ASYMPTOTIC EXPANSIONS ABOUT INFINITY FOR SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH COHERENTLY DECAYING FORCING FUNCTIONS

LUAN HOANG

Abstract. This paper studies, in fine details, the long-time asymptotic behavior of decaying solutions of a general class of dissipative systems of nonlinear differential equations in complex Euclidean spaces. The forcing functions decay, as time tends to infinity, in a coherent way expressed by combinations of the exponential, power, logarithmic and iterated logarithmic functions. The decay may contain sinusoidal oscillations not only in time but also in the logarithm and iterated logarithm of time. It is proved that the decaying solutions admit corresponding asymptotic expansions, which can be constructed concretely. In the case of the real Euclidean spaces, the real-valued decaying solutions are proved to admit real-valued asymptotic expansions. Our results unite and extend the theory investigated in many previous works.

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1. Introduction

This paper broadens the theory of the asymptotic expansions, as time tends to infinity, for solutions of systems of nonlinear ordinary differential equations (ODEs) that was studied in previous work [10]. The paper [10] itself was motivated by Foias–Saut’s work [21] for the Navier–Stokes equations (NSE). In Foias and Saut’s paper [21], the NSE are written in the functional form as an ODE

$$u_t + Au + B(u, u) = 0,$$

(1.1)

where $A$ is a linear operator, and $B$ is a bi-linear form. They proved that every solution $u(t)$ of (1.1) admits the following asymptotic expansion, as $t \to \infty$,

$$u(t) \sim \sum_{k=1}^{\infty} q_k(t)e^{-\mu_k t},$$

(1.2)

where $q_k(t)$’s are polynomials in $t$ valued in some functional spaces, and $\mu_k$’s are positive numbers increasing strictly to infinity. Briefly speaking, the solution $u(t)$ can be approximated, as $t \to \infty$, by the finite sum

$$s_N(t) := \sum_{k=1}^{N} q_k(t)e^{-\mu_k t},$$

for $N \in \mathbb{N}$, in the sense that the remainder $u(t) - s_N(t)$, as $t \to \infty$, decays exponentially faster than the fastest decaying mode $e^{-\mu_N t}$ in $s_N(t)$. Because $\mu_k \to \infty$, the asymptotic expansion (1.2) provides very fine details about the asymptotic behavior of $u(t)$.

The result (1.2) has been developed in many directions. One can study its associated normalization map, normal form, invariant nonlinear spectral manifold, etc., see [13–17, 19, 20, 22] and references therein. The interested reader is referred to the survey [18] for more information on the subject. A similar asymptotic expansion is obtained for the NSE of rotating fluids [27]. The asymptotic expansions for the associated Lagrangian trajectories are derived in [24] based on (1.2).

The asymptotic expansions of the type (1.2) can also be established for other partial differential equations (PDEs) and ODEs. They are obtained in [30] for a class of PDEs including dissipative wave equations, in [29] for a system of analytic ODEs, and in [11] for non-smooth ODEs. These results are for homogeneous equations which have more general nonlinearity than the bi-linear form in (1.1).

Concerning with inhomogeneous equations instead, the papers [8, 9, 26] study the NSE (1.1) with a forcing function $f(t)$ added to its right-hand side. It is proved that if $f(t)$ decays to zero in a coherent way, then any solution $u(t)$ admits an asymptotic expansion with a correspondingly coherent decay [8, 9, 26].

For more general inhomogeneous systems of ODEs, paper [10] investigates

$$y' = -Ay + G(y) + f(t) \text{ in } \mathbb{R}^n,$$

(1.3)
Nonlinear Differential Equations with Coherently Decaying Forcing Functions

where \( A \) is a real diagonalizable \( n \times n \) real matrix with positive eigenvalues, and \( G(y) \) has the Taylor’s expansion about the origin starting the quadratic monomials. It proves that if

\[
f(t) \sim \sum p_k(\phi(t))\psi(t)^{-\gamma_k}, \text{ then } y(t) \sim \sum q_k(\phi(t))\psi(t)^{-\gamma_k},
\]

where, roughly speaking, \( p_k \)'s and \( q_k \)'s are functions of the same type. For example,

- \( \psi(t) = e^t, \phi(t) = t \), \( p_k \)'s and \( q_k \)'s are polynomials. This yields the same asymptotic expansions as (1.2).
- \( \psi(t) = t, \phi(t) = (\ln t, \ln \ln t) \), \( p_k \)'s and \( q_k \)'s are real power functions of two variables.
- \( \psi(t) = \ln t, \phi(t) = (\ln \ln t, \ln \ln \ln t, \ln \ln \ln \ln t) \), \( p_k \)'s and \( q_k \)'s are real power functions of three variables.

Although the asymptotic expansions in (1.4) are rather complicated, some common elements are still missing. For instance, because of the assumptions on the matrix \( A \) and the real powers in the functions \( p_k \)'s and \( q_k \)'s, no oscillations are present in (1.4). This paper will overcome this deficiency. In fact, its new features are the following:

(a) Studying (1.3) in both \( \mathbb{C}^n \) and \( \mathbb{R}^n \).
(b) Treating a general matrix \( A \), namely, \( A \) is only required to have eigenvalues with positive real parts.
(c) Allowing \( p_k \)'s and \( q_k \)'s to have complex powers, therefore, allowing the asymptotic expansions in (1.4) to have sinusoidal oscillations not only in \( t \) but also in \( \ln t, \ln \ln t \), etc.
(d) Providing more general but still concrete constructions for the \( q_k \)'s.

Note that previous papers [29, 30] consider complex exponential functions and, hence, already obtain asymptotic expansions with sinusoidal functions \( \cos(\omega t) \) and \( \sin(\omega t) \). These functions are both oscillating and periodic. (See also [27] when the oscillation occurs due to the Coriolis effect.) However, (c) will allow the asymptotic expansions to contain functions such as \( \cos(\omega \ln t) \) and \( \sin(\omega \ln \ln t) \), which are oscillating but not periodic.

Even for the homogeneous system, i.e. \( f = 0 \), the current work still adds some contributions to the current literature. In [29], although the condition on \( A \) is the same as in (b), the function \( F(y) := -Ay + G(y) \) is required to be analytic. The constructions of \( q_k \)'s are not explicit and depend crucially on the analytic flows associated with the vector field \( F(y) \). On contrary, this paper and [10] do not require the analyticity and, hence, cannot use the constructions in [29]. The work [30], when viewed in the ODE context, requires \( A \) to be complex diagonalizable, which is more stringent than (b) above. Our constructions in (d) will be more concrete than [29] and more general than [10, 30].

There are other asymptotic expansion theories for analytic ODEs such as Lyapunov’s First Method, see e.g. [1, Chapter I, §4] or [28, Chapter V, §3], and the Poincaré–Dulac normal form. The latter normal form theory has been developed much more by Bruno into the theory of power geometry, see [2–7] and the references therein. However, his classes of equations, results and methods are very different from ours.

The paper is organized as follows. Section 2 presents the notation that we use throughout. Section 3 gives definitions of the asymptotic expansions that we study, see Definitions 3.8 and 3.10. For the latter, they involve specific classes of power functions which allow the bases to be logarithmic and iterated logarithmic functions, and the powers to be complex numbers. Therefore, they are more general and sophisticated than the previously studied expansions in [9,10,29,30]. Section 4 imposes the main conditions in Assumption 4.1 for the matrix \( A \) and in Assumption 4.2 for the nonlinear function \( G \). In Section 5 we establish the
asymptotic approximations, as \( t \to \infty \), for the solutions of the linearized equations of \((1.3)\).

Theorem 5.1 treats the case of exponentially decaying forcing functions. The constructions of the approximating functions in Theorem 5.1 are concrete, rather simple and close the original ideas of Foias and Saut. Theorem 5.5 deals with the case of power, logarithmically, and iterated logarithmically decaying forcing functions. The constructions in Theorem 5.5 are subtler than those in [10], see operator \( Z_A \) in Definition 5.2. Besides their own merits, the results are the key building blocks for the nonlinear problem later. Section 6 deals with the nonlinear systems. Theorem 6.1 establishes the global existence and uniqueness for the solutions under the smallness conditions on the initial data and forcing functions. Solutions that decay to zero are also obtained. The main asymptotic estimate for large time is in Theorem 6.2. It provides a specific decaying rate for the solution corresponding to the asymptotic behavior of the forcing function. This estimate will be needed in Sections 7–9. Our main results for the systems in \( \mathbb{C}^n \) are in Sections 7–9 where we establish the asymptotic expansions for the decaying solutions. Theorem 7.7 is for exponentially decaying forcing functions, Theorem 8.3 is for power decaying forcing functions, and Theorem 9.3 is for logarithmically and iterated logarithmically decaying forcing functions. The case of systems in \( \mathbb{R}^n \) is dealt with in Section 10. The counterparts of the results in Sections 7–9 are Theorems 10.5 and 10.8. The proofs use the complexification technique but guarantee that the approximating functions still stay in \( \mathbb{R}^n \). These results are expressed even more clearly using only real-valued functions in Theorems 10.10 and 10.12. Appendix A contains elementary proofs of Lemma 3.3 and Proposition 3.4.

We end this Introduction with a note that the ideas and techniques in the current paper can be applied to problems in PDEs either straightforwardly or with some sophisticated adjustments.

2. Notation

We use the following notation throughout the paper.

- \( \mathbb{N} = \{1, 2, 3, \ldots\} \) denotes the set of natural numbers, and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \).
- Number \( i = \sqrt{-1} \).
- For any vector \( x \in \mathbb{C}^n \), we denote by \( |x| \) its Euclidean norm, and by \( x^{(k)} \) the \( k \)-tuple \((x, \ldots, x)\) for \( k \in \mathbb{N} \), and \( x^{(0)} = 1 \). The real part, respectively, imaginary part, of \( x \) is denoted by \( \text{Re} \, x \), respectively, \( \text{Im} \, x \).
- For an \( m \times n \) matrix \( M \) of complex numbers, its Euclidean norm in \( \mathbb{R}^{mn} \) is denoted by \( |M| \).
- We will use the convention
  \[
  \sum_{k=1}^{0} a_k = 0 \quad \text{and} \quad \prod_{k=1}^{0} a_k = 1.
  \]
- Let \( f \) be a \( \mathbb{C}^m \)-valued function and \( h \) be a non-negative function, both are defined in a neighborhood of the origin in \( \mathbb{C}^n \). We write
  \[
  f(x) = \mathcal{O}(h(x)) \quad \text{as} \quad x \to 0,
  \]
  if there are positive numbers \( r \) and \( C \) such that \( |f(x)| \leq Ch(x) \) for all \( x \in \mathbb{C}^n \) with \( |x| < r \).
- Let \( f : [T_0, \infty) \to \mathbb{C}^n \) and \( h : [T_0, \infty) \to [0, \infty) \) for some \( T_0 \in \mathbb{R} \). We write
  \[
  f(t) = \mathcal{O}(h(t)) \quad \text{implicitly meaning as} \quad t \to \infty,
  \]
if there exist numbers $T \geq T_0$ and $C > 0$ such that $|f(t)| \leq Ch(t)$ for all $t \geq T$.

In the case $h(t) > 0$ for all $t \geq T_0$, we write

$$f(t) = o(h(t)),$$

if $\lim_{t \to \infty} \frac{|f(t)|}{h(t)} = 0.$

Let $T_0 \in \mathbb{R}$, functions $f, g : [T_0, \infty) \to \mathbb{C}^n$, and $h : [T_0, \infty) \to [0, \infty)$. We will conveniently write

$$f(t) = g(t) + O(h(t))$$
to indicate $f(t) - g(t) = O(h(t)).$

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If $m \in \mathbb{N}$ and $\mathcal{L}$ is an $m$-linear mapping (over $\mathbb{K}$) from $(\mathbb{K}^n)^m$ to $\mathbb{K}^n$, then the norm $||\mathcal{L}||$ is defined by

$$||\mathcal{L}|| = \max\{|\mathcal{L}(y_1, y_2, \ldots, y_m)| : y_j \in \mathbb{K}^n, |y_j| = 1, \text{ for } 1 \leq j \leq m\}.$$

Then $||\mathcal{L}||$ is a number in $[0, \infty)$, and

$$|\mathcal{L}(y_1, y_2, \ldots, y_m)| \leq ||\mathcal{L}|| \cdot |y_1| \cdot |y_2| \cdots |y_m|$$

for any $y_1, y_2, \ldots, y_m \in \mathbb{K}^n$.

**Definition 2.1.** Let $S$ be a subset of $\mathbb{C}$.

- We say $S$ preserves the addition if $x + y \in S$ for all $x, y \in S$.
- We say $S$ preserves the unit increment if $x + 1 \in S$ for all $x \in S$.
- The additive semigroup generated by $S$ is defined by

$$\langle S \rangle = \left\{ \sum_{j=1}^{N} z_j : N \in \mathbb{N}, z_j \in S \text{ for } 1 \leq j \leq N \right\}.$$

- The real part of $S$ is $\text{Re} S = \{ \text{Re } z : z \in S \}$.

Regarding Definition 2.1 it is obvious that $\langle S \rangle$ preserves the addition, and $\text{Re} \langle S \rangle = \langle \text{Re } S \rangle$.

If $1 \in S$, then $\langle S \rangle$ preserves the unit increment.

3. **Classes of Functions and Types of Asymptotic Expansions**

3.1. **The complex power function.** In this paper, we only deal with single-valued complex functions. To avoid any ambiguity we recall basic definitions and properties of elementary complex functions.

For $z \in \mathbb{C}$ and $t > 0$, the exponential and power functions are defined by

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \text{ and } t^z = \exp(z \ln t).$$

When $t = e = \exp(1)$ in (3.1), one has the usual identity $e^z = \exp(z)$.

If $z = a + ib$ with $a, b \in \mathbb{R}$, then

$$t^z = t^a (\cos(b \ln t) + i \sin(b \ln t)) \text{ and } |t^z| = t^a.$$

The standard properties of the power functions still hold, namely,

$$t^{z_1} t^{z_2} = t^{z_1 + z_2}, \quad (t_1 t_2)^z = t_1^z t_2^z, \quad (t^z)^m = t^{mz} = (t^m)^z,$$

$$\frac{d}{dz}(t^z) = z t^{z-1},$$

for any $t, t_1, t_2 > 0, z, z_1, z_2 \in \mathbb{C}$, and $m \in \mathbb{Z}_+$. 
3.2. Iterated exponential and logarithmic functions.

**Definition 3.1.** Define the iterated exponential and logarithmic functions as follows:

- \( E_0(t) = t \) for \( t \in \mathbb{R} \), and \( E_{m+1}(t) = e^{E_m(t)} \) for \( m \in \mathbb{Z}_+ \), \( t \in \mathbb{R} \),
- \( L_{-1}(t) = e^t \), \( L_0(t) = t \) for \( t \in \mathbb{R} \), and
- \( L_{m+1}(t) = \ln(L_m(t)) \) for \( m \in \mathbb{Z}_+ \), \( t > E_m(0) \).

For \( k \in \mathbb{Z}_+ \), define

\[
L_k = (L_1, L_2, \ldots, L_k) \quad \text{and} \quad \hat{L}_k = (L_{-1}, L_0, L_1, \ldots, L_k).
\]

Explicitly,

\[
\hat{L}_k(t) = (e^t, t, \ln t, \ln \ln t, \ldots, L_k(t)).
\]

The function \( \hat{L}_k \) is used to formulate the results in the previous work [10]. Instead, this paper will use \( \hat{L}_k \), which is more general than \( L_k \).

For \( m \in \mathbb{Z}_+ \), note that \( L_m(t) \) is positive and increasing for \( t > E_m(0) \), \( (3.2) \)

\[
L_m(E_{m+1}(0)) = 1, \quad \lim_{t \to \infty} L_m(t) = \infty. \quad (3.3)
\]

It is also clear that

\[
\lim_{t \to \infty} \frac{L_k(t) \lambda}{L_m(t)} = 0 \quad \text{for all} \quad k > m \geq -1 \quad \text{and} \quad \lambda \in \mathbb{R}. \quad (3.4)
\]

For \( m \in \mathbb{N} \), the derivative of \( L_m(t) \) is

\[
L'_m(t) = \frac{1}{t \prod_{k=1}^{m-1} L_k(t)} = \frac{1}{\prod_{k=0}^{m-1} L_k(t)}. \quad (3.5)
\]

By L'Hopital’s rule and induction, one can verify that

\[
\lim_{t \to \infty} \frac{L_m(T + t)}{L_m(t)} = 1 \quad \text{for any} \quad m \in \mathbb{Z}_+ \quad \text{and} \quad T \in \mathbb{R}. \quad (3.6)
\]

3.3. Classes of functions. Our results will cover a wide range of asymptotic expansions which involve the following types of functions.

**Definition 3.2.** Let \( X \) be a linear space over \( \mathbb{C} \).

(i) Define \( \mathcal{F}_E(X) \) to be the collection of functions \( g : \mathbb{R} \to X \) of the form

\[
g(t) = \sum_{\lambda \in S} p_{\lambda}(t) e^{\lambda t} \quad \text{for} \quad t \in \mathbb{R}, \quad (3.7)
\]

where \( S \) is some finite subset of \( \mathbb{C} \), and each \( p_{\lambda} \) is a polynomial from \( \mathbb{R} \) to \( X \).

(ii) For \( \mu \in \mathbb{R} \), define

\[
\mathcal{F}_E(\mu, X) = \left\{ \text{function} \ g(t) = \sum_{\lambda \in S} p_{\lambda}(t) e^{\lambda t} \in \mathcal{F}_E(X) : \Re \lambda = \mu \quad \text{for all} \quad \lambda \in S \right\}.
\]
Clearly, \( \mathcal{F}_E(X) \) is a linear space over \( \mathbb{C} \) and \( \mathcal{F}_E(\mu, X) \) is a subspace of \( \mathcal{F}_E(X) \).

In particular, when \( \mu = 0 \),

\[
\mathcal{F}_E(0, X) = \left\{ \text{function } g(t) = \sum_{\omega \in \Omega} p_\omega(t) e^{i\omega t} : \ \Omega \text{ is a finite subset of } \mathbb{R}, \right. \\
\left. \text{each } p_\omega \text{ is an } X\text{-valued polynomial on } \mathbb{R} \right\}.
\]

It is clear that \( g \in \mathcal{F}_E(\mu, X) \) if and only if \( g(t) = h(t) e^{\mu t} \) for some \( h \in \mathcal{F}_E(0, X) \).

Also, if \( (X, \| \cdot \|_X) \) is a normed space and \( g \in \mathcal{F}_E(\mu, X) \), then

\[
\| g(t) \|_X = \mathcal{O}(e^{(\mu+\delta)t}) \quad \text{for all } \delta > 0.
\]

Regarding the uniqueness of the representation of \( g(t) \) in (3.7), we have the following lemma.

**Lemma 3.3.** Let \( X \) be a normed space over \( \mathbb{C} \). Suppose two functions \( g \) and \( h \) in \( \mathcal{F}_E(\mu, X) \)

\[
\lim_{t \to \infty} e^{-\mu t} (g(t) - h(t)) = 0,
\]

and are given by

\[
g(t) = \sum_{\lambda \in S} p_\lambda(t) e^{\lambda t} \quad \text{and} \quad h(t) = \sum_{\lambda \in S} q_\lambda(t) e^{\lambda t}, \quad \text{for } t \in \mathbb{R},
\]

where \( S \) is a finite subset of \( \mathbb{C} \), \( \Re \lambda = \mu \) for all \( \lambda \in S \), and \( p_\lambda \)'s and \( q_\lambda \)'s are polynomials from \( \mathbb{R} \) to \( X \).

Then \( p_\lambda = q_\lambda \) for all \( \lambda \in S \).

We give an elementary proof of Lemma 3.3 in Appendix A.

Concerning the assumption (3.11), the two functions \( g \) and \( h \) initially may not have the same set \( S \) in their representations. For example,

\[
g(t) = \sum_{\lambda \in S_g} p_\lambda(t) e^{\lambda t} \quad \text{and} \quad h(t) = \sum_{\lambda \in S_h} q_\lambda(t) e^{\lambda t}, \quad \text{for } t \in \mathbb{R}.
\]

By setting \( S = S_g \cup S_h \), and adding the zero functions to the sums for \( g(t) \) and \( h(t) \) when needed, we can rewrite (3.12) as (3.11).

**Proposition 3.4.** Suppose \( g \in \mathcal{F}_E(X) \) is given by (3.7) for some non-empty set \( S \). Then the polynomials \( p_\lambda \)'s are unique for \( \lambda \in S \). If, in addition, \( p_\lambda \neq 0 \) for all \( \lambda \in S \), then the set \( S \) is unique.

The proof of Proposition 3.4 will be presented in Appendix A.

Next, we consider the power functions of several variables and complex exponents. For

\[
z = (z_{-1}, z_0, z_1, \ldots, z_k) \in (0, \infty)^{k+2} \text{ and } \alpha = (\alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_k) \in \mathbb{C}^{k+2},
\]

define

\[
z^\alpha = \prod_{j=-1}^{k} z_j^{\alpha_j}.
\]
Definition 3.5. For \( \mu \in \mathbb{R}, m, k \in \mathbb{Z} \) with \( k \geq m \geq -1 \), denote by \( \mathcal{E}(m, k, \mu) \) the set of vectors \( \alpha \) in (3.13) such that
\[
\text{Re}(\alpha_j) = 0 \quad \text{for} \ -1 \leq j < m \quad \text{and} \quad \text{Re}(\alpha_m) = \mu.
\]
In particular, when \( m = -1, k \geq -1, \mu = 0 \), the set \( \mathcal{E}(-1, k, 0) \) is the collection of vectors \( \alpha \in \mathbb{C}^{k+2} \) as in (3.13) with \( \text{Re}(\alpha_{-1}) = 0 \).

Let \( k \geq m \geq -1, \mu \in \mathbb{R}, \) and \( \alpha = (\alpha_{-1}, \alpha_0, \ldots, \alpha_k) \in \mathcal{E}(m, k, \mu) \). Consider \( \hat{\mathcal{L}}_k(t)^\alpha \) for \( t > E_k(0) \).

For \( j < m \), the power \( \alpha_j \) is imaginary, hence \( |L_j(t)^{\alpha_j}| \leq 1 \).

