Self-gravitational solitary waves in astrophysical compact objects

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The condition for the existence of self-gravitational solitary waves (SGSWs), and their polarity in any astrophysical compact object are theoretically found for the first time. The pseudo-potential approach, which is valid for arbitrary amplitude SGSWs, is employed. The general analytical results are applied in white dwarfs and neutron stars to identify the basic features (polarity, amplitude, and width) of the SGSWs formed in them. It found for the first time that the SGSWs exist with negative self-gravitational potential in perturbed states of white dwarfs and neutron stars. It is also estimated that for their typical degenerate plasma parameters, the amplitude and the width of the SGSWs (moving with the speed 2 cm/s) in white dwarfs are \( \sim -1.5 \) ergs/gm and \( \sim 50 \) cm, respectively, and those of the SGSWs (moving with the speed \( \sim 27 \) m/s) in neutron stars are \( \sim -7.76 \times 10^4 \) Joules/kg, and 5.5 km, respectively.

The astrophysical compact objects (ACOs) are completely different from other terrestrial bodies because of their (ACOs') extremely low temperate and extremely high density. They are, in fact, degenerate quantum plasma systems containing degenerate light particle species (viz. electron species with/without positron or/and quark (non-zero mass \([13]\)) species depending on the class of ACOs under consideration), degenerate heavy particle (compared to electron mass) species \([1, 3, 14, 15]\) (viz. \( ^{3}He \) or/and \( ^{16}O \) or/and \( ^{12}C \) or/and \( ^{16}O \) species with/without proton or/and neutron species depending on the class of ACOs under consideration), and non-degenerate species of low dense heavy elements \([3, 16, 17]\) (viz. \( ^{56}Fe \) or/and \( ^{96}Rd \) or/and \( ^{92}Mo \) depending again on the class of ACOs under consideration).

The degeneracy of the light and heavy particle species (viz. electron or proton or neutron or \( ^{3}He \) or \( ^{16}O \) or \( ^{12}C \) or \( ^{16}O \) species) arises due to Heisenberg's uncertainty principle, \( \Delta p \Delta x \geq \hbar/2 \) (where \( \hbar \) is the reduced Planck constant, \( \Delta p \) is the uncertainty in momentum of the species, and \( \Delta x \) is the uncertainty in position of the species). This indicates that the momenta of highly compressed particles are extremely uncertain, since the particles are located in a very confined space. Therefore, even though the degenerate plasma is very cold, the plasma particles must move very fast on average, and give rise to a very high pressure (known as degenerate pressure), which does not depend on thermal temperature, but on degenerate particle number density. This leads us to a conclusion that in order to compress degenerate particles to an object into a very small space, a tremendous force (which is the self-gravitational force in ACOs like white dwarfs, neutron stars, black holes, etc. \([2]\)) is required to control its particles' momenta. Thus, if any ACO is disturbed/perturbed due any reason (viz. merging \([18]\) of two small ACOs, fragmentation \([3]\) of a large ACO, gravitational interaction \([3]\) among neighboring ACOs, etc.), the self-gravitational perturbation mode [in which the compression (rarefaction) is provided by the self-gravitational attraction pressure (degenerate pressure)] is developed. The propagation of this self-gravitational perturbation (SGPM) cannot be usually explained by linear analysis. Therefore, for the first time a theoretical investigation is made on the nonlinear propagation of this SGPM by employing the pseudo-potential approach, which is valid for arbitrary amplitude SGPM. It is first time shown here that the SGPM nonlinearly propagate as self-gravitational solitonic potential signals (SGSWs) with negative potential in the ACOs like white dwarfs and neutron stars.

We consider a general degenerate quantum plasma system like an ACO containing arbitrary number of degenerate species \( s \) of non-inertial light particles \([3, 8]\) (viz. electrons or/and positrons or/and non-zero mass quarks \([13]\), etc.), degenerate species \( j \) of inertial heavy (compared to electron mass) particles \([2, 3, 14, 15]\) (viz. \( ^{3}He \) or \( ^{16}O \) or \( ^{12}C \) or \( ^{16}O \) or/and protons or/and neutrons, etc.), and non-degenerate species \( h \) of inertial heavy elements \([2, 16, 17]\) (viz. \( ^{56}Fe \) or/and \( ^{92}Rd \) or/and \( ^{92}Mo \), etc.). The heavy elements, viz. \( ^{56}Fe \), \( ^{92}Rd \), \( ^{92}Mo \), etc., are assumed to be non-degenerate just because of their low density and high mass.

