic problems in the data which we have not further explored. When

\(^2\)The \( \chi^2 \) would reduce to 370 when adding quadratically 3\% to the Stanford error bars, with an increase of \( r_{\text{rms}} \) of 0.002 \( \text{ fm} \). A norm change of 1\% of \(^9\) would increase \( r_{\text{rms}} \) by 0.007 \( \text{ fm} \) and decrease \( \chi^2 \) by 60.
repeating their fit with the \( \langle r^4 \rangle \)-moment given by a fit that explains the data to larger \( q \), e.g. the one from the CF fit, one finds a radius that agrees with the one we find.

The fits based on dispersion relations and the VDM are strongly constrained by theory and the need to fit all four nucleon form factors. When looking at the ratio of experimental and VDM cross sections with the resolution employed in fig. 2, the systematic deviations of the fits \cite{22, 23} from the data at low \( q \) are immediately obvious.

Rosenfelder \cite{27} (\( r_{\text{rms}} = 0.880 \text{fm} \)), whose primary interest was the exploration of the effect of Coulomb distortion, also used the polynomial expansion, with the \( \langle r^4 \rangle \) term taken from a low-\( q \) fit quoted in the literature. When correcting his value for a better \( \langle r^4 \rangle \) value from a good fit to the higher-\( q \) data and accounting for differences in the data set, one arrives at the value of the proton \( \text{rms} \)-radius we find.

**Conclusions.** From an analysis of the world-data on e-p scattering we determine the proton \( \text{rms} \)-radius and find a value that is significantly larger than previous values. The change is understood as a consequence of treating properly the higher moments \( \langle r^n \rangle \).

**Acknowledgment.** The author acknowledges discussions with Savely Karshenboim which triggered this study.

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