For \( j = m \), surely \( |L_m(t)^{\alpha_m}| = L_m(t)^\mu \).

For \( j > m \), one has, thanks to (3.4), \( |L_j(t)^{\alpha_j}| = o(L_m(T_\ast + t)^s) \) for all \( s > 0 \).

Therefore,
\[
\lim_{t \to \infty} \frac{\hat{\mathcal{L}}_k(t)^\alpha}{L_m(t)^{\mu + \delta}} = 0 \quad \text{for any} \ \delta > 0. \tag{3.15}
\]

Definition 3.6. Let \( \mathbb{K} \) be \( \mathbb{C} \) or \( \mathbb{R} \), and \( X \) be a linear space over \( \mathbb{K} \).

(i) For \( k \geq -1 \), define \( \mathcal{P}(k, X) \) to be the set of functions of the form
\[
p(z) = \sum_{\alpha \in S} z^\alpha \xi_\alpha \quad \text{for} \quad z \in (0, \infty)^{k+2}, \tag{3.16}
\]
where \( S \) is some finite subset of \( \mathbb{R}^{k+2} \), and each \( \xi_\alpha \) belongs to \( X \).

(ii) Let \( \mathbb{K} = \mathbb{C}, k \geq m \geq -1 \) and \( \mu \in \mathbb{R} \).

Define \( \mathcal{P}_m(k, \mu, X) \) to be set of functions of the form (3.16), where \( S \) is a finite subset of \( \mathcal{E}(m, k, \mu) \) and each \( \xi_\alpha \) belongs to \( X \).

Define
\[
\mathcal{P}_m(k, \mu, X) = \left\{ p \circ \hat{\mathcal{L}}_k : p \in \mathcal{P}_m(k, \mu, X) \right\}.
\]

Explicitly, if \( f \in \mathcal{P}_m(k, \mu, X) \) then it is a function from \( (E_k(0), \infty) \) to \( X \) and
\[
f(t) = \sum_{\alpha \in S} \hat{\mathcal{L}}_k(t)^\alpha \xi_\alpha \quad \text{for} \quad t > E_k(0),
\]
where \( S \) is a finite subset of \( \mathcal{E}(m, k, \mu) \), and all \( \xi_\alpha \)'s belong to \( X \).

Below are immediate observations about Definition 3.6.

(a) \( \mathcal{P}(k, X) \) contains all polynomials from \( \mathbb{R}^{k+2} \) to \( X \), in the sense that, if \( p : \mathbb{R}^{k+2} \to X \) is a polynomial, then its restriction on \( (0, \infty)^{k+2} \) belongs to \( \mathcal{P}(k, X) \).

(b) Each \( \mathcal{P}(k, X) \) is a linear space over \( \mathbb{K} \).

(c) If \( m > k \geq -1 \), then \( \mathcal{P}(k, X) \) can be embedded into \( \mathcal{P}(m, X) \), see Remark (c) after Definition 2.7 in [10].

(d) One has
\[
q \in \mathcal{P}_m(k, \mu, X) \quad \text{if and only if} \quad q(t) = p(t)L_m(t)^\mu \quad \text{for some} \quad p \in \mathcal{P}_m(k, 0, X). \tag{3.17}
\]

(e) For any \( k \geq m \geq 0 \) and \( \mu \in \mathbb{R} \), one has
\[
\mathcal{P}_m(k, \mu, X) \subset \mathcal{P}_{-1}(k, 0, X). \tag{3.18}
\]

Lemma 3.7. In this lemma \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), and \( X, Y \) and \( X_j \)'s are linear spaces over \( \mathbb{K} \).

The following statements hold true.
(i) If \( p_j \in \mathcal{P}(k, X_j) \) for \( 1 \leq j \leq m \), where \( m \geq 1 \) and \( L \) is an \( m \)-linear mapping from \( \prod_{j=1}^{m} X_j \) to \( X \), then \( L(p_1, p_2, \ldots, p_m) \in \mathcal{P}(k, X) \).

(ii) If \( p \in \mathcal{P}(k, X) \) and \( L : X \to Y \) is a linear mapping, then \( Lp \in \mathcal{P}(k, Y) \).

(iii) If \( p \in \mathcal{P}(k, \mathbb{K}^n) \) and \( 1 \leq j \leq n \), then the canonical projection \( \pi_jp \), that maps \( p \) to its \( j \)-th component, belongs to \( \mathcal{P}(k, \mathbb{K}) \).

(iv) If \( p \in \mathcal{P}(k, \mathbb{K}) \) and \( q \in \mathcal{P}(k, X) \), then the product \( pq \in \mathcal{P}(k, X) \).

Consequently, if \( p_j \in \mathcal{P}(k, \mathbb{K}) \) for \( 1 \leq j \leq m \), then \( p_1p_2 \ldots p_m \in \mathcal{P}(k, \mathbb{K}) \).

(v) If \( p \in \mathcal{P}(k, \mathbb{K}^n) \) and \( q \) is a polynomial from \( \mathbb{K}^n \) to \( X \), then the composition \( q \circ p \) belongs to \( \mathcal{P}(k, X) \).

(vi) In case \( X \) is a normed space, if \( p \in \mathcal{P}(k, X) \), then so is each partial derivative \( \partial p(z)/\partial z_j \), for \( z = (z_1, z_2, \ldots, z_k) \in (0, \infty)^k+2 \) and \( -1 \leq j \leq k \).

Lemma 3.7 is stated and proved in [10, Lemma 2.8] for \( \mathbb{K} = \mathbb{R} \). However, it is equally true for \( \mathbb{K} = \mathbb{C} \) with the same proof.

3.4. Types of asymptotic expansions. First, we give a definition for the asymptotic expansions which have the exponential functions as the main decaying modes.

**Definition 3.8.** Let \( (X, \| \cdot \|_X) \) be a normed space over \( \mathbb{C} \), and \( g \) be a function from \((T, \infty)\) to \( X \) for some \( T \in \mathbb{R} \).

(i) Let \((\gamma_k)_{k=1}^{\infty}\) be a divergent, strictly increasing sequence of nonnegative numbers. We say

\[
g(t) \sim \sum_{k=1}^{\infty} g_k(t), \quad \text{where } g_k \in \mathcal{F}_E(-\gamma_k, X) \quad \text{for } k \in \mathbb{N},
\]

if for any \( N \geq 1 \), there exists \( \mu > \gamma_N \) such that

\[
\left\| g(t) - \sum_{k=1}^{N} g_k(t) \right\|_X = O(e^{-\mu t}).
\]

(ii) Let \( N \in \mathbb{N} \), and \((\gamma_k)_{k=1}^{N}\) be nonnegative and strictly increasing. We say

\[
g(t) \sim \sum_{k=1}^{N} g_k(t), \quad \text{where } g_k \in \mathcal{F}_E(-\gamma_k, X) \quad \text{for } 1 \leq k \leq N,
\]

if

\[
\left\| g(t) - \sum_{k=1}^{N} g_k(t) \right\|_X = O(e^{-\mu t}) \quad \text{for any } \mu > 0.
\]

In case \( X \) is a finite dimensional normed space, all norms on \( X \) are equivalent. Hence, the above definitions of (3.19) and (3.20) are independent of the particular norm \( \| \cdot \|_X \).

Thanks to either [27, Lemma 2.3] or Lemma 3.3 [see also 29, page 195], the asymptotic expansion (3.19) for a function \( g(t) \) is unique.

**Remark 3.9.** The meaning of the asymptotic expansions (3.19) and (3.20) can be seen more clearly when they are stated with the use of the equivalence form in (3.3) for \( g_k(t) \). For example, an equivalence of (3.19) is the following.

We say

\[
g(t) \sim \sum_{k=1}^{\infty} \tilde{g}_k(t)e^{-\gamma_k t}, \quad \text{where } \tilde{g}_k \in \mathcal{F}_E(0, X) \quad \text{for } k \in \mathbb{N},
\]

\[
(3.21)
\]
if for any \( N \geq 1 \), there exists \( \mu > \gamma_N \) such that

\[
\left\| g(t) - \sum_{k=1}^{N} \hat{g}_k(t)e^{-\gamma_k t} \right\|_X = O(e^{-\mu t}). \tag{3.22}
\]

Note that the function \( \hat{g}_k(t) \) in (3.22) does not contribute any extra exponential decay to the decaying mode \( e^{-\gamma_k t} \).

Next, we define the asymptotic expansions in which the power or logarithmic or iterated logarithmic functions are the main decaying modes.

**Definition 3.10.** Let \((X, \| \cdot \|_X)\) be a normed space over \( \mathbb{C} \). Suppose \( g \) is a function from \((T, \infty)\) to \( X \) for some \( T \in \mathbb{R} \), and \( m_* \in \mathbb{Z}_+ \).

(i) Let \((\gamma_k)_{k=1}^{\infty}\) be a divergent, strictly increasing sequence of positive numbers, and \((n_k)_{k=1}^{\infty}\) be a sequence in \( \mathbb{N} \cap [m_*, \infty) \). We say

\[
g(t) \sim \sum_{k=1}^{\infty} g_k(t), \text{ where } g_k \in \mathcal{F}_{m_*}(n_k, -\gamma_k, X) \text{ for } k \in \mathbb{N},
\]

if, for each \( N \in \mathbb{N} \), there is some \( \mu > \gamma_N \) such that

\[
\left\| g(t) - \sum_{k=1}^{N} g_k(t) \right\|_X = O(L_{m_*}(t)^{-\mu}).
\]

(ii) Let \( N \in \mathbb{N} \), \((\gamma_k)_{k=1}^{N}\) be positive and strictly increasing, and \( n_* \in \mathbb{N} \cap [m_*, \infty) \). We say

\[
g(t) \sim \sum_{k=1}^{N} g_k(t), \text{ where } g_k \in \mathcal{F}_{m_*}(n_*, -\gamma_k, X) \text{ for } 1 \leq k \leq N,
\]

if it holds for all \( \mu > 0 \) that

\[
\left\| g(t) - \sum_{k=1}^{N} g_k(t) \right\|_X = O(L_{m_*}(t)^{-\mu}).
\]

Similar to Remark 3.9, by using the equivalence (3.17), we have the following equivalent form of (3.23)

\[
g(t) \sim \sum_{k=1}^{\infty} \hat{g}_k(\hat{L}_{n_k}(t))L_{m_*}(t)^{-\gamma_k}, \text{ where } \hat{g}_k \in \mathcal{F}_{m_*}(n_k, 0, X) \text{ for } k \in \mathbb{N}. \tag{3.25}
\]

For example, when \( m_* = 0 \) the asymptotic expansion (3.25) reads as

\[
g(t) \sim \sum_{k=1}^{\infty} \hat{g}_k(\hat{L}_{n_k}(t))t^{-\gamma_k}, \text{ where } \hat{g}_k \in \mathcal{F}_0(n_k, 0, X) \text{ for } k \in \mathbb{N}.
\]

Same equivalence also applies to (3.24).
4. The main assumptions

Let \( n \in \mathbb{N} \) be fixed throughout the paper. Consider the following system of nonlinear ODEs in \( \mathbb{C}^n \):

\[
y' = -Ay + G(y) + f(t),
\]

where \( A \) is an \( n \times n \) constant matrix of complex numbers, \( G \) is a vector field on \( \mathbb{C}^n \), and \( f \) is a function from \((0, \infty)\) to \( \mathbb{C}^n \).

The following Assumptions 4.1 and 4.2 will be imposed throughout Sections 4–9.

Assumption 4.1. All eigenvalues of the matrix \( A \) have positive real parts.

This assumption is as general as [29], and more general than [30]. It is very often used to prove the asymptotic stability of an equilibrium.

Assumption 4.2. Function \( G : \mathbb{C}^n \to \mathbb{C}^n \) has the following properties.

(i) \( G \) is locally Lipschitz.

(ii) There exist functions \( G_m : \mathbb{C}^n \to \mathbb{C}^n \), for \( m \geq 2 \), each is a homogeneous polynomial of degree \( m \), such that, for any \( N \geq 2 \), there exists \( \delta > 0 \) so that

\[
\left| G(x) - \sum_{m=2}^{N} G_m(x) \right| = O(|x|^{N+\delta}) \text{ as } x \to 0. \tag{4.2}
\]

For the sake of brevity, we write Assumption 4.2(ii) as

\[
G(x) \sim \sum_{m=2}^{\infty} G_m(x) \text{ as } x \to 0. \tag{4.3}
\]

Obviously, any \( C^\infty \)-function \( G \) with \( G(0) = 0 \) and derivative matrix \( DG(0) = 0 \) satisfies Assumption 4.2. However, the function \( G \), in general, is not required to analytic in a neighborhood of the origin.

We examine Assumptions 4.1 and 4.2 more now.

4.1. The linear part. Denote by \( \Lambda_k \), for \( 1 \leq k \leq n \), the eigenvalues of \( A \) counting the multiplicities. The spectrum of \( A \) is

\[
\sigma(A) = \{ \lambda_k : 1 \leq k \leq n \} \subset \mathbb{C}.
\]

Thanks to Assumption 4.1, \( \Re \Lambda_j > 0 \) for all \( j \). We order the set \( \Re \sigma(A) \) by strictly increasing numbers \( \lambda_j \)'s, with \( 1 \leq j \leq d \) for some \( d \leq n \). Of course,

\[
0 < \lambda_1 \leq \Re \Lambda_k \leq \lambda_d \quad \text{for } k = 1, 2, \ldots, n.
\]

It follows that for any \( \varepsilon > 0 \), there exists a positive constant \( c_\varepsilon \) such that

\[
|e^{-tA}| \leq c_\varepsilon e^{-(\lambda_1-\varepsilon)t} \text{ for all } t \geq 0. \tag{4.4}
\]

In particular, there is \( C_0 > 0 \) such that

\[
|e^{-tA}| \leq C_0 e^{-\lambda_1 t/2} \quad \text{for all } t \geq 0. \tag{4.5}
\]
4.2. The nonlinear part. Consider condition (4.3). For each \( m \geq 2 \), there exists an \( m \)-linear mapping \( G_m \) from \((C^n)^m\) to \( C^n \) such that

\[
G_m(x) = G_m(x, x, \ldots, x) \quad \text{for} \quad x \in C^n. \tag{4.6}
\]

By (2.1), one has, for any \( x_1, x_2, \ldots, x_m \in C^n \), that

\[
|G_m(x_1, x_2, \ldots, x_m)| \leq \|G_m\| \cdot |x_1| \cdot |x_2| \cdots |x_m|. \tag{4.7}
\]

In particular,

\[
|G_m(x)| \leq \|G_m\| \cdot |x|^m \quad \text{for all} \quad x \in C^n. \tag{4.8}
\]

It follows (4.2), when \( N = 2 \), and (4.8), for \( m = 2 \), that

\[
|G(x)| = O(|x|^2) \quad \text{as} \quad x \to 0. \tag{4.9}
\]

5. Asymptotic approximations for solutions of the linear system

This section is focused on the linearization of system (4.1) around the origin. The linearized system contains a forcing function which consists of two parts: the coherently decaying part \( f \) and the faster decaying part \( g \). The task is to approximate its solutions by the coherently decaying solutions of the same linear system but with \( f \) only.

5.1. Forcing function with exponential decay. We recall an useful, elementary integral formula. If \( B \) is an invertible \( k \times k \) matrix of complex numbers, and \( p : \mathbb{R} \to C^k \) is a \( C^k \)-valued polynomial, then integration by parts repeatedly yields

\[
\int e^{tB}p(t)dt = \sum_{k=0}^{\text{deg}(p)} (-1)^k e^{tB}B^{-k-1}\frac{d^k p(t)}{dt^k} + C, \tag{5.1}
\]

where \( C \) is any constant vector in \( C^k \).

**Theorem 5.1.** Given \( \mu > 0 \), \( f \in \mathcal{F}_E(-\mu, C^n) \) and a function \( g \in C([T, \infty), C^n) \), for some \( T \geq 0 \), that satisfies

\[
g(t) = O(e^{-(\mu+\delta)t}) \quad \text{for some} \quad \delta > 0. \tag{5.2}
\]

Assume \( y \in C([T, \infty), C^n) \) is a solution of

\[
y'(t) + Ay(t) = f(t) + g(t), \quad \text{for} \quad t > T, \tag{5.3}
\]

and it holds for any \( \lambda \in \text{Re} \sigma(A) \) with \( \lambda < \mu \) and any number \( m \in \mathbb{N} \) that

\[
\lim_{t \to \infty} t^m e^{\lambda t} |y(t)| = 0. \tag{5.4}
\]

Then there exists a function \( z \in \mathcal{F}_E(-\mu, C^n) \) and a number \( \varepsilon > 0 \) such that

\[
z'(t) + Az(t) = f(t) \quad \text{for} \quad t \in \mathbb{R}, \tag{5.5}
\]

and

\[
|y(t) - z(t)| = O(e^{-(\mu+\varepsilon)t}). \tag{5.6}
\]
Proof. Note from equation (5.3) and the stated assumptions that, in fact, \( y \in C^1([T, \infty), \mathbb{C}^n) \).

Observe that the function \( t \mapsto f(T + t) \) belongs to \( \mathcal{F}_E(-\mu, \mathbb{C}^n) \) and the function \( t \mapsto g(T + t) \) is of \( \mathcal{O}(e^{-(\mu+\delta)t}) \). Moreover, as a consequence of (5.4),

\[
\lim_{t \to \infty} p(t)e^{\lambda_j t}|y(t)| = 0 \text{ for any polynomial } p : \mathbb{R} \to \mathbb{R}.
\]  

(5.7)

Let \( m \in \mathbb{N} \) and \( \tau = T + t \). We write

\[
t^m e^{\lambda_j t}|y(T + t)| = e^{-\lambda_j T(\tau - T)}m^\mathcal{E} e^{\lambda_j \tau}|y(\tau)|.
\]  

(5.8)

Thanks to (5.7), the right-hand side of (5.8) goes to zero as \( \tau \to \infty \). Thus,

\[
\lim_{t \to \infty} t^m e^{\lambda_j t}|y(T + t)| = 0.
\]

Therefore, we can translate the time variable and assume that \( T = 0 \).

By projecting equation (5.3) to the invariant subspaces corresponding to the Jordan normal form of the matrix \( A \), we can reduce the problem to the following

\[
y'(t) + By(t) = f(t) + g(t), \quad t > 0,
\]

where \( y(t) \in \mathbb{C}^{n'} \), \( B = B_j \) is a Jordan matrix of size \( n' \times n' \), for some \( n' \leq n \), corresponding to an eigenvalue \( \Lambda_j \) of \( A \), \( f \in \mathcal{F}_E(-\mu, \mathbb{C}^{n'}) \) and \( g \in C([0, \infty), \mathbb{C}^{n'}) \) satisfies (5.2).

Assume

\[
f(t) = \sum_{\Re \lambda = \mu} f_\lambda(t) \text{ with } f_\lambda(t) = p_\lambda(t)e^{-\lambda t},
\]  

(5.9)

where each \( p_\lambda \) is a polynomial from \( \mathbb{R} \) to \( \mathbb{C}^{n'} \). In (5.9) and throughout this proof, the sum \( \sum_{\Re \lambda = \mu} \) is understood to be over finitely many \( \lambda \)'s.

It is known that

\[
e^{-tB} = e^{-\Lambda_j t}p(t),
\]  

(5.10)

where \( p(t) \) is an \( n' \times n' \) matrix-valued polynomial in \( t \in \mathbb{R} \). Consequently, for any \( \varepsilon > 0 \), then, there exist positive constants \( \tilde{C}_\varepsilon \) and \( \tilde{C}_\varepsilon \) such that one has, for all \( t \geq 0 \),

\[
|e^{tB}| \leq \tilde{C}_\varepsilon e^{(\Re \Lambda_j + \varepsilon)t},
\]  

(5.11)

\[
|e^{-tB}| \leq \tilde{C}_\varepsilon e^{-(\Re \Lambda_j - \varepsilon)t}.
\]  

(5.12)

By (5.2) and the continuity of \( g \) on \([0, \infty)\), we can assume that there is a constant \( C_1 > 0 \) such that

\[
|g(t)| \leq C_1 e^{-(\mu+\delta)t} \text{ for all } t \geq 0.
\]  

(5.13)

**Case 1**: \( \Re \Lambda_j > \mu \). For \( t \in \mathbb{R} \), let

\[
z(t) = \int_{-\infty}^{t} e^{-(t-\tau)B}f(\tau)d\tau.
\]  

(5.14)

Then \( z(t) \) solves equation (5.5) with \( A := B \). Clearly, the difference \( u := y - z \) satisfies the linear equation

\[
u'(t) + Bu(t) = g(t), \quad t > 0.
\]

Hence, we have

\[
y(t) - z(t) = u(t) = e^{-tB}u(0) + \int_{0}^{t} e^{-(t-\tau)B}g(\tau)d\tau.
\]  

(5.15)
Fix a number $\varepsilon$ such that
\[ 0 < \varepsilon < \min \left\{ \delta, \frac{\Re \Lambda_j - \mu}{2} \right\}. \]

Note with this choice that $\Re \Lambda_j - \varepsilon > \mu + \varepsilon$. Using (5.12) to estimate the first exponential term on the right-hand side of (5.15) gives
\[ |e^{-tB}u(0)| \leq \tilde{C}_\varepsilon e^{-(\Re \Lambda_j - \varepsilon)t}|u(0)| \leq \tilde{C}_\varepsilon e^{-(\mu + \varepsilon)t}|u(0)|. \] (5.16)

Similarly, thanks to (5.12) and (5.13), the integral in (5.15) is bounded by
\[ \left| \int_0^t e^{-(t-\tau)B}g(\tau)d\tau \right| \leq \tilde{C}_\varepsilon C_1 \int_0^t e^{-(\mu + \varepsilon)(t-\tau)}e^{-(\mu + \delta)\tau}d\tau = \tilde{C}_\varepsilon C_1 \frac{e^{-(\mu + \varepsilon)t}}{\delta - \varepsilon}(1 - e^{-(\delta - \varepsilon)t}). \]

Because $\delta > \varepsilon$, we deduce
\[ \left| \int_0^t e^{-(t-\tau)B}g(\tau)d\tau \right| \leq \frac{\tilde{C}_\varepsilon C_1}{\delta - \varepsilon}e^{-(\mu + \varepsilon)t}. \] (5.17)

Thus, we obtain (5.6) from formula (5.15) and estimates (5.16), (5.17). It remains to prove $z \in \mathcal{F}_E(-\mu)$. We rewrite (5.14) as
\[ z(t) = \sum_{\Re \lambda = \mu} z_\lambda(t), \]
where
\[ z_\lambda(t) = \int_{-\infty}^t e^{-(t-\tau)B}f_\lambda(\tau)d\tau = e^{-tB} \int_{-\infty}^t e^{\tau(B-\lambda I_n')}p_\lambda(\tau)d\tau. \]

Applying formula (5.1), we can compute
\[ z_\lambda(t) = e^{-tB} \left\{ \sum_{k=0}^{\deg(p_\lambda)} (-1)^k e^{t(B-\lambda I_n')} (B - \lambda I_n')^{-k-1} \frac{d^k p_\lambda(t)}{dt^k} \right. \]
\[ \left. - \lim_{s \to -\infty} \sum_{k=0}^{\deg(p_\lambda)} (-1)^k e^{s(B-\lambda I_n')} (B - \lambda I_n')^{-k-1} \frac{d^k p_\lambda(s)}{ds^k} \right\}. \] (5.18)

Because $B - \lambda I_n'$ has the sole eigenvalue $\Lambda_j - \lambda$, and
\[ \Re(\Lambda_j - \lambda) = \Re \Lambda_j - \mu > 0, \]
the norm $|e^{s(B-\lambda I_n')}|$ decays exponentially as $s \to -\infty$. Meanwhile, each $d^k p_\lambda(s)/ds^k$ is a polynomial. Therefore, the limit as $s \to -\infty$ in (5.18) is zero, and, consequently,
\[ z_\lambda(t) = e^{-\lambda t} \sum_{k=0}^{\deg(p_\lambda)} (-1)^k (B - \lambda I_{n'})^{-k-1} \frac{d^k p_\lambda(t)}{dt^k}. \] (5.19)

Thanks to the facts $\Re \lambda = \mu$ and each $d^k p_\lambda(t)/dt^k$ in (5.19) is a $\mathbb{C}^n'$-valued polynomial, one obtains $z_\lambda \in \mathcal{F}_E(-\mu, \mathbb{C}^n')$ for each $\lambda$. Hence, $z \in \mathcal{F}_E(-\mu, \mathbb{C}^n')$.