The nonlinear dynamics of the SGPM, whose phase speed is much smaller than \( C_{e} \) (where \( C_{e} = \sqrt{\hbar n_{e0}/m_{e}} \) in which \( m_{e} \) is the electron mass, and \( n_{e0} \) is the equilibrium degenerate electron number density) is described by

\[
\frac{\partial \psi}{\partial x} = -\frac{3}{2} \frac{\partial^{2} \rho_{s}^{2}}{\partial x^{2}},
\]

\[
\frac{\partial \rho_{s}}{\partial t} + \frac{\partial \rho_{s}}{\partial x} (\rho_{s} u_{j}) = 0,
\]

\[
\frac{\partial u_{j}}{\partial t} + u_{j} \frac{\partial u_{j}}{\partial x} = -\frac{\partial \psi}{\partial x} - \frac{3}{2} \frac{\partial \rho_{s}^{2}}{\partial x},
\]

\[
\frac{\partial^{2} \psi}{\partial x^{2}} = \sum_{s} \sigma_{s} (\rho_{s} - 1) + \sum_{j} \mu_{j} (\rho_{j} - 1),
\]

where \( \rho_{s} \) (\( \rho_{j} \)) is the number density of the degenerate light (heavy) particle species \( s \) (\( j \)), and is normalized by \( \rho_{s0} \) (\( \rho_{j0} \)). 

$u_j$ is the degenerate fluid speed of the species $j$, and is normalized by $C_q$ in which $C_q = (\sqrt{\pi} h n_{0}^{1/3}/m_q)$, and $q$ represents the heaviest particle species among all the degenerate particle species (e.g. $m_q$ is the mass of $^{12}$C for white dwarfs and of $^3$He for neutron stars); $\psi$ is the self-gravitational potential, and is normalized by $C_q^2 \alpha_s' = (m_q/m_s)^2 (n_{0}/n_{00})^{2/3}$, $\beta_j = (m_j/m_s)^2 (n_{0}/n_{00})^{2/3}$, in which $m_s$ ($m_j$) is the mass of the degenerate particle species $s$ ($j$); $n_{00}$ ($n_{0}$) is the equilibrium number density of the degenerate particle species $s$ ($j$); $\tau_q$ is the time variable normalized by $\tau_q = (4\pi G \rho_{00})^{-1/2}$; $x$ is the space variable normalized by $L_q = C_q \tau_q$; $\sigma_s = \rho_{00}/\rho_{0}$, and $\mu_j = \rho_{00}/\rho_0$ in which $\rho_{0}$ is the density of the degenerate species $q$ (e.g. $^{12}$C for white dwarfs and $^3$He for neutron stars). We note that $j = h$ when $\beta_j = 0$. The typical approximate values of the physical quantities, viz. $m_q, m_h, \rho_{00}, \rho_{0}, \rho_{s}, C_q, \omega_{j}, \omega_{q}$ corresponding to white dwarfs and neutron stars are tabulated in Table I. We note that $\rho_{00}$ in both white dwarfs and neutron stars are determined by the charge neutrality condition, and that $m_{n0} \simeq m_p$.

We now study the possibility for the existence of the SGSSs, and their basic features by the pseudo-potential approach [13, 20], which is valid for arbitrary amplitude SGSSW. To do so, we first assume that all dependent variables in (1)-(4) depend only on a single variable $\xi = x - Mt$, where $M$ is the speed of the the SGSSWs, and is normalized by $C_q$ and $\xi$ is normalized by $L_q$. On using (1)-(4), this assumption leads us to express $\rho_{s}$ and $\rho_{j}$ (under the steady state condition) as

$$\rho_{s} = \left(1 - \frac{2}{3} \alpha_s \psi\right)^{\frac{1}{3}}, \tag{5}$$

$$\rho_{j} = \mu_j \left[1 - b_j + b_j \sqrt{1 - \frac{2 \psi}{M^2 b_j^2}}\right]^{-1}. \tag{6}$$

$$\frac{\partial^2 \psi}{\partial \xi^2} = \sum_s \sigma_s (\rho_{s} - 1) + \sum_j \mu_j (\rho_{j} - 1), \tag{7}$$

where $\alpha_s = 1/\alpha_s'$, and $b_j = (1 - \beta_j/M^2)/(1 + 5\beta_j/3)$. We note that the species $j$ includes the species $q$ and $h$, and that $\rho_j = \rho_q$ when $j$ represents the heaviest degenerate species (e.g. $^{12}$C for white dwarfs [14, 15] and $^3$He for neutron stars [2, 3]), and $\rho_j = \rho_h$ when $j$ represents non-degenerate species ($\beta_j = 0$) of heavy elements (e.g. $^{56}$Fe in white dwarfs \[2, 17\] and $^{85}$Rd in neutron stars [2, 16]).

We first substitute (5) and (6) into (7), and then integrate the resulting equation over $\xi$ to obtain an energy integral in the form \[19, 20\]

$$\frac{1}{2} \left(\frac{d\psi}{d\xi}\right)^2 + V(\psi) = 0. \tag{8}$$

This energy integral represents the motion of a pseudo particle of unit mass moving with pseudo-speed $d\psi/d\xi$ in a pseudo-potential $V(\psi)$, where $\psi$ is the pseudo-position and $\xi$ is the pseudo-time [19]. The pseudo-potential $V(\psi)$ appeared in (8) is given by

$$V(\psi) = - \sum_s \delta_s \left[ \frac{3}{30 \alpha_s} - \psi - \frac{3}{30 \alpha_s} \left(1 - \frac{2}{3} \alpha_s \psi\right)^{\frac{2}{3}} \right] - \sum_j \mu_j \left[ b_j, M^2 - \psi - b_j, M^2 Y_j - (1 - b_j), M^2 Z_j \right] \tag{9}$$

where $b_j = (1 - \beta_j/M^2)/(1 + 5\beta_j/3)$, $Y_j = [1 - 2\psi/(b_j^2, M^2)]^{1/2}$, $Z_j = \ln(1 - b_j + b_j Y_j)$, and the integration constant is chosen in such a way that $V(\psi) = 0$ at $\psi = 0$. We note that the energy integral defined by (8) with the pseudo-potential $V(\psi)$ defined by (9) is valid for any ACO containing arbitrary number of degenerate species $s$ of light particles (viz. electrons, positrons, etc.), arbitrary number of degenerate species $j$ of heavy particles (viz. protons, neutrons, $^3$He nuclei, $^{12}$C nuclei, etc.), and arbitrary number of non-degenerate species $h$ ($\beta_h = 0$) of heavy elements (viz. $^{56}$Fe, $^{85}$Rd, $^{96}$Mo, etc.). The Taylor series expansion of $V(\psi)$ around $\psi = 0$ is

$$V(\psi) = \frac{1}{2} C_2 \psi^2 + \frac{1}{2} C_3 \psi^3 + \cdots, \tag{10}$$

where

$$C_2 = \sum_s \delta_s \alpha_s \sum_j \mu_j \left[ \frac{1 + \frac{2}{3} \beta_j}{M^2 - \beta_j} \right], \tag{11}$$

$$C_3 = -\frac{1}{3} \sum s \delta_s \alpha_s^2 \left[ \frac{1 - \beta_j}{M^2 - \beta_j} \frac{M^2 (3 + \frac{2}{3} \beta_j) - 2 \beta_j}{(M^2 - \beta_j)^3} \right]. \tag{12}$$

It is obvious from (10) that $V(0) = 0$ and $(dV/d\psi)|_{\psi=0} = 0$. This means that (i) the energy integral defined by (8) with the pseudo-potential $V(\psi)$

| Physical quantity | White dwarfs | Neutron stars |
|-------------------|-------------|---------------|
| $m_q$ (gm)        | 12$m_p$    | 4$m_p$        |
| $m_h$ (gm)        | 1.82 $\times$ 10$^3$ | 85$m_p$ |
| $\rho_{00}$ (gm cm$^{-3}$) | 9.10 $\times$ 10$^{11}$ | 1.67 $\times$ 10$^{13}$ |
| $\rho_0$ (gm cm$^{-3}$) | 0 | 1.53 $\times$ 10$^{14}$ |
| $\rho_{0}$ (gm cm$^{-3}$) | 6.35 $\times$ 10$^{10}$ | 3.17 $\times$ 10$^{15}$ |
| $C_q$ (em$^{-1}$) | 1.17 $\times$ 10$^6$ | 2.78 $\times$ 10$^6$ |
| $\omega_{j}$ (s$^{-1}$) | 2.31 $\times$ 10$^{-2}$ | 5.15 $\times$ 10$^2$ |
| $L_q$ (cm) | 5.07 $\times$ 10$^4$ | 5.40 $\times$ 10$^6$ |

$^a$ where $m_p = 1.67 \times 10^{-24}$ gm is the proton mass.
defined by \( \psi \) gives rise to the existence of SGSWs if and only if \( (\partial^2 U/\partial y^2)_{y=0} < 0 \), i.e. \( C_2 < 0 \), and thus \( C_2 |_{M=M_c} = 0 \) yields \( M_c \) (the critical value of \( M \) below which the SGSWs exist); (ii) the SGSWs exist with \( \psi > 0 \) if \( C_3 |_{M=M_c} > 0 \), and with \( \psi < 0 \) if \( C_3 |_{M=M_c} < 0 \).