**Case 2:** $\Re \Lambda_j = \mu$. Denote $y_0 = y(0)$. By the variation of constants formula, we can write solution $y(t)$ as
\[ y(t) = e^{-tB}y_0 + \int_0^t e^{-(t-\tau)B}f(\tau)d\tau + e^{-tB} \int_0^\infty e^{\tau B}g(\tau)d\tau - \int_0^\infty e^{-(t-\tau)B}g(\tau)d\tau. \] (5.20)
Let $\varepsilon = \delta/2$. By using inequalities (5.11) and (5.13), we have

$$|e^{tB}g(t)| \leq \tilde{C}_e C_1 e^{(\mu+\delta/2)t}e^{-(\mu+\delta)t} = \tilde{C}_e C_1 e^{-\delta/2t} \quad \forall t \geq 0.$$  

Thus, by defining $Y_0 = \int_0^\infty e^{tB}g(\tau)d\tau$, one has $Y_0$ is a vector in $\mathbb{C}^n$.

For $t \in \mathbb{R}$, let

$$z(t) = e^{-tB}(y_0 + Y_0) + \int_0^t e^{-(t-\tau)B}f(\tau)d\tau = e^{-tB}\xi + J(t), \quad (5.21)$$

where $\xi = y_0 + Y_0$ and $J(t) = \int_0^t e^{-(t-\tau)B}f(\tau)d\tau$.

We rewrite $y(t)$ from (5.20) as

$$y(t) = z(t) - \int_t^\infty e^{(\tau-t)B}g(\tau)d\tau, \quad (5.22)$$

Clearly, the function $z(t)$ satisfies equation (5.5) with $A := B$. For $t > 0$, one has, thanks to (5.22) and inequalities (5.11), (5.13),

$$|y(t) - z(t)| = \left|\int_t^\infty e^{(\tau-t)B}g(\tau)d\tau\right| \leq \tilde{C}_e C_1 \int_t^\infty e^{(\mu+\delta/2)(\tau-t)}e^{-(\mu+\delta)\tau}d\tau = \frac{2\tilde{C}_e C_1}{\delta}e^{-(\mu+\delta)t}.$$  

Therefore, we obtain estimate (5.6).

We prove $z \in \mathcal{F}_E(-\mu, \mathbb{C}^n)$ now. Thanks to formula (5.10) and the fact $\text{Re } \Lambda_j = \mu$, one has

the term $e^{-tB}\xi$ in (5.21) belongs to $\mathcal{F}_E(-\mu, \mathbb{C}^n)$. \hspace{1cm} (5.23)

It remains to calculate the remaining term $J(t)$ in (5.21). With $f(t)$ given in (5.9), we have

$$J(t) = \sum_{\text{Re } \lambda = \mu} J_\lambda(t), \quad \text{where } J_\lambda(t) = e^{-tB} \int_0^t e^{(B-\lambda I_n^*)\tau} p_\lambda(\tau)d\tau. \quad (5.24)$$

For each $J_\lambda(t)$, we consider the two cases $\lambda \neq \Lambda_j$ and $\lambda = \Lambda_j$ separately.

**Case 2a**: $\lambda \neq \Lambda_j$. Then $B - \lambda I_n^*$ is invertible, and applying the integral formula (5.1) gives

$$J_\lambda(t) = e^{-tB} \left\{ \sum_{k=0}^{\deg(p_\lambda)} (-1)^k e^{t(B-\lambda I_n^*)}(B - \lambda I_n^*)^{-k-1} \frac{d^kp_\lambda(t)}{dt^k} \right. \left. - \sum_{k=0}^{\deg(p_\lambda)} (-1)^k (B - \lambda I_n^*)^{-k-1} \frac{d^kp_\lambda(0)}{dt^k} \right\}. \quad (5.25)$$

It follows that

$$J_\lambda(t) = e^{-\lambda t} \sum_{k=0}^{\deg(p_\lambda)} (-1)^k (B - \lambda I_n^*)^{-k-1} \frac{d^kp_\lambda(t)}{dt^k} \left. - e^{-tB} \sum_{k=0}^{\deg(p_\lambda)} (-1)^k (B - \lambda I_n^*)^{-k-1} \frac{d^kp_\lambda(0)}{dt^k} \right. \quad (5.26)$$
The first term (including the first sum) on the right-hand side of (5.26) is the right-hand side of (5.19), hence, it belongs to $\mathcal{F}_E(-\mu, \mathbb{C}^n)$. Same as (5.23), the second term (including the second sum) on the right-hand side of (5.26) also belongs to $\mathcal{F}_E(-\mu, \mathbb{C}^n)$. Therefore,

$$J_\lambda \in \mathcal{F}_E(-\mu, \mathbb{C}^n).$$

(5.27)

\textbf{Case 2b: $\lambda = \Lambda_j$.} Note that $B - \lambda I_{n'}$ is a Jordan matrix having zero as its only eigenvalue. Then $e^{(B - \lambda I_{n'})\tau}$ is just a matrix-valued polynomial in $\tau$. Hence, $\tilde{J}_\lambda(t) \overset{\text{def}}{=} \int_t^\infty e^{(B - \lambda I_{n'})\tau} p_\lambda(\tau)d\tau$ is a polynomial in $t$. Combining this with equation $J_\lambda(t) = e^{-tB}\tilde{J}_\lambda(t)$ from (5.24), formula (5.10) for $e^{-tB}$, and the fact $\text{Re} \lambda_j = \mu$, we obtain (5.27) again.

By (5.21), (5.23), (5.24), and property (5.27) for both cases 2a and 2b, we obtain $z \in \mathcal{F}_E(-\mu, \mathbb{C}^n)$.

\textbf{Case 3: $\text{Re} \Lambda_j < \mu$.} For any $s, t > T$, we have

$$y(t) = e^{-(t-s)B}y(s) - e^{-tB}\int_t^s e^{tB}(f(\tau) + g(\tau))d\tau. \quad (5.28)$$

Thanks to (5.10) and condition (5.4) applied to $\lambda = \text{Re} \Lambda_j$, which belongs to $\text{Re} \sigma(A)$ and is less than $\mu$, one has $e^{sB}y(s) \to 0$ as $s \to \infty$. Then letting $s \to \infty$ in (5.28) gives

$$y(t) = -e^{-tB}\int_t^\infty e^{\tau B}(f(\tau) + g(\tau))d\tau. \quad (5.29)$$

For $t \in \mathbb{R}$, let

$$z(t) = -e^{-tB}\int_t^\infty e^{\tau B}f(\tau)d\tau. \quad (5.30)$$

It is obvious that $z(t)$ satisfies (5.30) with $A := B$. For the difference between $y(t)$ and $z(t)$, we estimate

$$|y(t) - z(t)| = \left| \int_t^\infty e^{-(t-\tau)B}g(\tau)d\tau \right| \leq \int_t^\infty |e^{(\tau-t)B}||g(\tau)||d\tau. \quad (5.31)$$

Let $\varepsilon = \mu - \text{Re} \Lambda_j > 0$. We apply inequality (5.11) to estimate $|e^{(\tau-t)B}|$ and utilize estimate (5.13) of $|g(\tau)|$. It results in

$$|y(t) - z(t)| \leq \tilde{C}_\varepsilon C_1 \int_t^\infty e^{\mu(\tau-t)}e^{-(\mu+\delta)\tau}d\tau = \frac{\tilde{C}_\varepsilon C_1}{\delta}e^{-(\mu+\delta)t},$$

which proves (5.6).

We compute $z(t)$ now. Again, rewrite (5.29) as $z(t) = \sum_{\text{Re} \lambda = \mu} z_\lambda(t)$, where

$$z_\lambda(t) = -e^{-tB}\int_t^\infty e^{\tau B}f_\lambda(\tau)d\tau = -e^{-tB}\int_t^\infty e^{\tau(B - \lambda I_{n'})}p_\lambda(\tau)d\tau. \quad (5.32)$$

Since $B - \lambda I_{n'}$ is invertible and, similar to (5.18) and (5.25), we have

$$z_\lambda(t) = -e^{-tB}\left\{ \lim_{s \to \infty} \left( \sum_{k=0}^{\deg(p_\lambda)} (-1)^k e^{s(B - \lambda I_{n'})} (B - \lambda I_{n'})^{-k-1} \frac{d^k p_\lambda(s)}{ds^k} \right) \right. \right. \left. \left. - \sum_{k=0}^{\deg(p_\lambda)} (-1)^k e^{t(B - \lambda I_{n'})} (B - \lambda I_{n'})^{-k-1} \frac{d^k p_\lambda(t)}{dt^k} \right\}. \quad (5.33)$$
In this case, \( |e^{s(B-\lambda \alpha)}| \) decays exponentially as \( s \to \infty \), while \( d^k p_\lambda(s)/ds^k \) is a polynomial in \( s \). As a result, the limit as \( s \to \infty \) in (5.30) is zero. Then simplifying the remaining part of \( z(t) \) in (5.30) yields the same formula (5.19) for \( z_\lambda(t) \). This formula, again, implies \( z_\lambda \in \mathcal{F}_E(-\mu, \mathbb{C}^n) \) and, consequently, \( z \in \mathcal{F}_E(-\mu, \mathbb{C}^n) \). 

### 5.2. Forcing function with power or logarithmic or iterated logarithmic decay

The following linear transformations will play crucial roles in our presentation.

**Definition 5.2.** Given an integer \( k \geq -1 \), let \( p \in \mathcal{P}(k, \mathbb{C}^n) \) be given by (3.16) with \( \mathbb{K} = \mathbb{C} \), \( X = \mathbb{C}^n \), and \( z \) and \( \alpha \) as in (3.13).

Define, for \( j = -1, 0, \ldots, k \), the function \( \mathcal{M}_{jp} : (0, \infty)^{k+2} \to \mathbb{C}^n \) by

\[
(\mathcal{M}_{jp})(z) = \sum_{\alpha \in S} \alpha_j z^\alpha \xi_\alpha.
\]

(5.31)

In the case \( k \geq 0 \), define the function \( \mathcal{R}_p : (0, \infty)^{k+2} \to \mathbb{C}^n \) by

\[
(\mathcal{R}_p)(z) = \sum_{j=0}^k z_0^{-1} z_1^{-1} \ldots z_j^{-1} (\mathcal{M}_{jp})(z).
\]

(5.32)

In the case \( p \in \mathcal{P}_-(k, 0, \mathbb{C}^n) \), define the function \( \mathcal{Z}_{AP} : (0, \infty)^{k+2} \to \mathbb{C}^n \) by

\[
(\mathcal{Z}_{AP})(z) = \sum_{\alpha \in S} z^\alpha (A + \alpha_{-1} I_n)^{-1} \xi_\alpha.
\]

(5.33)

In particular,

\[
\mathcal{M}_{-1} p(z) = \sum_{\alpha \in S} \alpha_{-1} z^\alpha \xi_\alpha \quad \text{and} \quad \mathcal{M}_0 p(z) = \sum_{\alpha \in S} \alpha_0 z^\alpha \xi_\alpha.
\]

An equivalent definition of \( (\mathcal{R}_p)(z) \) in (5.32) is

\[
(\mathcal{R}_p)(z) = \frac{\partial p(z)}{\partial z_0} + \sum_{j=1}^k z_0^{-1} z_1^{-1} \ldots z_{j-1}^{-1} \frac{\partial p(z)}{\partial z_j}.
\]

Note in (5.33) that \( \text{Re}(\alpha_{-1}) = 0 \) for all \( \alpha \in S \). The eigenvalues of \( A + \alpha_{-1} I_n \) are \( \Lambda_j + \alpha_{-1} \) for \( 1 \leq j \leq n \). With \( \text{Re}(\Lambda_j + \alpha_{-1}) = \text{Re} \Lambda_j > 0 \), we have \( A + \alpha_{-1} I_n \) is invertible and definition (5.33) is valid.

From (5.33), one has \( \mathcal{Z}_{AP} \in \mathcal{P}_-(k, 0, \mathbb{C}^n) \). Clearly,

\[
(A + \mathcal{M}_{-1})(\mathcal{Z}_{AP}) = \mathcal{Z}_{A}((A + \mathcal{M}_{-1})p) = p \quad \forall p \in \mathcal{P}_-(k, 0, \mathbb{C}^n).
\]

(5.34)

By the mappings \( p \mapsto \mathcal{M}_{jp} \), \( p \mapsto \mathcal{R}_p \), and \( p \mapsto \mathcal{Z}_{AP} \), one can define linear operator \( \mathcal{M}_j \), for \(-1 \leq j \leq k \), on \( \mathcal{P}(k, \mathbb{C}^n) \), linear operator \( \mathcal{R} \) on \( \mathcal{P}(k, \mathbb{C}^n) \) for \( k \geq 0 \), and linear operator \( \mathcal{Z}_A \) on \( \mathcal{P}_-(k, 0, \mathbb{C}^n) \) for \( k \geq -1 \).

The powers \( \alpha \)'s in (5.31) for \( \mathcal{M}_{jp}(z) \), and in (5.33) for \( (\mathcal{Z}_{AP})(z) \) are the same as those that appear in (3.16) for \( p(z) \). Consequently, we have the following properties.

(a) For \( k \geq m \geq 0 \) and \( \mu \in \mathbb{R} \), if \( p \in \mathcal{P}_m(k, \mu, \mathbb{C}^n) \), then all \( \mathcal{M}_{jp} \)'s, for \(-1 \leq j \leq k \), and \( \mathcal{Z}_{AP} \) are also in \( \mathcal{P}_m(k, \mu, \mathbb{C}^n) \). Here, \( \mathcal{Z}_{AP} \) is validly defined thanks to the inclusion (3.18).

(b) \( \mathcal{R}_p(z) \) has the same powers of \( z_{-1} \) as \( p(z) \).

(c) If \( p \in \mathcal{P}_0(k, \mu, \mathbb{C}^n) \), then \( \mathcal{R}_p \in \mathcal{P}_0(k, \mu - 1, \mathbb{C}^n) \).

We recall a useful inequality from [10].
Lemma 5.3 ([10] Lemma 2.5). Let \( m \in \mathbb{Z}_+ \) and \( \lambda > 0, \gamma > 0 \) be given. For any number \( T_0 > E_m(0) \), there exists a number \( C > 0 \) such that

\[
\int_0^t e^{-\gamma(t-\tau)} L_m(T_0 + \tau)^{-\lambda} d\tau \leq CL_m(T_0 + t)^{-\lambda} \quad \text{for all} \quad t \geq 0. \tag{5.35}
\]

The next lemma contains the asymptotic approximations of the integrals that appear in the variation of constants formula for the solutions of many linear ODE systems.

Lemma 5.4. Let \( m \in \mathbb{Z}_+, \mu > 0, N \geq m, p \in \mathcal{P}_m(N, -\mu, \mathbb{C}^n) \) and \( T_0 > E_N(0) \).

If \( m = 0 \), then

\[
\int_0^t e^{-(t-\tau)} p(\hat{L}_N(T_0 + \tau)) d\tau = (\mathcal{Z}_p)(\hat{L}_N(T_0 + t)) + O((T_0 + t)^{-\mu-\gamma}) \tag{5.36}
\]

for any \( \gamma \in (0, 1) \).

If \( m \geq 1 \), then

\[
\int_0^t e^{-(t-\tau)} p(\hat{L}_N(T_0 + \tau)) d\tau = (\mathcal{Z}_p)(\hat{L}_N(T_0 + t)) + O(L_m(T_0 + t)^{-\gamma}) \tag{5.37}
\]

and, consequently,

\[
\int_0^t e^{-(t-\tau)} p(\hat{L}_N(T_0 + \tau)) d\tau = (\mathcal{Z}_p)(\hat{L}_N(T_0 + t)) + O(L_m(T_0 + t)^{-\gamma}) \tag{5.38}
\]

for all \( \gamma > 0 \).

Proof. It suffices to prove (5.36) and (5.37) for \( p(z) = z^\alpha \xi \), with

\[
z = (z_{-1}, z_0, \ldots, z_k) \in (0, \infty)^{k+2}, \quad \alpha = (\alpha_{-1}, \alpha_0, \ldots, \alpha_k) \in \mathcal{E}(m, N, -\mu),
\]

and some \( \xi \in \mathbb{C}^n \).

Since \( m \geq 0 \), we have \( \alpha_{-1} = i\beta \) for some \( \beta \in \mathbb{R} \). By defining \( F(t) = \prod_{j=0}^N L_j(T_0 + t)^{\alpha_j} \), we can write

\[
p(\hat{L}_N(T_0 + t)) = e^{i\beta(T_0 + t)} F(t).
\]

Denote \( I(t) = \int_0^t e^{-(t-\tau)} p(\hat{L}_N(T_0 + \tau)) d\tau \). Then

\[
I(t) = \int_0^t e^{i\beta(T_0 + \tau)} e^{-(t-\tau)} A F(\tau) d\tau = e^{i\beta(T_0 + t)} \int_0^t e^{-(t-\tau)} (A + i\beta I_n) F(\tau) d\tau.
\]

Note that the function \( \tau \mapsto e^{-(t-\tau)} (A + i\beta I_n) \) has an anti-derivative \( (A + i\beta I_n)^{-1} e^{-(t-\tau)} F(\tau) \). Then using integration by parts gives

\[
I(t) = e^{i\beta(T_0 + t)} \left\{ (A + i\beta I_n)^{-1} e^{-(t-\tau)} (A + i\beta I_n) F(\tau) \right|_{\tau=0}^{\tau=t}
\]

\[
- \int_0^t (A + i\beta I_n)^{-1} e^{-(t-\tau)} (A + i\beta I_n) \frac{dF(\tau)}{d\tau} d\tau \right\}
\]

\[
= e^{i\beta(T_0 + t)} (A + i\beta I_n)^{-1} F(t) - e^{i\beta T_0} (A + i\beta I_n)^{-1} e^{-tA} F(0) - J(t),
\]

where

\[
J(t) = \int_0^t e^{i\beta(T_0 + \tau)} (A + i\beta I_n)^{-1} e^{-(t-\tau)} A \frac{dF(\tau)}{d\tau} d\tau.
\]

Hence,

\[
I(t) = (A + i\beta I_n)^{-1} p(\hat{L}_N(T_0 + t)) - (A + i\beta I_n)^{-1} e^{-tA} p(\hat{L}_N(T_0)) - J(t). \tag{5.39}
\]
We consider each term on the right-hand side of (5.39). For the first term, it is obvious that
\[(A + i\beta I_n)^{-1} p(\hat{L}_N(T_* + t)) = (Z_{AP})(\hat{L}_N(T_* + t)).\] (5.40)

For the second term, one has, thanks to (4.5),
\[(A + i\beta I_n)^{-1} e^{-tA} p(\hat{L}_N(T_*)) = O(e^{-\lambda_1 t/2}).\] (5.41)

For the third term, we estimate $J(t)$ by applying inequality (4.5) again to have
\[|J(t)| \leq C_1 \int_0^t e^{-\lambda_2 (t-\tau)/2} \left| \frac{dF(\tau)}{d\tau} \right| d\tau, \text{ where } C_1 = C_0 |(A + i\beta I_n)^{-1}|.\] (5.42)

Taking the derivative of $F(t)$, by using the product rule and (3.5), yields
\[
\frac{dF(t)}{dt} = \alpha_0 (T_* + t)^{-1} \prod_{j=0}^N (L_j(T_* + \tau))^{\alpha_j} \xi \\
+ \sum_{k=1}^N \left[ \alpha_k L_k(T_* + t)^{-1} \prod_{j=0}^N L_j(T_* + \tau)^{\alpha_j} \left( (T_* + t) \prod_{\ell=1}^{k-1} L_\ell(T_* + t) \right)^{-1} \right] \xi \\
= (T_* + t)^{-1} \left\{ \alpha_0 + \sum_{k=1}^N \left[ \alpha_k \prod_{\ell=1}^k (L_\ell(T_* + t))^{-1} \right] \right\} F(t).
\]

Same as (3.15),
\[F(t) = O(L_m(T_* + t)^{-\mu+s}) \text{ for all } s > 0.
\]

Additionally, note, for $\ell \geq 1$, that $L_\ell(T_* + t)^{-1} = O(1)$ as $t \to \infty$. Therefore, it holds, for any $s > 0$, that
\[
\left| \frac{dF(t)}{dt} \right| = O((T_* + t)^{-\mu+s})
\] (5.43).

Consider $m = 0$. Let $\gamma$ be an arbitrary number in $(0, 1)$. Taking $s = 1 - \gamma > 0$ in (5.43) yields
\[
\left| \frac{dF(t)}{dt} \right| = O((T_* + t)^{-\mu-\gamma}).
\]

By the continuity of $dF(t)/dt$ and $(T_* + t)^{-\mu-\gamma}$ on $[0, \infty)$, we deduce
\[
\left| \frac{dF(t)}{dt} \right| \leq C_2 (T_* + t)^{-\mu-\gamma} \text{ for all } t \geq 0,
\] (5.44)

for some positive constant $C_2$. Using (5.44) in (5.42) and then applying Lemma 5.3 to estimate the resulting integral gives
\[
|J(t)| \leq C_1 C_2 \int_0^t e^{-\lambda_2 (t-\tau)/2} (T_* + \tau)^{-\mu-\gamma} d\tau = O((T_* + t)^{-\mu-\gamma}).
\] (5.45)

Combining (5.39), (5.40), (5.41), and (5.45) yields (5.36).
Consider $m \geq 1$. Taking $s = \mu > 0$ in (5.43) yields
\[
\left| \frac{dF(t)}{dt} \right| = O((T_* + t)^{-1}).
\]
We obtain, similar to (5.45), that
\[ J(t) = \mathcal{O}((T_0 + t)^{-1}). \]  
(5.46)

Combining (5.39), (5.40), (5.41) and (5.46) yields (5.37). Then (5.36) for \( m \geq 1 \) follows (5.37).

Finally, inequality (5.38) clearly is a consequence of (5.37).

The main asymptotic approximation result for this subsection is the following.

**Theorem 5.5.** Given integers \( m, k \in \mathbb{Z}_+ \) with \( k \geq m \), and a number \( t_0 > E_k(0) \). Let \( \mu > 0 \), \( p \in \mathcal{P}_m(k, -\mu, \mathbb{C}^n) \), and let function \( g \in C([t_0, \infty), \mathbb{C}^n) \) satisfy
\[ |g(t)| = \mathcal{O}(L_m(t)^{-\alpha}) \text{ for some } \alpha > \mu. \]  
(5.47)

Suppose \( y \in C([t_0, \infty), \mathbb{C}^n) \) is a solution of
\[ y' = -Ay + p(\mathcal{L}_k(t)) + g(t) \text{ on } (t_0, \infty). \]  
(5.48)

Then there exists \( \delta > 0 \) such that
\[ |y(t) - (Z_{Ap})(\mathcal{L}_k(t))| = \mathcal{O}(L_m(t)^{-\mu-\delta}). \]  
(5.49)

**Proof.** By the variation of constant formula,
\[ y(t_0 + t) = e^{-tA}y(t_0) + \int_0^t e^{-(t-\tau)A}p(\mathcal{L}_k(t_0 + \tau))d\tau + \int_0^t e^{-(t-\tau)A}g(t_0 + \tau)d\tau. \]  
(5.50)

For the first term on the right-hand side of (5.50), we have, by (1.5),
\[ e^{-tA}y(t_0) = \mathcal{O}(e^{-\lambda_{1t}/2}) = \mathcal{O}(L_m(t_0 + t)^{-\mu-1}). \]

For the second term on the right-hand side of (5.50), we apply Lemma 5.4 to \( N = k \), \( T_0 = t_0 \), using equation (5.36) with \( \gamma = 1/2 \) when \( m = 0 \), and equation (5.38) with \( \gamma = \mu + 1/2 \) when \( m \geq 1 \). It results in
\[ \int_0^t e^{-(t-\tau)A}p(\mathcal{L}_k(t_0 + \tau))d\tau = (Z_{Ap})(\mathcal{L}_k(t_0 + t)) + \mathcal{O}(L_m(t_0 + t)^{-\mu-1/2}). \]

Consider the third term on the right-hand side of (5.50). By (5.47) and the continuity of \( g(t) \) and \( L_m(t) > 0 \) on \([t_0, \infty)\), there is \( C > 0 \) such that
\[ |g(t)| \leq C L_m(t)^{-\alpha} \text{ for all } t \geq t_0. \]  
(5.51)

Then combining inequalities (1.5), (5.51) with Lemma 5.3 yields
\[ \left| \int_0^t e^{-(t-\tau)A}g(t_0 + \tau)d\tau \right| \leq C_0 C \int_0^t e^{-\lambda_1(t-\tau)/2}L_m(t_0 + t)^{-\alpha}d\tau = \mathcal{O}(L_m(t_0 + t)^{-\alpha}). \]

Let \( \delta = \min\{1/2, \alpha - \mu\} > 0 \). We obtain
\[ y(t_0 + t) = (Z_{Ap})(\mathcal{L}_k(t_0 + t)) + \mathcal{O}(L_m(t_0 + t)^{-\mu-\delta}), \]
which proves (5.49).

Note that \( q = Z_{Ap} \) in (5.49) belongs to \( \mathcal{P}_m(k, -\mu, \mathbb{C}^n) \), the same as \( p \) on the right-hand side of equation (5.48). Estimate (5.49) shows that \( q(\mathcal{L}_k(t)) \) is an asymptotic approximation of \( y(t) \). Such function \( q(\mathcal{L}_k(t)) \) turns out to satisfy a specific ODE as shown in the next lemma.
Lemma 5.6. If \( k \in \mathbb{Z}_+ \) and \( q \in \mathcal{P}(k, \mathbb{C}^n) \), then
\[
\frac{d}{dt} q(\hat{L}_k(t)) = \mathcal{M}_{-1} q(\hat{L}_k(t)) + \mathcal{R} q(\hat{L}_k(t)) \quad \text{for} \quad t > E_k(0). \tag{5.52}
\]

In particular, when \( k \geq m \geq 1, \mu \in \mathbb{R}, \) and \( q \in \mathcal{P}_m(k, \mu, \mathbb{C}^n) \), one has
\[
\frac{d}{dt} q(\hat{L}_k(t)) = \mathcal{M}_{-1} q(\hat{L}_k(t)) + \mathcal{O}(t^{-\gamma}) \quad \text{for all} \quad \gamma \in (0, 1). \tag{5.53}
\]

**Proof.** Because formulas (5.52) and (5.53) depend linearly on \( q \), it suffices to prove them for \( q(z) = z^\alpha \xi \), for a constant vector \( \xi \in \mathbb{C}^n \), where \( z = (z_1, z_0, \ldots, z_k) \) and \( \alpha = (\alpha_1, \alpha_0, \ldots, \alpha_k) \).

By the chain rule and (3.5), one has
\[
\frac{d}{dt} q(\hat{L}_k(t)) = \left. \alpha \xi + t^{-1} \left( \frac{\alpha_0 z^\alpha}{z_0} + \sum_{j=1}^{k} \frac{1}{L_j(t)} \sum_{j_1 \cdots j_{j-1}} \frac{\alpha_{j_1} z_1^{j_1-1} \cdots z_j^{j_j^{-1}}}{z_j} \right) \right|_{z=\hat{L}_k(t)} \xi
\]
\[
= \alpha_{-1} q(\hat{L}_k(t)) + t^{-1} \left( \mathcal{M}_0 q(z) + \sum_{j=1}^{k} \frac{1}{z_1 \cdots z_j} \mathcal{M}_j q(z) \right) \right|_{z=\hat{L}_k(t)}
\]
Therefore, we obtain (5.52).

Consider \( k \geq m \geq 1 \) and \( q \in \mathcal{P}_m(k, \mu, \mathbb{C}^n) \) now. In this case, \( \text{Re} \alpha_{-1} = \text{Re} \alpha_0 = 0 \), which implies that \( \mathcal{M}_j q \), for \( 0 \leq j \leq k \), belongs to \( \mathcal{P}_0(k, 0, \mathbb{C}^n) \). Then, by applying (5.15) to \( m = \mu = 0 \), one has
\[
R(t) \overset{\text{def}}{=} \left( \mathcal{M}_0 q(z) + \sum_{j=1}^{k} \frac{z_1^{-1} z_2 \cdots z_j^{-1}}{z_j} \mathcal{M}_j q(z) \right) \right|_{z=\hat{L}_k(t)} = \mathcal{O}(t^s) \quad \text{for any} \quad s > 0.
\]

Therefore,
\[
\mathcal{R} q(\hat{L}_k(t)) = t^{-1} R(t) = \mathcal{O}(t^{-\gamma}) \quad \text{for all} \quad \gamma \in (0, 1). \tag{5.54}
\]

Then (5.53) follows (5.52) and (5.54). \qed

Note that if \( k = -1 \), then one can define \( \mathcal{R} q = 0 \) and formula (5.52) still holds true.

6. Global existence and asymptotic estimates

In this section, we establish the existence and uniqueness of a solution \( y(t) \) of (4.1) for all \( t \geq 0 \) when the initial data and the forcing function are small. For a decaying solution, we obtain its specific decaying rates corresponding to that of the forcing function.

The following Theorems 6.1 and 6.2 are counterparts of Theorems 3.1 and 3.2 in [10]. However, because of the changed assumption on the matrix \( A \), the proofs presented below are different. They are based on a standard technique using the variation of constants formula instead of differential inequalities. (See also [30, Lemma 3] for an autonomous system.)

**Theorem 6.1.** There are positive numbers \( \varepsilon_0 \) and \( \varepsilon_1 \) such that if \( y_0 \in \mathbb{C}^n \) and \( f \in C([0, \infty)) \) satisfy
\[
|y_0| < \varepsilon_0 \quad \text{and} \quad \|f\|_\infty := \sup\{|f(t)| : t \in [0, \infty)\} < \varepsilon_1, \tag{6.1}
\]
then there exists a unique solution \( y \in C^1([0, \infty)) \) of (4.1) on \([0, \infty)\) with \( y(0) = y_0 \). Moreover,
\[
\limsup_{t \to \infty} |y(t)| \leq \frac{4C_0}{\lambda_1} \limsup_{t \to \infty} |f(t)|. \tag{6.2}
\]
Consequently, if
\[ \lim_{t \to \infty} f(t) = 0, \quad (6.3) \]
then
\[ \lim_{t \to \infty} y(t) = 0. \quad (6.4) \]

**Proof.** Recall that \( C_0 \) is the positive constant in (4.5). We choose
\[ M = \min \left\{ r_*, \frac{\lambda_1}{12C_0c_*} \right\}, \quad \varepsilon_0 = \min \left\{ \frac{M}{2}, \frac{M}{6C_0} \right\}, \quad \varepsilon_1 = \frac{\lambda_1M}{12C_0}. \quad (6.5) \]
Suppose that solution \( y(t) \) exists on \([0, T)\), for some \( T \in (0, \infty] \), and
\[ |y(t)| < M \text{ for all } t \in [0, T). \quad (6.6) \]
For \( t \in [0, T) \), one has
\[ y(t) = e^{-tA}y_0 + \int_0^t e^{-(t-\tau)A}(G(y(\tau)) + f(\tau))d\tau. \quad (6.7) \]
Using (6.7), (4.5) and (4.9), we can estimate
\[ |y(t)| \leq C_0 e^{-\lambda_1 t/2} |y_0| + C_0 \int_0^t e^{-\lambda_1(t-\tau)/2}(c_* |y(\tau)|^2 + |f(\tau)|)d\tau. \quad (6.8) \]
By the assumptions in (6.1), we have
\[ |y(t)| \leq C_0 e^{-\lambda_1 t/2} \varepsilon_0 + C_0 \int_0^t e^{-\lambda_1(t-\tau)/2}(c_* M^2 + \varepsilon_1)d\tau \leq C_0 \varepsilon_0 + \frac{2C_0}{\lambda_1} (c_* M^2 + \varepsilon_1). \quad (6.9) \]
The last upper bounded in (6.9) can be further estimated, thanks to (6.5), by
\[ C_0 \varepsilon_0 + \frac{2C_0 c_* M}{\lambda_1} + 2C_0 \varepsilon_1 \leq \frac{M}{6} + \frac{M}{6} + \frac{M}{2} = M. \]
By the standard contradiction argument with the choice of parameters in (6.5), we obtain
\( T = \infty \), and (6.6) holds true.
We prove (6.2) now. Let \( F = \limsup_{t \to \infty} |f(t)| \) and \( L = \limsup_{t \to \infty} |y(t)| \). By properties (6.1) and (6.6) with \( T \) already being established to be \( \infty \), we have
\[ 0 \leq F \leq \|f\|_\infty \text{ and } 0 \leq L \leq M. \]
By taking the limit superior of (6.8) as \( t \to \infty \) and using [25], Lemma 3.9] to deal with integral, we obtain
\[ L \leq \frac{2C_0}{\lambda_1} \left( c_* \limsup_{t \to \infty} |y(t)|^2 + \limsup_{t \to \infty} |f(t)| \right) = \frac{2C_0}{\lambda_1} (c_* L^2 + F) \]
\[ \leq \frac{2C_0 c_* M}{\lambda_1} \cdot L + \frac{2C_0}{\lambda_1} F. \]
Because of (6.5), one has \( 2C_0 c_* M/\lambda_1 \leq 1/6 < 1/2 \). Hence, it follows that \( L \leq 4C_0 F/\lambda_1 \), which proves (6.2). Finally, under condition (6.3), one has \( F = 0 \) which, by (6.2), deduces (6.4).

Thanks to Theorem (6.1), the set of solutions \( y(t) \) that satisfy (6.4) is not empty. When more information about (6.3) is provided, more specific form of (6.4) will be obtained.
Theorem 6.2. Assume there is $T \geq 0$ such that $f \in C((T, \infty))$. Let $y \in C^1((T, \infty))$ be a solution of (4.1) on $(T, \infty)$ that satisfies
\[
\lim_{t \to \infty} |y(t)| = 0. \tag{6.10}
\]
(i) If there is a number $\alpha \in (0, \lambda_1)$ such that
\[
f(t) = O(e^{-\alpha t}), \tag{6.11}
\]
then
\[
y(t) = O(e^{-\alpha t}). \tag{6.12}
\]
(ii) If there are numbers $m \in \mathbb{Z}_+$ and $\alpha > 0$ such that
\[
f(t) = o(L_m(t)^{-\alpha}), \tag{6.13}
\]
then
\[
y(t) = O(L_m(t)^{-\alpha}). \tag{6.14}
\]
Proof. The proof is similar to that of Theorem 6.1 with some refinements.

Part (i). Take $\varepsilon = (\lambda_1 - \alpha)/2 > 0$. Then
\[
\alpha < \lambda_1 - \varepsilon = \alpha + \varepsilon = \frac{1}{2}(\lambda_1 + \alpha) < \lambda_1.
\]
Let $c_\varepsilon$ be a positive number such that inequality (4.4) holds true. Let
\[
M = \min \left\{ r_*, \frac{\varepsilon}{6c_\varepsilon c_*} \right\}, \quad \varepsilon_0 = \min \left\{ \frac{M}{2}, \frac{M}{6c_\varepsilon} \right\}, \quad \varepsilon_1 = \frac{\varepsilon M}{6c_\varepsilon}. \tag{6.15}
\]
Fix $T_0 > T$. Because of (6.11) and the continuity of $f$ on $[T_0, \infty)$, there is $C > 0$ such that
\[
|f(T_0 + t)| \leq Ce^{-\alpha(T+t)} \quad \text{for any } T_0 \geq T_0 \text{ and } t \geq 0. \tag{6.16}
\]
By the virtue of (6.10), there is a sufficiently large $T_0 \in [T, \infty)$ such that
\[
|y(T_0)| < \varepsilon_0 \text{ and } Ce^{-\alpha T_0} \leq \varepsilon_1.
\]
From this choice of $T_0$ and (6.10), it follows that
\[
|f(T_0 + t)| \leq \varepsilon_1 e^{-\alpha t} \quad \text{for all } t \geq 0. \tag{6.17}
\]
Set $z(t) = y(T_0 + t)$ and $\tilde{f}(t) = f(T_0 + t)$. Then
\[
z'(t) + Az(t) = G(z) + \tilde{f}(t), \text{ for } t \in [0, \infty). \tag{6.18}
\]
By the variation of constants formula, one has, for all $t \geq 0$,
\[
z(t) = e^{-tA}y(T_0) + \int_0^{t} e^{-(t-\tau)A}(G(z(\tau)) + \tilde{f}(\tau))d\tau. \tag{6.19}
\]
Suppose
\[
|z(t)| < Me^{-\alpha t} \quad \text{for all } t \in [0, T'), \text{ where } T' \in (0, \infty]. \tag{6.20}
\]
For $t < T'$, by using (6.19), inequality (4.4), property (4.9), (6.17) and (6.20), one can obtain, similar to (6.8) and (6.9),
\[
|z(t)| \leq c_\varepsilon e^{-(\lambda_1 - \varepsilon)t}|y(T_0)| + \int_0^{t} c_\varepsilon e^{-(\lambda_1 - \varepsilon)(t-\tau)}(c_* M^2 e^{-2\alpha \tau} + \varepsilon_1 e^{-\alpha \tau})d\tau
\]
\[
\leq c_\varepsilon e^{-\alpha t} \varepsilon_0 + c_\varepsilon \int_0^{t} e^{-(\alpha + \varepsilon)(t-\tau)}(c_* M^2 + \varepsilon_1)e^{-\alpha \tau}d\tau.
\]
This and our choice in (6.15) yield
\[ |z(t)| \leq \left( c_\varepsilon c_0 + \frac{c_\varepsilon (c_\varepsilon M^2 + \varepsilon_1)}{\varepsilon} \right) e^{-\alpha t} \leq \left( \frac{M}{6} + \frac{M}{6} + \frac{M}{6} \right) e^{-\alpha t} = \frac{M}{2} e^{-\alpha t}. \]

By a standard contradiction argument, it can be shown that (6.20) holds with \( T' = \infty \). By shifting the time variable, it follows from (6.20) that \( y(t) \) satisfies (6.12).

**Part (ii).** Set \( \psi = L_m \). We choose \( T_\ast > \max\{T, E_{m+1}(0)\} \). Thanks to properties (3.2) and (3.3), the function \( t \mapsto \psi(T_\ast + t) \) is increasing on \([0, \infty)\), and \( \psi(T_\ast + t) \geq 1 \) for all \( t \geq 0 \).

There is a constant \( C_\ast > 0 \) such that
\[ e^{-\lambda t t/2} \leq C_\ast \psi(T_\ast + t)^{-\alpha} \quad \text{for all} \; t \geq 0. \tag{6.21} \]

By inequality (5.35), there is \( C_\ast > 0 \) such that
\[ \int_0^t e^{-\lambda (t-\tau)/2} \psi(T_\ast + \tau)^{-\alpha} d\tau \leq C_\ast \psi(T_\ast + t)^{-\alpha} \quad \text{for all} \; t \geq 0. \tag{6.22} \]

Let \( C_0 > 0 \) be as in (4.5). Let
\[ M = \min \left\{ r_\ast, \frac{\varepsilon}{6C_0C_zc_\ast} \right\}, \quad \varepsilon_0 = \min \left\{ \frac{M}{2}, \frac{M}{6C_0C_z} \right\}, \quad \varepsilon_1 = \frac{\varepsilon M}{6C_0C_\ast}. \tag{6.23} \]

Thanks to conditions (6.10) and (6.13), we can choose \( T' > 0 \) sufficiently large so that
\[ |y(T_\ast + T')| < \varepsilon_0 \] and
\[ |f(T_\ast + T' + t)| \leq \varepsilon_1 \psi(T_\ast + T' + t)^{-\alpha} \quad \text{for all} \; t \geq 0. \tag{6.24} \]

Let \( z(t) = y(T_\ast + T' + t) \) and \( \tilde{f}(t) = f(T_\ast + T' + t) \). Then \( z(t) \) satisfies (6.18) again. Because of (6.24) and the increase of \( \psi(T_\ast + t) \) in \( t \geq 0 \), we have
\[ |\tilde{f}(t)| \leq \varepsilon_1 \psi(T_\ast + t)^{-\alpha} \quad \text{for all} \; t \geq 0. \]

Suppose
\[ |z(t)| \leq M \psi(T_\ast + t)^{-\alpha} \quad \text{for all} \; t \in [0, T'), \] where \( T' \in (0, \infty) \). \hfill (6.25)

Performing the same calculations as in part (i) with the use of (4.5) in place of (4.4), and inequality (6.21), one has, for \( t \in [0, T') \),
\[ |z(t)| \leq |e^{-tA}||y(T_\ast + T')| + \int_0^t |e^{-(\tau-t)A}|(c_\ast |z(\tau)|^2 + |\tilde{f}(\tau)|) d\tau \leq C_0 e^{-\lambda \ast t/2} \varepsilon_0 + C_0 \int_0^t e^{-\lambda \ast (t-\tau)/2} (c_\ast M^2 \psi(T_\ast + \tau)^{-2\alpha} + \varepsilon_1 \psi(T_\ast + \tau)^{-\alpha} - \varepsilon_1) d\tau \leq C_0 C_\ast \psi(T_\ast + t)^{-\alpha} \varepsilon_0 + C_0 \int_0^t e^{-\lambda \ast (t-\tau)/2} (c_\ast M^2 + \varepsilon_1) \psi(T_\ast + \tau)^{-\alpha} d\tau. \]

Applying (6.22) to estimate the last integral, we derive
\[ |z(t)| \leq C_0 (C_\ast \varepsilon_0 + C_\ast c_\ast M^2 + C_\ast \varepsilon_1) \psi(T_\ast + t)^{-\alpha}. \]

Together with (6.23), this inequality gives
\[ |z(t)| \leq \frac{M}{2} \psi(T_\ast + t)^{-\alpha}. \]

Then (6.25) holds true with \( T' = \infty \). By shifting the time and using property (3.6), we obtain (6.14) from (6.25).
Remark 6.3. Following the proof of Theorem 6.2, we, in fact, can replace condition (6.10) with a weaker one, namely,
\[ \lim \inf_{t \to \infty} |y(t)| < \varepsilon_0. \]

Another technical point is that the small oh condition (6.13) is slightly stricter than the corresponding big oh condition (3.13) in [10, Theorem 3.2]. Nonetheless, the result still serves our proofs well in Sections 7–9.