To find the expression for \( M_c \) and to examine whether \( C_3 |_{M=M_c} > 0 \) or \( C_3 |_{M=M_c} < 0 \), for example an example, we first consider white dwarfs, which can be assumed to consist of degenerate species \( s \) of light particles like electrons, degenerate species \( q \) of heavy (compared to electron mass) particles like \( ^{12}\text{C} \), and non-degenerate species \( h \) of heavy element like \( ^{56}\text{Fe} \). Thus, using (11) and (12), the conditions for the existence of SGSWs in white dwarfs become

\[
M^2 < M_c^2 = \frac{1}{2\alpha_e \delta_e} \left[ \frac{\beta_1 + \sqrt{\beta_1^2 - 4\alpha_e \beta_q \delta_e \mu_h}}{\alpha_e \delta_e} \right], \quad (13)
\]

\[
C_3 |_{M=M_c} = C_3^e = -\frac{1}{9} \delta_e \alpha_e^2 - \frac{\mu_h}{M_c^2} \left[ 1 + \frac{5}{3} \beta_1 \right] \frac{[M_c^2 (3 + 2\beta_q) - 2\beta_1]}{(M_c^2 - \beta_1)^3}, \quad (14)
\]

where \( \beta_1 = 1 + \mu_h \beta_q (\alpha_e \delta_e + 5/3) \). It is obvious from (14) that we cannot analytically predict \( C_3^e < 0 \) for any possible value of \( \beta_q \) which is 0.29 for \( ^{12}\text{C} \). The numerical variation of \( C_3^e \) with \( \beta_q \) (for its range from 0 to 1) implies that \( C_3^e < 0 \) for \( \beta_q = 0 - 1 \). Similarly, we can also use (11) and (12) to verify that the condition (viz. \( M < M_c \) and \( C_3^e < 0 \)) for the existence of the SGSWs in neutron stars is satisfied. However, the expressions of \( M_c \) and \( C_3^e \) for neutron stars are more complex than those for white dwarfs. This is because the neutron stars can be assumed to consist of a degenerate species of light particles (viz. electrons), three degenerate species of heavy (compared to electron mass) particles (viz. protons, neutrons, light nuclei like \( ^4\text{He} \)), and a non-degenerate (\( \beta_h = 0 \)) species of heavy element like \( ^{16}\text{O} \). We note that in neutron stars \( j = q \) for \( ^4\text{He} \) species, and that \( \beta_j \) is 0.74, 0.93, and 0.97 for \( ^{16}\text{He} \), proton, and neutron species, respectively.

To examine the basic features (amplitude and width) of these SGSWs, we limit \( V(\psi) \) up to the terms containing \( \psi^3 \) (since \( \psi \ll 1 \) always valid because of its normalization by \( C_q^2 \), which is \( 1.36 \times 10^{12} \text{cm}^2 \text{s}^{-2} \) for white dwarfs, and \( 7.73 \times 10^{18} \text{cm}^2 \text{s}^{-2} \) for neutron stars). This limit allows us to express (8) as

\[
\frac{d\psi}{d\xi} = \psi \sqrt{-C_2 - C_3 \psi}.
\]

We now look for the solitonic signal solution of this first order ordinary nonlinear differential equation \( \psi \rightarrow 0 \) at \( \xi \rightarrow \pm \infty \), its stationary solitonic signal solution can be written as

\[
\psi = -\left( \frac{C_2}{C_3} \right) \text{Sech}^2(\sqrt{-C_2} \xi).
\]

This represents a solitonic signal solution of (15), and means that the SGSWs with \( \psi < 0 \) exist in the degenerate plasma system under consideration since \( C_2 < 0 \) and \( C_3 < 0 \). We have graphically shown the SGSWs that can be formed for the parameters corresponding to white dwarfs and neutron stars. These are displayed in figures 1 and 2, which clearly indicates that

FIG. 1: Showing the SGSWs formed in white dwarfs for \( M = 1.0 \times 10^{-6} \) (solid curve), \( M = 2.0 \times 10^{-6} \) (dotted curve), and \( M = 3.0 \times 10^{-6} \) (dash curve).

FIG. 2: (Color online) Showing the SGSWs formed in neutron stars for \( M = 1.0 \times 10^{-6} \) (solid curve), \( M = 2.0 \times 10^{-6} \) (dotted curve), and \( M = 3.0 \times 10^{-6} \) (dash curve).

the SGSWs exist with \( \psi < 0 \) in ACOs (like white dwarfs and neutron stars) and that the magnitude of the amplitude and the width of the SGSWs increase with the rise of \( M \) within the range of \( M_c < M < 0 \). We have also numerically solved (8) with the pseudopotential \( V(\psi) \) defined by (9) for the parameters given in Table I, and used in figures 1 and 2 and have found the existence of the SGSWs, which are completely identical to those shown in figures 1 and 2.

To summarize, we have considered a general self-gravitating degenerate quantum plasma system containing an arbitrary number of degenerate species of non-inertial light particles, (viz. electrons, positrons, non-zero mass quarks, etc.), arbitrary number of degenerate species of inertial heavy (compared to electron mass) particles (viz. protons, neutrons, nuclei of light elements like \( ^4\text{He} \), \( ^{12}\text{C} \), \( ^{16}\text{O} \), etc.), and an
arbitrary number of non-degenerate species of inertial heavy elements (viz. \(^{56}\)Fe, \(^{85}\)Rd, \(^{96}\)Mo, etc.), and have found the conditions for the existence of the SGSWs in ACOs. The pseudo-potential approach, which is valid for the arbitrary amplitude SGSWs, is used. The general analytical results have been applied in white dwarfs (containing a degenerate electron species \([3]\), a degenerate \(^{12}\)C species \([3, 15]\), and a non-degenerate \(^{56}\)Fe species \([17]\) and neutron stars (containing degenerate electron, proton, neutron, and \(^{4}\)He species \([3]\), and a non-degenerate \(^{55}\)Rd species \([10]\)) to identify the basic features (viz. polarity, amplitude, width, etc.) of the SGSWs, which are found to be formed in them. It is shown that the SGSWs exist with negative self-gravitational potential in both white dwarfs and neutron stars for \(M_e < M < 0\). It is observed that the speed \((V_0)\), magnitude of the amplitude (\(|\psi_m|\)) and the width (\(\delta\)) of the SGSWs (in both white dwarfs and neutron stars) increase with the rise of their normalized speed (\(M\)) within the range \(0 < M < M_e\). It has been carefully checked that the analysis for arbitrary amplitude SGSWs gives the same results as that for small amplitude SGSWs does since \(|\psi| < 1\) always valid because of its normalization by \(C_0^2\), which is \(1.36 \times 10^{12}\) cm\(^2\)/s\(^2\) for white dwarfs, and \(7.73 \times 10^{10}\) cm\(^2\)/s\(^2\) for neutron stars. It is also observed that the dimensional (non-normalized) values of \(V_0, \psi_m, \) and \(\delta\) in neutron stars are much higher than those in white dwarfs. It has been estimated that for \(M = 10^{-6}\) and for the parameters given in Table I, \(V_0, |\psi_m|, \) and \(\delta\) are \(\sim 2\) cm/s, \(\sim 1.5\) ergs/gm, and \(\sim 50\) cm, respectively, in white dwarfs, and are \(\sim 27\) m/s, \(\sim 7.76 \times 10^4\) Joules/kg, and 5.5 km, respectively, in neutron stars.

Recent discovery \([18]\) of the gravitational waves \([18, 21, 22]\) produced by merging of two black holes, and the results of the present investigation indicate that the signatures of the nonlinear structures like SGSWs exist in ACOs like white dwarfs and neutron stars. The SGSWs are associated with the perturbation mode generated in ACOs (viz. white dwarfs and neutron stars) due to the departure from their equilibrium states by any event (viz. merging of two small ACOs \([18, 21, 22]\) splitting up a large ACO according to the Chandrasekhar limit \([1, 2]\), gravitational interaction among nearby ACOs \([3]\), etc.) or by any other reasons.

The degenerate quantum plasma system considered here is generalized to an arbitrary number of species of degenerate light non-inertial particles, degenerate heavy (compared to electron mass) inertial particles, and non-degenerate inertial heavy nuclear species of heavy elements. The degenerate quantum plasma system is also very general from the view of arbitrary mass densities of all the species comprising the system. Therefore, the investigation presented here can be applied for any ACO, and can be very useful in understanding the basic features of nonlinear localized structures (viz. SGSWs) in any ACO (viz. white dwarfs, neutron stars, and black holes \([3]\)).

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