7. CASE OF EXPONENTIAL DECAY

We study the system of nonlinear differential equations (4.1).

Assumption 7.1. There exists a number \( T_f \geq 0 \) such that \( f \) is continuous on \( [T_f, \infty) \).

More specific conditions on \( f \) will be specified later for each result.

Assumption 7.2. There exists a number \( T_0 \geq 0 \) such that \( y \in C^1((T_0, \infty)) \) is a solution of (4.1) on \( (T_0, \infty) \), and \( y(t) \to 0 \) as \( t \to \infty \).

The main assumption on \( f \) for this section is the following.

Assumption 7.3. The function \( f(t) \) admits the asymptotic expansion, in the sense of Definition 3.8 with \( X = \mathbb{C}^n \),
\[ f(t) \sim \sum_{k=1}^{\infty} f_k(t), \text{ where } f_k \in \mathcal{F}_E(-\mu_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N}, \] (7.1)
with \( (\mu_k)_{k=1}^{\infty} \) being a divergent, strictly increasing sequence of positive numbers. Moreover, the set \( S = \{ \mu_k : k \in \mathbb{N} \} \) preserves the addition and contains \( \Re \sigma(A) \).

Note from the last condition of Assumption 7.3 that \( \mu_1 \leq \lambda_1 \).

Under Assumption 7.3, denote \( \bar{f}_N(t) = \sum_{k=1}^{N} f_k(t) \).

Then, according to Definition 3.8 for any \( N \in \mathbb{N} \), one has
\[ |f(t) - \bar{f}_N(t)| = \left| f(t) - \sum_{k=1}^{N} f_k(t) \right| = O(\psi(t)^{-\mu_N - \varepsilon_N}) \text{ for some } \varepsilon_N > 0. \] (7.2)

The following Scenarios 7.4–7.6 are typical cases that Assumption 7.3 holds true. They are of the same nature as Scenarios 4.6–4.8 in [10].

Scenario 7.4. Suppose the forcing function has the following expansion, in the sense of Definition 3.8
\[ f(t) \sim \sum_{k=1}^{\infty} \tilde{f}_k(t), \text{ where } \tilde{f}_k \in \mathcal{F}_E(-\alpha_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N}, \] (7.3)
and \( (\alpha_k)_{k=1}^{\infty} \) is a divergent, strictly increasing sequence of positive numbers.

We define \( S \) to be the additive semigroup generated by the real parts of \( \Lambda_j \)'s and \( \alpha_j \)'s, i.e.,
\[ S = \langle \Re \{ \Lambda_j, \alpha_{j\ell} : 1 \leq j \leq n, \ell \in \mathbb{N} \} \rangle. \] (7.4)
We can re-arrange the set \( S \) as a sequence \((\mu_k)_{k=1}^{\infty}\) which is divergent and strictly increasing. The set \( S \) and sequence \((\mu_k)_{k=1}^{\infty}\) satisfy the properties in Assumption 7.3. Note that \( S \) contains the set \( \{\alpha_k : k \in \mathbb{N}\} \), and, hence, \((\alpha_k)_{k=1}^{\infty}\) is a subsequence of \((\mu_k)_{k=1}^{\infty}\). Then one can formally rewrite the sum in (7.3), after re-indexing \( \tilde{f}_k \)'s, as the sum in (7.1), and verify that expansion (7.1), indeed, holds true in the sense of Definition 3.8.

**Scenario 7.5.** Suppose \( f(t) \) has the finite asymptotic expansion, in the sense of Definition 3.8(ii)

\[
f(t) \sim \sum_{k=1}^{N} \tilde{f}_k(t), \quad \text{where} \quad \tilde{f}_k \in \mathcal{F}_E(-\alpha_k, \mathbb{C}^n) \quad \text{for} \quad 1 \leq k \leq N. \tag{7.5}
\]

Define the set \( S \) by formula (7.4) with 1 \( \leq \ell \leq N \). This set \( S \) is still infinite, and can be arranged as sequence \((\mu_k)_{k=1}^{\infty}\) as in Scenario 7.3. Then, again, we can obtain (7.1) from (7.3).

**Scenario 7.6.** Consider the case when the function \( f \) is zero, or, more generally, decays faster than any exponential functions, that is, \( e^{\alpha t}f(t) \to 0 \) as \( t \to \infty \) for any \( \alpha > 0 \).

Let \( S = (\text{Re} \sigma(A)) \). Again, arrange \( S \) as the sequence \((\mu_k)_{k=1}^{\infty}\). Then we have expansion (7.1) with \( f_k = 0 \) for all \( k \in \mathbb{N} \).

The main result of this section is the following.

**Theorem 7.7.** Under Assumptions 7.1, 7.3 and 7.2, there exist functions

\[
y_k \in \mathcal{F}_E(-\mu_k, \mathbb{C}^n) \quad \text{for} \quad k \in \mathbb{N}, \tag{7.6}
\]

such that the solution \( y(t) \) admits the asymptotic expansion

\[
y(t) \sim \sum_{k=1}^{\infty} y_k(t) \quad \text{in the sense of Definition 3.8} \tag{7.7}
\]

Moreover, for each \( k \in \mathbb{N} \), the functions \( y_k(t) \) solves the following equation

\[
y'_k + Ay_k = \sum_{\mu_1, \mu_2, \ldots, \mu_m = \mu_k} \mathcal{G}_m(y_{j_1}, y_{j_2}, \ldots, y_{j_m}) + f_k, \quad \text{for} \quad t \in \mathbb{R}. \tag{7.8}
\]

The followings are clarifications for equation (7.8).

(a) The indices \( j_1, j_2, \ldots, j_m \) are taken to be in \( \mathbb{N} \). The double summation is zero if there are no indices that satisfy the stated conditions.

(b) When \( k = 1 \), equation (7.8) reads as

\[
y'_1 + Ay_1 = f_1. \tag{7.9}
\]

(c) We recall identity (4.20) of [10], which states, for any numbers \( M \geq \mu_k/\mu_1 \) and \( Z \geq k - 1 \), that

\[
\sum_{2 \leq m \leq M} \sum_{\mu_1, \mu_2, \ldots, \mu_m = \mu_k} \mathcal{G}_m(y_{j_1}, y_{j_2}, \ldots, y_{j_m}) = \sum_{m \geq 2} \mu_1 + \mu_2 + \ldots + \mu_m = \mu_k. \tag{7.10}
\]

One can see from (7.10) that the double summation in (7.8) is a finite sum.

(d) Regarding the double summation in (7.8), note that \( m \geq 2 \) and \( \mu_{j_\ell} > 0 \) for \( \ell = 1, 2, \ldots, m \). Hence, \( \mu_{j_\ell} < \mu_k \) which implies \( j_\ell \leq k - 1 \). Therefore, the terms \( y_{j_\ell} \)’s in the summand come from the previous steps.
(e) Thanks to the previous properties (b) and (d) equation (7.8) is actually a recursive expression.

**Proof of Theorem 7.7.** Set $\psi(t) = e^t$ in this proof.

For $N \in \mathbb{N}$, denote by $(\mathcal{T}_N)$ the statement: There exist functions $y_k \in \mathcal{F}_E(-\mu_k, \mathbb{C}^n)$, for $k = 1, 2, \ldots, N$, such that

equation (7.8) holds true on $\mathbb{R}$ for $k = 1, 2, \ldots, N$, \hspace{1cm} (7.11)

and

$$
|y(t) - \sum_{k=1}^{N} y_k(t)| = \mathcal{O}(\psi(t)^{-\mu_N - \delta_N}) \quad \text{for some} \quad \delta_N > 0. \hspace{1cm} (7.12)
$$

We will prove, by induction, that $(\mathcal{T}_N)$ holds true for all $N \in \mathbb{N}$. In the calculations below, $t$ will be sufficiently large.

**First step.** Let $N = 1$. By the triangle inequality, (7.2) with $N = 1$ and the fact $f_1 \in \mathcal{F}_E(-\mu_1, \mathbb{C}^n)$, one has

$$
|f(t)| \leq |f(t) - f_1(t)| + |f_1(t)| = \mathcal{O}(\psi(t)^{-\mu_1 - \varepsilon_1}) + o(\psi(t)^{-\mu_1 + \delta})
$$

for all $\delta > 0$. This yields

$$
f(t) = o(\psi(t)^{-\mu_1 + \delta}) \quad \forall \delta > 0. \hspace{1cm} (7.13)
$$

Applying Theorem 6.4(1) yields

$$
y(t) = \mathcal{O}(\psi(t)^{-\mu_1 + \delta}) \quad \forall \delta \in (0, \mu_1), \text{ hence,} \quad \forall \delta > 0. \hspace{1cm} (7.14)
$$

Select $\delta < \mu_1$ in (7.14) so that $2(\mu_1 - \delta) > \mu_1 + \varepsilon_1$. Thanks to (7.14), $|y(t)| < r_*$ for $t$ sufficiently large, hence, we can apply inequality (4.9) to $x = y(t)$ and have

$$
G(y(t)) = \mathcal{O}(|y(t)|^2) = \mathcal{O}(\psi(t)^{-2(\mu_1 - \delta)}) = \mathcal{O}(\psi(t)^{-\mu_1 - \varepsilon_1}).
$$

We rewrite equation (1.1) as

$$
y'(t) + Ay(t) = f_1(t) + [G(y(t)) + (f(t) - f_1(t))] = f_1(t) + \mathcal{O}(\psi(t)^{-\mu_1 - \varepsilon_1}). \hspace{1cm} (7.15)
$$

By the virtue of Theorem 5.1 applied to equation (7.15), there exist a function $y_1 \in \mathcal{F}_E(-\mu_1, \mathbb{C}^n)$ and a number $\delta_1 > 0$ such that $y_1(t)$ satisfies (7.9), which is (7.8) for $k = 1$, on $\mathbb{R}$, and

$$
|y(t) - y_1(t)| = \mathcal{O}(\psi(t)^{-\mu_1 - \delta_1}).
$$

Therefore, statement $(\mathcal{T}_1)$ holds true. Notice that we did not check condition (5.4) because $\text{Re}\, \Lambda_j \geq \mu_1$ for all $j$.

**Induction step.** Let $N \geq 1$. Suppose there are function $y_k \in \mathcal{F}_E(-\mu_k, \mathbb{C}^n)$, for $1 \leq k \leq N$, such that (7.11) and (7.12) are true.

Let

$$
u(t) = \sum_{k=1}^{N} y_k(t) \quad \text{and} \quad v_N(t) = y(t) - u_N(t). \hspace{1cm} (7.16)
$$

The assumption (7.12) reads as

$$
v_N(t) = \mathcal{O}(\psi(t)^{-\mu_N - \delta_N}). \hspace{1cm} (7.17)
$$
For $1 \leq k \leq N + 1$, define

$$\mathcal{J}_k(t) = \sum_{m \geq 2} \sum_{\mu_j + \mu_j + \cdots + \mu_{jm} = \mu_k} \mathcal{G}_m(y_{j1}(t), y_{j2}(t), \ldots, y_{jm}(t)).$$

(7.18)

We claim that $v_N(t)$ satisfies

$$v_N'(t) + Av_N(t) = f_{N+1}(t) + \mathcal{J}_{N+1}(t) + \mathcal{O}(\psi(t)^{-\mu_{N+1} - \delta_*}),$$

(7.19)

for some number $\delta_* > 0$.

We accept (7.19) momentarily. By the fact that $y_{j\ell} \in \mathcal{F}_E(-\mu_{j\ell}, \mathbb{C}^n)$ for $\ell = 1, 2, \ldots, m$ and $\mathcal{G}_m$ being multi-linear, it is clear that

$$\mathcal{G}_m(y_{j1}(t), y_{j2}(t), \ldots, y_{jm}(t)) \in \mathcal{F}_E(-\sum_{\ell=1}^{m} \mu_{j\ell}, \mathbb{C}^n).$$

As a consequence, $\mathcal{J}_{N+1}(t) \in \mathcal{F}_E(-\mu_{N+1}, \mathbb{C}^n)$. We apply Theorem 5.1 to equation (7.19) and solution $v_N(t)$.

We check condition (5.4) for $\mu = \mu_{N+1}$. Suppose $\lambda = \text{Re} \Lambda_j < \mu_{N+1}$.

Since $\Lambda_j$ is an eigenvalue of $A$, we have, thanks to the last condition of Assumption 7.3 $\lambda \in \mathcal{S} = \{\mu_k : k \in \mathbb{N}\}$. This fact, together with $\lambda < \mu_{N+1}$ and $(\mu_k)_{k=1}^{\infty}$ being strictly increasing implies that $\lambda \leq \mu_N$. Combining this upper bound of $\lambda$ with the estimate of $v_N(t)$ in (7.17), we see that condition (5.4) is met for $y = v_N$ and any $m \in \mathbb{N}$.

Then, by the virtue of Theorem 5.1 there exist a function $y_{N+1} \in \mathcal{F}_E(-\mu_{N+1}, \mathbb{C}^n)$ and a number $\delta_{N+1} > 0$ such that

$$|v_N(t) - y_{N+1}(t)| = \mathcal{O}(\psi(t)^{-\mu_{N+1} - \delta_{N+1}})$$

and

$$y_{N+1}'(t) + Ay_{N+1}(t) = f_{N+1}(t) + \mathcal{J}_{N+1}(t) \text{ for } t \in \mathbb{R}.$$

Thus, the statement $(\mathcal{T}_{N+1})$ holds true.

**Conclusion of the proof of $(\mathcal{T}_N)$.** By the Induction Principle, the statement $(\mathcal{T}_N)$ is true for all $N \in \mathbb{N}$.

Note, for each $N \in \mathbb{N}$, that the function $y_{N+1}$ is constructed without changing the previous $y_k$ for $1 \leq k \leq N$. Therefore, the functions $y_k$’s exist for all $k \in \mathbb{N}$, and, for any $N \in \mathbb{N}$, (7.11) and (7.12) in $(\mathcal{T}_N)$ hold true. Consequently, we obtain the asymptotic expansion (7.7) with the functions $y_k$’s satisfying (7.8).

It remains to prove equation (7.19). Its proof is divided into three parts (a)–(c).

**Part (a).** Note from (3.9) that

$$|y_k(t)| = \mathcal{O}(\psi(t)^{-\mu_k + \delta}) \quad \forall \delta > 0, \quad k = 1, 2, \ldots, N.$$

(7.20)

Then

$$|u_N(t)| \leq \sum_{k=1}^{N} |y_k(t)| = \mathcal{O}(\psi(t)^{-\mu_1 + \delta}) \quad \forall \delta > 0.$$

(7.21)

Taking derivative of $v_N(t)$ and using (4.11) give

$$v'_N = y' - \sum_{k=1}^{N} y'_k = -Ay + G(y) + f(t) - \sum_{k=1}^{N} y'_k.$$
By writing
\[ Ay = \sum_{k=1}^{N} Ay_k + Av_N \] and \( f(t) = \sum_{k=1}^{N} f_k(t) + f_{N+1}(t) + \mathcal{O}(\psi(t)^{-\mu_{N+1} - \varepsilon_{N+1}}), \)
we have
\[ v'_N = -Av_N + G(y) + f_{N+1}(t) - \sum_{k=1}^{N} (Ay_k + y'_k - f_k(t)) + \mathcal{O}(\psi(t)^{-\mu_{N+1} - \varepsilon_{N+1}}). \quad (7.22) \]

By the induction hypothesis, \( y'_k = -Ay_k + \mathcal{J}_k(t) + f_k(t) \) for \( k = 1, \ldots, N \). Hence, we obtain from (7.22) that
\[ v'_N = -Av_N + f_{N+1} + G(y) - \sum_{k=1}^{N} \mathcal{J}_k(t) + \mathcal{O}(\psi(t)^{-\mu_{N+1} - \varepsilon_{N+1}}). \quad (7.23) \]

**Part (b).** We calculate \( G(y(t)) \). Letting \( \delta = \mu_1/2 \) in \( (7.14) \) yields
\[ y(t) = \mathcal{O}(\psi(t)^{-\mu_1/2}). \quad (7.24) \]

Let integer
\[ M_{N+1} \geq 2\mu_{N+1}/\mu_1. \quad (7.25) \]
Note that \( M_{N+1} \geq 2 \). By \( (4.2) \), there exists \( \theta_N > 0 \) such that
\[ |G(y) - \sum_{m=2}^{M_{N+1}} G_m(y)| = \mathcal{O}(|y|^{M_{N+1} + \theta_N}) \text{ as } y \to 0. \quad (7.26) \]

We calculate and estimate, using \( (7.26) \) and \( (7.24) \),
\[ G(y(t)) = \sum_{m=2}^{M_{N+1}} G_m(y(t)) + \mathcal{O}(|y(t)|^{M_{N+1} + \theta_N}) \]
\[ = \sum_{m=2}^{M_{N+1}} G_m(y(t)) + \mathcal{O}(\psi(t)^{-\mu_{N+1} + \theta_N \mu_1/2}). \]

Thus, thanks to \( (7.25) \),
\[ G(y(t)) = \sum_{m=2}^{M_{N+1}} G_m(y(t)) + \mathcal{O}(\psi(t)^{-\mu_{N+1} - \theta_N \mu_1/2}). \quad (7.27) \]

For each \( G_m(y(t)) \) in \( (7.27) \), we rewrite it, using \( (4.6) \), as
\[ G_m(y(t)) = G_m(u_N + v_N) = G_m((u_N + v_N)^{(m)}). \quad (7.28) \]

By the multi-linearity of \( G_m \) and inequality \( (1.7) \), we have
\[ G_m(y(t)) = G_m(u_N^{(m)}) + G_m(v_N^{(m)}) + \sum_{k=1}^{m-1} \mathcal{O}(|u_N(t)|^k|v_N(t)|^{m-k}) \]
\[ = G_m(u_N^{(m)}) + \mathcal{O}(|v_N(t)|^2) + \mathcal{O}(|u_N(t)||v_N(t)|). \]

The last two terms are estimated, by using \( (7.17) \) and \( (7.21) \) with \( \delta = \delta_N/2 \), by
\[ \mathcal{O}(\psi(t)^{-2(\mu_N + \delta_N)}) + \mathcal{O}(\psi(t)^{-\mu_1 + \delta_N/2 - \mu_N - \delta_N}). \quad (7.29) \]
Because $2\mu_N$ and $\mu_N + \mu_1$ are two numbers in $S$ which are greater than $\mu_N$, they, in fact, are greater or equal to $\mu_{N+1}$. Thus, the quantities in (7.29) can be estimated further by

$$O(\psi(t)^{-\mu_{N+1}-2\delta N}) + O(\psi(t)^{-\mu_{N+1}-\delta N/2}).$$

Therefore, we obtain

$$G_m(y(t)) = G_m((u_N(t))^{(m)}) + O(\psi(t)^{-\mu_{N+1}-\delta N/2}).$$

(7.30)

Summing up (7.30) in $m$ and combining with (7.27), we obtain

$$G(y(t)) = \sum_{m=2}^{M_{N+1}} G_m((u_N(t))^{(m)}) + O(\psi(t)^{-\mu_{N+1}-\delta N/2}) + O(\psi(t)^{-\mu_{N+1}-\theta_N\mu_1/2}).$$

(7.31)

We continue to manipulate

$$\sum_{m=2}^{M_{N+1}} G_m((u_N(t))^{(m)}) = \sum_{m=2}^{M_{N+1}} \sum_{1\leq j_1,j_2,\ldots,j_m \leq N} G_m(y_{j_1}(t), y_{j_2}(t), \ldots, y_{j_m}(t)).$$

(7.32)

Note from (4.7), the fact that each $y_{j_\ell} \in F_B(-\mu_{j_\ell}; \mathbb{C}^n)$ and the estimates in (7.20) that

$$|G_m(y_{j_1}(t), y_{j_2}(t), \ldots, y_{j_m}(t))| \leq \|G_m\| \cdot \prod_{\ell=1}^{m} |y_{j_\ell}(t)| = O(\psi(t)^{-\mu_{j_1}+\mu_{j_2}+\ldots+\mu_{j_m}+\delta}) \quad \forall \delta > 0.$$

(7.33)

Thanks to Assumption 7.3, the set $S$ preserves the addition. Hence, the sum

$$\mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m}$$

belongs to $S$,

and, thus, it must be $\mu_k$ for some $k \geq 1$. Therefore, we can split the sum in (7.32) into three parts:

$$\mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_k \quad \text{for} \quad k \leq N, \quad k = N + 1 \quad \text{and} \quad k \geq N + 2.$$

Corresponding to the last part, i.e., $\mu_k \geq \mu_{N+2}$, taking into account (7.33), the summand in (7.32) is

$$G_m(y_{j_1}(t), y_{j_2}(t), \ldots, y_{j_m}(t)) = O(\psi(t)^{-\mu_{N+2}+\delta}) \quad \forall \delta > 0.$$

Thus, we rewrite (7.32) as

$$\sum_{m=2}^{M_{N+1}} G_m((u_N(t))^{(m)}) = \sum_{k=1}^{N+1} Q_k(t) + O(\psi(t)^{-\mu_{N+2}+\delta}) \quad \forall \delta > 0,$$

(7.34)

where, for $1 \leq k \leq N + 1$,

$$Q_k(t) = \sum_{m=2}^{M_{N+1}} \sum_{1\leq j_1,j_2,\ldots,j_m \leq N \atop \mu_{j_1}+\mu_{j_2}+\ldots+\mu_{j_m}=\mu_k} G_m(y_{j_1}(t), y_{j_2}(t), \ldots, y_{j_m}(t)) \quad \text{for} \quad t \in \mathbb{R}.$$

Therefore, combining (7.31) with (7.34) for $\delta = (\mu_{N+2} - \mu_{N+1})/2$ yields

$$G(y(t)) = \sum_{k=1}^{N+1} Q_k(t) + O(\psi(t)^{-\mu_{N+1}-\delta_{N+1}}),$$

(7.35)

where $\delta_{N+1} = \min\{\delta_N/2, \theta_N \mu_1/2, (\mu_{N+2} - \mu_{N+1})/2\} > 0.$
Part (c). For $1 \leq k \leq N + 1$, we note that $N \geq k - 1$, and, thanks to (7.25), $M_{N+1} > \mu_{N+1}/\mu_1 > \mu_k/\mu_1$. By relation (7.10),

$$
\sum_{m=2}^{M_{N+1}} \sum_{1 \leq j_1, j_2, \ldots, j_m \leq N} \mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \sum_{m \geq 2} \mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_k.
$$

Thus, we have

$$
Q_k = \sum_{m \geq 2} \sum_{\mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_k} G_m(y_{j_1}, y_{j_2}, \ldots, y_{j_m}) = J_k \quad \text{for } 1 \leq k \leq N + 1. \quad (7.36)
$$

By (7.23), (7.35) and (7.36), we have

$$
v_N' + Av_N = f_{N+1}(t) + \sum_{k=1}^{N+1} J_k - \sum_{k=1}^{N} J_k + O(\psi(t)^{-\mu_{N+1}-\varepsilon_{N+1}}) + O(\psi(t)^{-\mu_{N+1}-\delta'_{N+1}}).
$$

Therefore, we obtain the desired equation (7.19) with $\delta_* = \min\{\varepsilon_{N+1}, \delta'_{N+1}\} > 0$, and completes the proof of Theorem 7.7.

Remark 7.8. In the case $G$ has only a finite sum approximation and/or $f(t)$ has a finite sum approximation, the solution $y(t)$ admits a corresponding finite sum approximation, see the treatments in [26, Theorem 2.6], [9, Section 4.1] and [11, Theorem 5.1].

8. Case of power-decay

In this section, we deal with the forcing functions that are power-decaying as time tends to infinity.

Assumption 8.1. The function $f(t)$ admits the asymptotic expansion, in the sense of Definition 3.10 with $X = \mathbb{C}^n$ and $m_* = 0$,

$$
f(t) \sim \sum_{k=1}^{\infty} f_k(t), \quad \text{where } f_k \in \mathcal{F}_0(n_k, -\mu_k, \mathbb{C}^n) \quad \text{for } k \in \mathbb{N}, \quad (8.1)
$$

with $(\mu_k)_{k=1}^{\infty}$ being a divergent, strictly increasing sequence of positive numbers, and $(n_k)_{k=1}^{\infty}$ being an increasing sequence in $\mathbb{Z}_+$. Moreover, the set $S = \{\mu_k : k \in \mathbb{N}\}$ preserves the addition and the unit increment.

Scenario 8.2. Suppose

$$
f(t) \sim \sum_{k=1}^{\infty} \tilde{f}_k(t), \quad \text{where } \tilde{f}_k \in \mathcal{F}_0(\tilde{n}_k, -\alpha_k, \mathbb{C}^n) \quad \text{for } k \in \mathbb{N}, \quad (8.2)
$$

with the sequence $(\alpha_k)_{k=1}^{\infty}$ being the same as in Scenario 7.4 and $(\tilde{n}_k)_{k=1}^{\infty} \subset \mathbb{Z}_+$.

Let $S$ be defined by

$$
S = \left\{ k + \sum_{j=1}^{m} \alpha_{\ell_j} : k \in \mathbb{Z}_+, \ m \in \mathbb{N}, \ \ell_j \in \mathbb{N} \right\}. \quad (8.3)
$$

Then the set $S$ is infinite, preserves the addition and the unit increment. We can arrange $S$ to be a sequence $(\mu_k)_{k=1}^{\infty}$ as in Assumption 8.1 and take

$$
n_k = \max\{\tilde{n}_j : 1 \leq j \leq k\}. \quad (8.4)
$$
Same as in Scenario [7.4] and taking into account the embedding in [c] after Definition [3.6] we obtain the asymptotic expansion (8.1) from (8.2).

Note in (8.3) that, because of the fact \( m \geq 1 \), there is the presence of at least one \( \alpha_{\ell_j} \), hence, \( \mu_1 \geq \alpha_1 \). Then taking \( k = 0 \) and \( m = j = \ell_j = 1 \) gives

\[
\mu_1 = \alpha_1. \tag{8.5}
\]

A counterpart of Scenario [7.5] can also be similarly demonstrated.

**Theorem 8.3.** Under Assumptions [7.1] [8.1] and [7.2], there exist functions

\[
y_k \in \mathcal{R}_0(n_k, -\mu_k, \mathbb{C}^n) \quad \text{for} \quad k \in \mathbb{N},
\]

such that the solution \( y(t) \) admits the asymptotic expansion

\[
y(t) \sim \sum_{k=1}^{\infty} y_k(t) \quad \text{in the sense of Definition} \quad [3.10] \quad \text{with} \quad m_* = 0. \tag{8.7}
\]

More specifically, assume, for all \( k \in \mathbb{N} \),

\[
f_k(t) = p_k(\hat{L}_{n_k}(t)) \quad \text{for some} \quad p_k \in \mathcal{R}_0(n_k, -\mu_k, \mathbb{C}^n). \tag{8.8}
\]

Then the functions \( y_k \)'s in (8.7) can be constructed recursively as follows. For each \( k \in \mathbb{N} \),

\[
y_k(t) = q_k(\hat{L}_{n_k}(t)), \tag{8.9}
\]

where

\[
q_k = \mathcal{Z}_A \left( \sum_{m \geq 2} \sum_{\mu_{j_1}+\mu_{j_2}+\cdots+\mu_{j_m} = \mu_k} \mathcal{G}_m(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k - \chi_k \right), \tag{8.10}
\]

with, recalling \( \mathcal{R} \) defined by (5.32),

\[
\chi_k = \begin{cases} 
\mathcal{R}q_\lambda & \text{if there exists } \lambda \leq k-1 \text{ such that } \mu_\lambda + 1 = \mu_k, \\
0 & \text{otherwise}. \end{cases} \tag{8.11}
\]

The following explanations and remarks are in order.

(a) Certainly, \( \chi_1 = 0 \) in (8.11) and \( q_1 = \mathcal{Z}_A p_1 \).

(b) Same as [c] after Theorem [7.7] equation (8.10) is a recursive formula.

(c) In (8.11), the index \( \lambda \), if exists, is unique. Moreover, if \( q_\lambda \in \mathcal{P}_0(n_\lambda, -\mu_\lambda, \mathbb{C}^n) \), then, by property (c) after Definition [5.2], then the \( \chi_k \) belongs to

\[
\mathcal{P}_0(n_\lambda, -\mu_\lambda - 1, \mathbb{C}^n) = \mathcal{P}_0(n_\lambda, -\mu_k, \mathbb{C}^n) \subset \mathcal{P}_0(n_k, -\mu_k, \mathbb{C}^n). \]

(d) By induction, one can verify, for all \( k \in \mathbb{N} \), that \( q_k \) and \( \chi_k \) defined by (8.10) and (8.11), respectively, belong to \( \mathcal{P}_0(n_k, -\mu_k, \mathbb{C}^n) \), and, hence, \( y_k \) defined by (8.9) belongs to \( \mathcal{P}_0(n_k, -\mu_k, \mathbb{C}^n) \).

**Proof of Theorem 8.3.** We follow the proof of Theorem [7.7] Let \( f_k \) be as in (8.8) and \( y_k \) be as in (8.9), for \( k \in \mathbb{N} \). Let \( m_* = 0 \) and \( \psi(t) = L_{m_*}(t) = t \) for \( t > 0 \).

We will prove by induction that, for any \( N \in \mathbb{N} \),

\[
\left| y(t) - \sum_{k=1}^{N} y_k(t) \right| = \mathcal{O}(\psi(t)^{-\mu_N - \delta_N}) \quad \text{for some} \quad \delta_N > 0. \tag{8.12}
\]
**First step.** Let $N = 1$. We similarly obtain estimate (7.13) for $f(t)$ and then, by Theorem 0.2 ii, estimate (7.14) for $y(t)$. Continuing with the proof of Theorem 7.7 after that, we obtain equation (7.15).

Note that $q_1 = Z_{AP_1}$ and $y_1(t) = q_1(\widehat{L}_{n_1}(t))$. Then $y_1 \in \mathcal{F}_0(-\mu_1, \mathbb{C}^n)$, and, according to Theorem 5.5 applied to $p = p_1$ and $k = n_1$,

$$|y(t) - y_1(t)| = \mathcal{O}(\psi(t)^{-\mu_1 - \delta_1}).$$

Thus, (8.12) is true for $N = 1$.

**Induction step.** Let $N \geq 1$. Suppose that (8.12) holds true. Let $u_N$ and $v_N$ be the same as in (7.16). Then (8.12) reads as $v_N = \mathcal{O}(\psi(t)^{-\mu_k - \delta_k})$.

We, again, obtain equation (7.22).

Let $J_k(t)$ be defined by (7.18) for $k \in \mathbb{N}$. Treating $G(y)$ in the same way as in parts (b) and (c) in the proof of Theorem 7.7 we obtain (7.35) and (7.36). They imply

$$G(y(t)) = \sum_{k=1}^{N+1} J_k(t) + \mathcal{O}(\psi(t)^{-\mu_{N+1} - \delta_{N+1}}).$$

(8.13)

By formula (5.32), it holds, for $k \in \mathbb{N}$, that

$$y'_k = (M_{-1}q_k + Rq_k) \circ \widehat{L}_{n_k} \text{ on } (E_{n_k}(0), \infty).$$

(8.14)

Summing up (8.14) in $k$ gives

$$\sum_{k=1}^{N} y'_k = \sum_{k=1}^{N} M_{-1}q_k \circ \widehat{L}_{n_k} + \sum_{\lambda=1}^{N} Rq_{\lambda} \circ \widehat{L}_{n_{\lambda}} \text{ on } (E_{n_N}(0), \infty).$$

(8.15)

Note that we already made a change of the index notation from $k$ to $\lambda$ for the last sum.

Regarding the last sum in (8.15), we observe that $Rq_{\lambda} \in \mathcal{F}_0(n_{\lambda}, -\mu_{\lambda} - 1, \mathbb{C}^n)$. Thanks to Assumption 8.1, $\mu_{\lambda} + 1 \in \mathcal{S}$. Hence, there exists a unique number $k \in \mathbb{N}$ such that $\mu_k = \mu_{\lambda} + 1$. Because $\mu_k > \mu_{\lambda}$, we have $\lambda \leq k - 1$. Thus, $Rq_{\lambda} = \chi_k$. Consider three possibilities $k \leq N$, $k = N + 1$ and $k \geq N + 2$, we rewrite, similar to (7.34),

$$\sum_{\lambda=1}^{N} Rq_{\lambda} \circ \widehat{L}_{n_{\lambda}} = \sum_{k=1}^{N} \chi_k \circ \widehat{L}_{n_k} + \chi_{N+1} \circ \widehat{L}_{n_{N+1}} + \mathcal{O}(\psi(t)^{-\mu_{N+2} - \delta}) \quad \forall \delta > 0.$$  

(8.16)

Combining (7.22), (8.13) and (8.16) yields

$$v'_N + Av_N = f_{N+1}(t) - \sum_{k=1}^{N} X_k(t) - \chi_{N+1} \circ \widehat{L}_{n_{N+1}}(t) + J_{N+1}(t) + \mathcal{O}(\psi(t)^{-\mu_{N+1} - \delta_*)},$$

(8.17)

for some $\delta_* > 0$, where

$$X_k(t) = (A(q_k + M_{-1}q_k + \chi_k - p_k) \circ \widehat{L}_{n_k}(t) - J_k(t)).$$

For $k \in \mathbb{N}$ and $z \in (0, \infty)^{n_k + 2}$, let

$$Q_k(z) = \sum_{m \geq 2} \mu_1 + \mu_2 \cdots + \mu_{j_m} = \mu_k \sum \mathcal{G}_m(q_{j_m,1}(z), q_{j_m,2}(z), \ldots, q_{j_m,n}(z)).$$

Obviously, $Q_k(\widehat{L}_{n_k}(t)) = J_k(t)$ for all $k \in \mathbb{N}$. By identity (5.34), we can write

$$\chi_k = (A + M_{-1})Z_A \chi_k, \quad p_k = (A + M_{-1})Z_A p_k, \quad J_k = ((A + M_{-1})Z_A Q_k) \circ \widehat{L}_{n_k}.$$
Therefore,
\[ X_k(t) = \left[(A + M_{-1})(q_k + Z_A(\chi_k - p_k - Q_k))\right] \circ \hat{L}_{n_k}(t). \]

For \( 1 \leq k \leq N \), one has from (8.10) that \( q_k = Z_A(Q_k + p_k - \chi_k) \), hence, \( X_k = 0 \). It follows from this fact and equation (8.17) that
\[ v_N' + Av_N = (p_{N+1} - \chi_{N+1} + Q_{N+1}) \circ \hat{L}_{n_{N+1}}(t) + O(\psi(t)^{-\mu_{N+1}-\delta_*}). \] (8.18)

Applying Theorem 5.5 to equation (8.18) yields
\[ |v_N(t) - y_{N+1}(t)| = O(\psi(t)^{-\mu_{N+1}-\delta_{N+1}}), \]
for some number \( \delta_{N+1} > 0 \), where
\[ y_{N+1} = (Z_A(Q_{N+1} + p_{N+1} - \chi_{N+1})) \circ \hat{L}_{n_{N+1}} = q_{N+1} \circ \hat{L}_{n_{N+1}}. \]

Thus, (8.12) is true for \( N := N + 1 \).

Conclusion of the proof of (8.12). By the Induction Principle, statement (8.12) is true for all \( N \in \mathbb{N} \).

Now, the asymptotic expansion (8.7) clearly follows (8.12). The proof of Theorem 8.3 is complete.\[ \square \]

9. Case of logarithmic or iterated logarithmic decay

We obtain the asymptotic expansions when the forcing function has logarithmic or iterated logarithmic decay.

**Assumption 9.1.** There exist a number \( m_* \in \mathbb{N} \), a divergent, strictly increasing sequence \( (\mu_k)_{k=1}^{\infty} \subset (0, \infty) \), and an increasing sequence \( (n_k)_{k=1}^{\infty} \subset \mathbb{N} \cap [m_*, \infty) \) such that the function \( f(t) \) admits the asymptotic expansion, in the sense of Definition 3.10 with \( X = \mathbb{C}^n \),
\[ f(t) \sim \sum_{k=1}^{\infty} f_k(t), \text{ where } f_k \in \mathcal{F}_{m_*}(n_k, -\mu_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N}. \] (9.1)

Moreover, the set \( S \) defined as \( \{ \mu_k : k \in \mathbb{N} \} \) preserves the addition.

**Scenario 9.2.** Assume, similar to (8.2) in Scenario 8.2, the following asymptotic expansion, in the sense of Definition 3.10 with \( X = \mathbb{C}^n \) and \( m_* \in \mathbb{N} \),
\[ f(t) \sim \sum_{k=1}^{\infty} \tilde{f}_k(t), \text{ where } \tilde{f}_k \in \mathcal{F}_{m_*}(\tilde{n}_k, -\alpha_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N}. \] (9.2)

Let \( \mathcal{S} = \{ \alpha_k : k \in \mathbb{N} \} \) and \( n_k \) be as in (8.4). Then, similar to Scenario 8.2, we obtain the asymptotic expansion (9.1) from (9.2). In this case, it is clear that relation (8.5) is true.

**Theorem 9.3.** Under Assumptions 7.1, 9.1 and 7.2, there exist functions
\[ y_k \in \mathcal{F}_{m_*}(n_k, -\mu_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N}, \] (9.3)
such that the solution \( y(t) \) admits the asymptotic expansion
\[ y(t) \sim \sum_{k=1}^{\infty} y_k(t) \text{ in the sense of Definition 3.10.} \] (9.4)
More specifically, suppose
\[ f_k(t) = p_k(\hat{L}_{n_k}(t)) \quad \text{with} \quad p_k \in \mathcal{P}_{m_k}(n_k, -\mu_k, C^n) \quad \text{for all} \ k \in \mathbb{N}. \] (9.5)
Then the functions \( y_k \)'s in (9.4) can be constructed recursively as follows. For each \( k \in \mathbb{N} \),

\[ y_k(t) = q_k(\hat{L}_{n_k}(t)), \]

where
\[ q_k = \mathcal{Z}_A \left( \sum_{m \geq 2} \sum_{\mu_{j_1} + \mu_{j_2} + \cdots + \mu_{j_m} = \mu_k} \mathcal{G}_m(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k \right). \] (9.6)

Proof. We follow the proof of Theorem 8.3. Set \( \psi(t) = L_{m_k}(t) \). By the virtue of (8.33), it holds, for \( t \in (E_{n_k}(0), \infty) \) and \( k \in \mathbb{N} \), that
\[ y_k'(t) = M^{-1}q_k(\hat{L}_{n_k}(t)) + \mathcal{O}(t^{-\gamma}) \quad \forall \gamma \in (0, 1). \] (9.7)

In [8.13], by using (9.7) instead of (8.14), we replace \( \sum_{k=1}^{N} \mathcal{R}q_k \circ \hat{L}_{n_k} \) with \( \mathcal{O}(t^{-\gamma}) \). Because a function of \( \mathcal{O}(t^{-\gamma}) \) is also of \( \mathcal{O}(\psi(t)^{-\mu N+1-\delta}) \) for any \( \delta > 0 \), then after neglecting [8.16], all terms \( \chi_k \)'s for \( 1 \leq k \leq N+1 \) in calculations thereafter can be taken to be 0. The proof goes through. During the process, formula (8.10) has the term \( \chi_k \) dropped, and, hence, becomes (9.6).

10. Results for systems in the real linear spaces

In this section, we focus on the case when the system of ODEs is given in \( \mathbb{R}^n \). Naturally, it is expected that the (real-valued) decaying solutions can be approximated by real-valued functions. However, our constructions in the previous Sections 7.9 rely heavily on calculations with complex numbers. Hence, the approximating functions are not necessarily real-valued. Nonetheless, we will prove that it is, in fact, still true.

We will use the idea of complexification, which we recall below in a brief and convenient form. For more details, see, e.g., [23, section 77].

Let \( X \) be a linear space over \( \mathbb{R} \). Its complexification is \( X_C = X + iX \) with the following natural addition and scalar multiplication. For any \( z = x + iy \) and \( z' = x' + iy' \) in \( X_C \) with \( x, x', y, y' \in X \), and any \( c = a + ib \) in \( \mathbb{C} \) with \( a, b \in \mathbb{R} \), define
\[ z + z' = (x + x') + i(y + y'), \]
\[ cz = (ax - by) + i(bx + ay). \]

Then \( X_C \) is a linear space over \( \mathbb{C} \) and, of course, \( X \subset X_C \).

For \( z = x + iy \in X_C \), with \( x, y \in X \), its conjugate is defined by \( \overline{z} = x - iy = x + i(-y) \). When more explicit notation is needed, we denote this \( \overline{z} \) by \( \overline{z}^{X_C} \). Obviously, \( z + \overline{z} \in X \).

Also, \( z = \overline{\overline{z}} \) if and only if \( z \in X \). One can verify that
\[ c\overline{z} = \overline{cz} \quad \text{for all} \ c \in \mathbb{C}, z \in X_C. \]

Suppose \( (X, \langle \cdot, \cdot \rangle_X) \) is an inner product space over \( \mathbb{R} \). Then \( X_C \) is an inner product space over \( \mathbb{C} \) with the corresponding inner product \( \langle \cdot, \cdot \rangle_{X_C} \) defined by
\[ \langle x + iy, x' + iy' \rangle_{X_C} = \langle x, x' \rangle_X + \langle y, y' \rangle_X + i(\langle y, x' \rangle_X - \langle x, y' \rangle_X) \quad \text{for} \ x, x', y, y' \in X. \]

Denote by \( \| \cdot \|_X \) and \( \| \cdot \|_{X_C} \) the norms on \( X \) and \( X_C \) induced from their respective inner products. Then
\[ \|x + iy\|_{X_C} = (\|x\|_X^2 + \|y\|_X^2)^{1/2} \quad \text{and} \quad \|z\|_{X_C} = \|z\|_{X_C} \quad \text{for all} \ x, y \in X \text{ and } z \in X_C. \]
For any $k \in \mathbb{N}$, the complexification of $X = \mathbb{R}^k$ is $\mathbb{C}^k$ as both a linear space and an inner product space. For $z = (z_1, \ldots, z_k) \in \mathbb{C}^k = X_\mathbb{C}$, $\overline{z}$ is the standard conjugate vector of $z$ and $\|z\|_{X_\mathbb{C}}$ is the standard Euclidean norm $|z| = (|z_1|^2 + \ldots + |z_k|^2)^{1/2}$.

Let $S$ be a subset of $\mathbb{C}^k$ with $k \in \mathbb{N}$. We say $S$ preserves the conjugation if the conjugate $\overline{z}$ of any $z \in S$ also belongs to $S$. Define the conjugate set of $S$ to be $\overline{S} = \{\overline{\lambda} : \lambda \in S\}$.

Because we deal with real-valued solutions and forcing functions now, we need to restrict the classes of functions in Definition 3.2.

**Definition 10.1.** Let $X$ be a linear space over $\mathbb{R}$, and $X_\mathbb{C}$ be its complexification. Define

$$
\mathcal{F}_E(X_\mathbb{C}, X) = \{g \in \mathcal{F}_E(X_\mathbb{C}) : g(t) \in X \text{ for all } t \in \mathbb{R}\}.
$$

For $\mu \in \mathbb{R}$, define

$$
\mathcal{F}_E(\mu, X_\mathbb{C}, X) = \mathcal{F}_E(X_\mathbb{C}, X) \cap \mathcal{F}_E(\mu, X_\mathbb{C}).
$$

The next lemma provides the characteristics for the representation (3.7) and the asymptotic expansions of the functions in $\mathcal{F}_E(X_\mathbb{C}, X)$.

**Lemma 10.2.** Let $X$ be a linear space over $\mathbb{R}$, and $X_\mathbb{C}$ be its complexification. The following statements hold true.

(i) Assume $g \in \mathcal{F}_E(X_\mathbb{C})$. Then $g(t) \in X$ for all $t \in \mathbb{R}$ if and only if

$$
g(t) = \sum_{\lambda \in S} g_\lambda(t) \text{ with } g_\lambda(t) = p_\lambda(t)e^{\lambda t}, \tag{10.1}
$$

where $S$ is a finite subset of $\mathbb{C}$ that preserves the conjugation, $p_\lambda$’s are polynomials from $\mathbb{R}$ to $X_\mathbb{C}$, and

$$
g_\lambda = \overline{p_\lambda} \text{ for all } \lambda \in S. \tag{10.2}
$$

(ii) Assume, in addition, that $X$ is an inner product space over $\mathbb{R}$. Let $g : (T, \infty) \to X$ for some $T \in \mathbb{R}$. Suppose $g$, as a $X_\mathbb{C}$-valued function, has an asymptotic expansion, in the sense of Definition 3.8

$$
g(t) \sim \sum_{k=1}^{\infty} g_k(t), \quad g_k \in \mathcal{F}_E(-\mu_k, X_\mathbb{C}). \tag{10.3}
$$

Then $g_k(t) \in X$ for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$.

**Proof.** We prove part (i) first. The sufficient condition is obvious thanks to the facts that $S$ preserves the conjugation and, by (10.2),

$$
g_\lambda(t) + g_\overline{\lambda}(t) = g_\lambda(t) + \overline{g_\lambda(t)} \in X \text{ for all } \lambda \in S, t \in \mathbb{R}.
$$

We prove the necessary condition now. Consider a function $g$ in $\mathcal{F}_E(X_\mathbb{C})$ given by (10.1) with a finite set $S$ first. By replacing $S$ with the union $S \cup \overline{S}$ as well as adding the zero functions if needed, we can assume (10.1) with $S$ preserving the conjugation. Then

$$
\overline{g} = \sum_{\lambda \in S} \overline{g_\lambda}, \quad \text{with } \overline{g_\lambda(t)} = \overline{p_\lambda(t)}e^{\lambda t}.
$$

Because $g$ is $X$-valued, we have

$$
g = \overline{g} = \sum_{\lambda \in S} \overline{g_\lambda}. \tag{10.4}
$$
By the uniqueness in Proposition 7.4, the two functions corresponding to $e^{\lambda t}$ in (10.1) and (10.4) must be the same, i.e., $g_{\lambda} = \overline{g_{\lambda}}$ for all $\lambda \in S$. Hence, we obtain (10.2).

We prove part (ii) now. Clearly, the conjugate function $\overline{g}$ has the following asymptotic expansion

$$
\overline{g}(t) \sim \sum_{k=1}^{\infty} \overline{g_k}(t),
$$

where $\overline{g_k}$ is the conjugate function of $g_k$, and, obviously, $\overline{g_k} \in \mathcal{F}_E(-\mu_k, X_C)$ for all $k \in \mathbb{N}$.

Because $g$ is $X$-valued, we have $g = \overline{g}$, thus, it follows (10.5) that

$$
g(t) \sim \sum_{k=1}^{\infty} \overline{g_k}(t), \quad \overline{g_k} \in \mathcal{F}_E(-\mu_k, X_C).
$$

By the uniqueness of the asymptotic expansion of $g$, see the second remark after Definition 3.8, we deduce from (10.3) and (10.6) that $g_k(t) = \overline{g_k}(t)$ for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Hence, $g_k(t) \in X$ for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$. \hfill \Box

Next, we have the complexification of multi-linear mappings. For the sake of simplicity, we consider only the particular space $\mathbb{R}^n$.

**Definition 10.3.** Let $\{e_j : 1 \leq j \leq n\}$ be the canonical basis for $\mathbb{R}^n$ and $\mathbb{C}^n$. Let $M$ be an $m$-linear mapping from $(\mathbb{R}^n)^m$ to $\mathbb{R}^n$. The complexification of $M$ is $M_C : (\mathbb{C}^n)^m \to \mathbb{C}^n$ defined by

$$
M_C \left( \sum_{j_1=1}^{n} z_{1,j_1} e_{j_1}, \ldots, \sum_{j_m=1}^{n} z_{m,j_m} e_{j_m} \right)
= \sum_{j_1,j_2,\ldots,j_m=1}^{n} z_{1,j_1} z_{2,j_2} \ldots z_{m,j_m} M(e_{j_1}, e_{j_2}, \ldots, e_{j_m})
$$

(10.7)

for all $z_{k,j_k} \in \mathbb{C}$, $k = 1, 2, \ldots, m$ and $j_k = 1, 2, \ldots, n$.

Then $M_C$ is the unique $m$-linear mapping (over $\mathbb{C}$) from $(\mathbb{C}^n)^m$ to $\mathbb{C}^n$ that extends $M$. Because $M(e_{j_1}, e_{j_2}, \ldots, e_{j_m})$ in (10.7) are real-valued, one has

$$
M_C(z_1, z_2, \ldots, z_m) = M_C(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_m) \quad \forall z_1, z_2, \ldots, z_m \in \mathbb{C}^n.
$$

(10.8)

We now impose the main assumptions for system (4.1) in $\mathbb{R}^n$.

**Assumption 10.4.** The following are assumed to hold in this section.

(i) The matrix $A$ is an $n \times n$ matrix of real numbers that satisfies Assumption 4.4.

(ii) The function $G$ is from $\mathbb{R}^n$ to $\mathbb{R}^n$, satisfies Assumption 4.2 with $G_m$’s being functions from $\mathbb{R}^n$ to $\mathbb{R}^n$.

(iii) The multi-linear mappings $G_m$’s in (4.6) are from $(\mathbb{R}^n)^m$ to $\mathbb{R}^n$.

(iv) The forcing function $f(t)$ and solution $y(t)$ are $\mathbb{R}^n$-valued and satisfy Assumptions 7.1 and 7.2.

In the following proofs, for the $m$-linear mapping $G_m : (\mathbb{R}^n)^m \to \mathbb{R}^n$ given in Assumption 10.4 let $G_{m,C} : (\mathbb{C}^n)^m \to \mathbb{C}^n$ be its complexification.

**Theorem 10.5.** Theorem 7.1 holds true when we replace $\mathcal{F}_E(-\mu_k, \mathbb{C}^n)$ in (7.1) and (7.6) with $\mathcal{F}_E(-\mu_k, \mathbb{C}^n, \mathbb{R}^n)$. 
Proof. We adjust the proof of Theorem 7.7. First, we replace $T_{11}$ in the statement $(T_N)$ by the following

$$y_k' + Ay_k = \sum_{m \geq 2} \sum_{\mu_j, \mu_j+\mu_j+...=\mu_k} G_{m,C}(y_{j_1}, y_{j_2}, \ldots, y_{j_m}) + f_k,$$

for $t \in \mathbb{R}$ and for $k = 1, 2, \ldots, N$.

In identity $(7.28)$, we can equate the last term with $G_{m,C}((u_N + v_N)^m)$. Then proceed the proof afterward with $G_{m,C}$ replacing $G_m$. At the end of that proof, we obtain the asymptotic expansion $(7.7)$ for the $\mathbb{R}$-valued solution $y(t)$ with each $y_k \in F_E(-\mu_k, \mathbb{C}^n)$ satisfying $(10.9)$. By Lemma 10.2, each $y_k$ belongs to $F_E(-\mu_k, \mathbb{C}^n, \mathbb{R}^n)$. Because $y_k(t)$ are real-valued now, the term $G_{m,C}$ in $(10.9)$ is equal to $G_m$, and, hence, we obtain $(7.8)$. \hfill \square

It is worth pointing out that we do not complexify equation $(1.1)$ in the above proof of Theorem 10.3. In fact, we do not have sufficient information about $G(y)$ to complexify it. We only approximate $G(y)$ by the finite sum $\sum_{m=2}^{M_N+1} G_m(y^{(m)})$, then use the complexified mapping $G_{m,C}$ to replace $G_m$ and continue the computations.

In dealing with power, logarithmic and iterated logarithmic functions valued in real linear spaces, we have the following counterpart of Definition 3.6.

**Definition 10.6.** $X$ be a linear space over $\mathbb{R}$, and $X_C$ be its complexification.

- Define $P(k, X_C, X)$ to be set of functions of the form
  $$p(z) = \sum_{\alpha \in S} z^\alpha \xi_\alpha$$
  for $z \in (0, \infty)^{k+2}$,
  \hspace{1cm} (10.10)

  where $S$ is a finite subset of $\mathbb{C}^{k+2}$ that preserves the conjugation, and each $\xi_\alpha$ belongs to $X_C$, with
  $$\xi_\bar{\alpha} = \overline{\xi_\alpha} \ \forall \alpha \in S.$$

- Define $P_m(k, \mu, X_C, X)$ to be set of functions in $P(k, X_C, X)$ with the restriction that the set $S$ in $(10.10)$ is also a subset of $\mathcal{E}(m, k, \mu)$.

- For $k \geq m \geq -1$ and $\mu \in \mathbb{R}$, define
  $$\mathfrak{P}_m(k, \mu, X_C, X) = \{ p \circ \hat{L}_k : p \in P_m(k, \mu, X_C, X) \}.$$

Note, for any $t > 0$ and $z \in \mathbb{C}$, that $t^z = \overline{t^\bar{z}}$. Then, referring to $(3.14)$, we have

$$z^\bar{\alpha} = \overline{z^\alpha} \text{ for all } z \in (0, \infty)^{k+2}, \ \alpha \in \mathbb{C}^{k+2}.$$

Because of $(10.12)$ and the conjugation condition $(10.11)$, each function $p$ in the class $P(k, X_C, X)$ is, in fact, $X$-valued.

If we rewrite $(10.10)$ as

$$p(z) = \sum_{\alpha \in S} p_\alpha(z), \ \text{where } p_\alpha(z) = z^\alpha \xi_\alpha,$$

then condition $(10.11)$ is equivalent to

$$p_\alpha(z) = \overline{p_\alpha(z)} \text{ for all } z \in (0, \infty)^{k+2}, \ \alpha \in S.$$

We remark that the classes $P(k, X_C, X)$ and $P_m(k, \mu, X_C, X)$ are (additive) subgroups of $P(k, X_C)$, but not linear spaces over $\mathbb{C}$.

We examine the restrictions of the mappings $M_j$’s, $\mathcal{R}$ and $\mathcal{Z}_A$ on the new classes in Definition 10.6.
Lemma 10.7. The following statements hold true.

(i) Each $M_j$, for $-1 \leq j \leq k$, maps $P(k, \mathbb{C}^n, \mathbb{R}^n)$ into itself, $R$ maps $P(k, \mathbb{C}^n, \mathbb{R}^n)$, for $k \geq 0$, into itself, and $Z_A$ maps $P_{-1}(k, 0, \mathbb{C}^n, \mathbb{R}^n)$, for $k \geq -1$, into itself.

(ii) All $M_j$'s, for $-1 \leq j \leq k$, and $Z_A$ map $P_m(k, \mu, \mathbb{C}^n, \mathbb{R}^n)$ into itself for any integers $k \geq m \geq 0$ and real number $\mu$.

(iii) $R$ maps $P_0(k, \mu, \mathbb{C}^n, \mathbb{R}^n)$ into $P_0(k, \mu - 1, \mathbb{C}^n, \mathbb{R}^n)$ for any $k \in \mathbb{Z}_+$ and $\mu \in \mathbb{R}$.

Proof. We prove part (i), while parts (ii) and (iii) are consequences of (i) and properties (a) (c) stated after Definition 5.2.

We look into $M_j$ first. Let $p \in P(k, \mathbb{C}^n, \mathbb{R}^n)$. Assume $p(z)$ is given by (10.10) with a finite set $S$ in $\mathbb{C}^{k+2}$ preserving the conjugation, and $\xi_\alpha \in \mathbb{C}^n$, $\xi_\alpha = \overline{\xi_\alpha}$ for all $\alpha \in S$. Write

$$M_j p(z) = \sum_{\alpha \in S} f_\alpha(z),$$

where

$$f_\alpha(z) = \alpha_j z^\alpha \xi_\alpha = \frac{\alpha_j}{\overline{z}^\alpha} \overline{\xi_\alpha} = \overline{f_\alpha(z)},$$

we obtain, thanks to the equivalent criterion (10.13), $M_j p \in P(k, \mathbb{C}^n, \mathbb{R}^n)$.

We now check with $R$ defined by (5.32) for $k \geq 0$. We write $R p = \sum_{j=0}^k N_j p$, where $N_j$, for each $0 \leq j \leq k$, is the linear transformation on $P(k, \mathbb{C}^n)$ defined by

$$(N_j p)(z) = z_0^{-1} z_1^{-1} \ldots z_j^{-1} M_j p(z)$$

for $p \in P(k, \mathbb{C}^n)$. For $p \in P(k, \mathbb{C}^n, \mathbb{R}^n)$, we write $p$ as the finite sum $p = \sum_{\alpha} p_\alpha$, where each $p_\alpha$ belongs to $\mathbb{P}(k, \mathbb{C}^n, \mathbb{R}^n)$ and is of the form $p_\alpha(z) = z^\alpha \xi_\alpha + z^{\bar{\alpha}} \overline{\xi_\alpha}$. We have

$$N_j p_\alpha(z) = z_0^{-1} z_1^{-1} \ldots z_j^{-1} (\alpha_j z^\alpha \xi_\alpha + \overline{\alpha_j z^\alpha \xi_\alpha}) = g(z) + h(z),$$

where

$$g(z) = \alpha_j z^\beta \xi_\alpha \quad \text{with} \quad \beta = \alpha - (e_0 + e_1 + \ldots + e_j),$$

$$h(z) = \overline{\alpha_j z^{\bar{\gamma}} \overline{\xi_\alpha}} \quad \text{with} \quad \gamma = \bar{\alpha} - (e_0 + e_1 + \ldots + e_j).$$

Above, $\{e_1, e_0, \ldots, e_k\}$ is the canonical basis of $\mathbb{C}^{k+2}$.

Note that $\gamma = \beta$ and $h(z) = \overline{\alpha_j z^{\bar{\beta}} \overline{\xi_\alpha}} = \overline{g(z)}$. Thus, by criterion (10.13), $N_j p_\alpha \in \mathbb{P}(k, \mathbb{C}^n, \mathbb{R}^n)$.

Then summing up over finitely many $\alpha$'s yields $N_j p \in \mathbb{P}(k, \mathbb{C}^n, \mathbb{R}^n)$. Finally, thanks to the group property of $\mathbb{P}(k, \mathbb{C}^n, \mathbb{R}^n)$, summing up in $j$ gives $R p = \sum_{j=0}^k N_j p \in P(k, \mathbb{C}^n, \mathbb{R}^n)$.

Consider formula (5.33) of $Z_A$ with $p \in P_{-1}(k, 0, \mathbb{C}^n, \mathbb{R}^n)$, for $k \geq -1$, given by (10.10) and (10.11) and $\alpha \in E(-1, k, 0)$ for any $\alpha \in S$. We write

$$Z_A p(z) = \sum_{\alpha \in S} f_\alpha(z),$$

where

$$f_\alpha(z) = z^\alpha (A + \alpha_{-1} I_n)^{-1} \xi_\alpha.$$  \hfill (10.14)

Noticing that $A = \bar{A}$, one has

$$f_\alpha(z) = z^\alpha (A + \alpha_{-1} I_n)^{-1} \xi_\alpha = \overline{z^\alpha (A + \alpha_{-1} I_n)^{-1} \overline{\xi_\alpha}} = \overline{f_\alpha(z)}.$$

Together with formula (10.14) and criterion (10.13), this implies $Z_A p \in P_{-1}(k, 0, \mathbb{C}^n, \mathbb{R}^n)$.

\hfill \Box

Theorem 10.8. We have the following counterparts of Theorems 8.3 and 9.3.
(i) Theorem 8.3 holds true when we replace $P_0(n_k, -\mu_k, \mathbb{C}^n)$ with $P_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n)$ in (8.1) and (8.6), and replace $P_0(n_k, -\mu_k, \mathbb{C}^n)$ with $P_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n)$ in (8.8).

(ii) Theorem 9.3 holds true when we replace $P_0(n_k, -\mu_k, \mathbb{C}^n)$ with $P_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n)$ in (9.11) and (9.3), and replace $P_0(n_k, -\mu_k, \mathbb{C}^n)$ with $P_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n)$ in (9.5).

Proof. We prove part (i). Assume (8.8) with $p_k \in P_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n)$ for all $k \in \mathbb{N}$. By combining the proof of Theorem 8.3 with the argument in the proof of Theorem 10.5, we obtain asymptotic expansion (8.7) with (8.9), where

$$q_k = Z_A \left( \sum_{m \geq 2} \sum_{\mu_j + \mu_j + \ldots + \mu_{jm} = \mu_k} G_{m, C}(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k - \chi_k \right),$$

(10.15)

for $\chi_k$ being given by (8.11). That is, $G_{m, C}$ in (10.15) replaces $G_m$ in (8.10).

We examine the construction of $q_k$’s in (10.15). By property (a) after Theorem 8.3, $q_1 = Z_A \mathbb{P}_1$. Because $p_1 \in P_0(n_1, -\mu_1, \mathbb{C}^n, \mathbb{R}^n)$, then by applying Lemma 10.7 we have $q_1 \in P_0(n_1, -\mu_1, \mathbb{C}^n, \mathbb{R}^n)$.

Let $k \geq 2$. Suppose $q_j \in P_0(n_j, -\mu_j, \mathbb{C}^n, \mathbb{R}^n)$ for $1 \leq j \leq k - 1$.

In (10.15), we already know $p_k \in P_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n)$.

By Lemma 10.7 $\mathbb{P}_k$ in (8.11) belongs to $P_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n)$. Therefore, $\chi_k$ in (10.15) belongs to $P_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n)$.

Consider each $G_{m, C}(q_{j_1}, q_{j_2}, \ldots, q_{j_m})$ in (10.15). By the distributive property of $G_{m, C}$ with respect to the addition and the form (10.10) of each $q_{j_k}$, it suffices to examine the following pairs

$$g(z) = G_{m, C}(\zeta^{(1)} \xi, \zeta^{(2)} \xi, \ldots, \zeta^{(m)} \xi),$$

and

$$h(z) = G_{m, C}(\zeta^{m(1)} \xi, \zeta^{m(2)} \xi, \ldots, \zeta^{m(m)} \xi).$$

We calculate

$$g(z) = z^{\beta} G_{m, C}(\xi^{(1)} \zeta, \xi^{(2)} \zeta, \ldots, \xi^{(m)} \zeta),$$

where $\beta = \alpha^{(1)} + \alpha^{(2)} + \ldots + \alpha^{(m)}$, and

$$h(z) = z^{\gamma} G_{m, C}(\xi^{(1)} \zeta, \xi^{(2)} \zeta, \ldots, \xi^{(m)} \zeta),$$

where $\gamma = \alpha^{(1)} + \alpha^{(2)} + \ldots + \alpha^{(m)}$.

Clearly, $\gamma = \beta$, and by (10.8),

$$h(z) = z^{\beta} G_{m, C}(\xi^{(1)} \zeta, \ldots, \xi^{(m)} \zeta) = z^{\beta} g(z).$$

Thus, $G_{m, C}(q_{j_1}, q_{j_2}, \ldots, q_{j_m})$ belongs to $P_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n)$.

Summing up, we obtain

$$\sum_{m \geq 2} \sum_{\mu_j + \mu_j + \ldots + \mu_{jm} = \mu_k} G_{m, C}(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k - \chi_k \in P_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n).$$

Applying $Z_A$ to this element and using Lemma 10.7, we have $q_k \in P_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n)$.

By the Induction Principle, $q_k \in P_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n)$ for all $k \in \mathbb{N}$.

Now that all $q_k$’s are real-valued, we can replace $G_{m, C}$ in (10.15) with $G_m$ and obtain (8.10). This completes the proof of part (i).

The proof of part (ii) is similar by neglecting the terms $\chi_k$’s. □
In the above proof of Theorem 10.8, formulas (8.10) and (10.15) are the same. However, formula (10.15) is preferred in manipulating the complex powers in order to perform the operator $Z_A$.

In Theorems 10.5 and 10.8, the functions in the asymptotic expansions are still expressed by the use of complex numbers. Below, we will remove such expressions and write the results in terms of real-valued functions only.

**Definition 10.9.** Let $X$ be a linear space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

(i) A function $g : \mathbb{R} \rightarrow X$ is an $X$-valued $S$-polynomial if it is a finite sum of the functions in the set

$$\left\{ t^m \cos(\omega t)Z, t^m \sin(\omega t)Z : m \in \mathbb{Z}_+, \omega \in \mathbb{R}, Z \in X \right\}.$$ 

(ii) Denote by $F_0(X)$, respectively, $F_1(X)$ the set of all $X$-valued polynomials, respectively, $S$-polynomials.

By the virtue of Lemma 10.2(i) and Euler’s formula, one has

$$F_E(0, \mathbb{C}^n, \mathbb{R}^n) = F_1(\mathbb{R}^n). \quad (10.16)$$

Let $(X, \| \cdot \|_X)$ be a normed space, and $(\gamma_k)_{k=1}^\infty$ be the same as in Definition 3.8(i) we define the asymptotic expansion

$$g(t) \sim \sum_{k=1}^\infty \widehat{g}_k(t)e^{-\gamma_k t}, \text{ where } \widehat{g}_k \in F_1(X) \text{ for } k \in \mathbb{N}, \quad (10.17)$$

in the same way as (3.21)–(3.22).

Therefore, one can use relation (10.16) and asymptotic expansion (10.17) to restate Theorem 10.5 as the following.

**Theorem 10.10.** Let $(\mu_k)_{k=1}^\infty$ be the same as in Assumption 7.3. If $f(t)$ has the following asymptotic expansion

$$f(t) \sim \sum_{k=1}^\infty \widehat{f}_k(t)e^{-\mu_k t}, \text{ where } \widehat{f}_k \in F_1(\mathbb{R}^n) \text{ for } k \in \mathbb{N}, \quad (10.18)$$

then the solution $y(t)$ admits the asymptotic expansion

$$y(t) \sim \sum_{k=1}^\infty \widehat{q}_k(t)e^{-\mu_k t}, \text{ with } \widehat{q}_k \in F_1(\mathbb{R}^n) \text{ for } k \in \mathbb{N}. \quad (10.19)$$

Note that even if $\widehat{f}_k \in F_0(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ in (10.18) we can only conclude $\widehat{q}_k \in F_1(\mathbb{R}^n)$ in (10.19).

We turn to Theorem 10.8 now. We will characterize the classes $F_m(k, 0, \mathbb{C}^n, \mathbb{R}^n)$ more explicitly using real-valued functions only.

**Definition 10.11.** Given integers $k \geq m \geq 0$. Define the class $P^m_{\cdot k}(k, \mathbb{R}^n)$ to be the collection of functions which are the finite sums of the following functions

$$z = (z_-, z_0, \ldots, z_k) \in (0, \infty)^{k+2} \mapsto z^\alpha \prod_{j=0}^k \sigma_j(\omega_j z_j)\xi, \quad (10.20)$$
where $\xi \in \mathbb{R}^n$, $\alpha \in \mathcal{E}(m, k, 0) \cap \mathbb{R}^{k+2}$, $\omega_j$’s are real numbers, and, for each $j$, either $\sigma_j = \cos$ or $\sigma_j = \sin$.

Define the class $\mathcal{P}_m^0(k, \mathbb{R}^n)$ to be the subset of $\mathcal{P}_m^1(k, \mathbb{R}^n)$ when all $\omega_j$’s in (10.20) are zero.

The results with the classes $\mathcal{P}_m^0(k, \mathbb{R}^n)$, for a real diagonalizable matrix $A$, were established in [10]. We focus on the classes $\mathcal{P}_m^1(k, \mathbb{R}^n)$ here.

Note in (10.20) that $\text{Re}(\alpha_1) = \text{Re}(\alpha_0) = \ldots = \text{Re}(\alpha_m) = 0$. Hence, the class $\mathcal{P}_m^1(k, \mathbb{R}^n)$ is related to $\mathcal{P}_m(k, 0, \mathbb{C}^n, \mathbb{R}^n)$.

Let $m \in \mathbb{Z}_+$, $k \geq m, -1 \leq j \leq k$ and $\omega \in \mathbb{R}$. For $\xi = x + iy \in \mathbb{C}^n$ with $x, y \in \mathbb{R}^n$, one has

$$L_j(t)^\omega \xi + L_j(t)^{-\omega} \bar{\xi} = 2(\cos(\omega L_{j+1}(t))x - \sin(\omega L_{j+1}(t))y).$$

Consequently, one can prove by induction that if $p \in \mathcal{P}_m(k, 0, \mathbb{C}^n, \mathbb{R}^n)$ then

$$p \circ \widehat{\mathcal{L}}_k = q \circ \widehat{\mathcal{L}}_{k+1}$$

for some $q \in \mathcal{P}_m(k+1, \mathbb{R}^n)$. (10.21)

In particular,

$$q \in \mathcal{P}_m^1(k, \mathbb{R}^n)$$

provided $p(z) = \sum z^n \xi_\alpha$, where $\alpha = (\alpha_{-1}, \alpha_0, \ldots, \alpha_k)$ with $\text{Im}(\alpha_k) = 0$.

(For, there is no term $L_{k+1}(t)^\alpha$ in $p \circ \widehat{\mathcal{L}}_k(t)$, and there is no term $L_k(t)^\omega$ in $p \circ \widehat{\mathcal{L}}_k(t)$ to contribute to $\cos(\omega L_{j+1}(t))$ and $\sin(\omega L_{j+1}(t))$ in $q \circ \widehat{\mathcal{L}}_k(t)$.)

We now observe that

$$\cos(\omega L_j(t)) = \frac{1}{2} (e^{i \omega L_j(t)} + e^{-i \omega L_j(t)}) = \frac{1}{2} (L_{j-1}(t)^\omega + L_{j-1}(t)^{-\omega}),$$

and, similarly,

$$\sin(\omega L_j(t)) = \frac{1}{2i} (L_{j-1}(t)^i - L_{j-1}(t)^{-i}).$$

Therefore,

$$\cos(\omega L_j(t)) = g(\widehat{\mathcal{L}}_k(t)) \quad \text{and} \quad \sin(\omega L_j(t)) = h(\widehat{\mathcal{L}}_k(t))$$

where

$$g(z) = \frac{1}{2} (z^\omega + z^{-\omega}) \quad \text{and} \quad h(z) = \frac{1}{2i} (z^{i} - z^{-i}).$$

(10.24)

Obviously,

$$g, h \in \mathcal{P}_m(k, 0, \mathbb{C}, \mathbb{R}).$$

(10.25)

Using properties (10.23), (10.24), (10.25) and the same arguments in Theorem 10.8 to prove $\mathcal{G}_{m, \mathbb{C}}(q_{j_1}, q_{j_2}, \ldots, q_{j_m})$ belongs to $\mathcal{P}_0(n_k, -\mu_k, \mathbb{C}^n, \mathbb{R}^n)$, one can verify that if $p \in \mathcal{P}_m^1(k, \mathbb{R}^n)$ then

$$p \circ \widehat{\mathcal{L}}_k = q \circ \widehat{\mathcal{L}}_k$$

for some $q \in \mathcal{P}_m(k, 0, \mathbb{C}^n, \mathbb{R}^n)$. (10.26)

More specifically,

$$q(z) = \sum z^n \xi_\alpha,$$

where $\alpha = (\alpha_{-1}, \alpha_0, \ldots, \alpha_k)$ with $\text{Im}(\alpha_k) = 0$. (10.27)

The last condition is due to the fact that the functions $\cos(\omega L_k(t))$ and $\sin(\omega L_k(t))$ can be converted via (10.23) and (10.24), when $j = k$, using the functions of the variable $z_{k-1}$.

Let $m_\ast$, $(\gamma_k)_{k=1}^\infty$ and $(n_k)_{k=1}^\infty$ be the same as in Definition 3.10(1). We say a function $g : (T, \infty) \to \mathbb{R}^n$, for some $T \in \mathbb{R}$, has an asymptotic expansion

$$g(t) \sim \sum_{k=1}^\infty \hat{g}_k(\widehat{\mathcal{L}}_{n_k}(t)) L_{m_\ast}(t)^{-\gamma_k}$$

where $\hat{g}_k \in \mathcal{P}_{m_\ast}^1(n_k, \mathbb{R}^n)$ for $k \in \mathbb{N}$, (10.28)
If, for each $N \in \mathbb{N}$, there is some $\mu > \gamma_N$ such that
\[
|g(t) - \sum_{k=1}^{N} \tilde{g}_k(\mathcal{L}_{n_k}(t)) L_{m_*}(t)^{-\gamma_k}| = O(L_{m_*}(t)^{-\mu}).
\]

We can now restate Theorem 10.8 using the class $\mathcal{P}_m^1(k, \mathbb{R}^n)$ and the expansion form (10.28) as the following.

**Theorem 10.12.** Given $m_* \in \mathbb{Z}_+$. Let $(\mu_k)_{k=1}^{\infty}$ and $(n_k)_{k=1}^{\infty}$ be the same as in Assumption 3.1 if $m_* = 0$, and be the same as in Assumption 9.7 if $m_* \geq 1$. If $f(t)$ has the asymptotic expansion
\[
f(t) \sim \sum_{k=1}^{\infty} \tilde{p}_k(\mathcal{L}_{n_k}(t)) L_{m_*}(t)^{-\mu_k}, \quad \text{where } \tilde{p}_k \in \mathcal{P}_m(n_k, \mathbb{R}^n) \text{ for } k \in \mathbb{N},
\]
then the solution $y(t)$ admits the asymptotic expansion
\[
y(t) \sim \sum_{k=1}^{\infty} \tilde{q}_k(\mathcal{L}_{n_k}(t)) L_{m_*}(t)^{-\mu_k}, \quad \text{with } \tilde{q}_k \in \mathcal{P}_m(n_k, \mathbb{R}^n) \text{ for } k \in \mathbb{N}.
\]

**Proof.** For each $k \in \mathbb{N}$, thanks to (10.26), we have $\tilde{p}_k(\mathcal{L}_{n_k}(t)) = \tilde{p}_k(\mathcal{L}_{n_k}(t))$ for some $\tilde{p}_k \in \mathcal{P}_m(n_k, 0, \mathbb{C}^n, \mathbb{R}^n)$. Applying Theorem 10.8, we obtain the asymptotic expansion
\[
y(t) \sim \sum_{k=1}^{\infty} \tilde{q}_k(\mathcal{L}_{n_k}(t)) L_{m_*}(t)^{-\mu_k}, \quad \text{where } \tilde{q}_k \in \mathcal{P}_m(n_k, 0, \mathbb{C}^n, \mathbb{R}^n) \text{ for } k \in \mathbb{N}.
\]

Thanks to property (10.21), we have $\tilde{q}_k(\mathcal{L}_{n_k}(t)) = \tilde{q}_k(\mathcal{L}_{n_k+1}(t))$ for some $\tilde{q}_k \in \mathcal{P}_m(n_k + 1, \mathbb{R}^n)$.

We examine $\tilde{q}_k$ more closely. In fact, $\tilde{p}_k$ has the representation as in (10.27) with $\alpha = (\alpha_{-1}, \ldots, \alpha_{n_k})$ satisfying $\text{Im}(\alpha_{n_k}) = 0$. By the recursive formula (8.10) for $m_* = 0$ or (9.6) for $m_* \geq 1$, each $\tilde{q}_k$ has the same property. By the virtue of (10.22), we have $\tilde{q}_k \in \mathcal{P}_m^1(n_k, \mathbb{R}^n)$, and hence, obtain (10.30). □

Similar to the remark after Theorem 10.10, even if $\tilde{p}_k \in \mathcal{P}_m^0(n_k, \mathbb{R}^n)$ in (10.29), we, in general, can only have $\tilde{q}_k \in \mathcal{P}_m^1(n_k, \mathbb{R}^n)$ in (10.30).

Our general results – Theorems 7.7, 8.3, 9.3, 10.5, 10.8, 10.10 and 10.12 – can have more specific forms in many different situations, see, for instance, [10] Example 5.9–Example 5.12. Below, we demonstrate the last theorem with some examples (for the systems in $\mathbb{R}^n$.)

**Example 10.13.** If, for large $t > 0$,
\[
f(t) = \frac{\cos(\alpha t)(\ln t)(\ln \ln t)^{-1/3} \xi}{t^{m}} \quad \text{for some } m \in \mathbb{N} \text{ and } \xi \in \mathbb{R}^n,
\]
then the solution $y(t)$ admits the asymptotic expansion
\[
y(t) \sim \sum_{k=0}^{\infty} \frac{q_k(t)}{t^{m+k}},
\]
where $q_k(t) = \tilde{q}_k(\mathcal{L}_2(t))$ with $\tilde{q}_k \in \mathcal{P}_0^1(2, \mathbb{R}^n)$. Roughly speaking, the functions $q_k(t)$’s are composed by
\[
\cos(\omega L_j(t)), \sin(\omega L_j(t)), L_\ell(t)^\alpha,
\]
(10.31)
for \( j = 0, 1, 2 \) and \( \ell = 1, 2 \), with some real numbers \( \omega \)'s and \( \alpha \)'s.

**Example 10.14.** If, for large \( t > 0 \),

\[
f(t) = \frac{\cos(2t) \sin(3 \ln t) (\ln \ln t)^2 \sin(5 \ln \ln t)}{(\ln t)^{1/2}} \xi \quad \text{for some} \quad \xi \in \mathbb{R}^n,
\]

then the solution \( y(t) \) admits the asymptotic expansion

\[
y(t) \sim \sum_{k=1}^{\infty} q_k(t) (\ln t)^{k/2},
\]

where, roughly speaking, \( q_k(t) \)'s are functions composed by the functions in (10.31) for \( j = 0, 1, 2 \) and \( \ell = 2, 3 \).

**Appendix A.**

*Proof of Lemma 3.3.* This proof follows [27, Lemmas 2.3 and A.1].

Suppose the conclusion is not true. Denote \( s_\lambda = p_\lambda - q_\lambda \). Then \( s_\lambda \neq 0 \) for some \( \lambda \in S \).

We write each \( \lambda \in S \) as \( \lambda = \mu + i\omega_\lambda \) with \( \omega_\lambda \in \mathbb{R} \). We have from (3.10) that

\[
\lim_{t \to \infty} \sum_{\lambda \in S} s_\lambda(t) e^{i\omega_\lambda t} = 0.
\]

Take \( d_* \) to be the maximum of the degrees of the polynomials \( s_\lambda \)'s, for \( \lambda \in S \). For \( \lambda \in S \), write

\[
s_\lambda(t) = t^{d_*} \xi_\lambda + \text{lower powers of } t, \quad \text{with } \xi_\lambda \in X.
\]

Note that

\[
\exists \lambda \in S : \xi_\lambda \neq 0.
\]

Dividing (A.1) by \( t^{d_*} \) yields

\[
\lim_{t \to \infty} \sum_{\lambda \in S} \xi_\lambda e^{i\omega_\lambda t} = 0.
\]

Let \( \{Y_j : 1 \leq j \leq N\} \), for some \( N \geq 1 \), be a basis of the linear span of \( \{\xi_\lambda : \lambda \in S\} \).

Write \( \xi_\lambda = \sum_{j=1}^{N} c_{\lambda,j} Y_j \) where \( c_{\lambda,j} \) are complex numbers. For each \( j = 1, 2, \ldots, N \), we have from (A.3) that

\[
\lim_{t \to \infty} \sum_{\lambda \in S} c_{\lambda,j} e^{i\omega_\lambda t} = 0.
\]

**Claim.** \( c_{\lambda,j} = 0 \) for all \( \lambda \in S \).

Accepting this Claim momentarily, we then have, for each \( \lambda \in S \), that \( c_{\lambda,j} = 0 \) for all \( j \). Hence, \( \xi_\lambda = 0 \) for all \( \lambda \in S \), which contradicts (A.2). Therefore, the conclusion of Lemma 3.3 must be true.

Fix an integer \( j \in [1, N] \), we prove the Claim from (A.4).

Let \( \omega_* = 1 + \max \{|\omega_\lambda| : \lambda \in S\} \). Multiplying (A.3) by \( e^{i\omega_* t} \), we rephrase the problem as the following. There are complex numbers \( a_k \)'s and strictly increasing positive numbers \( r_k \)'s, for \( 1 \leq k \leq m \), such that

\[
\lim_{t \to \infty} \sum_{k=1}^{m} a_k e^{i r_k t} = 0, \quad \text{(A.5)}
\]

\[
\{c_{\lambda,j} : \lambda \in S\} = \{a_k : 1 \leq k \leq m\}, \quad \text{(A.6)}
\]
and

\[ \{ \omega_\lambda + \omega_k : \lambda \in S \} = \{ r_k : 1 \leq k \leq m \}. \]

We will establish that

\[ a_k = 0 \text{ for all } k = 1, 2, \ldots, m. \] (A.7)

In fact, we consider a more general statement, namely,

\((H_m)\) “If \((A.5)\) holds for complex numbers \(a_k\)’s and strictly increasing positive numbers \(r_k\)’s, for \(1 \leq k \leq m\), then \((A.7)\) is true.”

We prove, by induction, that \((H_m)\) is true for all \(m \in \mathbb{N}\).

Suppose \(m = 1\) and \((A.5)\) holds. Then it is clear that \(a_1 = 0\). Therefore, \((H_1)\) is correct.

Let \(m \geq 1\). Suppose \((H_m)\) holds true. Now, assume

\[ \lim_{t \to \infty} \sum_{k=1}^{m+1} a_k e^{ir_k t} = 0, \] (A.8)

for some complex numbers \(a_k\)’s and strictly increasing positive numbers \(r_k\)’s, for \(1 \leq k \leq m + 1\).

Integrating the sum in \((A.8)\) from \(t\) to \(t + 2\pi/r_{m+1}\) and taking \(t \to \infty\) give

\[ \lim_{t \to \infty} \left( \sum_{j=1}^{m} \frac{a_j}{ir_j} e^{ir_j t} (e^{2\pi i r_j t/r_{m+1}} - 1) + 0 \right) = 0. \]

By the Induction Hypothesis \((H_m)\), we have

\[ \frac{a_k}{ir_k} (e^{2\pi i r_k t/r_{m+1}} - 1) = 0 \text{ for } 1 \leq k \leq m. \] (A.9)

Note in \((A.9)\) that \(e^{2\pi i r_k t/r_{m+1}} \neq 1\). Hence, \(a_k = 0\) for \(1 \leq k \leq m\). Returning to \((A.8)\), we then have

\[ \lim_{t \to \infty} (a_{m+1} e^{ir_{m+1} t}) = 0, \]

which yields \(a_{m+1} = 0\). Therefore, we obtain \((H_{m+1})\).

By the Induction Principle, the statement \((H_m)\) is true for all \(m \in \mathbb{N}\).

Consequently, we have \((A.7)\), which, together with \((A.6)\), implies that the Claim is true.

The proof of Lemma 3.3 is complete. \(\Box\)

**Proof of Proposition 3.4.** Because \(X\) is not assumed to be a normed space, we cannot apply Lemma 3.3 straightforwardly.

**Part 1.** Suppose \(g\) has two representations

\[ g(t) = \sum_{\lambda \in S} p_\lambda(t) e^{\lambda t} \text{ and } g(t) = \sum_{\lambda \in S} q_\lambda(t) e^{\lambda t}, \text{ for } t \in \mathbb{R}. \] (A.10)

By subtracting the above two formulas of \(g(t)\), it suffices to assume now \(g = 0\) has the form \((3.7)\), and then prove that \(p_\lambda = 0\) for all \(\lambda \in S\).

Suppose \(p_\lambda(t) = \sum_{j=0}^{d_\lambda} t^j \xi_{\lambda,j}\) for \(\lambda \in S\), and some vectors \(\xi_{\lambda,j}\)’s in \(X\). Let \(E\) be the finite dimensional linear subspace of \(X\) spanned by the vectors \(\xi_{\lambda,j}\)’s for \(\lambda \in S\) and \(0 \leq j \leq d_\lambda\). Let \(\{Y_k : 1 \leq k \leq N\}\) be a basis of \(E\). For any \(\lambda \in S\) and integer \(j \in [1, d_\lambda]\), one has \(\xi_{\lambda,j} = \sum_{k=1}^{N} c_{\lambda,j,k} Y_k\) for some complex numbers \(c_{\lambda,j,k}\). We then have, for all \(t \in \mathbb{R}\),

\[ 0 = \sum_{\lambda \in S} \sum_{j=1}^{d_\lambda} e^{\lambda t} j \xi_{\lambda,j} = \sum_{\lambda \in S} \sum_{j=1}^{d_\lambda} \sum_{k=1}^{N} e^{\lambda t} j c_{\lambda,j,k} Y_k = \sum_{k=1}^{N} \left( \sum_{\lambda \in S} Q_{\lambda,k}(t) e^{\lambda t} \right) Y_k, \]
where \( Q_{\lambda,k}(t) = \sum_{j=1}^{d_{\lambda}} t^j c_{\lambda,j,k} \). This implies, for each \( k \in [1, N] \) and all \( t \in \mathbb{R} \), that
\[
\sum_{\lambda \in S} Q_{\lambda,k}(t)e^{\lambda t} = 0. \tag{A.11}
\]

Note that each \( Q_{\lambda,k} \) is a polynomial from \( \mathbb{R} \) to \( \mathbb{C} \). Suppose \( \text{Re} \, S = \{ \mu_1 > \mu_2 > \ldots > \mu_s \} \).

We decompose
\[
S = \bigcup_{m=1}^{s} S_m \quad \text{with} \quad S_m = \{ \lambda \in S : \text{Re} \, \lambda = \mu_m \}.
\]

We then write the sum in (A.11) as
\[
\sum_{\lambda \in S} = \sum_{\lambda \in S_1} + \sum_{\lambda \in S_2} + \ldots + \sum_{\lambda \in S_m}.
\]

Dividing (A.11) by \( e^{\mu_1 t} \) and taking \( t \to \infty \) give
\[
\lim_{t \to \infty} e^{-\mu_1 t} \sum_{\lambda \in S_1} Q_{\lambda,k}(t)e^{\lambda t} = 0.
\]

Applying Lemma (3.3) to \( X = \mathbb{C} \), \( S = S_1 \) and \( p_{\lambda} = Q_{\lambda,k} \), \( \mu = \mu_1 \), \( q_{\lambda} = 0 \), we have the polynomials \( Q_{\lambda,k} = 0 \) for all \( \lambda \in S_1 \). With this fact, the sum in (A.11) is reduced to \( \sum_{\lambda \in S_2} + \ldots + \sum_{\lambda \in S_m} \). Repeating the above arguments with \( S_2 \) replacing \( S_1 \), we obtain \( Q_{\lambda,k} = 0 \) for all \( \lambda \in S_2 \). By this recursive reasoning, we obtain \( Q_{\lambda,k} = 0 \) for all \( \lambda \in S_m \) for \( 1 \leq m \leq s \), that is, \( Q_{\lambda,k} = 0 \) for all \( \lambda \in S \). Because each \( Q_{\lambda,k}(t) \) is a polynomial, this yields that its coefficients \( c_{\lambda,j,k} \)'s, for \( 0 \leq j \leq d_{\lambda} \), are zeros. Now that \( c_{\lambda,j,k} = 0 \) for all \( \lambda, j, k \), we infer \( \xi_{\lambda,j} = 0 \) for all \( \lambda, j \), and hence \( p_{\lambda} = 0 \) for all \( \lambda \).

Part 2. Now, assume \( g \) has two representations
\[
g(t) = \sum_{\lambda \in S'} p_{\lambda}(t)e^{\lambda t} \quad \text{and} \quad g(t) = \sum_{\lambda \in S''} q_{\lambda}(t)e^{\lambda t}, \quad \text{for} \ t \in \mathbb{R},
\]

with \( p_{\lambda} \neq 0 \) for all \( \lambda \in S' \) and \( q_{\lambda} \neq 0 \) for all \( \lambda \in S'' \).

Set \( S = S' \cup S'' \), define \( p_{\lambda} = 0 \) for \( \lambda \in S \setminus S' \), and \( q_{\lambda} = 0 \) for \( \lambda \in S \setminus S'' \). Then we have (A.10). Thanks to Part 1, \( p_{\lambda} = q_{\lambda} \) for all \( \lambda \in S \). Thus, \( S' = S'' = S \).

\[ \square \]

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**Department of Mathematics and Statistics, Texas Tech University, 1108 Memorial Circle, Lubbock, TX 79409–1042, U. S. A.**

*Email address: luang.hoang@ttu.edu*